A Fast Active Set Block Coordinate Descent Algorithm for \( \ell_1 \)-regularized least squares

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Abstract
The problem of finding sparse solutions to underdetermined systems of linear equations arises in several real-world problems (e.g. signal and image processing, compressive sensing, statistical inference). A standard tool for dealing with sparse recovery is the \( \ell_1 \)-regularized least-squares approach that has been recently attracting the attention of many researchers.
In this paper, we describe an efficient active set block coordinate descent algorithm that at each iteration uses a bunch of variables (i.e. those variables which are non-active and violate the most some specific optimality conditions) to improve the objective function. We further analyze the convergence properties of the proposed method. Finally, we report some numerical results showing the effectiveness of the approach.

Keywords. \( \ell_1 \)-regularized least squares, active set, sparse optimization

AMS subject classifications. 65K05, 90C25, 90C06

1 Introduction
The problem of finding sparse solutions to large underdetermined linear systems of equations has received a lot of attention in the last decades. This is due to the fact that several real-world problems (e.g. signal and image processing, compressive sensing, statistical inference) can be formulated as linear inverse problems. A standard approach to these problems is the so called \( \ell_2-\ell_1 \) unconstrained optimization problem:

\[
\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|_2^2 + \tau \|x\|_1,
\]

(1)
where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ ($m < n$) and $\tau \in \mathbb{R}^+$. We denote by $\| \cdot \|_2$ the standard $\ell_2$ norm and by $\| \cdot \|_1$ the $\ell_1$ norm defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$.

Several classes of algorithms have been proposed for the solution of Problem (1). Among the others, we would like to remind Iterative Shrinkage/Thresholding (IST) methods (see e.g. [3, 4, 8, 9, 24]), Augmented Lagrangian Approaches (see e.g. [2]), Second Order Methods (see e.g. [5, 14]), Sequential Deterministic (see e.g. [22, 23, 28]) and Stochastic (see e.g. [19] and references therein) Block Coordinate Approaches, Parallel Deterministic (see e.g. [13] and references therein) and Stochastic (see e.g. [20] and references therein) Block Coordinate Approaches, and Active-set strategies (see e.g. [25, 26]).

In particular, identifying the active set (i.e. the subset of zero-components in an optimal solution) for Problem (1) is becoming a very crucial task in the context of Big Data Optimization since it can guarantee relevant savings in terms of CPU time. As a very straightforward example, we can consider a huge scale problem having a solution with just a few nonzero components. In this case, the correct identification of the active set can considerably reduce the complexity of the problem, thus giving us the chance to use more sophisticated optimization methods than the ones usually adopted.

In [25, 26] Wen et al. proposed a two-stage algorithm, FPC-AS, where an estimate of the active variable set is driven, by using a first-order iterative shrinkage method. The $\ell_1$-norm $\|x\|_1$ is then reduced to a linear function of $x$ by fixing the components of $x$ estimated to be active and fixing their signs at their current values. This yields to the second stage, when a subspace problem involving the minimization of a smaller and smooth quadratic function is solved by means of a second-order method.

In a recent paper [5], Nocedal et al. described an interesting family of second order methods for $\ell_1$-regularized convex problems. Those methods combine a semi-smooth Newton approach with a mechanism to identify the active manifold in the given problem. An efficient version of the two-block nonlinear constrained Gauss-Seidel algorithm that at each iteration fixes some variables to zero according to a simple active-set rule has been proposed in [18] for solving $\ell_1$-regularized least squares.

Anyway, in the case one wants to solve very large problems, Block Coordinate Descent Algorithms (both Sequential and Parallel) represent a very good alternative and, sometimes, the best possible answer. An interesting Coordinate Descent algorithm combining a Newton steps with a line search technique was described by Yuan et al. in [27]. In this context, the authors also proposed a shrinking technique (i.e. a heuristic strategy that tries to fix to zero a subset of variables according to a certain rule), which can be seen as a way to identify the active variables.

In [28], some ideas on how to speed up their Block Coordinate Descent Algorithm by including an active-set identification strategy are described, but no theoretical analysis is given of the resulting approach.

In this work, inspired by the papers [23, 27, 28], we describe a new Block Coordinate Descent Algorithm where only a bunch of variables is analyzed at each iteration. In particular, we use at iteration $k$ a subset of the non-active variables which violate the most some kind of optimality condition. The main difference with respect to the methods described in [23, 27, 28] is in the way we choose the subset of variables to be analyzed. Indeed, at each iteration $k$ we first identify the subset of active variables according to a certain rule, somehow related to the ones proposed in [5, 27], and fix those ones to zero. Then, we analyze those variables which are non-
active and violate the most some specific optimality conditions. Thanks to this new variables selection strategy, we can ensure a sufficient decrease of the function both when fixing to zero the active variables and when minimizing with respect to a single block of non-active variables, thus guaranteeing convergence without the use of any line search technique.

In practice, when minimizing with respect to the non-active variables, we use blocks of dimension 1 and 2, thus obtaining a closed-form solution for the related subproblem. Furthermore, in order to accelerate convergence, we propose a modified version of the algorithm that, taking into account the properties of our active set strategy, identifies a somehow good estimate of the active set, and solves in the subspace of the non-active variables a smaller and smooth quadratic problem by means of a suitably chosen method.

The paper is organized as follows. In Section 3 we introduce our active set strategy. In Section 4 we describe the active set coordinate descent algorithm, and prove its convergence. In Section 5 we report some numerical results showing the effectiveness of the approach. Finally, we draw some conclusions in Section 6.

2 Notations and Preliminary Results

Throughout the paper we denote by $q(x)$, $g(x)$ and $H(x)$ the quadratic term of the objective function in Problem (1), the $n$ gradient vector and the $n \times n$ hessian matrix of $\frac{1}{2} \| Ax - b \|^2$ respectively. Explicitly

$$q(x) = \frac{1}{2} \| Ax - b \|^2, \quad g(x) = A^T (Ax - b), \quad H(x) = A^T A.$$ 

Given a matrix $Q$, we further denote by $\lambda_{\text{max}}(Q)$ and $\lambda_{\text{min}}(Q)$ the maximum and the minimum eigenvalue of the matrix $Q$, respectively.

We also report the optimality conditions for Problem (1):

**Proposition 1.** $x^* \in \mathbb{R}^n$ is an optimal solution of Problem (1) if and only if

$$\begin{cases}
    x_i^* > 0, & g_i(x^*) + \tau = 0 \\
    x_i^* < 0, & g_i(x^*) - \tau = 0 \\
    x_i^* = 0, & -\tau \leq g_i(x^*) \leq \tau.
\end{cases}$$  \hspace{1cm} (2)

Finally, we recall the concept of strict complementarity.

**Definition 1.** Strict complementarity holds if, for any $x_i^* = 0$, we have

$$-\tau < g_i(x^*) < \tau.$$  \hspace{1cm} (3)

3 Active set estimate

All the algorithms that adopt active set strategies need to estimate a particular subset of components of the optimal solution $x^*$. In nonlinear constrained minimization problems, for example, using an active set strategy usually means to correctly identify the set of active constraints at the solution. In our context, we deal with Problem (1) and the active set is considered as the subset of zero-components of $x^*$. 

