Edge-disjoint double rays in infinite graphs:
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DOI:
10.1016/j.jctb.2014.08.005

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Citation for published version (Harvard):
Bowler, N, Carmesin, J & Pott, J 2015, 'Edge-disjoint double rays in infinite graphs: a Halin type result', Journal of Combinatorial Theory. Series B, vol. 111, pp. 1-16. https://doi.org/10.1016/j.jctb.2014.08.005

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Download date: 16. Sep. 2023
Edge-disjoint double rays in infinite graphs: 
 a Halin type result

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Abstract
We show that any graph that contains \( k \) edge-disjoint double rays for any \( k \in \mathbb{N} \) contains also infinitely many edge-disjoint double rays. This was conjectured by Andreae in 1981.

1 Introduction

We say a graph \( G \) has arbitrarily many vertex-disjoint \( H \) if for every \( k \in \mathbb{N} \) there is a family of \( k \) vertex-disjoint subgraphs of \( G \) each of which is isomorphic to \( H \). Halin’s Theorem says that every graph that has arbitrarily many vertex-disjoint rays, also has infinitely many vertex-disjoint rays [5]. In 1970 he extended this result to vertex-disjoint double rays [6]. Jung proved a strengthening of Halin’s Theorem where the initial vertices of the rays are constrained to a certain vertex set [7].

We look at the same questions with ‘edge-disjoint’ replacing ‘vertex-disjoint’. Consider first the statement corresponding to Halin’s Theorem. It suffices to prove this statement in locally finite graphs, as each graph with arbitrarily many edge-disjoint rays contains a locally finite union of tails of these rays. But the statement for locally finite graphs follows from Halin’s original Theorem applied to the line-graph.

This reduction to locally finite graphs does not work for Jung’s Theorem or for Halin’s statement about double rays. Andreae proved an analog of Jung’s Theorem for edge-disjoint rays in 1981, and conjectured that a Halin-type Theorem would be true for edge-disjoint double rays [1]. Our aim in the current paper is to prove this conjecture.

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More precisely, we say a graph $G$ has *arbitrarily many edge-disjoint* $H$ if for every $k \in \mathbb{N}$ there is a family of $k$ edge-disjoint subgraphs of $G$ each of which is isomorphic to $H$, and our main result is the following.

**Theorem 1.** *Any graph that has arbitrarily many edge-disjoint double rays has infinitely many edge-disjoint double rays.*

Even for locally finite graphs this theorem does not follow from Halin’s analogous result for vertex-disjoint double rays applied to the line graph. For example a double ray in the line graph may correspond, in the original graph, to a configuration as in Figure 1.

![Figure 1: A graph that does not include a double ray but whose line graph does.](image)

A related notion is that of ubiquity. A graph $H$ is *ubiquitous* with respect to a graph relation $\leq$ if $nH \leq G$ for all $n \in \mathbb{N}$ implies $\aleph_0 H \leq G$, where $nH$ denotes the disjoint union of $n$ copies of $H$. For example, Halin’s Theorem says that rays are ubiquitous with respect to the subgraph relation. It is known that not every graph is ubiquitous with respect to the minor relation [2], nor is every locally finite graph ubiquitous with respect to the subgraph relation [8, 9], or even the topological minor relation [2, 3]. However, Andreae has conjectured that every locally finite graph is ubiquitous with respect to the minor relation [2]. For more details see [3]. In Section 6 (the outlook) we introduce a notion closely related to ubiquity.

The proof is organised as follows. In Section 3 we explain how to deal with the cases that the graph has infinitely many ends, or an end with infinite vertex-degree. In Section 4 we consider the ‘two ended’ case: That in which there are two ends $\omega$ and $\omega'$ both of finite vertex-degree, and arbitrarily many edge-disjoint double rays from $\omega$ to $\omega'$.

The only remaining case is the ‘one ended’ case: That in which there is a single end $\omega$ of finite vertex-degree and arbitrarily many edge-disjoint double rays from $\omega$ to $\omega$. One central idea in the proof of this case is to consider 2-rays instead of double rays. Here a 2-ray is a pair of vertex-disjoint rays. For example, from each double ray one obtains a 2-ray by removing a finite path. The remainder of the proof is subdivided into two parts: In Subsection 5.3 we show that if there are arbitrarily many edge-disjoint 2-rays into $\omega$, then there are infinitely many such 2-rays. In Subsection 5.2 we show that if there are infinitely
many edge-disjoint 2-rays into $\omega$, then there are infinitely many edge-disjoint double rays from $\omega$ to $\omega$.

We finish by discussing the outlook and mentioning some open problems.

2 Preliminaries

All our basic notation for graphs is taken from [4]. In particular, two rays in a graph are equivalent if no finite set separates them. The equivalence classes of this relation are called the ends of $G$. We say that a ray in an end $\omega$ converges to $\omega$. A double ray converges to all the ends of which it includes a ray.

2.1 The structure of a thin end

It follows from Halin’s Theorem that if there are arbitrarily many vertex-disjoint rays in an end of $G$, then there are infinitely many such rays. This fact motivated the central definition of the vertex-degree of an end $\omega$: the maximal cardinality of a set of vertex-disjoint rays in $\omega$.

An end is thin if its vertex-degree is finite, and otherwise it is thick. A pair $(A, B)$ of edge-disjoint subgraphs of $G$ is a separation of $G$ if $A \cup B = G$. The number of vertices of $A \cap B$ is called the order of the separation.

Definition 2. Let $G$ be a locally finite graph and $\omega$ a thin end of $G$. A countable infinite sequence $((A_i, B_i))_{i \in \mathbb{N}}$ of separations of $G$ captures $\omega$ if for all $i \in \mathbb{N}$

- $A_i \cap B_{i+1} = \emptyset$,
- $A_{i+1} \cap B_i$ is connected,
- $\bigcup_{i \in \mathbb{N}} A_i = G$,
- the order of $(A_i, B_i)$ is the vertex-degree of $\omega$, and
- each $B_i$ contains a ray from $\omega$.

