A CW complex homotopy equivalent to spaces of locally convex curves

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Abstract

Locally convex (or nondegenerate) curves in the sphere $S^n$ (or the projective space) have been studied for several reasons, including the study of linear ordinary differential equations of order $n+1$. Taking Frenet frames allows us to obtain corresponding curves $\Gamma$ in the group $\text{Spin}_{n+1}$; recall that $\Pi : \text{Spin}_{n+1} \rightarrow \text{Flag}_{n+1}$ is the universal cover of the space of flags. Determining the homotopy type of spaces of such curves $\Gamma$ with prescribed initial and final points appears to be a hard problem. Due to known results, we may focus on $L_n$, the space of (sufficiently smooth) locally convex curves $\Gamma : [0,1] \rightarrow \text{Spin}_{n+1}$ with $\Gamma(0) = 1$ and $\Pi(\Gamma(1)) = \Pi(1)$. Convex curves form a contractible connected component of $L_n$; there are $2^{n+1}$ other components, corresponding to non convex curves, one for each endpoint. The homotopy type of $L_n$ has so far been determined only for $n = 2$ (the case $n = 1$ is trivial). This paper is a step towards solving the problem for larger values of $n$.

The itinerary of a locally convex curve $\Gamma : [0,1] \rightarrow \text{Spin}_{n+1}$ belongs to $W_n$, the set of finite words in the alphabet $S_{n+1} \setminus \{e\}$. The itinerary of a curve lists the non open Bruhat cells crossed by the curve. Itineraries yield a stratification of the space $L_n$. We construct a CW complex $D_n$ which is a kind of dual of $L_n$ under this stratification: the construction is similar to Poincaré duality. The CW complex $D_n$ is homotopy equivalent to $L_n$. The cells of $D_n$ are naturally labeled by words in $W_n$ so that $D_n$ is infinite but locally finite. Explicit glueing instructions are described for lower dimensions.

As an application, we describe an open subset $Y_n \subset L_n$, a union of strata of $L_n$. In each non convex component of $L_n$, the intersection with $Y_n$ is connected and dense. Most connected components of $L_n$ are contained in $Y_n$. For $n > 3$, in the other components the complement of $Y_n$ has codimension at least 2. We prove that $Y_n$ is homotopically equivalent to the disjoint union of $2^{n+1}$ copies of $\Omega \text{Spin}_{n+1}$. In particular, for all $n \geq 2$, all connected components of $L_n$ are simply connected.
1 Introduction

Let $J \subset \mathbb{R}$ be an interval. A sufficiently smooth curve $\gamma : J \to S^n \subset \mathbb{R}^{n+1}$ is \textit{(positive) locally convex} \cite{3,24,25} or \textit{(positive) nondegenerate} \cite{13,18,20,22} if it satisfies
\[ \forall t \in J, \; \det(\gamma(t), \gamma'(t), \ldots, \gamma^{(n)}(t)) > 0. \]
Such a curve $\gamma$ can be associated with $\Gamma = \mathcal{F}_{\gamma} : J \to \text{SO}_{n+1}$ where the column-vectors of $\Gamma(t)$ are the result of applying the Gram-Schmidt algorithm to the ordered basis $(\gamma(t), \gamma'(t), \ldots, \gamma^{(n)}(t))$ of $\mathbb{R}^{n+1}$. The curve $\Gamma$ can then be lifted to the double cover $\text{Spin}_{n+1}$. The curve $\Gamma : J \to \text{Spin}_{n+1}$ is also called \textit{locally convex}; a direct description is in order.

For $j \in \llbracket n \rrbracket = \{1, 2, \ldots, n\}$, consider the skew-symmetric tridiagonal matrices $a_j = e_{j+1}^T e_j - e_j e_{j+1}^T \in \mathfrak{so}_{n+1}$ as elements of the Lie algebra $\mathfrak{spin}_{n+1} = \mathfrak{so}_{n+1}$. The identification between the two Lie algebras is induced by the covering map $\Pi : \text{Spin}_{n+1} \to \text{SO}_{n+1}$. A sufficiently smooth curve $\Gamma : J \to \text{Spin}_{n+1}$ is called \textit{locally convex} if its logarithmic derivative is of the form
\[ (\Gamma(t))^{-1} \Gamma'(t) = \sum_{j \in [n]} \kappa_j(t) a_j, \tag{1} \]
for positive functions $\kappa_1, \ldots, \kappa_n : J \to (0, +\infty)$. Recall that $\Pi_{\text{Flag}} : \text{Spin}_{n+1} \to \text{Flag}_{n+1}$ is the universal cover of the flag space. Let $\text{Quat}_{n+1} = \Pi_{\text{Flag}}^{-1}[\Pi_{\text{Flag}}(\{1\})] \subset \text{Spin}_{n+1}$: this is a subgroup of order $2^{n+1}$, isomorphic to $\pi_1(\text{Flag}_{n+1})$.

Given sufficiently large $r \in \mathbb{N}^* = \{1, 2, 3, \ldots\}$ and $z_0, z_1 \in \text{Spin}_{n+1}$, let $\mathcal{L}^{C^r}(z_0; z_1)$ denote the space of locally convex curves $\Gamma : [0, 1] \to \text{Spin}_{n+1}$ of differentiability class $C^r$ with endpoints $\Gamma(0) = z_0$ and $\Gamma(1) = z_1$. We endow this space with the usual $C^r$ topology and consider the problem of describing its homotopy type. It is easy to see that $\mathcal{L}^{C^r}(z_0; z_1)$ is homeomorphic to $\mathcal{L}^{C^r}(1; z_0^{-1} z_1)$ so we assume $z_0 = 1$. As discussed in \cite{14} (among others), the value of $r$ does not affect the homotopy type so we drop it and write $\mathcal{L}_n(1; z)$. It is proved in \cite{15,25} that for every $z \in \text{Spin}_{n+1}$ there exists $q \in \text{Quat}_{n+1}$ such that $\mathcal{L}_n(1; z)$ and $\mathcal{L}_n(1; q)$ are homotopy equivalent (the map from $z$ to $q$ is explicitly described). We aim to study the homotopy type of $\mathcal{L}_n(1; q)$, $q \in \text{Quat}_{n+1}$, or equivalently that of the disconnected space
\[ \mathcal{L}_n = \bigsqcup_{q \in \text{Quat}_{n+1}} \mathcal{L}_n(1; q). \]
In this paper we construct and study a CW complex $\mathcal{D}_n$ which is homotopy equivalent to $\mathcal{L}_n$. We shall see that $\mathcal{D}_n$ is infinite but locally finite. Cells are naturally labeled by finite words in the alphabet $S_{n+1} \setminus \{e\}$ ($e \in S_{n+1}$ being the identity); such words correspond to \textit{itineraries} of curves $\Gamma \in \mathcal{L}_n$ (itineraries are discussed at length in \cite{14}). We proceed to discuss the symmetric group $S_{n+1}$ and itineraries of curves. In the process we review the main results from \cite{14}. 

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We consider the symmetric group $S_{n+1}$ as a Coxeter-Weyl group, that is, we use as generators the transpositions $a_j = (j, j+1)$, $1 \leq j \leq n$. The number of inversions $\text{inv}(\sigma)$ of a permutation $\sigma$ is its length with these generators. Notice that for $\sigma \in S_{n+1}$ and $j \in \left[n + 1\right] = \{1, 2, \ldots, n, n + 1\}$ we write $j^\sigma$ (instead of $\sigma(j)$); composition is defined by $(j^\sigma_0)^{\sigma_1} = j^{(\sigma_0\sigma_1)}$. Let $\eta \in S_{n+1}$ be the top permutation, i.e., the permutation with the maximum number of inversions: $\text{inv}(\eta) = n(n + 1)/2$, $j^n = n + 2 - j$. The element $\eta$ is the longest element of $S_{n+1}$ with the above generators and is often denoted in the literature by $w_0$. Permutations are often written in the generators $a_j$; for $n \leq 4$ we also write $a = a_1$, $b = a_2$, $c = a_3$ and $d = a_4$. Thus, for instance, for $n = 2$ we have $\eta = aba = bab$; for $n = 3$ we have $\eta = abaca$.

Let $B_{n+1}$ be the group of signed permutation matrices; this is also Coxeter-Weyl group but we shall not use such generators. Let $B^+_{n+1} = B_{n+1} \cap SO_{n+1}$. Let $\tilde{B}^+_{n+1} \subset \text{Spin}_{n+1}$ be the preimage of $B^+_{n+1}$ under the covering map $\Pi : \text{Spin}_{n+1} \to SO_{n+1}$. The subgroup $\text{Quat}_{n+1} \subset \tilde{B}^+_{n+1}$ is the preimage (under $\Pi$) of the subgroup $\text{Diag}_{n+1} \subset B^+_{n+1}$ of diagonal matrices. We have a short exact sequence

$$1 \to \text{Quat}_{n+1} \to \tilde{B}^+_{n+1} \to S_{n+1} \to 1. \tag{2}$$

The spin group is stratified by contractible Bruhat cells $\text{Bru}_z \subset \text{Spin}_{n+1}$, $z \in \tilde{B}^+_{n+1}$. The unsigned (and more usual) Bruhat cells are

$$\text{Bru}_\sigma = \bigcup_{z \in \Pi^{-1}([\sigma])} \text{Bru}_z, \quad \sigma \in S_{n+1}, \tag{3}$$

where $\Pi : \tilde{B}^+_{n+1} \to S_{n+1}$ is the homomorphism in the exact sequence above (we shall say more about the subsets $\Pi^{-1}([\sigma]) \subset \tilde{B}^+_{n+1}$ below). For $z \in \tilde{B}^+_{n+1}$ with $\Pi(z) = \sigma$, $\text{Bru}_z \subset \text{Bru}_\sigma$ is the connected component of $\text{Bru}_\sigma$ containing $z$.

The set $\text{Bru}_\eta$ is a dense open subspace of the spin group with $2^{n+1}$ contractible connected components. Let $\text{Sing}_{n+1} = \text{Spin}_{n+1} \setminus \text{Bru}_\eta$ be the union of the non open Bruhat cells. We define the \textit{singular set} of a locally convex curve $\Gamma : [0, 1] \to \text{Spin}_{n+1}$ as $\text{sing}(\Gamma) = \Gamma^{-1}([\text{Sing}_{n+1}] \setminus \{0, 1\})$. It turns out that $\text{sing}(\Gamma)$ is finite and a continuous function of $\Gamma \in \mathcal{L}_n$ (with the Hausdorff metric in compact subsets of $[0, 1]$); this is Theorem 1 in [14]. The elements of $\text{sing}(\Gamma)$ are sometimes called the \textit{moments of non-transversality} of $\Gamma$ (as in [27]). Given $\Gamma \in \mathcal{L}_n$, write $\text{sing}(\Gamma) = \{\tau_1 < \cdots < \tau_\ell\}$. Let $\sigma_1, \ldots, \sigma_\ell \in S_{n+1}$ be such that $\Gamma(\tau_j) \in \text{Bru}_{\eta \sigma_j}$; the itinerary of $\Gamma$ is the word $\text{iti}(\Gamma) = (\sigma_1, \ldots, \sigma_\ell)$. Figure 1 shows itineraries of a few locally convex curves in $\mathbb{S}^2$.

Let $W_n$ be the set of finite words in the alphabet $S_{n+1} \setminus \{e\}$. A simplified notation for the word $w = (\sigma_1, \ldots, \sigma_\ell) \in W_n$ is often used, which we exemplify for a few words in $W_3$:

$$acbac = (a, c, b, a, c), \quad [ac][b[ac]] = (ac, b, ac), \quad [acb] = (acb).$$
For \( w = (\sigma_1, \ldots, \sigma_\ell) \in W_n \), define

\[
\dim(w) = \dim(\sigma_1) + \cdots + \dim(\sigma_\ell), \quad \dim(\sigma) = \text{inv}(\sigma) - 1.
\] (4)

Let \( \mathcal{L}_n[w] \subset \mathcal{L}_n \) be the set of curves \( \Gamma \) with \( \text{iti}(\Gamma) = w \). As in [14], we prefer to work with Hilbert spaces, more precisely with the Sobolev space \( H^r \). For large \( r \in \mathbb{N} \), let \( \mathcal{L}_n[H^r] \) be the Hilbert manifold of locally convex curves \( \Gamma \) of class \( H^r \) with the usual boundary conditions. From now on, assume \( \mathcal{L}_n = \mathcal{L}_n[H^r] \); recall that this does not affect the homotopy type of \( \mathcal{L}_n \). Theorem 2 in [14] tells us that \( \mathcal{L}_n[w] \subset \mathcal{L}_n \) is a (non empty) contractible embedded submanifold of codimension \( \dim(w) \) and of class \( C^{r-1} \). Notice that [14] discusses the case \( H^1 \) in considerable detail: here we consistently prefer the case \( H^r \), \( r \) large.

This stratification by itineraries of \( \mathcal{L}_n \) is not as well behaved as might be desired. We do not, for instance, have the Whitney property. Indeed, for \( n = 3 \) and \( w_0 = [ac]b[ac] = (a_1a_3, a_2, a_1a_3) \) and \( w_1 = [acb] = (a_1a_3a_2) \) we have

\[
\mathcal{L}_n[w_1] \cap \overline{\mathcal{L}_n[w_0]} \neq \emptyset, \quad \mathcal{L}_n[w_1] \not\subseteq \overline{\mathcal{L}_n[w_0]}.
\]

Notice that \( \mathcal{L}_3[w_0], \mathcal{L}_3[w_1] \subset \mathcal{L}_3 \) both have codimension 2: this example is extensively discussed in Section 9 of [14]. On the other hand, the stratification of \( \mathcal{L}_n \) is well behaved enough for the construction of a homotopically equivalent CW complex \( D_n \). In a sense, the CW complex is a dual of the stratification.
The maps acute, grave : $S_{n+1} \to \tilde{B}_{n+1}^+$ and hat : $S_{n+1} \to \text{Quat}_{n+1}$ are discussed in [15] and revised in Section 2 below. For $w \in W_n$ define $\hat{w} \in \text{Quat}_{n+1}$ by

$$w = \sigma_1 \cdots \sigma_\ell = (\sigma_1, \ldots, \sigma_\ell) \in W_n, \quad \hat{w} = \hat{\sigma}_1 \cdots \hat{\sigma}_\ell \in \text{Quat}_{n+1};$$

here $\hat{\sigma} = \text{hat}(\sigma) \in \text{Quat}_{n+1}$. Theorem 2 in [14] tells us that $L_n[w] \subset L_n(1; \hat{\eta} \hat{w} \hat{\eta}) \subset L_n$. Thus, words with different values of hat do not interact directly and can be studied separately.

There is a form of duality (a variant of Poincaré duality) which allows us to relate the stratification of $L_n$ with a rather explicit CW complex $D_n$. We now present a short statement; a more detailed one is given in Theorem 4. The statement of Theorem 4 assumes familiarity with the partial order $\sqsubseteq$ in $W_n$ (discussed in Section 4), a variant of another partial order $\preceq$ in $W_n$ (introduced in [14], and based on previous work by Shapiro and Shapiro).

**Theorem 1.** There exists a CW complex $D_n$ with one cell $c_w$ of dimension $\dim(w)$ for each word $w \in W_n$. Furthermore, there exists a continuous map $i : D_n \to L_n$ which is a homotopy equivalence.

A similar but simpler example is presented in [4]: there we have a finite dimensional manifold $BL_\sigma \subset L_{n+1}^1 (\sigma \in S_{n+1})$ which is stratified by finitely many contractible submanifolds $BLS_\varepsilon \subseteq BL_\sigma$. A finite homotopically equivalent CW complex $BLC_\sigma$ is then constructed. The result in [4] most relevant for the above description is Theorem 2 (for the homotopy equivalence $i_\sigma : BLC_\sigma \to BL_\sigma$), which in turn relies on Lemma 14.1 (a topological result which validates the inductive step). The present paper does not presuppose these results.

Glueing maps for cells of dimensions 1 and 2 are described explicitly. This allows us to construct the 1 and 2-skeletons of $D_n$ in Sections 7 and 8. The following result is then proved.

**Theorem 2.** Let $z_0, z_1 \in \text{Spin}_{n+1}$. Every connected component of $L_n(z_0; z_1)$ is simply connected.

In order to state our third main result we need a few algebraic definitions. A permutation $\sigma \in S_{n+1}$ is parity alternating if $k^\sigma \not\equiv (k + 1)^\sigma (\text{mod } 2)$ for all $k$, $1 \leq k \leq n$. We shall see that $\sigma$ is parity alternating if and only if $\hat{\sigma} \in Z(\text{Quat}_{n+1})$ (the center of Quat$_{n+1}$). For $n > 3$, if $\sigma \not\equiv e$ is parity alternating then $\text{inv}(\sigma) \geq 3$. Let $I_Y \subset W_n$ be the set of words containing at least one letter which is not parity alternating. It is not hard to verify that $I_Y \subset W_n$ is a lower set with respect to the partial orders $\preceq$ and $\sqsubseteq$. It then follows from [14] (or from Theorem 4) that $Y_n = L_n[I_Y] \subset L_n$ is an open subset. For $z \in \text{Spin}_{n+1}$, let $\Omega \text{Spin}_{n+1}(1; z)$ be the space of all continuous curves $\Gamma : [0, 1] \to \text{Spin}_{n+1}, \Gamma(0) = 1$ and $\Gamma(1) = z$. The space $\Omega \text{Spin}_{n+1}(1; z)$ is homeomorphic to the loop space $\Omega \text{Spin}_{n+1} = \Omega \text{Spin}_{n+1}(1; 1)$ and we have a natural inclusion $L_n(1; z) \subset \Omega \text{Spin}_{n+1}(1; z)$. 

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Theorem 3. Let \( Y_n \subset \mathcal{L}_n \) be the open subset constructed above. For \( q \in \text{Quat}_{n+1} \) let \( Y_n(1; q) = Y_n \cap \mathcal{L}_n(1; q) \). For any \( q \in \text{Quat}_{n+1} \), the open subset \( Y_n(1; q) \subset \mathcal{L}_n(1; q) \) is dense in the connected component of non convex curves and the inclusion \( Y_n(1; q) \subset \Omega \text{Spin}_{n+1} \) is a homotopy equivalence.

The case \( n = 2 \) of the above theorem is the hardest result in [24]. The set \( Y_2 \) is there called the set of complicated curves and the complement \( \mathcal{L}_2 \setminus Y_2 \) is the closed subset of multiconvex curves. We may therefore say that the basic structure of the arguments in [24], in the case \( n = 2 \), seems to survive in the cases \( n > 2 \), except that the complement \( \mathcal{L}_n \setminus Y_n \) does not admit a correspondingly direct description. We could also say that we have to study what is the correct generalization of the concept of a multiconvex curve to the cases \( n > 2 \). We expect to address these issues in future work.

The case \( n = 3 \) deserves special attention; in Section 15 we state and prove an alternative to Theorem 3 for this special case. We also state in Theorem 6 a full description of the homotopy type of \( \mathcal{L}_3 \); this is the main result in [2].

The following is an easy consequence of Theorem 3. Notice that, for large \( n \), it gives us the homotopy type for most of the connected components of \( \mathcal{L}_n \).

Corollary 1.1. If \( q \in \text{Quat}_{n+1} \) does not belong to its center \( Z(\text{Quat}_{n+1}) \subset \text{Quat}_{n+1} \) then the inclusion \( \mathcal{L}_n(1; q) \subset \Omega \text{Spin}_{n+1}(1; q) \) is a homotopy equivalence.

We expect to discuss higher dimensional skeletons and glueing maps in future work. Some conjectures are listed in Section 15; in particular, we believe that if \( q \in Z(\text{Quat}_{n+1}) \) then \( \mathcal{L}_n(1; q) \) is not homotopy equivalent to \( \Omega \text{Spin}_{n+1} \).

Our problem is related to the study of linear ordinary differential operators. This point of view was the original motivation of V. Arnold, B. Khesin, V. Ovsienko, B. Shapiro and M. Shapiro for considering this class of questions in the early nineties [18, 19, 20, 30]. The second author was first led to consider this subject while studying the critical sets of nonlinear differential operators with periodic coefficients, in a series of works with D. Burghelea and C. Tomei [5, 6, 7, 28].

In Section 2 we see several results concerning the combinatorics of the symmetric group \( S_{n+1} \): these results will of course be useful later. In Section 3 we review classical results about Bruhat cells, some recent results from our previous papers ([15, 14]) and a few other results which will be needed later. In Section 4 we present the two partial orders \( \preceq \) and \( \subseteq \) in \( W_n \): we see some examples and conjectures. In Section 5 we continue to study \( W_n \) as a poset: we are now interested in lower and upper sets. At this point, in Section 6 we are ready to state and prove Theorem 4, a stronger and more detailed version of Theorem 1, and to construct the CW complex \( D_n \). After we define some auxiliary concepts and state
Lemma 6.3 (a technical result, essentially the inductive step in the proof of Theorem 4), most of the section is dedicated to the proof of Lemma 6.3. In Section 7 we study the 1-skeleton of our newly obtained CW complex $D_n$: as an exercise, we give a new proof of the well known classification of the connected components of $L_n$. In Section 8 we study the 2-skeleton of $D_n$: this is the highest dimension to be completely described. Some partial results concerning higher dimensions are discussed in Section 9; this is where the combinatorial results from Section 2 are applied. The concept of loose maps has been discussed in previous papers: it is revised and adapted in Section 10. In Section 11 this concept is adapted to ideals (subsets of the poset $W_n$): Proposition 11.3 is a pivotal result with a long and technical proof. In Section 12 we finally prove Theorem 2. In Section 13 we prepare for the proof of Theorem 3 by stating and proving Proposition 13.2, which in a sense is an improvement of Proposition 11.3. Theorem 3 is proved in Section 14. Section 15, the final remarks, includes the statement of Theorem 6, the main result in [2]; this gives the homotopy type of $L_3$. Theorem 5, a retouched version of Theorem 3 for the case $n = 3$, aims specifically at the proof of Theorem 6. We also state as conjectures other results which we hope to prove in future work.

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2 Permutations

Consider the symmetric group $S_{n+1}$ (acting on $[n + 1] = \{1, 2, \ldots, n + 1\}$) as the Coxeter-Weyl group of type $A_n$, i.e., use the $n$ generators $a = a_1 = (12)$, $b = a_2 = (23), \ldots, a_n = (n, n + 1)$. For $\sigma \in S_{n+1}$ and $k \in [n+1]$ we use the notation $k^\sigma$ (rather than $\sigma(k)$) so that $(k^\sigma)^{\sigma_2} = k^{(\sigma_1 \sigma_2)}$. Given $\sigma \in S_{n+1}$, let $P_\sigma$ be the $(n+1) \times (n+1)$ permutation matrix satisfying $e_i^\top P_\sigma = e_{\sigma(i)}^\top$; notice that $P_{\sigma_0} P_{\sigma_1} = P_{\sigma_0 \sigma_1}$. For $\sigma \in S_{n+1}$, let inv($\sigma$) be the length of $\sigma$ with the generators $a_i, 1 \leq i \leq n$.

For $\sigma_0, \sigma_1 \in S_{n+1}$, write $\sigma_0 \equiv \sigma_1$ (mod 2) if $i^{\sigma_0} \equiv i^{\sigma_1}$ (mod 2) for all $i \in [n+1]$. A permutation $\sigma \in S_{n+1}$ is parity preserving if $i^\sigma \equiv i$ (mod 2) for all $i \in [n+1]$, i.e., if $\sigma \equiv e$ (mod 2). A permutation $\sigma \in S_{n+1}$ is parity alternating if $i^\sigma \not\equiv (i+1)^\sigma$ (mod 2) for all $i \in [n]$. The sets $S_{PP}$ and $S_{PA}$
of parity preserving and parity alternating permutations, respectively, are both subgroups: \( S_{PP} \leq S_{PA} \leq S_{n+1} \). If \( n \) is even, \( S_{PP} = S_{PA} \); if \( n \) is odd, \( S_{PP} \) is a subgroup of index 2 of \( S_{PA} \). We have \( \sigma_0 \equiv \sigma_1 \pmod{2} \) if and only if \( \sigma_1^{-1} \sigma_0 \in S_{PP} \), or, equivalently, if \( \sigma_0 S_{PP} = \sigma_1 S_{PP} \).

**Example 2.1.** For \( n = 2 \) we have \( S_{PA} = S_{PP} = \{ e, \eta = aba \} \). For \( n = 3 \) we have \( S_{PP} = \{ e, aba, bcb, bacb \} \) and \( S_{PA} = S_{PP} \cup \{ ac, abc, cba, \eta = abacba \} \). For \( n = 4 \) we have \( S_{PA} = S_{PP} \) and

\[
S_{PP} = \{ e, aba, bacb, bcb, cbdc, abacdc, abacdc, abedcba, cdacb, cbdcaba, \eta \},
\]

with \( \eta = abacbadcba \).

