Orbital stability of two-component peakons

Cheng He\textsuperscript{1}, Xiaochnuan Liu\textsuperscript{2} & Changzheng Qu\textsuperscript{1,*}

\textsuperscript{1}School of Mathematics and Statistics, Ningbo University, Ningbo 315211, China; \textsuperscript{2}School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China

Email: 1811071003@nbu.edu.cn, liuxiaochuan@mail.xjtu.edu.cn, quchangzheng@nbu.edu.cn

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Abstract We prove that the two-component peakon solutions are orbitally stable in the energy space. The system concerned here is a two-component Novikov system, which is an integrable multi-component extension of the integrable Novikov equation. We improve the method for the scalar peakons to the two-component case with genuine nonlinear interactions by establishing optimal inequalities for the conserved quantities involving the coupled structures. Moreover, we also establish the orbital stability for the train-profiles of these two-component peakons by using the refined analysis based on monotonicity of the local energy and an induction method.

Keywords Novikov equation, two-component Novikov system, peakons, orbital stability, conservation law, Camassa-Holm equation

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1 Introduction

In this paper, we are devoted to the orbital stability of the two-component peakon solutions and the corresponding configuration of train-profiles. The system we are concerned with is the following integrable two-component Novikov system [32]:

\begin{equation}
\begin{aligned}
& m_t + uvm_x + (2vu_x + uw_x)m = 0, \quad m = u - u_{xx}, \\
& n_t + uvn_x + (2uv_x + v^2)n = 0, \quad n = v - v_{xx}.
\end{aligned}
\end{equation}

Note that this system reduces to the well-studied integrable Novikov equation [28,45]

\begin{equation}
m_t + u^2m_x + 3uu_xm = 0, \quad m = u - u_{xx},
\end{equation}

when \( v = u \).

Since the celebrated work [4] by Camassa and Holm, in which they first discovered the non-smooth peaked soliton solutions (called peakons) to the Camassa-Holm (CH) equation, the existence of peakons

*Corresponding author
and multi-peakons is one of the significant properties of the integrable CH-type equations. The basic wave profile of a peakon takes a quite compact form

\[ \varphi_c(x - ct) = a(c)e^{-|x - ct|}, \]  

(1.3)

where \( c \in \mathbb{R} \) is the wave speed and the amplitude \( a(c) \) is a function related to \( c \). Remarkably, the seemingly simple form of the peakon displays the deep relationship with some important phenomena of wave propagation in shallow water waves. Indeed, due to the discussion in [12,13,51], the feature of the peakons that their profile (1.3) is smooth, except at the crest where it is continuous but the lateral tangents differ, is similar to that of the so-called Stokes waves of the greatest height, i.e., traveling waves of the largest possible amplitude which are solutions to the governing equations for irrotational water waves. There is no closed form available for these waves, and the peakons can capture these described essential features. It is well understood that the CH equation

\[ m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx} \]

and the Degasperis-Procesi (DP) equation

\[ m_t + um_x + 3u_x m = 0, \quad m = u - u_{xx} \]

both arise as the appropriate asymptotic approximations of the Euler equations for the free-surface shallow water waves in the moderately nonlinear regime [15], and admit the following peakon solutions (\( c > 0 \), and the anti-peakon for \( c < 0 \)) on the line [4–6,18,31,38]:

\[ u(t,x) = \varphi_c(x - ct) = ce^{-|x - ct|}, \quad c \neq 0. \]  

(1.4)

On the other hand, the peakons and the corresponding multi-peakons admit a rich mathematical structure related to the underlying integrable features. Indeed, both the CH and DP equations are integrable equations with Lax-pair formulations, and the inverse scattering approach can be used to derive explicitly these peakons (1.4) and the related multi-peakon solutions [1,19,39]. In the past ten years, two typical CH-type integrable equations with cubic nonlinearity that support peakon dynamics attracted much attention. One is the Novikov equation (1.2), whose peakon solutions take the form of [28]

\[ u(t,x) = \varphi_c(x - ct) = \sqrt{c}e^{-|x - ct|}, \quad c > 0. \]

Another one is the modified Camassa-Holm (mCH) equation [46]

\[ m_t + ((u^2 - u_x^2)m)_x = 0, \quad m = u - u_{xx}, \]

which has the following peakon structure [25]:

\[ u(t,x) = \varphi_c(x - ct) = \sqrt{\frac{3c}{2}}e^{-|x - ct|}, \quad c > 0. \]

Although the physical background of the Novikov and mCH equations is not so clear, their peakon and multi-peakon dynamics are demonstrated to have several non-trivial properties in the framework of Lax integrability (see the discussion in [8,27]).

In this paper, special concern for these peakon solutions is the issue of their stability, which lies in the fact that they are the explicit weak solutions in the sense of distribution to the corresponding equations. The peakon equations exhibit different features in contrast to the classical integrable equations such as the KdV equation, the modified KdV equation and the Schrödinger equation that admit the smooth solitons, especially in the study of qualitative properties related to stability and instability. There are a huge number of papers to study the stability and instability of solitons for classical integrable systems. We do not attempt to exhaust all the literature, and one can refer to [2,3,7,23,24,47] for the stability issue and [42,43,48] for the issue of asymptotic stability, as well as the references therein. Note that
the peakons do not admit the classical second-order derivatives and the linearized operators at peakons appear to be degenerate. So the classical methods based on the spectral analysis are not available in the case of peakons. In an intriguing paper due to Constantin and Strauss [17], they proved the orbital stability of peakons for the CH equation by discovering several precise optimal inequalities relating to the maximum value of the approximate solutions and the conserved quantity of the quadratic form and higher-order conserved quantities (see also [16] for a variational argument and [30] for the periodic case).

This approach in [17] was further developed by El Dika and Molinet [20] to study orbital stability of the train of peakons to the CH equation. Orbital stability of peakons and the train of peakons of other CH-type integrable equations were investigated in [29, 34–37, 50]. More recently, the issues of instability of peakons for the CH equation and the Novikov equation were addressed in [44] and [11], respectively.

To study of orbital stability of multi-component peakons for integrable multi-component CH-type equations. Recently, another kind of two-component integrable generalization (1.1) of the Novikov equation (1.2) was introduced in [32] and we find that this system admits the two-component peakon structure, which is given by

\[
\begin{align*}
    m_t + um_x + 2u_x m + \rho P_x &= 0, \quad m = u - u_{xx}, \\
    \rho_t + (\rho u)_x &= 0.
\end{align*}
\]  

However, the system (1.5) does not admit the peaked solitons [26]. The orbital stability of a kind of reduced peakon profile was studied via the variational method [10]. The Novikov equation (1.2) admits the following two-component integrable extension, called the Geng-Xue system [22];

\[
\begin{align*}
    m_t + uv m_x + 3uv_x m &= 0, \quad m = u - u_{xx}, \\
    n_t + uv n_x + 3uv_x n &= 0, \quad n = v - v_{xx},
\end{align*}
\]  

which has been paid much attention recently [33, 40, 41]. Although the system (1.6) supports the multi-peakon structures, they are derived in the framework of the Lax-pair formulation of (1.6) (see [40]), which are not weak solutions in the sense of distribution. To the best of our knowledge, there is no work on the study of orbital stability of multi-component peakons for integrable multi-component CH-type equations.

Recently, another kind of two-component integrable generalization (1.1) of the Novikov equation (1.2) was introduced in [32] and we find that this system admits the two-component peakon structure, which is given by

\[
(u(t, x), v(t, x)) = (\varphi_c(x - ct), \psi_c(x - ct)) = (ae^{-|x-ct|}, be^{-|x-ct|}),
\]  

(1.7)

traveling at a constant speed \(c = ab \neq 0\). It is demonstrated that these peakons (1.7) are indeed the weak solutions of (1.1) in the distribution form

\[
\begin{align*}
    u_t + uv u_x + P_x * \left(\frac{1}{2} u^2 x + uv_x u_x + u^2 v_x\right) + \frac{1}{2} P * (u^2 v_x) &= 0, \\
    v_t + uv v_x + P_x * \left(\frac{1}{2} v^2 u + uv_x u_x + v^2 u_x\right) + \frac{1}{2} P * (v^2 u_x) &= 0,
\end{align*}
\]  

(1.8)

where \(P(x) = e^{-|x|}/2\) and \(\ast\) stands for convolution with respect to the spatial variable \(x \in \mathbb{R}\). Here, a question arises: are these multi-component peakons and the corresponding train-profiles for the system (1.1) stable in the energy space?

