New spectral approach for community structure analysis on complex networks

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A simple but efficient spectral approach for analyzing the community structure of complex networks is introduced. It works the same way for all types of networks by spectrally splitting the adjacency matrix into a “unipartite” and a “multipartite” component whose entries provide measures of the affinity or antagonism between the nodes in a manner that clearly reveals the communities and also provides additional information, including a ranking of the links and nodes and the highlighting of the “gateway” links that are important in connecting the communities together. Two algorithms are then proposed to achieve the actual partition into communities. These algorithms can be tuned to discard the overlaps and produce the most meaningful non-overlapping community structure.

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I. INTRODUCTION

Community structure detection has been one of the most important research topics in network science in recent years. Although no exact definition exists, a community is broadly understood as a set of nodes that “work together to achieve a certain function of the network”. It is usually assumed that there is a correlation between the density of connections and function, namely that subsets of the network whose nodes are more densely connected than in a random “null model” are likely to perform some function together [1–4]. Alternatively, especially in the case of bipartite and directed networks, a frequently used assumption is that nodes that share many connections are likely to perform a common task [1–3]. The two assumptions have essentially the same meaning in the case of very densely connected communities, but are otherwise distinct. The method presented in this paper naturally identifies communities defined according to either assumption as well as the “gateway” nodes that play an important role in connecting the communities together.

Various methods have been proposed to identify the community structure, most of them applying only to unipartite undirected networks [1–3, 6–24]. They include divisive algorithms [2], graph partitioning [10], hierarchical clustering [12], partitional clustering [13], spectral clustering [14–18] as well as more unusual methods [19–21]. However, the most commonly used methods are those based on the maximization of a goal function called modularity, introduced by Newman and Girvan [2–4, 7]. The maximization is achieved using different heuristic approaches like greedy search [7], extremal optimization [9], simulated annealing [8], or spectral optimization [3, 4]. The latter has developed into more sophisticated algorithms which increase performance [22, 23] or are specifically designed for bipartite networks [2–24, 26], directed networks [27], or networks with overlapping communities [28–31]. Although community detection algorithms that use modularity as a goal function are known to suffer from a resolution problem which prevents them from detecting communities below a certain size [32–34], they are so far the most frequently used in the case of undirected networks with non-overlapping communities because modularity is based on a clear working definition of what it means for such a network to be modular [1]. However, in the case of bipartite or directed networks and especially for networks with overlapping communities there is no universally accepted definition of modularity [1, 2, 26–31] and there is no way to directly compare the quality of partitions that have been obtained by maximizing different modularity functions. For this reason, it is important to have a community detection method that is independent of a definition of modularity, works for both uni- and bipartite networks, but produces results compatible with modularity-based methods wherever comparison is meaningful.

A first step in this direction was taken in [23]. They use a singular value decomposition of the unsigned Laplacian matrix for unipartite networks or of the rectangular adjacency sub-matrix for bipartite networks, followed by the application of a k-means clustering algorithm in the subspace spanned by the left and right singular vectors corresponding to the largest singular values. In this sense they are still very close to the spectral algorithms of Refs. [13–16]. Their algorithm has the drawback of using different matrices for uni- and bipartite networks and can only identify “unipartite”-type communities on bipartite or multipartite networks, where nodes from the two (or more) parties are lumped together into communities. In addition, Ref. [23] lacks a performance comparison with modularity-based methods in terms of ensemble averages. The community detection method presented in this paper is simpler and works the same way for all types of networks. It generates two matrices in which the “unipartite” and the “multipartite” community structures are immediately visible. The entries of these matrices provide a measure of the affinity or antagonism between the different nodes which can be useful by itself (and likely sufficient for many purposes), but can also be used to

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generate either overlapping or non-overlapping community structures.

Finally, with the exception of Ref. [22], all spectral algorithms currently employed to maximize modularity perform recursive bisections of the network and its communities by using only the leading eigenvalue of the modularity matrix. These bisections must be combined with additional “fine-tuning” [3, 4] and “final tuning” [22] steps without which the performance of the algorithms would be insufficient. These steps do not increase the complexity of the algorithms but require significant extra effort to program. A question of both theoretical and practical importance is whether a different type of spectral algorithm that uses multiple eigenvectors of the adjacency matrix and is not specifically designed to maximize modularity still needs such additional steps to produce good modularity maxima. We present results showing that, except for low-degree or weakly modular networks, the algorithm proposed in this paper produces good to excellent modularity values without additional steps.

