Similarity reductions, new traveling wave solutions, conservation laws of 
(2+1)-dimensional Boiti-Leon-Pempinelli system

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Abstract

In this article we obtain exact solutions of (2+1)-dimensional Boiti-Leon-Pempinelli system of nonlinear partial differential equations which describes the evolution of horizontal velocity component of water waves propagating in two directions. We perform the Lie symmetry analysis to the given system and construct one-dimensional optimal subalgebra which involves some arbitrary functions of spatial variables. Several new exact solutions are obtained by symmetry reduction using each of the optimal subalgebra. We then study the physical behavior of some exact solutions by numerical simulations and observed many interesting phenomena such as traveling waves, lump type solitons, kink and anti-kink type solitons, breather solitons, singular kink type solitons and etc. We construct several conservation laws of the system by using multipliers method. As an application, we study the nonlocal conservation laws of the system by constructing potential systems and appending gauge constraints.

Keywords: Lie symmetry; Conservation laws; Boiti-Leon-Pempinelli system; Exact solution; Traveling wave solution; Nonlocally related system; Nonlocal conservation law

1. Introduction

A wide range of nonlinear physical phenomena in the vast areas of scientific disciplines are depicted by the nonlinear coupled partial differential equations (PDEs). Many significant phenomena in physics and engineering are represented by such nonlinear PDEs. These systems describe multiple behaviors in various fields such as mathematical physics, fluid dynamics, chemistry, condensed matter, biophysics, plasma physics, optical fibers, biology and other areas of engineering. The exact solutions of such system of nonlinear PDEs play an important role in nonlinear science, especially in nonlinear physics, since they can yield very much physical information and more insight into the physical aspects of the problem and thus lead to applications like understanding the behavior of the physics associated with the problem and also to test and analyze numerical schemes. The exact solutions of nonlinear PDEs are very interesting and popular area of research in nonlinear mathematical physics. However, no effective method has been proposed till date to derive the general solution of nonlinear PDEs; only special solutions can be obtained by a few methods such as inverse scattering transformation, Bäcklund transformation, Darboux transformation, Hirota’s direct method, Painleve analysis, symmetry reductions, variable separation approach, homogeneous balance method, F-expansion method and etc.

Symmetry analysis is one of the most efficient tool and easy to implement when searching for some particular exact solutions to differential equations. A symmetry of system of PDEs is one-parameter Lie group of transformations which leaves the given system invariant, or more precisely, a symmetry of PDE system leaves the solution manifold of that system invariant and it maps one solution to another solution of the given PDE system. Once one has determined the symmetry group of a system of differential equations, based on it variety of applications are available. One of the most important and useful application of symmetry method is to obtain systematically some classes of exact solutions. A particular solution obtained from symmetry group G is group invariant solution corresponding to the group G. For a

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given PDE system there may exist infinitely many particular solutions, so one needs to minimize the search for exact solutions. In this context, the concept of classification of optimal subalgebras was introduced by Ovsiannikov[7] where one needs to find a set of inequivalent subalgebras. Later, Olver[8] improved this method by introducing adjoint representations. Recently, many mathematicians[9, 10, 11, 12, 13, 14] contributed in this direction and obtained exact solutions of various physically relevant systems.

Conservation laws[15, 16] describe many essential physical properties of a given PDE system and have also applications in existence, uniqueness and stability analysis for the development of numerical methods. Moreover, one can construct nonlocally related PDE systems of the original PDE system by introducing some potential (nonlocal) variables through conservation laws. A PDE may have more than one conservation law which arises by multiplying appropriate multipliers[17, 18, 19] to the given PDE. Recently, Sil et al. [20, 21] applied direct multipliers method to construct conservation laws and applied them to study nonlocal symmetry analysis.

For the case where the associated Lie algebra is of infinite dimensional, the classification of optimal subalgebra is of special interest. Here one obtains an infinite number of exact solutions as the corresponding symmetries involve some arbitrary functions of independent variables or dependent variables.

The (2+1)-dimensional Boiti-Leon-Pempinelli (BLP) system[22]

\[ u_{ty} - (u^2 - u_x)_{xy} - 2v_{xxx} = 0, \]
\[ v_t - v_{xx} - 2uv_x = 0 \]

is actually a generalization of the (2+1)-dimensional sinh-Gordon equations. The Hamiltonian structure, Painlevé property[23], Lax pair[22], and Bäcklund transformation[24, 25] have been studied for BLP system[11] and moreover various exact solutions[26, 27, 28] were obtained by using tanh-coth method[29], CTE solvability[30], improved projective equation approach and a linear variable separation approach[31]. Later Kumar and Kumar[32] studied the BLP system[11] in terms of Lie symmetry analysis and obtained one family of solutions consisting of arbitrary function. Very recently Wang et. al[33] proposed the modified BLP system

\[ R_1 : u_{ty} = a(u^2 - u_x)_{xy} + bv_{xxx} \]
\[ R_2 : v_t = cv_{xx} + guv_x \]

by introducing some new parameters a, b, c and g where a, b, c and g are real numbers. It describes the evolution of the horizontal velocity component of water waves propagating in x and y directions in an infinite narrow channel of constant depth. In fact, it demonstrates the evolution of the horizontal component of the velocity of water waves propagating through an infinite narrow channel that maintains constant depth of the x – y plane. Here t denotes time, x, y represents spatial variables, the dependent variables u and v demonstrate the velocity component in x and y directions respectively. In[33], the authors obtained only the stationary domain walls solution of (2). There is a major research gap in the direction of obtaining exact solutions of the BLP system[22]. Since Lie symmetry analysis is the most powerful tool to construct exact solutions of nonlinear system of PDEs, therefore our aim is to study the BLP system[22] by means of Lie symmetry analysis and obtain several new exact solutions by constructing set of optimal subalgebras. Moreover, we construct conservation laws of the BLP system[22] by direct multiplier method. The outline of our work is as follows:

In section 2 we apply the Lie symmetry method to the BLP system[22] and compute the infinite dimensional Lie algebra. We perform the optimal classification of one-dimensional subalgebra consisting of arbitrary functions in section 3. Section 4 deals with obtaining several new exact solutions systematically from each subalgebra which are reported first time in the literature and also discuss the physical significance of the solution profiles geometrically. We obtain several new traveling wave solutions these indicate various important physical properties in section 5. In section 6 we construct conserved vectors of the BLP system[22]. We study the nonlocal conservation laws of system[22] as an application of those conserved vectors in section 7. Finally we provide concluding remarks in section 8.
2. Lie symmetry analysis

We apply the Lie symmetry analysis to the given system (2). Let us consider a one-parameter($\epsilon$) infinitesimal Lie group of point transformations of the form

$$
\begin{align*}
t^* &= t + \epsilon \tau(t, x, y, u, v) + O(\epsilon^2), \\
x^* &= x + \epsilon \xi(t, x, y, u, v) + O(\epsilon^2), \\
y^* &= y + \epsilon \pi(t, x, y, u, v) + O(\epsilon^2), \\
u^* &= u + \epsilon \eta(t, x, y, u, v) + O(\epsilon^2), \\
v^* &= v + \epsilon \phi(t, x, y, u, v) + O(\epsilon^2),
\end{align*}
$$

where $\epsilon$ is a parameter and $\tau, \xi, \pi, \eta$ and $\phi$ are unknown infinitesimals which are to be determined. The associated Lie symmetry generator takes the form

$$
\Psi = \tau(t, x, y, u, v) \frac{\partial}{\partial t} + \xi(t, x, y, u, v) \frac{\partial}{\partial x} + \pi(t, x, y, u, v) \frac{\partial}{\partial y} + \eta(t, x, y, u, v) \frac{\partial}{\partial u} + \phi(t, x, y, u, v) \frac{\partial}{\partial v}.
$$