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Definition 2. Let \( x^* \in \mathbb{R}^n \) be an optimal solution for Problem (1). We define as active set the following:
\[
\bar{A}(x^*) = \{ i \in \{1, \ldots, n\} : x^*_i = 0 \}.
\] (4)
We further define as non-active set the complementary set of \( \bar{A}(x^*) \):
\[
\bar{N}(x^*) = \{ 1, \ldots, n \} \setminus \bar{A}(x^*) = \{ i \in \{1, \ldots, n\} : x^*_i \neq 0 \}
\] (5)

In order to get an estimate of the active set we rewrite Problem (1) as a box constrained programming problem and we use similar ideas of those proposed in [10]. Problem (1) can be equivalently rewritten as follows:
\[
\min_{u \geq 0} \begin{array}{c}
\frac{1}{2} \| A(u - v) - b \|^2 + \tau \sum_{i=1}^{n} (u_i + v_i) \\
u \geq 0
\end{array}
\] (6)
where \( u, v \in \mathbb{R}^n \). Indeed, we can transform a solution \( x^* \in \mathbb{R}^n \) of Problem (1) into a solution \( (u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n \) of (6) by using the following transformation:
\[
u^*_i = \max(0, -x^*_i),
\]
Equivalently, we can transform a solution \( (u^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^n \) of (6) into a solution \( x^* \in \mathbb{R}^n \) of Problem (1) by using the following transformation:
\[x^*_i = u^*_i - v^*_i.\]

The Lagrangian function associated to (6) is
\[
\mathcal{L}(u, v, \lambda, \mu) = \frac{1}{2} \| A(u - v) - b \|^2 + \tau \sum_{i=1}^{n} (u_i + v_i) - \lambda^T u - \mu^T v,
\]
with \( \lambda, \mu \in \mathbb{R}^n \) vectors of lagrangian multipliers.
Let \( (u^*, v^*, \lambda^*, \mu^*) \) be an optimal solution of Problem (6). Then, from necessary optimality conditions, we have
\[
\nabla_u \mathcal{L}(u^*, v^*, \lambda^*, \mu^*) = 0
\] (7)
\[
\nabla_v \mathcal{L}(u^*, v^*, \lambda^*, \mu^*) = 0
\] (8)
\[
\lambda^* \geq 0
\] (9)
\[
\mu^* \geq 0
\] (10)
\[
\lambda^*^T u^* = 0
\] (11)
\[
\mu^*^T v^* = 0
\] (12)

Remark 1. Let \( x^* \) be an optimal solution of (7) and let \( (u^*, v^*, \lambda^*, \mu^*) \) be an optimal solution of Problem (6). Then
\[
x^*_i > 0 \iff u^*_i > 0,
\]
\[
x^*_i < 0 \iff v^*_i > 0.
\]
We have that the two equations (7) and (8) imply:

\[
\begin{align*}
\lambda_i^* &= g_i(u^* - v^*) + \tau = g_i(x^*) + \tau; \\
\mu_i^* &= \tau - g_i(u^* - v^*) = \tau - g_i(x^*).
\end{align*}
\]  

From (13), we can introduce the following two multiplier functions

\[
\begin{align*}
\lambda_i(u, v) &= g_i(u - v) + \tau; \\
\mu_i(u, v) &= \tau - g_i(u - v).
\end{align*}
\]

By means of the multiplier functions, we can recall the non-active set estimate \(N(u, v)\) and active set estimate \(A(u, v)\) proposed in the field of constrained smooth optimization (see [12] and references therein):

\[
N(u, v) = \{i : u_i > \epsilon \lambda_i(u, v)\} \cup \{i : v_i > \epsilon \mu_i(u, v)\}
\]

and

\[
A(u, v) = \{1, \ldots, n\} \setminus N(u, v),
\]

where \(\epsilon\) is a positive scalar.

We draw inspiration from (15) and (16) to propose the new estimates of active and non-active set for Problem (1). Indeed, by using the relations

\[
\begin{align*}
u &= \max(0, x); \\
v &= \max(0, -x),
\end{align*}
\]

we can give the following definitions.

**Definition 3.** Let \(x \in \mathbb{R}^n\). We define the following sets as estimate of the non-active and active variables sets:

\[
N(x) = \{i : \max(0, x_i) > \epsilon (\tau + g_i(x))\} \cup \{i : \max(0, -x_i) > \epsilon (\tau - g_i(x))\}
\]

and

\[
A(x) = \{1, \ldots, n\} \setminus N(x),
\]

The following result can be proved for the new estimates.

**Theorem 1.** Let \(x^* \in \mathbb{R}^n\) be a solution of Problem (1). Then, there exists a neighborhood of \(x^*\) such that, for each \(x\) in this neighborhood, we have

\[
\bar{A}^+(x^*) \subseteq A(x) \subseteq \bar{A}(x^*),
\]

with \(\bar{A}^+(x^*) = A(x^*) \cup \{i : -\tau < g_i(x^*) < \tau\}\).
**Proof.** We first prove that
\[ \mathcal{N}(x) \supseteq \bar{\mathcal{N}}(x^*) \] holds for each \( x \) in a suitably chosen neighborhood of \( x^* \). From Definition 2 and the optimality of \( x^* \), we have, for all \( h \in \bar{\mathcal{N}}(x^*) \),
\[ x^*_h \neq 0 \text{ and } \tau + g_h(x^*) = 0. \]
Then, either
\[ \max(0, x^*_h) > \epsilon (\tau + g_{i}(x^*_h)) \]
or
\[ \max(0, -x^*_h) > \epsilon (\tau - g_{i}(x^*_h)) \]
must be satisfied. By continuity of \( g(\cdot) \), there exists a neighborhood of \( x^* \) such that, for each \( x \) in this neighborhood the same inequality is satisfied. Then we have that (19) holds, and
\[ \bar{\mathcal{A}}(x^*) \supseteq \mathcal{A}(x). \] (20)
For any index \( h \in \bar{\mathcal{A}}^+(x^*) \), we have
\[ x^*_h = 0, \tau - g_h(x^*) > 0 \text{ and } \tau + g_h(x^*) > 0. \]
Then, we can write
\[ \max(0, x^*_h) \leq \epsilon (\tau + g_{i}(x^*_h)) \]
and
\[ \max(0, -x^*_h) \leq \epsilon (\tau - g_{i}(x^*_h)). \]
Once again, by continuity of \( g(\cdot) \), there exists a neighborhood of \( x^* \) such that, for each \( x \) in this neighborhood the same inequality is satisfied, and
\[ \bar{\mathcal{A}}^+(x^*) \subseteq \mathcal{A}(x). \]
\[ \square \]
If strict complementarity holds, we can state the following result:

**Corollary 1.** Let \( x^* \in \mathbb{R}^n \) be a solution of Problem (11) where strict complementarity (11) holds. Then, there exists a neighborhood of \( x^* \) such that, for each \( x \) in this neighborhood, we have
\[ \mathcal{A}(x) = \bar{\mathcal{A}}(x^*). \] (21)

Our active set estimate is somehow related to those proposed respectively by Byrd et al. in [5] and by Yuan et al. in [27]. Here, we would like to point out the similarities and the differences between those two strategies and the one we propose in the present paper.
In the block active set algorithm for quadratic \( \ell_1 \)-regularized problems proposed in [5], the active set estimate, at a generic iteration \( k \), can be rewritten in the following way:
\[ A_{Byrd}^k = \{ i : x^k_i = 0; g_i(x^k) \in (-\tau, \tau) \} \cup \{ i : x^k_i < 0; g_i(x^k) = -\tau \} \cup \{ i : x^k_i > 0; g_i(x^k) = \tau \}. \]
Let $x^k \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$ be an index estimated active by our estimate, that is,

$$i \in A(x^k) = \{i : \max(0, x^k_i) \leq \epsilon (\tau + g_i(x^k))\} \cap \{i : \max(0, -x^k_i) \leq \epsilon (\tau - g_i(x^k))\}.$$ 

Then, necessarily, we have

$$\tau + g_i(x^k) \geq 0$$
$$\tau - g_i(x^k) \geq 0.$$ 

This is equivalent to say that $g_i(x^k) \in [-\tau, \tau]$. Then, in the case $x^k_i = 0$, $i \in A_{Byrd}$ implies $i \in A_{Byrd}$. The other way around is also true. In fact, let $i \in A_{Byrd}$. If $x^k_i = 0$ we have $g_i(x^k) \in (-\tau, \tau)$ so that $i \in A(x^k)$.