II. METHOD

Let A be the adjacency matrix of a sparse network with N nodes. At first we will assume that the network is undirected, but there is no restriction on whether the network is uni- or bipartite, weighted or unweighted. If the network is weighted, A is understood to be the weights matrix. The goal is to partition the network into a set of communities \( \{C_k\} \) with \( k = 1, K \) that makes sense in light of the criteria mentioned in the first paragraph of the Introduction. Although the adjacency matrix is the most straightforward representation of a network, it has so far been considered unfit for the purpose of determining the community structure. The reason for this apparent inability and the way to deal with it are discussed below.

Community detection algorithms have been proposed that use either the stochastic matrix [16, 17] or different forms of the network Laplacian [18-23] but the most popular algorithms start with the definition of a modularity function. In the case of unipartite undirected networks, modularity is defined as

\[
Q = \sum_{k=1}^{K} \sum_{i,j \in C_k} \left( A_{ij} - \frac{d_i d_j}{2m} \right),
\]

where \( d_i \) is the degree of node \( i \) and \( 2m = \sum_{i=1}^{N} d_i \). Modularity is then expressed as

\[
Q = \frac{1}{2m} S^T M S
\]

where \( M \) is the modularity matrix defined by

\[
M_{ij} = A_{ij} - \frac{d_i d_j}{2m}
\]

and \( S \) is a binary \( N \times K \) matrix with \( S_{ik} = 1 \) if node \( i \) belongs to community \( k \) and zero otherwise.

In the standard spectral algorithm due to Newman [3, 4] as well as in its versions [5, 26, 27, 31] which are in use today, \( S \) is a column matrix and the network is recursively bisectioned according to the signs of the components of the eigenvector corresponding to the largest eigenvalue of the modularity matrix and then of its modified community-wide versions until the modularity function can no longer be increased. There are also “fine-tuning” [3, 4] and “final tuning” [22] steps that can be added at the end of each bisection and at the end of the bisectioning process respectively and which improve the performance of the algorithm.

Of particular interest are the versions introduced by Guimera [5] and Barber [26], which are specifically designed to deal with bipartite networks. The two algorithms detect different types of communities, comprising nodes from only the same party (Ref. [5]) and cross-party communities (Ref. [26]). As will be seen, the algorithm presented in this paper is capable of detecting both types of communities on bipartite and therefore potentially on directed networks [5].

In Ref. [3], Newman points out the possibility of using more than one eigenvector of the modularity matrix but this idea has not been pursued until recently [22, 23]. The algorithm proposed in Ref. [22] uses orthonormal rotations in a space spanned by the eigenvectors corresponding to the \( K \) largest eigenvalues of the modularity matrix while [23] uses a singular value decomposition of the unsigned network Laplacian followed by \( K \)-means clustering in a similar subspace.

On the other hand, it is obvious that the community structure can be regarded as a “coarse-graining” of the network under analysis. The intuition behind the method proposed in this paper is to translate the coarse-graining algebraically into a representation of a community as a square sub-matrix whose entries are all positive or greater than a certain positive threshold, centered on the main diagonal of a simplified adjacency matrix. This makes sense if belonging to a community is viewed as being under the influence of a “center of power”, with all members interacting with each-other through it. The problem of identifying the community structure (including the case of overlapping communities) then translates into finding all such sub-matrices that are maximal (not contained within larger ones).

Sub-matrices of the kind described above are nowhere to be found in the adjacency matrices of typical real-world or model networks. Networks composed of sparsely interconnected cliques come closest to this picture but even they have all diagonal elements equal to zero unless self-loops are allowed. In order to obtain a coarse-grained version of the adjacency matrix it seems natural to perform a singular value decomposition \( A = U \Sigma V^T \) [32] and then retain only the terms corresponding to the largest
Community detection has been noted before and all methods revealing a community structure.