The group \( S_{PP} \) is isomorphic to \( S_{\frac{n+1}{2}} \times S_{\frac{n+1}{2}} \). For odd \( n \), the top element \( \eta \) and \( \sigma = a_1 a_3 \cdots a_{n-2} a_n \) are both elements of \( S_{PA} \setminus S_{PP} \). \( \sigma \) is the element in this set with minimal number of inversions, equal to \( \frac{n+1}{2} \).

The maps acute, grave: \( S_{n+1} \to \tilde{B}_{n+1}^+ \) and hat: \( S_{n+1} \to \text{Quat}_{n+1} \) are discussed in [15] (these are not homomorphisms!). We also write \( \acute{\sigma} = \text{acute}(\sigma) \), \( \grave{\sigma} = \text{grave}(\sigma) \), \( \hat{\sigma} = \text{hat}(\sigma) \). If \( \Pi : \tilde{B}_{n+1}^+ \to S_{n+1} \) is as in the exact sequence (2) above we have \( \Pi(\hat{\sigma}) = \Pi(\acute{\sigma}) = \sigma \). In order to define \( \acute{\sigma}_i \in \tilde{B}_{n+1}^+ \), consider the two preimages under the surjective homomorphism \( \tilde{B}_{n+1}^+ \to B_{n+1} \) of the signed permutation matrix \( P \) with entries \( P_{i+1,i} = 1 \), \( P_{i,i+1} = -1 \) and \( P_{j,j} = 1 \) for \( j \notin \{ i, i + 1 \} \): the element \( \acute{\sigma}_i \) is the preimage closest to 1 \( \in \text{Spin}_{n+1} \). We have \( \acute{\sigma}_i = (\acute{\sigma})^{-1} \). For a reduced word \( \sigma = a_{i_1} \cdots a_{i_\ell} \), set

\[
\acute{\sigma} = \acute{\sigma}_{i_1} \cdots \acute{\sigma}_{i_\ell}, \quad \grave{\sigma} = \grave{\sigma}_{i_1} \cdots \grave{\sigma}_{i_\ell}, \quad \hat{\sigma} = \hat{\sigma}(\acute{\sigma})^{-1}.
\]

(5)

For all \( \sigma \in S_{n+1} \) we have \( \Pi^{-1}([\sigma]) = \text{Quat}_{n+1} \hat{\sigma} = \hat{\sigma} \text{Quat}_{n+1} \).

**Lemma 2.2.** A permutation \( \sigma \in S_{n+1} \) is parity alternating if and only if \( \hat{\sigma} \in Z(\text{Quat}_{n+1}) \). Also, \( \sigma \in S_{PP} \) if and only if \( \hat{\sigma} = \pm 1 \).

**Remark 2.3.** It is easy to verify that \( Z(\text{Quat}_{n+1}) \) (the center of the group) equals \( \{ \pm 1 \} \) for \( n \) even and \( \{ \pm 1, \pm (\hat{a}_1 \hat{a}_3 \cdots \hat{a}_n) \} \) for \( n \) odd. For even \( n \) we have \( \eta \in S_{PP} \); for odd \( n \) we have \( \eta \in S_{PA} \setminus S_{PP} \). Remark 3.7 in [15] confirms that \( \hat{\eta} = \pm 1 \) for even \( n \) and \( \hat{\eta} = \pm (\hat{a}_1 \hat{a}_3 \cdots \hat{a}_n) \) for odd \( n \).

A computation verifies that \( \hat{\eta} \) commutes with every element of \( Z(\text{Quat}_{n+1}) \). In particular, \( q \in Z(\text{Quat}_{n+1}) \) if and only if \( \hat{\eta} q \hat{\eta} \in Z(\text{Quat}_{n+1}) \).

The **multiplicity** of \( \sigma \in S_{n+1} \) is a vector \( \text{mult}(\sigma) \in \mathbb{N}^n \) (where \( \mathbb{N} = \{ 0, 1, 2, \ldots \} \)) with coordinates

\[
\text{mult}_j(\sigma) = (1^\sigma + \cdots + j^\sigma) - (1 + \cdots + j), \quad 1 \leq j \leq n;
\]

(6)

see [15] for the reason of the name.
Proof of Lemma 2.2. Clearly, \( \sigma \in S_{PP} \) if and only if \( \text{mult}_k(\sigma) \) is even for all \( k \). Similarly, \( \sigma \in S_{PA} \setminus S_{PP} \) if and only if \( \text{mult}_k(\sigma) \equiv k \pmod{2} \) for all \( k \). We know from Lemma 3.3 in [15] that \( \hat{\sigma} = \pm \hat{a}_1 \text{mult}_1(\sigma) \cdots \hat{a}_n \text{mult}_n(\sigma) \). Thus, \( \sigma \in S_{PP} \) if and only if \( \hat{\sigma} = \pm 1 \) and, for odd \( n \), \( \sigma \in S_{PA} \setminus S_{PP} \) if and only if \( \hat{\sigma} = \pm (\hat{a}_1 \hat{a}_3 \cdots \hat{a}_n) \) for odd \( n \). The result now follows from Remark 2.3. \( \square \)

Remark 2.4. It follows directly from Lemma 3.5 in [15] that \( \sigma_0 \triangleleft \sigma_1 \) implies \( \hat{\sigma}_0 = \hat{\sigma}_1 \); also, if \( \sigma_0 \triangleleft \sigma_1 \) and \( \hat{\sigma}_0 = \hat{\sigma}_1 \) then \( \sigma_0 \triangleleft \sigma_1 \).

Lemma 2.5. If \( \sigma_0, \sigma_1 \in S_{n+1} \) satisfy
\[
\sigma_0 \prec \sigma_1, \quad 2 \text{mult}(\sigma_0) \leq \text{mult}(\sigma_1)
\]
then we have one of the cases:
\[
\sigma_0 = e \prec \sigma_1 = a_i, \quad \sigma_0 = a_i \prec \sigma_1 = \sigma_0 a_i \pm 1, \quad \sigma_0 = a_i a_{i+2} \prec \sigma_1 = \sigma_0 a_{i+1}.
\]
In particular, if \( \sigma_0 \triangleleft \sigma_1 \) then \(2 \text{mult}(\sigma_0) \not\leq \text{mult}(\sigma_1)\).

Proof. If \( \sigma_0 \prec \sigma_1 = (i_0 i_1) \sigma_0 = \sigma_0 (j_0 j_1) \) then
\[
\text{mult}_k(\sigma_1) = \text{mult}_k(\sigma_0) + (j_1 - j_0)[i_0 \leq k < i_1],
\]
as is shown in [15], Lemma 2.4. Thus, \( 2 \text{mult}(\sigma_0) \leq \text{mult}(\sigma_1) \) implies \( \text{mult}_k(\sigma_0) = 0 \) for \( k < i_0 \) or \( k > i_1 \). Thus \( i_0 < i < i_1 \) implies \( i_0 \leq i^\sigma_0 \leq i_1 \). We must then have \( j_1 - j_0 = 1 \). Indeed, if \( j_1 - j_0 > 1 \) take \( j \) with \( j_0 < j < j_1 \) and \( i \) with \( i^\sigma_0 = j \); we have \( i_0 < i < i_1 \), contradicting \( \sigma_0 \prec \sigma_1 \). We thus have \( \text{mult}_k(\sigma_0) \leq 1 \) for \( i_0 \leq k < i_1 \). This implies that \( \sigma_0 \) is a product of commuting generators \( a_i \), \( i_0 \leq i < i_1 \); a case by case analysis completes the proof. \( \square \)

Define the subset \( Y \subset S_{n+1} \) by
\[
Y = S_{n+1} \setminus S_{PA} = \{ \sigma \in S_{n+1} \mid \hat{\sigma} \notin Z(\text{Quat}_{n+1}) \}
\]
(7)
The equivalence follows from Lemma 2.2. For \( \sigma \in Y \), let \( k \) be the smallest integer with \( k^\sigma \equiv (k+1)^\sigma \pmod{2} \). Let \( \sigma' = a_k \sigma \). Notice that \( \sigma' \equiv \sigma \pmod{2} \) and that \( \sigma'' = \sigma \) so that we have involutions in \( Y \) and in each lateral class \( \sigma S_{PP} \subset S_{n+1} \), \( \sigma \in Y \). We call \( \sigma \in Y \) low if \( \sigma \triangleleft \sigma' \) and high otherwise. Thus, \( \sigma \in Y \) is high if and only if \( \sigma' \triangleleft \sigma \).
Lemma 2.6. Let $\sigma_0, \sigma_1 \in Y$ with $\sigma_0$ high, $\sigma_1 \bowtie \sigma_0$ and $\sigma_1 \not= \sigma_0'$; then $\sigma_1$ is high.

Proof. We have $\sigma_0 \equiv \sigma_1 \pmod{2}$. Let $k$ be the smallest integer with $k^{\sigma_0} \equiv (k + 1)^{\sigma_0} \pmod{2}$: we have $\sigma'_0 = ak\sigma_0 \bowtie \sigma_0$ and $\sigma'_1 = ak\sigma_1$. We need to prove that $k^{\sigma_1} > (k + 1)^{\sigma_1}$. Write $\sigma_1 = (i_0i_1\sigma_0)$, $i_0 < i_1$. If $i_0 = k$ and $i_1 = k + 1$ then $\sigma_1 = \sigma'_0$, contradicting the statement. If $\{i_0, i_1\}$ is disjoint from $\{k, k + 1\}$ then $k^{\sigma_1} = k^{\sigma_0} > (k + 1)^{\sigma_0} = (k + 1)^{\sigma_1}$ and therefore $\sigma_1$ is high. If $i_0 = k$ and $i_1 > k + 1$ we must have $(k + 1)^{\sigma_0} < i_1^{\sigma_0} < k^{\sigma_0}$ and therefore again $\sigma_1$ is high. If $i_0 = k + 1$ we must have $i_1^{\sigma_0} < (k + 1)^{\sigma_0} < k^{\sigma_0}$ and therefore also in this case $\sigma_1$ is high. The two remaining cases are: $i_1 = k$ and $i_1 = k + 1$ with $i_0 < k$; both are similar to the cases above and imply $\sigma_1$ high. \hfill \Box

Remark 2.7. A $\bowtie$-matching of a set $X \subset S_{n+1}$ is a partition of $X$ into subsets of cardinality 2 of the form $\{\sigma_-, \sigma_+\}$ with $\sigma_- \bowtie \sigma_+$. For instance, for $n = 2$ and $X = S_3 \setminus \{e, \eta = aba\}$ the following is a $\bowtie$-matching:

$$\{\{a, ba\}, \{b, ab\}\}.$$ 

Indeed, we have $a \bowtie ba$ and $b \bowtie ab$.

Partition $S_{n+1}$ into lateral classes $\sigma_{SP} \subset S_{n+1}$. The involution $\sigma \leftrightarrow \sigma'$ above defines a $\bowtie$-matching in each class except $\sigma_{SP}$ and, for $n$ odd, $\sigma_{PA} \setminus \sigma_{SP} = \eta_{SP}$. See the case $n = 2$ above.

For $n = 3$, $S_4$ is partitioned into 6 classes of cardinality 4 each. For instance, $b_{SP} = \{b, ab, cb, acb\}$ is partitioned as $\{\{b, ab\}, \{cb, acb\}\}$. The classes $\sigma_{SP} = \{e, aba, bcb, bacb\}$ and $\eta_{SP} = \{ac, abc, cba, \eta\}$ admit no $\bowtie$-matching.

For $n = 4$, we have $\sigma_{PA} = \sigma_{SP}$ and $|\sigma_{SP}| = 12$; the list of elements is given both in Example 2.1 and in Figure 2. The group $S_5$ is then partitioned into 10 classes of cardinality 12. The class $\sigma_{SP}$ admits no $\bowtie$-matching; see Figure 2.

![Figure 2: The $\bowtie$-pairs in $\sigma_{SP}$ of $S_5$.](image-url)

For any value of $n$, there is no permutation $\sigma \in S_{n+1}$ for which $e \bowtie \sigma$ or $\sigma \bowtie e$. Similarly, there is no permutation $\sigma \in S_{n+1}$ for which $\eta \bowtie \sigma$ or $\sigma \bowtie \eta$. Thus, there is no $\bowtie$-matching of either $\sigma_{SP}$ or of $\eta_{SP}$.

Removing $e$ and $\eta$ does not help much. Figure 2 shows the $\bowtie$-connected components of $aba$ and of $abcdcba \in \sigma_{SP}$ for $n = 4$. The figure for the connected component of $aba$ is similar for other values of $n$. The number of elements is odd and trying to construct a $\bowtie$-matching always leaves out an element.
Remark 2.8. Let $Y \subset S_{n+1}$ be as in Equation (7). For $k \geq 2$, let $Y_k \subseteq Y$ be the set of permutations $\sigma \in Y$ which are either low with $\text{inv}(< k$ or high with $\text{inv}(\sigma) \leq k$. For instance, for $n \geq 2$:

$$Y_2 = \{a_1, a_1' = a_2 a_1, a_2, a_2' = a_1 a_2, \ldots, a_{n-1}, a_{n-1}' = a_{n-2} a_{n-1}, a_n, a_n' = a_{n-1} a_n\}.$$ 

The correspondence $\sigma \leftrightarrow \sigma'$ defines an involution in $Y_k$ and a $\heartsuit$-matching of $Y_k$. It follows from Lemma 2.6 that if $\sigma_0 \in Y_k$ and $\sigma_1 \heartsuit \sigma_0$ then $\sigma_1 \in Y_k$; if we also have $\sigma_1 \neq \sigma_0'$ then $\sigma_1 \in Y_{k-1}$.

## 3 Bruhat cells

In this section we revise some notation and results and prove some results concerning Bruhat cells and related constructions.

Two invertible matrices $A_0, A_1$ are (unsigned) Bruhat equivalent if there exist $U_0, U_1 \in U_{p_{n+1}}$ with $A_0 = U_0 A_1 U_1$ (recall that $U_{p_{n+1}}$ is the group of invertible real upper triangular matrices). Any invertible matrix $A$ is Bruhat equivalent to a unique permutation matrix $P_{\rho}$, $\rho \in S_{n+1}$. We then say that $A$ is Bruhat equivalent to $\rho$. Given $A \in GL_{n+1}$ and a pair $(i, j)$, let

$$\text{SW}(A, i, j) = \text{SubMatrix}(A, i \ldots n + 1, 1 \ldots j),$$  

(8)

a $(n + 2 - i) \times j$ submatrix of $A$. As is well known, two matrices $A_0$ and $A_1$ are Bruhat equivalent if for every pair $(i, j)$ the ranks of $\text{SW}(A_0, i, j)$ and of $\text{SW}(A_1, i, j)$ are equal. Let $(A_k)_{k \in \mathbb{N}}$ be a sequence of matrices in $GL_{n+1}$ converging to $A_\infty \in GL_{n+1}$; if, for all $k \in \mathbb{N}$, $A_k$ is Bruhat equivalent to $\rho_0$ and $A_\infty$ is Bruhat equivalent to $\rho_1$ then $\rho_1 \leq \rho_0$ in the strong Bruhat order. Indeed, this condition is often used to define the Bruhat order.

Given a signed permutation matrix $Q_0 \in B^+_{n+1}$ corresponding to the permutation $\rho \in S_{n+1}$, let $\sigma = \eta \rho$ and consider the affine space of matrices

$$M_{Q_0} = Q_0 \text{Lo}_{n+1}^1 \cap \text{Lo}_{n+1}^1 Q_0 = U_{p_{\sigma}} Q_0 = Q_0 \text{Lo}_{\sigma^{-1}} \subset GL_{n+1}.$$ 

The subgroups $U_{p_{\sigma}} \subseteq U^1_{p_{n+1}}$ and $\text{Lo}_{\sigma} \subseteq \text{Lo}^1_{n+1}$ (for $\sigma \in S_{n+1}$) are implicitly defined by this equation and are discussed in Section 2 of [15]. We have $\dim(M_{Q_0}) = \dim(\text{Lo}_{\sigma^{-1}}) = \text{inv}(\sigma) = \text{inv}(\eta) - \text{inv}(\rho)$. In order to construct a parametrization of $M_{Q_0}$, start with $Q_0$ and run through the zero entries. An entry $(i, j)$ satisfies both $j < i^\rho$ and $j^{\rho^{-1}} < i$ if and only if it is both below a non zero entry of $Q_0$ and to the left of another non zero entry of $Q_0$. In this case, replace it by a free real variable $x_k$. This space of matrices is constructed in the proof of Theorem 2 in [15] (where its generic element is called $M$) and discussed in Section 7 in [14] (where its generic element is called $\tilde{M}$). We have
Every element $A \in Q_0 L_{n+1}$ can be uniquely written as a product $A = UM$ with $U \in Up_{\rho}$ and $M \in M_{Q_0}$, and $U \in Up_{\rho}$ and $M \in M_{Q_0}$ imply $UM \in Q_0 L_{n+1}$. Clearly, $M$ is Bruhat equivalent to $A = UM$.

**Example 3.1.** Consider $n = 5$ and the matrix

$$Q_1 = \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \in B_{n+1}^+$$

so that the corresponding permutation is $\rho_1 = [326154]$ and $\sigma_1 = \eta \rho_1 = [451623]$.

The affine space $M_{Q_1}$ is the set of matrices $\tilde{M}_1$ of the form

$$\tilde{M}_1 = \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & x_1 & x_2 & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot \\ x_3 & x_4 & x_5 & \cdot & 1 \\ x_6 & x_7 & x_8 & 1 & \cdot \end{pmatrix}, \quad x_1, \ldots, x_8 \in \mathbb{R}.$$ 

Notice that $\text{inv}(\rho_1) = 7$ and $\dim(M_{Q_1}) = \text{inv}(\sigma_1) = 8$.

**Lemma 3.2.** Consider $Q_1 \in B_{n+1}^+$ corresponding to $\rho_1 \in S_{n+1}$ and the affine space $M_{Q_1}$.

1. If $M \in M_{Q_1}$ is equivalent to $\rho \in S_{n+1}$ then $\rho_1 \preceq \rho$. The only such matrix $M$ equivalent to $\rho_1$ is $M = Q_1$.

2. Consider $\rho_0 \in S_{n+1}$, $\rho_1 \prec \rho_0$, $\rho_0 = (i_0 j_0) \rho_1 = \rho_1 (j_0 i_1)$, $i_0 < i_1$, $j_0 < j_1$. Take $X = e_{i_0} e_{j_0}^T$ be the matrix with a single non zero entry equal to 1 in position $(i_1, j_0)$. Then $Q_1 + tX \in M_{Q_1}$ for all $t \in \mathbb{R}$. Moreover, $M \in M_{Q_1}$ is equivalent to $\rho_0$ if and only if $M$ is of the form $M = Q_1 + tX$ for $t \in \mathbb{R} \setminus \{0\}$.

**Example 3.3.** Consider $Q_1$ and $\rho_1$ and in Example 3.1. Take $\rho_0 = [356124]$ so that $\rho_1 \prec \rho_0$, $i_0 = 2$, $i_1 = 5$, $j_0 = 2$ and $j_1 = 5$. The matrices in $M_{Q_1}$ which are equivalent to $\rho_0$ are precisely

$$\tilde{M}_1 = Q_1 + x_4 X = \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_4 & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}, \quad x_4 \in \mathbb{R} \setminus \{0\}.$$ 

Notice that for $x_4 = 0$ we have $\tilde{M}_1 = Q_1$, which is equivalent to $\rho_1$.
Corollary 3.4. Let $\rho_0$, $\rho_1$ and $Q_1$ be as in Lemma 3.2. The matrices $A \in Q_1 L_{n+1}$ which are Bruhat equivalent to $\rho_1$ are precisely $A \in U_{\rho_1} Q_1 = Q_1 L_{n+1}$. The matrices $A \in Q_1 L_{n+1}$ which are Bruhat equivalent to $\rho_0$ are precisely those of the form $U(Q_1 + tX)$ for $U \in U_{\rho_1}$ and $t \in \mathbb{R} \setminus \{0\}$ (and $X$ as in Lemma 3.2).

Proof. This follows from Lemma 3.2 and $Q_1 L_{n+1} = U_{\rho_1} M_{Q_1}$. \hfill \Box

Proof of Lemma 3.2. Clearly $Q_1$ is equivalent to $\rho_1$ and $Q_1 + tX$ is equivalent to $\rho_0$ if $t \neq 0$. Set

$$D_0(\lambda) = \text{diag}(\lambda^{i_1}, \ldots, \lambda^{i_n}), \quad D_1(\lambda) = \text{diag}(\lambda^{-1}, \ldots, \lambda^{-i}),$$

Given $M_1 \in M_{Q_1}$, set $M_\lambda = D_0(\lambda) M_1 D_1(\lambda)$. For $\lambda \in (0, 1)$ we have that $M_\lambda \in M_{Q_1}$ is Bruhat equivalent to $M_1$. Also,

$$\lim_{\lambda \downarrow 0} M_\lambda = Q_1;$$

from the characterization of the strong Bruhat order above we have $\rho_1 \leq \rho$.

Assume that $M \in M_{Q_1}$ is Bruhat equivalent to $Q_1$. If $M \neq Q_1$, take $(i,j)$ such that $M_{i,j} \neq (Q_1)_{i,j}$ with minimal $j - i$, that is, the position furthest to the southwest direction. The submatrices $S_Q = \text{SW}(Q_1, i, j)$ and $S_M = \text{SW}(M, i, j)$ (with the notation of Equation (8)) have the same rank and are equal except for the northeast entry. We must have $j < i^{\rho_1}$ and therefore the first row of $S_Q$ is the zero vector. The first row of $S_M$ is therefore of the form $(0, \ldots, 0, c)$, $c \neq 0$. By equality of rank, it has to be a linear combination of other rows of $S_Q$. But we must have $j^{\rho_1} - 1 < i$, so that the last column of $S_Q$ is the zero vector, a contradiction.

Assume that $M \in M_{Q_1}$ is Bruhat equivalent to $\rho_0$. If $M$ is not of the form $Q_1 + tX$, take $(i,j)$ to be a position which violates that form. We cover the entries in a convenient order: assume $(i,j)$ to be the first violation in this order. The cases $i > i_1$ and $j < j_0$ are just as in the $\rho_1$ case, discussed in the previous paragraph. The case $(i,j) = (i_1, j_0)$ is easily ruled out. The case $i = i_1$, $j_0 < j < j_1$ is divided in two subcases: if $j^{\rho_1} - 1 > i_1$ then the position is zero and therefore $M_{i,j} = (Q_1)_{i,j} = 0$, so there is no violation. Otherwise, notice that the ranks of $\text{SW}(Q_1, i_0, j - 1)$ and of $\text{SW}(Q_1, i_0, j)$ are equal and that therefore the $j$-th column of $\text{SW}(M, i_0, j)$ is a linear combination of previous columns. Notice also that for $j' < j$, $j' \neq j_0$ we have $M_{i_0,j'} = 0$: for $j' < j_0$ this follows from the previous case; for $j' > j_0$ this follows from $M \in M_{Q_1}$. In particular, $M_{i_0,j} = 0$. Furthermore, for $j' < j$, $j' \neq j_0$, we have $M_{i_0,j} = 0$, by the minimality hypothesis. Thus, the linear combination of columns $1$ to $j - 1$ of $\text{SW}(M, i_0, j)$ which produces column $j$ has zero coefficient for $j_0$, implying $M_{i_1,j} = 0$, as desired. The case $j = j_0$, $i_0 < i < i_1$ is similar. The cases $i = i_1$, $j \geq j_1$ and $j = j_0$, $i \leq i_0$ are trivial. The cases $i_0 < i < i_1$, $j_0 < j < j_1$ are handled similarly to the $\rho_1$ case, but with
\( \rho_0 \) instead. The cases \( i = i_0, j_0 < j \leq j_1 \) and \( j = j_1, i_0 \leq i < i_1 \) are trivial. The case \( j > j_1, i_0 < i < i_1 \) is also similar, but for clarity we describe it. The rank of \( SW(M, i, j) \) equals that of \( SW(M, i, j - 1) \) and therefore the \( j \)-th column of \( SW(M, i, j) \) is a linear combination of the first \( j - 1 \) columns. But we have \( M_{i,j'} = 0 \) for all \( j' < j \), implying \( M_{i,j} = 0 \). The case \( i < i_0, j_0 < j < j_1 \) is similar. The final case is \( i < i_0, j > j_1 \): this is again similar to the \( \rho_1 \) case, completing the proof. \( \Box \)

Below we will make heavy use of notation and results from \([15]\); we briefly recall some basic facts. Given \( \sigma \in S_{n+1} \), let \( \text{Bru}_\sigma \subset SO_{n+1} \) be the set of matrices \( Q \in SO_{n+1} \) which are Bruhat equivalent to \( \sigma \). For \( \Pi : \text{Spin}_{n+1} \to SO_{n+1} \), the inverse image \( \Pi^{-1}[\text{Bru}_\sigma] \) is also called \( \text{Bru}_\sigma \subset \text{Spin}_{n+1} \). We have \( \text{Bru}_\sigma \subset \text{Bru}_\rho \) if and only if \( \sigma \leq \rho \) (in the strong Bruhat order). The connected components of \( \text{Bru}_\sigma \subset \text{Spin}_{n+1} \) are all contractible submanifolds of dimension \( inv(\sigma) \).

