Weakly Inhomogeneous models for the Low-redshift Universe

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We analyze two different algorithms for constructing weakly inhomogeneous models for the low-redshift Universe, in order to provide a tool for testing the photon dynamics, within the sphere of validity for the Universe acceleration.

We first implement the so-called quasi-isotropic solution in the late Universe, when a pure dark energy equation of state for the cosmological perfect fluid is considered. We demonstrate, that a solution exists only if the physical scale of the inhomogeneities is larger then the Hubble scale of the microphysics. Then, we analyze a weakly deformed isotropic Universe toward a spherically symmetric model, thought as the natural metric framework of the $\Lambda$CDM model. The obtained picture offer a useful scenario to investigate the influence of the inhomogeneity spectrum (left free in the obtained solution), on the photon propagation at low redshift values.

Keywords : inhomogeneous universe, low-redshift universe, quasi-isotropic solution

1 Introduction

The Standard Cosmological model [1] [2], is based on the homogeneous and isotropic Robertson-Walker metric, and bases its reliability on the high isotropy of the Cosmic Microwave Background Radiation [3] [4].

Actually, the estimates from the galaxy surveys of the spatial scale at which the present Universe reaches homogeneity, provides a value of about $60\,\text{Mpc}$ [5], which implies that, at lower scales, significant deviations from the Robertson-Walker geometry may be observed, at least as higher order corrections.

The presence of such small scale deviations, could influence the information we get from extragalactic sources and, in general, they affect the photon paths from distant regions up to our detectors.

In this work, we investigate the spatial metric which admit inhomogeneous corrections to the flat isotropic model (the present contribution of the spatial curvature is clearly negligible with respect to the matter terms).

As first step, we analyze the so-called quasi-isotropic solution [6] (see also [7] [8]), but implemented in the low redshift Universe.

In particular, we study this solution in the presence of a perfect fluid, having a dark energy equation of state, i.e. $p = w\rho$ ($p$ and $\rho$ being the pressure and energy density of the fluid respectively), where $-1 < w < -1/3$.

The interesting feature of the obtained solution, relies in the fact that the considered inhomogeneities, included in the model as small corrections, must correspond, in order the solution...
be consistent, to physical scales much greater than the Hubble (microphysical) scale. This fact makes such inhomogeneous corrections of pure curvature nature and, over all, they can not affect the physical processes taking place in the Hubble sphere.  

Then, we consider the case of a Lemaitre-Tolmann-Bondi spacetime \[ \text{[10]}, \] which describes a spherically symmetric Universe in the presence of a matter source, and a non-zero cosmological constant, an appropriate scenario to account for the so-called \( \Lambda \text{CDM} \) model \[ \text{[11]}. \]  

Clearly, in order to describe the local behavior of the actual Universe, we consider the inhomogeneous perturbations again, as first order modification of the flat Robertson-Walker geometry. We demonstrate the existence of a consistent solution of the linearized Einstein equations, which does not fixes the radial dependence of the inhomogeneities, but only their time scaling. Moreover, we specialize the obtained solution to the case of the \( \Lambda \text{CDM} \) model, by tuning the values of the parameters in order to obtain, that the matter be the 30\% and the constant energy density the 70\% of the Universe critical parameter respectively. The obtained time profile for the perturbations, together with the arbitrariness of their specific spatial morphology, offer an interesting arena to study the effects on the photon propagation, due to the local deviations of the actual Universe from homogeneity. 

Furthermore, the results of our analysis suggest an intriguing issue: while the inhomogeneities allowed by a \( \Lambda \text{CDM} \) model are physically observable, living in principle, in the present Hubble sphere, the dark energy dominated Universe appears incompatible with the physical scale of inhomogeneity, the microphysics processes remain essentially concerned by the homogeneity restriction. 

The different behavior of the two considered equations of state (we recall that the cosmological constant is associated to the relation \( \rho = -p \)), could become a qualitative discrimination property when incoming missions, like Euclid \[ \text{[12]} \] will be able to test the large scale properties of the Universe, detecting details of the matter distribution across the cosmological space. 

This paper, is structured as follows: on section \[ 5] the Lifshitz-Khalatnikov quasi-isotropic solution for a pure radiation high-redshift universe will be described, so that on section \[ 6] the same procedure will be applied to the case of a low-redshift dark energy universe. 

Following the same steps, on section \[ 4] is introduced the Lemaitre-Tolman-Bondi model for spherically symmetric universes in the generic case, while in section \[ 5] the previous solution will be extended to the case of a low redshift universe, filled with both a matter and a cosmological constant perfect fluid. 

On section \[ 3] the weakly inhomogeneous model derived in the previous section, will be fitted with the actual observational data, in order to describe as best as possible the behaviour of our \( \Lambda \text{CDM} \) universe. 

Lastly, the article will be closed with concluding remarks, that are reported in section \[ 7]. 

2 The Lifshitz-Khalatnikov quasi-isotropic solution for high redshift radiation universes 

The Lifshitz-Khalatnikov quasi-isotropic solution \[ \text{[6]} \], is a generalization of the FRW cosmology \[ \text{[13]} \], in which a certain degree of dishomogeneity, and so anisotropy, is introduced. 

