A NOTE ON THE INDEX OF CONE DIFFERENTIAL OPERATORS

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Abstract. We prove that the index formula for \( b \)-elliptic cone differential operators given by Lesch in [6] holds verbatim for operators whose coefficients are not necessarily independent of the normal variable near the boundary. We also show that, for index purposes, the operators can always be considered on weighted Sobolev spaces.

1. Introduction

Let \( M \) be a smooth compact manifold with boundary, \( \mathfrak{m} \) a smooth positive \( b \)-measure on \( M \). With respect to a suitable choice of a collar neighborhood \( \pi : U \to \partial M \) of the boundary and globally defined defining function \( x \) we may assume \( \mathfrak{m} = \frac{1}{x} dx \otimes \mathfrak{m}_{\partial M} \) where \( \mathfrak{m}_{\partial M} \) is a smooth positive density on \( \partial M \). Let \( X \) be a vector field defined near \( \partial M \) such that \( X \) is vertical with respect to \( \pi \) and \( Xx = 1 \). Let \( E \) be a vector bundle over \( M \) and \( A \in x^{-\nu} \text{Diff}^m_b(M; E) \) be \( b \)-elliptic, \( \nu > 0 \) (see Lesch [6] or Melrose [7] for the notation). Fix a hermitian metric on \( E \) and compatible connection \( \nabla \). Write \( D_x = -i \nabla_X \). The operator \( A = x^{-\nu}P \) is said to have coefficients independent of \( x \) near the boundary if \( xD_xP = PxD_x \) near \( \partial M \).

Regard \( A \) as an unbounded operator \( A : C_c^\infty(M; E) \subset x^\mu L^2_b(M; E) \to x^{\mu+\nu} L^2_b(M; E) \)

and denote by \( D_{\text{min}} \) the domain of the closure of \( A \). It is convenient to assume \( \mu = -\nu/2 \); we can always reduce to this case by conjugation with \( x^{\mu+\nu/2} \). In Section 2.4 of [6] Lesch gives an analytic formula for the index of \( A \) on \( D_{\text{min}} \), assuming that it has coefficients independent of \( x \) near \( \partial M \). This index formula is derived via heat trace asymptotics, also obtained in [6]. In this note we show that the formula remains true also without the assumption on the coefficients.

The results presented here represent a significant simplification of various aspects of the analysis of \( b \)-elliptic cone operators and show that simplifying assumptions made by various authors can be used in the general case. For instance, operators with coefficients independent of \( x \) have a certain scalability property which was crucial in the works of Cheeger [2] and Lesch [3]. This independence was also assumed by Schulze, Shatalov and Sternin in [4] although, in the case they considered, a much simpler homotopy invariance argument can be used to avoid this assumption.

From the technical point of view, the simplification comes about by observing first that for the purpose of index calculations, one can reduce to the case where the operator has coefficients independent of \( x \) near \( \partial M \) (Theorem 2) and second, that one can assume that the domain is the weighted Sobolev space \( x^{\nu/2-\varepsilon} H^m_{\text{reg}}(M; E) \) with suitable small \( \varepsilon > 0 \) rather than \( D_{\text{min}} \) (Theorem 3) by replacing \( A \) with \( x^\varepsilon A \).

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2. The index formula

Let $A \in x^{-\nu} \text{Diff}^m_b(M; E)$ be $b$-elliptic. We shall need the following lemma which also establishes the notation.

**Lemma 1.** On $\mathcal{D}_{\text{min}}(A)$, with small enough $\varepsilon > 0$, the operator norm

$$
\|u\|_A = \|u\|_{x^{-\nu/2}L^2_b} + \|Au\|_{x^{-\nu/2}L^2_b}
$$

and the norm

$$
\|u\|_{A,\varepsilon} = \|u\|_{x^{\nu/2-\varepsilon}L^2_b} + \|Au\|_{x^{\nu/2-\varepsilon}L^2_b}.
$$

are equivalent.

**Proof.** Recall that the embedding $x^{\nu/2-\varepsilon}L^2_b \hookrightarrow x^{-\nu/2}L^2_b$ is continuous for $\varepsilon < \nu$. The equivalence of the norms follows from the continuity of $(\mathcal{D}_{\text{min}}(A), \| \cdot \|_A) \hookrightarrow x^{\nu/2-\varepsilon}L^2_b$ which is a consequence of the closed graph theorem. \hfill \square

Write $A = A_0 + xA_1$ with $A_0$ having coefficients independent of $x$ near $\partial M$. Let $\omega \in C_c^\infty(\R)$, $\omega = 1$ near $0$. Furthermore, for $\tau > 0$ let $\omega_\tau = \omega(x/\tau)$ and let

$$
A^\tau = \omega_\tau A_0 + (1 - \omega_\tau)A.
$$

Clearly, $A$ and $A^\tau$ have the same conormal symbol.

