REGULARITY PROPERTIES OF SOME PERTURBATIONS OF NON-DENSELY DEFINED OPERATORS WITH APPLICATIONS

DELIANG CHEN

ABSTRACT. This paper is to study some conditions on semigroups, generated by some class of non-densely defined operators in the closure of its domain, in order that certain bounded perturbations preserve some regularity properties of the semigroup such as norm continuity, compactness, differentiability and analyticity. Furthermore, we study the critical and essential growth bound of the semigroup under bounded perturbations. The main results generalize the corresponding results in the case of Hille-Yosida operators. As an illustration, we apply the main results to study the asymptotic behaviors of a class of age-structured population models in $L^p$ spaces $(1 \leq p < \infty)$.

1. Introduction

The main goal of this paper is to study the preservation of the regularity properties of some class of non-densely defined operators under bounded perturbations. Let $A : D(A) \subset X \to X$ be a linear operator on some Banach Space $X$ and $B$ a perturbing linear operator. Assume that $A$ has some good regularity properties. Under which conditions can $A + B$ keep the properties of $A$? When $A$ is the generator of a $C_0$ semigroup $T_A$, or equivalently $A$ is a Hille-Yosida operator and densely defined, i.e., $D(A) = X$, many classes of operator $B$ allow $A + B$ to generate a $C_0$ semigroup $T_{A+B}$, e.g., $B$ is a bounded operator, Desch-Schappacher perturbation, or Miyadera-Voigt perturbation; see [EN00]. If $T_A$ has higher regularity properties, such as immediate/eventual norm continuity, immediate/eventual compactness, immediate/eventual differentiability and analyticity, then one may ask naturally that under what kinds of compactness, immediate differentiability and analyticity are stable under bounded perturbation, and Iley [Ile07] pointed out that this condition is also necessary. Mátrai [Mát08] showed that immediate norm continuity is preserved under Desch-Schappacher perturbation and Miyadera-Voigt perturbation.

However, in many applications, the operator $A$ may be not densely defined or even not a Hille-Yosida operator (see, e.g., [DPS87, PS02, MR07, DMP10]); see also Section 6. Let $A$ be a Hille-Yosida operator [DPS87, ABHN11], $X_0 := \overline{D(A)}$, $A_0 := A_{X_0}$, where $A_{X_0}$ denotes the part of $A$ in $X_0$, i.e.,

$$A_{X_0} x = Ax, \quad x \in D(A_{X_0}) := \{x \in D(A) : Ax \in X_0\}.$$ 

It is well known that $A_0$ generates a $C_0$ semigroup $T_{A_0}$ in $X_0$, and $A + B$ is also a Hille-Yosida operator if $B \in \mathcal{L}(X_0, X)$ [KH89]. A natural question may be asked: Are the regularities of $T_{(A+B)_0}$ the same

\begin{itemize}
  \item [2010 Mathematics Subject Classification.] Primary 47A55, 34D10; Secondary 34K12, 47N20, 47D62.
  \item [Key words and phrases.] regularity, perturbation, non-densely defined operators, critical growth bound, essential growth bound, integrated semigroup, age-structured population model.
\end{itemize}

Part of this work was done at East China Normal University. The author would like to thank Shigui Ruan, Ping Bi and Dongmei Xiao for their useful discussions and encouragement. The author is grateful to the referee(s) for useful comments and suggestions and particularly pointing out lemma 6.9 and a mistake in theorem 7.4, which improved significantly the presentation of the original manuscript.
as \( T_{A_0} \)? Bátkai, Maniar and Rhandi [BMR02] dealt with this problem by using extrapolation theory and obtained similar results as in [NP98].

Magal and Ruan [MR07] studied a more general class of non-densely defined operators which in this paper are called MR operators and quasi Hille-Yosida operators (see Definition 2.5 and Definition 2.7). These operators turn out to be important for the study of certain abstract Cauchy problems, such as age-structured population models, parabolic differential equations and delay equations (see, e.g., [PS02, MR07, DMP10, MR18, Che18]). Let \( A \) be an MR operator (resp. quasi Hille-Yosida operator) and \( T_{A_0} \) the \( C_0 \) semigroup generated by \( A_0 \) in \( X_0 \). It was shown in [MR07, Thi08] that \( A \) is stable under the bounded perturbation, that is \( A + B \) is still an MR operator (resp. quasi Hille-Yosida operator) for all \( B \in \mathcal{L}(X_0, X) \). We are interested in that under which conditions \( T_{(A+B)_0} \), generated by \( (A + B)_0 \) in \( X_0 \), can preserve the regularity properties of \( T_{A_0} \). The first part of the paper addresses the problem and obtains analogous and generalized results as in [NP98, BMR02], i.e., norm continuity, compactness, differentiability and analyticity (see Section 4). We use the method developed in [NP98, BMR02] and the integrated semigroups theory.

The second part of the paper is to study the stability of the critical growth bound and essential growth bound (see Definition 2.18 and Definition 2.20) of a \( C_0 \) semigroup generated by the part of an MR operator in the closure of its domain under certain bounded perturbations. Critical spectrum was introduced independently by Nagel and Poland [NP00] and Blake [Bla01]. The authors used this notion to obtain the partial spectral mapping theorem, which could characterize the stability of \( C_0 \) semigroups very well, see [NP00] for details. Brendle, Nagel and Poland [BNP00] studied the stability of critical growth bound of a \( C_0 \) semigroup under some class of Miyadera-Voigt perturbation. In particular, they obtained, under appropriate assumptions, a partial spectrum mapping theorem for the perturbed semigroup. Similar result was also obtained by Boulite, Hadd and Maniar [BHM05] for the Hille-Yosida operators. The stability of the essential growth bound were considered by many authors, e.g., Voigt [Voi80] and Andreu, Martínez and Mazón [AMM91] (the generators of \( C_0 \) semigroups), Thieme [Thi97] (Hille-Yosida operators), Ducrot, Liu and Magal [DLM08] (quasi Hille-Yosida operators). Such results have wide applications, e.g., to study the stability of equilibriums, the existence of center manifolds and Hopf bifurcation, see [EN00, BNP00, ABHN11, MR09b, MR09a]. See also [Sbi07, Bre01] for an approach based on the resolvent characterization in Hilbert spaces and [MS16, Section 2 and 3] for some new partial spectral mapping theorems in an abstract framework. We consider the perturbation of the critical and essential growth bound in the case of MR operators and give a unified treatment of the two problems (see Section 5). Our proof is close to [BNP00].

Some simple version of our main results in Section 4 and Section 5 may be summarized below; see those sections for more detailed results and see Section 2 for definitions and notations. For a linear operator \( A : D(A) \subset X \rightarrow X \), let \( X_0 = D(A) \), \( A_0 := A_{X_0} \). \( T_{A_0} \) denotes the \( C_0 \) semigroup (if it exists) generated by \( A_0 \).

**Theorem A** (norm continuity and compactness). Let \( A \) be an MR operator (see Definition 2.5), \( L \in \mathcal{L}(X_0, X) \).

(a) Assume \( LT_{A_0} \) is norm continuous on \((0, \infty)\). Then, \( T_{A_0} \) is eventually (resp. immediately) norm continuous if and only if \( T_{(A+L)_0} \) is eventually (resp. immediately) norm continuous.

(b) Assume \( LT_{A_0} \) is norm continuous and compact on \((0, \infty)\). Then, \( T_{A_0} \) is eventually (resp. immediately) compact if and only if \( T_{(A+L)_0} \) is eventually (resp. immediately) compact.

(c) Suppose \( A \) is a quasi Hille-Yosida operator (see Definition 2.7) and \( LT_{A_0} \) is compact on \((0, \infty)\). Then, \( T_{A_0} \) is eventually norm continuous (resp. eventually compact) if and only if \( T_{(A+L)_0} \) is eventually norm continuous (resp. eventually compact); the result also holds when “eventually” is replaced by “immediately”.

The following result can be proved very simply if one uses the corresponding characterization of the resolvent.
Theorem B (differentiability and analyticity). (a) Let $A$ be a Hille-Yosida operator. Then, $T_{(A+L)_h}$ is eventually differentiable for all $L \in \mathcal{L}(X_0, X)$ if and only if $A_0$ satisfies the Pazy-Iley condition (i.e., the condition of Theorem 4.7 in (b) or (c)).

(b) If $A$ is a $p$-quasi Hille-Yosida operator (see Definition 2.7) and $A_0$ is a Crandall-Pazy operator satisfying

$$\| R(iy, A_0) \|_{\mathcal{L}(X_0)} = O(|y|^{-\beta}), \ |y| \to \infty, \ \beta > 1 - \frac{1}{p},$$

then $(A + L)_h$ is still a Crandall-Pazy operator (see, e.g., [Ile07, CP68]) for any $L \in \mathcal{L}(X_0, X)$.

(c) Let $A$ be an MR operator (see Definition 2.5). Then, $T_{A_0}$ is analytic if and only if for any $L \in \mathcal{L}(X_0, X)$, $T_{(A+L)_h}$ is analytic. In particular, if $A$ is an almost sectorial operator (see Definition 2.10), so is $A + L$ for any $L \in \mathcal{L}(X_0, X)$.

Let $\omega_{\text{crit}}(T_{A_0})$ and $\omega_{\text{ess}}(T_{A_0})$ denote the critical growth bound and the essential growth bound of $T_{A_0}$ respectively (see Definition 2.18 and Definition 2.20).

Theorem C. (a) Let $A$ be an MR operator (see Definition 2.5) and $L \in \mathcal{L}(X_0, X)$. If $LT_{A_0}$ is norm continuous on $(0, \infty)$, then $\omega_{\text{crit}}(T_{(A+L)_h}) = \omega_{\text{crit}}(T_{A_0})$. If $LT_{A_0}$ is norm continuous and compact on $(0, \infty)$, then $\omega_{\text{ess}}(T_{(A+L)_h}) = \omega_{\text{ess}}(T_{A_0})$.

(b) If $A$ is a quasi Hille-Yosida operator (see Definition 2.7) and $LT_{A_0}$ is compact on $(0, \infty)$, then $\omega_{\text{ess}}(T_{(A+L)_h}) = \omega_{\text{ess}}(T_{A_0})$, $\omega_{\text{crit}}(T_{(A+L)_h}) = \omega_{\text{crit}}(T_{A_0})$.

Remark 1.1. (a) The above results are known at least for the Hille-Yosida operators; see, e.g., [BMR02, BHM05, Thi97].

(b) Theorem C (b) is basically due to [DLM08] but the result is strengthened as $\omega_{\text{ess}}(T_{(A+L)_h}) = \omega_{\text{ess}}(T_{A_0})$; also our proof in some sense simplifies [DLM08].

As an illustration, in the third part of this paper (see Section 6), we apply the main results in Section 4 and Section 5 to study the asymptotic behaviors of a class of age-structured population models in $L^p$ ($1 \leq p < \infty$). This problem was investigated extensively by many authors, see, e.g., [Web85, Web08, Thi91, Thi98, Rha98, BHM05, MR09a] etc in the $L^1$ case. It seems that the $L^p$ ($p > 1$) case was first investigated in [MR07].

As a motivation, in control theory (and approximation theory), age-structured population models can be considered as boundary control systems and in this case the state space is usually taken as $L^p$ ($p > 1$); see, e.g., [CZ95]. Basically, in order to give the asymptotic behaviors of the age-structured population model (6.1) in $L^p$ ($p > 1$), the results given in Section 4 and Section 5 are necessary. Concrete examples are given in Example 6.13 to verify different cases in the main result (Theorem 6.12).

The paper is organized as follows. In section 2, we recall some definitions and results about integrated semigroups, some classes of non-densely defined operators (with a detailed summary of their basic properties), critical spectrum and essential spectrum. In section 3, we consider the regularities of $S_A \circ V$ (see section 2 for the symbol’s meaning). In section 4, we deal with the regularity properties of the perturbed semigroups generated by the part of MR operators in their closure domains. In section 5, we study the perturbation of critical and essential growth bound. Section 6 contains an application of our results to a class of age-structured population models in $L^p$. In the final section, we give a relatively bounded perturbation for MR operators associated with their perturbed regularities, and some comments that how the results could be applied to (nonlinear) differential equations.

2. Preliminaries

In this section, we recall some definitions and results about what we need in the following, such as integrated semigroups, some classes of non-densely defined operators, the critical spectrum and essential spectrum of $C_0$ semigroups.
2.1. **Integrated semigroup.** The integrated semigroup was introduced by W. Arendt [Are87]. The systematic treatment based on techniques of Laplace transforms was given in [ABHN11].

Let $X$ and $Z$ be Banach spaces. Denote by $\mathcal{L}(X, Z)$ the space of all bounded linear operators from $X$ into $Z$ and by $\mathcal{L}(X) = \mathcal{L}(X, X)$. Let $A : D(A) \subset X \to X$ be a linear operator and assume $\rho(A) \neq \emptyset$. If $Z \to X$ (i.e., $Z \subset X$ and the imbedding is continuous), $A_Z$ denotes the part of $A$ in $Z$, i.e.,

$$A_Z x = Ax, \ x \in D(A_Z) : = \{ x \in D(A) \cap Z : Ax \in Z \}.$$

By the closed graph theorem, $A_Z$ is a closed operator in $Z$ since $A$ is closed. Here is the relationship between $A$ and $A_Z$, see [ABHN11, Proposition B.8, Lemma 3.10.2].

**Lemma 2.1.** Let $Z \to X$.

(a) If $\mu \in \rho(A)$ and $R(\mu, A)Z \subset Z$, then $D(A_Z) = R(\mu, A)Z$, and $R(\lambda, A)|_Z = R(\lambda, A_Z)$, $\forall \lambda \in \rho(A)$.

(b) If $D(A) \subset Z$, then $R(\lambda, A)Z \subset Z$, $\forall \lambda \in \rho(A)$ and $\rho(A) = \rho(A_Z)$.

**Definition 2.2** (integrated semigroup [ABHN11]). Let $A$ be an operator on a Banach space $X$. We call $A$ the generator of (once non-degenerate) integrated semigroup if there exist $\omega \geq 0$ and a strongly continuous function $S : [0, \infty) \to \mathcal{L}(X)$ satisfying $\| \int_0^t S(s) \, ds \| \leq M e^{\omega t} (t \geq 0)$ for some constant $M > 0$ such that $(\omega, \infty) \subset \rho(A)$ and

$$R(\lambda, A)x = \lambda \int_0^\infty e^{-\lambda s} S(s)x \, ds, \forall \lambda > \omega, \forall x \in X.$$

In this case, $S$ is called the (once non-degenerate) integrated semigroup generated by $A$.

The equivalent definition is the following, see [ABHN11, Proposition 3.2.4].

**Lemma 2.3.** Let $S : [0, \infty) \to \mathcal{L}(X)$ be a strongly continuous function satisfying $\| \int_0^t S(s) \, ds \| \leq M e^{\omega t} (t \geq 0)$, for some $M$, $\omega \geq 0$. Then, the following assertions are equivalent.

(a) There exists an operator $A$ such that $(\omega, \infty) \subset \rho(A)$, and (2.1) holds.

(b) For $s$, $t \geq 0$,

$$S(t)S(s) = \int_0^t (S(r + s) - S(r)) \, dr,$$

and $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$.

For some basic properties of integrated semigroups, see [ABHN11, Section 3.2]. From now on, we set

$$X_0 := D(A), \ A_0 := A_{X_0}.$$

Note that, by Lemma 2.1, $\rho(A) = \rho(A_0)$.

**Lemma 2.4.** If $A_0$ generates a $C_0$ semigroup $T_{A_0}$ in $X_0$, then $A$ generates an integrated semigroup $S_A$ in $X$. Furthermore, the following hold.

(a) $T_{A_0}(t)x = \frac{d}{dt} S_A(t)x, \forall x \in X_0$;

(b) $S_A$ can be represented as the following,

$$S_A(t)x = (\mu - A_0) \int_0^t T_{A_0}(s) R(\mu, A)x \, ds$$

$$= \mu \int_0^t T_{A_0}(s) R(\mu, A)x \, ds - T_{A_0}(t) R(\mu, A)x + R(\mu, A)x, \forall x \in X,$$

for all $\mu \in \rho(A)$.

(c) $S_A$ is exponentially bounded, i.e., $\| S_A(t) \| \leq M e^{\omega t}$, $\forall t \geq 0$, for some constants $M \geq 0$ and $\omega \in \mathbb{R}$;

(d) $T_{A_0}(t)S_A(s) = S_A(t + s) - S_A(t)$, for all $t, s \geq 0$;
Lemma 2.3. (d), we

\[ \|S_A(t)\| \leq \delta(t), \]

for any \( t \in [0, \infty) \) and \( \|S_A(t)\| \leq \delta(t) \).

Proof. It suffices to show (b), but this directly follows from the definition of integrated semigroup. Others follow from (b) and Lemma 2.3. \( \square \)

2.2. Non-densely defined operator. Here, we discuss a general class of non-densely defined operators (i.e., \( D(A) \neq X \)), developed by Magal and Ruan [MR07]. Since the importance of the operators and for the sake of our reference, we call them MR operator and \( p \)-quasi Hille-Yosida operator below.

Definition 2.5 (MR operator [MR07]). We call a closed linear operator \( A \) an MR operator, if the following two conditions hold.

\( \text{(a)} \) \( A_0 \) generates a \( C_0 \) semigroup \( T_{A_0} \) in \( X_0 \), i.e., \( A_0 \) is a Hille-Yosida operator in \( X_0 \), and \( D(A_0) = X_0 \).

\( \text{(b)} \) There exists an increasing function \( \delta : [0, \infty) \rightarrow [0, \infty) \) such that \( \delta(t) \rightarrow 0 \), as \( t \rightarrow 0 \), and for any \( f \in C^1 (0, \tau; X) \), \( \tau > 0 \),

\[
(2.4) \quad \left\| \frac{d}{dt} \int_0^t S_A(t-s)f(s) \, ds \right\| \leq \delta(t) \sup_{s \in [0,t]} \|f(s)\|, \quad \forall t \in [0, \tau].
\]

In the following, we set

\[
(S_A \ast f)(t) := \int_0^t S_A(t-s)f(s) \, ds, \quad (S_A \circ f)(t) := \frac{d}{dt}(S_A \ast f)(t).
\]

We collect some basic properties about MR operators basically due to [MR07, MR09b]; we give a direct proof for the sake of readers.

Lemma 2.6. Let \( A \) be an MR operator.

\( \text{(a)} \) \( S_A \) is norm continuous on \( [0, \infty) \) and \( \|S_A(t)\| \leq \delta(t) \).

