Supergravity Theories in $D \geq 12$
Coupled to Super p-Branes

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Abstract

We construct supergravity theories in twelve and thirteen dimensions with the respective signatures $(10, 2)$ and $(11, 2)$ with some technical details. Starting with $N = 1$ supergravity in 10+2 dimensions coupled to Green-Schwarz superstring, we give $N = 2$ chiral supergravity in 10+2 dimensions with its couplings to super $(2 + 2)$-brane. We also build an $N = 1$ supergravity in 11+2 dimensions, coupled to supermembrane. All of these formulations utilize scalar (super)fields intact under supersymmetry, replacing the null-vectors introduced in their original formulations. This method makes all the equations $SO(10, 2)$ or $SO(11, 2)$ Lorentz covariant, up to modified Lorentz generators. We inspect the internal consistency of these formulations, in particular with the usage of the modified Lorentz generators for the extra coordinates.

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1. Introduction

There has been strong indication that higher-dimensional theories of extended objects in higher than eleven dimensions (11D) [1], such as F-theory [2] or S-theory [3], all with multiple time coordinates [4] have many promising features. In particular, these theories may well provide the guiding principle for understanding non-perturbative features or vacuum structures of superstring [5] or supermembrane [6] theories via M-theory [7][8][9][10] in terms of supersymmetry algebra, e.g., in \( D = 10 + 2 \) [3] or \( D = 11 + 3 \) [11].

Motivated by this philosophy, explicit field theoretic formulations of a supersymmetric Yang-Mills theory in \( D = 10 + 2 \) [12], or in \( D = 11 + 3 \) [13], and of an \( N = 1 \) supergravity theory [14], or of an \( N = 2 \) supergravity theory [15] have been recently presented. Further developed are an invariant lagrangian for supersymmetric Yang-Mills theory in \( D = 10 + 2 \), as well as a set of Lorentz covariant field equations for the first time [16], in all the even dimensions higher than \( D = 12 \) [17], or more general supersymmetry algebras [18].

All of these previous formulations were based on null-vectors that are common in these dimensions with more than a single time coordinate. The existence of such supergravity theories had been also vaguely predicted in various contexts, such as the categorization of Clifford algebra in arbitrary dimensions [19], due to the smallness of the freedoms in the Majorana-Weyl spinors in 12D, when there are two time coordinates [19][20]. However, there is also a certain no-go theorem [21][22] that prohibits ‘conventional’ supergravity theories in such higher-dimensions. A recent new technique in [16] introducing scalar fields intact under supersymmetry, seems to bypass (but not overcome) this no-go theorem by making the higher-dimensional supergravity/supersymmetry formulations manifestly \( SO(10, 2) \) Lorentz covariant, up to modified Lorentz generators.

In this present paper, we give improved formulations of higher-dimensional supergravity in which all the null-vectors in the previous formulations [12][14][15] are replaced by the gradients of scalar (super)fields which are invariant under supersymmetry, both in superspace and component. By this prescription, all of these higher-dimensional supergravity theories will become formally Lorentz covariant, leaving the non-covariant nature to the modified Lorentz generators.

This paper is organized as follows. We start with the \( N = 1 \) supergravity in \( D = 10 + 2 \) in superspace [14], where the consistency of the system is guaranteed by the satisfaction of all the Bianchi identities, based on the improved method using the gradient of scalar superfields making the system \( SO(10, 2) \) Lorentz covariant as manifest as possible. We give rather detailed construction of this theory which was not given enough in our previous paper [14], that will be common features of other supergravity theories. As a

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2In the expression \( D = s + t \) the number \( s \) (or \( t \)) denotes that of spacial (or time) coordinates. When clear from the context, we also use the expression \( 12D \) or \( D = 12 \).
probe for the validity of this supergravity theory, we confirm fermionic invariance of Green-Schwarz superstring put in this 12D supergravity background. We next give the component formulation of $N = 2$ supergravity in $D = 10 + 2$ [15] predicted by F-theory [2], which now looks straightforward, once the $N = 1$ case is understood. We further confirm the fermionic symmetry on the $(2 + 2)$-dimensional world-supervolume of super $(2 + 2)$-brane coupled to this $N = 2$ supergravity theory in 12D. Based on the experience in 12D supergravity, we build an $N = 1$ supergravity in $D = 11 + 2$, which can be consistently coupled to supermembranes [6] with fermionic symmetries. Appendix A and B are devoted for useful identities in 12D and 13D, while in Appendix C, we inspect the consistency of our modified Lorentz generators. In Appendix D, we study the consistency of our extra constraints in component with supersymmetry.

2. $N = 1$ Supergravity in $D = 10 + 2$

2.1 Notations

We first establish all the notational foundation, in order to deal with our $N = 1$ supergravity in $D = 10 + 2$. We first fix our metric to be $(\eta_{ab}) = \text{diag.} \ (-,+,\cdots,+,+,−)$, where we use the indices $a, b, \cdots = (0), (1), \cdots, (9), (11), (12)$ for local Lorentz coordinates, while $m, n, \cdots = 0, 1, \cdots, 9, 11, 12$ for curved coordinates, in this section of superspace. Accordingly, our Clifford algebra is $\{\gamma_a, \gamma_b\} = +2\eta_{ab}$. Relevantly, we have $\epsilon^{012\cdots91112} = +1$, and $\gamma_{(13)} \equiv \gamma_0\gamma_{(1)}\cdots\gamma_9\gamma_{(10)}\gamma_{(12)}$. Compared with the notation in ref. [14], the only difference is the usage of $\gamma^a$ instead of $\sigma^a$ for $\gamma$-matrices. We next setup two null-vectors, which have zero norm, and are orthogonal to each other [12] :

$$(n^a) = (0,0,\cdots,0, +\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \ , \quad (n_a) = (0,0,\cdots,0, +\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}) \ ,$$

$$(m^a) = (0,0,\cdots,0, +\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}) \ , \quad (m_a) = (0,0,\cdots,0, +\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \ . \quad (2.1.1)$$

It is also convenient to use $\pm$-indices [14][15], in order to handle the extra dimensions:

$$V_\pm \equiv \frac{1}{\sqrt{2}}(V_{(11)} \pm V_{(12)}) \ . \quad (2.1.2)$$

Accordingly, we have $n_+ = m_+ = +1$, $n_- = m_- = 0$, and therefore

$$n^an_a = m^am_a = 0 \ , \quad m^an_a = m^+n_+ = m_-n_- = +1 \ . \quad (2.1.3)$$

Now the necessity of $\pm 1/\sqrt{2}$ in (2.1.1) is now obvious, maintaining the normalization $m^an_a = +1$.

The reason we use two different index systems in superspace and component formulations in this paper is due to their proper advantages. For example, the indices $\alpha, \beta, \cdots$ are more convenient for frequently-used spinorial components in superspace, while in component formulation these spinorial indices are usually implicit and suppressed.
Other important quantities to be defined are the projection operators in the space of two extra coordinates, satisfying the usual ortho-normality conditions [14]:

\[ P_\uparrow \equiv \frac{1}{2} \eta \bar{\eta} = \frac{1}{2} \gamma^+ \gamma^-, \quad P_\downarrow \equiv \frac{1}{2} \eta \bar{\eta} = \frac{1}{2} \gamma^- \gamma^+, \quad (2.1.4a) \]
\[ P_\uparrow P_\uparrow = +P_\uparrow, \quad P_\downarrow P_\downarrow = +P_\downarrow, \quad P_\uparrow + P_\downarrow = +I, \quad (2.1.4b) \]
\[ P_\uparrow \downarrow \equiv P_\uparrow - P_\downarrow = \gamma^+, \quad (2.1.4c) \]

where as usual \( \eta \bar{\eta} \equiv m^a \gamma_a \) and \( \eta \bar{\eta} \equiv n^a \gamma_a \). The following symmetry properties are also useful for the manipulations of \( \gamma \)-matrices:

\[ (\eta \bar{\eta})_{\alpha \beta} = - (\eta \bar{\eta})_{\beta \alpha}, \quad (\eta \bar{\eta})_{\alpha \beta} = - (\eta \bar{\eta})_{\beta \alpha}, \quad (2.1.5) \]

Note that in our signature convention in 12D, the dotted (or undotted) spinors have positive (or negative) chirality under \( \gamma_{13} \) [14], as opposed to the usual convention. We also use the collective spinorial indices \( \underline{a} \equiv (a, \hat{a}), \underline{b} \equiv (\beta, \hat{\beta}), \ldots \) to symbolize both chiralities, for the chiral spinorial indices \( \alpha, \beta, \ldots = 1, 2, \ldots, 32 \) and \( \hat{a}, \hat{\beta}, \ldots = \hat{1}, \hat{2}, \ldots, \hat{32} \).

We next study various features of our modified Lorentz generators introduced in [14]. These modified Lorentz generators are defined by [14]

\[ (\widetilde{M}_{ab})^{cd} \equiv +\delta_{[a}^c \delta_{b]}^d, \quad (2.1.6a) \]
\[ (\widetilde{M}_{ab})_{\alpha \beta} \equiv + \frac{1}{2} (\gamma_{ab} P_\uparrow)_{\alpha \beta}, \quad (\widetilde{M}_{a\underline{b}})_{\underline{a} \underline{b}} \equiv + \frac{1}{2} (P_\downarrow \gamma_{ab})_{\underline{a} \underline{b}}. \quad (2.1.6b) \]

Here \( \delta \) is defined by

\[ \delta_{ab} \equiv \delta_{ab} - m_a n_b = \begin{cases} 
\delta_i^j & \text{(for } a = i, \ b = j) \\
\delta_+^+ = 1 & \text{(for } a = +, \ b = +) \\
0 & \text{(otherwise)} \end{cases}, \quad (2.1.7) \]

Here \( i, j, \ldots \) are purely 10D indices, and in particular, \( \delta_-^- = 0 \). This is to be consistent with the spinorial representation (2.1.6b) satisfying \( (\widetilde{M}_{-\alpha})_{\underline{a} \underline{b}} = 0 \). The vectorial representation (2.1.6a) implies that \( (\widetilde{M}_{++})^{cd} = 0 \), causing no problem with \( (\widetilde{M}_{++})_{\underline{a} \underline{b}} \neq 0 \), because as long as \( \widetilde{M}_{ab} \) is always accompanied by \( \phi_A^{ab} \), the combination \( \phi_A^{+-} \widetilde{M}_{+-} \) vanishes due to the extra constraint \( \phi_A^{+-} = 0 \), to be systematically given in (2.3.6). Note also that the only effect of (2.1.7) is to get rid of the unwanted generators \( (\widetilde{M}_{-\alpha})^{cd} \) in the vectorial representation which does not vanish in the combination \( \phi_A^{-b} \widetilde{M}_{-b} \) even with the extra constraint \( \phi_A^{+-b} = 0 \) on \( \phi \). As is seen in (C.3) in Appendix C, we emphasize that these modified Lorentz generators satisfy the usual Jacobi identities among \( \widetilde{M} \)'s, which is the foundation of the Bianchi identities in superspace. In the next subsection, we will confirm these Bianchi identities at engineering dimensions \( d = 1 \) and \( d = 3/2 \).

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4 The engineering dimensions are defined in the usual way in superspace, i.e., we put dimension 1/2 (or 1) for a spinorial derivative \( \nabla_a \) (or bosonic derivative \( \nabla_a \)), which determine all the dimensions of torsion/curvatures, e.g., the dimension of \( T_{\alpha \beta \gamma} \) is 1/2. In this paper, we do not use differential forms in order to avoid confusing expressions especially for index-contractions.
In order to see the internal consistency of our modified Lorentz generators, we first show that all the null-vectors are really ‘constant’ under our superspace covariant derivatives \[14\]:

\[
\nabla_M n^- = \partial_M n^- + \frac{1}{2} \phi_M^{ab}(\widehat{\mathcal{M}}_{ba})^{+\,n_+} = 0 ,
\]

\[
\nabla_M m^+ = \partial_M m^+ + \frac{1}{2} \phi_M^{ab}(\widehat{\mathcal{M}}_{ba})^{\,m_+} = 0 .
\] (2.1.8)

We next establish the action of the Lorentz generators on spinorial components \[14\]:

\[
\widehat{\mathcal{M}}_{ab} \Psi_{\alpha} = + (\widehat{\mathcal{M}}_{ab})_\alpha^\beta \Psi_\beta , \quad \widehat{\mathcal{M}}_{ab} \Psi^\alpha = - \Psi^\beta (\widehat{\mathcal{M}}_{ab})^\alpha_\beta ,
\]

\[
\widehat{\mathcal{M}}_{ab} \bar{\Psi}_{\dot{\alpha}} = + (\widehat{\mathcal{M}}_{ab})_{\dot{\alpha}}^{\dot{\beta}} \bar{\Psi}_{\dot{\beta}} , \quad \widehat{\mathcal{M}}_{ab} \bar{\Psi}^{\dot{\alpha}} = - \bar{\Psi}^{\dot{\beta}} (\widehat{\mathcal{M}}_{ab})^{\dot{\alpha}}_{\dot{\beta}} .
\] (2.1.9)

Accordingly, we get the commutators involving the charge conjugation matrices \[14\]:

\[
[\widehat{\mathcal{M}}_{ij}, C_{\alpha\beta}] = + \frac{1}{2} (\gamma_{ij} P_1)_{\alpha\beta} , \quad [\widehat{\mathcal{M}}_{ij}, C^{\alpha\beta}] = - \frac{1}{2} (\gamma_{ij} P_1)^{\alpha\beta} ,
\]

\[
[\widehat{\mathcal{M}}_{ij}, C^\dagger_{a\beta}] = + \frac{1}{2} (\gamma_{ij} P_1)_{a\beta} , \quad [\widehat{\mathcal{M}}_{ij}, C_{a\beta}^\dagger] = - \frac{1}{2} (\gamma_{ij} P_1)^{a\beta} ,
\]

\[
[\widehat{\mathcal{M}}_{-i}, C_{a\beta}] = [\widehat{\mathcal{M}}_{-i}, C^{a\beta}] = [\widehat{\mathcal{M}}_{-i}, C_{a\beta}^\dagger] = [\widehat{\mathcal{M}}_{-i}, C^\dagger_{a\beta}] = 0 ,
\] (2.1.10)

which can be easily confirmed using our definitions.

The significance of (2.1.10) is that the charge conjugation matrices transform under our Lorentz generators, but are no longer constants. Even though this sounds disastrous, we can easily see that all the $\gamma$-matrices used in our superspace constraints in (2.3.4) are shown to be constant, as desired. In order to see this, we give the important relations

\[
[\widehat{\mathcal{M}}_{ab}, (\gamma^e)_{\gamma}^\delta] = \delta_{[a}^e (\gamma_{b])_{\gamma}^\delta + \frac{1}{2} (\gamma_{ab} P_1 \gamma^e)_{\gamma}^\delta - \frac{1}{2} (\gamma_{ab} P_1 \gamma^e)_{\gamma}^\delta ,
\]

\[
[\widehat{\mathcal{M}}_{ab}, (\gamma^e)_{\gamma}^\delta] = \delta_{[a}^e (\gamma_{b])_{\gamma}^\delta + \frac{1}{2} (\gamma_{ab} P_1 \gamma^e)_{\gamma}^\delta - \frac{1}{2} (\gamma_{ab} P_1 \gamma^e)_{\gamma}^\delta ,
\] (2.1.11)

which can help us to prove that

\[
[\widehat{\mathcal{M}}_{ab}, (\hat{\phi})_{\gamma}^\delta] = [\widehat{\mathcal{M}}_{ab}, (\hat{\phi})_{\gamma}^\delta] = 0 ,
\]

\[
[\widehat{\mathcal{M}}_{ab}, (\hat{\eta})_{\gamma}^\delta] = [\widehat{\mathcal{M}}_{ab}, (\hat{\eta})_{\gamma}^\delta] = 0 .
\] (2.1.12)

These relations yield the desirable commutativity of $\nabla_A$ with $\hat{\phi}$ and $\hat{\eta}$:

\[
[\nabla_A, (\hat{\phi})_{\alpha}^\beta] = [\nabla_A, (\hat{\eta})_{\alpha}^\beta] = [\nabla_A, (P_1)_{\alpha}^\beta] = [\nabla_A, (P_1)_{\alpha}^\beta] = 0 ,
\] (2.1.13a)

\[
[\nabla_A, (\gamma^a P_1)_{\alpha}^\beta] = [\nabla_A, (P_1^a)_{\alpha}^\beta] = 0 ,
\] (2.1.13b)

\[
[\nabla_A, (\hat{\phi}^a P_1)_{\alpha}^\beta] = [\nabla_A, (P_1^a)_{\alpha}^\beta] = 0 .
\] (2.1.13c)

As will be seen, these relations can help us to see, e.g.,

\[
[\nabla_A, T_{\alpha}^\beta] = \frac{1}{5} 0 ,
\] (2.1.14)
as desired for the total consistency of our constraints. See Appendix C for other notes.

2.2 Scalar Superfields Intact under Supersymmetry

In ref. [14], we formulated our 12D supergravity using the null-vectors introduced above. However, this formulation had a drawback of breaking the manifest \( SO(10,2) \) Lorentz covariance in 12D. In ref. [16], we have improved this for global supersymmetry by introducing a scalar field \( \varphi \) whose gradient replaces the null-vector: \( n_a = \nabla_a \varphi \). In this section, we use this prescription to re-formulate our \( N = 1 \) supergravity in [14], avoiding the usage of null-vectors, to make the \( SO(10,2) \) Lorentz covariance as manifestly as possible. In our supergravity formulation, we need an additional scalar superfield \( \tilde{\varphi} \) in addition to \( \varphi \) [16], satisfying the constraints [16]

\[
\nabla_a \nabla_b \varphi = 0 \ , \quad \nabla_a \nabla_b \tilde{\varphi} = 0 \ , \quad \nabla_a \varphi = 0 \ , \quad \nabla_a \tilde{\varphi} = 0 \ , \\
(\nabla_a \varphi)^2 = 0 \ , \quad (\nabla_a \tilde{\varphi})^2 = 0 \ , \quad (\nabla_a \varphi)(\nabla^a \varphi) = 1 
\]

(2.2.1)

As is easily seen, a set of non-trivial solutions to these differential equations is

\[
\nabla_a \varphi = n_a \ , \quad \nabla_a \tilde{\varphi} = m_a 
\]

(2.2.2)

where \( n_a, m_a \) are the same null-vectors we have already introduced. The novel feature of the superfield equations (2.2.1) is their manifest \( SO(10,2) \) Lorentz covariance in 12D with no usage of null-vectors any longer, unless we consider their solutions. These scalar superfields enable us to re-formulate the whole supergravity systems given in [14][15]. Until we replace these gradient superfields by the null-vectors, all the superfield equations in our system are manifestly \( SO(10,2) \) Lorentz covariant, except for the modified Lorentz generators. Note also that if we choose a set of solutions different from (2.2.2), e.g., \( \varphi = \tilde{\varphi} = 0 \), then the vacuum of the system will collapse to non-supersymmetric vacuum, even without the supersymmetry algebra \( \{Q_\alpha, Q_\beta\} = (\gamma^c)^{\alpha\beta} P_c n_d \). To put it differently, this feature is also understood as choosing non-trivial solutions for BPS condition \( \det \{Q_\alpha, Q_\beta\} = 0 \) for these higher-dimensional algebra of supersymmetry [3][11].

In an ordinary supersymmetric theory, there will arise a problem for introducing any non-constant field invariant under supersymmetry. For example, the commutator \( \{\nabla_a, \nabla_b\} \varphi = (\gamma^c)_{\alpha\beta} \nabla_c \varphi \) does not hold for a non-constant but superinvariant scalar field \( \varphi(x) \), because the l.h.s. is zero due to \( \nabla_a \varphi = 0 \), while the r.h.s. is non-zero due to \( \nabla_c \varphi \neq 0 \). In our formulation, however, thanks to (2.2.1) and \( T_{\alpha\beta} \) in (2.3.4a), the r.h.s. also vanishes due to \( T_{\alpha\beta} \nabla_c \varphi = 0 \) and \( T_{\alpha\beta} \nabla_c \tilde{\varphi} = 0 \), as will be seen in (2.3.4a), consistently with the vanishing l.h.s. both for \( \varphi \) and \( \tilde{\varphi} \). This is one of the most important features of our formulation based on superinvariant scalar fields, replacing the original null-vectors.

The prescription of replacing all the null-vectors in [14] by the gradients of scalars superfields is transparent in superspace formulation, where the superfield equations are directly
associated with Bianchi identities. For example, the projection operators (2.1.4) can be re-expressed as

$$P_\uparrow \equiv \frac{1}{2} (\gamma^a \gamma^b)(\nabla_a \varphi)(\nabla_b \bar{\varphi}) , \quad P_\downarrow \equiv \frac{1}{2} (\gamma^a \gamma^b)(\nabla_a \bar{\varphi})(\nabla_b \varphi) , \quad P_{\uparrow\downarrow} \equiv P_\uparrow - P_\downarrow . \quad (2.2.3)$$

This is because even though these gradients seem ‘non-constant’ as they stand, once those additional superfield equations (2.2.1) are taken into account, they are effectively ‘constant’, and these operators play exactly the same role as the projection operators in (2.1.4). The same is also true for Lorentz covariant derivatives $\nabla_A$.

In (2.1.8) we were concerned with the null-vectors. We can repeat the same analysis with the gradients $\nabla_A \varphi$ and $\nabla_A \bar{\varphi}$, now with $n_a$ replaced by $\nabla_a \varphi$ and $m_a$ replaced by $\nabla_a \bar{\varphi}$:

$$\nabla_M \nabla^- \varphi = \partial_M \nabla^- \varphi + \frac{1}{2} \phi_M^{ab} (\tilde{\mathcal{M}}_{ba})^- \nabla_+ \varphi = 0 ,$$
$$\nabla_M \nabla^+ \bar{\varphi} = \partial_M \nabla^+ \bar{\varphi} + \frac{1}{2} \phi_M^{ab} (\tilde{\mathcal{M}}_{ba})^+ \nabla_- \bar{\varphi} = 0 . \quad (2.2.4)$$

In principle, we can use unmodified Lorentz generators $\mathcal{M}_{ab}$ everywhere in these equations. However, as we will see in (2.3.10), a Bianchi identity at dimension one requires the modified form of Lorentz generators with $P_\uparrow$ inserted like in (2.1.6). Or to put it differently, only a particular set of solutions for the scalar fields $\varphi, \bar{\varphi}$ achieves the satisfaction of all the Bianchi identities in a non-trivial way. In this sense, the original $SO(10,2)$ Lorentz covariance is broken at the level of solutions, or equivalently by the particular choice of modified Lorentz generators (2.1.6).

