Well-posedness for the FENE dumbbell model of polymetric flows in Besov spaces

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Abstract

In this paper we mainly investigate the Cauchy problem of the finite extensible nonlinear elastic (FENE) dumbbell model with dimension $d \geq 2$. We first proved the local well-posedness for the FENE model in Besov spaces by using the Littlewood-Paley theory. Then by an accurate estimate we get a blow-up criterion. Moreover, if the initial data is perturbation around equilibrium, we obtain a global existence result. Our obtained results generalize recent results in \cite{8}.

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1 Introduction

In this paper we consider the finite extensible nonlinear elastic (FENE) dumbbell model [3]:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla P &= \text{div} \tau, \quad \text{div} u = 0, \\
\partial_t \psi + (u \cdot \nabla) \psi &= \text{div} R [-\nabla u \cdot R \psi + \beta \nabla R \psi + \nabla U \psi], \\
\tau_{ij} &= \int_B R_i \otimes \nabla_j \psi dR, \\
|u|_{t=0} &= u_0, |\psi|_{t=0} = \psi_0, \\
(\beta \nabla R \psi + \nabla U \psi) \cdot n &= 0 \quad \text{on} \quad \partial B(0, R_0).
\end{align*}
\]

(1.1)

In (1.1) \(\psi(t, x, R)\) denotes the distribution function for the internal configuration and \(u(t, x)\) stands for the velocity of the polymeric liquid, where \(x \in \mathbb{R}^d\) and \(d \geq 2\) means the dimension. Here the polymer elongation \(R\) is bounded in ball \(B = B(0, R_0)\) of \(\mathbb{R}^d\) which means that the extensibility of the polymers is finite. \(\beta\) is a constant related to the temperature and \(\nu > 0\) is the viscosity of the fluid. \(\tau\) is an additional stress tensor and \(P\) is the pressure.
This model describes the system coupling fluids and polymers. The system is of great interest in many branches of physics, chemistry, and biology, see [3][5]. In this model, a polymer is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring that can be modeled by a vector \( R \). At the level of liquid, the system couples the Navier-Stokes equation for the fluid velocity with a Fokker-Planck equation describing the evolution of the polymer density. This is a micro-macro model (For more details, one can refer to [3], [8] and [9]).

In the paper we will take \( \beta = 1 \) and \( R_0 = 1 \). Notice that \( (u, \psi) \) with \( u = 0 \) and

\[
\psi_\infty(R) = \frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R)}dR},
\]

is a stationary solution of (1.1). Thus we can rewrite (1.1) for the following system:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla P &= \text{div} \tau, \quad \text{div} u = 0, \\
\partial_t \tau_{ij} &= \int_B R_i \otimes \nabla_j \mathcal{U} \psi dR, \\
\tau_{ij} &= \int_B R_i \otimes \nabla_j \mathcal{U} \psi dR, \\
u|_{t=0} &= u_0, \psi|_{t=0} = \psi_0, \\
\psi_{\infty} \nabla R \frac{\psi}{\psi_{\infty}} \cdot n &= 0 \quad \text{on} \quad \partial B(0, R_0).
\end{align*}
\]

Moreover the potential \( \mathcal{U}(R) = -k \log(1 - |R|^2) \) for some constant \( k > 0 \). We have to add a boundary condition to insure the conservation of \( \psi \), namely, \( (-\nabla u \cdot R \psi + \psi_{\infty} \nabla R \frac{\psi}{\psi_{\infty}}) \cdot n = 0 \). The second equation in (1.2) can be understood in the weak sense: for any function \( g(R) \in C^1(B) \), we have

\[
\partial_t \int_B g \psi dR + (u \cdot \nabla) \int_B g \psi dR = -\int_B \nabla R g \left[ -\nabla u \cdot R \psi + \psi_{\infty} \nabla R \frac{\psi}{\psi_{\infty}} \right] dR.
\]

**Definition 1.1.** Assume that \( u_0 \in S'(\mathbb{R}^d) \) and \( \psi_0 \in S'(\mathbb{R}^d; D'(B)) \). A couple of functions \( (u, \psi) \in C([0, T]; S'(\mathbb{R}^d)) \times C([0, T]; S'(\mathbb{R}^d; D'(B))) \) with \( \text{div} u = 0 \) is called a solution for (1.2) if for each \( (v, \phi) \in C^1([0, T]; S(\mathbb{R}^d)) \times C^1([0, T]; S(\mathbb{R}^d); C^\infty(B)) \) with \( v(T) = 0, \phi(T) = 0 \) we have

\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^d} u \partial_t v + (u \otimes u) : \nabla v - \nu u \Delta v + P \cdot \text{div} v &= \int_0^T \int_{\mathbb{R}^d} \tau : v + \int_{\mathbb{R}^d} u_0 v_0, \\
\int_0^T \int_{\mathbb{R}^d \times B} \psi \partial_t \phi + \psi u \cdot \nabla x \phi &= \int_0^T \int_{\mathbb{R}^d \times B} \left[ -\nabla u \cdot R \phi + \psi_{\infty} \nabla R \frac{\psi}{\psi_{\infty}} \right] \cdot \nabla R \phi + \int_{\mathbb{R}^d \times B} \psi_0 \phi_0.
\end{align*}
\]

Let us mention that the earliest local well-posedness for (1.1) was established by Renardy in [10], where the author considered the Dirichlet problem with \( d = 3 \) for smooth boundary and proved local existence for (1.1) in \( C^1([0, T]; H^{4-i}) \times \bigcap_{i=0}^3 \bigcap_{j=0}^{3-i} C^1([0, T]; H^j) \) with potential \( \mathcal{U}(R) = (1 - |R|^2)^{1-\sigma} \) for \( \sigma > 1 \). Later, Jourdain, Lelièvre, and Le Bris [5] proved local existence of a stochastic differential equation with potential \( \mathcal{U}(R) = -k \log(1 - |R|^2) \) in the case \( k > 3 \) for a Couette flow. Zhang and Zhang
proved local well-posedness of (1.1) with $d = 3$ in $\bigcap_{i=0}^{2} H^i([0, T); H^{4-2i}) \times \bigcap_{i=0}^{1} H^i([0, T); H^{3-2i})$ for $k > 38$. Lin, Zhang, and Zhang \cite{7} proved global well-posedness of (1.2) with $d = 2$ for $k > 6$ in $C([0, T]; C\big([0, T]; H^s(R^2; H^1_0(D))\big))$, where $s \geq 3$. Masmoudi \cite{8} proved local well-posedness of (1.2) in $C([0, T]; H^s) \times C([0, T]; H^s(R^d))$ and global well-posedness of (1.2) when the initial data is perturbation around equilibrium for $k > 0$. In the case $d = 2$, the author \cite{8} obtained a global result for $k > 0$. Kreml and Pokorný \cite{6} proved local well-posedness in Sobolev spaces $W^{1,p}$ with $p > d$.

Recently, global existence of weak solutions in $L^2$ was proved by Masmoudi \cite{9} under some entropy conditions.

To our best knowledge, there were no results about the well-posedness of (1.1) in Besov spaces. In this paper we investigate the well-posedness of (1.1) in Besov spaces $B^{s}_{p,r}$, which requires more elaborate techniques. In FENE model (1.1), the most difficult term is the additional stress tensor, however, Mousmoudi \cite{9} proved a lot of useful lemmas to deal with this term. By using Mousmoudi’s lemmas, we can easily get a corollary to solve the problem. Thus the remain difficulties are the product term and the pressure term. In order to obtain the well-posedness of (1.1) in Besov spaces, one can apply $\Delta_j$ to (1.2) and get a localization of the equations. The product term leads to the commutator. If $p \neq 2$, the energy method doesn’t work. However, by using the Littlewood-Paley theory and Bony’s decomposition, we split the commutator into 8 terms and for each term one can easily deal with by the basic Hölder’s inequality. Then we obtain the commutator estimates which lead us to obtain a priori estimates. In order to deal with the pressure, one can apply $div$ to (1.2), since $div u = 0$ it follows that $p$ satisfies an elliptic equation. There is an explicit formula giving the pressure in terms of the velocity field, then by using some techniques in Fourier analysis, we can deal with the pressure term. Thanks to the viscosity coefficient $\nu > 0$, the energy decays as time grows. If the $H^s$-norm of initial data is small, one can get the $H^s$-norm of the corresponding solution to (1.1) is smaller than that of initial data for any $t > 0$. Then by an iteration argument, one can obtain the global well-posedness of (1.1) in $H^s$. But for Besov spaces $B^{s}_{p,r}$, if $p \neq 2$, we can’t obtain this property. However, we can use a continuous argument mentioned in \cite{11} to show that if the initial data is small then the corresponding solution is uniformly bounded by the initial data independent with $t$, which leads to the global well-posedness of (1.1) in $B^{s}_{p,r}$.

The paper is organized as follow. In Section 2 we introduce some notations and our main results. In Section 3 we give some preliminaries which will be used in this paper. In Section 4 we investigate the linear problem of (1.1) and give some a priori estimates for solutions to (1.1). In Section 5 we prove
the local well-posedness of (1.1) by using approximate argument. Section 6 is devoted to the study of a blow-up criterion. In Section 7 we prove the global well-posedness of (1.1) by a contradiction argument.

2 Notations and main results

In this section we introduce our main results and the notations that we shall use throughout the paper.

For $p \geq 1$, we denote by $L^p$ the space

$$L^p = \{ \psi | |\psi| \|_{L^p} = \int \psi \|_{\infty}^{\psi} |dR| < \infty \}.$$ 

We will use the notation $L^p_x(L^q)$ to denote $L^p([0,R])$ : 

$$L^p_x(L^q) = \{ \psi | |\psi| \|_{L^p(L^q)} = (\int \psi \|_{\infty}^{\psi} (y) dR)^{\frac{1}{p}} < \infty \}.$$ 

Next we introduce the Littlewood-Paley decomposition and Besov spaces (see [2] for more details).

Let $C$ be the annulus $\{ \xi \in R^d | \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \}$. There exist radial functions $\chi$ and $\varphi$, valued in the interval $[0, 1]$, belonging respectively to $D(B(0, \frac{3}{4}))$ and $D(C)$, and such that 

$$\forall \xi \in R^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,$$

$$\forall \xi \in R^d \setminus \{0\}, \quad \sum_{j \in Z} \varphi(2^{-j} \xi) = 1,$$

$$|j - j'| \geq 2 \Rightarrow \text{Supp} \ \varphi(2^{-j} \xi) \cap \text{Supp} \ \varphi(2^{-j'} \xi) = \emptyset,$$

$$j \geq 1 \Rightarrow \text{Supp} \ \chi(\xi) \cap \text{Supp} \ \varphi(2^{-j} \xi) = \emptyset.$$ 

Define the set $\tilde{C} = B(0, \frac{3}{4}) + C$. And we have 

$$|j - j'| \geq 5 \Rightarrow 2^j \tilde{C} \cap 2^j C = \emptyset.$$ 

Further, we have 

$$\forall \xi \in R^d, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) \leq 1,$$

$$\forall \xi \in R^d, \quad \frac{1}{2} \leq \sum_{j \in Z} \varphi(2^{-j} \xi) \leq 1.$$ 

Denote $F$ by the Fourier transform and $F^{-1}$ by its inverse. From now on, we write $h = F^{-1} \varphi$ and $\tilde{h} = F^{-1} \chi$. The nonhomogeneous dyadic blocks $\Delta_j$ are defined by 

$$\Delta_j u = 0 \quad \text{if} \quad j \leq -2, \quad \Delta_{-1} u = \chi(D) u = \int_{R^d} \tilde{h}(y) u(x - y) dy,$$
and, \( \Delta_j u = \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} h(2^jy)u(x-y)dy \) if \( j \geq 0 \),

\[
S_j u = \sum_{j' \leq j-1} \Delta_{j'} u.
\]

And the homogeneous dyadic blocks \( \hat{\Delta}_j \) are defined by

\[
\hat{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^d} h(2^jy)u(x-y)dy,
\]

\[
\hat{S}_j u = \chi(2^{-j}D)u = \int_{\mathbb{R}^d} \tilde{h}(2^jy)u(x-y)dy.
\]

The homogeneous and nonhomogeneous Besov spaces are denote by \( \dot{B}^s_{p,r} \) and \( \dot{B}^s_{p,r} \)

\[
\dot{B}^s_{p,r} = \left\{ u \in S' ||u||_{B^s_{p,r}} = \left( \sum_{j \in \mathbb{Z}} 2^{js ||\hat{\Delta}_j u||_{L^p_{L_r}}} \right)^{\frac{1}{r}} < \infty \right\},
\]

\[
\dot{B}^s_{p,r} = \left\{ u \in S' ||u||_{B^s_{p,r}} = \left( \sum_{j \geq -1} 2^{js ||\hat{\Delta}_j u||_{L^p_{L_r}}} \right)^{\frac{1}{r}} < \infty \right\}.
\]

Also we denote \( C_T(B^s_{p,r}) \) by \( C([0,T]; B^s_{p,r}) \) and \( L^p_T(B^s_{p,r}) \) by \( L^p([0,T]; B^s_{p,r}) \) respectively. Moreover we use the spaces \( \dot{L}^p_T(B^s_{p,r}) \) and \( B^s_{p,r}(\mathcal{L}^q) \)

\[
\dot{L}^p_T(B^s_{p,r}) = \left\{ u \in S' ||u||_{B^s_{p,r}} = \left( \sum_{j \geq -1} 2^{js ||\hat{\Delta}_j u||_{L^p_{L_r}}} \right)^{\frac{1}{r}} < \infty \right\},
\]

\[
B^s_{p,r}(\mathcal{L}^q) = \left\{ \phi \in S' ||\phi||_{B^s_{p,r}(\mathcal{L}^q)} = \left( \sum_{j \geq -1} 2^{js ||\hat{\Delta}_j \phi||_{L^q_{L^q}}} \right)^{\frac{1}{r}} < \infty \right\}.
\]

The dyadic blocks \( \Delta_j \) are related to the variable \( x \) and independent with variable \( R \). It means that

\[
\mathcal{F}(\psi(t,x,R)) = \int_{\mathbb{R}^d} \psi(t,x,R)e^{-i\xi \cdot x} dx.
\]

Next we define a special space \( E^s_{p,r} \) which is useful in this paper,

\[
E^s_{p,r}(T) = \left\{ \psi : ||\psi||_{E^s_{p,r}(T)} = \left( \sum_{j \geq -1} 2^{js} \left( \int_0^T \int_{\mathbb{R}^d} \left| \nabla_R \left( \frac{\psi}{\psi_{\infty}} \right) \right|^{\frac{2}{r}} \psi_{\infty} \, dx \, dt \right)^{\frac{1}{r}} \right)^{\frac{1}{s}} < \infty \right\}.
\]

Now we state our main results.

**Theorem 2.1.** Assume that \( s > \frac{d}{p} + 1 \), \( 2 \leq p < \infty \), \( r \geq p \) and \( u_0 \in B^s_{p,r}(\mathbb{R}^d) \), \( \psi_0 \in B^s_{p,r}(\mathbb{R}^d; \mathcal{L}^p) \).

Then there exist some \( T^* > 0 \) and a unique solution \( (u, \psi) \) of (1.2) in

\[
C([0,T^*]; B^s_{p,r}) \times C([0,T^*]; B^s_{p,r}(\mathbb{R}^d; \mathcal{L}^p)), \quad \text{if } r < \infty,
\]

\[
C_w([0,T^*]; B^s_{p,\infty}) \times C_w([0,T^*]; B^s_{p,\infty}(\mathbb{R}^d; \mathcal{L}^p)).
\]

Moreover \( u \in L^2_T(B^{s+1}_{p,r}) \) and \( \psi \in E^s_{p,r}(T) \).
Theorem 2.2. Let \((u_0, \psi_0)\) be as in Theorem 2.1, and let \(T^*\) be the lifespan of the solution to (1.2). If \(T^* < \infty\), then we have that
\[
\int_0^{T^*} \|u\|_{L^\infty}^2 = \infty,
\]
\[
\|\psi\|_{L_{T^*}^p(B_{p,r}^s(L^p))}^2 + \|\psi\|_{E_{T^*}^s(B_{p,r}^s(L^p))}^2 = \infty.
\]

Theorem 2.3. Under the assumption of Theorem 2.1 and assume that \(\int_B \psi_0 dR = 1\) a.e. in \(x\). If there exists a constant \(c_0\) such that
\[
\|u_0\|_{B_{p,r}^s} + \|\psi_0 - \psi_\infty\|_{B_{p,r}^s(L^p)} \leq c_0,
\]
then the solution constructed in Theorem 2.1 is global. Moreover, there exist a constant \(M\) such that
\[
\|u(t)\|_{B_{p,r}^s} + \|\psi(t) - \psi_\infty\|_{B_{p,r}^s(L^p)} \leq M c_0.
\]

Remark 2.4. Thanks to \(B_{2,2}^s = H^s\) for \(s > 0\), if we take \(p = 2, r = 2\) in Theorem 2.1 and Theorem 2.3, then our theorems cover the recent results obtained by Masmoudi in [8].

3 Preliminaries

In this section we introduce some useful lemmas which will be used in the sequel. For more details, one can refer to Section 2 in [2].

3.1. Propositions of Besov spaces

Firstly we introduce the Bernstein inequalities.

Lemma 3.1. [2] Let \(C\) be an annulus and \(B\) a ball. A constant \(C\) exists such that for any nonnegative integer \(k\), any couple \((p, q)\) in \([1, \infty)^2\) with \(q \geq p \geq 1\), and any function \(u\) of \(L^p\), we have
\[
\text{Supp} \, \hat{u} \subseteq \lambda B \Rightarrow \|D^k u\|_{L^q} \triangleq \sup_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p},
\]
\[
\text{Supp} \, \hat{u} \subseteq \lambda C \Rightarrow C^{-k-1} \lambda^{d} \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}.
\]

Lemma 3.2. [2] Let \(C\) be an annulus. Positive constants \(c\) and \(C\) exist such that for any \(p\) in \([1, \infty]\) and any couple \((t, \lambda)\) of positive real numbers, we have
\[
\text{Supp} \, \hat{u} \subseteq \lambda C \Rightarrow \|e^{t \Delta} u\|_{L^p} \leq C e^{-ct} \|u\|_{L^p}.
\]

The following proposition is about the embedding for Besov spaces.
Proposition 3.3. [2] Let \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( 1 \leq r_1 \leq r_2 \leq \infty \), and let \( s \) be a real number. Then we have

\[
B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}.
\]

If \( s > \frac{d}{p} \) or \( s = \frac{d}{p}, \ r = 1 \), then we have

\[
B_{p,r}^s \hookrightarrow L^\infty.
\]

Proposition 3.4. [2] The set \( B_{p,r}^s \) is a Banach space and satisfies the Fatou property, namely, if \( (u_n)_{n \in \mathbb{N}} \) is a bounded sequence of \( B_{p,r}^s \), then an element \( u \) of \( B_{p,r}^s \) and a subsequence \( (u_{\psi(n)}) \) exist such that

\[
\lim_{n \to \infty} u_{\psi(n)} = u \text{ in } S' \quad \text{and} \quad \|u\|_{B_{p,r}^s} \leq C \liminf_{n \to \infty} \|u_{\psi(n)}\|_{B_{p,r}^s}.
\]

Next we introduce the Bony decomposition.

