The group Aut and Out of the fundamental group of a closed Sol 3-manifold

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Abstract

Let $E$ be the fundamental group of a closed Sol 3-manifold. We describe the groups $\text{Aut}(E)$ and $\text{Out}(E)$. We first consider the case where $E$ is the fundamental group of a torus bundle, and then the case where $E$ is the fundamental group of a closed Sol 3-manifold which is not a torus bundle. The groups are described in terms of some iterated semi-direct products of well known groups, where one exception is for $\text{Out}(E)$ for some $E'$s, in which case $\text{Out}(E)$ is described as an extension and a presentation is given.

Keywords: Sol 3-manifold, torus bundle, Anosov, Aut and Out.

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1 Introduction

The family of the closed 3-manifolds which admit the Sol-geometry is one of the 8 families which appear in the Thurston geometrization conjecture. All elements of this family are $K(\pi, 1)$ and they are known to be Haken, i.e., irreducible and containing incompressible surfaces. So a homotopy equivalence is homotopic to a homeomorphism. Further, if two homeomorphisms are homotopic, they are in fact isotopic. The isotopy classes of base

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point preserving homeomorphisms of a Sol-manifold $N$ are into one-to-one correspondence with $\text{Aut}(E)$, where $E = \pi_1(N)$. Furthermore, the free isotopic classes of homeomorphisms of $N$ are into one-to-one correspondence with $\text{Out}(E)$. See more about the mapping class group of a 3-manifold and $\text{Aut}(E)$, $\text{Out}(E)$ for $E$ the fundamental group of the manifold in [14]. Therefore, the computation of $\text{Aut}(E)$ and $\text{Out}(E)$, besides having interest in their own right as an algebraic problem, is also of interest in the study of 3-dimensional manifolds. Also other aspects of this family of manifolds have been explored. For example, in [19], the involutions on such spaces have been studied in detail. In [11] the Nielsen fixed point theory has been studied. In [20] the degrees of self-maps are classified. The present work is related with these three works. In this work we compute the groups $\text{Aut}(E)$ and $\text{Out}(E)$ for $G$ the fundamental group of a closed Sol-manifold. It turns out that $\text{Out}(G)$ is finite. The calculation uses extensively properties of the linear group $GL_2(\mathbb{Z})$, some well known, others we explored, although they should be known for the experts. As a by-product an application for the fixed point theory of self maps on Sol-manifolds, related with the work [11], is obtained, see remark at the end of subsection 5.1.

The groups in question can be divided into two classes. One class consists of the groups of the form $E = (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}$, where $\theta(1) \in GL_2(\mathbb{Z})$ is an Anosov matrix. Let $M_0$ be a primitive root of $\theta(1)$. Depending on the matrix $M_0$ these groups are classified into 4 types. The main results about these groups are:

**Theorem 3.5** The group $\text{Aut}(E)$ is given by:

I) Suppose that $\theta(1)$ is conjugate to a matrix of the form $\begin{pmatrix} x & y \\ z & x \end{pmatrix}$. Then $\text{Aut}(E) \cong \text{Aut}_0(E) \rtimes_{w} \mathbb{Z}_2$, where the action of the semi-direct product is given by the automorphism $w = w(1_2) : \left( (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\phi_{M_0}} \mathbb{Z}_2 \right) \rightarrow \left( (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\phi} \mathbb{Z}_2 \right)$ defined by $w(\gamma_-) = \gamma_-$, $w(m,n) = -\theta(1)B_0(m,n)$, and

$$w(\gamma_+) = \begin{cases} \gamma_+, & \text{if } B_0M_0B_0^{-1} = M_0^{-1}, \\ \gamma_-\gamma_+, & \text{if } B_0M_0B_0^{-1} = -M_0^{-1}. \end{cases}$$

II) Suppose that $\theta(1)$ is conjugate to a matrix of the form $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Then $\text{Aut}(E) \cong \left( (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\phi_{M_0}} \mathbb{Z}_4 \right)$, where the action of the semi-direct product is given by the automorphism $w = w(1_4) : \left( (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\phi_{M_0}} \mathbb{Z}_4 \right) \rightarrow \left( (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\phi} \mathbb{Z}_4 \right)$ defined by $w(m,n) =$
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\[ -\theta(1)B_0(m, n) \]

and

\[ w(\gamma_) = \begin{cases} 
\gamma+ , & \text{if } B_0M_0B_0^{-1} = M_0^{-1}, \\
\gamma-\gamma+ , & \text{if } B_0M_0B_0^{-1} = -M_0^{-1}.
\end{cases} \]

III) Suppose that \( \theta(1) \) is conjugate to a matrix of the form \( A = \begin{pmatrix} x & y \\ w - x & w \end{pmatrix} \). Then \( \text{Aut}(E) \cong \text{Aut}_0(E) \rtimes_\phi \mathbb{Z}_2 \), where the action of the semi-direct product is given by the automorphism \( w = w(1_2) : ((\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z})_M \to ((\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z})_M \) defined by \( w(\gamma) = \gamma, w(m, n) = -\theta(1)B_0(m, n) \), and

\[ w(\gamma+) = \begin{cases} 
\gamma+ , & \text{if } B_0M_0B_0^{-1} = M_0^{-1}, \\
\gamma-\gamma+ , & \text{if } B_0M_0B_0^{-1} = -M_0^{-1}.
\end{cases} \]

We also show that \( \text{Aut}_0(E) \cong ((\mathbb{Z} \oplus \mathbb{Z})_M \rtimes \phi \mathbb{Z}_2) \).

**Theorem 3.6** Let \( H = \langle \alpha, \beta \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \rangle / \langle \kappa_\ell, \kappa_b \rangle \), which is a finite group since \( \theta(1) \) is Anosov. Considering the four cases listed above, we have:

I) a) If \( \theta(1) = M_0^\ell \), \( \text{Out}(E) \cong (H \rtimes M_0 \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \), where the generator of \( \mathbb{Z}_2 \) acts on \( H \) by multiplication by \(-1\) and on \( \mathbb{Z}_2 \) as the identity.

b) If \( \theta(1) = -M_0^\ell \), \( \text{Out}(E) \cong H \rtimes M_0 \mathbb{Z}_2 \).

II) a) If \( \theta(1) = M_0^\ell \), \( \text{Out}(E) \cong ((H \rtimes M_0 \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \), where the actions of the two \( \mathbb{Z}_2 \) factors are induced by the conjugations by \( \gamma- \) and \( \xi \) in \( \text{Aut}(E) \).

b) If \( \theta(1) = -M_0^\ell \), \( \text{Out}(E) \cong (H \rtimes M_0 \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \), where the action of \( \mathbb{Z}_2 \) is induced by the conjugation by \( \xi \) in \( \text{Aut}(E) \).

III) a) If \( \theta(1) = M_0^\ell \), \( \text{Out}(E) = (H \rtimes M_0 \mathbb{Z}_2) \rtimes \mathbb{Z}_4 \), where the action of \( \mathbb{Z}_2 \) is induced by the conjugation by \( \xi \) in \( \text{Aut}(E) \).

b) If \( \theta(1) = -M_0^\ell \), \( \text{Out}(E) = (H \rtimes M_0 \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \), where the action of \( \mathbb{Z}_2 \) is induced by the conjugation by \( \xi \) in \( \text{Aut}(E) \).

IV) a) If \( \theta(1) = M_0^\ell \), \( \text{Out}(E) \cong ((H \rtimes M_0 \mathbb{Z}_2) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \), where the actions of the two \( \mathbb{Z}_2 \) factors are induced by the conjugations by \( \gamma- \) and \( \xi \) in \( \text{Aut}(E) \).

b) If \( \theta(1) = -M_0^\ell \), \( \text{Out}(E) \cong (H \rtimes M_0 \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \), where the action of \( \mathbb{Z}_2 \) is induced by the conjugation by \( \xi \) in \( \text{Aut}(E) \).
About the groups of the second class, also known either as sapphire or semi-torus bundles, we have the following results:

**Theorem 4.10** The short exact sequence

\[
1 \longrightarrow \text{Aut}_0(E) \longrightarrow \text{Aut}(E) \longrightarrow \mathbb{Z}_2 \longrightarrow 1
\]

splits. Hence

\[
\text{Aut}(E) \cong \text{Aut}_0(E) \rtimes \mathbb{Z}_2 \cong [(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}_2] \rtimes_i \mathbb{Z} \rtimes_z \mathbb{Z}_2.
\]

and

**Theorem 4.11**

I) If \( \text{Aut}_0^1(E) = \emptyset \) we have

\[
\text{Out}(E) = \langle \alpha, \beta, \rho, | \alpha^2 = \beta^{2s} = \rho^2 = 1, \alpha \beta = \beta \alpha, \rho \alpha \rho = \alpha, \rho \beta \rho = \beta^{-1} \rangle.
\]

Hence \( \text{Out}(E) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2 s) \rtimes_{-1} \mathbb{Z}_2 \).

II) If \( \text{Aut}_0^1(E) \neq \emptyset \) and \( \theta(1) = M_0^t = (M_0^t)^2, \det(M_0^t) = 1, M_0^t = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ -s & r \end{pmatrix} \), we have \( \text{Out}(E) \cong [(\mathbb{Z}_2 \oplus \mathbb{Z}_2 t) \rtimes_{-1} \mathbb{Z}_2] \rtimes_{-1} \mathbb{Z}_2 \), where

\[
\omega \alpha \omega^{-1} = \alpha^r \beta^s, \omega \beta \omega^{-1} = \alpha \beta^r, \omega \rho \omega^{-1} = \alpha^t \beta^{t(r+1)} \rho.
\]

III) If \( \text{Aut}_0^1(E) \neq \emptyset \) and \( \theta(1) = -M_0^t = -(M_0^t)^2, \det(M_0^t) = -1, M_0^t = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ s & -r \end{pmatrix} \), we have a short exact sequence

\[
1 \longrightarrow (\mathbb{Z}_2 \oplus \mathbb{Z}_2 t) \rtimes_{-1} \mathbb{Z}_2 \longrightarrow \text{Out}(E) \longrightarrow \mathbb{Z}_2 \longrightarrow 1
\]

and a presentation of \( \text{Out}(E) \) is given by

\[
\text{Out}(E) = \langle \alpha, \beta, \rho, \omega | \alpha^2 = \beta^{2i} = \rho^2 = 1, \alpha \beta = \beta \alpha, \rho \alpha \rho = \alpha, \rho \beta \rho = \beta^{-1}, \omega^2 = \beta^t \rho, \omega \alpha \omega^{-1} = \alpha^r \beta^t, \omega \beta \omega^{-1} = \alpha \beta^r, \omega \rho \omega^{-1} = \alpha^t \beta^{t(r+1)} \rho \rangle.
\]
Finally, one remark about the calculations we had to work through: specifically in Section 4, we used the computer algebra system Maxima (maxima.sourceforge.net) to verify many of the identities that we present, since working them by hand would be extremely error-prone.

2 Preliminaries

The family of closed Sol 3-manifolds can be divided into two types. One type is the subfamily of the torus bundles given by an Anosov homeomorphism of the torus. Therefore any such manifold is determined by a matrix $A \in GL_2(\mathbb{Z})$ and is denoted by $M_A$. Furthermore, from [19, Lemma 1.1 item 4] we have that $M_A$ is isomorphic to $M_B$ if and only if $A$ is either conjugate to $B$ or to $B^{-1}$.

The second type of closed Sol 3-manifolds are the ones which are not torus bundle and belong to the subfamily of the so-called sapphires, following [17]. They are constructed as follows: for any Anosov matrix $A$ given by

$$
\begin{pmatrix}
  r & s \\
  t & u
\end{pmatrix},
$$

we obtain a sapphire by gluing two $I$-bundles over the Klein bottle by a homeomorphism of the boundary, which is a torus, defined by the matrix $A$. Denote by $K_A$ the resulting manifold. From [17] we obtain that $K_A$ is not a torus bundles if and only if $rstu \neq 0$. Also, [17, Theorem 1] gives the precise description when two such matrices $A$ and $B$ lead to two homeomorphic manifolds $K_A, K_B$.

Now we recall the main result from [23]. Let us consider a short exact sequence of groups

$$
1 \longrightarrow G \overset{i}{\longrightarrow} E \overset{n}{\longrightarrow} \Pi \longrightarrow 1. \tag{2}
$$

Denote by: $\text{Aut}(E; G)$ the set of automorphisms of $E$ which, when restricted to $G$ define an isomorphism of $G$; $Z(G)$ the centre of $G$; $\tilde{\alpha}: \Pi \rightarrow \text{Out}(G)$ the homomorphism given by the above extension; $Z^1_{\tilde{\alpha}}(\Pi, ZG)$ the crossed homomorphisms.