The differences between the two estimates, come out when considering indices $i$ such that $x^k_i \neq 0$. Let $i \in A_{Byrd}$ and, in particular, $i \in \{i : x^k_i < 0; g_i(x^k) = -\tau\}$. It can happen that

$$\max(0, -x^k_i) = -x^k_i > \epsilon 2\tau = \epsilon (\tau - g_i(x^k)),$$

so that $i \notin A^k$. Using the same reasoning we can see that, in the case $i \in A^k_{Byrd}$ and, in particular, $i \in \{i : x^k_i > 0; g_i(x^k) = \tau\}$, it can happen

$$\max(0, x^k_i) = x^k_i > \epsilon 2\tau = \epsilon (\tau + g_i(x^k)),$$

so that $i \notin A^k$.

In [27], the active set estimate is defined as follows

$$A_{Yuan}^k = \{i : x^k_i = 0; g_i(x^k) \in (-\tau + M^{k-1}, \tau - M^{k-1})\}, \quad (22)$$

where $M^{k-1}$ is a positive scalar that measures the violation of the optimality conditions. It is easy to see that our active set contains the one proposed in [27]. Furthermore, we have that variables contained in our estimate are not necessarily contained in the estimate (22).

### 4 A Fast Active Set Block Coordinate Descent Algorithm

In this section, we describe our Fast Active SeT Block Coordinate Descent Algorithm (FAST-BCDA) and analyze its theoretical properties. The main idea behind the algorithm is that of exploiting as much as possible the good properties of our active set estimate, namely:

- the ability to identify, for $k$ sufficiently large, the “strong” active variables (namely, those variables satisfying the strict complementarity, see Theorem 1);

- the ability to obtain, at each iteration, a sufficient decrease of the objective function, by fixing to zero those variables belonging to the active set estimate (see Proposition 2 of the next section).

At each iteration $k$, the algorithm defines two sets $\mathcal{N}^k = \mathcal{N}(x^k)$, $\mathcal{A}^k = A(x^k)$ and executes two steps:

1) it sets to zero all the active variables;
2) it minimizes only a subset of the non-active variables, which violate the most the optimality conditions.

More specifically, we consider a measure related to the violation of the optimality conditions in \( x^k \) (which is somehow connected to the Gauss-Southwell-r rule proposed in [23]), that is

\[
|g_i(x^k) + \tau| \quad \text{if } x_i^k > 0; \\
|g_i(x^k) - \tau| \quad \text{if } x_i^k < 0; \\
\max\{0, - (g_i(x^k) + \tau), g_i(x^k) - \tau\} \quad \text{if } x_i^k = 0.
\]  

(23)

We then sort in decreasing order the indices of non-active variables (i.e. the set of indices \( N^k \)) with respect to this measure and define the subset \( \bar{N}^k_{ord} \subseteq N^k \) containing the first \( s \) sorted indices.

The set \( \bar{N}^k_{ord} \) is then partitioned into \( q \) subsets \( I_1, \ldots, I_q \) of cardinality \( r \), such that \( s = qr \). Then the algorithm performs \( q \) subiterations. At the \( j \)-th subiteration the algorithm considers the set \( I_j \subseteq \bar{N}^k_{ord} \) and solves to optimality the subproblem we get from (1), by fixing all the variables but the ones whose indices belong to \( I_j \).

The scheme of the proposed algorithm is reported below (see Algorithm 1).

**Algorithm 1 Fast Active SeT Block Coordinate Descent Algorithm (FAST-BCDA)**

1. Choose \( x^0 \in \mathbb{R}^n \), Set \( k = 0 \).
2. For \( k = 0, 1 \ldots \)
3. Compute \( A^k, N^k, \bar{N}^k_{ord} \);
4. Set \( y^{0,k}_{A^k} = 0 \) and \( y^{0,k}_{N^k} = x^k_{N^k} \);
5. For \( j = 1, \ldots, q \)
6. Compute \( y^{j,k}_{I_j} \), with \( I_j \subseteq \bar{N}^k_{ord} \), solution of problem

\[
\min_{w \in \mathbb{R}^r} g_{I_j}(y^{j-1,k})^T (w - y^{j-1,k}) + \frac{1}{2} (w - y^{j-1,k})^T H_{I_j, I_j} (w - y^{j-1,k}) + \tau \|w\|_1
\]

7. Set \( y^{j,k}_i = y^{j-1,k}_i \) if \( i \notin I_j \);
8. End For
9. Set \( x^{k+1} = y^{q,k} \);
10. End For

The convergence of FAST-BCDA is based on two important results.

The first one completes the properties of the active-set identification strategy proposed in the previous section. More specifically, it shows that, for a suitably chosen value of the parameter \( \epsilon \) appearing in Definition 3 it is possible to obtain, at each iteration, a significant decrease of the objective function by fixing to zero one or more variables whose indices belong to the active set estimate.

**Proposition 2.** Assume that the parameter \( \epsilon \) appearing in Definition 3 satisfies the following conditions

\[
0 < \epsilon < \frac{1}{\lambda_{max}(A^T A)}.
\]  

(24)
Given the point $x^k$ and the set $A^k$, let $A^y$ and $A^z$ be two sets of indices and let $y$ and $z$ two points such that

\[
A^y \subseteq \{ i \in A^k : x^k_i \neq 0 \} \quad A^z \subseteq \{ i \in A^k : x^k_i \neq 0 \} \quad A^z \subseteq A^y
\]

\[
y_{A^y} = 0 \quad y_{I \setminus A^y} = x^k_{I \setminus A^y}
\]

\[
z_{A^z} = 0 \quad z_{I \setminus A^z} = x^k_{I \setminus A^z}
\]

with $I = 1, \ldots, n$. Then,

\[
f(y) - f(z) \leq -\frac{1}{2\epsilon} \|y - z\|^2.
\]

**Proof.** We first define the following set

\[
A = A^y \setminus A^z.
\]

By taking into account the definitions of the sets $A^y$ and $A^z$ and the points $y$ and $z$, it is possible to write:

\[
f(y) = q(y) + \tau \sum_{i=1}^{n} \text{sign}(y_i) y_i = q(y) + \tau \sum_{i \in I \setminus A} \text{sign}(y_i) y_i + \tau \sum_{i \in A} \text{sign}(z_i) y_i. \tag{25}
\]

from which

\[
f(y) = f(z) + (g_A(z) + \tau S_A)^T(y - z)_A + \frac{1}{2}(y - z)_A^T H_{AA}(y - z)_A
\]

where $S_A$ is the diagonal matrix defined as

\[
S_A = \text{Diag}(\text{sign}(z_A)),
\]

where the function $\text{sign}(\cdot)$ is intended componentwise, and $H_{AA}$ is the Hessian matrix of the quadratic part restricted to the variables with indices belonging in $A$.

Since $H = A^T A$ we have that the following inequality holds

\[
f(y) \leq f(z) + (g_A(z) + \tau S_A)^T(y - z)_A + \frac{\lambda_{\text{max}}(A^T A)}{2} \|(y - z)_A\|^2.
\]

Recalling (24) we obtain:

\[
f(y) \leq f(z) + (g_A(z) + \tau S_A)^T(y - z)_A + \frac{1}{2\epsilon} \|(y - z)_A\|^2. \tag{26}
\]

Then, we can write

\[
f(y) \leq f(z) + \left(g_A(z) + \tau S_A + \frac{1}{\epsilon}(y - z)_A\right)^T(y - z)_A - \frac{1}{2\epsilon} \|(y - z)_A\|^2.
\]

In order to prove the Proposition, we need to show that

\[
\left(g_A(z) + \tau S_A + \frac{1}{\epsilon}(y - z)_A\right)^T(y - z)_A \leq 0 \tag{27}
\]
Inequality (27) follows from the fact that \( \forall i \in A \):

\[
\left( g_i(z) + \tau \text{sign}(z_i) + \frac{1}{\epsilon}(y_i - z_i) \right)^T (y_i - z_i) \leq 0.
\]

We distinguish two cases:

a) If \( z_i > 0 \), we have that \( \text{sign}(z_i) = +1 \) and, since \( y_i = 0 \), \((y_i - z_i) \leq 0\).

