Relating Green’s Functions in Axial and Lorentz Gauges using Finite Field-Dependent BRS Transformations

Satish. D. Joglekar,* A. Misra †
Department of Physics, Indian Institute of Technology,
Kanpur 208 016, UP, India

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Abstract

We use finite field-dependent BRS transformations (FFBRS) to connect the Green functions in a set of two otherwise unrelated gauge choices. We choose the Lorentz and the axial gauges as examples. We show how the Green functions in axial gauge can be written as a series in terms of those in Lorentz gauges. Our method also applies to operator Green’s functions. We show that this process involves another set of related FFBRS transformations that is derivable from infinitesimal FFBRS. We suggest possible applications.

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1 Introduction

Strong, weak and electromagnetic interactions are known to be described very well by the standard model [SM] [1] which is a nonabelian gauge the-
ory. Calculations in nonabelian gauge theories require a choice of gauge. These can be chosen in many ways. There are many families of gauges that have been used in practical calculations. Lorentz-type gauges have been used in a large number of calculations in SM on account of their covariance and availability of a free gauge parameter that helps in the check of gauge-independence. Another family of gauges, the axial gauges, \( \eta \cdot A = 0 \) have also been used extensively. These have the formal advantage of being free of ghosts, which leads to simplifications in calculations of Green’s functions, anomalous dimensions, etc. Special cases such as those with \( \eta^2 = 0 \), viz the light cone gauges have been used extensively in perturbative QCD calculations. Analogous gauges, the planar gauges have the advantages of the axial gauges and in addition have simpler propagator and hence have also found favor. The radial gauges have found widespread use in the context of QCD sum rules and operator product expansions in QCD. Certain quadratic gauges have been found to simplify Feynman rules and calculations of diagrams in spontaneously broken gauge theories (SBGT). \( R_\xi \) gauge have extensively been used in performing practical calculations and in formal arguments in SBGT. Thus, to summarize, various descriptions of gauge theories have been found useful in various different contexts. It therefore becomes an important question how the calculations in various (families of) gauge choices are related to each other.

Now, we expect the physical results to be independent of the choice of gauge. Indeed, gauge-independence in a limited framework, has been proven in early days. For example, within the Lorentz-type of gauges, one establishes the \( \lambda \)-independence of the physical observable, etc. Such proofs utilize the infinitesimal gauge transformations responsible for gauge-parameter change. Ways of connecting Green’s functions in a family of gauges (and establishing explicitly gauge-independence) has not been done until recently. Indeed, recently discrepancies have been reported in anomalous dimension calculations in the Lorentz-type and axial-type gauges.

Thus, for these and further reasons detailed below, we consider it valuable to obtain a procedure to connect the Green’s functions in different families of gauges. Certain progress along these lines has already been made. In \([1]\), we established a general procedure for obtaining a field transformation that connects the vacuum-to-vacuum amplitude \( W \) (and also the vacuum-expectation values of gauge invariant observables) in two sets of gauges. This was elaborated by a set of examples given there. These transformations turned out to
be a generalization of the usual BRS transformations in which the anticommuting global parameter is (i) field-dependent, but $x$-independent, and (ii) finite rather than infinitesimal. These were thus named finite field-dependent BRS (FFBRS) transformations. In view of the importance of the two families of gauges, viz the Lorentz-type and the axial-type practical in calculations, we established a similar FFBRS connections between these set of gauges [10], [11].

In this work, we establish a connection between arbitrary Green’s functions (or operator Green’s functions) in two sets of gauges, and in view of their practical importance, we choose these to be the Lorentz and the axial-type gauges. Of course, once an FFBRS is established between any two sets of gauges, an identical procedure would go through. We show that the required procedure involves another FFBRS. We establish finally a compact result expressing an arbitrary Green’s function/ operator Green’s function in axial gauges with a closed expression involving similar Green’s functions in Lorentz gauges. The expression can then be evaluated in principle, as a power series in $g$ to desired order.

We shall mention in passing a number of applications of this result. We can use the result for the axial gauge propagator in terms of the Lorentz gauge propagator as a way for obtaining the prescription for the $\frac{1}{g^2}$-singularity. This is so since we understand how to deal with the Green’s functions in Lorentz-type gauges. We should also be able to eliminate the possible reported discrepancy between the anomalous dimensions of physical observables [6] in the two sets of gauges. Such and other possible applications are under progress.

We now summarize the plan of the paper. In Section 2, we review the background needed together with the results of references [9]-[11] on FFBRS transformations. In Section 3, we show how Green’s functions in the two sets of gauges can be related. We show how this involves the use of another FFBRS. Appendix A deals with FFBRS along the lines of [9]. Section 3 gives a compact formula relating the Green’s functions in the two gauges. Section 4 gives a simple example of the compact formula obtained in Section 3. Section 5 deals with some future intended applications and conclusions.
2 Summary of Results on FFBRS Transformation between Lorentz and Axial-type Gauges

2.1 Notations and Conventions

We start with the Faddeev-Popov effective action (FPEA) in linear Lorentz-type gauges:

\[ S_{\text{eff}}^{L}[A, c, \overline{c}] = \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^{\alpha} F^{\alpha,\mu\nu} \right) + S_{\text{gf}} + S_{\text{gh}}, \]

(1)

where the gauge-fixing action \( S_{\text{gf}} \) is given by:

\[ S_{\text{gf}}^{L} = -\frac{1}{2\lambda} \int d^4x \sum_{\alpha} (\partial \cdot A^{\alpha})^2 \equiv -\frac{1}{2\lambda} \int d^4x \sum_{\alpha} (f_{L}^{\alpha}[A])^2, \]

(2)

and the ghost action \( S_{\text{gh}} \) is given by:

\[ S_{\text{gh}}^{L} = -\int d^4x \overline{c}^{\alpha} M^{\alpha\beta} c^{\beta}, \]

(3)

where

\[ M^{\alpha\beta}[A(x)] \equiv \partial^{\mu} D^{\alpha\beta}_{\mu}(A, x). \]