As shrewd readers may have already noticed, we can even make our modified Lorentz generators (2.1.6) themselves more ‘covariant’, by

$$\tilde{\delta}_a^b \equiv \delta_a^b - (\nabla_a \bar{\varphi})(\nabla_b \varphi) , \quad (2.2.5)$$

replacing (2.1.7). As has been already mentioned, since the $P_\uparrow, P_\downarrow$ can be replaced by (2.2.3), this prescription removes all the non-covariant ingredients in our 12D supergravity formulation. However, since this sort of field-dependent Lorentz generators, which becomes constant only on-shell, might be controversial, we do not claim that this method makes the whole system totally $SO(10,2)$ covariant, leaving this note just as another important ingredient of our supergravity formulations.

## 2.3 Bianchi Identities and Superspace Constraints

We next study the Bianchi identities in our system to be satisfied, which are $T$-, $G$- and $R$-Bianchi identities:

$$\frac{1}{2} \nabla_{[A} T_{BC]} D - \frac{1}{2} T_{[AB]} E_{T(C]} - \frac{1}{4} R_{[AB]} E^f (\mathcal{M}^e_f)_{|C]} D \equiv 0 , \quad (2.3.1)$$
$$\frac{1}{6} \nabla_{[A} G_{BCD]} - \frac{1}{4} T_{[AB]} E G_{|CD]} \equiv 0 , \quad (2.3.2)$$
$$\frac{1}{2} \nabla_{[A} R_{BC]} d^e - \frac{1}{2} T_{[AB]} E R_{|C]} d^e \equiv 0 . \quad (2.3.3)$$
In all the sections for superspace, we use the symbol \([\alpha\beta]\) for (anti)symmetrization without normalization, \(i.e.,\) \(A_{\alpha\beta} \equiv A_{\alpha\beta} + A_{\beta\alpha}\). We sometimes call (2.3.1) - (2.3.3) respectively the \((ABC, D)\), \((ABCD)\), and \((ABC, de)\)-type Bianchi identities for convenience sake.

One subtlety to be examined is the satisfaction of the \(R\)-Bianchi identities (2.3.3). In the usual supergravity system, the \(R\)-Bianchi identities are automatically satisfied, once the \(T\)-Bianchi identities hold [23]. In ref. [14], this was non-trivial due to the modified Lorentz generators. The point there was that all the \(R\)-Bianchi identities are still satisfied, if we follow the same proof in the manifestly covariant case in [23], even though the whole \(SO(10, 2)\) Lorentz covariance was lost.

With these preliminaries at hand, we now present our superspace constraints. Our field content is formally the same as the \(N = 1\) supergravity in 10D [24], namely \((e_m^a, \psi_m^a, B_{mn}, \bar{\chi} \cdot \Phi; \varphi, \bar{\varphi})\), where \(e_m^a\) is the zwölfbein, \(\psi_m^a\) is the Majorana-Weyl gravitino, \(B_{mn}\) is a real antisymmetric tensor, \(\bar{\chi} \cdot \Phi\) is an anti-chiral Majorana-Weyl dilatino, and \(\Phi\) is a real dilaton. Our results for constraints in superspace are [14]

\[
\begin{align*}
T_{\alpha\beta}^\gamma &= (\gamma^{cd})_{\alpha\beta} \nabla_d \varphi + (\gamma^{de})_{\alpha\beta} (\nabla^c \varphi) (\nabla_e \bar{\varphi}) = (\gamma^{cd})_{\alpha\beta} \nabla_d \varphi + (P_\gamma)_{\alpha\beta} \nabla^c \varphi, \\
G_{\alpha\beta \gamma} &= T_{\alpha\beta}^\gamma, \\
T_{\alpha\beta} = (P_\gamma)_{\alpha\beta} (\gamma^{\gamma} \bar{\chi} \cdot \beta) \nabla_e \varphi - (\gamma^{ab})_{\alpha\beta} (P_{\gamma} \gamma_a \bar{\chi}) \nabla_b \varphi, \\
\nabla \Phi &= (\gamma^{\gamma} \bar{\chi} \cdot \alpha) \nabla_e \varphi, \\
\nabla \chi \cdot \beta &= - \frac{1}{24} (\gamma^{de} P_{\gamma})_{\alpha\beta} G_{cde} + \frac{1}{2} (\gamma^{e} P_{\gamma})_{\alpha\beta} \nabla_e \Phi - (\gamma^{\gamma} \bar{\chi} \cdot \alpha) \nabla_e \varphi, \\
T_{\alpha\beta}^\gamma &= 0, \quad T_{\alpha\beta} = 0, \quad G_{\alpha\beta \gamma} = 0, \\
R_{\alpha\beta \gamma \delta} &= + (\gamma^{ef})_{\alpha\beta} G_{f \gamma \delta} \nabla_e \varphi, \\
\nabla \nabla \bar{\chi} \cdot \beta &= \frac{1}{2} \gamma^{ef} \gamma_{[b} T_{c d]} \gamma \nabla_e \varphi = - \nabla \alpha T_{\beta c d}, \\
R_{\alpha \beta c d} &= + (\gamma^{e})_{\gamma} (\gamma [c T_{d \beta}] \gamma \nabla_e \varphi, \\
\nabla \nabla T_{\beta c} &= - \frac{1}{4} (\gamma^{de} P_{\gamma})_{\alpha} \delta R_{b c d e} + T_{\beta c} \gamma (\gamma^{e} \bar{\chi} \cdot \alpha) \nabla_e \varphi + (P_{\gamma})_{\alpha} \delta (\nabla^{e} T_{\beta c}) \nabla_e \varphi \\
&\quad + (\gamma^{de} T_{\beta c})_{\alpha} (P_{\gamma} \gamma_d \bar{\chi}) \delta \nabla_e \varphi, \\
\nabla_{\alpha} \varphi &= \nabla_{\alpha} \bar{\varphi} = 0, \quad (\nabla_{\alpha} \varphi)^2 = (\nabla_{\alpha} \bar{\varphi})^2 = 0, \quad (\nabla_{\alpha} \varphi)(\nabla_{\alpha} \bar{\varphi}) = 1, \\
\nabla_{\alpha} \nabla_{\beta} \varphi &= \nabla_{\alpha} \nabla_{\beta} \bar{\varphi} = 0.
\end{align*}
\]

Here \(P_{\gamma}, P_{\alpha}, P_{\alpha \gamma}\) are exactly the same as in (2.2.3). Our short-hand notation, such as \((\gamma^{\gamma} \bar{\chi} \cdot \beta) \equiv (\gamma^{\epsilon} \bar{\chi} \cdot \beta) \gamma_{\alpha\beta}\), and \((\gamma^{e} \gamma_{b} T_{c d}) \equiv (\gamma^{e} \gamma_{b}) \gamma_{\alpha\beta} (\gamma^{\epsilon} \bar{\chi} \cdot \beta) \gamma_{\alpha\beta} T_{c d}\), is always taken for granted. As they stand, these equations are formally \(SO(10, 2)\) Lorentz covariant.
As usual in higher-dimensional supergravity, we have extra constraints [14]:

$$T_{AB} \nabla_c \varphi = 0 \ , \ G_{ABc} \nabla_c \varphi = 0 \ , \ T_{aB}^c \nabla^a \varphi = 0 \ , \quad (2.3.5)$$

$$R_{ABC}^d \nabla_d \varphi = R_{ABC}^d \nabla^a \varphi = 0 \ , \quad (2.3.6)$$

$$(\nabla^a \varphi) \nabla_a \Phi = 0 \ , \ (\nabla^a \varphi) \nabla_a \Phi_\beta = 0 \ , \quad (2.3.7)$$

$$(\gamma^\alpha)_{\alpha \beta \gamma} \Phi = 0 \ , \ T_{ab}^c (\gamma^d)_{\alpha \beta \gamma} \nabla_d \varphi = 0 \ ,$$

$$\phi_{Ab}^c \nabla_c \varphi = \phi_{ab}^c \nabla^a \varphi = 0 \ . \quad (2.3.8)$$

Note that $\nabla_c \varphi$ appears instead of $\nabla_c \varphi$ in (2.3.8). Notice that not all the extra components in these fields are deleted by these constraints (2.3.5) - (2.3.9). For example, if we had imposed also $G_{ABC} \nabla^c \varphi = 0$, then there would be no extra component left over for the superfield $G_{abc}$, and therefore the system is totally reduced to the conventional 10D theory [24]. We stress that the non-trivial feature of our 12D theory is that not all extra components are deleted by these conditions, while all the Bianchi identities are satisfied.

We give next some details in the derivation of these results. We first mention the second term in $T_{\alpha \beta} \nabla^c \varphi$ in (2.3.4a), which is additional compared with the globally supersymmetric result in [12]. As will be seen, this additional term will be also important, when we confirm $\kappa$-fermionic symmetry in the Green-Schwarz superstring coupled to our supergravity background. In fact, the crucial relationship $\Pi^{\alpha a} \nabla_{\alpha a} \varphi = 0$ will hold, only when the second term in (2.3.4a) is present in the system. Another important ingredient to be mentioned is the modification of our Lorentz generators (2.1.6). This was required by the $(\alpha \beta \gamma, \delta)$-type Bianchi identity at dimension $d = 1$. We found that terms like $(\sigma^{ab}_{\alpha \beta})(\sigma^{cd}_{\gamma \delta}) G_{abcd} (\nabla_f \varphi) (\nabla_b \varphi) (\nabla_d \varphi)$ would be left over, if we did not have the modification of $\mathcal{M}_{ab}$, and these terms are completely cancelled, when the Lorentz generator are modified like (2.1.6) with $P_{\uparrow}$, via the term

$$R_{(\alpha \beta \gamma)} = -\frac{1}{4} R_{(\alpha \beta)} \delta G_{cd}(\gamma_{cd} P_{\uparrow}(\gamma^\delta) \ , \quad (2.3.10)$$

with $P_{\uparrow}$ inserted in the Lorentz generator, instead of the original one $(\gamma^{cd})_{\gamma \delta}$.

We next describe the derivation of our other constraints in (2.3.4). As usual in supergravity theory, we put some unknown coefficients $a_1$, $a_2$, $c_1$, $c_2$, $g$, like

$$T_{\alpha \beta}^\gamma = a_1 (P_{\uparrow})_{(\alpha} \gamma (\gamma^d \Phi)_{\beta)} \nabla_{b \varphi} + a_2 (\gamma^a)_{\alpha \beta} (P_{\uparrow})_{\gamma a} \nabla_{b \varphi} \ ,$$

$$\nabla_{\alpha} \Phi_\beta = c_1 (\gamma^{e \delta} P_{\uparrow})_{\alpha \beta} \Phi + c_2 (\gamma^{e \delta} P_{\uparrow})_{\alpha \beta} \nabla_{e \varphi} + g (\gamma^c \Phi)_{\alpha} \Phi_\beta \nabla_{c \varphi} \ , \quad (2.3.11)$$

and require the satisfaction of all the Bianchi identities. First of all, $c_2$ is fixed by the closure on $\Phi$,

$$\{\nabla_\alpha, \nabla_\beta\} \Phi = \nabla_{\alpha} [ (\gamma^c \Phi)_{\beta} \nabla_{c \varphi}] = +2 c_2 (\gamma^{cd})_{\alpha \beta} (\nabla_{c \varphi})(\nabla_{d \varphi}) \ , \quad (2.3.12)$$
as $c_2 = +1/2$, compared with $T_{\alpha\beta}\nabla\varphi$. Next step is to go to dimensions $d = 0$ Bianchi identity of $(\alpha\beta\gamma\delta)$-type, which is easily satisfied by help of relation (A.5) in Appendix, as well as the properties of null-vectors like $(\partial/\partial\varphi)(\partial/\partial\varphi) \equiv 0$.

The less trivial sector arises at $d = 1/2$ for the $(\alpha\beta\gamma d)$ and $(\alpha\beta\gamma, \delta)$ Bianchi identities. The former yields only three sorts of terms with at least two $\nabla\varphi$: If we denote the l.h.s. of the $(ABCD)$-Bianchi identity by $X_{ABCD}$, then some appropriate manipulations yield

$$X_{\alpha\beta\gamma d} = 2(a_1 + a_2) \left[ + 2(\gamma_d^a)_{(\alpha\beta\gamma|} (\gamma^b)_{\gamma)(\nabla_a\varphi)(\nabla_b\varphi) - (\gamma^{ab})_{(\alpha\beta|} (\gamma^c)_{\gamma)} (\nabla_b\varphi)(\nabla_c\varphi)(\nabla_d\varphi)(\nabla_\varphi) \right].$$

(2.3.13)

Since these two terms are independent, we get the condition

$$a_1 = -a_2.$$  

(2.3.14)

Fortunately, the $(\alpha\beta\gamma, d)$-type Bianchi identity at $d = 1/2$ is automatically satisfied, once this $(\alpha\beta\gamma\delta)$-type Bianchi identity holds.

Next final non-trivial Bianchi identities are at $d = 1$, which are of (i) $(\alpha\beta\gamma\delta)$, (ii) $(\alpha\beta\gamma, d)$ and (iii) $(\alpha\beta\gamma, \delta)$-types. Among these the first one is straightforward, while (ii) gives the relation (2.3.4h), that in turn is used in (iii), which is now composed of three sorts of terms: $\nabla\Phi$-terms, $G$-terms, and $\chi^2$-terms. Here the $\nabla\Phi$-terms are arranged as

$$(\nabla\Phi\text{-terms}) = -c_2 (a_1 + a_2) \left[ (\gamma^{ab})_{(\alpha\beta\gamma|} (\gamma^c)_{\gamma)(\nabla_a\Phi)(\nabla_b\varphi)(\nabla_c\varphi)(\nabla_d\tilde{\varphi}) + (\gamma^{ab})_{(\alpha\beta\gamma|} (\gamma^d)_{\gamma)(\nabla_a\Phi)(\nabla_b\varphi)(\nabla_d\tilde{\varphi}) \right].$$

(2.3.15)

This gives $a_1 = -a_2$, consistently with (2.3.13). The $G$-terms are

$$(G\text{-terms}) = (-3c_1a_2 + \frac{1}{8}) \times \left[ (\gamma^{ab})_{(\alpha\beta\gamma)\delta} (\gamma^c)_{\gamma\delta} G_{\alpha\beta\gamma\delta}(\nabla_b\varphi) - 2(\gamma^{ab})_{(\alpha\beta\gamma)\delta} (\gamma^f)_{\gamma\delta} G_{\alpha\beta\gamma\delta}(\nabla_f\varphi)(\nabla_b\varphi)(\nabla_d\tilde{\varphi}) + (\gamma^{ab})_{(\alpha\beta\gamma)\delta} (\gamma^{f\delta\gamma})_{\gamma\delta} G_{\alpha\beta\gamma\delta}(\nabla_f\varphi)(\nabla_b\varphi)(\nabla_g\tilde{\varphi}) \right].$$

(2.3.16)

This yields the condition

$$c_1a_2 = \frac{1}{24}\ .$$

(2.3.17)

The remaining terms in (iii) are the $\chi^2$-terms which are after appropriate manipulations:

$$(\chi^2\text{-terms}) = a_2(g - a_1 - 2a_2) (\gamma^{ab})_{(\alpha\beta\gamma)\delta} \left[ (\gamma^c)_{\gamma\delta} (\gamma^d)_{\gamma\delta} (\nabla_c\varphi)(\nabla_b\varphi)(\nabla_d\tilde{\varphi})(\nabla_a\tilde{\varphi}) - (\gamma^a)_{(\alpha\beta\gamma)\delta} (\gamma^d)_{\gamma\delta} (\nabla_a\varphi)(\nabla_b\varphi)(\nabla_d\tilde{\varphi}) \right],$$

(2.3.18)

yielding

$$g = a_1 + 2a_2.$$  

(2.3.19)
We now collect all the conditions on the unknown coefficients:

\[a_1 = -a_2, \quad c_1 a_2 = \frac{1}{24}, \quad g = a_1 + 2a_2,\] (2.3.20)

which fortunately have a set of consistent solutions

\[a_1 = -a_2, \quad c_1 = \frac{1}{24}a_2^{-1}, \quad g = -a_1.\] (2.3.21)

We can choose \(a_1\) to be \(a_1 = +1\), in order also to accord with the 10D result after the dimensional reduction to be performed later. This fix all the coefficients in (2.3.11), and therefore our constraints (2.3.4) have been confirmed. Due to the limited resource of the publisher, as well as the interest of the majority of readers who need few technical details, we are to skip further details here.

Our superfield equations in our system are much similar to those in 10D [25]:

\[(\gamma^{bc})_{\alpha\beta} T^{\beta}_{ab} \nabla^c \varphi - 2(\gamma^c)_{\alpha} (\nabla_{a} \nabla_{\beta}) \nabla^c \varphi = 0,\] (2.3.22)

\[R_{a[b]} \nabla_{|c|} \varphi + 4(\nabla_{a} \nabla_{[b]} \Phi) \nabla_{|c|} \varphi - 4(\nabla_{c} T_{a[b]}(\nabla_{|c|} \varphi) \nabla_{d} \varphi = 0,\] (2.3.23)

\[R_{[ab]} = -\nabla_{c} G_{ab}^c.\] (2.3.24)

These are obtained from Bianchi identities at \(d \geq 3/2\) [14]. Since this procedure is similar to the usual procedure, and nothing essential is peculiar to our system, we skip the details, except for the results. First, at \(d = 3/2\), the \((abcd)\) Bianchi identity gives (2.3.4i), while \((abc,d)\) Bianchi identity gives (2.3.4j). Now out of \((a,\beta\gamma,\delta)\) Bianchi identity \(X_{a\beta\gamma}^{\delta} = 0\), we take the contraction \(X_{a\beta\gamma}^{\delta}\), to get

\[X_{a\beta\gamma}^{\delta} = -\frac{7}{2} \left[ (\gamma^{bc})_{\alpha\beta} T_{ab} \nabla^c \varphi + 2a_1 (\gamma^c)_{\alpha} (\nabla_{a} \nabla_{\beta}) \nabla^c \varphi \right] = 0,\] (2.3.25)

for the gravitino field equation (2.3.22).

At \(d = 2\), we have (i) \((a\beta\gamma,\delta)\) and (ii) \((abc,d)\) and (iii) \((abcd)\)-type Bianchi identities. The first one gives (2.3.4k), which in turn can be combined with the gravitino superfield equation (2.3.22), as

\[0 = + (\gamma_{de})^{\beta\gamma} \nabla_\beta \left[ (\gamma^{bc})_{\gamma} \delta T_{abd} \nabla^c \varphi + 2a_1 (\gamma^b)_{\gamma} (\nabla_{a} \nabla_{\delta}) \nabla^b \varphi \right]
\]

\[= + 8 \left[ R_{a[d]n_c} + 8a_1 c_2 (\nabla_{a} \nabla_{[d]} \Phi) \nabla_{n_c} \varphi - 4a_1 (\nabla^b T_{a[d]}(\nabla_{d} \varphi) \nabla_{b} \varphi \right],\] (2.3.26)

yielding the gravitational superfield equation (2.3.23) much like that for \(N = 1\) supergravity in 10D [25]. Now the \((abc,d)\)-type Bianchi identity gives (2.3.24) after the contraction of \(c\) and \(d\) indices:

\[0 = X_{abc}^c = -R_{[ab]} - \nabla_{c} G_{ab}^c.\] (2.3.27)

The \((abcd)\)-type Bianchi identity gives no information, as usual. This concludes the satisfaction of all the Bianchi identities in our superspace, and therefore the confirmation of the consistency.
We do not repeat the same remark as in [25] about the peculiar structure of our constraint system with no separate field equations for the dilaton or dilatino, but mixed up with the zwölfbiein or gravitino field equations (2.3.23) and (2.3.22): There is no loss of degree of freedom for all the physical fields for the same reason as in 10D [25] after the dimensional reduction. As a matter of fact, in 10D the equivalence between the constraint set in [25] to the canonical set [24] was easily confirmed by super-Weyl rescalings [26].

Before concluding this subsection, we give the component transformation rule that can be easily obtained from our superspace constraints, by the aid of the standard technique in pages 321 - 327 of [27]:

\[
\begin{align*}
\delta_Q e^a_m &= + (\epsilon \gamma^{ab} \psi_m) D_b \varphi + (\epsilon P_{\uparrow} \psi_m) D^a \varphi, \\
\delta_Q \psi_m^a &= D_m e^a + (P_\downarrow \epsilon) (\chi \psi_m) \partial_n \varphi + (P_\downarrow \psi_m)^a (\epsilon \gamma^n \chi) \partial_n \varphi \\
&\quad - (P_\downarrow \gamma_a \chi \psi_m) \partial_n \varphi, \\
\delta_Q B_{mn} &= + (\epsilon \gamma^r \psi_n) \partial_m \varphi - (\epsilon P_{\uparrow} \psi_m) \partial_n \varphi, \\
\delta_Q \bar{\chi}_a &= + \frac{1}{24} (P_\downarrow \gamma^{mnr} \epsilon) \chi G_{mnr} + \frac{1}{2} (P_\downarrow \gamma^m \epsilon) \partial_m \Phi - \bar{\chi}_a (\epsilon \gamma^m \chi) \partial_m \varphi, \\
\delta_Q \varphi &= 0.
\end{align*}
\]

As is easily seen, the second term in (2.3.28a) is not important in component formulation, because it can be interpreted as an extra transformation for $e^a_m$ proportional to $\nabla^a \varphi$, like supersymmetric Yang-Mills in 12D [12]. Note also that the common factor $P_\downarrow$ in front of the last three terms in (2.3.28b) is consistent with the constraint (2.3.8) for the gravitino field strength. The same is also true with the first two terms in (2.3.28c).

