DIMENSION FILTRATION ON LOOPS

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Abstract. We show that the graded group associated to the dimension filtration on a loop acquires the structure of a Sabinin algebra after being tensored with a field of characteristic zero. The key to the proof is the interpretation of the primitive operations of Umirbaev and Shestakov in terms of the operations on a loop that measure the failure of the associator to be a homomorphism.

1. Introduction.

Let \( G \) be a group and
\[
G = G_1 \triangleright G_2 \triangleright G_3 \triangleright \ldots
\]
— its lower central series. Then the graded group \( \bigoplus G_i/G_{i+1} \otimes \mathbb{Q} \) is a Lie algebra with the Lie bracket induced by the commutator on \( G \). Its universal enveloping algebra can be identified with the algebra associated to the filtration of the group ring \( \mathbb{Q} G \) by the powers of its augmentation ideal \( \mathfrak{I} \).

In this note we generalize these facts to arbitrary loops. It will be convenient to speak of dimension series rather than the lower central series. For groups, both series give rise to the same Lie algebra. However, while there are several inequivalent ways of defining the lower central series for loops, the definition of the dimension series extends to loops without any change.

Now it has become clear that appropriate generalization of Lie algebras to the non-associative context are the Sabinin algebras (called “hyperalgebras” in \cite{2} and \cite{6}). Sabinin algebras were initially introduced in \cite{5} as tangent algebras to local loops. Later, Shestakov and Umirbaev proved that in any (not necessarily associative) bialgebra the set of primitive elements has the structure of a Sabinin algebra \cite{6}. In fact, it was shown in \cite{1} that any Sabinin algebra can be described as the set of primitive elements of some bialgebra.

Our constructions shed some light on the nature of the primitive operations introduced by Shestakov and Umirbaev in \cite{6}. It turns out that the primitive operations in the free non-associative algebra are induced by the associator deviations (in the sense of \cite{3}) in the free loop.

2. Dimension subloops.

Let \( L \) be a loop, \( \mathcal{R} \) — a commutative unital ring and \( \mathcal{R} L \) — the loop ring of \( L \) over \( \mathcal{R} \). Denote by \( IL \) the augmentation ideal, that is, the kernel of the map \( \mathcal{R} L \to \mathcal{R} \) that sends \( \sum a_i x_i \) with \( a_i \in \mathcal{R} \) and \( x_i \in L \) to \( \sum a_i \). The ideal \( IL \) (or simply \( I \)) is spanned over \( \mathcal{R} \) by elements of the form \( x - 1 \) with \( x \in L \). Let \( I^n L \)


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\end{itemize}
be $m$th power of $I$, that is, the submodule of $RL$ spanned over $R$ by products of at least $m$ elements of $J$ with any arrangement of the brackets.

**Lemma 1.** Let $u \in I^m$ and $a \in L$. Then $au$, $u/a$ and $a \backslash u$ all lie in $I^m$.

**Proof.** First, $au = (a - 1)u + u \in I^m$. Next, $u/a \cdot ((a - 1) + 1) \in I^m$ implies $u/a \in I$ as $u/a \cdot (a - 1)$ is in $I$. This, in turn, implies $u/a \in I^2$ etc. The same argument works for $a \backslash u$. □

**Definition 1.** The $n$th dimension subloop of $L$ over $R$ is the intersection $D_n(L, R) = L \cap (1 + I)^n$.

We shall sometimes write $D_n$ instead of $D_n(L, R)$.

The set $D_n$ is indeed a subloop of $L$. Let $a = 1 + u$, $b = 1 + v$ with $u, v \in I^n$. Clearly, $ab$ is in $D_n$. To see that $a/b$ belongs to $D_n$ take $x = a/b - 1$. Then $1 + u = (1 + v)(1 + x)$ and $x = (u - v)/b \in I^n$. In the same manner one shows that $D_n$ is closed with respect to the left division.

It is clear that the dimension subloops are fully invariant subloops of $L$ since the augmentation ideal is mapped into itself by any endomorphism of $R L$.

### 3. The main theorem.

In this section we will assume that $R = k$ is a field of characteristic zero. We will denote by $D$ (or $DL$) the graded vector space

$$\oplus D_n(L, k)/D_{n+1}(L, k) \otimes k$$

and by $I$ (or $IL$) — the graded algebra $\oplus I^nL/I^{n+1}L$.