Suppose $\Psi^{(3)}$ is a 3rd prolongation of $\Psi$, then the symmetry determining equations are

$$
\begin{align*}
\Psi^{(3)}(R_1)|_{\{R_1=0, R_2=0\}} &= 0, \\
\Psi^{(3)}(R_2)|_{\{R_1=0, R_2=0\}} &= 0,
\end{align*}
$$

where the given PDE system is denoted as $R_1 = 0, R_2 = 0$ which results an overdetermined linear system of PDEs. After solving the determining system we obtain the unknown infinitesimals as

$$
\begin{align*}
\tau &= c_3 - 2c_4 t, \\
\xi &= c_1 - 2c_4 x, \\
\pi &= c_2 - F_1(y), \\
\eta &= c_4 u, \\
\phi &= F_2(y) + v F'_1(y)
\end{align*}
$$

where $c_1, c_2, c_3$ and $c_4$ are arbitrary constants while $F_1(y)$ and $F_2(y)$ are arbitrary functions of $y$. Consequently the Lie symmetry generators are listed as

$$
\begin{align*}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial y}, \\
X_3 &= \frac{\partial}{\partial t}, \\
X_4 &= -2t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, \\
X_5 &= F_2(y) \frac{\partial}{\partial v}, \\
X_6 &= -F_1(y) \frac{\partial}{\partial y} + v F'_1(y) \frac{\partial}{\partial v}.
\end{align*}
$$

Now, the corresponding one-parameter Lie group of transformations can be obtained by solving the
following initial value problem:

\[
\begin{align*}
    \frac{d\tau}{d\epsilon} &= \tau(t, x, y, u, v), \\
    \frac{d\xi}{d\epsilon} &= \xi(t, x, y, u, v), \\
    \frac{d\pi}{d\epsilon} &= \pi(t, x, y, u, v), \\
    \frac{d\eta}{d\epsilon} &= \eta(t, x, y, u, v), \\
    \frac{d\phi}{d\epsilon} &= \phi(t, x, y, u, v),
\end{align*}
\]

with the initial data \(\tau = t, \xi = x, \pi = y, \eta = u\) and \(\phi = v\) when \(\epsilon = 0\). On solving the above system of ODEs for each infinitesimal transformation, we obtain the corresponding one-parameter Lie group of transformations

\[
\begin{align*}
    G_1 : (t^*, x^*, y^*, u^*, v^*) &= (t, x + \epsilon, y, u, v), \\
    G_2 : (t^*, x^*, y^*, u^*, v^*) &= (t, x, y + \epsilon, u, v), \\
    G_3 : (t^*, x^*, y^*, u^*, v^*) &= (t + \epsilon, x, y, u, v), \\
    G_4 : (t^*, x^*, y^*, u^*, v^*) &= (t e^{-2\epsilon}, x e^{-\epsilon}, y, u e^{\epsilon}, v).
\end{align*}
\]

It is difficult to compute the group of transformations associated with the symmetries \(X_5\) and \(X_6\) since they involve some arbitrary functions and thus cannot be integrated. In the view of the above discussion on one-parameter Lie group of point transformations we have the following result:

**Theorem 1.** Let \(u = g_1(t, x, y)\) and \(v = g_2(t, x, y)\) be a solution of the given system (2). The group actions \(G_1, G_2, G_3\) and \(G_4\) acting on the solution surface \(u - g_1 = 0, v - g_2 = 0\) provide one-parameter family of solutions

\[
\begin{align*}
    G_1 : u &= g_1(t, x - \epsilon, y), v = g_2(t, x - \epsilon, y), \\
    G_2 : u &= g_1(t, x, y - \epsilon), v = g_2(t, x, y - \epsilon), \\
    G_3 : u &= g_1(t - \epsilon, x, y), v = g_2(t - \epsilon, x, y), \\
    G_4 : u &= e^\epsilon g_1(te^{2\epsilon}, xe^{\epsilon}, y), v = g_2(te^{2\epsilon}, xe^{\epsilon}, y).
\end{align*}
\]

3. Classification of optimal subalgebras

In this section, we discuss the structure of the infinite dimensional Lie algebra \(\mathfrak{g}\). The commutator table, corresponding to the symmetries \(X_i\) for \(i = 1, ..., 6\), is presented in the Table 1 where the entry in the \(ij\)-th position of the table is defined as

\[
[X_i, X_j] = X_i X_j - X_j X_i \text{ for } i, j = 1, ..., 6.
\]

| \([X_i, X_j]\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) | \(X_5(F_2)\) | \(X_6(F_1)\) |
|-----------------|-------|-------|-------|-------|-------------|-------------|
| \(X_1\)        | 0     | 0     | 0     | \(-X_1\) | 0           | 0           |
| \(X_2\)        | 0     | 0     | 0     | 0     | \(X_5(F_2(y))\) | \(X_6(F_1(y))\) |
| \(X_3\)        | 0     | 0     | \(-2X_3\) | 0     | 0           | 0           |
| \(X_4\)        | \(X_3\) | 0     | 2\(X_3\) | 0     | 0           | 0           |
| \(X_5(F_2)\)   | 0     | \(-X_5(F_2(y))\) | 0     | 0     | \(X_5(F_1F_2 + F_2F_1)\) | 0           |
| \(X_6(F_1)\)   | 0     | \(-X_6(F_1(y))\) | 0     | 0     | \(-X_5(F_1F_2 + F_2F_1)\) | 0           |

Table 1: Commutator table corresponding to the Lie algebra \(\mathfrak{g}\)

4
For the construction of inequivalent set of optimal subalgebras, we need to find the adjoint representation of the symmetries. The adjoint action on $\mathfrak{L}$ is defined by the adjoint operator as

$$\text{Ad}_{\exp(\xi X_i)} X_j = e^{-\epsilon X_i} X_j e^{\epsilon X_i},$$

where $\epsilon$ being the small parameter.

| $\text{Ad}_{\exp(\xi X_i)} (X_j)$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5(F_2)$ | $X_6(F_1)$ |
|---|---|---|---|---|---|---|
| $X_1$ | $X_1$ | $X_2$ | $X_3$ | $X_4 + \epsilon X_1$ | $X_5(F_2)$ | $X_6(F_1)$ |
| $X_2$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5(F_2(y - \epsilon))$ | $X_6(F_1(y - \epsilon))$ |
| $X_3$ | $X_1$ | $X_2$ | $X_3$ | $X_4 + 2\epsilon X_3$ | $X_5(F_2)$ | $X_6(F_1)$ |
| $X_4$ | $e^{-\epsilon X_1}$ | $X_2$ | $e^{-2\epsilon X_3}$ | $X_4$ | $X_5(F_2)$ | $X_6(F_1)$ |
| $X_5(F_2)$ | $X_1$ | $X_2 + \epsilon X_5(F_2')$ | $X_3$ | $X_4$ | $X_5(F_2)$ | $X_6(F_1) - \epsilon X_5(F_1 F_2' + F_2 F_1')$ |
| $X_6(F_1)$ | $X_1$ | $X_2 + \epsilon X_6(F_1')$ | $X_3$ | $X_4$ | $X_5(F_2 + \epsilon (F_1 F_2'))$ | $X_6(F_1)$ |

