Logarithmic Regret for Learning
Linear Quadratic Regulators Efficiently

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February 20, 2020

Abstract

We consider the problem of learning in Linear Quadratic Control systems whose transition parameters are initially unknown. Recent results in this setting have demonstrated efficient learning algorithms with regret growing with the square root of the number of decision steps. We present new efficient algorithms that achieve, perhaps surprisingly, regret that scales only (poly)logarithmically with the number of steps in two scenarios: when only the state transition matrix \( A \) is unknown, and when only the state-action transition matrix \( B \) is unknown and the optimal policy satisfies a certain non-degeneracy condition. On the other hand, we give a lower bound that shows that when the latter condition is violated, square root regret is unavoidable.

1 Introduction

The linear-quadratic regulator model (LQR) is a classic model in optimal control theory. In this model, the dynamics of the environment are given as

\[ x_{t+1} = A_x x_t + B_x u_t + w_t, \]

where \( x_t \) and \( u_t \) are the state and the action vectors at time \( t \), \( A_x \) and \( B_x \) are transition matrices, and \( w_t \) is a zero-mean i.i.d. Gaussian noise. The cost function is quadratic in both the state and the action. An interesting property of LQR systems is that a linear control policy minimizes the cost while keeping the system at a steady-state (stable) position.

In this work, we study the problem of designing an adaptive controller that regulates the system while learning its parameters. This problem has recently been approached through the lens of regret minimization, beginning in the work of Abbasi-Yadkori and Szepesvári (2011) that established an \( O(\sqrt{T}) \) regret bound for this setting albeit with a computationally inefficient algorithm. The problem of designing an efficient algorithm that enjoys \( O(\sqrt{T}) \) was later resolved by Cohen et al. (2019) and Mania et al. (2019). The former work relied on the “optimism in the face of uncertainty” principle and a reduction to an online semi-definite problem, and the latter work used a simpler greedy strategy.

Following this line of work, it has been believed that an \( O(\sqrt{T}) \) regret is tight for the problem. This appears natural as it is the typical rate for many imperfect information (bandit) optimization problems (e.g., Shamir, 2013). On the other hand, one could suspect that better, polylogarithmic regret bounds, are possible in the LQR setting thanks to the strongly convex structure of the cost functions. Often in optimization, this structure gives rise to faster convergence/regret rates, and indeed, in a recent work, Agarwal et al. (2019b) have demonstrated that such fast rates are attainable in the related, yet full-information online LQR problem endowed with any strongly convex loss functions.

In this paper, we show two interesting scenarios of learning unknown LQR systems in which an expected regret of \( O(\log^2 T) \) is, in fact, achievable. In the first, we assume that the matrix \( B_x \) is known and show...
that polylogarithmic regret can be attained by harnessing the intrinsic noise in the system dynamics for exploration. In the second, we assume that $A_*$ is known and that the optimal control policy $K_*$ is given by a full-rank matrix. Both results are attained using simple and efficient algorithms whose runtime per time step is polynomial in the natural parameters of the problem.

We complement our results with a lower bound showing that our assumptions are indeed necessary for obtaining improved regret guarantees. Specifically, we show that when $B_*$ is unknown and the optimal policy $K_*$ is near-degenerate (i.e., with very small singular values), any online algorithm, whether efficient or not, must suffer at least $\Omega(\sqrt{T})$ regret. To the best of our knowledge, this is the first $\Omega(\sqrt{T})$ lower bound for learning linear quadratic regulators (that particularly holds even when the learner knows the entire set of system parameters but the matrix $B_*$).

### 1.1 Setup: Learning in LQR

We consider the problem of regret minimization in the LQR model. At each time step $t$, a state $x_t \in \mathbb{R}^d$ is observed and action $u_t \in \mathbb{R}^k$ is chosen. The system evolves according to

$$x_{t+1} = A_* x_t + B_* u_t + w_t,$$

where the state-state $A_* \in \mathbb{R}^{d \times d}$ and state-action $B_* \in \mathbb{R}^{d \times k}$ matrices form the transition model and the $w_t$ are i.i.d. noise terms, each is a zero mean Gaussian with covariance matrix $\sigma^2 I$. At time $t$, the instantaneous cost is

$$c_t = x_t^T Q x_t + u_t^T R u_t,$$

where $Q, R > 0$ are positive definite.

A policy of the learner is a mapping from a state $x \in \mathbb{R}^d$ to an action $u \in \mathbb{R}^k$ to be taken at that state. Classic results in linear control establish that, given the system parameters $A_*, B_*, Q$ and $R$, the optimal policy is a linear mapping from the state space $\mathbb{R}^d$ to the action space $\mathbb{R}^k$ in an infinite-horizon setup. We thus consider policies of the form $u_t = K x_t$ and define the infinite horizon expected cost,

$$J(K) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^{T} x_t^T (Q + K^T R K) x_t \right],$$

where the expectation is taken with respect to the random noise variables $w_t$. Let $K_* = \arg \min_K J(K)$ be an (unique) optimal policy and $J_* = J(K_*)$ denote the optimal infinite horizon expected cost, which are both well defined under mild assumptions.\(^2\) We are interested in minimizing the regret over $T$ decision rounds, defined as

$$R_T = \sum_{t=1}^{T} (x_t^T Q x_t + u_t^T R u_t - J_*).$$

We focus on the setting where the learner does not have a full a-priori description of the transition parameters $A_*$ and $B_*$, and has to learn them while controlling the system and minimizing the regret.

Throughout, we assume that the learner has knowledge of the cost matrices $Q$ and $R$, and that there are constants $\alpha_0, \alpha_1 > 0$ such that

$$\|Q\|, \|R\| \leq \alpha_1, \text{ and } \|Q^{-1}\|, \|R^{-1}\| \leq \alpha_0^{-1}.$$

We further assume that the learner has bounds on the transition matrices, as well as on the optimal cost; that is, there are known constants $\vartheta, \nu > 0$ such that

$$\|A_*\|, \|B_*\| \leq \vartheta, \text{ and } J_* \leq \nu.$$

Finally, we assume that there is a known stable (not necessarily optimal) policy $K_0$ and $\nu_0 > 0$ such that $J(K_0) \leq \nu_0$.\(^3\)

\(^2\)These hold under standard, very mild controllability assumptions (see Bertsekas, 1995) that we implicitly assume throughout.

\(^3\)Regarding the necessity of this assumption, see the discussion in Mania et al. (2019); Cohen et al. (2019).
1.2 Main results

Our first result focuses on the case where the state-action transition matrix $B_*$ is known (but the matrix $A_*$ is unknown).

**Theorem 1.** There exists an efficient online algorithm (see Algorithm 1 in Section 3.1) that, given the matrix $B_*$ as input, has expected regret
\[ E[R_T] = \text{poly}(\sigma_0^{-1}, \sigma_1, \vartheta, \nu, \nu_0, d, k) O(\log^2 T). \]

Next, we consider the dual setup in which only the state-state matrix $A_*$ is known. Here we require an additional non-degeneracy assumption for obtaining polylogarithmic regret.

**Theorem 2.** Suppose that the optimal policy of the system satisfies $K_* K_*^T \succeq \mu_* I$ for some constant $\mu_* > 0$ that is unknown to the learner. Then there exists an efficient online algorithm (see Algorithm 2 in Section 3.2) that, given the matrix $A_*$ as input, has expected regret
\[ E[R_T] = \text{poly}(\mu_*^{-1}, \sigma_0^{-1}, \sigma_1, \vartheta, \nu, \nu_0, d, k) O(\log^2 T). \]

Finally, we show that our assumption regarding the non-degeneracy of the optimal policy $K_*$ is necessary. Our next result shows that without it, the expected regret of any algorithm is unavoidably at least $\Omega(\sqrt{T})$, even in simple one-dimensional (single input, single output) systems.

**Theorem 3.** For any learning algorithm and any $\sigma > 0$, there exists an LQR system (in dimensions $d = k = 1$) which is stabilized by the policy $K_0 = 0$ and for which $\sigma_1 = \sigma_0 = 1$, $\vartheta = 1$ and $\nu = 2\sigma^2$, such that the expected regret of the algorithm is at least $\Omega(\sigma^2 \sqrt{T})$. This is true even if the algorithm receives the matrix $A_*$ as input.

1.3 Discussion

Our results could be interpreted as a proof-of-concept that faster, polylogarithmic rates for learning in LQRs are possible under more limited uncertainty assumptions. This is perhaps surprising in light of the aforementioned work of Shamir (2013), that established $\Omega(\sqrt{T})$ regret lower bounds for online (bandit) optimization, even with quadratic and strongly convex objectives (as is the case in our LQR setup). The question of whether polylogarithmic regret guarantees are possible under more general, or even full uncertainty (of both $A_*$ and $B_*$) remains open. Our lower bound, however, shows that more assumptions are required for obtaining stronger positive results.

Our results focused on the expected regret compared to the infinite-horizon performance of the optimal policy $K_*$. As far as we know, this is the first analysis that bounds the regret in expectation rather than in high-probability. Indeed, in previous analyses we are aware of, there was always a small probability where the algorithm fails and incurs very large (possibly exponentially large) regret. Here, we address this low-probability event by employing a novel “abort procedure” when our algorithms suspect the system has been destabilized; this ensures that the expected regret remains controlled. The question of whether our regret bounds hold with high probability remains for future investigation. We remark that in the analogous multi-armed bandit setting, it is well-known that the logarithmic expected regret bounds of UCB-type algorithms can be converted into high probability ones, and so it is a natural question whether the same holds for LQRs.

We also remark that the infinite-horizon cost of the optimal policy can be easily replaced in the definition of the regret with the finite-time cost of $K_*$ (up to additional additive low order terms). This is since the expected costs of any (strongly) stable policy converge exponentially fast to its expected steady-state cost. One could also consider a different definition of the regret, akin to that of “pseudo-regret” in multi-armed bandits, where the learner has to commit at each time step to a linear policy $K_t$ and incurs its mean infinite-horizon cost, $J(K_t)$. (This is the type of notion considered in several recent papers, e.g., Fazel et al., 2018; Malik et al., 2019.) We note, however, that in the unbounded LQR setting there are subtleties that make this definition potentially weaker than the actual expected regret that we focus on; for example, the learner could choose $K_t$ so as to deliberately blow up the magnitude of the states and thereby boost the estimation rates of the unknown system parameters, but at the same time, $J(K_t)$ would remain controlled and no significant penalty in the regret will be incurred.
1.4 Related work

The topic of learning in linear control has been attracting considerable attention in recent years. Since the early work of Abbasi-Yadkori and Szepesvári (2011), a long line of research has focused on obtaining improved regret bounds for learning in LQRs with a variety of algorithms (Ibrahimi et al., 2012; Faradonbeh et al., 2017; Abeille and Lazaric, 2018; Dean et al., 2018; Fazel et al., 2018; Cohen et al., 2019; Ibrahimi et al., 2018). To the best of our knowledge, our results are the first to exhibit logarithmic regret rates for LQRs albeit in a more restrictive setting.

A closely related line of work considered a non-stochastic variant of online control in which the cost functions can change arbitrarily from round to round (Cohen et al., 2018; Agarwal et al., 2019a). Other notable works have studied the sample complexity of estimating the unknown parameters of linear dynamical systems (Dean et al., 2017; Simchowitz et al., 2018; Sarkar and Rakhlin, 2019), improper prediction of linear systems (Hazan et al., 2017, 2018), as well as model-free learning of LQRs via policy gradient methods Fazel et al. (2018); Malik et al. (2019).

2 Preliminaries

2.1 Linear Quadratic Control

We give a brief background on several basic properties and results in linear quadratic control that we require in the paper. For a given LQR system $(A, B)$ with cost matrices $Q, R > 0$, the optimal (infinite horizon) feedback controller is given by

$$K(A, B, Q, R) = -(R + B^T PB)^{-1} B^T PA,$$

where $P$ is the positive definite solution to the discrete Riccati equation

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA.$$

(1)

(2)

In particular, for the system $(A_\star, B_\star)$ we have $K_\star = K(A_\star, B_\star, Q, R)$. For more background on linear control and derivation of the relations above, see Bertsekas (1995).

The following lemma, proved in Mania et al. (2019), relates the error in estimating a system’s parameters to the deviation of the corresponding estimated controller from the optimal one. This relation is given in terms of cost as well as in terms of distance in operator norm.

**Lemma 4.** There are explicit constants $C_0, \varepsilon_0 = \text{poly}(\alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k)$ such that, for any $0 \leq \varepsilon \leq \varepsilon_0$ and matrices $A, B$ such that $\|A - A_\star\| \leq \varepsilon$ and $\|B - B_\star\| \leq \varepsilon$, the policy $K = K(A, B, Q, R)$ satisfies

$$J(K) - J_\star \leq C_0 \varepsilon^2, \quad \text{and} \quad \|K - K_\star\| \leq C_0 \varepsilon.$$

Importantly, the lemma shows that the performance scales *quadratically* in the estimation error. This served Mania et al. (2019) as a key feature in showing that an $\varepsilon$-greedy algorithm obtains $O(\sqrt{T})$ regret. Here, we use this lemma to show that considerably improved regret bounds are achievable in certain scenarios.

Next, we recall the notion of strong stability (Cohen et al., 2018). This is essentially a quantitative version of classic stability notions in linear control.

**Definition 5** (strong stability). A matrix $M$ is $(\kappa, \gamma)$–strongly stable (for $\kappa \geq 1$ and $0 < \gamma \leq 1$) if there exists matrices $H \succ 0$ and $L$ such that $M = HLH^{-1}$ with $\|L\| \leq 1 - \gamma$ and $\|H\|\|H^{-1}\| \leq \kappa$. A controller $K$ for the system $(A, B)$ is $(\kappa, \gamma)$–strongly stable if $\|K\| \leq \kappa$ and the matrix $A + BK$ is $(\kappa, \gamma)$–strongly stable.

We remark that Cohen et al. (2018) also introduced the notion of sequential strong stability that is an analogous definition for an adaptive strategy that changes its linear policy over time. Here, we avoid this notion by ensuring that each linear policy is played in our algorithms for a sufficiently long duration.
2.2 Confidence bounds for least-squares estimation

Our algorithms use regularized least squares methods in order to estimate the system parameters. An analysis of this method for a general, possibly-correlated sample, was introduced in the context of linear bandit optimization (Abbasi-Yadkori et al., 2011), and was first used in the context of LQRs by Abbasi-Yadkori and Szepesvári (2011). We state the results in terms of a general sequence, since the estimation procedures differ between our two algorithms.

Let \( \Theta \in \mathbb{R}^{d \times m} \), \( \{y_{t+1}\}_{t=1}^{\infty} \in \mathbb{R}^d \), \( \{z_t\}_{t=1}^{\infty} \in \mathbb{R}^m \), \( \{w_t\}_{t=1}^{\infty} \in \mathbb{R}^d \) such that \( y_{t+1} = \Theta z_t + w_t \), and \( \{w_t\}_{t=1}^{\infty} \) are i.i.d.

with distribution \( N(0, \sigma^2 I) \). Denote by

\[
\hat{\Theta}_t \in \arg\min_{\Theta \in \mathbb{R}^{d \times m}} \left\{ \sum_{s=1}^{t-1} \|y_{s+1} - \Theta z_s\|^2 + \lambda \|\Theta\|_F^2 \right\},
\]

(3)

the regularized least squares estimate of \( \Theta^* \) with regularization parameter \( \lambda \).

**Lemma 6** (Abbasi-Yadkori and Szepesvári, 2011). Let \( V_t = \lambda I + \sum_{s=1}^{t-1} z_t z_s^T \) and \( \Delta_t = \Theta^* - \hat{\Theta}_t \). With probability at least \( 1 - \delta \), we have for all \( t \geq 1 \)

\[
\text{Tr}(\Delta_t^T V_t \Delta_t) \leq 4\sigma^2 d \log \left( \frac{d \det(V_t)}{\delta \det(\hat{V}_t)} \right) + 2\lambda \|\Theta^*\|_F^2.
\]

3 Proofs and Algorithms

In this section we present our algorithms and illustrate the main ideas of our upper and lower bounds. The complete versions of the proofs are deferred to Appendices A to C.

3.1 Upper Bound for Unknown \( A^* \)

We start with the setting where \( A^* \) is unknown, and show an efficient algorithm that achieves regret at most \( O(\log^2 T) \). To that end, we propose **Algorithm 1**. The algorithm begins by playing the stable controller \( K_0 \) for a \( \tau_0 \)-long warm-up period. It subsequently operates in phases whose length grows exponentially (quadrupling). Each phase begins by estimating the system parameters using Eq. (3) and computing the greedy controller with respect to said parameters using Eq. (1). It then proceeds to play greedily as long as a fail condition is not reached.

**Algorithm 1**

1. **input:** parameters \( \tau_0, x_b, \kappa, \lambda \), a strongly stable controller \( K_0 \), and the action-state transition matrix \( B_* \).
2. **initialize:** \( n_T = \lceil \log_4(T/\tau_0) \rceil \), \( \tau_{n_T+1} = T + 1 \)
3. **set:** \( \tau_i \leftarrow \tau_0 4^i \) for all \( 0 \leq i \leq n_T \).
4. **for** \( t = 1, \ldots, \tau_0 - 1 \) **do**
    - play \( u_t = K_0 x_t \).
5. **for phase** \( i = 0, \ldots, n_T \) **do**
    - \( A_{t_i} = \arg\min_{A} \sum_{s=1}^{t_i-1} \| (x_{s+1} - B_* u_s) - A x_s \|_F^2 + \lambda \|A\|_F^2 \)
    - \( K_{t_i} = K(A_{t_i}, B_*, \hat{Q}, \hat{R}) \).
8. **for** \( t = \tau_i, \ldots, \tau_i+1 - 1 \) **do**
10. **if** \( \|x_t\|^2 > x_b \) **or** \( \|K_t\| > \kappa \) **then**
11. **abort** and play \( K_0 \) forever.
12. play \( u_t = K_{t_i} x_t \).