Then, from the fact that \( i \in A \), we have

\[
y_i = 0 \\
z_i \leq \epsilon (g_i(z) + \tau) \\
(y_i - z_i) \leq \epsilon (g_i(z) + \tau) \\
\frac{1}{\epsilon}(y_i - z_i) \leq g_i(z) + \tau
\]

so that

\[ g_i(z) + \tau + \frac{1}{\epsilon}(y_i - z_i) \geq 0. \]

and (28) is satisfied.

b) If \( z_i < 0 \), we have that \( \text{sign}(z_i) = -1 \) and, since \( y_i = 0 \), \((y_i - z_i) \geq 0\).

Then, by reasoning as in case a), since \( i \in A \), we can write

\[
y_i = 0 \\
-z_i \leq \epsilon (\tau - g_i(z)) \\
(y_i - z_i) \leq \epsilon (\tau - g_i(z)) \\
\frac{1}{\epsilon}(y_i - z_i) \leq \tau - g_i(z)
\]

from which we have:

\[ g_i(z) - \tau + \frac{1}{\epsilon}(y_i - z_i) \leq 0. \]

Again, we have that (28) is satisfied. \( \square \)

The second proposition shows that, despite the presence of the nonsmooth term, an exact minimization of Problem (1) with respect to a subset \( J \) of the variables gives a relevant decrease of the objective function (which is needed to guarantee the global convergence of the algorithm) in case \( \lambda_{\min}(H_{JJ}) > 0 \).

**Proposition 3.** Given a point \( z \in \mathbb{R}^n \) and a set \( J \subseteq I = \{1, \ldots, n\} \), let \( w^* \in \mathbb{R}^{|J|} \) be the solution of Problem (1), where all variables but the ones whose indices belong to \( J \) are fixed to \( z_{I \setminus J} \).

Let \( y \in \mathbb{R}^n \) be defined as

\[ y_J = w^*, \quad y_{I \setminus J} = z_{I \setminus J}. \]

Then we have

\[ f(y) - f(z) \leq -\frac{1}{2}\lambda_{\min}(H_{JJ})\|y - z\|^2. \]
\textbf{Proof.} Let \( w^* \in \mathbb{R}^{|J|} \) be the solution of Problem (11), where all variables but the ones whose indices belong to \( J \) are fixed.

We consider the set \( J = \{ j_1, \ldots, j_{|J|} \} \) as the union of two sets

\[
J = J_E \cup J_D,
\]

where

\[
J_E = J_{E+} \cup J_{E-}, \quad J_D = J_{D+} \cup J_{D-}
\]

and

\[
J_{D+} = \{ j_i \in J : w^*_i \neq 0; \text{sign}(w^*_i) > 0 \};
\]

\[
J_{D-} = \{ j_i \in J : w^*_i \neq 0; \text{sign}(w^*_i) < 0 \};
\]

\[
J_{E+} = \{ j_i \in J : w^*_i = 0; \text{sign}(z_{j_i}) > 0 \};
\]

\[
J_{E-} = \{ j_i \in J : w^*_i = 0; \text{sign}(z_{j_i}) < 0 \}.
\]

Let \( \tilde{f} : \mathbb{R}^{|J|} \to \mathbb{R} \) and \( w \in \mathbb{R}^{|J|} \) be the following function:

\[
\tilde{f}(w) = q(z) + \tau \sum_{j \in I \setminus J} \text{sign}(z_j) z_j + g_J(z) (w - z_J)^\top + \frac{\tau}{2} (w - z_J)^\top H_{J,J}(z) (w - z_J)
\]

\[
+ \tau \sum_{j_i \in J_E} \text{sign}(z_{j_i}) w_i + \tau \sum_{j_i \in J_D} \text{sign}(w^*_i) w_i.
\]

Then, \( w^* \) can be equivalently seen as the solution of the following problem

\[
\text{min } \tilde{f}(w) \quad \text{s.t. } w_i \geq 0 \text{ for } j_i \in J_{D+} \cup J_{E+},
\]

\[
\text{and } w_i \leq 0 \text{ for } j_i \in J_{D-} \cup J_{E-},
\]

(30)

By introducing the diagonal matrix \( S = \text{Diag}(s_i) \in \mathbb{R}^{|J| \times |J|} \), where \( s \in \{-1, 0, 1\}^{|J|} \) is the vector defined as

\[
s_i = \begin{cases} 
\text{sign}(w^*_i) & \text{if } j_i \in J_D \\
\text{sign}(z_{j_i}) & \text{if } j_i \in J_E,
\end{cases}
\]

Problem (30) can be written in a more compact form as

\[
\text{min } \tilde{f}(w) \quad \text{s.t. } Sw \geq 0.
\]

(31)

From the KKT condition for Problem (31) in the point \( w^* \) we have:

\[
g_J(z) + H_{J,J}(z)^\top (w^* - z_J) + \tau s - S \lambda = 0;
\]

(32)

where \( \lambda \in \mathbb{R}^{|J|} \) is the vector of multipliers with respect to the constraints \( Sw \geq 0 \).

We now analyze (32) for each index \( i \in J \). We distinguish two cases:
- $j_i \in J_D$. In this case we have that $s_i = \text{sign}(w_i^*)$ and $\lambda_i = 0$. Then, from (32) we have
\[ g_{j_i}(z) + H_{j_i,j_i}(z)(w_i^* - z_{j_i}) + \tau s_i = 0. \tag{33} \]
- $j_i \in J_E$. In this case we have that $s_i = \text{sign}(z_{j_i})$ and $\lambda_i \geq 0$. Therefore
\[ g_{j_i}(z) + H_{j_i,j_i}(z)(w_i^* - z_{j_i}) + \tau s_i \geq 0 \quad \text{if } s_i = \text{sign}(z_{j_i}) \geq 0, \]
and
\[ g_{j_i}(z) + H_{j_i,j_i}(z)(w_i^* - z_{j_i}) + \tau s_i \leq 0 \quad \text{if } s_i = \text{sign}(z_{j_i}) \leq 0. \]
The previous inequalities and the fact that $w_i^* = 0$ for all $j_i \in J_E$ imply that, whatever is the sign of $z_i$, we have
\[ (g_{j_i}(z) + H_{j_i,j_i}(z)(w_i^* - z_{j_i}) + \tau s_i)(w_i^* - z_{j_i}) \leq 0. \tag{34} \]
Taking into account (33) and (34), we have that
\[
\left( g_J(z) + H_{J,J}(z)^\top (w^* - z_J) + \tau s \right)^\top (w^* - z_J) \leq 0. \tag{35}
\]
Now, consider the difference between $\tilde{f}(w^*)$ and $\tilde{f}(z_J)$. We have that
\[
\tilde{f}(w^*) - \tilde{f}(z_J) = g_J(z)^\top (w^* - z_J) + \frac{1}{2}(w^* - z_J)^\top H_{J,J}(z)(w^* - z_J) + \tau \sum_{j_i \in J_E} \text{sign}(z_{j_i})(w_i^* - z_{j_i}) + \tau \sum_{j_i \in J_D} \text{sign}(w_i^*) (w_i^* - z_{j_i}),
\]
which can be rewritten as
\[
\tilde{f}(w^*) - \tilde{f}(z_J) = \left( g_J(z) + H_{J,J}(z)(w^* - z_J) + \tau s \right)^\top (w^* - z_J) - \frac{1}{2}(w^* - z_J)^\top H_{J,J}(z)(w^* - z_J).
\]
Recalling (35) we have
\[
\tilde{f}(w^*) - \tilde{f}(z_J) \leq -\frac{1}{2}(w^* - z_J)^\top H_{J,J}(z)(w^* - z_J) \leq -\frac{1}{2}\lambda_{\text{min}}(H_{J,J})\|y - z\|^2. \tag{36}
\]
Since
\[ q(y) = q(z) + g_J(z)^\top (y - z)J + \frac{1}{2}(y - z)^\top H_{J,J}(z)(y - z)J, \]
by definition of $\tilde{f}$ we have that
\[
f(y) = q(y) + \tau \sum_{j=1}^n \text{sign}(y_j) y_j = q(y) + \tau \sum_{j_i \in I \setminus J_D} \text{sign}(z_{j_i}) z_{j_i} + \tau \sum_{j_i \in J_D} \text{sign}(w_i^*) w_i^* = \tilde{f}(w^*) \tag{37}
\]
and
\[
\tilde{f}(z_J) = q(z) + \tau \sum_{j_i \in I \setminus J_D} \text{sign}(z_{j_i}) z_{j_i} + \tau \sum_{j_i \in J_D} \text{sign}(w_i^*) z_{j_i} \leq q(z) + \tau \|z\|_1 = f(z) \tag{38}
\]
Now (36), (37) and (38) prove the Proposition. \hfill \square

