Abstract

When counting subgraphs, well-known algebraic identities arise, which can be addressed by the theory of combinatorial Hopf algebras. For the many notions of subgraphs, e.g., restricted, connected, or induced, there appear several definitions of graph counting operations in the literature. For the same reason, there exist several combinatorial Hopf algebras on graphs. At first sight, the connections between those Hopf algebras are not clear. However, we will show that they are isomorphic to the symmetric Hopf algebra. In addition, the transformation formulas between different counting operations, many of which are known in the literature, are in fact Hopf algebra isomorphisms. This article approaches the counting problem of subgraphs in terms of signature-type functionals. These functionals satisfy character properties as well as a version of Chen’s identity. We explain also how the transformations interact with the functionals.

Keywords: Combinatorial Hopf algebras, graph theory, (quasi-)shuffle products, graph counting, graph products, signature, Chen’s identity

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1 Introduction

We consider finite simple undirected graphs. Fixing such a graph $\Lambda$, we denote the number of times another graph $\sigma$ appears as a subgraph of $\Lambda$ by $c_\sigma(\Lambda)$. It is known that the product of two such quantities is again a linear combination of counting occurrences

$$c_\sigma(\Lambda)c_\tau(\Lambda) = \sum_\gamma a(\gamma; \sigma, \tau)c_\gamma(\Lambda). \quad (1)$$

See for example the references [Mau+20] and [Ruc88]. Due to the fact that the coefficients $a(\gamma; \sigma, \tau)$ in (1) do not depend on the graph $\Lambda$, such identities can be expressed by defining a product on graphs. For details, we refer the reader to [Bor15], [Mau+20] and [Pen20]. Namely, one has what we call a quasi-shuffle product directly defined on graphs

$$\sigma \sqshuffle \tau = \sum_\gamma a(\gamma; \sigma, \tau)\gamma.$$

The graphs $\gamma$ on the right are certain combinations (to be made precise) of the two graphs, $\sigma$ and $\tau$, and the coefficients $a(\gamma; \sigma, \tau)$ encode multiplicities associated to each triple. This product defines a (unital, commutative and associative) algebra on the space spanned by graphs. In the work at hand, we propose to consider the number of occurrences of a subgraph $\sigma$ in a graph $\Lambda$ as a feature of the latter. Those features, i.e., the numbers $c_\sigma(\Lambda)$, are stored in the so-called graph-counting signature of the graph $\Lambda$. More precisely, the graph-counting signature is a functional $\text{GC}$ that associates to any graph $\Lambda$ a particular linear map on the quasi-shuffle (Hopf) algebra of graphs

$$\text{GC}(\Lambda) := \sum_\gamma c_\gamma(\Lambda)\gamma^*.$$

Seen as a function, the occurrences of subgraphs in the graph $\Lambda$ are extracted by pairing $\langle \text{GC}(\Lambda), \sigma \rangle = c_\sigma(\Lambda)$. We mention that the notion of signature was originally developed for solving differential equations. See for instance [Lyo98]. Recent applications in other fields include feature extraction from time series or machine learning [BF20; CO18; DET20b; DET20a; KO19]. The idea of associating a certain sequence
of numbers (features) to a graph is present in the literature. For instance, in [Lov67, Thm. 3.6], [Lov12, Thm. 5.29] and [Bor+06], the notion of graph profile appears: a graph \( \Lambda \) can be characterized by the sequence with the cardinalities of homomorphisms from all graphs to \( \Lambda \) as entries.

Key to computing with \( \mathcal{GC}(\Lambda) \) are its (Hopf) algebraic properties. Indeed, it is multiplicative (i.e., a character) over the quasi-shuffle (Hopf) algebra defined on graphs, which means that it is compatible with identity (1)

\[
\langle \mathcal{GC}(\Lambda), \sigma \rangle \langle \mathcal{GC}(\Lambda), \tau \rangle = \sum_{\gamma} a(\gamma; \sigma, \tau) \langle \mathcal{GC}(\Lambda), \gamma \rangle.
\]

Emphasizing the Hopf algebra perspective on graphs, we will also show that the functional \( \mathcal{GC} \) satisfies a Chen-type identity.

A relevant result, found already in [Bor15] and [Pen20], is that the quasi-shuffle algebra is freely generated by connected graphs. That means that for any graph \( \sigma \), there exists a finite number of connected graphs \( \tau_1, \ldots, \tau_n \) and a polynomial \( p_\sigma(x_1, \ldots, x_n) \) such that, when evaluated with the product \( \text{qs} \), it yields \( \sigma = p_\sigma(\tau_1, \ldots, \tau_n) \). From the multiplicativity property (2) of \( \mathcal{GC}(\Lambda) \) and \( \langle \mathcal{GC}(\Lambda), \sigma \rangle = c_\sigma(\Lambda) \) one deduces

\[
c_\sigma(\Lambda) = p_\sigma(c_{\tau_1}(\Lambda), \ldots, c_{\tau_n}(\Lambda)).
\]

This identity says that we only need to know the occurrences of connected graphs to count subgraphs inside another graph. The existence of such polynomials is known in the literature. See, for instance [Pen20; Mau+20; GB20]. Here are two examples of such counting relations

\[
c_1(\Lambda) = \frac{1}{2}(c_1(\Lambda)c_1(\Lambda) - c_1(\Lambda)) - c_\Lambda(\Lambda)
\]

\[
c_1(\Lambda) = c_1(\Lambda)c_\Lambda(\Lambda) - 3c_\Lambda(\Lambda) - 3c_1(\Lambda) - 2c_\Lambda(\Lambda).
\]

We remark that similar relations can be traced back to Whitney’s seminal 1932 paper [Whi32].

Regarding Chen’s identity, we are interested in the comultiplicative properties of \( \mathcal{GC} \). After we endow graphs with a multiplication of combinatorial nature, it is natural to search for a comultiplication of combinatorial nature, which splits a given graph into simpler graphs. The compatible combination of multiplication and comultiplication of combinatorial objects, such as graphs or permutations, gives rise to combinatorial bi- and Hopf algebras [CP21; GR20; Sch94; Haz+10].

The study of co- and bialgebras on combinatorial objects can be traced back to the works of Rota and his school in the 1960-70s, in particular to his article with Joni [JR79, Sect. III]. An example of a well-studied combinatorial Hopf algebra is the Butcher–Connes–Kreimer Hopf algebra on non-planar rooted trees [CK99]. Similarly, an equally popular Hopf algebra on graphs is the one given by Schmitt [Sch94; Sch95] defined in the general framework of incidence Hopf algebras.

Another Hopf algebra of graphs is defined in [Bor15]. It shows that the quasi-shuffle product on graphs is compatible with a coproduct obtained by dualizing the disjoint union of graphs. Notice that, contrary to Schmitt’s Hopf algebra, the product in [Bor15] works on finite simple graphs without isolated vertices. However, the major difference is that Schmitt defined his coproduct with a focus on vertices while the operations in [Bor15] are based on edges. This distinction is made precise in Section 2.3. However, all these bialgebras are, up to a certain extent, equivalent as is shown in Section 3.

Because of the plethora of algebraic structures and the ambiguous definition of the notion of subgraph, several concepts for graph counting are available in the literature [GB20; Bra+21; Mau+20; Pen20; Bor+06; Lov12]. Lovász’ influence [Lov67; Lov12; Bor+06] is well-known for studying this distinction as well as providing so-called translations between different ways of counting subgraphs. This was also mentioned by [CDM17]. In Section 4.4, it is shown that these translations correspond to Hopf algebra isomorphisms.

The paper is structured as follows. In Section 2, we endow graphs with graded linear structures and then introduce the products and coproducts that we use in the rest of the text. In particular, Section 2.9 shows which of these algebraic structures for graphs form filtered or graded Hopf algebras. In Section 3, we then proceed to show that those Hopf algebras are isomorphic to the polynomial Hopf algebra on connected graphs. This is an application of Samnelson–Leray’s theorem as stated in the book [CP21]. As
a consequence, we also obtain that the underlying algebras are free over connected graphs. In Section 4, we explore the notion of signature in the setting of graph counting. We start by introducing different counting operations in Section 4.1, i.e., counting of restricted subgraphs, induced subgraphs, and graph homomorphisms. We then introduce in Section 4.2 four signature-type objects denoted $GC^e$, $GC^{vi}$, $GC^{hom}$ and $GC^{hom;dp}$. They encode the counting operations as linear functionals over graphs and satisfy both character properties as well as a form of Chen’s identity. Finally, in Section 4.4, the translations between counting characters are formulated in terms of Hopf algebra isomorphisms, including those already known in the computer science literature.

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2 Hopf algebras of graphs

In this section, we introduce the polynomial, shuffle, and quasi-shuffle Hopf algebras of graphs. To avoid confusion, we also present the well-known (incidence) Hopf algebra of graphs introduced by Schmitt [Sch94] and explain why these new Hopf algebras are different. We will define vector spaces and then endow them with coalgebra and algebra structures. We start by defining graphs and the notion of induced and restricted subgraphs. For a brief introduction on Hopf algebras and related concepts used throughout the text, we refer to Appendix A.2.

2.1 Graphs

The combinatorial objects under consideration in this work are graphs. Recall that a graph $\tau$ consists of a pair $(V, E)$ where $V = V(\tau)$ is the set of vertices and $E = E(\tau) \subseteq [V]^2$ is the set of edges; $[V]^2$ denotes the set of all 2-element subsets of $V$. We denote the empty graph $e := (\emptyset, \emptyset)$. The graphs considered in this paper are finite, i.e., $|V| < \infty$. A graph homomorphism is a map preserving edges. More precisely, given two graphs $\sigma, \tau$, the map $f : V(\sigma) \to V(\tau)$ is a graph homomorphism if for all vertices $u, v \in V(\sigma)$:

$$\{u, v\} \in E(\sigma) \implies \{f(u), f(v)\} \in E(\tau).$$

An isomorphism between graphs is a bijective homomorphism such that the inverse map is also a graph homomorphism. An automorphism of a graph $\tau$ is a graph isomorphism onto itself, which is defined in terms of a permutation on the vertex set $V(\tau)$. The set $\text{Aut}(\tau)$ of all automorphisms of a graph $\tau$ is closed under composition and forms the automorphism group of the graph. We refer to Section 4 for other specific notions of graph morphisms used throughout the paper. In particular, the notion of isomorphism allows us to consider the set of equivalence classes of graphs, denoted $G$. We will refer to graphs when considering both graphs or elements in the set of equivalence classes. The meaning will become clear from the context.

Many combinatorial operations on graphs can be defined using set theory. For a graph $\sigma$ we consider the notions of induced and restricted subgraph, corresponding to $U \subseteq V(\sigma)$ and $A \subseteq E(\sigma)$, respectively, such that:

$$\sigma_U := \left(U, \bigcup_{b \in E(\sigma)} \{b\}_{b \in U}\right) \quad \sigma_{|A} := \left(\bigcup_{b \in A} b, A\right).$$

---

1Note that this definition allows for isolated vertices.
Given two sets $S_1$ and $S_2$, we write $S_1 \cup S_2$ for their external disjoint union. We define the **disjoint union of graphs**, $\sigma$ and $\tau$, as follows:

$$\sigma \sqcup \tau := (V(\sigma) \cup V(\tau), E(\sigma) \cup E(\tau))$$

(4)

We will write $\sqcup$, for example, or simply $\sqcup \sqcup$.

### 2.2 Free vector spaces on graphs

Let $G$ be the set of equivalence classes of graphs. We denote by $\mathbb{R}\langle G \rangle$ the **free vector space** spanned by the elements in $G$, i.e., formal finite linear combinations of graphs with coefficients in $\mathbb{R}$. We will also consider $\mathbb{R}\langle \tilde{G} \rangle$, where $\tilde{G} \subset G$, is the set of graphs with no isolated vertices. Let $G_n$ be the set of equivalence classes of graphs with $n$ vertices. We can define a grading on $\mathbb{R}\langle G \rangle$ by considering the number of vertices

$$\mathbb{R}\langle G \rangle = \bigoplus_{n \geq 0} \mathbb{R}\langle G_n \rangle.$$  

(5)

Similarly, considering edges instead of vertices, we have

$$\mathbb{R}\langle \tilde{G} \rangle = \bigoplus_{n \geq 0} \mathbb{R}\langle \tilde{G}_n \rangle,$$

(6)

where $\tilde{G}_n$ contains graphs with $n$ edges and no isolated vertices. We can also define a grading on $\mathbb{R}\langle \tilde{G} \rangle$ in terms of the number of vertices

$$\mathbb{R}\langle \tilde{G} \rangle = \bigoplus_{n \geq 0} \mathbb{R}\langle \tilde{G}_n \rangle,$$

(7)

where $\tilde{G}_n$ contains those elements of $\tilde{G}$ with $n$ vertices. These vector spaces are connected since the zero level is spanned by the empty graph $G_0 := \emptyset$ alone. Moreover, they are all **locally finite**, i.e., each vector space appearing in the direct sum is finite-dimensional. Note that other gradings are possible. Indeed, we could define a grading $\mathbb{R}\langle G \rangle$ by the number of connected components, for example. However, it would not be locally finite.

In the following we will also consider the dual space $\mathbb{R}\langle G \rangle^* = (\bigoplus_{n \geq 0} \mathbb{R}\langle G_n \rangle)^*$. Note that $\mathbb{R}\langle G \rangle^* \cong \mathbb{R}[\mathbb{G}]$ and that for $\sum_{\tau} c_{\tau} \tau^* \in \mathbb{R}\langle G \rangle^*$ and $\sum_{\sigma} d_{\sigma} \sigma \in \mathbb{R}\langle G \rangle$ the pairing

$$\left\langle \sum_{\tau} c_{\tau} \tau^*, \sum_{\sigma} d_{\sigma} \sigma \right\rangle = \sum_{\gamma} c_{\gamma} d_{\gamma},$$

(8)

is defined since only finitely many $d_{\sigma}$ are non-zero. The graded dual $\mathbb{R}\langle G \rangle^* := \bigoplus_{n \geq 0} (\mathbb{R}\langle G_n \rangle)^* \subset \mathbb{R}\langle G \rangle^*$ will be used too. Notice that in (8) we consider the graphs in $\sum_{\tau} c_{\tau} \tau^* \in \mathbb{R}\langle G \rangle^*$ as linear maps over $\mathbb{R}\langle G \rangle$ and the fundamental dual pairing $\langle \tau, \sigma \rangle = \delta_{\tau, \sigma}$, for $\sigma \in \mathbb{R}\langle G \rangle$, is in place. This is usually called the **Kronecker delta** $\delta_{\tau, \sigma}$, which is one if $\tau = \sigma$ and zero else.

While we sometimes denote elements in $\mathbb{R}\langle G_n \rangle^*$ as $\tau^*$ or $\delta_{\tau}$, we prefer to avoid cluttered notation and when it is clear from the context, we will simply use $\tau$. 

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Figure 1: Looking at the graph on the left, the triangle graph in the middle is an induced subgraph, while the one on the right would be a restricted subgraph, but not an induced subgraph.
2.3 Algebras and coalgebras on graphs

Introducing the algebraic structure we need, we will adopt three perspectives. First, we define the usual polynomial algebra via the disjoint union of graphs.

2.4 Symmetric algebra of graphs

Consider (4) and extend it linearly so that we have the product
\[ \sqcup : \mathbb{R}(G) \otimes \mathbb{R}(G) \to \mathbb{R}(G). \]
Since the disjoint union is a graded map, we can use (21) in Appendix A.2.6 to define a corresponding coproduct:

**Definition 2.1 (co-disjoint union).**

\[
\Delta_{\sqcup} : \mathbb{R}(G) \to \mathbb{R}(G) \otimes \mathbb{R}(G)
\]

\[
\Delta_{\sqcup}(\tau) := \sum_{(\tau_1, \tau_2) \in G \times G} (\tau_1 \sqcup \tau_2) \tau_1 \otimes \tau_2.
\]

**Example 2.2.**

\[
\Delta_{\sqcup}(1) = e \otimes 1 + 1 \otimes e
\]
\[
\Delta_{\sqcup}(11) = e \otimes 1 + 1 \otimes 1 + 1 \otimes e
\]
\[
\Delta_{\sqcup}(11) = e \otimes 1 \Lambda + 1 \otimes 1 \Lambda + 1 \otimes 1 \Lambda \otimes e.
\]

**Remark 2.3.** Note that \( \Delta_{\sqcup} \) only sees connected components. A more explicit definition can be given as follows. By a slight abuse of notation we can write a graph \( \tau = \tau_1 \sqcup q_1 \sqcup \cdots \sqcup \tau_m \sqcup q_m \in G \), where the \( q_i \)’s tell us the multiplicity of the respective connected graph. Then, we can define
\[
\Delta_{\sqcup}(\tau) := \sum_{0 \leq p_1 \leq q_1} \cdots \sum_{0 \leq p_m \leq q_m} \tau_1^{p_1} \sqcup \cdots \sqcup \tau_m^{p_m} \otimes \tau_1^{(q_1-p_1)} \sqcup \cdots \sqcup \tau_m^{(q_m-p_m)}.
\]
(9)
Here \( \tau^{p_i} := e \) and each term has coefficient equal to one. An analogous definition holds for \( \Delta_{\sqcup} : \mathbb{R}(G) \to \mathbb{R}(G) \otimes \mathbb{R}(G) \).

We now introduce the usual coproduct that turns the polynomial algebra into a bialgebra.

**Definition 2.4 (Divided powers coproduct).** On \( \mathbb{R}(G) \) as well as \( \mathbb{R}(G) \otimes \mathbb{R}(G) \) define for connected graphs \( \tau \)

\[
\Delta_{DP}(\tau) := \tau \otimes e + e \otimes \tau,
\]
and extend as an algebra morphism for the \( \sqcup \) product.

**Example 2.5.**

\[
\Delta_{DP}(1) = e \otimes 1 + 1 \otimes e
\]
\[
\Delta_{DP}(11) = \Delta_{DP}(1) \sqcup \Delta_{DP}(1) = (e \otimes 1 + 1 \otimes e) \sqcup (e \otimes 1 + 1 \otimes e) = e \otimes 11 + 1 \otimes 11 + 11 \otimes e.
\]

**Remark 2.6.** Like \( \Delta_{\sqcup} \), the coproduct \( \Delta_{DP} \) only “sees” connected components. Again we write \( \tau = \tau_1^{p_1} \sqcup \cdots \sqcup \tau_m^{p_m} \in G \), where the \( q_i \)’s tell us the multiplicity of the respective connected graph. It follows then

\[
\Delta_{DP}(\tau_1^{p_1} \sqcup \cdots \sqcup \tau_m^{p_m}) = (e \otimes \tau_1 + \tau_1 \otimes e)^{p_1} \sqcup \cdots \sqcup (e \otimes \tau_m + \tau_m \otimes e)^{p_m}.
\]

More explicitly:
\[
\Delta_{DP}(\tau) := \sum_{0 \leq p_1 \leq q_1} \cdots \sum_{0 \leq p_m \leq q_m} \left( \begin{array}{c} q_1 \\ p_1 \\ \vdots \\ q_m \\ p_m \end{array} \right) \tau_1^{p_1} \sqcup \cdots \sqcup \tau_m^{p_m} \otimes \tau_1^{(q_1-p_1)} \sqcup \cdots \sqcup \tau_m^{(q_m-p_m)}.
\]
Here τ ⊔ 0 := e. Note that the coefficients differ but the terms are the same as the ones of the disjoint union coproduct.

The coproduct ∆_{DP} is graded, therefore we can define a corresponding product ·_{DP} : R⟨G⟩ ⊗ R⟨G⟩ → R⟨G⟩

τ₁ ·_{DP} τ₂ := \sum_{γ \in G} (τ₁ ⊗ τ₂, ∆_{DP}(γ))γ.