**Theorem 2.** For small enough $\tau$, $A^\tau$ is also $b$-elliptic, and therefore $\mathcal{D}_{\text{min}}(A^\tau) = \mathcal{D}_{\text{min}}(A)$. Moreover, $A^\tau \rightarrow A$ in the graph norm of $A$ as $\tau \rightarrow 0$. Thus on $\mathcal{D}_{\text{min}}(A)$

$$
\text{ind } A^\tau = \text{ind } A \quad \text{for every small } \tau.
$$

**Proof.** Let $\sigma(A)$ denote the totally characteristic principal symbol of $A$ order $m$. Then

$$
\sigma(A^\tau) = \omega_\tau \sigma(A_0) + (1 - \omega_\tau)\sigma(A) = \omega_\tau \sigma(A) + (1 - \omega_\tau)\sigma(A) - x\omega_\tau \sigma(A_1) = \sigma(A) - \tau \tilde{\omega}_\tau \sigma(A_1)
$$

with $\tilde{\omega}_\tau = (x/\tau)^2 \omega(x/\tau)$. Since $\tilde{\omega}_\tau$ is bounded, $\tau \tilde{\omega}_\tau$ is small for $\tau$ small, and thus the invertibility of $\sigma(A)$ implies that of $\sigma(A) - \tau \tilde{\omega}_\tau \sigma(A_1)$ for such $\tau$. Hence $A^\tau$ is $b$-elliptic too. Since $A$ and $A^\tau$ have the same conormal symbol, part 1 of [5, Proposition 4.1] gives that $\mathcal{D}_{\text{min}}(A^\tau) = \mathcal{D}_{\text{min}}(A)$.

From the $b$-ellipticity of $A$ it follows that there is a bounded parametrix $Q : x^\gamma H^s_b \rightarrow x^{\gamma+\nu}H^{s+m}_b$ such that

$$
R = I - QA : x^\gamma H^s_b \rightarrow x^\gamma H^\infty_b
$$

is bounded for all $s$ and $\gamma$. Write

$$
A - A^\tau = x\omega_\tau A_1 = x\omega_\tau A_1 QA + x\omega_\tau A_1 R
$$

$$
= \tau \tilde{\omega}_\tau A_1 QA + x\omega_\tau A_1 R.
$$

Now, $A_1 Q : x^{-\nu/2}L^2_b \rightarrow x^{-\nu/2}L^2_b$ is bounded, so if $u \in \mathcal{D}_{\text{min}}(A)$, then

$$
\|\tau \tilde{\omega}_\tau A_1 QAu\|_{x^{-\nu/2}L^2_b} \leq c \tau \|Au\|_{x^{-\nu/2}L^2_b} \leq c \tau \|u\|_A.
$$

Write $x\omega_\tau A_1 R = \tau^{1-\varepsilon}(x^{\varepsilon})^{-1}\omega_\gamma x^\gamma A_1 R$ and note that

$$
x^\tau A_1 R : x^{\nu/2-\varepsilon}L^2_b \rightarrow x^{-\nu/2}L^2_b
$$
in continuous. Then, using Lemma 1 we get

$$\|x\omega_A \cdot A_1 Ru\|_{x^{-\nu} \mathbb{L}_b^2} \leq \tilde{c} \tau^{1-\nu} \|u\|_{x^{\nu/2} \cdot \mathbb{L}_b^2} \leq c \tau^{1-\nu} \|u\|_A.$$ 

Altogether,

$$\| (A - A^\tau) u \|_{x^{-\nu} \mathbb{L}_b^2} \leq C \tau^{1-\nu} \|u\|_A$$

and thus $A^\tau \to A$ as $\tau \to 0$. 

