Canonical Quantization of Non-Einsteinian Gravity and the Problem of Time

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Abstract

For a 1+1 dimensional theory of gravity with torsion different approaches to the formulation of a quantum theory are presented. They are shown to lead to the same finite dimensional quantum system. Conceptual questions of quantum gravity like e.g. the problem of time are discussed in this framework.

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1 Introduction

The canonical quantization of general relativity is, though considerably simplified by the introduction of the Ashtekar variables [1] and the loop variables [2], still an unsolved problem. Among the main open issues are [3]: regularizing and solving the quantum constraints to get (all) the physical states, finding all Dirac observables and the correct inner product, interpreting the obtained theory and, as part of this, (re)introducing the notion of space and time.

In [4] the action for 2–dim gravity with torsion was given:

\[ L = e \left( -\frac{\gamma}{4} R^2 + \frac{\beta}{2} T^2 - \lambda \right). \] (1)

The classical solutions of the equations of motion were calculated in [4], [5] and a Hamiltonian formulation of the theory was provided in [6], [7].

Quantizing the Hamiltonian system derived from (1), one faces the same conceptual problems as in a four-dimensional theory of quantum gravity. The calculations, however, are much simpler. In particular, all classical solutions of the theory are known locally, and, similarly to 2 + 1–dim gravity [8], the phase space is found to be finite dimensional. Thus we think this theory to provide a good scenario for testing general concepts of quantum gravity. This is the motivation behind the present paper.

Those aspects of the classical theory, which are important for the quantization of the system are comprised in section 2. In order to have the presentation self-contained and to avoid confusion about the notation, material is included in this section which is already contained in previous publications on the subject. But also new aspects are provided, among them local analytic solutions around points of vanishing torsion, which are missing in the literature. Furthermore we show that the phase space is two dimensional if one assumes the space time manifold \( M \) to be of the form \( M = S^1 \times R \). For \( \gamma \lambda < 0 \) and spacelike \( S^1 \) the reduced phase space has a simple topology and a quantum theory on it can be formulated easily (section 3.1).

In section 3.2 the Dirac method [11] for the quantization of constrained systems is employed: The variables of the unconstrained phase space are quantized in a canonical way. The space of physical wave functions is then identified with the kernel of the quantized constraints. Finally an inner product is to be introduced in the space of physical wave functions. We present two ways to achieve that: One proposed in [10] starts from a measure in the unconstrained phase space, which is reduced by gauge conditions. Since the Faddeev-Popov determinant turns out to be inadequate to guarantee gauge independence in our case, the method is altered somewhat. No gauge conditions are needed for constraints which act multiplicatively. The gauge conditions implicitly introduce an internal time into the system. A somewhat different approach [3] defines the inner product without any reference to a measure in the unconstrained phase space by requiring a sufficiently large set of Dirac observables to be hermitian. We conclude the subsection by applying also the simple quantization scheme used in [8] to quantize 2 + 1 dim gravity with zero cosmological constant.
Under the restriction $M = S^1 \times R$ with spacelike $S^1$ the quantum theories resulting from the reduced phase space quantization and the Dirac method are shown to be equivalent. In the Dirac approach, however, one need not know the topology of the reduced phase space. The method is still applicable, if the above restrictions are lessened.

An elegant alternative is presented in section 3.3: The constraints are abelianized before quantization. It is most remarkable that this is possible in a relatively simple way. Exploiting the fact that the abelianized constraints may serve as canonical variables, the quantization becomes simple now. The resulting quantum system is completely equivalent to the one obtained in the preceding sections.

Physical questions usually refer to space–time events characterized by coordinates $x^\mu$. To answer them in a quantum theory of gravity they have to be reformulated in terms of Dirac observables — the space time coordinates $x^\mu$ enter as parameters. As in the classical theory the choice of a coordinate system is equivalent to the choice of a gauge. This idea is realized in section 4: A one parameter family of gauge conditions allows to express gauge dependent quantities in terms of the Dirac observables. Once the parameter in this family is interpreted as an intrinsic time, these quantities become dynamical (i.e. time dependent). With respect to this dynamics, a Schroedinger picture is formulated. The gauge independent formulation of the quantum states then corresponds to the according Heisenberg picture. In either picture it is possible to predict quantities such as $\langle g_{\mu\nu}(x^\mu) \rangle$ or $\langle T^2(x^\mu) \rangle$.

2 The Classical Theory

In the Lorentz-bundle over the space-time manifold we will use the metric

$$\eta_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with $a, b \in \{+, -\}$. In (1) $R$ is the Ricci scalar and $T^a$ is the Hodge dual of the torsion two–form ($T^2 \equiv T^a T_a$). Thus they are (Lorentz vector valued) functions on the space time manifold. A simple rescaling allows us to set $\beta = 1$ in the following.

The calculation of the canonical conjugate momenta to the components $e_\mu^\pm$ and $\omega_\mu$, $\mu \in \{0, 1\}$, of the zweibein and the connection yields the relations ($\dot{\varphi} = \partial \varphi / \partial x^0, \partial \varphi = \partial \varphi / \partial x^1$)

$$\pi_a := \frac{\partial L}{\partial \dot{e}_1^a} = -T_a$$

$$\pi_\omega := \frac{\partial L}{\partial \dot{\omega}_1} = \gamma R$$

and the primary constraints

$$\frac{\partial L}{\partial \dot{e}_0^a} = \frac{\partial L}{\partial \dot{\omega}_0^a} = 0.$$
It is remarkable that torsion and curvature serve as canonical momenta for the 1-components of zweibein and connection. The canonical Hamiltonian is

$$H = - \int \varepsilon_0^a G_a + \omega_0 G_\omega$$

(4)

with the secondary constraints ($\varepsilon_{+-} = 1$)

$$G_a = \varepsilon^{ab} E e_1^b - \varepsilon^b_a \pi_b \omega_1 + \partial \pi_a \approx 0$$

(5a)

$$G_\omega = \varepsilon^a_b \pi_a e_1^b + \partial \pi_\omega \approx 0,$$

(5b)

in which we used the abbreviations

$$E \equiv \frac{1}{4\gamma} (\pi_\omega)^2 - \frac{1}{2} \pi^2 - \lambda, \quad \pi^2 \equiv \pi^a \pi_a.$$  

(6)

Anticipating our later restriction to $x^1 \in S^1$, we have omitted a surface term in (4) resp. (5). Because of (3) $e_0^a$ and $\omega_0$ act as mere Lagrange multipliers within the Hamiltonian (4). They resemble lapse and shift in 4d gravity. The constraints (5) are first class:

$$\{G_a, G_\omega\} = -\varepsilon^b_a G_b \delta$$

(7a)

$$\{G_a, G_b\} = \varepsilon_{ab} \left( -\pi^c G_c + \frac{1}{2\gamma} \pi_\omega G_\omega \right) \delta.$$  