**Example 2.7.**

1 ·_{DP} 1 = 2 
Λ ·_{DP} 1 = Λ 1 = 1 Λ = 1 ·_{DP} Λ.

The following remark shows that ·_{DP} and ⊔ are equal up to a scaling factor.

**Remark 2.8.** One sees that for all τ₁, τ₂ ∈ G, the following holds:

τ₁ ·_{DP} τ₂ = \sum_{γ \in G} (τ₁ ⊗ τ₂, ∆_{DP}(γ))(τ₁ ⊔ τ₂) = \frac{|Aut(τ₁ ⊔ τ₂)|}{|Aut(τ₁)||Aut(τ₂)|} τ₁ ⊔ τ₂.

Recall that Aut(τ) denotes the automorphism group of a graph τ.

### 2.5 Graph operations based on edges

We now consider the edges of a graph and define the following coproducts, which are decomposing graphs. Recall the notions of induced and restricted subgraph given in (3).

**Definition 2.9 (edge-restricted co-shuffle).** On R⟨G⟩ we define

\[ \Delta_{w}(\tau) := \sum_{A \subseteq E(\tau)} \tau|_A \otimes \tau|_{E(\tau) \setminus A}. \]

**Example 2.10.**

\[ \Delta_{w}(1) = 1 \otimes e + e \otimes 1, \]
\[ \Delta_{w}(Λ) = Λ \otimes e + e \otimes Λ + 2 1 \otimes 1. \]

Since as a coproduct it is graded according to the number of edges, we define the corresponding shuffle product:

\[ w : R(\vec{G}) \otimes R(\vec{G}) \to R(\vec{G}). \]

\[ τ₁ \, w \, τ₂ := \sum_{γ \in R(\vec{G})} (τ₁ ⊗ τ₂, \Delta_{w}(γ))γ, \]

**Example 2.11.**

\[ 1 \, w \, 1 = 2 1 + 2 Λ, \]
\[ 1 \, w \, Λ = 1 Λ + 3 Λ + 3 Λ + 2 Λ. \]

We now consider a version of the shuffle product that also admits overlapping sets. Inspired by the works of [Bor15], [Man+20, Def. 5] and trying to frame [GB20] into an algebraic setting, we define on R(\vec{G}) the notions of quasi-shuffle product and coproduct.
Definition 2.12 (edge-restricted co-quasi-shuffle).

\[ \Delta_{qs, \tau} := \sum_{\emptyset \subseteq A \subseteq E(\tau) \subseteq (E(\tau) \setminus A) \subseteq B \subseteq E(\tau)} \tau \big|_{A} \otimes \tau \big|_{B}. \]

Example 2.13.

\[ \Delta_{qs, \emptyset} = \mathds{1} \otimes e + e \otimes \mathds{1} \]

\[ \Delta_{qs, \emptyset} (\Lambda) = \Lambda \otimes e + e \otimes \Lambda + 2 \mathds{1} \otimes \mathds{1} + 2 \Lambda \otimes \mathds{1} + 2 \mathds{1} \otimes \Lambda + \Lambda \otimes \Lambda \]

Remark 2.14. We note that

\[ \Delta_{qs, \tau} = \Delta_{\omega} + \Delta_{ol}, \]

where the overlapping part

\[ \Delta_{ol}(\tau) := \sum_{\emptyset \subseteq A, B \subseteq E(\tau) \setminus A \cap B \neq \emptyset} \tau \big|_{A} \otimes \tau \big|_{B}. \]

The co-quasi shuffle is not graded. To obtain the quasi-shuffle product

\[ \mathcal{Q} \triangleright \mathcal{Q} : \mathbb{R}(\mathcal{G}) \otimes \mathbb{R}(\mathcal{G}) \rightarrow \mathbb{R}(\mathcal{G}), \]

observe that the quasi-shuffle co-product satisfies the assumptions of Lemma A.25. Therefore:

\[ \tau_1 \mathcal{Q} \triangleright \mathcal{Q} \tau_2 := \sum_{\gamma \in \mathbb{R}(\mathcal{G})} \langle \tau_1 \otimes \tau_2, \Delta_{qs, \tau}(\gamma) \rangle \gamma. \]

Example 2.15.

\[ \mathds{1} \mathcal{Q} \triangleright \mathcal{Q} \mathds{1} = 2 \mathds{1} \mathds{1} + 2 \Lambda + \mathds{1} \]

\[ \mathds{1} \mathcal{Q} \triangleright \mathcal{Q} \Lambda = \mathds{1} \Lambda + 3 \Delta + 3 \mathcal{U} + 2 \mathcal{U} + 2 \Lambda \]

Remark 2.14 implies that the shuffle and quasi-shuffle products are related.

2.6 Graph operations based on vertices

We can now define co-shuffle and co-quasi-shuffle operations analogously and the corresponding products for vertex induced subgraphs as well. This coproduct on \( \mathbb{R}(\mathcal{G}) \) appears in the context of the graph coalgebra introduced in [Sch94, Sec. 12].

Definition 2.16 (vertex-induced co-shuffle).

\[ \Delta_{\omega, \tau} := \sum_{\emptyset \subseteq U \subseteq V(\tau)} \tau_U \otimes \tau_{V(\tau) \setminus U}. \]

Example 2.17.

\[ \Delta_{\omega, \bullet} = \bullet \otimes e + e \otimes \bullet \]

\[ \Delta_{\omega, \mathds{1}} = \mathds{1} \otimes e + e \otimes \mathds{1} + 2 \bullet \otimes \bullet \]

Again we can define a product

\[ \tau_1 \mathcal{W} \triangleright \mathcal{W} \tau_2 := \sum_{\gamma \in \mathbb{R}(\mathcal{G})} \langle \tau_1 \otimes \tau_2, \Delta_{\omega, \tau}(\gamma) \rangle \gamma. \]
Example 2.18.

\[ \begin{align*}
\bullet \sqcup \bullet &= 2 \bullet + 2 \\
1 \sqcup 1 &= 2 \bullet + 2 \hfill + 4 \hfill + 4 \hfill + 6 \\
\end{align*} \]

For completeness and symmetry, we also define a co-quasi-shuffle on \( \mathcal{R}(G) \).

**Definition 2.19** (vertex-induced co-quasi-shuffle).

\[ \Delta_{qi}(\tau) := \sum_{\emptyset \subseteq I \subseteq V(\tau)} \tau_I \otimes \tau_J \]

This coproduct is the object of study of [Sch95] and [Pen20].

Example 2.20.

\[ \begin{align*}
\Delta_{qi}(\bullet) &= \bullet \otimes e + e \otimes \bullet + \bullet \otimes \bullet \\
\Delta_{qi}(1) &= 1 \otimes e + e \otimes 1 + 2 \bullet \otimes \bullet + 2 \bullet \otimes 1 + 2 1 \otimes \bullet \\
\end{align*} \]

The corresponding quasi-shuffle product with respect to vertex induced subgraphs is given by

\[ \tau_1 \sqcup_{qi} \tau_2 := \sum_{\gamma \in \mathcal{R}(G)} \langle \tau_1 \otimes \tau_2, \Delta_{qi}(\gamma) \rangle \gamma. \]

Example 2.21.

\[ \begin{align*}
\bullet \sqcup_{qi} \bullet &= 2 \bullet + 2 \hfill + \\
1 \sqcup_{qi} 1 &= 2 \hfill + 2 \hfill + 4 \hfill + 4 \hfill + 6 \\
\end{align*} \]

Remark 2.22. As noted above, the coproducts, \( \Delta_{qi} \) and \( \Delta_{qi} \), and therefore the corresponding products are tightly related. Note that one can order graphs according to the number of connected components. According to the definitions we gave, when we take the products of two graphs, one has that the “leading term” is the disjoint union of the two graphs. For \( \cdot_{DP}, \sqcup, \) and \( \sqcup_{qi} \) the coefficients of the leading term coincide, for instance:

\[ \begin{align*}
1_{\cdot_{DP}} \cdot &= 2 \bullet + 2 \hfill + \\
1 \sqcup \cdot &= 1_{\cdot_{DP}} \cdot + 2 \hfill + \\
1 \sqcup_{qi} \cdot &= 1 \sqcup \cdot + 2 \hfill + \\
\end{align*} \]

### 2.7 Coalgebras, algebras

Note that all the (co-)products defined on graphs are (co-)commutative. And all of them share the same (co-)unit.

**Definition 2.23.** The unit map is given by \( u : \mathbb{R} \rightarrow \mathcal{R}(G) \) with \( u(1_{\mathbb{R}}) = e \). The counit map is defined by

\[ \varepsilon(\tau) = \begin{cases} 1, & \text{if } \tau = e \\ 0, & \text{else.} \end{cases} \]

The proof of the following result is given in Appendix A.3.

**Proposition 2.24.** Coproducts in Section 2.4, Section 2.5, and Section 2.6 are counital, coassociative and cocommutative. Dually, the corresponding products are unital, associative and commutative.
Remark 2.25. We defined $\Delta_I, \Delta_q : \mathbb{R}(\hat{G}) \to \mathbb{R}(\hat{G}) \otimes \mathbb{R}(\hat{G})$. Note that neither $\Delta_I(\mathbb{R}(\hat{r}G))$ nor $\Delta_q(\mathbb{R}(\hat{r}G))$ are in $\mathbb{R}(\hat{G}) \otimes \mathbb{R}(\hat{G})$. For example

$$\Delta_I(1) = 1 \otimes e + e \otimes 1 + 2 \cdot \otimes \cdot.$$ 

On the other hand, we have that both $\Delta_I(\mathbb{R}(\hat{r}G))$ and $\Delta_q(\mathbb{R}(\hat{r}G))$ are in $\mathbb{R}(\hat{G})$. Therefore we can restrict the two products, $\Delta_I, \Delta_q : \mathbb{R}(\hat{G}) \otimes \mathbb{R}(\hat{G}) \to \mathbb{R}(\hat{G})$.

2.8 Behavior with respect to the gradings

We now display how coproducts and products behave with respect to the grading. Recall that $G_n$ consists of graphs with $n$ vertices, if we include isolated vertices. If there are no isolated vertices, $\hat{G}_n \subset \hat{G}$ contains all graphs with $n$ edges, while $\hat{G}_n'$ are the graphs in $\hat{G}$ with $n$ vertices.

Lemma 2.26. Products and coproducts behave as follows according to the gradings and the induced filtrations

|                  | $\bigoplus_{n \geq 0} \mathbb{R}(G_n)$ | $\bigoplus_{n \geq 0} \mathbb{R}(\hat{G}_n)$ | $\bigoplus_{n \geq 0} \mathbb{R}(\hat{G}_n')$ |
|------------------|--------------------------------------|---------------------------------------------|-----------------------------------------------|
| $\Delta_u$      | graded                               | graded                                      | graded                                        |
| $\Delta_{DP}$   | graded                               | graded                                      | graded                                        |
| $\Delta_{\hat{u}}$ | graded                           | non-filtered                               | non-filtered                                 |
| $\Delta_{\hat{u}}'$ | graded                           | non-filtered                               | non-filtered                                 |
| $\Delta_{\hat{q}}$ | non-filtered                       |                                             |                                               |
| $\Delta_{\hat{q}}'$ | non-filtered                       |                                             |                                               |

|                  | $\bigoplus_{n \geq 0} \mathbb{R}(\hat{G}_n)$ | $\bigoplus_{n \geq 0} \mathbb{R}(\hat{G}_n)$ | $\bigoplus_{n \geq 0} \mathbb{R}(\hat{G}_n')$ |
|------------------|---------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $\hat{u}$        | graded                                      | graded                                        | graded                                        |
| $\hat{DP}$       | graded                                      | graded                                        | graded                                        |
| $\hat{u}$        | graded                                      | filtered                                     | filtered                                     |
| $\hat{q}$        | filtered                                    | filtered                                     | filtered                                     |
| $\hat{q}'$       | graded                                      | filtered                                     | filtered                                     |
| $\hat{q}'$       | filtered                                    |                                                |                                               |

Proof. The disjoint union and the divided power coproduct are obviously graded. It is also easy to see that: $\Delta_u$ is graded by the number of edges, but not filtered by the number of vertices and that $\Delta_{\hat{u}}$ is graded by the number of vertices. For the quasi-shuffle cases, the following examples may serve to make the point:

$$\Delta_{\hat{q}}(1) = 1 \otimes e + e \otimes 1 + 2 \cdot \otimes \cdot 1 + 2 \cdot \otimes 1 + 2 \cdot \otimes \cdot.$$  

$$\Delta_{\hat{q}}'(1) = e \otimes 1 + 1 \otimes e + 1 \otimes 1.$$ 

Now consider the products. The statements are immediate for the disjoint union, the divided power, and the shuffle products. We only show the quasi-shuffle product is filtered by the number of edges. Recall that by definition

$$\tau_1 \hat{u} \hat{q} \tau_2 := \sum_{\gamma \in \Psi} (\tau_1 \otimes \tau_2, \Delta(\gamma)) \gamma.$$
Let $\tau_1 \in \tilde{G}_{n_1}$ and $\tau_2 \in \tilde{G}_{n_2}$, if $\gamma \in \tilde{G}_m$ with $m > n_1 + n_2$, then $\langle \tau_1 \otimes \tau_2, \Delta_M^q(\gamma) \rangle = 0$. □

### 2.9 Bialgebras and Hopf Algebras

In the following, we state which coproducts are compatible (and which are not) with which products. Note that the results do not depend on whether we deal with vertex or edge-based operations.

**Lemma 2.27 (Bialgebra property).** Considering all possible product-coproduct combinations:

| Bialgebra | $\sqcup$ | $\cdot_{DP}$ | $\sqcup$ | $\sqcap_{qr}$ |
|-----------|---------|------------|---------|---------|
| $\Delta_{\sqcup}$ | no | yes | yes | yes |
| $\Delta_{DP}$ | yes | no | no | no |
| $\Delta_{\sqcup}$ | yes | no | no | no |
| $\Delta_{\sqcap}$ | yes | no | no | no |

Table 1: Compatibility in $R\langle G \rangle$ (left) and $R\langle \tilde{G} \rangle$ (right). In the last two columns, on the right, $\sqcap$ and $\sqcup$ appear because of Remark 2.25.

**Proof.** For proving that $(R\langle \tilde{G} \rangle, \sqcup, \Delta_{\sqcup})$ is a bialgebra, suppose that $\sigma, \tau \in \tilde{G}$. Then

$$\Delta_{\sqcup}^q(\sigma \sqcup \tau) = \sum_{A \sqcup B = E(\sigma \sqcup \tau)} (\sigma \sqcup \tau)_{|A} \otimes (\sigma \sqcup \tau)_{|B}$$

$$= \sum_{A \sqcup B = E(\sigma) \sqcup E(\tau)} (\sigma_{|A \cap E(\sigma)} \sqcup (\tau_{|A \cap E(\tau)}) \otimes (\sigma_{|B \cap E(\sigma)} \sqcup (\tau_{|B \cap E(\tau)}))$$

$$= \sum_{A \sqcup B = E(\sigma) \sqcup E(\tau)} \left( (\sigma_{|A \cap E(\sigma)} \otimes (\sigma_{|B \cap E(\sigma)}) \sqcup (\tau_{|A \cap E(\tau)} \otimes (\tau_{|B \cap E(\tau)}) \right)$$

$$= \sum_{A_1 \sqcup B_1 = E(\sigma)} (\sigma_{|A_1} \otimes (\tau_{|B_1}) \sqcup (\sigma_{|B_2} \otimes (\tau_{|A_2})$$

$$= \Delta_{\sqcup}^q(\sigma) \sqcup \Delta_{\sqcup}^q(\tau).$$

Where, for the first equality, we used another way of writing the quasi-shuffle coproduct. For the rest of the proof, we refer to Lemma A.28 in the Appendix. □

We now look at the bialgebras and verify which are also Hopf Algebras. Recall that all of them are commutative and cocommutative.

**Proposition 2.28.**

We have polynomial Hopf algebras on connected graphs:

- $(R\langle \tilde{G} \rangle, \sqcup, \Delta_{DP})$ and $(R\langle \tilde{G} \rangle, \cdot_{DP}, \Delta_{\sqcup})$ are Hopf algebras, connected and graded by (5).
- $(R\langle \tilde{G} \rangle, \sqcup, \Delta_{DP})$ and $(R\langle \tilde{G} \rangle, \cdot_{DP}, \Delta_{\sqcup})$ are Hopf algebras, connected and graded by both (6) or (7).

We have shuffle Hopf algebras on vertices and edges:
• \((\mathbb{R}(\mathcal{G}), \sqcup, \Delta_{\sqcup})\) and \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) are Hopf algebras, connected and graded by (5). Note that this appears already in [Sch94, Section 12].

• \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) is a Hopf algebra, connected and graded by (7).

• \((\mathbb{R}(\mathcal{G}), \sqcup, \Delta_{\sqcup})\) and \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) are Hopf algebras, connected and graded by (6).

We have quasi-shuffle Hopf algebras on vertices and edges, but only on the “product side”:

• \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) is a Hopf algebra, connected and graded by (7).

• \((\mathbb{R}(\mathcal{G}), \sqcup, \Delta_{\sqcup})\) and \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) are Hopf algebras, connected and graded by (6).

On the other hand:

• \((\mathbb{R}(\mathcal{G}), \sqcup, \Delta_{\sqcup})\) and \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) fail to be Hopf algebras.

Proof. For connected, filtered bialgebras we can use Proposition A.22. Recall that \((\mathbb{R}(\mathcal{G}), \sqcup, \Delta_{\sqcup})\) and \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) are not filtered by the number of edges. We now show for \((\mathbb{R}(\mathcal{G}), \sqcup, \Delta_{\sqcup})\) that the antipode does not exist. An analogous statement works for \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\). Suppose the antipode exists. A direct calculation of (18) on the single edge graph, \(\mathbf{1}\), yields:

\[
S(\mathbf{1}) = \sum_{n \geq 0} (-1)^n \mathbf{1}^n = -1 + 1 - 1 + 1 - 1 + \cdots,
\]

which is not an element of \(\mathbb{R}(\mathcal{G})\). \(\square\)

| Hopf algebra | \(\sqcup\) | \(\hat{\sqcup}\) | \(\Delta_{\sqcup}\) | \(\Delta_{\hat{\sqcup}}\) |
|-------------|----------|----------|-----------------|-----------------|
| \(\Delta_{\sqcup}\) | no       | yes      | yes             | yes             |
| \(\Delta_{\hat{\sqcup}}\) | yes      | no       | no              | no              |
| \(\Delta_{\hat{\sqcup}}\) | yes      | no       | no              | no              |
| \(\Delta_{\hat{\sqcup}}\) | no       | no       | no              | no              |

Table 2: Hopf Algebras on \(\mathbb{R}(\mathcal{G})\) (left) and \(\mathbb{R}(\hat{\mathcal{G}})\) (right)

3 Isomorphisms to the polynomial Hopf algebra on connected graphs

In this section, we investigate some of the properties of the products introduced in Section 2. We will show that the respective Hopf algebras are isomorphic to the polynomial Hopf algebra on connected graphs. For some of the Hopf algebras that we consider, this result is already known. Schmitt used in [Sch94, Thm. 10.2] the theory of incidence Hopf algebras to prove this result for \((\mathbb{R}(\mathcal{G}), \sqcup, \Delta_{\sqcup})\). More recently, the product \(\hat{\sqcup}\) appeared as a working example in reference [Pen20], where it was interpreted in the context of the theory of species. We also give a proof that is basically equivalent to the one used by Borie in [Bor15], to prove the freeness of the associative algebra \((\mathbb{R}(\mathcal{G}), \sqcup)\). We will use a well-known result in the literature on Hopf algebras, based on the universal enveloping algebra on primitive elements: [CP21, Thms. 4.3.1 and 4.3.3], that immediately shows the isomorphism to the polynomial Hopf algebra.

