The Lorentz-invariant boundary action of the confining string and its universal contribution to the inter-quark potential

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Abstract: We study the boundary contribution to the low energy effective action of the open string describing the confining flux tube in gauge theories. The form of the boundary terms is strongly constrained by the requirement of Lorentz symmetry, which is spontaneously broken by the formation of a long confining flux tube in the vacuum. Writing the boundary action as an expansion in the derivatives of the Nambu-Goldstone modes describing the transverse fluctuations of the string, we single out and put in a closed form the first few Lorentz invariant boundary terms. We also evaluate the leading deviation from the Nambu-Goto string produced by the boundary action on the vacuum expectation value of the Wilson loop and we test this prediction in the 3d Ising gauge model. Our simulation attains a level of precision which is sufficient to test the contribution of this term.

Keywords: Bosonic Strings, Lattice Gauge Field Theories, Wilson loops.
1. Introduction

The confinement of quarks is one of the most fundamental features of the strong interactions and our understanding of it is far from being complete. However, even if the origin of this non-perturbative phenomenon remains unclear, it is still possible to quantitatively understand some of its consequences. In particular the vacuum state in presence of a pair of quark sources is believed to be dominated, for large quark separations, by the formation of a thin flux tube which generates the linear rising of the confining potential. Clearly this configuration breaks spontaneously some of the space-time symmetries of the underlying gauge theory. The resulting massless Nambu-Goldstone bosons are the natural degrees of freedom to be taken into account in the low-energy regime of the theory. They define the embedding of the string-like flux tube in the space-time. As we shall see, one may obtain important consequences of spontaneous breaking of the space-time symmetries, even if the dynamics leading to the formation of the confining flux tube is unknown.

As first proposed by Lüscher, Symanzik and Weisz [1], the effective action of the color flux tube connecting a static quark-antiquark pair in a $d$-dimensional confining Yang-Mills theory is a two-dimensional action describing the world-sheet swept out by the flux string in its time evolution. The leading term in the infrared limit is a free-field action for the $d - 2$ Nambu-Goldstone modes resulting from the spontaneous breaking of the translation invariance in the transverse directions. The ensuing universal string fluctuation effects, namely, the leading correction to the linear term in the static potential [2] and the growth of the flux tube cross-section with the logarithm of the interquark distance [3] were first unambiguously observed many years ago in the $\mathbb{Z}_2$ gauge theory in three dimensions [4, 5, 6] and more recently in non-abelian Yang-Mills theories [7, 8, 9, 10, 11].
In recent years, the great numerical accuracy of Monte Carlo simulations allowed to check and analyze also the subleading corrections of the free-field action. Actually the Lagrangian of the low-energy effective theory consists of all terms respecting the internal and space-time symmetries of the system. The action of the effective string theory can be written as low energy expansion in the number of derivatives of the Nambu-Goldstone fields. In particular the first few terms of the effective action $S$ for an open string stretched between fixed ends, for instance Polyakov lines, is [12]

$$S = S_{cl} + \frac{\sigma}{2} \int d^2 \xi \left[ \partial_\alpha X \cdot \partial^\alpha X + c_2 (\partial_\alpha X \cdot \partial^\alpha X)^2 + c_3 (\partial_\alpha X \cdot \partial_\beta X)^2 + \ldots \right] + S_b, \quad (1.1)$$

where the classical term $S_{cl}$ describes the usual perimeter-area term. The Nambu-Goldstone fields $X_i(\xi_0, \xi_1)$ ($i = 1, \ldots, d - 2$) parametrize the displacements orthogonal to the surface of minimal area representing the configuration around which we expand and $\xi_0, \xi_1$ are the world-sheet coordinates. $S_b$ is the boundary contribution characterizing the open string. If the boundary is a Polyakov line in the $\xi_0$ direction placed at $\xi_1 = 0$, on which we assume Dirichlet boundary conditions $X_i(\xi_0, 0) = 0$, the first few terms are

$$S_b = \int d\xi_0 \left[ b_1 \partial_1 X \cdot \partial_1 X + b_2 \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X + b_3 (\partial_1 X \cdot \partial_1 X)^2 + \ldots \right]. \quad (1.2)$$

As first observed in 2004 by Lüscher and Weisz [12] the coefficients $c_i$ and $b_i$ of the above expansion should satisfy some consistency constraints; they were obtained by the comparison of the string partition function in different channels ("open-closed string duality"). These results were further generalized by Ref. [13]. It was also realized [14, 15, 16] that the crucial ingredient of these constraints is the Lorentz symmetry of the underlying Yang-Mills theory. Indeed, even if the complete $SO(1, d - 1)$ invariance is broken by the classical configuration around which we expand, the effective action should still respect this symmetry through a non-linear realisation in terms of transverse fields $X_i$. In this way it was shown [15, 16, 17] that the terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion. The first allowed correction to the Nambu-Goto action turns out to be the the six derivative term [13]

$$c_4 \left( \partial_\alpha \partial_\beta X \cdot \partial^\alpha \partial^\beta X \right) (\partial_\gamma X \cdot \partial^\gamma X) \quad (1.3)$$

