THE ISOTROPY GROUP OF A FOLIATION: THE LOCAL CASE.

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Abstract. Given a holomorphic singular foliation $F$ of $(\mathbb{C}^n, 0)$ we define $\text{Iso}(F)$ as the group of germs of biholomorphisms on $(\mathbb{C}^n, 0)$ preserving $F$: $\text{Iso}(F) = \{\Phi \in \text{Diff}(\mathbb{C}^n, 0) \mid \Phi^*(F) = F\}$. The normal subgroup of $\text{Iso}(F)$, of biholomorphisms sending each leaf of $F$ into itself, will be denoted as $\text{Fix}(F)$. The corresponding groups of formal biholomorphisms will be denoted as $\hat{\text{Iso}}(F)$ and $\hat{\text{Fix}}(F)$, respectively. The purpose of this paper will be to study the quotients $\text{Iso}(F)/\text{Fix}(F)$ and $\hat{\text{Fix}}(F)/\hat{\text{Fix}}(F)$, mainly in the case of codimension one foliation.

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1. INTRODUCTION

Let $F$ be a germ at $0 \in \mathbb{C}^n$ of a singular codimension $p$ holomorphic foliation, where $1 \leq p \leq n - 1$. It is known that $F$ can be defined by a germ of holomorphic $p$-form $\Omega \in \Omega^p(\mathbb{C}^n, 0)$, which is integrable in the following sense:

1. $\Omega$ is locally completely decomposable outside its singular set: if $m \notin \text{Sing}(\Omega)$, that is $\Omega(m) \neq 0$, then the germ $\Omega_m$ of $\Omega$ at $m$, is completely decomposable; i.e., there exist germs of holomorphic 1-forms at $m$, say $\omega_1, ..., \omega_p$, such that $\Omega_m = \omega_1 \wedge ... \wedge \omega_p$.
2. $\omega_1, ..., \omega_p$ satisfy the Frobenius integrability condition: $d\omega_j \wedge \Omega \equiv 0$, $\forall 1 \leq j \leq p$.

Condition (1) implies that we can define a codimension $p$ distribution $D$ outside $\text{Sing}(\Omega)$ by

$$D(m) = \{v \in T_m \mathbb{R}^n \mid i_v \Omega(m) = 0\} := \ker(\Omega(m)),$$

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Condition (2) implies that the distribution $D$ is integrable.

**Remark 1.1.** If $\text{Cod}_C(\text{Sing}(\Omega)) \geq 2$ and $\Omega$ is another germ of $p$-form that represents $\mathcal{F}$ then there exists an unity $u \in \mathcal{O}_n^*$ such that $\Omega = u \cdot \Omega$. From now on, we will assume that $\text{Sing}(\mathcal{F}) = \text{Sing}(\Omega)$ has codimension $\geq 2$.

Associated to the foliation $\mathcal{F}$ we introduce the following group

$$\text{Iso}(\mathcal{F}) = \{ \phi \in \text{Diff}(\mathbb{C}^n, 0) \mid \phi^*(\mathcal{F}) = \mathcal{F} \},$$

the subgroup of $\text{Diff}(\mathbb{C}^n, 0)$ of germs of holomorphic diffeomorphisms fixing the foliation $\mathcal{F}$.

In terms of a germ of $p$-form $\Omega$ defining $\mathcal{F}$ relation (1) means that $\phi^*(\Omega)$ also defines the foliation $\mathcal{F}$, and so $\phi^*(\Omega) = u \cdot \Omega$, where $u \in \mathcal{O}_n^*$.

We will consider also

$$\text{Iso}_u(\mathcal{F}) = \{ \phi \in \text{Diff}(\mathbb{C}^n, 0) \mid \phi^*(\Omega) = \widehat{u} \cdot \Omega, \widehat{u} \in \mathcal{O}_n^* \},$$

the group of formal diffeomorphisms "fixing" $\mathcal{F}$.

**Remark 1.2.** Let $\mathcal{F}$ and $\Omega$ be as before. The reader can check that a germ $\phi \in \text{Iso}(\mathcal{F})$ if, and only if, there are neighborhoods $U$ and $V$ of $0 \in \mathbb{C}^n$, and representatives of $\Omega$ and $\phi$, denoted by the same symbols, such that $\Omega \in \Omega^p(U \cup V)$, $\phi : U \to V$, and

a. $\phi(\text{Sing}(\Omega|_U)) = \text{Sing}(\Omega|_V)$,

b. $\phi$ sends leaves of $\mathcal{F}|_U$ onto leaves of $\mathcal{F}|_V$.

Let us see some examples.

**Example 1.** The simplest example of a germ $\mathcal{F}$ of codimension $p$ foliation is the regular foliation, when $\Omega(0) \neq 0$. In this case, the definition and Frobenius theorem implies that there are local coordinates $z = (z_1, ..., z_n)$ around $0 \in \mathbb{C}^n$ and an unity $v \in \mathcal{O}_n^*$ such that $\Omega = v(z) dz_1 \wedge ... \wedge dz_p$. In this case, it is easy to see that $\phi = (\phi_1, ..., \phi_n) \in \text{Iso}(\mathcal{F})$ if, and only if $\frac{\partial \phi_i}{\partial z_j} = 0, \forall 1 \leq i \leq p, \forall j > p$. In other words, if we set $\Psi_1 = (\phi_1, ..., \phi_p)$, $\Psi_2 = (\phi_{p+1}, ..., \phi_n)$, $\zeta_1 = (z_1, ..., z_p)$ and $\zeta_2 = (z_{p+1}, ..., z_n)$ then

$$\phi(z) = (\Psi_1(\zeta_1), \Psi_2(\zeta_1, \zeta_2)).$$

**Example 2.** Let $\mathcal{F}$ and $\Omega$ be as above. We say that a germ of holomorphic (resp. formal) vector field $X \in \mathcal{X}(\mathbb{C}^n, 0)$ (resp. $X \in \tilde{\mathcal{X}}(\mathbb{C}^n, 0)$) is tangent to the foliation $\mathcal{F}$ if $i_X \Omega = 0$. Let us denote the local flow of $X$ by $t \in (\mathbb{C}, 0) \to X_t$. If $X$ is holomorphic and $t \in (\mathbb{C}, 0)$ then $X_t$ defines a germ of diffeomorphism $\mathcal{X}_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, X_t(0))$. In particular, if $X(0) = 0$ then $\mathcal{X}_t \in \text{Diff}(\mathbb{C}^n, 0), \forall t \in (\mathbb{C}, 0)$. The reader can verify directly that in this case $\mathcal{X}_t \in \text{Iso}(\mathcal{F}), \forall t \in (\mathbb{C}, 0)$.

In this example, $X_t$ fixes the leaves of $\mathcal{F}$, in the sense that if $L$ is a leaf of $\mathcal{F}$ then $X_t(L) \cap L \neq \emptyset$ is an open subset of $L$.

Motivated by example 2 we define $\text{Fix}(\mathcal{F})$ as the subgroup of $\text{Iso}(\mathcal{F})$ of germs of diffeomorphisms that "fix" the leaves of $\mathcal{F}$. In other words, if $\Phi \in \text{Fix}(\mathcal{F})$, $U$ is the domain of $\Phi$ and $L \subset U$ is a leaf of $\mathcal{F}|_U$ then $\Phi(L) \cap L \neq \emptyset$ is an open subset of $L$. For instance, if $L$ accumulates in the origin $0 \in \mathbb{C}^n$ then, as a germ, we have $\Phi(0) = L$.

The formal completion of $\text{Fix}(\mathcal{F})$ in $\text{Diff}(\mathbb{C}^n, 0)$ will be denoted by $\widehat{\text{Fix}}(\mathcal{F})$. 
Remark 1.3. $\text{Fix}(\mathcal{F})$ (resp. $\hat{\text{Fix}}(\mathcal{F})$) is a normal subgroup of $\text{Iso}(\mathcal{F})$ (resp. $\hat{\text{Iso}}(\mathcal{F})$). We leave the proof to the reader.

One of the goals of this paper is to describe the groups $\text{Iso}(\mathcal{F})$ (resp. $\hat{\text{Iso}}(\mathcal{F})$) and the quotient $\text{Iso}(\mathcal{F})/\text{Fix}(\mathcal{F})$ (resp. $\hat{\text{Iso}}(\mathcal{F})/\hat{\text{Fix}}(\mathcal{F})$) in certain cases. Let us begin by a simple example.

Example 3. Let $\omega \in \Omega^1(\mathbb{C}^n, 0)$ be an integrable 1-form and $\Phi \in \text{Diff}(\mathbb{C}^n, 0)$ be an involution $(\Phi^2 = I)$, which is not in $\text{Iso}(\mathcal{F}_\omega)$. In particular, $\eta := \omega \wedge \Phi^*(\omega) \neq 0$ and if $n \geq 3$ then $\eta$ is integrable and defines a foliation $\mathcal{F}_\eta$ of codimension two of $(\mathbb{C}^n, 0)$. Since $\Phi^2 = I$ we have

$$\Phi^*(\eta) = \Phi^*(\omega \wedge \Phi^*(\omega)) = \Phi^*(\omega) \wedge \omega = -\eta \implies \Phi \in \text{Iso}(\mathcal{F}_\eta).$$

We would like to observe that $\Phi \notin \text{Fix}(\mathcal{F}_\eta)$, so that its class in $\text{Iso}(\mathcal{F}_\eta)/\text{Fix}(\mathcal{F}_\eta)$ is non-trivial.

Another simple example is the following:

Example 4. We say that $\mathcal{F}$ is a homogeneous foliation on $\mathbb{C}^n$ if there exists a p-form $\Omega$ defining $\mathcal{F}$ with all coefficients homogeneous of the same degree. Let $R = \sum_j z_j \frac{\partial}{\partial z_j}$ be the radial vector field. The p-form $\Omega$ is homogeneous if, and only if, $L_R \Omega = (d + p) \Omega$, where $d$ is the degree of the coefficients. In particular, the group of dilatations of $\mathbb{C}^n$

$$H = \{ \eta(z) = \rho \cdot z \mid \rho \in \mathbb{C}^* \}$$

is contained in $\text{Iso}(\mathcal{F})$: $h^*_\rho(\Omega) = \rho^{d+p} \cdot \Omega$ and so $h^*_\rho \in \text{Iso}(\mathcal{F})$.

We note however that $i_R \Omega \equiv 0$ if, and only if, $H \subset \text{Fix}(\mathcal{F})$. When $i_R \Omega = 0$ we say that the foliation $\mathcal{F}$ is conic.

If $i_R \Omega \neq 0$ then the generic element of $H$ does not fix the leaves of $\mathcal{F}$ and the quotient $H/H \cap \text{Fix}(\mathcal{F})$ can be very complicated in general, as in the next example.

Example 5. Let $\mathcal{F}$ be the germ of codimension one logarithmic foliation given by $\Omega = x_1 \ldots x_n \sum_{j=1}^n \lambda_j \frac{dx_j}{x_j^2}$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are linearly independent over $\mathbb{Z}$.

In this case the holonomy group of the leaf $L := (x_1 = 0) \setminus \bigcup_{j>1} (x_j = 0)$ in the transversal section $\Sigma := \{(\tau, 1, \ldots, 1) \mid \tau \in \mathbb{C}\}$ is linear and its multipliers are in the sub-group $G$ of $\mathbb{C}^*$ generated by $e^{2\pi i \lambda_j/\lambda_1}$, $2 \leq j \leq n$. It can be shown that $H/H \cap \text{Fix}(\mathcal{F})$ is isomorphic to $\mathbb{C}^*/G$.

Example 6. Let $\omega_1, \ldots, \omega_p$ be $p$ germs of integrable 1-forms on $(\mathbb{C}^n, 0)$, where $2 \leq p < n$. Let $\eta := \omega_1 \wedge \ldots \wedge \omega_p$. We will assume that $\eta \neq 0$. Note that $\eta$ is integrable. We will assume also that:

1. If $\omega = \sum_{j=1}^p f_j \cdot \omega_j$ is integrable, where $f_1, \ldots, f_n \in \mathcal{O}_n$, and there is $i$ such that $f_i \in \mathcal{O}_n^*$ then $\omega = f_i \cdot \omega_i$.

2. If $i \neq j$ then there is no $\Phi \in \text{Iso}(\mathcal{F}_\eta)$ such that $\omega_i \wedge \Phi^*(\omega_j) = 0$.

With the above hypothesis we have $\text{Iso}(\mathcal{F}_\eta) = \bigcap_{j=1}^p \text{Iso}(\mathcal{F}_{\omega_j})$.

In fact, if $\Phi \in \text{Iso}(\mathcal{F}_\eta)$ then $\Phi^*(\eta) = u \cdot \eta$, $u \in \mathcal{O}_n^*$, implies that $\Phi^*(\omega_j) = \sum_{i=1}^p f_{ji} \cdot \omega_i$, $\forall 1 \leq j \leq p$, where $f_{ji} \in \mathcal{O}_n^*$ for some $i$. Therefore, from (1) we get $\Phi^*(\omega_j) = f_{ji} \cdot \omega_i$ and by (2) we get $i = j$ and $\Phi^*(\omega_j) = f_{jj} \cdot \omega_j$, so that $\Phi \in \text{Iso}(\mathcal{F}_{\omega_j})$, $\forall j$. The converse statement is immediate and is left to the reader.
A concrete example of this situation is the following: let \( f, g \in \mathcal{O}_2 \) be such that \( f^{-1}(0) \) and \( g^{-1}(0) \) are not homeomorphic. Define \( \omega_1, \omega_2 \in \Omega^1(\mathbb{C}^4, 0) \) as \( \omega_1 = z_1 \, dz_2 + df(z_1, z_2) \) and \( \omega_2 = z_3 \, dz_4 + dg(z_3, z_4) \).

Using that \( dw_1 = dz_1 \wedge dz_2 \) and \( dw_2 = dz_3 \wedge dz_4 \) it is possible to prove that \( \omega_1 \) and \( \omega_2 \) satisfy hypothesis (1) and (2) above. In particular, we get \( Iso(\mathcal{F}_{\omega_1 \wedge \omega_2}) = \{ (\phi(z_1, z_2), \psi(z_3, z_4)) \mid \phi \in Iso(\mathcal{F}_{\omega_1}) \text{ and } Iso(\mathcal{F}_{\omega_2}) \} \simeq Iso(\mathcal{F}_{\omega_1}) \times Iso(\mathcal{F}_{\omega_2}) \).

Another example in \((\mathbb{C}^3, 0)\), satisfying (1) and (2), is the following: let \( \omega_1 = a(x, y) \, dx + b(x, y) \, dy \) and \( \omega_2 = dz \), where we assume that \( \omega_1 \) has no meromorphic integrating factor. Here we have \( Iso(\mathcal{F}_{\omega_1 \wedge \omega_2}) = \{ (\phi(x, y), \psi(z)) \mid \phi \in Iso(\mathcal{F}_{\omega_1}) \text{ and } \psi \in Diff(\mathbb{C}, 0) \} \simeq Iso(\mathcal{F}_{\omega_1}) \times Diff(\mathbb{C}, 0) \).

We begin in §2 studying the case of codimension one foliations. In [4] the authors study some special cases of this situation: when the dimension is \( n = 2 \) and when \( \mathcal{F} \) has an holomorphic first integral, related with a known Briançon-Skoda theorem [3]. In particular, we intend here to precise and generalize some of the results of [3]. Our first result in this direction is the following:

**Theorem 1.** Let \( \mathcal{F} \) be a germ at \( 0 \in \mathbb{C}^n \) of codimension one foliation defined by a germ of integrable 1-form \( \Omega \). Suppose that the quotient \( \widehat{Iso}(\mathcal{F})/\widehat{Fix}(\mathcal{F}) \) has an element of infinite order. Then \( \mathcal{F} \) is formally Liouville integrable, that is, there exists \( f \in \mathcal{O}_n \) such that \( \frac{1}{f} \Omega \) is closed.

In §2.3 we will apply theorem [3] in the case of foliations on \((\mathbb{C}^2, 0)\) that in the process of resolution have a non-dicritical irreducible component with non-abelian holonomy. In theorem [4] we will prove that for such a foliation \( \mathcal{F} \) then \( \widehat{Iso}(\mathcal{F})/\widehat{Fix}(\mathcal{F}) \) is isomorphic to a finite subgroup of the linear group \( GL(2, \mathbb{C}) \).