The dishomogeneity of space, is reflected to the presence of three phisically arbitrary functions of the coordinates in the metric of the system. In a isotropic solution in fact, isotropy and homogeneity implies the vanishing of the off-diagonal metric components \( g_{\alpha \beta} \), while if isotropy and homogeneity assumption are dropped, is always possible to move to a frame where the previous condition may be imposed. 

To do so, must be defined a Synchronous reference frame \[ \text{[14]} \], with the following choice for the metric tensor : 

\[
\begin{align*}
 g_{00} & = 1 \\
 g_{0\alpha} & = 0
\end{align*}
\] 

(1)

such that the metric reduces to the form : 

\[
ds^2 = dt^2 - h_{\alpha \beta}(x,t)dx^\alpha dx^\beta
\] 

(2)

where the term \( h_{\alpha \beta} \), is called Trimetric, and represent the pure spatial component of the metric. 

The original Lifshitz-Khalatnikov model, was developed for a pure radiation universe, and for the ultrarelativistic matter, the equation of state reads as \( P = \rho/3 \), while the trimetric \( h_{\alpha \beta} \) is linear in \( t \) at the first order. 

When searching for a quasi-isotropic extension of the Robertson-Walker geometry, the metric should be expandable in integer powers of \( t \), asymptotically as \( t \to 0 \), following the Taylor-like expansion:
After a suitable rescaling, and introducing the adimensional time \( t = \frac{t}{t_0} \), the trinmetric gets the form:

\[
h_{\alpha\beta} = a_{\alpha\beta}^0 t_0^n + a_{\alpha\beta} (t/t_0)^2 + \ldots
\]

it is possible to write:

\[
h_{\alpha\beta} = a_{\alpha\beta}^0 t_0^n + a_{\alpha\beta} (t/t_0)^2 + \ldots
\]

After a suitable rescaling, and introducing the adimensional time \( t = t/t_0 \), the trinmetric gets the form:

\[
\begin{align*}
    h_{\alpha\beta} &= a_{\alpha\beta}^0 t_0^n + a_{\alpha\beta} (t/t_0)^2 + \ldots, \\
    h^{\alpha\beta} &= a_{\alpha\beta} (t/t_0)^{-1} - b_{\alpha\beta} + \ldots
\end{align*}
\]

the system moreover, assume a much easier form by defining the auxiliary tensors \( K_{\alpha\beta} \) as:

\[
\begin{align*}
    K_{\alpha\beta} &= \partial_t h_{\alpha\beta}, \\
    K_\alpha &= h^{\alpha\beta} K_{\beta\gamma} = t^{-1} \delta_\alpha^\gamma + b_\alpha, \\
    K &= \partial_t \ln(h) = 3t^{-1} + b
\end{align*}
\]

where from the third equation, it is possible to get:

\[
h = \det(h_{\alpha\beta}) \sim t^3 (1 + \tilde{b} t) \det(a_{\alpha\beta})
\]

The last thing needed to write down the Einstein Equations, is the Energy-Momentum Tensor \[15\], that in the case of ultrarelativistic matter gets the form:

\[
T_{ij} = \rho(t) (4u_i u_j - g_{ij})
\]

leading to the following Einstein equations:

\[
\begin{align*}
    \rho(t) = \frac{1}{2} \partial_t K_\alpha + \frac{1}{2} K_\beta K^{\beta}_\alpha = -k^2 (4u_i u_i - 1) \\
    R_0^0 &= \frac{1}{2} (K^\beta_\alpha - K_\alpha) = \frac{4}{5} k^2 (u_t u_t - u_i u_i) \\
    R_\alpha^\gamma &= \frac{1}{2} \Gamma_\nu^\alpha (\sqrt{h} K^\nu_\gamma) + 3 R_\alpha^\gamma = -k^2 (4u_\beta u_\alpha + \delta_\alpha^\beta)
\end{align*}
\]

In the last equations, the term \( 3 R_\beta^\alpha \) is the Tridimensional curvature Ricci's Tensor, which analytically assume the form:

\[
3 R_\beta^\alpha = \Gamma_\gamma^\alpha \gamma - \Gamma_\delta^\delta^\alpha + \Gamma_\sigma^\alpha \Gamma^\lambda_\sigma - \Gamma_\alpha^\nu \Gamma_\nu^\beta
\]

Recalling the 4-velocity relation:

\[
1 = u_i u^i \sim u_0^2 - \tilde{t}^{-1} a^\alpha u_\alpha
\]

and assuming that the approximation \( u_0^2 \sim 1 \) is valid, the system \[10\] may be solved up to the zeroth order \( O(1/t^2) \) and first-order \( O(1/t) \). The solutions obtained are:

\[
\begin{align*}
    k^2 (u_t) &= \frac{3}{2 \tilde{t}^2} - \frac{b}{2} \\
    u_\alpha &= \frac{2}{\tilde{t}} (b_\alpha - b^\beta_{\alpha,\beta})
\end{align*}
\]