(4)

The covariant derivative is defined by:

\[ D^{\alpha\beta}_{\mu} \equiv \delta^{\alpha\beta}\partial_{\mu} + gf^{\alpha\beta\gamma}A^{\gamma}_{\mu}. \]

(5)

In a similar manner, the FPEA in axial-type gauges, is given by:

\[ S_{\text{gf}}^{A} \equiv -\frac{1}{2\lambda} \int d^4x \sum_{\alpha} (\eta \cdot A^{\alpha})^2 \equiv -\frac{1}{2\lambda} \sum_{\alpha} \int d^4x (f^{\alpha}[A])^2. \]

(6)

We require \( \eta_{\mu} \) to be real, but otherwise unrestricted. and and

\[ S_{\text{gh}}^{A} = -\int d^4x \overline{c}^{\alpha} \tilde{M}^{\alpha\beta} c^{\beta}, \]

(7)

with

\[ \tilde{M}^{\alpha\beta} = \eta^{\mu} D^{\alpha\beta}_{\mu}. \]

(8)

In the \( \lambda \to 0 \) limit,

\[ e^{iS_{\text{gf}}^{A}} \sim \prod_{\alpha, x} \delta \left( \eta \cdot A^{\alpha}(x) \right). \]

(9)
Thus, in the presence of the delta function, the $A$-dependent term in $\tilde{M}$ can be dropped leading to the formally ghost-free matrix. As is well known, $S_{\text{eff}}^L$ and $S_{\text{eff}}^A$ are invariant under the BRS transformations:

\begin{align*}
\delta A_\mu^\alpha (x) &= D^\alpha_\mu c^\beta (x) \delta \Lambda \\
\delta c^\alpha (x) &= -\frac{g}{2} f^{\alpha \beta \gamma} c^\beta (x) c^\gamma (x) \delta \Lambda \\
\delta \bar{c}^\alpha (x) &= \frac{f^\alpha [A]}{\lambda} \delta \Lambda ,
\end{align*}

(10)

where $f^\alpha [A] = \partial \cdot A^\alpha$ or $\eta \cdot A^\alpha$, depending on whether one has written the action in the Lorentz or the axial-type gauges.

### 2.2 FFBRS Transformations

As observed by Joglekar and Mandal [9], in (10), $\delta \Lambda$ need not be infinitesimal nor need it be field-independent as long as it does not depend on $x$ explicitly for (10) to be a symmetry of FPEA In fact, the following finite field-dependent BRS (FFBRS) transformations were introduced:

\begin{align*}
A'_\mu^\alpha &= A_\mu^\alpha + D_\mu^{\alpha \beta} c^\beta (x) \Theta [\phi] \\
c'^\alpha &= c^\alpha - \frac{g}{2} f^{\alpha \beta \gamma} c^\beta (x) c^\gamma (x) \Theta [\phi] \\
\bar{c}'^\alpha &= \bar{c}^\alpha + \frac{f^\alpha [A]}{\lambda} \Theta [\phi] ,
\end{align*}

(11)

or generically

\begin{equation}
\phi'_i (x) = \phi_i (x) + \delta_{\text{BRS}} \phi_i (x) \Theta [\phi] ,
\end{equation}

(12)

where $\Theta [\phi]$ is an $x$-independent functional of $A$, $c$, $\bar{c}$ (generically denoted by $\phi_i$) and these were also the symmetry of the FPEA. The transformations of the form (11) were used to connect actions of different kinds for Yang-Mills theory in [9] and [10]. The FPEA is invariant under (11), but the functional measure is not invariant under the (nonlocal) transformations (11). The Jacobian for the FFBRS transformations can be expressed (in special cases dealt with in [9], [10]) effectively as $\exp (i S_1)$ and this $S_1$ explains the difference between the two effective actions. Such FFBRS transformations were constructed in [9], [10] by integration of an infinitesimal field-dependent
BRS (IFBRS) transformation:

\[
\frac{d\phi_i(x, \kappa)}{d\kappa} = \delta_{\text{BRS}}[\phi(x, \kappa)]\Theta'[\phi(x, \kappa)] \tag{13}
\]

The integration of (13) from \(\kappa = 0\) to 1, leads to the FFBRS transformation of (12) with \(\phi(\kappa = 1) \equiv \phi'\) and \(\phi(\kappa = 0) = \phi\). Further \(\Theta\) in (12) was related to \(\Theta'\) by:

\[
\Theta[\phi] = \Theta'[\phi]\frac{\exp[f[\phi]] - 1}{f[\phi]}, \tag{14}
\]

where

\[
f[\phi] = \sum_i \int d^4x \frac{\delta \Theta'}{\delta \phi_i(x)} \delta_{\text{BRS}} \phi_i(x) \tag{15}
\]

FFBRS transformations of the type (12) were used to connect the FPEA in Lorentz-type gauges with gauge parameter \(\lambda\) to (i) the most general BRS/anti-BRS symmetric action in linear gauges, (ii) FPEA in quadratic gauges, (iii) the FPEA in Lorentz-type gauges with another gauge parameter \(\lambda'\) in [9]. It was also used to connect the former to FPEA in axial-type gauges in [10]. We shall now summarize the results of [10] in II C.

