Quasi-periodic and almost periodic homogenizations of integro-differential equations with Lévy operators.

Mariko Arisawa
DAMTP
University of Cambridge
E-mail: M.Arisawa@damtp.cam.ac.uk

1 Introduction.

The quasi-periodic homogenization and the almost periodic homogenization of a class of integro-differential equations with Lévy operators are studied in this paper. First, the quasi-periodic homogenization is the following. For $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, consider

$$ u_\varepsilon(x) + \sup_{\alpha \in \mathcal{A}} \{ - b(x, \alpha), \nabla u_\varepsilon \} - a(x) \int_{\mathbb{R}^N} [u_\varepsilon(x + z) - u_\varepsilon(x)] \frac{1}{|z|^{N+a}} dz - g_1\left(\frac{x}{\varepsilon_1}\right) - g_2\left(\frac{x}{\varepsilon_2}\right) = 0 \quad x \in \Omega, \quad (1) $$

$$ u_\varepsilon(x) = h(x) \quad x \in \Omega^c, \quad (2) $$

where $\Omega$ is an open domain in $\mathbb{R}^N$, $\mathcal{A}$ is a compact subset of a metric space, $a(\cdot)$ a bounded continuous function, $b(x, \alpha)$ is a bounded function from $\mathbb{R}^N \times \mathcal{A}$ to $\mathbb{R}^N$ such that there exists a constant $L > 0$

$$ |b(x, \alpha) - b(y, \alpha)| \leq L|x - y| \quad \forall x, y \in \mathbb{R}^N, \quad \alpha \in \mathcal{A}, \quad (3) $$

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$a(y), g_i(y)$ ($i = 1, 2$) are real valued periodic functions in $T^N$ ($N$ dimensional torus with periods 1), two parameters $\varepsilon_1, \varepsilon_2$ satisfy
\[
\frac{\varepsilon_2}{\varepsilon_1} = \gamma \in \mathbb{R} \setminus \mathbb{Q},
\]
the nonlocal (integral) term is the Lévy operator with the $\alpha$-stable symmetric measure
\[
\frac{1}{|z|^{N+\alpha}} dz \quad \text{with} \quad \alpha \in (0, 2),
\]
and $h(x)$ is a bounded continuous function defined in $\Omega \subset \mathbb{R}^N$. The existence and the uniqueness of the solution $u_\varepsilon$ is known in the framework of the viscosity solution. We are interested in the asymptotic limit of $u_\varepsilon$ as $\varepsilon \to 0$ while satisfying the relationship (4). We shall show in below in more generality the unique existence of the limit $u = \lim_{\varepsilon_1, \varepsilon_2 \to 0} u_\varepsilon$ and its characterization by an effective integro-differential equation.

Next, the following is an example of the almost periodic homogenization. Let $\varepsilon > 0$, and consider
\[
u_\varepsilon + \sup_{\alpha \in A} \{-b(x, \alpha, \nabla u_\varepsilon) \} - a(x) \int_{\mathbb{R}^N} [u_\varepsilon(x+z) - u_\varepsilon(x)]
- 1_{|z| < 1} \langle z, \nabla u_\varepsilon(x) \rangle \frac{1}{|z|^{N+\alpha}} dz - g(x) = 0 \quad x \in \mathbb{R}^N
\]
with (2), where $a(y), g(y)$ are real valued functions defined in $\mathbb{R}^N$, uniformly almost periodic in the sense of Bohr [14].

(Uniformly almost periodic function) A real valued function $f(y)$ defined in $\mathbb{R}^N$ is uniformly almost periodic in the sense of Bohr if and only if the set of functions
\[
\{ f(y+z) \mid z \in \mathbb{R}^N \}
\]
is relatively compact in the space of the bounded functions in $\mathbb{R}^N$ with the norm $\|f\|_\infty = \sup_{x \in \mathbb{R}^N} |f(x)|$.

Remark that a quasi-periodic function (for example $g_1(x_{\varepsilon_1}) + g_2(x_{\varepsilon_2})$ in (1)) is a uniformly almost periodic function. We refer the readers to Besicovitch [12] for the rich informations on the uniformly almost periodic function, some of which we utilize in below. As before, we are interested in the asymptotic
limit of the solution \( u_\varepsilon \) of (5) as \( \varepsilon \to 0 \), and in characterizing the limit by finding an effective integro-differential equation for it.

The present work is a straightforward generalization of the periodic homogenization for the integro-differential equation with the Lévy operator in Arisawa [7] to the quasi-periodic and the almost periodic homogenizations. In the case of the partial differential equation (PDE in short), such generalizations were done in Arisawa [1] (the quasi-periodic homogenization for first-order PDEs with non-convex Hamiltonians), [2] (the almost periodic homogenization for second-order elliptic PDEs), and in Ishii [21] (the almost periodic homogenization for first-order PDEs with the convex Hamiltonian), and then more generally treated in the stationally ergodic setting by Caffarelli, Souganidis and Wang in [16] (the stochastic homogenization for second-order uniformly elliptic PDEs). It is known that for non-convex first-order Hamilton-Jacobi equations, the almost-periodic homogenization is not well-posed in general. It is also known in Lions and Souganidis [24] that the stochastic homogenization for the first-order Hamilton-Jacobi equation is not necessarily well-posed. In the integro-differential problems (1) and (5), the \( \alpha \)-stable Lévy operator is the fractional power of Laplacian: \( \Delta^{\frac{\alpha}{2}} \). According to whether \( \alpha \in (0, 1] \) or \( \alpha \in (1, 2) \), the operator can be considered to be close to the first-order operator or to the second-order elliptic operator. Therefore, the quasi-periodic and the almost periodic homogenizations are natural to be studied for the integro-differential equations.

Now, let us explain the outline of this paper. We generalize the quasi-periodic problem to the following.

\[
\begin{align*}
  u_\varepsilon + \sup_{\alpha \in A} \{ -b(x, \alpha), \nabla u_\varepsilon \} - a(x) \int_{\mathbb{R}^N} [u_\varepsilon(x + z) - u_\varepsilon(x)] dz = 0 \\
  -1_{|z| \leq 1} \langle z, \nabla u_\varepsilon(x) \rangle \frac{1}{|z|^{N+\alpha}} dz - g_M\left(\frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_M}\right) = 0
\end{align*}
\]  

(6)

where such that

\[
a(y_1) \text{ is periodic in } T^N; \quad a(y_1) \geq a_0 \quad \forall y \in T^N,
\]  

(7)

where \( a_0 > 0 \) is a constant, and \( g_M(y_1, \ldots, y_M) \) \( (M \in \mathbb{N}) \) is a real valued periodic functions in \( (y_1, \ldots, y_M) \in T^{MN}, \varepsilon_i > 0 \) \( (1 \leq i \leq M) \) satisfy the following
non-resonance condition.

(Non-resonance condition) A countable set of real numbers $E = \{\varepsilon_i\} (i \in \mathbb{N})$ is said to satisfy the non-resonance condition if for any $k \in \mathbb{N}$ and for $E_k = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_k\}$ the only rational numbers $a_1, a_2, ..., a_k$ to satisfy

$$\sum_{i=1}^{k} a_i \varepsilon_i = 0$$

are $a_i = 0 (1 \leq i \leq k)$.

The above condition is taken from [12], where the finite version was used in Arisawa and Lions [8]. We assume also that there exists a constant $\theta_0 \in (0, 1]$ such that

$$|a(y) - a(y')| \leq C|y - y'|^{\theta_0} \quad \forall y, y' \in T^\mathbb{N}, \quad 1 \leq i \leq M, \quad (9)$$

$$|g_M(y_1, ..., y_{i-1}, \overline{y}_i, y_{i+1}, ..., y_M) - g_M(y_1, ..., y_{i-1}, \overline{y}_i', y_{i+1}, ..., y_M)| \leq C|\overline{y}_i - \overline{y}_i'|^{\theta_0}$$

$$\forall \overline{y}_i, \overline{y}_i' \in T^\mathbb{N}, \quad 1 \leq i \leq M, \quad (10)$$

where $C > 0$ is a constant which depends only on $\theta_0$.

Our method is based on the relationship between the formal asymptotic expansion and the ergodic problem. The formal asymptotic expansion was introduced by Bensoussan, Lions and Papanicolaou in [13], and developed rigorously by Lions, Papanicolaou and Varadhan [23], Evans [18], [19], and others. In §2, we utilize the formal asymptotic expansion method to obtain the ergodic cell problem. The key ingredient to solve the ergodic cell problem is the strong maximum principle for the Lévy operator. In §3, we prove the strong maximum principle for some general class of Lévy operators

$$\int_{\mathbb{R}^N} [u(x + z) - u(x) - 1_{|z| \leq 1} \langle z, \nabla u(x) \rangle] dq(z),$$

which includes the $\alpha$-stable symmetric operators as special cases. In §4, by using the result in §3 the quasi-periodic ergodic cell problems are solved. In §5, the almost periodic ergodic cell problems are solved. In §6, we give our main results on the quasi-periodic and the almost periodic homogenizations.
For an upper semi-continuous (USC in short) function $u$ and a lower semi-continuous (LSC in short) function $v$ in $\mathbb{R}^N$, $J_{\Omega}^{2,+}u(x)$ and $J_{\Omega}^{2,-}v(x)$ represent respectively the set of second-order subdifferentials and the set of superdifferentials of $u$ and $v$ at $x \in \Omega$. That is, for $u \in USC(\mathbb{R}^N)$, $(p,Q) \in J_{\Omega}^{2,+}u(x)$ means that $(p,Q) \in \mathbb{R}^N \times \mathbb{S}^N$, and for any $\delta > 0$ there exists $\nu > 0$ such that

$$u(x + z) \leq u(x) + \langle p, z \rangle + \frac{1}{2} \langle Qz, z \rangle + \delta |z|^2 \quad \forall |z| \leq \nu. \quad (11)$$

For $v \in LSC(\mathbb{R}^N)$, $(p,Q) \in J_{\Omega}^{2,-}v(x)$ means that $(p,Q) \in \mathbb{R}^N \times \mathbb{S}^N$, and for any $\delta > 0$ there exists $\nu > 0$ such that

$$v(x + z) \geq v(x) + \langle p, z \rangle + \frac{1}{2} \langle Qz, z \rangle - \delta |z|^2 \quad \forall |z| \leq \nu. \quad (12)$$

We use the notation $I[u](x) = \int_{\mathbb{R}^N} [u(x + z) - u(x) - 1_{|z| \leq \nu} \langle \nabla u(x), z \rangle ] dq(z)$,

$$I_{\nu,\delta}^{1,+}[u,p,X](x) = \int_{|z| \leq \nu} \frac{1}{2} \langle (X + 2\delta I)z, z \rangle dq(z),$$

(resp.

$$I_{\nu,\delta}^{1,-}[u,p,X](x) = \int_{|z| \leq \nu} \frac{1}{2} \langle (X - 2\delta I)z, z \rangle dq(z),$$

$$I_{\nu,\delta}^{2}[u,p,X](x) = \int_{|z| > \nu} [u(x + z) - u(x) - 1_{|z| \leq \nu} \langle p, z \rangle ] dq(z).$$

Consider

$$A(x,u(x),\nabla u(x),\nabla^2 u(x),I[u](x)) = 0 \quad x \in \Omega, \quad (13)$$

where $A(x,u,p,Q,I) \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \times \mathbb{R})$.

