Cyclic Identities Involving Jacobi Elliptic Functions. II

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Abstract: Identities involving cyclic sums of terms composed from Jacobi elliptic functions evaluated at $p$ equally shifted points on the real axis were recently found. These identities played a crucial role in discovering linear superposition solutions of a large number of important nonlinear equations. We derive four master identities, from which the identities discussed earlier are derivable as special cases. Master identities are also obtained which lead to cyclic identities with alternating signs. We discuss an extension of our results to pure imaginary and complex shifts as well as to the ratio of Jacobi theta functions.
1 Introduction

In a recent paper [1], we have given many new mathematical identities involving the Jacobi elliptic functions \( sn(x, m) \), \( cn(x, m) \), \( dn(x, m) \), where \( m \) is the elliptic modulus parameter \((0 \leq m \leq 1)\). The functions \( sn(x, m) \), \( cn(x, m) \), \( dn(x, m) \) are doubly periodic functions with periods \((4K(m), i2K'(m)), (4K(m), i4K'(m)), (2K(m), i4K'(m))\), respectively [2]. Here, \( K(m) \) denotes the complete elliptic integral of the first kind, and \( K'(m) = K(1-m) \). The \( m = 0 \) limit gives \( K(0) = \pi/2 \) and trigonometric functions: \( sn(x, 0) = \sin x, \ cn(x, 0) = \cos x, \ dn(x, 0) = 1 \). The \( m \to 1 \) limit gives \( K(1) \to \infty \) and hyperbolic functions: \( sn(x, 1) \to \tanh x, \ cn(x, 1) \to \sech x, \ dn(x, 1) \to \sech x \). For simplicity, from now on we will not explicitly display the modulus parameter \( m \) as an argument of the Jacobi elliptic functions.

The cyclic identities discussed in ref. [1] play an important role in showing that a kind of linear superposition is valid for many nonlinear differential equations of physical interest [3, 4]. In all identities, the arguments of the Jacobi functions in successive terms are separated by either \( 2K(m)/p \) or \( 4K(m)/p \), where \( p \) is an integer. Each \( p \)-point identity of rank \( r \) involves a cyclic homogeneous polynomial of degree \( r \) (in Jacobi elliptic functions with \( p \) equally spaced arguments) related to other cyclic homogeneous polynomials of degree \( r - 2 \) or smaller. In ref. [1], explicit algebraic proofs were given for specific small values of \( p \) and \( r \) by using standard properties of Jacobi elliptic functions. However, identities corresponding to higher values of \( p \) and \( r \) were only verified numerically using advanced mathematical software packages. In this article, we present rigorous mathematical proofs valid for arbitrary \( p \) and \( r \). As a useful byproduct, we determine explicit forms for the constants appearing in various identities. All the identities in ref. [1] corresponded to real shifts of multiples of \( 2K(m)/p \) or \( 4K(m)/p \). Here, we discuss how to obtain new identities corresponding to pure imaginary shifts by multiples of \( i2K'(m)/p \) or \( i4K'(m)/p \), as well as identities corresponding to complex shifts by multiples of \( 2[K(m) + iK'(m)]/p \) or \( 4[K(m) + iK'(m)]/p \). We also discuss the identities for the nine secondary Jacobi elliptic functions like \( cd(x, m), \ ns(x, m), \ ds(x, m) \). Also, we give results for several identities involving Weierstrass elliptic functions and ratios of Jacobi theta functions, both of which are intimately related with the Jacobi elliptic functions [2]. In our proofs, we classify the identities into four types, each with its own “master identity” which we prove using a combination of the Poisson summation formula and the special properties of elliptic functions.

All our identities involve sums of the following generic form:

\[
S_p(x_0) = \sum_{j=1}^{p} f(x_j), \tag{1}
\]

where \( f(x) \) is composed from Jacobi elliptic functions with arguments corresponding to \( p \) equally spaced points

\[
x_j = x_0 + (j-1)T/p, \quad j = 1, \ldots, p,
\]

2
where $T$ is a period of $f(x)$ and the base point $x_0$ is an arbitrary complex number. We define the quantities $P, Q$ by:

$$f(z + 2iK') = (-1)^P f(z), \quad f(z + 2K) = (-1)^Q f(z), \quad P, Q = 0, 1.$$  (2)

Note that $Q = 0, 1$ correspond to real periods $2K(m), 4K(m)$ and $P = 0, 1$ correspond to pure imaginary periods $i2K'(m), i4K'(m)$ respectively. We denote the four possibilities as $(+, +), (-, +), (+, -)$ and $(-, -)$, where the first sign refers to the sign of $(-1)^P$ and the second to that of $(-1)^Q$. We will derive master identities for each of these four possibilities.

For example, one of the simplest identities discussed in ref. [1] reads

$$\sum_{j=1}^{p} \text{dn}(x_j) \text{dn}(x_{j+1}) = A,$$  (3)

where $A$ is a constant independent of the base point $x_0$, $T = 2K$, and $p$ is any integer. In this case, we have $f(z) = \text{dn}(z)\text{dn}(z + T/p)$ which corresponds to $P = 0, Q = 0$, since $\text{dn}(z + 2K) = \text{dn}(z)$ and $\text{dn}(z + 2iK') = -\text{dn}(z)$. Liouville’s theorem can be used to prove the above identity, since $\text{dn}(z)$ has simple poles within its fundamental region $(0, 2K, 2K + 4iK', 4iK')$ at $iK'$ and $3iK'$ both of which we collectively refer to as $z^*$. The identity $\text{dn}(z^* + u) + \text{dn}(z^* - u) = 0$ for arbitrary complex $u$ then implies that every pole in the sum is cancelled exactly by a zero of the same order. Thus the sum is an analytic function without any poles in the finite part of the complex plane and by Liouville’s theorem must be a constant $[3]$. This is an explicit illustration of the general principle underlying the identities, namely that the orders of poles in a higher order polynomial are reduced by some zeros leading to simpler sums. However this method does not yield the constants like $A$ explicitly. In fact, using the Poisson summation formula and special properties of Jacobi elliptic functions, we show below that the constant $A$ in Eq. (3) is given by

$$A = \frac{p}{2K} \int_0^{2K} \text{dn}(x)\text{dn}(x + T/p) \, dx = \frac{p Z(\beta_2K)}{\text{sn}(2K/p)},$$  (4)

where $Z$ is the Jacobi zeta function ($Z \equiv Z(\beta_q, m)$) $[2]$ with $\beta_q \equiv \arcsin(\text{sn}(q/p, m))$ being the Jacobi Amplitude function.

Identities analogous to Eq. (3) also hold for $\text{sn}$ and $\text{cn}$. For instance,

$$\sum_{j=1}^{p} \text{sn}(x_j) \text{sn}(x_{j+1}) = \frac{p Z(\beta_2K)}{m \text{sn}(2K/p)}.$$  (5)

Here $T$ is $2K$ as this is the periodicity of $f(z) = \text{sn}(z)\text{sn}(z + T/p)$. The expression as given above is valid for all integer values of $p > 2$, with both sides vanishing when $p = 2$.

A further generalization that can be easily treated with the techniques developed below is to sums that involve $r$-th neighbors. We consider the case when $r$ and $p$ are coprime integers and
1 \leq r < p - 1$, the other cases being included since identities for any choice of $p$ also include the identities for the factors of $p$. Such a generalization of say Eq. (11) above is

$$\sum_{j=1}^{p} \text{dn}(x_j)\text{dn}(x_{j+r}) = p \left( \text{dn}(2rK/p) - \frac{\text{cn}(2rK/p)Z(\beta rK)}{\text{sn}(2rK/p)} \right). \quad (6)$$

Another easy generalization is to identities involving terms consisting of a product of an arbitrary number of Jacobi elliptic functions.

The plan of this article is as follows. In Sec. 2, we derive the master identities which form the basis for obtaining all the identities in this paper, and are tabulated in Appendices A and B. Sec. 3 contains a derivation of identities involving alternating signs. In Sec. 4, we present a collection of comments, including some which permit a generalization of all the identities to incorporate pure imaginary and complex shifts and to present the identities for Weierstrass functions as well as for ratios of Jacobi theta functions.

## 2 The Master Identities

In this section, we derive the four master identities corresponding to $Q, P$ taking on values 0, 1, which effectively encompass most of the cyclic identities discussed in ref. [1]. The remaining identities in ref. [1] correspond to master identities with alternating signs, and these are discussed in the next section.

For completeness we first derive the finite version of the Poisson summation formula [6] that fully exploits the equally spaced nature of the sampling points, and which plays a crucial role in subsequent derivations. Since $f(x)$ has a period $T$, we may expand it in a Fourier series:

$$f(x) = \frac{1}{T} \sum_{k=-\infty}^{\infty} a_k e(kx/T), \quad a_k = \int_{0}^{T} f(x) e(-kx/T) \, dx, \quad (7)$$

where we have introduced the convenient notation $e(x) \equiv \exp(2\pi ix)$. The required sum may then be written as:

$$S_p(x_0) = \sum_{j=1}^{p} f(x_j) = \frac{1}{T} \sum_{k=-\infty}^{\infty} a_k e(kx_0/T) \sum_{j=1}^{p} e(kj/p). \quad (8)$$

Using the simple identity

$$\sum_{j=1}^{p} e(kj/p) = \begin{cases} p & \text{if } p|k \\ 0 & \text{otherwise} \end{cases}, \quad (9)$$

we get

$$S_p(x_0) = \frac{p}{T} \sum_{k, p|k} a_k e(kx_0/T). \quad (10)$$

Note that we need to evaluate only those Fourier coefficients $a_k$ for which $k$ is a multiple of $p$. 

4
2.1 Cases corresponding to $Q = 0$

We first derive the two master identities corresponding to $Q = 0$ [or equivalently $T = 2K$], allowing $P$ to be either 1 or 0. Consider the rectangle $ABCD \equiv (-K, K, K + 2iK', -K + 2iK')$. We assume that $f(z)$ has a finite number of poles inside $ABCD$ situated at points $z_w^* = iK' + wT/p$, where $w = 0, \pm 1, \pm 2, \ldots$ and $|w| < p$. Let the principal part of $f(z)$ about the pole $z_w^*$ be

$$\sum_{l=1}^{L_w} \frac{\alpha_l^{(w)}}{(z - z_w^*)^l},$$

which makes this a pole of order $L_w$.

We now use the fact that $f(z)$ is composed of elliptic functions, and this essentially allows evaluation of $a_k$. To evaluate $a_k$, for $k \neq 0$, consider the integral over the rectangle $ABCD$:

$$\oint f(z) e(-kz/T) \, dz = a_k + \int_{-K+2iK'}^{-K} f(z) e(-kz/T) \, dz$$

$$= a_k + \int_{K}^{K+2iK'} f(z) e(-kz/T) \, dz$$

$$= a_k + (-1)^{P+1} q^{-2k} \int_{-K}^{K} f(z) e(-kz/T) \, dz = a_k[1 + (-1)^{P+1} q^{-2k}],$$

where $q = \exp(-\pi K'/K)$ is the Jacobian nome $\tau$. The contributions of the vertical segments of the integration contour are equal and opposite, and cancel each other.

On the other hand, the sum of the residues of $f(z) e(-kz/T)$ may also be calculated. The residue at the pole $z_w^*$ is:

$$\text{Res} [f(z) e(-kz/T)] = \text{Res} \left[ f(z) e(-k[z - (iK' + wT/p)]/T) q^{-k} e(-kw/p) \right]$$

$$= q^{-k} \text{Res} \left[ \sum_{l=1}^{L_w} \frac{\alpha_l^{(w)}}{(z - z_w^*)^l} \left( \sum_{n=0}^{\infty} (-2\pi i k)^n \frac{z - z_w^*}{n!} \right) \right]$$

$$= q^{-k} \sum_{l=1}^{L_w} \frac{\alpha_l^{(w)}}{(l-1)!} \left( \frac{-2\pi i}{T} \right)^{l-1} k^{l-1}. \quad (13)$$

For the second equality, we have made use of the fact that only those $a_k$ for which $k/p$ is an integer need to be evaluated, thanks to the Poisson summation formula Eq. (10).

