Light bullets in coupled nonlinear Schrödinger equations with variable coefficients and a trapping potential

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Abstract: We analyze three-dimensional (3D) vector solitary waves in a system of coupled nonlinear Schrödinger equations with spatially modulated diffraction and nonlinearity, under action of a composite self-consistent trapping potential. Exact vector solitary waves, or light bullets (LBs), are found using the self-similarity method. The stability of vortex 3D LB pairs is examined by direct numerical simulations; the results show that only low-order vortex soliton pairs with the mode parameter values \( n \leq 1, l \leq 1 \) and \( m = 0 \) can be supported by the spatially modulated interaction in the composite trap. Higher-order LBs are found unstable over prolonged distances.

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1. Introduction

Three-dimensional optical spatiotemporal solitary waves, also known as light bullets (LBs), are self-sustained wave packets that are localized in both spatial dimensions and time [1–4]. They may form in dielectric media in which diffraction and dispersion are balanced by medium’s nonlinearity. The generation of LBs is a nontrivial task from the analytical and numerical points of view, and even more complicated in real experimental settings [5,6].

It is well known that optical solitons appear in the nonlinear Schrödinger equation (NLSE) with cubic self-focusing optical nonlinearity, which models a wealth of phenomena in nonlinear science in general and physics in particular. Owing to the occurrence of optical beam collapse, optical solitons are unstable in two and three dimensions (2D, 3D) [7,8]. However, in the past decades, the stability of solitons in 2D and 3D has been improved, using different media and methods. One typically uses quadratic and cubic nonlinear optical media that support stable optical solitons regardless the physical dimension in which they form and propagate [8,9]. In 1999, a seminal experimental study was reported by the group of Wise [10], in which, using the achromatic phase matching technique and generating the necessary anomalous GVD, robust (2 + 1)D nonlinear LBs were formed in quadratic nonlinear crystals. We also mention several other physical settings which are adequate for getting robust LBs. These include (a) either saturable [11,12] or nonlocal [13,14] optical materials, (b) optical media with competing quadratic, cubic or cubic-quintic nonlinearities [15,16], (c) confining by two- or three-dimensional optical lattices [17–20], and (d) periodic waveguide structures with controlled diffraction and/or GVD [21–23].

Spatial solitons can be generally divided into scalar solitons (single-component) and vector solitons (multicomponent), according to the number of field components and the nature of the medium [24]. Here, we are interested in the vector type, arising in (3 + 1) dimensions in a Kerr medium with variable coefficients and a trapping potential. It is well known that vector solitons commonly form in the absence of interference between components. In general, there exist many ways to generate vector solitons, such as cross-phase modulation (XPM) [25] or considering beams of different wavelengths [26] and using mutually incoherent beams which consist of two-component azimuthons [27]. In nonlinear optics, the XPM-mediated interaction between mutually incoherent or orthogonally polarized waves leads to the formation of bound states that are also vector solitons.

A field in which spatiotemporal solitons appear regularly is the Bose-Einstein condensation [28]. Recently, a theoretical analysis and numerical study of quantized vortices in a rotating Bose-Einstein condensate (BEC) with spatiotemporally modulated interaction in a harmonic potential, was reported in [29]. In [30], 2D vector matter waves in the form of soliton-vortex and vortex-vortex pairs have been investigated for the case of attractive intra-component interaction in two-component BECs. Further, in two-component BECs, the superflow of atomic spinor BECs optically trapped in a ring-shaped geometry were considered in [31,32]. In our previous studies, 3D approximate but still analytical spatiotemporal vector LBs were built with the help of spherical harmonics, including multipole solutions and necklace rings [33]. In [34], we constructed exact self-similar soliton solutions of 3D coupled Gross-Pitaevskii equations for two-species BECs in a combined time-dependent harmonic-lattice potential. We have investigated the control and manipulation of solitary waves for three kinds of BECs with changing diffraction and nonlinearity coefficients; the solutions include Ma breathers, and Peregrine and Akhmediev solitons [34].

