Relativistic Coulomb $S$-factor of Two Spinor Particles with Arbitrary Masses

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Abstract

A new resummation of the $S$-factor of a composite system of two relativistic spin-1/2 particles of arbitrary masses interacting via a Coulomb-like chromodynamical potential is presented. The analysis is performed in the framework of a relativistic quasipotential approach based on the Hamiltonian formulation of the covariant quantum field theory in the relativistic configuration representation. The pseudoscalar, vector, and pseudovector systems are considered and the behaviour of the $S$-factor near the threshold and in the relativistic limit is investigated in detail. The spin dependence of the $S$-factors is discussed as well. It is argued that at the threshold the contribution of spins significantly reduces the Sommerfeld effect, while at ultrarelativistic velocities their role diminishes and the $S$-factor becomes basically the same as for the spinless systems. A connection between the new and previously obtained $S$-factors for spinless particles of arbitrary masses and for relativistic spinor particles of equal masses is established.

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I. INTRODUCTION

It is well known from quantum electrodynamics (QED) that the so-called Coulomb resummation factor plays an important role in describing the system of two charged particles near the threshold [1]. The resummation performed on the basis of the nonrelativistic Schrödinger equation with the Coulomb potential, \( V(r) = -\alpha / r \), where \( \alpha \) is the fine structure constant, leads to the Sommerfeld–Gamov–Sakharov factor [2–4] which, for two particles of equal masses, reads as:

\[
S_{nr} = \frac{X_{nr}}{1 - \exp(-X_{nr})}, \quad X_{nr} = \frac{\pi \alpha}{v_{nr}}, \tag{1}
\]

where \( v_{nr} \) denotes the velocity of each particle in the center-of-mass frame. At small values of \( \alpha / v_{nr} \), the \( S \)-factor (1) behaves as \( S_{nr} \sim 1 + \pi \alpha / (2 v_{nr}) \), which implies that at low velocities the corresponding cross sections increase. As originally mentioned by Sommerfeld [2], this enhancement is directly related to the wave function of the relative motion in the Coulomb field evaluated at the origin, \( |\psi(0)|^2 \), i.e. to the probability to find two interacting particles close to each other. Such an increase of the \( S \)-factor is often referred to as the Sommerfeld effect. A similar phenomenon was also predicted and then confirmed by Gamov [3] in nuclear reactions when interacting particles overcome the Coulomb barrier and by Sakharov [4] in electron-positron pair production. Intensive studies of the Sommerfeld-effect demonstrate the crucial role it plays in understanding a large variety of inelastic processes with heavy particles moving slowly in a thermal environment, see, e.g., Refs. [5–11]. These processes are encountered in many physical situations, and nowadays there is renewed interest in detailed investigations of the Sommerfeld effect. A classic example is given by astrophysical nuclear reactions taking place within the electromagnetic plasma of stars. In particle physics, one may mention heavy Dark Matter particles produced by pair annihilation in the early Universe. If Dark Matter in the Universe contains interactions with the Standard Model particles, the annihilation may give visible imprints in the cosmic rays [5]. Within quantum chromodynamics (QCD), an increase in the \( S \)-factor may occur in heavy quark and anti-quark pair annihilation into light quarks and gluons within a quark-gluon plasma environment generated in heavy ion collision experiments. Bound states contribute to the annihilation process, and enhancement factors of up to \( \sim 100 \) can be encountered, see, e.g., Ref. [6]. Other examples where the Sommerfeld effect is crucial, include threshold production of heavy states at colliders and partial decay rates when the products have large
phase space suppression. The Sommerfeld effect in boosting the dark matter signal with Coulomb resonances has been considered, to some extent, in Ref. [7, 8], where the $S$-factor was computed within a nonrelativistic approach for attractive and repulsive Coulomb and Yukawa (between neutral particles) potentials. Details of the theoretical consideration of the Sommerfeld enhancement in Dark Matter can be found in, e.g., Refs. [9, 10].

Another manifestation of the enhancement of the Sommerfeld-Gamelow-Sakharov factor can be clearly seen in abundant creations of di-lepton pairs with low masses in heavy-ion collisions [11], which is directly related to the physics at the NICA and LHC colliders. It should be noted that most of the theoretical studies of the $S$-factor have been performed within nonrelativistic approaches originating from the Schrödinger equation. Coming back to Eq. (1) one shall stress that, in spite of the expansion of the $S$-factor in a power series $(\alpha/v)^n$ with respect to the coupling constant $\alpha$ reproduces the threshold singularities of the Feynman diagrams, in the threshold-near region, $v \to 0$, the parameter $\alpha/v$ can no longer be adequate to cut off the perturbative series. Consequently, the $S$-factor should be taken into account in its entirety.

A similar situation also arises also in QCD because in describing a quark-antiquark system near the threshold, $s_{th} = (m_q + m_{\bar{q}})^2$, the expansion parameter $\alpha_s/v$ becomes singular when $v \to 0$ [12, 13]. Here $\alpha_s$ denotes the strong coupling constant, and the quark velocity $v$ for the case of equal masses, $m_q = m_{\bar{q}} = m$, reads as