(7b)

Let us analyze briefly the geometrical meaning of the constraints. $G_\omega(x^1)$ generates local Lorentz transformations, thus corresponding exactly to the Gauss constraint in the Ashtekar formulation of general relativity: The first term of (5b), $e_1^a \pi_\omega - e_1^a \pi_+ + \pi^a$, is the generator of O(1,1)–transformations in the 4 dim. phasespace $(e_1^a, \pi^a)$, the second term reflects the transformation property of the connection $\omega$ under local Lorentz transformations on a slice of the space time manifold with $x^0 = \text{const}$. The combination

$$e_1^a G_a + \omega_1 G_\omega = e_1^a \partial \pi_a + \omega_1 \partial \pi_\omega$$

(8)

of the constraints can easily be shown to generate diffeomorphisms $\delta x^1 = \epsilon(x^1)$ [cf. also (4)], thus being the analogue of the vector constraint of the usual 3 + 1–theory. The combination

$$\pi^a G_a - E G_\omega = \partial (\pi_+ \pi_-) - E \partial \pi_\omega = \frac{1}{4\gamma} \exp(-\pi_\omega) \partial Q$$

(9)

with

$$Q := \exp(\pi_\omega) [2\gamma \pi^2 - (\pi_\omega - 1)^2 - 1 + \Lambda], \quad \Lambda = 4\gamma \lambda$$

(10)

generates, up to local Lorentz transformations, diffeomorphisms in the direction of constant curvature and torsion squared, since it is a polynomial in the momenta only. The quantity $Q$ defined in (10) has vanishing Poisson brackets with all the constraints, and therefore it is a constant on any classical solution of the field equations [5], [6]. As a consequence of this and (10) lines of constant curvature always coincide with lines of constant $T^2$.  

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Another combination of the constraints, generating diffeomorphisms in the $x^0$ direction, is the Hamiltonian \( \mathcal{H} \). The gauge choice $e_0^+ = 1, e_0^- = 0, \omega_0 = 0$ in \( \mathcal{H} \) (light cone gauge) identifies it with $-G_+$. As this choice corresponds to a space time metric with $g_{00} = 0$, it is obvious that $G_+$ generates diffeomorphisms in a lightlike direction (up to local Lorentz transformations, again). A similar argument holds for $G_-$. To complete the analogy to the $3+1$ dim. theory, one can choose an appropriate linear combination of $G_+$ and $G_-$ to play the role of the scalar constraint.

To find the local behaviour of a solution in the neighbourhood of a point $P$ on the space time manifold where $T_+(P) = -\pi_+(x^\mu(P)) \neq 0$, we can satisfy the constraints by inverting them algebraically to express $\pi_-, \omega_1,$ and $e_1^+$ in terms of $\pi_\omega, \pi_+$, and $e_1^-$, as well as $Q(x^1) = Q_0 = \text{const}$ [because of (3) using (1) instead of $G_-$]. The general solution in a neighbourhood of $P$ can then be determined by applying $-G_+$ to the remaining three fields, generating the $x^0$ dependence of the solution in the light cone gauge. Thus integrating $\{\pi_+, fG_+\} = 0$, $\{\pi_\omega, fG_\omega\} = -\pi_+$, and $\{e_1^-, fG_+\} = -e_1^- \pi_+$, we end up with

\[
\begin{align*}
\pi_\omega &= \pi_+ x^0 + B(x^1) \\
\pi_+ &= A(x^1) \\
\pi_- &= \frac{1}{4\gamma\pi_+} [Q_0 \exp(-\pi_\omega) + (\pi_\omega - 1)^2 + 1 - \Lambda] \\
\omega_1 &= \frac{1}{\pi_+} (e_1^- E + \partial \pi_+) \\
e_1^+ &= \frac{1}{\pi_+} (e_1^- \pi_- + \partial \pi_\omega) \\
e_1^- &= D(x^1) \exp(\pi_\omega),
\end{align*}
\]

in which $A, B, D \ [A(x^1(P)) \neq 0]$ are still arbitrary functions. Nevertheless, locally it is possible to gauge them away: Not changing the values of $e_0^+$ and $\omega_0$, it is possible to obtain $A(x^1) = 1$ by the choice of an appropriate Lorentz frame, $B(x^1) = 0$ by an $x^1$-dependent shift of the origin of the $x^0$-variable, and $D(x^1) = 1$ by an $x^0$ independent transformation of the $x^1$ variable. This ‘normalized solution’ may be taken also to represent a solution in the neighbourhood of a point $P$ with $\pi_+(P) = 0, \pi_-(P) \neq 0$ since we still have not made use of the discrete Lorentz transformation which exchanges ’+’ and ’-’. Thus locally the space of solutions to the field eqs. is parametrized completely by the value of $Q_0 [4, 9].$

The solution in the neighbourhood of a point with $\pi_+(P) = \pi_-(P) = 0$ can be obtained by first calculating the flow of $G_-$ starting from $P$ (we label this null line $x^0 = 0$):

\[
\begin{align*}
\pi_+(x^1) &= E_0 x^1, \pi_-(x^1) = 0, \quad (12a) \\
\pi_\omega(x^1) &= B(x^1(P)) = \text{const} \text{ and } E(x^1) = E_0 = \text{const} \text{ being determined implicitly by } Q_0 \text{ [cf. (3) and (4)]. For } E_0 \neq 0 \text{ the Gauss constraint then yields }
\end{align*}
\]

\[
e_1^+(x^1) = 0, \quad (12b)
\]
whereas locally a Lorentz gauge representative of the solutions to (5a) is

$$\omega_1(x^1) = -E_0, \ e_1^-(x^1) = E_0 x^1 - 1. \quad (12c)$$

To avoid $e = 0$ at $x^1 = 1/E_0$ one can also ‘deform’ $\pi_+(x^1)$ into $E_0 \arctan x^1$, when changing (12c) correspondingly. Finally acting with $-G_+$ on (12) gives the (local) $x^0$-dependence.

Note that according to (10) such solutions (and all solutions with points where $\pi_2 = 2\pi_+\pi_- = 0$) are possible only for a certain range of $Q_0$-values. An analysis of (10) shows the existence of a function $h(\Lambda)$ with the following property: $\pi_2$ has zeros on $M$ for $Q_0 < h(\Lambda)$ and does not for $Q_0 > h(\Lambda)$; one finds $h$ to be zero for $\Lambda \leq 1$ and to increase monotonically for $\Lambda > 1$.

A solution with constant curvature and vanishing torsion (de Sitter or Liouville solution, $\pi_\omega = \pm \sqrt{\Lambda}$) is included in (12); it is obtained by choosing

$$Q_0 = Q_{des} := 2 \exp(\pm \sqrt{\Lambda}) (\mp \sqrt{\Lambda} - 1) \iff E_0 = 0. \quad (13)$$

In this case the eqs. (12b), (12c) are not a consequence of the constraints, which are trivially fulfilled, but they represent a (local) gauge choice along the line $x^0 = 0$. Note that (11) gives a different solution for $Q_0 = Q_{des}$.