We now give a quantified restatement of Theorem 1.
Theorem (Theorem 1 restated). Suppose Algorithm 1 is run with parameters

\[
\kappa_0 = \sqrt{\frac{\nu_0}{\alpha_0\sigma^2}}, \quad \kappa = \sqrt{\frac{\nu + \varepsilon_0^2 C_0}{\alpha_0\sigma^2}}, \quad \tau_0 = \left[ \frac{80d\lambda(1 + \vartheta^2)}{\sigma^2\varepsilon_0^2} \right],
\]

\[
\lambda = x_b = 135d\kappa^2\sigma^2 \max \{\kappa_0^6, 4\kappa^6\} \log(3T).
\]

Then for \( T \geq \text{poly}(\alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k) \) we have \( \mathbb{E}[R_T] \leq \text{poly}(\alpha_0^{-1}, \alpha_1, \vartheta, \nu, \nu_0, d, k) \log^2 T \).

We start by quantifying a high probability event on which the regret of the algorithm is small. The event holds when the error of the algorithm’s estimate of \( A \) scales as \( t^{-1/2} \), the states are bounded, and all control policies generated by the algorithm are strongly-stable. This is formally given by the following lemma.

Lemma 7. Let \( \gamma = 1/2\kappa^2 \). With probability at least \( 1 - T^{-2} \),

(i) \( K_{\tau_i} \) is \( (\kappa, \gamma) \)-strongly stable, for all \( 0 \leq i \leq n_T \);
(ii) \( \|x_t\|^2 \leq x_b \), for all \( 1 \leq t \leq T \);
(iii) \( \|\Delta_{A_i}\| \leq \varepsilon_0 2^{-1} \), for all \( 0 \leq i \leq n_T \).

Here we give a sketch of the proof of Lemma 7, deferring technical details to Appendix A.

Proof (sketch). Consider Lemma 6 with \( z_t = x_t, y_{t+1} = x_{t+1} - B_s u_t, V_t = \lambda I + \sum_{s=1}^{t-1} x_s x_s^T \) and \( \Delta_{A_i} = A_i - A_\star \), then we have with probability at least \( 1 - 1/3T^{-2} \)

\[
\text{Tr}(\Delta_{A_i}^T V_i \Delta_{A_i}) \leq 4\sigma d \log \left( 3dT^2 \frac{\det(V_t)}{\det(V_i)} \right) + 2\lambda d \vartheta^2,
\]

for all \( t \geq 1 \). Transforming Eq. (4) into the desired bound requires that we bound \( V_t \) from above and below. In what follows we show \( \|V_t\| \leq \lambda t \) on one hand, and \( V_t \geq \frac{\varepsilon_0^2}{40} I \) on the other hand. Using the upper bound, one can show that simplifying the right-hand side of Eq. (4) yields \( \text{Tr}(\Delta_{A_i}^T V_i \Delta_{A_i}) \leq \sigma^2 \varepsilon_0^2 \tau_0 / 40 \). Complementing this with the lower bound gets us

\[
\|\Delta_{A_i}\|^2 \leq \frac{40}{\sigma^2 t} \text{Tr}(\Delta_{A_i}^T V_i \Delta_{A_i}) \leq \frac{\varepsilon_0^2 \tau_0}{t},
\]

and taking the square root, we obtain the desired estimation error bound that indeed scales as \( t^{-1/2} \) (up to logarithmic factors).

For a lower bound on \( V_t \), notice that the system noise \( w_t \) ensures that we have a sufficient exploration of the state space. Formally, we have

\[
\mathbb{E}[V_t] \geq \lambda I + \sum_{s=1}^{t-1} \mathbb{E}[x_s x_s^T] \geq t\sigma^2 I,
\]

where we used \( \mathbb{E}[x_s x_s^T] \geq \mathbb{E}[w_s w_s^T] \geq \sigma^2 I \) and \( \lambda \geq \sigma^2 \). Applying a measure concentration argument yields the sought-after high-probability lower bound on \( V_t \).

Now, for an upper bound on \( V_t \), notice that

\[
\|V_t\| \leq \lambda + \sum_{s=1}^{t-1} \|x_s\|^2
\]

thus it suffices to show that \( \|x_t\|^2 \leq x_b = \lambda \). The proof of the lemma now follows inductively by the following argument. If the parameter estimation at time \( \tau_i \) holds then \( K_{\tau_i} \) is strongly-stable. This implies that the states throughout phase \( i \) satisfy \( \|x_t\|^2 \leq x_b \) which in turn implies the upper bound on \( V_{\tau_{i+1}} \). Thus we can bound the parameter estimation error at time \( \tau_{i+1} \). We note that the initial parameter estimation, i.e., at time \( \tau_0 \), follows from the strong-stability of \( K_0 \) and by taking the warm-up duration \( \tau_0 \) to be sufficiently long.
Proof of Theorem 1. Let \( \mathcal{E}_A \) be the event where Lemma 7 hold, and notice that the algorithm does not abort on \( \mathcal{E}_A \). Defining \( J_i = \sum_{t=\tau_i}^{\tau_{i+1}-1} x_t^T(Q + K_{\tau_i}^T R K_{\tau_i})x_t \), we have the following decomposition of the regret:

\[
\mathbb{E}[R_T] = R_1 + R_2 + R_3 - TJ_s,
\]

where

\[
R_1 = \mathbb{E} \left[ \sum_{i=0}^{n_T} 1(\mathcal{E}_A) J_i \right]; \quad R_2 = \mathbb{E} \left[ \mathbb{I}\{\mathcal{E}^c_A\} \sum_{t=\tau_0}^T c_t \right]; \quad R_3 = \mathbb{E} \left[ \sum_{t=1}^{\tau_0-1} c_t \right],
\]

are the costs due to success, failure, and warm-up respectively. We now bound each of \( R_1, R_2, R_3 \) to conclude the proof.

Starting with \( R_1 \), the following lemma uses the strong-stability of \( K_{\tau_i} \) (whenever \( \mathcal{E}_A \) holds) to show that \( J_i \) is closely related to the steady-state cost of \( K_{\tau_i} \).

Lemma 8. Fix some \( i \) such that \( 0 \leq i \leq n_T \), and define the event \( E_i = \{ \| \Delta_{A_{\tau_i}} \| \leq \varepsilon_0 2^{-i} \} \). We have

\[
\mathbb{E}[\mathbb{I}\{\mathcal{E}_A\} J_i] \leq (\tau_{i+1} - \tau_i) \mathbb{E}[\mathbb{I}\{E_i\} J(K_{\tau_i})] + 4\alpha_1 \kappa^6 x_b.
\]

We further relate the lemma’s bound to the cost of the optimal policy using Lemma 4. This gets us

\[
(\tau_{i+1} - \tau_i) \mathbb{E}[\mathbb{I}\{E_i\} J(K_{\tau_i})] \leq (\tau_{i+1} - \tau_i) (J_s + C_0 \varepsilon_0^2 4^{-i}) \leq (\tau_{i+1} - \tau_i) J_s + 3C_0 \varepsilon_0^2 \tau_0.
\]

Next, summing over \( i \), noticing that \( \sum_{i=0}^{n_T} \tau_{i+1} - \tau_i \leq T \), and simplifying the arguments yields

\[
R_1 \leq T \cdot J_s + n_T (6C_0 \varepsilon_0^2 \tau_0 + 8\alpha_1 \kappa^6 x_b).
\]

Moving to \( R_2 \), let \( \tau_{\text{abort}} \) be the time when the algorithm decides to abort, formally,

\[
\tau_{\text{abort}} = \min \{ t \geq \tau_0 \mid \| x_t \|^2 > x_b \text{ or } \| K_t \| > \kappa \},
\]

where we treat \( \min \emptyset = T + 1 \). Then we have the following bound on \( R_2 \).

\[
R_2 \leq \mathbb{E} \left[ \mathbb{I}\{\mathcal{E}^c_A\} \sum_{t=\tau_0}^{\tau_{\text{abort}}-1} c_t \right] + \mathbb{E} \left[ \sum_{t=\tau_{\text{abort}}}^T c_t \right].
\]

Now, the state and control policy before \( \tau_{\text{abort}} \) are bounded by \( x_b \) and \( \kappa \) respectively hence \( c_t \leq 2\alpha_1 \kappa^2 x_b \). Further recalling that \( \mathbb{P}(\mathcal{E}^c_A) \leq T^{-2} \) bounds the first term. After \( \tau_{\text{abort}} \) the stable controller \( K_0 \) is played for the remaining period. This ensures that the state will not keep growing however some care is required as the state at \( \tau_{\text{abort}} \), \( x_{\tau_{\text{abort}}} \), is not bounded. The above is made formal in the following lemma.

Lemma 9. \( R_2 \leq J(K_0) + 2\alpha_1 \kappa^2 x_b + o(1) \).

Last, for \( R_3 \), the strongly stable controller \( K_0 \) is played throughout warm-up. Unlike \( R_2 \), here the initial state \( x_1 = 0 \) is clearly bounded and thus it is not difficult to show that \( R_3 \) scales linearly with the warm-up duration \( \tau_0 \). Since the latter behaves as \( O(\log T) \), the desired result is obtained. This is made formal in the following lemma.

Lemma 10. \( R_3 \leq \tau_0 J(K_0) \).

The final bound now follows by combining the bounds of \( R_1, R_2, \) and \( R_3 \) and from \( n_T, x_b, \tau_0 \) being \( O(\log T) \).

For a full proof of Lemmas 8 to 10, see Appendix A.
3.2 Upper Bound for Unknown $B$*

We move to a setting where $A_*$ is known, $B_*$ is unknown, but $K_*K_*^T \succeq \mu_*I$ for some unknown constant $\mu_* > 0$. We show an efficient algorithm that achieves regret at most $O(\mu_*^2 \log^2 T)$. We propose Algorithm 2 to that end. The algorithm operates in a similar fashion to Algorithm 1 with warm-up with $K_0$ and then greedy with fail-safe, but with two main differences:

1. It adds artificial noise to the action during warm-up;
2. The warm-up length is not predetermined and implicitly depends on $\mu_*$.

The first change ensures that the action space is explored uniformly during warm-up, and the second ensures that exploration continues at the same rate during the main loop where noise is not added. The specifics of these are made clear in what follows.

Algorithm 2

1: input: parameters $\tau_0, x_b, \kappa, \lambda, \mu_0$, a strongly stable controller $K_0$, and the state transition matrix $A_*$.
2: initialize: $n_T = \lceil \log_2(T/\tau_0) \rceil$, $n_s = n_T + 1$, $\tau_{n_s+1} = T + 1$.
3: set: $\tau_i \leftarrow \tau_0 4^i$, $\mu_i \leftarrow \mu_0 2^{-i}$ for all $0 \leq i \leq n_T$
4: for $t = 1, \ldots, \tau_0 - 1$ do $\triangleright$ initial warm-up
5: play $u_t \sim \mathcal{N}(K_0x_t, \sigma^2 I)$
6: for phase $i = 0, \ldots, n_T$ do $\triangleright$ adaptive warm-up
7: $B_{\tau_i} = \arg \min_B \sum_{s=1}^{n_s-1} \| (x_{s+1} - A_*x_s) - Bu_s \|^2 + \lambda \| B \|^2_F$
8: $K_{\tau_i} = \mathcal{K}(A_*, B_{\tau_i}, Q, R)$.
9: if $K_{\tau_i}^T K_{\tau_i} \geq 3/2$ then
10: save $n_s = i$ and break.
11: for $t = \tau_i, \ldots, \tau_{i+1} - 1$ do $\triangleright$ main loop
12: play $u_t \sim \mathcal{N}(K_0x_t, \sigma^2 I)$
13: for phase $i = n_s, \ldots, n_T$ do $\triangleright$ fail, abort
14: $B_{\tau_i} = \arg \min_B \sum_{s=1}^{n_s-1} \| (x_{s+1} - A_*x_s) - Bu_s \|^2 + \lambda \| B \|^2_F$
15: $K_{\tau_i} = \mathcal{K}(A_*, B_{\tau_i}, Q, R)$.
16: for $t = \tau_i, \ldots, \tau_{i+1} - 1$ do
17: if $\| x_t \|^2 > x_b$ or $\| K_{\tau_i} \| > \kappa$ then
18: abort and play $K_0$ forever.
19: play $u_t = K_{\tau_i}x_t$.

We now give a quantified restatement of Theorem 2.

Theorem (Theorem 2 restated). Suppose Algorithm 2 is run with parameters

$$\kappa_0 = \sqrt{v_0 / \alpha_0 \sigma^2}, \quad \kappa = \sqrt{v + \varepsilon_0^2 C_0 / \alpha_0 \sigma^2}, \quad \tau_0 = \left\lceil \frac{80 \kappa \lambda (1 + \vartheta^2)}{\sigma^2 \varepsilon_0^2} \right\rceil,$$

$$x_b = 135 \kappa \sigma^2 \max \left\{ (1 + \vartheta)^2 \kappa^2, 4 \kappa_0 \right\} \log(4T),$$

$$\lambda = \kappa^2 x_b, \quad \mu_0 = 4 \kappa C_0 \varepsilon_0.$$

Then for $T \geq \text{poly} \left( \alpha_0^{-1}, \alpha_1, \vartheta, \nu, n_0, d, k, \mu_*^{-1} \right)$ we have $\mathbb{E}[R_T] \leq \text{poly} \left( \alpha_0^{-1}, \alpha_1, \vartheta, \nu, n_0, d, k, \mu_*^{-1} \right) \log^2 T$.

We provide the main ideas required to prove Theorem 2. As in Algorithm 1, we first quantify the high probability event under which the regret of the algorithm is small. Let us first consider the parameter estimation error during warm-up, which is bounded by the following lemma.

Lemma 11. With probability at least $1 - T^{-2}$, it holds that $\| \Delta_{B_{\tau_i}} \| \leq \varepsilon_0 2^{-i}$ for all $0 \leq i \leq n_s$.

Here we only give a sketch of the proof; for the full technical details, see Appendix B.
Proof (sketch). Consider Lemma 6 with \( z_t = u_t, y_{t+1} = x_{t+1} - A_s x_t, V_t = \lambda I + \sum_{s=1}^{t-1} u_t u_t^T \) and \( \Delta_{B_t} = B_t - B_s \), then with probability at least \( 1 - \frac{4}{T^2} \),

\[
\text{Tr}(\Delta_{B_t} V_t \Delta_{B_t}) \leq 4d T^2 \log \left( \frac{\det(V_t)}{\det(V_1)} \right) + 2\lambda k \theta^2,
\]

for all \( t \geq 1 \). Hence, bounding \( V_t \) from above and below as in Lemma 7 yields the desired parameter estimation error bound.

Now, during warm-up \( u_t \sim \mathcal{N}(K_0 x_t, \sigma^2 I) \) which is equivalent to having \( u_t = K_0 x_t + \eta_t \) where \( \eta_t \sim \mathcal{N}(0, \sigma^2 I) \) are i.i.d. random variables. Note that just as \( w_t \) provided exploration for \( x_t \), here \( \eta_t \) provides exploration for \( u_t \). Indeed, for the lower bound, we have

\[
\mathbb{E}[V_t] \geq \lambda I + \sum_{s=1}^{t-1} \mathbb{E}[u_s u_s^T] \geq \lambda I + \sum_{s=1}^{t-1} \mathbb{E}[\eta_s \eta_s^T] \geq t \sigma^2 I,
\]

and thus a measure concentration argument yields the desired high probability lower bound. For the upper bound, notice that

\[
\|V_t\| \leq \lambda + \sum_{s=1}^{t-1} \|u_s\|^2 \leq \lambda + 2 \sum_{s=1}^{t-1} (\|K_0\|^2 \|x_s\|^2 + \|\eta_s\|^2),
\]

and so the strong-stability of \( K_0 \) together with a high probability bound on the system and artificial noises yields the desired upper bound on \( V_t \). Combining both upper and lower bounds concludes the proof. \( \blacksquare \)

While the estimation rate during warm-up is desirable, adding constant magnitude noise to the action incurs regret that is linear in the warm-up length, even if \( K_0 = K_\star \), and as such we avoid this strategy during the main loop. Nonetheless, the following lemma shows that the estimation rate continues into the main loop albeit with slightly different constants.

Lemma 12. Let \( \gamma = 1/2\kappa^2 \). With probability at least \( 1 - T^{-2} \),

(i) \( K_{\tau_i} \) is \((\kappa, \gamma)\)-strongly stable, \( \forall \, n_s \leq i \leq n_T \);
(ii) \( \|x_t\|^2 \leq x_b, \, \forall \, 1 \leq t \leq T \);
(iii) \( \|\Delta_{B_{\tau_i}}\| \leq \varepsilon_0 \min\{2^{-n_s}, 2\mu^{-1/2} \} \), \( \forall \, n_s < i \leq n_T \).

We proceed with a proof sketch and defer details to Appendix B.

Proof (sketch). The proof follows inductively by similar arguments to those of Lemma 7, yet with the caveat that the lower bound on \( V_t \) may not hold when the controller is rank deficient.