The adjacency matrix can be used to discern a (nearly-)bipartite structure directly from the adjacency matrix and that, in addition, two types of community structure, similar to those described in Refs. [5] and [26] can be obtained in Refs. [5] and [26] can be obtained in the case of bipartite or directed networks.

To understand the root of the problem, note first that for real symmetric matrices the singular value decomposition is closely related to the eigenvalue decomposition $A = U \Lambda U^T$: the singular values are the absolute values of the eigenvalues, $\sigma_i = |\lambda_i|$, and any negative eigenvalue signs are transferred to the columns of $U$ on the right to form $V$. Retaining the largest $K$ singular values in a SVD is the same as retaining the largest $K$ eigenvalues in absolute value. However, individual rank-1 terms of the form $\lambda_i U_i U_i^T$ in the eigenvalue expansion of $A$ tell different stories when interpreted in terms of community structure depending on the sign of $\lambda_i$.

If $\lambda_i > 0$, the matrix has two blocks with positive entries on the main diagonal and two off-diagonal blocks with negative entries. This corresponds to a partition of the network into two unipartite-style communities, with the positive matrix elements quantifying affinity and the negative ones quantifying antagonism between the nodes.

If $\lambda_i < 0$, the blocks with positive entries are off-diagonal, which corresponds to a bipartite approximation of the network, with two same-party communities appearing in the negative blocks and the connections between the nodes in the positive ones. This is reminiscent of Newman’s observation that the eigenvector corresponding to the largest negative eigenvalue of the modularity matrix can be used to discern a (nearly-)bipartite structure.

A simple example is the network shown in Fig. 1 which is nearly bipartite except for the link between nodes 9 and 10. The network is shown in two different layouts which emphasize the unipartite and bipartite communities respectively. The first term of the expansion in Eq. (4) does contain information about the relative importance of the nodes within the network, which is not surprising, since $U_1$ defines the eigenvector centrality measure. As more terms are added, though, the singular value expansion simply converges towards the adjacency matrix without ever revealing a community structure.

The fact that the adjacency matrix seems unfit for community detection has been noted before and all methods introduced so far avoid it. In the following we show that it is possible to obtain a meaningful community structure directly from the adjacency matrix and that, in addition, two types of community structure, similar to those described in Refs. [5] and [26] can be obtained in the case of bipartite or directed networks.

\[ A_{(1-K)} = \sum_{k=1}^{K} \sigma_k U_k V_k^T. \]  

Here $U$ and $V$ are orthogonal matrices whose columns are the singular vectors of matrix $A$ while $\Sigma$ is diagonal with non-negative entries. This is reminiscent of approaches used in some lossy image compression and face recognition algorithms as well as of the principal component analysis method used in statistics. A low-rank approximation of the adjacency matrix is expected to retain only its most important features, enhancing sets of similar rows or columns, introducing additional links within the densely connected subsets and weakening the links between them. This is exactly what is needed in order to reveal communities defined either by high density of links or by similarity of connection, as discussed in the first paragraph of the Introduction. Moreover, it is known that retaining the first $K$ singular values from an SVD leads to the best rank-$K$ approximation of the original matrix in terms of Frobenius norm. Everything seems right and yet, if the method is applied as described above, it gives fair results on some networks but completely fails to identify a meaningful community structure on many.

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FIG. 1. A simple nearly-bipartite network.

FIG. 2. Split eigenvalue expansions of the adjacency matrix for the network in Fig. 1. Red and blue mean positive and negative matrix entries respectively and the dot in the legend box has unity diameter.
It is known\cite{36} that bipartite networks have symmetric positive and negative eigenvalues of the adjacency matrix. In addition, many unipartite networks have large negative eigenvalues, of magnitude comparable to the largest positive ones. This means that two mutually exclusive types of community description interfere if one simply performs a singular value decomposition of the adjacency matrix. The key to correctly revealing the community structure of the network based on the adjacency matrix is to spectrally split it into two separate matrices, one constructed using exclusively the eigenvectors with positive eigenvalues and the other the eigenvectors with negative eigenvalues,

\[
A_U = \sum_{\lambda_k > 0} \lambda_k U_k U_k^T \\
A_M = \sum_{\lambda_k < 0} \lambda_k U_k U_k^T.
\]