Recall from Equation (5) that if \( \sigma \in S_{n+1} \) then \( \sigma \in \tilde{B}_{n+1}^+ \); also, \( \text{Quat}_{n+1} \subset \tilde{B}_{n+1}^+ \) is a normal subgroup of order \( 2^{n+1} \), defined in Equation (2). Each element \( q\tilde{\sigma}, q \in \text{Quat}_{n+1} \), belongs to a different connected component of \( \text{Bru}_\sigma \), called \( \text{Bru}_{q\tilde{\sigma}} \), so that

\[
\text{Bru}_\sigma = \bigsqcup_{q \in \text{Quat}_{n+1}} \text{Bru}_{q\tilde{\sigma}}, \quad q\tilde{\sigma} \in \text{Bru}_{q\tilde{\sigma}},
\]

as in Equation (3). The following lemma discusses some special cases of the relation \( \text{Bru}_{z_1} \subset \text{Bru}_{z_0}, z_0, z_1 \in \tilde{B}_{n+1}^+ \).

**Lemma 3.5.** Consider \( \rho_1 < \rho_0, \rho_1 = \rho_a \rho_b, \rho_0 = \rho_a \hat{a}_i \rho_b, \) \( \text{inv}(\rho_1) = \text{inv}(\rho_a) + \text{inv}(\rho_b) \). Consider \( z_1 = q\tilde{\rho}_1 = q\hat{a}_i \tilde{\rho}_b \in \tilde{B}_{n+1}^+, q \in \text{Quat}_{n+1} \). Set \( z_0^+ = q\tilde{\rho}_a \tilde{\rho}_b, z_0^- = q\tilde{\rho}_a \hat{a}_i \tilde{\rho}_b \). Then there exists a tubular neighborhood \( \mathcal{U}_{z_1} \) of \( \text{Bru}_{z_1} \) such that

\[
\mathcal{U}_{z_1} \cap \text{Bru}_{\rho_0} = \mathcal{U}_{z_1} \cap (\text{Bru}_{z_0^+} \cup \text{Bru}_{z_0^-}).
\]

Moreover, \( N = \text{Bru}_{z_0^+} \cup \text{Bru}_{z_1} \cup \text{Bru}_{z_0^-} \subset \text{Spin}_{n+1} \) is a smooth submanifold of dimension \( \text{inv}(\rho_0) \).

**Remark 3.6.** The open subsets \( \mathcal{U}_z \subset \text{Spin}_{n+1}, z \in \tilde{B}_{n+1}^+ \), have been discussed in Section 4 of \([15]\); Theorem 2 of \([15]\) gives a system of coordinates for \( \mathcal{U}_z \). The set \( \text{Bru}_\eta \subset \text{Spin}_{n+1} \) is an open neighborhood of \( \eta \in \text{Spin}_{n+1} \). For \( z \in \tilde{B}_{n+1}^+ \), the set \( \mathcal{U}_z = z\hat{\eta} \text{Bru}_\eta \subset \text{Spin}_{n+1} \) is an open neighborhood of \( z \) and an open tubular neighborhood of \( \text{Bru}_z \subset \text{Spin}_{n+1} \). \( \diamond \)

**Proof of Lemma 3.5.** Set \( Q_1 = \Pi(z_1) \) (where \( \Pi : \text{Spin}_{n+1} \to SO_{n+1} \)) and consider the affine space \( Q_1 \text{Lo}^1_{n+1} \subset GL_{n+1} \). Define \( Q : Q_1 \text{Lo}^1_{n+1} \to \mathcal{U}_{z_1} \) by \( Q(A) = z \) if and only if there exists \( R \in U_{\text{Spin}_{n+1}^+} \) with \( A = \Pi(z)R \). The map \( Q \) is a
diffeomorphism. Furthermore, if $A$ is Bruhat equivalent to $\rho$ then $Q(A) \in \text{Bru}_\rho$. It therefore follows from Corollary 3.4 that

$$U_{z_1} \cap \text{Bru}_{\rho_1} = Q[U_{p_1} Q_1], \quad U_{z_1} \cap \text{Bru}_{\rho_0} = Q[U_{p_1} (Q_1 + (\mathbb{R} \setminus \{0\}) X)].$$

In particular, $U_{z_1} \cap \text{Bru}_{\rho_1}$ is connected and $U_{z_1} \cap \text{Bru}_{\rho_0}$ has at most two connected components. We present another system of coordinates which identifies the two components. In particular, $\Phi$ restrictions are diffeomorphisms: $\Phi |_{U_{z_1} \cap \text{Bru}_{\rho_1}} = \text{Bru}_{\rho_1}$, $\Phi |_{U_{z_1} \cap \text{Bru}_{\rho_0}} = \text{Bru}_{\rho_0}$. It therefore follows from Corollary 3.4 that $Q(U_{z_1} \cap \text{Bru}_{\rho_0}) \cap Q(U_{z_1} \cap \text{Bru}_{\rho_1}) = \emptyset$.

Let $D = \text{Bru}_{q\rho_a} \times (-\pi, \pi) \times \text{Bru}_{\rho_b}$. Consider the smooth map $\Phi : D \to \text{Spin}_{n+1}$, $\Phi(z_0, \theta, z_b) = z_0 \alpha_1(\theta) z_b$. Consider the following subsets of the domain $D$: $D_- = \text{Bru}_{q\rho_a} \times (-\pi, 0) \times \text{Bru}_{\rho_b}$, $D_0 = \text{Bru}_{q\rho_a} \times \{0\} \times \text{Bru}_{\rho_b}$, $D_+ = \text{Bru}_{q\rho_a} \times (0, \pi) \times \text{Bru}_{\rho_b}$. It follows from Theorem 1 in [13] that the following restrictions are diffeomorphisms: $\Phi |_{D_-} : D_- \to \text{Bru}_{\rho_0}$, $\Phi |_{D_0} : D_0 \to \text{Bru}_{z_1}$, $\Phi |_{D_+} : D_+ \to \text{Bru}_{z_0}$, completing the proof. \qed

4 Partial orders

Recall that $W_n$ is the set of finite words in the alphabet $S_{n+1} \setminus \{e\}$. The set $W_n$ plays a very important part in our discussion. In this section we discuss a few partial orders in $W_n$.

A partial order $\preceq$ on $W_n$ is defined and discussed in [14]; we recall the definition. For $\sigma \in S_{n+1} \setminus \{e\}$, take $\sigma = q\rho$ and consider $z_0 = q\rho_a \in \text{Bru}_{\rho_a}$. Consider the following subsets of the domain $D$: $D_- = \text{Bru}_{q\rho_a} \times (-\pi, 0) \times \text{Bru}_{\rho_b}$, $D_0 = \text{Bru}_{q\rho_a} \times \{0\} \times \text{Bru}_{\rho_b}$, $D_+ = \text{Bru}_{q\rho_a} \times (0, \pi) \times \text{Bru}_{\rho_b}$. The set $\text{Bru}_{\rho_0}$ is connected and $\text{Bru}_{\rho_0}$ has at most two connected components. We present another system of coordinates which identifies the two connected components of $U_{q\rho_0}$ as being contained in $\text{Bru}_{z_0 \pm}$. Define the open neighborhood $U_{z_1} \subset \text{Spin}_{n+1}$ as in Remark 3.6. Define $z_1 \in \text{Bru}_{\rho}$ by $z_1 = q\rho_a \eta$, $z_1 = q\rho_b$. Consider a short convex curve $\Gamma_1 : [-1, +1] \to U_{z_0} \subset \text{Spin}_{n+1}$ with

$$\Gamma_1(0) = z_0, \quad t \in [-1, 0) \to \Gamma_1(t) \in \text{Bru}_{z_1}, \quad t \in (0, +1] \to \Gamma_1(t) \in \text{Bru}_{z_1}.$$

Notice that the itinerary of $\Gamma_1$ is $\sigma$. We have $w \preceq \sigma$ if and only if there exists a convex curve $\Gamma_0 : [-1, +1] \to U_{z_0} \subset \text{Spin}_{n+1}$ with $\Gamma_0(-1) = \Gamma_1(-1)$, $\Gamma_0(+1) = \Gamma_1(+1)$ and itinerary $w$. It is not hard to verify that the above definition does not depend on the choice of $\Gamma_1$. For $w_1 = \sigma_1 \cdots \sigma_\ell$, write $w_0 \preceq w_1$ if there exist words $w_{0,1}, \ldots, w_{0,\ell}$ with $w_0 = w_{0,1} \cdots w_{0,\ell}$ and $w_{0,i} \preceq \sigma_i$ (for all $i$). We prove in [14] that

$$L_n[w_1] \cap \bar{L}_n[w_0] \neq \emptyset \quad \Rightarrow \quad w_0 \preceq w_1.$$ (9)

This implication also holds for topologies which we are not considering in the present paper, such as $H^1$. Some basic properties of the partial order $\preceq$ are proved in Theorem 3 of [14].

Unfortunately, the partial order $\preceq$ is complicated: there are basic open questions concerning it, including a conjecture by Shapiro and Shapiro [29, 30]. This conjecture is essentially equivalent to Conjecture 2.2 of [14]. In order to state
it, we need to introduce the concept of multiplicity of a word. Recall that the multiplicity of a word $w \in S_{n+1}$ is a vector \( \text{mult}(w) \in \mathbb{N}^n \), defined in Equation (6): for $w = \sigma_1 \cdots \sigma_r = (\sigma_1, \ldots, \sigma_r)$, define $\text{mult}(w) = \text{mult}(\sigma_1) + \cdots + \text{mult}(\sigma_r)$. Conjecture 2.2 of [14] asks: does $w \leq \sigma$ imply $\text{mult}(w) \leq \text{mult}(\sigma)$? Theorem 4 in [14] proves a related result. See also [27] and [26] for other new related results. The original conjecture, however, remains open as of this writing.

In the present paper, we prefer therefore to work with a more manageable partial order $\sqsubseteq$ (also on $W_n$). For $w \in W_n$ and $\sigma \in S_{n+1}$, we define

\[(w \sqsubseteq \sigma) \iff (w \neq ()) \land (\text{mult}(w) \leq \text{mult}(\sigma)) \land (w = \hat{\sigma}) ;\]

here () denotes the empty word. For words $w_0$ and $w_1 = (\sigma_1, \ldots, \sigma_r) \in W_n$, we have $w_0 \sqsubseteq w_1$ if and only if there exists nonempty words $w_{00}, \ldots, w_{0k}$ such that $w_0$ equals the concatenation $w_{00} \cdots w_{0k}$ and $w_{0j} \sqsubseteq \sigma_j$ for every $j$, $1 \leq j \leq r$. It follows from Theorems 3 and 4 from [14] that, if we work in $H^r$ and $r > ((n+1)/2)^2$ we have an implication similar to the one in Equation (9):

$$
\mathcal{L}_n[w_1] \cap \mathcal{L}_n[w_0] \neq \emptyset \implies w_0 \sqsubseteq w_1 .
$$

Notice that given $w_0 \in W_n$ both sets

$$
\{w \in W_n, w \sqsubseteq w_0\}, \quad \{w \in W_n, w_0 \sqsubseteq w\}
$$

are finite. Indeed, $w_0 \sqsubseteq w_1$ implies $\text{mult}(w_0) \leq \text{mult}(w_1)$ and $\ell(w_0) \geq \ell(w_1)$ (where $\ell(w)$ is the length of the word $w$). A similar result for $\leq$ follows from [24].

It is natural to ask at this point how the partial orders $\leq$ and $\sqsubseteq$ are related. We first consider the implication $(w_0 \leq w_1) \implies (w_0 \sqsubseteq w_1)$. We know that $w \leq \sigma$ implies $w \neq ()$ and $\hat{\sigma} = 1$. We do not know, however, whether $w \leq \sigma$ implies $\text{mult}(w) \leq \text{mult}(\sigma)$: this is essentially Conjecture 1.2 in [14]. It is relatedly also an open problem whether the implication $(w_0 \leq w_1) \implies (w_0 \sqsubseteq w_1)$ holds. The next example shows that the converse implication is false.

**Example 4.1.** Let $n = 4$, $\sigma_0 = aba$ and $\sigma_1 = \eta = abacabadeba$. It follows from Example 3.7 in [15] that $\hat{\sigma}_0 = \hat{\sigma}_1 = -1$ and a simple computation then verifies that $(\sigma_0) \sqsubseteq (\sigma_1)$. We now prove, however, that $\sigma_0 \not\sqsubseteq \sigma_1$ and therefore, from Equation (9),

$$
\mathcal{L}_4[(\sigma_1)] \cap \mathcal{L}_4[(\sigma_0)] = \emptyset.
$$

As in [14], we work in the nilpotent group $L_0^5$ and its subsets $\text{Pos}_{\eta}, \text{Neg}_{\eta} \subset L_0^5$. The only point in $\text{Pos}_{\eta} \cap \text{Neg}_{\eta}$ is the identity. Consider a locally convex curve $\Gamma_L : [-1,1] \rightarrow L_0^5$ with $\Gamma_L(-1) \in \text{Neg}_{\eta}$ and $\Gamma_L(1) \in \text{Pos}_{\eta}$. Set $\Gamma(t) = qQ(\Gamma_L(t))$, $q \in \text{Quat}_{n+1}$. If $\Gamma_L(t) = I$ for some $t$ then $\text{iti}((I)) = \eta$. Otherwise, $\Gamma_L$ must cross $\partial \text{Neg}_{\eta}$ and $\partial \text{Pos}_{\eta}$ at two distinct times $t_- < t_+$, $t_- < t_+ \in \text{sing}(\Gamma)$: thus $\text{iti}((I)) \neq \sigma_0$.

In the following examples we also have $(\sigma_0) \sqsubseteq (\sigma_1)$ and $(\sigma_0) \not\sqsubseteq (\sigma_1)$:
1. \( n = 6, \sigma_0 = a_1a_2a_1a_4a_5a_4, \sigma_1 = \eta; \)
2. \( n = 7, \sigma_0 = a_1a_3a_5a_7, \sigma_1 = \eta; \)
3. \( n = 8, \sigma_0 = a_1a_2a_1, \sigma_1 = a_1a_2a_1a_4a_5a_4a_7a_8a_7. \)

In all cases the verification of the \( \sqsubseteq \) condition is an easy computation. For the first two items, the proof of the \( \preceq \) claim is similar to the one given above; the third item requires a modification of the argument (which we neither present nor need in this paper).

\[ \text{\textcircled{}} \]

**Remark 4.2.** Neither partial order \( \preceq \) nor \( \sqsubseteq \) respects dimension of words (defined in Equation (14)). For instance, we have \( [ac]b[ac] \preceq [acb], [ac]b[ac] \sqsubseteq [acb] \) and \( \dim([ac]b[ac]) = \dim([acb]) = 2 \): this is the example discussed in detail in Section 9 in [14].

\[ \text{\textcircled{}} \]

## 5 Lower and upper sets

A subset \( I \subseteq W_n \) is a lower set (for \( \sqsubseteq \)) if, for all \( w_1 \in I \) and \( w_0 \in W_n \), \( w_0 \sqsubseteq w_1 \) implies \( w_0 \in I \). In particular, \( \emptyset \) and \( W_n \) are lower sets. If \( I \) is a lower set then it follows from Equation (10) that

\[ \mathcal{L}_n[I] = \bigsqcup_{w \in I} \mathcal{L}_n[w] \subseteq \mathcal{L}_n \]

is an open subset. Given \( w_0 \in W_n \), let \( I(w_0) = \{ w \in W_n, w \sqsubseteq w_0 \} \) and \( I^*(w_0) = I(w_0) \setminus \{ w_0 \} \): both are lower sets. The map \( \phi : \mathbb{D}^k \to \mathcal{L}_n \) transversal section to \( \mathcal{L}_n[\sigma] \) constructed in Lemma 7.1 of [14] is of the form \( \phi : \mathbb{D}^k \to \mathcal{L}_n[I(\sigma)] \subseteq \mathcal{L}_n \).

Figure 3 is a diagram of

\[ I([aba]) = \{ aa, abab, bb, baba, [ba]a, a[ba], [ab]b, b[ab], [aba] \}; \]

compare with Figure [1]

![Figure 3: The lower set I([aba]).](image)
As another example of a lower set, let \( I_{(\omega)} \) be the set of words of dimension 0, i.e., words whose letters are generators \( a_k \). (The subscript is an ordinal, a relic of notation used in [13], starting in Section 14. The notation shall not be defined or needed in its general form.) The set \( L_n[I_{(\omega)}] \) is a disjoint union of contractible open sets \( L_n[w], w \in I_{(\omega)} \).

Similarly, \( U \subseteq W_n \) is an upper set (for \( \sqsubseteq \)) if, for all \( w_0 \in U \) and \( w_1 \in W_n \), \( w_0 \sqsubseteq w_1 \) implies \( w_1 \in U \). If \( U \) is an upper set then

\[
L_n[U] = \bigcup_{w \in U} L_n[w] \subseteq L_n
\]

is a closed subset (again from Equation (10)).

A few examples are in order.

**Example 5.1.** The set \( I_Y \subseteq W_n \) of words containing at least one letter belonging to \( Y = S_{n+1} \setminus S_{PA} \) is also a lower set. Indeed, consider \( \sigma_0 \in Y \) and a word \( w_1 \subseteq \sigma_0 \), \( w_1 = \sigma_1, 1 \cdots \sigma_1, \ell \). From Lemma 2.2 \( \hat{\sigma}_0 \notin Z(\text{Quat}_{n+1}) \). We thus have \( \hat{\sigma}_1, 1 \cdots \hat{\sigma}_1, \ell = \hat{\sigma}_0 \notin Z(\text{Quat}_{n+1}) \) and therefore \( \hat{\sigma}_1, k \notin Z(\text{Quat}_{n+1}) \) for some \( k \): again from Lemma 2.2 we have \( \sigma_1, k \in Y \), as desired. The open subset \( Y_n = L_n[I_Y] \subseteq L_n \) is studied in Theorem 3.

**Example 5.2.** Recall from Remark 2.8 that for \( n \geq 2 \) we have

\[
Y_2 = \{a_1, a'_1 = a_2 a_1, a_2, a'_2 = a_1 a_2, \ldots, a_{n-1}, a'_n-1 = a_{n-2} a_{n-1}, a_n, a'_n = a_{n-1} a_n\}.
\]

The set \( I_{Y_2} \subseteq W_n \) of words containing at least one letter belonging to \( Y_2 \) is also a lower set. Indeed, if \( \sigma \in Y_2 \), \( \text{inv}(\sigma) > 1 \) and \( w \sqsubseteq \sigma \) then either \( w = \sigma \) or \( w \) is a word of dimension 0 and length 1 or 3. The same conclusion holds assuming instead \( w \preceq \sigma \), implying the \( I_{Y_2} \subseteq W_n \) is also a lower subset for \( \preceq \). It follows that \( L_n[I_{Y_2}] \subseteq L_n \) is an open subset. The lower set \( I_{Y_2} \subseteq I_Y \) is considered again in Proposition 13.2.

**Example 5.3.** Let \( I_{[0]} \subseteq W_n \) be the set of words containing at least one letter of dimension 0. As in the previous examples, \( I_{[0]} \) is a lower set for both \( \preceq \) and \( \sqsubseteq \). The lower set \( I_{[0]} \subseteq I_{Y_2} \) is considered again in Proposition 11.3.

**Example 5.4.** The following are examples of upper sets:

- \( U_{2,2} = \{[aba]\} \subseteq W_2 \)
- \( U_{2,3} = \{[aba], [bacb], [bcb]\} \subseteq W_3 \)
- \( \{[abc], [ac], [cba]\} \subseteq W_3 \)
- \( U_{4,4} = \{[34521], [32541], [52341], [52143], [54123]\} \subseteq W_4 \)

The example \( U_{2,2} \) is easy: there is no letter in \( S_3 \) of greater dimension than \( [aba] \). For \( U_{2,3} \), verify that there are no other letters \( \sigma \in S_4 \) with \( \hat{\sigma} = \text{hat}([aba]) = -1 \) (notice also that \( [aba] \sqsubseteq [bacb], [bcb] \sqsubseteq [bacb] \)). Also, the only letters \( \sigma \in S_4 \) with \( \hat{\sigma} = \hat{a} \hat{c} \) are \([abc],[ac]\) and \([cba]\) (notice that \([ac] \sqsubseteq [abc], [ac] \sqsubseteq [cba]\)). The only letters \( \sigma \in S_5 \setminus \{e\} \) with \( \hat{\sigma} = 1 \) are the elements of \( U_{4,4} \).
Example 5.5. Consider $n = 4$ and the set

$$U_{2,4} = \{ [aba], [bacb], [bcb], [cbdc], [cdc] \} \subset W_4.$$ 

Notice that these are the letters in Figure 2. The set $U_{2,4}$ is not an upper set for $\sqsubseteq$: we have $\sigma \sqsubseteq \eta$ for all $\sigma \in U_{2,4}$. It is, however, a double ($\sqsubseteq, \preceq$) upper set, meaning that

$$\forall w_0 \in U_{2,4}, \forall w_1 \in W_4, ((w_0 \sqsubseteq w_1) \land (w_0 \preceq w_1)) \implies w_1 \in U_{2,4};$$

this follows from Example 4.1. This is sufficient to prove (with Equations (9) and (10)) that the subset $\mathcal{L}_4[U_{2,4}] \subset \mathcal{L}_4$ is closed. 

6 Construction of $D_n$ and Theorems 1 and 4

We now describe the construction of the CW complex $D_n$. In a nutshell, we proceed as follows. For each word $w$, there is a cell $c_w$ of dimension $\dim(w)$. The CW complex $D_n$ can be constructed in increasing order (with respect to $\sqsubseteq$ in $W_n$). For each new word $w$, the glueing map $g_w$ for the new cell $c_w$ takes $S^{d-1}$ (for $d = \dim(w)$) to previously constructed cells $c_{\tilde{w}}$, $\tilde{w} \sqsubseteq w$. The glueing map is determined by studying a transversal section to $\mathcal{L}_n[w] \subset \mathcal{L}_n$, as constructed in Section 7 of [14]. Notice that the image of such a transversal section is contained in the union of the strata $\mathcal{L}_n[\tilde{w}]$, $\tilde{w} \sqsubseteq w$.

More precisely, consider a lower set $I \subseteq W_n$ (for the partial order $\sqsubseteq$). Let

$$\mathcal{L}_n[I] = \bigcup_{w \in I} \mathcal{L}_n[w] \subseteq \mathcal{L}_n.$$ 

The set $\mathcal{L}_n[I] \subset \mathcal{L}_n$ is an open subset and therefore a Hilbert manifold (recall that we work in the Hilbert space $H^r$ of curves). For each lower set $I$ we have a CW complex $D_n[I]$ and a homotopy equivalence $\mathcal{D}_n[I] \to \mathcal{L}_n[I]$. Furthermore, we may assume the homotopy equivalence to be an inclusion $\mathcal{D}_n[I] \subset \mathcal{L}_n[I]$. Indeed, the induction step is as follows. Let $\tilde{I} \subset I \subseteq W_n$ be lower sets with $I = \tilde{I} \cup \{ w_0 \}$. Assume that a CW complex $\mathcal{D}_n[I] \subset \mathcal{L}_n[I]$ has been previously constructed and that the inclusion is a homotopy equivalence. We show in Lemma 6.3 how to add a cell $c_{w_0}$ to $\mathcal{D}_n[I]$ to obtain the desired CW complex $\mathcal{D}_n[I] \subset \mathcal{L}_n[I]$, the inclusion also being a homotopy equivalence. It then suffices to take the union of a chain of CW complexes to obtain $\mathcal{D}_n$ (see Lemma 6.2).

We sum up our construction in a theorem.

Theorem 4. There exists a CW complex $\mathcal{D}_n$ and a continuous map $i : \mathcal{D}_n \to \mathcal{L}_n$ with the following properties:
1. For each word \( w \in \mathbb{W}_n \) there exists a cell \( c_w \) of \( \mathcal{D}_n \) of dimension \( \dim(w) \).