System (1.1) adopts the following conserved densities:

\[
\begin{align*}
    E_0[u,v] &= \int_{\mathbb{R}} (mn)^2 dx, \\
    E_u[u] &= \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad E_v[v] = \int_{\mathbb{R}} (v^2 + v_x^2) dx, \quad H[u,v] = \int_{\mathbb{R}} (uv + u_x v_x) dx
\end{align*}
\]
and

$$F[u, v] = \int_{\mathbb{R}} \left( u^2 v^2 + \frac{1}{3} u^2 v_x^2 + \frac{1}{3} v^2 u_x^2 + \frac{4}{3} w u_x v_x - \frac{1}{3} u_x^2 v_x^2 \right) dx,$$

which will play a prominent role in proving stability of peakons, while the corresponding three conserved quantities of the Novikov equation (1.2) are

$$H_0[u] = \int_{\mathbb{R}} m \frac{\dot{m}}{2} dx, \quad \mathcal{E}[u] = \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad F[u] = \int_{\mathbb{R}} (u^4 + 2u^2u_x^2 - \frac{1}{3} u_x^4) dx.$$

If we choose $u = v$, then we have $H_0 = E_0[u, u]$, $\mathcal{E}[u] = E_0[u] = E_v[v] = H[u, v]$ and $F[u, v] = F[u]$. Due to the existence of the $H^1$ conservation law respectively for the $u$- and $v$-components as well as the mutual interaction conservation law $H[u, v]$, it is expected to prove stability for the two-component Novikov system in the sense of the energy space of $H^1 \times H^1$-norm. In general, a small perturbation of a solitary wave can yield another one with a different speed and a different phase shift. It is appropriate to define the orbit of traveling-wave solutions $(\varphi_c, \psi_c)$ to be the set $U(\varphi, \psi) = \{(a\varphi(\cdot + x_1), b\psi(\cdot + x_2)), x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$. However, if $x_1 \neq x_2$, the functionals $F[\varphi_c, \psi_c]$ and $H[\varphi_c, \psi_c]$ are not conserved in the time evolution for $(\varphi_c, \psi_c)$ in this set $U(\varphi, \psi)$. Thus, we consider here a suitable orbit of the traveling-wave solutions $\varphi_c$ and $\psi_c$ to be the set $U_0(\varphi, \psi) = \{(a\varphi(\cdot + x_0), b\psi(\cdot + x_0)), x_0 \in \mathbb{R}\}$ and the peakon solutions of the two-component Novikov equation are called orbitally stable if a wave standing close to the peakon remains close to the orbit $U_0(\varphi, \psi)$ at all the later existence time.

The first main result is stated as follows. Here, we only consider the case of peakons traveling to the right, i.e., the case of $a > 0, b > 0$ and then $c = ab > 0$ in (1.7).

**Theorem 1.1.** Let $(\varphi_c, \psi_c)$ be the peaked solutions in (1.7), traveling with a speed $c = ab > 0$. Then $(\varphi_c, \psi_c)$ are orbitally stable in the following sense. Assume that $u_0, v_0 \in H^s(\mathbb{R})$ for some $s \geq 3$, $(1 - \partial_x^2)u_0(x)$ and $(1 - \partial_x^2)v_0(x)$ are nonnegative, and there is a $\delta > 0$ such that

$$\|(u_0, v_0) - (\varphi_c, \psi_c)\|_{H^1(\mathbb{R}) \times H^1(\mathbb{R})} \leq \|u_0 - \varphi_c\|_{H^1(\mathbb{R})} + \|v_0 - \psi_c\|_{H^1(\mathbb{R})} < \delta.$$

Then the corresponding solution $(u(t, x), v(t, x))$ of the Cauchy problem for the two-component Novikov system (1.1) with the initial data $u(0, x) = u_0(x)$ and $v(0, x) = v_0(x)$ satisfies

$$\sup_{t \in [0, T]} \|(u(t, \cdot), v(t, \cdot)) - (\varphi_c(\cdot - \xi(t)), \psi_c(\cdot - \xi(t)))\|_{H^1(\mathbb{R}) \times H^1(\mathbb{R})} \leq \sup_{t \in [0, T]} \|(u(t, \cdot) - \varphi_c(\cdot - \xi(t)), v(t, \cdot) - \psi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} = A\delta,$$

where $T > 0$ is the maximal existence time, $\xi(t) \in \mathbb{R}$ is the maximal point of the function $u(t, x)v(t, x)$, and the constant $A$ depends only on $a$ and $b$ as well as the norms $\|u_0\|_{H^1(\mathbb{R})}$ and $\|v_0\|_{H^1(\mathbb{R})}$.

To prove orbital stability of the two-component peakons $(\varphi_c, \psi_c)$, some new insights are developed. We aim to obtain for the components $u$ and $v$ the dynamical estimates $|u(t, \xi(t)) - a|$ and $|v(t, \xi(t)) - b|$ along some trajectory $t \mapsto \xi(t)$, where $a$ and $b$ are the maximal values of the components $\varphi_c$ and $\psi_c$, respectively. Here, due to the nonlinear interaction between $u$ and $v$ involved in the system (1.1), the key obstacle is how to find the suitable location of $\xi(t)$ in order to derive the precise estimates for $|u(t, \xi(t)) - a|$ and $|v(t, \xi(t)) - b|$ (note that in the case of scalar peakons such a $\xi(t)$ is always chosen to locate at the maximal point of the perturbed solution, which is no longer valid in the multi-component case considered here and $\xi(t)$ must change according to the appearance of the characteristic speed of the nonlinear interaction $uv$). Moreover, the dynamical energy identities and energy inequalities should involve the nonlinear interaction of the two components. In addition, the conservation law $F[u, v]$ is much more complicated than $F[u]$ of the Novikov equation. Therefore, the stability issue of the two-component peakon solutions is subtler. To overcome the difficulties, two observations will be crucial. System (1.1) has not only the separated $H^1$ conserved quantities $\int (u^2 + u_x^2) dx$ and $\int (v^2 + v_x^2) dx$ with which one can derive the pointwise estimates separately for the components $u$ and $v$, but also the second-order interacting conserved quantity $H[u, v] = \int (uv + u_x v_x) dx$ with which we are motivated to find the exact
location of $\xi(t)$ as the maximal point of the multiplication function $u(t,\cdot)v(t,\cdot)$ of two components. This argument is quite different from the case for the scalar CH-type equations. On the other hand, new optimal energy identities for $H[u,v]$ and $F[u,v]$ are established. The new insight is that the precise control of two components is involved in one optimal energy identity. This point is also different from the scalar case. Based on these observations together with corresponding refined analysis, we are able to prove orbital stability of the two-component peakons $(\varphi,\psi)$ on the line $\mathbb{R}$.

**Remark 1.2.** For the Geng-Xue system (1.6), even though the peakon solutions in the Lax-pair sense are considered, the system (1.6) does not admit sufficient conserved quantities to establish the corresponding estimates for the stability.

For the issue of orbital stability of train-profiles of these two-component peakons, we have the following result.

**Theorem 1.3.** Given $N$ velocities $c_1, c_2, \ldots, c_N$ such that $0 < a_1 < a_2 < \cdots < a_N$, $0 < b_1 < b_2 < \cdots < b_N$ and $c_i = a_ib_i$ for any $i \in \{1, \ldots, N\}$, there exist $A > 0$, $L_0 > 0$ and $\epsilon_0 > 0$ such that if the initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for some $s \geq 3$ with $(1 - \partial_x^2)u_0(x)$ and $(1 - \partial_x^2)v_0(x)$ being nonnegative satisfy

$$\left\|u_0 - \sum_{i=1}^{N}\varphi_i(\cdot - z_i^0)\right\|_{H^1} + \left\|v_0 - \sum_{i=1}^{N}\psi_i(\cdot - z_i^0)\right\|_{H^1} \leq \epsilon$$

for some $0 < \epsilon < \epsilon_0$ and $z_i^0 - z_{i-1}^0 \geq L$ with $L > L_0$, then there exist $x_1(t), \ldots, x_N(t)$ such that the corresponding strong solution $(u(t,x), v(t,x))$ satisfies

$$\left\|u(t,\cdot) - \sum_{i=1}^{N}\varphi_i(\cdot - x_i(t))\right\|_{H^1} + \left\|v(t,\cdot) - \sum_{i=1}^{N}\psi_i(\cdot - x_i(t))\right\|_{H^1} \leq A(\epsilon^\frac{1}{4} + L^{-\frac{1}{4}}),$$

$\forall t \in [0,T)$, where $x_j(t) - x_{j-1}(t) > L/2$.

