Faster sparse polynomial interpolation of straight-line programs over finite fields

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Abstract

We present a faster Monte Carlo algorithm for the interpolation of a straight-line program to find a sparse polynomial \( f \) over an arbitrary finite field of size \( q \). We assume \textit{a priori} bounds \( D \) and \( T \) are given on the degree and number of terms of \( f \). The approach presented in this paper is a hybrid of the diversified and recursive interpolation algorithms, the two previous fastest known probabilistic methods for this problem. By making effective use of the information contained in the coefficients themselves, this new algorithm improves on the bit complexity of previous methods by a “soft-\( O \)h” factor of \( T, \log D, \) or \( \log q \).

1 Introduction

Let \( \mathbb{F}_q \) be a finite field of size \( q \) and consider a “sparse” polynomial

\[
 f = \sum_{i=1}^{t} c_i z^{e_i} \in \mathbb{F}_q[z],
\]
where the \( c_1, \ldots, c_t \in \mathbb{F}_q \) are nonzero and the exponents \( e_1, \ldots, e_t \in \mathbb{Z}_{\geq 0} \) are distinct. Suppose \( f \) is provided as a straight-line program, a simple branch-free program which evaluates \( f \) at any point (see below for a formal definition). Suppose also that we are given upper bounds on the degree \( D \geq \text{deg}(f) \) and “sparsity” \( T \geq t \). Our goal is to recover the standard form (1.1) for \( f \), that is, to recover the coefficients \( c_i \) and their corresponding exponents \( e_i \) as in (1.1). Our main result is as follows.

**Theorem 1.1.** Let \( f \in \mathbb{F}_q[z] \) with at most \( T \) terms and degree at most \( D \), and let \( 0 < \epsilon \leq 1/2 \). Suppose we are given a division-free straight-line program \( S_f \) of length \( L \) that computes \( f \). Then there exists an algorithm (presented below) that interpolates \( f \) with probability at least \( 1 - \epsilon \) with a cost of

\[
\tilde{O}(LT \log^2 D (\log D + \log q) \log \frac{1}{\epsilon})
\]

bit operations.

This cost improves on previous methods by a factor of \( T \), \( \log D \), or \( \log q \), and may lays the groundwork for even further improvements. See Table 1 below for a detailed comparison.

### 1.1 Straight-line programs and sparse polynomials

The interpolation algorithm presented in this paper is for straight-line programs, though it could be adapted to other more traditional models of interpolation. Informally, a straight-line program is a very simple program, with no branches or loops, which evaluates a polynomial at any point, possibly in an extension ring or field. Straight-line programs serve as a very useful model to capture features of the complexity of algebraic problems (see, e.g., Strassen (1990)) especially with respect to evaluation in extension rings, as well as having considerable efficacy in practice (see, e.g., Freeman et al. (1988)).

More formally, a division-free *Straight-Line Program* over a ring \( R \), henceforth abbreviated as an *SLP*, is a branchless sequence of arithmetic instructions that represents a polynomial function. It takes as input a vector \((a_1, \ldots, a_K)\) and outputs a vector \((b_1, \ldots, b_L)\) by way of a series of instructions \( \Gamma_i : 1 \leq i \leq L \) of the form \( \Gamma_i : b_i \leftarrow \alpha \star \beta \), where \( \star \) is an operation \( '+' \), \( '-' \), or \( '\times' \), and \( \alpha, \beta \in R \cup \{a_1, \ldots, a_K\} \cup \{b_0, \ldots, b_{i-1}\} \). The inputs and outputs may belong to \( R \) or a ring extension of \( R \). We say a straight-line program *computes* \( f \in R[x_1, \ldots, x_K] \) if it sets \( b_L \) to \( f(a_1, \ldots, a_K) \).

The straight line programs in this paper compute over finite fields \( \mathbb{F}_q \) with \( q \) elements, and ring extensions of \( \mathbb{F}_q \). We assume that elements of \( \mathbb{F}_q \) are stored in some reasonable representation with \( O(\log q) \) bits, and that each field operation requires \( \tilde{O}(\log q) \) bit operations. Similarly, we assume that elements in a field extension \( \mathbb{F}_{q^s} \) of...

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†For functions \( \phi, \psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \), we say \( \phi \in \tilde{O}(\psi) \) if and only if \( \phi \in O(\psi(\log \psi)^c) \) for a constant \( c \geq 0 \).
\( \mathbb{F}_q \) can be represented with \( O(s \log q) \) bits, and that operations in \( \mathbb{F}_q^* \) require \( \tilde{O}(s \log q) \) bit operations.

Each of the algorithms described here determines \( f \) by probing its SLP, executing it at an input of our choosing and observing the output. This is analogous to evaluation of the polynomial at a point in an extension of the ground field in the more traditional interpolation model. To fairly account for the cost of such a probe we define the probe degree as the degree of the ring extension over \( \mathbb{F}_q \) in which the probe lies. A probe of degree \( u \) costs \( \tilde{O}(Lu) \) field operations, or \( \tilde{O}(Lu \log q) \) bit operations. The total probe size is the sum of the probe degrees of all the probes used in a computation.

Polynomials are also stored with respect to the power basis (powers of \( x \)) in a sparse representation. In particular, \( f \) as in (1.1) would be stored as a list \( [(c_i, e_i)]_{1 \leq i \leq t} \). Given such a polynomial \( f \), we let \( \text{coeff}(f, k) \), denote the coefficient of the term of \( f \) of degree \( k \) (which may well be zero). We let \( \text{exponents}(f) \) denote the sorted list of exponents of the nonzero terms in \( f \), and \( \#f \) denote the sparsity of \( f \), the number of nonzero terms of \( f \). When we write \( r(z) = f(z) \mod d(z) \), we assume that the modular image \( r \) is reduced, i.e., it is stored as the remainder of \( f \) divided by \( d \), and \( \deg r < \deg d \). We frequently refer to such \( r \) as “images” of the original polynomial \( f \), as they reveal some limited amount of information about \( f \).