Finally we are ready to prove the main result concerning the global convergence of FAST-BCDA.
Theorem 2. Assume that the parameter \( \varepsilon \) appearing in Definition 3 satisfies the following condition

\[
0 < \varepsilon < \frac{1}{\lambda_{\text{max}}(A^T A)},
\]

(39)

and that the matrix \( A \in \mathbb{R}^{m \times n} \) satisfies the following condition

\[
\min_J \lambda_{\text{min}}(A^T A)_J \geq \sigma > 0,
\]

(40)

where \( J \) is any subset of \( \{1, \ldots, n\} \) such that \( |J| = r \).

Let \( \{x^k\} \) be the sequence produced by Algorithm FAST–BCDA.

Then, either an integer \( \bar{k} \geq 0 \) exists such that \( x^{\bar{k}} \) is an optimal solution for Problem (1), or the sequence \( \{x^k\} \) is infinite and every limit point \( x^\star \) of the sequence is an optimal point for Problem (1).

Proof. Let \( \{y^{h,k}\} \), with \( h = 0, \ldots, q \) be the sequence of points produced by Algorithm FAST–BCDA. By setting \( y = y^{0,k} \) and \( z = x^k \) in Proposition 2 we have:

\[
f(y^{0,k}) \leq f(x^k) - \frac{1}{2\varepsilon} \|y^{0,k} - x^k\|^2.
\]

(41)

By setting \( y = y^{h+1,k} \) and \( z = y^{h,k} \), for \( h = 0, \ldots, q - 1 \) in Proposition 3 we have:

\[
f(y^{h+1,k}) \leq f(y^{h,k}) - \frac{\sigma}{2} \|y^{h+1,k} - y^{h,k}\|^2.
\]

(42)

By using (11) and (12), we can write

\[
f(x^{k+1}) \leq f(y^{g-1,k}) \leq \cdots \leq f(y^{0,k}) \leq f(x^k),
\]

(43)

from which we have:

\[
x^k \in L^0 = \{x \in \mathbb{R}^n : f(x) \leq f(x^0)\}.
\]

From the coercivity of the objective function of Problem (1) we have that the level set \( L^0 \) is compact and, hence, the sequence \( \{x^k\} \) has at least a limit point and

\[
\lim_{k \to \infty} (f(x^{k+1}) - f(x^k)) = 0.
\]

(44)

Now let \( x^\star \) be any limit point of the sequence \( \{x^k\} \), let \( \{x^k\}_K \) be the subsequence such that

\[
\lim_{k \to \infty, k \in K} x^k = x^\star
\]

(45)

and let us assume, by contradiction, that \( x^\star \) is not an optimal point of Problem (1).

By recalling (11), (12) and (13) we have

\[
f(x^{k+1}) \leq f(y^{0,k}) \leq f(x^k) - \frac{1}{2\varepsilon} \|y^{0,k} - x^k\|^2,
\]

(46)

\[
f(x^{k+1}) \leq f(y^{h,k}) \leq f(x^k) - \frac{\sigma}{2} \|y^{h,k} - x^k\|^2.
\]

(47)
with $h = 1, \ldots, q$.

Now, (44), (45), (46) and (47) imply
\[
\lim_{k \to \infty, k \in K} y_h^{h,k} = x^*,
\]
for $h = 0, \ldots, q$.

For every index $j \in A_k$, we can define the point $\tilde{y}_j^{j,k}$ as follows:
\[
\tilde{y}_j^{j,k} = \begin{cases} 
0 & \text{if } i = j \\
x_k^j & \text{otherwise}
\end{cases}
\]
\[\text{(49)}\]

Recalling the definition of points $\tilde{y}_j^{j,k}$ and $y_0^{0,k}$, we have
\[
\| \tilde{y}^{j,k} - x^k \|^2 = (\tilde{y}^{j,k} - x^k)_j^2 = (y_0^{0,k} - x^k)_j^2 + \sum_{i \in A_k, i \neq j} (x_k^j)^2 = \| y_0^{0,k} - x^k \|^2.
\]
From the last inequality and (48) we obtain
\[
\lim_{k \to \infty, k \in K} \tilde{y}_j^{j,k} = x^*,
\]
for all $j \in A^k$.

To conclude the proof, we define a function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, that measures the violation of the optimality conditions for a variable $x_i$:
\[
\Phi_i(x) = \min \left\{ \max\{|g_i(x) - \tau|, \max(0, x_i)|, \max\{|g_i(x) + \tau|, \max(0, -x_i)\}, \max\{|x_i|, -(g_i(x) + \tau), (g_i(x) - \tau)\} \right\}
\]
Since, by contradiction, we assume that $x^*$ is not an optimal point there must exists an index $\hat{i}$ such that
\[
\Phi_i(x^*) > 0.
\]
Taking into account that the number of subsets of $\{1, \ldots, n\}$ is finite and therefore also the number of possible different choices of $A^k$ and $N^k$ is finite, we can find a subset $\hat{K} \subseteq K \subseteq \{1, 2, 3, \ldots \}$ such that $A^k = \hat{A}$ and $N^k = \hat{N}$ for all $k \in \hat{K}$.

We can have two different cases:

- $\hat{i} \in \hat{A}$ for $k$ sufficiently large. Then by Definition 2 we have for all $k \in \hat{K}$:
\[
\max\{0, x_i^k\} \leq \epsilon (g_i(x^k) + \tau)
\]
and
\[
\max\{0, -x_i^k\} \leq \epsilon (\tau - g_i(x^k))
\]
For all $k \in \hat{K}$, we can recall the point $\tilde{y}_i^{i,k}$ defined by (49). By construction we have that
\[
\tilde{y}_i^{i,k} = 0.
\]
Now we consider three different subcases:
i) \( x_i^k > 0 \). In this case, (49) and (52) imply
\[
(\hat{y}_i^{j,k} - x_i^k) \leq 0.
\]
Recalling (39) of Definition 3, there exists \( \rho \geq 0 \), such that
\[
\epsilon \leq \frac{1}{H_{ii}(x^k) + \rho}.
\]
Using again Definition 3 and \( \hat{i} \in A \), we can write
\[
\begin{align*}
    x_i^k & \leq \epsilon (g_i(x^k) + \tau) \\
    x_i^k - \tilde{y}_i^{i,k} & \leq \epsilon (g_i(x^k) + \tau) \\
    1 & \leq \frac{1}{H_{ii}(x^k) + \rho} (g_i(x^k) + \tau)
\end{align*}
\]
Then we have:
\[
(H_{ii}(x^k) + \rho)(x_i^k - \tilde{y}_i^{i,k}) \leq g_i(x^k) + \tau,
\]
which can be rewritten as follows
\[
g_i(x^k) + H_{ii}(x^k)(\tilde{y}_i^{i,k} - x_i^k) + \tau \geq \rho(x_i^k - \tilde{y}_i^{i,k}) \geq 0,
\]
that is
\[
g_i(\tilde{y}_i^{i,k}) + \tau \geq 0. \tag{53}
\]
On the other hand, since
\[
0 \leq \max\{0, -x_i^k\} \leq \epsilon (\tau - g_i(x^k))
\]
we have that
\[
g_i(x^k) - \tau \leq 0
\]
and, as \( H_{ii}(x^k) \geq 0 \), we get
\[
g_i(\tilde{y}_i^{i,k}) - \tau = g_i(x^k) + H_{ii}(x^k)(\tilde{y}_i^{i,k} - x_i^k) - \tau \leq 0. \tag{54}
\]
By (52), (53) and (54), we have that
\[
\Phi_i(\tilde{y}_i^{i,k}) = 0.
\]
Furthermore, by (51) and the continuity of \( \Phi \), we can write
\[
\Phi_i(x^*) = 0.
\]
Thus we get a contradiction with (51).