$$v = \sqrt{1 - \frac{4m^2}{s}},$$

where $\sqrt{s}$ is the total energy of the considered two particles in their c.m. system. Therefore, in order to obtain a meaningful result, threshold singularities of the form $(\alpha_s/v)^n$ should be summed, which can be achieved by solving the corresponding relativistic problem. It is important to note that the problem of accounting for the relativistic effects in a few body system is a more general and longstanding task. Up to now there are no reliable relativistic equations derived from the first principles to describe few body bound and/or continuum states near the threshold. For this reason, one is forced to use phenomenological or quasi-phenomenological formalisms to relativistically describe these systems. One can mention several approaches defining the relativistic wave functions of few-body systems. One of the approaches is based on the fully covariant and Lorentz-invariant Bethe-Salpeter (BS) formalism [14]. As is well known, in Minkowski space the BS equation is singular, hence in order
to avoid uncertainties in numerical calculations, one usually performs the Wick rotation of the equation to Euclidean space and solves the BS equation along the imaginary energy axis. Such a representation is extremely useful for covariant calculations of diagonal matrix elements which are the same in both Euclidean and Minkowski spaces. For instance, within the BS formalism one can relativistically describe the main properties of few-body bound states such as glueballs [15, 16], quark-antiquark systems (mesons) [17–19], two-nucleon bound state (deuteron) [20], etc. However, not all physical observables can be obtained from calculations in Euclidean space by the inverse Wick rotation back to Minkowski space. Consequently, one needs to solve the BS equation directly in Minkowski space by finding new mathematical methods allowing one to overcome the difficulties resulting from the singularities of the BS equation, c.f. [21, 22]. Also, within the BS formalism one encounters difficulties related to the problem of probabilistic interpretation of the BS amplitudes and BS vertex functions. This problem can be partially solved by considering different approximations to the BS equation like the Gross Covariant Spectator Equation [23].

Another strategy of relativization of calculations consists in elaborating relativistic analogues of the usual three-dimensional Schrödinger equation. These approaches are mainly based on the 1949 seminal paper [24], where P.A.M. Dirac proposed several peculiar representations of the Poincaré group, in strict relation with the choice of possible space-time hyper surfaces without a time-like direction. Among them the most popular is the approach based on the Light Front dynamics which allows for a specific description of the dynamics of relativistic interacting systems [25, 26].

Eventually, a tempting and rather popular approach is the one based on the quasipotential (QP) equations in Lobachevsky space [27–30]. The QP equations are differential ones with the structure very similar to the Schrödinger equation; however, the interaction quasipotential is now energy-dependent. Note that at vanishing curvature the solution of the QP equation reduces to the known nonrelativistic approaches. Albeit the QP wave functions, being three-dimensional, admit a probabilistic interpretation, they at the same time have the main advantages (renormalizability, analyticity, etc.) of the completely covariant field theory, see e.g. Ref. [29].

For the first time, the relativization of the S-factor (1) in the case of two particles with equal masses was obtained in Refs. [31, 32], and the result is reduced to the replacement of the nonrelativistic velocity $v_{nr}$ in Eq. (1) by the relativistic one, $v_{nr} \rightarrow v$. Exactly the
same form of the $S$-factor but with the change $v_{nr} \rightarrow v/(1 + v^2)$ was later on suggested in Ref. [33]. Note here that the relativistic generalization of the $S$-factor is obviously not unique, since there are numerous ways of expressing the nonrelativistic velocity in terms of the invariant energy $\sqrt{s}$. Another form of the relativistic generalization of the $S$-factor was obtained in Ref. [34]. The relativistic $S$-factor for two particles of arbitrary masses was for the first time presented in Ref. [35] (see also Refs. [36, 37]). It was considered within the framework of a version of the relativistic quantum mechanics, on the basis of the Schrödinger equation for the wave function $\Psi(t, x_1, x_2)$ in a specific frame of reference, $p_2 = -p_1 m_2/m_1$, and treating $p_{1,2}$ as differential operators.

In the present paper, we employ the relativistic quasipotential (RQP) approach [28, 29, 38, 39], which, in our opinion, is the most suitable one for the goal of investigations of the effects of relativisation of the $S$-factor for systems consisting of two relativistic, spin 1/2, particles with arbitrary masses $m_1$ and $m_2$. The pseudoscalar, vector, and pseudovector systems are considered, and the behavior of the corresponding $S$-factors is analyzed in the nonrelativistic, relativistic and ultrarelativistic limits.

Our paper is organized as follows. In the next section, we present the $S$-factor calculated within the RQP approach for two spinless relativistic particles. Explicit, analytical expressions for the $S$-factor of two spinor particles with equal and arbitrary masses are presented in Secs. III and IV, respectively. The results of numerical calculations for pseudo-scalar, vector and pseudo-vector systems are presented in Secion V. The conclusions and summary are collected in Secs. VI and VII.

II. RELATIVISTIC $S$-FACTOR FOR SPINLESS PARTICLES

In this section we consider the $S$-factor corresponding to systems of two relativistic spinless particles with equal masses $m_1 = m_2 = m$. It is important to note that the use of the RQP approach for our task is justified by that the BS amplitude $\Phi_{BS}(x), x = (x_0, \mathbf{x})$, that parameterizes, e.g., the Drell ratio $R(s)$ in QCD (see also Ref. [40]), is evaluated at $x = 0$ [41]. Consequently, the BS amplitude can be related to the RQP wave function in momentum space, $\Psi_\chi(p)$, and in the configuration representation, $\psi_\chi(r)$, as

$$\Phi_{BS}(x = 0) = \frac{1}{(2\pi)^3} \int d^3p \Psi_\chi(p) = \psi_\chi(r)\bigg|_{r = i\lambda}, \quad (3)$$
where $\chi$ is the rapidity related to the total c.m. energy of particles $\sqrt{s}$ as $\sqrt{s} = 2m \cosh \chi$, $\lambda = 1/m$ is the Compton wavelength of the particle of mass $m$, $d\Omega_p = (mdp)/p_0$ is the invariant space volume in the Lobachevsky space realized on the hyperboloid $p_0^2 - p^2 = q_0^2 - q^2 = m^2$. According to Eq. (3), the $S$-factor within the RQP approach is defined in terms of the corresponding wave function in the continuum, $\psi_\chi(r)$, as follows [42]

$$S_{\text{RQP}}(\chi) = \lim_{r \to i\lambda} |\psi_\chi(r)|^2. \tag{4}$$

The nonrelativistic replacement of the amplitude $\chi_{\text{BS}}(x)$ by the wave function, which obeys the Schrödinger equation with the Coulomb potential, leads to formula (1).