However, thus far no exact 3D spatial vector LBs in two-component NLSE with spatially modulated coefficients and a trapping potential have been found, either theoretically or experimentally. Here, we extend the analysis performed in [33–36], to demonstrate such exact solutions of the coupled NLSE in (3 + 1)D with varying coefficients and a self-consistent trapping potential.

The paper is organized as follows. The coupled (3 + 1)D NLSE with the modulated coefficients and a composite trapping potential are analyzed in Sec. II, where the nonlinearity
coefficients are allowed to vary in space. Also presented in Sec. II are the vortex-vortex LB pairs, obtained using the self-similarity method. The stability of these vortex LB pairs is discussed in Sec. III. Characteristic distributions of LBs are provided in Sec. IV. The concluding remarks with a simple summary are given in Sec. V.

2. The model and exact soliton solutions

In this paper we consider vector solitons consisting of \( N \) mutually incoherent optical components in a nonlinear Kerr medium. The propagation of the slowly-varying field components \( u_j(r, \varphi, \theta, z) \) (\( j=1,2,3\ldots N \)) is described by a system of coupled dimensionless \((3+1)D\) NLSEs with changing coefficients, under the influence of an optical trapping potential [37, 38]:

\[
\frac{1}{i} \frac{\partial u_j}{\partial z} + \frac{1}{2} \nabla^2 u_j + \chi'(r) |u_j|^2 V(r) u_j = 0,
\]

where \( \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left( \sin \theta \frac{\partial}{\partial \varphi} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \) is the 3D Laplacian in spherical coordinates, \( r = \sqrt{x^2 + y^2 + \tau^2} \) is the spatiotemporal radius, and \( \varphi \) is the azimuthal coordinate in the transverse \((x, y)\) plane. The evolution variable is the propagation distance \( z \), not time. The retarded time \( \tau \) is the third ‘transverse’ coordinate in this coordinate system.

Since \( \cos(\theta) = \frac{\tau}{r} \), the colatitude angle \( \theta \) is also spatiotemporal in nature. The self-consistent composite trapping potential \( V \) is specified below. The symbol \( I \) in coupled Eq. (1) represents the total intensity, \( I = \sum |u_j|^2 \), thus the nonlinearity is of the Kerr type.

We assume the solutions of Eq. (1) in a form that separates the angular variables from the spatial ones \( Y_j(\theta, \varphi) \). When the self-consistency condition \( \sum_{j=1}^N |Y_j(\theta, \varphi)|^2 = 1 \) is imposed, substituting \( u_j \) into (1) leads to:

\[
\frac{1}{Y_j} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) Y_j + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_j}{\partial \varphi^2} \right] = \frac{l(l+1)}{2}\psi \left[ \frac{\partial \psi}{\partial z} + \frac{1}{2} \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{r \partial r^2} \right] + \chi(r) |\psi|^2 \psi + V \psi = \frac{l(l+1)}{2},
\]

where \( l \) is the separation constant, cast in a convenient form. As a consequence of the self-consistency condition, the total intensity \( I = I(r, z) \) should be radially symmetric at any \( z \); hence, in our search for vector solutions of Eq. (1) we are restricted to the solutions that maintain this property.

Equation (2) allows solutions in the form of spherical harmonics \( Y_n = K \Phi_n(\varphi) P_l^m(\cos \theta) \), where \( K = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \) and \( P_l^m(\cos \theta) \) are the associated Legendre polynomials with \( l \geq m \geq 0 \). The azimuthal part of \( Y_n \) is of the form \( \Phi_n(\varphi) = a_n \cos(m\varphi) + b_n \sin(m\varphi) \) where \( m \) is the topological charge (TC) and \( a_n \) and \( b_n \) are the coefficients satisfying the conditions specified below [35, 36]; index \( n \) here enumerates the number of components. In this manner, the angular part of the solutions to Eq. (1) carries two indices, \( l \) and \( m \). For a fixed \( l \) the index
\( m \) takes the values 0, 1, \ldots, \( l - 1, l \). We chose the simplest case of 3D vector solitons, with two components, \( N = 2 \). The complex coefficients \( a_1, a_2 \) and \( b_1, b_2 \) are calculated from Eqs. (4) and (5) below, and satisfy the following relations [38,39]:

\[
a_1 = (1 + q^2)^{1/2}, \quad b_1 = iqa_1, \tag{4}
\]

\[
a_2 = qa_1, \quad b_2 = \pm ia_1, \tag{5}
\]

where the parameter \( q \in [0, 1] \) determines the modulation depth of the beam [29]. Hence, the coefficients \( a_n \) and \( b_n \) depend on \( q \) only.