For the case $Q_0 \neq Q_{des}$ (and $Q_0$ within the range of the existence of $T_a = 0$) one may construct the solutions (12) from (11), if one allows for zeros of $A(x^1)$ (but $\partial A(P) \neq 0$) and chooses $B$ and $D$ such that singularities are avoided. Since in the coordinate system chosen above the curvature has the form $\pi_\omega = -E_0 x^1 x^0 + B_0(Q_0)$, points with vanishing $T_a$ are saddle points of the curvature (and vice versa). Local solutions around such points have not been obtained in the previous literature; nevertheless, within the conformal gauge saddle points of $R$ appeared by gluing together solutions [9].

To do quantum mechanics purely local considerations are not sufficient. A complete analysis, however, treating all the topological aspects of the theory is beyond the scope of the present paper. Instead we will restrict our considerations to the case that the space time manifold $M$ can be written as $M = S^1 \times R$, the $S^1$ being spacelike. Further we will regard the case $\Lambda < 0$ so as to exclude $Q_0 = Q_{des}$.

Let us prove that under these assumptions there are no solutions to the field eqs. which allow for a zero of $T_+$ or $T_-$. The cylinder we are to consider may be covered by one chart with periodic boundary conditions in the $x^1$ direction. Now, having e.g. a point $P$ with $\pi_-(P) = 0$, the constraint equation (5a) as well as (13) and the existence of a spacelike $S^1$ through $P$ tell us that

$$\partial \pi_-(P) = E_0 e_1^+(P) \neq 0. \quad (14)$$

Since $\pi_-$ is a periodic function in $x^1$ there has to exist at least another point $P'$ on this line $x^0 = const$ with vanishing $\pi_-$ and $\text{sgn} \partial \pi_-(P') \neq \text{sgn} \partial \pi_-(P)$. But then eq. (14) for $P$ and $P'$ implies that the sign of $e_1^+$ changes along the curve $x^0 = const$; this contradicts the assumption that it is spacelike. Let us note in parenthesis that when regarding $M = \Sigma \times R^1$ with $\Sigma = R^1$ spacelike, the above argumentation shows us only
that there cannot exist more than one (null) line with \( \pi_- = 0 \) for \( Q_0 \neq Q_{dS} \); the same is true for \( \pi_+ = 0 \). For \( Q_0 = Q_{dS} \) the above reasoning does not go through as (14) is not true.

Thus in our topological setting we know all the classical solutions. They are given by (11) with \( A(x^1) \neq 0 \) and \( Q_0 \geq 0 \); the latter restriction comes from the requirement \( \pi_- \neq 0 \) (cf. the paragraph following eq. (12)). But how can we possibly do quantum mechanics, if there is only one parameter \( Q_0 \) labelling the gauge inequivalent solutions? Actually for the case of our topology there is a second one as a simple consideration shows: For a particular fixed value \( R_0 \) of \( R \) the metric on the space time manifold induces a metric on the curve generated by \( R = R_0 \). As this curve is compact (cf. the first eq. of (11)), it has a finite length. This length is (though \( R_0 \) dependent) by construction gauge-invariant and obviously not determined by \( Q_0 \). (We can e.g. change the interval of periodicity at will without changing the integrand, which can be made \( x^1 \)-independent). Thus there is a quantity \( P_0 > 0 \) characteristic for the 'size of the universe'.

This fact and that there are no further gauge invariant quantities can be seen also from a more formal point of view: As before it is always possible to find a gauge such that \( A(x^1) = 1 \) and \( B(x^1) = 0 \). But now, normalizing the interval of periodicity of \( x^1 \) to \([0, 1]\), a diffeomorphism \( x^1 \rightarrow f(x^1) \), \( D(x^1) \rightarrow \partial f D(f(x^1)) \) cannot change the zero mode of the arbitrary (periodic) function \( D(x^1) \); therefore it is possible only to make \( D \) constant: \( D(x^1) = D_0 =: -P_0/4\gamma \). The identification \( P_0 \leftrightarrow -P_0 \) then is obtained by the gauge transformation \( x^1 \rightarrow -x^1 \); and \( P_0 \neq 0 \) since we required the \( S^1 \) to be spacelike. Thus in our topological setting the space of solutions of the eq.o.m. (and thus the reduced phase space of the theory) is a two parameter space:

\[
\pi_\omega = x^0, \quad \pi_+ = 1
\]

\[
\pi_- = \frac{1}{4\gamma} [Q_0 \exp(-x^0) + (x^0 - 1)^2 + 1 - \Lambda]
\]

\[
\omega_1 = -e_1^+ E, \quad e_1^+ = e_1^- \pi_-
\]

\[
e_1^- = -4\gamma P_0 \exp(x^0),
\]

with

\[
Q_0 \geq 0, \quad P_0 > 0.
\]

In this gauge the quantity \( E \), defined in (3), becomes:

\[
E = \frac{1}{4\gamma} (x^0)^2 - \pi_- - \lambda = \frac{1}{4\gamma} [-Q_0 \exp(-x^0) + 2(x^0 - 1)].
\]

Because of \( g_{11} = 2(e_1^-)^2 \pi_- \) the requirement '\( S^1 \) be spacelike' is compatible only with \( \gamma > 0 \), whereas for the case \( \gamma < 0 \) there exist no such solutions to the field equations. Requiring \( M = S^1 \times R^1 \), the \( S^1 \) being timelike, on the other hand, one obtains (13) for \( \gamma > 0 \) and no solutions for \( \gamma < 0 \) (\( \Lambda < 0 \)). This result holds irrespective of any gauge as is obvious from (5b) multiplied by \( e_1^-/\pi_+ \) and the fact that \( \partial \pi_\omega \) cannot be definite on a circle. As will be shown in sec. 4, furthermore, the evolution parameter \( x^0 \) in (15) can be taken to be purely timelike.
For the case \( \Lambda \geq 0 \) the requirement of the existence of a spacelike closed section leads to (13) with \( Q_0 = Q_{\text{deS}} \) or \( Q_0 > h(\Lambda) \), \( h \) having been defined in the paragraph following eq. (12), as well as to the Liouville solution. A gauge representative of the latter depends also on one (topological) quantity \( P_0 \), as can be shown by considerations similar to the ones leading to (13). Although we do know all the classical solutions in this extended situation, too, the construction of a consistent quantum theory on these classical solutions seems hardly manageable due to the existence of a discrete part in the spectrum of \( Q_0 \) (cf. sec. 3.1 below) — except when assigning these points the measure zero, certainly. Thus for \( \Lambda \geq 0 \) the classical requirement that the \( S^1 \) be spacelike cannot be maintained within a quantum theory.

