Particle Creation in Bose-Einstein Condensates: Numerical Analysis of the Bogoliubov-de Gennes Equation for Trapped Ultracold Atoms

M. Kobayashi, Y. Kurita, T. Morinari, M. Tsubota, and H. Ishihara

Department of Physics, The University of Tokyo, Tokyo 113-0033, JAPAN
Department of Physics, Kwansei Gakuin University, Hyogo 669-1337, JAPAN
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, JAPAN
Department of Mathematics and Physics, Osaka City University, Osaka 558-8585, JAPAN

E-mail: michikaz@cat.phys.s.u-tokyo.ac.jp

Abstract. In this paper, we formulate quantum field theory on an analogue curved spacetime in a Bose-Einstein condensate where quantum particles on curved spacetime are equated with Bogoliubov quasi-particle excitations on the Bose-Einstein condensate. By using our formulation, we numerically investigate a particle creation in analogue expanding Universe which can be expressed as Bogoliubov quasiparticles in an expanding Bose-Einstein condensate and obtain its spectrum which shows the thermal Maxwell-Boltzmann distribution, the temperature of which can be experimentally accessible. This is the first direct numerical simulation for the particle creation and its spectrum on analogue curved spacetime by using the system of Bose-Einstein condensates.

1. Introduction

Since the establishment of general relativity by A. Einstein, the study of curved spacetime and cosmology has shown its great development. When noticing quantum effects on such a curved spacetime, however, we face the great mystery and it is very active research field. One of the most important subjects of such quantum effects is quantum fluctuation and particle creation on curved spacetime in the inflationary Universe. However, this is just a theoretical scenario for explaining the origin of the cosmological structure without experimental evidence. Another example is the very mysterious objects called black holes. According to S. W. Hawking’s prediction, black holes are not completely black but have black body radiation through particle creations in curved spacetime around them [1]. This surprising phenomenon also has great difficulty to be confirmed experimentally because of the extreme weakness of the radiation.

With such a background, new attempts to look for the analogous phenomena in fluid systems have appeared [2]. The basic idea is to relate velocity field and sound waves in moving fluid with curved spacetime and scalar fields in it. This analogy has been applied for many other theoretical phenomena on curved spacetime and tested in some condensed matter systems [3].

When considering such an analogy, we are interested not only in the structure of curved spacetime but also in its quantum effects. For the purpose of investigating quantum effects exactly, we should consider quantum fluids. Bose-Einstein condensates (BEC) in trapped cold
atoms are one of the best system in this context, because microscopic theory can be easily studied, which is different from other quantum fluids like superfluid \(^4\)He [4]. In BEC, analogue spacetime and quanta on that should correspond to nonuniform velocity field of the condensate and its low energy Bogoliubov quasiparticle excitations. Basic equation are known to be the Gross-Pitaevskii (GP) equation and Bogoliubov-de Gennes (BdG) equations.

In this paper, we formulate the spacetime analogy in terms of the BdG equations by explicitly identifying the Bogoliubov quasiparticle with quantum particles on curved spacetime. After that, we perform a numerical simulation for the simplest quantum effect on analogue curved spacetime; the particle creation by analogue expanding Universe as an expanding BEC. Obtained spectrum is consistent with the thermal Maxwell-Boltzmann distribution, the temperature of which has already been achieved.

2. Formulation

In a BEC, the bosonic field \( \psi = \Psi + \phi \) consists of the condensate \( \Psi \) and fluctuations \( \phi \). The condensate wave function \( \Psi = \sqrt{n_0} e^{iS} \) obeys the Gross-Pitaevskii equation:

\[
\partial_t n_0 + \nabla \cdot (n_0 \mathbf{v}_0) = 0, \quad \hbar \partial_t S + n_0^{-1/2} K n_0^{1/2} + V_{\text{ext}} - \mu + U_0 n_0 + \frac{m}{2} \nabla^2 n_0 = 0, \tag{1}
\]

where \( K = -\hbar^2 \nabla^2 / 2m \) is the kinetic energy operator, \( V_{\text{ext}} \) is the external trapping potential, \( \mu \) is the chemical potential, and \( U_0 \) is the coupling constant of the atomic interaction. The condensate contributions to the density and the current density are \( n_0 = |\Psi|^2 \) and \( j_0 = n_0 \hbar \nabla S = n_0 \mathbf{v}_0 \).

