Abstract

In this work for the first time we enumerate unlabelled maps on orientable genus $g$ surfaces with respect to all homeomorphisms, including both orientation-preserving and orientation-reversing. We show that in the latter case as an intermediate step one has to enumerate rooted maps of a special kind (quotient maps) on orientable and non-orientable surfaces possibly having a boundary and a certain number of branch points. In this work we develop a special technique for enumerating such maps.

Keywords: maps on surfaces; orbifold; unlabelled enumeration
Enumeration of Unsensed Orientable Maps on Surfaces of a Given Genus

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Introduction

By a topological map M on a surface we will mean an embedding of a connected graph $G$, loops and multiple edges allowed, into a compact 2-dimensional surface $S$, such that $G$ is as a subset of $S$, and its complement $S \setminus G$ is homeomorphic to a set of topological polygons. These polygons are the faces of the map $M$ which also has some amount $v$ of vertices (points on the surface $S$) and $n$ of edges (nonintersecting curves on the surface that have no common points other than the vertices of the graph). A map is rooted if one of its semi-edges is distinguished and one of two possible local orientations of the surface in the neighborhood of this semi-edge is chosen. The vertex this semi-edge is incident to will often be referred to as the root vertex.

William Tutte noted (see, for example [1]) that a map with a distinguished root semi-edge always has a trivial symmetry group. In the series of his “census” papers he derived formulas for enumerating rooted planar maps with $n$ edges [1], rooted 3-regular maps [2], rooted eulerian maps [3], and some other classes maps on the sphere. The first work devoted to the enumeration of rooted maps on surfaces of an arbitrary genus $g > 0$ was the paper [4] of Walsh and Lehman. Using Tutte’s approach for enumerating planar maps, the authors derived a recurrence relation for the numbers of rooted maps and calculated the first terms of the corresponding sequences. In addition, they provided an explicit expression for the number of one-face maps with $n$ edges on a surface of genus $g$, as well as a formula for the number of such maps with a prescribed list of vertex degrees.

The next step in enumerating rooted maps on surfaces was done by Bender and Canfield. In [5] they derived a system of equations for generation functions that enumerates rooted maps on orientable and non-orientable surfaces, and found the asymptotics of the corresponding sequences. In [6] D. Arques used the method of [5] to obtain closed-form formulas for the number of rooted toroidal maps with $n$ edges and for the number of rooted toroidal maps with $v$ vertices and $f$ faces. In [7] Bender, Canfield and Robinson independently from Arques enumerated rooted maps on the torus and the projective plane, derived some explicit expressions for the corresponding generation functions and analyzed the asymptotic behavior of their coefficients. In [8] a generation function was obtained for the number of rooted maps of genera 2 and 3 with $n$ edges. Some recent results regarding enumeration of rooted maps on surfaces of higher genera can be found in [9] and [10].
Two topological maps $M_1$ and $M_2$ on some surface are said to be equivalent if there exists a homeomorphism $h$ of the surface into itself that transforms the edges, vertices and faces of one map into the corresponding elements of the other map. By a sensed (unsensed) map we mean an equivalence class of maps where the equivalence relation is given by sense-preserving (sense-preserving or sense-reversing) map homeomorphisms.

A general technique for counting sensed maps on orientable surfaces was developed in the papers [11, 12] and applied to enumeration of arbitrary maps [12], regular maps [13], hypermaps [14] and one-face regular maps [15]. This technique reduces the enumerating problem for sensed maps to counting so called quotient maps on orbifolds, maps on quotients of a surface under a finite group of automorphisms. Consider as an example a map on a torus symmetric with respect to a rotation by $90^\circ$ (see Figure 1(a)). It is convenient to depict such map using the representation of the torus as a square with its opposite sides identified pairwise (Figure 1(b)). The rotation by $90^\circ$ is the simplest example of a homeomorphism with the period $L = 4$ which preserves the surface orientation. This rotation splits the set of the torus’ points into two subsets, an infinite set of points in the general position and a finite set of singular points (Figure 2(a)). Points in the general position are those that lie on an orbit of length $L = 4$. Singular points are the remaining ones, and they necessary lie on an orbit of smaller length. In our example there are four singular points: $x$, $z$, $y_1$ and $y_2$. The former two of them are fixed, and the latter two are transformed into each other by the rotation by $90^\circ$. Identifying points that lie on the same orbit of this action, we obtain a sphere (Figure 2(b)). The branch points (Figure 2(b), points $x, y, z$) on this sphere are the images of torus’ singular points. The above-described construction can be thought of as a 4-fold branched covering of the sphere $O$ from the topological point of view. This transformation maps the symmetric map on the torus onto a so-called quotient map on the sphere (see Figure 2(b)), and the problem of enumerating toroidal maps possessing a certain symmetry is reduced to the problem of enumerating rooted quotient maps on the sphere with additional branch points, which is called an orbifold.

Now consider a map $M$ on the torus depicted on the Figure 3(a). This map is transformed into itself under the action of the reflection with respect to the axis $i$, which is the simplest example of a periodic orientation-reversing homeomorphism. The corresponding quotient map $\mathfrak{M}$ on an orbifold
$O$ is actually a quotient map on the annulus (Figure 3(b)). It can be seen that the boundary of an orbifold may contain vertices (vertices $x_1$, $x_2$ and $x_3$ on Figure 3(b)) and/or edges (edges $e_3$ and $e_4$ on Figure 3(b)). In addition, the edges $e_1$ and $e_2$ on the torus that cross the axis of symmetry correspond to so-called halfedges of the quotient map $\mathcal{M}$ on the annulus. Consequently, taking periodic orientation-reversing homeomorphisms into account makes it necessary to enumerate quotient maps on surfaces with a boundary, and in more complex cases on orientable and non-orientable surfaces with or without a boundary and with possible branch points.