Definition 3.5. [2] The nonhomogeneous paraproduct of \( v \) and \( u \) is defined by

\[
T_u v \triangleq \sum_j S_{j-1} u \Delta_j v.
\]

The nonhomogeneous remainder of \( v \) and \( u \) is defined by

\[
R(u, v) \triangleq \sum \Delta_k u \Delta_j v.
\]

We have the following Bony decomposition

\[
u v = T_u v + R(u, v) + T_v u.
\]

Proposition 3.6. [2] For any couple of real numbers \( (s, t) \) with \( t \) negative and any \( (p, r_1, r_2) \) in \([1, \infty]^3\), there exists a constant \( C \) such that:

\[
\|T_u v\|_{B_{p,r}^s} \leq C \|u\|_{L^\infty} \|D^k v\|_{B_{p,r}^{s-k}},
\]

\[
\|T_u v\|_{B_{p,r}^{s+t}} \leq C \|u\|_{B_{p,r_1}^t} \|D^k v\|_{B_{p,r_2}^{s-k}},
\]

where \( r = \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\} \).

Proposition 3.7. [2] A constant \( C \) exists which satisfies the following inequalities. Let \( (s_1, s_2) \) be in \( \mathbb{R}^2 \) and \( (p_1, p_2, r_1, r_2) \) be in \([1, \infty]^4\). Assume that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.
\]

If \( s_1 + s_2 > 0 \), then we have, for any \( (u, v) \) in \( B_{p_1,r_1}^{s_1} \times B_{p_2,r_2}^{s_2} \),

\[
\|R(u, v)\|_{B_{p,r}^{s_1+s_2}} \leq C \frac{s_1 + s_2 + 1}{s_1 + s_2} \|u\|_{B_{p_1,r_1}^{s_1}} \|v\|_{B_{p_2,r_2}^{s_2}}.
\]
If \( r = 1 \) and \( s_1 + s_2 = 0 \), then we have, for any \((u, v)\) in \( B_{p_1, r_1}^{s_1} \times B_{p_2, r_2}^{s_2} \),
\[
\| R(u, v) \|_{B_{p, \infty}} \leq C \| u \|_{B_{p_1, r_1}^{s_1}} \| v \|_{B_{p_2, r_2}^{s_2}}.
\]

**Corollary 3.8.** [2] For any positive real number \( s \) and any \((p, r)\) in \([1, \infty]^2\), the space \( L^\infty \cap B_{p, r}^s \) is an algebra, and a constant \( C \) exists such that
\[
\| uv \|_{B_{p, r}^s} \leq C (\| u \|_{L^\infty} \| v \|_{B_{p, r}^s} + \| u \|_{B_{p, r}^s} \| v \|_{L^\infty}).
\]

If \( s > \frac{d}{p} \) or \( s = \frac{d}{p}, r = 1 \), we have
\[
\| uv \|_{B_{p, r}^s} \leq C \| u \|_{B_{p, r}^s} \| v \|_{B_{p, r}^s}.
\]

### 3.2. Propositions of \( \mathcal{L}^p \) and \( B_{p, r}^s (\mathcal{L}^p) \) spaces

Now we introduce some propositions about \( \mathcal{L}^p \) and \( B_{p, r}^s (\mathcal{L}^p) \).

**Proposition 3.9.** For any \( 1 \leq p \leq \infty \). We have \( \mathcal{L}^p(B) \hookrightarrow \mathcal{L}^p(B) \).

**Proof.** By the definition we have
\[
\psi_\infty = \frac{e^{-u(R)}}{\int_B e^{-u(R)} dR} = \frac{(1 - |R|^2)^k}{\int_B (1 - |R|^2)^k dR}.
\]

Since \( |R| \leq 1 \), it follows that \( \psi_\infty^p \leq C \psi_\infty \), where \( C \) is a constant independent of \( R \). Then
\[
\int_B |\psi|^p = \int_B \psi_\infty^p |\psi|^p \psi_\infty \leq C \int_B \psi_\infty |\psi|^p \psi_\infty.
\]

**Remark 3.10.** If \( 1 \leq p < \infty \), then \( C_0^\infty (B) \) is dense in \( \mathcal{L}^p(B) \).

**Proposition 3.11.** If \( 1 < p < \infty \) then \( \mathcal{L}^p \) is a reflexive Banach space.

**Proof.** By the definition of \( \mathcal{L}^p \), we have
\[
\mathcal{L}^p = L^p (\frac{dR}{\psi_\infty^{p-1}}).
\]

Since \( \frac{dR}{\psi_\infty^{p-1}} \) is a \( \sigma \) finitely additive measure, it follows that
\[
(\mathcal{L}^p)^* = (L^p (\frac{dR}{\psi_\infty^{p-1}}))^* = L^{p'} (\frac{dR}{\psi_\infty^{p-1}}),
\]

and
\[
(L^{p'} (\frac{dR}{\psi_\infty^{p-1}}))^* = L^p (\frac{dR}{\psi_\infty^{p-1}}).
\]

So \( \mathcal{L}^p \) is a reflexive Banach space.
Next we will introduce an inequality which is used to estimate the stress tensor $\text{div}\, \tau$, and we set $x = 1 - |R|$.

**Lemma 3.12.** For all $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that

$$
(\int_B \frac{|\psi|}{x} dR)^2 \leq \varepsilon \int_B \psi_\infty \left|\nabla_R \frac{\psi}{\psi_\infty}\right|^2 dR + C_\varepsilon \int_B \frac{|\psi|^2}{\psi_\infty} dR.
$$

**Corollary 3.13.** Assume that $2 \leq p < \infty$, for all $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that

$$
(\int_B \frac{|\psi|}{x} dR)^p \leq \varepsilon \int_B \psi_\infty \left|\nabla_R \left(\frac{\psi}{\psi_\infty}\right)^{\frac{p}{2}}\right|^2 dR + C_\varepsilon \int_B \frac{|\psi|^p}{\psi_\infty} dR.
$$

**Proof.** By a direct calculation and the Hölder inequality, we have

$$
(\int_B \frac{|\psi|}{x} dR)^p = (\int_B x^{\frac{2}{p}(k-1) - k} |\psi|^\frac{2}{p} dR)^p \leq (\int_B x^{k-1} |\psi|^\frac{2}{p} dR)^p \leq C(\int_B x^{k-1} |\psi|^\frac{2}{p} dR)^2
$$

(3.1)

So by Lemma (3.12) we obtain

$$
(3.1) \leq \varepsilon \int_B \psi_\infty \left|\nabla_R \frac{\psi}{\psi_\infty}\right|^2 dR + C_\varepsilon \int_B \frac{|\psi|^2}{\psi_\infty} dR
$$

$$
= \varepsilon \int_B \psi_\infty \left|\nabla_R \left(\frac{\psi}{\psi_\infty}\right)^{\frac{2}{p}}\right|^2 dR + C_\varepsilon \int_B \frac{|\psi|^p}{\psi_\infty} dR.
$$

\qed

Next we will introduce some propositions about the space $B^s_{p,r}(\mathcal{L}^q)$. The following proposition is similar to Proposition (3.4). For more details, one can refer to Section 2.3 in [2].

**Proposition 3.14.** The set $B^s_{p,r}(\mathcal{L}^q)$ is a Banach space. Moreover if $1 < p, q < \infty$, the set $B^s_{p,r}(\mathcal{L}^q)$ satisfies the Fatou property, namely, if $(\psi_n)_{n \in \mathbb{N}}$ is a bounded sequence of $B^s_{p,r}(\mathcal{L}^q)$, then an element $\psi$ of $B^s_{p,r}(\mathcal{L}^q)$ and a subsequence $(\psi_{n_k})$ exist such that

$$
\lim_{k \to \infty} \psi_{n_k} = \psi \text{ in } S' \text{ and } \|\psi\|_{B^s_{p,r}(\mathcal{L}^q)} \leq C \liminf_{k \to \infty} \|\psi_{n_k}\|_{B^s_{p,r}(\mathcal{L}^q)}.
$$

**Proof.** Assume that the sequence $(\psi_n)$ is bounded in $B^s_{p,r}(\mathcal{L}^q)$. Then for any $j \geq -1$, the sequence $(\Delta_j \psi_n)$ is bounded in $L^p_S(\mathcal{L}^q)$. Because $\mathcal{L}^q$ is a reflexive Banach space. Cantors diagonal process thus supplies a subsequence $(\Delta_j \psi_{n_k})$ and a sequence $(\tilde{\psi}_j)$ of $C^\infty(\mathbb{R}^d; \mathcal{L}^q)$ functions with Fourier transform supported in $2^j\mathbb{C}$ such that, for any $j \in \mathbb{Z}, \phi \in \mathcal{S},$

$$
\lim_{k \to \infty} \langle \Delta_j \psi_{n_k}, \phi \rangle = \langle \tilde{\psi}_j, \phi \rangle \text{ and } \|\tilde{\psi}_j\|_{L^p(\mathcal{L}^q)} \leq \liminf_{n \to \infty} \|\psi_n\|_{L^p(\mathcal{L}^q)}.
$$

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Now, the sequence \( (2^{j^*} \| \Delta_j \psi_n \|_{L^p(L^q)})_j \) is bounded in \( L^r \). Hence, there exists an element \( \tilde{c}_j \) of \( L^r \) such that (up to an extraction) for any sequence \( (d_j) \) of nonnegative real numbers different from 0 for only a finite number of indices \( j \),

\[
\lim_{k \to \infty} \sum_{j \geq -1} 2^{j^*} \| \Delta_j \psi_n \|_{L^p(L^q)} d_j = \sum_{j \geq -1} \tilde{c}_j d_j \quad \text{and} \quad \|\tilde{c}_j\|_{L^r} \leq \liminf_{k \to \infty} \|\psi_n\|_{B_{p,r}^s(L^q)}.
\]

Passing to the limit in the sum gives that \( (2^{j^*} \| \psi_j \|_{L^p(L^q)})_j \) belongs to \( L^r \). The Fourier transform of \( \langle \tilde{\psi}_j \rangle \) is supported in \( 2^j \mathcal{C} \). So the series \( \sum_{j \geq -1} \psi_j \) converges to some \( \psi \) in \( S' \). For all the \( N \) and \( \phi \in S \) we have

\[
\langle \psi, \phi \rangle = \langle \psi, \phi \rangle = \langle \sum_{j \geq -1} \psi_j, \phi \rangle,
\]

then we obtain

\[
\sum_{j \geq -1} \Delta_j \psi = \lim_{k \to \infty} \sum_{j \geq -1} \Delta_j \psi_n \quad \text{in} \quad S',
\]

and \( (\text{Id} - S_N) \psi_n \) tends to 0 in \( B_{p,r}^s \). So \( \psi \) is indeed the limit of \( \psi_n \) in \( S' \), which completes the proof of the Fatou property.

We will now check that \( B_{p,r}^s(L^q) \) is complete. Consider a Cauchy sequence \( \psi_n \). This sequence is of course bounded, so there exist some \( \psi \) and a subsequence \( \psi_{n_k} \), such that \( \psi_{n_k} \) converges to \( \psi \) in \( S' \). For any \( \varepsilon > 0 \) there exist a \( N_\varepsilon \) such that

\[
n_k \geq m_k \geq N_\varepsilon \Rightarrow \| \psi_{n_k} - \psi_{m_k} \|_{B_{p,r}^s(L^q)} < \varepsilon.
\]

The Fatou property ensures that

\[
n_k \geq N_\varepsilon, \quad \| \psi_{n_k} - \psi \|_{B_{p,r}^s(L^q)} < \varepsilon.
\]

Hence \( \psi_{n_k} \) tends to \( \psi \) in \( B_{p,r}^s(L^q) \), as \( k \) goes to infinity.

\[
\square
\]

The next lemma is very useful to deal with the product of \( u(t, x) \) and \( \psi(t, x, R) \).

**Lemma 3.15.** For any positive real number \( s \) and any \( (p, r) \) in \([1, \infty]^2\), if \( u = u(t, x) \in L^\infty \cap B_{p,r}^s \) and \( \psi = \psi(t, x, R) \in L^\infty(L^p) \cap B_{p,r}^s(L^q) \), then a constant \( C \) exists such that

\[
\| u\psi \|_{B_{p,r}^s(L^q)} \leq C(\| u \|_{L^\infty} \| \psi \|_{B_{p,r}^s(L^q)} + \| u \|_{B_{p,r}^s(L^p)} \| \psi \|_{L^\infty(L^q)}).
\]

If \( s > \frac{d}{p} \) or \( s = \frac{d}{p}, \ r = 1 \), we have

\[
\| u\psi \|_{B_{p,r}^s(L^q)} \leq C \| u \|_{B_{p,r}^s} \| \psi \|_{B_{p,r}^s(L^q)}.
\]
Proof. We can write
\[ u\psi = T_u\psi + T_\psi u + R(u, \psi). \]

Firstly, we consider the term \( T_u\psi \), by definition and the proposition about \( \Delta_j \) and \( S_j \) we deduce that
\[
\Delta_j T_u \psi = \Delta_j \sum_{|j-j'| \leq 2} S_{j'-1} u \Delta_j' \psi.
\]

So using Hölder’s inequality, we infer that
\[
2^{js} \| \Delta_j T_u \psi \|_{L^p(\mathcal{L}^p)} \leq 2^{js} \sum_{|j-j'| \leq 2} \| S_{j'-1} u \Delta_j' \psi \|_{L^p(\mathcal{L}^p)}
\]
\[
= \sum_{|j-j'| \leq 2} 2^{js} \left( \int_{\mathbb{R}^d \times B} \left| S_{j'-1} u(t, x) \frac{\Delta_j' \psi(t, x, R)}{\psi_\infty} \right|^p \psi_\infty \, dx \, dR \right)^{\frac{1}{p}}
\]
\[
\leq C \sum_{|j-j'| \leq 2} 2^{js} \| S_{j'-1} u(t, x) \|_{L^\infty} \left( \int_{\mathbb{R}^d \times B} \left| \frac{\Delta_j' \psi(t, x, R)}{\psi_\infty} \right|^p \psi_\infty \, dx \, dR \right)^{\frac{1}{p}}
\]
\[
\leq C \sum_{|j-j'| \leq 2} 2^{(j-j')s} \| u(t, x) \|_{L^\infty} 2^{j's} \| \Delta_j' \psi \|_{L^p(\mathcal{L}^p)}
\]
\[
\leq C c_j \| u \|_{L^\infty} \| \psi \|_{B^{p,1}(\mathcal{L}^p)}.
\]

Where \( c_j \) denotes an element of the unit sphere of \( U \). Taking \( U \)-norm for both sides of the above inequality, we obtain
\[
(3.2) \quad \| T_u \psi \|_{B^{p,1}(\mathcal{L}^p)} \leq C \| u \|_{L^\infty} \| \psi \|_{B^{p,1}(\mathcal{L}^p)}.
\]

Next, we consider the second term \( T_\psi u \). Similarly we deduce that
\[
\Delta_j T_\psi u = \Delta_j \sum_{|j-j'| \leq 2} S_{j'-1} \psi \Delta_j' u.
\]

So using Fubini’s theorem and Hölder’s inequality, we infer that
\[
2^{js} \| \Delta_j T_\psi u \|_{L^p(\mathcal{L}^p)} \leq 2^{js} \sum_{|j-j'| \leq 2} \| S_{j'-1} \psi \Delta_j' u \|_{L^p(\mathcal{L}^p)}
\]
\[
= \sum_{|j-j'| \leq 2} 2^{js} \left( \int_{\mathbb{R}^d \times B} \left| \Delta_j' u(t, x) \frac{S_{j'-1} \psi(t, x, R)}{\psi_\infty} \right|^p \psi_\infty \, dx \, dR \right)^{\frac{1}{p}}
\]
\[
= \sum_{|j-j'| \leq 2} 2^{js} \left( \int_{\mathbb{R}^d} \left| \Delta_j' u(t, x) \right|^p \int_{B} \left| \frac{S_{j'-1} \psi(t, x, R)}{\psi_\infty} \right|^p \psi_\infty \, dR \, dx \right)^{\frac{1}{p}}
\]
\[
\leq C \sum_{|j-j'| \leq 2} 2^{js} \| \Delta_j' u(t, x) \|_{L^p} \sup_x \left( \int_{B} \left| \frac{S_{j'-1} \psi(t, x, R)}{\psi_\infty} \right|^p \psi_\infty \, dR \right)^{\frac{1}{p}}
\]
\[
\leq C \sum_{|j-j'| \leq 2} 2^{(j-j')s} 2^{j's} \| \Delta_j' u(t, x) \|_{L^p} \| \psi \|_{L^p(\mathcal{L}^p)}
\]
\[ \leq C c_j \|u\|_{B^p_{r,r}} \|\psi\|_{L^\infty(\mathcal{L}^r)}. \]

Where \( c_j \) denotes an element of the unit sphere of \( l^r \). Taking \( l^r \)-norm for both sides of the above inequality, we get

\[ \|T\psi u\|_{B^p_{r,r}(\mathcal{L}^r)} \leq C \|u\|_{B^p_{r,r}} \|\psi\|_{L^\infty(\mathcal{L}^r)}. \]

(3.3)

Finally, we consider the last term \( R(u, \psi) \). By definition, we can write

\[ R(u, \psi) = \sum_{j'} R_{j'}, \quad \text{with} \quad R_{j'} = \sum_{|k| \leq 1} \Delta_{j'-k} u(t, x) \Delta_{j'} \psi(t, x, R). \]

By the construction of the dyadic partition of unity, there exists an integer \( N_0 \) such that

\[ j > j' + N_0 \Rightarrow \Delta_j R_{j'} = 0. \]

From this we deduce that

\[ \Delta_j R(u, \psi) = \Delta_j \sum_{j' \geq j-N_0} \sum_{|k| \leq 1} \Delta_{j'-k} u \Delta_{j'} \psi. \]

So using Hölder’s inequality and due to \( s > 0 \), we infer that

\[ 2^{j's} \|\Delta_j R(u, \psi)\|_{L^p_\infty(\mathcal{L}^r)} \leq 2^{j's} \sum_{j' \geq j-N_0} \sum_{|k| \leq 1} \|\Delta_{j'-k} u \Delta_{j'} \psi\|_{L^p_\infty(\mathcal{L}^r)} \]

\[ \quad = \sum_{j' \geq j-N_0} \sum_{|k| \leq 1} 2^{j's} \left( \int_{\mathbb{R}^d \times B} \left| \Delta_{j'-k} u(t, x) \frac{\Delta_{j'} \psi(t, x, R)}{\psi_\infty} \right|^p \psi_\infty dx dR \right)^{\frac{1}{p}} \]

\[ \leq C \sum_{j' \geq j-N_0} 2^{j's} \|u(t, x)\|_{L^\infty} \left( \int_{\mathbb{R}^d \times B} \left| \Delta_{j'} \psi(t, x, R) \right|^p \psi_\infty dx dR \right)^{\frac{1}{p}} \]

\[ \leq C \sum_{j' \geq j-N_0} 2^{(j-j')s} \|u(t, x)\|_{L^\infty} 2^{j's} \|\Delta_{j'} \psi\|_{L^p_\infty(\mathcal{L}^r)} \]

\[ \leq C c_j \|u\|_{L^\infty} \|\psi\|_{B^p_{r,r}(\mathcal{L}^r)}. \]