For the purpose of revealing the community structure, we can retain the largest \(K_p\) positive eigenvalues and the largest \(N - K_n + 1\) negative eigenvalues. Assuming the eigenvalues are listed in decreasing order, the matrices are

\[
A_{(1-K_p)} = \sum_{k=1}^{K_p} \lambda_k U_k U_k^T \\
A_{(K_n-N)} = \sum_{k=K_n}^{N} \lambda_k U_k U_k^T.
\]

The results of such a split for the network in Fig.\text{11} are shown in Fig.\text{12}(a) and (b). The first matrix reveals communities in “unipartite” mode: even in the case of bi- or multi-partite networks, nodes in one party that are densely connected as second-order neighbors are lumped together with the first-order neighbors through which they are connected into cross-party communities. The negative entries of the second matrix reveal communities in “bipartite” mode, with nodes from only one party that share many neighbors in other parties lumped by themselves. On the other hand, even in the case of unipartite networks that have a few prominent negative eigenvalues a “bipartite” type community structure might be discernible.

The interpretation of the eigenvectors of the adjacency matrix as “community modes” is best understood as generalizing the definition of the eigenvector centrality: the eigenproblem \(Au = \lambda u\) is interpreted as a self-consistent way of quantifying the centrality of the nodes on a network such that the centrality \(u_i\) of node \(i\) is proportional to the sum of the centralities of its neighbors \(\sum_{j=1}^{N} A_{ij} u_j\). Since centrality measures are assumed to be non-negative, only the eigenvector corresponding to the largest eigenvalue is used to define the classical centrality. On the other hand, if negative eigenvector elements are allowed, the negative signs can be transferred to the elements of \(A\). We thus end up with two groups of nodes, all with positive centrality measures, but the centrality of one node is proportional to the sum of the centralities of the nodes from the same group that are connected to it minus the sum of the centralities of the nodes from the opposite group to which it is connected. This leads to meaningful bisections of the network into communities.

To better understand the way the method works, let us analyze in detail what it does to a bipartite network. The eigenproblem for a bipartite adjacency matrix

\[
A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}
\]

with \(B\) of dimensions \(m \times n\) is equivalent with

\[
(BB^T)u = \lambda^2 u \\
(B^TB)v = \lambda^2 v
\]

and if we perform a singular value decomposition

\[
B = U \Sigma V^T
\]

we find

\[
BB^T = U \Sigma^2 U^T \\
B^TB = V \Sigma^2 V^T.
\]
where the form components of $A$ of a positive-definite matrix $M$ and negative matrix entries respectively.

The eigensystem of $A$ (nullspace excluded) is thus of the form

$$\left\{ \pm \sigma_i, \frac{1}{\sqrt{2}} \begin{pmatrix} U_i \\ \pm V_i \end{pmatrix} \right\}, \ i = 1, \ldots, r$$

where $r \leq \min(m, n)$ is the rank of $B$.

The full (non-truncated) unipartite and multipartite components of $A$ are then

$$A_U = \frac{1}{2} \sum_{i=1}^{r} \sigma_i \begin{pmatrix} U_i \\ V_i \end{pmatrix} \begin{pmatrix} U_i^T & V_i^T \end{pmatrix}$$

$$A_M = -\frac{1}{2} \sum_{i=1}^{r} \sigma_i \begin{pmatrix} V_i \\ -U_i \end{pmatrix} \begin{pmatrix} U_i^T & -V_i^T \end{pmatrix}$$

or, in terms of $B$,

$$A_U = \frac{1}{2} \begin{pmatrix} \sqrt{BB^T} & B \\ B^T & \sqrt{B^TB} \end{pmatrix}$$

$$A_M = \frac{1}{2} \begin{pmatrix} -\sqrt{BB^T} & B \\ B^T & -\sqrt{B^TB} \end{pmatrix}$$

where $\sqrt{M}$ denotes the principal, positive-definite root of a positive-definite matrix $M$.

The elements of the matrices $BB^T$ and $B^TB$ count the number of ways one can travel in two steps from a node in one party to another (or the same) node in the same party. The roots of these matrices, and especially their low-rank approximations, turn out to be excellent tools for highlighting “bipartite” (same-party) communities since they tend to highlight similar connections of the nodes from only one party. Bipartite communities appear as negative entries in $A_M$.