**Definition 2.1** A pair $(\sigma, \tau) \in \text{Aut}(\Pi) \times \text{Aut}(G)$ is called compatible if $\sigma$ fixes $\text{Ker}(\tilde{\alpha})$ and the automorphism induced by $\sigma$ on $\Pi \tilde{\alpha}$ is the same as that induced by the inner automorphism of $\text{Out}(G)$ determined by $\tau$.

Let $C$ be the set of all compatible pairs. The following Theorem is proved in [23]:
Theorem 2.2 There is a set map $C \to H^2(\Pi, \mathcal{Z}(G))$ such that the sequence

$$1 \to \mathcal{Z}_1^1(\Pi, \mathcal{Z}(G)) \to \text{Aut}(E; G) \to C \to H^2(\Pi, \mathcal{Z}(G))$$

is exact. Sometimes it is more convenient to use the following exact sequence, which promptly follows from the Theorem above:

$$1 \to \mathcal{Z}_1^1(\Pi, \mathcal{Z}(G)) \to \text{Aut}(E; G) \xrightarrow{\psi} \text{Aut}(\Pi) \times \text{Aut}(G),$$

where the last map is the natural obvious one, and it is also a homomorphism. For more details about the maps and homomorphisms that appear in the theorem above see [23].

Corollary 2.3 If the group $G$ has trivial center then we have an isomorphism

$$1 \to \text{Aut}(E; G) \xrightarrow{\cong} C \to 1.$$

Corollary 2.4 If the extension (2) is characteristic then we have the following short exact sequence

$$1 \to \mathcal{Z}_1^1(\Pi, \mathcal{Z}(G)) \to \text{Aut}(E) \xrightarrow{\psi} \text{Im}(\psi) \to 1.$$  

Let us recall how the action of $\text{im}(\psi)$ on $\mathcal{Z}_1^1(\Pi, \mathcal{Z}(G))$ is defined: given an element of $\lambda \in \text{im}(\psi)$, then there is an automorphism $\alpha : E \to E$ which corresponds to the given element. Then we must compute $\lambda * (\beta) = \alpha \circ \beta \circ (\alpha)^{-1}$ where $\beta = Id + \beta_0$ and $\beta_0$ is a derivation. The result is an automorphism of $E$ that when restricted to the subgroup $G$ is the identity and induces the identity on the quotient. So $\lambda * (\beta) = Id + \beta_0'$ and it is easy to see that $\beta_0' = \alpha|_G^{-1} \circ \beta_0 \circ \bar{\alpha}$.

3 The groups Aut and Out of the fundamental group of a torus bundle

Here we consider the fundamental group of torus bundles where the homeomorphism used to construct it is Anosov. It is well known that these fundamental groups are semi-direct products $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z}$ where the automorphism $\theta(1)$ is a $2 \times 2$ Anosov matrix. To simplify, sometimes by abuse of notation we denote $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z}$ by $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta(1)} \mathbb{Z}$. 
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So let $E = (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z}$, where $\theta(1) \in \text{GL}_2(\mathbb{Z})$ is an Anosov matrix. The short exact sequence

$$1 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow E \longrightarrow \mathbb{Z} \longrightarrow 1$$

splits and the action of $\mathbb{Z}$ on $\mathbb{Z} \oplus \mathbb{Z}$ is given by $\theta(1)$.

If we call $d$ and $b$ the generators of $\mathbb{Z} \oplus \mathbb{Z} \leq E$ which correspond to generators of the summands, respectively, and $v$ an element of $E$ that projects to the generator $1 \in \mathbb{Z}$, then a presentation of $E$ is given by

$$E = \left\langle d, b, v \mid db = bd, \ vdv^{-1} = \theta(1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ vbv^{-1} = \theta(1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle,$$

(6)

where we make the identification $d^m b^n = \begin{pmatrix} m \\ n \end{pmatrix}$ for the elements of $\mathbb{Z} \oplus \mathbb{Z}$.

Lemma 3.1 The subgroup $\mathbb{Z} \oplus \mathbb{Z}$ is characteristic with respect to automorphisms of $E$. Hence we obtain the exact sequence

$$1 \longrightarrow \mathbb{Z}^1(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}) \longrightarrow \text{Aut}(E) \xrightarrow{\psi} \text{Aut}(\mathbb{Z}) \times \text{Aut}(\mathbb{Z} \oplus \mathbb{Z}).$$

Proof: The first part follows from [11]. The second part is a consequence of Theorem [22]. More precisely, it follows readily from equation [1].

In order to compute $\text{Aut}(E)$ we first determine all automorphisms that induce the identity $\text{id}_\mathbb{Z}$ on the quotient $\mathbb{Z}$. Next we determine all the automorphisms that induce $-\text{id}_\mathbb{Z}$ on the quotient, and finally we describe the group structure of $\text{Aut}(E)$ and of $\text{Out}(E)$.

For $S \in \text{GL}_2(\mathbb{Z})$ let $C(S) = \{M \in \text{GL}_2(\mathbb{Z}) : MS = SM\}$. It is known (see [19, Lemma 1.7]) that there is a matrix $M_0 \in \text{GL}_2(\mathbb{Z})$ such that

$$C(S) = \{\pm M_0^\ell : \ell \in \mathbb{Z} \} \cong \mathbb{Z}_2 \oplus \mathbb{Z},$$

(7)

if $S$ is neither the identity matrix $I_2$ nor $-I_2$. If $S$ is of infinite order then the number of possible matrices $M_0$ are exactly 4. Namely, given $M_0$, they are $\{\pm M_0^{\pm 1}\}$. 
Proposition 3.2 The intersection of image of \( \psi \) with the subgroup \( \{ \text{id}_Z \} \times \text{Aut}(\mathbb{Z} \oplus \mathbb{Z}) \) is the subgroup generated by \( (\text{id}_Z, M_0) \) and \( (\text{id}_Z, - \text{id}_Z \oplus 2) \). If we denote by \( \text{Aut}_0(E) \) the elements of \( \text{Aut}(E) \) which project to elements of the form \( (\text{id}_Z, \tau) \), we have the short exact sequence

\[
1 \rightarrow (\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{Z} \rightarrow \text{Aut}_0(E) \rightarrow \mathbb{Z}_2 \rightarrow 1,
\]

where the map \( \text{Aut}_0(E) \rightarrow \mathbb{Z}_2 \) is determined by the map sending \( (\text{id}_Z, M_0) \) to 02 and \( (\text{id}_Z, - \text{id}_Z \oplus 2) \) to 12. Further, the sequence splits, so \( \text{Aut}_0(E) \cong ((\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{Z}) \rtimes \phi \mathbb{Z}_2 \), and with respect to the splitting given by \( s(12) = (\text{id}_Z, - \text{id}_Z \oplus 2) \), the action is given by \( \phi(m, n, t) = (-m, -n, t) \).

Proof: The equation \( M\theta(1) = \theta(1)M \) is precisely the condition for the existence of an automorphism on the semi-direct product which induces the automorphisms \( \text{id}_Z \in \text{Aut}(\mathbb{Z}) \) and \( M \in \text{Aut}(\mathbb{Z} \oplus \mathbb{Z}) \). The short exact sequence given by Theorem 2.2 and the fact that \( Z^1_0(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \), together with equation (7), give us the short exact sequence

\[
1 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \text{Aut}_0(E) \xrightarrow{\psi|_{\text{Aut}_0(E)}} \mathbb{Z}_2 \oplus \mathbb{Z} \rightarrow 1.
\]

The automorphisms \( \alpha, \beta \in \text{Aut}_0(E) \) defined by

\[
\alpha(d) = d, \quad \beta(d) = d, \\
\alpha(b) = b, \quad \beta(b) = b, \\
\alpha(v) = dv, \quad \beta(v) = bv
\]

generate \( \ker(\psi|_{\text{Aut}_0(E)}) \cong \mathbb{Z} \oplus \mathbb{Z} \). The automorphism \( \gamma_+ \in \text{Aut}_0(E) \) given by

\[
\gamma_+(d^m b^n) = M_0 \begin{pmatrix} m \\ n \end{pmatrix}, \quad \gamma_(v) = v
\]

is projected by \( \psi \) onto \( (\text{id}_Z, M_0) \) and the automorphism \( \gamma_- \in \text{Aut}_0(E) \) defined by

\[
\gamma_-(d^m b^n) = d^{-m} b^{-n}, \quad \gamma_-(v) = v
\]

is projected onto \( (\text{id}_Z, - \text{id}_Z \oplus 2) \). The subgroup of \( \text{Aut}_0(E) \) generated by \( \alpha, \beta \) and \( \gamma_+ \) is isomorphic to \( (\mathbb{Z} \oplus \mathbb{Z}) \rtimes M_0 \mathbb{Z} \). This can be readily seen since the subgroup generated by \( \gamma_+ \) is infinite cyclic and \( (\gamma_+) (\alpha^x \beta^y) (\gamma_+)^{-1} = M_0 (\alpha^x \beta^y) \). Finally, the subgroup generated by \( \alpha, \beta \) and \( \gamma_+ \) has index 2 in \( \text{Aut}_0(E) \), and this gives the short exact sequence of the statement. If one wants a presentation for \( \text{Aut}_0(E) \), a straightforward computation shows that \( (\gamma_-) \alpha (\gamma_-)^{-1} = \alpha^{-1}, \quad (\gamma_-) \beta (\gamma_-)^{-1} = \beta^{-1}, \quad (\gamma_-) (\gamma_+)(\gamma_-)^{-1} = \gamma_+ \) and, of course, \( (\gamma_-)^2 = 1 \). The “further” part is straightforward.
Although the explicit calculation of a generator $M_0$ may not be straightforward, the elements $(\text{id}_Z, \theta(m))$, for $m \in Z \setminus \{0\}$ are non-trivial elements which belong to the image of $\psi$ besides $(\text{id}_Z, -\text{id}_{Z\oplus Z})$. This is not the case for the elements where the first coordinate is $-\text{id}_Z$, since the set of such elements may or may not be empty. In order to find elements in the image of $\psi$ in this case, this is equivalent to ask if for the given $\theta(1) \in GL_2(Z)$ we can find $B \in GL_2(Z)$ such that $B\theta(1)B^{-1} = \theta^{-1}(1)$. Following [19] we denote by $R(\theta(1))$ the set of $B \in GL_2(Z)$ such that $B\theta(1)B^{-1} = \theta(1)^{-1}$. See the Appendix for a detailed analysis of the set $R(\theta(1))$, where we give necessary and sufficient condition for the set $R(\theta(1))$ to be not empty.

**Lemma 3.3** There is a short exact sequence

$$1 \longrightarrow \text{Aut}_0(E) \longrightarrow \text{Aut}(E) \overset{\psi}{\longrightarrow} Z_2 \longrightarrow 1,$$

if $R(\theta(1))$ is not empty, otherwise we have $\text{Aut}_0(E) \cong \text{Aut}(E)$.

**Proof:** The proof is straightforward.

Now we calculate a presentation of $\text{Aut}(E)$ assuming that $R(\theta(1))$ is not empty, and let $B_0 \in R(\theta(1))$. We use [15] Proposition 1, page 139, and consider an automorphism $\xi$ such that restricted to $Z \oplus Z$ is given by a matrix $B_0 \in R(\theta(1))$, and such that induces $-\text{id}_Z$ on the quotient $Z$. The matrix $B_0^2$ commutes with $\theta(1)$. Therefore if we choose another element $\xi_1$ with similar properties as $\xi$, it must be of the form $\xi_1 = \varepsilon M_0^k \xi$ where $\varepsilon \in \{I_2, -I_2\}$. It follows that

$$\xi_1^2 = \varepsilon M_0^k \xi \varepsilon M_0^k \xi = (\varepsilon)^2 M_0^k \xi M_0^k \xi^{-1} \xi^2 = \xi^2,$$

so $\xi^2$ is independent of the choice of $B_0 \in R(\theta(1))$. In fact we can show:

**Lemma 3.4** The set $R(\theta(1))$ is non empty if and only if the matrix $\theta(1)$ has determinant 1 and is conjugate to one of the matrices of the form $A = \begin{pmatrix} x & y \\ z & x \end{pmatrix}$, or to a matrix of the form $A = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$, or to a matrix of the form $A = \begin{pmatrix} x & y \\ w-x & w \end{pmatrix}$. In the first and third cases we have $B_0^2 = I_2$ and in the second case we have $B_0^2 = -I_2$. 
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Proof: This follows promptly from Proposition 5.8 since $B_0$ is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, in the first and third cases, respectively, and conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, in the second case.

\[ \]

So to complete a presentation of the group it suffices to describe a choice of a matrix $B_0$ and the action of the automorphism $\xi \in \text{Aut}(E)$ on $\text{Aut}_0(E)$ by conjugation. The choice of $B_0$ can be made using the appendix and then we can get a presentation of $\text{Aut}(E)$.

