A Novel Method for Basic Reproduction Ratio of a Size-Structured Population Model with Delay

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A novel method for basic reproduction ratio of a size-structured population model with delay

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Abstract

In this paper, we discuss a size-structured population model with a delay in birth process. The focus is to discuss the existence and uniqueness results along with basic reproduction ratio ($R_0$). The birth function $h$ is not necessarily a Ricker type of function but expression of $R_0$ obtained in this paper works for a more general function $h$. This analysis covers a larger class of models. The problem is converted into an abstract form and then theory of semigroup is used to obtain the results. The basic reproduction number is constructed as spectral radius of the next generation operator. The form for basic reproduction ratio is very general and can be applied to other problems as well. We also solve our model numerically to study the effect of delay on population distribution.

Keywords: Size structured PDE, Basic reproduction ratio, Delay model.
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1 Introduction

One of the most important terms in an infectious disease model is the basic reproduction ratio. It gives immediate information about the spread of the disease. For the compartmental models of infectious diseases, basic reproduction number ($R_0$) is defined as the number of new infections produced by an infected individual. Similarly, in size-structured population models, the basic reproduction number is defined as the expected number of offspring that an individual has throughout his whole life. Therefore, in a given environment a small population will start to establish if basic reproduction number is strictly greater than unity and will become extinct if $R_0$ is strictly less than unity.

Continuously structured population models have an extensive history. G.F. Webb [1] discussed various methods for the qualitative analysis of age and size-structured population models. Techniques of operator semigroup theory are used to study the dynamics of population under consideration. Inaba [2] considered age-structured population models in demography and epidemiology. One of the standard works on size-structured population models with spatial diffusion is done by Kato [3]; the techniques used are semigroup of operators and fixed point theorems. Size-structured population models with diffusion allow movement of individuals in a given spatial environment, which also includes spatial structure in a given population.
It is natural to consider the delay in a size-structured population model (see [4]) as it represents the maturation period. In fact, it is more general and realistic to consider a model with delay. For example, the characteristic equation corresponding to delay equation is transcendental which contains much difficulty for analysis. There is enough literature available on basic reproduction number in structured population models. Barril et al. [5] gave a practical approach to derive the expression of basic reproduction number in continuous size-structured population models. They also applied this approach to three population models with birth rate concentrated on a single point. For reaction diffusion epidemic model, Magal et al. [6] computed the basic reproduction number to study the influence of spatial heterogeneity and population mobility on the disease transmission. Barril et al. [7] showed an approach to derive basic reproduction number to structured population models with distributed birth rate and population models with birth rate concentrated on size zero individuals. Inaba [8] gave a new definition of basic reproduction number based on the generation evolution operator that can be applied to structured population models in any heterogeneous environment. There is enough literature available on size-structured population models (for more details see [10–14]).

Let us consider a population of individuals classified based on their size and their freedom to move in a spatial environment. The spatial movement is controlled by Laplacian, and the individuals are not allowed to move out of a given region \( \Omega \subset \mathbb{R}^n \) through the boundary \( \partial \Omega \). We use the following notations in this work.

\[
Q_T = (0, T) \times \Omega, Q_s = (0, s_f) \times \Omega, Q_{Ts} = (0, s_f) \times (0, T) \times \Omega, \partial Q_{Ts} = (0, s_f) \times (0, T) \times \partial \Omega.
\]

The evolution of population density \( u(s, t, x) \) of size \( s \) individuals at time \( t \) in position \( x \) with delay is given in the following model

\[
\frac{\partial u(s, t, x)}{\partial t} + \frac{\partial}{\partial s} (\gamma(s)u(s, t, x)) = k\Delta u(s, t, x) - \mu(s, t, x)u(s, t, x) + f(s, t, x) \text{ in } Q_{Ts}
\]

\[
\gamma(0)u(0, t, x) = \alpha h \left( \int_0^{s_f} \int_{-\tau}^{0} \beta(s, \theta, x)u(s, t + \theta, x)d\theta ds \right) \text{ in } Q_T
\]

\[
\frac{\partial u}{\partial \nu}(s, t, x) = 0 \text{ in } \partial Q_{Ts}, \quad u(s, \theta, x) = u_0(s, \theta, x), \quad \theta \in [-\tau, 0]
\]

where \( h \) is the birth function, and \( f \) is the inflow of \( s \) size individuals from outside the environment; \( s_f \) is the final size which individuals can attain. \( \gamma \) is the size dependent growth rate i.e. the change in size with respect to time. \( \mu \) and \( \beta \) are mortality and birth rate, respectively, of the size \( s \) individuals at time \( t \) in position \( x \). \( k \) is diffusion coefficient, \( \alpha \) is positive constant, and \( u_0 \) is the initial population distribution.