### 2.4 Dimensional Reduction

As the first important confirmation of the validity of our result, we perform simple dimensional reduction [14] into 10D [25]. This process is the standard one, namely we require all the dependence of the superfields on the extra coordinates to vanish, truncating all the extra components as well, except those for null-vectors. To be more specific, our 64 x 64 $\gamma$-matrices in 12D will be dimensionally reduced as

\[
\bar{\gamma}_a = \begin{cases} 
\hat{\gamma}_a = \gamma_a \otimes \tau_3, \\
\hat{\gamma}_{(11)} = I \otimes \tau_1, \\
\hat{\gamma}_{(12)} = -I \otimes i\tau_2,
\end{cases}
\]

while our charge conjugation matrix is to be

\[
\bar{C} = C \otimes \tau_1, \quad \hat{\gamma}_{13} = \gamma_{11} \otimes \tau_3.
\]

The supercovariant derivative $\nabla_a$ corresponds to the component supercovariant derivative $D_a$. 

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5The supercovariant derivative $\nabla_a$ corresponds to the component supercovariant derivative $D_a$. 

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12
Here \( \gamma_a, I, C \) and \( \gamma_{11} \) are all \( 32 \times 32 \) matrices for the 10D Clifford algebra, while \( \tau_1, \tau_2 \) and \( \tau_3 \) are the standard \( 2 \times 2 \) Pauli matrices. Only in the sections for dimensional reductions, we use the hats for the quantities and indices in 12D, distinguished from non-hatted quantities and indices are in 10D. We next replace all the gradients of \( \varphi \) and \( \tilde{\varphi} \) by the null-vectors as in (2.2.2). Accordingly, we have the dimensional reductions for the null-vectors and projection operators:

\[
(\hat{\nu})^{\dot{\alpha}}_{\dot{\beta}} = (\hat{\gamma}^+)_{\dot{\alpha}^\beta} = \sqrt{2} I \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad (\hat{\eta})^{\dot{\alpha}}_{\dot{\beta}} = (\hat{\gamma}^-)_{\dot{\alpha}^\beta} = \sqrt{2} I \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,
\]

\[
\hat{P}_\uparrow = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad \hat{P}_\downarrow = I \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .
\]

(2.4.3)

Similarly, the dimensional reduction for our spinorial superfield goes as

\[
(\hat{\chi}_{\dot{\alpha}}) = \left( \frac{\hat{\chi}}{\hat{\chi}}_{\alpha} = \begin{pmatrix} 0 \\ \chi_\alpha \end{pmatrix} \right) , \quad \left( \hat{T}_{\dot{a}\dot{b}}^{\gamma} \right) = \left( \hat{T}_{\dot{a}\dot{b}}^{\alpha} \right) \gamma \left( \hat{T}_{\dot{a}\dot{b}}^{\dot{\gamma}} \right) ^\gamma = \left( T_{\dot{a}\dot{b}}^{\gamma} , 0 \right) ,
\]

(2.4.4)

where conveniently, we use also \( \uparrow \) or \( \downarrow \) to denote the upper or lower eigen-components of \( P_\uparrow \) or \( P_\downarrow \) in the spinors in the dimensional reductions. Note that the components in \( \hat{T}_{\dot{a}\dot{b}}^{\gamma} \) are given as a row vector to be multiplied by 12D \( \gamma \)-matrices from the right, in accordance with our multiplication rule. This is crucial when it comes to the extra constraint (2.3.8).

A typical example illustrating our process is

\[
\hat{T}_{\alpha\beta\gamma} \rightarrow T_{\alpha\beta\gamma} = \sqrt{2} \delta(\alpha^\gamma \chi_\beta) - (\gamma^a)_{\alpha\beta} (\gamma_a \chi)^\gamma .
\]

(2.4.5)

The dimensional reduction for our superfield equations is straightforward. We start with the gravitino field equation (2.3.22) by rewriting it as

\[
(\hat{\gamma}^{\dot{\alpha}} \hat{\gamma}^+) \hat{T}_{\dot{a}\dot{b}}^{\dot{\beta}} + 2(\hat{\gamma}^+)_{\dot{a}\dot{b}} \cdot \hat{\nabla}_{\dot{a}} \hat{\chi}_{\dot{b}} = 0 .
\]

(2.4.7)

There are in total four options for the free indices \( \dot{a}\dot{a} \): (i) \( \dot{a}\dot{a} = \alpha\alpha \), (ii) \( \dot{a}\dot{a} = \alpha\dot{\alpha} \), (iii) \( \dot{a}\dot{a} = \alpha\dot{\alpha} \), (iv) \( \dot{a}\dot{a} = \dot{\alpha}\dot{\alpha} \). This is because when \( \dot{a} = - \), (2.4.7) is trivially satisfied by \( \hat{T}_{\dot{a}\dot{b}}^{\dot{\beta}} = 0 \), \( \hat{\nabla}_{\dot{a}} \hat{\chi}_{\dot{b}} = 0 \) of extra constraints (2.3.5) and (2.3.7). Among these four cases, the case (i) yields

\[
\gamma^b T_{ab} + \nabla_a \chi = 0 ,
\]

(2.4.8)
under our dimensional reduction rules (2.4.1) - (2.4.4), giving nothing but the 10D gravitino field equation in ref. [25]. All the other cases (ii) - (iv) can be easily satisfied by our dimensional reduction rules, such as \( T_{ab}^{\beta\gamma} = 0 \).

We next perform the dimensional reduction of our zwölfbein field equation (2.3.23). There are in total six possibilities for the free indices \( \hat{a}[\hat{b}\hat{c}]: (i) \, a[b+], \, (ii) \, a[-], \, (iii) \, +[b+], \, (iv) \, +[-], \, (v) \, -[b+], \, (vi) \, -[-] \). First, the case (i) yields

\[
0 = \hat{R}_{ab}\hat{n} + 4\tilde{\nabla}_a\tilde{\nabla}_b\tilde{\Phi}\hat{n} - 4(\tilde{\gamma}^+\tilde{T}_{ab})\hat{n} = -R_{ab} + 4\nabla_a\nabla_b\Phi - 4\tilde{T}_{ab}^{\gamma\gamma}(\tilde{\gamma}^+)\tilde{\chi}\delta^{\delta}\delta^{\delta}, \quad (2.4.9)
\]

which gives the zwölfbein field equation

\[
R_{ab} + 4\nabla_a\nabla_b\Phi - 4\sqrt{2}\tilde{T}_{ab}\chi = 0 \quad (2.4.10)
\]

in [25]. All other sectors (ii) - (vi) turn out to be satisfied easily by our dimensional reduction prescription and constraints, such as \( R_{a-} = 0, \, \nabla_-\Phi = 0, \, etc. \) In a similar fashion, eq. (2.3.24) is easily reduced to 10D, yielding

\[
R_{[ab]} = -2\nabla_cG_{ab}^c \quad (2.4.11)
\]

as in [25], which is formally of the same form as in 12D. This concludes the description of our dimensional reduction, as an important confirmation of our original 12D supergravity.

3. Superstring on Background of \( D = 12, \, N = 1 \) Supergravity

Once our superspace formulation is established for \( N = 1 \) supergravity in 12D, then the next natural task is to put some probe for the background, such as superstring. The existence of consistent superstring on such background is also naturally expected from the viewpoint of F-theory [2], namely all the superstring theories such as heterotic or type IIB string, that are not directly from 11D M-theory [7][8][10], are from 12D F-theory [2]. The first natural trial is to put Green-Schwarz superstring on our supergravity background.

We start with the postulate for the total action for Green-Schwarz superstring [14]:

\[
S \equiv S_\sigma + S_B + S_\Lambda, \quad (3.1)
\]

\[
S_\sigma \equiv \int d^2\sigma \left[ V^{-1}\eta_{ab}\Pi_+^a\Pi_-^b \right], \quad (3.2)
\]

\[
S_B \equiv \int d^2\sigma \left[ V^{-1}\Pi_+^A\Pi_-^B\Pi^B_{BA} \right], \quad (3.3)
\]

\[
S_\Lambda \equiv \int d^2\sigma \left[ V^{-1}\Lambda_{++}(\Pi_-^a\nabla_a\varphi)\left(\Pi_-^b\nabla_b\tilde{\varphi}\right) + V^{-1}\tilde{\Lambda}_{++}\left\{ (\Pi_-^a\nabla_a\varphi)^2 + (\Pi_-^a\nabla_a\tilde{\varphi})^2 \right\} \right], \quad (3.4)
\]
where \( V \equiv \det (V_{\pm i}) \) is the determinant of the zweibein \( V_{\pm i} \), and the indices \( i, j, \ldots = 0, 1 \) are for the curved 2D coordinates \( \sigma^i \), while \( \pm \) are for the local Lorentz light-cone coordinates.[6]

Due to our extra coordinates, we expect some symmetry that will get rid of non-physical components associated with them. In fact, we have not only the usual \( \kappa \)-symmetry [5], but also an additional fermionic \( \eta \)-symmetry in our total action, dictated by [14]

\[
\delta V_+^i = \pi_+^i \delta (\gamma^c) \delta \Pi_+ (\nabla_c \varphi) (\nabla_v \varphi) \Pi_+ \Pi_+^i V_-^j = (\pi_+^i \gamma^c \Pi_+^j) V_-^j \nabla_c \varphi , \\
\delta \Pi_+ = (\pi_+^i \gamma^c \Pi_+^i) (\nabla_c \varphi), \\
\delta \Pi_+ = (\pi_+^i \gamma^c \Pi_+^i) (\nabla_v \varphi) = 0 , \\
\delta (V^{-1}) = 0 , \\
\delta E^a = \delta E^a = 0 , \\
\delta E^a = \frac{1}{2} (\pi_+^i \gamma^c \Pi_+^i) (\nabla_v \varphi) = 0 , \\
\delta \Pi_+ = -2 (\pi_+^i \gamma^c \Pi_+^i) (\nabla_v \varphi) , \\
\delta \Pi_+ = 0 , \\
\delta \varphi = 0 , \\
\delta \bar{\varphi} = 0 ,
\]

where \( \kappa \) and \( \eta \) are infinitesimal arbitrary \( \sigma \)-dependent fermionic parameters. Notice the important significance of (3.5a) through (3.5c) that the effective 2D gravitational field is \( h_{++} \approx V^{-1} g_{++} \) in the light-cone coordinates, and \( \delta h_{++} \approx (\pi_+ \nabla \varphi \Pi_+) \), so that the only non-trivial component for the energy-momentum tensor will be \( T_{--} \). This feature will be important, when we later study the contributions of the extra string coordinates to the conformal anomaly.

We first confirm the \( \kappa \)-invariance of the total action. To this end, we need the basic relations for variations in Green-Schwarz \( \sigma \)-model, such as

\[
\delta_\kappa \Pi_+^A = V_{\pm}^i D_i (\delta_\kappa E^A) + (\delta_\kappa V_{\pm}^i) \Pi_+^A - \Pi_+^D \delta_\kappa (T_{CD}^A - \phi_{CD}^A) ,
\]

where the explicit Lorentz connection \( \phi \) will automatically disappear or will be absorbed into covariant derivatives, as in the usual 10D case. Using this, we get that

\[
\delta_\kappa (S_\sigma + S_B) = + (\pi_+ \gamma^c \Pi_+) (\Pi_-^a)^2 \nabla_v \varphi - \frac{1}{2} (\pi_+ \gamma^d \gamma^a \gamma^b \Pi_+) \Pi_-^e \Pi_-^b (\nabla_v \varphi) (\nabla_v \varphi) \\
- \frac{1}{2} (\pi_+ \gamma^c \Pi_+) (\Pi_-^a \nabla_v \varphi) (\nabla_v \varphi) \\
+ 2V^{-1} (\pi_+ \gamma^c \Pi_+) (\Pi_-^a \nabla_v \varphi) (\Pi_-^b \nabla_v \varphi) \nabla_v \varphi .
\]

From the second toward the third line, we have performed \( \gamma \)-matrix manipulations, such as \( \gamma^c \gamma^d \gamma^b = \gamma^d \gamma^b + \eta^{e[b} \gamma^{c]} \), as well as the constraint (3.5b), so that \( \pi_+ \gamma^a \gamma^b (\nabla_v \varphi) (\nabla_b \varphi) = +2\pi_+ \), under the null-vector conditions (2.3.4\ell). We see that the first term in the second line is cancelled by other like terms, while we are left only with one term in the third line. Similarly, we get

\[
\delta_\kappa S_\lambda = -2V^{-1} (\pi_+ \gamma^c \Pi_+) (\Pi_-^a \nabla_v \varphi) (\Pi_-^b \nabla_v \varphi) \nabla_v \varphi ,
\]

\[\text{We try to avoid the simultaneous usage of the } \pm \text{-indices for the 12D target space-time, and these 2D light-cone indices.}\]
by the use of the relations such as \( \delta_\kappa (\Pi^a a \nabla_a \varphi) = \delta_\kappa (\Pi^a a \nabla_a \bar{\varphi}) = 0 \), which are easily confirmed first by showing that \( T_{a\alpha}^\beta c \nabla_c \bar{\varphi} = 0 \). Here technically we need also the feature of the Lorentz generators, such as \((\tilde{M}_{bc})^{-d} = 0\). After all, we have

\[
\delta_\kappa S = \delta_\kappa (S_\sigma + S_B + S_\Lambda) = 0 .
\]

As for the second fermionic symmetry, we can similarly confirm the invariance [14]:

\[
\delta_\eta S = 0 .
\]

The two fields \( \Lambda_{++} \) and \( \bar{\Lambda}_{++} \) are playing roles of Lagrange multipliers, yielding the two field equations

\[
(\Pi^a a \nabla_a \varphi)(\Pi^b b \nabla_b \bar{\varphi}) = 0 , \quad (\Pi^a a \nabla_a \varphi)^2 + (\Pi^a a \nabla_a \bar{\varphi})^2 = 0 ,
\]

which in turn are equivalent to the two equations\(^7\)

\[
\Pi^a a \nabla_a \varphi = 0 , \quad \Pi^a a \nabla_a \bar{\varphi} = 0 .
\]

Note that (3.12) is a consequence of field equations out of the Lagrange multiplier action \( S_\Lambda \), but not imposed by hand. The reason is that for the invariance check of the total action, we should not impose by hand the constraint with the first-order derivative such as (3.12) which can be interpreted as unidexterous field equations in 2D. This is a particular caution needed for action invariance in 2D. Interestingly, the action \( S_\Lambda \) also cancels the unwanted term in (3.7). As mentioned after eq. (3.5), the only non-trivial component of the energy-momentum tensor coupled to 2D zweibein field is \( T_{--} \), therefore the deletion of the components \( \Pi^a a \nabla_a \bar{\varphi} = \Pi^a a m_a \) and \( \Pi^a a \nabla_a \varphi = \Pi^a a n_a \) removes any additional contribution from the extra string variables to the conformal anomaly. Accordingly, the usual 2D conformal anomaly cancellation works in the same way as in the 10D Green-Schwarz superstring [5].

There is another crucial point related to eq. (3.12). Note that these constraints effectively force the string variables \( X^\pm \) to depend only on \( \sigma^+ \). In other words, there are non-vanishing extra components \( X^\pm (\sigma^+) \) which distinguish our system from just a ‘rewriting’ of the conventional 10D superstring theory [5]. Due to these non-trivial components, our Green-Schwarz superstring [5] coupled to 12D supergravity is by no means just a rewriting of the conventional \( N = 1 \) superstring coupled to 10D supergravity ‘in disguise’. To put it differently, our system cleverly maintains the conformal anomaly cancellation of the conventional 10D superstring, while keeping new variables inherent in the theory.

We mention that there is an alternative form of our \( \kappa \)-symmetry. This can be obtained by the replacement \( \bar{\kappa}_{+\ast} = (\gamma^c \lambda_+)_{\ast} \nabla_c \varphi \), with the constraint \( \gamma^c \lambda_+ \nabla_c \varphi = 0 \). This is merely a

\(^7\)Some ideas similar to these constraints have been suggested in various contexts [28].
rewriting of the original $\kappa$-symmetry, with nothing significant, reflecting just the nilpotency of $\gamma^a \nabla_a \tilde{\varphi}$ and $\gamma^a \nabla_a \varphi$.

The counting of the physical degrees of freedom can be easily done, by considering the components deleted by these fermionic symmetries. First of all, the $\eta$-symmetry deletes half of the original 32 components of the fermionic chiral coordinates $\theta^a$ in superspace in 12D, and thus at most 16 components can be physical. Subsequently, the usual $\kappa$-symmetry [5] deletes further half of 16 components, and we are left with the usual 8 components in accordance with the light-cone gauge in Green-Schwarz superstring [5].

We finally stress that the null-vector conditions in (2.3.4$\ell$) are also required by these fermionic invariances on the Green-Schwarz superstring world-sheet. Therefore these world-sheet fermionic symmetries provide an independent validity confirmation of our 12D supergravity constraints in superspace.

4. $N = 2$ Supergravity in $D = 10 + 2$

4.1 Notations

We have so far worked on $N = 1$ supergravity in $D = 10 + 2$ and its related features. We now turn to $N = 2$ chiral supergravity which is supposed to be the strong coupling limit of F-theory [2]. Once we have understood how our peculiar Lorentz generators work for $N = 1$, it is easier to handle the $N = 2$ theory in component language, where we can get directly the transformation rules and field equations.

Our basic conventions are consistent with the notation in the preceding sections, except for minor differences peculiar to the component formulation. One of them is the index convention such as \( \mu, \nu, \ldots = 0, 1, \ldots, 9, 11, 12 \) used for curved indices, while \( m, n, \ldots = (0), (1), \ldots, (9), (11), (12) \) for local Lorentz indices. Another difference from the superspace notation is the normalized anti-symmetrization, such as $A_{[\mu \nu]} \equiv (1/2)(A_{\mu \nu} - A_{\nu \mu})$, and the component covariant derivative $D_\mu$, etc. Other than these, we use the same null-vectors $\partial/\partial \varphi$, $\partial/\partial \tilde{\varphi}$ or the operators $P_\uparrow$, $P_\downarrow$, $P_{\uparrow\downarrow}$, as in (2.2.1) and (2.2.2). Due to the chiral nature of our system, we need to distinguish the chiralities for the $N = 2$ case. The explicit representations for $\mathcal{M}_{mn}$ is the exactly the same as (2.1.6).

Similarly to the $N = 2$ chiral supergravity in 10D [29], our system also has the coset $SU(1,1)/U(1)$ parametrized by the scalar fields playing roles of coordinates on this manifold. The scalar fields $V_{\pm}^\alpha$ are $SU(1,1)$ group matrix-valued, transforming as

$$
\delta V_{\pm}^\alpha = m_{\alpha}^\beta V_{\pm}^\beta \pm i \Sigma V_{\pm}^\alpha.
$$

Here the indices $\alpha, \beta, \ldots = 1, 2$ should not be confused with the 12D spinorial indices in (2.1.6), as long as they are clear from the context. The explicit matrix representations for
\( V_{\pm}^{\alpha} \), such as
\[
\begin{pmatrix}
V_{-}^{-1} & V_{+}^{-1} \\
V_{-}^{-2} & V_{+}^{-2}
\end{pmatrix} = \exp \left( i \varphi A - i \varphi A^{*} \right) = \begin{pmatrix}
\cosh \rho + i \varphi \frac{\sinh \rho}{\rho} & A \frac{\sinh \rho}{\rho} \\
A^{*} \frac{\sinh \rho}{\rho} & \cosh \rho - i \varphi \frac{\sinh \rho}{\rho}
\end{pmatrix}, \tag{4.1.2}
\]
are sometimes useful, where \( \rho^2 \equiv A^{*} A - \varphi^2 \). The constant parameter
\[
(m_{\alpha \beta}^\prime) = \begin{pmatrix} i \gamma & \alpha \\ \alpha^* & - i \gamma \end{pmatrix}, \tag{4.1.3}
\]
is for the global \( SU(1, 1) \) group, while \( \Sigma \) is a real parameter for the \( U(1) \) transformation.

The \( V^\prime s \) satisfy the relationships
\[
\epsilon_{\alpha \beta} V_{-}^{\alpha} V_{+}^{\beta} = \det V = 1, \quad V_{-}^{\alpha} V_{+}^{\beta} - V_{+}^{\alpha} V_{-}^{\beta} = \epsilon^{\alpha \beta}, \tag{4.1.4}
\]
so that we do not need their inverse matrices. The composite \( U(1) \) connection defined by
\[
Q_{\mu} = - i \epsilon_{\alpha \beta} V_{-}^{\alpha} \partial_{\mu} V_{+}^{\beta}, \tag{4.1.5}
\]
transforms as \( \delta Q_{\mu} = \partial_{\mu} \Sigma \). The \( SU(1, 1) \) invariant field strength \( P_{\mu} = - \epsilon_{\alpha \beta} V_{-}^{\alpha} \partial_{\mu} V_{+}^{\beta} \) transforms as \( \delta P_{\mu} = 2 \Sigma P_{\mu} \). Among the fields in our supergravity multiplet \( (\epsilon_{\mu}^{m}, \psi_{\mu}, A_{\mu \nu \rho \sigma}, \lambda, A_{\mu \nu}^{\alpha}, V_{\pm}^{\alpha}; \varphi, \bar{\varphi}) \), the following fields transform under \( SU(1, 1) \otimes U(1) \):
\[
\delta A_{\mu \nu}^{\alpha} = m_{\beta}^{\alpha} A_{\mu \nu}^{\beta}, \quad \delta \psi_{\mu} = \frac{i}{2} \Sigma \psi_{\mu}, \quad \delta \lambda = \frac{3i}{2} \Sigma \lambda. \tag{4.1.6}
\]

Useful relation associated with (anti)self-duality are such as (A.20) in Appendix A, and
\[
\gamma^{[6]} \psi_{\pm} S_{[6]} \equiv 0, \quad \gamma^{[6]} \psi_{-} A_{[6]} \equiv 0, \quad \gamma_{13} \psi_{\pm} \equiv \pm \psi_{\pm},
\]
\[S_{[6]} = \pm \frac{1}{6!} \epsilon_{[6]}^{[6]} S_{[6]}, \quad A_{[6]} = - \frac{1}{6!} \epsilon_{[6]}^{[6]} A_{[6]}, \tag{4.1.7}\]
Here \( [\alpha] \) denotes the normalized antisymmetrization of \( n \) indices.