2.3 FFBRS Transformation for Lorentz to Axial Gauge

\(S_{\text{eff}}\)

We give results for the FFBRS transformation that connects the Lorentz-type gauges (See [4]) with gauge parameter \(\lambda\) to axial gauges (See [4]) with same gauge parameter \(\lambda\). [The same calculation can be used to connect it to axial gauges with another gauge parameter \(\lambda'\): one simply rescales \(\eta\) suitably.] They are obtained by integrating:

\[
\frac{d\phi_i(\kappa)}{d\kappa} = \delta_{\text{BRS}}[\phi] \Theta'[\phi], \tag{16}
\]

with

\[
\Theta' = i \int d^4x \bar{c}^\alpha (\partial \cdot A^\alpha - \eta \cdot A^\alpha). \tag{17}
\]

the consequent \(\Theta[\phi]\) is given by (14) with

\[
f[\phi] = i \int d^4x \left[ \frac{\partial \cdot A^\alpha}{\lambda} (\partial \cdot A^\alpha - \eta \cdot A^\alpha) + \bar{c}(\partial \cdot D - \eta \cdot D)c^\alpha \right]. \tag{18}
\]
The meaning of these field transformations is as follows. Suppose we begin with the vacuum expectation value of a gauge invariant functional \( G[\phi] \) in the Lorentz-type gauges:

\[ \langle \langle G[\phi] \rangle \rangle = \int \mathcal{D}\phi G[\phi] e^{iS_{\text{eff}}^{L}[\phi]}. \quad (19) \]

Now, we perform the transformation \( \phi \rightarrow \phi' \) given by (12). Then we have [with \( G[\phi'] = G[\phi] \) by gauge invariance]

\[ \langle \langle G[\phi] \rangle \rangle = \equiv \langle \langle G[\phi'] \rangle \rangle = \int \mathcal{D}\phi' J[\phi'] G[\phi'] e^{iS_{\text{eff}}^{L}[\phi']} \]

(20)
on account of the BRS invariance of \( S_{\text{eff}}^{L} \). Here \( J[\phi'] \) is the Jacobian

\[ \mathcal{D}\phi = \mathcal{D}\phi' J[\phi']. \quad (21) \]

As was shown in [9], for the special case \( G[\phi] \equiv 1 \), the Jacobian \( J[\phi'] \) in (21), can be replaced by \( e^{iS[\phi']} \) where

\[ S_{\text{eff}}^{L}[\phi'] + S_{1}[\phi'] = S_{\text{eff}}^{A}[\phi']. \quad (22) \]

As shown in Section 3, this replacement is valid for any gauge invariant \( G[\phi] \) functional of \( A \). If one were to live with vacuum expection values of gauge invariant observables, the FFBRS in [9] would be sufficient. But as seen in Section 3, general Green’s functions need a modified treatment.

### 3 Relation between Green’s Functions for Axial-type and Lorentz-type Gauges

In [9], we established a general procedure for writing down an FFBRS that transforms the \( W \) in one kind of a gauge choice to \( W \) in another kind of a gauge choice. This procedure was applied to the concrete example of the construction of an FFBRS connecting the axial-type gauges and Lorentz-type gauges in [10]. In order to bring out the need for a further treatment, we first elaborate on the meaning of this statement in some detail: We note that

\[ W^{L} = \int \mathcal{D}\phi e^{iS_{\text{eff}}^{L}[\phi]} \quad (23) \]
of the Lorentz-type gauges is formally carried over (without altering its "value") to

$$W^A \equiv \int D\phi' e^{iS^A_{\text{eff}}[\phi']} = W^L$$

by the FFBRS transformation

$$\phi'(x) = \phi(x) + \delta_{\text{BRS}}[\phi]\Theta[\phi]$$

constructed explicitly in (14) - (18). We now want to use this transformation to understand how the Green’s functions in the two gauges, and not just the vacuum-to-vacuum amplitudes, are related to each other. This may at first sight seem trivial. We may expect a relation of the kind (condensed notation used):

$$G^A_{i_1 \ldots i_n} \equiv \int D\phi' \prod_{r=1}^n \phi'_r e^{iS^A_{\text{eff}}[\phi']}$$

$$\approx \int D\phi \prod_{r=1}^n \left( \phi'_r + \delta_{\text{BRS}}[\phi]\Theta[\phi] \right) e^{iS^L_{\text{eff}}[\phi]}$$

$$\equiv G^L_{i_1 \ldots i_n} + \Delta G^L_{i_1 \ldots i_n},$$

where $\Delta G^L_{i_1 \ldots i_n}$, containing the terms on the right-hand involving $\Theta$’s, gives the difference between the $n$-point Green’s functions $G_{i_1 \ldots i_n}$ in the two sets of gauges. This $\Delta G^L_{i_1 \ldots i_n}$ then would be expressed in terms of the Green’s functions of the Lorentz-type gauges and may involve additional vertices corresponding to insertions of operators $\delta_{\text{BRS}}[\phi]$. This however, turns out to be incorrect and the technical reason for this is explained below.

In [9, 10], we showed that the Jacobian for the FFBRS (25), could be replaced by a factor $\exp(iS_1)$ within the expression for $W$ if the condition

$$\int D\phi(\kappa) \left( \frac{1}{\mathcal{J}} \frac{dJ}{d\kappa} - i \frac{dS_1[\phi(\kappa), \kappa]}{d\kappa} \right) \exp[i(S^L_{\text{eff}} + S_1)] = 0$$

was fulfilled. This replacement then became valid for $W$ (i.e. without additional operators in the integrand of the path integral). A priori, it is not obvious that if (27) holds, an equation of the type

$$\int D\phi(\kappa) \mathcal{O}[\phi(\kappa)] \left( \frac{1}{\mathcal{J}} \frac{dJ}{d\kappa} - i \frac{dS_1[\phi(\kappa), \kappa]}{d\kappa} \right) \exp[i(S^L_{\text{eff}} + S_1)] = 0,$$