\( \text{(b)} \) \( \|R(\lambda, A)\| \rightarrow 0 \), as \( \lambda \rightarrow \infty, \lambda \in \mathbb{R} \); particularly \( S_A(t)x = \lim_{\mu \rightarrow \infty} \int_0^t T_{A_0}(s)\mu R(\mu, A)x \, ds \).

\( \text{(c)} \) For any \( f \in C(0, \tau; X) \), \( (S_A \ast f)(t) \in D(A) \), \( t \mapsto (S_A \ast f)(t) \) is differentiable, \( (S_A \circ f)(t) \in X_0, t \mapsto (S_A \circ f)(t) \) is continuous and (2.4) also holds. Furthermore, the following equations hold.

\[
(2.5) \quad (S_A \circ f)(t) = \lim_{\mu \rightarrow \infty} \int_0^t T_{A_0}(t-s)\mu R(\mu, A)f(s) \, ds
\]

\[
(2.6) \quad = T_{A_0}(t-s)(S_A \circ f)(s) + (S_A \circ f(s + \text{.})(t-s)
\]

\[
(2.7) \quad = \lim_{h \rightarrow 0} \frac{1}{h} \int_0^t T_{A_0}(t-s)S(h)f(s) \, ds.
\]

Proof. (a) To show \( \|S_A(t)\| \leq \delta(t) \), it suffices to take \( f(s) = x \) in (2.4). Then, by Lemma 2.4 (d), we know \( S_A \) is norm continuous on \( [0, \infty) \).

(b) Note that for \( \lambda > 0 \) and \( \varepsilon = \lambda^{-1/2} \), we have

\[
\|R(\lambda, A)\| = \|\lambda \int_0^\infty e^{-As}S_A(s) \, ds\|
\]

\[
\leq \lambda \|\int_0^\varepsilon e^{-As}S_A(s) \, ds\| + \lambda \|\int_\varepsilon^\infty e^{-As}S_A(s) \, ds\|
\]

\[
\leq \delta(\varepsilon) + \lambda \|\int_\varepsilon^\infty S_A(r) \, dr \| + \lambda \|e^{-As} \int_0^\varepsilon S_A(r) \, dr\|
\]

\[ \rightarrow 0, \text{ as } \lambda \rightarrow \infty. \]

The representation of \( S_A \) now immediately follows from (2.3).
Lemma 2.8. Let $A$ be a $p$-quasi Hille-Yosida operator.

(a) 1-quasi Hille-Yosida operators are Hille-Yosida operators.

(b) For all $f \in L^p(0, \tau; X)$, $(S_A \ast f)(t) \in D(A)$, $t \mapsto (S_A \ast f)(t)$ is differentiable, $(S_A \circ f)(t) \in X_0$, $t \mapsto (S_A \circ f)(t)$ is continuous and (2.8) also holds.

(c) There is $M \geq 0$ such that

\[
\|R(\lambda, A)\| \leq \frac{M}{(\lambda - \widehat{\omega})^p}, \quad \forall \lambda > \widehat{\omega}.
\]

Proof. (a) See [Thi08, Section 4] for more general results.

(b) See [MR07, Thi08].

(c) This is a corollary of [MR07, Theorem 4.7 (iii) and Remark 4.8]. Indeed, by

\[
\|x^\ast \circ [\lambda - (A - \widehat{\omega})]^{-n}\|_{X^\ast} \leq \frac{1}{(n-1)!} \int_0^\infty \int_0^s e^{-\lambda s} \chi_{X^\ast}^n(s) \, ds, \quad \lambda > 0,
\]

where $X^\ast \in \{X_0\}^\ast$, $\sup_{x^\ast \in \{X_0\}^\ast, \|x^\ast\|_L^p(0, \infty; \mathbb{R}) < \infty, n \in \mathbb{N}} \|\chi_{X^\ast}^n\|_{L^p(0, \infty; \mathbb{R})}$, and by the Hölder inequality for the case $n = 1$, we obtain the result. The proof is complete.

Remark 2.9. A beautiful characterization of $A$ which is an MR operator or a $p$-quasi Hille-Yosida operator by using the regularity of the integrated semigroup $S_A$ was given by Thieme [Thi08]. We state here for the convenience of readers. Suppose for the operator $A : D(A) \subset X \to X$, $A_0$ generates a $C_0$ semigroup $T_{A_0}$ in $X_0$. Let $S_A$ be the integrated semigroup generated by $S_A$.
(a) \( A \) is an MR operator if and only if \( S_A \) is of bounded semi-variation on \([0, b]\) for some \( b > 0 \) with its semi-variation \( V_D(S_A; 0, b) \rightarrow 0 \) as \( b \rightarrow 0 \). Here we mention that if \( T_{A_0} \) is norm continuous on \((0, b)\), then \( V_D(S_A; 0, b) \rightarrow 0 \) as \( b \rightarrow 0 \). Note also that \( S_A \) is of bounded semi-variation on \([0, b]\) if and only if \( x' S_A \) is of bounded variation for all \( x' \in (X_0)^* \) (see, e.g., [Mon14, Theorem 2.12]).

(b) \( A \) is a \( p \)-quasi Hille-Yosida operator if and only if \( S_A \) is of bounded semi-\( p \)-variation on \([0, b]\) for some \( b > 0 \), and if and only if \( x' S_A \) is bounded \( p' \)-variation on \([0, b]\) for all \( x' \in (X_0)^* \) where \( 1/p' + 1/p = 1 \). We notice that for the semi-\( p \)-variation \( V_p(S_A; 0, b) \), one has \( V_D(S_A; 0, b) \leq b^{1/p} V_p(S_A; 0, b) \).

For the notions of (semi-) \( (p-) \) variation, see [Thi08, Mon14] for details. Additional remark should be made: for a function \( f : [a, b] \rightarrow X \) with \( X \) having Radon-Nikodym property, then \( f \) is of bounded \( p \)-variation if and only if \( f \in W^{1, p}((a, b), X) \). Also note that for MR operator \( A \), \((S_A \circ f)(t)\) can be written in the convolution form by using the Stieltjes-type integral, i.e., (see, [Thi08, Theorem 3.2])

\[
(S_A \circ f)(t) = \int_0^t ds_A(s)f(s - t).
\]

Quasi Hille-Yosida operators are related to almost sectorial operators.

**Definition 2.10** \((\frac{1}{p}, \text{almost sectorial operator}) \) [PS02, DMP10]). We call a closed linear operator \( A \) a \( \frac{1}{p} \)-almost sectorial operator \( (p \geq 1) \), if there exist \( \hat{M} > 0, \hat{\omega} \in \mathbb{R}, \theta \in (\frac{\pi}{2}, \pi) \) such that

\[
||R(\lambda, A)|| \leq \frac{\hat{M}}{||\lambda - \hat{\omega}||^p},
\]

for all \( \lambda \in \{ \lambda \in \mathbb{C} : \lambda \neq \hat{\omega}, |\arg(\lambda - \hat{\omega})| < \theta \} \).

**Lemma 2.11.** (a) If \( A \) is a \( \frac{1}{p} \)-almost sectorial operator, then \( A_0 \) is a generator of an analytic \( C_0 \) semigroup, and \( A \) is a \( p \)-quasi Hille-Yosida operator for any \( \hat{p} > p \).

(b) If \( A \) is a \( p \)-quasi Hille-Yosida operator and \( A_0 \) is a generator of an analytic \( C_0 \) semigroup, then \( A \) is a \( \frac{1}{p} \)-almost sectorial operator.

(c) Let \( A_0 \) be a generator of an analytic \( C_0 \) semigroup. Then, \( A \) is a Hille-Yosida operator if and only if \( A \) is \( 1 \)-almost sectorial operator (i.e., the generator of a holomorphic semigroup, see [ABHN11, Definition 3.7.1]).

**Proof.**  (a) That \( A_0 \) is a generator of an analytic \( C_0 \) semigroup was shown in [PS02, Proposition 3.12, Theorem 3.13], and that \( A \) is a \( \hat{p} \)-quasi Hille-Yosida operator for any \( \hat{p} > p \) was proved in [DMP10, Theorem 3.11].

(b) Combine [DMP10, Proposition 3.3] and Lemma 2.8 (c).

(c) See [ABHN11, Theorem 3.7.11 and Example 3.5.9 c)]. \( \square \)

Because of the above lemma, we may interpret almost sectorial operators as an analytic version of quasi Hille-Yosida operators. We refer to [MR07, MR09b, MR09a] for more results and examples on MR operators (quasi Hille-Yosida operators), particularly in applications to abstract Cauchy problems, and to [PS02, DMP10, CDN08] on almost sectorial operators; see also Section 6 and Section 7.

**Remark 2.12.** For a generator \( A \) of an integrated semigroup \( S_A \), unlike the case of Hille-Yosida operator (where \( A \) is a Hille-Yosida operator if and only if \( S_A \) is locally Lipschitz, and if and only if \( S_A \) is of locally bounded semi-1-variation) and \( \frac{1}{p} \)-almost sectorial operator, even \( S_A \) is of bounded semi-\( p \)-variation \((p > 1)\) or semi-variation, in general, \( \underbrace{A_{D(A)}}_{\text{Def}} \) might not generate a \( C_0 \) semigroup; see, e.g., [Thi08, Example 5.2 and 5.3] where the operators given in that paper are densely defined.

Assume that \( A \) is an MR operator. Denote by \( C_c([0, \infty), \mathcal{L}(X_0, X)) \) the space of strongly continuous map from \([0, \infty)\) into \( \mathcal{L}(X_0, X) \). Set

\[
(S_A \circ V)(t)x := (S_A \circ V(\cdot)x)(t), \quad \forall x \in X_0,
\]
Lemma 2.6. \textit{Obviously, by the uniqueness, (2.10) holds.}

Theorem 2.14. \textit{Suppose that }$A$\textit{ is an MR operator. Then, for any }$V \in C_s([0, \infty), \mathcal{L}(X_0, X))$\textit{, the equation (2.9) }

\[ W = V + \mathcal{B}(W) \]

\textit{has a unique solution }$W \in C_s([0, \infty), \mathcal{L}(X_0, X))$. \textit{Furthermore,}

\[ W = \sum_{n=0}^{\infty} \mathcal{B}^n(V), \]

\textit{where the series is uniformly convergent on any finite interval in the uniform operator topology.}

\textit{Proof.} The result was already stated in the proof of [MR07, Theroem 3.1]. For the sake of readers, we give a detailed proof here. Since \( \delta(t) \to 0, \ t \to 0 \), there exists \( \delta_0 > 0 \) such that \( \delta(t)\|L\| \leq r < 1 \) for \( t \in [0, \delta_0] \). Fix \( x \in X_0 \). Consider the operator

\[ (\mathcal{B} f)(t) = (A \circ L f)(t), \ f \in C([0, \delta_0], X), \]

which is contractive and \((\mathcal{B} V)(\cdot)(t) = (\mathcal{B} V)(t)x\). Thus, we have a unique \( W(\cdot)x \in C([0, \delta_0], X) \) such that

\[ \tilde{W}(\cdot)x + V(\cdot)x = W(\cdot)x, \]

and

\[ W(t)x = \sum_{n=0}^{\infty} \mathcal{B}^n(V)(x)(t) = \sum_{n=0}^{\infty} \mathcal{B}^n(V)(t)x. \]

Obviously, by the uniqueness, \( x \mapsto W(t)x \) is linear for each \( t \). In addition, there is \( \tilde{M} > 0 \) such that

\[ \|(\mathcal{B}^m V)(\cdot)(t) - (\mathcal{B}^n V)(\cdot)(t)\| \leq \frac{r^n}{1 - r} \sup_{t \in [0, \delta_0]} \|(S_A \circ LV)(t)x - V(t)x\| \leq \frac{r^n}{1 - r} \tilde{M}\|x\|, \]

whenever \( m \geq n \), which implies (2.10) is uniformly convergent on \([0, \delta]\) in the uniform operator topology. Therefore \( W \in C_s([0, \delta_0], \mathcal{L}(X_0, X)) \). Next, there is \( W(\cdot) \) satisfying

\[ (S_A \circ LW)(s) + T_{\delta_0}(s)(S_A \circ LW)(\delta_0) + V(s + \delta_0) = \tilde{W}(s), \ \ s \in [0, \delta_0]. \]

(That is we take \( T_{\delta_0}(\cdot)(S_A \circ LW)(\delta_0) + V(\cdot + \delta_0) \) as the initial function.) Let

\[ W(t) = \tilde{W}(t - \delta_0), \ t \in [\delta_0, 2\delta_0]. \]

Then,

\[ W(t) = (S_A \circ LW(\delta_0 + \cdot))(t - \delta_0) + T_{\delta_0}(t - \delta_0)(S_A \circ LW)(\delta_0) + V(t) = (\mathcal{B} W)(t) + V(t). \]

So \( W(\cdot) \) is well defined on \([\delta_0, 2\delta_0]\) and satisfies (2.9). Since \( W(\cdot) \) is uniquely constructed in this way, this completes the proof. \( \square \)

It was shown in [MR07, Thi08] that MR operators (quasi Hille-Yosida operators) are stable under the bounded perturbation.

Theorem 2.14 ([MR07, Thi08]). \textit{Let }$A$\textit{ be an MR operator (resp. p-quasi Hille-Yosida operator). For any }$L \in \mathcal{L}(X_0, X)$, \( A + L \) is still an MR operator (resp. p-quasi Hille-Yosida operator), and the following fixed equations hold.

\[ T_{(A+L)\varnothing} = T_{\delta_0} + S_A \circ LT_{(A+L)\varnothing}, \]

\[ = T_{\delta_0} + S_A \circ LT_{\delta_0} + S_A \circ LS_A \circ LT_{(A+L)\varnothing} = \cdots, \]
\[ S_{A+L} = S_A + S_A \circ LS_{A+L} = \cdots. \]

Set

\[
S_0 := T_{A_0}, \quad R_0 := T_{(A+L)_0}, \quad S_n := B^n(T_{A_0}), \quad R_n := B^n(T_{(A+L)_0}), \quad n = 1, 2, \cdots.
\]

Combining with Lemma 2.13, we have

\[
S_{n+1} = S_A \circ L S_n,
\]

\[
R_k = \sum_{n=k}^{\infty} S_n,
\]

\[
R_k - S_k = B \circ R_k,
\]

\[
T_{(A+L)_0} = \sum_{n=0}^{\infty} S_n + R_k,
\]

\[
= \sum_{n=0}^{\infty} S_n,
\]

where (2.13) and (2.16) are uniformly convergent on finite interval in the uniform operator topology. The following is a result about \( S_n \), which is similar as Dyson-Phillips series ([EN00]).

**Lemma 2.15.** \( S_n(t + s) = \sum_{k=0}^{n} S_k(t)S_{n-k}(s) \).

**Proof.** The case \( n = 0 \) is clear. Consider

\[
S_{n+1}(t + s) = [S_A \circ LS_n(\cdot)](t + s)
= T_{A_0}(t)[S_A \circ LS_n(\cdot)](s) + [S_A \circ LS_n(s + \cdot)](t)
= T_{A_0}(t)S_{n+1}(s) + [S_A \circ L \sum_{k=0}^{n} S_k(\cdot)S_{n-k}(s)](t)
= T_{A_0}(t)S_{n+1}(s) + \left[ \sum_{k=0}^{n} S_{k+1}(\cdot)S_{n-k}(s) \right](t)
= \sum_{k=0}^{n+1} S_k(t)S_{n+1-k}(s),
\]

the second equality being a consequence of (2.6), and the third one by induction. \( \square \)

Using the above lemma, we obtain the following corollary which generalizes the corresponding case of \( C_0 \) semigroups; Corollary 2.16 (e) was also given in [Bre01, Theorem 3.2] in the context of \( C_0 \) semigroups.

**Corollary 2.16.** (a) If \( S_0, S_1, \cdots, S_n \) are norm continuous (resp. compact, differentiable) at \( t_0 > 0 \), then \( S_n(t) \) is norm continuous (resp. compact, differentiable) for all \( t \geq t_0 \).

(b) If \( S_0, S_1, \cdots, S_n \) are compact at \( t_0 > 0 \), then \( S_n \) is norm continuous on \( (t_0, \infty) \).

(c) If \( S_n(t) \) is compact for all \( t > t_1 (\geq 0) \), then \( S_n \) is norm continuous on \( (t_1, \infty) \).

(d) If \( R_n(t) \) is compact for all \( t > t_1 (\geq 0) \), then \( R_n \) is norm continuous on \( (t_1, \infty) \).

(e) \( S_n(t) \) is compact for all \( t > 0 \) if and only if \( S_n \) is norm continuous on \( (0, \infty) \) and \((R(\lambda + i\mu, A_0) L)^n R(\lambda + i\mu, A_0)\) is compact for all \( \mu \in \mathbb{R} \) and for some/all large \( \lambda > 0 \).

**Proof.** (a) and (b) are direct consequences of Lemma 2.15.

(c) Let \( t_0 > t_1 \). Take a constant \( M > 0 \) such that \( \sup_{t \in [0, 2t_0]} \|S_k(t)\| \leq M, \ k = 0, 1, 2, \cdots, n \).

For any small \( \varepsilon > 0 \), by Lemma 2.6 (a), we know there is \( 0 < \delta < t_0 \) such that \( t_0 - \delta > t_1 \) and

\[
\begin{align*}
\sup_{t_0 - \delta < t < t_0} \|S_k(t)\| & \leq M, \quad k = 0, 1, 2, \cdots, n. \\
\end{align*}
\]
sup_{\sigma \in [0,2\delta]} \| S_n(\sigma) \| \leq \varepsilon / M. \ So
\sup_{\sigma \in [0,2\delta]} \sum_{k=1}^{n} \| S_k(\sigma) S_{n-k}(t_0-\delta) \| \leq \varepsilon.

Now for \( 0 < \delta' < \delta \), we see
\[ \| S_n(t_0) - S_n(t_0 + \delta') \| \]
\[ = \| (T_{A_0}(\delta) - T_{A_0}(\delta + \delta'))S_n(t_0 - \delta) + \sum_{k=1}^{n} (S_k(\delta) - S_k(\delta + \delta'))S_{n-k}(t_0 - \delta) \| \]
\[ \leq \| (T_{A_0}(\delta) - T_{A_0}(\delta + \delta'))S_n(t_0 - \delta) \| + 2\varepsilon. \]

If we take \( \delta' \) sufficiently small, then \( \| (T_{A_0}(\delta) - T_{A_0}(\delta + \delta'))S_n(t_0 - \delta) \| \leq \varepsilon \) as \( S_n(t_0 - \delta) \) is compact, and so \( \| S_n(t_0) - S_n(t_0 + \delta') \| \leq 3\varepsilon \). This shows \( S_n \) is norm continuous at \( t_0 > 0 \).

