THE MINCUT GRAPH OF A GRAPH

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Abstract. In this paper we introduce an intersection graph of a graph $G$, with vertex set the minimum edge-cuts of $G$. We find the minimum cut-set graphs of some well-known families of graphs and show that every graph is a minimum cut-set graph, henceforth called a mincut graph. Furthermore, we show that non-isomorphic graphs can have isomorphic mincut graphs and ask the question whether there are sufficient conditions for two graphs to have isomorphic mincut graphs. We introduce the $r$-intersection number of a graph $G$, the smallest number of elements we need in $S$ in order to have a family $F = \{S_1, S_2, \ldots, S_i\}$ of subsets, such that $|S_i| = r$ for each subset. Finally we investigate the effect of certain graph operations on the mincut graphs of some families of graphs.

Keywords: Connectivity, edge-cut set, mincut, intersection graph.

1. Introduction

Given a set $S$ and a family $F = \{S_1, S_2, \ldots, S_i\}$ of subsets of $S$, an intersection graph of $F$ is a graph with vertices $v_i$ corresponding to each of the $S_i$ and two vertices $v_i$ and $v_j$ are adjacent if $S_i \cap S_j \neq \emptyset$, see [3, 6]. In 1945 Szpilrajn-Marczewski proved that every graph is an intersection graph, [3]. One of the first class of intersection graphs to be widely studied was the line graph, generalised as $(X,Y)$-intersection graphs in [1], while in the 1970’s chordal graphs were first characterised in terms of intersection graphs. Other intersection graphs that are studied intensively are interval and circular-arc graphs, competition graphs, $p$-intersection and tolerance graphs, to name but a few, see [4, 7, 8]. Problems involving intersection graphs often have real world applications in topics like biology, computing, matrix analysis and statistics, see [4, 7].

In this paper we introduce the intersection graph of a graph $G$, with vertex set the minimum edge-cuts of $G$, called a mincut graph. We then study some of its properties and characteristics. We conclude with some topics for further discussion resulting from the properties identified and related topics in intersection graph theory. Unless otherwise stated, we follow the conventions and notation in [2].
Definition 1.1. Let $G$ be a simple connected graph, then an edge-cut of $G$ is a subset $X$ of $E(G)$, such that $G - X$ is disconnected. An edge-cut of minimum cardinality in $G$ is a minimum edge-cut and this cardinality is the edge-connectivity of $G$, denoted $\lambda(G)$. We will call such a minimum edge-cut a mincut of $G$.

Example 1.1. We illustrate this concept with an example. The diagrams in Figure 1 are the Wheel graph, $W_6$, and the Peterson graph, both with minimal edge cuts labeled $\{e_1,\ldots, e_5\}$. However, $\{e_1,\ldots, e_5\}$ is not a mincut for either of the graphs, since in each of the graphs the removal of any of the three edges incident on a single vertex of degree 3 will also give a disconnected graph.

![Figure 1. The edge sets $\{e_1,\ldots, e_5\}$ are minimal, but not mincuts.](image)

Definition 1.2. Let $X = \{X_1, X_2, \ldots, X_i\}$ be the set of all mincuts of a simple connected graph $G$. Represent each of the $X_i$ with a vertex $v_i$ such that two vertices $v_i$ and $v_j$ are adjacent if $X_i \cap X_j \neq \emptyset$, and call this intersection graph the mincut graph of $G$, denoted by $X(G)$.

Given that the vertex set of $X(G)$ is exactly the set of mincuts of $G$ and the edges of $X(G)$ are determined by the edge intersection of those mincuts, it is clear that $X(G)$ is unique. That is, there is only one mincut graph for a graph $G$, although we can have an infinite number of graphs with isomorphic mincut graphs, see Proposition 4.1 and Corollary 4.1.1.

2. Mincut graphs of certain families of graphs

In this section we study the mincut graphs of some well-known families of graphs.

Proposition 2.1. Let $T_n$ be a tree on $n$ vertices. Then the mincut graph of $T_n$ 

$$X(T_n) \cong \bigcup_{n-1} K_1.$$ 


Proof. Every edge of $T_n$ is a bridge and $\lambda(T_n) = 1$. Each of the $n - 1$ edges of $T_n$ is a mincut and hence $X(T_n)$ has $n - 1$ vertices. But none of the singleton mincuts intersect and thus $X(T_n)$ is the empty graph on $n - 1$ vertices.