**Theorem 3.5** The group $\text{Aut}(E)$ is given by:

I) Suppose that $\theta(1)$ is conjugate to a matrix of the form $\begin{pmatrix} x & y \\ z & x \end{pmatrix}$. Then $\text{Aut}(E) \cong \text{Aut}_0(E) \rtimes_w \mathbb{Z}_2$, where the action of the semi-direct product is given by the automorphism $w = w(1_2): ((\mathbb{Z} \oplus \mathbb{Z})_M \rtimes \mathbb{Z}_2) \to ((\mathbb{Z} \oplus \mathbb{Z})_M \rtimes \mathbb{Z}_2)$ defined by $w(\gamma) = \gamma$, $w(m, n) = -\theta(1)B_0(m, n)$, and

\[
w(\gamma) = \begin{cases} 
\gamma_+, & \text{if } B_0M_0B_0^{-1} = M_0^{-1}, \\
\gamma_-, & \text{if } B_0M_0B_0^{-1} = -M_0^{-1}.
\end{cases}
\]

II) Suppose that $\theta(1)$ is conjugate to a matrix of the form $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$. Then $\text{Aut}(E) \cong ((\mathbb{Z} \oplus \mathbb{Z})_M \rtimes \mathbb{Z}_2) \rtimes_w \mathbb{Z}_4$, where the action of the semidirect product is given by the automorphism $w = w(1_4): ((\mathbb{Z} \oplus \mathbb{Z})_M \rtimes \mathbb{Z}_4) \to ((\mathbb{Z} \oplus \mathbb{Z})_M \rtimes \mathbb{Z}_4)$ defined by $w(m, n) = -\theta(1)B_0(m, n)$ and

\[
w(\gamma) = \begin{cases} 
\gamma_+, & \text{if } B_0M_0B_0^{-1} = M_0^{-1}, \\
\gamma_-, & \text{if } B_0M_0B_0^{-1} = -M_0^{-1}.
\end{cases}
\]

III) Suppose that $\theta(1)$ is conjugate to a matrix of the form $A = \begin{pmatrix} x & y \\ \frac{w-x}{w} & \frac{y}{w} \end{pmatrix}$. Then $\text{Aut}(E) \cong \text{Aut}_0(E) \rtimes_w \mathbb{Z}_2$, where the action of the semi-direct product is given by
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the automorphism \( w = w(1_2) : ((\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}) \rtimes_{\phi} \mathbb{Z}_2 \to ((\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}) \rtimes_{\phi} \mathbb{Z}_2 \) defined by \( w(\gamma_-) = \gamma_- \), \( w(m, n) = -\theta(1)B_0(m, n) \), and

\[
w(\gamma_+) = \begin{cases} 
\gamma_+, & \text{if } B_0M_0B_0^{-1} = M_0^{-1}, \\
\gamma_-\gamma_+, & \text{if } B_0M_0B_0^{-1} = -M_0^{-1}.
\end{cases}
\]

Proof:

I) Choose \( B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( \xi(d^mb^n) = d^mb^{-n}, \xi(v) = v^{-1} \). Then

\[
\xi(\alpha^m\beta^n)\xi^{-1}(v) = d^rb^sv,
\]

where \( (r, s) = -\theta(1)B_0(m, n) \), and \( \xi\gamma_+\xi^{-1}(d^mb^n) = \pm M_0^{-1}(d^mb^n) \), hence \( \xi\gamma_+\xi^{-1} \in \{\gamma_+^{-1}, \gamma_+\gamma_-^{-1}\} \). We have \( \xi^2 = \text{id}_E \) and \( \xi\gamma_-\xi^{-1} = \gamma_- \), hence Aut(\( E \)) = Aut_0(\( E \)) \rtimes \mathbb{Z}_2 \) with the action given in the statement.

II) Choose \( B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \xi(d^mb^n) = d^{-n}b^m, \xi(v) = v^{-1} \). Then

\[
\xi(\alpha^m\beta^n)\xi^{-1}(v) = d^rb^sv,
\]

where \( (r, s) = -\theta(1)B_0(m, n) \), and we have \( \xi\alpha\xi^{-1} = \theta(b^{-1})v \) and \( \xi\beta\xi^{-1} = \theta(d)v \). We also have \( \xi\gamma_+\xi^{-1}(d^mb^n) = \pm M_0^{-1}(d^mb^n) \) and \( \xi\gamma_+\xi^{-1}(v) = v \), hence \( \xi\gamma_+\xi^{-1} \in \{\gamma_+^{-1}, \gamma_+\gamma_-^{-1}\} \). Finally, \( \xi^2 = \gamma_- \) and we have Aut(\( E \)) \cong ((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{M_0} \mathbb{Z}_2 \rtimes Z_4 \), where \( Z_4 \) is generated by the class of \( \xi \).

III) Choose \( B_0 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \) and \( \xi(d^mb^n) = d^{m+n}b^{-n}, \xi(v) = v^{-1} \). Then

\[
\xi(\alpha^m\beta^n)\xi^{-1}(v) = d^rb^sv,
\]

where \( (r, s) = -\theta(1)B_0(m, n) \), and \( \xi\gamma_+\xi^{-1}(d^mb^n) = \pm M_0^{-1}(d^mb^n) \), hence \( \xi\gamma_+\xi^{-1} \in \{\gamma_+^{-1}, \gamma_+\gamma_-^{-1}\} \). We have \( \xi^2 = \text{id}_E \) and \( \xi\gamma_-\xi^{-1} = \gamma_- \), hence Aut(\( E \)) = Aut_0(\( E \)) \rtimes \mathbb{Z}_2 \) with the action given in the statement.

\[\blacksquare\]
Now we calculate Out$(E)$. For $g \in E$, we denote by $\kappa_g \in \text{Inn}(E)$ the conjugation by $g$, that is, $\kappa_g(x) = gxg^{-1}$. Let $\theta(1) = \frac{r}{t} \begin{pmatrix} s & \alpha \beta \\ t & u \end{pmatrix}$; an easy calculation shows that $\kappa_d = \alpha^{1-r} \beta^{-t}$, $\kappa_b = \alpha^{-s} \beta^{1-u}$, so the subgroup $\langle \kappa_d, \kappa_b \rangle \leq \langle \alpha, \beta \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$ is isomorphic to $(I_2 - \theta(1))(\mathbb{Z} \oplus \mathbb{Z})$.

We also have $\kappa_v = \begin{cases} \gamma^\ell, & \text{if } \theta(1) = M_0^\ell, \\ \gamma - \gamma^\ell, & \text{if } \theta(1) = -M_0^\ell, \end{cases}$

Observe that every inner automorphism of $E = (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z}$ induces the identity on the quotient $\mathbb{Z}$. Now we compute Out$_0(E) = \text{Aut}_0(E)/\text{Inn}(E)$ and Out$(E) = \text{Aut}(E)/\text{Inn}(E)$. We consider 4 cases.

I) $\text{Aut}_0(E) = \text{Aut}(E)$;

II) $\text{Aut}_0(E) \neq \text{Aut}(E)$ and $\theta(1)$ is conjugate to $\begin{pmatrix} x & y \\ z & x \end{pmatrix}$.

III) $\text{Aut}_0(E) \neq \text{Aut}(E)$ and $\theta(1)$ is conjugate to $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$.

IV) $\text{Aut}_0(E) \neq \text{Aut}(E)$ and $\theta(1)$ is conjugate to $\begin{pmatrix} x & y \\ w - x & w \end{pmatrix}$.

**Theorem 3.6** Let $H = \frac{\langle \alpha, \beta \rangle}{\langle \kappa_d, \kappa_b \rangle} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{(I_2 - \theta(1))(\mathbb{Z} \oplus \mathbb{Z})}$, which is a finite group since $\theta(1)$ is Anosov. Considering the four cases listed above, we have:

I) a) If $\theta(1) = M_0^\ell$, Out$(E) \cong (H \rtimes_{M_0} \mathbb{Z}_\ell) \rtimes \mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ acts on $H$ by multiplication by $-1$ and on $\mathbb{Z}_\ell$ as the identity.

b) If $\theta(1) = -M_0^\ell$, Out$(E) \cong H \rtimes_{M_0} \mathbb{Z}_{2\ell}$.

II) a) If $\theta(1) = M_0^\ell$, Out$(E) \cong ((H \rtimes_{M_0} \mathbb{Z}_\ell) \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, where the actions of the two $\mathbb{Z}_2$ factors are induced by the conjugations by $\gamma$ and $\xi$ in Aut$(E)$.

b) If $\theta(1) = -M_0^\ell$, Out$(E) \cong (H \rtimes_{M_0} \mathbb{Z}_{2\ell}) \rtimes \mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is induced by the conjugation by $\xi$ in Aut$(E)$.

III) a) If $\theta(1) = M_0^\ell$, Out$(E) = (H \rtimes_{M_0} \mathbb{Z}_\ell) \rtimes \mathbb{Z}_4$, where the action of $\mathbb{Z}_2$ is induced by the conjugation by $\xi$ in Aut$(E)$. 
b) If \( \theta(1) = -M_0^t \), \( \text{Out}(E) = (H \times_{M_0} \mathbb{Z}_{2^t}) \times \mathbb{Z}_2 \), where the action of \( \mathbb{Z}_2 \) is induced by the conjugation by \( \xi \) in \( \text{Aut}(E) \).

IV) a) If \( \theta(1) = M^t \), \( \text{Out}(E) \cong ((H \times_{M_0} \mathbb{Z}_2) \times \mathbb{Z}_2) \times \mathbb{Z}_2 \), where the actions of the two \( \mathbb{Z}_2 \) factors are induced by the conjugations by \( \gamma_- \) and \( \xi \) in \( \text{Aut}(E) \).

b) If \( \theta(1) = -M^t \), \( \text{Out}(E) \cong (H \times_{M_0} \mathbb{Z}_{2^t}) \times \mathbb{Z}_2 \), where the action of \( \mathbb{Z}_2 \) is induced by the conjugation by \( \xi \) in \( \text{Aut}(E) \).

Proof: The analyses of the four cases are similar.

I) Let \( \theta(1) = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \).

a) A presentation for \( \text{Out}(E) \) is given by

\[
\text{Out}(E) = \langle \alpha, \beta, \gamma_+, \gamma_- \mid \alpha \beta = \beta \alpha, \gamma_+ (\alpha^m \beta^n) \gamma_-^{-1} = M_0 (\alpha^m \beta^n), \\
\gamma_+^2 = 1, \gamma_- \alpha \gamma_-^{-1} = \alpha^{-1}, \gamma_- \beta \gamma_-^{-1} = \beta^{-1}, \gamma_- \gamma_+ \gamma_-^{-1} = \gamma_+^{-1}, \\
\alpha^{1-r} \beta^{-t} = \alpha^{-s} \beta^{1-u} = 1, \gamma_+^t = 1 \rangle,
\]

from which the statement follows.

b) In this case, a presentation of \( \text{Out}(E) \) is given by

\[
\text{Out}(E) = \langle \alpha, \beta, \gamma_+, \gamma_- \mid \alpha \beta = \beta \alpha, \gamma_+ (\alpha^m \beta^n) \gamma_-^{-1} = M_0 (\alpha^m \beta^n), \\
\gamma_+^2 = 1, \gamma_- \alpha \gamma_-^{-1} = \alpha^{-1}, \gamma_- \beta \gamma_-^{-1} = \beta^{-1}, \gamma_- \gamma_+ \gamma_-^{-1} = \gamma_+^{-1}, \\
\alpha^{1-r} \beta^{-t} = \alpha^{-s} \beta^{1-u} = 1, \gamma_+ = \gamma_+^t \rangle
\]

= \langle \alpha, \beta, \gamma_+ \mid \alpha \beta = \beta \alpha, \gamma_+^{2t} = 1, \gamma_+ (\alpha^m \beta^n) \gamma_-^{-1} = M_0 (\alpha^m \beta^n), \\
\alpha^{1-r} \beta^{-t} = \alpha^{-s} \beta^{1-u} = 1 \rangle,
\]

from which the statement follows.

II) This case is analogous to the previous one.

III) a) This item is also analogous to case I.
b) Let \( \theta(1) = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \) and \( B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). In this case, a presentation of Out\((E)\) is

\[
\text{Out}(E) = \langle \alpha, \beta, \gamma_+, \xi \mid \alpha \beta = \beta \alpha, \alpha^{1-x}\beta^{-y} = \alpha^{-y}\beta^{1-z} = 1, \\
\gamma_+(\alpha^m\beta^n)\gamma_+^{-1} = M_0(\alpha^m\beta^n), \\
\xi(\alpha^m\beta^n)\xi^{-1} = -\theta B_0(\alpha^m\beta^n), \xi\gamma_+\xi^{-1} = \xi^2\gamma_+^{-1}, \\
\xi^4 = 1, \xi^2\gamma_+ = 1 \rangle
\]

from which the statement follows, since the exact sequence

\[
1 \longrightarrow H \rtimes M_0 \mathbb{Z}_{2\ell} \longrightarrow \text{Out}(E) \longrightarrow \mathbb{Z}_2 = \langle \bar{\xi} \rangle \longrightarrow 1
\]

splits, with a section given by \( \bar{\xi} \mapsto \xi\gamma_+^{-1} \).

