Extensors in Geometric Algebra

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Abstract

This paper, the third in a series of eight introduces some of the basic concepts of the theory of extensors needed for our formulation of the differential geometry of smooth manifolds. Key notions such as the extension and generalization operators of a given linear operator (a \((1,1)\)-extensor) acting on a real vector space \(V\) are introduced and studied in details. Also, we introduce the notion of the determinant of a \((1,1)\)-extensor and the concepts of standard and metric Hodge (star) operators, disclosing a non trivial and useful relation between them.

Contents

1 Introduction 2

2 General \(k\)-Extensors 3

2.1 \((p,q)\)-Extensors .......................................................... 4
2.2 Extensors ................................................................. 5
2.3 Elementary \(k\)-Extensors ................................................... 6
1 Introduction

In this paper, the third in a series of eight dealing formulation of the differential geometry of smooth manifolds with Clifford (geometric) algebra methods we recall some basic notions of the theory of extensors thus completing the presentation of the algebraic notions necessary for the remaining papers of the series. Here, in Section 2 we introduce $k$-extensors and $(p,q)$-extensors. Section 3 deals with projector operators. Section 4 introduces the key concept of the extension operator (or exterior power operator) of a given linear operator $t$ (i.e., a $(1,1)$-extensor) on a real vector space $V$. In section 5 the standard and metric adjoint operators are recalled as extensor operators. Section 6 introduces the generalization operator of $t$, an important concept in our formulation of the differential geometry of arbitrary (smooth) manifolds which is presented in sequel papers of this series. Section 7 introduces the concept of the determinant of a $(1,1)$-extensor and Section 8 shows through some applications the concept of extensors in action. There, the standard and metric Hodge operators are introduced and an explicit formula connecting them is derived. Moreover, another important formula is derived which
relates the metric Hodge operators corresponding to two distorted metric structures on \( V \). In Section 9 we present our conclusions.

2 General \( k \)-Extensors

Let \( \Lambda_1^\circ V, \ldots, \Lambda_k^\circ V \) be \( k \) subspaces of \( \Lambda V \) such that each of them is \textit{any} sum of homogeneous subspaces of \( \Lambda V \), and \( \Lambda^\circ V \) is either \textit{any} sum of homogeneous subspaces of \( \Lambda V \) or the trivial subspace \{0\}. A multilinear mapping from the cartesian product \( \Lambda_1^\circ V \times \cdots \times \Lambda_k^\circ V \) to \( \Lambda^\circ V \) will be called a general \( k \)-extensor over \( V \), i.e., 

\[
t : \Lambda_1^\circ V \times \cdots \times \Lambda_k^\circ V \to \Lambda^\circ V
\]

such that for any \( \alpha_j, \alpha_j' \in \mathbb{R} \) and \( X_j, X_j' \in \Lambda_j^\circ V \),

\[
t(\ldots, \alpha_jX_j + \alpha_j'X_j', \ldots) = \alpha_jt(\ldots, X_j, \ldots) + \alpha_j't(\ldots, X_j', \ldots), \tag{1}
\]

for each \( j \) with \( 1 \leq j \leq k \).

It should be noticed that the linear operators on \( V, \Lambda^p V \) or \( \Lambda V \) which appear in ordinary linear algebra are particular cases of \( 1 \)-extensors over \( V \). Note also that a covariant \( k \)-tensor over \( V \) is just a \( k \)-extensor over \( V \). On this way, the concept of general \( k \)-extensor generalizes and unifies both of the concepts of linear operator and of covariant \( k \)-tensor. These mathematical objects are of the same nature!

The set of general \( k \)-extensors over \( V \), denoted by \( k\text{-ext}(\Lambda_1^\circ V, \ldots, \Lambda_k^\circ V; \Lambda^\circ V) \), has a natural structure of real vector space. Its dimension is clearly given by

\[
\dim k\text{-ext}(\Lambda_1^\circ V, \ldots, \Lambda_k^\circ V; \Lambda^\circ V) = \dim \Lambda_1^\circ V \cdots \dim \Lambda_k^\circ V \dim \Lambda^\circ V. \tag{2}
\]

We shall need to consider only some particular cases of these general \( k \)-extensors over \( V \). So, special names and notations will be given for them.

We will equip \( V \) with an arbitrary (but fixed once and for all) euclidean metric \( G_E \), and denote the scalar product of multivectors \( X, Y \in \Lambda V \) with respect to the euclidean metric structure \( (V, G_E) \), \( X \cdot Y \) instead of the more detailed notation \( X \cdot_{G_E} Y \).

Let \( \{e_j\} \) be any basis for \( V \), and \( \{e_j\} \) be its euclidean reciprocal basis for \( V \), i.e., \( e_j \cdot e^k = \delta_j^k \).
2.1 \((p, q)\)-Extensors

Let \(p\) and \(q\) be two integer numbers with \(0 \leq p, q \leq n\). A linear mapping which sends \(p\)-vectors to \(q\)-vectors will be called a \((p, q)\)-extensor over \(V\). The space of these objects, namely \(1\text{-ext}(\wedge^p V; \wedge^q V)\), will be denoted by \(\text{ext}_{p}^q(V)\) for short. By using Eq. (2) we get

\[
\dim \text{ext}_{p}^q(V) = \binom{n}{p} \binom{n}{q}.
\]

For instance, we see that the \((1, 1)\)-extensors over \(V\) are just the well-known linear operators on \(V\).

The \(\binom{n}{p} \binom{n}{q}\) extensors belonging to \(\text{ext}_{p}^q(V)\), namely \(\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}\), defined by

\[
\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}(X) = (e^{j_1} \wedge \ldots \wedge e^{j_p}) \cdot X e^{k_1} \wedge \ldots \wedge e^{k_q}
\]

define a \((p, q)\)-extensor over \(V\).

Indeed, the extensors given by Eq. (4) are linearly independent, and for each \(t \in \text{ext}_{p}^q(V)\) there exist \(\binom{n}{p} \binom{n}{q}\) real numbers, say \(t_{j_1 \ldots j_p; k_1 \ldots k_q}\), given by

\[
t_{j_1 \ldots j_p; k_1 \ldots k_q} = t(e^{j_1} \wedge \ldots \wedge e^{j_p}) \cdot (e^{k_1} \wedge \ldots \wedge e^{k_q})
\]

such that

\[
t = \frac{1}{p!q!} t_{j_1 \ldots j_p; k_1 \ldots k_q} \varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q},
\]

Such \(t_{j_1 \ldots j_p; k_1 \ldots k_q}\) will be called the \(j_1 \ldots j_p; k_1 \ldots k_q\)-th contravariant components of \(t\) with respect to the \((p, q)\)-extensor basis \(\{\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}\}\).

Of course, there are still other kinds of \((p, q)\)-extensor bases for \(\text{ext}_{p}^q(V)\) besides the one given by Eq. (4) which can be constructed from the vector bases \(\{e_j\}\) and \(\{e^j\}\). The total number of these different kinds of \((p, q)\)-extensor bases for \(\text{ext}_{p}^q(V)\) are \(2^{p+q}\).

