On effective solutions of the nonlinear Schrödinger equation

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Abstract. Cubic nonlinear Schrödinger type equation with specific initial-boundary conditions in the infinite domain is considered. The equation is reduced to an equivalent system of partial differential equations and studied in the case of solitary waves. The system is modified by introducing new functions, one of which belongs to the class of functions of negligible fifth order and vanishing at infinity exponentially. For this class of functions the system is reduced to a nonlinear elliptic equation which can be solved analytically, thereby allowing us to present nontrivial approximated solutions of nonlinear Schrödinger equation. These solutions describe a new class of symmetric solitary waves. Graphics of modulus of the corresponding wave function are constructed by using Maple.

1. Introduction
Nonlinear Schrödinger equation describes a lot of physical processes, such as electron plasmatic waves, electromagnetic ion cyclotron waves, waves in cosmic gases, non-linear optics phenomena etc. [1-2, 17-22]. As a result, this equation has been considered by numerous authors and from different points of view (see for example [3-17, 20-22]).

In this paper the nonlinear Schrödinger equation equation is considered in the infinite domain and with specific initial-boundary conditions. We study the case of solitary waves vanishing at infinity and with the modulus of the wave function of the class $0(Re^{-D})$, $D \geq 3$, where $D$ and $R$ are some definite constants. By introducing new variables the equation is modified for this class of waves and non-trivial solutions are derived analytically. Consequently, a new class of localized approximated solutions is obtained.

2. Statement of the problem
In the coordinate system $0xyz$ we consider the following problem.

Problem 1. In the domain $Q_T = R^3 \times \{0 < t < T\}$, find continuous function $\Psi$ satisfying the Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} + \Delta \Psi + \lambda |\Psi|^2 \Psi = 0,$$

and the conditions

$$\lim_{|x|+|y|+|z| \to \infty} \Psi = 0, \quad |\Psi|_{t=0} = r(x, y, z),$$
where $\Psi$ is a wave function, $\Psi = U + iV$, ($U, V$ are real continuous functions having second order derivatives), $\lambda$ is some parameter, and $r(x, y, z) \geq 0$ is a function also to be defined.

We will find the solutions of Equation (1) for which $|\Psi|$ is independent of $t$, vanishing at infinity exponentially, and $|\Psi|^5$ is negligible. This class describes some class of solitary waves. To achieve our goal, we will examine the auxiliary problem first. To this, let us consider the infinite domain

$$G_0 = \{x > 0, y > 0, z > 0\}$$

and the following problem.

**Problem 2.** In the domain $Q_0^T = G_0 \times \{0 < t < T\}$, find continuous function $\Psi$ with the modulus of the class $0(Re^{-D})$, $D \geq 3$, satisfying the equation (1) and the condition (2).

It is obvious that equation (1) is equivalent to the following system of partial differential equations

$$\frac{\partial U}{\partial t} = -\Delta V - \lambda V (U^2 + V^2), \quad \frac{\partial V}{\partial t} = \Delta U + \lambda U (U^2 + V^2). \quad (3)$$

Consequently, Problem 2 can be reduced to the following problem.

**Problem 3.** In the domain $Q_0^T = G_0 \times \{0 < t < T\}$, find continuous functions $U$ and $V$ of the class $0(Re^{-D})$, $D \geq 3$, having second order derivatives, satisfying the system (3) and the condition

$$|U|^2 + |V|^2 = r^2(x, y, z).$$

3. **Modification of the Schrodinger equation and its solution**

Let us introduce the notation

$$U = r \cos \varphi; \quad V = r \sin \varphi, \quad (4)$$

where $r, r \geq 0$, and $\varphi$ are unknown functions of the variables $x, y, z, t$. Taking into account (4), the system (3) becomes [7]

$$\frac{\partial r}{\partial t} \cos \varphi - r \sin \varphi \frac{\partial \varphi}{\partial t} = -\sin \varphi \Delta r - 2 \cos \varphi \left( \frac{\partial r}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial r}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{\partial r}{\partial z} \frac{\partial \varphi}{\partial z} \right) +$$

$$-r \left\{ - \sin \varphi \left( \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 + \left( \frac{\partial \varphi}{\partial z} \right)^2 \right) + \cos \varphi \Delta \varphi \right\} - \lambda r^3 \sin \varphi, \quad (5)$$

$$\frac{\partial r}{\partial t} \sin \varphi + r \cos \varphi \frac{\partial \varphi}{\partial t} = \cos \varphi \Delta r - 2 \sin \varphi \left( \frac{\partial r}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial r}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{\partial r}{\partial z} \frac{\partial \varphi}{\partial z} \right) +$$

$$-r \left\{ \cos \varphi \left( \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 + \left( \frac{\partial \varphi}{\partial z} \right)^2 \right) + \sin \varphi \Delta \varphi \right\} + \lambda r^3 \cos \varphi. \quad (6)$$