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modified to include an operator $O[\phi(\kappa)]$ would also hold. That it does not, in fact, hold generally arises from the following fact. The verification of (27) in [9, 10], made use of the antighost equation of motion (See e.g. the discussion below equation (3.20) of [10]). Thus, it is clear that if $O[\phi(\kappa)]$ contains $\bar{c}$, then the procedure would fail as it involves integration by parts. In fact, the procedure does not work for any finite interval of $\kappa$ for any operator $O$. This is so since $O$ is evolving in form (as $\kappa$ is varied) on account of the IFBRS transformation of (16) which will always induce $\bar{c}$-dependence as $\kappa$ is varied even if at $\kappa = 0$, one started out with $O[A, \epsilon]$, an operator independent of $\bar{c}$ (barring the gauge-invariant case as discussed in Section 2). For this reason, the construction of relation between Green’s functions for the two type of gauges requires as elaborate a treatment as the original FFBRS construction itself. We begin with a general Green’s function in one of the gauges, say, the axial gauge:

$$G = \int \mathcal{D}\phi' O[\phi'] e^{iS_{\text{eff}}[\phi']}. \quad (29)$$

Here, the form of $O$ is unrestricted; so that (29) covers arbitrary operator Green’s functions as well as arbitrary ordinary Green’s functions. For example, with:

$$O_1 = A_\mu^\alpha A_\nu^\beta$$

one has the gauge boson propagator; whereas with

$$O_2 = A_\mu^\alpha(x) e_\beta^{\ '}(y) \bar{c}_\gamma^{\ '}(z) \quad (31)$$

one has a 3-point Green’s function; or with

$$O_3 = F_{\mu\nu}^\alpha(x) F_{\sigma\tau}^{\nu,\alpha}(x) A_\rho^\beta(u) A^{\gamma,\sigma'}(w) \quad (32)$$

one has the two-point Green’s function of an operator insertion of the twist two local operator $F_{\mu\nu}^\alpha F_{\sigma,\alpha}^{\nu,\alpha}$, etc. We want to express $G$ entirely in terms of the Lorentz-type gauge Green’s functions (and possibly involving vertices from $\delta_{\text{BRS}}[\phi]$). We, therefore construct the quantity:

$$G(\kappa) \equiv \int \mathcal{D}\phi(\kappa) O[\phi(\kappa), \kappa] e^{iS_{\text{eff}}[\phi(\kappa)] + iS_1[\phi(\kappa), \kappa]} \quad (33)$$

and define the form of $O[\phi(\kappa), \kappa]$ such that

$$\frac{dG}{d\kappa} = 0. \quad (34)$$
This is exactly analogous to the equation \( \frac{dW}{d\kappa} = 0 \) of [4] that related \( W(1) \equiv W^A = W(0) \equiv W^L \) together. Then

\[
G(1) = \int \mathcal{D}\phi' \mathcal{O}[\phi', 1] e^{iS_{\text{eff}}[\phi']}
\]

\[\phi' \equiv \phi(1)\] with \( \mathcal{O}[\phi', 1] \equiv \mathcal{O}[\phi'] \) gives the Green’s function (29), whereas it is alternately expressed as

\[
G(0) = \int \mathcal{D}\phi \hat{\mathcal{O}}[\phi] e^{iS_{\text{eff}}[\phi]} = G(1),
\]

where \( \hat{\mathcal{O}}[\phi] \equiv \mathcal{O}[\phi(0), 0] \). Equation (36) gives the same quantity in terms of Lorentz gauge quantities. We, thus, need to determine how \( \mathcal{O}[\phi(\kappa), \kappa] \) of (33) should evolve so as to keep \( G(\kappa) \) independent of \( \kappa \) [equation (34)]. To determine this, we perform the field transformation from \( \phi(\kappa) \) to \( \phi(\kappa + d\kappa) \) via the IFBRS of (16). We write, making due use of the BRS invariance of \( S_{\text{eff}}^L \),

\[
G(\kappa) = \int \mathcal{D}\phi(\kappa + d\kappa) \frac{J(\kappa + d\kappa)}{J(\kappa)} \mathcal{O}[\phi(\kappa + d\kappa), \kappa + d\kappa] \n\]

\[-\delta_{\text{BRS}}[\phi] \Theta \frac{\delta \mathcal{O}}{\delta \phi_i} d\kappa - \frac{\partial \mathcal{O}}{\partial \kappa} d\kappa \n\]

\[\times e^{iS_{\text{eff}}^L[\phi(\kappa + d\kappa)] + iS_1[\phi(\kappa + d\kappa), \kappa + d\kappa]} \times \left(1 + i \frac{dS_1}{d\kappa} d\kappa\right)\n\]

\[= \int \mathcal{D}\phi[\kappa + d\kappa] \left(1 + \frac{1}{J} J d\kappa\right) \mathcal{O}[\phi(\kappa + d\kappa), \kappa + d\kappa] \n\]

\[-\delta_{\text{BRS}}[\phi] \Theta \frac{\delta \mathcal{O}}{\delta \phi_i} d\kappa - \frac{\partial \mathcal{O}}{\partial \kappa} d\kappa \times \left(1 + i \frac{dS_1}{d\kappa} d\kappa\right)\n\]

\[\times e^{iS_{\text{eff}}^L[\phi(\kappa + d\kappa)] + iS_1[\phi(\kappa + d\kappa), \kappa + d\kappa]}\n\]

\[\equiv G[\kappa + d\kappa] \quad (37)\]

iff

\[
\int \mathcal{D}\phi(\kappa) \left(\left[1 + \frac{dJ}{J} d\kappa\right] \mathcal{O}[\phi(\kappa), \kappa] - \delta_{\text{BRS}}[\phi] \Theta \frac{\delta \mathcal{O}}{\delta \phi_i} - \frac{\partial \mathcal{O}}{\partial \kappa}\right)\n\]

\[\times e^{iS_{\text{eff}}^L[\phi(\kappa)] + iS_1[\phi(\kappa), \kappa]} \equiv 0. \quad (38)\]

[We have replaced \( \phi(\kappa + d\kappa) \rightarrow \phi(\kappa) \) in (38) in view of the fact that the quantity on the left-hand side is multiplied by \( d\kappa \).] Thus the condition (incorrect one) (28) is replaced by the correct condition (38).
We shall now simplify the condition (38) and show that this condition is fulfilled if a certain evolution equation is satisfied by $O[\phi(\kappa), \kappa]$. We then show how it can be solved. The procedure for the solution to the evolution equation will pertain to the introduction of another field transformation and we shall show that this is an FFBRS too.