IV) This case is analogous to the case II, which in turn is analogous to I.

\[\blacksquare\]

4 The groups Aut and Out for the Sapphire Sol-manifold

Here we follow the classification of the sapphire manifolds given by Morimoto in \[17\]. For each matrix

\[
B = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL_2(\mathbb{Z}),
\]

a 3-manifold is constructed using a homeomorphism \( h \) of the torus \( T \), where \( h \) induces on the fundamental group of \( T \) the homomorphism given by the matrix \( B \). We know from \[17\] Theorem 1] when two matrices \( B_1, B_2 \) provide, up to homeomorphism, the same manifold, and it follows that we can assume without loss of generality that \( \det(B) = ru - st = 1 \).
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So, from now on, our matrix $B$ is fixed and of determinant 1. Also, if the sapphire has the Sol geometry and is not a torus bundle, from [17] we must have $rstu \neq 0$.

Let $E$ be the fundamental group of the sapphire defined by the matrix $B$. The group $\text{Aut}(E)$ is closely related to $\text{Aut}(E')$, where $E'$ is the fundamental group of a certain torus bundle. More precisely, by [11, Lemma 3.3] we have the following short exact sequence where the kernel is characteristic with respect to automorphism of $E$:

$$1 \to (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z} \to E \to \mathbb{Z}_2 \to 1.$$ 

The matrix $\theta = \theta(1)$ is given by

$$\begin{pmatrix} ru + st & -2rt \\ -2su & ru + st \end{pmatrix}.$$ \hspace{1cm} (9)

One presentation of $E$ is

$$E = \langle d, b, v, a \mid db = bd, vdv^{-1} = \theta(d), vbv^{-1} = \theta(b), a^2 = d, ab = b^{-1}a, ava^{-1} = d^{r-u-st}b^{s-2su}v^{-1} \rangle,$$ \hspace{1cm} (10)

where $d$ and $b$ are the generators of the subgroup $(\mathbb{Z} \oplus \mathbb{Z})$ of $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$, $v \in (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$ is such that its class generates the quotient $\mathbb{Z} = ((\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z})/(\mathbb{Z} \oplus \mathbb{Z})$, and $a$ is a remaining generator of $E$ that projects onto the generator of $\mathbb{Z}_2$.

**Lemma 4.1** The map $\psi : \text{Aut}(E) \to \text{Aut}((\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z})$ is injective.

**Proof:** This follows promptly from the exact sequence given by Theorem 2.2 and the fact that the center of $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$ is trivial.

Our task is to decide which automorphisms of $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$ extend to the group $E$, which amounts to deciding which automorphisms of $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$ also preserve the last three relations of (10). We will consider two types of automorphisms of $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$: Type I are the automorphisms which induce the identity map on the quotient $[(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}] / (\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}$, and Type II are the automorphisms which induce minus the identity map on the same quotient.

**Remark:** The torus bundles $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}$ where $\theta(1)$ is given by (9), by Proposition 5.8, always admit automorphisms of Type II since the matrix of the gluing map of the torus bundles has the two elements of the diagonal equal.
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We first address the case of Type I maps. In this case it follows from the classification of the automorphisms, of \((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z}\), that a such automorphism when restricted to \(\mathbb{Z} \oplus \mathbb{Z}\) is an automorphism of the form \(\delta M_0^\ell\) for some \(\ell \in \mathbb{Z}\), \(\delta \in \{\pm 1\}\) and some primitive Anosov matrix \(M_0\) given by [19, Lemma 1.7]. Recall that the matrix \(B\) used to construct the Sapphire has det\((B) = 1\). So let \(\varphi \in \text{Aut}((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z})\) be a Type I automorphism that we attempt to extend to an automorphism of \(E\) (that we also call \(\varphi\) by an abuse of notation) by setting

\[
\varphi(d^x b^y) = \delta M_0^\ell (d^x b^y), \quad \delta \in \{\pm 1\},
\]

\[
\varphi(v) = d^p b^q v, \quad (11)
\]

\[
\varphi(a) = d^m b^n v^k a,
\]

for certain integers \(\ell, p, q, m, n, k\). The next lemma answers when such a map preserves the relation \(ab = b^{-1}a\).

**Lemma 4.2 (Fundamental I)** Let \(\varphi \in \text{Aut}((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z})\) be a Type I automorphism. If we attempt to extend \(\varphi\) to an automorphism of \(E\) as in equation (11), then the extension \(\varphi\) satisfies \(\varphi(ab) = \varphi(b^{-1}a)\) if, and only if, \(2\ell = k\ell_0\) and \(\varepsilon \det(M_0)^\ell = 1\), where \(\ell_0 \in \mathbb{Z}\) is the integer such that \(\theta(1) = \varepsilon M_0^{\ell_0}\), \(\varepsilon \in \{\pm 1\}\).

**Proof:** Let \(P = \begin{pmatrix} x & z \\ y & w \end{pmatrix} \in SL_2(\mathbb{R})\) be a matrix such that \(P^{-1}M_0P\) is diagonal. We claim that \(xw + yz = 0\) (which readily implies \(xw = 1/2\) and \(yz = -1/2\)). One way to see this is the following: if \(P^{-1}M_0P\) is diagonal, then so is \(P^{-1}\theta P\). Now a simple computation of the eigenvectors of \(\theta\) (observing that the diagonal entries of \(\theta\) are equal) yields the claim.

We have

\[
\varphi(ab) = \varphi(b^{-1}a) \iff d^m b^n v^k a \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} M_0^\ell (b) \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} M_0^\ell (b^{-1}) d^m b^n v^k a \iff
\]

\[
v^k a \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} M_0^\ell (b) \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} M_0^\ell (b^{-1}) v^k a \iff
\]

\[
\theta^k \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix} M_0^\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} M_0^\ell \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \iff
\]

\[
\left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_0^{-\ell} \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \theta^k \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix} M_0^\ell \right] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Let $N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_0^\ell \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \theta^k \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix} M_0^\ell$. We have $\det(N) = \det(\theta)^k = 1^k = 1$ and, since 1 is an eigenvalue of $N$, the characteristic polynomial of $N$ is $(\lambda - 1)^2$. We also have that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 1, so we actually have

$$N = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$$

for some $e \in \mathbb{Z}$. Hence we can write

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_0^{-\ell} \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \theta^k \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix} M_0^\ell = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}.$$  

Now let $P = \begin{pmatrix} x & x & z & w \\ y & 0 & w & 0 \end{pmatrix} \in SL_2(\mathbb{R})$ be such that $P^{-1}M_0P = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Multiplying the equation above by $P^{-1}$ on the left and by $P$ on the right, we get

$$e^k \left( \begin{array}{cc} \lambda_1^\ell \lambda_2^{k\ell_0-\ell} & 0 \\ 0 & \lambda_2 \lambda_1^{k\ell_0-\ell} \end{array} \right) = \begin{pmatrix} (1 - exz) & -ez^2 \\ ex^2 & (1 + exz) \end{pmatrix},$$

which implies $e = 0$ and $\lambda_1^\ell \lambda_2^{k\ell_0-\ell} = \lambda_2 \lambda_1^{k\ell_0-\ell} = \pm 1$. But $M_0$ is an Anosov matrix, so $|\lambda_1| \neq 1 \neq |\lambda_2|$ and $|\lambda_1 \lambda_2| = 1$. Since we have

$$|\lambda_1^\ell \lambda_2^{k\ell_0-\ell}| = 1 \Rightarrow |\lambda_2^{k\ell_0-2\ell}| = 1,$$

we get $k\ell_0 - 2\ell = 0$. Now, from the above matrix equality, we must have $e^k(\lambda_1 \lambda_2)^\ell = 1 \Leftrightarrow e^k \det(M_0)^\ell = 1$.

Next we look at the other two relations that $\varphi$ in equation (11) must satisfy to be an element of $\text{Aut}(E)$. We first analyze the case $k$ even, starting when $k = 0$. Still using the notation of equation (11), let’s denote by $\text{Aut}_0(E)$ the automorphisms of $E$ that induce a type I automorphism of $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}$, and let

$$\text{Aut}_0^k(E) = \{ \varphi \in \text{Aut}_0(E) \mid \varphi(a) = d^m b^nu^k a \text{ for some } m, n \in \mathbb{Z} \}. \quad (12)$$

It is easy to see that $\varphi_{k_1} \in \text{Aut}_0^{k_1}(E)$, $\varphi_{k_2} \in \text{Aut}_0^{k_2}(E) \Rightarrow \varphi_{k_1} \varphi_{k_2} \in \text{Aut}_0^{k_1+k_2}(E)$ and $\varphi \in \text{Aut}_0^k(E) \Rightarrow \varphi^{-1} \in \text{Aut}_0^{-k}(E)$, hence $\text{Aut}_0^k(E)$ is a normal subgroup of $\text{Aut}_0(E)$. It is this subgroup that we describe now. By Lemma 4.2, we know that $k = 0$ implies $\ell = 0$. 


Proposition 4.3 An element $\varphi \in \mathrm{Aut}_0^0(E)$ has one of the following two forms:

1. 
   \[
   \varphi(dx^yb^c) = dx^yb^c, \\
   \varphi(a) = b^na, \\
   \varphi(v) = d^{rn+ct}b^{-stn-cu}v,
   \]
   where $c$, $n$ are any integers.

2. 
   \[
   \varphi(dx^yb^c) = d^{-x}b^{-y}, \\
   \varphi(a) = d^{-1}b^na, \\
   \varphi(v) = d^{r(tn-1+u)+ct}b^{-s(tn-1+u)-cu}v,
   \]
   where $c$, $n$ are any integers.

Proof: The relation $a^2 = d$ yields $d^{2m+1} = d^{\pm 1}$, hence $m = 0$ or $m = -1$ according to the sign. In the first case, corresponding to $m = 0$, the relation $ava^{-1} = d^{r-u-sts-2su}v^{-1}$ yields the linear system (in the unknowns $p$, $q$)

\[
\begin{pmatrix}
\theta^{-1} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
p \\ q
\end{pmatrix}
= (\theta^{-1} - I_2)
\begin{pmatrix}
0 \\ n
\end{pmatrix},
\]

which is equivalent to $up + tq = tn$ and has $p = rtn + ct$, $q = -stn - cu$ as the general integer solution.

In the second case, corresponding to $m = -1$, the linear system obtained is

\[
\begin{pmatrix}
\theta^{-1} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
p \\ q
\end{pmatrix}
= -2
\begin{pmatrix}
(r - ru - st) \\ s - 2su
\end{pmatrix}
+ (\theta^{-1} - I_2)
\begin{pmatrix}
-1 \\ n
\end{pmatrix},
\]

which is equivalent to $up + tq = nt - 1 + u$ and has $p = r(tn-1+u) + ct$, $q = -s(tn-1+u) - cu$ as the general integer solution.
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In the previous proposition, an element \( \varphi \in \text{Aut}_0^0(E) \) of the first type is entirely determined by the integers \( c \) and \( n \). It is immediate to check that the map

\[
\Xi: \mathbb{Z} \oplus \mathbb{Z} \to \text{Aut}_0^0(E)
\]

\[
(n, c) \mapsto \varphi: E \to E
\]

\[
\varphi(d^x b^y) = d^x b^y \\
\varphi(a) = b^n a \\
\varphi(v) = d^{rn+ct} b^{-stn-cu} v
\]

is an injective group homomorphism.

**Theorem 4.4** There is a split exact sequence

\[
0 \rightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\Xi} \text{Aut}_0^0(E) \rightarrow \mathbb{Z}_2 \rightarrow 0
\]

and \( \text{Aut}_0^0(E) \cong (\mathbb{Z} \oplus \mathbb{Z}) \rtimes -I_2 \mathbb{Z}_2 \).

**Proof:** The homomorphism \( \text{Aut}_0^0(E) \to \mathbb{Z}_2 \) is defined by mapping to the generator of \( \mathbb{Z}_2 \) exactly those automorphisms \( \varphi \) such that \( \varphi(d^x b^y) = d^{-x} b^{-y} \), so the existence of the short exact sequence is clear. Also, a simple calculation shows that any \( \rho \in \text{Aut}_0^0(E) \) that satisfies \( \rho(d^x b^y) = d^{-x} b^{-y} \) also satisfies \( \rho^2 = 1_E \), so the sequence splits. In particular, for

\[
\rho(d^x b^y) = d^{-x} b^{-y} \\
\rho(a) = d^{-1} a \\
\rho(v) = d^{ru-1} b^r(1-u) v
\]

and

\[
\varphi(d^x b^y) = d^x b^y \\
\varphi(a) = b^n a \\
\varphi(v) = d^{rn+ct} b^{-stn-cu} v,
\]

we have

\[
\rho \varphi \rho(d^x b^y) = d^x b^y, \\
\rho \varphi \rho(a) = b^{-n} a, \\
\rho \varphi \rho(v) = d^{-rn-ct} b^{stn+cu} v,
\]

so the map \( \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z} \oplus \mathbb{Z}) \) takes the generator of \( \mathbb{Z}_2 \) to the matrix \(-I_2 \).
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Now we investigate the cosets $\text{Aut}_k^0(E)$ of $\text{Aut}_0^0(E)$ in $\text{Aut}_0(E)$. The next result shows that $\text{Aut}_k^0(E) \neq \emptyset$ when $k$ is even.

