EXAMPLES OF ANOSOV DIFFEOMORPHISMS

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Abstract. We prove that if \( n \) is any graded rational Lie algebra, then the simply connected nilpotent Lie group \( N \times N \) with Lie algebra \( (n \otimes \mathbb{R}) \oplus (n \otimes \mathbb{R}) \) has a lattice \( \Gamma \) such that the corresponding nilmanifold \( (N \times N)/\Gamma \) admits an Anosov diffeomorphism. This gives a procedure to construct easily several explicit examples of nilmanifolds admitting an Anosov diffeomorphism, and shows that a reasonable classification up to homeomorphism (or even up to commensurability) of such nilmanifolds would not be possible.

1. Introduction

Anosov diffeomorphisms play an important and beautiful role in dynamics as the notion represents the most perfect kind of global hyperbolic behavior, giving examples of structurally stable dynamical systems (see §2 for a precise definition).

In [18], S. Smale raised the problem of classifying the compact manifolds (up to homeomorphism) which admit an Anosov diffeomorphism. At this moment, the only known examples are of algebraic nature, namely Anosov automorphisms of nilmanifolds and infranilmanifolds. It is conjectured that any Anosov diffeomorphism is topologically conjugate to an Anosov automorphism of an infranilmanifold. J. Franks [6] and A. Manning [14] proved the conjecture for Anosov diffeomorphisms on infranilmanifolds themselves.

All this certainly highlights the problem of classifying all nilmanifolds which admit Anosov automorphisms, which are easily seen in correspondence with hyperbolic automorphisms of nilpotent Lie algebras over \( \mathbb{Z} \). Nevertheless, not too much is known on the question since it is not so easy for an automorphism of a (real) nilpotent Lie algebra being hyperbolic and unimodular at the same time. We propose the following

Definition 1.1. A rational Lie algebra \( n \) (i.e. with structure constants in \( \mathbb{Q} \)) of dimension \( d \) is said to be Anosov if it admits a hyperbolic automorphism \( A \) (i.e. all their eigenvalues have absolute value different from 1) such that \( [A]_\beta \in GL(d, \mathbb{Z}) \) for some basis \( \beta \) of \( n \), where \( [A]_\beta \) denotes the matrix of \( A \) with respect to \( \beta \).

It is well known that any Anosov Lie algebra is necessarily nilpotent, and it is easy to see that the classification of nilmanifolds which admit an Anosov automorphism is essentially equivalent to that of Anosov Lie algebras (see [2]). If \( n \) is a rational Lie algebra, we call the real Lie algebra \( n \otimes \mathbb{R} \) the real completion of \( n \).

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Curiously enough, the only explicit examples of Anosov Lie algebras in the literature so far are an infinite family of 6-dimensional rational Lie algebras all having real completion isomorphic to $h_3 \oplus h_3$ (see [12, 9]) —where $h_3$ is the 3-dimensional Heisenberg Lie algebra—, the free $k$-step nilpotent Lie algebras on $n$ generators with $k < n$ (see [2], and also [4, 3] for a different approach) and certain $k$-step nilpotent Lie algebras of dimension $d + \binom{d}{2} + \ldots + \binom{d}{k}$ with $d \geq k^2$ (see [5]). For the known examples of infranilmanifolds which are not nilmanifolds and admit Anosov automorphisms we refer to [17, 15, 13].

This motivates the following natural questions:

(i) The minimal possible dimension for an Anosov non-abelian Lie algebra is 6 (see [12, 9]). Is there an Anosov Lie algebra (without a nonzero abelian factor) in every dimension $d \geq 6$?

(ii) It is proved in [12, Theorem 3.2] that any Anosov $k$-step nilpotent Lie algebra $n$ has $\dim n \geq 2k + 2$. Is there for any $k \geq 2$ an Anosov $n$ with $\dim n = 2k + 2$?

(iii) Does there exist a dimension $d$ for which there are infinitely many Anosov Lie algebras such that their real completions are pairwise non-isomorphic?

(iv) Is it feasible a more or less explicit classification of all Anosov Lie algebras (or at least of their real completions) up to isomorphism?.

(v) Does every Anosov automorphism on a nilmanifold come from the Anosov action of some lattice in a semisimple Lie group? (see [2, 10]).