After simple transformations, Eqs. (5)-(6) yield [7]

$$\Delta r + \lambda r^3 = r \frac{\partial \varphi}{\partial t} + r \left\{ \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 + \left( \frac{\partial \varphi}{\partial z} \right)^2 \right\}, \quad (7)$$

$$\Delta \varphi = -\frac{\partial}{\partial t} \ln r - 2 \left\{ \frac{\partial}{\partial x} (\ln r) \frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial y} (\ln r) \frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial z} (\ln r) \frac{\partial \varphi}{\partial z} \right\} \quad (8)$$
Here we consider the case $\varphi = A_0 t + A_1$, where $A_0; A_1; A_0 > 0$; are some constants. From Eqs. (7)-(8) we get

$$\Delta r + \lambda r^3 - A_0 r = 0. \quad (9)$$

We seek the symmetric solutions of (9) in the form

$$r = R \sin \psi, \quad (10)$$

where $R$ is a constant and $\psi(x, y, z)$ is a sufficiently small unknown function such that $\psi^5$ is negligible (in one dimensional case the exact solution of (9) can be obviously given in terms of elliptic functions [9, 21-22]). By virtue of (10), Equation (9) then turns into a non-linear partial differential equation for $\psi(x, y, z)$, i.e.

$$\cos \psi \Delta \psi - \sin \psi \left\{ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 \right\} + \lambda R^2 \sin^3 \psi - A_0 \sin \psi = 0. \quad (11)$$

Taking into the account that $\psi^5$ is sufficiently small, and putting into (11) the approximated formulas $\sin \psi \approx \psi - \frac{\psi^3}{6}$, and $\cos \psi \approx 1 - \frac{\psi^2}{2} + \frac{\psi^4}{24}$, it then results

$$\left( 1 - \frac{\psi^2}{2} + \frac{\psi^4}{24} \right) \Delta \psi - \left( \psi - \frac{\psi^3}{6} \right) \left\{ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 \right\} +$$

$$+ \lambda R^2 \left( \psi - \frac{\psi^3}{6} \right)^3 - A_0 \left( \psi - \frac{\psi^3}{6} \right) = 0. \quad (12)$$

For the equation (12), we will solve the following problem.

**Problem 3.** In the domain $\mathcal{G}_0$ to find symmetric continuous function $\psi$ of the class $0(e^{-D})$, $D \geq 3$, having second order derivatives and satisfying the equation (12).

Let us consider the function

$$\psi = e^{-\alpha x - \beta y - \gamma z - D}, \quad (13)$$

where $D, \alpha, \beta, \gamma > 0$ are some constants satisfying the following conditions

$$\alpha^2 + \beta^2 + \gamma^2 = A_0, \quad \lambda R^2 = \frac{4}{3} A_0, \quad (14)$$

and $D$ is the constant chosen for the desired accuracy, in such a way that $e^{-5D}$ is negligible (for example for $D = 3$, $e^{-15} \approx 10^{-7}$). It is clear from (13) that $\psi$ belongs to the class $0(e^{-D}), D \geq 3$, and $|\psi| \leq e^{-D}$. Insertion of (13) into the left hand side of (12) yields

$$\left( 1 - \frac{\psi^2}{2} + \frac{\psi^4}{24} \right) \Delta \psi - \left( \psi - \frac{\psi^3}{6} \right) \left\{ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 \right\} +$$

$$+ \lambda R^2 \left( \psi - \frac{\psi^3}{6} \right)^3 - A_0 \left( \psi - \frac{\psi^3}{6} \right) = A_0 \left( -\frac{\psi^5}{72} + \frac{\psi^7}{27} - \frac{\psi^9}{162} \right).$$

Since $\psi^5$ is negligible we can ignore the rhs term in this equation and the function $\psi$ given by the formula (13) will solve equation (12). Hence we conclude that, if $D$ is chosen accordingly, the function (13) is an effective solution of the equation (12) with the accuracy

$$\left| \left( 1 - \frac{\psi^2}{2} + \frac{\psi^4}{24} \right) \Delta \psi - \left( \psi - \frac{\psi^3}{6} \right) \left\{ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 \right\} +$$

$$+ \lambda R^2 \left( \psi - \frac{\psi^3}{6} \right)^3 - A_0 \left( \psi - \frac{\psi^3}{6} \right) \right| \leq A_0 \frac{\psi^5}{72}.$$
Consequently, the solution of the Problem 2 will be given by
\[
\Psi = R e^{iA_0 t + iA_1} e^{-\alpha x - \beta y - \gamma z - D},
\]
where $R$ is some constant, $e^{-5D}$ is sufficiently small and the constants $\lambda, \alpha, \beta, \gamma > 0$ satisfy the conditions (14).

Having solved the Problem 2 it is now easy to find the solutions for the Problem 1. By direct verification, we recognize indeed that the symmetric solutions of the Problem 1 will be identified by means of the formula
\[
\Psi = R e^{iA_0 t + iA_1} e^{-\alpha |x| - \beta |y| - \gamma |z| - D},
\]
where the constants $A_0, \lambda, \alpha, \beta, \gamma > 0$ satisfy the conditions (14). Note that the first order derivatives of the function (15) have discontinuities at the planes $x = 0, y = 0,$ and $z = 0,$ but their squares and the second order derivatives are continuous at this planes and the equation (12) holds.
4. Conclusions
There exist solitary waves for which the wave function $\Psi$ is of the form (15) and its modulus satisfies the inequality

$$|\Psi| \leq R e^{-D},$$

where $R$ is the amplitude and $D$ is the constant for which $e^{-5D}$ is negligible. For given $R$ and $A_0$, the constant $\lambda$ is uniquely defined. The constants $R, A_0$ and any two of $\alpha, \beta, \gamma > 0$ can be chosen arbitrary. Representative graphics of $r = |\Psi|$ plotted using Maple are given in Figures 1-2.

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