**Theorem 4.5** If $k = 2$, an extension $\varphi$ as in equation (11) exists.

**Proof:** If the extension $\varphi$ exists, we have $\ell = \ell_0$, $M_0^\ell = \varepsilon \theta$ ($\varepsilon = \pm 1$) and the relation $a^2 = d$ gives us

\[
d^m b^n v^2 ad^m b^n v^2 a = \varepsilon \theta(d) \iff d^m b^n v^2 d^n b^{-n} (ava^{-1})^2 a^2 = \varepsilon \theta(d)
\]

\[
d^m b^n \theta^2 (d^n b^{-n}) v^2 (d^{r-ru-st} b^{s-2su} v^{-1})^2 d = \varepsilon \theta(d)
\]

\[
\begin{pmatrix}
I_2 + \theta^2 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
m \\
n
\end{pmatrix}
+ (\theta + \theta^2)
\begin{pmatrix}
r - ru - st \\
2s - 2su
\end{pmatrix}
= (\varepsilon \theta - I_2)
\begin{pmatrix}
1 \\
0
\end{pmatrix}.
\]

We put the above system (with unknowns $m$, $n$) in the matrix form $A \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and analyze two cases, corresponding to $\varepsilon = \pm 1$. In both cases, we have

\[
A = \begin{pmatrix}
(ru + st)^2 + 4rstu + 1 & 4rt(ru + st) \\
-4su(ru + st) & -(ru + st)^2 - 4rstu + 1
\end{pmatrix},
\]

and note that $\det(A) = 0$. Hence the system will have (rational) solutions if, and only if, $A_{11}b_2 - A_{21}b_1 = 0$. Now we write $d = ru - st$ and notice that, in both cases, corresponding to the $+$ or $-$ signs, we have

\[
A_{11}b_2 - A_{21}b_1 = 2(d - 1)s(2st - 1) = 0.
\]

Assuming $\varepsilon = 1$, the linear system is equivalent to

\[(2st + 1)m + (2rt)n = 1 - r,
\]

which has integer solutions since $\gcd(2st + 1, 2rt) = 1$. The general integer solution $(m, n)$ is given by

\[
m = (ru + st - r) + c(2rt)
\]

\[
n = (s - 2su) - c(2st + 1).
\]

Still assuming $\varepsilon = 1$, we now consider the relation $ava^{-1} = d^{r-ru-st} b^{s-2su} v^{-1}$, which yields

\[
d^m b^n v^2 ad^m b^n b^q v^{-1} v^2 d^{-m} b^{-n} = \theta(d^{r-ru-st} b^{s-2su} v^{-1}) d^{-p} b^{-q}.
\]
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from which we obtain the following linear system in the unknowns $p$ and $q$:

$$(I_2 - \theta^{-1}) \begin{pmatrix} m \\ n \end{pmatrix} + \left( \theta^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \theta^{-1} \right) \begin{pmatrix} p \\ q \end{pmatrix} = (\theta - \theta^2) \begin{pmatrix} r - ru - st \\ s - 2su \end{pmatrix}.$$  

Writing the above system in the matrix form $C \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, we have $\det(C) = 0$ and hence the system has rational solutions if, and only if, $C_{11}e_2 - C_{21}e_1 = 0$. We get

$$C_{11}e_2 - C_{21}e_1 = (8s^2t^2 + 10st + 2)(2msu - 2su + 4s^2t + 2nyst + 2s) - ((2 - 4r)st + 2nyst + 2nrt)(-8s^2t - 2s)u.$$  

Substituting the expressions for $m$ and $n$ in the above expression, it factors as $4s(4st + 1)(u + 2cst^2 - 3st + ct - 1)(ru - st - 1) = 0$, so the linear system has rational solutions and is equivalent to

$$(8s^2t^2 + 10st + 2)p + (8rst^2 + 6rt)q = t(-4rs - 2cr)$$  

$$u(4ru - 3)p + t(4ru - 1)q = -t(2s + c),$$  

that has integer solutions for any $c \in \mathbb{Z}$ since $\gcd(u(4ru - 3), t(4ru - 1)) = 1$. In fact

$$\begin{array}{c|c|c|c|c|c} 
| d & 4ru - 3u \Rightarrow & d & 4ru^2t - 3ut \Rightarrow & d & 2ut \Rightarrow & d & 4rut \Rightarrow \\
| d & 4rut - t \Rightarrow & d & 4rut - t \Rightarrow & d & 4rut - t \Rightarrow & \\
| d & t \Rightarrow & d & 4rut - 3 \Rightarrow & d & t \Rightarrow & d & 4(1 + st) - 3 = 4st + 1 \Rightarrow d & 1. \\
| d & u(4ru - 3) \Rightarrow & d & 4ru - 3 \Rightarrow & d & t \Rightarrow & \\
\end{array}$$

Finally, in the case $\varepsilon = -1 \iff M_\theta^k = -\theta$, we also have extensions $\varphi$ since they are all obtained by composing with an appropriate element of $\text{Aut}^0(E)$.

The investigation of the cosets $\text{Aut}^k(E)$ for $k$ odd is greatly simplified now that the even case is understood. For $k$ odd, we have $\text{Aut}^k_0(E) \neq \emptyset \iff \text{Aut}^1_0(E) \neq \emptyset$. We start with a very simple lemma.

**Lemma 4.6** If neither $\theta(1)$ nor $-\theta(1)$ has a square root, there is no automorphism $\varphi$ with $k$ odd.

**Proof:** By the fundamental lemma, if there is an automorphism with $k$ odd, then we have $\ell_0$ even. Since $\theta(1) = \pm M_0^\ell$, there follows that $M_0^\ell$ is a square root of either $\theta(1)$ nor $-\theta(1)$. So the result follows.
Let $k = 1$, which by the fundamental lemma implies $\ell_0 = 2\ell$. As a consequence of the previous lemma, we need to consider two cases, the first being when $\theta(1)$ admits square root, and the second when $-\theta(1)$ admits square root.

Let us set up some notation. Because the two elements of the diagonal of $\theta(1)$ are the same, this is also the case with $M_0$. So let us write $M_0 = \begin{pmatrix} r & s \\ t & r \end{pmatrix}$, where $M_0$ is Anosov, so

$M_0^{\ell_0} = \begin{pmatrix} r^2 + st & 2rs \\ 2rt & r^2 + st \end{pmatrix}.$

According to Lemma 4.2, the two cases to analyze are as follows, knowing that the relation $ab = b^{-1}a$ is satisfied:

1) $\theta(1) = M_0^{\ell_0}$ (which implies $\det(M_0) = 1$), $\varphi(d^x b^y) = \delta M_0^{\ell_0} (d^x b^y)$, $\delta = \pm 1$.

2) $\theta(1) = -M_0^{\ell_0}$ (which implies $\det(M_0) = -1$), $\varphi(d^x b^y) = \delta M_0^{\ell_0} (d^x b^y)$, $\delta = \pm 1$.

We will show below that in the two cases above we have extensions $\varphi \in \text{Aut}_0^1(E)$ for $\delta = 1$. Composing with elements of $\text{Aut}_0^0(E)$, this means we also have extensions with $\delta = -1$. So we will assume that $\delta = 1$ in our analysis.

Consider the second relation $a^2 = d$. To study the equation $\varphi(a)^2 = \varphi(d)$, we need to have our sapphire defined, since we use the relation $ava^{-1}$, which was not the case for the first relation. Let $\theta(1)$ be as given in case 1 and consider the sapphire given by the matrix

$B = \begin{pmatrix} r & -t \\ -s & r \end{pmatrix}.$

The two fold cover is the torus bundle given by $\theta(1)$ and observe that $\det(N) = 1$.

**Theorem 4.7** In case 1, there is a $\varphi \in \text{Aut}_0^1(E)$.

**Proof:** Let $\varphi$ be as in (11) with $k = \delta = 1$. The relation $\varphi^2(a) = \varphi(d)$ is equivalent to the homogeneous linear system

$\begin{pmatrix} r^2 + st + 1 & -2rs \\ 2rt & -r^2 - st + 1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$

so we take the obvious solution $m = n = 0$. The last relation to be verified is then

$\varphi(ava^{-1}) = \varphi(d^{r^2 - r - 2r t + 2rt} v^{-1}),$
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which is equivalent to \(2(r^2 + st)p - (4rs)q = 2 - 2r\) and admits \(p = r^2 - r + st,\ q = t(2r - 1)\) as a solution. One explicit element \(\omega \in \text{Aut}^1_0(E)\) is then given by

\[
\begin{align*}
\omega(d^x b^y) &= M(d^x b^y), \\
\omega(v) &= d^{r^2 - r + st} b^{(2r - 1)} v, \\
\omega(a) &= va,
\end{align*}
\]

where \(M = M_0^t = \begin{pmatrix} r & s \\ t & r \end{pmatrix}\).

Case 2 is similar: we consider the sapphire given by the matrix

\[
B = \begin{pmatrix} r & -t \\ s & -r \end{pmatrix}
\]

and then we can state:

**Theorem 4.8** In case 2, there is also a \(\varphi \in \text{Aut}^1_0(E)\).

The proof of case 2 is completely analogous and will be omitted. We will, however, write down an explicit element \(\omega \in \text{Aut}^1_0(E)\) obtained in this case:

\[
\begin{align*}
\omega(d^x b^y) &= M(d^x b^y), \\
\omega(v) &= d^{-t} b^{-t} v, \\
\omega(a) &= va,
\end{align*}
\]

where \(M = M_0^t = \begin{pmatrix} r & s \\ t & r \end{pmatrix}\).

The coset \(\text{Aut}^1_0(E)\) of \(\text{Aut}^0_0(E)\) may be empty or not. In any case, we have a homomorphism \(\text{Aut}_0(E) \to \mathbb{Z}\) given by

\[
\varphi \in \text{Aut}^k_0(E) \mapsto k,
\]

whose image is either \(\mathbb{Z}\) or \(2\mathbb{Z} \cong \mathbb{Z}\), so there is a split short exact sequence

\[
1 \longrightarrow \text{Aut}^0_0(E) \longrightarrow \text{Aut}_0(E) \longrightarrow \mathbb{Z} \longrightarrow 1
\]

and we have
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**Theorem 4.9** \( \text{Aut}_0(E) \cong \text{Aut}_0^0(E) \times_\omega \mathbb{Z} \cong [(\mathbb{Z} \oplus \mathbb{Z}) \times_{-I_2} \mathbb{Z}_2] \times_\omega \mathbb{Z} \).