Based on the RQP approach, Milton and Solovtsov obtained the following expression [42]:

$$S(\chi) = \frac{X(\chi)}{1 - \exp[-X(\chi)]}, \quad X(\chi) = \frac{\pi \tilde{\alpha}}{\sinh \chi}, \tag{5}$$

where $\tilde{\alpha}$ denotes the coupling constant in the Coulomb-like potential $V(r) = -\tilde{\alpha}/r$. The use of Eq. (5) in QCD requires the replacement of $\tilde{\alpha} \rightarrow 4\alpha_s/3$, where $4/3$ is due to the SU(3) color factor. Here, it is worth emphasizing that in the above mentioned method [42], the possibility of transformation of the RQP equation from the momentum space to the relativistic configurational representation in the case of two particles of equal masses [38] has been widely employed. Note also that the Coulomb quasipotential formally has the same form as the nonrealistic Coulomb potential since its behavior in the momentum Lobachevsky space corresponds to the static quark-antiquark potential $V_{q\bar{q}} \sim \tilde{\alpha}_s(Q^2)/Q^2$ for which $\tilde{\alpha}_s(Q^2) \sim 1/\ln Q^2$ plays a role of effective charge [43]. Thereby, one accumulates a dominant effects induced by the running QCD coupling.

The function $X(\chi)$ in Eq. (5) can be expressed in terms of the velocity $v$ defined by Eq. (2) as $X(v) = \pi \tilde{\alpha} \sqrt{1 - v^2}/v$. It becomes obvious that the shape of the $S$-factor in the relativistic case remains the same as in Eq. (1), albeit with a replacement

$$v_{nr} \rightarrow \frac{v}{\sqrt{1 - v^2}}.$$

The presence of the square root $\sqrt{1 - v^2}$ in the denominator is essential since it provides the correct expected relativistic limit ($v \rightarrow 1$) of the $S$-factor in Eq. (5) [42, 44]. Notice that the relativistic limits of the $S$-factors from Refs. [33, 34] differ substantially from the relativistic limit ($v \rightarrow 1$) of the $S$-factor corresponding to Eq. (5). A relativistic Coulomb-like resummation factor for arbitrary masses and orbital moment $\ell \geq 1$, called the $L$-factor, was investigated in Ref. [45].
Applications of the relativistic $S$-factor (5) in describing some hadronic processes can be found in Refs. [46–48]. Also, the factor (5) was applied in Ref. [49] to reanalyze the mass limits obtained for magnetic monopoles, which might have been produced at the Fermilab Tevatron.

Generalization of the relativistic $S$-factor (5) to the case of two relativistic particles of unequal masses $m_1$ and $m_2$ can be written in terms of the velocity $u$ as (for details see Refs. [44, 45])

$$S_{\text{uneq}}(u) = \frac{X_{\text{uneq}}(u)}{1 - \exp[-X_{\text{uneq}}(u)]}; \quad X_{\text{uneq}}(u) = \frac{\pi \tilde{\alpha}\sqrt{1 - u^2}}{u},$$

(6)

where the subscript “uneq” indicates the quantities related to the case of unequal masses, and the velocity $u$ is determined by the expression

$$u = \sqrt{1 - \frac{4m'^2}{s - (m_1 - m_2)^2}}.$$

(7)

Here $m' = \sqrt{m_1m_2}$ is the mass of an effective particle with 3-momentum $\Delta_{k',m'\lambda_Q}$ and energy $\Delta_{k',m'\lambda_Q}^0$ proportional to the total c.m. energy of particles, $\sqrt{s}$, see Refs. [29, 39]:

$$\sqrt{s} = \sqrt{\left(k_1 + k_2\right)^2} = \frac{m'}{\mu} \Delta_{k',m'\lambda_Q}^0,$$

(8)

$$\Delta_{k',m'\lambda_Q}^0 = \sqrt{m'^2 + \Delta_{k',m'\lambda_Q}^2};$$

where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass of a composite particle with 4-momentum $Q = q_1 + q_2$ and total mass $M_Q = \sqrt{Q^2}$; $\Delta_{k',m'\lambda_Q}^0$ and $\Delta_{k',m'\lambda_Q}$ are the time and spatial components of the 4-vector $\Lambda_{\lambda_Q}^{-1}k' = \Delta_{k',m'\lambda_Q}$ of the effective relativistic particle in Lobachevsky space corresponding to the Lorentz boost $L = \Lambda_{\lambda_Q}^{-1}$, where $\lambda_Q$ is the 4-velocity vector of the composite system, $\lambda_Q = (\lambda_Q^0; \lambda_Q) = Q/\sqrt{Q^2}$ (see Refs. [29, 39, 50] for details):

$$\Delta_{k',m'\lambda_Q}^0 = (\Lambda_{\lambda_Q}^{-1}k')^0 = k'^0\lambda_Q^0 - k' \cdot \lambda_Q$$

(9)

$$= \sqrt{m'^2 + \Delta_{k',m'\lambda_Q}^2};$$

$$\Delta_{k',m'\lambda_Q} = \Lambda_{\lambda_Q}^{-1}k' = k'(-)m'\lambda_Q$$

$$= k' - \lambda_Q \left(k'^0 - \frac{k' \cdot \lambda_Q}{1 + \lambda_Q^0}\right).$$

In the case when $m_1 = m_2 \equiv m$, the velocity (7) of the effective particle reduces to the velocity $v$, Eq. (2). The 3-momentum $k'$ of the effective particle in Lobachevsky space
is related to the relative relativistic velocity $v$ of the composite system by the following
expression [29]

$$k^2 = 2\mu^2 \left( \frac{1}{\sqrt{1 - v^2}} - 1 \right). \tag{10}$$

Recall that within the RQP approach [29, 39] the modulus of the relative relativistic
velocity $|v|$ of two particles can be expressed in terms of their total c.m. energy $\sqrt{s}$ (see
Ref. [44]) as

$$|v| = 2 \sqrt{\frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2}} \left[ 1 + \frac{s - (m_1 + m_2)^2}{s - (m_1 - m_2)^2} \right]^{-1}, \tag{11}$$

that is exactly the same as in, e.g., Ref. [35]. From Eqs. (7) and (11) one can infer that

$$|v| = \frac{2u}{1 + u^2}. \tag{12}$$

It is obvious that the relativistic limit ($u \to 1$) of the $S$-factor considered in Ref. [35] \(^1\)

$$S(|v|) = \frac{2\pi\alpha/|v|}{1 - \exp(-2\pi\alpha/|v|)}, \tag{13}$$

differs from the $S$-factor defined by Eq. (6) for which the relativistic limit is $S_{\text{uneq}}(u) \to 1$. However, for small values of $u$ the $S$-factors defined by Eqs. (6) and Eq. (13) provide the
same result.