We will derive the expression of the basic reproduction number for our proposed model. To the best of our knowledge, there is no work available on computing basic reproduction number for size-structured population models with spatial diffusion term. We allow the movement of individuals in a given environment. Firstly, we converted our problem into an abstract framework, and then by applying the method of semigroups, we show the existence and uniqueness of a mild solution. In this work the basic reproduction number is defined as the spectral radius of the next generation operator. We obtain estimates on the basic reproduction ratio under some assumptions on birth function \( h \). These estimates show the dependence of \( R_0 \) on diffusion and other parameters. We apply this technique to two population models and compare the expression of \( R_0 \) derived in [4,17]. For one of the models, expression of \( R_0 \) computed from our method and method given in [4] matches. This makes sense because the reproduction ratio may not have a unique expression.
The article is divided in the following manner. Section 2 explains the procedure to derive the expression of basic reproduction number for size-structured population models with spatial diffusion. Existence and uniqueness of mild solution is also shown in the same section. We also show the positivity and boundedness of mild solution. In section 3, we apply this process to derive the expression of basic reproduction ratio to various models. We compared our approach with the approach given in [4] for the purpose of better understanding. Section 4 is devoted to numerical simulation. In the last section, we provide concluding remarks.

2 Main results

In this section, we prove the existence and uniqueness of our solution and also derive an expression for the basic reproduction number. We also show that under some assumptions, the mild solution is positive and bounded. Firstly, we give a setup in which we study our problem.

Let us consider the distribution of individuals at time \( t \geq 0 \) given by

\[
\frac{du}{dt} = B(u(t))u(t) - M(u(t))u(t), \quad u(0) = u_0 \in X - \text{Banach Lattice}. \tag{2.2}
\]

Linearizing around \( u^* = 0 \) gives

\[
\frac{du}{dt} = B(u(t))u(t) - M(u(t))u(t) \quad B := B(0), \quad M := M(0). \tag{2.3}
\]

Then \( BM^{-1} \) is interpreted as the next generation operator.

For structured population models with concentrated state at birth, consider

\[
\frac{du}{dt} = B_m u(t) - Mu(t), \quad u(0) = u_0 \in X, \quad k \in \mathbb{N}. \tag{2.4}
\]

Where \( -M \) is the generator of strongly continuous semigroup such that spectral bound \( s(-M) < 0 \).

\( B_m : D(B_m) \subset X \mapsto X \) are positive bounded operators, all of finite rank \( m < \infty \) such that \( B_m M^{-1} \) is bounded and \( B_m - M \) is the generator of strongly continuous semigroup.

Let us assume that \( m = 1 \), \( B_m u = (Lu)\phi_m \) for some positive \( \phi_m \in X \). \( \phi_m \) represents the distribution of offsprings, and \( L \) is a positive linear functional. Moreover, we assume that \( LM^{-1}\phi_m \) converges to \( L\psi_\infty \).

Then

\[
R_0 = \lim_{m \to \infty} R_{0,m} = \lim_{m \to \infty} \rho(B_m M^{-1}) = \lim_{m \to \infty} \rho((LM^{-1})\phi_m) = \lim_{m \to \infty} LM^{-1}\phi_m = L\psi_\infty \tag{2.5}
\]

where we assume that \( \text{range}(B_m) = \text{span}\{\phi_m\} \).

2.1 Existence and uniqueness

Let \( X \) be a Banach space with norm \( ||| \) , and let \( A \) be realization of Laplacian in \( L^q(\Omega) \), \( q \) in \((1, \infty)\) with the boundary conditions of Neumann type i.e.

\[
D(A) = \left\{ v \in W^{2,q}(\Omega) \mid \frac{\partial v}{\partial n} = 0 \text{ a.e on } \partial\Omega \right\}
\]

\[
A\psi = k\Delta\psi \quad \text{for} \quad \psi \in D(A).
\]
It is well known that $A$ will generate an analytic semigroup $S(t), t \geq 0$, and there exists a positive constant $C$ and $\omega \in \mathbb{R}$, such that $\|S(t)\psi\|_{L^q(\Omega)} \leq Ce^{\omega t}\|\psi\|_{L^q(\Omega)}$ for $\psi \in L^q(\Omega)$.