Where \( c_j \) denotes an element of the unit sphere of \( l^r \). Taking \( l^r \)-norm for both sides of the above inequality, we obtain

\[ \|R(u, \psi)\|_{B^p_{r,r}(\mathcal{L}^r)} \leq C \|u\|_{L^\infty} \|\psi\|_{B^p_{r,r}(\mathcal{L}^r)}. \]

(3.4)

Combining (3.2), (3.3) and (3.4), we complete the proof. \( \square \)

### 3.3. Commutator estimates

This section is devoted to various commutator estimates which enable us to establish a priori estimates. The proof is similar to the commutator estimates for Besov spaces. For more details, one can refer to Section 2.10 in [2].
Lemma 3.16. Let \( \theta \) be a \( C^1 \) function on \( \mathbb{R}^d \) such that \( |\cdot|F^{-1}(\theta) \in L^1 \). There exists a constant \( C \) such that for any Lipschitz function \( u(x) \) and any function \( \psi(x,R) \) in \( L^p_\mathcal{L}(\mathcal{L}^p) \), we have, for any positive \( \lambda \),

\[
\| \theta(\lambda^{-1} D), u \psi \|_{L^p_\mathcal{L}(\mathcal{L}^p)} \leq C \lambda^{-1} \| \nabla u \|_{L^\infty} \| \psi \|_{L^p_\mathcal{L}(\mathcal{L}^p)}.
\]

Proof. Indeed,

\[
(\theta(\lambda^{-1} D), u \psi) = \lambda^d \int_{\mathbb{R}^d} k(\lambda(x-y))(u(y) - u(x))\psi(y,R)dy \quad \text{with} \quad k = F^{-1}.\theta.
\]

Let \( k_1(z) = |z||k(z)| \). From the first order Taylor formula, we deduce that

\[
(\theta(\lambda^{-1} D), u \psi)(x, R) \leq \lambda^{-1} \int_{[0,1] \times \mathbb{R}^d} \lambda^d k_1(\lambda z) (\nabla |u(x - \tau z)|) \| \psi(x - z, R) \| dz.
\]

Taking the \( L^p_\mathcal{L}(\mathcal{L}^p) \)-norm of the above inequality and using H"older’s inequality, we infer that

\[
\| \theta(\lambda^{-1} D), u \psi \|_{L^p_\mathcal{L}(\mathcal{L}^p)} \leq \lambda^{-1} \int_{[0,1] \times \mathbb{R}^d} \lambda^d k_1(\lambda z) (\nabla |u(x - \tau z)|) \| \psi(x - z, R) \| L^p_\mathcal{L}(\mathcal{L}^p) dz
\]

\[
\leq C \lambda^{-1} \| k_1 \|_{L^1} \| \nabla u(x) \psi(x, R) \|_{L^p_\mathcal{L}(\mathcal{L}^p)}
\]

\[
\leq C \lambda^{-1} \int_{\mathbb{R}^d \times B} \left| \nabla u(x) \frac{\psi(x, R)}{\psi_\infty} \right|^p \psi_\infty dxdR \frac{1}{\| \psi \|_{L^p_\mathcal{L}(\mathcal{L}^p)}}
\]

\[
\leq C \lambda^{-1} \| \nabla u \|_{L^\infty} \| \psi \|_{L^p_\mathcal{L}(\mathcal{L}^p)}.
\]

\[ \square \]

Lemma 3.17. Let \( s > 0 \) and \( (p, r) \in [1, \infty]^2 \), and let \( u \) be a vector field on \( \mathbb{R}^d \) with \( \nabla u \in L^\infty \cap B^{s-1}_{p,r} \). Define \( R_j = [u \cdot \nabla, \Delta_j] \psi(t, x, R) \). There exists a constant \( C \), depending on \( p, s, r \) and \( d \), such that

\[
\| (2^j)^s R_j \|_{L^p_\mathcal{L}(\mathcal{L}^p)} \|_{L^r} \leq C (\| \nabla u \|_{L^\infty} \| \psi \|_{B^{s-1}_{p,r}(\mathcal{L}^p)} + \| \nabla_x \psi \|_{L^\infty(\mathcal{L}^p)} \| \nabla u \|_{B^{s-1}_{p,r}}) \). \]

If \( s > 1 + \frac{d}{p} \) or \( s = 1 + \frac{d}{p}, \ r = 1 \), we have

\[
\| (2^j)^s R_j \|_{L^p_\mathcal{L}(\mathcal{L}^p)} \|_{L^r} \leq C \| \nabla u \|_{B^{s-1}_{p,r}} \| \psi \|_{B^{s}_{p,r}(\mathcal{L}^p)}. \]

If \( s \leq 1 + \frac{d}{p} \), we have

\[
\| (2^j)^s R_j \|_{L^p_\mathcal{L}(\mathcal{L}^p)} \|_{L^r} \leq C \| \nabla u \|_{B^{s}_{p,\infty} \cap L^\infty} \| \psi \|_{B^{s}_{p,r}(\mathcal{L}^p)}. \]

Proof. We shall split \( u \) into low and high frequencies: \( u = S_0 u + \bar{u} \). Obviously, we have

\[
(3.5) \quad \| S_0 \nabla u \|_{L^p} \leq C \| \nabla u \|_{L^p} \quad \text{and} \quad \| \nabla \bar{u} \|_{L^p} \leq C \| \nabla u \|_{L^p}. \]
Further, as $\tilde{u}$ is spectrally supported away from the origin, Lemma 3.1 ensures that

\[(3.6) \quad \forall j \geq -1, \quad \|\Delta_j \nabla \tilde{u}\|_{L^p} \approx 2^j \|\Delta_j \tilde{u}\|_{L^p}.
\]

Using Bony’s decomposition, we end up with $R_j = \sum_{i=1}^{8} R^i_j$, where

\[
R^1_j = [\tilde{u}^k, \Delta_j \partial_k \psi], \quad R^2_j = \Delta_j \partial_k \psi \tilde{u}^k, \quad R^3_j = -\Delta_j \partial_k \psi \tilde{u}^k, \quad R^4_j = \partial_k R(\Delta_j \psi, \tilde{u}^k), \quad R^5_j = -R(\Delta_j \psi, \text{div} \, \tilde{u}), \quad R^6_j = -\partial_k \Delta_j R(\psi, \tilde{u}^k), \quad R^7_j = \Delta_j R(\psi, -\text{div} \, \tilde{u}), \quad R^8_j = |\bar{S}_0 u, \Delta_j \partial_k \psi|.
\]

In the following computations, we denote by $(c_j)_{j \geq -1}$ a sequence such that $\|c_j\|_{L^p} \leq 1$.

**Bounds for $2^{js} \|R^1_j\|_{L^p(L^p)}$.** By the construction of the dyadic partition of unity, we have

\[
R^1_j = \sum_{|j-j'| \leq 4} [S_{j'-1} \tilde{u}^k, \Delta_j \partial_k \Delta_j' \psi].
\]

Hence, according to Lemma 3.16 and (3.5), we have

\[
2^{js} \|R^1_j\|_{L^p(L^p)} \leq C \|\nabla u\|_{L^\infty} \sum_{|j-j'| \leq 4} 2^{js} \|\Delta_j' \psi\|_{L^p(L^p)} \leq C c_j \|\nabla u\|_{L^\infty} \|\psi\|_{B^p_{p,r}(L^p)}.
\]

**Bounds for $2^{js} \|R^2_j\|_{L^p(L^p)}$.** By the construction of the dyadic partition of unity, we obtain

\[
R^2_j = \sum_{|j-j'| \leq 3} S_{j'-1} \partial_k \Delta_j \psi \Delta_j' \tilde{u}^k.
\]

Hence, using (3.5), (3.6) and Hölder’s inequality, we deduce that

\[
2^{js} \|R^2_j\|_{L^p(L^p)} \leq C c_j \|\nabla u\|_{L^\infty} \|\psi\|_{B^p_{p,r}(L^p)}.
\]

**Bounds for $2^{js} \|R^3_j\|_{L^p(L^p)}$.** By the definition, we have

\[
R^3_j = \sum_{|j-j'| \leq 4} \Delta_j (S_{j'-1} \partial_k \psi \Delta_j' \tilde{u}^k).
\]

Hence, using (3.5), (3.6) and Hölder’s inequality, we obtain

\[
2^{js} \|R^3_j\|_{L^p(L^p)} \leq C \sum_{|j-j'| \leq 4} 2^{j(j'-1)} \|\nabla S_{j'-1} \psi\|_{L^\infty(L^p)} 2^{j'(s-1)} \|\Delta_j' \nabla u\|_{L^p},
\]

from which it follows that

\[
2^{js} \|R^3_j\|_{L^p(L^p)} \leq C c_j \|\nabla u\|_{B^p_{p,r}(L^p)} \|\nabla \psi\|_{L^\infty(L^p)}.
\]

If $s < 1 + \frac{d}{p}$, we may write $R^3_j$ as follow:

\[
R^3_j = \sum_{|j'-j| \leq 4} \sum_{j'' < j'-2} \Delta_j (\Delta_j' \partial_k \psi \Delta_j' \tilde{u}^k).
\]
Hence, using (3.5), (3.6) and Hölder’s inequality, we get
\[
2^{js} \| R_j^3 \|_{L^p_x(L^p)} \leq C \sum_{|j-j'| \leq 4, j' < j - 2} 2^{j''s} \| \Delta_{j''} \psi \|_{L^p_x(L^p)} 2^{-j'} \| \Delta_{j'} \nabla u \|_{L^p}.
\]
\[
\leq C \sum_{|j-j'| \leq 4, j' < j - 2} 2^{j''(s - 1 - \frac{d}{p})} 2^{j''s} \| \Delta_{j''} \psi \|_{L^p_x(L^p)} 2^{-j'} \| \Delta_{j'} \nabla u \|_{L^p}.
\]
Since \( s < 1 + \frac{d}{p} \), it follows that
\[
2^{js} \| R_j^3 \|_{L^p_x(L^p)} \leq C c_j \| \nabla u \|_{B^{s}_{p,\infty}} \| \psi \|_{B^{s}_{p,r}(L^p)}.
\]
Bounds for \( 2^{js} \| R_j^4 \|_{L^p_x(L^p)} \), \( 2^{js} \| R_j^5 \|_{L^p_x(L^p)} \), \( 2^{js} \| R_j^6 \|_{L^p_x(L^p)} \) and \( 2^{js} \| R_j^7 \|_{L^p_x(L^p)} \) are similar. We only treat with \( R_j^4 \). Defining \( \Delta_{j'} = \Delta_{j-1} + \Delta_j + \Delta_{j+1} \), we have
\[
R_j^4 = \sum_{|j-j'| \leq 2} \partial_k (\Delta_j \Delta_{j'} \psi \Delta_{j'} \overline{u}^k).
\]
Hence, using (3.5), (3.6) and Hölder’s inequality, we deduce that
\[
2^{js} \| R_j^4 \|_{L^p_x(L^p)} \leq C c_j \| \nabla u \|_{L^\infty} \| \psi \|_{B^{s}_{p,r}(L^p)}.
\]
Bounds for \( 2^{js} \| R_j^8 \|_{L^p_x(L^p)} \). A direct calculation yields
\[
R_j^8 = \sum_{|j-j'| \leq 1} [\Delta_j, \Delta_{-1} u] \cdot \nabla \Delta_{j'} \psi.
\]
So by Lemma 3.16 we have
\[
2^{js} \| R_j^8 \|_{L^p_x(L^p)} \leq C \sum_{|j-j'| \leq 1} \| \nabla \Delta_{-1} u \|_{L^\infty} 2^{js} \| \Delta_{j'} \psi \|_{L^p_x(L^p)} \leq C c_j \| \nabla u \|_{L^\infty} \| \psi \|_{B^{s}_{p,r}(L^p)}.
\]
So combining the bounds for \( R_j^1 \) to \( R_j^8 \) yields the result.

**Remark 3.18.** For homogeneous Besov spaces, Propositions 3.6, 3.7, 3.8, 3.15 and Lemmas 3.16, 3.18 still hold true. The proofs are similar, just replace \( \Delta_j \) by \( \tilde{\Delta}_j \).
4 Linear problem and a priori estimates

In this section we will consider the following linearized equations for (1.2):

\[
\begin{align*}
\partial_t u + (v \cdot \nabla) u + \nu \Delta u + \nabla P &= f, \quad \text{div} u = 0, \\
\partial_t \psi + (v \cdot \nabla) \psi &= \text{div}_R[-\nabla v R \psi + \psi_\infty \nabla R \frac{\psi}{\psi_\infty}], \\
\tau_{ij} &= \int_B R_i \otimes \nabla_j u dR, \\
\psi|_{t=0} &= \psi_0, \\
\psi_\infty \nabla R \frac{\psi}{\psi_\infty} \cdot n &= 0 \quad \text{on} \quad \partial B(0, R_0).
\end{align*}
\]

(4.1)

4.1. Solutions to the linear equations in \( \mathbb{R} \)

Using Proposition 3.9 proved by Masmoudi in [5], we can solve the following linear problem in \( \mathbb{R} \).

**Proposition 4.1.** Assume that \( A(t) \in C([0, T]) \) is a matrix-valued function and \( \psi_0 \in \mathcal{L}^p \) with \( p \in [2, +\infty) \), then

\[
\begin{align*}
\partial_t \psi &= \text{div}_R[-A(t) R \psi + \psi_\infty \nabla R \frac{\psi}{\psi_\infty}], \\
\psi|_{t=0} &= \psi_0, \\
\psi_\infty \nabla R \frac{\psi}{\psi_\infty} \cdot n &= 0 \quad \text{on} \quad \partial B(0, R_0),
\end{align*}
\]

(4.2)

has a unique weak solution \( \psi \) in \( C([0, T]; \mathcal{L}^p) \). Moreover, we have

\[
\sup_{t \in [0, T]} \int_B \left| \frac{\psi}{\psi_\infty} \right|^p \psi_\infty dR + \frac{2(p-1)}{p} \int_0^T \int_B \left| \nabla R \left( \frac{\psi}{\psi_\infty} \right) \right|^2 \psi_\infty dR dt \leq C e^{C \int_0^t \int_B \left| \psi_0 \right|^p \psi_\infty dR dt}.
\]

**Proof.** Firstly we smooth out the initial data \( \psi_0 \). Since \( C^\infty(B) \) is dense in \( \mathcal{L}^p \), it follows that there exists \( \psi_0^N \in C^\infty(B) \) such that

\[
\psi_0^N \rightarrow \psi_0 \quad \text{in} \quad \mathcal{L}^p.
\]

Assume that \( \psi^N \) is the solution of

\[
\begin{align*}
\partial_t \psi^N &= \text{div}_R[-A(t) R \psi^N + \psi_\infty \nabla R \frac{\psi^N}{\psi_\infty}], \\
\psi^N|_{t=0} &= \psi_0^N, \\
\psi_\infty \nabla R \frac{\psi^N}{\psi_\infty} \cdot n &= 0 \quad \text{on} \quad \partial B(0, R_0).
\end{align*}
\]

(4.3)
As $p \geq 2$, we have $\mathcal{L}^p \hookrightarrow \mathcal{L}^2$. So by Proposition 3.9 in [1], we can exactly find $\psi^N \in C^\infty$. By multiplying both sides of (4.3) by $\text{sgn}(\psi^N) \frac{\psi^N}{\psi^\infty}$ and integrating over $B$, we deduce that

$$
\frac{1}{p} \partial_t \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^p \psi^\infty dR = (p-1) \left[ \int_B A(t)R \left| \frac{\psi^N}{\psi^\infty} \right|^{p-1} \nabla R \frac{\psi^N}{\psi^\infty} \psi^\infty dR - \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^{p-2} \left( \nabla R \frac{\psi^N}{\psi^\infty} \right)^2 \psi^\infty dR \right]
$$

$$
\leq (p-1) \left[ C |A(t)|^2 \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^p \psi^\infty dR - \frac{1}{2} \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^{p-2} \left( \nabla R \frac{\psi^N}{\psi^\infty} \right)^2 \psi^\infty dR \right]
$$

$$
= C(p-1) |A(t)|^2 \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^p \psi^\infty dR - \frac{2(p-1)}{p^2} \int_B \left| \nabla R \frac{\psi^N}{\psi^\infty} \right|^2 \psi^\infty dR.
$$

Then we obtain

$$
\int_B \left| \frac{\psi^N}{\psi^\infty} \right|^p \psi^\infty dR + \frac{2(p-1)}{p} \int_0^t \int_B \left| \nabla R \left( \frac{\psi^N}{\psi^\infty} \right) \right|^2 \psi^\infty dR \leq C(p-1) |A(t)|^2 \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^p \psi^\infty dR.
$$

Using Gronwall’s inequality, we have

$$
(4.4) \quad \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^p \psi^\infty dR + \frac{2(p-1)}{p} \int_0^t \int_B \left| \nabla R \left( \frac{\psi^N}{\psi^\infty} \right) \right|^2 \psi^\infty dR \leq C e^{C \int_0^t |A^2(t') dt'} \int_B \left| \frac{\psi^N}{\psi^\infty} \right|^p \psi^\infty dR.
$$

Because (4.3) is a linear equation, $\psi^N - \psi^M$ has the same estimate as above, that is

$$
\sup_{t \in [0,T]} \| \psi^N - \psi^M \|_{L^p} \leq C e^{C \int_0^T A^2(t') dt'} \| \psi_0^N - \psi_0^M \|_{L^p}.
$$

Then $\psi^N$ is a Cauchy sequence in $L^\infty_\|^p$. There exists a $\psi \in L^\infty([0,T]; L^p)$ such that

$$
\psi^N \rightarrow \psi \quad \text{in} \quad L^\infty([0,T]; L^p).
$$

Passing to the limit in the equation (4.3) we can see that $\psi$ is a solution of (4.2). Passing to the limit in (4.4), we obtain

$$
\sup_{t \in [0,T]} \left| \frac{\psi}{\psi^\infty} \right|^p \psi^\infty dR + \frac{2(p-1)}{p} \int_0^T \int_B \left| \nabla R \left( \frac{\psi}{\psi^\infty} \right) \right|^2 \psi^\infty dR \leq C e^{C \int_0^T A^2(t') dt'} \int_B \left| \frac{\psi}{\psi^\infty} \right|^p \psi^\infty dR.
$$

Assume that $\psi, \phi$ are two solutions of (4.2) with the same initial data. From the above estimate we have

$$
\| \psi - \phi \|_{L^p} \leq 0,
$$

which leads to the uniqueness.