Table 2: Adjoint table corresponding to the Lie algebra $\mathfrak{L}$

It can be defined as an infinite series form involving the Lie bracket which is given below:

$$\text{Ad}_{\exp(\xi X_i)} X_j = X_j - \epsilon [X_i, X_j] + \frac{\epsilon^2}{2} [X_i, [X_i, X_j]] - ....$$

The adjoint actions are summarized in the Table 2. It is to be noted that, due to complexity of the calculations we have used first two terms of the infinite series for computation of $\text{Ad}_{\exp(\xi X_0)} X_5$ without affecting the mathematical analysis for constructing the inequivalent optimal subalgebra. Now we perform the classification of inequivalent subalgebra using the adjoint table (refer Table 2). First consider a general element

$$E = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + X_5(F_2) + X_6(F_1)$$

where $a_1, a_2, a_3$ and $a_4$ are constants. The basic idea [34] is to make $E$ more simpler element, say, $E'$ by choosing appropriate constants while using suitable adjoint actions. Suppose we apply the adjoint action of $X_i$ on $E$, then the updated element is of the form

$$E' = \text{Ad}_{\exp(\xi X_i)} E = a_1' X_1 + a_2' X_2 + a_3' X_3 + a_4' X_4 + X_5 + X_6,$$  \hspace{1cm} (3)

where $a_i'$s are functions of $a_i$ and $\epsilon$. Here we perform the following cases:

**Case-I:** We set $a_1 \neq 0$ and let other constants be unrestricted. Without loss of generality we assume that $a_1 = 1$. By choosing $X_i = X_1$ in (3) we have

$$E' = (1 + a_4) X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + X_5(F_2) + X_6(F_1).$$

We cancel the $X_1$ term by choosing $\epsilon = -\frac{1}{a_4}$ and consequently we have

$$E' = a_2 X_2 + a_3 X_3 + a_4 X_4 + X_5(F_2) + X_6(F_1).$$

Then we apply the adjoint action of $X_3$ on $E'$, which yields $E'' = a_2 X_2 + (a_3 + 2\epsilon a_4) X_3 + a_4 X_4 + X_5(F_2) + X_6(F_1)$. Again by choosing $\epsilon = -\frac{a_3}{2a_4}$, we are having with

$$E'' = a_2 X_2 + a_4 X_4 + X_5(F_2) + X_6(F_1).$$

Similarly applying the successive adjoint action of $X_5, X_6$ and canceling the $X_5, X_6$ terms, we have $E''' = a_2 X_2 + a_4 X_4$, where $a_2 \in \{-1, 0, 1\}$ since generalized BLP system [2] admits the discrete symmetries $(t, x, y, u, v) \rightarrow (t, x, -y, u, -v)$. Hence we assume $a_2 = 0, a_2 = 1$ which results the optimal set in this case as $E_1 = X_2 + a_4 X_4$ and $E_2 = X_4$.

**Case-II:**
The optimal system of one-dimensional subalgebras of the generalized BLP system

**Theorem 2.** The optimal system of one-dimensional subalgebras of the generalized BLP system \( \mathcal{B} \) consists of the following vector fields:

\[
E_1 = < X_2 + a_4 X_4 >, \quad E_2 = < X_4 >, \quad E_3 = < a_4 X_4 + X_6(F_1) >, \quad E_4 = < X_6(F_1) >.
\]

4. Similarity reductions and invariant solutions

In this section, we obtain some group invariant solutions of the governing system \( \mathcal{B} \) by using each subalgebra in the optimal set.

4.1. Reduction using \(< X_2 + a_4 X_4 >\)

In this case, the corresponding characteristic equations are

\[
\frac{dt}{-2a_4t} = \frac{dx}{-a_4x} = \frac{dy}{1} = \frac{du}{a_4u} = \frac{dv}{0}.
\]
The similarity variables are \( m = \frac{t}{x} \), \( n = \frac{\ln(t) + 2a_4 y}{2a_4} \) and the corresponding similarity forms are

\[
\begin{align*}
  u(t, x, y) &= \frac{U(m, n)}{\sqrt{t}}, \quad v(t, x, y) = V(m, n)
\end{align*}
\]

where \( U(m, n) \) and \( V(m, n) \) are functions of \( m \) and \( n \) which are to be determined. Now, with this expression of \( u \) and \( v \), the governing system (3) reduces to the following system with fewer independent variables

\[
\begin{align*}
  a_4 m U_{mn} - U_{nn} + a_4 U_n + 4a_4U_n U_n - 4a_4 U U_{mmn} + 2a_4 U_{mnm} - 2b a_4 V_{mmn} = 0, \\
  a_4 m V_m - V_n + 2ca_4 V_{mm} + 2a_4 g U_V = 0.
\end{align*}
\]

In general it is not feasible to solve so we apply the Lie symmetry technique to compute some exact solutions of (4). Using the Lie symmetry reduction method, we have the following ansatz:

\[
\begin{align*}
  U(m, n) &= f_1(m), \quad V(m, n) = \frac{m}{s} + f_2(m)
\end{align*}
\]

where \( p, s \) are arbitrary constants and \( f_1(m), f_2(m) \) are unknown functions which are to be determined. With substitution of \( U \) and \( V \) in the above reduced system and solving the reduced system of ODEs we have

\[
\begin{align*}
  f_1(m) &= \frac{1}{2} \left[ \frac{p - 2a_4 C_1 - a_4 m (C_1 m + C_2)}{a_4 g (C_1 m + C_2)} \right], \\
  f_2(m) &= \frac{1}{2} C_1 m^2 + C_2 m + C_3
\end{align*}
\]

where \( C_1, C_2 \) and \( C_3 \) are integration constants. Thus, we have the solution for the given system (3) as follows:

\[
\begin{align*}
  u_1 &= -\frac{1}{2} \frac{a_4 s C_1 x^2 + a_4 s C_2 x \sqrt{t} + (2c C_1 a_4 s - p) t}{a_4 s g (C_1 x + C_2 \sqrt{t})}, \\
  v_1 &= \frac{1}{2} \frac{a_4 s C_1 x^2 + 2a_4 s C_2 x \sqrt{t} + (2a_4 C_3 + p \ln(t) + 2a_4 p y) t}{a_4 s \sqrt{t}}.
\end{align*}
\]

The physical behavior of the solution profile is illustrated in the Figure 1 by choosing \( a_4 = 1, s = 1, C_1 = 1, C_2 = 1, g = 1 \) and \( p = 1 \). Figure 1A represents single-lump soliton or 1-lump soliton for \( u_1 \). We illustrate the solution profile of \( v_1 \) in the Figure 1B by fixing \( y = 1 \). We noticed that (see, Figure 1B) rapid increase of \( v_1 \) near the initial time when we approach far away from the origin that corresponds to two peaks. As time evolves, it gradually decreases uniformly when increasing the values of \( x \). We depict the 2-dimensional plot of \( v_1 \) by fixing \( y = 1 \) with respect to \( x \) for various values of \( t \) in the Figure 1c. It represents an upward parabola with vertex at the origin and as time evolves the parabola started to flatten and tends to a straight line.