In general, two main ingredients in the proof of orbital stability for the train-profiles of peakons are involved [20,37,43]. One is orbital stability of the single peakons, and the another one is the property of almost monotonicity of the local energy on the right-hand side of the peakons. For the two-component peakons, more difficulties come from the interaction of the two components $u(t,x)$ and $v(t,x)$. The first one is how to establish inequalities among the localized conserved quantities to verify the orbital stability of the two-component peakons separately. For the train-profiles of peakons, we need to apply $N$ inequalities to control $2N$ estimates. The second one is to establish the monotonicity result of the functionals $J_{j,k}^{u,v}(t)$ since we cannot identify the sign for the term $u_{\cdot}v_{\cdot\cdot}$ in the conserved density $H[u,v]$. To overcome the difficulties, we use the conserved densities $E_u[u]$, $E_v[v]$, $H[u,v]$ and $F[u,v]$, and establish the delicate inequalities relating to the conserved densities and the maximal value of the two components $u(t,x)$ and $v(t,x)$. In the case of train-profiles of two-component peakons, we apply the proof of the single peakons, the modulation theory, the accurate estimates on the conserved densities $E_u[u]$, $E_v[v]$, $H[u,v]$ and $F[u,v]$ and the induction method to obtain the desired result.

The rest of this paper is organized as follows. In Section 2, we provide a brief discussion on the integrability, conservation laws, the sign invariant property of $m(t,x)$ and $n(t,x)$ of the system (1.1), and the local well-posedness result on the Cauchy problem of the system (1.1). In Section 3, we prove orbital stability of single peakons on the line. Finally in Section 4, we verify orbital stability of the train of peakons.

## 2 Preliminaries

In the present section, the issue of well-posedness is discussed. First of all, we call that $(u,v) \in C([0,T);H^1(\mathbb{R})) \times C([0,T);H^1(\mathbb{R}))$ is a solution of the system (1.1), if $(u,v)$ is a solution of (1.8) in the sense of distribution and $E_u[u]$, $E_v[v]$, $H[u,v]$ and $F[u,v]$ are conserved quantities.
Consider the following Cauchy problem of the system (1.1) on the whole line $\mathbb{R}$:

$$
\begin{aligned}
&\left\{\begin{array}{l}
m_t + um_x + (2vu_x + uv_x)m = 0, \\
n_t + vn_x + (2wu_x + vu_x)n = 0,
\end{array}\right. \\
&u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \mathbb{R}.
\end{aligned}
$$

(2.1)

First, similar to the results given in [21, 49], one can establish the following local well-posedness result.

**Theorem 2.1.** Given $z_0 = (u_0, v_0)^T \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ ($s > 3/2$), there exist a maximal $T = T(\|z_0\|_{H^s(\mathbb{R}) \times H^s(\mathbb{R})}) > 0$ and a unique strong solution $z = (u, v)^T$ to (2.1) such that

$$
z = z(\cdot, z_0) \in C([0, T]; H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R})).
$$

In addition, the solution depends continuously on the initial data, i.e., the mapping

$$
z_0 \mapsto z(\cdot, z_0) : H^s(\mathbb{R}) \times H^s(\mathbb{R}) \to C([0, T]; H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-1}(\mathbb{R}))
$$

is continuous. Furthermore, the quantities $E_u[u], E_v[v], H[u, v]$ and $F[u, v]$ are all conserved along the solution $z = (u, v)^T$.

Consider the flow governed by $(uv)(t, x)$, i.e.,

$$
\begin{aligned}
&\frac{d}{dt} q(t, x) = (uv)(t, q(t, x)), \quad x \in \mathbb{R}, \quad t \in [0, T), \\
&q(0, x) = x, \quad x \in \mathbb{R}.
\end{aligned}
$$

(2.2)

Just as in the case for the Novikov equation (1.2), the following lemma can be proved.

**Lemma 2.2.** Assume that $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > 3/2$, and let $T > 0$ be the maximal existence time of the corresponding strong solution $(u, v)$ to the Cauchy problem of the system (2.1). Then the problem (2.2) has a unique solution $q(t, x) \in C^1([0, T) \times \mathbb{R})$. Furthermore, the map $q(t, \cdot)$ is an increasing diffeomorphism over $\mathbb{R}$ with

$$
q_x(t, x) = \exp\left(\int_0^t (uv)_x(s, q(s, x))ds\right), \quad (t, x) \in [0, T) \times \mathbb{R}.
$$

(2.3)

**Lemma 2.3.** Let $u_0, v_0 \in H^s(\mathbb{R})$ and $s \geq 3$. Assume that $m_0 = u_0 - u_{0xx}$ and $n_0 = v_0 - v_{0xx}$ are nonnegative on the line. Then for the corresponding strong solution $(u, v) \in C([0, T]; H^s(\mathbb{R}) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ of the Cauchy problem of the two-component Novikov system (2.1) with the initial data $u_0$ and $v_0$, we see that for all $t \in [0, T)$, $m(t, x)$ and $n(t, x)$ are both nonnegative functions. In addition, $u(t, \cdot) \geq 0, v(t, \cdot) \geq 0$ and $[u_x(t, \cdot)] \leq u(t, \cdot), [v_x(t, \cdot)] \leq v(t, \cdot)$ on the line.

**Proof.** It follows from the Cauchy problem (2.1) of the two-component Novikov system that along the flow (2.2), $m$ and $n$ satisfy

$$
\begin{aligned}
&m' + (2vu_x + uv_x)(t, q(t, x))m(t, q(t, x)) = 0, \\
n' + (2wu_x + vu_x)(t, q(t, x))n(t, q(t, x)) = 0,
\end{aligned}
$$

(2.4)

where $'$ denotes the derivative with respect to $t$ along the flow (2.2). Define

$$
\gamma_1(t, x) = \exp\left(\int_0^t (2vu_x + uv_x)(s, q(s, x))ds\right), \quad \gamma_2(t, x) = \exp\left(\int_0^t (2wu_x + vu_x)(s, q(s, x))ds\right).
$$

Then they satisfy

$$
\gamma_1'(t, x) = (2vu_x + uv_x)(t, q(t, x))\gamma_1, \quad \gamma_2'(t, x) = (2wu_x + vu_x)(t, q(t, x))\gamma_2.
$$

Let

$$
\tilde{m}(t, x) = \gamma_1(t, x)m(t, q(t, x)), \quad \tilde{n}(t, x) = \gamma_2(t, x)n(t, q(t, x)).
$$
(2.4) becomes
\[ \tilde{m}'(t, x) = 0, \quad \tilde{n}'(t, x) = 0. \]  
(2.5)
The equations
\[ \tilde{m}(t, x) = m_0 \quad \text{and} \quad \tilde{n}(t, x) = n_0 \]
lead to
\[ m(t, q(t, x)) = \exp\left( -\int_0^t (2u(t, q(t, x)) + \int_0^\infty e^{-y} m(y)dy \right) m_0, \]
\[ n(t, q(t, x)) = \exp\left( -\int_0^t (2u(t, q(t, x)) + \int_0^\infty e^{-y} m(y)dy \right) n_0. \]
Thus, for all \( t \in [0, T] \), we have \( m(t, \cdot) \geq 0 \) and \( n(t, \cdot) \geq 0 \). Then \( u(t, \cdot) \geq 0 \) and \( v(t, \cdot) \geq 0 \).
Formally regarding \( m(x) = u(x) - u_{xx}(x) \), we see that for all \( x \in \mathbb{R} \),
\[ u(x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{y} m(y)dy + \frac{e^{x}}{2} \int_{x}^{\infty} e^{-y} m(y)dy \]
and
\[ u_x(x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{y} m(y)dy + \frac{e^{x}}{2} \int_{x}^{\infty} e^{-y} m(y)dy. \]
Then we infer that
\[ u(x) \geq |u_x(x)|, \quad \forall x \in \mathbb{R}. \]
Similarly, we find
\[ v(x) \geq |v_x(x)|, \quad \forall x \in \mathbb{R}. \]
This completes the proof of the lemma. \( \square \)

