We study theoretically the generation of photon pairs with controlled spectral correlations via the four-wave mixing (FWM) process in graded-index multimode optical fibers (GIMFs). We show that the quantum correlations of the generated photons in GIMFs can be preserved over a wide spectral range for a tunable pump source. Therefore, GIMFs can be utilized as quantum-state-preserving tunable sources of photons. In particular, we have shown that it is possible to generate factorable two-photon states, which allow for heralding of single-photon pure-state photons without the need for spectral post filtering. We also elaborate on the possibility of simultaneously generating correlated and uncorrelated photon pairs in the same optical fiber.

The ability to control the coherent dynamics of photons and correlations among them paves the way to generate a wide range of quantum states from heralded single photons to entangled photon pairs. FWM in optical fibers has been used to spontaneously generate correlated photon pairs [1–9]. These signal-idler photons, sometimes referred to as daughter photons, share some information due to the conservation of energy and momentum. The photon pair correlations can be manipulated via tailoring the joint spectral amplitude (JSA) of the photon pair [5, 10–14], giving access to a variety of quantum states, from a highly correlated photon pair (entangled) [15] to an uncorrelated pair (factorable) that can be used as a heralded pure-state single photon source [16, 17].

The extra degrees of freedom available in multimode optical fibers can be used to control the physical attributes of the photon pairs generated via the spontaneous FWM process. The presence of multiple spatial modes allows for the intermodal FWM (IM-FWM) process, which can result in signal and idler photons that have large spectral separations from the pump [18–20]; therefore, they are minimally contaminated by the scattered and residual pump and Raman photons. Different spatial mode combinations can also result in different IM-FWM processes that allow for the simultaneous generation of multiple photon pairs in the same fiber [7, 14]. The diversity of the group velocities of different spatial modes can also be used as an effective tool to manipulate the JSA of the photons pairs.

Among the general class of MMFs, GIMFs exhibit unique dispersive, nonlinear, and spatiotemporal properties [21–24]. For example, their modes can be classified into mode-groups with identical intra-group and equally-spaced inter-group phase velocities [21, 25]. All modes can also be designed to have nearly identical group velocities near a special wavelength [21]. Therefore, short high peak power laser pulses do not easily disintegrate due to intermodal group velocity dispersion and the laser-pulses go through a rapid submillimeter-length self-imaging pattern when propagating along these fibers [26]. These unique properties make GIMFs an appealing platform for observing novel nonlinear optic phenomena. Examples include the observation of multimode solitons [27, 28], supercontinuum generation [29, 30], multimode saturable absorption [31], self-induced beam cleanup [29, 32], and geometric parametric instability [33–35]. In this paper, we leverage the unique dispersive and nonlinear properties of GIMFs for the generation of photon-pair states with a high degree of control over their spectral correlations for an ultra-broad spectral range.

We recently explored the tailoring of the JSA of the photon pair generated in a commercial multimode step-index optical fiber via the IM-FWM process [14]. We showed that it is possible to generate factorable two-photon states exhibiting minimal spectral correlations between the photon pair components in conventional multimode fibers using commonly available pump lasers. In this paper, we extend our studies to the case of photon pair generation in GIMFs. We show that a commercial GIMF can be used as a robust medium to generate factorable two-photon states exhibiting minimal spectral correlations under a wide range of pump power and wavelength for a tunable pump.

In this paper, we leverage the unique dispersive and nonlinear properties of GIMFs for the generation of photon-pair states with a high degree of control over their spectral correlations for an ultra-broad spectral range.

We first present a brief overview of GIMFs and establish the notation that will be used in the rest of the paper. The refractive index profile of a GIMF is given by

\[ n^2(\rho) = n_0^2 \left[ 1 - 2\Delta \left( \frac{\rho}{R} \right)^a \right]. \]  

(1)

\( R \) is the core radius, \( n_0 \) is the refractive index in the center of the core, \( \Delta \) is the relative index difference between the core and
which is parameterized using two dimensionless parameters \( r_1 \) and \( r_2 \) defined as

\[
\begin{align*}
\eta & = 0.193, \\
\mathcal{P} & = \sqrt{\frac{2r_i^2(X_1-1)X_1}{(2r_i^2X_i+\eta r_1)(2r_i^2X_i+\eta/r_1)}}, \end{align*}
\]

(a) fiber length for fixed pump bandwidth of 0.5 THz, and (b) pump bandwidth for a 50 cm long fiber. Both contours are for the \([1,1,1,3]\) process. In Fig. 2, it can be seen that for the fixed value of \( \sigma_p \), \( \mathcal{P} \approx 1 \) can be achieved by tuning the value of \( L \). Moreover, even if the value of \( L \) is kept fixed, it is possible to maintain high purity over a reasonably large pump frequency range, making it highly desirable from a device standpoint. A similar argument can be made in Fig. 2(b) by reversing the role of \( L \) and \( \sigma_p \).