Theorem [3] is stated in a formal context. A germ of meromorphic closed 1-form \( \omega/f, \omega \in \Omega^1(\mathbb{C}^n, 0) \), \( f \in \mathcal{O}_n \), can be written in a normal form of the following type: if \( f = f_1^{n_1+1} \cdots f_r^{n_r+1} \) is the decomposition of \( f \) into irreducible factors, then it can be proved that (cf. [3])

\[
\frac{\omega}{f} = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} + d \left( \frac{H}{f_1^{n_1} \cdots f_r^{n_r}} \right),
\]

where \( H \in \mathcal{O}_n \) and \( \lambda_j \in \mathbb{C} \), \( 1 \leq j \leq r \).

In the formal case, when \( \omega \in \widehat{\Omega}^1(\mathbb{C}^n, 0) \) and \( f_j \in \widehat{\mathcal{O}}_n \), \( 1 \leq j \leq r \), there is a similar result, proved in [23], with \( H \in \widehat{\mathcal{O}}_n \) in [22]. In our case, \( \omega \in \Omega^1(\mathbb{C}^n, 0) \) and \( f \in \widehat{\mathcal{O}}_n \), but there are conditions, on the residues \( \lambda_j \), assuring the convergence of \( f \) (see [3]).

About the cardinality of the quotient group and the existence of holomorphic integrating factor, by using results of Pérez Marco [20] it is possible to construct examples of germs of foliations \( \mathcal{F} \) on \((\mathbb{C}^2, 0)\) in which \( Iso(\mathcal{F})/Fix(\mathcal{F}) \) is non-countable, contains elements of infinite order, but \( \mathcal{F} \) is not Liouville integrable: it cannot be defined by a meromorphic closed form (see theorem [3] in §2.4).

In §3 we will study \( Iso(\mathcal{F}) \) and \( Fix(\mathcal{F}) \) when \( \mathcal{F} \) is homogeneous or has an integrating factor.
Definition 1. We say that a p-form $\Omega$ on $\mathbb{C}^n$ is conical if it is homogeneous and $i_\xi \Omega = 0$, where $R$ is the radial vector field on $\mathbb{C}^n$. A holomorphic foliation of codimension $p$ on $\mathbb{C}^n$ is conical if it can be defined by a conical p-form.

If $\mathcal{F}$ is a conical foliation on $\mathbb{C}^n$, as above, then $\mathcal{F}$ induces a foliation $\tilde{\mathcal{F}}$, of the same codimension, on the projective space $\mathbb{P}^{n-1}$. In theorem 4 we prove that, in this case, $\text{Iso}(\tilde{\mathcal{F}})/\text{Fix}(\mathcal{F})$ is isomorphic to $\text{Aut}(\tilde{\mathcal{F}})$, the subgroup of $\text{Aut}(\mathbb{P}^{n-1})$ of automorphisms of $\mathbb{P}^{n-1}$ preserving $\tilde{\mathcal{F}}$. Let us see an example.

Example 7. Jouanolou’s example of degree $d \geq 2$ on $\mathbb{P}^n$ (see [9] and [12]) is defined in homogeneous coordinates of $\mathbb{C}^{n+1}$ by the homogeneous $(n - 1)$-form $\Omega = i_R \omega$, where $R$ is the radial vector field on $\mathbb{C}^{n+1}$, $\omega = dx_1 \wedge \ldots \wedge dx_{n+1}$ and

$$X = x_{n+1}^d \frac{\partial}{\partial x_1} + \sum_{j=2}^n x_j^{d-1} \frac{\partial}{\partial x_j}.$$

The foliation $\tilde{\mathcal{F}}$, induced on $\mathbb{P}^n$ by $\mathcal{F}_\Omega$, can be defined in the affine coordinate $x_{n+1} = 1$ by the vector field

$$Y = (1 - x_{n+1}^d, x_1) \frac{\partial}{\partial x_1} + \sum_{j=2}^n (x_j^{d-1} - x_{n+1}^d) \frac{\partial}{\partial x_j}.$$

Let $D = \frac{d^{n+1}}{n+1}$ and $\lambda$ be a primitive $D$th-root of unity. It is known that $\text{Aut}(\tilde{\mathcal{F}})$ is isomorphic to the finite sub-group $G(n, d)$ of $\text{PSL}(n+1, \mathbb{C})$ generated by the transformations $\tau$ and $\rho$, where

$$\tau(x_1, \ldots, x_n, x_{n+1}) = \left(\lambda. x_1, \lambda^{d+1}. x_2, \ldots, \lambda^{d^{n-1}}. x_j, \ldots, \lambda^{d^{n-1}}. x_n, x_{n+1}\right)$$

and

$$\rho(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_{n+1}, x_1, x_2, \ldots, x_n).$$

In particular, $\text{Iso}(\mathcal{F}_\Omega)/\text{Fix}(\mathcal{F}_\Omega) \simeq G(n, d)$ by theorem 4.

On the other hand, if $\omega \in \Omega^1(\mathbb{C}^n)$ is integrable and homogeneous, but is non-conical, then the following facts are known (see [3]):

a. If $i_\xi \omega = f \neq 0$ then $f$ is an integrating factor of $\omega$: $d(f^{-1}. \omega) = 0$.

b. $f$ is homogeneous: $R(f) = \ell \xi$, for some $\ell \in \mathbb{N}$.

c. If $f = f_1^{r_1+1} \ldots f_r^{r_r+1}$ is the decomposition of $f$ into homogeneous factors then the hypersurfaces $(f_j = 0)$ are $\mathcal{F}_\xi$-invariant, $\forall 1 \leq j \leq r$. Moreover, the form $f^{-1}. \omega$ can be written as in [2].

d. When $f$ is reduced, $f = f_1 \ldots f_r$, then $\Omega := f^{-1}. \omega$ is logarithmic:

$$\Omega = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j}.$$

When $\Omega = f^{-1}. \omega$ is like in [2] or [3] and $f_1, \ldots, f_r \in \mathcal{O}_n$ are relatively prime, but not necessarily homogeneous, we will consider in §3.3 the case in which $\mathcal{F}_\Omega$ has no meromorphic first integral. In proposition 4 we will prove that if $\Omega$ is a closed meromorphic 1-form, without meromorphic first integral, and if $\Phi \in \text{Iso}(\mathcal{F}_\Omega)$ then $\Phi^* (\Omega) = \delta \Omega$, where $\delta$ is a root of unity. As we will see there, in this case $\Phi$ permutes the "separatrices" $(f_j = 0)$, and if $\Phi(f_j = 0) = (f_j = 0)$, $1 \leq j \leq r$, then $\delta = 1.$
In theorem 3 we will prove that if $\Omega$ is logarithmic, has no meromorphic first integral and $Iso(\mathcal{F}_1)$ contains a transformation $\Phi$ with $D\Phi(0) = \rho, I$, where $\rho$ is not a root of unity, then there exists $h \in Diff(C^n, 0)$ such that $h^* \Omega$ is homogeneous.

In corollaries 2, 3, 4 and 5 we apply theorem 3 in some special situations.

**Example 8.** Let $f_1, f_2 \in \mathcal{O}_2$ be such that:

1. $f_1$ is irreducible in $\mathcal{O}_n$ and its first non-vanishing jet at $0 \in \mathbb{C}$ is $J^1_0(f_1) = z_1^* z_2^*$.
2. $J^1_0(f_2) = z_1 + z_2$.

Let $\Omega = \frac{df_1}{f_1} + \lambda \frac{df_2}{f_2}$, where $\lambda \notin \mathbb{Q}$. Then $Iso(\mathcal{F}_1) = Fix(\mathcal{F}_1)$.

This example is an application of corollary 4 as we will see in § 3.4.

**Remark 1.4.** Let $\omega \in \Omega^1(C^n, 0)$ be an integrable form inducing the germ of foliation $\mathcal{F}_\omega$. Note that $Iso(\mathcal{F}_\omega)/Fix(\mathcal{F}_\omega)$ can be considered as a subgroup of $\tilde{Iso}(\mathcal{F}_\omega)/\tilde{Fix}(\mathcal{F}_\omega)$.

In fact, there is a natural group inclusion $In: Iso(\mathcal{F}_\omega)/Fix(\mathcal{F}_\omega) \rightarrow \tilde{Iso}(\mathcal{F}_\omega)/\tilde{Fix}(\mathcal{F}_\omega)$, as the reader can check. This map is injective, but not surjective in general.

For instance, in the case of 1-forms $\omega \in \Omega^1(C^2, 0)$ that are formally linearizable, but not holomorphically linearizable, studied in theorem 3 the map is not surjective. In contrast, in the case of conical foliations, to be studied in § 3.2 the map is an isomorphism. A natural problem is the following:

**Problem 1.** When $In: Iso(\mathcal{F}_\omega)/Fix(\mathcal{F}_\omega) \rightarrow \tilde{Iso}(\mathcal{F}_\omega)/\tilde{Fix}(\mathcal{F}_\omega)$ is an isomorphism?

Another natural problem is the following:

**Problem 2.** Let $\mathcal{F}$ be a germ of foliation such that any element $\Phi \in Iso(\mathcal{F})/Fix(\mathcal{F})$ has finite order. Is $Iso(\mathcal{F})/Fix(\mathcal{F})$ finite? The same question can be posed for $\tilde{Iso}(\mathcal{F})/\tilde{Fix}(\mathcal{F})$.

In theorem 2 we prove that the answer of problem 2 is positive in a particular case in dimension two. In corollary 3 of theorem 5 we prove a similar statement in the case of logarithmic foliations in $(C^n, 0)$ (see § 3.1).

**Remark 1.5.** Let $S = \sum_{j=1}^{n} k_j z_j \frac{\partial}{\partial z_j}$ be a semi-simple linear vector field with eigenvalues $k_1, ..., k_n \in \mathbb{Z}_{>0}$ and $gcd(k_1, ..., k_n) = 1$. We say that a $p$-form $\eta \in \Omega^p(C^n, 0)$ is $S$ quasi-homogeneous if $L_S \eta = k \eta$, where $k \in \mathbb{N}$. In the case of a germ of function $f \in \mathcal{O}_n$ the identity $S(f) = L_S(f) = k \cdot f$ means that

$$f(\tau^{k_1} z_1, ..., \tau^{k_n} z_n) = \tau^k f(z_1, ..., z_n), \forall (z_1, ..., z_n) \in (C, 0), \forall \tau \in C.$$ 

We would like to observe the following facts:

1. $\eta$ is homogeneous if, and only if, $\eta$ is quasi-homogeneous with respect to $R$, the radial vector field in $C^n$.
2. If $\eta$ is $S$ quasi-homogeneous then their coefficients are $S$ quasi-homogeneous. As a consequence all coefficients of $\eta$ are polynomials.
3. The flow $exp(t \cdot S)$ of $S$ induces a $C^*$ action $\Psi: C^* \times C^n \rightarrow C^n$:

$$\Psi_{(z_1, ..., z_n)} := \Psi(\tau, z_1, ..., z_n) = (\tau^{k_1} z_1, ..., \tau^{k_n} z_n).$$
The relation $L_S \eta = k \eta$ is equivalent to $\Psi^*(\eta) = \tau^k \eta$.

We say that $\eta$ is a $S$ conical foliation if $\eta$ is $S$ quasi-homogeneous and $i_S \eta = 0$. A $S$ conical and integrable form induces a foliation on the weighted projective space $\mathbb{P}^{n-1}_k$, associated to the weights $k = (k_1, ..., k_n)$ (cf. [7]).

A natural question is the following:

**Problem 3.** Let $S = \sum_{j=1}^n k_j z_j \frac{\partial}{\partial z_j}$ be as above. Are there statements concerning $S$ conical and $S$ quasi-homogeneous foliations, similar to theorems 4 and 5?

### 2. Theorem and Correlated Facts

**2.1. Preliminaries.** This section is devoted to the statement of some well known results that will be used along the text. We will prove also theorem 1 in § 2.2 and give an application in § 2.3. In § 2.4 we construct examples of foliations $F$ in dimension two for which $Iso(F)/Fix(F)$ is non-countable. These examples are not Liouville integrable, but formally Liouville integrable, in the sense that they admit a formal integrating factor.

We begin recalling the concept of unipotent germ of formal diffeomorphism.

**Definition 2.** We say that $\phi \in \hat{Diff}(\mathbb{C}^n, 0)$ is unipotent if the linear part $D\phi(0)$ is unipotent. A formal vector field $X \in \hat{X}(\mathbb{C}^n, 0)$ is nilpotent if its linear part $DX(0)$ is nilpotent.

In fact, the following proposition is known (cf. [1] and [4]):

**Proposition 1.** A formal germ $\phi \in \hat{Diff}(\mathbb{C}^n, 0)$ is unipotent if, and only if, there exists a nilpotent formal vector field $X \in \hat{X}(\mathbb{C}^n, 0)$ such that $\phi = \exp(X)$, where $t \mapsto \exp(t X)$ denotes the formal flow of $X$. Moreover, if $X \in \hat{X}(\mathbb{C}^n, 0)$ is nilpotent then the formal flow is polynomial in $t$, in the sense that

$$
\exp(t X)(z) = \sum_\sigma P_\sigma(t) z^\sigma,
$$

where $P_\sigma(t)$ is a polynomial in $t$ for all $\sigma = (\sigma_1, ..., \sigma_n)$.

As a consequence, we have the following:

**Corollary 2.1.** If $\phi \in \hat{Iso}(F)$ is unipotent and $\phi = \exp(X)$ then $\phi_t = \exp(t X) \in \hat{Iso}(F)$, $\forall t \in \mathbb{C}$.

**Proof.** Let $\Omega$ be a germ of integrable $p$-form that represents $F$. Since $\phi \in \hat{Iso}(F)$ we have

$$
\phi^*(\Omega) = \hat{u} \cdot \Omega, \quad \hat{u} \in \mathcal{O}^*_n.
$$

From the above relation, we get by induction on $k \in \mathbb{N}$ that

$$
(\phi^k)^*(\Omega) = \hat{u}_k \cdot \Omega,
$$

where $\hat{u}_k$ is defined inductively by $\hat{u}_1 = \hat{u}$ and $\hat{u}_{k+1} = \hat{u}_k \circ \phi \cdot \hat{u}$, if $k \geq 1$.

Write

$$
\Omega = \sum_I F_I(z) dz^I,
$$

where $I = (1 \leq i_1 < ... < i_p \leq n)$, $dz^I = dz_{i_1} \wedge ... \wedge dz_{i_p}$ and $F_I(z) = \sum_\sigma A_{I,\sigma} z^\sigma$. 


Let $X \in \tilde{\mathcal{X}}(\mathbb{C}^n, 0)$ be such that $\phi = \exp(X)$. Since $\exp(tX)(z) = \sum P_\sigma(t) z^\sigma$, where $P_\sigma \in \mathbb{C}[t]$, we get by direct substitution that

$$\exp(tX)^*(\Omega) = \sum I F_i(t, z) dz^I,$$

where $F_i(0, z) = F_i(z)$ and

$$F_i(t, z) = \sum Q_{I, \sigma}(t) z^\sigma, \quad Q_{I, \sigma} \in \mathbb{C}[t], \forall I, \sigma \label{eq:6}$$

Now, we can write $\phi^k = \exp(k X)$ if $k \in \mathbb{Z}$, so that, from (6) and (5) we obtain that for all $I_1, \sigma_1$ and $I_2, \sigma_2$ such that $A_{I_1, \sigma_1} A_{I_2, \sigma_2} \neq 0$ we have

$$\frac{Q_{I_1, \sigma_1}(k)}{A_{I_1, \sigma_1}} = \frac{Q_{I_2, \sigma_2}(k)}{A_{I_2, \sigma_2}}, \forall k \in \mathbb{Z}.$$ 

Since all $Q_{I, \sigma}(t)$ are polynomials in $t$, from the above relation that

$$\frac{Q_{I_1, \sigma_1}(t)}{A_{I_1, \sigma_1}} = \frac{Q_{I_2, \sigma_2}(t)}{A_{I_2, \sigma_2}}, \forall t \in \mathbb{C}$$

which implies the corollary. \qed

Another well known fact is the following:

**Proposition 2.** Let $\phi \in \text{Diff}(\mathbb{C}^n, 0)$ be a formal diffeomorphism. Then $\phi$ admits a formal Jordan decomposition: $\phi = \phi_S \circ \phi_U$, where $\phi_S$ and $\phi_U$ are commuting formal diffeomorphisms, $\phi_S$ is semi-simple (formally conjugated to its linear part $D\phi_S(0)$) and $\phi_U$ is unipotent.