The purpose of this paper is to give a procedure to construct explicit examples of Anosov Lie algebras, which is inspired on the famous 6-dimensional example given by S. Smale in [18] (due to A. Borel), but from the point of view of L. Auslander and J. Scheuneman in [1, Section 4]. We prove that $n \oplus n$ is Anosov for any graded rational nilpotent Lie algebra $n$. The construction is very direct and simple, but it produces examples enough to answer positively questions (i) (for $d$ even), (ii) and (iii), and negatively questions (iv) and (v). Also, two examples of Anosov Lie algebras which are not of the form $n \oplus n$ are given. One of them has dimension 9, and so by considering direct sums we get examples for question (i) for any $d \neq 7, 11, 13$.

We note that any Anosov Lie algebra is necessarily graded as they always admit semisimple hyperbolic automorphisms (see [3]).

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2. ANOSOV DIFFEOMORPHISMS ON NILMANIFOLDS

A diffeomorphism $f$ of a compact differentiable manifold $M$ is called Anosov if the tangent bundle $TM$ admits a continuous invariant splitting $TM = E^+ \oplus E^-$ such that $df$ expands $E^+$ and contracts $E^-$ exponentially, that is, there exist constants $0 < c$ and $0 < \lambda < 1$ such that

$$||df^n(X)|| \leq c\lambda^n||X||, \quad \forall X \in E^-, \quad ||df^n(Y)|| \geq c\lambda^{-n}||Y||, \quad \forall Y \in E^+,$$

for all $n \in \mathbb{N}$. The condition is independent of the Riemannian metric.

Let $N$ be a real simply connected nilpotent Lie group with Lie algebra $n$. Let $\varphi$ be a hyperbolic automorphism of $N$, that is, all the eigenvalues of its derivative $A = (d\varphi)_e : n \rightarrow n$ have absolute value different from 1. If $\varphi(\Gamma) = \Gamma$ for some lattice $\Gamma$ of $N$ then $\varphi$ defines an Anosov diffeomorphism on the nilmanifold $N/\Gamma$, which shall be called an Anosov automorphism. The subspaces $E^+$ and $E^-$ are
obtained by left translation of the eigenspaces of eigenvalues of $A$ of absolute value greater than 1 and less than 1, respectively.

We now review the well-known relationship with Definition 1.1 (see [2, 7, 3 for more detailed expositions). The lattice $\Gamma$ contains, as a subgroup of finite index, some full lattice $\Gamma_1$, that is, $\log(\Gamma_1)$ is a $\mathbb{Z}$-Lie subalgebra of $\mathfrak{n}$ (i.e. with integer structure constants). Thus $\varphi^m(\Gamma_1) = \Gamma_1$ for some $m$ and hence $A^m$ is an hyperbolic automorphism of the $\mathbb{Z}$-Lie algebra $\log(\Gamma_1)$. This implies that the rational completion $\log(\Gamma_1) \otimes \mathbb{Q}$ is an Anosov Lie algebra.

Conversely, let $A$ be an automorphism of a rational Lie algebra $\mathfrak{n}$ satisfying the conditions in Definition 1.1. Then there exists a $\mathbb{Z}$-Lie subalgebra $\mathfrak{n}_1$ of $\mathfrak{n}$ such that $\Gamma = \exp(\mathfrak{n}_1)$ is a lattice in $N$, where $N$ is the simply connected nilpotent Lie group with Lie algebra $\mathfrak{n} \otimes \mathbb{R}$, the real completion of $\mathfrak{n}$. Now for some $m$, $A^m(\mathfrak{n}_1) = \mathfrak{n}_1$ and hence $A^m$ determines an Anosov automorphism of the nilmanifold $N/\Gamma$. The second condition on $A$ in Definition 1.1 is equivalent to the fact that the characteristic polynomial of $A$ has integer coefficients and constant term equal to $\pm 1$ (see [3]).

Two lattices $\Gamma_1, \Gamma_2$ of $N$ are called commensurable if $\Gamma_1 \cap \Gamma_2$ has finite index in both $\Gamma_1$ and $\Gamma_2$, or equivalently, the Lie algebras of their rational Malcev completions are isomorphic as rational Lie algebras. It is proved in [2, 7, 3 that if $\Gamma_1$ and $\Gamma_2$ are commensurable then $N/\Gamma_1$ admits an Anosov automorphism if and only if $N/\Gamma_2$ does. This fact is the main reason for defining the notion of Anosov in the context of rational Lie algebras.