For any loop $L$ the graded algebra $I$ has a comultiplication $I \rightarrow I \otimes I$ that is a non-trivial algebra homomorphism. Indeed, the loop algebra $kL$ has a comultiplication $\delta$ that sends $g$ in $L$ to $g \otimes g$. Under $\delta$ the element $g - 1$ is sent to

$$(g - 1) \otimes 1 + 1 \otimes (g - 1) + (g - 1) \otimes (g - 1)$$

and, hence, there is an induced comultiplication on the algebra $\oplus I^n/I^{n+1}$.

There is an inclusion map of $D$ into $I$ given by sending $x \in D_nL$ to $x - 1 \in I^nL$. The image of the class of $g - 1$ is primitive for any $g \in L$.

Here is our main result:

**Theorem 2.** The image of $D$ in $I$ coincides with the subspace of primitive elements in $I$.

The set of the primitive elements of any bialgebra has the structure of a Sabinin algebra [6]. In fact, if the bialgebra in question is primitively generated, it can be identified with the universal enveloping algebra of the Sabinin algebra of its primitive elements [4]. For any loop the algebra $I$ is primitively generated since it is generated by the classes of $g - 1 \in I$. Therefore, Theorem 2 can be re-stated as follows:

**Theorem 3.** The graded vector space $D$ is a Sabinin algebra whose universal enveloping algebra is $I$.

For groups, Theorem 3 was first proved by Quillen [7]. Quillen’s result involves the Lie algebra associated to the lower central series rather than the dimension series; however, for groups these are isomorphic.
4. The dimension subloops of a free loop.

Let \( x_i \leftrightarrow x'_i \) be a bijection between two sets of variables \( V \) and \( V' \). Denote by \( R[[V']] \) the \( R \)-algebra of formal power series in \( m \) non-associative variables \( x'_1, \ldots, x'_m \). The power series that start with 1 are readily seen to form a loop \( R_0[[V']] \) under multiplication.

**Definition 2.** The Magnus expansion is the homomorphism of the free loop \( F(V) \) on the generators \( x_1, \ldots, x_m \) into \( R_0[[V']] \) that sends the generator \( x_i \) to the power series \( 1 + x'_i \).

We denote the Magnus expansion of \( x \in F \) by \( M(x) \). It follows from the definition that
\[
M(x_i|1) = 1 - x'_i + x'_i^2 - x'_i x''_i + x'_i (x'_i x''_i)^2 - \ldots
\]
and
\[
M(1/x_i) = 1 - x'_i + x'_i^2 - x'_i x''_i + (x'_i x''_i) x'_i - \ldots
\]
The elements of \( F(V) \) whose Magnus expansion has no terms of non-zero degree less than \( n \) form a normal subloop of \( F(V) \). These subloops, in fact, are precisely the dimension subloops:

**Lemma 4.** The Magnus expansion of \( x \in F(V) \) begins with terms of degree \( n \) if and only if \( x \in D_n(F(V), R) \) and \( x \notin D_{n+1}(F(V), R) \).

It is clear that the Magnus expansion of any element of \( I^n F(V) \) starts with terms of degree at least \( k \). The converse is established with the help of the Taylor formula for free loops.

Define \( u_i \in I \) by \( u_i = x_i - 1 \). We shall call the \( u_i \) monomials of degree 1. A monomial of degree \( k \) is a product (with any arrangement of the brackets) of \( k \) monomials of degree 1.

**Lemma 5.** Given a positive integer \( n \), any \( x \in F(V) \) can be uniquely written as
\[
x = 1 + \sum_{j \leq n, \alpha} a_{j, \alpha} \mu_{j, \alpha} + r,
\]
where \( \mu_{j, \alpha} \) are monomials of degree \( j \), \( a_{j, \alpha} \) are elements of \( R \) and \( r \in I^{n+1} F(V) \).

