STEINBERG GROUPS AS AMALGAMS

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Abstract. For any root system and any commutative ring we give a relatively simple presentation of a group related to its Steinberg group $\mathcal{G}t$. For classical root systems and many others, it equals $\mathcal{G}t$. When this holds, it establishes a Curtis-Tits style presentation: $\mathcal{G}t$ is the direct limit of the Steinberg groups coming from 1- and 2-node subdiagrams of the Dynkin diagram. Over non-fields this is new even for classical root systems. Our presentation is concrete, with no implicit coefficients or signs. It is defined in terms of the Dynkin diagram rather than the full root system. It allows one to show that many Kac-Moody groups over finitely generated rings are finitely presented. And it makes visible the exceptional diagram automorphisms in characteristics 2 and 3. Some of these results are new even over fields. The details for some of these applications are deferred to later papers.

1. Introduction

In this paper we give a presentation for a Steinberg-like group, for any classical or non-classical root system and any commutative ring $R$. For many root systems, including all classical ones (those with finite Weyl group), it is the same as the Steinberg group $\mathcal{G}t$. This is the case of interest, for then it gives a new presentation of $\mathcal{G}t$;

(i) it is concrete, with no coefficients or signs left implicit;
(ii) it is defined in terms of the Dynkin diagram rather than the full root system;
(iii) it generalizes the Curtis-Tits presentation, without requiring $R$ to be a field;
(iv) it is finite when $R$ is finitely generated as an abelian group;
(v) it is often rewritable as a finite presentation when $R$ is merely finitely generated as a ring;
(vi) it makes visible the exceptional diagram automorphisms that lead to the Suzuki and Ree groups.

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all $i$ $\ i \neq j \ $unjoined $\ i \neq j \ $joined

\[ S_i S_j = S_j S_i \quad S_i S_j S_i = S_j S_i S_j \]
\[ S_i^2 S_j S_i^{-2} = S_j^{-1} \]

\[
\begin{array}{ll}
[S_i^2, X_i] = 1 & [S_i, X_j] = 1 \\
X_i S_j S_i = S_j S_i X_j & S_i^2 X_j S_i^{-2} = X_j^{-1} \\
S_i = X_i S_i X_i S_i^{-1} X_i \\
[X_i, X_j] = 1 & [X_i, X_j] = S_i X_j S_i^{-1}
\end{array}
\]

\[ [X_i, S_i X_j S_i^{-1}] = 1 \]

Table 1. Presentation of the Steinberg groups over $\mathbb{Z}$ of types $A_{n>1}$, $D_{n>2}$, and $E_n = 6, 7, 8$, on generators $S_i$ and $X_i$, with $i$ varying over the nodes of the Dynkin diagram. See the text for simplifications and generalizations, and for how the batches of relations correspond to the stages of our construction.

To illustrate, we give the $ADE$ presentations over $\mathbb{Z}$ in table 1; one can even omit most of the third and fourth batches of relations. The presentations with a commutative ring $R$ in place of $\mathbb{Z}$ replace $X_i$ by $X_i(t)$ with $t$ varying over $R$. When the Dynkin diagram has double or triple bonds then the presentation is more complicated but has the same general form.

Using work of Splitthoff [15], we will show in [4] and [5] that the Steinberg groups for most affine and some hyperbolic root systems are finitely presented over any finitely generated ring $R$. If $R$ also satisfies some mild $K$-theoretic conditions then the associated Kac-Moody groups are also finitely presented. In the special case $R = \mathbb{Z}$, the affine Steinberg groups $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$ and some hyperbolic groups are presented just as in table 1. This includes the group $E_{10}(\mathbb{Z})$, of interest in physics.

In [3] we will construct Kac-Moody-like groups over fields of characteristics 2 and 3 which generalize the Suzuki and Ree groups. In the current paper our main application is theorem 1 below, a generalization of the Curtis-Tits presentation from fields to rings.

Like the Steinberg group, our “group” is a functor from rings to groups and is defined in terms of a generalized Cartan matrix $A$. Tits defined his version of the Steinberg group by starting with the free product of one copy of the additive group $R$ for each real root. Non-real roots only occur for non-classical root systems, and even then they play no role in our work. So we follow the convention of Tits’ paper
that “root” means “real root”. Whenever two roots $r, s$ form a “prenilpotent pair”, Tits imposes relations involving the corresponding root groups. These relations are the ones discovered by Chevalley if $r, s$ are “classically prenilpotent”, defined below, and otherwise they are relations of a similar but more general form due to Tits.

There is another version of the Steinberg group in the literature, due to Morita and Rehmann [13]. They adjoin additional relations to get a better-behaved group; in particular they recover Steinberg’s original group in the $A_1$ case. So we take theirs as the official definition of the Steinberg group $\mathfrak{St} = \mathfrak{St}_{A_1}$. But the definitions coincide in most interesting cases; see step 4 of section 4 for the differences.

To define our group $\mathfrak{St}^{cl} = \mathfrak{St}^{cl}_{A_1}$ we mimic Tits’ presentation, but impose only the relations coming from classically prenilpotent pairs of roots, and then impose the additional Morita-Rehmann relations. We call a pair of roots $\alpha, \beta$ classically prenilpotent if they are linearly independent and $\mathbb{Q}\alpha \oplus \mathbb{Q}\beta$ contains only finitely many roots. The second condition is equivalent to $\alpha, \beta$ lying in some $A_2$, $A_2$, $B_2$ or $G_2$ root system.

We call $\mathfrak{St}^{cl}$ the “classical Steinberg group” to reflect the fact that we only use the classical Chevalley relations. This term already has a meaning: $\mathfrak{St}$ itself in the classical case that $A$ is a Cartan matrix rather than a generalized Cartan matrix. In this case all prenilpotent pairs are classically prenilpotent. So $\mathfrak{St}^{cl} = \mathfrak{St}$ and our meaning generalizes rather than conflicts with the older meaning.

Our description of $\mathfrak{St}^{cl}$ makes visible its relation to $\mathfrak{St}$, but lacks the virtues enumerated above. The main purpose of the paper is to write down a fairly simple presentation for $\mathfrak{St}^{cl}$, in section 4. Our main result is theorem [4] that it indeed describes $\mathfrak{St}^{cl}$. If the natural map $\mathfrak{St}^{cl} \to \mathfrak{St}$ is an isomorphism then we get a presentation of $\mathfrak{St}$. One can establish this isomorphism for many $A$ beyond the classical ones [3][5], but it does not hold universally [10, rk. 2]. There is no reason to expect $\mathfrak{St}^{cl}$ to be interesting when $\mathfrak{St}^{cl} \to \mathfrak{St}$ is not an isomorphism.

By inspecting table 1 one observes that each relation involves only one or two nodes of the Dynkin diagram. This establishes the following theorem in the $ADE$ cases. The proof in the general case is the same, once the presentation for $\mathfrak{St}^{cl}$ is established. For classical root systems one can of course replace $\mathfrak{St}^{cl}_{A_1}$ by $\mathfrak{St}_{A_1}$ in the statement. It is surprising that such a simple and natural property of the Steinberg groups of classical root systems remained undiscovered.

**Theorem 1** (Curtis-Tits presentation for Steinberg groups). *Let $A$ be a generalized Cartan matrix and $R$ a commutative ring. Consider the*
groups $\mathcal{Gt}_B^c(R)$ and the obvious maps between them, as $B$ varies over the $1 \times 1$ and $2 \times 2$ submatrices of $A$ coming from singletons and pairs of nodes of the Dynkin diagram. The direct limit of this family of groups is $\mathcal{Gt}_A^c(R)$.

The Curtis-Tits presentation is usually discussed only when $R$ is a field, because it is formulated in terms of buildings. See [16] for the case of spherical buildings and [1][14][10] for generalizations to the case of 2-spherical twin buildings. One result from [10] is that if $A$ is 2-spherical and $R$ is a field (with a few tiny exceptions) then the theorem above holds with $\mathcal{Gt}$ in place if $\mathcal{St}$. This meshes nicely with our results, because it implies $\mathcal{Gt}^c \cong \mathcal{Gt}$, so our concrete presentation in terms of the Dynkin diagram becomes available. For example, for any simply-laced root system, a presentation of $\mathcal{Gt}$ over any field is got from table 1 by replacing $X_i$’s by $X_i(t)$’s in a simple way. And the non-simply-laced case is not much more complicated.

We now explain the four batches of relations in table 1. We begin with the free group on generators $S_i$ and $X_i$, with $i$ varying over the set $I$ of nodes of the Dynkin diagram. Imposing the first batch of relations reduces the group generated by the $S_i$ to a group we call $\hat{W}$. One can think of this as a sort of Weyl group of the Steinberg group; it maps to the usual Weyl group $W$ with kernel a class 2 nilpotent group. (There is a well-known extension of the Weyl group $W$ by an elementary abelian 2-group, sometimes called the Tits-Dieudonné group. Our $\hat{W}$ is a central extension of this.) Imposing the first-batch relations yields $\left(\ast_{i \in I} \mathbb{Z}\right) \rtimes \hat{W}$.

The second set of relations is our main contribution: imposing them gives $\left(\ast_{\alpha \in \Phi} \mathbb{Z}\right) \rtimes \hat{W}$. Here $\Phi$ is the set of all roots, $\hat{W}$ permutes the factors of the free product via $W$’s action on $\Phi$, and the stabilizer of a factor acts on it by $\pm 1$. It is surprising that this semidirect product has such a simple presentation. We regard it as saying that the Curtis-Tits presentation holds not only for $\mathcal{Gt}^c$ but even for $\left(\ast_{\alpha \in \Phi} \mathbb{Z}\right) \rtimes \hat{W}$. The key tool in the proof is Brink’s theorem on reflection centralizers in Coxeter groups [7]. This part of our construction could be generalized easily to any group with a root group datum (see [11] for a definition).

Recall that Tits began with $\ast_{\alpha \in \Phi} \mathbb{Z}$ when defining the Steinberg group. Our third batch of relations are the Chevalley relations for a few specific classically prenilpotent pairs of roots. Because of the $\hat{W}$ factor in $\left(\ast_{\alpha \in \Phi} \mathbb{Z}\right) \rtimes \hat{W}$, imposing these relations imposes all their $\hat{W}$-conjugates. These turn out to be all of the classical Chevalley relations.
And in the cases in the table, the Morita-Rehmann relations follow automatically. So after the third batch of relations we have $\mathfrak{St}^cl \rtimes \hat{W}$. (The $\hat{W}$ action also allows one to economize by omitting most of these relations; see theorem 24.) We can avoid the notorious problem of signs in the structure constants because we are “letting $\hat{W}$ keep track of the signs for us”. That this is possible in principle has been clear since 20, but achieving it seems to be new.

The fourth batch of relations eliminates the factor $\hat{W}$ by folding it into $\mathfrak{St}^cl$. One can also economize here by imposing just one of these relations; see the remark after theorem 14.

In section 2 we recall some notation and language from Tits 18. In section 3 we recall from Tits 18 20 the group $W^*$ of automorphisms of the complex Kac-Moody algebra $\mathfrak{g}$. It is an extension of the Weyl group $W$ by an elementary abelian 2-group. We also refine Brink’s results 7 on reflection centralizers in Coxeter groups, to describe the $W^*$-stabilizer of a root space and its action on that root space. This is essential for our whole approach. In section 4 we review the definition of $\mathfrak{St}$, define $\mathfrak{St}^cl$ precisely, give our presentation, and prove that it defines $\mathfrak{St}^cl$. Some proofs are deferred to sections 5 7.

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2. The Kac-Moody algebra

All group actions are on the left. We will use the following general notation.

$\langle ., \rangle$ some bilinear pairing
$\langle \ldots \rangle$ a group generated by the elements enclosed
$\langle \ldots | \ldots \rangle$ a group presentation
$[x,y]$ $xyx^{-1}y^{-1}$ if $x$ and $y$ are group elements
* free product of groups (possibly with amalgamation)

The Steinberg group is built from a generalized Cartan matrix $A$:

$I$ an index set (the nodes of Dynkin diagram)
i, j will always indicate elements of $I$

$A=(A_{ij})$ the generalized Cartan matrix, i.e., an integer matrix satisfying $A_{ii} = 2$, $A_{ij} \leq 0$ if $i \neq j$, and $A_{ij} = 0 \iff A_{ji} = 0$

$m_{ij}$ numerical edge labels of the Dynkin diagram: $m_{ii} = 1$ and when $i \neq j$, $m_{ij} = 2, 3, 4, 6$ or $\infty$ according to whether $A_{ij}A_{ji} = 0, 1, 2, 3$ or $\geq 4$.
W the Coxeter group \(<s_{i \in I} \mid (s_is_j)^{m_{ij}} = 1 \text{ if } m_{ij} \neq \infty>\)
\[Z^I\] the free abelian group with basis \(\alpha_{i \in I}\) (the simple roots).
\(W\) acts on \(Z^I\) by \(s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i\). This action is faithful
by the theory of the Tits cone [9, V§4.4].
\(\Phi\) the set of \(w\alpha_i\) with \(w \in W\) and \(i \in I\) (the real roots)
The Kac-Moody algebra \(\mathfrak{g} = \mathfrak{g}_A\) associated to \(A\) means the complex
Lie algebra with generators \(e_i, f_i, h_i \in \mathfrak{g}\) and defining relations
\[
[\bar{h}_i, e_j] = A_{ij}e_j, \quad [\bar{h}_i, f_i] = -A_{ij}f_j, \quad [\bar{h}_i, \bar{h}_j] = 0, \quad [e_i, f_i] = -\bar{h}_i,
\]
for \(i \neq j\): \([e_i, f_j] = 0\), \((\text{ad } e_i)^{1-A_{ij}}(e_j) = (\text{ad } f_i)^{1-A_{ij}}(f_j) = 0\).
(Note: \((\text{ad } x)(y)\) means \([x, y]\). Also, Tits’ generators differ from Kac’s [12] by a sign on \(f_i\).) For any \(i\) the linear span of \(e_i, f_i\) and \(\bar{h}_i\) is
isomorphic to \(\mathfrak{sl}_2\mathbb{C}\), via
\[
(1) \quad e_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_i = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \bar{h}_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
We equip \(\mathfrak{g}\) with a grading by \(Z^I\), with \(\bar{h}_i \in \mathfrak{g}_0, e_i \in \mathfrak{g}_1, f_i \in \mathfrak{g}_{-1}\).
For \(\alpha \in Z^I\) we refer to \(\mathfrak{g}_\alpha\) as its root space, and abbreviate \(\mathfrak{g}_{\alpha_i}\) to \(\mathfrak{g}_i\).
As mentioned in the introduction, we follow Tits [13] in saying “root”
(meaning an element of \(\Phi\)) for “real root”.

