How to obtain a cosmological constant from small exotic $\mathbb{R}^4$

Torsten Asselmeyer-Maluga  
German Aerospace Center (DLR), 10178 Berlin, Germany  
torsten.asselmeyer-maluga@dlr.de  
and  
Jerzy Król  
Institute of Physics, University of Silesia, 40-007 Katowice, Poland  
erzy.krol@us.edu.pl

December 7, 2017

Abstract

In this paper we determine the cosmological constant as a topological invariant by applying certain techniques from low dimensional differential topology. We work with a small exotic $\mathbb{R}^4$ which is embedded into the standard $\mathbb{R}^4$. Any exotic $\mathbb{R}^4$ is a Riemannian smooth manifold with necessary non-vanishing curvature tensor. To determine the invariant part of such curvature we deal with a canonical construction of $\mathbb{R}^4$ where it appears as a part of the complex surface $K3\#CP(2)$. Such $\mathbb{R}^4$’s admit hyperbolic geometry. This fact simplifies significantly the calculations and enforces the rigidity of the expressions. In particular, we explain the smallness of the cosmological constant with a value consisting of a combination of (natural) topological invariant. Finally, the cosmological constant appears to be a topologically supported quantity.


1 Introduction

One of the great mysteries in modern cosmology is the accelerated expansion of the universe as driven by dark energy. After the measurements of the Planck satellite (PLANCK) were completed, the model of a cosmological constant (CC) has been favored among other models explaining the expansion, like quintessence. In 1917, the cosmological constant $\Lambda$ was introduced by Einstein (and later discarded) in his field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu}$$

($g_{\mu\nu}$ is a metric tensor, $R_{\mu\nu}$ the Ricci tensor and $R$ the scalar curvature). By now it seems to be the best explanation of dark energy. However, the entire mystery of the cosmological constant lies in its extremely small value (necessarily non-zero, seen as energy density of the vacuum) which remains constant in an evolving universe and is a driving force for its accelerating expansion. These features justify the search for the very reasons explaining their occurrences, among them the understanding of the small value of the cosmological constant is particularly challenging. Our strategy in this paper is to compute the value of a cosmological constant as a topological invariant in dimension 4.

Such an attempt is far from being trivial or even recognized as possible. As a motivation to demonstrate the possibility, let us consider the trace of the Einstein’s field equations

$$R = -4\Lambda$$

with a strictly negative but constant scalar curvature for a spacetime. It follows that the underlying spacetime must be a manifold of constant negative curvature or admitting an Einstein metric (as solution of $R_{\mu\nu} = \lambda g_{\mu\nu}$) with negative constant $\lambda = -\Lambda < 0$. The evolution of the cosmos from the Big Bang up to now determines a spacetime of finite volume. The interior of this finite-volume spacetime can be seen as compact manifold with negative Ricci curvature. That is why the corresponding spacetime manifold is diffeomorphic to the hyperbolic 4-manifold (see the appendix about Mostow-Prasad rigidity and hyperbolic manifolds for this uniqueness result). By Mostow-Prasad rigidity, every hyperbolic 4-manifold with finite volume is rigid, i.e. geometrical expressions like volume, scalar curvature etc. are topological
invariants. Then the discussion above indicates that Λ might be a topological invariant. In fact in this paper we show how to calculate the CC as a topological invariant based on some features of hyperbolic manifolds of dimension 3 and 4.

It is a rather well-founded and powerful approach in various branches of physics to look for the explanations of observed phenomena via underlying topological invariants. There are many examples of such invariant quantities known from particle physics to solid state physics as well from the history of physics. Let us mention just two recent examples, i.e. topological phases in strong electron interactions and emerging Kondo insulators as heavy fermions \[3\], or the search for experimental realizations of topological chiral superconductors with nontrivial Chern numbers (e.g. \[4\]).

The distinguished feature of differential topology of manifolds in dimension 4 is the existence of open 4-manifolds carrying a plenty of non-diffeomorphic smooth structures. In the computation of the CC value presented here, the special role is played by the topologically simplest 4-manifold, i.e. \(\mathbb{R}^4\), which carries a continuum of infinitely many different smoothness structures. Each of them except one, the standard \(\mathbb{R}^4\), is called exotic \(\mathbb{R}^4\). All exotic \(\mathbb{R}^4\) are Riemannian smooth open 4-manifolds homeomorphic to \(\mathbb{R}^4\) but non-diffeomorphic to the standard smooth \(\mathbb{R}^4\). The standard smoothness is distinguished by the requirement that the topological product \(\mathbb{R} \times \mathbb{R}^3\) is a smooth product. There exists only one (up to diffeomorphisms) smoothing, the standard \(\mathbb{R}^4\), where the product above is smooth. In the following, an exotic \(\mathbb{R}^4\), presumably small if not stated differently, will be denoted as \(\mathbb{R}^4\).

But why are we dealing with \(\mathbb{R}^4\)? As we mentioned already any \(\mathbb{R}^4\) (small or big) has necessarily non-vanishing Riemann curvature. However, the non-zero value of the curvature depends crucially on the embedding (the curvature is not a diffeomorphism invariant) of \(\mathbb{R}^4\). That is why our strategy is to look for natural embeddings of exotic \(\mathbb{R}^4\)’s in some manifold \(M^n\) and estimate the corresponding curvature of this \(\mathbb{R}^4\). This curvature depends on the embeddings in general. However, we can try to work out an invariant part of this embedded \(\mathbb{R}^4\). If we are lucky enough we will be able to construct the invariant part of \(\mathbb{R}^4\) (with respect to some natural embeddings into certain 4-manifold \(M^4\)) with the topologically protected curvature. We would expect that this curvature would reflect the realistic value of CC for some (canonical) \(M^4\).

There are canonical 4-manifolds into which some exotic \(\mathbb{R}^4\) are embeddable. Here we will use the defining property of small exotic \(\mathbb{R}^4\): every small
exotic $R^4$ is embeddable in the the standard $\mathbb{R}^4$ (or in $S^4$). We analyze these embeddings in Secs. 3 and 4. There exists a chain of 3-submanifolds of $R^4$ $Y_1 \to \cdots \to Y_\infty$ and the corresponding infinite chain of cobordisms

$$\text{End}(R^4) = W(Y_1, Y_2) \cup Y_2 W(Y_2, Y_3) \cup \cdots$$

where $W(Y_k, Y_{k+1})$ denotes the cobordism between $Y_k$ and $Y_{k+1}$ so that $R^4 = K \cup Y_1 \text{End}(R^4)$ where $\partial K = Y_1$. The $\text{End}(R^4)$ is the invariant part of the embedding $R^4 \subset \mathbb{R}^4$ mentioned above. In the first part of the paper we will show that the embedded $R^4$ admits a negative curvature, i.e. it is a hyperbolic 4-manifold. This follows from the fact that $Y_k, k = 1, 2, \ldots$ are 3-manifolds embedded into hyperbolic 4-cobordism $W(Y_k, Y_{k+1})$ and the curvature of $Y_{k+1}$, $\text{curv}(Y_{k+1})$, is determined by the curvature of $Y_k$:

$$\text{curv}(Y_{k+1}) = \text{curv}(Y_k) \exp(-2\theta) = \frac{1}{L^2} \exp(-2\theta)$$

where $L$ is the invariant length of the hyperbolic structure of $Y_k$ induced from $W(Y_k, Y_{k+1})$, i.e. $L^3 = \text{vol}(Y_{k+1})$, and $\theta$ is the topological parameter $\theta = -\frac{3}{2\text{cs}(Y_{k+1})}$. The induction over $k$ leads to the expression for the constant curvature of the cobordism $W(Y_1, Y_\infty)$ as the function of $\text{CS}(Y_\infty)$

$$\text{curv}(W(Y_1, Y_\infty)) = \frac{1}{L^2} \exp \left( - \frac{3}{\text{CS}(Y_\infty)} \right).$$

This is precisely what we call the cosmological constant of the embedding $R^4 \subset \mathbb{R}^4$. It is the topological invariant. However, in case of the embedding into the standard $\mathbb{R}^4$, $Y_\infty$ is a (wildly embedded) 3-sphere and thus its Chern-Simons invariant vanishes. This leads to the vanishing of the cosmological constant as far as the embedding into $\mathbb{R}^4$ is considered. We should work more globally, namely $R^4 \subset \mathbb{R}^4 \subset M^4$ and look for the suitable (still canonical) $M^4$ into which $R^4$ embeds and the corresponding cosmological constant of the cobordisms, determined by the embedding, assumes realistic value. Now the discussion is along the line of the argumentation at the beginning of the section where we considered a hyperbolic geometry on certain Einstein manifold. At first, one could think that it is not possible at all, that such miracle can happen, and one can find suitable $M^4$ giving the correct value of CC, and even it can, this $M^4$ could not be canonical. A big surprise of Sec. 6 is the existence of canonical $M^4$, which is the $K3\#CP^2$ where $K3$ is the elliptic surface $E(2)$, such that the embedding into it of certain
(also canonical) small exotic $R^4$, generates the realistic value of CC as the
curvature of the hyperbolic cobordism of the embedding. Again the curvature
is constant which is supported by hyperbolic structure and thus it can be
a topological invariant. In this well recognized case the boundary of the
Akbulut cork of $K3\#\overline{\mathbb{C}P^2}$ lies in the compact submanifold $K$ generating $R^4$.
The boundary is a certain homology 3-sphere (Brieskorn sphere $\Sigma(2, 5, 7)$)
which is neither topologically nor smoothly $S^3$, contrary to the previously
considered case of the embedding into the standard $\mathbb{R}^4$, and the CS invariant
of $\Sigma(2, 5, 7)$ does not vanish. Exotic $R^4$ as embedded in $K3\#\overline{\mathbb{C}P^2}$ lies between
this Brieskorn sphere and the sum of two Poincare spheres $P\#P$. Thus,
starting from the 3-sphere (wildly embedded) in $K$ and fixing the size of $S^3$
to be of the Planck length, the subsequent two topology changes take place
which allow for the embedding $R^4 \rightarrow K3\#\overline{\mathbb{C}P^2}$. Namely
\[ S^3 \rightarrow \Sigma(2, 5, 7) \rightarrow P\#P. \]
Now the ratio of the curvature of the (wildly embedded) $S^3$ and the curva-
ture of $P\#P$ is a topological invariant. Still there is a freedom to include
quantum corrections to this expression. The corrections are also represented
by topological invariants (Pontryagin and Euler classes of the Akbulut cork).
The numerical calculations of the resulting invariant show a good agreement
with the Planck result for the dark energy density. All details are presented
in Sec. 6.