**Proof:** Following theorem 4.4 and considering the morphism $\Xi$ just before it, the standard generators of \((\mathbb{Z} \oplus \mathbb{Z}) \times_{-I_2} \mathbb{Z}_2\) corresponding to \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\) are $\alpha, \beta, \rho \in \text{Aut}_0^0(E)$ given by

\[
\begin{align*}
\alpha(b^y) &= d^r b^y \\
\beta(b^y) &= d^r b^y \\
\rho(b^y) &= d^{-r} b^{-y}
\end{align*}
\]

\[
\begin{align*}
\alpha(v) &= d^r b^{-st} v \\
\beta(v) &= d^r b^{-u} v \\
\rho(v) &= d^{(u-1)} b^{s(1-u)} v
\end{align*}
\]

\[
\begin{align*}
\alpha(a) &= b a \\
\beta(a) &= a \\
\rho(a) &= d^{-1} a.
\end{align*}
\]

We have two cases to consider. If $\text{Aut}_0^1(E) = \emptyset$, then $\omega$ is determined by an element of $\text{Aut}_0^2(E)$. One such element, that we also call $\omega$, is given by

\[
\begin{align*}
\omega(d^r b^y) &= \theta(d^r b^y) \\
\omega(v) &= d^{rst-2} b^{2su} v \\
\omega(a) &= d^{u+st-r} b^{2su} v^2 a.
\end{align*}
\]

Its inverse is given by

\[
\begin{align*}
\omega^{-1}(d^r b^y) &= \theta^{-1}(d^r b^y) \\
\omega^{-1}(v) &= b^{2s} v \\
\omega^{-1}(a) &= d^{-8rst+1} b^{4u-8rst+2s} v^{-2} a.
\end{align*}
\]

The calculation of $\omega \alpha \omega^{-1}$, $\omega \beta \omega^{-1}$ and $\omega \rho \omega^{-1}$ is extensive but straightforward. We get

\[
(\omega \alpha \omega^{-1})(d^r b^y) = d^r b^y
\]

\[
(\omega \alpha \omega^{-1})(v) = d^{rt(4st+1)} b^{-st(4st+3)} v \\
(\omega \alpha \omega^{-1})(a) = b^{ru+st} (a)
\]

\[
(\omega \beta \omega^{-1})(d^r b^y) = d^r b^y
\]

\[
(\omega \beta \omega^{-1})(v) = d^{r(4st+3)} b^{-u(4st+1)} v \\
(\omega \beta \omega^{-1})(a) = b^{2u} a
\]

\[
(\omega \rho \omega^{-1})(d^r b^y) = d^{-r} b^{-y}
\]

\[
(\omega \rho \omega^{-1})(v) = d^{4s^2 + 4rst + 5st - r + 1} b^{-s(1+4st+3u+4stu)} v \\
(\omega \rho \omega^{-1})(a) = d^{-1} b^{2s+2su} a
\]

\[
\Rightarrow \omega \alpha \omega^{-1} = \alpha^{ru+st} \beta^{2rst}
\]

\[
\Rightarrow \omega \beta \omega^{-1} = \alpha^{2u} \beta^{ru+st}
\]

\[
\Rightarrow \omega \rho \omega^{-1} = \alpha^{2s(u+1)} \beta^{2rs(u+1)} \rho
\]
The second case is \( \text{Aut}^1_0(E) \neq \emptyset \), which breaks into two subcases. The first is \( \theta(1) = M_0^{\ell_0} = (M_0^{\ell})^2 \), \( \det(M_0^{\ell}) = 1 \), where \( M_0^{\ell} = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ -s & r \end{pmatrix} \). The maps \( \alpha, \beta, \rho \) and \( \omega \) analogous to the ones considered in the first case are given by

\[
\begin{align*}
\alpha(d^x b^y) &= d^x b^y \\
\alpha(v) &= d^{-r} b^{-st} v \\
\alpha(a) &= ba \\
\omega(d^x b^y) &= M_0^{\ell}(d^x b^y) \\
\omega(v) &= d^{r-2st} b^{t(2r-1)} v \\
\omega(a) &= va
\end{align*}
\]

\[
\begin{align*}
\beta(d^x b^y) &= d^x b^y \\
\beta(v) &= d^{-s} b^{-r} v \\
\beta(a) &= a \\
\omega^{-1}(d^x b^y) &= M_0^{-\ell}(d^x b^y) \\
\omega^{-1}(v) &= d^{1-r} b^{-t} v \\
\omega^{-1}(a) &= d^{-2st+r+1} b^{t(2r-1)} v^{-1} a
\end{align*}
\]

What we get in this case is

\[
\begin{align*}
\omega \alpha \omega^{-1} &= \alpha^r \beta^{st} \\
\omega \beta \omega^{-1} &= \alpha \beta^r \\
\omega \rho \omega^{-1} &= \alpha^{-t} \beta^{-t(r+1)} \rho.
\end{align*}
\]

Finally, the second subcase is \( \theta(1) = -M_0^{\ell_0} = -(M_0^{\ell})^2 \), \( \det(M_0^{\ell}) = -1 \), where \( M_0^{\ell} = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ s & -r \end{pmatrix} \). The maps \( \alpha, \beta, \rho \) and \( \omega \) are now given by

\[
\begin{align*}
\alpha(d^x b^y) &= d^x b^y \\
\alpha(v) &= d^x b^{st} v \\
\alpha(a) &= ba \\
\omega(d^x b^y) &= M_0^{\ell}(d^x b^y) \\
\omega(v) &= d^{-r} b^{-t} v \\
\omega(a) &= va
\end{align*}
\]

\[
\begin{align*}
\beta(d^x b^y) &= d^x b^y \\
\beta(v) &= d^s b^{-r} v \\
\beta(a) &= a \\
\omega^{-1}(d^x b^y) &= M_0^{-\ell}(d^x b^y) \\
\omega^{-1}(v) &= dv \\
\omega^{-1}(a) &= d^{2st-1} b^{-2rt} v^{-1} a
\end{align*}
\]
Once again we get

\[ \omega_1 \omega^{-1} = \alpha^r \beta^t \]
\[ \omega_2 \omega^{-1} = \alpha \beta^r \]
\[ \omega \rho^{-1} = \alpha^{-t} \beta^{-t(r+1)} \rho. \]

Let’s now consider the type II automorphisms. First we observe that the group \((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}\), where

\[ \theta(1) = \begin{pmatrix} ru + st & -2rt \\ -2su & ru + st \end{pmatrix}, \]

always admits automorphisms that induces \(-id_\mathbb{Z}\) on the quotient \([\mathbb{Z} \oplus \mathbb{Z}] / (\mathbb{Z} \oplus \mathbb{Z})\), and the restrictions to \((\mathbb{Z} \oplus \mathbb{Z})\) can always be written on the form \(\pm B_0 M_0 \ell\) for some \(\ell \in \mathbb{Z}\), where \(B_0\) satisfies \(B_0 \theta(1) B_0^{-1} = \theta^{-1}\). A solution for \(B_0\) on the latter equation is given by

\[ B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

see Proposition 5.8 in the Appendix.

If \(\varphi_1\) and \(\varphi_2\) are two automorphisms of \(E\) whose restrictions to \((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}\) are Type II automorphisms, then the restriction of \(\varphi_1 \varphi_2^{-1}\) to \((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}\) is a Type I automorphism. Hence, if we denote by Aut\(_0(\mathcal{E})\) the elements of Aut\((\mathcal{E})\) that induce Type I automorphisms on \((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}\), we have the short exact sequence of groups

\[ 1 \longrightarrow \text{Aut}_0(\mathcal{E}) \longrightarrow \text{Aut}(\mathcal{E}) \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{13} \]

The generator of the quotient \(\mathbb{Z}_2\) is the class of any element of Aut\((\mathcal{E})\) that induces a Type II automorphism on \((\mathbb{Z} \oplus \mathbb{Z}) \rtimes_\theta \mathbb{Z}\). One such family of automorphisms is given by

\[ \zeta : E \rightarrow E \]
\[ \zeta(d^p b^q) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \]
\[ \zeta(v) = d^p b^q v^{-1}, \]
\[ \zeta(a) = a, \]
where we must find all $p$ and $q$ for which the map above extends to an automorphism. The calculation leads to

\[
p = r - (ru + st) + \lambda \frac{(ru + st - 1)}{\gcd\{-2su, ru + st - 1\}} = r - (ru + st) + \lambda t,
\]

\[
q = (s - 2su) - \lambda \frac{(-2su)}{\gcd\{-2su, ru + st - 1\}} = (s - 2su) + \lambda u,
\]

for any $\lambda \in \mathbb{Z}$, since $ru + st - 1 = 2st$ and $\gcd(2su, 2st) = 2s$.

For $\lambda = s$, we have $p = r - ru$ and $q = s - su$ and a direct computation shows that $\zeta^2 = 1_E$. Hence we have

**Theorem 4.10** The short exact sequence

\[
1 \to \text{Aut}_0(E) \to \text{Aut}(E) \to \mathbb{Z}_2 \to 1
\]

splits, hence

\[
\text{Aut}(E) \cong \text{Aut}_0(E) \rtimes_\alpha \mathbb{Z}_2 \cong [(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{-I_2} \mathbb{Z}_2] \rtimes_\omega \mathbb{Z} \rtimes_\zeta \mathbb{Z}_2.
\]

**Proof:** The structure of $\text{Aut}(E)$ is clear. We just need to compute the action of $\zeta$, where

\[
\zeta(d^x b^y) = d^x b^{-y},
\]

\[
\zeta(v) = d^{r-ru}b^{s-su}v^{-1}
\]

\[
\zeta(a) = a
\]

is an automorphism of type II of order 2. We will follow the notation from theorem 4.9.

When $\text{Aut}_1(E) = \emptyset$, we get

\[
\zeta\alpha \zeta = \alpha^{-1}, \quad \zeta\beta \zeta = \beta, \quad \zeta\rho \zeta = \rho, \quad \zeta\omega \zeta = \alpha^{-2s(u+1)}\beta^{2rs(u+1)}\omega^{-1}.
\]

When $\text{Aut}_0(E) \neq \emptyset$, we have the same two subcases considered in theorem 4.9. In the first subcase, we have $\theta(1) = M_0^0 = (M_0^0)^2$, $\det(M_0^1) = 1$, where $M_0^1 = \begin{pmatrix} r & s \\ t & r \end{pmatrix}$ and the sapphire is defined by $B = \begin{pmatrix} r & -t \\ s & r \end{pmatrix}$. The map $\zeta$ is given by

\[
\zeta(d^x b^y) = d^x b^{-y},
\]

\[
\zeta(v) = d^{(r-1)}b^{(r-1)}v^{-1},
\]

\[
\zeta(a) = a.
\]
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and we get
\[ \zeta \alpha \zeta = \alpha^{-1}, \quad \zeta \beta \zeta = \beta, \quad \zeta \rho \zeta = \rho, \quad \zeta \omega \zeta = \alpha^t \beta^{-(r+1)} \omega^{-1}. \]

In the second subcase, we have \( \theta(1) = -M_0^{\ell_0} = -(M_0^{\ell_0})^2 \), \( \det(M_0^{\ell_0}) = -1 \), where \( M_0^{\ell} = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ s & -r \end{pmatrix} \). The map \( \zeta \) is given by
\[
\zeta(d^x b^y) = d^x b^y - y, \\
\zeta(v) = d^{r(r+1)} b^{-t(r+1)} v^{-1} - t, \\
\zeta(a) = a,
\]
and we get
\[ \zeta \alpha \zeta = \alpha^{-1}, \quad \zeta \beta \zeta = \beta, \quad \zeta \rho \zeta = \rho, \quad \zeta \omega \zeta = \alpha^t \beta^{rt} \rho \omega^{-1}. \]

Now we compute Out\((E)\). If \( g \in E \), let's denote by \( \kappa_g \in \text{Inn}(E) \) the conjugation by \( g: \kappa_g(x) = g x g^{-1} \). We will use the notations from theorems 4.9 and 4.10.

If Aut\(_1^0(E) = \emptyset\), we have
\[
\kappa_d = \beta^{-2s}, \quad \kappa_b = \alpha^2, \quad \kappa_v = \alpha^{-2s} \beta^{-2rs} \omega, \quad \kappa_a = \beta^{-s} \zeta,
\]
hence a presentation of Out\((E)\) is given by
\[
\text{Out}(E) = \langle \alpha, \beta, \rho, \omega, \zeta \mid \alpha \beta = \beta \alpha, \rho^2 = 1, \rho \alpha \rho = \alpha^{-1}, \rho \beta \rho = \beta^{-1}, \\
\omega \alpha \omega^{-1} = \alpha^r u + s + t \beta^2 r s t, \omega \beta \omega^{-1} = \alpha^{2s(u+1)} \beta^2 r s (u+1) \rho, \\
\zeta^2 = 1, \zeta \alpha \zeta = \alpha^{-1}, \zeta \beta \zeta = \beta, \\
\zeta \rho \zeta = \rho, \zeta \omega \zeta = \alpha^{-2s(u+1)} \beta^2 r s (u+1) \omega^{-1}, \\
\beta^{-2s} = \alpha^2 = \alpha^{-2s} \beta^{-2rs} \omega = \beta^{-s} \zeta = 1 \rangle.
\]

We have \( \zeta = \beta^s \), and \( \omega = \alpha^{2s} \beta^{2rs} = 1 \), so
\[
\text{Out}(E) = \langle \alpha, \beta, \rho, \omega \mid \alpha^2 = \beta^{2s} = \rho^2 = 1, \\
\alpha \beta = \beta \alpha, \rho \alpha \rho = \alpha, \rho \beta \rho = \beta^{-1} \rangle
\]
and Out\((E) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_{2s}) \rtimes_{-1} \mathbb{Z}_2\).