3. THE EXTENSION \(W^* \subseteq \text{Aut } \mathfrak{g}\) OF THE WEYL GROUP

The Weyl group \(W\) does not necessarily act on \(\mathfrak{g}\), but a certain extension
of it called \(W^*\) does. In this section we review its basic properties.
The results through theorem 6 are due to Tits. The last result is new:
it describes the root stabilizers in \(W^*\). The proof relies on Brink’s study of reflection centralizers in Coxeter groups [17], in the form given in [2].

It is standard [12, lemma 3.5] that \(\text{ad } e_i\) and \(\text{ad } f_i\) are locally nilpotent
on \(\mathfrak{g}\), so their exponentials are automorphisms of \(\mathfrak{g}\). Furthermore,
\[
(\exp \text{ad } e_i)(\exp \text{ad } f_i)(\exp \text{ad } e_i) = (\exp \text{ad } f_i)(\exp \text{ad } e_i)(\exp \text{ad } f_i).
\]
We write \(s_i^*\) for this element of \(\text{Aut } \mathfrak{g}\) and \(W^*\) for \(<s_i^* \mid i \in I> \subseteq \text{Aut } \mathfrak{g}\). It turns out [12, lemma 3.8] that \(s_i^*(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i(\alpha)}\) for all \(\alpha \in Z^I\). This
defines a \(W^*\)-action on \(Z^I\), with \(s_i^*\) acting as \(s_i\). Since \(W\) acts faithfully
on \(Z^I\) this yields a homomorphism \(W^* \rightarrow W\). Using \(W^*\), the general
theory [12, prop. 5.1] shows that \(\mathfrak{g}_\alpha\) is 1-dimensional for any \(\alpha \in \Phi\).

Let \(Z^\vee\) be the free abelian group with basis the formal symbols \(\alpha_{i \in I}\),
and define a bilinear pairing \(Z^\vee \times Z^I \rightarrow \mathbb{Z}\) by \(\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}\). We define
an action of \(W\) on \(Z^\vee\) by \(s_i(\alpha_j^\vee) = \alpha_j^\vee - A_{ij}\alpha_i^\vee\). One can check that
this action satisfies \(\langle w\alpha^\vee, wb \rangle = \langle \alpha^\vee, b \rangle\). There is a homomorphism
Ad : $\mathbb{Z}^{I^V} \to \text{Aut} \mathfrak{g}$, with $\text{Ad}(\alpha^\vee)$ acting on $\mathfrak{g}_\beta$ by $(-1)^{\langle \alpha^\vee, \beta \rangle}$, where $\beta \in \mathbb{Z}^I$. The proof of the next lemma is easy and standard.

**Lemma 2.** $\text{Ad} : \mathbb{Z}^{I^V} \to \text{Aut} \mathfrak{g}$ is $W^*$-equivariant in the sense that $w^* \cdot \text{Ad}(\alpha^\vee) \cdot w^{-1} = \text{Ad}(w\alpha^\vee)$, where $\alpha^\vee \in \mathbb{Z}^{I^V}$ and $w$ is the image in $W$ of $w^* \in W^*$.

**Lemma 3.**

(i) $s_i^2 = \text{Ad}(\alpha_i^\vee)$.

(ii) $s_i^*(s_j^*)^2s_i^{*-1} = (s_j^*)^2(s_i^*)^{-2}A_{ji}$.

**Proof sketch.** (i) Identifying the span of $e_i, f_i, h_i$ with $\mathfrak{sl}_2\mathbb{C}$ as in (1) identifies $s_i^2$ with $(-1^0 \; 0^1) \in \text{SL}_2\mathbb{C}$. One uses the representation theory of $\text{SL}_2\mathbb{C}$ to see how this acts on $\mathfrak{g}$'s weight spaces.

(ii) uses (i) to identify $s_j^2$ with $\text{Ad}(\alpha_j^\vee)$, then lemma 2 to identify $s_i^*\text{Ad}(\alpha_j^\vee)s_j^{*-1}$ with $\text{Ad}(s_i(\alpha_j^\vee))$, then the formula defining $s_i(\alpha_j^\vee)$, and finally (i) again to convert back to $s_i^*$ and $s_j^*$. □

To understand the relations satisfied by the $s_i^*$ it will be useful to have a characterization of them in terms of the choice of $e_i$ (together with the grading on $\mathfrak{g}$). This is essentially part of Tits’ “trijection” [17] §1.1. In the notation of the following lemma, $s_i^*$ is $s_{e_i}^*$ (or equally well $s_{f_i}^*$).

**Lemma 4.** If $\alpha \in \Phi$ and $e \in \mathfrak{g}_{\alpha} - \{0\}$ then there exists a unique $f \in \mathfrak{g}_{-\alpha}$ such that

$$s_e^* := (\exp \text{ad} e)(\exp \text{ad} f)(\exp \text{ad} e)$$

exchanges $\mathfrak{g}_{\pm \alpha}$. If $\phi \in \text{Aut} \mathfrak{g}$ permutes the $\mathfrak{g}_{\beta \in \Phi}$ then $\phi s_e^*\phi^{-1} = s_{\phi(e)}^*$.

□

**Lemma 5.**

(i) If $m_{ij} = 3$ then $s_j^*s_i^*(e_j) = e_i$.

(ii) If $m_{ij} = 2, 4$ or 6 then $e_j$ is fixed by $s_i^*$, $s_i^*s_j^*s_i^*$ or $s_i^*s_j^*s_i^*s_j^*s_i^*$ respectively.

**Proof.** (i) follows from direct calculation in $\mathfrak{sl}_3\mathbb{C}$. In the $m_{ij} = 2$ case of (ii) we have $(\text{ad} e_i)(e_j) = (\text{ad} f_i)(e_j) = 0$, and $s_i^*(e_j) = e_j$ follows immediately. The remaining cases involve careful tracking of signs. We will write $(\mathfrak{sl}_2\mathbb{C})_i$ for the span of $e_i, f_i, h_i$.

If $m_{ij} = 4$ then $\{A_{ij}, A_{ji}\} = \{-1, -2\}$ and $\alpha_i$ and $\alpha_j$ are simple roots for a $B_2$ root system. Using lemma 4

$$s_i^*s_j^*s_i^*(e_j) = s_i^*s_j^*s_i^{*-1}s_i^{*2}(e_j) = s_{e_j}^*((\text{Ad} \alpha_i^\vee)(e_j))$$
(2) \[ = (-1)^{A_{ij}} s_i^* (e_j). \]

Suppose first that \( A_{ij} = -2 \). Then \( \alpha_i \) is the short simple root, \( \alpha_j \) the long one, and \( s_i(\alpha_j) \) is a long root orthogonal to \( \alpha_j \). We have

\[
s_{i}^*(e_j) = (\exp \text{ ad} s_i^*(e_j))(\exp \text{ ad} s_j^*(f_j))(\exp \text{ ad} s_i^*(e_j)) \in \exp \text{ ad}(s_i^*((\mathfrak{sl}_2\mathbb{C})_j)).
\]

Now, \( s_i^*((\mathfrak{sl}_2\mathbb{C})_j) \) annihilates \( g_j \) because its root string through \( \alpha_j \) has length 1. So \( s_{i}^*(e_j) \) fixes \( e_j \) and (2) becomes

\[
s_i^* e_j = (-1)^{A_{ij}} e_j = (-1)^{-2} e_j = e_j.
\]

On the other hand, if \( A_{ij} = -1 \) then \( \alpha_j \) and \( s_i(\alpha_j) \) are orthogonal short roots. Now the root string through \( \alpha_j \) for \( s_i^*((\mathfrak{sl}_2\mathbb{C})_j) \) has length 3, so the \( s_i^*((\mathfrak{sl}_2\mathbb{C})_j) \)-module generated by \( e_j \) is a copy of the adjoint representation. In particular, \( s_i^*(e_j) = s_i^* s_j^* s_i^{*-1} \) acts on \( g_j \) by the same scalar as on the Cartan subalgebra \( s_i^*(\mathfrak{h}) \) of \( s_i^*((\mathfrak{sl}_2\mathbb{C})_j) \). This is the same scalar by which \( s_j^* \) acts on \( \mathfrak{h} \), which is \(-1\). So \( s_{i}^*(e_j) \) negates \( e_j \) and (2) reads

\[
s_i^* s_j^* s_i^*(e_j) = (-1)^{A_{ij}} (-e_j) = (-1)^{-1}(-e_j) = e_j.
\]

Now suppose \( m_{ij} = 6 \), so that \( \{A_{ij}, A_{ji}\} = \{-1, -3\} \), \( \alpha_i \) and \( \alpha_j \) are simple roots for a \( G_2 \) root system, and \( s_i s_j(\alpha_i) \perp \alpha_j \). Then

\[
s_i^* s_j^* s_i^* s_j^* s_i^*(e_j) = (s_j^* s_i^* s_j^* s_i^* s_i^{*-1})s_i^* s_j^* s_i^*(e_j)
\]

\[
= s_i^* s_j^* s_i^*(e_j) \circ (s_i^* s_j^* s_i^{*-1}) \circ s_i^2(e_j)
\]

\[
= s_i^* s_j^* s_i^*(e_j) \circ s_i^2 s_i^{*-2A_{ij}} \circ s_i^2(e_j)
\]

\[
= s_i^* s_j^* s_i^*(e_j) \circ s_i^2 s_i^{4\text{or} 8}(e_j)
\]

\[
= s_i^* s_j^* s_i^*(e_j).
\]

The root string through \( \alpha_j \) for \( s_i^*(\mathfrak{sl}_2\mathbb{C})_i \) has length 1, so arguing as in the \( B_2 \) case shows that \( s_{i}^* s_j^*(e_j) \) fixes \( e_j \).

**Theorem 6** (Tits [20], §4.6). The \( s_i^* \) satisfy the Artin relations of \( M \).

That is, if \( m_{ij} \neq \infty \) then \( s_i^* s_j^* \cdots = s_j^* s_i^* \cdots \), where there are \( m_{ij} \) factors on each side, alternately \( s_i^* \) and \( s_j^* \).

**Proof.** For \( m_{ij} = 3 \) we start with \( e_j = s_i^* s_j^*(e_i) \) from lemma [4][4]. Using lemma [4] yields

\[
s_j^* s_j^* s_i^*(e_i) = s_i^* s_j^* s_i^* s_j^* s_i^{*-1} s_i^{*-1} = s_i^* s_j^* s_i^* s_j^* s_i^{*-1} s_i^{*-1}.
\]

The other cases are the same.
We will need to understand the $W^*$-stabilizer of a simple root $\alpha_i$ and how it acts on $q_i$. The first step is to quote from [2] a refinement of a theorem of Brink [7] on reflection centralizers in Coxeter groups. Then we will “lift” this result to $W^*$ by keeping track of signs.

Both theorems refer to the “odd Dynkin diagram” $\Delta^{\text{odd}}$, which means the graph with vertex set $I$ where vertices $i$ and $j$ are joined just if $m_{ij} = 3$. For $\gamma$ an edge-path in $\Delta^{\text{odd}}$, with $i_0, \ldots, i_n$ the vertices along it, we define

$$p_{\gamma} := (s_{i_{n-1}}s_{i_n}) (s_{i_{n-2}}s_{i_{n-1}}) \cdots (s_{i_1}s_{i_2}) (s_{i_0}s_{i_1}).$$

(If $\gamma$ has length 0 then we set $p_\gamma = 1$.) For $i \in I$ we write $\Delta^{\text{odd}}_i$ for its component of $\Delta^{\text{odd}}$.