2. For every lower set \( \mathcal{I} \subseteq \mathbb{W}_n \) (with respect to the partial order \( \sqsubseteq \)) the union of the cells \( c_w, w \in \mathcal{I}, \) is a subcomplex \( \mathcal{D}_{n}[\mathcal{I}] \subseteq \mathcal{D}_n \).

3. For every lower set \( \mathcal{I} \subseteq \mathbb{W}_n \) the union of the strata \( \mathcal{L}_n[w], w \in \mathcal{I}, \) is an open subset \( \mathcal{L}_{n}[\mathcal{I}] \subseteq \mathcal{L}_n \).

4. For every lower set \( \mathcal{I} \subseteq \mathbb{W}_n \) the restriction \( i|_{\mathcal{D}_{n}[\mathcal{I}]} : \mathcal{D}_{n}[\mathcal{I}] \to \mathcal{L}_{n}[\mathcal{I}] \) is a homotopy equivalence.

Clearly, Theorem \([1]\) follows from Theorem \([4]\). We remind the reader that a similar but simpler construction (for a finite stratification of another space) is presented in \([4]\) (the proof of that Theorem 2 is particularly relevant).

We state a few definitions aimed at the proof of Theorem \([4]\). Here \( \mathbb{D}^{k} \) is the closed disk of dimension \( k \). Let \( \mathcal{I} \subseteq \mathbb{W}_n \) be a lower set (for \( \sqsubseteq \)). A valid (CW) complex \( \mathcal{D}_{n}[\mathcal{I}] \) (for \( \mathcal{I} \)) has the following ingredients and properties:

(i) The CW complex \( \mathcal{D}_{n}[\mathcal{I}] \) has one cell of dimension \( \dim(w) \) for each \( w \in \mathcal{I} \).

(ii) For each \( w \in \mathcal{I} \), we have a continuous map \( c_w : \mathbb{D}_{\dim(w)} \to \mathcal{L}_{n}[\mathcal{I}_w] \) (which will also be called the \( w \) cell).

(iii) The maps \( c_w \) are compatible: if \( i_{w_0}(p_0) = i_{w_1}(p_1) \) then \( c_{w_0}(p_0) = c_{w_1}(p_1) \).

(iv) For any \( w \in \mathcal{I} \), the cell \( c_w \) is a topological embedding in the interior of \( \mathbb{D}_{\dim(w)} \), smooth in the ball of radius \( \frac{1}{2} \), and intersects \( \mathcal{L}_n[w] \) transversally at \( c_w(0) \in \mathcal{L}_n[w] \).

(v) If \( s \in \mathbb{D}_{\dim(w)}, s \neq 0 \), then \( c_w(s) \in \mathcal{L}_n[\mathcal{I}^*(w)] \) (and therefore \( c_w(s) \notin \mathcal{L}_n[w] \)).

We abuse notation and identify a valid complex with the corresponding family of cells and write \( \mathcal{D}_{n}[\mathcal{I}] \) = \( (c_w)_{w \in \mathcal{I}} \) (notice that this family yields all the necessary information). Given a valid complex \( \mathcal{D}_{n}[\mathcal{I}] \) = \( (c_w)_{w \in \mathcal{I}} \) and a lower set \( \tilde{\mathcal{I}} \subseteq \mathcal{I} \), the subcomplex \( (c_w)_{w \in \tilde{\mathcal{I}}} \) is also valid (for \( \tilde{\mathcal{I}} \)). We call this subcomplex \( \mathcal{D}_{n}[\tilde{\mathcal{I}}] \) the restriction of \( \mathcal{D}_{n}[\mathcal{I}] \) to \( \tilde{\mathcal{I}} \).
Remark 6.1. It follows from (iv) and (v) that the map \( c_w \) intersects a tubular neighborhood \( \mathcal{A}_w \supseteq \mathcal{L}_n[w] \) transversally around \( c_w(0) \in \mathcal{L}_n[w] \). Such a tubular neighborhood \( \mathcal{A}_w \) is constructed in the proof of Theorem 2 in [13], Section 6. A map \( \hat{F}_w : \mathcal{A}_w \to \mathbb{D}^{\dim(w)} \) with \( \hat{F}_w^{-1}[0] = \mathcal{L}_n[w] \) is also constructed. The map \( \hat{F}_w \circ c_w \) is then a homeomorphism from an open neighborhood of \( 0 \in \mathbb{D}^{\dim(w)} \) to an open neighborhood of \( 0 \in \mathbb{D}^{\dim(w)} \). The map \( \hat{F}_w \circ c_w \) also yields a transversal orientation to \( \mathcal{L}_n[w] \subset \mathcal{L}_n \) (the preimage orientation for \( \hat{F}_w \); see [13] after Lemma 7.1). We assume that the map \( c_w \) respects this orientation, that is, that the composition \( \hat{F}_w \circ c_w \) respects orientation. Lower dimensional cases shall be explicitly presented in examples.

A valid complex \( \mathcal{D}_n[I] \) is good if for every lower set \( \tilde{I} \subseteq I \) the map \( c_{\tilde{I}} : \mathcal{D}_n[\tilde{I}] \to \mathcal{L}_n[\tilde{I}] \) is a weak homotopy equivalence. Clearly, a restriction of a good complex is also a good complex.

Let \( J \) be a totally ordered set. Let \( (I_j)_{j \in J} \) be a family of lower sets such that \( j_0 < j_1 \) implies \( I_{j_0} \subseteq I_{j_1} \). Let \( I = \bigcup_{j \in J} I_j \), which is therefore also a lower set. For each \( j \in J \), let \( \mathcal{D}_n[I_j] \) be a valid complex. Assume that if \( j_0 < j_1 \) then \( \mathcal{D}_n[I_{j_0}] \) is the restriction of \( \mathcal{D}_n[I_{j_1}] \) to \( I_{j_0} \). Then the union \( \mathcal{D}_n[I] \) of all the valid complexes \( \mathcal{D}_n[I_j] \) (for all \( j \in J \)) is a valid complex for \( I \). A family \( (\mathcal{D}_n[I_j])_{j \in J} \) of valid complexes is a chain if it satisfies the conditions above.

Lemma 6.2. Let \( (\mathcal{D}[I_j])_{j \in J} \) be a chain of good complexes. Let \( I = \bigcup_{j \in J} I_j \) and \( \mathcal{D}_n[I] \) be the union of the complexes \( \mathcal{D}_n[I_j] \). Then \( \mathcal{D}_n[I] \) is a good complex.

Proof. Let \( \alpha : S^k \to \mathcal{D}_n[I] \) be a continuous map which is homotopically trivial in \( \mathcal{L}_n[I] \), i.e., there exists a map \( A_0 : \mathbb{D}^{k+1} \to \mathcal{L}_n[I] \), \( A_0|_{S^k} = c_I \circ \alpha \). By compactness, there exists \( j \in J \) such that the image of \( A_0 \) is contained in \( \mathcal{L}_n[I_j] \) so that we may write \( A_0 : \mathbb{D}^{k+1} \to \mathcal{L}_n[I_j] \). Since \( \mathcal{D}_n[I_j] \) is good, there exists \( A_1 : \mathbb{D}^{k+1} \to \mathcal{D}_n[I_j] \) such that \( A_1|_{S^k} = \alpha \).

Let \( \alpha_0 : S^k \to \mathcal{L}_n[I] \) be a continuous map. By compactness, there exists \( j \in J \) such that the image of \( \alpha_0 \) is contained in \( \mathcal{L}_n[I_j] \). Since \( \mathcal{D}_n[I_j] \) is good, there exists \( \alpha_1 : S^k \to \mathcal{D}_n[I_j] \subset \mathcal{D}_n[I] \) such that \( c_{I_j} \circ \alpha_1 \) is homotopic to \( \alpha_0 \) in \( \mathcal{L}_n[I_j] \) (and therefore also in \( \mathcal{L}_n[I] \)).

This completes the proof that \( c_I : \mathcal{D}_n[I] \to \mathcal{L}_n[I] \) is a weak homotopy equivalence. The proof for \( I \subseteq \tilde{I} \) is similar. □

Lemma 6.3. Let \( \tilde{I} \subset I \subseteq W_n \) be lower sets with \( I \setminus \tilde{I} = \{w_0\} \). Let \( \mathcal{D}_n[\tilde{I}] \) be a good complex (for \( \tilde{I} \)). Then this complex can be extended to a valid complex \( \mathcal{D}_n[I] \) (for \( I \)). Moreover, all such valid complexes are good.

The proof of this lemma is the longest and most technical part of the present section: we postpone it for a while. Before proving Lemma 6.3 we present the main conclusion for this section.
**Lemma 6.4.** For any lower set \( I \subseteq W_n \) there exists a valid complex \( D_n[I] \subseteq L_n[I] \). All such valid complexes are good.

**Proof.** The partial order \( \sqsubseteq \) is well founded and can therefore be extended to a (total) well-ordering. Let \( \gamma^+ \) be the set of all initial segments of \( I \) under this well-ordering. This defines a chain of lower sets \( (I_{(\beta)})_{\beta \in \gamma^+} \) with a top element \( I_{(\gamma)} = I \). From now on in this proof we identify such \( \beta, \gamma \) and \( \gamma^+ \) with ordinal numbers; in particular, \( \gamma^+ = \gamma + 1 \).

We prove by transfinite induction on \( \alpha \leq \gamma \) that there exists a chain of good complexes \( (D_n[I_{(\beta)}])_{\beta < \alpha} \). For \( \alpha = 0 \) there is nothing to do. For \( \alpha = \bar{\alpha} + 1 \) (a successor ordinal), apply Lemma 6.3. For \( \alpha \) a limit ordinal apply Lemma 6.2.

The fact that all valid complexes are good is likewise proved by transfinite induction. We again use Lemma 6.3 for successor ordinals and Lemma 6.2 for limit ordinals. \( \square \)

Theorems 1 and 4 are now easy.

**Proof of Theorems 1 and 4** All claims in the statement of Theorem 1 follow from Lemma 6.4, together with the definitions of valid and good complexes. Theorem 4 is a direct consequence of Theorem 1. \( \square \)

**Example 6.5.** Following the above construction for \( I_{[aba]} \) we have the cell shown in Figure 4: the transversal map \( \tilde{c} \) in the proof above is shown in Figure 1. Notice that \( c_{[bcb]} \) is very similar. The immediate predecessors of \( [aba] \) are \( [ba]a, b[ab], [ab]b \) and \( a[ba] \) (see Figure 3) and we have

\[
\partial[aba] = [ba]a + b[ab] - [ab]b - a[ba], \quad \partial[bcb] = [cb]b + c[bc] - [bc]c - b[cb].
\]

Here we use homological notation. The cells are oriented as in Remark 6.1. \( \diamond \)

![Figure 4: The cells \( c_{[aba]} \) and \( c_{[bcb]} \).](image-url)
Remark 6.6. As we shall see, for cells of very low dimension the complex can be taken to be polyhedric; it is not clear whether this is true in general. We shall not attempt to clarify this and related issues here.

Proof of Lemma 6.3. Assume \( \mathcal{D}_n[I] \) given (and good): we construct the cell \( c_{w_0} \) so that \( \mathcal{D}_n[I] \) is also good. Let \( k = \dim(w_0) \). For a small ball \( B \subset \mathbb{R}^k \) around the origin, construct as in Section 7 of [14] (after Lemma 7.1) a smooth map \( \phi : B \to L_n[I] \subseteq L_n \) with \( \phi(0) = \Gamma_0 \in L_n[w_0] \) and transversal to \( L_n[w_0] \subset L_n \) at this point. By taking \( r > 0 \) sufficiently small, the image under \( \phi \) of a ball \( B(2r) \subset B \) of radius \( 2r \) around the origin satisfies the following condition: \( s \in B(2r) \) and \( s \neq 0 \) imply \( \phi(s) \in L_n[I^*_{w_0}] \subseteq L_n[I] \).

Let \( S(r) \subset B(2r) \) be the sphere of radius \( r \) around the origin so that \( \phi|_{S(r)} : S(r) \to L_n[I^*_{w_0}] \). Since \( \mathcal{D}_n[I] \) is good, the map \( c_{I_{w_0}} : \mathcal{D}_n[I^*_{w_0}] \to L_n[I^*_{w_0}] \) is a weak homotopy equivalence. There exists therefore a map \( g_{w_0} : S^{k-1} \to \mathcal{D}_n[I^*_{w_0}] \) such that \( c_{I_{w_0}} \circ g_{w_0} \) is homotopic (in \( L_n[I^*_{w_0}] \)) to \( \phi|_{S(r)} \). In other words, there exists \( c_{w_0} : \mathbb{D}^k \to L_n[I^*_{w_0}] \) coinciding with \( \phi \) in \( B(r) \), assuming values in \( L_n[I^*_{w_0}] \) outside \( 0 \) and with boundary \( c_{I_{w_0}} \circ g_{w_0} \). This completes the construction of a valid complex \( \mathcal{D}_n[I] \) and the proof of the first claim.

We now prove the second claim. More precisely, let \( c_{w_0} \) be such that extending the good complex \( \mathcal{D}_n[I] \) by \( c_{w_0} \) defines a valid complex \( \mathcal{D}_n[I] \) (for \( I \)): we prove that this is also a good complex. In other words, we prove that the map \( c_{I} : \mathcal{D}_n[I] \to L_n[I] \) is a weak homotopy equivalence for any lower set \( I \subseteq I \). Again, we present the proof for \( I = I \); the general case is similar. The construction is similar to that of certain classical results involving CW complexes; here \( L_n[w_0] \subseteq L_n[I] \) is a submanifold of class \( C^{r-1} \). Recall that \( L_n[w_0] \subseteq L_n[I] \) is a contractible closed subset (of \( L_n[I] \)) and a (globally) collared submanifold of codimension \( d = \dim(w_0) \). Indeed, in the proof of Theorem 2 in [14] (Section 6, near Remark 6.8) we constructed a tubular neighborhood of \( \tilde{A}_{w_0} \supset L_n[w_0] \), a projection \( \Pi : \tilde{A}_{w_0} \to L_n[w_0] \) and a map \( \tilde{F} = \tilde{F}_{w_0} : \tilde{A}_{w_0} \to \mathbb{B}^d \) (where \( \mathbb{B}^d \) is the unit open ball) such that \( (\Pi, \tilde{F}) : \tilde{A}_{w_0} \to (L_n(w_0), \mathbb{B}^d) \) is a homeomorphism. By construction, the map \( c_{w_0} \) intersects \( \tilde{A}_{w_0} \supset L_n[w_0] \) transversally so that \( c_{w_0}^{-1}[\tilde{A}_{w_0}] \) is an open neighborhood of \( 0 \in \mathbb{B}^d \). The map \( \tilde{F} \circ c_{w_0} \) (where defined) is a homeomorphism from a neighborhood of 0 to a neighborhood of 0; indeed, in the example constructed in the first part of the proof, it is the multiplication by a positive constant. We may assume that \( c_{w_0} \) is such that \( \Pi(c_{w_0}(p)) = \Gamma_0 \) for \( p \) in an open neighborhood \( B_0 \subset \mathbb{B}^d \), \( 0 \in B_0 \) (this is actually true for the maps constructed in [14]). Let \( b_{B_0} : \mathbb{B}^d \to [0, 1] \) be a bump function with support contained in the neighborhood \( B_0 \) above and constant equal to 1 in a smaller open ball \( B_1 \subset B_0 \), \( 0 \in B_1 \).

Consider a compact manifold \( M_0 \) of dimension \( k \) and continuous map \( \alpha_0 : M_0 \to L_n[I] \). We prove that there exists \( \alpha_1 : M_0 \to \mathcal{D}_n[I] \) such that \( \alpha_0 \) is
homotopic to $c_1 \circ \alpha_1$. First deform $\alpha_0$ to obtain a map $\alpha_{\frac{1}{3}}$ which intersects $\mathcal{L}_n[w_0]$ transversally. By using the contractibility of $\mathcal{L}_n[w_0]$, we may deform $\alpha_{\frac{1}{3}}$ to obtain a map $\alpha_{\frac{2}{3}}$ which intersects a thinner tubular neighborhood contained in $\mathcal{A}_{w_0}$ only along the image of $c_{w_0}$. More precisely, take $\Gamma_0 = c_{w_0}(0) \in \mathcal{L}_n[w_0]$ as a base point; let $H : [0, 1] \times \mathcal{L}_n[w_0] \to \mathcal{L}_n[w_0]$ be a homotopy such that, for all $\Gamma \in \mathcal{L}_n[w_0]$, $H(0, \Gamma) = \Gamma$ and $H(1, \Gamma) = \Gamma_0$. For $s \in \left[\frac{1}{3}, \frac{2}{3}\right]$, we define $\alpha_{s}$ to coincide with $\alpha_{\frac{1}{3}}$ outside $\mathcal{A}_{w_0}$; if $\alpha_{\frac{1}{3}}(p) \in \mathcal{A}_{w_0}$ define $\alpha_{s}(p) \in \mathcal{A}_{w_0}$ to satisfy $\hat{F}_{w_0}(\alpha_s(p)) = \hat{F}_{w_0}(\alpha_{\frac{1}{3}}(p))$ and

$$\Pi(\alpha_s(p)) = H((3s - 1) \text{ bump}(\hat{F}_{w_0}(\alpha_{\frac{1}{3}}(p))), \Pi(\alpha_{\frac{1}{3}}(p))).$$

Thus, the set of points $p \in M_0$ such that $\alpha_{\frac{2}{3}}(p) \in \mathcal{A}_{w_0}$ and $\hat{F}_{w_0}(\alpha_{\frac{2}{3}}(p)) \in B_1$ consists of open tubes taken by $\alpha_{\frac{2}{3}}$ to the image of the cell $c_{w_0}$. By removing these open tubes we obtain $M_1 \subseteq M_0$, a manifold with boundary, also of dimension $k$. Let $D_n[I] = D_n[I] \setminus \omega[I]$ so that both the map $c_1|D_n[I] : D_n[I] \to \mathcal{L}_n[I]$ and the inclusion $D_n[I] \subset D^*_n[I]$ are weak homotopy equivalences (since $D_n[I]$ is assumed to be good). The restriction $\alpha_{\frac{2}{3}}|M_1 : M_1 \to \mathcal{L}_n[I]$ has boundary $\alpha_{\frac{2}{3}}|\partial M_1 = c_1 \circ \beta_1$ for $\beta_1 : \partial M_1 \to D^*_n[I]$. There exists $\beta : M_1 \to D^*_n[I]$ with $\beta_1 = \beta|\partial M_1$ and such that $c_1 \circ \beta : M_1 \to \mathcal{L}_n[I]$ is homotopic to $\alpha_{\frac{2}{3}}|M_1 : M_1 \to \mathcal{L}_n[I]$ (in $\mathcal{L}_n[I]$). Define $\alpha_1$ to coincide with $\beta$ in $M_1$ and such that $c_{w_0} \circ \alpha_1$ coincides with $\alpha_{\frac{2}{3}}$ in the tubes $M_0 \setminus M_1$. This is our desired map.

Conversely, consider a compact manifold with boundary $M$ and its boundary $\partial M = N$. Consider a maps $\alpha_0 : M \to \mathcal{L}_n[I]$ and $\beta_0 : N \to D_n[I]$ such that $c_1 \circ \beta_0 = \alpha_0|N$. We prove that there exists a map $\alpha_1 : M \to D_n[I]$ with $\alpha_1|N = \beta_0$. Furthermore, $\alpha_0$ and $c_1 \circ \alpha_1$ are homotopic in $\mathcal{L}_n[I]$, with the homotopy constant in the boundary. Again, we may assume without loss of generality that $\alpha_0$ is transversal to $\mathcal{A}_{w_0}$. As in the previous paragraph, use the contractibility of $\mathcal{L}_n[w_0]$ to deform $\alpha_0$ in $\mathcal{A}_{w_0}$ towards $c_{w_0}$, thus defining $\alpha_{\frac{1}{2}}$; notice that this keeps the boundary fixed, as required. Again remove tubes around points taken to $c_{w_0}$, thus defining $M_1 \subseteq M$, also a manifold with boundary. By hypothesis, $\alpha_{\frac{1}{2}}$ can be deformed in $M_1$ to obtain $\alpha_1 : M_1 \to D_n[I]$; more precisely, $\alpha_{\frac{1}{2}} : M_1 \to \mathcal{L}_n[I]$ is homotopic to $c_1 \circ \alpha_1$. As in the previous paragraph we fill in the tubes to define $\alpha_1 : M \to D_n[I]$, the desired map.

7 Examples of lower sets

In this section we consider a few simple examples of lower sets and valid complexes. Recall that $I_{(\omega)} \subset W_n$ is the lower set of words of dimension 0. Two other examples will be $I_{(\omega, 2)}$, the lower set of words of dimension at most 1 and $I_{(\omega^2)}$, the lower set of words with all letters of dimension at most 1. We clearly
have $I_{(\omega)} \subset I_{(\omega^2)} \subset I_{(\omega^2)}$: we shall construct the corresponding valid complexes $D_n[I_{(\omega)}] \subset D_n[I_{(\omega^2)}] \subset D_n[I_{(\omega^2)}]$.

For each $w \in I_{(\omega)}$, let $c_w \in L_n[w]$ be an arbitrary curve: we think of $c_w$ as a vertex of the complex $D_n$. In other words, the 0-skeleton of $D_n$ is the infinite countable set of vertices $D_n[I_{(\omega)}] = \{c_w, w \in I_{(\omega)}\}$; the inclusion $D_n[I_{(\omega)}] \subset L_n[I_{(\omega)}]$ is a homotopy equivalence.

A word $w \in I_{(\omega^2)}$ of dimension 1 is of the form $w = w_0[a_k a_l] w_1$ where $w_0, w_1 \in I_{(\omega)} \subset W_n$ are (possibly empty) words of dimension 0 and $k \neq l$ so that $a_k a_l \in S_{n+1}$ is an element of dimension 1 (i.e., $\text{inv}(a_k a_l) = 2$). There are three cases: case (i) is $l = k - 1$, case (ii) is $l = k + 1$ and case (iii) is $l > k + 1$ (if $l < k - 1$ we write $a_l a_k$ instead; notice that in this case the permutations $a_k$ and $a_l$ commute). Recall from Theorem 2 in [14] that in each case the stratum $L_n[w]$ is a hypersurface with an open stratum on either side. From Remark 6.1 there is a transversal orientation to $L_n[w]$. Call the two open strata $L_n[w^+]$ and $L_n[w^-]$ in such a way that the transversal orientation is from $L_n[w^-]$ to $L_n[w^+]$. The words $w^+, w^- \in W_n$ have dimension 0; their values according to case are:

(i) $w^- = w_0 a_l w_1$, $w^+ = w_0 a_k a_l a_k w_1$;

(ii) $w^- = w_0 a_k a_l w_1$, $w^+ = w_0 a_l w_1$;

(iii) $w^- = w_0 a_k a_l w_1$, $w^+ = w_0 a_l a_k w_1$.

From computing multiplicities in each case, it follows easily that the only two words $\bar{w} \in W_n$ such that $\bar{w} \subseteq w$ are $\bar{w} = w^\pm$. These words also satisfy $w^\pm \preceq w$, and are the only two words apart from $w$ to do so. In order to construct the 1-skeleton $D_n[I_{(\omega^2)}]$ of $D_n$, we add an oriented edge $c_w$ from $c_{w^-}$ to $c_{w^+}$. The edge may be assumed to be contained in $L_n[w^-] \cup L_n[w] \cup L_n[w^+]$ and to cross $L_n[w]$ transversally and precisely once; edges are also assumed to be simple, and disjoint except at endpoints. Again, the inclusion $D_n[I_{(\omega^2)}] \subset L_n[I_{(\omega^2)}]$ is a homotopy equivalence.

The four sides of Figure 1 above provide examples of edges of cases (i) and (ii). Figure 5 shows the edges of $D_n$. Here we omit the initial and final words $w_0$ and $w_1$ (since they are not involved anyway). We also prefer to present examples (instead of spelling out conditions as above; a more formal discussion is given in Section 7 from [14] and in Section 6 above). Thus, cases (i), (ii) and (iii) are represented by $[ba]$, $[ab]$ and $[ac]$, respectively.

![Figure 5: Examples of edges of $D_n$.](image-url)
In a homological language, we write

\[ \partial[ba] = bab - a, \quad \partial[ab] = b - aba, \quad \partial[ac] = ca - ac. \]

Notice that in all cases \( \hat{\omega} = \hat{\omega}^+ = \hat{\omega}^- \in \text{Quat}_{n+1} \), consistently with Equation 10.