The problem of (sparse) interpolation can then be seen as one of conversion between representations: Efficiently transform a polynomial given by a straight-line program into a “standard” sparse representation with respect to a power basis in \( x \) as in (1.1), given “size” constraints \( D \) and \( T \) as above.

### 1.2 Previous results

The following table gives a comparison of existing algorithms for the sparse interpolation of straight-line programs.

|          | Bit complexity                     | Type            |
|----------|------------------------------------|-----------------|
| Dense    | \( LD \log q \)                    | deterministic   |
| Garg & Schost | \( LT^4 \log^2 D \log q \) | deterministic   |
| Las Vegas G & S | \( LT^3 \log^2 D \log q \) | Las Vegas       |
| Diversified | \( LT^2 \log^2 D (\log D + \log q) \) | Las Vegas       |
| Recursive | \( LT \log^2 D \log q \cdot \log(1/\epsilon) \) | Monte Carlo     |
| This paper | \( LT \log^2 D (\log D + \log q) \cdot \log(1/\epsilon) \) | Monte Carlo     |

Table 1: A “soft-Oh” comparison of interpolation algorithms for SLPs

Most of these are probabilistic algorithms. By a Las Vegas algorithm we mean one that runs in the expected time stated but produces an output of guaranteed correctness. A Monte Carlo algorithm takes an additional parameter \( \epsilon \in (0, 1) \) and produces an output guaranteed to be correct with probability at least \( 1 - \epsilon \).
The algorithm described in Garg and Schost (2009) finds a **good prime**, that is, a prime that keeps the terms of $f$ separate. For $f$ given by (1.1),

$$ f \mod (z^p - 1) = \sum_{i=1}^{t} c_i z^{e_i \mod p}, \quad (1.2) $$

and so the terms of the **good image** $f \mod (z^p - 1)$ remain distinct provided the exponents $e_i$ are distinct modulo $p$. This good image gives us the sparsity $t = \#f$, which in turn makes it easy to identify other good primes. Their algorithm then constructs the symmetric polynomial $\chi(y) = \prod_{i=1}^{t} (y - e_i)$ by Chinese remaindering of images $\chi(y) \mod q$, for sufficiently many good primes $q$. Note that the image $f \mod (z^q - 1)$ gives the values $e_i \mod q$ and hence $\chi(y) \mod q$. Given $\chi$, the algorithm then factors $\chi$ to obtain the exponents $e_i$, with which it is straightforward to ascertain the coefficients of $f$. Their algorithm can be made faster, albeit Monte Carlo, by using randomness; we probabilistically search for a good prime by selecting primes at random over a specified range, choosing as our good prime $p$ one for which the image $f \mod (z^p - 1)$ has maximally many terms.

An information-theoretic lower bound on the total probe size required is $O(T(\log D + \log q))$ bits, the number of bits used to encode $f$ in (1.1). This bound is met by Prony’s (1795) original algorithm, which requires a total probe size of $O(2T \log q)$ under the implicit assumption that $q > D$. Much of the complexity of the sparse interpolation problem appears to arise from the requirement to accommodate any finite field. Prony’s algorithm is dominated by the cost of discrete logarithms in $\mathbb{F}_q$, for which no polynomial time algorithm is known in general. However, when there is a choice of fields (say, as might naturally arise in a modular scheme for interpolating integer or rational polynomials) more efficient interpolation methods have been developed. Kaltofen (1988) demonstrates a method for sparse interpolation over $\mathbb{F}_p$ when $p - 1$ is smooth; see Kaltofen (2010) for further exposition. In our notation, this algorithm would require an essentially optimal $\widetilde{O}(LT(\log D + \log q))$ bit operations. Parallel algorithms and implementations for this case of chosen characteristic are also presented in Javadi and Monagan (2010).

### 1.2.1 Diversified Interpolation

We say a polynomial $g(z)$ is **diverse** if its non-zero terms have distinct coefficients. The interpolation algorithm given in Giesbrecht and Roche (2011) distinguishes between images of distinct terms by **diversifying** $f$, that is, choosing an $\alpha$ such that $f(\alpha z)$ is diverse. This Monte Carlo algorithm entails three probabilistic steps. First, it determines $t = \#f$ by searching for a probable good prime. Second, it finds an $\alpha$ that diversifies $f \mod (z^p - 1)$. Finally, it looks for more good primes $p$, and constructs the exponents $e_i$ of $f$ by way of Chinese remaindering on the congruences $e_i \mod p$. This gives $f(\alpha z)$, from which it is straightforward to recover $f(z)$.

The diversified interpolation algorithm was initially described for a “remainder black box” representation of $f$, a weaker model than a straight-line program. To search for an appropriate $\alpha \in \mathbb{F}_q$, the algorithm requires that the field size $q$ is greater than
$T(T-1)D$. Under the SLP model we can easily adapt the algorithm to work when $q \leq T(T-1)D$; simply choose $\alpha$ from a sufficiently large field extension. This slight adaptation increases the cost of a probe by a factor of $\tilde{O}((\log D + \log T) / \log q)$.

The cost of interpolation using the diversified algorithm is then $\tilde{O}(\log D + \log \frac{1}{\epsilon})$ probes of degree $\tilde{O}((T^2 \log D) / [(\log D) / (\log q)])$, where $\epsilon \in (0, 1)$ is a given bound on the probability of failure. Since each operation in $\mathbb{F}_q$ costs $\tilde{O}(\log q)$ bit operations, the total cost in bit operations is

$$\tilde{O} \left( LT^2 \log D (\log D + \log \frac{1}{\epsilon}) (\log D + \log q) \right).$$

(1.3)

For fixed $\epsilon$ this cost becomes $\tilde{O}(LT^2 \log^2 D (\log D + \log q))$.

The diversified and probabilistic Garg-Schost interpolation algorithms may be made Las Vegas (i.e., guaranteed error free) by way of a deterministic polynomial identity testing algorithms. The fastest-known method for certifying polynomials (given by algebraic circuits) equal over an arbitrary field, by Bläser et al. (2009), requires $\tilde{O}(LT^2 \log^2 D)$ field operations in our notation. The gains in faster interpolation algorithms would be dominated by the cost of this certification; hence they are fast Monte Carlo algorithms whose unchecked output may be incorrect with controllably small probability.

