Dark quantum droplets in beyond-mean-field Bose-Einstein condensate mixtures

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Quantum liquid-like states of matter have been realized in an ongoing series of experiments with ultracold Bose gases. By means of analytical and theoretical methods we identify the specific criteria for the existence of dark solitons in beyond-mean-field condensates, revealing how these excitations exist for both repulsive and attractive interactions, the latter leading to dark quantum droplets with properties intermediate between a dark soliton and a quantum droplet. The dark quantum droplet’s physical characteristics are investigated, including calculation of the integrals of motion, revealing their sensitive dependence on physical parameters relevant to the current generation of experiments with quantum gases in the beyond-mean-field limit.

I. INTRODUCTION

Liquid states of matter give rise to a plethora of fluidic phenomena caused by the interaction of atoms with each other, external forces and other matter [1]. For classical fluids, intermolecular potentials give rise to macroscopic consequences such as surface tension and viscosity, as well as transient effects like the Rayleigh-Taylor instability and turbulence, phenomena that can be observed on terrestrial [2] and astronomical [3] scales. The intrinsic properties of fluids depend critically on their thermodynamic environment, quantum liquids can also share some of the properties of their classical counterparts while also exhibiting unique and unexpected phenomena with no classical analogue [4].

The last few years have seen a series of groundbreaking experiments with degenerate atomic Bose-Einstein condensates which have demonstrated the capacity of these intrinsically weakly correlated systems to manifest liquid-like states of matter in the form of quantum droplets, made from highly magnetic [5–8] or mixtures [9–13] of quantum gases. These surprising discoveries have been attributed to purely quantum mechanical effects in the form of the Lee-Huang-Yang (LHY) correction [14], which provides the stabilization required to avoid instability originating from collisional forces.

While there has been intense focus on understanding the ground states of many-body systems, their excitations also play a crucial role in understanding their fundamental behaviour. Recently there has been renewed experimental interest in realizing nonlinear excitations with quantum gases such as dark solitons [15–17] and domain walls [18, 19] which provide insight into reduced dimensionality topology in a highly controllable setting. Such states could provide an important resource for future applications in atromtronics [20] as well as providing fundamental insight into the physics of lower dimensional quantum systems [21]. Quantum gases possessing internal degrees of freedom represent an important testing ground for many body phenomena. These additional degrees of freedom can facilitate unique quantum states that sensitively depend on the nature of the atomic interactions [22]. The presence of attractive interactions in these systems can ordinarily lead to the collapse of the quantum state; however it was shown theoretically that such a system can in principle be stabilized by beyond-mean-field effects [23]. This stimulated an intense interest in the phenomenology of beyond-mean-field physics in these systems – here fundamental questions such as the role of dimensionality [24–26], confinement [27, 28], dynamical [29, 30], collective [31], coherent [32] and gauge couplings [33], as well as non-equilibrium [34, 35] effects have provided key insight into the unusual liquid-like properties of these ultra-dilute droplets [36, 37]. Complementary to their existence in degenerate atomic systems, droplet states have also been investigated in other systems such as photonic [38], optomechanical [39], as well as in the Helium liquids [40].

While previous works have addressed aspects of the fundamental nature of liquid-like ground states in quantum gases, recent work has focussed on investigating the

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FIG. 1. (color online) Dark quantum droplet schematic. The three-dimensional tubes represent individual condensate mixtures. The grey stripes represent the dark soliton (background) and dark quantum droplet (foreground). The red-white-blue shading indicates the local phase of the excitation.
excitations in these systems which possess non-trivial phase windings such as kinks [41], vortices [42], and dark solitons in the dipolar system [43].

The purpose of this work is to elucidate the fundamental criteria for the existence of dark quantum droplets (DQDs) – dark soliton-like excitations that exist in the beyond-mean-field model with attractive, rather than repulsive interactions in the cubic-quadratic Schrödinger system, as well as characterising their fundamental properties with a complimentary combination of numerical and analytical approaches. A schematic representation of the dark quantum droplet is presented in Fig. 1.

