Subelliptic Resolvent Estimates for Non-self-adjoint Semiclassical Schrödinger Operators

Ben Bellis
Department of Mathematics
UCLA
Los Angeles, CA, 90095
bbellis@math.ucla.edu

Abstract

In this paper we prove a subelliptic resolvent estimate for a broad class of semiclassical non-self-adjoint Schrödinger operators with complex potentials when the spectral parameter is in a parabolic neighborhood of the imaginary axis.

1 Introduction

Non-self-adjoint Schrödinger operators can appear in a variety of settings. These settings can range physical problems to purely mathematical ones. Such examples include the study of the Ginzburg-Landau equation in superconductivity [1], [4], the Orr-Sommerfeld operator in fluid dynamics [11], [12], the theory of scattering resonances [14], or non-self-adjoint perturbations of self-adjoint operators [7]. In the self-adjoint case, the spectral theorem provides a powerful tool to control the resolvent of Schrödinger operators. However, there is no suitable analog to this for non-self-adjoint operators.

In this paper, we study semiclassical non-self-adjoint differential operators, and are thus concerned with the behavior of the resolvent as the semiclassical parameter $h$ tends towards 0. The general difficulty is that for non-self-adjoint semiclassical operators the spectrum does not control the resolvent, which may become very large far away from the spectrum as $h \to 0$. By a theorem of Davies [2] and Dencker, Sjöstrand, and Zworski [3], for a non-self-adjoint semiclassical Schrödinger operator of the form $P = -h^2 \Delta + V(x)$, for $V \in C^\infty(\mathbb{R}^n)$, and any $z$ of the form $z = \xi_0^2 + V(x_0)$ where $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and $\text{Im} \xi_0 \cdot V'(x_0) \neq 0$, $z$ is an “almost eigenvalue” of $P$, in the sense that there exists a family of functions $u(h) \in L^2$ for which $\| (P - z) u(h) \|_{L^2} = O(h^\infty) \| u(h) \|_{L^2}$. Thus, when $\text{Re} V \geq 0$ we should not generally expect to have much control over the resolvent of such an operator in the interior of the right half-plane. So instead
we will study resolvent estimates of such operators when the spectral parameter $z$ is near the boundary of this region.

In this paper we show that for a broad class of non-self-adjoint semiclassical Schrödinger operators there is an unbounded parabolic region near the imaginary axis where the resolvent is well controlled. Let us now introduce the precise assumptions on our operators.

Let $p \in C^\infty (\mathbb{R}^{2n})$ be such that

$$p (X) = |\xi|^2 + V (x),$$

where $V = V_1 + iV_2$ with $V_1, V_2$ real valued and $X = (x, \xi)$, with $x, \xi \in \mathbb{R}^n$.

We place the following conditions on the potential $V$:

$$V_1 (x) \geq 0, \quad x \in \mathbb{R}^n$$

$$|V_2 (x)| \leq 1 + |V_2' (x)|^2, \quad x \in \mathbb{R}^n,$$

$$\partial^\alpha V \in L^\infty (\mathbb{R}^n), \quad |\alpha| \geq 2.$$  \hfill (4)

Here, and throughout the paper, we use the notation “$f \lesssim g$” to denote that there exists a constant $c > 0$ such that $f \leq cg$. We define the Weyl quantization of a symbol $a (x, \xi)$ by

$$a^w (x, D_x) u (x) = \int_{\mathbb{R}^{2n}} e^{2\pi i (x-y) \cdot \xi} a \left( \frac{x+y}{2}, \xi \right) u (y) dy d\xi$$

and the semiclassical Weyl quantization by

$$a^w (x, hD_x) u (x) = \int_{\mathbb{R}^{2n}} e^{2\pi i (x-y) \cdot \xi} a \left( \frac{x+y}{2}, h\xi \right) u (y) dy d\xi,$$

where $0 < h \leq 1$. Note that

$$p^w = -\frac{h^2}{4\pi^2} \Delta + V (x).$$

We first prove the following a priori estimate for this operator.

**Theorem 1.** For such $p$, let $T \geq 0$ be such that

$$|V_2 (x)| - T \lesssim |V_2' (x)|^2, \quad x \in \mathbb{R}^n,$$

and choose any $K \in \mathbb{R}, K > 1$. Then there exist positive constants $h_0, A, \text{ and } M$ such that for all $0 < h < h_0$, $z \in \mathbb{C}$ with $|z| \geq KT + Mh$ and $\text{Re } z \leq Ah^{2/3} (|z| - T)^{1/3}$, and $u \in S$,

$$\| (p^w (x, hD_x) - z) u \|_{L^2} \gtrsim h^{2/3} (|z| - T)^{1/3} \| u \|_{L^2}.$$  \hfill (5)

We then use this to get a resolvent estimate on $L^2$. 

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**Theorem 2.** For \( p \) as above, \( P \), the \( L^2 \)-graph closure of \( p^w(x,hD_x) \) on \( S \) is the maximal realization of \( p^w(x,hD_x) \) equipped with the domain \( D_{\text{max}} = \{ u \in L^2 : p^w u \in L^2 \} \). For \( T, h \) and \( z \) as above we have the resolvent estimate

\[
\| (P - z)^{-1} \|_{L^2 \to L^2} \lesssim h^{-2/3} (|z| - T)^{-1/3}.
\]

**Remark.** For such \( P \), we have that \( P \) is accretive because

\[
\text{Re} \ (Pu,u)_{L^2} = \left( -\frac{h^2}{4\pi^2} \Delta + V_1 \right) u,u_{L^2} \geq 0, \quad u \in D_{\text{max}}.
\]

Thus Theorem 2 implies that \( P \) is maximally accretive.

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![Figure 1: The shaded region indicates the values of \( z \) for which the Theorems 1 and 2 apply.](image)

Similar resolvent estimates have been attained for different classes of semiclassical non-self-adjoint operators. Herau, Sjöstrand, and Stolk proved a similar resolvent estimate for the Kramers-Fokker-Planck operator under certain conditions [5]. We use a multiplier method inspired by one used in [5], but our proof proceeds quite differently. Theirs uses the FBI transform in a compact region of phase space and and Weyl-Hörmander calculus with a suitable metric near infinity, while ours works globally using the Wick quantization and some standard Weyl calculus. Hitrik and Sjöstrand attained a similar estimate for certain one-dimensional non-self-adjoint Schrödinger operators [7], with ellipticity assumptions on the potential. Also, Dencker, Sjöstrand, and Zworski showed that for non-self-adjoint semiclassical operators, under suitable assumptions including ellipticity at infinity, the resolvent can be similarly estimated in a small region near a boundary point of the range of the symbol, away from critical values of the symbol [3]. What distinguishes our result, in addition to the relatively direct proof, is that we have fairly loose conditions on the potential, with no requirement of ellipticity, and we attain a resolvent estimate for \( z \) in an unbounded region.

To demonstrate the applicability of this result, here are some examples of cases where it can be used.
Example 1. Let $V(x) = q(x)$ for $q$ any quadratic form with $\Re q(x) \geq 0$. By diagonalization we can see that $|\Im q(x)| \lesssim |\Im q'(x)|^2$, and $q''$ is constant so we can apply the above theorems to $p = |\xi|^2 + q(x)$ with $T = 0$. Thus for some $h_0$, $A$, and $M$,

$$\left\| \frac{-h^2}{4\pi^2} \Delta + q(x) - z \right\|_{L^2 \to L^2} \lesssim h^{-2/3} |z|^{-1/3},$$

for all $z \in \mathbb{C}$ with $|z| > Mh$ and $\Re z \leq Ah^{2/3} |z|^{1/3}$ and $0 < h \leq h_0$.