**Definition 1.1.** Let $u \in USC(\mathbb{R}^N)$ (resp. $v \in LSC(\mathbb{R}^N)$). We say that $u$ (resp. $v$) is a viscosity subsolution (resp. supersolution) of (13), if for any $\hat{x} \in \Omega$, any $(p,X) \in J_{\mathbb{R}^N}^{2,+}u(\hat{x})$ (resp. $J_{\mathbb{R}^N}^{2,-}v(\hat{x})$), and for any pair of numbers $(\varepsilon,\delta)$ satisfying (11) (resp. (12)), the following holds

$$A(\hat{x},u(\hat{x}),p,X,I_{\nu,\delta}^{1,+}[u,p,X](\hat{x}) + I_{\nu,\delta}^{2}[u,p,X](\hat{x})) \leq 0.$$

(resp.

$$A(\hat{x},v(\hat{x}),p,X,I_{\nu,\delta}^{1,-}[v,p,X](\hat{x}) + I_{\nu,\delta}^{2}[v,p,X](\hat{x})) \geq 0.$$
If \( u \) is a viscosity subsolution and a viscosity supersolution at the same time, it is called a viscosity solution.

The above Definition 1.1 is equivalent to the following (see Arisawa [6]).

**Definition 1.2.** Let \( u \in USC(\mathbb{R}^N) \) (resp. \( v \in LSC(\mathbb{R}^N) \)). We say that \( u \) (resp. \( v \)) is a viscosity subsolution (resp. supersolution) of (13), if for any \( \hat{x} \in \Omega \), any \( \phi \in C^2(\mathbb{R}^N) \) such that \( u(\hat{x}) = \phi(\hat{x}) \) and \( u - \phi \) takes a global maximum (resp. minimum) at \( \hat{x} \),

\[
A(\hat{x}, u(\hat{x}), \nabla \phi(\hat{x}), \nabla^2 \phi(\hat{x}), I[\phi](\hat{x})) \leq 0.
\] (14)

(resp.

\[
A(\hat{x}, v(\hat{x}), \nabla \phi(\hat{x}), \nabla^2 \phi(\hat{x}), I[\phi](\hat{x})) \geq 0.
\] (15)

If \( u \) is a viscosity subsolution and a viscosity supersolution at the same time, it is called a viscosity solution.

The existence and the uniqueness of the solution \( u_\varepsilon \) of (5)-(2) and (6)-(2) are established in the framework of the viscosity solution. We refer the readers to Arisawa [3], [4], Barles, Buckdahn and Pardoux [9], Barles and Imbert [10], etc...

2 **Formal asymptotic expansions.**

We treat the quasi-periodic homogenization [6]. The almost periodic homogenization can be treated similarly, which we mention in §5 below. Put \( \varepsilon = (\varepsilon_1, ..., \varepsilon_M) \), and \( \gamma_i = \frac{\varepsilon_i}{\varepsilon_1} \) \((1 \leq i \leq M)\). We devide the situation into three cases.

- **I.** \( \alpha \in (0, 1) \) and \( b(x, \alpha) \not\equiv 0 \).
- **II.** \( \alpha = 1 \) and \( b(x, \alpha) \not\equiv 0 \).
- **III.** \( \alpha \in (1, 2) \), or \( \alpha \in (0, 1] \) and \( b(x, \alpha) \equiv 0 \).
The formal asymptotic expansions are: for the case of I and II
\[ u_{\varepsilon}(x) = \bar{u}(x) + \varepsilon_1 v(\frac{x}{\varepsilon_1}), \quad (16) \]
and for the case of III
\[ u_{\varepsilon}(x) = \bar{u}(x) + \varepsilon_1^\alpha v(\frac{x}{\varepsilon_1}). \quad (17) \]

We introduce in (6) the formal derivatives of the above expansions for each case.

**Case I.** If \( \alpha \in (0,1) \) and \( b(x, \alpha) \neq 0 \), by introducing the derivatives of (16) into (6), ignoring the \( o(1) \) terms, and rewriting \( y = \frac{x}{\varepsilon_1}, \quad p = \nabla \bar{u}(x) \), we get the following relationship.
\[
\bar{u}(x) + \sup_{\alpha \in \mathcal{A}} \{ -b(x, \alpha), p + \nabla_y v(y) \} - a(y) I[\bar{u}](x) - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) = 0.
\]

As in [23] and other works, for each fixed \( (x, p, I) \in \Omega \times \mathbb{R}^N \times \mathbb{R} \) we intend to get a unique constant \( d_{x,p,I} \) such that there exists at least a viscosity solution \( v(y) \), bounded in \( \mathbb{R}^N \),
\[
d_{x,p,I} + \sup_{\alpha \in \mathcal{A}} \{ -b(x, \alpha), p + \nabla_y v(y) \} - a(y) I - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) = 0 \quad y \in \mathbb{R}^N,
\]
which is the ergodic cell problem. In fact, if \( d_{x,p,I} \) exists for any \( (x, p, I) \), then by putting \( \bar{T}(x, p, I) = -d_{x,p,I} \) the limit \( \bar{u} \) formally satisfies
\[
\bar{u} + \bar{T}(x, \nabla \bar{u}(x), I[\bar{u}](x)) = 0 \quad x \in \Omega, \quad (19)
\]
which will be verified rigorously below in §6.

**Case II.** If \( \alpha = 1 \) and \( b(x, \alpha) \neq 0 \), the introduction of the derivatives of (16) into (6) leads
\[
\bar{u} + \sup_{\alpha \in \mathcal{A}} \{ -b(x, \alpha), p + \nabla_y v(y) \} - a(y) \int_{\mathbb{R}^N} [v(y+z) - v(y)] - a(y) I[u](x) - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) = 0,
\]
where \( y, p \) are same as above. The following ergodic cell problem is thus derived. For each fixed \((x, p, I) \in \Omega \times \mathbb{R}^N \times \mathbb{R}\), find a unique number \( d_{x,p,I} \) such that the following problem has at least a viscosity solution \( v(y) \), bounded in \( \mathbb{R}^N \),

\[
d_{x,p,I} + \sup_{\alpha \in A} \{ -b(x, \alpha), p + \nabla y v(y) \} - a(y) \int_{\mathbb{R}^N} [v(y+z) - v(y)] - a(y) I - g_M(\gamma^{-1}_1 y, ..., \gamma^{-1}_M y) = 0 \quad y \in \mathbb{R}^N. \tag{20}
\]

As before, if \( d_{x,p,I} \) exists for any \((x, p, I)\), then by putting \( I(x, p, I) = -d_{x,p,I} \) the limit \( \overline{u} \) formally satisfies (19), which will be shown rigorously later.

**Case III.** If \( \alpha \in (1, 2) \), or \( \alpha \in (0, 1) \) and \( b(x, \alpha) \equiv 0 \), the introduction of the derivatives of (17) into (6) leads

\[
\overline{u} + \sup_{\alpha \in A} \{ -b(x, \alpha), \nabla \overline{u} \} - a(y) I[\overline{u}](x) - a(y) \int_{\mathbb{R}^N} [\overline{u}(x+z) - \overline{u}(x)] - a(y) I \overline{u}(x) - g_M(\gamma^{-1}_1 y, ..., \gamma^{-1}_M y) = 0, \quad y \in \mathbb{R}^N.
\]

where \( y = \frac{x}{\varepsilon_1}, p = \nabla \overline{u}(x), \) and \( o(1) \) terms are neglected. Then, we are interested in the following ergodic cell problem. For each fixed \((x, p, I) \in \Omega \times \mathbb{R}^N \times \mathbb{R}\), find a unique constant \( d_{x,p,I} \) such that there exists at least a viscosity solution \( v(y) \), bounded in \( \mathbb{R}^N \),

\[
d_{x,p,I} - a(y) \int_{\mathbb{R}^N} [v(y+z) - v(y)] - 1_{|z| \leq 1} \langle z, \nabla y v(y) \rangle \frac{1}{|z|^{N+\alpha}} dz - a(y) I \overline{u}(x) - g_M(\gamma^{-1}_1 y, ..., \gamma^{-1}_M y) = 0 \quad y \in \mathbb{R}^N. \tag{21}
\]

If \( d_{x,p,I} \) exists for any \((x, p, I)\), then by defining \( \overline{T}(x, p, I) = -d_{x,p,I} \), the limit \( \overline{u} \) formally satisfies (19), which will be rigorously proved in below.

Instead of (16) and (17), the following expansions are also possible. For the cases of I and II

\[
u_\varepsilon(x) = \overline{u}(x) + \varepsilon_1 w\left( \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, ..., \frac{x}{\varepsilon_M} \right), \tag{22}
\]

\[8\]
and for the case of III

\[ u_\varepsilon(x) = \overline{u}(x) + \varepsilon^\alpha w\left(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \ldots, \frac{x}{\varepsilon_M}\right). \]

(23)

Let \( B(x, \alpha) = (\gamma_1^{-1}b(x, \alpha), \ldots, \gamma_M^{-1}b(x, \alpha)) \), \( \Gamma z = (\gamma_1^{-1}z, \ldots, \gamma_M^{-1}z) \). By introducing the derivatives of (22), (23) into (6), by putting \( p = \nabla_x \overline{u}, \ y_i = \frac{x}{\varepsilon_i}, \ I = I[\overline{u}](x) \), we get the ergodic cell problems: find a unique number \( d_{x,p,I} \) such that there exists at least a periodic viscosity solution \( w(y) \) \( (y = (y_1, \ldots, y_M) \in T^{MN}, \ y_i \in T^N, \ 1 \leq i \leq M) \) which satisfies the following. For the case I,

\[
d_{x,p,I} + \sup_{\alpha \in A} \left\{ -b(x, \alpha) - \langle B(x, \alpha), \nabla w(y) \rangle - a(y_1)I - g_M(y) \right\} = 0 \quad \forall y \in T^{MN}. \]

(24)

For the case II,

\[
d_{x,p,I} + \sup_{\alpha \in A} \left\{ -b(x, \alpha) - \langle B(x, \alpha), \nabla w(y) \rangle - a(y_1)I - a(y_1) \int_{\mathbb{R}^N} \left[ w(y + \Gamma z) - w(y) - 1_{|z| \leq 1} \langle \Gamma z, \nabla w(y) \rangle \right] dq(z) - g_M(y) \right\} = 0 \quad \forall y \in T^{MN}. \]

(25)

For the case III,

\[
d_{x,p,I} - a(y_1) \int_{\mathbb{R}^N} \left[ w(y + \Gamma z) - w(y) - 1_{|z| \leq 1} \langle \Gamma z, \nabla w(y) \rangle \right] dq(z) - a(y_1)I - g_M(y) = 0 \quad \forall y \in T^{MN}. \]

(26)

**Remark 2.1.** The two types of the ergodic cell problems (18), (20), (21) and (24), (25), (26) are respectively connected by the relationship

\[ v(y) = w(y, \gamma_2^{-1}y, \ldots, \gamma_M^{-1}y). \]

(27)

We use (24)-(26) to complement the informations of (18), (20) and (21) in below.