Proposition 2.2. For two positive integers $n > m$, let $K_{m,n}$ be a complete bipartite graph with vertex partition sets of size $m$ and $n$ respectively. Then the mincut graph of $K_{m,n}$ is $$X(K_{m,n}) \cong \bigcup_{n} K_1.$$ 

Proof. If $m = 1$, $K_{1,n}$ is a star and the proof follows from Proposition 2.1 since $K_{1,n}$ is a tree on $n + 1$ vertices. If $m > 1$ then the mincuts of $K_{m,n}$ are exactly the $m$ edges incident on each of the $n$ vertices of degree $m$ in the larger of the two vertex partitions. Since none of these intersect, we have a mincut graph with $n$ vertices and no edges.

Recall that the line graph, $L(G)$, of a graph $G$ has the edges of $G$ as its vertices, such that two vertices in $L(G)$ are adjacent if their corresponding edges in $G$ have a vertex in common.

Proposition 2.3. Let $C_n$ be the cycle on $n$ vertices and $L(K_n)$ the line graph of $K_n$. Then the mincut graph of $C_n$ is $$X(C_n) \cong L(K_n).$$

Proof. The edge connectivity of $C_n$, $\lambda(C_n) = 2$, and any choice of two edges is a mincut. Thus the set of all mincuts, $X$, of $C_n$ is the set of all two element subsets of $E(C_n)$, where $|E(C_n)| = n$. But the vertex set of $L(K_n)$ is the set of all two element subsets of the $n$-element vertex set of $K_n$, the edges of $K_n$, and the proof follows.

Proposition 2.4. Let $W_n$ be the wheel on $n$ vertices and set $n > 4$. Then the mincut graph of $W_n$ $X(W_n) \cong C_{n-1}.$

Proof. For $n = 4$, $W_4 \cong K_4$. We set $n > 4$, since complete graphs are dealt with in Corollary 3.1. Then, for any $W_n$, $\lambda(W_n) = 3$ and the mincuts are exactly the three edges incident on every vertex $v_i$ on the “rim” of the wheel. By labeling the $n - 1$ vertices on the rim in sequence, we see that every mincut $X_i$ has non-empty intersection with $X_{i-1}$ and $X_{i+1}$, thus giving the cycle $C_{n-1}$ as the mincut graph.
3. Super edge-connected graphs

In this section we give a sufficient condition for a mincut graph of a graph \( G \) to be isomorphic to \( G \).

**Definition 3.1.** A graph \( G \) is maximally edge connected when \( \lambda = \delta \), where \( \lambda \) is the cardinality of the minimum edge-cut and \( \delta \) is the minimum vertex degree of \( G \).

**Definition 3.2.** A maximally edge-connected graph is super-\( \lambda \) if every minimum edge-cut set is trivial; that is, consists of the edges incident on a vertex of minimum degree.

**Proposition 3.1.** If \( G \) is \( r \)-regular and super-\( \lambda \), then \( X(G) \cong G \).

**Proof.** By definition the mincuts of \( G \) are exactly the edges incident on every vertex and since \( G \) is regular, these are the only mincuts. Hence, there is a one-to-one correspondence between the vertices of \( X(G) \) and the vertices of \( G \) and the adjacencies are preserved. \( \square \)

**Corollary 3.1.1.** Let \( K_n \) be the complete graph on \( n \) vertices, \( K_{n,n} \) the complete bipartite graph with equal vertex partitions and \( L(K_n) \) the line graph of the complete graph. If \( n > 2 \), then

1. \( X(K_n) \cong K_n \)
2. \( X(K_{n,n}) \cong K_{n,n} \)
3. \( X(L(K_n)) \cong L(K_n) \).

**Proof.** If \( n = 2 \), \( K_2 \) is a tree, \( K_{2,2} \cong C_4 \) and \( L(K_2) \cong K_1 \). If \( n > 2 \) all three graphs are clearly regular and super-\( \lambda \). \( \square \)

4. Every graph is a mincut graph

In this section we show that every graph is a mincut graph. We show this by constructing a graph from a given mincut graph.

Every graph is an intersection graph, see [3, 6]. We show in the following proposition that every graph is also the mincut graph of not just one graph, but a number of other, not necessarily isomorphic, graphs.

**Proposition 4.1.** Every graph \( G \) is the mincut graph of a family of graphs.

**Proof.** Let \( G \) be a connected graph with \( |V(G)| = n \) and \( \Delta(G) = k \). We will construct a supergraph \( H \) such that \( G \subseteq H \), \( \lambda(H) = k \), \( |X(H)| = n \) and \( X(H) \cong G \). Suppose \( G \) is not one of the mincut graphs identified in Sections [2] and [3] since the proposition already holds for these.