Finally we shall prove that $\psi \in C([0,T]; L^p)$. For any $t, s \in [0,T]$

$$
\| \psi(t) - \psi(s) \|_{L^p} \leq \int_s^t \frac{1}{p} \partial_t \int_B \left| \frac{\psi}{\psi^\infty} \right|^p \psi^\infty dR dt'
$$
Then we have the following estimate

\[ \psi \leq (p - 1) \left| C \int_s^t \int_B |A(t')|^2 \left| \frac{\psi}{\psi_{\infty}} \right|^p \psi_{\infty} dR + \frac{1}{2} \int_s^t \int_B \left| \frac{\psi}{\psi_{\infty}} \right|^{p-2} \left( \nabla R \frac{\psi}{\psi_{\infty}} \right)^2 \psi_{\infty} dt' \right| \]

\[ \leq C(|t - s| \sup_{t \in [0,T]} A^2(t) \| \psi \|_{L^p} + \int_s^t \int_B \left| \nabla R \left( \frac{\psi}{\psi_{\infty}} \right) \right|^2 \psi_{\infty} dR dt'). \]

Since

\[ \int_0^T \int_B \left| \nabla R \left( \frac{\psi}{\psi_{\infty}} \right) \right|^2 \psi_{\infty} dt < +\infty, \]

it then follows that

\[ \| \psi(t) - \psi(s) \|_{L^p} \to 0, \text{ as } t \to s. \]

Then \( \psi \in C([0,T]; L^p) \).

Now we give a priori estimate for the Fokker-Planck equation.

**Lemma 4.2.** Assume that \( \psi_0 \in B^s_{p,r}(L^p) \) and \( f, g \in L^2([0,T]; B^s_{p,r}(L^p)) \), \( u \in L^\infty([0,T]; B^s_{p,r}) \cap L^2([0,T]; B^{s+1}_{p,r}) \) and \( \text{div} \ u = 0 \), where \( s > 1 + \frac{d}{p} \), \( s - 1 \leq \alpha \leq s \), \( p \in [2, +\infty) \), \( r \geq p \). If \( \psi \) is a solution of

\[
\begin{align*}
\partial_t \psi + (u \cdot \nabla) \psi &= \text{div} R [-\nabla u R \psi + \psi_{\infty} \nabla R \frac{\psi}{\psi_{\infty}} + g] + f, \\
\psi|_{t=0} &= \psi_0, \\
\psi_{\infty} \nabla R \frac{\psi}{\psi_{\infty}} \cdot n &= 0 \quad \text{on} \quad \partial B(0,R_0).
\end{align*}
\]

(4.5)

Then we have the following estimate

\[ \sup_{t \in [0,T]} \| \psi \|_{B^s_{p,r}(L^p)} + \| \psi \|_{E^{s}_{p,r}(T)} \leq Ce^{CU(T)} \left( \| \psi_0 \|_{B^s_{p,r}(L^p)} + \left( \int_0^T e^{-CU(t')} (\| g(t') \|_{B^s_{p,r}(L^p)} + \| f(t') \|_{B^s_{p,r}(L^p)}) dt' \right)^{\frac{1}{2}} \right), \]

where \( U(t) = \int_0^t (\| \nabla u \|_{B^s_{p,r}}^2 + 1) dt' \).

**Proof.** Applying \( \Delta j \) to (4.5) yields

\[
\begin{align*}
\partial_t \Delta_j \psi + (u \cdot \nabla) \Delta_j \psi &= \text{div} R [-\Delta_j (\nabla u R \psi) + \psi_{\infty} \nabla R \Delta_j \frac{\psi}{\psi_{\infty}} + \Delta_j g] + \Delta_j f + R_j, \\
\Delta_j \psi|_{t=0} &= \Delta_j \psi_0, \\
\psi_{\infty} \nabla R \frac{\psi}{\psi_{\infty}} \cdot n &= 0 \quad \text{on} \quad \partial B(0,R_0),
\end{align*}
\]

(4.6)

where \( R_j = [u \cdot \nabla, \Delta_j] \psi \). Since \( u \in C([0,T]; C^{0,1}) \), we can define the flow of \( u \), namely \( \Phi(t,x) \) such that

\[
\begin{align*}
\partial_t \Phi(t,x) &= u(t, \Phi(t,x)), \\
\Phi(t,x)|_{t=0} &= x.
\end{align*}
\]

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For any function $a(t, x, R)$, we let $\tilde{a}(t, x, R) = a(t, \Phi(t, x), R)$. Then (4.6) is equivalent to

\[
\begin{aligned}
\frac{\partial t}{\partial t} \Delta \tilde{\psi} &= \text{div}_R[-\Delta_j(\nabla u R \tilde{\psi}) + \psi_\infty \nabla R \Delta_j \tilde{\psi} + \tilde{\Delta}_j g] + \tilde{\Delta}_j f + \tilde{R}_j, \\
\Delta_j \tilde{\psi}|_{t=0} &= \Delta_j \psi_0, \\
\psi_\infty \nabla R \Delta_j \tilde{\psi} &\cdot n = 0 \quad \text{on} \quad \partial B(0, R_0).
\end{aligned}
\] (4.7)

By multiplying both sides of (4.7) by $\text{sgn}(\Delta_j \psi) \frac{|\Delta_j \psi|^{p-1}}{p^{p-1}}$ and integrating over $\mathbb{R}^d \times B$, we have

\[
\begin{aligned}
\frac{1}{p} \frac{1}{p} \int_{\mathbb{R}^d \times B} \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^p \psi_\infty \text{d}x \text{d}R &= (p - 1) \int_{\mathbb{R}^d \times B} \left[ \frac{\Delta_j (\nabla u R \psi)}{\psi_\infty} + \frac{\tilde{\Delta}_j g}{\psi_\infty} \right] \frac{|\Delta_j \psi|^{p-2}}{p^{p-1}} \nabla R \frac{\Delta_j \psi}{\psi_\infty} \psi_\infty \text{d}x \text{d}R \\
&\quad - \int_{\mathbb{R}^d \times B} \left[ \frac{\Delta_j (\nabla u R \psi)}{\psi_\infty} \right]^2 \psi_\infty \text{d}x \text{d}R + \int_{\mathbb{R}^d \times B} \left( \tilde{\Delta}_j f + \tilde{R}_j \right) \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^{p-1} \psi_\infty \text{d}x \text{d}R \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^d \times B} \left[ \frac{\Delta_j (\nabla u R \psi)}{\psi_\infty} \right]^2 \psi_\infty \text{d}x \text{d}R + \int_{\mathbb{R}^d \times B} \left( \tilde{\Delta}_j f + \tilde{R}_j \right) \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^{p-1} \psi_\infty \text{d}x \text{d}R \\
&\quad - \frac{2(p - 1)}{p^2} \int_{\mathbb{R}^d \times B} \nabla R \left( \frac{\Delta_j (\nabla u R \psi)}{\psi_\infty} \right)^2 \psi_\infty \text{d}x \text{d}R + \int_{\mathbb{R}^d \times B} \left( \tilde{\Delta}_j f + \tilde{R}_j \right) \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^{p-1} \psi_\infty \text{d}x \text{d}R.
\end{aligned}
\]

Since $u$ is divergence-free, the flow of $u$ is measure-preserving. From the above inequality we obtain

\[
\begin{aligned}
\frac{1}{p} \frac{1}{p} \int_{\mathbb{R}^d \times B} \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^p \psi_\infty \text{d}x \text{d}R + \frac{2(p - 1)}{p} \int_{\mathbb{R}^d \times B} \nabla R \left( \frac{\Delta_j \psi}{\psi_\infty} \right)^2 \psi_\infty \text{d}x \text{d}R \leq \\
C \int_{\mathbb{R}^d \times B} \left[ \left( \frac{\Delta_j (\nabla u R \psi) + |\Delta_j g|}{\psi_\infty} \right)^2 \psi_\infty \text{d}x \text{d}R + \int_{\mathbb{R}^d \times B} \left( \tilde{\Delta}_j f + \tilde{R}_j \right) \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^{p-1} \psi_\infty \text{d}x \text{d}R.
\end{aligned}
\] (4.8)

Using Hölder’s inequality, we deduce that

\[
\begin{aligned}
\frac{1}{p} \frac{1}{p} \int_{\mathbb{R}^d \times B} \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^p \psi_\infty \text{d}x \text{d}R + \frac{2(p - 1)}{p^2} \int_{\mathbb{R}^d \times B} \nabla R \left( \frac{\Delta_j \psi}{\psi_\infty} \right)^2 \psi_\infty \text{d}x \text{d}R \leq \\
C \int_{\mathbb{R}^d \times B} \left[ \left( \frac{\Delta_j (\nabla u R \psi)}{\psi_\infty} \right)^2 + |\Delta_j g|^2 \right] \frac{|\Delta_j \psi|^{p-2}}{p^{p-1}} \psi_\infty \text{d}x \text{d}R + \int_{\mathbb{R}^d \times B} \left( \tilde{\Delta}_j f + \tilde{R}_j \right) \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^{p-1} \psi_\infty \text{d}x \text{d}R.
\end{aligned}
\]

Thus we have

\[
\begin{aligned}
\frac{1}{p} \frac{1}{p} \int_{\mathbb{R}^d \times B} \left| \frac{\Delta_j \psi}{\psi_\infty} \right|^2 \psi_\infty \text{d}x \text{d}R \leq C \left[ \left( \frac{\Delta_j (\nabla u R \psi)}{\psi_\infty} \right)^2 + |\Delta_j g|^2 \right] + \left( \| \Delta_j f \|_{L^2(\mathbb{R}^d)} + \| \tilde{R}_j \|_{L^2(\mathbb{R}^d)} \right) \| \Delta_j \psi \|_{L^2(\mathbb{R}^d)}.
\end{aligned}
\]
(4.9) \[ \| \Delta_j \psi \|^2_{L^2_x(L^p)} \leq \| \Delta_j \psi_0 \|^2_{L^2_x(L^p)} + \int_0^t \| \Delta_j (\nabla u R \psi) \|^2_{L^2_x(L^p)} + \| \Delta_j g \|^2_{L^2_x(L^p)} \]

\[ + \left( \| \Delta_j f \|^2_{L^2_x(L^p)} + \| R_j \|^2_{L^2_x(L^p)} \right) \| \Delta_j \psi \|^2_{L^2_x(L^p)} dt'. \]

Let \((c_j)_{j \geq -1}\) denotes an element of the unit sphere of \(l^r\). By Lemma (3.17) we have

(4.10) \[ \| R_j \|^2_{L^2_x(L^p)} \leq Cc_j 2^{-j\sigma} \| \nabla u \|^2_{B^{\sigma-1}_r} \| \psi \|^2_{B^r_r(L^p)}. \]

Plugging (4.10) into (4.9) and by the Cauchy-Schwarz inequality we have

(4.11) \[ \| \Delta_j \psi \|^2_{L^2_x(L^p)} \leq \| \Delta_j \psi_0 \|^2_{L^2_x(L^p)} + \int_0^t \| \Delta_j (\nabla u R \psi) \|^2_{L^2_x(L^p)} + \| \Delta_j g \|^2_{L^2_x(L^p)} \]

\[ + \| \Delta_j f \|^2_{L^2_x(L^p)} + \| \Delta_j \psi \|^2_{L^2_x(L^p)} + Cc_j 2^{-j\sigma} \| \nabla u \|^2_{B^{\sigma-1}_r} \| \psi \|^2_{B^r_r(L^p)} \| \Delta_j \psi \|^2_{L^2_x(L^p)} dt'. \]

By multiplying both sides \(2^{2j\sigma}\) and taking \(L^2\) norm (here we use the fact that \(r \geq 2\)), we deduce that

\[ \| \psi \|^2_{B^r_r(L^p)} \leq C \{ \| \psi_0 \|^2_{B^r_r(L^p)} + \int_0^t \| \nabla u R \psi \|^2_{B^r_r(L^p)} + \| g \|^2_{B^r_r(L^p)} + C \| \Delta_j \psi \|^2_{L^2_x(L^p)} dt' \}

\[ \leq C \{ \| \psi_0 \|^2_{B^r_r(L^p)} + \int_0^t \| \nabla u \|^2_{B^r_r(L^p)} \| \psi \|^2_{B^r_r(L^p)} + (1 + \| \nabla u \|^2_{B^{\sigma-1}_r}) \| \psi \|^2_{B^r_r(L^p)} dt' \]

\[ + \int_0^t \| f \|^2_{B^r_r(L^p)} + \| g \|^2_{B^r_r(L^p)} dt' \}

\[ \leq C \{ \| \psi_0 \|^2_{B^r_r(L^p)} + \int_0^t \| \nabla u \|^2_{B^r_r(L^p)} + (1 + \| \nabla u \|^2_{B^{\sigma-1}_r}) \| \psi \|^2_{B^r_r(L^p)} dt' \]

\[ + \int_0^t \| f \|^2_{B^r_r(L^p)} + \| g \|^2_{B^r_r(L^p)} dt' \}

\[ \leq C \{ \| \psi_0 \|^2_{B^r_r(L^p)} + \int_0^t \| f \|^2_{B^r_r(L^p)} + (1 + \| \nabla u \|^2_{B^{\sigma-1}_r}) \| \psi \|^2_{B^r_r(L^p)} dt' \} \]

Using Gronwall’s inequality, we deduce that

(4.12) \[ \sup_{t \in [0, T]} \| \psi \|^2_{B^r_r(L^p)} \leq Ce^{CU(T)} \left( \| \psi_0 \|^2_{B^r_r(L^p)} + \left( \int_0^T e^{-CU(T')} (\| f(t') \|^2_{B^r_r(L^p)} + \| g(t') \|^2_{B^r_r(L^p)}) dt' \right)^{\frac{1}{2}} \right), \]

where \(U(t) = \int_0^t \| \nabla u \|^2_{B^r_r(L^p)} + 1 dt'\). Now integrating (4.8) with respect to \(t\) over \([0, T]\), we obtain

(4.13) \[ \int_0^T \int_{\mathbb{R}^d \times B} \left| \nabla R \left( \frac{\psi}{\psi_\infty} \right) \right|^2 \psi_\infty dx \, dR \, dt \leq C \| \Delta_j \psi_0 \|^p_{L^p_x(L^p)} \]

\[ + C \int_0^T \left( \| \Delta_j (\nabla u R \psi) \|^p_{L^p_x(L^p)} + \| \Delta_j g \|^p_{L^p_x(L^p)} \right) \| \Delta_j \psi \|^p_{L^p_x(L^p)} \]

\[ + \left( \| \Delta_j f \|^p_{L^p_x(L^p)} + \| R_j \|^p_{L^p_x(L^p)} \right) \| \Delta_j \psi \|^p_{L^p_x(L^p)} dt'. \]

Multiplying both sides of (4.13) by \(2^{j\sigma}\) and taking \(L^p\)-norm (here we use the fact that \(r \geq p\), and
using the inequality (4.10) we deduce that

\begin{equation}
\|\psi\|_{E_{p,r}(T)} \leq Ce^{CU(T)} \left( \|\psi_0\|_{B_{p,r}(L^p)} + \left( \int_0^T e^{-CU(t')} (\|f(t')\|_{B_{p,r}(L^p)} + \|g(t')\|_{B_{p,r}(L^p)}) dt' \right)^{\frac{1}{2}} \right).
\end{equation}

Combining (4.12) and (4.14), we thus complete the proof. \qed

**Proposition 4.3.** Assume that \( \psi_0 \in B_{p,r}^s(\mathbb{L}^p) \) and \( u \in L^\infty([0,T]; B_{p,r}^s) \cap L^2([0,T]; B_{p,r}^{s+1}) \), where \( s > 1 + \frac{d}{p} \), \( p \in [2, +\infty) \), \( r \geq p \). Then

\begin{equation}
\begin{cases}
\partial_t \psi + (u \cdot \nabla) \psi = \text{div}_R[-\nabla u R \psi + \psi_\infty \nabla_R \frac{\psi}{\psi_\infty}], \\
\psi|_{t=0} = \psi_0, \\
\psi_\infty \nabla_R \frac{\psi}{\psi_\infty} \cdot n = 0 \quad \text{on} \quad \partial B(0, R_0),
\end{cases}
\end{equation}

has a unique solution \( \psi \) in \( C([0,T]; B_{p,r}^s(\mathbb{L}^p)) \) if \( r < \infty \) or in \( C_w([0,T]; B_{p,r}^s(\mathbb{L}^p)) \) if \( r = \infty \). Where \( C_w([0,T]; X) \) denotes the weak continuous space over \([0,T]\) on the Banach space \( X \). Moreover, we have the following estimate

\[ \sup_{t \in [0,T]} \|\psi\|_{B_{p,r}^s(\mathbb{L}^p)} + \|\psi\|_{E_{p,r}(T)} \leq Ce^{CU(T)} \|\psi_0\|_{B_{p,r}^s(\mathbb{L}^p)}, \]

where \( U(t) = \int_0^t (\|\nabla u\|_{B_{p,r}^s}^2 + 1) dt' \).

**Proof.** As in Lemma 4.2, defining the flow of \( u \), and letting \( a(t, x, R) = a(t, \Phi(t, x, R)) \) so we can write (4.15) as

\begin{equation}
\begin{cases}
\partial_t \tilde{\psi} = \text{div}_R[-\nabla u \tilde{\psi} + \psi_\infty \nabla_R \frac{\tilde{\psi}}{\psi_\infty}], \\
\tilde{\psi}|_{t=0} = \psi_0, \\
\psi_\infty \nabla_R \frac{\tilde{\psi}}{\psi_\infty} \cdot n = 0 \quad \text{on} \quad \partial B(0, R_0).
\end{cases}
\end{equation}

Then using Proposition (4.1) for each fixed \( x \), we deduce the existence and uniqueness of \( \psi(t, x, R) \in C([0,T]; \mathbb{L}^p) \). Thus using Lemma 4.2 with \( f, g = 0, \sigma = s \), we get the following estimate

\[ \sup_{t \in [0,T]} \|\psi\|_{B_{p,r}^s(\mathbb{L}^p)} + \|\psi\|_{E_{p,r}(T)} \leq Ce^{CU(T)} \|\psi_0\|_{B_{p,r}^s(\mathbb{L}^p)}. \]

So we have \( \psi \in L^\infty([0,T]; B_{p,r}^s(\mathbb{L}^p)) \). Since the equation is a linear equation, using the above estimate we thus prove the uniqueness in the space \( \psi \in L^\infty([0,T]; B_{p,r}^s(\mathbb{L}^p)) \).