In operator form our model reduces to

$$\frac{\partial u}{\partial t}(s,t) = -\frac{\partial}{\partial s}(\gamma(s)u(s,t)) + [A - \mu(s,t)]u(s,t) + f(s,t), \quad (s,t) \in \mathbb{R}_T \times [0, T]$$

$$\gamma(0)u(0, t) = \alpha h \left( \int_0^{s_f} \int_{-\tau}^0 \beta(s,\theta)u(s,t+\theta)d\theta ds \right)$$

$$u(s,\theta) = u_0(s,\theta), \quad s \in [0, s_f], \quad \theta \in [-\tau, 0].$$

(2.6)

Now, in this section, instead of working with our original problem, we will work with problem (2.6). In problem (2.6), there is no spatial variable, so $u$ depends only on two variables $s$ and $t$.

**Theorem 2.1.** The problem (1.1) has a unique mild solution.

**Proof:** Let $\mathcal{X} = L^1(0, s_f)$, and define

$$(\mathcal{A}_1\psi)(s) = [A - \mu(s)]\psi(s) - (\gamma(s)\psi(s))' + f(s)$$

with

$$D(\mathcal{A}_1) = W^{2,1}(0, s_f), \quad \overline{D(\mathcal{A}_1)} = \mathcal{X} = L^1(0, s_f).$$

Also, let us define $\mathcal{B} : D(\mathcal{A}_1) \mapsto \mathbb{C}$ by

$$\mathcal{B}\psi = \gamma(0)\psi(0).$$

Now, we will handle the birth operator, for this purpose on $\mathcal{X}_1 = L^1([-\tau, 0], \mathcal{X})$, let us define

$$\Psi(\phi) = h \left( \int_0^{s_f} \int_{-\tau}^0 \beta(s,t)\phi(s,t)d\theta ds \right).$$

(2.8)

Then our problem will reduce into the following Cauchy problem

$$\frac{du}{dt} = \mathcal{A}_1u(t)$$

$$\mathcal{B}u(t) = \alpha \Psi(u^t)$$

$$u^0 = u_0$$

(2.9)

where $u^t(x) = u(t + x), \quad x \in [-\tau, 0]$. Now, let us define the operator

$$(\mathcal{B}_1\xi)(\theta) = \frac{d}{d\theta}\xi(\theta)$$

with $D(\mathcal{B}_1) = W^{1,1}([-\tau, 0], \mathcal{X})$ and $\overline{D(\mathcal{B}_1)} = \mathcal{X}_1 = L^1([-\tau, 0], \mathcal{X})$. Let us define the boundary operator by

$$\Phi \psi = \psi(0)$$

and $\mathcal{A}_2 : D(\mathcal{A}_2) \subset \mathcal{X} \times \mathcal{X}_1 \mapsto \mathcal{X} \times \mathcal{X}_1$ by

$$\begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{B}_1 \end{bmatrix}$$
with
\[ D(A_2) = \left\{ \left( \begin{array}{c} \varphi \\ \psi \end{array} \right) \in D(A_1) \times D(B_1) : \Phi \psi = \varphi, B_\psi = \alpha \Psi(\psi) \right\}. \]

Then our problem further reduced into
\[
\begin{align*}
\frac{dU}{dt} &= A_2 U(t) \\
U(0) &= (u(0), u^0)
\end{align*}
\] (2.10)
where \( U(t) = (u(t), u'(-)) \). Define
\[ A = \begin{bmatrix} 0 & -B & 0 & \alpha \Psi \\ 0 & A_1 & 0 & 0 \\ 0 & I & 0 & -\Phi \\ 0 & 0 & 0 & B_1 \end{bmatrix} \]
with \( D(A) = \{0\} \times D(A_1) \times \{0\} \times D(B_1) \). Then on \( \mathbb{R} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X}_1 \), our problem reduced into
\[
\begin{align*}
\frac{dv}{dt} &= Av(t) \\
v(0) &= (0, u(0), 0, u^0)^T, \quad v(t) = (0, u(t), 0, u'(-))^T
\end{align*}
\] (2.11)
It is clear that \( A \) generates a \( C_0 \) semigroup; hence, problem (2.11) has a unique mild solution. For more details on semigroup, we refer to Pazy [15].

