POHOZAEV-TYPE IDENTITIES FOR A PSEUDO-RELATIVISTIC SCHröDINGER OPERATOR AND APPLICATIONS

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ABSTRACT. In this paper we prove a Pohozaev-type identity for both the problem \((-\Delta + m^2)^s u = f(u)\) in \(\mathbb{R}^N\) and its harmonic extension to \(\mathbb{R}^{N+1}_+\) when \(0 < s < 1\). So, our setting includes the pseudo-relativistic operator \(\sqrt{-\Delta + m^2}\) and the results showed here are original, to the best of our knowledge. The identity is first obtained in the extension setting and then “translated” into the original problem. In order to do that, we develop a specific Fourier transform theory for the fractional operator \((-\Delta + m^2)^s\), which lead us to define a weak solution \(u\) of the original problem if the identity

\[
\int_{\mathbb{R}^N} (-\Delta + m^2)^{s/2} u (-\Delta + m^2)^{s/2} v dx = \int_{\mathbb{R}^N} f(u) v dx
\]

is satisfied by all \(v \in H^s(\mathbb{R}^N)\). The obtained Pohozaev-type identity is then applied to prove both a result of nonexistence of solution to the case \(f(u) = |u|^{p-2} u\) if \(p \geq 2^*_s\) and a result of existence of a ground state, if \(f\) is modeled by \(\kappa u^3/(1 + u^2)\), for a constant \(\kappa\). In this last case, we apply the Nehari-Pohozaev manifold introduced by D. Ruiz. Finally, we prove that positive solutions of \((-\Delta + m^2)^s u = f(u)\) are radially symmetric and decreasing with respect to the origin, if \(f\) is modeled by functions like \(t^\alpha\), \(\alpha \in (1, 2^*_s - 1)\) or \(t \ln t\).

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1. INTRODUCTION

The pseudo-relativistic Schrödinger operator \(\sqrt{-\Delta + m^2}\) is associated with the Hamiltonian \(H = \sqrt{p^2 c^2 + m^2 c^4}\) of a free relativistic particle of mass \(m\) by the usual quantization \(p \to i\hbar \Delta\), changing units so that \(\hbar = 1\) and \(c = 1\). A good
exposition of the basic properties of the operator $\sqrt{-\Delta + m^2}$ can be found in [28], see also [39].

If $m > 0$ denotes the mass of bosons in units, the equation
\begin{align}
\begin{cases}
    i\partial_t \psi &= (\sqrt{-\Delta + m^2} - m) \psi + (W * |\psi|^2) \psi &\text{in } \mathbb{R}^N, \\
    u(x, 0) &= \phi(x),
\end{cases}
\end{align}

where $N \geq 2$ and $*$ denotes convolution, was used to describe the dynamics of pseudo-relativistic boson stars in astrophysics. See [13, 15, 21, 29] for more details.

The existence of solitary waves solutions for (1.1) $\psi(x) = e^{i\mu t} \varphi(x)$ with $\varphi$ satisfying the following pseudo-relativistic Hartree equation
\begin{align}
\sqrt{-\Delta + m^2} \varphi - m \varphi - (W * |\varphi|^2) \varphi &= -\mu \varphi
\end{align}
was first established by Lieb and Yau [29], in the case $W(x) = |x|^{-1}$.

For generalizations or variations on (1.2), the existence of ground state solutions was obtained by various authors [3, 14, 15, 17, 18, 31, 34]. A good survey on equations like (1.2) is given in [35].

Comparison between the operators $(-\Delta)^s$ and $(-\Delta + m^2)^s$. At first sight, one supposes that the treatment of both operators might be similar. In fact, there are huge differences between them.

(a) $(-\Delta)^s$ is $2s$-homogeneous with respect to dilatations, that is, $(-\Delta)^s u(\lambda x) = \lambda^{2s} (-\Delta)^s u(x)$, while such a property is not valid for $(-\Delta + m^2)^s$.

(b) As will see, $(-\Delta + m^2)^s$ generates a norm in $H^s(\mathbb{R}^N)$ and this is not the case for $(-\Delta)^s$. In consequence, the adequate spaces to handle both operators are quite different.

(c) Some results about fractionary Laplacian spaces are now standard (see [19, 20, 33]), but not so easy to find for $(-\Delta + m^2)^s$. See, however, [1, 23].

Why to handle $(-\Delta + m^2)^s$ instead of $\sqrt{-\Delta + m^2}$.

In this paper we deal with a generalized version of the operator $\sqrt{-\Delta + m^2}$, namely the operator $T(u) = (-\Delta + m^2)^s u$, $0 < s < 1$. We study the problem
\begin{align}
(-\Delta + m^2)^s u &= f(u), & x \in \mathbb{R}^N.
\end{align}

Concerning the applications of equation (1.3), we recall that fractionary Laplacian operators are the infinitesimal generators of Lévy stable diffusion processes. In particular, $(-\Delta + m^2)^s - m^2s$ is called the $2s$-stable relativistic process, see [11, 10, 37]. Stable diffusion processes have application in several areas such as anomalous diffusion of plasmas, probability, finances and populations dynamics, see [2].

Our approach applies the Dirichlet-to-Neumann operator, that is, we consider the extension problem naturally related to (1.3) for the operator $(-\Delta + m^2)^s$, thus resting on the celebrated papers by Cabré and Solà-Morales [6] and Caffarelli and Silvestre [8].

We state a general result about the extension problem:

**Theorem** (Stinga-Torrea [40]) Let $h \in \text{Dom}(L^s)$ and $\Omega$ be an open subset of $\mathbb{R}^N$. A solution of the extension problem
\begin{align}
\begin{cases}
    -L_x u + \frac{1-2s}{y} u_y + u_{yy} &= 0 & \text{in } \Omega \times (0, \infty) \\
    u(x, 0) &= h(x) & \text{on } \Omega \times \{y = 0\}
\end{cases}
\end{align}
is given by
\[ u(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tL}(L^sh)(x)e^{-\frac{y^2}{4t}} \frac{dt}{t^{1-s}} \]
and satisfies
\[ \lim_{y \to 0} y^{1-2s}u_y(x, y) = \frac{2s\Gamma(-s)}{4s\Gamma(s)} (L^sh)(x). \]

Applying this result to (1.13), the extension problem produces, for \((x, y) \in \mathbb{R}^N \times (0, \infty) = \mathbb{R}^{N+1}_+\),
\[
( P ) \begin{cases}
\Delta_x w + \frac{1-2s}{y}w_y + w_{yy} - m^2w = 0 \quad \text{in } \mathbb{R}^{N+1}_+ \\
\lim_{y \to 0^+} (-y^{1-2s} \frac{\partial w}{\partial y}) = k_s f(w(x, 0)), \quad \text{in } \mathbb{R}^N \times \{0\} \simeq \mathbb{R}^N,
\end{cases}
\]
where
\[ k_s = \frac{2s\Gamma(-s)}{4s\Gamma(s)} = \frac{21-2s\Gamma(1-s)}{\Gamma(s)}. \]

Applying Fourier transforms with respect to \(x \in \mathbb{R}^N\), we are lead to the Bessel function attached to this problem, which is trivial in the case of \(s = 1/2\), but not for an arbitrary \(s \in (0, 1)\). (See discussion below.)

The natural setting for problem (P) is the Hilbert space
\[ H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) = \left\{ u: \mathbb{R}^{N+1}_+ \to \mathbb{R}: \iint_{\mathbb{R}^{N+1}_+} (|\nabla w|^2 + |w|^2) y^{1-2s} \, dx \, dy < \infty \right\} \]
endowed with the norm
\[ ||w||_s = \left( \iint_{\mathbb{R}^{N+1}_+} (|\nabla w(x, y)|^2 + |w(x, y)|^2) y^{1-2s} \, dx \, dy \right)^{\frac{1}{2}}, \]
see [5] for details.

Observe that, in the case \(s = 1/2\), the problem (P) is set in a much simpler space, since the weight \(y^{1-2s}\) does not appear in the definition of \(H^1(\mathbb{R}^{N+1}_+)\). We will return to this difference later on.

**The definition of solution.** Although the definition of solution for (S) is easily obtainable in terms of Fourier transforms as in M. M. Fall and V. Felli [23], we do think that our definition is more natural, since it deals with a partial integration and remembers the one used in fractional spaces.

This lead us to develop a specific Fourier theory for the fractional operator \((-\Delta + m^2)^s\) in the case \(0 < s < 1\), with an approach influenced by Stinga and Torrea [40] and also by Stein [39].

Therefore, we justify the definition
\[ (-\Delta + m^2)^s f(x) = \mathcal{F}^{-1}\left( (m^2 + 4\pi^2 |\cdot|^2)^{\frac{s}{2}} \hat{f} \right)(x), \]
see Section 2.

Proceeding with our analysis, we show that \((-\Delta + m^2)^s\) is symmetric in the space \(\mathcal{S}(\mathbb{R}^N)\) and we finally achieve the natural definition of a weak solution to the problem (1.3).
Definition 1.1. A function $u \in H^s(\mathbb{R}^N)$ is a solution of (6.1) if

$$ (1.5) \quad \int_{\mathbb{R}^N} (-\Delta + m^2)^{s/2} u (-\Delta + m^2)^{s/2} v dx = \int_{\mathbb{R}^N} f(u) v dx $$

for all $v \in H^s(\mathbb{R}^N)$.

The space $H^s(\mathbb{R}^N)$ will be considered with the norm generated by the inner product defined by the left-hand side of (1.5):

$$ \|u\|^2 = \int_{\mathbb{R}^N} \left| (m^2 - \Delta)^{s/2} u(x) \right|^2 dx = \int_{\mathbb{R}^N} \left( m^2 + 4\pi^2 |\xi|^2 \right)^{s/2} |\hat{u}(\xi)|^2 d\xi. $$

A Pohozaev-type identity for problem (1.7).

After that, we establish a Pohozaev-type identity for the extension problem. With additional (but natural) hypotheses, it is not difficult to consider $f(x,u)$ instead of $f(u)$. We recall that a local approach, based on integration in balls $B_x^r \subset \mathbb{R}^{N+1}_+$, was obtained in [23].

The Pohozaev-type identity is a valuable tool when proving results of non-existence of non-trivial solutions for non-linear problems. It is also associated with the Pohozaev manifold generated by this identity, which is a precious technique in solving problems when either the (PS)-condition or the mountain pass geometry are difficult to be verified, see [27, 38].

Although arguments leading to Pohozaev-type identities are beginning to be standard, we present its proof in the case of the extension problem (1.7) (see by X. Chang and Z-Q. Wang [11]).

$$ (1.6) \quad \frac{N - 2s}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w|^2 dx + m^2 \frac{N + 2 - 2s}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} w^2 dx = Nk_s \int_{\mathbb{R}^{N+1}_+} F(w(x,0)) dx dy,$$

where $F(t) = \int_0^t f(\tau) d\tau$.

Observe the constant $k_s$ in this equation. Usually, changing scales, the constant $k_s$ is assumed to be equal to 1. We decide not to do so in order to better understand how this constant appears implicitly in the left-hand side of the equation above.

A Pohozaev-type identity for problem (1.3).

Our next step was to “translate” the Pohozaev-type identity in $\mathbb{R}^{N+1}_+$ into the original setting in $\mathbb{R}^N$. The Fourier transform is our main technique, following Brändle, Colorado, de Pablo and Sánchez [4]. This “reinterpretation” is as consequence of the Fourier transform theory developed for the operator $(-\Delta + m^2)^s$ and the study of the Bessel function attached to the extension problem.

To interpret the last integral in (1.6) as a integral in $\mathbb{R}^N$, we observe that a solution of problem (1.7) satisfies

$$ (1.7) \quad \left\{ \begin{array}{ll} \Delta_x w + \frac{1 - 2s}{y} w_y + w_{yy} - m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
 w(x,0) = u(x), & x \in \mathbb{R}^N, \end{array} \right. $$

since problem (1.7) is a extension of problem (1.3). In particular, $F(w(x,0)) = F(u)$ and the right-hand side of (1.6) causes no problem.
To proceed with the translation, we take the Fourier transform (in the variable $x \in \mathbb{R}^N$) of problem (1.7). Its solution is given in terms of the Bessel function $\Phi_s$ by

$$\hat{w}(\xi, y) = \hat{\nu}(\xi) \Phi_s \left( \sqrt{m^2 + 4\pi^2|\xi|^2} y \right),$$

where $\Phi_s$ solves (see [1, 4, 9, 23])

$$-\Phi + \frac{1-2s}{y} \Phi' + \Phi'' = 0, \quad \Phi(0) = 1, \quad \lim_{y \to \infty} \Phi(y) = 0. \tag{1.9}$$

The asymptotic behavior satisfied by $\Phi_s$ is well-known:

$$\Phi_s(s) \sim \begin{cases} 1 - c_1s^2, & \text{when } y \to 0, \\ c_2s^{(2s-1)/2}e^{-y}, & \text{when } y \to \infty, \end{cases} \tag{1.10}$$

where

$$c_1(s) = 2^{1-2s} \frac{\Gamma(1-s)}{2s\Gamma(s)} \quad \text{and} \quad c_2(s) = \frac{2^{(1-s)/2}\pi^{1/2}}{\Gamma(s/2)}, \tag{1.11}$$

(see [1, 4]) and $\Phi_s$ is a minimum of the functional

$$\mathcal{K} (\Phi) = \int_0^\infty (|\Phi(y)|^2 + |\Phi'(y)|^2) y^{1-2s} dy. \tag{1.12}$$

In [4] is stated that $\mathcal{K}(\Phi_s) = k_s$ and that this value can be obtained applying integration by parts. We were not able to do so. The method we could find to prove that $\mathcal{K}(\Phi_s) = k_s$ was tricky.

Observe that, in the case $s = 1/2$, the Bessel function $\Phi_{1/2}(t) = e^{-t}$ is easy to manipulate. This fact motivates our exposition in the general setting $(-\Delta + m^2)^s u$.