Lemma 3. Let $G$ be a locally finite graph with a thin end $\omega$. Then there is a sequence that captures $\omega$.

Proof. Without loss of generality $G$ is connected, and so is countable. Let $v_1, v_2, \ldots$ be an enumeration of the vertices of $G$. Let $k$ be the vertex-degree of $\omega$. Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a set of vertex-disjoint rays in $\omega$ and let $S$ be the set of their start vertices. We pick a sequence $((A_i, B_i))_{i \in \mathbb{N}}$ of separations and a sequence $(T_i)$ of connected subgraphs recursively as follows. We pick $(A_i, B_i)$ such that $S$ is included in $A_i$, such that there is a ray from $\omega$ included in $B_i$, and such that $B_i$ does not meet $\bigcup_{j \leq i} T_j$ or $\{v_j \mid j \leq i\}$; subject to this we minimise the size of the set $X_i$ of vertices in $A_i \cap B_i$. Because of this minimization $B_i$ is connected and $X_i$ is finite. We take $T_i$ to be a finite connected subgraph of $B_i$ including $X_i$. Note that any ray that meets all of the $B_i$ must be in $\omega$. 

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By Menger’s Theorem \([4]\) we get for each \(i \in \mathbb{N}\) a set \(P_i\) of vertex-disjoint paths from \(X_i\) to \(X_{i+1}\) of size \(|X_i|\). From these, for each \(i\) we get a set of \(|X_i|\) vertex-disjoint rays in \(\omega\). Thus the size of \(X_i\) is at most \(k\). On the other hand it is at least \(k\) as each ray \(R_j\) meets each set \(X_i\).

Assume for contradiction that there is a vertex \(v \in A_i \cap B_{i+1}\). Let \(R\) be a ray from \(v\) to \(\omega\) inside \(B_{i+1}\). Then \(R\) must meet \(X_i\), contradicting the definition of \(B_{i+1}\). Thus \(A_i \cap B_{i+1}\) is empty.

Remark 4. Every infinite subsequence of a sequence capturing \(\omega\) also captures \(\omega\).

The following is obvious:

Remark 5. Let \(G\) be a graph and \(v, w \in V(G)\). If \(G\) contains arbitrarily many edge-disjoint \(v \rightarrow w\) paths, then it contains infinitely many edge-disjoint \(v \rightarrow w\) paths.

We will need the following special case of the theorem of Andreae mentioned in the Introduction.

Theorem 6 (Andreae \([1]\)). Let \(G\) be a graph and \(v \in V(G)\). If there are arbitrarily many edge-disjoint rays all starting at \(v\), then there are infinitely many edge-disjoint rays all starting at \(v\).

3 Known cases

Many special cases of Theorem 7 are already known or easy to prove. For example Halin showed the following.

Theorem 7 (Halin). Let \(G\) be a graph and \(\omega\) an end of \(G\). If \(\omega\) contains arbitrarily many vertex-disjoint rays, then \(G\) has a half-grid as a minor.

Corollary 8. Any graph with an end of infinite vertex-degree has infinitely many edge-disjoint double rays.

Another simple case is the case where the graph has infinitely many ends.

Lemma 9. A tree with infinitely many ends contains infinitely many edge-disjoint double rays.

Proof. It suffices to show that every tree \(T\) with infinitely many ends contains a double ray such that removing its edges leaves a component containing infinitely many ends, since then one can pick those double rays recursively.

There is a vertex \(v \in V(T)\) such that \(T - v\) has at least 3 components \(C_1, C_2, C_3\) that each have at least one end, as \(T\) contains more than 2 ends. Let
$e_i$ be the edge $vw_i$ with $w_i \in C_i$ for $i \in \{1, 2, 3\}$. The graph $T \setminus \{e_1, e_2, e_3\}$ has precisely 4 components $(C_1, C_2, C_3$ and the one containing $v$), one of which, $D$ say, has infinitely many ends. By symmetry we may assume that $D$ is neither $C_1$ nor $C_2$. There is a double ray $R$ all whose edges are contained in $C_1 \cup C_2 \cup \{e_1, e_2\}$. Removing the edges of $R$ leaves the component $D$, which has infinitely many ends.

Corollary 10. Any connected graph with infinitely many ends has infinitely many edge-disjoint double rays.

4 The ‘two ended’ case

Using the results of Section 3 it is enough to show that any graph with only finitely many ends, each of which is thin, has infinitely many edge-disjoint double rays as soon as it has arbitrarily many edge-disjoint double rays. Any double ray in such a graph has to join a pair of ends (not necessarily distinct), and there are only finitely many such pairs. So if there are arbitrarily many edge-disjoint double rays, then there is a pair of ends such that there are arbitrarily many edge-disjoint double rays joining those two ends. In this section we deal with the case where these two ends are different, and in Section 5 we deal with the case that they are the same. We start with two preparatory lemmas.

Lemma 11. Let $G$ be a graph with a thin end $\omega$, and let $R \subseteq \omega$ be an infinite set. Then there is an infinite subset of $R$ such that any two of its members intersect in infinitely many vertices.

Proof. We define an auxiliary graph $H$ with $V(H) = R$ and an edge between two rays if and only if they intersect in infinitely many vertices. By Ramsey’s Theorem either $H$ contains an infinite clique or an infinite independent set of vertices. Let us show that there cannot be an infinite independent set in $H$. Let $k$ be the vertex-degree of $\omega$: we shall show that $H$ does not have an independent set of size $k + 1$. Suppose for a contradiction that $X \subseteq R$ is a set of $k + 1$ rays that is independent in $H$. Since any two rays in $X$ meet in only finitely many vertices, each ray in $X$ contains a tail that is disjoint to all the other rays in $X$. The set of these $k + 1$ vertex-disjoint tails witnesses that $\omega$ has vertex-degree at least $k + 1$, a contradiction. Thus there is an infinite clique $K \subseteq H$, which is the desired infinite subset.