We can apply these theorems to many other classes of potentials. Note that the condition $|V_2(x)| - T \lesssim |V_2'(x)|^2$ implies that $T$ will be at least as large as the maximum absolute value of a critical value of $V_2$.

Example 2. Let $V \in C^\infty (\mathbb{R}^2)$ be given by $V(x_1, x_2) = ix_1^2 + i\sin(x_2)$. Then $|V(x_1, x_2)| - 1 \lesssim |V'(x_1, x_2)|^2$ so applying the above to $p = |\xi|^2 + V$ with $T = 1$ and any $K > 1$ yields, for some $h_0$, $A$, and $M$,

$$\left\| \frac{-h^2}{4\pi^2} \Delta + i (x_1^2 + \sin(x_2)) - z \right\|_{L^2} \lesssim h^{-2/3} (|z| - 1)^{-1/3},$$

for all $z \in \mathbb{C}$ with $|z| > K + Mh$ and $\Re z \leq Ah^{2/3} (|z| - 1)^{1/3}$ and $0 < h \leq h_0$.

For a broader example we also have the following:

Example 3. Let $V_2 \in C^\infty (\mathbb{R}^n; \mathbb{R})$ be a Morse function with finitely many critical points that satisfies (4). Furthermore suppose that $|V'_2(x)| \gtrsim |x|$ for all $x \in \mathbb{R}^n$ with $|x| > R$ for some $R > 0$. Let $x_1, ... , x_N \in \mathbb{R}^n$ be the critical points of $V$, and let $T = \max_{1 \leq j \leq N} |V_2(x_j)|$. Since $V_2$ is Morse, in a neighborhood of each $x_j$, $V_2(x) = V_2(x_j) + q_j(x - x_j) + O(|x - x_j|^3)$ for some nondegenerate quadratic form $q_j$. So $V'_2(x) = q'_j(x - x_j) + O(|x - x_j|^2)$ and $|q'_j(x - x_j)| \sim |x - x_j|$. Then, locally near $x_j$ we have

$$|V_2(x)| - T \lesssim |q_j(x - x_j)| + O(|x - x_j|^3) \lesssim |x - x_j|^2 \lesssim |V_2'(x)|^2.$$

Thus $|V_2(x)| - T \lesssim |V_2'(x)|^2$ in a neighborhood of each critical point. For $x$ away from critical points and $|x| \leq R$, $|V'_2(x)|$ is bounded below away from 0 and $|V_2(x)|$ is bounded above, so $|V_2(x)| - T \lesssim |V_2'(x)|^2$ here as well. Lastly (4) implies that $|V_2(x)| \lesssim 1 + |x|^2$ so $|V_2(x)| \lesssim |V_2'(x)|^2$ for $|x| > R$, and we see that the preceding theorems can be applied to $p = |\xi|^2 + V_1(x) + iV_2(x)$ for any such $V_2$ and any $V_1$ satisfying (2) and (4).

The plan of the paper is as follows. In Section 2 we will construct a bounded weight function $g$ to be used in proving Theorem (4). Then in Section 3 we will provide a brief overview of the Wick quantization. In Section 4 we prove
Theorem 1 by using the weight function as a bounded multiplier to prove an estimate for the Wick quantization of $p$ and use the relationship between the Wick and Weyl quantizations as well as some Weyl symbol calculus to get the desired estimate. In Section 5 we prove Theorem 2 by showing the estimate from Theorem 1 can be extended to the maximal domain of $P$. In Section 6, we show how the preceding proofs can be modified to prove a similar result for a larger class of potential functions if we additionally require that $|z|$ be bounded above.

2 The Weight Function

Let

$$\lambda (X) := (|\xi|^2 + V_1 (x) + |V_2' (x)|^2)^{1/2}.$$ 

It is worth noting that for this $p$ we have that

$$\lambda (X)^2 \lesssim \text{Re} \, p + H_{\text{Im} \, p} \text{Re} \, p \lesssim \lambda (X)^2,$$

because this motivates our choice of weight function. Here, for $f \in C^1 (\mathbb{R}^2n)$, we use the notation $Hf$ to denote the Hamiltonian vector field of $f$, i.e. given $f (x, \xi), \ g (x, \xi) \in C^1 (\mathbb{R}^2n)$ we define

$$Hf g = \{f, g\} = \partial_\xi f \cdot \partial_x g - \partial_x f \cdot \partial_\xi g.$$ 

Lemma 1. Let $p \in C^\infty (\mathbb{R}^{2n})$ be given by $p (x, \xi) = |\xi|^2 + V (x)$ with $V = V_1 + iV_2, V_1, V_2$ real valued, $V'' \in L^\infty$, and $V_1 \geq 0$. Let $\psi \in C^\infty_c (\mathbb{R}; [0, 1])$ be a cutoff with $\psi (t) = 1$ for $|t| \leq 1$ and $\psi (t) = 0$ for $|t| \geq 2$.

There exist $0 < \epsilon < 1$ and $0 < h_0 \leq 1$ depending on $p$ such that for all $0 < h \leq h_0$ and $X$ with $\lambda (X) \geq h^{1/2}$, the smooth weight function $G$ given by

$$G (X) = \epsilon h^{-1/3} \frac{H_{\text{Im} \, p} \text{Re} \, p}{\lambda (X)^{4/3}} \psi \left( \frac{4 \text{Re} \, p}{(h \lambda (X))^{2/3}} \right)$$

satisfies

$$|G (X)| = O (\epsilon),$$

$$|G' (X)| = O \left( \epsilon h^{-1/2} \right),$$

and

$$\text{Re} \, p (X) + h \text{Im} \, p G (X) \gtrsim h^{2/3} \lambda (X)^{2/3}.$$ 

Proof. The support of $G$ is contained in the region where $|\xi|^2 \leq \frac{1}{2} (h \lambda (X))^{2/3}$, so we see that since $\psi \leq 1$ we have

$$|G (X)| \leq \epsilon h^{-1/3} \frac{2 |V_2' (x)||\xi|}{\lambda (X)^{4/3}} \psi \left( \frac{\text{Re} \, p (X)}{(h \lambda (X))^{2/3}} \right).$$
\[
\frac{\zeta}{\epsilon} \preceq Ch^{-1/3} \lambda(X) (h \lambda(X))^{1/3} \frac{\lambda(X)^{4/3}}{\lambda(X)} \preceq \epsilon,
\]

which verifies that \(G\) satisfies (6). Note that as \(V_1'' \in L^\infty\) and \(V_1 \geq 0\) we have, using a standard inequality (Lemma 4.31 of [13]), that

\[
|V_1''(x)| \preceq V_1(x)^{1/2}.
\]

This and (4) then imply that

\[
\partial^\alpha \lambda^2 = O(\lambda), \quad |\alpha| = 1.
\]

