Inflation and dark energy are two of the most relevant aspects of modern cosmology. These different epochs provide the universe is passing through accelerated phases soon after the Big-Bang and at present stage of its evolution. In this review paper, we discuss that both eras can be, in principle, described by a geometric picture, under the standard of $f(R)$ gravity. We give the fundamental physics motivations and outline the main ingredients of $f(R)$ inflation, quintessence and cosmography. This wants to be a quick summary of $f(R)$ paradigm without claiming of completeness.

**Keywords**: Alternative Theories of Gravity; Quantum Cosmology; Dark Energy; Observational Cosmology.

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1. Introduction

The huge amount of present cosmological data has led to new perspectives and scenarios in the field of modern cosmology. For the first time, one refers to current-time cosmology as Precision Cosmology, i.e. the cosmological models precisely reproduce the universe expansion history, showing robust bounds which well match cosmic data. Relevant consequences of using cosmic data to constrain the correct cosmological models were carried forward from the end of last century. Indeed, before 1998 cosmologists assumed that the total content of the cosmic energy budget was filled by standard pressureless matter density. However, after 1998, several evidences pointed out that the universe is currently undergoing an accelerated expansion. Soon, it was evident that this experimental outcome could not be interpreted by using baryons and dark matter only and so the corresponding standard cosmological model was definitively modified, re-including a cosmological constant term, $\Lambda$, within Einstein’s energy momentum tensor. The cosmological constant likely represents a first explanation of current universe speeding up.

The physical nature of cosmological constant can be related to the existence of non-zero vacuum energy and can be computed in the context of quantum field theory. Unfortunately, theoretical predictions and cosmological observations differ from a huge amount of orders of magnitude, leading to a severe fine-tuning problem. Moreover, matter density and $\Lambda$ density are extremely close to each other in order of magnitudes, leading to a further issue named the coincidence problem. It consists in the fact that there is no reasons to expect that matter and $\Lambda$ densities have to be comparable at present time, since matter evolves as the universe expands, while $\Lambda$ is constant at all stages of the universe evolution. Conversely, to differently assess the observed acceleration, one may assume that the fluid responsible for the speeding up of the universe cannot be a pure constant along the universe expansion history.

In turn, any possible extensions lie on the existence of some additional fluids, whose physical properties, e.g. particle masses, thermal and electromagnetic interactions, and so forth, are not known a priori. Consequently, we do not have a final experimental evidence for the existence of those fluids at a fundamental level. Thus, cosmologists interpreted such a dynamical fluid in terms of a dark energy counterpart. Afterwards, the need of comparing both late and early-phases of the universe evolution has brought one to wonder whether introducing scalar fields, capable of describing de-Sitter-like phases, would be useful to relate both inflation and current acceleration to a single unified description. Hence, alternative approaches have been proposed in terms of curvature invariants and geometric corrections. Those schemes, supported by evidences at ultra violet scales, usually involve the use of additional curvature terms into the Hilbert-Einstein action. This recipe enables one to assume that "scalar fields" are derived from geometrical properties of space-time and also provides viable interpretations of dark energy and inflation as geometric effects at large scales (weak energies) and small scales (high
In other words, this geometric view represents a way to generalize and extend the standard General Relativity, aimed to consistently describe the early-time inflation and late-time acceleration without introducing other by hand dark components. More practically, a set of extended theories of gravity, containing additional curvature terms, can be relevant at very high energies, naturally producing inflation. During the cosmic evolution, the curvature decreases and General Relativity gives a sufficiently good approximation at intermediate scales. Afterwards, infra-red corrections start to work at very large scales. Rephrasing it differently, the curvature decreasing permits sub-dominant terms to start growing and then transition from deceleration to acceleration to happen. An important consequence is that this phenomenon roughly fixes the critical points of the whole cosmic evolution. Thus, the early-time as well as the late-time cosmic speed-up can be addressed by the fact that some curvature corrections to the Ricci scalar provide significative consequences at large and small curvatures.

In summary, a Lagrangian like

\[ f(R) \approx \ldots + \alpha(-2)R^{-2} + \alpha(-1)R^{-1} + \alpha(0)R^0 + \alpha(1)R + \alpha(2)R^2 + \ldots \]  

or, in general,

\[ f(R) \approx \sum_{i=-n}^{i=n} \alpha(i)R^i, \]  

with \( n \in \mathbb{N} \), could grossly fit the whole universe expansion history starting from the high energy regimes, \( (n > 0) \), recovering the cosmological constant, \( (n = 0) \), and General Relativity \( (n = 1) \) at intermediate scales, and evolving towards infra-red limit at large scales \( (n < 0) \). Clearly, the role played by \( \Lambda \equiv \alpha(0)R^0 \) is only formally equivalent to the case of a pure cosmological constant model. Indeed, according to \( f(R) \) gravity there is no a priori reason to consider the cosmological constant as associated to the vacuum energy density. In other words, one may recover a dynamical effective \( \Lambda \), assuming it as a limiting case of a more general solution of the above \( f(R) \) expansion series. In doing so, the dynamics of dark energy is not mimicked by \( \Lambda \), which appears only as a zero order term of \( f(R) \) gravity. This may be clearer if one assumes the effective gravitational action coming from some fundamental theory. In fact, we do not need to add the cosmological constant as a further term, put by hand into the gravitational action, but to reproduce it from first principles, with a completely different physical interpretation. In fact, it can be derived for \( f(R) \) gravity, a class of models capable of producing viable cosmology (different from the \( \Lambda \)CDM) where the cosmological constant is zero in flat spacetime, but appears in a curved one for sufficiently large curvatures. A smoking gun for these models could be the slope of primordial perturbation power spectrum determined from CMB fluctuations.

On the other hand, Lagrangians such as Eq. show several defects and issues that need to be necessarily addressed (see for a detailed discussion). Unfortunately, this class of drawbacks remains one of the main open problems of modern
high-energy physics. However, in the absence of a final quantum gravity theory, modified gravities can be viewed as practicable approaches built up to comply observational data with space-time phenomenology. In addition, actually as a by-product of this framework, modified gravities even provide a self-consistent dark matter explanation. It is also possible to describe, in fact, the observed evidence at galactic and extragalactic scales of dark matter distribution in terms of geometric modifications. In this review, we underline how inflation and dark energy can be encompassed within a single geometric approach, offered by \( f(R) \) gravity which can be considered as the simplest geometrical extension of General Relativity. We summarize, with no claims of completeness, the most relevant clues related to \( f(R) \) theories and get hints on possible future developments. This approach does not exhaust the possibilities of extended theories where more general curvature invariants can be employed albeit \( f(R) \) models may be assumed as a useful paradigm.

The paper is structured as follows. Sec. 2 is a quick summary on the emergence of curvature corrections as soon as a quantum field theory is formulated on curved space. In Sec. 3, we consider a realization of such an approach: the Starobinsky model capable of naturally producing an inflationary scenario. In Sec. 4 we develop the variational principles and the field equations of \( f(R) \) gravity. The basic equations of \( f(R) \) cosmology are presented in Sec. 5 where some toy models, in view of dark energy, are discussed. Sec. 6 is a wide discussion of cosmography where cosmographic parameters are constructed starting from \( f(R) \) functions and their derivatives. The goal is to recover viable phenomenological models in agreement with observational data. Conclusions are drawn in Sec. 7.