3 Strong maximum principle.

The strong maximum principle is the key to solve the ergodic cell problem. The present result concerns with a general class of the Lévy operators

\[
\int_{\mathbb{R}^N} \left[ w(y + \Gamma z) - w(y) - 1_{|z| \leq 1} \langle \Gamma z, \nabla w(y) \rangle \right] dq(z) - a(y_1)I - g_M(y) = 0 \quad \forall y \in T^{MN}. \]

(26)
including the $\alpha$-stable symmetric operator. This is an improvement of our previous result in [4]. Consider

$$H(x, \nabla u, \nabla^2 u) - \int_{\mathbb{R}^N} [u(x+z) - u(x) - \mathbf{1}_{|z| \leq 1} \langle z, \nabla u(x) \rangle] dq(z) = 0 \quad x \in \mathbb{R}^N,$$

(28)

where $H \in C(\Omega \times \mathbb{R}^N \times \mathbb{S}^N)$, $dq(z)$ is a positive Radon measure such that

$$\int_{|z| < 1} |z|^2 dq(z) + \int_{|z| \geq 1} |z|^2 dq(z) < \infty. \quad (29)$$

Assume that

$$H(x, 0, O) \geq 0 \quad \forall x \in \mathbb{R}^N, \quad (30)$$

and that there exists a ball in $\mathbb{R}^N$, $B = B_r(0)$, centered at the origin with radius $r > 0$, for which the following holds

$$\int_B 1 dq(z) > 0 \quad \forall x \in \mathbb{R}^N. \quad (31)$$

Theorem 3.1.

Let $u$ be a viscosity subsolution of (28), and assume that (30) and (31) hold. Assume also that there exists a maximum point of $u$, $\hat{x}$ in $\mathbb{R}^N$. Then, $u$ is constant almost everywhere in $\mathbb{R}^N$.

Proof. Let $M = \max_{\mathbb{R}^N} u(x)$. Put $D = \{x \in \mathbb{R}^N | \ u(x) = M\}$, which is non-empty and closed from the assumption. If $D = \mathbb{R}^N$, the claim is clear. So, assume that there exists a point $y_1 \in D^c$ such that $\text{dist}(y_1, D) = \inf_{x \in D} |y_1 - x|^2$. Take $x_1 \in D$ such that $|x_1 - y_1|^2 < s$. Since $D^c$ is open, there exists $0 < s < \frac{\nu}{2}$ such that

$$u(y) < u(x_1) = M \quad \text{if} \quad |y - y_1| < s. \quad (32)$$

Since $x_1$ is a maximum point of $u$, $(0, O) \in J_{\mathbb{R}^N}^{2,+} u(x_1)$, i.e. for any $\delta > 0$ there exists $\nu > 0$ such that

$$u(x_1 + z) \leq u(x_1) + \langle 0, z \rangle + \frac{1}{2} \langle Oz, z \rangle + \delta |z|^2 \quad \forall |z| \leq \nu.$$

From the definition of the viscosity subsolution

$$H(x_1, 0, O) - \int_{|z| \leq \nu} \frac{1}{2} \langle (O + 2\delta I)z, z \rangle dq(z)$$
\[-\int_{|z|>\nu} [u(x_1 + z) - u(x_1) - 1_{|z|\leq 1} (0, z)] dq(z) \leq 0.\]

Put \( E = \{ z | x_1 + z \in B(y_1, s) \} \). Remark that \( E \subset B \). Since \( u(x_1 + z) - u(x_1) \leq 0 \) for any \( z \in \mathbb{R}^N \), from (30), (31) and (32), the above inequality leads

\[
0 < -\int_{E \cap \{ |z|>\nu \}} [u(x_1 + z) - u(x_1) - 0, z] dq(z)
\leq -\int_{|z|>\nu} [u(x_1 + z) - u(x_1) - 0, z] dq(z) \leq O(\delta).
\]

By choosing \( \delta > 0 \) small enough we get a contradiction, and \( D = \mathbb{R}^N \) must hold.

**Remark 3.1.**

1. In [4], instead of (31), the following condition was assumed.

\[
\int_D 1 dq(z) > 0 \quad \forall x \in \mathbb{R}^N, \quad \forall D \subset \mathbb{R}^N \text{ open}. \quad (33)
\]

Various generalization is possible beyond Theorem 2.1, which we shall visit in our future work.

2. The \( \alpha \)-stable symmetric operator satisfies the conditions (29) and (31) assumed in Theorem 3.1.

For the later purpose, we are also interested in the following "degenerate" Lévy operator in \( \mathbb{T}^{2N} \). Let \( v(x_1, x_2) \) be a periodic function in \( \mathbb{R}^{2N} \), a solution of

\[
H(x, \nabla v, \nabla^2 v) - \int_{\mathbb{R}^{2N}} [v(x_1 + z, x_2 + \gamma^{-1} z) - v(x_1, x_2)] dq(z) = 0 \quad x = (x_1, x_2) \in \mathbb{T}^{2N},
\]

where \( \gamma \in \mathbb{R} \setminus \mathbb{Q}, \ H(x, p, R) \in C(\mathbb{R}^{2N} \times \mathbb{R}^{2N} \times \mathbb{S}^{2N}) \) satisfies

\[
H(x, 0, O) \geq 0 \quad \forall x \in \mathbb{R}^{2N}, \quad (35)
\]

and \( dq(z) \) satisfies

\[
\int_D 1 dq(z) > 0 \quad \forall x \in \mathbb{R}^{2N}, \quad \forall D \subset \mathbb{R}^N \text{ open}. \quad (36)
\]

We claim the following.

**Proposition 3.2.**
Let (29), (35), and (36) hold. Let \( u \) be a periodic viscosity subsolution of (34) in \( \mathbb{R}^{2N} \). Assume that there exists a maximum point \( \hat{x} = (\hat{x}_1, \hat{x}_2) \in \mathbb{R}^{2N} \). Then, \( u \) is constant almost everywhere in \( \mathbb{R}^{2N} \).

**Proof.** We use the argument by contradiction. Put \( D_0 = \{ x \in \mathbb{R}^{2N} | u(x) < u(\hat{x}) \} \). Assume that \( D_0 \) is non-empty and thus open. Since \( \gamma \) is irrational, the set \( \{ \hat{x} + (z, \gamma^{-1}z) | z \in \mathbb{R}^N \} \) is dense in \( \mathbb{T}^{2N} \), and in particularly in \( D_0/\mathbb{Z}^N \). Thus, \( D_1 = \{ z \in \mathbb{R}^N | u(\hat{x}_1 + z, \hat{x}_2 + \gamma^{-1}z) < u(\hat{x}_1, \hat{x}_2) \} \) is non-empty and open. On the other hand, since \( \hat{x} \) is a maximum point of \( u \), \( (0, O) \in J_{\mathbb{R}^{2N}}^0 u(\hat{x}) \), i.e. for any \( \delta > 0 \) there exists \( \nu > 0 \) such that

\[
u_0 > 0 \text{ such that } |(z, \gamma^{-1}z)| \leq \nu \text{ for any } |z| < \nu_0.
\]

From the definition of the viscosity subsolution,

\[
\begin{align*}
H(\hat{x}, 0, O) - \int_{|z| < \nu_0} \frac{1}{2} \langle (O + 2\delta I)(z, \gamma^{-1}z), (z, \gamma^{-1}z) \rangle dq(z) \\
- \int_{|z| \geq \nu_0} [u(\hat{x} + (z, \gamma^{-1}z)) - u(\hat{x}) - 1_{|z| \leq 1} \langle 0, (z, \gamma^{-1}z) \rangle] dq(z) \leq 0.
\end{align*}
\]

Now, (35), (36), the definition of \( D_1 \), and the above inequality lead

\[
0 < -\int_{D_1 \cap \{|z| \geq \nu_0\}} [u(\hat{x}_1 + z, \hat{x}_2 + \gamma^{-1}z) - u(\hat{x}_1, \hat{x}_2)] dq(z)
\]

\[
\leq -\int_{|z| \geq \nu_0} [u(\hat{x}_1 + z, \hat{x}_2 + \gamma^{-1}z) - u(\hat{x}_1, \hat{x}_2) - 1_{|z| \leq 1} \langle (z, \gamma^{-1}z), 0 \rangle] dq(z) \leq O(\delta),
\]

which is a contradiction for \( \delta > 0 \) sufficiently small. Therefore, \( D_0 \) must be measure zero.

**Remark 3.2.** Proposition 3.2 can be generalized to the operator in \( \mathbb{R}^{MN} \).

4 Ergodic problems for the quasi-periodic homogenizations.

First, we study the following general ergodic problem, including the cases I and II as special examples.
(P) Find a unique constant $d$ such that the following problem has at least a viscosity solution $v(y)$, bounded in $\mathbb{R}^N$:

$$
d + \sup_{\alpha \in A} \{ -\langle -\beta(\alpha), \nabla v(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [v(y + z) - v(y)]$$

$$- 1_{|z| \leq 1} \langle z, \nabla v(y) \rangle dq(z) - a(y) I - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) = 0 \quad y \in \mathbb{R}^N,$$

where $\beta(\alpha) \in \mathbb{R}^N (\alpha \in A)$, and

$$|c(\alpha)| \leq C \quad \forall \alpha \in A. \quad (38)$$

In some cases, the number $d$ can only be characterized by the following (see [8]): for any $\mu > 0$ there exist $v$ and $\overline{v}$ such that

$$d + \sup_{\alpha \in A} \{ -\langle -\beta(\alpha), \nabla v(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [v(y + z) - v(y)]$$

$$- 1_{|z| \leq 1} \langle z, \nabla v(y) \rangle dq(z) - a(y) I - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) \leq \mu \quad y \in \mathbb{R}^N,$$

and

$$d + \sup_{\alpha \in A} \{ -\langle -\beta(\alpha), \nabla \overline{v}(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [\overline{v}(y + z) - \overline{v}(y)]$$

$$- 1_{|z| \leq 1} \langle z, \nabla \overline{v}(y) \rangle dq(z) - a(y) I - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) \geq -\mu \quad y \in \mathbb{R}^N. \quad (39)$$

We need also the following formulation.