We shall now simplify the first term on the right hand side of (38). To do this, we note that (27) is fulfilled and that we can use the explicit form of $S_1[\phi(\kappa), \kappa]$ and $\Theta'$ of (10) to simplify the combination $\frac{1}{\lambda} \int dJ d\kappa - i \delta S_1 d\kappa$. When this is done, this term reads:

$$-i \int \mathcal{D}\phi O[\phi(\kappa), \kappa] e^{i S_{\text{eff}} + i S_1[\phi(\kappa), \kappa]}$$

$$\times \int d^4x \left( \frac{1}{\lambda} \kappa (\partial \cdot A - \eta \cdot A) \left[ (1 - \kappa) M c + \kappa \tilde{M} c \right](x) \Theta' \right.$$  

$$- \kappa \left( \frac{\partial \cdot A - \eta \cdot A}{\lambda} \right)^2 \right).$$

(39)

Now we note that

$$\int \mathcal{D}\phi(\kappa) O[\phi(\kappa), \kappa] \int d^4x \mathcal{F}[A(x)] \left[ (1 - \kappa) M c + \kappa \tilde{M} c \right](x) \Theta'$$

$$\times e^{i S_{\text{eff}} + i S_1[\phi(\kappa), \kappa]}$$

$$= \int \mathcal{D}\phi(\kappa) O[\phi(\kappa), \kappa] \int d^4x \mathcal{F}[A(x)] i \frac{\delta}{\delta \bar{c}(x)} e^{i S_{\text{eff}} + i S_1} \Theta'.$$

(40)

$[\mathcal{F}[A(x)] \equiv \frac{\kappa}{\lambda} (\partial \cdot A - \eta \cdot A)]$ Integrating by parts with respect to $\bar{c}$ and taking due account of the anticomuting nature of $\bar{c}$ and possibly $O$, the above expression equals,

$$\int \mathcal{D}\phi(\kappa) \int d^4x \left( -i O \frac{\delta}{\delta \bar{c}} \mathcal{F}[A(x)] \Theta' \right.$$  

$$- O[\phi(\kappa), \kappa] \mathcal{F}[A(x)] \frac{\delta \Theta'}{\delta \bar{c}} \right) e^{i S_{\text{eff}} + i S_1}$$

(41)

Now, the term $\int d^4x \frac{\delta \Theta'}{\delta \bar{c}(x)} \mathcal{F}[A(x)]$ is precisely the factor that also arose in fulfillment of (38) (i.e. when $O[\phi(\kappa), \kappa]$ was absent) and such a term, in presence of $O$ cancels precisely with the last term in (39) just as it did in (27) in absence of $O$. Using the above information in (38), the required
condition for $\kappa$-independence of $G$ reads:

$$
\int D\phi(\kappa)e^{is_{\kappa}^{\mu}} + iS_{1[\phi(\kappa),\kappa]} \left( \frac{\partial O}{\partial \kappa} + \int D_\mu \Theta_0 \frac{\delta O}{\delta A_\mu} \right)
- \frac{g}{2} \int (fcc) \Theta_0 \frac{\delta O}{\delta c} + \int \left[ \frac{\partial \cdot A}{\lambda} + \kappa \left( \eta \cdot A - \partial \cdot A \right) \right] \Theta_0 \frac{\delta O}{\delta c} = 0
$$

(42)

So far we have not spelled out the $\kappa$-dependence of $O$. Now, if we construct an $O[\phi(\kappa),\kappa]$ which satisfies:

$$
\frac{\partial O}{\partial \kappa} + \int D_\mu \Theta_0 \frac{\delta O}{\delta A_\mu} \left[ \frac{\partial \cdot A}{\lambda} + \kappa \left( \eta \cdot A - \partial \cdot A \right) \right] \Theta_0 \frac{\delta O}{\delta c} = 0,
$$

then (42) would automatically be satisfied, thus leading to (36). Thus, we have to know how to solve (43) to obtain $O[\phi(\kappa),\kappa]$. To this end, consider the same function $O$ with a different argument $\tilde{\phi}(\kappa)$, $O[\tilde{\phi}(\kappa),\kappa]$ where $\tilde{\phi}(\kappa)$ is defined via a new set of evolution equations:

$$
\frac{d\tilde{A}_\mu(\kappa)}{d\kappa} = D_\mu [\tilde{A}] \tilde{c} \Theta'[\tilde{\phi}(\kappa)]
$$

$$
\frac{d\tilde{c}(\kappa)}{d\kappa} = -\frac{g}{2} f \tilde{c}(\kappa) \tilde{c}(\kappa) \Theta'[\tilde{\phi}(\kappa)]
$$

$$
\frac{d\tilde{\phi}(\kappa)}{d\kappa} = \frac{\partial \cdot \tilde{A}(\kappa) + \kappa (\eta \cdot \tilde{A} - \partial \cdot \tilde{A})}{\lambda} \Theta'[\tilde{\phi}(\kappa)],
$$

(44)

or in short,

$$
\frac{d\tilde{\phi}(\kappa)}{d\kappa} \equiv \tilde{\delta}[\tilde{\phi}(\kappa),\kappa] \Theta'[\tilde{\phi}(\kappa)],
$$

(45)

together with the boundary condition:

$$
\tilde{\phi}(1) = \phi' = \phi(1).
$$

(46)

The the condition (43) when expressed for $O[\tilde{\phi}(\kappa),\kappa]$ instead of $O[\phi(\kappa),\kappa]$ as:

$$
\frac{\partial O[\tilde{\phi}(\kappa),\kappa]}{\partial \kappa} + \int \tilde{\delta}_i \Theta'[\tilde{\phi}(\kappa)] \frac{\delta O[\tilde{\phi}(\kappa),\kappa]}{\delta \tilde{\phi}_i(\kappa)} = 0,
$$

(47)
i.e.
\[ \frac{d\mathcal{O}[\tilde{\phi}(\kappa), \kappa]}{d\kappa} \equiv 0. \] (48)