Recall that if $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are two non-isomorphic Anosov Lie algebras, then the corresponding Anosov diffeomorphisms are not topologically conjugate since the nilmanifolds $N_1/\Gamma_1$ and $N_2/\Gamma_2$ can never be homeomorphic. Indeed, $\Pi_1(N_i/\Gamma_1) = \Gamma_i$ and $\mathfrak{n}_i$ is the Lie algebra of the rational Malcev completion of $\Gamma_i$.

Let $\mathfrak{n}$ be a real nilpotent Lie algebra of dimension $d$. We note that $\mathfrak{n}$ has a rational form which is Anosov (or equivalently, $N$ is the simply connected cover of a nilmanifold admitting an Anosov automorphism) if and only if there exists a $\mathbb{Z}$-basis $\beta$ of $\mathfrak{n}$ and a hyperbolic $A \in \text{Aut}(\mathfrak{n})$ such that $[A]_{\beta} \in GL(d, \mathbb{Z})$, where $[A]_{\beta}$ is the matrix of $A$ in terms of $\beta$.

3. Construction of Examples

A rational Lie algebra $\mathfrak{n}$ is said to be graded if there exist (rational) subspaces $\mathfrak{n}_i$ of $\mathfrak{n}$ such that

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus ... \oplus \mathfrak{n}_k$$

and

$$[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_{i+j}.$$ 

Equivalently, $\mathfrak{n}$ is graded when there are nonzero (rational) subspaces $\mathfrak{n}_{d_1}, ..., \mathfrak{n}_{d_r}$, $d_1 < ... < d_r$, such that $\mathfrak{n} = \mathfrak{n}_{d_1} \oplus ... \oplus \mathfrak{n}_{d_r}$, and if $0 \neq [\mathfrak{n}_{d_i}, \mathfrak{n}_{d_j}]$ then $d_i + d_j = d_k$ for some $k$ and $[\mathfrak{n}_{d_i}, \mathfrak{n}_{d_j}] \subset \mathfrak{n}_{d_k}$. Recall that any graded Lie algebra is necessarily nilpotent.

**Theorem 3.1.** Let $\mathfrak{n}$ be a graded rational nilpotent Lie algebra. Then the direct sum $\bar{\mathfrak{n}} = \mathfrak{n} \oplus \mathfrak{n}$ is Anosov.

**Proof.** Let $\{X_1, ..., X_d\}$ be a $\mathbb{Z}$-basis of $\mathfrak{n}$ compatible with the gradation $\mathfrak{n} = \mathfrak{n}_{d_1} \oplus ... \oplus \mathfrak{n}_{d_r}$, that is, each $X_i \in \mathfrak{n}_{d_j}$ for some $j$. We denote by $\{Y_i\}$ a copy of the basis $\{X_i\}$, so that $\{X_1, ..., X_d, Y_1, ..., Y_d\}$ is a basis of $\bar{\mathfrak{n}}$ and

$$[X_i, X_j] = \sum_{k=1}^{d} m_{ij}^k X_k, \quad [Y_i, Y_j] = \sum_{k=1}^{d} m_{ij}^k Y_k, \quad m_{ij}^k \in \mathbb{Z}.$$
Every nonzero $\lambda \in \mathbb{R}$ defines an automorphism $A_\lambda$ of $\mathfrak{n}$ by

$$A_\lambda|_{\mathfrak{n}_d} = \lambda^d I,$$

and also an automorphism $\tilde{A}_\lambda$ of $\tilde{\mathfrak{n}}$ by

(2) $$\tilde{A}_\lambda = \begin{bmatrix} A_\lambda & 0 \\ 0 & A_\lambda^{-1} \end{bmatrix}.$$ 

If $a \in \mathbb{Z}$, $a \geq 2$, then the roots $\lambda, \lambda^{-1}$ of $x^2 - 2ax + 1 \in \mathbb{Z}[x]$ are