2. Curvature corrections from fundamental physics

Let us start our discussion with some fundamental physics considerations. At high energies and small scales, an accurate description of matter requires quantum field theory formulated on curved spaces. Since the matter should be quantized, one can assume a semi-classical description of gravitation where Einstein’s equations take the form

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \langle T_{\mu\nu} \rangle .
\] (3)

Here, we are far from the full quantum gravity regime and \( \langle T_{\mu\nu} \rangle \) is the expectation value of the quantum stress-energy tensor acting as a source in the gravitational field. Hereafter, we assume physical units where conventionally \( 8\pi G = k_B = c = 1 \). The l.h.s of the above field equations is assumed to classically evolve. The simplest case is the homogeneous and isotropic universe, characterized by a Friedmann-Robertson-Walker metric

\[
ds^2 = dt^2 - a^2(t) d\Omega_k^2 ,
\] (4)

where \( d\Omega_k^2 \) is the metric on the 3-space whose topology depends on the space-curvature parameter \( k \). In a curved space-time, also in the case in which both matter
and radiation fluids are zero, quantum fluctuations of fields determine non-trivial contributions to the whole energy-momentum tensor 68, 69 and may arise. In the presence of conformal invariant, massless and free-matter fields, those corrections can be framed as:

$$< T_{\mu \nu} > = k_1 H_{\mu \nu} + k_3 H_{\mu \nu},$$

(5)

with $k_1$ and $k_3$ numerical coefficients and

$$(1) H_{\mu \nu} = 2 R_{\mu \nu} - 2 g_{\mu \nu} \Box R + 2 R R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R^2,$$

(6)

$$(3) H_{\mu \nu} = R_{\mu \sigma} R_{\nu \sigma} - \frac{2}{3} R R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R^{\sigma \tau} R_{\sigma \tau} + \frac{1}{4} g_{\mu \nu} R^2.$$

(7)

An important remark is useful at this point. The masses of the matter fields and their mutual interactions can be neglected in the high curvature limit because $R >> m^2$. The matter-graviton interactions generate non-minimal coupling terms in the effective Lagrangian. The one-loop contributions of such terms are comparable to the ones coming from $\frac{1}{3} H_{\mu \nu}$ and generate, from the conformal point of view, the same effects on gravity. The simplest effective Lagrangian that takes into account these corrections is

$$\mathcal{L}_{NMC} = -\frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) - \frac{\xi}{2} R \phi^2,$$

(8)

where $\xi$ is a dimensionless coupling constant between the scalar and the gravitational fields. The scalar field stress-energy tensor will be modified accordingly but a conformal transformation can be found such that the modifications due to curvature terms can, at least formally, be cast in the form of a matter-curvature interaction. 47

The same argument holds for the trace anomaly, as we will see below.

The tensor $(1) H_{\mu \nu}$ is conserved, since $(1) H_{\mu \nu} = 0$. This tensor is obtained by varying a quadratic contribution of the Ricci scalar $R$ in the local action,

$$\mathcal{L}_{NMC} = -\frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) - \frac{\xi}{2} R \phi^2,$$

(9)

The infinities coming from $< T_{\mu \nu} >$ should be neglected somehow. To do so and to obtain a re-normalized theory, one might add an infinite number of many counterterms in the Lagrangian density of gravity. One of those terms is, for example, $C R^2 \sqrt{-g}$, where $C$ represents a diverging parameter in terms of a logarithm. In addition, one has to consider

$$(2) H_{\mu \nu} = 2 R_{\mu \nu} - \Box R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \Box R + R_{\mu \sigma} R_{\nu \sigma} - \frac{1}{2} R^{\sigma \tau} R_{\sigma \tau} g_{\mu \nu},$$

(10)

where the relation

$$\mathcal{L}_{NMC} = -\frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) - \frac{\xi}{2} R \phi^2,$$

(9)

holds in conformally flat space-times. In these cases, only the first and the third $H_{\mu \nu}$ terms of Eq. (5) do not vanish. Since one can add to the term $C \sqrt{-g} R^2$
an arbitrary constant, the coefficient \( k_1 \) may assume any value and, in principle, should be determined experimentally. On the other hand, the tensor \((3) \ H_{\mu \nu} \) is conserved only in conformally flat space-time and it cannot be obtained by varying a local action. Finally, one has:

\[
k_3 = \frac{1}{1440 \pi^2} \left( N_0 + \frac{11}{2} N_{1/2} + 31 N_1 \right),
\]

where the coefficients \( N_i \)'s \( (i = 0, 1/2, 1) \) are given by the number of quantum fields with spin 0, 1/2, and 1 present into the dynamics. Moreover, vector fields would give their contributions highly to \( k_3 \) due to the larger coefficient 31 showed in \( N_1 \). The cited massless fields, as well as the spinorial case, are even described by conformally invariant equations. They are present in \( < T_{\mu \nu} > \) in the form. The energy-momentum tensor trace goes to zero for conformally invariant classical fields whereas, owing to the term weighted by \( k_3 \), one infers that the outcome derived from the tensor \((5) \) provides a non-vanishing trace. This leads to the existence of the trace anomaly which may show serious consequences in cosmology. The matter field masses and the corresponding mutual interactions may be neglected as \( R >> m^2 \), i.e. at high curvature regime as discussed above. In addition, interactions between matter and gravitons lead to non-minimally coupled terms in the effective field Lagrangian. Summing up, what we have found definitively forecast that, as one quantizes matter fields on curved space-times, higher-order curvature corrections naturally arise as a corresponding effect. The paradigm deals with the fact that generic higher-order curvature corrections to the Hilbert-Einstein action can be easily framed at fundamental levels and the corresponding effects are highly relevant both at ultra-violet and at infra-red energy scales.

3. The case of Starobinsky inflation

A realization of the above effective theory is the Starobinsky inflation where higher curvature terms give the possibility to realize a de-Sitter behavior for the early universe. The Starobinsky model represents a prototype of any \( f(R) \) cosmology that, in principle, can track the whole cosmic history as soon as the cosmological solutions fit dynamics of the various epochs (e.g. transit from accelerated to decelerated behaviors and viceversa). Let us define the quantities:

\[
H_0 = (k_3)^{-\frac{1}{2}}, \quad M = (6k_1)^{-\frac{1}{2}},
\]

which have the obvious meaning of the Hubble parameter \( H_0 \) related to the number of quantum fields and an effective mass \( M \). The above tensor, defined in Eq. \((5) \), can be re-written as

\[
< T_{\mu \nu} > = \frac{1}{H_0^2} (3) H_{\mu \nu} + \frac{1}{6M^2} (1) H_{\mu \nu},
\]

where, for physical compatibility, we place \( H_0 > 0 \) and \( M > 0 \). Even though the trace of the energy-momentum tensor is null for conformally invariant fields, the
expected value of Eq. (14) has a non-zero trace, that is

\[ < T_\nu^\nu > = \frac{1}{H_0^2} \left( \frac{1}{3} R^2 - R_{\nu\sigma} R^{\nu\sigma} \right) - \frac{1}{M^2} R_{\nu}^{\nu}. \] (15)

This trace anomaly means that the conformal invariance is broken by the regularization of infinities in the energy-momentum tensor.\(^6\) Eq. (8), with \( < T_{\mu\nu} > \) given by Eq. (14), contains a de-Sitter space-time \( R_{\mu\nu} = 1/4 g_{\mu\nu} R \), \( R = \text{const} \), \( R_{\mu\nu} \), \( \nu \), \( \nu \).