**Theorem 7 ([2]).** Suppose $i \in I$, $Z$ is a set of edge-loops based at $i$ that generate $\pi_1(\Delta^{\text{odd}}_i, i)$, and $\delta_j$ is an edge-path in $\Delta^{\text{odd}}_i$ from $i$ to $j$, for each vertex $j$ of $\Delta^{\text{odd}}_i$. For each such $j$ and each $k \in I$ with $m_{jk}$ finite and even, define

$$r_{jk} := p_{\delta_j^{-1}} \cdot \left\{ \begin{array}{c} s_k \\ s_k s_j s_k \\ s_k s_j s_k s_j s_k \end{array} \right\} \cdot p_{\delta_j}$$

according to whether $m_{jk} = 2$, 4 or 6. Then the $W$-stabilizer of the simple root $\alpha_i$ is generated by the $r_{jk}$ and the $p_{z \in Z}$. \hfill \square

It is easy to see that the $r_{jk}$ and $p_z$ stabilize $\alpha_i$. In fact this is the “image under $W^* \to W$” of the corresponding part of the next theorem.

**Theorem 8.** Suppose $i$, $Z$ and the $\delta_j$ are as in theorem [7]. Define $p^*_\gamma$ and $r^*_jk$ by attaching *’s to the $s$’s, $p$’s and $r$’s in (3) and (4). Then the $p^*_{z \in Z}$ and $r^*_jk$ fix $e_i$, and together with the $s^*_{i \in I}$ they generate the $W^*$-stabilizer of $\alpha_i$. (By lemma [4(i)], $s^*_{i \in I}$ acts on $e_i$ by $(-1)^{A_i}$).

**Proof.** The $W^*$-stabilizer of $\alpha_i$ is generated by ker($W^* \to W$) and any set of elements of $W^*$ whose projections to $W$ generate the $W$-stabilizer of $\alpha_i$. Now, the $s^*_{i \in I}$ normally generate the kernel because of the Artin relations. Lemma [4(ii)] shows that the subgroup they generate is normal, hence equal to this kernel. Since and the $p^*$’s and $r^*$’s project to the $p$’s and $r$’s of theorem [7] our generation claim follows from that theorem. To see that the $p^*_{z \in Z}$’s fix $e_i$, apply lemma [4(i)] repeatedly. The same argument proves $p^*_{\delta_j}(e_i) = e_j$. Then using lemma [4(ii)] shows that $e_j$ is fixed by $s^*_k$, $s^*_k s^*_j s^*_k$ or $s^*_k s^*_j s^*_k s^*_j s^*_k$ according to whether $m_{jk}$ is 2, 4 or 6. Applying $p^*_{\delta_j^{-1}}$ sends $e_j$ back to $e_i$, proving $r^*_jk(e_i) = e_i$. \hfill \square
4. Presentation of $\mathbf{St}_A$

In this section we develop the main result of the paper: after some introductory material we give a group presentation and show in theorem [13] that it equals $\mathbf{St}_A$. This presentation has generators $S_i$ and $X_i(t)$ with $i \in I$ and $t \in R$, and relators (5)–(30). For continuity some proofs are deferred to later sections.

$\mathbb{Add}$ denotes the additive group, regarded as a group scheme over $\mathbb{Z}$. That is, it assigns to each commutative ring $R$ the abelian group underlying $R$. The Lie algebra of $\mathbb{Add}$ is canonically isomorphic to $\mathbb{Z}$.

For each $\alpha \in \Phi$, $\mathfrak{g}_\alpha \cap W^*(\{e_i \in I\})$ consists of either one vector or two antipodal vectors. This is [18, 3.3.2] and its following paragraph, which relies on [19, §13.31]. Alternately, it follows from our theorem 8.

We write $\mathfrak{g}_\alpha$, $\mathbb{Z}$ for the $\mathbb{Z}$-span of this element (or antipodal pair) in $\mathfrak{g}_\alpha$. We define $\mathcal{U}_\alpha$ as the group scheme over $\mathbb{Z}$ which is isomorphic to $\mathbb{Add}$ and has Lie algebra $\mathfrak{g}_\alpha, \mathbb{Z}$. This is just a fancy way to say that $\mathcal{U}_\alpha$ is the functor assigning to $R$ the abelian group $\mathfrak{g}_\alpha, \mathbb{Z} \otimes R \cong R$. For $i \in I$ we abbreviate $\mathcal{U}_\alpha_i$ to $\mathcal{U}_i$.

If $\alpha \in \Phi$ and $e$ is either of the two generators of $\mathfrak{g}_\alpha, \mathbb{Z}$, then we define $r_e$ as the isomorphism $\mathbb{Add} \to \mathcal{U}_\alpha$ whose corresponding Lie algebra isomorphism identifies $1 \in \mathbb{Z}$ with $a \in \mathfrak{g}_\alpha, \mathbb{Z}$. Once $R$ is fixed, this amounts to

$$r_e(t) := e \otimes t \in \mathfrak{g}_\alpha, \mathbb{Z} \otimes R = \mathcal{U}_\alpha.$$

When $R = \mathbb{R}$ or $\mathbb{C}$, one may think of $r_e(t)$ as $\exp(te)$. For $i \in I$ we abbreviate $r_{e_i}$ to $r_i$.

Next we give Tits’ version of the Steinberg group. We call it the Tits-Steinberg group and write $\mathcal{T}_i = \mathcal{T}_i_A$ to distinguish it from the Morita-Rehmann version of the Steinberg group, which we will denote $\mathcal{G}_i = \mathcal{G}_i_A$ (see step 4 below). If $A$ is 2-spherical ($m_{ij} < \infty$ for all $i, j \in I$) with no $A_1$ components (for all $i$ there exists $j \in I$ with $m_{ij} > 2$) then the two versions agree.

Tits calls a set of roots $\Psi \subseteq \Phi$ prenilpotent if some element of the open Tits cone lies on the positive side of all their mirrors and some other element of it lies on the negative side of all of them. It follows that $\Psi$ is finite. If $\Psi$ is also closed under addition then it is called nilpotent. In this case $\mathfrak{g}_\Psi := \oplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$ is a nilpotent Lie algebra [18, p. 547].

**Lemma 9** (Tits [18, sec. 3.4]). If $\Psi \subseteq \Phi$ is a nilpotent set of roots, then there is a unique unipotent group scheme $\mathcal{U}_\Psi$ over $\mathbb{Z}$ with the properties

$(i)$ $\mathcal{U}_\Psi$ contains all the $\mathcal{U}_\alpha \in \Psi$;

$(ii)$ $\mathcal{U}_\Psi(\mathbb{C})$ has Lie algebra $\mathfrak{g}_\Psi$;
For any ordering on $\Psi$, the product morphism $\prod_{\alpha \in \Psi} U_\alpha \to U_\Psi$ is an isomorphism of the underlying schemes.

$\Xi_i = \Xi_{i1}$ is defined as follows. For each prenilpotent pair $\alpha, \beta$ of roots, $\theta(\alpha, \beta)$ is defined as $(N\alpha + N\beta) \cap \Phi$. (Note: $\mathbb{N}$ contains 0.) If $\gamma \in \theta(\alpha, \beta)$ then there is a natural injection $U_\gamma \to U_{\theta(\alpha, \beta)}$. Letting $\alpha, \beta$ vary over all prenilpotent pairs, and for each pair letting $\gamma$ vary over $\theta(\alpha, \beta) \subseteq \Phi$, one obtains a diagram of inclusions of group functors. $\Xi_i$ is the direct limit of this diagram. Every automorphism of $g$ that permutes the $g_\alpha, Z$ induces an automorphism of the diagram of inclusions of group functors, hence an automorphism of $\Xi_i$. In particular, $W^*$ acts on $\Xi_i$.

As Tits points out, a helpful but less-canonical way to think about $\Xi_i(R)$ is to begin with the free product $\ast_{\alpha \in \Phi} U_\alpha(R)$ and impose relations of the form $[e \in\mathbb{N} \gamma(t), e^{-1}(u)] = \prod e^{\gamma}(v_\gamma)$ for each prenilpotent pair $\alpha, \beta$. Here $e$ and $e'$ are generators for $g_{\alpha, Z}$ and $g_{\beta, Z}, \gamma$ varies over $\theta(\alpha, \beta) - \{\alpha, \beta\}$ and $e''$ is a generator for $g_{\gamma, Z}$. The coefficients $v_\gamma$ depend on $t, u$, the position of $\gamma$ relative to $\alpha$ and $\beta$, the choices of $e, e', e''$, and the chosen ordering of the product; cf. (3) of [18]. These are the Chevalley relations for $\alpha$ and $\beta$.

Recall that we call two linearly independent roots $\alpha, \beta$ classically prenilpotent if $(Q\alpha + Q\beta) \cap \Phi$ is finite. In this case they lie in some $A_1, A_2, B_2$ or $G_2$ root system. We define the classical Tits-Steinberg group $\Xi_i^{cl} = \Xi_{i1}^{cl}$ exactly as we did $\Xi_i$, except that $\alpha, \beta$ vary only over the classically prenilpotent pairs rather than all prenilpotent pairs. Of course, if $\Phi$ is finite then these are the same. $W^*$ acts on $\Xi_i^{cl}$ for the same reason it acts on $\Xi_i$.

Now we begin our presentation of $\hat{\Xi}_i^{cl}$. We begin with a group $\hat{W}$, defined as the quotient of the free group on formal symbols $S_{i \in I}$ by the subgroup normally generated by the words

\begin{align*}
(5) \quad (S_i S_j \cdots) \cdot (S_j S_i \cdots)^{-1} & \text{ if } m_{ij} \neq \infty \\
(6) \quad S_i^2 S_j S_i^{-2} \cdot S_j^{-1} & \text{ if } A_{ij} \text{ is even} \\
(7) \quad S_i^2 S_j S_i^{-2} \cdot S_j & \text{ if } A_{ij} \text{ is odd}
\end{align*}

where $i, j$ vary over $I$, and $[5]$ has $m_{ij}$ terms inside each pair of parentheses, alternating between $S_i$ and $S_j$. These are the Artin reators, for example $S_i S_j S_i \cdot (S_j S_i S_j)^{-1}$ if $m_{ij} = 3$. Relators $[5] - [7]$ were chosen so that $\hat{W}$ is the simplest possible extension of $W$ that is guaranteed to map to $\hat{\Xi}_i^{cl}$. But all we need is the following weaker result.

**Lemma 10** (Basic properties of $\hat{W}$).
(i) \( S_i \mapsto s_i^* \) defines a surjection \( \widehat{W} \to W^* \).
(ii) \( S_j S_i^2 S_j^{-2} = S_i^2 \) resp. \( S_j^2 S_i^2 \) if \( A_{ij} \) is even resp. odd.
(iii) The \( S_i^2 \) generate the kernel of the composition \( \widehat{W} \to W^* \to W \).

Proof. We saw in lemma (ii) that the \( s_i^* \) satisfy the Artin relations. Rewriting lemma (i) gives a relation in \( W^* \) with \( i \) and \( j \) reversed gives
\[
s_i^* (s_j^*)^2 s_j^{*-1} = (s_i^*)^2 (s_j^*)^{-2 A_{ij}}.
\]
Multiplying on the left by \( s_i^{*-1} \) and on the right by \( (s_i^*)^{-2} \), then inverting, gives
\[
(s_i^*)^2 s_j^* (s_i^*)^{-2} = (s_i^*)^2 (s_j^*)^{2 A_{ij}} (s_i^*)^{-2} s_j^* = (s_i^*)^{1+2 A_{ij}}
\]
In the second step we used the fact that \( s_i^{*2} \) and \( s_j^{*2} \) commute. Using \( s_i^{*4} = 1 \), the right side is \( s_j^* \) if \( A_{ij} \) is even and \( s_j^*^{-1} \) if \( A_{ij} \) is odd. This shows that \( S_i \mapsto s_i^* \) sends the relations (5)–(7) to the trivial element of \( W^* \), proving (i).

One can manipulate (5)–(7) in a similar way, yielding (ii). It follows immediately that the subgroup generated by the \( S_i^2 \) is normal. Because of the Artin relations, this is the kernel of \( \widehat{W} \to W \). So we have proven (iii) \( \Box \)

Remark. Though we don’t need them, the following relations in \( \widehat{W} \) show that \( \widehat{W} \) is “not much larger” than \( W^* \). First (6)–(7) imply the centrality of every \( S_i^4 \). Second, if some \( A_{ij} \) is odd then (7) shows that \( S_j^{*4} \) are conjugate; since both are central they must be equal, so \( S_j^8 = 1 \). Third, the relation obtained at the end of the proof implies \( [S_j^2, S_i^2] = 1 \) or \( S_j^4 \), according to whether \( A_{ij} \) is even or odd. In particular, these commutators are central. Finally, we can use this two ways:
\[
\begin{cases} 1 & \text{if } A_{ij} \text{ is even} \\ S_i^4 & \text{if } A_{ij} \text{ is odd} \end{cases} = [S_j^2, S_i^2] = [S_i^2, S_j^2]^{-1} = \begin{cases} 1 & \text{if } A_{ji} \text{ is even} \\ S_i^{*-4} & \text{if } A_{ji} \text{ is odd} \end{cases}
\]
In particular, if both \( A_{ij} \) and \( A_{ji} \) are odd then \( S_i^4 \) and \( S_j^4 \) are equal. If \( A_{ij} \) is even while \( A_{ji} \) is odd then we get \( S_i^4 = 1 \).