Now, to check (7), one can use (4), (10), (11), and the fact that \(|\xi| \preceq (h \lambda(X))^{1/3}\) on the support of \(G\) to get the following estimates on the support of \(G\):

\[
\left|H_{V_2}|\xi|^2 \lambda(X)^{3/4}\right| = O\left(h^{1/3}\right),
\]

\[
\left|\partial^\alpha \frac{H_{V_2}|\xi|^2 \lambda(X)^{3/4}}{\lambda(X)}\right| = O\left(\lambda(X)^{-1/3}\right) = O\left(h^{-1/6}\right), \quad |\alpha| = 1,
\]

\[
\left|\partial^\alpha \left(\psi \left(\frac{4(|\xi|^2 + V_1(x))}{(h \lambda(X))^{2/3}}\right)\right)\right| = O\left(\frac{|\xi| + |V_1'(x)|}{(h \lambda(X))^{2/3}} + \lambda(X)^{-1}\right)
\]

\[
= O\left((h \lambda(X))^{-1/3} + \lambda(X)^{-1}\right) = O\left(h^{-1/2}\right), \quad |\alpha| = 1.
\]

Thus by (12), (13), and (14),

\[
|G'(x)| \preceq \zeta h^{-1/3} \left(O\left(h^{-1/6}\right) + O\left(h^{1/3}h^{-1/2}\right)\right) = O\left(Ch^{-1/2}\right),
\]

which verifies (7).

Now we shall attain (8) in the case where \(|\xi|^2 + V_1(x) \leq \frac{1}{4} (h \lambda(X))^{2/3} \leq \frac{1}{4} \lambda(X)^2\), and so \(|V_1''(x)| \geq \frac{3}{4} \lambda(X)^2\). In this region \(\psi \left(\frac{4\text{Re} p}{(h \lambda(X))^{2/3}}\right) \equiv 1\), and so

\[
G(X) = \zeta h^{-1/3} \frac{H_{V_2}|\xi|^2}{\lambda(X)^{3/4}}. \text{ Now we get}
\]

\[
H_{V_2}G = \zeta h^{-1/3} \left(\frac{2|V_1'(x)|^2}{\lambda(X)^{4/3}} - \frac{8 (V_1'(x) \cdot \xi)^2}{3\lambda(X)^{10/3}}\right).
\]

Thus

\[
\text{Re} p(X) + h \text{Im} p(G(X)) = \text{Re} p(X) + \zeta h^{2/3} \left(\frac{2|V_1'(x)|^2}{\lambda(X)^{4/3}} - \frac{8 (V_1'(x) \cdot \xi)^2}{3\lambda(X)^{10/3}}\right) \geq \text{Re} p(X) + \zeta h^{2/3} \left(\frac{2|V_1'(x)|^2}{\lambda(X)^{4/3}} - \frac{2|V_1'(x)|^2}{3\lambda(X)^{4/3}}\right) \geq \zeta h^{2/3} \frac{4|V_1'(x)|^2}{3\lambda(X)^{4/3}}
\]

Thus
\[ \geq c h^{2/3} |V_2'(x)|^2 \lambda(X)^{-4/3} \geq c h^{2/3} \lambda(X)^{2/3}. \]

It remains to show the bound in the region where \( |\xi|^2 + V_1(x) \geq \frac{1}{4} (h \lambda(X))^{2/3} \).

Using (12), (13), and (14) we get that
\[
|h H_{V_2} G| \leq c h^{2/3} \lambda(X) O \left( \lambda(X)^{-1/3} \right) \\
+ c h^{2/3} \lambda(X) O \left( h^{1/3} \left( (h \lambda(X))^{-1/3} + \lambda(X)^{-1} \right) \right) \\
= O \left( \epsilon (h \lambda(X))^{2/3} \right).
\]

Here, fixing \( \epsilon \) sufficiently small yields
\[ |\xi|^2 + V_1(x) + h H_{V_2} G \geq h^{2/3} \lambda(X)^{2/3} - O \left( c h^{2/3} \lambda(X)^{2/3} \right) \geq h^{2/3} \lambda(X)^{2/3}. \]

This completes the proof of the lemma.

**Corollary 1.** For such \( p \) as above, there exists a bounded real weight function \( g \in C^\infty(\mathbb{R}^{2n}) \) and constants \( C_0, \lambda_0 > 0 \) such that for all \( 0 < h \leq \lambda_0 \) and all \( X \in \mathbb{R}^{2n} \) \( |g(X)| \leq 1 \), \( |g'(X)| = O(h^{-1/2}) \) and
\[ \text{Re} \, p(X) + h H_{\text{im} \, p} g(X) + C_0 h \geq h^{2/3} \lambda(X)^{2/3}. \] (16)

**Proof.** Let \( G \) be a weight function for \( p \) as constructed in Lemma 1 and set \( \epsilon \) small enough that \( |G| \leq 1 \). Now we extend \( G \) to all of \( \mathbb{R}^{2n} \) by defining
\[ g(X) = \left( 1 - \psi \left( \frac{2 \lambda(X)^2}{h} \right) \right) G(X), \]
where \( \psi \in C_c(\mathbb{R}; [0, 1]) \), \( \psi(t) = 1 \) for \( |t| \leq 1 \), \( \psi(t) = 0 \) for \( |t| \geq 2 \), as before. By (7) and (11),
\[ |g'| \lesssim \frac{\lambda(X)}{h} \left| \psi' \left( \frac{2 \lambda(X)^2}{h} \right) \right| |G(X)| + \left( 1 - \psi \left( \frac{2 \lambda(X)^2}{h} \right) \right) |G'(X)| \]
\[ \lesssim h^{-1/2}. \]

By Lemma 1 (16) holds in the region where \( \lambda(X) > h^{1/2} \) for \( h \) sufficiently small since \( g = G \) there. When \( \lambda(X) < \frac{1}{2} h^{1/2} \) we have \( H_{V_2} g(X) = 0 \) and \( h^{2/3} \lambda(X)^{2/3} < h \) so the inequality holds in this region as well. When \( \frac{1}{2} h^{1/2} \leq \lambda(X) \leq h^{1/2} \), using (17) we get
\[ |\xi|^2 + V_1(x) + h H_{V_2} g(X) + C_0 h \geq C_0 h - O \left( h^{1/2} \lambda(X) \right) \geq h^{2/3} \lambda(X)^{2/3}, \]
for \( C_0 \) sufficiently large. \( \square \)
3 Wick quantization overview

Before proving Theorem 1 we first will note some facts about the Wick quantization. For $Y = (y, \eta) \in \mathbb{R}^{2n}$ and $x \in \mathbb{R}^n$ define

$$
\phi_Y(x) = \frac{a_n}{4} e^{-\pi|x-y|^2} e^{2\pi i \eta \cdot (x-y)}.
$$

Then for $u \in L^2(\mathbb{R}^n)$ define the wave packet transform of $u$ by

$$
Wu(Y) = (u, \phi_Y),
$$

where $(\cdot, \cdot)$ denotes the $L^2$ scalar product. As proven in [9], $W$ is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ and continuous from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^{2n})$. The function $\phi_Y$ is $L^2$ normalized, so the rank-one orthogonal projection of $u$ onto $\phi_Y$ is given by

$$
\Pi_Y u = (u, \phi_Y) \phi_Y.
$$

For a symbol $a(x, \xi) \in L^\infty(\mathbb{R}^{2n})$ the Wick quantization of $a$ is given by

$$
a_{\text{Wick}} = W^* a^\mu W, \quad (18)
$$

where $a^\mu$ denotes multiplication by $a$ and $W^*: L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^n)$ is the adjoint of $W$, or equivalently