\( (16) \)

as a possible solution. Substituting Eq. (16) into Eqs. (3) and (14) and discarding the trivial solution \( R = 0 \), we obtain \( R = 12H_0 \). The corresponding de-Sitter solutions are

\[ a(t) = H_0^{-1} \cosh(H_0 t), \quad k = +1, \]
\[ a(t) = a_0 \exp(H_0 t), \quad k = 0, \]
\[ a(t) = H_0^{-1} \sinh(H_0 t), \quad k = -1, \] (17)

for closed, flat, and open models, respectively. These solutions describe inflationary phases driven by quantum curvature corrections of Einstein’s equations. The \( H_0 \) value depends on the numbers of fields involved in Eq. (12). Typically, it is not so much different from the Planck mass \( m_p \). For example, in the minimal \( SU(5) \) model, it is \( N_0 = 34, N_{1/2} = 45, N_1 = 24, 8\pi k_3 = 1.8 \), and then \( H_0 = 0.7m_p \). Clearly the value of \( H_0 \) evolves according to the number of quantum fields present into dynamics. In particular, it changes after the inflation and for any phase transition.

The evolution equation for the scale factor, obtained from Eqs. (3) and (14), by inserting the Friedmann-Robertson-Walker metric (4), is:

\[ \dot{a}^2 + k a^2 = \frac{1}{H_0^2} \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 - \frac{1}{M^2} \left[ 2 \frac{\ddot{a}}{a} - \frac{\ddot{a}}{a^2} + 2 \frac{\dot{a}^2}{a^3} - 3 \left( \frac{\dot{a}}{a} \right)^4 - 2k \frac{\dot{a}^2}{a^4} + k^2 \right]. \] (18)

It is worth noticing that the source of the Friedmann equation, (i.e. the r.h.s), is totally geometric. In Eq. (18), we indicate with \( k \) the spatial curvature scalar, i.e. the curvature of the spatial part of Einstein’s equations. In \( \Lambda \)CDM cosmology and in inflation, the scalar curvature \( k \) is negligibly small and it is usually neglected, albeit it is not completely clear if its role may influence the dark energy evolution.\(^7\)

The de-Sitter solutions (17) implies that the universe scale factor exponentially grows and the \( k \)-dependent terms in (18) become negligible. It is, therefore, sufficient to study the flat space model with \( k = 0 \). Introducing \( H(t) = \dot{a}/a \), we can rewrite Eq. (18) as

\[ H^2 (H^2 - H_0^2) = \frac{H_0^2}{M^2} \left( 2H \ddot{H} + 6H^2 \dot{H} - \dot{H}^2 \right). \] (19)
The de-Sitter solution (17) corresponds to $H = H_0$. From a physical point of view, such a solution has to be unstable in order to allow the transition of the universe to the radiation dominated era. To show that this solution is unstable, consider a small deviation from $H = H_0$:

$$H = H_0(1 + \delta).$$

Substituting this in Eq. (19) and linearizing in $\delta$ we obtain

$$\ddot{\delta} + 3H_0\dot{\delta} - M^2\delta = 0.$$  (21)

The two solutions of (21) are given by $\delta = \exp(\alpha t)$ with

$$\alpha = -\frac{3H_0}{2} \pm \sqrt{\frac{9H_0^2}{4} + M^2}.\quad (22)$$

The existence of a growing mode for $\alpha > 0$ indicates the instability of the de-Sitter solution (17). We have to stress that the flat space-time, $H = 0$, is a stable solution of Eq. (19). The linearization in $H$ gives $2\ddot{H} = -M^2 H$, which has no growing solutions for $M^2 > 0$. The linear approximation (21) breaks down when $\delta$ becomes $\sim 1$. The nonlinear evolution is achieved by studying approximate solutions of Eq. (19) in various regimes. We assume that, at the beginning, $H$ is near $H_0$ and $\dot{H}$ is small, $\dot{H} \ll H_0^2$. If $H > H_0$, then $H$ grows without bounds. Such solutions are non-physical, so one has to take into account the case $H < H_0$.

This situation is not satisfactory since it implies a fine-tuning for initial conditions. In some sense, this is a sort of anthropic principle where dynamics has to be selected a priori. Despite of this shortcoming, the problem can be addressed and solved in the framework of Quantum Cosmology where unphysical initial conditions give rise to non-observable universes. The case $H > H_0$ falls into this set of conditions.

With these considerations in mind, assuming $H(t)$ slowly varying, we have

$$\dot{H} \ll H^2, \quad \ddot{H} \ll H\dot{H}.$$  (23)

The solution of Eq. (19) is

$$H = H_0 \tanh \left( \gamma - \frac{M^2 t}{6H_0} \right),$$

where $\gamma = \frac{1}{2} \ln \left( \frac{1}{\delta_0} \right)$ and $\delta_0$ is the magnitude of $|H - H_0|/H_0$ for $t = 0$. From Eq. (24), $H(t)$ changes on a time scale of the order $\sim 6H_0/M^2$. Sufficiently long inflation is obtained for $M^2 \ll 6H_0^2$. The solution (24) is valid until the neglected terms become comparable to those we kept in Eq. (19). This happens for $H \sim M$. This means that during the inflation, the expansion rate gradually changes from $H_0$ to
\( f(R) \) cosmology

\( \sim M << H_0 \). The scale factor \( a(t) \) is found by integrating Eq. (24), that is

\[
a(t) = \frac{1}{H_0} \left[ \frac{\cosh \gamma}{\cosh \gamma - \frac{M^2}{6H_0}} \right]^{\frac{6H_0^2}{M^2}}. \tag{25}
\]

For \( t_\ast - t \geq 6H_0/M^2 \), this gives

\[
a(t) = \frac{1}{H_0} \exp(H_0 t), \tag{26}
\]

The scale factor is given by

\[
a(t) = \text{const} \times t^\frac{2}{3} \left[ 1 + \left( \frac{2}{3Mt} \right) \sin Mt + O \left( \frac{1}{t^3} \right) \right]. \tag{30}
\]

The expansion rate averaged over the oscillation period is

\[
\bar{H} = \frac{2}{3t}, \tag{31}
\]

and corresponds to the expansion law \( \bar{a}(t) \propto t^{2/3} \). The oscillations of the expansion rate in Eq. (29) can be thought as coherent oscillations of a massive field describing scalar particles of mass \( M \) (the scalarons). The gravitational effect of such particles is similar to that of pressureless gas and leads to the expansion law \( a \propto t^{2/3} \), that is to a matter dominated universe.

However, after inflation, one expects a radiation-dominated epoch. The case of Eq. (31) has been obtained in the context of a homogeneous and isotropic universe without including a radiation term. Since radiation evolves as \( a^{-4} \), it will dominate over the effective dynamics due to \( f(R) \) corrections driving the universe expansion after inflation. Thus, for the sake of completeness, one needs to include an additional \( \propto a^{-4} \) term as soon as the Starobinky inflationary phase terminates. The phenomenology of such a model is richer than that described here. One should consider also thermalization effects, generation of gravitational waves, structure formation. All these aspects are well discussed in literature. Here we want to
stress again that the shortcomings of early Standard Cosmological Model can be suitably solved by considering curvature terms generated by quantum effects in curved space. The paradigm is to consider general classes of theories non-linear in the Ricci scalar (the so-called $f(R)$ gravity) and try to track the whole cosmic history up to dark energy epoch.