To see this, recall that the algorithm plays \( u_t = K_{\tau_i} x_t \) during the main loop as long as the abort state is not triggered, so we have

\[
\mathbb{E}[u_t u_t^T | K_{\tau_i}] = K_{\tau_i} \mathbb{E}[x_t x_t^T | K_{\tau_i}] K_{\tau_i}^T \geq \sigma^2 K_{\tau_i} K_{\tau_i}^T.
\]

This means that transforming the exploration of states \( x_t \), provided for by the system noise \( w_t \), into exploration of actions \( u_t \) depends on the controller \( K_{\tau_i} \) being strictly non-degenerate. We show that with high probability, \( K_{\tau_i} K_{\tau_i}^T \geq (\mu_*/2)I \) thus ensuring the exploration and the parameter estimation rate.

First, suppose that the learner had knowledge of \( \mu_* \) and recall that \( \mu_0 = 4\kappa C_0 \varepsilon_0 \). Taking \( n_s \geq \max\{0, \log_2 (\mu_0/\mu_*)\} \), Lemma 11 implies that \( \|\Delta_{B_{\tau_i}}\| \leq \min\{\varepsilon_0, \frac{\mu_*}{4\kappa C_0} \} \) and applying Lemma 4 we get that \( \|K_{\tau_i} - K_{\star}\| \leq \mu_*/4\kappa \). Further assuming that \( \|K_{\tau_i}\| \leq \kappa \), which is ensured by strong-stability, simple algebra yields that \( K_{\tau_i} K_{\tau_i}^T \geq (\mu_*/2)I \).

Now, when \( \mu_* \) is unknown, we show that the break condition of the warm-up loop ensures that with high probability

\[
\max\left\{0, \log_2 \frac{\mu_0}{\mu_*} \right\} \leq n_s \leq 2 + \max\left\{0, \log_2 \frac{\mu_0}{\mu_*} \right\}, \tag{5}
\]

a proof of which may be found in Appendix B. The lower bound on \( n_s \) ensures the desired non-degeneracy of \( K_{\tau_i} \), and proceeding by induction, the same follows for subsequent controllers. We note that the purpose of the upper bound on \( n_s \) is to ensure that the warm-up is not so long as to incur more than \( O\left(\mu_*^{-2} \log^2 T \right) \) regret. \( \blacksquare \)
Proceeding from Lemma 12, we obtain a regret decomposition similar to that of Algorithm 1 with an added dependence on the random number of warm-up phases $n_s$. While this randomness introduces some additional technical challenges, the proof ideas remain largely the same. For the full proof of Theorem 2, see Appendix B.

3.3 Lower Bound for Degenerate $K_*$

In this section we prove an $\Omega(\sqrt{T})$ lower bound for systems with a (nearly) degenerate optimal policy, stated in Theorem 3. By Yao’s principle, to establish the theorem it is enough to demonstrate a randomized construction of an LQR system such that the expected regret of any deterministic learning algorithm is large.

Fix $d = k = 1$ and consider the system

$$
\begin{align*}
    x_{t+1} &= ax_t + bu_t + w_t; \\
    c_t &= x_t^2 + u_t^2.
\end{align*}
$$

(6)

Here, $w_t \sim N(0, \sigma^2)$ are i.i.d. Gaussian random variables, $a = 1/\sqrt{5}$ and $b = \chi \sqrt{\epsilon}$ where $\chi$ is a Rademacher random variable (drawn initially) and $\epsilon > 0$ is a parameter whose value will be chosen later. For simplicity, we assume that $x_1 = 0$. Notice that for this system, we have the bounds $\alpha_1 = \alpha_0 = 1$, $\vartheta = 1$ and, as we will see below, the optimal cost of the system is bounded by $\nu = 2\sigma^2$. Further, note that the system is controllable and $k_0 = 0$ is a stabilizing policy. Our goal is to lower bound the regret, given by

$$
R_T = \sum_{t=1}^{T} (x_t^2 + u_t^2 - J(k_*)).
$$

Theorem 3 follows directly from the following:

**Theorem 13.** Assume that $T \geq 12000$ and set $\epsilon = T^{-1/2}/4$. Then the expected regret of any deterministic learning algorithm on on the system in Eq. (6) satisfies

$$
E[R_T] \geq \frac{1}{3100} \sigma^2 \sqrt{T} - 4\sigma^2.
$$

Here, the expectation is taken with respect to both the stochastic noise terms as well as the random variable $\chi$.

For the proof, we use the following notation. We use $k_*$ to denote the optimal policy for the system, which (recalling Eqs. (1) and (2)) is given by

$$
k_* = -\frac{ab p_*}{1 + b^2 p_*},
$$

where $p_* > 0$ is a positive solution to the Riccati equation

$$
p_* = 1 + a^2 p_* - \frac{a^2 b^2 p_*}{1 + b^2 p_*} = 1 + \frac{a^2 p_*}{1 + b^2 p_*}.
$$

Observe that for our choice of $\epsilon \leq 1/400$ we have that $|b| \leq 1/20$, and so

$$
1 \leq p_* \leq 1/(1 - a^2) = 5/4,
\quad 0.99 \sqrt{\epsilon}/5 \leq |k_*| \leq \sqrt{\epsilon}/3.
$$

In particular, this means that the cost of the optimal policy is at most $\sigma^2 p_* \leq 2\sigma^2$. Further, the sign of $k_*$ is solely determined by the sign of $\chi$.

Now, fix any deterministic learning algorithm. Let $x^{(t)} = (x_1, \ldots, x_t)$ denote the trajectory generated by the learning algorithm up to and including time step $t$. Denote by $P_+$ and $P_-$ the probability laws with respect to the trajectory generated conditioned on $\chi = 1$ and $\chi = -1$ respectively.
First, we lower bound the expected regret in terms of the cumulative magnitude of the algorithm’s actions \( u_t \). The proof first relates the regret to the overall deviation of \( u_t \) from the actions of the optimal policy \( k_\star \) by using the fact that the action played by \( k_\star \) at any state minimizes the Q-function of the system. Since the actions of \( k_\star \) are small in expectation, the latter quantity can be in turn related to the total magnitude of the \( u_t \).

**Lemma 14.** Suppose \( \epsilon \leq 1/400 \). The expected regret is lower bounded as

\[
E[R_T] \geq 0.99 E \left[ \sum_{t=1}^{T} (u_t - k_\star x_t)^2 \right] - 4\sigma^2,
\]

and consequently,

\[
E[R_T] \geq \frac{1}{3} E \left[ \sum_{t=1}^{T} u_t^2 \right] - \frac{5}{6} \sigma^2 k_\star^2 T - 4\sigma^2.
\]

Note that for the last bound to be meaningful, \( k_\star \) indeed has to be very small so that the additive term that scales with \( k_\star^2 T \) does not dominate the right hand side. The proofs of this as well as subsequent lemmas are deferred to Appendix C.

Next, by standard information theoretic arguments, we obtain an upper bound on the statistical distance between the probability laws of \( x^{(T)} \) under \( \mathbb{P}_+ \) and \( \mathbb{P}_- \), that scales with the total magnitude of the actions \( u_t \).

**Lemma 15.** For the trajectory \( x^{(T)} \), it holds that

\[
TV(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \frac{\sqrt{\epsilon}}{\sigma} \sqrt{E \left[ \sum_{t=1}^{T} u_t^2 \right]}.
\]

Our final lemma shows that most of the states visited by the algorithm have a non-trivial (constant) magnitude. This is a straightforward consequence of the added Gaussian noise at each time step.

**Lemma 16.** Assume that \( T \geq 12000 \). With probability \( \geq \frac{T}{8} \), at least \( \frac{2}{3} T \) of the states \( x_1, \ldots, x_T \) satisfy \( |x_t| \geq 2\sigma/5 \).

We are now ready to prove the main result of this section.

**Proof of Theorem 13.** Notice that if \( E[\sum_{t=1}^{T} u_t^2] > \frac{1}{4} \sigma^2 \sqrt{T} \), then the desired lower bound is directly implied by the second inequality in Lemma 14, as \( k_\star^2 \leq \epsilon/3 = T^{-1/2}/12 \), so \( E[R_T] \geq \frac{1}{1000} \sigma^2 \sqrt{T} - 4\sigma^2 \). We henceforth assume that \( E[\sum_{t=1}^{T} u_t^2] \leq \frac{1}{4} \sigma^2 \sqrt{T} \). Plugging this into the bound of Lemma 15 for the total variation distance between \( \mathbb{P}_+ \) and \( \mathbb{P}_- \), and using our choice \( \epsilon = T^{-1/2}/4 \), we obtain that

\[
TV(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \sqrt{\frac{\epsilon}{\sigma^2}} \cdot \frac{\sigma^2}{4} \sqrt{T} = \frac{1}{4}.
\]

Now, let \( N_T \) denote the number of time steps in which \( u_t k_\star x_t \leq 0 \), i.e., the number of times in which the learner has guessed the sign of \( \chi \) incorrectly. We claim that \( \mathbb{P}[N_T \geq T/2] \geq 3/8 \). To see this, denote by \( N'_T \) the number of time steps \( t \) in which \( u_t x_t \leq 0 \). Using the fact that \( N'_T \) is a deterministic function of the trajectory \( x^{(T)} \) together with the bound on the total variation gives

\[
|\mathbb{P}_+[N_T \geq T/2] - \mathbb{P}_-[N_T \geq T/2]| \leq TV(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \frac{1}{4}.
\]

Now, recall that the sign of \( k_\star \) is determined by that of \( \chi \). Thus, \( \mathbb{P}_-[N_T \geq T/2] = \mathbb{P}_-[N'_T < T/2] \) and \( \mathbb{P}_+[N_T \geq T/2] = \mathbb{P}_+[N'_T \geq T/2] \) from which

\[
\mathbb{P}[N_T \geq T/2] = \frac{1}{2} \mathbb{P}_+[N_T \geq T/2] + \frac{1}{2} \mathbb{P}_-[N_T \geq T/2]
\]

\[
= \frac{1}{2} \left( 1 + \mathbb{P}_+[N'_T \geq T/2] - \mathbb{P}_-[N'_T \geq T/2] \right)
\]

\[
\geq 3/8.
\]
On the other hand, Lemma 16 implies that with probability at least 7/8, no less than \( \frac{2T}{3} \) of the states \( x_1, \ldots, x_T \) satisfy \( |x_t| > 2\sigma/5 \). Then by a union bound, with probability at least 1/4, at least \( \frac{T}{6} \) instances of \( x_1, \ldots, x_T \) satisfy \( |x_t| \geq 2\sigma/5 \) and \( u_t k_* x_t \leq 0 \). For these instances, we have

\[
(u_t - k_* x_t)^2 \geq k_*^2 x_t^2 \geq 0.99^2 \cdot \frac{4}{125} \epsilon \sigma^2,
\]

where we have bounded \( k_* \) as in Eq. (7). Hence, we can lower bound the regret using the first inequality in Lemma 14 as follows:

\[
\mathbb{E}[R_T] \geq 0.99 \cdot \mathbb{E} \left[ \sum_{t=1}^{T} (u_t - k_* x_t)^2 \right] - 4\sigma^2 \\
\geq 0.99^3 \cdot \frac{1}{4} \cdot \frac{T}{6} \cdot \frac{4}{125} \epsilon \sigma^2 - 4\sigma^2 \\
\geq \frac{1}{3100} \sigma^2 \sqrt{T} - 4\sigma^2,
\]

where the last transition used our choice of \( \epsilon \).

\[\blacksquare\]

Acknowledgements

We thank Yishay Mansour for many helpful discussions. This work was partially supported by the Israeli Science Foundation (ISF) grant 2549/19.

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A Algorithm 1 Proofs

A.1 The Good Event

We begin with an explicit statement of the probabilistic events that comprise $\mathcal{E}_A$. Recall that

$$A_t = \arg \min_A \sum_{s=1}^{t-1} \|x_{s+1} - B_s u_s - A x_s\|^2 + \lambda \|A\|_F^2,$$

and denote $\Delta_{A_t} = A_t - A_s$, $V_t^x = \lambda I + \sum_{i=1}^{t-1} x_i x_i^T$. Now, define the following events

$$\mathcal{E}_{A_{t+1}} = \left\{ \left( \text{Tr}(\Delta_{A_t}^T V_t^x \Delta_{A_t}) \right) \leq 4\sigma^2 d \log \left( \frac{3T^3 \det(V_t^x)}{\det(V_t^{x_t})} \right) + 2\lambda d \delta^2, \text{ for all } t \geq 1 \right\},$$

$$\mathcal{E}_A = \left\{ \sum_{i=1}^{\tau-1} x_i x_i^T \geq \frac{(\tau_i - 1)\sigma^2}{40} I, \text{ for all } 0 \leq i \leq n_T \right\},$$

$$\mathcal{E}_{A_w} = \left\{ \max_{1 \leq t \leq T} \|w_t\| \leq \sigma \sqrt{15d \log 3T} \right\}.$$

Then we have the following lemma.

**Lemma 17.** Let $\mathcal{E}_A = \mathcal{E}_{A_{t+1}} \cap \mathcal{E}_A \cap \mathcal{E}_{A_w}$, and suppose that $T \geq 600d \log 3T$. Then we have that $\mathbb{P}(\mathcal{E}_A) \geq 1 - T^{-2}$.

**Proof.** First, we describe the parameter estimation error in terms of Lemma 6. To that end, let $z_t = x_t$, $y_{t+1} = x_{t+1} - B_s u_t$, $V_t^x = \lambda I + \sum_{i=1}^{t-1} x_i x_i^T$, and $\Delta_{A_t} = A_t - A_s$. Indeed, we have $y_{t+1} = A_s x_t + w_t$, $w_t \sim \mathcal{N}(0, \sigma^2 I)$, and $\|A_s\|_F \leq d \|A_s\|^2 \leq d \delta^2$ and so taking Lemma 6 with $\delta = \frac{1}{3} T^{-2}$, recalling that $T \geq d$, and simplifying, we get that $\mathbb{P}(\mathcal{E}_{A_{t+1}}) \geq 1 - \frac{1}{3} T^{-2}$.

Next, for $\mathcal{E}_A$, we apply Lemma 36 to the sequence $x_t$ with the filtration $\mathcal{F}_t = \sigma(x_1, u_1, \ldots, x_t, u_t)$. Notice that given $x_{t-1}, u_{t-1}$ we have $x_t \sim \mathcal{N}(A_s x_{t-1} + B_s u_{t-1}, \sigma^2 I)$ and hence we also get

$$\mathbb{E}[x_t x_t^T | \mathcal{F}_{t-1}] \succeq (A_s x_{t-1} + B_s u_{t-1})(A_s x_{t-1} + B_s u_{t-1})^T + \sigma^2 I \succeq \sigma^2 I.$$

Finally, our choice of $\tau_0$ ensures the minimal sum size assumption. We thus apply Lemma 36 $n_T + 1$ times with $\delta = \frac{1}{3} T^{-3}$ and apply a union bound. Since $n_T + 1 \leq T$ we conclude that $\mathbb{P}(\mathcal{E}_A) \geq 1 - \frac{1}{3} T^{-2}$.

Finally, for $\mathcal{E}_{A_w}$ we apply Lemma 34 with $\delta = \frac{1}{3} T^{-2}$ to get $\mathbb{P}(\mathcal{E}_{A_w}) \geq 1 - \frac{1}{3} T^{-2}$. The final result is obtained by taking a union bound over the three events.

A.2 Proof of Lemma 7

We first need the following two lemmas.

**Lemma 18 (Bounded warm-up).** On $\mathcal{E}_A$ we have that $\|x_t\| \leq \sigma \kappa_0^3 \sqrt{60d \log 3T} \leq \sqrt{x_0}$, for all $1 \leq t \leq \tau_0$.

**Proof.** First, by Lemma 41, $J(K_0) \leq \nu_0$ implies that $K_0$ is $(\kappa_0, \gamma_0)$–strongly stable with $\gamma_0^{-1} = 2 \kappa_0^2$. So, applying Lemma 38 with $x_1 = 0$ we get that for all $1 \leq t \leq \tau_0$

$$\|x_t\| \leq 2 \kappa_0^3 \max_{1 \leq s \leq t} \|w_t\|,$$

and applying the noise bound in Eq. (11) we obtain the desired result.

**Lemma 19 (Conditional parameter estimation).** On $\mathcal{E}_A$ fix some $i$ such that $0 \leq i \leq n_T$ and suppose that $\|x_t\|^2 \leq x_b$ for all $1 \leq t \leq \tau_i$. Then we have that $\|\Delta_{A_{\tau_i}}\| \leq \varepsilon_0 2^{-i}$.

**Proof.** First, on $\mathcal{E}_A$ by Eq. (10) we have that

$$V_{\tau_i}^x = \lambda I + \sum_{t=1}^{\tau_i-1} x_t x_t^T \succeq \left( \lambda + \frac{(\tau_i - 1)\sigma^2}{40} \right) I \succeq \frac{\tau_0 \sigma^2}{40} I,$$
and so we conclude that
\[
\text{Tr}\left(\Delta_{A_i}^T V_{\tau}^2 \Delta_{A_i} \right) \geq \text{Tr}\left(\Delta_{A_i}^T \Delta_{A_i} \right) \frac{\tau_i \sigma^2}{40} \geq \|\Delta_{A_i}\| \frac{2\tau_i \sigma^2}{40}.
\]
Rearranging and applying Eq. (9) we obtain
\[
\|\Delta_{A_i}\|^2 \leq \frac{1}{\tau_i} \left( 160d \log \left( 3T^3 \frac{\det(V_{\tau i})}{\det(V_{\tau})} \right) + 80 \frac{\lambda d \delta^2}{\sigma^2} \right).
\]
Now, since we assumed \(\|x_t\|^2 \leq x_b = \lambda\), we can apply Lemma 37 to conclude that
\[
\log \frac{\det(V_{\tau i})}{\det(V_{\tau})} \leq d \log T,
\]
and plugging this into the above we get that
\[
\|\Delta_{A_i}\|^2 \leq \frac{1}{\tau_i} \left( 640d^2 \log (3T) + 80 \frac{\lambda d \delta^2}{\sigma^2} \right) \leq \frac{1}{\tau_i} \frac{80 \lambda d (1 + \delta^2)}{\sigma^2} \leq \frac{\varepsilon_0 \tau_0}{\tau_i} \leq \varepsilon_0 4^{-1},
\]
where all transitions are due to our choice of parameters.