Now, if we take the basis \((p, q)\)-extensors \(\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}\) and the real numbers \(t_{j_1 \ldots j_p; k_1 \ldots k_q}\) defined by

\[
\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}(X) = (e^{j_1} \wedge \ldots \wedge e^{j_p}) \cdot X e^{k_1} \wedge \ldots \wedge e^{k_q},
\]

\[
t_{j_1 \ldots j_p; k_1 \ldots k_q} = t(e^{j_1} \wedge \ldots \wedge e^{j_p}) \cdot (e^{k_1} \wedge \ldots \wedge e^{k_q}),
\]

we get an expansion formula for \(t \in \text{ext}_{p}^q(V)\) analogous to that given by Eq. (6), i.e.,

\[
t = \frac{1}{p!q!} t_{j_1 \ldots j_p; k_1 \ldots k_q} \varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q},
\]

Such \(t_{j_1 \ldots j_p; k_1 \ldots k_q}\) are called the \(j_1 \ldots j_p; k_1 \ldots k_q\)-th contravariant components of \(t\) with respect to the \((p, q)\)-extensor basis \(\{\varepsilon^{j_1 \ldots j_p; k_1 \ldots k_q}\}\).
2.2 Extensors

A linear mapping which sends multivectors to multivectors will be simply called an **extensor** over \( V \). They are the linear operators on \( \bigwedge V \). For the space of extensors over \( V \), namely \( 1\text{-}\text{ext}(\bigwedge V; \bigwedge V) \), we will use the short notation \( \text{ext}(V) \). By using Eq.\((2)\) we get

\[
\text{dim ext}(V) = 2^n 2^n. \tag{10}
\]

For instance, we will see that the so-called Hodge star operator is just a well-defined extensor over \( V \) which can be thought as an exterior direct sum of \((p, n-p)\)-extensor over \( V \). The extended (or exterior power) of \( t \in \text{ext}^1_1(V) \) is just an extensor over \( V \), i.e., \( t \in \text{ext}(V) \).

There are \( 2^n 2^n \) extensors over \( V \), namely \( \varepsilon^{J;K} \), given by\(^1\)

\[
\varepsilon^{J;K}(X) = (e^J \cdot X)e^K \tag{11}
\]

which can be used to introduce an extensor bases for \( \text{ext}(V) \).

In fact they are linearly independent, and for each \( t \in \text{ext}(V) \) there exist \( 2^n 2^n \) real numbers, say \( t_{J;K} \), given by

\[
t_{J;K} = t(e_J) \cdot e_K \tag{12}
\]

such that

\[
t = \sum_J \sum_K \frac{1}{\nu(J)!} \frac{1}{\nu(K)!} t_{J;K} \varepsilon^{J;K}. \tag{13}
\]

Such \( t_{J;K} \) will be called the \( J;K \)-th covariant components of \( t \) with respect to the extensor bases \( \{\varepsilon^{J;K}\} \).

We notice that exactly \( (2^n+1)^2 \) extensor bases for \( \text{ext}(V) \) can be constructed from the basis vectors \( \{e_J\} \) and \( \{e^J\} \). For instance, whenever the basis extensors \( \varepsilon_{J;K} \) and the real numbers \( t_{J;K} \) defined by

\[
\varepsilon_{J;K}(X) = (e_J \cdot X)e_K, \tag{14}
\]

\[
t_{J;K} = t(e^J) \cdot e^K \tag{15}
\]

\(^1\) \( J \) and \( K \) are collective indices. Recall, for example, that: \( e_J = 1, e_j, e_j \wedge e_j, \ldots, (e^J = 1, e^j, e^j \wedge e^j, \ldots) \) and \( \nu(J) = 0,1,2,\ldots \) for \( J = \emptyset, j_1, j_1j_2, \ldots, \) where all index \( j_1, j_2, \ldots \) runs from 1 to \( n \).
are used, an expansion formula for \( t \in \text{ext}(V) \) analogous to that given by Eq.\((13)\) can be obtained, i.e.,

\[
t = \sum_{J} \sum_{K} \frac{1}{\nu(J)!} \frac{1}{\nu(K)!} t^{J;K} \varepsilon_{J;K}.
\]  

(16)

Such \( t^{J;K} \) are called the \( J; K \)-th contravariant components of \( t \) with respect to the extensor bases \( \{ \varepsilon_{J;K} \} \).

### 2.3 Elementary \( k \)-Extensors

A multilinear mapping which takes \( k \)-uple of vectors into \( q \)-vectors will be called an elementary \( k \)-extensor over \( V \) of degree \( q \). The space of these objects, namely \( k\text{-ext}(V, \ldots, V; \bigwedge^q V) \), will be denoted by \( k\text{-ext}^q(V) \). It is easy to verify (using Eq.\((2)\)) that

\[
\dim k\text{-ext}^q(V) = n^k \binom{n}{q}.
\]  

(17)

It should be noticed that an elementary \( k \)-extensor over \( V \) of degree 0 is just a covariant \( k \)-tensor over \( V \), i.e., \( k\text{-ext}^0(V) \equiv T_k(V) \). It is easily realized that \( 1\text{-ext}^q(V) \equiv \text{ext}^q_1(V) \).

The elementary \( k \)-extensors of degrees 0, 1, 2, \ldots etc. are sometimes said to be scalar, vector, bivector, \ldots etc. elementary \( k \)-extensors.

The \( n^k \binom{n}{q} \) elementary \( k \)-extensors of degree \( q \) belonging to \( k\text{-ext}^q(V) \), namely \( \varepsilon^{j_1, \ldots, j_k; k_1, \ldots, k_q} \), given by

\[
\varepsilon^{j_1, \ldots, j_k; k_1, \ldots, k_q}(v_1, \ldots, v_k) = (v_1 \cdot e^{j_1}) \ldots (v_k \cdot e^{j_k}) e^{k_1} \wedge \ldots \wedge e^{k_q}
\]  

(18)

define elementary basis vectors, (i.e., \( k \)-extensor of degree \( q \)) for \( k\text{-ext}^q(V) \).

In fact they are linearly independent, and for all \( t \in k\text{-ext}^q(V) \) there are \( n^k \binom{n}{q} \) real numbers, say \( t_{j_1, \ldots, j_k; k_1, \ldots, k_q} \), given by

\[
t_{j_1, \ldots, j_k; k_1, \ldots, k_q} = t(e_{j_1}, \ldots, e_{j_k}) \cdot (e_{k_1} \wedge \ldots \wedge e_{k_q})
\]  

(19)

such that

\[
t = \frac{1}{q!} t_{j_1, \ldots, j_k; k_1, \ldots, k_q} \varepsilon^{j_1, \ldots, j_k; k_1, \ldots, k_q}.
\]  

(20)

Such \( t_{j_1, \ldots, j_k; k_1, \ldots, k_q} \) will be called the \( j_1, \ldots, j_k; k_1, \ldots, k_q \)-th covariant components of \( t \) with respect to the bases \( \{ \varepsilon^{j_1, \ldots, j_k; k_1, \ldots, k_q} \} \).
We notice that exactly \(2^{k+q}\) elementary \(k\)-extensors of degree \(q\) bases for \(k\text{-ext}^q(V)\) can be constructed from the vector bases \(\{e_j\}\) and \(\{e^j\}\). For instance, we may define \(\varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}\) (the basis elementary \(k\)-extensor of degree \(q\)) and the real numbers \(t^{j_1,\ldots,j_k;k_1\ldots k_q}\) by

\[
\varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}(v_1,\ldots, v_k) = (v_1 \cdot e_{j_1}) \cdots (v_k \cdot e_{j_k}) e_{k_1} \wedge \cdots \wedge e_{k_q}, \quad (21) \\
t^{j_1,\ldots,j_k;k_1\ldots k_q} = t(e^{j_1},\ldots, e^{j_k}) \cdot (e^{k_1} \wedge \cdots \wedge e^{k_q}). \quad (22)
\]

Then, we also have other expansion formulas for \(t \in k\text{-ext}^q(V)\) besides that given by Eq.\((21)\), e.g.,

\[
t = \frac{1}{q!} t^{j_1,\ldots,j_k;k_1\ldots k_q} \varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}. \quad (23)
\]

Such \(t^{j_1,\ldots,j_k;k_1\ldots k_q}\) are called the \(j_1,\ldots,j_k;k_1\ldots k_q\)-th contravariant components of \(t\) with respect to the bases \(\{\varepsilon_{j_1,\ldots,j_k;k_1\ldots k_q}\}\).