If Aut\(_1^0(E) \neq \emptyset\), as usual we have two subcases: in the first one, when \( \theta(1) = M_0^{\ell_0} = (M_0^{\ell_0})^2 \), \( \det(M_0^{\ell_0}) = 1 \), \( M_0^{\ell} = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ s & -r \end{pmatrix} \), we get
\[
\kappa_d = \beta^{2t}, \quad \kappa_b = \alpha^2, \quad \kappa_v = \alpha^{2t} \beta^{2rt} \omega^2, \quad \kappa_a = \beta^t \zeta,
\]
hence a presentation of $\text{Out}(E) \cong [(\mathbb{Z}_2 \oplus \mathbb{Z}_2t) \rtimes_{-1} \mathbb{Z}_2] \rtimes \omega \mathbb{Z}_2$ is given by

$$\text{Out}(E) = \langle \alpha, \beta, \rho, \omega \mid \alpha^2 = \beta^{2t} = \rho^2 = \omega^2 = 1, \alpha \beta = \beta \alpha, \rho \alpha \rho = \alpha, \rho \beta \rho = \beta^{-1}, \omega \alpha \omega^{-1} = \alpha^r \beta^s, \omega \beta \omega^{-1} = \alpha \beta^r, \omega \rho \omega^{-1} = \alpha^t \beta^{(r+1) \rho} \rangle.$$ 

Finally, when $\theta(1) = -M_0^e = -(M_0^e)^2$, $\det(M_0^e) = -1$, $M_0^e = \begin{pmatrix} r & s \\ t & r \end{pmatrix}$ and the sapphire is defined by $B = \begin{pmatrix} r & -t \\ s & -r \end{pmatrix}$, we have

$$\kappa_d = \beta^{2t}, \quad \kappa_b = \alpha^2, \quad \kappa_v = \beta^{-1} \rho \omega^2, \quad \kappa_a = \beta^t \zeta,$$

hence $\text{Out}(E)$ fits in the short exact sequence

$$1 \longrightarrow (\mathbb{Z}_2 \oplus \mathbb{Z}_2t) \rtimes_{-1} \mathbb{Z}_2 \longrightarrow \text{Out}(E) \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

and a presentation is given by

$$\text{Out}(E) = \langle \alpha, \beta, \rho, \omega \mid \alpha^2 = \beta^{2t} = \rho^2 = 1, \alpha \beta = \beta \alpha, \rho \alpha \rho = \alpha, \rho \beta \rho = \beta^{-1}, \omega^2 = \beta^t \rho, \omega \alpha \omega^{-1} = \alpha^r \beta^s, \omega \beta \omega^{-1} = \alpha \beta^r, \omega \rho \omega^{-1} = \alpha^t \beta^{(r+1) \rho} \rangle.$$ 

As a result of the above we obtain:

**Theorem 4.11**

1) If $\text{Aut}_0^1(E) = \emptyset$ we have

$$\text{Out}(E) = \langle \alpha, \beta, \rho, \omega \mid \alpha^2 = \beta^{2s} = \rho^2 = 1, \alpha \beta = \beta \alpha, \rho \alpha \rho = \alpha, \rho \beta \rho = \beta^{-1} \rangle,$$

hence $\text{Out}(E) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_2s) \rtimes_{-1} \mathbb{Z}_2$. 

II) If \( \text{Aut}_1(E) \neq \emptyset \) and \( \theta(1) = M_{0}^{f_{0}} = (M_{0}^{f})^2 \), \( \text{det}(M_{0}^{f}) = 1 \), \( M_{0}^{f} = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ -s & r \end{pmatrix} \), we have \( \text{Out}(E) \cong [(\mathbb{Z}_2 \oplus \mathbb{Z}_{2t}) \times_{-1} \mathbb{Z}_2] \times_{\omega} \mathbb{Z}_2 \)
where
\[
\omega \alpha \omega^{-1} = \alpha^r \beta^{st}, \quad \omega \beta \omega^{-1} = \alpha \beta^r, \quad \omega \rho \omega^{-1} = \alpha^t \beta^{t(r+1)} \rho.
\]

III) If \( \text{Aut}_1(E) \neq \emptyset \) and \( \theta(1) = -M_{0}^{f_{0}} = -(M_{0}^{f})^2 \), \( \text{det}(M_{0}^{f}) = -1 \), \( M_{0}^{f} = \begin{pmatrix} r & s \\ t & r \end{pmatrix} \) and the sapphire is defined by \( B = \begin{pmatrix} r & -t \\ s & -r \end{pmatrix} \), we have a short exact sequence
\[
1 \longrightarrow (\mathbb{Z}_2 \oplus \mathbb{Z}_{2t}) \times_{-1} \mathbb{Z}_2 \longrightarrow \text{Out}(E) \longrightarrow \mathbb{Z}_2 \longrightarrow 1
\]
and a presentation of \( \text{Out}(E) \) is given by
\[
\text{Out}(E) = \langle \alpha, \beta, \rho, \omega \mid \alpha^2 = \beta^{2t} = \rho^2 = 1, \quad \alpha \beta = \beta \alpha, \quad \rho \alpha \rho = \alpha, \quad \rho \beta \rho = \beta^{-1}, \quad \omega^2 = \beta^t \rho, \quad \omega \alpha \omega^{-1} = \alpha^r \beta^{st}, \quad \omega \beta \omega^{-1} = \alpha \beta^r, \quad \omega \rho \omega^{-1} = \alpha^t \beta^{t(r+1)} \rho \rangle.
\]

5 Appendix

The main goal of this appendix is to provide a classification of the Torus bundles \( E \) which admits a homeomorphism such that the induced automorphism on the quotient of the fundamental group \( \pi_1(E)/(\mathbb{Z} \oplus \mathbb{Z}) \cong \mathbb{Z} \) is multiplication by \(-1\). We also show some results about \( \text{Aut}(E) \).

5.1 Equations on \( GL_2(\mathbb{Z}) \)

We begin by recalling some results from [19] which are relevant for our purpose. Given \( A \in GL_2(\mathbb{Z}) \), \( M_A \) denotes the torus bundles associated to the homeomorphism of the torus determined by the matrix \( A \).
Lemma 5.1 [19, Lemma 1.1] Let $A$ and $B$ be matrices in $GL_2(\mathbb{Z})$. Then the following conditions are equivalent:

1. $M_A$ is homeomorphic to $M_B$.
2. $\pi_1(M_A)$ is isomorphic to $\pi_1(M_B)$.
3. The $\mathbb{Z}\langle t \rangle$-module $H_A$ is isomorphic or anti-isomorphic to the $\mathbb{Z}\langle t \rangle$-module $H_B$.
4. $A$ is conjugate to $B$ or $B^{-1}$.

For a matrix $A$ in $GL_2(\mathbb{Z})$ we define the following sets:

Definition 5.2 (1) For a matrix $A$ in $GL_2(\mathbb{Z})$ let $C(A) = \{B \in GL_2(\mathbb{Z}) | BAB^{-1} = A \}$ and let $R(A) = \{B \in GL_2(\mathbb{Z}) | BAB^{-1} = A^{-1} \}$.

2. $A$ is called exceptional, if one of the following conditions is satisfied:
   (i) $\det(A) = 1$ and $|\text{tr}(A)| \leq 2$.
   (ii) $\det(A) = -1$ and $\text{tr}(A) = 0$.

3. $A$ is called Anosov if $A$ is not exceptional.

One way to provide the classification of the matrices $A$ such that $R(A) \neq \emptyset$ is by means of Lemma 1.7, item 2, in [19]. Unfortunately the criterion is not too practical to be used in our present work. Once we know that for a given Torus bundle determined by a matrix $A$ there is a $B_0 \in R(A)$, then it is easy to see that $R(A) = \{B \in GL_2(\mathbb{Z}) : B\theta B^{-1} = \theta^{-1} \}$ is the set of elements of the form $B_0C$ as $C$ runs over $C(A)$.

We have seen that given $\theta$ if $R(\theta) = \emptyset$, then $\text{Aut}(E) = \text{Aut}_0(E)$ and this group is given by Proposition 3.2.

So suppose that the equation $B\theta B^{-1} = \theta^{-1}$ has a solution $B_0$. We provide an explicit classification, up to conjugacy class, of the matrices $\theta$ where this equation admits at least one solution $B_0$. Then we use the result to describe $\text{Aut}(E)$, $\text{Out}(E)$ which is given in Theorems 3.5, 3.6, respectively.

Let $A$ be an Anosov matrix. We quote the result that $C(A)$ is isomorphic $\mathbb{Z} \oplus \mathbb{Z}_2$. The generator of $\mathbb{Z}_2$ corresponds to the matrix $-I_2$. Let $A_0$ be a generator of the summand $\mathbb{Z}$, which is certainly a primitive element, i.e. it is not a proper power of another matrix. There exists an integer $\ell$ such that $A_0^\ell$ is either $A$ or $-A$, but we can not guarantee the equality $A_0^\ell = A$. For suppose that the equality holds and $\ell$ is even. Take $A_1 = -A_0$. 
Both elements $A_0$ and $-A_0$ to the power $\ell$ give $A$. In any case if $A$ has infinite order then the elements of $C(A)$ are of the form $\pm M_0^k$ as $k$ runs over the integers, where $M_0 \in C(A)$ is a primitive root of $A$. So we have to consider the two possibilities.

We quote now Lemma 1.7 from [19].

**Lemma 5.3** If $A$ is exceptional then it is conjugate to one and only one of the following matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} (n \geq 0), \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Moreover we have the following:

- $C\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \left\{ \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathbb{Z}_4$
- $R\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$
- $C\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\} \cong \mathbb{Z}_6$
- $R\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C\left(\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right)$
- $C\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) = \left\{ \pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} (k \in \mathbb{Z}) \right\} \cong \mathbb{Z} \oplus \mathbb{Z}_2$ ($n \neq 0$)
- $R\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C\left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right)$ ($n \neq 0$)
- $C\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = R\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$
- $C\left(\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right) = R\left(\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\right) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \right\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

**Corollary 5.4** Let $A = \varepsilon A_0^n$, $\varepsilon \in \{\pm 1\}$, be an Anosov matrix where $A_0$ is a primitive root of $A$. Then $C(A) = \{ \pm A_0^i \mid i \in \mathbb{Z} \} \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

From the results above we will show:

**Corollary 5.5** Suppose that $A$ has infinite order. If $B_0 A B_0^{-1} = A^{-1}$ and $B_0$ is of finite order then $B_0$ is conjugate to one of the following 3 matrices:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$
which have orders equal to 4, 2 and 2, respectively. Furthermore, $C(A) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ if and only if $A$ has infinite order.

Proof: By the Lemma 5.3 it suffices to show that $B_0$ cannot have order either 3 or 6. If $B_0^3 = I_2$ it follows that

$$A = B_0^3 AB_0^{-3} = (B_0(B_0(B_0AB_0^{-1})B_0^{-1})B_0^{-1} = (B_0(B_0(A^{-1})B_0^{-1})B_0^{-1} = (B_0AB_0^{-1}) = A^{-1}. $$

So $A = A^{-1}$ or $A^2 = I_2$, which is a contradiction since $A$ has infinite order. If $B_0^6 = I_2$ it follows that $B_0$ is conjugate of $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, so $B_0^3$ is conjugate of the cube of $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, which is $-I_2$. Therefore $B_0^3 AB_0^{-3} = (-I_2)A(-I_2) = A$. But as before, $B_0^3 AB_0^{-3} = A^{-1}$ and we get a contradiction. The “furthermore” part follows promptly from the two Lemmas above.

Observe that given any matrix, if it has order 4, then it is conjugate to the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If it has order 2, then we look at the matrix mod 2 and then, if it is the identity, it is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, otherwise it has order two and it is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$.

**Lemma 5.6** If $A \in GL_2(\mathbb{Z})$ has infinite order, $C(A) = C(-A) = C(A^k)$ for $k \neq 0$. In particular, this holds if $A$ is Anosov.

Proof: Straightforward.

Using the lemma above we show the following proposition, which is interesting in its own right.

**Proposition 5.7** Let $A$ be Anosov and $B_0$ a matrix such that $B_0AB_0^{-1} = A^{-1}$. Then $B_0^2 = \pm I_2$ and $B_0 \neq -I_2$. 

Proof: The first observation is that $B_0$ is not Anosov. In fact, if $B_0$ is Anosov, then $B_0^3 A B_0^{-2} = A \Rightarrow A \in C(B_0^3) = C(B_0)$, hence $B_0 A B_0^{-1} = A = A^{-1}$, which cannot happen for $A$ Anosov.

Given that $B_0$ is exceptional, suppose first that $B_0$ has infinite order, that is, it is conjugate to $\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for some $n \neq 0$. So for an appropriate matrix $A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, conjugate of $A$, and for $B_1 = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, we have $B_1 A_1 B_1^{-1} A_1 = I_2$ if, and only if,

\[
\begin{vmatrix}
  a^2 - n^2 c^2 + bc + ncd \\
  c(a - nc + d) \\
  ab + nbc - nad - n^2 cd + bd + nd^2 \\
  bc - ncd + d^2
\end{vmatrix} = 1
\]

If $c = 0$, then we have $a^2 = d^2 = 1$ and $A$ is not Anosov. Hence $c \neq 0$ and the system is equivalent to

\[
\begin{vmatrix}
  (a - nc)(a + nc) + bc + ncd \\
  a - nc = -d \\
  ab + nbc - nad - n^2 cd + bd + nd^2 = 0 \\
  bc - ncd + d^2 = 1
\end{vmatrix} \iff \begin{vmatrix}
  ad - bc = -1 \\
  a - nc = -d \\
  ab + n - n^2 cd + bd + nd^2 = 0 \\
  bc - ncd + d^2 = 1
\end{vmatrix}
\]

Solving the third and fourth equations for $b$ making use of equation (2) yields to $b = \frac{1 - d^2 + ncd}{c} = \frac{-1 - d^2 + ncd}{c}$, which is a contradiction.