Low rank approximations of the unipartite component obtained by retaining only the first $K < r$ terms highlight situations when there is similarity of connection from either side to the other and nodes from one party together with those from the other party through which they are connected are placed in the same community.

Note that, especially when the bipartite division of the network is not previously known and the bipartite adjacency matrix has not been already written in the standard form of Eq. (9), the best way to reveal the same-party communities is to use

$$A_U - A_M = \begin{pmatrix} \sqrt{BB^T} & 0 \\ 0 & \sqrt{B^TB} \end{pmatrix}$$

instead of $A_M$. This prevents the off-diagonal blocks in Eq. (19) from interfering with the community detection process and also shows the communities with positive entries, as can be seen in Fig. 2 (d).

Finally, discarding the first term of the unipartite component $A_U$ can be useful for revealing high modularity community structures which are also less likely to exhibit overlaps. This is because the matrix

$$A_{\{2-N\}} = A - \lambda_1 U_1 U_1^T$$

has similar properties with the modularity matrix defined in Eq. (3). Since the components $U_1$ are the eigenvector centralities of the nodes, they are expected to be fairly correlated with the node degrees. Matrix $A_{\{2-N\}}$ is in fact a modularity-type matrix with a different null model, which uses the eigenvector centralities instead of the degrees, and $A_U - \lambda_1 U_1 U_1^T$ is its unipartite component. The matrix depicted in Fig. 2 (c) represents $A_{\{2-3\}}$ for the network in Fig. 1.

In light of the meaning of the first term in Eq. (5) as an outer product of the classical centrality eigenvector and best rank-1 approximation of the adjacency matrix we see that the elements of $A_{\{1-K\}}$ provide information about the importance of the nodes and links on the network as a whole, while $A_{\{2-K\}}$ is more focused on the division into communities, the importance of the nodes and links within their communities, and the antagonism between the different communities. It should also be noted that keeping the first term does help with the detection of overlapping communities.

In Fig. 2 (a) the “unipartite” component of the adjacency matrix reveals three communities comprising nodes.
{1-4}, {5-8} and {7, 9, 10}. This is consistent with the visual analysis of the network and the overlap between the latter two communities is clearly indicated. Moreover, the importance of the “gateway” link between nodes 3 and 5 as well as the central importance of node 7 are clearly indicated. Other smaller but significant entries indicate the stronger relationship between node 3 and nodes {6, 8} as well as between node 5 and nodes {2, 4}. Finally, the relatively close interaction between sets {6, 8} and {9, 10} is also indicated. Figs. (b) and (d) reveal “bipartite” communities {1, 3}, {2, 4}, {5, 7} and {6, 8} defined based on similarity of connection. These figures show nodes 9 and 10 each in a community by itself, indicating that they are not part of the “bipartite” community picture. The matrix depicted in Fig. (c) represents the “unipartite” component of A with the first term in Eq. (5) not included.

The essence of the spectral split method can be seen from the example presented above but two more ingredients are needed in order to have an algorithm that can consistently and automatically produce near-optimal results. The first ingredient is an algorithm to choose the number of eigenvalues \( K \). The second is an algorithm to assign the nodes to communities.

A. Choosing the eigenvalue threshold

The most important features of the community structure are revealed by the most prominent of the eigenvalues, positive or negative. The spectra of all graphs examined exhibit, at least at the positive end, a few prominent eigenvalues separated by much larger eigengaps than the rest of them. This is reminiscent of properties observed in the spectrum of the unsigned Laplacian matrix \( A_U \). An example is shown in Fig. 2. It is found that the highest modularity partitions can be obtained if exactly these eigenvalues are retained.

It is important to emphasize that retaining more eigenvalues can be very useful, shedding additional light on the interactions between the nodes on the network, but if more eigenvalues are used to partition the network into communities the modularity will typically be lower.

For the purpose of automatic computation, it is found that a simple rule that gives high modularity results at least for the unipartite component is to choose the threshold at the rightmost of the four most prominent eigengaps.