Lemma 12. Let $G$ be a graph consisting of the union of a set $R$ of infinitely many edge-disjoint rays of which any pair intersect in infinitely many vertices. Let $X \subseteq V(G)$ be an infinite set of vertices, then there are infinitely many edge-disjoint rays in $G$ all starting in different vertices of $X$.

Proof. If there are infinitely many rays in $R$ each of which contains a different vertex from $X$, then suitable tails of these rays give the desired rays. Otherwise there is a ray $R \in R$ meeting $X$ infinitely often. In this case, we choose the desired rays recursively such that each contains a tail from some ray in $R - R$. 

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Having chosen finitely many such rays, we can always pick another: we start at some point in $X$ on $R$ which is beyond all the (finitely many) edges on $R$ used so far. We follow $R$ until we reach a vertex of some ray $R'$ in $R - R$ whose tail has not been used yet, then we follow $R'$.

**Lemma 13.** Let $G$ be a graph with only finitely many ends, all of which are thin. Let $\omega_1, \omega_2$ be distinct ends of $G$. If $G$ contains arbitrarily many edge-disjoint double rays each of which converges to both $\omega_1$ and $\omega_2$, then $G$ contains infinitely many edge-disjoint double rays each of which converges to both $\omega_1$ and $\omega_2$.

**Proof.** For each pair of ends, there is a finite set separating them. The finite union of these finite sets is a finite set $S \subseteq V(G)$ separating any two ends of $G$.

For $i = 1, 2$ let $C_i$ be the component of $G - S$ containing $\omega_i$.

There are arbitrarily many edge-disjoint double rays from $\omega_1$ to $\omega_2$ that have a common last vertex $v_1$ in $S$ before staying in $C_1$ and also a common last vertex $v_2$ in $S$ before staying in $C_2$. Note that $v_1$ may be equal to $v_2$. There are arbitrarily many edge-disjoint rays in $C_1 + v_1$ all starting in $v_1$. By Theorem 6 there is a countable infinite set $R_1 = \{ R_i \mid i \in \mathbb{N} \}$ of edge-disjoint rays each included in $C_1 + v_1$ and starting in $v_1$. By replacing $R_1$ with an infinite subset of itself, if necessary, we may assume by Lemma 11 that any two members of $R_1$ intersect in infinitely many vertices. Similarly, there is a countable infinite set $R_2 = \{ R_i \mid i \in \mathbb{N} \}$ of edge-disjoint rays each included in $C_2 + v_2$ and starting in $v_2$ such that any two members of $R_2$ intersect in infinitely many vertices.

Let us subdivide all edges in $\bigcup R_1$ and call the set of subdivision vertices $X_1$. Similarly, we subdivide all edges in $\bigcup R_2$ and call the set of subdivision vertices $X_2$. Below we shall find double rays in the subdivided graph, which immediately give rise to the desired double rays in $G$.

Suppose for a contradiction that there is a finite set $F$ of edges separating $X_1$ from $X_2$. Then $v_i$ has to be on the same side of that separation as $X_i$ as there are infinitely many $v_i - X_i$ edges. So $F$ separates $v_1$ from $v_2$, which contradicts the fact that there are arbitrarily many edge-disjoint double rays containing both $v_1$ and $v_2$. By Remark 5 there is a set $P$ of infinitely many edge-disjoint $X_1 - X_2$ paths. As all vertices in $X_1$ and $X_2$ have degree 2, and by taking an infinite subset if necessary, we may assume that each end-vertex of a path in $P$ lies on no other path in $P$.

By Lemma 12 there is an infinite set $Y_1$ of start-vertices of paths in $P$ together with an infinite set $R_1'$ of edge-disjoint rays with distinct start-vertices whose set of start-vertices is precisely $Y_1$. Moreover, we can ensure that each ray in $R_1'$ is included in $\bigcup R_1$. Let $Y_2$ be the set of end-vertices in $X_2$ of those paths in $P$ that start in $Y_1$. Applying Lemma 12 again, we obtain an infinite set $Z_2 \subseteq Y_2$ together with an infinite set $R_2'$ of edge-disjoint rays included in $\bigcup R_2$ with distinct start-vertices whose set of start-vertices is precisely $Z_2$.

For each path $P$ in $P$ ending in $Z_2$, there is a double ray in the union of $P$ and the two rays from $R_1'$ and $R_2'$ that $P$ meets in its end-vertices. By construction, all these infinitely many double rays are edge-disjoint. Each of
those double rays converges to both $\omega_1$ and $\omega_2$, since each $\omega_i$ is the only end in $C_i$.

**Remark 14.** Instead of subdividing edges we also could have worked in the line graph of $G$. Indeed, there are infinitely many vertex-disjoint paths in the line graph from $\bigcup \mathcal{R}_1$ to $\bigcup \mathcal{R}_2$.

5 The ‘one ended’ case

We are now going to look at graphs $G$ that contain a thin end $\omega$ such that there are arbitrarily many edge-disjoint double rays converging only to the end $\omega$. The aim of this section is to prove the following lemma, and to deduce Theorem 1.

**Lemma 15.** Let $G$ be a countable graph and let $\omega$ be a thin end of $G$. Assume there are arbitrarily many edge-disjoint double rays all of whose rays converge to $\omega$. Then $G$ has infinitely many edge-disjoint double rays.

We promise that the assumption of countability will not cause problems later.

5.1 Reduction to the locally finite case

A key notion for this section is that of a 2-ray. A 2-ray is a pair of vertex-disjoint rays. For example, from each double ray one obtains a 2-ray by removing a finite path.

In order to deduce that $G$ has infinitely many edge-disjoint double rays, we will only need that $G$ has arbitrarily many edge-disjoint 2-rays. In this subsection, we illustrate one advantage of 2-rays, namely that we may reduce to the case where $G$ is locally finite.

**Lemma 16.** Let $G$ be a countable graph with a thin end $\omega$. Assume there is a countable infinite set $\mathcal{R}$ of rays all of which converge to $\omega$. Then there is a locally finite subgraph $H$ of $G$ with a single end which is thin such that the graph $H'$ includes a tail of any $R \in \mathcal{R}$.