The existence of such an expansion follows from the Taylor formula with \( n = 1 \). For all the generators \( x_i \), it is obviously true. Assume now that the set of such \( x \in F(V) \) that \( x - 1 \) cannot be written as a linear combination of the \( u_i \) modulo \( I^2 \), is non-empty. Let \( w \) be a reduced word of minimal possible length (number of operations used to form the word) representing such \( x \). If \( w = ab \) with \( a, b \) reduced words then
\[
w - 1 = ab - 1 = (a - 1)(1 + (b - 1)) + (b - 1) \equiv (a - 1) + (b - 1) \mod I^2
\]
and we come to a contradiction since \( a \) and \( b \) are of smaller length then \( w \). If \( w = a/b \) with \( a, b \) reduced words then
\[
w - 1 = a/b - 1 = a(1 - (b - 1)) - 1 \equiv (a - 1) - (b - 1) \mod I^2
\]
since
\[
a/b - a(1 - (b - 1)) = (a(1 - b))(1 - b) \in I^2
\]
and we have a contradiction again. Similarly \( w \) cannot be of the form \( b/a \); hence \( w - 1 \) is a linear combination of the \( u_i \) modulo \( I^2 \) for all \( w \).
The coefficients $a_{i,\alpha}$ are uniquely defined as the Magnus expansion of a monomial $\mu$ starts with $\mu'$. This proves Lemma 5. Lemma 4 follows immediately from Lemma 5.

5. PRIMITIVE OPERATIONS AND ASSOCIATOR DEVIATIONS.

Let $A$ be a bialgebra over a field of characteristic zero, with non-trivial comultiplication. Assume that $A$ is generated by a set $S$ of its primitive elements. Then the space of all primitive elements of $A$ is the minimal vector subspace of $A$ that contains $S$ and is closed with respect to the commutators and the primitive operations $p_{r,s}$ of Umirbaev and Shestakov [6].

Let $u = (\ldots(x_1x_2)\ldots)x_r$ and $v = (\ldots(y_1y_2)\ldots)y_s$ where $r, s \geq 1$ and $x_i, y_i$ and $z$ are in $A$. Specifying the products $u$ and $v$ together with the numbers $r$ and $s$ is equivalent to giving the sequences $x_i$ and $y_i$. The operations

$$p_{r,s}(x_1, \ldots, x_r; y_1, \ldots, y_s; z) = p_{r,s}(u; v; z)$$

are defined by the formula

$$(uv)z - u(vz) = \sum u(1) v(1) \cdot p_{\alpha,\beta}(u(2); v(2); z)$$

where the sum is taken over all decompositions of the sequences $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ into complementary subsequences $u(1) = (\ldots(x_{i_1}x_{i_2})\ldots)x_{i_r}$, $u(2) = (\ldots(x_{j_1}x_{j_2})\ldots)x_{j_s}$, and $v(1) = (\ldots(y_{k_1}y_{k_2})\ldots)y_{k_r}$, $v(2) = (\ldots(y_{l_1}y_{l_2})\ldots)y_{l_s}$.

For example, if $r = s = 1$ the operation $p_{1,1}(x_1; y_1; z)$ is just the associator

$$p_{1,1}(x_1, y_1; z) = (x_1y_1)z - x_1(y_1z).$$

Also,

$$p_{2,1}(x_1, x_2; y_1; z) = (x_1x_2, y_1, z) - x_1(x_2, y_1, z) - x_2(x_1, y_1, z),$$

$$p_{1,2}(x_1; y_1, y_2; z) = (x_1, y_1y_2, z) - y_1(x_1, y_2, z) - y_2(x_1, y_1, z).$$

Let $\mathcal{R}(V')$ be the free non-associative algebra on $m$ generators $x'_1, \ldots, x'_m$. There is a map

$$\mathcal{M} : F(V) \to \mathcal{R}(V')$$

defined by taking the lowest-degree term of the Magnus expansion. Our method of proving Theorem 2 consists in finding operations in the free loop $F(V)$ which correspond to the operations $p_{r,s}$ in the free algebra under the above map.

Such operations were introduced in [3] under the name of associator deviations. Associator deviations (or simply deviations) are functions from $L^{n+3}$ to $L$ where $L$ is an arbitrary loop and $n$ is a non-negative integer called the level of the deviation. There exists one deviation of level zero, namely the loop associator:

$$(a, b, c) = (a(bc)) \setminus ((ab)c).$$

In general, there are $(n+2)!/2$ associator deviations of level $n$. Given $n > 0$ and an ordered set $\alpha_1, \ldots, \alpha_n$ of not necessarily distinct integers satisfying $1 \leq \alpha_k \leq n+2$, the deviation $(a_1, \ldots, a_{n+3})_{\alpha_1, \ldots, \alpha_n}$ is defined inductively by

$$(a_1, \ldots, a_{n+3})_{\alpha_1, \ldots, \alpha_n} := (A(a_{\alpha_n}) A(a_{\alpha_{n+1}})) \setminus A(a_{\alpha_n} a_{\alpha_{n+1}}),$$