## 3 Projectors

Let \(\Lambda^\circ V\) be either any sum of homogeneous subspaces\(^2\) of \(\Lambda V\) or the trivial subspace \(\{0\}\). Associated to \(\Lambda^\circ V\), a noticeable extensor from \(\Lambda V\) to \(\Lambda^\circ V\), namely \(\langle \rangle_{\Lambda^\circ V}\), can defined by

\[
\langle X \rangle_{\Lambda^\circ V} = \begin{cases} 
\langle X \rangle_{\Lambda^{p_1} V + \cdots + \Lambda^{p_\nu} V}, & \text{if } \Lambda^\circ V = \Lambda^{p_1} V + \cdots + \Lambda^{p_\nu} V \\
0, & \text{if } \Lambda^\circ V = \{0\}. 
\end{cases} \quad (24)
\]

Such \(\langle \rangle_{\Lambda^\circ V} \in 1\text{-ext}(\Lambda V; \Lambda^\circ V)\) will be called the \(\Lambda^\circ V\)-projector extensor.

We notice that if \(\Lambda^\circ V\) is any homogeneous subspace of \(\Lambda V\), i.e., \(\Lambda^\circ V = \Lambda^p V\), then the projector extensor is reduced to the so-called \(p\)-part operator, i.e., \(\langle \rangle_{\Lambda^\circ V} = \langle \rangle_p\).

We now summarize the fundamental properties for the \(\Lambda^\circ V\)-projector extensors.

Let \(\Lambda_1^\circ V\) and \(\Lambda_2^\circ V\) be two subspaces of \(\Lambda V\). If each of them is either any sum of homogeneous subspaces of \(\Lambda V\) or the trivial subspace \(\{0\}\), then

\[
\langle \langle X \rangle_{\Lambda_1^\circ V} \rangle_{\Lambda_2^\circ V} \rangle_{\Lambda_2^\circ V} = \langle X \rangle_{\Lambda_1^\circ V \cap \Lambda_2^\circ V} \quad (25)
\]

\[
\langle X \rangle_{\Lambda_1^\circ V} + \langle X \rangle_{\Lambda_2^\circ V} = \langle X \rangle_{\Lambda_1^\circ V \cup \Lambda_2^\circ V}. \quad (26)
\]

\(^2\)Note that for such a subspace \(\Lambda^\circ V\) there are \(\nu\) integers \(p_1,\ldots,p_\nu\) (\(0 \leq p_1 < \cdots < p_\nu \leq n\)) such that \(\Lambda^\circ_1 V = \Lambda^{p_1} V + \cdots + \Lambda^{p_\nu} V\).
Let $\wedge^p V$ be either any sum of homogeneous subspaces of $\wedge V$ or the trivial subspace $\{0\}$. Then, it holds

$$\langle X \rangle_{\wedge^p V} \cdot Y = X \cdot \langle Y \rangle_{\wedge^p V}.$$  \hfill (27)

We see that the concept of $\wedge^p V$-projector extensor is just a natural generalization of the concept of $p$-part operator.

### 4 The Extension Operator

Let $\{e_j\}$ be any basis for $V$, and $\{\varepsilon^j\}$ be its dual basis for $V^*$. As we know, $\{\varepsilon^j\}$ is the unique 1-form basis associated to the vector basis $\{e_j\}$ such that $\varepsilon^j(e_i) = \delta_i^j$. The linear mapping $\text{ext}(V) \ni t \mapsto \underline{t} \in \text{ext}(V)$ such that for any $X \in \wedge V$ and $X = X_0 + \sum_{k=1}^n X_k$, then

$$\underline{t}(X) = X_0 + \sum_{k=1}^n \frac{1}{k!} X_k (\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k})$$  \hfill (28)

will be called the extension operator. We call $\underline{t}$ the extended of $t$. It is the well-known outermorphism of $t$ in ordinary linear algebra.

The extension operator is well-defined since it does not depend on the choice of $\{e_j\}$.

We summarize now the basic properties satisfied by the extension operator.

**e1** The extension operator is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^p V, \text{ then } t(X) \in \bigwedge^p V.$$  \hfill (29)

It is an obvious result which follows from Eq. (28).

**e2** For any $\alpha \in \mathbb{R}$, $v \in V$ and $v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V$,

$$\underline{t}(\alpha) = \alpha,$$  \hfill (30)

$$\underline{t}(v) = t(v),$$  \hfill (31)

$$\underline{t}(v_1 \wedge \ldots \wedge v_k) = t(v_1) \wedge \ldots \wedge t(v_k).$$  \hfill (32)

**Proof**

8
The first statement trivially follows from Eq. (28). The second one can easily be deduced from Eq. (28) by recalling the elementary expansion formula for vectors and the linearity of extensors. In order to prove the third statement we use the formulas:

\[ v_1 \wedge \ldots \wedge v_k (\omega^1, \ldots, \omega^k) = \varepsilon^{i_1 \ldots i_k} \omega^1(v_{i_1}) \ldots \omega^k(v_{i_k}) \]

and

\[ w_i_1 \wedge \ldots \wedge w_i_k = \varepsilon_{i_1 \ldots i_k} w_1 \wedge \ldots \wedge w_k, \]

where \( v_1, \ldots, v_k \in V, w_1, \ldots, w_k \in V \) and \( \omega^1, \ldots, \omega^k \in V^* \), and the combinatorial formula \( \varepsilon^{i_1 \ldots i_k} \varepsilon_{i_1 \ldots i_k} = k! \). From Eq. (28) by recalling the elementary expansion formula for vectors and the linearity of extensors we have that

\[ t(v_1 \wedge \ldots \wedge v_k) = \frac{1}{k!} v_1 \wedge \ldots \wedge v_k (\varepsilon^{j_1}, \ldots, \varepsilon^{j_k}) t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k}) \]

\[ = \frac{1}{k!} \varepsilon^{i_1 \ldots i_k} \varepsilon^{j_1}(v_{i_1}) \ldots \varepsilon^{j_k}(v_{i_k}) t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k}) \]

\[ = \frac{1}{k!} \varepsilon^{i_1 \ldots i_k} t(v_{i_1}) \wedge \ldots \wedge t(v_{i_k}), \]

\[ = t(v_1) \wedge \ldots \wedge t(v_k). \]

\[ e3 \] For any \( X, Y \in \bigwedge V \),

\[ t(X \wedge Y) = t(X) \wedge t(Y). \]

Eq. (33) an immediate result which follows from Eq. (32).