Step 1. We define \( \mathcal{G}_1 \) as the free product of the group functors \( \mathcal{U}_{i \in I} \).
That is,
\[
\mathcal{G}_1(R) := \ast_{i \in I} \mathcal{U}_i(R).
\]
Note that we take only the root groups for the simple roots, not all roots. For \( t \in R \) we write \( X_i(t) \) for \( (r_i(R))(t) \in \mathcal{U}_i(R) \). Taking the \( X_i(t) \) as generators for \( \mathcal{G}_1(R) \), defining relations are
\[
X_i(t) X_i(u) X_i(t+u)^{-1} \quad (8)
\]
for all $i \in I$ and $t, u \in R$.

Step 2. We define $\mathfrak{G}_2$ as a certain quotient of the free product $\mathfrak{G}_1 \ast \hat{W}$ of group functors. Namely, $\mathfrak{G}_2(R)$ is the quotient of $\mathfrak{G}_1(R) \ast \hat{W}$ by the subgroup normally generated by the following relators, with $i$ and $j$ varying over $I$ and $t$ over $R$.

\begin{align}
(9) \quad & S_i^2 X_j(t) S_i^{-2} \cdot \left( X_j((-1)^{A_{ij} t}) \right)^{-1} \\
(10) \quad & [S_i, X_j(t)] \quad \text{if } m_{ij} = 2 \\
(11) \quad & S_j S_i X_j(t) \cdot (X_i(t) S_j S_i)^{-1} \quad \text{if } m_{ij} = 3 \\
(12) \quad & [S_i S_j S_i X_j(t)] \quad \text{if } m_{ij} = 4 \\
(13) \quad & [S_i S_j S_i S_j S_i X_j(t)] \quad \text{if } m_{ij} = 6
\end{align}

The next theorem is the key ingredient in our development. Although it is not at all obvious, we have presented $(\ast_{\alpha \in \Phi} \mathfrak{U}_\alpha) \rtimes \hat{W}$. This sets us up for imposing the Chevalley relations in the next step. See section 5 for the proof.

**Theorem 11.** $\mathfrak{G}_2$ is the semidirect product of $\ast_{\alpha \in \Phi} \mathfrak{U}_\alpha$ by $\hat{W}$, where $\hat{W}$ acts on the free product via its homomorphism to $W^*$ and $W^*$'s action on the $\mathfrak{U}_\alpha$'s, induced by its action on the $\mathfrak{g}_{\alpha, Z}$'s.

Step 3. Now we adjoin Chevalley relations corresponding to finite edges in the Dynkin diagram. That is, we define $\mathfrak{G}_3(R)$ as the quotient of $\mathfrak{G}_2(R)$ by the subgroup normally generated by the relators (14)–(27) below, for all $t, u \in R$. When $i, j \in I$ with $m_{ij} = 2$,

\begin{align}
(14) \quad & [X_i(t), X_j(u)]
\end{align}

When $i, j \in I$ with $m_{ij} = 3$,

\begin{align}
(15) \quad & [X_i(t), S_j X_j(u) S_i^{-1}] \\
(16) \quad & [X_i(t), X_j(u)] \cdot S_i X_j(-t u) S_i^{-1}
\end{align}

When $s, l \in I$, $m_{sl} = 4$ and $s$ is the shorter root of the $B_2$,

\begin{align}
(17) \quad & [S_s X_l(t) S_s^{-1}, S_l X_s(u) S_l^{-1}] \\
(18) \quad & [X_l(t), S_s X_l(u) S_s^{-1}] \\
(19) \quad & [X_s(t), S_l X_s(u) S_l^{-1}] \cdot S_s X_l(2 t u) S_s^{-1} \\
(20) \quad & [X_s(t), X_l(u)] \cdot S_s X_l(-t^2 u) S_s^{-1} \cdot S_l X_s(t u) S_l^{-1}
\end{align}

When $s, l \in I$, $m_{sl} = 6$ and $s$ is the shorter root of the $G_2$,

\begin{align}
(21) \quad & [X_l(t), S_s S_l X_l(u) S_s^{-1} S_l^{-1}] \\
(22) \quad & [S_s S_l X_s(t) S_s^{-1} S_s^{-1}, S_l S_s X_l(u) S_s^{-1} S_l^{-1}]
\end{align}
These are standard Chevalley relators, but their form is unusual. Our reasons for writing them this way are that there are no signs to puzzle over, and that the presentation refers only to the Dynkin diagram, rather than all of \( \Phi \). One can convert them to a more standard form by working out what root groups contain the terms on the “right hand sides” of the relators. For example, the last term of theorem 11, the kernels of \( W \) coincide. More precisely, under the identification \( \widehat{W} \)-conjugates, for example \( [S_s X_i(t) S_s^{-1}, X_s(u)] \). We chose the listed form because it behaves well under the exceptional diagram automorphism when \( R \) is a field of characteristic 2; see [3] for further development. Similar considerations guided our choice of relators \( (24) - (23) \).

In section 6 we give the fairly easy proof of the following theorem, that the group presented so far is \( \Xi^{cl}_A \rtimes \widehat{W} \). We also show in proposition 24 that in practice most of relators \( (14) - (16) \) can be omitted. Usually this reduces the size of the presentation greatly.

**Theorem 12.** \( \mathfrak{g}_3 \cong \Xi^{cl}_A \rtimes \widehat{W} \). More precisely, under the identification \( \mathfrak{g}_2 \cong (\ast_{\alpha \in \Phi} \mathfrak{u}_{\alpha}) \rtimes \widehat{W} \) of theorem 17 the kernels of \( \mathfrak{g}_2 \to \mathfrak{g}_3 \) and \( (\ast_{\alpha \in \Phi} \mathfrak{u}_{\alpha}) \rtimes \widehat{W} \to \Xi^{cl}_A \rtimes \widehat{W} \) coincide.

**Step 4.** In this step we impose the additional relations \( (28) - (29) \) required to get the Steinberg group \( \mathfrak{St} \) from the Tits-Steinberg group \( \mathfrak{T} \). These were included in Tits’ paper [18, eqn. (6)], but at a later stage in his construction of Kac-Moody groups. Morita and Rehmann [13] observed that they should really be included in the definition of the Steinberg group. Two reasons for this are (1) under suitable restrictions on \( R \) the resulting group is centrally closed, and (2) one recovers Steinberg’s original group in the \( A_1 \) case. If the root system is 2-spherical without \( A_1 \) components then the natural maps \( \Xi^{cl}_A \to \mathfrak{st}^{cl} \), \( \mathfrak{T} \to \mathfrak{St} \) and \( \mathfrak{g}_3 \to \mathfrak{g}_4 \) are isomorphisms and this step can be skipped. First
we will describe the relations in an intrinsic manner, following Morita-Rehmann, and then we will give the quicker description in terms of our generators.

Recall from lemma 2 and its preceding remarks that $\mathbb{Z}^I$ is the free abelian group generated by formal symbols $\alpha\in I$ and that there is a $W$-invariant bilinear pairing $\mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z}$. We defined a map $\operatorname{Ad} : \mathbb{Z}^I \to \operatorname{Aut} \mathfrak{g}$, which we generalize to $\operatorname{Ad} : (\mathbb{R}^\ast \otimes \mathbb{Z}^I) \to \operatorname{Aut}(*_{\alpha\in\Phi} \mathfrak{U}_\alpha)$ as follows. For any $\alpha\in\mathbb{Z}^I$, $r \in \mathbb{R}^\ast$ and $\beta \in \Phi$, $\operatorname{Ad}(r \otimes \alpha^\vee)$ acts on $\mathfrak{U}_\beta \cong \mathbb{R}$ by multiplication by $r^{(\alpha^\vee, \beta)} \in \mathbb{R}^\ast$. One recovers the original $\operatorname{Ad}$ by taking $r = -1$.

It is standard that there is a $W$-equivariant bijection $\alpha \mapsto \alpha^\vee$ from the roots $\Phi \subseteq \mathbb{Z}^I$ to their corresponding coroots in $\mathbb{Z}^I$. For $\alpha \in \Phi$ and $r \in \mathbb{R}^\ast$ we define $h_\alpha(r) \in \operatorname{Aut}(*_{\alpha\in\Phi} \mathfrak{U}_\alpha)$ as $\operatorname{Ad}(r \otimes \alpha^\vee)$. For the simple roots $\alpha_i$ their coroots are the basis $\alpha_i^\vee$ for $\mathbb{Z}^I$. In general one chooses $w \in W$ sending some $\alpha_i$ to $\alpha$ and defines $\alpha^\vee$ as $w(\alpha_i^\vee)$.

We recall from lemma 4 that if $\alpha \in \Phi$ and $e$ is a generator for $\mathfrak{g}_{\alpha,\mathbb{Z}}$ then there is a distinguished generator $f$ for $\mathfrak{g}_{-\alpha,\mathbb{Z}}$. As the notation suggests, if $e = e_i$ then $f = f_i$. For any $r \in \mathbb{R}^\ast$ we define

$$\tilde{s}_e(r) := r_e(r)r_f(1/r)r_e(r)$$

which we regard either as an element of $\mathfrak{U}_\alpha \ast \mathfrak{U}_{-\alpha} \subseteq \ast_{\beta\in\Phi} \mathfrak{U}_\beta$ or its image in $\mathfrak{Ti}^d$ or $\mathfrak{Ti}$. We use the abbreviation $\tilde{s}_e(r)$ for $\tilde{s}_{e_i}(r)$, which equals $\tilde{s}_{f_i}(r)$. The Steinberg group $\mathfrak{St}$ is the quotient of the Tits-Steinberg group $\mathfrak{Ti}$ got by forcing every $\tilde{s}_e(r)$ to act on every $\mathfrak{U}_\beta$ by $s_e^* \circ h_\alpha(r)$. Formally, it is the quotient of $\mathfrak{Ti}$ by the smallest subgroup normally generated by the elements

$$\tilde{s}_e(r) u \tilde{s}_e(r)^{-1} \cdot \left( (s_e^* \circ h_\alpha(r))(u) \right)^{-1}$$

as $\alpha$ varies over $\Phi$, $e$ over the two generators of $\mathfrak{g}_{\alpha,\mathbb{Z}}$, $r$ over $\mathbb{Z}^I$, $u$ over $\mathfrak{U}_\beta$ and $\beta$ over $\Phi$. In the “right hand side” $s_e^* \circ h_\alpha(r)$ is not a word in the generators of $\mathfrak{Ti}$ but rather an automorphism of $*_{\beta\in\Phi} \mathfrak{U}_\beta$.

Because this normal subgroup is defined in a $W^*$-invariant manner, $W^*$ acts on $\mathfrak{St}$. We define $\mathfrak{St}^d$ as we did for $\mathfrak{St}$, with $\mathfrak{Ti}^d$ in place of $\mathfrak{Ti}$, and $W^*$ acts on it also, for the same reason.

Now we return to our presentation by defining $\mathfrak{G}_4$, which by theorem 13 below will be isomorphic to $\mathfrak{St}^d \rtimes \widehat{W}$. We first regard $\tilde{s}_i(r)$ as the following word in the generators for $\mathfrak{G}_3$:

$$\tilde{s}_i(r) := X_i(r)S_iX_i(1/r)S_i^{-1}X_i(r)$$
This is compatible with the definition of $\tilde{s}_i(r)$ above because $S_i$ acts on $g_i$ as $s_i^*$, hence sends $e_i$ to $f_i$. Then we define

$$\tilde{h}_i(r) := \tilde{s}_i(r)\tilde{s}_i(-1)$$

We define $G_4$ by forcing $\tilde{h}_i(r)$ to act on each $U_{\pm\alpha_j}$ as $h_i(r)$ does. Formally, $G_4$ is the quotient of $G_3$ by the subgroup normally generated by the relators

\begin{align}
\tilde{h}_i(r)X_j(t)\tilde{h}_i(r)^{-1} & \cdot X_j(r^{A_{ij}}t)^{-1} \\
\tilde{h}_i(r)S_jX_j(t)S_j^{-1}\tilde{h}_i(r)^{-1} & \cdot S_jX_j(r^{-A_{ij}}t)^{-1}S_j^{-1}
\end{align}

where $r$ varies over $R^*$, $t$ over $R$ and $(i,j)$ over all ordered pairs in $I$ satisfying either of the following. First, if $m_{ij} = \infty$. Second, if $i = j$ and there is no $k \in I$ with $m_{ik} \in \{3, 4, 6\}$. Here we are just economizing: Steinberg showed that these relators for all other pairs $(i,j)$ are already trivial in $\Xi^{cl}$; see the proof of lemma 27. If $A$ is 2-spherical without $A_1$ components then we are imposing no relations at all, and $G_4$ is just $G_3$. The proof of the following theorem appears in section 7.

**Theorem 13.** The identification $G_3 \cong \Xi^{cl} \rtimes \hat{W}$ of theorem 12 induces an isomorphism $G_4 \cong \St^{cl} \rtimes \hat{W}$.

**Step 5.** Our final step is to get rid of the outer $\hat{W}$ factor. We define $G_5$ as the quotient of $G_4$ by the subgroup normally generated by the

$$S_i \cdot \tilde{s}_i(1)^{-1}$$

with $i$ varying over $I$. This equates $S_i$ with an element of $\St^{cl}$, and the next theorem shows that it doesn’t force any further collapsing.