**Remark 2.1.** If $\phi \in \tilde{\text{Isol}}(\mathcal{F})$ then $\phi_S, \phi_U \in \tilde{\text{Isol}}(\mathcal{F})$. The proof can be found in [4].

### 2.2. Proof of theorem \[1\] and complements.

Let $\mathcal{F}$ be a germ of codimension one foliation defined by an integrable germ $\Omega \in \Omega^1(\mathbb{C}^n, 0)$. We will assume that $\tilde{\text{Isol}}(\mathcal{F})$ has an element of infinite order in the quotient $\tilde{\text{Isol}}(\mathcal{F})/\tilde{\text{Fix}}(\mathcal{F})$, say $\phi$. The proof will be based in several remarks. The first is elementary:

**Proposition 3.** Let $\phi \in \tilde{\text{Isol}}(\mathcal{F})$ be unipotent and $X \in \tilde{\mathcal{X}}(\mathbb{C}^n, 0)$ be such that $\phi = \exp(X)$. Let $f := i_X \Omega \in \tilde{\mathcal{O}}_n$. We have two possibilities:

(a). $f \neq 0$ and $\omega := \frac{1}{i} \Omega$ is closed.

(b). $f \equiv 0$ and in this case $\phi \in \tilde{\text{Fix}}(\mathcal{F})$.

**Proof.** From corollary [2.1] we know that $\exp(tX) \in \tilde{\text{Isol}}(\mathcal{F})$, which means that $\exp(tX)^* \Omega = u_t \Omega$, where $u_t \in \tilde{\mathcal{O}}_n^* \forall t \in \mathbb{C}$. Using that $\frac{d}{dt}\exp(tX)^* \Omega|_{t=0} = L_X \Omega$, where $L_X$ denotes the Lie derivative in the direction of $X$, and taking the derivative in both members of the above relation we get

$$L_X \Omega = v \Omega, \quad v = \frac{d}{dt} u_t|_{t=0}.$$ 

Since $v \Omega = L_X \Omega = i_X d\Omega + d i_X \Omega = i_X d \Omega + d f$ we get

$$0 = L_X \Omega \wedge \Omega = i_X d \Omega \wedge \Omega + df \wedge \Omega \implies df \wedge \Omega = \Omega \wedge i_X d \Omega.$$ 

On the other hand, the integrability condition $\Omega \wedge d\Omega = 0$ implies that

$$0 = i_X (\Omega \wedge d\Omega) = i_X \Omega \wedge d\Omega - \Omega \wedge i_X d\Omega = f d\Omega - \Omega \wedge i_X d \Omega \implies$$
This proves (a).

If $X$ is holomorphic and $i_X\Omega = 0$ then the orbits of the flow $exp(tX)$ are contained in the leaves of $\mathcal{F}$, so that $\phi = exp(X) \in Fix(\mathcal{F})$.

In the general case the formal flow $t \mapsto exp(tX)$ is tangent to $\mathcal{F}$.

Let us finish the proof of theorem 1. Let $\mathcal{F}$ be defined by the germ of integrable 1-form $\Omega$ and $\phi \in \tilde{Iso}(\mathcal{F})$ be of infinite order in $\tilde{Iso}(\mathcal{F})/\tilde{Fix}(\mathcal{F})$. By proposition 2 we can decompose $\phi = \phi_S \circ \phi_U$, where $\phi_S$ is semi-simple and $\phi_U$ unipotent. By remark 2.1 $\phi_S$ and $\phi_U$ are in $\tilde{Iso}(\mathcal{F})$. Let $X \in \mathcal{X}(\mathbb{C}^n, 0)$ be such that $\phi_U = exp(X)$. If $i_X \Omega \neq 0$ we are done by proposition 3. If $i_X \Omega \equiv 0$ then $\phi_U \in \tilde{Fix}(\mathcal{F})$ and this implies that the classes of $\phi_S$ and $\phi$ in $\tilde{Iso}(\mathcal{F})/\tilde{Fix}(\mathcal{F})$ coincide. Hence, $\phi_S$ is of infinite order in $\tilde{Iso}(\mathcal{F})/\tilde{Fix}(\mathcal{F})$. Now, $\phi_S$ is linearizable: there exists $\varphi \in \tilde{Diff}(\mathbb{C}^n, 0)$ such that $\varphi^{-1} \circ \phi_S \circ \varphi = L$, where $L = D\phi_S(0)$ is linear and semi-simple: in some base of $\mathbb{C}^n$ the matrix of $L$ is diagonal and the subgroup of powers of $L$, $H = \{L^k | k \in \mathbb{Z}\} \subset GL(n, \mathbb{C})$, is abelian. Note that $L$ is not of finite order, for otherwise $\phi_S$ would be also of finite order. In particular $H$ is infinite.

Let $G$ be the Zariski closure of $H$. Since $H$ is infinite and abelian, $G$ is an abelian algebraic Lie group of dimension $\geq 1$. Let

$$\hat{G} = \varphi \circ G \circ \varphi^{-1} = \{\varphi \circ g \circ \varphi^{-1} | g \in G\} \subset \tilde{Diff}(\mathbb{C}^n, 0).$$

Note that $\hat{G}$ is abelian. We assert that $\hat{G} \subset \tilde{Iso}(\mathcal{F})$. In fact, let $\hat{\Omega} = \varphi^*(\Omega)$. From $\phi_S^*(\Omega) = u_\Omega$, where $u \in \mathcal{O}_{\mathbb{C}^n}$, we get $L^*(\hat{\Omega}) = \hat{u}_\Omega$ and $(L^k)^*(\hat{\Omega}) = \hat{u}_k \hat{\Omega}$, $\hat{u}_k \in \mathcal{O}_{\mathbb{C}^n}$, $\forall k \in \mathbb{Z}$. Since $G$ is the Zariski closure of $H$, for all $g \in G$ we must have $g^*(\hat{\Omega}) = \hat{u}_g \hat{\Omega}$, $\hat{u}_g \in \mathcal{O}_{\mathbb{C}^n}$. Hence, if $\hat{g} = \varphi \circ g \circ \varphi^{-1} \in \hat{G}$ then $\hat{g}^*(\hat{\Omega}) = u_g \Omega$, $u_g = \hat{u}_g \circ \varphi^{-1}$, so that $\hat{g} \in \tilde{Iso}(\mathcal{F})$.

Now, since $G$ is a complex Lie group of dimension $\geq 1$ the Lie algebra $\mathcal{G}$ of $G$ has dimension $\geq 1$. Moreover, if $X \in \mathcal{G}$ then $exp(tX)^*(\hat{\Omega}) = u_t \hat{\Omega}$, so that $L_X \hat{\Omega} = h \hat{\Omega}$, where $h = \frac{d}{dt} u_t|_{t=0}$. We assert that there exists $X \in \mathcal{G}$ such that $i_X \hat{\Omega} \neq 0$.

In fact, if not then $0 \in \mathcal{G}$ has a neighborhood $V$ such that $i_X \hat{\Omega} = 0 \forall X \in V$. In particular, if $G^o$ is the connected component of $G$ containing the identity, then for any $g \in G^o$ we have $g = exp(tX)$ where $t \in \mathbb{C}$ and $X \in V$. This implies that the conjugate $\hat{g} := \varphi \circ g \circ \varphi^{-1} \in \tilde{Fix}(\mathcal{F})$.

On the other hand, since $G$ has a finite number of connected components there exists $k \in \mathbb{N}$ such that $L^k \in G^o$. But, this would imply that the class of $\phi_S^k$ in $\tilde{Iso}(\mathcal{F})/\tilde{Fix}(\mathcal{F})$ would be trivial, contradicting the hypothesis.

Finally, if $X \in \mathcal{G}$ is such that $i_X \hat{\Omega} \neq 0$ then the vector field $Y = \varphi_*(X) \in \tilde{X}(\mathbb{C}^n, 0)$ satisfies $f := i_Y \hat{\Omega} \neq 0$, so that $d\left(\frac{1}{f} \hat{\Omega}\right) = 0$.

### 2.3. Application of theorem 1

A consequence of theorem 1 is that when a germ of holomorphic form that has no formal integrating factor then all elements of $\tilde{Iso}(\mathcal{F})/\tilde{Fix}(\mathcal{F})$ are of finite order.

We will apply the above remark in the case of germs of foliations on $(\mathbb{C}^2, 0)$.
Is is known that any germ at $0 \in \mathbb{C}^2$ of foliation by curves, say $\mathcal{F}$, has a resolution by a sequence of punctual blowing-ups (cf. [22] and [19]). After the resolution process $\Pi: (M, E) \to (\mathbb{C}^2, 0)$, where $\Pi$ denotes the blowing-up map and $E \subset M$ the exceptional divisor, we obtain the resolved foliation $\Pi^*(\mathcal{F}) := \tilde{\mathcal{F}}$. All the irreducible components of $E$ are biholomorphic to $\mathbb{P}^1$. Some of them, the non-dicritical components, are $\tilde{\mathcal{F}}$-invariant, whereas others, the dicritical ones, are not invariant. The foliation $\tilde{\mathcal{F}}$ has only simple singularities (see [19]). A singularity of a germ of holomorphic vector field $Z$ at $(\mathbb{C}^2, 0)$ is simple if:

a. The derivative $DZ(0)$ is semi-simple and not identically zero. Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $DZ(0)$.

b. If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, or vice-versa, the singularity is called a saddle-node.

c. If $\lambda_1, \lambda_2 \neq 0$ then $\lambda_2/\lambda_1 \notin \mathbb{Q}_+.$

A non-dicritical component $D$ of $E$ contains necessarily singularities of $\tilde{\mathcal{F}}$, say $S = \{p_1, ..., p_k\}$. If we fix a transverse section $\Sigma$ to $\tilde{\mathcal{F}}$ at a non-singular point $p \in D$ then the holonomy group of $\tilde{\mathcal{F}}$ at $D$ is a representation $\rho$ of $\Pi_1(D \setminus S)$ on the group of germs of biholomorphisms $Diff(\Sigma, p) \simeq Diff(\mathbb{C}, 0)$. The image $\rho(\Pi_1(D \setminus S))$ is called the holonomy group of $D$.

We would like to observe that, given a finitely generated subgroup $H$ of $Diff(\mathbb{C}, 0)$ then it is possible to construct examples of foliations such that $\tilde{\mathcal{F}}$ has a non-dicritical divisor $D$ with holonomy group isomorphic to $H$ (see [13]). As an application of theorem [4] we have the following result:

**Theorem 2.** Let $\mathcal{F}$ be a germ at $0 \in \mathbb{C}^2$. Assume that after the resolution process, $\Pi: (M, E) \to (\mathbb{C}^2, 0)$, the total divisor $E$ has a non-dicritical irreducible component $D$ for which the holonomy of the strict transform $\Pi^*(\mathcal{F})$ is non-abelian. Then $\text{Iso}(\mathcal{F})/\text{Fix}(\mathcal{F})$ is isomorphic to a finite subgroup of the linear group $\text{GL}(2, \mathbb{C})$.

**Proof.** Assume that $\mathcal{F}$ is defined by the germ of vector field $X = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y}$, or equivalently, by the dual form $\Omega = P \, dy - Q \, dx$, where $P, Q \in \mathcal{O}_2$ and $P(0) = Q(0) = 0$.

**Lemma 2.1.** $\Omega$ has no formal integrating factor. In particular, any $\phi \in \text{Iso}(\mathcal{F})/\text{Fix}(\mathcal{F})$ has finite order.

**Proof.** Suppose by contradiction that $\Omega$ has a formal integrating factor $\hat{f}$: $d \left( \frac{\hat{f}}{\hat{f}} \Omega \right) = 0$. Let $\hat{f} = \prod_{j=1}^k \hat{f}_j$ be the decomposition of $\hat{f}$ into formal irreducible factors. In this case, we can write (cf. [3] and [23]):

\[
\frac{1}{\hat{f}} \, \Omega = \sum_{j=1}^k \lambda_j \, \frac{df_j}{\hat{f}_j} + d \left( \frac{g}{\hat{f}_1 \cdots \hat{f}_k} \right)
\]

where $\lambda_j \in \mathbb{C}$ and $g \in \hat{O_2}$. Let $\Pi: (M, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution process of $\mathcal{F}$ and $\mathcal{F}^*$ be the strict transform of $\mathcal{F}$ by $\Pi$. Let $D \subset E$ be the irreducible component with non abelian holonomy group.

**Claim 2.1.** Let $p \in D \setminus \text{Sing}(\mathcal{F}^*)$. Then there are formal coordinates $(t, \hat{x})$ around $p$ such that

a. $\{\hat{x} = 0\} \subset D$ and $p = (0, 0)$.

b. $\Pi^* \left( \frac{1}{\hat{f}} \, \Omega \right) = \phi(\hat{x}) \, d\hat{x}$, where $\phi$ is of one of the following types:
1. \( \phi(\bar{x}) = \bar{x}^m \), where \( m \geq 0 \).
2. \( \phi(\bar{x}) = \lambda/\bar{x} \), where \( \lambda \in \mathbb{C}^* \).
3. \( \phi(\bar{x}) = (1 + \lambda \bar{x}^{\ell-1})/\bar{x}^\ell \), where \( \lambda \in \mathbb{C} \) and \( \ell \geq 2 \).

**Proof.** Since \( p \notin \text{Sing}(\mathcal{F}^*) \) and \( D \) is \( \mathcal{F}^* \)-invariant there exists a holomorphic coordinate system \( (U, (t, x)) \) such that:

i. \( D \cap U = (x = 0) \) and \( p = (0, 0) \).

ii. \( \mathcal{F}^*|_{U} \) is defined by \( dx = 0 \), or equivalently their leaves are the levels \( x = cte \).

Let \( \bar{f}_j := f_j \circ \Pi \) and \( \bar{g} = g \circ \Pi \). Since \( f_j(0) = 0 \) and \( p \notin \text{Sing}(\mathcal{F}^*) \) we must have \( \bar{f}_j(t, x) = x^{m_j}, u_j(t, x) \), where \( m_j \geq 1 \) and \( u_j \) is a formal unity, \( 1 \leq j \leq k \). Since \( \mathcal{F}^* \) is defined by \( dx = 0 \), we must have \( \Pi^* (\Omega) = (v(t, x), x^\mu \, dx) \), where \( \mu \geq 1 \) and \( v \) is a holomorphic unity. In particular, we get

\[ \Pi^* \left( \frac{1}{f} \Omega \right) = \frac{\tilde{v}}{x^\ell}, \]

where \( \tilde{v} = v/u_1^\ell \ldots u_k^\ell \) is a formal unity, and \( \ell = \sum_j r_j m_j - \mu \in \mathbb{Z} \). Since \( \frac{1}{f} \Omega \) is closed, we must have \( d\tilde{v} \wedge dx = 0 \), so that \( \tilde{v} = \tilde{v}(x) \).

1. If \( \ell = -m \leq 0 \) then \( \tilde{\phi}(x) dx = \tilde{v}(x), x^m \, dx \). Let \( H(x) = x^{m+1}, w(x) \) be a formal series with \( H'(x) = x^m \tilde{v}(x) \) and \( w \in \mathcal{O}_a^n \). Since \( w(0) \neq 0 \) there exists \( u \in \mathcal{O}_a^n \) with \( u^{m+1} = (m+1) \). If \( \bar{x} = x \) then \( H = (m+1)^{-1} \bar{z}^{m+1} \) and \( dH = \bar{z}^m \, d\bar{x} \).