3 Stability of two-component peakons
In this section, we prove Theorem 1.1, which will be based on a series of lemmas. Note that the assumptions on the initial profile guarantee the existence of a unique positive solution for the Cauchy problem (2.1) of the two-component Novikov system. In general, for \( a > 0 \) and \( b > 0 \), the profile functions of peakon solutions \( \varphi_c(x) = ae^{-|x|} \) and \( \psi_c(x) = be^{-|x|} \) are in \( H^1(\mathbb{R}) \), which have peaks at \( x = 0 \), and thus
\[ \max_{x \in \mathbb{R}} \varphi_c(x) = \varphi_c(0) = a \quad \text{and} \quad \max_{x \in \mathbb{R}} \psi_c(x) = \psi_c(0) = b. \]
A direct calculation gives
\[ E_u[\varphi_c(x)] = \| \varphi_c \|_{H^1}^2 = 2a^2, \quad E_v[\psi_c(x)] = \| \psi_c \|_{H^1}^2 = 2b^2. \]
and
\[ H[\varphi_c(x), \psi_c(x)] = 2ab, \quad F[\varphi_c(x), \psi_c(x)] = \frac{4}{3}a^2 b^2. \]
Due to the conservation of the \( H^1 \)-norm of the components \( u \) and \( v \), the following pointwise identities still hold for the two-component Novikov system as in the scalar CH and Novikov cases.

**Lemma 3.1.** For any \( u, v \in H^1(\mathbb{R}) \) and \( \xi \in \mathbb{R} \), we have
\[ E_u[u] - E_u[\varphi_c] = \| u - \varphi_c \|_{H^1(\mathbb{R})}^2 + 4a(u(\xi) - a), \]
\[ E_v[v] - E_v[\psi_c] = \| v - \psi_c \|_{H^1(\mathbb{R})}^2 + 4b(v(\xi) - b). \]
(3.1)
Lemma 3.2. Let the invariants with the global maximal value of the multiplication function \( u(x)v(x) \) be defined. Consider \( 0 \neq u, v \in H^s(\mathbb{R}), s \geq 3 \) and \( u, v \geq 0 \). Then \( u, v \in C^2 \) due to the Sobolev imbedding theory. Since \( u \) and \( v \) decay at infinity, it must have a point \( \xi \in \mathbb{R} \) where
\[
M = \max_{x \in \mathbb{R}} \{ u(x)v(x) \} = u(\xi)v(\xi). \tag{3.2}
\]

**Proof.** To show (3.5), we evaluate the integral of \( u(x)v(x) \) on \( \mathbb{R} \). Thus,
\[
\int_{\mathbb{R}} g_1(x)g_2(x)dx = \int_{-\infty}^{\xi} (u(x) - u_x(x))(v(x) - v_x(x))dx + \int_{\xi}^{\infty} (u(x) + u_x(x))(v(x) + v_x(x))dx
\]
\[
= \int_{-\infty}^{\xi} (uv - (uv)_x + u_xv_x)dx + \int_{\xi}^{\infty} (uv + (uv)_x + u_xv_x)dx
\]
\[
= H[u, v] - 2M. \tag{3.5}
\]

**Proof.** To show (3.5), we evaluate the integral of \( g_1(x)g_2(x) \) on \( \mathbb{R} \). Thus,
\[
\int_{\mathbb{R}} g_1(x)g_2(x)dx
\]
\[
= \int_{-\infty}^{\xi} (u(x) - u_x(x))(v(x) - v_x(x))dx + \int_{\xi}^{\infty} (u(x) + u_x(x))(v(x) + v_x(x))dx
\]
\[
= \int_{-\infty}^{\xi} (uv - (uv)_x + u_xv_x)dx + \int_{\xi}^{\infty} (uv + (uv)_x + u_xv_x)dx
\]
\[
= H[u, v] - 2M. \tag{3.5}
\]

This completes the proof. \( \square \)

The construction of the auxiliary function \( h(x) \) in the following lemma is crucial in the proof of stability of peakons, which is different from the scalar cases of CH and DP equations. This new defined function is a nontrivial refinement of the case in the Novikov equation.

**Lemma 3.3.** With the same assumptions and notations as in Lemma 3.2, define the function \( h(x) \) by
\[
h(x) = \begin{cases} 
  uv - \frac{1}{3}(uv)_x - \frac{1}{3}u_xv_x, & x < \xi, \\
  uv + \frac{1}{3}(uv)_x - \frac{1}{3}u_xv_x, & x > \xi. 
\end{cases} \tag{3.6}
\]

Then
\[
\int_{\mathbb{R}} h(x)g_1(x)g_2(x)dx = F[u, v] - \frac{4}{3}M^2. \tag{3.7}
\]

**Proof.** To show (3.7), we evaluate the integral of \( h(x)g_1(x)g_2(x) \) on \( \mathbb{R} \). Thus,
\[
\int_{\mathbb{R}} h(x)g_1(x)g_2(x)dx = \int_{-\infty}^{\xi} \left[ uv - \frac{1}{3}(uv)_x - \frac{1}{3}u_xv_x \right] [uv - (uv)_x + u_xv_x]dx
\]
\[
+ \int_{\xi}^{\infty} \left[ uv + \frac{1}{3}(uv)_x - \frac{1}{3}u_xv_x \right] [uv + (uv)_x + u_xv_x]dx
\]
\[
\triangleq I + II.
\]
Proof. First, by the sign-invariant property, the solution $(u, v)$ is a positive solution and fulfills all the conditions assumed in Lemmas 3.2 and 3.3. Hence, we have the

$$M = \max_{t,x} |u|.$$  

In fact, due to the definition of $M$, it follows that

$$\Pi = \int_{\xi}^{\infty} \left( u^2 v^2 + \frac{1}{3} u^2 v_x^2 + \frac{1}{3} v^2 u_x^2 + \frac{4}{3} u v u_x v_x - \frac{1}{3} u_x^2 v_x^2 \right) dx = - \frac{2}{3} M^2.$$  

Combining the above, we have

$$\int_{\xi}^{\infty} h(x) g_1(x) g_2(x) dx = \int_{\infty}^{\infty} \left( u^2 v^2 + \frac{1}{3} u^2 v_x^2 + \frac{1}{3} v^2 u_x^2 + \frac{4}{3} u v u_x v_x - \frac{1}{3} u_x^2 v_x^2 \right) dx - \frac{4}{3} M^2 = F[u, v] - \frac{4}{3} M^2.$$  

This completes the proof.

With the two energy identities (3.5) and (3.7) in hand, one can derive the following delicate relation between the second-order conserved quantity $H[u, v]$ and the higher-order conserved quantity $F[u, v]$ for the strong solution $(u, v)$.

**Lemma 3.4.** Assume that $0 \not\equiv u_0, v_0 \in H^s$, $s \geq 3$ and $m_0 = u_0 - u_0 x \geq 0$, $n_0 = v_0 - v_0 x \geq 0$. For the corresponding strong solution $(u(t, x), v(t, x))$ with initial data $(u_0, v_0)$ in the lifespan $[0, T)$, it holds that

$$F[u, v] - \frac{4}{3} M(t) H[u, v] - \frac{4}{3} M(t)^2 \leq 0, \quad \forall t \in [0, T),$$  

where $M(t) = \max_{x \in \mathbb{R}} \{u(t, x)v(t, x)\} = u(t, \xi(t))v(t, \xi(t))$ for some trajectory $(\xi(t)) \in \mathbb{R}$ in $[0, T)$.