with arbitrary coefficient $c_4$; however this term is non-trivial only when $d > 3$. For $d = 3$ the first non-trivial deviation of the Nambu-Goto action is an eight-derivative term and it has been recently shown [18], using the recursion relations generated by the non-linear Lorentz transformations, that it generates a geometric term proportional to the squared curvature of the induced metric on the world-sheet. The fact that the first deviations from the Nambu-Goto string are of high order, especially in $d = 3$, explains why in early Monte Carlo calculations [19, 20, 21] a good agreement with the Nambu-Goto string was observed. The leading deviations of the effective string spectrum were explicitly calculated in [22] and they turn out to be consistent with lattice simulations for long closed confining strings [23, 24].
In this paper we deal with open effective strings both from the point of view of theoretical investigation and numerical calculations. In [16] it was already observed that in the boundary action (1.2) only the $b_2$ term passes the first test of Lorentz invariance (the vanishing of $b_1$ was previously proven in [12]). Here we complete the proof by solving the recursion relations dictated by the requirement of Lorentz invariance. The resulting expansion can be written in a closed form, yielding

$$b_2 \int d\xi_0 \left[ \frac{\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} \right].$$

(1.4)

Using the same technique we worked out also the next few boundary invariants. For instance, the complete form of that generated by the six derivative term $\partial_0^2 \partial_1 X \cdot \partial_0^2 \partial_1 X$ in $d = 3$ is $b_4^{(1)} \int d\xi_0 \mathcal{L}_{b,4}$, where

$$\mathcal{L}_{b,4} = \frac{(\partial_0^2 \partial_1 X)^2}{(1 + (\partial_1 X)^2)^2} - 4 \frac{\partial_1 X \partial_0^2 \partial_1 X (\partial_0 \partial_1 X)^2}{(1 + (\partial_1 X)^2)^3} + 4 \frac{(\partial_0 \partial_1 X)^4 (\partial_1 X)^2}{(1 + (\partial_1 X)^2)^4},$$

(1.5)

with an arbitrary coefficient $b_4^{(1)}$ (this nomenclature will become clear later).

The lowest order term in (1.4) represents the first deviation from the Nambu-Goto string in $d = 3$, it is thus important to look for the contribution of this term to physical observables in lattice gauge theories. A first test of the effect of this term on the open string energy levels has been already performed in [25], in the case of SU(2) gauge theory. In the present paper we probe the contribution of this boundary term to the vacuum expectation value of Polyakov loop correlators and of Wilson loops, using the three-dimensional $\mathbb{Z}_2$ gauge model. We shall see that this contribution is of the same order of magnitude of the subleading effective string corrections to the interquark potential which were recently studied both using Polyakov loop correlators [20, 26] and Wilson loops [27] and thus cannot be neglected when performing high precision tests of the effective string. We use the data obtained from our simulation of Polyakov loop correlators to fit the coefficient $b_2$ of the boundary correction. We then use this value to predict the Wilson loop behaviour; we have no free parameter left in this comparison, so the remarkable agreement that we find with the data represents a strong consistency check of our treatment of the boundary correction given in eq. (1.4).

As it is easy to guess, these boundary corrections are more important in the Wilson case than in the Polyakov loop correlators case. In particular, in the Wilson loop case they perfectly explain the deviations recently observed in [27] between effective string predictions and numerical results. It is possible that they could also explain the subtle, higher order, deviations recently observed using Polyakov loop correlators [26], however such a test, which would require a much stronger numerical effort is beyond the scope of the present paper.

2. The boundary action

The confining string action can be thought of as the effective low energy action which is obtained integrating over all the massive degrees of freedom of Yang-Mills theory in pres-
ence of a pair of static sources. As already emphasized in the Introduction, the formation of a confining flux tube spontaneously breaks the transverse translational as well as the Lorentz (or rotational) invariance of the bulk space-time. This symmetry is not manifest in the static gauge, where one uses only the physical degrees of freedom, associated to the transverse fluctuations $X_i(\xi_0, \xi_1)$ ($i = 1, \ldots, d - 2$). However the effective string action should still respect this symmetry through a non-linear realisation.

$$X^i P^*(0) \overset{\text{string}}{\longrightarrow} P(R) \overset{\text{string}}{\longrightarrow} X^i \xi_1 = R$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Rotation of the Polyakov correlator around the $\xi_0$ axis.}
\end{figure}

To make the discussion concrete and explicit, we specialize now to the case where the string action describes the vacuum expectation value of the correlator of a pair of Polyakov loops $\langle P(0)^* P(R) \rangle$ parallel to the $\xi_0$ axis and located respectively at $\xi_1 = 0$ and $\xi_1 = R$. The string stretched between these two Polyakov loops fluctuates in the $d - 2$ transverse directions, orthogonal to the plane $[\xi_0, \xi_1]$. Rotate now this system of an infinitesimal angle $\epsilon$ around the $\xi_0$ axis in the plane $[\xi_1, \hat{i}]$ as shown in Figure 1. Of course this transformation in a Lorentz invariant theory keeps the physics of the system invariant; however, it modifies the world-sheet coordinates $\xi_0, \xi_1$. The linear transformation of the transverse fields $X_j$ generated by the rotation should therefore be followed by a reparametrization which puts again the system in the static gauge. As a result the string action should be invariant under the infinitesimal transformation

$$\delta X^j = -\epsilon \delta^j \xi_1 - \epsilon X^i \partial_1 X^j, \quad (i, j = 1, \ldots, d - 2). \quad (2.1)$$