Some of the material seems to be very similar to our previous work [5].
Therefore we will comment about the differences between [5] and this work.
Main idea of [5] is a new description of the inflation process by using exotic
smoothness. Then, inflation as a process is generated by a change in the spa-
tial topology. In particular we studied a model with two inflationary phases
which will produce a tiny cosmological constant (CC). But the approach in
the paper misses many important points: it was never shown why CC is a
constant, the model uses a very special Casson handle (so that the attach-
ment is the sum of two Poincare spheres) and it assumed the embedding of
the Akbulut cork in the small exotic $R^4$. With the results of this paper,
these arbitrary assumptions will be no longer needed. CC is really a con-
stant and we will present the reason for the constancy (the Mostow-Prasad
rigidity of the spacetime). The model is natural, i.e. there are topological
changes starting with the 3-sphere to Brieskorn sphere $\Sigma(2, 5, 7)$ and finally
the change to the sum of two Poincare spheres. In contrast to [5], there is
no freedom for other topology changes in this paper. Part of the previous
work is the calculation of expansion factor which was identified with CC. The previous calculation depends strongly on the embedding. In this paper we will use a general approach via hyperbolic geometry which will produce a generic result identical to the previous work. Therefore, some results of this paper are similar to the previous work but obtained with different methods for a more general case. We will comment on it in the last two sections.

Secondly, we have to comment about the relation between causality and topology change in our model. As shown by Andersen and DeWitt [6] the singularities of the spatial topology change imply infinite particle and energy production under reasonable laws of quantum field propagation. Here, the concept of causal continuity is central. Causal continuity of a spacetime means, roughly, that the volume of the causal past and future of any point in the spacetime increases or decreases continuously as the point moves continuously around the spacetime. In a series of papers Sorkin, Dowker et.al. [7, 8, 9] analyzed possible topology changes. In particular they showed that causal discontinuity occurs if and only if the Morse index is 1 or \( n - 1 \), i.e. the 4D spacetime has to contain 1- and 3-handles in its description. By a result of Laudenbach and Poenaru [10], the number of 3-handles is determined by the smoothness structure. For \( \mathbb{R}^4 \), no 3-handles are needed [11]. Furthermore, by a method of Akbulut [12] any 1-handle can be described by removing a 2-handle. Therefore, for our spacetime \( \mathbb{R}^4 \) there is no causal discontinuity. Secondly, the spacetime is topologically trivial (homeomorphic to \( \mathbb{R}^4 \) with end \( S^3 \times \mathbb{R} \)). Any closed time-like curve will be canceled by a continuous transformation.

Finally, let us comment briefly on the physical meaning of the embedding of exotic \( \mathbb{R}^4 \) into some 'big' 4-manifold like \( E(2) \# \overline{\mathbb{C}P^2} \). The observed local part of the universe allows for embeddings like \( \mathbb{R}^4 \subset \mathbb{R}^4 \). However, a more global picture can exhibit the embeddings like \( \mathbb{R}^4 \subset E(2) \# \overline{\mathbb{C}P^2} \) as having observed consequences and explaining the CC value. The factor \( \overline{\mathbb{C}P^2} \) in 4-dimensional topology of manifolds is distinguished by itself (blowing up process). However, \( \overline{\mathbb{C}P^2} \) can be given independent meaning relating quantum field theory contributions of \( \mathbb{R}^4 \) into CC. In the case of the embedding \( \mathbb{R}^4 \subset \mathbb{R}^4 \) the \( \overline{\mathbb{C}P^2} \) factor disappears and the QFT effects of \( \mathbb{R}^4 \) detected in the ambient standard \( \mathbb{R}^4 \) disappear either. This is the case of the vanishing of the curvature contributions to CC, derived in Sec. 4 from the embedding \( \mathbb{R}^4 \to \mathbb{R}^4 \). The QFT contributions are thus canceled topologically. One recovers these 'QFT contributions' just by adding the \( \overline{\mathbb{C}P^2} \) factor. Such an enlarging the target manifold of the embedding of \( \mathbb{R}^4 \) reproduces the correct
value of the vacuum energy density, which is precisely the result of Sec. 6. We close the main body of the paper with the brief discussion of the obtained results. There are three appendixes attached explaining hyperbolic manifolds together with Mostow-Prasad rigidity, the concept of a wild embedding and two models (Poincare disk and half-space model) of hyperbolic geometry used in the paper.

We strongly acknowledge the critical remarks of the anonymous referee. The response to these remarks increases significantly the readability of the paper.

2 Small exotic $\mathbb{R}^4$

In 4-manifold topology [13], a homotopy-equivalence between two compact, closed, simply-connected 4-manifolds implies a homeomorphism between them (so-called h cobordism). But Donaldson [14] provided the first smooth counterexample, i.e. both manifolds, being h-cobordant, are generally non-diffeomorphic each to the other. The failure can be localized in some contractible submanifold (Akbulut cork) so that an open neighborhood of this submanifold is a small exotic $\mathbb{R}^4$. The whole procedure implies that this exotic $\mathbb{R}^4$ can be embedded in the 4-sphere $S^4$. Below we will discuss more details of the construction.

To be more precise, consider a pair $(X_+, X_-)$ of homeomorphic, but non-diffeomorphic, smooth, closed, simply-connected 4-manifolds. The transformation from $X_-$ to $X_+$ can be described by the following construction.

**Let W be a smooth h-cobordism between closed, simply connected 4-manifolds $X_-$ and $X_+$. Then there is an open subset $U \subset W$ homeomorphic to $[0, 1] \times \mathbb{R}^4$ with a compact subset $K \subset U$ such that the pair $(W \backslash K, U \backslash K)$ is diffeomorphic to a product $[0, 1] \times (X_- \backslash K, U \cap X_- \backslash K)$. The subsets $R_{\pm} = U \cap X_{\pm}$ (homeomorphic to $\mathbb{R}^4$) are diffeomorphic to open subsets of $\mathbb{R}^4$. Since $X_-$ and $X_+$ are non-diffeomorphic, there is no smooth 4-ball in $R_{\pm}$ containing the compact set $Y_{\pm} = K \cap R_{\pm}$, so both $R_{\pm}$ are exotic $\mathbb{R}^4$’s.**

Thus, first remove a certain contractible, smooth, compact 4-manifold $Y_- \subset X_-$ (called an Akbulut cork) from $X_-$, and then re-glue it by an involution of $\partial Y_-$, i.e. a diffeomorphism $\tau : \partial Y_- \to \partial Y_-$ with $\tau \circ \tau = Id$ and $\tau(p) \neq \pm p$ for all $p \in \partial Y_-$. This argument was modified above so that it works for a contractible open subset $R_- \subset X_-$ with similar properties, such that $R_-$ will be an exotic $\mathbb{R}^4$ if $X_+$ is not diffeomorphic to $X_-$. Furthermore $R_-$ lies in
a compact set, i.e. a 4-sphere and $R_-$ is a small exotic $R^4$. Freedman and DeMichelis [15] constructed a continuous family of small exotic $R^4$'s.

3 How to embed small exotic $R^4$ into the standard $\mathbb{R}^4$

In this section we will construct the embedding of the exotic $R^4$ into the standard $\mathbb{R}^4$ as well the sequence of non-trivial 3-manifolds $Y_1 \rightarrow \cdots \rightarrow Y_\infty$ characterizing the exotic $R^4$. This section is a little bit technical and all readers who accept these facts can switch to the next section.

One of the characterizing properties of an exotic $R^4$, which is present in all known examples, is the existence of a compact subset $K \subset \mathbb{R}^4$ which cannot be surrounded by any smoothly embedded 3-sphere (and homology 3-sphere bounding a contractible, smooth 4-manifold), see sec. 9.4 in [16] or [17]. The topology of this subset $K$ depends strongly on the $R^4$. In the example below, $K$ is constructed from the Akbulut cork of the compact 4-manifold $E(2)\#\mathbb{C}P^2$. Let $R^4$ be the standard $\mathbb{R}^4$ (i.e. $R^4 = \mathbb{R}^3 \times \mathbb{R}$ smoothly) and let $R^4$ be a small exotic $R^4$ with compact subset $K \subset \mathbb{R}^4$ which cannot be surrounded by a smoothly embedded 3-sphere. Then every completion $N(K)$ of an open neighborhood $N(K) \subset \mathbb{R}^4$ of $K$ is not bounded by a smooth embedded 3-sphere $S^3 \neq \partial N(K)$. But $R^4$ being small, allows for a smooth embedding $E : R^4 \rightarrow \mathbb{R}^4$ in the standard $\mathbb{R}^4$. Then the completion of the image $E(R^4)$ has the boundary $S^3 = \partial E(R^4)$ as subset of $\mathbb{R}^4$. So, we have the strange situation that an open subset of the standard $\mathbb{R}^4$ represents a small exotic $R^4$.

Now we will describe the construction of this exotic $R^4$. Historically it emerged as a counterexample of the smooth h-cobordism theorem [14, 18]. The compact subset $K$ as above is given by a non-canceling 1-/2-handle pair. Then, the attachment of a Casson handle $CH$ cancels this pair only topologically. A Casson handle is is a 4-dimensional topological 2-handle constructed by an infinite procedure. In this process one uses disks with self-intersections (so-called kinky handles) and arrange them along a tree $T_{CH}$: every vertex of the tree is the kinky handle and the number of branches in the tree are the number of self-intersections. Freedman [13] was able to show that every Casson handle is topologically the standard open 2-handle $D^2 \times \mathbb{R}^2$. As the result to attach the Casson handle $CH$ to the subset $K$,
one obtains the topological 4-disk $D^4$ with interior $\mathbb{R}^4$ o the 1-/2-handle pair was canceled topologically. The 1/2-handle pair cannot cancel smoothly and a small exotic $\mathbb{R}^4$ must emerge after gluing the $CH$. It is represented schematically as $\mathbb{R}^4 = K \cup CH$. Recall that $\mathbb{R}^4$ is a small exotic $\mathbb{R}^4$, i.e. $\mathbb{R}^4$ is embedded into the standard $\mathbb{R}^4$, and the completion $\bar{\mathbb{R}}^4$ of $\mathbb{R}^4 \subset \mathbb{R}^4$ has a boundary given by certain 3-manifold $Y_r$. One can construct $Y_r$ directly as the limit $n \to \infty$ of the sequence $\{Y_n\}$ of some 3-manifolds $Y_n$, $n = 1, 2, \ldots$. To construct this sequence $[17]$, one represents, by the use of Kirby calculus of handles, the compact subset $K$ by 1- and 2-handles pictured by a link say $L_K$ where the 1-handles are represented by a dot (so that surgery along this link gives $K$) $[16]$. Then one attaches a Casson handle to this link $[18]$. As an example see Figure 1.

The Casson handle is given by a sequence of Whitehead links (where the unknotted component has a dot) which are linked according to the tree (see the right figure of Figure 2 for the building block and the left figure for the simplest Casson handle given by the unbranched tree).