2. If \( \ell = 1 \) then \( \tilde{\phi}(x) dx \) has a pole of order one and we can write

\[ \tilde{\phi}(x) = \frac{\lambda}{x} + \varphi(x), \]

where \( \lambda \in \mathbb{C}^* \) and \( \varphi(x) \) is a formal power series. If we set \( u(x) = \text{exp}(\varphi(x)/\lambda) \) then \( d\varphi = \lambda \frac{d\bar{x}}{x} \). In particular, if \( \bar{x} = u(x) \) then

\[ \tilde{\phi}(x) dx = \lambda \frac{d\bar{x}}{x}. \]

3. If \( \ell > 1 \) and \( \tilde{\phi}(x) = \sum_{j \geq 1} a_j x^j \) then we can write

\[ \tilde{\phi}(x) dx = \left( \frac{\lambda}{x} + \frac{\varphi(x)}{x^\ell} \right) dx = \frac{\varphi(x) + \lambda x^{\ell-1}}{x^\ell} dx, \]

where \( \lambda = a_{\ell-1} \) and \( \varphi(0) = a_0 \neq 0 \). In this case we consider the formal vector field \( X = \bar{x}^\ell \frac{\partial}{\partial x} \), for which \( i_X(\tilde{\phi}(x) dx) = 1 \). It is known that there exists \( \bar{x} = \psi(x) \in \text{Diff}(\mathcal{C}, 0) \) such that \( \psi^*(X) = \frac{\bar{x}^\ell}{\bar{x}^{\ell-1}} \frac{\partial}{\partial x} \) (see [17] and [6]). It can be checked that \( \psi^*(\tilde{\phi}(x) dx) = \frac{1+\lambda x^{\ell-1}}{x^\ell} d\bar{x} \), which proves the claim. □

**Claim 2.2.** Let \( h \in \text{Diff}(\mathcal{C}, 0) \) be such that \( h^*(\phi(x) dx) = \phi(x) dx \), where \( \phi(x) \) is like (1), (2) or (3) of claim [2.1]. Then:

i. If \( \phi \) is like in (1) then \( h(x) = \delta x \), where \( \delta^{m+1} = 1 \).

ii. If \( \phi \) is like in (2) then \( h(x) = \rho x \), \( \rho \in \mathbb{C}^* \).

iii. If \( \phi \) is like in (3) then \( h(x) = \delta \exp(t Z) \) for some \( t \in \mathbb{C} \), where \( \delta^{\ell-1} = 1 \) and \( Z = \frac{\bar{x}^\ell}{1+\lambda x^{\ell-1}} \frac{\partial}{\partial x} \).

In particular, in any case, the group \( G = \left\{ h \in \text{Diff}(\mathcal{C}, 0) \mid h^*(\phi(x) dx) = \phi(x) dx \right\} \) is abelian.
The proof of claim 2.2 can be found in [6] or [11].

Now, let \( p \in D \setminus \text{Sing}(F^*) \) and \( \Sigma \) be a parameterization section to \( F^* \) with \( p \in \Sigma \). Let \( x \in \mathbb{C}, 0 \mapsto \rho(x) \in (\mathbb{C}, p) \) be a parametrization of \( \Sigma \), so that, we can consider \( \text{Diff}(\mathbb{C}, \Sigma) = \text{Diff}(\mathbb{C}, 0) \) and the holonomy group of \( D \) in the section as a subgroup \( H \subset \text{Diff}(\mathbb{C}, 0) \). Let \( \tilde{x} = x u(x) \) be a formal change of variables as in claim 2.1.

The definition of holonomy and claim 2.2 implies that any \( h \in H \) satisfies \( h^*(\eta) = \eta \). In particular, if \( \Omega \) has a formal integrating factor then the holonomy group of \( D \) must be abelian, contradicting the hypothesis, which proves lemma 2.1.

Let us continue the proof of theorem 2.3.

**Claim 2.3.** Any \( \phi \in \widehat{\text{Isom}}(F) / \widehat{\text{Fix}}(F) \) has a semi-simple representative \( \phi_S \in \widehat{\text{Isom}}(F) \). Moreover:

- a. If \( \rho: \widehat{\text{Isom}}(F) \rightarrow \text{GL}(2, \mathbb{C}) \) is the group homomorphism \( \rho(\phi) = D\phi(0) \) then \( \ker(\rho) \subset \widehat{\text{Fix}}(F) \).
- b. If \( \phi \in \widehat{\text{Isom}}(F) \setminus \widehat{\text{Fix}}(F) \) then \( \rho(\phi) \) has finite order in \( \text{GL}(2, \mathbb{C}) \).

**Proof.** Given \( \phi \in \widehat{\text{Isom}}(F) \) let \( \phi = \phi_S \circ \phi_U \) be the the Jordan decomposition of \( \phi \), where \( \phi_S \) is semi-simple and \( \phi_U \) unipotent. As we have seen \( \phi_S, \phi_U \in \widehat{\text{Isom}}(F) \). By proposition 2.4 there exists a formal nilpotent vector field \( Y \in \widehat{X}_2 \) such that \( \phi_U = \exp(Y) \). On the other hand, by proposition 3. \( \phi_U \in \widehat{\text{Fix}}(F) \), because otherwise \( \Omega \) would have a formal integrating factor, which contradicts lemma 2.1. In particular, the classes of \( \phi \) and \( \phi_S \) in \( \widehat{\text{Isom}}(F) / \widehat{\text{Fix}}(F) \) are the same, which proves the first assertion of the claim.

Assume that \( \phi \in \widehat{\text{Isom}}(F) \) and \( D\phi(0) = I \) then there exists \( Y \in \widehat{X}_2 \) such that \( \phi = \exp(Y) \). Therefore, as above, \( \phi \in \widehat{\text{Fix}}(F) \). This proves (a).

It remains to prove (b). Let \( \phi \in \widehat{\text{Isom}}(F) \setminus \widehat{\text{Fix}}(F) \). By the first assertion we can suppose that \( \phi \) has a semi-simple representative in \( \widehat{\text{Isom}}(F) \), that we call \( \phi_S \). In this case, there exists \( \nu \in \text{Diff}(\mathbb{C}^2, 0) \) such that \( \nu^{-1} \circ \phi \circ \nu = D\phi(0) := L \) is linear and diagonal in some base of \( \mathbb{C}^2 \).

Suppose by contradiction that \( L \) has infinite order. Let \( \mathcal{H} \) be the Zariski closure in \( \text{GL}(2, \mathbb{C}) \) of the group \( H = \{ L^k \mid k \in \mathbb{Z} \} \). As we have seen in the proof of theorem 1 \( \mathcal{H} \) is abelian and \( dim(\mathcal{H}) \geq 1 \). Let \( \mathcal{H} = \varphi \circ \mathcal{H} \circ \varphi^{-1} \), so that \( \phi_S \in \mathcal{H} \). Since \( \phi_S \) has finite order in \( \widehat{\text{Isom}}(F) / \widehat{\text{Fix}}(F) \) there exists \( \ell \in \mathbb{N} \) such that \( \phi_S^n \in \widehat{\text{Fix}}(F) \), \( \forall n \in \mathbb{Z} \). Note that \( \mathcal{H} \) is also the Zariski closure of \( \{ L^n \mid n \in \mathbb{Z} \} \). As a consequence, if \( \mathcal{H} \) is the Lie algebra of \( \mathcal{H} \) and \( \mathcal{H} = \varphi_\ast \mathcal{H} \), then for any \( Y \in \mathcal{H} \) we have \( iY \Omega = 0 \) and \( \exp(Y) \in \widehat{\text{Fix}}(F) \). However, since \( L = \exp(A) \) for some \( A \in \mathcal{H} \), this would imply that \( \phi_S = \exp(\hat{A}) \in \widehat{\text{Fix}}(F) \), where \( \hat{A} = \varphi_\ast(A) \), a contradiction.

By the argument of the proof of claim 2.3 we can construct an injective representation \( \rho: \widehat{\text{Isom}}(F) / \widehat{\text{Fix}}(F) \rightarrow \text{GL}(2, \mathbb{C}) \): \( \rho(\phi) = D\phi_S(0) \). Denote \( G := \rho \left( \widehat{\text{Isom}}(F) / \widehat{\text{Fix}}(F) \right) \), the image of \( \rho \). Observe that any element \( T \in G \) has finite order, say \( o(T) \in \mathbb{N} \).
The idea is to prove that there exists $k \in \mathbb{N}$ such that $o(T) \leq k$, $\forall T \in G$. It is known that any subgroup of $GL(2, \mathbb{C})$ with this property is finite and this will finish the proof of theorem \[2\]

**Lemma 2.2.** There exists $k \in \mathbb{N}$ such that $o(T) \leq k$, $\forall T \in G$.

**Proof.** We will assume first that the resolution process of $\mathcal{F}$ involves just one blowing-up $\Pi: (M, D) \to (\mathbb{C}^2, 0)$. In this case, $D \simeq \mathbb{P}^1$ is not dicritical and contains at least three singularities of $\Pi^*(\mathcal{F}) := \mathcal{F}^*$, because the holonomy group of $D$ is not abelian. Let $\{p_1, ..., p_k\}, k \geq 3$, and $\ell = k!$. We assert that for any $T \in G$ then

$$T^\ell \in \{\lambda I | \lambda \in \mathbb{C}^*\}.$$  

In fact, fix $T \in G$ and $\phi \in \tilde{I}so(\mathcal{F})$ semi-simple such that $D\phi(0) = T$. Assume that $\mathcal{F}$ is represented by the germ of vector field $X$, with Taylor series $X = \sum_{j \geq n} X_j$, where $X_j$ is homogeneous of degree $j$, $\forall j$, and $X_n \neq 0$. From $\phi^*(X) = u X$, $u \in \mathcal{O}_2^*$, we get $T^*(X_n) = u(0) X_n$. The singularities $p_1, ..., p_k$ of $\mathcal{F}^*$ in $D$ correspond to the $X_n$-invariant directions of $\mathbb{C}^2$. In particular, $T$ induces a permutation of the set $\{p_1, ..., p_k\}$ and so $T^\ell(p_j) = p_j$, $1 \leq j \leq k$. A linear isomorphism of $\mathbb{C}^2$ that preserves more than two directions is of the form $\lambda I, \lambda \in \mathbb{C}^*$, which proves the assertion.

Let $H = \langle T^\ell | T \in G \rangle \subset \{\lambda I | \lambda \in \mathbb{C}^*\}$ and let $S \in H$ with $o(S) = r$.

**Claim 2.4.** We assert that there exists $h \in \text{Diff}^\infty(\mathbb{C}^2, 0)$ and $v \in \hat{\mathcal{O}}_2^*$ such that $h^*(v X) := \tilde{X}$ has the Taylor series of the form

$$\tilde{X} = \sum_{m \geq 1} X_{n+m,r}, \text{ where } X_n = X_n.$$  

In fact, let $\phi \in \tilde{I}so(\mathcal{F})$ be semi-simple with $D\phi(0) = S$. Let $\phi^*(X) = u X$, where $u \in \mathcal{O}_2^*$. Since $\phi$ is formally linearizable there exists $f \in \text{Diff}^\infty(\mathbb{C}^2, 0)$ such that $f^{-1} \circ \phi \circ f = S$. Let $Y = f^*(X)$, so that $S^*(Y) = \tilde{u} Y$, where $\tilde{u} = u \circ f$. Note that $u(0) = \delta$ where $\delta^r = 1$. This follows from

$$(S^j)^*(Y) = \tilde{u}_j Y, \quad \tilde{u}_j = \Pi_{i=0}^{j-1} \tilde{u} \circ S^i.$$  

Since $S^r = I$ we must have $\tilde{u}_r = 1 \implies \delta^r = \tilde{u}_r(0) = 1$.

We assert that there exists $\tilde{v} \in \tilde{\mathcal{O}}_2^*$ such that $\tilde{v}(0) = 1$ and

$$S^*(\tilde{v} Y) = \tilde{u}(0) \tilde{v} Y.$$  

As the reader can check, the existence of $\tilde{v}$ as in (8) is equivalent to solve the functional equation

$$(9) \quad \tilde{v} \circ S = \frac{\tilde{u}(0)}{\tilde{u}}. \tilde{v} := w \tilde{v}.$$  

Note that $\Pi_{j=0}^{r-1} w \circ S^j = 1$. If $\varphi \in \hat{\mathcal{O}}_2$ is such that $\exp(\varphi) = w$ then

$$\sum_{j=0}^{r-1} \varphi \circ S^j = 0$$  

The reader can check that relation (10) implies that it is possible to solve the functional equation

$$(10) \quad \theta \circ S - \theta = \varphi \implies \tilde{v} := \exp(\theta) \text{ is a solution of (9).}$$
We leave the details for the reader.

Set $\tilde{X} = \tilde{v} \cdot Y$. Let $\tilde{X} = \sum_{j \geq n} \tilde{X}_j$, where $\tilde{X}_j$ is homogeneous of degree $j$, $j \geq k$. Note that $\tilde{X}_n = X_n$.

We have seen that $S^*(\tilde{X}) = \delta \tilde{X}$, where $S = \lambda I$, $\lambda$ is a primitive $r$th root of the unity and $\delta^r = 1$. A direct computation shows that

$$S^*(\tilde{X}) = \sum_{j \geq k} \lambda^{j-1} \tilde{X}_j = \delta \tilde{X} = \delta \sum_{i = 0}^{r-1} Z_i,$$

where

$$Z_i = \sum_{m = 0}^{\infty} \tilde{X}_{i+1+m,r}, \text{ } 1 \leq i \leq r - 1.$$

Therefore,

$$\sum_{i = 0}^{r-1} (\lambda^i - \delta) Z_i \equiv 0.$$  

Since $Z_{n-1} = X_n + h.o.t. \neq 0$ we have $\delta = \lambda^{n-1}$ and $Z_i = 0$ if $i \neq n - 1$. In particular, we get

$$\tilde{X} = Z_{n-1} = X_n + \sum_{m \geq 1} \tilde{X}_{n+m,r},$$

which proves claim $\square$

Let $\tilde{X}_j = P_j \frac{\partial}{\partial x} + Q_j \frac{\partial}{\partial y}$, where $P_j$ and $Q_j$ are homogeneous of degree $j$. In the chart $\Pi(t, x) = (x, t x) = (x, y)$ of the blow-up $\Pi$ we obtain

$$\Pi^*(X_j) = x^{j-1} \left( f_j(t) \frac{\partial}{\partial t} + x P_j(1,t) \frac{\partial}{\partial x} \right),$$

where $f_j(t) = Q_j(1,t) - t P_j(1,t)$.

This implies $\Pi^*(\tilde{X}) = x^{n-1} \left[ x G(t, x^r) \frac{\partial}{\partial x} + F(t, x^r) \frac{\partial}{\partial t} \right]$, where

$$\begin{cases}
G(t, x^r) = \sum_{m \geq 0} f_{n+mr}(t) x^{mr} \\
F(t, x^r) = \sum_{m \geq 0} p_{n+mr}(1,t) x^{mr}.
\end{cases}$$

The strict transform of $\Pi^*(F)$ is therefore defined by the form $\Omega^* := x G(t, x^r) dt - F(t, x^r) dx$.

Let $\gamma : [0, 1] \to D \setminus \{p_1, ..., p_k\}$ be a $C^1$ closed curve with $\gamma(0) = \gamma(1) = t_o \in D$. The holonomy $h_\gamma$ of the curve $\gamma$ calculated in the section $(t = t_o)$ is the solution of the differential equation

$$(11) \quad \frac{dx}{ds} = x, \quad \frac{G(\gamma(s), x^r)}{F(\gamma(s), x^r)} \gamma'(s)$$

with initial condition $x(0) = x$. If $X(s, x)$ is this solution then $h_\gamma(x) = X(1, x)$.

Note that $h_\gamma$ is necessarily of the form

$$h_\gamma(x) = \lambda x (1 + H_\gamma(x^r)).$$

In fact, if we consider the ramification $y = x^r$ applied in (11) we obtain

$$\frac{dy}{ds} = r x^{r-1} \frac{dx}{ds} = r x^r \frac{G(\gamma(s), x^r)}{F(\gamma(s), x^r)} \gamma'(s) = r \frac{G(\gamma(s), y)}{F(\gamma(s), y)} \gamma'(s).$$

If $Y(s, y)$ is the solution of the above equation with initial condition $Y(0, y) = y$ then
(1). $Y(s, y) = y \cdot U(s, y)$ where

$$U(s, 0) = \exp \left( \int_0^s \frac{G(\gamma(s), 0)}{F(\gamma(s), 0)} \gamma'(s) \, ds \right).$$

(2). $X(s, x) = x \left( U(s, x^m) \right)^{1/r}$. This implies the assertion.