(d) This is very similar as \( S_n \) by using the following equality:
\[ \| R_n(t_0) - R_n(t_0 + \delta') \| = \| (T_{A_0}(\delta) - T_{A_0}(\delta + \delta'))R_n(t_0 - \delta) + (S_A \circ LR_{n-1}(\cdot + t_0 - \delta))(\delta) - (S_A \circ LR_{n-1}(\cdot + t_0 - \delta))(\delta + \delta') \|.

(c) This follows from the Sublemma 2.17 and conclusion (c) (letting \( t_1 = 0 \), since \( (R(\lambda + i\mu, A)L)^n R(\lambda + i\mu, A_0) = \mathcal{L}(S_n)(\lambda + i\mu) \) (the Laplace transform of \( S_n \) at \( \lambda + i\mu \)). The proof is complete. \( \square \)

**Sublemma 2.17.** For norm continuous \( F : \mathbb{R}_+ \to \mathcal{L}(X_0, X) \) such that \( \| F(t) \| \leq C_1 e^{\omega t} \) for all \( t > 0 \) and some constants \( C_1 \geq 1, \omega \in \mathbb{R}, \) \( F(t) \) is compact for all \( t > 0 \) if and only if \( \mathcal{L}(F)(\lambda + i\mu) \) (the Laplace transform of \( F \) at \( \lambda + i\mu \)) is compact for all \( \mu \in \mathbb{R} \) and some/all \( \lambda > \omega \).

**Proof.** If \( F(t) \) is compact for all \( t > 0 \), then
\[ \mathcal{L}(F)(\lambda + i\mu) = \int_{0}^{\infty} e^{-(\lambda+i\mu)t} F(t) \, dt = \lim_{r \to \infty} \int_{0}^{r} e^{-(\lambda+i\mu)t} F(t) \, dt, \]
is compact. And if \( \mathcal{L}(F)(\lambda + i\mu) \) is compact for all \( \mu \in \mathbb{R} \) and some \( \lambda > \omega \), then by the complex inversion formula of Laplace transform (see, e.g., [ABHN11, Theorem 2.3.4.]), one gets
\[ \int_{0}^{t} e^{-\omega s} F(s) \, ds = \lim_{\mu \to \infty} \frac{1}{2\pi i} \int_{\lambda-i\omega}^{\lambda+i\mu} e^{rt} \frac{\mathcal{L}(F)(r + \omega)}{r} \, dr, \]
where the limit is uniform for \( t \) belonging to compact intervals and exists in the uniform operator topology. This shows that \( \int_{0}^{t} e^{-\omega s} F(s) \, ds \) is compact and consequently \( F(t) \) is compact for all \( t > 0 \). The proof is complete. \( \square \)

**Problem:** for Hilbert space \( X \), is it true that if \( \| (R(\lambda + i\mu, A)L)^n R(\lambda + i\mu, A_0) \| \to 0 \) as \( |\mu| \to \infty \), then \( S_{n-2} \) is norm continuous? (See [Bre01].) When \( LR(\lambda, A_0) = R(\lambda, A)L \), the proof is very similar.

### 2.3. Critical spectrum and essential spectrum
In this subsection, we recall some known results about spectral theory. For a complex Banach algebra \( Y \), \( \sigma(x) \) denotes the spectrum of \( x \in Y \), i.e.,
\[ \sigma(x) = \{ \lambda \in \mathbb{C} : \lambda - x \text{ is not invertible in } Y \}, \]
and \( r(x) \) denotes the spectral radius of \( x \), i.e.,
\[ r(x) = \sup \{ |\lambda| : \lambda \in \sigma(x) \}. \]

Critical spectrum was discovered by Nagel and Poland [NP00]. Let \( T \) be a \( C_0 \) semigroup on a Banach space \( X \), whose generator is \( B \). Set
\[ I^{\infty}(X) := \{ (x_n) : \| (x_n) \|_{\infty} \triangleq \sup_{n} \| x_n \| < \infty \}. \]
$T$ can be naturally extended to $l^\infty(X)$ by

$$\tilde{T}(t)(x_n) := (T(t)x_n), \ (x_n) \in l^\infty(X).$$

Then, $\|\tilde{T}(t)\| = \|T(t)\|$. Define

$$l^\infty_T(X) := \{(x_n) \in l^\infty(X) : \sup_n \|T(t)x_n - x_n\| \to 0, \text{ as } t \to 0^+\}$$

$$= \{(x_n) \in l^\infty(X) : \tilde{T}(t)(x_n) \to (x_n), \text{ as } t \to 0^+\}$$

$$= \{(x_n) \in l^\infty(X) : \sup_n \|T(t)x_n - x_n\| \to 0, \text{ as } t \to 0^+\}.$$

$l^\infty_T(X)$ is the closed space of strong continuity of $\tilde{T}$. Note that $\tilde{T}(t)l^\infty_T(X) \subset l^\infty_T(X)$, for each $t \geq 0$. So $\tilde{T}$ can induce $\hat{T}$, defined on the quotient space $l^\infty(X)/l^\infty_T(X)$ as

$$\hat{T}(t)[(x_n)] := [\tilde{T}(t)(x_n)], \text{ where } [(x_n)] := (x_n) + l^\infty_T(X).$$

In general, if $F \in \mathcal{L}(X)$, define $\hat{F}$ on $l^\infty(X)$ as

$$\hat{F}(x_n) := (F(x_n)),$$

with $\|\hat{F}\| = \|F\|$. If $\hat{F}(l^\infty_T(X)) \subset l^\infty_T(X)$, then $\hat{F}$ can be defined on $l^\infty(X)/l^\infty_T(X)$ by

$$\hat{F}[(x_n)] := [\hat{F}(x_n)],$$

with $\|\hat{F}\| \leq \|F\|$. If $G \in \mathcal{L}(X)$ and $\hat{G}(l^\infty_T(X)) \subset l^\infty_T(X)$, then

$$F \circ G = \tilde{F} \circ \hat{G}, \ \hat{F} \circ \hat{G} = \hat{F} \circ \hat{G}.$$  

Note that $l^\infty(X)/l^\infty_T(X)$ is a Banach algebra. Now we give the following definitions.

**Definition 2.18 ([NP00]).** For a $C_0$ semigroup $T$, we call

$$\sigma_{\text{crit}}(T(t)) := \sigma(\tilde{T}(t)),$$

the critical spectrum of $T(t)$,

$$r_{\text{crit}}(T(t)) := r(\tilde{T}(t)),$$

the critical spectral radius of $T(t)$, and

$$\omega_{\text{crit}}(T) := \inf\{\omega \in \mathbb{R} : \exists M \geq 0, \ ||\tilde{T}(t)|| \leq Me^{\omega t}, \forall t \geq 0\}$$

$$= \lim_{t \to \infty} \frac{1}{t} \ln ||\tilde{T}(t)|| = \inf_{t > 0} \frac{1}{t} \ln ||\tilde{T}(t)|| = \frac{1}{t_0} \ln r_{\text{crit}}(T(t_0)) (t_0 > 0),$$

the critical growth bound of semigroup $T$.

With these definitions, the following theorem holds.

**Theorem 2.19 ([NP00]).** For a $C_0$ semigroup $T$ with generator $B$, the following statements hold.

(a) $\sigma_{\text{crit}}(T(t)) \subset \sigma(T(t))$.

(b) (Partial spectral mapping theorem) For each $t > 0$,

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(B)} \cup \sigma_{\text{crit}}(T(t)) \setminus \{0\},$$

and

$$\sigma(T(t)) \cap \{r \in \mathbb{C} : |r| > r_{\text{crit}}(T(t))\} = e^{t\sigma(B)} \cap \{r \in \mathbb{C} : |r| > r_{\text{crit}}(T(t))\}.$$

(c) $\omega(T) = \max\{s(B), \omega_{\text{crit}}(T)\}$,

where we denote by $\omega(T)$ the growth bound of $T$, and $s(B)$ the spectral bound of $B$, i.e.,

(2.17) $\omega(T) = \inf\{\omega \in \mathbb{R} : \exists M \geq 0, \ ||T(t)|| \leq Me^{\omega t}, \forall t \geq 0\}, \ s(B) = \sup\{\Re \lambda : \lambda \in \sigma(B)\}.$
See [Bla01] for a different but equivalent definition of critical spectrum. We refer to [NP00, BNP00, Shi07] for more details and applications on critical spectrum. Next, we turn to the essential spectrum. We follow the presentation of [GGK90]. Consider the Calkin algebra \( \mathcal{L}(X)/\mathcal{K}(X) \), where \( \mathcal{K}(X) \) denotes the closed ideal of bounded compact linear operators on \( X \). Define \( \overline{T} \) on \( \mathcal{L}(X)/\mathcal{K}(X) \) by

\[
\overline{T}(t) := [T(t)] := T(t) + \mathcal{K}(X).
\]

Now we have the following definitions.

**Definition 2.20.** For a \( C_0 \) semigroup \( T \), we call

\[
\sigma_{\text{ess}}(T(t)) := \sigma(\overline{T}(t)),
\]

the essential spectrum of \( T(t) \),

\[
r_{\text{ess}}(T(t)) := r(\overline{T}(t)),
\]

the essential spectral radius of \( T(t) \) and

\[
\omega_{\text{ess}}(T) := \inf \{ \omega \in \mathbb{R} : \exists M > 0, \|\overline{T}(t)\| \leq M e^{\omega t}, \forall t \geq 0 \}
\]

\[
= \lim_{t \to +\infty} \frac{1}{t} \ln \|\overline{T}(t)\| = \inf_{t > 0} \frac{1}{t} \ln \|\overline{T}(t)\| = \frac{1}{t_0} \ln r_{\text{ess}}(T(t_0)) (t_0 > 0),
\]

the essential growth bound of semigroup \( T \).

**Theorem 2.21** ([NP00, EN00]). For a \( C_0 \) semigroup \( T \) with generator \( B \), the following statements hold.

(a) \( \sigma_{\text{ess}}(T(t)) \subset \sigma_{\text{crit}}(T(t)) \subset \sigma(T(t)) \). Thus, \( \omega_{\text{ess}}(T) \leq \omega_{\text{crit}}(T) \leq \omega(T) \), \( \omega(T) = \max \{s(B), \omega_{\text{crit}}(T)\} \).

(b) \( \forall \gamma > \omega_{\text{ess}}(T) \), the set \( \{ \lambda \in \sigma(B) : \text{Re} \lambda \geq \gamma \} \) is finite. And if it is not empty, then its elements are the finite-order poles of \( R(\cdot, B) \), in particular the point spectrum of \( B \).

For other definitions of essential spectrum, see [Dei85]. Although different definitions of essential spectrum may not be coincided, all the essential spectral radiuses are equal.

3. General arguments

From now on, we assume that \( A \) is an MR operator, \( V \in \mathcal{C}_b((0, \infty), \mathcal{L}(X_0, X)) \). Then, \( A_0 := A_{X_0} \) generates a \( C_0 \) semigroup \( T_{A_0} \) in \( X_0 \), and \( A \) generates an integrated semigroup \( S_A \), where \( X_0 := D(A) \).

In this section, we consider the regularity of \( S_A \circ V \), when \( T_{A_0} \) or \( V \) has higher regularity. Note that \( t \mapsto (S_A \circ V)(t) \) is norm continuous at zero (see Lemma 2.6 (a)).

**Lemma 3.1.** (a) If \( T_{A_0} \) is norm continuous on \((0, \infty)\), then \( S_A \circ V \) is norm continuous on \((0, \infty)\) (hence \([0, \infty)\)). More generally, if \( L \in \mathcal{L}(X_0, X) \) and \( LT_{A_0} \) is norm continuous on \((0, \infty)\), so is \( LS_A \circ V \).

(b) If \( V \) is norm continuous on \((0, \infty)\), so is \( S_A \circ V \).

(c) If \( T_{A_0} \) in norm continuous on \((\alpha, \infty)\), and \( V \) is norm continuous on \((\beta, \infty)\), then \( S_A \circ V \) is norm continuous on \((\alpha + \beta, \infty)\).

**Proof.** Let \( t > s > 0 \) in \( U(t_0; \eta) = \{ r \in \mathbb{R} : |r - t_0| < \eta \} \) (a neighborhood of \( t_0 > 0 \)), and \( 0 < \varepsilon < s \), where \( t_0 > \eta > 0 \).

(a) By (2.6), we have

\[
L(S_A \circ V)(t) - L(S_A \circ V)(s) = LT_{A_0}(t - s)(S_A \circ V)(s) + L(S_A \circ V)(s + \cdot)(t - s) - L(S_A \circ V)(s)
\]

\[
= (LT_{A_0}(t - s + \varepsilon) - LT_{A_0}(\varepsilon))(S_A \circ V)(s - \varepsilon) + L(T_{A_0}(t - s) - I)(S_A \circ V(s - \varepsilon + \cdot))(\varepsilon) + L(S_A \circ V(s + \cdot))(t - s)
\]

\[
:= (LT_{A_0}(t - s + \varepsilon) - LT_{A_0}(\varepsilon))(S_A \circ V)(s - \varepsilon) + R,
\]

where

\[
R := L(T_{A_0}(t - s) - I)(S_A \circ V(s - \varepsilon + \cdot))(\varepsilon) + L(S_A \circ V(s + \cdot))(t - s),
\]
and
\[ \|R\| \leq \tilde{M}(\delta(e) + \delta(t-s)), \]
for some constant \(\tilde{M} > 0\). Let \(\varepsilon, \eta\) be sufficiently small (\(\eta\) depending on \(e\)), then \(\|L(S_A \circ V)(t) - L(S_A \circ V)(s)\|\) can be sufficiently small, which shows that \(LS_A \circ V\) is norm continuous at \(t_0\).

(b) By (2.6), we see
\[
\|(S_A \circ V)(t) - (S_A \circ V)(s)\|
= \|T_{A_0}(s - e)(S_A \circ V)(t - s + e) + (S_A \circ V(t - s + e + \cdot))(s - e)
- T_{A_0}(s - e)(S_A \circ V)(e) - (S_A \circ V(e + \cdot))(s - e)\|
\leq \tilde{M}(\delta(t - s + e) + \delta(e)) + \tilde{M} \sup_{r \in [0, s-e]} \|V(t - s + e + r) - V(e + r)\|,
\]
for some constant \(\tilde{M} > 0\). Since \(V\) is uniformly continuous on \([\varepsilon, 2t_0]\), the last inequality can be sufficiently small.

(c) When \(t > \alpha + \beta\), by (2.6), we get
\[
(S_A \circ V)(t) = T_{A_0}(t - \beta)(S_A \circ V)(\beta) + (S_A \circ V(\beta + \cdot))(t - \beta).
\]
The first term of the right side is norm continuous by norm continuity of \(T_{A_0}\), and the second is norm continuous by (b). The proof is complete. \(\square\)

For \(W \in C_c([0, \infty), \mathcal{L}(X_0, X))\), we say \(W\) is compact on \((a, b)\) if for each \(t \in (a, b)\), \(W(t)\) is compact.

Lemma 3.2. (a) If \(T_{A_0}\) is compact on \((0, \infty)\), so is \(S_A \circ V\). More generally, if \(L \in \mathcal{L}(X_0, X)\) and \(LT_{A_0}\) is compact on \((0, \infty)\), so is \(LS_A \circ V\).
(b) If \(V\) is compact and norm continuous on \((0, \infty)\), so is \(S_A \circ V\).
(c) If \(T_{A_0}\) is compact on \((\alpha, \infty)\), and \(V\) is compact and norm continuous on \((\beta, \infty)\), then \(S_A \circ V\) is compact and norm continuous on \((\alpha + \beta, \infty)\).

Proof. (a) For \(t > \varepsilon > 0\), by
\[
L(S_A \circ V)(t) = LT_{A_0}(\varepsilon)(S_A \circ V)(t - \varepsilon) + L(S_A \circ V(t - \varepsilon + \cdot))(\varepsilon),
\]
we see \(LT_{A_0}(\varepsilon)(S_A \circ V)(t - \varepsilon)\) converges to \(L(S_A \circ V)(t)\) in the operator norm topology, as \(\varepsilon \to 0^+\). Since \(LT_{A_0}(\varepsilon)\) is compact, the result follows.

(b) Since \(V\) is norm continuous on \((0, \infty)\), so is \(S_A \circ V\) (Lemma 3.2 (b)). Hence,
\[
(S_A \circ V)(t) = \lim_{h \to 0} \frac{1}{h} \int_0^t T_{A_0}(t - s)S_A(h)V(s) \, ds,
\]
where the convergence is in the uniform operator topology; see the proof of equation (2.7). Since \(V(t)\) is compact for all \(t > 0\), \(\int_0^t T_{A_0}(t - s)S_A(h)V(s) \, ds\) is compact (see, e.g., [EN00, Theorem C.7]), yielding \((S_A \circ V)(t)\) is compact.

(c) Use (3.1) and (b). The proof is complete. \(\square\)

The following result is a simple relation of differentiability in strong topology and uniform operator topology; the proof is omitted.

Lemma 3.3. Let \(f : I \to \mathcal{L}(X, Y)\), where \(I\) is an interval of \(\mathbb{R}\).

(a) If \(t \mapsto f(t)x\) is continuously differentiable for each \(x \in X\), then \(t \mapsto f(t)\) is norm continuous.

(b) If \(t \mapsto f(t)x\) is differentiable for each \(x \in X\), let \(G(t)x = f'(t)x, \forall x \in X\). If \(t \mapsto G(t)\) is norm continuous, then \(t \mapsto f(t)\) is norm continuously differentiable.

Lemma 3.4. (a) If \(V\) is strongly continuously differentiable (resp. norm continuously differentiable) on \([0, \infty)\), and \(V(0) = 0\), so is \(S_A \circ V\). Moreover, \(S_A \circ V = S_A \ast V'\).
(b) If \(T_{A_0}\) is strongly continuously differentiable on \([\alpha, \infty)\) and \(V\) is strongly continuously differentiable on \([\beta, \infty)\), then \(S_A \circ V\) is strongly continuously differentiable on \([\alpha + \beta, \infty)\).
Lemma 2.6

(b), Lemma 3.2

(b), Lemma 3.3

Lemma 3.1

(resp. Lemma 2.4)

(R)

Proof. (a) Since

\[(S_A \circ V)(t) = S_A(t)V(0) + \int_0^t S_A(s)V'(t-s) \, ds = S_A * V',\]

by the assumption of \(A\), it shows \(S_A * V\) is strongly continuously differentiable (see Lemma 2.6 (c)). Thus, \(S_A \circ V\) is strongly continuously differentiable. For the case of norm continuously differentiable, by Lemma 3.1 (b), \(S_A \circ V\) is norm continuous. The result follows from Lemma 3.3 (b).