where $A(x)$ stands for $(a_1, \ldots, a_{n-1}, x, a_{n+2}, \ldots, a_{n+3})_{\alpha_1, \ldots, \alpha_{n-1}}$. In particular, there are three deviations of level one:

$$(a, b, c, d)_1 = ((a, c, d)(b, c, d)) \setminus (ab, c, d),$$

$$(a, b, c, d)_2 = ((a, b, d)(a, c, d)) \setminus (a, bc, d),$$

$$((a, b, c)(b, c, d)) \setminus (ab, b, c).$$
(a, b, c, d)₃ = ((a, b, c)(a, b, d)) \setminus (a, b, cd).

Let us write \( P_m,n(x_1, \ldots, x_m, y_1, \ldots, y_n, z) \) for the deviation

\[
(x_1, \ldots, x_m, y_1, \ldots, y_n, z)_{1, \ldots, m, 1, \ldots, m, 1}^{m-1, n-1}.
\]

**Proposition 6.** With the above notation

\[
\mathcal{M}(P_m,n(x_1, \ldots, x_m, y_1, \ldots, y_n, z)) = 1 + p_m,n(x'_1, \ldots, x'_m; y'_1, \ldots, y'_n; z') + O(n + m + 2)
\]

where \( O(n + m + 2) \) contains no terms of degree less than \( n + m + 2 \).

The proof of Proposition \( \text{6} \) is given in the next section. Here we show how Proposition \( \text{6} \) implies Theorem \( \text{2} \).

First, let us establish Theorem \( \text{2} \) for finitely generated free loops. Consider the filtration on \( k(V') \) by be subspaces of elements of degree at least \( i \). The graded algebra associated to this filtration is clearly \( k(V') \) itself. It follows from Lemma \( \text{5} \) that the map \( \mathcal{M} \) induces an isomorphism between the algebra \( IF(V) \) and the algebra \( k(V') \).

Now, assume that all primitive elements on \( IF(V) \) of degree less than \( k \) are contained in \( D \); this is certainly true for \( k = 2 \). Any primitive element of degree \( k \) is a linear combination of terms of the form \( p_{\alpha, \beta}(u_1, \ldots, u_\alpha; v_1, \ldots, v_\beta; w) \) and \( [u, v] \) where \( u, v, u_1, v_1, w \) are primitive elements of \( IF(V) \) of degree smaller than \( k \).

Without loss of generality we can assume that these elements belong to

\[
\oplus D_i/D_{i+1} \subset DF(V) \subset IF(V);
\]

this implies that there exist \( \hat{u}, \hat{v}, \hat{u}_i, \hat{v}_i \) and \( \hat{w} \) in the loop \( F(V) \) such that \( \mathcal{M}(\hat{u}) = u \) and similarly for \( v, u_i, v_i \) and \( w \). Now, by Proposition \( \text{6} \)

\[
\mathcal{M}(p_{\alpha, \beta}(\hat{u}_1, \ldots, \hat{u}_\alpha; \hat{v}_1, \ldots, \hat{v}_\beta; \hat{w})) = p_{\alpha, \beta}(u_1, \ldots, u_\alpha; v_1, \ldots, v_\beta; w).
\]

It is also easy to see that

\[
\mathcal{M}(([\hat{u}] \setminus (\hat{u} \hat{v}))) = [u, v]
\]

and, hence, any primitive element of degree \( k \) also belongs to \( D \).

Now, let \( L \) be an arbitrary loop. Any primitive element \( u \) of \( I_L \) can be obtained from a finite number of elements (say, \( m \)) of the form \( g_i - 1 \in IL/I^2L \) with \( g_i \in L \), by applying commutators, the \( p_{\alpha, \beta} \)'s and taking linear combinations. Consider the homomorphism of the free loop \( F(V) \) on \( m \) generators to \( L \) that sends the generators of \( F(V) \) to the \( g_i \). It is clear that \( u \) is the image of a primitive element \( w \in IF(V) \) under the induced map \( IF(V) \to IL \). However, since \( 1 + w \in F(V) \) we see that \( 1 + u \in L \) and therefore \( u \) is in the image of \( DL \) inside \( IL \).