By choosing the ansatz as \( U(m, n) = f_3(n) \) and \( V(m, n) = f_4(n) \) and plugging into the reduced system of PDEs (3) we solve for the unknowns \( f_3, f_4 \) and obtain

\[
\begin{align*}
  f_3(n) &= C_4 \exp(a_4 n) + C_5, 
  f_4(n) &= C_6,
\end{align*}
\]

where \( C_4, C_5 \) and \( C_6 \) are integration constants. So, we have the following exact solution for the given system (2)

\[
\begin{align*}
  u_2 &= \frac{C_4 \exp \left[ \frac{1}{2} (\ln(t) + 2a_4 y) \right] + C_5}{\sqrt{t}}, 
  v_2 &= C_6.
\end{align*}
\]

4.2. Reduction using \(< X_1 >\)

In this case, the similarity variables are \( y \) and \( m = \frac{x}{\sqrt{t}} \) which leads to the invariant solution of the following form

\[
\begin{align*}
  u(t, x, y) &= \frac{1}{\sqrt{t}} U(m, y), 
  v(t, x, y) &= V(m, y).
\end{align*}
\]
Figure 1: Solution profile of (2) for a solution (5): (a) 3d profile of $u_1$  
(b) 3d profile of $v_1$ when $y = 1$  
(c) 2d profile of $v_1$ when $y = 1$ for various values of $t$.

Substituting this form of $u$ and $v$ into the given system (2), we derive the reduced system

\[
m U_{my} + U_y + 4aU_m U_y + 4aUU_{my} - 2aU_{mmy} + 2bV_{mmm} = 0, \tag{7}
m V_m + 2cV_{mm} + 2gUV_m = 0.
\]

After solving the above system (7), we have the following solution

\[
U(m, y) = \frac{1}{2g} \frac{C_7(-m^2 - 2c) - C_8 m}{C_7 m + C_8},
\]

\[
V(m, y) = \frac{C_7}{2} f_3(y) m^2 + C_8 f_3(y) + g_3(y),
\]

where $C_7$ and $C_8$ are arbitrary constants and $f_3(y), g_3(y)$ are arbitrary functions of $y$, which in turn the solution of the given system (2) as

\[
u_3 = \frac{1}{2} \frac{C_7 x^2 \sqrt{t} + 2cC_7 t^{\frac{3}{2}} + C_8 t x}{g t^2 (C_7 x + C_8 \sqrt{t})}, \tag{8}
\]

\[
v_3 = \frac{1}{2} \frac{C_7 x^2 \sqrt{t} f_3(y) + 2C_8 x t f_3(y) + 2t f_3(y)}{t^2}.
\]

By choosing the parameters $c = 1, g = 1, C_7 = 1, C_8 = 1$ and considering $f_3(y) = y, g_3(y) = y$ we depict the solution profile which represents (8) (see, Figure 2) for $u_3$ and $v_3$. The physical behavior of $u_3$ is demonstrated
in the Figure 2a which represents a multiple breather soliton. On the other hand, the 3-dimensional profile of $v_3$ is presented in the Figure 2b by fixing $y = 10$ and the 2-dimensional profile is illustrated in the Figure 2c with respect to $x$ at various values of $t$ by fixing $y = 10$ which demonstrates an upward parabola.

We have another solution of (7) as

$$U(m, y) = \frac{-m}{4a} \cosh(y - m), V(m, y) = f_5(y)$$

where $f_5(y)$ is an arbitrary function of $y$, which yields a solution of the given system as

$$u_4 = \frac{1}{4} \frac{-x + 4a\sqrt{t} \cosh(y - \frac{x}{\sqrt{t}})}{a t}, v_4 = f_5(y).$$

4.3. Reduction using $<a_4X_4 + X_6(F_1) >$

The basis associated with this subalgebra is $-2a_4t \frac{\partial}{\partial t} - a_4x \frac{\partial}{\partial x} - F_1(y) \frac{\partial}{\partial y} + a_4u \frac{\partial}{\partial u} + vF_1'(y) \frac{\partial}{\partial v}$ and the corresponding similarity variables are $m = \frac{t}{4a}$ and $n = -\ln(x) + a_4 \int \frac{1}{F_1(y)}dy$. With the help of these similarity variables we have the ansatz for $u$ and $v$ as

$$u(t, x, y) = \frac{U(m, n)}{x}, v(t, x, y) = \frac{V(m, n)}{F_1(y)}.$$
Using this ansatz for $u, v$ and substituting in the given system (2) we have the following reduced PDE system

\begin{align*}
a_4 U_{mn} + 4a_4 U_{m} U_{n} + 2a_4 U_{n}^2 + 4a_4 U_{U_{mn}} + 2a_4 U_{U_{n}n} + 4a_4 U U_{mn} + 4a_4 U_{mn} U_{n} + 10a_4 U_{m} U_{mn} + aa_4 U_{mn} + 3aa_4 U_{n} + 2aa_4 U_{n} + 8bm^2 V_{m} V_{mn} + 12bm^2 V_{m} V_{nn} + 36m^2 V_{m} V_{mn} + 6m V_{m} V_{nn} + 24bm V_{m} + 24bm V_{m} + 2b V_{m} + 28 V_n = 0,
+ 10aa_4 U_{mn} + 4aa_4 U_{m} U_{mn} + 2aa_4 U_{mn} + 4aa_4 U_{mn} U_{n} + 8bm^2 V_{m} V_{mn} + 12bm^2 V_{m} V_{nn} + 36m^2 V_{m} V_{mn} + 6m V_{m} V_{nn} + 24bm V_{m} + 24bm V_{m} + 2b V_{m} + 28 V_n = 0.
\end{align*}

In order to find the solution of the reduced system (10), we again apply the Lie symmetry analysis and derive the ansatz as $U(m, n) = f_6(m)$, $V(m, n) = C_9 + f_7(m)$ where $C_9$ is an arbitrary constant and $f_6(m), f_7(m)$ are unknown functions those have to be determined. By exploiting $U$ and $V$ into the reduced system (10) we have the following ODE system

\begin{align*}
4C_9 m^3 f_7''' - 18C_9 m^2 f_7'' + 12C_9 mf_7' + 1 = 0, \\
-C_9 f_7' + 4C_9 m^2 f_7' - 6C_9 m f_7' - 2gC_9 m f_6 f_7' - g f_6 + c = 0.
\end{align*}

Figure 3: Solution profile of (2) for a solution (11): (a) 3d profile of $u_5$ (b) 3d profile of $v_5$ when $y = 10$ (c) 2d profile of $v_5$ when $y = 10$ for various values of $t$.

By solving the above system of ODEs yields

\begin{align*}
f_6(m) &= -\frac{1}{4} \frac{-2C_9 C_{11} \sqrt{m} + 4C_9 C_{10} + (1 + 8C_9 C_{10}) m}{C_9 g \sqrt{m} (2C_{10} \sqrt{m} - C_{11} m)}, \quad f_7(m) = \frac{2C_{11}}{\sqrt{m}} + \frac{2C_{10}}{m} \ln(m) + C_{12},
\end{align*}
where $C_{10}, C_{11}$ and $C_{12}$ are arbitrary constants. Thus we have the exact solution of the given system (2) as

$$u_5 = -\frac{1}{4} x \left( -2C_9 C_{11} x \sqrt{t} + 4C_9 C_{10} x^2 + t + 8C_9 C_{10} t \right),$$

$$v_5 = -\frac{1}{2} \frac{2 t \ln(x) - 2 t a_4 \int \left( \frac{1}{F_1(y)} \right) dy + 4C_9 C_{11} x \sqrt{t} - 4C_{10} x^2 + t \ln \left( \frac{1}{x^2} \right) - 2C_9 C_{12} t}{C_9 t F_1(y)}.$$ (11)

We now discuss the physical significance of the solution profile (11) (see, Figure 3) which represents $u_5$ and $v_5$ by considering $c = 1, g = 1, a_4 = 1, C_9 = 1, C_{10} = 1, C_{11} = 1, C_{12} = 1$ and by choosing $F_1(y) = 1$. The solution profile of $u_5$ is illustrated in the Figure 3a which represents a 3-lump type soliton. The 3-dimensional profile 3c we observed that $v_5$ increases with respect to $x$ for different values of $t$. Here Figure 3b shows that $v_5$ increases rapidly if $x$ increases at initial time period and then it decreases gradually as time evolves. From the corresponding 2-dimensional profile 3d we observed that $v_5$ increases with respect to $x$ for any $t$ but as $t$ increases, the rate of increasing of $v_5$ gradually decreases and after some time (say, $t=5$) the curve becomes almost horizontal line.