(b) For \(t \geq \alpha + \beta\),

\[(S_A \circ V)(t) = T_{A_0}(t - \beta)(S_A \circ V)(\beta) + (S_A \circ V(\beta + \cdot))(t - \beta) = T_{A_0}(t - \beta)(S_A \circ V)(\beta) + S_A(t - \beta)V(\beta) + S_A * V(\beta + \cdot)'\]

The first term of the right side is strongly continuously differentiable. By Lemma 2.4 (e), \(S_A\) is strongly continuously differentiable on \([\alpha, \infty)\). By the assumption on \(A\), the third term is also strongly continuously differentiable (see Lemma 2.6 (c)). This completes the proof. □

4. Regularity of perturbed semigroups

If \(A\) is an MR operator, \(L \in \mathcal{L}(X_0, X)\), then \(A + L\) is also an MR operator. In this section, we study the regularity properties of the perturbed semigroup \(T_{(A + L)0}\), generated by \((A + L)0\) in \(X_0\). We make some regularity assumptions similar as [NP98, BMR02] to ensure \(T_{(A + L)0}\) preserves the regularity of \(T_{A_0}\).

Lemma 4.1. Suppose that \(V \in C_s([0, \infty), \mathcal{L}(X_0, X))\) has one of the following properties,

(a) norm continuity on \((0, \infty)\);
(b) norm continuity and compactness on \((0, \infty)\);
(c) \(V(0) = 0\) and strongly continuous differentiability on \([0, \infty)\).

Then, the solution \(W\) of the equation

\[W = V + B(W),\]

has the corresponding property.

Proof. If \(V\) has the property of (a) (resp. (b)), by Lemma 3.1 (resp. Lemma 3.2), \(B^k(V)\) has the property of (a) (resp. (b)) for \(k = 0, 1, 2, \cdots\). By Lemma 2.13, \(W = \sum_{k=0}^{\infty} B^k(V)\) is internally closed uniformly norm convergent. So \(W\) has property of (a) (resp. (b)).

If \(V(0) = 0\) and \(V\) is strongly continuously differentiable on \([0, \infty)\), then by Lemma 3.4 (a), \([B(V)']' = B(V')\), and \([B^k(V)']' = B^k(V')\) (\(k \geq 1\)). (Note that \(B^k(V)(0) = 0\) for \(k \geq 1\).) Since \(\sum_{k=0}^{\infty} B^k(V)\) and \(\sum_{k=0}^{\infty} (B^k(V)') = \sum_{k=0}^{\infty} B^k(V')\) are internally closed uniformly convergent, we have \(W' = \sum_{k=0}^{\infty} B^k(V')\). That is \(W' = V' + B(W')\), which completes the result. □

Next, we consider the relation between the regularity of \(S_n, R_n\) (see (2.11)).

Corollary 4.2. The following are equivalent.

(a) \(S_n\) has property \(P\);
(b) \(S_k\) has property \(P\), \(\forall k \geq n\);
(c) \(R_n\) has property \(P\);
(d) \(R_k\) has property \(P\), \(\forall k \geq n\).

The property \(P\) stands for norm continuity, or compactness on \((0, \infty)\). If \(n \geq 1\), \(P\) also stands for strongly continuous differentiability on \([0, \infty)\).
Proof. (a) \(\Rightarrow\) (b) By Lemma 3.1 (or Lemma 3.2, or Lemma 3.4) and (2.12). (b) \(\Rightarrow\) (c) By (2.12) and Lemma 4.1. (c) \(\Rightarrow\) (d) The same reason as (a) \(\Rightarrow\) (b). (d) \(\Rightarrow\) (a) By (2.14) and Lemma 3.1 (or Lemma 3.2, or Lemma 3.4). Here note that if \(S_n\) (or \(R_n\)) is compact on \((0, \infty)\), then it is norm continuous on \((0, \infty)\) by Corollary 2.16 (c) (d).

\[ \square \]

**Theorem 4.3.** Let \(A\) be an MR operator, \(L \in \mathcal{L}(X_0, X)\). Then, the following statements hold.

(a) \(T_{A_0}\) is immediately norm continuous if and only if \(T_{(A+L)0}\) is immediately norm continuous.

(b) \(T_{A_0}\) is immediately compact if and only if \(T_{(A+L)0}\) is immediately compact.

Proof. It's a direct result of Corollary 4.2 for the case \(n = 0\). Note that \(C_0\) semigroup is immediately compact, then it is also immediately norm continuous (see, e.g., [EN00, Lemma II.4.22]).

\[ \square \]

**Theorem 4.4.** Let \(A\) be an MR operator, \(L \in \mathcal{L}(X_0, X)\), \(\alpha \geq 0\). Then, the following statements hold.

(a) If \(T_{A_0}\) is norm continuous on \((a, \infty)\), and there exists \(S_n\) which is norm continuous on \((0, \infty)\), then \(T_{(A+L)0}\) is norm continuous on \((na, \infty)\).

(b) If \(T_{A_0}\) is compact on \((a, \infty)\), and there exists \(S_n\) which is compact on \((0, \infty)\), then \(T_{(A+L)0}\) is compact (and norm continuous) on \((na, \infty)\).

(c) If \(T_{A_0}\) is strongly continuously differentiable on \([a, \infty)\), and there exists \(S_n\) \((n \geq 1)\) which is strongly continuously differentiable on \([0, \infty)\), then \(T_{(A+L)0}\) is strongly continuously differentiable on \([na, \infty)\).

Proof. If \(T_{A_0}\) is norm continuous (resp. compact (hence norm continuous), or differentiable) on \((a, \infty)\), then \(S_n\) is norm continuous (resp. compact and norm continuous, or strongly continuously differentiable) on \((ma, \infty)\) by Lemma 3.1 (c) (resp. Lemma 3.2 (c), or Lemma 3.4 (b)). Combining with Corollary 4.2 and (2.15), we obtain the results. Here note that if \(S_n\) is compact on \((0, \infty)\), then it is also norm continuous on \((0, \infty)\) by Corollary 2.16 (c).

\[ \square \]

In applications, we need to calculate the \(S_n\) to show it has higher regularity, see some examples in [BNP00, BMR02]. Here we give some conditions such that \(S_n\) has higher regularity. The following result can be obtained directly by Lemma 3.1 (a), Lemma 3.2 (a) and Corollary 4.2.

**Corollary 4.5.** Let \(A\) be an MR operator, \(L \in \mathcal{L}(X_0, X)\).

(a) If \(LT_{A_0}\) is norm continuous on \((0, \infty)\), then \(S_1\) and \(T_{(A+L)0} - T_{A_0}\) are norm continuous on \((0, \infty)\). Particularly, in this case, \(T_{A_0}\) is eventually norm continuous if and only if \(T_{(A+L)0}\) is eventually norm continuous.

(b) If \(LT_{A_0}\) is norm continuous and compact on \((0, \infty)\), then \(S_1\) and \(T_{(A+L)0} - T_{A_0}\) are norm continuous and compact on \((0, \infty)\). Particularly, in this case, \(T_{A_0}\) is eventually compact if and only if \(T_{(A+L)0}\) is eventually compact.

For \(p\)-quasi Hille-Yosida operator \(A\), the compactness of \(LT_{A_0}\) can make \(S_2\) be higher regularity, which is discovered by Ducrot, Liu, Magal [DLM08]; the key of the proof used the following fact: \(\forall x^* \in (X_0)^*, x \in X, x^*S_A(\cdot)x \in W^{1,p'}_{loc}(0, \infty)\) (as \(x^*S_A(\cdot)x\) is of bounded \(p'\)-variation), where \(1/p' + 1/p = 1\) (see also Remark 2.9).

**Corollary 4.6.** Let \(A\) be a quasi Hille-Yosida operator. If \(LT_{A_0}\) is compact on \((0, \infty)\), then \(S_2\) is norm continuous and compact on \((0, \infty)\). Particularly, in this case, \(T_{A_0}\) is eventually norm continuous (resp. eventually compact) if and only if \(T_{(A+L)0}\) is eventually norm continuous (resp. eventually compact).

Proof. Since \(LT_{A_0}\) is compact on \((0, \infty)\), we have \(LS_A \circ LT_{A_0}\) is norm continuous ([DLM08, Proposition 4.8]) and compact (Lemma 3.2 (a)) on \((0, \infty)\). Thus, \(S_2\) is norm continuous and compact on \((0, \infty)\) (Lemma 3.2 (b)). Next we show that if \(LT_{A_0}\) is compact on \((0, \infty)\), so is \(LT_{(A+L)0}\). Since by Theorem 2.14, we have

\[ LT_{(A+L)0} = LT_{A_0} + LS_A \circ LT_{A_0}, \]

and since by Lemma 3.2 (a), \(LS_A \circ LT_{(A+L)0}\) is compact on \((0, \infty)\), it yields the compactness of \(LT_{(A+L)0}\). Then, the last statement follows from Theorem 4.4 and the previous argument.

\[ \square \]
Problem: is it true that if $LS_1$ is compact, then $S_3$ is compact and norm continuous?

The above results don’t show that whether $T_{(A + L)\nu}$ is immediately differentiable if $T_{A\nu}$ is immediately differentiable. In fact, though $A$ is a generator of a $C_0$ semigroup, the result may not hold [Ren95]. We need more detailed characterization of differentiability. Set

$$D_{\beta, c} = \{ \lambda \in \mathbb{C} : \Re\lambda \geq c - \beta \log |\Im\lambda| \},$$

where $\beta > 0$, $c \in \mathbb{R}$. The following deep result about the characterization of differentiability is due to Pazy and Iley, see [Ile07, Theorem 4.1, Theorem 4.2].

**Theorem 4.7** (Pazy-Iley). Let $B$ be the generator of a $C_0$ semigroup $T_B$. The following are equivalent.

(a) For every $C \in \mathcal{L}(X)$, $T_{B+C}$ generated by $B + C$, is eventually strongly differentiable (resp. immediately strongly differentiable).

(b) There is some (resp. all) $\beta > 0$, and a constant $k$, such that $D_{\beta, c} \subset \rho(B)$, $\|R(\lambda, B)\| \leq k$, $\forall \lambda \in D_{\beta, c}$, for some $c \in \mathbb{R}$.

(c) There is some (resp. all) $\beta' > 0$, such that $D_{\beta', c'} \subset \rho(B)$, $\|R(\lambda, B)\| \to 0$, as $\lambda \to \infty$, $\forall \lambda \in D_{\beta', c'}$, for some $c' \in \mathbb{R}$.

**Remark 4.8.** That (a) implies (c) was proved by Pazy [Paz68]. Other’s equivalence was proved by Iley [Ile07]. See more results on characterization of differentiability in [BK10, Section 3].

Pazy [Paz68] characterized the generators of eventually and immediately strongly differentiable semigroups as follows.

**Theorem 4.9** (Pazy criterion of differentiability). Let $B$ be the generator of the $C_0$ semigroup $T$, $\|T(t)\| \leq Me^{\alpha t}$. Then, $T$ is eventually (resp. immediately) strongly differentiable if and only if there is some (resp. all) $\beta > 0$, $c \in \mathbb{R}$, $\epsilon \in \mathbb{N}$, $k > 0$, such that $D_{\beta, c} \subset \rho(B)$ and

$$\|R(\lambda, B)\| \leq |\Im\lambda|^m,$$

for all $\lambda \in D_{\beta, c}$, $\Re \lambda \leq \omega$. In this case, $T$ is strongly differentiable on $((m + 2)/\beta, \infty)$.

Now the question arises: For what an MR operator $A$, is $T_{(A + L)\nu}$ eventually differentiable for all $L \in \mathcal{L}(X_0, X)$? By Theorem 4.7, we know $A_0$ at least satisfies the condition of Theorem 4.7 (b) or (c) in $X_0$ (i.e., consider the case $L \in \mathcal{L}(X_0)$).

When $\|R(\lambda, A)\|_{\mathcal{L}(X)} \|L\|_{\mathcal{L}(X_0, X)} < 1$, by Lemma 2.1, we have

$$R(\lambda, (A + L)\nu) = R(\lambda, A + L)|_{X_0} = (I - R(\lambda, A)L)^{-1}R(\lambda, A)|_{X_0} = (I - R(\lambda, A)L)^{-1}R(\lambda, A_0),$$

and so

$$(4.1) \quad \|R(\lambda, (A + L)\nu)\|_{\mathcal{L}(X_0)} \leq \frac{\|R(\lambda, A_0)\|_{\mathcal{L}(X_0)}}{1 - \|R(\lambda, A)\|_{\mathcal{L}(X)}\|L\|_{\mathcal{L}(X_0, X)}}.$$

We will use this estimate to characterize the differentiability of $T_{(A + L)\nu}$.

**Theorem 4.10.** Let $A$ be an MR operator satisfying $\limsup_{\lambda \to \infty} \|\lambda R(\lambda, A)\| < \infty$ (e.g., $A$ is a Hille-Yosida operator). Then, $T_{(A + L)\nu}$ is eventually strongly differentiable for all $L \in \mathcal{L}(X_0, X)$ if and only if $A_0$ satisfies the Pazy-Iley condition, i.e., the condition of Theorem 4.7 in (b) or (c).

**Proof.** Only need to consider the sufficiency. Assume $L \neq 0$ and $A_0$ satisfies the condition of Theorem 4.7 (b) or (c). Let $\epsilon = \frac{\|L\|^{-1}}{M}$, $\lambda \in D_{\beta, c}$, $\mu = \omega + |\lambda|$, where $\omega > 0$ is sufficiently large such that (by the assumption limsup $\|\lambda R(\lambda, A)\| < \infty$)

$$\|R(\mu, A)\| \leq \frac{M}{\mu} \leq \frac{M}{\omega} < \epsilon.$$

Then,

$$\|R(\lambda, A)\| = \|(\mu - \lambda)R(\lambda, A_0)R(\mu, A) + R(\mu, A)\|.$$
\[ \leq M \frac{|\mu - \lambda|}{\mu} \|R(\lambda, A_0)\| + \epsilon \]
\[ \leq M \frac{2|\lambda| + \omega}{|\lambda| + \omega} \|R(\lambda, A_0)\| + \epsilon. \]

Let \( K \) be sufficiently large, \( D_{\beta', c'} \subset D_{\beta, c} \), such that
\[ \|R(\lambda, A_0)\| \leq \frac{\epsilon}{2M}, \]
for \( \lambda \in D_{\beta', c'}, |\lambda| > K \). Hence by (4.1), we get
\[ \|R(\lambda, (A + L)_0)\|_{L(X_0)} \leq 2\|R(\lambda, A_0)\|_{L(X_0)}. \]

The result now follows from Theorem 4.9. \( \square \)

Next we consider another class of operators. A \( C_0 \) semigroup \( T \) with the generator \( B \) is called the Crandall-Pazy class if \( T \) is strongly immediately differentiable and
\[ \|BT(t)\| = O(t^{-\alpha}), \text{ as } t \to 0^+, \]
for some \( \alpha \geq 1 \). We also call the generator \( B \) a Crandall-Pazy operator. It was shown in [CP68] that this is equivalent to
\[ \|R(\lambda, B)\| = O(|\lambda|^{-\beta}), \text{ as } |\lambda| \to \infty, \]
for some \( 1 \geq \beta > 0 \). Indeed, (4.2) implies (4.3) with \( \beta = \frac{1}{\alpha} \), and (4.3) implies (4.2) for any \( \alpha > \frac{1}{\beta} \).

See [Ile07, Theorem 5.3] for a new characterization of the Crandall-Pazy class.

**Theorem 4.11.** Let \( A \) be an MR operator satisfying \( \limsup_{\mu \to \infty} \|\mu - \tilde{\omega}\|^{\frac{1}{p}} R(\mu, A)\| < \infty \), where \( p \geq 1 \), \( \tilde{\omega} \in \mathbb{R} \) (e.g., \( A \) is a \( p \)-quasi Hille-Yosida operator, see Lemma 2.8 in (c)). In addition, \( A_0 \) is a Crandall-Pazy operator satisfying
\[ \|R(iy, A_0)\|_{L(X_0)} = O(|y|^{-\beta}), \text{ as } |y| \to \infty, \beta > 1 - \frac{1}{p}. \]

Then, \((A + L)_0\) is still a Crandall-Pazy operator for any \( L \in L(X_0, X) \).

**Proof.** Assume \( L \neq 0 \). Let \( K_1 > 0 \) be sufficiently large such that for all \( |y| > K_1 \),
\[ \|R(\lambda, A_0)\| \leq \frac{M}{|y|^\beta} \|R(\mu, A)\| \leq \frac{M}{|\mu - \tilde{\omega}|^\frac{1}{p}}, \]
where \( \mu = \tilde{\omega} + |y| \). Then,
\[ \|R(iy, A)\| = \|(\mu - iy)R(iy, A_0)R(\mu, A) + R(\mu, A)\| \]
\[ \leq M \frac{2|y| + \tilde{\omega}}{|y|^\beta \cdot |y|^\frac{1}{p}} + \frac{M}{|y|^\frac{1}{p}} \]
\[ \leq \frac{M}{|y|^\alpha}, \]
where \( \alpha = \min\{\beta + 1/p - 1, 1/p\} > 0 \). Let \( K_2 > K_1 \) be sufficiently large such that
\[ \|R(iy, A)\| \leq \frac{\|L\|^{-1}}{2}, \]
provided \( |y| > K_2 \). By (4.1), we have
\[ \|R(iy, (A + L)_0)\| \leq 2\|R(iy, A_0)\|. \]

The proof is complete. \( \square \)

Finally, we consider the analyticity.
Theorem 4.12. Suppose that $A$ is an MR operator and $T_{A_0}$ is analytic. Then, for any $L \in \mathcal{L}(X_0, X)$, $T_{(A + L)_0}$ is analytic.