When it is applied to a term with $m$ derivatives and $n$ transverse fields $X_j$, schematically $\partial^m X^n$, it generates terms with the same value of the difference $m - n$, called “scaling” of the given term [18]. When we apply the transformation (2.1) to the boundary term $\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X$ of (1.2), which has scaling 2, it generates a tower of all possible terms of the same scaling value. Thus the Lagrangian density $L_{b,2}$ of all boundary terms of scaling 2 contributing to the boundary action $S_{b,2} = b_2 \int d\xi_0 L_{b,2}$

$$L_{b,2} = \sum_{k=0}^{\infty} \left[ \alpha_k \partial_1 \partial_0 X \cdot \partial_1 \partial_0 X (\partial_1 X \cdot \partial_1 X)^k + \beta_{k+1} (\partial_1 \partial_0 X \cdot \partial_1 X)^2 (\partial_1 X \cdot \partial_1 X)^k \right]. \quad (2.2)$$

In listing these terms we eliminated, as is usual in this context [12, 13, 16, 18], terms proportional to the equation of motion or its derivatives, since these terms can be reabsorbed by a field redefinition; it is sufficient to use the equation of motion of the free field theory,
in view of the fact that corrections to this will generate terms of higher order that can be ignored since we list the most general terms at each order anyway.

The invariance of $L_{b,2}$ under (2.1) dictates the following recursion relations among the $\alpha_k$ and $\beta_k$ coefficients:

$$\alpha_k + \alpha_{k+1} = 0, \quad (k + 1) \beta_k + k \beta_{k+1} = 0, \quad \alpha_k + \beta_k + \beta_{k+1} = 0.$$  \hspace{1cm} (2.3)

The solution of these recursion relations depends on a single free parameter and yields at once Eq.(1.4), which for $d = 3$ becomes simply

$$L_{b,2}^{d=3} = \frac{(\partial_0 \partial_1 X)^2}{(1 + (\partial_1 X)^2)^2}. \hspace{1cm} (2.4)$$

It is easy to verify that eq. (1.4), hence in particular eq. (2.4), is not a total derivative, because this is possible only if each term in (2.2) is a total derivative, but direct inspection shows that this is not the case. An indirect proof of this fact is that the vacuum expectation value of the first term of (2.2), calculated in section 3, is different from zero.

It is not difficult to enlarge this game to terms of higher scaling. There is no invariant term of scaling 3; in fact, it is possible to prove that there are no invariant boundary terms of odd scaling. There are two invariant terms of scaling 4. One is the invariant $L_{b,4}$ whose form in $d = 3$ was anticipated in (1.5); we sketch its derivation in appendix B. The other is simply $(L_{b,2})^2$, the square of the scaling 2 invariant (2.4). There are five invariants of scaling 6, but two of them are simply the product of Lorentz invariants of lower degree. In conclusion, the most general boundary action, up to terms of scaling 8, can be written in the form

$$S_b = \int d\xi \left[ b_2 L_{b,2} + b_4^{(1)} L_{b,4} + b_4^{(2)} (L_{b,2})^2 + \sum_{i=1}^{3} b_6^{(i)} L_{b,6}^i + b_6^{(4)} (L_{b,2})^3 + b_6^{(5)} L_{b,2} L_{b,4} \right], \hspace{1cm} (2.5)$$

with $b_i^{(k)}$ arbitrary coefficients. The general expressions of $L_{b,k}$ for arbitrary space-time dimensions $d$ are reported in appendix B.

3. Effects of the first boundary correction in specific geometries

In this section we investigate the effect on Polyakov loop correlators and Wilson loop expectation values of the first term in the boundary component of the string effective action eq. (1.4). We aim for comparison with the lattice simulations discussed in next Section, hence we focus on the $d = 3$ case, where there is a single transverse embedding field and the action reduces, as noted in eq. (2.4), to

$$S_{b,2} = b_2 \int_{\partial \Sigma} \frac{(\partial_0 \partial_1 X)^2}{(1 + (\partial_1 X)^2)^2} = b_2 \int_{\partial \Sigma} \left\{ (\partial_0 \partial_1 X)^2 - 2(\partial_0 \partial_1 X)^2(\partial_1 X)^2 + \ldots \right\}. \hspace{1cm} (3.1)$$

In our simulations we will be able to check the effect of the leading term in the derivative expansion of this boundary action, namely

$$S_{b,2}^{(1)} = b_2 \int_{\partial \Sigma} (\partial_0 \partial_1 X)^2 \hspace{1cm} (3.2)$$
and to fit reliably the value of $b_2$. The first subleading correction in eq. (3.1) appears at the same order in the derivative expansion as the leading term arising from the possible scaling-weight 4 invariant lagrangian of eq. (1.5); at this order, whose effects in the simulations is very tiny, we would therefore have to fit the independent parameter $b^{(1)}_4$, and this is at the moment beyond the reach of our precision.