**Theorem 14** (Main theorem). $G_5 \cong \St^{cl}$. In more detail, the natural map from $G_4 \cong \St^{cl} \rtimes \hat{W}$ to $G_5$ maps the visible subgroup $\St^{cl}$ isomorphically onto $G_5$.

**Summary.** $\St_A^{cl}(R)$ is a quotient of the free group on symbols $S_i$ and $X_i(t)$, where $i \in I$ and $t \in R$. It is the quotient by the subgroup normally generated by the words (5)–(30). When $\St_A^{cl}(R) \to \St_A(R)$ is an isomorphism, in particular for every classical root system, this also presents the Steinberg group $\St_A(R)$.

**Remarks.** (1) One need only impose the relators (30) for a single $i$ in each component $\Omega$ of $\Delta^{\text{odd}}$, because if $m_{ij} = 3$ then $S_iS_j$ conjugates $S_i$ to $S_j$ and $X_i(t)$ to $X_j(t)$. 
(2) We got the presentations for the classical ADE groups over \( \mathbb{Z} \) in table 1 by discarding obviously redundant relations, eliminating all \( X_i(t) \) with \( t \neq 1 \) from the above presentation by Tietze transformations, and writing \( X_i \) for \( X_i(1) \). Taking into account the previous remark and proposition 24 our presentation for \( E_8(\mathbb{Z}) \) has 16 generators and 124 relators. With more effort one can discard a few more of the relators.

(3) The same analysis shows that the presentation in table 1 gives \( \mathrm{St}_{\mathrm{cl}}(\mathbb{Z}) \) for any simply laced Dynkin diagram. This becomes interesting when one can verify that \( \mathrm{St}_{\mathrm{cl}}(\mathbb{Z}) \to \mathrm{St}(\mathbb{Z}) \) is an isomorphism. This is the extra ingredient in our claim in the introduction that this presentation gives the affine Steinberg groups \( \tilde{A}_{n>1}(\mathbb{Z}), \tilde{D}_{n>2}(\mathbb{Z}) \) and \( \tilde{E}_{n=6,7,8}(\mathbb{Z}) \) and some hyperbolic groups like \( E_{10}(\mathbb{Z}) \). See [4] and [5] for this isomorphism. In any case, one obtains \( \mathrm{St} \) from \( \mathrm{St}_{\mathrm{cl}} \) by adjoining Tits' relations for prenilpotent pairs that are not classically prenilpotent.

(4) Once one has \( \mathrm{St} \), one obtains the simply-connected form of Tits' Kac-Moody group functor by imposing the relations \( \tilde{s}_i(a)\tilde{s}_i(b) = \tilde{s}_i(ab) \) and \( [\tilde{h}_i(a), \tilde{h}_j(b)] = 1 \). After doing this, it follows from (34) and lemma 27 that \( \tilde{W} \) acts on them by \( S_j\tilde{h}_i(a)S_j^{-1} = \tilde{h}_j(a^{-A_{ij}})\tilde{h}_i(a) \).

(5) To reduce the risk of typographical errors, we constructed our generators as explicit matrices in standard representations of the \( A_1^2, A_2, B_2 \) and \( G_2 \) Chevalley groups over a polynomial ring in the variables \( r^{\pm 1}, t, u \). Then we checked on the computer that they satisfy the defining relations, and also the relations in the previous remark. We only typed in the relations once, for both typesetting and this check, so the paper is typeset using exactly the relations we checked.

(6) As promised in the introduction, the proof of theorem 1 is trivial: every relator (5)–(30) involves only one or two subscripts.

Proof of theorem 1. We are imposing the relations \( S_i = \tilde{s}_i \). Using them in Tietze transformations to eliminate the \( S_i \), we find the \( \mathfrak{g}_5 \) is the quotient of \( \mathfrak{g}^{cl} \) by the normal subgroup generated by the words got by replacing \( S_i \) by \( \tilde{s}_i \) in each of the relators (5)–(29). All of these relations are already trivial in \( \mathfrak{g}^{cl} \), so \( \mathfrak{g}_5 = \mathfrak{g}^{cl} \).

In more detail, (5) requires the \( \tilde{s}_i \) to satisfy the Artin relations, which they do by [18] (d) on p. 551]. (6)–(7) require that \( \tilde{s}_i^2 \) act on each \( \tilde{s}_j \) in a certain way. This follows from theorem 13 which says that \( \tilde{s}_i \) acts on every \( \mathfrak{u}_\alpha \) as does \( s_i^* \). The same argument works for all remaining relators (9)–(29), because every occurrence of an \( S_i \) in them is by conjugating some element of some \( \mathfrak{u}_\alpha \). (Conjugation by \( S_i \) and \( \tilde{s}_i \) both act as \( s_i^* \) does, by theorems 11 and 13.) \( \square \)
5. The isomorphism $\mathcal{G}_2 \cong \left( \ast_{\alpha \in \Phi} \mathcal{U}_\alpha \right) \rtimes \hat{W}$

In this section we will suppress the dependence of group functors on the base ring $R$, always meaning the group of points over $R$. Our goal is to prove theorem 11, namely that the group $\mathcal{G}$ with the base ring $R$.

In this section we will suppress the dependence of group functors on $X$ and $t$.

Lemma 15. Suppose $G = \left( \ast_{\alpha \in \Phi} U_\alpha \right) \rtimes H$, where $\Phi$ is some index set, the $U_\alpha$’s are groups isomorphic to each other, and $H$ is a group whose action on the free product permutes the displayed factors transitively. Then $G \cong (U_\infty \rtimes H_\infty) \ast_{H_\infty} H$, where $\infty$ is some element of $\Phi$ and $H_\infty$ is its $H$-stabilizer.

Proof. The idea is that $U_\infty \rtimes H_\infty \rightarrow (U_\infty \rtimes H_\infty) \ast_{H_\infty} H$ is a sort of free-product analogue of inducing a representation from $H_\infty$ to $H$. We suppress the subscript $\infty$ from $U_\infty$. Take a set $Z$ of left coset representatives for $H_\infty$ in $H$, and for $u \in U$ and $z \in Z$ define $u_z := zu z^{-1} \in G$. The $u_z$ for fixed $z$ form the free factor $zUz^{-1} = U_{z(\infty)}$ of $\left( \ast_{\alpha \in \Phi} U_\alpha \right) \subseteq G$. Assuming $U \neq 1$, every free factor occurs exactly once this way, since $H$’s action on $\Phi$ is the same as on $H_\infty$’s left cosets. So the maps $u_z \mapsto zu z^{-1} \in (U \rtimes H_\infty) \ast_{H_\infty} H$ define a homomorphism $\left( \ast_{\alpha \in \Phi} U_\alpha \right) \rightarrow (U \rtimes H_\infty) \ast_{H_\infty} H$. This homomorphism is obviously $H$-equivariant, so it extends to a homomorphism $G \rightarrow (U \rtimes H_\infty) \ast_{H_\infty} H$. It is easy to see that this is inverse to the obvious homomorphism $(U \rtimes H_\infty) \ast_{H_\infty} H \rightarrow G$. 

Now we begin proving theorem 11 by reducing it to lemma 16 below, a sort of analogue of it for a single component of $\Delta_{\text{odd}}$. It is well-known that two generators $s_i$, $s_j$ of $W$ ($i, j \in I$) are conjugate in $W$ if and only if $i$ and $j$ lie in the same component of $\Delta_{\text{odd}}$. Let $\Omega$ be one of these components, and write $\Phi(\Omega) \subseteq \Phi$ for the roots whose reflections are conjugate to some (hence any) $s_i \in \Omega$. Because $\Phi(\Omega)$ is a $W$-invariant subset of $\Phi$, we may form the group $\left( \ast_{\alpha \in \Phi(\Omega)} \mathcal{U}_\alpha \right) \rtimes \hat{W}$ just as we did $\left( \ast_{\alpha \in \Phi} \mathcal{U}_\alpha \right) \rtimes \hat{W}$.

We will write $\mathcal{G}_2(\Omega)$ for the group having generators $S_i$, with $i \in I$, and $X_i(t)$, with $i \in \Omega$ and $t \in R$, modulo the subgroup normally generated by the relators (3)–(7), and to those relators (8)–(13) with $i \in \Omega$. Note that (11) applies only if $m_{ij} = 3$, in which case $i \in \Omega$ implies $j \in \Omega$, so the relator makes sense. Caution: the subscripts on $S$ vary over all of $I$ while those on $X$ vary only over $\Omega$. 


**Lemma 16.** For any component $\Omega$ of $\Delta^{\text{odd}}$,

$$\mathcal{G}_2(\Omega) \cong \left( \star_{\alpha \in \Phi(\Omega)} \mathcal{U}_\alpha \right) \rtimes \hat{W}.$$

**Proof of theorem [11] given lemma [10]** An examination of the presentation of $\mathcal{G}_2$ reveals that the $X$’s corresponding to different components of $\Delta^{\text{odd}}$ don’t interact. Precisely: $\mathcal{G}_2$ is the amalgamated free product of the $\mathcal{G}_2(\Omega)$’s, where $\Omega$ varies over the components of $\Delta^{\text{odd}}$ and the amalgamation is that the copies of $\hat{W}$ in the $\mathcal{G}_2(\Omega)$’s are identified in the obvious way. Lemma 16 shows that $\mathcal{G}_2(\Omega) = \left( \star_{\alpha \in \Phi(\Omega)} \mathcal{U}_\alpha \right) \rtimes \hat{W}$ for each $\Omega$. Taking their free product, amalgamated along their copies of $\hat{W}$, obviously yields $\left( \star_{\alpha \in \Phi} \mathcal{U}_\alpha \right) \rtimes \hat{W}$. □

The rest of the section is devoted to proving lemma 16. So we fix a component $\Omega$ of $\Delta^{\text{odd}}$ and phrase our problem in terms of the free product $F := \left( \star_{j \in \Omega} \mathcal{U}_j \right) \rtimes \hat{W}$. The heart of the proof of lemma 16 is to define normal subgroups $M, N$ of $F$ and show they are equal. $M$ is normally generated by those relators (9)–(13) with $i \in \Omega$, and $\mathcal{G}_2(\Omega)$ was defined as $F/M$. The other set of relations leads to a presentation like the one in lemma 15. But it requires some preparation even to define, so we begin with an informal overview.

Start with the presentation of $\mathcal{G}_2(\Omega)$, and distinguish some point $\infty$ of $\Omega$ and a spanning tree $T$ for $\Omega$. We will use the relators (11) coming from the edges of $T$ to rewrite the $X_j \Omega_{\infty}(t)$ in terms of $X_{\infty}(t)$, and then eliminate the $X_j \Omega_{\infty}(t)$ from the presentation. This “uses up” those relators and makes the other relators messier because each $X_j \Omega_{\infty}(t)$ must be replaced by a word in $X_{\infty}(t)$ and elements of $\hat{W}$. We studied the $W^*$-stabilizer of $a_{\infty}$ in theorem 8 and how it acts on $g_{\infty}$, hence on $\mathcal{U}_{\infty}$. It turns out that the remaining relations in $\mathcal{G}_2(\Omega)$ are exactly the relations that the $\hat{W}$-stabilizer $\hat{W}_{\infty}$ of $a_{\infty}$ acts on $\mathcal{U}_{\infty}$ via $\hat{W}_{\infty} \to \hat{W} \to W^* \subseteq \text{Aut} g$. That is, $\mathcal{G}_2(\Omega) \cong \left( \mathcal{U}_{\infty} \times \hat{W}_{\infty} \right) \rtimes \hat{W}_{\infty}$. Then lemma 15 identifies this with $\left( \star_{\alpha \in \Phi(\Omega)} \mathcal{U}_\alpha \right) \rtimes \hat{W}$.

Now we proceed to the formal proof, beginning by defining some elements of $F$. For $\gamma$ an edge-path in $\Omega$, with $i_0, \ldots, i_n$ the vertices along it, define $\alpha(\gamma) = i_0$ and $\omega(\gamma) = i_n$ as its initial and final endpoints, and define $P_\gamma$ by (3) with $S$’s in place of $s$’s. For $k \in I$ evenly joined to the end of $\gamma$ (i.e., $m_{k\omega(\gamma)}$ finite and even), define

$$R_{\gamma,k} = P^{-1}_\gamma \cdot \begin{cases} S_k & S_kS_{\omega(\gamma)}S_k \\ S_kS_{\omega(\gamma)}S_kS_k & \end{cases} \cdot P_\gamma.$$
according to whether $m_{k\omega(\gamma)} = 2, 4$ or 6. $(R_{\gamma,k}$ is got from (4) by replacing $s$’s and $p$’s by $S$’s and $P$’s and $j$ by $\omega(\gamma)$.) Next, for $t \in R$ we define

$$C_\gamma(t) := P_\gamma X_{\alpha(\gamma)}(t) \cdot \left( X_{\omega(\gamma)}(t) P_\gamma \right)^{-1}$$

and for $k \in I$ evenly joined to $\omega(\gamma)$ we define

$$D_{\gamma,k}(t) := [R_{\gamma,k}, X_{\alpha(\gamma)}(t)].$$

For ease of reference we will also give the name

$$B_{ij}(t) := S^2_i X_j(t) S_i^{-2} \cdot X_j((-1)^{A_{ij}}t)^{-1}$$

to the word (9), where $i \in I$ and $j \in \Omega$. We will suppress the dependence of the $X_j$, $B_{ij}$, $C_\gamma$ and $D_{\gamma,k}$ on $t$ except where it plays a role.