For the construction of the 3-manifold $Y_n$ which surrounds the compact $K$, one considers $n-$stages of the Casson handle and transforms the diagram to a real link, i.e. the dotted components are changed to usual components with framing 0. By a handle manipulations one obtains a knot so that the $n$th (untwisted) Whitehead double of this knot represents the desired 3-manifold (by surgery along this knot). Then our example in Figure 1 with the $n$-th
Figure 3: pretzel knot \((-3, 3, -3)\) or the knot 9_{46} in Rolfsen notation producing the 3-manifold \(Y_1\) by 0–framed Dehn surgery

stages Casson handle, will result in the \(n\)th untwisted Whitehead double of the pretzel knot \((-3, 3, -3)\) as in Figure 3 (see [17] for the details of handle manipulations).

Then the entire sequence of 3-manifolds

\[ Y_1 \to Y_2 \to \cdots \to Y_\infty = Y_r \]

characterizes the exotic smoothness structure of \(R^4\). Every \(Y_n\) is embedded in \(R^4\) and into \(\mathbb{R}^4\). An embedding is a map \(i : Y_n \hookrightarrow \mathbb{R}^4\) so that \(i(Y_n)\) is diffeomorphic to \(Y_n\). Usually, the image \(i(Y_n)\) represents a manifold which is given by a finite number of polyhedra (seen as triangulation of \(Y_n\)). Such an embedding is tame. In contrast, the limit of this sequence \(n \to \infty\) gives an embedded 3-manifold \(Y_r\) which must be covered by an infinite number of polyhedra. Then, \(Y_r\) is called a wild embedded 3-manifold (see the appendix 9 and the book [19]). Then 0–framed surgery along the pretzel knot for \(n = 1\) produces \(Y_1\) whereas the \(n\)th untwisted Whitehead double will give \(Y_n\). For large \(n\), the structure of the Casson handle is coded in the topology of \(Y_n\) and in the limit \(n \to \infty\) we obtain \(Y_r\) (which is now a wildly embedded \(Y_r \subset \mathbb{R}^4\) in the standard \(\mathbb{R}^4\)). But what do we know about the structure of \(Y_n\) and \(Y_r\) in general? The compact subset \(K\) is a 4-manifold constructed by a pair of one 1-handle and one 2-handle which cancel only topologically. The boundary of \(K\) is a compact 3-manifold having the first Betti number \(b_1 = 1\). This feature is preserved by taking the limit \(n \to \infty\) and characterizes \(Y_r\) as well. By the work of Freedman [13], every Casson handle is topologically \(D^2 \times \mathbb{R}^2\) (relative to the attaching region) and therefore \(Y_r\) must be the boundary of \(D^4\) (the Casson handle trivializes \(K\) to be \(D^4\)), i.e. \(Y_r\) is a wild embedded 3-sphere \(S^3\).
4 Geometric properties of the embedding

As we have just seen, the main restriction of the embedding \( E : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) is the sequence of 3-manifolds \( Y_1 \rightarrow \cdots \rightarrow Y_\infty \). \( Y_1 \) was described as the boundary of the compact subset \( K \) whereas \( Y_n \) is given by 0-framed surgeries along \( n \)th untwisted Whitehead double of the pretzel knot \( 9_{46} \). The entire embedding \( E : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) and the sequence \( \{ Y_n \} \) were determined from the failure of the h-cobordism and the explicit example of the non-smooth-cobordant pair of 4-manifolds. Thus we have a sequence of inclusions

\[ \ldots \subset Y_{n-1} \subset Y_n \subset Y_{n+1} \subset \ldots \subset Y_\infty \]

with the 3-manifold \( Y_\infty \) as limit. Let \( K_+ \) be the corresponding (wild) knot, i.e. the \( \infty \)th untwisted Whitehead double of the pretzel knot \((-3, 3, -3)\) (\( 9_{46} \) knot in Rolfsen notation). The surgery description of \( Y_\infty \) induces the decomposition

\[ Y_\infty = C(K_+) \cup (D^2 \times S^1) \quad C(K_+) = S^3 \setminus (K_+ \times D^2) \quad (1) \]

where \( C(K_+) \) is the knot complement of \( K_+ \). In [20], the splitting of the knot complement was described. Let \( K_{9_{46}} \) be the pretzel knot \((-3, 3, -3)\) and let \( L_{Wh} \) be the Whitehead link (with two components). Then the complement \( C(K_{9_{46}}) \) has one torus boundary whereas the complement \( C(L_{Wh}) \) has two torus boundaries. Now according to [20], one obtains the splitting

\[ C(K_+) = C(L_{Wh}) \cup_{T^2} \cdots \cup_{T^2} C(L_{Wh}) \cup_{T^2} C(K_{9_{46}}) \]

This splitting allows us to characterize every right-hand-side factor (hence left-hand-side either) as a hyperbolic manifold (see Figure 4).

At first the knot \( K_{9_{46}} \) is a hyperbolic knot, i.e. the interior of the 3-manifold \( C(K_{9_{46}}) \) admits a hyperbolic metric. In general, every link \( L \) is hyperbolic if the interior of its complement \( C(L) \) is a hyperbolic 3-manifold (with negative sectional curvature), see [21] for a recent survey. There is a deep result of Menasco [22], that every link having a knot diagram with alternating crossings along each components is a hyperbolic link if it is non-split (every component is linked) and prime (it cannot be decomposed into sums) unless it is a torus link. This theorem can be applied to the knot \( 9_{46} \) (see Fig. 3), i.e. it is an alternating knot and therefore this knot is a hyperbolic knot. The same argumentation can be used to show that the
Whitehead link is an alternating link as well the $n$th Whitehead link. Using the result [20], every Whitehead double is also a hyperbolic link because of the splitting of the knot/link complements. The complement of a Whitehead double of $9_{46}$ is the sum of $C(K_{9_{46}})$ and $C(L_{Wh})$ both admitting a hyperbolic structure. Finally, the iteration of this case leads to the general case that $C(K_+)$ admits also hyperbolic structure, i.e. it is a homogeneous space of constant negative curvature. Therefore we obtained the first condition: the sequence of 3-manifolds is geometrically a sequence of hyperbolic 3-manifolds.

As a second condition we will remark that the sequence of 3-manifolds is infinite but the embedding space can be a compact space ($S^4$) or a compact subset of a non-compact space ($R^4$). By the first condition, we have an infinite sequence of hyperbolic 3-manifolds with increasing size where the maximal size of $Y_\infty$ is bounded from above. Finally, $Y_\infty$ has the topology of a 3-sphere (using Freedman’s famous result). Therefore we will look for a compact subset with boundary a topological 3-sphere, i.e. a 4-disk $D^4$. The embedding of the infinite sequence into a compact subset enforces us to choose a 4-ball with metric

$$ds_{4D}^2 = \frac{dr^2 + r^2 d\Omega^2}{(1 - r^2)^2}$$

(2)

with the angle coordinates $\Omega$ (a tuple of 3 angles) and the radius $r$, i.e. the Poincare hyperbolic 4-ball. At first we will remark that for a fixed $r = \text{const.}$
the corresponding submanifold in the 4-disk $D^4$ is a 3-sphere. The embedding of the sequence $Y_1 \to Y_2 \to \cdots \to Y_\infty$ used the fact that $Y_n \subset Y_{n+1}$. Therefore we embed this sequence by mapping every $Y_n$ to some 3-sphere of radius $r$. This mapping is possible by a deep result of Freedman [13] showing the embedding of every Casson handle in the standard open 2-handle given by the interior of a 4-disk. As shown above, every $Y_n$ is part of this Casson handle and by construction $Y_n$ can be embedded into $S^3 \times (-\epsilon, \epsilon)$ for small $\epsilon$. Then the limit manifold $Y_\infty$ must be mapped to $r = 1$, representing the boundary of the Poincare disk (which is the ‘sphere at infinity’ for the metric above). Then we choose that $Y_1$ is mapped to the 3-sphere of radius $r = \frac{1}{2}$. All other $Y_n$ have to be mapped in the right order to spheres of radius between $r = \frac{1}{2}$ and $r = 1$. In general, every embedding is given by a strictly increasing function of the radius $r(n)$ with respect to the integer $n$. In a particular choice of the embedding, the number $n$ of the Whitehead double (numbering the 3-manifolds $Y_n$) is related to the radius by

$$r = 1 - \frac{1}{n+1} \quad \text{(3)}$$

for the simplest case. In general one has to choose

$$r = f(n) \quad f(1) = \frac{1}{2}, \quad f(\infty) = 1 \quad \text{(4)}$$

Finally the embedded $R^4$, i.e. $E(R^4)$, has negative curvature or (more strongly) admits a hyperbolic 4-metric.

There is also another argument for the negative curvature of the embedded $R^4$ which uses the exotic smoothness structure directly. Namely, the construction of the small exotic $R^4$ referred to the h-cobordism directly and leads to the smooth embedding $R^4 \hookrightarrow S^4$ or $R^4 \hookrightarrow R^4$. A sufficient condition for this construction is the exoticness of the compact 4-manifolds $X_+, X_-$ which appear in the h-cobordism, i.e. $X_+, X_-$ are homeomorphic but non-diffeomorphic (see Sec. 2). In the construction of the example above, it was the pair of compact 4-manifolds with the topology $E(2) = K3\#\mathbb{C}P^2$ both ($K3$ for K3-surface, sometimes also called elliptic surface $E(2)$, see [16]) but non-diffeomorphic as a smooth manifolds. This pair can be simply constructed by using the Fintushel-Stern knot surgery [23], i.e. one starts with $X_+ = E(2)$ and modifies $X_+$ by using a knot to obtain $X_-$. If the knot has a non-trivial Alexander polynomial then $X_+$ is non-diffeomorphic to $X_-$. In our case, there is a non-trivial solution of the Seiberg-Witten equation on $X_-$
implying the existence of a negatively curved submanifold. For completeness we will present this argument.