To translate the left-hand side of (1.6) into $\mathbb{R}^N$, we first note that it can be written as

$$\frac{N-2s}{2} \int_{\mathbb{R}^N} y^{1-2s} \left[ |\nabla w|^2 + m^2 w^2 \right] dxdy + m^2 \int_{\mathbb{R}^{N+1}_+} y^{1-2s} w^2 dxdy.$$

We show that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s} \left[ |\nabla w|^2 + m^2 w^2 \right] dxdy = k_s \int_{\mathbb{R}^N} \left( m^2 - \Delta \right)^{s/2} u(x)^2 

\text{thus obtaining}

\frac{N-2s}{2} k_s \int_{\mathbb{R}^N} \left( m^2 - \Delta \right)^{s/2} u(x)^2 

\text{and the constant } k_s \text{ already appears in the first integral of the left-hand side of the above equation.}

By applying Plancherel’s identity and changing variables, we obtain

$$m^2 \int_{\mathbb{R}^{N+1}_+} y^{1-2s} w^2 dxdy = m^2 \int_{\mathbb{R}^N} \frac{|\hat{\nu}(\xi)|^2 \xi^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \int_0^\infty |\Phi_s(t)|^2 t^{1-2s} dt.$$

Our last obstacle in the “translation” was the evaluation of the integral en $\Phi_s$:

$$\int_0^\infty |\Phi_s(t)|^2 t^{1-2s} dt = s k_s.$$
This evaluation was obtained as a consequence of $K(\Phi_s) = k_s$. As expected, the Pohozaev-type identity does not depend on $k_s$:

**Theorem 1.** A solution $u \in H^s(\mathbb{R}^N)$ of problem (1.3) satisfies
\[
\frac{N - 2s}{2} \int_{\mathbb{R}^N} \left( (m^2 - \Delta)^{s/2} u(x) \right)^2 \, dx + sm^2 \int_{\mathbb{R}^N} \frac{|\hat{u}(\xi)|^2 \xi^s}{(m^2 + 4\pi^2 |\xi|^2)^{1-s}} \, d\xi
\]
\[= N \int_{\mathbb{R}^N} F(u) \, dx.\]

A non-existence result. Once the Pohozaev-type identity in $\mathbb{R}^N$ was obtained, we prove a result of non-existence of non-trivial solutions. Observe that our result is valid not only for positive solutions.

**Theorem 2.** The problem
\[(-\Delta + m^2)^s u = |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N\]
has no non-trivial solution if $p \geq 2^*_s$, where
\[2^*_s = \frac{2N}{N - 2s}.\]

It follows from Theorem 2 that the constant
\[0 < \Lambda = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \int_{\mathbb{R}^N} \left( \frac{(m^2 + 4\pi^2 |\xi|^2)^{s/2} |\hat{u}(\xi)|^2 |\xi|^s}{\left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{2/2^*_s}} \right) \, d\xi < \infty\]
is not attained.

However, Cotsiolis and Tavoularis [16] proved that the Sobolev constant
\[S = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \int_{\mathbb{R}^N} \left( \frac{(2\pi |\xi|)^{2^*_s} |\hat{u}(\xi)|^2 |\xi|^s}{\left( \int_{\mathbb{R}^N} |u|^{2^*_s} \, dx \right)^{2/2^*_s}} \right) \, d\xi\]
is attained by the function $U(x) = c(\mu^2 + (x - x_0)^2)^{-(N-2s)/2}$, where $c$ and $\mu$ are constants, with $c$ chosen so that $|U|_{2^*_s} = 1$. The function $U$ is in $H^s(\mathbb{R}^N)$, if $N > 4s$.

We prove that $\Lambda = S$.

**Solution to a asymptotic linear problem.** In the sequel, we handle the problem
\[(1.13) \quad (-\Delta + m^2)^s u = f(u) \quad \text{in} \quad \mathbb{R}^N,\]
with the non-linearity $f$ having as a model (see specific hypotheses in Section 4)
\[f(t) = c \frac{t^3}{1 + t^2},\]
where $c$ is a constant greater than $m^{2s}$.

Considering the behavior of the non-linearity $f$ at infinity, it is natural to consider Cerami sequences and to apply the Ghoussoub-Preiss theorem. As a closed manifold, we consider a Nehari-Pohozaev manifold, as introduced by Ruiz [38], making use of the Pohozaev-type identity obtained for the space $\mathbb{R}^N$. 
integrating by parts, we conclude that
\[ \in \alpha, \gamma \]
the operator \( L \) fractionary operator (our approach being based on the action of the heat semigroup \( e \)).
A transform can be found in many texts about the subject, see, e.g., [25].

(1.14) \((\alpha + m^2)^s u = f(u) \) in \( \mathbb{R}^N \),
when \( f \) satisfies

1) \( f: \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function such that \( f(t)/t \) is increasing if \( t > 0 \) and decreasing if \( t < 0 \);
2) \[ \lim_{t \to 0} \frac{f(t)}{t} = 0 \] and \[ \lim_{t \to \infty} \frac{f(t)}{t} = k \in (m^{2s}, \infty) ; \]
3) \( \lim_{|t| \to \infty} tf(t) - 2F(t) = \infty \), where \( F(t) = \int_0^t f(\tau) d\tau \),
has a symmetry result. In Section 7 we present some simple facts about a modified Bessel kernel. Then, applying the moving planes method in integral form as introduced by W. Chen, C. Li and B. On [12], we prove the following result:

**Theorem 4.** Let \( f: [0, \infty) \to \mathbb{R} \) be a continuous function that satisfies

1) \( f(t) \geq 0 \) and \( f''(t) \geq 0 \) for all \( t \in [0, \infty) \).
2) For any \( \beta \in (1, 2_s^* - 1) \), there exists \( q \in [2, 2^*_s] \) with \( q > \max \{ \beta, \frac{N(\beta - 1)}{2s} \} \)
   such that \( f'(w) \in L^{\beta/(\beta - 1)}(\mathbb{R}^N), \forall w \in H^s(\mathbb{R}^N) \).

For any \( 0 < s < 1, N > 2s \) and \( m \in \mathbb{R} \setminus \{ 0 \} \), if \( u(x) \) is a positive solution of
\[ (-\Delta + m^2)^s u = f(u) \) in \( \mathbb{R}^N \),
then \( u \) is radially symmetric and decreasing with respect to the origin.

Although hypothesis (s2) is not standard, in that section we show that it is satisfied by functions like: a) \( f(t) = t^\alpha \), if \( \alpha \in (1, 2_s^* - 1) \); b) \( f(t) = t^\alpha + t^\gamma \), if \( \alpha, \gamma \in (1, 2_s^* - 1) \); c) \( f(t) = t \ln(1 + t) \).

2. A Fourier transform theory for \((\alpha + m^2)^s\)

In the case \( 0 < s < 1 \), aiming to present a specific Fourier theory for the fractionary operator \((\alpha + m^2)^s\), we follow the work of Stinga and Torrea [40], our approach being based on the action of the heat semigroup \( e^{tL} \) generated by the operator \( L \) acting on \( L^s \), for \( h \in \text{Dom}(L^s) \). Some results about the Fourier transform can be found in many texts about the subject, see, e.g., [23].

For any \( \lambda > 0 \) and \( 0 < s < 1 \), making the change of variables \( s = \lambda t \) and integrating by parts, we conclude that
\[ \lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\lambda t} - 1) \frac{dt}{t^{1+s}}. \]

Therefore, for an operator \( L \), we have an expression for \( L^s f(x) \), if \( f: \mathbb{R}^N \to \mathbb{R} \):
\[ L^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-\lambda t} f(x) - f(x)) \frac{dt}{t^{1+s}}, \forall x \in \mathbb{R}^N, \forall s \in (0, 1). \]

This formula makes use of the classical heat-diffusion semigroup generated by \( L \), see [40] Equation (1.11)].

Let us consider the problem
\[ \begin{cases} v_t(x, t) = (\Delta - m^2) v(x, t), & (x, t) \in \mathbb{R}^N \times (0, \infty) \\ v(x, 0) = f(x), & x \in \mathbb{R}^N, \end{cases} \]
which has the solution
\[ v(x, t) = e^{(\Delta - m^2)t}f(x), \quad \forall x \in \mathbb{R}^N, \; \forall t \in (0, \infty). \]

Taking the Fourier transform in (2.3) with respect to the variable \( x \), we obtain
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} \hat{v}(\xi, t) = -\left( m^2 + 4\pi^2|\xi|^2 \right) \hat{v}(\xi, t), \quad (\xi, t) \in \mathbb{R}^N \times (0, \infty)\\
\hat{v}(\xi, 0) = \hat{f}(\xi), \quad \xi \in \mathbb{R}^N, \end{array} \right.
\end{align*}
\]
its solution being given by
\[ \hat{v}(\xi, t) = e^{-(m^2 + 4\pi^2|\xi|^2)t} \hat{f}(\xi), \quad (\xi, t) \in \mathbb{R}^N \times (0, \infty). \]

We conclude that
\[ e^{(\Delta - m^2)t}f(x) = \mathcal{F}^{-1} (\hat{v}(\xi, t)) (x) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} \hat{v}(\xi, t) d\xi \]
(2.4)
\[ = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} e^{-(m^2 + 4\pi^2|\xi|^2)t} \hat{f}(\xi) d\xi. \]

On the other hand, substituting \( L = (\Delta + m^2) \) into (2.2), we obtain
\[ (-\Delta + m^2)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{(\Delta - m^2)t}f(x) - f(x) \right) \frac{dt}{t^{1+s}}. \]

Since \( \mathcal{F}^{-1}(\mathcal{F}f) = f \), it follows from (2.4) that
\[ (-\Delta + m^2)^s f(x) \]
\[ = \frac{1}{\Gamma(-s)} \int_0^\infty \left[ \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} e^{-(m^2 + 4\pi^2|\xi|^2)t} \hat{f}(\xi) d\xi - \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi \right] \frac{dt}{t^{1+s}} \]
\[ = \frac{1}{\Gamma(-s)} \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} \hat{f}(\xi) \left[ \int_0^\infty \left( e^{-(m^2 + 4\pi^2|\xi|^2)t} - 1 \right) \frac{dt}{t^{1+s}} \right] d\xi. \]

Since (2.1) yields
\[ \int_0^\infty \left( e^{-(m^2 + 4\pi^2|\xi|^2)t} - 1 \right) \frac{dt}{t^{1+s}} = \Gamma(-s) (m^2 + 4\pi^2|\xi|^2)^s, \]
we obtain
\[ (-\Delta + m^2)^s f(x) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} (m^2 + 4\pi^2|\xi|^2)^s \hat{f}(\xi) d\xi \]
(2.5)
\[ = \mathcal{F}^{-1} \left( (m^2 + 4\pi^2|\xi|^2)^s f \right) (x). \]

Therefore, we conclude that
\[ \mathcal{F} \left[ (-\Delta + m^2)^s f \right] (\xi) = (m^2 + 4\pi^2|\xi|^2)^s \mathcal{F}(f)(\xi). \]

In the case \( s = 1/2 \), formula (2.6) can be found in [28]. In the general case, it is no surprise, see Stein [39] or Garofalo [25]. The same happens with the next result:

**Lemma 2.1.** For any \( f \in \mathcal{S}(\mathbb{R}^N) \) and \( s \in (0, 1) \) we have
\[ (-\Delta + m^2)^s f(\xi) = \mathcal{F} \left[ (m^2 + 4\pi^2|\xi|^2)^s \mathcal{F}^{-1}(f) \right] (\xi). \]
Proof. Since \( g = \mathcal{F}^{-1}(\mathcal{F}(g)) \) for any \( g \in \mathcal{S}(\mathbb{R}^N) \), it follows from \([2.5]\) that
\[
(-\Delta + m^2)^s f(\xi) = \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} (m^2 + 4\pi^2 |x|^2)^s \hat{f}(x) dx
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{2\pi i x \cdot \xi} (m^2 + 4\pi^2 |x|^2)^s e^{-2\pi i x \cdot z} f(z) dz dx
\]
The change of variables \( x \to -x \) yields
\[
(-\Delta + m^2)^s f(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x \cdot \xi} (m^2 + 4\pi^2 |x|^2)^s \left( \int_{\mathbb{R}^N} e^{2\pi i x \cdot z} f(z) dz \right) dx
\]
\[
= \mathcal{F} \left[ (m^2 + 4\pi^2 |\cdot|^2)^s \mathcal{F}^{-1}(f) \right](\xi)
\]
and the proof is complete. \( \square \)

We recall that, for any \( f, g \in L^1(\mathbb{R}^N) \) it holds
\[ (2.7) \quad \int_{\mathbb{R}^N} \hat{f}(x)g(x)dx = \int_{\mathbb{R}^N} f(x)\hat{g}(x)dx. \]

We now prove the symmetry of the operator \((-\Delta + m^2)^s\):

**Lemma 2.2.** For any \( s \in (0, 1) \) and \( u, v \in \mathcal{S}(\mathbb{R}^N) \) we have
\[
\int_{\mathbb{R}^N} \left[ (-\Delta + m^2)^s u(x) \right] v(x)dx = \int_{\mathbb{R}^N} u(x) \left[ (-\Delta + m^2)^s v(x) \right] dx.
\]

**Proof.** We have
\[
\int_{\mathbb{R}^N} \left[ (-\Delta + m^2)^s u(x) \right] v(x)dx = \int_{\mathbb{R}^N} (-\Delta + m^2)^s u(x) \mathcal{F} \left( \mathcal{F}^{-1}(v) \right)(x) dx
\]
\[
= \int_{\mathbb{R}^N} \mathcal{F} \left( (-\Delta + m^2)^s u \right)(\xi) \left( \mathcal{F}^{-1}(v) \right)(\xi) d\xi
\]
\[
= \int_{\mathbb{R}^N} (m^2 + 4\pi^2 |\xi|^2)^s \mathcal{F}(u) \left( \mathcal{F}^{-1}(v) \right)(\xi) d\xi,
\]
where the second equality follows from \([2.7]\) and the third by \([2.6]\). Since Lemma \([2.1]\) guarantees that \((m^2 + 4\pi^2 |\xi|^2)^s \mathcal{F}^{-1}(v)(\xi) = \mathcal{F}^{-1} \left[ (-\Delta + m^2)^s v \right](\xi)\), we obtain
\[
\int_{\mathbb{R}^N} \left[ (-\Delta + m^2)^s u(x) \right] v(x)dx = \int_{\mathbb{R}^N} \mathcal{F}(u)(\xi) \mathcal{F}^{-1} \left[ (-\Delta + m^2)^s v \right](\xi)
\]
\[
= \int_{\mathbb{R}^N} u(x) \left[ (-\Delta + m^2)^s v(x) \right] dx,
\]
the last equality being a consequence of \([2.7]\). \( \square \)