Perturbatively, and up to the order discussed above, the correction to the partition function for a given world-sheet geometry due to the boundary terms is given by

$$
\delta^{(1)}_{\text{bound}} Z = -Z_{\text{free}} \langle S^{(1)}_{b,2} \rangle ,
$$

(3.3)

where $Z_{\text{free}}$ is the partition function obtained using the free action (the first term in eq. (1.1))

$$
S_{\text{free}} = \frac{\sigma}{2} \int_\Sigma \left[ (\partial_0 X)^2 + (\partial_1 X)^2 \right]
$$

(3.4)

and

$$
\langle S^{(1)}_{b,2} \rangle = \frac{1}{Z_{\text{free}}} \int DX S^{(1)}_{b,2} e^{-S_{\text{free}}} = \frac{1}{Z_{\text{free}}} \int DX b_2 \int_{\partial \Sigma} (\partial_0 \partial_1 X)^2 e^{-S_{\text{free}}}
$$

$$
= b_2 \int_{\partial \Sigma} \lim_{\xi_0' \to \xi_0, \xi_1' \to \xi_1} \frac{\partial}{\partial \xi_0'} \frac{\partial}{\partial \xi_1'} \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial \xi_1} G(\vec{\xi}', \vec{\xi}) .
$$

(3.5)

In the last line we used Wick theorem to relate the result to the free propagator $G(\vec{\xi}, \vec{\xi})$ of the field $X$, which depends on the conditions imposed at the boundary $\partial \Sigma$. In the following, we shall distinguish with a subscript $P$ or $W$ the boundary conditions corresponding to Polyakov loop correlators or Wilson loops.

In [16] the cylinder geometry corresponding to the correlator of two Polyakov loops was considered, with the following result:

$$
\langle S^{(1)}_{b,2} \rangle_P = -b_2 \frac{\pi^3 L}{60 R^4} E_4(i \frac{L}{2R})
$$

(3.6)

where $R$ is the distance between the two Polyakov loops and $L$ the length of the compactified time direction (i.e. the inverse temperature).

Here we consider $\partial \Sigma$ to be a rectangular Wilson loop of sides $R$ and $L$, as in figure 2. It has four components corresponding to the sides of the rectangle.
The Green function with Dirichlet boundary conditions on this loop is given by

\[ G_W(\zeta, \bar{\zeta}) = \frac{4}{RL} \sum_{m,n=1}^{\infty} \frac{\sin\left(\frac{n\pi\xi}{R}\right) \sin\left(\frac{m\pi\xi_0}{L}\right) \sin\left(\frac{n\pi\xi'}{R}\right) \sin\left(\frac{m\pi\xi_0'}{L}\right)}{\pi^2 \left(\frac{m^2}{L^2} + \frac{n^2}{R^2}\right)}. \]  

(3.7)

In order to compute eq. (3.5) we focus our attention on one side of the Wilson loop, for example the one with coordinates \((0, \xi_1)\), with \(\xi_1 \in [0, R]\). We have to compute the integral

\[ \int_0^R d\xi' d\xi_0 d\eta d\eta_0 G_W(\zeta, \bar{\zeta})|_{\xi_1=\xi_1', \xi_0=\xi_0'=0} = \int_0^R d\xi' \frac{4\pi^2}{R^3L^3} \sum_{m,n=1}^{\infty} \frac{m^2n^2(\cos\left(\frac{n\pi\xi_1}{R}\right))^2}{\left(\frac{m^2}{L^2} + \frac{n^2}{R^2}\right)}, \]

(3.8)

With some manipulations we can rewrite the sum in eq. (3.8) as follows:

\[ \sum_{m,n=1}^{\infty} \frac{m^2n^2}{\left(\frac{m^2}{L^2} + \frac{n^2}{R^2}\right)^2} = R^2 \sum_{m=1}^{\infty} m^2 \sum_{n=1}^{\infty} \left(1 - \frac{m^2}{L^2} \frac{1}{\left(\frac{m^2}{L^2} + \frac{n^2}{R^2}\right)}\right) = -\frac{\pi R^3}{2L} \sum_{m=1}^{\infty} m^2 \coth\left(\frac{m\pi R}{L}\right), \]

(3.9)

where we used eq. (A.1) and regularized the divergent sum using eq. (A.2), taking into account that \(\zeta(0) = -1/2\).

Introducing \(q = \exp(-2\pi R/L)\) we can rewrite the above expression as

\[ \sum_{m,n=1}^{\infty} \frac{m^2n^2}{\left(\frac{m^2}{L^2} + \frac{n^2}{R^2}\right)^2} = -\frac{\pi R^3}{2L} \sum_{m=1}^{\infty} m^3 \left(1 + \frac{2q^m}{1 - q^m}\right) = -\frac{\pi R^3}{2L} \left(\zeta(-3) + 2 \sum_{m=1}^{\infty} \frac{m^3q^m}{1 - q^m}\right) = -\frac{\pi R^3}{2L} \zeta(-3) E_4 \left(\frac{iR}{L}\right). \]

(3.10)

Here we used eq. (A.2) to regularize the divergent series and eq. (A.7) to rewrite the remaining sum in terms of the Eisenstein series \(E_4\).

Substituting the last expression in eq. (3.8), and using the explicit value \(\zeta(-3) = 1/120\), we find thus that the contribution to the boundary term eq. (3.5) from the \((0, \xi_1)\) side of the rectangle is given by

\[ \int_0^R d\xi' d\xi_0 d\eta d\eta_0 G_W(\zeta, \bar{\zeta})|_{\xi_1=\xi_1', \xi_0=\xi_0'=0} = -\frac{\pi^3 R}{120L^4} E_4 \left(\frac{iR}{L}\right). \]

(3.11)

The symmetry of the boundary implies that the contribution from the other side in the same direction, the one parametrized by \((L, \xi_1)\), with \(\xi_1 \in [0, R]\), must be the same. The contribution from the two sides of the rectangle in the orthogonal direction are obtained exchanging \(R\) and \(L\). Altogether we have thus

\[ \langle S^{(1)}_{b,2} \rangle_W = -b_2 \frac{\pi^3}{60} \left[ \frac{R}{L^4} E_4 \left(\frac{1}{L}\right) + \frac{L}{R^4} E_4 \left(\frac{1}{R}\right) \right]. \]