Now, in view of the fact that
\[ \mathcal{O}[\tilde{\phi}(1), 1] = \mathcal{O}[\phi(1), 1] = \mathcal{O}[\phi'], \] (49)
we find
\[ \mathcal{O}[\tilde{\phi}(\kappa), \kappa] = \mathcal{O}[\phi']. \] (50)

Equation (50) tells us how the function \( \mathcal{O}[\phi(\kappa), \kappa] \) should evolve: we solve (45) for \( \phi' \) in terms of \( \tilde{\phi}(\kappa) \), express \( \mathcal{O}[\phi'] \equiv \mathcal{O}[\phi'(\tilde{\phi}(\kappa), \kappa)] = \mathcal{O}[\tilde{\phi}(\kappa), \kappa] \). This gives us the function \( \mathcal{O} \). In this we replace the argument \( \tilde{\phi} \rightarrow \phi \) to obtain \( \mathcal{O}[\phi(\kappa), \kappa] \) which then will solve (43). The value of \( \mathcal{O}[\phi(\kappa), \kappa] \) at \( \kappa = 0 \), i.e., \( \mathcal{O}[\phi(0), 0] \) will then give us the function \( \tilde{\mathcal{O}}[\phi] \) of (36) involved in the expression of \( G(1) \) in terms of the Lorentz gauge quantities. Thus the evolution of \( \mathcal{O}[\phi(\kappa), \kappa] \) with \( \kappa \) is easy to obtain if the IFBRS (45) is solved. The IFBRS of (45) differs from the IFBRS of (13) in that the transformation for \( \tilde{\phi} \) involves the \( \tilde{\delta}[\phi(\kappa)] \) and is explicitly \( \kappa \)-dependent. The integration the IFBRS proceeds the same way as the basic IFBRS (13) as done in [9]; the only complication being the \( \kappa \)-dependent \( \delta_{\text{BRS}}[\tilde{\phi}(\kappa), \kappa] \) involved in \( \frac{d\tilde{\phi}}{d\kappa} \). The integration is given in appendix A. The result is
\[ \phi' = \phi + \left( \delta_1[\phi][\Theta_1[\phi]] + \delta_2[\phi][\Theta_2[\phi]] \right) \Theta'[\phi] \]
\[ \equiv \phi + \delta \phi[\phi] \] (51)

Using (51), (36), (19) and (50), we obtain the following result:
\[ G(1) = G(0) = \int \mathcal{D}[\phi] \mathcal{O}[\phi] e^{iS_{\text{eff}}[\phi]} \]
\[ \int \mathcal{D}[\phi] \mathcal{O} \left( \phi + \delta \phi[\phi] \right) e^{iS_{\text{eff}}[\phi]} = \int \mathcal{D}[\phi] \mathcal{O}[\phi] e^{iS_{\text{eff}}[\phi]} \] (52)

In view of the nilpotency of \( \delta[\phi] \), this leads to
\[ \int \mathcal{D}[\phi] \mathcal{O}[\phi] e^{iS_{\text{eff}}[\phi]} + \int \mathcal{D}[\phi] \delta \phi[\phi] \frac{\delta \mathcal{O}}{\delta \phi[\phi]} e^{iS_{\text{eff}}[\phi]} \] (53)
Further, as done in appendix A, the last term can be cast in a neat form; so that (53) can be written as:

$$\langle O \rangle_A = \langle O \rangle_L + \int_0^1 d\kappa \int D\phi \left( \tilde{\delta}_1[\phi] + \kappa \tilde{\delta}_2[\phi] \right) \Theta'[\phi] \frac{\delta O}{\delta \phi} e^{iS_{\text{eff}}^{M}}. \tag{54}$$

Our aim in this work was to establish formally the link between two gauges considered. This has been done in (53) and (54). In this work, we shall content ourselves with some comments on concrete calculations. Concrete evaluation of (54) can be carried out in two ways. While the one based on (54) is much superior, we shall enumerate both for formal reasons. (I) We can look upon the integrand on the right hand side as an expansion in $\kappa$ (and carry out the $\kappa$ integration). Then each term gives a Green's function of the operator $O$ (and its BRS variation) in Lorentz-type gauges. We can further regard each term in the expansion as an expansion in $g$. Then to a given desired order only a finite number of terms in each need be kept. To any given order in $g$, the infinite terms have however to be summed. This however can be avoided with the help of the alternate expression (54) which turns out to be much superior for practical purposes. (II) We can alternately regard the evaluation of the functional integral on the right hand side of (54) in terms of the vertices and the propagators of the interpolating mixed gauge action $S_{\text{eff}}^{M}$. This approach has many technical advantages. The last term on the right hand side of (54) now consists of usual Feynman diagrams, with one difference: the propagators of ghost and gauge fields are now $\kappa$-dependent and a final (overall) $\kappa$ integration is to be preformed. To any given order in $g$, there are only a finite number of Feynman diagrams to be evaluated on the right hand side. If, for example $O$ is a local polynomial operator, these are the Feynman diagrams with one insertion each of two local polynomial operators (or integrated local density), $\delta_{\text{BRS}}[\phi_i] \frac{\delta O}{\delta \phi_i}$ and $\int d^4x \bar{c} (x) (\partial \cdot A - \eta \cdot A)(x)$, and can be evaluated by usual techniques. If, on the other hand, $O$ is a product of $n$ elementary fields at distinct space time points (such as in (30) and (31)), then the right hand side has (a finite number of) Feynman diagrams corresponding to the $(n - 1)$-point functions with one insertion each of $\delta_{\text{BRS}}[\phi_i]$ and $\int d^4x \bar{c} (\partial \cdot A - \eta \cdot A)(x)$. [We shall give a simple example of this calculation in Section 4] We can use such an expansion (especially the approach II) to correlate the axial gauge propagator in terms of Lorentz gauge quantities. Knowing how to deal with the Lorentz gauge calculations should throw direct light on how to deal with axial gauge calculations especially
the prescription for the $\frac{1}{\eta k}$-type singularities in axial propagator. It should also help in resolving number of existing problems with light cone gauge calculations. This work is in progress \[12, 13\].