Another solution of the reduced PDE system (10) is $U(m,n) = f_8(m), V(m,n) = C_{13}$ where $C_{13}$ is an arbitrary constant and $f_8(m)$ is an arbitrary function. Thus we have another solution of (2) as follows:

$$u_6 = f_8 \left( \frac{t}{x^2} \right), v_6 = \frac{C_{13}}{F_1(y)}.$$ (12)

We obtain another two solutions of the reduced system (10) as

$$U(m,n) = \frac{1}{2} 2C_{15} + C_{16} \sqrt{m} + 4C_{15} m,$$

$$V(m,n) = C_{14} + \frac{C_{15}}{m} + \frac{C_{16}}{\sqrt{m}}$$

and

$$U(m,n) = c(2 + n),$$

$$V(m,n) = \frac{1}{n}, \text{ if } a_4 = \frac{1}{2} \frac{b g^2}{ac(2c + g)}$$

where $C_{14}, C_{15}$ and $C_{16}$ are arbitrary constants.

Using the above two solutions of the reduced PDE system (10) for $U(m,n)$ and $V(m,n)$ we obtain two more exact solutions for the given system (2) as follows:

$$u_7 = -\frac{x}{2t} \frac{2C_{15} x^2 + C_{16} x \sqrt{t} + 4C_{15} t}{g(2C_{15} x^2 + C_{16} x \sqrt{t})},$$

$$v_7 = \frac{C_{14} t + C_{15} x^2 + C_{16} x \sqrt{t}}{t F_1(y)}.$$ (13)

and

$$u_8 = \frac{c(2 + \ln(x) - a_4 \left( \int \frac{1}{F_1(y)} \right) dy)}{g x \left( \ln(x) - a_4 \left( \int \frac{1}{F_1(y)} \right) dy \right)},$$

$$v_8 = -\frac{1}{\left( \ln(x) - a_4 \left( \int \frac{1}{F_1(y)} \right) dy \right) F_1(y)}, \text{ if } a_4 = \frac{1}{2} \frac{b g^2}{ac(2c + g)}.$$ (14)
Now we discuss the physical behavior of the solution profile (13) in the Figure 4 for $u_7$ and $v_7$ by choosing the parameters $c = 1$, $g = 1$, $C_{14} = 1$, $C_{15} = 1$, $C_{16} = 1$ and considering $F_1(y) = 2y^2$. The Figure 4a indicates a multiple breather soliton type solution for $u_7$. The 3-dimensional profile of $v_7$ is depicted in the Figure 4b by choosing $y = 10$ and Figure 4c illustrates the 2-dimensional profile of $v_7$ at fixed $y = 10$ with respect to $x$ for varying $t$ which indicates an upward parabola.

The physical significance of the solution profile (14) is depicted in the Figure 5 by considering the
parameters \( a = 1, b = 1, c = 1, g = 1, a_4 = \frac{1}{2} \frac{bg}{ac(2c + g)} = \frac{1}{6} \) and by setting \( F_1(y) = \frac{1}{y} \). The surface profiles of \( u_8 \) and \( v_8 \) are illustrated in the Figure 5(a) and Figure 5(b) respectively those indicate a multiple breather soliton type solutions.

4.4. Reduction using \( <X_6(F_1)> \)

The representative for this class of subalgebra is given by
\[
X_6(F_1) = -F_1(y) \frac{\partial}{\partial y} + vF_1'(y) \frac{\partial}{\partial v}.
\]
The governing system (2) with the similarity transformations
\[
u(t, x, y) = U(t, x), v(t, x, y) = \frac{V(t, x)}{F_1(y)}
\]

Figure 6: Solution profile of (2) for a solution (16): (a) \( u_9 \), (b) \( v_9 \) at \( t = 1 \), (c) \( v_9 \) at \( t = 3 \), (d) \( v_9 \) at \( t = 4 \), (e) \( v_9 \) at \( t = 5 \), (f) \( v_9 \) at \( t = 10 \), (g) \( v_9 \) at \( t = 50 \)
reduces to the following PDE system

\[ V_{xxx} = 0, \]
\[ V_t - cV_{xx} - gUV_x = 0. \]  

We easily solve this system (15) and obtain

\[ U(t, x) = \frac{1}{2} f_9'(t)x^2 + 2 f_{10}'(t)x + 2 f_{11}'(t) - 2c f_9(t), \]
\[ V(t, x) = \frac{1}{2} f_9'(t)x^2 + f_{10}(t)x + f_{11}(t) \]

where \( f_9, f_{10} \) and \( f_{11} \) are arbitrary functions of \( t \). Thus finally we have the solution for the given system of PDEs of the form

\[ u_9 = \frac{1}{2} f_9'(t)x^2 + 2 f_{10}'(t)x + 2 f_{11}'(t) - 2c f_9(t), \]
\[ v_9 = \frac{1}{2} f_9'(t)x^2 + f_{10}(t)x + f_{11}(t) \]

Now by choosing the parameters \( c = 1, g = 1 \) and considering \( F_1(y) = y, f_9(t) = t, f_{10}(t) = t^2 \) and \( f_{11}(t) = t \) we study the physical behavior of \( u_9 \) and \( v_9 \) given in (16) (see, Figure 6). The Figure 6 demonstrates a 5-lump type soliton or multiple lump type soliton profile for \( u_9 \). On the other hand we draw the 3-dimensional surface for \( v_9 \) with respect to the spatial variables \( x \) and \( y \) for different values of \( t \) in the Figure 6. It is very interesting to observe that initially at \( t = 1 \), the corresponding Figure 6 indicates 2-soliton profile where both the peaks are in the same (positive) direction. After some time, say \( t = 5 \), it annihilates into a 1-soliton or single soliton (see, Figure 6) and further later, say at \( t = 50 \), the profile \( v_9 \) again behaves like a 2-soliton solution (see, Figure 6) but in the opposite direction.

5. Traveling wave solutions

The exploration of the traveling wave solutions, in particular soliton solutions of nonlinear system of PDEs play a vital role in describing the characters of nonlinear problems in the area of engineering, applied science and mathematical physics. It also describes many interesting physical phenomena in the study of dynamical systems. Here we consider the traveling wave solution of the form

\[ u(t, x, y) = U \left( \frac{l_1x - l_2t, l_1y - l_3t}{l_1} \right), \]
\[ v(t, x, y) = V \left( \frac{l_1x - l_2t, l_1y - l_3t}{l_1} \right) \]

which is invariant under the symmetry \( l_2X_1 + l_3X_2 + l_1X_3 \). Using this form of \( u, v \) and exploiting them into the given system (2) which yields the reduced system of PDEs

\[ l_2U_{mn} + l_3U_{nn} + 2al_1U_mU_n + 2al_1UU_{mn} - al_1U_{mmn} + bl_1V_{mnn} = 0, \]
\[ l_2V_m + l_3V_n + cl_1V_{mn} + gl_1VV_m = 0 \]

where \( m = \frac{l_1x - l_2t}{l_1} \) and \( n = \frac{l_1y - l_3t}{l_1} \). In order to solve the system (17), we use the Lie symmetry approach and obtain the solutions of (17) in the form of \( U(m, n) = f_{12}(m), V(m, n) = \frac{C_{18}m}{C_{18}n} + f_{13}(m) \) and \( U(m, n) = f_{14}(n), V(m, n) = \frac{C_{19}m}{C_{19}n} + f_{15}(n) \) where \( C_{17}, C_{18} \) and \( C_{19} \) are arbitrary constants and \( f_{12}(m), f_{13}(m), f_{14}(n) \) and \( f_{15}(n) \) are unknown functions.