Proposition 5.8 Let $B_0 A B_0^{-1} = A^{-1}$. The solutions of this equation for $A$ are:

I) If $B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, where $ad - b^2 = 1$ or $\det(A) = 1$.

II) If $B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, where $a^2 - bc = 1$ or $\det(A) = 1$. 

Now if we assume that $B_0$ is any of the 3 matrices of the statement of Corollary 5.5, we can find all matrices $A$ of infinite order such that $B_0 A B_0^{-1} = A^{-1}$. 


III) If \( B_0 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \), then either

\[
A = \begin{pmatrix} a & b \\ d-a & d \end{pmatrix}
\]
for arbitrary \( a, b, d \) such that \( \det(A) = 1 \) or

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}
\]
or

\[
A = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

Proof: Case I: \( B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). In this case by solving the system \( B_0 A B_0^{-1} = A^{-1} \) we find that the solution for the matrices \( A \) are of the form \( A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \), where \( ad - b^2 = 1 \) or \( \det(A) = 1 \).

Case II: \( B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). In this case by solving the system \( B_0 A B_0^{-1} = A^{-1} \) we find that \( A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \), where \( a^2 - bc = 1 \) or \( \det(A) = 1 \).

Case III: \( B_0 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \). In this case by solving the system \( B_0 A B_0^{-1} = A^{-1} \) for \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) we find that \( c = 0 \) or \( c = d - a \). For \( c = 0 \) we have the following matrices as solutions:

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}
\]

for arbitrary \( b \).

For \( c = d - a \) we have the matrix \( A = \begin{pmatrix} a & b \\ d-a & d \end{pmatrix} \). By straightforward calculation the equation \( B_0 A B_0^{-1} = A^{-1} \) holds if and only if \( ad - bd + ab = 1 \), i.e. if \( \det(A) = 1 \) and \( d^2 - ab = 1 \). So the result follows.

\[\blacksquare\]

Remark: a) The Proposition above defines three families \( F_1, F_2, F_3 \) matrices in \( GL_2(\mathbb{Z}) \) given by the items I), II) and III), respectively. Namely \( F_i \) is the family of matrices which are conjugated to one of the matrices given by the item \( i \). Certainly the families are not disjoint since \( \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \) belong to \( F_1 \) and \( F_3 \).
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b) We claim that $F_3$ is not contained in the union of the other two families. For let $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, which belongs to the family $F_3$. The matrix $A$ cannot belong to the family $F_2$ since its trace $A$ is odd. We also claim that that $A$ cannot belong to the family $F_1$, i.e. $A$ is not conjugated to a matrix of the form $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ having determinant $1$. This can be verified is by directly calculation. Suppose that $A$ is conjugated to a matrix of the form $A_1 = \begin{pmatrix} a & b \\ b & 5-a \end{pmatrix}$, det$(A_1) = 1$ i.e. $a^2 - 5a + b^2 + 1 = 0$. But we claim that the quadratic equation $a^2 - 5a + b^2 + 1 = 0$ does not admit an integer solution. For the discriminant of the equation is $\Delta = 25 - 4b^2 = 21 - 4b^2$ and must be $\geq 0$. This implies that $|b| \leq 2$ so we have three possibilities which are $|b| = 0, |b| = 1, |b| = 2$. But for $|b| = 0$ we have $\Delta = 21$, for $|b| = 1$ we have $\Delta = 21$, for $|b| = 2$ we have $\Delta = 5$. So in all these 3 cases the square root of $\Delta$ is not an integer and the result follows.

We do not know if the same happens for $F_1$ and for $F_2$, i.e., if one of these families are contained in the union of the other two families.

Now we prove the major result.

**Theorem 5.9** If $A$ has infinite order and $B_0 A B_0^{-1} = A^{-1}$ then $B_0$ is conjugate to one of the three matrices given by Corollary 5.5 and $A$ is conjugate to one of the matrices $A$ given by Proposition 5.8. Further, if $A$ is Anosov then $A$ is conjugate to one of the matrices of the form

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \text{ where det}(A) = 1 \text{ and } |a + d| > 2, \text{ or}$$

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}, \text{ where det}(A) = 1 \text{ and } |2a| > 2, \text{ or}$$

$$A = \begin{pmatrix} a & b \\ d-a & d \end{pmatrix}, \text{ where det}(A) = 1 \text{ and } |a + d| > 2.$$

**Proof:** The proof follows immediatly from Proposition 5.8.

Remark: We do not know a practical algorithm to decide if an arbitrary matrix is conjugate to one of the matrices of the families $F_1, F_2$ and $F_3$. In [10] examples of Anosov
matrices $A$ where $R(A)$ is non empty were given, to illustrate some result in fixed point theory, where the examples are matrices which belong to $\mathbb{F}_1$.

Based on the result above, we can deduce the following consequence for fixed point theory of the spaces mapping torus: It follows from [10] and the Theorem 5.9 that in order to have an automorphism of $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_A \mathbb{Z}$ which has Reidemeister finite, where $A$ is Anosov, $A$ is conjugated to one of the matrices of the form given by Theorem 5.9, i.e.

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \text{ where } ad - b^2 = 1, \text{ i.e. } \det(A) = 1,$$

or

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}, \text{ where } a^2 - bc = 1, \text{ i.e. } \det(A) = 1,$$

or

$$A = \begin{pmatrix} a & b \\ d-a & d \end{pmatrix}, \text{ where } ad - bd + ab = 1, \text{ i.e. } \det(A) = 1.$$

We claim that in the former case we are able to construct an automorphism with Reidemeister number finite. In fact they are the automorphisms which when restricted to $\mathbb{Z} \oplus \mathbb{Z}$ have matrix $B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

For the second case we claim that it is not possible, since the product of the matrix

$$B_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

by any matrix of the form

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

is a matrix which has Reidemeister number infinite.

For the third case we also claim that it is not possible to find such examples in fixed point theory. For the product of the matrix $B_0 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

by matrix of the form

$$A = \begin{pmatrix} a & b \\ d-a & d \end{pmatrix}$$

is the matrix $\begin{pmatrix} d & b+d \\ a-d & -d \end{pmatrix}$ which has Reidemeister infinite since $\det(A) = 1$. The conclusion is that the type of examples provided in [10] are the only ones which have Reidemeister finite. By type we mean that the elements of the anti-diagonal are the same.

### 5.2 About $\text{Aut}_0^1(E)$

We have seen that our main result about $\text{Aut}(E)$ and $\text{Out}(E)$ depends if $\text{Aut}_0^1(E)$ is empty or not, which in turn is equivalent to say that a certain matrix has a square root or not. Here we show a lemma which describes for certain matrices, closely related to our study,
its square roots. Then this can be used to decide if $\text{Aut}^1_0(E)$ is empty or not.

A practical criterion to decide if a matrix has a square root was given by Paulo Agozzini, [1]. He shows:

**Lemma 5.10** The square roots of a matrix $A$ are either $\pm(A + \text{Id})/\sqrt{T + 2}$ or $\pm(A - \text{Id})/\sqrt{T - 2}$ where the matrices which are the square roots are in $GL_2(\mathbb{Z})$. Otherwise $A$ does not admit a square root.

It follows from the criterion given by Lemma 5.10 that if a matrix has a square root, then at most one of the two numbers $T + 2$, $T - 2$ has integral square root. In our case we have $T = 2x$ and we have the two numbers $2x + 2$ and $2x - 2$. Besides one of the two numbers having integral root square, all the enters of the matrix $2y$ and $2z$ must be divisible by the square root.

Now we apply the lemma above in order to describe all matrices of the form $\begin{pmatrix} x & 2y \\ 2z & x \end{pmatrix}$, which admit square root.

**Lemma 5.11** Consider the matrices of the form $\begin{pmatrix} 2\lambda^2 + 1 & 2\lambda y_1 \\ 2\lambda z_1 & 2\lambda^2 + 1 \end{pmatrix}$, where $\lambda$ runs over $\mathbb{Z}$ and all $y_1, z_1$ such that $y_1z_1 = \lambda^2 + 1$

or $\begin{pmatrix} 2\lambda^2 - 1 & 2\lambda y_1 \\ 2\lambda z_1 & 2\lambda^2 - 1 \end{pmatrix}$, where $\lambda$ runs over $\mathbb{Z}$ and all $y_1, z_1$ such that $y_1z_1 = \lambda^2 - 1$.

If a matrix of the form $\begin{pmatrix} x & 2y \\ 2z & x \end{pmatrix}$, admits a square root, then it is one of the matrices above.

**Proof:** Certainly the matrices of the form given by Lemma 5.11 admit a square root. Now using Lemma 5.10 we obtain the result.

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**References**

[1] Paulo Agozzini, Private communication.

[2] Alexandre P. Barreto, Daciberg L. Gonçalves and Daniel Vendrúscolo, Free involutions on torus semi-bundles and the Borsuk-Ulam Theorem for maps into $R^n$. *Hiroshima Math. J.*, 46: 255–270, 2016.
The group $\text{Aut}$ and $\text{Out}$ of the fundamental group of a closed Sol 3-manifold

[3] Lucídia D. Borsari and Daciberg L. Gonçalves, The first group (co)homology of a group $G$ with coefficients in some $G$-modules. *Quaestiones Mathematicae*, 31:89–100, 2008.

[4] Kenneth S. Brown, *Cohomology of groups*. Springer, 1982.

[5] J. Buckley, Automorphisms groups of isoclinic $p$-groups. *J. London Math. Soc.*, 12:37–44, 1975.

[6] Edward Fadell and Suffian Husseini, The Nielsen number on surfaces. *Topological Methods in nonlinear functional analysis, Contemporary Mathematics*, 21:59–98, 1983.

[7] R. H. Fox and R. H. Crowell, *Introduction to Knot Theory*. Ginn and Co., Boston, Mass., 1963.

[8] Marek Golasinski and Daciberg L. Gonçalves, On automorphisms of split metacyclic groups. *Manuscripta Math.*, 128:251–273, 2009.

[9] Daciberg L. Gonçalves and Edson de Oliveira, The Lefschetz coincidence number for maps among compact surfaces. *Far East Journal of Math Science*, 2:147–166, 1997.

[10] Daciberg L. Gonçalves and Peter Wong, Twisted conjugacy classes in exponential growth groups. *Bull. London Math. Soc.*, 35:261–268, 2003.

[11] Daciberg L. Gonçalves and Peter Wong, Nielsen numbers of self homeomorphisms of Sol 3-manifolds. *Top. and its Applic.*, 159(18):3729–3737, 2012.

[12] Daciberg L. Gonçalves and John Guaschi, The Borsuk-Ulam theorem for maps into a surface. *Top. and its Applic.*, 157:1742–1759, 2010.

[13] Jonathan A. Hillman. *Four-manifolds, geometries and knots*, University of Warwick, Mathematics Institute: Geometry and Topology Publications, 2002.

[14] Sungbok Hong and Darryl McCullough, Mapping class groups of 3-manifolds, then and now. *Geometry and topology down under, Contemp. Math.*, 597: 53–63, Amer. Math. Soc., Providence, RI, 2013.

[15] D. L. Johnson, *Presentations of groups*. London Mathematical Society Student Texts 15, 2nd edition, Cambridge University Press, Cambridge, 1997.
[16] Roger C. Lyndon, Cohomology theory of groups with a single defining relation. *Annals of Mathematics*, 52(3):650–665, 1950.

[17] K. Morimoto, Some orientable 3-manifolds containing Klein bottles. *Kobe J. Math.*, 2:37–44, 1985.

[18] C. A. Robinson, Moore-Postnikov systems for non-simple fibrations. *Illinois J. Math.*, 16(2):234–242, 1972.

[19] M. Sakuma, Involutions on torus bundles over $S^1$. *Osaka J. Math.*, 22:163–185, 1985.

[20] H. Sun, S. Wang and J. Wu, Self-mapping degrees of torus bundles and torus semi-bundles. (English summary) *Osaka J. Math.* 47 no. 1: 131–155, 2010.

[21] Satoshi Tomoda and Peter Zvengrowski, Remarks on the cohomology of finite fundamental groups of 3-manifolds. *Geometry & Topology Monographs*, 14:519–556, 2008.

[22] C. T. C. Wall, Resolutions for extensions of groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 57:251–255, 1961.

[23] C. Wells, Automorphisms of group extensions. *Trans. Amer. Math. Soc.*, 155:189–194, 1971.

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