We expect such relations to resolve the discrepancy reported between the anomalous dimensions of physical observables in the two sets of gauges \[13\]. We leave the issue to a further publication.

4 An Example

In this section, we shall give a simple example of the relation (54). Consider for example,

$$O[\phi] = A_{\mu}^\alpha(x) A_{\nu}^\beta(y).$$

Then

$$\langle O[\phi] \rangle_A \equiv \langle A_{\mu}^\alpha(x) A_{\nu}^\beta(y) \rangle = iG^{A \alpha\beta}_{\mu\nu}(x - y)$$

is (for the connected part) the axial gauge propagators. In obvious notations

$$iG^{A \alpha\beta}_{\mu\nu}(x - y) = iG^{L \alpha\beta}_{\mu\nu}(x - y) + i \int_0^1 d\kappa \int D\phi e^{iS_{\text{eff}}[\phi, \kappa] - i\int (A^2/2 - \bar{c}c)d^4x} \times \left((D_{\mu}c)^\alpha(x) A_{\nu}^\beta(y) + A_{\mu}^{\alpha}(x)(D_{\nu}c)^\beta(y)\right) \int d^4z \bar{c}(z)(\partial \cdot A^\gamma - \eta \cdot A^\gamma)$$

The right hand side consists of one point functions of one insertion each of two local operators (or integrated local density) $D_{\mu}c$ and $\int d^4x \bar{c}(\partial \cdot A - \eta \cdot A)d^4x$. To any finite order, such terms can be evaluated by drawing the appropriate Feynman diagram whose propagators and vertices arise from $S_{\text{eff}}[\phi, \kappa]$. The propagators are now $\kappa$-dependent. We expect the results (53) and (54) to be useful in this manner to be able to solve the number of problems mentioned in the Introduction (last-but-one paragraph). We now make brief comments on one such application as an example.

Equation (57) leads to, for zero loop case,

$$G^{0A}_{\mu\nu} \alpha\beta(x - y) = G^{0L}_{\mu\nu} \alpha\beta(x - y)$$

$$-i \int_0^1 d\kappa \left[-i\partial_{\mu}^{\tau} G^{0M}_{\mu\nu}(x - y)(\partial_{\nu}^{\tau} - \eta^\sigma)G^{0M}_{\sigma\nu} \alpha\beta + (\mu, x, \alpha) \leftrightarrow (\nu, y, \beta)\right].$$

$$\text{(58)}$$
The last term on the right hand side involves \( \kappa \)-dependent functions for ghost and gauge fields:

\[
\tilde{G}^{0M}(x - y) = \int d^4q \frac{e^{-i\mathbf{q}\cdot(x-y)}}{(\kappa - 1)q^2 - i\kappa \mathbf{q} \cdot \eta - i\epsilon}
\]

and

\[
\tilde{G}^{0M}_{\alpha\beta}(x - y) = \delta^{\alpha\beta} \int d^4ke^{-ik\cdot(x-y)} \tilde{G}^{0M}(k)
\]

with

\[
\tilde{G}^{0M}_{\mu\nu}(k) = -\frac{1}{k^2 + i\epsilon} \left[ g_{\mu\nu} + \frac{\kappa^2}{(k^2 + i\epsilon)} \left( (1 - \kappa)^2 - \eta^2k^2 + (k^2 + \eta^2)(k^2 + i\epsilon) \right) \right].
\]

[It should be emphasized that (59), (60) are only intermediate objects occurring in calculations and are not the actual ghost and gauge propagators (even in intermediate gauges) as the latter must be evaluated ultimately with a term like \( \epsilon O(0, \kappa) \) in the exponent.] We obtain:

\[
\tilde{G}^{0A}_{\mu\nu} - \tilde{G}^{0L}_{\mu\nu} = -\frac{i}{(k^2 + i\epsilon)^2(1 - i\xi_1 - i\xi_2)(1 - i\xi_2 + \xi_1^2 + i\xi_2\xi_3)}
\]

\[
\times \int_0^1 d\kappa \left[ k_\mu k_\nu \left( \kappa + \left[ \frac{i\lambda - \xi_1(1 - \lambda)}{\xi_1 + i\xi_3} \right] \right) (1 + i\xi_3) + \eta_\mu k_\nu \left( \kappa + \left[ \frac{1 - i\xi_2(1 - \lambda)}{-1 - i\xi_1 + i\xi_2} \right] \right) \right] (-1 - i\xi_1 + i\xi_2)
\]

\[
+ (k \rightarrow -k, \mu \leftrightarrow \nu)
\]

with

\[
\xi_1 \equiv \frac{\eta \cdot k}{k^2 + i\epsilon}; \quad \xi_2 \equiv \frac{\epsilon}{k^2 + i\epsilon}; \quad \xi_3 \equiv \frac{\eta^2}{k^2 + i\epsilon};
\]

\[
a_1 \equiv \frac{1}{1 - i\xi_1 - i\xi_2};
\]

\[
\gamma \equiv \frac{1 - i\xi_2}{1 - i\xi_2 + \xi_1^2 + i\xi_2\xi_3} \equiv \frac{1 - i\xi_2}{D};
\]

\[
\beta \equiv \frac{1 + i\xi_2(\lambda - 1)}{1 - i\xi_2 + \xi_1^2 + i\xi_2\xi_3} = \gamma + \frac{i\xi_2\lambda}{D}.
\]

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For $|\eta \cdot k| >> \epsilon$ one can show that (62) leads to the usual behavior of the axial propagator (See [12, 13]), which then reads:

$$
\bar{G}^{0A}_{\mu\nu} - \bar{G}^{0L}_{\mu\nu} = -\frac{1}{k^2}k_\mu k_\nu \left( \frac{\lambda k^2 + \eta^2}{(\eta \cdot k)^2} + \frac{1 - \lambda}{k^2} \right) + \frac{k_\mu \eta_\nu}{k^2 \eta \cdot k}. 
$$

Equation (62) has been used to deal with the singularity structure near $\eta \cdot k = 0$ [12, 13].