II. BEYOND-MEAN-FIELD MODEL

The energy of $N = N_\uparrow + N_\downarrow$ Bose particles with mass $m$ forming a two-component homogeneous atomic Bose-Einstein condensate can be written as

$$E = \int d^3r \left[ \frac{\hbar^2}{2m} \sum_j |\nabla \psi_j(r)|^2 +\sum_{j,k} \frac{g_{jk}}{2} n_j(r)n_k(r) \right] \quad (1)$$

here $j,k \in \{\uparrow, \downarrow\}$, $g_{jk} = 4\pi\hbar^2 a_{jk}/m$ defines the scattering parameter between atoms and $n_j(r) \equiv |\psi_j(r)|^2$ defines the atomic density for component $j$. In order to understand the effect of beyond-mean-field effects, the underlying many body Hamiltonian is diagonalized within the standard Bogoliubov de-Gennes formalism for the weakly interacting limit, from which the one-dimensional ground state energy density is [24]

$$E_{1D} = \left( \sqrt{g_{\uparrow\uparrow}} n_\uparrow - \sqrt{g_{\downarrow\downarrow}} n_\downarrow \right)^2/2 + \delta g \left( \sqrt{g_{\uparrow\uparrow}} n_\uparrow + \sqrt{g_{\downarrow\downarrow}} n_\downarrow \right)^2 \left( g_\uparrow + g_\downarrow \right)^2$$

$$+ \frac{2\sqrt{m}}{3\pi\hbar} \left( g_\uparrow n_\uparrow + g_\downarrow n_\downarrow \right)^{3/2}, \quad (2)$$

here $g = (g_{\uparrow\uparrow} n_\uparrow + g_{\downarrow\downarrow} n_\downarrow)/n$, $\delta g = g_{\uparrow\downarrow} \pm \sqrt{g_{\uparrow\uparrow} g_{\downarrow\downarrow}}$, and $n = n_\uparrow + n_\downarrow$ [25]. Assuming an equal number of atoms in the spin mixture such that $n_\uparrow = n_\downarrow \equiv n$ and equal inter-component interaction strengths $g_{\uparrow\uparrow} = g_{\downarrow\downarrow}$, Eq. (2) simplifies to $E_{1D} = \delta gn^2 - 4\sqrt{2m}(gn)^{3/2}/(3\pi\hbar)$. Then within the local density approximation an effective equation of motion can be derived from the chemical potential $\mu_{QF}[\psi] = \partial E_{1D}/\partial N$ giving

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - \frac{\sqrt{2m}}{\pi\hbar} g^{3/2}|\psi|^2 + \delta g |\psi|^2 \psi. \quad (3)$$

Equation (3) describes the dynamics of the binary system in the equal (miscible) spin limit in the form of a cubic-quadratic nonlinear Schrödinger system. Let us consider the fundamental solutions of Eq. (3) in the limits of interest, $g \to 0$ with $\psi(\mu, x \to \pm \infty) = \pm \sqrt{n_0}$ and for $g \neq 0$ with $\psi(\mu, x \to \pm \infty) = 0$. In the first limit the system is integrable with the well known family of dark soliton solutions $\psi_{DS}(\mu_{DS}, x) = \sqrt{n_0} \beta \tanh(\sqrt{\beta}/2 x/\xi_{DS}) + i \sqrt{1 - \beta^2}$, where the healing length is $\xi_{DS} = h/\sqrt{mn_0}\delta g$ with velocity $u_\beta = \sqrt{1 - u^2}$ where $0 < u < c$, is the speed of sound and $n_0 = \lim_{x \to \pm \infty} |\psi(\mu, x)|^2$ defines the constant asymptotic density. Then we consider the second situation where Eq. (3) possesses instead a quantum droplet solution [24]