We emphasize that the three fundamental properties as given by Eq. (30), Eq. (31) and Eq. (33) together are completely equivalent to the extension procedure as defined by Eq. (28).

We present next some important properties for the extension operator.

\[ e4 \] Let us take \( s, t \in ext^1_1(V) \). Then, the following result holds

\[ s \circ t = s \circ t. \]

**Proof**

It is enough to present the proofs for scalars and simple \( k \)-vectors.

For \( \alpha \in \mathbb{R} \), by using Eq. (30) we get

\[ s \circ t(\alpha) = \alpha = s(\alpha) = s(t(\alpha)) = s \circ t(\alpha). \]

For a simple \( k \)-vector \( v_1 \wedge \ldots \wedge v_k \in \bigwedge^k V \), by using Eq. (32) we get

\[ s \circ t(v_1 \wedge \ldots \wedge v_k) = s \circ t(v_1) \wedge \ldots \wedge s \circ t(v_k) = s(t(v_1)) \wedge \ldots \wedge s(t(v_k)) \]

\[ = s(t(v_1)) \wedge \ldots \wedge t(v_k) = s(t(v_1 \wedge \ldots \wedge v_k)), \]

\[ = s \circ t(v_1 \wedge \ldots \wedge v_k). \]
Next, we can easily generalize to multivectors due to the linearity of extensors. It yields

\[ s \circ t(X) = s \circ t(X). \]  

\( \text{e5} \) Let us take \( t \in ext^1_1(V) \) with inverse \( t^{-1} \in ext^1_1(V) \), i.e., \( t^{-1} \circ t = t \circ t^{-1} = i_V \). Then, \( (t^{-1}) \in ext(V) \) is the inverse of \( t \in ext(V) \), i.e.,

\[ (t)^{-1} = (t^{-1}). \]  

(35)

Indeed, by using Eq.(34) and the obvious property \( i_V = i \wedge V \), we have that

\[ t^{-1} \circ t = t \circ t^{-1} = i_V \Rightarrow (t^{-1}) \circ t = t \circ (t^{-1}) = i_V, \]

which means that the inverse of the extended of \( t \) equals the extended of the inverse of \( t \).

In accordance with the above corollary we use in what follows a more simple notation \( t^{-1} \) to denote both \( (t)^{-1} \) and \( (t^{-1}) \).

Let \( \{e_j\} \) be any basis for \( V \), and \( \{e^j\} \) its euclidean reciprocal basis for \( V \), i.e., \( e_j \cdot e^k = \delta^k_j \). There are two interesting and useful formulas for calculating the extended of \( t \in ext^1_1(V) \), i.e.,

\[ t(X) = 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!} (e^{j_1} \wedge \ldots \wedge e^{j_k}) \cdot X t(e_{j_1}) \wedge \ldots \wedge t(e_{j_k}) \]  

(36)

\[ = 1 \cdot X + \sum_{k=1}^{n} \frac{1}{k!} (e_{j_1} \wedge \ldots \wedge e_{j_k}) \cdot X t(e^{j_1}) \wedge \ldots \wedge t(e^{j_k}). \]  

(37)

5 Adjoint Operator

5.1 Standard Adjoint Operator

Let \( \Lambda^1_1 V \) and \( \Lambda^2_2 V \) be two subspaces of \( \Lambda V \) such that each of them is any sum of homogeneous subspaces of \( \Lambda V \). Let \( \{e_j\} \) and \( \{e^j\} \) be two euclidean reciprocal bases to each other for \( V \), i.e., \( e_j \cdot e^k = \delta^k_j \).

We call standard adjoint operator of \( t \) the linear mapping \( 1-ext(\Lambda^1_1 V; \Lambda^2_2 V) \in \)
t \rightarrow t^\dagger \in 1\text{-}ext(\bigwedge^\circ_2 V; \bigwedge^\circ_1 V) \text{ such that for any } Y \in \bigwedge^\circ_2 V:\]

\begin{align*}
\ t^\dagger(Y) & = \ t(\langle 1 \rangle_{\bigwedge^\circ_1 V}) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} t(\langle e_{j_1} \wedge \ldots \wedge e_{j_k} \rangle_{\bigwedge^\circ_1 V}) \cdot Y e_{j_1} \wedge \ldots \wedge e_{j_k} \quad (38) \\
& = \ t(\langle 1 \rangle_{\bigwedge^\circ_1 V}) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} t(\langle e_{j_1} \wedge \ldots \wedge e_{j_k} \rangle_{\bigwedge^\circ_1 V}) \cdot Y e_{j_1} \wedge \ldots \wedge e_{j_k}. \quad (39)
\end{align*}

Using a more compact notation by employing the 	extit{collective index} \( J \) we can write

\begin{align*}
\ t^\dagger(Y) & = \sum_{J} \frac{1}{\nu(J)!} t(\langle e^J \rangle_{\bigwedge^\circ_1 V}) \cdot Y e^J \\
& = \sum_{J} \frac{1}{\nu(J)!} t(\langle e^J \rangle_{\bigwedge^\circ_1 V}) \cdot Y e^J, \quad (40)
\end{align*}

We call \( t^\dagger \) the \textit{standard adjoint} of \( t \). It should be noticed the use of the \( \bigwedge^\circ_1 V \)-projector extensor.

The standard adjoint operator is well-defined since the sums appearing in each one of the above places do not depend on the choice of \( \{ e_j \} \).

Let us take \( X \in \bigwedge^\circ_1 V \) and \( Y \in \bigwedge^\circ_2 V \). A straightforward calculation yields

\begin{align*}
X \cdot t^\dagger(Y) & = \sum_{J} \frac{1}{\nu(J)!} t(\langle e^J \rangle_{\bigwedge^\circ_1 V}) \cdot Y (X \cdot e^J) \\
& = t(\sum_{J} \frac{1}{\nu(J)!} \langle (X \cdot e^J) e^J \rangle_{\bigwedge^\circ_1 V}) \cdot Y \\
& = t(\langle X \rangle_{\bigwedge^\circ_1 V}) \cdot Y,
\end{align*}

i.e.,

\begin{equation}
X \cdot t^\dagger(Y) = t(X) \cdot Y. \quad (42)
\end{equation}

It is a generalization of the well-known property which holds for linear operators.