**Proof of Lemma 7.** First recall that by Lemma 42, if \(\|\Delta_{A_i}\| \leq \varepsilon_0\) then \(K_i\) is \((\kappa, \gamma)\)–strongly stable. We now show by induction on \(n\) that for all \(0 \leq i \leq n\), \(K_i\) is \((\kappa, \gamma)\)–strongly stable. Note that \(0 \leq n \leq n_T\).

For the base case, \(n = 0\), Lemma 18 shows that \(\|x_t\|^2 \leq x_b\) for all \(1 \leq t \leq \tau_0\), which in turn satisfies Lemma 19, i.e., \(\|\Delta_{A_{\tau 0}}\| \leq \varepsilon_0\) and so the required strong stability of \(K_{\tau_0}\) is obtained.

Now, suppose the induction holds up to \(n - 1\) and we show for \(n\). By the strong stability of the controllers up to time \(\tau_n - 1\), and since \(\tau_0 \geq \frac{\log \kappa}{\delta}\), we can apply Lemma 39 to conclude that
\[
\|x_t\| \leq 3\kappa \max \left\{ \frac{\|x_{\tau_0}\|}{2}, \frac{\kappa}{\gamma} \max_{1 \leq i \leq T} \|w_i\| \right\}, \quad \text{for all } \tau_0 \leq t \leq \tau_i.
\]
recalling that \(\gamma^{-1} = 2\kappa^2\), bounding the noise with Eq. (11), and bounding \(\|x_{\tau_0}\|\) by Lemma 18 we get that
\[
\|x_t\| \leq 3\kappa \max \left\{ \frac{\sigma \kappa^3 \sqrt{d} \log (3T)^2}{2}, 2\kappa^3 \sqrt{d} \log (3T)^2 \right\} \leq \sigma \kappa \max \left\{ \kappa_0^3, \kappa^3 \right\} \sqrt{15d \log (3T)} = \sqrt{x_b},
\]
and as for the base case, we can now invoke Lemmas 19 and 42 to conclude the strong stability of \(K_{\tau_n}\) and finish the induction. Notice that this together with the above equation also show the algorithm does not abort.

The induction proves the first part of the lemma, i.e., all controller are strongly-stable. Now, we can apply Lemma 39 once more to conclude that \(\|x_t\|^2 \leq x_b\) for all \(\tau_0 \leq t \leq T\) and together with Lemma 18 this concludes the second claim of the lemma.

Finally, the third claim is now an immediate corollary of the Lemma 19.

### A.3 Proof of Lemma 8

Recall that \(E_i = \{\|\Delta_{A_i}\| \leq \varepsilon_0 2^{-i}\}\), and further denote \(S_i = \{\|x_{\tau_i}\|^2 \leq x_b\}\). Trivially, we have that \(E_A \subseteq E_i \cap S_i\).

Now, define \(\tilde{x}_{\tau_i} = x_{\tau_i}\) and for \(\tau_i < t \leq \tau_{i+1} - 1\)
\[
\tilde{x}_t = (A_\ast + B_\ast K_{\tau_i})\tilde{x}_{t-1} + w_t.
\]
Since on \(E_A\) the algorithm does not abort, we have that
\[
\mathbb{1}_{\{E_A\}} J_i = \mathbb{1}_{\{E_A\}} \sum_{t = \tau_i}^{\tau_{i+1} - 1} \tilde{x}_t^T (Q + K_{\tau_i}^T R K_{\tau_i}) \tilde{x}_t \leq \mathbb{1}_{\{E_i \cap S_i\}} \sum_{t = \tau_i}^{\tau_{i+1} - 1} \tilde{x}_t^T (Q + K_{\tau_i}^T R K_{\tau_i}) \tilde{x}_t.
\]
Noticing that $E_t$, $S_t$, and $K_t$ are completely determined by $x_t, A_t$, we use total expectation to get that
\[
\mathbb{E}[1 \{ E_t \} J_t] \leq \mathbb{E} \left[ 1 \{ E_t \cap S_t \} \mathbb{E} \left[ \sum_{t=1}^{\tau_{t+1} - 1} x_t^T (Q + K_t^T R K_t) x_t \mid x_t, A_t \right] \right].
\]

Now, by Lemma 42, $E_t$ implies that $K_t$ is $(\kappa, \gamma)$–strongly stable and so we can use Lemma 40 to get that
\[
\mathbb{E}[1 \{ E_t \} J_t] \leq (\tau_{t+1} - \tau_t) \mathbb{E}[1 \{ E_t \} J(K_t)] + \frac{2\alpha_t \kappa^4}{\gamma} \mathbb{E}[1 \{ S_t \} \| x_t \|^2]
\]
\[
\leq (\tau_{t+1} - \tau_t) \mathbb{E}[1 \{ E_t \} J(K_t)] + 4\alpha_t \kappa^6 x_b,
\]
where the second transition also used that $\gamma^{-1} = 2\kappa^2$ and the third used our choice of $x_b \geq \sigma^2 \kappa^4$.

### A.4 Proof of Lemma 9 (R2 upper bound)

We first need the following lemma.

**Lemma 20** (Expected abort state). Suppose that $\mathbb{P}(\tau_{\text{abort}} \leq T) \leq T^{-2}$. Then we have that
\[
\mathbb{E} \left[ \| x_{\tau_{\text{abort}}} \|^2 1 \{ \tau_{\text{abort}} < T \} \right] \leq (1 + 8\theta^2)(\kappa^2 + \kappa_0^2) x_b T^{-2}.
\]

**Proof.** First, by the lemmas assumption, we can apply Lemma 35 to get that
\[
\mathbb{E} \left[ 1 \{ \tau_{\text{abort}} \leq T \} \max_{1 \leq t \leq T} \| x_t \|^2 \right] \leq 5d\sigma^2 T^{-2} \log 3T.
\]

Now, notice that $\| A_t + B_t K \| \leq 2\theta \| K \|$ and split into two cases. First, if $\tau_{\text{abort}} > \tau_0$ then by definition of $\tau_{\text{abort}}$ we have that
\[
\| x_{\tau_{\text{abort}}} \| = \| (A_t + B_t K_{\tau_{\text{abort}} - 1}) x_{\tau_{\text{abort}} - 1} + w_{\tau_{\text{abort}} - 1} \| \leq 2\theta \kappa x_b + \max_{1 \leq s \leq T} \| x_t \|,
\]
and taking expectation we get that
\[
\mathbb{E} \left[ 1 \{ \tau_0 < \tau_{\text{abort}} \leq T \} \| x_{\tau_{\text{abort}}} \|^2 \right] \leq 8\theta^2 \kappa^2 x_b T^{-2} + 5d\sigma^2 T^{-2} \log 3T \leq (1 + 8\theta^2) \kappa^2 x_b T^{-2}.
\]

On the other hand if $\tau_{\text{abort}} = \tau_0$ then
\[
\| x_{\tau_{\text{abort}}} \| = \| (A_t + B_t K_0) x_{\tau_0 - 1} + w_{\tau_0 - 1} \| \leq 2\theta \kappa_0 \| x_{\tau_0 - 1} \| + \max_{1 \leq t \leq T} \| x_t \| \leq (4\theta + 1) \kappa_0 \| x_{\tau_0} \|,
\]
where the last transition used Lemma 38 and $\gamma_0^{-1} = 2\kappa_0^2$. Taking expectation we get that
\[
\mathbb{E} \left[ 1 \{ \tau_{\text{abort}} = \tau_0 \} \| x_{\tau_{\text{abort}}} \|^2 \right] \leq 20(1 + 8\theta^2) \kappa_0^8 d\sigma^2 T^{-2} \log 3T \leq (1 + 8\theta^2) \kappa_0^2 x_b T^{-2},
\]
and combining both cases yields the final bound.

**Proof of Lemma 9.** First, recall the decomposition of $R_2$.
\[
R_2 \leq \mathbb{E} \left[ 1 \{ E_A^c \} \sum_{t=\tau_0}^{\tau_{\text{abort}} - 1} c_t \right] + \mathbb{E} \left[ \sum_{t=\tau_{\text{abort}}}^T c_t \right].
\]
For $\tau_0 \leq t < \tau_{\text{abort}}$ we have that $\| x_t \|^2 \leq x_b$ and $\| K_t \| \leq \kappa$ and so we get that
\[
c_t = x_t^T (Q + K_t^T R K_t) x_t \leq \| Q \| + \| R \| \| K_t \|^2 \leq 2\alpha_t \kappa^2 x_b.
\]
By Lemma 7 we have that \(\mathbb{P}(E^*_A) \leq T^{-2}\) and so we get that
\[
\mathbb{E}\left[\mathbbm{1}\{E^*_A\} \sum_{t=\tau_0}^{\tau_0-1} c_t\right] \leq \mathbb{E}\left[\mathbbm{1}\{E^*_A\} \cdot 2\alpha_1\kappa^2 x_0 T\right] = 2\alpha_1\kappa^2 x_0 T \mathbb{P}(E^*_A) \leq 2\alpha_1\kappa^2 x_0 T^{-1},
\]
bounding the first term of \(R_2\). Next, for \(t \geq \tau_0\) we have that \(K_t = K_0\) and so we can apply Lemma 40 to relate the expected cost of this period to that of the steady state cost of \(K_0\). We get that
\[
\mathbb{E}\left[\sum_{t=\tau_0}^{T} c_t\right] = \mathbb{E}\left[\sum_{t=\tau_0}^{T} x_t^T (Q + K_t^T R K_0) x_t \mid \tau_0 \leq T, \tau_0 \leq \tau_{\text{ab}, \text{end}}\right]
\leq \mathbb{E}\left[\mathbbm{1}\{\tau_0 \leq T\} \left(J(K_0) + \frac{2\alpha_1\kappa_0^4}{\gamma_0} \|x_0\|^2\right)\right]
= T J(K_0) \mathbb{P}(\tau_0 \leq T) + 4\alpha_1\kappa_0^6 \mathbb{E}\left[\|x_0\|^2 \mathbbm{1}\{\tau_0 \leq T\}\right],
\]
where the last transition used \(\gamma_0^{-1} = 2\kappa_0^3\). Now, by Lemma 7 we know that on \(E_A\) the algorithm does not abort. We conclude that \(\{\tau_0 \leq T\} \subseteq E_A\) which in turn implies \(\mathbb{P}(\tau_0 \leq T) \leq \mathbb{P}(E_A) \leq T^{-2}\). We get that
\[
\mathbb{E}\left[\sum_{t=\tau_0}^{T} c_t\right] \leq J(K_0) T^{-1} + 4\alpha_1\kappa_0^6 \mathbb{E}\left[\|x_0\|^2 \mathbbm{1}\{\tau_0 \leq T\}\right],
\]
Finally, we use Lemma 20 and simplify to get that
\[
R_2 \leq 2\alpha_1\kappa^2 x_0 T^{-1} + J(K_0) T^{-1} + 4\alpha_1\kappa_0^6 (1 + 8\theta^2) (\kappa^2 + \kappa_0^2) x_0 T^{-2}
= (J(K_0) + 2\alpha_1\kappa^2 x_0) T^{-1} + 4\alpha_1\kappa_0^6 (1 + 8\theta^2) (\kappa^2 + \kappa_0^2) x_0 T^{-2},
\]
as desired.

**A.5 Proof of Lemma 10**

Notice that for \(t < \tau_0\) we have that \(K_t = K_0\). Moreover, we have that \(x_1 = 0\). Applying Lemma 40 we get that
\[
R_3 = \mathbb{E}\left[\sum_{t=1}^{\tau_0-1} x_t^T (Q + K_0^T R K_0) x_t\right] \leq \tau_0 J(K_0).
\]

**B Algorithm 2 Proofs**

**B.1 The Good Event**

We begin by stating the probabilistic events that guarantee the “good” operation of the algorithm. To that end, it will be convenient to specify how the randomized actions during the warm-up stage are generated. For \(t = 1, \ldots, T\) let \(\eta_t \sim \mathcal{N}(0, \sigma^2 I)\) be i.i.d. samples generated before the algorithm starts. Define \(u_t = K_0 x_t + \eta_t\)
and if at time \(t\) the algorithm chooses at random, i.e., during warm-up, then choose \(u_t = \tilde{u}_t\). These virtual actions are a convenient technical tool as they do not directly depend on the action chosen by the algorithm.

Now, recall that
\[
B_t = \arg\min_B \sum_{s=1}^{t-1} \| (x_{s+1} - A_x x_s) - B u_s \|^2 + \lambda \|B\|^2_F,
\]
and denote $\Delta_{B_t} = B_t - B_s$, $V_t^u = \lambda I + \sum_{s=1}^{t-1} u_s u_s^T$. Further recalling that $\tau_i = \tau_0 4^i$ for $0 \leq i \leq n_T$ and $\tau_{n_T+1} = T + 1 \leq \tau_0 4^{n_T+1}$, we define the following events

$$
\mathcal{E}_{B_{\text{ols}}} = \left\{ \text{Tr}(\Delta_{B_t}^T V_t^u \Delta_{B_t}) \leq 4\sigma^2 d \log(4T \frac{\det(V_t^u)}{\det(V_\tau^u)}) + 2\lambda k\delta^2, \text{ for all } t \geq 1 \right\},
$$

$$
\mathcal{E}_{B_s} = \left\{ \sum_{t=\tau_{i-1}}^{\tau_i-1} x_t x_t^T \geq \frac{(\tau_i - \tau_{i-1})\sigma^2}{40} I, \text{ for all } 1 \leq i \leq n_T \right\},
$$

$$
\mathcal{E}_{B_w} = \left\{ \max_{1 \leq t \leq T} \|w_t\| \leq \sigma \sqrt{15d \log 4T} \right\},
$$

$$
\mathcal{E}_{B_u} = \left\{ \sum_{t=1}^{\tau_{i-1}} \hat{u}_t \hat{u}_t^T \geq \frac{(\tau_i - 1)\sigma^2}{40} I, \text{ for all } 0 \leq i \leq n_T \right\},
$$

$$
\mathcal{E}_{B_\eta} = \left\{ \max_{1 \leq t \leq T} \|\eta_t\| \leq \sigma \sqrt{15d \log 4T} \right\}.
$$

Then we have the following lemma.

**Lemma 21.** Let $\mathcal{E}_B = \mathcal{E}_{B_{\text{ols}}} \cap \mathcal{E}_{B_s} \cap \mathcal{E}_{B_w} \cap \mathcal{E}_{B_u} \cap \mathcal{E}_{B_\eta}$, and suppose that $T \geq 600d \log 48T$. Then we have that $\mathbb{P}(\mathcal{E}_B) \geq 1 - T^{-2}$.

**Proof.** First, we describe the parameter estimation error in terms of Lemma 6. To that end, let $z_t = u_t$, $y_{t+1} = x_{t+1} - A_s x_t$, $V_t^u = \lambda I + \sum_{s=1}^{t-1} u_s u_s^T$, and $\Delta_{B_t} = B_t - B_s$ Indeed, we have $y_{t+1} = B_s x_t + w_t$, $w_t \sim \mathcal{N}(0, \sigma^2 I)$, and $\|B_t\|^2 \leq k \|B_u\|^2 \leq k\delta^2$ and so taking Lemma 6 with $\delta = \frac{1}{4} T^{-2}$, recalling that $T \geq d$, and simplifying, we get that $\mathbb{P}(\mathcal{E}_{B_{\text{ols}}}) \geq \frac{1}{1 - \frac{1}{4} T^{-2}}$.

Next, for $\mathcal{E}_{B_s}$, we apply Lemma 36 to the sequence $x_t$ with the filtration $\mathcal{F}_t = \sigma(x_1, u_1, \ldots, x_t, u_t)$. Notice that given $x_{t-1}, u_{t-1}$ we have $x_t \sim \mathcal{N}(A_s x_{t-1} + B_s u_{t-1}, \sigma^2 I)$ and hence we also get

$$
\mathbb{E}[x_t x_t^T \mid \mathcal{F}_{t-1}] \succeq (A_s x_{t-1} + B_s u_{t-1})(A_s x_{t-1} + B_s u_{t-1})^T + \sigma^2 I \succeq \sigma^2 I.
$$

Notice that our choice of $\tau_0$ ensures the minimal sum size assumption. We thus apply Lemma 36 for each $1 \leq i \leq n_T$ with $\delta = \frac{1}{4} T^{-3}$ and apply a union bound to get that $\mathbb{P}(\mathcal{E}_{B_s}) \geq 1 - \frac{1}{4} n_T T^{-3}$. Repeating the same process for $\hat{u}_t$ we also have that $\mathbb{P}(\mathcal{E}_{B_u}) \geq 1 - \frac{1}{4} n_T T^{-3}$.

Finally, for $\mathcal{E}_{B_w}, \mathcal{E}_{B_\eta}$ we apply Lemma 34 with $\delta = \frac{1}{4} T^{-2}$ to get that $\mathbb{P}(\mathcal{E}_{B_w}) \geq 1 - \frac{1}{4} T^{-2}$ and $\mathbb{P}(\mathcal{E}_{B_\eta}) \geq 1 - \frac{1}{4} T^{-2}$.