Proof. Assume $L \neq 0$. Since $T_{A_0}$ is analytic and $A$ is an MR operator, by [ABHN11, Corollary 3.7.18] and Lemma 2.6 (b), there is a constant $K > 0$, such that

$$
\|R(iy, A_0)\| \leq \frac{M}{|y|}, \quad \|R(\mu, A)\| \leq \varepsilon,
$$

provided $|y| > K$, where $\mu = |y|, \varepsilon = \frac{\|L\|^{-1}}{2}$. Hence,

$$
\|R(iy, A)\| = \|(\mu - iy)R(iy, A_0)R(\mu, A) + R(\mu, A)\|
\leq \frac{2|\mu|}{M} \varepsilon + \varepsilon = \|L\|^{-1}/2.
$$

The result follows from (4.1) and [ABHN11, Corollary 3.7.18]. \hfill \Box

Combining with Lemma 2.11 (b) and Theorem 4.12, we obtain the following perturbation of almost sectorial operators. See also [DMP10] for more results on the relatively bounded perturbations of almost sectorial operators.

Corollary 4.13. If $A$ is a $p$-almost sectorial operator ($p \geq 1$), so is $A + L$ for any $L \in \mathcal{L}(X_0, X)$.

5. Critical and essential growth bound of perturbed semigroups

We still assume $A$ is an MR operator, $L \in \mathcal{L}(X_0, X)$. In this section, we study the critical and essential growth bound of a perturbed semigroup $T_{(A + L)_0}$. In many applications, one would like to hope the following hold:

$$
\omega_{\text{crit}}(T_{(A + L)_0}) \leq \omega_{\text{crit}}(T_{A_0}) \quad \omega_{\text{ess}}(T_{(A + L)_0}) \leq \omega_{\text{ess}}(T_{A_0}),
$$

see Section 7 for a short discussion. The following elementary lemma seems well known, especially for the case $j = 0$, see also the proof of [BNP00, Proposition 3.7].

Lemma 5.1. Suppose $f_j : [0, \infty) \to [0, \infty)$, $j = 0, 1, 2, \ldots$, satisfy

$$
f_j(t + s) \leq \sum_{k=0}^{j} f_k(t)f_{j-k}(s), \quad \forall t, s > 0, \quad j = 0, 1, 2, \ldots,
$$

and each $f_j$ is bounded on any compact intervals. Set $(\ln 0 := -\infty)$

\begin{equation}
(\ast) \quad \omega := \lim_{t \to \infty} \frac{1}{t} \ln f_0(t) = \inf_{t > 0} \frac{1}{t} \ln f_0(t) = \inf\{\omega \in \mathbb{R} : \exists M \geq 0, f_0(t) \leq Me^{\omega t}, \forall t \geq 0\}.
\end{equation}

Then, for any $\gamma > \omega$, $\forall j \geq 0$, $\exists M_j \geq 0$, such that $f_j(t) \leq M_j e^{\gamma t}$, $\forall t \geq 0$.

Proof. Note that by a standard result (see, e.g., [EN00, Lemma IV.2.3]), the assumption on $f_0$ gives that $(\ast)$ holds. By the definition of $\omega$, for any $\gamma > \omega$, there is $t_0 > 0$, such that $f_0(t_0) \leq \frac{1}{2} e^{\gamma t_0}$. By induction, one gets

$$
f_j(t) = f_j(t - t_0 + t_0) \leq \sum_{k=0}^{j} f_k(t - t_0)f_{j-k}(t_0)
\leq \frac{1}{2} e^{\gamma t_0} f_j(t - t_0) + M'_j e^{\gamma t},
$$

where $t \geq t_0$, and the constant $M'_j$ doesn’t depend on $t$. Set

$$
a_n(s) := f_j(nt_0 + s), \quad s \in [0, t_0).
$$
Using (5.1), we have

\[ a_n(s) \leq \frac{1}{2}e^{\gamma_0}a_{n-1}(s) + M_j e^{\gamma(n_0+s)} \]
\[ \leq \frac{1}{2}e^{2\gamma_0}a_{n-2}(s) + (\frac{1}{2} + 1)M_j e^{\gamma(n_0+s)} \]
\[ \leq \cdots \]
\[ \leq \frac{1}{2^n}e^{n\gamma_0}a_0(s) + (\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \cdots + 1)M_j e^{\gamma(n_0+s)} \]
\[ \leq M_j e^{\gamma(n_0+s)}, \]

for some constant \( M_j \geq 0 \). This completes the proof. \( \square \)

Since the definition of critical spectrum depends on the space \( \ell_0^\infty(X) \), we need some preliminaries. See [BNP00] for similar results. Recall the meaning of \( S_A \circ V, S_n, R_n \) (see Section 2.2), \( \bar{S}_n, \hat{S}_n, \bar{T}_n \) (which are defined similarly as for a \( C_0 \) semigroup \( \hat{T} \) in Section 2.3) and so on in Section 2.

**Lemma 5.2.** \( \ell_0^\infty(T_{A+L_0}(X_0)) = \ell_0^\infty(T_{A_0}(X_0)) \).

**Proof.** This follows from

\[ \|T_{A+L_0}(h) - T_{A_0}(h)\| = \|(S_A \circ LT_{A+L_0})(h)\| \leq M\delta(h) \rightarrow 0, \text{ as } h \rightarrow 0^+. \]

\( \square \)

**Lemma 5.3.** \( \bar{S}_n(t)\ell_0^\infty(X_0) \subset \ell_0^\infty(T_{A_0}(X_0), \forall t \geq 0. \)

**Proof.** Consider

\[ T_{A_0}(h)S_n(t) - S_n(t) \]
\[ = T_{A_0}(h)(S_A \circ LS_{n-1})(t) - (S_A \circ LS_{n-1})(t) \]
\[ = (S_A \circ LS_{n-1})(t + h) - (S_A \circ LS_{n-1})(t) - (S_A \circ LS_{n-1}(t + h))(h) \]
\[ = (S_A \circ LS_{n-1}(h + \cdot))(t) - (S_A \circ LS_{n-1})(t) + T_{A_0}(t)(S_A \circ LS_{n-1})(h) + W_1(h) \]
\[ = (S_A \circ L \sum_{k=0}^{n-1} S_k(\cdot)S_{n-1-k}(h))(t) - (S_A \circ LS_{n-1})(t) + W_2(h) \]
\[ = (S_A \circ LS_{n-1}(\cdot)[T_{A_0}(h) - I])(t) + W_3(h) \]
\[ = S_n(t)(T_{A_0}(h) - I) + W_3(h), \]

the second and third equality being a consequence of (2.6), and the fourth one being a consequence of Lemma 2.15, where

\[ W_1(h) := -(S_A \circ LS_{n-1}(t + \cdot))(h), \]
\[ W_2(h) := W_1(h) + T_{A_0}(t)(S_A \circ LS_{n-1})(h), \]
\[ W_3(h) := W_2(h) + \sum_{k=0}^{n-2} (S_A \circ LS_k(\cdot)S_{n-1-k}(h))(t), \]
\[ = W_2(h) + \sum_{k=0}^{n-2} S_{k+1}(t)S_{n-1-k}(h). \]

Note that for \( k \geq 1, \|S_k(h)\| \leq \bar{M}_k\delta(h), \) for some constant \( \bar{M}_k > 0. \) Hence, \( \|W_3(h)\| \leq \bar{M}\delta(h), \) which shows the result. \( \square \)

**Lemma 5.4.** If \( (S_A \circ V)(t) \) is (right) norm continuous at \( t_0, \) then \( (S_A \circ V)(t_0)\ell_0^\infty(X_0) \subset \ell_0^\infty(T_{A_0}(X_0), \text{ where } V \in C_{\hat{L}}([0, \infty), \bar{L}(X_0, X)). \)
Proof. We need to show \(|T_{A_0}(h)(S_A \diamond V)(t_0) - (S_A \diamond V)(t_0)| \to 0\), as \(h \to 0^+\), which follows from
\[T_{A_0}(h)(S_A \diamond V)(t_0) - (S_A \diamond V)(t_0) = (S_A \diamond V)(t_0 + h) - (S_A \diamond V)(t_0) - (S_A \diamond V(t_0 + \cdot))(h)\].

The following are main results.

**Lemma 5.5.** (a) In \(\mathcal{L}(X_0)/\mathcal{K}(X_0), \forall \gamma > \omega_{\text{ess}}(T_{A_0}), \forall j \geq 0\), there is \(M_j^1 \geq 0\), such that \(\|\overline{S}_j(t)\| \leq M_j^1 e^{\gamma t}\).

(b) In \(l^\infty(X_0)/l^\infty_{T_{A_0}}(X_0), \forall \gamma > \omega_{\text{crit}}(T_{A_0}), \forall j \geq 0\), there is \(M_j^2 \geq 0\), such that \(\|\overline{S}_j(t)\| \leq M_j^2 e^{\gamma t}\).

**Proof.** By Lemma 5.3, \(\overline{S}_j\) can be defined in \(l^\infty(X_0)/l^\infty_{T_{A_0}}(X_0)\). Thus, by Lemma 2.15, we have
\[\overline{S}_j(t + s) = \sum_{k=0}^{j} \overline{S}_k(t)\overline{S}_{j-k}(s)\text{ in } l^\infty(X_0)/l^\infty_{T_{A_0}}(X_0),\]
and
\[\overline{S}_j(t + s) = \sum_{k=0}^{j} \overline{S}_k(t)\overline{S}_{j-k}(s)\text{ in } \mathcal{L}(X_0)/\mathcal{K}(X_0).\]

The proof is complete by Lemma 5.1.

**Theorem 5.6.** Let positive sequence \(\{t_n\}\) satisfy \(t_n \to \infty\), as \(n \to \infty\).

(a) If there is \(k \in \mathbb{N}\) such that \(R_k(t_n)\) is compact, \(n = 1, 2, 3, \ldots\), then \(\omega_{\text{ess}}(T_{(A+L_0)}) \leq \omega_{\text{ess}}(T_{A_0}).\)

(b) If there is \(k \in \mathbb{N}\) such that \(R_k\) is norm continuous at \(t_1, t_2, t_3, \ldots\), then \(\omega_{\text{crit}}(T_{(A+L_0)}) \leq \omega_{\text{crit}}(T_{A_0}).\)

(c) If \(R_k\) is compact (resp. norm continuous) at \(t_1, t_2, t_3, \ldots\), then \(\omega_{\text{ess}}(T_{(A+L_0)}) = \omega_{\text{ess}}(T_{A_0})\) (resp. \(\omega_{\text{crit}}(T_{(A+L_0)}) = \omega_{\text{crit}}(T_{A_0})\)).

**Proof.** (a) Since \(R_k(t_n)\) is compact, in \(\mathcal{L}(X_0)/\mathcal{K}(X_0), \overline{R}_k(t_n) = 0, \text{ for } n = 1, 2, 3, \ldots\). Hence,
\[\overline{T}_{(A+L_0)}(t_n) = \sum_{j=0}^{k-1} \overline{S}_j(t_n),\]
for \(n = 1, 2, 3, \ldots\). By Lemma 5.5, we have
\[\omega_{\text{ess}}(T_{(A+L_0)}) = \lim_{n \to \infty} \frac{1}{t_n} \ln \|\overline{T}_{(A+L_0)}(t_n)\| \leq \omega_{\text{ess}}(T_{A_0}),\]
which completes the proof of (a).

(b) Note that due to Lemma 5.2, \(l^\infty(X_0)/l^\infty_{T_{(A+L_0)}}(X_0) = l^\infty(X_0)/l^\infty_{T_{A_0}}(X_0)\). Thus, by Lemma 5.4, \(\overline{R}_k(t_n) = 0, \text{ for } n = 1, 2, 3, \ldots\). The following proof is similar as (a).

(c) In this case \(\overline{T}_{(A+L_0)}(t_n) = \overline{T}_{A_0}(t_n)\) (resp. \(\overline{T}_{(A+L_0)}(t_n) = \overline{T}_{A_0}(t_n)\)), for \(n = 1, 2, 3, \ldots\), which shows the result.

In many cases, since we don’t know the explicit expression of \(T_{(A+L_0)}\), calculating \(R_k\) is hard. In the following, we will give more elaborate result than Theorem 5.6. Consider \(T_{A_0}\) as the perturbation of \(T_{(A+L_0)}\) under \(-L\). Set
\[\mathcal{D}(W) := -S_{A+L} \diamond LW, \ W \in C([0, \infty), \mathcal{L}(X_0, X)).\]

**Lemma 5.7.** The following statements are equivalent, where \(P\) stands for norm continuity, or compactness on \((0, \infty)\).

(a) \(B^k(T_{(A+L_0)}) = R_k\) has property \(P\) (\(\Leftrightarrow B^k(T_{A_0}) = S_k\) has property \(P\)).

(b) \(D^k(T_{(A+L_0)})\) has property \(P\) (\(\Leftrightarrow D^k(T_{A_0})\) has property \(P\)).
Proof. Here note that if $S_a$ (or $R_a$) is compact on $(0, \infty)$, then it is norm continuous on $(0, \infty)$ by Corollary 2.16 (c) (or (d)). So if $P$ stands for compactness on $(0, \infty)$, then it stands for norm continuity and compactness on $(0, \infty)$. It suffices to consider one direction, e.g., (a) $\Rightarrow$ (b). Since

$$S_{A+L} = S_A + S_A \circ L S_{A+L},$$

we have

$$\mathcal{D} = -\mathcal{B} + \mathcal{B} \mathcal{D}.$$  

We prove that $\mathcal{B}^{-1} \mathcal{D}'(T_{(A+L)_k})$ has property $P$ by induction, where $l = 0, 1, 2, \ldots, k$. The case $l = 0$ is (a). Since

$$\mathcal{B}^{-1} \mathcal{D}' = \mathcal{B}^{-1} \mathcal{D}' - \mathcal{B}^{-1} \mathcal{D} = -\mathcal{B}^{k-1} \mathcal{D}'(T_{(A+L)_k}) + \mathcal{B}^{-1} \mathcal{D},$$

and by induction, $\mathcal{B}^{k-1-1} \mathcal{D}'(T_{(A+L)_k})$ has property $P$, it yields $\mathcal{B}^{k-1} \mathcal{D}'(T_{(A+L)_k})$ has property $P$ by Lemma 4.1. Note that the equivalences in the brackets are Corollary 4.2. This completes the proof. □

**Theorem 5.8.** (a) If there is $S_k$ (or $R_k$) such that it is compact on $(0, \infty)$, then $\omega_{\text{ess}}(T_{(A+L)_k}) = \omega_{\text{ess}}(T_{A_k})$.

(b) If there is $S_k$ (or $R_k$) such that it is norm continuous on $(0, \infty)$, then $\omega_{\text{crit}}(T_{(A+L)_k}) = \omega_{\text{crit}}(T_{A_k})$.

**Proof.** Use Lemma 5.7 and apply Theorem 5.6 twice. □

Using Corollary 4.5 and Theorem 5.8, we have the following corollary.

**Corollary 5.9.** Let $A$ be an MR operator, $L \in \mathcal{L}(X_0, X)$.

(a) If $LT_{A_0}$ is norm continuous on $(0, \infty)$, then $\omega_{\text{crit}}(T_{(A+L)_k}) = \omega_{\text{crit}}(T_{A_k})$.

(b) If $LT_{A_0}$ is norm continuous and compact on $(0, \infty)$, then $\omega_{\text{ess}}(T_{(A+L)_k}) = \omega_{\text{ess}}(T_{A_k})$.

Using Corollary 4.6 and Theorem 5.8, we have the following corollary due to [DLM08].

**Corollary 5.10 ([DLM08, Theorem 1.2]).** Suppose that $A$ is a quasi Hille-Yosida operator and $LT_{A_0}$ is compact on $(0, \infty)$, then $\omega_{\text{ess}}(T_{(A+L)_k}) = \omega_{\text{ess}}(T_{A_k})$.

**Remark 5.11.** It was only shown in [DLM08, Theorem 1.2] that $\omega_{\text{ess}}(T_{(A+L)_k}) \leq \omega_{\text{ess}}(T_{A_k})$. However, using [DLM08, Theorem 1.2], it also yields $\omega_{\text{ess}}(T_{(A+L)_k}) = \omega_{\text{ess}}(T_{A_k})$, because $LT_{(A+L)_k}$ is also compact on $(0, \infty)$, see the proof in Corollary 4.6.

6. Age-structured population models in $L^p$ spaces

Consider the following age-structured population model in $L^p$ (see [Web85, Web08, Thi91, Thi98, Rha98, BH05, MR09a]):

$$
\begin{aligned}
(\partial_t + \partial_a)u(t, a) &= A(a)u(t, a), \quad t > 0, a \in (0, c), \\
\frac{d}{dt}u(t, 0) &= \int_0^c C(a)u(t, a) \, da, \quad t > 0, \\
\frac{d}{dt}u(0, a) &= u_0(a), \quad a \in (0, c), \quad u_0 \in L^p((0, c), E),
\end{aligned}
$$

where $1 \leq p < \infty$, $c \in (0, \infty]$ and $E$ is a Banach space ($u(t, a) \in E$); for all $a \in (0, c)$, $C(a) : E \to E$ are bounded linear operators and $A(a) : E \to E$ are closed linear operators (the detailed assumptions are given in Section 6.2). The approach given in this section to the study of model (6.1) goes back to Thieme [Thi91] as early as 1991. It seems that this model in $L^p((0, c), E)$ ($p > 1$) was first investigated by Magal and Ruan in [MR07] (see also [Thi08]). We notice that, in control theory (and approximation theory), age-structured population models can be considered as boundary control systems and in this case the state space is often taken as $L^p((0, c), E)$ ($p > 1$); see, e.g., [CZ95]. In addition, when $c = \infty$, since $L^p((0, \infty), E)$ does not include in $L^1((0, \infty), E)$, this gives in some sense more solutions of (6.1) by taking initial data $u_0 \in \bigcup_{1 \leq p < \infty} L^p((0, \infty), E)$. Finally, we mention that the geometric property of $L^2((0, c), E)$ is usually better than $L^1((0, c), E)$. 