### 1.2.2 Recursive interpolation

The algorithm presented in Arnold et al. (2013) is faster than diversified interpolation when $T$ asymptotically dominates either $\log D$ or $\log q$. The chief novelty behind that algorithm is to use smaller primes $p$ with relaxed requirements. Instead of searching for good primes separating all of the terms of $f$, we probabilistically search for an ok prime $p$ which separates most of the terms of $f$. Given this prime $p$ we then construct images of the form $f \mod (z^p - 1)$, for a set of coprime moduli $\{q_1, \ldots, q_\ell\}$ whose product exceeds $D$, in order to build those non-colliding terms of $f$.

The resulting polynomial $f^*$ contains these terms, plus possibly a small number of deceptive terms not occurring in $f$, such that $f - f^*$ now has at most $T/2$ terms. The algorithm then updates the sparsity bound $T \leftarrow T/2$ and recursively interpolates the difference $g = f - f^*$.

The total cost of the recursive interpolation algorithm is $\tilde{O}(\log D + \log \frac{1}{\epsilon})$ probes of degree $\tilde{O}(T \log^2 D)$, for a total cost of

$$\tilde{O} \left( LT \log^2 D (\log D + \log \frac{1}{\epsilon}) \log q \right)$$

(1.4)

bit operations, which is $\tilde{O}(LT \log^3 D \log q)$ when $\epsilon \in (0, 1)$ is a constant.

### 1.3 Outline of the new algorithm and this paper

As in Arnold et al. (2013), our new algorithm recursively builds an approximation $f^*$, initially zero, to the input polynomial $f$ given by an SLP. The algorithm interpolates $g = f - f^*$ with bounds $D \geq \deg(g)$ and $T \geq \#g$. We update $T$ as we update $g$. 

5
We begin with the details of our straight-line program model in Section 2. Next, Section 3 describes how we choose a set of \( \ell \in \tilde{O}(L \log D) \) ok primes \( p_i \in \tilde{O}(T \log D) \), \( 1 \leq i \leq \ell \). Given these primes \( p_i \), we then compute images \( g_{ij} = g(\alpha_j z) \mod (z^{p_i} - 1) \) for choices of \( \alpha_j \) that will (probably) allow us to identify images of like terms of \( g \). This approach relies on a more general notion of diversification. We explain how we choose the values \( \alpha_j \) and give a probabilistic analysis in Section 4. Section 5 details how we use information from the images \( g_{ij} \) to construct at least half of the terms of \( g \).

Section 6 describes how the algorithm iteratively builds \( f^* \), followed by a probabilistic and cost analysis of the algorithm as a whole in Section 7. Conclusions are in Section 8.

2 Probing a straight-line program

In this paper we will only consider a single-input, division-free SLP \( S_f \) that computes a univariate polynomial \( f \) over a finite field. We can reduce multivariate \( f \in \mathbb{F}_q[x_1, \ldots, x_n] \) of total degree less than \( D \), by way of a Kronecker substitution. Interpolating the univariate polynomial \( f(z, z^D, z^{D^2}, \ldots, z^{D^{n-1}}) \) preserves the sparsity of \( f \) and easily allows the recovery of the original multivariate terms, but increases the degree bound to \( D^n \). See Arnold and Roche (2014) for recent advances on this topic.

We will probe \( S_f \), that is, execute the straight-line program on inputs of our choosing. One could naively input an indeterminant \( z \) into \( S_f \), and expand each resulting polynomial \( b_i \). This sets \( b_L \) to \( f(z) \). Unfortunately, intermediate polynomials \( b_i \) may have sparsity and degrees which grow exponentially in terms of \( L, T, \) and \( D \).

Instead, we limit the size of intermediate results by selecting a symbolic \( p \)-th root of unity \( z \) as input, for different choices of \( p \). At each instruction we expand \( b_i \in \mathbb{F}_q[z] \), and reduce modulo \( (z^p - 1) \). This sets \( b_L \) to \( f(z) \mod (z^p - 1) \). By Cantor and Kaltofen (1991), the cost of one such instruction becomes \( \tilde{O}(p) \) field operations. More generally we will produce images of the form \( f(\alpha z) \mod (z^p - 1) \), where \( \alpha \) is algebraic with degree \( s \) over \( \mathbb{F}_q \). That is, we choose as input \( \alpha z \), where \( z \) again is a \( p \)-th root of unity. Here the instruction cost becomes \( \tilde{O}(p) \) operations in \( \mathbb{F}_{q^p} \), or \( \tilde{O}(ps) \) operations in \( \mathbb{F}_q \). We call \( ps \) the probe degree of a probe with input \( \alpha z \).

We will also have to produce such images of polynomials given explicitly by sparse representations rather than implicitly in a SLP. Given a single term \( cz^e, e < D \), we can produce \( c(\alpha z)^e \mod (z^p - 1) \) by computing \( \alpha^e \) via a square-and-multiply approach and reducing the exponent of \( z^e \) modulo \( p \). The former step dominates this cost and may entail \( \tilde{O}(s \log D) \) operations in \( \mathbb{F}_q \). For \( f^* \), a sum of at most \( T \) such terms, the cost becomes \( \tilde{O}(sT \log D) \).