The final result is obtained by taking a union bound over the events and noticing that $2n_T \leq T$. 

**B.2 Proof of Lemma 11**

The proof is implied by the last part of the following lemma.

**Lemma 22** (Algorithm 2 good warm-up). On $\mathcal{E}_B$ we have that

1. $\|x_t\| \leq \sigma \sqrt{d}(1 + \delta) \sqrt{60d \log 4T}$, for all $1 \leq t \leq \tau_{n_s};$
2. $\|u_t\|^2 \leq \lambda$, for all $1 \leq t < \tau_{n_s};$
3. $V_t^u \geq \frac{\sigma^2}{40} I$, for all $0 \leq i \leq n_s;$
4. $\|\Delta_{B_t}\| \leq \varepsilon_0 2^{-i}$, for all $0 \leq i \leq n_s.$

**Proof.** Recall the definition of $\eta_t$ from Appendix B.1 and define $\hat{w}_t = w_t + B_t \eta_t$. then for $t \leq \tau_{n_s}$ we have that

$$
x_t = A_s x_{t-1} + B_s \hat{u}_{t-1} + w_{t-1} = A_s x_{t-1} + B_s K_0 \hat{x}_{t-1} + w_{t-1} + B_s \eta_{t-1},
$$
i.e., we can consider $x_t$ as a sequence generated from running the controller $K_0$ on a linear system with noise sequence $\tilde{w}_t$. We can then apply Lemma 38 to get that

$$ \|x_t\| \leq \frac{K_0}{\gamma_0} \max_{1 \leq s \leq T} \|\tilde{w}_s\| $$

for all $1 \leq t \leq \tau_n$.

Now, on $\mathcal{E}_B$ we have the noise bounds in Eq. (17) and Eq. (15) and so we have that

$$ \max_{1 \leq s \leq T} \|\tilde{w}_s\| \leq \max_{1 \leq s \leq T} \|w_s\| + \|B_s\| \max_{1 \leq s \leq T} \|\eta_s\| \leq \sigma(1 + \vartheta)\sqrt{15d \log 4T}. $$

Combining the above and recalling that $\gamma_0^{-1} = 2\kappa_0^2$ we conclude that

$$ \|x_t\| \leq \sigma\kappa_0^2(1 + \vartheta)\sqrt{60d \log 4T} $$

for all $1 \leq t \leq \tau_n$, proving the first claim of the lemma. Next, for $1 \leq t < \tau_n$ we have that $u_t = \bar{u}_t = K_0 x_t + \eta_t$ and so

$$ \|u_t\| = \kappa_0 \|x_t\| + \|\eta_t\| \leq \sigma\kappa_0^2(2 + \vartheta)\sqrt{60d \log 4T} \leq \sqrt{\lambda}, $$

proving the second claim. Next, notice that for $0 \leq i \leq n_s$ we have that $V^u_{\tau_i} = \lambda I + \sum_{s=1}^{\tau_i-1} \bar{u}_s \bar{u}_s^T$. Since $\mathcal{E}_B$ holds, we can use the warm-up actions lower bound in Eq. (16) to get that

$$ V^u_{\tau_i} \geq \left(\lambda + \frac{(\tau_i - 1)\sigma^2}{40}\right) I \geq \frac{\tau_i \sigma^2}{40} I $$

for all $0 \leq i \leq n_s$.

proving the third claim. For the final claim, we first use the lower bound on $V^u_{\tau_i}$ to get that

$$ \|\Delta_{B_{\tau_i}}\|^2 \leq \text{Tr}\left(\Delta_{B_{\tau_i}}^T \Delta_{B_{\tau_i}}\right) \leq \frac{40}{\tau_i \sigma^2} \text{Tr}\left(\Delta_{B_{\tau_i}}^T V^u_{\tau_i} \Delta_{B_{\tau_i}}\right). $$

Next, we apply Eq. (13) to get that

$$ \|\Delta_{B_{\tau_i}}\|^2 \leq \frac{1}{\tau_i} \left(160d \log \left(4T^3 \frac{\det(V^u_{\tau_i})}{\det(V^u)\sigma^2}\right) + \frac{80\lambda \kappa \vartheta^2}{\sigma^2}\right). $$

Now, using the second claim of the lemma, we can use Lemma 37 to get that $\log \frac{\det(V^u_{\tau_i})}{\det(V^u)} \leq k \log T$, and applying it to the above and simplifying we get that

$$ \|\Delta_{B_{\tau_i}}\|^2 \leq \frac{1}{\tau_i} \left(160dk \log(4T^4) + \frac{80\lambda \kappa \vartheta^2}{\sigma^2}\right) \leq \frac{1}{\tau_i} \left(640dk \log(4T) + \frac{80\lambda \kappa \vartheta^2}{\sigma^2}\right) \leq \frac{1}{\tau_i} \frac{80\lambda \kappa (1 + \vartheta^2)}{\sigma^2} \leq \frac{\epsilon_0^2 \tau_0}{\tau_i} = \epsilon_0^2 4^{-i}, $$

thus concluding the proof.

\subsection*{B.3 Proof of Lemma 12}

The proof is broken into the following lemmas. The first two claims are concluded by putting together Lemmas 22 and 25 and the third is given by Lemma 26.

Before proceeding, we need the two following lemmas.

\textbf{Lemma 23} (Algorithm 2 warm-up length). On $\mathcal{E}_B$ we have that $\max\left\{0, \log_2 \frac{\mu_0}{\mu_s}\right\} \leq n_s \leq 2 + \max\left\{0, \log_2 \frac{\mu_0}{\mu_s}\right\}$. 

19
Proof. First recall that by Lemma 22, we have that \( \| \Delta_{B_t} \| \leq \varepsilon_0 2^{-i} \) for all \( 0 \leq i \leq n_s \). Now, our choice of \( \mu_0 \) implies that \( \varepsilon_0 = \frac{\mu_0}{\delta t c_0} \) and further recalling that \( \mu_i = \mu_0 2^{-i} \), we apply Lemma 42 to get that
\[
K_t K^T_{\tau_i} \geq K_s K^T_{\tau_i} - \frac{\mu_i}{2} I, \quad \text{for all } 0 \leq i \leq n_s
\]
and
\[
K_s K^T_{\tau_i} \geq K_s K^T_{\tau_i} - \frac{\mu_i}{2} I, \quad \text{for all } 0 \leq i \leq n_s.
\]
Now, suppose in contradiction that \( n_s > 0 \) and \( \mu_{n_s} < \frac{\mu_0}{2} \). This means that \( \mu_{n_s-1} \geq \frac{\mu_0}{2} \) and so we can apply Eq. (18) to get that
\[
K_{\tau_{n_s-1}} K^T_{\tau_{n_s-1}} \geq \left( \mu_s - \frac{\mu_{n_s-1}}{2} \right) I \geq \left( 2\mu_{n_s-1} - \frac{\mu_{n_s-1}}{2} \right) I = \frac{3}{2} \mu_{n_s-1} I,
\]
which contradicts the fact that \( n_s \) is the first time the warm-up condition is satisfied. We conclude that either \( n_s = 0 \) or \( \mu_{n_s} \geq \frac{\mu_0}{2} \). Plugging \( \mu_{n_s} = \mu_0 2^{-n_s} \), the latter condition implies \( n_s \leq 2 + \log_2 \frac{\mu_0}{\mu_s} \) thus giving the lemma’s upper bound.

Now for the lower bound, suppose in contradiction that \( \mu_{n_s} > \mu_s \) then by Eq. (19) we get that
\[
K_s K^T_{\tau_i} \geq \left( \frac{3}{2} \mu_{n_s} - \frac{\mu_s}{2} \right) I > \mu_s I,
\]
which contradicts the fact that \( \mu_s \) is the tight lower bound on the eigenvalues of \( K_s K^T_{\tau_i} \). We conclude that \( \mu_{n_s} \leq \mu_s \) which in turn implies the desired lower bound. ■

Lemma 24 (Algorithm 2 conditional control). Suppose \( \mathcal{E}_B \) holds and fix some \( i \) such that \( n_s \leq i \leq n_T \). If \( \| u_t \|^2 \leq \lambda \) for all \( 1 \leq t \leq \tau_i - 1 \), then \( K_{\tau_i} \) is \((\kappa, \gamma)-\)strongly stable and \( K_{\tau_i} K^T_{\tau_i} \geq \frac{\mu_0}{2} I \).

Proof. If \( \| \Delta_{B_{\tau_i}} \| \leq \min \left\{ \varepsilon_0, \frac{\mu_0}{\delta t c_0} \right\} \) then Lemma 42 immediately implies the desired result. We prove this estimation error bound thus concluding the proof.

To that end, notice that for \( t \geq s \) we have \( V^u_{t, \tau_{n_s}} \geq V^u_{s, t} \). Using the lower bound on \( V^u_{t, \tau_{n_s}} \) in Lemma 22 we get that
\[
\text{Tr} \left( \Delta^T_{B_{\tau_i}} V^u_{\tau_i, \tau_{n_s}} \Delta_{B_{\tau_i}} \right) \geq \text{Tr} \left( \Delta^T_{B_{\tau_i}} V^u_{\tau_{n_s}, \tau_{n_s}} \Delta_{B_{\tau_i}} \right) \geq \| \Delta_{B_{\tau_i}} \|^2 \frac{T_{\tau_{n_s}} \sigma^2}{40},
\]
and by changing sides and applying the parameter estimation bound in Eq. (13) we get that
\[
\| \Delta_{B_{\tau_i}} \|^2 \leq \frac{1}{T_{\tau_{n_s}}} \left( 160 d \log \left( 4 \log \frac{\text{det}(V^u_{\tau_{n_s}})}{\text{det}(V^u_{\tau_{n_s}})} \right) + \frac{80 \lambda k \sigma^2}{\sigma^2} \right)
\]
Now, using the assumption on \( u_t \), we can apply Lemma 37 to get that \( \log \frac{\text{det}(V^u_{\tau_{n_s}})}{\text{det}(V^u_{\tau_T})} \leq k \log T \). Plugging this back into Eq. (20) and simplifying we get that
\[
\| \Delta_{B_{\tau_i}} \| \leq \varepsilon_0 2^{-n_s},
\]
and plugging in the lower bound on \( n_s \) from Lemma 23 gives the desired bound on the estimation error thus concluding the proof. ■

Lemma 25 (Algorithm 2 bounded operation). On \( \mathcal{E}_B \) we have that
1. \( \| x_t \|^2 \leq x_b \), for all \( \tau_{n_s} \leq t \leq T \);
2. \( K_{\tau_i} \) is \((\kappa, \gamma)-\)strongly stable, for all \( n_s \leq i \leq n_T \);
3. \( K_{\tau_i} K^T_{\tau_i} \geq \frac{1}{T_{\tau_i}} \mu_s I \), for all \( n_s \leq i \leq n_T \).
Proof. First, recall the bounds on \( x_t, u_t \) from Lemma 22, i.e.,
\[
\|x_{\tau_{n_s}}\| \leq \sigma \kappa_0^3 (1 + \vartheta) \sqrt{60d \log 4T} \leq \sqrt{x_b}, \\
\|u_t\|^2 \leq \lambda, \text{ for all } 1 \leq t < \tau_{n_s}.
\]

We prove by induction on \( n \) where \( n_s \leq n \leq n_T \) that the claims of the lemma hold up to time \( \tau_n \) and phase \( n \) respectively.

For the base case, \( n = n_s \) the bounds above satisfy Lemma 24 and so we conclude that \( K_{\tau_0} \) is strongly stable and that \( K_{\tau_0} K_{\tau_0}^T \geq \frac{1}{2} \mu_\star I \) thus satisfying the induction base.

Next, assume the induction hypothesis holds for \( n - 1 \) and we show for \( n \). By the induction hypothesis, the algorithm does not abort up to (including) time \( \tau_{n-1} \). Moreover, it means that for all \( n_s \leq i \leq n - 1 \) the controllers \( K_{\tau_i} \) are \((\kappa, \gamma)\)-strongly stable and so we can use Lemma 39 to get that
\[
\|x_t\| \leq 3\kappa \max \left\{ \frac{\|x_{\tau_{n_s}}\|}{\kappa}, \gamma \max_{1 \leq s \leq T} \|w_t\| \right\}, \quad \text{for all } \tau_{n_s} \leq t \leq \tau_n,
\]
and plugging in that \( \gamma^{-1} = 2\kappa^2 \), the bound for \( \|x_{\tau_{n_s}}\| \) and the bound for the noise in Eq. (15) we get that
\[
\|x_t\| \leq \kappa \sigma \max \left\{ \kappa_0^3 (1 + \vartheta), 2\kappa^2 \right\} \sqrt{135d \log 4T} \leq \sqrt{x_b}, \quad \text{for all } \tau_{n_s} \leq t \leq \tau_n,
\]
as desired for \( x_t \). Notice that this ensures that the algorithm does not abort up to time \( \tau_{n-1} \). So, for \( \tau_{n_s} \leq t \leq \tau_{n-1} \) we have that \( \|u_t\| = \|K_t x_t\| \leq \|K_t\| \|x_t\| \leq \kappa \sqrt{x_b} = \sqrt{\lambda} \), and thus Lemma 24 establishes the desired strong-stability and non-degeneracy of \( K_{\tau_{n_s}} \), finishing the induction.

Finally, using the strong stability of all controllers we apply Lemma 39 a final time to obtain the bound on \( x_t \) for all \( \tau_{n_s} \leq t \leq T \). \( \blacksquare \)

**Lemma 26 (Algorithm 2 parameter estimation).** On \( E_B \) we have that \( \|\Delta_{B,v}\| \leq \varepsilon_0 \min \left\{ 2^{-n_s}, \mu_\star^{-1/2} 2^{-1/2} \right\}, \forall n_s < i \leq n_T. \)

**Proof.** Recall that by Lemma 25, the algorithm does not abort on \( E_B \) and so for \( \tau_i \leq t \leq \tau_{i+1} \) we have that \( K_t = K_{\tau_i} \). This means we can decompose \( V^u_{\tau_i} \) as
\[
V^u_{\tau_i} = V^u_{\tau_{n_s}} + \sum_{t=\tau_{n_s}}^{\tau_i-1} u_t u_t^T = V^u_{\tau_{n_s}} + \sum_{j=n_s}^{i-1} \sum_{t=\tau_j}^{\tau_{j+1}-1} u_t u_t^T = V^u_{\tau_{n_s}} + \sum_{j=n_s}^{i-1} K_{\tau_j} \left( \sum_{t=\tau_j}^{\tau_{j+1}-1} x_t x_t^T \right) K_{\tau_j}^T.
\]

Next, we lower bound \( V^u_{\tau_{n_s}} \) using Lemma 22 and the states using Eq. (14) and get that
\[
V^u_{\tau_i} \geq \tau_{n_s} \sigma^2 40 I + \sum_{j=n_s}^{i-1} \left( \frac{\tau_{j+1} - \tau_j}{40} \right) K_{\tau_{j+2}} K_{\tau_{j+2}}^T,
\]
and recalling that \( K_{\tau_j} K_{\tau_j}^T \geq \frac{\mu_\star}{2} I \) (see Lemma 25) we get that, assuming \( i > n_s \),
\[
V^u_{\tau_i} \geq \frac{\sigma^2}{40} \left( \tau_{n_s} + \frac{\mu_\star}{2} \sum_{j=n_s}^{i-1} (\tau_{j+1} - \tau_j) \right) I = \frac{\sigma^2}{40} \left( \tau_{n_s} + \frac{\mu_\star}{2} (\tau_i - \tau_{n_s}) \right) I \geq \frac{\sigma^2}{40} \max \left\{ \tau_{n_s}, \frac{\mu_\star}{4} \tau_i \right\} I.
\]

Now, apply this together with the parameter estimation bound in Eq. (13) to get that
\[
\|\Delta_{B,v}\|^2 \leq \text{Tr} \left( \Delta_{B,v}^T \Delta_{B,v} \right) \leq \frac{40}{\sigma^2} \max \left\{ \tau_{n_s} + \frac{\mu_\star}{4} \tau_i \right\} \text{Tr} \left( \Delta_{B,v}^T V^u_{\tau_i} \Delta_{B,v} \right) \leq \frac{1}{\max \left\{ \tau_{n_s} + \frac{\mu_\star}{4} \tau_i \right\}} \left( 160d \log \left( 4T^3 \frac{\text{det}(V^u_{\tau_i})}{\text{det}(V^u_{\tau_i})} \right) + \frac{80\lambda \kappa \vartheta^2}{\sigma^2} \right).
\]
Finally, from Lemma 22 we have that \( \|u_t\|^2 \leq \lambda \) for \( 1 \leq t \leq T \), and from Lemma 25 we have that \( \|x_t\|^2 \leq x_b \) for \( \tau_n \leq t \leq T \) and so \( \|u_t\|^2 = \|K_x x_t\|^2 \leq \kappa^2 x_b = \lambda \). Combining both claims, we apply Lemma 37 to get that \( \log \det V_{\tau_i}^x \leq k \log T \) and plugging this into the above equation we get

\[
\|\Delta_{\tau_i}\|^2 \leq \frac{1}{\max\{\tau_n, \mu_s, \mu^2_T - \tau_i\}} \left( 640dk \log(4T) + \frac{80\lambda^2}{\sigma^2} \right) \leq \frac{\tau_0 \varepsilon_0^2}{\max\{\tau_n, \mu_s, \mu^2_T - \tau_i\}} = \varepsilon_0^2 \min\left\{4^{-n_s}, \frac{4}{\mu_s} 4^{-i}\right\},
\]

where the second transition follows from our choice of \( \tau_0 \).