**Lemma 2.3.** For any \( s_1, s_2 \in (0, 1) \) such that \( s_1 + s_2 < 1 \) it holds
\[
(-\Delta + m^2)^{s_1} \cdot (-\Delta + m^2)^{s_2} = (-\Delta + m^2)^{s_1 + s_2}.
\]

**Proof.** For any \( u \in \mathcal{S}(\mathbb{R}^N) \) we have
\[
\mathcal{F} \left( (-\Delta + m^2)^{s_1 + s_2} u \right)(\xi) = (m^2 + 4\pi^2 |\xi|^2)^{s_1 + s_2} \mathcal{F}(u)(\xi)
\]
\[
= (m^2 + 4\pi^2 |\xi|^2)^{s_1} (m^2 + 4\pi^2 |\xi|^2)^{s_2} \mathcal{F}(u)(\xi)
\]
\[
= (m^2 + 4\pi^2 |\xi|^2)^{s_1} \mathcal{F} \left( (-\Delta + m^2)^{s_2} u \right)(\xi)
\]
\[
= \mathcal{F} \left( (-\Delta + m^2)^{s_1} \left( (-\Delta + m^2)^{s_2} u \right) \right)(\xi).
\]
Taking \( \mathcal{F}^{-1} \), we conclude. \( \square \)
By applying Lemmas 2.3 and 2.2 we immediately obtain:

**Corollary 2.4.** For any \( s \in (0, 1) \) and \( u, v \in \mathcal{S}(\mathbb{R}^N) \) it holds

\[
\int_{\mathbb{R}^N} \left[ (-\Delta + m^2)^{s} u(x) \right] v(x) \, dx = \int_{\mathbb{R}^N} (-\Delta + m^2)^{s/2} u(x) (-\Delta + m^2)^{s/2} v(x) \, dx.
\]

3. A Pohozaev-type identity for the extension problem

We consider the extension problem in \( \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty) \)

\[
\begin{cases}
-\text{div}(y^{1-2s} \nabla w) + m^2 y^{1-2s} w = 0, & (x, y) \in \mathbb{R}^{N+1}_+,
\end{cases}
\]

(3.1)

\[
\lim_{y \to 0^+} \left(-y^{1-2s} \frac{\partial w}{\partial y}(x, y)\right) = k_s f(w(x, 0)), \quad x \in \mathbb{R}^N.
\]

Usually, changing scales, the constant \( k_s \) is assumed to be equal to 1. However, in order to better understand the behavior while obtaining the Pohozaev-type identity, we will not change scales.

For all \( R > 0 \) and \( \delta \in (0, R) \), define

\[
\begin{align*}
D_{R, \delta}^+ &= \{ z = (x, y) \in \mathbb{R}^N \times [\delta, \infty) : |z|^2 \leq R^2 \} \\
\partial D_{R, \delta}^1 &= \{ z = (x, y) \in \mathbb{R}^N \times \{ y = \delta \} : |z|^2 \leq R^2 - \delta^2 \} \\
\partial D_{R, \delta}^2 &= \{ z = (x, y) \in \mathbb{R}^N \times [\delta, \infty) : |z|^2 = R^2 \}
\end{align*}
\]

and note that \( \partial D_{R, \delta}^+ = \partial D_{R, \delta}^1 \cup \partial D_{R, \delta}^2 \).

Denoting \( \eta \) the unit outward normal vector to \( \partial D_{R, \delta}^+ \), then

\[
\eta = \begin{cases} 
(0, \ldots, 0, -1), & \text{in } \partial D_{R, \delta}^1 \\
\frac{z}{R}, & \text{in } \partial D_{R, \delta}^2
\end{cases}
\]

Since

\[
\text{div}[y^{1-2s}\nabla w](z \cdot \nabla w)
\]

\[
= \text{div}[y^{1-2s}\nabla w(z \cdot \nabla w)] - [y^{1-2s}\nabla w \cdot \nabla(z \cdot \nabla w)]
\]

\[
= \text{div} \left[ y^{1-2s}\nabla w(z \cdot \nabla w) - y^{1-2s}z \frac{|\nabla w|^2}{2} \right] + \frac{N - 2s}{2} y^{1-2s}|\nabla w|^2,
\]

multiplication of (3.1) by \( w \cdot \nabla w \) and integration on \( D_{R, \delta}^+ \) give

\[
0 = -\iint_{D_{R, \delta}^+} \text{div} \left[ y^{1-2s}\nabla w(z \cdot \nabla w) - y^{1-2s}z \frac{|\nabla w|^2}{2} \right] \, dx \, dy
\]

\[
- \frac{N - 2s}{2} \iint_{D_{R, \delta}^+} y^{1-2s}|\nabla w|^2 \, dx \, dy + \iint_{D_{R, \delta}^+} m^2 y^{1-2s} w(z \cdot \nabla w) \, dx \, dy
\]

so that the application of the divergence theorem yields

\[
0 = -\iint_{\partial D_{R, \delta}^+} \left[ y^{1-2s}(\nabla w \cdot \eta)(z \cdot \nabla w) - y^{1-2s}(z \cdot \eta) \frac{|\nabla w|^2}{2} \right] \, d\sigma
\]

\[
- \frac{N - 2s}{2} \iint_{D_{R, \delta}^+} y^{1-2s}|\nabla w|^2 \, dx \, dy + \iint_{D_{R, \delta}^+} m^2 y^{1-2s} w(z \cdot \nabla w) \, dx \, dy
\]

\[
= J_1(R, \delta) + J_2(R, \delta) + J_3(R, \delta) + J_4(R, \delta),
\]
where

\[ J_1(R, \delta) = -\int_{\partial D_{R, \delta}} y^{1-2s} \left[ (-\partial_y w) (z \cdot \nabla w) + y \frac{\left| \nabla w \right|^2}{2} \right] d\sigma \]

\[ J_2(R, \delta) = -\int_{\partial D_{R, \delta}} y^{1-2s} \left[ \frac{1}{R} (z \cdot \nabla w)^2 - R \frac{\left| \nabla w \right|^2}{2} \right] d\sigma \]

\[ J_3(R, \delta) = -\frac{N - 2s}{2} \int_{D_{R, \delta}} y^{1-2s} \left| \nabla w \right|^2 dxdy \]

\[ J_4(R, \delta) = \int_{D_{R, \delta}} m^2 y^{1-2s} w (z \cdot \nabla w) dxdy. \]

Since \( u(x) = w(x, 0) \), then \( f(u)(x \cdot \nabla u) = \text{div}[F(u)x] - NF(u) \).

Denote \( B_R = \{(x, 0) \in \mathbb{R}^{N+1} : |x|^2 \leq R^2\} \). Then, (3.1) and the divergence theorem imply that

\[ \lim_{\delta \to 0} \int_{\partial D_{R, \delta}} y^{1-2s} (-\partial_y w) (z \cdot \nabla w) d\sigma \]

\[ = \int_{B_R} k_s f(u)(x \cdot \nabla u) dx = k_s \int_{B_R} [\text{div}[xF(u)] - NF(u)] dx \]

\[ = k_s \int_{\partial B_R} F(u)(x \cdot \eta) d\sigma - k_s N \int_{B_R} F(u) dx. \]

It follows that

\[ (3.2) \lim_{\delta \to 0} J_1(R, \delta) = - \left[ k_s \int_{\partial B_R} F(u)(x \cdot \eta) d\sigma - k_s N \int_{B_R} F(u) dx \right]. \]

Now we observe that

\[ \left| \int_{\partial B_R} F(u)(x \cdot \eta) d\sigma \right| \leq R \int_{\partial B_R} |F(u)| d\sigma \]

and

\[ \left| \int_{\partial D_{R, \delta}^+} y^{1-2s} \left[ \frac{1}{R} (z \cdot \nabla w)^2 - R \frac{\left| \nabla w \right|^2}{2} \right] d\sigma \right| \leq 2R \int_{\partial D_{R, \delta}^+} y^{1-2s} \left| \nabla w \right|^2 d\sigma. \]

We claim that there exists a sequence \( (R_n) \) such that, when \( R_n \to \infty \),

\[ \lim_{n \to \infty} R_n \int_{\partial B_{R_n}} |F(u)| d\sigma = 0 = \lim_{n \to \infty} R_n \int_{\partial D_{R, \delta}^+} y^{1-2s} \left| \nabla w \right|^2 d\sigma. \]

Supposing the contrary, there exist \( \tau > 0 \) \( R_1 > 0 \) such that, for all \( R \geq R_1 \),

\[ \int_{\partial B_{R_1}} |F(u)| d\sigma \geq \frac{\tau}{R}, \]

from what follows

\[ \int_{\mathbb{R}^N} |F(u)| dx \geq \int_{R_1} \int_{\partial B_{R_1}} |F(u)| d\sigma dR \geq \int_{R_1} \frac{\tau}{R} dR = \infty, \]
a contradiction. The same argument also applies to the second integral in (3.3) and proves the claim, which yields not only that

$$\lim_{n \to \infty} \lim_{\delta \to 0} J_1(R, \delta) = -k_s N \int_{\mathbb{R}^N} F(u)dx$$

but also

$$\lim_{n \to \infty} \lim_{\delta \to 0} J_2(R, \delta) = 0.$$  

By considering the same sequence $(R_n)$, it holds

$$\lim_{n \to \infty} \lim_{\delta \to 0} J_3(R_n, \delta) = -\frac{N - 2s}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 dxdy.$$

We now analyze $J_4(R, \delta)$:

$$J_4(R, \delta) = \int_{D_{R, \delta}^+} m^2 y^{1-2s} (z \cdot \nabla w) dxdy$$

$$= m^2 \int_{D_{R, \delta}^+} y^{1-2s} \left( z \cdot \nabla \left( \frac{w^2}{2} \right) \right) dxdy.$$

For this, we consider the field $\Phi = y^{1-2s}w^2 z$. Since

$$\text{div } \Phi = \frac{N + 2 - 2s}{2} y^{1-2s} w^2 + y^{1-2s} \left( z \cdot \nabla \left( \frac{w^2}{2} \right) \right),$$

the divergence theorem yields

$$m^2 \int_{D_{R, \delta}^+} y^{1-2s} \left( z \cdot \nabla \left( \frac{w^2}{2} \right) \right) dxdy$$

$$= -m^2 \frac{N + 2 - 2s}{2} \int_{D_{R, \delta}^+} y^{1-2s} w^2 dxdy - m^2 \int_{\partial D_{R, \delta}^+} y^{1-2s} w^2 \frac{w^2}{2} d\sigma$$

$$+ m^2 \int_{\partial D_{R, \delta}^+} y^{1-2s} R \frac{w^2}{2} d\sigma.$$

The same argument applied before shows that

$$\lim_{n \to \infty} \lim_{\delta \to 0} J_4(R, \delta) = -m^2 \frac{N + 2 - 2s}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} w^2 dxdy.$$

Collecting our results, we conclude the Pohozaev-type identity in $\mathbb{R}^{N+1}_+$

$$\frac{N - 2s}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 dxdy + m^2 \frac{N + 2 - 2s}{2} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} w^2 dxdy$$

$$= N k_s \int_{\mathbb{R}^N} F(w(x, 0))dx.$$  

Of course, equation (3.4) is direct applicable to problem (2). As mentioned before, it is not difficult to consider $f(x, u)$ instead of $f(u)$.  


4. A Pohozaev-type identity for the problem in $\mathbb{R}^N$

In this section we obtain a Pohozaev-type identity for the problem

$$(-\Delta + m^2)^s u = f(u) \text{ in } \mathbb{R}^N.$$ 

Since problem $(P)$ is an extension of problem $(1.3)$, it satisfies

\begin{align}
\Delta_x w + \frac{1 - 2s}{y} w_y + w_{yy} - m^2 w &= 0 \quad \text{in } \mathbb{R}^{N+1}_+ \\
w(x, 0) &= u(x), \quad x \in \mathbb{R}^N,
\end{align}

In particular, $F(w(x, 0)) = F(u)$ and the right-hand side of $(3.4)$ causes no problem.

We now interpret the integrals in $\mathbb{R}^{N+1}_+$ as integrals in $\mathbb{R}^N$. We start with a technical result, which has a tricky proof.