(Q) Find a unique constant $d$ such that the following problem has at least a viscosity solution $w(y)$ ($y = (y_1, ..., y_M)$), periodic in $\mathbb{T}^M$,

$$d + \sup_{\alpha \in A} \{ -B(\alpha), \nabla w(y) \} - c(\alpha) \} - a(y_1) \int_{\mathbb{R}^N} [w(y + \Gamma^{-1} z) - w(y)]$$

$$- 1_{|z| \leq 1} \langle \Gamma^{-1} z, \nabla w(y) \rangle dq(z) - a(y_1) I - g_M(y_1, ..., y_M) = 0 \quad \overline{y} \in \mathbb{T}^M,$$

where $B(\alpha) = (\gamma_1^{-1} \beta(\alpha), ..., \gamma_M^{-1} \beta(\alpha))$, $\Gamma^{-1} = (\gamma_1^{-1}, ..., \gamma_M^{-1})$.

In fact, (P) and (Q) are related by $v(y) = w(\gamma_1^{-1} y, ..., \gamma_M^{-1} y)$. We abbreviate the weaker version of (Q).
As in [8], we approximate (P) by
\[ lv_l + \sup_{\alpha \in A} \{ -\beta(\alpha), \nabla v_l(y) \} - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [v_l(y + z) - v_l(y) \)
\[ - 1_{|z| \leq 1} \langle \nabla v_l(y) \rangle \rangle dq(z) - a(y) I - g_M(\gamma_1^{-1} y, \ldots, \gamma_M^{-1} y) = 0 \quad y \in \mathbb{R}^N, \quad (40) \]
and (Q) by
\[ lw_l + \sup_{\alpha \in A} \{ -B(\alpha), \nabla w_l(y) \} - c(\alpha) \} - a(y_1) \int_{\mathbb{R}^N} [w_l(y + \Gamma^{-1} z) - w_l(y) \)
\[ - 1_{|z| \leq 1} \langle \Gamma^{-1} z, \nabla w_l(y) \rangle \rangle dq(z) - a(y) I - g_M(y) = 0 \quad y \in \mathbb{T}_{MN}, \quad (41) \]
for \( l \in (0, 1). \)

**Remark 4.1.** If \( dq(z) = \frac{1}{|z|^{N+1}} dz \) (\( \alpha = 1 \) in (40)), then the Lévy operator is close to the first-order partial differential operator. Certainly, a condition is necessary between \((\beta, c)\) and \(a(\cdot)\) to determine which term: \( \sup_{\alpha \in A} \{ -B(\alpha), \nabla w_l(y) \} - c(\alpha) \} \), and \( -a(y_1) \int_{\mathbb{R}^N} [w_l(y + \Gamma^{-1} z) - w_l(y) \)
\[ - 1_{|z| \leq 1} \langle \Gamma^{-1} z, \nabla w_l(y) \rangle \rangle dq(z) \), in major, serves for the ergodicity. This is not a trivial question, and to avoid the complexity we assume that \( a(\cdot) \equiv a \), if \( \alpha = 1. \)

Our claim is the following.

**Theorem 4.1.**

Assume that [7], [8], [9], [10], [22], [31], and [38] hold. Assume also that either \( a(\cdot) \equiv a \) (\( a > 0 \) is a constant), or \( \beta(\alpha) \equiv 0 \) (\( \forall \alpha \in A \)). Let \( v_l \) be the solution of (40). Then, for any \( \theta \in (0, \theta_0] \), there exists a constant \( C_\theta > 0 \) independent on \( l > 0 \), such that
\[ |v_l(y) - v_l(y')| \leq \frac{C_\theta}{l} |y - y'|^\theta \quad \forall y, y' \in \mathbb{R}^N. \quad (42) \]

We prepare some lemmas.

**Lemma 4.2.**
Consider (40) (resp. (41)). The following hold.

(i) Let $v_l$ (resp. $w_l$) be a bounded USC subsolution of (40) (resp. (41)). Let $v_l$ (resp. $w_l$) be a bounded LSC supersolution of (40) (resp. (41)). Then, $v_l < v_l$ (resp. $w_l < w_l$) holds in $\mathbb{R}^N$ (resp. $\mathbb{T}^{MN}$).

(ii) There exists a unique bounded viscosity solution $v_l$ (resp. $w_l$) of (40) (resp. (41)).

**Proof.** (i) The proof of the comparison principle can be done by a standard way. We refer the reader to [3], [4], [9], [10].

(ii) The existence of the solutions can be shown by the Perron’s method (see Crandall, Ishii and Lions [17]). We refer the reader to [3], [4], [9], [10] for details.

We multiply (40) by $l > 0$, put $m_l = lv_l$, $f(y) = g_M(\gamma_1^{-1}y, ..., \gamma_M^{-1}y) + a(y) I$ to have

$$lm_l(y) + \sup_{\alpha \in A}\{(-\beta(\alpha), \nabla m_l(y)) - lc(\alpha)\} - a(y) \int_{\mathbb{R}^N} [m_l(y + z) - m_l(y)] 1_{|z| < 1}\langle z, \nabla m_l(y)\rangle dq(z) = lf(y) \quad y \in \mathbb{R}^N.$$  

**Lemma 4.3.**

There exists a constant $M > 0$ such that

$$|m_l| \leq M \quad \forall \lambda \in (0, 1), \quad (44)$$

and for any $\theta \in (0, \theta_0]$ there exists a constant $C_\theta > 0$ independent on $l > 0$ such that

$$|m_l(y) - m_l(y')| \leq C_\theta |y - y'|^\theta \quad \forall y, y' \in \mathbb{R}^N. \quad (45)$$

**Proof.** Since $m_l = lv_l$, from the comparison principle for (40) (Lemma 4.2 (i)), (41) is clear. Fix $\theta \in (0, \theta_0]$, and let $r_0 > 0$ be a constant to be determined later. Put

$$C_\theta = \frac{2M}{r_0^\theta}. \quad (46)$$

We use the argument by contradiction to prove the claim. So, assume that there exist $\tilde{y}, \tilde{y'}$ such that

$$m_l(\tilde{y}) - m_l(\tilde{y'}) > C_\theta |\tilde{y} - \tilde{y'}|^\theta. \quad (47)$$
From (44), (46), \( |\tilde{y} - y'| < r_0 \) must hold. We regularize \( m_t \) by the sup-convolution and the inf-convolution: for \( l > 0 \)

\[
m^r(y) = \sup_{y' \in \mathbb{R}^N} \{ m_t(y') - \frac{r}{2}|y - y'|^2 \}, \quad m_r(y) = \inf_{y' \in \mathbb{R}^N} \{ m_t(y') + \frac{r}{2}|y - y'|^2 \}.
\]

Remark that (see [4]) for any \( \nu > 0 \) we can take \( r > 0 \) small enough so that

\[
\begin{align*}
lm^r + \sup_{\alpha \in A} \{ -\beta(\alpha), \nabla m^r(y) \} - lc(\alpha) & - a(y) \int_{\mathbb{R}^N} [m^r(y + z) - m^r(y)] \\
- 1_{|z| < 1} \langle z, \nabla m^r(y) \rangle dq(z) & \leq |f(y) + \nu|,
\end{align*}
\]

\[
\begin{align*}
lm_r + \sup_{\alpha \in A} \{ -\beta(\alpha), \nabla m_r(y) \} - lc(\alpha) & - a(y) \int_{\mathbb{R}^N} [m_r(y + z) - m_r(y)] \\
- 1_{|z| < 1} \langle z, \nabla m_r(y) \rangle dq(z) & \geq |f(y) - \nu|,
\end{align*}
\]
in the sense of the viscosity solution. Since \( m_r \leq m_t \leq m^r \), from (47)

\[
m^r(\hat{y}) - m_r(\hat{y}') > C_0|\tilde{y} - y'|^\theta. \tag{48}
\]

Define

\[
\Phi(y, y') = m^r(y) - m_r(y') - C_0|y - y'|^\theta \quad \forall (y, y') \in \mathbb{R}^{2N}.
\]

Since \( m^r, m_r \) are the sup and the inf convolutions of \( lv = lw(\gamma_1^{-1}y, \ldots, \gamma_M^{-1}y) \), and since \( w_l \) is periodic, the maximum point of \( \Phi \) exists. Let \((\hat{y}, \hat{y}')\) be the maximum point of \( \Phi \).

Put \( p = \nabla_y \phi(\hat{y}, \hat{y}') \), \( Q = \nabla_{y'} \phi(\hat{y}, \hat{y}') \). In particular, we may assume that \((\hat{y}, \hat{y}')\) is a global strict maximum point of \( \Phi \). We can take an open precompact subset \( \mathcal{O} \subset \mathbb{R}^{2N} \) such that \((\hat{y}, \hat{y}') \in \mathcal{O}, \sup_{\mathcal{O}} \Phi(y, y') - \sup_{\partial \mathcal{O}} \Phi(y, y') > 0 \). Then, from the Alexandrov’s maximum principle and the Jensen’s lemma (see Fleming and Soner [20]), the following holds (4).

**Lemma A.** (4) Lemma 1.3.)

(i) There exists a sequence \((y_j, y'_j)\) in \( \mathcal{O} \) which converges to \((\hat{y}, \hat{y}')\) as \( j \to \infty \), and \((p_j, Y_j) \in J_1^{2+} m^r(y_j) \), \((p'_j, Y'_j) \in J_1^{-} m_r(y'_j) \) such that

\[
\lim_{j \to \infty} p_j = \lim_{j \to \infty} p'_j = p, \quad Y_j \leq Y'_j \quad \forall j \in \mathbb{Z}.
\]
(ii) For \( P_j = (p_j - p, -(p'_j - p)) \), \( \Phi_j(y, y') = \Phi(y, y') - \langle P_j, (y, y') \rangle \) takes a maximum at \((y_j, y'_j)\) in \( \mathcal{O} \).