In particular, given an SLP \( S_f \) and a sparse polynomial \( f^* \), we will need to compute \( f(\alpha z) - f^*(\alpha z) \mod (z^p - 1) \), as described above. Procedure ComputeImage is a subroutine to perform such a computation, and its cost is \( \tilde{O}(Lsp + sT \log D) \) operations in \( \mathbb{F}_q \). When \( p \) is at least of magnitude \( \Omega((T \log D)/L) \), the SLP probe dominates the cost, and it becomes simply \( \tilde{O}(Lsp \log q) \) bit operations.
Procedure \texttt{ComputeImage}(S_f, f^\ast, \alpha, p)

\begin{itemize}
\item \textbf{Input:} \(S_f\), an SLP computing \(f \in \mathbb{F}_q[z]\); \(f^\ast \in \mathbb{F}_q[x]\) given explicitly; \(\alpha \in \mathbb{F}_{q^\ast}\); integer \(p \geq 1\).
\item \textbf{Result:} \(f(\alpha z) - f^\ast(\alpha z) \mod (z^p - 1)\)
\end{itemize}

\begin{algorithmic}
\State \(g_{p,\alpha} \leftarrow f(\alpha z) \mod (z^p - 1)\), by probing \(S_f\) at \(\alpha z\) over \(\mathbb{F}_{q^\ast}[z]/\langle z^p - 1 \rangle\)
\ForEach {\(e \in \text{exponents of } f^\ast\)}
\State \(g_{p,\alpha} \leftarrow g_{p,\alpha} - \text{coeff}(f^\ast, e) \cdot \alpha^e \cdot z^e \mod p\)
\EndFor
\State \textbf{return} \(g_{p,\alpha}\)
\end{algorithmic}

3 Selecting primes

The aim of Sections 3–5 is to build a polynomial \(f^{**}\) that contains at least half of the terms of \(g = f - f^\ast\). For notational simplicity, in these sections we write \(g = \sum_{i=1}^\ell c_i z^e_i\) and let \(T\) and \(D\) bound \(#g\) and \(\text{deg}(g)\) respectively.

We will use images of the form \(g \mod (z^p - 1)\), \(p\) a prime, in order to extract information about the terms of \(g\). We say two terms \(c z^e\) and \(c' z^{e'}\) of \(g\) collide modulo \((z^p - 1)\) if \(p\) divides \(e - e'\), that is, if their reduced sum modulo \((z^p - 1)\) is a single term. If a term \(c z^e\) of \(g\) does not collide with any other term of \(g\), then its image \(c z^e \mod p\) contained in \(g \mod (z^p - 1)\) gives us its coefficient \(c\) and the image of the exponent \(e \mod p\).

We let \(C_g(p)\) denote the number of terms of \(g\) that collide with any other term modulo \((z^p - 1)\). We need \(C_g(p)\) to be small so that the next approximation \(f^\ast\) of \(f\) contains many terms of \(f\), but the primes \(p\) must also be small so that the cost of computing each \(f^\ast\) is not too great. In this technical section, we show how to bound the size of the primes \(p\) to balance these competing concerns and minimize the total cost of the algorithm.

In the first phase of the algorithm, we will look for a set of primes \(p_i \in \tilde{O}(T \log D)\), and corresponding images

\[ g_i = g \mod (z^{p_i} - 1), \quad 1 \leq i \leq \ell, \]

such that, in each of the images \(g_i\), most of the terms of \(g\) are not in any collisions. More specifically, we want the primes \((p_i)_{1 \leq i \leq \ell}\) and images \((g_i)_{1 \leq i \leq \ell}\) to meet the following criteria:

(i) At least half of the terms of \(g\) do not collide in at least \(\lceil \ell/2 \rceil\) of the images \(g_i\);

(ii) Any fixed sum of two or more terms of \(g\) collides together in fewer than \(\lceil \ell/2 \rceil\) of the images \(g_i\);

(iii) Any \(\lceil \ell/2 \rceil\) of the primes \(p_i\) have a product exceeding \(D\).

Provided we are able to collect terms of the reduced images \(g_i\) that are images of the same sum of terms of \(g\), then from (ii), any such collection containing a term at least
\( \ell/2 \) images \( g_i \) must be an image of a single term of \( g \). By (i), at least half of the terms of \( g \) will correspond to such a collection. By (iii), these collections will contain sufficient information to reconstruct its corresponding term of \( g \). Thus, given a means of collecting terms, we will be able to construct half of the terms of \( g \).

We note, since any two terms of \( g \) have degree differing by at most \( D \), that (iii) implies (ii). To satisfy (i), it suffices that \( C_g(p_i) \leq T/2 \) for each prime \( p_i \). We will accordingly call \( p \) an \textit{ok prime} in this paper if \( C_f(p) < T/2 \). To this end we to establish a range in which most primes \( p \) have \( C_g(p) < T/4 \), half the desired bound.

**Lemma 3.1** (Arnold et al. (2013)). Let \( g \in \mathbb{F}_q[z] \) be a polynomial with \( t \leq T \) terms and degree \( d \leq D \). Let
\[
\lambda = \max \left(21, \left\lceil \frac{40}{3}(T-1) \ln D \right\rceil \right).
\]
Then fewer than half of the primes \( p \) in the range \([\lambda, 2\lambda]\) satisfy \( C_g(p) \geq T/4 \).

We will look for ok primes in the range \([\lambda, 2\lambda]\). To satisfy (iii), we will select \( \ell = 2[\log_\lambda D] \) primes. Corollary 3.2 gives us a means of selecting a set of primes for which a constant proportion of those primes \( p \) have fewer than \( T/4 \) colliding terms.

**Corollary 3.2.** Suppose \( g, D, T, \) and \( \lambda \) are as in Lemma 3.1. Let \( 0 < \mu < 1 \), and suppose
\[
\ell = 2[\log_\lambda D], \quad \gamma = \left\lceil \max(8 \log_\lambda D, 8 \ln \frac{1}{\mu}) \right\rceil.
\]
If we randomly select \( \gamma \) distinct primes from \([\lambda, 2\lambda]\), then \( C_g(p) \leq T/4 \) for at least \( \ell \) of the chosen primes \( p \), with probability at least \( 1 - \mu \).

**Proof.** By Lemma 3.1, at least half of the primes \( p \) in \([\lambda, 2\lambda]\) have \( C_g(p) < T/4 \). We require that a proportion of \( \frac{1}{4} \) of the \( \gamma \) primes randomly selected to have this property.

Then, by Theorems 1 and 4 of Hoeffding (1963), the probability that fewer than \( 2[\log_\lambda D] \) of the primes \( p \) chosen randomly from \([\lambda, 2\lambda]\) have \( C_g(p) < T/4 \) is less than \( e^{-2\gamma(1/2-1/4)^2} = e^{-\gamma/8} \). For \( \gamma \geq 8 \ln \frac{1}{\mu} \), this is less than \( \mu \).