**III. S-FACTOR FOR SPINORS OF EQUAL MASSES**

For the case of two spin-1/2 particles interacting via a Coulomb-like potential in $s$-state
($\ell = 0$), the wave function in the configuration representation and the corresponding $S$-
factor were thoroughly investigated within the RQP approach in Refs. [51–53]. Furthermore,
a closed analytical expression for the $S$-factor has been found

$$S_{\text{eq}}(\chi) = \frac{X_{\text{eq},s}(\chi)}{1 - \exp[-X_{\text{eq},s}(\chi)]} e^{-\pi\tilde{\rho}} \left| \Gamma(2 + i\tilde{\rho})_2 F_1(1 + iB, -i\tilde{\rho}; 2; 1 - e^{-2\chi}) \right|^2, \tag{14}$$

where $\Gamma(2 + i\tilde{\rho})$ is the familiar Euler gamma function, $_2F_1(1 + iB, -i\tilde{\rho}; 2; 1 - e^{-2\chi})$ is the
hypergeometric function and the subscript “s” refers to spinors. In Eq. (14) the quantities

\(^1\) In Ref. [35] the notation $T(|v|)$ is used instead of $S(|v|)$ adopted in the present paper.
\( X_{eq,s}(\chi) \) and \( \tilde{\rho}, B \) are defined as

\[
X_{eq,s}(\chi) = \frac{\pi \tilde{\alpha}(a \cosh^2 \chi + b)}{2 \sinh \chi},
\]

\[
\tilde{\rho} = \frac{\tilde{\alpha}a \cosh \chi}{4}, \quad B = \frac{\tilde{\alpha}(a \cosh^2 \chi + b)}{4 \sinh \chi},
\]

where the parameters \( a \) and \( b \) for different total spin of the system are

\[
a = \begin{cases} 
1 & \text{for } \hat{O} = \gamma_5 \text{ (pseudoscalar)}, \\
\frac{1}{2} & \text{for } \hat{O} = \gamma_\mu \text{ (vector)}, \\
-\frac{1}{2} & \text{for } \hat{O} = \gamma_5 \gamma_\mu \text{ (pseudovector)};
\end{cases}
\] (15)

\[
b = \begin{cases} 
0 & \text{for } \hat{O} = \gamma_5 \text{ (pseudoscalar)}, \\
\frac{1}{4} & \text{for } \hat{O} = \gamma_\mu \text{ (vector)}, \\
\frac{3}{4} & \text{for } \hat{O} = \gamma_5 \gamma_\mu \text{ (pseudovector)}.
\end{cases}
\] (16)

It can be shown that the relativistic limit, \( v \to 1 (\chi \gg 1) \), of the \( S \)-factor is

\[
S_{eq,s}(\chi)_{|\chi\gg 1} \simeq \frac{2\pi(B - \tilde{\rho})}{1 - \exp[-2\pi(B - \tilde{\rho})]}.
\] (17)

which is valid for all values of the spin parameters \( a \) and \( b \).

**IV. \( S \)-FACTOR FOR TWO SPINORS OF ARBITRARY MASSES**

**A. Relativistic Coulomb wave function**

In this subsection we consider, in some detail, the wave function of two relativistic spinor particles with arbitrary masses \( m_1 \) and \( m_2 \) interacting via a Coulomb-like chromodynamical potential

\[
V(\rho) = -\frac{\tilde{\alpha}_s}{\rho},
\] (18)
where \( \tilde{\alpha}_s \equiv 4\alpha_s / 3 > 0 \) and \( \rho = rm' \). Correspondingly, the RQP equation for the radial wave function, \( \phi_0(\rho, \chi') \), acquires the form

\[
\int_0^\infty d\chi \left[ (\cosh \chi' - \cosh \chi) \sin \rho \chi \right] \int_0^\infty d\rho' \sin(\rho' \chi) \phi_0(\rho', \chi') = -\frac{\tilde{\alpha}_s}{\rho} \int_0^\infty d\chi \left[ \hat{A}(\cosh \chi) \sin \rho \chi \right] \int_0^\infty d\rho' \sin(\rho' \chi) \phi_0(\rho', \chi').
\] (19)

Here the rapidity \( \chi' \) parametrizes the relative 3-momentum, \( \Delta_{k', m'\lambda, Q} \), and the total c.m. energy \( \sqrt{s} \), Eq. (8), as

\[
\Delta_{k', m'\lambda, Q} = m'\sinh \chi' n_{\Delta_{k', m'\lambda, Q}}, |n_{\Delta_{k', m'\lambda, Q}}| = 1,
\] (20)

\[
\sqrt{s} = \frac{m'}{\mu} \Delta_0^{0, \lambda, m', Q}, \Delta_0^{0, \lambda, m', Q} = m' \cosh \chi',
\] (21)

\[
\Delta_0^{2, m', \lambda, Q} - \Delta_2^{0, m', \lambda, Q} = m'^2.
\] (22)