3 The Quantum Theory

For the present model the simplest and most straightforward quantization is the reduced phase space quantization (sec. 3.1). It makes use of the fact that we know all the classical gauge inequivalent solutions (under the assumptions made in the preceding section). Nevertheless, in order to gain insight into theories where not all the classical solutions are known, such as general relativity, it is instructive to apply also other standard methods of quantization like for instance the Dirac procedure (sec. 3.2). Canonical transformations, obtainable also without knowing the classical solutions, can dramatically simplify the task of quantization. This shall be illustrated in sec. 3.3, where we succeed in describing the constraint surface by the vanishing of canonical coordinates.

3.1 Reduced Phase Space Quantization

Having the reduced phase space at our disposal, which is the quarter of a plane under our assumptions [cf. (16)], we need to find the symplectic structure it inherits from the unconstrained phase space. This can be achieved by first finding the Dirac observables which correspond to \( Q_0 \) and \( P_0 \) in (13) and then by calculating their Poisson bracket.

\( Q_0 \) is obviously just the constant mode of (10), i.e. (as we fixed the length of periodicity to one, factors \( 2\pi \) are avoided)

\[
Q_0 = \int_{S^1} Q. \tag{18a}
\]

To find the gauge independent quantity corresponding to \( P_0 \), one first makes the last eq. of (13) explicit in \( P_0 \). Dividing the obtained expression by \( \pi_1 \), which is one in the gauge of (13), it becomes Lorentz invariant. Integration of the resulting one-form yields the diffeomorphism invariant quantity

\[
P_0 = -\frac{1}{4\gamma} \int_{S^1} \exp(-\pi_\omega) \frac{e_1^-}{\pi_+} \tag{18b}
\]

as our second Dirac observable, commuting (weakly) with all the constraints. Now it is straightforward to verify that the symplectic form on the reduced phase space equals

\[
dQ_0 \wedge dP_0. \tag{19}
\]
However, $P_0$ in (18b) is not invariant against the discrete transformation $x^1 \to -x^1$, which is not included within the (continuous) flow of the constraints. Thus the completely gauge independent quantity corresponding to the normalized solution (15) subject to the restriction (16) is actually

$$|P_0| = \sqrt{P_0^*P_0}.$$  

(20)

Now the quantization is quite simple. The commutation relations (19), or better the corresponding Weyl algebra, as well as the first eq. (10) as a restriction to the spectrum of $\hat{Q}_0$ yield an $L^2(R_+)$ with Lebesgue measure as our Hilbert space (cf. e.g. [14]). In this $\hat{Q}_0$ acts as a multiplicative operator and $\hat{P}_0$ as the usual derivative operator (up to unitary equivalence). And since it is $\sqrt{\hat{P}_0^*\hat{P}_0}$ which corresponds to the classical quantity $P_0$ in (13), we also have no problem with self–adjointness and the second restriction (16) (as we would have with $\hat{P}_0$).

Note that there is so far no 'dynamics' present in this formulation of the quantum theory. As typical for theories formulated in a reparametrization invariant way our Hamiltonian (4) vanishes so that a priori there is no (naive) Schroedinger eq. or also no (naive) path integral. How and in how far we can introduce some notion of time into the canonical framework above shall be discussed in sec. 4. The corresponding problem in the path integral formulation shall be tackled elsewhere [13].

### 3.2 Dirac quantization

In this section we shall quantize the unconstrained phase space and then calculate physical wave functions as the kernel of the quantized constraints. The form of the primary constraints (3) allows to simply eliminate the zero components of our fields. So we are left with a phase space $\Gamma$ spanned by the variables $\omega_1, e_1^a, \pi_\omega, \pi_a$. Since our constraints (5) are linear in the coordinates but quadratic in the momenta, we will work in the momentum representation:

$$\omega_1 \to \imath\hbar \frac{\delta}{\delta \pi_\omega}, \quad e_1^a \to \imath\hbar \frac{\delta}{\delta \pi_a}.$$  

(21)

Thus the quantum constraints become

$$\hat{G}_\omega \Psi = \hat{G}_+ \Psi = \hat{G}_- \Psi = 0$$  

(22)

with ($[,]_+$ denotes the anticommutator)

$$\hat{G}_\omega = \partial \pi_\omega + \imath\hbar (\pi_- \frac{\delta}{\delta \pi_-} - \pi_+ \frac{\delta}{\delta \pi_+})$$  

(23a)

$$\hat{G}_+ = \partial \pi_+ + \imath\hbar (\frac{1}{2}[E, \frac{\delta}{\delta \pi_-}]_+ + \pi_+ \frac{\delta}{\delta \pi_\omega})$$  

(23b)

$$\hat{G}_- = \partial \pi_- - \imath\hbar (\frac{1}{2}[E, \frac{\delta}{\delta \pi_+}]_+ + \pi_- \frac{\delta}{\delta \pi_\omega}).$$  

(23c)
As already proven in [7] the quantized constraints (23) form a close d algebra. This is crucial for the consistency of the simple quantization scheme used here. Otherwise we would rely on more elaborate techniques like e.g. BRST quantization (cf. also [15] and sec. 5). Note that the first replacement in (21) breaks the local Lorentz covariance present in the classical theory: whereas the lefthand side of that eq. transforms as usual for a connection of the Lorentz group \( \pi \pm \rightarrow \exp[\pm \alpha(x)] \pi \pm \Rightarrow \omega \mu \rightarrow \omega \mu - \partial_\mu \alpha \), the righthand side remains unchanged. This is a feature which should prevail also in the Ashtekar formulation of the 3 + 1 theory.

Up to purely multiplicative terms our quantum constraints contain only Lie derivatives. Thus the calculation of the kernel of the constraint operators will simplify considerably, if, instead of some of the momenta, we use other variables which commute strongly with the classical constraints. Because of (16) our wave functions have their support only in an area where \( \pi^+ \) and \( \pi^- \) are different from zero [cf. (10) and remember \( \Lambda < 0 \)]. In such an area the map from either \( \pi^- \) or \( \pi^+ \) to \( Q \) is bijective. Therefore to start with we will write our wave functions as

\[
\Psi = \Psi[Q(\pi^\omega, \pi^\omega), \pi^+, \pi^-].
\]  

(24)

With this general ansatz the integration of the first two eqs. (22) is straightforward, yielding

\[
\Psi = \exp(-i \frac{\hbar}{\bar{h}} \int_{S^1} \partial \pi^\omega \ln |\pi^+|) \exp(\frac{1}{2} \delta(0) \int_{S^1} \pi^\omega) \tilde{\Psi}[Q],
\]  

(25)

whereas the last eq. (22) becomes

\[
\partial Q \tilde{\Psi}[Q] = 0,
\]  

(26)

as is also clear from (9) which is valid also in the quantum case. The \( \delta(0) \) is understood to be defined in an appropriate regularization. We could e.g. discretize the \( x^1 \) variable according to \( x_i - x_{i-1} = l \). \( \delta(0) \) appears to be \( \frac{1}{l} \) in this regularization.