**Proof.** Let $(R_i \mid i \in \mathbb{N})$ be an enumeration of $\mathcal{R}$. Let $(v_i \mid i \in \mathbb{N})$ be an enumeration of the vertices of $G$. Let $U_i$ be the unique component of $G \setminus \{v_1, \ldots, v_i\}$ including a tail of each ray in $\omega$.

For $i \in \mathbb{N}$, we pick a tail $R'_i$ of $R_i$ in $U_i$. Let $H_1 = \bigcup_{i \in \mathbb{N}} R'_i$. Making use of $H_1$, we shall construct the desired subgraph $H$. Before that, we shall collect some properties of $H_1$.

As every vertex of $G$ lies in only finitely many of the $U_i$, the graph $H_1$ is locally finite. Each ray in $H_1$ converges to $\omega$ in $G$ since $H_1 \setminus U_i$ is finite for every $i \in \mathbb{N}$. Let $\Psi$ be the set of ends of $H_1$. Since $\omega$ is thin, $\Psi$ has to be finite: $\Psi = \{\omega_1, \ldots, \omega_n\}$. For each $i \leq n$, we pick a ray $S_i \subseteq H_1$ converging to $\omega_i$.

Now we are in a position to construct $H$. For any $i > 1$, the rays $S_1$ and $S_i$ are joined by an infinite set $\mathcal{P}_i$ of vertex-disjoint paths in $G$. We obtain $H$ from...
by adding all paths in the sets $P_i$. Since $H_1$ is locally finite, $H$ is locally finite.

It remains to show that every ray $R$ in $H$ is equivalent to $S_1$. If $R$ contains infinitely many edges from the $P_i$, then there is a single $P_i$ which $R$ meets infinitely, and thus $R$ is equivalent to $S_1$. Thus we may assume that a tail of $R$ is a ray in $H_1$. So it converges to some $\omega_i \in \Psi$. Since $S_i$ and $S_1$ are equivalent, $R$ and $S_1$ are equivalent, which completes the proof.

**Corollary 17.** Let $G$ be a countable graph with a thin end $\omega$ and arbitrarily many edge-disjoint 2-rays of which all the constituent rays converge to $\omega$. Then there is a locally finite subgraph $H$ of $G$ with a single end, which is thin, such that $H$ has arbitrarily many edge-disjoint 2-rays.

**Proof.** By Lemma 16 there is a locally finite graph $H \subseteq G$ with a single end such that a tail of each of the constituent rays of the arbitrarily many 2-rays is included in $H$.

5.2 Double rays versus 2-rays

A connected subgraph of a graph $G$ including a vertex set $S \subseteq V(G)$ is a connector of $S$ in $G$.

**Lemma 18.** Let $G$ be a connected graph and $S$ a finite set of vertices of $G$. Let $H$ be a set of edge-disjoint subgraphs $H$ of $G$ such that each connected component of $H$ meets $S$. Then there is a finite connector $T$ of $S$, such that at most $2|S| - 2$ many graphs from $H$ contain edges of $T$.

**Proof.** By replacing $H$ with the set of connected components of graphs in $H$, if necessary, we may assume that each member of $H$ is connected. We construct graphs $T_i$ recursively for $0 \leq i < |S|$ such that each $T_i$ is finite and has at most $|S| - i$ components, at most $2i$ graphs from $H$ contain edges of $T_i$, and each component of $T_i$ meets $S$. Let $T_0 = (S, \emptyset)$ be the graph with vertex set $S$ and no edges. Assume that $T_i$ has been defined.

If $T_i$ is connected let $T_{i+1} = T_i$. For a component of $C$ of $T_i$, let $C'$ be the graph obtained from $C$ by adding all graphs from $H$ that meet $C$.

As $G$ is connected, there is a path $P$ (possibly trivial) in $G$ joining two of these subgraphs $C_1'$ and $C_2'$ say. And by taking the length of $P$ minimal, we may assume that $P$ does not contain any edge from any $H \in H$. Then we can extend $P$ to a $C_1$–$C_2$ path $Q$ by adding edges from at most two subgraphs from $H$ — one included in $C_1'$ and the other in $C_2'$. We obtain $T_{i+1}$ from $T_i$ by adding $Q$.

$T = T_{|S|-1}$ has at most one component and thus is connected. And at most $2|S| - 2$ many graphs from $H$ contain edges of $T$. Thus $T$ is as desired.

Let $d, d'$ be 2-rays. $d$ is a tail of $d'$ if each ray of $d$ is a tail of a ray of $d'$. A set $D'$ is a tailor of a set $D$ of 2-rays if each element of $D'$ is a tail of some element of $D$ but no 2-ray in $D$ includes more than one 2-ray in $D'$.
Lemma 19. Let $G$ be a locally finite graph with a single end $\omega$, which is thin. Assume that $G$ contains an infinite set $D = \{d_1, d_2, \ldots \}$ of edge-disjoint 2-rays.

Then $G$ contains an infinite tailor $D'$ of $D$ and a sequence $((A_i, B_i))_{i \in \mathbb{N}}$ capturing $\omega$ (see Definition 3) such that there is a family of vertex-disjoint connectors $T_i$ of $A_i \cap B_i$ contained in $A_{i+1} \cap B_i$, each of which is edge-disjoint from each member of $D'$.

Proof. Let $k$ be the vertex-degree of $\omega$. By Lemma 3 there is a sequence $((A'_i, B'_i))_{i \in \mathbb{N}}$ capturing $\omega$. By replacing each 2-ray in $D$ with a tail of itself if necessary, we may assume that for all $(r, s) \in D$ and $i \in \mathbb{N}$ either both $r$ and $s$ meet $A'_i$ or none meets $A'_i$. By Lemma 18 there is a finite connector $T'_i$ of $A'_i \cap B'_i$ in the connected graph $B'_i$ which meets in an edge at most $2k - 2$ of the 2-rays of $D$ that have a vertex in $A'_i$.