**Proof.** First, by the sign-invariant property, the solution $(u(t, x), v(t, x))$ satisfies $m(t, x) = u(t, x) - \partial_x^2 u(t, x) \geq 0$ and $n(t, x) = v(t, x) - \partial_x^2 v(t, x) \geq 0$ for all $(t, x) \in [0, T) \times \mathbb{R}$. It follows that $(u(t, x), v(t, x))$ is a positive solution and fulfills all the conditions assumed in Lemmas 3.2 and 3.3. Hence, we have the following energy identities of the dynamical forms in $[0, T)$:

$$\int_{\mathbb{R}} g_1(t, x) g_2(t, x) dx = H[u, v] - 2M(t)$$  

and

$$\int_{\mathbb{R}} h(t, x) g_1(t, x) g_2(t, x) dx = F[u, v] - \frac{4}{3} M(t)^2.$$  

We now claim that for any $(t, x) \in [0, T) \times \mathbb{R}$,

$$h(t, x) = \begin{cases} \left( u v - \frac{1}{3} (u v)_x - \frac{1}{3} u_x v_x \right)(t, x), & x < \xi(t), \\ \left( u v + \frac{1}{3} (u v)_x + \frac{1}{3} u_x v_x \right)(t, x), & x > \xi(t) \end{cases} \leq \frac{4}{3} u(t, x)v(t, x).$$  

In fact, due to the definition of $h(t, x)$ and from the fact that $u(t, x) \geq |u_x(t, x)|$ and $v(t, x) \geq |v_x(t, x)|$, it follows that

$$h(t, x) = \frac{4}{3} u(t, x)v(t, x) - \frac{1}{3} (u \pm u_x(t, x))(v \pm v_x(t, x)) \leq \frac{4}{3} u(t, x)v(t, x).$$  


Thus, the combination of the above inequalities yields
\[ h(t, x) \leq \frac{4}{3} u(t, x) v(t, x) \leq \frac{4}{3} \max_{x \in \mathbb{R}} \{ u(t, x) v(t, x) \} = \frac{4}{3} M(t), \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \]

Now, using the estimates \(|u_x| \leq u\) and \(|v_x| \leq v\) again in the expressions (3.3) and (3.4), we obtain (3.8) from (3.9) and (3.10).

In the following lemma, we study the perturbation of the conserved quantities around the profile functions \(\varphi_c(x)\) and \(\psi_c(x)\) of peakon solutions, in the case of time independence.

**Lemma 3.5.** For \(u, v \in H^s(\mathbb{R})\), \(s \geq 3\), if \(||u - \varphi_c||_{H^1(\mathbb{R})} < \delta\) and \(||v - \psi_c||_{H^1(\mathbb{R})} < \delta\) with \(0 < \delta < 1/2\), then
\[ |E_u[u] - E_u[\varphi_c]| < 2\sqrt{2}a\delta + \delta^2, \quad |E_v[v] - E_v[\psi_c]| < 2\sqrt{2}b\delta + \delta^2 \]
and
\[ |H[u, v] - H[\varphi_c, \psi_c]| < 2\sqrt{2}(a+b)\delta + 6\delta^2, \quad |F[u, v] - F[\varphi_c, \psi_c]| < C\delta + O(\delta^2), \]
where \(C > 0\) is a constant depending on \(a, b\), \(||u||_{H^s}\) and \(||v||_{H^s}\).

**Proof.** Let \(\tilde{u} = u - \varphi_c\) and \(\tilde{v} = v - \psi_c\) for convenience. Since \(||\tilde{u}||_{H^1(\mathbb{R})} < \delta\) and \(||\tilde{v}||_{H^1(\mathbb{R})} < \delta\), it follows that
\[ |E_u[u] - E_u[\varphi_c]| = ||u_0||_{H^1(\mathbb{R})}^2 - ||\varphi_c||_{H^1(\mathbb{R})}^2 \]
\[ \leq ||u_0||_{H^1(\mathbb{R})}^2 - ||\varphi_c||_{H^1(\mathbb{R})}^2||u_0||_{H^1(\mathbb{R})} + ||\varphi_c||_{H^1(\mathbb{R})} \]
\[ \leq 2\sqrt{2}a\delta + \delta^2. \]

Similarly, we have \(|E_v[v] - E_v[\psi_c]| < 2\sqrt{2}b\delta + \delta^2\).

Next, we estimate
\[ |H[u, v] - H[\varphi_c, \psi_c]| = \int \left( uw + u_x v_x \right) dx - \int \left( \varphi_c \psi_c + \varphi_c' \psi_c' \right) dx \]
\[ = \int \left( \tilde{u} \tilde{v} + \tilde{u} \tilde{v} + u \tilde{v} \right) dx + \int \left( u_x (v_x - \psi_c') + u_x (v_x - \psi_c') + v_x (u_x - \varphi_c') \right) dx \]
\[ \leq \int |\tilde{u} |dx + \int |\tilde{v} |dx + \int \left| (u - \varphi_c) u \right| dx \]
\[ + \int |\tilde{u} |dx + \int |\tilde{v} |dx + \int |\tilde{v}' | dx. \]

Using Hölder’s inequality,
\[ \int |\tilde{u} |dx \leq \left( \int \tilde{u}^2 dx \right)^{\frac{1}{2}} \left( \int v^2 dx \right)^{\frac{1}{2}} \leq ||\tilde{u}||_{H^1(\mathbb{R})} (||\tilde{v}||_{H^1(\mathbb{R})} + ||\varphi_c||_{H^1(\mathbb{R})}) \leq \delta^2 + \sqrt{2}a\delta, \]
we deduce that \(|H[u, v] - H[\varphi_c, \psi_c]| < 2\sqrt{2}(a+b)\delta + 6\delta^2. \)