6. **Proof of Proposition \( \text{6} \)**

Given a subset \( S' \subseteq V' \), we shall say that a monomial in \( k[[V']] \) is balanced (with respect to \( S' \)) if it contains each element in \( S' \) at least once. For \( S \subseteq V \), we shall say that an element \( \phi \in F(S) \) is balanced (with respect to \( S \)) if any nonzero term in \( M(\phi) - 1 \) is balanced with respect to \( S' \).
Lemma 7. Given a balanced φ ∈ F(S), x ∈ S (so φ = φ(x)) and y ∈ V \ S then the expression
\[\phi(x, y) = (\phi(x)\phi(y))\phi(xy) \in F(S \cup \{y\})\]
is balanced too.

Proof.
\[\mathcal{M}(\phi(xy)) - 1 = (\mathcal{M}(\phi(x)) - 1) + (\mathcal{M}(\phi(y)) - 1) + M'\]
where \(M'\) is an (infinite) sum of balanced monomials. Hence,
\[\mathcal{M}(\phi(x, y)) - 1 = (\mathcal{M}(\phi(x))\mathcal{M}(\phi(y)))\mathcal{M}(\phi(y)) - (\mathcal{M}(\phi(x)) - 1)(\mathcal{M}(\phi(y)) - 1)).\]
It follows that all the monomials contained in \(\mathcal{M}(\phi(x, y)) - 1\) with non-zero coefficients are balanced. Indeed, all of the monomials in the lowest-degree term of \(\mathcal{M}(\phi(x, y)) - 1\) are balanced since every monomial in
\[M' - (\mathcal{M}(\phi(x)) - 1)(\mathcal{M}(\phi(y)) - 1)\]
is. It follows that all the other terms of \(\mathcal{M}(\phi(x, y)) - 1\) are balanced since they are expressed via the lowest-degree term with the help of a recurrent relation.

Given \(f \in k[[V']]\), let \(L_{S'}(f)\) be the part of \(f\) which contains all the variables in \(S'\) with multiplicity one, but no variables in \(V' \setminus S'\). Similarly, given \(\phi \in F(V)\), \(L_S(\phi)\) will stand for \(L_{S'}(\mathcal{M}(\phi))\).

Lemma 8. Let \(S \subseteq \tilde{S} \subseteq V\) with \(|S| \geq 2\), \(x \in S\) and \(y \in V \setminus \tilde{S}\). Given \(\phi = \phi(x) \in F(S)\) balanced, \(\phi(x, y)\) as above and \(w \in F(\tilde{S})\) then
\[L_{\tilde{S} \cup \{y\}}(\phi(w, y)) = L_{\tilde{S} \cup \{y\}}(\phi(wy)).\]

Proof. First we decompose \(S = \{x\} \sqcup S_0\) (\(\sqcup\) denotes disjoint union), \(\mathcal{M}(\phi(x)) = \sum_I A_I(x)\) with \(I\) the multidegree of \(A_I(x)\) on \(S'_0\) and \(\mathcal{M}(\phi(x, y)) = \sum_K A_K(x, y)\). With this notation we have that
\[A_M(xy) = \sum_{I+J+K=M} (A_I(x)A_J(y))A_K(x, y).\]

Hence,
\[A_M(x, y) = A_M(xy) - A_M(x) - A_M(y) - \sum_{I+J+K=M, I, J, K \neq M} (A_I(x)A_J(y))A_K(x, y)\]
and, since \(\phi(x)\) and \(\phi(x, y)\) are balanced,
\[A_{\{1, \ldots, 1\}}(x, y) = A_{\{1, \ldots, 1\}}(xy) - A_{\{1, \ldots, 1\}}(x) - A_{\{1, \ldots, 1\}}(y).\]

Therefore, using that \(L_{\tilde{S} \cup \{y'\}}(A_{\{1, \ldots, 1\}}(w)) = 0\) and \(L_{\tilde{S} \cup \{y'\}}(A_{\{1, \ldots, 1\}}(y)) = 0\), we get
\[L_{\tilde{S} \cup \{y\}}(\phi(w, y)) = L_{\tilde{S} \cup \{y'\}}(A_{\{1, \ldots, 1\}}(w, y)) = L_{\tilde{S} \cup \{y\}}(A_{\{1, \ldots, 1\}}(wy)) = L_{\tilde{S} \cup \{y\}}(\phi(wy))\]
as desired.