Note that several different homeomorphisms may yield the same orbifold. For example, the sphere with three branch points depicted on Figure 2(b) corresponds to rotations of the torus by $90^\circ$ and $270^\circ$. A technique for counting different orientation-preserving homeomorphisms that correspond to the same orbifold was developed by Mednykh and Nedela in [12]. In that paper the problem was reduced to determining the numbers of order-preserving epimorphisms from fundamental groups of orbifolds to cyclic groups. The case of orientation-reversing homeomorphisms was analyzed in the paper [16]. In the present article we extend these results and obtain the formulas for the numbers of order-and-orientation-preserving epimorphisms from fundamental groups of orbifolds to cyclic groups with sign structure. Then, using Tutte’s approach [17], we obtain a system of recurrence relations for counting quotient maps on orientable and non-orientable surfaces with a boundary that arise as orbifolds when considering symmetries that reverse surface orientation. These results allow us to solve the problem of enumerating maps on orientable surfaces of a given genus $g$ with respect to all homeomorphisms, including orientation-reversing. To the best of our knowledge there are no published analytic results on the enumeration of genus $g$ maps with respect to all homeomorphisms.

1 Combinatorial and algebraic descriptions of maps

Along with a topological definition of a map we will also need a combinatorial one. Since we will have to deal with both orientable and non-orientable surfaces, we will use the approach described in [18]. Namely by a combinatorial unsensed map we mean a quadruple $(F; l, r, t)$, where $(l, r, t)$ is a transitive subgroup of the symmetric group $S_F$, and $l^2 = r^2 = t^2 = (tl)^2 = 1$. The elements of $F$ are called flags, the respective orbits of $(r, t)$, $(t, l)$ and $(r, l)$ are vertices, edges and faces.

Such maps also admit an algebraic description [19]. Namely, consider the group

$$\Delta = \Delta(\infty, \infty, 2) = \langle \lambda, \rho, \tau \mid \lambda^2 = \rho^2 = \tau^2 = (\lambda \cdot \tau)^2 = 1 \rangle.$$  

For a given map $(F; l, r, t)$ the mapping $\lambda \to l$, $\rho \to r$ and $\tau \to t$ extends to an epimorphism $\Phi$ of the group $\Delta$ to the group $(l, r, t)$, acting of $F$ according to the rule $z \circ x = \Phi(z)[x]$, $z \in \Delta$, $x \in F$. The
stabilizer \( K := \text{St}_\Delta(x) \) of an element \( x \), \( K \leq \Delta \), has an index equal to \( |F| \). Conversely, any subgroup \( K \leq \Delta \) of a finite index \( |F| = [\Delta : K] \) defines a rooted unsensed map \( K := (F, x_0; l, r, t) \), assigning to each left conjugacy class \( xK \), \( x \in \Delta \) some element of the set \( F \) with the action \( r(xK) := (\rho \circ x)K \), \( l(xK) := (\tau \circ x)K \), \( t(xK) := (\lambda \circ x)K \), \( x_0 = K \). Torsion free subgroups correspond to maps having all edges complete, that is, containing 4 flags. Finally, isomorphism classes of unsensed maps on \( n \) flags (so called unrooted maps) are in \( 1-1 \) correspondence with conjugacy classes of subgroups of index \( n \).

By a group with sign structure we denote a pair \((G, \omega)\), where \( G \) is a group and \( \omega \) is a homomorphism from \( G \) to \((\{+1, -1\}, \times)\). Another way to define a group with sign structure is to take a pair \((G, G^+)\) where \( G^+ \) is a subgroup of \( G \) that defines positive elements. For the group \( \Delta \) as the group \( \Delta^+ \) we will use the subgroup \((\rho \tau, \tau \lambda)\) of index 2. Note that subgroups \( K^+ \) of index \( n \) in \( \Delta^+ \) define sensed maps with \( n \) darts.

In 2008 Mednykh [20] obtained the following important formula for counting conjugacy classes of subgroups of a given index.

**Theorem 1.1.** Let \( \Gamma \) be a finitely generated group, \( \mathcal{P} \) be some set of subgroups of \( \Gamma \) closed under conjugation. Denote by \( \text{Epi}_\mathcal{P}(K, \mathbb{Z}_l) \) the number of epimorphisms from a subgroup \( K \) of \( \Gamma \) into \( \mathbb{Z}_l \) with kernel in \( \mathcal{P} \). Then the number \( c_n^\mathcal{P}(\Gamma) \) of conjugacy classes of index \( n \) subgroups in \( \mathcal{P} \) is equal to

\[
c_n^\mathcal{P}(\Gamma) = \frac{1}{n} \sum_{\mid l \mid n} \sum_{\mid l \mid m = n} \frac{\text{Epi}_\mathcal{P}(K, \mathbb{Z}_l)}{\text{Epi}_\mathcal{P}(\Gamma, \mathbb{Z}_l)}.
\]

Theorem 1.1 used in conjunction with the above-mentioned correspondence between conjugacy classes of subgroups of index \( n \) and unrooted maps allows to obtain an explicit formula for enumerating sensed and unsensed maps. Namely, in the articles [14], [12] it is shown how to apply theorem 1.1 to derive the following formula for counting sensed maps on genus \( g \) surfaces:

\[
\tilde{\nu}_g(n) = \frac{1}{n} \sum_{\mid l \mid n} \sum_{\mid l \mid m = n} \nu_O(m) \cdot \text{Epi}_\omega(\pi_1(O), \mathbb{Z}_l).
\]  

(1)

Here \( \nu_O(m) \) is the number of rooted quotient maps on the orbifold \( O \) with \( m \) darts, corresponding to maps on the surface \( X_g \) of genus \( g \), \( O \) runs through all orientation preserving cyclic orbifolds \( \text{Orb}(X_g/\mathbb{Z}_l) \) of \( X \) with period \( l \), and \( \text{Epi}_\omega(\pi_1(O), \mathbb{Z}_l) \) is the number of order preserving epimorphisms from fundamental group \( \pi_1(O) \) of the orbifold \( O \) onto the cyclic group \( \mathbb{Z}_l \).

In [16] it was shown how to generalize these results to enumerate unsensed maps. We will write \( K^- \) to denote a subgroup of \( \Delta \) which is not a subgroup of \( \Delta^+ \). Then a result analogous to 1.1, but for the group \((\Delta, \Delta^+\)) with a sign structure, is the formula

\[
c_n^\mathcal{P}(\Delta) = \frac{1}{2n} \sum_{\mid l \mid n} \sum_{\mid l \mid m = n} \frac{\text{Epi}_\mathcal{P}(K, \mathbb{Z}_l)}{\text{Epi}_\mathcal{P}(\Delta, \mathbb{Z}_l)} + \frac{1}{2n} \sum_{\mid l \mid n} \sum_{\mid l \mid m = n} \frac{\text{Epi}_\mathcal{P}(K^-, \mathbb{Z}_l)}{\text{Epi}_\mathcal{P}(\Delta, \mathbb{Z}_l)}.
\]  

(2)

Here we assume that the group \( \mathbb{Z}_2l \) is endowed with a sign structure \((\mathbb{Z}_2l, \mathbb{Z}_l)\) that defines the sign based on element parity.