(3.12)
4. Numerical results

In order to test our prediction for the boundary terms we performed a set of simulations in the three-dimensional \( \mathbb{Z}_2 \) gauge model. We used the following strategy. We first evaluated a set of Polyakov loop correlators in the low temperature regime. Comparing our results with eq. (3.6) we could verify the correctness of the effective string prediction and extract an estimate of the coefficient \( b_2 \). To check the correct scaling behaviour of \( b_2 \) we performed this analysis for three different values of the bare coupling \( \beta \). Then we inserted this estimate for \( b_2 \) into eq. (3.12) to predict the boundary correction in the Wilson loop case and compared it with the simulation results. In this last comparison there was no free parameter to fit. The details of the simulation algorithm that we employed can be found in [28, 27]. As a basic update mechanism we used a multispin coded version of the standard metropolis algorithm. Let us now describe the various steps of our analysis.

**Polyakov loop correlators**

In order to eliminate the non-universal perimeter and constant terms from the expectation value of Polyakov loop correlators \( P(R, L) \) (where \( L \) is the length of the two loops and \( R \) their distance) we measured the following ratio:

\[
R_P(R, L) = \frac{P(R + 1, L)}{P(R, L)}. \tag{4.1}
\]

Due to the peculiar nature of our algorithm, based on the dual transformation to the 3d spin Ising model, this ratio can be evaluated for large values of \( R \) and \( L \) with very high precision.

The effective string prediction for this observable reads, up to the second loop order,

\[
R_P(R, L) = e^{-\sigma L} \eta\left(\frac{L}{1152\sqrt{2}R}\right) \left(1 + F_2(R + 1, L) + F_P(R + 1, L) - F_2(R, L) - F_P(R, L)\right), \tag{4.2}
\]

where \( \eta \) is Dedekind’s function, \( \sigma \) is the string tension, \( F_2 \) the standard two loop effective string contribution to the Polyakov loop correlator (see for instance [20])

\[
F_2(R, L) = \frac{\pi^2 L}{1152R^3\sigma} (2E_4(\frac{L}{2R}) - E_2^2(\frac{L}{2R})) \tag{4.3}
\]

and \( F_P = \langle S_b^{(1)} \rangle_P \) is the leading correction coming from the boundary [16] given in eq. (3.6), which we report here for completeness:

\[
F_P(R, L) = -b_2 \frac{\pi^3 L}{60R^4} E_4(\frac{L}{2R}). \tag{4.4}
\]

We can use the fact that \( \sigma \) is known with very high precision to define a new observable which allows to isolate the boundary term:

\[
R'_P(R, L) = e^{\sigma L} \eta\left(\frac{L}{1152\sqrt{2}R}\right) R_P(R, L) - F_2(R + 1, L) + F_2(R, L) - 1 \tag{4.5}
\]
From eq. (4.4) we expect for this observable, when \( L \gg 2R \),

\[
R'_P(R, L) = b_2 \frac{L \pi^3}{15 R^5}
\]  
(4.6)

We measured this observable for three values of the bare coupling \( \beta \). Table 1 displays the values of \( \beta \) and the corresponding string tension, as well as the values of \( L \) and of the inverse of the critical temperature \( 1/T_c \), for the three data sets. We always chose \( L \gg 1/T_c \) to avoid the additional complication of dealing with finite temperature correction terms.

| data set | \( \beta \) | \( L \) | \( \sigma \) | \( 1/T_c \) |
|----------|----------|--------|--------|--------|
| 1        | 0.743543 | 68     | 0.0228068(15) | 5      |
| 2        | 0.751805 | 100    | 0.0105255(11)  | 8      |
| 3        | 0.754700 | 125    | 0.0067269(17)  | 10     |

Table 1: Some informations on the data sample

We report in table 2 the numerical results for the various datasets. We fitted the

| R      | data set 1 | R      | data set 2 | R      | data set 3 |
|--------|------------|--------|------------|--------|------------|
| 11     | 0.00611(15)| 16     | 0.00515(18)| 20     | 0.00446(13)|
| 12     | 0.00431(15)| 17     | 0.00395(18)| 21     | 0.00355(13)|
| 13     | 0.00290(16)| 18     | 0.00333(18)| 22     | 0.00304(13)|
| 14     | 0.00182(16)| 19     | 0.00207(18)| 23     | 0.00226(14)|
| 15     | 0.00159(16)| 20     | 0.00132(18)| 24     | 0.00190(14)|
| 16     | 0.00092(16)| 21     | 0.00119(18)| 25     | 0.00162(14)|
| 17     | 0.00088(16)| 22     | 0.00133(18)| 26     | 0.00127(14)|
| 18     | 0.00062(16)| 23     | 0.00083(18)| 27     | 0.00134(14)|
| 19     | 0.00020(16)| 24     | 0.00071(18)| 28     | 0.00104(14)|
| 20     | 0.00016(16)| 25     | 0.00088(18)| 29     | 0.00074(14)|
|        |            |        |            | 26     | 0.00067(18)|
|        |            |        |            |        | 30     | 0.00060(14) |
|        |            |        |            |        | 31     | 0.00078(14) |
|        |            |        |            |        | 32     | 0.00062(14) |
|        |            |        |            |        | 33     | 0.00053(14) |
|        |            |        |            |        | 34     | 0.00033(14) |