We finally mention the important relations with respect to inner products of spinors in our \( N = 2 \) system, \textit{e.g.}, for two Weyl spinors \( \psi_{1} \) and \( \psi_{2} \), we have
\[
\left( \overline{\psi}_{1} \gamma^{\mu_{1} \cdots \mu_{N}} \psi_{2} \right) = \left( \overline{\psi}_{2} \gamma^{\mu_{1} \cdots \mu_{N}} \psi_{1} \right) = (-)^{N(N-1)/2} \left( \overline{\psi}_{2} \gamma^{\mu_{1} \cdots \mu_{N}} \psi_{1} \right), \tag{4.1.8a}
\]
\[
\left( \overline{\psi}_{1} \gamma^{\mu_{1} \cdots \mu_{N}} \psi_{2} \right)^{\dagger} = \overline{\psi}_{2}^{\dagger} \left( \gamma^{\mu_{1} \cdots \mu_{N}} \right)^{\dagger} \overline{\psi}_{1} = \left( \overline{\psi}_{2} \gamma^{\mu_{1} \cdots \mu_{N}} \psi_{1} \right) \quad \text{(4.1.8b)}
\]
where the dagger \( \dagger \) denote a hermitian conjugate, and \( * \)-symbol is a complex conjugation of a Weyl spinor, such as \( \psi^{\dagger} = (\psi^{(1)} + i \psi^{(2)})^{*} = \psi^{(1)} - i \psi^{(2)} \) for two Majorana-Weyl spinors \( \psi^{(1)} \) and \( \psi^{(2)} \) forming a Weyl spinor \( \psi. \)

### 4.2 \( N = 2 \) Supergravity in \( D = 10 + 2 \)

We first give our result for supersymmetry transformation for our multiplet of supergravity \( (\epsilon_{\mu}^{m}, \psi_{\mu}, A_{\mu \nu \rho \sigma}, \lambda, A_{\mu \nu}^{\alpha}, V_{\pm}^{\alpha}; \varphi, \bar{\varphi}) \) [15]:
\[ \delta Q e^m_\mu = [(\bar{\psi}^m \gamma^m \psi_\mu) D_n \varphi + (\bar{\psi} P_{\uparrow} \psi_\mu) D^m \varphi] + \text{c.c.} \, , \]  
\(\delta_Q \psi_\mu = \ddot{D}_\mu \epsilon - \frac{i}{480} (P_\downarrow [\gamma_\mu \epsilon]) \hat{F}[5] + \frac{1}{96} P_\downarrow (\gamma_\mu [3] \hat{G}[3] - 9 \gamma_\mu [2] \hat{G}_\mu [2]) \epsilon^* \, , \]  
\[ \delta_Q A^{\mu \nu \alpha} = V_+^{\alpha} (\bar{\tau} \gamma_\mu \rho \lambda^*) \partial_\rho \varphi + V_-^{\alpha} (\bar{\tau} \gamma_\mu \rho \lambda) \partial_\rho \varphi - 4V_+^{\alpha} (\bar{\tau} \gamma_\mu \rho \psi_\mu [\alpha]) \partial_\rho \varphi - 4V_-^{\alpha} (\bar{\tau} \gamma_\mu \rho \psi_\mu [\alpha]) \partial_\rho \varphi \, , \]  
\[ \delta_Q A^{\mu \nu \rho \sigma} = i(\bar{\tau}_\gamma [\mu_\rho \rho] \varphi) \partial_\nu \varphi - i(\bar{\tau}_\gamma [\mu_\rho \rho] \varphi^* \gamma_\nu \varphi) \partial_\nu \varphi - \frac{3i}{8} \epsilon_{\alpha \beta} A^{(\mu \nu \alpha} \delta \kappa A^{\rho \sigma) \beta} \, , \]  
\[ \delta_Q \lambda = - (P_\gamma [\mu \epsilon^*]) \hat{P}_\mu - \frac{1}{24} (P_\gamma [\mu \rho \epsilon]) \hat{G}_{\mu \rho} \, , \]  
\[ \delta_Q V^{\alpha}_+ = V_-^{\alpha} (\bar{\tau} \gamma_\mu \lambda^*) \partial_\mu \varphi \, , \]  
\[ \delta_Q V^{\alpha}_- = V_-^{\alpha} (\bar{\tau} \gamma_\mu \lambda^*) \partial_\mu \varphi \, , \]  
\[ \delta_Q \varphi = 0 \, , \quad \delta_Q \hat{\varphi} = 0 \, , \]  
where \( e^m_\mu \) is the zwölfbein, \( \psi_\mu \) is a pair of two Majorana-Weyl spinors of the same chirality: \( \gamma_1 3 \psi_\mu = - \psi_\mu \), or equivalently a Weyl spinor for \( N = 2 \) supersymmetry, \( A^{\mu \nu \alpha} \) is a pair of complex vector fields, \( A^{\mu \nu \rho \sigma} \) is a real fourth-rank antisymmetric tensor, \( \lambda \) is a Weyl spinor sometimes called gravitello satisfying \( \gamma_1 3 \lambda = + \lambda \), and \( V^{\alpha}_+ \) is a scalar field parametrizing the coset \( SU(1, 1)/U(1) \). The field strengths with the hat-symbols are meant to be supercovariantization [30][29] of the field strengths defined by [15] \[ G^{\mu \nu \rho \sigma} = - \epsilon_{\alpha \beta} V^{\alpha}_+ F^{\mu \nu \rho \sigma} \, , \quad P_\mu \equiv - \epsilon_{\alpha \beta} V^{\alpha}_+ \partial_\mu V^\beta_+ \, , \quad Q_\mu \equiv - i \epsilon_{\alpha \beta} V_-^{\alpha} \partial_\mu V^\beta_+ \, , \quad F^{\mu \nu \rho \sigma} = 3 \partial_{[\mu} A_{\nu \rho \sigma]} \alpha \, , \quad F^{\mu \nu \rho \sigma \tau} = 5 \partial_{[\mu} V_{\nu \rho \sigma \tau]} + \frac{5i}{8} \epsilon_{\alpha \beta} A^{(\mu \nu \alpha \beta} F_{\rho \sigma \tau)} \, , \]  
which satisfy useful identities such as [15] \[ D_{[\mu} P_\nu] = 0 \, , \quad D_{[\mu} G^{\nu \rho \sigma]} = + P_{[\mu} G^{\rho \sigma]}_\nu \, , \quad D_{[\mu} F^{\nu \rho \sigma \tau]} = 0 \, , \quad D_{[\mu} Q_\nu] = - i P_{[\mu} P^\alpha_\nu] \]  
parallel to the 10D case [29].

As in the \( N = 1 \) supergravity theory [14], we have the extra constraints imposed on the field strengths and spinors [15] \[ \hat{G}^{\mu \nu \rho \sigma} \partial_\rho \varphi = 0 \, , \quad \hat{F}^{\mu \nu \rho \sigma} \partial_\rho \varphi = 0 \, , \quad \hat{R}^{\mu \nu \rho \sigma \tau} \partial_\rho \varphi = 0 \, , \quad \hat{R}^{\mu \nu \rho \sigma \tau} D_m \varphi = 0 \, , \]  
\[^8\text{We will not confuse} \quad P_{\mu} \quad \text{with the projectors} \quad P_{\uparrow}, \quad P_{\downarrow} \quad \text{of} \quad P_{\uparrow \downarrow}, \quad \text{as long as we are careful about the context.}\]

\[^9\text{We use} \quad D_m \varphi \equiv e^m_\mu \partial_\mu \varphi \quad \text{in} \quad e.g. \quad (4.2.4e), \quad \text{avoiding the misleading expression} \quad \partial_m \varphi \quad \text{with the local Lorentz index} \quad m.\]
\[ \hat{F}^\mu \partial_\nu \varphi = 0 \quad , \quad Q^\mu \partial_\mu \varphi = 0 \quad , \quad (4.2.4b) \]
\[ \hat{R}_\mu^{\nu} \partial_\nu \varphi = 0 \quad , \quad \hat{R}_\mu^{\nu} \gamma^m D_m \tilde{\varphi} = 0 \quad , \quad (4.2.4c) \]
\[ \gamma^\mu \lambda \partial_\mu \tilde{\varphi} = 0 \quad , \quad (D_m \lambda)(D^m \varphi) = 0 \quad . \quad (4.2.4d) \]
\[ D_m D_n \varphi = 0 \quad , \quad D_m D_n \tilde{\varphi} = 0 \quad , \quad (D_m \varphi)(D^m \tilde{\varphi}) = 1 \quad , \quad (D_m \varphi)^2 = 0 \quad , \quad (D_m \tilde{\varphi})^2 = 0 \quad . \quad (4.2.4e) \]

The conditions in (4.2.4e) are the same as (2.2.1) for our scalar fields, whose non-trivial solutions are (2.2.2). Note the involvement of \( D_m \tilde{\varphi} \) in (4.2.4c) and (4.2.4d).

Our component fields undergo also extra transformations in addition to our supersymmetry, translation and Lorentz rotations, or gauge transformations, dictated symbolically for a general component field \( \phi_{[m]}^{[n]} \) in our multiplet by

\[ \delta_E \phi_{\mu_1 \ldots \mu_m}^{r_1 \ldots r_n} = \Omega_{[\mu_1 \ldots \mu_m]}^{r_1 \ldots r_n} \partial_{\mu_1 \ldots \mu_m} \varphi + \Omega'_{\mu_1 \ldots \mu_m}^{[r_1 \ldots r_n-1]} D^{[r_n]} \varphi \quad . \quad (4.2.5) \]

For example, for \( A_{\mu \nu \rho \sigma} \) with \( m = 4 \), \( n = 0 \):

\[ \delta_E A_{\mu \nu \rho \sigma} = \Omega_{[\mu \nu \rho]} \partial_\sigma \varphi \quad , \quad (4.2.6) \]

where \( \Omega_{[3]} \) is an infinitesimal local parameter. Since each component field has different index structure, eq. (2.4.5) expresses collectively all of these extra transformations. As will be shortly mentioned, these extra transformations are needed also for the closure of supersymmetries.

We now list up our field equations

\[ \hat{D}_\mu \hat{G}^\mu + \frac{1}{24} \hat{G}^2 + O(\psi^2) = 0 \quad , \quad (4.2.7a) \]
\[ (\hat{D}_\mu \hat{G}^\mu_{[\nu \rho]} \partial_\sigma) \varphi + \hat{P}^\mu \hat{G}^*_{[\nu \rho]} \partial_\sigma \varphi + \frac{2i}{3} \hat{F}^{\tau \omega \lambda}_{[\nu \rho]} \hat{G}_{\omega \lambda} \partial_\sigma \varphi + O(\psi^2) = 0 \quad , \quad (4.2.7b) \]
\[ \left( \hat{R}_{[\mu]} - \hat{P}_\rho \hat{G}^{*[\mu]}_{\nu \rho} - \hat{P}_{[\mu]} \hat{F}^{*}_{\nu} - \frac{1}{6} \hat{F}_{[4]} \hat{F}_{[4]} \right) \left( \hat{G}^{[\mu \sigma \tau]}_{\rho} \right) + O(\psi^2) = 0 \quad , \quad (4.2.7d) \]
\[ \hat{F}^{[\mu_1 \ldots \mu_5]} \partial_{\mu_6} \varphi = - \frac{1}{6!} \epsilon_{\mu_1 \ldots \mu_6} \hat{F}^{[\mu_1 \ldots \mu_5]} \tilde{\varphi} \quad , \quad (4.2.7d) \]
\[ \gamma^\sigma \left( \gamma^\rho \hat{R}_{[\rho]} + \lambda^* \hat{P}_{[\rho]} - \frac{1}{48} \gamma^{[3]}_{[\rho]} \hat{G}^{*}_{[3]} - \frac{1}{48} \gamma_{[\rho]} \hat{G}^{*}_{[3]} \right) \left( \partial_\rho \varphi \right) = 0 \quad , \quad (4.2.7e) \]
\[ \gamma^\sigma \left( \gamma^\mu \hat{D}_\mu \lambda - \frac{i}{240} \gamma^{[5]} \hat{F}^{[5]} \right) \partial_\sigma \varphi = 0 \quad . \quad (4.2.7f) \]

The terms denoted by \( O(\psi^2) \) are fermionic terms in bosonic field equations, that are skipped in this paper as in ref. [29]. Note that the usual self-duality condition on \( F_{[5]} \) in 10D [29]
corresponds to anti-self-duality condition (4.2.7d) in 12D. This is merely due to our notation related to the \( \varepsilon \)-tensor (A.20).

We now give the detailed derivation of our transformation rule and field equations. We first confirm the transformation rule (4.2.1), by taking a closure of two supersymmetry transformations on all the bosonic component fields, relying on the useful relationships in (4.1.8). As a typical example, we give the case on \( A_{\mu \nu} \): Using (4.1.8), we get

\[
\begin{align*}
[&\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]A_{\mu \nu} = \left[ -\frac{3}{4}V_+^\alpha (\tau_2 \gamma^{\rho} \varepsilon_1) G_{\sigma \mu \nu}^* \partial_\rho \varphi + \frac{1}{4}V_-^\alpha (\tau_2 \gamma^{\rho} \varepsilon_1) G_{\rho \mu \nu} \partial_\rho \varphi \\
&\quad + \frac{1}{24}V_-^\alpha (\tau_2 \gamma_{\mu \nu}^{\rho \sigma \tau \omega} \varepsilon_1) G_{\rho \sigma \tau \omega} \partial_\omega \varphi \\
&\quad - \frac{1}{24}V_+^\alpha (\tau_2 \gamma_{\mu \nu}^{\rho \sigma \tau \omega} \varepsilon_1) G_{\sigma \mu \nu}^* \partial_\rho \varphi \right] - (1\leftrightarrow 2) + \text{c.c.} \quad (4.2.8)
\end{align*}
\]

as the \( G \)-linear terms, where \( \xi^\rho \equiv (\tau_2 \gamma^{\rho \sigma} \varepsilon_1) \partial_\sigma \varphi \), and we have used the relationship

\[
\left[ \frac{1}{4}V_-^\alpha (\tau_2 \gamma^{\rho} \varepsilon_1) G_{\rho \mu \nu} \partial_\rho \varphi - \frac{3}{4}V_+^\alpha (\tau_2 \gamma^{\rho \sigma} \varepsilon_1) G_{\sigma \mu \nu}^* \partial_\rho \varphi \right] - (1\leftrightarrow 2) + \text{c.c.} = +\xi^\rho F_{\rho \mu \nu}^\alpha . \quad (4.2.9)
\]

All other terms like those proportional to \( F_{\mu \nu \rho \sigma} \) cancel themselves. Needless to say, this closure is up to the terms understood as extra transformations.

We next outline the derivation of our field equations. The main ingredient in this process is much like the \( N = 2 \) chiral supergravity in 10D [29], except for the involvement of the gradients \( \partial \varphi \) and \( \partial \tilde{\varphi} \) which are sometimes subtle. We first postulate our gravitello \( \lambda \)-field equation as

\[
P_{\lambda \gamma} \tilde{D}_\mu \lambda - ia_1 P_{\lambda \gamma}[5] \lambda \tilde{F}[5] = 0 \quad , \quad (4.2.10)
\]

and take its variation under supersymmetry as in 10D [29]. Note that the supercovariantization of the derivative and field strength is also crucial. After the variation, all the terms are categorized either into \( \varepsilon \)-terms or \( \varepsilon^* \)-terms. The former is further composed of three sorts of terms: (i) \( FG \)-terms, (ii) \( DG \)-terms, and (iii) \( G^*P \)-terms, where \( D \) in \( DG \) denote derivative acting on \( G \). After appropriate algebra, the (i) \( FG \)-terms are arranged as

\[
\begin{align*}
(\text{FG-terms}) &= \quad + i \left( \frac{1}{320} - \frac{3}{4}a_1 \right) (\partial \tilde{\varphi})_{\gamma^{\rho \sigma \tau \omega}} \epsilon F[^{\nu_1 \ldots \nu_5}] G_{\rho \sigma \tau \omega} \partial_{[\nu] \varphi} \\
&\quad - i \left( \frac{1}{16} + 5a_1 \right) P_{\lambda \gamma} \epsilon F[^{[3]}[3] G[3] . \quad (4.2.11)
\end{align*}
\]

As in the 10D case [29], we require the first line to vanish, getting the condition

\[
a_1 = +\frac{1}{240} , \quad (4.2.12)
\]

while the second line contributes to the \( A_{\mu \nu}^\alpha \)-field equation, as will be seen later. The (ii) \( DG \)-terms and the (iii) \( G^*P \)-terms talk to each other under the Bianchi identity (4.2.3a).
After all, we get

\[ (\epsilon\text{-terms}) = -\frac{1}{8} P_\psi \gamma^{\mu \nu} \epsilon \left[ D_\tau G_{\mu \nu} + P_\tau G^*_{\mu \nu} + \frac{2i}{3} F_{\mu \nu}^{[3]} G_{[3]} \right] \] (4.2.13a)

\[ = -\frac{1}{4} \gamma^\sigma \gamma^{\rho \mu \nu} \epsilon \left[ D_\tau G^*_{[\mu \nu}(\partial_\rho)\varphi) + P^* G^*_{[\mu \nu}(\partial_\rho)\varphi) \right. \]
\[ + \left. \frac{2i}{3} F_{[\mu \nu}]^{[3]} G_{[3]}(\partial_\rho)\varphi) \right] \partial_\sigma \varphi = 0 \ , \] (4.2.13b)

which yields our \( A_{\mu \nu}^\alpha \)-field equation (4.2.7b). Note that it is too strong to require the vanishing of the square bracket in (4.2.13a), because of the multiplication of \( P_\psi \) in front.

We next look into the \( \epsilon^* \)-terms. They consist of three sectors: (i) \( PF \)-terms, (ii) \( DP \)-terms, and (iii) \( G^2 \)-terms. Here the (i) \( PF \)-terms are arranged as

\[ (PF\text{-terms}) = i \left( \frac{1}{24} - 10 a_1 \right) P_\psi \gamma^4 \epsilon^* P^\mu F_{[4]A} \ , \] (4.2.14)

yielding the condition

\[ a_1 = +\frac{1}{240} \ , \] (4.2.15)

consistently with (4.2.12). Now the remaining (ii) \( DP \)- and (iii) \( G^2 \)-terms are arranged under (4.2.3a) to give

\[ (\epsilon^*\text{-terms}) = P_\psi \epsilon^* \left( D_\mu P^\mu - \frac{1}{24} G_{\rho \sigma \tau}^* G_{\rho \sigma \tau}^2 \right) \ , \] (4.2.16)

resulting in the scalar field equation (4.2.7a). This concludes all the variation of the gravitello \( \lambda \)-field equation.

We next postulate the gravitino field equation as

\[ \gamma^\lambda \left[ \gamma^\rho R_{\rho[\mu] + b_2 \left( \gamma^\rho \gamma_{[\mu] \lambda} \right) P_\rho + b_3 \left( \gamma_{[\mu] \gamma^\rho \lambda} \right) \tilde{P}_\rho \right. \]
\[ + \left. b_4 \left( \gamma^\rho \gamma_{[\mu] \lambda} \right) \tilde{G}_{\rho \sigma \tau}^* + b_5 \left( \gamma_{[\mu] \gamma^\rho \sigma \tau \lambda} \right) \tilde{G}_{\rho \sigma \tau}^* \right] (\partial_{[\nu] \varphi})(\partial_{\lambda} \varphi) = 0 \ , \] (4.2.17)

with the constants \( b_2, \cdots, b_5 \) to be fixed. Note that the supercovariantization of the gravitino field strength is crucial, while that of \( P_\mu \) is not, due to the higher dimensions of the latter, affecting only fermionic terms in bosonic field equations that we skip in this paper. The basic structure of this form is fixed after some trial and error process we performed in order to produce the \( e_{\mu}^m \) and \( A_{\mu \nu \rho \sigma \tau} \)-field equations, as in the \( N = 2 \) chiral supergravity in 10D [29]. To be more specific, our first guiding principle was to rely on the anti-self-duality equation (4.2.7d), and we take its variation under supersymmetry. It basically yields the equation

\[ \gamma_{[\mu_1 \mu_2 \mu_3} \gamma^\nu R_{\mu_4 \mu_5}(\partial_{\nu_6} \varphi)(\partial_{\nu} \varphi) + \frac{1}{240} \epsilon_{\mu_1 \cdots \mu_6 \nu_1 \cdots \nu_6} \gamma_{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6} (\partial_{\nu_6} \varphi)(\partial_{\nu} \varphi) + O(\varphi^2) = 0 \ , \] (4.2.18)

which is arranged after some manipulations of \( \gamma \)-matrices to be \( (\partial \varphi)^* \gamma^\rho R_{\rho[\mu \partial_{\nu}] \varphi} = O(\varphi^2) \), leading us to the postulate (4.2.17). Here \( O(\varphi^2) \) denotes the bilinear or higher-order terms in physical fields, e.g., the \( \varphi \)-field is not physical.
The variation of the gravitino field equation consists of two parts: The $\epsilon$-terms and $\epsilon^*$-terms. Now we first see that the $\epsilon^*$-terms consists further of three parts: (i) $DG^*$-terms, (ii) $PG^*$-terms, and (iii) $FG$-terms. Due to the Bianchi identity (4.2.3a), the first two sectors talk to each other, yielding

\[
(DG\text{-terms}) + (PG^*\text{-terms}) = + \left( -\frac{1}{96} - b_5 \right) \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} \epsilon^* P_{\rho} G_{\sigma \tau \omega}^* (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ \left( \frac{1}{32} - \frac{1}{12} b_2 - b_5 \right) \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} \epsilon^* P_{\mu} G_{\rho \sigma \tau \omega}^* (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ \left( -\frac{3}{32} - 9 b_5 \right) \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} \epsilon^* P_{\rho \sigma \tau \omega} (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ \left( +\frac{3}{16} + 18 b_5 \right) \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} \epsilon^* P_{\rho \sigma \tau \omega}^* (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ \left( +\frac{1}{12} + 3 b_5 \right) \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} \epsilon^* P_{\rho \sigma \tau \omega}^* (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
- \frac{\gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega}}{8} F_{\rho}[\mu \nu \lambda \omega]^3 (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ \frac{\gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega}}{48} F_{\rho \sigma \tau \omega} (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ \left( (b_2 - b_3)\text{-terms} \right) + \left[ (b_4 - 2 b_5)\text{-terms} \right] . \tag{4.2.19}
\]

Even though we skip the explicit structure of the last line, the important point is that these two sorts of terms are independent, yielding two conditions \( b_2 - b_3 = 0 \) and \( b_4 - 2 b_5 = 0 \). Now requiring the vanishing of each \( PG^*\text{-terms} \) in (4.2.19), we can fix the values

\[
b_2 = +\frac{1}{2} , \quad b_3 = +\frac{1}{2} , \quad b_4 = -\frac{1}{48} , \quad b_5 = -\frac{1}{96} . \tag{4.2.20}
\]

Even though we do not give the details here, we stress the usage of various relationships based on the properties of $\partial \varphi$, $\partial \bar{\varphi}$ together with $P_\uparrow$, $P_\downarrow$, in addition to our extra constraints (4.2.4). For instance, we can show the relationship

\[
\gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} P_{\nu} G^*_{\rho \sigma \tau \omega} (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi) = - \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} P_{\nu} G^*_{\rho \sigma \tau \omega} (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
- 3 \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} P_{\nu} G^*_{\rho \sigma \tau \omega} (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ 3 \gamma^\lambda \gamma_{[\mu} \rho^{\sigma \tau \omega} P_{\nu} G^*_{\rho \sigma \tau \omega} (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi)
+ 6 \gamma^\lambda \gamma_{\rho \sigma \tau \omega} \epsilon^* P_{\rho \sigma \tau \omega} (\partial_{\nu} \varphi)(\partial_{\lambda} \varphi) . \tag{4.2.21}
\]

For example, in the left hand side, we can push the particular combination $\gamma^\lambda \partial_{\lambda} \varphi$ all the way to the left side of $P_\downarrow$, making this projection operator redundant. This is because of the constraint $G^*_{\rho \sigma \tau \omega} \partial_{\lambda} \varphi = 0$ and the antisymmetrization $[\mu \nu]$. Once the projection operator $P_\downarrow$ is deleted, we see that the $\gamma$-matrix algebra is parallel to the $N = 2$ chiral supergravity in 10D [29], and even the coefficient turns out to be the same, except for $\partial \bar{\varphi}$ in front as an overall factor. In our 12D computation, we frequently use this technique of moving around the combination $\partial \varphi$, using the constraints (4.2.4) on the fields.