The formula (2) used together with the correspondence between conjugacy classes of subgroups of index \( n \) and unrooted maps with \( n \) flags allows us to obtain a result analogous to (1) for unsensed
maps on a surface $X$ of a given genus $g$. In order to do that, as the subset $\mathcal{P}$ of subgroups of index $n$ in $\Delta$ we will take torsion free subgroups corresponding to maps $M$ on a surface $X_g$ of genus $g$. Considerations analogous to those used to prove [21] Theorem 3.3 allow to derive from (2) the following expression for the number $\tilde{\nu}_g(n)$ of unrooted maps with $n$ edges:

$$\tilde{\nu}_g(n) = \frac{\nu_g(n)}{2} + \frac{1}{2n} \sum_{\nu \leq n} \sum_{O \in \text{Orb}^-(X_g/\mathbb{Z}_2)} \nu_O(m) \cdot \text{Epi}^+(\pi_1(O), \mathbb{Z}_2). \tag{3}$$

Here $\text{Epi}^+(\pi_1(O), \mathbb{Z}_2)$ is the number of order-and-orientation-preserving epimorphisms from the fundamental group $\pi_1(O)$ of the orbifold $O$ which is endowed with the sign structure defined by some homomorphism $\sigma$ into $(\mathbb{Z}_2, \mathbb{Z}_4)$, $\text{Orb}^-(X_g/\mathbb{Z}_2)$ is the set cyclic orbifolds of $X$ with period $2l$ arising from orientation-reversing homomorphisms, and $\nu_O(m)$ is the number of quotient maps with $2m$ flags on the orbifold $O$.

From [21] it follows that for an orientable surface $X_g$ without a boundary such an orbifold $O$ can have $r$ branch points $x_1, \ldots, x_r$ of branch indices $m_1, \ldots, m_r$ which are divisors of $n$, and be either an orientable surface with $h > 0$ boundary components or a non-orientable surface with $h \geq 0$ boundary components. Orientable orbifold has $g$ handles, and non-orientable orbifold has $g$ cross-caps. The fundamental group $\pi_1(O)$ with sign structure for an orientable orbifold $O$ is generated by the elements (see [21] Proposition 2)

$$x_1, \ldots, x_r, c_1, \ldots, c_h, e_1, \ldots, e_h, a_1, b_1, \ldots, a_g, b_g,$$

subject to relations

$$\prod_{i=1}^{r} x_i^{h} \prod_{j=1}^{g} e_j \prod_{k=1}^{g} [a_k, b_k] = 1, \quad x_i^{m_i} = 1, \quad c_j^2 = 1, \quad [c_j, e_j] = 1, \tag{4}$$

and is endowed with a sign structure defined by a homomorphism $\sigma : \pi_1(O) \to (\{+1, -1\}, \times)$, such that

$$\sigma(x_i) = \sigma(e_j) = \sigma(a_k) = \sigma(b_k) = +1, \quad \sigma(c_j) = -1. \tag{5}$$

For a non-orientable orbifold the list of generators has the form

$$x_1, \ldots, x_r, c_1, \ldots, c_h, e_1, \ldots, e_h, a_1, \ldots, a_g,$$

the relations are

$$\prod_{i=1}^{r} x_i^{h} \prod_{j=1}^{g} e_j \prod_{k=1}^{g} a_k^2 = 1, \quad x_i^{m_i} = 1, \quad c_j^2 = 1, \quad [c_j, e_j] = 1, \tag{6}$$

and the sign structure is defined by $\sigma$ as

$$\sigma(x_i) = \sigma(e_j) = +1, \quad \sigma(c_j) = \sigma(a_k) = -1. \tag{7}$$

Finally, the parameters of an orbifold $O$ and the genus $g$ of the original surface $X_g$ are related by the Riemann–Hurwitz formula

$$2g - 2 = 2l \left[ \alpha g - 2 + h + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) \right], \tag{8}$$

where $\alpha = 2$ if $O$ is orientable and $\alpha = 1$ if $O$ is non-orientable.
2 Enumeration of epimorphisms

In order to enumerate unrooted maps by the formula \(3\) we should solve three sub-problems: find the set \(\text{Orb}^-(X_g/\mathbb{Z}_{2l})\) of suitable cyclic orbifolds for a given orientable surface \(X_g\). determine the number \(\text{Epi}_o^+(\pi_1(O), \mathbb{Z}_{2l})\) of order- and orientation-preserving epimorphisms from the fundamental group \(\pi_1(O)\) with a sign structure into \((\mathbb{Z}_{2l}, \mathbb{Z}_d)\), and also find the number of quotient maps \(\nu_O(m)\) on these orbifolds. In this section we will solve the first two problems.

The first problem can be solved with the Riemann–Hurwitz formula \(8\) and with some additional constraints on the number \(l\) that appears in the formula. For \(g \geq 2\) it will be sufficient to use the following upper bound for this parameter \((21, \text{Corollary 2})\):

\[
2l \leq 4g + 4, \quad \text{g even;} \quad 2l \leq 4g - 4, \quad \text{g odd.}
\]

For a sphere and a torus the parameter \(2l\) can be bounded by the number \(2n\) of darts of a map \(M\) on the surface. Having this upper bound on \(l\), we can iterate over all suitable values of \(g, h, r\) and \(m_i\) and hence obtain a finite list of signatures of possible orbifolds which will contain some excess. Then for each such signature of an orbifold \(O\) we should calculate the number \(\text{Epi}_o^+(\pi_1(O), \mathbb{Z}_{2l})\) of order-and-orientation-preserving epimorphisms from \(\pi_1(O)\) to \(\mathbb{Z}_{2l}\) and retain only those that have a non-zero number of such epimorphisms.