$$\psi_{QD}(\mu, x) = \frac{\sqrt{n_0} \mu/\mu_{QD}}{1 + \sqrt{1 - \mu/\mu_{QD}} \cosh(\sqrt{2\mu n_0} x/\hbar)}, \quad (4)$$
here the flat-topped droplet state forms as \( \mu \rightarrow \mu_{\text{QD}} \) where \( \mu_{\text{QD}} = -4mg^2/\pi^2\hbar^2g \). We consider the general situation where both \( g \neq 0 \) and \( \delta g \neq 0 \). As such the model Eq. (3) has a number of important symmetries. From a physical point of view we consider the regularized versions of the atom number, momentum and energy given respectively by

\[
N_{\text{QD}} = \int dx \left[ n_0 - |\psi(x)|^2 \right], \quad (5a)
\]

\[
P_{\text{QD}} = \frac{i\hbar}{2} \int dx \left[ \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right] - \hbar n_0 \Delta \phi, \quad (5b)
\]

\[
E_{\text{QD}} = \int dx \left[ \frac{\hbar^2}{2m} \left| \psi \right|^2 + \frac{\delta g}{2} (n_0 - |\psi|^2)^2 \right.
- \left. \frac{2\sqrt{2m}}{3\hbar} g^{1/2} \left\{ \left| \psi \right|^3 - \frac{3}{2} \sqrt{n_0} |\psi|^2 + \frac{1}{2} \sqrt{n_0^3} \right\} \right]. \quad (5c)
\]

here the phase difference \( \Delta \phi = \arg(+\infty) - \arg(-\infty) \). As well as the three integrals of motion Eqs. (5), the model Eq. (3) accommodates distinct dilation invariances in the limits \( \delta g = 0 \) and \( g = 0 \). A dilation transformation is a scaling such that \( x \rightarrow \sqrt{\alpha} x \) and \( t \rightarrow \alpha t \) for \( \alpha \in \mathbb{R}_{>0} \), and will in general leave a Schrödinger system with a single nonlinearity \( |\psi|^n \) invariant if \( \psi(x,t) \rightarrow \alpha^{1/n} \psi(\sqrt{\alpha}x,\alpha t) \). Then, we can see that when \( \delta g = 0 \) the dark soliton solution obeys \( \psi_0 \rightarrow \sqrt{\alpha} \psi_0(\sqrt{\alpha}x,\alpha t) \) while for \( g = 0 \) the quantum droplet undergoes the dilation \( \psi_{\text{QD}} \rightarrow \alpha \psi_{\text{QD}}(\sqrt{\alpha}x,\alpha t) \). The competition between the two length scales associated with the interaction parameter \( g \) and \( \delta g \) facilitates unusual phenomenology in this nonlinear system.

From Eq. (3) we can define a set of dimensionless units appropriate for numerical simulations. The healing length \( \xi_0 = \hbar/\sqrt{m|\mu_0|} \) with \( \mu_0 = -2m n_0 g^2/\pi \hbar + \delta gn_0 \) defines the intrinsic length of the system, from this a time scale \( \hbar/|\mu_0| \) follows. The resulting dimensionless interaction strength used in the numerical simulations is \( \sqrt{9/\mu_{\text{QD}}}/2n_0 \delta g \).

### III. DARK QUANTUM DROPLETS

#### A. Dark soliton to dark quantum droplet crossover

In this section we explore the nature of the solutions to Eq. (3). Since we are interested in the excited states, we use an iterative (Newton-Raphson) approach to compute these states. An overview of the numerical procedure is given in the Appendix. We explore the transition from a dark soliton excitation to the dark quantum droplet in Fig. 2. In panel (a) we solve the cubic-quadratic Schrödinger equation (Eq. (3)) as a function of the interaction strength for both the excited (dark soliton-like excitation) and quantum droplet ground state. Each dark soliton solution is computed using a Newton-Raphson method with fixed von Neumann boundary conditions. From this, the atom number Eq. (5a) is calculated. This in turn is used as the input for the ground state quantum droplet’s atom number. Each quantum droplet’s ground state is computed using an imaginary time Fourier split-step method. The chemical potential is plotted for both situations, and for each fixed boundary condition, it is found that the chemical potential of the dark soliton eventually meets that of the droplet state. We can calculate the critical point at which this occurs by equating the quantum droplet’s chemical potential \( \mu_{\text{QD}} \) with the homogeneous chemical potential \( \mu_0 \), which leads to the