Next we will check that \( \psi \in C([0,T]; B_{p,r}^s(\mathbb{L}^p)) \) when \( r < \infty \). Applying \( \Delta_j \) to (4.15) and by a similar calculation as in Lemma 4.2 with \( f, g = 0 \), we can deduce that

\[ \partial_t \|\Delta_j \psi\|_{L^p(L^p)}^2 \leq C \left( \|\Delta_j(\nabla u R \psi)\|_{L_x^p(L^p)}^2 + \|R_j\|_{L_x^p(L^p)} \|\Delta_j \psi\|_{L_x^p(L^p)} \right). \]
Let \((c_j)_{j \geq 1}\) denote an element of the unit sphere of \(l^r\). By the definition of \(B^s_{p,r}(L^p)\) and Lemma 3.17. We see that
\[
\partial_t \|\Delta_j \psi\|^2_{L^2(L^p)} \leq C \|\Delta_j \psi\|^2_{L^2(L^p)} + \sum_{t \in [0,T]} \|\Delta_j \psi\|^2_{L^2(L^p)} \\
\leq C \|\Delta_j \psi\|^2_{L^2(L^p)} + \sum_{t \in [0,T]} \|\Delta_j \psi\|^2_{L^2(L^p)} \\
= C \|\Delta_j \psi\|^2_{L^2(L^p)} + \frac{\varepsilon}{2}.
\]
Due to \(\psi \in L^\infty([0,T]; B^s_{p,r}(L^p))\), for any \(\varepsilon > 0\), there exists \(N\) such that
\[
\sup_{t \in [0,T]} \sum_{j \geq N} 2^{jsr} \|\Delta_j \psi\|^2_{L^2(L^p)} \leq \frac{\varepsilon}{4},
\]
hence, for any \(t_1, t_2 \in [0,T]\), we have
\[
\|\psi(t_1) - \psi(t_2)\|_{B^s_{p,r}(L^p)} \leq (\sum_{-1 \leq j < N} 2^{jsr} \|\Delta_j \psi(t_1) - \Delta_j \psi(t_2)\|^2_{L^2(L^p)} + 2 \sum_{t \in [0,T]} \|\Delta_j \psi\|^2_{L^2(L^p)})^{\frac{1}{2}} \\
\leq (\sum_{-1 \leq j < N} 2^{jsr} \|\Delta_j \psi(t_1) - \Delta_j \psi(t_2)\|^2_{L^2(L^p)} + \frac{\varepsilon}{2})^{\frac{1}{2}} \\
= (\sum_{-1 \leq j < N} 2^{jsr} \int_{t_1}^{t_2} \partial_t \|\Delta_j \psi(t')\|^2_{L^2(L^p)} \Delta_j \psi(t'))^{\frac{1}{2}} + \frac{\varepsilon}{2}.
\]
By the inequality (4.17) and the definition of \(B^s_{p,r}(L^p)\), we obtain
\[
\|\psi(t_1) - \psi(t_2)\|_{B^s_{p,r}(L^p)} \leq (\sum_{-1 \leq j < N} \frac{1}{2} \int_{t_1}^{t_2} \|\Delta_j \psi\|^2_{L^2(L^p)} + \frac{\varepsilon}{2})^{\frac{1}{2}} \\
\leq C(N + 1) \int_{t_1}^{t_2} \|\Delta_j \psi\|^2_{L^2(L^p)} + \frac{\varepsilon}{2}.
\]
Since \(u \in L^\infty([0,T]; B^s_{p,r}(L^p)) \cap L^2([0,T]; B^{s+1}_{p,r})\), it follows that \(\nabla u \in L^\infty([0,T]; B^{s-1}_{p,r}) \cap L^2([0,T]; B^{s}_{p,r})\).
Then we have
\[
\|\psi(t_1) - \psi(t_2)\|_{B^s_{p,r}(L^p)} \to 0 \text{ as } t_1 \to t_2.
\]
Thus we prove that \(\psi \in C([0,T]; B^s_{p,r}(L^p))\) when \(r < \infty\). Finally we consider \(r = \infty\). Since \(s > 1 + \frac{d}{p}\), there exist \(s' < s\) such that \(s' > 1 + \frac{d}{p}\). Using the fact that \(B^s_{p,\infty} \to B^{s'}_{p,r}\) for any \(r \geq 1\). Fix some \(r \in [p, \infty)\), by the previous argument, we have \(\psi \in C([0,T]; B^{s'}_{p,r}(L^p))\). Now we need to check that \(\psi \in C_w([0,T]; B^{s'}_{p,r})\). Indeed, for any \(\phi \in S(R^d; C_0^\infty(B))\)
\[
\langle \psi(t_1) - \psi(t_2), \phi \rangle \leq C\|\psi(t_1) - \psi(t_2)\|_{B^{s'}_{p,r}(L^p)} \|\phi\|_{B^{-s'}_{p',r}(L^{p'}); \psi^*_{p}dR}.
\]
We thus complete the proof. \(\square\)
4.2. A priori estimates for the Navier-Stokes equations

In the following lemma, we give a priori estimates for the linear Navier-Stokes equations.

**Lemma 4.4.** Assume that \( u_0 \in B^{s}_{p,r}, v \in L^\infty([0,T]; B^{s}_{p,r}) \) with \( \text{div} \, v = 0 \), \( f \in L^{2}([0,T]; B^{s-1}_{p,r}) \) and \( \psi \in L^\infty([0,T]; E^{s}_{p,r}(T)) \) where \( s > 1 + \frac{4}{p}, p \in [2; \infty) \), \( r \geq p \), \( u \) is the solution of

\[
\begin{cases}
\partial_t u + (v \cdot \nabla) u - \nu \Delta u + \nabla P = \text{div} \, f, \quad \text{div} \, u = 0, \\
\tau_{ij} = \int_B R_i \otimes \nabla_j \psi dR, \\
u_t |_{t=0} = u_0.
\end{cases}
\tag{4.19}
\]

Then for any \( \varepsilon > 0 \), we have following estimates:

\[
\sup_{t \in [0,T]} \| u \|^2_{B^{s}_{p,r}} \leq C e^{CV(T)} \left[ \| u_0 \|^2_{B^{s}_{p,r}} + (\nu^{-1} + T)(\| \psi \|^2_{E^{s}_{p,r}(T)} + \int_0^T e^{-CV(t')} (\| f \|^2_{B^{s-1}_{p,r}} + \| \psi \|^2_{B^{s}_{p,r}(L^p)}) dt') \right],
\]

\[
\nu \int_0^T \| u \|^2_{B^{s-1}_{p,r}} dt \leq C(1 + \nu \varepsilon) e^{CV(T)} \left[ \| u_0 \|^2_{B^{s}_{p,r}} + (\| \psi \|^2_{E^{s}_{p,r}(T)} + \int_0^T e^{-CV(t)} (\| \psi \|^2_{B^{s}_{p,r}(L^p)} + \| f \|^2_{B^{s-1}_{p,r}}) dt) \right],
\]

where \( V(t) = \int_0^t \| u \|^2_{B^{s}_{p,r}} dt' \).

**Proof.** Applying \( \Delta_j \) to (4.19) yields

\[
\begin{cases}
\partial_t \Delta_j u - \nu \Delta \Delta_j u + \nabla \Delta_j P = \text{div} \Delta_j \tau + \Delta_j f - \Delta_j (v \cdot \nabla u), \\
\text{div} \Delta_j u = 0, \\
\Delta_j u |_{t=0} = \Delta_j u_0.
\end{cases}
\tag{4.20}
\]

So we can write that

\[
\Delta_j u = e^{\nu \Delta t} \Delta_j u_0 + \int_0^t e^{\nu (t-t') \Delta} (\text{div} \Delta_j \tau + \Delta_j f - \Delta_j (v \cdot \nabla u) - \nabla \Delta_j P) dt'.
\tag{4.21}
\]

If \( j \geq 0 \), by Lemma 3.2 we have

\[
\| \Delta_j u \|_{L^p} \leq C \left( e^{\nu \Delta t 2^{j}} \| \Delta_j u_0 \|_{L^p} + \int_0^t e^{\nu (t-t') 2^{2j}} (\| \text{div} \Delta_j \tau \|_{L^p} + \| \Delta_j f \|_{L^p} + \| \Delta_j (v \cdot \nabla u) \|_{L^p} + \| \nabla \Delta_j P \|_{L^p} dt') \right).
\tag{4.22}
\]

Firstly we deal with the pressure term \( \| \nabla \Delta_j P \|_{L^p} \). Taking \( \text{div} \) for (4.20), we deduce that

\[
\Delta_j P = \text{div}(\text{div} \Delta_j \tau + \Delta_j f - \Delta_j (v \cdot \nabla u)).
\]

Then we see that \( \Delta_j P \) is a solution of an elliptic equation. Thus, we obtain

\[
\nabla \Delta_j P = \nabla (\Delta)^{-1} \text{div}(\text{div} \Delta_j \tau + \Delta_j f - \Delta_j (v \cdot \nabla u)).
\]
Thanks to $\nabla(\Delta)^{-1}\mathbf{div}$ is a Calderon-Zygmund operator and $p < \infty$, we have
\[
\|\nabla\Delta_j P\|_{L^p} \leq C(\|\mathbf{div}\Delta_j \tau\|_{L^p} + \|\Delta_j f\|_{L^p} + \|\Delta_j (v \cdot \nabla u)\|_{L^p}).
\]
Plugging into (4.22), we deduce that
\[
(4.23) \qquad \|\Delta_j u\|_{L^p} \leq C \left( e^{-c\nu T} \|\Delta_j u_0\|_{L^p} + \left( \int_0^T e^{-c\nu(t-t')} 2^j (\|\mathbf{div}\Delta_j \tau\|_{L^p} + \|\Delta_j f\|_{L^p} + \|\Delta_j (v \cdot \nabla u)\|_{L^p} dt') \right)^{\frac{1}{2}} \right).
\]
Notice that
\[
(4.24) \quad \int_0^T e^{-c\nu(t-t')} 2^j (\|\mathbf{div}\Delta_j \tau\|_{L^p} + \|\Delta_j f\|_{L^p} + \|\Delta_j (v \cdot \nabla u)\|_{L^p} dt' = e^{-c\nu T} 1_{[0,T]} * (\|\mathbf{div}\Delta_j \tau(t)\|_{L^p} + \|\Delta_j f(t)\|_{L^p} + \|\Delta_j (v(t) \cdot \nabla u(t)\|_{L^p}) 1_{[0,T]}.
\]
Using Young's inequality, we deduce that
\[
(4.25) \qquad \|\Delta_j u\|_{L^p} \leq C \left[ e^{-c\nu T} \|\Delta_j u_0\|_{L^p} + \left( \int_0^T (\|\mathbf{div}\Delta_j \tau\|_{L^p} + \|\Delta_j f\|_{L^p} + \|\Delta_j (v \cdot \nabla u)\|_{L^p} dt') \right)^{\frac{1}{2}} \right].
\]
Since $\mathbf{div} v = 0$, it follows that $v \cdot \nabla u = \mathbf{div}(v \otimes u)$. And by Bernstein’s inequality, we get
\[
(4.26) \qquad \|\Delta_j u\|_{L^p} \leq C \left[ e^{-c\nu T} \|\Delta_j u_0\|_{L^p} + \nu^{-\frac{1}{2}} \left( \int_0^T (\|\Delta_j \tau\|_{L^p} + 2^{-2j} \|\Delta_j f\|_{L^p} + \|\Delta_j (v \otimes u)\|_{L^p} dt') \right)^{\frac{1}{2}} \right].
\]
That is
\[
(4.27) \qquad \|\Delta_j u\|_{L^p} \leq C \left[ e^{-c\nu T} \|\Delta_j u_0\|_{L^p} \nu^{-1} \left( \int_0^T (\|\Delta_j \tau\|_{L^p} + 2^{-2j} \|\Delta_j f\|_{L^p} + \|\Delta_j (v \otimes u)\|_{L^p} dt') \right) \right].
\]
Now we deal with the stress tensor $\tau$. By Corollary (3.13), we obtain
\[
\|\Delta_j \tau\|_{L^p} \leq C \left[ \varepsilon \int_{\mathbb{R}^d} \left| \nabla_R \left( \Delta_j \frac{\psi}{\psi_\infty} \right) \right|^2 \psi_\infty dxdR + \|\nabla \Delta_j \psi\|_{L^2(\mathbb{R}^d)} \right].
\]
Plugging into (4.27), we deduce that
\[
(4.28) \quad \|\Delta_j u\|_{L^p} \leq C \left[ e^{-c\nu T} \|\Delta_j u_0\|_{L^p} \nu^{-1} \left( \int_0^T \left( \varepsilon \int_{\mathbb{R}^d} \left| \nabla_R \left( \Delta_j \frac{\psi}{\psi_\infty} \right) \right|^2 \psi_\infty dxdR \right)^{\frac{1}{2}} \right) + \|\Delta_j \psi\|_{L^2(\mathbb{R}^d)} \right].
\]
If $j = -1$, applying $\Delta_{-1}$ to (4.19) yields

$$
\begin{aligned}
\partial_t \Delta_{-1} u + v \cdot \nabla \Delta_{-1} u - \nu \Delta_{-1} u + \nabla \Delta_{-1} P &= div \Delta_{-1} \tau + \Delta_{-1} f - [v \cdot \nabla, \Delta_{-1}] u, \\
\Delta_{-1} u|_{t=0} &= \Delta_{-1} u_0.
\end{aligned}
$$

(4.29)

Multiplying both sides of (4.29) by $\sgn(\Delta_{-1} u)|\Delta_{-1} u|^{p-1}$, and using the fact that $div v = 0$, we obtain

$$
\begin{aligned}
\|\Delta_{-1} u\|_{L^p} &\leq \|\Delta_{-1} u_0\|_{L^p} + C \int_0^T \|div \Delta_{-1} \tau\|_{L^p} + \|\Delta_{-1} f\|_{L^p} + \|[v \cdot \nabla, \Delta_{-1}] u\|_{L^p} dt' \\
&\leq \|\Delta_{-1} u_0\|_{L^p} + CT^{\frac{j}{p}} \left[ \int_0^T \left( \epsilon \int_{B^{2j+1}} \left| \nabla \left( \frac{\psi}{\psi_{\infty}} \right) \right|^2 \psi_{\infty}^2 dx dR \right)^{\frac{j}{2}} + \|\Delta_{-1} \psi\|_{L^p}^2 + \|\Delta_{-1} f\|_{L^p}^2 + \|v\|_{L^\infty}^2 \|\Delta_{-1} u\|_{L^p}^2 \right]^{\frac{j}{2}}.
\end{aligned}
$$

(4.30)

Multiplying both sides of (4.28) by $2^{2j}s$, and taking $l^\frac{j}{2}$-norm, and combining the inequality (4.30), we deduce that

$$
\|u\|_{B^{s}_{p,r}}^2 \leq C \|u_0\|_{B^{s}_{p,r}}^2 + (\nu^{-1} + T) \int_0^T \|\psi\|_{B^{s}_{p,r}(L^p)}^2 + \|f\|_{B^{s-1}_{p,r}}^2 + \|v \otimes u\|_{B^{s}_{p,r}}^2 + \|\nabla v\|_{L^\infty}^2 \|u\|_{B^{s}_{p,r}}^2 dt + \epsilon \|\psi\|_{E^{s}_{p,r}(T)}^2.
$$

(4.31)

Since $s > 1 + \frac{d}{p}$, it follows that $B^{s}_{p,r} \hookrightarrow C^{0,1}$ is an algebra. Then we have

$$
\sup_{t \in [0, T]} \|u\|_{B^{s}_{p,r}}^2 \leq C \|u_0\|_{B^{s}_{p,r}}^2 + (\nu^{-1} + T) \int_0^T \|\psi\|_{B^{s}_{p,r}(L^p)}^2 + \|f\|_{B^{s-1}_{p,r}}^2 + \|\nabla v\|_{B^{s}_{p,r}}^2 \|u\|_{B^{s}_{p,r}}^2 dt + \|\psi\|_{E^{s}_{p,r}(T)}^2.
$$

(4.32)

Using Gronwall’s inequality, we obtain

$$
\sup_{t \in [0, T]} \|u\|_{B^{s}_{p,r}}^2 \leq Ce^{CV(T)} \left[ \|u_0\|_{B^{s}_{p,r}}^2 + (\nu^{-1} + T) (\|\psi\|_{E^{s}_{p,r}(T)}^2 + \int_0^T e^{-cV(t)} \|\psi\|_{B^{s}_{p,r}(L^p)}^2 + \|f\|_{B^{s-1}_{p,r}}^2) dt \right].
$$

(4.33)

Now we consider the second estimate. Mutiplying both sides of (4.23) by $2^{j+1}s$, and taking $l^r$-norm with $j > 0$, we deduce that

$$
\|(2^{(j+1)s}) u\|_{L^r|_{j \geq 0}} \leq C \left( (e^{-c\nu 2^{2j} 2^{j+1} s}) \|\Delta_j u_0\|_{L^r} + \int_0^T (e^{-c\nu (t-t') 2^{2j} 2^{j+1} s}) \|\Delta_j \tau\|_{L^r} + \|\Delta_j f\|_{L^p} + \|\Delta_j (v \cdot \nabla u)\|_{L^p}) dt' \right).
$$

Using H"older’s inequality in $l^r$, and by the previous argument for $div \Delta_j \tau$ and $\Delta_j (v \cdot \nabla u)$, we infer that
\begin{align}
\|2j+1\|u\|_{L^p}\|_{\mathcal{L}^r} &\leq C \left( \sup_{j \geq 0} (e^{-cTv^{2j}} 2^j) \right) \|2j\|u\|_{L^p}\|_{\mathcal{L}^r} + \int_0^t \sup_{j} (e^{-cT(t-t')^{2j}} 2^j) \\
&\quad \|2j\|u\|_{L^p}\|_{\mathcal{L}^r} + 2^{(j-1)s}\|\Delta_j f\|_{L^p} + 2^{(j-1)s}\|\Delta_j (v \otimes u)\|_{L^p} \right) \right) \right).
\end{align}

Note that for any \(j \geq 0\), and sufficiently small \(\delta > 0\)

\[
e^{-cTv^{2j}} 2^j \leq \frac{C}{(\nu t)^{\frac{1}{2} - \delta}}, \quad e^{-cT(t-t')^{2j}} 2^j \leq \frac{C}{\nu(t-t')}.
\]

Plugging into (4.34), then we have

\[
\|2j+1\|u\|_{L^p}\|_{\mathcal{L}^r} \leq C \left( \frac{1}{(\nu t)^{\frac{1}{2} - \delta}} \right) \|2j\|u\|_{L^p}\|_{\mathcal{L}^r} + \int_0^t \frac{1}{\nu(t-t')} \\
&\quad \|2j\|u\|_{L^p}\|_{\mathcal{L}^r} + 2^{(j-1)s}\|\Delta_j f\|_{L^p} + 2^{(j-1)s}\|\Delta_j (v \otimes u)\|_{L^p} \right) \right) \right).
\]

Taking \(L^2\)-norm over \([0, T]\), and using Young’s inequality \(\|u \ast v\|_{L^2} \leq C\|u\|_{\mathcal{L}^\infty} \|v\|_{L^2}\), we obtain

\[
\int_0^T \|2j+1\|u\|_{L^p}\|_{\mathcal{L}^r} \|_2^2 dt \leq C \left( \nu^{2\delta-1} T^{2\delta} \|2j\|u\|_{L^p}\|_{\mathcal{L}^r} + \int_0^T \nu^{-1} \right) \\
&\quad \|2j\|u\|_{L^p}\|_{\mathcal{L}^r} + 2^{(j-1)s}\|\Delta_j f\|_{L^p} + 2^{(j-1)s}\|\Delta_j (v \otimes u)\|_{L^p} \right) \right) \right).
\]

Taking \(L^2\)-norm over \([0, T]\) for both sides of (4.30), and combining with the previous argument dealing with \(\|\Delta_j\|_{L^s}\), we deduce that

\[
\nu \int_0^T \|u\|_{B_{p,r}^{s+1}}^2 dt \leq C(1 + \nu T) \left( \|u_0\|_{B_{p,r}^{s+1}}^2 + \varepsilon \|\psi\|_{B_{p,r}^{s+1}}^2 + \\
&\quad \int_0^T \|\psi\|_{B_{p,r}^{s+1}(L^p)}^2 + \|f\|_{L^p}^2 + \|u\|_{B_{p,r}^{s+1}}^2 dt' \right)
\]

Then by the inequality (4.33), we have the following estimate

\[
\nu \int_0^T \|u\|_{B_{p,r}^{s+1}}^2 dt \leq C(1 + \nu T) e^{-cV(T)} \left[ \|u_0\|_{B_{p,r}^{s+1}}^2 + \varepsilon \|\psi\|_{E_{p,r}^{s+1}(T)}^2 \right] + \int_0^T e^{-cV(t)} \left[ \|\psi\|_{B_{p,r}^{s+1}(L^p)}^2 + \|f\|_{B_{p,r}^{s+1}}^2 dt' \right)
\]

\begin{remark}
If \(u = v\), then Lemma 4.4 holds true for \(V(T) = \int_0^T \|u\|_{E_{p,r}^{s+1}}^2 dt\). The proof is similar to that of Lemma 4.4, the only difference is treating with the term \(u \cdot \nabla u\). Using the fact that

\[
\|u \cdot \nabla u\|_{B_{p,r}^{s+1}} = \|\text{div}(u \otimes u)\|_{B_{p,r}^{s+1}} \leq C \|u \otimes u\|_{B_{p,r}^{s+1}} \leq C \|u\|_{L^{\infty}} \|u\|_{B_{p,r}^{s+1}}
\]

then we get the desired result.
\end{remark}
5 Local well-posedness

5.1. Approximate solutions

First, we construct approximate solutions which are smooth (for $x$ variable) solutions of some linear equations.