Then substituting these forms of \( U, V \) into the reduced system (17) and solving the corresponding ODE systems we obtain

\[ f_{12}(m) = \frac{l_2C_{18}C_{20}m + l_3C_{18}C_{21} + l_3C_{19} + cl_1C_{18}C_{20}}{gl_1C_{18}(C_{20}m + C_{21})}, \]
\[ f_{13}(m) = \frac{1}{2} C_{20}m^2 + C_{21}m + C_{22} \]

and

\[ f_{14}(n) = C_{24}n + C_{25}, \]
\[ f_{15}(n) = \frac{1}{2} - gl_1C_{19}C_{24}n^2 - 2C_{19}n(l_2 + gl_1C_{25}) + 2l_3C_{17}C_{23}}{l_3C_{17}} \]
where $C_{20}, \ldots, C_{25}$ are integration constants which in turn yields the following solutions of the given system

$$u_{10} = \frac{l_2 C_{18} C_{20} (l_1 x - l_2 t) + l_1 l_2 C_{18} C_{21} + l_1 l_3 C_{19} + \frac{c_1^2}{2} C_{18} C_{20} - l_1 C_{21}}{l_1 g C_{18} (l_2 t - l_1 x) - l_1 C_{21}},$$

$$v_{10} = \frac{1}{2} \frac{2 l_1 C_{19} (l_1 y - l_3 t) + C_{18} C_{20} (l_1^2 t^2 + l_1^2 x^2) - 2 l_1 l_2 C_{18} C_{20} t x + 2 l_1 C_{18} C_{21} (l_1 x - l_2 t) + 2 l_1^2 C_{18} C_{22}}{l_1^2 C_{18}}$$

and

$$u_{11} = \frac{C_{24} (l_1 y - l_3 t) + l_1 C_{25}}{l_1},$$

$$v_{11} = \frac{1}{2}\frac{2 l_1 C_{19} (l_1 x - l_2 y) - g C_{18} C_{24} (l_1^2 t^2 + l_1^2 y^2) + 2 g l_1 C_{19} C_{25} (l_3 t - l_1 y) + 2 l_1 l_3 C_{17} C_{23}}{l_1 l_3 C_{25}}.$$

We now discuss the physical significance of the solution profile of (18) by choosing the parameters $c = 1, g = 1, l_1 = 1, l_2 = 1, l_3 = 1, C_{18} = 1, C_{19} = 1, C_{20} = 1, C_{21} = 1$ and $C_{22} = 1$ in the Figure 7. Figure 7a demonstrates the singular kink traveling wave solution profile for $u_{10}$. While we illustrate the surface profile of $v_{10}$ in the Figure 7b by choosing $y = 10$ which indicates an upward parabola.

We compute some other solutions of the reduced PDE system (17) as given by

$$U(m, n) = \frac{2 c C_{27} \tanh \left( C_{26} + C_{27} m - \frac{t_s C_{27} n}{l_3} \right)}{g}, V(m, n) = C_{28} - \frac{2 a c l_2 C_{27} (2 c + g) \tanh \left( C_{26} + C_{27} m - \frac{t_s C_{27} n}{l_3} \right)}{b g^2 l_3}$$

and

$$U(m, n) = -\frac{1}{2} \frac{l_2 C_{31} + l_3 C_{32}}{a l_1 C_{31}} - C_{31} \tanh (C_{31} m + C_{32} n + C_{30}), V(m, n) = C_{29}$$

where $C_{26}, \ldots, C_{32}$ are arbitrary constants. This again yields to new exact solutions of the given system (2) as follows:

$$u_{12} = \frac{2 c C_{27} \tanh \left( \frac{C_{26} (l_1 x - l_2 y) + l_s C_{26}}{l_3} \right)}{g},$$

$$v_{12} = \frac{b g^2 l_3 C_{28} - 2 a c C_{27} l_2 (2 c + g) \tanh \left( \frac{C_{26} (l_1 x - l_2 y) + l_s C_{26}}{l_3} \right)}{b g^2 l_3}$$

and

$$u_{13} = -\frac{1}{2} \frac{l_2 C_{31} + l_3 C_{32} + 2 a l_1 C_{31} \tanh \left( \frac{l_3 C_{30} - (l_2 C_{31} + l_3 C_{32}) t + l_1 C_{31} C_{27} + l_1 C_{32} y}{l_3} \right)}{a l_1 C_{31}},$$

$$v_{13} = C_{29}.$$
We study the physical significance of the stationary solution profile given in (20) by considering the parameters \( a = 1, c = 1, g = 1, l_2 = 1, l_3 = 1, C_{26} = 1, C_{27} = 1, C_{28} = 1 \) and we demonstrate their 3-dimensional profiles in the Figure 8. Here we observed that \( u_{12} \) behaves like an anti-kink type soliton profile (see, Figure 8a) and \( v_{12} \) satisfies the properties of kink type soliton profile (see, Figure 8b).

In [33] authors obtained only one stationary domain walls solution of (2) of the following form

\[
\begin{align*}
  u(t, x, y) &= A_1 \tanh[B_1 x + B_2 y], \\
  v(t, x, y) &= A_2 \tanh[B_1 x + B_2 y]
\end{align*}
\]

where \( bA_2B_2^2 = aA_1(A_1 + B_1)B_2 \) and \( gA_1 = 2cB_1 \). One can observe that the solution (22) of (2) is a particular case of our solution (20) by considering \( C_{26} = 0, C_{28} = 0 \) in (20) and letting \( A_1 = \frac{2cC_{27}}{g}, A_2 = -\frac{2acC_{27}l_2(2c+g)}{bg^2l_3}, B_1 = C_{27}, B_2 = -\frac{C_{27}l_2}{l_3} \) in (22).