The results presented in Fig. 2 are for the \([1,1,1,3]\) process. In
the following, we will show that the value of purity in GIMFs, in general, depends mainly on the spectral separation of the signal/idler from the pump ($\Omega$). This is an important result from an experimental standpoint because one will not even need to examine the mode profiles and can evaluate the value of the purity expected from the process only by a simple spectral measurement. From a theoretical standpoint, this result is interesting because the frequency separation is analytically calculable knowing the mode-group numbers involved in the FWM process \[25\].

In order to show that the purity in GIMFs depends mainly on $\Omega$, we expand Eq. (2) to the first order in the small parameter $\Delta$ and obtain

$$\beta_{g} \approx n_{0}(\lambda)k - \frac{\sqrt{2\Delta}}{\Lambda} g.$$  \hspace{1cm} (9)

Because terms of higher orders in $\Delta$ have been ignored, this approximation is only reliable if the length of the fiber $L$ satisfies $L \ll \frac{L_{0}}{g^2}$, where $L_{0} = \frac{n_{0}pR^2}{\Delta}$ \[21\]. For the commercial GIMF in Refs. \[21, 25\], we have $L_{0} \approx 2.2$ m at $\lambda = 850$ nm. Therefore, this approximation holds for the typical fiber lengths considered in this paper for photon pair generation. Using Eq. (9) and ignoring the frequency dependence of $\Delta$, we have $\beta^{(1)} = \omega_{p}(n\omega/c)$; therefore, the group velocity is independent of the group number $g$. Furthermore, we can expand the $\beta^{(1)}$ belonging to the signal (idler) around the $\omega_{p}$ and only keep the terms up to the first order in the signal/idler-pump frequency separation $\Omega$ to get even a more simplified equation for the group delay $\tau$. We will discuss the validity of these approximations later in this paper. In this case, the group delays can be written as

$$\tau_{i} \approx +\frac{\Omega}{c} \frac{L}{2} \frac{\partial_{n}^{2} \omega}{\partial_{n} \omega} |_{\omega_{p}} , \hspace{1cm} \tau_{i} \approx -\frac{\Omega}{c} \frac{L}{2} \frac{\partial_{n}^{2} \omega}{\partial_{n} \omega} |_{\omega_{p}} .$$  \hspace{1cm} (10)

In GIMFs, $\Omega$ can be accurately estimated using the material dispersion and physical parameters of the fiber by \[25\]

$$\Omega^{2} \approx \frac{\sqrt{2\Delta}}{R} \frac{Gc}{\partial_{n}^{2} \omega(n\omega)},$$  \hspace{1cm} (11)

where $G = g_{s} + g_{i} - s_{p}^{(1)} - s_{p}^{(2)}$. $G$ is determined by the modes involved in the FWM process, e.g. $G = 2$ is used for the FWM process identified by \{1,1,1,3\}. Using Eqs. (10) and (11), $r_{1}$ and $r_{2}$ simplify to

$$r_{1} \approx -1 , \hspace{1cm} r_{2} \approx \frac{1}{c_{p}L} \sqrt{\frac{R c}{G\sqrt{8\Delta}}} \left[ \frac{\partial_{n}^{2} \omega(n\omega)}{\omega_{p}} \right]^{-\frac{1}{2}} .$$  \hspace{1cm} (12)

from which the purity is calculated as

$$\mathcal{P} = \frac{4 \eta}{r_{2}} \frac{r_{2}}{r_{2}^{2} + \eta} .$$  \hspace{1cm} (13)

We emphasize that in order to use these approximations, it is important for the modes involved in the FWM process to be far from their cutoff frequencies such that the material dispersion dominates over the waveguide contribution.