### B.4 Proof of Theorem 2

As in Algorithm 1, denote \( J_i = \sum_{t=\tau_i}^{\tau_{i+1}-1} x_t^T (Q + K_{x_t}^T R K_{x_t}) x_t \). Recalling that warm-up lasts until phase \( n_s \), we have the following decomposition of the regret:

\[
E[R_T] = R_1 + R_2 + R_3 - TJ_x,
\]

where

\[
R_1 = E \left[ \sum_{i=n_s}^{n_T} 1 \{\mathcal{E}_B\} J_i \right], \quad R_2 = E \left[ 1 \{\mathcal{E}_B^c\} \sum_{t=\tau_n}^T c_t \right], \quad R_3 = E \left[ \sum_{i=1}^{\tau_{n_s}-1} c_t \right],
\]

are the costs due to success, failure, and warm-up respectively. The following lemmas bound each of \( R_1, R_2, R_3 \) thus concluding the proof. The proofs for \( R_1, R_2 \) remain nearly the same but are provided for completeness. The proof of \( R_3 \) contains a few technical challenges, introduced by the randomness of the warm-up period duration.

**Lemma 27.** \( R_1 - TJ_x \leq n_T \left( 6C_0 \varepsilon_0^3 \max\{1, 4 \mu_s^{-1}\} \tau_0 + 8 \alpha_1 \kappa^6 x_b \right) \).

**Lemma 28.** \( R_2 \leq (J(K_0) + 2 \alpha_1 \kappa^2 x_b) T^{-1} + 4 \alpha_1 \kappa^2 (1 + \theta^2) (\kappa^2 + \kappa^6_0) x_b T^{-2} \).

**Lemma 29.** \( R_3 \leq (1 + \theta^2) \left( 65 J(K_0) \max\{1, \frac{\mu_s^2}{\rho^2}\} \tau_0 + 8 \alpha_1 \sigma^2 \kappa^4 \log^2 3T \right) \).

#### B.4.1 Proof of Lemma 27

**Proof.** We begin by bounding \( E[1 \{\mathcal{E}_B\} J_i \mid n_s] \) for \( n_s \leq i \leq n_T \). This follows exactly as in Lemma 8 but with some changes to the events \( E_i \), and thus is repeated here. For \( n_s \leq i \leq n_T \) define the events \( S_i = \left\{ \|x_t\|^2 \leq x_b \right\} \) and

\[
E_{n_s} = \left\{ \|\Delta_{B_{n_s}}\| \leq \varepsilon_0 2^{-n_s} \right\}, \quad E_i = \left\{ \|\Delta_{B_t}\| \leq \varepsilon_0 \min\{2^{-n_s}, 2 \mu_s^{-1} 2^{-i}\} \right\}, \quad \forall \ n_s < i \leq n_T.
\]

By Lemma 12, we have that \( \mathcal{E}_B \subseteq E_i \cap S_i \). Now, define \( \tilde{x}_{t_i} = x_{t_i} \) and for \( \tau_i < t \leq \tau_{i+1} - 1 \) define

\[
\tilde{x}_t = (A_x + B_x K_{x_t}) \tilde{x}_{t-1} + w_t.
\]

Since on \( \mathcal{E}_B \) the algorithm does not abort, we have that

\[
1 \{\mathcal{E}_B\} J_i = 1 \{\mathcal{E}_B\} \sum_{t=\tau_i}^{\tau_{i+1}-1} \tilde{x}_t^T (Q + K_{x_t}^T R K_{x_t}) \tilde{x}_t \leq 1 \{E_i \cap S_i\} \sum_{t=\tau_i}^{\tau_{i+1}-1} \tilde{x}_t^T (Q + K_{x_t}^T R K_{x_t}) \tilde{x}_t.
\]

Noticing that \( E_i, S_i \), and \( K_{x_t} \) are completely determined by \( x_{\tau_i}, B_{\tau_i} \) we use total expectation to get that

\[
E[1 \{\mathcal{E}_B\} J_i \mid n_s] \leq E \left[ 1 \{E_i \cap S_i\} \left( \sum_{t=\tau_i}^{\tau_{i+1}-1} \tilde{x}_t^T (Q + K_{x_t}^T R K_{x_t}) \tilde{x}_t \mid x_{\tau_i}, B_{\tau_i} \right) \right] \mid n_s,
\]

22
where in the inner expectation we removed the conditioning on \( n_s \) since the \( \tilde{x}_i \) are conditionally independent of \( n_s \) given \( x_{r_i} \). Now, by Lemma 42, \( E_i \) implies that \( K_{r_i} \) is \((\kappa, \gamma)\)-strongly stable and so we can use Lemma 40 to get that

\[
E[1 \{ E_B \} J_i] \leq (\tau_{i+1} - \tau_i)E[1 \{ E_i \} J(K_{r_i}) \mid n_s] + \frac{2\alpha_1\kappa^4}{\gamma}E[1 \{ S_i \} \| x_{r_i} \|^2 \mid n_s]
\]

\[
\leq (\tau_{i+1} - \tau_i)E[1 \{ E_i \} J(K_{r_i}) \mid n_s] + 4\alpha_1\kappa^6 x_b,
\]

where the second transition also used that \( \gamma^{-1} = 2\kappa^2 \).

Now, by Lemma 4, on \( E_{n_s} \), we have that \( J(K_{r_i}) \leq J_* + C_0 \varepsilon_0^2 4^{-n_s} \) and on \( E_i \), where \( n_s < i \leq n_T \), we have that \( J(K_{r_i}) \leq J_* + C_0 \varepsilon_0^2 \min\{4^{-n_s}, 4\mu_*^{-1}4^{-i}\} \). Combining both cases we conclude that

\[
1 \{ E_i \} J(K_{r_i}) \leq J_* + C_0 \varepsilon_0^2 \max\{1, 4\mu_*^{-1}\} 4^{-i}, \quad \forall \ n_s \leq i \leq n_T,
\]

and plugging this back into Eq. (21) and recalling that \( \tau_{i+1} - \tau_i \leq 3\tau_i = 3\tau_0 4^i \) we have that

\[
E[1 \{ E_B \} J_i \mid n_s] \leq (\tau_{i+1} - \tau_i)J_* + 3C_0 \varepsilon_0^2 \max\{1, 4\mu_*^{-1}\} \tau_0 + 4\alpha_1\kappa^6 x_b.
\]

Finally, we sum over \( i \) to conclude that

\[
R_1 = \mathbb{E} \left[ \sum_{i=n_0}^{n_T} E[1 \{ E_B \} J_i \mid n_s] \right] \leq \mathbb{E} \left[ \sum_{i=n_0}^{n_T} (\tau_{i+1} - \tau_i)J_* + 3C_0 \varepsilon_0^2 \max\{1, 4\mu_*^{-1}\} \tau_0 + 4\alpha_1\kappa^6 x_b \right]
\]

\[
\leq \mathbb{E}\left[ (\tau_{n_T+1} - \tau_{n_s})J_* + (n_T - n_s)(3C_0 \varepsilon_0^2 \max\{1, 4\mu_*^{-1}\} \tau_0 + 4\alpha_1\kappa^6 x_b) \right]
\]

\[
\leq T J_* + n_T (6C_0 \varepsilon_0^2 \max\{1, 4\mu_*^{-1}\} \tau_0 + 8\alpha_1\kappa^6 x_b),
\]

thus concluding the proof.

**B.4.2 Proof of Lemma 28**

The proof is identical to that of Lemma 9 where the initial warm-up duration \( \tau_0 \) is replaced with \( \tau_{n_s} \) and the uses of Lemmas 17 and 20 are replaced with Lemmas 21 and 30 respectively. We thus conclude by proving Lemma 30. To that end, recall that \( \tau_{\text{abort}} \) is the time when the algorithm decides to abort, formally,

\[
\tau_{\text{abort}} = \min\{t \geq \tau_{n_s} \mid \| x_t \|^2 > x_b \text{ or } \| K_t \| > \kappa\},
\]

where we treat \( \min \emptyset = T + 1 \).

**Lemma 30 (Expected abort state).** Suppose that \( \mathbb{P}(\tau_{\text{abort}} \leq T) \leq T^{-2} \). Then we have that

\[
\mathbb{E}\left[ \| x_{\tau_{\text{abort}}} \|^2 \mathbb{1}\{\tau_{\text{abort}} < T\} \right] \leq (1 + 8\theta^2)(\kappa^2 + \kappa_0^2) x_b T^{-2}.
\]

**Proof.** First, by the lemmas assumption, we can apply Lemma 35 to get that

\[
\mathbb{E}\left[ 1 \{ \tau_{\text{abort}} \leq T \} \max_{1 \leq t \leq T} \| w_t \|^2 \right] \leq 5d\sigma^2 T^{-2} \log 3T,
\]

\[
\mathbb{E}\left[ 1 \{ \tau_{\text{abort}} \leq T \} \max_{1 \leq t \leq T} \| B_* \eta_t + w_t \|^2 \right] \leq 5d\sigma^2 (1 + \vartheta^2) T^{-2} \log 3T.
\]

Now, notice that \( \| A_* + B_* K \| \leq 2\vartheta \| K \| \) and split into two cases. First, if \( \tau_{\text{abort}} > \tau_{n_s} \) then by definition of \( \tau_{\text{abort}} \) we have that

\[
\| x_{\tau_{\text{abort}}} \| = \| (A_* + B_* K_{\tau_{\text{abort}}} - 1)x_{\tau_{\text{abort}} - 1} + w_{\tau_{\text{abort}} - 1} \| \leq 2\vartheta \kappa \sqrt{x_b} + \max_{1 \leq t \leq T} \| w_t \|,
\]

and taking expectation and applying Eq. (22) we get that

\[
\mathbb{E}\left[ 1 \{ \tau_{n_s} < \tau_{\text{abort}} \leq T \} \| x_{\tau_{\text{abort}}} \|^2 \right] \leq 8\vartheta^2 \kappa^2 x_b T^{-2} + 5d\sigma^2 T^{-2} \log 3T \leq (1 + 8\theta^2)\kappa^2 x_b T^{-2}.
\]
On the other hand if \( \tau_{\text{abort}} = \tau_{n_s} \) then \( u_{\tau_{\text{abort}} - 1} = K_0 x_{\tau_{n_s} - 1} + \eta_{\tau_{n_s} - 1} \) and so we have that
\[
\|x_{\tau_{\text{abort}}}\| = \|(A_s + B_s K_0) x_{\tau_{n_s} - 1} + (B_s \eta_{\tau_{n_s} - 1} + w_{\tau_{n_s} - 1})\|
\leq 2 \vartheta \kappa_0 \|x_{\tau_{n_s} - 1}\| + \max_{1 \leq t \leq T} \|B_s \eta_t + w_t\|
\leq (4 \vartheta + 1) \kappa_0 \|x_{\tau_{n_s} - 1}\|,
\]
where the last transition used Lemma 38 and \( \gamma_0^{-1} = 2 \kappa_0^2 \). Taking expectation and applying Eq. (23) we get that
\[
\mathbb{E}\left[ \mathbf{1}\{\tau_{\text{abort}} = \tau_{n_s}\}\|x_{\tau_{\text{abort}}}\|^{2} \right] \leq 80(1 + \vartheta)^{2}(1 + \vartheta^2) \kappa_0^8 \sigma^2 T^{-2} \log 3T \leq (1 + \vartheta^2) \kappa_0^2 \tau x_{\tau} T^{-2},
\]
and combining both cases yields the final bound.

### B.4.3 Proof of Lemma 29

**Proof.** We begin by decomposing \( R_3 \). Notice that \( n_s \leq n_T + 1 \) and so we have that
\[
R_3 = \mathbb{E}\left[ \sum_{t=1}^{n_T+1} c_t \right] + \mathbb{E}\left[ \sum_{t=0}^{n_T} \sum_{t=\tau_i}^{\tau_i+1} c_t \right] = \mathbb{E}\left[ \sum_{t=1}^{n_T+1} c_t \right] + \mathbb{E}\left[ \mathbf{1}\{n_s > i\} \sum_{t=\tau_i}^{\tau_i+1-1} c_t \right].
\]
Now, define \( J(K, W) \) to be the infinite horizon cost of playing controller \( K \) on the LQ system \( (A_s, B_s) \) whose system noise has covariance \( W \in \mathbb{R}^{d \times d} \). In terms of our notation so far, this means that \( J(K) = J(K, \sigma^2 I) \). It is well known that \( J(K, W) = \text{Tr}(PW) \) where \( P \) is a positive definite solution to
\[
P = Q + K^T R K + (A_s + B_s K)^T P (A_s + B_s K),
\]
and thus does not depend on \( W \).

Now, for \( 1 \leq t < \tau_{n_s} \) we have that \( x_{t+1} = (A_s + B_s K_0) x_t + (B_s \eta_t + w_t) \), i.e., this is equivalent to an LQ system \( (A_s, B_s) \) with noise covariance \( \sigma^2 (I + B_s B_s^T) \leq (1 + \vartheta^2) \sigma^2 I \) and controller \( K_0 \) and so we have that
\[
J(K_0, \sigma^2 (I + B_s B_s^T)) = \text{Tr}(\sigma^2 (I + B_s B_s^T) P) \leq (1 + \vartheta^2) \text{Tr}(\sigma^2 P) = (1 + \vartheta^2) J(K_0, \sigma^2) = (1 + \vartheta^2) J(K_0).
\]
With the above in mind, we bound the first term in the decomposition of \( R_3 \) using Lemma 40. We get that
\[
\mathbb{E}\left[ \sum_{t=1}^{n_T+1} c_t \right] \leq \tau_0 J(K_0, \sigma^2 (I + B_s B_s^T)) + \frac{2\alpha_1 \kappa_0^2}{\gamma_0} \|x_1\|^2 \leq (1 + \vartheta^2) J(K_0) \tau_0. \tag{24}
\]

Next, recall that \( \gamma_0^{-1} = 2 \kappa_0^2 \), denote the filtration of the history, \( \mathcal{F}_t = \sigma(x_1, u_1, w_1, \ldots, x_t, u_t, w_t) \) and similarly apply Lemma 40 to get that
\[
\mathbb{E}\left[ \sum_{t=\tau_i}^{\tau_{i+1}-1} c_t \middle| \mathcal{F}_{\tau_{i-1}} \right] \leq (1 + \vartheta^2) J(K_0)(\tau_{i+1} - \tau_i) + 4\alpha_1 \kappa_0^6 \|x_{\tau_i}\|^2.
\]
Now, using Lemmas 35 and 38 we get that
\[
\mathbb{E}\left[ \mathbf{1}\{n_s > i\}\|x_{\tau_i}\|^2 \right] \leq \mathbb{E}\left[ \frac{\kappa_0^2}{\gamma_0} \max_{1 \leq t \leq T} \|w_t + B_s \eta_t\|^2 \right] \leq 20d(1 + \vartheta^2) \sigma^2 \kappa_0^8 \log 3T.
\]
Combining the last two inequalities and noticing that \( \mathbf{1}\{n_s > i\} \) is \( \mathcal{F}_{\tau_{i-1}} \) measurable we further have that
\[
\mathbb{E}\left[ \mathbf{1}\{n_s > i\} \sum_{t=\tau_i}^{\tau_{i+1}-1} c_t \right] = \mathbb{E}\left[ \mathbf{1}\{n_s > i\} \mathbb{E}\left[ \sum_{t=\tau_i}^{\tau_{i+1}-1} c_t \middle| \mathcal{F}_{\tau_{i-1}} \right] \right] \leq \mathbb{E}\left[ \mathbf{1}\{n_s > i\} \left( (1 + \vartheta^2) J(K_0)(\tau_{i+1} - \tau_i) + 4\alpha_1 \kappa_0^6 \|x_{\tau_i}\|^2 \right) \right] \leq (1 + \vartheta^2) (\mathbb{E}(n_s > i) J(K_0)(\tau_{i+1} - \tau_i) + 80\alpha_1 d \sigma^2 \kappa_0^4 \log 3T).
\]
Now, from Lemma 23 we know that $\mathbb{P}\left(n_s > 2 + \max\left\{0, \log_2 \frac{M}{\mu R}\right\}\right) \leq \mathbb{P}(\mathcal{E}_b^c) \leq T^{-2}$, and recalling that $\tau_i = \tau_0 4^i$ we get that

$$\tau_0 + \sum_{i=0}^{n_T} (\tau_{i+1} - \tau_i) \mathbb{P}(n_s > i) \leq \tau_0 + \sum_{i=0}^{\left[2 + \max\left\{0, \log_2 \frac{M}{\mu R}\right\}\right]} (\tau_{i+1} - \tau_i) + \sum_{i=0}^{n_T} (\tau_{i+1} - \tau_i) T^{-2}
= \tau_0 4^{\left[3 + \max\left\{0, \log_2 \frac{M}{\mu R}\right\}\right]} + (\tau_{n_T+1} - \tau_0) T^{-2}
\leq 64 \tau_0 \max\left\{1, \frac{\mu_0^2}{\mu_*^2}\right\} + 4T^{-1}. \tag{26}$$

Finally, combining Eqs. (24) to (26) we get that

$$R_T \leq (1 + \vartheta^2) \left(J(K_0) \left(\tau_0 + \sum_{i=0}^{n_T} (\tau_{i+1} - \tau_i) \mathbb{P}(n_s > i)\right) + 80 \alpha_1 \sigma^2 \kappa_0^{14} (n_T + 1) \log 3T\right)
\leq (1 + \vartheta^2) \left(64 J(K_0) \max\left\{1, \frac{\mu_0^2}{\mu_*^2}\right\} \tau_0 + 4 J(K_0) T^{-1} + 80 \alpha_1 \sigma^2 \kappa_0^{14} \log^2 3T\right)
\leq (1 + \vartheta^2) \left(65 J(K_0) \max\left\{1, \frac{\mu_0^2}{\mu_*^2}\right\} \tau_0 + 80 \alpha_1 \sigma^2 \kappa_0^{14} \log^2 3T\right),$$

where the second transition also used $n_T + 1 \leq \log 3T$. \hfill \blacksquare

C Lower bound proofs

The next lemma requires the following well known results in LQRs (see, e.g., Bertsekas, 1995). Consider the Q-function of the system with respect to $k_*$, that in the one-dimensional case takes the form $F(x, u) = x^2 + u^2 + (ax + bu)^2 p_*$. Using the form of $k_*$ given in Eq. (1), and by simple algebra we obtain

$$F(x_t, u_t) - F(x_t, k_* x_t) = (1 + b^2 p_*)(u_t - k_* x_t)^2. \tag{27}$$

Further, we have $F(x_t, k_* x_t) = x_t^2 p_*$ as both sides are equal to the value of the optimal policy $k_*$ starting from state $x_t$. Finally, also recall that $J(k_*) = \sigma^2 p_*$. The following explains Eq. (27):

$$F(x_t, u_t) = x_t^2 + ((u_t - k_* x_t) + k_* x_t)^2 + ((a + bk_*) x_t + b(u_t - k_* x_t))^2 p_*$$

$$= F(x_t, k_* x_t) + (u_t - k_* x_t)^2 + 2(u_t - k_* x_t)k_* x_t + b^2 p_*(u_t - k_* x_t)^2 + 2bp_*(u_t - k_* x_t)(a + bk_*) x_t$$

$$= F(x_t, k_* x_t) + (1 + b^2 p_*)(u_t - k_* x_t)^2 + 2x_t(u_t - k_* x_t)(k_* + b p_*(a + bk_*))$$

$$= F(x_t, k_* x_t) + (1 + b^2 p_*)(u_t - k_* x_t)^2 + 2x_t(u_t - k_* x_t)(k_*(1 + b^2 p_*) + bp_* a)$$

$$= F(x_t, k_* x_t) + (1 + b^2 p_*)(u_t - k_* x_t)^2,$$

where the last transition used $k_*(1 + b^2 p_*) = -bp_* a$ (see Eq. (1)).