We will generate some \( \gamma \) primes, of which \( \ell \) primes \( p \) have \( C_g(p) < T/4 \). In order to identify some primes \( p \) for which \( C_g(p) \) is low, we have the following lemma.

**Lemma 3.3** (Arnold et al. (2013)). Suppose \( g \mod (z^n - 1) \) has sparsity \( s_q \), and \( g \mod (z^p - 1) \) has sparsity \( s_p \geq s_q \). Then \( C_g(p) \leq 2C_g(q) \).

Corollary 3.2 guarantees with high probability that at least \( \ell \) primes \( p_i \) of the \( \gamma \) selected satisfy \( C_g(p_i) < T/4 \). We cannot necessarily determine which \( \ell \) primes meet this condition, but by ordering the primes in decreasing order of \#\( g_i \), Lemma 3.3 guarantees that the first \( \ell \) of the primes \( p_i \) have \( C_g(p_i) < T/2 \). This method is described in procedure \texttt{FindPrimes} below.

The statement of Corollary 3.2 assumes that there are a certain number of primes in \([\lambda, 2\lambda]\), in order to pick \( \gamma \) of them. Letting \( n \) be the actual number of primes in this
range, we require that \( n \geq \gamma = \lceil \max(8 \log_\lambda D, \ln \frac{1}{\mu}) \rceil \). This puts further constraints on \( \lambda \). By Corollary 3 of Rosser and Schoenfeld (1962), and using the definition of \( \lambda \) from Lemma 3.1,

\[
n \geq 3\lambda / (5 \ln \lambda) \geq 8(T - 1) \log_\lambda D,
\]

which is at least \( 8 \log_\lambda D \) for \( T \geq 2 \). Thus, for \( T > 1 \), we only require that \( n \geq \lceil 8 \ln \frac{1}{\mu} \rceil \).

If the number of primes \( n \) is even smaller than this, one could simply compute all \( n \) primes in the interval and use them all instead of picking a random subset. Since at least half these primes have \( C_g(p) < T/4 \), we would only require that \( n \geq 2\ell \), which must be true since \( n \geq 8 \log_\lambda D \).

To ensure that computing all \( n \) primes in the interval does not increase the overall cost of the algorithm, consider that in this case \( \gamma \) exceeds \( 3\lambda / (5 \ln \lambda) \). Then the upper bound of Corollary 3 of Rosser and Schoenfeld (1962) gives

\[
n \leq \lceil 7\lambda / (5 \ln \lambda) \rceil \in O(\gamma).
\]

Therefore whether we choose only \( \gamma \) primes from the interval or are forced to compute all \( n \) of them, the number of primes used is always \( O(\gamma) \).

We make one further note on the cost of producing \( \gamma \) primes at random. In a practical implementation, one would likely choose numbers at random (perhaps in a manner that avoids multiples of small primes), and perform Monte Carlo primality testing to verify whether such a number is prime. Problems could arise if the algorithm produced pseudoprimes \( p_i \) and \( p_j \) that are not coprime. Thus one would also have to consider the failure probability of primality testing in the analysis of such an approach.

For the purposes of our analysis, we generate all the primes up to \( 2\lambda \) with \( \tilde{O}(\lambda) = \tilde{O}(T \log D) \) bit operations using a wheel sieve (Pritchard, 1982), which does not dominate the cost in (3.3), and guarantees that all of the chosen \( p \) are actually prime.

The total cost is \( \gamma \) probes of degree at most \( 2\lambda \), where

\[
\lambda \in \tilde{O}(T \log D), \quad \text{and} \quad \gamma \in \tilde{O}\left( \frac{\log D}{\log T} + \log \frac{1}{\mu} \right),
\]

for a field-operation cost of

\[
\tilde{O}(L\gamma \lambda) = \tilde{O}\left( L \left( \log D + \log \frac{1}{\mu} \right) T \log D \right). \tag{3.3}
\]
Procedure FindPrimes($S_f, f^*, T, D, \mu$)

**Input:** $S_f$, an SLP computing $f \in \mathbb{F}_q[z]$; $f^* \in \mathbb{F}_q[x]$ given explicitly. 
$T \geq \max(#g, 2)$; $D \geq \deg(g)$; $0 < \mu < 1/3$; where $g$ is defined as the unknown polynomial $f - f^*$.

**Result:** A list of primes $p_i$ and images $g_i(z)$ mod $(z^{p_i} - 1)$ such that $\prod_{i=1}^{\ell} p_i \geq D$ and, with probability exceeding $1 - \mu$, $C_g(p_i) < T/2$ for $1 \leq i \leq \ell$.

$\lambda \leftarrow \max\left(21, \left\lceil \frac{40}{3}(T - 1) \ln D \right\rceil \right)$

$\gamma \leftarrow \left\lceil \max(8 \log_3 D, 8 \ln \frac{1}{\mu}) \right\rceil$

$\ell = 2\lceil \log_3 D \rceil$

if $\gamma \leq 3\lambda/(5 \ln \lambda)$ then

- $\mathcal{P} \leftarrow \gamma$ primes chosen at random from $[\lambda, 2\lambda]$

else

- $\mathcal{P} \leftarrow$ all primes from $[\lambda, 2\lambda]$

foreach $p \in \mathcal{P}$ do 

- $g_p \leftarrow \text{ComputeImage}(S_f, f^*, 1, p)$

(p1, p2, ...) ← $\mathcal{P}$ sorted in decreasing order of $\#g_p$

return (p1, ..., p$\ell$), (g$p_1$, ..., g$p_\ell$)

4. Detecting deception amongst images

At this stage we have probably found primes $p_i$, $C_g(p_i) \leq T/2$, and their corresponding images $g_i$, for $1 \leq i \leq \ell$. The challenge now remains to collect terms amongst the images $g_i$ that are images of the same term of $g$. For our purposes we need a more general notion of diversification than that introduced in Giesbrecht and Roche (2011).

To this end we will construct images

$$g_{ij} = g(\alpha_j z) \mod (z^{p_i} - 1),$$

where $\alpha_j \neq 0$ belongs to $\mathbb{F}_q$ or a field extension $\mathbb{F}_{q^*}$.