4. The field equations of $f(R)$ gravity

Having in mind the above results, we can now discuss a generic $f(R)$ function in the metric formalism. The geometric approach adopted for the inflation (ultra-violet regime) may work even at current epoch (infra-red regime). Even if energy and size scales are completely different, an accelerating behavior is recovered again by curvature corrections as it was first shown by Capozziello in 2002\textsuperscript{78} and Carroll et al. in 2004.\textsuperscript{79}

Let us consider the action

$$A_{(\text{curv})} = \int d^4x \sqrt{-g} f(R).$$

(32)

The vanishing of the variation gives us the vacuum field equations:

$$f'(R) R_{\mu\nu} - \frac{f(R)}{2} g_{\mu\nu} = \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R),$$

(33)

where the prime indicates the derivative with respect to the Ricci scalar $R$. The above equations can be re-framed according to the Einstein-like form

$$G_{\mu\nu} = \frac{1}{f'(R)} \left\{ \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R) + g_{\mu\nu} \frac{[f(R) - f'(R)R]}{2} \right\}. $$

(34)

The r.h.s. of Eq. (34) is thus reviewed as an effective energy-momentum tensor. We name it as curvature energy-momentum tensor $T_{\mu\nu}^{(\text{curv})}$. This tensor fuels the modified Einstein equations in terms of curvature corrections. Even though this interpretation is questionable, since the field equations depict a theory different from General Relativity, and one is forcing upon them the interpretation as effective Einstein equations, the scheme becomes therefore fruitful as it will be better clarified later. Further, considering the standard matter contribution, we get

$$G_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\mu\nu} [f(R) - R f'(R)] + \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \Box f'(R) \right\} + \frac{T_{\mu\nu}^{(m)}}{f'(R)}.$$

(35)

In the case of General Relativity, $T_{\mu\nu}^{(\text{curv})}$ becomes zero whereas the standard minimal coupling is easily reobtained for the matter contribution. Thus, let us recast, for our convenience:

$$T_{\mu\nu}^{(\text{curv})} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\mu\nu} [f(R) - R f'(R)] + f'(R)\gamma_{\mu\nu}(g_{\alpha\beta}g_{\nu\mu} - g_{\alpha\beta}g_{\mu\nu}) \right\}. $$

(36)
Obviously this quantity satisfies the Bianchi identities. Afterwards, our purpose is to demonstrate that it provides all the requirements we need to tackle with the dark components of our cosmos. Depending on the precise scales, the curvature component may reproduce the dark energy\(^80\) and dark matter\(^66,81–83\) roles respectively. More precisely, even the coupling term \(1/f'(R)\), entering the matter energy-momentum tensor, plays a crucial role in the whole dynamics. This happens because it affects all the physical processes (\(e.g.\) the nucleo-synthesis) and all the observable quantities (luminous, clustered, baryonic). In other words, the entire problem of understanding the universe dark components is naturally addressed, employing a self consistent theory where the interplay between geometry and matter is reconsidered assuming non-linear contributions and non-minimal couplings in curvature invariants.

5. Cosmology and curvature quintessence

Reducing the action (32) to a point-like, Friedmann-Robertson-Walker one, we can write the corresponding geometrical part as\(^78\)

\[
A_{\text{curve}} = \int dt \mathcal{L}(a, \dot{a}, R, \dot{R}) ,
\]

where, again, the dot indicates the derivative with respect to the cosmic time \(t\). The scale factor \(a = a(t)\) and the Ricci scalar \(R\) may be assumed to be the canonical variables. This appears as an arbitrary position since \(R\) depends upon \(a, \dot{a}, \ddot{a}\), but it is commonly employed in canonical quantization procedures\(^84\).

The Ricci definition in terms of \(a, \dot{a}, \ddot{a}\) involves a constraint that eliminates second and higher order derivatives in the action (37), providing a system of second order differential equations in terms of \(\{a, R\}\). The action (37) can be recast as

\[
A_{\text{curve}} = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[ R + 6 \left( \frac{\dot{a}}{a} + \frac{\ddot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\} ,
\]

in which the Lagrange multiplier \(\lambda\) has been obtained by varying with respect to the Ricci scalar \(R\), giving

\[
\lambda = a^3 f'(R) .
\]

It follows that the total point-like Lagrangian becomes

\[
\mathcal{L} = \mathcal{L}_{\text{curve}} + \mathcal{L}_{(m)}
= a^3 \left[ f(R) - Rf'(R) \right] + 6a\dot{a}^2 f'(R) + 6a^2 \dot{a}\ddot{R}f''(R) - 6ka^3 f'(R) + a^3 p_{(m)} ,
\]

which shows a canonical form in terms of the variables \(\{a, \dot{a}, R, \dot{R}\}\). Here, the contribution of standard matter reduces to a pure pressure term. Hence, the Euler-Lagrange equations are

\[
2 \left( \frac{\ddot{a}}{a} \right) + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = -p_{(tot)} ,
\]
and

\[
\frac{f''(R)}{f'(R)} \left\{ R + 6 \left[ \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \right\} = 0. \tag{42}
\]

In particular, Eq. (42) is interpreted in terms of the Lagrange multiplier definition, guaranteeing the consistency of the approach. Further, the dynamical system is completed by involving the following energy condition:

\[
\left( \frac{\ddot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{1}{3} \rho_{(\text{tot})}. \tag{43}
\]

In the above equations, we have

\[
p_{(\text{tot})} = p_{(\text{curv})} + p_{(m)} \quad \rho_{(\text{tot})} = \rho_{(\text{curv})} + \rho_{(m)}, \tag{44}
\]

in which we put in evidence both curvature and matter contributions to the whole cosmic fluid. We also inserted the above non-minimal coupling factor $1/f''(R)$ into the matter term definition. From $T^{(\text{curv})}_{\mu\nu}$, it is easy to get a curvature pressure definition

\[
p_{(\text{curv})} = \frac{1}{f'(R)} \left\{ 2 \left( \frac{\dot{a}}{a} \right) \ddot{R} f''(R) + \ddot{R} f''(R) + \dddot{R} f''(R) - \frac{1}{2} [f(R) - R f'(R)] \right\}, \tag{45}
\]

and a corresponding curvature density

\[
\rho_{(\text{curv})} = \frac{1}{f'(R)} \left\{ \frac{1}{2} \left[ f(R) - R f'(R) \right] - 3 \left( \frac{\dot{a}}{a} \right) \dddot{R} f''(R) \right\}. \tag{46}
\]

Starting from the above formalism, the dark energy drawbacks and the phenomenon of the universe speed up can be described assuming this effective curvature term. Combining Eq. (41) and Eq. (43), we get the Friedmann equation

\[
\left( \frac{\ddot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{1}{3} \rho_{(\text{tot})}. \tag{47}
\]

in which it is evident that the acceleration depends upon the corresponding r.h.s. and then the acceleration is achieved for

\[
\rho_{(\text{tot})} + 3p_{(\text{tot})} < 0, \tag{48}
\]

which means, from Eq. (47):

\[
p_{(\text{curv})} \gg p_{(m)}. \tag{49}
\]

We assume that the ordinary matter components provide non-negative pressure. Moreover, we assume that they are represented by standard fluids, defined as $0 \leq w_{(m)} \leq 1$. Rephrasing it differently, viable conditions to observe cosmic acceleration depend on the relation

\[
\rho_{(\text{curv})} + 3p_{(\text{curv})} = \frac{3}{f'(R)} \left\{ \dddot{R} f'''(R) + \left( \frac{\dot{a}}{a} \right) \ddot{R} f''(R) - \frac{1}{3} [f(R) - R f'(R)] \right\}, \tag{50}
\]

and
where has to be compared with matter contribution which is not dominant, according to the observations. It has to be
\[ \frac{p_{(\text{curv})}}{\rho_{(\text{curv})}} = w_{(\text{curv})}, \quad -1 \leq w_{(\text{curv})} < 0. \tag{51} \]

Particularly, the functional form of \( f(R) \) represents the main ingredient to obtain curvature quintessence. Soon, it is clear that the simplest choice to obtain the above prescriptions is to take into account a class of power-law solutions:
\[ f(R) = f_0 R^n. \tag{52} \]

Inserting Eqs. (52) into the above dynamical system, we obtain, by Noether’s symmetries, the exact solutions
\[ n = -1, \quad \frac{3}{2}, \quad \text{for} \quad k = 0. \tag{53} \]

In both the cases, the deceleration parameter is
\[ q_0 = -\frac{1}{2}, \tag{54} \]
in perfect agreement with the expected values permitted in the case of cosmic acceleration. However, those solutions cannot fit the whole cosmic history, together with some present shortcomings if confronted with data. However, they can be considered as useful toy models to clarify how the problem of accelerating the universe can be addressed directly by \( f(R) \) gravity.