(iii) For any \( z \in \mathbb{R}^n \) such that \((y_j + z, y'_j + z) \in \mathcal{O}\)

\[ m^r(y_j + z) - m^r(y_j) - \langle p_j, z \rangle \leq m_r(y'_j + z) - m_r(y'_j) - \langle p'_j, z \rangle. \]

Take a pair of positive numbers \((\nu_j, \delta_j)\) such that

\[ m^r(y_j + z) \leq m^r(y_j) + \langle z, p_j \rangle + \frac{1}{2} \langle Y_j z, z \rangle + \delta_j |z|^2 \quad \forall |z| \leq \nu_j, \]

\[ m_r(y'_j + z) \geq m_r(y'_j) + \langle z, p'_j \rangle + \frac{1}{2} \langle Y'_j z, z \rangle - \delta_j |z|^2 \quad \forall |z| \leq \nu_j. \]

From the definition of the viscosity solution, by remarking that \( a(y_j), a(y'_j) \geq a_0 > 0 \),

\[
\begin{align*}
\frac{lm^r(y_j)}{a(y_j)} + \sup_{\alpha \in A} \{ -\frac{\beta(\alpha)}{a(y_j)} p_j - \frac{lc(\alpha)}{a(y_j)} \} &- \int_{|z| < \nu_j} \frac{1}{2} \langle (Y_j + 2\delta_j I) z, z \rangle dq(z)
\end{align*}
\]

\[
- \int_{|z| > \nu_j} \{ m^r(y_j + z) - m^r(y_j) - 1_{|z| \leq 1} \langle z, p_j \rangle \} dq(z) \leq \frac{lf(y_j) + \nu}{a(y_j)},
\]

\[
\frac{lm_r(y'_j)}{a(y'_j)} + \sup_{\alpha \in A} \{ -\frac{\beta(\alpha)}{a(y'_j)} p'_j - \frac{lc(\alpha)}{a(y'_j)} \} - \int_{|z| < \nu_j} \frac{1}{2} \langle (Y'_j - 2\delta_j I) z, z \rangle dq(z)
\]

\[
- \int_{|z| > \nu_j} \{ m_r(y'_j + z) - m_r(y'_j) - 1_{|z| \leq 1} \langle z, p'_j \rangle \} dq(z) \geq \frac{lf(y'_j) - \nu}{a(y'_j)}. \]

We take the difference of two inequalities. Put

\[ \mathcal{O}_j^c = \{|z| > v_j\} \cap \{z| (y_j + z, y'_j + z) \in \mathcal{O}^c\}. \]

By remarking \( Y_j \leq Y'_j \), Lemma A (iii), by choosing \( \alpha \in A \) appropriately

\[
\begin{align*}
\frac{la(y'_j) m^r(y_j)}{a(y_j)a(y'_j)} - \langle \frac{\beta(\alpha)}{a(y_j)}, p_j \rangle + \langle \frac{\beta(\alpha)}{a(y'_j)}, p'_j \rangle
\end{align*}
\]

\[
\leq - \int_{z \in \mathcal{O}_j^c} \{ m^r(y_j + z) - m^r(y_j) - m_r(y'_j + z) + m_r(y'_j) \} dq(z)
\]

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+ \frac{lc(\alpha)|a(y_j) - a(y'_j)|}{a(y_j)a(y'_j)} + \frac{l(a(y'_j)f(y_j) - a(y_j)f(y'_j)) + \nu(a(y'_j) - a(y_j))}{a(y_j)a(y'_j)}.\]

We may assume \( \nu_j \to 0 \) as \( j \to \infty \). Then, \( \mathcal{O}_j^c \to \mathcal{O}_c^c \). We let \( j \to \infty \) in the above inequality, by remarking

\[
m^r(\hat{y}) - m_r(\hat{y}') - C_\theta|\hat{y} - \hat{y}'|^\theta \geq m^r(\hat{y}) - m_r(\hat{y}') - C_\theta|\hat{y} - \hat{y}'|^\theta,
\]

and by multiplying by \( a(\hat{y})a(\hat{y}') \)

\[
l(a(\hat{y}')m^r(\hat{y}) - a(\hat{y})m_r(\hat{y}')) + (a(\hat{y}) - a(\hat{y}')) \langle \beta(\alpha), p \rangle
\]

\[
\leq lc(\alpha)|a(y) - a(y')| + l(a(y)f(y) - a(\hat{y})f(\hat{y}')) + \nu(a(\hat{y}) - a(y)).
\]

Since either \( a(\cdot) \equiv a \), or \( \beta(\alpha) \equiv 0 \ (\forall \alpha \in \mathcal{A}) \), and since \( \nu > 0 \) is arbitrary, deviding the both hands side by \( l > 0 \),

\[
a_0(m^r(\hat{y}) - m_r(\hat{y}')) \leq C(|a(\hat{y}') - a(\hat{y})| + |f(\hat{y}') - f(\hat{y})|).
\]

From the Hölder continuity of \( a, f \) and (48), we get

\[
C_\theta|\hat{y} - \hat{y}'|^\theta \leq C|\hat{y} - \hat{y}'|^\theta_0.
\]

From (46), \( \frac{2M}{r_0^\theta} \leq C|\hat{x} - \hat{y}|^\theta_0 - \theta \), that is

\[
2M \leq C|\hat{x} - \hat{y}|^{\theta_0 - \theta} r_0^{\theta_0} \leq Cr_0^{\theta_0}.
\]

Therefore, for \( r_0 > 0 \) small enough such that \( r_0^{\theta_0} < \frac{M}{C} \), we get a contradiction. For such \( r_0 > 0 \), by defining \( C_\theta \) as in (46), we proved our claim.

**Proof of Theorem 4.1.** Since \( m_\ell = lv_\ell \) in Lemma 4.2, the claim of Theorem 4.1 is clear.

**Theorem 4.4.**

Assume that (7), (8), (9), (10), (29), (31), and (38) hold. Assume also that either \( a(\cdot) \equiv a \ (a > 0 \ is \ a \ constant) \), or \( \beta(\alpha) \equiv 0 \ (\forall \alpha \in \mathcal{A}) \). Let \( v_\ell \) be the solution of (40). Then, there exists a unique number \( d \) such that

\[
\lim_{\ell \to 0} lv_\ell(y) = d \quad \text{uniformly in } \mathbb{R}^N,
\]

which is characterized by (39).
Proof. From Theorem 4.1, by the Ascoli-Arzela theorem, there exists a subsequence \( l' \to 0 \) such that
\[
\lim_{l' \to 0} l' v_{l'}(y) = d(y) \quad y \in \mathbb{R}^N.
\]
We still use \( l \) instead of \( l' \) to simplify the notation. Remark that \( d(y) \) is Hölder continuous. Since \( v_{l'}(y) = w_l(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) \) and \( w_l \) is periodic, the above convergence is uniform in \( \mathbb{R}^N \). Multiplying (40) by \( l > 0 \), passing \( l \to 0 \), we get
\[
\sup_{\alpha \in \mathcal{A}} \{-\beta(y, \alpha), \nabla d(y)\} - a(y) \int_{\mathbb{R}^N} [d(y+z)-d(y)-1_{|z| \leq 1} \langle z, \nabla d(y) \rangle] dq(z) = 0.
\]
Since \( d(y) \) is a uniform limit of a sequence of quasi-periodic functions \( lv_l(y) \), it takes a maximum at some point \( \hat{y} \in \mathbb{R}^N \). From the strong maximum principle in Theorem 3.1, \( d(y) \equiv d \). The uniqueness of \( d \) can be proved by the standard argument (see [8] for example). Let \( \mu > 0 \) be arbitrary. From the uniform convergence of \( lv_l \) as \( l \) goes to 0, for \( l > 0 \) small enough if we put \( \underline{v} = v_l \) and \( \overline{v} = v_l \) they satisfy (39). The claims in Theorem 4.4 are thus proved.

Next, we study the ergodic problem of the first-order PDE, which includes the case I as a special example. Let us consider the following deterministic system
\[
\frac{dy_\alpha}{dt} = \beta(\alpha(t)) \quad t > 0, \quad y_\alpha(0) = y \in \mathbb{T}^N, \tag{49}
\]
where \( \alpha(\cdot) \) is a measurable function from \([0, \infty)\) to \( \mathcal{A} \), which we call a control. We assume the following.

(A) A controlled dynamical system (49) is approximately controllable if for any \( y, y' \in \mathbb{T}^N \), and for any \( \delta > 0 \), there exists a control \( \alpha(\cdot) \) and \( T_\delta > 0 \) such that the solution \( y_\alpha(t) \) of (49) satisfies \( |y' - y(T_\delta)| < \delta \).

Under the above condition, we intend to solve the following.

(R) Find a unique constant \( d \) such that the following problem has at least a viscosity solution \( v(y) \), bounded in \( \mathbb{R}^N \):
\[
d + \sup_{\alpha \in \mathcal{A}} \{-\beta(\alpha), \nabla v(y)\} - c(\alpha) \} - a(y_1)I - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) = 0 \quad y \in \mathbb{R}^N.
\]
Find a unique constant $d$ such that the following problem has at least a viscosity solution $w(y)$ ($y = (y_1, ..., y_M)$), periodic in $T^{MN}$:

$$d + \sup_{\alpha \in A} \{ \langle -B(y, \alpha), \nabla w(y) \rangle - c(\alpha) \} - a(y_1) I - g_M(y) = 0 \quad \forall y \in T^{MN},$$

where $B(\alpha) = (\gamma_1^{-1} \beta(\alpha), ..., \gamma_M^{-1} \beta(\alpha))$, $\Gamma^{-1} = (\gamma_1^{-1}, ..., \gamma_M^{-1})$.

The problems (R) and (S) are related by $v(y) = w(\gamma_1^{-1} y, ..., \gamma_M^{-1} y)$. We abbreviate the weaker versions of (R) and (S). As before, we approximate the problems, for $l \in (0, 1)$

$$lv_l + \sup_{\alpha \in A} \{ \langle -\beta(\alpha), \nabla v_l(y) \rangle - c(\alpha) \} - a(y_1) I - g_M(y) = 0 \quad y \in \mathbb{R}^N. \quad (50)$$

$$lw_l + \sup_{\alpha \in A} \{ \langle -B(y, \alpha), \nabla w_l(y) \rangle - c(\alpha) \} - a(y_1) I - g_M(y) = 0, \quad y \in T^{MN}. \quad (51)$$

**Theorem 4.5.**

Assume that (8), (9), (10), and (38) hold. Assume also that (49) is approximately controllable. Let $v_l$ be the solution of (50). Then, there exists a unique number $d$ such that

$$\lim_{l \to 0} lv_l(y) = d \quad \text{uniformly in} \quad \mathbb{R}^N.$$

Moreover, the number $d$ is characterized by (39).