The following formally-meaningless intuition may help the reader; lemma [17] below gives it some support. The relation $C_\gamma = 1$ declares that the path $\gamma$ conjugates the $X$ “at” the beginning of $\gamma$ to the $X$ “at” the end. And the relation $D_{\gamma,k} = 1$ declares that the $X$ “at” the beginning of $\gamma$ commutes with a certain word that corresponds to going along $\gamma$, going around some sort of “small loop near the endpoint of $\gamma$”, and then retracing $\gamma$.

Our first normal subgroup $M$ of $F$ is normally generated by all the $B_{ij}$, the $C_\gamma$ for all $\gamma$ of length 1, and the $D_{\gamma,k}$ for all $\gamma$ of length 0. Unwinding the definitions shows that these elements of $F$ are exactly the ones we used in defining $G_2(\Omega)$. For example, if $\gamma$ is the length 1 path from one vertex $j$ of $\Omega$ to an adjacent vertex $i$ then $P_\gamma = S_j S_i$ and $C_\gamma$ is the word (11). And if $i \in \Omega$ is evenly joined to $j \in I$ then we take $\gamma$ to be the zero-length path at $i$, and $D_{\gamma,j}$ turns out to be the relator (10), (12) or (13). Which one depends on $m_{ij} \in \{2, 4, 6\}$. So $F/M \cong G_2(\Omega)$.

Before defining the other normal subgroup $N$ we explain how to work with the $C$’s and $D$’s by thinking in terms of paths rather than complicated words.

**Lemma 17.** Suppose $\gamma_1$ and $\gamma_2$ are paths in $\Omega$ with $\omega(\gamma_1) = \alpha(\gamma_2)$, and let $\gamma$ be the path which traverses $\gamma_1$ and then $\gamma_2$.

(i) Any normal subgroup of $F$ containing two of $C_{\gamma_1}$, $C_{\gamma_2}$ and $C_\gamma$ contains the third.

(ii) Suppose $k \in I$ is evenly joined to $\omega(\gamma_2)$. Then any normal subgroup of $F$ containing $C_{\gamma_1}$ and one of $D_{\gamma_2,k}$ and $D_{\gamma,k}$ contains the other as well.
Proof. Both identities
\[ C_\gamma = (P_{\gamma_2} C_{\gamma_1} P_{\gamma_2}^{-1}) C_{\gamma_2} \]
\[ D_{\gamma,k} = P_{\gamma_1}^{-1} \left( (R_{\gamma_2,k} C_{\gamma_1} R_{\gamma_2,k}^{-1}) D_{\gamma_2,k} C_{\gamma_1}^{-1} \right) P_{\gamma_1} \]
unravel to tautologies, using \( P_{\gamma} = P_{\gamma_2} P_{\gamma_1} \). These imply (i) and (ii) respectively.

To define \( N \) we refer to the base vertex \( \infty \) and spanning tree \( T \) that we introduced above. For each \( j \in \Omega \) we take \( \delta_j \) to be the backtrackingle-free path in \( T \) from \( \infty \) to \( j \). For each edge of \( \Omega \) not in \( T \), choose an orientation of it, and define \( E \) as the corresponding set of paths of length 1. For \( \gamma \in E \) we write \( z(\gamma) \) for the corresponding loop in \( \Omega \) based at \( \infty \). That is, \( z(\gamma) = \delta_\alpha(\gamma) \) followed by \( \gamma \) followed by the reverse of \( \delta_\omega(\gamma) \). We define \( Z \) as \( \{ z(\gamma) : \gamma \in E \} \), which is a free basis for \( \pi_1(\Omega, \infty) \). We define \( N \) as the subgroup of \( F \) normally generated by all \( B_{i\infty} \) with \( i \in I \), all \( C_{z \in Z} \), the \( C_{\delta_j} \) with \( j \in \Omega \), and all \( D_{\delta_j,k} \) where \( j \in \Omega \) and \( k \in I \) are evenly joined. We will show \( M = N \); one direction is easy:

**Lemma 18.** \( M \) contains \( N \).

Proof. Since \( M \) contains \( C_\gamma \) for every length 1 path \( \gamma \), repeated applications of lemma [17(i)] show that it contains the \( C_{\delta_k} \) and \( C_{z \in Z} \). Since \( M \) contains \( D_{\gamma,k} \) for every \( \gamma \) of length 0, part (ii) of the same lemma shows that \( M \) also contains the \( D_{\delta_j,k} \). Since \( M \) contains all the \( B_{ij} \), not just the \( B_{i\infty} \), the proof is complete.

Now we set about proving the reverse inclusion. For convenience we use \( \equiv \) to mean “equal modulo \( N \)”. We must show that each generator of \( M \) is \( \equiv 1 \).

**Lemma 19.** \( C_\gamma \equiv 1 \) for every length 1 subpath \( \gamma \) of every \( \delta_j \).

Proof. This follows from lemma [17(i)] because \( \delta_\alpha(\gamma) \) followed by \( \gamma \) is \( \delta_\omega(\gamma) \). □

**Lemma 20.** \( B_{ik} \equiv 1 \) for all \( i \in I \) and \( k \in \Omega \).

Proof. We claim: if \( \gamma \) is a length 1 path in \( \Omega \), such that \( C_\gamma \equiv 1 \) and \( B_{i\alpha(\gamma)} \equiv 1 \) for every \( i \in I \), then also \( B_{i\omega(\gamma)} \equiv 1 \) for every \( i \in I \). Assuming this, we use the fact that \( B_{i\infty} \equiv 1 \) for all \( i \in I \) and also \( C_\gamma \equiv 1 \) for every length 1 subpath \( \gamma \) of every \( \delta_k \) (lemma [19]). Since every \( k \in \Omega \) is the end of chain of such \( \gamma \)’s starting at \( \infty \), the lemma follows by induction.
So now we prove the claim, writing $i$ for some element of $I$ and $j$ and $k$ for the initial and final endpoints of $\gamma$. We use $C_\gamma \equiv 1$, i.e., $S_j S_k X_j(t) \equiv X_k(t) S_j S_k$, to get

$$S_i^2 X_k(t) S_i^{-2} \equiv S_i^2 S_j S_k X_j(t) S_k^{-1} S_i^{-1} S_j^{-1}$$

(31)

$$= S_j S_k \left[ (S_k^{-1} S_j^{-1} S_i^2 S_j S_k) X_j(t) (S_k^{-1} S_j^{-1} S_i^{-2} S_j S_k) \right] S_k^{-1} S_j^{-1}$$

We rewrite the relation from lemma [10(ii)] as $S_j^{-1} S_i^2 S_j = S_j^{(-1)^{A_{ij}}} - S_i^2$. Then we use it and its analogues with subscripts permuted to simplify the first parenthesized term in (31). We also use $A_{jk} = -1$, which holds since $j$ and $k$ are joined. The result is

$$S_k^{-1} S_j^{-1} S_i^2 S_j S_k = S_k^{1-(-1)^{A_{ij}}} S_j^{(-1)^{A_{ij}}} - S_i^{(-1)^{A_{ik}}} - S_i^2$$

Note that each exponent is 0 or ±2.

The bracketed term in (31) is the conjugate of $X_j(t)$ by this. We work this out in four steps, using our assumed relations $B_{ij} \equiv B_{jj} \equiv B_{kj} \equiv 1$. Conjugation by $S_i^2$ changes $X_j(t)$ to $X_j((-1)^{A_{ij}} t)$. Because $A_{kj} = -1$, conjugating $X_j((-1)^{A_{ij}} t)$ by $S_k^{(-1)^{A_{ik}}} - 1$ sends it to itself if $A_{ik}$ is even, because $(-1)^{A_{ik}} - 1 = 0$ and

$$X_j((-1)^{A_{ij}} t)$$

if $A_{ik}$ is odd, because $(-1)^{A_{ik}} - 1 = -2$.

We write this as $X_j((-1)^{A_{ik}} (-1)^{A_{ij}} t)$. In the third step we conjugate by an even power of $S_j$, which does nothing. The fourth step is like the second, and introduces a second factor $(-1)^{A_{ij}}$. The net result is that the bracketed term of (31) equals $X_j((-1)^{A_{ik}} t)$ modulo $N$.

Plugging this into (31) and then using the conjugacy relation $C_\gamma \equiv 1$ between $X_j$ and $X_k$ yields

$$S_i^2 X_k(t) S_i^{-2} \equiv S_j S_k X_j((-1)^{A_{ik}} t) S_k^{-1} S_j^{-1} \equiv X_k((-1)^{A_{ik}} t).$$

We have established the desired relation $B_{ik} \equiv 1$. □

Lemma 21. Suppose $\gamma$ is a length 1 path in $\Omega$ with $C_\gamma \equiv 1$. Then $C_{\text{reverse}(\gamma)} \equiv 1$ also.

Proof. Suppose $\gamma$ goes from $j$ to $k$. We begin with our assumed relation $C_\gamma \equiv 1$, i.e., $S_j S_k X_j(t) \equiv X_k(t) S_j S_k$, rearrange and apply the relation from lemma [10(ii)] with $A_{jk} = \text{odd}$.

$$X_k(t) \equiv S_j S_k X_j(t) S_k^{-1} S_j^{-1}$$

$$S_k S_j X_k(t) \equiv (S_k S_j S_k^{-1}) S_k^2 X_j(t) S_k^{-1} S_j^{-1}$$

$$= (S_k S_j) S_k^2 X_j(t) S_k^{-1} S_j^{-1}. $$
Then we use lemma $\text{B_{0}}$ $B_{jj} \equiv B_{kj} \equiv 1$ with $A_{kj} = \text{odd}$:

$\equiv S_{k}^{2}S_{j}^{2}X_{j}(-t)S_{k}^{2}S_{j}^{-1}S_{j}^{-1}$

$\equiv S_{k}^{2}X_{j}(-t)S_{j}^{2} \cdot S_{k}^{2}S_{j}^{-1}S_{j}^{-1}$

$\equiv X_{j}(t)S_{k}^{2}S_{j}^{2} \cdot S_{k}^{2}S_{j}^{-1}S_{j}^{-1}$

$\equiv X_{j}(t)S_{k}S_{j}^{-1}$

We have shown $C_{\text{reverse(γ)}} \equiv 1$, as desired. $\square$

Lemma 22. $M = N$. In particular, $\mathfrak{G}_{2}(\Omega)$ is the quotient of $F = (\ast_{\gamma \in \Omega} U_{\gamma}) \ast \hat{W}$ by $N$.

Proof. We showed $N \subseteq M$ in lemma $\text{B_{1}}$. To show the reverse inclusion, recall that $M$ is normally generated by all $B_{ij}$, the $C_{\gamma}$ for all $\gamma$ of length 1, and the $D_{\gamma,k}$ for all $\gamma$ of length 0. We must show that each of these $\equiv 1$. We showed $B_{ij} \equiv 1$ in lemma $\text{B_{0}}$.

Next we show that $C_{\gamma} \equiv 1$ for every length 1 path $\gamma$ in $T$. Either $\gamma$ is part of one of the paths $\delta_j$ in $T$ based at $\infty$, or its reverse is. In the first case we know $C_{\gamma} \equiv 1$ by lemma $\text{B_{1}}$. In the second the same reasoning shows $C_{\text{reverse(γ)}} \equiv 1$, and then lemma $\text{B_{1}}$ gives $C_{\gamma} \equiv 1$.

Lemma $\text{B_{1}}(\gamma)$ now shows $C_{\gamma} \equiv 1$ for every path $\gamma$ in $T$.

Next we show $C_{\gamma} \equiv 1$ for every length 1 path $\gamma$ not in $T$. Recall that we chose a set $E$ of length 1 paths, one traversing each edge of $\Omega$ not in $T$. For $\gamma \in E$ we wrote $z(\gamma)$ for the corresponding loop in $\Omega$ based at $\infty$, namely $\delta_{\alpha(\gamma)}$ followed by $\gamma$ followed by $\text{reverse}(\delta_{\omega(\gamma)})$. Recall that $N$ contains $C_{z(\gamma)}$ by definition, and contains $C_{\delta_{\alpha(\gamma)}}$ and $C_{\text{reverse}(\delta_{\omega(\gamma)})}$ by the previous paragraph. So a double application of lemma $\text{B_{1}}(\gamma)$ proves $C_{\gamma} \in N$. And another use of lemma $\text{B_{1}}$ shows that $N$ also contains $C_{\text{reverse(γ)}}$. This finishes the proof that $C_{\gamma} \equiv 1$ for all length 1 paths $\gamma$ in $\Omega$.

It remains only to show $D_{\gamma,k} \equiv 1$ for every length 0 path $\gamma$ in $\Omega$ and each $k \in I$ joined evenly to the unique point of $\gamma$, say $j$. Since $N$ contains $C_{\delta_j}$ and $D_{\delta_j,k}$ by definition, and $\delta_j$ followed by $\gamma$ is trivially equal to $\delta_j$, lemma $\text{B_{1}}(\gamma)$ shows that $N$ contains $D_{\gamma,k}$ also. $\square$

We now review the “shape” of the presentation $\mathfrak{G}_{2}(\Omega) \cong F/N$. The generators are the $S_{\gamma \in I}$ and the $X_{j \in \Omega}(t \in R)$. The relations are the addition rules defining the $U_{\gamma}$, the relations on the $S_{i}$’s defining $\hat{W}$, and the $B_{i \in I}$, $C_{z \in Z}$, $C_{\delta_j}$ and $D_{\delta_j,k}$ where $i$ varies over $I$, $j$ over $\Omega$ and $k \in I$ is evenly joined to $j$. The relations $B_{i \in I} \equiv 1$ say that the $S_{i}^{2}$ centralize or invert the $X_{\infty}$.