As shown in \(22\) (see also \(16\)), the numbers \(\text{Epi}_o^+(\pi_1(O), \mathbb{Z}_{2l})\) can be expressed using inclusion-exclusion principle through the numbers \(\text{Hom}_o^+(\pi_1(O), \mathbb{Z}_{2l})\) of sign-preserving homomorphisms from the group \(\pi_1(O)\) to the group \((\mathbb{Z}_{2d}, \mathbb{Z}_d)\):

\[
\text{Epi}_o^+(\pi_1(O), \mathbb{Z}_{2l}) = \sum_{d|l, \ l/d \text{ odd}} \mu(l/d) \cdot \text{Hom}_o^+(\pi_1(O), \mathbb{Z}_{2d}).
\]

The problem of determining the numbers \(\text{Hom}_o^+(\pi_1(O), \mathbb{Z}_{2l})\) in its turn can be solved much easier than the analogous problem for epimorphisms.

We should consider three different cases for an orbifold \(O\): the case of a non-orientable surface without a boundary, the case of an orientable surface with \(h > 0\) boundary components, and the case of a non-orientable surface with \(h\) boundary components. We begin with the first case. The fundamental group \(\pi_1(O)\) of a non-orientable surface without a boundary has the following form:

\[
\prod_{i=1}^{r} x_i \prod_{k=1}^g a_k^2 = 1, \quad x_i^{m_i} = 1; \quad \sigma(x_i) = +1, \quad \sigma(a_k) = -1.
\]

Under a homomorphism \(\psi: \pi_1(O) \to \mathbb{Z}_{2d}\) that preserves the orders \(m_i\) of branch points, the first two equalities are transformed into relations

\[
\sum_{i=1}^{r} \psi(x_i) + 2 \sum_{k=1}^{g} \psi(a_k) \equiv 0 \pmod{2d}, \quad \gcd(\psi(x_i), 2d) = \frac{2d}{m_i}.
\]

Since the homomorphism \(\psi\) also preserves the sign, equalities \(\sigma(x_i) = +1\) and \(\sigma(a_k) = -1\) rewritten for the group \(\mathbb{Z}_{2d}\) are transformed into conditions

\[
z_i := \psi(x_i) \text{ even,} \quad \psi(a_k) \text{ odd.}
\]
Each $z_i$ must be even, so the equality $\gcd(z_i, 2d) = 2d/m_i$ can be rewritten for $z'_i = z_i/2$ and simplified as

$$\gcd(z'_i, d) = \frac{d}{m_i}. \quad (9)$$

The number of different values of $z'_i$ that satisfy it, is equal to $\varphi(m_i)$, where $\varphi(n)$ is the Euler’s function. It remains to satisfy the equality

$$2 \sum_{k=1}^{g} \psi(a_k) \equiv 2z' \pmod{2d},$$

where $z' := -\sum_{i=1}^{r} z'_i$ for arbitrary $z'$. The latter equation, in its turn, can be rewritten as

$$\sum_{k=1}^{g} \psi(a_k) \equiv s \pmod{2d}, \quad (10)$$

where $s \in \{z', z' + d\}$. All $\psi(a_k)$ are odd, so the sum in the left-hand side has a fixed parity equal to the parity of $g$.

Further considerations depend on the parity of $d$. If $d$ is odd, the numbers $z'$ and $z' + d$ have different parity. Consequently, exactly one of them has the same parity as $g$. Then, choosing the first $g - 1$ numbers $\psi(a_k)$ arbitrarily, we uniquely identify the last number $\psi(a_g)$ from (10). It follows that the equation (10) has $d^{g-1}$ solutions for any $z'$, and

$$\text{Hom}^+_{\mathbb{O}}(\pi_1(O), \mathbb{Z}_{2d}) = d^{g-1} \prod_{i=1}^{r} \varphi(m_i), \quad d \text{ odd}. \quad (11)$$

Since all $m_i$ divide $d$,

$$m := \text{lcm}(m_1, m_2, \ldots, m_r) \mid d.$$

Consequently, for an odd $l$

$$\text{Epi}^+_{\mathbb{O}}(G, \mathbb{Z}_{2l}) = \prod_{i=1}^{r} \varphi(m_i) \sum_{m|d|l} \mu(l/d) \cdot d^{g-1} = \prod_{i=1}^{r} \varphi(m_i) \sum_{d'|d} \mu\left(\frac{l}{md'}\right) \cdot m^{g-1} \cdot (d')^{g-1},$$

so for the number of order-and-orientation-preserving epimorphisms we have the formula

$$\text{Epi}^+_{\mathbb{O}}(\pi_1(O), \mathbb{Z}_{2l}) = m^{g-1} \cdot J_{g-1}\left(\frac{l}{m}\right) \cdot \prod_{i=1}^{r} \varphi(m_i), \quad l \text{ odd}, \quad (11)$$

where $J_k(n)$ is the Jordan’s totient function.

For an even $d$ the numbers $z'$ and $z' + d$ have the same parity. Consequently, if they have the same parity as $g$, we have

$$\text{Hom}^+_{\mathbb{O}}(\pi_1(O), \mathbb{Z}_{2d}) = 2d^{g-1} \prod_{i=1}^{r} \varphi(m_i). \quad (12)$$

Otherwise, the number of such homomorphisms is equal to 0.

It turns out that the parity of $z'$ and $z' + d$ is determined by the set of numbers $m_i$. Indeed, from the formula (9) it follows that for an even $d$ the parities of $z'_i$ and $d/m_i$ coincide. So the parity of
z’ and z’ + d is determined by the parity of the sum \( \sum_{i=1}^{r} \frac{d}{m_i} \), and the number of homomorphisms is defined by (12) if
\[
\sum_{i=1}^{r} \frac{d}{m_i} \equiv q \pmod{2}.
\]
The number of epimorphisms for an even \( l \) is equal to
\[
\text{Epi}^+(\pi_1(O), \mathbb{Z}_{2l}) = \prod_{i=1}^{r} \varphi(m_i) \sum_{d|l} \mu(l/d) \cdot 2 \cdot d^{q-1}, \quad l \text{ — even.}
\]
As opposed to the case of an odd \( l \), the condition of \( l/d \) being odd is now satisfied not for all \( d \). Rewrite \( l \) as \( 2^q \cdot k' \) for an odd \( k' \). The fraction \( l/d \) is odd if and only if \( d = 2^q \cdot d' \) and \( d' \mid k' \). Assume that these conditions do hold. Then \( l/m' \), where
\[
m' := \text{lcm}(2^q, m_1, m_2, \ldots, m_r),
\]
is an odd number, and the number of epimorphisms is equal to
\[
\text{Epi}^+(\pi_1(O), \mathbb{Z}_{2l}) = \prod_{i=1}^{r} \varphi(m_i) \sum_{d'|m'} \mu(l/d') \cdot 2 \cdot (m')^{q-1} \cdot (d')^{q-1} = 2 \cdot (m')^{q-1} \cdot J_{g-1}\left(\frac{l}{m'}\right) \cdot \prod_{i=1}^{r} \varphi(m_i),
\]
where \( l \) is even.