ii) \( x_i^k < 0 \). It is a verbatim repetition of the previous case.
iii) \( x_i^k = 0 \). Since \( \hat{i} \in \hat{A} \) we have
\[
g_{\hat{i}}(x^k) + \tau \geq 0 \\
-(g_{\hat{i}}(x^k) - \tau) \geq 0,
\]
which imply that
\[
\Phi_{\hat{i}}(x^k) = 0.
\]
Then, by the continuity of \( \Phi(\cdot) \) and the fact that
\[
\lim_{k \to \infty, k \in \hat{K}} x^k = x^*,
\]
we get a contradiction with (51).

• \( \hat{i} \in \hat{N} \) for \( k \) sufficiently large. We can choose a further subsequence \( \{x^k\}_{\hat{K}} \) with \( \hat{K} \subseteq \hat{K} \) such that
\[
\Phi_i(x^k) = \max_{i \in \hat{N}} \Phi_i(x^k), \quad \forall \ k \in \hat{K}.
\]
We then have that
\[
\Phi_i(x^k) \geq \Phi_\hat{i}(x^k) \geq 0, \quad \forall \ k \in \hat{K},
\]
which, by using (55) and the continuity of \( \Phi(\cdot) \), implies
\[
\Phi_i(x^*) \geq \Phi_\hat{i}(x^*) > 0.
\]
Furthermore, the instructions of Algorithm \textsc{Fast-BCDA} guarantee that, for all \( k \in \hat{K} \), a set of indices \( I_{\hat{h}_k} \) exists such that
\[
\hat{i} \in I_{\hat{h}_k} \subseteq \hat{N}_{ord}^k.
\]
Then, Algorithm \textsc{Fast-BCDA}, for all \( k \in \hat{K} \), produces a vector \( y^{\hat{h}_k,k} \) by minimizing Problem (1) with respect to all the variables whose indices belong to \( I_{\hat{h}_k} \). Therefore, the point \( y^{\hat{h}_k,k} \) satisfies
\[
\Phi_{\hat{i}}(y^{\hat{h}_k,k}) = 0.
\]
Furthermore, by (48) and the continuity of \( \Phi(\cdot) \), we can write
\[
\Phi_i(x^*) = 0,
\]
which contradicts (56).
\[ \square \]

In order to guarantee the convergence of the algorithm we need to properly set the parameter \( \epsilon \), so that condition (39) is satisfied. This condition requires the evaluation of \( \lambda_{\max}(A^T A) \), which is not always easily computable for large scale problems.

Hence, we introduce a variation of \textsc{Fast-BCDA}, namely \textsc{Fast-BCDA-\( \epsilon \)}, that includes an updating rule for the parameter \( \epsilon \), thus avoiding any “a priori” assumption on \( \epsilon \). We report below the scheme of \textsc{Fast-BCDA-\( \epsilon \)} (see Algorithm 2) and we prove its global convergence.
Algorithm 2 FAST-BCDA-ε

1. Given $\tilde{\epsilon}, \gamma > 0$ and $\theta \in (0, 1)$, Choose $x^0 \in \mathbb{R}^n$, Set $k = 0$.
2. For $k = 0, 1, \ldots$
3. Find the smallest index $h = 0, 1, \ldots$ such that
4. the value $\epsilon = \theta^h \tilde{\epsilon}$ and the corresponding sets $A^k, N^k$;
5. produce a point $y_{A^k}^{0,k} = 0$ and $y_{N^k}^{0,k} = x_{N^k}^k$ which satisfies
   \[ f(y_{A^k}^{0,k}) \leq f(x^k) - \gamma \| y_{A^k}^{0,k} - x^k \|^2; \]  \hspace{1cm} (57)
6. Compute $\tilde{N}^k_{ord}$;
7. For $j = 1, \ldots, q$
8. Compute $y_{I_j}^{j,k}$, with $I_j \subseteq \tilde{N}^k_{ord}$, solution of problem
   \[ \min_{w \in \mathbb{R}^r} g_{I_j}(y_{I_j}^{j-1,k})^T (w - y_{I_j}^{j-1,k}) + \frac{1}{2} (w - y_{I_j}^{j-1,k})^T H_{I_j,I_j} (w - y_{I_j}^{j-1,k}) + \tau \| w \|_1; \]
9. Set $y_{i}^{j,k} = y_{i}^{j-1,k}$ if $i \not\in I_j$;
10. End For
11. Set $x^{k+1} = y_{q,k}$;
12. End For

Theorem 3. Assume that the matrix $A \in \mathbb{R}^{m \times n}$ satisfies the following condition
   \[ \min_{J} \lambda_{\min}(A^T A)_{JJ} \geq \sigma > 0, \] \hspace{1cm} (58)

where $J$ is any subset of $\{1, \ldots, n\}$ such that $|J| = r$.

Let $\{x^k\}$ be the sequence produced by Algorithm FAST-BCDA-ε.

Then, either an integer $\bar{k} \geq 0$ exists such that $x^{\bar{k}}$ is an optimal solution for Problem \((\text{I})\),

or the sequence $\{x^k\}$ is infinite and every limit point $x^\star$ of the sequence is an optimal point for Problem \((\text{I})\).

Proof. First, we prove that an index $\bar{h}$ exists such that (57) holds.

Assume, by contradiction, that for all $h = 0, 1, \ldots$ the values $\epsilon = \theta^h \tilde{\epsilon}$ and the corresponding sets $A^k, N^k$ produce points $y_{A^k}^{h,k} = 0$ and $y_{N^k}^{h,k} = x_{N^k}^k$ for which
\[ f(y_{A^k}^{h,k}) > f(x^k) - \gamma \| y_{A^k}^{h,k} - x^k \|^2. \]

By Proposition 2 for $h$ sufficiently large, we have
\[ f(x^k) - \frac{1}{2 \theta^h \tilde{\epsilon}} \| y_{A^k}^{h,k} - x^k \|^2 \geq f(y_{A^k}^{h,k}) > f(x^k) - \gamma \| y_{A^k}^{h,k} - x^k \|^2, \]

and we get a contradiction with the fact that $\theta \in (0, 1)$.

The rest of the proof follows by repeating the same arguments of the proof of Theorem 2 by replacing the relation (41) with (57). \(\square\)
We finally would like to remark that the assumptions (40) and (58) we need to satisfy in order to guarantee convergence of FAST-BCDA and FAST-BCDA-\(\epsilon\) are not a big deal in practice if we consider blocks of 1 or 2 variables. Indeed, when solving blocks of 1 variable, we need to guarantee that any column \(A_j\) of matrix \(A\) is such that
\[
\|A_j\|^2 \geq \sigma > 0,
\]
which is often the case when dealing with overcomplete dictionaries for signal/image reconstruction (as the columns of matrix \(A\) are usually normalized, see e.g. [1]). When using 2-dimensional blocks, we want no parallel columns in the matrix \(A\). This is also a quite common requirement in the context of overcomplete dictionaries (as it corresponds to ask that mutual coherence is lower than 1, see e.g. [1]). Furthermore, the solution of 1-dimensional block subproblems can be determined in closed form by means of the well-known scalar soft-threshold function (see e.g. [3, 24]). Similarly, we can express in closed form also the solution of 2-dimensional block subproblems.