Each relation $C_{z} \equiv 1$ says that a certain word in $\hat{W}$ conjugates each $X_{\infty}$ to itself. The relations $D_{\delta_j,k} \equiv 1$
say that certain other words in \( \hat{W} \) commute with each \( X_\infty(t) \). Finally, for each \( j \), the relations \( C_{\delta_j} \equiv 1 \) express the \( X_j(t) \) as conjugates of the \( X_\infty(t) \) by still more words in \( \hat{W} \). The obvious way to simplify the presentation is to use this last batch of relations to eliminate the \( X_{j \neq \infty}(t) \) from the presentation. We make this precise in the following lemma.

**Lemma 23.** Define \( F_\infty = \mathcal{U}_\infty * \hat{W} \) and let \( N_\infty \) be the subgroup normally generated by the \( B_{i\infty} (i \in I) \), the \( C_z \ (z \in \mathbb{Z}) \), and the \( D_{\delta_j,k} \ (j \in \Omega \text{ and } k \in I \text{ evenly joined}) \). Then the natural map \( F_\infty/N_\infty \to F/N \) is an isomorphism.

**Proof.** We begin with the presentation \( F/N \) from the previous paragraph and apply Tietze transformations. The relation \( C_{\delta_j}(t) \equiv 1 \) reads:

\[
X_j(t) \equiv P_{\delta_j} X_\infty(t) P_{\delta_j}^{-1}
\]

For \( j = \infty \) this is the trivial relation \( X_\infty(t) = X_\infty(t) \), which we may discard. For \( j \neq \infty \) we use it to replace \( X_j(t) \) by \( P_{\delta_j} X_\infty(t) P_{\delta_j}^{-1} \) everywhere else in the presentation, and then discard \( X_j(t) \) from the generators and \( C_{\delta_j}(t) \) from the relators.

The only other occurrences of \( X_{j \neq \infty}(t) \) in the presentation are in the relators defining \( \mathcal{U}_j \). After the replacement of the previous paragraph, these relations read

\[
P_{\delta_j} X_\infty(t) P_{\delta_j}^{-1} \cdot P_{\delta_j} X_\infty(u) P_{\delta_j}^{-1} \equiv P_{\delta_j} X_\infty(t + u) P_{\delta_j}^{-1}.
\]

These relations can be discarded because they are the \( P_{\delta_j} \)-conjugates of the relations \( X_\infty(t) X_\infty(u) \equiv X_\infty(t + u) \). What remains is the presentation \( F_\infty/N_\infty \). \( \Box \)

**Proof of lemma 10.** The previous lemma shows \( \mathfrak{G}_2(\Omega) \cong F_\infty/N_\infty \). This describes \( \mathfrak{G}_2(\Omega) \) as the quotient of \( \mathcal{U}_\infty * \hat{W} \) by relations asserting that certain elements of \( \hat{W} \) act on \( \mathcal{U}_\infty \) by automorphisms. The relations \( B_{i\infty} = 1 \) make \( S_i^2 \) act on \( \mathcal{U}_\infty \) by \((-1)^{A_\infty}\). The relations \( C_z \equiv D_{\delta_j,k} = 1 \) make the words \( P_z \) and \( R_{\delta_j,k} \) centralize \( \mathcal{U}_\infty \).

By lemma 10(iii) the \( S_i^2 \) generate the kernel of \( \hat{W} \to W \). By theorem 8 the images of the \( P_z \) and \( R_{\delta_j,k} \) in \( W \) generate the \( W \)-stabilizer of the simple root \( \infty \in I \). Therefore the \( S_i^2, P_z \) and \( R_{\delta_j,k} \) generate the \( \hat{W} \)-stabilizer \( \hat{W}_\infty \) of \( \infty \). Their actions on \( \mathcal{U}_\infty \) are the same as the ones given by the homomorphism \( \hat{W} \to W^* \), by theorem 8. Therefore \( \mathfrak{G}_2(\Omega) = (\mathcal{U}_\infty \times \hat{W}_\infty) * \hat{W}_\infty \). And lemma 13 identifies this with \( (\ast_{\alpha \in \Phi(\Omega)} \mathcal{U}_\alpha) \times \hat{W} \), as desired. \( \Box \)
6. The isomorphism $G_3 \cong \mathfrak{T}^d \rtimes \hat{W}$

We have two goals in this section. The first is to prove theorem [12] that $G_3 \cong \mathfrak{T}^d \rtimes \hat{W}$, beginning from theorem [11] that $G_2 \cong (\ast \alpha \in \Phi U_\alpha) \rtimes \hat{W}$. The second is to explain how one may discard many of the Chevalley relators, for example for $E_6, 7,$ or $8$ one can get away with imposing the relators for a single unjoined pair of nodes of the Dynkin diagram, and for a single joined pair. The latter material is not necessary for our main results.

Several times we will refer to elements of $W^*$ or $\hat{W}$ conjugating the Chevalley relations for a given prenilpotent pair of roots to those for another such pair. To make this precise we specify that the phrase “the Chevalley relators for $\alpha, \beta$” means the kernel of $\ast \gamma \in \Theta(\alpha, \beta) U_\gamma \rightarrow U_{\Theta(\alpha, \beta)}$, rather than any particular set of words normally generating this kernel. The definition of $\mathfrak{T}$ shows that if $\alpha, \beta \in \Phi$ and $w^* \in W^*$ then the $w^*$-conjugates of the Chevalley relators for $\alpha, \beta$ are those of their $w^*$-images.

**Proof of theorem [12]** First note that the relators (14)–(27) have the form of the Chevalley relators. We will discuss (20) in detail and the others very briefly. In (20), $s$ and $l$ are short and long simple roots of a $B_2$ root system, and $X_s(t) \in U_s$ and $X_l(u) \in U_l$. There are two roots in $(\mathbb{N}s + \mathbb{N}l) - \{s, l\}$, namely $s + l$ and $2s + l$. Because $s_l(l) = 2s + l$ and $s_l(s) = s + l$, we have $S_sX_l(-t^2u)S_s^{-1} \in U_{2s+l}$ and $S_lX_s(tu)S_l^{-1} \in U_{s+l}$. Killing the word (20) adjoins a relation

$$[X_s(t), X_l(u)] = \text{(element of } U_{2s+l})\text{(element of } U_{s+l}),$$

to $G_2$, which does indeed have the form of the Chevalley relations.

In the other cases we simply mention the relative positions of the roots appearing in the commutator:

- eqn. (14): orthogonal roots in $A_2^2$
- eqn. (15): roots in $A_2$ with angle $\pi/3$
- eqn. (16): roots in $A_2$ with angle $2\pi/3$
- eqn. (17): a short and a long root in $B_2$ with angle $\pi/4$
- eqn. (18): orthogonal long roots in $B_2$
- eqn. (19): orthogonal short roots in $B_2$
- eqn. (20): a short and a long root in $B_2$ with angle $3\pi/4$ (details above)
- eqn. (21): long roots in $G_2$ with angle $\pi/3$
- eqn. (22): a short and a long root in $G_2$ with angle $\pi/6$
- eqn. (23): orthogonal roots in $G_2$
- eqn. (24): long roots in $G_2$ with angle $2\pi/3$
- eqn. (25): short roots in $G_2$ with angle $\pi/3$
- eqn. (26): short roots in $G_2$ with angle $2\pi/3$
eqn. (27): a short and a long root in $G_2$ with angle $5\pi/6$.

Next we claim that the words (14)–(27) are (or in terms of our formal definition: normally generate) the Chevalley relators for the listed pairs of roots. If $\alpha, \beta$ are two roots in one of these configurations then $U_{\theta(\alpha,\beta)}$ is a subgroup of one of the Chevalley groups $SL_2 \times SL_2$, $SL_3$, $Sp_4$ or $G_2$. As mentioned in section 4 we used a computer to verify that the relators map to the trivial element of these groups over $R = \mathbb{Z}[t,u]$, hence over any ring. It follows from lemma 9(iii) that they are the Chevalley relators. (The isomorphism there implies that the only relations having the form of the Chevalley relations that can hold are the Chevalley relations themselves.)

Next we show that the Chevalley relators become trivial in $G_3$ for any classically prenilpotent pair $\alpha', \beta' \in \Phi$. By classical prenilpotency, $\Phi'_0 := (Q\alpha' \oplus Q\beta') \cap \Phi$ is an $A_1^2$, $A_2$, $B_2$ or $G_2$ root system. There exists $w \in W$ sending $\Phi'_0$ to the root system $\Phi_0 \subseteq \Phi$ generated by some pair of simple roots. (Choose simple roots for $\Phi'_0$. Then choose a chamber in the open Tits cone which has facets lying in the mirrors of those roots, and which lies on the positive sides of these mirrors. Take $w$ sending this chamber to the standard one.)

Some pair $\alpha, \beta \in \Phi_0$ have (i) the same configuration relative to each other as $\alpha', \beta'$ have relative to each other, and (ii) normal generators for their Chevalley relators appearing among (14)–(27). This is just the assertion that the above list of possible relative positions is complete, which is easy to check. By refining the choice of $w$, we may suppose that it sends $\{\alpha', \beta'\}$ to $\{\alpha, \beta\}$. Now choose $\hat{w} \in \hat{W}$ lying over $w$. The Chevalley relators for $\alpha', \beta'$ are the $\hat{w}^{-1}$-conjugates of the Chevalley relators for $\alpha, \beta$. Since the latter are trivial in $S_3$, so are the former. \quad \square

The proof of theorem 12 amounted to exploiting the $\hat{W}$-action on $*_{\alpha \in \Phi} U_\alpha$ to obtain all the classical Chevalley relators from those listed explicitly in (14)–(27). One can further exploit this idea to omit many of these relators. The key ingredient is the notion of an ordered pair of simple roots being associate to another. In Brink-Howlett [8] and Borcherds [6] this means conjugacy by $W$ of the ordered pair of their reflections. We adopt a slightly more flexible definition, which amounts to the Chevalley relations for one ordered pair being “the same as” those for another pair.

The $A_1^2$ Chevalley relators (14) are indexed by a set we call $E_2$: all ordered pairs $(i, j)$ in $I$ with $m_{ij} = 2$. Consider the equivalence relation on $E_2$ generated by $(i, j) \sim (j, i)$ and by $(i, j) \sim (i', j')$ whenever $i, j, i', j'$ all lie in some $A_1A_2$ subdiagram of the Dynkin diagram. Two elements of $E_2$ are called associates if they are equivalent in this sense.
As examples we note that $E_2$ falls into
3 associate classes for $D_4$
2 for $(B$ or $C)_{n \geq 4}$ or $D_{n \geq 5}$, and
1 for $A_{n \geq 3}$, $(B$ or $C)_3$, $E_n$ or $F_4$.

The $A_2$ Chevalley relators (15)–(16) are indexed by the set $E_3$ of ordered pairs defined in the same way using the condition $m_{ij} = 3$. We define associates differently, in particular non-symmetrically, using the equivalence relation generated by $(i, j) \sim (j, k)$ whenever $i, j, k$ form an $A_3$ diagram. There are

4 associate classes for $F_4$
2 for $A_{n \geq 2}$ or $(B$ or $C)_{n \geq 3}$
1 for $D_{n \geq 4}$ or $E_n$.

The following proposition usually greatly reduces the number of Chevalley relators that must be imposed on $G_2$ to get $G_3$. There does not seem to be any similar reduction possible for $m_{ij} = 4$ or 6.

**Proposition 24.** If $(i, j) \in E_2$ or $E_3$ then the normal closure in $G_2$ of $(i, j)$’s Chevalley relators also contains the Chevalley relators of $(i, j)$’s associates.

**Proof.** We write $N$ for this normal closure. First we treat the $E_2$ case, so $N$ is the normal closure of $[\mathcal{U}_i, \mathcal{U}_j]$. Obviously $N$ contains the Chevalley relators $[\mathcal{U}_j, \mathcal{U}_i]$ for $(j, i)$ . Next we show that $N$ contains the Chevalley relators for $(i, j')$ if $m_{ij'} = 2$ and $m_{jj'} = 3$. This holds because $\mathcal{U}_j$ and $\mathcal{U}_{j'}$ are conjugate under $\langle S_j, S_{j'} \rangle$, which centralizes $\mathcal{U}_i$. This uses relator (11) in $G_2$.

The $E_3$ case is similar but fancier. Fix $(i, j) \in E_3$, so $N$ is normally generated by the Chevalley relators for $(\alpha_i, \alpha_j)$ and $(\alpha_i, \alpha_i + \alpha_j)$. We suppose $m_{ik} = 2$ and $m_{jk} = 3$ and must show that $N$ contains the Chevalley relators for $(\alpha_j, \alpha_k)$ and $(\alpha_j, \alpha_j + \alpha_k)$. It suffices to exhibit $\hat{w} \in \hat{W}$ conjugating the “known” Chevalley relators to the desired ones, or equivalently sending $(\alpha_i, \alpha_j)$ to $(\alpha_j, \alpha_k)$.