**Lemma 4.1.** If $K(\Phi)$ is the functional given by $(1.12)$, then

$$K(\Phi_s) = k_s.$$ 

**Proof.** Since $\Phi_s$ satisfies $(1.9)$, we have

$$\Phi_s^2 = \frac{1 - 2s}{y} \Phi_s' \Phi_s + \Phi_s'' \Phi_s,$$

from what follows

$$\Phi_s^2 y^{1-2s} = (\Phi_s' \Phi_s) \frac{d}{dy} y^{1-2s} + (\Phi_s'' \Phi_s) y^{1-2s} = \frac{d}{dy} (\Phi_s' \Phi_s y^{1-2s}) - (\Phi_s')^2 y^{1-2s},$$

thus showing that

$$[\Phi_s^2 + (\Phi_s')^2] y^{1-2s} = \frac{d}{dy} (\Phi_s' \Phi_s y^{1-2s}).$$

Therefore,

$$\int_0^\infty [\Phi_s^2 + (\Phi_s')^2] y^{1-2s} dy = \lim_{y \to 0} -\Phi_s' \Phi_s y^{1-2s} = -\lim_{y \to 0} \Phi_s' y^{1-2s} = 2sc_1 = k_s$$

and we are done. \qed

A second technical result that will be necessary in our analysis is the following:

**Lemma 4.2.** It holds

$$\int_0^\infty [\Phi_s(t)]^2 t^{1-2s} dt = sk_s.$$ 

**Proof.** Integration by parts yields

$$\int_0^\infty [\Phi_s(t)]^2 t^{1-2s} dt$$
as a consequence of Lemma 4.1.\index{Lemma 4.1}

\[ \begin{align*}
&= -\frac{1}{1-s} \int_{0}^{\infty} \Phi'_s(t) \left( \frac{1-2s}{t} \Phi'_s(t) + \Phi''_s(t) \right) t^{2-2s} dt \\
&= -\frac{1-2s}{1-s} \int_{0}^{\infty} \Phi'_s(t)^2 t^{1-2s} dt - \frac{1}{1-s} \int_{0}^{\infty} \Phi'_s(t) \Phi''_s(t) t^{1-2s} dt \\
&= -\frac{1}{1-s} \int_{0}^{\infty} \Phi'_s(t)^2 t^{1-2s} dt - \int_{0}^{\infty} \Phi'_s(t)^2 t^{1-2s} dt \\
&= -\frac{1}{s} \int_{0}^{\infty} \Phi'_s(t)^2 t^{1-2s} dt,
\end{align*} \]

from what follows our result. \hfill \Box

We are now in position to translate the Pohozaev-type identity in $\mathbb{R}_{+}^{N+1}$ into terms of integrals in $u$. We begin by writing the left-hand side of (3.4) as

\[ \frac{N-2s}{2} \int_{\mathbb{R}_{+}^{N+1}} y^{1-2s} \left[ |\nabla w|^2 + m^2 w^2 \right] \, dx dy + m^2 \int_{\mathbb{R}_{+}^{N+1}} y^{1-2s} u'^2 \, dx dy. \]

Observe that

\[ \begin{align*}
\int_{\mathbb{R}^N} |\nabla w(x,y)|^2 \, dx &= \int_{\mathbb{R}^N} \left( |\nabla x w(x,y)|^2 + \left| \frac{\partial w}{\partial y}(x,y) \right|^2 \right) \, dx \\
&= \int_{\mathbb{R}^N} \left( 4\pi^2 |\xi|^2 |\hat{w}(\xi,y)|^2 + \left| \frac{\partial \hat{w}}{\partial y}(\xi,y) \right|^2 \right) \, d\xi.
\end{align*} \]

By making use of the expression for $\hat{w}(\xi,y)$ given by \ref{eq:1.8} and denoting $c = \sqrt{m^2 + 4\pi^2|\xi|^2}$, by multiplying the last equality by $y^{1-2s}$ and integrating in $y$ we obtain

\[ \begin{align*}
&\int_{\mathbb{R}_{+}^{N+1}} (|\nabla w(x,y)|^2 + m^2 w(x,y))^2 \, y^{1-2s} \, dx dy \\
&= \int_{0}^{\infty} \int_{\mathbb{R}^N} |\nabla w(x,y)|^2 y^{1-2s} \, dx dy + \int_{0}^{\infty} \int_{\mathbb{R}^N} m^2 |w(x,y)|^2 y^{1-2s} \, dx dy \\
&= \int_{0}^{\infty} \int_{\mathbb{R}^N} \left( c^2 |\hat{w}(\xi,y)|^2 + \left| \frac{\partial \hat{w}}{\partial y}(\xi,y) \right|^2 \right) y^{1-2s} \, d\xi dy \\
&= \int_{0}^{\infty} \int_{\mathbb{R}^N} \left( c^2 |\hat{w}(\xi,y)|^2 |\Phi_s(c y)|^2 + |\hat{w}(\xi,y) c \Phi'_s(c y)|^2 \right) y^{1-2s} \, d\xi dy \\
&= \int_{0}^{\infty} \int_{\mathbb{R}^N} c^2 |\hat{w}(\xi,y)|^2 \left( |\Phi_s(c y)|^2 + |\Phi'_s(c y)|^2 \right) y^{1-2s} \, d\xi dy, \\
&= \int_{\mathbb{R}^N} c^{2s} |\hat{w}(\xi)|^2 \, d\xi \left( \int_{0}^{\infty} \left( |\Phi_s(t)^2 + |\Phi'_s(t)|^2 \right) t^{1-2s} \, dt \right) \\
&= \mathcal{K}(\Phi_s) \int_{\mathbb{R}^N} c^{2s} |\hat{w}(\xi)|^2 \, d\xi = \mathcal{K}_s \int_{\mathbb{R}^N} \left( 4\pi^2 |\xi|^2 + m^2 \right)^s |\hat{w}(\xi)|^2 \, d\xi,
\end{align*} \]

as a consequence of Lemma \ref{lemma:4.1}.

We conclude that

\[ \int_{\mathbb{R}_{+}^{N+1}} y^{1-2s} \left[ |\nabla w|^2 + m^2 w^2 \right] \, dx dy = \mathcal{K}_s \int_{\mathbb{R}^N} \left( m^2 - \Delta \right)^{s/2} u(x) \right|^2 \, dx. \]
By applying Plancherel’s identity, we interpret the last integral in \( w \) in (5.3) as a integral in \( \mathbb{R}^N \).
\[
\frac{N - 2s}{2} k_s \int_{\mathbb{R}^N} \left| (m^2 - \Delta)^{s/2} u(x) \right|^2 \, dx + sk_r m^2 \int_{\mathbb{R}^N} \frac{\left| \hat{u}(\xi) \right|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi = \mathcal{N}_k \int_{\mathbb{R}^N} F(u) \, dx,
\]
(4.3)
thus showing that the Pohozaev-type identity does not depend on \( k_s \).

5. A NON-EXISTENCE RESULT

In this section we show that the problem
\[
(-\Delta + m^2)^s u = |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N
\]
has no solution \( u \neq 0 \) if \( p \geq 2^*_s \).

Applying the Pohozaev-type identity (4.3) to the problem (5.1), we obtain
\[
\frac{N - 2s}{2} \int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{s/2} u \right|^2 \, dx + m^2 \int_{\mathbb{R}^N} \frac{\left| \hat{u}(\xi) \right|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi = \frac{N}{p} \int_{\mathbb{R}^N} |u|^p \, dx.
\]
(5.2)

Since \( u \) is a solution of (5.1), it satisfies Corollary 2.4 for \( f(u) = |u|^{p-2} u \). Thus,
\[
\int_{\mathbb{R}^N} (-\Delta + m^2)^{s/2} u (-\Delta + m^2)^{s/2} v \, dx = \int_{\mathbb{R}^N} |u|^{p-2} u v \, dx
\]
for any \( v \in H^s(\mathbb{R}^N) \). Choosing \( v = u \), we have
\[
\int_{\mathbb{R}^N} (-\Delta + m^2)^{s/2} u^2 \, dx = \int_{\mathbb{R}^N} |u|^p \, dx.
\]
Substituting (5.1) into (5.2), we obtain
\[
\left( \frac{N}{p} - \frac{N - 2s}{2} \right) \int_{\mathbb{R}^N} |u|^p \, dx = m^2 \int_{\mathbb{R}^N} \frac{\left| \hat{u}(\xi) \right|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi > 0,
\]
from what follows that \( \left( \frac{N}{p} - \frac{N+2s}{2} \right) > 0 \) and thus \( p < 2N/(N-2s) = 2^*_s \).

As a consequence, the constant

\[
0 < \Lambda = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (m^2 + 4\pi^2|\xi|^2)^s \hat{|u}(\xi)|^2 d\xi}{\left( \int_{\mathbb{R}^N} |u|^{2^*_s} dx \right)^{2/2^*_s}} < \infty
\]

is not attained.

It is well-known (see Cotsiolis and Tavoularis [10]) that the function \( U(x) = c(\mu^2 + (x-x_0)^2)^{-(N-2s)/2} \), where \( c \) and \( \mu \) are constants, with \( c \) chosen so that \( |U|_{2^*_s} = 1 \), attains the Sobolev constant

\[
S = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} \hat{|u}(\xi)|^2 d\xi}{\left( \int_{\mathbb{R}^N} |u|^{2^*_s} dx \right)^{2/2^*_s}} = \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} \hat{|U}(\xi)|^2 d\xi.
\]

If \( N > 4s \), then \( U \in H^s(\mathbb{R}^N) \).

We will show that \( \Lambda = S \).

Of course, we have \( \Lambda \geq S \), since

\[
\int_{\mathbb{R}^N} (m^2 + 4\pi^2|\xi|^2)^s \hat{|u}(\xi)|^2 d\xi \geq \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} \hat{|u}(\xi)|^2 d\xi, \quad \forall u \in H^s(\mathbb{R}^N).
\]

In order to show the opposite inequality, we define \( v_t(x) = U(tx) \) for \( t > 0 \). Changing variables, we obtain:

\[(i) \int_{\mathbb{R}^N} |\hat{v}_t(\xi)|^2 d\xi = t^{-2N} \int_{\mathbb{R}^N} |\hat{U}(\xi/t)|^2 d\xi = t^{-N} \int_{\mathbb{R}^N} |\hat{U}(\xi)|^2 d\xi;\]

\[(ii) \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} |\hat{v}_t(\xi)|^2 d\xi = t^{-2N} \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} |\hat{U}(\xi/t)|^2 d\xi\]

\[= t^{-N+2s} \int_{\mathbb{R}^N} (2\pi)^{2s} |\xi|^s |\hat{U}(\xi)|^2 d\xi;\]

\[(iii) \int_{\mathbb{R}^N} |v_t(x)|^{2^*_s} dx = t^{-N} \int_{\mathbb{R}^N} |U(x)|^{2^*_s} dx = t^{-N}.\]

It immediately follows from \((iii)\) that

\[
\left( \int_{\mathbb{R}^N} |v_t(x)|^{2^*_s} dx \right)^{2/2^*_s} = t^{-N+2s} \left( \int_{\mathbb{R}^N} |U(x)|^{2^*_s} dy \right)^{2/2^*_s} = t^{-N+2s}.
\]
Thus,
\[
\Lambda \leq \int_{\mathbb{R}^N} \left( m^2 + 4\pi^2 |\hat{\psi}(\xi)|^2 \right) |\hat{\psi}(\xi)|^2 d\xi
\]
\[
\left( \int_{\mathbb{R}^N} |\psi|^2 \right)^{2/2^*}
\]
\[
\leq \frac{m^{2s}}{t-N+2s} \int_{\mathbb{R}^N} |\hat{\psi}(\xi)|^2 d\xi + \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} |\hat{\psi}(\xi)|^2 d\xi
\]
\[
= \frac{m^{2s} t^{-N}|U|^2}{t-N+2s} + \frac{t^{-N+2s}}{t-N+2s} \int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} |\hat{\psi}(\xi)|^2 d\xi
\]
\[
= \frac{m^{2s}}{t^{2s}} |U|^2 + S.
\]
Making \( t \to \infty \), we obtain \( \Lambda \leq S \), completing the proof of \( \Lambda = S \).

6. The solution of a asymptotic linear problem

In this section we will prove existence of solution for the problem
\[
(-\Delta + m^2)^s u = f(u) \quad \text{in} \quad \mathbb{R}^N.
\]
when \( f \) satisfies

(\( f_1 \)) \( f: \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function such that \( f(t)/t \) is increasing if \( t > 0 \) and decreasing if \( t < 0 \);

(\( f_2 \)) \( \lim_{t \to 0} \frac{f(t)}{t} = 0 \) and \( \lim_{t \to \infty} \frac{f(t)}{t} = k \in (m^{2s}, \infty] \);

(\( f_3 \)) \( \lim_{|\xi| \to \infty} |\xi|^s f(t) - 2F(t) = \infty \), where \( F(t) = \int_0^t f(\tau) d\tau \).

Our hypotheses on \( f \) imply that the non-quadraticity condition is satisfied by our problem, that is, \( tf(t) - 2F(t) > 0 \) for all \( t \neq 0 \) and \( (f_3) \). A model problem is given by
\[
f(t) = c \frac{t^3}{1+t^2},
\]
where \( c > m^{2s} \) is a constant.

It follows from our hypotheses that
\[
|f(t)| \leq \epsilon|t| + C_\epsilon|t|^{p-1} \quad \text{and} \quad F(t) \leq \epsilon|t|^2 + C_\epsilon|t|^p,
\]
for all \( 2 < p < 2^*_s \), where \( 2^*_s = 2N/(N-2s) \).