Define $L' \equiv \text{Max}\{L_1, L_2, \ldots, L_w\}$ and $\gamma_l = \sum_w \alpha_l^{(w)}$, $l = 1, \ldots, L'$, where we set nonexistent $\alpha_l^{(w)}$ to be zero. We also set $L$ to be the maximum integer such that $\gamma_L$ is nonzero. If $L = 0$, there are no nonvanishing $\gamma$ and the function is regular. Using $T = 2K$ the sum of the residues at all the interior poles may be written as:

$$\text{Res} = q^{-k} \sum_{l=1}^{L} \frac{\gamma_l}{(l-1)!} \left( \frac{-2\pi i}{2K} \right)^{l-1} k^{l-1}. \quad (14)$$

5
Thus

\[ a_k = \frac{2\pi i q^{-k}}{1 + (-1)^{p+1} q^{-2k}} \sum_{l=1}^{L} \left[ \frac{\gamma_l}{(l-1)!} \left( -\frac{2\pi i}{2K} \right)^{l-1} k^{l-1} \right] \quad k \neq 0. \]  

(15)

Therefore, Eq. (10) now becomes

\[ S_p(x_0) = \frac{p}{2K} \left[ a_0 + 2\pi i \sum_{l=1}^{L} \frac{\gamma_l}{(l-1)!} \left( -\frac{2\pi i}{2K} \right)^{l-1} \sum_{k \neq 0,p \mid k} \frac{k^{l-1} q^{-k}}{1 + (-1)^{p+1} q^{-2k}} e^{\left( \frac{k x_0}{2K} \right)} \right]. \]  

(16)

We are in a position to derive two master identities (MI) corresponding to the \( P = 1 \) and \( P = 0 \) cases, which we call MI - I and MI - II respectively. We state for convenience the following well-known symmetry properties:

\[ \begin{align*}
\text{sn}(z + 2K) &= -\text{sn}(z), \quad \text{cn}(z + 2K) = -\text{cn}(z), \quad \text{dn}(z + 2K) = \text{dn}(z), \\
\text{sn}(z + 2iK') &= \text{sn}(z), \quad \text{cn}(z + 2iK') = -\text{cn}(z), \quad \text{dn}(z + 2iK') = -\text{dn}(z).
\end{align*} \]

\subsection*{2.1.1 MI - I: Case \( Q = 0, P = 1 \)}

MI - I identities result if there are an odd total number of \( \text{dn} \) and \( \text{cn} \), and an even total number of \( \text{sn} \) and \( \text{cn} \) functions in \( f(z) \), i.e. if one considers terms of the form \( \text{dn}^a \text{sn}^b \text{cn}^c \), then \( a + c \) is odd and \( b + c \) is even. The primitive function of this type is \( \text{dn}(z) \) and we consider an “archetypal sum” \( \sigma_1 \) from which identities in this class can be derived:

\[ \sigma_1(x_0) = \sum_{j=1}^{p} \text{dn}(x_j), \]  

(17)

where \( p \) is any (odd or even) integer. We note that in the case of MI - I

\[ a_0 = \int_{0}^{T=2K} f(x) \, dx = i\pi \gamma_1, \]  

(18)

as can be seen on integrating \( f(z) \) around \( ABCD \), and making use of the antisymmetry about \( 2iK' \), since we are considering the case \( P = 1 \).

Since \( \text{dn}(z) \) has a single simple pole at \( iK' \) interior to \( ABCD \) with \( \gamma_1 = -i \), using the Poisson summation formula yields

\[ \sigma_1(x_0) = \frac{p\pi}{2K} \left[ 1 + 2 \sum_{k \neq 0,p \mid k} \left( \frac{q^{-k}}{1 + q^{-2k}} e^{\left( \frac{k x_0}{2K} \right)} \right) \right]. \]  

(19)

The above expression for \( \sigma_1(x_0) \) now allows us to re-write \( S_p(x_0) \) as given in Eq. (16), yielding our first master identity:

\[ S_p(x_0) = i \sum_{l=1}^{L} \frac{(-1)^{l-1} \gamma_l}{(l-1)!} \frac{d^{l-1}}{dx_0^{l-1}} \sigma_1(x_0). \]  

(20)
Thus all the sums in this class can be written as sums over the higher order derivatives of the function \( n(z) \). The highest derivative order is one less than the maximum of the orders of the function \( f(z) \) at all the interior poles. We see that the sums involving the Jacobi functions are intimately related to their singularity structure in the complex plane.

As an illustration, consider the sum

\[
S_p(x_0) = \sum_{j=1}^{p} \text{dn}(x_j) \text{dn}(x_{j+1}) \text{dn}(x_{j+2}) .
\]  

(21)

The relevant function is \( f(z) = \text{dn}(z) \text{dn}(z+2K/p) \text{dn}(z+4K/p) \), with poles at \( iK' \), \( iK'-2K/p \), and \( iK'-4K/p \) within \( ABCD \). The principal part of the function \( \text{dn}(z)[\text{dn}(z+2K/p)\text{dn}(z+4K/p) + \text{dn}(z-2K/p)\text{dn}(z+2K/p) + \text{dn}(z-4K/p)\text{dn}(z-2K/p)] \) around \( z = iK' \) determines the \( \gamma_l \). The singularity of \( \text{dn}(z) \) is simple, therefore \( L = 1 \). Using the identity \( \text{dn}(z+iK') = -i \text{cs}(z) \) we get that \( \gamma_1 = -i [\text{cs}^2(2K/p) - 2\text{cs}(2K/p)\text{cs}(4K/p)] \). Substituting this result in Eq. (20) gives the identity

\[
\sum_{j=1}^{p} \text{dn}(x_j) \text{dn}(x_{j+1}) \text{dn}(x_{j+2}) = \left[ \text{cs}^2(2K/p) - 2\text{cs}(2K/p)\text{cs}(4K/p) \right] \sum_{j=1}^{p} \text{dn}(x_j) .
\]  

(22)

As another example consider

\[
S_p(x_0) = \sum_{j=1}^{p} \text{sn}(x_j) \text{cn}(x_j) \text{dn}(x_j) \left[ \text{dn}(x_{j+1}) + \text{dn}(x_{j-1}) \right] .
\]  

(23)

The relevant function now is \( f(z) = \text{sn}(z) \text{cn}(z) \text{dn}(z)[\text{dn}(z+2K/p) + \text{dn}(z-2K/p)] \). There are three poles, one at \( iK' \) and the others at \( iK' \pm 2K/p \). To get the quantities \( \gamma_l \), it is convenient to consider the principal part of \( f(z) + f(z+2K/p) + f(z-2K/p) \) around \( z = iK' \). At \( iK' \), while the product of the three functions \( \text{sn}, \text{cn} \) and \( \text{dn} \) gives an order three singularity, it is reduced by one due to the vanishing of a constant term in the expansion of \( \text{dn}(z+2K/p) + \text{dn}(z-2K/p) \) around the same point. Thus we get that the maximum order of \( f(z) \) is \( L = 2 \) while \( \gamma_1 = 0 \) and \( \gamma_2 = (-2i/m)\text{ds}(2K/p)\text{ns}(2K/p) \). Substitution in Eq. (20) leads to the identity:

\[
\sum_{j=1}^{p} \text{sn}(x_j) \text{cn}(x_j) \text{dn}(x_j) \left[ \text{dn}(x_{j+1}) + \text{dn}(x_{j-1}) \right] = 2\text{ds}(2K/p)\text{ns}(2K/p) \sum_{j=1}^{p} \text{cn}(x_j) \text{sn}(x_j) .
\]  

(24)

Several other identities of this type are given in Appendix A.

2.1.2 MI - II: Case \( Q = 0, P = 0 \)

This case results when there are an even total number of \( \text{dn} \) and \( \text{cn} \) and an even total number of \( \text{sn} \) and \( \text{cn} \) i.e. if one considers terms of the form \( \text{dn}^a \text{sn}^b \text{cn}^c \), then both \( a + c \) and \( b + c \) must be even.
In this case the relevant primitive function can be taken as \( \text{dn}^2(z) \) and we define and evaluate the following archetypal sum:

\[
\sigma_2(x_0) = \sum_{j=1}^{p} \text{dn}^2(x_j) = \frac{pE}{K} + \frac{p\pi^2}{K^2} \sum_{k \neq 0, p|k} \frac{kq^k}{1 - q^{2k}} e\left(\frac{kx_0}{2K}\right). \tag{25}
\]

Here \( E \) is the complete elliptic integral of the second kind \([2]\). We note that in this case \( \gamma_1 \) is zero, as the integral of \( f(z) \) around the rectangle \( ABCD \) vanishes.

Substituting Eq. (25) in Eq. (16), yields the second master identity:

\[
S_p(x_0) = \frac{p}{2K} \left[ \int_0^{2K} f(x) dx + 2\gamma_2 E \right] + \sum_{l=2}^{L} \frac{(-1)^{l-1} \gamma_l}{(l-1)!} \frac{d^{l-2}}{dx_0^{l-2}} \sigma_2(x_0). \tag{26}
\]

Thus all MI - II identities have derivatives of \( \text{dn}^2(z) \) upto order \( L - 2 \). This is also the only master identity that has a non-vanishing “constant” (independent of \( x_0 \)) term on the right hand side. The simplest member of this class has already been discussed in Eq. (3). We note that the relevant function for this identity is \( f(z) = \text{dn}(z) \text{dn}(z + 2K/p) \). There are two poles, one at \( iK' \) and other at \( iK' - 2K/p \). Thus we can construct \( \text{dn}(z + iK')[\text{dn}(z + iK' + 2K/p) + \text{dn}(z + iK' - 2K/p)] = -\text{cs}(z)[\text{cs}(z + 2K/p) + \text{cs}(z - 2K/p)] \) and its principal part around \( z = 0 \) will give us the \( \gamma_l \). The principal part of \( \text{cs}(z) \) around \( z = 0 \) is \( 1/z \). Therefore the only relevant number is \( \gamma_1 = -[\text{cs}(2K/p) + \text{cs}(-2K/p)] = 0 \). Anyway we have already observed above that for this class \( \gamma_1 = 0 \) from the fact that \( ABCD \) is a period parallelogram for \( f(z) \). Thus the sum of the principal parts cancel and so do all the \( \gamma_l \). Hence using Eq. (26), we obtain the identity (4). In fact we can easily generalize using the same argument to a cyclic sum of any even number of \( \text{dn} \) or \( \text{sn} \) or \( \text{cn} \).

For instance,

\[
\sum_{j=1}^{p} \text{dn}(x_j) \text{dn}(x_{j+r}) \text{dn}(x_{j+s}) \text{dn}(x_{j+t}) = \frac{p}{2K} \int_0^{2K} f(x) dx, \tag{27}
\]

where \( f(x) = \text{dn}(x) \text{dn}(x + r2K/p) \text{dn}(x + s2K/p) \text{dn}(x + t2K/p) \).

As another example, we establish the identity

\[
\sum_{j=1}^{p} \text{dn}^2(x_j) = A \sum_{j=1}^{p} \text{dn}^2(x_j) + B. \tag{28}
\]

Writing \( f(z) = \text{dn}^2(z) \text{dn}^2(z + 2K/p) \), there are two poles of order two at \( iK' \) and \( iK' - 2K/p \) within \( ABCD \). We find that \( L = 2 \) and \( \gamma_2 = 2\text{cs}^2(2K/p) \). Thus applying the master identity leads to

\[
A = -2\text{cs}^2(2K/p), \quad B = \frac{p}{2K} \left( \int_0^{2K} \text{dn}^2(t) \text{dn}^2(t + 2K/p) dt + 4E \text{cs}^2(2K/p) \right). \tag{29}
\]

For an example of this class with \( L = 3 \), we prove the identity

\[
\sum_{j=1}^{p} \text{cn}(x_j) \text{sn}(x_j)[\text{dn}^3(x_{j+1}) + \text{dn}^3(x_{j-1})] = A \sum_{j=1}^{p} \text{cn}(x_j) \text{sn}(x_j) \text{dn}(x_j). \tag{30}
\]
We can derive this using the master identity with \( f(z) = \text{cn}(z) \text{sn}(z)[\text{dn}^3(z+2K/p)+\text{dn}^3(z-2K/p)] \), and we find that \( L = 3 \) with \( \gamma_2 = 0 \) and \( \gamma_3 = (2/m) \text{ds}(2K/p) \text{ns}(2K/p) \). Thus the first derivative of \( \text{dn}^2(z) \) will appear in the RHS, which indeed leads to the above identity with the constant \( A = -2 \text{ns}(2K/p) \text{ds}(2K/p) \). Note that

\[
\int_0^{2K} f(t) \, dt = \int_0^K [f(t) + f(-t)] \, dt = 0 ,
\]  

since \( f(t) \) is an odd function of \( t \).

### 2.2 Cases corresponding to \( Q = 1 \)

When \( Q = 1 \), the function \( f(z) \) has a real period \( 4K \). We consider the rectangle \( ABCD \equiv (-\epsilon, 4K - \epsilon, 4K - \epsilon + 2iK', -\epsilon + 2iK') \), where \( \epsilon \) is a small positive number, and integrate around this rectangle. Poles occur at \( iK' + w4K/p \) and \( iK' + 2K + w4K/p \) inside the rectangle \( ABCD \). If the principal part around \( iK' + w4K/p \) is given by the set of coefficients \( \{\gamma_l\} \), the set around \( iK' + 2K + w4K/p \) is \( \{-\gamma_l\} \), since \( f(z + 2K) = -f(z) \). Also note that

\[
a_0 = \int_0^{4K} f(x) \, dx = 0 ,
\]

due to antisymmetry about \( 2K \). Applying the Poisson summation formula and following the same procedures as for the previous cases, we get the equivalent of Eq. (35):

\[
S_p(x_0) = \frac{2\pi i p}{4K} \sum_{l=1}^{L} \frac{\gamma_l}{(l-1)!} \left( \frac{-2\pi i}{4K} \right)^{l-1} \sum_{k\neq 0,p;k} \frac{k^{l-1}[1 - (-1)^k]q^{-k/2}}{1 + (-1)^{p+1}q^{-k}} e \left( \frac{Kx_0}{4K} \right) .
\]  

We note that \( S_{2p}(x_0) = 0 \), i.e. the sums in these cases vanish for even values of \( p \). This is however a trivial identity since \( f(x_j) = -f(x_{j+p/2}) \) for \( j = 1, \ldots, p/2 \). Thus, for \( Q = 1 \), it is sufficient to only consider identities where \( p \) is odd.