The operator ordering in (23) guarantees that the quantum constraints are hermitian with respect to the Lebesgue measure \( \int [d\pi^\omega][d\pi^a] \). We could avoid the \( \delta(0) \) term by a different choice of the operator ordering in (22): Putting all derivatives to the right, the constraint algebra still closes and as the constraints vanish on physical states, they are automatically hermitian in the physical sector, whatever operator ordering we choose. We will find, however, that the \( \delta(0) \) term plays quite a crucial role in the reduction of the Lebesgue measure to an inner product in the space of physical states.

Starting from (24) with '+' and '-' exchanged, we obtain analogously

\[
\Psi = \exp(i \frac{\hbar}{\bar{h}} \int \partial \pi^\omega \ln |\pi^-|) \exp(\frac{1}{2} \delta(0) \int \pi^\omega) \tilde{\Psi}[Q]
\]  

(27)

as well as (26). Due to the latter eq., which is equivalent to setting (10) equal to some constant \( Q_0 \), it is obvious that the transition amplitude

\[
\exp(i \frac{\hbar}{\bar{h}} \int_{S^1} \partial \pi^\omega \ln |\pi^2| dx^1) = 1
\]  

(28)
so that the $\tilde{\Psi}[Q]$ in (23) and (27) do indeed coincide.

Note that in the above considerations we made use of our restrictions on $\Lambda$ and the topology only when we restricted the support of the physical wave functions to positive values of (10). Within the Dirac quantization everything else is the same also in the completely general case: (23) and (27) fulfilling (29) give the general solution to (22) on charts of the phase space with $\pi_+ \neq 0$ and $\pi_- \neq 0$, respectively. Because of (28) they can be patched together to give the wave functions fulfilling (22) on all of the phase space except for points with simultaneous zeros of $\pi_+$ and $\pi_-$. To extend this solution to all of the phase space except for points with $\pi_a = E = 0$ ($\Leftrightarrow Q = Q_{\text{deS}}$) by a further ansatz $\Psi = \Psi[Q, \pi_a]$ does not seem to be so easy though: A calculation analogously to the above ones yield a difficult differential eq. of first order, and to make a good guess is aggravated by the fact that the expected phase factor will definitely be not locally Lorentz covariant due to (21) or (23a), as can be seen explicitely from (23) or (27).

However, our solutions (23) or (27) extend to a much more general situation anyway: The phase factors in $\Psi$ can be integrated as long as the torsion does not vanish on an interval of the $S^1$. Thus with (23) we exclude only functional distributions solving the quantum constraints (22) such as

$$\delta[\pi_a] \delta[\pi_+ \mp \sqrt{\Lambda}], \quad (29)$$

which obviously corresponds to the Liouville or de Sitter solution.

Still we have to define an inner product in the space of wave functions (23). To this end we may first realize that $\Psi^* \Psi$ gives a factor $\prod_{x^1} \exp(\pi_\omega(x^1))$ and that the product of this factor and the formal Lebesgue measure $[d\pi_\omega][d\pi_a]$ on the unconstrained momentum space yields an expression being invariant under the classical flow of the constraints. The integral of $\Psi^* \Psi$ with the Lesbegue measure, however, will diverge, as the wave functions are roughly speaking constant in the direction of $G_+$ and $G_\omega$. Note that having implemented these two constraints $G_-$ is purely multiplicative and is of no relevance for the considerations at this stage. $G_+$ and $G_\omega$ are the generators of a non–abelian group [cf. (7a)], the (infinite) volume of which has to be ‘devided out’ from the integral. As this group acts freely and transitively on the $(\pi_\omega, \pi_+)–$plane (or more strictly speaking the half plane with positive values of $\pi_+$), it is suggestive to restrict the values of $\pi_\omega$ and $\pi_+$ by the gauge conditions

$$\pi_+(x^1) = c(x^1) \quad \pi_\omega(x^1) = t(x^1). \quad (30)$$

These gauge conditions may be realized by the introduction of $\delta[\pi_+ - c] \delta[\pi_\omega - t]$ into the measure. This expression is, however, not invariant under the flow of $G_+$ and $G_\omega$. Thus the resulting expression for $\langle \Psi^* \Psi \rangle$ will become gauge dependent. In our simple model this is not really disturbing, as the gauge dependence can be reabsorbed in the normalization of the wave function. Nevertheless, to get insight into similar problems in more complicated theories, it is interesting how a gauge independent measure can be constructed. The introduction of a Faddeev-Popov determinant, which is the determinant of $\delta$ denotes the delta function)

$$
\begin{pmatrix}
\{\pi_\omega, G_+\} & \{\pi_\omega, G_\omega\} \\
\{\pi_+, G_+\} & \{\pi_+, G_\omega\}
\end{pmatrix}
= \begin{pmatrix}
-\pi_+ & 0 \\
0 & \pi_+
\end{pmatrix} \delta,
\quad (31)
$$

10
will not lead to a satisfactory result. To our mind this seems to be correlated to the fact that the group generated by $G_\omega, G_+$ does not allow for an invariant, non-degenerate bilinear form on its algebra.

To find an invariant measure let us calculate the action of $G_\omega$ and $G_+$ on $\Omega = [d\pi][d\pi_\omega]$. We find [cf. (31)]

\[
\{\Omega, \int G_\omega\} = \Omega, \quad \{\Omega, \int G_+\} = 0. \tag{32}
\]

As this coincides with the transformation of $\pi_+$, it is obvious that the expression $\prod (1/\pi_+) \Omega$ and thus its dual $\prod (\pi_+) \delta[\pi_+ - c] \delta[\pi_\omega - t]$ is invariant. Realizing the constraint (9) by a further delta functional, we end up with

\[
\langle \Psi, \Phi \rangle \propto \int dQ_0 \bar{\psi}(Q_0) \bar{\Phi}(Q_0), \tag{33a}
\]

Changing the variables of integration from $\pi_-(x^1)$ to $Q(x^1)$ we find

\[
\langle \Psi, \Phi \rangle = \int dQ_0 \bar{\psi}(Q_0) \bar{\Phi}(Q_0), \tag{33b}
\]

with the normalized $\bar{\psi}(Q_0) \propto \bar{\psi}[\delta Q = 0, Q_0]$. Note that all divergent factors are compensated by the transformation of the variable of integration. We may now remove the regularisation introduced after (26) and remain again with a one dimensional quantum mechanical system described by an $L^2(R_+)$ [or $L^2(R)$] with Lebesgue measure. A solution such as (29) could be also implemented at this stage when assigning some (arbitrary) weight to the point(s) $Q_0 = Q_{des}$. This does not seem very rewarding, though. To complete the equivalence with sec. 3.1 we have to apply the Dirac observable (18b) to our wave function (25); we indeed find:

\[
\langle \Psi, \hat{P}_0 \Phi \rangle = \int dQ_0 \bar{\psi}(Q_0) \frac{\hbar}{i} \frac{d}{dQ_0} \bar{\Phi}(Q_0). \tag{34}
\]

The constraint $P_0 > 0$ is then implemented such as in the preceding subsection ($P_0 \leftrightarrow \sqrt{\hat{P}_0^* \hat{P}_0}$).