Lemma 6.1. For each \( t > 0 \) and \( u \in H^s(\mathbb{R}^N) \) it holds
\[
\frac{t^2}{2} f(u)u - F(tu) \leq \frac{1}{2} f(u)u - F(u).
\]

Proof. Define \( \psi(t) = \frac{t^2}{2} f(u)u - F(tu) \). Then \( \psi'(t) = tf(u)u - f(tu)u \), from what follows \( \psi'(1) = 0 \). Since
\[
\psi'(t) = tu^2 \left[ \frac{f(u)}{u} - \frac{f(tu)}{tu} \right], \quad u \neq 0,
\]
we have \( \psi'(t) > 0 \) if \( 0 < t < 1 \) and \( \psi'(t) < 0 \) if \( t > 1 \). Thus, \( \psi(1) = \max_{t \geq 0} \psi(t) > 0 \) and our claim follows. \( \square \)
Observe that Definition 1.1 is satisfied by critical points of the functional

$$
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{s/2} u \right|^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} f(u) \, dx
$$

and

$$
\Phi'(u) \cdot u = \|u\|^2 - \int_{\mathbb{R}^N} f(u) \, dx. 
$$

(6.3)

We denote by \( \mathcal{N} \) the Nehari manifold naturally attached to \( \Phi \):

$$
\mathcal{N} = \{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : \Phi'(u) \cdot u = 0 \}
$$

and by \( P(u) \) the functional generated by the Pohozaev-type identity (4.3):

$$
P(u) = \frac{N - 2s}{2} \|u\|^2 + m^2s \int_{\mathbb{R}^N} \frac{|\hat{u}(\xi)|^2}{|\xi|^{2s}} \, d\xi - N \int_{\mathbb{R}^N} F(u) \, dx.
$$

Now, for each \( u \in H^s(\mathbb{R}^N) \) and \( t > 0 \), denote by

$$
u_t(x) = tu \left( \frac{x}{t^2} \right) \in H^s(\mathbb{R}^N)
$$

and consider

$$
h_u(t) = \Phi(u_t) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta + m^2)^{s/2} tu \left( \frac{x}{t^2} \right) \right|^2 \, dx - \int_{\mathbb{R}^N} F \left( tu \left( \frac{x}{t^2} \right) \right) \, dx
$$

$$
= \frac{t^{4N+2}}{2} \int_{\mathbb{R}^N} (m^2 + 4\pi^2|\xi|^2)^s |\hat{u}(t^2\xi)|^2 \, d\xi - t^{2N} \int_{\mathbb{R}^N} F(tu) \, dx
$$

$$
= \frac{t^{2N+2-4s}}{2} \int_{\mathbb{R}^N} (t^4m^2 + 4\pi^2|\xi|^2)^{1/2} |\hat{u}(\xi)|^2 \, d\xi - t^{2N} \int_{\mathbb{R}^N} F(tu) \, dx
$$

(6.4)

$$
t^{2N+2} \left[ \frac{1}{2} \int_{\mathbb{R}^N} \left( m^2 + \frac{4\pi^2|\xi|^2}{t^4} \right)^s |\hat{u}(\xi)|^2 \, d\xi - \int_{\mathbb{R}^N} \frac{F(tu)}{(tu)^2} u^2 \, dx \right].
$$

According to \((f_2)\), the expression between brackets in (6.4) converges to

$$
\frac{m^{2s} - k}{2} \int_{\mathbb{R}^N} |u|^2 \, dx,
$$

from what follows

$$
\lim_{t \to \infty} h_u(t) = -\infty. 
$$

(6.5)

But we also have that

$$
\frac{h_u(t)}{t^{2N+2}} = \frac{1}{2t^{4s}} \int_{\mathbb{R}^N} (t^4m^2 + 4\pi^2|\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi - \int_{\mathbb{R}^N} \frac{F(tu)}{(tu)^2} u^2 \, dx,
$$

so that \((f_2)\) yields \( \lim_{t \to 0} \frac{h_u(t)}{t^{N+2}} = \infty \), from what follows that

$$
h_u(t) > 0, \quad \text{if} \quad t > 0 \quad \text{is small enough}.
$$

Therefore, \( h_u(t) \) attains a maximum point since, for each fixed \( u \in H^s(\mathbb{R}^N) \setminus \{0\} \), \( h_u \in C^1(\mathbb{R}_+, \mathbb{R}) \).
Taking the derivative of $h_u$, we obtain
\begin{align*}
  h_u'(t) &= (N + 1 - 2s)t^{2N+1-4s} \int_{\mathbb{R}^N} (t^4 m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \\
  &\quad + 2st^{2N+5-4s} \int_{\mathbb{R}^N} \frac{m^2 |\hat{u}(\xi)|^2}{(t^4 m^2 + 4\pi^2 |\xi|^2)^{1-s}} d\xi \\
  &\quad - 2N t^{2N-1} \int_{\mathbb{R}^N} F(tu) dx - t^{2N} \int_{\mathbb{R}^N} f(tu) u dx.
\end{align*}

(6.6)

So, when $t = 1$,
\begin{align*}
  h_u'(1) &= (N + 1 - 2s) \int_{\mathbb{R}^N} (m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \\
  &\quad + 2sm^2 \int_{\mathbb{R}^N} \frac{|\hat{u}(\xi)|^2}{(m^2 + 4\pi^2 |\xi|^2)^{1-s}} d\xi - 2N \int_{\mathbb{R}^N} F(u) dx - \int_{\mathbb{R}^N} f(u) u dx \\
  &= \Phi'(u) \cdot u + 2P(u) =: J(u).
\end{align*}

(6.7)

This motivates to consider the Nehari-Pohozaev manifold
\[ \mathcal{M} = \{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : J(u) = 0 \} . \]

Changing variables, we observe that
\begin{equation}
  h_{u_t}'(1) = th_u'(t),
\end{equation}

It follows from the Pohozaev-type identity (4.3) that any solution $u \in H^s(\mathbb{R}^N)$ of (6.1) belongs to $\mathcal{M}$, since $P(u) = 0$ and $\Phi'(u) \cdot u = 0$.

Furthermore, taking into account (6.8),
\begin{equation}
  u_t \in \mathcal{M} \iff J(u_t) = 0 \iff h_{u_t}' = 1 \iff h_u'(t) = 0.
\end{equation}

(6.9)

We now show that there exists a unique point $t$ where $h_u(t)$ attains its maximum.

**Lemma 6.2.** For each $u \in H^s(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u = t(u) > 0$ such that $h_u(t)$ attains its maximum at $t_u$. The function $h_u(t)$ is positive and increasing for $t \in (0, t_u]$ and decreasing for $t > t_u$.

Furthermore, the function
\[ u \mapsto t_u \]

is continuous and
\[ u_{t_u} \in \mathcal{M} \quad \text{and} \quad \Phi(u_{t_u}) = \max_{t > 0} \Phi(u_t) > 0. \]

**Proof.** We have already shown that $h_u(t)$ attains its maximum at a point $t_u$. Since $h_u \in C^1(\mathbb{R}_+, \mathbb{R})$, we have $h_u'(t_u) = 0$. 

According to (6.6) we have

\[ h_u'(t) = 0 \iff t^{2N+1} \left[ \frac{(N+1-2s)}{t^{4s}} \int_{\mathbb{R}^N} (t^4 m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \
+ 2sm^2 \int_{\mathbb{R}^N} \frac{t^{4(1-s)}|\hat{u}(\xi)|^2}{(m^2 t^4 + 4\pi^2 |\xi|^2)^{1-s}} d\xi \right] 
- t^{2N+1} \left[ 2N \int_{\mathbb{R}^N} \frac{F(tu)}{t^2} dx + \int_{\mathbb{R}^N} \frac{f(tu)u}{t} dx \right] = 0 \]

(6.10a)

\[ \iff \frac{(N+1-2s)}{t^{4s}} \int_{\mathbb{R}^N} (t^4 m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi 
+ 2sm^2 \int_{\mathbb{R}^N} \frac{t^{4(1-s)}|\hat{u}(\xi)|^2}{(m^2 t^4 + 4\pi^2 |\xi|^2)^{1-s}} d\xi 
- \frac{1}{t^2} \int_{\mathbb{R}^N} [2NF(tu) + f(tu)tu] dx = 0 \]

(6.10b)

We denote

\[ I_1(t) = \frac{1}{t^2} \int_{\mathbb{R}^N} 2NF(tu) dx, \quad I_2(t) = \frac{1}{t^2} \int_{\mathbb{R}^N} f(tu)tudx \]

and

\[ g(t) = \frac{(N+1-2s)}{t^{4s}} \int_{\mathbb{R}^N} (t^4 m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi 
+ 2sm^2 \int_{\mathbb{R}^N} \frac{t^{4(1-s)}|\hat{u}(\xi)|^2}{(m^2 t^4 + 4\pi^2 |\xi|^2)^{1-s}} d\xi. \]

It follows from (f1) that

\[ \frac{d}{dt} I_1(t) = \frac{2N}{t^3} \int_{\mathbb{R}^N} [f(tu)tu - 2F(tu)] dx > 0 \]

and also

\[ \frac{d}{dt} I_2(t) = \frac{d}{dt} \int_{\mathbb{R}^N} f(tu) |u|^2 d\xi = \int_{\mathbb{R}^N} \frac{d}{dt} \left( \frac{f(tu)}{tu} \right) |u|^2 dx > 0. \]

We conclude that \( I_1(t) + I_2(t) \) is a strictly increasing function.

We will now show that \( g(t) \) is strictly decreasing. In fact,
such that \( u (N + 1 - 2s) \) implies that

\[
g' (t) = \frac{-4s(N + 1 - 2s)}{t^{4s+1}} \int_{\mathbb{R}^N} \left( t^4 m^2 + 4\pi^2 |\xi|^2 \right)^s |\hat{u}(\xi)|^2 d\xi + 4s(N + 1 - 2s) \int_{\mathbb{R}^N} \frac{t^{4s} m^2 |\hat{u}(\xi)|^2}{(t^4 m^2 + 4\pi^2 |\xi|^2)^{1-s}} d\xi
\]

+ 2sm^2(1-s)t^{4(1-s)-1} \int_{\mathbb{R}^N} \frac{|\hat{u}(\xi)|^2}{(t^4 m^2 + 4\pi^2 |\xi|^2)^{1-s}} d\xi

- 2sm^2(1-s)t^{4(1-s)+3} \int_{\mathbb{R}^N} \frac{|\hat{u}(\xi)|^2}{(t^4 m^2 + 4\pi^2 |\xi|^2)^{2-s}} d\xi

= \frac{-4s(N + 1 - 2s)}{t^{4s+1}} \int_{\mathbb{R}^N} \frac{4\pi^2 |\hat{u}(\xi)|^2}{(t^4 m^2 + 4\pi^2 |\xi|^2)^{1-s}} d\xi

+ 2sm^2(1-s)t^{4(1-s)-1} \int_{\mathbb{R}^N} \frac{|\hat{u}(\xi)|^2}{(t^4 m^2 + 4\pi^2 |\xi|^2)^{1-s}} d\xi

- \int_{\mathbb{R}^N} \left[ 2N F(t_n u_n) + f(t_n u_n) u_n \right] dx = 0.

Thus, \( g(t) - (I_1(t) + I_2(t)) \) is strictly decreasing, proving the uniqueness of \( t_u \).

To prove that the function \( u \mapsto t_u \) is continuous, let us consider a sequence \((u_n)\) such that \( u_n \to u \) in \( H^s (\mathbb{R}^N) \). We denote \( t_n = t_{u_n} \). We claim that \( t_n \) is bounded.

To prove our claim, we observe that (6.10b) implies that

\[
h'_{u_n} (t_n) = 0 \iff \frac{(N + 1 - 2s)}{t^{4s}_n} \int_{\mathbb{R}^N} \left( t^n m^2 + 4\pi^2 |\xi|^2 \right)^s |\hat{u}_n(\xi)|^2 d\xi
\]

+ 2sm^2 \int_{\mathbb{R}^N} \frac{t^{4(1-s)} |\hat{u}_n(\xi)|^2}{(m^2 t^n + 4\pi^2 |\xi|^2)^{1-s}} d\xi

- \int_{\mathbb{R}^N} \left[ 2N F(t_n u_n) + f(t_n u_n) u_n \right] dx = 0.

Since \( t_n > 0 \) for all \( n \), suppose that \( t_n \to \infty \). Application of the dominated convergence theorem and hypotheses \((f_1)\) and \((f_2)\) yield

\[
\int_{\mathbb{R}^N} \left[ 2N \frac{F(t_n u_n)}{t_n^2} + \frac{f(t_n u_n) u_n}{t_n} \right] dx = \int_{\mathbb{R}^N} \left[ 2N \frac{F(t_n u_n)}{|t_n u_n|^2} + \frac{f(t_n u_n)}{|t_n u_n|} \right] |u_n|^2 dx

\to \infty \int_{\mathbb{R}^N} \left[ 2N \frac{k}{2} + k \right] |u|^2 dx

= (N + 1)k \int_{\mathbb{R}^N} |u|^2 dx.

On the other side, since

\[
\frac{(N + 1 - 2s)}{t^{4s}_n} \int_{\mathbb{R}^N} \left( t^n m^2 + 4\pi^2 |\xi|^2 \right)^s |\hat{u}_n(\xi)|^2 d\xi + 2sm^2 \int_{\mathbb{R}^N} \frac{t^{4(1-s)} |\hat{u}_n(\xi)|^2}{(m^2 t^n + 4\pi^2 |\xi|^2)^{1-s}} d\xi
\]
converges to
\[
\int_{\mathbb{R}^N} (N + 1) m^2 \lvert \hat{u}(\xi) \rvert^2 d\xi = \int_{\mathbb{R}^N} (N + 1) m^2 \lvert u(x) \rvert^2 dx,
\]

it follows from (6.11) that
\[
(N + 1)(m^{2s} - k) \int_{\mathbb{R}^N} \lvert u \rvert^2 dx = 0,
\]
and we have reached a contradiction.