Table 2: Numerical results for \( R'_P(R, L) \)

We report in table 2 the numerical results for the various datasets. We fitted the numerical data in table 2 with the function in eq. (4.6), where the only free parameter is \( b_2 \). The results of the fit are given in table 3. For each value of \( \beta \) we chose the lowest value for \( R \) in the fit to be roughly twice the value of the inverse critical temperature. Previous analysis on this model [20] showed that this threshold is well within the regime of validity of the effective string picture. Looking at table 3 we see that in the fits obtained from all the data sets the \( \chi^2 \) values are of order one. This strongly supports the correctness of eq. (3.6), obtained in [16]. It is interesting to note that the value of \( b_2 \) changes dramatically with \( \beta \). This is due to the fact that \( b_2 \) has the dimensions of (length)\(^3\). In fact, a remarkable
consistency check of our results is that the adimensional combination \( b_2 \sqrt{\sigma^3} \) is almost constant. This combination can be used to define the continuum limit value of \( b_2 \) which can thus be considered as a new physical scale of the model, on the same ground as the string tension \( \sigma \) or the glueball mass \( m_g \). A slight scaling deviation is still visible between the data set 2 and the data set 3. Since we have no hint of the behaviour of these scaling corrections we propose as our final estimate for the continuum limit value of the combination \( b_2 \sqrt{\sigma^3} \) the value obtained in the data set 3 (which is the nearest to the continuum limit) with a systematic error equal to the difference (within statistical errors) between the value obtained in the data set 3 and that of data set 2. In this way we obtain \( b_2 \sqrt{\sigma^3} \sim 0.032(2) \). It is interesting to notice that our result for \( b_2 \) in the 3d \( \mathbb{Z}_2 \) gauge model is very similar (but ten times more precise) to the same parameter measured in the 3d \( SU(2) \) gauge model in [25].

| data set | \( b_2 \) | \( b_2 \sqrt{\sigma^3} \) | \( \chi^2 \) |
|----------|---------|----------------|--------|
| 1        | 7.25(15)| 0.0250(5)      | 1.2    |
| 2        | 26.8(8 )| 0.0289(9)      | 1.8    |
| 3        | 57.9(12)| 0.0319(7)      | 1.3    |

Table 3: Some informations on the data sample

Wilson loops.

Using the above value of \( b_2 \) we can now test the leading boundary contribution to the Wilson loop reported in eq. (3.12). Since we know \( \sigma \) and \( b_2 \) from independent simulations, there are no free parameters left, thus we may perform, so to speak, an "absolute" test of our predictions.

In order to eliminate the non-universal perimeter and constant terms we introduce (as in [27]) the following ratio of Wilson loops:

\[
R_W(L, R) = \frac{W(L, R)}{W(L + 1, R - 1)} \tag{4.7}
\]

where \( W(R, L) \) denotes the expectation value of a Wilson loop of size \( R \times L \) (see fig. 2). Then in order to eliminate the dominant confining term, using again the fact the \( \sigma \) is known with very high precision we define

\[
R'_W(L, uL) = R_W(L, uL) - \exp\{-\sigma(1 + L(1 - u))\}, \tag{4.8}
\]

where we introduced the asymmetry parameter \( u = R/L \) (with \( R > L \)).

In table 4 we report the numerical results for this observable evaluated for the same three values of \( \beta \) reported in table 1.

In figs. 3, 4 and 5 we plotted the effective string prediction including the one loop and two loop corrections from the bulk string action, which we report here for completeness (see [27] for a detailed derivation)

\[
W(L, R) = e^{-\sigma RL + p(R + L) + k} \left( \frac{\eta(\imath u)}{\sqrt{L}} \right)^{-\frac{d-2}{2}} \left\{ 1 + \frac{L_2(u)}{\sigma RL} + \ldots \right\}, \tag{4.9}
\]
Table 4: Numerical results for $R'_W(L, \frac{4}{3}L)$

| R  | data set 1       | R  | data set 2       | R  | data set 3       |
|----|------------------|----|------------------|----|------------------|
| 12 | -0.00642(8)      | 12 | -0.00687(4)      | 21 | -0.00356(10)     |
| 15 | -0.00468(9)      | 15 | -0.00512(9)      | 24 | -0.00292(10)     |
| 18 | -0.00376(10)     | 18 | -0.00414(10)     | 27 | -0.00246(11)     |
| 21 | -0.00285(11)     | 21 | -0.00317(10)     | 30 | -0.00223(12)     |
| 24 | -0.00247(11)     | 24 | -0.00282(11)     | 33 | -0.00182(12)     |
| 27 | -0.00204(12)     | ----|------------------|----|------------------|

\[ L_2(u) = \left( \frac{\pi}{24} \right)^2 \left[ 2(d-2) u^2 E_4(iu) - \frac{(d-2)(d-6)}{2} E_2(iu) E_2(i/u) \right]. \] \hspace{1cm} (4.10)

We also report in the figures the curve obtained adding to the bulk string terms the leading correction coming from the boundary, given in eq. (3.12), and the numerical results of our simulations.

The agreement between the data and the prediction is impressive and, with the precision of our data, the boundary correction is absolutely necessary to correctly describe them. Indeed, looking at the similar analysis discussed in [27] (which did not keep into account boundary terms) it is clear that the boundary correction discussed in this paper exactly fills the gap between data and prediction found, for instance, in fig. 2 of [27].