\(^2\)One could work with the dual coalgebra of [Rad11, Section 2.5] to dualize \((\mathbb{R}(\mathcal{G}), \hat{\sqcup}, \Delta_{\hat{\sqcup}})\) to a Hopf algebra. But we shall not need this here.
3.1 Bicommutative unipotent Hopf algebras

We work with a particular class of Hopf algebras which are suitable for studying combinatorial objects.

**Definition 3.1.** A bialgebra $\mathcal{H}$ is said to be unipotent if for all $a \in \mathcal{H}$, there exists $N(a) \geq 0$ such that for all $n \geq N(a)$:

$$(\text{id} - u \circ \varepsilon)^n(a) = 0.$$ 

If a bialgebra is associative, coassociative, unital, counital, commutative, and cocommutative, we say that it is bicommutative.

Note that any filtered connected bialgebra is unipotent. Indeed, we can use a similar argument as the one used in the proof of Proposition A.22. We present a technical definition for constructing algebra homomorphisms.

**Definition 3.2.** Suppose that $(A_1, m_1)$ and $(A_2, m_2)$ are unital associative algebras and $(A_1, m_1)$ is freely generated by $\{x_i\}$. For any linear function $\varphi : \mathbb{R}\langle \{x_i\}\rangle \to A_2$, we define its natural extension $\tilde{\varphi} : (A_1, m_1) \to (A_2, m_2)$ by the linear extension of

$$\tilde{\varphi} \circ m_1^{n-1}(x_i \cdots z x_{n}) := m_2^{n-1}(\varphi(x_i) \cdots \varphi(x_{n})).$$

Since $(A_1, m_1)$ is free, $\tilde{\varphi}$ is well defined. It is also an algebra homomorphism by definition.

We now state a result which is an application of Samuelson–Leray theorem, found in [CP21, Thm. 4.3.3]. This is a particular case of Cartier’s theorem [CP21, Thm. 4.3.1] for commutative and cocommutative unipotent Hopf algebras.

**Theorem 3.3** (Samuelson–Leray–Cartier). Suppose that $\mathcal{H}$ is a commutative, cocommutative, and unipotent Hopf algebra over $\mathbb{R}$. Let $\text{Prim}(\mathcal{H})$ be the set of primitive elements in $\mathcal{H}$. The inclusion of $\text{Prim}(\mathcal{H})$ in $\mathcal{H}$ naturally extends to an isomorphism of Hopf algebras

$$\Phi : (\mathbb{R}[\text{Prim}(\mathcal{H})], \cdot, \Delta) \to \mathcal{H},$$

where $(\mathbb{R}[\text{Prim}(\mathcal{H})], \cdot, \Delta)$ is the polynomial Hopf algebra on the primitive elements.

This theorem is paramount for the current text. We extend this result into a practical tool for verifying that a map is a Hopf algebra isomorphism.

**Lemma 3.4.** Suppose that $(A_1, m_1, \Delta_1)$ is a bicommutative unipotent Hopf algebra, and $(A_2, m_2, \Delta_2)$ is a bialgebra. If $\varphi : (A_1, m_1) \to (A_2, m_2)$ is an algebra homomorphism and

$$\varphi(\text{Prim}(A_1, \Delta_1)) \subset \text{Prim}(A_2, \Delta_2),$$

then $\varphi$ is a coalgebra homomorphism, i.e. $\Delta_2 \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_1$.

**Proof.** From Theorem 3.3, we know that $(A_1, m_1)$ is free with generators $\{x_i\} \subset \text{Prim}(A_1, \Delta_1)$. Let us then prove the comultiplicative property of $\varphi$ using induction on the size $n$ of monomials $x_{i_1} \cdots x_{i_n}$. Note that when $n = 0$, $\varphi(1_{A_1}) = 1_{A_2}$, then $(\varphi \otimes \varphi) \circ \Delta_1(1_{A_1}) = (\varphi \otimes \varphi)(1_{A_1} \otimes 1_{A_1}) = \varphi(1_{A_1}) \otimes \varphi(1_{A_1}) = 1_{A_2} \otimes 1_{A_2} = \Delta_2(1_{A_2}) = \Delta_2(\varphi(1_{A_1}))$. We now use induction with base case $n = 1$. From the hypothesis, we know that any primitive element $a \in \text{Prim}(A_1, \Delta_1)$ has a primitive image. Hence

$$\Delta_2 \circ \varphi(a) = \varphi(a) \otimes 1_{A_2} + 1_{A_2} \otimes \varphi(a)$$

$$= (\varphi \otimes \varphi)(a \otimes 1_{A_1} + 1_{A_1} \otimes a)$$

$$= (\varphi \otimes \varphi) \circ \Delta_1(a).$$

Suppose now that for $b = x_{i_1} \cdots x_{i_n} \in A_1$, it holds that

$$\Delta_2 \circ \varphi(b) = (\varphi \otimes \varphi) \circ \Delta_1(b).$$
For \( a \in \text{Prim}(A_1) \), we use the multiplicative property of \( \varphi \), and that \((A_2, m_2, \Delta_2)\) is a bialgebra.

\[
\Delta_2 \circ \varphi \circ m_1 (a \otimes b) = \Delta_2 \circ m_2 \circ (\varphi \otimes \varphi) (a \otimes b) = m_{A_2 \otimes A_2} \circ (\Delta_2 \otimes \Delta_2) \circ (\varphi \otimes \varphi) (a \otimes b).
\]

We use now that \( \Delta_2 \circ \varphi(a) = (\varphi \otimes \varphi) \Delta_1(a) \) and also the induction hypothesis. Then, the last expression equals

\[
m_{A_2 \otimes A_2} \circ ((\varphi \otimes \varphi) \otimes (\varphi \otimes \varphi)) \circ (\Delta_1(a) \otimes \Delta_1(b)).
\]

Recall now that \( \varphi \otimes \varphi : (A_1 \otimes A_1, m_{A_1 \otimes A_1}) \to (A_2 \otimes A_2, m_{A_2 \otimes A_2}) \) is an algebra homomorphism. The former expression reduces to the following by the bialgebra compatibility of \( \Delta_1 \) and \( m_1 \).

\[
(\varphi \otimes \varphi) \circ m_{A_1 \otimes A_1} \circ (\Delta_1(a) \otimes \Delta_1(b)) = (\varphi \otimes \varphi) \circ \Delta_1 \circ m_1 (a \otimes b).
\]

\[\square\]

An important consequence of these results is that we only need to define a function in the primitive elements to obtain Hopf algebra homomorphisms. Even more, linear isomorphisms on primitive elements extend to Hopf algebra isomorphisms.

**Proposition 3.5.** Suppose that \((H_1, m_1, \Delta_1)\) and \((H_2, m_2, \Delta_2)\) are bicommutative unipotent Hopf algebras. If \( \varphi : \text{Prim}(H_1, \Delta_1) \to \text{Prim}(H_2, \Delta_2) \) is an invertible linear map, then its natural extension is a Hopf algebra isomorphism \( \bar{\varphi} : (H_1, m_1, \Delta_1) \to (H_2, m_2, \Delta_2) \).

**Proof.** Theorem 3.3 implies that \((H_1, m_1),(H_2, m_2)\) are freely generated by \( \{x_i\} \subset \text{Prim}(H_1, \Delta_1) \) and \( \{y_j\} \subset \text{Prim}(H_2, \Delta_2) \), respectively. Since \( \varphi \) is a linear isomorphism \( \varphi(\{x_i\}) \subset \text{Prim}(H_2, \Delta_2) \) is also a basis for \( \text{Prim}(H_2, \Delta_2) \) and freely generates \( (H_2, m_2) \). Therefore, \( \bar{\varphi} \) is injective and surjective by construction. Lemma 3.4 implies that \( \bar{\varphi} \) is a bialgebra homomorphism. And Lemma A.5 that it is a bialgebra isomorphism, thus by Proposition A.15, a Hopf algebra isomorphism. \[\square\]

In fact, Proposition 3.5 can be generalized to cocommutative unipotent bialgebras if we use a more general version of Theorem 3.3 by Cartier [CP21, Theorem 4.3.1].

### 3.2 The symmetric Hopf algebra of graphs

We now want to use Theorem 3.3 to understand the Hopf algebras on graphs introduced in Section 2.3. One way is to compare them to a very well-known object. The simplest of our Hopf algebras is \((R(\tilde{G}), \sqcup, x)\). This is commonly called the symmetric, polynomial, or free commutative Hopf algebra [Haz+10; CP21; GR20]. In [Sch94, Section 9], Schmitt even calls \(x\) “the usual coproduct”. Some also call it the *shuffle* Hopf algebra in the commutative alphabet of graphs [DET20a; CP21]. We will avoid the name in this work to avoid confusion with the products \(\omega\) and \(\sqcup\). Before using Theorem 3.3, it is necessary to determine which elements are primitive for \(x\).

**Lemma 3.6.** Let us denote the connected graphs by \(G^0 := \{\tau \in \mathcal{G} | \tau \text{ is connected}\}\) and \(\tilde{G}^0 := G^0 \cap \tilde{G}\). The primitive elements of both \(x\), and \(x\) are given by the span of connected graphs, i.e.

\[
\text{Prim}(R(\tilde{G}), x) = R\langle \tilde{G}^0 \rangle = \text{Prim}(R(\tilde{G}), x),
\]

\[
\text{Prim}(R(\tilde{G}), x) = R\langle \tilde{G}^0 \rangle = \text{Prim}(R(\tilde{G}), x).
\]

**Proof.** For \(x\), this follows from its definition.

It is a direct consequence from the definition that \(R(\tilde{G}^0) \subset \text{Prim}(R(\tilde{G}), x)\).

Suppose now that \(\tau \notin R(\tilde{G}^0)\), then it can be written as a linear combination of graphs that are not all connected, i.e., \(\tau = \sum_{i \in I} s_i \tau_i\). Then using (9) we have

\[
\Delta_{x} \left( \sum_{i \in I} s_i \tau_i \right) = \sum_{i \in I} s_i \sum_{0 \leq p_{1,i} \leq q_{1,i}} \tau_{1,i}^{\text{Le}(p_{1,i}, r_{1,i})} \cdots \sum_{0 \leq p_{m,i} \leq q_{m,i}} \tau_{m,i}^{\text{Le}(q_{m,i}, r_{m,i})} \tau_{m,i}^{\text{Le}(q_{m,i}, r_{m,i})} \cdots \tau_{1,i}^{\text{Le}(p_{1,i}, r_{1,i})}.
\]
Analogously: The inclusion

\[ R \langle G \rangle \rightarrow R(\mathcal{G}) \]

extends naturally –in the sense of Definition 3.2– to different Hopf algebra isomorphisms,

- \( \hat{\psi}_t : (R(\mathcal{G}), \cup, \Delta_{DP}) \rightarrow (R(\mathcal{G}), \hat{\cup}, \Delta_{\cup}) \), where \( \hat{\cup} = \cdot_{DP}, \ \hat{\cup}, \ \hat{\cup} \).

Analogously: The inclusion

\[ \phi : R(\mathcal{G}^0) \rightarrow R(\mathcal{G}) \]

extends to the isomorphisms,

- \( \hat{\phi}_s : (R(\mathcal{G}), \cup, \Delta_{DP}) \rightarrow (R(\mathcal{G}), \hat{s}, \Delta_{\cup}) \), where \( \hat{s} = \cdot_{DP}, \ , \ , \ , \ , \).

In particular, the isomorphism is true on the side of the algebras: the connected graphs freely generate the algebra under the shuffle and the quasi-shuffle product as well.

**Remark 3.8.** Note that the isomorphism between \( (R(\mathcal{G}), \cdot_{DP}, \Delta_{\cup}) \) and \( (R(\mathcal{G}), \cup, \Delta_{DP}) \) is the well-known self-duality of the polynomial Hopf algebra. The isomorphisms of \( (R(\mathcal{G}), \hat{\cup}, \Delta_{\cup}) \) and \( (R(\mathcal{G}), \hat{s}, \Delta_{\cup}) \) to \( (R(\mathcal{G}), \cup, \Delta_{DP}) \) were studied by [Sch94] and [Pen20], respectively. These last two articles only treat these particular isomorphisms, while Proposition 3.7 relates a broader family of Hopf algebras.

**Proof.** All these Hopf algebras are commutative, cocommutative, and unipotent, so we apply Theorem 3.3. Using Lemma 3.6, we know that \( \text{Prim}(\mathcal{H}) = R(\mathcal{G}^0) \). Therefore the inclusion

\[ \psi : R(\mathcal{G}^0) \rightarrow R(\mathcal{G}) \]

extends to an isomorphism between Hopf algebras

\[ \hat{\psi}_t : (R(\mathcal{G}), \cup, \Delta_{DP}) \rightarrow (R(\mathcal{G}), \hat{\cup}, \Delta_{\cup}) \]

where \( \hat{\cup} = \cdot_{DP}, \ , \ , \ , \ , \). And we have analogous arguments for \( R(\mathcal{G}) \).

**3.3 Connected graphs and the quasi-shuffle algebra**

In this section, we study the Hopf algebra \( (R(\mathcal{G}), \hat{\cup}, \Delta_{\cup}) \) and give a more explicit description of the isomorphism \( \hat{\phi}_s : (R(\mathcal{G}), \cup, \Delta_{DP}) \rightarrow (R(\mathcal{G}), \hat{\cup}, \Delta_{\cup}) \). This will turn out to be useful for counting subgraphs, when combined with results appearing in Section 4.2. From the proof of Proposition 3.7, one immediately obtains the following corollary. Instead, we now prove it directly using a procedure common in the literature, see for instance [Pen20] and [Bor15].
Corollary 3.9. The extension of the inclusion $\phi_n^\Delta$,

$$\tilde{\phi}_n : (\mathbb{R}(G^q), \sqcup) \to (\mathbb{R}(G^q), \sqcup)$$

is an algebra isomorphism.

Proof. Assume that $\tau \in G^q$ with $n \geq 2$ connected components. We can write $\tau = \tau_1 \sqcup \tau_2$ where $\tau_1, \tau_2$ have $n_1, n_2 \geq 1$ connected components, respectively ($n = n_1 + n_2$). Recall that

$$\tau_1 \sqcup \tau_2 = \sum_{\gamma \in G} \langle \tau_1 \otimes \tau_2, \Delta^\Delta(\gamma) \rangle \gamma.$$ 

Since $\langle \tau_1 \otimes \tau_2, \Delta^\Delta(\tau_1 \sqcup \tau_2) \rangle \neq 0$, it holds the recursion

$$\tau = \tau_1 \sqcup \tau_2 = \frac{1}{\langle \tau_1 \otimes \tau_2, \Delta^\Delta(\tau_1 \sqcup \tau_2) \rangle} \left( \tau_1 \sqcup \tau_2 - \sum_{\eta \in G : cc(\eta) < n} \langle \tau_1 \otimes \tau_2, \Delta^\Delta(\eta) \rangle \eta \right),$$

where we denote the number of connected components of $\eta$ with $cc(\eta)$. Iterating the same procedure on the right-hand side will yield an expression of the form

$$\tau = \sum_{i \in I} a_i \bigcup_j \sigma_{i,j}$$

where $a_i \in \mathbb{R}$ and $\sigma_{i,j}$ is a connected graph. Therefore we have

$$\tau = \tilde{\phi}_n^\Delta \left( \sum_{i \in I} a_i \bigcup_j \sigma_{i,j} \right) = \sum_{i \in I} a_i \tilde{\phi}_n^\Delta \left( \bigcup_j \sigma_{i,j} \right).$$

For injectivity, we use proof by contradiction. Assume

$$\tilde{\phi}_n^\Delta \left( \sum_{i \in I} a_i \bigcup_j \sigma_{i,j} \right) = \sum_{i \in I} a_i \bigcup_j \sigma_{i,j} = 0,$$

where $a_i \neq 0$, $\forall i \in I$. Now, for each quasi-shuffle monomial, we can write $\bigcup_j \sigma_{i,j} = c \bigcup_j \sigma_{i,j} + \cdots$, where $c$ is a non-zero constant depending on $\sigma_{i,j}, i \in I$. Comparing with $\bigcup_j \sigma_{i,j}$, any other graph appearing in the product $\bigcup_j \sigma_{i,j}$ has a strictly smaller number of connected components. Then, among the $\bigcup_j \sigma_{i,j}, i \in I$, pick the graph having the largest number of connected components, say $\bigcup_j \sigma_{\text{max},j}$. The corresponding term cannot be canceled out by any other term and we are done. 

We can think of these algebra isomorphisms as linear isomorphisms. The theorem then implies that the shuffle and the quasi-shuffle algebras amount to a change of basis of the polynomial algebra on connected graphs. Consider the following example, which is comparable to that in [Big78, p. 162].

Example 3.10. Consider on $(\mathbb{R}(G^q), \sqcup)$ the span of all graphs with at most 3 edges.

$$a = a_0 \mathbf{e} + a_1 \mathbf{i} + a_2 \bigtriangleup + a_3 \bigcup \mathbf{i} + a_4 \bigtriangledown + a_5 \bigcirc \bigcup \mathbf{i} + a_6 \bigcirc \bigcup \mathbf{i} + a_7 \bigcup \mathbf{i} + a_8 \bigcup \mathbf{i}.$$

We want to write:

$$a = b_0 \mathbf{e} + b_1 \mathbf{i} + b_2 \bigtriangleup + b_3 \bigcirc \bigcup \mathbf{i} + b_4 \bigtriangledown + b_5 \bigcirc \bigcup \mathbf{i} + b_6 \bigcup \mathbf{i} + b_7 \bigcup \mathbf{i} + b_8 \bigcup \mathbf{i} + b_9 \bigcup \mathbf{i} + b_{10} \bigcup \mathbf{i}$$

which is

$$a = b_0 \mathbf{e} + b_1 \mathbf{i} + b_2 \bigtriangleup + b_3 (\mathbf{i} + 2 \mathbf{i} + 2 \bigtriangleup) + b_4 \bigtriangledown + b_5 \bigcirc \bigcup \mathbf{i} + b_6 \bigcup \mathbf{i}.$$
Consider the matrix associated to the linear map \( \tilde{\phi}_{\mathbb{M}} : (\mathbb{R}(\tilde{G}), \sqcup) \to (\mathbb{R}(\tilde{G}), \sqcup_{\mathbb{M}}) \). Regarding the last example, we can change coordinates \((a_i) \in \mathbb{R}^9\) into \(\sqcup_{\mathbb{M}}\)-polynomials by solving the following equation for \((b_i) \in \mathbb{R}^9\).

\[
\begin{pmatrix}
1 & 1 & 1 & b_0 & a_0 \\
1 & 2 & 2 & 6 & b_1 & a_1 \\
2 & 6 & b_2 & a_2 \\
1 & 3 & 6 & b_3 & a_3 \\
1 & 3 & 6 & b_4 & a_4 \\
1 & 3 & 6 & b_5 & a_5 \\
1 & 2 & 6 & b_6 & a_6 \\
1 & 6 & b_7 & a_7 \\
6 & b_8 & a_8 \\
\end{pmatrix}
\]

The inverse is:

\[
\begin{pmatrix}
1 & 1/3 \\
1 & -1/2 \\
1 & -1 \\
1/2 & -1/2 \\
1 & -3/2 \\
1 & -3/2 \\
1 & -2 \\
1 & -1 \\
1/6 \\
\end{pmatrix}
\]

as an example, on the fourth column we read \( \mathbb{M} = -1/2 \mathbf{1} + 1/2 \mathbb{M} \mathbf{1} \) which can be checked.