$$
a_{\text{Wick}} = \int_{\mathbb{R}^{2n}} a(Y) \Pi_Y dY.
$$

We can see from [13] that for $a \in L^\infty(\mathbb{R}^{2n})$ then $a_{\text{Wick}}$ is a bounded operator on $L^2(\mathbb{R}^n)$ with

$$
\|a_{\text{Wick}}\|_{L^2 \to L^2} \leq \|a\|_{L^\infty} \quad (19)
$$

and that

$$
(a_{\text{Wick}})^* = (\overline{a})_{\text{Wick}}. \quad (20)
$$

More generally we can define the Wick quantization for symbols in the space of tempered distributions, $S'(\mathbb{R}^{2n})$. For $a \in S'(\mathbb{R}^{2n})$, $a_{\text{Wick}}$ is a map from $S(\mathbb{R}^n)$ to $S'(\mathbb{R}^n)$ defined by

$$
a_{\text{Wick}} u(\overline{\tau}) = a(Wu \overline{\tau}),
$$

for $u, \tau \in S(\mathbb{R}^n)$. As long as the symbol $a \in L^\infty_{\text{loc}}$ satisfies $|a(X)| \lesssim (1 + |X|)^N$ for some $N$ then $a^\mu$ is continuous as a map from $S(\mathbb{R}^{2n})$ to $L^2(\mathbb{R}^{2n})$, and thus [13] implies that $a_{\text{Wick}}$ is continuous from $S(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Furthermore, we have that for such symbols $a$ and $u \in S(\mathbb{R}^n)$

$$
a \geq 0 \Rightarrow (a_{\text{Wick}} u, u)_{L^2} \geq 0. \quad (21)
$$

Let $S(m)$ denote the symbol space

$$
S(m) = \{ f \in C^\infty(\mathbb{R}^{2n}) : |\partial^\alpha f(X)| \leq C_\alpha m(X), \forall \alpha \in \mathbb{N}^{2n} \},
$$
where $m$ is an order function on $\mathbb{R}^{2n}$ (cf. section 4.4 of [13]). Another fact we will need from [9] is that for $a \in S(m)$,

$$a^{Wick} = a^w + r(a)^w,$$

where

$$r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1-t) a''(X + tY) Y^2 e^{-2\pi|Y|^2} 2^n dY dt.$$

For smooth symbols $a$ and $b$ with $a \in L^\infty(\mathbb{R}^{2n})$ and $\partial^\alpha b \in L^\infty(\mathbb{R}^{2n})$ for $|\alpha| = 2$ we have the following composition formula proven in [10],

$$a^{Wick}b^{Wick} = (ab - \frac{1}{4\pi} a' \cdot b' + \frac{1}{4\pi i} \{a, b\})^{Wick} + R,$$

where $\|R\|_{L^2 \rightarrow L^2} \lesssim \|a\|_{L^\infty} \sup_{|\alpha| = 2} \|\partial^\alpha b\|_{L^\infty}$. We can see that the right-hand side is well defined as an operator $S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ because for $|\alpha_1| = |\alpha_2| = 1$,

$$(\partial^{\alpha_1} a)(\partial^{\alpha_2} b) = \partial^{\alpha_1} (a \partial^{\alpha_2} b) - a \left(\partial^{\alpha_1 + \alpha_2} b\right).$$

As $|a(X)\partial^{\alpha_2} b(X)| \lesssim 1 + |X|$ we can see that the symbol on the right-hand side of (24) is indeed a tempered distribution.

### 4 Proving the a priori estimate

Now we will use the Wick quantization and the weight function from Lemma 1 to prove Theorem 1.

**Proof of Theorem 1.** We will now follow a multiplier method based on section 4 of [6]. Let $g$ be a bounded real weight function for $p$ as constructed in Corollary 1. We first note that for $u \in S$, by (20),

$$\text{Re} \left( [p \left( \sqrt{h}X \right) - z]^{Wick} u, [2 - g \left( \sqrt{h}X \right)]^{Wick} u \right) = \text{Re} \left( [2 - g \left( \sqrt{h}X \right)]^{Wick} \left( p \left( \sqrt{h}X \right) - z \right)^{Wick} u, u \right) = \left( \text{Re} \left( [2 - g \left( \sqrt{h}X \right)]^{Wick} \left( p \left( \sqrt{h}X \right) - z \right)^{Wick} \right) u, u \right).$$

From (20) it follows that

$$\text{Re} a^{Wick} = \frac{1}{2} \left( a^{Wick} + (a^{Wick})^* \right) = \frac{1}{2} \left( a^{Wick} + (\pi^{Wick}) \right) = (\text{Re} a)^{Wick}.$$

Using this fact and the composition formula for the Wick quantization (24),

$$\text{Re} \left( [2 - g \left( \sqrt{h}X \right)]^{Wick} \left( p \left( \sqrt{h}X \right) - z \right)^{Wick} \right) =$$

9
\[
\text{Re} \left[ \left( 2 - g \left( \sqrt{h} X \right) \right) \left( p \left( \sqrt{h} X \right) - z \right) + \frac{1}{4\pi} \nabla \left( g \left( \sqrt{h} X \right) \right) \cdot \nabla \left( p \left( \sqrt{h} X \right) \right) \right]
\]

\[
- \frac{1}{4\pi i} \left\{ g \left( \sqrt{h} X \right), p \left( \sqrt{h} X \right) \right\} \text{Wick} + S_h
\]

\[
= \left[ \left( 2 - g \left( \sqrt{h} X \right) \right) \left( \text{Re} \ p \left( \sqrt{h} X \right) - \text{Re} \ z \right) \right]
\]

\[
+ \frac{\hbar}{4\pi} g' \left( \sqrt{h} X \right) \cdot \text{Re} p' \left( \sqrt{h} X \right) + \frac{\hbar}{4\pi} H_{V_2} g \left( \sqrt{h} X \right)
\]

\text{Wick} + S_h,
\]

where \( \| S_h \|_{L^2 \rightarrow L^2} = O(\hbar) \). Using (10) and (17) we have

\[
\left| h g' \left( \sqrt{h} X \right) \cdot \text{Re} p' \left( \sqrt{h} X \right) \right| \lesssim \hbar^{1/2} \left( \text{Re} p \left( \sqrt{h} X \right) \right)^{1/2}
\]

\[
\lesssim r \hbar + \frac{1}{r} \text{Re} p \left( \sqrt{h} X \right),
\]

for arbitrary \( r > 0 \). By taking \( r \) large enough the \( \frac{1}{r} \text{Re} p \left( \sqrt{h} X \right) \) term can be absorbed by \( \left( 2 - g \left( \sqrt{h} X \right) \right) \text{Re} \ p \left( \sqrt{h} X \right) \).