Now we consider the solution of (2) of the form \( u(t, x, y) = U(k_1 x + k_2 y + k_3 t + k_4), v(t, x, y) = V(k_1 x + k_2 y + k_3 t + k_4) \) which yields a reduced system of ODEs of the form

\[
\begin{align*}
  k_2k_3U''(h) - 2al_1l_2U''(h)^2 - 2al_1l_2UU'''(h) + al_1^2l_2U''(h) - bl_1^3V'''(h) &= 0, \\
  l_3V''(h) - cl_1^2V''(h) - l_1gUV' &= 0
\end{align*}
\]

where \( h = k_1 x + k_2 y + k_3 t + k_4 \) and \( k_1, ..., k_4 \) are arbitrary constants. In general it is difficult to solve the above ODE system (23) but we can obtain some particular class of exact solutions. One of them is as follows:

\[
\begin{align*}
  U(h) &= \frac{k_3C_{33}h + k_3C_{34}gk_1^2C_{33}}{k_1g(C_{33}h + C_{34})}, \\
  V(h) &= \ln(C_{33}h + C_{34})
\end{align*}
\]

provided some restrictions on the parameters involved in the given system (2) such as \( c = -g \) and \( a = \frac{1}{2} \frac{g(bk_2^2 + 2k_2k_4)}{k_2k_3} \). In this context, we obtain the following exact solution of

\[
\begin{align*}
  u_{14} &= \frac{C_{33}k_3(k_1 x + k_2 y + k_3 t + k_4) + k_3C_{34} - gk_1^2C_{33}}{k_1g(C_{33}k_1 x + C_{33}k_2 y + C_{33}k_3 t + k_4 + C_{34})}, \\
  v_{14} &= \ln(C_{33}(k_1 x + k_2 y + k_3 t + k_4) + C_{34})
\end{align*}
\]

where \( C_{33} \) and \( C_{34} \) are arbitrary constants.

On the other hand we have another solution of the reduced system of ODEs (23) as

\[
\begin{align*}
  U(h) &= \frac{k_3 \cosh(C_{35}h + C_{36}) + 2ck_1^2C_{35} \sinh(C_{35}h + C_{36})}{gk_1 \cosh(C_{35}h + C_{36})}, \\
  V(h) &= \tanh(C_{35}h + C_{36})
\end{align*}
\]
which yields another exact solution for the given system (2) as follows:

\[ u_{15} = \frac{k_3 \cosh(C_{35}(k_1 x + k_2 y + k_3 t + k_4) + C_{36}) + 2ck_2^2 C_{35} \sinh(C_{35}(k_1 x + k_2 y + k_3 t + k_4) + C_{36})}{gk_1 \cosh(C_{35}(k_1 x + k_2 y + k_3 t + k_4) + C_{36})} \]

\[ v_{15} = \tanh(C_{35}(k_1 x + k_2 y + k_3 t + k_4) + C_{36}) \]

where \( C_{35} \) and \( C_{36} \) are arbitrary constants provided that \( a = \frac{c^2 k_2 C_{35}}{b - ck_2 C_{35}} \) and \( g = \frac{2c^2 k_2 C_{35}}{b - ck_2 C_{35}} \).

Now we demonstrate the physical significance of the solution profile for \( u_{15} \) and \( v_{15} \) given by (25) in the Figure 9 with respect to \( x \) and \( y \) at fixed \( t = 1 \) by considering the parameters \( b = 3, c = 1, k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1, C_{35} = 1, C_{36} = 1, a = \frac{c^2 k_2 C_{35}}{b - ck_2 C_{35}} = \frac{1}{2} \) and \( g = \frac{2c^2 k_2 C_{35}}{b - ck_2 C_{35}} = 1 \). Here we noticed that the solution profile for \( u_{15} \), illustrated in the Figure 9a, represents a kink type soliton whilst the solution profile for \( v_{15} \) displayed in the Figure 9b, represents an anti-kink type soliton profile.

Now, by imposing the condition \( a = \frac{c^2 k_2 C_{35}}{b - ck_2 C_{35}} \) and \( g = c \), we have another exact solution of the above reduced system of ODEs (23) as

\[ U(h) = \frac{k_3 C_{38} - 2ck_2^2 C_{37} + 2k_3 C_{37}h}{ck_1(C_{38} + 2C_{37}h)} \]

\[ V(h) = C_{37}h^2 + C_{38}h + C_{39} \]

which results another exact solution of the given system (2) of the form as follows:

\[ u_{16} = \frac{k_3 C_{38} - 2ck_2^2 C_{37} + 2k_3 C_{37}(k_1 x + k_2 y + k_3 t + k_4)}{ck_1(C_{38} + 2C_{37}(k_1 x + k_2 y + k_3 t + k_4))} \]

\[ v_{16} = C_{37}(k_1 x + k_2 y + k_3 t + k_4)^2 + C_{38}(k_1 x + k_2 y + k_3 t + k_4) + C_{39} \]

where \( C_{37}, C_{38} \) and \( C_{39} \) are arbitrary constants.

We investigate the physical behavior of the solution profile of (26) for \( u_{16} \) and \( v_{16} \) in the Figure 10 by choosing \( c = 1, k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1, C_{37} = 1, C_{38} = 1 \) and \( C_{39} = 1 \). We noticed from the Figure 10 that \( u_{16} \) represents a singular kink traveling wave solution profile for different values of \( t \). It is very interesting to observe that as time evolves, the singular kink waves (see, Figure 10a and 10b) travel towards the negative x-direction and after certain time the kink property disappears. On the other hand, we demonstrate the physical behavior of \( v_{16} \) at \( t = 1 \) in the Figure 10c and observed bowl shaped profile.

6. Conservation laws

Conservation laws deal with essential physical properties of the process modeled by a given PDE system and have also wide applications in existence, uniqueness and stability analysis for the development of
numerical methods. Moreover, one can construct a nonlocally related PDE systems of the original PDE system by introducing some potential (nonlocal) variables through conservation laws and thus possibility of finding nonlocal symmetries and hence new exact solutions. A conservation law of the given PDE system can be represented in divergence form as $D_t \phi_t + D_x \phi_x + D_y \phi_y = 0$ which holds true for the solution manifold of the same system. Recently, Anco and Bluman [17, 18, 19] presented a systematic procedure to construct conservation law multipliers using direct multiplier method in terms of Euler operator which
annihilates divergence expression. The advantage of this method over that of Noether’s method is that this method does not require that the given PDE system to admit variational symmetry. In their approach, the authors applied direct multipliers method to obtain conservation law multipliers by considering $a = 1, b = 2, c = 1$ and $g = 2$, consequently constructed the corresponding conserved vectors. Authors claimed that

$$Q = f(y)v + g_1(t)x^2 + g_2(t)x + F(y) + g_3(t)$$

is the multiplier of where $f(y), g_1(t), g_2(t), g_3(t)$ and $F(y)$ are arbitrary functions. This claim lacks proper sense of understanding as there should be a set of multipliers where each set consists of two quantities instead of just one quantity. As a result, the associated conserved vectors are also incorrect.

So, in this section, we apply the direct multiplier technique to the original system and construct conservation laws systematically. The Euler operator with respect to dependent variables is the operator defined by

$$E_{u^j} = \frac{\partial}{\partial u^j} - D_{i_1}\frac{\partial}{\partial u^i_1} + ... + (-1)^jD_{i_1} ... D_{i_l}\frac{\partial}{\partial u_{i_1...i_l}} + ...$$

So, using the multipliers method we obtain the following sets of conservation law multipliers

$$\Lambda_1^2 = \frac{1}{2}\alpha(t)x^2, \quad \Lambda_2^3 = 0,$$
$$\Lambda_3^3 = \beta(t)x, \quad \Lambda_4^3 = 0,$$
$$\Lambda_3^1 = \gamma(t), \quad \Lambda_4^1 = 0$$

where $\alpha(t), \beta(t), \gamma(t)$ and $\mu(y)$ are arbitrary functions. Using these multipliers we obtain the conservation laws of the form $D_t \phi_j^1 + D_x \phi_j^2 + D_y \phi_j^3 = 0, \quad j = 1, 2, 3, 4$ whose conserved vectors are given by

$$\phi_1^1 = \frac{1}{2}x^2u_y\alpha(t), \phi_1^2 = \left(-auu_y - axu_y + \frac{1}{2}ax^2u_{xy} - bv + bxv_x - \frac{1}{2}bx^2v_{xx}\right)\alpha(t),$$

$$\phi_1^3 = v\alpha(t), \phi_1^4 = \left(-2auu_y + axu_y - bxv_x - 2auu_y - 2axu_y + 2auu_y + axu_y - bxv_x - 2auu_y + axu_y - bxv_x\right)\alpha(t)$$

Conservation laws are useful for various applications including construction of nonlocally related PDE systems. Also, one can perform nonlocal symmetry analysis and further construct nonlocal conservation laws those are very challenging and recent topic of research. We show the existence of nonlocal conservation laws of the given system in the succeeding section.