As a simple application of our methods, we compute the connected components of \( D_n \) (i.e., we are reproving in our notation the main result of [29]). We need only consider words of dimension 0 (the vertices of \( D_n \)), i.e., words in the generators \( a_k, 1 \leq k \leq n \). Write \( w_0 \sim w_1 \) if \( c_{w_0} \) and \( c_{w_1} \) are in the same connected component; in particular, if \( w_0 \sim w_1 \) then \( \hat{w}_0 = \hat{w}_1 \). Figure 5 illustrates that

\[ a \sim bab, \quad aba \sim b, \quad ac \sim ca. \]

For the empty word, the vertex \( c() \) is not attached to any edge and thus forms a connected component in the complex \( D_n \) (see also Remark 7.2 below).

**Proposition 7.1.** Consider two non-empty words \( w_0, w_1 \in D_n \) of dimension 0. Then \( w_0 \sim w_1 \) if and only if \( \hat{w}_0 = \hat{w}_1 \in \text{Quat}_{n+1} \).

**Remark 7.2.** For \( q \in \text{Quat}_{n+1} \), the set \( I_q = \{ w \in \mathcal{W}_n \mid \hat{w} = q \} \) is both a lower set and an upper set. Theorem 2 in [14] tells us that \( D_n[I_q] \subset \mathcal{L}_n(1; \hat{\eta}\hat{\eta}) \). For \( q = 1 \), we can further split \( I_q \) into \( I_{q,\text{convex}} = \{()\} \) and \( I_{q,\text{non-convex}} = I_q \setminus I_{q,\text{convex}} \). It follows from Proposition 7.1 that the connected components of \( D_n \) are \( D_n[I_q] \) \( (q \in \text{Quat}_{n+1}, q \neq 1) \), \( D_n[I_{1,\text{convex}}] \) and \( D_n[I_{1,\text{non-convex}}] \). Thus, from Theorem 4, \( \mathcal{L}_n(1; \hat{\eta}) \) is connected for \( q \neq \hat{\eta} \). The set \( \mathcal{L}_n(1; \hat{\eta}) \) has two connected components: one of them is \( \mathcal{L}_{n,\text{convex}}(1; \hat{\eta}) = \mathcal{L}_n[()] \), which is contractible; its complement \( \mathcal{L}_n(1; \hat{\eta}) \setminus \mathcal{L}_n[()] = \mathcal{L}_n[I_{1,\text{non-convex}}] \) is also connected.

These results are of course well known but also follow from our methods. \( \diamond \)

**Proof of Proposition 7.1.** We have already seen that \( w_0 \sim w_1 \) implies \( \hat{w}_0 = \hat{w}_1 \). We prove the other implication for non-empty words. A **basic word** is either:

(i) a non-empty word \( a_{k_1}a_{k_2} \cdots a_{k_l} \) with \( k_1 < k_2 < \cdots < k_l \);

(ii) of the form \( aaa_{k_1}a_{k_2} \cdots a_{k_l} \) with \( k_1 < k_2 < \cdots < k_l \) (here the words \( aa \) and \( aaa \) are allowed: they correspond to \( l = 0 \) and \( l = 1, k_1 = 1 \));

(iii) \( aaaa \).

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Clearly, for each \( z \in \text{Quat}_{n+1} \) there exists a unique basic word \( w \) with \( z = \hat{w} \). Using the edges in Figure 5 first notice that
\[
\begin{align*}
aa & \sim abab \sim bb \sim bcbc \sim cc \sim \cdots \sim a_k a_k; \\
also \quad a_k+1 a_k & \sim a_k+1 a_k+1 a_k+1 a_k+1 \sim aab a_k a_k+1; \\
\text{furthermore, } baa & \sim babab \sim aab \text{ and, for } k > 2, a_k a a \sim a_a a_k. \text{ Also, }
\end{align*}
\]
\[a \sim bab \sim ababababa \sim aaaa.\]
Thus \( aa \) commutes with all generators \( a_k \) and can be brought to the beginning
of the word; other generators either commute (\( a_k a_l \sim a_l a_k \) if \( |k - l| \neq 1 \)) or satisfy
\( a_k+1 a_k \sim aaa a_k a_k+1 \) (corresponding to the fact that \( \hat{a} \) and \( \hat{a}_{k+1} \) anticommute in \( \text{Quat}_{n+1} \), and that \( \hat{a} \hat{a} = -1 \)). Thus, for an arbitrary non-empty word \( w \),
generators can be arranged in increasing order of index, at the price of creating copies of \( aa \) which are
taken to the beginning of the word. Duplicate generators can also be transformed into further copies of \( aa \). Finally, if there are more than 4 copies of \( a \), they can be removed 4 by 4 thus arriving at a basic word.

\textbf{Example 7.3.} Take \( n = 4 \) and \( w = abacadaba \in \text{W}_4 \). Applying the procedure in the proof of Proposition 7.1 we have
\[abacadaba \sim abacadb \sim abacadb \sim abacabd \sim abaacbd \sim aaabcbd \sim aaacd,\]
and \( w_1 = aaacd \) is a basic word, with \( \hat{w} = \hat{w}_1 = -\hat{a} \hat{c} \hat{d}.\)

\textbf{Example 7.4.} It follows from Proposition 7.1 that for every non-empty word \( w \in \text{W}_n \) of dimension 0 there exists a path \( w^H \) in the 1-skeleton of \( D_n \) joining \( w \) and \( aaaaaw \). (Or more precisely, the vertices \( c_w \) and \( c_{aaaaw} \); we omit the \( c \)'s for conciseness.) We proceed to construct explicit paths which shall be extensively used later.

The simplest example is:
\[
a^H = (a \xleftarrow{ba} bab \xleftarrow{[ab]ab} \xrightarrow{aba[ba]} \xrightarrow{[ab]ab[ab]} abababa \xrightarrow{a[ba]ababa} a[ba][ba]a \xrightarrow{aaa[ba]a} aaaa),
\]
which already appeared implicitly in the proof of Proposition 7.1. If the word starts with \( a \), we fall back on this example:
\[
(aw)^H = a^H w = (aw \rightarrow babw \leftarrow \cdot \rightarrow \cdot \leftarrow abababaw \leftarrow \cdot \leftarrow aaaaaw).
\]
Next, define
\[
b^H = (b \xleftarrow{ab} aba \xrightarrow{a^H ba} aaaaaba \xrightarrow{aaa[ab]} aaaaab)
\]
\[
= (b \leftarrow aba \rightarrow babba \leftarrow \cdot \rightarrow \cdot \leftarrow ababababa \leftarrow \cdot \leftarrow aaaaaba \rightarrow aaaaab)
\]
and $(bw)^H = b^H w$. Define recursively
\[
a_{k+1}^H = (a_{k+1} \xleftarrow{[a_k a_{k+1}]} a_k a_{k+1} a_k \xrightarrow{a_{k+1} a_k} aaaaa_k a_{k+1} a_k \xrightarrow{aaaa[a_k a_{k+1}]} aaaa a_{k+1}).
\]
Notice that for every $k$, the path $a_k^H$ begins as
\[
a_k^H = a_k \xleftarrow{a_k^*} a_k^* \cdots.
\]
The permutation $a_k^*$ is defined in Section 2 (above Lemma 2.6); the words $a_k^*$ are given by
\[
a' = [ba], \quad a^* = bab, \quad a_{k+1}^* = [a_k a_{k+1}], \quad a_{k+1}^* = a_k a_{k+1} a_k.
\]
For longer words, set $(a_k w)^H = a_k^H w$.

We now construct the complex $D_n[\mathbb{I}(\omega^2)]$. Notice that $w \in \mathbb{I}(\omega^2)$ if and only if $w$ is of the form $w = w_0 \sigma_1 w_1 \cdots \sigma_l w_l$ where $\dim(w_j) = 0$ and $\dim(\sigma_j) = 1$ (some of the $w_j$ may equal the empty word). Set $c_w$ to be a product cell of dimension $l$, i.e., the $l$-th dimensional cube
\[
c_w = c_{w_0} \times c_{\sigma_1} \times c_{w_1} \times \cdots \times c_{\sigma_l} \times c_{w_l}.
\]
In homological notation we have
\[
\partial(w_0 \sigma_1 w_1 \cdots \sigma_l w_l) = w_0 (\partial \sigma_1) w_1 \cdots \sigma_l w_l - w_0 \sigma_1 w_1 (\partial \sigma_2) \cdots \sigma_l w_l + \cdots
\]
\[
\cdots + (-1)^{l+1} w_0 \sigma_1 w_1 \cdots (\partial \sigma_l) w_l.
\]
See Figure 6 for the following examples of the previous formula:
\[
\partial([ba][ab]) = bab[ab] - [ba]b - [ab] + [ba]aba,
\]
\[
\partial([ac][b][ac]) = cab[ac] - [ac]bca - acb[ac] + [ac]bac.
\]

Figure 6: The 2-cells $[ba][ab]$ and $[ac][b][ac]$. 
Remark 7.5. The construction of product cells works in greater generality. If \( w \in W_n \) is a word containing more than one letter of positive dimension, define \( c_w \) as a product cell. As before, write \( w = w_0 \sigma_1 w_1 \cdots \sigma_l w_l \) where \( \dim(w_j) = 0 \) and \( \dim(\sigma_j) > 0 \) (some of the \( w_j \) may equal the empty word). Set

\[
c_w = c_{w_0} \times c_{\sigma_1} \times c_{w_1} \times \cdots \times c_{\sigma_l} \times c_{w_l}.
\]

Here we assume the cells \( c_{\sigma_j} \) to have been previously constructed.

Remark 7.6. The reader will agree that an important part in the study of the CW complex \( D_n \) is the construction of the boundary maps for permutations \( \sigma \in S_{n+1} \), or, equivalently, of the cells \( c_\sigma \). The basic blocks in this construction are permutations \( \sigma \) for which \( 1^\sigma > 1 \). For instance, the construction of the boundary map for \( \sigma = bcb \) is easy if we already discussed \( \tilde{\sigma} = aba \).

Indeed, let \( \sigma \in S_{n+1} \) be a letter of dimension \( k \) such that \( 1^\sigma = 1 \). Equivalently, a reduced word for \( \sigma \) does not use the generator \( a_1 \). Write a reduced word \( \sigma = a_{n_1} \cdots a_{n_{k+1}} \) and \( s = -1 + \min n_j > 0 \). Set \( \tilde{\sigma} = a_{n_1-s} \cdots a_{n_{k+1}-s} \in S_{n-s+1} \); the cell \( c_{\tilde{\sigma}} \) is assumed to be already constructed. Define \( c_\sigma \) from \( c_{\tilde{\sigma}} \) by adding \( s \) to the index of every generator of every letter. Notice that this fits with our construction of the 1-skeleton; see also Figure 4 for \( c_{[bcb]} \).

8 Cells of dimension 2

Given \( n \) and Remarks 7.5 and 7.6, there is a finite (and short) list of possibilities of letters of dimension 2. In \( S_3 \), the only letter of dimension 2 is \([aba]\): Figure 3 shows the elements of \( W_2 \) (or \( W_n \)) below \([aba]\), i.e., the smallest lower set containing \([aba]\) (in this case the partial orders \( \preceq \) and \( \subseteq \) agree). See also Figures 1 and 4 for a transversal surface to the submanifold \( L_2[[aba]] \) and for the cell \( c_{[aba]} \).

In \( S_4 \) we also have \([bcb]\) (which, from Remark 7.6, is similar to \([aba]\), as in Figure 4). The example \([acb]\) deserves special attention: it is discussed in detail in Section 9 of [14], valid cells are given in Example 8.2 and in Figure 8 below. The three remaining cells are and \([abc]\), \([bac]\) and \([cba]\), for which transversal sections and valid cells are shown in Figure 7 in a homological notation:

\[
\begin{align*}
\partial[abc] &= abc[ab] + a[bc] + [ac] - [bc]a - [ab]cba + ab[ac]ba, \\
\partial[bac] &= [ac]b - [bc]ab - bc[ba] - b[ac] - ba[bc] - [ba]cb, \\
\partial[cba] &= [ac] + c[ba] + cba[cb] + cb[ac]bc - [cb]abc - [ba]c.
\end{align*}
\]

In each figure, we consider the family of paths constructed in Section 7 of [14] so that the functions \( m_j \) are polynomials in the real parameters \( x_1 \) and \( x_2 \).
Figure 7: The cells $c_{(abc)}$, $c_{(bac)}$ and $c_{(cba)}$.

and in the variable $t$. The semialgebraic curves in the figure indicate regions for which the itinerary has positive dimension and therefore separate open regions for which the itinerary has dimension 0.

Consider $\sigma \in S_{n+1} \setminus \{e\}$, $\dim(\sigma) = k$, and the map $\phi : \mathbb{D}^k \rightarrow \mathcal{L}_n$ transversal to $\mathcal{L}_n[\sigma]$ constructed in Lemma 7.1 of [14] (and discussed above in Section 6). By construction of $\phi$, if $x \in \mathbb{D}^k \setminus \{0\}$ and $\text{iti}(\phi(x)) = w$ there exists a continuous map $h : [0, 1] \rightarrow \mathbb{D}^k$ such that $h(0) = 0$, $h(1) = x$ and $\text{iti}(\phi(h(s))) = w$ for all $s \in (0, 1]$. In particular, we have both $w \sqsubseteq \sigma$ and $w \preceq \sigma$. As we shall see in Example 8.2, the reciprocal does not hold.

Example 8.1. Take $n = 3$ and $\sigma = abc$: we discuss the first diagram of Figure 7. Following the construction in [14], we have

$$M = \begin{pmatrix} -\frac{t^2}{2} & -t & -1 & 0 \\ t & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{t^3}{6} + x_2t + x_1 & \frac{t^2}{2} + x_2 & t & 1 \end{pmatrix}.$$  

and therefore

$$m_1(t) = \frac{t^3}{6} + x_2t + x_1, \quad m_2(t) = \frac{t^2}{2} + x_2, \quad m_3(t) = -t.$$  

The vertical line, corresponding to $[ac]$, has equation

$$\text{resultant}(m_1, m_3; t) = x_1 = 0.$$  

More precisely, the positive semiaxis corresponds to itinerary $[ac]$ and the negative semiaxis corresponds to itinerary $ab[ac]ba$. The horizontal line, corresponding to $[bc]$, has equation

$$\text{discriminant}(m_2; t) = -2x_2 = 0;$$

the positive semiaxis corresponds to itinerary $a[bc]$ and the negative semiaxis to $[bc]a$. Finally, the cusp-like curve in the figure, corresponding to $[ab]$, has equation

$$\text{resultant}(m_1, m_2; t) = \frac{x_2^3}{9} + \frac{x_1^2}{8} = 0;$$

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in the third quadrant the itinerary is \([ab]cba\); in the fourth quadrant it is \(abc[ab]\). \(\diamondsuit\)

**Example 8.2.** Take \(n = 3\) and \(\sigma = acb\). The transversal section constructed in Example 7.3 of [14] implies that

\[ [ac]b[ac] \subseteq [acb], \quad [ac]b[ac] \preceq [acb]. \]

Notice that \(\dim([ac]b[ac]) = \dim([acb]) = 2\). By transitivity we have

\[ acbac, cabca, [ac]bac, [ac]bca, acb[ac], cab[ac] \subseteq [acb] \]

(and similarly for \(\preceq\)) but none of the itineraries on the left hand side appear in the transversal section constructed in Example 7.3 of [14].

It is easy, on the other hand, to perturb the map \(\phi\) to obtain other maps transversal to \(L_n[[ac]b[ac]]\). Take

\[
M = \begin{pmatrix}
-t & -1 & 0 & 0 \\
\frac{t^2}{6} + xt & \frac{t^2}{2} + x & t & 0 \\
\frac{ut}{2} + 1 & u & 0 & 0 \\
\frac{t^2}{2} + y & t & 1 & 0
\end{pmatrix}
\]

where \(u\) is to be thought of as a fixed real number of small absolute value: say, \(|u| < 1/4\). Figure 8 shows the resulting sections. The construction in Example 7.3 of [14] corresponds to \(u = 0\). See Example 7.4 of [14] for a slightly different construction; see also Section 9 of [14].

![Figure 8: Two other transversal sections to \(L_3[[acb]]\) (\(u < 0\) and \(u > 0\)).](image-url)

Notice that the two diagrams differ combinatorially. For \(u < 0\), the itinerary \(acbac\) appears and \(cabca\) does not; for \(u > 0\) it is the other way round. Provided \(I \subset W_3\) contains the 2-dimensional word \([ac]b[ac]\), the two boundary maps (from \(S^1\) to the 1-skeleton) are homotopic in \(D_3[I]\), thus guaranteeing that \(D_3[I \cup \{[acb]\}]\) is well defined, consistently with the results from Section 6. \(\diamondsuit\)
In $\mathcal{D}_4$ there exist faces (i.e., cells of dimension 2) equivalent to those above in the sense of Remark 7.6 (such as $[bcd]$, which is equivalent to $[abc]$) but also a few genuinely new ones: $[abd]$, $[acd]$, $[adc]$ and $[bad]$, all shown in Figure 9; more generally, we have

$$\partial[aba_k] = ab[aa_k] + [aa_k]ba + a_k[ab] - [ba_k] - [ab]a_k, \quad k \geq 4;$$
$$\partial[aak_k+1] = a[a_{k+1}] + [aa_{k+1} ] - [ak_{k+1} ]a$$
$$- a_{k+1}[aa_k] - a_k[aa_{k+1}]a_k - [aa_{k+1}]a_{k+1}a_k, \quad k \geq 3;$$
$$\partial[aak+1k] = a[a_{k+1}a_k] + [aa_{k+1}] [aa_{k+1}]a_{k+1} + a_{k+1}[aa_{k+1}]a_{k+1}$$
$$+ a_{k+1}[aa_{k+1}]a_{k+1} - [aa_{k+1}]a_k - [aa_{k+1}]a_k, \quad k \geq 3;$$
$$\partial[baa_k] = [aa_k] + a_k[ba] - [ba_k]ab - b[aa_k]b - ba[ba_k] - [ba]a_k, \quad k \geq 4.$$  

The only genuinely new face in $\mathcal{D}_5$ is $[a_1a_3a_5]$, shown in Figure 10, we have

$$\partial[aak_k] = a[a_{k+l}] + [aa_{k+l}]a_k + a_{l}[aa_{k+l}] - [a_{k+l}]a_k - [aa_{k+l}]a_{k+l}, \quad 3 \leq k < l - 1.$$

9 Boundaries

The previous sections contain several examples of boundary maps of cells $c_\sigma$, $\sigma \in S_{n+1}$. We do not know a simple and general description of such boundary maps in higher dimension. In this section we prove some special cases.

Recall from Section 2 that $\sigma_0 \mathrel{\triangleleft} \sigma_1$ if $\sigma_0 \triangleleft \sigma_1$ and $\sigma_0 \equiv \sigma_1 \mod 2$, i.e., if $j_0 \equiv j_1 \mod 2$. The following proposition is the main result of this section.
In particular, we have

**Proposition 9.1.** Let \( c \) a valid boundary map for the cell \( c \) one copy of \( c \). We can present examples and a preliminary, more technical lemma. It follows of \( \partial \) cases \( \sigma \). In these examples we take \( \sigma \) the cases \( \sigma \). Consider also \( \sigma \) is a word of length one; see Figure 7. Also, \( \sigma \), \( \sigma \), \( \sigma \). The case \( \sigma \) is illustrated in Figure 5, as is the case \( \sigma \). The proof of this proposition is postponed to the end of this section so that we can present examples and a preliminary, more technical lemma. It follows easily from Remark 2.4 that if \( \sigma \) and \( \sigma \) then \( \sigma \).

**Example 9.2.** In these examples we take \( w_0 \) and \( w_1 \) both empty. The case \( \sigma_0 = a \) is illustrated in Figure 5 as is the case \( \sigma_0 = b \). The case \( \sigma_0 = a \) is also illustrated in Figure 7 (the top side is \([ac]\)). The case \( \sigma_0 = a \) are both illustrated in Figure 8. Figure 9 illustrates the cases \( \sigma_0 = a \) and \( \sigma_0 = a \). Notice that there is no \( \sigma_0 \in S_4 \) with \( \sigma_0 \) and consistently no side of \( \partial \) is a word of length one; see Figure 7. Also, \( \sigma_1 \in Y = S_{n+1} \setminus S_{PA} \) is low, with \( \sigma_1 \) with \( \sigma_1 ' = abac \).

Consider also \( \sigma_1 = a_1a_3a_5 \in S_6 \); no side of the boundary of \( c_{a_1} \) is a word of length 1, see Figure 10. In this case we have \( \sigma_1 \in S_{PA} \) and \( \sigma_1 ' \) is undefined. There is no \( \sigma_0 \in S_6 \) with \( \sigma_0 \) and \( \sigma_1 ' = a_1a_2a_3a_5 \).

**Lemma 9.3.** Let \( \sigma_0 \prec \sigma_1 \in S_{n+1} \) with \( \sigma_1 = (i_0i_1) \) and \( \sigma_0 \). For \( \Gamma \in \mathcal{L}_n[\sigma_1] \) there exist 0-dimensional words \( w_0^+, w_0^-, w_1^+, w_1^- \in W_n \) such that \( \Gamma \in \mathcal{L}_n[w_0^+\sigma_0w_1^+] \cap \mathcal{L}_n[w_0^-\sigma_0w_1^-] \).

In particular, we have

\[
\begin{align*}
&w_0^+\sigma_0w_1^+ \subseteq \sigma_1, \\
&w_0^+\sigma_0w_1^- \preceq \sigma_1, \\
&w_0^-\sigma_0w_1^- \subseteq \sigma_1, \\
&w_0^-\sigma_0w_1^- \preceq \sigma_1.
\end{align*}
\]
If $\sigma_0 \triangleright \sigma_1$ then $w_0^+ \text{ and } w_1^+$ are both empty and
\[ \text{mult}_k(w_0^-) = \text{mult}_k(w_1^-) = [i_0 \leq k < i_1]. \]

If $\sigma_0 \triangleleft \sigma_1$ then
\[ \begin{align*}
\text{mult}_k(w_0^-) &= \text{mult}_k(w_1^+), \\
\text{mult}_k(w_0^+) &= \text{mult}_k(w_1^-), \\
\text{mult}_k(w_0^-) + \text{mult}_k(w_0^+) &= [i_0 \leq k < i_1].
\end{align*} \]

Remark 9.4. Notice that we are not claiming that the two words $w_0^+ \sigma_0 w_1^+$ are the only words of the form $\tilde{w}_0 \sigma_0 \tilde{w}_1$ for which we have $\Gamma \in L_n[\tilde{w}_0 \sigma_0 \tilde{w}_1]$, $\tilde{w}_0 \sigma_0 \tilde{w}_1 \subseteq \sigma_1$ or $\tilde{w}_0 \sigma_0 \tilde{w}_1 \preceq \sigma_1$. In full generality such a claim is not correct. Indeed, for $\sigma_1 = [acb]$ and $\sigma_0 = [ac]$ we have
\[ abc[ac], cab[ac], [ac]bac, [ac]bca, [ac]b[ac] \subsetneq [acb] \]
and similarly for $\preceq$.

\[ \diamond \]

The following proof makes systematic use of both Theorem 4 and Lemma 7.2 in [15]. We recall Theorem 4: consider a smooth locally convex curve $\Gamma : (-\epsilon, \epsilon) \to SO_{n+1}$ with $\Gamma(0) \in \text{Bru}_n$. Let $m_j(t)$ be the determinant of the southwest $j \times j$ minor of $\Gamma(t)$. Then $t = 0$ is a zero of multiplicity $\text{mult}_j(\sigma)$ of $m_j$.

Proof of Lemma 9.3. Consider $\sigma_0 \triangleleft \sigma_1$ and let $i_*, j_*$ be as in the statement. For $s \in \mathbb{R}$ (small), take $M_s : \mathbb{R} \to \text{GL}_{n+1}$ with nonzero entries
\[ (M_s)_{n+2-i,1} = \begin{cases} 
 t(i_1-1) + st(j_0-1), & i = i_0; \\
 t(i_2-1), & i \neq i_0; 
\end{cases} \quad (M_s)_{ij+1} = \frac{d}{dt}(M_s)_{ij}. \]

This matrix $M_0$ has the form considered in Lemma 7.2 in [15] so its determinant is constant (in $t$), with the sign given by the parity of $\eta_0 \sigma_1$ ($\eta$ is the top permutation, the longest element of $S_{n+1}$ in the generators $a_j$). The matrix $M_s$ is obtained from $M_0$ by row operations so it has the same determinant. If necessary, change the sign of the first row of $M_s$ so that $M_s : \mathbb{R} \to \text{GL}_{n+1}^+$. Perform a QR factorization $M_s(t) = \Gamma_s(t)R_s(t)$ to obtain $\Gamma_s : \mathbb{R} \to SO_{n+1}$: the curve $\Gamma_s$ is locally convex.