An alternative way to formulate an inner product in the space of physical wave functions is to first recognize that there is a natural bijective map $\Psi \leftrightarrow \hat{\Psi}(Q_0)$ between this space and the space of functions over the variable $Q_0$. An inner product on this space $L^2(R_+)$ [or $L^2(R)$] is then implicitly defined by the condition that a basic set of Dirac observables, in our model $\hat{Q}_0$ and $\hat{P}_0$, should be hermitian with respect to this inner product [3]. Because of $\hat{P}_0 := (\hbar/i)(d/dQ_0)$ [cf. (19)] the hermiticity requirement obviously fixes the measure $\mu$ within the general ansatz

\[
\langle \Psi, \Phi \rangle = \int dQ_0 \mu(Q_0) \bar{\psi}(Q_0) \bar{\Phi}(Q_0) \tag{35}
\]

to be independent of $Q_0$; so again we end up with (33b). Whereas this approach to find an inner product is more straightforward than the first one, a generalization of
it to models (such as general relativity), where a basic set of Dirac observables is not (yet) known, seems to be difficult. The approach leading to (33), on the other hand, is applicable whenever one finds good gauge conditions.

Choosing \( t \) to be \( x^1 \)-independent, it is suggestive to regard it as an ‘intrinsic’ time. In analogy to the classical case one can then denote this time parameter by \( x^0 \). With this interpretation the second equation of (22) can be regarded as a kind of Schroedinger equation [with a time dependent Hamiltonian — cf. (23b)], and the (time dependent) coordinate transformation from \( Q(\pi_a, \pi_\omega = t) \) to the variable \( Q \) in (33) as a shift to a Heisenberg representation of quantum mechanics (cf. also sec. 4 for more details). An obvious generalization would be \( \pi_+ = c_+(x^\mu), \pi_\omega = c_\omega(x^\mu) \). Due to our construction of (33) the resulting quantum theories are independent of the choice of \( c_+ \) and \( c_\omega \). To get ‘physical’ results we can e.g. calculate the expectation value of the torsion: Plugging the multiplicative operator \( \pi^2 \) into (33a) we find

\[
\langle \pi^2 \rangle(t) = \frac{1}{4\gamma} \left[ \langle Q_0 \rangle \exp(-t) + (t - 1)^2 + 1 - \Lambda \right].
\] (36)

Although we obtained some nontrivial dynamics by reinterpreting and generalizing the gauge choice (30) and although we could calculate e.g. (36), we are not yet ready to determine \( \langle g_{11} \rangle \) etc., for having not fixed the corresponding gauge freedom. This will be done in sec. 4.

There is also another related approach [8] leading to the correct Hilbert space: Since the constraints are linear in the coordinates, the momenta are transformed into momenta under the action of the constraints. Thus we could regard the functionals \( \Psi[\pi_\omega, \pi_a] \) on the constraint surface modulo the flow of the constraints as the physical wave functions. With the general ansatz (24) \((\pi_+ \neq 0)\) the Lie derivative of the constraints yield the dependence on \( Q(x^1) \) which reduces to the dependence on its zero mode due to (4). The inner product is constructed as two paragraphs above.

### 3.3 Abelianization

It is well known that any system of first class constraints allows a formulation, where the constraints are abelian. A system of canonical coordinates may then be found such that the abelianized constraints are part of it. Unfortunately, in general this canonical coordinates are defined locally only and they are non polynomial in the original coordinates. Moreover, it is difficult to find them. For these reasons they are of minor practical use in most systems. In our system, however, the abelianization will turn out a powerful tool.

Again let us first assume \( \pi_+ \neq 0 \). We already know the quantity \( Q \) to commute with all the constraints and \( \partial Q \) to be a linear combination of the constraints. It is thus clear that \( Q \) will play a crucial role in the abelianization. As \( Q \) is a combination of the momenta, it commutes with all the momenta and the Poisson bracket with any of the coordinates on the configuration space yields a function of the momenta times the delta-function. So take a configuration space coordinate, devide it by the function of the momenta on the right hand side of its commutator with \( Q \) to end up with a canonical
conjugate; e.g. for $e_1^+$ one obtains in this way:

$$\left\{ \frac{1}{4\gamma} \exp(-\pi \omega) \frac{e_1^-}{\pi_+}, Q(y^1) \right\} = \delta(x^1 - y^1). \quad (37)$$

There are no obvious canonical conjugates to the other constraints. But a glance at (31) suggests to reformulate $G_\omega$ and $G_\pi^+$ by multiplying them with a factor $(1/\pi_+)$. So we are led to

$$(\tilde{\omega}_1, \tilde{e}_1^+, Q; \pi_\omega, \pi_+, P) \quad (38a)$$

with

$$\tilde{\omega}_1 = \frac{G_+}{\pi_+}, \quad \tilde{e}_1^+ = -\frac{G_\omega}{\pi_+} \quad (38b)$$

$$P = -\frac{1}{4\gamma} \exp(-\pi \omega) \frac{e_1^-}{\pi_+} \quad (38c)$$

Since $\tilde{\omega}_1$ and $P$ are Lorentz invariant and commute with $\pi_+$ they obviously commute with $\tilde{e}_1^+$. Checking finally that also $\tilde{\omega}_1$ and $P$ commute, we indeed find that (38a) forms a complete set of canonical coordinates (in a region where $\pi_+ \neq 0$).

In a region where $\pi_- \neq 0$ we may, up to signs, exchange the role of ‘+’ and ‘-’ in the above considerations. We thus find

$$(\tilde{-G}_-, \tilde{G}_\omega, Q; \pi_\omega, \pi_-, \tilde{-1}/4\gamma \exp(-\pi \omega) \frac{e_1^+}{\pi_-}) \quad (39)$$

to form a set of canonical coordinates.

Within our topological framework it is near at hand to further Fourier transform $Q(x^1)$ and $P(x^1)$. This then completes the canonical splitting of our theory into the gauge sector and the Dirac sector, the latter being spanned by the conjugates $Q_0 = \int_{S^1} Q(x^1)$, $P_0 = \int_{S^1} P(x^1)$. [Note that the zero modes of (38c) and the corresponding variable in (39) coincide due to (5b) and (10)]. The quantization is now obvious. Any quantization scheme will lead to a system equivalent to that of sec. 3.1.