#### 2.2.1 MI - III: Case \( Q = 1, P = 0 \)

This case applies when \( f(z) \) has an even total number of \( \text{dn} \) and \( \text{cn} \) and there are an odd total number of \( \text{sn} \) and \( \text{cn} \) i.e. if one considers terms of the form \( \text{dn}^a \text{sn}^b \text{cn}^c \), then \( a + c \) is even and \( b + c \) is odd. The relevant primitive function here is \( \text{sn}(z) \) and the archetypal sum is

\[
\sigma_3(x_0) = \sum_{i=j}^{p} \text{sn}(x_j) = \frac{2\pi i p}{4K\sqrt{m}} \sum_{l=1}^{L} \frac{[1 - (-1)^l]q^{-l/2}}{1 - q^{-l}} e \left( \frac{Kx_0}{4K} \right) .
\]  

Therefore, using Eq. (34) in Eq. (35), we get the third master identity:

\[
S_p(x_0) = \sqrt{m} \sum_{l=1}^{L} \frac{(-1)^{l-1}\gamma_l}{(l-1)!} \frac{d^{l-1}}{dx_0^{l-1}} \sigma_3(x_0) .
\]
As an illustration take \( f(z) = sn^2(z)[sn(z + 4K/p) + sn(z - 4K/p)] \). This gives \( L = 1 \) with \( \gamma_1 = (2/m^{3/2})[ns^2(4K/p) - ds(4K/p)cs(4K/p)] \). The resulting identity is

\[
\sum_{j=1}^{p} sn^2(x_j)[sn(x_{j+1}) + sn(x_{j-1})] = (2/m) \left[ ns^2(4K/p) - ds(4K/p)cs(4K/p) \right] \sum_{j=1}^{p} sn(x_j) .
\]

(36)

An example with \( L = 2 \) is provided by \( f(z) = sn(z)dn(z)[sn(z + 4K/p)cn(z + 4K/p) + sn(z - 4K/p)cn(z - 4K/p)] \). This results in \( \gamma_1 = 0 \) while \( \gamma_2 = -(2/m^{3/2})ns(4K/p)[cs(4K/p) + ds(4K/p)] \). Therefore the first derivative of \( sn \) will appear on the right hand side of the identity which we write as:

\[
\sum_{j=1}^{p} sn(x_j)dn(x_j)[sn(x_{j+1}) cn(x_{j+1}) + sn(x_{j-1}) cn(x_{j-1})] \\
= (2/m)ns(4K/p)[cs(4K/p) + ds(4K/p)] \sum_{j=1}^{p} cn(x_j) dn(x_j) .
\]

(37)

2.2.2 MI - IV: Case \( Q = 1, \ P = 1 \)

This case applies when there are an odd total number of \( dn \) and \( cn \) and there are an odd total number of \( sn \) and \( cn \) in \( f(z) \), i.e. if one considers terms of the form \( dn^a sn^b cn^c \), then both \( a + c \) and \( b + c \) must be odd. The relevant primitive function here is \( cn(z) \) and the archetypal sum is

\[
\sigma_4(x_0) = \sum_{i=1}^{p} cn(x_i) = \frac{2\pi p}{4K/m} \sum_{k\neq 0, p|k} \frac{1 - (-1)^k q^{-k/2}}{1 + q^{-k}} e \left( \frac{kx_0}{4K} \right) .
\]

(38)

Therefore, using Eq. (38) in Eq. (38), we get the fourth and final master identity:

\[
S_p(x_0) = i\sqrt{m} \sum_{l=1}^{L} \frac{(-1)^{l-1}\gamma_l}{(l-1)!} \frac{d^{l-1}}{dx_0^{l-1}} \sigma_4(x_0) .
\]

(39)

As an illustration, consider \( f(z) = cn^2(z) [cn(z + 4K/p) + cn(z - 4K/p)] \). This gives \( L = 1 \) with \( \gamma_1 = (2i/m^{3/2})[ds^2(4K/p) - ns(4K/p)cs(4K/p)] \). The resulting identity is

\[
\sum_{j=1}^{p} cn^2(x_j)[cn(x_{j+1}) + cn(x_{j-1})] = (2/m) \left[ ns(4K/p)cs(4K/p) - ds^2(4K/p) \right] \sum_{j=1}^{p} cn(x_j) .
\]

(40)

For \( f(z) = cn^2(z) dn(z)[sn(z + 4K/p) + sn(z - 4K/p)] \), we get \( L = 2 \) and \( \gamma_1 = 0 \) while \( \gamma_2 = (-2i/m^{3/2})cs(4K/p)ds(4K/p) \). The resultant identity therefore involves the first derivative of \( cn \):

\[
\sum_{j=1}^{p} cn^2(x_j)dn(x_j)[sn(x_{j+1}) + sn(x_{j-1})] = (2/m) cs(4K/p)ds(4K/p) \sum_{j=1}^{p} sn(x_j)dn(x_j) .
\]

(41)

This completes our enumeration of master identities for ordinary sums. Many additional examples are given in Appendix A.
3 Master Identities With Alternating Signs

Alternating sums provide an immediate and important extension of the master identities discussed above. We consider sums of the form:

$$S_p^A(x_0) = \sum_{j=1}^{p} (-1)^{j-1} f(x_j) . \tag{42}$$

where again $f(x)$ has the properties discussed for the ordinary sums in Eq. (1). We are however forced to restrict $p$ to be even in this section and as a result we only have MI-I and MI-II master identities with alternating signs. One of the consequence of having an alternating sum, as we will see below, is the appearance of the simply periodic Jacobi zeta function \[7\] as an important player.

To clarify the differences that arise between ordinary and alternating sums we first work out an example:

$$S_p^A(x_0) = \sum_{j=1}^{p} (-1)^{j-1} \delta^2(x_j)[\delta(x_{j+r}) + \delta(x_{j-r})]$$

Here the spacing is $r2K/p$. Since $r$ and $p$ are coprimes and $p$ is restricted to be an even integer, hence for alternating sums $r$ can only take odd integral values.

To prove the above identity, in the case $r = 1$, consider the sums $S_p^+$ and $S_p^-$, corresponding to the positive and negative signed terms in $S_p^A$. We have to take $f(z) = \delta^2(z)[\delta(z+T/p) + \delta(z-T/p)]$ with $T = 2K$.

$$S_p^+(x_0) = \delta^2(x_1)[\delta(x_2) + \delta(x_p)] + \delta^2(x_3)[\delta(x_4) + \delta(x_2)] + \cdots + \delta^2(x_{p-1})[\delta(x_p) + \delta(x_{p-2})] , \tag{44}$$

and

$$S_p^-(x_0) = \delta^2(x_2)[\delta(x_3) + \delta(x_1)] + \delta^2(x_4)[\delta(x_5) + \delta(x_3)] + \cdots + \delta^2(x_p)[\delta(x_1) + \delta(x_{p-1})] . \tag{45}$$

We see that

$$S_p^-(x_0) = S_p^+(x_0 + T/p)$$

and

$$S_p^+(x_0) = \sum_{j=1}^{\tilde{p}} \delta^2[x_0 + jT/\tilde{p}][\delta(x_0 + jT/\tilde{p} + T/p) + \delta(x_0 + jT/\tilde{p} - T/p)] , \tag{46}$$

where we have defined $\tilde{p} = p/2$. The important point to note is that while the above sum appears to be in the form of an ordinary sum considered earlier by simply replacing $p$ with $\tilde{p}$, it is not so, as
the function \( f(x) \) (which usually depends on \( p \)) has remained the same, or equivalently the position of the symmetric poles is still at \( iK' \pm T/p \), rather than \( iK' \pm T/\tilde{p} \).

Applying the Poisson summation formula we get

\[
S_p^+(x_0) = \frac{\tilde{p}}{2K} \left[ a_0 + \sum_{k \neq 0, \tilde{p}|k} a_k e \left( \frac{kx_0}{2K} \right) \right].
\] (47)

Note that we now need \( a_k \) for \( k \) that is a multiple of \( \tilde{p} = p/2 \) and not merely those that are multiples of \( p \). For such \( k \) we get upon integrating over the same rectangle \( ABCD \) that is relevant for type I ordinary identities,

\[
a_k = 4\pi [\text{ns}(2K/p)\text{ds}(2K/p) - (-1)^{k/\tilde{p}}\text{cs}^2(2K/p)] \frac{q^k}{1 + q^{2k}}.
\] (48)

This is because at the poles \( iK' \pm 2K/p \), \( f(x) e(kx/2K) \) has a residue of \( 2i\text{cs}^2(2K/p)q^{-k}(-1)^{k/\tilde{p}} \).

Now the negative signed sum \( S_p^- \) is related to the positive signed one, by merely a shift in the argument by an amount \( 2K/p \). Thus subtracting the two sums leads to a cancellation of the zero mode term involving \( a_0 \) and also restricts \( k/\tilde{p} \) to be odd integers. We then finally get

\[
S_p^A(x_0) = \frac{8\pi}{2K} [\text{ns}(2K/p)\text{ds}(2K/p) + \text{cs}^2(2K/p)] \sum_{k/\tilde{p} = \text{odd}} \frac{q^k}{1 + q^{2k}} e \left( \frac{kx_0}{2K} \right).
\] (49)

A similar evaluation of the archetypal alternating sum can be done with \( f(z) = \text{dn}(z) \) which is simpler as there is only one pole at \( iK' \) within \( ABCD \):

\[
\sigma_1^A(x_0) = \sum_{j=1}^p (-1)^{j-1} \text{dn}(x_j) = \frac{2\pi}{K} \sum_{k/\tilde{p} = \text{odd}} \frac{q^k}{1 + q^{2k}} e \left( \frac{kx_0}{2K} \right).
\] (50)

Therefore the stated alternating identity \[43\] follows.

We can now generalize these arguments and provide master identities for alternating sums. Consider an elliptic function \( f(z) \) of real period \( T \) satisfying Eq. (1) that has poles at \( iK' + wT/p \) where \( w = 0, \pm 1, \pm 2, \ldots \) and \( |w| < p \). For both MI-I and MI-II classes we have

\[
S_p^A(x_0) = \frac{\tilde{p}}{K} \sum_{k/\tilde{p} = \text{odd}} a_k e \left( \frac{kx_0}{2K} \right),
\] (51)

thus there are no constant terms, even for type II alternating identities.

For type I and II identities we can write the \( a_k \), the counterpart of Eq. (15) as

\[
a_k = \frac{2\pi iq^{-k}}{1 + (-1)^{p+1}q^{-2k}} \sum_{l=1}^L \left[ \frac{\tilde{\gamma}_l}{(l-1)!} \left( \frac{-2\pi i}{2K} \right)^{l-1} k^{l-1} \right].
\] (52)

12
The difference between the two Eqs. (15) and (52) that is crucial, is that \( \tilde{\gamma}_l = \sum_w (-1)^w \alpha_{l(w)} \). Thus at the pole \( wT/p \), the coefficient of the order \( l \) principal part gets weighted by a factor of \( (-1)^w \), as the residue calculation is restricted to those \( k \) where \( k/\tilde{\rho} \) is an odd integer. Therefore for instance \( \tilde{\gamma}_1 \) does not in general have the meaning of sum of residues at all the poles. This in turn implies that it need not vanish for type II alternating identities.

Defining the first archetypal alternating sum as in Eq. (50), we then see that for type I identities:

\[
S^A_p(x_0) = i \sum_{l=1}^{L} (-1)^{l-1} \tilde{\gamma}_l \frac{d^{l-1}}{dx_0^{l-1}} \sigma^A_1(x_0).
\]

Some alternating sum identities of type I are:

\[
\sum_{j=1}^{p} (-1)^{j-1} \text{sn}(x_j) \{ \text{cn}(x_{j+1}) + \text{cn}(x_{j-1}) \} = 0,
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} \text{sn}(x_j) \text{cn}(x_j) \{ \text{dn}(x_{j+1}) + \text{dn}(x_{j-1}) \} = 2 \text{ns}(2K/p) \text{ds}(2K/p) \sum_{j=1}^{p} (-1)^{j-1} \text{sn}(x_j) \text{cn}(x_j).
\]

Turning to the type II alternating identities, it turns out that the primitive function in this case is Jacobi zeta function \( Z(x) \) rather than \( \text{dn}^2(x) \). To see this, consider the second archetypal sum as

\[
\sigma^A_2(x_0) = \sum_{j=1}^{p} (-1)^{j-1} Z(x_j).
\]

On using the Fourier series expansion [8, 9] and the Poisson summation formula for \( Z(x) \) we get:

\[
\sum_{j=1}^{p} (-1)^{j-1} Z(x_j) = \frac{2\tilde{\rho} \pi i}{K} \sum_{k/\tilde{\rho}=\text{odd}} \frac{q^{-k}}{1-q^{-2k}} e \left( \frac{k x_0}{2K} \right).
\]

On following the steps carried out above then leads to the second master identity:

\[
S^A_p(x_0) = \sum_{l=1}^{L} (-1)^{l-1} \tilde{\gamma}_l \frac{d^{l-1}}{dx_0^{l-1}} \sigma^A_2(x_0).
\]

Some alternating sum identities of type II are

\[
\sum_{j=1}^{p} (-1)^{j-1} \text{dn}(x_j) \{ \text{cn}(x_{j+1}) \text{sn}(x_{j+1}) + \text{cn}(x_{j-1}) \text{sn}(x_{j-1}) \} = -(4/m) \text{ds}(2K/p) \text{ns}(2K/p) \sum_{j=1}^{p} (-1)^{j-1} Z(x_j).
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} \cn^3(x_j) [\cn(x_{j+1}) + \cn(x_{j-1})] = (2/m^2) \cs(2K/p) \ns(2K/p) \sum_{j=1}^{p} (-1)^{j-1} \dn^2(x_j), \tag{60}
\]

\[
\begin{align*}
\sum_{j=1}^{p} (-1)^{j-1} \cn^2(x_j) \sn(x_j) \dn(x_j) [\cn(x_{j+1}) + \cn(x_{j-1})] \\
= -(4/m^2) \ds^2(2K/p) \cs(2K/p) \ns(2K/p) \sum_{j=1}^{p} (-1)^{j-1} Z(x_j) \\
+ (2/m) \cs(2K/p) \ns(2K/p) \sum_{j=1}^{p} (-1)^{j-1} \cn(x_j) \sn(x_j) \dn(x_j). \tag{61}
\end{align*}
\]

Summarizing, for functions of the form \( f(z) = h(z)[g(z + T/p) + g(z - T/p)] \), which occur in ordinary sums, we may use the symmetrized form \( h(z)[g(z + T/p) + g(z - T/p)] + g(z)[h(z + T/p) + h(z - T/p)] \) and evaluate its principal part at \( iK' \). On the other hand, for alternating sums, we may use the anti-symmetrized form \( h(z)[g(z + T/p) + g(z - T/p)] - g(z)[h(z + T/p) + h(z - T/p)] \) and consider its principal part at \( iK' \). Its generalization to more complex forms of \( f(z) \) is straightforward. Using the master identities derived in this and the previous section and this methodology, we have obtained a large number of identities, some of which are given in Appendices A and B.