Let $(M_s(t))_k$ and $(\Gamma_s(t))_k$ be the $k \times k$ southwest minors of $M_s(t)$ and $\Gamma_s(t)$, respectively. Consider the smooth functions
\[ m_{s;k}(t) = \det((M_s(t))_k), \quad m_{\gamma_s;k}(t) = \det((\Gamma_s(t))_k); \]
the QR factorization shows that one is a positive multiple of the other. In order to study the multiplicities of zeroes we may as well use $m_{s;k}(t)$.

Let
\[ C_{j,k} = \prod_{i_a < i_b \leq k} (i^\sigma_a - i^\sigma_b), \quad \tilde{C}_k = \frac{C_{1,k}}{C_{0,k}}, \quad d_{j,k} = \frac{k(k-1)}{2} + \sum_{i \leq k} i^\sigma_j. \]

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Notice that for \( k < i_0 \) and \( k \geq i_1 \) we have \( d_{1,k} = d_{0,k} \) but for \( i_0 \leq k < i_1 \) we have \( d_{1,k} = d_{0,k} + (j_1 - j_0) \). Also, for \( k < i_0 \) we have \( C_{1,k} = 1 \); for \( i_0 \leq k < i_1 \), \( C_{1,k} \) has the same sign as \( C_{i_0} \) (this is where we use the condition \( \sigma_0 < \sigma_1 \)).

For \( k < i_0 \) Lemma 7.2 in [15] gives \( \tilde{m}_{s,k}(t) = C_{1,k} t^{d_{1,k}} \). For \( k \geq i_1 \), \( (M_s(t))_k \) is obtained from \( (M_0(t))_k \) by row operations and Lemma 7.2 in [15] again implies \( \tilde{m}_{s,k}(t) = C_{1,k} t^{d_{1,k}} \). Finally, for \( i_0 \leq k < i_1 \), we use linearity of the determinant on row \( n + 2 - i_0 \); more precisely, let \( \hat{M} \) be defined by

\[
(\hat{M})_{n+2-i+1} = t^{i_0}, \quad (\hat{M})_{i,j+1} = \frac{d}{dt} (\hat{M})_{i,j},
\]

with southwest \( k \times k \) minor \( \hat{M} \) and let \( \hat{m}_k(t) = \det((\hat{M}(t)))_k \). By linearity and Lemma 7.2 in [15] we have

\[
\tilde{m}_{s,k}(t) = \det((M_s(t))_k) = \det(M_0(t))_k + s \det((\hat{M}(t)))_k \\
= \tilde{m}_{0,k}(t) + s \hat{m}_k(t) = C_{1,k} t^{d_{1,k}} + C_{0,k} s t^{d_{0,k}} = C_{0,k} s t^{d_{0,k}} (\tilde{C}_k (j_1 - j_0) + s).
\]

The multiplicities imply that \( \sigma(\gamma_0,0) = \sigma_1 \) and \( \sigma(\gamma_s,0) = \sigma_0 \) for \( s \neq 0 \) (here \( \sigma(\gamma, t) \in S_{n+1} \) denotes the singularity type of \( \gamma \) at time \( t \), as in the definition of itinerary). If \( j_1 - j_0 \) is odd there are extra roots with \( t \neq 0 \); if \( j_1 - j_0 \) is even and \( s \) has the same sign as \( \tilde{C}_{i_0} \) then there are no other real roots. This completes the proof. \( \square \)

**Proof of Proposition 9.1**. We first consider the case when both words \( w_0 \) and \( w_1 \) are empty. Set \( \rho_1 = \eta \sigma_1 \), \( z_1 = \hat{\rho}_1 \) and \( d_1 = \text{inv}(\sigma_1) - 1 \). In order to construct a section, we start with a locally convex curve \( \Gamma : (-\epsilon, \epsilon) \rightarrow \text{Spin}_{n+1} \) with \( \Gamma(0) = z_1 \).

We then translate the curve \( \Gamma \) to define a family \( \Gamma_s \) of curves, \( s \) in a small ball around \( 0 \in \mathbb{R}^{d_1} \). From Lemma 2.5, if \( \sigma_0 \pלו \sigma_1 \) then \( 2 \text{mult}(\sigma_0) \leq \text{mult}(\sigma_1) \). From Theorem 4 in [14], each curve \( \Gamma_s \) intersects the manifold \( N \) (as in Lemma 3.5) at most once. This implies that there exists a curve of values of \( s \) through 0 of curves passing through a (single) point of \( N \). Thus, for a nice ball, the boundary map contains two points with itinerary including \( \sigma_0 \): these are the two points identified by Lemma 9.3.

The general case (arbitrary \( w_0 \) and \( w_1 \)) is handled as in Remark 7.5. \( \square \)

### 10 Loose and tight maps

In this section \( * \) denotes the concatenation of curves: if \( \gamma_0 \in \mathcal{L}_n(1) \) and \( \gamma_1 \in \mathcal{L}_n(z) \) then \( \gamma_0 * \gamma_1 \in \mathcal{L}_n(z) \) is given by

\[
(\gamma_0 * \gamma_1)(t) = \begin{cases} 
\gamma_0(2t), & t \leq 1/2; \\
\gamma_1(2t - 1), & t \geq 1/2.
\end{cases}
\]
Clearly, the curve $\gamma_0 * \gamma_1$ can fail to be smooth at $t = 1/2$: as discussed in previous occasions (in several papers), apply a smoothening procedure. For $\gamma_0 \in \mathcal{L}_n(1)$ and $\gamma_1 \in \mathcal{L}_n$, the itinerary of $\gamma_0 * \gamma_1$ is $w_{\gamma_0*\gamma_1} = w_{\gamma_0}\eta w_{\gamma_1}$ (where $\eta \in S_{n+1}$ is the top permutation).

There exists a related construction in the CW complex $\mathcal{D}_n$, a continuous product $m : \mathcal{D}_n \times \mathcal{D}_n \to \mathcal{D}_n$. Given two words $w_0, w_1 \in \mathcal{W}_n$, we saw in Remark 7.5 that there exists a bijection from $c_{w_0} \times c_{w_1}$ to $c_{w_0w_1}$. The product is defined by taking this bijection for every pair of cells, so that $m[c_{w_0} \times c_{w_1}] = c_{w_0w_1}$. Thus, for instance, if $w_0$ is a word of dimension 2, the map $m_{w_0} : \mathcal{D}_n \to \mathcal{D}_n$ defined by $m_{w_0}(w_1) = m(w_0, w_1)$ is a homeomorphism taking cells to cells from $\mathcal{D}_n$ to the open subset $\mathcal{D}_n[I_{w_0}]$, where $I_{w_0} \subseteq \mathcal{W}_n$ is the lower set of words $w$ of the form $w = w_0\bar{w}$, $\bar{w} \in \mathcal{W}_n$. For brevity, for $w_0 \in \mathcal{W}_n$, $\dim(w_0) = 0$, and $f : K \to \mathcal{D}_n$ we write $w_0 f : K \to \mathcal{D}_n$, $(w_0 f)(p) = m(w_0, f(p))$. Similarly, for $p_0 \in \mathcal{D}_n$ we write $p_0 f : K \to \mathcal{D}_n$ for $(p_0 f)(p) = m(p_0, f(p))$.

Let $\gamma_0 \in \mathcal{L}_n(1)$ be a fixed non-convex closed curve. Let $M$ be a compact manifold and consider a map $\phi : M \to \mathcal{L}_n(z)$: $\phi$ is loose if $\phi$ is homotopic to

$$\gamma_0 * \phi : M \to \mathcal{L}_n(z)$$

$$s \mapsto \gamma_0 * \phi(s); \quad (11)$$

otherwise, $\phi$ is tight. These definitions generalize those in [24]; there, $n = 2$ and $\gamma_0$ is a fixed curve. It is easy to see, however, that the definition does not depend on the choice of $\gamma_0$. Indeed, if $\gamma_1 \in \mathcal{L}_n(1)$ is another non-convex closed curve then $\gamma_0$ and $\gamma_1$ are homotopic (from Proposition 7.1). Also, $\gamma_0$ is homotopic to $\gamma_0^2 = \gamma_0 * \gamma_0$ and to $\gamma_0^N = \gamma_0 * \gamma_0^{N-1} = \gamma_0 * \cdots * \gamma_0$.

The following lemma gives a characterization of loose maps $f : K \to \mathcal{D}_n$. \textbf{Lemma 10.1}. A map $f_0 : K \to \mathcal{D}_n$ is loose if and only if $f_0$ is homotopic to the map $f_1 = \text{aaaa} f_0$.

\textbf{Proof}. Let $w_0 = \text{iti}(\gamma_0)$. Take a short convex curve $\gamma_1 : [-1, 1] \to \text{Spin}_{n+1}$ with $\gamma_1(0) = 1$ so that $\text{iti}(\gamma_1) = \eta$. Slightly perturb $\gamma_1$ with fixed endpoints so that its itinerary $w_1$ has dimension 0. Thus, for $\gamma \in \mathcal{L}_n$ with $\text{iti}(\gamma) = w$ we have $\text{iti}(\gamma_0 * \gamma) = w_0 \eta w_1$ but a small perturbation of $\gamma_0 * \gamma$ has itinerary $w_0 w_1 w$.

Let $f_2 = \gamma_0 * f_0$. Perturb each curve $f_2$ near the glueing point to define a homotopic function $f_3 : K \to \mathcal{L}_n$ with $\text{iti}(f_3(p)) = w_0 w_1 \text{iti}(f_0(p))$. Take $f_4 = w_0 w_1 f_0 : K \to \mathcal{D}_n \subset \mathcal{L}_n$. The functions $f_3$ and $f_4$ are homotopic. Indeed, up to details in the construction of $\mathcal{D}_n$ we might have $f_3 = f_4$. Thus, $f_0$ is loose if and only if $f_0$ and $f_4$ are homotopic.

We claim that $f_1$ and $f_4$ are homotopic, completing the proof. Indeed, apply Proposition 7.1 to obtain a path $\delta : [0, 1] \to \mathcal{D}_n$ assuming values in the 1-skeleton of $\mathcal{D}_n$ with $\delta(0) = \text{aaaa}$, $\delta(1) = w_0 w_1$. Define $H : [0, 1] \times K \to \mathcal{D}_n$, $H(s, p) = \delta(s)f_0(p) = m(\delta(s), f_0(p))$: $H$ is the desired homotopy between $f_1$ and $f_4$. \qed
The following result partially justifies the interest in considering loose maps. Recall that $\Omega \text{Spin}_{n+1}(1; z)$ is the space of continuous maps $\Gamma : [0, 1] \to \text{Spin}_{n+1}$ with $\Gamma(0) = 1$ and $\Gamma(1) = z$, so that we have a continuous inclusion map $i : \mathcal{L}_n(z) \to \Omega \text{Spin}_{n+1}(1; z)$.

**Lemma 10.2.** Let $M$ be a compact manifold and consider maps $\phi_0, \phi_1 : M \to \mathcal{L}_n(z)$. If $\phi_0$ and $\phi_1$ are both loose and $i \circ \phi_0, i \circ \phi_1$ are homotopic (in the space $\Omega \text{Spin}_{n+1}(1; z)$) then $\phi_0$ and $\phi_1$ are homotopic (in $\mathcal{L}_n(z)$).

**Proof.** This result is proved in [25] and, for $n = 2$, in [24]. Since this is such a crucial result we present here a brief sketch of proof.

![Figure 11: Any base curve becomes locally convex if we add loops.](image)

If $\phi_0$ is loose then it is homotopic to $\gamma_0 \ast \phi_0$ and therefore to $\gamma_0^N \ast \phi_0$ (for a very large natural number $N$). The copies of $\gamma_0$ are then spread along $\phi_0$. At this stage our curve looks like a phone wire (see Figure 11); the many loops allow us to follow the homotopy in $\Omega \text{Spin}_{n+1}$ and keep our curves locally convex. □

One important construction in [29] and [24] is the add-loop procedure, which, in certain cases, is used to loosen up compact families of nondegenerate curves through a homotopy in $\mathcal{L}_n$. The resulting families of curly curves are then maleable: if a homotopy exists in the space of immersions, another homotopy exists in the space of locally convex curves. In [24], for instance, open dense subsets $\mathcal{Y}_+ \subset \mathcal{L}_2(\pm 1)$ are shown to be homotopy equivalent to the space of loops $\Omega S^3$. Thus, certain questions regarding homotopies can be transferred to the space of continuous paths in the group Spin$_{n+1}$. This approach is reminiscent of classical constructions such as the proof of Hirsch-Smale Theorem [17, 31] and Thurston’s eversion of the sphere by corrugations [21]. It can be considered as an elementary instance of the h-principle of Eliashberg and Gromov [9, 16].

We now restate Lemma 6.6 in [24], with minor changes. This is in a sense a more general version of Lemma 10.2 above (with a more complicated statement). Compared to the situation in [24], we are now in arbitrary dimension. Also, the concept of “adding two loops” has to be adapted. The proof, however, is very
much the same; we give a more succinct version here, the proof given in [24] is more careful.

If \( \gamma \in \mathcal{L}_n(1) \) and \( z \in \text{Spin}_{n+1} \), then \( z\gamma \) denotes the closed curve \( z\gamma : [0, 1] \to \text{Spin}_{n+1} \), \((z\gamma)(t) = z(\gamma(t))\).

**Lemma 10.3.** Let \( K \) be a compact manifold and \( f : K \to \mathcal{L}_n \) a continuous map. Let \( J \) be a positive integer and let \( (\gamma_j)_{1 \leq j \leq J} \) be a family of non-convex closed curves \( \gamma_j \in \mathcal{L}_n(1) \). Assume that:

1. \( t_{1,-}, t_{1,+}, t_{2,-}, \ldots, t_{J,+} : K \to (0, 1) \) are continuous functions with \( 0 < t_{1,-} < t_{1,+} < t_{2,-} < \cdots < t_{J,+} < 1 \);
2. the family of open sets \( \{U_j\}_{j \leq J} \) covers \( K \), that is, \( K = \bigcup_{1 \leq j \leq J} U_j \);
3. for \( p \in U_j \), \( f(p)(t_{j,-}(p)) = f(p)(t_{j,+}(p)) \) and the restriction \( f(p)\big|_{[t_{j,-}(p), t_{j,+}(p)]} \) is a reparametrization in time of the closed curve \( (f(p)(t_{j,-}(p)))\gamma_j \).

Then the map \( f \) is loose.

From Proposition 7.1 each curve \( \gamma_j \) is homotopic to \( \gamma_0 \). Notice that in Lemma 6.6 in [24] the curves \( \gamma_j \), \( 1 \leq j \leq J \), and \( \gamma_0 \) are all equal to \( \nu_2 \), a circle traversed twice, and the closed curve \( f(p)\big|_{[t_{j,-}(p), t_{j,+}(p)]} \) is a pair of loops.

**Proof of Lemma 10.3.** Recall that \( \gamma_j \sim \gamma_0^N \ast \gamma_j \), that is, there exists a path in \( \mathcal{L}_n(1) \) from \( \gamma_j \) to \( \gamma_0^N \ast \gamma_j \) (here \( N \) is a large natural number, to be chosen later). Apply this path near the boundary of each open set \( U_j \) so that in a slightly smaller open set \( \bar{U}_j \) the closed curve \( f(p)\big|_{[t_{j,-}(p), t_{j,+}(p)]} \) is now (a reparametrization of) \( (f(p)(t_{j,-}(p)))\gamma_0^N \ast \gamma_j \). We may assume that the open sets \( \bar{U}_j \) still cover \( K \). For each \( p \in K \), spread the copies of \( \gamma_0 \) which appear along \( f(p) \) in time, so that now we have “phone wire” curves as in Figure 11. As in the proof of Lemma 10.2, apply the homotopy which is known to exist in the space \( \Omega \text{Spin}_{n+1} \) to the base curves. For sufficiently large \( N \), the “phone wire” structure is sufficient to give us local convexity, thus obtaining the desired homotopy in \( \mathcal{L}_n \).

\[ \square \]

11 Loose ideals

A non empty lower set \( I \subset \mathcal{W}_n \) for both \( \subseteq \) and \( \preceq \) is an **ideal** if \( w_1 \in I \) and \( w_2 \in \mathcal{W}_n \) imply \( w_1w_2 \), \( w_2w_1 \in I \). The lower sets \( I_{[0]} \), \( I_{Y_2} \) and \( I_Y \) (discussed in Examples 5.3, 5.2 and 5.1) are ideals. A lower set \( I \) is a **weakly loose ideal** if it is an ideal and every map \( f : K \to \mathcal{L}_n[I] \subset \mathcal{L}_n \) is loose. Notice that we do not claim that the homotopy between \( f \) and \( \gamma_0 \ast f \) in Equation (11) has image contained in \( \mathcal{L}_n[I] \). A weakly loose ideal \( I \) is **(strongly) loose** if for every map
f : K → L_n[I] ⊂ L_n there exists a homotopy H : [0, 1] × K → L_n[I] between f and γ₀ ∗ f.

A nice loop is a periodic locally convex but non convex curve γ : [0, 1] → Spin_{n+1} with γ(0) = γ(1) ∈ Bru and image contained in Bru ∪ ⋃ Bru_{ηa}. For arbitrary n there exist nice loops: take an arbitrary periodic locally convex curve ˜γ ∈ L_n(1) and take γ = z ˜γ for generic z ∈ Spin_{n+1}. The itinerary iti(γ) ∈ W_n of a nice loop γ is defined similarly to itineraries of curves in L_n. More precisely, γ−1[⋃ Bru_{ηa}] ⊂ (0, 1) is a finite set \{t₁ < \cdots < t_ℓ\}. For j ≤ ℓ, take i_j such that γ(t_j) ∈ Bru_{ηa_{i_j}} and set iti(γ) = a_{i_1} \cdots a_{i_ℓ} ∈ W_n.

A word w ∈ W_n is a loop word if there exists a nice loop γ with iti(γ) = w. Notice that if w is a loop word then dim(w) = 0 and ˆw = 1. For n = 2, aaaa is a loop word. Investigating exactly which words are loop words is an interesting question which we shall not pursue.

**Lemma 11.1.** Let w ∈ W_n be a loop word. There exist γ⋆ ∈ L_n(η) with the following properties:

1. The itinerary of γ⋆ is w = iti(γ⋆).
2. The arcs γ⋆|[0, \frac{1}{3}] and γ⋆|[\frac{2}{3}, 1] are convex.
3. We have γ⋆(\frac{1}{3}) = γ⋆(\frac{2}{3}) = ˆη.

**Proof.** Take a nice loop ˜γ : [\frac{1}{3}, \frac{2}{3}] → Spin_{n+1}. Multiplication by an element of Quat_{n+1} allows us to assume that ˜γ(\frac{1}{3}) ∈ Bru. By applying a projective transformation (see Remark 11.2), we may also assume that ˜γ(\frac{1}{3}) = ˆη. Extend the curve ˜γ by two convex arcs (from 1 to ˆη and from ˆη to ˆη) to define the desired curve γ⋆.

**Remark 11.2.** For γ⋆ as in Lemma 11.1 it is the restriction ˜γ = γ⋆|[\frac{1}{3}, \frac{2}{3}] which is a closed curve, not γ⋆ (unless n is such that ˆη = 1).

In the above proof we used projective transformations, which are discussed in other papers, particularly in Section 2.1 of [14]. We try to avoid introducing extra notation for such transformations and will use them in other proofs without further comments.

**Proposition 11.3.** The set Iᵰ ⊂ W_n of words containing at least one letter of dimension 0 is a weakly loose ideal. In particular, if K is a compact manifold and f : K → D_n[Iᵰ] ⊂ L_n is a continuous map then f is loose.

**Remark 11.4.** Take K = S⁰ = {+1, −1} and f₀, f₁ : K → D_n[I₀], f₁ = aaaa f₀ so that

\begin{align*}
    f₀(+1) &= a, & f₀(−1) &= f₁(+1) &= aaaa, & f₁(−1) &= aaaaaaaa.
\end{align*}
Proposition 7.1 implies that \( f_0 \) and \( f_1 \) are homotopic in \( D_n \), consistently with Proposition 11.3 which says that \( f_0 \) is loose and Lemma 10.1 which says that \( f_0 \) and \( f_1 \) are therefore homotopic in \( D_n \). On the other hand, \( f_0 \) and \( f_1 \) are not homotopic in \( D_n[I_0] \) (or in \( L_n[I_0] \)): the cell \( c_a \) is an isolated vertex in \( D_n[I_0] \). This proves that \( I_0 \) is not (strongly) loose.

\[ \diamond \]

**Proof of Proposition 11.3** The set \( I_0 \) is clearly a lower set for both \( \preceq \) and \( \sqsubset \); it is also an ideal. All we have to prove is therefore the last claim in the statement: that the map \( f \) is loose. The idea of the proof is to start with \( f_0 = f \) as in the statement and deform it (i.e., apply a homotopy) first in \( D_n \) and then in \( L_n \) to obtain a finite sequence of functions \( f_i : K \to L_n \). The function \( f_3 \) satisfies the hypothesis of Lemma 10.3 and is therefore loose: thus, so is the original \( f \). Let \( \gamma \in L_n(1) \) be a periodic non convex curve (as \( \gamma_0 \) in Equation (11)). Let \( w_0 \) be a loop word, the itinerary of the nice loop \( \gamma_0 = z\gamma \), where \( z \in \text{Spin}_{n+1} \) is a generic element.

We first describe the strategy of the proof. The function \( f_1 : K \to D_n \) has the property that for every \( p \in K \) if \( f_1(p) \in c_w \) then \( w_0 \) contains a subword \( w_0 \) \( (\text{where } N \text{ is a large positive integer}) \). We then prove that the function \( f_1 \) satisfies conditions similar (but not identical) to those in Lemma 10.3. More precisely, there exists a finite open cover \( (U_j)_{1 \leq j \leq J} \) of \( K \) and continuous functions \( t_{1,2} < t_{1,2} < \cdots < t_{j,2} < t_{j,2} : K \to (0,1) \) such that if \( p \in U_j \) then the arc \( f_1(p)[t_{j,-2}(p), t_{j,+2}(p)] \) has itinerary \( w_0 \). From \( f_1 \) to \( f_2 \) the homotopy takes place in \( L_n \). The function \( f_2 : K \to L_n \) satisfies yet another variation of the conditions in Lemma 10.3. More precisely, there exist continuous functions \( t_{j,-}, t_{j,+} : U_j \to (0,1) \) with \( t_{j,-} < t_{j,2} < t_{j,-} < t_{j,2} < t_{j,2} \) such that \( p \in U_j \) implies that \( f(p)(t_{j,-}) = f(p)(t_{j,2}) \) and the arc \( f(p)[t_{j,-}(p), t_{j,+}(p)] \) is obtained from \( \gamma_0 \) by a projective transformation. Finally, a homotopy takes \( f_2 \) to \( f_3 : K \to L_n \) satisfying the original conditions: the arcs \( f(p)[t_{j,-}(p), t_{j,+}(p)] \) are obtained from \( \gamma_0 \) by multiplication.

For a function \( f_0 : K \to L_n \) we have a finite open cover \( (U_j)_{1 \leq j \leq J} \) of \( K \) such that each \( U_j \) is connected and for \( p \in U_j \), the itinerary of \( f_0(p) \) is of the form \( w_0(p) \alpha_i, w_0(p) \). Similarly, a function \( f_0 : K \to D_n \) can be slightly modified by a homotopy so that we have a finite open cover \( (U_j)_{1 \leq j \leq J} \) of \( K \) such that each \( U_j \) is connected and \( p \in U_j \) implies that \( f(p) \in c_w(p) \) where the word \( w(p) \) is of the form \( w_0(p) \alpha_i w_0(p) \). Moreover, we can take the cover so that there are continuous functions \( f_{\pm j} : U_j \to D_n \) such that, for all \( p \in U_j \), \( f(p) = f_{\pm j}(p) \alpha_i f_{\pm j}(p) \). Take a family of compact manifolds with boundary \( W_j \subset U_j \) such that the sets \( U_j^{(0)} = \text{int}(W_j) \) cover \( K \). Take tubular neighborhoods of \( \partial W_j \), i.e., local homeomorphisms \( \Phi_j : \partial W_j \times [-1,1] \to U_j \) such that \( \Phi_j(p,0) = p \) and \( \Phi_j(p,t) \in \text{int}(W_j) \) for all \( p \in \partial W_j \) and \( t > 0 \). For \( s \in [0,1] \), set \( U_j^{(s)} = \text{int}(W_j) \setminus \Phi_j[\partial W_j \times [-1,s]] \) so
that \( s_0 < s_1 \) implies \( U_j \supset U_j^{(s_0)} \supset U_j^{(s_1)} \). We also assume that the open sets \((U_j^{(1)})_{1 \leq j \leq n}\) cover \( K \).