Note that for \( n \geq 1 \),

\[\langle \mathbf{1}, \tilde{\phi}_{\mathbb{M}}(\mathbb{M}^{\mathbb{M} n}) \rangle = \langle \mathbf{1}, \mathbb{M}^{\mathbb{M} n} \rangle = 1.\]

That would mean that the matrix representation for \( \tilde{\phi}_{\mathbb{M}} \), as given above, has columns with finitely many non-zero entries, but some rows have infinitely many non-zero entries.

### 4 Counting subgraphs

Different works on graph counting defined their objects of interest –subgraphs– according to the application of interest. Due to this, one can also find various classes of definitions for subgraph counting, see for instance [CDM17; Big78; Bor+06; Bra+21; Pen20; Bor15]. For the most part, such definitions of counting are based on enumerative combinatorics, either from set restrictions or graph homomorphisms. In the current work, we aim at relating those definitions not only to each other but also to combinatorial Hopf algebra objects. With this purpose in mind, we consider restricted and induced subgraphs, see (3).
Let us first recall some definitions. Let $\sigma, \tau$ be graphs, a map $f : V(\sigma) \to V(\tau)$ is a **graph homomorphism** if for all $i, j \in V(\sigma)$

$$\{i, j\} \in E(\sigma) \Rightarrow \{f(i), f(j)\} \in E(\tau).$$

For a graph homomorphism $f$ we also write $f : \sigma \to \tau$ instead of $f : V(\sigma) \to V(\tau)$. Then $\text{im} f = f(V(\sigma))$ is the image of the vertex set. By slight abuse of notation we also denote by $f$ the induced map on edges, in particular

$$f(E(\sigma)) = \{\{f(i), f(j)\} \mid \{i, j\} \in E(\sigma)\} \subset E(\tau).$$

We call $f$ a **graph isomorphism** if it is a bijection and for all $i, j \in V(\sigma)$

$$\{i, j\} \in E(\sigma) \iff \{f(i), f(j)\} \in E(\tau).$$

Some notation:

- $\text{Hom}(\sigma, \tau)$ is the set of all graph homomorphisms.
- $\text{Mon}(\sigma, \tau)$ is the set of all injective (on vertices, and then necessarily on edges) graph homomorphisms.
- $\text{RegMon}(\sigma, \tau)$ is the set of all graph embeddings, that is, $\phi \in \text{RegMon}(\sigma, \tau)$ if $\phi \in \text{Hom}(\sigma, \tau)$ and $\phi : \sigma \to \tau_{\text{im} \phi}$ is an isomorphism, where $\tau_{\text{im} \phi}$ is the subgraph of $\tau$ induced by $\text{im} \phi \subset V(\tau)$.
- $\text{Epi}(\sigma, \tau)$ the set of all surjective (on vertices) graph homomorphisms.
- $\text{RegEpi}(\sigma, \tau)$ the set of all surjective (on vertices) graph homomorphisms such that the induced function on edges is also surjective.
- $\text{Iso}(\sigma, \tau)$ the set of all graph isomorphisms between $\sigma$ and $\tau$.

See Appendix A.5 for comments on nomenclature. We also write $\text{Aut}(\sigma) = \text{Iso}(\sigma, \sigma)$ for the group of all graph isomorphisms on $\sigma$.

![Graph Homomorphism Diagram](image-url)

**Figure 2**: Examples of the classes of graph homomorphisms.

### 4.1 Counting functions

The following theorems state the nature of different counting functions in the literature. This in turn allows us to develop useful definitions in combinatorial algebra. We start by formalizing the common notion of subgraph counting.

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Definition 4.1. Suppose that \( \tau, \Lambda \in \mathcal{G} \). Define the edge-restricted counting of \( \tau \) inside of \( \Lambda \) as,

\[
\mathcal{c}^e_{\tau}(\Lambda) := \left| \left\{ A \subseteq E(\Lambda) \mid \Lambda \upharpoonright_A \cong \tau \right\} \right|
\]

We first show that edge-restricted counting can also be expressed using graph homomorphisms.

Proposition 4.2. Suppose that \( \Lambda \) is a fixed graph in \( \mathcal{G} \). If \( \tau \in \mathcal{G} \), then

\[
\mathcal{c}^e_{\tau}(\Lambda) = \frac{\left| \text{Mon}(\tau, \Lambda) \right|}{|\text{Aut}(\tau)|}.
\]

Proof. For every \( A \subseteq E(\Lambda) \) with \( \Lambda \upharpoonright_A \cong \tau \) there exist \( |\text{Aut}(\tau)| \) elements in \( f \in \text{Mon}(\tau, \Lambda) \) with \( \text{im} f = A \). This gives the identity

\[
\mathcal{c}^e_{\tau}(\Lambda) = \frac{\left| \text{Mon}(\tau, \Lambda) \right|}{|\text{Aut}(\tau)|}.
\]

To show the second equality, we first notice that every isomorphism is a monomorphism. Since \( \tau \) and \( \Lambda \) do not have isolated vertices, \( |V(\tau)| = |V(\Lambda|_{\phi(E(\tau))})| = |V(\phi(\tau))| \) for every monomorphism \( \phi : \tau \rightarrow \Lambda \). Hence, any monomorphism \( \phi : \tau \rightarrow \Lambda \) is also a regular epimorphism on \( \Lambda|_{\phi(E(\sigma))} \). Therefore \( \phi \in \text{Iso}(\sigma, M|_{\phi(E(\sigma))}) \).

In addition to these rewriting results, the edge-restricted counting can be computed algebraically. Once fixed the parameter \( \tau \), we can read \( \mathcal{c}^e_{\tau}(\Lambda) \) as a function of \( \Lambda \), which leads to a Hopf algebraic interpretation. For this, we use the zeta function \( \zeta \) that maps each graph on \( \mathcal{G} \) to one, see Definition A.24.

Proposition 4.3. If \( \tau, \Lambda \in \mathcal{G} \), then edge-restricted counting can be obtained from convolutions.

\[
\mathcal{c}^e_{\tau}(\Lambda) = \left( \zeta *_{\mathbb{Z}} \tau^* \right)(\Lambda) = \frac{\left( \zeta *_{\mathbb{Z}} \tau^* \right)(\Lambda)}{2^{\left| E(\tau) \right|}}.
\]

Proof. From the definition of convolution,

\[
(\zeta *_{\mathbb{Z}} \tau^*)(\Lambda) = m_{\mathbb{Z}} \circ (\zeta \otimes \tau^*) \circ \Delta_{\mathbb{Z}}(\Lambda)
\]

\[
= \sum_{F \subseteq E(\Lambda)} m_{\mathbb{Z}} \circ (\zeta \otimes \tau^*)|_{E(\Lambda)} \otimes \Lambda|_{E(\Lambda)}
\]

\[
= \sum_{F \subseteq E(\Lambda)} m_{\mathbb{Z}} \circ (\zeta \otimes \tau^*)|_{E(\Lambda)} \otimes \Lambda|_{E(\Lambda)}
\]

\[
= \sum_{F \subseteq E(\Lambda)} \tau^*(\Lambda|_F)
\]

\[
= \left| \left\{ F \subseteq E(\Lambda) \mid \Lambda|_F \cong \tau \right\} \right|
\]

\[
= \mathcal{c}^e_{\tau}(\Lambda).
\]

\[
\left( \zeta *_{\mathbb{Z}} \tau^* \right)(\Lambda) = m_{\mathbb{Z}} \circ (\zeta \otimes \tau^*) \circ \Delta_{\mathbb{Z}}(\Lambda)
\]
\[
\begin{align*}
\tau^*(A_A) &= 2^{\left|E(\tau)\right|} \sum_{A \subseteq E(\Lambda)} \tau^*(A_A) \\
\tau^*(\Lambda_{|\Lambda}) &= 2^{\left|E(\tau)\right|} \tau^*(\Lambda_{|\Lambda}).
\end{align*}
\]

**Remark 4.4.** The edge-restricted counting is also known as subgraph counting [Pas21]. It is not explicitly stated what counting is used in [GB20], but their (unnormalized) counting \(c_{er}\) seems to be of edge-restricted type. Even though it is claimed in [Bra+21, Section 5.1] that counting of injective homomorphisms is used, this does not match the quasi-shuffle identity used in their equation (18) for example.

We have analogous results for induced subgraphs. To address these results, we should shift our focus to the vertices rather than the edges.

**Definition 4.5.** For \(\sigma\) and \(\Upsilon\) in \(\mathcal{G}\), define the vertex-induced counting of \(\sigma\) inside of \(\Upsilon\) as

\[
c_{vi}^\sigma(\Upsilon) := \left| \{A \subseteq V(\Upsilon) \mid \Upsilon|_A \cong \sigma\} \right|.
\]

We have again a reformulation in terms of graph homomorphisms.

**Proposition 4.6.** Suppose that \(\Upsilon \in \mathcal{G}\) is fixed. If \(\sigma \in \mathcal{G}\), then

\[
c_{vi}^\sigma(\Upsilon) = \frac{\left|\text{RegMon}(\sigma, \Upsilon)\right|}{\left|\text{Aut}(\sigma)\right|}.
\]

**Proof.** Notice that

\[
c_{vi}^\sigma(\Upsilon) = \left| \{A \subset V(\Upsilon) \mid M_A \cong \sigma\} \right| = \sum_{A \subseteq V(\Upsilon)} \frac{\left|\{\phi \in \text{Hom}(\sigma, \Upsilon_A) \mid \phi \in \text{Iso}(\sigma, \Upsilon_A)\}\right|}{\left|\text{Aut}(\sigma)\right|}.
\]

The corresponding result to vertices of Proposition 4.3 is the following.

**Proposition 4.7.** Suppose that \(\Upsilon \in \mathcal{G}\). If \(\sigma \in \mathcal{G}\), then

\[
c_{vi}^\sigma(\Upsilon) = \left(\zeta \ast_{i_{\mathcal{U}}} \sigma^\ast\right)(\Upsilon) = \left(\zeta \ast_{i_{\mathcal{U}}} \sigma^\ast\right)(\Upsilon).
\]

The proof is analogous to that of Proposition 4.3.

Finally, let us mention that some works, like [CDM17; Bor+06], and indirectly [Bra+21], count homomorphisms instead of subgraphs.

**Definition 4.8** (Homomorphisms counting). For \(\sigma, \Upsilon \in \mathcal{G}\), define

\[
c_{\text{hom}}^\sigma(\Upsilon) := \left|\text{Hom}(\sigma, \Upsilon)\right|.
\]

It is worth noting that the homomorphism counting function \(c_{\text{hom}}\) and its relation to \(c_{er}\) is covered in [CDM17].
4.2 Graph counting signatures

Motivated by the literature on iterated-sums/-integrals, in particular [DET20b], and the literature on graph motifs [Mil+02], in particular on their cumulants [Rec+19; GB20], we define a subgraph-counting signature. For a concrete, finite graph $\Lambda \in \mathcal{G}$ and any $\tau \in \mathcal{G}$, we define:

**Definition 4.9.** Assume that $\Lambda \in \mathcal{G}$ is a fixed graph, sometimes called the sample graph. Its graph counting signature is the linear functional $\text{GC}^\text{er}: \mathbb{R}(\mathcal{G})^\circ \to \mathbb{R}(\mathcal{G})^\circ$ defined by

$$\text{GC}^\text{er}(\Lambda) := \sum_{\tau \in \mathcal{G}} c^\text{er}_\tau(\Lambda) \tau.$$

**Remark 4.10.** This functional was defined in a fashion that resembles the iterated-integrals signature.

$$\langle \text{GC}^\text{er}(\Lambda), \tau \rangle = c^\text{er}_\tau(\Lambda).$$

Similarly, we define $\text{GC}^\text{vi}$

$$\text{GC}^\text{vi}(\Upsilon) := \sum_{\sigma \in \mathcal{G}} c^\text{vi}_\sigma(\Upsilon) \sigma.$$

We also define $\text{GC}^\text{hom}$:

$$\text{GC}^\text{hom}(\Upsilon) := \sum_{\sigma \in \mathcal{G}} c^\text{hom}_\sigma(\Upsilon) \sigma$$

and finally $\text{GC}^\text{hom;dp}$:

$$\text{GC}^\text{hom;dp}(\Upsilon) := \sum_{\sigma \in \mathcal{G}} \frac{1}{|\text{Aut}(\sigma)|} c^\text{hom}_\sigma(\Upsilon) \sigma.$$

Associating a graph to a sequence encoding occurrences of subgraphs or homomorphisms is a common procedure in the literature: $\text{GC}^\text{er}$ and $\text{GC}^\text{vi}$ appear in [CDM17] and $\text{GC}^\text{hom}$ is mentioned in [Bor+06].

**Example 4.11.**

\[
\begin{align*}
\text{GC}^\text{er}(\begin{array}{c}
\bullet
\end{array}) &= e + 6 \hat{\bullet} + 12 \ \hat{\Lambda} + 3 \hat{\bullet} \hat{\bullet} + 4 \ \hat{\Lambda} + \# \hat{\Lambda} + 14 \ \hat{\Lambda} + 6 \ \hat{\Lambda} + \# \\
\text{GC}^\text{vi}(\begin{array}{c}
\bullet
\end{array}) &= e + 3 \bullet + 1 + 2 \bullet + \bullet \bullet \\
\text{GC}^\text{vi}(\begin{array}{c}
\bullet
\end{array}) &= e + 4 \bullet + 3 \hat{\bullet} + 4 \ \hat{\Lambda} + \# \\
\text{GC}^\text{hom}(\begin{array}{c}
\bullet
\end{array}) &= e + 4 \bullet + 6 \hat{\bullet} + 36 \ \hat{\Lambda} + 144 \hat{\Lambda} \hat{\bullet} + \cdots \\
\end{align*}
\]

Notice that $\text{GC}^\text{er}(\mathbb{R}(\mathcal{G})) \subset \mathbb{R}(\mathcal{G})^\circ$ and $\text{GC}^\text{vi}(\mathbb{R}(\mathcal{G})) \subset \mathbb{R}(\mathcal{G})^\circ$. However, $\text{GC}^\text{hom}(\mathbb{R}(\mathcal{G})) \not\subset \mathbb{R}(\mathcal{G})^\circ$ since to each graph corresponds an infinite sum of terms.

$\text{GC}^\text{er}$ and $\text{GC}^\text{vi}$ can be considered as a change of basis:

**Lemma 4.12.** The linear functions $\text{GC}^\text{er}: \mathbb{R}(\mathcal{G})^\circ \to \mathbb{R}(\mathcal{G})^\circ$ and $\text{GC}^\text{vi}: \mathbb{R}(\mathcal{G})^\circ \to \mathbb{R}(\mathcal{G})^\circ$ are linear automorphisms.

**Proof.** The graphs in $\mathcal{G}$ are partially ordered by the number of edges. For this order, the matrix of $\text{GC}^\text{er}$ is upper-triangular.

i. $\langle \text{GC}^\text{er}(\Lambda), \sigma \rangle \neq 0$ implies that $|E(\sigma)| \leq |E(\Lambda)|$.

ii. $\langle \text{GC}^\text{er}(\Lambda), \sigma \rangle \neq 0$ and $|E(\sigma)| = |E(\Lambda)|$ implies $\sigma = \Lambda$.
It is invertible since the diagonal satisfies
\[ \langle \text{GC}^e(\Lambda), \Lambda \rangle = 1. \]
\[ \square \]

As a consequence of this lemma, \( \text{GC}^e(\Lambda) = \text{GC}^e(\Lambda') \) implies \( \Lambda = \Lambda' \). And the same holds for \( \text{GC}^v(\Upsilon) \). Notice that \( \text{GC}^e \) can be defined for graphs without isolated vertices as well:
\[ \text{GC}^e(\Upsilon) := \sum_{\sigma \in G} c^e(\Upsilon) \sigma. \]

but \( \text{GC}^e(\Upsilon) \) does not characterize \( \Upsilon \) anymore. For example,
\[ \text{GC}^e(1) = e + \mathbb{1}, \]
\[ \text{GC}^e(1 \cdot) = e + \mathbb{1}. \]

These equalities imply that \( \text{GC}^e(1 - 1 \cdot) = 0. \)

We now express the counting functionals using convolutions, as in Proposition 4.3. In this statement only, we use the notation \( \Lambda^* \) to clarify the space in which each operation is occurring.

**Lemma 4.13.** The map \( \text{GC}^e : (R(\tilde{G})^*, \sqcup, \Delta_{\mathbb{1}}) \rightarrow (R(\tilde{G})^*, \sqcup, \Delta_{\mathbb{1}}) \), as a linear function on the dual of \( \Lambda \in R(\mathcal{G}) \), is a convolution map

\[ \text{GC}^e(\Lambda^*) = \sqcup \circ (g \otimes \zeta) \circ \Delta_{\mathbb{1}}(\Lambda^*), \]

where we define \( g : R(\tilde{G})^* \rightarrow R(\mathcal{G})^* \) to be the linear extension of

\[ g(\tau^*) := \left( \frac{1}{2} \right)^{|E(\tau)|} \tau^*. \]

respectively, for \( h(\sigma) = 2^{-|V(\sigma)|} \sigma \), it holds

\[ \text{GC}^v(\Upsilon^*) = \sqcup \circ (h \otimes (u \circ \zeta)) \circ \Delta_{\mathbb{1}}(\Upsilon^*). \]

**Proof.** Using the same ideas as in Proposition 4.3,

\[ \sqcup \circ (g \otimes (u \circ \zeta)) \circ \Delta_{\mathbb{1}}(\Lambda^*) = \sum_{A \subseteq E(\Lambda)} \sqcup \circ (g \otimes (u \circ \zeta))(\Lambda_{\lfloor A}^* \otimes \Lambda_{\lfloor B}^*), \]

\[ = \sum_{A \subseteq E(\Lambda)} \sqcup \left( \left( \frac{1}{2} \right)^{|E(\Lambda)|} (\Lambda_{\lfloor A}^*) \otimes e^* \right), \]

\[ = \sum_{A \subseteq E(\Lambda)} \left( \frac{1}{2} \right)^{|E(\Lambda)|} (\Lambda_{\lfloor A}^*) \cdot \left( \# \{ B \mid E(\Lambda) \setminus A \subseteq B \subseteq E(\Lambda) \} \right), \]

\[ = \sum_{A \subseteq E(\Lambda)} \left( \frac{1}{2} \right)^{|E(\Lambda)|} (\Lambda_{\lfloor A}^*) \cdot 2^{|E(\Lambda)|}, \]

\[ = \sum_{A \subseteq E(\Lambda)} (\Lambda_{\lfloor A}^*) = \text{GC}^e(\Lambda^*). \]

The proof for \( \text{GC}^v \) follows analogously. \[ \square \]
4.3 Multiplicative and comultiplicative properties

We will now show that $GC^\sigma(\Lambda), GC^\iota(\Upsilon), GC^{\text{hom}}(\Upsilon), GC^{\text{hom,dp}}(\Upsilon)$ are characters over the appropriate Hopf algebras and furthermore satisfy a version of Chen’s identity.