**Proof.** Let $w_l$ be the periodic solution of (51).

(Step 1) We first show that there exists a constant $C > 0$ such that

$$|lw_l(y) - lw_l(y')| \leq C|y - y'|^{\theta_0} \quad \forall y, y' \in T^{MN}, \quad \forall l \in (0, 1). \quad (52)$$

Let $\alpha(t)$ be an arbitrary measurable function from $(0, \infty)$ to $A$. Let $Y_\alpha(t)$, $Y'_\alpha(t)$ be respectively the solution of

$$\frac{dY_\alpha}{dt} = B(\alpha(t)) \quad t > 0, \quad Y_\alpha(0) = y; \quad \frac{dY'_\alpha}{dt} = B(\alpha(t)) \quad t > 0, \quad Y'_\alpha(0) = y'.$$
Remark that there exists a constant \( L > 0 \) such that
\[
|Y_\alpha(t) - Y'_\alpha(t)| \leq L|\overline{y} - \overline{y}| \quad \forall t \geq 0. \tag{53}
\]

Put \( f(\overline{y}) = a(y_1)I + g_M(\overline{y}) \). Since (see for example [17])
\[
w_t(\overline{y}) = \inf_{\alpha(\cdot)} \{ \int_0^\infty e^{-lt}(f(Y_\alpha(t)) + c(\alpha(t)))dt \},
\]
\[
w_t(\overline{y}') = \inf_{\alpha(\cdot)} \{ \int_0^\infty e^{-lt}(f(Y'_\alpha(t)) + c(\alpha(t)))dt \},
\]
for any \( \nu > 0 \), we can take a control \( \alpha(\cdot) \) such that
\[
w_t(\overline{y}) - w_t(\overline{y}') \leq \int_0^\infty e^{-lt}|f(Y_\alpha(t)) - f(Y'_\alpha(t))|dt + \nu \leq \frac{L}{l}|\overline{y} - \overline{y}|^{\theta_0} + \nu,
\]
where we used \([2], [10], (53)\) to derive the last inequality. Since \( \nu > 0 \) is arbitrary, \([52]\) is shown.

(Step 2) From \([52]\), we can extract a subsequence \( l' \to 0 \) such that
\[
\lim_{l' \to 0} l'w_{l'}(\overline{y}) = d(\overline{y}) \text{ uniformly in } \overline{y} \in T^{MN},
\]
where \( d(\overline{y}) \) is Hölder continuous and periodic. Multiplying \([54]\) by \( l' > 0 \), and tending \( l' \) to zero, we deduce that \( d(\overline{y}) \) satisfies
\[
\sup_{\alpha \in A}\{ \langle -B(\alpha), \nabla d(\overline{y}) \rangle \} = 0 \quad \forall \overline{y} \in T^{MN}. \tag{54}
\]
Now,
\[
\langle -B(\alpha), \nabla d(\overline{y}) \rangle \leq 0 \quad \forall \alpha \in A.
\]
Since \( d(\cdot) \) is Lipshitz, the above holds almost everywhere in \( T^{MN} \). Then, since \( d(\cdot) \) is periodic,
\[
\langle -B(\alpha), \nabla d(\overline{y}) \rangle = 0 \quad \forall \alpha \in A.
\]
Thus,
\[
d(\overline{y}) + \int_0^t B(\alpha(s))ds = d(\overline{y}) \quad \forall t \geq 0, \quad \forall \alpha(\cdot) : [0, \infty) \to A.
\]
Remark that \( B(\alpha) = (\gamma_1^{-1}\beta(\alpha), ..., \gamma_M^{-1}\beta(\alpha)) \), and the set \( \{ \gamma_i \} (1 \leq i \leq M) \) satisfies the non-resonance condition. From (A), the set
\[
\bigcup_{\alpha(\cdot), t \geq 0} \{ y_1 + \int_0^t \beta(\alpha(s))ds \} \quad \text{is dense in } T^N.
\]

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Thus, the set
\[ \bigcup_{\alpha(\cdot), t \geq 0} \{ \overline{y} + \int_0^t B(\alpha(s)) ds \} \]
is dense in \( T^{MN} \).

Therefore, \( d(\overline{y}) \equiv d \) in \( \overline{y} \in T^{MN} \).

(Step 3) The number \( d \) is unique, and it does not depend on the choice of the subsequence \( l' \to 0 \). The proof is standard, and we refer the reader to \[8\]. By defining \( v_l(y) = w_l(y, \gamma_1^{-1} y, ..., \gamma_M^{-1} y) \), we have shown the claim.

**Proposition 4.6.**

Assume that (7), (8), (9), and (10) hold. Let \((x,p,I)\in \Omega \times \mathbb{R}^N \times \mathbb{R}\) be arbitrarily fixed. Then, the following hold.

(i) Let \( \alpha \in (0,1) \) and \( b(x,\alpha) \neq 0 \). Let (49) with \( \beta(\alpha) = b(x,\alpha) \) satisfy (A). There exists a unique number \( d_{x,p,I} \) which satisfies (18) in the sense of (39).

(ii) Let \( \alpha = 1 \), \( b(x,\alpha) \neq 0 \), and \( a(y) \equiv a \) (\( y \in \mathbb{R}^N \)). There exists a unique number \( d_{x,p,I} \) which satisfies (20) in the sense of (39).

(iii) Let \( \alpha \in (1,2) \), or \( \alpha \in (0,1] \) and \( b(x,\alpha) \equiv 0 \). There exists a unique number \( d_{x,p,I} \) which satisfies (21) in the sense of (39).

**Proof.** (i) Put \( \beta(\alpha) = b(x,\alpha) \) in (50), \( c(\alpha) = \langle b(x,\alpha), p \rangle \). Then, from Theorem 4.5, the statement follows.

(ii) In (40), put \( \beta(\alpha) = b(x,\alpha) \), \( c(\alpha) = \langle b(x,\alpha), p \rangle \). The claim follows from Theorem 4.4.

(iii) In (40), put \( \beta(\alpha) = 0 \), \( c(\alpha) = 0 \). From Theorem 4.4, the claim follows.

5 Ergodic problems for the almost periodic homogenizations.

Next, we solve the ergodic cell problem for the almost periodic homogenizations. Similar to the case of the quasi-periodic homogenizations, the situation is divided into the following.
• I'. \( \alpha \in (0,1) \) and \( b(x, \alpha) \not\equiv 0 \).
• II'. \( \alpha = 1 \) and \( b(x, \alpha) \not\equiv 0 \).
• III'. \( \alpha \in (1,2) \), or \( \alpha \in (0,1] \) and \( b(x, \alpha) \equiv 0 \).

The formal asymptotic expansion (see §2) leads, for the case I’
\[
\begin{align*}
\quad d_{x,p,I} + \sup_{\alpha \in A} \{ \langle -b(x, \alpha), p + \nabla v(y) \rangle \} - a(y)I - g(y) &= 0 \quad y \in \mathbb{R}^N. \\
\end{align*}
\]  
(55)

For the case II’,
\[
\begin{align*}
\quad &d_{x,p,I} + \sup_{\alpha \in A} \{ \langle -b(x, \alpha), p + \nabla v(y) \rangle \} - a(y)\int_{\mathbb{R}^N} [v(y + z) - v(y)] \\
&\quad - 1_{\mid z \mid \leq 1} \langle z, \nabla v(y) \rangle \frac{1}{\mid z \mid^{N+\alpha}} d\mu - a(y)I - g(y) = 0 \quad y \in \mathbb{R}^N. \\
\end{align*}
\]  
(56)

And for the case III’,
\[
\begin{align*}
\quad &d_{x,p,I} - a(y)\int_{\mathbb{R}^N} [v(y + z) - v(y) - 1_{\mid z \mid \leq 1} \langle z, \nabla v(y) \rangle \frac{1}{\mid z \mid^{N+\alpha}} d\mu - a(y)I \\
&\quad - g(y) = 0 \quad y \in \mathbb{R}^N. \\
\end{align*}
\]  
(57)

Our claim is the following.

**Proposition 5.1.**

Assume that (7), (9), and (10) hold, and that \( g \) in (5) is uniformly almost periodic in the sense of Bohr in \( \mathbb{R}^N \). Let \( (x,p,I) \in \Omega \times \mathbb{R}^N \times \mathbb{R} \) be arbitrarily fixed. Then, the following hold.

(i) Let \( \alpha \in (0,1) \) and \( b(x, \alpha) \not\equiv 0 \). Let (49) with \( \beta(\alpha) = b(x, \alpha) \) satisfy (A). There exists a unique number \( d_{x,p,I} \) which satisfies (55) in the sense of (39).

(ii) Let \( \alpha = 1 \), \( b(x, \alpha) \not\equiv 0 \), and \( a(y) \equiv a \ (y \in \mathbb{R}^N) \). There exists a unique number \( d_{x,p,I} \) which satisfies (56) in the sense of (39).

(iii) Let \( \alpha \in (1,2) \), or \( \alpha \in (0,1] \) and \( b(x, \alpha) \equiv 0 \). There exists a unique number \( d_{x,p,I} \) which satisfies (57) in the sense of (39).
We consider the following general ergodic problem, which includes (56) and (57) as special cases. Find a unique constant $d > 0$ such that there exists a bounded viscosity solution $v$ of

$$d + \sup_{\alpha \in A} \{ \langle \beta(\alpha), \nabla_y v(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} \left[ v(y + z) - v(y) \right] - 1_{|z| \leq 1} \langle z, \nabla_y v(y) \rangle \, dq(z) - a(y) I - g(y) = 0 \quad y \in \mathbb{R}^N. \tag{58}$$

For (55), we are interested in finding a unique constant $d > 0$ such that there exists a bounded viscosity solution $v$ of

$$d + \sup_{\alpha \in A} \{ \langle \beta(\alpha), \nabla_y v(y) \rangle - c(\alpha) \} - a(y) I - g(y) = 0 \quad y \in \mathbb{R}^N. \tag{59}$$

In some cases, the number $d$ satisfies (58) (resp. (59)) in the sense of (39). We use a useful characterization of the uniformly almost periodic function in Braides [15], which was first shown by Bohr (see [12]) for the one dimensional case.

**Lemma B.** ([15] Definition A.1, Theorem A.6.) If a continuous function $f(x)$ defined in $\mathbb{R}^N$ is uniformly almost periodic in the sense of Bohr, then $f$ is the uniform limit of a sequence of trigonometric polynomials. The converse is also true.

We refer the readers to [12] and [15] for details.

**Lemma 5.2.**

(i) Assume that (7), (9), (10), (29), (31), and (38) hold. Assume also that $g$ is uniformly almost periodic in $\mathbb{R}^N$. There exists a unique constant $d$ which satisfies (58) in the sense of (39).

(ii) Assume that (9), (10), and (38) hold, that (49) is approximately controllable. Assume also that $g$ is uniformly almost periodic in $\mathbb{R}^N$. There exists a unique constant $d$ which satisfies (59) in the sense of (39).