![FIG. 3. (color online) Dark quantum droplet root-mean-squared width. Panel (a) shows Eq. (9), \( \sqrt{\langle x^2 \rangle/\xi_{\text{QD}}} \) as a function of \( \mu/\mu_{\text{QD}} \), the minima occurs for \( \mu \approx 0.8306 \mu_{\text{QD}} \), while the inset shows Eq. (10) for several values of \( n_0 \xi_{\text{QD}} \). Panel (b) shows \( \sqrt{\langle x^2 \rangle/\xi_{\text{QD}}} \) instead as a function of \( N_{\text{QD}} \), while the inset displays a log-log plot for \( n_0 \xi_{\text{QD}} = 1/2 \). The minima of \( \sqrt{\langle x^2 \rangle/\xi_{\text{QD}}} \) in (b) are plotted in (c) (green solid) with the red circles correspond to the locations of the four curves individual minima.](image)
with corresponding critical chemical potential $\mu_{\text{crit}} = -n_0 \delta g/2$. Very close to this point, the dark soliton acquires a profile resembling an inverted quantum droplet with a hollow central region, but with an asymmetric wave function. The final simulation point is chosen by including the pre-factor $f(\lambda) = 1 - 10^{-3}$ in Eq. (6) with $\lambda = 3$ for Fig. 2. Panels (i-iv) show a number of example density profiles taken from the red-dotted data ($n_{\text{DS}}^0 \xi_0 \sim 0.29$). Far from the transition point at weak attractive (repulsive) atomic interactions a broad quantum droplet (narrow dark soliton) is observed (panels (i) and (ii) respectively). Then, very close to the point at which the chemical potentials cross, the dark soliton develops a wide hollow region around its core, while the droplet state at this point becomes narrow and tall (panels (iii) and (iv) respectively). Following this panels (b) and (c) compare the solutions $\psi_S$, scaled to the asymptotic spatial values and the accompanying phase $\phi(x)$ respectively for $\mu \approx 0.26 \mu_0$ (dark soliton) and $\mu \approx 0.74 \mu_0$ (dark quantum droplet). A heat map of the red-dotted transition data from (a) is shown in (d), along with the accompanying quantum droplet ground sate data in (e), while the dashed lines in panels (d) and (e) correspond to the solutions (iii) and (iv) discussed above. The final panel (f) presents the atom number $N_{\text{DQD}}$ (Eq. (5a)) for both situations, showing the gradual increase that occurs as the transition point is approached.

**B. Root-mean-squared width**

The results presented in Fig. 2 reveal that as the chemical potential of the dark soliton approaches that of the quantum droplet, the soliton’s profile resembled that of an inverted droplet. Previous experimental studies of the soliton to droplet crossover [11, 44] have established that one can define a soliton at relatively small atom numbers, while for large atom numbers a quantum droplet emerges, we can perform a similar distinction here to understand the crossover from a dark soliton to a dark quantum droplet. From Fig. 2 (iii-iv) we can infer that

$$\lim_{\mu \to \mu_{\text{QD}}} \left( n_{\text{QD}}(\mu, x) + n_{\text{DQD}}(\mu, x) \right) = n_0. \quad (7)$$