Since the population considered in our model is a biological population, it does not make sense to have a negative population density. So, our task is to prove that the solution of system (1.1) is positive under some conditions. Let \( X_+ \) be a positive cone in ordered Banach space \( X \). The order relation defined in \( X_+ \) is
\[ x_1 \leq x_2 \text{ if } x_1 - x_2 \in X_+ \text{ for } x_1, x_2 \in X. \]
It is known from the Pazy [15] that \( A \) generates an analytic semigroup \( \{S(t) \mid t \geq 0\} \) in Banach space \( X \). Hence, it is positive in \( X_+ \) as it can be written in the form \( e^{tA} \).

**Lemma 2.2.** Suppose that the birth function \( h \) and birth rate \( \beta \) takes only positive values, inflow of s-size individuals and initial population density are positive. Then the mild solution of system (1.1) is positive.

**Proof.** We know that system (1.1) can be reduced into the form
\[
\begin{align*}
\frac{dv}{dt} &= Av(t) \\
v(0) &= (0, u(0), 0, u^0)^T = v_0, \quad v(t) = (0, u(t), 0, u'(-))^T
\end{align*}
\] (2.12)
Using semigroup of operators, (see [3][15]) the solution can be written as
\[ v(t) = T(t)v_0 \] (2.13)
where \( \{T(t) \mid t \geq 0\} \) is the semigroup generated by operator \( A \). Now, under the hypothesis of lemma and from the form of matrix \( A \), it is clear that the solution is positive.

Since, it does not make sense to have an infinite population, we prove the boundedness of the solution to our model.
Lemma 2.3. Let $\mu, \beta \in L^\infty(Q_T s)$, $f \in L^1(Q_T s)$ and $u_0 \in L^1(Q_s)$. Also, let us suppose that $h$ is a bounded function, then solution of system (1.1) is bounded.

Proof. Since Laplacian is a bounded operator on $D(A)$; therefore, $A_1$ is a bounded operator under the hypothesis of theorem. Similarly, $B, \Psi, \Phi$ are bounded operators. It is clear that upper bound of semigroup generated by $A$ depends on vital rates (mortality rate, birth rate, and growth rate) and initial population distribution. So, if $h$ is a bounded function, then the mild solution will be bounded. \qed

2.2 Basic reproduction number

In this subsection, we derive the expression for the basic reproduction number. The idea is to construct the basic reproduction number as the spectral radius of the next generation operator. Since in our model, the birth rate is concentrated at age zero, the next generation operator is defined as the limit of the next generation operator of size-structured population models with distributed states at the birth.

Theorem 2.4. Assume that $h$ is a monotonically increasing function then population will extinct if

$$h \left[ \tau \beta_\infty \int_0^{s_f} e^{\int_0^r \mu(t) dt} \left( \frac{M + \frac{1}{k}}{(4\pi \frac{cs}{k})^\frac{3}{2}} \right) \right] < \frac{1}{\alpha}$$

and if $h$ is a monotonically decreasing function, then the population will go to extinction if

$$h \left[ -\tau C \int_0^{s_f} \beta_\infty e^{\int_0^r \mu(t) dt} \left( M + \frac{1}{k} \right) \right] < \frac{1}{\alpha},$$

where $\beta_\infty$ is supremum of $\beta$, and $M, C$ are positive constants.

Proof. Let us define $M$ as the linear operator corresponding to non birth terms, that is,

$$\text{define } (Mu)(s, x) = -\frac{\partial}{\partial s} (\gamma(s)u(s, x)) + k\Delta u - \mu(s)u(s, x).$$

We handle the diffusion term by introducing a new function:

$$F_\tau(\phi) = \int_{-\tau}^0 \beta(s, \theta, x)\phi(s, \theta, x)d\theta.$$

Now, we define the birth operator:

$$B_m u = \alpha h \left( \int_0^{s_f} F_\tau(u_t(s, \theta, x))ds \right) .m1[0, \frac{1}{m}],$$

where $u_t(s, \theta, x) = u(s, t + \theta, x)$. Now, our task is to find $M^{-1}$ i.e. we need to solve

$$-\frac{\partial}{\partial s} (\gamma(s)u(s, x)) + k\Delta u - \mu(s)u(s, x) = v(s, x).$$