Thus, there are at most $2k - 2$ 2-rays in $D$ that meet all but finitely many of the $T'_i$ in an edge. By throwing away these finitely many 2-rays in $D$ we may assume that each 2-ray in $D$ is edge-disjoint from infinitely many of the $T'_i$. So we can recursively build a sequence $N_1, N_2, \ldots$ of infinite sets of natural numbers such that $N_i \supseteq N_{i+1}$, the first $i$ elements of $N_i$ are all contained in $N_{i+1}$, and $d_i$ only meets finitely many of the $T'_j$ with $j \in N_i$ in an edge. Then $N = \bigcap_{i \in \mathbb{N}} N_i$ is infinite and has the property that each $d_i$ only meets finitely many of the $T'_j$ with $j \in N$ in an edge. Thus there is an infinite tailor $D' = D'$ of $D$ such that no 2-ray from $D'$ meets any $T'_j$ for $j \in N$ in an edge.

We recursively define a sequence $n_1, n_2, \ldots$ of natural numbers by taking $n_i \in N$ sufficiently large that $B'_{n_i}$ does not meet $T'_{n_j}$ for any $j < i$. Taking $(A_i, B_i) = (A'_{n_i}, B'_{n_i})$ and $T_i = T'_{n_i}$ gives the desired sequences. \qed

Lemma 20. If a locally finite graph $G$ with a single end $\omega$ which is thin contains infinitely many edge-disjoint 2-rays, then $G$ contains infinitely many edge-disjoint double rays.

Proof. Applying Lemma 19 we get an infinite set $D$ of edge-disjoint 2-rays, a sequence $((A_i, B_i))_{i \in \mathbb{N}}$ capturing $\omega$, and connectors $T_i$ of $A_i \cap B_i$ for each $i \in \mathbb{N}$ such that the $T_i$ are vertex-disjoint from each other and edge-disjoint from all members of $D$.

We shall construct the desired set of infinitely many edge-disjoint double rays as a nested union of sets $D_i$. We construct the $D_i$ recursively. Assume that a set $D_i$ of $i$ edge-disjoint double rays has been defined such that each of its members is included in the union of a single 2-ray from $D$ and one connector $T_j$. Let $d_{i+1} \in D$ be a 2-ray distinct from the finitely many 2-rays used so far. Let $C_{i+1}$ be one of the infinitely many connectors that is different from all the finitely many connectors used so far and that meets both rays of $d_{i+1}$. Clearly, $d_{i+1} \cup C_{i+1}$ includes a double ray $R_{i+1}$. Let $D_{i+1} = D_i \cup \{R_{i+1}\}$. The union $\bigcup_{i \in \mathbb{N}} D_i$ is an infinite set of edge-disjoint double rays as desired. \qed
5.3 Shapes and allowed shapes

Let $G$ be a graph and $(A, B)$ a separation of $G$. A shape for $(A, B)$ is a word $v_1x_1v_2x_2\ldots x_{n-1}v_n$ with $v_i \in A \cap B$ and $x_i \in \{l, r\}$ such that no vertex appears twice. We call the $v_i$ the vertices of the shape. Every ray $R$ induces a shape $\sigma = \sigma_R(A, B)$ on every separation $(A, B)$ of finite order in the following way: Let $<_R$ be the natural order on $V(R)$ induced by the ray, where $v <_R w$ if $w$ lies in the unique infinite component of $R - v$. The vertices of $\sigma$ are those vertices of $R$ that lie in $A \cap B$ and they appear in $\sigma$ in the order given by $<_R$. For $v_i, v_{i+1}$ the path $v_iRv_{i+1}$ has edges only in $A$ or only in $B$ but not in both. In the first case we put $l$ between $v_i$ and $v_{i+1}$ and in the second case we put $r$ between $v_i$ and $v_{i+1}$.

Let $(A_1, B_1), (A_2, B_2)$ be separations with $A_1 \cap B_2 = \emptyset$ and thus also $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$. Let $\sigma_i$ be a nonempty shape for $(A_1, B_i)$. The word $\tau = v_1x_1v_2\ldots x_{n-1}v_n$ is an allowed shape linking $\sigma_1$ to $\sigma_2$ with vertices $v_1\ldots v_n$ if the following holds:

- $v$ is a vertex of $\tau$ if and only if it is a vertex of $\sigma_1$ or $\sigma_2$,
- if $v$ appears before $w$ in $\sigma_i$, then $v$ appears before $w$ in $\tau$,
- $v_i$ is the initial vertex of $\sigma_1$ and $v_n$ is the terminal vertex of $\sigma_2$,
- $x_i \in \{l, m, r\}$,
- the subword $v_iw$ appears in $\tau$ if and only if it appears in $\sigma_1$,
- the subword $v_mw$ appears in $\tau$ if and only if it appears in $\sigma_2$,
- $v_i \neq v_j$ for $i \neq j$.

Each ray $R$ defines a word $\tau = \tau_R[(A_1, B_1), (A_2, B_2)] = v_1x_1v_2\ldots x_{n-1}v_n$ with vertices $v_i$ and $x_i \in \{l, m, r\}$ as follows. The vertices of $\tau$ are those vertices of $R$ that lie in $A_1 \cap B_1$ or $A_2 \cap B_2$ and they appear in $\tau$ in the order given by $<_R$. For $v_i, v_{i+1}$ the path $v_iRv_{i+1}$ has edges either only in $A_1$, only in $A_2 \cap B_1$, or only in $B_2$. In the first case we set $x_i = l$ and $\tau$ contains the subword $v_iv_{i+1}$. In the second case we set $x_i = m$ and $\tau$ contains the subword $v_iv_{i+1}$. In the third case we set $x_i = r$ and $\tau$ contains the subword $v_iv_{i+1}$.

For a ray $R$ to induce an allowed shape $\tau_R[(A_1, B_1), (A_2, B_2)]$ we need at least that $R$ starts in $A_2$. However, each ray in $\omega$ has a tail such that whenever it meets an $A_i$ it also starts in that $A_i$. Let us call such rays lefty. A 2-ray is lefty if both its rays are.