Any term of an image $g_i$ is either an image of a single term of $g$, or a sum of multiple terms of $g$. Our algorithm needs to identify and discard the terms in $g_i$ corresponding to multiple terms of $g$, using only the single terms to reconstruct the actual terms in $g$.

To analyse this situation, consider the bivariate polynomials

$$g(yz) \mod (z^{p_i} - 1) = h_{i,0} + h_{i,1} z + \cdots + h_{i,p_i-1} z^{p_i-1},$$

where each $h_{i,u} \in \mathbb{F}_q^*[y]$ is the sum of the terms in $g$ with degrees congruent to $u$ modulo $p_i$. Each $h_{i,u}$ has between 0 and $T$ terms and degree at most $D$ in $y$. For a given $\alpha$, computing $g_i(\alpha z)$ gives the univariate polynomial whose coefficient of degree $u$ is $h_{i,u}(\alpha)$.

Consider a single term in the unknown polynomial $g$. If that term does not collide with any others mod $p_i$, then for some $u$, $h_{i,u}$ consists of that single term. If the same term does not collide modulo $p_k$, then there exists some $v$ such that $h_{i,u} = h_{k,v}$. Obviously, for any $\alpha \in \mathbb{F}_{q^*}$, we will have $h_{i,u}(\alpha) = h_{k,v}(\alpha)$, and our algorithm can use
this correlation to reconstruct the term, since its exponent equals \( u \mod p_i \) and \( v \mod p_k \). Based on the previous section, most of the terms in \( g \) will not collide modulo most of the \( p_i \)'s, and so there will be sufficient information here to reconstruct those terms.

The problem is that we may have \( h_{i,u}(\alpha) = h_{k,v}(\alpha) \), but \( h_{i,u} \neq h_{k,v} \). We call this a deception, since it may fool our algorithm into reconstructing a single term in \( g \) that does not actually exist. Our algorithm will evaluate with multiple choices for deception \( \alpha \), and we “hope” that, whenever \( h_{i,u} \neq h_{k,v} \), at least one of \( \alpha_j \)'s gives \( h_{i,u}(\alpha_j) = h_{k,v}(\alpha_j) \). In this case we say \( \alpha_j \) detects the deception. The following lemma provides this hope.

**Lemma 4.1.** Let \( m = \left\lceil \log \frac{2}{\mu} + 2 \log T + 2 \log (1 + \ell/4) \right\rceil \) and \( s = \lceil \log_q (2D + 1) \rceil \).

Choose \( \alpha_1, \ldots, \alpha_m \) at random from \( \mathbb{F}_q^* \). Then, with probability at least \( 1 - \mu \), every deception amongst the images \( g_1, \ldots, g_s \) is detected by at least one of the \( \alpha_j \).

**Proof.** Consider \( h_{i,u}, h_{k,v} \) as above. As \( (h_{i,u} - h_{k,v})(y) \) has degree at most \( D \), there are at most \( D \) choices of \( \alpha \) for which \( h_{i,u}(\alpha) = h_{k,v}(\alpha) \).

Thus, if \( q^s > 2D \) and \( \deg(h_1 - h_2) \leq D \), then at most half of the \( \alpha \in \mathbb{F}_q^* \) can comprise a root of a \( h_1 - h_2 \). Thus, if we then select \( \alpha_1, \ldots, \alpha_m \) at random from \( \mathbb{F}_q^* \) and construct images \( g(\alpha_i) \mod (z^{p_k} - 1) \) for \( j = 1, \ldots, m, k = 1, 2 \), we will fail to detect a single given deception with probability at most

\[
(\frac{1}{2})^m = \left(\frac{1}{2}\right)^{\left\lceil \log(2/\mu) + 2 \log(T) + 2 \log(1 + \ell/4) \right\rceil} = \frac{\mu}{2^T(1 + \ell/4)^2}. \tag{4.2}
\]

As \( g \) has at most \( T \) terms, there are at most \( 2^T - 1 \) possible choices for \( h_{i,u} \neq 0 \); however, only a small proportion of these choices may correspond to a term in an image \( g_i \). Each \( g_i, 1 \leq i \leq \ell \) contains at most \( T/4 \) terms that is an image some sum of at least two terms of \( g \), as well as at least \( T/2 \) terms that are images of some of the \( T \) terms of \( g \). Thus there are fewer than \( T(1 + \ell/4) \) distinct sums corresponding to a nonzero polynomial \( h_{i,u} \neq 0 \).

A deception occurs between a pair of such sums, thus there are fewer than \( \frac{1}{2} T^2 (1 + \ell/4)^2 \) deceptions. It then follows from (4.2) that the probability that at least one deception is not detected by any of \( \alpha_1, \ldots, \alpha_m \) is bounded above by \( \mu \).

It is important to note that \( m \) is logarithmic in \( T \) and \( \ell \), so that a multiplicative factor of \( m \) will not affect the “soft-\( \text{Oh} \)” cost of the algorithm in terms of \( T \) or \( D \). If \( q \leq 2D \), then we need instead to work in an extension of \( \mathbb{F}_q \) of degree \( s = \lceil \log_q (2D + 1) \rceil \).

The cost of computing \( g_{ij}, 1 \leq i \leq \ell, 1 \leq j \leq m \), in terms of \( \mathbb{F}_q \)-operations, becomes

\[
\tilde{O}(L\ell m \lambda s) = \tilde{O} \left( LT \log^2 D \left\lceil \frac{\log D}{\log q} \right\rceil \log \frac{1}{\mu} \right),
\]

which dominates the cost (3.3) of Section 3.
5 Identifying images of like terms

As we construct the images $g_{ij}$, we will build vectors of coefficients of images $g_{ij}$. Namely, for every congruence class $e \mod p_i$ for which there exists a nonzero term of degree $e$ in at least one image $g_{ij}$, we will construct a vector $v^{i,e}_j \in \mathbb{F}_{q^m}$, where $v^{i,e}_j$ contains the coefficient (possibly zero) of the term of $g_{ij}$ of degree $e$.