Let us assume that $\gamma$ is differentiable and positive, then

$$-(\gamma(s))'u(s, x) - \gamma(s)\frac{\partial}{\partial s} u(s, x) + k\Delta u - \mu(s)u(s, x) = v(s, x).$$
\[ \frac{\partial u}{\partial s} = \frac{k}{\gamma(s)} \Delta u - \left( \frac{(\gamma(s))'}{\gamma(s)} + \frac{\mu(s)}{\gamma(s)} \right) u(s, x) - \frac{v(s, x)}{\gamma(s)}. \]

For convenience, let us replace \( s \) by \( t \) and consider

\[ \frac{\partial u}{\partial t} = f_1(t) \Delta u + f_2(t) u(t, x) + f_3(t, x) \]

Let \( w(t, x) = e^{\int_0^t f_2(\tau)d\tau} u(t, x) \), then

\[ \frac{\partial w}{\partial t} = f_1(t) \Delta w + f_3(t, x) \]

Now, using the transformation \( \tau = \int_0^t f_1(z)dz \), we get

\[ \frac{\partial w}{\partial \tau} = \Delta w + F_3(\tau, x). \]

Fundamental solution will be given by

\[ w(\tau, x) = \int_{\mathbb{R}^n} \phi(x - y, \tau) u_0(y)dy + \int_0^\tau \int_{\mathbb{R}^n} \phi(x - y, \tau - z) F_3(y, z)dzdy, \quad (2.14) \]

where

\[ \phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0. \end{cases} \]

Therefore,

\[ u(\tau, x) = e^{-\int_0^\tau f_2(\tau)d\tau} \left( \int_{\mathbb{R}^n} \phi(x - y, \tau) u_0(y)dy + \int_0^\tau \int_{\mathbb{R}^n} \phi(x - y, \tau - z) F_3(y, z)dzdy \right) \quad (2.15) \]

Now,

\[ R_0 = \lim_{k \to \infty} \rho(B_m M^{-1}). \]

For a particular case, if we take \( f_1(z) = c \neq 0 \), then \( \tau = \frac{cs}{k} \) and

\[ u(s, x) = e^{\int_0^s \frac{cs}{k}\mu(\tau)d\tau} \left( \int_{\mathbb{R}^n} \phi\left(x - y, \frac{cs}{k}\right) u_0(y)dy + \frac{1}{c} \int_0^{\frac{cs}{k}} \int_{\mathbb{R}^n} \phi\left(x - y, \frac{cs}{k} - z\right) \frac{c}{k} v(y, z)dzdy \right). \quad (2.16) \]

Therefore,

\[ M^{-1} v(s, x) = e^{\int_0^s \frac{cs}{k}\mu(\tau)d\tau} \left( \int_{\mathbb{R}^n} \phi\left(x - y, \frac{cs}{k}\right) u_0(y)dy + \frac{1}{c} \int_0^{\frac{cs}{k}} \int_{\mathbb{R}^n} \phi\left(x - y, \frac{cs}{k} - z\right) \frac{c}{k} v(y, z)dzdy \right). \quad (2.17) \]

We can also write

\[ M^{-1} v(s, t, x) = e^{\int_0^s \frac{cs}{k}\mu(\tau)d\tau} \left( \int_{\mathbb{R}^n} \phi\left(x - y, \frac{cs}{k}\right) u_0(y)dy + \frac{1}{c} \int_0^{\frac{cs}{k}} \int_{\mathbb{R}^n} \phi\left(x - y, \frac{cs}{k} - z\right) \frac{c}{k} v(y, t, z)dzdy \right). \quad (2.18) \]
Now,
\[
(B_m M^{-1}v)(s, t, x) = \alpha h \left( \int_0^s F_\tau(M^{-1}v_t(s, \theta, x)) ds \right) . m \mathbb{I}_{[0,1]}.
\]  
(2.19)

This implies
\[
(B_m M^{-1}v)(s, t, x) = \alpha h \left( \int_0^s e^{f_\tau^0 \int_0^\tau \mu(\tau) d\tau} \left( \int_{\mathbb{R}^n} \phi(x - y, \frac{cs}{k}) u_0(y) dy + \frac{1}{c} \int_0^\frac{cs}{k} \int_{\mathbb{R}^n} \phi(x - y, \frac{cs}{k} - z) \frac{c}{k} v_t(y, \theta, z) dz dy \right) ds \right) . m \mathbb{I}_{[0,1]}.
\]  
(2.20)