Note also that

$$h'_{\gamma}(0) = \exp \left( \int_{\gamma} \frac{G(z, 0)}{F(z, 0)} \, dz \right).$$

Since the holonomy of $D$ is non abelian, there are $\alpha, \beta \in \Pi_1(D \setminus \{p_1, \ldots, p_k\}, t_0)$ such that if $\gamma = [\alpha, \beta] := \alpha \beta \alpha^{-1} \beta^{-1}$, then $h_\gamma \neq id$, but is tangent to the identity. Therefore, the formal diffeomorphism is necessarily of the form

$$h_{\gamma}(x) = x + a \cdot x^{m \cdot r} + h.o.t.,$$

where $a \neq 0$ and $m \geq 1$.

The integer $m \cdot r$ is the order of tangency of $h_\gamma$ with the identity. It is a formal invariant of $h_\gamma$, in the sense that if $g = f^{-1} \circ h_\gamma \circ f$, where $f \in \text{Diff}(\mathbb{C}, 0)$ then the order of tangency of $g$ with the identity is also $m \cdot r$.

Suppose by contradiction that the assertion of lemma [2] is false. This implies that the set $\{o(T) \mid T \in G\}$ is unbounded, so that there is $T \in G$ such that $o(T) = r' > m \cdot r$. But, as we have seen above, this implies that the order of tangency of $h_\gamma$ with the identity is a multiple $m', r' > m \cdot r$, a contradiction. Therefore, the set $\{o(T) \mid T \in G\}$ is finite, as we wished.

Suppose now that the resolution process $\Pi: (M, E) \to (\mathbb{C}^2, 0)$ involves more than one blowing-up. Let $\text{Diff}(M, E)$ be the set of germs at $E$ of formal diffeomorphims of $M$.

We assert that for any $\phi \in \hat{\text{Iso}}(F)$ there is $\bar{\phi} \in \hat{\text{Diff}}(M, E)$ such that the diagram below commutes

$$
\begin{array}{ccc}
(M, E) & \xrightarrow{\bar{\phi}} & (M, E) \\
\Pi \downarrow & & \downarrow \Pi \\
(\mathbb{C}^2, 0) & \xrightarrow{\phi} & (\mathbb{C}^2, 0)
\end{array}
$$

(12)

The proof is done following the resolution process. To give an idea, we will do the two first steps. Since $\phi(0) = 0$, when we perform the first blow-up $\Pi_1: (M_1, D_1) \to (\mathbb{C}^2, 0)$ then there is $\phi_1 \in \hat{\text{Diff}}(M_1, D_1)$ that lifts $\phi$: $\Pi_1 \circ \phi_1 = \phi \circ \Pi_1$.

The formal diffeomorphism $\phi_1$ converges in the divisor and $\phi_1|_{D_1}$ is an automorphism of $D_1$. Moreover, it preserves the germ of the strict transform of $F$, say $F_1$, along $D_1$. In particular, it induces a permutation in the set of singularities $\text{Sing}(F_1) \subset D_1$.

Let $p_1 \in \text{Sing}(F_1)$ be a non-simple singularity of $F_1$. Its orbit by $\phi_1$ is periodic: $p_2 = \phi_1(p_1)$, $p_3 = \phi_1(p_2)$, ..., $p_i = \phi_1(p_{i-1})$. Since $\phi_1^i(F_1) = F_1$ the germs of $F_1$ at all $p_j$ are equivalent and to continue the blowing-up process we have to blow-up in all these singularities, thus obtaining another step of the process $\Pi_2: (M_2, E_2) \to (M_1, D_1)$, with $E_2 = \bar{D}_1 \cup D_1' \cup \ldots \cup D_{l}',$ where $\bar{D}_1$ is the strict transform of $D_1$ and $D_j' = \Pi_2^{-1}(p_j)$.

In this case, we can lift $\phi_1$ to a formal germ of diffeomorphism $\phi_2 \in \hat{\text{Diff}}(M_2, E_2)$ such that

$$\Pi_2 \circ \phi_2 = \phi_1 \circ \Pi_2.$$
(4). \( \phi_2|_{E_2} \) is an automorphism of \( E_2 \).

(5). \( \phi_2(D_1) = D_1 \) and \( \phi_2(D'_j) = D'_{j+1}, \forall 1 \leq j \leq \ell - 1 \).

(6). If \( \mathcal{F}_2 = \Pi^2_1(\mathcal{F}_1) \) then \( \phi^*_2(\mathcal{F}_2) = \mathcal{F}_2 \).

Continuing this process inductively, at the end we will find a germ of formal diffeomorphism \( \tilde{\phi} \in \text{Diff}(M, E) \) that makes the diagram \( \text{(12)} \) to commute. Moreover, if \( \tilde{\mathcal{F}} = \Pi^*(\mathcal{F}) \) then \( \tilde{\phi}^*(\tilde{\mathcal{F}}) = \tilde{\mathcal{F}} \). The restriction \( \tilde{\phi}^*|_E \) is an automorphism of \( E \) and permutes the irreducible components of \( E \) and also \( \tilde{\phi}(\text{Sing}(\tilde{\mathcal{F}})) = \text{Sing}(\tilde{\mathcal{F}}), \tilde{\mathcal{F}} = \Pi^*(\mathcal{F}) \).

Let \( D \) be an irreducible component of \( E \) with non abelian holonomy. Since \( E \) has a finite number \( k \) of irreducible components, if \( m = k! \) then for any \( \phi \in \text{Iso}(\mathcal{F}) \) we have \( \tilde{\phi}^m(D) = D \), where \( \tilde{\phi} \) lifts \( \phi \). Let \( \#(\text{Sing}(\tilde{\mathcal{F}}) \cap D) = k' \). Since the holonomy of \( D \) is non abelian we have \( k' \geq 3 \). In particular, if \( m' = k!k'! \) then \( \tilde{\phi}^{m'}|_D = \text{id}_D \), the identity of \( D \).

Assume that the divisor \( D \) was obtained blowing-up a singularity \( q \) in a previous step of the process. Taking local coordinates at \( q \) we can assume that the blowing-up is \( \Pi : (M, D) \rightarrow (\mathbb{C}^2, q = 0) \) and the germ at \( q \) of the previous foliation in the process was \( \mathcal{F}' \), so that \( \Pi^*(\mathcal{F}') \) is the germ of \( \mathcal{F} \) along \( D \). The group \( G := \langle \tilde{\phi}^m|_{M'} \mid \phi \in \text{Iso}(\mathcal{F}) \rangle \subset \text{Diff}(M', D) \) is a subgroup of \( \text{Iso}(\tilde{\mathcal{F}}|_{M'}) \).

Repeating the previous argument it can be shown that \( \text{Iso}(\tilde{\mathcal{F}}|_{M'})/\text{Fix}(\tilde{\mathcal{F}}|_{M'}) \) is finite. In particular, there exists \( k \in \mathbb{N} \) such that for any semi-simple \( \phi \in \text{Iso}(\mathcal{F}) \backslash \text{Fix}(\mathcal{F}) \) we have \( \tilde{\phi}^k|_{M'} = \text{id}_{M'} \), which implies that \( \tilde{\phi}^k = \text{id}_{M} \) and \( \phi^k = \text{id}_{\mathbb{C}^2} \), proving lemma \( \text{(22)} \) and theorem \( \text{(2)} \). \( \square \)

2.4. **Examples with formal, but without meromorphic integrating factor.**

The purpose of this section is to prove the following result:

**Theorem 3.** There exist germs of foliations \( \mathcal{F} \) on \( (\mathbb{C}^2, 0) \) with the following properties:

(a). \( \text{Iso}(\mathcal{F})/\text{Fix}(\mathcal{F}) \) is non-countable and contains elements of infinite order.

(b). \( \mathcal{F} \) is not liouvillé integrable: it cannot be defined by a closed meromorphic 1-form.

**Proof.** The proof is based in \( \text{(20)} \) and in the fact that any germ \( h \in \text{Diff}(\mathbb{C}, 0) \) can be realized as the holonomy of a separatrix of a germ holomorphic vector field \( X \in \mathcal{X}(\mathbb{C}^2, 0) \) such that \( DX(0) \) is linear and diagonal:

**Claim 2.5.** Let \( h \in \text{Diff}(\mathbb{C}, 0) \) be a germ of biholomorphism with \( h'(0) = \lambda \in \mathbb{C}^* \). Then there exists a holomorphic vector field \( X \in \mathcal{X}(\Delta), \Delta = \{(x, y) \in \mathbb{C}^2 \mid |x| < 2, |y| < \epsilon\} \) of the form

\[
X(x, y) = x \frac{\partial}{\partial x} + y b_h(x, y) \frac{\partial}{\partial y},
\]

where \( \lambda = e^{2\pi i \alpha_h}, \alpha_h = b_h(0, 0), \) and the holonomy of the curve \( \beta(t) = (e^{2\pi i t}, 0), t \in [0, 1], \) contained in the leaf \( (y = 0) \backslash \{(0, 0)\} \), in the transversal section \( \Sigma := (x = 1) \) is \( y \mapsto h(y) \).

**Proof.** When \( h \) is linearizable theorem \( \text{(35)} \) is immediate. When \( h \) is non-linearizable then \( |\lambda| = 1 \). When \( \lambda \) is a root of unity the proof can be found in \( \text{(18)} \), whereas when \( \lambda \) is not a root of unity the proof can be found in \( \text{(21)} \). \( \square \)
Let \( F = F_X \) be the germ of foliation defined by the vector field \( X \). The foliation \( F \) satisfies the following properties:

1. Outside the axis \( (x = 0) \) it is transverse to the vertical fibration \( (x = ct) \).
2. The leaf \( L_y \) of \( F \) through the point \((1, y)\) cuts the fiber \( (x = 1) \) exactly at the points of the form \((1, \tilde{y})\), where \( \tilde{y} \) belongs to the pseudo-orbit of \( h \): \( \tilde{y} = h^n(y), \ n \in \mathbb{Z} \).

More precisely, given \( n \in \mathbb{Z} \) let \( Dom(h^n) \) = the connected component of \( 0 \in \mathbb{C} \) of the set \( \{ y \in \mathbb{C} \mid |y| < \epsilon \} \) and \( h^j(y) \) is defined for all \( j \) with \( 0 \leq |n - j| \leq |n| \). If \( y \in Dom(h^n) \), then \((1, h^j(y)) \in L_y \cap (x = 1) \) for all \( j \) with \( 0 \leq |n - j| \leq |n| \).

We will assume also that \( \alpha_h \notin \mathbb{R}^+ \). With this condition, the saturation by \( F \) of the set \( \Sigma_\epsilon = \{(1, y) \mid |y| < \epsilon \} \) contains a set of the form \( D_\delta \times D_\epsilon \setminus \{ x = 0 \} \), where \( D_\epsilon \) denotes the disc \( \{ z \in \mathbb{C} \mid |z| < \epsilon \} \).

Given \( h_1 \in Diff(\mathbb{C}, 0) \) a germ of diffeomorphism commuting with \( h \), we will construct a germ \( \Phi_{h_1} \in Iso(F) \subset Diff(\mathbb{C}^2, 0) \). The construction will be done in such a way that:

3. \( \Phi_{h_1}(x, y) = (x, f(x, y)) \), so that \( \Phi_{h_1} \) preserves the fibers \( (x = ct) \).
4. \( \Phi_{h_1}(1, y) = (1, h_1(y)) \) \( \{ f(1, y) = h_1(y) \} \).

**Remark 2.2.** We will see that there is only one \( \Phi_{h_1} \in Iso(F) \) satisfying (3) and (4).

In order to formalize the construction of \( \Phi_{h_1} \) we consider the universal covering of \( \Delta \setminus \{ x = 0 \} \), \( \Pi: B \times D_\epsilon \rightarrow \Delta \setminus \{ x = 0 \} \),

\[ \Pi(z, y) = (e^z, y), \]

where \( B = \{ z \in \mathbb{C} \mid Re(z) \in (-\infty, log(2)) \} \) and \( \Delta = D_2 \times D_\epsilon \).

The pull-back foliation \( \tilde{F} := \Pi^*(F) \) is defined in \( B \times D_\epsilon \) by the vector field

\[ \tilde{X}(z, y) = \Pi^*(X)(z, y) = \frac{\partial}{\partial z} + y \cdot b_h(e^z, y) \frac{\partial}{\partial y}. \]

Note that this foliation has no holonomy, in the sense that if \( \tilde{L} \) is a leaf of \( \tilde{F} \) then it cuts any transversal \( (z = ct) \) in at most one point. Denote by \( \tilde{L}_y \) the leaf of \( \tilde{F} \) such that \( \tilde{L}_y \cap (z = 0) = \{ 0, y \} \).

Note that, by the definition of holonomy (of \( F \)), if \( y \in Dom(h^n) \) then \( \tilde{L}_y \cap (z = 2n \pi i, h^n(y)) \).

Now, we define the covering \( \tilde{\Phi}_{h_1} \) of (the future) \( \Phi_{h_1} \) as the germ along \( (y = 0) = B \times \{ 0 \} \) of map that satisfies:

5. \( \tilde{\Phi}_{h_1} \) preserves the fibers \( (z = ct) \): \( \tilde{\Phi}_{h_1}(z = z_0) \subset (z = z_0) \).
6. \( \tilde{\Phi}_{h_1}(\tilde{L}_y) = \tilde{L}_{h_1(y)} \).

Since \( \tilde{F} \) has no holonomy (5) and (6) define an unique germ of holomorphic diffeomorphism \( \tilde{\Phi}_{h_1} \) along \( B \times \{ 0 \} \) such that \( \tilde{\Phi}_{h_1}(B \times \{ 0 \}) = B \times \{ 0 \} \).

Now, we will see that there exists a germ \( \Phi_{h_1} \in Iso(F) \subset Diff(\mathbb{C}^2, 0) \) such that \( \Phi_{h_1} \circ \Pi = \Pi \circ \tilde{\Phi}_{h_1} \). By (5) and (6), the extension of \( \tilde{\Phi}_{h_1} \) to the fiber \( (t = t_0) \) is done
using the holonomy of \( \tilde{F} \), say \( H_t : (t = t_o) \to (t = 0) \), so that

\[
\tilde{F}_{h_1}(t_o, y) = (t_o, H_{t_o}^{-1}(h_1(H_{t_o}(y)))) .
\]

For instance, \( H_{2\pi i}(y) = h^{-1}(y) \) and so

\[
\tilde{F}_{h_1}(2\pi i, y) = (2\pi i, h(h_1(h^{-1}(y)))) = (2\pi i, h_1(y)) ,
\]

because \( h \) and \( h_1 \) commute. In particular,

\[
\Pi \circ \tilde{F}_{h_1}(2\pi i, y) = (1, h_1(y)) = \Pi(0, h_1(y)) = \Pi \circ \tilde{F}_{h_1}(0, y) .
\]

Similarly, \( \Pi \circ \tilde{F}_{h_1}(2k\pi i, y) = \Pi \circ \tilde{F}_{h_1}(0, y) \) for all \( k \in \mathbb{Z} \) and \( y \in \text{Dom}(h^k) \). This implies that we can define \( \Phi_{h_1}(1, y) = (1, h_1(y)) \) for all \( y \in \text{Dom}(h_1) \).

The extension of \( \Phi_{h_1} \) to the fibers \( (x = x_o) \), \( x_o \neq 0 \), can be done by using (13). We leave the details to the reader. It remains to prove that \( \Phi_{h_1} \) can be extended to the fiber \( (x = 0) \).

When \( \alpha = \alpha_h \notin \mathbb{R} \) then, by Poincaré’s linearization theorem, we can assume that \( X \) is linear. In this case, \( h(y) = \lambda y \), where \( \lambda = \exp(2\pi i \alpha) \). If \( h_1 \) commutes with \( h \) then \( h_1 \) is also linear and \( \Phi_{h_1}(x, y) = (x, h_1(y)) \), as the reader can check.