From the second equation, it may be observed that the assumption \( u_0^2 \sim 1 \) is valid at the first order, the second term of \[12\] is proportional to \( \tilde{t}^3 \), and for \( t \to 0 \) is negligible respect to the first term. The density contrast moreover, may be defined as the ratio between the perturbed term and the zeroth order term of the density, giving:

\[
\delta = -\frac{2}{3} b \tilde{t} \sim t
\]

This behavior implies that, as expected in the standard cosmological model, the zeroth-order term of the energy density diverges more rapidly than the perturbation and the singularity is naturally approached with a vanishing density contrast. Considering the last remaining equation of \[10\] \( 3 R_\beta^\alpha \) at leading order may be written as:

\[
3 R_\beta^\alpha = A^\alpha_\beta / t
\]

on which \( A^\alpha_\beta \) are pure functions of spatial coordinates, constructed in terms of \( a_{\alpha\beta} \). The third equation of \[10\] so, upon using relation \[15\] reduces to:
\[ A_\beta^\alpha + \frac{3}{4} b_\beta^\alpha + \frac{5}{12} b^\beta \alpha = 0 \]  (16)

which admits the following trace :

\[ b_\beta^\alpha = \frac{4}{3} A_\beta^\alpha + \frac{5}{18} A^\delta \beta \]  (17)

Finally, using the tridimensional Bianchi identity, that read as :

\[ A^\alpha_{\beta,\alpha} = 1/2 A_{\alpha,\alpha} \]  (18)

The condition (22) imposes that the perturbation scale factor \( b(t) \) is negligible respect to the main scale factor \( a(t) \) at early stage time, so that the inhomogeneous perturbation, will result in a small correction over the background homogeneous and isotropic universe.

Let’s now consider the energy-momentum tensor, the equation of state for a dark energy fluid is

\[ w = \frac{p}{\rho} \]  

\[ P = \rho w \]  

\[ w \in (-1, -1/3) \] 

while the Einstein equations, that in a synchronous reference frame gets the form :

\[ R^\alpha_\beta - \frac{1}{2} K^\alpha_\beta K_\beta^\gamma = k (T^\alpha_\beta - \frac{1}{2} T) \]  

\[ R^\alpha_\beta - \frac{1}{2} (K_{\alpha,\beta} - K_{\beta,\alpha}) = k T^\alpha_\beta \]  

\[ R^\alpha_\beta \delta^\beta_\alpha + \frac{1}{2\sqrt{K}} (\sqrt{K} K_\beta^\gamma) ; \alpha = k (T^\alpha_\beta - \frac{1}{2} \sqrt{K}^\alpha_\beta T) \]  (24)

\[ u_\alpha = \frac{t^2}{9} h_\alpha \]  (20)

It may be worth observing, that the metric [5] allows an arbitrary spatial coordinate transformation, while the above solution contains only 3 arbitrary space functions arising from \( a_\alpha \).

### 3 The *Lifshitz-Khalatnikov* quasi-isotropic solution for low-redshift Dark Energy universes

In this section, will now be shown how the *Lifshitz-Khalatnikov* quasi-isotropic solution, may be applied for the analysis of low-redshift, dark energy [9] universes.

Let’s start by introduce the scale factors ratio \( \eta(t) = a^2(t)/b^2(t) \) in the trimetric of the system, obtaining :

\[ h_{\alpha,\beta}(t, x) = a^2(t) \xi_{\alpha,\beta}(x^\gamma) + b^2(t) \theta_{\alpha,\beta}(x^\gamma) + ... = a^2(t) [\xi_{\alpha,\beta}(x^\gamma) + \eta(t) \theta_{\alpha,\beta}(x^\gamma) + ...] \]  (21)

where \( \xi_{\alpha,\beta}(x^\gamma) \) is the *Minkowskian* unperturbed metric, and \( \theta_{\alpha,\beta}(x^\gamma) \) are free functions of spatial coordinates, that represent the inhomogeneous perturbation of the metric.

To analyze the system at low redshift, it must be imposed that the ratio \( t/t_0 >> 1 \), where this time \( t_0 \) was used to denote an arbitrary time smaller than the present one, while the contrast \( \eta(t) \) must respect the limit :

\[ \lim_{t \to \infty} \eta(t) = 0 \]  (22)

\[ \frac{\alpha(t)}{a(t)} = 7/9 \]  

\[ \frac{\beta(t)}{b(t)} = 5/9 \]  

\[ \frac{\gamma(t)}{c(t)} = 12/9 \]  

\[ \frac{\alpha(t)}{a(t)} \]  

\[ \frac{\beta(t)}{b(t)} \]  

\[ \frac{\gamma(t)}{c(t)} \]  

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

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\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]

\[ \frac{\alpha(t)}{a(t)} \]

\[ \frac{\beta(t)}{b(t)} \]

\[ \frac{\gamma(t)}{c(t)} \]
The tri-velocity \( a(t) \), in fact, that already has a temporal dependence at least of \( O(1/t^2) \), appears in the equations 25 at least multiplied by the zeroth order term of the density, that, in a FRW cosmology goes as \( O(1/t^2) \).

The validity of the approximation anyway, will be verified later, by calculating the analytical form of tri-velocity \( u_\alpha \).