Using Lesch’s formula [6, Cor. 2.4.7] and Proposition 3.14 in [5] we get

**Corollary 3.** If $A_D$ is an arbitrary closed extension of $A$, then

$$\text{ind } A_D = \int_M \omega_A + \frac{1}{\nu} \hat{\eta}(A) + \dim D/D_{\text{min}}.$$ 

The various terms occurring in the index formula are defined as follows. First, $\omega_A$ denotes the local index density of $A$. One way to define this density is as the constant term in the difference of the small time fiberwise trace asymptotics of the heat operators for $A_D^* A_D$ and $A_D A_D^*$. Actually, $\omega_A$ is determined from the homogeneous terms in $A$, and hence is defined independent of $D$. In general, $\omega_A$ diverges like $1/x$ at $\partial M$. The integral $f_M \omega_A$ denotes the regularized integral of $\omega_A$, cf. [6, Def. 2.1.3], and is defined as the usual integral of $\omega_A$ off the collar $\pi: U \to \partial M$ of $\partial M$; and on the collar, $f_U \omega_A$ is defined as $\int_U \omega_A$, where $\omega_A = \pi^* \omega_0 1/2 dx + \omega_1$ on $U$, with $\omega_0$ a smooth density on $\partial M$ and with $\omega_1$ a smooth density on $U$. If $A$ has coefficients independent of $x$ near $\partial M$, then $\omega_A$ vanishes identically near $\partial M$. In particular, in this case, $f_M \omega_A = f_M \omega_A$ is just the usual integral of $\omega_A$.

In order to define $\hat{\eta}(A)$ let us first recall the definition of the indicial operator of $A$. On the collar, we can write $A = x^{-\nu} \sum_{k=0}^m P_{m-k}(x)(xD_x)^k$, where $P_{m-k}(x)$ is a family of differential operators on $\partial M$ depending smoothly on $x$. The indicial operator of $A$ is defined as the operator

$$A_\lambda = x^{-\nu} \sum_{k=0}^m P_{m-k}(0)(xD_x)^k$$
on the model cone $N^\lambda = \mathbb{R}_+ \times \partial M$. The eta invariant $\hat{\eta}(A)$ is defined as the regular value at $z = 0$ of

$$\Gamma(z)(\zeta(A_\lambda, \min ; A_\lambda, \min ; z) - \zeta(A_\lambda, \min ; A_\lambda, \min ; z)),$$ 

where $A_\lambda, \min$ denotes the closure of $A_\lambda$, $\Gamma(z)$ is the gamma function, and where the zeta functions are defined in the following way. Take for instance $L = A_\lambda, \min A_\lambda, \min$. In Section 2.2 of [5], it is shown that

$$k = \int_{\partial M} \text{tr}(e^{-tL}(x, p, x, p))m_{\partial M}(p),$$ 

is a function only of the singular coordinate $s = t/x^\nu$. Then, $\zeta(L, z)$ is defined as the transform

$$\hat{\zeta}(L, z) := \frac{1}{\Gamma(z)} \int_0^\infty s^{z-1} k(s) ds.$$ 

By Proposition 2.2.6 of [5], $\hat{\zeta}(L, z)$ is meromorphic on a half plane $\Re z < \delta$ for some $\delta > 0$, with only possible simple poles at the points $(\dim M - k)/m, k = 0, 1, \ldots$. Note that a priori, the function defined by (1) may have a simple pole at $z = 0$; however, as shown in [5, Cor. 2.4.7], this function is in fact regular there.
As already mentioned, the index formula of Corollary 3 is an extension of Lesch’s result [5, Cor. 2.4.7] to operators which do not necessarily have coefficients independent of \( x \) near the boundary. This index formula also generalizes the theorem of Brüning and Seeley [7] for first order regular singular operators. Index formulas for cone differential operators were first proved by Cheeger [2] for the Gauss-Bonnet and signature operators on a conic manifold, and were later generalized by Chou [8] to Dirac operators. For formulas in the pseudodifferential situation see Fedosov, Schulze and Tarkhanov [4]. For formulas in the totally characteristic case (\( \nu = 0 \)), see Melrose [7] and Piazza [8]. We should point out, however, that the results of this note only apply when \( \nu > 0 \).