Let

\[
y = |z| - T \geq (K - 1) T + M \hbar.
\]

By using (16) we get that for some \( C_1, C_2 > 0 \) and arbitrary \( A > 0 \),

\[
\left( 2 - g \left( \sqrt{h} X \right) \right) \left( \text{Re} \ p \left( \sqrt{h} X \right) - \text{Re} \ z \right)
\]

\[
+ \frac{\hbar}{4\pi} g' \left( \sqrt{h} X \right) \cdot \text{Re} p' \left( \sqrt{h} X \right) + \frac{\hbar}{4\pi} H_{V_2} g \left( \sqrt{h} X \right)
\]

\geq \text{Re} p \left( \sqrt{h} X \right) - 3 \max (0, \text{Re} z) + \frac{\hbar}{4\pi} H_{V_2} g \left( \sqrt{h} X \right) + O(\hbar)
\]

\geq \hbar^{2/3} \left( \sqrt{h} X \right)^{2/3} - C_1 \max (0, \text{Re} z) - C_2 \hbar
\]

\geq \hbar^{2/3} \left( \sqrt{h} X \right)^{2/3} - 2AC_1 y^{1/3} + AC_1 \hbar^{2/3} y^{1/3}
\]

\[
+ C_1 \left( Ah^{2/3} y^{1/3} - \max (0, \text{Re} z) \right) - C_2 \hbar.
\]

As we required that \( \text{Re} z \leq Ah^{2/3} y^{1/3} \) we have that

\[
\hbar^{2/3} \left( \sqrt{h} X \right)^{2/3} - 2AC_1 y^{1/3} + C_1 \left( Ah^{2/3} y^{1/3} - \max (0, \text{Re} z) \right)
\]

\geq -2AC_1 \hbar^{2/3} y^{1/3} \psi \left( \frac{B \lambda \left( \sqrt{h} X \right)^{2}}{y} \right),
\]

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where
\[ B = \frac{1}{(2AC_1)^\frac{3}{2}}, \]  
and \( \psi \) is the same cutoff as before. Fix the value of \( A \) by choosing it small enough such that we can use that \( |V_2(x)| - T \lesssim |V'_2(x)|^2 \) to get
\[ |p(X)| - T \leq \frac{B\lambda(X)^2}{4}, \quad X \in \mathbb{R}^{2n}. \]  
(30)

Substituting (28) into (27) gives
\[ \left( 2 - g\left(\sqrt{h}X\right) \right) \left( \text{Re}p\left(\sqrt{h}X\right) - \text{Re} z \right) + \frac{h}{4\pi}g'\left(\sqrt{h}X\right) \cdot \text{Re}p'\left(\sqrt{h}X\right), \]  
(31)
\[ + \frac{h}{4\pi}Hv_2g\left(\sqrt{h}X\right) \]
\[ \gtrsim -2AC_1h^{2/3}y^{1/3}\psi\left(\frac{B\lambda\left(\sqrt{h}X\right)^2}{y}\right) - C_2h + AC_1h^{2/3}y^{1/3}. \]

Now (21), (25), (26), and (31) imply that, for \( h \) sufficiently small and \( \text{Re} z \leq Ah^{2/3}y^{1/3}, \)
\[ \text{Re} \left( [p\left(\sqrt{h}X\right) - z]^\text{Wick} u, [2 - g\left(\sqrt{h}X\right)]^\text{Wick} u \right) + h\|u\|_{L^2} + 
\]
\[ h^{2/3}y^{1/3} \left( \psi\left(\frac{B\lambda\left(\sqrt{h}X\right)^2}{y}\right)^\text{Wick} u, u \right) \gtrsim h^{2/3}y^{1/3}\|u\|_{L^2}. \]

By the Cauchy-Schwarz inequality and (19) we get that
\[ \left\| [p\left(\sqrt{h}X\right) - z]^\text{Wick} u \right\|_{L^2} + h\|u\|_{L^2} + h^{2/3}y^{1/3} \left\| \psi\left(\frac{B\lambda\left(\sqrt{h}X\right)^2}{y}\right)^\text{Wick} u \right\|_{L^2} \]
\[ \gtrsim h^{2/3}y^{1/3}\|u\|_{L^2}. \]

Now we pick \( M \) sufficiently large so that the \( h\|u\|_{L^2} \) term can be absorbed by the right-hand side to get
\[ \left\| [p\left(\sqrt{h}X\right) - z]^\text{Wick} u \right\|_{L^2} + h^{2/3}y^{1/3} \left\| \psi\left(\frac{B\lambda\left(\sqrt{h}X\right)^2}{y}\right)^\text{Wick} u \right\|_{L^2} \]
\[ \gtrsim h^{2/3}y^{1/3}\|u\|_{L^2}. \]  
(32)
This resembles the desired inequality, but we still need to switch from the Wick quantization to the Weyl quantization, and we need to deal with the term involving $\psi$. First we will switch to the Weyl quantization. The Calderón-Vaillancourt Theorem (Theorem 4.23 in [13]) states that for $a \in S(1)$ there exists a universal constant $\lambda$ such that

$$\|a^w(x,D_x)\|_{L^2 \to L^2} \lesssim \sup_{|\alpha| \leq \lambda n} \|\partial^\alpha a\|_{L^\infty}.$$  (33)

From (4) we have that

$$\partial^\alpha \left( p \left( \sqrt{h} X \right) \right) = O \left( h^{\left| \alpha \right| / 2} \right), |\alpha| \geq 2,$$

so we can apply the Calderón-Vaillancourt theorem to the remainder term in (22) with $a(X) = p \left( \sqrt{h} X \right) - z$ to get

$$\left\| p \left( \sqrt{h} X \right)^{Wick} u - zu \right\|_{L^2} = \left\| p \left( \sqrt{h} X \right)^w u - zu \right\|_{L^2} + O \left( h \|u\|_{L^2} \right).$$  (34)

To do the same thing to the other term on the left side of (32) we need to estimate the derivatives of $\psi \left( \frac{B\lambda(\sqrt{h} X)^2}{y} \right)$.

**Lemma 2.**

$$\left| \partial^\alpha \left( \psi \left( \frac{B\lambda(\sqrt{h} X)^2}{y} \right) \right) \right| \lesssim \frac{h^{1/2}}{y^{1/2}}, |\alpha| \geq 1.$$  (35)

**Proof.** First, note that because $V'' \in S(1)$ we have

$$\partial^\alpha \lambda(x)^2 = \partial^\alpha \left( |\xi|^2 + V_1(x) + |V'_2(x)|^2 \right) \lesssim 1 + |V'_2| \lesssim 1 + \lambda, |\alpha| \geq 2.$$  (36)

Also, for $X$ in the support of $\psi \left( \frac{B\lambda(\sqrt{h} X)^2}{y} \right)$ we have

$$\lambda(\sqrt{h} X) \lesssim y^{1/2},$$

and so, by (11)

$$\left| \partial^\alpha \left( \lambda \left( \frac{\sqrt{h} X}{y} \right)^2 \right) \right| \lesssim \frac{h^{1/2} \lambda \left( \frac{\sqrt{h} X}{y} \right)}{y^{1/2}} \lesssim \frac{h^{1/2}}{y^{1/2}}, |\alpha| = 1,$$

and by (36)

$$\left| \partial^\alpha \left( \frac{\lambda \left( \frac{\sqrt{h} X}{y} \right)^2}{y} \right) \right| \lesssim \frac{h^{1/2} \left( 1 + \lambda \left( \frac{\sqrt{h} X}{y} \right) \right)}{y} \lesssim \frac{h}{y} + \frac{h}{y^{1/2}} \lesssim \frac{h^{1/2}}{y^{1/2}}, |\alpha| \geq 2.$$
We can express $\partial^\alpha \left( \psi \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right) \right)$ as a linear combination of terms of the form

$$\psi^{(k)} \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right) \partial^{\gamma_1} \left( \frac{\lambda (\sqrt{hX})^2}{y} \right) \ldots \partial^{\gamma_k} \left( \frac{\lambda (\sqrt{hX})^2}{y} \right),$$

where $\alpha = \gamma_1 + \ldots + \gamma_k$, $|\gamma_i| \geq 1$ for all $i$, $1 \leq k \leq |\alpha|$. Each such term is of size $O \left( \left( \frac{h}{y} \right)^{k/2} \right)$, proving the lemma.