Let us take \( t \in 1\text{-}ext(\bigwedge^\circ_1 V; \bigwedge^\circ_2 V) \) and \( u \in 1\text{-}ext(\bigwedge^\circ_2 V; \bigwedge^\circ_3 V) \). We can note that \( u \circ t \in 1\text{-}ext(\bigwedge^\circ_1 V; \bigwedge^\circ_3 V) \) and \( t^\dagger \circ u^\dagger \in 1\text{-}ext(\bigwedge^\circ_3 V; \bigwedge^\circ_1 V) \). Then, let us take \( X \in \bigwedge^\circ_1 V \) and \( Z \in \bigwedge^\circ_3 V \), by using Eq. (42) we have that

\begin{equation}
X \cdot (u \circ t)^\dagger(Z) = (u \circ t)(X) \cdot Z = t(X) \cdot u^\dagger(Z) = X \cdot (t^\dagger \circ u^\dagger)(Z). \quad (43)
\end{equation}
Hence, we get
\[(u \circ t) \dagger = t^\dagger \circ u^\dagger. \tag{43}\]

Let us take \(t \in 1\text{-}ext(\Lambda^\circ V; \Lambda^\circ V)\) with inverse \(t^{-1} \in 1\text{-}ext(\Lambda^\circ V; \Lambda^\circ V)\), i.e., \(t^{-1} \circ t = t \circ t^{-1} = i_{\Lambda^\circ V}\), where \(i_{\Lambda^\circ V} \in 1\text{-}ext(\Lambda^\circ V; \Lambda^\circ V)\) is the so-called identity function for \(\Lambda^\circ V\). By using Eq.\!(43)\! and the obvious property 
\(i_{\Lambda^\circ V} = i_{\Lambda^\circ V}^\dagger\), we have that 
\[t^{-1} \circ t = t \circ t^{-1} = i_{\Lambda^\circ V} \Rightarrow t^\dagger \circ (t^{-1})^\dagger = (t^{-1})^\dagger \circ t^\dagger = i_{\Lambda^\circ V},\]
hence,
\[(t^\dagger)^{-1} = (t^{-1})^\dagger, \tag{44}\]
i.e., the inverse of the adjoint of \(t\) equals the adjoint of the inverse of \(t\). In accordance with the above corollary it is possible to use a more simple symbol, say \(t^*\), to denote both \((t^\dagger)^{-1}\) and \((t^{-1})^\dagger\).

Let us take \(t \in ext_1(V)\). We note that \(t \in ext(V)\) and \((t^\dagger) \in ext(V)\). A straightforward calculation by using Eqs.\!(36)\! and \!(37)\! yields
\[
(t^\dagger)(Y) = 1 \cdot Y + \sum_{k=1}^{\infty} \frac{1}{k!} (e_{j_1} \wedge \ldots \wedge e_{j_k}) \cdot Y t^\dagger(e_{j_1}) \wedge \ldots \wedge t^\dagger(e_{j_k}) \\
= 1 \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} (e_{j_1} \wedge \ldots \wedge e_{j_k}) \cdot Y t^\dagger(e_{j_1}) \cdot e_{p_1} e_{p_1} \wedge \ldots \wedge t^\dagger(e_{j_k}) \cdot e_{p_k} e_{p_k} \\
= 1 \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} (e_{j_1} \cdot t(e_{p_1}) e_{j_1} \wedge \ldots \wedge e_{j_k} \cdot t(e_{p_k}) e_{j_k}) \cdot Y e_{p_1} \wedge \ldots \wedge e_{p_k} \\
= t(1) \cdot Y + \sum_{k=1}^{n} \frac{1}{k!} (e_{p_1} \wedge \ldots \wedge e_{p_k}) \cdot Y e_{p_1} \wedge \ldots \wedge e_{p_k} \\
= (t^\dagger)(Y).
\]
Hence, we get
\[(t^\dagger) = (t^\dagger)^\dagger. \tag{45}\]

This means that the extension operator commutes with the adjoint operator. In accordance with the above property we may use a more simple notation \(t^\dagger\) to denote without ambiguities both \(t^\dagger\) and \((t^\dagger)^\dagger\).
5.2 Metric Adjoint Operator

As we know [6] whenever \( V \) is endowed with another metric \( G \) (besides \( G_E \)) there exists an unique \((1,1)\)-extensor \( g \) such that the \( G \)-scalar product of \( X, Y \in \bigwedge V \), namely \( X \cdot_g Y \), is given by

\[
X \cdot_g Y = g(X) \cdot Y. \quad (46)
\]

Such \( g \in \text{ext}_1^1(V) \) is symmetric and non-degenerate, and has signature \((p,q)\), i.e., \( g = g^\dagger \), \( \det[g] \neq 0 \), \( g \) has \( p \) positive and \( q \) negative \((p+q = n)\) eigenvalues. It is called the metric extensor for \( G \).

To each \( t \in 1-\text{ext}(\bigwedge^2 V; \bigwedge^1 V) \) we can assign \( t^{(g)} \in 1-\text{ext}(\bigwedge^2 V; \bigwedge^1 V) \) defined as follows

\[
t^{(g)} = g^{-1} \circ t^\dagger \circ g. \quad (47)
\]

It will be called the metric adjoint of \( t \).

As we can easily see, \( t^{(g)} \) is the unique extensor from \( \bigwedge^2 V \) to \( \bigwedge^1 V \) which satisfies the fundamental property

\[
X \cdot_g t^{(g)}(Y) = t(X) \cdot_g Y, \quad (48)
\]

for all \( X \in \bigwedge^1 V \) and \( Y \in \bigwedge^2 V \).

The noticeable property given by Eq.(48) is the metric version of the fundamental property given by Eq.(42).

6 The Generalization Operator

6.1 Standard Generalization Operator

Let \( \{e_k\} \) be any basis for \( V \), and \( \{e^k\} \) be its euclidean reciprocal basis for \( V \), as we know, \( e_k \cdot e^l = \delta^l_k \).

The linear mapping \( \text{ext}_1^1(V) \ni t \mapsto \sim t \in \text{ext}(V) \) such that for any \( X \in \bigwedge V \)

\[
\sim t(X) = t(e^k) \wedge (e_k \perp X) = t(e_k) \wedge (e^k \perp X) \quad (49)
\]

will be called the generalization operator. We call \( \sim t \) the generalized of \( t \).

The generalization operator is well-defined since it does not depend on the choice of \( \{e_k\} \).
We present now some important properties which are satisfied by the generalization operator.

**g1** The generalization operator is grade-preserving, i.e.,

\[ \text{if } X \in \bigwedge_k V, \text{ then } \sim(t(X)) \in \bigwedge_k V. \quad (50) \]

**g2** The grade involution \( \hat{\sim} \in \text{ext}(V) \), reversion \( \tilde{\sim} \in \text{ext}(V) \), and conjugation \( \hat{\sim} \in \text{ext}(V) \) commute with the generalization operator, i.e.,

\[
\begin{align*}
\tilde{\sim}(X) &= \tilde{\sim}(\sim(X)), \\
\hat{\sim}(X) &= \hat{\sim}(\sim(X)), \\
\hat{\sim}(X) &= \hat{\sim}(\sim(X)).
\end{align*}
\]

They are immediate consequences of the grade-preserving property.

**g3** For any \( \alpha \in \mathbb{R}, v \in V \) and \( X, Y \in \bigwedge V \) it holds

\[
\begin{align*}
\sim(\alpha) &= 0, \\
\sim(v) &= \sim(t(v)), \\
\sim(t(X \wedge Y)) &= \sim(t(X) \wedge Y + X \wedge \sim(t(Y)).
\end{align*}
\]

The proof of Eq. (54) and Eq. (55) are left to the reader. Hint: \( v \cdot \alpha = 0 \) and \( v \cdot w = v \cdot w \). Now, the identities: \( a \wedge (X \wedge Y) = (a \wedge X) \wedge Y + \hat{X} \wedge (a \wedge Y) \) and \( a \wedge X = \hat{X} \wedge a \), with \( a \in V \) and \( X, Y \in \bigwedge V \), allow us to prove the property given by Eq. (56).

We can prove that the basic properties given by Eq. (54), Eq. (55) and Eq. (56) together are completely equivalent to the generalization procedure as defined by Eq. (49).