6.1. **evolution family: mostly review.** Before giving some naturally standard assumptions on (6.1), we first recall some backgrounds about evolution family. We say \( \{U(a,s)\}_{0 \leq s \leq a < c} \) is an exponentially bounded linear evolution family (or for short **evolution family**) if it satisfies the following:

1. \( U(a,s) \in \mathcal{L}(E,E), U(a,s) = U(a,\tau)U(\tau,s) \) and \( U(a,a) = I \) for all \( 0 \leq s \leq \tau \leq a < c \);
2. \( (a,s) \mapsto U(a,s)x \) is continuous in \( 0 \leq s \leq a < c \) and \( x \in E \);
3. there exist constants \( C \geq 1 \) and \( \omega \in \mathbb{R} \) such that

\[
|U(a,s)| \leq Ce^{\omega(a-s)}, \quad 0 \leq s \leq a < c.
\]

Define the Howland semigroup \( T_0 \) on \( L^p((0,c),E) \) (with respect to \( \{U(a,s)\}_{0 \leq s \leq a < c} \)) as

\[
(T_0(t)\varphi)(a) = \begin{cases} 
U(a,a-t)\varphi(a-t), & a \geq t, \\
0, & a \in [0,t],
\end{cases}
\]

which is a \( C_0 \) semigroup, and let \( B_0 \) denote its generator. Note that \( D(B_0) \subset C([0,c), E) \). In fact, a simple computation shows that if \( \lambda \in \rho(B_0) \), then

\[
(R(\lambda,B_0)f)(a) = \int_0^a e^{\lambda(a-s)}U(a,s)f(s) \, ds, \quad f \in L^p((0,c),E).
\]

**Proof.** Let \( R(\lambda,B_0)f = \varphi \). Note that we have

\[
e^{-\lambda t}T_0(t)\varphi - \varphi = (\lambda - B_0) \int_0^t e^{-\lambda s}T_0(s)\varphi \, ds = \int_0^t e^{-\lambda s}T_0(s)f \, ds.
\]

For \( t \geq a \), we see \( (e^{-\lambda t}T_0(t)\varphi - \varphi)(a) = -\varphi(a) \) and

\[
\int_0^a (e^{-\lambda s}T_0(s)f)(a) \, ds = \int_0^a e^{-\lambda s}U(a,a-s)f(a-s) \, ds = -\int_0^a e^{\lambda(a-s)}U(a,s)f(s) \, ds,
\]

completing the proof. \( \square \)

**Theorem 6.1** (See [CL99, Section 3.3]). \( \omega(T_0) = s(B_0) = \omega(U) \), where \( \omega(U) \) is the growth bound of \( \{U(a,s)\} \), i.e.,

\[
\omega(U) := \inf \{ \omega : \text{there is } M(\omega) \geq 1 \text{ such that } |U(a,s)| \leq M(\omega)e^{\omega(a-s)} \text{ for all } a \geq s \}.
\]

**Remark 6.2.** The sufficient and necessary conditions such that \( B_0 \) is a generator of Howland semigroup (6.2) induced by an evolution family were given by Schnaubelt [Sch96, Theorem 2.6] (see also [RSRV00, Theorem 2.4]).

Let us use the evolution family \( \{U(a,s)\}_{0 \leq s \leq a < c} \) to study the sum \( -\frac{d}{da} + A(\cdot) \) on \( L^p((0,c),E) \) with

\[
D(-\frac{d}{da} + A(\cdot)) = D(\frac{d}{da}) \cap D(A(\cdot)),
\]

where

\[
\frac{d}{da} : f \mapsto \frac{df}{da}, \quad D(\frac{d}{da}) = \{ f \in W^{1,p}((0,c),E) : f(0) = 0 \},
\]

and \( A(\cdot) \) is the multiplication operator, i.e.,

\[
(A(\cdot)f)(a) = A(a)f(a), \quad a \in (0,c) \text{ with } D(A(\cdot)) = \{ f \in L^p((0,c),E) : f(a) \in D(A(a)) \text{ a.e. } a \in (0,c), \text{ and } f(0) \in L^p((0,c),E) \}.
\]

**Definition 6.3.** We say \( \{A(a)\}_{0 \leq a < c} \) generates an exponentially bounded linear evolution family \( \{U(a,s)\}_{0 \leq s \leq a < c} \) in \( L^p \)-sense if \( B_0 \) is an invertible closure of \( -\frac{d}{da} + A(\cdot) \) (with \( D(-\frac{d}{da} + A(\cdot)) = D(\frac{d}{da}) \cap D(A(\cdot)) \)) on \( L^p((0,c),E) \).

**Remark 6.4.** Consider the following abstract non-autonomous (linear) Cauchy problems:

\[\text{(ACP)} \quad \dot{x}(a) = A(a)x(a), \quad x(s) = x_s \in D(A(s)), \quad a \geq s,\]

where \( D(A(s)) = X \) for all \( s \in [0,c) \). An evolution family \( \{U(a,s)\}_{0 \leq s \leq a < c} \) is said to solve the above **Cauchy problem** (ACP) if for each \( s \in [0,c) \), there is a dense linear space \( Y_s \subset D(A(s)) \) such that for each \( x_s \in Y_s \), \( x(a) = U(a,s)x_s \) is \( C^1 \) and satisfies (ACP) point-wisely; see, e.g., [Sch96]
Consider the following examples.

**Example 6.5.** (a) Assume \( A_0 \) is a generator of a \( C_0 \) semigroup \( T_0 \) and \( b(\cdot) \in C(\mathbb{R}, \mathbb{R}_+) \) (such that \( \inf b(\cdot) > 0 \)). Let

\[
A(a) = b(a)A_0, \quad 0 \leq a < c.
\]

Then, for \( U(a, s) = T_0(\int_s^0 b(r) \, dr) \), \( \{U(a, s)\}_{0 \leq s \leq a < c} \) solves (ACP).

(b) Assume \( A_0 \) is a generator of a \( C_0 \) semigroup and \( L : (0, c) \mapsto L(E, E) \) is strongly continuous and \( \sup_{t \in [0, c]} |L| < \infty \). Let

\[
A(a) = A_0 + L(a), \quad 0 \leq a < c.
\]

Then, it is easy to see \( \{A(a)\}_{0 \leq a < c} \) generates an evolution family \( \{U(a, s)\}_{0 \leq s \leq a < c} \); see, e.g., the proof of [CL99, Proposition 6.23].

For the above examples, using the classical solutions of (ACP) to give evolution family sometimes is limited; other way by using the mild solutions in the sense of [ABHN11, Definition 3.1.1] can be found in [Che19, Section 3]. See also [Paz83, Section 5.6–5.7] for the “parabolic” type (in the sense of Tanabe) and [Paz83, Section 5.3–5.5] for the “hyperbolic” type (in sense of Kato) which in some contexts \( \{A(a)\}_{0 \leq a < c} \) can generate evolution family (see also [Sch02] for a survey).

6.2. **standard assumptions and main results.** Hereafter, we make the following assumptions.

**H1** (about \( \{A(a)\} \)) Assume \( \{A(a)\}_{0 \leq a < c} \) generates an exponentially bounded linear evolution family \( \{U(a, s)\}_{0 \leq s \leq a < c} \); see Definition 6.3. Let \( T_0 \) be its corresponding Howland semigroup (see (6.2)) with the generator \( B_0 \).

**H2** (about \( \{C(a)\} \)) Assume \( C(\cdot) : (0, c) \mapsto L(E, E) \) is strongly continuous. In addition, there is a positive function \( \gamma \in L^{p'}(0, c) \) such that \( |C(a)| \leq \gamma(a) \) where \( 1/p' + 1/p = 1 \).

Set \( X = E \times L^p((0, c), E), \ X_0 = \{0\} \times L^p((0, c), E), \)

\[
\mathcal{A} : D(\mathcal{A}) = \{0\} \times D(B_0) \to X : (0, \varphi) \mapsto (-\varphi(0), B_0\varphi),
\]

(here \( \mathcal{A} \) is well defined due to \( D(B_0) \subseteq C([0, c), E) \)) and

\[
L : X_0 \to X, (0, \varphi) \mapsto (L\varphi, 0), \quad L\varphi = \int_0^c C(a)\varphi(a) \, da.
\]

Note that by the assumption **H2** (on \( \{C(a)\} \)), we see \( L \) and so \( \mathcal{L} \) are bounded. Now, the solutions of the age-structured population model (6.1) can be interpreted as the mild solutions of the following abstract Cauchy problem:

\[
(CP) \quad \begin{cases} 
\dot{U}(t) = (\mathcal{A} + \mathcal{L})U(t), \\
U(0) = U_0 \in X_0.
\end{cases}
\]

Recall that \( U \in C([0, \tau], X) \) is called a mild solution of (CP) if \( \int_0^\tau U(r) \, dr \in D(\mathcal{A} + \mathcal{L}) \) and

\[
U(s) = U(0) + (\mathcal{A} + \mathcal{L}) \int_0^s U(r) \, dr, \quad s \in [0, \tau],
\]

see, e.g., [ABHN11, Definition 3.1.1].

**Definition 6.6.** If \( U = (0, u) \) is a mild solution of (CP), then \( u \) is called a mild solution of the age-structured population model (6.1).

Note that if \( u \in W^{1,p}([0, \tau] \times [0, c], E) \) satisfies (6.1) a.e. (or \( u \in C^1([0, \tau] \times [0, c], E) \) with \( c < \infty \) satisfies (6.1) point-wisely), then \( u \) is a mild solution of (6.1). See Lemma 6.9 for a characterization of mild solutions of the model (6.1) in terms of themselves.
By a simple computation, we see \( \rho(\mathcal{A}) = \rho(\mathcal{B}_0), \mathcal{A}_0 := \mathcal{A}_{X_0} = 0 \times \mathcal{B}_0 \) and for \( \lambda \in \rho(\mathcal{A}) \),

\[
R(\lambda, \mathcal{A})^n(y, f) = (0, \varphi_n), \quad \varphi_n(a) = \frac{1}{(n-1)!} a^{n-1} e^{-\lambda a} U(a, 0)y + (R(\lambda, \mathcal{B}_0)^n f)(a).
\]

Particularly, we obtain

Lemma 6.7. If \( p = 1 \), then \( \mathcal{A} \) is a Hille-Yosida operator and so is \( \mathcal{A} + \mathcal{L} \).

Consider the case \( p > 1 \). Since \( \mathcal{A}_0 (\cong \mathcal{B}_0) \) generates a \( C_0 \) semigroup \((0, T_0)\), we know \( \mathcal{A} \) generates an integrated semigroup \( S_{\mathcal{A}} \) defined by (see (2.3))

\[
S_{\mathcal{A}}(t)y, f) = (0, U_0(t)y + \int_0^t T_0(s)f \, ds),
\]

where

\[
(U_0(t)y)(a) = \begin{cases} 
U(a, 0)y, & a \leq t, \\
0, & a > t.
\end{cases}
\]

For \( f_1 \in L^p((0, t_1), E) \) and \( f_2 \in L^p((0, t_1), L^p((0, c), E)) \) \((\equiv L^p((0, t_1) \times (0, c), E))\), we have

\[
(S_{\mathcal{A}} \circ (f_1, f_2))(t)(a) = \begin{cases} 
(0, U(a, 0)f_1(t - a) + (T_0 * f_2)(t)(a)), & a \leq t, \\
(0, (T_0 * f_2)(t)(a)), & a > t.
\end{cases}
\]

So, it’s clear to see that there exists \( \hat{M} \geq 1 \) independent of \((f_1, f_2)\) and \( t_1 \) such that

\[
\left( \int_0^c |(S_{\mathcal{A}} \circ (f_1, f_2))(t)(a)|^p \, da \right)^{1/p} \leq \hat{M} \left\| e^{\omega(t_1-1)/p} (f_1, f_2)(\cdot) \right\|_{L^P(0,t_1;X)}, \quad \forall t \in [0, t_1].
\]

That is, we have the following; see also the proof of [MR07, Theorem 6.6] and the following of Proposition 5.6 in [Thi08].

Lemma 6.8. If \( p > 1 \), then \( \mathcal{A} \) is a \( p \)-quasi Hille-Yosida operator and so is \( \mathcal{A} + \mathcal{L} \).

As \( \mathcal{A} \) is a quasi Hille-Yosida operator and thus the mild solutions of the linear equation (CP) are given by

\[
U(t) = T_{\mathcal{A}_0}(t)U_0 + (S_{\mathcal{A}} \circ L U(\cdot))(t), \quad U(0) \in X_0,
\]

returning to the second component of \( U = (0, u) \), by using \( T_{\mathcal{A}_0} = (0, T_0) \) and (6.5), we obtain the following representation of the mild solutions of (6.1) defined in Definition 6.6; see also [Thi91, Section IV].

Lemma 6.9. \( u \) is a mild solution of (6.1) with \( u(0, \cdot) = u_0 \in L^p((0, c), E) \) if and only if \( u \) satisfies

\[
u(t, a) = \begin{cases} 
U(a, a - t)u_0(a - t), & a \geq t, \\
U(a, 0) \int_0^c C(s)u(t - a, s) \, ds, & a \in [0, t].
\end{cases}
\]

Remark 6.10. If there are a function \( h : (0, \infty) \to \mathbb{R}_+ \) and \( y \in E \) such that \( |U(a, 0)y| \geq h(a) \) and

\[
\limsup_{\lambda \to \infty} \lambda^p \int_0^{\infty} e^{-\lambda a} h^p(a) \, da > 0,
\]

then from (6.4), we know \( \mathcal{A} \) is not a Hille-Yosida operator when \( p > 1 \) and \( c = \infty \). For instance, \( U(a, 0) = I \) for all \( a \in (0, \infty) \).

Next, let us compute \( \mathcal{L} S_{\mathcal{A}} \circ (LT_0, 0) \). Write \( \mathcal{L} S_{\mathcal{A}} \circ (LT_0, 0) = (F, 0) \), then by (6.5), we get

\[
F(t) = \begin{cases} 
\int_{t-c}^t C(t-a)U(t-a, 0)LT_0(a) \, da, & t \leq c, \\
\int_{t-c}^c C(t-a)U(t-a, 0)LT_0(a) \, da, & t > c.
\end{cases}
\]
Lemma 6.11. (a) Suppose $U(\cdot, 0)$ is norm continuous on $(0, c)$, and $C(\cdot)$ is norm measurable on $(0, c)$ if $p > 1$ and is (essentially) norm continuous if $p = 1$. Then, $F$ (i.e., $\mathcal{L}S_A \circ (LT_0, 0)$) is norm continuous on $(0, \infty)$.
(b) If $U(\cdot, 0)$ is compact on $(0, c)$, then $F$ (i.e., $\mathcal{L}S_A \circ (LT_0, 0)$) is norm continuous and compact on $(0, \infty)$.

Proof. We only consider the case $t < c$. Set

$$F_x(t) = \int_0^{t+\epsilon} C(t-a)U(t-a, 0)LT_0(a) \, da.$$ 

Note that by the condition on $\{C(a)\}$ and the boundedness of $\{U(t-a, 0)LT_0(a)\}_{0 \leq a \leq t}$, we see $F_x(t) \to F(t)$ as $\epsilon \to 0$ in the uniform operator topology. So it suffices to consider $F_x(t)$. Let $0 < |h| < \epsilon$ (with $\epsilon$) be small and let $p'$ satisfy $1/p' + 1/p = 1 (1 < p' \leq \infty)$. In the following, denote by $\tilde{C} > 0$ the universal constant independent of $h$ which might be different line by line.

$$|F_x(t+h) - F_x(t)| = |\int_0^{t+h-\epsilon} C(t-h-a)U(t+h-a, 0)LT_0(a) \, da - \int_0^{t-\epsilon} C(t-a)U(t-a, 0)LT_0(a) \, da|$$

$$= |\int_0^{t+h-\epsilon} C(t-h-a)U(t+h-a, 0)LT_0(a) \, da - \int_0^{t-\epsilon} C(t-a)U(t-a, 0)LT_0(a) \, da|$$

$$+ \int_0^{t-\epsilon} \{C(t-h-a)U(t+h-a, 0) - C(t-a)U(t-a, 0)\}LT_0(a) \, da|$$

$$\leq \tilde{C}h + R_1 + R_2,$$

where

$$R_1 = \int_0^{t-\epsilon} |\{C(t+h-a) - C(t-a)\}U(t-a, 0)LT_0(a)| \, da,$$

$$\leq \left( \int_0^{t-\epsilon} |\{C(t+h-a) - C(t-a)\}U(t-a, 0)|^{p'} \, da \right)^{1/p} \left( \int_0^{t-\epsilon} |LT_0(a)|^p \, da \right)^{1/p},$$

and

$$R_2 = \int_0^{t-\epsilon} |C(t+h-a)(U(t+h-a, 0) - U(t-a, 0))LT_0(a)| \, da$$

$$\leq \left( \int_0^{t-\epsilon} |C(t+h-a)|^{p'} \, da \right)^{1/p'} \left( \int_0^{t-\epsilon} |U(t+h-a, 0) - U(t-a, 0)|^p \, da \right)^{1/p},$$

$$\leq \tilde{C} \left( \int_0^{t-\epsilon} |U(t+h-a, 0) - U(t-a, 0)|^p \, da \right)^{1/p}.$$

To prove (a), by the condition on $\{U(a, 0)\}$, we know $R_2 \to 0$ as $h \to 0$, and by the condition on $\{C(a)\}$, we get

$$R_1 \leq \tilde{C} \left( \int_0^{t-\epsilon} |C(t+h-a) - C(t-a)|^{p'} \, da \right)^{1/p'} \to 0, \text{ as } h \to 0,$$

since $\mathcal{L}(C(\cdot) : (0, c) \to \mathcal{L}(E, E))$ is Bochner $p'$-integrable (in the uniform operator topology) if $p' < \infty$ and is (essentially) norm continuous if $p' = \infty$. This shows that $F_x$ is norm continuous at $t$.

To prove (b), note first that if $U(\cdot, 0)$ is compact on $(0, c)$, then it is also norm continuous on $(0, c)$. Indeed, if $s > 0$ and $a > 0$, then

$$U(a+s, 0) - U(a, 0) = (U(a+s, a) - I)U(a, 0) \to 0,$$
as \( s \to 0^+ \) in the uniform operator topology due to the compactness of \( U(a, 0) \); the left norm continuity can be considered similarly. In particular, \( R_k \to 0 \) as \( h \to 0 \). Since \( \{ |C(a)| \}^\infty_{a=0} \) is bounded (due to the strong continuity of \( C(\cdot) \) on \( (0, c) \)) and \( U(\cdot, 0) \) is compact on \([\varepsilon, t]\), by the Lebesgue dominated convergence theorem (see, e.g., [ABHN11, Theorem 1.1.8]), we get

\[
\lim_{h \to 0} R_1 \leq C \left( \int_0^{t-\varepsilon} \lim_{h \to 0} |\{C(t+h-a) - C(t-a)\}U(t-a, 0)|^{p'} \, da \right)^{1/p'} = 0.
\]

Thus, \( F_\varepsilon \) is norm continuous at \( t \). The compactness of \( F_\varepsilon(t) \) follows from [EN00, Theorem C.7] since \( U(\cdot, 0) \) is compact on \([\varepsilon, t]\). The proof is complete. \( \square \)

Now we can give some asymptotic behaviors of the age-structured population model (6.1); see Theorem 6.1 for the characterization of \( \omega(T_0) \).