Finally, we estimate
\[ |F[u, v] - F[\varphi_c, \psi_c]| \]
\[ = \int \left( u^2 v^2 + \frac{1}{3} u^2 v^2 + \frac{1}{3} v^2 u^2 + \frac{4}{3} u^2 v^2 - \frac{1}{3} u^2 v^2 \right) dx \]
\[ + \int \left[ \varphi_c^2 \psi_c' + \frac{1}{3} \varphi_c^2 \psi_c' \right] dx \]
\[ \leq \int \left( u^2 v^2 - \varphi_c^2 \psi_c' \right) dx + \frac{1}{3} \int \left( u^2 v^2 - \psi_c^2 \psi_c' \right) dx \]
\[ + \frac{4}{3} \int \left( u^2 v^2 - \varphi_c \psi_c \psi_c' \right) dx \]
\[ \triangleq I_1 + \frac{1}{3} I_2 + \frac{1}{3} I_3 + \frac{4}{3} I_4 + \frac{1}{3} I_5. \]
For the first term $I_1$, we obtain
\[
I_1 \leq \int_{\mathbb{R}} |(u^2 - \varphi_c^2)v^2| \, dx + \int_{\mathbb{R}} |(v^2 - \psi_c^2)| \varphi_c^2 \, dx \\
\leq \|v\|_{L^\infty}^2 \left( \int_{\mathbb{R}} \tilde{u}^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} (u + \varphi_c)^2 \, dx \right)^{\frac{1}{2}} + \|\varphi_c\|_{L^\infty}^2 \left( \int_{\mathbb{R}} \tilde{v}^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} (v + \psi_c)^2 \, dx \right)^{\frac{1}{2}} \\
\leq \|v\|_{L^\infty}^2 \|\tilde{u}\|_{H^1} \left( \|\tilde{u}\|_{H^1} + 2\|\varphi_c\|_{H^1} \right) + \|\varphi_c\|_{L^\infty}^2 \|\tilde{v}\|_{H^1} \left( \|\tilde{v}\|_{H^1} + 2\|\psi_c\|_{H^1} \right) \\
\leq \frac{1}{2} (\delta + \sqrt{2b})^2 \delta (\delta + 2\sqrt{2a}) + b^2 \delta (\delta + 2\sqrt{2b}) \leq 2\sqrt{2b}^2 (a + b) \delta + O(\delta^2).
\]
For the second term $I_2$, we have
\[
I_2 \leq \int_{\mathbb{R}} |(u^2 - \varphi_c^2)v^2| \, dx + \int_{\mathbb{R}} |(v^2 - \psi_c^2)| \varphi_c^2 \, dx \\
\leq \|v_x\|_{L^\infty}^2 \left( \int_{\mathbb{R}} (u_x + \varphi_x)^2 \, dx \right)^{\frac{1}{2}} + \|\varphi_x\|_{L^\infty}^2 \left( \int_{\mathbb{R}} (v_x + \psi_x)^2 \, dx \right)^{\frac{1}{2}} \\
\leq \|v_x\|_{L^\infty}^2 \|u_x\|_{H^1} \left( \|u_x\|_{H^1} + 2\|\varphi_x\|_{H^1} \right) + \|\varphi_x\|_{L^\infty}^2 \|v_x\|_{H^1} \left( \|v_x\|_{H^1} + 2\|\psi_x\|_{H^1} \right) \\
\leq \frac{1}{2} (\delta + \sqrt{2b})^2 \delta (\delta + 2\sqrt{2a}) + b^2 \delta (\delta + 2\sqrt{2b}) \leq 2\sqrt{2b}^2 (a + b) \delta + O(\delta^2).
\]
Similarly, for the term $I_3$, we get $I_3 < 2\sqrt{2} a^2 (a + b) \delta + O(\delta^2)$. For the fourth term $I_4$, we estimate
\[
I_4 \leq \int_{\mathbb{R}} |\tilde{u} u_x v_x| \, dx + \int_{\mathbb{R}} |\varphi_c \tilde{u} u_x v_x| \, dx + \int_{\mathbb{R}} |\varphi_c \psi \tilde{u} u_x v_x| \, dx + \int_{\mathbb{R}} |\varphi_c \psi \varphi \tilde{u} u_x v_x| \, dx \\
\leq \frac{1}{2} \|\tilde{u}\|_{L^\infty}^2 \left( \int_{\mathbb{R}} u_x^2 + v_x^2 \, dx \right) + \frac{1}{2} \|\tilde{v}\|_{L^\infty}^2 \|u\|_{L^\infty} \left( \int_{\mathbb{R}} u_x^2 + v_x^2 \, dx \right) \\
+ \|\varphi_c\|_{L^\infty} \|\psi\|_{L^\infty} \left( \int_{\mathbb{R}} \tilde{u}^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \tilde{v}^2 \, dx \right)^{\frac{1}{2}} + \|\varphi_c\|_{L^\infty} \|\psi\|_{L^\infty} \left( \int_{\mathbb{R}} \tilde{u}^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} \tilde{v}^2 \, dx \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} \|\tilde{u}\|_{H^1(\mathbb{R})} \|v\|_{H^1(\mathbb{R})} + \|\tilde{v}\|_{H^1(\mathbb{R})} \|u\|_{H^1(\mathbb{R})} + \|\varphi_c\|_{H^1(\mathbb{R})} \|\psi\|_{H^1(\mathbb{R})} (\|\tilde{u}\|_{H^1(\mathbb{R})}^2 + \|\tilde{v}\|_{H^1(\mathbb{R})}^2) \\
+ \frac{1}{2} \|\varphi_c\|_{H^1(\mathbb{R})} \|\psi\|_{H^1(\mathbb{R})} (\|\tilde{u}\|_{H^1(\mathbb{R})} + \|\tilde{v}\|_{H^1(\mathbb{R})})^2 + \|u\|_{H^1(\mathbb{R})}^2 \\
\leq \sqrt{2}(a + b) (a^2 + b^2 + ab) \delta + O(\delta^2).
\]
For the fifth term $I_5$, we have
\[
I_5 \leq \left| \int_{\mathbb{R}} u_x^2 (v_x^2 - (\psi_c')^2) \, dx \right| + \left| \int_{\mathbb{R}} (\psi_c')^2 (u_x^2 - (\varphi_c')^2) \, dx \right| \\
\leq \|u_x\|_{L^\infty}^2 \left( \int_{\mathbb{R}} v_x^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} (v_x + \psi_c')^2 \, dx \right)^{\frac{1}{2}} + \|\psi_c'\|_{L^\infty}^2 \left( \int_{\mathbb{R}} u_x^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} (u_x + \varphi_c')^2 \, dx \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} \|u_x\|_{H^1(\mathbb{R})}^2 \|v_x\|_{H^1(\mathbb{R})} (\|\tilde{v}\|_{H^1(\mathbb{R})} + 2\|\psi_c\|_{H^1(\mathbb{R})} + \|\psi_c'\|_{H^1(\mathbb{R})}) \|\tilde{u}\|_{H^1(\mathbb{R})} (\|\tilde{u}\|_{H^1(\mathbb{R})} + 2\|\phi_c\|_{H^1(\mathbb{R})}) \\
\leq 2\sqrt{2} ab (a + b) \delta + O(\delta^2).
\]
Accordingly, for $0 < \delta < 1/2$, we have $|F(u, v) - F(\varphi_c, \psi_c)| \leq C \delta + O(\delta^2)$, where $C$ is a constant depending on $a, b, \|u\|_{H^s},$ and $\|v\|_{H^s}$, which completes the proof of this lemma. \hfill \Box

Now, we are in a position to prove that the strong solution satisfies the novel error estimates at some kind of critical point under the assumption of the small perturbation of initial data around the profile of peakon solutions.

**Lemma 3.6.** Assume that $(u(t, x), v(t, x)) \ (t \in [0, T])$ is the corresponding strong solution of the Cauchy problem (2.1) with initial data $(u_0(x), v_0(x))$ satisfying $0 \neq u_0, v_0 \in H^s(\mathbb{R})$, $s \geq 3$ and $m_0 = u_0 - u_{0xx} \geq 0$, $n_0 = v_0 - v_{0xx} \geq 0$. If $(u_0(x), v_0(x))$ satisfies
\[
\|u_0 - \varphi_c\|_{H^s(\mathbb{R})} < \delta \quad \text{and} \quad \|v_0 - \psi_c\|_{H^s(\mathbb{R})} < \delta
\]
with $0 < \delta < 1/2$, then there exists a constant $C > 0$ such that

$$|u(t, \xi(t)) - a| < C\delta^{1/2} \quad \text{and} \quad |v(t, \xi(t)) - b| < C\delta^{1/2}, \quad \forall t \in [0, T),$$

where $\xi(t) \in \mathbb{R}$ is located such that $u(t, \xi(t))v(t, \xi(t)) = \max_{x \in \mathbb{R}} \{u(t, x)v(t, x)\} = M(t)$.

**Proof.** In view of the conservation of the functionals $H[u, v]$ and $F[u, v]$, as well as the invariant property established in Lemma 2.3 of the strong solution (lemma), it follows that for any $t \in [0, T)$,

$$|H[u(t, \cdot), v(t, \cdot)] - H[\varphi_c, \psi_c]| < 2\sqrt{2}(a + b)\delta + 6\delta^2 \quad (3.11)$$

and

$$|F[u(t, \cdot), v(t, \cdot)] - F[\varphi_c, \psi_c]| < C\delta + O(\delta^2), \quad (3.12)$$

where the constants involved depend on $a, b, \|u_0\|_{H^s}$ and $\|v_0\|_{H^s}$.

Furthermore, we have obtained in Lemma 3.4 the following dynamical inequality:

$$F[u, v] - \frac{4}{3}M(t)H[u, v] + \frac{4}{3}M(t)^2 \leq 0.$$ 

Then define the polynomial $P(y)$ to be

$$P(y) = \frac{4}{3}y^2 - \frac{4}{3}H[u, v]y + F[u, v].$$

For the peakons, we have $H[\varphi_c, \psi_c] = 2ab$ and $F[\varphi_c, \psi_c] = 4a^2b^2/3$, and thus the above polynomial with respect to peakons is defined by

$$P_0(y) = \frac{4}{3}y^2 - \frac{4}{3}H[\varphi_c, \psi_c]y + F[\varphi_c, \psi_c] = \frac{4}{3}(y - ab)^2.$$ 

Note that

$$P(y) = P_0(y) - \frac{4}{3}(H[u, v] - H[\varphi_c, \psi_c])y + (F[u, v] - F[\varphi_c, \psi_c]).$$

It follows that for any $t \in [0, T)$,

$$\frac{4}{3}(M(t) - ab)^2 = P_0(M(t)) \leq \frac{4}{3}|H(u, v) - H(\varphi_c, \psi_c)|M(t) + |F(u, v) - F(\varphi_c, \psi_c)|,$$

which together with (3.11) and (3.12) leads to

$$|M(t) - ab| \leq C\delta^{1/2}. \quad (3.13)$$

On the other hand, since $2u^2(t, \xi(t)) \leq \|u_0\|^2_{H^1(\mathbb{R})}$ and $2v^2(t, \xi(t)) \leq \|v_0\|^2_{H^1(\mathbb{R})}$, we have

$$0 < u(t, \xi(t)) \leq a + \tilde{C}\delta^{1/2}, \quad 0 < v(t, \xi(t)) \leq b + \tilde{C}\delta^{1/2}.$$ 

It follows from (3.13) and the above estimates that

$$ab - \tilde{C}\delta^{1/2} \leq u(\xi)v(\xi) \leq (a + \tilde{C}\delta^{1/2})v(\xi),$$

which implies $v(t, \xi(t)) \geq b - \tilde{C}\delta^{1/2}$. Similarly, we have $u(t, \xi(t)) \geq a - \tilde{C}\delta^{1/2}$. This finishes the proof of the lemma. \hfill $\Box$