**g4** The generalization operator commutes with the adjoint operator, i.e.,

\[ (\sim(t))^\dagger = (t^\dagger), \quad (57) \]

or put it on another way, the adjoint of the generalized of \( t \) is just the generalized of the adjoint of \( t \).
Proof

A straightforward calculation by using Eq.(42) and the multivector identities: \( X \cdot (a \wedge Y) = (a \wedge X) \wedge Y \) and \( X \cdot (a \perv X Y) = (a \wedge X) \cdot Y \), with \( a \in V \) and \( X, Y \in \wedge V \), gives

\[
(t)\hat{\dagger}(X) \cdot Y = X \cdot t(Y) = (e_j \wedge (t(e^j) \perv X)) \cdot Y = (t^\dagger e_k \wedge (e_k \perv X)) \cdot Y = (t^\dagger)(X) \cdot Y.
\]

Hence, by the non-degeneracy property of the euclidean scalar product, the required result follows. ■

In agreement with the above property we use in what follows a more simple symbol, \( t^\dagger \) to denote both \( (t)\hat{\dagger} \) or \( (t^\dagger) \).

**g5** The symmetric (skew-symmetric) part of the generalized of \( t \) is just the generalized of the symmetric (skew-symmetric) part of \( t \), i.e.,

\[
(t)_{\pm} = (t_{\pm}). \tag{58}
\]

This property follows immediately from Eq.(57).

We see also that it is possible to use a more simple notation, \( t_{\pm} \) to denote \( (t)_{\pm} \) or \( (t^\dagger)_{\pm} \).

**g6** The skew-symmetric part of the generalized of \( t \) can be factorized by the noticeable formula

\[
t_{\perp}(X) = \frac{1}{2} \text{biv}[t] \times X, \tag{59}
\]

where \( \text{biv}[t] \equiv t(e^k) \wedge e_k \) is a characteristic invariant of \( t \), called the bivector of \( t \).

Proof

By using Eq.(58), the well-known identity \( t_{\perp}(a) = \frac{1}{2} \text{biv}[t] \times a \) and the multivector identity \( B \times X = (B \times e^k) \wedge (e_k \perv X) \), with \( B \in \wedge^2 V \) and \( X \in \wedge V \), we have that

\[
t_{\perp}(X) = t_{\perp}(e^k) \wedge (e_k \perv X) = \frac{1}{2} \text{biv}[t] \times e^k \wedge (e_k \perv X) = \frac{1}{2} \text{biv}[t] \times X. \tag{59}
\]

\(^3\)Recall that \( X \times Y \equiv \frac{1}{2}(XY - YX) \).
A noticeable formula holds for the skew-symmetric part of the generalized of $t$. For all $X, Y \in \wedge V$

$$t_\sim (X \ast Y) = t_\sim (X) \ast Y + X \ast t_\sim (Y), \quad (60)$$

where $\ast$ is any product either $(\wedge), (\cdot), (\ll, \ll)$ or (Clifford product).

In order to prove this property we must use Eq.(59) and the multivector identity $B \times (X \ast Y) = (B \times X) \ast Y + X \ast (B \times Y)$, with $B \in \wedge^2 V$ and $X, Y \in \wedge V$. By taking into account Eq.(54) we can see that the following property for the euclidean scalar product of multivectors holds

$$t_\sim (X) \cdot Y + X \cdot t_\sim (Y) = 0. \quad (61)$$

It is consistent with the well-known property: the adjoint of a skew-symmetric extensor equals minus the extensor!

### 7 Determinant

Let $t$ be any $(1, 1)$-extensor. We define the determinant$^4$ of $t$ as the unique real number, denoted by $\det[t]$, such that for all non-zero pseudoscalar $I$

$$t(I) = \det[t]I. \quad (62)$$

It is a well-defined scalar invariant since it does not depend on the choice of $I$.

We present now some of the most important properties satisfied by the determinant.

- **d1** Let $t$ and $u$ be two $(1, 1)$–extensors. It holds

$$\det[u \circ t] = \det[u] \det[t]. \quad (63)$$

**Proof**

Take a non-zero pseudoscalar $I \in \wedge^n V$. By using Eq.(51) and Eq.(62) we can write that

$$\det[u \circ t]I = u \circ t(I) = u \circ t(I) = u(t(I)) = u(\det[t]I) = \det[t]u(I),$$

$$= \det[t] \det[u]I. \blacksquare$$

$^4$The concept of determinant of a $(1, 1)$-extensor is related, but distinct from the well-known determinant of a square matrix. For details the reader is invited to consult [10].
\textbf{d2} Let us take \( t \in \text{ext}^1_1(V) \) with inverse \( t^{-1} \in \text{ext}^1_1(V) \). It holds
\[
\det[t^{-1}] = (\det[t])^{-1}. \tag{64}
\]
Indeed, by using Eq. (63) and the obvious property \( \det[i_V] = 1 \), we have that
\[
t^{-1} \circ t = t \circ t^{-1} = i_V \Rightarrow \det[t^{-1}] \det[t] = \det[t] \det[t^{-1}] = 1,
\]
which means that the determinant of the inverse equals the inverse of the determinant.

Due to the above corollary it is often convenient to use the short notation \( \det^{-1} [t] \) for both \( \det[t^{-1}] \) and \( (\det[t])^{-1} \).

\textbf{d3} Let us take \( t \in \text{ext}^1_1(V) \). It holds
\[
\det[t^\dagger] = \det[t]. \tag{65}
\]
Indeed, take a non-zero \( I \in \bigwedge^n V \). Then, by using Eq. (62) and Eq. (42) we have that
\[
\det[t^\dagger] I \cdot I = t^\dagger (I) \cdot I = I \cdot t(I) = I \cdot \det[t] I = \det[t] I \cdot I,
\]
whence, the expected result follows.

Let \( \{e_j\} \) be any basis for \( V \), and \( \{e^j\} \) be its euclidean reciprocal basis for \( V \), i.e., \( e_j \cdot e^k = \delta_j^k \). There are two interesting and useful formulas for calculating \( \det[t] \), i.e.,
\[
\det[t] = t(e_1 \wedge \ldots \wedge e_n) \cdot (e^1 \wedge \ldots \wedge e^n), \tag{66}
\]
\[
= t(e^1 \wedge \ldots \wedge e^n) \cdot (e_1 \wedge \ldots \wedge e_n). \tag{67}
\]
They follow from Eq. (62) by using \( (e_1 \wedge \ldots \wedge e_n) \cdot (e^1 \wedge \ldots \wedge e^n) = 1 \) which is an immediate consequence of the formula for the euclidean scalar product of simple \( k \)-vectors and the reciprocity property of \( \{e_k\} \) and \( \{e^k\} \).

Each of Eq. (66) and Eq. (67) is completely equivalent to the definition of determinant given by Eq. (62).

We will end this section presenting an useful formula for the inversion of a non-singular \((1, 1)\)-extensor.

Let us take \( t \in \text{ext}^1_1(V) \). If \( t \) is non-singular, i.e., \( \det[t] \neq 0 \), then there exists its inverse \( t^{-1} \in \text{ext}^1_1(V) \) which is given by
\[
t^{-1}(v) = \det^{-1} [t] t^\dagger (vI) I^{-1}, \tag{68}
\]
\text{17}
where $I \in \bigwedge^n V$ is any non-zero pseudoscalar.