### 4 Comments and Discussion

In this section, we give some general comments and extensions in several new directions.

(i) **Identities for Auxiliary Functions**: Until now, we have discussed identities for the three basic Jacobi elliptic functions \( \sn, \cn, \dn \). However, nine auxiliary functions are also frequently found in the literature. They are:

\[
\begin{align*}
\nd u & \equiv \frac{1}{\dn u}; \quad \cd u \equiv \frac{\cn u}{\dn u}; \quad \sd u \equiv \frac{\sn u}{\dn u}; \\
\ns u & \equiv \frac{1}{\sn u}; \quad \cs u \equiv \frac{\cn u}{\sn u}; \quad \ds u \equiv \frac{\dn u}{\sn u}; \\
\nc u & \equiv \frac{1}{\cn u}; \quad \dc u \equiv \frac{\dn u}{\cn u}; \quad \sc u \equiv \frac{\sn u}{\cn u}. \tag{62}
\end{align*}
\]

Identities for these auxiliary functions are readily obtained via the following relations \[2, 8\]:

\[
\begin{align*}
\dn(u, m) &= \sqrt{1 - m} \, \nd(u - K, m) = -i \cs(u - iK', m) = i \sqrt{1 - m} \, \sc(u - K - iK', m), \tag{63}
\end{align*}
\]

\[
\begin{align*}
\sn(u, m) &= \cd(u - K, m) = \frac{1}{\sqrt{m}} \ns(u - iK', m) = \frac{1}{\sqrt{m}} \dc(u - K - iK', m). \tag{64}
\end{align*}
\]
\[ \text{cn}(u, m) = -\sqrt{1 - m} \text{sd}(u - K, m) = \frac{-i}{\sqrt{m}} \text{ds}(u - iK', m) = \frac{-i\sqrt{1 - m}}{\sqrt{m}} \text{nc}(u - K - iK', m). \] (65)

As an example, consider the identity \( \text{dn}(u, m) \text{dn}(u + K, m) = \sqrt{1 - m} \). Using Eq. (63), we obtain

\[
\begin{align*}
\text{nd}(x, m) \text{nd}(x + K, m) &= \frac{1}{\sqrt{1 - m}}, \\
\text{cs}(x, m) \text{cs}(x + K, m) &= -\sqrt{1 - m}, \\
\text{sc}(x, m) \text{sc}(x + K, m) &= -\frac{1}{\sqrt{1 - m}}. 
\end{align*}
\] (66)

(ii) Identities for Pure Imaginary Shifts: So far, we have focused our attention on identities involving Jacobi elliptic functions evaluated at points separated by real gaps \( T/p \), with real \( T \). As mentioned in ref. [1], since Jacobi functions are doubly periodic, we can convert each identity to another one involving points separated by pure imaginary gaps \( iT'/p \), with real \( T' \). The procedure consists of taking any given identity, writing it for modulus \( 1 - m \) [noting that \( K(1 - m) = K'(m) \)], using the standard results [2, 8]

\[
\begin{align*}
\text{sn}(x, 1 - m) &= \frac{-1}{\sqrt{1 - m}} \text{dn}(ix + K(m) + iK'(m), m), \\
\text{cn}(x, 1 - m) &= \frac{i\sqrt{m}}{\sqrt{1 - m}} \text{cn}(ix + K(m) + iK'(m), m), \\
\text{dn}(x, 1 - m) &= \sqrt{m} \text{sn}(ix + K(m) + iK'(m), m), 
\end{align*}
\] (67)

and changing to a new variable \( u = ix + K(m) + iK'(m) \).

For instance, again consider the simple identity \( \text{dn}(x, m) \text{dn}(x + K, m) = \sqrt{1 - m} \). Re-writing with modulus \( 1 - m \), using Eq. (67), and changing to the new variable \( u = ix + K(m) + iK'(m) \) gives a simple identity involving a pure imaginary shift

\[ \text{sn}(u, m) \text{sn}(u + iK'(m), m) = 1/\sqrt{m}. \] (68)

A more non-trivial example consists of identity (42) in ref. [1]:

\[
\begin{align*}
\text{sn}(x, m) \text{sn}(x + 4K(m)/3, m) \text{sn}(x + 8K(m)/3, m) \\
= \frac{-1}{1 - q^2} [\text{sn}(x, m) + \text{sn}(x + 4K(m)/3, m) + \text{sn}(x + 8K(m)/3, m)], 
\end{align*}
\] (69)

where \( q \equiv \text{dn}(2K(m)/3, m) \).
The corresponding identity with pure imaginary shifts is
\[
\text{dn}(u, m) \text{dn}(u + 4iK'(m)/3, m) \text{dn}(u + 8iK'(m)/3, m)
= \frac{-(1-m)}{1-q'^2} [\text{dn}(u, m) + \text{dn}(u + 4iK'(m)/3, m) + \text{dn}(u + 8iK'(m)/3, m)],
\]
where \(q' \equiv \text{dn}(2K'(m)/3, 1-m)\).

(iii) Identities for Complex Shifts: Just as we have derived identities containing pure imaginary shifts, we can also derive new identities involving complex shifts. Here, the procedure consists of taking any given identity for real shifts, writing it for modulus 1/m (noting that \(K(1/m) = \sqrt{m[K(m) + iK'(m)]} \[2\] ), using the standard results
\[
\begin{align*}
\text{sn}(x, \frac{1}{m}) &= \sqrt{m} \text{sn}(\frac{x}{\sqrt{m}}, m) ; \\
\text{cn}(x, \frac{1}{m}) &= \text{dn}(\frac{x}{\sqrt{m}}, m) ; \\
\text{dn}(x, \frac{1}{m}) &= \text{cn}(\frac{x}{\sqrt{m}}, m),
\end{align*}
\]
and changing to a new variable \(u = x/\sqrt{m}\).

As a simple example, let us once more take the simple identity \(\text{dn}(x, m) \text{dn}(x + K, m) = \sqrt{1-m}\).
It now transforms to
\[
\text{cn}(u, m) \text{cn}(u + K + iK'(m), m) = -i\frac{\sqrt{1-m}}{\sqrt{m}}.
\]

As a second example, take identity (45) in ref. [1]:
\[
\begin{align*}
\text{cn}(x, m) \text{sn}(x + 4K/3, m) \text{sn}(x + 8K/3, m) + \text{cn}(x + 4K/3, m) \text{sn}(x + 8K/3, m) \text{sn}(x, m) \\
+ \text{cn}(x + 8K/3, m) \text{sn}(x, m) \text{sn}(x + 4K/3, m)
= \frac{-(1+q'^2)}{m} [\text{cn}(x, m) + \text{cn}(x + 4K/3, m) + \text{cn}(x + 8K/3, m)],
\end{align*}
\]
where \(q \equiv \text{dn}(2K(m)/3, m)\).

The corresponding identity with complex shifts is
\[
\begin{align*}
\text{dn}(u, m) \text{sn}(u + 4(K + iK')/3, m) \text{sn}(u + 8(K + iK')/3, m) \\
+ \text{dn}(u + 4(K + iK')/3, m) \text{sn}(u + 8(K + iK')/3, m) \text{sn}(u, m) \\
+ \text{dn}(u + 8(K + iK')/3, m) \text{sn}(u, m) \text{sn}(u + 4(K + iK')/3, m)
= -(1+r)^2 [\text{dn}(u, m) + \text{dn}(u + 4(K + iK')/3, m) + \text{dn}(u + 8(K + iK')/3, m)],
\end{align*}
\]
where \(r \equiv q(1/m) = \text{cn}(2\{K(m) + iK'(m)\}/3, m)\).

(iv) Identities for Ratios of Jacobi Elliptic Functions: In applications involving linear superposition of solutions of nonlinear differential equations [3,4], one often needs identities for ratios
of Jacobi elliptic functions like \( \text{cndn}/\text{sn} \). These can be obtained from the identities derived in this paper. For example, noting that
\[
\frac{\text{cn} \ x \ \text{dn} \ x}{\text{sn} \ x} = \frac{\text{dn} \ 2x + \text{cn} \ 2x}{\text{sn} \ 2x} = i \left[ \sqrt{m} \text{cn}(2x + iK', m) + \text{dn}(2x + iK', m) \right],
\]
gives rise to
\[
\frac{\text{cn} \ (x + 2K/3) \ \text{dn} \ (x + 2K/3)}{\text{sn} \ (x + 2K/3)} \quad \text{and} \quad \frac{\text{cn} \ (x + 4K/3) \ \text{dn} \ (x + 4K/3)}{\text{sn} \ (x + 4K/3)}
\]
\[
+ \frac{\text{cn} \ (x + 4K/3) \ \text{dn} \ (x + 4K/3)}{\text{sn} \ (x + 4K/3)} \frac{\text{cn} \ x \ \text{dn} \ x}{\text{sn} \ x} \quad \text{for Weierstrass function for shifts in the units of } \omega \ \text{[7]. Using this relationship and identities obtained by us, we can immediately write down identities}
\]
\[
P \quad \text{where} \ A, B \quad \text{are the constants appearing in Eq. (28).}
\]
\[
\text{gives rise to}
\]
\[
- m[\text{cn}(2u) \ \text{cn}(2u + 4K/3) + \text{cn}(2u + 4K/3) \ \text{cn}(2u + 8K/3) + \text{cn}(2u + 8K/3) \ \text{cn}(2u) - \text{dn}(2u) \ \text{dn}(2u + 4K/3) + \text{dn}(2u + 4K/3) \ \text{dn}(2u + 8K/3) + \text{dn}(2u + 8K/3) \ \text{dn}(2u)] ,
\]
where \( u = x + iK'(m)/2 \). In the above derivation, the \( \text{cn} \ \text{dn} \) terms cancel in view of identity (33) in ref. [1]. Further, the right hand side of Eq. (76) has the constant value \( q(2 + q)(m - (1 + q)^2)/(1 + q)^2 \), \( q \equiv \text{dn}(2K(m)/3, m) \), due to identities (32) in ref. [1].