Our aim in finding analytical solutions of Eq. (3) is to connect them with the stationary NLSE

\[
\frac{1}{2} \frac{dU^2}{dR^2} + EU + \chi_0 U^3 = 0 \tag{6}
\]

where \( U = U(R) \) depends only on \( R = R(z, r) \), both of which are real functions. Here, \( E \) is the eigenvalue of the NLS equation, and \( \chi_0 \) the nonlinearity coefficient. This equation has many known solutions; for example, when \( \chi_0 = 1 \) and \( E < 0 \), the bright soliton solution of Eq. (6) is found,

\[
U(R) = \sqrt{-2E} \sec h(\sqrt{-2E} R). \tag{7}
\]

To connect solutions of Eq. (3) with those of Eq. (6), one introduces \( \psi = \rho U[R(r)]e^{i\phi} \), where the amplitude \( \rho(z, r) \) and the phase \( \phi(z, r) \) are real functions of \( z \) and \( r \) [27]. Combining with Eq. (6), a closed system of equations is obtained. By analyzing these equations, one finds:

\[
\chi_0 = -\frac{\chi}{2r^4 \rho^6}, \quad \frac{\partial R}{\partial z} + \frac{\partial R}{\partial r} = 0 \tag{8}
\]

\[
\frac{\partial \rho}{\partial z} + \frac{1}{2} \left( \rho \frac{\partial^2 \phi}{\partial r^2} + 2 \frac{\partial \rho}{\partial r} \frac{\partial \phi}{\partial r} + 2 \frac{\partial \phi}{\partial r} \frac{\partial \phi}{\partial r} \right) = 0, \tag{9}
\]

\[
-\rho \frac{\partial \phi}{\partial z} + \frac{1}{2} \left[ \frac{\partial^2 \rho}{\partial r^2} - \rho \left( \frac{\partial \phi}{\partial r} \right)^2 + 2 \frac{\partial \rho}{\partial r} \frac{\partial \phi}{\partial r} - \rho \frac{m^2}{r^2} \right] + V \rho = 0, \tag{10}
\]

\[
2 \frac{\partial \rho}{\partial r} \frac{\partial R}{\partial r} + \rho \frac{\partial^2 R}{\partial r^2} + 2 \frac{\partial \rho}{\partial r} \frac{\partial R}{\partial r} = 0, \tag{11}
\]

To find exact solutions of Eqs. (8)-(11), we introduce a self-similarity transformation [33,35,40] \( \rho(z, r) = \kappa F(\theta)w^{1/2}(z), \quad \phi(z, r) = a(z)r^2 + b(z) \). Here, \( \kappa \) is the normalization constant, \( w(z) \) is the beam width, \( \theta(z, r) \) is the similarity variable to be determined, and \( a(z) \) and \( b(z) \) are the wave front curvature and the phase offset, respectively. These variables vary with the distance \( z \). Substituting the presumed solutions for the amplitude and the phase into Eqs. (8)-(11), one obtains \( \theta(z, r) = r^2/w^2(z), a(z) = -\frac{dw}{2w dz} \),

\[
R(z, r) = \int_0^{r^2/w^2} \frac{1}{\tau^2 F^2(\tau)} d\tau, \quad \text{and} \quad \chi(z, r) = -2\frac{\dot{\chi}_0}{\rho^2} \left( \frac{\partial R}{\partial r} \right)^2. \tag{12}
\]

Thus, the nonlinearity coefficient is specified in terms of the beam characteristics.
Further, we introduce a variable transformation $F(\theta) = \frac{m}{\theta^2} e^{-\frac{\eta}{\theta^2}} f(\theta)$, so that Eq. (10) transforms into:

$$\frac{d^2 f}{d\theta^2} + \left( m + \frac{3}{2} - \theta \right) \frac{df}{d\theta} + nf = 0,$$