□
Now we are in the position to prove Proposition 7. It follows from Lemma 8 that $P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, z)$ is balanced with respect to the set

$$S = \{x_1, \ldots, x_m, y_1, \ldots, y_n, z\}.$$ 

Therefore it suffices to show that

$$L_S(P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, z)) = p_{m,n}(x'_1, \ldots, x'_m, y'_1, \ldots, y'_n, z').$$

As a corollary of Lemma 8 we have

$$L_S(P_{m,n}(x_1, \ldots, x_m, y_1, \ldots, y_n, z)) = L_S((x, y, z)).$$

Set $x' = ((x'_1 x'_2) \cdots) x'_m$ and $y' = ((y'_1 y'_2) \cdots) y'_m$ and recall that $x = ((x_1 x_2) \cdots) x_m$ and $y = ((y_1 y_2) \cdots) y_m$. Since $(xy)z = x(yz) \cdot (x, y, z)$, we have

$$L_S((xy)z) = \sum_{S_1 \cup S_2 = S} L_{S_1}(x(yz)) L_{S_2}((x, y, z))$$

with the convention $L_\emptyset(\cdot) = 1$. The right–hand side of this equality is

$$x'(y' z') + \sum_{S_1 \cup S_2 = S \setminus \{z\}} L_{S_1}(x y) L_{S_2 \cup \{z\}}((x, y, z))$$

since $z'$ appears in any term of positive degree of $\mathcal{M}((x, y, z))$. Moreover, setting

$$\hat{\rho}(x'_i, \ldots, x'_i, y'_j, \ldots, y'_j, z') = L_{\{x'_i, \ldots, x'_i, y'_j, \ldots, y'_j\}}(\mathcal{M}((x, y, z)))$$

and $\hat{\rho}(1, \ldots, z') = \hat{\rho}(1, 1, z') = 0$, we obtain

$$(x' y') z' - x'(y' z') = \sum_{S_1 \cup S_2 = S \setminus \{z\}} L_{S_1}(x y) L_{S_2 \cup \{z\}}((x, y, z))$$

$$= \sum (x'_1 y'_1) \hat{\rho}(x'_{(2)}, y'_{(2)}, z'),$$

which shows that the operators $\hat{\rho}$ agree with the primitive operations of Umirbaev and Shestakov since they satisfy the same recurrence and initial conditions. Therefore,

$$L_S((x, y, z)) = \hat{\rho}(x'_1, \ldots, x'_m, y'_1, \ldots, y'_n, z') = p_{m,n}(x'_1, \ldots, x'_m, y'_1, \ldots, y'_n, z').$$

7. Miscellaneous remarks.

1. All the results of this paper can be stated without change for left loops (binary systems with left division and a two-sided unit) since the definition of the associator deviations does not involve right division.

2. If all elements of a loop $L$ satisfy some identity, then the bialgebra $\mathcal{Z}$ satisfies a “linearized” version of the same identity. In particular, if $L$ is Moufang loop, $\mathcal{D}L$ is readily seen to be a Malcev algebra. Similarly, if $L$ is a Bol loop, $\mathcal{D}L$ is Bol algebra.

3. Our results are only stated over fields of characteristic zero since the necessary general results from the theory of non-associative algebras are only available for such fields. In particular, the notions corresponding to Lie rings and restricted Lie algebras are not yet axiomatized. It is clear, however, that the direct sum $\mathcal{D}L_{(1)} / \mathcal{D}L_{(2)}$ always has a rich algebraic structure. Using the same argument as in the proof of Lemma 8 one sees that

- $[D_p, D_q] \subset D_{p+q}$
- $(D_p, D_q, D_r) \subset D_{p+q+r}$.
\[ (D_{p_1}, \ldots, D_{p_n})_{\alpha_1, \ldots, \alpha_{n-3}} \subset D_{p_1 + \ldots + p_n} \] for all combinations \( \alpha_1, \ldots, \alpha_{n-3} \).

Therefore, \( \oplus D_i(L, R)/D_{i+1}(L, R) \) carries multilinear operations induced by the commutator and the associator deviations, and all these operations respect the grading.

4. For groups, the dimension filtration is closely related to the lower central series. In particular, the Lie algebras over the field of rational numbers, associated to both filtrations, are isomorphic. This is no longer true for general loops, at least if one uses Bruck’s definition of lower central series \([1]\). Instead, the dimension filtration is related to the commutator-associator filtration \([3]\); this connection will be discussed elsewhere.

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