We will use the vectors $v^{i,e}_j$ to identify terms of the images $g_i$ that are images of like terms of $g$. We use these vectors $v^{i,e}_j$ as keys in a dictionary tuples. Each value in the dictionary is comprised of a list of those tuples $(e, i)$ for which $v^{i,e}_j = v$.

Provided the probabilistic steps of Sections 3 and 4 succeeded, if a key $v$ is found more than $\ell/2$ times, then it corresponds to a single, distinct term of $g$, as opposed to a sum of terms of $g$. This is indicated by the size of the list tuples$(v)$ being at least $\ell/2$.

The dictionary tuples should be an ordered dictionary that supports logarithmic-time insertion and retrieval. Any balanced binary search tree, such as a red-black tree will be suitable (see, e.g., Cormen et al. (2001), Chapter 13). To set tuples$(v)$, we first search to see if $v$ is an existing key; if not, an empty list is first inserted as the value of tuples$(v)$.

A red-black tree of size $n$ requires $O(\log n)$ comparisons for insert and search operations. We compare keys $v \in \mathbb{F}_{q^m}$ lexicographically, which may entail $m$ comparisons of elements in $\mathbb{F}_{q^m}$. Each comparison therefore requires $O(ms \log q)$ bit operations.

The number of different vectors $v^{i,d}$ that will appear is bounded above by the number of distinct subsets of terms of $g$ which can collide modulo $(z_{p_i} - 1)$, for any of the primes $p_i$. Since there are $\ell$ primes, and at most $T$ terms in $g$, there are no more than $T\ell$ vectors which will be inserted as keys into tuples. Thus, the cost of constructing this tree is

$$O(T\ell ms \cdot \log q \cdot \log(T\ell)) = \tilde{O} \left( T \log D (\log D + \log q) \log \frac{1}{\mu} \right)$$

bit operations. This cost is dominated by that of constructing the $g_{ij}$. Each term in each $g_{ij}$ contains an element of $\mathbb{F}_{q^m}$ and an exponent at most $2\lambda$. This requires $\tilde{O}(s \log q + \log \gamma)$ bits, which is $\tilde{O}(\log D + \log q + \log T + \log \frac{1}{\mu})$. The additional bit-cost of traversing the images $g_{ij}$ and appending to the lists in tuples$(v)$ is reflected in the cost of their construction in Section 4.

After we have constructed the dictionary tuples, we traverse it again to build terms of $g$. For every key $v$ whose corresponding list has size at least $\ell/2$, we have all the pairs $(i, d)$ such that $v^{i,d} = v$. What remains is to construct the term corresponding to the key $v$. We reconstruct the exponent by Chinese remaindering on the first $\ell/2$ congruences $d \mod p_i$. As each exponent is at most $D$, the cost of constructing one exponent is bounded by $\tilde{O}(\log^2 D)$ bit operations. Thus the total bit-cost of Chinese remaindering becomes $\tilde{O}(T \log^2 D)$. As the bit cost of $s$ operations in $\mathbb{F}_q$ is $\tilde{O}(\log q + \log D)$, this cost of Chinese remaindering is bounded asymptotically by (5.1), the cost of constructing tuples itself.

We obtain the coefficient by inspection of $g_i$. We sum all of these constructed terms into a polynomial $f^*$ approximating $g = f - f^*$. 

"\text{12}"
Procedure **BuildApproximation** restates the method described to construct $f^{**}$. For the sake of brevity, this procedure does not detail the data structures used for the polynomials $g_{ij}$ and $f^{**}$ that it constructs. In order to achieve the stated complexity bounds, these sparse polynomials must be implemented by dictionaries mapping exponents to coefficients that support logarithmic-time insertion and retrieval. As with tuples, a red-black tree, hash table, or similar standard data structure could be used. Converting between such a representation and the usual list of coefficient-exponent pairs is trivial.

**Procedure BuildApproximation**($S_f, f^*, T, D, \mu$)

**Input:** $S_f$, an SLP computing $f \in \mathbb{F}_q[z]$; $f^* \in \mathbb{F}_q[x]$ given explicitly; $T \geq \#g$; $D \geq \deg(g)$; $0 < \mu < 1/3$; where $g$ is defined as the unknown polynomial $f - f^*$.

**Result:** $f^{**}$ such that $g - f^{**}$ has at most $T/2$ terms, with probability greater than $1 - \mu$.

$(p_1, \ldots, p_\ell), (g_1, \ldots, g_\ell) \leftarrow \text{FindPrimes} (S_f, f^*, T, D, \mu)$

$(m, s) \leftarrow \left(\left\lfloor \log_2 \frac{2}{\mu} + 2 \log T + 2 \log(1 + \ell/4) \right\rfloor, \left\lfloor \log_q (2D + 1) \right\rfloor\right)$

$(\alpha_1, \ldots, \alpha_m) \leftarrow m$ randomly chosen nonzero elements from $\mathbb{F}_q^*$

**tuples** $\leftarrow$ dictionary mapping $\mathbb{F}_q^m$ to lists of pairs of integers

for $i \leftarrow 1$ to $\ell$ do

foreach $e \in \bigcup_{1 \leq j \leq m} \text{exponents}(g_{ij})$ do

$v \leftarrow (\text{coeff}(g_{i1}, e), \text{coeff}(g_{i2}, e), \ldots, \text{coeff}(g_{im}, e))$

**tuples**(v) $\leftarrow$ **tuples**(v), (i, e)

end

d $\leftarrow$ solution to congruences $\{e \mod p_i \mid (i, e) \in \text{tuples}(v)\}$

c $\leftarrow \text{coeff}(g_i, e)$, for any of the $(i, e) \in \text{tuples}(v)$

$f^{**} \leftarrow f^{**} + cz^e$

end for

return $f^{**}$

---

6 Updating our approximation

Recall we have a polynomial $f^*$ approximating $f$ given by our straight-line program. We construct a polynomial $f^{**}$ that is comprised of at least $T/2$ terms of $g = f - f^*$. Once we have $f^{**}$, we set $f^* \leftarrow f^* + f^{**}$, $T \leftarrow \lfloor T/2 \rfloor$, update $g$ accordingly, and repeat the process until $T$ is 1, at which point $g$ consists of (at most) a single nonzero term.