(12)

with $-\frac{w^2}{2} \frac{db}{dz} - \frac{m}{2} = n$, $-\frac{w}{2} \frac{dw^2}{dz} - \eta w^2 + \frac{1}{2} w^2 = 0$, $V = E \left( \frac{dR}{dr} \right)^2 + \eta r^2 - \frac{m^2}{2r}$. Here, $\eta$ is constant and $n$ is assumed to be a non-negative integer, which can be considered as the radial mode number. Finally, the trapping potential $V$ is defined more closely; it is composed of a harmonic and an inverse harmonic term, in addition to a term depending on the transformed radial function $R$. Thus, $V$ is determined self-consistently with the solution presumed. Differential Eq. (12) [41] is the well-known Kummer confluent hypergeometric function, whose solutions are known as Kummer’s functions, namely

$$f(\theta) = \sum_{k=0}^{n} \frac{(-1)^k (2m+2n+1)!2^k}{(m+n)!(2k+2n+1)!} \theta^k.$$  

To find simple soliton solutions, we take $w(z)|_{z=0} = w_0$ and $\frac{dw(z)}{dz}|_{z=0} = 0$, where the subscript '0' denotes the value of the corresponding quantity at $z = 0$. Hence, the exact soliton solution is obtained with $w = w_0$, when the optical diffraction is exactly balanced by the nonlinearity. Other parameters in this case are: $a(z) = 0$, and $b(z) = b_0 - (2n + m + 1) \frac{z}{w_0}$. Plugging these results into Eq. (2), one obtains the following analytical solution for the beam components:

$$u_i = \frac{K}{w_0^\frac{3\delta}{2}} \left( \frac{r}{w_0} \right)^m \left( a_n \cos m\phi + b_n \sin m\phi \right) P_n^m(\cos \phi) f(\theta) U(R)e^{-\frac{r^2}{2w_0^2} + \left( b_0 - \frac{2n+m+1}{w_0^2} \right) \frac{z}{w_0^2}}. (13)$$

In Eq. (1), the nonlinearity coefficient and the external potential are given in terms of the radial solution. This requires a self-consistent solution procedure, in which the conditions put on the coefficients and the potential can be considered as integrability constraints. It is an inverted solution procedure, in which one first finds the solution of the equation and then defines the coefficients and the potential in the equation in terms of the solution itself. Thus, it is a self-consistent procedure which offers the convenience of localized solutions with desirable features of equations with changing coefficients and potentials that are related to the solutions self-consistently. The procedure might lead to interesting models with realistic space-varying coefficients and potentials. While one may argue about the availability of such coefficients and potentials, we still believe that there is enough freedom in the solution method, to provide interesting solutions to viable material models.

3. Stability of the localized nonlinear vector LBs

Next, we discuss the linear stability of solutions of Eq. (1) [35]. The perturbations in the solutions are assumed of the form [42]:

$$u_i(x, y, r, z) = e^{-\lambda z} \left[ u_{0i}(x, y, r) + u_i(x, y, r)e^{i\delta z} + v_i(x, y, r)e^{-i\delta z} \right],$$

(14)

where $u_{0i}(x, y, r) = \psi_i(r, \theta)$ is the steady-state component solution of Eq. (1), $\lambda = (2n + m + 1) / w_0^2$ is the propagation constant, $\mu_i(x, y, r)$ and $\nu_i(x, y, r)$ are the real and imaginary parts of the perturbation, and $\delta$ stands for the perturbation growth rate. The
superscript * denotes the complex conjugate quantity. Substituting the perturbed solution 
\( u(x, y, \tau, z) \) into Eq. (1) and linearizing around the steady-state solution, one finds the 
following eigenvalue equations:

\[
\begin{pmatrix}
L_j & B & A & A \\
-B & -L_j & -A & -A \\
A & A & L_{1-j} & C \\
-A & -A & -C & -L_{3-j}
\end{pmatrix}
\begin{pmatrix}
\mu_1 \\
v_1 \\
\mu_2 \\
v_2
\end{pmatrix}
= \delta
\begin{pmatrix}
\mu_1 \\
v_1 \\
\mu_2 \\
v_2
\end{pmatrix},
\] (15)

where 
\( L_j = \frac{1}{2}(\partial_{xx} + \partial_{yy} + \partial_{zz}) - 2\chi(u_{oj}^2 + u_{oj_3}^2) - V + \lambda \), 
\( A = -\chi u_{w0} u_{w2} \), 
\( B = -\chi u_{o1}^2 \), 
\( C = -\chi u_{o2}^2 \).