When \( \alpha \in \mathbb{R} \) and \( h \) is non-linearizable, then we have to use that the saturation of the transversal \( (x = 1) \) by \( F \) contains a set of the form \( D_\delta \times D_\epsilon \setminus (x = 0) \). In this case, it can be proved that \( \Phi_{h_1} \) is bounded in the set \( D_\delta \times D_\epsilon \setminus (x = 0) \) and so \( \Phi_{h_1} \) can be extended to \( \{0\} \times D_\epsilon \) by Riemann’s extension theorem (see [10]). We leave the details to the reader.

Now, we use a construction of Perez Marco. In [20] he proves the following result:

**Theorem.** There exists non-linearizable germs of diffeomorphisms \( h \in Diff(\mathbb{C}, 0) \) of the form \( h(y) = \lambda y + h.o.t., \) with \( |\lambda| = 1 \) and not a root of unity, whose centralizer

\[
C(h) = \{ g \in Diff(\mathbb{C}, 0) \mid g \circ h \circ g^{-1} \circ h^{-1} = Id \}
\]

is a Cantor set, in the sense that the set \( \{ g'(0) \mid g \in C(h) \} \) is a Cantor set of \( S^1 = \{ \mu \in \mathbb{C} \mid |\mu| = 1 \} \). In particular, \( C(h) \) is non-countable. In fact, in \( C(h) \) there are infinitely many elements of finite order and a non-countable set of elements of infinite order.

Now, we take a vector field \( X \) like in claim (2.3) associated to \( h \) like in Perez Marco’s theorem. The foliation \( F_X \), associated to \( X \), cannot be defined by a closed meromorphic 1-form, and \( Iso(F_X)/\text{Fix}(F_X) \) is non-countable.

This finishes the proof of theorem [11].

3. Conical and logarithmic foliations

In this section we study \( Iso(F) \) when \( F \) is a conical or a logarithmic foliation.

3.1. Preliminaries and statement of the results. A conical foliation \( F \) on \( \mathbb{C}^n \) induces a foliation of the same codimension, say \( \tilde{F} \), on the projective space \( \mathbb{P}^{n-1} \). Denote by \( Aut(\tilde{F}) \) the subgroup of \( Aut(\mathbb{P}^{n-1}) \), of automorphisms of \( \mathbb{P}^{n-1} \) preserving \( \tilde{F} \):

\[
Aut(\tilde{F}) = \left\{ \Phi \in Aut(\mathbb{P}^{n-1}) \mid \Phi^*(\tilde{F}) = \tilde{F} \right\} .
\]

We have the following:
Theorem 4. If $\mathcal{F}$ is conical and $\tilde{\mathcal{F}}$ is the foliation induced by $\mathcal{F}$ on the projective space $\mathbb{P}^{n-1}$ then $\text{Iso}(\mathcal{F})/\text{Fix}(\mathcal{F})$ and $\text{Iso}(\tilde{\mathcal{F}})/\tilde{\text{Fix}}(\tilde{\mathcal{F}})$ are isomorphic to $\text{Aut}(\tilde{\mathcal{F}})$.

When $\mathcal{F}$ is a conical homogeneous foliation of dimension one and the degree of the foliation $\tilde{\mathcal{F}}$, as a foliation of $\mathbb{P}^{n-1}$, is $d$ then the degree of the homogeneous form $\Omega$ on $\mathbb{C}^n$ inducing $\mathcal{F}$ is $d + 1$ (see [14]). Denote by $\text{Fol}(d, n - 1)$ the set of 1-dimensional foliations on $\mathbb{P}^{n-1}$ of degree $d$. It is known that $\text{Fol}(d, n - 1)$ can be identified with a Zariski open and dense subset of some $\mathbb{P}^{nN}$. As a consequence of theorem 4 we have the following:

Corollary 1. If $d \geq 2$ then $\text{Fol}(d, n - 1)$ contains a Zariski open and dense subset, say $G$, such that for any $\mathcal{G} \in G$ we have $\text{Iso}(\Pi^*(\mathcal{G})) = \text{Fix}(\Pi^*(\mathcal{G}))$, where $\Pi: \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is the canonical projection.

When $\Omega \in \Omega^1(\mathbb{C}^n)$ is homogeneous and non-conical, then it has a holomorphic integrating factor: if $f := i_\theta \Omega \neq 0$ then $f^{-1}\Omega$ is closed. In particular, if the decomposition of $f$ into irreducible factors is $f_1^{d_1}...f_r^{d_r}$, then

$$\frac{\Omega}{f} = \sum_{j=1}^r \lambda_j d\frac{f_j}{j} + d\left(\sum_{k=1}^r \frac{H}{f_1^{k_1-1}...f_r^{k_r-1}}\right),$$

where $\lambda_j \in \mathbb{C}$, $1 \leq j \leq r$, and $f_1,...,f_r,H$ are all quasi-homogeneous with respect to $S$. When $f = f_1...f_r$, is reduced, then in the above formula we have $H = 0$ and $\lambda_1...\lambda_r \neq 0$. In this case, the form $f^{-1}\Omega$ is logarithmic.

In section 3.3 we will consider germs of closed logarithmic 1-forms in general, that is when the divisor of poles is reduced and not necessarily quasi-homogeneous. In this case, if $\Omega$ has pole divisor $F = f_1...f_r$ then

$$\Omega = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j} + dh,$$

where $\lambda_1,...,\lambda_r \in \mathbb{C}^*$ and $h \in \mathcal{O}_n$.

Remark 3.1. Multiplying $f_1$ by an unity $u \in \mathcal{O}_n$, we can suppose that $dh = 0$. In fact, since $\lambda_1 \neq 0$, if we set $\tilde{f}_1 = u f_1$, where $u = \exp(-h/\lambda_1)$, then

$$\Omega = \lambda_1 \frac{df_1}{\tilde{f}_1} + \sum_{j \geq 2} \lambda_j \frac{df_j}{f_j}.$$

Other remarks about a 1-form $\Omega$ as above are the following

a. $\Omega$ has a holomorphic (or formal) non-constant first integral if, and only if, there exists $\lambda \in \mathbb{C}^*$ such that $(\lambda_1/\lambda,...,\lambda_n/\lambda) \in \mathbb{N}^n$.

b. $\Omega$ has a meromorphic (or formal meromorphic) non-constant first integral if, and only if, there exists $\lambda \in \mathbb{C}^*$ such that $(\lambda_1/\lambda,...,\lambda_n/\lambda) \in \mathbb{Z}^n$.

Another fact is the following:

Proposition 4. Let $\Omega$ be a germ at $0 \in \mathbb{C}^n$ of closed meromorphic 1-form, without non-constant meromorphic first integral. If $\Phi \in \text{Iso}(\mathcal{F}_\Omega)$ then $\Phi^*(\Omega) = \delta \Omega$, where $\delta$ is a root of unity.

Example 9. In general $\delta = 1$ in proposition 4 as we will see in the proof. Let us see an example in which $\delta = e^{2\pi i/3}$, we set $\Omega = \frac{dx}{y} + \delta \frac{dy}{x} + \delta^2 \frac{dz}{z}$ and $\Phi(x,y,z) = (z,x,y)$. As the reader can check we have $\Phi^*(\Omega) = \delta \Omega$. 


In the next result we give conditions implying that a logarithmic 1-form is holomorphically equivalent to a homogeneous form.

**Theorem 5.** Let $\Omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}$, where $f_1, \ldots, f_r$ are irreducible and $F_{\Omega}$ has no meromorphic first integral. Suppose also that:

(a) There exists $\Phi \in \text{Iso}(F_{\Omega})$ such that $D\Phi(0) = \rho.I$ and, either $\rho$ is not a root of unity, or the class of $\Phi$ in $\text{Iso}(F_{\Omega})/\text{Fix}(F_{\Omega})$ has infinite order.

(b) The first non-zero jet of $f_j$ is $J_0^{k_j}(f_j) = h_j$ and $\sum_{j=1}^{r} k_j \lambda_j \neq 0$.

Then there exists $\phi \in \text{Diff}(\mathbb{C}^n, 0)$ such that

$$\phi^*(\Omega) = \sum_{j=1}^{r} \lambda_j \frac{dh_j}{h_j}.$$ 

In particular, $F_{\Omega}$ is holomorphically equivalent to a homogeneous foliation.

As a consequence we have:

**Corollary 2.** Let $\Omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}$, where $f_1, \ldots, f_r \in \mathcal{O}_n$ are irreducible. Suppose further that:

(a) $F_{\Omega}$ has no meromorphic first integral.

(b) The first non-zero jet of $f_j$ is $J_0^{k_j}(f_j) = h_j$ and $\sum_{j=1}^{r} k_j \lambda_j \neq 0$.

(c) There exists a formal diffeomorphism $\psi \in \text{Diff}(\mathbb{C}^n, 0)$ such that $\psi^*(\Omega) = \Omega_h$, where $\Omega_h = \sum_{j=1}^{r} \lambda_j \frac{dh_j}{h_j}$.

Then there exists $\psi \in \text{Diff}(\mathbb{C}^n, 0)$ such that $\psi^*(\Omega) = \Omega_h$.

The proof of corollary 2 is based in the fact that if $T = \rho.I$ then $T \in \text{Iso}(F_{\Omega_h})$. This of course, implies that hypothesis (a) of theorem 5.

We state below a condition implying that for any $\Phi \in \text{Iso}(F_{\Omega})$ there exists $N \in \mathbb{N}$ such that $D\Phi^N(0) = \rho.I$, where $\rho \in \mathbb{C}^*$.

**Definition 3.** Let $H = \langle h_1, \ldots, h_\ell \rangle$ be a set of homogeneous polynomials, not necessarily of the same degree. Set

$$\mathcal{I}(H) = \{T \in GL(n, \mathbb{C}) \mid h_j \circ T = \alpha_j, h_j, \alpha_j \in \mathbb{C}^*, 1 \leq j \leq \ell\}.$$ 

Note that:

I. $\mathcal{I}(H)$ is a closed sub-group of $GL(n, \mathbb{C})$.

II. $\mathcal{I}(H) \supset \mathcal{C}.I = \{\rho.I \mid \rho \in \mathbb{C}^*\}$, where $I$ is the identity in $GL(n, \mathbb{C})$.

We say that $H = \langle h_1, \ldots, h_\ell \rangle$ is rigid if $\mathcal{I}(H) = \mathcal{C}.I$.

**Example 10.** If $H = \langle z_1, \ldots, z_n \rangle \subset \mathcal{O}_n$ then $\mathcal{I}(H) = \{T \mid T(z_1, \ldots, z_n) = (\lambda_1, z_1, \ldots, \lambda_n, z_n), \lambda_1, \ldots, \lambda_n \in \mathbb{C}^*\}$.

Two examples of rigid sets are $H_1 = \langle z_1, \ldots, z_n, z_1 + z_2 + \ldots + z_n \rangle$ and $H_2 = \langle z_1, z_2^2, z_3^n, z_4 + \ldots + z_n \rangle$.

**Corollary 3.** Let $\Omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}$ and assume that:

(a) $f_1, \ldots, f_r$ are irreducible and $F_{\Omega}$ has no meromorphic first integral.

(b) The first non zero jet of $f_j$ at $0 \in \mathbb{C}^n$ is $J_0^{k_j}(f_j) := h_j$, where $k_j \geq 1$, and the set $H := \langle h_1, \ldots, h_r \rangle$ is rigid.

(c) $\sum_{j=1}^{r} k_j \lambda_j \neq 0$.
There exists $\Phi \in \tilde{Iso}(\mathcal{F}_\Omega)$ whose class in $\tilde{Iso}(\mathcal{F}_\Omega)/\tilde{Fix}(\mathcal{F}_\Omega)$ has infinite order.

Then there exists $\phi \in Diff(C^n,0)$ such that $\phi^*(\Omega) = \sum_{j=1}^{r} \lambda_j \frac{dh_j}{h_j}$.

Another consequence of theorem 3 is the following:

**Corollary 4.** Let $\Omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}$ and assume that:

(a) $f_1, ..., f_r$ are irreducible and $\mathcal{F}_\Omega$ has no meromorphic first integral.

(b) The first non-zero jet of $f_j$ at $0 \in C^n$ is $J_0^1(f_j) = h_j$, where $k_j \geq 1$, and the set $H := (h_1, ..., h_r)$ is rigid.

(c) $h_1$ is not irreducible.

(d) $\sum_{j=1}^{r} k_j \lambda_j \neq 0$.

Then $Iso(\mathcal{F}_\Omega)/\tilde{Fix}(\mathcal{F}_\Omega) \simeq \tilde{Iso}(\mathcal{F}_\Omega)/\tilde{Fix}(\mathcal{F}_\Omega)$ and both are isomorphic to the same finite sub-group of $GL(n,C)$.

**Example 11.** An example for which $Iso(\mathcal{F}_\Omega) = Fix(\mathcal{F}_\Omega)$ and $\tilde{Iso}(\mathcal{F}_\Omega) = \tilde{Fix}(\mathcal{F}_\Omega)$ is $\Omega = \lambda_1 \frac{dt_1}{t_1} + \lambda_2 \frac{dt_2}{t_2} + \lambda_3 \frac{dt_3}{t_3} + \lambda_4 \frac{dt_4}{t_4}$, where $\lambda_j/\lambda_i \notin \mathbb{Q}$, $\forall i \neq j$, $f_1 = t_1^2 + t_2^2$, $f_2 = t_1$, $f_3 = t_3$ and $f_4 = t_1 + t_2$.

Another example was done in example 8.

In the case of dimension two we have the following:

**Corollary 5.** Let $\Omega = \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j} \in \Omega_1(C^n,0)$, where $f_1, ..., f_r \in O_2$ are all irreducible and relatively primes two by two. Suppose that:

(a) $\mathcal{F}_\Omega$ has no meromorphic first integral.

(b) The first non-zero jet of $f_j$ is $J_0^1(f_j) = h_j$, $1 \leq j \leq r$, and the set $(h_1, ..., h_r)$ is rigid.

(c) $\sum_{j=1}^{r} k_j \lambda_j \neq 0$.

(d) There exists $\Phi \in \tilde{Iso}(\mathcal{F}_\Omega)$ whose class in $\tilde{Iso}(\mathcal{F}_\Omega)/\tilde{Fix}(\mathcal{F}_\Omega)$ has infinite order.

Then $k_1 = ... = k_r = 1$, $h_1, ..., h_r$ are linear forms and there coordinates $(z_1, z_2)$ such that $\Omega$ is holomorphically equivalent to

$$\lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2} + \sum_{j=3}^{r} \lambda_j \frac{dh_j}{h_j}.$$ (15)

Corollary 5 is a direct consequence of corollary 3 and the fact that any homogeneous polynomial in $\mathbb{C}[z_1, z_2]$ can be decomposed into linear factors. We leave the details to the reader.

### 3.2. Conical foliations: proof of theorem 4

Let $\omega$ be a conic integrable $p$-form on $C^n$. We will assume that $\text{cod}(\text{Sing}(\omega)) \geq 2$. The form $\omega$ is homogeneous and defines a codimension $p$ foliation $\mathcal{F}_\omega$ on $C^n$ and a foliation $\mathcal{F}_\omega$ on $\mathbb{P}^{n-1}$. We want to prove that $\text{Aut}(\mathcal{F}_\omega) \simeq Iso(\mathcal{F}_\omega)/Fix(\mathcal{F}_\omega)$.

Given $\Phi \in Iso(\mathcal{F}_\omega)$ we denote as $\Phi$ its class in $Iso(\mathcal{F}_\omega)/Fix(\mathcal{F}_\omega)$.

**Lemma 3.1.** Let $\Phi \in \tilde{Iso}(\mathcal{F}_\omega)$ and $\Phi_1 = D\Phi(0)$ be the linear part of $\Phi$ at 0. Then:

(a) $\Phi_1 \in Iso(\mathcal{F}_\omega)$.

(b) $\Phi$ and $\Phi_1$ define the same class at $\tilde{Iso}(\mathcal{F}_\omega)/\tilde{Fix}(\mathcal{F}_\omega)$. 