Thus, we can suppose that \( t_n \to t_0 \in (0, \infty) \). (Observe that we already know that \( t_0 \neq 0 \).) By applying once again the dominated convergence theorem to (6.11), we obtain both
\[
\int_{\mathbb{R}^N} \left[ 2N \frac{F(t_n u_n)}{t_n^2} + \frac{f(t_n u_n) u_n}{t_n} \right] dx \xrightarrow{n \to \infty} \int_{\mathbb{R}^N} \left[ 2N \frac{F(t_0 u)}{t_0^2} + \frac{f(t_0 u) u}{t_0} \right] dx
\]
and
\[
\frac{(N + 1 - 2s)}{t_n^{4s}} \int_{\mathbb{R}^N} (t_n^4 m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}_n(\xi)|^2 d\xi + 2sm^2 \int_{\mathbb{R}^N} \frac{t_n^4 |\hat{u}_n(\xi)|^2}{m^2 t_n^4 + 4\pi^2 |\xi|^2} d\xi \xrightarrow{n \to \infty} \int_{\mathbb{R}^N} \frac{(N + 1 - 2s)}{t_0^{4s}} (t_0^4 m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi + 2sm^2 \int_{\mathbb{R}^N} \frac{t_0^4 |\hat{u}(\xi)|^2}{m^2 t_0^4 + 4\pi^2 |\xi|^2} d\xi.
\]
Thus, by passing to the limit in (6.10a) yields
\[
t_0^{2N+1} \left[ (N + 1 - 2s) \int_{\mathbb{R}^N} \frac{(t_0^4 m^2 + 4\pi^2 |\xi|^2)^s |\hat{u}(\xi)|^2}{t_0^4} d\xi + 2sm^2 \int_{\mathbb{R}^N} \frac{t_0^4 |\hat{u}(\xi)|^2}{m^2 t_0^4 + 4\pi^2 |\xi|^2} d\xi \right] - n_0^{2N+1} \left[ 2N \int_{\mathbb{R}^2} \frac{F(t_0 u)}{t_0^2} dx + \int_{\mathbb{R}^N} \frac{f(t_0 u) u}{t} dx \right] = 0,
\]
and it follows from (6.10a) that \( h'_u(t_0) = 0 \). Uniqueness of \( t_u \) imply \( t_u = t_0 \) and so \( t_n \to t_u \). We are done. \( \square \)

**Lemma 6.3.** For all \( u \in H^s(\mathbb{R}^N) \setminus \{0\} \) we have
\[
\Phi(u) - \frac{1}{2N+2} J(u) > 0.
\]

**Proof.** We have
\[
\Phi(u) - \frac{1}{2N+2} J(u) =
\]
If \( \epsilon \) is chosen such that \( \frac{1}{2} \| u \|^2 - \int_{\mathbb{R}^N} F(u) \, dx \geq \frac{(N+1-2s)}{2N+2} \| u \|^2 - \frac{2sm^2}{2N+2} \int_{\mathbb{R}^N} \frac{\| \hat{u}(\xi) \|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi \)

\[ \leq \frac{2N}{2N+2} \int_{\mathbb{R}^N} F(u) \, dx + \int_{\mathbb{R}^N} f(u) \, dx \]

\[ = \frac{2s}{2N+2} \left[ \int_{\mathbb{R}^N} (m^2 + 4\pi^2|\xi|^2) \| \hat{u}(\xi) \|^2 \, d\xi - \int_{\mathbb{R}^N} \frac{m^2|\hat{u}(\xi)|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi \right] \]

\[ + \frac{1}{2N+2} \int_{\mathbb{R}^N} [f(u) - 2F(u)] \, dx \]

(6.12)

\[ = \frac{2s}{2N+2} \left[ \int_{\mathbb{R}^N} \frac{4\pi^2|\xi|^2|\hat{u}(\xi)|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi + \int_{\mathbb{R}^N} [f(u) - 2F(u)] \, dx \right] \]

\[ > \frac{2s}{2N+2} \int_{\mathbb{R}^N} \frac{4\pi^2|\xi|^2|\hat{u}(\xi)|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi > 0, \text{ if } u \neq 0. \]

\[ \square \]

**Lemma 6.4.** If \( f \) satisfies hypotheses \((f_1)\) and \((f_2)\), then

(i) \( J(u) > 0 \) for all \( 0 < \| u \| < \rho \) and \( \mathcal{M} \) is a closed subset of \( H^s(\mathbb{R}^N) \);

(ii) For all \( u \in \mathcal{M} \) we have \( \Phi(u) > 0 \);

(iii) \( \mathcal{M} \) is a \( C^1 \)-manifold.

**Proof.** Since there exist constants \( \gamma_p \) such that \( |u|_p \leq \gamma_p \| u \| \) for all \( 2 < p < 2^*, \) it follows from the definition of \( J \) - see (1.1) - and (6.2) that

\[ J(u) \geq (N+1-2s)\| u \|^2 - 2N \int_{\mathbb{R}^N} F(u) \, dx - \int_{\mathbb{R}^N} f(u) \, dx \]

\[ \geq (N+1-2s)\| u \|^2 - (2N+1) \left[ \epsilon \| u \|_2 + C_s \| u \|_p^2 \right] \]

\[ \geq \left[ (N+1-2s) - (2N+1)\gamma_p^2 \right] \| u \|^2 - (2N+1)C_s \gamma_p^p \| u \|^p \]

\[ = \frac{(N+1-2s)}{2} \| u \|^2 - (2N+1)C_s \gamma_p^p \| u \|^p, \]

if we choose \( \epsilon = (N+1-2s)/(2(2N+1)\gamma_p^2) \). Now, taking

\[ \rho = \| u \| = \left( \frac{(N+1-2s)}{p(2N+1)C_s \gamma_p^p} \right)^{1/(p-2)}, \]

we obtain that

\[ J(u) \geq \frac{(N+1-2s)(p-2)}{2p} \rho^2 > 0 \]

for all \( 0 < \| u \| < \rho \). Thus, \( u = 0 \) is an isolated point of \( J^{-1}(0) \) and \( \mathcal{M} \subset \mathcal{M} \cup \{0\} = J^{-1}(0) \) is closed.

For all \( u \in \mathcal{M} \) we have \( J(u) = 0 \), so that

\[ \Phi(u) = \Phi(u) - \frac{1}{2N+2} J(u) \]

and (ii) follows from Lemma (6.3).

Since

\[ J'(u) \cdot u = 2(N+1-2s)\| u \|^2 + 4sm^2 \int_{\mathbb{R}^N} \frac{\| \hat{u}(\xi) \|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}} \, d\xi \]

\[ - 2N \int_{\mathbb{R}^N} f(u) \, dx - \int_{\mathbb{R}^N} [f'(u)u^2 + f(u)u] \, dx, \]
substitution of $J(u) = 0$ in the last equation and using hypothesis $(f_1)$ we obtain
\[
J'(u) \cdot u = -2N \int_{\mathbb{R}^N} [f(u)u - 2F(u)] \, dx - \int_{\mathbb{R}^N} [f'(u)u^2 - f(u)u] \, dx < 0,
\]
as consequence of our hypotheses. We are done.

We now define the minimax value
\[
c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi(\gamma(t)),
\]
where
\[
\Gamma = \{ \gamma \in C([0,1], H^s(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \Phi(\gamma(1)) < 0 \},
\]
and the infimum in the Nehari-Pohozaev manifold
\[
\tilde{c} = \inf_{u \in \mathcal{M}} \Phi(u) = \inf_{u \in H^s} \max_{t > 0} \Phi(u_t) \geq 0.
\]

**Lemma 6.5.** The level $c$ is well-defined and $c = \tilde{c}$.

**Proof.** Taking into account (6.5), for all $u \in H^s(\mathbb{R}^N) \setminus \{0\}$ there exists $t_1 = t_1(u)$ such that $\Phi(u_{t_1}) < 0$, where $u_{t_1}(x) = tu(x/t^2)$. Defining $\gamma_0(t) = u_{tt_1}$, if $t > 0$ and $\gamma_0(0) = 0$, we have
\[
\Phi(\gamma_0(1)) = \Phi(u_{t_1}) < 0,
\]
from what follows that $\gamma_0 \in \Gamma$.

Furthermore,
\[
\max_{0 \leq t \leq q} \Phi(u_{tt_1}) \geq \max_{0 \leq t \leq 1} \Phi(u_{tt_1}) \geq \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \Phi(\gamma(t)) = c,
\]
proving that $\tilde{c} \geq c$.

But we also know the existence of $\rho > 0$ such that $J(u) \geq 0$ for all $u \in H^s(\mathbb{R}^N)$, if $\|u\| \leq \rho$. Lemma 6.3 yields
\[
\Phi(u) \geq \frac{1}{2N+2} J(u) \geq 0, \quad \text{if } \|u\| \leq \rho.
\]

If $\gamma \in \Gamma$, since $\Phi(\gamma(1)) < 0$, a new application of Lemma 6.3 gives
\[
J(\gamma(1)) \leq (2N + 2)\Phi(\gamma(1)) < 0,
\]
so that $J(\gamma(0)) = 0$, $J(\gamma(t)) > 0$ if $\gamma(t) < \rho$ and $J(\gamma(1)) < 0$. We conclude the existence of $t$ such that $J(\gamma(t)) = 0$, proving that $\gamma$ intercepts $\mathcal{M}$. Therefore,
\[
\max_{0 \leq t \leq 1} \Phi(\gamma(t)) \geq \inf_{u \in \mathcal{M}} \Phi(u) = \tilde{c},
\]
so that $c \geq \tilde{c}$ and completing the proof of $c = \tilde{c}$. \hfill \square

**Definition 6.6.** A sequence $(u_n) \in H^s(\mathbb{R}^N)$ is a Cerami sequence for $\Phi$ in the level $\theta$ if
\[
\Phi(u_n) \to \theta \quad \text{and} \quad \Phi'(u_n)||_s (1 + ||u_n||) \to 0,
\]
where $|| \cdot ||_s$ stands for the norm in $(H^s(\mathbb{R}^N))^*$.

**Lemma 6.7.** Let $(u_n)$ be a Cerami sequence for $\Phi$ at the level $\theta > 0$. Then, passing to a subsequence if necessary, $(u_n)$ is bounded in $H^s(\mathbb{R}^N)$. 

Proof. Since \(|\Phi'(u_n)|_*(1 + \|u_n\|) \to 0\), we have
\[\|\Phi'(u_n)|_* \leq \|\Phi'(u_n)|_*(1 + \|u_n\|) \leq \frac{1}{n}\]
for \(n\) big enough. Thus, we can suppose that
\[
\frac{1}{n} < \Phi'(u_n) \cdot u_n = \|u_n\|^2 - \int_{\mathbb{R}^N} f(u_n) u_n \, dx < \frac{1}{n}.
\] (6.13)

The last inequality and Lemma 6.1 imply that
\[
\Phi(tu_n) = \frac{t^2}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} F(tu_n) \, dx
\] (6.14)
\[
\leq \frac{t^2}{2} \left[ \frac{1}{n} + \int_{\mathbb{R}^N} f(u_n) u_n \, dx \right] - \int_{\mathbb{R}^N} F(u_n) \, dx
\]
\[
\leq \frac{t^2}{2n} + \int_{\mathbb{R}^N} \left[ \frac{1}{2} f(u_n) u_n - F(u_n) \right] \, dx.
\]

But it also follows from (6.13) that
\[
\Phi(u_n) = \frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} F(u_n) \, dx
\] (6.15)
\[
\geq - \frac{1}{2n} + \int_{\mathbb{R}^N} \left[ \frac{1}{2} f(u_n) u_n - F(u_n) \right] \, dx.
\]

Thus, it follows from (6.14) and (6.15),
\[
\Phi(tu_n) \leq \frac{t^2}{2n} + \left[ \frac{1}{2n} + \Phi(u_n) \right].
\] (6.16)

Since \(\Phi(u_n) = \theta + O_n(1)\), (6.16) yields
\[
\frac{t^2}{2} \|u_n\|^2 \leq \frac{t^2}{2n} + \frac{1}{2n} + \theta + O_n(1) + \int_{\mathbb{R}^N} F(tu_n) \, dx.
\] (6.17)

Taking \(t_n = \frac{2\sqrt{\theta}}{\|u_n\|}\) and substituting \(t_n\) into (6.17), yields
\[
2\theta \leq \frac{4\theta}{n\|u_n\|^2} + O_n(1) + \theta + \int_{\mathbb{R}^N} F(tu_n) \, dx
\]
and so
\[
\theta \leq \frac{4\theta}{n\|u_n\|^2} + O_n(1) + 4\epsilon \theta \int_{\mathbb{R}^N} \left( \frac{u_n}{\|u_n\|} \right)^2 \, dx + C\epsilon(2\sqrt{\theta})^p \int_{\mathbb{R}^N} \left( \frac{u_n}{\|u_n\|} \right)^p \, dx.
\] (6.18)

Now, by contradiction, suppose that \(\|u_n\| \to \infty\) for a subsequence and consider the bounded sequence
\[\tilde{u}_n = \frac{u_n}{\|u_n\|}.
\]

Since \(H^s(\mathbb{R}^N)\) is reflexive, passing to a subsequence if necessary, we can suppose \(\tilde{u}_n \to \tilde{u}\) for some \(\tilde{u} \in H^s(\mathbb{R}^N)\). There are two possible cases:

(i) \(\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 \, dx = 0\);

(ii) \(\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |\tilde{u}_n|^2 \, dx > 0\).
Thus, we can suppose that

\[
\theta \leq O_n(1) + 4\epsilon \theta \int_{\Omega} |\tilde{u}_n|^2 \, dx \leq O_n(1) + \frac{4\epsilon \theta}{m^{2s}} \int_{\Omega} (m^2 + 4\pi^2 |\xi|^2 |F(\tilde{u}_n)|^2) \, d\xi
\]

(6.19)

\[
\leq O_n(1) + \frac{4\epsilon \theta}{m^{2s}} \|\tilde{u}_n\|^2 = O_n(1) + \frac{4\epsilon \theta}{m^{2s}}
\]

and we have reached a contradiction by taking \( \epsilon = m^{2s}/8 \).

Now suppose that case (ii) occurs. If \( \delta = \limsup_{n \to \infty} \sup_{y \in \Omega} \int_{B_1(y)} |\tilde{u}_n|^2 \, dx > 0 \), passing to a subsequence if necessary, we have

\[
\int_{B_1(y_n)} |\tilde{u}_n|^2 \, dx > \frac{\delta}{2} > 0.
\]

We define

\[\tilde{v}_n(x) = \tilde{u}_n(x + y_n).\]

Since \( \tilde{v}_n \) is a translation of \( \tilde{u}_n \), we have \( \|\tilde{v}_n\| = 1 \). Thus, passing to a subsequence we can suppose that

\[\tilde{v}_n \to v \text{ in } H^s(\mathbb{R}^N), \quad \tilde{v}_n \to \tilde{v} \text{ in } L^2_{loc}(\mathbb{R}^N) \quad \text{and} \quad \tilde{v}_n(x) \to \tilde{v}(x) \text{ a.e. in } \mathbb{R}^N.\]

We now consider two cases: \((y_n)\) unbounded and \((y_n)\) bounded. In the first case, since

\[\frac{\delta}{2} < \int_{B_1(y_n)} |\tilde{v}_n|^2 \, dx = \int_{B_1(0)} |\tilde{v}_n|^2 \, dx \to \int_{B_1(0)} |\tilde{v}|^2 \, dx,
\]

we conclude that \( \tilde{v} \neq 0 \). Thus, there exists \( \Omega \subset B_1(0) \), such that \( |\tilde{v}(x)| > 0 \) for all \( x \in \Omega \), with \( \Omega \) satisfying \( |\Omega| > 0 \). (Observe that \( \Omega \) does not depend on \( n \).)