(3) $$\lambda = a + (a^2 - 1)^{\frac{1}{2}}, \quad \lambda^{-1} = a - (a^2 - 1)^{\frac{1}{2}},$$

and hence $0 < \lambda^{-1} < 1 < \lambda$. Consider the new basis of $\tilde{\mathfrak{n}}$ defined by

$$\beta = \{X_1 + Y_1, (a^2 - 1)^{\frac{1}{2}}(X_1 - Y_1), \ldots, X_d + Y_d, (a^2 - 1)^{\frac{1}{2}}(X_d - Y_d)\}.$$ 

It follows from (1) that

$$[X_i + Y_i, X_j + Y_j] = \sum_{k=1}^{d} m_{ij}^k (X_k + Y_k),$$

$$[X_i + Y_i, (a^2 - 1)^{\frac{1}{2}}(X_j - Y_j)] = \sum_{k=1}^{d} m_{ij}^k (a^2 - 1)^{\frac{1}{2}}(X_k - Y_k),$$

$$[(a^2 - 1)^{\frac{1}{2}}(X_i - Y_i), (a^2 - 1)^{\frac{1}{2}}(X_j - Y_j)] = \sum_{k=1}^{d} (a^2 - 1)m_{ij}^k (X_k + Y_k).$$

This implies that $\beta$ is also a $\mathbb{Z}$-basis. On the other hand, it is easy to see by using (3) that if $T$ is the transformation whose matrix in terms of the basis $\{X_i, Y_i\}$ equals

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix},$$

then the matrix of $T$ with respect to the basis $\{X_i + Y_i, (a^2 - 1)^{\frac{1}{2}}(X_i - Y_i)\}$ is given by

$$B = \begin{bmatrix} a & (a^2 - 1) \\ 1 & a \end{bmatrix} \in SL(2, \mathbb{Z}).$$

Therefore the matrix of $\tilde{A}_\lambda \in \text{Aut}(\tilde{\mathfrak{n}})$ (see (2)) in terms of the basis $\beta$ is given by

$$\begin{bmatrix} B^{d_1} & \cdots & \cdots \\ \cdots & B^{d_1} & \cdots \\ \cdots & \cdots & B^{d_r} \end{bmatrix} \in SL(2d, \mathbb{Z}).$$

Thus $\tilde{A}_\lambda$ determines a hyperbolic automorphism of $\tilde{\mathfrak{n}}$ whose matrix in terms of $\beta$ is in $SL(2d, \mathbb{Z})$, and hence $\tilde{\mathfrak{n}}$ is Anosov.

**Remark 3.2.** In spite of the simply connected cover of the examples given by Theorem 3.1 are all of the form $N \times N$, the nilmanifolds $(N \times N)/\Gamma$ which admit Anosov automorphisms are not a direct product of nilmanifolds when $N$ is irreducible.
We will now answer questions (i)-(v) in the Introduction. Theorem 3.1 will be applied in all cases without any further reference.

**Question (iv).** We first note that any two-step nilpotent Lie algebra is graded. Therefore, the classification of all Anosov nilpotent Lie algebras contains the classification of all rational two-step nilpotent Lie algebras. This is equivalent to the classification of all alternating bilinear maps

\[ \mu : \mathbb{Q}^n \times \mathbb{Q}^n \mapsto \mathbb{Q}^m \]

under the equivalence relation \( \mu \simeq \lambda \) if and only if \( \mu(gX, gY) = h \lambda(X, Y) \) for some \( g \in GL(n, \mathbb{Q}), h \in GL(m, \mathbb{Q}) \), which is considered for \( m > 2 \) a wild problem (see \[8\]). Moreover, the classification up to isomorphism of the real two-step nilpotent Lie algebras admitting a rational form is also completely open.

**Question (iii).** We can answer this question positively for \( d = 14 \) in the following way. For each \( 0 < t < 1 \) consider the 7-dimensional nilpotent Lie algebra \( \mathfrak{n}_t \) with Lie bracket \( \mu_t \) defined by

\[
\begin{align*}
\mu_t(X_1, X_2) &= (1-t)^{\frac{1}{2}}X_3, & \mu_t(X_2, X_3) &= X_5, \\
\mu_t(X_1, X_3) &= X_4, & \mu_t(X_2, X_4) &= X_6, \\
\mu_t(X_1, X_4) &= t^2 X_5, & \mu_t(X_2, X_5) &= t^2 X_7, \\
\mu_t(X_1, X_5) &= X_6, & \mu_t(X_3, X_4) &= (1-t)^{\frac{1}{2}}X_7. \\
\mu_t(X_1, X_6) &= X_7,
\end{align*}
\]