Remark 21. Let $(A_1, B_1)$, and $(A_2, B_2)$ be two separations of finite order with $A_1 \subseteq A_2$, and $B_2 \subseteq B_1$. For every lefty ray $R$ meeting $A_1$, the word $\tau_R[(A_1, B_1), (A_2, B_2)]$ is an allowed shape linking $\sigma_R(A_1, B_1)$ and $\sigma_R(A_2, B_2)$.
From now on let us fix a locally finite graph $G$ with a thin end $\omega$ of vertex-degree $k$. And let $((A_i,B_i))_{i \in \mathbb{N}}$ be a sequence capturing $\omega$ such that each member has order $k$.

A 2-shape for a separation $(A,B)$ is a pair of shapes for $(A,B)$. Every 2-ray induces a 2-shape coordinatewise in the obvious way. Similarly, an allowed 2-shape is a pair of allowed shapes.

Clearly, there is a global constant $c_1 \in \mathbb{N}$ depending only on $k$ such that there are at most $c_1$ distinct 2-shapes for each separation $(A_i,B_i)$. Similarly, there is a global constant $c_2 \in \mathbb{N}$ depending only on $k$ such that for all $i,j \in \mathbb{N}$ there are at most $c_2$ distinct allowed 2-shapes linking a 2-shape for $(A_i,B_i)$ with a 2-shape for $(A_j,B_j)$.

For most of the remainder of this subsection we assume that for every $i \in \mathbb{N}$ there is a set $D_i$ consisting of at least $c_1 \cdot c_2 \cdot i$ edge-disjoint 2-rays in $G$. Our aim will be to show that in these circumstances there must be infinitely many edge-disjoint 2-rays.

By taking a tailor if necessary, we may assume that every 2-ray in each $D_i$ is lefty.

**Lemma 22.** There is an infinite set $J \subseteq \mathbb{N}$ and, for each $i \in \mathbb{N}$, a tailor $D'_i$ of $D_i$ of cardinality $c_2 \cdot i$ such that for all $i \in \mathbb{N}$ and $j \in J$ all 2-rays in $D'_i$ induce the same 2-shape $\sigma[i,j]$ on $(A_j,B_j)$.

**Proof.** We recursively build infinite sets $J_i \subseteq \mathbb{N}$ and tailors $D'_i$ of $D_i$ such that for all $k \leq i$ and $j \in J_i$ all 2-rays in $D'_k$ induce the same 2-shape on $(A_j,B_j)$.

For all $i \geq 1$, we shall ensure that $J_i$ is an infinite subset of $J_{i-1}$ and that the $i-1$ smallest members of $J_i$ and $J_{i-1}$ are the same. We shall take $J$ to be the intersection of all the $J_i$.

Let $J_0 = \mathbb{N}$ and let $D'_0$ be the empty set. Now, for some $i \geq 1$, assume that sets $J_k$ and $D'_k$ have been defined for all $k < i$. By replacing 2-rays in $D_i$ by their tails, if necessary, we may assume that each 2-ray in $D_i$ avoids $A_i$, where $\ell$ is the $(i-1)$st smallest value of $J_{i-1}$. As $D_i$ contains $c_1 \cdot c_2 \cdot i$ many 2-rays, for each $j \in J_{i-1}$ there is a set $S_j \subseteq D_i$ of size at least $c_2 \cdot i$ such that each 2-ray in $S_j$ induces the same 2-shape on $(A_j,B_j)$. As there are only finitely many possible choices for $S_j$, there is an infinite subset $J_i$ of $J_{i-1}$ on which $S_j$ is constant. For $D'_i$ we pick this value of $S_j$. Since each $d \in D'_i$ induces the empty 2-shape on each $(A_k,B_k)$ with $k \leq \ell$ we may assume that the first $i-1$ elements of $J_{i-1}$ are also included in $J_i$.

It is immediate that the set $J = \bigcap_{k \in \mathbb{N}} J_i$ and the $D'_i$ have the desired property. 

**Lemma 23.** There are two strictly increasing sequences $(n_i)_{i \in \mathbb{N}}$ and $(\ell_i)_{i \in \mathbb{N}}$ with $n_i \in \mathbb{N}$ and $\ell_i \in J$ for all $i \in \mathbb{N}$ such that $\sigma[n_i,\ell_i] = \sigma[n_{i+1},\ell_i]$ and $\sigma[n_i,\ell_i]$ is not empty.

**Proof.** Let $H$ be the graph on $\mathbb{N}$ with an edge $vw \in E(H)$ if and only if there are infinitely many elements $j \in J$ such that $\sigma[v,j] = \sigma[w,j]$. 

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As there are at most \( c_1 \) distinct 2-shapes for any separator \((A_i, B_i)\), there is no independent set of size \( c_1 + 1 \) in \( H \) and thus no infinite one. Thus, by Ramsey’s theorem, there is an infinite clique in \( H \). We may assume without loss of generality that \( H \) itself is a clique by moving to a subsequence of the \( D'_i \) if necessary. With this assumption we simply pick \( n_i = i \).

Now we pick the \( j_i \) recursively. Assume that \( j_i \) has been chosen. As \( i \) and \( i + 1 \) are adjacent in \( H \), there are infinitely many indices \( \ell \in \mathbb{N} \) such that \( \sigma[i, \ell] = \sigma[i + 1, \ell] \). In particular, there is such an \( \ell > j_i \) such that \( \sigma[i + 1, \ell] \) is not empty. We pick \( j_{i+1} \) to be one of those \( \ell \).

Clearly, \((j_i)_{i\in \mathbb{N}} \) is an increasing sequence and \( \sigma[i,j_i] = \sigma[i+1,j_i] \) as well as \( \sigma[i,j_i] \) is non-empty for all \( i \in \mathbb{N} \), which completes the proof.