Let us start with the generic case: let $M$ be a compact Riemannian 4-manifold and let $P \to M$ be a Spin$_C$ structure for $M$, i.e. a lift of the principal $SO(4)$ bundle (frame bundle, associated to the tangent bundle $TM$) to a Spin$_C$ principal bundle $P$ with $Spin_C = (SU(2) \times SU(2)) \times_{\mathbb{Z}_2} U(1)$. The introduction of a Spin$_C$ structure do not restrict possible 4-manifolds, because each 4-manifold admits a Spin$_C$ structure. Now one can introduce the following objects: a Spin$_C$ bundle $S_C(P)$ as associated vector bundle to $P$ with splitting $S_C(P) = S_C^+(P) \oplus S_C^-(P)$ (induced by the $\gamma^5$ matrix), a canonical line bundle $\det(P)$ with connection $A$, and a Dirac operator $D_A : \Gamma(S^+_C) \to \Gamma(S^-_C)$. $D_A$ depends on the Levi-Civita connection and on the connection $A$ in $\det(P)$-bundle, and acts on spinors which are sections $\psi \in \Gamma(S^+_C)$. Then the Seiberg-Witten equations are formulated as

\begin{align*}
F_A^+ &= q(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2} \text{id} \\
D_A\psi &= 0
\end{align*}

where $F_A = dA$ is the curvature of the connection $A$ and $F_A^+ = \frac{1}{2}(F_A - \ast F_A)$ is the self-dual part. A trivial (or reducible) solution of these equations is given by $\psi = 0$ (and then $A = d\phi$). To get a non-trivial Seiberg-Witten invariant, one needs a non-trivial solution $(A, \psi)$ of the equations. For that purpose we consider the second equation $D_A\psi = 0$ and obtain for the square of the Dirac operator (Weizenböck formula)

\begin{align*}
0 &= D_A D_A \psi = \nabla_A^* \nabla_A (\psi) + \frac{R}{4} \psi + \frac{F_A^+}{2} \psi
\end{align*}

where $R$ is the scalar curvature of $M$. Using the first equation we obtain

\begin{align*}
0 &= \nabla_A^* \nabla_A (\psi) + \frac{R}{4} \psi + \frac{1}{2} \left( \psi \otimes \psi^* - \frac{|\psi|^2}{2} \text{id} \right) \psi
\end{align*}

which simplifies to

\begin{align*}
0 &= \nabla_A^* \nabla_A (\psi) + \frac{R}{4} \psi + \frac{|\psi|^2}{4} \psi.
\end{align*}

Now we multiply both sides with $\psi^*$ and integrate over $M$ to get

\begin{align*}
\int_M \left( |\nabla_A \psi|^2 + \frac{R}{4} |\psi|^2 + \frac{|\psi|^4}{4} \right) \sqrt{g} d^4x = 0
\end{align*}
Except for the scalar curvature term, all other terms are positive definite. Furthermore, a non-trivial solution of the SW equations is given by $\psi \neq 0$. Therefore the exotic smoothness structure enforces the manifold $M$ to have a negative scalar curvature $R < 0$. Or, there must exist a submanifold with $R < 0$ which dominates the curvature of $M$. Interestingly, this behavior is unchanged for generic perturbations (see [24]).

Now we will use this result by setting $M = X_\ast$. For $X_\ast$ we can choose the standard smoothness structure allowing only the trivial solution of the Seiberg-Witten equation ($\psi = 0, A = d\phi$). By using the h-cobodism theorem (see section 2), the h-cobordism between $X_\ast \setminus Y_\ast$ and $X_\ast \setminus Y_\ast$ is trivial and therefore the non-trivial solution of the Seiberg-Witten equation is localized at $Y_\ast$ and by using $Y_\ast = K \cap R_\ast$ also at $R_\ast$, the small exotic $R^4$ embedded into $S^4$ (or $R^4$). This general argument together with the particular embedding described above with (4) gives the main result of this section:

The small exotic $R^4$ embeds into the standard $R^4$ by a smooth map $E : R^4 \hookrightarrow R^4$. This submanifold $E(R^4) \subset R^4$ admits always a negative scalar curvature $R < 0$ and a hyperbolic structure with isometry group $SO(4, 1)$.

5 Cosmological constant

So far, we have studied the geometric properties of the embedding $E : R^4 \hookrightarrow R^4$ resulting from the topology of certain infinite chain of 3-manifolds $Y_1 \rightarrow \cdots \rightarrow Y_\infty$. As explained above this sequence defines $R^4$ as embedded in $R^4$. Equivalently, the exotic $R^4$ is given by an infinite chain of cobordisms

$$\text{End}(R^4) = W(Y_1, Y_2) \cup W(Y_2, Y_3) \cup \cdots$$

so that $R^4 = K \cup Y_1 \text{End}(R^4)$ (remember $\partial K = Y_1$). Obviously, there is also an embedding $\tilde{E} : \text{End}(R^4) \hookrightarrow R^4$ which will be studied now. The geometry of this embedding is also given by hyperbolic geometry. This hyperbolic geometry of the cobordism is best expressed by the metric

$$ds^2 = dt^2 - a(t)^2 h_{ik} dx^i dx^k$$

also called the Friedmann-Robertson-Walker metric (FRW metric) with the scaling function $a(t)$ for the (spatial) 3-manifold (denoted as $\Sigma$ in the following). As explained in the previous section, this spatial 3-manifold in the case of embedding $E : R^4 \hookrightarrow R^4$ admits (at least for the pieces) a homogenous metric of constant curvature.
The cosmological constant is defined via Einstein's theory as the eigenvalue of the Ricci tensor w.r.t. metric, i.e. $\text{Ric} = -\Lambda g$. Then one obtains the equation

$$
\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} - \frac{k}{a^2}
$$

having the solutions $a(t) = a_0 \sqrt{3|k|/\Lambda} \sinh(t\sqrt{\Lambda/3})$ for $k < 0$, $a(t) = a_0 \exp(t\sqrt{\Lambda/3})$ for $k = 0$ and $a(t) = a_0 \sqrt{3|k|/\Lambda} \cosh(t\sqrt{\Lambda/3})$ for $k > 0$ all with exponential behavior. At first we will consider this equation for constant topology, i.e. for the spacetime $Y_n \times [0,1]$. But as explained above, the subspace $E(R^4) \subset R^4$ and every $Y_n$ admits a hyperbolic structure. Now taking Mostow-Prasad rigidity seriously, the scaling function $a(t)$ must be constant, or $\dot{a} = 0$. Therefore we will get

$$
\Lambda = \frac{3k}{a^2}
$$

by using (7) for the parts of constant topology. From the topological point of view, $R^4$ is homeomorphic to $S^3 \times [0,1)$ outside of the subset $K \subset R^4$ (also called end of $R^4$). From this point of view, we have to choose $k = 1/3$ to reflect the topological $S^3$ structure and normalize the curvature so that $3R = \Lambda$. One can determine the local structure of this space by considering the isometry groups of the hyperbolic 3- and 4-space. They are $Isom(\mathbb{H}^3) = SO(3,1)$ and $Isom(\mathbb{H}^4) = SO(4,1)$, respectively. In case of $R^4$, the 3-manifolds $Y_n$ admit also hyperbolic structures with isometry group $SO(3,1)$. This group acts freely on the isometry group of the 4-space $SO(4,1)$. The quotient space $SO(4,1)/SO(3,1)$ describes the corresponding symmetry reduction but at the same time, it reflects the local structure of the embedding $E : R^4 \hookrightarrow \mathbb{R}^4$. This quotient space is known as de Sitter space $S^3 \times [0,1)$ which is compatible with the embedding, $E$, and its geometry represents physical CC.

Formula (8) can be now written in the form

$$
\Lambda = \frac{1}{a^2} = ^3 R
$$

so that CC is related to the curvature of the 3D space. This relation is true for a fixed spatial space but, as explained above, we have a process $Y_0 \rightarrow \cdots \rightarrow Y_\infty$ ending at $Y_\infty = S^3$. We described the formation of the CC as inflationary process in our previous work [5]. Therefore we define

$$
\Lambda = \text{curv}(Y_\infty)
$$
by using (9), i.e. CC is related to the curvature of the (wild) 3-sphere surrounding $R^4$. Here, there is an unusual behavior with the following interpretation. We have a series of static Einstein universes. Since Eddington [25], it is known that the static Einstein universe is unstable. This instability motivates the series of transitions $Y_n \to Y_{n+1}$ in our model leading to a larger scale of the resulting (spatial) manifold. In the literature, there is the model of Ellis and Maartens [26] which have the static Einstein universe in common with our model. Main difference to our model is the inclusion of topological transitions. Interestingly, the stability of the static Einstein universe filled with perfect fluids of different types is currently under investigations, see for instance [27].

By using $a_n = \sqrt[3]{\text{vol}(Y_n)}$, we are able to define a scaling parameter for every $Y_n$. By Mostow-Prasad rigidity, $a_n$ is also constant, $a_n = \text{const.}$. But the change $Y_n \to Y_{n+1}$ increases the volumes of $Y_n$, $\text{vol}(Y_{n+1}) > \text{vol}(Y_n)$, by adding complements of the Whitehead links. Thus the smoothness structure of $R^4$ determines the spaces (Whitehead link complements) which have to be added. Therefore we have the strange situation that the spatial space changes by the addition of new (topologically non-trivial) spaces. To illustrate the amount of the change, we have to consider the embedding of $\text{End}(R^4)$ directly. It is given by the embedding of the Casson handle $CH$ as represented by the corresponding infinite tree $T_{CH}$. As explained above, this tree must be embedded into the hyperbolic space. For the tree, it is enough to use a 2D model, i.e. the hyperbolic space $\mathbb{H}^2$. There are many isometric models of $\mathbb{H}^2$(see the appendix 10 for two models). Above we used the Poincare disk model but now we will use the half-plane model with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

(10)

to simplify the calculations. The infinite tree must be embedded along $y-$axis and we set $dx = 0$. The tree $T_{CH}$, as the representative for the Casson handle, can be seen as metric space instead of a simplicial tree. In case of a simplicial tree, one is only interested in the structure given by the number of levels and branches. The tree $T_{CH}$ as a metric space (so-called $\mathbb{R}-$tree) has the property that any two points are joined by a unique arc isometric to an interval in $\mathbb{R}$. Then the embedding of $T_{CH}$ is given by the identification of the coordinate $y$ with the coordinate of the tree $a_T$ representing the distance from the root. This coordinate is a real number and we can build the new distance function
after the embedding as 
\[ ds_T^2 = \frac{da^2_T}{a^2_T}. \]

But as discussed above, the tree \( T_{CH} \) grows with respect to a time parameter so that we need to introduce an independent time scale \( t \). From the physics point of view, the time scale describes the partition of the tree into slices. Then a natural choice seems to be the setting 
\[ ds_T^2 \sim dt^2 \]
where the time scale is related to the hyperbolic distance
\[ \frac{da^2_T}{a^2_T} = \frac{1}{L^2} dt^2 \] 
via the scale \( L \) of the hyperbolic structure for the 3-manifold \( Y_n \) where \( n \) denotes the \( n \)th level of the tree \( T_{CH} \). The branching of the tree into the level \( n \) is related to a length, denoted by \( a_T(n) \). Then the level \( n \) corresponds to the distance \( a_T(n) \) from the root where the tree branches into the \( n \)th level of \( T_{CH} \). In other words, the tree branches at particular values of \( a_T \) but the details are not important for the following discussion. But this equality is only a heuristic argument. A rigorous mathematical argumentation is based on the embedding of the tree \( T_{CH} \) for a Casson handle \( CH \). The Casson handle is a branched surface and it can be described by quadratic differentials as follows. Let \( X \) be a Riemannian surface then a quadratic differential is a section of \( T^*X \otimes T^*X \) which is locally given by
\[ q = q(z) dz^2 = q(z) dz \otimes dz \]
with the holomorphic function \( q(z) \). Away from the zeros of \( q(z) \) we can choose a canonical conformal coordinate \( \xi(z) = \int^z \sqrt{q} \) so that \( q = d\xi^2 \).