In the second case, suppose that \( |y_n| \leq R \) for all \( n \). We can suppose that \( R > 1 \). Thus,

\[\frac{\delta}{2} < \int_{B_1(0)} |\tilde{u}_n(x + y_n)|^2 \, dx \leq \int_{B_{2R}(0)} |\tilde{u}_n(x + y_n)|^2 \, dx \to \int_{B_{2R}(0)} |\tilde{v}(x)|^2 \, dx,
\]

since \( \tilde{v}_n \to \tilde{v} \) in \( L^2(B_{2R}(0)) \). So, as before, we conclude the existence of \( \Omega \subset B_{2R}(0) \) such that \( \tilde{v} > 0 \) in \( \Omega \).

In both cases, seeing that

\[0 < |\tilde{v}(x)| = \lim_{n \to \infty} \frac{|u_n(x + y_n)|}{\|u_n\|}, \quad \forall x \in \Omega,
\]

we conclude that

\[\lim_{n \to \infty} |u_n(x + y_n)| = \infty, \text{ if } x \in \Omega.
\]
By applying hypothesis \((f_3)\) and Fatou’s lemma, we have
\[
\liminf_{n \to \infty} \int_{\mathbb{R}^N} [(1/2)f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))] \, dx \\
\geq \liminf_{n \to \infty} \int_{\Omega} [(1/2)f(u_n(x + y_n))u_n(x + y_n) - F(u_n(x + y_n))] \, dx = \infty.
\]

But
\[
\int_{\mathbb{R}^N} [(1/2)f(u_n)u_n - F(u_n)] \, dx = \Phi(u_n) - \frac{1}{2}\Phi'(u_n) = \theta + O_n(1)
\]
and once again we reached a contradiction, and we are done.

The existence of a Cerami sequence for \(\Phi\) at the level \(c\) is a consequence of the Ghoussoub-Preiss theorem, that we now recall, for the convenience of the reader, using our notation. A good exposition of this result can be found in one of Ekeland’s books, see [22, Theorem 6, p. 140], see also [26].

**Theorem 5** (Ghoussoub-Preiss). Let \(X\) be a Banach space and \(\Phi: X \to \mathbb{R}\) a continuous, Gateaux-differentiable function, such that \(\Phi': X \to X\) is continuous from the norm topology of \(X\) to the weak* topology of \(X^*\). Take two points \(z_0, z_1\) in \(X\) and consider the set \(\Gamma\) of all continuous paths from \(z_0\) to \(z_1\):
\[
\Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = z_0, \gamma(1) = z_1 \}.
\]
Define a number \(c\) by
\[
c := \inf \max_{0 \leq t \leq 1} \Phi(\gamma(t)).
\]
Assume that there is a closed subset \(\mathcal{M}\) of \(X\) such that
\[
\mathcal{M} \cap \Phi_c \text{ separates } z_0 \text{ and } z_1,
\]
with \(\Phi_c = \{ x \in X : \Phi(x) \geq c \}\).

Then, there exists a sequence \((x_n)\) in \(X\) such that
\[
(i) \text{ dist } (x_n, \mathcal{M}) \to 0;
\]
\[
(ii) \Phi(x_n) \to c;
\]
\[
(iii) (1 + \|x_n\|)\|\Phi'(x_n)\|_* \to 0.
\]

In the original Ghoussoub-Preiss theorem, we have \(\delta(x_n, \mathcal{M}) \to 0\), where \(\delta\) stands for the geodesic distance. If \(z_0 = 0\), then \(\delta(0,x) = \ln(1 + \|x\|)\) and we can change \(\delta\) by dist, see [22, p. 138]. A closed subset \(\mathcal{F} \subset X\) separates two points \(z_0\) and \(z_1\) in \(X\) if \(z_0\) and \(z_1\) belong to disjoint connected components in \(X \setminus \mathcal{F}\).

Of course, in our case \(X = H^s(\mathbb{R}^N)\). If we take \(z_0 = 0\) and \(z_1\) such that \(\Phi(z_1) < 0\), then
\[
H^s(\mathbb{R}^N) \setminus \mathcal{M} = \{0\} \cup \{ u \in H^s(\mathbb{R}^N) : J(u) > 0 \} \cup \{ u \in H^s(\mathbb{R}^N) : J(u) < 0 \}
\]
(remember that \(0 \notin \mathcal{M}\) and \(J(0) = 0\)). According to Lemma 6.3, \(B_p(0)\) belongs to a connected component of \(\{0\} \cup \{ u \in H^s(\mathbb{R}^N) : J(u) > 0 \}\). Since \(\Phi(z_1) < 0\), it follows from Lemma 6.3 that \(J(z_1) < 0\). Thus, \(\mathcal{M}\) separates \(z_0\) and \(z_1\). But we also have \(\mathcal{M} \cap \Phi_c = \mathcal{M}\), since \(\inf_{u \in \mathcal{M}} \Phi(u) = c\), as consequence of Lemma 6.3. So, the assumptions of the Ghoussoub-Preiss theorem are fulfilled.

**Proof of Theorem** Since the Ghoussoub-Preiss theorem guarantees the existence of a Cerami sequence \(\{u_n\} \subset H^s(\mathbb{R}^N)\), which is bounded by Lemma 6.7, we can suppose that
\[
u_n \rightharpoonup u \text{ in } H^s(\mathbb{R}^N), \quad u_n(x) \to u(x) \text{ a.e. and } u_n \to u \text{ in } L^p_{loc}(\mathbb{R}^N)
\]
for $p \in [2, 2^*_s)$.

We define, mimicking the proof of Lemma 6.7,

$$\delta = \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 \, dx.$$ 

If $\delta = 0$, by the principle of concentration-compactness of Lions we have $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for any $p \in (2, 2^*_s)$. Since $(u_n)$ is bounded in $L^2(\mathbb{R}^N)$, there exists $M_0 > 0$ such that $|u_n|^2 \leq M$ for all $n \in \mathbb{N}$. Thus, for any $\eta > 0$, by taking $\epsilon = \eta/M_0$, we conclude that

$$\int_{\mathbb{R}^N} |F(u_n)| \, dx \leq \epsilon |u_n|^2 + C|u_n|^p = \eta + C|u_n|^p$$

and $|u_n|^p \to 0$ implies that

$$\int_{\mathbb{R}^N} F(u_n) \, dx \to 0, \text{ when } n \to \infty.$$ 

Similarly,

$$\int_{\mathbb{R}^N} f(u_n)u_n \, dx \to 0, \text{ when } n \to \infty,$$

and, since $\|u_n\|^2 - \int_{\mathbb{R}^N} f(u_n)u_n = \Phi'(u_n) \cdot u_n \to 0$ if $n \to \infty$, we have $\|u_n\|^2 \to 0$.

Thus,

$$0 < \tilde{c} = \lim_{n \to \infty} \Phi(u_n) = \lim_{n \to \infty} \left[ \frac{1}{2} \|u_n\|^2 - \int_{\mathbb{R}^N} F(u_n) \, dx \right] = 0,$$

a contradiction.

If, however, $\delta > 0$, there exists a sequence $(y_n)$ such that, for all $n \in \mathbb{N},$

(6.20) \hspace{1cm} \int_{B_1(y_n)} |\tilde{u}_n|^2 \, dx > \frac{\delta}{2} > 0.

We define $w_n = u_n(x+y_n)$. Then $\|w_n\| = \|u_n\|, J(w_n) = J(u_n), \Phi(w_n) = \Phi(u_n)$ and $\Phi'(w_n) \to 0$ when $n \to \infty$. Passing to a subsequence we can suppose that, for $p \in [2, 2^*_s)$, we have

$$w_n \to w \text{ in } H^s(\mathbb{R}^N), \quad w_n \to w \text{ in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ and } w_n(x) \to w(x) \text{ a.e. in } \mathbb{R}^N.$$ 

From (6.20) follows $w \neq 0$. Furthermore, for all $\varphi \in H^s(\mathbb{R}^N)$, we have

$$\Phi'(w) \cdot \varphi = \lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} (-\Delta + m^2)^{s/2} w_n(-\Delta + m^2)^{s/2} \varphi \, dx - \int_{\mathbb{R}^N} f(w_n) \varphi \, dx \right]$$

$$= \lim_{n \to \infty} \Phi'(w_n) \cdot \varphi = 0,$$

and we conclude that

$$\Phi'(w) = 0,$$

that is, $w$ is a weak solution of (6.1) and, therefore, satisfies the Pohazaev identity.

But

$$J(w) = \Phi'(w) \cdot w + 2P(w) = 0$$

proves that $w \in \mathcal{M}$. Therefore, $\Phi(w) \geq \tilde{c}$, as consequence of Lemma 6.5.
However, by applying Fatou’s Lemma to (6.12) with $w$ instead of $u$, we obtain

$$\Phi(w) = \Phi(w) - \frac{1}{2N+2}J(w)$$

$$= \frac{2s}{2N+2} \left[ \int_{\mathbb{R}^N} \frac{4\pi^2|\xi|^2|\hat{w}(\xi)|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}}\,d\xi + \int_{\mathbb{R}^N} [f(w)w - 2F(w)]\,dx \right]$$

$$\leq \liminf_{n \to \infty} \frac{2s}{2N+2} \left[ \int_{\mathbb{R}^N} \frac{4\pi^2|\xi|^2|\hat{w}_n(\xi)|^2}{(m^2 + 4\pi^2|\xi|^2)^{1-s}}\,d\xi \right.$$

$$+ \int_{\mathbb{R}^N} [f(w_n)w_n - 2F(w_n)]\,dx \left. \right]$$

$$= \liminf_{n \to \infty} \left[ \Phi(w_n) - \frac{1}{2N+2}J(w_n) \right] = \liminf_{n \to \infty} \Phi(w_n) = \tilde{c}.$$ 

Thus, we have $\Phi(w) \leq \tilde{c}$ and conclude that $\Phi(w) = \tilde{c}$. We are done. 

7. Radial Symmetry

We commence this section presenting some basic results about a modified Bessel kernel, defined for any $s > 0$ by

$$g_s(x) = \frac{1}{(4\pi)^s\Gamma(s)} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-m^2\delta/(4\pi)^s\delta^{(2s-N)/2}}\,d\delta.$$ 

The results follow simply by adapting the proofs presented in [32] or [39].

**Proposition 7.1.** For every $s > 0$ we have

(i) $g_s \in L^1(\mathbb{R}^N)$;

(ii) $\hat{g}_s(\xi) = (m^2 + 4\pi^2|\xi|^2)^{-s}$.

The proof of Proposition 7.1 is a consequence of the identity

$$\int_{\mathbb{R}^N} e^{-\pi|x|^2/\delta}\,dx = \delta^{N/2},$$

and the application of Fubini’s theorem.

The next result follows immediately by considering the Fourier transform of $g_{s_1+s_2}$, applying Proposition 7.1 and then the inversion formula.

**Corollary 7.2.** For every $s_1, s_2 > 0$ it holds

$$g_{s_1} * g_{s_2} = g_{s_1+s_2}.$$ 

**Definition 7.3.** For a given $f \in L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$, we define the Bessel potential $I_s(f)$ by

$$I_s(f) = \begin{cases} 
 g_s * f, & \text{if } s > 0 \\
 f, & \text{if } s = 0
\end{cases}$$

The next result is a consequence of Corollary 7.2.

**Proposition 7.4.** We have

(i) For any $f \in L^p(\mathbb{R}^N)$ with $1 \leq p \leq \infty$,

$$I_s(f) \in L^p(\mathbb{R}^N) \quad \text{and} \quad |I_s(f)|_p \leq \frac{1}{m^{2s}}|f|_p;$$

(ii) $I_{s_1} \circ I_{s_2} = I_{s_1+s_2}$. 

Definition 7.5. For any $s > 0$ and $1 \leq p \leq \infty$, we define
\[ L^{s,p}({\mathbb R}^N) = \{ g_s \ast f : f \in L^p({\mathbb R}^N) \}. \]
If $u = g_s \ast f \in L^p({\mathbb R}^N)$, we also define
\[ \|u\|_{s,p} = |f|_p. \]

The space $L^{s,p}({\mathbb R}^N)$ is Banach, see [32].

Remark 7.6. Since
\[ \mathcal F \left( (-\Delta + m^2)u \right)(\xi) = (m^2 + 4\pi^2|\xi|^2)^s \hat{u}(\xi), \]
it follows from Proposition 7.1 that
\[ \hat{u}(\xi) = (m^2 + 4\pi^2|\xi|^2)^{-s} \mathcal F \left( (-\Delta + m^2)^s u \right)(\xi) = \hat{g}_s(\xi) \mathcal F \left( (-\Delta + m^2)^s u \right)(\xi), \]
from what follows
\[ u = g_s \ast (-\Delta + m^2)^s u. \]

Therefore, $u$ solves \[ (-\Delta + m^2)^s u = f(u) \]
if, and only if,
\[ u = g_s \ast f(u). \]

We now state a result proven in [30] Theorem 9:

Theorem 6. Let $q > \max\{\beta, \frac{N(\beta - 1)}{\alpha}\}$. If $f \in L^{q/\beta}({\mathbb R}^N)$, then $I_s(f) \in L^q({\mathbb R}^N)$. Moreover, we have the estimate
\[ |I_s(f)|_q \leq C|f|_{q/\beta}, \]
where $C = C(\alpha, \beta, N, q)$

Let us consider the problem
\[ (-\Delta + m^2)^s u = f(u) \quad \text{in} \quad {\mathbb R}^N, \]
where $0 < s < 1$, $N > 2s$, $m \in {\mathbb R} \setminus \{0\}$ and $f : [0, \infty) \to {\mathbb R}$ a continuous function that satisfies
(s1) $f'(t) \geq 0$ and $f''(t) \geq 0$ for all $t \in [0, \infty)$,
(s2) For any given $\beta \in (1, 2^*_s - 1)$, there exists $q \in [2, 2^*_s]$ with $q >\max\{\beta, \frac{N(\beta - 1)}{2s}\}$ such that $f'(w) \in L^{q/(\beta - 1)}({\mathbb R}^N)$, $\forall w \in H^s({\mathbb R}^N)$.