**Lemma 31.** The expected regret can be written as

$$\mathbb{E}[R_T] = \mathbb{E}\left[\sum_{t=1}^{T} (1 + b^2 p_*)(u_t - k_* x_t)^2\right] - \mathbb{E}\left[x_{T+1}^2 p_*\right].$$
Proof. Using the expressions for the Q-function of the system with respect to \( k_* \), we have that

\[
R_T = \sum_{t=1}^{T} \mathbb{E}[x_t^2 + u_t^2 - J(k_*)]
\]

\[
= \sum_{t=1}^{T} \mathbb{E}[F(x_t, u_t) - ((ax_t + bu_t)^2 + w_t^2)p_*] 
= \sum_{t=1}^{T} \mathbb{E}[F(x_t, u_t) - x_{t+1}^2p_*]
\]

\[
= \sum_{t=1}^{T} \mathbb{E}[F(x_t, u_t) - F(x_t, k_* x_t)] + \sum_{t=1}^{T} \mathbb{E}[x_{t+1}^2p_* - x_{t+1}^2x_t^2] 
= \mathbb{E}\left[\sum_{t=1}^{T} (1 + b^2p_*)(u_t - k_*x_t)^2\right] + \mathbb{E}[x_{T+1}^2] - \mathbb{E}[x_{T+1}^2].
\]

The lemma now follows from our assumption that \( x_1 = 0 \).

Lemma 32. We have \( \mathbb{E}[x_{T+1}^2] \leq \frac{b^2}{2} \mathbb{E}\left[\sum_{t=1}^{T} (u_t - k_* x_t)^2\right] + \sigma^2 \).

Proof. Denote \( m = a + bk_* \) and \( u_t = u_t - k_* x_t \) for all \( t \geq 1 \). Then, \( x_{t+1} = ax_t + b(u_t - k_* x_t + k_* x_t) + w_t = mx_t + bv_t + w_t \), and by unfolding the recursion and using \( x_1 = 0 \) we obtain

\[
x_{T+1} = \sum_{t=1}^{T} m^{T-t} bv_t + \sum_{t=1}^{T} m^{T-t} w_t,
\]

hence

\[
\mathbb{E}[x_{T+1}^2] \leq 2b^2 \mathbb{E}\left(\sum_{t=1}^{T} m^{T-t} v_t^2\right)^2 + 2\mathbb{E}\left(\sum_{t=1}^{T} m^{T-t} w_t\right)^2,
\]

Now, observe that

\[
|m| = |a + bk_*| = \left| a - b \cdot \frac{abp_*}{1 + b^2 p_*}\right| = \frac{|a|}{1 + b^2 p_*} \leq \frac{1}{\sqrt{5}}.
\]

Using this bound and the Cauchy-Schwartz inequality, we have

\[
\mathbb{E}\left(\sum_{t=1}^{T} m^{T-t} v_t^2\right)^2 \leq \sum_{t=1}^{T} m^{2(T-t)} \cdot \mathbb{E}\left[\sum_{t=1}^{T} v_t^2\right] \leq \frac{1}{1 - m^2} \mathbb{E}\left[\sum_{t=1}^{T} v_t^2\right] \leq \frac{5}{4} \mathbb{E}\left[\sum_{t=1}^{T} v_t^2\right].
\]

Further, as the noise terms \( w_1, \ldots, w_T \) are i.i.d. and have variance \( \sigma^2 \),

\[
\mathbb{E}\left(\sum_{t=1}^{T} m^{T-t} w_t\right)^2 = \sum_{t=1}^{T} m^{2(T-t)} \mathbb{E}[w_t^2] \leq \frac{1}{1 - m^2} \sigma^2 \leq \frac{5}{4} \sigma^2.
\]

Combining inequalities, the lemma follows.

Proof of Lemma 14. Since \( 1 + b^2 p_* \geq 1 \) and \( p_* \leq 5/4 \) (see Eq. (7)), Lemma 31 lower bounds the regret as

\[
\mathbb{E}[R_T] \geq \mathbb{E}\left[\sum_{t=1}^{T} (u_t - k_* x_t)^2\right] - \frac{5}{4} \mathbb{E}[x_{T+1}^2].
\]

Plugging in the bound of Lemma 32 and the assumption that \( b^2 = \epsilon \leq 1/400 \), we obtain

\[
\mathbb{E}[R_T] \geq \frac{99}{100} \mathbb{E}\left[\sum_{t=1}^{T} (u_t - k_* x_t)^2\right] - 4\sigma^2.
\]

(28)
On the other hand, note that $u_t^2 \leq 2(u_t - k_* x_t)^2 + 2k_*^2 x_t^2$, and so

$$\mathbb{E} \left[ \sum_{t=1}^{T} u_t^2 \right] \leq 2 \mathbb{E} \left[ \sum_{t=1}^{T} (u_t - k_* x_t)^2 \right] + 2k_*^2 \mathbb{E} \left[ \sum_{t=1}^{T} x_t^2 \right].$$

Further, since $J(k_*) = \sigma^2 p_* \leq \frac{2}{5} \sigma^2$ we have

$$\mathbb{E} \left[ \sum_{t=1}^{T} x_t^2 \right] \leq \mathbb{E} \left[ \sum_{t=1}^{T} (x_t^2 + u_t^2) \right] = \mathbb{E}[R_T] + T \mathbb{E}[J(k_*)] \leq \mathbb{E}[R_T] + \frac{5}{4} \sigma^2 T.$$

Therefore,

$$\mathbb{E} \left[ \sum_{t=1}^{T} u_t^2 \right] \leq 2 \mathbb{E} \left[ \sum_{t=1}^{T} (u_t - k_* x_t)^2 \right] + 2k_*^2 \mathbb{E}[R_T] + \frac{5}{2} \sigma^2 k_*^2 T. \quad (29)$$

Combining Eqs. (28) and (29) and recalling that $2k_*^2 \leq \epsilon \leq 1$ (see Eq. (7)), results with

$$\mathbb{E} \left[ \sum_{t=1}^{T} u_t^2 \right] \leq 2 \left( \frac{100}{99} \mathbb{E}[R_T] + 5 \sigma^2 \right) + 2k_*^2 \mathbb{E}[R_T] + \frac{5}{2} \sigma^2 k_*^2 T \leq 3 \mathbb{E}[R_T] + \frac{5}{2} \sigma^2 k_*^2 T + 12 \sigma^2,$$

and changing sides yields the second part of the lemma, thus concluding the proof.

**Proof of Lemma 16.** Let $Z$ be a standard Gaussian random variable. Then, using a standard Gaussian tail lower bound,

$$\mathbb{P} \left[ |w_{t-1}| \geq \frac{2 \sigma}{5} \right] = \mathbb{P} \left[ |Z| \geq \frac{2}{5} \right] \geq \frac{17}{25}.$$

Now, recall that $x_t = ax_{t-1} + bu_{t-1} + w_{t-1}$ and notice that, as the learning algorithm is deterministic, both $x_{t-1}$ and $u_{t-1}$ are determined conditioned on $x_1, \ldots, x_{t-1}$. We next aim to lower bound $\mathbb{P}(|x_t| > 2\sigma/5 \mid x_1, \ldots, x_{t-1})$ which we claim that, as $w_{t-1}$ is a zero-mean Gaussian random variable, is minimized when $ax_{t-1} + bu_{t-1} = 0$. Therefore,

$$\mathbb{P} \left[ |x_t| > \frac{2 \sigma}{5} \mid x_1, \ldots, x_{t-1} \right] \geq \mathbb{P} \left[ |w_{t-1}| > \frac{2 \sigma}{5} \right] \geq \frac{17}{25}.$$

Denote by $I_t = 1 \{ |x_t| > 2\sigma/5 \}$. Then, by Azuma’s concentration inequality we have that with probability at least 7/8,

$$\sum_{t=1}^{T} I_t \geq \sum_{t=1}^{T} \mathbb{E}[I_t \mid x_1, \ldots, x_{t-1}] - \sqrt{\frac{T}{2} \log 8} \geq \frac{17}{25} T - \sqrt{2T} \geq \frac{2}{3} T,$$

where for the last inequality we used the assumption that $T \geq 12000$.

**Proof of Lemma 15.** First, using Pinsker’s inequality yields

$$\text{TV}(\mathbb{P}_+ \mid x(T)), \mathbb{P}_- \mid x(T)) \leq \sqrt{\frac{1}{2} \text{KL}(\mathbb{P}_+ \mid x(T)) || \mathbb{P}_- \mid x(T))},$$

and by the chain rule of the KL divergence

$$\text{KL}(\mathbb{P}_+ \mid x(T)) || \mathbb{P}_- \mid x(T)) = \sum_{t=1}^{T} \mathbb{E} \left[ \text{KL}(\mathbb{P}_+ \mid x_t \mid x(t-1)) || \mathbb{P}_- \mid x_t \mid x(t-1)) \right]. \quad (31)$$

Next, let $E_+$ and $E_-$ denote the expectations conditioned on whether $\chi = 1$ or $\chi = -1$ respectively. Observe that as the learning algorithm is deterministic, the sequence of actions $u_1, \ldots, u_{t-1}$ is determined given $x_{(t-1)}$.}
As such, given \( x^{(t-1)} \), the random variable \( x_t \) is Gaussian with variance \( \sigma^2 \) and expectation \( ax_{t-1} + \sqrt{\epsilon} u_{t-1} \). Therefore, by a standard formula for the KL divergence between Gaussian random variables, we have

\[
\text{KL}(\mathbb{P}_+[x_t \mid x^{(t-1)}] \parallel \mathbb{P}_-[x_t \mid x^{(t-1)}]) = \frac{1}{2\sigma^2} E_+ \left[ (ax_{t-1} + \sqrt{\epsilon} u_{t-1}) - (ax_{t-1} - \sqrt{\epsilon} u_{t-1}) \right]^2
\]

\[
= \frac{1}{2\sigma^2} E_+ \left( 2\sqrt{\epsilon} u_{t-1} \right)^2
\]

\[
= \frac{2\epsilon}{\sigma^2} E_+ [u_{t-1}^2],
\]

unless \( t = 1 \) in which case \( \text{KL}(\mathbb{P}_+[x_1] \parallel \mathbb{P}_-[x_1]) = 0 \) since \( x_1 \) is fixed. Using this bound in Eq. (31) and substituting into Eq. (30) yields

\[
\text{TV}(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \sqrt{\frac{\epsilon}{\sigma^2}} E_+ \left[ \sum_{t=1}^T u_t^2 \right].
\]

Similarly, switching the roles of \( \mathbb{P}_+ \) and \( \mathbb{P}_- \), we get the bound

\[
\text{TV}(\mathbb{P}_+[x^{(T)}], \mathbb{P}_-[x^{(T)}]) \leq \sqrt{\frac{\epsilon}{\sigma^2}} E_- \left[ \sum_{t=1}^T u_t^2 \right].
\]

Averaging the two inequalities, using the concavity of the square root, and since \( E[\cdot] = \frac{1}{2} E_+ [\cdot] + \frac{1}{2} E_- [\cdot] \), we obtain our claim.

\[\blacksquare\]

### D Technical Lemmas

#### D.1 Noise Bounds

The following theorem is a variant of the Hanson-Wright inequality (Hanson and Wright, 1971; Wright, 1973) which can be found in Hsu et al. (2012).

**Theorem 33.** Let \( x \sim \mathcal{N}(0, I) \) be a Gaussian random vector,, let \( A \in \mathbb{R}^{m \times n} \) and define \( \Sigma = A^T A \). Then we have that

\[
\mathbb{P} \left( \|Ax\|^2 > \text{Tr}(\Sigma) + 2\sqrt{\text{Tr}(\Sigma)z} + 2\|z\|z \right) \leq \exp(-z), \quad \text{for all } z \geq 0.
\]

The following lemma is a direct corollary of Theorem 33.

**Lemma 34.** Let \( w_t \in \mathbb{R}^d \) for \( t = 1, \ldots, T \) be i.i.d. random variables with distribution \( \mathcal{N}(0, \sigma^2 I) \). Suppose that \( T > 2 \), then with probability at least \( 1 - \delta \) we have that

\[
\max_{1 \leq t \leq T} \|w_t\| \leq \sigma \sqrt{5d \log \frac{T}{\delta}}.
\]

**Proof.** Consider Theorem 33 with \( A = \sigma I \) and thus \( \Sigma = \sigma^2 I \). We then have that \( \text{Tr}(\Sigma) = d\sigma^2, \|z\| \leq \sigma^2 \) and \( \text{Tr}(\Sigma^2) \leq \|z\| \text{Tr}(\Sigma) \leq d\sigma^4 \). We conclude that for \( z \geq 1 \) we have that

\[
\text{Tr}(\Sigma) + 2\sqrt{\text{Tr}(\Sigma^2)z} + 2\|z\|z \leq \sigma^2 d + 2\sigma^2 \sqrt{dz} + 2\sigma^2 z \leq 5\sigma^2 dz.
\]

Now, for \( x \sim \mathcal{N}(0, I) \) we have that \( w_t \overset{d}{=} Ax \) (equals in distribution). We thus have that for \( z \geq 1 \)

\[
\mathbb{P} \left( \|w_t\| > \sigma \sqrt{5d z} \right) \leq \mathbb{P} \left( \|Ax\| > \sqrt{\text{Tr}(\Sigma) + 2\sqrt{\text{Tr}(\Sigma^2)z} + 2\|z\|z} \right) \leq \exp(-z).
\]

Denoting \( z = \log \frac{T}{\delta} \), the assumption \( T > 2 \) ensures that \( z \geq 1 \) and thus \( \mathbb{P} \left( \|w_t\| > \sigma \sqrt{5d \log \frac{T}{\delta}} \right) \leq \frac{1}{T} \).