We now look into the remaining (iii) $FG$-terms in the $\epsilon^*$-sector. This sector is actually the most involved, as in 10D case [29]. However, we can categorize all the terms with
respect to the number of $\gamma$-matrices involved, so in total we have (1) $\partial \varphi \gamma[1] \epsilon$-terms, (2) $\partial \varphi \gamma[3] \epsilon^*$-terms, (3) $\partial \varphi \gamma[5] \epsilon^*$-terms, (4) $\partial \varphi \gamma[7] \epsilon^*$-terms. After some manipulations, we soon notice that (1) $\partial \varphi \gamma[1] \epsilon$-terms cancel themselves, while (2) $\partial \varphi \gamma[3] \epsilon^*$-terms and (4) $\partial \varphi \gamma[7] \epsilon^*$-terms cancel each other, due to the self-duality of the fifth-rank antisymmetric field strength (4.2.7d), expressed more symmetrically as

$$F[6] = -\frac{1}{6!} \epsilon_{[6]}^\rho \epsilon_{\rho} F[6] \equiv F[\mu_{1} \ldots \mu_{6}] \partial_{\rho \sigma} \varphi \quad (4.2.22)$$

combined with the technical relations associated with the $\epsilon$-tensor in (A.20). Finally, we see that the (3) $\partial \varphi \gamma[5] \epsilon^*$-terms cancel themselves, by help of the relations such as $F[3] \epsilon_{[\rho | \sigma | \nu]} \varphi = +3 F[3] \epsilon_{\rho \sigma \nu}$. After all, all the (iii) $FG^*$-terms cancel themselves consistently, and they do not yield any field equations, as expected also from the experience of $N = 2$ supergravity in 10D [29]. This concludes all the $\epsilon^*$-terms in the variation of (4.2.17).

We next compute the $\epsilon$-terms in the supersymmetric variation of the gravitino field equation. They are categorized as (i) $R$-terms, (ii) $PP^*$-terms, (iii) $DF$-terms, (iv) $F^2$-terms, (v) $GG^*$-terms. Here the (i) $R$-terms contain the Riemann tensor, coming from the commutator $[D_{\mu}, D_{\nu}]\epsilon$. These $R$-terms give the leading Ricci tensor term in the zw"olfbein $e^{\mu}_{m}$-field equation, as will be in (4.2.34). The (ii) $PP^*$-terms are arranged as

$$(PP^*-\text{terms}) = \gamma^\lambda \gamma^\rho \epsilon \left[ -\frac{1}{2} P_{\rho} P_{\mu}^* + \left( \frac{1}{2} - 2 b_2 \right) P_{\nu} P_{\rho}^* \right] \left( \partial_{[\nu]} \varphi \right) \left( \partial_{\lambda} \varphi \right)
+ (b_2 - b_3) \gamma^\lambda \gamma^\rho \gamma^\mu \rho \gamma^\sigma \epsilon P_{\mu} P_{\sigma}^* \left( \partial_{[\nu]} \varphi \right) \left( \partial_{\lambda} \varphi \right) \quad (4.2.23)$$

Here the first two terms will contribute to the energy-momentum tensor, while the last line is to vanish, yielding the result

$$b_2 = b_3 = + \frac{1}{2} \quad (4.2.24)$$

consistently with (4.2.20). Now the (iii) $DF$-terms turn out to be equivalent to the (v) $GG^*$-terms by the use of the Bianchi identity (4.2.3b). In fact, we get

$$(DF-\text{terms}) = - \frac{1}{192} \gamma^\lambda \gamma^\rho \gamma^\mu \gamma_{[\nu]} \epsilon \left[ G_{[3]} \gamma \gamma_{[2]} \epsilon_{\mu \nu \lambda} \left( \partial_{[\nu]} \varphi \right) - G_{[2]} \gamma \gamma_{[\mu \nu \lambda]} \left( \partial_{\nu \nu} \varphi \right) \right] \left( \partial_{\lambda} \varphi \right)
- \frac{1}{576} \gamma^\lambda \gamma_{[\nu]} \gamma_{[3]} \epsilon G_{[3]} \gamma \gamma_{[\mu \nu \lambda]} \left( \partial_{\nu \nu} \varphi \right) \left( \partial_{\lambda} \varphi \right) \quad (4.2.25)$$

which will be combined with the explicit (v) $GG^*$-terms below. We will combine these terms with the explicit $GG^*$-terms of the category (v) later. Now (iv) $F^2$-terms are arranged to be

$$(F^2-\text{terms}) = - \frac{1}{12} \gamma^\lambda \gamma^\rho \epsilon F_{[4]} \epsilon_{\nu \mu \lambda} \left( \partial_{\nu \nu} \varphi \right) \left( \partial_{\lambda} \varphi \right) \quad (4.2.26)$$

which contributes to the energy-momentum tensor in the $e^{\mu}_{m}$-field equation. For these complicated $F^2$-terms, we have used the following important technique. Notice, that e.g., in (4.2.26) all the indices on $F$ including the contracted ones take purely 10D values. This is because the $\rho$-index can take only the 10D value, because of $\partial \varphi$ in front, while the contracted ones [1] stay also within 10D, due to the constraint (4.2.4a). This implies that
we can basically use the purely 10D relations for simplifications of these $F^2$-terms. In fact, the anti-self-duality (4.2.22) in 12D implies, as desired, the self-duality in 10D [29]:

$$F_{i_1 \cdots i_6} = + \frac{1}{5!} \varepsilon_{i_1 \cdots i_6 j_1 \cdots j_5} F_{j_1 \cdots j_5} ,$$

where $i_1, i_2, \ldots$ are purely 10D indices. This relation in turn leads to other identities, such as

$$\gamma^{[2]} [ \varepsilon F^{[3]} ij F_{[2][3]} ] \equiv 0 , \quad \gamma^{[3]} [ \varepsilon F_{[3]ij} F_{[4]} ] \equiv 0 .$$

(4.2.28)

Fortunately, we found that all of these relevant $F^2$-terms always have purely 10D indices on $F$’s, and we can keep this technique in this sector.

We finally arrange all the $GG^*$-terms which are explicitly from the (v) $GG^*$-terms and from the (iii) $DF$-terms via (4.2.25). These terms are rather involved, but we can further categorize them by the number of $\gamma$-matrices, as (1) $\partial \varphi \gamma^7 \varepsilon$-terms, (2) $\partial \varphi \gamma^5 \varepsilon$-terms, (3) $\partial \varphi \gamma^3 \varepsilon$-terms, (4) $\partial \varphi \gamma^1 \varepsilon$-terms. Here (1) $\partial \varphi \gamma^7 \varepsilon$-terms turn out to be

$$\left( \partial \varphi \gamma^7 \varepsilon \text{-terms} \right) = \frac{1}{768} \left[ 32 (-b_4 + b_5) + \frac{1}{3} \right] \gamma^\lambda \gamma_{[\mu]} [3][3]’ \varepsilon G_{[3]} G^*_{[3]’} (\partial_{[\mu]} \varphi) (\partial_{\lambda} \varphi) .$$

(4.2.29)

The $\gamma$-matrix structure here is different from the leading Ricci-tensor term in the $e^{\mu m}$-field equation, so that these terms should vanish, yielding the condition

$$b_4 - b_5 = - \frac{1}{96} ,$$

(4.2.30)

consistent with all our previous values. Next the (2) $\partial \varphi \gamma^5 \varepsilon$-terms can be arranged after some algebra into

$$\left( \partial \varphi \gamma^5 \varepsilon \text{-terms} \right) = + \frac{1}{512} \left[ 256 (b_4 - b_5) + 1 \right] \gamma^\psi \gamma_{[\mu]} [\rho \lambda \omega \varepsilon G_{[\rho \sigma \tau]} G^*_{[\lambda \omega \tau]} (\partial_{[\mu]} \varphi) (\partial_{[\psi]} \varphi)$$

$$+ \frac{1}{3072} \left[ -12 - 384 (b_4 + b_5) \right] \gamma^\lambda \gamma_{[3][2]} [\varepsilon G_{[3]} G^*_{[2][\mu]} (\partial_{[\mu]} \varphi) (\partial_{\lambda} \varphi)$$

$$+ \frac{1}{3072} \left[ -4 - 384 (b_4 - b_5) \right] \gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\tau \varepsilon G_{[\mu \lambda \omega \varepsilon G^*_{[\rho \sigma \tau]} (\partial_{[\mu]} \varphi) (\partial_{\omega} \varphi) .$$

(4.2.31)

Fortunately, each line vanishes, for the same values of $b_4$ and $b_5$ as in (4.2.20). In a similar fashion, the (3) $\partial \varphi \gamma^3 \varepsilon$-terms are arranged to

$$\left( \partial \varphi \gamma^3 \varepsilon \text{-terms} \right) = + \frac{1}{512} \left[ 4 + 384 (b_4 - b_5) \right] \gamma^\psi \gamma_{[\mu]} [\rho \omega \varepsilon G_{[\rho \sigma \tau]} G^*_{[\omega \sigma \tau]} (\partial_{[\mu]} \varphi) (\partial_{[\psi]} \varphi)$$

$$+ \frac{1}{512} \left[ -12 - 384 (b_4 + b_5) \right] \gamma^\psi \gamma^\rho \gamma^\sigma \varepsilon G_{[\rho \sigma \tau \varepsilon G^*_{[\omega \tau]} (\partial_{[\mu]} \varphi) (\partial_{\sigma} \varphi)$$

$$+ \frac{1}{512} \left[ +4 + 384 (b_4 - b_5) \right] \gamma^\mu \gamma^\rho \gamma^\lambda \varepsilon G_{[\mu \lambda \omega \varepsilon G^*_{[\rho \sigma \tau]} (\partial_{[\mu]} \varphi) (\partial_{[\omega]} \varphi) .$$

(4.2.32)

Fortunately, we see that each of these line vanish consistently with the values of $b_4, b_5$, as in (4.2.20). Finally we look into the (4) $\partial \varphi \gamma^1 \varepsilon$-terms which contribute to the energy-momentum tensor:

$$\left( \partial \varphi \gamma^1 \varepsilon \text{-terms} \right)$$

$$= \frac{1}{16} \gamma^\lambda \gamma^\tau \varepsilon \left[ - G_{[\rho \sigma \tau]} G^*_{[\rho \sigma \tau]} - G_{[\rho \tau]} G^*_{[\rho \tau]} + \frac{1}{6} g_{[\rho \tau]} |G_{[\rho \tau]}|^2 \right] (\partial_{[\mu]} \varphi) (\partial_{[\lambda]} \varphi) .$$

(4.2.33)
This completes all the $GG^*$-terms.

We now collect all the terms to contributing to the $e_\mu^m$-field equation. They are all with $\partial /\phi^\gamma$, arranged as

$$
\frac{1}{2} \gamma^\lambda \gamma^\rho \epsilon \left[ R_{\mu\rho} - P_{\mu}P^*_\rho - P_{\rho}P^*_\mu - \frac{1}{6} F_{[\eta\rho]}F^{[\eta\rho]} + \frac{1}{8} g_{[\rho]}G^{[\mu]}G^*_{[\mu]} + \frac{1}{48} g_{[\rho]}G_{[\mu]}G_{[\nu]}G_{[\lambda]}G^*_{[\mu]}G^*_{[\nu]}G^*_{[\lambda]} \right] (\partial_{[\nu]}\varphi)(\partial_{\lambda}\varphi) = 0 .
$$

yielding nothing but (4.2.7c). This concludes the internal consistency check, as well as the derivation of all of our field equations, after the supersymmetric variation of our fermionic field equations.

### 4.3 Dimensional Reduction into $D = 9 + 1$

As before, we can perform dimensional reduction into $D = 9 + 1$ in order to see the validity of our $D = 12$, $N = 2$ supergravity. The most frequently-used relationships are the same as (2.4.1) - (2.4.4), and the only new ones for $N = 2$ case are

$$
\hat{\mathcal{R}}_{\mu\nu} = (\mathcal{R}_{\mu\nu}, 0) , \quad \hat{\lambda} = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} .
$$

Since most of the dimensional reduction prescription is the same as in the $N = 1$ case, we skip the details to get 10D field equations in agreement with ref. [29]. As a matter of fact, their forms can be easily figured out, due to our 12D field equations which already resemble those in 10D.

### 5. Super $(2 + 2)$-Brane on Background of $D = 12$, $N = 2$ Supergravity

As we have studied the Green-Schwarz superstring on the background of $D = 12$, $N = 1$ supergravity to see the validity of our theory as a probe, we can try to put super $(2 + 2)$-brane on the $N = 2$ supergravity in 12D [15]. Here we choose the super $(2 + 2)$-brane [31], because the super $(2 + 2)$-brane is the right p-brane action [32] on such a background which is supposed to be the strong coupling limit of F-theory [2] with 12D space-time with two time coordinates. The existence of the forth-rank antisymmetric tensor $A_{[4]}$ also suggests this is the natural super p-brane [32].

Our postulate for the total action for the super $(2 + 2)$-brane is [15]

$$
S = S_\sigma + S_A ,
$$

$$
S_\sigma \equiv \int d^4\sigma \left( \frac{1}{2} \sqrt{g} g^{ij} \eta_{ab} \Pi_i^a \Pi_j^b - \sqrt{g} \right) ,
$$

$$
S_A \equiv \int d^4\sigma \left( - \frac{i}{6} \varepsilon^{i_1 \cdots i_4} \Pi_i^{B_1} \cdots \Pi_i^{B_4} A_{B_1 \cdots B_4} \right) .
$$
Here the indices $i, j, \ldots = 1, 2, 3, 4$ stand for curved coordinates in $D = 2 + 2$, and $\Pi_i^a \equiv (\partial_i Z^M) E_M^a$, etc., similarly to section 3. Note the extra imaginary unit factor for $S_A$ understood as Wick rotation from Minkowskian $D = 1 + 3$ to our $D = 2 + 2$. This feature is similar to the case for the topological $F\tilde{F}$-terms in Euclidean $D = 4 + 0$ from the usual Minkowskian $D = 3 + 1$ [33]. Accordingly, $g \equiv \det (g_{ij})$ is positive definite, so we use $\sqrt{g}$ instead of $\sqrt{-g}$.

Our relevant superspace constraints are [15]

$$T_{a \beta}^c = (\gamma^{cd})_{a \alpha} \nabla^\alpha \varphi + (P_{\bar{\gamma}})_{a \alpha} (\nabla^\alpha \varphi) , \quad T_{\alpha \beta}^c = (\gamma^{cd})_{\alpha \beta} \nabla^\alpha \varphi + (P_{\bar{\gamma}})_{\alpha \beta} (\nabla^\alpha \varphi) , \quad (5.2a)$$

$$F_{\alpha \beta cde} = - \frac{i}{4} (\gamma_{cde})_{\alpha \beta} \nabla^\beta \varphi , \quad F_{\bar{\alpha} \bar{\beta} cde} = + \frac{i}{4} (\gamma_{cde})_{\alpha \beta} \nabla^\beta \varphi , \quad (5.2b)$$

$$\nabla_a \nabla^a \varphi = 0 , \quad \nabla_a \nabla^a \tilde{\varphi} = 0 , \quad (\nabla_a \varphi)(\nabla^a \tilde{\varphi}) = 1 , \quad (5.2c)$$

These expressions are easily obtained from the component results using the standard technique in [27]. The barred spinorial indices $\bar{\pi}, \bar{\varphi}, \ldots$ denote the complex conjugations in superspace, corresponding to the star-operations in component in (4.1.8b).

As usual in general $p$-brane formulation [32], we have the fermionic symmetries [15]

$$\delta E^\alpha = (I + \Gamma)^{a \beta} k^\beta + \frac{1}{2} \left[ (\nabla \varphi)(\nabla \tilde{\varphi}) \right]^{a \alpha} \eta_\beta \equiv - \left[ (I + \Gamma)^{a \beta} (P_{\bar{\alpha}, \eta})^{a} \right] , \quad (5.3a)$$

$$\delta E^{\bar{\alpha}} = (I + \Gamma)^{a \beta} k^{\bar{\beta}} + \frac{1}{2} \left[ (\nabla \varphi)(\nabla \tilde{\varphi}) \right]^{a \alpha} \eta^{a \bar{\beta}} \equiv - \left[ (I + \Gamma)^{a \beta} (P_{\bar{\alpha}, \eta})^{a} \right] , \quad (5.3b)$$

$$\delta E^a = 0 , \quad \delta \varphi = 0 , \quad \delta \tilde{\varphi} = 0 , \quad (5.3c)$$

with the fermionic parameters $\kappa$ and $\eta$, where $\nabla \varphi \equiv \gamma^a \nabla_a \varphi$, and $\Gamma$ defined by [31]

$$\Gamma \equiv \frac{1}{24 \sqrt{g}} \epsilon^{ijkl} \Pi_i^a \Pi_j^b \Pi_k^c \Pi_l^d (\gamma_{abcd}) , \quad (5.4)$$

satisfies relations such as

$$\Gamma^2 = I , \quad (5.5a)$$

$$\epsilon_i^{jkl} \Pi_j^a \Pi_k^b \Pi_l^c \gamma_{abc} \Gamma = + 6 \sqrt{g} \Pi_i^a \gamma_a , \quad (5.5b)$$

under the algebraic $g_{ij}$-field equation $g_{ij} = \Pi_i^a \Pi_j^a$, which are useful for the invariance check of our total action $S$. In fact, the variation of $S$ under (5.3) takes the form

$$\delta (S_\sigma + S_A) = \left[ + \sqrt{g} \epsilon^{ijkl} \Pi_i^a (\gamma_\alpha \gamma_\beta) \gamma_\gamma (\nabla \varphi)(\nabla \tilde{\varphi})(\delta E^\beta) \Pi_j^a + \frac{1}{6} \epsilon^{ijkl} \Pi_i^a (\gamma_\alpha \gamma_\beta \gamma_\gamma) (\nabla \varphi)(\nabla \tilde{\varphi})(\delta E^\alpha) \Pi_j^b \Pi_k^c \Pi_l^d \right] + (\delta E^\alpha \rightarrow \delta E^{\bar{\alpha}}) . \quad (5.6)$$

For the $\eta$-transformation in (5.3), by the aid of (5.4) we see that two sorts of terms cancel themselves, if we impose the extra condition

$$\Pi_i^a \nabla \varphi = 0 , \quad (5.7)$$
together with the null-ness condition \((\nabla_a \phi)^2 = 0\) in (5.2c). The extra condition (5.7) is formally the same as the first equation in (3.12) for the Green-Schwarz superstring on \(N = 1\) supergravity background. However, the difference is that for the Green-Schwarz superstring, we could not impose such condition from outside, because these conditions are of the first order, interpreted as unidexterous field equations on 2D world-sheet. On the other hand, in the present case of super \((2 + 2)\)-brane, since the world-supervolume is 4D, the condition (5.7) can be imposed as constraint from outside, for the invariance check of our total action. As for the \(\kappa\)-transformation in (5.3) applied to (5.6), we see that eq. (5.5) helps us to rearrange two sorts of terms cancelling each other under (5.7), but now without (5.2c). This is more natural than the \(N = 1\) case, because the \(\eta\)-symmetry is associated with the extra dimensions governing the null-vector condition, while the \(\kappa\)-symmetry governs the physical freedom within 10D. In other words, our null-ness condition (5.2c) is not artificially put by hand, but required by one of the fermionic symmetries of the super \((2 + 2)\)-brane action.

As in the case of Green-Schwarz superstring for the \(D = 12, N = 1\) supergravity, we can easily understand the ordinary 16+16 degrees of freedom come out of the super \((2 + 2)\)-brane by these fermionic symmetries: First, the \(\eta\)-symmetry deletes half of the original 64 components of the fermionic coordinates \(\theta^\mu\) in \(N = 2\) superspace in 12D, and thus at most 32 components can be physical. Next, the \(\kappa\)-symmetry deletes further half of 32 components, leaving the usual 16 components in the light-cone coordinates in \(N = 2\) Green-Schwarz superstring [5].