\( \rho = r / \lambda' \), and \( \lambda' = 1 / m' \) is the Compton wavelength associated with the effective relativistic particle of mass \( m' = \sqrt{m_1 m_2} \), the operator \( \hat{A} \) is given by the expression

\[
\hat{A}(\cosh \chi') = \frac{1}{4} \left[ a' \cosh^2 \chi' + b' \right],
\] (23)

where

\[
a' = \begin{cases} 
g'^2 & \text{for } \hat{O} = \gamma_5 \text{ (pseudoscalar)}, \\
\frac{1}{2} g'^2 & \text{for } \hat{O} = \gamma_{\mu} \text{ (vector)}, \\
-\frac{1}{2} g'^2 & \text{for } \hat{O} = \gamma_5 \gamma_{\mu} \text{ (pseudovector)};
\end{cases}
\] (24)

\[
b' = \begin{cases} 
1 - g'^2 & \text{for } \hat{O} = \gamma_5 \text{ (pseudoscalar)}, \\
\frac{3}{4} - \frac{1}{2} g'^2 & \text{for } \hat{O} = \gamma_{\mu} \text{ (vector)}, \\
\frac{1}{4} + \frac{1}{2} g'^2 & \text{for } \hat{O} = \gamma_5 \gamma_{\mu} \text{ (pseudovector)},
\end{cases}
\]

and the factor \( g' \) is defined as

\[
g' = \frac{m'}{2 \mu} = \frac{m_1 + m_2}{2 \sqrt{m_1 m_2}}.
\] (25)

Obviously, the values of the parameters \( a' \) and \( b' \) in Eq. (24) at \( m_1 = m_2 = m \) coincide with the corresponding values in Eq. (15). It is worth mentioning that the rapidity \( \chi' \), velocity \( u \)
and factor $g'$ are interrelated by the expression

$$\sinh \chi' = \frac{u}{g'\sqrt{1-u^2}}.$$  \hfill (26)

The solution of Eq. (19) is sought in the form

$$\varphi_0(\rho, \chi') = \int_{\alpha_-}^{\alpha_+} d\zeta e^{i\rho\zeta} R_0(\zeta, \chi'),$$  \hfill (27)

where the integration is carried out in the complex plane along a contour between the end points $\alpha_{\pm} = -R \pm i\varepsilon, R \to +\infty, \varepsilon \to +0$, the values $\pm \chi + 2\pi ni \ (n = 0, \pm 1, \ldots)$ are the branch points of the function $R_0(\zeta, \chi)$, and the integration contour must not intersect the cuts which we take from $-\infty + 2\pi ni$ to $\pm \chi + 2\pi ni$ (see Fig. 1), that is, just in the same manner as before, c.f. Refs. [42, 44, 45, 54].

Substituting Eq. (27) into Eq. (19), and taking into account that

$$\frac{1}{i\pi} \int_{0}^{\infty} d\rho' \sin(\rho' \chi') e^{i\rho'\zeta} = \frac{1}{i\pi} \frac{\chi'}{\chi'^2 - \zeta^2}, \quad \text{Im} \zeta > 0,$$

and performing integration by parts, we arrive at the differential equation

$$\frac{d}{d\zeta} \left[ (\cosh \chi' - \cosh \zeta) R_0(\zeta, \chi') \right] - i\tilde{\alpha} \hat{A}(\cosh \zeta) R_0(\zeta, \chi') = 0$$  \hfill (28)

with the boundary condition

$$e^{i\rho \zeta} (\cosh \chi' - \cosh \zeta) R_0(\zeta, \chi') \bigg|_{\zeta=\alpha_+} = 0.$$

(29)

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**FIG. 1**: Integration contour in Eq. (27) and singularities of the function (30) in the complex $\zeta$-plane.
Then the solution of Eq. (28) with such a boundary condition can be written as

\[ R_0(\zeta, \chi') = \frac{C_0(\chi')}{(e^\zeta - e^{\chi'})^2} \exp \left[ -i \tilde{\alpha}_s' \frac{a'}{4} \sinh \zeta + (1 - i \tilde{\rho}') \zeta + i B' \chi' \right] \left[ \frac{e^\zeta - e^{-\chi'}}{e^\zeta - e^{\chi'}} \right]^{-1+iB'}, \quad (30) \]

where \( C_0(\chi') \) is an arbitrary function of \( \chi' \), the parameters \( a', b' \) are defined in Eqs. (24), and \( \tilde{\rho}' \) and \( B' \) are

\[ \tilde{\rho}' = \frac{\tilde{\alpha}_s' a' \cosh \chi'}{4}, \quad B' = \frac{\tilde{\alpha}_s' (a' \cosh^2 \chi' + b')}{4 \sinh \chi'}. \quad (31) \]

At \( \chi' = i \kappa_n \), the parameter \( B' \) is related to the quantization condition of the energy levels

\[ \frac{\tilde{\alpha}_s' (a' \cos^2 \kappa_n + b')}{4 \sin \kappa_n} = n \]

with \( n = 1, 2, \ldots \) and \( 0 < \kappa_n < \pi/2 \).

It should be emphasized that for vanishing interaction, \( \tilde{\alpha}_s \to 0 \), the asymptotics of the wave function \( \varphi_0(\rho, \chi') \) must reproduce the known free wave function,

\[ \lim_{\tilde{\alpha}_s \to 0} \varphi_0(\rho, \chi') \xrightarrow{\rho \to \infty} \frac{\sin(\rho \chi')}{\sinh \chi'}. \quad (32) \]

Substituting the solution (30) into Eq. (27) and performing the \( \zeta \)-integration in the complex plane along the contour displayed in Fig. 1 with the end points \( \alpha_{\pm} \) (c.f. Refs. [42, 44, 45, 54]), we obtain the resulting solution which does not contain the \( i \)-periodic constant:

\[ \varphi_0(\rho, \chi') = 2 C_0(\chi') e^{iB' \chi'} \sinh \left[ \pi (\rho - \tilde{\rho}') \right] \]

\[ \times \int_{-\infty}^{\infty} \exp \left[ \frac{i \tilde{\alpha}_s' a'}{4} \sinh x + (1 + i (\rho - \tilde{\rho}')) x \right] \left[ \frac{e^x + e^{-\chi'}}{e^x + e^{\chi'}} \right]^{-1+iB'} dx, \quad (33) \]