Equation (7) will allow us to calculate observables of the dark quantum droplet state. The mean-squared width is an important characteristic which can be used to characterise the behaviour of the dark droplet as the chemical potential approaches that of the quantum droplet’s. Similarly to the regularized forms of the atom number, momentum and energy (Eqs. (5a)-(5c)) we can also compute the mean-squared width from

$$\langle x^2 \rangle = \frac{1}{N_{\text{DQD}}(\mu)} \int_{-\infty}^{\infty} dx x^2 \left[n_0 - n_{\text{DQD}}(\mu, x)\right],$$

$$= \frac{1}{N_{\text{DQD}}(\mu)} \int_{-\infty}^{\infty} dx x^2 \lim_{\mu \to \mu_{\text{QD}}} n_{\text{QD}}(\mu, x) \quad (8)$$

here the second line, Eq. (8) has been written using Eq. (7). From here the known solution for the droplet $n_{\text{QD}}(\mu, x) \equiv |\psi_{\text{QD}}(\mu, x)|^2$ (Eq. (4)) can be used to obtain an expression for both the atom number $N_{\text{DQD}}(\mu)$ and the mean-squared width $\langle x^2 \rangle$ using the inversion formulæ for the polylogarithms for the latter, yielding

$$\langle x^2 \rangle^2 = \frac{N_0}{N_{\text{DQD}}(\mu)} \mu_{\text{QD}} \left[ \frac{1}{3} \left( \text{arcsech}^2 \sqrt{1 - \frac{\mu}{\mu_{\text{QD}}}} + \pi^2 \text{arcsech} \sqrt{1 - \frac{\mu}{\mu_{\text{QD}}}} - \frac{\mu}{\mu_{\text{QD}}} \left( \text{arcsech}^2 \sqrt{1 - \frac{\mu}{\mu_{\text{QD}}}} \right) \right) \right],$$

$$\xi_{\text{QD}} = h/\sqrt{m|\mu_{\text{QD}}|}$$

with the constant $N_0 = 2n_0 \xi_{\text{QD}} = \sqrt{2n_0 \hbar^2/m|\mu_{\text{QD}}|}$. Then the atom number $N_{\text{DQD}}(\mu)$ appearing in Eq. (9) can be evaluated in a sim-
in a similar manner, giving

\[
\frac{N_{\text{DQD}}(\mu)}{N_0} = 2\text{arctanh} \left[ \frac{\sqrt{\mu / \mu_{\text{DQD}}}}{1 + \sqrt{1 - \mu / \mu_{\text{DQD}}}} \right] - \sqrt{\frac{\mu}{\mu_{\text{DQD}}}}. \tag{10}
\]

Using Eqs. (9) and (10) we can understand the intrinsic properties of the dark quantum droplet. First, let us derive the asymptotic behaviour of Eqs. (9) and (10) when \(N_{\text{DQD}} \gg 1\). For the atom number, one finds the relationship between the chemical potential and \(N_{\text{DQD}}\) is

\[
N_{\text{DQD}}(\mu) / N_0 = \psi(2/(\sqrt{1 - \mu / \mu_{\text{DQD}}})).
\]

Hence the atom number \(N_{\text{DQD}}(\mu)\) diverges logarithmically as \(\mu \to \mu_{\text{DQD}}\). Expanding Eq. (9) for \(\mu \to \mu_{\text{DQD}}\) and using the asymptotic form of Eq. (10), the root-mean square width in the limit \(N_{\text{DQD}} \gg 1\) is

\[
\frac{\sqrt{\langle x^2 \rangle}}{\xi_{\text{DQD}}} = \frac{N_{\text{DQD}}}{\mu - \mu_{\text{DQD}}} \sqrt{3} N_0, \tag{11}
\]

showing that the effective width of the dark quantum droplet diverges linearly in a fashion qualitatively similar to the quantum droplet [29].