6. \(N = 1\) Supergravity in \(D = 11 + 2\)

6.1 Notations

Our metric in \(D = 11 + 2\) is \((\eta_{mn}) = \text{diag.} (-, +, \cdots, +, +, +, -)\), with the local Lorentz indices \(m, n, \cdots = (0), (1), \cdots, (9), (10), (12), (13)\), and our \(\epsilon\)-tensor is defined by \(\epsilon^{012\cdots101213} = +1\), and accordingly \(\gamma^{(13)} = \gamma_{(0)}\gamma^{(1)}\gamma^{(2)}\cdots\gamma^{(9)}\gamma^{(10)}\gamma^{(12)}\). Our null-vectors are defined in the same way as in 12D [14][15]:

\[
(n_m) = \left(0, 0, \cdots, 0, +\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right), \quad (m_m) = \left(0, 0, \cdots, 0, +\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right). \tag{6.1.1}
\]

As in 12D, we define the \(\pm\)-indices by

\[
V_\pm \equiv \frac{1}{\sqrt{2}}(V_{(12)} \pm V_{(13)}), \quad \tag{6.1.2}
\]

so that we have \(n_+ = m^+ = +1, \ n_- = m^- = 0, \ n_+m^+ = n^-m^- = +1\) as in 12D. Other important relations such as (2.1.5) are formally the same as in 12D [14][15]. Therefore we can identify these null-vectors with the gradients of superinvariant scalars \(\varphi, \bar{\varphi}\), as in (2.2.2):

\[
n_\mu \equiv \partial_\mu \varphi, \quad m_\mu \equiv \partial_\mu \bar{\varphi}, \quad D_m D_n \varphi = 0, \quad D_m D_n \bar{\varphi} = 0. \tag{6.1.3}
\]
As for the modified Lorentz generators in 13D, since its structure is to be exactly parallel
to the 12D case in subsection 2.1, we do not repeat the details here. We also skip other
relations that are exactly and formally parallel to 12D.

6.2. \( N = 1 \) Supergravity in \( D = 11 + 2 \)

Based on our accumulated experience with 12D supergravity, the construction of 13D
supergravity is now straightforward. The field content of our 13D supergravity is the same
as in 11D supergravity \([1]\), namely \( (e^m_\mu, \psi_\mu, A_{\mu \nu \rho}; \varphi, \tilde{\varphi}) \), with the dreizehnbein, a gravitino and the third-rank antisymmetric tensor, in addition to the supercovariant scalars. The
supersymmetry transformation rule is fixed by the requirement of closure of gauge algebra
\([30]\).

We first present our result of supersymmetry transformation rule:

\[
\begin{align*}
\delta Q e^m_\mu &= (\bar{\tau} \gamma^{m \nu} \psi_\nu) \partial_\nu \varphi , \quad (6.2.1a) \\
\delta Q \psi_\mu &= D_\mu \epsilon + \frac{1}{144} P_\downarrow (\gamma_\mu^{[4]} \epsilon \hat{F}_4 - 8 \gamma_3^{[3]} \epsilon \hat{F}_3) , \quad (6.2.1b) \\
\delta Q A_{\mu \nu \rho} &= +\frac{1}{2} (\bar{\tau} \gamma_{[\mu \nu} \psi_{\rho]} \partial_\sigma \varphi , \quad (6.2.1c) \\
\delta Q \varphi &= 0 , \quad \delta Q \tilde{\varphi} = 0 , \quad (6.2.1d)
\end{align*}
\]

where as usual all the hatted field strengths are supercovariantized \([30]\). Our extra constraints
are similar to the 12D case \((4.2.4)\):

\[
\begin{align*}
\hat{F}_{\mu \rho \rho} \varphi & = 0 , \quad \hat{R}_\nu^{\nu \rho \rho} \partial_\nu \varphi = 0 , \quad \hat{R}_{\mu \nu}^{\nu \rho \rho} D_m \varphi = 0 , \quad (6.2.2a) \\
\hat{R}_\mu^{\nu \rho \rho} \partial_\nu \varphi & = 0 , \quad \hat{R}_{\mu \nu}^{\nu \rho \rho} D_m \tilde{\varphi} = 0 , \quad (6.2.2b) \\
(D_m \varphi)^2 &= 0 , \quad (D_m \tilde{\varphi})^2 = 0 , \quad (D_m \varphi)(D_m \tilde{\varphi}) = 1 . \quad (6.2.2c)
\end{align*}
\]

The \( R_{\mu \nu} \) is the field strength for the gravitino.

We outline the derivations of this rule. Based on the experience with the 12D theories,
we first postulate the form of the transformation rule as \((6.2.1b)\) and \((6.2.1c)\) with three
unknown coefficients \( a_1, a_2 \) and \( a_3 \), as

\[
\begin{align*}
\delta Q \psi_\mu &= P_\downarrow (a_1 \gamma_\mu^{[4]} \epsilon F_4 + a_2 \gamma_3^{[3]} \epsilon F_3) , \quad (6.2.3a) \\
\delta Q A_{\mu \nu \rho} &= a_3 (\bar{\tau} \gamma_{[\mu \nu} \psi_{\rho]} \partial_\sigma \varphi . \quad (6.2.3b)
\end{align*}
\]

This structure is similar to 12D supergravities \([14][15]\), where the null-vector is involved
in the zwölfbein transformation \((6.2.1a)\), while the projector \( P_\downarrow \) is needed in \((6.2.1b)\).
The involvement of the null-vector in \((6.2.1c)\) is expected also from superspace as will be
mentioned shortly, or as an analog of the second-rank tensor transformation rule \((2.3.28c)\).
for the $D = 12$, $N = 1$ supergravity. Similarly to globally supersymmetric Yang-Mills [12], our system has also local extra symmetries:

$$\delta_E e_\mu^m = \alpha_\mu (D^m \varphi) + \bar{\alpha}^m \partial_\mu \varphi \quad , \quad (6.2.4a)$$

$$\delta_E A_{\mu \nu \rho} = \beta_{[\mu \nu} \partial_{\rho]} \varphi \quad . \quad (6.2.4b)$$

The requirement of closure of supersymmetry on all the bosonic fields, up to the gauge, Lorentz transformations, or the above-mentioned extra symmetries, fixes the unknown coefficients. However, we do not need to confirm the closure on the gravitino, as long as we confirm the consistency among field equations.

First, the closure on the dreizehnbein fixes the parameter of translation $\xi^m \equiv \langle \tilde{\tau}_2 \gamma^{m \nu} \epsilon_1 \rangle \partial_\nu \varphi$, up to local Lorentz transformation. Second, the closure on $A_{\mu \nu \rho}$ yields

$$[\delta_1, \delta_2] A_{\mu \nu \rho} = - 12 a_2 a_3 \xi^\sigma F_{\sigma \mu \nu \rho} + \partial_{[\mu} A_{\nu \rho]}$$

$$- 2 a_3 (8 a_1 + a_2) \langle \tilde{\tau}_2 \gamma_{[\mu \nu]} [3] m \epsilon_1 \rangle F_{[\rho] [3]} D_m \varphi$$

$$+ a_1 \left[ - 2 \langle \tilde{\tau}_2 \gamma_{[\mu} P_{1} \gamma_{(\nu]} [4] \epsilon_1 \rangle F_{[\rho]} - \langle \tilde{\tau}_2 \gamma_{[\mu} \gamma_{\rho \sigma \tau} [4] \epsilon_1 \rangle F_{[\nu_\sigma \tau]} \right] \partial_{[\rho]} \varphi - (1 \leftrightarrow 2)$$

$$+ a_2 \left[ - 2 \langle \tilde{\tau}_2 \gamma_{[\mu} P_{1} \gamma_{\rho \sigma \tau} [3] \epsilon_1 \rangle F_{[\nu_\sigma \tau]} + 6 \langle \tilde{\tau}_2 \gamma_{[\mu} \gamma_{\rho \sigma \tau} [2] \epsilon_1 \rangle F_{[\nu_\sigma \tau]} \right] \partial_{[\rho]} \varphi - (1 \leftrightarrow 2) \quad , \quad (6.2.5)$$

where $A_{\mu \nu \rho}$ is the parameter of gauge transformation, while the last three lines with $\partial_{[\rho]} \varphi$ at the end are interpreted as the extra transformation (6.2.4b). The normalization of translation parameter in the first term and the vanishing of the last term require the conditions

$$a_2 a_3 = - \frac{1}{12} \quad , \quad (6.2.6a)$$

$$8 a_1 + a_2 = 0 \quad . \quad (6.2.6b)$$

We can choose $a_3 = +3/2$ as the normalization to get the solutions

$$a_1 = + \frac{1}{144} \quad , \quad a_2 = - \frac{1}{18} \quad , \quad a_3 = + \frac{3}{2} \quad , \quad (6.2.7)$$

yielding (6.2.1), also in agreement with their corresponding terms in the 11D supergravity [1], as desired.

We next give the list of our field equations:

$$\left[ \hat{R}_{\rho [\mu} + \frac{1}{3} \hat{F}_{\rho [3]} \hat{F}_{[\mu]} - \frac{1}{30} g_{\rho [\mu} \hat{F}_{[4]} \right] \partial_{[\nu]} \varphi = \mathcal{O}(\psi^2) \quad , \quad (6.2.8)$$

$$\left( \hat{D}_\mu \hat{F}_{\nu \rho \sigma \tau} \right) \partial_{[\nu} \varphi = + \frac{1}{2304} \epsilon_{\nu \rho \sigma \tau} [4] [4] \hat{F}_{[\mu]} \hat{F}_{[4]} \partial_{[\rho]} \partial_{[\sigma]} \varphi \quad , \quad (6.2.9)$$

$$\gamma^\sigma \gamma^\rho \hat{R}_{\rho [\mu} \partial_{[\nu]} \varphi \partial_{[\omega]} \varphi = 0 \quad . \quad (6.2.10)$$

These field equations are derived in the same way as in $D = 12$, $N = 2$ supergravity [15]: We first postulate the gravitino field equation as in (6.2.10), and vary it under supersymmetry. There are three types of terms generated: (i) $R$-terms, (ii) $DF$-terms, and
(iii) $F^2$-terms. The (i) $R$-terms are going to give the leading Ricci tensor term in the dreizehnbein (gravitational) field equation (6.2.8), while the (iii) $F^2$-terms correspond to its energy-momentum tensor terms, and (ii) $DF$-terms give the $F$-field equation (6.2.9). However, these original terms talk to each other after the use of the $F$- and dreizehnbein field equations. To clarify this point, we use the unknown coefficients $\alpha, \beta$ and $\delta$ also for the $F$- and the gravitational field equations:

\begin{align*}
(D_{\mu} F_{\nu \rho \sigma}^{\mu \nu \rho \sigma}) \partial_{\gamma} \phi &= \alpha e^{-1} \epsilon_{\nu \rho \sigma \tau}^{[4][4]} \epsilon_{\mu}^{\epsilon} \tilde{F}_{[4]}^{\epsilon \mu} \tilde{F}_{[4]}^{\epsilon \mu} \partial_{\mu} \phi, \\
\left[ \tilde{R}_{\rho \mu} + \delta F_{\rho [3]} F_{[\mu [3]} + \beta g_{\rho [\mu} F_{[4]}^{2} \right] \partial_{\nu} \phi &= 0.
\end{align*}

(6.2.11) (6.2.12)

For example, a typical arrangement by the use of the $F$-field equation is

\begin{align*}
-3a_2 (\partial \phi) \gamma^{[2]} \epsilon_{(D_{\mu} F_{\rho [2]}^{\mu \rho}) \partial_{\nu} \phi} &= -24a_2 \alpha (\partial \phi) \gamma^{[4][4]} \epsilon_{\mu \epsilon} F_{[4]}^{\epsilon \mu} \tilde{F}_{[4]}^{\epsilon \mu} \partial_{\nu} \phi \\
&- 6a_2 (\partial \phi) \gamma^{\sigma \rho} \epsilon \left( (D_{\tau} F_{\sigma \tau}) \partial_{\nu} \phi - \alpha \epsilon_{\rho \sigma \nu}^{[4][4]} F_{[4]}^{\epsilon \mu} \partial_{\epsilon} \phi \right).
\end{align*}

(6.2.13)

where the last line vanishes by the $F$-field equation (6.2.9). Similarly, we have

\begin{align*}
-4a_1 (\partial \phi) \gamma^{[3]} \epsilon_{(D_{\mu} F_{\rho [3]}^{\mu \rho}) \partial_{\nu} \phi} &= +768a_1 \alpha (\partial \phi) \gamma^{[4][4]} \epsilon F_{[4]}^{\epsilon \mu} \tilde{F}_{[4]}^{\epsilon \mu} \partial_{\nu} \phi \\
&- 8a_1 (\partial \phi) \gamma^{\tau \omega} \epsilon \ell (D_{\mu} F_{\tau \omega}) \partial_{\nu} \phi - \alpha \epsilon_{\tau \omega \nu}^{[4][4]} F_{[4]}^{\epsilon \mu} \partial_{\epsilon} \phi \right) - (\mu = \nu),
\end{align*}

(6.2.14)

with the vanishing second line. Similarly, by the use of the gravitational field equation (6.2.8), we have

\begin{align*}
\frac{1}{2} (\partial \phi) \gamma^{\rho} \epsilon \tilde{R}_{\rho \mu \nu} \partial_{\nu} \phi &= +\frac{1}{2} (\partial \phi) \gamma^{\rho} \epsilon \left( \delta F_{\rho [3]} F_{[\mu [3]} \partial_{\nu] \phi} - \beta g_{\rho [\mu} F_{[4]}^{2} \partial_{\nu] \phi} \right) \\
&+ \frac{1}{2} (\partial \phi) \gamma^{\rho} \epsilon \left( R_{\rho \mu \nu} \partial_{\nu} \phi + \delta F_{\rho [3]} F_{[\mu [3]} \partial_{\nu] \phi} + \beta g_{\rho [\mu} F_{[4]}^{2} \partial_{\nu] \phi} \right).
\end{align*}

(6.2.15)

where the second line vanishes on-shell. In order to see the consistency of our transformation rule and field equations, we keep the coefficients $a_1, a_2, a_3$. After these manipulations we get

\begin{align*}
\delta \left[ (\partial \phi) \gamma^{\rho} \tilde{R}_{\rho \mu \nu} \partial_{\nu} \phi \right] &= (768a_1 + 32a_1^2 - 6a_1 a_2 - 2a_2^2) N_{\mu \nu} \\
&+ (-24a_2 \alpha + 4a_1^2 + 2a_1 a_2) W_{\mu \nu} + \left( \frac{1}{2} \beta + 288a_1^2 \right) S_{\mu \nu} + \left( -\frac{1}{2} \delta + 1152a_1^2 + 36a_2^2 \right) P_{\mu \nu} \\
&- 36(8a_1 + a_2)(2a_1 - a_2) Q_{\mu \nu} - 6(8a_1 + a_2)^2 T_{\mu \nu} - 72a_1 (8a_1 + a_2) U_{\mu \nu}.
\end{align*}

(6.2.16)

Here $N, P, Q, S, T, U, W$ stand for different structures for the $F^2$-terms:

\begin{align*}
N_{\mu \nu} &\equiv (\partial \phi) \gamma^{[4][3]} \epsilon F_{[4]}^{\epsilon \mu} F_{[3]}^{\epsilon \mu} \partial_{\nu] \phi}, & P_{\mu \nu} &\equiv (\partial \phi) \gamma^{\rho} \epsilon F_{\rho [4]}^{\epsilon \mu} F_{[4]}^{\epsilon \mu} \partial_{\nu] \phi}, \\
Q_{\mu \nu} &\equiv (\partial \phi) \gamma^{[2]2} \epsilon F_{[2]}^{\epsilon \mu} F_{[2]}^{\epsilon \mu} \partial_{\nu] \phi}, & S_{\mu \nu} &\equiv (\partial \phi) \gamma^{[4][4]} \epsilon F_{[4]}^{\epsilon \mu} \partial_{\nu] \phi}, \\
T_{\mu \nu} &\equiv (\partial \phi) \gamma^{[3][2]} \epsilon F_{[3]}^{\epsilon \mu} F_{[2]}^{\epsilon \mu} \partial_{\nu] \phi}, & U_{\mu \nu} &\equiv (\partial \phi) \gamma^{[2][2]} \epsilon F_{[2]}^{\epsilon \mu} F_{[2]}^{\epsilon \mu} \partial_{\nu] \phi}, \\
W_{\mu \nu} &\equiv (\partial \phi) \gamma^{[4][4]} \epsilon F_{[4]}^{\epsilon \mu} \partial_{\nu] \phi}.
\end{align*}

(6.2.17)

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The \( \alpha \)-dependent coefficients in \( N \)- and \( W \)-terms are the result of using the \( F \)-field equation (6.2.9), while the \( \beta \) and \( \delta \)- dependent coefficients in the \( S \)- and \( P \)-terms are from the dreizehnbein field equation (6.2.8). The vanishing of these \( N \) and \( W \)-terms fixes the coefficient

\[
\alpha = \frac{1}{2304} ,
\]

(6.2.18) after the use of (6.2.7). The vanishing of the \( S \) and \( P \)-terms fix the coefficients

\[
\beta = \frac{1}{36} \quad \delta = \frac{1}{3}
\]

(6.2.19) after the use of (6.2.7). Finally in (6.2.16), the \( Q, T \) and \( U \)-terms vanish under (6.2.7). At this stage, all the coefficients \( \alpha, \beta \) and \( \delta \) in the field equations are fixed, consistently with the values for \( a_1, a_2 \) and \( a_3 \) in (6.2.7).

### 6.3. Superspace for \( N = 1 \) Supergravity in \( D = 11 + 2 \)

In this subsection, we briefly provide helpful information for the superspace formulation of our \( D = 13, N = 1 \) supergravity. The structures for constraints in superspace are easily read from the component transformation rules (6.2.1) [27]. However, there is one caveat about the extra transformation terms which are implicit in component, but manifest themselves in superspace. There is also some caution needed for \( d = 1 \) Bianchi identities to be mentioned shortly.

To clarify these points, we give first our results for superspace constraints at \( 0 \leq d \leq 1/2 \), which will be of relevance also for supermembrane formulation in the next section:

\[
T_{\alpha\beta}^{\;c} = (\gamma^{cd})_{\alpha\beta} \nabla_d \phi + (P_{\perp})_{\alpha\beta} \nabla^c \phi ,
\]

(6.3.1a)

\[
F_{\alpha\beta\gamma} = -\frac{1}{2}(\gamma^{cd})_{\alpha\beta} \nabla^c \phi - \frac{1}{2}(P_{\perp})_{\alpha\beta} \nabla^d \phi ,
\]

(6.3.1b)

\[
(\nabla_a \phi)(\nabla^a \phi) = 0 \quad (\nabla_a \phi)(\nabla^a \tilde{\phi}) = 0 \quad (\nabla_a \phi)(\nabla^a \tilde{\phi}) = 1 \quad \nabla_a \phi = \nabla_a \tilde{\phi} = 0 .
\]

(6.3.1c)

As usual all other supertorsion components at \( d \leq 1/2 \), such as \( T_{\alpha\beta}^\gamma \) are all zero. The range of spinorial indices in 13D is \( \alpha, \beta \ldots = 1, 2, \ldots, 64 \). The first terms in (6.3.1a) and (6.3.1b) are obtained from the component transformation rules (6.2.1a) and (6.2.1c), while the second terms can be understood as maintaining the conditions \( T_{\alpha\beta} \nabla_c \tilde{\phi} \equiv 0 \) and \( F_{\alpha\beta\gamma} \nabla^d \phi \equiv 0 \), in addition to the trivial ones: \( T_{\alpha\beta} \nabla_c \phi \equiv 0 \), \( F_{\alpha\beta\gamma} \nabla^d \phi \equiv 0 \). This can be understood as in general terms: Let any vector \( V_a \) be modified to be \( \tilde{V}_a \), that satisfies the condition \( \tilde{V}_a \nabla^a \phi \equiv 0 \) by the simple shift:

\[
\tilde{V}_a \equiv V_a - (\nabla_a \phi)(\nabla^b \tilde{\phi})V_b .
\]

(6.3.2)

For example, we easily see that the second term in (6.3.1b) is nothing else than this modification of the first term for the two indices \( c, d \) directly obtained from the component...
transformation rule (6.2.1c). The condition \( F_{\alpha\beta cd} \nabla^d \phi \equiv 0 \) is also required by superspace Bianchi identities at \( d = 1/2 \). Technically, it is sometimes useful to use the purely 11D indices \( i, j, \ldots = (0), (1), \ldots, (9), (10) \) and the extra ones \( \pm \) for the bosonic indices in \( T_{\alpha\beta c}, F_{\alpha\beta cd} \), like \( T_{\alpha\beta i}, F_{\alpha\beta ij} \) in order to make superspace computation easier.

For the rest of this subsection, instead of going through all the details of Bianchi identities of \( 0 \leq d \leq 1/2 \), we give important Fierz identities needed for these Bianchi identities. The first important Fierz identity is associated with the \( d = 0 \) Bianchi identity of \( (\alpha\beta\gamma de) \)-type:

\[
(\gamma^{ab})_{(\alpha\beta|}(\gamma_{ad}\mid)(\nabla_b \phi)(\nabla_c \phi) - 2(\gamma^{ab})_{(\alpha\beta|}(P_{\gamma\delta}(\nabla_b \phi)(\nabla_d \phi) = 0 . \tag{6.3.3}
\]

There are two principal methods to confirm this identity: The first method is the direct one, namely using more basic Fierz identities in 13D in Appendix B. The second method uses the dimensional reduction from 13D into 11D. For example, if (6.3.3) is easily seen to hold, if we assign explicit index ranges for all the bosonic indices in (6.3.3). To be more specific, we see first that the \( a \)-index can take only purely 11D values due to the factor of \( \nabla \phi \), while there are three options for the index \( d = i, +, - \). It can be easily seen that for each of these cases, (6.3.3) holds, in particular, the case of \( d = i \) corresponds to the familiar 11D Fierz identity [34]:

\[
(\gamma^{i})_{(\alpha\beta|}(\gamma_{ij}\mid)(\nabla_j \phi) \equiv 0 , \tag{6.3.4}
\]

where all the spinorial indices are 11D ones. Note that even though we used dimensional reduction, there is no extra component overlooked in this method, as long as we scan all the possibilities of the range of indices.