Then the set \( \{Re(\xi) = \text{const.}\} \) defines a foliation, called the vertical measured foliation. The holomorphic function \( q(z) \) can be locally expressed as a polynomial. The zeros of the polynomial are the branching points of the surface, i.e. \( z^n \) branches into \( n \) pieces at \( z = 0 \). Using this result, we are able to generate the tree \( T_{CH} \) by a polynomial. Furthermore we identify the coordinate \( t \) with \( t = Re(\xi) \) so that \( t = \text{const.} \) defines a vertical foliation (into slices of the constant time). By the deep theorem of Hubbard and Masur [28], for every measured foliation on \( X \) (of genus \( g > 1 \)) there exists a unique
quadratic differential so that its vertical measured foliation is equivalent to the measured foliation. In our case, the infinite tree seen as branched surface is the covering space of a Riemannian surface of infinite genus. Therefore the quadratic differential is unique by lifting it to the covering.

In our construction of the $\text{End}(R^4)$, the infinite tree is given by the process $Y_0 \to \cdots \to Y_\infty$ ending at $Y_\infty = S^3$. Every change $Y_n \to Y_{n+1}$ defines the branching into the tree (as given by the branching of the link complements, see sec. [4]). To every $Y_n$, one has a hyperbolic structure by choosing a homomorphism $\pi_1(Y_n) \to SO(3,1) = Isom(\mathbb{H}^3)$ (up to conjugation). Because of Mostow-Prasad rigidity, the volume is a topological invariant and we obtain a natural scale $L$ for every $Y_n$. This scale $L$ changes during the transition $Y_n \to Y_{n+1}$ from $L|_{Y_n}$ to $L|_{Y_{n+1}}$. But the change is given by the cobordism $W(Y_n, Y_{n+1})$ and the scales varies as a smooth variable at the cobordism. Therefore, the foliation with $t = \text{const.}$ is enough. One has to consider the time with respect to the corresponding scale. Therefore we are enforced to make the identification

$$Re(\xi) = \frac{t}{L}$$

in the quadratic differential (defining the tree $T_{CH}$). Then the length in the embedded tree is given by $d(Re(\xi))$, the measure of the vertical foliation. Now the growing $ds_T^2$ of the tree with respect to the hyperbolic structure is given by the measure $d(Re(\xi))^2$ of the vertical foliation, or

$$ds_T^2 = \frac{da_T^2}{a_T^2} = d\left(\frac{t}{L}\right)^2$$

in agreement with our heuristic, dimensional argument above. This equation agrees with the Friedman equation for a (flat) deSitter space, i.e. the current model of our universe with a CC. This equation can be formally integrated yielding the expression

$$a_T(t, L) = a_0 \cdot \exp\left(\frac{t}{L}\right)$$

and we are enforced to determine the ratio $t/L$. Therefore let

$$|\beta R| = \frac{1}{L^2}$$

be the absolute value of the scalar curvature of some $Y_n$ with respect to the scale $L$. By a simple integration with respect to the metric $h$ of the
\begin{align*}
\frac{1}{L^2} = \frac{\int_{Y_n} |^3 R| \sqrt{\hat{h}} d^3 x}{\int_{Y_n} \sqrt{\hat{h}} d^3 x}
\end{align*}

for the constant scalar curvature. Finally we will show that the integral is proportional to the Chern-Simons invariant \(CS(Y_n)\).

As a motivation, let us consider the cobordism \(W(\Sigma_1, \Sigma_2)\) between two 3-manifolds \(\Sigma_1\) and \(\Sigma_2\). One important invariant of a cobordism is the signature \(\sigma(W)\), i.e. the number of positive minus the number of negative eigenvalues of the intersection form. Using the Hirzebruch signature theorem, it is given by the first Pontryagin class

\begin{align*}
\sigma(W(\Sigma_1, \Sigma_2)) = \frac{1}{3} \int_{W(\Sigma_1, \Sigma_2)} tr(R \wedge R)
\end{align*}

with the curvature 2-form \(R\) of the tangent bundle \(TW\). By Stokes theorem, this expression is given by the difference

\begin{align*}
\sigma(W(\Sigma_1, \Sigma_2)) = \frac{1}{3} CS(\Sigma_2) - \frac{1}{3} CS(\Sigma_1)
\end{align*} \quad (13)

of two boundary integrals where

\begin{align*}
\int_{\Sigma} tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) = 8\pi^2 CS(\Sigma)
\end{align*}

is known as Chern-Simons invariant of a 3-manifold \(\Sigma\). Using ideas of Witten \cite{29, 30, 31} we will interpret the connection \(A\) as \(ISO(2, 1)\) connection. Note that \(ISO(2, 1)\) is the Lorentz group \(SO(3, 1)\) by Wigner-Inönü contraction or the isometry group of the hyperbolic geometry. For that purpose we choose

\begin{align*}
A_i = \frac{1}{\ell} e_i^a P_a + \omega_i^a J_a
\end{align*} \quad (14)

with the length \(\ell\) and 1-form \(A = A_i dx^i\) with values in the Lie algebra \(ISO(2, 1)\) so that the generators \(P_a, J_a\) fulfill the commutation relations

\begin{align*}
[J_a, J_b] = \epsilon_{abc} J^c \quad [P_a, P_b] = 0 \quad [J_a, P_b] = \epsilon_{abc} P^c
\end{align*}
with pairings $\langle J_a, P_b \rangle = Tr(J_a P_b) = \delta_{ab}$, $\langle J_a, J_b \rangle = 0 = \langle P_a, P_b \rangle$. This choice was discussed in \cite{32} in the context of Cartan geometry. The appearance of the length $\ell$ can be understood by considering the generators $P_a$ and $J_a$. $P_a$ generates translations in units of a length, with scale $\ell$, whereas $J_a$ generates rotations in units of an angle. Remember, that every transition $Y_n \rightarrow Y_{n+1}$ is described by a cobordism $W(Y_n, Y_{n+1})$. The coordinate normal to $Y_n$ and in direction to $Y_{n+1}$ will be denoted by $t$, called time. Because of the volume growing $\text{vol}(Y_n) < \text{vol}(Y_{n+1})$, the corresponding length $\ell$ (of the generator $P_a$) varies by every transition $Y_n \rightarrow Y_{n+1}$. Then it has the meaning of the time coordinate parametrizing the transitions. Therefore we identify $t = \ell$, see below for the consequences.

Then we obtain for the curvature

$$F_{ij} = \frac{1}{\ell} P_a \left( \partial_i e^a_j - \partial_j e^a_i + e^{abc} (\omega_i b e_{j c} + e_{i b} \omega_{j c}) \right) + J_a \left( \partial_i \omega^a_j - \partial_j \omega^a_i + e^{abc} \omega_{i b} \omega_{j c} \right)$$

In what follows we start with the expression $A \wedge F$ and use the pairing $\langle \cdot, \cdot \rangle = Tr(\cdot)$ (following the MacDowell Mansouri approach, see \cite{33}). Then we will get

$$\text{tr} (A \wedge F) = \frac{1}{\ell} e \wedge (d\omega + \omega \wedge \omega) + \frac{1}{\ell} \omega \wedge (de + \omega \wedge e)$$

where the second expression is given by $\omega \wedge T$ with torsion form $T$. The first expression has the structure $e \wedge R$ with the curvature $2$-form $R$, which agrees with the scalar curvature $3 R$ multiplied by the volume form in the first order formalism. Therefore for vanishing torsion $T = 0$, we obtain

$$\int_\Sigma \text{tr} (A \wedge F) = \frac{1}{\ell} \int_\Sigma 3 R \sqrt{h} d^3 x$$

By using a simple scaling $\omega \rightarrow \frac{3}{2} \omega$, we get the new connection

$$\frac{2}{3} A_i = \frac{2}{3 \ell} e^a_i P_a + \omega^a_i J_a$$

and finally the relation

$$\text{tr} \left( A \wedge (dA + \frac{2}{3} A \wedge A) \right) = \frac{3}{2 \ell} e \wedge (d\omega + \omega \wedge \omega)$$

21
or

\[ 8\pi^2 \cdot \ell \cdot CS(\Sigma) = \frac{3}{2} \int_{\Sigma} 3R\sqrt{\hat{h}}d^3x. \]  \hspace{1cm} (15) 

From (13) it follows that

\[ \sigma(W(\Sigma_1, \Sigma_2)) = \frac{1}{3}CS(\Sigma_1) - \frac{1}{3}CS(\Sigma_2) + \frac{1}{3}CS(\Sigma_3) - \ldots - \frac{1}{3}CS(\Sigma_k) = \frac{1}{3}CS(\Sigma_k) - \frac{1}{3}CS(\Sigma_1). \]

Taking $\Sigma_2$ as $\Sigma_2 = Y_\infty$ we can define the cobordism $End(R^4) = W(Y_0, Y_\infty)$ as $W(\Sigma_0, \Sigma_2)$ above, where the limiting 3-manifold $Y_\infty$ results from the resolution of the 1-/2-handle pair by gluing the Casson handle to $K$ as before.

Thus we will concentrate on the properties of $Y_\infty = \Sigma_2$ in the following. To relate this cobordism with the CS invariant as above we need only one assumption: $\Sigma_2$ must admit a metric of constant curvature. After the Thurston’s geometrization conjecture was proved, it is not a strong restriction. $\Sigma_2$ must be a prime manifold with no incompressible torus submanifold. Thus the formula (13) and the expression above show that the Chern-Simons invariant of $\Sigma_2$ is directly related to the 4-dimensional cobordism $W(\Sigma_0, \Sigma_2)$.