To integrate the system 25, must be defined the form that the auxiliary tensors assume, in relation to our current choice for the metric 21.

Following the definitions given in 7, at the first order they reduces to :

\[
K_{\alpha\beta} = h_{\alpha\beta,0} = 2\dot{a}a [\xi_{\alpha\beta} + \eta(t)\theta_{\alpha\beta}] + \ddot{\eta}(t)a^2(t)\theta^\alpha_{\beta} \tag{26}
\]

\[
K^\alpha_{\beta} = h^{\alpha\delta}K_{\delta\beta} \approx 2\frac{\ddot{a}}{a}\delta^\alpha_{\beta} + \eta(t)\theta^\alpha_{\beta} \tag{27}
\]

\[
K \approx 6\frac{\ddot{a}}{a} + \ddot{\eta}(t)\theta = \partial_t\ln(h) \tag{28}
\]

Furthermore, taking into account equation 28 the determinant of the trimetric \( h_{\alpha\beta} \) get the form :

\[
h = ja^6(t)e^{\eta(t)\theta} \tag{29}
\]

that for \( t \gg 0 \), under the assumption 22 may be approximated as :

\[
h \approx ja^6(t)(1+\eta(t)\theta) \Rightarrow \sqrt{h} = a^3(t) \left(1 + \frac{\eta(t)\theta}{2}\right) \tag{30}
\]

Moreover, working in the regime of \( t \gg t_0 \), and assuming that \( t_0 > 0 \), the Tridimensional curvature Ricci’s Tensor may be approximated as :

\[
3R^\alpha_{\beta} \approx \frac{\eta(t)}{a^2(t)}\rho^\mu_{\alpha,\beta,\mu} \tag{31}
\]

By assuming that both \( a(t) \) and \( b(t) \) follow a power law relation in function of time, from equations 25,26,27 and 28, it may be observed that the system has \( O(1/t^2) \) zeroth order terms, while has \( O(\eta(t)/t^2) \) perturbation terms.

Knowing that, in a flat dark energy universe, the scale factor \( a(t) \) evolve as :

\[
a(t) = t \frac{\eta(t)}{t} \tag{32}
\]

the tridimensional curvature Ricci’s tensor 31 may be negligible in the equations 25 when the condition :

\[
a(t) \gg t \Rightarrow t^2 \frac{\eta(t)}{t} \gg t^1 \Rightarrow w < -\frac{1}{3} \tag{33}
\]

is verified.

However, a condition must be imposed also on the pure spatial part of 31 and it may be done by introducing the scale of the perturbation, that may be derived from the equation :

\[
\frac{\eta(t)}{a^2(t)}\theta^\mu_{\alpha,\beta,\mu} < \frac{\eta(t)}{t^2} \tag{34}
\]

Infact, considering that \( \theta^\mu_{\alpha,\beta,\mu} \approx \theta_\alpha^\mu/\lambda^2 \) and \( L_H \sim t \)

where \( L_H \) stands for the Hubble Lenght, equation 34 up to constants, leads to the relation :

\[
\lambda^2_{phys} \gg L^2_H \tag{35}
\]

where \( \lambda_{phys} = \lambda a(t) \).

The inhomogeneous perturbations so, must be at scale greater than the Hubble Horizon, hence theoretically they can’t actually be observed.

Imposing that our \( \delta_w \) is now in the range \((0, 2/3)\), assumption 43 is valid, and the system 25 reduces to :

\[
\begin{align*}
R_0^\alpha & = \frac{3}{2}k\rho\delta_w = 3\frac{\alpha}{a} + \frac{4(t)\theta}{2} + \frac{\dot{\alpha}}{a}\eta(t)\theta \\
R_0^\beta & = \frac{1}{2}\delta_w \left(\delta_w - \theta^\alpha_{\alpha,\gamma}\right) = k\rho \left(\frac{3}{2} - \delta_w\right) u_\alpha \\
R^0_\beta & = \frac{1}{2\sqrt{h}} \left(\sqrt{h}K^\alpha_{\alpha,\beta}\right)_{\beta} = k\rho \left[\left(\frac{5}{2} + \frac{3\delta_w}{2}\right) \delta^\alpha_{\beta}\right]
\end{align*}
\tag{36}
\]

Taking into account the trace of \( R^\alpha_\beta \) equation :

\[
R : \frac{1}{2\sqrt{h}} \left(\sqrt{h}K\right)_{\alpha} = k\rho \left[2 + 3\delta_w\right] \tag{37}
\]

the system may be solved by introducing 37 in the first equation of 36.