**Proof of Theorem 1.1.** Let $u \in C([0, T); H^s)$ and $v \in C([0, T); H^s)$ ($s \geq 3$) be the unique strong solution for the Cauchy problem (2.1) of the two-component Novikov system (1.1) with initial data $u(0, x) = u_0(x)$ and $v(0, x) = v_0(x)$. Since $E_u[u]$ and $E_v[v]$ are both conserved, we deduce that

$$E_u[u(t, \cdot)] = E_u[u_0] \quad \text{and} \quad E_v[v(t, \cdot)] = E_v[v_0], \quad \forall t \in (0, T). \quad (3.14)$$
By (3.14) and Lemma 3.1, we infer that for any $t \in [0, T)$,
\[
\|u(t, \cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} = E_u[u_0] - E_u[\varphi_c] - 4a(u(t, \xi(t)) - a),
\]
\[
\|v(t, \cdot) - \psi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} = E_v[v_0] - E_v[\psi_c] - 4b(v(t, \xi(t)) - b).
\]
Combining the above two dynamical pointwise identities with Lemmas 3.5 and 3.6, we conclude that for any $t \in [0, T)$,
\[
\|u - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} < C\delta^\frac{1}{2} \quad \text{and} \quad \|v - \psi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} < C\delta^\frac{1}{2},
\]
where as in Lemma 3.6, $C$ is a constant depending on $a, b, \|u\|_{H^s}$ and $\|v\|_{H^s}$. This thus completes the proof of Theorem 1.1.

4 Stability of train-profiles

In this section, we are devoted to proving orbital stability of train-profiles of two-component peakons $(\varphi_c, \psi_c)$. For $\alpha > 0$ and $L > 0$, we define the following neighborhood of all the sums of $N$ peakons of speeds $c_1, \ldots, c_N$ with spatial shifts $x_1$ that satisfy $x_i - x_{i-1} \geq L$:
\[
U(\alpha, L) = \left\{(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) : \inf_{x_1 - x_{i-1} \geq L} \left(\left\|u - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - x_i)\right\|_{H^1} + \left\|v - \sum_{i=1}^{N} \psi_{c_i}(\cdot - x_i)\right\|_{H^1}\right) \leq \alpha \right\}.
\]
By the continuity of the map $t \mapsto (u(t), v(t))$ from $[0, T]$ into $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, to prove Theorem 1.3, it suffices to verify that there exist $A > 0, \epsilon_0 > 0$ and $L_0 > 0$ such that for any $L > L_0$ and $0 < \epsilon < \epsilon_0$, if $(u_0, v_0)$ satisfies (1.9) and for some $0 < t_0 < T$, we have
\[
(u(t), v(t)) \in U\left(A(\epsilon^\frac{1}{2} + L^{-\frac{1}{2}}), \frac{L}{2}\right), \quad \forall t \in [0, t_0],
\]
then
\[
(u(t_0), v(t_0)) \in U\left(\frac{L}{2}(\epsilon^\frac{1}{2} + L^{-\frac{1}{2}}), \frac{2L}{3}\right).
\]
Therefore, in the sequel of this section, we will assume (4.1) for some $0 < \epsilon < \epsilon_0$ and $L > L_0$ with $A, \epsilon_0$ and $L_0$ to be specified later.

4.1 Control of the distance between the peakons

In this subsection, we prove that the different bumps of $u$ and $v$ are individually close to their own peakons and get away from each other as time increases. This is crucial in our analysis since we do not know how to manage strong interactions.

**Proposition 4.1.** Assume that $(u_0, v_0)$ satisfies (1.9), and there exist $\alpha_0 > 0, L_0 > 0$ and $C_0 > 0$ such that for all $0 < \alpha < \alpha_0$ and $L > L_0 > 0$, if $(u(t), v(t)) \in U(\alpha, \frac{L}{2})$ on $[0, t_0]$ for some $0 < t_0 < T$, then there exist $C^1$ functions $\tilde{x}_1, \ldots, \tilde{x}_N$ defined on $[0, t_0]$ such that
\[
\frac{d}{dt} \tilde{x}_i = c_i + O(\sqrt{\alpha}) + O(L^{-1}), \quad i = 1, \ldots, N, \tag{4.3}
\]
\[
\left\|u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t))\right\|_{H^1} + \left\|v(t) - \sum_{i=1}^{N} \psi_{c_i}(\cdot - \tilde{x}_i(t))\right\|_{H^1} = O(\sqrt{\alpha}), \tag{4.4}
\]
\[
\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{c_i - c_{i-1}}{2} t, \quad i = 2, \ldots, N. \tag{4.5}
\]
Moreover, setting $J_i := [y_i(t), y_{i-1}(t)]$, $i = 1, \ldots, N$ with
\[ y_1 = -\infty, \quad y_{N+1} = +\infty \text{ and } y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \quad i = 2, \ldots, N, \] (4.6)
we have
\[ |\tilde{x}_i(t) - x_i(t)| \leq \frac{L}{12}, \quad i = 1, \ldots, N, \] (4.7)
where $x_1(t), \ldots, x_N(t)$ are any points such that
\[ u(t, x_i(t))v(t, x_i(t)) = \max_{x \in J_i} u(t)v(t), \quad i = 1, \ldots, N. \] (4.8)

To prove this proposition, we use a modulation argument. The strategy is to construct $C^1$ functions $\tilde{x}_1, \ldots, \tilde{x}_N$ on $[0, t_0]$ satisfying a suitable orthogonality condition (see (4.12)). Thanks to this orthogonality condition, we are able to prove that the speed of $\tilde{x}_i$ stays close to $c_i$ on $[0, t_0]$. 

**Proof of Proposition 4.1.** For $Z = (z_1, \ldots, z_N) \in \mathbb{R}^N$ satisfying $z_i - z_{i-1} > L/2$, we set
\[ R_Z(\cdot) = \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i) \quad \text{and} \quad S_Z(\cdot) = \sum_{i=1}^{N} \psi_{c_i}(\cdot - z_i). \] (4.9)

For $\alpha_0 > 0$ and $L_0 > 0$, we define the function
\[ Y : (-\alpha, \alpha)^N \times B_{H^1 \times H^1}((R_Z, S_Z), \alpha) \to \mathbb{R}^N, \]
\[ (y_1, \ldots, y_N, u, v) \mapsto (Y^1(y_1, \ldots, y_N, u, v), \ldots, Y^N(y_1, \ldots, y_N, u, v)) \]
with
\[ Y^i(y_1, \ldots, y_N, u, v) = \int_{\mathbb{R}} \left( \left( u - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - z_j - y_j) \right) \partial_x \varphi_{c_i}(\cdot - z_i - y_i) \right. \]
\[ \left. + \left( v - \sum_{j=1}^{N} \psi_{c_j}(\cdot - z_j - y_j) \right) \partial_x \psi_{c_i}(\cdot - z_i - y_i) \right) dx, \]
and $Y$ is clearly of class $C^1$. For $i = 1, \ldots, N,$
\[ \frac{\partial Y^i}{\partial y_i}(y_1, \ldots, y_N, u, v) = \int_{\mathbb{R}} \left( \left( u_x - \sum_{j \neq i}^{N} \partial_x \varphi_{c_j}(\cdot - z_j - y_j) \right) \partial_x \varphi_{c_i}(\cdot - z_i - y_i) \right. \]
\[ \left. + \left( v_x - \sum_{j \neq i}^{N} \partial_x \psi_{c_j}(\cdot - z_j - y_j) \right) \partial_x \psi_{c_i}(\cdot - z_i - y_i) \right) dx, \]
and $\forall j \neq i$,
\[ \frac{\partial Y^i}{\partial y_j}(y_1, \ldots, y_N, u, v) = \int_{\mathbb{R}} \left( \partial_x \varphi_{c_j}(\cdot - z_j - y_j) \partial_x \varphi_{c_i}(\cdot - z_i - y_i) \right. \]
\[ \left. + \partial_x \psi_{c_j}(\cdot - z_j - y_j) \partial_x \psi_{c_i}(\cdot - z_i - y_i) \right) dx. \]