Using Lemma 2 and (22) we get

$$\left\| \psi \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right) \right\|_{L^2} = \left\| \psi \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right) \right\|_{L^2} + O \left( M^{-1/2} \right) \|u\|_{L^2}. \quad (37)$$

By substituting (34) and (37) into (32) and taking $M$ sufficiently large we get

$$\left\| \left[ p \left( \frac{\sqrt{hX}}{y} \right) - z \right] u \right\|_{L^2} + h^{2/3} y^{1/3} \left\| \psi \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right) \right\|_{L^2} \geq h^{2/3} y^{1/3} \|u\|_{L^2}. \quad (38)$$

Now all that remains is to deal with the $\psi$ term, which we will accomplish by showing, with some basic Weyl calculus, that it can be absorbed by the other two terms.

Since $\psi$ is real valued $\psi^w$ is self-adjoint. Therefore

$$\left\| \psi \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right) \right\|_{L^2}^2 = \left( \psi \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right) \right)^2 \left( u, u \right).$$

For the sake of brevity we will henceforth use the notation

$$\Psi (X) := \psi \left( \frac{B\lambda (\sqrt{hX})^2}{y} \right).$$

Lemma 2 can then be rephrased as:

$$\Psi' (X) \in S \left( \frac{h^{1/2}}{y^{1/2}} \right).$$
Let us now recall some basic Weyl calculus. For symbols $a$ and $b$ in $S(1)$, we have the following composition formula for their Weyl quantizations [9],

$$a^w b^w = (a \# b)^w = \left(ab + \frac{1}{4\pi i} \{a, b\} + R\right)^w,$$  \hspace{1cm} (39)

where

$$R = -\frac{1}{16\pi^2} \int_0^1 (1 - t) e^{\frac{\pi i}{4}(D_\xi \cdot D_y - D_x \cdot D_\eta)} (D_\xi \cdot D_y - D_x \cdot D_\eta)^2 a(x, \xi) b(y, \eta) dt \bigg|_{(y, \eta) = (x, \xi)}.$$

Thus, using that $\{\Psi, \Psi\} = 0$,

$$\Psi (X) \# \Psi (X) = \Psi^2 (X) - \frac{1}{16\pi^2} \int_0^1 (1 - t) e^{\frac{\pi i}{4}(D_\xi \cdot D_y - D_x \cdot D_\eta)} (D_\xi \cdot D_y - D_x \cdot D_\eta)^2 (x, \xi) (y, \eta) \Psi (x, \xi) \Psi (y, \eta) dt \bigg|_{(y, \eta) = (x, \xi)}.$$

By Lemma [2]

$$(D_\xi \cdot D_y - D_x \cdot D_\eta)^2 \Psi (x, \xi) \Psi (y, \eta) = O_{S(1)} \left(\frac{h}{y}\right),$$

where “$F_1 = O_{S(1)} (F_2)$” means $\partial^\alpha F_1 = O (F_2)$, for all $\alpha$. By Theorem 4.17 in [13] the operator $e^{\frac{\pi i}{4}(D_\xi \cdot D_y - D_x \cdot D_\eta)}$ maps $S(m)$ to $S(m)$ continuously for any order function $m$, so by the above we get that

$$(\Psi (X)^w)^2 = \Psi^2 (X)^w + \frac{h}{y} R_1^w,$$

for some $R_1 = O_{S(1)} (1)$. Thus by applying (33) we get

$$\|\Psi (X)^w u\|_{L^2}^2 = (\Psi^2 (X)^w u, u) + O \left(\frac{h}{y}\right) \|u\|_{L^2}^2. \hspace{1cm} (40)$$

To control the first term on the right-hand side we follow a method similar to Lemma 8.2 from [5].

**Lemma 3.** $(\Psi^2 (X)^w u, u) \leq \left(\frac{4|p(\sqrt{\Lambda} X) - z|^2}{y^2}\Psi^2 (X)^w u, u\right) + O \left(\frac{h^{1/2}}{y^{1/2}}\right) \|u\|_{L^2}^2.$

**Proof.** Recalling (30), we see that on the support of $\Psi (X)$ we have that

$$|p(\sqrt{\Lambda} X)| - T \leq \frac{BA \left(\sqrt{\Lambda} X\right)^2}{4} \leq \frac{y}{2}. \hspace{1cm} (41)$$
Thus
\[
\frac{1}{y} \left| \rho \left( \sqrt{h}X \right) - z \right| \geq \frac{1}{y} \left( |z| - \left| \rho \left( \sqrt{h}X \right) \right| \right) = \frac{1}{y} \left( y + T - \left| \rho \left( \sqrt{h}X \right) \right| \right) \geq \frac{1}{2},
\]
and so
\[
\Psi^2 (X) \leq 4 \frac{\left| \rho \left( \sqrt{h}X \right) - z \right|^2}{y^2} \Psi^2 (X). \tag{42}
\]
Let
\[
Q (X) = \frac{4}{y^2} \left| \rho \left( \sqrt{h}X \right) - z \right|^2 \Psi^2 (X) - \Psi^2 (X) \geq 0. \tag{43}
\]
By (21), (22), and (23) we get that
\[
\left( Q^w (x, D_x) u, u \right)_{L^2} + \left\| \int_0^1 \int_{\mathbb{R}^n} (1 - t) Q'' (X + tY) Y^2 e^{-2\pi |Y|^2} 2^n dY dt \right\|_{L^2} \geq 0. \tag{44}
\]
To estimate the second term, (33) implies that we need to estimate the derivatives of order two and higher of \( Q \).
As \( |z| > KT + Mh \) and \( K > 1 \),
\[
y = |z| - T > (K - 1) T \geq T.
\]
So, for \( X \) in the support of \( \Psi \), using (41), \( y \geq T \), and \( y \geq |z| \), we get the following
\[
\left| \frac{\rho \left( \sqrt{h}X \right) - z}{y} \right| \lesssim \frac{1}{y} (y + T + |z|) \lesssim 1.
\]
For such \( X \), using (10) we also have
\[
\left| \partial^\alpha \frac{\rho \left( \sqrt{h}X \right) - z}{y} \right| \lesssim \frac{h^{1/2}}{y} \lambda \left( \sqrt{h}X \right) \lesssim \frac{h^{1/2}}{y^{1/2}}, \ |\alpha| = 1 \tag{45}
\]
and
\[
\left| \partial^\alpha \frac{\rho \left( \sqrt{h}X \right) - z}{y} \right| \lesssim \frac{h^{|\alpha|/2}}{y}, \ |\alpha| \geq 2. \tag{46}
\]
By the above and (35), for \( |\alpha| \geq 1 \),
\[
\left| \partial^\alpha Q (X) \right| \lesssim \frac{h^{1/2}}{y^{1/2}}.
\]
Thus by applying the Calderón-Vaillancourt theorem we can bound the latter term of as follows.