5. Conclusions

In this paper we used the requirement of Lorentz invariance to constrain the effective boundary action of the confining string which describes the behaviour of the Wilson loops...
and Polyakov correlators in any confining gauge theory. We wrote explicitly all the possible Lorentz invariants up to terms of scaling $8$. We also presented the analytic calculation of the contribution of the leading term of the boundary action to the mentioned observables (the calculation for the Polyakov correlator was already done in [16]). Although the boundary term depends only on (derivatives) of string coordinates evaluated on the boundary, its contribution due to quantum fluctuations depends on the whole shape of the observable and can be expressed in a closed form in terms of the Eisenstein series $E_4$. The coefficient $b_2$ in front of this boundary term has the dimensions of $(\text{length})^3$, thus it defines a new physical scale in any confining gauge theory. It is interesting to observe that in the derivative expansion of the string action, $b_2$ is the first term where one can find a dependence on the
gauge group, while all the terms of lower scaling are completely fixed and are the same for any 3D gauge theory. We evaluated the coefficient $b_2$ in the three-dimensional $\mathbb{Z}_2$ gauge model using the data of Monte Carlo simulations for the Polyakov correlator corresponding to three different values of the bare coupling in order to check the correct scaling behaviour of $b_2$. We then inserted this estimate to predict the boundary correction in the Wilson loop case and compared it with the data of the numerical simulation. Although there are no adjustable parameters, the agreement between the prediction and the data is impressive, as it is clearly visible in fig.s 3, 4 and 5.

Acknowledgements
The MC simulations were performed at the INFN Pisa GRID DATA center and on the INFN cluster CSN4. We thank Mattia Bruno for discussions, and in particular for pointing out a few misprints in the first version of this paper.

A. Useful formulae
In section 3 we make use of the following series for the hyperbolic cotangent:

$$
\sum_{n=1}^{\infty} \frac{1}{(\frac{n^2}{L^2} + \frac{n^2}{R^2})} = \frac{\pi RL}{2m} \coth \left( \frac{m\pi R}{L} \right) - \frac{L^2}{2m^2}.
$$

(A.1)

We also encounter divergent sums which we regularize by means of the zeta function

$$
\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}.
$$

(A.2)

Dedekind’s $\eta$-function is defined as

$$
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
$$

(A.3)

where $q = \exp(2\pi i \tau)$. The Eisenstein series $E_{2k}$, with $k = 1, 2, \ldots$, are defined by

$$
E_{2k}(\tau) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,
$$

(A.4)

where $\sigma_p(n)$ is the sum of the $p$-th powers of the divisors of $n$:

$$
\sigma_p(n) = \sum_{m|n} m^p.
$$

(A.5)

In section 3 we encounter the sum

$$
2 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^m} = 2 \sum_{m=1}^{\infty} \sum_{s=1}^{\infty} m^3 q^{ms} = 2 \sum_{n=1}^{\infty} \sigma_3(n) q^n,
$$

(A.6)
where in the last step we changed summation variables to \( n = ms \) and \( m \), so that \( m \) is a divisor of \( n \), and employed the definition eq. (A.5). From eq. (A.4) we have then

\[
2 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^m} = \zeta(-3) (E_4(\tau) - 1) .
\]  

(A.7)

In section 3 the parameter \( \tau \) is in fact \( iR/L \).

B. Boundary Lorentz invariants of higher scaling

In this appendix we list the Lorentz-invariant terms of scaling 4 and 6 which contribute to the boundary action of the open string. Since the infinitesimal transformation (2.1) when applied to a generic term \( \partial^m X^n \) keeps invariant the scaling \( m - n \), it suffices to list the set of all possible terms of definite scaling with arbitrary coefficients and then solve the recursion relations dictated by the request of invariance. For instance in \( d = 3 \) there are three different sequences of terms of scaling 4, namely

\[
\mathcal{L}_{b,4}[d=3] = \sum_n \left[ \alpha_n (\partial_1 X)^{2n} (\partial_0^2 \partial_1 X)^2 + \beta_n (\partial_1 X)^{2n+1} \partial_0^2 \partial_1 X (\partial_0 \partial_1 X)^2 + \gamma_n (\partial_1 X)^{2n+2} (\partial_0 \partial_1 X)^4 \right].
\]

(B.1)

Demanding \( \delta \mathcal{L}_{b,4} = 0 \) generates the following set of recursion relations

\[
0 = (n + 2)\alpha_n + (n + 1)\alpha_{n+1}, \quad \text{(B.2)}
\]
\[
0 = 4\alpha_{n+1} + (2n + 7)\beta_n + (2n + 3)\beta_{n+1}, \quad \text{(B.3)}
\]
\[
0 = \beta_{n+1} + (n + 5)\gamma_n + (n + 2)\gamma_{n+1}. \quad \text{(B.4)}
\]

They can be solved in terms of a single free parameter \( c = b_1^4 \), yielding

\[
\alpha_n = (-1)^n (n+1)c, \quad \beta_n = -2(-1)^n (n+1)(n+2)c, \quad \gamma_n = \frac{2}{3} (-1)^n (n+1)(n+2)(n+3)c. \quad \text{(B.5)}
\]