By moving to a subsequence of \((D'_i)\) and \((\{A_j, B_j\})\), if necessary, we may assume by Lemma 22 and Lemma 23 that for all \( i, j \in \mathbb{N} \) all \( d \in D'_i \) induce the same 2-shape \( \sigma[i,j] \) on \((A_j, B_j)\), and that \( \sigma[i,i] = \sigma[i+1,i] \), and that \( \sigma[i,i] \) is non-empty.

**Lemma 24.** For all \( i \in \mathbb{N} \) there is \( D''_i \subseteq D'_i \) such that \( |D''_i| = i \), and all \( d \in D''_i \) induce the same allowed 2-shape \( \tau[i] \) that links \( \sigma[i,i] \) and \( \sigma[i,i+1] \).

**Proof.** Note that it is in this proof that we need all the 2-rays in \( D'_i \) to be lefty as they need to induce an allowed 2-shape that links \( \sigma[i,i] \) and \( \sigma[i,i+1] \) as soon as they contain a vertex from \( A_i \). As \( |D'_i| \geq i \cdot c_2 \) and as there are at most \( c_2 \) many distinct allowed 2-shapes that link \( \sigma[i,i] \) and \( \sigma[i,i+1] \) there is \( D''_i \subseteq D'_i \) with \( |D''_i| = i \) such that all \( d \in D''_i \) induce the same allowed 2-shape.

We enumerate the elements of \( D''_i \) as follows: \( d'_1, d'_2, \ldots, d'_i \). Let \((s'_i, t'_i)\) be a representation of \( d'_i \). Let \( S'_i = s'_i \cap A_{j+1} \cap B_j \), and let \( S_i = \bigcup_{j \geq i} S'_j \). Similarly, let \( T'_i = t'_i \cap A_{j+1} \cap B_j \), and let \( T_i = \bigcup_{j \geq i} T'_j \).

Clearly, \( S_i \) and \( T_i \) are vertex-disjoint and any two graphs in \( \bigcup_{i \in \mathbb{N}} \{S_i, T_i\} \) are edge-disjoint. We shall find a ray \( R_i \) in each of the \( S_i \) and a ray \( R'_i \) in each of the \( T_i \). The infinitely many pairs \((R_i, R'_i)\) will then be edge-disjoint 2-rays, as desired.

**Lemma 25.** Each vertex \( v \) of \( S_i \) has degree at most 2. If \( v \) has degree 1 it is contained in \( A_i \cap B_i \).

**Proof.** Clearly, each vertex \( v \) of \( S_i \) that does not lie in any separator \( A_j \cap B_j \) has degree 2, as it is contained in precisely one \( S'_j \), and all the leaves of \( S'_j \) lie in \( A_j \cap B_j \) and \( A_{j+1} \cap B_{j+1} \) as \( d'_i \) is lefty. Indeed, in \( S'_j \) it is an inner vertex of a path and thus has degree 2 in there. If \( v \) lies in \( A_i \cap B_i \) it has degree at most 2, as it is only a vertex of \( S'_i \) for one value of \( j \), namely \( j = i \).

Hence, we may assume that \( v \in A_j \cap B_j \) for some \( j > i \). Thus, \( \sigma[j,j] \) contains \( v \) and \( l : \sigma[j,j] : r \) contains precisely one of the four following subwords:

\[
\text{le}l, \text{lv}r, \text{rel}, \text{rv}r
\]
(Here we use the notation $p : q$ to denote the concatenation of the word $p$ with the word $q$.) In the first case $\tau[j - 1]$ contains $mvm$ as a subword and $\tau[j]$ has no $m$ adjacent to $v$. Then $S_i^{j - 1}$ contains precisely 2 edges adjacent to $v$ and $S_i^j$ has no such edge. The fourth case is the first one with $l$ and $r$ and $j$ and $j - 1$ interchanged.

In the second and third cases, each of $\tau[j - 1]$ and $\tau[j]$ has precisely one $m$ adjacent to $v$. So both $S_i^{j - 1}$ and $S_i^j$ contain precisely 1 edge adjacent to $v$.

As $v$ appears only as a vertex of $S_i^j$ for $\ell = j$ or $\ell = j - 1$, the degree of $v$ in $S_i$ is 2.

**Lemma 26.** There are an odd number of vertices in $S_i$ of degree 1.

**Proof.** By Lemma 25 we have that each vertex of degree 1 lies in $A_i \cap B_i$. Let $v$ be a vertex in $A_i \cap B_i$. Then, $\sigma[i, i]$ contains $v$ and $l : \sigma[i, i] : r$ contains precisely one of the four following subwords:

$$lvr, lrl, rvl, rrv$$

In the first and fourth case $v$ has even degree. It has degree 1 otherwise. As $l : \sigma[i, i] : r$ starts with $l$ and ends with $r$, the word $lvr$ appear precisely once more than the word $rvl$. Indeed, between two occurrences of $lvr$ there must be one of $rvl$ and vice versa. Thus, there are an odd number of vertices with degree 1 in $S_i$. 

**Lemma 27.** $S_i$ includes a ray.

**Proof.** By Lemma 25 every vertex of $S_i$ has degree at most 2 and thus every component of $S_i$ has at most two vertices of degree 1. By Lemma 25, $S_i$ has a component $C$ that contains an odd number of vertices with degree 1. Thus $C$ has precisely one vertex of degree 1 and all its other vertices have degree 2, thus $C$ is a ray.

**Corollary 28.** $G$ contains infinitely many edge-disjoint 2-rays.

**Proof.** By symmetry, Lemma 27 is also true with $T_i$ in place of $S_i$. Thus $S_i \cup T_i$ includes a 2-ray $X_i$. The $X_i$ are edge-disjoint by construction.

Recall that Lemma 15 states that a countable graph with a thin end $\omega$ and arbitrarily many edge-disjoint double rays all whose subrays converge to $\omega$ also has infinitely many edge-disjoint double rays. We are now in a position to prove this lemma.