The case \( n = 3/2 \) deserves a further discussion. Considering conformal transformation from Jordan frame to Einstein frame, it is possible to give an explicit form for the scalar field potential that leads to the accelerated expansion. It is
\[ \tilde{g}_{\alpha\beta} \equiv f'(R)g_{\alpha\beta}, \quad \varphi = \sqrt{\frac{3}{2}} \ln f'(R). \tag{55} \]

The conformal equivalence of the Lagrangians gives
\[ \mathcal{L} = \sqrt{-g} f_0 R^{3/2} \longleftrightarrow \tilde{\mathcal{L}} = \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2} + \frac{1}{2} \nabla_{\mu} \varphi \nabla^{\mu} \varphi - V_0 \exp \left( \sqrt{\frac{2}{3}} \varphi \right) \right], \tag{56} \]
in our physical units. This kind of model is particularly interesting to get inflation. However, for the sake of completeness, it is relevant to notice that, in the Jordan frame, Eq. (56) may show problems with basic Solar System bounds. Further details may be found in literature.

For \( n = 3/2 \), and \( k = 0 \), the general solution of the system is
\[ a(t) = a_0 \sqrt{c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0}. \tag{57} \]

The integration constants \( c_i \) are given by combining different initial conditions and their values definitively fix the cosmological evolution. For example, if we consider
$c_4 \neq 0$, we obtain a power law inflation, whereas if the regime is dominated by the linear term in $c_1$, we find a radiation-dominated epoch. \cite{10}

More realistic models can be worked out as reported in literature\cite{11, 12} but the general question is that the form of $f(R)$ function should be reconstructed by observational data. In next section, we will discuss in detail this problem.

6. Cosmography

In this section, we introduce the basic demands of cosmography, giving particular emphasis to its standard usage to fix cosmographic bounds on $f(R)$ and derivatives. In particular, to fix cosmological constraints on curvature quintessence, it is important to find out a strategy which permits to reconstruct the universe expansion history at present time. Indeed, cosmography represents a method to constrain current time cosmology, without postulating any cosmological model \textit{a priori}. In so doing, dark energy’s evolution can be directly framed in terms of cosmic data and $f(R)$ gravity can be featured by reconstructing numerical outcomes from the cosmographic coefficients. The corresponding \textit{cosmographic method} stands for a coarse grained technique to infer bounds on late time universe expansion history, rewriting quantities under interest in terms of cosmographic coefficients. Furthermore, cosmography is capable of discriminating among competing $f(R)$ models that are compatible with cosmographic predictions. From now on, we fix spatial curvature to be negligibly small, in order to get cosmography as a pure model independent treatment to bound the universe today.\cite{13, 14} If scalar curvature is not fixed \textit{a priori}, a degeneracy problem occurs between the variation of acceleration and the spatial curvature density parameter $\Omega_k$.\cite{15}

For our purposes, we simply use the $f(R)$ equation of state given by a geometrical fluid with curvature pressure $p_{\text{curv}}$ and we expand all quantities of interest into Taylor series around the present epoch, \textit{i.e.} $z = 0$. Typically, one may expand the Hubble parameter, the cosmological distances, the apparent magnitude modulus, the net pressure, and so forth.\cite{16, 17} All cosmographic coefficients are thus related to the derivatives of such Taylor expansions and can be bounded by cosmic data. In order to baptize such cosmographic coefficients and to permit one to handle cosmographic observables, it is possible to expand the scale factor $a(t)$ as

$$a(t) = 1 + \sum_{n=1}^{\infty} \left. \frac{d^n a(t)}{dt^n} \right|_0 \Delta t^n,$$

or more practically

$$\frac{1 - a(t)}{H_0} \sim \Delta t - \frac{q_0}{2} H_0 \Delta t^2 + \frac{j_0}{6} H_0^2 \Delta t^3 + \frac{s_0}{24} H_0^3 \Delta t^4 + \ldots,$$

which displays the $a(t)$ Taylor series around $\Delta t \equiv t - t_0$, truncated at the fourth order. Usually, $q_0, j_0, s_0, \ldots$ are named the \textit{cosmographic series} (CS), representing scale factor derivatives evaluated at present time, \textit{i.e.} at the redshift $z = 0$. In particular, $q_0$ is the deceleration parameter that quantifies how much the universe
accelerates today, \( j_0 \) is the jerk parameter and it is related to the variation of \( q(t) \) in the past, whereas \( s_0 \) measures the change of \( j(t) \) and it is commonly referred to as the snap parameter.

Usually, the today Hubble rate \( H_0 \) enters the definition of the CS. However, since all coefficients may be expressed in terms of \( H(t) \), at all stages of the universe evolution, it would be better to consider \( H_0 \) as the parameter to set the CS. In so doing, the cosmographic approach does not involve the definition of any cosmological model, becoming a powerful model independent method to fix \( f(R) \) limits at late times. In other words, \( H_0 \) is a prior set from observational data. Therefore, the CS is defined as follows

\[
\frac{H_0}{H_0^2} = -(1 + q), \quad \frac{H_0}{H_0^3} = j + 3q + 2, \quad \frac{H_0^{(3)}}{H_0^4} = s - 4j - 3q(q+4) - 6. \tag{60}
\]

All quantities are evaluated at present time \( t = t_0 \). In principle, the coefficients can be defined at all epochs by considering more general definitions as

\[
H(t) = \frac{1}{a} \frac{da}{dt}, \quad q(t) = -\frac{1}{aH^2} \frac{d^2a}{dt^2}, \quad j(t) = \frac{1}{aH^3} \frac{d^3a}{dt^3}, \quad s(t) = \frac{1}{aH^4} \frac{d^4a}{dt^4}. \tag{61}
\]

Thus, cosmography enables one to get a snapshot of the observable universe in terms of the CS, in order to reconstruct the universe cosmic evolution at different epochs. However, possible drawbacks are essentially based on the fact that current data are not accurate enough to fit significant intervals of convergence for \( z \gg 1 \). Moreover, there exist no physical arguments to employ a particular cosmological distance than others, since all distances are physically well supported. In fact, all standard definitions implicitly postulate that the universe is currently speeding up, since they are built up in terms of the photon distance \( r_0 \), i.e. the length that a photon travels from a light source at \( r = r_0 \) to a given reference point placed at \( r = 0 \). The photon length definition leads to \( r_0 = \int_0^{t_0} \frac{dt'}{a(t')} \) and depends on the scale factor only. Rephrasing those two problems differently, cosmographic expansions are plagued by a convergence problem due to truncated series, fitted with data in the interval \( z \gg 1 \), and by a duality problem, since the correct cosmological distance to fit data is not known a priori. To alleviate the convergence problem, an alternative approach can provides the construction of different redshift definitions, i.e. ad hoc functions of the redshift \( z \). Those re-parameterizing functions reduce the redshift intervals to tighter ranges and fulfill the conditions that all distance curves should not behave too steeply in the interval \( z < 1 \). Once re-parameterized functions are built up to be one-to-one invertible, they can be directly compared with data.