We start the construction with \( G \cup K_m \), the disjoint union of \( G \) and a complete graph \( K_m \), such that \( \lambda(K_m) > k \). To every \( v_i \in V(G) \) such that \( \text{deg}(v) < k \) we add edges between
G and \(K_m\) until \(\text{deg}(v) = k\). Without loss of generality, let \(v \in V(G)\) be a vertex such that \(\text{deg}(v) = l < k\). We connect \(v\) to \(k - l\) vertices of \(K_m\). Thus the \(k\) edges incident on \(v\) will be an edge-cut in \(H\). Once this has been done for all \(v \in V(G)\), we have \(G \subseteq H\), such that \(G\) is \(k\)-regular and the edges incident on the vertices of \(G\) are edge-cuts of \(H\). Since the edge-cuts in \(K_m\) have cardinality greater than \(k\), by our choice of \(m\), the edge-cuts incident on the vertices of \(G\) are mincuts in \(H\) and, since their adjacency is preserved in \(H\), their intersection graph will be \(G\).

If \(G\) is not connected we follow the same procedure, connecting each component of \(G\) to \(K_m\).

However, this construction depends on the structure of \(G\). It may be necessary to construct \(H\) such that \(\lambda(H) > \Delta(G)\). We note that the edges between \(G\) and \(K_m\) may form an edge cut of cardinality \(k\) in \(H\) which is not accounted for by the vertices of \(G\). To avoid this we need to increase \(\Delta(G)\) by adding an edge between all vertices \(v \in V(G)\) of degree \(k\) and \(K_m\) and starting the procedure again with appropriate choice of \(m\). See Example 4.2 for such a scenario.

We note that by a process of contraction of edges in \(K_m\) we can decrease the number of vertices in \(H\) and still have \(X(H) \cong G\), see Example 4.1.

**Example 4.1.** The graph \(K_{1,3}\) is the mincut graph of a family of graphs \(\mathcal{P}(H)\) such that \(G \subseteq H\). Label the vertices of \(K_{1,3}\) as shown in Figure 2. We note \(\Delta(K_{1,3}) = 3\), there is one \(v_i \in V(K_{1,3})\) such that \(\text{deg}(v_i) = 3\) and there are three vertices \(v_j \in V(K_{1,3})\) such that \(\text{deg}(v_j) < 3\).

![Figure 2](image)

**Figure 2.** Graphs \(H\) and \(H_1\), constructed such that \(X(H) \cong X(H_1) \cong K_{1,3}\).

**Example 4.2.** The graph \(G\) formed by the one point join of \(C_3\) and \(P_2\), sometimes called “the Paw”, is the mincut graph of a graph \(H\) such that \(\lambda(H) > \Delta(G)\). We note that in Figure 3, \(H'\) is formed as per the description in Proposition 4.1, but that it has an extra mincut with edges labeled \(\{e_1, e_2, e_3\}\). By increasing \(\Delta(G)\) and hence \(\lambda(H)\) by one, we have the mincuts exactly the edges incident with \(V(G)\).

![Figure 3](image)
Corollary 4.1.1. For any connected graph $G$ there is a non-isomorphic graph $H$ such that $X(G) \cong X(H)$.

Proposition 4.2. Let $G$ and $H$ be super-$\lambda$ graphs and $X$ a vertex-induced subgraph of $G$ and $H$ such that $V(X) = V_\delta(G)$ and $V(X) = V_\delta(H)$, where $V_\delta$ is the set of minimum degree vertices of a graph. Then $X$ is the mincut graph of both $G$ and $H$ and $X(G) \cong X(H)$.

Proof. The proof follows from the definition of a super-$\lambda$ graph and the construction in Proposition 4.1. □

Corollary 4.1.1 leads naturally to ask the question whether there are non-isomorphic graphs $G$ and $H$ such that $X(G) \cong H$ and $X(H) \cong G$. We will call such graphs mincut duals. We note that any graph such that $X(G) \cong G$ is mincut self-dual.

5. Mincut graphs and some graph operations.

In this section we link two graph operations, the cartesian product and vertex join, by the mincut graph.

The following proposition links the mincut of the cartesian product of $K_n$ and $K_2$ and the vertex join of the product.