### 4 Space–Time and Observables

As the symmetries of a theory of gravity include diffeomorphisms in space and time, any Dirac observable (i.e. any function on the phase space invariant under the action of the constraints) is space and time independent. This is the reason for the lack of any dynamics within the (classical) reduced phase space or the corresponding quantum system (cf. sec. 3.1). In order to reintroduce the notion of space and time into the theory we have to break the according symmetries. This is most easily done by gauge conditions. Measurable quantities are then defined by the requirement to be invariant under the remaining symmetries. There is, of course, some arbitrariness in the choice of gauge conditions. This arbitrariness reflects the fact that different observers may have different means to measure quantities.
We have already seen an example in section 3.2: $\pi_\omega$ is a function on the space time manifold. Under the restrictions specified in section 2 the lines where $\pi_\omega$ is constant provide a foliation of space-time. From (13) (or also (14)) we find the leaves to be spacelike. This might encourage us to choose $\pi_\omega$ as a time variable $x^0 := t$. Functions on the constraint surface which depend on $t$ and the Dirac observables alone, like e.g. $\pi^2$ [cf. (10)], are invariant under the flow of those linear combinations of the constraints which leave the gauge condition

$$\pi_\omega - t = 0$$

invariant. They thus are measurable quantities in this setting and we may calculate their expectation values etc. for a given quantum state. In this way we regain results like (36).

Let us mention that the choice of a time variable $t$ does not determine its flow $(\partial/\partial t)$: A $(t$–dependent) diffeomorphism in the direction of constant time will leave the choice of time unchanged while varying the flow of time and thus the Hamiltonian generating it. In our example the condition (40a) implies

$$\{\pi_\omega, H\} = 1.$$  \hspace{1cm} (41)

With (4) we find that the values of the Lagrange multipliers $e_0^a, \omega_0$ are restricted by (40a), but certainly not completely determined.

In order to quantize quantities like e.g. the components of the metric, we have to fix the coordinate system of the observer by further gauge conditions. The form of the canonical coordinates (38) suggests the choice

$$\pi_\omega + 1 = 1, \quad \partial P = 0.$$  \hspace{1cm} (40b)

Due to (40a) and the first eq. of (40b), which have been implemented already within the approach of sec. 3.2 for the special case $c = 1$, the second eq. of (40b) is equivalent to $\partial e_1^- = 0$. These gauge conditions together with our Dirac observables uniquely determine all the quantities $(\omega_1, e_1^a, \pi_\omega, \pi_a)$ on the constraint surface $\hat{\Gamma}$. A simple algebraic manipulation yields (13) with $x^0 = t$. (Note that the choice of good gauge conditions, turning all first class constraints into second class constraints, saves one the integration of the flow of the Hamiltonian; in more complicated systems this can be a decisive advantage). Antisymmetrizing these classical relations, we obtain a one parameter family of hermitian operators on our Hilbert space:

$$\pi_-(t) = (1/4\gamma) [Q_0 \exp(-t) + (t - 1)^2 + 1 - \Lambda]$$

$$e_1^-(t) = -4\gamma P_0 \exp t$$

$$e_1^+(t) = (1/2) [e_1^-(t), \pi_-(t)]_+$$

$$\omega_1(t) = -(1/2) [e_1^-(t), E(t)]_+$$

Thus we can now predict mean values, standard deviations etc. for $g_{11}$ and $\omega_1$.

The time evolution of the operator relations (42) is unitary. To show this let us first specify the relation between (33a) and (33b). By means of the second eq. (4) we can,
instead of performing the shift of variables from $\pi_-(x^1)$ to $Q(x^1)$, also directly integrate out all the delta functions within (33a). This yields:

$$\langle \Psi, \Phi \rangle = \int d\pi_- \Psi^*_S(\pi_-, t) \Phi_S(\pi_-, t)$$

with

$$\Psi_S(\pi_-, t) \equiv \exp\left(\frac{t}{2}\right) \tilde{\Psi}(Q_0(\pi_-, t)),$$

the latter function coinciding with $\tilde{\Psi}$ in (33b) and $Q_0(\pi_-, t)$ being the function at the righthand side of (10) in the gauge (40); $\pi_-$ is the zero mode of $\pi_-(x^1)$, projected out within the inner product (33). Due to our gauge choice (40) (with $t \equiv x^0$) we have been allowed to drop the phase factors in (25), provided we do not consider derivative operators of some higher order in $\pi_\omega$ and $\pi_+$. Since the inner product $\langle \Psi, \Phi \rangle$ does not depend on $t$ [cf. (33b)], there exists a unitary operator $U(t)$ satisfying

$$\Psi_S(\pi_-, t) = U(t)\Psi_S(\pi_-, 0).$$

Because of $\Psi_H(\pi_-) = \Psi_S(\pi_-, 0) = \tilde{\Psi}(-Q_0 + \frac{1}{2\gamma} - \lambda)$, (33b) differs from the usual Heisenberg representation only by the trivial bijection

$$\pi_- \leftrightarrow -Q_0 + \frac{1}{2\gamma} - \lambda.$$

Since (42a) is the result of the transition from (33a) to (33b) applied to $\pi_-$, it is obvious that $\pi_-(t)$ is a unitary evolution in $t$. It differs from the Heisenberg operator $(\pi_-)_H(t) = U^*(t)\pi_- U(t)$, corresponding to the time independent operator $\pi_-$ in the effective Schrödinger picture, just by this bijection.

In the Dirac-approach of section 3.2 we had to partially fix the gauge in order to define an inner product in the space of physical states. It was not necessary to formulate a gauge condition for the multiplicative constraint $\partial Q = 0$. Nevertheless, wanting to obtain unique results for $\langle e_1^-(x^1) \rangle$ etc., this is necessary. The corresponding gauge condition is implemented as an operator condition on the wave functions in this approach:

$$\partial P \Psi = \frac{\delta}{\delta \partial Q} \Psi = 0.$$

This guarantees that when operators such as $e_1^-(x^1)$ act on $\Psi$ only its (physical) dependence on $Q_0(\pi_-, t)$ contribute to the result in (33a). Thus effectively the operator $e_1^-(x^1)$ acting on wave functions in the Schrödinger picture can be replaced by its constant mode $e_1^-$. The eqs. (18b) and (34) show that the time independent Schrödinger operator $e_1^-$ is transformed into the righthand side of (42b) within the transition from (33a) to (33b). Since the latter has been shown to be practically a Heisenberg picture, the unitary evolution of $e_1^-(t)$ is also obvious. Now the unitarity of the remaining two eqs. of (42) is a trivial consequence: (42c) and (42d) are the 'Heisenberg evolution' of $(1/2)[e_1^-, \pi_-]_+$ and $-(1/2) [e_1^-, E_S(t)]_+$, respectively. The operator $E_S(t)$ is given by the first eq. of (17); it is explicitly time dependent in the Schrödinger picture.
So, having chosen a gauge (40), there is a natural Schroedinger picture associated to the operator evolution in the $\tilde{\Psi}(Q_0)$-representation. The wave functions (44) obey a usual Schroedinger equation [cf. second eq. of (22) in our gauge] with a hermitian, time dependent Hamiltonian

$$h_S(t) = -\frac{1}{2}[e_1^-, E_S(t)]_+.$$  \hspace{1cm} (48)