Essentially, any viable re-parameterizations need, as basic requirements, to satisfy the following two properties:

\[
Z \to 1 \quad z \to \infty, \tag{62a}
\]

\[
Z \to 0 \quad z \to 0. \tag{62b}
\]
A simple example of reparametrization is offered by \( Z(z) \equiv z(1+z)^{-1} \). Afterwards, we list below three relevant definitions, in terms of \( r_0 \), as possible examples of cosmic distances:

\[
\begin{align*}
  d_L &= a_0 r_0 (1 + z) = r_0 a(t)^{-1}, \\
  d_F &= \frac{d_L}{(1 + z)^{1/2}} = r_0 a(t)^{-\frac{1}{2}}, \\
  d_A &= \frac{d_L}{(1 + z)^2} = r_0 a(t),
\end{align*}
\]

respectively the luminosity, flux and angular distance. All the different cosmological distances assume the total number of photons is preserved\[117\] and all reduce at first order to

\[
d_i \sim \frac{z}{H_0},
\]

where \( d_i \) represents the generic distance, i.e. \( i = L; F; A \). Notice that all the above distances can be rewritten in terms of auxiliary variables \( Z(z) \). Moreover, once \( H_0 \) is fixed, the series better converges, since its shape increases or decreases as \( H_0 \) decreases or increases respectively. Thus, fixing \( H_0 \) leads to determine a low redshift cosmographic setting value, since all distances reduce to Eq. (64) at first order of Taylor expansions.

This technique enables one to get viable cosmographic constraints that should be related somehow to \( f(R) \) function and its derivatives. To do so, it is straightforward to start from the definition of \( R \) in terms of \( H \), i.e. \( R = -6 \left( \dot{H} + 2H^2 \right) \) and by means of

\[
\frac{d \log(1 + z)}{dt} = -H(z),
\]

having

\[
R = 6 \left[ (1 + z)H H_z - 2H^2 \right],
\]

where the subscript indicates the derivative with respect to the redshift \( z \). We are assuming that the spatial curvature is \( k = 0 \). Deriving \( R \) at different orders allows to relate \( R \) to \( H \) and derivatives. Thus, since \( H \) can be expanded as

\[
H = H_0 + \sum_{n=1}^{\infty} \frac{d^n H}{dz^n} z^n,
\]

it is easy to show

\[
\begin{align*}
  H_{z0}/H_0 &= 1 + q_0, \\
  H_{2z0}/H_0 &= j_0 - q_0^2, \\
  H_{3z0}/H_0 &= -3j_0 - 4j_0q_0 + q_0^2 + 3q_0^3 - s_0,
\end{align*}
\]
where we adopted the convention $H_{n=0} \equiv \left. \frac{d^n H}{dz^n} \right|_{z=0}$. Considering a pure matter term, evolving as dust, the cosmological Eqs. (41) and (43) can be recast, by a little algebra, as

$$H^2 = \frac{1}{3} \left[ \rho_{\text{curv}} + \frac{\rho_{(m)}}{f(R)} \right],$$

(69)

$$2\dot{H} + 3H^2 = -p_{(\text{curv})}.$$  

(70)

From the above expressions, deriving the $f(R)$ function, one gets

$$f'(R) = R^{-1} f_z,$$

$$f''(R) = (f_{zz} R_z - f_z R_{zz}) R_z^{-3},$$

$$f'''(R) = \frac{f_{3z}}{R_z^3} - \frac{f_z R_{3z} + 3f_{2z} R_{2zz}}{R_z^4} + \frac{3f_z R_{2zz}^2}{R_z^5},$$

(71)

where we introduced the definition of $f(z)$. Indeed, since $R = R(z)$, there exists a direct correspondence between $f(R)$ and $f(z)$ functions: knowing $f(z)$ is equivalent to know $f(R)$ and vice versa. In cosmographic treatments, it is much easier to handle $f(z)$ than $f(R)$, due to the complexity of the modified Friedmann equations. Afterwards, since

$$\dot{H} = -(1 + z)HR_z,$$

(72)

and

$$\ddot{H} = (1 + z)H \left[ HR_z + (1 + z)(H_z R_z + HR_{zz}) \right],$$

(73)

we easily get

$$\frac{f_0}{2H_0^2} = -2 + q_0,$$

$$\frac{f_{z0}}{6H_0} = -2 - q_0 + j_0,$$

$$\frac{f_{zz0}}{6H_0^2} = -2 - 4q_0 - (2 + q_0)j_0 - s_0,$$

(74)

which to calculate $f(R)$ by a simple inverse procedure, once $f(z)$ and its derivatives are numerically known. Hence, determining numerical outcomes derived by cosmography, it is possible to bound $f(z)$ and its corresponding derivatives. In particular, if $f(z)$ is somehow constrained by cosmography, it naturally follows that $f(R)$ is bounded as well, since $R = R(z)$. 
In particular, to obtain $f(R)$ and derivatives, one needs to know $R$ as a function of the cosmographic parameters. Thus, we have

\[
\frac{R_0}{6H_0^2} = q_0 - 1,
\]
\[
\frac{R_{z=0}}{6H_0^2} = j_0 - q_0 - 2,
\]
\[
\frac{R_{z=2}}{6H_0^2} = - (2 + 4q_0 + 2q_0^2 + j_0(2 + q_0) + s_0). \tag{75}
\]

Moreover, it is possible to demonstrate that $f(z)$ and $f(R)$ decrease as the redshift increases. Analogously, the corresponding first derivatives negatively evolve as the redshift expands.

To get constraints on $f(z)$ and derivatives, one needs experimental procedures able to fix numerical outcomes on the CS. Two relevant data sets are for example given by the Union 2.1 compilation and the baryonic acoustic oscillation (BAO) . To fix viable constraints, it is possible to perform a Monte Carlo analysis based on minimizing the $\chi$ square functions. In the case of supernovae, we have as distance modulus,

\[
\mu = 25 + 5 \log_{10} \frac{d_L}{Mpc}, \tag{76}
\]

and $\chi$ square function

\[
\chi^2_{SN} = \sum_i \frac{(\mu_i^{\text{theor}} - \mu_i^{\text{obs}})^2}{\sigma_i^2}, \tag{77}
\]

while in case of BAO, we employ the measurable $A$, defined as

\[
A = \sqrt{\Omega_m} \left[ \left( \frac{H(z_{BAO})}{H(z_{BAO})} \right)^{1/3} \left[ \frac{1}{z_{BAO}} \int_0^{z_{BAO}} \frac{H_0}{H(z)} dz \right]^{\frac{2}{3}} \right], \tag{78}
\]

with $z_{BAO} = 0.35$ and the corresponding $\chi$ square:

\[
\chi^2_{BAO} = \frac{1}{\nu} \left( \frac{A - A_{\text{obs}}}{\sigma_A} \right)^2. \tag{79}
\]