Proof. The condition $\Phi \in \tilde{Iso}(F_\omega)$ is equivalent to $\Phi^*(\omega) = u.\omega$, where $u \in \hat{O}_n^\ast$. Since the coefficients of $\omega$ are homogeneous of the same degree, the first non-zero jet of $u.\omega$ is $u(0).\omega$, whereas the first non-zero jet of $\Phi^*(\omega)$ is $\Phi^*(\omega)$. Hence $\Phi^*(\omega) = u(0).\omega$ and $\Phi_1 \in Iso(F_\omega)$.

In particular, we can write $\Phi = \Phi_1 \circ \varphi$, where $\varphi \in \tilde{Iso}(F_\omega)$ and $D\varphi(0) = I$. We assert that $\varphi \in \hat{Fix}(F_\omega)$.

To see this, let us blow-up once at the origin $0 \in \mathbb{C}^n$. If we denote this blow-up by $\Pi: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ then the exceptional divisor $E = \Pi^{-1}(0) \simeq \mathbb{P}^{n-1}$ and there exists a germ $\hat{\varphi} \in Diff(\mathbb{C}^n, 0)$ such that $\Pi \circ \hat{\varphi} = \varphi \circ \Pi$.

Recall also that $\mathbb{C}^n$ is a linear bundle over $E$, say $P: \tilde{\mathbb{C}}^n \rightarrow E$, where $E$ is the zero section and the fibers $P^{-1}(a)$, $a \in E$, project by $\Pi$ onto the lines of $\mathbb{C}^n$ through the origin.

Let $\tilde{F}_\omega = \Pi^*(F_\omega)$. Since $F_\omega$ is conical with respect to the radial vector field $R = \sum_j z_j \frac{\partial}{\partial z_j}$, the lines through the origin of $\mathbb{C}^n$ are contained in the leaves or in the singular set of $F_\omega$, so that the fibers of $P$ are contained in the leaves or in the singular set of $\tilde{F}_\omega$. In particular the leaves and the singular set of $\tilde{F}_\omega$ are transverse to $E$.

Finally, we note that $\tilde{F}_\omega = i_E^*(\tilde{F}_\omega)$, where $i_E: E \rightarrow \tilde{\mathbb{C}}^n$ is the inclusion. In other words, $\tilde{F}_\omega$ can be viewed as the intersection of $\tilde{F}_\omega$ with the zero section $E$. Since $d\varphi(0) = I$ we get $\hat{\varphi}|_E = id_E$, the identity map of $E$, which implies that $\hat{\varphi}$ preserves the leaves and the singular set of $\tilde{F}_\omega$. This implies that $\varphi \in \hat{Fix}(F_\omega)$. \(\square\)

In particular, any $\tilde{\Phi} \in \tilde{Iso}(F)/\tilde{Fix}(F)$ has a linear representative $\Phi_1 \in GL(n, \mathbb{C})$.

On the other hand, any $T \in GL(n, \mathbb{C})$ induces an automorphism $\tilde{T} \in Aut(\mathbb{P}^{n-1})$. Let $Q: GL(n, \mathbb{C}) \rightarrow Aut(\mathbb{P}^{n-1})$ be the group homomorphism given by $Q(T) = \tilde{T}$, where $\tilde{T}$ is as before. Recall that $Q$ is surjective and

\begin{equation}
ker(Q) = \mathbb{C}^\ast \cdot I = \{ \rho \cdot I \mid \rho \in \mathbb{C}^\ast \}.
\end{equation}

Finally, we have seen in lemma 3.1 that $Q(Iso(F)) \subset Aut(\tilde{F})$ and that $\Phi_1 \in Fix(\tilde{F})$ if, and only if, $Q(\Phi_1) = id$, where $id$ is identity of $Aut(\mathbb{P}^{n-1})$. Therefore [10] implies that

$$Aut(F) \simeq \tilde{Iso}(F)/ker(Q|_{\tilde{Iso}(F)}) = \tilde{Iso}(F)/\tilde{Fix}(F) = Iso(F)/Fix(F).$$

This proves theorem [4] \(\square\)

3.3. Logarithmic foliations. In this section we prove the results stated before concerning the isotropy group of logarithmic foliations.

3.3.1. Proof of proposition [4]. Let $\omega \in \Omega^1(\mathbb{C}^n, 0)$ be integrable and having an integrating factor $f \in O_n$, so that, $\Omega := f^{-1} \cdot \omega$ is closed. We will assume that, $F_\omega$ has no meromorphic first integral and we want to prove that for any $\Phi \in \tilde{Iso}(F_\omega)$ then $\Phi^*(\Omega) = \delta \cdot \Omega$, where $\delta$ is a root of unity.

First of all, recall that $\Phi^*(\omega) = u.\omega$ where $u \in \hat{O}_n^\ast$. We assert that $\Phi^*(f) = v. f$, where $v \in \hat{O}_n^\ast$.
In fact, since \( f^{-1}, \omega \) is closed we have \( \frac{df}{f} \wedge \omega = d\omega \). Applying \( \Phi^* \) in both members of this relation we get

\[
\frac{d(f \circ \Phi)}{f \circ \Phi} \wedge \Phi^*(\omega) = d(\Phi^*(\omega)) \iff \left( \frac{d(f \circ \Phi)}{f \circ \Phi} - \frac{du}{u} \right) \wedge \omega = d\omega
\]

If we set \( g = \frac{f \circ \Phi}{u} \) then the above relation becomes

\[
\frac{dg}{g} \wedge \omega = d\omega \implies \left( \frac{dg}{g} - \frac{df}{f} \right) \wedge \omega = 0 \implies d(g/f) \wedge \omega = 0.
\]

Therefore, \( g/f \) is a meromorphic first integral of \( \mathcal{F}_\Omega \) and so \( f/g = c \), where \( c \in \mathbb{C}^* \).

Hence,

\[
\Phi^*(\Omega) = \Phi^* \left( \frac{\omega}{f} \right) = C. \Omega,
\]

where \( C = 1/c \).

It remains to prove that \( C \) is a root of unity. Let \( f = \Pi_{j=1}^r f_j^{k_j}, \ k_j \geq 1 \), be the decomposition of \( f \) into irreducible factors, so that

\[
\Omega = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j} + d\left( \frac{H}{f_1^{k_1-1} \cdots f_r^{k_r-1}} \right),
\]

where \( \lambda_1, \ldots, \lambda_r \in \mathbb{C} \). Note that:

i. \( \lambda_j \neq 0 \) for some \( j \), because otherwise \( \frac{df_j}{f_j} \) would be a meromorphic first integral. Without lost of generality, we will assume that \( j = 1 \).

ii. \( \Phi^* \) permutes the factors of \( f \); there exists a permutation \( \sigma \in S_r \) such that \( f_j \circ \Phi = u_j \cdot f_{\sigma(j)} \), where \( u_j \in \hat{O}_n^* \).

In particular, we have

\[
C. \Omega = \Phi^*(\Omega) = \sum_{j=1}^r \lambda_j \left( \frac{df_{\sigma(j)}}{f_{\sigma(j)}} + \frac{du_j}{u_j} \right) + d\Phi^* \left( \frac{H}{f_1^{k_1-1} \cdots f_r^{k_r-1}} \right)
\]

Comparing the residues, we get

\[
\lambda_{\sigma(i)} = C. \lambda_i, \ 1 \leq i \leq r.
\]

Let \( m \in \{1, \ldots, r\} \) be such that \( \sigma^m(1) = 1 \). From the above relation we get

\[
\lambda_1 = \lambda_{\sigma^m(1)} = C. \lambda_{\sigma^{m-1}(1)} = C^2. \lambda_{\sigma^{m-2}(1)} = \ldots = C^m. \lambda_1 \implies C^m = 1. \quad \Box
\]

**Remark 3.2.** It follows from the proof of proposition 4 that if \( \Phi^*(f_j) = u_j \cdot f_j \), \( \forall 1 \leq j \leq r \), then \( \delta = 1; \Phi^*(\Omega) = \Omega \).

3.3.2. **Proof of theorem** Let \( \Omega = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j} \) satisfying the hypothesis of theorem \( \Box \) and let \( \Phi \in \widetilde{Iso}(\mathcal{F}_\Omega) \) be such that \( D\Phi(0) = \rho \cdot I \), where either \( \rho \) is not a root of unity, or the class of \( \Phi \) in \( \widetilde{Iso}(\mathcal{F}_\Omega)/\text{Fix}(\mathcal{F}_\Omega) \) has infinite order. Since \( \Phi \) permutes the hypersurfaces \( (f_j = 0), \ 1 \leq j \leq r \), there exists \( N \leq r! \) such that \( \Phi^N(f_j = 0) = (f_j = 0), \ 1 \leq j \leq N \). Note that \( \Phi := \Phi^N \) fixes all hypersurfaces \( (f_j = 0) \) and \( D\Phi(0) = \rho^N \cdot I \).

**Lemma 3.2.** Let \( \phi \in \widetilde{Iso}(\mathcal{F}_\Omega) \) be such that \( D\phi(0) = I \). Then \( \phi \in \text{Fix}(\mathcal{F}_\Omega) \).
Proof. We will consider the more general case where \( D\phi(0) = \beta. I \).

Let \( \Pi: (\mathbb{C}^n, E) \rightarrow (\mathbb{C}^n, 0) \) be a blow-up at \( 0 \in \mathbb{C}^n \), where \( E \simeq \mathbb{P}^{n-1} \) is the exceptional divisor. Denote by \( \widetilde{F}_\Omega \) the foliation \( \Pi^*(\mathcal{F}_\Omega) \).

In the chart \( U_1 = \{(x, t) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid x \in \mathbb{C}, \ t = (t_2, \ldots, t_n) \in \mathbb{C}^{n-1} \} \subset \mathbb{C}^n \) we have
\[
\Pi(x, t) = (x, x t_2, \ldots, x t_n) = (x, x t) \in \mathbb{C}^n,
\]
and \( E \cap U_1 = \{x = 0\} \).

Moreover, since the order of \( f_j \) at \( 0 \in \mathbb{C}^n \) is \( k_j \), we can write
\[
f_j \circ \Pi(x, t) = x^{k_j} \tilde{f}_j(x, t),
\]
where \( \tilde{f}_j \) is the strict transform of \( f_j \) in this chart. Note that:

iii. The foliation \( \widetilde{F}_\Omega \) is defined in this chart by
\[
(17) \quad \tilde{\Omega} := \Pi^*(\Omega) = \alpha. \frac{dx}{x} + \sum_{j=1}^{r} \lambda_j \frac{df_j}{\tilde{f}_j},
\]
where \( \alpha = \sum_{j=1}^{r} k_j \). \( \lambda_j \neq 0 \).

In particular, \( \tilde{\Omega} \) is non-dicritical and \( \Lambda_E := E \setminus \bigcup_j (\tilde{f}_j = 0) \) is a leaf of \( \tilde{F}_\Omega \).

Let \( p = (0, t_0) \in \Lambda_E \) and fix. We assert that there exists a local chart \( (U, (\tilde{x}, t) \in (\mathbb{C}, 0) \times (\mathbb{C}^{n-1}, 0)) \) such that
\[
\tilde{\Omega}|_U = \alpha \frac{d\tilde{x}}{\tilde{x}}.
\]

In fact, since \( \tilde{f}_j(0, t_0) \neq 0, 1 \leq j \leq r \), the form \( \theta := \sum_j \lambda_j \frac{df_j}{f_j} \) is exact in a neighborhood \( U \) of \( (0, t_0) \). In particular, there exists \( h \in \mathcal{O}(U) \) such that \( \theta|_U = dh \).

Therefore, if \( \tilde{x} = x.e^h \) then \( \tilde{\Omega}|_U = \alpha \frac{d\tilde{x}}{\tilde{x}} \).

Now, there exists \( \tilde{\phi} \in Diff(\mathbb{C}^n, E) \) such that \( \Pi \circ \tilde{\phi} = \phi \circ \Pi \). The reader can check that in the chart \( (\tilde{U}, (\tilde{x}, t)) \) the transformation \( \tilde{\phi} \) is of the form
\[
\tilde{\phi}(\tilde{x}, t) = (g_1(\tilde{x}, t), g_2(\tilde{x}, t), \ldots, g_n(\tilde{x}, t)) = (g_1(x, t), t + \tilde{x}, H(\tilde{x}, t))
\]
where \( H \in \mathcal{O}(U) \). In particular, \( \tilde{\phi}|_E = id_E \), the identity map of \( E \).

On the other hand, by proposition \( 3 \) we have \( \phi^*(\Omega) = \delta \Omega \), where \( \delta \) is a root of unity. From \( \Pi \circ \tilde{\phi} = \phi \circ \Pi \) we get
\[
\tilde{\phi}^*(\tilde{\Omega}) = \tilde{\phi}\circ \Pi^*(\Omega) = (\Pi \circ \tilde{\phi})^*(\Omega) = \Pi^* \circ \phi^*(\Omega) = \Pi^* \delta \Omega = \delta \tilde{\Omega} \quad \Rightarrow \quad \\
\tilde{\phi}^*(\alpha. \frac{d\tilde{x}}{\tilde{x}}) = \delta \alpha. \frac{dx}{x} \quad \Rightarrow \quad \delta = 1 \text{ and } \frac{dg_1}{g_1} = \frac{d\tilde{x}}{\tilde{x}} \quad \Rightarrow \quad g_1(x, t) = \beta. x.
\]

In the chart \( (\tilde{x}, t) \) the leaves of \( \tilde{F}_\Omega \) are of the levels \( \tilde{x} = \text{constant} \). If \( \beta = 1 \) then \( \tilde{\phi}(\tilde{x}, t) = (\tilde{x}, t + x, H(\tilde{x}, t)) \), so that \( \tilde{\phi} \) preserves the leaves of \( \tilde{F}_\Omega \), which implies that \( \phi \in Fix(\mathcal{F}_\Omega) \). \qed

**Corollary 3.1.** Let \( \Phi \in \widehat{Iso}(\mathcal{F}_\Omega) \) such that \( D\Phi(0) = \rho. I \), where \( \rho \neq 1 \). Then the class of \( \Phi \) in \( \widehat{Iso}(\mathcal{F}_\Omega)/Fix(\mathcal{F}_\Omega) \) has a formally linearizable representative \( \widehat{\Phi} \).

**Proof.** Let \( \Phi \in \widehat{Iso}(\mathcal{F}_\Omega) \) be such that \( D\Phi(0) = \rho. I \). Let \( \Phi = \Phi_S \circ \Phi_U \) be the decomposition of \( \Phi \) as in proposition \( 2 \). Recall that \( \Phi_S, \Phi_U \in \widehat{Iso}(\mathcal{F}_\Omega) \), \( \Phi_S \) is formally linearizable and \( \Phi_U \) is unitary. Since \( \rho. I = D\Phi_S(0) \circ D\Phi_U(0) \) we have
the other hand, the Zariski closure of the group root of unity. 

linearization theorem: there exists an unique \( \Phi \) formally linearizable representative of \( \hat{g} \) for some \( k \). In particular, \( g^*(\Omega) \) is homogeneous.

Proof. It follows from lemma 3.2 that \( \rho \neq 1 \). Therefore by corollary 3.1, there is a formally linearizable representative of \( \Phi \) in \( Iso(F \Omega) \) say \( \Phi \). If \( \rho^k = 1 \), for some \( k \in \mathbb{N} \) then \( \Phi^k = I \), so that its class has finite order. Therefore, \( \rho \) is not a root of unity.

In particular, \( D\Phi(0) \) is non-resonant and \( \Phi \) is formally linearizable by Poincaré’s linearization theorem: there exists an unique \( g \in Diff(\mathbb{C}^n, 0) \) such that \( Dg(0) = I \) and \( g^{-1} \circ \Phi \circ g = D\Phi(0) \).