3. Index on Sobolev spaces

In general, \( \mathcal{D}_{\text{min}} \) is not a Sobolev space. The problem lies in the possible presence of elements of \( \text{spec}_0(A) \) along the line \( \Im \sigma = -\nu/2 \). However, for index purposes, one can conveniently reduce the analysis to a slightly modified operator whose closure has a Sobolev space as its domain:

**Theorem 4.** Let \( A \) be \( b \)-elliptic. Let \( A_x = x^\sigma A \), and regard it as an unbounded operator on \( x^{-(\nu-\varepsilon)/2}L^2_b(M;E) \). If \( \varepsilon > 0 \) is sufficiently small, then

\[
A_x : x^{(\nu-\varepsilon)/2}H^m_b(M;E) \to x^{-(\nu-\varepsilon)/2}L^2_b(M;E)
\]

is Fredholm, and

\[
\text{ind } A_x = \text{ind}(A, \mathcal{D}_{\text{min}}(A)).
\]

**Proof.** Write \( A = x^{-\nu}P \) with \( P \in \text{Diff}^m_b(M;E) \). Let \( \eta > 0 \) be so small that there is no \( \sigma \in \text{spec}_b(A) \) with \( \nu/2 - \eta \leq \Im \sigma < \nu/2 \) or \( -\nu/2 < \Im \sigma \leq -\nu/2 + \eta \). The kernel of \( A \) on tempered distributions \( x^{-\infty}H_b^{-\infty}(M;E) \) is the same as that of \( P \), which we’ll denote \( K(P) \). Recall that \( \mathcal{D}_{\text{max}}(A) = \{ u \in x^{-\nu/2}L^2_b | Au \in x^{-\nu/2}L^2_b \} \). The kernel \( K_{\text{max}}(A) \) of \( A : \mathcal{D}_{\text{max}}(A) \subset x^{-\nu/2}L^m_b \to x^{-\nu/2}L^m_b \) consists those elements of \( K(P) \) whose Mellin transforms are holomorphic in \( \Im \sigma \geq \nu/2 \) since these elements of \( K(P) \) belong to \( x^{-\nu/2}L^2_b \) and \( Au \in x^{-\nu/2}L^2_b \) trivially. That is, their Mellin transforms are holomorphic in \( \Im \sigma > \nu/2 - \eta \). Thus \( K_{\text{max}}(A) \) is \( K_{\text{max}}(A_x) \) if \( 0 < \varepsilon < \eta \). On the other hand, the kernel \( K_{\text{min}}(A) \) of \( A : \mathcal{D}_{\text{min}}(A) \subset x^{-\nu/2}L^m_b \to x^{-\nu/2}L^m_b \) consists those elements of \( K(P) \) whose Mellin transforms are holomorphic in \( \Im \sigma > -\nu/2 \); indeed in part 1 of [5], Proposition 3.6] it is shown show that \( \mathcal{D}_{\text{min}}(A) = \mathcal{D}_{\text{max}}(A) \cap x^{\nu/2-\eta}H^m_b \). Thus if \( \varepsilon < \eta \) then \( K_{\text{min}}(A) = K_{\text{min}}(A_x) \). Thus \( \dim K_{\text{min}}(A) = \dim K_{\text{min}}(A_x) \).

Finally, note that the formal adjoint of \( A \) in \( x^{-\nu/2}L^2_b \) is \( A^* = x^{-\nu}P^* \), where \( P^* \) is the formal adjoint of \( P \) in \( L^2_b \), and likewise \( A^*_x = x^{-\nu+\varepsilon}P^* \). Now recall that the Hilbert adjoint of \( A \mid_{\mathcal{D}_{\text{min}}} \) is \( A^* \) with domain \( \mathcal{D}_{\text{max}}(A^*) \) so the first part of the argument yields \( K_{\text{max}}(A^*) = \dim K_{\text{max}}(A^*_x) \).

**References**

[1] J. Brüning and R. Seeley, *An index theorem for first order regular singular operators*, Amer. J. Math. **110** (1988), no. 4, 669–714.

[2] J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), no. 4, 575–657 (1984).

[3] A. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. Amer. Math. Soc. **289** (1985), no. 1, 1–40.
[4] B. Fedosov, B.-W. Schulze and N. Tarkhanov, *The index of elliptic operators on manifolds with conical points*, Selecta Math. (N.S.) 5 (1999), no. 4, 467–506.

[5] J. B. Gil and G. A. Mendoza, *Adjoints of elliptic cone operators*, preprint, Temple University, August 2001.

[6] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Math. vol 136, B.G. Teubner, Stuttgart, Leipzig, 1997.

[7] R. B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, A K Peters, Ltd., Wellesley, MA, 1993.

[8] P. Piazza, *On the index of elliptic operators on manifolds with boundary* J. Funct. Anal. 117 (1993), no. 2, 308–359.

[9] B.-W. Schulze, B. Sternin and V. Shatalov, *On the index of differential operators on manifolds with conical singularities*, Ann. Global Anal. Geom. 16 (1998), no. 2, 141–172.

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