Each Lie algebra \( \mu_t \) is isomorphic to the Lie algebra \( \mathfrak{g}_t \), denoted by \( 1, 2, 3, 4, 5, 7_t : t \) in \[16\] pp.494 (see also the curve \( \tilde{g}(0, t, 1, 0, 1, 0, 0, 0) \) in \[11\] 5.2.3). The isomorphism \( \varphi_t : \mathfrak{g}_t \mapsto \mathfrak{n}_t \) is given by

\[
\begin{align*}
\varphi_t X_1 &= X_1, & \varphi_t X_2 &= t^\frac{1}{2} X_2, & \varphi_t X_4 &= t^\frac{1}{2} (1-t)^{\frac{1}{2}}X_i, & i &= 3, 4, \\
\varphi_t X_j &= t(1-t)^{\frac{1}{2}} X_j, & j &= 5, 6, 7.
\end{align*}
\]

This proves that \( \mathfrak{n}_t \) \( (0 < t < 1) \) is a curve of pairwise non-isomorphic real Lie algebras. It is easy to check that \( \{X_1, ..., X_7\} \) is a \( \mathbb{Q} \)-basis of \( \mathfrak{n}_t \) for every

\[
t_k = \frac{4k^2}{(k^2 + 1)^2}, \quad k \in \mathbb{N},
\]

and that the subspaces \( \mathfrak{n}_k = \mathbb{Q}X_1 \) determines a gradation \( \mathfrak{n}_{t_k} = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_7 \) for any \( k \). Thus \( \tilde{\mathfrak{n}}_k = \mathfrak{n}_1 \oplus ... \oplus \mathfrak{n}_k \) is Anosov for all \( k \in \mathbb{N} \), and \( \tilde{\mathfrak{n}}_k \oplus \mathbb{R} \equiv \tilde{\mathfrak{n}}_{k'} \oplus \mathbb{R} \) if and only if \( k = k' \), since the function \( f(x) = \frac{4x^2}{(x^2 + 1)^2} \) is strictly decreasing in \( [1, \infty) \).

We can also get examples for this question from any pairwise non-isomorphic infinite sequence \( \{\mathfrak{n}_k\} \) of real two-step nilpotent Lie algebras all admitting a rational form. There is an explicit example of such a sequence of dimension 10 in \[7\], and so \( d = 20 \).

**Question (ii).** One can easily answer affirmatively this question by considering for each \( k \geq 2 \) the following \( k \)-step nilpotent Lie algebra \( \mathfrak{n} \) of dimension \( k + 1 \):

\[
[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad ..., \quad [X_1, X_k] = X_{k+1},
\]

for which \( \{X_1, ..., X_{k+1}\} \) is clearly a \( \mathbb{Z} \)-basis, and \( \mathfrak{n} \) is graded by \( \mathfrak{n} = \mathbb{Q}X_1 \oplus ... \oplus \mathbb{Q}X_{k+1} \). Thus \( \tilde{\mathfrak{n}} = \mathfrak{n} \oplus \mathfrak{n} \) is Anosov, \((2k + 2)\)-dimensional and \( k \)-step nilpotent. Actually, any \( d \)-dimensional graded rational Lie algebra which is filiform (i.e. \((d-1)\)-step nilpotent) provides an example as required in this question.
**Question (v).** It is easy to prove that $\text{Aut}(n \oplus n)$ is solvable for any $n$ defined in question (ii) with $k \geq 3$ (and for any filiform example with $d \geq 4$ as well). Thus $\text{Aut}(n \oplus n)$ does not contain any semisimple Lie subgroup, which implies that the Anosov automorphism on $(N \times N)/\Gamma$ determined by $\tilde{A}_\lambda$ (see (3)) does not come from the Anosov action of any lattice in a semisimple Lie group (compare with [2]).