In (14) we are enforced to introduce the length $\ell$ for the translation (represented by the generator $P_a$). But as discussed above, this translation is parametrized by the coordinate $t$. Above we identify $t = \ell$ with the time and using (15) we will obtain the expression

\[ t \cdot CS(\Sigma_2) = \frac{3}{2} \int_{\Sigma_2} 3R_{ren}\sqrt{\hat{h}}d^3x \]  \hspace{1cm} (16) 

where the extra factor $8\pi^2$ (equals $4 \cdot \text{vol}(S^3)$) is the normalization of the curvature integral. This normalization of the curvature changes the absolute value of the curvature into

\[ |^3R_{ren}| = \frac{1}{8\pi^2 L^2} \]  \hspace{1cm} (17) 

and we choose the scaling factor by the relation to the volume $L = \sqrt[3]{\text{vol}(\Sigma_2)/(8\pi^2)}$. Then we will obtain formally

\[ \int_{\Sigma_2} |^3R_{ren}|\sqrt{\hat{h}}d^3x = \int_{\Sigma_2} \frac{1}{8\pi^2 L^2}\sqrt{\hat{h}}d^3x = L^3 \cdot \frac{1}{L^2} = L \]  \hspace{1cm} (18)
by using

\[ L^3 = \frac{vol(\Sigma_2)}{8\pi^2} = \frac{1}{8\pi^2} \int_{\Sigma_2} \sqrt{h} \, d^3x \]

in agreement with the normalization above. Let us note that Mostow-Prasad rigidity enforces us to choose a rescaled formula

\[ vol_{hyp}(\Sigma_2) \cdot L^3 = \frac{1}{8\pi^2} \int_{\Sigma_2} \sqrt{h} \, d^3x , \]

with the hyperbolic volume (as a topological invariant). The volume of all other 3-manifolds can be arbitrarily scaled. In case of hyperbolic 3-manifolds, the scalar curvature \( 3R < 0 \) is negative but above we used the absolute value \(|3R|\) in the calculation. Therefore we have to modify (16), i.e. we have to use the absolute value of the curvature \(|3R|\) and of the Chern-Simons invariant \(|CS(\Sigma_2)|\). By (16) and (18) using

\[
    \frac{t}{L} = \begin{cases} 
        \frac{3}{2CS(\Sigma_2)} & \Sigma_2 \text{ non-hyperbolic 3-manifold} \\
        \frac{3\cdot vol_{hyp}(\Sigma_2)}{2|CS(\Sigma_2)|} & \Sigma_2 \text{ hyperbolic 3-manifold} 
    \end{cases}
\]

a simple integration (12) gives the following exponential behavior

\[
    a(t) = a_0 \cdot e^{t/L} = \begin{cases} 
        a_0 \cdot \exp \left( \frac{3}{2CS(\Sigma_2)} \right) & \Sigma_2 \text{ non-hyperbolic 3-manifold} \\
        a_0 \cdot \exp \left( \frac{3\cdot vol_{hyp}(\Sigma_2)}{2|CS(\Sigma_2)|} \right) & \Sigma_2 \text{ hyperbolic 3-manifold} .
    \end{cases}
\]

For the following, we will introduce the shortening

\[
    \vartheta = \begin{cases} 
        \frac{3}{2CS(\Sigma_2)} & \Sigma_2 \text{ non-hyperbolic 3-manifold} \\
        \frac{3\cdot vol_{hyp}(\Sigma_2)}{2|CS(\Sigma_2)|} & \Sigma_2 \text{ hyperbolic 3-manifold} 
    \end{cases}
\]

Finally we can state:

*Let \( W(\Sigma_1, \Sigma_2) \) be a cobordism which is embedded into a hyperbolic 4-manifold (succeeding this structure). Let \( \Sigma_1, \Sigma_2 \) admit metrics of constant curvature. Let the curvature of \( \Sigma_1 \) be \( curv(\Sigma_1) \) (up to a sign), then \( \Sigma_2 \) admits the curvature

\[ curv(\Sigma_2) = curv(\Sigma_1) \cdot \exp(-2\vartheta) \] (19)

which we call the cosmological constant for the cobordism. The ratio of the two curvatures

\[ \frac{curv(\Sigma_2)}{curv(\Sigma_1)} = \exp(-2\vartheta) \]

is a topological invariant of \( \Sigma_2 \).*
6 A realistic model for the cosmological constant

Main problem in the calculation of the cosmological constant value above was the chain of topology changes. In the formula (19) above, we were able to restrict the analysis to the final 3-manifold $\Sigma$. Let us consider again the embedding $\mathbb{R}^4 \hookrightarrow \mathbb{R}^4$. We have a chain of 3-manifolds $Y_0 \to \cdots \to Y_\infty$ which end with $Y_\infty$, the wild $S^3$. Thus the value of the cosmological constant induced by the embedding $\mathbb{R}^4 \hookrightarrow \mathbb{R}^4$ is given by

$$\frac{1}{8\pi^2 L^2} \exp \left( -\frac{3}{CS(Y_\infty)} \right)$$

using the normalization (17) and we need to calculate the Chern-Simons invariant $CS(Y_\infty)$. But the Chern-Simons invariant $CS(Y_\infty)$ vanishes for any 3-sphere (wild or not). Finally we have:

The embedding $\mathbb{R}^4 \hookrightarrow \mathbb{R}^4$ induces a vanishing cosmological constant.

This result is interesting by itself and important for many different reasons. The example are [34, 35, 36, 37] where we used directly the embedding $\mathbb{R}^4 \hookrightarrow \mathbb{R}^4$ and obtained a relation to quantum field theory where the cosmological constant is given by the vacuum expectation value. This is the embedding and the hyperbolic geometry which made the short-length modes damped and the sum over all modes vanishes.

However the vanishing of CC certainly does not explain its small non-zero value. That is why in the search for the topological origins for the realistic CC value we look for other, presumably more global, embeddings. Fortunately, in the case of our small exotic $\mathbb{R}^4$ there is a natural embedding as given by the construction based on h-cobordism theorem. Namely, one starts with two non-diffeomorphic but homeomorphic, compact, simply-connected, closed 4-manifolds $X_\pm$ and their 5-dimensional h-cobordism $W$. Now there are submanifolds $Y_\pm \subset X_\pm$ (called also Akbulut corks) reflecting the non-triviality of the h-cobordism i.e. the h-cobordism between $X_+ \setminus Y_+$ and $X_- \setminus Y_-$ is trivial or $X_+ \setminus Y_+$ is diffeomorphic to $X_- \setminus Y_-$. Then an open neighborhood of $Y_\pm$ inside of the remaining h-cobordism is the small exotic $\mathbb{R}^4$. It contains the submanifold $K$ having $Y_\pm$ as submanifolds.

In the simplest and well-described example [15], the small exotic $\mathbb{R}^4$ was constructed by using the compact 4-manifold $E(2)\#\overline{CP^2}$ (with the K3 surface $E(2)$) where the standard $\mathbb{R}^4$ is a part of this compact 4-manifold and serves
as the embedding space for $R^4$. Therefore we expect that the real embedding should be finally given by $R^4 \hookrightarrow E(2)\#\overline{CP^2}$. Such embedding determines also modified topology changes compared to the $R^4 \to R^4$ case. The Akbulut cork of $E(2)\#\overline{CP^2}$ is a contractible 4-manifold with the boundary $\Sigma(2, 5, 7)$ (a Brieskorn sphere). Then the small exotic $R^4$ can be understood as lying between the Akbulut cork and the topological part of the 4-manifold $E(2)\#\overline{CP^2}$. Here we will comment on the similarity of this construction to those from our previous work [3]. In contrast to our previous work the current model is generic and do not use any special choices (like a special Casson handle or the embedding of the Akbulut cork). Instead, it is natural model where $R^4$ is the part of a larger spacetime given by $E(2)\#\overline{CP^2}$ from which $R^4$ inherits its unique data.

As we remember, every subset $K', K' \subset K \subset R^4$, is surrounded by a 3-sphere. Now we take it as Planck-size 3-sphere $S^3$ inside of the compact subset $K \subset R^4$. This is the initial point where our cosmos starts to evolve. By the construction of $R^4$, as mentioned above, there exists the homology 3-sphere $\Sigma(2, 5, 7)$ inside of $K$ which is the boundary of the Akbulut cork for $E(2)\#\overline{CP^2}$ (see chapter 9, [16]). If $S^3$ is the starting point of the cosmos as above, then $S^3 \subset \Sigma(2, 5, 7)$. But then we will obtain the first topological transition

$$S^3 \to \Sigma(2, 5, 7)$$

inside $R^4$. The construction of $R^4$ was based on the topological structure of $E(2)$ (the K3 surface). $E(2)$ splits topologically into a 4-manifold $|E_8 \oplus E_8|$ with intersection form $E_8 \oplus E_8$ (see [16]) and the sum of three copies of $S^2 \times S^2$. The 4-manifold $|E_8 \oplus E_8|$ has a boundary which is the sum of two Poincare spheres $P\#P$. Here we used the fact that a smooth 4-manifold with intersection form $E_8$ must have a boundary (which is the Poincare sphere $P$), otherwise it would contradict the Donaldson’s theorem. Then any closed version of $|E_8 \oplus E_8|$ does not exist and this fact is the reason for the existence of $R^4$. To express it differently, the $R^4$ lies between this 3-manifold $\Sigma(2, 5, 7)$ and the sum of two Poincare spheres $P\#P$. Therefore we have two topological transitions resulting from the embedding into $E(2)\#\overline{CP^2}$

$$S^3 \xrightarrow{\text{cork}} \Sigma(2, 5, 7) \xrightarrow{\text{gluing}} P\#P.$$

Each of these two transitions is connected to a different embedding and therefore we will obtain two contributions (in contrast to the chain of 3-manifolds for one embedding leading to the single factor as before). Finally
we obtain the two contributions as arranged in one expression

\[ a = a_0 \cdot \exp \left( \frac{3}{2 \cdot CS(\partial A_{\text{cork}})} + \frac{3}{2 \cdot CS(P\#P)} \right). \]

As mentioned above, the 3-sphere is assumed to be of Planck-size

\[ a_0 = L_P = \sqrt{\frac{\hbar G}{c^3}} \]

and one obtains for the first transition

\[ a_1 = L_P \cdot \exp \left( \frac{3}{2 \cdot CS(\Sigma(2, 5, 7))} \right). \]