Now assume that the orbifold \( O \) is an orientable surface of genus \( g \) with \( h > 0 \) boundary components. Under a homomorphism \( \psi: \pi_1(O) \rightarrow \mathbb{Z}_{2d} \) the first two equalities in (4) are transformed into
\[
\sum_{i=1}^{r} \psi(x_i) + \sum_{j=1}^{h} \psi(e_j) \equiv 0 \pmod{2d}, \quad \gcd(\psi(x_i), 2d) = \frac{2d}{m_i}, \quad (15)
\]
the condition \( [c_j, e_j] = 1 \) is satisfied in \( \mathbb{Z}_{2d} \) trivially, and the equality \( c_j^2 = 1 \) can be rewritten as \( \psi(c_j) = d \) where \( d \) is odd. Then from the first equality in (15) it follows that all \( \psi(x_i) \) and also \( \psi(e_j), j = 1, \ldots, h - 1 \), can be chosen arbitrarily. Then if the left hand side of the equation is equal to 0, \( \psi(e_h) \) is uniquely defined. Consequently,
\[
\text{Hom}^+(\pi_1(O), \mathbb{Z}_{2d}) = \prod_{i=1}^{r} \varphi(m_i) \cdot d^{2g+h-1}, \quad d \text{ odd,}
\]
and hence
\[
\text{Epi}^+(\pi_1(O), \mathbb{Z}_{2l}) = m^{2g+h-1} \cdot J_{2g+h-1}\left(\frac{l}{m}\right) \cdot \prod_{i=1}^{r} \varphi(m_i), \quad l \text{ odd.} \quad (16)
\]
For a non-orientable surface with \( g \) crosscaps and \( h \) boundary components, the first relation in (6) under a homomorphism \( \psi: \pi_1(O) \rightarrow \mathbb{Z}_{2d} \) gets transformed into
\[
\sum_{i=1}^{r} \psi(x_i) + \sum_{j=1}^{h} \psi(e_j) + 2 \sum_{k=1}^{g} \psi(a_k) \equiv 0 \pmod{2d}.
\]
Table 1: Orientation-reversing orbifolds of genus $g$ surfaces

As in the previous case, $\psi(x_i)$ and $\psi(a_k)$, as well as the first $h - 1$ numbers $\psi(e_j)$ can be chosen arbitrarily. It follows that the number of epimorphisms can also be expressed as

$$Epi_+^+(\pi_1(O), \mathbb{Z}_{2l}) = m^{g + h - 1} \cdot J_{g+h-1}(\frac{l}{m}) \cdot \prod_{i=1}^{r} \varphi(m_i), \quad l \text{ odd.} \quad (17)$$

It’s easy to see that the formulas (11), (16) and (17) have the same form. Consequently, they all can be unified and written as

$$Epi_+^+(\pi_1(O), \mathbb{Z}_{2l}) = m^{l-\chi} \cdot J_{1-\chi}(\frac{l}{m}) \cdot \prod_{i=1}^{r} \varphi(m_i), \quad l \text{ odd,} \quad (18)$$

where $\chi$ is the Euler’s characteristics of the orbifold $O$.
Using the results obtained in this section one can generate a list of orbifolds and the corresponding numbers of order-and-orientation-preserving epimorphisms for a given genus $g$ of the orientable surface $X$. In the table we provide the corresponding lists for surfaces of genera 2, 3, 4, 5. In addition, the authors have implemented a program which can extend this list up to a given genus $g$. This program can be found at https://github.com/krasko/o_r_orbifolds.

3 Quotient maps on bordered orbifolds

In this section we turn to the problem of determining the numbers $\nu_O(n)$ of quotient maps on an orbifold $O$. Relatively to maps on closed surfaces, quotient maps may possess a number of peculiarities [16]. Namely, a boundary component of an orbifold may contain vertices (vertices $x_1$, $x_2$ and $x_3$ on Figure 4) and/or edges (edge $e_1$ on Figure 4) of the map. In addition, some edges may end on the boundary (edges $e_5$, $e_6$ on Figure 4); they’ll be called halfedges. Next, some edges may end with branch points of index 2; they’ll be called semiedges. Normal edges ($e_2$, $e_3$, $e_4$ on Figure 4) will be called complete edges, and edges lying entirely on the boundary ($e_1$ on Figure 4) will be called boundary edges. Darts of boundary edges will be called boundary darts. When counting darts for a given quotient map, each semiedge, halfedge and boundary edge counts as 1 dart, and every complete edge edge counts as 2 darts.

![Figure 4](image)

Quotient maps have one more important property related to their faces. In the general case, each face must be a simply connected domain that intersects with not more than one boundary component by not more than one segment. This limitation is needed to ensure that on the map $M$ on the original orientable surface $X$ each face is topologically homeomorphic to a disc.

It will be convenient to consider the case of orientable orbifolds first. Next we will extend the solutions obtained on this step to solve the problem of enumerating quotient maps on non-orientable orbifolds.