We thus execute this process at most $\log T$ times, where $T$ is the initial bound on
the sparsity of $f$. Recall that the steps of sections 3 and 5 each succeed with probability greater than $1 - \mu$. Thus, if we would like the algorithm to succeed with probability greater than $1 - \epsilon > 1/2$, we can set

$$\mu = \epsilon/(2 \log T).$$

(6.1)

As $T \geq 2$ and $\epsilon \leq 1/2$, we will always have $\mu \leq 1/4$, which satisfies the constraint $\mu < 1/3$ from Section 3.

When $T = 1$, $g$ is a single term and its coefficient is given by $g(1)$. The exponent of the term comprising $g$ may be computed from $g \mod (z^p - 1)$ for the first $\log D$ primes $p$. This cost is $\log D$ probes of degree $\tilde{O}(\log D)$, or a cost of $\tilde{O}(\log^2 D \log q)$ bit operations.

We give the interpolation algorithm in procedure MajorityVoteSparseInterpolate. We call the algorithm “majority-vote” sparse interpolation, as we effectively require a majority of the images $f_i$, $1 \leq i \leq \ell$, to vote on whether a sum of terms of $f$ is in fact a single term.

Procedure MajorityVoteSparseInterpolate($S_f, T, D, \epsilon$)

| Input: $S_f$, an SLP computing $f \in \mathbb{F}_q[z]$; $T \geq \# f$; $D \geq \deg(f)$; $0 < \epsilon < 1/2$. |
| Result: The sparse representation of $f$, with probability at least $1 - \epsilon$. |

$(f^*, \mu) \leftarrow (0, \epsilon/(2 \log T))$

while $T > 1$ do

$f^{**} \leftarrow \text{BuildApproximation}(S_f, f^*, T, D, \mu/2)$

$f^* \leftarrow f^* + f^{**}$

$T \leftarrow \lfloor T/2 \rfloor$

$(c, e) \leftarrow (\text{ComputeImage}(S_f, f^*, 1, 1), 0)$

if $c = 0$ then return $f^*$ for the first $\log D$ primes $p$ do

$e' \leftarrow \text{degree of the single term in ComputeImage}(S_f, f^*, 1, p)$

Update $e$ with $e'$ modulo $p$ by Chinese remaindering.

return $f^* + cz^e$

7 Cost analysis

We now are ready to analyze the cost of the algorithm MajorityVoteSparseInterpolate and verify Theorem 1.1. As we have argued in Sections 3–5, the cost of one iteration of the algorithm is dominated by the cost of constructing the images $g_{ij}$. Recall that there are $\ell$ primes $p_i$, each less than or equal to $2\lambda$, and there are $m$ elements $\alpha_j$’s, each in a extension $\mathbb{F}_{q^*}$. The total number of field operations is thus $\tilde{O}(L\ell m\lambda s)$. The values of $L, T, D,$ and $\epsilon$ are specified in the input. Recall the following parameters from equation (6.1), Corollary 3.2 and Lemma 4.1:
\[ \mu = \epsilon/(2 \log T) \Rightarrow \log \frac{1}{\mu} \in \tilde{O}(\log \frac{1}{\epsilon} \log T), \]
\[ \lambda = \max (21, \lceil \frac{40}{3} (T - 1) \ln D \rceil) \in \tilde{O}(T \log D), \]
\[ \ell = 2\lceil \log_2 D \rceil \in \tilde{O}((\log D)/(\log T)), \]
\[ m = \lceil \log_2 \frac{2}{\mu} + 2 \log T + 2 \log(1 + \ell/4) \rceil, \]
\[ \in \tilde{O}(\log \frac{1}{\epsilon} \log \log T + \log T + \log \log D), \]
\[ s = \lceil \log_q (2D + 1) \rceil \in \tilde{O}(1 + (\log D)/(\log q)). \]

Therefore the total cost of the \( \tilde{O}(L\ell m\lambda s) \) operations in \( \mathbb{F}_q \) is
\[ \tilde{O} \left( LT \log^2 D (\log D + \log q) \log \frac{1}{\epsilon} \right) \]
bit operations. The multiplicative \( \log T \) factor due to the number of iterations does not affect the “soft-Oh” cost above. This cost is a multiplicative factor of \( \tilde{O}(T) \) improvement over the cost (1.3) of diversified interpolation (Giesbrecht and Roche, 2011). It also improves over the cost (1.4) of recursive interpolation Arnold et al. (2013) by a multiplicative factor of \( \tilde{O}(\min(\log D, \log q)) \).

8 Conclusions

We have presented a new algorithm for sparse interpolation over an arbitrary finite field that is asymptotically faster than those previously known. In terms of bit operations it improves on previous methods by a factor “soft-Oh” multiplicative factor of \( T, \log D, \) or \( \log q \).

In this “majority-vote” interpolation, we combine the main ideas the previous two algorithms. Namely, we reduce the probe degree by only aiming to reconstruct some of the terms of \( f \) every iteration; and we distinguish images of distinct terms (and subsets of terms) by way of diversification.

We mention a few open and motivating problems. First, we believe the algorithm in this paper has considerable potential in the more traditional numerical (floating point) domain. There, an unknown sparse polynomial is reconstructed from a small number of evaluations in \( \mathbb{C} \) under a standard backward-error model of precision and stability. See Giesbrecht and Roche (2011) for an example of a straight-line program interpolation algorithm adapted to floating point computation. Our hope is that by evaluating at lower-order roots of unity (such as used in this paper) we can provably increase the numerical stability over the Prony-like algorithm of Giesbrecht et al. (2009), while maintaining its near-optimal efficiency.

Second, the algorithm presented here is Monte Carlo. It remains unknown whether there exist faster deterministic or Las Vegas polynomial identity tests that may render recursive or majority-vote interpolation Las Vegas of the same complexity.
Finally, as noted earlier, an information theoretic lower-bound on sparse interpolation suggests a minimum bit-complexity of $\tilde{O}(LT(\log q + \log D))$ bit operations. While this paper gets closer, some considerable improvements remain to be found.

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