Starting for $(u^0, \psi^0) \triangleq (S_0u_0, S_0\psi_0)$ we define by induction a sequence $(u^n, \psi^n)_{n \in \mathbb{N}}$ by solving the following linear equations:

\[
\begin{aligned}
&\partial_t u^{n+1} + (u^n \cdot \nabla)u^{n+1} - \nu \Delta u^{n+1} + \nabla P^{n+1} = \text{div} \, r^n, \text{div} \, u^{n+1} = 0, \\
&\partial_t \psi^{n+1} + (u^n \cdot \nabla)\psi^{n+1} = \text{div}_R[-\nabla u^n R\psi^{n+1} + \psi_{\infty} \nabla R \frac{\psi^{n+1}}{\psi_{\infty}}], \\
&u^{n+1}|_{t=0} = S_{n+1} u_0, \psi^{n+1}|_{t=0} = S_{n+1} \psi_0, \\
&\psi_{\infty} \nabla_R \frac{\psi^{n+1}}{\psi_{\infty}} = 0 \quad \text{on} \quad \partial B(0, R_0).
\end{aligned}
\]

(5.1)

Now assume that $u^n \in L^\infty([0, T]; B^{s}_{p,r}) \cap L^2([0, T]; B^{s+1}_{p,r})$ and $\psi^n \in C([0, T]; B^s_{p,r}(L^p)) \cap E^s_{p,r}(T)$ for some positive $T$. By the previous section’s argument we can find a $\psi^{n+1} \in C([0, T]; B^s_{p,r}(L^p)) \cap E^s_{p,r}(T)$. Notice that the initial data are smooth, for the linear Stokes equations, there exists a smooth solution $u^{n+1}$. Then Lemma 4.4 guarantees that $u^{n+1} \in L^\infty([0, T]; B^{s}_{p,r}) \cap L^2([0, T]; B^{s+1}_{p,r})$.

5.2. Uniform bounds

Next, we are going to find some positive $T$ such that for which the approximate solutions are uniformly bounded. Define $U^n_1(t) = \int_0^t \|u(t')\|_{B^s_{p,r}(L^p)}^2 dt'$ and $U^n_2(t) = \int_0^t \|
abla u(t')\|_{B^s_{p,r}(L^p)}^2 dt'$. Then by Lemma 4.2 with $f, g = 0$, we have

\[
\sup_{t \in [0, T]} \|\psi^{n+1}\|_{B^s_{p,r}(L^p)} + \|\psi^{n+1}\|_{E^s_{p,r}(T)} \leq Ce^{T e^{U^n_2(T)}} \|\psi_0\|_{B^s_{p,r}(L^p)}.
\]

(5.2)

And by Lemma 4.4 with $f = 0$, we obtain

\[
\sup_{t \in [0, T]} \|u^{n+1}\|_{B^s_{p,r}}^2 \leq Ce^{U^n_1(T)} \left[\|u_0\|_{B^s_{p,r}}^2 + (\nu^{-1} + T) (\|\psi^n\|_{E^s_{p,r}(T)} + \int_0^T e^{-CU^n_1(t')} \|\psi^n\|_{E^s_{p,r}(L^p)}^2 dt')\right],
\]

(5.3)

\[
\nu \int_0^T \|u^{n+1}\|_{B^{s+1}_{p,r}}^2 dt \leq C(1 + \nu T) e^{U^n_1(T)} \left[\|u_0\|_{B^s_{p,r}}^2 + (\epsilon \|\psi^n\|_{E^s_{p,r}(L^p)}^2 + \int_0^T e^{-CU^n_1(t')} \|\psi^n\|_{E^s_{p,r}(L^p)}^2 dt')\right].
\]

(5.4)
Now fix a $T > 0$, such that

$$T < \min \left\{ \nu^{-1}, \ln 2, \frac{1}{C^2\|u_0\|_{B_{p,r}^\infty}^2 + \|\psi_0\|_{B_{p,r}^\infty(L^p)}^2}, \tilde{T} \right\},$$

where $\tilde{T}$ denotes the maximal time such that $\frac{4C}{\nu}te^{\frac{2C^2}{\nu}A(t)} \leq \frac{1}{2}$, with

$$A(t) = \frac{\|u_0\|_{B_{p,r}^\infty}^2 + \|\psi_0\|_{B_{p,r}^\infty(L^p)}}{1 - C^2(t(\|u_0\|_{B_{p,r}^\infty}^2 + \|\psi_0\|_{B_{p,r}^\infty(L^p)})}.$$

And choose an $\varepsilon$ such that $\frac{4C}{\nu}e^{\frac{2C^2}{\nu}A(t)} \leq \frac{1}{2}$. We claim that for any $n$ and $t \in [0, T]$:

\begin{equation}
\|u^n(t)\|_{B_{p,r}^\infty}^2 \leq CA(t), \quad \int_0^T \|u^n\|_{B_{p,r}^\infty}^2 dt \leq \frac{2C}{\nu}A(T), \quad \|\psi^n\|_{B_{p,r}^\infty(L^p)} + \|\psi^n\|_{E_{p,r}^s(T)} \leq Ce^{\frac{2C^2}{\nu}A(T)}\|\psi_0\|_{B_{p,r}^\infty}.
\end{equation}

By induction, when $n = 0$, (5.5) holds true for a fixed $T$. Now suppose that (5.5) is true for $n$. Plugging (5.5) into (5.2), and using the fact that $\|\nabla u\|_{B_{p,r}^\infty} \leq C\|u\|_{B_{p,r}^{s+1}}$, then we see that

$$\|\psi^{n+1}\|_{B_{p,r}^\infty(L^p)} + \|\psi^{n+1}\|_{E_{p,r}^s(T)} \leq Ce^{\frac{2C^2}{\nu}A(T)}\|\psi_0\|_{B_{p,r}^\infty}.$$

Plugging (5.5) into (5.3) we obtain

\begin{equation}
\|u^{n+1}(t)\|_{B_{p,r}^\infty}^2 \leq \frac{C[\|u_0\|_{B_{p,r}^\infty}^2 + (\nu^{-1} + T)(\varepsilon + T)Ce^{\frac{2C^2}{\nu}A(T)}\|\psi_0\|_{B_{p,r}^\infty}^2]}{1 - C^2(t(\|u_0\|_{B_{p,r}^\infty}^2 + \|\psi_0\|_{B_{p,r}^\infty(L^p)})}.
\end{equation}

The choices of $T$ and $\varepsilon$ ensure that $(\nu^{-1} + T)(\varepsilon + T)Ce^{\frac{2C^2}{\nu}A(T)} < 1$. Then we deduce that $\|u^{n+1}(t)\|_{B_{p,r}^\infty}^2 \leq CA(t)$.

Plugging (5.5) into (5.4), and by a similar estimate, we have $\int_0^T \|u^n\|_{B_{p,r}^\infty}^2 dt \leq \frac{2C}{\nu}A(T)$. Therefore, $u^n$ is uniformly bounded in $L^\infty([0, T]; B_{p,r}^s) \cap L^2([0, T]; B_{p,r}^{s+1})$, and $\psi^n$ is uniformly bounded in $L^\infty([0, T]; B_{p,r}^s(L^p)) \cap E_{p,r}^s(T)$.

5.3. Convergence

We are going to show that $(u^n, \psi^n)$ is a Cauchy sequence in $L^\infty([0, T]; B_{p,r}^{s-1}) \times L^\infty([0, T]; B_{p,r}^{s-1}(L^p)) \cap E_{p,r}^{s-1}$. By (5.1) we have

\begin{equation}
\begin{cases}
\partial_t(u^{n+1} - u^n) + (u^n \cdot \nabla)(u^{n+1} - u^n) - \nu\Delta(u^{n+1} - u^n) + \nabla(P^{n+1} - P^n) = \text{div}(\tau^n - \tau^{n-1}) + F^n, \\
\partial_t(\psi^{n+1} - \psi^n) + (u^n \cdot \nabla)(\psi^{n+1} - \psi^n) = \text{div}_R[\nabla u^nR(\psi^{n+1} - \psi^n) + \psi^n \nabla R(\frac{\psi^{n+1} - \psi^n}{\psi_\infty}) + g^n] + f^n, \\
u^{n+1} - u^n |_{t=0} = \Delta u_0, \psi^{n+1} - \psi^n |_{t=0} = \Delta \psi_0,
\end{cases}
\end{equation}
where \( F^n = -(u^n - u^{n-1}) \nabla u^n, g^n = -\nabla (u^n - u^{n-1}) R \psi^n, f^n = -(u^n - u^{n-1}) \nabla \psi^n \). By Lemma 4.2 with \( \sigma = s - 1 \), we deduce that

\[
(5.8) \quad \| \psi^{n+1} - \psi^n \|_{L^2(B_{p,r}^{-1}(L^p))}^2 + \| \psi^{n+1} - \psi^n \|_{E^{s-1}_{p,r}(T)}^2 \leq C e^{U_r(T)} \left( \| \Delta_n \psi_0 \|_{L^2(B_{p,r}^{-1}(L^p))}^2 + \int_0^T \| \nabla (u^n - u^{n-1}) R \psi^n \|_{L^2(B_{p,r}^{-1}(L^p))}^2 + \| (u^n - u^{n-1}) \nabla \psi^n \|_{L^2(B_{p,r}^{-1}(L^p))}^2 \right) dt.
\]

By a similar calculation as in Lemma 4.4, we obtain

\[
(5.9) \quad \| u^{n+1} - u^n \|_{L^\infty(B_{p,r}^{-1})}^2 + \int_0^T \| u^{n+1} - u^n \|_{B_{p,r}^{-1}}^2 dt \leq C_T e^{U_r(T)} \left( \| \Delta_n u_0 \|_{B_{p,r}^{-1}}^2 + \| \psi^n - \psi^{n-1} \|_{E^{s-1}_{p,r}(T)}^2 \right.
+ \left. \int_0^T \| \psi^n - \psi^{n-1} \|_{B_{p,r}^{-1}}^2 \right) + \left. \| (u^n - u^{n-1}) \nabla u^n \|_{B_{p,r}^{-1}}^2 dt \right).
\]

Thanks to the choice of \( T \) and that \((u^n, \psi^n)\) is uniformly bounded in \( L^\infty([0, T]; B_{p,r}^s) \cap L^2([0, T]; B_{p,r}^{s+1}) \times L^\infty([0, T]; B_{p,r}^{s+1}(L^p)) \cap E_{p,r}^s(T) \), we get a constant \( C_T \) independent of \( n \), such that

\[
(5.10) \quad \| u^{n+1} - u^n \|_{L^\infty(B_{p,r}^{-1})}^2 + \int_0^T \| u^{n+1} - u^n \|_{B_{p,r}^{-1}}^2 dt \leq C_T \left( \| \Delta_n u_0 \|_{B_{p,r}^{-1}}^2 + \| \psi^n - \psi^{n-1} \|_{E^{s-1}_{p,r}(T)}^2 \right)
+ \left. \int_0^T \| \psi^n - \psi^{n-1} \|_{B_{p,r}^{-1}}^2 \right) + \left. \| (u^n - u^{n-1}) \nabla u^n \|_{B_{p,r}^{-1}}^2 dt \right).
\]

And

\[
(5.11) \quad \| \psi^{n+1} - \psi^n \|_{L^\infty(B_{p,r}^{-1}(L^p))}^2 + \| \psi^{n+1} - \psi^n \|_{E^{s-1}_{p,r}(T)}^2 \leq C_T \left( \| \Delta_n \psi_0 \|_{L^2(B_{p,r}^{-1}(L^p))}^2 + \int_0^T \| \nabla (u^n - u^{n-1}) \|_{B_{p,r}^{-1}}^2 \| \psi^n \|_{L^2(B_{p,r}^{-1}(L^p))}^2 \right)
+ \left. \int_0^T \| (u^n - u^{n-1}) \nabla \psi^n \|_{L^2(B_{p,r}^{-1}(L^p))} dt \right) \leq C_T \left( \| \Delta_n \psi_0 \|_{L^2(B_{p,r}^{-1}(L^p))}^2 + \int_0^T \| (u^n - u^{n-1}) \|_{B_{p,r}^{-1}}^2 \right) \leq C_T \left( \| \Delta_n \psi_0 \|_{L^2(B_{p,r}^{-1}(L^p))} + \int_0^T \| (u^n - u^{n-1}) \|_{B_{p,r}^{-1}}^2 dt \right).
\]

By a direct calculation, we obtain

\[
(5.12) \quad \| \Delta_n u_0 \|_{B_{p,r}^{-1}}^2 = \sum_{|\beta| = n} 2^{j r(s-1)} \| \Delta_j \Delta_n u_0 \|_{L^p}^2 \leq C 2^{2ns} \| \Delta_n u_0 \|_{L^p}^2 \leq C 2^{-2n} \| u_0 \|_{B_{p,r}^{-1}}^2.
\]

By a similar argument, we have \( \| \Delta_n \psi_0 \|_{B_{p,r}^{-1}} \leq C 2^{-2n} \| \psi_0 \|_{B_{p,r}^{-1}}^2 \). If we define

\[
(5.13) \quad A_n(T) = \| u^{n+1} - u^n \|_{L^\infty(B_{p,r}^{-1})}^2 + \int_0^T \| u^{n+1} - u^n \|_{B_{p,r}^{-1}}^2 dt,
\]
\[ B_n(T) = \| \psi^{n+1} - \psi^n \|_{L^p(E_{p,r}^s([0,T]))}^2 + \| \psi^{n+1} - \psi^n \|_{E_{p,r}^{s-1}(T)}^2. \]

Plugging (5.12) into (5.10), we have

\[ A_n(T) \leq C_T [2^{-2n} \| u_0 \|_{B_{p,r}^s}^2 + (\varepsilon + T)(A_{n-1}(T) + B_{n-1}(T))]. \]

(5.16)

\[ B_n(T) \leq C_T (2^{-2n} \| \psi_0 \|_{B_{p,r}^s(L^p)}^2 + A_{n-1}(T)). \]

Plugging (5.16) into (5.15), we deduce that

\[ A_n(T) \leq C_T 2^{-2n} (\| u_0 \|_{B_{p,r}^s}^2 + \| \psi_0 \|_{B_{p,r}^s(L^p)}^2) + C_T^2 (\varepsilon + T)(A_{n-1}(T) + A_{n-2}(T)). \]

(5.18)

Now let \( \varepsilon \) and \( T \) be sufficiently small such that \( C_T^2 (\varepsilon + T) < \frac{1}{2} \), then

\[ A_n(T) \leq C_T 2^{-2n} (\| u_0 \|_{B_{p,r}^s}^2 + \| \psi_0 \|_{B_{p,r}^s(L^p)}^2) + \frac{1}{4} (A_{n-1}(T) + A_{n-2}(T)). \]

If \( n < 0 \), we may set \( A_n = 0 \), (5.18) still holds true. Then we have

\[ \sum_{n=1}^m A_n(T) \leq C_T \sum_{n=1}^m 2^{-2n} (\| u_0 \|_{B_{p,r}^s}^2 + \| \psi_0 \|_{B_{p,r}^s(L^p)}^2) + \frac{1}{4} \sum_{n=1}^m (A_{n-1}(T) + A_{n-2}(T)). \]

(5.19)

Since \( A_n(T) > 0 \) and \( A_n = 0 \) if \( n < 0 \), it follows that

\[ \sum_{n=1}^m (A_{n-1}(T) + A_{n-2}(T)) \leq \frac{1}{2} \sum_{n=1}^m A_n(T). \]

Then

\[ \sum_{n=1}^m A_n(T) \leq C_T \sum_{n=1}^m 2^{-2n} (\| u_0 \|_{B_{p,r}^s}^2 + \| \psi_0 \|_{B_{p,r}^s(L^p)}^2). \]

(5.20)

This implies that \( \sum_{n=1}^\infty A_n(T) \) is convergent. Hence, by (5.16), \( \sum_{n=1}^\infty B_n(T) \) is also convergent. Then we deduce that \( (u^n, \psi^n) \) is a Cauchy sequence in \( L^\infty([0, T]; B_{p,r}^{s-1}) \times L^\infty([0, T]; E_{p,r}^{s-1}) \). Thus, there exists \( (u, \psi) \in L^\infty([0, T]; B_{p,r}^{s-1}) \times L^\infty([0, T]; E_{p,r}^{s-1}(L^p)) \) such that

\[ u^n \to u \text{ in } L^\infty([0, T]; B_{p,r}^{s-1}) \text{ and } \psi^n \to \psi \text{ in } L^\infty([0, T]; E_{p,r}^{s-1}(L^p)). \]

Since \( u^n \) and \( \psi^n \) are uniform bounded in \( L^\infty([0, T]; B_{p,r}^{s}) \cap L^2([0, T]; B_{p,r}^{s+1}) \times L^\infty([0, T]; B_{p,r}^{s}(L^p)) \). The Fatou property for Besov spaces ensures that \( (u, \psi) \in L^\infty([0, T]; B_{p,r}^{s}) \cap L^2([0, T]; B_{p,r}^{s+1}) \times L^\infty([0, T]; B_{p,r}^{s}(L^p)). \) An interpolation argument ensures that the convergence holds true for any \( s' < s \). Passing to the limit in (5.1) in the weak sense, we conclude that \((u, \psi)\) is indeed a solution of (1.2).
5.4. Regularity