7. Applications

Conservation laws are very useful in constructing nonlocally related PDE systems, developing mathematical theory of nonlocal conservation laws and nonlocal symmetry analysis and thus new exact solutions. It would be very interesting and challenging to perform nonlocal symmetry analysis of (2+1)-dimensional nonlinear system of PDEs. This kind of problems are stated as open problems in . For PDE systems with $n > 3$ independent variables, the situation for obtaining and using nonlocally related PDE systems is considerably more complex than the case of $n = 2$. In particular, every divergence-type conservation law gives rise to several potential variables, which are only defined to within arbitrary functions of the independent variables. The corresponding potential system is thus under-determined, and is said to have gauge...
freedom. Additional equations involving potential variables, called gauge constraints, are needed to make such potential systems determined.

For example, consider a divergence-type conservation law in three-dimensional space

$$\text{div} \phi = \phi_x^1 + \phi_y^2 + \phi_z^3 = 0$$

with flux vector $\phi = (\phi^1(x, y, z), \phi^2(x, y, z), \phi^3(x, y, z))$ and independent variables $x, y, z$. It immediately follows that there exists a vector potential $\psi = (\psi^1(x, y, z), \psi^2(x, y, z), \psi^3(x, y, z))$, such that $\phi = \text{curl}\psi$. Consequently, the potential system in this case becomes

\[
\begin{align*}
\psi_y^3 - \psi_x^2 &= \phi^1, \\
\psi_z^1 - \psi_y^3 &= \phi^2, \\
\psi_x^2 - \psi_z^1 &= \phi^3.
\end{align*}
\]

However, unlike in the two-dimensional situation, the potential system is under-determined. An additional equation involving the potential variables is required in order to complete the potential system to eliminate its gauge freedom. For example, one can have the gauges:

- divergence (Coulomb) gauge: $\text{div} \psi = \psi_x^1 + \psi_y^2 + \psi_z^3 = 0$,
- spatial gauge: $\psi^k = 0$, $k = 1$ or 2 or 3,
- Poincaré gauge: $x\psi^3 + y\psi^2 + z\psi^3 = 0$,

provided that all solutions of the potential system can be obtained from the solution of the corresponding gauge-constrained (determined) potential system. If one of the coordinates in a given PDE system is time $t$, special gauges are frequently used, such as

- Lorentz gauge (in (2+1)-dimensional): $\psi_x^3 - \psi_y^2 - \psi_z^3 = 0$,
- Cronstrom gauge (in (2+1)-dimensional): $t\psi_1^3 - x\psi_2^2 - y\psi_3^3 = 0$.

Here we write down the potential systems associated to the given system (2) by making use of the conserved vectors given in preceding section.

Potential system I:

\[
\begin{align*}
\psi_x^3 - \psi_y^2 &= \frac{1}{2} x^2 u_y \alpha(t), \\
\psi_y^1 - \psi_t^3 &= \left(-auu_y - axu_y + \frac{1}{2} ax^2 u_{xy} - bv + bxx_x - \frac{1}{2} b v_{xx} \right) \alpha(t), \\
\psi_t^1 - \psi_z^1 &= -\frac{1}{2} \mu(-2axu \alpha(t) - 2a\alpha(t) + x^2 \alpha'(t));
\end{align*}
\]

Potential system II:

\[
\begin{align*}
\psi_x^3 - \psi_y^2 &= xu_y \beta(t), \\
\psi_y^1 - \psi_t^3 &= (-2axuu_y - au_y + axu_{xy} + bv_x - bv_{xx}) \beta(t), \\
\psi_t^1 - \psi_z^1 &= -u(-au \beta(t) + x \beta'(t));
\end{align*}
\]

Potential system III:

\[
\begin{align*}
\psi_x^3 - \psi_y^2 &= u_y \gamma(t), \\
\psi_y^1 - \psi_t^3 &= (-2auu_y + au_{xy} - bv_{xx}) \gamma(t), \\
\psi_t^1 - \psi_z^1 &= -u \gamma'(t);
\end{align*}
\]

and

Potential system IV:

\[
\begin{align*}
\psi_x^3 - \psi_y^2 &= u_y \mu(y), \\
\psi_y^1 - \psi_t^3 &= (-2auu_y + au_{xy} - bv_{xx}) \mu(y), \\
\psi_t^1 - \psi_z^1 &= 0.
\end{align*}
\]
For the potential system I, let us consider the spatial gauge $\psi^3 = 0$ given in (31) and by choosing $\alpha(t) = 2$, the corresponding conservation law multipliers are

$$\delta^1 = \left(\frac{\psi^2}{x^2} + u\right) k_1(t), \quad \delta^2 = \frac{H_1(t)x + H_2(t)}{x^3},$$

(35)

and

$$\delta^3 = 0, \quad \delta^4 = 0,$$

$$\delta^5 = \left(\frac{\psi^2}{x^2} + u\right) k_2(t), \quad \delta^6 = \frac{H_3(t)x + H_4(t)H_5(y) + H_6(t)}{x^3},$$

(36)

$$\delta^7 = -\frac{1}{2}\frac{H_4(t)H_5'(y)}{x^2}, \quad \delta^8 = 0,$$

where $k_1(t), k_2(t), H_1(t), H_2(t), H_3(t), H_4(t)$ and $H_5(y)$ are arbitrary functions. Here we observe that the multiplier components $\delta^1$ and $\delta^3$ have an essential dependence on the potential variable $\psi^2$. Hence they yield nonlocal conservation laws of the given system (2). Similarly, it is easy to show that potential system II and potential system IV also yield nonlocal conservation laws for (2) by considering $\beta(t) = 1$ and $\mu(y) = 1$.

In future study, it will be very interesting to analyze nonlocal symmetries of the given system (2) arising from potential systems (I-IV) as well as from inverse potential systems and to obtain some new exact solutions.

8. Conclusions

The (2+1)-dimensional BLP system is studied in the context of classical Lie symmetry analysis. It is observed that the system admits infinite dimensional Lie algebra. We performed the classification of optimal subalgebras and using each subalgebra we obtained several new exact solutions (5), (6), (8), (9), (11), (12), (13), (14), (16), (18), (19), (20), (21), (24), (25) and (26) of the BLP system. In addition to that, we noticed that the only solution (22) presented in [33] was recovered as a particular case of the obtained solution (20).

The computed solutions are reported first time in the literature. Physical behavior of some of the solutions are exhibited geometrically with the help of numerical simulations which consists of traveling waves, lump type solitons, kink and anti-kink type solitons, breather solitons, singular kink type solitons and etc. We constructed some conservation laws of the given system by using the direct multipliers method those may be used to further study on the nonlocal symmetry analysis of BLP system. Finally, as an application, we study the nonlocal conservation laws of the given system by using direct multiplier technique to the corresponding potential systems and appending spatial gauge constraints on them.

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Conflict of interest

The authors declare that they have no conflict of interest.
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