The two equations obtained, respectively for the background and perturbed term, are :
\[
\frac{4}{\delta_w} \frac{\ddot{a}}{a} = 6 \left(\frac{\dot{a}}{a}\right)^2 \quad (38)
\]
Furthermore, from the first equation of [36] the energy density reduces to:

\[
k\rho = \frac{2}{\delta_w t^2} \frac{1}{1 - \frac{3}{2} \delta_w} \left(\frac{1}{1 - \frac{3}{2} \delta_w} - 1\right) - \frac{2}{3 \delta_w} \frac{t_0}{t^3} \eta \left(\frac{1}{1 - \frac{3}{2} \delta_w} - 1\right) \quad (46)
\]

assuming that both \(a(t)\) and \(\eta(t)\), follow a power law relation in respect of time:

\[
a(t) = \left(\frac{t}{t_0}\right)^x \quad (40)
\]

\[
\eta(t) = \left(\frac{t}{t_0}\right)^y \quad (41)
\]

the first equation [38] reduces to:

\[
4x(x - 1) = 6\delta_w x^2 \quad \Rightarrow \quad x = 0, \quad x = \frac{1}{1 - \frac{3}{2} \delta_w} \quad (42)
\]

Since the background solution must be an isotropic \(FRW\) universe, the first solution will be excluded, so that \(a(t)\) will assume the same form obtainable with a standard Friedmann approach.

Taking now into account equation [39] and considering the solution determined for \(a(t)\), the equation for \(\eta(t)\) reduces to:

\[
y(y - 1) + 2y \frac{1}{1 - \frac{3}{2} \delta_w} \left(1 - \frac{3}{2} \delta_w\right) = 0 \Rightarrow
\]

\[
\Rightarrow \quad y = 0, \quad y = -1
\]

of which, only the second solution may be accepted due to the limit [22].

Considering the solution obtained for \(\eta(t)\), the form of the perturbation scale factor \(b(t)\) may be determined from the relation:

\[
\eta(t) = \frac{b^2(t)}{a^2(t)} \Rightarrow b^2(t) = \left(\frac{t}{t_0}\right)^{\frac{1+3\delta_w/2}{1-3\delta_w/2}} \quad (44)
\]

giving the following form, to the metric of the system:

\[
h_{\alpha\beta}(t,x) = \left(\frac{t}{t_0}\right)^{\frac{2}{1-3\delta_w/2}} \xi_{\alpha\beta} + \left(\frac{t}{t_0}\right)^{\frac{1+3\delta_w/2}{1-3\delta_w/2}} \theta_{\alpha\beta}(x) \quad (45)
\]

Lastly, it may be observed that by keeping the tridimensional curvature Ricci's tensor \(3R^\alpha_\beta\) in the equations, the system may be solved also for \(\delta_w = 0\). The solution obtained though, considering by a pure temporal point of view, is just a continuous extension of the solution actually derived, as it goes as \(a(t) \propto t\) and \(\eta(t) \propto 1/t\). The value \(\delta_w = -2/3\) instead, will lead to a complete different case, as it represent the case of a pure \(Cosmological\ Constant\ universe\ [11]\).
The **Lemaitre-Tolman-Bondi** model for spherically symmetric universes

The **Lemaitre-Tolman-Bondi** model \[10\], can be thought of as a generalization of the **RW** line element in which the requirement of homogeneity is dropped, while that of isotropy is kept. The **LTB** model in fact, may be isotropic but not homogeneous, basically due to the fact that by adopting a spherical symmetry, a preferred point is singled out, allowing the space to appear isotropic just by observing the universe from that particular point.

For this reason, it may be said that the **LTB** approach describes the evolution of a zero-pressure spherical overdensity in the mass distribution, resulting in a spherically symmetric, inhomogeneous solution of the **Einstein** equations, though the resulting solution is different from the **Schwarzschild** one \[16\] because of the non-stationarity.

In the synchronous reference system \[2\], the spherically symmetric line element for a **LTB** model can be written as:

$$ds^2 = dt^2 - e^{2\alpha}dr^2 - e^{2\beta}(d\theta^2 + \sin^2\theta d\phi^2) \quad (51)$$

where both \(\alpha\) and \(\beta\) were function of time \(t\), and radial distance from the preferred point \(r\).

The metric \[51\] will lead to just 3 independent **Einstein field equations**, that are:

\[
\begin{align*}
kt_1^{\alpha} &= -2\dot{\beta}' - 2\ddot{\beta}' + 2\dot{\alpha}\beta' \\
kt_1^{\beta} &= 2\ddot{\beta} + 3\dot{\beta}^2 + e^{-2\beta} - (\beta')^2 e^{-2\alpha} \\
kt_0^{\alpha} &= \ddot{\beta} + 2\dot{\beta} + e^{-2\beta} - e^{-2\alpha}[2\beta'' + 3(\beta')^2 - 2\alpha'\beta'] 
\end{align*}
\quad (52)
\]

while the other equations, will be related as:

\[
\begin{align*}
G_3^{\alpha} &= G_2^{\alpha} \\
G_2^{\beta} &= G_1^{\alpha} + [G_1^{\beta}]/2\beta'
\end{align*}
\quad (53)
\]

Originally, this kind of solution was solved under the assumption that the perfect fluid energy-momentum tensor is dominated by pressure-less dust \(P = 0\) and a cosmological constant term \(\Lambda\).