Inserting these solutions in (B.1) gives at once (1.5). It is not difficult to prove that its generalization to \( d \) space-time dimensions is

\[
\mathcal{L}_{b,4} = \frac{\partial_0^2 \partial_1 X \cdot \partial_0^2 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} - \frac{(\partial_0^2 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} + 4 \left( \frac{\partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} \right) (\partial_0 \partial_1 X \cdot \partial_1 X)
\]
\[
+ 4 \left( \frac{\partial_0 \partial_1 X \cdot \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} \right) \left( \partial_1 X \cdot \partial_0 \partial_1 X + \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right) \right] - 4 \left( \frac{\partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} \right)^4 .
\]

(B.6)

Using the same procedure it is possible to show that there are in total five Lorentz invariants of scaling 6; three of them cannot be written as products of invariants of lower scaling. We list them for the sake of completeness:
\[ \mathcal{L}_{b,6}^1 = \frac{1}{(1 + \partial_1 X \cdot \partial_1 X)^3} \left\{ \partial_0^3 \partial_1 X \cdot \partial_0 \partial_1 X \left[ \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X (1 + \partial_1 X \cdot \partial_1 X) - (\partial_0 \partial_1 X \cdot \partial_1 X)^2 \right] \\
- \left( \partial_0^3 \partial_1 X \cdot \partial_1 X \right) \left( \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^3 \left( 1 + \partial_1 X \cdot \partial_1 X \right)^3 \right\} \\
+ \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X)^3}{(1 + \partial_1 X \cdot \partial_1 X)^4} - 3 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X) (\partial_0 \partial_1 X \cdot \partial_1 X)}{(1 + \partial_1 X \cdot \partial_1 X)^3} \\
- 3 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^3} + 3 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^3}{(1 + \partial_1 X \cdot \partial_1 X)^4} \\
+ \frac{6 \left( \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right)^6}{(1 + \partial_1 X \cdot \partial_1 X)^6} + 9 \frac{\left( \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right) \left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^4} \\
- \frac{6 \left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^4}{(1 + \partial_1 X \cdot \partial_1 X)^5} + 6 \frac{\left( \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right)^2 (\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^4} \\
- 12 \frac{\left( \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^4}{(1 + \partial_1 X \cdot \partial_1 X)^5} \right) , \tag{B.7} \]

\[ \mathcal{L}_{b,6}^2 = \frac{\partial_0^3 \partial_1 X \cdot \partial_0 \partial_1 X}{1 + \partial_1 X \cdot \partial_1 X} \left\{ \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} - 6 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_1 X \cdot \partial_0 \partial_1 X)}{(1 + \partial_1 X \cdot \partial_1 X)^2} \\
- 6 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_1 X \cdot \partial_0 \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} + 12 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X) (\partial_0 \partial_1 X \cdot \partial_1 X)}{(1 + \partial_1 X \cdot \partial_1 X)^2} \\
+ 12 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^3} - 12 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^3}{(1 + \partial_1 X \cdot \partial_1 X)^4} \\
+ 9 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right)^2 (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X)}{(1 + \partial_1 X \cdot \partial_1 X)^3} + 9 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^3} \\
+ 18 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) \left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X)}{(1 + \partial_1 X \cdot \partial_1 X)^3} - 36 \frac{\left( \partial_0^2 \partial_1 X \cdot \partial_1 X \right)^2 (\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^4} \\
- 36 \frac{\left( \partial_0 \partial_1 X \cdot \partial_1 X \right)^2 \left[ \left( \partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0 \partial_1 X \cdot \partial_1 X) + \left( \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right) (\partial_0^2 \partial_1 X \cdot \partial_1 X) \right]}{(1 + \partial_1 X \cdot \partial_1 X)^4} \\
+ 36 \frac{\left( \partial_0 \partial_1 X \cdot \partial_1 X \right)^4 \left[ 2 \partial_0^2 \partial_1 X \cdot \partial_1 X + \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X \right]}{(1 + \partial_1 X \cdot \partial_1 X)^5} - 36 \frac{\left( \partial_0 \partial_1 X \cdot \partial_1 X \right)^6}{(1 + \partial_1 X \cdot \partial_1 X)^6}, \tag{B.8} \right. \]
\[ \mathcal{L}_{b,6} = \frac{(\partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^2} + 4 \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^6}{(1 + \partial_1 X \cdot \partial_1 X)^6} \]

\[ -4 \frac{(\partial_0 \partial_1 X \cdot \partial_1 X)^4 [\partial_0^2 \partial_1 X \cdot \partial_1 X + 2 \partial_0 \partial_1 X \cdot \partial_0 \partial_1 X]}{(1 + \partial_1 X \cdot \partial_1 X)^5} \]

\[ + 4 \frac{(\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X)^2 [\partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X] (\partial_0 \partial_1 X \cdot \partial_1 X) + (\partial_0 \partial_1 X \cdot \partial_0 \partial_1 X) (\partial_0 \partial_1 X \cdot \partial_1 X)}{(1 + \partial_1 X \cdot \partial_1 X)^4} \]

\[ + \frac{(\partial_0^2 \partial_1 X \cdot \partial_1 X)^2 (\partial_0 \partial_1 X \cdot \partial_1 X)^2}{(1 + \partial_1 X \cdot \partial_1 X)^4} - 2 \frac{(\partial_0^2 \partial_1 X \cdot \partial_0 \partial_1 X) (\partial_0^2 \partial_1 X \cdot \partial_1 X) (\partial_0 \partial_1 X \cdot \partial_1 X)}{(1 + \partial_1 X \cdot \partial_1 X)^3} \]  

(B.9)

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