Performing a union bound over \( 1 \leq t \leq T \) we conclude that

\[
\mathbb{P} \left( \max_{1 \leq t \leq T} \|w_t\| > \sigma \sqrt{5d \log \frac{T}{\delta}} \right) \leq \delta,
\]

and taking the complement we obtain the desired.
Lemma 35 (Expected maximum noise). Let \( E \) be an event such that \( \mathbb{P}(E) \leq \delta \) for some \( \delta \in [0,1] \) and let \( w_t \in \mathbb{R}^d \) for \( t = 1, \ldots, T \) be i.i.d. random variables with distribution \( \mathcal{N}(0, \sigma^2 I) \). Suppose \( T > 2 \), then we have that

1. \( \mathbb{E}\left[ \max_{1 \leq t \leq T} \|w_t\|^2 \right] \leq 5\sigma^2 d \log 3T; \)

2. \( \mathbb{E}\left[ \mathbb{I}\{E\} \max_{1 \leq t \leq T} \|w_t\|^2 \right] \leq 5\sigma^2 d \delta \log \frac{3T}{\delta}. \)

**Proof.** Recall that from Lemma 34 we have that for all \( x > 5\sigma^2 d \log T \)

\[
\mathbb{P}\left( \max_{1 \leq t \leq T} \|w_t\|^2 > x \right) \leq T \exp\left( -\frac{x}{5\sigma^2 d} \right).
\]

Applying the tail sum formula we get that

\[
\mathbb{E}\left[ \max_{1 \leq t \leq T} \|w_t\|^2 \right] = \int_0^\infty \mathbb{P}\left( \max_{1 \leq t \leq T} \|w_t\|^2 > x \right) dx \\
\leq 5\sigma^2 d \log T + \int_{5\sigma^2 d \log T}^\infty T \exp\left( -\frac{x}{5\sigma^2 d} \right) dx \\
\leq 5\sigma^2 d \log 3T,
\]

proving the first part of the lemma. For the second part notice that \( \mathbb{P}\left( \mathbb{I}\{E\} \max_{1 \leq t \leq T} \|w_t\|^2 > x \right) \leq \min\{\mathbb{P}(E), \mathbb{P}\left( \max_{1 \leq t \leq T} \|w_t\|^2 > x \right)\}. \) So, applying the tail sum formula we get that

\[
\mathbb{E}\left[ \mathbb{I}\{E\} \max_{1 \leq t \leq T} \|w_t\|^2 \right] = \int_0^\infty \mathbb{P}\left( \mathbb{I}\{E\} \max_{1 \leq t \leq T} \|w_t\|^2 > x \right) dx \\
\leq \int_0^{5\sigma^2 d \log \frac{T}{\delta}} \mathbb{P}(E) dx + \int_{5\sigma^2 d \log \frac{T}{\delta}}^\infty \mathbb{P}\left( \max_{1 \leq t \leq T} \|w_t\|^2 > x \right) dx \\
\leq 5\sigma^2 d \delta \log \frac{T}{\delta} + \int_{5\sigma^2 d \log \frac{T}{\delta}}^\infty T \exp\left( -\frac{x}{5\sigma^2 d} \right) dx \\
= 5\sigma^2 d \delta \left( 1 + \log \frac{T}{\delta} \right) \\
\leq 5\sigma^2 d \delta \log \frac{3T}{\delta},
\]

proving the second part and concluding the proof.

### D.2 Estimation auxiliary lemmas

The following is due to Cohen et al. (2019). Here we state the result for a general sequence of conditionally Gaussian vectors but the proof follows without change.

**Lemma 36** (Theorem 20 of Cohen et al., 2019). Let \( z_t \) for \( t = 1, 2, \ldots \) be a sequence random variables that is adapted to a filtration \( \{F_t\}_{t=1}^\infty \). Suppose that \( z_t \) are conditionally Gaussian on \( F_{t-1} \) and that \( \mathbb{E}[z_t z_T^T | F_{t-1}] \succeq \sigma^2 z z_T^T \) for some fixed \( \sigma^2 > 0 \). Then for \( t \geq 200d \log \frac{120T}{\delta} \) we have that with probability at least \( 1 - \delta \)

\[
\sum_{s=1}^t z_s z_s^T \succeq \frac{t\sigma^2}{40} I.
\]

**Lemma 37.** Let \( z_s \in \mathbb{R}^m \) for \( s = 1, \ldots, t - 1 \) be such that \( \|z_s\|^2 \leq \lambda \). Define \( V_t = \lambda I + \sum_{s=1}^{t-1} z_s z_s^T \) then we have that

\[
\log \frac{\det(V_t)}{\det(V_1)} \leq m \log t.
\]
Proof. First we have that
\[ \|V_t\| \leq \lambda + \sum_{s=1}^{t-1} \|z_s z_s^T\| \leq \lambda + \sum_{s=1}^{t-1} \|z_s\|^2 \leq \lambda t. \]
Now, recall that \( \det(V_t) \leq \det(\|V_t\|^m) \) and so we have that
\[ \log \frac{\det(V_t)}{\det(V_1)} \leq \log \frac{\det(\|V_t\|^m)}{\lambda^m} \leq \log \frac{\lambda^{mt}}{\lambda^m} = m \log t, \]
as desired. \[\square\]

### D.3 Strong Stability Lemmas

The following lemma bounds the norm of the state when playing a strongly stable controller. Its proof adapts techniques from Cohen et al. (2019).

**Lemma 38.** Suppose \( K \) is a \((\kappa, \gamma)\)-strongly stable controller and \( s_0, s_1 \) are integers such that \( 1 \leq s_0 < s_1 \leq T \). Let \( x_s \) for \( s = s_0, \ldots, s_1 \) be the sequence of states generated under the control \( K \) starting from \( x_{s_0} \), i.e., \( x_{s+1} = (A_s + B_s K)x_s + w_s \) for all \( s_0 \leq s < s_1 \). Then we have that
\[ \|x_t\| \leq \kappa (1 - \gamma)^{t-s_0} \|x_{s_0}\| + \kappa \max_{1 \leq t \leq T} \|w_t\|, \quad \text{for all } s_0 \leq t \leq s_1. \]

**Proof.** Denote \( M = A_s + B_s K \) then for \( s_0 < t \leq s_1 \) we have that \( x_t = M x_{t-1} + w_{t-1} \) and by expanding this equation we have
\[ x_t = M^{t-s_0} x_{s_0} + \sum_{s=s_0}^{t-1} M^{t-(s+1)} w_s. \]
Recall that by strong stability we have that
\[ \|M^s\| = \|H L^s H^{-1}\| \leq \kappa (1 - \gamma)^s. \]
To ease notation denote \( W = \max_{1 \leq t \leq T} \|w_t\| \). Then for \( s_0 < t \leq s_1 \) we have that
\[ \|x_t\| \leq \|M^{t-s_0}\| \|x_{s_0}\| + \sum_{s=s_0}^{t-1} \|M^{t-(s+1)}\| \|w_s\| \]
\[ \leq \kappa (1 - \gamma)^{t-s_0} \|x_{s_0}\| + \sum_{s=s_0}^{t-1} \kappa (1 - \gamma)^{t-(s+1)} W \]
\[ \leq \kappa (1 - \gamma)^{t-s_0} \|x_{s_0}\| + \frac{\kappa W}{\gamma}. \]
\[\square\]

The following lemma bounds the norm of the state when playing a sequence of strongly stable controllers.

**Lemma 39.** Suppose \( K_1, \ldots, K_t \) are \((\kappa, \gamma)\)-strongly stable controllers and \( \{t_i\}_{i=1}^{t_1+1} \) are integers such that \( 1 \leq t_1 < \ldots < t_{i+1} \leq T \). Let \( x_t \) for \( t = t_1, \ldots, t_{i+1} \) be the sequence of states generated by starting from \( x_{t_i} \) and playing controller \( K_i \) at times \( t_i \leq t < t_{i+1} \), i.e., \( x_{t+1} = (A_s + B_s K_i)x_t + w_t \) for all \( t_i \leq t < t_{i+1} \). Denote \( \tau = \min_i \{t_{i+1} - t_i\} \) and suppose that \( \tau \geq \gamma^{-1} \log(2\kappa) \), then we have that
\[ \|x_t\| \leq 3 \kappa \max \left\{ \frac{1}{2} \|x_{t_i}\|, \frac{\kappa}{\gamma} \max_{1 \leq t \leq T} \|w_t\| \right\}, \quad \forall \ t_i \leq t \leq t_{i+1}. \]

**Proof.** For \( 0 < \gamma \leq 1 \) it is a well known fact that \( \gamma \leq -\log 1 - \gamma \). Plugging this into the lower bound on \( \tau \) and rearranging we get that \( \kappa (1 - \gamma)^\tau \leq \frac{1}{2} \). Now, applying Lemma 38 with \( s_0 = t_i \) and \( s_1 = t_{i+1} \), and
taking \( t = t_i + 1 \) we have that

\[
\|x_{t_i+1}\| \leq \kappa (1 - \gamma)^{t_i+1-t_i} \|x_{t_i}\| + \frac{\kappa}{\gamma} W
\]

\[
\leq \kappa (1 - \gamma)^T \|x_{t_i}\| + \frac{\kappa}{\gamma} W
\]

\[
\leq \frac{1}{2} \|x_{t_i}\| + \frac{\kappa}{\gamma} W,
\]

and solving this difference equation we get that

\[
\|x_t\| \leq \frac{2\kappa}{\gamma} W + \left( \|x_{t_i}\| - \frac{2\kappa}{\gamma} W \right) 2^{1-i} \leq \max \left\{ \|x_{t_i}\|, \frac{2\kappa}{\gamma} W \right\}.
\]

Plugging this result back into Lemma 38 we have that for \( t_i < t \leq t_i + 1 \)

\[
\|x_t\| \leq \kappa (1 - \gamma)^{t-t_i} \max \left\{ \|x_{t_i}\|, \frac{2\kappa}{\gamma} W \right\} + \frac{\kappa}{\gamma} W
\]

\[
\leq \kappa \max \left\{ \|x_{t_i}\|, \frac{2\kappa}{\gamma} W \right\} + \frac{\kappa}{\gamma} W
\]

\[
\leq \kappa \max \left\{ \frac{3}{2} \|x_{t_i}\|, \frac{3\kappa}{\gamma} W \right\},
\]

where the last inequality used the fact that \( \kappa \geq 1 \). This is true for all \( i \) and thus for all \( t_1 \leq t \leq t_{i+1} \).

The next two lemmas require the following well known result in linear control theory (see, e.g., Bertsekas, 1995). We have that \( J(K) = \sigma^2 \text{Tr}(P) \) where \( P \) is a positive definite solution of

\[
P = Q + K^T RK + (A_* + B_* K)^T P(A_* + B_* K).
\]

The following lemma relates the expected cost of playing controller \( K \) for \( t \) rounds to the infinite horizon cost of \( K \).

**Lemma 40.** Suppose \( K \) is a \((\kappa, \gamma)\)-strongly stable controller and let \( x_s \) for \( s = 1, \ldots, t \) be the sequence of states generated under the control \( K \) starting from \( x_1 \), i.e., \( x_{s+1} = (A_* + B_* K)x_s + w_s \) for all \( 1 \leq s < t \). Then we have that

\[
E \left[ \sum_{s=1}^{t} x_s^T (Q + K^T RK)x_s \bigg| x_1 \right] \leq tJ(K) + \frac{201\kappa^4}{\gamma} \|x_1\|^2.
\]

**Proof.** To ease notation, assume, without loss of generality, that \( x_1 \) is deterministic. We thus omit the conditioning on \( x_1 \) in all expectation arguments.

First, recall that \( x_{s+1} = (A_* + B_* K)x_s + w_s \) and \( J(K) = \sigma^2 \text{Tr}(P) \) where \( P \) satisfies Eq. (32). Then we have that

\[
E[x_{s+1}^T P x_{s+1}] = E \left[ ((A_* + B_* K)x_s + w_s)^T P((A_* + B_* K)x_s + w_s) \right]
\]

\[
= E \left[ (A_* + B_* K)x_s^T P((A_* + B_* K)x_s) \right] + E[w_s^T P w_s]
\]

\[
= E \left[ x_s^T (A_* + B_* K)^T P(A_* + B_* K)x_s \right] + J(K).
\]

Now, multiplying Eq. (32) by \( x_s \) from both sides and taking expectation we get that

\[
E[x_s^T P x_s] = E[x_s^T (Q + K^T RK)x_s] + E[x_s^T (A_* + B_* K)^T P(A_* + B_* K)x_s]
\]

\[
= E[x_s^T (Q + K^T RK)x_s] + E[x_{s+1}^T P x_{s+1}] - J(K),
\]

31
We conclude the proof by bounding $\|P\|$. To that end, recall that the strong stability of $K$ implies that $A_s + B_s K = HLH^{-1}$ where $\|L\| \leq 1 - \gamma$ and $\|H\| \|H^{-1}\| \leq \kappa$. Applying Eq. (32) recursively we then have that

$$
\|P\| = \sum_{s=0}^{\infty} \left( (A_s + B_s K)^s \right)^T (Q + K^T RK) (A_s + B_s K)^s
$$


$$
= \sum_{s=0}^{\infty} (HL^s H^{-1})^T (Q + K^T RK) HL^s H^{-1}
$$


$$
\leq \|H\|^2 \|H^{-1}\|^2 \|Q + K^T RK\| \sum_{s=0}^{\infty} \|L\|^2 s
$$


$$
\leq 2\alpha_1 \kappa^4 \sum_{s=0}^{\infty} (1 - \gamma)^s = \frac{2\alpha_1 \kappa^4}{\gamma},
$$

thus concluding the proof.

The following lemma relates the infinite horizon cost of a controller to its strong stability parameters. Its proof is an adaptation of Lemma 18 in Cohen et al. (2019) that fits our assumptions.

**Lemma 41.** Suppose $J(K) < J$ then $K$ is $(\kappa, \gamma)$—strongly stable with $\kappa = \sqrt{\frac{J}{\alpha_0 \sigma^2}}$ and $\gamma = \frac{\alpha_0 \sigma^2}{2J}$.

**Proof.** Recall that $J(K) = \sigma^2 \text{Tr}(P)$ where $P$ satisfies Eq. (32). Using the bound $J(K) \leq J$ we have that $\text{Tr}(P) \leq J/\sigma^2$ and thus also that $P \leq (J/\sigma^2)I$. Recalling that $Q \geq \alpha_0 I$ we get that $Q \geq \frac{\alpha_0 \sigma^2}{2J} = 2\gamma P$. Recalling that $R$ is positive definite and plugging back into Eq. (32) we get that

$$
P \geq 2\gamma P + (A_s + B_s K)^T P (A_s + B_s K),
$$

rearranging the equation we get that

$$
P^{-1/2} (A_s + B_s K)^T P (A_s + B_s K) P^{-1/2} \preceq (1 - 2\gamma)I.
$$

Now, denote $H = P^{-1/2}$ and $L = P^{1/2} (A_s + B_s K) P^{-1/2}$ and notice that indeed $HLH^{-1} = A_s + B_s K$. Plugging into the above we get that

$$
P^{-1/2} (A_s + B_s K)^T P (A_s + B_s K) P^{-1/2} = H (HLH^{-1})^T H^{-1} (HLH^{-1}) H = L^T L \preceq (1 - 2\gamma)I,
$$

and thus $\|L\| \leq \sqrt{1 - 2\gamma} \leq 1 - \gamma$. Now recall that $P \preceq (J/\sigma^2)I$ and thus $\|H^{-1}\| = \|P^{1/2}\| \leq \sqrt{J/\sigma^2}$. Going back to Eq. (32) we also have that $P \preceq Q \geq \alpha_0 I$ and thus $\|H\| = \|P^{-1/2}\| \leq \sqrt{1/\alpha_0}$. All together, we get that $\|H\| \|H^{-1}\| \leq \sqrt{J/\alpha_0 \sigma^2} = \kappa$. Finally, recall that $R \geq \alpha_0 I$ and thus going back to Eq. (32) we have that $P \preceq K^T RK \geq \alpha_0 K^T K$ and thus $\|K\| \leq \sqrt{\|P\|/\alpha_0} \leq \sqrt{J/\alpha_0 \sigma^2} = \kappa$, as desired.

The following lemma relates system parameter estimation bounds to properties of the resulting greedy controller.
Lemma 42. Let $A \in \mathbb{R}^{d \times d}, B \in \mathbb{R}^{d \times K}$ and denote $\Delta = \max\{\|A - A_*\|, \|B - B_*\|\}$. Taking $K = K(A, B, Q, R)$ and denoting $\kappa = \sqrt{\frac{\nu + C_0 \varepsilon_0^2}{\alpha_0 \sigma^2}}$ and $\gamma = \frac{1}{2\kappa^2}$ we have that

1. If $\Delta \leq \varepsilon_0$ then $K$ is $(\kappa, \gamma)$-strongly stable;

2. If $\Delta \leq \min\{\varepsilon_0, \frac{\mu_*}{4\kappa C_0}\}$ then $KK^T \succeq K_* K_*^T - \frac{\mu}{2} I$ and $K_* K_*^T \succeq KK^T - \frac{\mu}{2} I$;

3. If $\Delta \leq \min\{\varepsilon_0, \frac{\mu_*}{4\kappa C_0}\}$ then $KK^T \succeq \frac{\mu_*}{2} I$.

Proof. First, if $\Delta \leq \varepsilon_0$ we can invoke Lemma 4 to get that $J(K) \leq J_* + C_0 \varepsilon_0^2 \leq \nu + C_0 \varepsilon_0^2$ and so by Lemma 41, $K$ is $(\kappa, \gamma)$-strongly stable, proving the first part of the lemma.

Second, if $\Delta \leq \min\{\varepsilon_0, \frac{\mu_*}{4\kappa C_0}\}$ then we can invoke Lemma 4 to get that $\|K - K_*\| \leq \frac{\mu}{4\kappa}$. Moreover, by the first claim of the lemma, $K, K_*$ are $(\kappa, \gamma)$-strongly stable and thus upper bounded by $\kappa$. Combining the above we get that

$$KK^T = K_* K_*^T - \frac{1}{2} \left((K_* + K)(K_* - K)^T + (K_* - K)(K_* + K)^T\right)$$

$$\geq K_* K_*^T - (\|K_*\| + \|K\|)\|K_* - K\| I$$

$$\geq K_* K_*^T - \frac{2\kappa \mu}{4\kappa} I = K_* K_*^T - \frac{\mu}{2} I,$$

and reversing the roles of $K$ and $K_*$ in the above yields $K_* K_*^T \succeq KK^T - \frac{\mu}{2} I$, thus proving the second part of the lemma.

Finally, if $\Delta \leq \min\{\varepsilon_0, \frac{\mu_*}{4\kappa C_0}\}$, then recalling that $K_* K_* \succeq \mu_* I$ and continuing from the second part we get that

$$KK^T \succeq K_* K_*^T - \frac{\mu_*}{2} I \succeq \mu_* I - \frac{\mu_*}{2} I = \frac{\mu_*}{2} I,$$

thus concluding the third and final part of the lemma.