The third variable \( z \) in the Laplacian, in this case is \( \tau \). This eigenvalue problem is computed 
by the Fourier Collocation Method for discrete eigenvalues, which allows the whole spectrum 
to be calculated at once [43]. Once parameters \( m, n, l, q \) and \( \eta \) are specified, one gets the 
imaginary and real parts of \( \delta \) from Eqs. (13) and (14). If imaginary parts of all eigenvalues \( \delta \) 
are equal to zero or are positive, the soliton solutions can be stable; otherwise, the perturbed 
solution would grow exponentially with \( z \), and thus, the corresponding matter waves would 
become linearly unstable [35,43].

4. Characteristic distributions of LBs

From Eq. (13), it is evident that the matter wave solitons are characterized by five parameters: 
the mode numbers \( n, m, l \), the harmonic potential width \( \eta \), and the modulation depth \( q \). In this 
section, we present typical distributions of LBs, for some values of these parameters. When 
\( w = w_0 \), the auxiliary function \( R \), the nonlinearity coefficient \( \chi \), and the trapping potential \( V \) 
are the functions of the radial variable \( r \) only.

![Fig. 1. (a), (b) Distributions of the nonlinearity coefficient \( \chi(r) \) and the trapping potential 
\( V(r) \) for \( n = 1 \). (c) and (d) Amplitude of the radial field distributions \( \psi_j \) for different mode 
numbers, \( m = 0, 1, 2 \) (c) and \( n = 2, 3, 4 \) (d). Other parameters: \( q = 0 \) and \( \eta = 0.4 \).](image)

Typical distributions of the nonlinearity coefficients \( \chi(r) \) for different \( m \) and of the 
potential \( V(r) \) for different \( E \) are shown in Figs. 1 (a) and 1(b). When \( m = 0 \), the nonlinearity
coefficient $\chi(r)$ is localized in the form of asymmetric double annuluses; when $m = 1$, localized three amplitude peaks can be seen in Fig. 1(a). In form, the external potential $V(r)$ is similar to the 2D pattern in [35]. Figures 1(c) and 1(d) illustrate the amplitude of the radial field distributions $\psi_j$ for different mode numbers $m = 0, 1, 2$ (c) and $n = 1, 2, 3$ (d). The optical field of LBs approaches 0 when $r = 0$ and $r = \infty$, forming different vortex structures. The ring size depends on the mode number $n$. The results show that the solutions in Eq. (13) indeed are localized.

Figure 2 displays the intensity distributions of 3D LBs with $n = m = 1$, $\eta = 0.4$, and $q = 0.1$, for different $l$ ($l = 1, 2, 3$) from top to bottom, at $z = 70$. A variety of intensity distributions for solitons with different $l$ are shown. For $l = 1$ there are two layers (two pulses) along the vertical $\tau$-axis, and the soliton is shaped as a pair of deformed ellipsoids. In general, by increasing $l$, the solution forms $l + 1$ layers in the vertical direction, forming multipole multipulse solitons. However, the total intensity distribution (the right column in Fig. 2) is radially symmetric. Obviously, when both parameters ($l$ and $m$) are 0, the soliton is the fundamental spherical LB. The physical origin of this arrangement can qualitatively be understood from the nature of the self-focusing cubic nonlinearity as a third-order susceptibility $\chi^{(3)}$; it yields the nonlinearity polarization of the medium. Owing to the assumption of strong dispersion along the vertical $\tau$ axis, resulting in spherical harmonics, the associated Legendre polynomial $P_1^0$ will appear in the solution; it possesses a maximum value (plus one) and a minimum value (minus one). Therefore, the spherical harmonics will force a change of the sign near the zero point.