Estimations of the CS may be performed through standard Bayesian techniques, i.e. maximizing the likelihood function: $\mathcal{L}_i \propto \exp(-\chi^2_i/2)$, where we define the total $\chi_t \equiv \chi_{SN} + \chi_{BAO}$. For the sake of completeness, one can expand the above cited causal distances $d_L, d_F$ and $d_A$, in terms of the redshift $z$, obtaining

\[
d_L = \frac{1}{H_0} \left[ z + z^2 \cdot \left( \frac{1}{2} - \frac{q_0}{2} \right) + z^3 \cdot \left( -\frac{1}{6} \frac{j_0}{6} + \frac{q_0}{6} + \frac{q_0^2}{2} \right) + z^4 \cdot \left( \frac{1}{12} + \frac{5j_0}{24} - \frac{q_0}{12} + \frac{5j_0q_0}{12} - \frac{5q_0^2}{8} - \frac{5q_0^3}{8} + \frac{s_0}{24} \right) + \ldots \right],
\]
\[ d_F = \frac{1}{H_0} \cdot \left[ z - z^2 \cdot \frac{q_0}{2} + z^3 \cdot \left( \frac{1}{24} \cdot \frac{j_0}{6} + \frac{5q_0}{12} + \frac{q_0^2}{2} \right) + \right. \]
\[ + z^4 \cdot \left( \frac{1}{24} + \frac{17j_0}{24} - \frac{17q_0}{12} - \frac{7q_0^2}{8} - \frac{5q_0^3}{8} + \frac{s_0}{24} \right) + \ldots \], \]
and
\[ d_A = \frac{1}{H_0} \cdot \left[ z + z^2 \cdot \left( -\frac{3}{2} - \frac{q_0}{2} \right) + z^3 \cdot \left( \frac{11}{6} - \frac{j_0}{6} + \frac{7q_0}{6} + \frac{q_0^2}{2} \right) + \right. \]
\[ + z^4 \cdot \left( \frac{25}{12} + \frac{13j_0}{12} - \frac{23q_0}{12} + \frac{5j_0q_0}{12} - \frac{13q_0^2}{8} - \frac{5q_0^3}{8} + \frac{s_0}{24} \right) + \ldots \].

Those expansions enter the definitions of \( \chi_t^2 \) and, after a numerical procedure, it is easy to bound the CS. Afterwards, keeping in mind \( q_0, j_0, s_0 \), and using Eqs. (74), it is possible to numerically fix \( f(z) \) and derivatives. Analogously, from Eqs. (71), it is possible to constrain \( f(R) \). This technique enables the determination of \( f(R) \) as the universe expands and consequently our numerical outcomes lead to frame the dark energy effects in terms of a pure curvature fluid. Cosmographic indications suggest that the cosmological standard model is extended by means of a logarithmic correction, as follows
\[ H(z) = H_0 \sqrt{\Omega_m(1 + z)^3 + \log(\alpha + \beta z)} , \] (80)
where \( \beta \) is a free constant, whereas \( \alpha = \exp(1 - \Omega_m) \). In other words, Eq. (80) represents an effective Hubble rate, numerically reconstructed, by employing \( f(R) \) corrections, set through cosmographic results.

This prescription provides an approximate form of \( f(z) \) given by
\[ f(z) = \tilde{f}_0 + \frac{1}{1 + z} + \tilde{f}_1 (1 + z)^\sigma_1 + \tilde{f}_2 (1 + z)^\sigma_2 , \] (81)
which well adapts its shape to data, with negligible departures from \( z \ll 1 \) to \( z \sim 2 \), for the intervals \( \tilde{f}_0 \sim -10, \tilde{f}_1 \sim 7, \tilde{f}_2 \sim -3.7, \sigma_1 = 1 \) and \( \sigma_2 = 2 \). Those results are consistent with the cosmographic ranges of \( f_0 \) and \( f_{z0} \). As example of direct fittings of \( f(z) \) and derivatives, we report some experimental results in Tab. I, while in Tab. II we report the corresponding cosmographic parameters.

All numerics provide a slightly lower dark energy pressure, with small corrections to a constant dark energy term. This fact suggests that the curvature dark energy is not described by a pure cosmological constant. The curvature quintessence seems to evolve in agreement with the following limits
\[ f_0 < 0 , \quad f_{z0} > 0 , \quad f_{z20} < 0 , \quad p_{\text{curv}} < p_{\Lambda CDM} , \] (82)
where \( p_{\Lambda CDM} \) is the pressure of the standard cosmological model. Moreover, it is a matter of fact that the absolute values of each variable increases as one performs fits using \( d_F \) and \( d_A \), giving reasonable departures from the cosmological standard model. Determining the limits on \( f(z) \) and derivatives, it is easy to show the corresponding bounds on \( f(R) \), listed in Tab. III.
Table I. Best fits of the parameters $H_0$, $j_0$, $f_{j0}$ and $f_{220}$ for three statistical models, i.e. A, B and C, corresponding to three different orders of the cosmographic Taylor expansion, respectively the second, third and fourth orders. $H_0$ is given in Km/s/Mpc.

| Parameter | Model A | Model B | Model C |
|-----------|---------|---------|---------|
| $H_0$    | $77.23^{+0.84}_{-1.82}$ | $75.69^{+2.03}_{-1.99}$ | $71.30^{+1.92}_{-1.91}$ |
| $10^{-4}j_0$ | $-3.324^{+0.227}_{-0.230}$ | $-3.144^{+0.320}_{-0.332}$ | $-2.669^{+0.287}_{-0.284}$ |
| $10^{-4}f_{j0}$ | $3.636^{+1.751}_{-1.735}$ | $-1.510^{+5.694}_{-5.656}$ | $-1.794^{+4.834}_{-4.800}$ |
| $10^{-5}f_{220}$ | $-2.292^{+0.965}_{-0.973}$ | $2.276^{+2.339}_{-2.032}$ | $0.499^{+2.192}_{-2.040}$ |

Table II. Table of numerical results for the CS; the numerical values are given at $z = 0$, corresponding to three different orders of the cosmographic Taylor expansion, respectively the second, third and fourth orders.

| Parameter | Model A | Model B | Model C |
|-----------|---------|---------|---------|
| $q_0$    | $-0.786^{+0.251}_{-0.324}$ | $-0.744^{+0.426}_{-0.434}$ | $-0.625^{+0.424}_{-0.420}$ |
| $j_0$    | $2.229^{+0.718}_{-0.761}$ | $0.817^{+2.106}_{-2.102}$ | $0.787^{+2.04}_{-1.83}$ |
| $s_0$    | $-7.713^{+4.997}_{-5.372}$ | $-6.671^{+11.15}_{10.295}$ | $-2.217^{+11.93}_{11.15}$ |

Table III. Values of $f(R)$ and its derivatives for three statistical models, i.e. A, B and C, corresponding to three different orders of the cosmographic Taylor expansion, respectively the second, third and fourth orders.

| Parameter | Model A | Model B | Model C |
|-----------|---------|---------|---------|
| $f(R_0)$ | $-3.324^{+0.227}_{-0.230}$ | $-3.144^{+0.320}_{0.332}$ | $-2.669^{+0.287}_{-0.284}$ |
| $f'(R_0)$ | $1.4^{+2.6}_{-2.7} \cdot 10^{-16}$ | $1.8^{+1.8}_{-1.8} \cdot 10^{-15}$ | $1.5 \cdot 10^{-15}$ |
| $f''(R_0)$ | $5.9 \cdot 10^{-20}$ | $-4.1 \cdot 10^{-19}$ | $-1.2 \cdot 10^{-19}$ |

From Tabs. I, II and III, we are able to describe the $f(z)$, $f(R)$ functions at late times, fixing the corresponding numerical bounds which represent the cosmographic settings (Tab. I) on $f(z)$ and derivatives. In other words, we are able to depict the universe expansion history in the observable limit of small redshift. In Tab. IV, we summarize the numerical outcomes inferred for $R$ and derivatives using the cosmographic results.