The fact that the eigenvalues separated by large eigengaps are sufficient to define the community structure is very important from a computational point of view. It is known that the first few eigenvectors of a symmetric matrix can be computed by using the Lanczos algorithm much faster than the \( O(N^3) \) time required to compute the complete set of eigenvectors if their eigenvalues are separated from the rest by a large eigengap.

B. Assigning the nodes

The following two algorithms were used for the purpose of obtaining high-modularity non-overlapping partitions once the low-rank approximations of \( A_U \) or \( A_U - A_M \) were computed.

1. Algorithm 1

1. For each eigenvector corresponding to a positive eigenvalue between 2 and \( K \) generate two communities, containing the nodes corresponding to the positive and negative components respectively. The strength of the tie between node \( i \) and a community defined using eigenvector \( k \) is equal to the average of the elements of matrix \( \lambda_k U_k^T \) connecting \( i \) with nodes in its community.

2. Assign each node to the community to which it is connected with the highest strength. For equal strengths, assign the node to the largest of the communities.

2. Algorithm 2

1. Set the negative entries of \( A_{(2-K)} \) to zero.

2. Perform a second eigenvalue decomposition of the resulting matrix, which has only a few large positive eigenvalues with eigenvectors whose positive components are typically much larger than the negative ones.

3. Assume that each eigenvector corresponding to a large eigenvalue represents a community and assign each node corresponding to a positive component to that community with a strength of the tie equal to the value of the component.

4. If non-overlapping communities are desired, assign each node to the community to which it is connected with the highest strength. For equal strengths, assign the node to the largest of the communities.

Statistically speaking, Algorithm 2 turns out to be the best at producing high-modularity structures. For the purpose of comparison, the spectral split method combined with Algorithm 2 was also applied with the classical modularity matrix \( M \) replacing \( A_{(2-N)} \).

III. RESULTS

We start by presenting results for the larger modular network in Fig. 4 which exhibits more features.
Fig. 5 (a) shows the low rank approximation $A_{(1-4)}$ based on the four prominent eigenvalues separated by large eigengaps from the other eigenvalues. In the upper-left corner we can clearly see a community formed primarily by nodes {1-6} but whose influence extends to nodes 7 and 8 as well. The central importance of nodes {1-3, 5} is clearly highlighted, with node 5 also highlighted as an important gateway node connected to communities {9-11, 14} and {15-17, 19} as well. Node 8, though not an important member of this community, appears as a gateway node towards community {18, 20, 21} to which it has stronger ties. Proceeding further down along the main diagonal, we find community {9-11, 14} with secondary nodes 12 and 13 attached to it and then the strong communities {15-17, 19} and {18, 20, 21}. The central importance of the pairs {14, 15} and {17, 18} as gateway nodes is also highlighted by significant off-community entries.

The full-rank unipartite component $A_{(1-8)}$ is shown in Fig. 5 (b). Here we see that adding more terms in Eqs. (7) and (8) indeed provides a more complete picture of the interactions between the nodes on the network even though it will likely cause community assignment algorithms to produce partitions of smaller modularity. At this stage, although communities are still visible, the information provided is more about the relative importance of the nodes and links on the network. The importance of the nodes can be inferred from the diagonal elements of the matrix. For example, within the first community, the importance of node 5 as a hub is emphasized in a way that distinguishes it from nodes {1-3}. Its connections with nodes 2, 4, 6, 9 and 17 are more clearly emphasized. The second community is resolved into two, {9, 11, 14} and {10, 12, 13}, with the link between 9 and 10 highlighted as an important gateway. A more detailed analysis is left to the reader, but it is clear that looking at high-rank approximations or at the full-rank unipartite matrix provides a much richer picture of the network’s structure than a simple partition into communities.

Finally, matrix $A_{(2-8)}$ shown Fig. 5 (c) highlights the antagonism between nodes {1-6} from the first community and community {15-17, 19} as well as between the latter and community {9-11, 14}.

Detailed results for two well-known benchmark networks, the unipartite karate network of Zachary [58] and the bipartite Southern women network [39, 40] are presented next.