**Proof**

We must prove that $t^{-1}$ given by the formula above satisfies both of conditions $t^{-1} \circ t = i_V$ and $t \circ t^{-1} = i_V$.

Let $I \in \bigwedge^n V$ be a non-zero pseudoscalar. Take $v \in V$, by using the extensor identities\footnote{These extensor identities follow directly from the fundamental identity $X \ast(Y) = t(t^\dagger(Y))X$ with $X, Y \in \bigwedge V$. For the first one: take $X = v, Y = I$ and use $(t^\dagger)^\dagger = t$, eq. (62) and $\det[t^\dagger] = \det[t]$. For the second one: take $X = vI, Y = I^{-1}$ and use eq. (62).} $t(t(v)I)I^{-1} = t(t^\dagger(vI)I^{-1}) = \det[t]v$, we have that

$$t^{-1} \circ t(v) = t^{-1}(t(v)) = \det^{-1}[t] t(t(v)I)I^{-1} = \det^{-1}[t] \det[t]v = i_V(v).$$

And

$$t \circ t^{-1}(v) = t(t^{-1}(v)) = \det^{-1}[t] t(t^\dagger(vI)I^{-1}) = \det^{-1}[t] \det[t]v = i_V(v).$$

\[\blacksquare\]

8 Hodge Extensor

8.1 Standard Hodge Extensor

Let $\{e_j\}$ and $\{e^j\}$ be two euclidean reciprocal bases to each other for $V$, i.e., $e_j \cdot e^k = \delta^k_j$. Associated to them we define a non-zero pseudoscalar

$$\tau = \sqrt{e_\Lambda \cdot e^\Lambda}, \tag{69}$$

where $e_\Lambda \equiv e_1 \wedge \ldots \wedge e_n \in \bigwedge^n V$ and $e^\Lambda \equiv e^1 \wedge \ldots \wedge e^n \in \bigwedge^n V$. Note that $e_\Lambda \cdot e^\Lambda > 0$, since an euclidean scalar product is positive definite. Such $\tau$ will be called a **standard volume pseudoscalar** for $V$.

The standard volume pseudoscalar has the fundamental property

$$\tau \cdot \tau = \tau \ast \tilde{\tau} = \tau \tilde{\tau} = 1, \tag{70}$$

which follows from the obvious result $e_\Lambda \cdot e^\Lambda = 1$.

From Eq. (70), we can easily get an expansion formula for pseudoscalars of $\bigwedge^n V$, i.e.,

$$I = (I \cdot \tau) \tau. \tag{71}$$

The extensor $\ast \in ext(V)$ which is defined by $\ast : \bigwedge V \to \bigwedge V$ such that

$$\ast X = \tilde{X} \circ \tau = \tilde{X} \tau, \tag{72}$$
will be called a \textit{standard Hodge extensor} on $V$.

It should be noticed that if $X \in p \bigwedge V$, then $\star X \in n-p \bigwedge V$.

That means that $\star$ can be also defined as a $(p, n-p)$-extensor over $V$.

The extensor over $V$, namely $\star^{-1}$, which is given by $\star^{-1} : \bigwedge V \rightarrow \bigwedge V$ such that

$$\star^{-1} X = \tau \ll X = \tau \tilde{X} \quad (74)$$

is the \textit{inverse extensor} of $\star$.

Let us take $X, Y \in \bigwedge V$. By Eq.(70), we have indeed that $\star^{-1} \circ \star X = \tau \tau X = X$, and $\star \circ \star^{-1} X = X \tau \tau = X$, i.e., $\star^{-1} \circ \star = \star \circ \star^{-1} = i_{\bigwedge V}$, where $i_{\bigwedge V} \in \text{ext}(V)$ is the so-called \textit{identity function} for $\bigwedge V$.

Let us take $X, Y \in \bigwedge V$. By using the multivector identity $(X A) \cdot Y = X \cdot (Y \tilde{A})$ and Eq.(70) we get

$$(\star X) \cdot (\star Y) = X \cdot Y. \quad (75)$$

That means that the standard Hodge extensor preserves the euclidean scalar product.

Let us take $X, Y \in \bigwedge^p V$. By using Eq.(71) together with the multivector identity $(X \ll Y) \cdot Z = Y \cdot (\tilde{X} \ll Z)$, and Eq.(75) we get

$$X \ll (\star Y) = (X \cdot Y) \tau. \quad (76)$$

This noticeable identity is completely equivalent to the definition of the standard Hodge extensor given by Eq.(72).

Let us take $X \in \bigwedge^p V$ and $Y \in \bigwedge^{n-p} V$. By using the multivector identity $(X \ll Y) \cdot Z = Y \cdot (\tilde{X} \ll Z)$ and Eq.(74) we get

$$(\star X) \cdot Y \tau = X \ll Y. \quad (77)$$

\section*{8.2 Metric Hodge Extensor}

Let $g$ be a metric extensor on $V$ with signature $(p, q)$, i.e., $g = g^\dagger \in \text{ext}_1^1(V)$ such that $g = g^\dagger$ and $\det[g] \neq 0$, and it has $p$ positive and $q$ negative eigenvalues. Associated to $\{e_j\}$ and $\{e^j\}$ we can define another non-zero pseudoscalar

$$\tau_g = \sqrt{|e_j \cdot e^j|} e^j = \sqrt{\|\det[g]\|} \tau. \quad (78)$$
It will be called a metric volume pseudoscalar for $V$. It has the fundamental property
\[
\tau_{g^{-1}g} \cdot \tau = \tau_{g^{-1}g} \cdot \tau = \tau = (-1)^q.
\] (79)

Eq. (79) follows from Eq. (70) by taking into account the definition of determinant of a $(1,1)$-extensor, and recalling that $\text{sgn}(\det[g]) = (-1)^q$.

An expansion formula for pseudoscalars of $\bigwedge^n V$ can be also obtained from Eq. (79), i.e.,
\[
I = (-1)^q (I \cdot \tau)_{g^{-1}g}.
\] (80)

The extensor $\star \in \text{ext}(V)$ which is defined by $\star : \bigwedge V \to \bigwedge V$ such that
\[
\star X = \tilde{X} = \tilde{X} \cdot \tau_{g^{-1}g}
\] (81)
will be called a metric Hodge extensor on $V$. It should be noticed that in general we need to use of both the $g$ and $g^{-1}$ metric Clifford algebras.

We see that if $X \in \bigwedge^p V$, then $\star X \in \bigwedge^{n-p} V$.

The extensor over $V$, namely $\star^{-1}$, which is given by $\star^{-1} : \bigwedge V \to \bigwedge V$ such that
\[
\star^{-1} X = (-1)^q (\tilde{X} \cdot \tau)_{g^{-1}g} = (-1)^q (\tilde{X} \cdot \tau)_{g^{-1}g}
\] (83)
is the inverse extensor of $\star$.

Let us take $X \in \bigwedge V$. By using Eq. (79), we have indeed that $\star^{-1} \circ \star X = (-1)^q (\tilde{X} \cdot \tau)_{g^{-1}g} = X$, and $\star \cdot \star^{-1} = (-1)^q X \cdot (\tilde{X} \cdot \tau)_{g^{-1}g}$, i.e., $\star^{-1} \circ \star = i \bigwedge V$.