Let us prove (b). Set \( \Omega^* := g^*(\Omega) \), so that \( T^*\Omega^* = \Omega^* \) by proposition [4]. On the other hand, the Zariski closure of the group \( H = \{ T^0 | n \in \mathbb{Z} \} \) is the group \( \mathbb{C}^*. I \subset GL(n, \mathbb{C}) \). This implies that, if \( T_\tau = e^{\tau}. I = exp(\tau R) \) then

\[
exp(\tau R)^* \Omega^* = \Omega^*, \quad \forall \tau \in \mathbb{C}^* \quad \implies \quad L_R \Omega^* = \frac{d}{d\tau} (exp(\tau R)^* \Omega^*)|_{\tau = 0} = 0 .
\]

Let us finish the proof that \( \Omega^* = \sum_{j=1}^r \lambda_j \frac{d h_j}{h_j} \). Set \( f_j^* := f_j \circ g \in \hat{\mathcal{O}}_\Omega \), \( 1 \leq j \leq r \), so that \( \Omega^* = \sum_{j=1}^r \lambda_j \frac{d f_j^*}{f_j^*} \). Note that \( j_0^0 j_0^1 f_j^* = h_j \), because \( Dg(0) = I \). Now,

\[
L_R \Omega^* = i_R d\Omega^* + d i_R \Omega^* = d i_R \Omega^* = 0 \quad \implies \quad i_R \Omega^* = c \quad \iff \quad \sum_{j=1}^r \lambda_j f_j^* \ldots R(f_j^*) \ldots f_r^* = c f_1^* \ldots f_r^* \quad \implies \quad R(f_j^*) = v_j, f_j^*, v_j \in \hat{\mathcal{O}}_\Omega ,
\]

because \( f_j^* \) is irreducible, \( 1 \leq j \leq r \). As the reader can check, the last relation implies that \( h_j \) divides \( f_j^* \): \( f_j^* = v_j, h_j \). Therefore,

\[
\Omega^* = \sum_{j=1}^r \lambda_j \frac{d h_j}{h_j} + \Theta , \quad \Theta = \sum_j \lambda_j \frac{d v_j}{v_j} \in \hat{\mathcal{O}}^1(\mathbb{C}^n, 0) .
\]

Since \( h_j \) is homogeneous of degree \( k_j \) we have \( R(h_j) = k_j h_j \) by Euler’s identity, so that \( L_R \left( \frac{d h_j}{h_j} \right) = 0 \). This implies that \( L_R \Theta = 0 \) and so \( \Theta = 0 \).

If \( \Phi \) is holomorphic and \( |\rho| \neq 1 \) or if \( |\rho| = 1 \) and \( \rho \) satisfies a small denominator condition then \( g \in Diff(\mathbb{C}^n, 0) \) (cf. [2]) and we are done.

In the general case however, when \( \Phi \) is only formal, the idea is to prove that there exists a holomorphic vector field \( \bar{R} \in \mathcal{X}(\mathbb{C}^n, 0) \) with \( D\bar{R}(0) = R \), the radial vector field, with \( i_{\bar{R}}\Omega = \alpha = \sum_j k_j \lambda_j \). In the proof of this fact, we will use Artin’s approximation theorem.

Let us finish the proof of theorem [5] assuming the existence of a vector field \( \bar{R} \) as above. If \( i_{\bar{R}}\Omega = \alpha \) then \( L_{\bar{R}}\Omega = i_{\bar{R}}d\Omega + d i_{\bar{R}}\Omega = 0 \), because \( \Omega \) is closed and \( i_{\bar{R}}\Omega \) is a constant.
On the other hand, since $D\tilde{R}(0) = R$, by Poincaré’s linearization theorem (cf. [2]), the vector field $\tilde{R}$ is holomorphically linearizable: there exists $\phi \in Diff(C^n,0)$ such that $D\phi(0) = I$ and $\phi^*(\tilde{R}) = R$. We assert that $\phi^*(\Omega)$ is homogenous:

$$\phi^*(\Omega) = \sum_{j=1}^{r} \lambda_j \frac{dh_j}{h_j}.$$ 

In fact, set $\tilde{f}_j = \phi^*(f_j)$ and $\tilde{\Omega} = \phi^*\Omega$. Then

$$R(\tilde{f}_j) = u_j \tilde{f}_j, \quad u_j \in \mathcal{O}_n \quad \text{and} \quad u_j(0) = k_j.$$ 

Writing the Taylor series of $\tilde{f}_j$ and of $u_j$, by an induction argument, we obtain that there exists an unity $v_j \in \mathcal{O}_n^*$ such that $\tilde{f}_j = v_j, h_j$. In particular, we get

$$\Omega_1 = \sum_{j=1}^{r} \lambda_j \frac{d\tilde{f}_j}{\tilde{f}_j} + \Theta,$$ 

where $\Theta = \sum_{j} \lambda_j \frac{dv_j}{v_j} \in \Omega^1(C^n,0)$.

Since $h_j$ is homogeneous, $1 \leq j \leq r$, we have

$$L_{\tilde{R}} \left( \sum_{j} \lambda_j \frac{d\tilde{f}_j}{\tilde{f}_j} \right) = 0 \implies L_{\tilde{R}} \Theta = 0 \implies \Theta = 0,$$

because $\Theta$ is holomorphic. Therefore, $\tilde{\Omega}$ is homogeneous.

It remains to prove the existence of $\tilde{R}$ with $i_{\tilde{R}} \Omega = \alpha$. We have proved that there exists $g \in \tilde{Diff}(C^n,0)$ such that

$$g^*(\Omega) = \sum_{j=1}^{r} \lambda_j \frac{dh_j}{h_j} := \Omega^*.$$ 

In particular, if $R$ is the radial vector field on $C^n$ then

$$i_{\tilde{R}} \Omega^* = \sum_{j=1}^{r} \lambda_j \frac{R(h_j)}{h_j} = \alpha.$$ 

Let $\tilde{R} = g_* R \in \tilde{X}(C^n,0)$ and note that $i_{\tilde{R}} \Omega = \alpha$, because $i_{\tilde{R}} g^* \Omega = \alpha$. In fact, let $\tilde{R} = \sum_{j=1}^{n} \varphi_j \frac{\partial}{\partial z_j}$, where $\varphi_1, ..., \varphi_n \in \tilde{O}_n$. Writing explicitly the relation $i_{\tilde{R}} \Omega = \alpha$ we get

$$\sum_{j=1}^{r} \lambda_j \frac{\tilde{R}(f_j)}{f_j} = \alpha \implies \sum_{j=1}^{r} f_1 ... \tilde{R}(f_j) ... f_r = \alpha. f_1 ... f_r \implies$$

$$\implies \sum_{1 \leq j \leq r} \varphi_j. f_1 ... \frac{\partial f_j}{\partial z_{i_1}} ... f_r = \alpha. f_1 ... f_r.$$ 

In particular, $\varphi = (\varphi_1, ..., \varphi_n) \in \tilde{O}_n^*$ is a formal solution of the analytic equation $F(z,w) = 0$, where

$$F(z,w) = \sum_{1 \leq j \leq r} \sum_{1 \leq i \leq n} w_j. f_1(z) ... \frac{\partial f_j(z)}{\partial z_{i_1}} ... f_r(z) - \alpha. f_1(z)... f_r(z).$$
It follows from Artin’s approximation theorem that \( F(z, w) = 0 \) has a convergent solution \( w = \phi(z) = (\phi_1(z), ..., \phi_n(z)) \) such that \( J^1_0(\phi) = J^1_0(\varphi) \). Since \( j^1_R(R) = R \), the radial vector field, we can conclude that the vector field \( \vec{R} = \sum_{j=1}^n \phi_j \frac{\partial}{\partial z_j} \) satisfies \( i^*_R \Omega = \alpha \) and \( D \vec{R}(0) = R \), as desired. This finishes the proof of theorem.

3.3.3. Proof of corollaries \( \text{[X]} \) and \( \text{[X]} \)

**Proof of corollary \( \text{[X]} \)** Let us sketch how theorem \( \text{[X]} \) implies corollary \( \text{[X]} \). Let \( \Phi \in \widetilde{Iso}(\mathcal{F}_\Omega) \) whose class in \( \widetilde{Iso}(\mathcal{F}_\Omega)/\widetilde{Fix}(\mathcal{F}_\Omega) \) has infinite order. Since \( \mathcal{F}_\Omega \) has no formal meromorphic first integral, then \( \Phi \) permutes the set of hypersurfaces \( (f_j = 0) \), \( 1 \leq j \leq r \). In particular, there exists \( N \in \mathbb{N} \) such that \( f_j \circ \Phi^N = u_j, f_j \), where \( u_j \in \mathcal{O}_n \). If \( T = D \Phi^N(0) \) then \( h_j \circ T = u_j(0), h_j, \forall \ 1 \leq j \leq r \). Since \( (h_1, ..., h_r) \) is rigid we must have \( D \Phi^N(0) = \rho I \). In particular \( \Phi := \Phi^N \) satisfies hypothesis (a) of theorem \( \text{[X]} \) as wished. This finishes the proof of corollary \( \text{[X]} \).

**Proof of corollary \( \text{[X]} \)** First of all we would like to observe that there exists \( N \in \mathbb{N} \) such that for any \( \Phi \in \widetilde{Iso}(\mathcal{F}_\Omega) \) then \( D \Phi^N(0) = \rho I \), where \( \rho \) is a root of unity.

In fact, there exists \( N \in \mathbb{N} \) with \( N \leq r! \) and such that \( f_j \circ \Phi^N = u_j, f_j, 1 \leq j \leq r \). Since the set \( (h_1, ..., h_r) \) is rigid, we have \( D \Phi^N(0) = \rho I \). If \( \rho \) was not a root of unity then by theorem \( \text{[X]} \) there would exist \( \psi \in \widetilde{Diff}(\mathbb{C}^n, 0) \) such that \( f_1 \circ \psi = u_1, h_1 \). However, this is impossible, because \( f_1 \) is irreducible and \( h_1 \) is not.

**Claim 3.1.** Let \( \Sigma = \{ k \in \mathbb{N} \mid \text{ there exists a primitive } k^{th}\text{-root of unity } \rho \text{ such that } D \Phi(0) = \rho I, \text{ for some } \Phi \in \widetilde{Iso}(\mathcal{F}_\Omega) \text{ such that } f_j \circ \Phi = u_j, f_j, \forall j \} \). We claim that \( \Sigma \) is bounded.

**Proof.** We will use that \( f_1 \) is irreducible and \( h_1 \) is not. First of all, we assert that for any \( \rho \in X \) there exists \( \Phi \in \widetilde{Iso}(\mathcal{F}_\Omega) \) such that \( D \Phi(0) = \rho I \) and \( \Phi \) is formally linearizable.

Fix \( \ell \in \Sigma \), so that there is \( \Phi \in \widetilde{Iso}(\mathbb{C}^n, 0) \) with \( D \Phi(0) = \rho I \), where \( \rho \) is a \( \ell^{th}\text{-root of unity and } f_j \circ \Phi = u_j, f_j, 1 \leq j \leq r \). Let \( \Phi = \Phi_S \circ \Phi_U \), where \( \Phi_S \) is semi-simple, \( \Phi_U \) unitary and \( \Phi_S \circ \Phi_U = \Phi_U \circ \Phi_S \). As we have seen in remark \( \text{[X]} \) \( \Phi_S, \Phi_U \in \widetilde{Iso}(\mathcal{F}_\Omega) \). On the other hand, \( D \Phi_U(0) = I \) and \( D \Phi_S(0) = \rho I \). Since \( \Phi_S \) is linearizable, this proves the assertion.

Let \( \ell \in \Sigma \) and \( \Phi \in \widetilde{Fix}(\mathcal{F}_\Omega) \) be formally linearizable with \( D \Phi(0) = \rho I \). Since \( \Phi \) is linearizable there exists \( \psi \in \widetilde{Diff}(\mathbb{C}^n, 0) \) such that \( \psi^{-1} \circ \Phi \circ \psi = \rho I \). Let \( f^*_1 = f_1 \circ \psi \). We assert that, if \( \ell > k_1 \) then \( h_1 \) divides the \((\ell + k_1 - 1)^{th}\)-jet of \( f^*_1 \).

In fact, let \( f^*_1 = \sum_{i \geq k_1} g_i \) be the Taylor series of \( f^*_1 \), where \( g_i \) is homogeneous of degree \( i \). Note that \( g_{k_1} = h_1 \), because \( D \psi(0) = I \). Since \( f_1 \circ \Phi = u_1, f_1 \), where \( u_1 \in \mathcal{O}_n^* \), we have

\[
(18) \quad f^*_1 \circ (\rho I) = u^*_1 \cdot f^*_1,
\]

where \( u^*_1 = u_1 \circ \psi \). Let \( u^*_1 = \sum_{i \geq 0} w_i \) be the Taylor series of \( u^*_1 \). Relation \( (18) \) can be written as

\[
\sum_{i \geq k_1} \rho^i g_i = \sum_{i \geq k_1} \sum_{s=k_1}^i w_{i-s} \cdot g_s \quad \implies \quad w_0 = \rho^{k_1} \text{ and } (\rho^i - \rho^{k_1}) g_i = \sum_{s=k_1}^{i-1} w_{i-s} \cdot g_s, \forall i > k_1
\]
Since \( \rho \) is a primitive \( \ell \)th root of unity, \( \rho^i - \rho^{k_1} \neq 0 \) if \( k_1 + 1 \leq i \leq \ell + k_1 - 1 \). Therefore, the above relation implies that \( h_1 | g_j \) if \( k_1 \leq i \leq \ell + k_1 - 1 \), as the reader can check. Hence, \( h_1 \) divides the \((\ell + k_1 - 1)\)th jet of \( f_1^* \), as asserted.

Now, suppose by contradiction that \( \Sigma \) was unbounded. In this case, since \( h_1 \) is reducible, the above argument implies that for any \( k \in \mathbb{N} \) there exists \( \psi \in Diff(C^n,0) \) such that \( J^k_0(f_1 \circ \psi) \) is reducible. On the other hand, we have

\[
J^k_0(f_1 \circ \psi) = J^k_0(f_1 \circ J_0^k \psi) \implies J^k_0 f_1 \text{ is reducible } \mod m_n^{k+1},
\]

in the sense that there exist \( f, g \in \mathcal{O}_n \) such that \( f(0) = g(0) = 0 \) and \( J^k_0 f_1 = J^k_0(f, g) \). Therefore, for any \( k \in \mathbb{N} \) then \( J^k_0(f_1) \) is reducible \( \mod m_n^{k+1} \), and this implies that \( f_1 \) is reducible. This proves the claim. \( \square \)

Now, let \( \Delta : \overset{\sim}{Iso}(\mathbb{F}_\Omega) \rightarrow GL(n, \mathbb{C}) \) be the homomorphism \( \Delta(\Phi) = D\Phi(0) \). Note that lemma 3.2 implies that \( ker(\Delta) \subset \overset{\sim}{Fix}(\mathbb{F}_\Omega) \). We assert that, in fact \( ker(\Delta) = \overset{\sim}{Fix}(\mathbb{F}_\Omega) \).

In fact, if \( \Phi \in \overset{\sim}{Fix}(\mathbb{F}_\Omega) \) then \( f_j | f_j \circ \Phi, 1 \leq j \leq r \), because \( (f_j = 0) \) is a leaf of \( \mathbb{F}_\Omega \). This implies that \( D\Phi(0) \in I(h_1, ..., h_r) \) (see definition 3). Since \( (h_1, ..., h_r) \) is rigid, we \( \Delta(\Phi) = I \). Hence, \( ker(\Delta) = \{ I \} \).

This implies that \( \overset{\sim}{Iso}(\mathbb{F}_\Omega)/\overset{\sim}{Fix}(\mathbb{F}_\Omega) \simeq Im(\Delta) \). On the other hand, since \( X \) is bounded, the subgroup \( G := \mathbb{C}^* I \cap Im(\Delta), \) of \( Im(\Delta) \), is finite. This subgroup is contained in the center of \( Im(\Delta) \), and so is a normal subgroup.

In order to conclude the proof of corollary 4 it is sufficient to prove that \( Im(\Delta)/G \) is finite. However, this group can be identified with a subgroup of the group \( S_r \) of permutations of \( \{1, ..., r\} \).

In fact, as we have seen before, if \( \Phi \in \overset{\sim}{Iso}(\mathbb{F}_\Omega) \) then there exists a permutation \( \sigma \in S_r \) such that \( f_j \circ \Phi = u_j f_{\sigma(j)}, \forall j \). Moreover, \( \Phi \in G \) if, and only if, \( \sigma = id \), which proves the assertion. This finishes the proof of corollary 4. \( \square \)

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