**Theorem 4.14** (Character property). Suppose that $\Lambda \in \tilde{\mathcal{G}}$ and $\Upsilon \in \mathcal{G}$ are fixed graphs.

i. The functional $GC^\sigma(\Lambda)$ is a character over $(\mathbb{R}(\tilde{\mathcal{G}}), \sqcup)$, i.e.

$$\langle GC^\sigma(\Lambda), \tau_1 \rangle \cdot \langle GC^\sigma(\Lambda), \tau_2 \rangle = \langle GC^\sigma(\Lambda), \tau_1 \sqcup \tau_2 \rangle$$

ii. The functional $GC^\iota(\Upsilon)$ is a character over $(\mathbb{R}(\tilde{\mathcal{G}}), \sqcup)$, i.e.

$$\langle GC^\iota(\Upsilon), \sigma_1 \rangle \cdot \langle GC^\iota(\Upsilon), \sigma_2 \rangle = \langle GC^\iota(\Upsilon), \sigma_1 \sqcup \sigma_2 \rangle$$

iii. The functional $GC^{\text{hom}}(\Upsilon)$ is a character over $(\mathbb{R}(\tilde{\mathcal{G}}), \sqcup)$, i.e.

$$\langle GC^{\text{hom}}(\Upsilon), \sigma_1 \rangle \cdot \langle GC^{\text{hom}}(\Upsilon), \sigma_2 \rangle = \langle GC^{\text{hom}}(\Upsilon), \sigma_1 \sqcup \sigma_2 \rangle$$

iv. The functional $GC^{\text{hom,dp}}(\Upsilon)$ is a character over $(\mathbb{R}(\tilde{\mathcal{G}}), \sqcup)$, i.e.

$$\langle GC^{\text{hom,dp}}(\Upsilon), \sigma_1 \rangle \cdot \langle GC^{\text{hom,dp}}(\Upsilon), \sigma_2 \rangle = \langle GC^{\text{hom,dp}}(\Upsilon), \sigma_1 \sqcup \sigma_2 \rangle$$

The result regarding $GC^\sigma(\Lambda)$ is known in the computer science literature, see for example [Mau+20, Lemma2]. See also [Bor15] and [Pen20]. The result for $GC^{\text{hom}}(\Lambda)$ appears in [Lov67, Thm. 3.6].

**Proof.**

i. This follows from

$$\left\{(E, F) \mid E, F \subseteq E(\Lambda), \Lambda \mid_E \equiv \sigma, \Lambda \mid_F \equiv \tau\right\}$$

$$= \bigcup_{\eta \in \mathcal{G}} \left\{(E, F) \mid E, F \subseteq E(\Lambda), \Lambda \mid_E \equiv \sigma, \Lambda \mid_F \equiv \tau, \Lambda \mid_{E \cup F} \equiv \eta\right\}$$

$$= \bigcup_{\eta \in \mathcal{G}} \bigcup_{\Lambda \subseteq E(\Lambda)} \left\{(E, F) \mid E, F \subseteq A, E \cup F = A, \Lambda \mid_E \equiv \sigma, \Lambda \mid_F \equiv \tau, \Lambda \mid_A \equiv \eta\right\}$$

ii. Analogously,

$$\left\{(E, F) \mid E, F \subseteq V(\Upsilon), M_E \equiv \sigma_1, M_F \equiv \sigma_2\right\}$$

$$= \bigcup_{\eta \in \mathcal{G}} \left\{(E, F) \mid E, F \subseteq V(\Upsilon), M_E \equiv \sigma_1, M_F \equiv \sigma_2, M_{E \cup F} \equiv \eta\right\}$$

$$= \bigcup_{\eta \in \mathcal{G}} \bigcup_{\Lambda \subseteq V(\Upsilon)} \left\{(E, F) \mid E, F \subseteq A, E \cup F = A, M_E \equiv \sigma_1, M_F \equiv \sigma_2, M_A \equiv \eta\right\}$$

iii. Immediate by construction.

iv. From 2.8, $\forall \tau_1, \tau_2 \in \mathcal{G}, \tau_1 \sqcup \tau_2 = \frac{|\text{Aut} \tau_1| |\text{Aut} \tau_2|}{|\text{Aut} \tau_1 \sqcup \tau_2|} \tau_1 \sqcup \tau_2$. Then

$$\langle GC^{\text{hom,dp}}(\Upsilon), \sigma_1 \sqcup \sigma_2 \rangle = \frac{|\text{Aut} \sigma_1 \sqcup \sigma_2|}{|\text{Aut} \sigma_1||\text{Aut} \sigma_2|} \langle GC^{\text{hom,dp}}(\Upsilon), \sigma_1 \sqcup \sigma_2 \rangle$$

$$= \frac{|\text{Aut} \sigma_1 \sqcup \sigma_2|}{|\text{Aut} \sigma_1||\text{Aut} \sigma_2|} \langle GC^{\text{hom,dp}}(\Upsilon), \sigma_1 \sqcup \sigma_2 \rangle$$
\[
\frac{|\text{Aut} \sigma_1 \sqcup \sigma_2|}{|\text{Aut} \sigma_1||\text{Aut} \sigma_2|} |\text{Hom}(\sigma_1 \sqcup \sigma_2, \Upsilon)| = \frac{|\text{Hom}(\sigma_1, \Upsilon)||\text{Hom}(\sigma_2, \Upsilon)|}{|\text{Aut} \sigma_1||\text{Aut} \sigma_2|} = \langle \mathcal{G}_{\text{hom,dp}}(\Upsilon), \sigma_1 \rangle \langle \mathcal{G}_{\text{hom,dp}}(\Upsilon), \sigma_2 \rangle
\]

We already know that connected graphs generate the quasi-shuffle algebra. This fact, together with Theorem 4.14, tells us that counting connected subgraphs is enough to count all subgraphs.

**Remark 4.15.** Let \( \Lambda \in \tilde{\mathcal{G}} \) and \( \tau \in \tilde{\mathcal{G}} \), then

\[
\mathcal{G}_{\text{er}}(\Lambda), \tau \rangle = \sum_{i \in I} a_i \prod_{j \in J} \langle \mathcal{G}_{\text{er}}(\Lambda), \theta_{ij} \rangle.
\]

where \( a_i \in \mathbb{R} \), \( \theta_{ij} \in \mathcal{G}_0 \), i.e., it is connected. This is folklore in the computer science community, and it is formalized by Corollary 3.9 and Theorem 4.14.

Analogously, as proven by [Pen20], for \( M \in \mathcal{G} \) and \( \sigma \in \mathcal{G} \), we also have

\[
\mathcal{G}_{\text{vi}}(\Upsilon), \sigma \rangle = \sum_{i \in I} b_i \prod_{j \in J} \langle \mathcal{G}_{\text{vi}}(\Upsilon), \gamma_{ij} \rangle.
\]

where again \( b_j \in \mathbb{R} \) and \( \gamma_{ij} \in \mathcal{G}_0 \) are connected.

The respective result for \( \mathcal{G}_{\text{hom}} \) is immediate given that

\[
\langle \mathcal{G}_{\text{hom}}(\Upsilon), \sigma \rangle = \prod_{i=1}^n \langle \mathcal{G}_{\text{hom}}(\Upsilon), \sigma_i \rangle.
\]

where \( \sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_n \).

Another property common for signatures is Chen’s identity. This is the compatibility of the signature with respect to a product in the parameter, not in the argument.

**Theorem 4.16** (Chen’s identity). The following holds

i. Let \( \Lambda, \Psi, \tau \in \tilde{\mathcal{G}} \)

\[
\langle \mathcal{G}_{\text{er}}(\Lambda \sqcup \Psi), \tau \rangle = \langle \mathcal{G}_{\text{er}}(\Lambda) \otimes \mathcal{G}_{\text{er}}(\Psi), \Delta \sqcup \tau \rangle.
\]

or dually

\[
\mathcal{G}_{\text{er}}(\Lambda \sqcup \Psi) = \mathcal{G}_{\text{er}}(\Lambda \sqcup \Psi).
\]

ii. Let \( \Upsilon, \Gamma, \sigma \in \mathcal{G} \)

\[
\langle \mathcal{G}_{\text{vi}}(\Upsilon \sqcup \Gamma), \sigma \rangle = \langle \mathcal{G}_{\text{vi}}(\Upsilon) \otimes \mathcal{G}_{\text{vi}}(\Gamma), \Delta \sqcup \sigma \rangle.
\]

iii.

\[
\langle \mathcal{G}_{\text{hom}}(\Upsilon \sqcup \Gamma), \sigma \rangle = \langle \mathcal{G}_{\text{hom}}(\Upsilon) \otimes \mathcal{G}_{\text{hom}}(\Gamma), \Delta_{DP} \sqcup \sigma \rangle.
\]

iv.

\[
\langle \mathcal{G}_{\text{hom,dp}}(\Upsilon \sqcup \Gamma), \sigma \rangle = \langle \mathcal{G}_{\text{hom,dp}}(\Upsilon) \otimes \mathcal{G}_{\text{hom,dp}}(\Gamma), \Delta \sqcup \sigma \rangle.
\]
Example 4.17.

\[ (\mathbb{G}^e(\Lambda \sqcup \Psi), \mathbf{1}) = (\mathbb{G}^e(\Lambda) \otimes \mathbb{G}^e(\Psi), \mathbf{e} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{i} + \mathbf{1} \otimes \mathbf{i} + \mathbf{i} \otimes \mathbf{e}) \]

\[ = (\mathbb{G}^e(\Psi), \mathbf{1}) + (\mathbb{G}^e(\Lambda), \mathbf{1})/\mathbb{G}^e(\Psi), \mathbf{1} + (\mathbb{G}^e(\Lambda), \mathbf{1}) + (\mathbb{G}^e(\Lambda), \mathbf{1}) \]

\[ = (\mathbb{G}^e(\Psi), \mathbf{1}) + (\mathbb{G}^e(\Lambda), \mathbf{1})/\mathbb{G}^e(\Psi), \mathbf{1} + (\mathbb{G}^e(\Lambda), \mathbf{1}) + (\mathbb{G}^e(\Lambda), \mathbf{1}) \]

Remark 4.18. Chen's identity for iterated sums (or integrals) yields a procedure of efficient calculation. Theorem 4.16 is very weak in this regard, since it does not allow to simplify the calculation of \( \mathbb{G}^e \) if \( \Lambda \) is connected.

Proof. Since \( g, u, \zeta : (\mathbb{R}(\tilde{G}), \odot) \to (\mathbb{R}(\tilde{G}), \odot) \) are characters for the product \( \sqcup \), from Lemma 4.13, we can use that in a commutative bialgebra the multiplication of characters is a character, as mentioned in Proposition A.26. As a consequence, the functional \( \mathbb{G}^e : (\mathbb{R}(\tilde{G}), \odot) \to (\mathbb{R}(\tilde{G}), \odot) \) is also an algebra homomorphism.

We give an alternative, combinatorial proof.

\[ \mathbb{G}^e(\Lambda \sqcup \Psi) = \sum_{A \subseteq B(\Lambda \sqcup \Psi)} |\{ A \subseteq E(\Lambda \sqcup \Psi) \} \cdot |A \cdot \gamma | \Gamma | \sigma | \tau \]

and

\[ \mathbb{G}^e(\Lambda \sqcup \Psi) = \sum_{B \subseteq E(\Lambda)} (\Lambda) \sqcup (\Psi) = \sum_{C \subseteq E(\Psi)} (\Psi) \]

\[ = \sum_{\sigma \in \tilde{G}} \sum_{\gamma \in \tilde{G}} \sum_{\Gamma} \sum \gamma \gamma \]

For \( \mathbb{G}^e \) both algebraical and combinatorial proofs are basically analogous.

First suppose that \( \sigma \in \tilde{G} \) is connected. By enumerative combinatorics,

\[ \langle \mathbb{G}^e(\Psi), \sigma \rangle = \langle \mathbb{G}^e(\Gamma), \sigma \rangle \]

Because of Theorem 4.14, for any \( \sigma_1, \ldots, \sigma_n \in \tilde{G} \) connected, we have that

\[ \langle \mathbb{G}^e(\Psi), \sigma_1 \sqcup \cdots \sqcup \sigma_n \rangle = \langle \mathbb{G}^e(\Psi), \sigma_1 \rangle \cdot \cdots \cdot \langle \mathbb{G}^e(\Psi), \sigma_n \rangle \]

\[ = \langle \mathbb{G}^e(\Gamma), \sigma_1 \rangle \cdot \cdots \cdot \langle \mathbb{G}^e(\Gamma), \sigma_n \rangle \]

\[ = \langle \mathbb{G}^e(\Gamma), \sigma_1 \rangle \cdot \cdots \cdot \langle \mathbb{G}^e(\Gamma), \sigma_n \rangle \]

\[ = \langle \mathbb{G}^e(\Gamma), \sigma_1 \rangle \cdot \cdots \cdot \langle \mathbb{G}^e(\Gamma), \sigma_n \rangle \]

For \( \mathbb{G}^{\text{hom}, \text{dp}} \), we have an analogous combinatorial argument in case \( \sigma \) is connected. For any \( \sigma_1, \ldots, \sigma_n \in \tilde{G} \) connected, we have \( \sigma_1 \sqcup \cdots \sqcup \sigma_n = c \sigma_1 \cdot \text{DP} \cdot \cdots \cdot \text{DP} \sigma_n \) for some positive integer \( c \). Then

\[ \langle \mathbb{G}^{\text{hom}, \text{dp}}(\Lambda \sqcup \Gamma), \sigma_1 \sqcup \cdots \sqcup \sigma_n \rangle = \langle \mathbb{G}^{\text{hom}, \text{dp}}(\Lambda \sqcup \Gamma), c \sigma_1 \cdot \text{DP} \cdot \cdots \cdot \text{DP} \sigma_n \rangle \]
\[ c\left< GC^{\text{hom;dp}}(\Upsilon \sqcup \Gamma), \sigma_1 \right> \cdots \left< GC^{\text{hom;dp}}(\Upsilon \sqcup \Gamma), \sigma_n \right> \]

\[ c\left< GC^{\text{hom;dp}}(\Upsilon) \otimes GC^{\text{hom;dp}}(\Gamma), \sigma_1 \otimes e + e \otimes \sigma_1 \right> \cdots \left< GC^{\text{hom;dp}}(\Upsilon) \otimes GC^{\text{hom;dp}}(\Gamma), \sigma_n \otimes e + e \otimes \sigma_n \right> \]

\[ c\left< GC^{\text{hom;dp}}(\Upsilon \sqcup \Gamma), \sigma_1 \otimes DP_{i=1}^n (\Delta_i) \right> \]

\[ c\left< GC^{\text{hom;dp}}(\Upsilon \sqcup \Gamma), \Delta_i (c \cdot \sigma_1 \otimes DP \cdots DP \sigma_n) \right> \]

\[ c\left< GC^{\text{hom;dp}}(\Upsilon \sqcup \Gamma), \Delta_i (c \cdot \sigma_1 \sqcup \cdots \sqcup \sigma_n) \right> \]

\[ \square \]

### 4.4 Translating between the counting characters

Here we show that counting subgraphs and counting homomorphisms is somehow equivalent: some maps allow to translate between these numbers. Additionally, it is possible to switch between the counting of induced and restricted subgraphs. These maps are well-known in computer science literature. Their relevance comes from the fact that counting graph homomorphisms is more efficient than counting subgraphs: see for instance [CDM17, Section 1.4]. We prove that these maps are Hopf isomorphisms.

**Definition 4.19.** Define

\[
\Phi_{\text{vi-er}}(\tau) : = \frac{1}{|\text{Aut}(\tau)|} \sum_{\sigma} \left| \text{Mon}(\tau, \sigma) \cap \text{Epi}(\tau, \sigma) \right| \sigma,
\]

\[
\Phi_{\text{er-hom}}(\tau) : = \sum_{\sigma} \left| \text{RegEpi}(\tau, \sigma) \right| \sigma,
\]

\[
\Phi_{\text{vi-hom}}(\tau) : = \sum_{\sigma} \left| \text{Epi}(\tau, \sigma) \right| \sigma.
\]

**Example 4.20.**

\[
\Phi_{\text{vi-er}}(1) = 1
\]

\[
\Phi_{\text{vi-er}}(\Lambda) = \Lambda + 3 \Delta
\]

\[
\Phi_{\text{er-hom}}(1) = 21
\]

\[
\Phi_{\text{er-hom}}(\Lambda) = 41 + 8 \Lambda + 8 \Omega
\]

\[
\Phi_{\text{vi-hom}}(1) = 21
\]

\[
\Phi_{\text{vi-hom}}(\Lambda) = 21 + 6 \Lambda
\]

**Remark 4.21.** If we take \( \Phi_{\text{vi-er}}, \Phi_{\text{er-hom}} \) as maps having \( R(\tilde{G}) \) as domain, then graphs with isolated vertices do not appear in the images.

*Note also that in the case where \( \Phi_{\text{vi-er}} : R(\tilde{G}) \rightarrow R(\tilde{G}) \), is easy to see that*

\[
\Phi_{\text{vi-er}}(\tau) = \sum_{\sigma : |V(\sigma)| = |V(\tau)|} \left< GC^{\sigma}(\sigma), \tau \right> \sigma
\]

These maps allow to translate between the different counting operations.

**Theorem 4.22.** For \( \tau, \Lambda \in \tilde{G} \), it holds that

\[
\left< GC^{\sigma}(\Lambda), \tau \right> = \left< GC^{\sigma}(\Lambda), \Phi_{\text{vi-er}}(\tau) \right>, \quad \text{(10)}
\]

\[
\left< GC^{\text{hom}}(\Lambda), \tau \right> = \left< GC^{\sigma}(\Lambda), \Phi_{\text{er-hom}}(\tau) \right>, \quad \text{(11)}
\]

\[
\left< GC^{\text{hom}}(\Lambda), \tau \right> = \left< GC^{\sigma}(\Lambda), \Phi_{\text{vi-er}} \circ \Phi_{\text{er-hom}}(\tau) \right>, \quad \text{(12)}
\]
Proof. Using Corollary A.33.i we have

\[ \langle GC^vixer(\Lambda), \tau \rangle = \frac{|\text{Mon}(\tau, \Lambda)|}{|\text{Aut}(\tau)|} = \frac{1}{|\text{Aut}(\tau)|} \sum_{\sigma} \left| \left\{ \phi \in \text{Mon}(\tau, \Lambda) \mid \sigma \cong \Lambda_{\text{im} \phi} \right\} \right| \]

\[ = \frac{1}{|\text{Aut}(\tau)|} \sum_{\sigma} |\text{Epi}(\tau, \sigma) \cap \text{Mon}(\tau, \sigma)| \frac{|\text{RegMon}(\sigma, \Lambda)|}{|\text{Aut}(\sigma)|} \]

\[ = \langle GC^vixer(\Lambda), \Phi_{\text{vix-er}}(\tau) \rangle. \]

This proves (10).