Other identities involving ratios follow from useful equations analogous to Eq. (75):
\[
\frac{\text{sn} \ x \ \text{dn} \ x}{\text{cn} \ x} = 1 - \frac{2x}{\text{sn} \ 2x} , \quad \frac{\text{sn} \ x \ \text{cn} \ x}{\text{sn} \ x} = \frac{1 - \text{dn} \ 2x}{\text{m} \ \text{sn} \ 2x} , \quad \frac{\text{cn} \ x}{\text{sn} \ x} = \frac{1 + \text{cn} \ 2x}{\text{sn} \ 2x} , \quad \frac{\text{dn} \ x}{\text{sn} \ x} = \frac{1 + \text{dn} \ 2x}{\text{sn} \ 2x} .
\]

(v) Cyclic Identities for Weierstrass Functions: Jacobi elliptic functions are closely related with the Weierstrass function \( \mathcal{P}(u) \) [27], the relations being:
\[
\text{sn} \ u = [\mathcal{P}(u) - e_3]^{-1/2} , \quad \text{cn} \ u = \left[ \frac{\mathcal{P}(u) - e_1}{\mathcal{P}(u) - e_3} \right]^{1/2} , \quad \text{dn} \ u = \left[ \frac{\mathcal{P}(u) - e_2}{\mathcal{P}(u) - e_3} \right]^{1/2} ,
\]
where
\[
e_1 = (2 - m)/3 , \quad e_2 = (2m - 1)/3 , \quad e_3 = -(1 + m)/3 .
\]
\( \mathcal{P}(u) \) has implicit arguments corresponding to its two periods \( 2\omega_1 = 2K(m) \) and \( 2\omega_3 = 2iK'(m) \) [7]. Using this relationship and identities obtained by us, we can immediately write down identities for Weierstrass function for shifts in the units of \( \omega_1/p, \omega_3/p \) and \( \omega_2/p \) where \( \omega_2 = -(\omega_1 + \omega_3) \). For example, using identity (28), one gets
\[
\sum_{j=1}^{p} \mathcal{P}(u + 2(j - 1)\omega_1/p) = (B + pAe_1 - pe_1^2) - (A - 2e_1) \sum_{j=1}^{p} \mathcal{P}(u + 2(j - 1)\omega_1/p)
\]
where \( A, B \) are the constants appearing in Eq. (28).
(vi) Cyclic Identities for Jacobi Theta Functions: The connection between the four Jacobi theta functions $\theta_1(z), \theta_2(z), \theta_3(z), \theta_4(z)$ and the Jacobi elliptic functions is given by

$$\text{sn} u = \frac{1}{m^{1/4}} \frac{\theta_1(z)}{\theta_4(z)}, \quad \text{cn} u = \frac{(1 - m)^{1/4} \theta_2(z)}{m^{1/4} \theta_4(z)}, \quad \text{dn} u = (1 - m)^{1/4} \frac{\theta_3(z)}{\theta_4(z)},$$

(81)

where $z \equiv \frac{\nu \pi}{2K(m)}$. Therefore, any of our cyclic identities for real, imaginary or complex shift can also be re-written as identities for the ratios of Jacobi elliptic functions for shifts in units of $\pi/p$, $\pi\tau/p$ or $\pi(1 + \tau)/p$ respectively where $\tau = iK'/K$. As an illustration, we consider identity (85). In terms of theta functions, one gets

$$\prod_{j=1}^{p} \frac{\theta_3(z + (j - 1)\pi/p)}{\theta_4(z + (j - 1)\pi/p)} = \left( \prod_{n=1}^{(p-1)/2} \frac{\theta_2^2(2nK/p)}{\theta_1^2(2nK/p)} \right) \sum_{j=1}^{p} \frac{\theta_3(z + (j - 1)\pi/p)}{\theta_4(z + (j - 1)\pi/p)}. \quad (82)$$

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Appendix A: Examples Using the Master Identities

In this Appendix, we present a collection of identities involving cyclic combinations of Jacobi elliptic functions. These identities are derived by using various choices of \( f(x) \) in the four master identities [Eqs. (20), (26), (35), (39)] developed in the text. We use the notation \( a \equiv r2K/p \) and \( b \equiv r4K/p \), where \( 1 \leq r < p - 1 \) and \((r,p) = 1\). We also use \( a' = s2K/p \), \( a'' = t2K/p \) and \( b' = s4K/p \), where \( s, r, t \) are all distinct. Also we use \( s_j \equiv \text{sn}(x_j) \) etc., where \( x_j = x_0 + (j - 1)T/p \).

Note that \( T = 2K \) for the first two master identities, while it is \( 4K \) for the remaining two. We note that these identities are not exhaustive (as indeed they cannot be) but are meant to be representative low \( L \) identities.

Examples belonging to the class MI - I

\( L = 0 \):

\[
\sum_{j=1}^{p} s_j(c_{j+r} + c_{j-r}) = \frac{p}{2K} \int_{0}^{2K} f(x)dx = 0 . \tag{83}
\]

\( L = 1 \):

\[
\sum_{j=1}^{p} d_j d_{j+r} \cdots d_{j+(l-1)r} = \left[ \prod_{k=1}^{(l-1)/2} \text{cs}^2(ka) + 2(-1)^{(l-1)/2} \sum_{k=1}^{(l-1)/2} \prod_{n=1, n \neq k}^{l} \text{cs}([n-k]a) \right] \sum_{j=1}^{p} d_j , \tag{84}
\]

which is valid for any odd integral \( l \leq p \). This is a generalization of Eq. (22) which corresponds to \( l = 3 \). In the special case of \( l = p \) this identity takes the simpler form

\[
\prod_{j=1}^{p} d_j = \prod_{n=1}^{(p-1)/2} \text{cs}^2\left( \frac{2Kn}{p} \right) \sum_{j=1}^{p} d_j . \tag{85}
\]

A special case of this identity for \( p = 3 \) has been worked out in [1].

\[
\sum_{j=1}^{p} d_j^2(d_{j+r} + d_{j-r}) = 2[\text{ds}(a) \text{ns}(a) - \text{cs}^2(a)] \sum_{j=1}^{p} d_j . \tag{86}
\]

\[
\sum_{j=1}^{p} c_j(c_{j+r}d_{j+r} + c_{j-r}d_{j-r}) = (-2/m)\text{cs}(a)[\text{ds}(a) - \text{ns}(a)] \sum_{j=1}^{p} d_j . \tag{87}
\]

\[
\sum_{j=1}^{p} s_j(s_{j+r}d_{j+r} + s_{j-r}d_{j-r}) = (-2/m)\text{cs}(a)[\text{ds}(a) - \text{ns}(a)] \sum_{j=1}^{p} d_j . \tag{88}
\]

19
\[ \sum_{j=1}^{p} d_j (d_{j+r}d_{j+s} + d_{j-r}d_{j-s}) = -2 \{ \text{cs}(a)\text{cs}(a') + \text{cs}(a-a')\{ \text{cs}(a) - \text{cs}(a') \} \} \sum_{j=1}^{p} d_j . \] (89)

\[ \sum_{j=1}^{p} d_j (c_{j+r}c_{j+s} + c_{j-r}c_{j-s}) = (-2/m) \{ \text{ds}(a)\text{ds}(a') + \text{ds}(a-a')\{ \text{cs}(a) - \text{cs}(a') \} \} \sum_{j=1}^{p} d_j . \] (90)

\[ \sum_{j=1}^{p} d_j (s_{j+r}s_{j+s} + s_{j-r}s_{j-s}) = (2/m) \{ \text{ns}(a)\text{ns}(a') + \text{ns}(a-a')\{ \text{cs}(a) - \text{cs}(a') \} \} \sum_{j=1}^{p} d_j . \] (91)

\[ \sum_{j=1}^{p} c_j (c_{j+r}d_{j+s} + c_{j-r}d_{j-s}) = (-2/m) \{ \text{cs}(a') + \text{cs}(a-a') \} \text{ds}(a) - \text{ds}(a-a')\text{ds}(a') \sum_{j=1}^{p} d_j . \] (92)

\[ \sum_{j=1}^{p} s_j (s_{j+r}d_{j+s} + s_{j-r}d_{j-s}) = (2/m) \{ \text{cs}(a') + \text{cs}(a-a') \} \text{ns}(a) - \text{ns}(a-a')\text{ns}(a') \sum_{j=1}^{p} d_j . \] (93)

\[ \sum_{j=1}^{p} s_j^2 [c_{j+r}c_{j+s}d_{j+t} + c_{j-r}c_{j-s}d_{j-t}] = (2/m^2) \{ \text{cs}(a)\text{cs}(a'')\text{ds}(a') \text{ns}(a) \\
+ \text{cs}(a')\text{cs}(a'')\text{ds}(a) \text{ns}(a') + \text{ds}(a)\text{ds}(a')\text{ds}(a'') \text{ns}(a'') - \text{cs}(a-a'')\text{ds}(a-a') \text{ns}^2(a) \\
- \text{cs}(a''-a') \text{ds}(a-a') \text{ns}^2(a') - \text{ds}(a-a'')\text{ds}(a'-a'') \text{ns}^2(a') \} \sum_{j=1}^{p} d_j . \] (94)

\[ \text{L} = 2 : \]

\[ \sum_{j=1}^{p} d_j^2 [c_{j+r}s_{j+r} + c_{j-r}s_{j-r}] = -2[\text{cs}^2(a) + \text{ds}(a)\text{ns}(a)] \sum_{j=1}^{p} c_j s_j . \] (95)

\[ \sum_{j=1}^{p} c_j s_j d_j [d_{j+r} + d_{j-r}] = 2\text{ds}(a)\text{ns}(a) \sum_{j=1}^{p} c_j s_j . \] (96)

\[ \sum_{j=1}^{p} s_j d_j (c_{j+r}d_{j+s} + c_{j-r}d_{j-s}) = -2\text{cs}(a)[\text{ns}(a) + \text{ds}(a)] \sum_{j=1}^{p} c_j s_j . \] (97)

\[ \sum_{j=1}^{p} c_j (s_j^3_{j+r} + s_j^3_{j-r}) = -(2/m)\text{cs}(a)\text{ns}(a) \sum_{j=1}^{p} c_j s_j . \] (98)

\[ \sum_{j=1}^{p} s_j (c_j^3_{j+r} + c_j^3_{j-r}) = (2/m)\text{cs}(a)\text{ds}(a) \sum_{j=1}^{p} c_j s_j . \] (99)
\[
\sum_{j=1}^{p} s_j c_j (d_{j+r} d_{j+s} + d_{j-r} d_{j-s}) = -2 \cos(a) \cos(a') \sum_{j=1}^{p} c_j s_j .
\] 
\[
\sum_{j=1}^{p} s_j c_j (c_{j+r} c_{j+s} + c_{j-r} c_{j-s}) = (-2/m) \sin(a) \sin(a') \sum_{j=1}^{p} c_j s_j .
\] 
\[
\sum_{j=1}^{p} s_j c_j (s_{j+r} s_{j+s} + s_{j-r} s_{j-s}) = (2/m) \sin(a) \sin(a') \sum_{j=1}^{p} c_j s_j .
\] 
\[
\sum_{j=1}^{p} d_j c_j (s_{j+r} d_{j+s} + s_{j-r} d_{j-s}) = -2 \sin(a) \sin(a') \sum_{j=1}^{p} c_j s_j .
\] 
\[
\sum_{j=1}^{p} d_j s_j (c_{j+r} d_{j+s} + c_{j-r} d_{j-s}) = -2 \sin(a) \sin(a') \sum_{j=1}^{p} c_j s_j .
\] 
\[
\sum_{j=1}^{p} d_j^2 (s_{j+r} c_{j+s} + s_{j-r} c_{j-s}) = 2 \left[ \cos(a) \sin(a - a') - \cos(a') \sin(a - a') \right] \sum_{j=1}^{p} c_j s_j .
\] 
\[
\sum_{j=1}^{p} d_j c_j s_j (d_{j+r}^3 + d_{j-r}^3)
\]
\[
= -2 \left[ \sin^2(a) \sin^2(a) + \sin^2(a) \sin^2(a) + \sin^2(a) \cos^2(a) + 3 \sin^2(a) \sin(a) \sin(a) \right] \sum_{j=1}^{p} c_j s_j .
\] 

**L = 3** :
\[
\sum_{j=1}^{p} d_j^4 (d_{j+r} + d_{j-r}) = 2 \sin(a) \sin(a) \sum_{j=1}^{p} d_j^3 + 2 \sin^2(a) \left[ \cos^2(a) - \sin(a) \sin(a) \right] \sum_{j=1}^{p} d_j .
\] 
\[
\sum_{j=1}^{p} d_j^3 (d_{j+r}^2 + d_{j-r}^2) = -2 \sin^2(a) \sum_{j=1}^{p} d_j^3
\]
\[
+ 2 \left[ \sin^2(a) \sin^2(a) + \sin^2(a) \sin^2(a) + \sin^2(a) \cos^2(a) - 3 \sin^2(a) \sin(a) \sin(a) \right] \sum_{j=1}^{p} d_j .
\] 
\[
\sum_{j=1}^{p} c_j s_j d_j (c_{j+r} s_{j+r} + c_{j-r} s_{j-r}) = (2/m^2) \sin(a) \sum_{j=1}^{p} d_j^3 + (2/m) \left[ \cos^2(a) \sin^2(a) + \sin^2(a) \sin^2(a) + \cos^2(a) - \sin(a) \sin(a) \sin(a) \right] \sum_{j=1}^{p} d_j .
\]
L = 4 :

\[ \sum_{j=1}^{p} d_j^4(s_{j+r}c_{j+r} + s_{j-r}c_{j-r}) = -2 \text{ns}(a)\text{ds}(a) \sum_{j=1}^{p} s_jc_jd_j^2 + 2\text{cs}^2(a)[\text{cs}^2(a) + 3 \text{ns}(a)\text{ds}(a)] \sum_{j=1}^{p} s_jc_j. \quad (110) \]

\[ \sum_{j=1}^{p} d_j^4(s_{j+r}c_{j+s} + s_{j-r}c_{j-s}) = -2 \text{ns}(a)\text{ds}(a') \sum_{j=1}^{p} s_jcjd_j^2 
+ 2 \left[ \text{cs}(a)\text{ds}(a)\text{cs}(a')\text{ns}(a') + \text{ns}(a)\text{ds}(a')\{\text{cs}^2(a) + \text{cs}^2(a')\} \right] \sum_{j=1}^{p} s_jc_j. \quad (111) \]

**Examples belonging to the class MI - II**

L = 0 :

\[ \sum_{j=1}^{p} d_jd_{j+r} = p \left( \text{dn}(a) - \text{cs}(a)Z(\beta_{2rK}) \right). \quad (112) \]

\[ \sum_{j=1}^{p} s_js_{j+r} = \frac{p}{m} \text{cs}(a)Z(\beta_{2rK}). \quad (113) \]

\[ \sum_{j=1}^{p} c_jc_{j+r} = p \left( \text{cn}(a) - \frac{\text{ds}(a)Z(\beta_{2rK})}{m} \right). \quad (114) \]