3.1 Orientable orbifolds

The simplest example of a bordered orientable orbifold is a disc, a sphere with a hole in it. On this example we will demonstrate the basic principles of deriving recurrence relations determining the numbers of quotient maps on bordered orientable surfaces. We will use the approach introduced by Tutte [1] which is based on contracting the root edge.
The recurrence relations will be different depending on the location of the root dart. First we consider the case when the root dart is incident to a vertex on the boundary, but does not lie on the boundary itself. Without loss of generality, we will assume that the root dart is the leftmost dart incident to the root vertex. The number \(d_{n,k}^{(0)}\) of such quotient maps with \(n\) darts and the root vertex of degree \(k\) can be expressed as follows:

\[
d_{n,k}^{(0)} = d_{n-1,k-1}^{(0)} + \sum_{k'=0}^{n-k-1} d_{n-2,k-1+k'}^{(0)} + \sum_{n'=0}^{n-2} \sum_{k'=0}^{k-2} d_{n',k'}^{(0)} \cdot s_{n-n'-2,k-k'-2} + \\
+ \sum_{n'=0}^{n-2} \sum_{k'=0}^{n-k-1} d_{n-2,n',k+k''-1}^{(i_1)} \cdot d_{n',k'-k''}^{(i_2)}
\] (19)

![Figure 5](image)

Here \(d_{n,k}^{(1)}\) is the number of quotient maps on a disc with the root dart that belongs to a boundary edge, \(k\) other darts incident to the root which all lie in the interior of the disc, \(n\) darts in total; \(s_{n,k}\) is the number of maps with \(n\) darts and the root vertex of degree \(k\) on a sphere.

Indeed, the first summand in (19) corresponds to the case of the root dart being a halfedge (Figure 5(a)). Contracting this dart yields a map with the number of darts and the degree of the root vertex decreased by 1. The second summand in (19) corresponds to contracting a root edge that joins the root vertex \(x\) and an internal vertex \(y\) (Figure 5(b)). The number of darts is decreased by two in this case, and the new degree of the root vertex becomes equal to the sum of those of \(x\) and \(y\) minus two. In the third case the root edge is a loop (Figure 5(c)). Contracting this loop splits the map into two maps, one on a disc and one on a sphere. The double summation iterates over all possible degrees of the new root vertices and the distribution of darts among the two maps. Finally, the third summand corresponds to contracting an edge that connects the root vertex \(x\) with another vertex \(y\) on the boundary (Figure 5(d)). In the general case, the non-root vertex \(y\) may have (1) or not
have (0) a boundary edge on either side. Consequently, we have 4 different options described by the
indices i₁ and i₂. The index k'' enumerates darts incident to y that lie to the left of the root edge;
the index k' enumerates all darts incident to y except for possible boundary edges and the dart that
belongs to the root edge.

The number \( d_{n,k}^{(1)} \) of maps in which the root dart is the only boundary dart incident to x (see Figure 5(c)) can be counted as

\[
d_{n,k}^{(1)} = \sum_{k' = 0}^{n-k-2} \sum_{i = 0}^{1} d_{n-1,k+k'}^{(i)}.
\]

Indeed, contracting the root boundary edge \( \{x, y\} \) yields, depending on the number of boundary
edges incident to y, a map with \( n-1 \) darts with \( (d_{n-1,k+k'}^{(1)}) \) or without \( (d_{n-1,k+k'}^{(0)}) \) a boundary edge incident to the root. The index k' stands for the number of darts incident to y lying in the interior
of the disc.

Finally, the number \( d_{n,k}^{(2)} \) of rooted quotient maps that have two boundary darts incident to the root
vertex x is calculated by the formula

\[
d_{n,k}^{(2)} = s_{n-1,k} + \sum_{k' = 0}^{n-k-2} \sum_{i = 1}^{2} d_{n-1,k+k'}^{(i)}.
\]

Here, the summand \( s_{n-1,k} \) corresponds to the case when the root boundary edge is a loop (see Figure 5(f)): contracting this loops yields a map on a sphere with \( n-1 \) darts and the root vertex of degree k.

It remains to derive a formula for the numbers \( d_{n,k} \) of rooted quotient maps with the root vertex
lying in the interior of the disk. Considerations similar to those used for deriving (19) show that the
expression for \( d_{n,k} \) has the form:

\[
d_{n,k} = d_{n-1,k-1}^{(0)} + \sum_{k' = 0}^{n-k-1} d_{n-2,k+k'-1}^{(1)} + 2 \sum_{n' = 0}^{n-2} \sum_{k' = 0}^{k-2} d_{n',k'} \cdot s_{n-n'-2,k-k'-2} + \sum_{k' = 0}^{n-k-1} \sum_{i_{1} = 0}^{1} \sum_{i_{2} = 0}^{1} (k' + 1) \cdot d_{n-2,k+k'-1}^{(i_{1}+i_{2})}.
\]

The multiplier 2 in the third summand appears because the root edge which is a loop can be oriented
in two possible ways. The final summand corresponds to contracting an edge \( \{x, y\} \) where y lies on
the boundary. The multiplier \( k' + 1 \) denotes the number of non-boundary darts, incident to y. Since
we assumed that the root dart incident to the new root vertex will always be the leftmost dart, \( k' + 1 \)
different quotient maps will be reduced to the same map with the root vertex lying on the boundary.

A disc is the simplest example of an orientable surface with a boundary. In order to use the approach
of Tutte in the general case of a sphere with \( g \) handles and \( h \) boundary components, we will need to
overcome a number of technical difficulties. To conclude what parameters the recurrence relations
should depend on, next we describe some of them.

First assume that a map contains a loop that wraps \( h_1 \) boundary components and \( g_1 \) handles (fig.
6(a)). Contracting this loop yields two surfaces with maps on them. We have encountered a special
case of this situation for the disc: contracting a loop yields a new disc and a sphere. Consequently,
besides the numbers \( n \) of darts and the degree \( k \) of the root vertex, we also need to keep track of the
numbers \( h \) of boundary components and \( g \) of handles.
Figure 6

Now assume that the root edge is a loop wrapping a handle (Figure 7(b)). Contracting this loop splits the root vertex of degree $k$ into two vertices with the sum of degrees equal to $k - 2$. For example, contracting a torus along its meridian yields a sphere with two distinguished vertices. In the general case, multiple applications of this procedure may yield $m$ internal distinguished vertices and we have to track their degrees.