Using Corollary A.33.iv we have

\[ \langle GC^\text{hom}(\Lambda), \tau \rangle = \left| \text{Hom}(\tau, \Lambda) \right| = \sum_{\sigma} \left| \left\{ \phi \in \text{Hom}(\tau, \Lambda) \mid \Lambda \cong \sigma_{\text{im} \phi} \right\} \right| \]

\[ = \sum_{\sigma} \frac{1}{|\text{Aut}(\sigma)|} |\text{RegEpi}(\tau, \sigma)| \left| \text{Mon}(\tau, \Lambda) \right| \]

\[ = \langle GC^\text{er}(\Lambda), \Phi_{\text{er-hom}}(\tau) \rangle. \]

This proves (11), and (12) follows by composition. \( \square \)

Example 4.23.

\[ \langle GC^vixer(\Lambda), \Lambda \rangle = 12 \]
\[ \langle GC^vixer(\Lambda), \Phi_{\text{vix-er}}(\Lambda) \rangle = \langle GC^vixer(\Lambda), \Lambda + 3 \Delta \rangle \]
\[ = \langle GC^vixer(\Lambda), \Lambda \rangle + 3 \langle GC^vixer(\Lambda), \Delta \rangle \]
\[ = 0 + 3 \cdot 4 = 12 \]
\[ \langle GC^\text{hom}(\Lambda), \Lambda \rangle = 144 \]
\[ \langle GC^vixer(\Lambda), \Phi_{\text{vix-er-hom}}(\Lambda) \rangle = \langle GC^vixer(\Lambda), 4\,\Lambda + 8 \Delta + 8 \Lambda \rangle \]
\[ = 4 \langle GC^vixer(\Lambda), \Lambda \rangle + 8 \langle GC^vixer(\Lambda), \Delta \rangle + 8 \langle GC^vixer(\Lambda), \Lambda \rangle \]
\[ = 4 \cdot 6 + 8 \cdot 12 + 8 \cdot 3 = 144 \]

Remark 4.24. Some of these results are already known in the literature for enumerative combinatorics.

i. A result traditionally attributed to Lovasz, which can be read in [CDM17, Equation (8)], is the same as (11). In [CDM17], Spasm(\tau) denotes the set of graphs \( \sigma \in \mathcal{G} \) such that RegEpi(\tau, \sigma) \( \neq \emptyset \).

ii. By Proposition 4.3 and then (10)

\[ |\text{Mon}(\tau, \Lambda)| = |\text{Aut}(\tau)| \langle GC^vixer(\Lambda), \tau \rangle \]
\[ = \langle GC^vixer(\Lambda), \sum_{\sigma} |\text{Mon}(\tau, \sigma) \cap \text{Epi}(\tau, \sigma)| \sigma \rangle \]
\[ = \sum_{\sigma} |\text{Mon}(\tau, \sigma) \cap \text{Epi}(\tau, \sigma)| \langle \text{RegMon}(\sigma, \Lambda), \rangle, \]

which is the same as [Lov12, Equation (5.15)].
These translating maps behave well with respect to the underlying Hopf structure.

**Theorem 4.25.** The maps

\[ \Phi_{\text{vi-e}} : (\mathcal{R}(\hat{G}), \sqcap, \sqcup, \Delta_{\square}) \rightarrow (\mathcal{R}(\hat{G}), \sqcap, \sqcup, \Delta_{\square}) \]

are Hopf algebra isomorphisms. Moreover,

\[ \Phi_{\text{vi-e}} \circ \Phi_{\text{er-hom}} = \Phi_{\text{vi-hom}}, \]

and in particular, \( \Phi_{\text{vi-hom}} \) is a Hopf algebra isomorphism as well.

**Proof.** We want to use Proposition 3.5, and divide the proof into three parts.

i. The functions are proved to be algebra homomorphisms.

ii. We verify that \( \Phi_{\text{vi-e}} \) and \( \Phi_{\text{er-hom}} \) are linear bijections when restricted to \( \mathcal{R}(\hat{G}^0) \).

iii. Since, in both cases, the domain is an algebra freely generated by connected graphs –as shown in Proposition 3.7, then every algebra homomorphism is the natural extension of a linear function defined on the connected graphs. This allows us to use Proposition 3.5.

We can restrict \( \mathcal{G}^{\text{ci}} : \mathcal{R}(\hat{G}) \rightarrow \mathcal{R}(\hat{G}) \) to the space of graphs without isolated vertices, we write \( \mathcal{G}^{\text{ci}} : \mathcal{R}(\hat{G}) \rightarrow \mathcal{R}(\hat{G}) \), and have that this is a linear automorphism using an argument analogous to the one used in the proof of Lemma 4.12. The algebra homomorphism property follows from applying first Theorem 4.22 and then Theorem 4.14 twice. Indeed, let \( \tau_1, \tau_2 \in \mathcal{R}(\hat{G}) \), hence

\[ \langle \mathcal{G}^{\text{ci}}(\Lambda), \Phi_{\text{vi-e}}(\tau_1 \sqcup \tau_2) \rangle = \langle \mathcal{G}^{\text{ci}}(\Lambda), \tau_1 \sqcup \tau_2 \rangle \]

Analogously, using again Theorem 4.14 and Theorem 4.22

\[ \langle \mathcal{G}^{\text{ci}}(\Lambda), \Phi_{\text{er-hom}}(\tau_1 \sqcup \tau_2) \rangle = \langle \mathcal{G}^{\text{ci}}(\Lambda), \Phi_{\text{er-hom}}(\tau_1) \sqcup \Phi_{\text{er-hom}}(\tau_2) \rangle. \]

Since \( \mathcal{G}^{\text{ci}}, \mathcal{G}^{\text{er}} : \mathcal{R}(\hat{G}) \rightarrow \mathcal{R}(\hat{G}) \) are linear automorphisms, we can conclude that \( \forall \tau_1, \tau_2 \in \mathcal{R}(\hat{G}) \)

\[ \Phi_{\text{vi-e}}(\tau_1 \sqcup \tau_2) = \Phi_{\text{vi-e}}(\tau_1) \sqcup \Phi_{\text{vi-e}}(\tau_2) \]

\[ \Phi_{\text{er-hom}}(\tau_1 \sqcup \tau_2) = \Phi_{\text{er-hom}}(\tau_1) \sqcup \Phi_{\text{er-hom}}(\tau_2). \]

Now, notice that

\[ \text{Prim}(\mathcal{R}(\hat{G}), \Delta_{\text{DP}}) = \mathcal{R}(\hat{G}^0) = \text{Prim}(\mathcal{R}(\hat{G}), \Delta_{\square}) \]

as proved in Lemma 3.6.

In order to prove that \( \Phi_{\text{er-hom}} \) is invertible, we define an order relation on \( \hat{G} \). We say that \( \sigma < \tau \) if and only if \( \text{RegEpi}(\tau, \sigma) \neq \emptyset \) and \( \text{RegEpi}(\sigma, \tau) = \emptyset \). We can take the closure of this relation due to the transitivity of regular epimorphisms in the sense of Appendix A.5.

Suppose that \( \phi : \tau \rightarrow \sigma \) is a regular epimorphism and \( \tau \) is connected. This implies that \( \sigma \) is connected. Now write

\[ \Phi_{\text{er-hom}}(\tau) = \sum_{\sigma \in \hat{G}^0, \sigma < \tau} |\text{RegEpi}(\tau, \sigma)| \sigma + |\text{Aut}(\tau)| \tau. \]
Since $|\text{Aut}(\tau)| \geq 1$ for all $\tau$, we can calculate $\Phi^{-1}_{\text{er-hom}}(\tau)$. This proves that $\Phi_{\text{er-hom}}$ restricted to the connected graphs is indeed a bijection. We now show that the restriction of $\Phi_{\text{vi-er}}$ to the connected graphs is invertible. Remember from Remark 4.21 that

$$\Phi_{\text{vi-er}}(\tau) = \sum_{\sigma:|V(\sigma)|=|V(\tau)|} \langle \text{GC}^\text{er}(\sigma), \tau \rangle \sigma$$

Clearly the image of a connected graph will be a linear combination of connected graphs. Moreover the map is clearly graded by the number of vertices and $\forall \tau \in \tilde{\mathcal{G}}^0$, it holds that $\langle \text{GC}^\text{er}(\sigma), \tau \rangle = 0$, if $E(\sigma) \geq E(\tau)$ if $\sigma \neq \tau$.

Finally, the composition in (13) follows from Corollary A.33.ii.

**Remark 4.26.** Note that these maps are not the same isomorphisms listed in Proposition 3.7, see Example 4.20.

**Remark 4.27.** Since these maps are invertible:

$$\langle \text{GC}^\text{vi}(\Lambda), \tau \rangle = \langle \text{GC}^\text{er}(\Lambda), \Phi_{\text{vi-er}}^{-1}(\tau) \rangle$$

$$\langle \text{GC}^\text{er}(\Lambda), \tau \rangle = \langle \text{GC}^\text{hom}(\Lambda), \Phi_{\text{er-hom}}^{-1}(\tau) \rangle$$

Notice that (13) can also be written more generally for $\sigma, \Upsilon \in \mathcal{G}$, and:

$$\langle \text{GC}^\text{hom}(\Upsilon), \sigma \rangle = \langle \text{GC}^\text{vi}(\Upsilon), \Phi_{\text{vi-hom}}(\sigma) \rangle$$

$$\langle \text{GC}^\text{vi}(\Upsilon), \sigma \rangle = \langle \text{GC}^\text{hom}(\Upsilon), \Phi_{\text{vi-hom}}^{-1}(\sigma) \rangle$$

5 Conclusions and outlook

This work explored different definitions of subgraph counting, ranging from connected components to edge restrictions and induced subgraphs, from the point of view of combinatorial Hopf algebra. These perspectives coincide in the sense that the corresponding Hopf algebras are isomorphic. We also showed that the maps translating between these counting procedures are Hopf isomorphism. The counting functions are parametrized by graphs and interpreted as signature-type objects. As observed in Remark 4.15, counting occurrences can be expressed as polynomials of the counting occurrences on connected graphs alone. This result can be traced back to the classical articles of Whitney [Whi32] and Biggs [Big78]. However, we remark that these two last articles express the counting of decomposable graphs in terms of indecomposable graphs. For instance, the following relation

$$e \Delta t(\Lambda) = e \Delta (t(\Lambda) - 3).$$

appears in [Big78], which cannot be obtained from the quasi-shuffle product. While the concepts of indecomposable and connected graphs are tightly related, the precise algebraic relations remain to be explored in further work. We also describe Chen’s identity – the comultiplicative property of graph counting. It permits the simplification of computations for disconnected graphs in terms of their connected components. However, we remark that it does not decompose connected graphs into smaller objects. This is deemed important in practice since graph counting problems usually appear on large connected sample graphs. It is an open problem to find a more practical type of Chen’s identity.

References

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Appendix

A Background on Hopf algebras

A.1 Background on Hopf algebras

A.2 Hopf algebras

This section is based on [GR20] and [Man08]. Here we give the definition of Hopf algebra. We start by recalling the basic notions necessary for their definition.

A.2.1 Algebras and coalgebras

Let $k$ be a field (of characteristic zero). All vector spaces considered here are $k$-vector spaces and all linear maps are $k$-linear maps. We now give the definition of (associative) algebra.

**Definition A.1 (Algebra).** A $k$-algebra $A$ is a vector space equipped with a bilinear product $m : A \otimes A \rightarrow A$ and a linear functional $u : k \rightarrow A$ called unit for which the following hold:

\[
m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)
\]
\[
m \circ (\text{id} \otimes u) \circ l = \text{id} = m \circ (u \otimes \text{id}) \circ l'.
\]

The first condition says that the product is **associative**, while the second means that there is a unit element in $A$. Here $l : A \rightarrow A \otimes k$ and $l' : A \rightarrow k \otimes A$, are isomorphisms sending $a \mapsto a \otimes 1_k$ and $a \mapsto 1_k \otimes a$, respectively. Note that in the following we will write $ab = a \cdot b := m(a \otimes b)$. The unit map is simply $u(1_k) = 1_A$.

The definition of coalgebra can be seen as the dual notion of algebra.

**Definition A.2 (Coalgebra).** A $k$-coalgebra $C$ is a vector space equipped with two linear maps: a coproduct $\Delta : C \rightarrow C \otimes C$ and a counit $\varepsilon : C \rightarrow k$ such that the following holds:

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta
\]
\[
t \circ (\text{id} \otimes \varepsilon) \circ \Delta = \text{id} = t' \circ (\varepsilon \otimes \text{id}) \circ \Delta.
\]

The first condition is called **coassociativity**. Here $t : C \otimes k \rightarrow C$, $t' : k \otimes C \rightarrow C$, are isomorphisms mapping $c \otimes 1_k \mapsto c$ respectively $1_k \otimes c \mapsto c$. Note that for a given coproduct $\Delta$ there is at most one map $\varepsilon$ such that the previous condition holds.
For the coproduct, we will use **Sweedler’s notation.**

\[ \Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_1 \otimes c_2. \]

We now introduce homomorphisms for algebras and coalgebras.

**Definition A.4.** A linear map \( \varphi : (A, m_A) \rightarrow (B, m_B) \) is an algebra homomorphism or an algebra map whenever

\[ \varphi \circ m_A = m_B \circ (\varphi \otimes \varphi). \]

Additionally, \( \varphi \circ u_A = u_B \) for unital associative algebras. A coalgebra map or coalgebra homomorphism is a linear function \( \psi : (C, \Delta_C) \rightarrow (D, \Delta_D) \) which preserves the coalgebra structure.

\[ (\psi \otimes \psi) \circ \Delta_C = \Delta_D \circ \psi. \]

If the coalgebra is counital, then \( \varepsilon_D \circ \psi = \varepsilon_C \).

Isomorphisms of algebras and coalgebras are bijective homomorphisms such that the inverses are also morphisms. We have the following result.

**Lemma A.5.** Let \((A_1, m_1), (A_2, m_2)\) be algebras and let \( \phi : A_1 \rightarrow A_2 \) be a homomorphism. In case \( \phi \) admits a compositional inverse, \( \phi \) is an isomorphism. Moreover, let \((C_1, \Delta_1), (C_2, \Delta_2)\) be coalgebras and let \( \psi : C_1 \rightarrow C_2 \) be a homomorphism. In case \( \psi \) admits a compositional inverse, \( \psi \) is an isomorphism.

**Proof.** The definition of algebra homomorphism implies that \( \phi \circ m_1 = m_2 \circ (\phi \otimes \phi) \). Composing by \( \phi^{-1} \) on the left yields \( m_1 = \phi^{-1} \circ \phi \circ m_1 = \phi^{-1} \circ m_2 \circ (\phi \otimes \phi) \). The result follows from composing the last equation with \( \phi^{-1} \otimes \phi^{-1} \) on the right.

\[ m_1 \circ (\phi^{-1} \otimes \phi^{-1}) = \phi^{-1} \circ m_2. \]

For coalgebras,

\[ \Delta_1 = (\psi^{-1} \otimes \psi^{-1}) \circ (\psi \otimes \psi) \circ \Delta_1 = (\psi^{-1} \otimes \psi^{-1}) \circ \Delta_2 \circ \psi, \]

and the result follows once we compose on the right with \( \psi^{-1} \).

### A.2.2 Bialgebras

**Definition A.6** (Tensor product of two algebras). Given two coalgebras \( A, B \), the tensor product \( A \otimes B \) becomes a \( k \)-algebra with multiplication defined as follows:

\[ m_{A \otimes B} : (A \otimes B) \otimes (A \otimes B) \rightarrow A \otimes B \]

\[ m((a \otimes b) \otimes (a' \otimes b')) := aa' \otimes bb' \]

and unit map:

\[ u_{A \otimes B} : k \rightarrow A \otimes B \]

\[ u_{A \otimes B}(1_k) = 1_A \otimes 1_B. \]

We are now ready to define bialgebras.

**Definition A.7** (Bialgebra). Let \( A \) be both a \( k \)-algebra and a \( k \)-coalgebra. We say that \( A \) is a bialgebra if for all \( x, y \in A \):

\[ \Delta(xy) = \Delta(x) \Delta(y), \]

i.e. the coproduct \( \Delta : A \rightarrow A \otimes A \) is an algebra morphism, and

\[ \varepsilon(xy) = \varepsilon(x) \varepsilon(y), \]

i.e. \( \varepsilon : A \rightarrow k \) is an algebra morphism.
**Definition A.8** (Bialgebra morphism). Let $A, B$ be $k$-bialgebras. Let $\varphi : A \to B$ be a $k$-linear map that is an algebra as well as a coalgebra homomorphism. Then we say that $\varphi$ is a bialgebra homomorphism.

**Definition A.9.** Let $x$ be an element in the coalgebra $C$. We say that $x$ is group-like if $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$.

**Definition A.10.** Let $A$ be a bialgebra, we say that $x \in A$ is primitive if $\Delta(x) = 1_A \otimes x + x \otimes 1_A$.

**Remark A.11.** Note that zero is not group-like. In fact, zero is primitive, $\varepsilon(0) = 0$.

**Definition A.12** (Ideal, Coideal). i) $J$ is a two-sided ideal of an algebra $A$, if it is a subspace of $A$ such that $m(J \otimes A) \subset A$ and $m(A \otimes J) \subset A$. The quotient $A/J$ is again an algebra.

ii) $J$ is a two-sided coideal of a coalgebra $C$, if it is a subspace of $C$ such that $\Delta(J) \subset J \otimes A + A \otimes J$ and $\varepsilon(J) = 0$. The quotient $C/J$ is again a coalgebra.

iii) In case $J$ is both a two-sided ideal as well as a coideal of a bialgebra $A$, then the quotient $A/J$ is a bialgebra.

### A.2.3 Hopf algebra

A Hopf algebra is a bialgebra with a particular linear map called antipode. The latter is defined to be the inverse of $id$ for the convolution product. Hence, to define a Hopf algebra we need to introduce the concept of convolution algebra. Let $C$ be a coalgebra and $A$ an algebra. Consider the vector space $\text{Hom}(C, A)$ of linear maps. It becomes a unital algebra by defining the convolution product:

$$f \ast g := m_A \circ (f \otimes g) \circ \Delta_C$$

for $f, g \in \text{Hom}(C, A)$. The unit is given by $u_A \circ \varepsilon_C$. Indeed, using (15), we find for every $f \in \text{Hom}(C, A)$:

$$\sum f(c_1)\varepsilon(c_2) = f(1_C) = \sum \varepsilon(c_1)f(c_2).$$

In particular, we can endow $\text{End}(A)$ with a convolution product when $A$ is a bialgebra.

**Definition A.13** (Hopf Algebra). A **Hopf algebra** is a bialgebra $(\mathcal{H}, m, u, \Delta, \varepsilon)$ with an antipode, i.e., there exists a linear map $S : \mathcal{H} \to \mathcal{H}$ such that

$$S \ast \text{id} = u \circ \varepsilon = \text{id} \ast S.$$  \hspace{1cm} (16)

The following statement tells us that if the antipode exists, it is unique. Moreover, it is an algebra anti-homomorphism.

**Theorem A.14.** Let $(\mathcal{H}, m, u, \Delta, \varepsilon)$ be a bialgebra. If $S$ and $S'$ are both antipodes, then $S = S'$. We have $S(1_{\mathcal{H}}) = 1_{\mathcal{H}}$. Moreover, the antipode is an anti-homomorphism.

$$S \circ m(a \otimes b) = m \circ (S(b) \otimes S(a)).$$

**Proof.** Uniqueness is guaranteed by the fact that inverse elements in a ring are unique. Let $S$ and $S'$ be inverses of $id$ under the convolution product then: $(S - S') \ast id = u \circ \varepsilon - u \circ \varepsilon = 0$. Since $\Delta(1_{\mathcal{H}}) = 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$, $S(1_{\mathcal{H}})1_{\mathcal{H}} = 1_{\mathcal{H}}$ means that $S(1_{\mathcal{H}}) = 1_{\mathcal{H}}$. For the anti-homomorphism property check the proof of [GR20, Proposition 1.4.10].

It is known, as noted in [CP21, p. 43], that given two Hopf Algebras, $\mathcal{H}_1$ and $\mathcal{H}_2$, any bialgebra homomorphism is automatically a Hopf algebra homomorphism.