It may be noted that the last two identities are valid for \( p > 2 \) while for \( p = 2 \) both sides vanish identically. Further, the product of any even number (< \( p \) ) of \( \text{dn} \)'s, \( \text{sn} \)'s or \( \text{cn} \)'s is also a constant, the constant being the integral of the corresponding function over the period \( 2K \). For example,

\[ \sum_{j=1}^{p} d_jd_{j+1} \cdots d_{j+r-1} = \frac{p}{2K} \int_{0}^{2K} \text{dn}(x)\text{dn}(x + 2K/p) \cdots \text{dn}(x + (r-1)2K/p) \, dx, \text{ for } r \text{ even}. \quad (115) \]

For \( \text{dn} \)'s of course even the product of all \( p \) of them (i.e. \( d_1d_2\ldots d_p \)) is a constant \( =(1 - m)^{p/4} \) in case \( p \) is even.

\[ \sum_{j=1}^{p} c_jd_j(s_{j+r} + s_{j-r}) = 0. \quad (116) \]

\[ \sum_{j=1}^{p} d_jd_j(c_{j+r} + c_{j-r}) = 0. \quad (117) \]
\[
\sum_{j=1}^{p} c_j s_j (d_{j+r} + d_{j-r}) = 0 .
\]

\[
\sum_{j=1}^{p} c_j (d_{j+r} s_{j+s} + d_{j-r} s_{j-s}) = 0 .
\]

\[
\sum_{j=1}^{p} s_j (d_{j+r} c_{j+s} + d_{j-r} c_{j-s}) = 0 .
\]

\[
\sum_{j=1}^{p} d_j (c_{j+r} s_{j+s} + c_{j-r} s_{j-s}) = 0 .
\]

**L = 2 :**

\[
\sum_{j=1}^{p} d_j^2 d_{j+r} = A \sum_{j=1}^{p} d_j^2 + B ,
\]

where \( A = -2cs^2(a) \), \( B = \frac{p}{2K} \left( \int_{0}^{2K} dn^2(t) dn^2(t + 2rK/p) dt + 4Ec^2(a) \right) \).

\[
\sum_{j=1}^{p} c_j s_j (c_{j+r} s_{j+r} + c_{j-r} s_{j-r}) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_{0}^{2K} f(x) dx + 2\gamma_2 E \right) ,
\]

where \( f(x) = cn(x) sn(x) [cn(x + a) sn(x + a) + cn(x - a) sn(x - a)] \) and \( \gamma_2 = -(4/m^2) ns(a) ds(a) \).

\[
\sum_{j=1}^{p} c_j d_j (c_{j+r} d_{j+r} + c_{j-r} d_{j-r}) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_{0}^{2K} f(x) dx + 2\gamma_2 E \right) ,
\]

where \( f(x) = cn(x)dn(x) [cn(x + a) dn(x + a) + cn(x - a) dn(x - a)] \) and \( \gamma_2 = (4/m) cs(a) ds(a) \).

\[
\sum_{j=1}^{p} s_j d_j (s_{j+r} d_{j+r} + s_{j-r} d_{j-r}) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_{0}^{2K} f(x) dx + 2\gamma_2 E \right) ,
\]

where \( f(x) = sn(x)dn(x) [sn(x + a) dn(x + a) + sn(x - a) dn(x - a)] \) and \( \gamma_2 = (-4/m) cs(a) ns(a) \).

\[
\sum_{j=1}^{p} d_j^3 (d_{j+r}^2 + d_{j+2r} d_{j-r} + d_{j+2r}^2) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_{0}^{2K} f(x) dx + 2\gamma_2 E \right) ,
\]

where \( f(x) = dn^3(x) [dn^2(x + a) dn(x + 2a) + dn^2(x - a) dn(x - 2a)] \) and \( \gamma_2 = 2 cs(a) [cs^3(a) + 2 cs(2a) ds(a) ns(a) + cs(a) ds(2a) ns(2a)] \).

\[
\sum_{j=1}^{p} d_j^3 (d_{j+r} + d_{j-r}) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_{0}^{2K} f(x) dx + 2\gamma_2 E \right) ,
\]

where \( f(x) = dn^3(x) [dn(x + a) + dn(x - a)] \) and \( \gamma_2 = -2 ns(a) ds(a) \).
\[
\sum_{j=1}^{p} s_j^3(s_{j+r} + s_{j-r}) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_0^{2K} f(x)dx + 2\gamma_2 E \right),
\]
where \( f(x) = \text{sn}^3(x)[\text{sn}(x + a) + \text{sn}(x - a)] \) and \( \gamma_2 = -2\text{cs}(a)\text{ds}(a)/m^2 \).

\[
\sum_{j=1}^{p} c_j^3(c_{j+r} + c_{j-r}) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_0^{2K} f(x)dx + 2\gamma_2 E \right),
\]
where \( f(x) = \text{cn}^3(x)[\text{cn}(x + a) + \text{cn}(x - a)] \) and \( \gamma_2 = -2\text{ns}(a)\text{cs}(a)/m^2 \).

\[
\sum_{j=1}^{p} d_j^3(d_{j+r}^2 + d_{j-r}^2) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_0^{2K} f(x)dx + 2\gamma_2 E \right),
\]
where \( f(x) = \text{dn}^3(x)[\text{dn}^3(x + a) + \text{dn}^3(x - a)] \) and \( \gamma_2 = 12\text{cs}^2(a)\text{ns}(a)\text{ds}(a) \).

\[
\sum_{j=1}^{p} s_j^3(s_{j+r}^2 + s_{j-r}^2) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_0^{2K} f(x)dx + 2\gamma_2 E \right),
\]
where \( f(x) = \text{sn}^3(x)[\text{sn}^3(x + a) + \text{sn}^3(x - a)] \) and \( \gamma_2 = -12\text{ns}^2(a)\text{ds}(a)\text{cs}(a)/m^3 \).

\[
\sum_{j=1}^{p} c_j^3(c_{j+r}^2 + c_{j-r}^2) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_0^{2K} f(x)dx + 2\gamma_2 E \right),
\]
where \( f(x) = \text{cn}^3(x)[\text{cn}^3(x + a) + \text{cn}^3(x - a)] \) and \( \gamma_2 = 12\text{ds}^2(a)\text{cs}(a)\text{ns}(a)/m^3 \).

\[
\sum_{j=1}^{p} c_j s_j d_j (c_{j+r} s_{j+r} + c_{j-r} s_{j-r} d_{j+r}) = -\gamma_2 \sum_{j=1}^{p} d_j^2 + \frac{p}{2K} \left( \int_0^{2K} f(x)dx + 2\gamma_2 E \right),
\]
where \( f(x) = \text{cn}(x)\text{sn}(x)\text{dn}(x)[\text{cn}(x + a)\text{sn}(x + a)\text{dn}(x + a) + \text{cn}(x - a)\text{sn}(x - a)\text{dn}(x - a)] \) and \( \gamma_2 = (4/m^2)[\text{ns}^2(a)\text{cs}^2(a) + \text{ds}^2(a)] + \text{cs}^2(a)\text{ds}^2(a) \).

\( L = 3 \) :

\[
\sum_{j=1}^{p} c_j s_j d_j (d_{j+r}^2 + d_{j-r}^2) = -2\text{cs}^2(a) \sum_{j=1}^{p} c_j s_j d_j,
\]
\[
\sum_{j=1}^{p} c_j s_j d_j (s_{j+r} + s_{j-r}) = -(2/m) \text{cs}(a)\text{ds}(a) \sum_{j=1}^{p} c_j s_j d_j,
\]
\[
\sum_{j=1}^{p} c_j d_j^2 s_j (d_{j+r} + d_{j-r}) = -2\text{ns}(a)\text{ds}(a) \sum_{j=1}^{p} c_j s_j d_j,
\]
\[
\sum_{j=1}^{p} s_j c_j^2 d_j (c_{j+r} + c_{j-r}) = (2/m) \text{cs}(a)\text{ns}(a) \sum_{j=1}^{p} c_j s_j d_j.
\]
\[
\sum_{j=1}^{p} s_j c_j d_j^2 (d_{j+r}^3 + d_{j-r}^3) = -4 \cos^2(a) \sin(a) \cos(s) \sum_{j=1}^{p} c_j s_j d_j .
\] (138)

\[
\sum_{j=1}^{p} c_j d_j s_j^2 (s_{j+r}^3 + s_{j-r}^3) = (-4/m^2) \cos(a) \cos(s) \sum_{j=1}^{p} c_j s_j d_j .
\] (139)

\[
\sum_{j=1}^{p} d_j s_j c_j^2 (c_{j+r}^3 + c_{j-r}^3) = (-4/m^2) \cos(a) \cos(s) \sum_{j=1}^{p} c_j s_j d_j .
\] (140)

\[
\sum_{j=1}^{p} c_j s_j d_j (d_{j+r}^4 + d_{j-r}^4) = 2 \left[ \cos^4(a) - \cos^2(a) \cos(s) - \cos^2(a) \cos^2(s) - \cos^2(a) \cos^2(s) \right] \sum_{j=1}^{p} c_j s_j d_j .
\] (141)

\[L = 4:\]
\[
\sum_{j=1}^{p} s_j^5 (s_{j+r} + s_{j-r}) = m^2 \gamma_4 \sum_{j=1}^{p} s_j^4 + \left[ m \gamma_2 - \frac{2m}{3} (m+1) \gamma_4 \right] \sum_{j=1}^{p} s_j^2 + \frac{p}{2K} \left[ \int_0^{2K} f(x) dx + 2 \gamma_2 E \right] + p \left( -\gamma_2 + \frac{m \gamma_4}{3} \right),
\] (142)

where \( \gamma_2 = -(1/3m^4) \cos(a) \cos(s) \left[ 5 + 5m + \cos^2(a) + \cos^2(a) + 4 \cos^2(a) \right] \), \( \gamma_4 = -(2/m^3) \cos(a) \cos(s) \) and \( f(x) = \sin^5(x) \left[ \sin(x + a) + \sin(x - a) \right] \).

\[
\sum_{j=1}^{p} d_j^4 (d_{j+r}^2 + d_{j-r}^2) = m^2 \gamma_4 \sum_{j=1}^{p} s_j^4 + \left[ m \gamma_2 - \frac{2m}{3} (m+1) \gamma_4 \right] \sum_{j=1}^{p} s_j^2 + \frac{p}{2K} \left[ \int_0^{2K} f(x) dx + 2 \gamma_2 E \right] + p \left( -\gamma_2 + \frac{m \gamma_4}{3} \right),
\] (143)

where \( f(x) = \sin^4(x) \left[ \sin^2(x + a) + \sin^2(x - a) \right] \), \( \gamma_4 = -2 \cos^2(a) \) and \( \gamma_2 = -(4/3) \cos^2(a)(m-2) - 2[\cos^4(a) + \cos^2(a) \cos^2(a) + \cos^2(a) \cos^2(a) \cos^2(a)] \).

**Examples belonging to the class MI - III**

\[L = 0:\]
\[
\sum_{j=1}^{p} c_j (d_{j+r} + d_{j-r}) = 0 .
\] (144)

\[L = 1:\]
There are identities in \( sn \) and \( cn \) analogous to the one given by Eq. (22) and its generalizations in Eq. (84) and Eq. (85). In particular, for odd \( l \leq p \) we have:

\[
\sum_{j=1}^{p} s_j s_{j+r} \cdots s_{j+(l-1)r} = \\
\frac{1}{m^{(l-1)/2}} \left( (-1)^{(l-1)/2} \prod_{k=1}^{(l-1)/2} \frac{ns^2(kb) + 2 \sum_{k=1}^{(l-1)/2} \prod_{n=1, n \neq k}^{l} ns([n - k]b)}{n^2(4Kn/p)} \right) \sum_{j=1}^{p} s_j.
\]  

(145)

When \( l = p \), the resulting identity takes the simpler form

\[
m^{(p-1)/2} \prod_{j=1}^{p} s_j = (-1)^{(p-1)/2} \left( \prod_{n=1}^{(p-1)/2} \frac{ns^2(b) - ds(b)\cs(b)}{n^2(p)} \right) \sum_{j=1}^{p} s_j.
\]

(146)

\[
\sum_{j=1}^{p} s_j^2(s_{j+r} + s_{j-r}) = (2/m)[ns^2(b) - ds(b)\cs(b)] \sum_{j=1}^{p} s_j.
\]

(147)

\[
\sum_{j=1}^{p} c_j(c_{j+r}s_{j+r} + c_{j-r}s_{j-r}) = (2/m) ns(b)[\cs(b) - ds(b)] \sum_{j=1}^{p} s_j.
\]

(148)

\[
\sum_{j=1}^{p} d_j(d_{j+r}s_{j+r} + d_{j-r}s_{j-r}) = 2 ns(b)[-\cs(b) + ds(b)] \sum_{j=1}^{p} s_j.
\]

(149)

\[
\sum_{j=1}^{p} s_j(c_{j+r}c_{j+s} + c_{j-r}c_{j-s}) = (-2/m) [ds(b)\ds(b') + ds(b - b')\{ns(b) - ns(b')\}] \sum_{j=1}^{p} s_j.
\]

(150)

\[
\sum_{j=1}^{p} s_j(d_{j+r}d_{j+s} + d_{j-r}d_{j-s}) = -2 [\cs(b)\cs(b') + \cs(b - b')\{ns(b) - ns(b')\}] \sum_{j=1}^{p} s_j.
\]