As careful readers may have already noticed, we have mentioned neither the \( d = 1 \) Bianchi identities, nor the constraints at \( d = 1 \). This is due to a problem with a Bianchi identity of \( (\alpha\beta cde) \)-type at \( d = 1 \) yet to be satisfied, associated with the extra symmetry (6.2.4). This is caused by the term \( (\gamma^{ef})_{\alpha\beta} F_{cde} + \nabla_f \phi \) which does not have any counter-terms in this Bianchi identity. To put it differently, due to the extra symmetry (6.2.4), there can be additional term proportional to the null-vector \( \nabla_a \phi \) in the anticommutator \( \{\nabla_\alpha, \nabla_\beta\} A_{abc} \), which has no direct geometrical interpretation in superspace. This problem seems peculiar to this 13D system, with no corresponding one in 12D supergravity. However, we also emphasize that this sort of problems for superspace formulation for supergravity with extra symmetries is not a new phenomenon at all, because we know similar supergravity/supersymmetry theories, such as the \( N = 4 \) Chern-Simons theory in 3D [35], which is possible only in component formulation, with some obstructions for superspace formulations caused by extra symmetries. As for the present 13D supergravity, notwithstanding this problem to be solved at \( d = 1 \) level by some possible modifications of

\[10^\text{This extra symmetry arose as the result of duality transformation performed to get this multiplet [35], and there seems to be a general close relationship between extra symmetries and duality transformations.} \]
Bianchi identities, e.g., by Chern-Simons modifications, we believe that our superspace constraints at $0 \leq d \leq 1/2$ are valid for supermembrane couplings with fermionic invariances based only on these lower-dimensional superspace constraints, that we will deal with next.

7. Supermembrane on Background of $D = 13$, $N = 1$ Supergravity

Our next step is to put some extended objects as a probe for the consistency of our 13D supergravity. The most natural extended object is the supermembrane [6] coupled to 11D supergravity [1], because our 13D supergravity is a higher-dimensional generalization of the former. The existence of the third-rank tensor $A_{[3]}$, also suggests the natural super p-brane [32] to be supermembrane [6]. As usual, our next task is to confirm the fermionic symmetries in the supermembrane [6] in our 13D supergravity background.

Our total action for the supermembrane is similar to that for the original supermembrane [6]:

$$S = S_\sigma + S_A \quad , \quad (7.1a)$$

$$S_\sigma \equiv \int d^3 \sigma \left( -\frac{1}{2} \sqrt{-g} g^{ij} \eta_{ab} \Pi_i^a \Pi_j^b + \frac{1}{2} \sqrt{-g} \right) \quad , \quad (7.1b)$$

$$S_A \equiv \int d^3 \sigma \left( +\frac{1}{3} \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C A_{CBA} \right) \quad . \quad (7.1c)$$

As in other sections of p-branes in this paper, we have switched to superspace notation. The indices $i, j, \ldots = 0, 1, 2$ are for curved coordinates in $D = 2 + 1$ world-volume, and $\Pi_i^A \equiv (\partial_i Z^M) E_{M}^A$, etc., as in the usual supermembrane formulation [6].

For the fermionic invariance of the action, we need superspace constraints (6.3.1) at $0 \leq d \leq 1/2$. We found the second terms in (6.3.1a) and (6.3.1b) have no contributions in our action invariance under the constraint (7.5) below.

As in the cases in 12D [14][15], we have two sorts of fermionic symmetries

$$\delta E^a = (I + \Gamma)^a \beta \kappa^\beta + (P_t)^{a\beta} \eta^\beta \quad , \quad (7.2a)$$

$$\delta E^a = 0 \quad , \quad (7.2b)$$

with the 64 component fermionic parameters $\kappa$ and $\eta$. The $\Gamma$ is defined similarly to [6] by

$$\Gamma \equiv \frac{1}{6\sqrt{-g}} \epsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^c (\gamma_{abc}) \quad , \quad (7.3)$$

satisfying the relations similar to (5.5):

$$\Gamma^2 = I \quad , \quad (7.4a)$$

$$\epsilon^{ijk} \Pi_j^a \Pi_k^b \gamma_{ab} \Gamma = +2\sqrt{-g} \Pi_i^a \gamma_i \quad , \quad (7.4b)$$
under the algebraic $g_{ij}$-field equation $g_{ij} = \eta_{ab}\Pi_i^a\Pi_j^b$.

The fermionic invariance of our action goes in the same way as in the $D = 12$, $N = 2$ supergravity. In particular, the important ingredient is the constraint condition on the pull-back, as in $D = 12$, $N = 2$ supergravity [15]:

$$\Pi_i^a\nabla_a \varphi = 0 ~. \tag{7.5}$$

In fact, the variation of our action under the fermionic $\eta$-symmetry yields the terms

$$\delta_{\eta}S = \sqrt{-g}\Pi_{ia}(\overline{\eta}P_{\gamma}^{\gamma ba})_{\beta}^i\Pi^{i\beta}\nabla_b \varphi + \frac{1}{4}e^{ijk}(\overline{\eta}P_{\gamma}^{\gamma dcb})_{\beta}^i\Pi_i^{\beta}\Pi_j^b\Pi_k^c\nabla_d \varphi ~. \tag{7.6}$$

Interestingly, the second terms both in (6.3.1a) and (6.3.1b) do not contribute under the constraint (7.5). Each of the terms in (7.6) further vanishes under (7.5), due to the identity: $P_{\gamma}^{\gamma} \nabla \varphi \equiv 0$. Similarly to the original supermembrane action [6], the $\kappa$-symmetry is also easily confirmed by the use of (7.3) as well as our constraint (7.5). This concludes our confirmation of fermionic invariance of our total action.

Similarly to the Green-Schwarz superstring [14] on $D = 12$, $N = 1$ supergravity or super $(2 + 2)$-brane [15] on $D = 12$, $N = 2$ backgrounds, the $\eta$-invariance and $\kappa$-symmetry reduce the total degrees of fermionic freedom into 16 which is a quarter of the original value of 64, agreeing with the conventional supermembrane formulation [6]: $64 \rightarrow 32 \rightarrow 16$.

### 8. Concluding Remarks

In this paper we have given rather detailed constructions of $N = 1$ and $N = 2$ supergravity theories in 12D and $N = 1$ supergravity in 13D, based on our recent technique of using scalar (super)fields that make the system manifestly $SO(10, 2)$ or $SO(11, 2)$ Lorentz covariant, up to modified Lorentz generators. We have also established the fermionic invariances of superstring on $D = 12$, $N = 1$, or super $(2 + 2)$-brane on $D = 12$, $N = 2$, and supermembrane on $D = 13$, $N = 1$ supergravity backgrounds, as confirmation of the consistency of our theories. We have seen new features of these supergravity theories in 12D as well as parallel structure to those in 10D or 11D. For example, we have found how the self-duality condition (4.2.27) in 10D is promoted to the anti-self-duality condition (4.2.22) in 12D in an elaborate fashion. We have noticed how the cancellation structures among terms for supersymmetry in 12D are parallel to the 10D case, based on similarities in gamma-matrix identities in these dimensions, as in Appendix A.

The important ingredient of our formulations is the role of the scalar (super)fields making our systems $SO(10, 2)$ or $SO(11, 2)$ Lorentz covariant, respectively for our 12D or 13D

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\footnote{We have shown even our modified Lorentz generators themselves can be also made ‘more’ covariant, in terms of the gradients of superinvariant scalar fields, using the definition (2.2.5).}
supergravities, up to modified Lorentz generators. This technique was first introduced in [16], where the scalar (super)fields are intact under supersymmetry. This feature is also consistent with the closure of local supersymmetry. All the (super)field equations are now made formally Lorentz covariant, but the systems realize the spontaneous $SO(10,2)$ or $SO(11,2)$ Lorentz symmetry breakings down to $SO(9,1)$ or $SO(10,1)$, when particular solutions (2.2.2) are used for these scalar (super)fields with null-vectors. To our knowledge, there has been no other example of this kind in other supersymmetric theories in lower-dimensions. The advantage of these superinvariant scalar (super)fields is more elucidated, when dealing with globally supersymmetric Yang-Mills theories in $D \geq 12$ [17][16], because we need no modified Lorentz generators, and these theories are entirely Lorentz covariant.

In addition to the gradients of scalar fields replacing the null-vectors, we have developed other practical techniques of formulating higher-dimensional supergravities. Using dimensional reduction for the confirmation of Fierz identities for 13D supergravity is one of them. This method utilizes the parallel structure between 13D and 11D supergravities, saving a lot of time for confirming complicated Fierz identities needed in practical computations in 13D. To our knowledge, this method is introduced into supergravity for the first time in this paper, and as such, this technique has potential applications to more higher-dimensional supergravities in $D \geq 14$. Armed with the lists of $\gamma$-matrix relations also in Appendices A and B, we have now great capability of controlling Bianchi identities in all of these higher-dimensions.

In our supergravity theories in 12D, since the Lorentz symmetry $SO(10,2)$ or $SO(11,2)$ is only formally recovered due to the modified Lorentz generators, some readers may wonder if this is just a reformulation of null-vectors violating the manifest Lorentz covariance. Even if we admit that the usage of scalar fields may be just a rewriting of null-vectors in supergravity theories, in the sense that modified Lorentz generators not totally Lorentz covariant, we stress important features of our supergravity formulations in these dimensions. There has been also some skepticism about the ‘uniqueness’ of these supersymmetry/supergravity theories, ever since the first construction of supersymmetric theory in 12D in [12]. According to such a claim, the lack of Lorentz invariance makes these supergravity theories arbitrary but not unique, unlike other conventional supergravity theories in $D \leq 11$. Even though this argument sounds convincing, it overlooks important features in supergravity theories. Such a claim is valid, only when we are dealing with non-supersymmetric theories without Lorentz invariance, because we can always put any null-vector to get rid of extra component in any term in a field equation at our will. Therefore the construction of non-supersymmetric theories is always ambiguous, when Lorentz invariance is not manifest. In supersymmetric theories, however, this is no longer the case due to the restriction by supersymmetry. As a matter of fact, the construction of a supergravity/supersymmetry theory satisfying all of the following conditions is extremely difficult:
(i) There exist extra non-vanishing components for physical fields.

(ii) These extra components have non-trivial dependence on coordinates.

(iii) Supersymmetry closes at least on-shell.

(iv) Supersymmetric p-Brane Formulations exist on such Supergravity Backgrounds.

For example, if we build a supergravity theory in which all of the extra components are deleted by constraints using null-vectors, then the system collapses into an ordinary supergravity theory in $D \leq 11$, leaving nothing new. The non-triviality of our formulation can be also seen from the fact that the coefficients in our supersymmetry transformation rules and field equations are so tightly fixed that we can not shift them even by small amounts, maintaining supersymmetry. One can try to build an arbitrarily new supersymmetric theories in these higher dimensions, and realize how our systems are selected, when supersymmetry is in the game. We emphasize that these supersymmetric theories are strictly fixed by their proper uniqueness, despite of the lack of total Lorentz invariance. We also mention that the loss of Lorentz invariance is not limited to our peculiar system of higher-dimensional supersymmetries. For example, the loss of Lorentz invariance seems also inevitable in the $SL(2, \mathbb{Z})$ duality symmetric formulation in ref. [36].

Another important point not to be overlooked is the existence of super p-brane [32] actions consistently coupled to our $D = 12, N = 1$ or $D = 12, N = 2$ and $D = 13, N = 1$ supergravity backgrounds. If we did not have such ‘probes’ on our backgrounds, then one could still say that these higher-dimensional supergravity theories were just ordinary supergravity in $D \leq 11$ ‘in disguise’. However, due to the non-vanishing components among string or membrane variables carrying the ‘extra’ components, we can see much more non-triviality in our total system formulated with these extended objects. For example, we saw that the constraints (3.12) for the extra string variables are required only for the components $\Pi_-^a \nabla^a \varphi$ and $\Pi_-^a \nabla^a \bar{\varphi}$, but not for $\Pi_+^a \nabla^a \varphi$ and $\Pi_+^a \nabla^a \bar{\varphi}$. In other words, the string variables $X^a(\sigma)$ in the extra dimensions can still have non-trivial dependence on the world-sheet coordinates $\sigma^+$. In this sense, the Green-Schwarz superstring [5] coupled to our $N = 1$ supergravity in 12D [14] has much more content than just a rewriting of 10D Green-Schwarz superstring. As a matter of fact, we have also found an important fact that the requirement of fermionic symmetries on the Green-Schwarz superstring world-sheet for $D = 12, N = 1$ supergravity, or on the super $(2 + 2)$-brane world-supervolume for $D = 12, N = 2$ supergravity leads to the null-vector conditions such as (2.3.4f) or (4.2.4e), respectively. In other words, our null-vector conditions are by no means artificially put by hand, but required by the fermionic symmetries of these extended objects as probes put in these backgrounds in 12D. The necessity of null-vectors is also understood as solutions for BPS conditions for supersymmetry algebra in higher-dimensions [3][11]. We have also seen how these fermionic symmetries reduce the degrees of freedom of these extended objects in

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12: ‘Physical’ field here means a field that has physical components in 10D.
12D down to the conventional 8+8 or 16+16 physical degrees of freedom in the light-cone coordinates for superstrings in 10D [5].

Our result for 13D supergravity will motivate also other interesting applications and directions to be explored, such as generalizing this result to more duality-symmetric way, like those in [37][38], looking for lagrangian formulation like that in ref. [16], or considering topological significance of the $F$-field equation, or possible duality connections with other higher-dimensional theories, etc.

As we have seen, it is just the beginning that higher-dimensional supergravity revealed various unexpected features, such as the modified Lorentz generators that had never been presented before in supersymmetric theories, or the power of superinvariant scalars making the system more covariant. There has been some indication that the success of our supergravity theories based on null-vectors [14][15] signals nothing but more fundamental theories where the null-vectors are replaced by more generalized momenta [3][4] in multi-local field theories. We are sure that the details given in this paper will provide us with the first step toward such directions, leading to more fundamental theories, with bi-local or multi-local fields. From this viewpoint, even though we performed rather detailed computation in this paper, we believe that such technicalities will be of practical importance, when we generalize our theories to more fundamental bi-local or multi-local theories, and we will realize that the structure of higher-dimensional supergravity is much deeper than we had initially expected.

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Appendix A: Useful Relationships in $D = 10 + 2$

In this Appendix, we list up important relationships, which are useful in practical manipulations. We start with the Fierz identities in our $D = 10 + 2$ for Weyl indices $\alpha, \beta, \gamma, \delta$:

\[
\delta_{(\alpha \beta \delta \gamma)} = \frac{1}{32} (\gamma^{ab})_{\alpha \gamma} (\gamma_{ab}^{\beta \delta}) + \frac{1}{32 (6!)} (\gamma^{[6]}_{\alpha \gamma}) (\gamma_{[6]}^{\beta \delta}), \quad (A.1)
\]
\[
\delta_{[\alpha \beta \delta \gamma]} = \frac{1}{16} C_{\alpha \gamma} C^{\beta \delta} + \frac{1}{16 (4!)} (\gamma^{[4]}_{\alpha \gamma}) (\gamma_{[4]}^{\beta \delta}). \quad (A.2)
\]

Based on these, we can further derive the following useful relations:

\[
(\gamma^{\alpha})_{(\alpha_1 \beta_1)} (\gamma^\alpha_{(\gamma)}) = \frac{1}{4} (\gamma^{bc})_{\alpha \gamma} (\gamma^{bc})_{\beta \gamma} \delta^{\beta \delta}, \quad (A.3)
\]
\[
(\gamma^{ab})_{(\alpha \beta)} (\gamma^\alpha_{(\gamma)}) = \frac{1}{12} (\gamma^{cd})_{\alpha \beta} (\gamma^{cd})_{\gamma \gamma} = \frac{1}{8} (\gamma^{cd})_{\alpha \beta} (\gamma^{ab})_{\gamma \gamma} \delta^{\beta \delta}, \quad (A.4)
\]
\[
(\gamma^{ab})_{(\alpha \beta)} (\gamma^{ac})_{\gamma \gamma} = \frac{1}{10} (\gamma^{de})_{\alpha \beta} (\gamma^{de})_{\gamma \gamma} + (\gamma^{bc})_{\alpha \beta} \delta_{\gamma \gamma}^{\beta \delta}
\]
\[
+ \frac{1}{10} \delta_{\gamma \gamma}^{\beta \delta} (\gamma^{de})_{\alpha \beta} (\gamma^{de})_{\gamma \gamma} - \frac{1}{5} (\gamma^{de})_{\alpha \beta} (\gamma^{de})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.5)
\]
\[
(\gamma^{ab})_{\alpha \beta} (\gamma^\alpha_{(\gamma)}) = - (\gamma^{ab})_{\alpha \beta} (\gamma^{bc})_{\gamma \gamma} - \frac{1}{20} (\gamma^{cd})_{\gamma \gamma} (\gamma^{bc})_{(\gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma}) \delta^{\beta \delta}, \quad (A.6)
\]
\[
(\gamma^{ab})_{(\alpha \beta)} (\gamma^{abde})_{\gamma \gamma} = - (\gamma^{abde})_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} - 2 (\gamma^{cd})_{(\alpha \beta)} (\gamma^{ef})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.7)
\]
\[
(\gamma^{ab})_{(\alpha \beta)} (\gamma^{abdef})_{\gamma \gamma} = - (\gamma^{abdef})_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.8)
\]
\[
(\gamma^{[5b]}_{(\alpha \beta)} (\gamma^{[5]})_{\gamma \gamma} = -720 (\gamma^{ab})_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.9)
\]
\[
(\gamma^{[5]}_{(\alpha \beta)} (\gamma^{[5]}_{\gamma \gamma} = -180 (\gamma^{ab})_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.10)
\]
\[
(\gamma^{[5]}_{(\alpha \beta)} (\gamma^{[5]}_{\gamma \gamma} = -120 \eta^{ab} (\gamma^{cd})_{(\alpha \beta)} (\gamma^{cd})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.11)
\]
\[
(\gamma^{[4a]}_{(\alpha \beta)} (\gamma^{[4]}_{\gamma \gamma} = +12 (\gamma^{ab})_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta} + 360 (\gamma^{ab})_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.12)
\]
\[
(\gamma^{[4]}_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} = +4 (\gamma^{[4]}_{(\alpha \beta)} (\gamma^{[6]}_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta} - 10 (\gamma^{ab})_{(\alpha \beta)} (\gamma^{ab})_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.13)
\]
\[
(\gamma^{a}_{(\alpha \beta)} (\gamma^{a})_{\gamma \gamma} = +\frac{2}{16} C_{\alpha \gamma} C^{\beta \delta} \delta_{\gamma \gamma}^{\beta \delta} + \frac{1}{10} (\gamma^{[2]}_{(\alpha \beta)} (\gamma^{[2]}_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta} + \frac{1}{92} (\gamma^{[4]}_{(\alpha \beta)} (\gamma^{[4]}_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.14)
\]
\[
(\gamma^{[3]}_{(\alpha \beta)} (\gamma^{[3]}_{\gamma \gamma} = +\frac{165}{4} C_{\alpha \gamma} C^{\beta \delta} \delta_{\gamma \gamma}^{\beta \delta} + \frac{15}{4} (\gamma^{[2]}_{(\alpha \beta)} (\gamma^{[2]}_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta} - \frac{3}{32} (\gamma^{[4]}_{(\alpha \beta)} (\gamma^{[4]}_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}, \quad (A.15)
\]
\[
(\gamma^{[5]}_{(\alpha \beta)} (\gamma^{[5]}_{\gamma \gamma} = +2970 C_{\alpha \gamma} C^{\beta \delta} \delta_{\gamma \gamma}^{\beta \delta} - 90 (\gamma^{[2]}_{(\alpha \beta)} (\gamma^{[2]}_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta} + \frac{5}{4} (\gamma^{[4]}_{(\alpha \beta)} (\gamma^{[4]}_{\gamma \gamma} \delta_{\gamma \gamma}^{\beta \delta}. \quad (A.16)
\]

Here since these relations are for superspace, all the (anti)symmetrizations are not normalized, e.g., $A_{(ab)} \equiv A_{ab} - A_{ba}$.\footnote{Notwithstanding this rule in superspace, the symbol $\{n\}$ is for normalized antisymmetrization, e.g., $A_{[3]} B^{[3]} \equiv (1/6) A_{[abc]} B^{[abc]}$ as in component notation, as has been already mentioned.} Needless to say, alternative identities obtained by exchanging all the dotted indices and undotted ones also hold. Even though this is not the exhaustive list of relations used in our calculation, one will still find it useful for practical computations.
There are also other useful relations with null-vectors. Some typical examples are

\[
(\gamma^\hat{c})_{\alpha\beta}(\gamma^\hat{d})_{\gamma} = -\frac{1}{2}(\gamma^\hat{c})_{\alpha\gamma}(\gamma^\hat{d})_{\beta} n_a n_b + \frac{1}{24}(\gamma^{[3]a})_{\alpha\gamma}(\gamma^{[3]}_{\beta} b) \delta n_a n_b ,
\]

(A.17)

\[
(\gamma^{[3]})_{\alpha\beta}(\gamma^{[3]})_{\gamma} = -15(\gamma^{ca})_{\alpha\gamma}(\gamma^{cb})_{\beta} n_a n_b + \frac{3}{4}(\gamma^{[3]a})_{\alpha\gamma}(\gamma^{[3]}_{\beta} b) \delta n_a n_b ,
\]

(A.18)

\[
(\gamma^{[5]})_{\alpha\beta}(\gamma^{[5]})_{\gamma} = +360(\gamma^{ca})_{\alpha\gamma}(\gamma^{cb})_{\beta} n_a n_b - 10(\gamma^{[3]a})_{\alpha\gamma}(\gamma^{[3]}_{\beta} b) \delta n_a n_b .
\]

(A.19)

Notice that these relations hold only under the null-vector condition \( n_a n^a = 0 \).

Another crucial relation is between the \( \epsilon \)-tensor and the \( \gamma \)-matrices:

\[
\gamma^{[n]} = \frac{(-1)^{n(n-1)/2}}{(12-n)!} \varepsilon^{[n]} \gamma^{[12-n]} \gamma_{13}^{[12-n]} (0 \leq n \leq 12) .
\]

(A.20)

In 12D we have convenient relations associated with duality of 6-th rank antisymmetric tensors. For example, we can easily show that the following combination is identically zero:

\[
S^{[6]}A^{[6]} \equiv 0 ,
\]

(A.21)

where \( S^{[6]} \equiv + (1/6!)\varepsilon^{[6]} S^{[6]} \) and \( A^{[6]} \equiv -(1/6!)(A^{[6]}) \) are respectively self-dual and anti-self-dual tensors. This is in a good contrast with the 6D case e.g., in [39], where we had \( S^{[3]}S^{[3]} \equiv A^{[3]}A^{[3]} \equiv 0 \). Even though there is prevailing tendency nowadays regarding these \( \gamma \)-matrix manipulations in supergravity as ‘out of date’ or ‘old-fashioned’ methods that we should not be bothered, we re-emphasize here their crucial importance for building supergravity theories not to be bypassed, with no other alternative ‘quick’ ways, even after 20 years since the first discovery of supergravity [30]. For example, the construction of 12D supersymmetric Yang-Mills theory [12], or our modified Lorentz generators [14] would have never been achieved without the crucial identities in this Appendix.