**Question (i).** We have required examples without a nonzero abelian factor in order to avoid Lie algebras of the form $n \oplus \mathbb{Q}^r$ with $n$ Anosov. For $r \geq 2$, the corresponding nilmanifolds $N/\Gamma \times T^r$, where $T^r$ is the $r$-dimensional torus, clearly admit Anosov automorphisms. The examples considered in question (ii) also give a positive answer to this question for any dimension $d \geq 6$ which is even. Thus, by using the 9-dimensional Anosov Lie algebra given in Example 3.4, we can consider direct sums and get examples of Anosov Lie algebras without an abelian factor for any dimension $d \geq 15$. Thus the question only remains open for $d = 7, 11, 13$. It should be noted that the examples of odd dimensions $d \geq 15$ are all direct products of nilmanifolds (compare with Remark 3.2).

We finally want to give two examples of Anosov two-step nilpotent Lie algebras which are not of the form $n \oplus n$. We note that the construction of the first one is definitely similar in spirit to that in the proof of Theorem 3.1, from where it can be deduced that such a construction might be just a particular case of a much more general procedure.

**Example 3.3.** Let $n$ be the two-step nilpotent Lie algebra with basis
\[
\{X_1, X_2, X_3, Y_1, Y_2, Y_3, Z_1, Z_2\}
\]
and Lie bracket defined by
\[
[X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2, \quad [Y_1, Y_2] = Z_1, \quad [Y_1, Y_2] = Z_2.
\]
Consider $A \in \text{Aut}(n)$ given by
\[
AX_1 = \lambda X_1, \quad AX_2 = \lambda X_2, \quad AX_3 = \lambda^{-3} X_3,
AY_1 = \lambda^{-1} Y_1, \quad AY_2 = \lambda^3 Y_2, \quad AY_3 = \lambda^{-1} Y_3,
AZ_1 = \lambda^2 Z_1, \quad AZ_2 = \lambda^{-2} Z_2,
\]
where $\lambda$ is as in (3). It is easy to see that
\[
\beta = \left\{X_1 + Y_1, (a^2 - 1)^{\frac{1}{2}}(X_1 - Y_1), X_2 + Y_3, (a^2 - 1)^{\frac{1}{2}}(X_2 - Y_3), X_3 + Y_2,
(a^2 - 1)^{\frac{1}{2}}(X_3 - Y_2), Z_1 + Z_2, (a^2 - 1)^{\frac{1}{2}}(Z_1 - Z_2)\right\}
\]
is a $\mathbb{Z}$-basis of $n$ and that if $B = \begin{bmatrix} a & a^2 - 1 \\ 1 & a \end{bmatrix}$ then
\[
[A]_\beta = \begin{bmatrix} B & B^{-3} \\ B^2 & B^2 \end{bmatrix} \in SL(8, \mathbb{Z}),
\]
showing that $n$ is Anosov.

The following example is just a modification of the free 2-step nilpotent Lie algebra on 3 generators (see [4]).
**Example 3.4.** Consider the \((3r+3)\)-dimensional \(\mathbb{Z}\)-Lie algebra
\[
n = \mathbb{Z}^3 \oplus \ldots \oplus \mathbb{Z}^3 \oplus \Lambda^2 \mathbb{Z}^3
\]
with Lie bracket defined by
\[
[v_1 + \ldots + v_r, w_1 + \ldots + w_r] = v_1 \wedge w_1 + \ldots + v_r \wedge w_r.
\]
If \(A \in GL(3, \mathbb{Z})\) has eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) and their products \(\lambda_i \lambda_j\) all of absolute value different from 1 then it easy to see that
\[
\begin{bmatrix}
A \\
\cdot \\
\cdot \\
A \Lambda^2 A
\end{bmatrix}
\in GL(3r+3, \mathbb{Z}),
\]
is an hyperbolic automorphism of \(n\). For instance, one can take
\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{bmatrix}
\]
(see [4]). Thus \(n \otimes \mathbb{Q}\) is Anosov of dimension \(3r+3\).

**Remark 3.5.** It was pointed me out by F. Grunewald that the construction given in Theorem [6] can be generalized by using the ring of integers in any totally real number field \(F\) over \(\mathbb{Q}\) (we have used \(F = \mathbb{Q} \oplus (a^2 - 1) \frac{1}{2} \mathbb{Q}\)). The proof is essentially the same, it only uses Dirichlet Unit Theorem (compare with [9, Proposition 3.2] and [5]). If the degree of \(F\) is \(m\) then we obtain Anosov Lie algebras of the form \(n \oplus \ldots \oplus n\) \((m\ times)\) for any graded rational nilpotent Lie algebra \(n\).

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