We give some examples of functions satisfying our hypotheses (s1) and (s2):

(1) For any $\alpha \in (1, 2^*_s - 1)$, the function $f(t) = t^\alpha$ clearly fulfill (s1). Taking $\beta = \alpha$, then
\[ \int_{\mathbb R^N} |f'(w)|^\frac{s}{2s - \alpha} dx = \alpha^{-\frac{s}{2s - \alpha}} \int_{\mathbb R^N} |w|^\alpha dx < \infty, \quad \forall q \in [2, 2^*_s]. \]

(2) Also for $f(t) = t^\alpha + t^\gamma$, where $\alpha, \gamma \in (1, 2^*_s - 1)$, condition (s1) is verified. Furthermore, choosing $\beta = \max\{\alpha, \gamma\} \in (1, 2^*_s - 1)$, since
\[ \int_{\mathbb R^N} |f'(w)|^\frac{s}{2s - \beta} dx \leq C \int_{\mathbb R^N} \left( |w|^q(\frac{s}{2s - \beta}) + |w|^q(\frac{s}{2s - \alpha}) \right) dx, \]

(7.3)
there exists \( q \in \left( \max\{\beta, \frac{N(\beta-1)}{2s}\}, 2_s^* \right) \) such that \( 2 < q \left( \frac{\alpha-1}{\beta} \right) < 2_s^* \) and \( 2 < q \left( \frac{\beta-1}{\alpha} \right) < 2_s^* \). Thus, if \( w \in H^s(\mathbb{R}) \), then (7.3) and the Sobolev immersions imply that \( f'(w) \in L^{q/(\beta-1)}(\mathbb{R}^N) \).

(3) Consider \( f(t) = t \ln(1 + t) \), for \( t \in [0, \infty) \). Since

\[
\begin{align*}
\int f'(t) &= \int \ln(1 + t) + \frac{t}{1 + t} \geq 0 \quad \text{and} \quad \int f''(t) = \frac{1}{1 + t} + \frac{1}{(1 + t)^2} \geq 0
\end{align*}
\]

for all \( t \in [0, \infty) \), we have \((s_i)\). Since \( f'(t) \leq 2t \) if \( t \geq 0 \), we have \( f'(w) \in L^{q/(\beta-1)}(\mathbb{R}^N) \) for any \( 1 < \beta < 2_s^* - 1 \) and \( q > \max\{\beta, \frac{N(\beta-1)}{2s}\} \).

We now apply the moving planes technique in integral form to show that any positive solution of (7.2) is radially symmetric. We start fixing some notation.

For any \( \lambda \in \mathbb{R} \), define

\[
\begin{align*}
\Sigma_{\lambda} &= \{ x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 \leq \lambda \}, \\
T_{\lambda} &= \{ x \in \mathbb{R}^N : x_1 = \lambda \}, \\
x_\lambda &= (2\lambda - x_1, x_2, \ldots, x_N), \quad \text{if} \ x \in \Sigma_{\lambda}, \\
u_{\lambda}(x) &= u(x_\lambda).
\end{align*}
\]

Lemma 7.7. For any positive solution \( u(x) \) of (7.2) we have

\[
(7.4) \quad u(x) - u_{\lambda}(x) = \int_{\Sigma_{\lambda}} (g_s(x - y) - g_s(x_\lambda - y)) (f(u(y)) - f(u_{\lambda}(y))) \, dy.
\]

Proof. Since \( u \) is a solution of (7.2), it holds

\[
\begin{align*}
u(x) &= \left[ g_s * (-\Delta + m^2)^s u \right](x) = (g_s * f(u))(x) = \int_{\mathbb{R}^N} g_s(x - y) f(u(y)) \, dy,
\end{align*}
\]

where \( g_s \) is the modified Bessel kernel (7.1).

Thus, the change of variables \( y \mapsto y_{\lambda} \) yields

\[
\begin{align*}
u(x) &= \int_{\Sigma_{\lambda}} g_s(x - y) f(u(y)) \, dy + \int_{\Sigma_{\lambda}} g_s(x - y) f(u(y)) \, dy \\
&= \int_{\Sigma_{\lambda}} g_s(x - y) f(u(y)) \, dy + \int_{\Sigma_{\lambda}} g_s(x_\lambda - y) f(u_{\lambda}(y)) \, dy \\
&= \int_{\Sigma_{\lambda}} (g_s(x - y) f(u(y)) + g_s(x_\lambda - y) f(u_{\lambda}(y))) \, dy
\end{align*}
\]

the last equality being a consequence of the fact that \( g_s \) is radially symmetric and \( |x_\lambda - y| = |x - y_{\lambda}| \).

In the last equality, changing \( x \) for \( x_\lambda \), since \( g_s \) is radial and \( |x - y| = |x_\lambda - y_{\lambda}| \), we obtain

\[
\begin{align*}
u(x) &= \int_{\Sigma_{\lambda}} (g_s(x_\lambda - y) f(u(y)) + g_s(x_\lambda - y) f(u_{\lambda}(y))) \, dy
\end{align*}
\]

and our proof is complete. \( \square \)

Proof of Theorem 4

In Step 1, we show that, for any negative \( \lambda \), if \( |\lambda| \) is big enough, then

\[
(7.5) \quad u(x) < u_{\lambda}(x).
\]
Therefore, we can move the plane $T_{\lambda}$ in the $x_1$-axis from a neighborhood of $-\infty$ to the right until (7.3) remains valid.

In Step 2 we will show that above translation of $T_{\lambda}$ is possible until we reach $x_1 = 0$. Thus, we conclude that $u(x) \leq u_0(x)$ for any $x \in \Sigma_0$.

However, the same process of steps 1 and 2 can be repeated, moving the plane $T_{\lambda}$ from a neighborhood of $+\infty$ to the left in the $x_1$-axis, thus concluding that $u(x) \geq u_0(x)$ for any $x \in \mathbb{R}^N \setminus \Sigma_0$.

Thus, we conclude that $u$ is symmetric in relation to the plane $T_0$. Since the direction $x_1$ was arbitrarily chosen, we conclude that $u$ is symmetric with respect to any axis. Changing coordinates, we conclude that $u$ is symmetric and decreasing with respect to the origin in any direction.

Step 1. For $\lambda$ negative enough we have

(7.6) \[ u(x) \leq u_\lambda(x), \forall x \in \Sigma_\lambda. \]

Let us denote

(7.7) \[ \Sigma^-_\lambda = \{ x \in \Sigma_\lambda; u_\lambda(x) < u(x) \}. \]

Lemma 7.7 yields

\[ u(x) - u_\lambda(x) = \int_{\Sigma_\lambda \setminus \Sigma^-_\lambda} (g_s(x - y) - g_s(x_\lambda - y)) (f(u(y)) - f(u_\lambda(y))) \, dy \]

\[ + \int_{\Sigma^-_\lambda} (g_s(x - y) - g_s(x_\lambda - y)) (f(u(y)) - f(u_\lambda(y))) \, dy. \]

Since $|x_\lambda - y| > |x - y|$, $g_s$ is both positive and radially decreasing, and $f$ satisfies $(f_1)$, we have $g_s(x_\lambda - y) \leq g_s(x - y)$ and $f(u(y)) \geq f(u_\lambda(y)), \forall y \in \Sigma \setminus \Sigma^-_\lambda$. Thus,

(7.8) \[ u(x) - u_\lambda(x) \leq \int_{\Sigma^-_\lambda} g_s(x - y) (f(u(y)) - f(u_\lambda(y))) \, dy. \]

It follows from the mean value theorem the existence of $\theta \in (0, 1]$ such that, for all $y \in \Sigma^-_\lambda$ we have

\[ f(u(y)) - f(u_\lambda(y)) = f'(u(y) + \theta[u_\lambda(y) - u(y)])(u(y) - u_\lambda(y)). \]

Since $(f_1)$ implies that $f'$ is increasing,

\[ f'(u(y) + \theta[u_\lambda(y) - u(y)]) \leq f'(u_\lambda(y)). \]

Substituting this estimate in (7.8) yields

\[ u(x) - u_\lambda(x) \leq \int_{\Sigma^-_\lambda} g_s(x - y) f'(u_\lambda(y))(u(y) - u_\lambda(y)) \, dy. \]

Now, fix $\beta = \frac{4s}{N} + 1$ and consider $q > \max\{\frac{4s}{N} + 1, 2\} = \max\{\beta, \frac{N(\beta-1)}{2s}\}$ (as given by hypothesis $(f_2)$). According to Theorem 4 we have

\[ |u - u_\lambda|_{L^q(\Sigma^-_\lambda)} \leq C |I_s(f'(u_\lambda)(u - u_\lambda))|_{L^q(\Sigma^-_\lambda)} \]

\[ \leq C |f'(u_\lambda)(u - u_\lambda)|_{L^q(\Sigma^-_\lambda)} \]

\[ \leq C |f'(u_\lambda)|_{L^{\frac{N}{N-\beta}}(\Sigma^-_\lambda)} |u - u_\lambda|_{L^q(\Sigma^-_\lambda)} \]

\[ \leq C |f'(u_\lambda)|_{L^{\frac{N}{N-\beta}}(\Sigma^-_\lambda)} |u - u_\lambda|_{L^q(\Sigma^-_\lambda)} \]

\[ = C |f'(u)|_{L^{\frac{N}{N-\beta}}(\Sigma^-_\lambda)} |u - u_\lambda|_{L^q(\Sigma^-_\lambda)}. \]
the last inequality being a consequence of the change of variables \( x \mapsto x_\lambda \).

Since \((f_2)\) implies that \( f'(u) \in L^{\frac{\beta - 1}{\beta}}(\mathbb{R}^N) \), there exists \( N_0 > 0 \) big enough so that

\[
\lambda \leq -N_0 \implies C |f'(u)|_{L^{\frac{\beta - 1}{\beta}}(\Sigma_\lambda^+)} \leq \frac{1}{2}.
\]

Applying \((7.10)\) in \((7.9)\) produces \( |u - u_\lambda|_{L^q(\Sigma_\lambda^+)} = 0 \) for any \( \lambda \leq -N_0 \). Thus, \( \Sigma_\lambda^+ \) has null measure for any \( \lambda \) negative enough.

**Step 2.** Let us suppose that the plane \( T_\lambda \) can be moved to the right until \( \lambda_0 < 0 \).

If there exists \( x^* \in \Sigma_{\lambda_0} \) such that \( u(x^*) = u_{\lambda_0}(x^*) \), then it follows from Lemma \((7,3)\) that

\[
0 = u(x^*) - u_{\lambda_0}(x^*)
\]

\[
= \int_{\Sigma_{\lambda_0}} \left( g_s(x^* - y) - g_s(x_{\lambda_0}^* - y) \right) (f(u(y)) - f(u_{\lambda_0})(y)) \, dy.
\]

Since \( g_s \) is radially decreasing and \( |x^* - y| > |x_{\lambda_0}^* - y| \) in \( \Sigma_{\lambda_0} \), then \( g_s(x^* - y) < g_s(x_{\lambda_0}^* - y) \), from what follows

\[
f(u(y)) = f(u_{\lambda_0})(y), \quad \forall y \in \Sigma_{\lambda_0}.
\]

According \((f_1)\), \((7.12)\) only happens if \( u(y) = u_{\lambda_0}(y) \) for all \( y \in \Sigma_{\lambda_0} \). In this case, \((7.11)\) yields

\[
u(x) = u_{\lambda_0}(x) \equiv 0 \quad \text{in} \quad \Sigma_{\lambda_0}.
\]

This implies that \( u(x) \equiv 0 \), contradicting the fact that \( u \) is a positive solution. By Step 1 we conclude that \( \Sigma_{\lambda_0} \) has null measure, thus yielding

\[
u(x) < u_{\lambda_0}(x), \quad \forall x \in \Sigma_{\lambda_0}.
\]

We claim that

\[
(7.13) \quad \lambda_0 = \sup \{ \lambda; \ u(x) \leq u_{\lambda}(x), \ \forall x \in \Sigma_{\lambda} \} = 0.
\]

Supposing the contrary, that is \( \lambda_0 < 0 \), we show that the plane \( T_{\lambda_0} \) can be moved to the right, contradicting the definition of \( \lambda_0 \).

Since \( f'(u) \in L^{q/(\beta - 1)}(\mathbb{R}^N) \) it follows that, for any \( \epsilon > 0 \) small enough, there exists \( R > 0 \) big enough so that

\[
\int_{\mathbb{R}^N \setminus B_R(0)} |f'(u)|^{\frac{\beta - 1}{q}} \, dx < \epsilon.
\]

Applying Lusin’s theorem, for any \( \delta > 0 \) there exists a closed set \( F_\delta \) such that \( (u - u_{\lambda_0})|_{F_\delta} \) is continuous, with \( F_\delta \subset B_R(0) \cap \Sigma_{\lambda_0} \) and \( \mu(E - F_\delta) < \delta \).

Since \( u(x) < u_{\lambda_0}(x) \) in the interior of \( \Sigma_{\lambda_0} \), we obtain that \( u(x) < u_{\lambda_0}(x) \) in \( F_\delta \).

Choose \( \epsilon_1 > 0 \) small enough so that, for any \( \lambda \in [\lambda_0, \lambda_0 + \epsilon_1) \),

\[
u - u_{\lambda} < 0, \quad \forall x \in F_\delta.
\]

It follows that

\[
\Sigma_{\lambda} \subset M := (\mathbb{R}^N \setminus B_R(0)) \cup (E \setminus F_\delta) \cup \left( \Sigma_{\lambda} \setminus \Sigma_{\lambda}^- \right) \setminus B_R(0).
\]

Now take \( \epsilon, \delta \) and \( \epsilon_1 \) small enough to that,

\[
C |f'(u)|_{L^{\frac{\beta - 1}{\beta}}(M)} \leq \frac{1}{2}.
\]
Thus,

\[
|u - u_\lambda|_{L^q(\Sigma^-)} \leq C |f'(u)|_{L^{\frac{q}{2}}(\Sigma^-)} |u - u_\lambda|_{L^q(\Sigma^-)} \leq \frac{1}{2} |u - u_\lambda|_{L^q(\Sigma^-)}.
\]

It follows that \(\Sigma^-\) has null measure and therefore \(u(x) \leq u_\lambda(x)\) for any \(x \in \Sigma_\lambda\), contradicting the definition of \(\lambda_0\). Thus, claim (7.13) is proved. We are done. \(\square\)

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