**Proof.** (i) From Lemma B, there exist a sequence of periodic functions $g_M (M = 1, 2, \ldots)$ defined in $(y_1, \ldots, y_M) \in \mathbb{T}^{MN}$, and a sequence of numbers $\gamma_i (i \in \mathbb{N})$ satisfying the non-resonance condition (8), such that

$$g(y) = \lim_{M \to \infty} g_M(\gamma_1^{-1} y, \ldots, \gamma_M^{-1} y) \quad \text{uniformly in} \quad \mathbb{R}^N. \tag{60}$$
From Theorem 4.4, for each $M$, there exist a constant $d_M$, $\underline{\varphi}_M \in USC(\mathbb{R}^N)$, and $\overline{\varphi}_M \in LSC(\mathbb{R}^N)$ which satisfy:

$$d_M + \sup_{\alpha \in A} \{ \langle \beta(\alpha), \nabla_y \underline{\varphi}_M(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [\underline{\varphi}_M(y + z) - \underline{\varphi}_M(y)] - 1_{|z| \leq 1} \langle z, \nabla_y \underline{\varphi}_M(y) \rangle dq(z) - g_M(y) \leq \mu \quad y \in \mathbb{R}^N,$$

$$d_M + \sup_{\alpha \in A} \{ \langle \beta(\alpha), \nabla_y \overline{\varphi}_M(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [\overline{\varphi}_M(y + z) - \overline{\varphi}_M(y)] - 1_{|z| \leq 1} \langle z, \nabla_y \overline{\varphi}_M(y) \rangle dq(z) - g_M(y) \geq -\mu \quad y \in \mathbb{R}^N.$$

Remarking that there exists a constant $C > 0$ such that

$$|d_M| \leq C \quad \forall M > 0,$$

for $g(y), g_M(\Gamma^{-1} y)$ ($M \in \mathbb{N}$) are uniformly bounded in $\mathbb{R}^N$. Thus, we can extract a sequence $M' \to \infty$ such that $\lim_{M' \to \infty} d_{M'} = d$. Define $v^*(y) = \lim_{M' \to \infty} \underline{\varphi}_{M'}(y)$, $v_*(y) = \lim_{M' \to \infty} \overline{\varphi}_{M'}(y)$. From Barles and Perthame [11], by passing $M' \to \infty$ in the above inequalities, we find that $v^*$ and $v_*$ respectively satisfy

$$d + \sup_{\alpha \in A} \{ \langle \beta(\alpha), \nabla_y v^*(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [v^*(y + z) - v^*(y)] - 1_{|z| \leq 1} \langle z, \nabla_y v^*(y) \rangle dq(z) - a(y) I - g(y) \leq \mu \quad y \in \mathbb{R}^N,$$

$$d + \sup_{\alpha \in A} \{ \langle \beta(\alpha), \nabla_y v_*(y) \rangle - c(\alpha) \} - a(y) \int_{\mathbb{R}^N} [v_*(y + z) - v_*(y)] - 1_{|z| \leq 1} \langle z, \nabla_y v_*(y) \rangle dq(z) - a(y) I - g(y) \geq -\mu \quad y \in \mathbb{R}^N.$$

The uniqueness of the number $d$ can be shown by the standard method (see [8]). Thus, the claim was proved.

(ii) The existence of the number $d$ in (59) can be shown in the same way to (i), by using Theorem 4.5.

**Proof of Proposition 5.1.** (i) Put $\beta(\alpha) = b(x, \alpha)$ in (59). Then, from Lemma 5.2, the statement follows.

(ii) In (58), put $a(y) \equiv a$ ($a > 0$ is a constant), $\beta(\alpha) = b(x, \alpha)$, $c(\alpha) = b(x, \alpha), p)$. The claim follows from Lemma 5.2.

(iii) In (58), put $\beta(\alpha) = 0$, $c(\alpha) = 0$. From Lemma 5.2, the claim follows.
6 Homogenizations.

First, we confirm that the effective integro-differential operator has the uniform subellipticity.

(Uniform subelliptic operator) An integro-differential operator $I(x, p, I)$ defined in $\Omega \times \mathbb{R}^N \times \mathbb{R}$ is uniformly subelliptic if there exists $\theta > 0$ such that

$$I(x, p, I + I') - \theta I' \quad \forall I' > 0, \quad \forall (x, p, I) \in \Omega \times \mathbb{R}^N \times \mathbb{R}. \quad (61)$$

Proposition 6.1.

(i) Let $d_{x,p,I}$ be given by (18) in the case I, by (20) in the case II, and by (21) in the case III, in the sense of (39). Put $I(x, p, I) = -d_{x,p,I}$ for any $(x, p, I) \in \Omega \times \mathbb{R}^N \times \mathbb{R}$. Then, $I$ is continuous and uniformly subelliptic.

(ii) Let $d_{x,p,I}$ be given by (24) in the case I', by (25) in the case II', and by (26) in the case III', in the sense of (39). Put $I(x, p, I) = -d_{x,p,I}$ for any $(x, p, I) \in \Omega \times \mathbb{R}^N \times \mathbb{R}$. Then, $I$ is continuous and uniformly subelliptic.

Proof. (i) We use the perturbed test function method in \cite{18}. We prove the claim for the case II. The other cases can be treated similarly. Take an arbitrary positive number $I' > 0$. From (20) and (39), for $\rho > 0$ there exist bounded functions $v_I$ and $v_{I+I'}$ which satisfy

$$d(x, p, I) + \sup_{\alpha \in \mathcal{A}} \left\{ -b(x, \alpha), p + \nabla v_I(y) \right\} - a(y) \int_{\mathbb{R}^N} [v_I(y + z) - v_I(y)] -1_{|z| \leq 1} \left\langle z, \nabla v_I(y) \right\rangle dq(z) - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) - a(y) I \leq \rho \quad y \in \mathbb{R}^N,$n

$$d(x, p, I+I') + \sup_{\alpha \in \mathcal{A}} \left\{ -b(x, \alpha), p + \nabla v_{I+I'}(y) \right\} - a(y) \int_{\mathbb{R}^N} [v_{I+I'}(y + z) - v_{I+I'}(y)] -1_{|z| \leq 1} \left\langle z, \nabla v_{I+I'}(y) \right\rangle dq(z) - g_M(\gamma_1^{-1} y, ..., \gamma_M^{-1} y) - a(y) (I+I') \geq -\rho \quad y \in \mathbb{R}^N. \quad (62)$$

By adding a constant if necessary, we may assume that $v_{I+I'} < v_I$. Let us prove (61) for $\theta = a_0$, where $a_0 > 0$ is given in (7). Assume that for fixed $x, p, I$ and $I'$, there exists a constant $l > 0$ such that

$$\mathcal{T}(x, p, I + I') > \mathcal{T}(x, p, I) - a_0 I' + l, \quad (63)$$
From (63), (ii) As shown in §5, for each case of I’, II’, and III’, the integro-differential
operator $\mathcal{T}(x, p, I)$ is the limit of the sequence of operators $\mathcal{T}_M(x, p, I)$ ($M = 1, 2, \ldots$). From (i),

$$\mathcal{T}_M(x, p, I + I') \leq \mathcal{T}_M(x, p, I) - a_0 I' \quad \forall I' > 0, \quad \forall (x, p, I) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}. $$

Therefore, (61) holds for $\mathcal{T}(x, p, I) = \lim_{M \to \infty} \mathcal{T}_M(x, p, I)$, too.

The following is the main result of the quasi-periodic homogenization.

**Theorem 6.2.**

Assume that (3), (7), (8), (9) and (10) hold. If $\alpha = 1$ and $\beta(x, \alpha) \neq 0$, assume that $a(y) \equiv a$ ($a > 0$ is a constant). Let $u_\varepsilon$ be the solution of (6)-(2). Then, there exists a function $\overline{u}$ such that

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = \overline{u}(x) \quad \text{uniformly in} \quad x \in \mathbb{R}^N.$$ 

The function $\overline{u}$ is the unique bounded solution of (19)-(2) with the effective integro-differential operator $\mathcal{T}$, given by

$$\mathcal{T}(x, p, I) = -d_{x, p, I} \quad \text{for any} \quad (x, p, I) \in \Omega \times \mathbb{R}^N \times \mathbb{R}. $$

The right-hand side $d_{x, p, I}$ is given by Proposition 4.6.

**Proof.** It is enough to prove the case of $\alpha = 1$. The other cases can be shown similarly.

(Step 1) First, remark that from Proposition 4.6, the effective integro-differential equation (19) is well-defined. From Proposition 6.1, $\mathcal{T}$ is subelliptic, and the comparison principle holds for (19)-(2) (see [3], [4], [7], [9] and [10] for example).

(Step 2) From the comparison for (6)-(2), there exists a constant $M > 0$ such that $|u_\varepsilon| \leq M$ for any $\varepsilon \in (0, 1)$. Therefore, we can take

$$u^*(x) = \overline{\lim}_{\varepsilon \to 0, x' \to x} u_\varepsilon(x'), \quad u_*(x) = \underline{\lim}_{\varepsilon \to 0, x' \to x} u_\varepsilon(x').$$

We prove that $u^*$ is a viscosity subsolution of (19) by the argument by the contradiction. So, assume that $u^*$ is not a subsolution of (19): there exists a function $\phi \in C^2$ such that $u^* - \phi$ takes a global strict maximum at a point $\hat{x} \in \mathbb{R}^N$, $u^*(\hat{x}) = \phi(\hat{x})$, and

$$\phi(\hat{x}) + \mathcal{T}(\hat{x}, \nabla \phi(\hat{x}), I[\phi](\hat{x})) = 3\gamma > 0,$$

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where $\gamma > 0$ is a constant. From the continuity of $\overline{T}$, for $r > 0$ small enough

$$\phi(x) + \overline{T}(x, \nabla \phi(x), I[\phi](x)) \geq \gamma \quad \text{in} \quad B_r(\hat{x}),$$

where $B_r(\hat{x}) = \{x \mid |x - \hat{x}| < r\}$. From the definition of $\overline{T}(\hat{x}, \nabla \phi(\hat{x}), I[\phi](\hat{x}))$, for the above $\gamma > 0$ there exist $\underline{\nu}, \overline{\nu}$ such that

$$-\overline{T}(\hat{x}, \nabla \phi(\hat{x}), I[\phi](\hat{x})) + \sup_{a \in A} \{\langle -b(\hat{x}, \alpha), \nabla \underline{\nu}(y) \rangle - a(y) \int \underline{\nu}(y + z) - \underline{\nu}(y)$$

$$-1_{|z| \leq 1} \langle z, \nabla \underline{\nu}(y) \rangle \frac{1}{|z|^{N+\alpha}} dz - g_M(\gamma_1 y, ..., \gamma_M y) - a(y) I[\phi](\hat{x}) \leq \gamma \quad y \in \mathbb{R}^N,$$

(64)

$$-\overline{T}(\hat{x}, \nabla \phi(\hat{x}), I[\phi](\hat{x})) + \sup_{a \in A} \{\langle -b(\hat{x}, \alpha), \nabla \overline{\nu}(y) \rangle - a(y) \int \overline{\nu}(y + z) - \overline{\nu}(y)$$

$$-1_{|z| \leq 1} \langle z, \nabla \overline{\nu}(y) \rangle \frac{1}{|z|^{N+\alpha}} dz - g_M(\gamma_1 y, ..., \gamma_M y) - a(y) I[\phi](\hat{x}) \geq -\gamma \quad y \in \mathbb{R}^N.$$  