5 Conclusions and Further Directions

In this work, we addressed the problem of relating calculations in two sets of uncorrelated gauges. We took for concreteness the axial and the Lorentz-type gauges from the point of view of their common usage. We used the results of [9] applied to the concrete case of FFBRS for axial and Lorentz-type gauges obtained in [10]. We established a procedure for relating arbitrary Green’s functions in the two sets of gauges. We showed that this involved another but related FFBRS, obtained by integration of an IFBRS as in [9]. We found that the final result could be put in a neat form (53) or (54). Form (53) is particularly useful from calculational point of view. We expect our results to be useful in (i) deriving the correct prescription for $\frac{1}{\eta k}$-type singularities in axial gauges; (ii) providing insights into problems associated with existing prescriptions in axial/light cone gauges; (iii) resolving existing discrepancies in the two sets of gauges.

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A Modified FFBRS

For the modified IFBRS, we wish to show that it can be integrated along the lines of [9] (Section 3). As done there, we can write with modification in $f$ of (3.6) of [9]:

$$
f[\phi, \kappa] \equiv f_1[\phi] + \kappa f_2[\phi]. 
$$

(A1)
Then,
\[
\frac{d\Theta'[\tilde{\phi}(\kappa)]}{d\kappa} = (f_1[\tilde{\phi}] + \kappa f_2[\tilde{\phi}])\Theta'[\tilde{\phi}(\kappa)].
\] (A2)

Following [9], we note
\[
f_i[\tilde{\phi}(\kappa)]\Theta'[\tilde{\phi}(\kappa)] = f_i[\tilde{\phi}(0)]\Theta'[\tilde{\phi}(\kappa)] \equiv f_i[\phi]\Theta'[\tilde{\phi}(\kappa)] (i = 1, 2),
\]
one gets:
\[
\frac{d\Theta'[\tilde{\phi}(\kappa)]}{d\kappa} = (f_1[\phi] + \kappa f_2[\phi])\Theta'[\tilde{\phi}(\kappa)].
\] (A3)

Integrating (A3) from \( \kappa = 0 \) to \( \kappa = \kappa \),
\[
\Theta'[\tilde{\phi}(\kappa)] = \Theta[\phi] \exp\left(\int_0^\kappa f[\phi'(\kappa)]d\kappa'\right)
\]
\[
= \Theta[\phi] \exp\left(\kappa f_1[\phi] + \frac{\kappa^2}{2} f_2[\phi]\right). \] (A4)

Similarly, one writes
\[
\frac{d\tilde{\phi}(\kappa)}{d\kappa} = (\tilde{\delta}_1[\tilde{\phi}(\kappa)] + \kappa \tilde{\delta}_2[\tilde{\phi}(\kappa)])\Theta'[\tilde{\phi}(\kappa)]
\]
\[
= (\tilde{\delta}_1[\phi] + \kappa \tilde{\delta}_2[\phi])\Theta'[\tilde{\phi}(\kappa)]. \] (A5)

Integrating (A5) from \( \kappa = 0 \) to \( \kappa = 1 \), one gets:
\[
\phi' = \phi + (\tilde{\delta}_1[\phi]\Theta_1[\phi] + \tilde{\delta}_2\Theta_2[\phi])\Theta'[\phi],
\] (A6)
where
\[
\Theta_{1,2}[\phi] \equiv \int_0^1 d\kappa (1, \kappa) \exp\left(\kappa f_1[\phi] + \frac{\kappa^2}{2} f_2[\phi]\right). \] (A7)

For the modified FFBRS of Section 3,
\[
f_1[\phi] \equiv i \int d^4x \left[ \frac{\partial \cdot A^\alpha}{\lambda} (\partial \cdot A^\alpha - \eta \cdot A^\alpha) + \bar{c}(\partial \cdot D - \eta \cdot D)c \right]
\]
\[
f_2[\phi] \equiv -\frac{i}{\lambda} \int d^4x (\partial \cdot A^\alpha - \eta \cdot A^\alpha)^2 \] (A8)

Now we apply the FFBRS of (A6) to the problem at hand. Consider the vev of \( \mathcal{O} \) in the axial gauge:
\[
\int D\phi' \mathcal{O}[\phi'] e^{iS_A}. \] (A9)
Now,\[
O[\phi'] \equiv O[\phi + (\tilde{\delta}_1 \Theta_1 + \tilde{\delta}_2 \Theta_2)\Theta'] = O[\phi] + (\tilde{\delta}_1 \Theta_1 + \tilde{\delta}_2 \Theta_2)\Theta' \frac{\delta O}{\delta \phi}.
\] (A10)

We substitute (A10) in (A9) to obtain\[
G^A_0 \equiv \int D\phi' O[\phi'] e^{iS_{\text{eff}}^A} \int D\phi O[\phi] e^{iS_{\text{eff}}^L} + \int D\phi (\tilde{\delta}_1 \Theta_1 + \tilde{\delta}_2 \Theta_2)\Theta' \frac{\delta O}{\delta \phi} e^{iS_{\text{eff}}^M}. \quad (A11)
\]

We now note the forms of $\Theta_1$ and $\Theta_2$ in (A7) and that\[
iS_{\text{eff}}^L + \kappa f_1[\phi] + \frac{\kappa^2}{2} f_2[\phi] \equiv iS_{\text{eff}}^M. \quad (A12)
\]

This leads us to\[
\langle O \rangle_A = \langle O \rangle_L + \int_0^1 dk \int D\phi (\tilde{\delta}_1[\phi] + \kappa \tilde{\delta}_2[\phi]) \Theta'[\phi] \frac{\delta O}{\delta \phi} e^{iS_{\text{eff}}^M}. \quad (A13)
\]

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