The third difficulty stems from the possibility of joining two boundary components or splitting a boundary component into two. The former case takes place when the root vertex connects two vertices on two different boundary components (Figure 6(c)). Contracting it merges these boundary components. In addition to the root vertex, a new distinguished vertex appears on the boundary, and we have to keep track of boundary edges incident to it. The latter case happens when the root edge wraps a handle, both vertices incident to it lie on the same boundary and they are distinct (Figure 6(d)). After contracting the root edge there appears a new distinguished vertex which lies on a separate boundary component. Consequently, we have to keep track of the list of distinguished vertices on each boundary component; each such vertex will be characterized by the presence or absence of boundary darts on each side.

The final difficulty is the possible presence of branch points on orbifolds. For surfaces without a boundary, branch points may fall into vertices, faces and dangling semi-edges of quotient maps. It is possible to get rid of branch points in dangling semi-edges using a combinatorial approach of double counting [12]. Using the Euler-Poincare formula one can then determine the total number of faces and vertices in a quotient map and reduce the problem of enumerating quotient maps to the problem of enumerating maps on surfaces without branch points.

As shown in [16], orbifolds with a boundary arising as quotients of orientable surfaces can’t have dangling semi-edges ending in branch points. However, the presence of a boundary makes it impossible to uniquely determine the number of faces and vertices which can coincide with branch points. Consequently, in this case we have to introduce one more parameter that would track the overall number of vertices and faces in the interior of the surface.

So, the recurrence relations for the numbers of quotient maps on orientable surfaces with $h$ boundary components and $g$ handles will also depend on the number $n$ of darts, the degree $k$ of the root vertex, the list of $m$ degrees of internal vertices, and also on the lists of distinguished vertices for each of $h$ boundary components together with the information about boundary darts for each such vertex. If
Next we will describe a schema for obtaining these recurrence relations. Conceptually, it will be sufficient to consider two recurrence relations differing in the position of the root vertex. The first relation corresponds to the case of the root vertex $x$ located in the interior of a surface. We have to consider five different situations:

(a) The root dart is a half edge, that is, it ends on a boundary (Figure 7(a));

(b) The root edge ends in an interior vertex $y$ different from $x$ (Figure 7(b));

(c) The root edge is a loop; contracting it splits the surface into two (Figure 7(c));

(d) The root edge is a loop; it wraps a handle so that contracting it decreases the surface genus (Figure 7(d));

(e) The root edge connects the root vertex $x$ with a vertex $y$ on a boundary (Figure 7(e)).

The second recurrence relation enumerates maps with the root vertex $x$ lying on a boundary (Figure 8). The following topologically distinct possibilities may take place:

(a) The root dart is a half edge ending on the boundary component that contains $x$ (Figure 8(a));

(b) The root edge ends in an interior vertex $y$ different from $x$ (Figure 8(b));

(c) The root edge is a loop; contracting it splits the surface into two (Figure 8(c));

(d) The root edge is a loop; it wraps a handle so that contracting it decreases the surface genus (Figure 8(d));

(e) The root edge connects the vertex $x$ with a different vertex $y$ on the same boundary component in a way that contracting the root edge $\{x, y\}$ splits the surface into two (Figure 8(e));
(f) The root edge connects the vertex $x$ with a different vertex $y$ on the same boundary component in a way that contracting the root edge \{x, y\} decreases the surface genus (Figure 8(f));

(g) The root edge connects vertices on two different boundary components (Figure 8(g));

(h) The root edge is a boundary edge (Figure 8(h)); it could be a loop or not.

3.2 Non-orientable orbifolds

A non-orientable surface is a sphere with $l + h$ boundary components, $l$ of which ($l > 0$) are glued with cross-caps. The number of different options for the position of the root edge is even bigger for non-orientable surfaces. As an example, consider maps on a M"obius band ($l = h = 1$) having the root vertex on the boundary and having no boundary edges incident to the root vertex (Figure 9; it will be convenient to use the representation of a M"obius band as an annulus with a cross-cap glued to its inner boundary). As before, we will assume that the root dart is the leftmost dart incident to the root vertex. The recurrence relation for the numbers $m_{n,k}^{(0)}$ of such maps with $n$ darts and the root of degree $k$ has the following form:

$$m_{n,k}^{(0)} = m_{n-1,k-1}^{(0)} + \sum_{k' = 0}^{n-k-1} m_{n-2,k-1+k'}^{(0)} + (k - 1) \cdot d_{n-2,k-2}^{(0)} + \sum_{k' = 0}^{n-1} \sum_{i_1 = 0}^{1} \sum_{i_2 = 0}^{1} d_{n-2,k-1+k'}^{(i_1,i_2)} +$$

$$+ \sum_{n' = 0}^{n-2} \sum_{k' = 0}^{k-2} (d_{n',k'}^{(0)} \cdot p_{n-n'-2,k-k'-2}^{(0)} + m_{n',k'}^{(0)} \cdot s_{n-n'-2,k-k'-2}) +$$

$$+ \sum_{n' = 0}^{n-2} \sum_{k' = 0}^{n-k-1} \sum_{k'' = 0}^{1} \sum_{i_1 = 0}^{1} \sum_{i_2 = 0}^{1} (m_{n-2-n',k+k''-1}^{(i_1)} \cdot d_{n',k''-k''}^{(i_2)} + d_{n-2-n',k+k''-1}^{(i_1)} \cdot m_{n',k''-k''}^{(i_2)}).$$
Indeed, the first two summands are analogous to the corresponding summands in (19) and describe the cases of the root dart being a half-edge and the root edge joining the root vertex with a vertex in the interior of the surface. The third summand in the right-hand side of (20) describes maps in which the root edge goes through the cross-cap (Figure 9(a)). In this case cutting along this edge, inverting one of the obtained parts and gluing these parts back yields a quotient map on a disc (Figure 9(b)). The multiplier \((d - 1)\) has the same combinatorial meaning as the analogous multiplier for maps on the projective plane (see [7]).