Proposition 5.1. Let $K_n \square K_2$ be the cartesian product of $K_n$ and $K_2$ and $\widehat{K_n \square K_2}$ be the vertex join of $K_n \square K_2$. Then the mincut graph of $K_n \square K_2$ is

$$X(K_n \square K_2) \cong \widehat{K_n \square K_2}.$$ 

Proof. From Corollary 3.1.1, $\lambda(K_n) = n - 1$ and each mincut of $K_n$ is exactly the set of $n - 1$ edges incident on every vertex. Now, in the cartesian product, each vertex is joined to a corresponding vertex in a copy of $K_n$. Thus $\lambda(X(K_n \square K_2)) = n$ and the vertices of $X(K_n \square K_2)$ are the vertices corresponding to vertices of $K_n \square K_2$ with one additional vertex.
representing the mincut on the $n$ edges between the two copies of $K_n$. Since each of the mincuts in the two individual copies of $K_n$ contains an element from the mincut joining the two copies, the additional mincut intersects with all the other mincuts.

Without loss of generality, let $n = 4$, then $K_4 \square K_2$ is as shown in Figure 4.

![Figure 4](image)

**Figure 4.** $K_4 \square K_2$.

The edges incident on each of the vertices $A_1 \ldots A_4$ and $B_1 \ldots B_4$ are mincuts and preserve the adjacencies of $K_4 \square K_2$. In addition, the edge set $\{e_1, e_2, e_3, e_4\}$ is also a mincut and intersects with all eight the other mincuts, hence giving a vertex join of the product graph. \qed

6. Further discussion

In this section we conclude by introducing and exploring further topics and questions raised by the properties and characteristics of the mincut graph.

6.1. A mincut operator on graphs. Are there graph operations other than the cartesian product and vertex join linked by the mincut graph? If we consider the mincut to be an operator on graphs, similar to the treatment of line graphs in [9], we see the following emerge from Propositions 2.1 to 3.1 and Corollary 3.1.1.

1. $X(K_n) \cong K_n$ and hence repeated application leaves the graph unchanged. Similarly for any other graph $G$ such that $X(G) \cong G$.

2. $X(W_n) \cong C_{n-1}$, $X(C_{n-1}) \cong L(K_{n-1})$ and $X(L(K_{n-1})) \cong L(K_{n-1})$ implies $XX(W_n) \cong L(K_{n-1})$, and similarly for any further application of $X$.

3. In general, however, it would seem that the edge connectivity of a mincut graph should be less than the graph itself and hence, in most cases, repeated application of $X$ should at some stage yield a disconnected graph $G$ with $\lambda(G) = 0$ and hence $X(G)$ is the null graph with no vertices and no edges.

From the observations above, it seems possible to define an $X$-operator index, the number of times the operator can be applied successively before the mincut graph is null. Thus a
connected graph with a disconnected mincut graph would have an index of two, since the second application of $X$ would yield the null graph, whereas the index of a complete graph would be infinite. Are there links between this index and other connectivity measures?

6.2. The mincut-intersection number of a graph. In 1945 Szpilrajn-Marczewski proved that every graph is an intersection graph, [3]. Thus we can meaningfully define a graph invariant, the intersection number of a graph $G$, denoted $i(G)$, to be the minimum cardinality of a set $S$ such that $G$ is the intersection graph of a family of subsets of $S$, see [5, 6]. In [3] it is proved that $i(G) \leq \lfloor \frac{n^2}{4} \rfloor$, where $|V(G)| = n$.

In view of Proposition 4.1, is it then possible to define a graph invariant similar to $i(G)$ from the mincut graph, say the mincut intersection number, $i_X(G)$? That is, what is the smallest set $S$, such that there is a family of subsets that represents $G$ as a mincut graph of some graph $H$?

Various conditions can be placed on the intersection of subsets in order to have the vertices adjacent in $G$, such as requiring the $|S_i \cup S_j| \geq p$ for some integer $p \geq 1$, in which $G$ would be the $p$-intersection graph. In order to define $i_X(G)$ for any graph we would need to set $p = 1$, and place a regularity condition on the sizes of the subsets $S_i \in F$ and define the $r$-intersection number of a graph $G$.

**Definition 6.1.** Let $S$ be a set and $F = \{S_1, S_2, \ldots, S_i\}$ a family of subsets of $S$ such that $|S_i| = r$ for each of the $S_i \in F$. The $r$-intersection number of $G$, denoted $i_r(G)$, is the minimum cardinality of $S$ such that $G$ is the intersection graph of an $r$-regular family of subsets of $S$.

If we now add the condition that the $S_i \in F$ should be mincuts of a graph we get the mincut intersection number, $i_X(G)$. What are the further conditions on $F$ that this requirement implies?

Since the mincuts are determined by the edges of $G$ and $G$ is simple and connected, each subset should be the same size, that is $\Delta(G)$, and distinct. Furthermore it can have only a one element intersection with any other subset and this intersection is unique; that is, a subset can intersec with two or more other subsets, but the intersections should be disjoint.