**Proposition A.15.** Suppose that $(\mathcal{H}_1, m_1, \Delta_1, S_1)$ and $(\mathcal{H}_2, m_2, \Delta_2, S_2)$ are Hopf algebras. For all bialgebra homomorphisms $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$, it holds that

$$\varphi \circ S_1 = S_2 \circ \varphi.$$  \hspace{1cm} (17)
Proof. Notice that both functions $\varphi \circ S_1$ and $S_2 \circ \varphi$ are convolutional inverses of $\phi$. We can first use the multiplicative property of $\varphi$, in order to see it.

\[
m_2 \circ (\varphi \otimes \varphi \circ S_1) \Delta_1 = m_2 \circ (\varphi \otimes \varphi) \circ (\text{id}_1 \otimes S) \circ \Delta_2 \\
= \varphi \circ (\text{id}_1 \otimes S_1) \circ \Delta_1 \\
= \varphi \circ (u_1 \otimes \varepsilon) \\
= u_2 \circ \varepsilon
\]

This is the same function we obtain by using the comultiplicative property.

\[
u_2 \circ \varepsilon_1 = (u_2 \circ \varepsilon_2) \circ \varphi \\
= m_2 \circ (S_2 \otimes \text{id}_2) \circ \Delta_2 \circ \varphi \\
= m_2 \circ (S_2 \otimes \text{id}_2) \circ (\varphi \otimes \varphi) \circ \Delta_1.
\]

The conclusion follows from the uniqueness of the inverse in associative unital algebras.

In addition to the abstract definitions, there are some natural examples of what is a Hopf algebra. Perhaps the most important of them, according to Theorem 3.3, is the polynomial Hopf algebra.

Example A.16. The polynomial Hopf algebra in one generator $x$ is defined in the space

\[
\mathbb{R}[x] := \left( \mathbb{R}\langle 1, x, x^2, x^3, \ldots \rangle, m, \Delta \right)
\]

where product and coproduct are

\[
m(x^m \otimes x^n) := x^{m+n} \text{ resp. } \Delta(x^n) := \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k.
\]

The counit and antipode are given by

\[
\varepsilon(a) = \begin{cases} 
1, & \text{if } a = 1 \\
0, & \text{else}
\end{cases} \text{ resp. } S(x^n) = (-1)^n x^n.
\]

Notice that if $e_a : \mathbb{R}[x] \to \mathbb{R}$ is the linear extension of the evaluation in $a \in \mathbb{R}$: $x^n \mapsto a^n$, then

\[
m \circ (e_a \otimes e_b) \circ \Delta = e_{a+b}.
\]

This is due to the binomial theorem.

A.2.4 Grading, filtration, and connectedness

We give the definition of a graded vector space, from which one can define graded algebras, coalgebras, and bialgebras. We will see that in some cases the weaker notion of filtered algebras and coalgebras can be used when these objects are not graded.

Definition A.17 (Graded vector space). A graded $\mathbf{k}$-vector space $V$ is a vector space that admits a decomposition as direct sum $V := \bigoplus_{n \geq 0} V_n$ of vector spaces $\{V_n\}_{n \geq 0}$. Note that for $V \otimes V$, one considers the induced grading given by $V 
\]

It is natural to consider graded linear maps.

Definition A.18 (Graded maps). Let $V, W$ be graded vector spaces. Then $\phi : V \to W$ is graded if $\phi(V_n) \subset W_n$, for all $n \geq 0$. 
Let $V$ be a graded vector space. In case $V_0 \cong k$ we say that $V$ is connected. This is a technical definition: its meaning will become clearer when introducing filtered connected bialgebras.

**Definition A.19 (Graded algebras and coalgebras).** A graded algebra and coalgebra, $A = \bigoplus_{n \geq 0} A_n$ respectively $C = \bigoplus_{n \geq 0} C_n$, are graded vector spaces such that for all $n, m \geq 0$

$$A_n A_m \subset A_{n+m} \text{ resp. } \Delta(C_n) \subset \bigoplus_{i+j=n} C_i \otimes C_j.$$ 

In other words, products and coproducts are graded maps.

A bialgebra is graded if it is graded as an algebra as well as a coalgebra.

**Definition A.20 (Filtered algebras and coalgebras).** We consider now an algebra $A = \bigcup_{n \geq 0} A_n$ such that $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots$. We say that $A$ is a filtered algebra if $A^n A^m \subset A^{n+m}$.

Similarly, a coalgebra $C = \bigcup_{n \geq 0} C_n$ is filtered if

$$\Delta(C_n) \subset \sum_{i+j=n} C^i \otimes C^j.$$ 

A bialgebra is filtered if it is filtered as an algebra as well as a coalgebra.

Note that a graded bialgebra is always filtered since we can define the filtration in terms of $A^n := \bigoplus_{i=0}^n A_i$.

We can define filtered maps (this is a very weak notion) between filtered vector spaces $V, W$ as linear maps $\phi : V \to W$ such that $\phi(V^n) \subset W^n$, for all $n \geq 0$.

**A.2.5 Connected filtered Hopf algebra**

Here we show that in the particular case where the bialgebra is connected and filtered, the antipode always exists. We start by introducing some notation. Let $A$ be an algebra, for $k \geq 0$ we define inductively the iterated product $m_k : A \otimes (k+1) \to A$

$$m^0 := \text{id}$$

$$m^k := m \circ (\text{id} \otimes m^{k-1}), \text{ if } k \geq 1.$$ 

Similarly, for a coalgebra $C$ we define the iterated coproduct $\Delta^k : C \to C \otimes (k+1)$

$$\Delta^0 := \text{id}$$

$$\Delta^k := (\text{id} \otimes \Delta^{k-1}) \circ \Delta, \text{ if } k \geq 1.$$ 

We can now define for any $f \in \text{Hom}(C, A)$ and $n \geq 1$, the $n$-th convolution power

$$f^{*0} := u_A \circ \varepsilon_C \quad \text{and} \quad f^{*n+1} := m^n (f \otimes \cdots \otimes f) \Delta^n.$$ 

**Lemma A.21.** [Man08, Section 4.2] Let $H$ be a connected filtered bialgebra, then for any $x \in H^n$ with $n \geq 1$, we can write:

$$\Delta(x) = x \otimes 1_H + 1_H \otimes x + \tilde{\Delta}(x),$$ 

where the reduced coproduct

$$\tilde{\Delta}(x) \in \bigoplus_{i+j=n, i \neq 0, j \neq 0} H^i \otimes H^j.$$
In particular, if $|x| = 1$, where $|x| := \min\{n \in \mathbb{N} : x \in \mathcal{H}^n\}$ then
\[
\Delta(x) = x \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes x.
\]
Moreover, $\Delta$ is coassociative on $\ker(\varepsilon)$ and $\Delta_k := (\text{id} \otimes^{k-1} \otimes \Delta) \circ (\text{id} \otimes^{k-2} \otimes \Delta) \circ \cdots \circ \Delta$ sends $\mathcal{H}^n$ to $(\mathcal{H}^{n-k}) \otimes k + 1$.

Note that $1_{\mathcal{H}}$ is group-like, i.e., $\Delta(1_{\mathcal{H}}) = 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$. We will use the following variation to Sweedler’s notation for the reduced coproduct $\tilde{\Delta}$:
\[
\tilde{\Delta}(x) := \sum_{\langle x \rangle} x' \otimes x''.
\]

From the next proposition, it follows that the antipode always exists in the connected filtered case.

**Proposition A.22.** [Man08, Section 4.3] Let $\mathcal{H}$ be a connected filtered bialgebra. Suppose $A$ is an algebra. Consider the convolution algebra $(\text{Hom}(\mathcal{H}, A), *)$ with unit $u_A \circ \varepsilon_\mathcal{H}$. The convolution product defines a group law on the set
\[
G(A) := \{\varphi \in \text{Hom}(\mathcal{H}, A), \varphi(1_{\mathcal{H}}) = 1_A\}.
\]

**Proof.** $G(A)$ is closed under the convolution product. Indeed
\[
\varphi * \varphi'(1_{\mathcal{H}}) = m_A \circ (\varphi \otimes \varphi') \circ \Delta(1_{\mathcal{H}}) = 1_A
\]
We need to show the existence of inverses. For this we consider
\[
\varphi^{*-1} := (u_A \circ \varepsilon_\mathcal{H} - (u_A \circ \varepsilon_\mathcal{H} - \varphi))^{*-1} = \sum_{n \geq 0} (u_A \circ \varepsilon_\mathcal{H} - \varphi)^n.
\]
The series always terminates. Indeed, first consider the elements of $\mathcal{H}^0$. We have $(u_A \circ \varepsilon_\mathcal{H})(1_{\mathcal{H}}) = 1_A$ and $(u_A \circ \varepsilon_\mathcal{H} - \varphi)^n(1_{\mathcal{H}}) = 0$ for $n \geq 1$, since $\Delta^n(1_{\mathcal{H}}) = 1_{\mathcal{H}}^\otimes (n+1)$. Due to the filtered bialgebra being connected, for all $x \in \mathcal{H}^0$, it is true that $\varphi^k(x) = 0$ for $k > 0$.

Now for $n \geq 0$ and $x \in \mathcal{H}^n$, suppose that $(u_A \circ \varepsilon_\mathcal{H} - \varphi)^k(x) = 0$ for $k > n$. If $y \in \mathcal{H}^{n+1}$, then it holds true that
\[
\Delta(y) = y \otimes 1_{\mathcal{H}} + 1_{\mathcal{H}} \otimes y + \sum_{\langle y \rangle} y' \otimes y'',
\]
where $y', y'' \in \mathcal{H}^n$ by Lemma A.21. Using coassociativity, one can observe that for $k > n$,
\[
(u_A \circ \varepsilon_\mathcal{H} - \varphi)^{k+1}(y) = m^k \circ (u_A \circ \varepsilon_\mathcal{H} - \varphi)^{k+1} \Delta^k(y)
\]
\[
= m^k \circ (\varphi^{\otimes k} \otimes \varphi) \left( \Delta^{k-1}(y) \otimes 1_{\mathcal{H}} + \Delta^{k-1}(1_{\mathcal{H}}) \otimes y + \sum_{\langle y \rangle} \Delta^{k-1}(y') \otimes y'' \right)
\]
\[
= \varphi^{*k}(y) \varphi(1_{\mathcal{H}}) + \varphi^{*k}(1_{\mathcal{H}}) \varphi(y) + \sum_{\langle y \rangle} \varphi^{*k}(y') \varphi(y'').
\]
Notice that, by induction hypothesis $y' \in \mathcal{H}^n$ and hence
\[
\varphi(1_{\mathcal{H}}) = \varphi^{*k}(1_{\mathcal{H}}) = \varphi^{*k}(y') = 0.
\]

In the special case in which we consider the convolution algebra on $\text{End}(\mathcal{H})$, recall that the antipode is the convolutional inverse of the identity:
\[
S = \sum_{n \geq 0} (u \circ \varepsilon - \text{id})^n.
\]

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Remark A.23. As pointed out in [Man08], when having a connected filtered Hopf algebra, we can compute the antipode recursively. Recall that $S(1_H) = 1_H$. Let $x \in H^n$, $n \geq 1$ then
\[
m \circ (S \otimes \text{id})\Delta(x) = u \circ \varepsilon(x) = 0
\]
and
\[
0 = m \circ (S \otimes \text{id})\Delta(x) = m \circ (S \otimes \text{id})(1_H \otimes x + x \otimes 1_H + \bar{\Delta}(x)) = x + S(x) + m \circ (S \otimes \text{id})\bar{\Delta}(x) = x + S(x) + \sum_{(x)} S(x')x''.
\]
This yields the recursion
\[
S(x) = -x - \sum_{(x)} S(x')x'', \tag{19}
\]
where in the sum on the righthand side, $S$ is evaluated on strictly smaller elements.

A.2.6 Dual vector spaces, adjoint maps

We introduce notation for the dual of a vector space and for the graded dual. The main reference for this section is [GR20]. The vector spaces we will be working on are of finite type, i.e. they are graded $V = \bigoplus_{n \geq 0} V_n$ and each subspace $V_n$ has finite dimension.

Definition A.24 (Graded dual). Let $V$ be a $k$-vector space. The dual of $V$ is the $k$-vector space $V^* := \text{Hom}(V, k)$. When $V = \bigoplus_{n \geq 0} V_n$ is a graded vector space, the graded dual is the subspace $V^o := \bigoplus_{n \geq 0} V^*_n \subset V^*$.

Note that $V^* = \prod_{n \geq 0} V^*_n$ is a larger space than $V^o$. Functions that are non-zero on infinitely many $V_n$ are also part of $V^*$. For instance, let $\zeta \in V^*$ be the functional sending each basis element of $V$ to 1. Then, $\zeta \in V^*$, but $\zeta \notin V^o$.

For a vector space $V$, we denote the bilinear pairing as follows:
\[
\langle \cdot, \cdot \rangle : V^* \otimes V \to k,\quad f \otimes v \mapsto f(v).
\]
Consider now a $k$-linear map $\phi : V \to W$. It induces an adjoint map $\phi_{\text{adj}} : W^* \to V^*$ defined as follows:
\[
\langle g, \phi(v) \rangle_W = \langle \phi_{\text{adj}}(g), v \rangle_V.
\]
In case $V, W$ are finite-dimensional, then $\text{Hom}(V, W)$ and $\text{Hom}(W^*, V^*)$ are isomorphic as vector spaces via the adjoint map. This connection is carried on by the matrix transpose on this finite case. However, there is not an equivalent notion of matrix transpose for the infinite-dimensional case [Sim97].

Fortunately, it is known [GR20; Haz+10; Car07] that this notion is well defined for the graded dual of finite type vector spaces and homomorphisms respecting that grading.

For a graded coalgebra $(C, \Delta_C)$, the dual product on $C^o \otimes C^o$ is uniquely defined as follows.
\[
\langle mc^o \otimes c^o(a \otimes b), c \rangle_C := \langle a \otimes b, \Delta_C(c) \rangle_{C^o \otimes C^o}.
\tag{20}
\]
The linear transformation $m_{C^o \otimes C^o}$ is then the adjoint map of $\Delta_C$ accordingly restricted to the graded dual. Note that this is only a subalgebra of the convolution algebra on $C^o$.

Moreover, for a graded algebra $(A, m_A)$ on a vector space of finite type, one can uniquely define its dual coproduct.
\[
\langle \Delta_A^e(c), a \otimes b \rangle_A := \langle c, m_A(a \otimes b) \rangle_{A^o \otimes A}.
\tag{21}
\]
These definitions allow us to dualize immediately graded bialgebras. In the following, we will consider bialgebras that are not graded and give sufficient conditions so that (20) and (21) can still be applied.
Lemma A.25. Suppose that \((C, \Delta)\) is a coalgebra where \(C = \bigoplus_{n \geq 0} C_n\) is of finite type. If
\[
\Delta(C_n) \subset \bigoplus_{i+j \geq n} C_i \otimes C_j,
\]
then its dual product can be defined as in Eq. (20). Assume that \((A, m_A)\) is an algebra where \(A = \bigoplus_n A_n\) is of finite type. If
\[
m(A_n \otimes A_k) \subset \bigoplus_{\ell \geq \max(n,k)} A_{\ell},
\]
then we can define its dual coalgebra \(\Delta_A^{-}(c)\) as in (21).

Proof. Fix \(a \otimes b \in C_i \otimes C_j\). Then \(\langle a \otimes b, \Delta(c) \rangle_{C \otimes C}\) can be non-zero only for finitely many basis elements of \(C\). Indeed, for \(c \in C_\ell\) where \(\ell > i + j\) it holds that \(\Delta(c) \in \bigoplus_{x+y \geq \ell} C_x \otimes C_y\). Since \(i+j<\ell\) and \(x+y \geq \ell\),
\[
\langle a \otimes b, \Delta(c) \rangle_{C \otimes C} = 0.
\]
Fix \(c \in A_i^n\). \(\langle c, m(a \otimes b) \rangle_{A \otimes A}\) can be non-zero only for a finite number of basis elements of \(A \otimes A\). Indeed:
\[
a \otimes b \in \bigoplus_{i > \ell \text{ or } j > \ell} A_i \otimes A_j \implies \langle c, m(a \otimes b) \rangle_{A \otimes A} = 0.
\]

A.2.7 Characters

Assume that \(\mathcal{H}\) is a Hopf Algebra. The characters form a group.

Proposition A.26. If \((A, m_A, u_A)\) is a commutative unital associative algebra. If \(f, g : (\mathcal{H}, m_{\mathcal{H}}) \to (A, m_A)\) are unital algebra homomorphisms, then \(f \circ (g \circ S)\) is also a unital algebra morphism.

Proof. Recall that \(h \circ m_{\mathcal{H}} = m_A \circ (h \otimes h)\) if \(h\) is an algebra morphisms.
\[
m_A \circ (f \otimes g) \circ \Delta_{\mathcal{H}} \circ m_{\mathcal{H}} = m_A \circ (f \otimes g) \circ (m_{\mathcal{H}} \otimes m_{\mathcal{H}}) \circ \tau_{(2,3)} \circ (\Delta_{\mathcal{H}} \otimes \Delta_{\mathcal{H}})
\]
\[
= m_A \circ (m_A \otimes m_A) \circ (f \otimes f \otimes g \otimes g) \circ \tau_{(2,3)} \circ (\Delta_{\mathcal{H}} \otimes \Delta_{\mathcal{H}})
\]
\[
= m_A \circ (m_A \otimes m_A) \circ \tau_{(2,3)} \circ (f \otimes g \otimes f \otimes g) \circ (\Delta_{\mathcal{H}} \otimes \Delta_{\mathcal{H}}).
\]
Here, \(\sigma_{(2,3)} := \text{id} \otimes \tau \otimes \text{id}\). Thanks to commutativity of \(A\), we have that
\[
m_A \circ (m_A \otimes m_A) \circ \tau_{(2,3)} = m_A \circ (m_A \otimes m_A).
\]
Therefore,
\[
m_A \circ (f \otimes g) \circ \Delta_{\mathcal{H}} \circ m_{\mathcal{H}} = m_A \circ (m_A \circ (f \otimes g) \circ \Delta_{\mathcal{H}}) \circ (m_A \circ (f \otimes g) \circ \Delta_{\mathcal{H}}).
\]
This shows that the set of algebra morphisms is closed under the convolution product. Now, regarding inverses, we see that
\[
m_A \circ (g \circ (g \circ S)) \circ \Delta_{\mathcal{H}} = m_A \circ (g \circ g) \circ (\text{id} \otimes S) \circ \Delta_{\mathcal{H}}
\]
\[
= g \circ m_{\mathcal{H}} \circ (\text{id} \otimes S) \circ \Delta_{\mathcal{H}}
\]
\[
= g \circ u_{\mathcal{H}} \circ \varepsilon_{\mathcal{H}}
\]
\[
= u_A \circ \varepsilon_{\mathcal{H}}.
\]

\[\square\]
Here we prove that all the coproducts are coassociative and cocommutative and that the respective products are associative and commutative.

**Theorem A.27.** Coproducts in Section 2.4, Section 2.5, and Section 2.6 are counital, coassociative and cocommutative. Dually, the corresponding products are unital, associative and commutative.

**Proof.** It is immediate to see that the coproducts are counital and the products are unital. Cocommutativity is also immediate.

We only show the coassociativity of the quasi-shuffle and the shuffle.

For the quasi-shuffle we need to show that,

\[(\Delta \circ \text{id}) \circ \Delta (g) = (\text{id} \circ \Delta) \circ \Delta (g)\]

The left-hand side equals

\[\sum_{(A,B) \in 2^E(g) \times 2^E(g): A \cup B = E(g)} g|_A \otimes g|_B\]

and also the RHS, yields:

\[\sum_{(A,B) \in 2^E(g) \times 2^E(g): A \cup B = E(g)} g|_A \otimes g|_B = (1)\]

We show that they coincide by showing that they both (1) and (2) are equal to:

\[\sum_{(A,B,C) \in 2^E(g) \times 2^E(g) \times 2^E(g): A \cup B \cup C = E(g)} g|_A \otimes g|_B \otimes g|_C = (3)\]

We now show that (3) and (1) are equal.