where \( C_0(\chi') \) is the normalization factor. In addition, it is seen that except for the oscillating factor \( \exp (i \tilde{\alpha}_s' a' \sinh x/4) \) the wave function \( \varphi_0(\rho, \chi') \) coincides in form with the spinless case. Moreover, for \( a' = 0 \) and \( b' = 2/g' \) Eq. (33) reproduces exactly the spinless wave function (see Refs. [44, 45]). This circumstance serves as a hint that we can approximate the wave function (33) by setting the oscillating factor \( \exp (i \tilde{\alpha}_s' a' \sinh x/4) \) equal to one; such an approximation preserves all the symmetry characteristics of the solution (33) and allows one to present the resulting wave function (for the \( s \)-wave state) in an explicit analytical form. Indeed, by a proper change of the integration variable \( x \),

\[ t = \frac{e^{\chi'}}{e^x + e^{\chi'}}, \quad \chi' < \infty \quad (34) \]
the integral in Eq. (33) leads to the well-known representation for the hypergeometric function $2F_1(a, b; c; z)$,

$$2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a}$$  \tag{35}

where \( Re c > Re b > 0 \),

exactly as in the case of spinors of equal masses (see Refs. [51–53]). The result is

$$\varphi_0(\rho, \chi') = 2\pi C_0(\chi') e^{iB'\chi' - \chi' + i(\rho - \bar{\rho})'\chi'} \times

\times (\rho - \bar{\rho}') 2F_1(1 - iB', 1 - i(\rho - \bar{\rho}'); 2; 1 - e^{-2\chi'}),$$  \tag{36}

where the parameters \( \bar{\rho}' \) and \( B' \) are given in Eq. (31) and the normalization factor \( 2\pi C_0(\chi') \) is defined by the expression:

$$|2\pi C_0(\chi')|^2 = e^{\pi B'} |\Gamma(1 - iB')|^2,$$  \tag{37}

which can be derived from the boundary condition (29) and the asymptotic behaviour (32)

$$\varphi_0(\rho, \chi')|_{\rho \to 1} \approx \frac{2\pi C_0(\chi') e^{-\pi B'/2}}{\sinh \chi' |\Gamma(1 - iB')|} \sin \left\{ (\rho - \bar{\rho})\chi' + B' \ln [2(\rho - \bar{\rho}) \sinh \chi'] + \arg \Gamma(1 - iB') \right\}.$$  \tag{38}

### B. Relativistic S-factor

Taking into account the relations (3) and (4), the Coulomb S-factor for a system of two relativistic spinor particles of arbitrary masses is defined as

$$S_{\text{uneq},s}(\chi') = \lim_{\rho \to 1} \left| e^{-\pi \bar{\rho}'/2} \Gamma(1 + i\bar{\rho}') \frac{\varphi_0(\rho, \chi')}{\rho} \right|^2.$$  \tag{39}

The additional factor \( \exp(-\pi \bar{\rho}'/2)\Gamma(1 + i\bar{\rho}') \) has been introduced to ensure the correct relativistic \( (\chi' \gg 1) \) behaviour of \( S_{\text{uneq},s}(\chi' \gg 1) \) and, for \( a' = 0 \) and \( b' = 2/g' \), to maintain the spinless results. Recall that in the spinless case, the ultrarelativistic limit \( (u \to 1) \) of the S-factor, Eq. (6), is \( S(u \to 1) = 1 \). In the case of spinors, the hypergeometric function is strictly defined for \( u < 1 \), c.f. Eq. (34), and consequently for \( u = 1 \) \( (\chi' \to \infty) \) the change of variables (34) becomes ill defined. However, the hypergeometric function can be further defined at \( u = 1 \) is such a way as to keep continuity with the previous results. This can be
achieved by substituting in the hypergeometric function its argument $1 - \exp(-2\chi')$ by unity and, at the same time, keeping the rest of the parameters as functions of $\chi'$. In this case, one can make use of the explicit expression of $\binom{2}{1}(a, b; c, 1)$ in terms of the Euler $\Gamma$-functions, $\binom{2}{1}(a, b; c, 1) = \Gamma(c)\Gamma(c - a - b)/\Gamma(c - a)\Gamma(c - b); \text{Re}(c - a - b) > 0$ and have the wave function (36) defined in the whole interval of velocities, including $u = 1$.

Thus, we see that the function

$$\psi_0(\rho, \chi') = e^{-\pi\tilde{\rho}/2}\Gamma(1 + i\tilde{\rho})\varphi_0(\rho, \chi')$$

is the sought physical spinor wave function for the Coulomb-like interaction (18). Then, by making use of relations (36)–(39), one obtains the $S$-factor for two spinors of arbitrary masses in the form

$$S_{uneq,s}(\chi') = \frac{X_{uneq,s}(\chi')}{1 - \exp[-X_{uneq,s}(\chi')]} e^{-\pi\tilde{\rho}/2}\Gamma(2 + i\tilde{\rho})\binom{2}{1}(1 + iB', -i\tilde{\rho}; 2; 1 - e^{-2\chi'})}^2 \quad (40)$$

with

$$X_{uneq,s}(\chi') = 2\pi B' = \frac{\pi\tilde{\alpha}_s (a'\cosh^2\chi' + b')}{2\sinh\chi'}, \quad (41)$$

where $\tilde{\rho}', B'$ are given by Eq. (31). Quantities (40) and (41) can be expressed in terms of the velocity $u$, Eq. (7), and the relative velocity $u_{rel}'$ of an effective relativistic particle with mass $m'$:

$$u_{rel}' = \frac{2u}{\sqrt{1 - u^2}}. \quad (42)$$

In particular, for Eq. (41) one has

$$X_{uneq,s}(u) = \frac{\pi\tilde{\alpha}_s \sqrt{1 - u^2}}{2g'u} \left[ g'^2 (a' + b') + \frac{a'u^2}{1 - u^2} \right] = \frac{\pi\tilde{\alpha}_s}{g'u_{rel}'} \left[ g'^2 (a' + b') + \frac{a'}{4}u_{rel}' \right]. \quad (43)$$

It is noteworthy that the kinematic factor $g'$ defined by expression (25) establishes the relationship between the total c.m. energy $\sqrt{s}$ of the system of two particles of arbitrary masses $m_1$ and $m_2$ and the energy $\Delta_{k',m',\lambda Q}^0$ of an effective particle with the mass $m'$ and 3-momentum $\Delta_{k',m',\lambda Q}$. Such an effective particle mimics the properties of the two-body system under consideration with the same total c.m. energy $\sqrt{s}$, see Eq. (8). Therefore, the kinematic factor $g'$ reflects the asymmetry between these two systems. In addition, the factor (25) reflects the relation of the arithmetic mean of the masses of two particles to their geometric mean. Consequently, $g' > 1$ for $m_1 \neq m_2$ and $g' = 1$ for symmetric systems.