We use the construction of $H$ such that $G \cong X(H)$ from Proposition 4.1 to explore this further. Let $G$ be the “Paw” from Example 4.2 and we determine the 3-intersection number with the condition that each subset of $S$ is distinct and has only one element unique intersections with other subsets. Then $F = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 7, 8\}\}$ is a $3$-regular representation of $G$ and $i_3(G) = 8$ is the smallest set $S$ that will give this representation. Clearly we just need to count the edges incident with the vertices of $G \subseteq H'$ in the example. However, $G$ is not a mincut graph of the supergraph constructed, as indicated in the
example. In order for $F$ to be a mincut representation of $G$ we need four 4-element subsets and a universal set of twelve elements corresponding to the graph $H$ in Figure 3.

Our construction of $H$ is such that $G \cong X(H)$ does not take into account any kind of minimising of the number of vertices or edges of $H$ or $G \subseteq H$. We should therefore be able to calculate an upper bound on $i_X(G)$ using the construction in Proposition 4.1.

We note that the number of edges of $G$ and the number of edges added to $G$ forms an upper bound for the cardinality of $S$ such that there is a family $F$ of subsets such that $F$ is a mincut-representation of $G$.

Let $G$ be a graph with $n$ vertices and $m$ edges. By construction, we increased the degrees of all vertices of $G$ to $\Delta(G)$. Therefore the number of edges added is $n\Delta(G) - \sum \deg(v)$, where $v \in V(G)$. Thus, counting the edges as elements of $S$ we have

$$i_X(G) \leq m + n\Delta(G) - \sum \deg(v).$$

But $\sum \deg(v) = 2m$ and hence we have

$$i_X(G) \leq n\Delta(G) - m.$$

As we saw in Example 4.2 it may be necessary to increase the maximum degree of $G$ by one and hence we have

$$i_X(G) \leq n(\Delta(G) + 1) - m.$$

Considering that the upper bound $i(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor = 4$ is significantly less than the upper bound $i_X(G) \leq 4(4) - 4 = 12$ for the Paw in Example 4.2 is it possible to decrease the upper bound on $i_X(G)$? A different construction of $H$ such that $X(H) \cong G$ that minimises the edges added would be a starting point, but a thorough examination of the characteristics of $F$ such that $F$ is a mincut-representation of $G$ is also needed.

6.3. Connectivity of $X(G)$. $X(G)$ will be connected if every mincut has non-empty intersection with at least one other mincut. What are the characteristics of a connected graph that will imply that its mincut graph is connected? Is it possible to find minimum bounds on $\kappa(G)$, $\lambda(G)$, and $\delta(G)$, (the vertex connectivity, edge connectivity and minimum degree values of $G$) that will guarantee a connected $X(G)$? Intuitively there should be some relationship between the vertices and the way the edges are distributed (some kind of “edge density” function?) and $\lambda(G)$ that will ensure connectivity of $X(G)$.

6.4. Duality and $X(G)$.

(1) Are there sufficient conditions for $X(G) \cong G$, other than that $G$ be regular and super-$\lambda$? What are the necessary conditions?
(2) Are there graphs other than regular and super-$\lambda$ graphs such that $H \cong X(G)$ and $G \cong X(H)$?

References

[1] L. Cai, D. Corneil, and A. Proskurowski. A generalization of line graphs: (X,Y)-intersection graphs. *Journal Graph Theory*, 21(3):267–287, 1996.

[2] D. Chartrand, L. Lesniak, and P. Zhang. *Graphs and Digraphs*. CRC Press, Boca Raton, 5th edition, 2011.

[3] P. Erdos, A. W. Goodman, and L. Posa. The representation of a graph by set intersections. *Can. J. Math.*, 18:106–112, 1966.

[4] M. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*. Elsevier, Amsterdam, 2nd edition, 2004.

[5] J. Gross and J. Yellen. *Handbook of Graph Theory*. CRC Press, Boca Raton, 2004.

[6] F. Harary. *Graph Theory*. Addison-Wesley, Reading, 1969.

[7] T. McKee and F. McMorris. *Topics in Intersection Graph Theory*. SIAM, Philadelphia, 1999.

[8] M. Pal. Intersection graphs: An introduction. *Annals of Pure and Applied Mathematics*, 4(1):43–91, 2013.

[9] A. Van Rooi and H. Wilf. The interchange graph of a finite graph. *Acta Mathematica Hungarica*, 16(3-4):263–269, 1965.