We give also basic relationships related to \( \gamma \)-matrices with their complex or hermitian conjugates. Using the same notation in [19], we list them up as

\[
\overline{\psi} = \psi^\dagger A \quad \text{(for Dirac conjugate)} ,
\]

\[
\psi = C\overline{\psi}^T \quad \text{(for Majorana-Weyl condition)} ,
\]

\[
\gamma^\mu = -A\gamma^\mu A^{-1} , \quad A \equiv \gamma^{(1)}_{(12)} , \quad A^\dagger = -A ,
\]

\[
A^T = -CA^{-1} , \quad \gamma^\mu = -B^{-1}\gamma^\mu B \quad (\eta = -1 \quad \text{for Majorana spinors}) ,
\]

\[
B \equiv (A^T)^{-1}C^T = -CA , \quad C^T = -C \quad (\epsilon = +1 , \quad \eta = -1) ,
\]

\[
\gamma^\mu = +C\gamma^\mu C^{-1} , \quad C^\dagger C = +I , \quad \psi^* = B\psi .
\]

(A.22)

By the aid of these expressions, we can easily confirm equations in (4.1.8).

Before concluding this Appendix, we mention other useful \( \gamma \)-matrix relations which are frequently used both in superspace and component computations in 12D. We give below a
list of such relations for readers’ convenience, even though this list is by no means exhaustive, because other relations may be also easily obtained by using these identities:

\[ \gamma_\mu \gamma_\nu = (1)^n (12 - 2n) \gamma_\nu \], \quad \gamma_\mu \gamma_\nu = 0 \]  \hfill (A.23)

\[ \gamma^\mu \gamma_\rho \gamma^\nu \gamma_\sigma = -52 \gamma^\rho \gamma_\sigma, \quad \gamma^\mu \gamma_\rho \gamma_\nu \gamma_\sigma = -88 \gamma_\rho \]  \hfill (A.24)

\[ \gamma^\nu \gamma_\mu \gamma_\nu = -9 \gamma_\mu - 11 g_\mu, \quad \gamma_\nu \gamma_\mu \gamma^\nu = -9 \gamma_\mu + 11 g_\mu \]  \hfill (A.25)

\[ \gamma_\mu \gamma_\sigma \gamma^{\mu \nu \rho} = -38 \gamma_\sigma \rho + 140 \delta^{[\rho \gamma_\tau]}, \]  \hfill (A.26)

\[ \gamma_\mu \gamma_\sigma \gamma^\nu \gamma_\nu = 16 \gamma_\mu \gamma^\sigma + 32 \gamma_\mu \gamma^\rho + 7 g_\mu \gamma^\sigma + 20 \delta_\mu \gamma_\sigma \]  \hfill (A.27)

\[ \gamma_\rho \gamma_\mu \gamma_\sigma \gamma^\nu \gamma_\rho = -7 \gamma_\mu \gamma^\sigma - 18 \delta_\mu \gamma_\nu \]  \hfill (A.28)

\[ \gamma_\mu \gamma^\sigma \gamma_\rho = +6 \gamma_\mu \gamma^\sigma + 32 \delta_\mu \gamma_\nu = 20 \delta_\mu \delta_\nu \]  \hfill (A.29)

\[ \gamma_\mu \gamma_\nu \gamma_\rho = (40 \gamma_\mu \gamma_\sigma - 216 \delta_\mu \gamma_\nu \]  \hfill (A.30)

\[ \gamma_\mu \gamma_\nu \gamma_\rho = -144 \delta_\mu \gamma_\nu + 720 \delta_\mu \gamma_\rho \]  \hfill (A.31)

\[ \gamma_\mu \gamma_\nu \gamma_\rho = -16 \gamma_\mu \gamma_\nu + 240 \delta_\mu \gamma_\nu \]  \hfill (A.32)

\[ \gamma_\mu \gamma_\nu \gamma_\rho = +4 \gamma_\mu \gamma_\rho + 48 \delta_\lambda \gamma_\nu \gamma_\rho \]  \hfill (A.33)

\[ \gamma_\mu \gamma_\nu \gamma_\rho = +5 \gamma_\mu \gamma_\nu + 12 \gamma_\nu \gamma_\rho - 140 \delta_\lambda \gamma_\nu \gamma_\rho \]  \hfill (A.34)

\[ \gamma_\mu \gamma_\nu \gamma_\rho = -26 \gamma_\mu \gamma_\nu - 216 \delta_\lambda \gamma_\nu \]  \hfill (A.35)

\[ \gamma_\nu \gamma_\sigma \gamma_\rho = +5 \gamma_\nu \gamma_\sigma - 42 \delta_\nu \gamma_\nu \]  \hfill (A.36)

\[ \gamma_\nu \gamma_\sigma \gamma_\rho \gamma_\sigma = \gamma_\lambda \gamma_\nu \gamma_\rho \]  \hfill (A.37)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma = +\gamma_\nu \gamma_\rho + 2 \delta_\nu \delta_\rho \]  \hfill (A.38)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.39)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.40)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.41)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.42)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.43)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.44)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.45)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.46)

\[ \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\nu = +\gamma_\nu \rho \gamma_\sigma \]  \hfill (A.47)
\[ \gamma_5 \gamma_4 \gamma_5^2 = +960 \gamma_4 \quad , \quad \gamma_6 \gamma_4 \gamma_5^2 = -20160 \gamma_4 \quad , \]

\[ \gamma_2 \gamma_5 \gamma_5^2 = +8 \gamma_5 \quad , \quad \gamma_3 \gamma_5 \gamma_5^3 = -60 \gamma_5 \quad , \quad \gamma_4 \gamma_5 \gamma_4^2 = +120 \gamma_5 \quad , \]

\[ \gamma_5 \gamma_5^2 \gamma_5^2 = -2400 \gamma_5 \quad , \quad \gamma_6 \gamma_5 \gamma_5^6 = 0 \quad , \]

\[ \gamma_2 \gamma_6 \gamma_6^2 = +12 \gamma_6 \quad , \quad \gamma_3 \gamma_6 \gamma_6^3 = 0 \quad , \quad \gamma_4 \gamma_6 \gamma_4^3 = +360 \gamma_6 \quad , \]

\[ \gamma_5 \gamma_6 \gamma_5^5 = 0 \quad , \quad \gamma_6 \gamma_6 \gamma_6^6 = +14400 \gamma_6 \quad . \]

Here our (anti)symmetrizations are normalized, prepared for component computations, as their indices show. We also mention a small point that these \( \gamma \)-matrix algebra depends only on the total space-time dimensions, but \emph{not} on the signature. Therefore, \( (A.23) - (A.49) \) are useful also for theories in \( D = 12 \) with other signatures, such as \( (+,+,\cdots,+,+) \) or the Euclidian one \( (+,+,\cdots,+) \).

**Appendix B: Useful Relationships in \( D = 11 + 2 \)**

Similarly to the previous Appendix A, we list up here some useful relationships, in \( D = 11 + 2 \), which are by no means exhaustive collection, but will be of great help for superspace manipulations. Note also that these relations for 13D should not be confused with those in Appendix A for 12D.

\[ \gamma_{[n]} = \frac{(-1)^{(n+1)(n+2)/2}}{(13-n)!} \epsilon_{[n]}^{[13-n]} \gamma_{[13-n]} \quad (0 \leq n \leq 13) \quad , \]

\[ \delta_{[\alpha \beta \gamma\delta]} = + \frac{1}{64} (\gamma_{[2]} \alpha \gamma)(\gamma_{[2]} \beta \gamma) - \frac{7}{192} (\gamma_{[3]} \alpha \gamma)(\gamma_{[3]} \beta \gamma) + \frac{1}{32(64)} (\gamma_{[6]} \alpha \gamma)(\gamma_{[6]} \beta \gamma) \quad , \]

\[ (\gamma^a)_{[\alpha \beta] \gamma \delta} = \frac{9}{64} (\gamma_{[2]} \alpha \gamma)(\gamma_{[2]} \beta \gamma) - \frac{7}{192} (\gamma_{[3]} \alpha \gamma)(\gamma_{[3]} \beta \gamma) + \frac{1}{32(64)} (\gamma_{[6]} \alpha \gamma)(\gamma_{[6]} \beta \gamma) \quad , \]

\[ (\gamma_{[2]} \gamma_{[2]} \gamma_{[2]})_{\gamma \delta} = + \frac{1}{3} (\gamma_{[3]} \gamma_{[3]})_{\gamma \delta} = - \frac{1}{6} (\gamma_{[6]} \gamma_{[6]})_{\gamma \delta} \quad , \]

\[ \gamma_{a \gamma [n]} \gamma_{a} = (-1)^n (13 - 2n) \gamma_{[n]} \quad , \]

\[ \gamma_{ab \gamma [cd]} \gamma_{ab} = -68 \gamma_{cd} \quad , \quad \gamma_{abc \gamma [de]} \gamma_{abc} = -396 \gamma_{de} \quad , \]

\[ \gamma_{ad \gamma [bc]} \gamma_{ad} = +8 \gamma_{a \gamma [bc]} + 10 \delta_{a \beta \gamma [c]} \gamma_{ad} = -8 \gamma_{a \gamma [bc]} + 10 \delta_{a \beta \gamma [c]} \gamma_{ad} \quad , \]

\[ \gamma_{ab \gamma [cd]} \gamma_{ab} = +10 \delta_{a \beta \gamma [c]} \gamma_{ab} + 11 \delta_{a \beta \gamma [c]} \gamma_{ab} \]

\[ \gamma_{ab \gamma [cd]} \gamma_{ab} = -7 \gamma_{a \gamma [cd]} - 8 \delta_{a \beta \gamma [c]} \gamma_{ab} + 9 \delta_{a \beta \gamma [c]} \gamma_{ab} - 11 \delta_{a \beta \gamma [c]} \gamma_{ab} \quad , \]

\[ \gamma_{ab \gamma [de]} \gamma_{abc} = +7 \gamma_{a \gamma [de]} - 9 \delta_{a \beta \gamma [c]} \gamma_{ab} - 11 \delta_{a \beta \gamma [c]} \gamma_{ab} \quad , \]

\[ \gamma_{a \gamma [de]} \gamma_{abc} = +7 \gamma_{a \gamma [de]} - 9 \delta_{a \beta \gamma [c]} \gamma_{ab} + 11 \delta_{a \beta \gamma [c]} \gamma_{ab} \quad , \]

\[ \gamma_{a \gamma [de]} \gamma_{abc} = +6 \gamma_{a \gamma [de]} + 4 \delta_{e \gamma [a \beta \gamma [c]} \gamma_{abc} \quad , \]

\[ \gamma_{a \gamma [de]} \gamma_{abc} = +6 \gamma_{a \gamma [de]} - 4 \delta_{e \gamma [a \beta \gamma [c]} \gamma_{abc} \quad , \]

\[ \gamma_{a \gamma [de]} \gamma_{abc} = +6 \gamma_{a \gamma [de]} - 4 \delta_{e \gamma [a \beta \gamma [c]} \gamma_{abc} \quad , \]

\[ \gamma_{a \gamma [de]} \gamma_{abc} = +6 \gamma_{a \gamma [de]} + 4 \delta_{e \gamma [a \beta \gamma [c]} \gamma_{abc} \quad , \]

\[ \gamma_{a \gamma [de]} \gamma_{abc} = +6 \gamma_{a \gamma [de]} + 4 \delta_{e \gamma [a \beta \gamma [c]} \gamma_{abc} \quad . \]
\[
\gamma^{ab}\gamma^{de}\gamma_{abc} = -52\gamma^{de}_{c} + 88\delta_{c}[^{d}]^{e}, \quad \gamma_{dea}\gamma^{bc}\gamma^{de} = -52\gamma^{bc}_{a} - 88\delta_{a}[^{b}]^{c}, \quad (B.14)
\]
\[
\gamma_{[3]}\gamma^{bc}\gamma_{[3]} = -240\gamma^{bc}_{a} + 660\delta_{a}[^{b}]^{c}, \quad (B.15)
\]
\[
\gamma_{[2]}\gamma^{cd}\gamma_{[2]} = -38\gamma^{cd}_{ab} + 70\delta_{a}[^{c}]^{d} - 110\delta_{a}[^{c}]^{d} - 52\eta_{ab}\gamma^{cd}, \quad (B.16)
\]
\[
\gamma_{[2]}\gamma^{cd}\gamma_{[2]} = -38\gamma^{cd}_{ab} - 70\delta_{a}[^{c}]^{d} + 110\delta_{a}[^{c}]^{d}, \quad (B.17)
\]
\[
\gamma_{[3]}\gamma^{ab}\gamma_{[3]} = -126\gamma^{ab}_{cd} - 450\delta_{c}[^{a}]^{b} + 990\delta_{c}[^{a}]^{b}, \quad (B.18)
\]
\[
\gamma^{b\gamma^{a_{1}...a_{6}}}_{[\beta_{1}...\beta_{6}]} = -\frac{1}{60}\delta_{c}[^{a_{1}}^{a_{2}...a_{6}}]. \quad (B.19)
\]

As their indices show, these formulae are prepared in the superspace notation, and as such, the antisymmetrization is like \(B_{(ab)} \equiv B_{ab} \mp B_{ba}\).

**Appendix C: Lorentz Algebra with Modified \(\tilde{M}_{ab}\)**

In this Appendix, we examine the significance of our algebra with \(\tilde{M}\) defined by (2.2.1), which is highly non-trivial. We know that the Lorentz covariance in the extra dimensions out of 12D is lost, and therefore we need to confirm at least the ordinary 10D. To this end, we use the local Lorentz indices \(i, j, \ldots = (0), (1), \ldots, (9)\) for the purely 10D part.

We first note the relationships

\[
R_{AB\gamma}^{\delta} = -\frac{1}{2}R_{ABij}(\tilde{M}_{ij})_{\gamma}^{\delta}, \quad R_{AB}^{ij} = -\frac{1}{2}R_{ABij}(\tilde{M}_{ij})_{\gamma}^{\delta}, \quad (C.1)
\]

which is confirmed also by the use of extra condition (2.3.6). This implies that \(R_{AB\gamma}^{\delta}\) behaves as if it were within the 10D sub-manifold, realizing only \(SO(9,1)\) subgroup of \(SO(10,2)\). This feature is also valid for the combination \(\phi_{M}^{ab}\tilde{M}_{ab}\), because only the combination \(\phi_{M}^{ij}(\tilde{M}_{ij})_{\alpha_{1}}^{\beta_{1}}\) survives, while other components \(\phi_{M}^{\pm i}(\tilde{M}_{\pm i})_{\alpha_{1}}^{\beta_{1}}\) and \(\phi_{M}^{\pm i}(\tilde{M}_{\pm i})_{\alpha_{1}}^{\beta_{1}}\) vanish\(^{14}\) due to either the definition (2.1.6), or the extra constraint (2.3.6). We now see how our system realizes only the \(SO(9,1)\) sub-algebra in the total 12D.

We next compute the commutators among \(\tilde{M}\)'s. Out of \([\tilde{M}_{ab}, \tilde{M}_{cd}]_{\gamma}^{\delta}\), there are six different combinations, when 10D indices are distinguished from the extra coordinates \(\pm: (i) [\tilde{M}_{ij}, \tilde{M}_{kl}]_{\alpha_{1}}^{\delta_{1}}, (ii) [\tilde{M}_{ij}, \tilde{M}_{\pm k}]_{\alpha_{1}}^{\delta_{1}}, (iii) [\tilde{M}_{ij}, \tilde{M}_{\pm j}]_{\alpha_{1}}^{\delta_{1}}, (iv) [\tilde{M}_{\pm i}, \tilde{M}_{\pm j}]_{\alpha_{1}}^{\delta_{1}}, (v) [\tilde{M}_{\pm i}, \tilde{M}_{\pm - j}]_{\alpha_{1}}^{\delta_{1}}, (vi) [\tilde{M}_{\pm i}, \tilde{M}_{\pm - j}]_{\alpha_{1}}^{\delta_{1}}\). The first combination is easy to satisfy the 10D algebra, when we use (2.2.6b). The sector (ii) is also straightforward, which agrees with the fully covariant 12D algebra

\[
[M_{ab}, M_{cd}] = -\delta_{[a}[^{c}]M_{b]}^{d}]. \quad (C.2)
\]

\(^{14}\)Here the indices \(\alpha_{1}, \beta_{1}, \ldots\) denote general spinorial indices both dotted and undotted.
All of the sectors (iii), (iv) and (vi) vanish, satisfying (C.2), while (v) yields the result $(1/2)\eta_{+}^{-1}(\tilde{M}_{++})^{-1}$, with the factor 2 discrepancy compared with (C.2), which, however, causes no problem, when we inspect the Jacobi identities next.

We can confirm the Jacobi identities for our non-trivially modified $\tilde{M}_{ab}$, which form the most important foundation for Bianchi identities in superspace:

$$[\tilde{M}_{ab}, [\tilde{M}_{cd}, \tilde{M}_{ef}]] + [\tilde{M}_{cd}, [\tilde{M}_{ef}, \tilde{M}_{ab}]] + [\tilde{M}_{ef}, [\tilde{M}_{ab}, \tilde{M}_{cd}]] \equiv 0.$$  \hspace{1cm} (C.3)

There are ten different combinations for the indices $[ab][cd][ef]$ when 10D indices are distinguished from the extra ones, symbolically categorized as (i) $[ij][kl][mn]$, (ii) $[ij][kl][+m]$, (iii) $[ij][+k][+l]$, (iv) $[+i][+j][+k]$, (v) $[ij][kl][+\ell]$, (vi) $[ij][+k][+\ell]$, (vii) $[ij][+\ell][+\ell]$, (viii) $[+i][+\ell][+\ell]$, (ix) $[+i][+\ell][+\ell]$, (x) $[+\ell][+\ell][+\ell]$. Among these (i) is easy to see, because $\tilde{M}_{ab}$ satisfies the $SO(9,1)$ sub-algebra, when all the indices are 10D. The sectors (ii) - (x) are all easily shown to vanish, when the results in the basic commutators are used. In particular, despite of the factor 2 discrepancy mentioned in the previous paragraph, we can confirm the satisfaction of the sectors (vi) and (ix).

There is, however, some caveat needed about the basic algebra structure in our formulation, associated with the Jacobi identities among our generators $\tilde{M}, P, Q$. As careful readers may have already noticed, the success of our superspace formulation does not necessarily corresponds to the satisfaction of all of these Jacobi identities among generators, because our superspace Bianchi identities hold only modulo our constraint, e.g., (2.3.9) for $D = 12, N = 1$. As a matter of fact, among the ten possible sectors (I) $\tilde{M}\tilde{M}\tilde{M}$, (II) $PPP$, (III) $QQQ$, (IV) $\tilde{M}\tilde{M}P$, (V) $\tilde{M}\tilde{M}Q$, (VI) $PP\tilde{M}$, (VII) $PPQ$, (VIII) $QQ\tilde{M}$, (IX) $QQP$, (X) $\tilde{M}PQ$ of Jacobi identities, we can easily confirm that all of these identities are satisfied, as long as $\tilde{M}$’s carry only purely 10D indices $i, j, \ldots$, while there are some non-vanishing components, e.g., in the cases of (V) $[\tilde{M}_{+i}, [\tilde{M}_{jk}, Q_{a}]] + (2\text{ perms.}) \neq 0$ and (VIII) $[\tilde{M}_{+i}, \{Q_{a}, Q_{b}\}] + (2\text{ perms.}) \neq 0$. This poses no problem, as we have stressed also in subsection 2.1 as well as in this Appendix, because these non-vanishing Jacobi identities become irrelevant under our constraints such as $\phi_{A}^{+1} = 0$, at the superspace Bianchi identity level $[\nabla_{A}, [\nabla_{B}, \nabla_{C}]] + (2\text{ perms.}) \equiv 0$ in terms of $\nabla_{A}$ instead of the generators. This feature is one of the most peculiar and important aspects in our formulation with no other analogs in other theories, which should be always kept in mind in future applications.

**Appendix D: Variation of Extra Constraints under Supersymmetry**

In this Appendix, we analyze the consistency between our extra constraints imposed on our fields and supersymmetry. Here we concentrate on the $D = 12, N = 2$ supergravity in
section 4.2, giving some typical examples. As such an example, we consider the variation of the following equation in (4.2.4a) under supersymmetry at the lowest order:

\[
\delta_Q(R_{\mu rs}^{\nu} \partial_\nu \varphi) = +(D^{\nu} \varphi)D_\mu (\delta \omega_{\nu rs}) - (D^{\nu} \varphi)D_\nu (\delta \omega_{\mu rs})
\]

\[
= -2(D^{\nu} \varphi)(\sigma_\nu [\tau D_{\nu}] R_{\mu rs}) \partial_\tau \varphi + \text{c.c.} = 0 \ .
\]

(D.1)

Here we have used another extra constraint (4.2.4c). Another interesting example involving the coset \(SU(1,1)/U(1)\) is the variation of the first equation in (4.2.4b):

\[
\delta_Q(P_\mu \partial_\mu \varphi) = - \epsilon_{\alpha\beta}(D_\mu \varphi)(\delta_Q V_{+}^\alpha)\partial_\mu V_{+}^\beta - \epsilon_{\alpha\beta}V_{+}^\alpha(D_\mu \varphi)\partial_\mu(\delta_Q V_{+}^\alpha)
\]

\[
= - \epsilon_{\alpha\beta}V_{+}^\alpha V_{-}^\beta(D_\mu \varphi)(\sigma^\gamma D_\mu \lambda)\partial_\gamma \varphi = 0 \ .
\]

(D.2)

Here use is made of the second equation in (4.2.4d). In a very similar fashion, we can see that the variations of all of our extra conditions (4.2.4) under supersymmetry vanish, upon using other extra constraints, up to higher-order terms which we skip in this paper. Even though we skipped similar analysis for the \(N = 1\) supergravity, it can be easily performed in a more direct manner in superspace.
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