To extrapolate the behavior of the $f(R)$ function at different stages of the universe evolution, one can use Eq. (71), Eqs. (80), and (81) to perform an inverse
The above procedure allows to reproduce the effective dynamics of dark energy without postulating the existence of a cosmological constant. Below, we report a reasonable cosmographic reconstruction that has been achieved by the following $f(R)$ reconstruction:\cite{120,121}

\begin{align}
 f(R) &= \frac{1}{2(a + b + c)e\pi R_0^4} \left\{ \Lambda R_0^2 \left[ 2\pi e^{R/R_0} ight] \\
 &\quad + e \left( 6b + (a + 2c)\pi + 8b \arctan \left( \frac{R}{R_0} \right) \right) \right\} \\
 &\quad + e\Lambda \left[ 2R_0 \left( (a + b + c)\pi R_0 - 4b\Lambda \right) \\
 &\quad + (2b - a\pi)\Delta R \right] - 2\pi e\Lambda (R - R_0)^2 \sin \left( \frac{2\pi R}{R_0} \right) \right\},
\end{align}

with $K_{cs}$ a cosmographic integration constant. Let us notice that $K_{cs}$ is not related to the cosmological constant. Instead, the constant $K_{cs}$ estimates the numerical difference between putting by hand $z$ in function of $R$ within $f(z)$ and evaluating the integral $\int \frac{df}{dz}(R_z)^{-1}dz$. Both the procedures enable one to extrapolate the numerical $f(R)$ function, knowing $f(z)$, although they differ from a constant, i.e. $K_{cs}$. Indeed, integrating the $f(z)$ function over the whole redshifts may increase or decrease the final shape of the numerical $f(R)$ function with respect to directly substitute $z$ in function of $R$ into $f(z)$. The increasing or decreasing factor is approximatively a constant, $K_{cs}$, which is due to the integration performed in Eq. \[83\].

It is also possible to determine the $f(R)$ shape, by keeping in mind the form of $f(z)$, for redshift intervals $z > 1$. A significative extension to higher redshifts employing Eq. \[80\] is possible by numerically solving the modified Friedmann equations. The above procedure allows to reproduce the effective dynamics of dark energy without postulating the existence of a cosmological constant. Below, we report a reasonable cosmographic reconstruction that has been achieved by the following $f(R)$ reconstruction:\cite{120,121}

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Parameter & Model A & Model B & Model C \\
\hline
$R_0$ & $-10.716^{+1.506}_{-1.944}$ & $-10.464^{+2.556}_{-2.604}$ & $-9.750^{+2.544}_{-2.520}$ \\
$R_0'$ & $6.090^{+2.802}_{-2.622}$ & $-2.634^{+10.079}_{-10.008}$ & $-3.528^{+9.696}_{-8.460}$ \\
$R_0''$ & $29.492^{+51.093}_{-41.695}$ & $33.082^{+74.670}_{-55.037}$ & $5.122^{+81.052}_{-101.575}$ \\
\hline
\end{tabular}
\end{center}

Table IV. Values of $R$ and its derivatives for three statistical models, i.e. A, B and C, corresponding to three different orders of the cosmographic Taylor expansion, respectively the second, third and fourth orders. The numerical results have been obtained, in power of $H_0^2$, using the numerical outcomes inferred from Tab. II and the expressions of Eqs. \[75\].

procedure and define a corresponding $f(R)$ function at our time. In fact, since

$$f(R) = \int \frac{df}{dz}(R_z)^{-1}dz + K_{cs},$$

(83)

with $K_{cs}$ a cosmographic integration constant. Let us notice that $K_{cs}$ is not related to the cosmological constant. Instead, the constant $K_{cs}$ estimates the numerical difference between putting by hand $z$ in function of $R$ within $f(z)$ and evaluating the integral $\int \frac{df}{dz}(R_z)^{-1}dz$. Both the procedures enable one to extrapolate the numerical $f(R)$ function, knowing $f(z)$, although they differ from a constant, i.e. $K_{cs}$. Indeed, integrating the $f(z)$ function over the whole redshifts may increase or decrease the final shape of the numerical $f(R)$ function with respect to directly substitute $z$ in function of $R$ into $f(z)$. The increasing or decreasing factor is approximatively a constant, $K_{cs}$, which is due to the integration performed in Eq. \[83\].

It is also possible to determine the $f(R)$ shape, by keeping in mind the form of $f(z)$, for redshift intervals $z > 1$. A significative extension to higher redshifts employing Eq. \[80\] is possible by numerically solving the modified Friedmann equations. The above procedure allows to reproduce the effective dynamics of dark energy without postulating the existence of a cosmological constant. Below, we report a reasonable cosmographic reconstruction that has been achieved by the following $f(R)$ reconstruction:\cite{120,121}

\begin{align}
 f(R) &= \frac{1}{2(a + b + c)e\pi R_0^4} \left\{ \Lambda R_0^2 \left[ 2\pi e^{R/R_0} \\
 &\quad + e \left( 6b + (a + 2c)\pi + 8b \arctan \left( \frac{R}{R_0} \right) \right) \right] \\
 &\quad + e\Lambda \left[ 2R_0 \left( (a + b + c)\pi R_0 - 4b\Lambda \right) \\
 &\quad + (2b - a\pi)\Delta R \right] - 2\pi e\Lambda (R - R_0)^2 \sin \left( \frac{2\pi R}{R_0} \right) \right\},
\end{align}

(84)
where $a, b, c$ represent three free parameters of the model related to integration constants. The model has been obtained by considering the cosmographic results on CS as initial settings, and fitting the numerical outcomes obtained from Eqs. (69), (70). The corresponding $f(R)$ function passes several experimental bounds at small redshifts\textsuperscript{122} with high agreement at higher redshifts. Comparing cosmographic results with the approximate function given in Eq. (84), we find

$$a \sim 145.5^{+11.64}_{-8.73}, \quad b \sim -148^{+11.44}_{-8.88}, \quad c \sim 1^{+0.08}_{-0.06},$$

(85)

which represent the numerical outcomes of the free parameters involved in Eq. (84). Typically the errors are estimated not to exceed the limit of $6 \div 8\%$. Thus for each coefficients, the errors bars $\delta a, \delta b, \delta c$ have been reported into the intervals $\delta a \sim [8.73, 11.64], \delta b \sim [8.88, 11.84], \delta c \sim [0.06, 0.08]$ respectively.

Other typologies of viable candidates can be determined shifting the cosmographic outcomes and consequently changing the initial settings due to cosmography. The scheme proposed here is able to describe the universe dynamics at small redshifts, by means of $f(R)$ cosmography. This procedure matches the early phases of $f(R)$ cosmology with present time and it is of great importance in order to reconstruct robust forms of $f(R)$ function. Future developments should allow to refine the $f(R)$ paradigm by relating late time results with high redshift data.

7. Outlooks and perspectives

In this paper, we have outlined some of the main features of $f(R)$ cosmology, with no claim to completeness. Our aim has been to show that such an alternative approach to cosmology directly derives from a natural extension of General Relativity and can be based on fundamental physics since comes out from quantum field theory formulated on curved space. In principle, $f(R)$ gravity could trace back from late type cosmology up to inflation, if reliable models are suitably constructed by matching observational data at various redshift regimes. Cosmographic analysis greatly aids in this task as soon as cosmographic parameters are derived from $f(R)$ functions and their derivatives, from one side, and self-consistently are matched with data, on the other side. In principle, this procedure could be extended up to inflation\textsuperscript{123} addressing also the large scale structure\textsuperscript{124} but the big challenge is to find out reliable and homogeneous datasets which allow realistic fittings at any cosmic epochs.

From a genuine theoretical point of view, there is no final $f(R)$, or alternative gravity model, today capable of addressing all the cosmological dynamics, however, despite of this lack, the approach seems very promising to encompass both problems of inflation and dark side. The forthcoming space experiments like EUCLID\textsuperscript{125} could realistically support or rule out definitely this view.

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