With (46) and the replacement $e_1^- \leftrightarrow -4\gamma P_0$ in the above, we obtain the explicit form of the evolution operator $U_0(t)$ generating the time dependence of (42):

$$U_0(t) = T \exp\left(-\frac{i}{\hbar} \int_0^t \frac{2 - t'^2}{2\gamma} P_0 - \frac{1}{2}[Q_0, P_0]_+ dt'\right).$$ \hspace{1cm} (49)

To find this operator the Dirac approach of sec. 3.2 was very helpful. Clearly, in order to have a (non–trivial) Schroedinger picture it has been necessary to break at least some of the gauge symmetries of the reparametrization invariant theory.

To calculate the zero components of the connection and the zweibein, we have to investigate the flow of time: with (41) and $\{\pi^+, H\} = \{\partial P, H\} = 0$ we find

$$e_0^+ = 1 + f \pi_-, \hspace{0.5cm} e_0^- = f(x^0), \hspace{0.5cm} \omega_0 = -f E,$$

in which $f(t)$ is an arbitrary function of $t$, $\pi_-$ is determined by (42a), and $E$ by the expression on the righthand side of (17). It is remarkable that a complete set of gauge conditions in the $(\omega_1, e_1^a, \pi_\omega, \pi_\alpha)$-space $\Gamma$ does not fix the flow of time completely. Irrespective of how we choose $f$ the classical solutions can be always brought into the form (50) (under the assumptions specified there). There is also another perspective to see this: Since (13) is independent of $x^1$ it is invariant under a coordinate transformation $x^1 \rightarrow x^1 + F(x^0)$, which is the most general invariance of (13). This transformation, on the other hand, induces $e_0^+ \rightarrow e_0^+ - e_1^+ \dot{F}$, and analogously for the other zero components. Starting from the light cone gauge $e_0^+ = 1, e_0^- = 0, \omega_0 = 0$, this transformation yields again (50) [with $f = -e_1^- \dot{F}$], when restricting it to the cross section $\bar{\Gamma}$ of the constraints and the gauge conditions. This generalization of the 'light cone gauge' allows for a strictly timelike flow of time: at least under our assumptions it is always possible to choose $f$ such that $g_{00} > 0$. It is interesting that through our prescription leading to (50) also the 'Lagrange multipliers' $e_0^a, \omega_0$ became operators in the Hilbert space for any gauge choice $f \neq 0$. Again different choices of $f$, which may be operator valued, correspond to different observers.

The mechanism described above also works in the opposite direction: One may first choose the values of the Lagrange multipliers. The according flow of time then restricts but not completely determines the choice of a gauge in the phase space $\Gamma$. For instance the light cone gauge leading to (11) does not allow for any $x^0$-dependence of $\pi_+$, the $x^1$-dependence of this function, however, is still arbitrary. Having chosen some combination of the constraints as our classical Hamiltonian $H$ and being able to integrate the flow of it, we may also introduce the affine parameter of $H$ as an 'extrinsic time' (14) (having factored out the action of the other constraints). This approach seems straightforward and suggestive, but it has the drawback that one has to be able to completely integrate
the equations of motion, what usually is not the case; moreover, any 'extrinsic' time is clearly equivalent to some 'intrinsic' one introduced by gauge conditions.

It is a special feature of our system that there exists a gauge such that the $x^1$--dependence of the solutions drops out completely. To obtain explicitly space–time dependent operators we could choose a gauge

$$\pi_\omega = x^0, \quad \pi_\pm = A(x^1), \quad \partial e_1^- = 0$$

in which $A(x^1)$ is an arbitrary nonvanishing periodic function, e.g. $A(x^1) = 2 + \sin(x^1/2\pi)$. Analogously to above the gauge conditions (51) determine uniquely the fields of $\bar{\Gamma}$ in terms of the Dirac observables:

$$\pi_-(x^1, t) = -\frac{1}{A(x^1)} [Q_0 \exp(-t) - \frac{1}{4\gamma} (t - 1)^2 + \lambda - \frac{1}{4\gamma}] ,$$

$$e_1^-(t) = \frac{P_0}{f_{S^1}} A \exp t , \quad e_1^+(x^1, t) = \frac{e_1^-(t) \pi_-(x^1, t)}{A(x^1)} ,$$

$$\omega_1(x^1, t) = -\frac{\partial A(x^1) + E(t) e_1^-(t)}{A(x^1)} ,$$

where $E$ is given by the righthandside of (17), and restrict the gauge choice for the zero components to:

$$e_0^+ = 1 + e_0^- \frac{\pi_-}{A} , \quad e_0^- = f(x^0) A , \quad \omega_0 = -\frac{e_0^- E}{A} .$$

Choosing some gauge for $f(x^0)$ and $A(x^1)$, we could now calculate $\langle g_{\mu\nu}(x^\mu) \rangle, \langle \omega_\mu(x^\mu) \rangle, \langle \Delta g_{\mu\nu}(x^\mu) \rangle, \langle \Delta \omega_\mu(x^\mu) \rangle$, etc.

5 Conclusion

We have succeeded in quantizing the model (1) under the assumption that the corresponding classical solutions lead to a space time $M$ of the 'physical' form $M = S^1 \times R^1$ with spacelike $S^1$ and the assumption $\Lambda \equiv 4\gamma \lambda < 0$ (in sec. 3.2 also under more general assumptions). The simple structure of the phase space in this restricted model allowed us to apply different methods of quantization, to compare the results, and to elucidate conceptual problems of quantum gravity like the relation between Dirac observables, gauge conditions, and measurable quantities — in a framework, where the connection between classical and quantum expressions becomes very clear.

From our considerations in chapter 2 we conjecture that the quantization of models with other values of $\Lambda$ and with a more general topology will come down to the quantization of some finite dimensional phase space with a more complicated topology. In this more general framework the Liouville theory, which is the de Sitter solution of our theory, would be included. Interesting questions like the one of topology changing could be addressed. We thus think that a detailed analysis of the general theory would be desirable.
It would be also interesting to compare our results to still further methods of quantization like e.g. the BRST quantization. In \[13\] the nilpotency of a quantum version of \( c^i G_i + \bar{c}^i C_{ijk} c^j c^k \) has been shown. \([G_i = (G_a, G_\omega), C\] are the structure functions, and \( c, \bar{c}\) the ghosts and antighosts, respectively]. The study of the cohomology problem of this operator would be the next step. Another promising area for investigations seems the coupling of (1) to matter fields \[17\].

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