5 Numerical results

In this section, we report our numerical experience related to FAST-BCDA algorithm described in the previous sections. We developed two MATLAB implementations, FAST-1CDA and FAST-2CDA, where blocks of dimension 1 and 2 are respectively considered. We first analyze, in subsection 5.1 the behavior of the algorithms when varying the parameter \(\epsilon\) within the active set estimate. Then, a second experiment is conducted in subsection 5.2 where we describe and use an “accelerated” version of FAST-BCDA. Finally, we report, in subsection 5.3 the results related to the comparison of FAST-1CDA and FAST-2CDA, both in the original and in the accelerated versions, with two state-of-the-art algorithms for \(\ell_1\)-regularized least squares problems, namely FPC_AS [25] and SpaRSA [24]. All the tests were performed on an Intel Core i7 with 8GB of RAM using MATLAB R2011b.

We considered two different testing problems of the form (1), commonly used for software benchmarking (see e.g. [25, 14]). In particular, we generated artificial signals of dimension \(n = 2^{14}\), with a number of observations \(m = n/4\) and we set the number of nonzeros \(T = \text{round}(\rho m)\), with \(\rho = \{0.01, 0.02, 0.03\}\).

The two test problems (P1 and P2) differ in the way matrix \(A\) is generated.

P1: Considering \(\bar{A}\) as the Gaussian matrix whose elements are generated independently and identically distributed from the normal distribution \(\mathcal{N}(0, 1)\), the matrix \(A\) was generated by scaling the columns of \(\bar{A}\).

P2: Considering \(A\) as the matrix generated by using the MATLAB command
\[
A = \text{sprand}(m, n, \text{density}),
\]
with density = 0.5, the matrix \(A\) was generated by scaling the columns of \(\bar{A}\).

We would like to notice that the Hessian matrices \(A^\top A\) related to instances of problem P1 have most of the mass on the diagonal. Then, those instances are in general easier to solve than the
ones of problem P2.

Once the matrix $A$ was generated, the true signal $x^*$ was built as a vector with $T$ randomly placed $\pm 1$ spikes, with zero in the other components. Finally, for all problems, the vector of observations $b$ was chosen as $b = Ax^* + \eta$, where $\eta$ is a Gaussian white noise vector, with variance $10^{-3}$. We set $\tau = 0.1\|A^T b\|_\infty$ as in [2,24]. In all the experiments, we performed ten runs for each problem, for a total of 60 runs. The comparison of the overall computational effort is carried out by using the performance profiles proposed by Dolan and Moré in [11], plotting graphs in a logarithmic (base 2) scale. In order to better analyze all the computational aspects related to FAST-BCDA, we report below a detailed version of the algorithm (Algorithm 3). As we can see, thanks to the particular structure of Problem (1), by suitably updating the residual ($res$), we can efficiently calculate the full gradient or a subset of its components (see Step 3 and 8), and the sets $A^k$, $N^k$, $\bar{N}^k_{ord}$. Furthermore, as we consider blocks of 1 and 2 variables, the problem related to a single block (Step 9) has closed-form solution, thus being very easy to solve and not expensive in terms of CPU time. For the value of $s$ (number of non-active variable to be used in $\bar{N}_{ord}$) we set $s = \text{round}(0.8 T)$ for FAST-1CDA and $s = \text{round}(0.65 T)$ for FAST-2CDA (these $s$ values are the ones that guarantee the best performances among the ones we tried).

### Algorithm 3 Detailed Scheme of FAST-BCDA

1. Choose $x^0 \in \mathbb{R}^n$, Set $k = 0$, $r = Ax^0 - b$.
2. For $k = 0, 1, \ldots$
   3. Compute $g(x^k) = A^T r$;
   4. Compute $A^k$, $N^k$, $\bar{N}^k_{ord}$;
   5. Set $y^{0,k}_A = 0$ and $y^{0,k}_{N^k} = x^{k}_{N^k}$;
   6. Update $res = res + \sum_{i \in A^k} A_i (y^{0,k}_i - x^{k}_i)$;
   7. For $j = 1, \ldots, q$
      8. Compute $g_{I_j}(y^{j-1,k}) = A_{I_j}^T r$;
      9. Compute $y^{j,k}_{I_j}$, with $I_j \subseteq \bar{N}^k_{ord}$, solution of problem
         \[
         \min_{w \in \mathbb{R}^r} g_{I_j}(y^{j-1,k})^T (w - y^{j-1,k}_{I_j}) + \frac{1}{2} (w - y^{j-1,k}_{I_j})^T H_{I_j} I_j (w - y^{j-1,k}_{I_j}) + \tau \|w\|_1
         \]
     10. Set $y^{j,k}_i = y^{j-1,k}_i$ if $i \notin I_j$;
     11. Update $res = res + A_{I_j} (y^{j,k}_{I_j} - x^{k}_{I_j})$;
    12. End For
   13. Set $x^{k+1} = y^{q,k}$;
   14. End For

### 5.1 Choice of $\epsilon$ in the active set estimate

From Proposition[2] we have that the value of $\epsilon$, used within the estimate of the active set, should be less or equal than $\frac{1}{\lambda_{\text{max}}(A^T A)}$. Since the matrix $A$ we are considering is huge, $\lambda_{\text{max}}(A^T A)$ cannot be computed in practice, then a suitable strategy for updating the parameter $\epsilon$ is needed.
The easiest choice is that of setting $\epsilon$ to a fixed value. In order to understand the behavior of the algorithms with respect to various choices of $\epsilon$, we compared the performance with respect to the CPU time of FAST-1CDA and FAST-2CDA for $\epsilon = \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$. We report in Figure 1 the performance profiles of FAST-1CDA and FAST-2CDA for different choices of $\epsilon$. We notice that both algorithms report several failures when $\epsilon$ is set to $10^{-1}$ and $10^{-2}$. In particular with $\epsilon = 10^{-1}$ both algorithms failed on all the problems P2. On the other hand, we notice that there are no failures when setting $\epsilon = \{10^{-3}, 10^{-4}, 10^{-5}\}$ and that the behavior of each algorithm is very similar. Nevertheless, we notice that for FAST-1CDA, setting $\epsilon = 10^{-4}$, gives the best results. While setting $\epsilon = 10^{-5}$ guarantees slightly better results when dealing with FAST-2CDA. We further tested an implementation of both FAST-1CDA-$\epsilon$ and FAST-2CDA-$\epsilon$. Since there were no significant improvements in the performance, we decided to keep the $\epsilon$ value fixed. We set $\epsilon = 10^{-4}$ and $\epsilon = 10^{-5}$ for FAST-1CDA and FAST-2CDA respectively.

Figure 1: Performance profiles (CPU time) of FAST-1CDA (upper figures) and FAST-2CDA (lower figures) with different $\epsilon$. 
5.2 Accelerated version of FAST-BCDA

By running our codes, we noticed that the cardinality of the set related to the non-active variables decreases quickly as the iterations go by. In general, very few iterations are needed to obtain the real non-active set. As an example, we report, in Figure 2, a plot showing the cardinality of $N^k$ as the iterations $k$ of FAST-2CDA go by. The figure refers to an instance of problem P2: the cardinality of $N^k$ (red line) decreases quickly, thus obtaining the cardinality of the real non-active set $|N(x^*)|$ in a few iterations. By this evidence, and keeping in mind the theoretical result reported in Section 3, we decided to develop an “accelerated” version of our algorithms, taking inspiration by the second stage of FPC-AS algorithm [25]. Once a “good” estimate $N^k$ of $N(x^*)$ is obtained, we solve the following smooth optimization subproblem

$$
\min \frac{1}{2} \|Ax - b\|^2 + \tau \text{sign}(x_{N^k})^\top x_{N^k}
\text{s.t.} \quad x_i = 0 \quad i \in A^k.
$$

In practice, we consider an estimate $N^k$ “good” if both there are no changes in the cardinality of the set with respect to the last two iterations, and $|N^k|$ is lower or equal than a certain threshold $\xi$ (we fixed $\xi = 0.05n$ in our experiments).