Density and current density fluctuations are described by

\[
\dot{\rho}' = \Psi^* \phi + \phi^* \Psi, \quad j'_0 = \frac{\hbar}{2m} [\Psi^* \nabla \phi + \phi^* \nabla \Psi - (\nabla \Psi^*) \phi - (\nabla \phi^*) \Psi], \tag{2}
\]

with the linearized approximation for fluctuations. Introducing \( U(1) \) gauge transformed fluctuations as \( \tilde{\phi} = e^{-iS} \phi \) and \( \tilde{\phi}^* = e^{iS} \phi^* \) and rewrite the density and current density fluctuations as \( \rho' = \sqrt{n_0} (\phi + \phi^*) \) and \( j'_0 = \rho' \mathbf{v}_0 + n_0 \dot{\mathbf{v}}' \), we obtain \( 1/\sqrt{g} \partial_\mu [\sqrt{-g} g^{\mu \nu} \partial_\nu \tilde{\phi}'] = 0 \) with the effective metric

\[
g_{\mu \nu} = \Lambda c_s \begin{pmatrix} (-c_s^2 - v_0^2) & -v_0^i \\ -v_0^i & \delta^{ij} \end{pmatrix}, \tag{3}
\]

where \( \Lambda \) is a constant, \( c_s = \sqrt{U_0 n_0 / m} \) is the local sound velocity, and the field \( \tilde{\Phi}' \) is defined as

\[
\tilde{\mathbf{v}}' = \nabla \tilde{\Phi}', \quad \tilde{\Phi}' = \frac{\hbar}{2m \sqrt{n_0}} (\phi - \phi^*). \tag{4}
\]

Formulating eq. (3), we assume that the density gradient is smooth over the local healing length \( \xi = \hbar / \sqrt{2m U_0 n_0} \).

Next, we introduce linear transformation of the fluctuation with defining annihilation and creation operators \( \tilde{\alpha}_j, \tilde{\alpha}_j^\dagger \) as

\[
\tilde{\phi} = \sum_j [u_j \alpha_j - v_j^* \alpha_j^\dagger], \quad \tilde{\phi}^\dagger = \sum_j [u_j^\dagger \alpha_j^\dagger - v_j \alpha_j]. \tag{5}
\]

Here we set the wave function \( u_j \) and \( v_j \) to obey

\[
i \hbar \partial_t \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} \tilde{W} & -U_0 n_0 \\ U_0 n_0 & -\tilde{W}^* \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix}. \tag{6}
\]
where $\tilde{W} = e^{-iS}Ke^{iS} + V_{\text{ext}} - \mu + 2U_0n_0 + h\partial S$. Then, the field $\Phi'$ and the wave function $u_j$ and $v_j$ are normalized as

$$\varphi = \sum_n \left[ \alpha_n f_n + \alpha_n^\dagger f_n^* \right] = \frac{m}{\sqrt{\Lambda\hbar U_0}} \Phi', \quad f_n = \frac{\sqrt{\Lambda}}{2i\sqrt{\Lambda U_0n_0}}(u_n + v_n).$$

(7)

The coefficient functions $\{f_n\}$ satisfies the following relativistic wave equation and Klein-Gordon (KG) inner product:

$$\partial_\mu [\sqrt{-g}g^{\mu\nu} \partial_\nu f_n] = 0, \quad (f_m, f_n)_{\text{KG}} = \int_\Sigma d^3x \left[ u_m u_n^* - v_m v_n^* \right] = \delta_{mn}. \quad (8)$$

Here, $\Sigma$ is the $t$-constant spacelike hypersurface.

Now we discuss a simple particle creation on curved spacetime. Let us consider a condensate which is static and having no excitation at initial time $t = t_1$. For $t_1 < t < t_2$, the condensate dynamically evolves and at $t = t_2$, it becomes static again. The corresponding effective spacetime evolves dynamically and quantum field theory on curved spacetime will predict particle creation. At initial and final time $t = t_i$ ($i = 1, 2$), we expand the field at each time as

$$\varphi = \sum_n \left[ \alpha_n^{(i)} f_n^{(i)} + \alpha_n^{(i)\dagger} f_n^{(i)*} \right]. \quad (9)$$

They are related each other by linear transformations

$$f_n^{(2)} = \sum_m \left[ A_{nm} f_m^{(1)} + B_{nm} f_m^{(1)*} \right], \quad \alpha_n^{(2)} = \sum_m (A_{nm}^* \alpha_m^{(1)} - B_{nm} \alpha_m^{(1)\dagger}). \quad (10)$$

Since the initial condensate has no excitations, the initial state for $\varphi$ is in the vacuum; $\alpha_n^{(1)} |0\rangle^{(1)} = 0$. Then, the expectation value of the final number operator is calculated as

$$|0\rangle^{(2)} = \sum_m |B_{nm}|^2.$$

(11)

Therefore, if $B_{nm}$’s take nonzero value, particle creation occurs.