The fourth summand in the right-hand side of (20) describes the situation when the root edge goes through the cross-cap and joins the root vertex \(x\), incident to \(k\) internal darts, with a different vertex \(y\) lying on the boundary (Figure 9(c)). The vertex \(y\) can have or not a boundary edge both on its right \((i_1 = 1 \text{ or } i_1 = 0)\) and on its left side \((i_2 = 1 \text{ or } i_2 = 0)\). It can also have some amount \(k'\) of interior darts lying to the right of the root edge \(\{x, y\}\). So, for the map depicted on Figure 9(c), \(i_1 = 1, i_2 = 0\), and \(k' = 1\). Contracting the root edge yields a map on a disc with a new root vertex \(x'\) and an additional distinguished vertex \(y'\) on its boundary (Figure 9(d)). All darts incident to \(y\) and lying on the right of the root edge will be incident to the new root vertex \(x'\), whereas darts that lied to the left of it will be incident to the distinguished vertex \(y'\). Note that after this operation the vertex \(y'\) cannot have an incident boundary edge on its right side. The number of maps with such properties in the formula (20) is denoted by \(d_{n-2,k-1+k'}^{(i_1,\{0,i_2\})}\).

The next summand in (20) enumerates maps with the root edge being a loop that doesn’t go through a cross-cap. If the crosscap lies inside this loop, contracting the loop yields a map on a disc and a map on a projective plane (Figure 9(e)). If the loop does not enclose the crosscap, contracting it yields a sphere and a Möbius band. These cases correspond to the summands \(d_{n',k'}^{(0)} \cdot p_{n-2,k-k'-2}\) and \(m_{n',k'}^{(0)} \cdot s_{n-2,k-k'-2}\) int (20), where \(p_{n,k}\) is the number of projective plane maps with \(k\) darts incident to the root and \(n\) darts in total.
Finally, the last two summands in the right-hand side of (20) describe the situation where the root edge joins the root vertex \( x \) with another vertex \( y \) on the boundary and does not pass through the cross-cap (Figure 9(f)). Depending on the side of the root edge which the cross-cap lies in, we obtain one or another summand in (20). Here the numbers \( m_{n,k}^{(1)} \) enumerate maps on a Möbius band which are defined analogously to the maps on a disc enumerated by \( d_{n,k}^{(1)} \).

As the example of a Möbius band shows, for non-orientable surface there is also an additional possibility that the root edge would go through a crosscap (Figure 9(a) and 9(c)). Consequently, instead of five cases of contracting the root edge for the root vertex \( x \) in the interior and eight cases for the root vertex \( x \) on the boundary, for non-orientable surfaces we will have six and ten cases, correspondingly.

It’s also important to note that for the case of maps on non-orientable surfaces it is convenient to derive recurrence relations which enumerate maps regardless of orientability, that is, enumerate maps on orientable and non-orientable surfaces together \( [23] \). The number of maps on non-orientable surfaces can then be obtained by subtracting the number of maps on non-orientable surfaces from this result.

Rewriting all these results in a form of a single mathematical formula is impractical, as it would be extremely cumbersome. Instead of that, authors have implemented a program that embodies all the technical details of the above-described considerations. It can be used to calculate the numbers of quotient maps on orbifolds of a given signature and, provided a list of orbifolds, enumerate reflexible maps on surfaces of a given genus up to a certain number of edges. This program can be found at https://github.com/krasko/reflexible_maps.

**Conclusion**

The results presented in this article allow to enumerate reflexible maps on orientable surfaces of a given genus \( g \). Using these results together with the results of enumeration of sensed maps which can be found in \( [12] \), we have obtained the numbers of unsensed maps with \( n \) edges on genus \( g \) orientable surfaces. Note that in \( [24] \) these numbers were obtained by explicit generation of the corresponding maps for \( g \in [1, 5] \) and \( n \in [2, 11] \). The method described in this work allows to advance in unsensed maps enumeration further. In the tables \( [2, 4] \) we’ve included the results of computations for \( g \in [1, 10] \) and \( n \in [2, 20] \).

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| $n \backslash g$ | 1 | 2 | 3 | 4 |
|----------------|---|---|---|---|
| 2             | 1 | 0 | 0 | 0 |
| 3             | 6 | 0 | 0 | 0 |
| 4             | 40| 4 | 0 | 0 |
| 5             | 320| 76| 0 | 0 |
| 6             | 2946| 1395| 82| 0 |
| 7             | 29364| 24950| 4348| 0 |
| 8             | 309558| 427336| 160500| 7258 |
| 9             | 3365108| 6987100| 4695604| 688976 |
| 10            | 37246245| 109761827| 118353618| 37466297 |
| 11            | 416751008| 2675297588| 1512650776|  |
| 12            | 4696232371| 55758114082| 50355225387|  |
| 13            | 53186743416| 1091344752470| 1461269893538|  |
| 14            | 604690121555| 20318440463052| 38236565513725|  |
| 15            | 6896534910612| 363171011546201| 92255232644030|  |
| 16            | 78867385697513| 6275111078422480| 20847359639841664|  |
| 17            | 904046279771682| 10536965796443204| 44629072818232620|  |
| 18            | 10384916465797240| 172659041710724316| 912923622886478458|  |
| 19            | 11952206378612992| 2769670730854989616| 179639061789993180|  |
| 20            | 1378014272286250059| 43624633648672487876| 34183667064441659877|  |

Table 2: Unsensed genus $g$ maps

| $n \backslash g$ | 5 | 6 | 7 |
|----------------|---|---|---|
| 10            | 1491629| 0 | 0 |
| 11            | 195728778| 0 | 0 |
| 12            | 14019733828| 506855279| 0 |
| 13            | 72464638784| 8493074344| 0 |
| 14            | 3020987171570| 7601322881752| 25411839668 |
| 15            | 107925389643492| 48247532533252| 5214804981864 |
| 16            | 34231899372185491| 24347701836204379| 5634797561708358 |
| 17            | 988157793188200998| 103882081135250668| 42649733168178767 |
| 18            | 26412878913430197293| 3892847895365580016| 2538894014717359412 |
| 19            | 66213303216309300424| 131462363823845390096| 12656232339138264624 |
| 20            | 15719783014093104131694| 40749347642026348171659| 54940200059090328012148 |

Table 3: Unsensed genus $g$ maps

| $n \backslash g$ | 8 | 9 | 10 |
|----------------|---|---|---|
| 16            | 176377605783906| 0 | 0 |
| 17            | 4305845711817178| 0 | 0 |
| 18            | 54779398722953288| 162019808170348933| 0 |
| 19            | 48357317128920541590| 46037869248765236030| 0 |
| 20            | 33299663456795126129156| 6762460437287955976080| 19037558741923108550 |

Table 4: Unsensed genus $g$ maps

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