So, here we have an explicit form of operator $B_m M^{-1}$ and the basic reproduction number will be spectral radius of this operator. If there is no delay term, then
\[
(B_m M^{-1}v)(s, x) = \alpha h \left[ \int_0^s \beta(s, x) e^{f_\tau^0 \int_0^\tau \mu(\tau) d\tau} \left( \int_{\mathbb{R}^n} \phi(x - y, \frac{cs}{k}) u_0(y) dy \\
+ \frac{1}{c} \int_0^\frac{cs}{k} \int_{\mathbb{R}^n} \phi(x - y, \frac{cs}{k} - z) \frac{c}{k} v(y, z) dz dy \right) ds \right] . m \mathbb{I}_{[0,1]}.
\]  
(2.21)

Now, our aim is to obtain an estimate on $R_0 = \rho(B_m M^{-1})$. We know that
\[
R_0 \leq \|B_m M^{-1}\|_X = \sup \left\{ \frac{\|B_m M^{-1} \phi\|_X}{\|\phi\|_X} : \phi \in X \setminus \{0\} \right\}.
\]  
(2.22)

We know that
\[
\phi(x, t) \leq \frac{1}{(4\pi t)^{\frac{n}{2}}},
\]

Now, choose $v$ such that
\[
\int_0^{\frac{cs}{k}} \int_{\mathbb{R}^n} |v(y, z)| dz dy = 1.
\]  
(2.23)

If we take any density function with a multiple of $\frac{k}{cs}$, then it will satisfy (2.23). For example, we can take
\[
v(y, z) = \begin{cases} 
\frac{k}{cs(4\pi t)^{\frac{n}{2}}} e^{-|x|^2} & y \in \mathbb{R}^n \\
0 & y \in \mathbb{R}^n.
\end{cases}
\]

So, assumption (2.23) makes sense.
Figure 1: Figure 1 depicts the plot of the Ricker type birth function for fixed value of parameters.

In Fig. 1, we have considered the plot of $h(x) = xe^{-\frac{x}{10}}$, which is increasing up to a fixed value of $x$, and then starts decreasing. A similar behavior will be obtained for $h(x) = xe^{-\alpha x}$ with $\alpha > 0$. The function $h$ is called the Ricker’s type birth function, and originally this function was introduced for population of fish. The idea to introduce this function was to describe the process of cannibalism of adult fish on lava during the season of reproduction. For $\alpha = 0$, we will get a linear birth function which is monotonically increasing.

So for the first case, assume that $h$ is monotonically increasing. Then

$$R_0 \leq \alpha h \left[ \int_0^{s_f} \beta(s, x)e^{-\tau} \frac{\tilde{\mu} (\tau)}{M + \frac{1}{k}} \right],$$

where $M$ depends on $u_0$. Let us assume that supremum of $\beta$ is $\beta_\infty$. Then

$$R_0 \leq \alpha h \left[ \beta_\infty \int_0^{s_f} e^{-\tau} \frac{\tilde{\mu} (\tau)}{M + \frac{1}{k}} \right].$$

So, here it is clear that the diffusion term is playing an important role on the bound of basic reproduction number. The bound of basic reproduction number depends on the birth rate, mortality rate, and the diffusivity coefficient. Now, if we assume that $h$ is monotonically decreasing, then for $\phi(x, t) \geq -C$, where $C$ is a positive constant, we have

$$R_0 \leq \alpha h \left[ -C \int_0^{s_f} \beta(s, x)e^{-\tau} \frac{\tilde{\mu} (\tau)}{M + \frac{1}{k}} \right].$$

Again using the upper bound $\beta$, we have

$$R_0 \leq \alpha h \left[ -C \int_0^{s_f} \beta_\infty e^{-\tau} \frac{\tilde{\mu} (\tau)}{M + \frac{1}{k}} \right].$$
So, in both cases, we have estimates on $R_0$. There may be cases where $h$ is neither monotonically increasing nor decreasing, but in this work those cases are not considered. Because we don’t have an explicit form of basic reproduction number, these estimates are useful which show us the dependence of $R_0$ on various parameters. We already know that in this model, the basic reproduction number is defined as the expected number of offspring an individual has throughout their lifetime. If $h$ is monotonically increasing, then $R_0 < 1$ if

$$\alpha h \left[ \beta_\infty \int_0^{s_f} e^{\int_0^s \xi \mu(\tau) d\tau} \left( \frac{M + \frac{1}{k}}{\left(4\pi \frac{c a}{k} \right)^{\frac{3}{2}}} \right) \right] < 1.$$ 