Equations (40)-(41) define the most general expression for the relativistic $S$-factor of systems of two spinor particles with arbitrary masses obtained within the quasi-potential
approach [28, 29]. Obviously, it includes all the previously considered cases. So for \( m_1 \neq m_2 \)
and \( a' = 0 \) and \( b' = 2/g' \) Eq. (40) reproduces exactly the spinless S-factor (6), while for
\( m_1 = m_2 = m \) it coincides with the equal-masses S-factor (14). The explicit analytical
form of the S-factor, Eq. (40), allows for a detailed investigation of the spin-dependent
Sommerfeld effect in the threshold-near region \((u \to 0)\) where, as already mentioned, one
predicts an essential enhancement of the cross sections in processes with spinless particles.
In this region the influence of spins on the scale of the effect is of particular interest. The
relativistic limit is also of great interest since, as will be shown below, the S-factor turns
out to be rather sensitive to the total spin and mass asymmetry of the system.

a. Threshold-near region: Using the known properties of the hypergeometrical functions

\[
_{2}F_{1}(a, b, c, z/b)|_{b \to \infty} \to _{1}F_{1}(a, c, z)
\]

and the known representation of the modulus of the \( \Gamma \) functions,

\[
|\Gamma(x + iy)| = |\Gamma(x)| \prod_{n=1}^{\infty} \left( 1 + \frac{y^2}{(x+n)^2} \right)^{-1/2}, \quad \text{or}
\]

\[
|\Gamma(x + iy)|^2 = |\Gamma(x)|^2 \frac{\pi y}{\sinh(\pi y)} \prod_{n=1}^{\infty} \left( 1 + \frac{y^2/n^2}{1 + y^2/(x+n)^2} \right),
\]

one finds the S-factor at the threshold \( u \to 0 \) \((\chi' \to +0)\) to be

\[
S_{uneq,s}(\chi')|_{\chi' \to +0} \approx \frac{\pi \tilde{\alpha}_s(a' + b')/2 \sinh \chi'}{1 - \exp[-\pi \tilde{\alpha}_s(a' + b')/2 \sinh \chi']} \times
\]

\[
\frac{(\pi \tilde{\alpha}_s a'/4) \exp(-\pi \tilde{\alpha}_s a'/4)}{\sinh(\pi \tilde{\alpha}_s a'/4)} \left( 1 + \frac{\tilde{\alpha}_s^2 a'^2}{16} \right) |_{1}F_{1}(-i\tilde{\alpha}_s a'/4; 2; i\tilde{\alpha}_s(a' + b')/2)|^2.
\]

b. Relativistic region: With the adopted definition of the wave function at \( u = 1 \) \((\chi' \to \infty)\),

\[
_{2}F_{1}(1 + iB'(\chi'), -i\bar{\rho}'(\chi'); 2; 1 - e^{-2\chi'})|_{\chi' \to \infty} \approx _{2}F_{1}(1 + iB'(\chi'), -i\bar{\rho}'(\chi'); 2; 1)
\]

the relativistic limit, \( \chi' \gg 1 \), of the S-factor is

\[
S_{uneq,s}(\chi')|_{\chi' \to +1} \approx \frac{2\pi (B' - \bar{\rho}')}{1 - \exp[-2\pi (B' - \bar{\rho})]}|_{\chi' \to +1} \approx 1 + \frac{\pi \tilde{\alpha}_s}{4} (a' + 2b') e^{-\chi'}.
\]

From Eq. (48) and Eqs. (24)-(16) it follows that in the pseudoscalar case \((\tilde{O} = \gamma_5)\) for
\( g' > \sqrt{2} \) and in the vector one \((\tilde{O} = \gamma_{\mu})\) for \( g' > \sqrt{3} \), the S-factor (40) in the relativistic
limit tends to unity from below, see also Figs. 2 and 3.
FIG. 2: (Color online) The behavior of the $S$-factor (40) for the pseudoscalar system, $a' = g'^2$ and $b' = 1 - g'^2$, as a function of the velocity $u$, Eq. (7), at $\tilde{\alpha}_s = 0.2$: solid curve corresponds to $g' = 1.5$, while the dashed one corresponds to $g' = 2.5$. The dot-dashed curve is for the spinless $S$-factor, Eq. (6).

V. RESULTS

We investigated numerically the $S$-factor for scalar, pseudo-scalar, vector and pseudovector two-particle systems with arbitrary masses by the exact formula (40). The results of calculations of the $S$-factor as a function of the velocity $u$, Eq. (7), are presented in Figs. 2, 3 and 4 for the pseudo-scalar, vector and pseudovector systems, respectively, for two values of the parameter $g'$. The solid curves corresponds to $g' = 1.5$, while the dashed ones corresponds to $g' = 2.5$. The dot-dashed curve is for the spinless $S$-factor, Eq. (6). For definiteness, numerical calculations have been performed for $\tilde{\alpha}_s = 0.2$. However, the general features of the $S$-factor do not depend on the concrete choice of $\tilde{\alpha}_s$. From these figures one infers that the results are rather sensitive to the value of $g'$ and, consequently, to the parameters $a'$ and $b'$.