\[
\left\| \left( \int_0^1 \int_{\mathbb{R}^{2n}} (1-t) Q''(X+tY) Y^2 e^{-2\pi|Y|^2} dY dt \right) w \ | u \right\|_{L^2} \lesssim \frac{h^{1/2}}{y^{1/2}} \|u\|_{L^2}.
\]

Therefore implies a variant of the sharp Gårding inequality (cf. Theorem 4.32 of [13]) for \( Q \),

\[
\langle Q^w(x, D_x) u, u \rangle_{L^2} + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \|u\|_{L^2}^2 \geq 0.
\]

And so by we attain the desired inequality,

\[
\left( \Psi^2(X)^w u, u \right) \leq \left( \left( 4 \frac{|p(\sqrt{\hbar}X) - z|^2}{y^2} \Psi^2(X) \right)^w u, u \right) + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \|u\|_{L^2}^2.
\] (47)

Finally, we have to understand the first term on the right side of (47). The estimates and imply that

\[
\partial^\alpha \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi(X) \right) = O \left( \left( \frac{h}{y} \right)^{1/2} \right), \ |\alpha| \geq 1.
\]

Thus, using this and and repeating the same Weyl calculus argument used to attain we get

\[
4 \frac{|p(\sqrt{\hbar}X) - z|^2}{y^2} \Psi^2(X) = 4 \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi(X) \# \frac{p(\sqrt{\hbar}X) - z}{y} \Psi(X) \right) - \frac{1}{\pi i} \left\{ \frac{p(\sqrt{\hbar}X) - z}{y} \Psi(X), \frac{p(\sqrt{\hbar}X) - z}{y} \Psi(X) \right\} + \frac{h}{y} R_2
\]

\[= 4 \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi(X) \# \frac{p(\sqrt{\hbar}X) - z}{y} \Psi(X) \right) + \frac{h}{y} R_3,
\]
where $R_2, R_3 = O_S(1)$. We also similarly get from (35), (45), (46) and (39) that

$$
\Psi (X) \# \left( \frac{p (\sqrt{h}X) - z}{y} \right) = \frac{p (\sqrt{h}X) - z}{y} \Psi (X) + \frac{h}{y} R_4,
$$

for $R_4 = O_S(1)$.

Now, using (40), Lemma 3, the fact that $\frac{h}{y} \leq \frac{1}{M}$, and that $\|\Psi^w u\|_{L^2} = O(1)$, we can conclude that

$$
\|\Psi^w u\|_{L^2}^2 \leq \left\| \Psi (X)^w \left( \frac{p (\sqrt{h}X) - z}{y} \right)^w u \right\|^2_{L^2} + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \|u\|^2_{L^2}.
$$

Plugging this in to (38) we get

$$
\left\| \left[ p \left( \sqrt{h}X \right) - z \right]^w u \right\|_{L^2} \leq \frac{1}{y^2} \left\| \left[ p \left( \sqrt{h}X \right) - z \right]^w u \right\|^2_{L^2} + O \left( \frac{1}{M^{1/2}} \right) \|u\|^2_{L^2}.
$$

Then taking $M$ sufficiently large yields

$$
\left\| \left[ p \left( \sqrt{h}X \right) - z \right]^w u \right\|_{L^2} \geq h^{2/3} y^{1/3} \|u\|_{L^2}.
$$

Finally, by making the symplectic change of coordinates $x \to \frac{x}{\sqrt{h}}, \xi \to \sqrt{h} \xi$ we obtain the desired estimate,

$$
\|(p^w (x, hD_x) - z) u\|_{L^2} \geq h^{2/3} y^{1/3} \|u\|_{L^2}.
$$

\[\square\]

5 From a priori to a resolvent estimate

Now we will use Theorem 1 to prove Theorem 2. To do so it will be convenient to work in the standard, or Kohn-Nirenberg, quantization rather than the Weyl quantization. In the semiclassical case, this quantization is defined by

$$
a^{KN} (x, hD_x) u (x) = \int_{\mathbb{R}^{2n}} e^{2\pi (x-y) \cdot \xi} a (x, h\xi) u (y) dy d\xi
$$

$$
= \mathcal{F}_{\xi \to x}^{-1} a (x, h\xi) \mathcal{F}_{y \to \xi} u (y),
$$

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where $\mathcal{F}$ denotes the Fourier transform. Note that just like in the Weyl quantization we have that

$$p^{KN} (x, hD_x) = -\frac{h^2}{4\pi^2} \Delta + V (x).$$

In this quantization we have the composition formula

$$a^{KN} (x, hD_x) b^{KN} (x, hD_x) = \left( ab + \frac{h}{2\pi i} D_\xi a \cdot D_x b + R \right)^{KN} (x, hD_x), \quad (48)$$

where

$$R = -\frac{h^2}{4\pi^2} \int_0^1 (1 - t) e^{\frac{i}{\pi} D_\xi \cdot D_y} (D_\xi \cdot D_y)^2 a (x, \xi) b (y, \eta) dt \bigg|_{(y, \eta) = (x, \xi)}.$$

The standard quantization of a symbol is equivalent to the Weyl quantization of a related symbol [13], specifically if $a \in S (m)$ for some order function $m$, we have

$$a^{KN} (x, hD_x) = \left( e^{\frac{i}{\pi} (D_\xi \cdot D_x)} a \right)^w (x, hD_x)$$

and

$$e^{\frac{i}{\pi} (D_\xi \cdot D_x)} a \in S (m).$$

This tells us that some properties of the Weyl quantization can be applied to the standard quantization as well, the Calderón-Vaillancourt theorem (33) among them.

**Proof of Theorem 2.** To show that $P$, the graph closure of $p^w (x, hD_x)$ on $S (\mathbb{R}^n)$ has domain $D_{max} = \{ u \in L^2 : p^w u \in L^2 \}$ we follow a method from Hörmander found in [8]. Let $\chi_\delta : L^2 \to S$ be a family of operators parametrized by $\delta > 0$ such that $\chi_\delta u \to u$ in $L^2$ as $\delta \to 0$ for all $u \in L^2$. If

$$P \chi_\delta - \chi_\delta P) u \to 0 \quad (49)$$

in $L^2$ as $\delta \to 0$ for all $u \in D_{max}$ then we have that $u_\delta := \chi_\delta u$ is a sequence of functions in $S$ converging to $u$ and that $Pu_\delta \to Pu$, thus the domain of $P$ is $D_{max}$.

To accomplish this, let $\phi \in C_c^\infty (\mathbb{R}^n, [0, 1])$ be a cutoff function with $\phi (x) = 1$ for $x$ in a neighborhood of 0. It suffices to consider the $h = 1$ case as $h$ is fixed independent of $\delta$ and thus does not affect issues of convergence. Then define

$$\chi_\delta u = (\phi (\delta x) \phi (\delta \xi))^{KN} u, \quad u \in L^2.$$ 

We then have that $\chi_\delta : L^2 \to S$ and $\chi_\delta u \to u$ in $L^2$ as $\delta \to 0$ for all $u \in L^2$ as desired. We then need to check (49). This can be accomplished using some standard quantization symbol calculus for the commutator $[P, \chi_\delta]$. By (48) we have

$$[P, \chi_\delta] = \left( \frac{1}{2\pi i} \{ p (x, \xi), \phi (\delta x) \phi (\delta \xi) \} + O_{S(1)} (\delta^2) \right)^{KN}$$
\[\frac{\delta}{\pi i} (\xi \cdot \phi' (\delta x) \phi (\delta x))^{KN} - \frac{\delta}{2\pi i} (V' (x) \cdot \phi' (\delta x) \phi (\delta x))^{KN} u + (O_{S(1)} (\delta^2))^{KN} \]

\[= I + II + III. \]