Figure 3 shows some further examples of vector vortex LBs with $l = 3$, $q = 0.1$, and for different $m = 0, 2, 3$ from top to bottom, also at $z = 70$. The first row just depicts the same three figures, because $m = 0$. For given $l$, the larger the $m$, the larger the LB radius in the horizontal plane. Looking at the field components, the number of solitary beads in the same layer is $2m$; it comes from the azimuthal angular dependence in Eq. (13). Because $q \neq 1$ and $m \neq 0$, there are $2m$ maxima and minima in the range $0 \leq m\varphi \leq 2\pi$. In this sense, the ring-necklace components are produced steadily as $m$ increases. The larger the $m$, the shorter the
pulse in the vertical direction. The total vector intensity is still radially symmetric, as it must be.

![Fig. 3. Same as Fig. 2 but for different \(m (m = 0, 2, 3)\), from top to bottom at \(z = 70\). The parameters are same as in Fig. 1, but for \(l = 3\), from left to right.](image)

Fig. 3 displays the vortex-shaped distributions of the LBs for \(m = l = 2, q = 0.5\) and \(n = 1, 2, 3\) from left to right. It is seen that the profiles of LBs possess polycyclic structure with several amplitude peaks covering the same ring. LBs are self-similar and composed of four symmetrical petals in each ring (the bottom row in Fig. 4). The number of outer bright rings is \(n + 1\), the optical intensity in the center is zero, and the intensity of rings surrounding the center decreases with the increasing radial distance.

Increasing the modulation depth \(q\) from 0 to 1, with fixed \(n, m,\) and \(l\), will lead to less azimuthally modulated vortex rings (Fig. 5). One can observe that the intensity and the phase of the beam is modulated by the modulation depth \(q\). When \(q = 0\), for each component we find a necklace LB, with the number of beads equal to \(2m\). By increasing \(q\), the distance
between petals decreases, and the multi-TC vortices change into vortex rings. It is apparent that the soliton has just one layer in the vertical direction.

Fig. 5. Same as Fig. 4 but for \( m = n = 3 \), and \( q = 0, 0.5, 1 \) from left to right.

Fig. 6. Evolution of 3D LB distributions and linear stability spectra of the condensate \( \psi \) against the perturbation with an initial random noise of level 10%, at different propagation distances. The first column \( z = 0 \), the second and the third columns \( z = 600 \). The fourth column presents the linear stability spectra. Top: \( m = 0, n = 1 \); middle: \( m = n = 1 \); bottom: \( m = 1, n = 2 \). Other parameters are the same as in Fig. 1.

The stability of the obtained exact solutions is a very important aspect of the present discussion, which has to be addressed numerically. In order to check the stability of Lbs, we perform direct numerical simulations, using the split-step Fourier method [44] and solve Eq. (1) by taking the analytical solution (13) at \( z = 0 \) as an initial condition. In Fig. 6, we present the comparison of analytical (the first column) and numerical (the second and the third columns) intensity contour plots. The linear-stability spectra are displayed in the fourth column of Fig. 6. It is found that only when the topological charge is \( m = 0,1 \) and \( n = l = 1 \), the
numerical solutions of LBs are stable against perturbation with an initial Gaussian noise level of up to 10%. But when the topological charge is \( m > 1 \), and mode numbers \( n \) and \( l \) are greater than one, the soliton solution (13) is unstable in propagation and splits into spreading quasi square-shaped band structures. Further, a quadruple complex eigenvalues \( \delta \) are visible in Figs. 6(h) and 6(i), thus these solitons might be linearly unstable.

5. Conclusion

In summary, we have studied the propagation of LBs in coupled nonlinear Schrödinger equations with spatially modulated coefficients and a self-consistent trapping potential. Using the self-similarity method, exact vector solitary waves, or light bullets, are obtained. The stability of vortex 3D LB pairs is examined by direct numerical simulation. Our results show that the only stable low-order vortex LB pairs with \( n \leq 1, l \leq 1 \) and \( m = 0 \) can be supported by the spatially modulated interaction in the composite trap. Higher-order LBs are found unstable over prolonged distances. Our approach may prove useful for other related problems, e.g., light propagation in Bose-Einstein condensates and plasmas.

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