As is seen from Figs. 2–4 at low and moderate velocities, the spinor $S$-factors are always smaller than the $S$-factor for spinless particles. Note that, to our knowledge, spin effects of spins in the relativistic $S$-factor have not yet been investigated. Therefore, a systematic comparison of spinless and spinor factors can be considered as the very first results of the
study of spin effects in the relativistic $S$-factor. For a better illustration of spin effects, we present in Fig. 5 the ratio of spinless to spinor $S$-factors for vector (solid), pseudo-vector (dash-dotted) and pseudo-scalar (dashed curve) systems. It is seen that the effects of spins significantly reduce the Sommerfeld effect at the threshold in comparison with the spinless case, while at relativistic velocities they become insignificant and all, spinor and spinless
FIG. 5: (Color online) The ratio of the S-factors of spinless systems to the spinor factors as a function of the velocity \( u \): solid curve corresponds to vector, dashed to pseudo-scalar and dot-dashed curve to pseudo-vector systems. The calculations have been performed for the same values of the parameters as in Figs. 2-4 for \( g' = 1.5 \).

...factors, are practically the same and close to unity.

Now let us focus on the regions of moderate and sufficiently large velocities, \( u \gtrsim 0.5 \). Figures 2 and 3 demonstrate that, according to Eqs. (48) and (24)-(16), the S-factor of pseudo-scalar and vector systems becomes less than unity and approaches its asymptotics, \( S \to 1 \), from below. Such an unexpected decrease of the S-factor, which reduces the cross sections, can have essential impacts in treating the measured experimental data at large \( u \). Figure 4 shows that the S-factor for a pseudovector system is always larger than unity, despite the fact that its ultrarelativistic limit is also unity. The common limit for different systems can be qualitatively understood if one considers the case of (at least one) super-light particles, \( m' \to 0 \). Since, as is well known, the S-factor characterizes the effects of the final (FSI) or initial (ISI) state interaction, complementing the main process, see e.g. [34], at ultrarelativistic velocities the FSI/ISI effects vanish and the main reactions are determined solely by the corresponding Feynman diagrams without any additional factors.

It should be stressed that since in our approach both the argument \( (r \equiv |\mathbf{r}|) \) of the Coulomb-like potential (18) and the relative velocity \( |\mathbf{v}| \) are relativistic invariants [29], the
S-factor (40) is manifestly relativistic invariant too. Therefore, due to the relation $|v| = 2u/(1 + u^2)$, the velocity $u$, Eq. (7), and the velocity $u'_{\text{rel}}$ of the effective particle, Eq. (42), are also relativistic invariants. Thus, instead of the previously used relative velocity $|v|$, an appropriate parameter in the $S$-factor (40) is the velocity (42) of an effective particle, which has the same properties as the considered two-particle system.

VI. CONCLUSION

In this paper, we have presented new calculations of the relativistic $S$-factor, Eq. (40), within the covariant quasi-potential approach in the three-dimensional relativistic configuration representation. For the first time, the most general expression for the manifestly relativistic $S$-factor of two-spinor systems with arbitrary masses has been derived explicitly. It is asserted that the relevant velocity parameter of the approach is the one defined by Eq. (42) instead of the relative velocity $|v|$ in so far commonly adopted in the literature. The spinless case can be obtained from the relativistic spinor $S$-factor by setting in Eqs. (40)–(41) the spin parameters $a' = 0$ and $b' = 4\sqrt{m_1m_2/(m_1 + m_2)} = 2/g'$. A comparison of the spinor factors with the spinless case persuades us that the spin effects play a significant role in the Sommerfeld effect making it, at small and moderate velocities, systematically smaller than the $S$-factor for spinless systems. Thus, one can predict a decrease of the cross sections in a large kinematic range of the relative velocity, the effect being amplified in the threshold-near region, see Fig. 5. However, at sizeably large velocities, $u \gtrsim 0.4$, the pseudo-vector and the spinless $S$-factors become compatible to each other; moreover, the pseudo-vector factor at large enough mass asymmetries ($g' \gtrsim 2.5$) becomes even larger than the spinless one, c.f. Fig. 4. Another interesting observation is that at large velocities, $u \gtrsim 0.4 - 0.5$, the pseudo-scalar and vector factors for certain asymmetric systems cross the unity from above and approach the ultrarelativistic limit from below. Since in this region the cross sections become smaller in comparison with the main cross section determined by the corresponding Feynman diagrams, this circumstance can be, in some sense, referred to as the "anti"-Sommerfeld effect, contrarily to the "true" Sommerfeld effect discussed above.\(^2\) The "anti"-Sommerfeld effect increases with the mass asymmetry, $g'$, of the system and can cause

\(^2\) Recall that in the present paper we consider the attractive Coulomb-like potentials.
new impacts in the cross section of processes with highly asymmetric relativistic particles.

An interrelation of the new $S$-factor (40) with the previously considered in the literature (symmetric spinors and spinless two-body systems of arbitrary masses) has been settled. Our analysis of the obtained results at the kinematic limits of the relative velocity $u$, i.e. in the deep nonrelativistic and ultrarelativistic regions, shows that the approach reproduces the well-known nonrelativistic behaviour of the spinless factor for the symmetric systems ($m_1 = m_2 = m$) as well as provides the expected correct ultrarelativistic limits for two-spinor systems.

VII. SUMMARY

In summary, we present the most general expression for the relativistic Sommerfeld–Gamov–Sakharov $S$-factors for spinless, pseudo-scalar, vector and pseudo-vector relativistic particles with arbitrary masses derived within the covariant, relativistic quasi-potential approach in the 3-dimensional configuration representation in Lobachevsky space. The new $S$-factor is thoroughly investigated in the whole kinematic region of the relative velocity $u$, including detailed studies in the deep threshold region and ultrarelativistic limit. The effects of spins and mass-asymmetry are discussed.

Acknowledgments

The authors gratefully acknowledge helpful discussions with S. M. Dorkin, V. I. Lashkevich and E. A. Tolkachev and their continuous interest in the research topic. This work was supported in part by the International Program of Cooperation between the Republic of Belarus and JINR.

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