Take $X, Y \in \bigwedge V$. The identity $(X \cdot A) \cdot Y = X \cdot (Y \cdot \tilde{A})$ and Eq. (79) yield
\[
(\star X) \cdot (\star Y) = (-1)^q X \cdot Y.
\] (84)

Take $X, Y \in \bigwedge^p V$. Eq. (80), the identity $(X \cdot Y) \cdot Z = Y \cdot (\tilde{X} \cdot Z)$ and Eq. (81) allow us to obtain
\[
X \cdot (\star Y) = (X \cdot Y) \tau.
\] (85)
This remarkable property is completely equivalent to the definition of the metric Hodge extensor given by Eq. (81).

Take $X \in \bigwedge^p V$ and $Y \in \bigwedge^{n-p} V$. The use of identity $(X \cdot g^{-1} Y) \cdot g^{-1} Z = Y \cdot g^{-1} (\tilde{X} \wedge Z)$ and Eq. (80) yield

$$ (\star g X) \cdot g^{-1} Y \tau g = (-1)^q X \wedge Y. \quad (86) $$

It might as well be asked what is the relationship between the standard and metric Hodge extensors as defined above by Eq. (72) and Eq. (81).

Take $X \in \bigwedge V$. By using Eq. (78), the multivector identity for an invertible $(1, 1)$-extensor $t^{-1}(X) \cdot Y = t^t(X \cdot t^*(Y))$, and the definition of determinant of a $(1, 1)$-extensor we have that

$$ \star g X = g^{-1}(\tilde{X}) \cdot \sqrt{|\det[g]|} \tau g = \sqrt{|\det[g]|} g(\tilde{X} \cdot g^{-1}(\tau)) $$

$$ = \frac{\sqrt{|\det[g]|}}{\det[g]} g(\tilde{X} \cdot \tau) = \frac{\text{sgn}(\det[g])}{\sqrt{|\det[g]|}} g \circ \star g (X), $$

i.e.,

$$ \star g = \frac{(-1)^q}{\sqrt{|\det[g]|}} g \circ \star g. \quad (87) $$

Eq. (87) is then the formula which relates the metric Hodge extensor $\star g$ with the standard Hodge extensor $\star$.

We know [6] that for any metric extensor $g \in \text{ext}^1_1(V)$ there exists a non-singular $(1, 1)$-extensor $h \in \text{ext}^1_1(V)$ such that

$$ g = h^t \circ \eta \circ h, \quad (88) $$

where $\eta \in \text{ext}^1_1(V)$ is an orthogonal metric extensor with the same signature as $g$. Such $h$ is called a gauge extensor for $g$.

We can also get a noticeable formula which relates the $g$-metric Hodge extensor with the $\eta$-metric Hodge extensor.

As we know, the $g$ and $g^{-1}$ contracted products $g \cdot$ and $g^{-1} \cdot$ are related to the $\eta$-contracted product $\eta \cdot$ (recall that $\eta = \eta^{-1}$) by the following golden formulas

$$ h(X \cdot g Y) = h(X) \cdot h(Y), \quad (89) $$

$$ h^*(X \cdot g^{-1} Y) = h^*(X) \cdot h^*(Y). \quad (90) $$

21
Now, take $X \in \bigwedge V$. By using Eq. (90), Eq. (78), the definition of determinant of a $(1,1)$-extensor, Eq. (88) and the obvious equation $\tau = \tau$ we have that

$$\star_{g} X = h^{\dagger}(h^{\ast}(\tilde{X}),_{\eta} h^{\ast}(\tau)) = \sqrt{\det[g]}h^{\dagger}(h^{\ast}(\tilde{X}),_{\eta} \det[h^{\ast}]\tau)$$

$$= |\det[h]| \det[h^{\ast}]h^{\dagger}(h^{\ast}(X),_{\eta-1} \eta) = sgn(\det[h])h^{\dagger} \circ \ast_{\eta} h^{\ast}(X),$$

i.e.,

$$\star_{g} = sgn(\det[h])h^{\dagger} \circ \ast_{\eta} h^{\ast}. \quad (91)$$

This formula which relates the $g$-metric Hodge extensor $\star$ with the $\eta$-metric Hodge extensor $\ast_{\eta}$ will play an important role in the applications we have in mind.

9 Conclusions

In this third paper in a series of eight we recalled some basic notions of the theory of extensors. Together with [5, 6] it completes the algebraic part of our enterprise. The $k$-extensors and $(p,q)$-extensors are introduced in Section 2 and projector operators are studied in Section 3. The extension operator (or exterior power operator) of a given linear operator $t$ (i.e., a $(1,1)$-extensor) on a real vector space $V$ has been studied in details in Section 4. In Section 5 the standard and metric adjoint operators have been given and Section 6 has been dedicated to the generalization operator of $t$ and its properties. Such generalization operator plays an important role in our formulation of the differential geometry of arbitrary (smooth) manifolds which is presented in sequel papers of this series. Section 8 showed some applications of the concept of extensors. In particular, the standard and metric Hodge operators are introduced and the non trivial relation between them is disclosed (Eq. (87)). A formula (Eq. (91)) relating metric Hodge operators corresponding to deformed metric structures is also derived. These formulas, every student of General Relativity and modern geometric theories of Physics will recognize as very useful ones, for they simplify many involved calculations (see, e.g., [7, 8, 9, 1, 2]). To end, we recall that the theory of extensors is a part of a more general theory of multivector functions and functionals, which are presented in [3, 4].
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References

[1] Fernández, V. V., Moya, A. M., and Rodrigues, W. A., Jr., Derivative Operators in Metric and Geometric Structures, submitted for publication.

[2] Fernández, V. V., Moya, A. M., and Rodrigues, W. A., Jr., Riemann and Ricci Fields in Geometric Structures, submitted for publication.

[3] Fernández, V. V., Moya, A. M., and Rodrigues, W. A., Jr., Multivector Functions of Multivector Variable, Adv. Appl. Clifford Algebras 11(S3), 79-91 (2001).

[4] Fernández, V. V., Moya, A. M., and Rodrigues, W. A., Jr., Multivector Functionals, Adv. Appl. Clifford Algebras 11(S3), 93-103 (2001).

[5] Moya, A. M., Fernández, V. V., and Rodrigues, W. A., Jr., Geometrical Algebras, submitted for publication.

[6] Moya, A. M., Fernández, V. V., and Rodrigues, W. A., Jr., Metric and Gauge Extensors, submitted to publication.

[7] Moya, A. M., Fernández, V. V., and Rodrigues, W. A., Jr. Multivector and Extensor Fields on Arbitrary Manifolds, submitted for publication.

[8] Moya, A. M., Fernández, V. V., and Rodrigues, W. A., Jr., Covariant Derivatives of Multivector and Extensor Fields and Intrinsic Cartan Theory, submitted for publication.

[9] Rodrigues, W. A., Jr., Fernández, V. V., and Moya, A. M., Metric Compatible Covariant Derivatives, submitted for publication.

[10] Rodrigues, W. A. Jr., and Oliveira, E. Capelas, The Many Faces of Maxwell, Dirac and Einstein Equations, RP 56-05 IMECC-UNICAMP, http://www.ime.unicamp.br/rel_pesoq/2005/rp56-05.html