From Proposition 7.3, \( a_i \sim a_i w_0^N \), where \( N \) is a large positive integer to be specified later. In other words, there exists a path \( \delta_i \) from \( [0, 1] \) to the 1-skeleton of \( D_n \) satisfying:

\[
\delta_i : [0, 1] \to D_n, \quad \delta_i(0) \in c_{a_i} \subset D_n, \quad \delta_i(1) \in c_{a_i w_0^N}.
\] (12)

The paths \( a_i^H \) constructed in Example 7.4 can be used here. Let \( a_i^H \) be, as in Section 2, a letter of dimension 1 (with abuse, a segment) joining \( a_i \) to a word \( a_i^* \) of dimension 0 and length 3:

\[
a' = [ba], a^* = bab, b' = [ab], b^* = aba, \ldots, a_i^H = [a_i a_i+1], a_i^* = a_i a_i+1 a_i, \ldots
\]

We may assume that \( \delta_i(t) \in c_{a_i} \) for \( t \in (0, \frac{1}{2}) \), \( \delta_i(\frac{1}{2}) \in c_{a_i^*} \), \( \delta_i(\frac{3}{4}) \in c_{a_i a_i+1 a_i} \) and \( \delta_i(1) \in c_{a_i w_0^N} \). Notice that for \( t \geq \frac{1}{2} \) if \( \delta_i(t) \in c_w \) then \( w \) has a letter of dimension 0.

Define a map \( f_1 : K \to D_n \) as follows. For \( p \in U_j^{(1/2)} \), \( f_1(p) \) is obtained from \( f_0(p) \) by replacing \( a_{ij} \) by \( a_i w_0^N \). More precisely: take \( p \) in \( K \) and \( w \in I_{[0]} \) such that \( f(p) \in c_w \). Let \( J_p \) be the set of \( j \) for which \( p \in U_j^{(0)} \). To each \( j \) in \( J_p \) corresponds a copy of the letter \( a_{ij} \) in \( w \). Notice that such copies need not be distinct in the word \( w \): for \( j_0 \neq j_1 \in J_p \), we may have \( i_{j_0} = i_{j_1} \) and the open sets \( U_{j_0} \) and \( U_{j_1} \) may both point to the exact same copy of \( a_{i_{j_0}} = a_{i_{j_1}} \) in \( w \). If this happens, we say that \( j_0 \) and \( j_1 \) are equivalent. Write \( w = w_1 a_{i_{j_1}} w_2 \ldots w_k a_{i_{j_k}} w_{k+1} \) where \( j_1, \ldots, j_k \) are representatives of the equivalent classes in \( J_p \).

For each \( j \in J_p \), define \( \bar{s}_j = 2 \sup\{s \in [0, 1/2] \mid p \in U_j^{(s)}\} \in [0, 1] \); notice that for any \( p \in K \) there exists \( j \in J_p \) such that \( \bar{s}_j = 1 \). For a representative \( j_\kappa \) \((1 \leq \kappa \leq k)\), define \( s_\kappa \) to be the maximum of all values of \( \bar{s}_j \), \( j \) equivalent to \( j_\kappa \). Write \( c_w = c_{w_1} \times a_{i_{j_1}} \times \cdots \times a_{i_{j_k}} \times c_{w_{k+1}} \) and

\[
f_0(p) = (p_1, a_{i_{j_1}}, \ldots, a_{i_{j_k}}, p_{k+1})
\] (13)

(so that, for instance, \( p_1 \in c_{w_1} \)). (Notice that, as we often do, we use a word of dimension 0 to denote both a point in \( D_n \) and the cell whose only element is said point.) Define

\[
f_1(p) = (p_1, \delta_{i_{j_1}}(s_{j_1}), \ldots, \delta_{i_{j_k}}(s_{j_k}), p_{k+1})
\] (14)

Similarly, define a homotopy \( H_1 : K \times [0, 1] \to D_n \) from \( f_0 \) to \( f_1 \) by

\[
H_1(p, s) = (p_1, \delta_{i_{j_1}}(\min\{s, s_{j_1}\}), \ldots, \delta_{i_{j_k}}(\min\{s, s_{j_k}\}), p_{k+1}).
\] (15)

This completes the construction of \( f_1 \). Notice that \( f_1 \) assumes values in \( D_n[I_{[0]}] \subset D_n \) and the homotopy \( H_1 \) assumes values in \( D_n \): nothing guarantees that the image of \( H_1 \) is contained in \( D_n[I_{[0]}] \) (see Remark 11.4). It is not clear at this point
that $f_1$ together with the cover $(U_j)$ satisfies the property stated above, that is, there are at this point no functions $t_{j,\pm 2}$. We adress this issue by modifying the cover (but not the function $f_1$).

A first difficulty is that in the intersection $U_{j_1} \cap U_{j_2}$ (with $j_1 \neq j_2$) the same copy of $w^N_0$ should not be used by the two open sets. This is easily addressed by taking $N$ sufficiently large. More precisely, so that each set $U_j$ has a copy of $w^N_0$ we perform the previous construction not with $N$ but with $\tilde{N} > JN$. The set $U_j$ then has the $j$-th copy of $w^N_0$ inside the copy of $w^\tilde{N}_0$. From now on we assume that this first difficulty has been taken care of.

A second difficulty is that copies of $w^N_0$ should appear in the itinerary of $f_1(p)$ in the same order as the indices $j$. Permuting the indices $j$ may not solve the problem, but taking $N$ large, followed by a refinement of the covering, does. Indeed, assume $N > J$. Define continuous functions $\tilde{t}_{j,-},\tilde{t}_{j,+} : U_j \to [0,1]$ such that, for $p \in U_j$, $t_{j,-}(p) < \tilde{t}_{j,+}(p)$ and the itinerary of the arc $f_1(p)[\tilde{t}_{j,-}(p),\tilde{t}_{j,+}(p)]$ is $w^N_0$. Set $\tilde{t}_j = (\tilde{t}_{j,-} + \tilde{t}_{j,+})/2 : U_j \to [0,1]$. The functions $\tilde{t}_j$ can be defined so as to be extendable to $K$; furthermore, we may assume that for any $p$ there exists at most one pair $(j_0, j_1)$ with $j_0 < j_1$ and $\tilde{t}_{j_0}(p) = \tilde{t}_{j_1}(p)$. Let $\tau_j(p) = \text{card}\{j' \mid \tilde{t}_{j'}(p) \leq \tilde{t}_j(p)\} \in [1,J]$. Let $U_{j,j'} \subseteq U_j$ be an open set such that $p \in U_j$ and $\tau_j(p) = j'$ imply $p \in U_{j,j'}$. The sets can be chosen so that if $j_0 \neq j_1$ then $U_{j_0,j'}$ and $U_{j_1,j'}$ are disjoint. Assign to $U_{j,j'}$ the $j'$-th copy of $w_0$ in $w^N_0 = \text{iti}(f_1(p)[\tilde{t}_{j,-}(p),\tilde{t}_{j,+}(p)])$. More precisely, define functions $\tilde{t}_{j,j'-}, \tilde{t}_{j,j'+ : U_{j,j'} \to [0,1]}$ with $t_{j,-} < t_{j,j'-} < t_{j,j'+} < t_{j,+}$ and such that $\text{iti}(f_1(p)[\tilde{t}_{j,-}(p),\tilde{t}_{j,j'-}(p)]) = w^{j'-1}_0$, $\text{iti}(f_1(p)[\tilde{t}_{j,j'-}(p),\tilde{t}_{j,j'+}(p)]) = w^N_0$. The functions can be chosen so that $p \in U_{j_0,j'} \cap U_{j_1,j''}$ and $j_0 < j_1$ imply $t_{j_0,j_0'}(p) < t_{j_1,j_1'}(p)$. Relabel the non empty sets $U_{j,j'}$ in increasing order of $j'$, allowing us to define the functions $t_{j,\pm 2}$. This completes the discussion of $f_1$.

We now discuss the construction of $f_2$ and of the homotopy from $f_1$ to $f_2$. For $p \in U_j$, the arc $f_1(p)[t_{j,-2}(p),t_{j,\pm 2}(p)]$ has itinerary $w_0$. For $f_2$, we want that arc to contain a closed curve, a copy of $\gamma_0$ (up to projective transformation). We show how to achieve this by a homotopy, working for one value of $j$ at a time. It will be noticed that the construction for $j = j_2$ does not spoil the previously performed construction for $j = j_1 < j_2$. For this, take families of subsets $U_{j,0} \subset W_{j,0} \subset U_{j,1} \subset W_{j,1} \subset U_j$ such that $U_{j,*}$ are open sets, $W_{j,*}$ are compact manifolds with boundary and the family $(U_{j,0})_{1 \leq j \leq J}$ covers $K$. As usual, after this step we drop the extra index and write $U_j$ instead of $U_{j,0}$.

Given $j$, assume $f_{1,j} \overset{\sim}{\to} f_{2,j}$ constructed with the property $f_{1,j} \overset{\sim}{\to} f_{2,j}$ of $f_{j,-1}(p)) = f_{1,j} \overset{\sim}{\to} f_{2,j}(p)(t_{j,-1}(p)) = f_{1,j} \overset{\sim}{\to} f_{2,j}(p)(t_{j,-1}(p))$ for $p \in U_{j,*}$ and $j' < j$. We want to construct a homotopic function $f_{1,j'}$ with the same property for all $j' \leq j$. Let $q_j \in \text{Quat}_{n+1}$ such that $f_1(p)(t_{j,\pm 2}(p)) \in \text{Bru}_{q_j}$. Let $t_{j,\pm 2}(p)$ be the times corresponding to the first and last letter in the arc $f_1(p)[t_{j,-2}(p),t_{j,\pm 2}(p)]$; notice that, from Theorem 1
in [14], these are continuously defined for \( p \in U_j \). Define functions \( t_{j,\pm 1}(p) = (2t_{j,\pm 1}(p) + 2t_{j,\pm 2}(p))/3 \) and \( t_{j,\pm \frac{3}{2}}(p) = (t_{j,\pm \frac{3}{2}}(p) + 2t_{j,\pm 2}(p))/3 \) so that \( t_{j,-2} < t_{j,\pm \frac{3}{2}} < t_{j,-1} < t_{j,-\frac{1}{2}} < t_{j,\frac{1}{2}} < t_{j,1} < t_{j,\frac{3}{2}} < t_{j,2} \). Draw a family of convex arcs from \( q_j \) to \( f_1(p)(t_{j,-\frac{3}{2}}(p)) \) (with domain \([0, t_{j,-\frac{3}{2}}(p)]\)) and from \( f_1(p)(t_{j,\frac{3}{2}}(p)) \) to \( q_j\hat{\eta} \). (with domain \([t_{j,\frac{3}{2}}(p), 1]\)). Multiplication by \( q_j^{-1} \) defines a continuous map \( g_{j,0} : W_{j,1} \to \mathcal{L}_n[w_0] \subset \mathcal{L}_n(\hat{\eta}) \) with \( g_{j,0}(p)(t) = f_1(p)(t) \). Since \( \mathcal{L}_n[w_0] \) is a contractible Hilbert manifold, there exists a homotopy from \( g_{j,0} \) to \( g_{j,1} : W_{j,1} \to \mathcal{L}_n[w_0] \) with the following properties:

1. The restrictions \( g_{j,0}|_{\partial W_{j,1}} \) and \( g_{j,1}|_{\partial W_{j,1}} \) are equal.

2. Up to reparametrization, the restriction \( g_{j,1}|_{W_{j,0}} \) is constant equal to \( \gamma_* \) (the curve introduced in Lemma [11.1]). The reparametrization takes \( t_{j,-1}(p) \) and \( t_{j,1}(p) \) to \( 1/3 \) and \( 2/3 \) so that \( g_{j,1}(p)(t_{j,-1}(p)) = g_{j,1}(p)(t_{j,1}(p)) = \hat{\eta} \).

3. For all \( p \in W_{j,1} \) and all \( s \in [0, 1] \), we have \( g_{j,s}(p)(t_{j,-2}(p)) \in \text{Bru}_{\hat{\eta}} \) and \( g_{j,s}(p)(t_{j,2}(p)) \in \text{Bru}_{\hat{\eta}} \).

Here of course \( g_{j,s}(p) = H(s, p) \), where \( H \) is the homotopy from \( g_{j,0} \) to \( g_{j,1} \). The careful reader will have noticed that we did not pay attention to differentiability of curves in \( \mathcal{L}_n \) at gluing points. Curves can be smoothened at such points, as has been amply discussed in several papers.

The maps \( g_{j,*} \) and the homotopy above guide us to construct \( f_{1+\frac{3}{2}} \) and the homotopy with \( f_{1+\frac{4}{3}} \). First, consider the arcs obtained by restricting \( f_{1+\frac{4}{3}}(p) \) to \([0, t_{j,-\frac{3}{2}}(p)]\). The corresponding arc in \( f_{1+\frac{2}{3}}(p) \) is obtained by a projective transformation taking \( f_{1+\frac{4}{3}}(p)(t_{j,-\frac{3}{2}}(p)) \in \text{Bru}_{\gamma,\hat{\eta}} \) to \( q_jg_{j,\gamma}(p)(t_{j,-\frac{3}{2}}(p)) \in \text{Bru}_{\gamma,\hat{\eta}} \). Consider now the restriction of \( f_{1+\frac{2}{3}}(p) \) to \([t_{j,\frac{3}{2}}(p), 1]\): the corresponding arc in \( f_{1+\frac{2}{3}}(p) \) is obtained by a projective transformation taking \( f_{1+\frac{4}{3}}(p)(t_{j,\frac{3}{2}}(p)) \in \text{Bru}_{\gamma,\hat{\eta}} \) to \( q_jg_{j,\gamma}(p)(t_{j,\frac{3}{2}}(p)) \in \text{Bru}_{\gamma,\hat{\eta}} \). Finally, the restriction of \( f_{1+\frac{2}{3}}(p) \) to \([t_{j,-\frac{3}{2}}(p), t_{j,\frac{3}{2}}(p)]\) is \( q_jg_{j,\gamma}(p) \). Again, smoothening procedures are implicit; notice that they have no effect on itineraries or on the condition \( f_2(p)(t_{j,-1}(p)) = f_2(p)(t_{j,1}(p)) \). This completes step \( j \) in the construction of \( f_2 \) and the discussion of \( f_2 \).

Notice that the closed loop \( f_2(p)|_{[t_{j,-1}(p), t_{j,1}(p)]} \) is obtained from \( \gamma_0 \) at step \( j \) in the construction of \( f_2 \), but suffers projective transformations at later steps. The group of projective transformations is contractible, however. In order to pass from \( f_2 \) to \( f_3 \) we apply multiplication and projective transformations to such closed loops, completing the construction of \( f_3 \) and the proof.
12 Simple connectivity

The aim of this section is to prove Theorem 12.1. This is essentially equivalent to the following result.

Proposition 12.1. Any continuous map $f_0 : S^1 \to D_n$ is loose.

Proof. We may assume that $f_0$ assumes values in the 1-complex of $D_n$. From Lemma 10.1 it suffices to prove that $f_0$ is homotopic to $f_1 = aaaa f_0$. We construct an explicit homotopy between the maps $f_0$ and $f_1$.

Recall that in Example 7.4 (consistently with Proposition 7.1) we constructed for every non-empty word $w \in W_n$ of dimension 0 an explicit path $w^H$ in the 1-skeleton of $D_n$ joining $w$ and $aaaa w$. We first define:

$$a^H = (a \xleftarrow{[ba]} bab \xleftarrow{[ab]ab} aba \xrightarrow{[ba][ba]b} abababa \xleftarrow{[ba][ba]a} aaaa).$$

Define recursively

$$a^H_{k+1} = (a_{k+1} \xleftarrow{a_k a_{k+1}} a_k a_{k+1} a_k \xrightarrow{a^H_{k+1} a_k} aaaa a_k a_{k+1} a_k \xrightarrow{aaa [a_k a_{k+1}]} aaaa a_k a_{k+1})$$

and $(a_k w)^H = a^H_k w$. We now define $w^H$ for words of dimension 1.

Let $w$ be a word of dimension 1 with $\partial w = w_1 - w_0$, where $w_0$ and $w_1$ are non-empty words of dimension 0. In order to complete the construction of the homotopy, it suffices to construct $w^H : [0, 1]^2 \to D_n$ with $w^H|_{[0, 1] \times [0, 1]} = w$, $w^H|_{[1] \times [0, 1]} = aaaa w$ $w^H|_{[0, 1] \times [1]} = w_0$, $w^H|_{[0, 1] \times \{1\}} = w_1$. If the first letter of $w$ has dimension 0 we may write $w = a_k \tilde{w}$ and $w^H = a^H_k \tilde{w}$. We are left with the case when the first letter of $w$ has dimension 1: if $w = \sigma \tilde{w}$ we set $w^H = \sigma^H \tilde{w}$. We are therefore left with the case $w = \sigma$, $\text{inv} (\sigma) = 2$.

This is done on a case by case basis: $[a_k a_{k+1}]$, $[a_{k+1} a_k]$ and $[a_k a_l]$ for $l > k + 1$.

We first observe that the case $[a_k a_{k+1}]$ is rather trivial: the square collapses to the segment in the definition of $a^H_{k+1}$. We define $[ac]^H$ in Figure 12.

The top and bottom hexagons (they look like triangles, but they have six vertices each) are $[abc]$ and $aaaaa [abc]$, as indicated. Each of the three central trapezoidal regions is actually composed of six product cells (since $a^H$ is not one edge, but six). The definition of $[a_k a_{k+2}]^H$ is similar: just substitute $a_k$, $a_{k+1}$ and $a_{k+2}$ for $a$, $b$ and $c$ in Figure 12 (a minor difference is that $a^H_k$ is actually 4 + 2k edges). We define $[ad]^H$ in Figure 13.

Notice that the second of the four central trapezoidal regions uses the previous construction of $[ac]^H$. The other three consist of several product cells. Again, the definition of $[a_k a_{k+3}]^H$ is similar. The definition of $[a_k a_{k+4}]^H$ uses that of $[a_k a_{k+3}]^H$ and so on, recursively.
This takes care of the cases \([a_k a_l]\) for \(l > k\); we are left with the case \([a_{k+1} a_k]\).

We now describe \([cb]^H\) in Figure 14.

This definition uses a specific cell for \([acb]\); a similar diagram works for the other cell. The definition of \([a_{k+1} a_k]^H\) for \(k > 2\) is similar, just substitute \(a_{k-1}\), \(a_k\) and \(a_{k+1}\) for \(a\), \(b\) and \(c\) (except for the initial \(aaaa\) in the lower part of the diagram).

Finally, in order to define \([ba]^H\) we have to fill in the trapezoidal region, indicated by (?) in Figure 15; notice that there is a partial collapse at the top.

This can be done explicitly but is rather messy; we prefer to use Lemma 10.2 and Proposition 11.3. The boundary of the trapezoidal region is a map from \(S^1\) to \(D_n[I_{(0)}]\) and therefore loose. As a map from \(S^1\) to \(\Omega Spin_{n+1}\) it is homotopically trivial since \(\Omega Spin_{n+1}\) is simply connected. Thus, from Lemma 10.2 the map is also homotopically trivial in \(D_n\), allowing us to fill the region, thus completing the construction of \([ba]^H\) and the proof of the proposition.
Proof of Theorem 12. From Proposition 12.1, any map from $\mathbb{S}^1$ to $\mathcal{D}_n$ is loose. As a map from $\mathbb{S}^1$ to $\Omega\Spin_{n+1}$ it is homotopically trivial ($\Omega\Spin_{n+1}$ being simply connected). Thus, from Lemma 10.2, any map from $\mathbb{S}^1$ to $\mathcal{D}_n$ is homotopically trivial, completing the proof of the theorem. \hfill $\Box$

Remark 12.2. It would be interesting to obtain an explicit solution to complete Figure 15, preferably with a small number of pieces. \hfill $\Diamond$

13 The subcomplexes $\mathcal{Y}_{n,2} \subseteq \mathcal{D}_n$

We briefly recall some notation and results from Section 2. We write $Y = S_{n+1} \setminus S_{PA}$ so that $\sigma \leftrightarrow \sigma'$ is an involution of $Y$: for $\sigma \in Y$ take $k$ to be the
smallest positive integer with \( k^n \equiv (k + 1)^n \pmod{2} \) and define \( \sigma' = a_k \sigma \). We call \( \sigma \in Y \) low if \( \sigma \prec \sigma' \) and high if \( \sigma' \nless \sigma \). As in Remark 2.8, let \( Y_k \subseteq Y \) be the set of permutations \( \sigma \) which are either low with \( \text{inv}(\sigma) < k \) or high with \( \text{inv}(\sigma) \leq k \). Let \( I_{Y_k} \subset W_n \) be the set of words with at least one letter in \( Y_k \).

**Remark 13.1.** We saw in Examples 5.1 and 5.2 that the subsets \( I_{Y_2}, I_{Y_2} \subset W_n \) are lower subsets for both \( \sqsubseteq \) and \( \preceq \). In this section we consider the lower set \( I_{Y_2} \), the open subset \( \mathcal{L}_n[I_{Y_2}] \subset \mathcal{L}_n \) and the corresponding subcomplex \( \mathcal{Y}_{n,2} = \mathcal{D}_n[I_{Y_2}] \subset \mathcal{D}_n \).

We believe that for all \( k > 2 \) the subset \( I_{Y_k} \subset W_n \) is a lower set under \( \preceq \), but we shall neither prove nor use this claim. Similarly, we believe (but do not prove) that \( \mathcal{L}_n[I_{Y_k}] \subset \mathcal{L}_n \) is an open subset. Still, in the next section we will construct and study subcomplexes \( \mathcal{Y}_{n,k} \) corresponding to the sets \( I_{Y_k} \). Lemma 14.2 below is good enough for our purposes.

A subcomplex \( \mathcal{X} \subseteq \mathcal{D}_n \) is **nice** if for every word \( w \in W_n \) if \( c_w \subseteq \mathcal{X} \) then \( c_{aaaaw} \subseteq \mathcal{X} \); the subcomplex \( \mathcal{Y}_{n,2} \) is clearly nice. Notice that if \( \mathcal{X} \) is nice and \( f : K \to \mathcal{X} \) is a continuous map then there exists a continuous map \( aaaaaf : K \to \mathcal{X} \). A nice subcomplex \( \mathcal{X} \subseteq \mathcal{D}_n \) is **loose** if for every compact manifold \( K \) and every continuous map \( f_0 : K \to \mathcal{X} \) there exists a homotopy \( H : [0,1] \times K \to \mathcal{X} \) from \( f_0 \) to \( f_1 = aaaaaf_0 \). More explicitly, we have \( H(0,s) = f_0(s) \) and \( H(1,s) = aaaaaf_0(s) \) for all \( s \in K \). Notice that we require the image of the homotopy \( H \) to be included in \( \mathcal{X} \). Contrast this with the definitions of a weakly loose ideal and of a (strongly) loose ideal in Section 11. Clearly, if \( I \subset W_n \) is a (strongly) loose ideal then \( \mathcal{D}[I] \subset \mathcal{D}_n \) is a loose subcomplex. It follows from Remark 11.4 that the subcomplex \( \mathcal{D}[I_{[0]}] \subset \mathcal{D}_n \) is not a loose subcomplex.

**Proposition 13.2.** The ideal \( I_{Y_2} \subset W_n \) is (strongly) loose.

The previous proposition is equivalent to saying that the subcomplex \( \mathcal{Y}_{n,2} \subseteq \mathcal{D}_n \) is loose. Its proof relies strongly on the proof of Proposition 11.3 above. A few remarks will hopefully make the proofs easier to follow.

**Remark 13.3.** For every \( k \in [n] \), we constructed in Example 7.4 a path \( a_k^H \) in the 1-skeleton of \( \mathcal{Y}_{n,2} \) from \( a_k \) to \( aaaaaa_k \). These paths were used in the proofs of Propositions 11.3 and 12.1.

Recall that \( a' = [ba] \) and \( a'_{k+1} = [a_ka_{k+1}] \); in all cases, the first edge in \( a_k^H \) is therefore \( a_k' \). The second vertex of \( a_k' \) is \( a_k^* \):

\[
a^* = bab, \quad b^* = aba, \quad a_{k+1}' = a_ka_{k+1}a_k.
\]

After that, every vertex and edge corresponds to a word with at least one letter of dimension 0. Thus, all vertices and edges in \( a_k^H \) are labeled by words which
contain at least one letter from $Y_2$: such words therefore belong to $I_{Y_2}$. Thus, all paths are in $Y_{n,2}$.