In Fig. 3, we examine the accuracy of the approximations made so far in detail. We consider the purity calculation for the \{1,1,1,3\} process ($G = 2$) for the fiber length of 50 cm and the pump bandwidth of 0.2 THz as a function of the pump frequency $\nu_{p}$. The solid (black) curve shows the full calculation using Eq. (2) and its derivatives, so no approximation is made. The dashed-dotted (red) curve uses the approximation in Eqs. (10), for which the value of $\Omega$ is calculated directly by using Eq. (2). The long-dashed (green) and short-dashed (cyan) curves are calculated in a similar way to the dashed-dotted (red) curve except 2nd and 3rd order Taylor expansion terms in Eq. (10) are retained. The dotted (purple) curve is plotted using the full approximation formula in Eq. (13) where $\Omega$ is calculated using the approximation in Eq. (11). Therefore, there is a good agreement between the exact solution and these approximate methods for a wide range of the pump frequency. Results presented in Fig. 3 also show more clearly the possibility of a tunable uncorrelated photon pair source in a GIMF.

According to the approximations presented earlier, we argued that the purity is determined chiefly by the spectral separation $\Omega$, and the dependence on the mode-groups numbers appears only indirectly through $\Omega$ and Eq. (11). Therefore, from a practical standpoint in an experiment, where one usually measures the FWM spectrum rather than the modal content, it is interesting to investigate the purity only as a function of $\Omega$ for various experimental scenarios. This is done in Fig. 4 where purity is plotted as a function of $\Omega$ for several different values of $L$ and $c_{p}$ for the photon pairs generated through the \{1,1,1,3\} pro-
cess using an 850 nm wavelength pump. In each subfigure, the dashed red line demonstrates the approximate calculation from Eq. (10), and the solid black line shows the calculation using the exact form of the group velocity. It must be noted that these should be treated as general reference plots; the exact modal content will fix the value of $\Omega$ at a particular phase-matched frequency shift. The fact that the approximate method (applicable to all values of $G$) and the exact method (only applicable to $[1,1,1,3]$ though ignoring the phase-matching) give nearly identical results prove the broad applicability of these curves. Moreover, different scenarios show that by a proper choice of $L$ and $c_p$, it is possible to have high purity over a broad spectral band; or obtain high purity in one frequency region and low purity in another. Therefore, it is possible to simultaneously generate mixed purity values using different multimode phase-matching processes in a single strand of optical fiber.

In conclusion, we have investigated GIMFs as an ultra-broadband source of photon pairs with controlled spectral correlations. We show that GIMFs can be used as a tunable source of uncorrelated (or correlated) photon pairs as their spectral correlations are relatively independent of the pump central wavelength. Our calculations indicate that while tuning the pump frequency will change the frequencies of the signal and idler, the signal (idler) photon state purity remains unchanged. Therefore, GIMFs can be used to make quantum-state-preserving tunable sources of photon pairs. We have also shown that the purity is mainly a function of the spectral separation of the idler-signal pair from the pump ($\Omega$), and its dependence on the spatial mode content is indirect and only through its dependence on $\Omega$. Finally, we have shown that by the right selection of the physical parameters of the system including the fiber length and/or the pump bandwidth, it is possible to simultaneously generate correlated and uncorrelated photon pairs in the same optical fiber.