### A. Zachary’s karate network

The adjacency matrix for the karate network is shown in Fig. 6 (a) and its eigenvalues are shown in Fig. 3. The four positive eigenvalues separated by a large eigen-gap from the others are again the ones that define the

| Network   | N   | $<d>$ | lev  | ss($A$)       | ss($M$)       | best   |
|-----------|-----|-------|------|---------------|---------------|--------|
| karate    | 34  | 4.59  | 0.3934 | 0.4174        | 0.4174        | 0.4197 |
| dolphins  | 62  | 5.13  | 0.4912 | 0.5190        | 0.5144        | 0.5285 |
| lesmis    | 77  | 6.60  | 0.5323 | 0.5526        | 0.5469        | 0.5600 |
| football  | 115 | 10.7  | 0.4926 | 0.5889        | 0.5817        | 0.6046 |
| jazz      | 198 | 27.7  | 0.3936 | 0.4328        | 0.4402        | 0.4450 |
| C. elegans| 453 | 8.97  | 0.3474 | 0.3394        | 0.3394        | 0.4520 |
| C. elegans| 453 | 8.97  | 0.3474 | 0.3394        | 0.3394        | 0.4520 |
| C. elegans| 453 | 8.97  | 0.3474 | 0.3394        | 0.3394        | 0.4520 |
| C. elegans| 453 | 8.97  | 0.3474 | 0.3394        | 0.3394        | 0.4520 |
correct community structure. The non-overlapping partition with the maximum modularity for this network is \{1-4, 8, 12-14, 18, 20, 22\}, \{5-7, 11, 17\}, \{9, 10, 15, 16, 19, 21, 23, 27, 30, 31, 33, 34\} and \{24-26, 28, 29, 32\} for which the modularity is $Q_{\text{max}} = 0.419790$. A quick inspection of Figs. 8 (b) and especially (c) reveals an overlap between the first two communities, with node 1 belonging to both of them, and an overlap between the last two at node 24. Both of these overlaps make sense in light of the way nodes 1 and 24 are connected. If either one of the two algorithms described in the previous section is applied in order to produce a non-overlapping community structure, the partition described above is reproduced with the exception of node 24 being assigned to the third community, which results in a very slight drop in modularity to $Q = 0.417406$. Note though that 24 is placed by maximizing modularity and to three nodes in the community where it is placed by maximizing modularity and to three nodes in the community where it is placed by this algorithm.

Finally, Fig. 8 (d) shows a bipartite component of the adjacency matrix that is split only by the term corresponding to the most prominent negative eigenvalue. This splits the network with nodes \{1, 2, 3, 17, 25, 26, 33, 34\} in one community and the rest of them in another, which is basically the two opposite centers of power connected through the other nodes.

**B. The Southern women network**

This is the most frequently used benchmark for bipartite community detection algorithms [5, 23, 26]. Its average degree is 5.56.

The adjacency matrix for this network is shown in Fig. 9 (a) while unipartite and bipartite components for $K = 2$ are shown in Fig. 9 (b-d). Nodes 1 through 18 represent women while nodes 19 through 32 represent events to which they participated. By inspection of Fig. 7 (c) we find the following two unipartite-style non-overlapping communities: \{1-7, 19-24\} and \{8-18, 25-32\}. Fig. 7 (b), which includes the largest eigenvalue term, reveals overlapping communities \{1-10, 19-27\} and \{1-3, 7-18, 25-32\}. These results are very close to the partition obtained in Ref. [5] and in Ref. [26] for the case of division in two communities, namely \{1-7, 9\}, \{8, 10-18\}, \{19-26\} and \{27-32\}. The original partition given by the authors of Ref. [39] pertains only to women and is an overlapping one, \{1-9\} and \{9-18\}.

Finally, Fig 7 (d) reveals overlapping communities in bipartite mode: \{1-10\}, \{12, 3-7, 18\}, \{19-27\} and \{25-32\}. Note that in this simple case when the network is bipartite and divided into only two unipartite communities (four bipartite), the information obtained from the bipartite component of the adjacency matrix simply duplicates the information from the unipartite component. This is not necessarily the case, however, if the network is multipartite or bipartite with more than two unipartite-style communities.