Figure 3 shows the root-mean-squared width of the dark quantum droplet, Eq. (9). Panel (a) shows the behaviour of the width \(\sqrt{\langle x^2 \rangle}/\xi_{\text{DQD}}\) as a function of the chemical potential \(\mu / \mu_{\text{DQD}}\), while the inset shows the atom number Eq. (10) for several values of the background density \(n_0\xi_{\text{DQD}} = 1/2, 1, 2, 4\); increasing \(n_0\xi_{\text{DQD}}\) has the effect of giving an overall increase to \(N_{\text{DQD}}\). The second panel (b) shows the root-mean-squared width \(\sqrt{\langle x^2 \rangle}/\xi_{\text{DQD}}\) as a function of the atom number, for the same values of background density shown in the inset of (a). Increasing \(n_0\xi_{\text{DQD}}\) has the effect of stretching \(\sqrt{\langle x^2 \rangle}/\xi_{\text{DQD}}\) such that the linear part \((N_{\text{DQD}}(\mu) \gg 1)\) associated with the dark quantum droplet occurs at larger values of \(N_{\text{DQD}}(\mu)\). The minima of \(\sqrt{\langle x^2 \rangle}/\xi_{\text{DQD}}\) also shift to larger values of \(N_{\text{DQD}}(\mu)\) as \(n_0\xi_{\text{DQD}}\) is increased. The inset of Fig. 3(b) shows the dataset for \(n_0\xi_{\text{DQD}} = 1/2\) in (b) in a log-log plot. The dashed lines show the asymptotic forms of Eq. (9) for \(\mu / \mu_{\text{DQD}} \to 0\), \(\sqrt{\langle x^2 \rangle} \sim N_{\text{DQD}}^{-1/3}\) and \(\mu / \mu_{\text{DQD}} \to 1\), \(\sqrt{\langle x^2 \rangle}/\xi_{\text{DQD}} = N_{\text{DQD}}/(\sqrt{3} N_0)\) [53], the second of these limits being appropriate to the dark quantum droplet. The minima of \(\sqrt{\langle x^2 \rangle}/\xi_{\text{DQD}}\) are plotted in (c) as a function of \(N_{\text{DQD}}\) (green solid) with the red circles corresponding to the locations of the four curves minima in (b). The shaded blue and green regions indicate the parameter regimes where we expect dark solitons and dark quantum droplets respectively.

A comparison of the dark quantum droplet’s analytical atom number and root-mean-squared width with the numerically obtained values is explored next in Fig. 4. Stationary state solutions to Eq. (3) are shown in (a) for \(n_0\xi_0 = 0.36\). The interaction strength is chosen using Eq. (6) again using the pre-factor \(f(\lambda) = 1 - 10^{-\lambda}\) with \(\lambda = 1, 2, \ldots, 9\). The analytic atom number of Eq. (10) (solid blue) is plotted along with the equivalent values computed from the numerical (orange pluses) data in panel (b), here good agreement is found as \(\mu \to \mu_{\text{DQD}}\). The root-mean-squared width \(\sqrt{\langle x^2 \rangle}/\xi_0\) is compared from Eq. (9) (solid green) and the numerical data (blue pluses). The agreement is found to improve as \(\lambda\) increases, and it was found that due to the underlying logarithmic divergence of \(N_{\text{DQD}}(\mu)\) as \(\mu \to \mu_{\text{DQD}}\) obtaining a convergence between the analytical and numerical results in general requires very large \(\lambda\), which becomes impractical for numerical simulations, but could be an interesting question to explore in a future experiment. The green solid and red dashed lines in (a) are computed...
from Eqs. (9) and (11) respectively.

C. Dark quantum droplet solutions and integrals of motion

Figure 5 presents numerical solutions to Eq. (3) as a function of the atomic interactions close to the point where the chemical potential of the dark quantum droplet approaches that of the quantum droplet, again using Eq. (6) with the pre-factor \( f(\lambda) = 1 - 10^{-\lambda} \) for \( \lambda = 4 \). Panel (a) shows a heat map of the solutions for \( 0.15 < \sqrt{g} |\mu_{\text{QD}}|/2n_0 \delta g < 1 \). As the interaction strength is increased the dark quantum droplet’s solutions change from being broad and shallow at weak interactions to narrow and deep at relatively stronger ones. Then in panel (b) the solutions are shown for \( 2.1 > \sqrt{g} |\mu_{\text{QD}}|/2n_0 \delta g > 1 \). Instead here the dark quantum droplets have a narrow profile at weaker interactions while for stronger interactions the profile approaches to a solution with approximately fixed width and slowly increasing depth. Some of the individual solutions from (a) (see dashed lines) are presented in (c), scaled such that asymptotic values are the same for ease of comparison. Note that these solutions show some similarity to recent work concerning vortices in two-component immiscible condensates [45–48], whose radial profiles can resemble an inverted droplet.