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REFERENCES

1. X. Li, P. L. Voss, J. E. Sharping, and P. Kumar, Phys. Rev. Lett. 94, 053601 (2005).
2. J. G. Rarity, J. Fulconis, J. Dulingall, W. J. Wadsworth, and P. S. J. Russell, Opt. Express 13, 534 (2005).
3. Q. Lin, F. Yaman, and G. P. Agrawal, Opt. Lett. 31, 1286 (2006).
4. E. A. Goldschmidt, M. D. Eisaman, J. Fan, S. V. Polyakov, and A. Migdall, Phys. Rev. A 78, 013844 (2008).
5. O. Cohen, J. S. Lundeen, B. J. Smith, G. Puentes, P. J. Mosley, and I. A. Walmsley, Phys. Rev. Lett. 102, 123603 (2009).
6. B. Fang, O. Cohen, and V. O. Lorenz, J. Opt. Soc. Am. B 31, 277 (2014).
7. D. Cruz-Delgado, J. Monroy-Ruz, A. M. Barragan, E. Ortiz-Ricardo, H. Cruz-Ramirez, R. Ramirez-Alecon, K. Garay-Palmett, and A. B. U’Ren, Opt. Lett. 39, 3583 (2014).
8. R. A. Smith, D. V. Reddy, D. L. Vitullo, and M. G. Raymer, Opt. Express 24, 5809 (2016).
9. R. J. A. Francis-Jones, R. A. Hoggart, and P. J. Mosley, Optica 3, 1270 (2016).
10. W. P. Grice, A. B. U’Ren, and I. A. Walmsley, Phys. Rev. A 64, 063815 (2001).
11. K. Garay-Palmett, H. J. Mcguinness, O. Cohen, J. S. Lundeen, R. Rangel-Rojo, A. B. Uren, M. G. Raymer, C. J. McKinstry, S. Radic, and I. A. Walmsley, Opt. Express 15, 14870 (2007).
12. J. B. Christensen, C. McKinstry, and K. Rottwitt, Physical Review A 94, 013819 (2016).
13. L. Cui, X. Li, and N. Zhao, New Journal of Physics 14, 123001 (2012).
14. H. Pourbeyram and A. Mafi, Phys. Rev. A 94, 023815 (2016).
15. X. Li, J. Chen, P. Voss, J. Sharping, and P. Kumar, Opt. Express 12, 3737 (2004).
16. C. Söller, O. Cohen, B. J. Smith, I. A. Walmsley, and C. Silberhorn, Phys. Rev. A 83, 031806 (2011).
17. B. Fang, O. Cohen, J. B. Moreno, and V. O. Lorenz, Opt. Express 21, 2707 (2013).
18. C. Lin and M. A. Bösch, Applied Physics Letters 38, 479 (1981).
19. H. Pourbeyram, E. Nazemosadat, and A. Mafi, Opt. Express 23, 14487 (2015).
20. R. Dupiol, A. Bendahmane, K. Krupa, A. Tonello, M. Fabert, B. Köbler, T. Sylvestre, A. Barthélémy, V. Couderc, S. Wabnitz, and G. Millot, Opt. Lett. 42, 1293 (2017).
21. A. Mafi, J. Lightwave Technol. 30, 2803 (2012).
22. L. G. Wright, D. N. Christodoulides, and F. W. Wise, Nature Photonics 9, 306 (2015).
23. S. Buch and G. P. Agrawal, J. Opt. Soc. Am. B 33, 2217 (2016).
24. L. G. Wright, D. N. Christodoulides, and F. W. Wise, Science 358, 94 (2017).
25. E. Nazemosadat, H. Pourbeyram, and A. Mafi, J. Opt. Soc. Am. B 36, 144–150 (2011).
26. A. Mafi, P. Hofmann, C. J. Salvin, and A. Schülzgen, Opt. Lett. 36, 3596 (2011).
27. L. G. Wright, W. H. Renninger, D. N. Christodoulides, and F. W. Wise, Opt. Express 23, 3492 (2015).
28. Z. Zhu, L. G. Wright, D. N. Christodoulides, and F. W. Wise, Opt. Lett. 41, 4819 (2016).
29. G. Lopez-Galmiche, Z. S. Ez naev et, M. A. Eftekhar, J. A. Lopez, L. G. Wright, F. Wise, D. Christodoulides, and R. A. Correa, Opt. Lett. 41, 2563 (2016).
30. K. Krupa, C. Louot, V. Couderc, M. Fabert, R. Guenard, B. M. Shalaby, A. Tonello, D. Pagnoux, P. Leproux, A. Bendahmane, R. Dupiol, G. Millot, and S. Wabnitz, Opt. Lett. 41, 5785 (2016).
31. E. Nazemosadat and A. Mafi, J. Opt. Soc. Am. B 30, 1357 (2013).
32. H. Pourbeyram, G. P. Agrawal, and A. Mafi, Applied Physics Letters 102, 201107 (2013).
33. S. Longhi, Opt. Lett. 28, 2363 (2003).
34. K. Krupa, A. Tonello, A. Barthélémy, V. Couderc, B. M. Shalaby, A. Bendahmane, G. Millot, and S. Wabnitz, Phys. Rev. Lett. 116, 183901 (2016).
35. U. Teğin and B. Ortaç, arXiv preprint arXiv:1705.09157 (2017).
36. B. J. Smith, P. Mahou, O. Cohen, J. S. Lundeen, and I. A. Walmsley, Opt. Express 17, 23589 (2009).
37. O. Cohen, “Generation of uncorrelated photon-pairs in optical fibres,” Ph.D. thesis, University of Oxford (2010).