We refer to FAST-1CDA-acc and FAST-2CDA-acc as the “accelerated” versions of FAST-1CDA and FAST-2CDA respectively.

First, we compare the four versions of our algorithm with respect to the number of iterations needed to find the target solution. In Figure 3, we report the performance profiles of FAST-1CDA (FAST1), FAST-2CDA (FAST2), FAST-1CDA-acc (FAST1-acc) and FAST-2CDA-acc (FAST2-acc). Since there is a big difference between the performance of the algorithms when dealing with P1 and P2 instances, we report, in three separated plots, the profiles related to all instances, to P1 instances and to P2 instances respectively. As we have already noticed, P1 instances are in general easier than P2 instances, and the number of iterations needed by all the algorithms is very similar. However, by taking a look at Figure 3 we can notice that, for P1 instances,
the original versions of our codes are better than the accelerated ones, and FAST-1CDA is the best among all versions of our algorithm. For what concerns P2 instances, we have a different scenario. Indeed, the accelerated version of FAST-2CDA is the best among the four algorithms. Furthermore, both FAST-1CDA-acc and FAST-2CDA-acc outperform the original ones. Similar comments can be given when considering the performances related to the CPU time. In order to better understand the differences between the algorithms, we analyze separately the algorithms using blocks of 1 and 2 variables. In Figure 4, we report the performance profiles related to the comparison between FAST-1CDA and FAST-1CDA-acc and the comparison between FAST-2CDA and FAST-2CDA-acc in terms of CPU time. By taking a look at Figure 4, we can notice that, for what concerns P1 instances, the original version of both algorithms is slightly faster, thus no speed up is obtained by minimizing in the subspace. This is probably due to the fact that the algorithms easily find a point very close to the solution in a few iterations, then the minimization in the subspace, which can be more costly in terms of CPU time than the block minimizations executed at each iteration, does not always give a significant improvement in terms of objective function value. On the other hand, when considering the P2 instances, we can notice that the accelerated versions clearly outperform the original ones. Finally, if we consider the overall performance profiles, we can conclude that the accelerated versions of the two algorithms we proposed usually give a good speed up.

5.3 Comparison with other algorithms

In this section, we report the numerical experience related to the comparison of FAST-1CDA (FAST-1CDA-acc) and FAST-2CDA (FAST-2CDA-acc) with FPCAS [25] and SpaRSA [24]. As we have already commented in subsection 5.1, we set $\epsilon = 10^{-4}$ and $\epsilon = 10^{-5}$ in the active set estimate for FAST-1CDA and FAST-2CDA algorithms respectively.

In our tests, we first ran FAST-2CDA to obtain a target objective function value, then ran the other algorithms until each of them reached the given target (see e.g. [24]). Any run exceeding
the limit of 1000 iterations is considered failure. In running SpaRSA and FPC-AS codes, default values were used for all parameters. In all codes, we considered the null vector as starting point and all matrices were stored explicitly.

In Figure 5, we report the performance profiles with respect to the CPU time for \texttt{FAST-1CDA}, \texttt{FAST-2CDA}, FPC-AS \cite{25} and SpaRSA \cite{24} algorithms. As we have already done in the previous figures, we report, in three separated plots, the profiles related to all instances, to P1 instances and to P2 instances respectively. We notice that both \texttt{FAST-1CDA} and \texttt{FAST-2CDA} outperform the other two algorithms and, in particular, \texttt{FAST-2CDA} is the best when considering the P2 instances. This seems to suggest that the use of 2-coordinates blocks, does give some speed up when the mass of the Hessian matrix is spread over its elements. By taking a look at the overall performance profiles, we further notice that \texttt{FAST-2CDA} is the best both in terms of efficiency and robustness.

In Figure 6, we report the performance profiles with respect to the CPU time for \texttt{FAST-1CDA-acc}, \texttt{FAST-2CDA-acc}, FPC-AS \cite{25} and SpaRSA \cite{24} algorithms. Again we report, in three separated plots, the profiles related to all instances, to P1 instances and to P2 instances respectively. We
can notice that both FAST-1CDA-acc and FAST-2CDA-acc outperform the other two algorithms and, that in particular, FAST-1CDA-acc is the best for P1 instances, while FAST-2CDA-acc is the best for P2 instances. By taking a look at the overall performance profiles, we also notice that FAST-1CDA-acc is slightly better than FAST-2CDA-acc in terms of efficiency but is less robust.

We report in Table 1 the average relative errors of the computed solutions with respect to the optimal solution. In case of failure, we consider the relative error related to the last point found by the algorithm. As we can easily see, for P1 instances, all the algorithms reach the same solutions, while, for the P2 instances, FAST-BCD Algorithms usually reach solutions with a higher precision.

We further report in Table 2 the average number on nonzero components of the computed
solutions. In case of failure, we consider the nonzero components related to the last point found by the algorithm. We can notice that all algorithms, for P1 instances, find the true number of nonzero components. This does not happen when solving P2 instances. In some case, we see that FAST-BCD Algorithms underestimate the true number of nonzero components, but guarantee overall better results than the others anyway.

|                | P1 instances | P2 instances |
|----------------|--------------|--------------|
|                | $\rho = 0.01$ | $\rho = 0.02$ | $\rho = 0.03$ | $\rho = 0.01$ | $\rho = 0.02$ | $\rho = 0.03$ |
| SpaRSA         | 0.0149       | 0.0193       | 0.0232       | 0.0641       | 0.5438       | 1.2574       |
| FPC-AS         | 0.0149       | 0.0194       | 0.0232       | 0.0654       | 0.2193       | 0.3924       |
| Fast-1CDA      | 0.0149       | 0.0194       | 0.0232       | 0.0640       | 0.2051       | 0.4049       |
| Fast-2CDA      | 0.0149       | 0.0194       | 0.0232       | 0.0640       | 0.2052       | 0.4050       |
| Fast-1CDA-acc  | 0.0149       | 0.0194       | 0.0232       | 0.0639       | 0.2053       | 0.4060       |
| Fast-2CDA-acc  | 0.0149       | 0.0194       | 0.0232       | 0.0639       | 0.2111       | 0.4049       |

Table 1: Average relative errors.

|                | $T = \text{round}(\rho m)$ |
|----------------|-----------------------------|
|                | 41  | 82  | 123 |
| SpaRSA         | 41  | 82  | 123 |
| FPC-AS         | 41  | 82  | 123 |
| Fast-1CDA      | 41  | 82  | 123 |
| Fast-2CDA      | 41  | 82  | 123 |
| Fast-1CDA-acc  | 41  | 82  | 123 |
| Fast-2CDA-acc  | 41  | 82  | 123 |

Table 2: Average number of non-zero components.

6 Conclusions

In this paper we devised an active set-block coordinate descent method (FAST-BCDA) for solving $\ell_1$-regularized least squares problems. The novelty of the method is in the identification, at each iteration, of a subset of the non-active variables to be analyzed. This selection strategy, let us ensure a sufficient decrease in the objective function at each iteration without the use of any linesearch technique. Global convergence of the method is established. Numerical results are presented to verify the practical efficiency of the method and they seem to indicate that FAST-BCDA compares favorably with other state-of-the-art techniques. We further would like to remark that the proposed active set strategy is independent from the specific algorithm we have designed and can be easily included into other algorithms, both sequential and parallel, for $\ell_1$-regularized least squares, to improve their performance. We also highlight that the algorithmic scheme we described can be easily modified in order to work in a parallel fashion. Future work
will be dedicated to adapt the presented approach to handle convex $\ell_1$-regularized problems.

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