On the support of \(\phi (\delta x) \phi (\delta x)\) we have that \(|x| \lesssim \delta^{-1}\) and \(|\xi| \lesssim \delta^{-1}\) so, as \(\delta \to 0\),

\[|\delta \partial^\alpha (\xi \cdot \phi' (\delta x) \phi (\delta x))| = O (1), \quad \forall \alpha \]

and, recalling (4),

\[|\delta \partial^\alpha (V' (x) \cdot \phi' (\delta x) \phi (\delta x))| = O (1), \quad \forall \alpha.\]

Thus by (33)

\[\|[P, \chi_\delta]\|_{L^2 \to L^2} = O (1).\]

It thus suffices to show that \([P, \chi_\delta]u \to 0\) for all \(u\) in a dense subset of \(L^2\). Term \(III\) is easily dealt with because as \(\delta \to 0\),

\[\|IIIu\|_{L^2} = O (\delta^2) ||u||_{L^2} \to 0.\]

To deal with terms \(I\) and \(II\), let \(u \in L^2\) be such that \(\mathcal{F}u \in C_\infty^\infty (\mathbb{R}^n)\). Then

\[IIu = -\frac{\delta}{2\pi i} \phi (\delta x) V' (x) \cdot \mathcal{F}^{-1} (\phi' (\delta x) (\mathcal{F}u) (\xi)).\]

Note that \(\phi' (\delta x)\) is supported where \(|\xi| \sim \delta^{-1}\) so for \(\delta\) sufficiently small

\[\phi' (\delta x) (\mathcal{F}u) (\xi) = 0\]

and so

\[\|IIu\|_{L^2} \to 0.\]

Also,

\[Iu = \frac{\delta}{\pi i} \phi' (\delta x) \cdot \mathcal{F}^{-1} (\xi \phi (\delta x) (\mathcal{F}u) (\xi)).\]

Because \(\mathcal{F}u (\xi)\) is compactly supported and \(\phi = 1\) in a neighborhood of 0, for \(\delta\) sufficiently small we have

\[\phi (\delta x) (\mathcal{F}u) (\xi) = (\mathcal{F}u) (\xi).\]

And then

\[Iu = \frac{\delta}{\pi i} \phi' (\delta x) \cdot \mathcal{F}^{-1} (\xi (\mathcal{F}u) (\xi))\]

\[= -\frac{\delta}{2\pi^2} \phi' (\delta x) \cdot u' (x).\]

Since \(\mathcal{F}u \in C_\infty^\infty\) we have \(u' \in L^2\) so

\[\|Iu\|_{L^2} \to 0.\]
Therefore (49) holds, which tells us that the graph closure of $p^w(x, hD_x)$ on $\mathcal{S}$, has the domain $D_{max}$. Thus, for $z$ and $h$ satisfying the conditions in Theorem 1 we have

$$\| (P - z) u \|_{L^2} \gtrsim h^{2/3} |z - T|^{1/3} \| u \|_{L^2} \quad \forall u \in D_{max}.$$  

We thus have that $P - z$ is injective on $D_{max}$ and has closed range. We can apply the same argument to the formal adjoint of $p^w$ on $\mathcal{S}$, $p^w - z = (\xi^2 + V(x))^{1/2} - \overline{\tau}$, and we similarly get its graph closure is $P - \overline{\tau} = -h^2/4 + \Delta + V(x) - \overline{\tau}$ with domain $\{ u \in L^2 : \overline{p}^uw \in L^2 \}$, which is also injective with closed range. As $P - \overline{\tau}$ has maximal domain we have that $P - z = (P - z)^*$. Thus $P - z$ is invertible, and we get the desired resolvent estimate,

$$\| (P - z)^{-1} u \|_{L^2} \lesssim h^{-2/3} (|z| - T)^{-1/3} \| u \|_{L^2}.$$  

6 The bounded $z$ case

In the preceding sections, the condition that was placed on $V$ in (5), that

$$|V_2(x)| - T \lesssim |V'_2(x)|^2, \quad \forall x \in \mathbb{R}^n,$$

is only used once. It is used so that we can get the inequality (30)

$$|p(X)| - T \leq \frac{B\lambda(X)^2}{4}, \quad \forall X \in \mathbb{R}^{2n},$$

which implies

$$\psi^2 \left( \frac{B\lambda \left( \sqrt{\tau} X \right)^2}{y} \right) \leq 4 \left| p \left( \sqrt{\tau} X \right) - z \right|^2 \psi^2 \left( \frac{B\lambda \left( \sqrt{\tau} X \right)^2}{y} \right) \lesssim 1.$$  

We see that the condition on $V_2$ in (5) is only needed in the region where $\lambda^2 \lesssim y$. Thus if we only consider values of $z$ such that $|z| - T \leq R$ for some $R > 0$ we do not need this condition on $V_2$ to apply globally. Instead we need there to exist some constant $L > 0$ such that

$$|V_2(x)| - T \lesssim |V'_2(x)|^2, \quad \forall x \in \{ x \in \mathbb{R}^n : |V'_2(x)| \leq L \}.$$  

(51)

Then by taking $B$ large enough (and hence $A$ small enough), we still get

$$\psi^2 \left( \frac{B\lambda \left( \sqrt{\tau} X \right)^2}{y} \right) \leq 4 \left| p \left( \sqrt{\tau} X \right) - z \right|^2 \psi^2 \left( \frac{B\lambda \left( \sqrt{\tau} X \right)^2}{y} \right) \lesssim 1.$$  

The rest of the proof can remain unchanged. This results in the following.
Theorem 3. Let $p$ be in $C^\infty (\mathbb{R}^{2n})$ be given by $p = \xi^2 + V(x)$ with $V = V_1 + iV_2$, $V_1, V_2$ real valued, $V_1 \geq 0$, $V'' \in S(1)$, and

$$V_2(x) - T \lesssim |V_2'(x)|^2, \quad \forall x \in \{ x \in \mathbb{R}^n : |V_2'(x)| \leq L \},$$

for some $L > 0$, $T > 0$. Then for any $R > 0$, $K > 1$, there exist positive constants $A$, $M$, and $h_0$ such that for all $0 < h \leq h_0$ and $z \in \mathbb{C}$ with $Mh \leq |z| - KT \leq R$ and $\Re z \leq Ah^{2/3} (|z| - T)^{1/3}$ we have

$$\| [p^w(x, \hbar D_x) - z] u \|_{L^2} \gtrsim h^{2/3} (|z| - T)^{1/3} \| u \|_{L^2}, \quad \forall u \in S(\mathbb{R}^n),$$

and taking $P$, the $L^2$-graph closure of $p^w$ on $S$, we have

$$\left\| (P - z)^{-1} u \right\|_{L^2} \lesssim h^{-2/3} (|z| - T)^{-1/3} \| u \|_{L^2}, \quad \forall u \in L^2.$$  

The set of potentials $V$ to which this can apply is very broad. Provided $V_1 \geq 0$ and $V'' \in S(1)$, then (51) will be satisfied for some $T$ and $L$ as long as there is no sequence of points $x_j$ along which $|V_2'(x_j)| \to 0$ and $V_2(x_j) \to \infty$.

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