So, the population will go to extinction if

$$h \left[ \beta_\infty \int_0^{s_f} e^{\int_0^s \xi \mu(\tau) d\tau} \left( \frac{M + \frac{1}{k}}{\left(4\pi \frac{c a}{k} \right)^{\frac{3}{2}}} \right) \right] < \frac{1}{\alpha}.$$ 

If $h$ is monotonically decreasing, then the population will go to extinction if

$$h \left[ -C \int_0^{s_f} e^{\int_0^s \xi \mu(\tau) d\tau} \left( \frac{M + \frac{1}{k}}{\left(4\pi \frac{c a}{k} \right)^{\frac{3}{2}}} \right) \right] < \frac{1}{\alpha}.$$ 

Some explicit forms of $h$ can be taken to get a sufficient condition for the extinction of a given population. Now, if we include a delay term in our model, then in similar manner, we have the following estimates

$$h \left[ \tau \beta_\infty \int_0^{s_f} e^{\int_0^s \xi \mu(\tau) d\tau} \left( \frac{M + \frac{1}{k}}{\left(4\pi \frac{c a}{k} \right)^{\frac{3}{2}}} \right) \right] < \frac{1}{\alpha},$$

where $h$ is a monotonically increasing function. However, if $h$ is monotonically decreasing, then population will go to extinction if

$$h \left[ -\tau C \int_0^{s_f} e^{\int_0^s \xi \mu(\tau) d\tau} \left( \frac{M + \frac{1}{k}}{\left(4\pi \frac{c a}{k} \right)^{\frac{3}{2}}} \right) \right] < \frac{1}{\alpha}.$$ 

**Remark 2.5.** Our results in this section are valid only if either $h$ is monotonically increasing or decreasing, that is, a monotonic function. The expression of $R_0$ is valid even for a non monotonic function. Also, similar estimates can be obtained for the case of the proliferation of a population.

### 3 Examples

In this section, we provide a few examples and compute the basic reproduction ratio using the method described in this paper. Moreover, we compute the expression using the approach given in [4] for comparison and better understanding.

#### 3.1 Example-1

Consider the following model
\[
\begin{align*}
\frac{\partial u(s, t)}{\partial t} + \frac{\partial u(s, t)}{\partial s} &= k \frac{\partial^2 u(s, t)}{\partial s^2} - \mu u(s, t) \\
u(0, t) &= \int_0^\infty (\tilde{\beta} \mu) u(s, t) ds \\
u(s, 0) &= u_0(s)
\end{align*}
\] (3.24)

We compute the \( R_0 \) using our method and the method given by Yan and Fu in [13]. Let us now consider the characteristic equation

\[
k \rho^2 \lambda - \rho \lambda - (\lambda + \mu) = 0
\]

\[
\Rightarrow \rho_+ = 1 \pm \sqrt{1 + 4k(\lambda + \mu)}
\]

Also,

\[
\chi = \int_0^\infty \tilde{b} \mu e^{\rho_0 s} ds = -\frac{\tilde{\beta} \mu}{\rho_0}
\]

\[
\Rightarrow R_0 = \frac{2\chi}{1 + \sqrt{1 + 4k\mu}} = -\frac{2\tilde{\beta} \mu}{\rho_0 (1 + \sqrt{1 + 4k\mu})} = \frac{\tilde{\beta}}{2k}
\]

So, by using the approach given in [4], we get

\[
R_0 = \frac{\tilde{\beta}}{2k}. \quad \text{(3.25)}
\]

By using our approach, we get

\[
R_0 = \tilde{\beta}. \quad \text{(3.26)}
\]

We can see the extra factor \( \frac{1}{2k} \) in the first expression.