Let \(\Phi : \{(A,B,C) \in 2^E(g) \times 2^E(g) \times 2^E(g): A \cup B \cup C = E(g)\} \to 2^E(g) \times 2^E(g)\) where

\[\Phi((A,B,C)) = (A \cup B, C)\]

and we claim that

\[\text{Im}(\Phi) = \{ (A,B) \in 2^E(g) \times 2^E(g): A \cup B = E(g) \}\]
We can then rewrite (3):

\[
\sum_{(A,B,C) \in 2^{E(g)} \times 2^{E(g)} \times 2^{E(g)}: A \cup B \cup C = E(g)} g|A \otimes g|B \otimes g|C
\]

\[
\sum_{(A,B) \in \text{Im}(\Phi)} \sum_{(L,M,N) \in \Phi^{-1}((A,B))} g|L \otimes g|M \otimes g|N
\]

\[
\sum_{(A,B) \in 2^{E(g)} \times 2^{E(g)}: A \cup B = E(g)} \sum_{L,M \in 2^{A}: L \cup M = A} g|L \otimes g|M \otimes g|B = (1)
\]

and similarly one can show that (2) and (3) are equal. Moreover plugging \( \cup \) instead of \( \cup \) shows the coassociativity of the shuffle. Associativity and commutativity of the products follows dually from the statements on coproducts. \( \square \)

### A.4 Bialgebras

We now prove which algebras and coalgebras are compatible in the sense of bialgebra.

**Lemma A.28 (Bialgebra property).** Considering all possible product-coproduct combinations:

The following are bialgebras on \( \mathbb{R}\langle G \rangle \):

i. \((\cup, \Delta_{\cup})\), \((\uplus, \Delta_{\cup})\)

ii. \((\cup, \Delta_{\cup})\), \((\uplus, \Delta_{\cup})\)

iii. \((\cup, \Delta_{DP})\), \((\cdot_{DP}, \Delta_{\cup})\)

The following are bialgebras on \( \mathbb{R}\langle \mathcal{G} \rangle \):

iv. \((\cup, \Delta_{\cup})\), \((\uplus, \Delta_{\cup})\)

v. \((\cup, \Delta_{\cup})\), \((\uplus, \Delta_{\cup})\)

vi. \((\cup, \Delta_{DP})\), \((\cdot_{DP}, \Delta_{\cup})\)

The following are not bialgebras on \( \mathbb{R}\langle G \rangle \):

vii. \((\cup, \Delta_{\cup})\), \((\cdot_{DP}, \Delta_{DP})\), \((\uplus, \Delta_{\cup})\), \((\uplus, \Delta_{\cup})\)

viii. \((\uplus, \Delta_{DP})\), \((\cdot_{DP}, \Delta_{\cup})\)

ix. \((\uplus, \Delta_{DP})\), \((\cdot_{DP}, \Delta_{\cup})\)

The following are not bialgebras on \( \mathbb{R}\langle \mathcal{G} \rangle \):

x. \((\cup, \Delta_{\cup})\), \((\cdot_{DP}, \Delta_{DP})\), \((\uplus, \Delta_{\cup})\), \((\uplus, \Delta_{\cup})\)

xi. \((\uplus, \Delta_{DP})\), \((\cdot_{DP}, \Delta_{\cup})\)

xii. \((\uplus, \Delta_{DP})\), \((\cdot_{DP}, \Delta_{\cup})\)
### Table 3: Compatibility in $R(G)$ and $R(\tilde{G})$

| Bialgebra | $\sqcup$ | $\sqcup_{DP}$ | $\sqcap_{\mathcal{U}}$ | $\sqcap_{\mathcal{W}}$ | $\sqcap_{\mathcal{V}}$ |
|-----------|---------|-------------|-------------|-------------|-------------|
| $\Delta_{\sqcup}$ | no | yes | yes | yes | yes |
| $\Delta_{\sqcup_{DP}}$ | yes | no | no | no | no |
| $\Delta_{\sqcap_{\mathcal{U}}}$ | yes | no | no | no | no |
| $\Delta_{\sqcap_{\mathcal{W}}}$ | yes | no | no | no | no |
| $\Delta_{\sqcap_{\mathcal{V}}}$ | yes | no | no | no | no |

**Proof.**

i. This is proven in [Sch94, Section 12]. It is also possible to give a proof analogous to the one in the edge disjoint union co-shuffle, changing $E(-)$ to $V(-)$ when appropriate.

ii. same argument as in v., but using induced subgraphs.

iii. immediate, since it is just a relabeling of the polynomial bialgebra on connected graphs.

iv.

$$
\Delta_{\sqcap_{\mathcal{W}}} (\sigma \sqcup \tau) = \sum_{A \cup B = E(\sigma \cup \tau)} (\sigma \sqcup \tau) |_{A} \otimes (\sigma \sqcup \tau) |_{B}
$$

$$
= \sum_{A \cup B = E(\sigma) \cup E(\tau)} (\sigma |_{A \cap E(\sigma)} \sqcup \tau |_{A \cap E(\tau)}) \otimes (\sigma |_{B \cap E(\sigma)} \sqcup \tau |_{B \cap E(\tau)})
$$

$$
= \sum_{A_1 \cup B_1 = E(\sigma)} (\sigma |_{A_1} \sqcup \tau |_{A_2}) \otimes (\sigma |_{B_1} \sqcup \tau |_{B_2})
$$

$$
= \Delta_{\sqcap_{\mathcal{W}}} (\sigma) \sqcup \Delta_{\sqcap_{\mathcal{W}}} (\tau),
$$

v.

$$
\Delta_{\sqcap_{\mathcal{W}}} (\sigma \sqcup \tau) = \sum_{A \cup B = E(\sigma \cup \tau)} (\sigma \sqcup \tau) |_{A} \otimes (\sigma \sqcup \tau) |_{B}
$$

$$
= \sum_{A \cup B = E(\sigma) \cup E(\tau)} (\sigma \sqcup \tau) |_{A} \otimes (\sigma \sqcup \tau) |_{B}
$$

$$
= \sum_{A \cup B = E(\sigma) \cup E(\tau)} \left( (\sigma |_{A \cap E(\sigma)}) \sqcup (\tau |_{A \cap E(\tau)}) \right) \otimes \left( (\sigma |_{B \cap E(\sigma)}) \sqcup (\tau |_{B \cap E(\tau)}) \right)
$$

$$
= \sum_{A \cup B = E(\sigma) \cup E(\tau)} \left( (\sigma |_{A \cap E(\sigma)}) \otimes (\sigma |_{B \cap E(\sigma)}) \right) \sqcup \left( (\tau |_{A \cap E(\tau)}) \otimes (\tau |_{B \cap E(\tau)}) \right)
$$

$$
= \sum_{A_1 \cup B_1 = E(\sigma)} (\sigma |_{A_1} \sqcup \sigma |_{B_1}) \sqcup (\tau |_{A_2} \sqcup \tau |_{B_2})
$$

$$
= \Delta_{\sqcap_{\mathcal{W}}} (\sigma) \sqcup \Delta_{\sqcap_{\mathcal{W}}} (\tau),
$$

The bialgebra property on $(R(\tilde{G}), \sqcup, \Delta_{\sqcap_{\mathcal{W}}})$ follows dually to the previous point.

vi. as for $R(G)$, it is a polynomial bialgebra.
vii. \((\sqcup, \Delta_{\sqcup})\) is \textit{not} a bialgebra.

\[
\Delta_{\sqcup}(\mathbf{1} \sqcup \mathbf{1}) = e \otimes \mathbf{1} \sqcup \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \sqcup \mathbf{1} \otimes e,
\]

but

\[
\Delta_{\sqcup}(\mathbf{1} \sqcup \Delta_{\sqcup}(\mathbf{1})) = e \otimes \mathbf{1} \sqcup 2 \mathbf{1} \sqcup \mathbf{1} \sqcup \mathbf{1} \otimes e.
\]

\((\star_{\text{DP}}, \Delta_{\text{DP}})\) is \textit{not} a bialgebra. Indeed

\[
\Delta_{\text{DP}}(\star_{\text{DP}} \cdot) = 2e \otimes \cdot + 4 \cdot \otimes \cdot + 2 \cdot \otimes e
\]
and

\[
\Delta_{\text{DP}}(\cdot) \cdot_{\text{DP}} \Delta_{\text{DP}}(\cdot) = 2e \otimes \cdot + 2 \cdot \otimes + 2 \cdot \otimes e.
\]

\((\sqcup_{i}, \Delta_{\sqcup_{i}})\) is \textit{not} a bialgebra. Indeed

\[
\Delta_{\sqcup_{i}}(\cdot, \sqcup_{i} \cdot) = 2e \otimes \cdot + 2e \otimes + 8 \cdot \otimes \cdot + 2 \cdot \otimes e + 2 \cdot \otimes e
\]
and

\[
\Delta_{\sqcup_{i}}(\cdot) \sqcup_{i} \Delta_{\sqcup_{i}}(\cdot) = 2e \otimes \cdot + 2e \otimes + 2 \cdot \otimes \cdot + 2 \cdot \otimes e + 2 \cdot \otimes e
\]

\((\sqcup_{i}, \Delta_{\sqcup_{i}})\) is \textit{not} a bialgebra. There is a simple counterexample as for \((\sqcup_{qs}, \Delta_{\sqcup_{qs}})\) in \(x\).

viii. simple counterexample similar to xi.

ix. simple counterexample similar to xii.

x. \((\sqcup, \Delta_{\sqcup})\) is \textit{not} a bialgebra. There is a simple counterexample as for \((\sqcup_{i}, \Delta_{\sqcup_{i}})\) in vii.

\((\sqcup_{i}, \Delta_{\sqcup_{i}})\) is \textit{not} a bialgebra. Indeed, first we compute: \(\Delta_{\sqcup_{i}}(\mathbf{1}) \sqcup_{i} \Delta_{\sqcup_{i}}(\mathbf{1})\). We have that:

\[
\Delta_{\sqcup_{i}}(\mathbf{1}) = \mathbf{1} \otimes e + e \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1}
\]
and

\[
\Delta_{\sqcup_{i}}(\mathbf{1} \sqcup_{i} \mathbf{1}) = (\mathbf{1} \otimes e + e \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1}) \sqcup_{i} (\mathbf{1} \otimes e + e \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1})
\]

\[
= \mathbf{1} \sqcup_{i} \mathbf{1} \otimes e + e \otimes \mathbf{1} \sqcup_{i} \mathbf{1} \otimes e + \mathbf{1} \otimes \mathbf{1} + 2 \mathbf{1} \otimes \mathbf{1}
\]

The basis element \(\mathbf{1} \otimes \mathbf{1}\) only appears in the last summand. And expanding the quasi-shuffle we see that its coefficient is 4.

We compute

\[
\Delta_{\sqcup_{i}}(\mathbf{1} \sqcup_{i} \mathbf{1}) = \Delta_{\sqcup_{i}}(\mathbf{1} + 2 \mathbf{1} \otimes \mathbf{1})
\]

and note that

\[
\Delta_{\sqcup_{i}}(\mathbf{1} \mathbf{1}) = \mathbf{1} \mathbf{1} \otimes e + e \otimes \mathbf{1} \mathbf{1} + 2 \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + 2 \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1}
\]

Therefore, in \(\Delta_{\sqcup_{i}}(\mathbf{1} \sqcup_{i} \mathbf{1})\), the basis element \(\mathbf{1} \otimes \mathbf{1}\) has coefficient 2. Therefore the two expressions differ.
\[ \Delta_{DP}(1 \sqcup 1) = \Delta_{DP}(2 \mathbf{1} \sqcup 2 \mathbf{1}) \]
\[ = 2(\mathbf{1} \otimes \mathbf{1} + 2 \mathbf{1} \otimes \mathbf{1} + 1 \otimes \mathbf{1}) + 2(\mathbf{1} \otimes 1 + 1 \otimes \mathbf{1}) \]
\[ = (2 \mathbf{1} \otimes \mathbf{1} + 2 \mathbf{1}) \otimes 1 + 4 \mathbf{1} \otimes \mathbf{1} + 1 \otimes (2 \mathbf{1} \otimes 2 \mathbf{1}) \]
\[ = \Delta_{DP}(1) \sqcup \Delta_{DP}(1) + 2 \mathbf{1} \otimes \mathbf{1}. \]

xii.
\[ \Delta_{DP}(1 \sqcup 1) = \Delta_{DP}(1) \sqcup \Delta_{DP}(1) + 2 \mathbf{1} \otimes \mathbf{1}. \]

\[ \square \]

A.5 Categorical background

The category of simple graphs consists of simple graphs \( G = (V(G), E(G)) \) as objects with graph homomorphisms as morphisms, i.e. \( f : G \to G' \) is a morphism if \( \{i, j\} \in E(G) \) implies \( \{f(i), f(j)\} \in E(G') \).

We recall basic notions from category theory. A morphism \( f \in \text{mor}(G, G') \) is

- a **monomorphism** if for all objects \( G'' \) and all \( g, h \in \text{mor}(G'', G) \)
  \[ f \circ g = f \circ h \Rightarrow g = h. \]

- a **regular monomorphism** if it is the equalizer of some parallel pair of morphisms, i.e. if there is a limit diagram of the form
  \[ G \xrightarrow{f} G' \xrightarrow{g} F. \]

  It is well-known that a regular monomorphism is a monomorphism.

- an **epimorphism**, if for all objects \( G'' \) and for all \( g, h \in \text{mor}(G', G'') \)
  \[ g \circ f = h \circ f \Rightarrow g = h. \]

- a **regular epimorphism** if it is the coequalizer of some parallel pair of morphisms, i.e. if there is a colimit diagram of the form
  \[ F \xrightarrow{g} G \xrightarrow{f} G'. \]

  It is well-known that a regular epimorphism is an epimorphism.

- an **isomorphism** if it has a two sided inverse: there is \( f^{-1} \in \text{mor}(G', G) \) with \( f^{-1} \circ f = \text{id}_G, f \circ f^{-1} = \text{id}_{G'} \).

**Proposition A.29.** In any category the following implications hold

i. **isomorphism** \( \Rightarrow \) **regular monomorphism** + **regular epimorphism**

ii. **monomorphism** + **regular epimorphism** \( \Rightarrow \) **isomorphism**

iii. **regular monomorphism** + **epimorphism** \( \Rightarrow \) **isomorphism**.

**Proof.**

i. Immediate.

ii. [Bor94, Corollary 2.1.5]
Proposition A.32. \( \text{In the category of simple graphs } f \in \text{mor}(G,G') \text{ is} \)

i. a monomorphism if and only if the function \( f : V(G) \to V(G') \) is injective (injective on vertices\(^3\)).

ii. a regular monomorphism if and only if the function \( f : V(G) \to V(G') \) is injective and
\[
\{i,j\} \in E(G) \Leftrightarrow \{f(i),f(j)\} \in E(G'),
\]
i.e. if \( G \cong G'_{\text{im } f} \).

iii. an epimorphism if and only if the function \( f : V(G) \to V(G') \) is surjective (surjective on vertices).

iv. a regular epimorphism if and only if the function \( f : V(G) \to V(G') \) is surjective and the induced function \( f : E(G) \to E(G') \) is surjective (surjective on vertices and edges).

Remark A.31. Recall that \( f : X \to Y \) in some category \( \mathcal{C} \) is a split monomorphism if there exists a morphism \( f_L \in \text{mor}(G',G) \) such that
\[
f_L \circ f = \text{id}_G.
\]
It is not clear to us whether there exists a useful alternative characterization of split monomorphisms in the category of simple graphs.

A split monomorphism is automatically a regular monomorphism, but the converse is not true in general. For example, in the category of simple graphs, the map taking an edge to one of the edges of a triangle is a regular monomorphism but not a split monomorphism.

Proof. For high-level proofs of these statements, see [nla]. We show ii. “by hand”. Assume that \( f : G \to H \) is injective such that moreover \( f : G \to H_{\text{im } f} \) is an isomorphism. Define the graph \( K \) as
\[
V(K) := \text{im } f \cup (V(H) \setminus \text{im } f) \cup (V(H) \setminus \text{im } f) \times \{\ast\}
\]
\[
E(K) := \{\{x,y\} \mid \{x,y\} \in H \text{ or } x = (x',\ast), y = (y',\ast) \in H \}
\]
\[
\text{or } x \in H, y = (y',\ast) \text{ with } \{x,y\} \in H
\}.
\]
Define \( g : H \to K \) as \( g(h) = h, \forall h \in H \) and
\[
g'(h) := \begin{cases} h & h \in \text{im } f \\ (h,\ast) & h \in V(H) \setminus \text{im } f \end{cases}.
\]
Then \( f \) is easily verified to be the equalizer of \( g,g' \).

Assume that \( f \) is the equalizer of some \( g,g' : H \to K \). Denote the injection \( \iota : H_{\text{im } f} \to H \). Then, since \( f \) is an equalizer
\[
g(f(h)) = g'(f(h)) \quad \forall h \in V(H).
\]
\[
g(\iota(h)) = g'(\iota(h)) \quad \forall h \in V(H_{\text{im } f}).
\]
Hence there exists a unique \( \phi : H_{\text{im } f} \to G \) such that \( \iota = f \circ \phi \). It is easily verified that \( \phi \) is an isomorphism. \( \square \)

Proposition A.32.

i. Every morphism \( f \in \text{mor}(G,G') \) can be factorized as \( f = b \circ a \) with \( a : G \to F \) a regular epimorphism and \( b : F \to G' \) a monomorphism. The factorization is unique in the following sense: if \( f = b' \circ a' \) is another such factorization with \( a' : G \to H, b' : H \to G' \) then there exists a unique isomorphism \( \psi : F \to H \) with \( a' = \psi \circ a, b' = b \circ \psi^{-1} \).

\(^3\).. and then automatically on edges ..
ii. Every morphism \( f \in \text{mor}(G, G') \) can be factorized as \( f = b \circ a \) with \( a : G \to F \) an epimorphism and \( b : F \to G' \) is a regular monomorphism, with analogously defined uniqueness.

Corollary A.33. In the category of finite simple graphs

i. 
\[
\left| \{ \phi \in \text{Mon}(\tau, \Lambda) \mid \Lambda_{\text{im}\phi} \cong \sigma \} \right| = \frac{1}{|\text{Aut}(\sigma)|} \left| \text{Mon}(\tau, \sigma) \cap \text{Epi}(\tau, \sigma) \right| \left| \text{RegMon}(\sigma, \Lambda) \right|,
\]

ii. 
\[
\left| \{ \phi \in \text{Epi}(\tau, \rho) \mid \rho \vert_{\phi\left(\text{E}(\tau)\right)} \cong \sigma \} \right| = \frac{1}{|\text{Aut}(\sigma)|} \left| \text{RegEpi}(\tau, \sigma) \right| \left| \text{Mon}(\sigma, \rho) \cap \text{Epi}(\sigma, \rho) \right|
\]

iii. 
\[
\left| \{ \phi \in \text{Hom}(\tau, \Lambda) \mid \Lambda_{\text{im}\phi} \cong \sigma \} \right| = \frac{1}{|\text{Aut}(\sigma)|} \left| \text{Epi}(\tau, \sigma) \right| \left| \text{RegMon}(\sigma, \Lambda) \right|
\]

iv. 
\[
\left| \{ \phi \in \text{Hom}(\tau, \Lambda) \mid \Lambda_{\phi(\text{E}(\tau))} \cong \sigma \} \right| = \frac{1}{|\text{Aut}(\sigma)|} \left| \text{RegEpi}(\tau, \sigma) \right| \left| \text{Mon}(\sigma, \Lambda) \right|
\]