The final panel (d) shows the chemical potential \( \mu/\mu_{\text{QD}} \) (left axis, blue pluses) and numerically computed root-mean-square width \( \sqrt{\langle x^2 \rangle / \xi_{\text{QD}}} \) (right axis, orange crosses) for the data presented in (a) and (b). As the interaction strength approaches zero, the corresponding chemical potential also approaches zero, resulting in the root-mean-square width becoming very large. Then for \( \sqrt{g} |\mu_{\text{QD}}|/2n_0 \delta g > 1 \), the chemical potential converges to a constant value of \( \mu \sim 0.9n_0 \), and consequently the root-mean-square width approaches a constant value.

The integrals of motion Eqs. (5a)-(5c) are plotted from the root-mean-square width approaches a constant value. For future studies, it would be intriguing to understand the behaviour of the dark quantum droplet in a harmonic trap, and how their oscillation frequency depends on the properties of the excitation [56]. The dynamical behaviour, such as constructing Toda-like lattices provides another future direction [57].

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Appendix: Newton-Raphson method

Here we give an overview of the numerical method used to procure the dark quantum droplet solutions to the cubic-quadratic Schrödinger equation (Eq. (3) of the text). This type of approach has been used previously to study excitations in superfluid systems such as vortices [49], solitons in dipolar [50, 51] and magnetic systems [52]. Our system differs from previous studies due to the presence of mixed nonlinearities. We consider a general scheme at finite velocity. First we write a function whose solutions we seek in the the Galilean-boosted frame as

\[
F[\psi] = \left( \hat{H}_{\text{cqGPE}} - u\hat{p}_x - \mu \right) \psi, \tag{A.1}
\]

where \( u \) is the excitations velocity. Then Eq. (A.1) can be translated into the iterative scheme

\[
F_u(\psi^{p+1}) \approx F(\psi^p) + \sum_{v=1}^{N} J_{u,v} \delta \psi_v \approx 0, \tag{A.2}
\]
Here \( \delta \phi = \psi^{p+1} - \psi^p \) and \( \mathcal{J}_{x,v} \) defines the matrix elements of the Jacobian. The solutions to Eq. (A.1) are in general complex valued, and since Newton-Raphson methods only work with real data we write the discrete \( \psi(x) \) comprising \( N \) complex numbers as \( 2N \) real numbers \( \delta \psi \) of \( \psi \) as \( \delta \psi = f_j + i g_j \) such that \( \text{Re}(\psi(x)) = f_j \) and \( \text{Im}(\psi(x)) = g_j \), the second subscript referring to the real and imaginary components. Then one can write the discrete form of Eq. (A.1) as

\[
f_{j,s} = -\frac{\hbar^2}{2m} \left[ \frac{\psi_{j-1,s} - 2\psi_{j,s} + \psi_{j+1,s}}{2\Delta x^2} \right] + (2s - 1)\hbar u \frac{\psi_{j+1,1-s} - \psi_{j-1,1-s}}{2\Delta x} + \frac{\sqrt{2m}}{\pi \hbar} g^{3/2} \sqrt{\psi_{j,0}^2 + \psi_{j,1}^2} \psi_{j,s} - \mu \psi_{j,s},
\]

(A.3)

The boundary conditions for the problem are treated as the von Neumann type, such that

\[
\left. \frac{d\psi}{dx} \right|_{x=\pm L} = 0,
\]