**Theorem 6.12.** Let one of the following conditions hold:

(a) \( U(\cdot, 0) \) is norm continuous on \((0, c)\), and \( C(\cdot) \) is norm measurable on \((0, c)\) if \( p > 1 \) and is (essentially) norm continuous if \( p = 1 \);

(b) \( U(\cdot, 0) \) is compact on \((0, c)\);

(c) \( L \) (see (6.3)) is compact.

Let \( T_{(\mathcal{A}+\mathcal{L})_0} \) be the \( C_0 \) semigroup generated by the part of \( \mathcal{A} + \mathcal{L} \) in \( L^p((0, c), E) \). Then, the following statements hold.

1. If \( c < \infty \) and condition (b) or (c) holds, then \( T_{(\mathcal{A}+\mathcal{L})_0} \) is eventually compact (and so eventually norm continuous). Particularly, \( \omega(T_{(\mathcal{A}+\mathcal{L})_0}) = s(T_{(\mathcal{A}+\mathcal{L})_0}) \) (see (2.17)).

2. If condition (b) or (c) holds, then \( \omega_{\text{ess}}(T_{(\mathcal{A}+\mathcal{L})_0}) = \omega_{\text{ess}}(T_0) \). Particularly, if \( \omega_{\text{ess}}(T_0) < 0 \), then \( T_{(\mathcal{A}+\mathcal{L})_0} \) is quasi-compact (see [EN00, Section V.3] for more consequences of this).

3. \( \omega_{\text{crit}}(T_{(\mathcal{A}+\mathcal{L})_0}) = \omega_{\text{crit}}(T_0) \).

**Proof.** If condition (a) (resp. (b)) holds, then by Lemma 6.11 and Lemma 3.1 (b) (resp. Lemma 3.2 (b)), we know \( S_2 \equiv S_{\mathcal{A}} \circ L(\mathcal{S}, \mathcal{A} \circ (LT_0, 0) \) is norm continuous (resp. compact) on \((0, \infty)\). If condition (c) holds (i.e., \( L \) is compact), then by Corollary 4.6, we have \( S_2 \) is compact on \((0, \infty)\).

To prove (1), as \( c < \infty \), we have for \( t > c \), \( T_0(t) = 0 \); so particularly \( T_0 \) is eventually compact. Conclusion (1) now follows Theorem 4.4 (b). Finally, conclusions (2) and (3) are direct consequence of Theorem 5.8. \( \square \)

**Example 6.13.** In order to verify Theorem 6.12, consider the following examples which might be not as general as possible.

(a) (See [MR09a, Section 5] and [Web85].) Let \( E = \mathbb{R}^n \). Then, \( C(a) \) and \( A(a) \) can be considered as matrices. Assume \( a \mapsto A(a) \) is continuous on \([0, c)\) and \( a \mapsto C(a) \) is continuous on \((0, c)\) with \( \sup_{0 \leq a \leq c} \|A(a)\|, \|C(a)\| < \infty \), then assumptions (H1) (H2) hold. Note also that in this case \( L \) is compact. Now all the cases in Theorem 6.12 are fulfilled. A very special case is \( A(a) \equiv -\mu I \) where \( \mu > 0 \); now \( U(t, s) = e^{-\mu(t-s)}I \) and so by Remark 6.10 in this case \( \mathcal{A} \) is not a Hille-Yosida operator if \( p > 1 \) and \( c = \infty \); in addition, \( \omega_{\text{ess}}(T_{(\mathcal{A}+\mathcal{L})_0}) < 0 \) as \( \omega(T_0) = \omega(U) \leq -\mu < 0 \).

(b) (See [BHM05, Section 5.1].) Consider the following model:

\[
\begin{align*}
\left( \partial_t + \Delta \right) u(t, a, s) &= -\mu(a) u(t, a, s) + k(a) \Delta_x u(t, a, s), \quad t > 0, \ a \in (0, c), s \in \mathbb{R}^n, \\
u(t, 0, s) &= \int_0^c \beta(a, s) u(t, a, s) \, da, \quad t > 0, \ s \in \mathbb{R}^n, \\
u(0, a, s) &= u_0(a, s), \quad a \in (0, c), s \in \mathbb{R}^n, \ u_0 \in L^p((0, c), L^q(\mathbb{R}^n)).
\end{align*}
\]

Let \( E = L^q(\mathbb{R}^n) (1 \leq q < \infty) \). Take \( A(a) = -\mu(a) I + k(a) \Delta_x \) and \( C(a) = \beta(a, \cdot) \), where

(i) \( \Delta_x \) denotes the Laplace operator on \( L^q(\mathbb{R}^n) \) (i.e., \( \Delta f = \sum_{j=1}^n D^2_{x_j} f \));

(ii) \( \mu(\cdot) \in L^\infty((0, c), \mathbb{R}_+) \), \( k(\cdot) \in C_b([0, c), \mathbb{R}_+) \) (with \( \inf_{a \geq 0} k(a) > 0 \)); and

(iii) \( \beta(\cdot, \cdot) \in C((0, c), L^\infty(\mathbb{R}^n)) \cap L^{p'}((0, c), L^\infty(\mathbb{R}^n)) \) \((1/p' + 1/p = 1)\).
Note that the evolution family generated by \{A(a)\} is \(U(t, s) = e^{-\int_s^t \mu(r) \, dr} G(\int_s^t k(r) \, dr)\) \((t \geq s)\), where \(G(\cdot)\) is the Gaussian semigroup generated by \(\Delta_x\), i.e.,

\[
(G(t)f)(x) := (4\pi t)^{-n/2} \int_{\mathbb{R}^n} f(x-y) e^{-|y|^2/(4t)} \, dy, \quad f \in L^q(\mathbb{R}^n).
\]

Clearly, \(U(t, s)\) is norm continuous on \(0 \leq s < t\) but \(U(a, 0)\) is not compact. The condition on \(\beta(\cdot, \cdot)\) implies \(C(\cdot)\) is norm continuous (but in general it is not uniformly norm continuous and also in this case \(L\) may be not compact). Therefore, Theorem 6.12 (a) holds; however the result given in [BHM05, Section 5] cannot be applied (even for \(p = 1\)). Particularly, we have

\[
\omega_{\text{crit}}(T_{(A+L)_0}) = \omega_{\text{crit}}(T_0) \leq \omega(U) \leq -|\mu(\cdot)|_{L^\infty}.
\]

Next, we show in this case \(A\) usually is not a Hille-Yosida operator if \(p > 1\) and \(c = \infty\). For simplicity, let \(n = 1, \mu(\cdot) \equiv 1\) and \(k(\cdot) \equiv 1\). Take \(f_1(x) = e^{-|x|^2}\). Then, \(f_1 \in L^p(\mathbb{R})\) and for \(x > 0\),

\[
(U(a, 0)f_1)(x) = e^{-a(4\pi a)^{-1/2}} \int_{-\infty}^{\infty} e^{-|x-y|^2} e^{-|y|^2/(4a)} \, dy \geq c_0 e^{-a} (1 + 4a)^{-1/2},
\]

where \(c_0 > 0\) is a constant. So

\[
|(U(a, 0)f_1)(\cdot)|_{L^q} = \left( \int_{-\infty}^{\infty} ((U(a, 0)f_1)(x))^q \, dx \right)^{1/q} \geq c_1 e^{-a} (1 + 4a)^{-1/2} = h(a),
\]

where \(c_1 > 0\) is a constant, and in particular, \(\lim_{\lambda \to \infty} \lambda^p \int_0^\infty e^{-\lambda^a} h^p(a) \, da = \infty\) (in fact \(\lim_{\lambda \to \infty} \lambda^p \int_0^\infty e^{-\lambda^a} \, da = 1/p\)). Hence, by Remark 6.10, in this case \(A\) is not a Hille-Yosida operator.

(c) (See [Rha98] and [Thi98, Section 5] in \(L^1\) case.) Let \(E = L^q(\Omega)\) \((1 \leq q < \infty)\) where \(\Omega\) is a bounded domain of \(\mathbb{R}^m\) with smooth boundary \(\partial \Omega\). Let \(A(\cdot)\) (with suitable boundary condition e.g., Dirichlet, Neumann, Robin, etc) and \(C(\cdot)\) as be in (b) with \(\mathbb{R}^n\) in (i) (ii) (iii) replaced by \(\Omega\). In this context, \(\{A(a)\}\) also generates exponentially bounded linear evolution family \{\(U(a, s)\)\} \(0 \leq s \leq a < c\) (see, e.g., [Paz83]) and the embedding theorem gives that \(U(t, s)\) is compact on \(0 \leq s < t < c\) and particularly Theorem 6.12 (b) holds. More generally, \(\{A(a)\}\) can be (uniformly strongly) elliptic differential operators with suitable boundary condition (see, e.g., [Paz83, Section 7.6]). Due to the positive setting in [Thi98, Section 5], \(\{C(a)\}\) can be unbounded which Theorem 6.12 cannot cover; the proof given here is different from [Thi98, Rha98].

(d) (See [BHM05, Section 6] in \(L^1\) case.) Consider the following model:

\[
\begin{cases}
(\partial_t + \partial_\alpha)u(t, a, s) = -\mu(a)u(t, a, s) + k(a)q(s)u(t, a, s), & t > 0, a \in (0, c), s \in \Omega, \\
u(t, 0, s) = \int_0^s \beta(a, s)u(t, a, s) \, da, & t > 0, s \in \Omega, \\
u(0, a, s) = u_0(a, s), & a \in (0, c), s \in \Omega, u_0 \in L^p((0, c), L^q(\Omega)).
\end{cases}
\]

Let \(E = L^q(\Omega)\) \((1 \leq q < \infty)\) measure where \(\Omega\) is a domain of \(\mathbb{R}^n\). Take \(A(a) = -\mu(a)I + k(a)q(\cdot)\) and \(C(a) = \beta(a, \cdot)\) where \(\mu(\cdot), k(\cdot), \beta(\cdot, \cdot)\) are the same as (b) with \(\mathbb{R}^n\) in (i), (ii), (iii) replaced by \(\Omega\). The measurable function \(q(\cdot) : \Omega \to \mathbb{C}\) satisfies the following:

1. \(\sup_{\Omega} \{\Re \lambda : \lambda \in q(\Omega)\} < \infty\) and
2. \(\forall d \in \mathbb{R}, \{q(\Omega) \cap \{\lambda \in \mathbb{C} : \Re \lambda \geq d\}\}\) is bounded.

Under this circumstance, the evolution family generated by \(\{A(a)\}\) is \(U(t, s) = e^{-\int_s^t \mu(r) - k(r)q(\cdot) \, dr}\) \((t \geq s)\). Note that by (a2), \(U(t, s)\) is norm continuous on \(0 \leq s < t < c\) (see, e.g., [EN00, p. 121]) but in general is not compact. So Theorem 6.12 (a) holds.
(e) (See [Web08, Section 1.4] in $L^1$ case.) Consider the following model:

$$
\begin{align*}
(\partial_t + \partial_a + k(a)\partial_s)u(t, a, s) &= -\mu(a)u(t, a, s), \quad t > 0, a \in (0, c), s \in (0, s_2), \\
\int_0^c \int_0^{s_2} \beta(a, \hat{s}, s)u(t, a, \hat{s}) \, d\hat{s} \, da, \quad t > 0, s \in (0, s_2), \\
u(t, 0, s) = u_0(a, s), \quad a \in (0, c), s \in (0, s_2), \quad u_0 \in L^p((0, c), L^q((0, s_2))), \\
u(t, a, 0) &= 0, \quad t > 0, a \in (0, c).
\end{align*}
$$

Let $E = L^q((0, s_2))$ ($1 \leq q < \infty$) where $0 < s_2 < \infty$. Let $p'$, $q'$ satisfy $1/p' + 1/p = 1$ and $1/q' + 1/q = 1$. The measurable function $\beta(\cdot, \cdot, \cdot)$ satisfies

$$
\beta(\cdot, \cdot, \cdot) \in C((0, c), L^{q'-q}(0, s_2)^2) \cap L^p((0, c) \times (0, s_2)^2), \quad P = (q, q', p'),
$$

where $L^p((0, c) \times (0, s_2)^2)$ (similarly for $L^{q'-q}(0, s_2)^2$) is the mixed-norm Lebesgue space (see, e.g., [BP61]) defined by

$$
L^p((0, c) \times (0, s_2)^2) := \{ f : (0, c) \times (0, s_2)^2 \to \mathbb{R} \text{ is measurable : } \left( \int_0^c \left( \int_0^{s_2} \left( \int_0^{s_2} |f(a, \hat{s}, s)|^{q'} \, d\hat{s} \right)^{q'/q} \, ds \right)^{1/p'} < \infty \}.
$$

In addition, if one of $p, q$ equals 1, then we assume

$$
\begin{align*}
\lim_{h \to 0^+} \sup_{(a, \hat{s}) \in (0, c) \times (0, s_2)} \left( \int_0^{s_2-h} |\beta(a, \hat{s}, s + h) - \beta(a, \hat{s}, s)| \, ds \right) &= 0, \quad \text{if } p = q = 1, \\
\lim_{h \to 0^+} \sup_{a \in (0, c)} \left( \int_0^{s_2-h} |\beta(a, \hat{s}, s + h) - \beta(a, \hat{s}, s)| \, ds \right)^{q/p'} \, da &= 0, \quad \text{if } p > 1, q = 1,
\end{align*}
$$

where $c < \infty$, then a typical case such that $\beta(\cdot, \cdot, \cdot)$ satisfies (b1) (b2) for all $1 \leq p, q < \infty$ is $\beta(\cdot, \cdot, \cdot) \in C([0, c] \times [0, s_2]^2)$.

Take $A(a) = -\mu(a)I - k(a)\frac{d}{ds}$ where $\mu(\cdot)$, $k(\cdot)$ are the same as (b) with $\mathbb{R}^n$ in (i), (ii), (iii) replaced by $(0, s_2)$; $\frac{d}{ds}$ is the first-order differential operator on $L^q(0, s_2)$, i.e.,

$$
\frac{d}{ds} : f \mapsto \frac{df}{ds}, \quad D(\frac{d}{ds}) = \{ f \in W^1,q(0, s_2) : f(0) = 0 \}.
$$

Let $C(a) = \beta(a, \cdot, \cdot)$ be defined by

$$
(C(a)\phi)(s) = \int_0^{s_2} \beta(a, \hat{s}, s)\phi(\hat{s}) \, d\hat{s}, \quad \phi \in L^q(0, s_2);
$$

By Minkowski integral inequality, we see $C(a)$ indeed is a bounded linear operator on $L^q(0, s_2)$. Clearly, $\{ A(a) \}$ generates the exponentially bounded linear evolution family $\{ U(a, s) \}_{0 \leq s \leq a < c}$ defined by $U(t, s) = e^{-\int_s^t \mu(r) \, dr} T_r (\int_s^t k(a) \, da)$, where $T_r(t)$ is the right translation on $L^q(0, s_2)$, i.e.,

$$
(T_r(t)\phi)(s) := \begin{cases} 
\phi(s - t), & s - t \geq 0, \\
0, & s - t < 0.
\end{cases}
$$

The condition on $\beta(\cdot, \cdot, \cdot)$ implies that $L$ is compact, and so in this case Theorem 6.12 (c) holds and especially, we have

$$
\omega_{\text{ess}}(T(\mathcal{A} + L_0)) = \omega_{\text{ess}}(T_0) \leq \omega(U) \leq -|\mu(\cdot)|_{L^\infty}.
$$

Notice also that $U(a, 0)$ is not compact for all $a > 0$. 

Proof. To show $L$ is compact, one can use the classical Kolmogorov theorem on the characterization of the relatively compact subsets in $L^q(0, s_2)$ which is standard; the details are as follows. We need to show $\{L\varphi : \varphi \in L^p((0, c), L^q(0, s_2))\}$ such that $|\varphi| \leq 1$ is relatively compact in $L^q(0, s_2)$, or equivalently,

$$\lim_{h \to 0^+} \int_0^{s_2-h} |(L\varphi)(s + h) - (L\varphi)(s)|^q \, ds = 0, \text{ uniformly for } |\varphi| \leq 1.$$ 

For $\varphi \in L^p((0, c), L^q(0, s_2)) = L^{(q,p)}((0, c) \times (0, s_2))$ such that $|\varphi| \leq 1$, i.e.,

$$\left( \int_0^c \left( \int_0^{s_2} |\varphi(a, \hat{s})|^q \, d\hat{s} \right)^{p/q} \, da \right)^{1/p} \leq 1,$$

using Minkowski integral inequality and Hölder inequality, we get

$$\int_0^{s_2-h} |(L\varphi)(s + h) - (L\varphi)(s)|^q \, ds$$

$$= \int_0^{s_2-h} \left| \int_0^c \int_0^{s_2} \{\beta(a, \hat{s}, s + h) - \beta(a, \hat{s}, s)\} \varphi(a, \hat{s}) \, d\hat{s} \, da \right|^q \, ds$$

$$\leq \int_0^c \int_0^{s_2} \left( \int_0^{s_2-h} |\beta(a, \hat{s}, s + h) - \beta(a, \hat{s}, s)|^q \, ds \right)^{1/q} |\varphi(a, \hat{s})| \, d\hat{s} \, da$$

$$\leq \int_0^c \left( \int_0^{s_2} \left( \int_0^{s_2-h} |\beta(a, \hat{s}, s + h) - \beta(a, \hat{s}, s)|^q \, ds \right)^{q'/q} \, d\hat{s} \right)^{1/q'} \left( \int_0^{s_2} |\varphi(a, \hat{s})|^q \, d\hat{s} \right)^{1/q} \, da$$

$$\leq \left( \int_0^c \left( \int_0^{s_2} \left( \int_0^{s_2-h} |\beta(a, \hat{s}, s + h) - \beta(a, \hat{s}, s)|^q \, ds \right)^{q'/q} \, d\hat{s} \right)^{1/q'} \, da \right)^{1/p'} \varphi(h).$$

Now we have $\rho(h) \to 0$ as $h \to 0^+$. Indeed, if $p \neq 1$ and $q \neq 1$, then by a standard argument we can obtain this (see, e.g., [BP61, Section 10 Theorem 1]); if one of $p, q$ equals 1, then this is condition (b2). The proof is complete. \hfill \Box

Remark 6.14. In Example 6.13 (b), (c), (d), if $p = 1$ and $\beta(\cdot, \cdot) \in BUC([0, c], L^\infty(\Omega))$ ($\Omega = \mathbb{R}^n$ or a domain of $\mathbb{R}^n$) as [Rha98, BHM05], then Theorem C can be applied directly. A more general model than Example 6.13 (e) has been considered in [Thi98, Section 6].