(65)

We can assume that $\underline{\nu}, \overline{\nu}$ are Lipschitz continuous, for if not we regularize them by the sup and the inf convolutions respectively. Put $\phi_\varepsilon = \phi(x) + \varepsilon_1 \overline{\nu}(x)$. We claim that $\phi_\varepsilon$ is a viscosity supersolution of

$$\phi_\varepsilon + \sup_{a \in A} \{\langle -b(x, \alpha), \nabla \phi_\varepsilon \rangle - a(x) \int_{\mathbb{R}^N} \phi_\varepsilon(x + z) - \phi_\varepsilon(x)$$

$$-1_{|z| \leq 1} \langle z, \nabla \phi_\varepsilon(x) \rangle \frac{1}{|z|^{N+\alpha}} dz - g_M(x, ..., x, x) \geq \gamma \quad \text{in} \quad B_r(\hat{x}),$$

(66)

for $r > 0$ small enough. To see this, assume that for $\psi \in C^2$ $\phi_\varepsilon - \psi$ attains its minimum at $\overline{\tau} \in U_r(\hat{x})$, and that $\phi_\varepsilon(\overline{\tau}) = \psi(\overline{\tau})$. We are to show

$$\phi_\varepsilon(\overline{\tau}) + \sup_{a \in A} \{\langle -b(\overline{\tau}, \alpha), \nabla \psi(\overline{\tau}) \rangle - a(\overline{\tau}) \int_{\mathbb{R}^N} \psi(\overline{\tau} + z) - \psi(\overline{\tau})$$

$$1_{|z| \leq 1} \langle z, \nabla \psi(\overline{\tau}) \rangle \frac{1}{|z|^{N+\alpha}} dz - \overline{\tau}(x, ..., x) \geq \gamma \quad \text{in} \quad B_r(\hat{x}).$$

Put $\beta(y) = \frac{1}{\varepsilon_1}(\psi - \phi)(\varepsilon_1 y)$. Since $(\overline{\tau} - \beta)(y)$ attains its minimum at $\overline{\gamma} = \frac{\overline{\tau}}{\varepsilon_1}$, and since $\overline{\gamma}$ is the viscosity supersolution of (65),

$$-\overline{T}(\hat{x}, \nabla \phi(\hat{x}), I[\phi](\hat{x})) + \sup_{a \in A} \{\langle -b(\hat{x}, \alpha), \nabla (\psi - \phi)(\overline{\tau}) \rangle$$

29
Therefore, for \( r > 0 \) and \( \varepsilon > 0 \) small enough

\[
\phi_\varepsilon(\bar{x}) + \sup_{\alpha \in A} \left\{ (-b(\bar{x}, \alpha), \nabla \psi(\bar{x})) \right\} - a\left( \frac{\bar{x}}{\varepsilon} \right) \int_{\mathbb{R}^N} \left[ \psi(\bar{x} - z) - \psi(\bar{x}) \right] - g_M \left( \frac{\bar{x}}{\varepsilon}, ... \right) \frac{1}{|z|^{N+\alpha}}dz - a\left( \frac{\bar{x}}{\varepsilon} \right) I[\phi](\hat{x}) \geq -\gamma.
\]

Therefore, for \( r > 0 \) and \( \varepsilon > 0 \) small enough

\[
\begin{align*}
\phi_\varepsilon(\bar{x}) + \sup_{\alpha \in A} \left\{ (-b(\bar{x}, \alpha), \nabla \psi(\bar{x})) \right\} - a\left( \frac{\bar{x}}{\varepsilon} \right) \int_{\mathbb{R}^N} \left[ \psi(\bar{x} - z) - \psi(\bar{x}) \right] \frac{1}{|z|^{N+\alpha}}dz - g_M \left( \frac{\bar{x}}{\varepsilon}, ... \right) \frac{1}{|z|^{N+\alpha}}dz + a\left( \frac{\bar{x}}{\varepsilon} \right) I[\phi](\hat{x}) &
\geq \phi_\varepsilon(\bar{x}) + \sup_{\alpha \in A} \left\{ (-b(\bar{x}, \alpha), \nabla \psi(\bar{x})) \right\} + \gamma + T(\bar{x}, \nabla \phi(\bar{x}), I[\phi](\bar{x})) - a\left( \frac{\bar{x}}{\varepsilon} \right) \int_{\mathbb{R}^N} \left[ \phi(\bar{x} + z) - \phi(\bar{x}) \right] - 1_{|z| \leq 1} \langle z, \nabla \phi(\bar{x}) \rangle \frac{1}{|z|^{N+\alpha}}dz + a\left( \frac{\bar{x}}{\varepsilon} \right) I[\phi](\hat{x}) \geq 2\gamma,
\end{align*}
\]

where we used \( \phi_\varepsilon(\bar{x}) = \psi(\bar{x}) \), and the fact that \( u \) is Lipschitz continuous. Hence, (66) was proved. From the comparison \( u_0 \leq \phi_\varepsilon - 2\gamma \), and since \( \gamma > 0 \) is arbitrary \( u_0 \leq \phi_\varepsilon \). Therefore,

\[
u(x) = \lim_{\varepsilon \to 0, x' \to x} u_\varepsilon(x') \leq \lim_{\varepsilon \to 0} \phi_\varepsilon(x) \in B_r(\hat{x}).
\]

However, this contradicts to the fact that \( u^* - \phi \) takes its strict maximum at \( \hat{x} \), and we have proved that \( u^* \) is a viscosity subsolution of (19). In parallel, we can prove that \( u_\ast \) is a viscosity supersolution of (19).

(Step 3) As we have confirmed in Step 1, the comparison principle holds for (19)-(2). The fact that \( u^* \) and \( u_\ast \) are respectively a subsolution and a supersolution of (19) leads \( u^* \leq u_\ast \). At the same time, from the definition \( u_\ast \leq u^* \). Therefore, \( u^* = u_\ast \), and \( \lim_{\varepsilon \to 0} u_\varepsilon = \bar{\pi} \) exists. From Step 2, \( \pi \) is the unique solution of (19)-(2).
The following is our main result of the almost periodic homogenizations.

**Theorem 6.3.**

Assume that (3), (7), (9) and (10) hold, and that $g$ is uniformly almost periodic in the sense of Bohr in $\mathbb{R}^N$. If $\alpha = 1$ and $\beta(x, \alpha) \not\equiv 0$, assume that $a(y) \equiv a$ ($a > 0$ is a constant). Let $u_\varepsilon$ be the solution of (5)–(2). Then, there exists a function $\overline{u}$ such that

$$
\lim_{\varepsilon \to 0} u_\varepsilon(x) = \overline{u}(x) \quad \text{uniformly in} \quad x \in \mathbb{R}^N.
$$

The function $\overline{u}$ is the unique bounded solution of (19)–(2) with the effective integro-differential operator $\overline{T}$, given by

$$
\overline{T}(x,p,I) = -d_{x,p,I} \quad \text{for any} \quad (x,p,I) \in \Omega \times \mathbb{R}^N \times \mathbb{R}.
$$

The right-hand side $d_{x,p,I}$ is given by Proposition 5.1.

**Proof.** From Lemma B, we can take a sequence of functions $g_M(y_1,\ldots,y_M)$ ($M = 1, 2, \ldots$) periodic in $T^{MN}$, and a sequence of positive numbers $\{\gamma_i\}$ ($i = 1, 2, \ldots$) satisfying the non-resonance condition (8), such that

$$
g(y) = \lim_{M \to \infty} g_M(\gamma_1^{-1}y,\ldots,\gamma_M^{-1}y) \quad \text{uniformly in} \quad y \in \mathbb{R}^N. \quad (67)
$$

We assume that $\gamma_1 = 1$. Let $\varepsilon_1 > 0$, and put $\varepsilon_i = \gamma_i \varepsilon_1$ for any $2 \leq i \leq M$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_M)$, let $u_\varepsilon^M$ be the solution of

$$
u_\varepsilon^M + \sup_{\alpha \in A} \left\{ -b(x, \alpha), \nabla u_\varepsilon^M \right\} - a(x) \int_{\mathbb{R}^N} [u_\varepsilon^M(x+z) - u_\varepsilon^M(x)] dz
$$

$$
- \mathbf{1}_{|z| < 1} \left( z, \nabla u_\varepsilon^M(x) \right) \frac{1}{|z|^{N+\alpha}} \cdot \frac{1}{\varepsilon_1} g_M(\gamma_1^{-1}x \varepsilon_1, \ldots, \gamma_M^{-1}x \varepsilon_1) = 0 \quad x \in \mathbb{R}^N,
$$

and (2). Then, by comparing the above equation with (5), there exists a sequence of constants $c_M > 0$ such that $\lim_{M \to \infty} c_M = 0$,

$$
u_\varepsilon^M - c_M \leq u_\varepsilon^M + c_M \quad \forall \varepsilon > 0, \quad \forall M \in \mathbb{N}. \quad (69)
$$

From Theorem 6.2, there exists a sequence of integro-differential operators $\overline{T}_M$ such that the limit $\lim_{\varepsilon \to 0} u_\varepsilon^M(x) = u^M(x)$ satisfies

$$
u^M(x) + \overline{T}_M(x,\nabla u^M(x),I[u^M](x)) = 0 \quad \text{in} \quad \mathbb{R}^N,
$$

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and [2]. Put \( u^*(x) = \lim_{M \to \infty, y \to x} u^M(y) \), \( u_*(x) = \lim_{M \to \infty, y \to x} u^M(y) \). Then, \( u^* \) and \( u_* \) respectively satisfy

\[
\begin{align*}
  u^* + T(x, \nabla u^*, I[u^*](x)) &\leq 0 \quad \text{in } \mathbb{R}^N, \\
  u_* + T(x, \nabla u_*, I[u_*](x)) &\geq 0 \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

where \( T(x, p, I) = \lim_{M \to \infty} T_M(x, p, I) \). From the comparison, we get \( u_* \leq u^* \leq u_* \).

Thus,

\[
\lim_{M \to \infty} u^M = \exists u(x) \quad \text{in } \mathbb{R}^N.
\]

Now, we first let \( \varepsilon \to 0 \) in (69) to have

\[
\begin{align*}
  u^M - c_M \leq \lim_{\varepsilon \to 0} u_\varepsilon \leq \lim_{\varepsilon \to 0} u_\varepsilon \leq u^M + c_M \quad \forall M \in \mathbb{N},
\end{align*}
\]

then let \( M \to \infty \) to have

\[
\begin{align*}
  \overline{u}(x) \leq \lim_{\varepsilon \to 0} u_\varepsilon \leq \underline{u}(x).
\end{align*}
\]

Thus, \( \lim_{\varepsilon \to 0} u_\varepsilon = \overline{u} \) exists which is the unique solution of (19)-(2).

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