In this scheme, equations \[52\] rewrite as:

\[
\begin{align*}
0 &= -2\dot{\beta}' - 2\ddot{\beta}' + 2\dot{\alpha}\beta' \\
\Lambda &= 2\ddot{\beta} + 3\dot{\beta}^2 + e^{-2\beta} - (\beta')^2 e^{-2\alpha} \\
k\rho + \Lambda &= \ddot{\beta} + 2\dot{\beta} + e^{-2\beta} - e^{-2\alpha}[2\beta'' + 3(\beta')^2 - 2\alpha'\beta'] 
\end{align*}
\quad (54)
\]

where, the \(\dot{}\) and \(\ddot{}\) denote respectively derivatives with respect to time \(t\), or radial coordinate \(r\). Since the first equation of \[51\] vanishes, a relation between the two functions \(\alpha(r, t)\) and \(\beta(r, t)\) may be defined, and it read as:

\[
\dot{\beta}'/\beta' = \partial_t\ln\beta' = \dot{\alpha} - \dot{\beta} \quad (55)
\]

which admits the solution:

\[
\beta' = f(r)e^{\alpha - \beta} \quad (56)
\]

and consequently:

\[
\beta' e^{\beta} = \partial_r e^\beta = f(r)e^\alpha \quad (57)
\]

In the last two equations, \(f(r)\) was introduced as a generic function of the pure spatial coordinate \(r\). The form of \(f(r)\), will be defined upon theoretical considerations about how the dishomogeneity of the model behave in function of \(r\), or simply by fitting the resulting model with observational data.

Let us now introduce the commonly used scale factor \(a(r, t)\), by adopting the following parametrization:

\[
e^\beta = ra(r, t), \quad f(r) = [1 - r^2K^2]^{1/2} \quad (58)
\]

where \(K = K(r)\), is another free function of the pure spatial coordinate \(r\).

It must be observed that although the function \(K^2\) has been written as a square, to conform with the standard notation for the isotropic models, \(K^2\) can be negative, like in the open isotropic universe.

Using the expressions \[57\] and \[58\] the **LTB** line element \[51\] rewrites as follows:

\[
\begin{align*}
ds^2 &= dt^2 - \frac{[(ar)^2]^2}{1 - r^2K^2} dr^2 - (ar)^2(d\theta^2 + \sin^2\theta d\phi^2) 
\end{align*}
\quad (59)
\]

It may be observed that, if \(a(r, t)\) and \(K(r)\) were independent of the radial coordinate \(r\), the line
element \([59]\) corresponds to the standard FRW line element.
The remaining field equations \([54]\) introducing the relations \([57]\) and \([58]\) rewrite now as:

\[
(k\rho + \Lambda)[(ar)^3]' = 3[a^2 ar^3 + ar^3 K^2]'
\]

\[
\Lambda = \frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K^2}{a^2}
\]

Considering in particular the equation \([61]\) it may be observed that if multiplied by \(a^2\ddot{a}\), it turns into a total time derivative, which can be integrated getting:

\[
\dot{a}^2 a + aK^2 - \frac{\Lambda}{3} a^3 = F(r)
\]

where \(F(r)\) is another free function of the pure spatial coordinate \(r\), resulting from integration respect to time.

5 The LTB approach for spherically symmetric universes in low-redshift regime

The LTB approach for spherically symmetric universes exposed in the last section, will now be applied in low-redshift regime.

By using a LTB approach over a Lifshitz-Khalatnikov Quasi-Isotropic solution, Einstein equations will appear a lot easier, as the anisotropy will be limited to the sole radial coordinate \(r\).

While working in low-redshift regime for a matter plus cosmological constant universe, Einstein equations remains the same written in \([52]\), moreover from the first of them, the relation between \(a(r, t)\) and \(\beta(r, t)\) may be determined, following the same procedure of the original model \([10]\).

Using the first equation so, the metric will reduce to the form \([59]\) reflecting to the form \([60]\) and \([61]\) for the two remaining Einstein field equation.

To define a small inhomogeneous perturbation, over a flat background Friedmann universe, the scale factor \(a(r, t)\) will be defined as:

\[
a(r, t) = a_0(t) + a_p(r, t)
\]

where \(a_0(t)\) is the FRW unperturbed scale factor, while \(a_p(r, t)\) represent the inhomogeneous radial perturbation term.

It will be imposed though, that the perturbative term satisfies the condition:

\[
\lim_{t \to \infty} \frac{a_p(r, t)}{a_0(t)} = 0
\]

so that by looking at odier stage or future universe, the perturbative term, will be a small correction over the homogeneous and isotropic background universe, in order to interpret the actually observed universe.

Once the scale factor \(a(r, t)\) has been defined, following the relation \([63]\), how scale factors \(a_0(t)\) and \(a_p(r, t)\) evolves may be obtained using equation \([61]\) getting the following 2 equations:

\[
\frac{2\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2} = \Lambda
\]

\[
\frac{2\ddot{a}_p}{a_0} - \frac{2\dot{a}_0 \dot{a}_p}{a_0^2} + \frac{2a_0 \ddot{a}_p}{a_0^3} + \frac{2\dot{a}_0^2 a_p}{a_0^3} + \frac{K^2}{a_0^2} = 0
\]

From now on, the subscript 0 in \(a_0\) will be dropped, so that \(a_0 \to a\), the first equation \([65]\) admit a class of solutions following the form:

\[
a(i) = ce^{\sqrt{\frac{2}{3}}i} \left(d - f e^{-\sqrt{\frac{2}{3}}i}\right)^{\frac{2}{3}}
\]

where \(i\) represent the ratio \(i = t/t_0\).