7. Comments

7.1. Unbounded perturbation. The unbounded perturbation theorem for MR operators (quasi Hille-Yosida operators) is few; see [TV09, Section 2] for certain unbounded perturbations of Hille-Yosida operators. It’s still a difficult task. For the case of the generators of integrated semigroups and almost sectorial operators, we refer the readers to see [ABHN11, KW03, Thi08] and the references therein.

Here, we give an unbounded perturbation theorem for MR operators, which is essentially due to Arendt et al. [ABHN11].

Theorem 7.1 ([ABHN11, Theorem 3.5.7]). Assume $A$ is an operator on a Banach space $X$ such that $(\omega, \infty) \subset \rho(A)$, $\lim_{\lambda \to \infty} \|R(\lambda, A)\| = 0$, and $B \in \mathcal{L}(D(A), D(A))$ ($D(A)$ is endowed with the graph norm). Then, there is $C \in \mathcal{L}(X)$ such that $A + B$ is similar to $A + C$.

Proof. The proof is essentially the same as [ABHN11, Theorem 3.5.7]. In fact, the condition $\sup \|AR(\lambda, A)\| < \infty$ in [ABHN11, Theorem 3.5.7] can be weakened by $\lim_{\lambda \to \infty} \|R(\lambda, A)\| = 0$, because we only need $I - BR(\lambda, A)$ is invertible for sufficiently large $\lambda$. We repeat the proof as follows for the sake of readers.
Take $\lambda_0 > \omega$. Since $(\lambda_0 - A)BR(\lambda_0, A) \in \mathcal{L}(X, X)$, we have $I - (\lambda_0 - A)BR(\lambda, A)R(\lambda_0, A) = I - (\lambda_0 - A)BR(\lambda_0, A)R(\lambda, A)$ is invertible for large $\lambda$ as $\limsup_{\lambda \to \infty} \|R(\lambda, A)\| = 0$. Now $I - BR(\lambda, A) = I - R(\lambda_0, A)(\lambda_0 - A)BR(\lambda, A)$ is invertible (here, we use the fact: for $U, V \in \mathcal{L}(X, X)$, $I - UV$ is invertible if and only if $I - VU$ is invertible).

Take $C := (\lambda - A)BR(\lambda, A)$ and $U := I - BR(\lambda, A)$ for sufficiently large $\lambda$. Then, as shown above $U$ is invertible (and $UD(A) = D(A)$). It is easy to verify that $U(A + C)U^{-1} = A + B$ (see also the proof [ABHN11, Theorem 3.5.7]). The proof is complete. $\square$

Since the condition $\limsup_{\lambda \to \infty} \|R(\lambda, A)\| = 0$ can be satisfied by MR operators (see Lemma 2.6 (b)), we obtain the following result.

**Corollary 7.2.** If $A$ is an MR operator (resp. $p$-quasi Hille-Yosida operator) and $B \in \mathcal{L}(D(A), D(A))$, so is $A + B$.

It’s possible for us to extend the results in sections 3-5 for this unbounded perturbation. We take $L = (\lambda - A)BR(\lambda, A)$. Here are some examples.

**Corollary 7.3.** Let $A$ be an MR operator, and $B \in \mathcal{L}(D(A), D(A))$.

(a) If there exists $\lambda_0 \in \rho(A)$ such that $(\lambda_0 - A)BR(\lambda_0, A)T_{A_0}$ is norm continuous (resp. norm continuous and compact) on $(0, \infty)$, then $\omega_{\text{crit}}(T_{A+B_0}) = \omega_{\text{crit}}(T_{A_0})$ (resp. $\omega_{\text{ess}}(T_{A+B_0}) = \omega_{\text{ess}}(T_{A_0})$).

(b) If $A$ is also a quasi Hille-Yosida operator and there is $\lambda_0 \in \rho(A)$ such that $(\lambda_0 - A)BR(\lambda_0, A_0)T_{A_0}$ is compact on $(0, \infty)$, then $\omega_{\text{ess}}(T_{A+B_0}) = \omega_{\text{ess}}(T_{A_0})$.

**Proof.** For (a) (b), it suffices to show if $(\lambda_0 - A)BR(\lambda_0, A_0)T_{A_0}$ is norm continuous (resp. compact) on $(0, \infty)$, so is $(\lambda - A)BR(\lambda, A_0)T_{A_0}$ for all $\lambda \in \rho(A)$. Note that

$$R(\lambda, A_0)T_{A_0} = T_{A_0}R(\lambda, A_0),$$

and

$$BR(\lambda_0, A)T_{A_0} = R(\lambda_0, A)(\lambda_0 - A)BR(\lambda_0, A)T_{A_0}$$

has the same regularity as $(\lambda_0 - A)BR(\lambda_0, A)T_{A_0}$. By

$$(\lambda - A)BT_{A_0}R(\lambda, A_0) = (\lambda - \lambda_0 + \lambda_0 - A)BT_{A_0}[R(\lambda_0, A_0) + (\lambda_0 - \lambda)R(\lambda_0, A_0)R(\lambda, A_0)]$$

$$= [(\lambda - \lambda_0)BT_{A_0}R(\lambda_0, A_0) + (\lambda_0 - \lambda)BT_{A_0}R(\lambda_0, A_0)][I + (\lambda_0 - \lambda)R(\lambda, A)],$$

we obtain the results. $\square$

### 7.2. Cauchy problems.

Our consideration is relevant to the following semilinear Cauchy problem:

(7.1) \[
\begin{align*}
\dot{u}(t) &= Au(t) + f(u(t)), \\
u(0) &= x,
\end{align*}
\]

where $A : D(A) \subset X \to X$ is an MR operator and $f : \overline{D(A)} \to X$ is $C^1$ and globally Lipschitz (for simplicity in order to avoid blowup). Many differential equations such as age-structured population models, parabolic differential equations, delay equations, and Cauchy problems with boundary conditions can be reformulated as Cauchy problems. However, in many cases, the operator $A$ is not densely defined or even not a Hille-Yosida operator, see, e.g., [DPS87, PS02, DMP10, MR18].

Assume zero is an equilibrium of (7.1) (i.e., $f(0) = 0$). Let $L := Df(0) \in \mathcal{L}(\overline{D(A)}, X)$. Consider the linearized equation of (7.1), i.e.,

(7.2) \[
\begin{align*}
\dot{u}(t) &= Au(t) + Lu(t), \\
u(0) &= 0.
\end{align*}
\]

It was shown in [MR09b] that solutions of (7.2) can reflect the properties of solutions of (7.1) in the neighborhood of 0. In general, the properties of $A$ (or $T_{A_0}$, $S_{A_0}$) would be known well. What we
need is the properties of $T_{A+L\lambda}$. Our results can be applied to this situation. For example, if $T_{(A+L\lambda)^n}$ is exponentially stable (i.e., $\omega(T_{(A+L\lambda)^n}) < 0$ (see (2.17))), then the zero solution of (7.1) is locally stable [MR09b, Proposition 7.1]. If $\omega_{\text{crit}}(T_{A\lambda}) < 0$ and $\omega_{\text{crit}}(T_{(A+L\lambda)^n}) \leq \omega_{\text{crit}}(T_{A\lambda})$, then $\omega_{\text{crit}}(T_{A\lambda}) < 0$. Therefore, all we need to consider is: $s(A + L) < 0$ (see (2.17))? (see Theorem 2.19 (c), and note that $s(A + L) = s(A + L\lambda)$. That is, the spectrum of $A + L$ can reflect the stability of the zero solution.

**Theorem 7.4.** (a) Assume the condition of Theorem 5.6 (a) (or Theorem 5.8 (a), or Corollary 5.9 (a)) is satisfied and $\omega_{\text{crit}}(T_{A\lambda}) < 0$, then the zero solution of (7.1) is locally exponentially stable, provided $s(A + L) < 0$.

(b) Assume the condition of Theorem 5.6 (b) (or Theorem 5.8 (b), or Corollary 5.9 (b), or Corollary 5.10) is satisfied and $\omega_{\text{ess}}(T_{A\lambda}) < 0$. Then, the zero solution of (7.1) is locally exponentially stable if $s(A + L) < 0$ and unstable if $s(A + L) > 0$.

**Proof.** (a) This follows from Theorem 2.19 (a) and [MR09b, Proposition 7.1].

(b) This follows from Theorem 2.21 (a) and [MR09b, Proposition 7.1, Proposition 7.4].

The existence of the Hopf bifurcation and the center manifold (of an equilibrium) for Cauchy problems needs the condition $\omega_{\text{ess}}(T_{(A+L\lambda)^n}) < 0$, see, e.g., [MR09a]; this condition was replaced by the exponential dichotomy condition instead in our paper [Che19] to give the invariant manifold theory around more general manifolds (in the sense of Hirsch, Pugh and Shub, and Fenichel). A more concrete application of the results in Section 5 and Section 4 to a class of delay equations with non-dense domains, see [Che18], which is very similar as [BMR02].

**References**

[ABHN11] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, 2nd ed., Monographs in Mathematics, vol. 96, Birkhäuser/Springer Basel AG, Basel, 2011. MR2798103 (2012b:47099)

[AMM91] F. Andreu, J. Martínez, and J. M. Mazón, *A spectral mapping theorem for perturbed strongly continuous semigroups*, Math. Ann. 291 (1991), no. 3, 453–462. MR1133342

[Are87] W. Arendt, *Resolvent positive operators*, Proc. London Math. Soc. (3) 54 (1987), no. 2, 321–349. MR872810 (88c:47074)

[BHM05] S. Boulite, S. Hadd, and L. Maniar, *Critical spectrum and stability for population equations with diffusion in unbounded domains*, Discrete Contin. Dyn. Syst. Ser. B 5 (2005), no. 2, 265–276. MR2129377

[BK10] C. J. K. Batty and S. Krol, *Perturbations of generators of \(C_0\)-semigroups and resolvent decay*, J. Math. Anal. Appl. 367 (2010), no. 2, 434–443. MR2607270

[Bla01] M. D. Blake, *A spectral bound for asymptotically norm-continuous semigroups*, J. Operator Theory 45 (2001), no. 1, 111–130. MR1823064

[BMR02] A. Bátkai, L. Maniar, and A. Rhandi, *Regularity properties of perturbed Hille-Yosida operators and retarded differential equations*, Semigroup Forum 64 (2002), no. 1, 55–70. MR1866316

[BNP00] S. Brendle, R. Nagel, and J. Poland, *On the spectral mapping theorem for perturbed strongly continuous semigroups*, Arch. Math. (Basel) 74 (2000), no. 5, 365–378. MR1753014

[BP01] A. Bátkai and S. Piazzera, *Semigroups and linear partial differential equations with delay*, J. Math. Anal. Appl. 264 (2001), no. 1, 1–20. MR1868323 (2003h:34161)

[BP61] A. Benedek and R. Panzone, *The space \(L^p\), with mixed norm*, Duke Math. J. 28 (1961), 301–324. MR0126155

[Bre01] S. Brendle, *On the asymptotic behavior of perturbed strongly continuous semigroups*, Math. Nachr. 226 (2001), 35–47. MR1839404

[CDN08] A. N. Carvalho, T. Dlotko, and M. J. D. Nascimento, *Non-autonomous semilinear evolution equations with almost sectorial operators*, J. Evol. Equ. 8 (2008), no. 4, 631–659. MR2460932 (2010a:47081)

[Che18] D. Chen, *A linear theory for a class of delay equations with non-dense domains*, 2018. in preparation.

[Che19] D. Chen, *The exponential dichotomy and invariant manifolds for some classes of differential equations*, arXiv e-prints (Mar. 2019), available at arXiv:1903.08040.

[CL99] C. Chicone and Y. Latushkin, *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, vol. 70, American Mathematical Society, Providence, RI, 1999. MR1707332

[CMP68] M. G. Crandall and A. Pazy, *On the differentiability of weak solutions of a differential equation in Banach space*, J. Math. Mech. 18 (1968/1969), 1007–1016. MR0242014 (39 #3349)

[CZ95] R. F. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*, Texts in Applied Mathematics, vol. 21, Springer-Verlag, New York, 1995. MR1351248
[Dei85] K. Deimling, Nonlinear functional analysis, Springer-Verlag, Berlin, 1985. MR787404 (86j:47001)

[DLM08] A. Ducrot, Z. Liu, and P. Magal, Essential growth rate for bounded linear perturbation of non-densely defined Cauchy problems, J. Math. Anal. Appl. 341 (2008), no. 1, 501–518. MR2394101

[DMP10] A. Ducrot, P. Magal, and K. Prevost, Integrated semigroups and parabolic equations. Part I: linear perturbation of almost sectorial operators, J. Evol. Equ. 10 (2010), no. 2, 263–291. MR2634797

[DPS87] G. Da Prato and E. Sinestrari, Differential operators with nondense domain, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 14 (1987), no. 2, 285–344 (1988). MR93631 (89f:47062)

[EN00] K.-J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR1721989 (2000i:47075)

[GGK90] I. Gohberg, S. Goldberg, and M. A. Kaashoek, Classes of linear operators. Vol. I, Operator Theory: Advances and Applications, vol. 49, Birkhäuser Verlag, Basel, 1990. MR1130394 (93d:47002)

[Ile07] P. S. Iley, Perturbations of differentiable semigroups, J. Evol. Equ. 7 (2007), no. 4, 765–781. MR2369679 (2008k:47092)

[KH89] H. Kellerman and M. Hieber, Integrated semigroups, J. Funct. Anal. 84 (1989), no. 1, 160–180. MR999494 (90h:47072)

[KWo03] C. Kaiser and L. Weis, Perturbation theorems for a-times integrated semigroups, Arch. Math. (Basel) 81 (2003), no. 2, 215–228. MR2009564 (2004i:47083)

[Mat08] T. Mátrai, On perturbations preserving the immediate norm continuity of semigroups, J. Math. Anal. Appl. 341 (2008), no. 2, 961–974. MR2398262 (2009c:47066)

[Mon14] G. A. Monteiro, On functions of bounded semivariation, Real Anal. Exchange 40 (2014/15), no. 2, 233–276. MR3499764

[MR07] P. Magal and S. Ruan, On integrated semigroups and age structured models in $L^p$ spaces, Differential Integral Equations 20 (2007), no. 2, 197–239. MR2294465

[MR09a] P. Magal and S. Ruan, Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models, Mem. Amer. Math. Soc. 202 (2009), no. 951, vi+71. MR2559965

[MR09b] P. Magal and S. Ruan, On semilinear Cauchy problems with non-dense domain, Adv. Differential Equations 14 (2009), no. 11-12, 1041–1084. MR2560868

[MR18] P. Magal and S. Ruan, Theory and applications of abstract semilinear Cauchy problems, Applied Mathematical Sciences, vol. 201, Springer, Cham, 2018. With a foreword by Glenn Webb. MR3887640

[MS16] S. Mischler and J. Scher, Spectral analysis of semigroups and growth-fragmentation equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2016), no. 3, 849–898. MR3489637

[NP98] R. Nagel and J. Poland, The critical spectrum of a strongly continuous semigroup, Adv. Math. 152 (2000), no. 1, 120–133. MR1762122 (2001e:47073)

[NP99] R. Nagel and S. Piazzera, On the regularity properties of perturbed semigroups, Rend. Circ. Mat. Palermo (2) Suppl. 56 (1998), 99–110. International Workshop on Operator Theory (Cefalù, 1997). MR1710826 (2001h:47059)

[Paz88] A. Pazy, On the differentiability and compactness of semigroups of linear operators, J. Math. Mech. 17 (1968), 1131–1141. MR0231242

[Paz83] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. MR710486

[Pi99] S. Piazzera, Qualitative properties of perturbed semigroups, Ph.D. Thesis, 1999.

[PS02] F. Periago and B. Straub, A functional calculus for almost sectorial operators and applications to abstract evolution equations, J. Evol. Equ. 2 (2002), no. 1, 41–68. MR1890881 (2003a:47040)

[Ren55] M. Renardy, On the stability of differentiability of semigroups, Semigroup Forum 51 (1995), no. 3, 343–346. MR1351960

[Rha98] A. Rhandi, Positivity and stability for a population equation with diffusion on $L^1$, Positivity 2 (1998), no. 2, 101–113. MR1656866

[RSV00] F. Räbiger, R. Schnaubelt, A. Rhandi, and J. Voigt, Non-autonomous Miyadera perturbations, Differential Integral Equations 13 (2000), no. 1-3, 341–368. MR1811962

[Sbi07] M. Shihi, A resolvent approach to the stability of essential and critical spectra of perturbed $C_0$-semigroups on Hilbert spaces with applications to transport theory, J. Evol. Equ. 7 (2007), no. 1, 35–58. MR2305725

[Sch02] R. Schnaubelt, Well-posedness and asymptotic behaviour of non-autonomous linear evolution equations, Evolution equations, semigroups and functional analysis (Milano, 2000), 2002, pp. 311–338. MR1944170

[Sch96] R. Schnaubelt, Exponential bounds and hyperbolicity of evolution families, Ph.D. Thesis, 1996.

[Thi08] H. R. Thieme, Differentiability of convolutions, integrated semigroups of bounded semi-variation, and the inhomogeneous Cauchy problem, J. Evol. Equ. 8 (2008), no. 2, 283–305. MR2407203

[Thi91] H. R. Thieme, Analysis of age-structured population models with an additional structure, Mathematical population dynamics (New Brunswick, NJ, 1989), 1991, pp. 115–126. MR1227358
[Thi97] H. R. Thieme, *Quasi-compact semigroups via bounded perturbation*, Advances in mathematical population dynamics—molecules, cells and man (Houston, TX, 1995), 1997, pp. 691–711. MR1634223 (99i:47070)

[Thi98] H. R. Thieme, *Positive perturbation of operator semigroups: growth bounds, essential compactness, and asynchronous exponential growth*, Discrete Contin. Dynam. Systems 4 (1998), no. 4, 735–764. MR1641201

[TV09] H. R. Thieme and J. Voigt, *Relatively bounded extensions of generator perturbations*, Rocky Mountain J. Math. 39 (2009), no. 3, 947–969. MR2505783

[Voi80] J. Voigt, *A perturbation theorem for the essential spectral radius of strongly continuous semigroups*, Monatsh. Math. 90 (1980), no. 2, 153–161. MR595321

[Web08] G. F. Webb, *Population models structured by age, size, and spatial position*, Structured population models in biology and epidemiology, 2008, pp. 1–49. MR2433574

[Web85] G. F. Webb, *Theory of nonlinear age-dependent population dynamics*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 89, Marcel Dekker, Inc., New York, 1985. MR772205

Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, People’s Republic of China

E-mail address: chernde@sjtu.edu.cn