To determine the Chern-Simons invariants of the Brieskorn spheres we can use the method of Fintushel and Stern [38, 39, 40]. The calculation can be found in [4]. The value of the Chern-Simons invariant \( CS(\Sigma(2, 5, 7)) \) is given by \( \frac{9}{280} \) and we obtain

\[ a_1 = L_P \cdot \exp \left( \frac{140}{3} \right) \approx 7.5 \cdot 10^{-15} m \]

which is interpreted as the size of the 'cosmos' represented here by the 3-manifold \( \Sigma(2, 5, 7) \) at the end of the first inflationary phase [5]. This size can be related to an energy scale by using it as Compton length and one obtains 165 MeV, comparable to the energy scale of the QCD lying between 217 MeV and 350 MeV (see [41, 42]). Thus starting with the 3-sphere \( S^3 \) of the Planck length size \( L_P \) and using formula (17) for the renormalized curvature \( 1/(8\pi^2 L_P^2) \) (see the discussion for formula (16) above), the corresponding expression for the CC reads

\[ \Lambda = \frac{1}{8\pi^2 L_P^2} \cdot \exp \left( -\frac{3}{CS(\Sigma(2, 5, 7))} - \frac{3}{CS(P\#P)} \right) \]

which, after introducing the exact values of the Chern-Simons invariants, \( CS(\Sigma(2, 5, 7)) = \frac{9}{280} \) and \( CS(P\#P) = \frac{1}{60} \), gives the value

\[ \Lambda \cdot L_P^2 = \frac{1}{8\pi^2} \exp \left( -\frac{280}{3} - 180 \right) \approx 2.5 \cdot 10^{-121} \]

in Planck units.
Finally we showed that the ratio between cosmological constant $\Lambda$ and the (normalized) curvature $\Lambda_{\text{BigBang}} = \frac{1}{8\pi^2 L^2}$ of the small $S^3$ at the Big Bang is a topological invariant

$$\frac{\Lambda}{\Lambda_{\text{BigBang}}} = \exp \left( -\frac{3}{CS(\Sigma(2,5,7))} - \frac{3}{CS(P\#P)} \right)$$

In cosmology one usually relates the cosmological constant to the Hubble constant $H_0$ (expressing the critical density) leading to the length scale

$$L_c^2 = \frac{c^2}{3H_0^2}.$$ 

The corresponding variable is denoted by $\Omega_\Lambda$ and for the expression above it gives the topological invariant value

$$\Omega_\Lambda = \frac{c^5}{24\pi^2 hGH_0^2} \cdot \exp \left( -\frac{3}{CS(\Sigma(2,5,7))} - \frac{3}{CS(P\#P)} - \chi(A_{\text{cork}}) \right)$$

in units of the critical density. Up to now everything went classical. The realistic model of CC would certainly require the inclusion of some kind of quantum corrections to the above calculations. Especially, the choice of the 3-sphere at the beginning of the Planck size requires quantum approach. One problem is certainly the lack of any final theory of quantum gravity. However in the approach to the structure of spacetime via exotic 4-smoothness (see e.g. [37]) there exist certain techniques (mainly based on the topology of the handle decompositions of exotic manifolds), which allow for grasping the corrections. These corrections are also topological invariants. Namely, for the first transition, i.e. $S^3 \to \Sigma(2,5,7)$, we have analyzed in [37] the corresponding gravitational action. It appears to be the linear combination of the Pontryagin and Euler classes of the Akbulut cork of the 5-dimensional non-trivial cobordism [37, p. 263]. Because of the contractibility of the cork, the Pontryagin part has to vanish but the Euler class gives a nonzero contribution

$$\exp \left( -\frac{\chi(A_{\text{cork}})}{4} \right) = \exp(-0.25)$$

with the Euler characteristics $\chi(A_{\text{cork}}) = 1$ of the Akbulut cork. Introducing this correction into Eq. (20) the final formula for $\Omega_\Lambda$ reads

$$\Omega_\Lambda = \frac{c^5}{24\pi^2 hGH_0^2} \cdot \exp \left( -\frac{3}{CS(\Sigma(2,5,7))} - \frac{3}{CS(P\#P)} - \frac{\chi(A_{\text{cork}})}{4} \right).$$
This additional factor can be also motivated by the short scale behavior of gravity as shown in [37]. At very small scales, one obtains a dimensional reduction from 4D to 2D. Then the 4D Einstein-Hilbert action will be reduced to the 2D Einstein-Hilbert action. But the 2D Einstein-Hilbert action is equal to the Euler characteristics. In the course of the dimensional reduction, the 4D contractable space (i.e. the Akbulut cork) will be reduced to a 2D contractable space. Finally, the 4D Einstein-Hilbert action of the Akbulut cork will be reduced to the Euler characteristics of the 2D contractable space. Interestingly, the Euler characteristics for a 2D and a 4D contractable space agree. This argumentation motivates the appearance of the contribution $\chi(A_{cork}) = 1$ in the formula above.

Current measurements [43, 44] of the Hubble constant in the PLANCK mission combined with other measurements like the Hubble telescope [45] give the value

$$H_0^{\text{Planck + Hubble}} = 69,2 \frac{km}{s \cdot Mpc}$$

when applied to (20), it gives rise to the following value of $\Omega_C$

$$\Omega_{\Lambda} \approx 0.7029$$

which is in excellent agreement with the measurements. In [5] we discussed some other possibilities of quantum corrections by using spin foam models or loop quantum gravity. However, the derivation given above rests on purely topological methods which is favored in this paper.

### 7 Discussion

We have presented the derivation of the value of the cosmological constant as a topological invariant based on low dimensional differential topology. Quantum corrections were also included as topological invariants. What is the meaning of such topologically supported cosmological constant derived from the embedding of exotic $R^4$ into $E(2)\#\mathbb{CP}^2$? One point was mentioned in the Introduction, namely the structure of the unobserved universe is more rich than the observed local one. However, everything is still happening entirely in dimension 4, i.e. $E(2)\#\mathbb{CP}^2$ is a 4-manifold which local structure is $R^4$ and exotic smoothness of $R^n$ is exclusively 4-dimensional phenomenon. Moreover, we focused on small exotic $R^4$ which embeds in the standard $R^4$. 
as open subset and it results in the chain of the embeddings $R^4 \hookrightarrow R^4 \hookrightarrow E(2)\#CP^2$ explaining global 4-dimensional structure of the universe. One indication that such structure exists is the topologically supported value of the dark energy density derived from it. As shown in this paper the value agrees with Planck mission data. Possibly, some other testable predictions could be drawn from the existence of such big, nontrivial structure of the 4-dimensional universe. Among results into this direction there is the derivation of the speed of the inflation of universe from exotic smooth structures on $R^4$. The calculations fit reasonably well with the data of PLANCK [5].

Alternatively, without assuming any big unobserved nontrivial universe structure, one could still consider the local differentiable structure of the universe as modeled by smooth exotic $R^4$ rather than by the standard $R^4$ (on which the Lorentz structure is to be subsequently introduced). This exotic $R^4$ can be precisely defined purely mathematically from the embedding into $E(2)\#CP^2$ which follows from the existence of the nontrivial h-cobordism. In this case one does not assume that universe is described globally by $E(2)\#CP^2$. However, in such a case there are left open physical questions regarding such choice and the embedding into the target manifold. On the other hand, this is the embedding which explains the results obtained and the uniqueness of the Casson handle structures on which they rest. Thus taking a global structure of the universe as $E(2)\#CP^2$ answers the questions and gives rise to interesting investigation perspective where the problem of global gains new meaning. We are leaning towards and favor the last option in the paper. However, as discussed in [16] there are inherent reasons that the exotic $R^4$ at cosmological scales emerged directly from the quantum mechanical structure itself. In addition exotic $R^4$’s can be described by operator algebras naturally appearing in QM [37]. Hence, if one agreed that the evolution of the universe began with a quantum regime where the standard QM description by a Hilbert space of states applies, the large scale differentiable structure should be an exotic $R^4$ rather than the standard 4-space as shown in [16]. Keeping this line of argumentation in mind, we conjecture that this exotic $R^4$ is central for the understanding of the small as well the large scale structure of the universe. It seems that one could further attempt to work out this $R^4$ directly from QM formalism. But currently, there is no unique connection. There is ongoing work on these fascinating topics and near future will yield more definite resolutions.
8 Appendix: Hyperbolic 3-/4-Manifolds and Mostow-Prasad rigidity

In short, Mostow–Prasad rigidity theorem states that the geometry of a complete, finite-volume hyperbolic manifold of dimension greater than two is uniquely determined by the fundamental group. The corresponding theorem was proven by Mostow for closed manifolds and extended by Prasad for finite-volume manifolds with boundary. In dimension 3, there is also an extension for non-compact manifolds also called ending lamination theorem. It states that hyperbolic 3-manifolds with finitely generated fundamental groups are determined by their topology together with invariants of the ends admitting a kind of foliation at surfaces in the end. The end of a 3-manifolds has always the form $S \times [0, 1)$ with the compact surfaces $S$. Then a lamination on the surface $S$ is a closed subset of $S$ that is written as the disjoint union of geodesics of $S$.

A general formulation of the Mostow-Prasad rigidity theorem is: Let $M, N$ be compact hyperbolic $n$–manifolds with $n \geq 3$. Assume that $M$ and $N$ have isomorphic fundamental groups. Then the isomorphism of fundamental groups is induced by a unique isometry.

An important corollary states that geometric invariants are topological invariants. The Mostow-Prasad rigidity theorem has special formulations for dimension 3 and 4. Both manifolds $M, N$ have to be homotopy-equivalent and every homotopy-equivalence induces an isometry. In dimension 3, the homotopy-equivalence of a 3-manifold of non-positive sectional curvature implies a homeomorphism (a direct consequence of the geometrization theorem, the exception are only the lens spaces) and a diffeomorphism (see Moise [17]). In dimension 4, compact homotopy-equivalent simply-connected 4-manifolds are homeomorphic (see Freedman [13]). This result can be extended to a large class of compact non-simply connected 4-manifolds (having a good fundamental group), see [48]. Therefore, if a 3- or 4-manifold admits a hyperbolic structure then this structure is unique up to isometry and all geometric invariants are topological invariants among them the volume and the curvature.

Then a hyperbolic 3-manifold $M^3$ is given by the quotient space $\mathbb{H}^3/\Gamma$ where $\Gamma \subset Isom(\mathbb{H}^3) = SO(3, 1)$ is a discrete subgroup (Kleinian group) so that $\Gamma \simeq \pi_1(M^3)$. A hyperbolic structure is a homomorphism $\pi_1(M^3) \to SO(3, 1)$ up to conjugacy (inducing the isometry). The analogous result holds.
for the hyperbolic 4-manifold which can be written as quotient $\mathbb{H}^4/\pi_1(M^4)$.

Let $X^4$ be a compact hyperbolic 4-manifold with metric $g_0$ and let $M^4$ be a compact manifold together with a smooth map $f : M \to X$. As shown in [49] or in the survey [50] (Main Theorem 1.1), the volumes of $X, M$ are related

$$\text{Vol}_r(M) \geq \text{deg}(f) \text{Vol}(X, g_0)$$

where $\text{deg}(f)$ denotes the degree of $f$. If equality holds, and if the infimum of the relation is achieved by some metric $g$, then $(M, g)$ is an isometric Riemannian covering of $(X, g_0)$ with covering map $M \to X$ homotopic to $f$. In particular, if $f$ is the identity map $X \to X$ (having degree $\text{deg}(f) = 1$) then it implies that $g_0$ is the only Einstein metric on $X$ up to rescalings and diffeomorphisms.