Assuming that in the following we will denote with \(t_0\) the value corresponding to the present stage time of the universe, constants of integration introduced in the form of \([67]\) may be defined imposing the two conditions:

\[
\lim_{t \to 0} a(i) = 0
\]

\[
\lim_{i \to 1} a(i) = 1
\]

reducing equation \([67]\) to:

\[
a(i) = \frac{e^{\sqrt{\frac{2}{3}}i} \left(1 - e^{-\sqrt{\frac{2}{3}}i}\right)^{\frac{2}{3}}}{e^{\sqrt{\frac{2}{3}}i} \left(1 - e^{\sqrt{\frac{2}{3}}i}\right)^{\frac{2}{3}}}
\]
The two conditions 68 and 69 represent respectively the assumption that the cosmologic singularity [17] is approached with a vanishing scale factor, and the standard notation used in astronomy which define our actual scale factor \( a_0 \approx 1 \). Moreover, it may be observed that under precedent assumptions, the limit:

\[
\lim_{\Lambda \to 0} a(\tilde{t}) \approx \tilde{t}^{2/3} \tag{71}
\]

shows that for low values of \( \Lambda \), the model behaves as a pure matter dominated universe. Now taking into account equation 66, and using a factorization as:

\[
a_p(r, \tilde{t}) = b(\tilde{t})K^2(r) \tag{72}
\]

where \( b(\tilde{t}) \) represent the temporal dipendance of the perturbation scale factor \( a(r, \tilde{t}) \), leads to the following equation:

\[
\ddot{b}(\tilde{t}) + 3\dot{b}(\tilde{t}) \left( \frac{\Lambda}{3} + \frac{2\sqrt{3}\Lambda e^{-\sqrt{3}\tilde{t}}}{1 - e^{-\sqrt{3}\tilde{t}}} \right) + \\
+ b(\tilde{t}) \left( -\frac{2\Lambda}{3} + \frac{2\Lambda e^{-\sqrt{3}\tilde{t}}}{1 - e^{-\sqrt{3}\tilde{t}}} - \frac{2\Lambda e^{-2\sqrt{3}\tilde{t}}}{3(1 - e^{-\sqrt{3}\tilde{t}})^2} \right) + \\
+ \frac{e\sqrt{3}}{2} \left( 1 - e^{-\sqrt{3}\tilde{t}} \right)^2 + 2e\sqrt{3}\Lambda \left( 1 - e^{-\sqrt{3}\tilde{t}} \right)^{3/2} = 0 
\]

Furthermore, equation 73 was solved using 3 different values of \( \Lambda \), in particular the values tested were \( \Lambda = 0.3, 0.7, 1 \), the results obtained from the simulation for \( a(\tilde{t}) \) and \( b(\tilde{t}) \), in function of \( \Lambda \), will be shown in the following figures 1 and 2:

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![Figure 1: Evolution of \( a(t) \) scale factor in function of \( \Lambda \) for a LTB model.](image1)

![Figure 2: Evolution of \( b(t) \) scale factor in function of \( \Lambda \) for a LTB model.](image2)

It may be observed from figure 2 that the perturbation scale factor \( b(t) \) is a monotonically increasing function as the background scale factor \( a(t) \), although compared to that, it appear to be just a small correction around 1%. What just said, appear clearer by observing the results obtained from the simulation for \( \eta(t) = b^2(t)/a^2(t) \), which behaves as in figure 3:
The last graph shows that assumption 64 is valid, in fact, $\eta(t)$ become smaller than $10^{-4}$ already at $3t/t_0$, reflecting to a ratio $a_p/a_0 < 10^{-2}$ in correspondance of the same value.

Moreover, another test of reliability of the solution, is that 3 shows that inhomogeneous perturbations damp faster in function of $\Lambda$, in particular, a more accelerating universe will dissolve inhomogeneous perturbations quicker respect to a universe with lower value of $\Lambda$.

A same behaviour was shown in the contest of an inflationary scenario [18], where an accelerating phase was introduced to stretch dishomogeneities of our primordial universe.

How the perturbation term $a(r, t)$ behaves in function of radial coordinate $r$, or taking into account equation 72, in term of $K^2(r)$, is not addressed here, and $K^2(r)$ will remain an arbitrary function in this analysis.

In general, by observing inhomogeneities of our local universe, the form of $K(r)$ may be determined upon opportune fits, then once $K(r)$ has been defined, the form of the metric and energy density of the model may be obtained using equations 59 and 60.

6 Comparison of the LTB analysis with the actual universe

The LTB model for matter and cosmological constant universes, will now be compared with observational data to fit the model that best describe our actual universe, even though in the following, the function $K(r)$ will remain a free function of spatial coordinate $r$.