The highest modularity division reported in Ref. [26] is \{1-6\}, \{7, 9, 10\}, \{8, 16-18\}, \{11-15\}, \{19-24\}, \{25, 26\}, \{27, 29\} and \{28, 30-32\}. Similar partitions can be obtained with Algorithm 2 if more eigenvalues are included. For example, for $K = 3$ we find using $A_U - A_M$ partitions \{1-7, 9, 10\}, \{8, 16-18\}, \{11-15\}, \{19-15, 27\}, \{26\} and \{28-32\}.

Figs. 9 (b) and (d) also show the higher importance of nodes \{25-27\} which represent events \{7-9\} and were commonly attended by many women from both groups [23, 39]. The event communities are actually shown to be overlapped.
FIG. 10. Ensemble averages of the modularity obtained using the leading eigenvector algorithm (black), spectral split of $A$ (red) and spectral split of $M$ (blue) versus the modularity of the built-in partition for $N = 1000$, $< k > = 20$, $k_{max} = 40$.

C. Other benchmark networks

Table I shows a detailed comparison of the modularities obtained with the spectral split method with the raw leading eigenvector method (without extra tuning) and with the highest modularity results known in literature [25]. Included are Zachary’s karate network [38], the dolphins network of Lusseau et al., the network of interactions between the characters in Victor Hugo’s “Les Miserables” [41], the American college football network first studied by Girvan and Newman [42], the network of jazz musicians [43] and the metabolic network of the worm C. elegans [44].

With the exception of the C. elegans metabolic network, both versions of the spectral split algorithm compare very well with their modularity maximization counterparts. The results in Table I suggest that the spectral split method works better for networks with higher average degree or higher modularity. They also seem to hint that the algorithm might not work well for larger networks.

D. Statistical ensemble results

To check the validity of these statements and to quantify the performance of the algorithm, tests were performed on ensembles of random benchmark networks generated using the algorithm in Ref. [13]. These are scale-free networks with a built-in community structure. They have a number of tunable parameters, which include the average degree, the maximum degree, the exponent of the degree distribution, and the mixing parameter $\mu$ which controls the average modularity of the network ensemble. The parameters not discussed here were kept at their default values.

Tests were performed on networks of size $N$ between 100 and 1000, average degree $< d >$ between 6 and 30 and maximum degree up to 100. Some of the results are presented in Figs. 8, 9 and 10. The data points in these figures represent modularity averages computed over ensembles of 100 networks with fixed values of $\mu$. The error bars represent the standard error of the mean. The modularity of the built-in partition is the abscissa, while the ordinate is the modularity of the computed partition. Results are presented for the leading eigenvector method (without additional tuning), the spectral split method applied on the adjacency matrix $A$ and the spectral split method applied to the modularity matrix $M$.

It is clear that the size of the network does not, statistically speaking, hinder the ability of the spectral split method to detect the correct community structure. The networks in the high-modularity ensembles routinely exhibit 10-20 communities. Spectral split is, generally speaking, vastly superior to the raw leading eigenvector method and it does not need any additional tuning to find the built-in community structure in the case of networks with significant modularity, which is exactly where the leading eigenvector method has its worst performance. On the other hand, it is true that the spectral split algorithm starts to slip in the case of low-modularity networks. These low modularity values, however, are comparable to those of random networks and therefore it is questionable whether the community structure is actually meaningful [1]. The results also indicate that the spectral split algorithm performs better on networks that are not extremely sparse. Finally, as far as modularity is concerned, the average modularity obtained by splitting the adjacency matrix is, in most cases, indistinguishable from that obtained by splitting the modularity matrix.

IV. CONCLUSIONS

A new method for analyzing the structure of complex networks was introduced. The method does more than help visualize the communities, providing information, at different levels of detail, about the strengths of the interactions between the nodes on the network. In this regard, the method is useful even without an actual grouping of the nodes in communities. The spectral split method introduced in this paper works best when applied to the adjacency matrix, in which case it can reveal both unipartite and bipartite community structures, but for unipartite networks it can also be applied to the modularity matrix. Two algorithms are also introduced for the purpose of constructing the communities. Tests on statistical ensembles of benchmark networks show that one of these algorithms produces excellent results, especially in the case of average to high modularity networks. It is possible that further research will produce faster and more accurate algorithms for assigning the nodes to communities based on the spectral split method.