The opposition involution of a spherical Coxeter system means the unique element of that Coxeter group which exchanges the fundamental chamber with its negative. It is also called the fundamental element or “the long word”. It is easy and standard that the opposition involution in $\langle s_i, s_j \rangle$ sends $\alpha_i, \alpha_j$ to $-\alpha_j, -\alpha_i$, and that the opposition involution in $\langle s_i, s_j, s_k \rangle$ sends $\alpha_i, \alpha_j, \alpha_k$ to $-\alpha_k, -\alpha_j, -\alpha_i$. So their composition $w$ sends $\alpha_i, \alpha_j$ to $\alpha_j, \alpha_k$. Choose any $\hat{w} \in \hat{W}$ lying over $w$. □
7. The isomorphism $\mathfrak{G}_4 \cong \mathfrak{G}t^d \ltimes \hat{W}$

In this section we prove theorem 13 that $\mathfrak{G}_4 \cong \mathfrak{G}t^d \ltimes \hat{W}$, starting from theorem 12 that $\mathfrak{G}_3 \cong \mathfrak{T}t^d \ltimes \hat{W}$. We use standard Steinberg-group manipulations, and a re-use of the proof of theorem 11 from section 5.

The argument can be carried out directly in $\mathfrak{G}_4$, but it becomes clearer if we use only some of the relations present in $\mathfrak{G}_4$. To describe which ones we want and why they are useful, we define a new group $\mathfrak{G}'_2$ and show that its structure is a lot like the group $\mathfrak{G}_2$ we studied in section 5. We will show that $\mathfrak{G}'_2$ maps to $\mathfrak{G}_4$, and then theorem 13 follows easily.

We define $\hat{N}$ as the group with generators $\tilde{S}_i(a)$ where $i \in I$ and $a \in R^*$, modulo the subgroup normally generated by the following relators. We abbreviate $\tilde{S}_i(1)$ to $\tilde{S}_i$ and write $\tilde{H}_i(a)$ for $\tilde{S}_i(a)\tilde{S}_i(-1)$.

\begin{align*}
(32) & \quad \tilde{S}_i(a) \cdot \tilde{S}_i(-a) \\
(33) & \quad (\tilde{S}_i \tilde{S}_j \cdots) \cdot (\tilde{S}_j \tilde{S}_i \cdots)^{-1} \quad \text{if } m_{ij} \neq \infty \\
(34) & \quad \tilde{S}_j(b)^{-1} \tilde{H}_i(a) \tilde{S}_j(b) \cdot \left(\tilde{H}_j(-b) \tilde{H}_j(-a^{A_{ij}}b)^{-1} \tilde{H}_i(a)^{-1}\right) \quad \text{if } m_{ij} = 2
\end{align*}

where the dots in (33) indicate the Artin relations just as in (5). This group is different from the group called $\hat{N}$ in [13] because we have omitted some relations that hold in the Steinberg group but are unneeded here.

**Lemma 25.** The subgroup $\hat{H}$ generated by the $\tilde{H}_i(a)$’s is normal, with quotient the Weyl group $W$.

**Proof.** The generators’ inverses normalize $\hat{H}$ by (34), and (32) shows that the generators themselves do too. Together with (32), the definition of $\tilde{H}_i(a)$ shows that every $\tilde{S}_i(a)$ differs from $\tilde{S}_i$ by an element of $\hat{H}$. Also, $\tilde{S}_i^2 = (\tilde{S}_i(-1)^{-1})^2 = (\tilde{S}_i(-1)^2)^{-1} = \tilde{H}_i(-1)^{-1}$. Because of the Artin relations (33) this proves $\hat{N}/\hat{H} = W$. \qed

Next we define $\mathfrak{G}'_2$ as the quotient of $(\ast_{i \in I} \mathfrak{U}_i) \ast \hat{N}$ by the subgroup normally generated by the relators

\begin{align*}
(35) & \quad \tilde{H}_i(r) X_j(t) \tilde{H}_i(r)^{-1} \cdot X_j(r^{A_{ij}} t)^{-1} \\
(36) & \quad [\tilde{S}_i, X_j(t)] \quad \text{if } m_{ij} = 2 \\
(37) & \quad \tilde{S}_j \tilde{S}_i X_j(t) \cdot (X_i(t) \tilde{S}_j \tilde{S}_i)^{-1} \quad \text{if } m_{ij} = 3 \\
(38) & \quad [\tilde{S}_i \tilde{S}_j \tilde{S}_i, X_j(t)] \quad \text{if } m_{ij} = 4
\end{align*}
Lemma 26. $\mathfrak{G}'_2$ is isomorphic to $\left( \underset{a \in \Phi}{\ast} \mathfrak{U}_a \right) \rtimes \hat{N}$, where each $\tilde{S}_i(a) \in \hat{N}$ acts on the free product as $s_i^a \circ h_i(a)$.

**Proof.** This is almost the same as theorem 11 proven in section 5. For a component $\Omega$ of $\Delta^{\text{odd}}$ one defines $\mathfrak{G}'_2(\Omega)$ by omitting the generators $X_{j \notin \Omega}(t)$ and the relations involving them, just as we did for $\mathfrak{G}_2(\Omega)$. The analogue of lemma 16 is

$$\mathfrak{G}'_2(\Omega) \cong \left( \underset{a \in \Phi(\Omega)}{\ast} \mathfrak{U}_a \right) \rtimes \hat{N}$$

and the current lemma follows from this just as theorem 11 follows from lemma 16.

To prove (40) one continues following section 5. The group $F$ there gets replaced by $\left( \underset{i \in \Omega}{\ast} \mathfrak{U}_i \right) \rtimes \hat{N}$. One defines a normal subgroup $M$ of $F$ just as before, and again it is immediate that $F/M \cong \mathfrak{G}'_2(\Omega)$. The only difference is that the relators $B_{ij}(t)$ are replaced by (35). One also defines the normal subgroup $\hat{N}$ of $F$ just as before, and shows $M = N$. The main difference is that the calculation for lemma 20 must be replaced by a more complicated calculation in $\mathfrak{G}'_2(\Omega)$. Writing $\equiv$ for equality modulo $N$ as before, one begins with

$$\hat{H}_i(a)X_k(t)\hat{H}_i(a)^{-1} \equiv \tilde{S}_j \hat{S}_k \left[ \left( \tilde{S}_k^{-1} \tilde{S}_j^{-1} \hat{H}_i(a) \tilde{S}_j \tilde{S}_k \right) X_j(t) \left( \tilde{S}_k^{-1} \tilde{S}_j^{-1} \hat{H}_i(a)^{-1} \tilde{S}_j \tilde{S}_k \right) \right] \tilde{S}_k^{-1} \tilde{S}_j^{-1}$$

Using (34) one “simplifies” the first parenthesized term to

$$\hat{H}_k(-1)\hat{H}_j(-1)\hat{H}_j(-a^{A_{ij}})^{-1} \hat{H}_k(a^{-A_{ij}}) \hat{H}_k(-a^{A_{ik}})^{-1} \hat{H}_i(a).$$

Then one conjugates $X_j(t)$ by this using (35), collapsing the bracketed terms to $X_j(a^{A_{ik}})$. Following the rest of the argument gives $\hat{H}_i(a)X_k(t)\hat{H}_i(a)^{-1} \equiv X_k(a^{A_{ik}})$, as desired.

The only other difference in the proof of $M = N$ is that one must replace $\tilde{S}_j^2$ by $\hat{H}_j(-1)^{-1}$ in the calculation for lemma 21. Once $M = N$ is known, one can follow section 5 verbatim to establish (40). \[ \square \]

**Remark.** One could avoid repeating the proofs from section 5 by using $\hat{N}$ in place of $\hat{W}$ in the definition of $\mathfrak{G}_2$. We prefer to leave the presentation less cluttered, particularly since step 4 of section 4 and the entire current section can be omitted in most cases of interest.

**Lemma 27.** There is a homomorphism $\mathfrak{G}'_2 \to \mathfrak{G}_4$ sending each $X_i(t)$ to itself and each $\tilde{S}_i(a)$ to $\tilde{s}_i(a)$.
Proof. We need to show that replacing $\tilde{S}_i(a)$’s by $\tilde{s}_i(a)$’s in any one of the relators (32)–(39) yields the trivial element of $\mathfrak{G}_4$. For (32) this follows from the definition of $\tilde{s}_i(a)$. For the Artin relations (33) we refer to Tits [18, (d) of p. 551]. We will come back to (34) later. For (35) we note that $\tilde{S}_i(a) \mapsto \tilde{s}_i(a)$ implies $\tilde{H}_i(a) \mapsto \tilde{h}_i(a)$, and refer to the relator (28) in the definition of $\mathfrak{G}_4$. For (36) we note that $\tilde{s}_i$ lies in $\langle \mathfrak{U}_i, S_i \rangle$ by definition, which commutes with $\mathfrak{U}_j$ by (10) from the definition of $\mathfrak{G}_2$ and (14) from the definition of $\mathfrak{G}_3$.

For (37)–(39) we rely on Tits’ study [18, (b) of p. 550] of the action of $\tilde{s}_i$ on the $\mathfrak{U}_\alpha$, where any $\alpha$ in the finite root system generated by $\alpha_i, \alpha_j$. He showed that if $m_{ij} = 3, 4, 6$ then already in $\mathfrak{G}_3$, $\tilde{s}_i$ acts same way $s_i^*$ does. This action is described in lemma 3 establishing (37)–(39).

It remains to establish that $\tilde{S}_i(r) \mapsto \tilde{s}_i(r)$ sends (34) to 1 $\in \mathfrak{G}_4$. We first establish the triviality of (28)–(29) in $\mathfrak{G}_4$ for all pairs $i, j$, not just the pairs specified there. If $m_{ij} = 2$ then we use the fact that $\tilde{h}_i(r) \in \langle S_i, \mathfrak{U}_i \rangle$, which commutes with $\langle S_j, \mathfrak{U}_j \rangle$ by (10) and (14). If $m_{ij} \in \{3, 4, 6\}$ then we refer to the previous paragraph—Tits showed that $\tilde{s}_i(r)$ acts on all those $\mathfrak{U}_\alpha$ just as $s_i^* \circ h_i(r)$ does. If follows that $\tilde{h}_i(r)$ acts on $\mathfrak{U}_{\pm \alpha_i}$ and $\mathfrak{U}_{\pm \alpha_j}$ the same way $h_i(r)$ does. The $\mathfrak{U}_{\pm \alpha_j}$ case establishes (28)–(29), and the $\mathfrak{U}_{\pm \alpha_i}$ case establishes the $i = j$ case of (28)–(29). It remains only to establish (28)–(29) in the case that $m_{ij} = \infty$ and in the case that $i = j$ and there is no $k \in I$ with $m_{ik} \in \{3, 4, 6\}$. These are exactly the cases in which we imposed these relators when defining $\mathfrak{G}_4$.

For any $i, j \in I$ and $a, b \in R^*$ it follows that

$$\tilde{h}_i(a)\tilde{s}_j(b)\tilde{h}_i(a)^{-1} = \tilde{s}_j(aA^{\alpha_j}b).$$

To see this, one just applies (28)–(29) and the definition of $\tilde{s}_j(a)$ to the left side. Now one multiplies on the left by $\tilde{s}_j(b)^{-1}$ and on the right by $\tilde{h}_i(a)$ to get

$$\tilde{s}_j(b)^{-1}\tilde{h}_i(a)\tilde{s}_j(b) = \tilde{s}_j(b)^{-1}\tilde{s}_j(aA^{\alpha_j}b)\tilde{h}_i(a) = \tilde{s}_j(-b)\tilde{s}_j(-aA^{\alpha_j}b)^{-1}\tilde{h}_i(a) = \tilde{s}_j(-b)\tilde{s}_j(-1) \cdot \tilde{s}_j(-1)^{-1}\tilde{s}_j(-aA^{\alpha_j}b)^{-1} \cdot \tilde{h}_i(a) = \tilde{h}_i(-b)\tilde{h}_j(-aA^{\alpha_j}b)^{-1}\tilde{h}_i(a)$$

That is, the remaining relator (34) for $\mathfrak{G}_2'$ maps to 1 $\in \mathfrak{G}_4$.  \qed
Proof of theorem 13. Lemma 26 shows that in $G', \tilde{S}_i(a)$ acts on each $U_{\beta} \in \Phi$ by $s_i^* \circ h_i(a)$. Taking the image in $G_4$ gives the relation
\[
\tilde{s}_{e_i}(a) u \tilde{s}_{e_i}(a)^{-1} = \left((s_i^* \circ h_i(a))(u)\right)
\]
for all $i \in I$, $a \in R^*$ and $u \in U_{\beta}$ with $\beta \in \Phi$. Conjugating these relations by $W^*$ gives us the corresponding relations with the simple roots $\alpha_i$ and the $e_i$ replaced by an arbitrary root $\alpha$ and generator $e$ for $g_{\alpha, Z}$. These are the relations that reduce $\Sigma^{cl}$ to $\mathcal{S}^{cl}$, yielding $G_4 = \mathcal{S}^{cl} \rtimes \widetilde{W}$. \qed

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