Now we check that \((u, \psi) \in C([0,T]; B_{p,r}^{s} \times C([0,T]; B_{p,r}^{s}(\mathcal{L}^{p})))\), when \(r\) is finite, and \((u, \psi) \in C_{w}([0,T]; B_{p,\infty}^{s} \times C_{w}([0,T]; B_{p,\infty}^{s}(\mathcal{L}^{p})))\). Proposition 4.3 guarantees that \(\psi\) is in the desired space. Since \(u \in L^{\infty}([0,T]; B_{p,r}^{s})\), it follows that \(u \cdot \nabla u \in L^{\infty}([0,T]; B_{p,r}^{s-1})\) and \(\Delta u \in L^{\infty}([0,T]; B_{p,r}^{s-2})\). Combining with \(\psi \in L^{\infty}([0,T]; B_{p,r}^{s}(\mathcal{L}^{p})) \cap E_{p,r}(T)\), we deduce that \(\text{div} \tau \in L^{\infty}([0,T]; B_{p,r}^{s-1})\). Applying \(\text{div}\) to (1.2) we obtain
\[
\Delta P = \text{div}[(u \nabla u) + \text{div} \tau],
\]
from which we deduce that
\[
\nabla P = \nabla(\Delta)^{-1}\text{div}[(u \nabla u) + \text{div} \tau].
\]
If \(p < \infty\), we have
\[
\|\nabla P\|_{B_{p,r}^{s-1}} \leq C\|u \nabla u\|_{B_{p,r}^{s-1}} + \|\text{div} \tau\|_{B_{p,r}^{s-1}}.
\]
So from the equations (1.1) we obtain \(\partial_{t} u \in L^{\infty}([0,T]; B_{p,r}^{s-2})\), then \(u \in C([0,T]; B_{p,r}^{s-2})\). An interpolation argument ensures that \(u \in C([0,T]; B_{p,r}^{s'})\) for any \(s' < s\). Notice that \(u \in L^{\infty}([0,T]; B_{p,r}^{s})\). If \(r < \infty\), for any \(t_1, t_2\), for any \(\varepsilon > 0\), there exists \(N\) such that
\[
(5.21) \quad \sup_{t \in [0,T]} \left( \sum_{j \geq N} 2^{j sr} \|u(t)\|_{L^{p}} \right)^{\frac{1}{s}} \leq \frac{\varepsilon}{4}.
\]
Hence
\[
\|u(t_1) - u(t_2)\|_{B_{p,r}^{s}} \leq \left( \sum_{-1 \leq j < N} 2^{j sr} \|u(t_1) - u(t_2)\|_{L^{p}} \right)^{\frac{1}{s}} + 2 \sup_{t \in [0,T]} \left( \sum_{j \geq N} 2^{j sr} \|u(t)\|_{L^{p}} \right)^{\frac{1}{s}}
\]
\[
\leq \left( \sum_{-1 \leq j < N} 2^{j sr} \|u(t_1) - u(t_2)\|_{L^{p}} \right)^{\frac{1}{s}} + \frac{\varepsilon}{4}
\]
\[
\leq 2^N \left( \sum_{-1 \leq j < N} 2^{j(s-1)r} \|u(t_1) - u(t_2)\|_{L^{p}} \right)^{\frac{1}{s}} + \frac{\varepsilon}{4}
\]
\[
\leq 2^N \|u(t_1) - u(t_2)\|_{B_{p,r}^{s}} + \frac{\varepsilon}{4},
\]
from which we deduce that \(u \in C([0,T]; B_{p,r}^{s})\). If \(r = \infty\), for any \(\phi \in S\)
\[
\langle u(t), \phi \rangle = \langle S_{j} u(t), \phi \rangle + \langle (Id - S_{j}) u(t), \phi \rangle = \langle S_{j} u(t), \phi \rangle + \langle u(t), (Id - S_{j}) \phi \rangle.
\]
Since \(S_{j} u(t) \in B_{p,r}^{s'}\), it follows that \(\langle S_{j} u(t), \phi \rangle \in C[0,T]\) and \(\|(Id - S_{j}) \phi\|_{L^{\infty}} \to 0, \ j \to \infty\)
Then \(\langle S_{j} u(t), \phi \rangle\) uniformly converges to \(\langle u(t), \phi \rangle\). So \(\langle u(t), \phi \rangle\) is continuous, which implies that \(u \in C_{w}([0,T]; B_{p,\infty}^{s})\).
5.5. Uniqueness

Assume that \((u, \psi)\) and \((v, \phi)\) are two solutions of (1.2) with the same initial data. Then we have

\[
\begin{align*}
\partial_t(u-v) + (v \cdot \nabla)(u-v) - \nu \Delta (u-v) + \nabla (P_1 - P_2) &= \text{div} \ (\tau_1 - \tau_2) + F, \\
\partial_t(\psi - \phi) + (v \cdot \nabla) (\psi - \phi) &= \text{div}_R[-\nabla u R(\psi - \phi) + \psi_\infty \nabla R \frac{(\psi - \phi)}{\psi_\infty} + g] + f,
\end{align*}
\]

(5.22)

where \(F = -(u-v)\nabla v, \ g = -\nabla (u-v) R\psi, \ f = -(u-v)\nabla \phi, \) and \(P_1\) corresponds to \(u, \ \tau_1\) corresponds to \(\psi, \ P_2\) corresponds to \(v, \ \tau_2\) corresponds to \(\phi\) respectively. By a similar calculation as in Section 5.3, we have

\[
\|u-v\|^2_{L^p_t(B^{r-1}_{p,r})} + \int_0^T \|u-v\|_{B^{r}_{p,r}}^2 dt \leq C_T \left( \varepsilon \|\psi - \phi\|_{E^{r-1}_{p,r}(T)}^2 + \int_0^T \|\psi - \phi\|_{E^{r-1}_{p,r}(L^p)}^2 + \|u-v\|_{B^{r-1}_{p,r}}^2 dt \right),
\]

(5.23)

\[
\|\psi - \phi\|_{L^p_t(B^{r-1}_{p,r}(L^p))} + \|\psi - \phi\|_{E^{r-1}_{p,r}(T)}^2 \leq C_T \left( \int_0^T \|u-v\|_{B^{r}_{p,r}}^2 + \|u-v\|_{B^{r-1}_{p,r}}^2 dt \right).
\]

(5.24)

Plugging (5.24) into (5.23), we deduce that

\[
\|u-v\|^2_{L^p_t(B^{r-1}_{p,r})} + \int_0^T \|u-v\|_{B^{r}_{p,r}}^2 dt \leq C_T^2 (\varepsilon + T) \left( \|u-v\|_{L^p_t(B^{r-1}_{p,r})}^2 + \int_0^T \|u-v\|_{B^{r}_{p,r}}^2 dt \right).
\]

(5.25)

So if \(\varepsilon\) and \(T\) are small enough such that \(C_T^2 (\varepsilon + T) < 1\), we obtain \(u = v\) for a.e. \((t,x)\). By the inequality (5.24), we have \(\psi = \phi\) for a.e. \((t,x,R)\). Splitting the interval \([0,T]\) into \([0,T_1], [T_1,T_2],...,[T_k,T_{k+1}]\), where each subinterval satisfies \(C_T^2 (\varepsilon + T) < 1\). Then we see that \((u,\psi) = (v,\phi)\) in the interval \(0,T_1]\). Since \(u(T_1) = v(T_1), \ \psi(T_1) = \phi(T_1)\), we deduce that \((u,\psi) = (v,\phi)\) in the interval \([T_1,T_2]\). Repeating the argument we have proved the uniqueness.

6 Blow-up criterion

In this section we give the proof of the blow-up criterion for (1.2).

Proof of Theorem 2.2: If \(T^* < \infty\), we have

\[
\|\psi\|^2_{L^\infty_t(B^{r-1}_{p,r}(L^p))} + \|\psi\|^2_{E^{r}_{p,r}(T^*)} = \infty.
\]

Since \(\|\psi\|^2_{L^\infty_t(B^{r-1}_{p,r}(L^p))} + \|\psi\|^2_{E^{r}_{p,r}(T^*)} < \infty\), it follows that \(T^*\) is not the maximal time. Now assume that

\[
\int_0^{T^*} \|u\|_{L^\infty}^2 < \infty.
\]
Using Remark 4.5 with \( f = 0 \), we deduce that

\[
\sup_{t \in [0,T^*)} \|u\|_{L^{p}_{\rho, \gamma}}^{2} \leq M(T^*)
\]

\[
\triangleq Ce^C \int_{0}^{T^*} \|u\|_{L^{p}_{\rho, \gamma}}^{2} \cdot dt \left[ \|u_0\|_{L^{p}_{\rho, \gamma}}^{2} + (\varepsilon \|\psi\|_{L^{p}_{\rho, \gamma}}^{2}) + \int_{0}^{T^*} \|\psi\|_{L^{p}_{\rho, \gamma}}^{2} \cdot dt' \right].
\]

By the assumption, we have that \( M(T^*) < \infty \). Let \( \delta \) be small enough such that

\[
\delta < \min\{\nu^{-1}, \ln 2, \frac{1}{C^2[M(T^*) + \|\psi\|_{L^{p}_{\rho, \gamma}}^{2}(B^{p}_{\rho, \gamma}(\mathcal{L}^{p}))]}\}.
\]

Then by the argument as in Section 5.1, we have a solution \( \tilde{u} \) of (1.1) with initial data \( u(T^* - \frac{\delta}{2}) \). By the uniqueness, we deduce that \( \tilde{u}(t) = u(t + T^* - \frac{\delta}{2}) \) on \( [0, \frac{\delta}{2}] \). So the solution \( \tilde{u} \) extends the solution \( u \) beyond \( T^* \). This contradicts the fact that \( T^* \) is the lifespan.

7 Global existence for small data

In this section, we proved that the solution is global in time if the initial data is close to equilibrium \((0, \psi_{\infty})\). Firstly we need the following Poincaré inequality with weight. The proof is similar as in Proposition 3.4 in [9].

**Lemma 7.1.** If \( \tilde{\psi} \) satisfy \( \int_{\mathcal{R}} \tilde{\psi} \cdot d\mathcal{R} = 0 \) and \( \int_{\mathcal{R}} \left| \nabla_{\mathcal{R}} \left( \frac{\tilde{\psi}}{\psi_{\infty}} \right) \right|^{2} \cdot \psi_{\infty} \cdot d\mathcal{R} < \infty \) with \( p \geq 2 \). There exists a constant \( C \) such that

\[
\int_{\mathcal{R}} \left| \frac{\tilde{\psi}}{\psi_{\infty}} \right|^{p} \cdot \psi_{\infty} \cdot d\mathcal{R} \leq C \int_{\mathcal{R}} \left| \nabla_{\mathcal{R}} \left( \frac{\tilde{\psi}}{\psi_{\infty}} \right) \right|^{2} \cdot \psi_{\infty} \cdot d\mathcal{R}.
\]

**Proof.** We argue by contradiction. Assume that the sequence \( \tilde{\psi}_n \) satisfies:

\[
\int_{\mathcal{R}} \tilde{\psi}_n \cdot d\mathcal{R} = 0, \quad \int_{\mathcal{R}} \left| \frac{\tilde{\psi}_n}{\psi_{\infty}} \right|^{p} \cdot \psi_{\infty} \cdot d\mathcal{R} = 1, \quad \int_{\mathcal{R}} \left| \nabla_{\mathcal{R}} \left( \frac{\tilde{\psi}_n}{\psi_{\infty}} \right) \right|^{2} \cdot \psi_{\infty} \cdot d\mathcal{R} \to 0.
\]

Hence, \( \sqrt[3]{\psi_{\infty}} \cdot \nabla_{\mathcal{R}} \left( \frac{\tilde{\psi}_n}{\psi_{\infty}} \right) \) tends to 0 in \( L^{2}(\mathcal{R}) \) and \( \nabla_{\mathcal{R}} \left( \frac{\tilde{\psi}_n}{\psi_{\infty}} \right) \) tends to 0 in \( L^{2}_{loc}(\mathcal{R}) \). Then we deduce that \( \left( \frac{\tilde{\psi}_n}{\psi_{\infty}} \right) \) tends to some constant \( c \) in \( L^{p}_{loc}(\mathcal{R}) \). Thus we obtain \( \frac{\tilde{\psi}_n}{\psi_{\infty}} \) tends to some constant \( c \) in \( L^{p}_{loc}(\mathcal{R}) \). Since \( \tilde{\psi}_n \) is bounded in \( L^{p}(\mathcal{R}) \), it follows that \( \int_{\mathcal{R}} \tilde{\psi}_n \cdot d\mathcal{R} \) tends to some constant \( c \). Using the fact that \( \int_{\mathcal{R}} \tilde{\psi}_n \cdot d\mathcal{R} = 0 \), we infer that \( c = 0 \). There exists a subsequence \( \tilde{\psi}_{n_k} \) such that \( \tilde{\psi}_{n_k} \to 0 \) almost every. Denote that \( x = 1 - |\mathcal{R}| \). By a similar calculation as in Section 3, we have \( \psi_{\infty} \sim x^{k} \). By the Hardy-type inequality (for more details, one can refer to Section 3.2 in [9]), we deduce that for some \( \beta > 0 \)

\[
\int_{\mathcal{R}} \left| \frac{\tilde{\psi}_n}{x^{\beta} \psi_{\infty}} \right|^{p} \cdot \psi_{\infty} \cdot d\mathcal{R} < C \left[ \int_{\mathcal{R}} \left| \frac{\tilde{\psi}_n}{\psi_{\infty}} \right|^{p} \cdot \psi_{\infty} \cdot d\mathcal{R} + \int_{\mathcal{R}} \left| \nabla_{\mathcal{R}} \left( \frac{\tilde{\psi}_n}{\psi_{\infty}} \right) \right|^{2} \cdot \psi_{\infty} \cdot d\mathcal{R} \right] < C.
\]
This gives some tightness of the sequence of \( \left| \frac{\psi_n}{\psi_\infty} \right|^p \), thus we have

\[
\lim_{n \to \infty} \int_B \left| \frac{\psi_n}{\psi_\infty} \right|^p \psi_\infty dR = \int_B \lim_{n \to \infty} \left| \frac{\psi_n}{\psi_\infty} \right|^p \psi_\infty dR = 0,
\]

which contradicts (7.1).

Now we are going to prove Theorem 2.3. Denote that \( \tilde{\psi} = \psi - \psi_\infty \). Since \((0, \psi_\infty)\) is the equilibrium of (1.2), it follows that \((u, \tilde{\psi})\) is also a solution of (1.2).

**Proof of Theorem 2.3:** We argue by contradiction, assume that the lifespan \( T^* \) is finite. Firstly we claim that there exists a constant \( M \) independent with \( T \), such that

\[
\sup_{t \in [0, T^*]} \| u(t) \|_{B^s_{p,r}} + \| u \|_{L^2_x(B_{p,r}^{s+1})} + \sup_{t \in [0, T^*]} \| \tilde{\psi}(t) \|_{B^s_{p,r}(L^p)} + \| \tilde{\psi} \|_{E^s_{p,r}(T^*)} \]

\[
\leq M(\| u_0 \|_{B^s_{p,r}} + \| \psi_0 - \psi_\infty \|_{B^s_{p,r}(L^p)}).
\]

If \( T \) is small enough, by the argument as in local well-posedness, we have

\[
\sup_{t \in [0, T]} \| u(t) \|_{B^s_{p,r}} + \| u \|_{L^2_x(B_{p,r}^{s+1})} + \sup_{t \in [0, T]} \| \tilde{\psi}(t) \|_{B^s_{p,r}(L^p)} + \| \tilde{\psi} \|_{E^s_{p,r}(T)}
\]

\[
\leq M(\| u_0 \|_{B^s_{p,r}} + \| \psi_0 - \psi_\infty \|_{B^s_{p,r}(L^p)}).
\]

Thus we can define \( T \) as follow:

\[
\mathcal{T} = \sup \left\{ T : \sup_{t \in [0, T]} \| u(t) \|_{B^s_{p,r}} + \| u \|_{L^2_x(B_{p,r}^{s+1})} + \sup_{t \in [0, T]} \| \tilde{\psi}(t) \|_{B^s_{p,r}(L^p)} + \| \tilde{\psi} \|_{E^s_{p,r}(T)} \right\}
\]

\[
\leq M(\| u_0 \|_{B^s_{p,r}} + \| \psi_0 - \psi_\infty \|_{B^s_{p,r}(L^p)}).
\]

We seek to prove that \( \mathcal{T} = T^* \), then the claim holds true. If \( \mathcal{T} < T^* \), for any \( t < \mathcal{T} \). By a similar calculation as in Lemma 4.2 with \( f, g = 0 \), we deduce that

\[
\frac{1}{p} \partial_t \int_{\mathbb{R}^d \times B} \left| \frac{\Delta_j \tilde{\psi}}{\psi_\infty} \right|^p \psi_\infty dxdR + \frac{2(p-1)}{p^2} \int_{\mathbb{R}^d \times B} \left| \nabla R \left( \frac{\Delta_j \tilde{\psi}}{\psi_\infty} \right) \right|^2 \psi_\infty dxdR \leq
\]

\[
C \int_{\mathbb{R}^d \times B} \left( \Delta_j \left( \nabla u R \frac{\tilde{\psi}}{\psi_\infty} \right) \right)^2 \left| \frac{\Delta_j \tilde{\psi}}{\psi_\infty} \right|^{p-2} \psi_\infty dxdR + \int_{\mathbb{R}^d \times B} R_j \left| \frac{\Delta_j \tilde{\psi}}{\psi_\infty} \right|^{p-1} dxdR,
\]

where \( R_j = [u \cdot \nabla, \Delta_j] \psi \).