3. Numerical simulation of the particle creation in an expanding BEC

Now, we show a numerical simulation for particle creation in expanding BEC in trapping potential $V_{\text{ext}} = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$, where $\omega_x$, $\omega_y$, and $\omega_z$ are the trapping frequencies of the potential along $x$, $y$, and $z$-axes. This situation can be interpreted as a cosmological expansion, because the sound velocity becomes small and the time interval during which a particle on the condensate can travel all over it, becomes long. We find the stationary solution of the GP equation for $\omega_z = \omega_1$ at $t = t_1 = 0$ and then decrease the frequency to $\omega_z = \omega_1 < \omega_1$. The spectrum of particle creation $\sum_m |B_{nm}|^2$ can be calculated as follows. First, we calculate the time evolution of $u_j^{(1)}$ and $v_j^{(1)}$ with that of the condensate wave function $\Psi$ by using the eq. (6), starting from the quasi-steady solution at $t = t_1$:

$$E_j^{(i)} \left( \begin{array}{c} u_j^{(i)} \\ v_j^{(i)} \end{array} \right) = \left( \begin{array}{cc} \tilde{W} & -U_0n_0 \\ U_0n_0 & -\tilde{W}^* \end{array} \right) \left( \begin{array}{c} u_j^{(i)} \\ v_j^{(i)} \end{array} \right). \quad (12)$$

Then, at $t = t_2$, we obtain $u_j^{(1)}$ and $v_j^{(1)}$ after the time evolution and $u_j^{(2)}$ and $v_j^{(2)}$ by using the eq. (12) again. Finally, we obtain the spectrum of particle creation from the eq. (11).
As the physical parameters, we use $m = 1.44 \times 10^{-25} \text{kg}$, $U_0 = 5.45 \times 10^{-51} \text{kg m}^2/\text{s}^2$, $\omega_i = 150 \times 2\pi \text{Hz}$, $\omega_f = \omega_i/\sqrt{2}$, and $\omega_0 = \omega_\perp = 50 \times 2\pi \text{Hz}$ considering the BEC experiment of $^{87}\text{Rb}$ atoms, the total number of which is set to be $N = 2.5 \times 10^6$.

Figure 1 (a) shows the time development of the sound velocity at the center $x = y = z = 0$. The sound velocity decreases with the expanding of the BEC, which corresponds to the expanding of the analogue spacetime. We calculate $u_j^{(i)}$ and $v_j^{(i)}$ at $t = t_2 \approx 2.55/\omega_i$ when the sound velocity becomes minimum, and investigate its particle creation. Figure 1 (b) shows the spectrum of the particle creation $\sum_m |B_{nm}|^2$. The spectrum takes the thermal Maxwell-Boltzmann distribution; $\sum_m |B_{nm}|^2 \propto \exp(-E_j^{(2)}/k_B T_e)$ with $k_B T_e \approx 0.777h\omega_i$.

During expansion, the analogue event horizon i.e. $c_s^2 - v_i^2 = 0$, is formed, which is closely related to the analogue Hawking radiation from the black hole [5, 6]. Therefore, the obtained thermal Maxwell-Boltzmann distribution has some relation with the formation of the analogue event horizon [7]. However, our result cannot be directly connected to analogue Hawking radiation because there is no event horizon at $t = t_2$ and the value of $T_e$ is different from what Hawking predicted. The origin of the Maxwell-Boltzmann distribution is one of future works including its dependence of $U_0$, $\omega_i$, and so on.

To confirm this experimentally, a BEC at the temperature lower than $T_e$ is needed not to be submerged by thermal noise. With the realistic parameter, $T_e$ is estimated as 5.60nK which has been already achieved in many experiments of BECs.

![Figure 1](image-url)

**Figure 1.** (a): Dependence of the the sound velocity $c_s = \sqrt{U_0 n_0(x = y = z = 0)/m}$ at the center. Here, $v_s = \sqrt{\hbar \omega_i/m}$ is the characteristic velocity defined by the trapping potential. (b): The spectrum of the particle creation at $t = t_2 = 2.55/\omega_i$.

4. Conclusion
We formulate quantum field theory on analogue spacetime based on the GP and BdG equations. Furthermore, we calculate the particle creation on the analogue expanding Universe for an expanding BEC. Obtained spectrum shows the thermal Maxwell-Boltzmann distribution, the temperature of which can be experimentally accessible.

References
[1] Hawking S W 1974 *Nature* **248** 30
[2] Unruh W G 1981 *Phys. Rev. Lett.* **46** 1351
[3] Novello M, Visser M and Volovik G (eds) 2002 *Artificial Black Holes* (World Scientific)
[4] Pethick C J and Smith H 2002 *Bose-Einstein Condensation in Dilute Gases* (Cambridge: Cambridge University Press)
[5] Barcelo C, Liberati S and Visser M 2005 *Living Rev. Rel.* **9** 12
[6] Uhlmann M, Xu Y and Schutzhold R 2005 *New J. Phys.* **7** 248
[7] Jain P, Weinfurtner S, Visser M and Gardiner C W 2007 *Phys. Rev. A* **76** 033616