In the proof of Proposition 12.1 we also constructed $(a'_k)^H$, a map from the square to $D_n$ which equals $a'_k$ at the top, $aaaaa_k$ at the bottom, $a_k^H$ at one side and $a_{k-1}^H a_k a_{k-1}$ at the other side. Since $a_k^H$ is constructed to be the concatenation of the other three sides, this map is rather trivial, a mere stretching of the path $a_k^H$. Again, all words belong to $I_{Y_2}$ and therefore the map assumes values in $Y_{n,2}$.

We completed this construction by considering $a' = [ba]$, with boundary points $a$ and $bab$. In Figure 15 we construct (or at least show how to construct) a map $(a')^H = [ba]^H$ from the square to $D_n$ which equals $a'$ at the top, $aaaaa'$ at the bottom, $a^H$ at one side and $b^H ab$ at the other side. The map also assumes values in $Y_{n,2}$. This is clear for the explicit part of the figure; in the remaining part we may also assume that at least one letter of dimension 0 is present.

Remark 13.4. Recall from Remark 11.4 the maps $f_0, f_1 : S^0 \to D_n[I_{[0]}] \subset D_n$ with $f_0(+1) = a, f_0(-1) = aaaaa, f_1 = aaaa f_0$. As we have seen, the maps $f_0$ and $f_1$ are homotopic in $D_n$ but not in $D_n[I_{[0]}]$. Indeed, the only way to move away from the vertex $a$ is through the edge $[ba]$, which joins it to the vertex $bab$. Since $[ba] \in Y_2$, it is clear that the obvious homotopy from $f_0$ to $f_1$ assumes values in $Y_{n,2}$, consistently with Proposition 13.2.

More generally, in the proof of Proposition 11.3 we show that there exists a homotopy $H$ from a continuous map $f_0 : K \to D_n[I_{[0]}]$ to $aaaa f_0$. The proof shows how to construct $H$, or at least parts of $H$: we pass from $f_0$ to $f_1$, then to $f_2$ and $f_3$. (The functions $f_*$ are as in the proof of Proposition 11.3.) It follows from Remark 11.4 that in general the image of the homotopy $H$ can not possibly be contained in $L_n[I_{[0]}]$. Indeed, in the passage from $f_0$ to $f_1$ we typically move from $(p_1, a_i, p_2)$ to $(p_1, a_i^*, p_2)$ through $c_{w_1} \times c_{a_i} \times c_{w_2}$. Unless $w_1$ or $w_2$ contain letters of dimension 0 (which we have no reason to expect) then $w_1 a'_i w_2 \notin I_{[0]}$.

On the other hand, from $f_1$ onwards, the homotopy $H$ assumes values in $L_n[I_{[0]}]$. Indeed, the curves then contain a large number of loops.

Also, the homotopy $f_0$ to $f_1$ assumes values in $L_n[I_{Y_2}] \supset L_n[I_{[0]}]$. Indeed, the paths $\delta_i$ assume values in $L_n[I_{Y_2}]$, as we saw in Remark 13.3.

We thus see that, for $f_0$ as above, there exists a homotopy $H : [0, 1] \times K \to L_n[I_{Y_2}]$ from $f_0$ to $aaaa f_0$.

Proof of Proposition 13.2 The idea is to follow the proof of Proposition 11.3 performing the necessary adaptations and verifying that the homotopies assume values in $Y_{n,2}$. We will notice that part of the proof which requires significant comments and adaptations is the passage from $f_0 : K \to D_n$ to $f_1$. The functions
$f_*$ are as in the proof of Proposition 11.3, thus, curves in the image of $f_1$ have multiple loops.

Recall that we begin the proof of Proposition 11.3 by taking a finite cover of $K$ by sets $U_j$. In that proof we have that $p \in U_j$ implies $f_0(p) \in c_w(p)$ with $w(p)$ of the form $w_-(p)a_iw_+(p)$. In the present proof we instead have that $p \in U_j$ implies $f_0(p) \in c_w(p)$ with $w(p)$ of one of the three forms: $w_-(p)a_iw_+(p)$, $w_-(p)a'_iw_+(p)$ or $w_-(p)a^*_iw_+(p)$. Construct maps $\delta_i : [0, 1] \rightarrow D_n$ as in Equation (12) (in the proof of Proposition 11.3) satisfying $\delta_i(t) \in c_{a'_i}$ for $t \in (0, \frac{1}{2})$, $\delta_i(\frac{1}{2}) \in c_{a'_i}$ and assuming values in $Y_{n,2}$. Instead of having $f_0$ as in Equation (13), we now have

$$f_0(p) = (p_1, \delta_{i_1}(s_{0,j_1}), \ldots, \delta_{i_k}(s_{0,j_k}), p_{k+1}),$$

with $s_{0,j_*} = s_{0,j_1}(p) \in [0, \frac{1}{2}]$. Recall that in the construction of $f_1$ in proof of Proposition 11.3 we define $s_{j_*} \in [0, 1]$, continuous functions of $p \in K$. In the old proof, the functions $s_{j_*}$ are bumps, equal to zero near the boundary of $U_j$; it is crucial that for every $p$ there exists at least one $j_*$ such that $p \in U_{j_*}$ and $s_{j_*}(p) = 1$. Define new function $s_{j_*}$ by $s_{j_*}(p) = \max\{s_{0,j_*}(p), s_{j_*}^{old}(p)\}$ so that we have $s_{0,j_*}(p) \leq s_{j_*}(p)$ (for all $p$). The new functions $s_{j_*}$ are also bumps, equal to $s_{0,j_*}$ near the boundary of $U_{j_*}$; the crucial property above still holds. The function $f_1$ is now defined as in Equation (14), using the new $s_{j_*}$:

$$f_1(p) = (p_1, \delta_{i_1}(s_{j_1}), \ldots, \delta_{i_k}(s_{j_k}), p_{k+1}).$$

Notice that (for all $p$) $f_1(p)$ has multiple loops, as desired. Finally, for the homotopy $H_1$ from $f_0$ to $f_1$, instead of Equation (15), we now write

$$H_1(p, s) = (p_1, \delta_{i_1}(\text{med}(s_{0,j_1}, s, s_{j_1})), \ldots, \delta_{i_k}(\text{med}(s_{0,j_k}, s, s_{j_k})), p_{k+1});$$

here med denotes the median among three real numbers. Notice that $f_0$, $f_1$ and $H_1$ all assume values in $Y_{n,2}$.

As discussed in Remark 13.4, the remainder of the proof (or construction) preserves at least one letter of dimension 0, so that the remaining homotopies assume values in $Y_{n,2}$, completing the proof.

\[\square\]

14 Proof of Theorem 3

Let $I \subseteq W_n$: if $I$ is a lower set (for either $\subseteq$ or $\preceq$) then $D_n[I] \subseteq D_n$ is a subcomplex (and therefore a closed subset). Here, of course, $D_n[I]$ is the subcomplex with cells $c_w$, $w \in I$. On the other hand, at least in principle, we do not need to prove that $I$ is a lower set to deduce that $D_n[I]$ is a subcomplex. Indeed, we presently define a weaker condition.
Two possibilities are shown in Figure 8: if we choose the left one then yes, it compatible? It depends on our choice of the cell \( c \)
\[ \text{set } I \text{ in such a way that if } w_0, w_1 \text{ are words with } \dim(w_1) = \dim(w_0) - 1 \text{ and } c_{w_1} \not\subseteq B_{w_0} \text{ then the cell } c_{w_1} \text{ is not necessary in order to glue } c_{w_0}. \]
More precisely, we must then have \( c_{w_1} \cap B_{w_0} \subseteq B_{w_1}. \)

Indeed, if the glueing map \( g_{w_0} \) touches \( c_{w_1} \) in a non-surjective manner then we may deform \( g_{w_0} \) so that \( g_{w_0} \) does not touch the interior of \( c_{w_1}. \)
Notice that such a deformation does not change the homotopy type of the complex. From now on we assume that \( D_n \) satisfies this condition.

The subset \( I \subseteq W_n \) is subcomplex compatible if for all \( w_0 \in I \) and for all \( w_1 \in W_n \) with \( \dim(w_1) = \dim(w_0) - 1 \) and \( c_{w_1} \subseteq B_{w_0} \) we have \( w_1 \in I. \)

If \( I \) is subcomplex compatible then \( D_n[I] \subseteq D_n \) is a subcomplex. Also, if \( I \) is a lower set (for either \( \subseteq \) or \( \preceq \)) then \( I \) is subcomplex compatible.

Remark 14.1. The concept of subcomplex compatible depends on arbitrary and unspecified choices in the construction of \( D_n. \)
For instance, the following set is a lower set and therefore subcomplex compatible:
\[ I_1 = \{b, aba, cbc, acbca, acbac, cabac, [ab], [cb], a[cb]a, c[ab]c, acb[ac], [ac]bac\}. \]

The set \( I_2 = I_1 \cup \{[acb]\} \) is not a lower set for either \( \subseteq \) or \( \preceq; \) is it subcomplex compatible? It depends on our choice of the cell \( c_{[acb]} \) (or of its glueing map).

Two possibilities are shown in Figure 8 if we choose the left one then yes, \( I_2 \) is subcomplex compatible, if we choose the right one then no.

\[ \diamond \]

Lemma 14.2. Consider \( k \geq 2 \) and, for \( \sigma_0 \in Y \), let \( B_{\sigma_0} \subset D_n \) be the image of its glueing map.

1. If \( \sigma_0 \in Y_k \) then \( B_{\sigma_0} \subseteq D_n[I_{Y_k}] \).

2. If \( \sigma_0 \in Y_{k+1} \) is low and \( \text{inv}(\sigma_0) = k \) then \( B_{\sigma_0} \subseteq D_n[I_{Y_k}] \).

3. If \( \sigma_0 \) is high and \( \text{inv}(\sigma_0) = k + 1 \) then \( B_{\sigma_0} \subseteq D_n[I_{Y_k}] \cup C_{c_0} \).

In particular, \( I_{Y_k} \subset W_n \) is subcomplex compatible.

Proof. Consider \( w_1 = \sigma_{1,1} \cdots \sigma_{1,\ell} \in W_n \) such that \( \dim(w_1) = \dim(\sigma_0) - 1 \) and \( c_{w_1} \subseteq B_{\sigma_0}. \)
We know from Example 9.1 that \( w_1 \in Y \) so that there exists \( j \) with \( \sigma_{1,j} \in Y. \)
If \( \text{inv}(\sigma_0) \leq k \) then \( \dim(w_1) \leq k - 2 \) and therefore \( \text{inv}(\sigma_{1,j}) < k \) so that \( \sigma_{1,j} \in Y_k \) and therefore \( w_1 \in I_{Y_k}. \)

If \( \sigma_0 \) is high and \( \text{inv}(\sigma_0) = k + 1 \) then we may still have \( \text{inv}(\sigma_{1,j}) < k \), which implies \( w_1 \in I_{Y_k}. \)
The other possibility is \( \text{inv}(\sigma_{1,j}) = k \) and therefore \( \sigma_{1,j} \preceq \sigma_0; \) it then follows from computing dimensions that all other letters in \( w_1 \) have dimension 0. If \( \ell > 1 \) there are other letters \( \sigma_{1,j'} \) with \( \text{inv}(\sigma_{1,j'}) = 1 \) and therefore again \( w_1 \in I_{Y_k}. \)
If \( \sigma_{1,j} \not\preceq \sigma_0 \) we have \( \hat{\sigma}_{1,j} = \hat{w}_1 \) and therefore \( \ell > 1 \) (compare with Lemma 9.3). If \( \sigma_{1,j} \not\preceq \sigma_0 \) and
\(\sigma_{1,j} \neq \sigma_0^j\) then (from Lemma 2.6) we have that \(\sigma_{1,j}\) is high and therefore \(\sigma_{1,j} \in Y_k\) and therefore also in this case \(w_1 \in I_{Y_k}\). The last possible case is \(j = \ell = 1\) and \(\sigma_{1,j} = \sigma_0^j\). This completes the proof of the itemized claims.

In order to prove the final claim we need to consider the more general case \(w_0 = w_a\sigma_0^jw_b\) and \(w_1 = w_a\tilde{w}_1w_b\). But this case is analogous, except that the words \(w_a\) and \(w_b\) may occasionally help us by containing a letter in \(Y_k\).

We are interested in the subcomplexes \(Y_{n,k} = D_n[I_{Y_k}]\). For \(m = \text{inv}(\eta)\), we have \(Y_m = Y\) and therefore \(Y_{n,m} = D_n[I_Y]\). Notice that from Theorem 4 the CW complex \(Y_{n,m}\) is homotopy equivalent to the Hilbert manifold \(Y_n\).

We are almost ready to prove Theorem 3: let us first state a lemma.

**Lemma 14.3.** For all \(k \geq 2\), the subcomplex \(Y_{n,k} \subseteq D_n\) is loose.

**Proof.** This proof is by induction on \(k\), the base case \(k = 2\) being Proposition 13.2 above. Assume therefore that \(k \geq 3\) and that the subcomplex \(Y_{n,k-1} \subseteq D_n\) is known to be loose. Consider a compact manifold \(K\) and a continuous map \(\alpha_0 : K \to Y_{n,k}\): we claim that \(\alpha_0\) is homotopic in \(Y_{n,k}\) to \(\alpha_1 : K \to Y_{n,k-1}\).

Proving the claim completes the proof of the lemma.

Consider \(Y_{n,k}\) as a CW complex. Let \(\sigma \in Y_k \setminus Y_{k-1}\) be a low permutation, so that \(\sigma' \in Y_k \setminus Y_{k-1}\) be a high permutation, \(\text{inv}(\sigma) = k - 1\) and \(\text{inv}(\sigma') = k\). Proposition 9.1 tells us that there is a valid collapse starting from \(Y_{n,k}\) and removing the cells \(c_\sigma\) and \(c_{\sigma'}\):

\[
Y_{n,k} \searrow (Y_{n,k} \setminus (c_\sigma \cup c_{\sigma'})).
\]

(We assume here only minimal knowledge of collapses; see [8] or [10] for much more on the subject.) In other words, by applying a homotopy with values in \(Y_{n,k}\) to the map \(\alpha_0\) we obtain a map which avoids the cells \(c_\sigma\) and \(c_{\sigma'}\). From Proposition 9.1 the new cells possibly touched by the new map are in \(Y_{n,k-1}\).

A similar collapse operation holds for pairs of the form \(c_{w_\sigma w_+}\) and \(c_{w_\sigma' w_+}\). Since \(K\) is compact, a finite number of such collapses obtains \(\alpha_1\) with image contained in \(Y_{n,k-1}\).

**Remark 14.4.** The above proof can be rewritten as defining maps \(\sigma^H\) and \((\sigma')^H\) for \(\sigma \in Y_k \setminus Y_{k-1}\) a low permutation. For instance, consider \(k = 3\), \(\sigma = cb = [1342\cdots]\) and \(\sigma' = acb = [3142\cdots]\). The construction of \(\sigma^H\) is illustrated in Figure 14.

More precisely, we take cells \(\sigma'\) next to \(\sigma\) and \(aaaa\sigma'\) next to \(aaaa\sigma\). For another face \(w\) of \(\sigma'\), we have \(w \in I_{Y_{k-1}}\) so that there is no difficulty in joining \(w\) to \(aaaaaw\): the map \(w^H\) has already been constructed. As with the pairs \((a_k, a_k')\) \((k > 1)\), the construction of \(\sigma^H\) gives us as a bonus \((\sigma')^H\).
Proof of Theorem 3. For \( q \in \text{Quat}_{n+1} \setminus Z(\text{Quat}_{n+1}) \) we have \( \mathcal{Y}_n(1; q) = \mathcal{L}_n(1; q) \). For \( q \in Z(\text{Quat}_{n+1}) \), the proper subset \( \mathcal{Y}_n(1; q) \subset \mathcal{L}_n(1; q) \) is clearly open. Apart from the contractible connected component of convex curves (with itinerary equal to the empty word), its closed complement \( M_n(1; q) = \mathcal{L}_n(1; q) \setminus \mathcal{Y}_n(1; q) \) is the union of the strata corresponding to the words formed by parity alternating permutations. For \( n \neq 3 \), non trivial parity alternating permutation has at least 3 inversions; for \( n = 3 \) at least 2 inversions. Thus \( M_n(1; q) \setminus \mathcal{L}_n,\text{convex}(1; q) \) is a closed subset of codimension at least 1 (at least 2 for \( n \neq 3 \)). This implies that \( \mathcal{Y}_n(1; q) \) is open and dense in \( \mathcal{L}_n(1; q) \setminus \mathcal{L}_n,\text{convex}(1; q) \), as claimed.

We now prove that the inclusion \( \mathcal{Y}_n(1; q) \subset \Omega \text{Spin}_{n+1}(1; q) \) is a weak homotopy equivalence. Indeed, consider a continuous map \( \alpha_0 : S^k \to \Omega \text{Spin}_{n+1}(1; q) \).

We already saw that the add-loop construction obtains a homotopic map \( \alpha_1 : S^k \to \mathcal{L}_n(1; q) \subset \Omega \text{Spin}_{n+1}(1; q) \). We may assume that after the add-loop construction, every locally convex curve \( \alpha_1(s) \) in the image of \( \alpha_1 \) begins with a non-convex closed arc. By homotopy (more precisely, by connectivity of the non-convex component of \( \mathcal{L}_n(1; 1) \), as in Proposition 7.1), this initial non-convex closed arc can be assumed to have an itinerary starting with \( aaaaaa \). This implies that \( \alpha_1 \) has the form \( \alpha_1 : S^k \to \mathcal{Y}_n(1; q) \subset \Omega \text{Spin}_{n+1}(1; q) \).

Similarly, consider \( \alpha_0 : S^k \to \mathcal{Y}_n(1; q) \). Assume there exists a \( \beta_0 : D^{k+1} \to \Omega \text{Spin}_{n+1}(1; q), \beta_0|_{S^k} = \alpha_0 \). By Lemma 14.3 \( \alpha_0 \) is homotopic in \( \mathcal{Y}_n(1; q) \) to \( \alpha_1 = aaaaaaaa \alpha_0 \). From the add-loop construction, \( aaaaaa \alpha_0 \) is homotopically trivial in \( \mathcal{L}_n(1; q) \), that is, there exists \( \beta_2 : D^{k+1} \to \mathcal{L}_n(1; q) \) with \( \beta_2|_{S^k} = aaaaaa \alpha_0 \). Define \( \beta_1 = aaaa \beta_2 \); we have \( \beta_1 : D^{k+1} \to \mathcal{Y}_n(1; q) \) with \( \beta_1|_{S^k} = \alpha_1 \). This completes the proof that the inclusion \( \mathcal{Y}_n(1; q) \subset \Omega \text{Spin}_{n+1}(1; q) \) is a weak homotopy equivalence. Since \( \mathcal{Y}_n(1; q) \) is a Hilbert manifold, the inclusion is also a homotopy equivalence.

\[ \square \]

15 Final Remarks

The present paper casts the foundations of a combinatorial approach in the study of spaces of locally convex curves. We proved a few results as a consequence but we believe this approach can be used to prove far more. We now state a conjecture another result we hope to prove using these methods in a forthcoming paper. First, however, we review an older result.

Recall that \( \text{Quat}_3 = \mathbb{Q}_8 = \{ \pm 1, \pm i, \pm j, \pm k \} \subset S^3 \subset \mathbb{H} \) where \( \mathbb{H} \) is the algebra of quaternions. The main result in [24] classifies the spaces \( \mathcal{L}_2(q), q \in \text{Quat}_3 \), into the following three weak homotopy types. For \( q \neq \pm 1 \), consistently with Corollary 1.1, we have \( \mathcal{L}_2(q) \cong \Omega S^3 \). For \( q \in \{ \pm 1 \} \) we have:

\[
\mathcal{L}_2(-1) \approx \Omega S^3 \vee S^0 \vee S^4 \vee S^8 \vee \cdots , \quad \mathcal{L}_2(1) \approx \Omega S^3 \vee S^2 \vee S^6 \vee S^{10} \vee \cdots . \tag{16}
\]
For \( n = 3 \) we have \( \text{Spin}_4 = S^3 \times S^3 \) for \( S^3 \subset \mathbb{H} \). We also have that \( \text{Quat}_4 \subset Q_8 \times Q_8 \) is generated by \((1, -1), (i, i)\) and \((j, j)\). This point of view is used in \([3]\) to obtain partial results concerning the homotopy type of \( L \).

Theorem \( \text{3} \) admits a minor improvement in the case \( n = 3 \) which should be helpful. Indeed, let

\[
Y = Y \cup \{ac, abc\} = S_4 \setminus \{e, aba, bach, abc, cba, abacba = \eta\}.
\]

Let \( \tilde{Y}_3 \subset L_3 \) be the open set of curves whose itinerary admits at least one letter in \( \tilde{Y} \); we clearly have \( Y_3 \subset \tilde{Y}_3 \subset L_3 \).

**Theorem 5.** For each \( q \in \text{Quat}_4 \), the inclusion \( \tilde{Y}_3 \cap L_3(1; q) = \Omega(\text{Spin}_4(1; q) \) is a weak homotopy equivalence.

**Proof.** Artificially define \([ac]' = [abc]\): notice that \([ac] \prec [abc]\), as expected. Proposition \( 9.1 \) holds for \( \sigma_0 = [ac] \) and \( \sigma_1 = [abc] \), as has been observed in Example \( 9.2 \) and can be verified from Figure \( 7 \): the cell \( c_{[abc]} \) has a side \( c_{[ac]} \); the five other sides all include a letter of dimension 0 (and are therefore in \( \tilde{Y}_{3,2} \)).

The proof of the present result from Theorem \( \text{3} \) (for \( n = 3 \)) is now similar to a step in the proof of Lemma \( 14.3 \): perform the necessary collapses for the pair \( \sigma = [ac] \prec \sigma' = [abc] \).

The following theorem is the main result in \([2]\).

**Theorem 6.** We have the following weak homotopy equivalences:

\[
\begin{align*}
L_3(1; 1) &\approx \Omega(S^3 \times S^3) \vee S^4 \vee S^8 \vee S^8 \vee S^{12} \vee S^{12} \vee \cdots, \\
L_3(1; -1) &\approx \Omega(S^3 \times S^3) \vee S^2 \vee S^6 \vee S^6 \vee S^{10} \vee S^{10} \vee \cdots, \\
L_3(1; -\hat{a} \hat{c}) &\approx \Omega(S^3 \times S^3) \vee S^0 \vee S^4 \vee S^4 \vee S^8 \vee S^8 \vee \cdots, \\
L_3(1; \hat{a} \hat{c}) &\approx \Omega(S^3 \times S^3) \vee S^2 \vee S^6 \vee S^6 \vee S^{10} \vee S^{10} \vee \cdots.
\end{align*}
\]

The above bouquets include one copy of \( S^k \); two copies of \( S^{(k+4)} \), \( \ldots, j + 1 \) copies of \( S^{(k+4)} \), \( \ldots \), and so on.

Recall that, in the notation from \([3]\), we have:

\[
\begin{align*}
L S^3((+1, +1)) & = L_3(1; 1), & L S^3((-1, -1)) & = L_3(1; -1), \\
L S^3((+1, -1)) & = L_3(1; -\hat{a} \hat{c}), & L S^3((-1, +1)) & = L_3(1; \hat{a} \hat{c}).
\end{align*}
\]

It seems to be perhaps within reach but certainly harder to use this combinatorial approach to determine the homotopy type of \( L_n(1; q) \) for \( n > 3 \) and \( q \in Z(\text{Quat}_{n+1}) \). We hope to be able to prove at least the following claim, which should be contrasted with Corollary \( 1.1 \).

**Conjecture 15.1.** Consider \( n > 3 \) and \( q \in Z(\text{Quat}_{n+1}) \). Then \( L_n(1; q) \) is not homotopically equivalent to \( \Omega(\text{Spin}_{n+1}) \). Moreover, if there are convex curves in \( L_n(1; q) \) then the connected component \( L_{n, \text{non-convex}}(1; q) \subset L_n(1; q) \) of non-convex curves is also not homotopically equivalent to \( \Omega(\text{Spin}_{n+1}) \).
Another interesting aspect of the subject is its relation with the theory of differential operators, mentioned in the Introduction. A linear differential operator can be canonically associated to a nondegenerate curve $\gamma : [0, 1] \to \mathbb{S}^n$ (see [18]). There is a Poisson structure in the space of the differential operators, given by the Adler-Gelfand-Dickey bracket [11, 12]. The identification above relates the spaces $\mathcal{L}_n(1; z)$ with symplectic leaves of this structure [18, 19]. Notice that the spheres appearing in the bouquets in Equation 16 and Theorem 6 are all even-dimensional. We wonder whether this is fortuitous or a manifestation of this symplectic structure; this question is worth clarification.

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