(A.4)

which translates into taking \( f_{1,s} = f_{0,s} = 0 \) and \( f_{N+1,s} = f_N,s = 0 \) for the kinetic term and \( f_{2,s} = f_{0,s} = 0 \) and \( f_{N+1} - f_{N-1,s} = 0 \) for the momentum operator. The matrix elements of the Jacobian appearing in Eq. (A.2) are found from \( \mathcal{J}_{j,s} = \partial f_{j,s} / \partial \psi_{k,r} \). Using \( \partial \psi_{j,s} / \partial \psi_{k,r} = \delta_{j,k}\delta_{s,r} \) we obtain

\[
\mathcal{J}_{j,s} = -\frac{\hbar^2}{2m} \delta_{j,r} \left[ \frac{\delta_{k,j-1} - 2\delta_{k,j} + \delta_{k,j+1}}{\Delta x^2} \right] + (2s - 1)\hbar u \frac{\delta_{k,j+1} - \delta_{k,j-1}}{2\Delta x} + \frac{\sqrt{2m}}{\pi \hbar} g^{3/2} \delta_{k,j} \frac{\psi_{j,s}^2 + \psi_{j,1,s}^2}{\sqrt{\psi_{j,0}^2 + \psi_{j,1}^2}} \delta_{s,0} - \mu \delta_{k,j} \delta_{s,r},
\]

(A.5)

which defines a \( 2N \times 2N \) matrix. Then stationary solutions can be obtained to Eq. (3) using Eqs. (A.1)-(A.5) using a tolerance based approach for \( \delta \psi \). As such we employ the Frobenius norm \( ||\delta \psi|| = \sum_{j=1}^{2N} \delta \psi \) as a measure which is deemed convergence after falling below a predefined value. Computation of \( \delta \psi \) at each step is accomplished by solving the linear system \( \mathcal{J} \delta \psi = -F \). The Newton-Raphson method requires an initial guess for \( \psi(x) \), which we take as the dark soliton solution to the cubic Schrödinger equation. An example Python script for generating a dark quantum droplet can be found here [58].

[1] P. M. Chaikin, and T. C. Lubensky, Principles of Condensed Matter Physics, Cambridge University Press, Cambridge (1995).
[2] Y. A. Cengel, Fluid Mechanics: Fundamentals and Applications McGraw Hill (2017).
[3] C. Clarke and B. Carswell, Principles of Astrophysical Fluid Dynamics Cambridge University Press, Cambridge (2007).
[4] A. J. Leggett, Rev. Mod. Phys. 71, S318 (1999).
[5] H. Kadau, M. Schmitt, M. Wenzel, C. Wink, T. Maier, I. F.-Barbut, and T. Pfau, Nature 530, 194 (2016).
[6] I. F.-Barbut, H. Kadau, M. Schmitt, M. Wenzel, and T. Pfau, Phys. Rev. Lett. 116, 215301 (2016).
[7] M. Schmitt, M. Wenzel, F. Böttcher, I. F.-Barbut, and T. Pfau, Nature 539, 259 (2016).
[8] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wüchtler, L. Santos, and F. Ferlaino, Phys. Rev. X 6, 041039 (2016).
[9] C. R. Cabrera, L. Tanzi, J. Sanz, B. Naylor, P. Thomas, P. Cheiney, and L. Terrull, Science 359, 301 (2018).
[10] G. Semeghini, G. Ferioli, L. Masi, C. R. Cabrera, L. Terrull, Phys. Rev. X 6, 041039 (2016).
[11] G. Semeghini, G. Ferioli, L. Masi, G. Giusti, G. Modugno, M. Inguscio, and M. Fattori, Phys. Rev. Lett. 120, 135301 (2018).
[12] G. Semeghini, G. Ferioli, L. Masi, G. Giusti, G. Modugno, M. Inguscio, A. Gallemí, A. Recati, and M. Fattori, Phys. Rev. Lett. 122, 090401 (2019).
In the limit $\mu/\mu_{QD} \to 0$ Eq. (10) becomes $\mu/\mu_{QD} = (3N_{DQD}/N_0)^{2/3}$ and the root-mean-squared width Eq. (9) is $\langle x^2 \rangle/\xi_{QD} = (\pi^2 - 6)/9(\sqrt{3}N_0/N_{DQD})^{1/3}$. This limit is appropriate for the quantum droplet [29].