By now, observation suggest that our universe is composed for $\approx 30\%$ by matter, and for the remaining $\approx 70\%$ by D.E/cosmological constant [19]. For first so, the model in the limit $t \to t_0$, where $t_0$ represent present stage time, must return the same density composition.

To impose such thing, it will be considered the unperturbed energy density for Friedmann matter plus cosmological constant universes, that read as:

$$kp = \frac{\rho_m}{a^3(t)} + \Lambda$$

where $\rho$ represents our universe total energy density, $\rho_m$ is the fraction of total density as matter, and $\Lambda$ is the fraction as cosmological constant.

Switching to normalized densities by using Critical density $\rho_c$ [19], the actual total energy density get the value $\rho(t_0) = 1$.

Moreover taking into account the condition imposed on our $a(t)$ [60], the values of $\rho_m$ and $\Lambda$ may be easily defined using 74 getting:

$$1 = \rho(t_0) = \frac{0.3}{a^3(t_0)} + 0.7$$

which implies $\rho_m = 0.3$ and $\Lambda = 0.7$.

In the following, in figures 4 and 5 will be shown how scales factors $a(t)$ and $b(t)$ evolves for $\Lambda = 0.7$:

$$kp = \frac{\rho_m}{a^3(t)} + \Lambda$$
reflecting to the profile for \( \eta(t) = b^2(t)/a^2(t) \):

Furthermore, as was impossible to define a analytical form for \( b(t) \) and \( \eta(t) \), an approximated form for them was estimated by using Origin software to fit the numerical data.

In particular, the best fit for \( b(t) \), is with exponential function of the form:

\[
 b(\tilde{t}) = \exp(a + b\tilde{t} + c\tilde{t}^2) \tag{76}
\]

on which the notation \( \tilde{t} = t/t_0 \) was used, and the coefficients that best fit our numerical data, were set to:

| Coefficients | Value       |
|--------------|-------------|
| \( a \)     | \(-3.36013 \pm 5.1236E - 4\) |
| \( b \)     | \(0.08426 \pm 1.8646E - 4\)  |
| \( c \)     | \(0.02364 \pm 1.4995E - 5\)  |

A comparison between the fit function defined by \( b(t) \) and \( \eta(t) \), and the numerical simulated data for \( b(t) \) function (in bold black), will be exposed in the following:

\[
 \eta(\tilde{t}) = \exp(a + b/(\tilde{t} + c)) \tag{77}
\]

with the following values for the coefficients of \( \eta(t) \):

| Coefficients | Value       |
|--------------|-------------|
| \( a \)     | \(-18.97617 \pm 0.0127\) |
| \( b \)     | \(116.54695 \pm 0.2353\) |
| \( c \)     | \(8.10923 \pm 0.0093\) |

As done with \( b(t) \), a comparison between the fit function defined by \( \eta(t) \) and \( \eta(\tilde{t}) \) (in red), and the numerical simulated data for \( \eta(t) \) function (in bold black), will be shown:
7 Concluding remarks

We analyze two different, but complementary algorithms to deal with small inhomogeneous corrections to the isotropic Universe: on one hand, we studied the so-called quasi-isotropic solution, as implemented to a late dynamics, on the other one, we study a Lemaitre-Tolman-Bondi spherically symmetric solution, containing only small deviations depending on the pure radial coordinate.

We considered in both cases, sources in the form of a perfect fluid, but, while for the quasi-isotropic case we consider a dark energy equation of state with \(-1 < w < -1/3\), the spherically symmetric solution contains two different contribution, a matter fluid and a cosmological constant, respectively.

The basic result of our analysis, is demonstrating that the presence of a real dark energy contribution, prevent the possibility to deal with physical scales of the inhomogeneous correction being smaller than the actual Hubble scale of the Universe.

This constraint, comes from the necessity to rule out of the solution the spatial curvature contribution (due to inhomogeneous corrections), and it has very deep implications: the obtained perturbed solution is characterized by perturbations evolving only from a kinematical point of view, but unaffected by microphysical processes and, de facto they are not observable at the present time.

The situation is different for the Lemaitre-Tolman-Bondi model, where, we actually consider spherically symmetric deviations to the background dynamics, underlying the \(ΛCDM\) model for the actual Universe.

We construct a inhomogeneous perturbation to the isotropic cosmology, whose spatial dependence, i.e. whose spectrum, is not fixed by the solution method, remaining a useful degree of freedom for fitting different physical situations.

Apart from the conceptual difference qualitatively emerging in the present study between the two used algorithms, the main merit of this work is outlining that in the LTB case, correspond to a consistent solution with late time sub-Hubble inhomogeneities. Infact, the possibility to check the photon dynamics on different weakly inhomogeneous Universe, offer an interesting tool to test some physical properties of the actual low redshift Universe.

In particular, we suggest that the Lemaitre-Tolmann-Bondi model studied above, could be adopted to try to account with weak inhomogeneity profiles, the discrepancy existing between the value of the Hubble constant \(H_0\) as it is measure by WMAP and Planck Satellites and by the ground based surveys.

The elimination of such a discrepancy by the proposed scenario, could put limit on the local inhomogeneity profile of the actual Universe, possibly tested by the incoming mission Euclid.

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