### 3.2 Example-2

Let us consider another model

\[
\begin{align*}
\frac{\partial u(s, t)}{\partial t} + \frac{\partial u(s, t)}{\partial s} &= -\mu(s) u(s, t) \\
u(0, t) &= \int_0^\infty \beta(s) u(s, t) ds \\
u(s, 0) &= u_0(s)
\end{align*}
\] (3.27)

Now, using the approach given in [4], we have

\[
\rho_\lambda = -(\lambda + \mu)
\]

\[
\chi = \int_0^\infty \beta(s) e^{-\mu s} ds.
\]

Therefore,

\[
R_0 = \frac{2\chi}{1 + 1} = \chi = \int_0^\infty \beta(s) e^{-\mu s} ds.
\]
Let us use our approach to calculate the basic reproduction number. Choose \( \phi_m(x) = m \mathbb{1}_{[0, \frac{1}{m}]} \) and

\[
(M\phi)(s) = -\phi'(s) - \mu(s)\phi(s)
\]

with domain \( D_M = \{ \phi \in L^1(0, \infty) : \phi(0) = 0 \} \). Also, let us define the birth operator by

\[
B_m\phi(s) = \int_0^\infty \beta(s)\phi(s)ds.m\mathbb{1}_{[0, \frac{1}{m}]}.
\]

It is known that \(-M\) is an infinitesimal generator of positive semigroup if \( \mu \) is bounded and takes only positive values (see [17]). Now, to find \( R_{0,m} \), we need to study the problem

\[
B_m\varphi - \lambda M \varphi = \xi, \quad \text{for } \varphi, \xi \in L^1(0, \infty).
\]

Solving it and taking the limit \( m \to \infty \), we get

\[
R_0 = \frac{2\chi}{1+1} = \chi = \int_0^\infty \beta(s)e^{-\mu s}ds.
\]

In this case, we see that the \( R_0 \) obtained from both methods are the same.

4 Numerical Simulation

In this section, we discuss some numerical results in order to achieve a better understanding of the distribution of the population. Forward in time and central in space finite difference scheme is applied to solve our proposed model numerically. Uniform mesh is taken for size variable \( s \), time \( t \) and spatial variable \( x \). The birth function \( h \) is taken linear function. We assume that zero size and \( s \)-size individuals are not coming from outside the environment. The growth rate and the mortality rate are taken, respectively as \( V(s) = \frac{1}{s+10}, \frac{1}{100+s} \). The birth rate is taken

\[
\beta(s,t,x) = \begin{cases} 
0 & 0 \leq s \leq 25 \\
\frac{7(s-30)\exp\left(-\frac{(s-30)}{36.3^3}\right)}{36.3^3} & 25 < s \leq 100.
\end{cases}
\]

The initial population density is taken as

\[
u_0(s,x) = \exp\left(\frac{(s-70)^2}{20} - (x-4.4)^2\right).
\]

The first two plots, Figure 2a, 2b are for the non-delayed case, that is for \( \tau = 0 \). It is clear that the peak becomes small as time passes and the population starts to disperse.
Figure 2: Size-space profile of population density at the fixed time $t = 0$ and $t = 0.8$. At $t = 1$, peak is attained at lower than at $t = 0$.

Figure 3: Dynamics of population density at the fixed spatial position $x = 1$ and $x = 2$.

Figure 3a and Figure 3b show the spread of population in a particular spatial region. The population distribution is random at a particular spatial region.
In Figure 4a, 4b we introduced the delay in our model. Starting from conception, an individual of size $s$ reproduces after a time lag of $\tau$. We proved the boundedness and positivity of a mild solution which is also clear from the plots given in Figure 2 - Figure 4. These numerical results also show the importance of incorporating size-structure in population models. The largest size individuals grow the slowest, and the delay shifts the peak to individuals with different size.

5 Discussion

$R_0$ can be considered as the threshold parameter for the extinction of a size-structured population. $R_0$ is a dimensionless quantity and not a rate. Computing the basic reproduction ratio for an ordinary differential equation model is well established and unified. However, there are different methods to compute the basic reproduction ratio for an age-structured population model. Interestingly, these values differ to some extent. This paper discusses an efficient method to compute the basic reproduction ratio. We assume the individuals are free to move in a particular spatial region, and the movement is controlled by Laplacian. The model considered here is of a general kind, and it contains the diffusion term. Comparison with another method is also done in order to get a better understanding. We have performed numerical simulations for delayed as well as non-delayed cases. We can see the high population density for size in a particular region and spatial variable. It is evident that the numerical simulations support our analytical findings.

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