9 Appendix: Wild and Tame embeddings

We call a map $f : N \to M$ between two topological manifolds an embedding if $N$ and $f(N) \subset M$ are homeomorphic to each other. From the differential-topological point of view, an embedding is a map $f : N \to M$ with injective differential on each point (an immersion) and $N$ is diffeomorphic to $f(N) \subset M$. An embedding $i : N \hookrightarrow M$ is tame if $i(N)$ is represented by a finite polyhedron homeomorphic to $N$. Otherwise we call the embedding wild. There are famous wild embeddings like Alexanders horned sphere [51] or Antoines necklace. In physics one uses mostly tame embeddings but as Cannon mentioned in his overview [52], one needs wild embeddings to understand the tame one.

10 Appendix: Models of Hyperbolic geometry

In the following we will describe two main models of hyperbolic geometry which were used in this paper. For simplicity we will concentrate on the two-dimensional versions.

The Poincare disk model also called the conformal disk model, is a model of 2-dimensional hyperbolic geometry in which the points of the geometry are inside the unit disk, and the straight lines consist of all segments of circles
contained within that disk that are orthogonal to the boundary of the disk, plus all diameters of the disk. The metric in this model is given by

$$ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$

which can be transformed to expression (2) by a radial coordinate transformation. In this model, the hyperbolic geometry is confined to the unit disk, where the boundary represents the 'sphere at infinity'.

The Poincare half-plane model is the upper half-plane, denoted by \( \mathbb{H}^2 = \{(x, y) \mid y > 0, \ x, y \in \mathbb{R}\} \), together with a metric, the Poincare metric,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

(see (10)) that makes it a model of two-dimensional hyperbolic geometry. Here the line \( y = 0 \) represents the infinity (so-called ideal points).

Both models are isometric to each other. A point \((x, y)\) in the disk model maps to the point

$$\left(\frac{2x}{x^2 + (1 - y)^2}, \frac{1 - x^2 - y^2}{x^2 + (1 - y)^2}\right)$$

in the half-plane model conversely a point \((x, y)\) in the half-plane model maps to the point

$$\left(\frac{2x}{x^2 + (1 + y)^2}, \frac{x^2 + y^2 - 1}{x^2 + (1 + y)^2}\right)$$

in the disk model. This transform is known as Cayley transform.

References

References

[1] G. D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, Publ. Math. IHES 34 (1968) 53–104.

[2] G. Prasad, Strong rigidity of q-rank 1 lattices, Inv. Math. 21 (1973) 255–286.
[3] P.-Y. Chang, O. Erten, P. Coleman, Möbius Kondo insulators, Nature Physics 13 (2017) 794–798.

[4] Y.-F. Ly, et all., Experimental signature of topological superconductivity and Majorana zero modes on β-Bi2Pd thin films, Science Bulletin 12(30) (2017) 852–856.

[5] T. Asselmeyer-Maluga, J. Król, Inflation and topological phase transition driven by exotic smoothness, Advances in High Energy Physics 2014 (2014) Article ID 867460. doi:10.1155/2014/867460.

[6] A. Anderson, B. DeWitt, Does the topology of space fluctuate?, Found. of Phys. 16 (1986) 91–105.

[7] H. Dowker, R. Garcia, A handlebody calculus for topology change, Class. Quant. Grav. 15 (1998) 1859.

[8] A. Borde, H. Dowker, R. Garcia, R. Sorkin, S. Surya, Causal continuity in degenerate spacetimes, Class. Quant. Grav. 16 (1999) 3457, arXiv:gr-qc/9901063.

[9] H. Dowker, R. Garcia, S. Surya, Morse index and causal continuity. a criterion for topology change in quantum gravity, Class. Quant. Grav. 17 (2000) 697.

[10] F. Laudenbach, V. Poenaru, A note on 4-diemnsional handlebodies, Bull. Soc. Math. France 100 (1972) 337–344.

[11] L. Taylor, An invariant of smooth 4-manifolds, Geom. Top. 1 (1997) 71–89.

[12] S. Akbulut, On 2-dimensional homology classes of 4-manifolds, Math. Camb. Phil. Soc. 82 (1977) 99–106.

[13] M. Freedman, The topology of four-dimensional manifolds, J. Diff. Geom. 17 (1982)) 357–454.

[14] S. Donaldson, Irrationality and the h-cobordism conjecture, J. Diff. Geom. 26 (1987) 141–168.

[15] S. DeMichelis, M. Freedman, Uncountable many exotic $R^4$'s in standard 4-space, J. Diff. Geom. 35 (1992) 219–254.
[16] R. Gompf, A. Stipsicz, 4-manifolds and Kirby Calculus, American Mathematical Society, 1999.

[17] S. Ganzell, Ends of 4-manifolds, Top. Proc., 30 (2006) 223–236.

[18] Ž. Bizaca, R. Gompf, Elliptic surfaces and some simple exotic $R^4$'s, J. Diff. Geom. 43 (1996) 458–504.

[19] R. Daverman, G. Venema, Embeddings in Manifolds, Vol. 106 of Graduate Studies in Math., AMS, Providence RI, 2009.

[20] R. Budney, JSJ-decompositions of knot and link complements in the 3-sphere, L’enseignement Mathématique 52 (2006) 319–359.

[21] D. Futer, E. Kalfagianni, J. Purcell, A survey of hyperbolic knot theory, arXiv:1708.07201 (2017).

[22] W. Menasco, Closed incompressible surfaces in alternating knot and link complements, Topology 23 (1984) 37–44.

[23] R. Fintushel, R. Stern, Knots, links, and 4-manifolds, Inv. Math. 134 (1998) 363–400.

[24] J. D. Moore, Lectures on Seiberg-Witten Invariants, Springer, New York, 2001.

[25] A. Eddington, On the instability of Einstein’s spherical world, Mont. Not. of the Royal Astron. Soc. 90 (1930) 668–678.

[26] G. Ellis, R. Maartens, The emergent universe: Inflationary cosmology with no singularity, Class. Quat. Grav. 21 (2004) 223–232, arxiv: gr-qc/0211082.

[27] S.-L. Li, H. Wei, Stability of the Einstein static universe in Eddington-inspired Born-Infeld theory, Phys. Rev, D 96 (2017) 023531.

[28] J. Hubbard, H. Masur, Quadratic differentials and foliations, Acta Math. 142 (1979) 221–274.

[29] E. Witten, 2+1 dimensional gravity as an exactly soluble system, Nucl. Phys., B 311 (1988/89) 46–78.
[30] E. Witten, Topology-changing amplitudes in 2+1 dimensional gravity, Nucl. Phys., B323 323 (1989) 113–140.

[31] E. Witten, Quantization of Chern-Simons gauge theory with complex gauge group, Comm. Math. Phys. 137 (1991) 29–66.

[32] D. Wise, Macdowell-Mansouri gravity and Cartan geometry, Class. Quantum Grav. 27 (2010) 155010, arXiv:gr-qc/0611154.

[33] S. MacDowell, F. Mansouri, Unified geometric theory of gravity and supergravity, Phys. Rev. Lett. 38 (1977) 739–742.

[34] T. Asselmeyer-Maluga, J. Król, Exotic smooth $\mathbb{R}^4$, noncommutative algebras and quantization. ArXiv:1001.0882.

[35] T. Asselmeyer-Maluga, J. Król, Constructing a quantum field theory from spacetime. ArXiv:1107.3458.

[36] T. Asselmeyer-Maluga, Exotic $R^4$ and quantum field theory, in: C.Burdik, et al. (Eds.), 7th International Conference on Quantum Theory and Symmetries (QTS7), IOP Publishing, Bristol, UK, 2012, p. 012011. doi:10.1088/1742-6596/343/1/012011.

[37] T. Asselmeyer-Maluga, At the Frontiers of Spacetime: Scalar-Tensor Theory, Bell’s Inequality, Mach’s Principle, Exotic Smoothness, Springer, Switzerland, 2016, Ch. Smooth quantum gravity: Exotic smoothness and quantum gravity, pp. 247–308, in honor of Carl Brans’s 80th birthday.

[38] R. Fintushel, R. Stern, Instanton homology of Seifert fibred homology three spheres, Proc. London Math. Soc. 61 (1990) 109–137.

[39] P. Kirk, E. Klassen, Chern-Simons invariants of 3-manifolds and representation spaces of knot groups, Math. Ann. 287 (1990) 343–367.

[40] D. Freed, R. Gompf, Computer calculation of Witten’s 3-manifold invariant, Comm. Math. Phys. 141 (1991) 79–117.

[41] R. Sekhar Chivukula, The origin of mass in QCD, arXiv:hep-ph/0411198 (2004).
[42] A. Deur, S. Brodsky, G. de Teramond, The QCD running coupling, Prog. Part. Nuc. Phys. 90 (2016) 1–74, arXiv:1604.08082.

[43] P. C. P. Ade, et.al., Planck 2013 results. XVI cosmological parameters., Astron. and Astrophys. 571 (2014) A16, arXiv:1303.5076[astro-ph.CO].

[44] P. C. P. Ade, et.al., Planck 2015 results. XIII cosmological parameters., Astron. and Astrophys. 594 (2016) A13, arXiv:1502.01589[astro-ph.CO].

[45] V. Bonvin, F. Courbin, S. H. e. A. Suyu, H0LiCOW-V. new COSMOGRAIL time delays of HE 0435-1223: H0 to 3.8 per cent precision from strong lensing in a flat $\Lambda$cdm model, Mon. Not. Roy. Astron. Soc. 465 (2016) 4914–4930, arXiv:1607.01790.

[46] J. Król, T. Asselmeyer-Maluga, K. Bielas, P. Klimasara, From quantum to cosmological regime. The role of forcing and exotic 4-smoothness, Universe 3(31) (2017). doi:10.3390/universe3020031.

[47] E. Moise, Affine structures on 3-manifolds, Ann. Math. 56 (1952) 96–114.

[48] M. Freedman, F. Quinn, Topology of 4-Manifolds, Princeton Mathematical Series, Princeton University Press, Princeton, 1990.

[49] G. Besson, G. Courtois, S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négatif, Geom. Funct. Anal. 5 (1995) 731–799.

[50] G. Besson, G. Courtois, S. Gallot, Minimal entropy and Mostow’s rigidity theorems, Ergod. Th. and Dynam. Sys. 16 (1996) 623–649.

[51] J. Alexander, An example of a simple-connected surface bounding a region which is not simply connected, Proceedings of the National Academy of Sciences of the United States 10 (1924) 8 – 10.

[52] J. Cannon, The recognition problem: What is a topological manifold?, BAMS 84 (1978) 832 – 866.