Using Hölder’s inequality, and by the definition of Besov spaces we have

\[
\partial_t \| \Delta_j \tilde{\psi} \|^p_{L^p_x(L^p)} + \frac{2(p-1)}{p} \int_{\mathbb{R}^d \times B} \left| \nabla R \left( \frac{\Delta_j \tilde{\psi}}{\psi_\infty} \right) \right|^2 \psi_\infty dxdR \leq
\]

\[
35
\]
\[ pC \left( c_j^{2 - 2js} \| \nabla u R \bar{\psi} \|_{B_{p,r}^s(C^r)}^2 \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{p - 2} + \| R_j^1 \bar{\psi} \|_{L^p_x(L^r)} \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{p - 1} \right). \]

By Lemmas 3.15 and 3.17, we obtain

\[ \partial_t \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^p + \frac{2(p - 1)}{p} \int_{\mathbb{R}^d \times \mathbb{B}} \left| \nabla R \left( \frac{\Delta_j \bar{\psi}}{\psi} \right) \right|^{\frac{p}{2}} \psi \infty dx dr \leq \]

\[ pC \left( c_j^{2 - 2js} \| \nabla u \|_{B_{p,r}^s}^2 \| \bar{\psi} \|_{B_{p,r}^s(C^r)}^2 \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{p - 2} + c_j 2^{-js} \| u \|_{B_{p,r}^s} \| \bar{\psi} \|_{B_{p,r}^s(C^r)} \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{p - 1} \right). \]

Let \( C_p \) denote a constant dependent on \( p \). By Lemma 7.1, we deduce that

\[ \partial_t \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^p + C_p \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^p \leq \]

\[ pC \left( c_j^{2 - 2js} \| \nabla u \|_{B_{p,r}^s}^2 \| \bar{\psi} \|_{B_{p,r}^s(C^r)}^2 \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{p - 2} + c_j 2^{-js} \| u \|_{B_{p,r}^s} \| \bar{\psi} \|_{B_{p,r}^s(C^r)} \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{p - 1} \right). \]

Thus we obtain

\[ \partial_t \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^r + C_p \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^r \leq \]

\[ pC \left( c_j^{2 - 2js} \| \nabla u \|_{B_{p,r}^s}^2 \| \bar{\psi} \|_{B_{p,r}^s(C^r)}^2 \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{r - 2} + c_j 2^{-js} \| u \|_{B_{p,r}^s} \| \bar{\psi} \|_{B_{p,r}^s(C^r)} \| \Delta_j \bar{\psi} \|_{L^p_x(L^r)}^{r - 1} \right). \]

Multiplying both sides of (7.9) by \( 2^{js} \), taking sum of \( j \) form \(-1 \) to \( \infty \), and using Hölder’s inequality, we deduce that

\[ \partial_t \| \bar{\psi} \|_{B_{p,r}^s(C^r)} + C_p \| \bar{\psi} \|_{B_{p,r}^s(C^r)} \leq pC \left( \| \nabla u \|_{B_{p,r}^s}^2 \| \bar{\psi} \|_{B_{p,r}^s(C^r)} + \| u \|_{B_{p,r}^s} \| \bar{\psi} \|_{B_{p,r}^s(C^r)} \right). \]

Since \( t < T \), it follows that

\[ \| u \|_{B_{p,r}^s} \leq M \left( \| u_0 \|_{B_{p,r}^s} + \| \psi_0 - \psi \|_{B_{p,r}^s(C^r)} \right) \leq M c_0. \]

So if \( pCM c_0 < \frac{C_p}{2} \), we get

\[ \partial_t \| \bar{\psi} \|_{B_{p,r}^s(C^r)} + \frac{C_p}{2} \| \bar{\psi} \|_{B_{p,r}^s(C^r)} \leq pC \| \nabla u \|_{B_{p,r}^s}^2 \| \bar{\psi} \|_{B_{p,r}^s(C^r)}, \]

or

\[ \partial_t \| \bar{\psi} \|_{B_{p,r}^s(C^r)} + \frac{C_p}{2} \| \bar{\psi} \|_{B_{p,r}^s(C^r)} \leq pC \| \nabla u \|_{B_{p,r}^s}^2 \| \bar{\psi} \|_{B_{p,r}^s(C^r)}. \]

Integrating over \([0, t]\) with respect to \( t \), we have

\[ \| \bar{\psi}(t) \|_{B_{p,r}(C^r)} \leq \| \psi_0 - \psi \|_{B_{p,r}(C^r)} + pC \int_0^T \| \nabla u \|_{B_{p,r}^s}^2 \| \bar{\psi} \|_{B_{p,r}^s(C^r)} dt \]

\[ \leq \| \psi_0 - \psi \|_{B_{p,r}(C^r)} + pCM c_0 \sup_{t \in [0,T]} \| \bar{\psi} \|_{B_{p,r}^s(C^r)}. \]
Taking $L^\infty$-norm for both sides of the above inequality, if $pCMc_0 < \frac{1}{2}$, we obtain

\begin{equation}
\sup_{t \in [0, T]} \| \tilde{\psi}(t) \|_{B^p_{p,r}(\mathbb{L}^p)} \leq 2 \| \psi_0 - \psi_\infty \|_{B^p_{p,r}(\mathbb{L}^p)}.
\end{equation}

Integrating (7.7) over $[0, T]$ with respect to $t$, and multiplying $2^{ip^s}$ for both sides of (7.7) and then taking the $t^{\tilde{p}}$-norm, we deduce that

\begin{align*}
\| \tilde{\psi} \|_{E^{p,s}_{p,r}(\mathbb{T})}^p &\leq \frac{2p}{p-1} \| \psi_0 - \psi_\infty \|_{B^p_{p,r}(\mathbb{L}^p)}^p + C_p \int_0^T \| \nabla u \|_{B^p_{p,r}(\mathbb{L}^p)}^p \| \tilde{\psi} \|_{B^p_{p,r}(\mathbb{L}^p)}^p dt + \int_0^T \| \psi \| \| \tilde{\psi} \|_{B^p_{p,r}(\mathbb{L}^p)}^p dt \\
&\leq \frac{2p}{p-1} \| \psi_0 - \psi_\infty \|_{B^p_{p,r}(\mathbb{L}^p)}^p + C_p (MC_0)^p \sup_{t \in [0, T]} \| \tilde{\psi}(t) \|_{B^p_{p,r}(\mathbb{L}^p)}^p + C_p Mc_0 \int_0^T \| \tilde{\psi} \|_{B^p_{p,r}(\mathbb{L}^p)}^p dt.
\end{align*}

By Lemma 7.1, we deduce that \( \int_0^T \| \tilde{\psi} \|_{E^{p,s}_{p,r}(\mathbb{T})}^p dt \leq C \| \tilde{\psi} \|_{E^{p,s}_{p,r}(\mathbb{T})}^p \) if $CC_p Mc_0 \leq \frac{1}{2}$, we infer that

\begin{equation}
\| \tilde{\psi} \|_{E^{p,s}_{p,r}(\mathbb{T})}^p \leq \frac{4p}{p-1} \| \psi_0 - \psi_\infty \|_{B^p_{p,r}(\mathbb{L}^p)}^p + 2C_p (MC_0)^p \sup_{t \in [0, T]} \| \tilde{\psi}(t) \|_{B^p_{p,r}(\mathbb{L}^p)}^p.
\end{equation}

So plugging (7.13) into (7.14) and choosing $c_0$ small enough we have

\begin{equation}
\| \tilde{\psi} \|_{E^{p,s}_{p,r}(\mathbb{T})}^p \leq \frac{8p}{p-1} \| \psi_0 - \psi_\infty \|_{B^p_{p,r}(\mathbb{L}^p)}^p.
\end{equation}

Since $p \in [2, \infty)$, it follows that

\begin{equation}
\| \tilde{\psi} \|_{E^{p,s}_{p,r}(\mathbb{T})} \leq \left( \frac{8p}{p-1} \right)^{\frac{1}{2}} \| \psi_0 - \psi_\infty \|_{B^p_{p,r}(\mathbb{L}^p)} \leq 4 \| \psi_0 - \psi_\infty \|_{B^p_{p,r}(\mathbb{L}^p)}.
\end{equation}

For the Navier-Stokes equations, we can write that

\begin{equation}
u = e^{\nu \Delta} u_0 + \int_0^t e^{(t-t')\nu \Delta} (\text{div} \tilde{\tau} - u \nabla u - \nabla P) dt'.
\end{equation}

Using the fact that $s > 0$, $B^s_{p,r} = B^s_{p,r} \cap L^p$. Applying $\hat{\Delta}_j$ to (7.17) we obtain

\begin{equation}
\hat{\Delta}_j u = e^{\nu \Delta} \hat{\Delta}_j u_0 + \int_0^t e^{(t-t')\nu \Delta} (\text{div} \hat{\Delta}_j \tilde{\tau} - \hat{\Delta}_j (u \nabla u) - \nabla \hat{\Delta}_j P) dt'.
\end{equation}

By Lemma 3.2, we deduce that

\begin{equation}
\| \hat{\Delta}_j u \|_{L^p} \leq C e^{-t \nu \Delta^2} \| \hat{\Delta}_j u_0 \|_{L^p} + \int_0^t e^{-(t-t') \nu \Delta} \| \text{div} \hat{\Delta}_j \tilde{\tau} \|_{L^p} + \| \hat{\Delta}_j (u \nabla u) \|_{L^p} + \| \nabla \hat{\Delta}_j P \|_{L^p}) dt'.
\end{equation}

By a similar argument as in Lemma 4.1, we deduce that $\| \nabla \hat{\Delta}_j P \|_{L^p} \leq C (\| \text{div} \hat{\Delta}_j \tilde{\tau} \|_{L^p} + \| \hat{\Delta}_j (u \nabla u) \|_{L^p})$. Thus

\begin{equation}
\| \hat{\Delta}_j u \|_{L^p} \leq e^{-t \nu \Delta^2} \| \hat{\Delta}_j u_0 \|_{L^p} + C \int_0^t e^{-(t-t') \nu \Delta} \| \text{div} \hat{\Delta}_j \tilde{\tau} \|_{L^p} + \| \hat{\Delta}_j (u \nabla u) \|_{L^p}) dt'.
\end{equation}
Using Young’s inequality, we get
\[ \|\Delta_j u\|_{L^p} \leq \|\Delta_j u_0\|_{L^p} + C(\int_0^t 2^{-2j} (\|\Delta_j \tilde{T}\|_{L^p}^2 + \|\Delta_j (u \nabla u)\|_{L^p}^2) dt')^{\frac{1}{2}} \]
\[ \leq \|\Delta_j u_0\|_{L^p} + C(\int_0^t (\|\Delta_j \tilde{T}\|_{L^p}^2 + 2^{-2j-2} \|\Delta_j (u \nabla u)\|_{L^p}^2) dt')^{\frac{1}{2}}. \]

By the definition of Besov spaces, we have
\[ \|\Delta_j u\|_{L^p}^2 \leq \|\Delta_j u_0\|_{L^p}^2 + C(\|\tilde{T}\|_{L^p}^2 + \int_0^t \|\nabla u\|_{L^p}^2 dt'), \]
where \( c_j \) denotes an element of the unit sphere of \( l^p \). Multiplying both sides of (7.21) by \( 2^{2js} \), taking the \( l^2 \)-norm, and by Lemma 7.1 and Corollary 3.13 we deduce that
\[ \|u\|_{B^{s,p}_{p,r}}^2 \leq \|u_0\|_{B^{s,p}_{p,r}}^2 + C(\|\tilde{T}\|_{L^p}^2 + \int_0^t \|\nabla u\|_{L^p}^2 dt'). \]
Since \( E^{s,p}_{p,r}(T) \hookrightarrow \dot{E}^{s,p}_{p,r}(T) \) and \( B^{s,p}_{p,r} \hookrightarrow B^{s,p}_{p,r} \), it follows that
\[ \|u\|_{B^{s,p}_{p,r}}^2 \leq \|u_0\|_{B^{s,p}_{p,r}}^2 + 16\|\psi_0 - \psi_{\infty}\|_{B^{s,p}_{p,r}}^2 + \int_0^t \|\nabla u\|_{B^{s,p}_{p,r}}^2 dt' \sup_{t \in T} \|u\|_{B^{s,p}_{p,r}} \]
\[ \leq \|u_0\|_{B^{s,p}_{p,r}}^2 + 16\|\psi_0 - \psi_{\infty}\|_{B^{s,p}_{p,r}}^2 + CMc_0 \sup_{t \in T} \|u\|_{B^{s,p}_{p,r}}. \]
If \( CMc_0 < \frac{1}{2} \), then we obtain
\[ \sup_{t \in T} \|u\|_{B^{s,p}_{p,r}}^2 \leq 2\|u_0\|_{B^{s,p}_{p,r}}^2 + 32\|\psi_0 - \psi_{\infty}\|_{B^{s,p}_{p,r}}^2. \]
Hence
\[ \sup_{t \in T} \|u\|_{B^{s,p}_{p,r}} \leq 2\|u_0\|_{B^{s,p}_{p,r}} + 8\|\psi_0 - \psi_{\infty}\|_{B^{s,p}_{p,r}}. \]
Multiplying both sides of (7.19) by \( 2^{2js+1} \) and taking the \( L^2([0,T]) \)-norm, by a similar calculation, we deduce that
\[ \nu \|u\|_{L^2(T B^{s,p}_{p,r})} \leq 2\|u_0\|_{B^{s,p}_{p,r}} + 8\|\psi_0 - \psi_{\infty}\|_{B^{s,p}_{p,r}}. \]
Now we estimate the \( L^p \)-norm. By (1.2), we write that
\[ \partial_t u^k + (u \cdot \nabla) u^k - \nu \Delta u^k + \partial_k P = \text{div} \tilde{k}, \]
where \( u^k \) is the \( k \) component of \( u \). Note that \( \text{div} u = 0 \). Multiplying \( \text{sgn}(u^k)|u^k|^{p-1} \) both sides of (7.26) and integrating over \( \mathbb{R}^d \), we have
\[ \frac{1}{p} \int_{\mathbb{R}^d} |u^k|^p dx + \nu(p-1) \int_{\mathbb{R}^d} |\nabla u^k|^2 |u^k|^{p-2} dx = (p-1) \int_{\mathbb{R}^d} P|\partial_k u^k| |u^k|^{p-2} dx - \int_{\mathbb{R}^d} \tilde{k} \nabla u^k |u^k|^{p-2} dx. \]
By Cauchy-Schwarz’s inequality, we deduce that
\begin{equation}
\frac{1}{p} \partial_t \int_{\mathbb{R}^d} |u(t)|^p dx + \frac{\nu(p-1)}{2} \int_{\mathbb{R}^d} |\nabla u|^2 |u|^{p-2} dx \leq C_p \left( \int_{\mathbb{R}^d} |u|^2 |u|^{p-2} dx + \int_{\mathbb{R}^d} |\tau|^2 |u|^{p-2} dx \right).
\end{equation}
Using Hölder’s inequality, we obtain
\begin{equation}
\partial_t \|u\|_{L^p}^p + \frac{\nu(p-1)}{2} \int_{\mathbb{R}^d} |\nabla u|^2 |u|^{p-2} dx \leq C_p \left( \|P\|_{L^p}^p \|u\|_{L^p}^{p-2} + \|\tau\|_{L^p}^2 \|u\|_{L^p}^{p-2} \right).
\end{equation}
Since $P = \Delta^{-1} \text{div}(u \nabla u + \text{div} \tilde{\tau}) = \Delta^{-1} \text{div} \text{div}(u \otimes u + \tilde{\tau})$, it follows that
\[ \|P\|_{L^p} \leq C \left( \|u \otimes u\|_{L^p} + \|\tilde{\tau}\|_{L^p} \right) \leq C \|u\|_{L^\infty} \|u\|_{L^p} + \|\tilde{\tau}\|_{L^p}. \]
By Lemma 7.1 and Corollary 3.13, we get
\begin{equation}
\|\tilde{\tau}\|_{L^p}^p \leq C \int_{\mathbb{R}^d \times B} |\nabla R \left( \frac{\tilde{\psi}}{\psi^\infty} \right) |^2 \psi^{\infty} dx dR \leq C \sum_{j \geq -1} \int_{\mathbb{R}^d \times B} \left| \nabla R \left( \frac{\Delta_j \tilde{\psi}}{\psi^\infty} \right) \right|^2 \psi^{\infty} dx dR.
\end{equation}
Plugging into (7.29), we deduce that
\begin{equation}
\partial_t \|u\|_{L^p}^p \leq C_p \left( \|u\|_{L^\infty}^2 \|u\|_{L^p}^p + 2\|\tilde{\tau}\|_{L^p}^2 \|u\|_{L^p}^{p-2} \right)
\end{equation}
\begin{equation}
\leq C_p \left( \|u\|_{L^\infty}^2 \|u\|_{L^p}^p + 2C \sum_{j \geq -1} \int_{\mathbb{R}^d \times B} \left| \nabla R \left( \frac{\Delta_j \tilde{\psi}}{\psi^\infty} \right) \right|^2 \psi^{\infty} dx dR \right)
\end{equation}
\begin{equation}
\leq C_p \left( \|u\|_{L^\infty}^2 \|u\|_{L^p}^p + 2C \sum_{j \geq -1} 2^{-2s} \int_{\mathbb{R}^d \times B} \left| \nabla R \left( \frac{\Delta_j \tilde{\psi}}{\psi^\infty} \right) \right|^2 \psi^{\infty} \|u\|_{L^p}^{p-2} \right).
\end{equation}
Note that $s > 0$. Integrating over $[0, t]$ with respect to $t$ and using Hölder’s inequality, we obtain
\begin{equation}
\|u\|_{L^p}^p \leq \|u_0\|_{L^p}^p + C_p \left( \int_0^T \|u\|_{L^\infty}^2 dt + \|\tilde{\psi}\|_{L^p(B^s_{p,p}(\mathcal{C}))} \sup_{t \in [0,T]} \|u\|_{L^p}^p \right).
\end{equation}
Since $B^s_{p,r} \hookrightarrow L^\infty$, it follows that
\begin{equation}
\|u\|_{L^p}^p \leq \|u_0\|_{L^p}^p + C_p M_c \sup_{t \in [0,T]} \|u\|_{L^p}^p.
\end{equation}
If $C_p M_c < \frac{1}{2}$, then we have
\begin{equation}
\sup_{t \in [0,T]} \|u\|_{L^p}^p \leq 2\|u_0\|_{L^p}^p \quad \text{or} \quad \sup_{t \in [0,T]} \|u\|_{L^p} \leq 2\|u_0\|_{L^p}.
\end{equation}
Combining with (13), (16), (24), (25), (35) and using the fact that $B^s_{p,r} = B^s_{p,r} \cap L^p$, we get
\begin{equation}
\sup_{t \in [0,T]} \|u(t)\|_{B^s_{p,r}} + \nu \|u\|_{L^p(B^s_{p,r-1})} + \sup_{t \in [0,T]} \|\tilde{\psi}(t)\|_{B^s_{p,r}(\mathcal{C})} + \|\tilde{\psi}\|_{L^p(B^s_{p,r}(\mathcal{C}))} \leq 16 \left( \|u_0\|_{B^s_{p,r}} + \|\psi_0 - \psi^\infty\|_{B^s_{p,r}(\mathcal{C})} \right).
\end{equation}
Note that $u \in C([0,T^*); B^{s}_{p,r}), \tilde{\psi} \in C([0,T^*); B^{s}_{p,r}(\mathcal{L}^p))$. If we set $M = 32$, then we can find a $T_1 \in [\mathcal{T}, T^*)$, such that

\begin{equation}
\sup_{t \in [0,T_1]} \|u(t)\|_{B^{s}_{p,r}} + \nu \|u\|_{L^{2}_{T^{'}}(B^{s+1}_{p,r})} + \sup_{t \in [0,T_1]} \|\tilde{\psi}(t)\|_{B^{s}_{p,r}(\mathcal{L}^p)} + \|\tilde{\psi}\|_{E^{s}_{p,r}(\mathcal{T})} \leq 32(\|u_0\|_{B^{s}_{p,r}} + \|\psi_0 - \psi_\infty\|_{B^{s}_{p,r}(\mathcal{L}^p)}) = M(\|u_0\|_{B^{s}_{p,r}} + \|\psi_0 - \psi_\infty\|_{B^{s}_{p,r}(\mathcal{L}^p)}).
\end{equation}

This contradicts the definition of $\mathcal{T}$. Thus we have $T = T^*$. Therefore we obtain

\begin{equation}
\sup_{t \in [0,T^*)} \|u(t)\|_{B^{s}_{p,r}} + \nu \|u\|_{L^{2}_{T^{'}}(B^{s+1}_{p,r})} + \sup_{t \in [0,T^*)} \|\tilde{\psi}(t)\|_{B^{s}_{p,r}(\mathcal{L}^p)} + \|\tilde{\psi}\|_{E^{s}_{p,r}(T^*)} \leq M(\|u_0\|_{B^{s}_{p,r}} + \|\psi_0 - \psi_\infty\|_{B^{s}_{p,r}(\mathcal{L}^p)}).
\end{equation}

Now set $T_\delta = T^* - \frac{\delta}{2}$. Then we can construct a solution $(\hat{u}, \hat{\psi})$ with initial data $\|u(T_\delta)\|_{B^{s}_{p,r}}$ and $\|\tilde{\psi}\|_{E^{s}_{p,r}(T_\delta)}$. This implies the solution can be extended outside $[0, T^*)$, which contradicts the lifespan $T^*$. Thus the solution is global. This completes the proof.

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