**Proof of Lemma 15.** By Lemma 20 it suffices to show that $G$ contains a subgraph $H$ with a single end which is thin such that $H$ has infinitely many edge-disjoint 2-rays. By Corollary 17, $G$ has a subgraph $H$ with a single end which is thin such that $H$ has arbitrarily many edge-disjoint 2-rays. But then by the argument above $H$ contains infinitely many edge-disjoint 2-rays, as required.
With these tools at hand, the remaining proof of Theorem 1 is easy. Let us collect the results proved so far to show that each graph with arbitrarily many edge-disjoint double rays also has infinitely many edge-disjoint double rays.

**Proof of Theorem 1.** Let \( G \) be a graph that has a set \( D_i \) of \( i \) edge-disjoint double rays for each \( i \in \mathbb{N} \). Clearly, \( G \) has infinitely many edge-disjoint double rays if its subgraph \( \bigcup_{i \in \mathbb{N}} D_i \) does, and thus we may assume without loss of generality that \( G = \bigcup_{i \in \mathbb{N}} D_i \). In particular, \( G \) is countable.

By Corollary 10 we may assume that each connected component of \( G \) includes only finitely many ends. As each component includes a double ray we may assume that \( G \) has only finitely many components. Thus, there is one component containing arbitrarily many edge-disjoint double rays, and thus we may assume that \( G \) is connected.

By Corollary 8 we may assume that all ends of \( G \) are thin. Thus, as mentioned at the start of Section 4 there is a pair of ends \((\omega, \omega')\) of \( G \) (not necessarily distinct) such that \( G \) contains arbitrarily many edge-disjoint double rays each of which converges precisely to \( \omega \) and \( \omega' \). This completes the proof as, by Lemma 13 \( G \) has infinitely many edge-disjoint double rays if \( \omega \) and \( \omega' \) are distinct and by Lemma 15 \( G \) has infinitely many edge-disjoint double rays if \( \omega = \omega' \).

\[ \square \]

### 6 Outlook and open problems

We will say that a graph \( H \) is edge-ubiquitous if every graph having arbitrarily many edge-disjoint \( H \) also has infinitely many edge-disjoint \( H \).

Thus Theorem 1 can be stated as follows: the double ray is edge-ubiquitous. Andreae’s Theorem implies that the ray is edge-ubiquitous. And clearly, every finite graph is edge-ubiquitous.

We could ask which other graphs are edge-ubiquitous. It follows from our result that the 2-ray is edge-ubiquitous. Let \( G \) be a graph in which there are arbitrarily many edge-disjoint 2-rays. Let \( v * G \) be the graph obtained from \( G \) by adding a vertex \( v \) adjacent to all vertices of \( G \). Then \( v * G \) has arbitrarily many edge-disjoint double rays, and thus infinitely many edge-disjoint double rays. Each of these double rays uses \( v \) at most once and thus includes a 2-ray of \( G \).

The vertex-disjoint union of \( k \) rays is called a \( k \)-ray. The \( k \)-ray is edge-ubiquitous. This can be proved with an argument similar to that for Theorem 1. Let \( G \) be a graph with arbitrarily many edge-disjoint \( k \)-rays. The same argument as in Corollaries 10 and 8 shows that we may assume that \( G \) has only finitely many ends, each of which is thin. By removing a finite set of vertices if necessary we may assume that each component of \( G \) has at most one end, which is thin. Now we can find numbers \( k_C \) indexed by the components \( C \) of \( G \) and summing to \( k \) such that each component \( C \) has arbitrarily many edge-disjoint \( k_C \)-rays. Hence, we may assume that \( G \) has only a single end, which is thin. By Lemma 16 we may assume that \( G \) is locally finite.
In this case, we use an argument as in Subsection 5.3. It is necessary to use $k$-shapes instead of 2-shapes but other than that we can use the same combinatorial principle. If $C_1$ and $C_2$ are finite sets, a $(C_1, C_2)$-shaping is a pair $(c_1, c_2)$ where $c_1$ is a partial colouring of $\mathbb{N}$ with colours from $C_1$ which is defined at all but finitely many numbers and $c_2$ is a colouring of $\mathbb{N}^{(2)}$ with colours from $C_2$ (in our argument above, $C_1$ would be the set of all $k$-shapes and $C_2$ would be the set of all allowed $k$-shapes for all pairs of $k$-shapes).

**Lemma 29.** Let $D_1, D_2, \ldots$ be a sequence of sets of $(C_1, C_2)$-shapings where $D_i$ has size $i$. Then there are strictly increasing sequences $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$ and subsets $S_n \subseteq D_{i_n}$ with $|S_n| \geq n$ such that

- for any $n \in \mathbb{N}$ all the values of $c_1(j_n)$ for the shapings $(c_1, c_2) \in S_{n-1} \cup S_n$ are equal (in particular, they are all defined).
- for any $n \in \mathbb{N}$, all the values of $c_2(j_n, j_{n+1})$ for the shapings $(c_1, c_2) \in S_n$ are equal.

Lemma 29 can be proved by the same method with which we constructed the sets $D_i'$ from the sets $D_i$. The advantage of Lemma 29 is that it can not only be applied to 2-rays but also to more complicated graphs like $k$-rays.

A *talon* is a tree with a single vertex of degree 3 where all the other vertices have degree 2. An argument as in Subsection 5.2 can be used to deduce that talons are edge-ubiquitous from the fact that 3-rays are. However, we do not know whether the graph in Figure 2 is edge-ubiquitous.

![Figure 2: A graph obtained from 2 disjoint double rays, joined by a single edge. Is this graph edge-ubiquitous?](image)

We finish with the following open problem.

**Problem 30.** Is the directed analogue of Theorem 1 true? More precisely: Is it true that if a directed graph has arbitrarily many edge-disjoint directed double rays, then it has infinitely many edge-disjoint directed double rays?

It should be noted that if true the directed analogue would be a common generalization of Theorem 1 and the fact that double rays are ubiquitous with respect to the subgraph relation.

## 7 Acknowledgement

We appreciate the helpful and accurate comments of a referee.
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