(151)

\[
\sum_{j=1}^{p} s_j(s_{j+r}s_{j+s} + s_{j-r}s_{j-s}) = (2/m) [ns(b)\ns(b') + ns(b - b')\{ns(b) - ns(b')\}] \sum_{j=1}^{p} s_j.
\]

(152)

\[
\sum_{j=1}^{p} d_j(s_{j+r}d_{j+s} + s_{j-r}d_{j-s}) = -2 [\{ns(b) - ns(b - b')\}\cs(b') + \cs(b - b')\cs(b)] \sum_{j=1}^{p} s_j.
\]

(153)

\[
\sum_{j=1}^{p} c_j(s_{j+r}c_{j+s} + s_{j-r}c_{j-s}) = -(2/m) [\{ns(b) - ns(b - b')\}\ds(b') + \ds(b - b')\ds(b)] \sum_{j=1}^{p} s_j.
\]

(154)
\( L = 2 \):

\[
\sum_{j=1}^{p} c_j d_j (s_{j+r}^2 + s_{j-r}^2) = (2/m) \left[ ns^2(b) + ds(b)cs(b) \right] \sum_{j=1}^{p} c_j d_j. 
\]

(155)

\[
\sum_{j=1}^{p} s_j d_j (s_{j+r} c_{j+r} + s_{j-r} c_{j-r}) = (2/m) ns(b) \left[ cs(b) + ds(b) \right] \sum_{j=1}^{p} c_j d_j.
\]

(156)

\[
\sum_{j=1}^{p} d_j c_j (s_{j+r} s_{j+s} + s_{j-r} s_{j-s}) = (2/m) ns(b) ns(b') \sum_{j=1}^{p} c_j d_j.
\]

(157)

\[
\sum_{j=1}^{p} d_j c_j (c_{j+r} c_{j+s} + c_{j-r} c_{j-s}) = (-2/m) ds(b) ds(b') \sum_{j=1}^{p} c_j d_j.
\]

(158)

\[
\sum_{j=1}^{p} d_j c_j (d_{j+r} d_{j+s} + d_{j-r} d_{j-s}) = -2cs(b) cs(b') \sum_{j=1}^{p} c_j d_j.
\]

(159)

\[
\sum_{j=1}^{p} s_j c_j (d_{j+r} s_{j+s} + d_{j-r} s_{j-s}) = (2/m) cs(b) ns(b') \sum_{j=1}^{p} c_j d_j.
\]

(160)

\[
\sum_{j=1}^{p} s_j d_j (c_{j+r} s_{j+s} + c_{j-r} s_{j-s}) = (2/m) ds(b) ns(b') \sum_{j=1}^{p} c_j d_j.
\]

(161)

\[
\sum_{j=1}^{p} c_j s_j d_j (s_{j+r} + s_{j-r}) = -(2/m) cs(b) ds(b) \sum_{j=1}^{p} c_j d_j.
\]

(162)

\[
\sum_{j=1}^{p} c_j (s_{j+r}^3 + s_{j-r}^3) = 2cs(b) ns(b) \sum_{j=1}^{p} c_j d_j.
\]

(163)

\[
\sum_{j=1}^{p} d_j (c_{j+r}^3 + c_{j-r}^3) = (2/m) ds(b) ns(b) \sum_{j=1}^{p} c_j d_j.
\]

(164)

\[
\sum_{j=1}^{p} d_j c_j s_j (s_{j+r}^3 + s_{j-r}^3) = -(2/m^2) \left[ cs^2(b) ns^2(b) 
+ ns^2(b)ds^2(b) + ds^2(b)cs^2(b) + 3 ns^2(b)cs(b)ds(b) \right] \sum_{j=1}^{p} c_j d_j.
\]

(165)

\( L = 3 \):

27
\[
\sum_{j=1}^{p} s_j^4(s_{j+r} + s_{j-r}) = -(2/m) \cos(b) \sin(b) \sum_{j=1}^{p} s_j^3 \\
\quad + (2/m^2) n \sin^2(b) [n \sin^2(b) - \cos(b) \sin(b)] \sum_{j=1}^{p} s_j, \quad (166)
\]

\[
\sum_{j=1}^{p} s_j^3(s_{j+r}^2 + s_{j-r}^2) = (2/m) n \sin^2(b) \sum_{j=1}^{p} s_j^3 \\
\quad + (2/m^2) \left[ \cos^2(b) n \sin^2(b) + \sin^2(b) \sin^2(b) + \sin^2(b) \cos^2(b) - 3 n \sin^2(b) \cos(b) \sin(b) \right] \sum_{j=1}^{p} s_j. \quad (167)
\]

\[
\sum_{j=1}^{p} c_j s_j d_j(c_{j+r} d_{j+r} + c_{j-r} d_{j-r}) = 2 \cos(b) \sin(b) \sum_{j=1}^{p} s_j^3 - (2/m) \left[ \cos^2(b) n \sin^2(b) \right. \\
\left. + n \sin^2(b) \sin^2(b) + \sin^2(b) \cos^2(b) - \cos(b) \sin(b) (\cos^2(b) + \sin^2(b)) \right] \sum_{j=1}^{p} s_j. \quad (168)
\]

\textbf{L = 4 :}

\[
\sum_{j=1}^{p} s_j^3(d_{j+r} c_{j+r} + d_{j-r} c_{j-r}) = (2/m) \cos(b) \sin(b) \sum_{j=1}^{p} c_j d_j s_j^2 \\
\quad + (2/m^2) n \sin^2(b) [n \sin^2(b) + 3 \cos(b) \sin(b)] \sum_{j=1}^{p} c_j d_j. \quad (169)
\]

\[
\sum_{j=1}^{p} s_j^3(d_{j+r} c_{j+s} + d_{j-r} c_{j-s}) = (2/m) \cos(b) \sin(b') \sum_{j=1}^{p} c_j d_j s_j^2 \\
\quad + (2/m^2) \left[ n \sin(b) \sin(b') \cos(b') + \cos(b) \sin(b') \{n \sin^2(b) + n \sin^2(b') \} \right] \sum_{j=1}^{p} c_j d_j. \quad (170)
\]

**Examples belonging to the class MI - IV**

\textbf{L = 0 :}

\[
\sum_{j=1}^{p} d_j (s_{j+r} + s_{j-r}) = 0. \quad (171)
\]
\( \text{L} = 1 \):

The generalizations of Eq. (83) and Eq. (84) pertinent to this class are for \( l \) odd \( (l \leq p) \):

\[
\sum_{j=1}^{p} c_j c_{j+r} \cdots c_{j+(l-1)r} = \left( \frac{(l-1)/2}{(l-1)/2} \sum_{k=1}^{(l-1)/2} d_s^2(kb) + 2(-1)^{(l-1)/2} \prod_{k=1}^{(l-1)/2} \prod_{n=1, n \neq k}^{(l-1)/2} ds([n-k]b) \right) \sum_{j=1}^{p} c_j .
\]

When \( l = p \), the resulting identity takes the simpler form

\[
m^{(p-1)/2} \prod_{j=1}^{p} c_j = \left( \frac{(p-1)/2}{(p-1)/2} \sum_{n=1}^{p} c_j \right) \prod_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} c_j^2 (c_{j+r} + c_{j-r}) = \left( 2/m \right) [ns(b)cs(b) - ds^2(b)] \sum_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} d_j (d_{j+r}c_{j+r} + d_{j-r}c_{j-r}) = -2ds(b)[cs(b) - ns(b)] \sum_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} s_j (s_{j+r}c_{j+r} + s_{j-r}c_{j-r}) = (-2/m)ds(b)[cs(b) - ns(b)] \sum_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} d_j (c_{j+r}d_{j+s} + c_{j-r}d_{j-s}) = -2 \left[ \{ds(b) - ds(b - b')\}cs(b') + cs(b - b')cs(b) \right] \sum_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} s_j (c_{j+r}s_{j+s} + c_{j-r}s_{j-s}) = \left( 2/m \right) \left[ \{ds(b) - ds(b - b')\} ns(b') + ns(b - b') ns(b) \right] \sum_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} c_j (s_{j+r}s_{j+s} + s_{j-r}s_{j-s}) = \left( 2/m \right) [ns(b) ns(b') + ns(b - b') \{ds(b) - ds(b')\}] \sum_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} c_j (d_{j+r}d_{j+s} + d_{j-r}d_{j-s}) = -2 \left[ cs(b)cs(b') + cs(b - b') \{ds(b) - ds(b')\} \right] \sum_{j=1}^{p} c_j .
\]

\[
\sum_{j=1}^{p} c_j (c_{j+r}c_{j+s} + c_{j-r}c_{j-s}) = \left( -2/m \right) \left[ ds(b)ds(b') + ds(b - b') \{ds(b) - ds(b')\} \right] \sum_{j=1}^{p} c_j .
\]
\( \mathbf{L} = 2: \)

\[
\sum_{j=1}^{p} c_j^2 [d_{j+r}s_{j+r} + d_{j-r}s_{j-r}] = -(2/m) \left[ ds^2(b) + cs(b) ns(b) \right] \sum_{j=1}^{p} s_j d_j . \tag{182}
\]

\[
\sum_{j=1}^{p} c_j s_j d_j [c_{j+r} + c_{j-r}] = (2/m) cs(b) ns(b) \sum_{j=1}^{p} s_j d_j . \tag{183}
\]

\[
\sum_{j=1}^{p} s_j c_j (d_{j+r}c_{j+r} + d_{j-r}c_{j-r}) = -(2/m) ds(b) \left[ ns(b) + cs(b) \right] \sum_{j=1}^{p} s_j d_j . \tag{184}
\]

\[
\sum_{j=1}^{p} c_j^2 d_j (s_{j+r} + s_{j-r}) = (2/m) cs(b) ds(b) \sum_{j=1}^{p} s_j d_j . \tag{185}
\]

\[
\sum_{j=1}^{p} d_j (s_{j+r}^3 + s_{j-r}^3) = -(2/m) ds(b) ns(b) \sum_{j=1}^{p} s_j d_j . \tag{186}
\]

\[
\sum_{j=1}^{p} c_j d_j s_j (c_{j+r}^3 + c_{j-r}^3) = -(2/m^2) \left[ cs^2(b) ns^2(b) + ns^2(b) ds^2(b) + ds^2(b) cs^2(b) + 3ds^2(b) ns(b) cs(b) \right] \sum_{j=1}^{p} s_j d_j . \tag{187}
\]

\[
\sum_{j=1}^{p} s_j d_j (c_{j+r}c_{j+s} + c_{j-r}c_{j-s}) = -(2/m) ds(b) ds(b') \sum_{j=1}^{p} s_j d_j . \tag{188}
\]

\[
\sum_{j=1}^{p} s_j d_j (d_{j+r}d_{j+s} + d_{j-r}d_{j-s}) = -2cs(b) cs(b') \sum_{j=1}^{p} s_j d_j . \tag{189}
\]

\[
\sum_{j=1}^{p} s_j d_j (s_{j+r} s_{j+s} + s_{j-r} s_{j-s}) = (2/m) ns(a) ns(a') \sum_{j=1}^{p} s_j d_j . \tag{190}
\]

\[
\sum_{j=1}^{p} c_j d_j (s_{j+r} c_{j+s} + s_{j-r} c_{j-s}) = -(2/m) ns(b) ds(b') \sum_{j=1}^{p} s_j d_j . \tag{191}
\]

\[
\sum_{j=1}^{p} c_j s_j (d_{j+r} c_{j+s} + d_{j-r} c_{j-s}) = -(2/m) cs(b) ds(b') \sum_{j=1}^{p} s_j d_j . \tag{192}
\]

\( \mathbf{L} = 3: \)
\[
\sum_{j=1}^{p} s_j^2 d_j^2 (c_{j+r} + c_{j-r}) = -2ns(b)cs(b) \sum_{j=1}^{p} c_j^3 \\
+ \frac{2}{m} cs(b) ns^3(b) \left[ m sn^2(b) + cn^2(b) - cn(b) \right] \sum_{j=1}^{p} c_j . \quad (193)
\]

\[
\sum_{j=1}^{p} c_j^3 (c_{j+r}^2 + c_{j-r}^2) = -(2/m) ds^2(b) \sum_{j=1}^{p} c_j^3 \\
+ \frac{2}{m^2} \left[ cs^2(b) ns^2(b) + ns^2(b) ds^2(b) + ds^2(b) cs^2(b) - 3ds^2(b) cs(b) ns(b) \right] \sum_{j=1}^{p} c_j . \quad (194)
\]

\[
\sum_{j=1}^{p} c_j s_j d_j (s_{j+r} d_{j+r} + s_{j-r} d_{j-r}) = 2cs(b)ns(b) \sum_{j=1}^{p} c_j^3 + \frac{2}{m} \left[ cs^2(b) ns^2(b) \\
+ ns^2(b) ds^2(b) + ds^2(b) cs^2(b) - cs(b) ns(b) (cs^2(b) + ds^2(b) + ns^2(b)) \right] \sum_{j=1}^{p} c_j . \quad (195)
\]

Appendix B: Examples Using Master Identities With Alternating Signs

Note that in this case the identities are only valid when \( p \) is even integer and since \( r \) and \( p \) are coprimes hence \( r \) is necessarily odd. The letters \( a, a' \) again stand for \( 2rK/p \) and \( 2sK/p \) respectively.

Examples belonging to the class MI - I

\( L = 1 \) :

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j (c_{j+r} + c_{j-r}) = 0 , \quad (196)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j d_{j+r} d_{j+2r} = -[cs^2(a) + 2cs(a)cs(2a)] \sum_{j=1}^{p} (-1)^{j-1} d_j \quad (197)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j d_{j+r} d_{j+s} = -[cs(a)cs(a') + cs(a)cs(a' - a) - cs(a')cs(a' - a)] \sum_{j=1}^{p} (-1)^{j-1} d_j \quad (198)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j^2 (d_{j+r} + d_{j-r}) = 2[ds(a) ns(a) + cs^2(a)] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (199)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} c_j (c_{j+r}d_{j+r} + c_{j-r}d_{j-r}) = -(2/m)cs(a)[ds(a) + ns(a)] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (200)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j (s_{j+r}d_{j+r} + s_{j-r}d_{j-r}) = (2/m)cs(a)[ns(a) + ds(a)] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (201)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j (d_{j+r}d_{j+s} + d_{j-r}d_{j-s}) = -2[cs(a)cs(a') - cs(a-a')(cs(a) - cs(a'))] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (202)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j (c_{j+r}c_{j+s} + c_{j-r}c_{j-s}) = -(2/m)[ds(a)ds(a') - ds(a-a')(cs(a) - cs(a'))] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (203)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j (s_{j+r}s_{j+s} + s_{j-r}s_{j-s}) = -(2/m)[ns(a)ns(a') - ns(a-a')(cs(a) - cs(a'))] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (204)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j (s_{j+r}d_{j+s} + s_{j-r}d_{j-s})
\]

\[
= (2/m)[ns(a)cs(a') - ns(a)cs(a-a') + ns(a')ns(a-a')] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (205)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} c_j (c_{j+r}d_{j+s} + c_{j-r}d_{j-s})
\]

\[
= -(2/m)[ds(a)cs(a') - ds(a)cs(a-a') + ds(a')ds(a-a')] \sum_{j=1}^{p} (-1)^{j-1} d_j . \quad (206)
\]

L = 2 :

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j^2 (c_{j+r}s_{j+r} + c_{j-r}s_{j-r}) = 2[cs^2(a) - ds(a)ns(a)] \sum_{j=1}^{p} (-1)^{j-1} c_js_j . \quad (207)
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j d_j (c_{j+r}d_{j+r} + c_{j-r}d_{j-r}) = -2cs(a)[-ns(a) + ds(a)] \sum_{j=1}^{p} (-1)^{j-1} c_js_j . \quad (208)
\]
\[ \sum_{j=1}^{p} (-1)^{j-1} c_j s_j (s_{j+r} s_{j+s} + s_{j-r} s_{j-s}) = (2/m) \text{ns}(a) \text{ns}(a') \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(209)

\[ \sum_{j=1}^{p} (-1)^{j-1} c_j s_j (c_{j+r} c_{j+s} + c_{j-r} c_{j-s}) = -(2/m) \text{ds}(a) \text{ds}(a') \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(210)

\[ \sum_{j=1}^{p} (-1)^{j-1} c_j s_j (d_{j+r} d_{j+s} + d_{j-r} d_{j-s}) = -2 \text{cs}(a) \text{cs}(a') \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(211)

\[ \sum_{j=1}^{p} (-1)^{j-1} c_j d_j (s_{j+r} d_{j+s} + s_{j-r} d_{j-s}) = -2 \text{ns}(a) \text{cs}(a') \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(212)

\[ \sum_{j=1}^{p} (-1)^{j-1} s_j d_j (c_{j+r} d_{j+s} + c_{j-r} d_{j-s}) = -2 \text{ds}(a) \text{cs}(a') \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(213)

\[ \sum_{j=1}^{p} (-1)^{j-1} d_j^2 (s_{j+r} c_{j+s} + s_{j-r} c_{j-s}) = 2[\text{cs}(a) \text{ds}(a-a') - \text{cs}(a') \text{ns}(a-a')] \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(214)

\[ \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j (d_{j+r} + d_{j-r}) = 2 \text{ds}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(215)

\[ \sum_{j=1}^{p} (-1)^{j-1} s_j c_j (c_{j+r}^3 + c_{j-r}^3) = (2/m) \text{cs}(a) \text{ds}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(216)

\[ \sum_{j=1}^{p} (-1)^{j-1} c_j (s_{j+r}^3 + s_{j-r}^3) = -(2/m) \text{cs}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j. \]  

(217)

**L = 3 :**

\[ \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j (s_{j+r} c_{j+r} + s_{j-r} c_{j-r}) = (2/m) \text{ds}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} d_j^3 \]

\[ - \left[ \text{cs}^2(a) \text{ns}^2(a) + \text{ds}^2(a) \text{ns}^2(a) + \text{cs}^2(a) \text{ds}^2(a) + \text{ds}(a) \text{ns}(a)(\text{cs}^2(a) + \text{ds}^2(a)) \right] \sum_{j=1}^{p} (-1)^{j-1} d_j. \]  

(218)

**Examples belonging to the class MI - II**

**L = 1 :**
While there are no ordinary identities of this class with \( L = 1 \), alternating identities abound. They are therefore unique and characterized by the appearance of the Jacobian zeta function. Also they have helped us in finding identities for the product of \( p \) sn’s as well as of \( p \) cn’s.

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j d_{j+r} = -2 \text{cs}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{219}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j s_{j+r} = \frac{2}{m} \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{220}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} c_j c_{j+r} = -\frac{2}{m} \text{ds}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{221}
\]

It may be noted that the above three identities are valid for \( p \geq 4 \).

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j d_{j+2r} d_{j+3r} = 2 \left[ \text{cs}(a) \text{cs}(2a) \text{cs}(3a) + \text{cs}^2(a) \text{cs}(2a) \right] \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{222}
\]

This generalizes for any even number \( l < p \) to:

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j d_{j+r} \cdots d_{j+(l-1)r} = 2(-1)^{l/2} \left( \sum_{k=1}^{l/2} (-1)^{k-1} \prod_{n=1, n \neq k}^{l} \text{cs}([n - k]a) \right) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{223}
\]

Similarly, for any even number \( l \leq p \), the sn and cn functions satisfy the identities \( (p \geq 4) \)

\[
m^{l/2} \sum_{j=1}^{p} (-1)^{j-1} s_j s_{j+r} \cdots s_{j+(l-1)r} = 2 \left( \sum_{k=1}^{l/2} (-1)^{k-1} \prod_{n=1, n \neq k}^{l} \text{ns}([n - k]a) \right) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{224}
\]

\[
m^{l/2} \sum_{j=1}^{p} (-1)^{j-1} c_j c_{j+r} \cdots c_{j+(l-1)r} = 2(-1)^{l/2} \left( \sum_{k=1}^{l/2} (-1)^{k-1} \prod_{n=1, n \neq k}^{l} \text{ds}([n - k]a) \right) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{225}
\]

When \( l = p \) \( (p \geq 4) \) the above identities reduce to:

\[
m^{p/2} \prod_{j=1}^{p} s_j = \left( \prod_{n=1}^{p/2-1} \text{ns}^2 \left( \frac{2Kn}{p} \right) \right) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{226}
\]

\[
m^{p/2} \prod_{j=1}^{p} c_j = \sqrt{1 - m} (-1)^{p/2} \left( \prod_{n=1}^{p/2-1} \text{ds}^2 \left( \frac{2Kn}{p} \right) \right) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{227}
\]
Thus these are the even product equivalents of Eqs. (146) and (173). It is worth reminding here that the product \( d_1d_2...d_p \) is on the other hand a constant being equal to \((1 - m)^{p/4}\).

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j [c_{j+r} s_{j+r} + c_{j-r} s_{j-r}] = -(4/m) ds(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{228}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j [c_{j+r} d_{j+r} + c_{j-r} d_{j-r}] = -(4/m) ds(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{229}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} c_j [s_{j+r} d_{j+r} + s_{j-r} d_{j-r}] = -(4/m) cs(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{230}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j [c_{j+r} s_{j+s} + c_{j-r} s_{j-s}] = (4/m) [cs(a) ds(a - a') - cs(a') ns(a - a')] \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{231}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j [d_{j+r} c_{j+s} + d_{j-r} c_{j-s}] = -(4/m) [ns(a) ds(a - a') - ns(a') cs(a - a')] \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{232}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} c_j [d_{j+r} s_{j+s} + d_{j-r} s_{j-s}] = -(4/m) [ds(a) ns(a - a') - ds(a') cs(a - a')] \sum_{j=1}^{p} (-1)^{j-1} Z_j . \tag{233}
\]

\( L = 2 : \)

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j^3 (d_{j+r} + d_{j-r}) = 2 ns(a) ds(a) \sum_{j=1}^{p} (-1)^{j-1} d_j^2 . \tag{234}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} c_j^3 (c_{j+r} + c_{j-r}) = (2/m^2) cs(a) ns(a) \sum_{j=1}^{p} (-1)^{j-1} d_j^2 . \tag{235}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} s_j^3 (s_{j+r} + s_{j-r}) = (2/m^2) cs(a) ds(a) \sum_{j=1}^{p} (-1)^{j-1} d_j^2 . \tag{236}
\]

\[
\sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j (c_{j+r}s_{j+r}d_{j+r} + c_{j-r}s_{j-r}d_{j-r})
\]

\[
= (4/m^2) [ns^2(a)(cs^2(a) + ds^2(a) + cs^2(a)ds^2(a))] \sum_{j=1}^{p} (-1)^{j-1} d_j^2 . \tag{237}
\]

\( L = 3 : \)

35
\[
\sum_{j=1}^{p} (-1)^{j-1} d_j^3 [c_{j+r} s_{j+r} + c_{j-r} s_{j-r}]
\]
\[
= (12/m) cs^2(a) ds(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j - 2 \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} s_j c_j d_j . \quad (238)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} c_j^2 s_j d_j [c_{j+r} + c_{j-r}]
\]
\[
= (2/m) \text{cs}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j - (4/m^2) \text{ds}^2(a) \text{cs}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \quad (239)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} s_j^2 c_j d_j [s_{j+r} + s_{j-r}]
\]
\[
= -(2/m) \text{cs}(a) \text{ds}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j + (4/m^2) \text{ns}^2(a) \text{cs}(a) \text{ds}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \quad (240)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} d_j^2 c_j s_j [d_{j+r} + d_{j-r}]
\]
\[
= 2 \text{ds}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j - (4/m) \text{cs}^2(a) \text{ds}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \quad (241)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} d_j c_j s_j [d_{j+r}^2 + d_{j-r}^2]
\]
\[
= -2 \text{cs}^2(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j
\]
\[
+ (4/m)[\text{ns}^2(a)(\text{cs}^2(a) + \text{ds}^2(a)) + \text{cs}^2(a)\text{ds}^2(a)] \sum_{j=1}^{p} (-1)^{j-1} Z_j . \quad (242)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} s_j c_j [d_{j+r}^3 + d_{j-r}^3]
\]
\[
= 2 \text{ds}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j - (12/m) \text{cs}^2(a) \text{ds}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \quad (243)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} s_j d_j [c_{j+r}^3 + c_{j-r}^3]
\]
\[
= (2/m) \text{cs}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j - (12/m^2) \text{ds}^2(a) \text{cs}(a) \text{ns}(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j . \quad (244)
\]
\[
\sum_{j=1}^{p} (-1)^{j-1} d_j c_j [s_{j+r}^3 + s_{j-r}^3]
\]
\[= -\left(\frac{2}{m}\right) ds(a) cs(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j + \left(\frac{12}{m^2}\right) ns^2(a) ds(a) cs(a) \sum_{j=1}^{p} (-1)^{j-1} Z_j. \quad (245)\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j c_j s_j^2 [s_{j+r}^3 + s_{j-r}^3] = -\left(\frac{8}{m^2}\right) ns^2(a) ds(a) cs(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j
\]
\[+ \left(\frac{4}{m^3}\right) ns^2(a) ds(a) cs(a) \left[3(ns^2(a) + ds^2(a) + cs^2(a)) + cs^2(a)ds^2(a)\right] \sum_{j=1}^{p} (-1)^{j-1} Z_j. \quad (246)\]

\[
\sum_{j=1}^{p} (-1)^{j-1} d_j s_j c_j^3 [c_{j+r}^3 + c_{j-r}^3] = -\left(\frac{8}{m^2}\right) ds^2(a) ns(a) cs(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j
\]
\[+ \left(\frac{4}{m^3}\right) ns(a) cs(a) \left[3ds^2(a)(ns^2(a) + ds^2(a) + cs^2(a)) + cs^2(a)ns^2(a)\right] \sum_{j=1}^{p} (-1)^{j-1} Z_j. \quad (247)\]

\[
\sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j^2 [d_{j+r}^3 + d_{j-r}^3] = -8 cs^2(a) ns(a) ds(a) \sum_{j=1}^{p} (-1)^{j-1} c_j s_j d_j
\]
\[+ \left(\frac{4}{m}\right) ns(a) ds(a) \left[3cs^2(a)(ns^2(a) + ds^2(a) + cs^2(a)) + ds^2(a)ns^2(a)\right] \sum_{j=1}^{p} (-1)^{j-1} Z_j. \quad (248)\]
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