Hence,
\[ \frac{\partial Y^i}{\partial y_i}(0, \ldots, 0, R_Z, S_Z) = \| \partial_x \varphi_{c_i} \|^2_{L^2} + \| \partial_x \psi_{c_i} \|^2_{L^2} \geq a_1^2 + b_1^2, \]
and $\forall j \neq i$, using the exponential decay of $\varphi_c$ and $z_i - z_{i-1} > L$, we infer for $L_0$ large enough that (recall that $L > L_0$)
\[ \frac{\partial Y^i}{\partial y_j}(0, \ldots, 0, R_Z, S_Z) = \int_{\mathbb{R}} \left( \partial_x \varphi_{c_j}(\cdot - z_j) \partial_x \varphi_{c_i}(\cdot - z_i) + \partial_x \psi_{c_j}(\cdot - z_j) \partial_x \psi_{c_i}(\cdot - z_i) \right) dx \]
\[ \leq O(e^{-\frac{L}{2}}). \]
We conclude that for $L > 0$ large enough, $D(\gamma_1, \ldots, \gamma_N) Y(0, \ldots, 0, R_{Z, S_{Z}}) = D + P$, where $D$ is an invertible diagonal matrix with $\|D^{-1}\| \leq (a_1^2 + b_1^2)^{-n}$ and $\|P\| \leq O(e^{-L/4})$. Hence, there exists an $L_0 > 0$ such that for $L > L_0$, $D(\gamma_1, \ldots, \gamma_N) Y(0, \ldots, 0, R_{Z, S_{Z}})$ is invertible with an inverse of the norm smaller than $2(a_1^2 + b_1^2)^{-n}$. The implicit function theorem implies that there exist $\beta_0 > 0$ and $C^1$ functions $y_1, y_2, \ldots, y_N$ from $B_{H^1 \times H^1}(R_{Z, S_{Z}}, \beta_0)$ to a neighborhood of $\{(0, 0, \ldots, 0)\}$ which are uniquely determined such that

$$Y(y_1, \ldots, y_N, u, v) = 0 \quad \text{for all } (u, v) \in B(R_{Z, S_{Z}}, \beta_0).$$

In particular, there exists a $C_0 > 0$ such that if $(u, v) \in B(R_{Z, S_{Z}}, \beta)$ with $0 < \beta \leq \beta_0$, then

$$\sum_{i=1}^N |y_i(u, v)| \leq C_0 \beta. \tag{4.10}$$

Note that $\beta_0$ and $C_0$ depend on only $a_1, b_1$ and $L_0$ and not on the point $(z_1, \ldots, z_N)$. For $(u, v) \in B(R_{Z, S_{Z}}, \beta_0)$, we set $\tilde{x}_i(u, v) = z_i + y_i(u, v)$. Assuming that $\beta_0 \leq L_0/(8C_0)$, we see that $\tilde{x}_1, \ldots, \tilde{x}_N$ are thus $C^1$ functions on $B(R_{Z, S_{Z}}, \beta)$ satisfying

$$\tilde{x}_j(u, v) - \tilde{x}_{j-1}(u, v) > \frac{L}{2} - 2C_0 \beta > \frac{L}{4}. \tag{4.11}$$

For $L > L_0$ and $0 < \alpha < \alpha_0 < \beta_0/2$ to be chosen later, we define the modulation of $(u, v) \in U(\alpha, L/2)$ in the following way: the trajectory of $(u, v)$ covered by a finite number of open balls, i.e.,

$$\{(u(t), v(t)), t \in [0, t_0]\} \subset \bigcup_{k=1, \ldots, M} B(R_{Z^k, S_{Z^k}}, 2\alpha).$$

It is worth noticing that since $0 < \alpha < \alpha_0 < \beta_0/2$, the functions $\tilde{x}_i(u, v)$ are uniquely determined for $(u, v) \in B(R_{Z^k, S_{Z^k}}, 2\alpha) \cap B(R_{Z^{k'}, S_{Z^{k'}}}, 2\alpha)$. We can thus define the functions $t \mapsto \tilde{x}_i(t)$ on $[0, t_0]$ by setting $\tilde{x}_i(t) = \tilde{x}_i(u(t), v(t))$. By construction,

$$\int_\mathbb{R} \left( u(t, \cdot) \sum_{j=1}^N \varphi_{c_j}(-\tilde{x}_j(t)) \right) \partial_x \varphi_{c_j}(-\tilde{x}_j(t)) \, dx = 0. \tag{4.12}$$

Moreover, on account of (4.10) and the fact that $\varphi''_c$ and $\psi''_c$ are the sums of an $L^1$ function and a Dirac mass, we claim

$$\|u(t), v(t) - (R_{X(t)}, S_{X(t)})\|_{H^1 \times H^1} \leq O(\sqrt{\alpha}), \quad \forall t \in [0, t_0]. \tag{4.13}$$

Indeed, one can calculate

$$\|u(t), v(t) - (R_{X(t)}, S_{X(t)})\|_{H^1 \times H^1} \leq \|u(t) - R_{X(t)}\|_{H^1} + \|v(t) - S_{X(t)}\|_{H^1}$$

$$\leq \alpha + \tilde{C} \sum_{i=1}^N \|\varphi(-z_i^k) - \varphi(-z_i^k - y_i(u, v))\|_{H^1}$$

$$\leq \alpha + \tilde{C} \sum_{i=1}^N (2(1 - e^{-y_i})^2 + 2g_i + O(y_i^2)) \frac{1}{2} \leq O(\sqrt{\alpha}).$$

Let us now prove that the speed of $\tilde{x}_i$ stays close to $c_i$. We set

$$R_j(t) = \varphi_{c_j}(-\tilde{x}_j(t)), \quad \tilde{u}(t) = u(t) - \sum_{j=1}^N R_j(t) = u(t, \cdot) - R_{X(t)},$$
\[ S_j(t) = \psi_{c_i}(\cdot - \hat{x}_j(t)), \quad \hat{v}(t) = v(t) - \sum_{j=1}^{N} S_j(t) = u(t, \cdot) - S_{\hat{x}(t)}, \]

Differentiating (4.12) with respect to \( t \), we get

\[
\int_{\mathbb{R}} (\hat{u}_t \partial_x R_i + \hat{v}_t \partial_x S_j) \, dx = \dot{\hat{x}}_i (\langle \partial_x^2 R_i \hat{u} \rangle_{H^{-1}, H^1} + \langle \partial_x^2 S_i \hat{v} \rangle_{H^{-1}, H^1}),
\]

and thus,

\[
\left| \int_{\mathbb{R}} (\hat{u}_t \partial_x R_i + \hat{v}_t \partial_x S_j) \, dx \right| \leq |\dot{\hat{x}}_i| (O(\|\hat{u}\|_{H^1}) + O(\|\hat{v}\|_{H^1})) + (O(\|\hat{u}\|_{H^1}) + O(\|\hat{v}\|_{H^1})). \tag{4.14}
\]

Substituting \( u \) by \( \hat{u} + \sum_{j=1}^{N} R_j(t) \) and \( v \) by \( \hat{v} + \sum_{j=1}^{N} S_j(t) \) into (1.8) and using

\[
\begin{align*}
\hat{u}_t + \sum_{i=1}^{N} \hat{x}_i(t) (\hat{u}_t - c_i) \partial_x R_i + \left( \hat{u} + \sum_{j=1}^{N} R_j \right) \left( \hat{v} + \sum_{j=1}^{N} S_j \right) - \sum_{j=1}^{N} (R_j \partial_x R_j) &= 0, \\
\hat{v}_t + \sum_{i=1}^{N} \hat{x}_i(t) (\hat{v}_t - c_i) \partial_x S_i + \left( \hat{u} + \sum_{j=1}^{N} R_j \right) \left( \hat{v} + \sum_{j=1}^{N} S_j \right) - \sum_{j=1}^{N} (S_j \partial_x S_j) &= 0,
\end{align*}
\tag{4.15}
\]

we infer that on \([0, t_0]\), \((\hat{u}, \hat{v})\) satisfies

\[
\begin{align*}
\hat{u}_t - \sum_{i=1}^{N} \hat{x}_i(t) (\hat{u}_t - c_i) \partial_x R_i + \left( \hat{u} + \sum_{j=1}^{N} R_j \right) \left( \hat{v} + \sum_{j=1}^{N} S_j \right) - \sum_{j=1}^{N} (R_j \partial_x R_j) &= 0, \\
\hat{v}_t - \sum_{i=1}^{N} \hat{x}_i(t) (\hat{v}_t - c_i) \partial_x S_i + \left( \hat{u} + \sum_{j=1}^{N} R_j \right) \left( \hat{v} + \sum_{j=1}^{N} S_j \right) - \sum_{j=1}^{N} (S_j \partial_x S_j) &= 0,
\end{align*}
\tag{4.161}
\]

Taking the \( L^2 \) scalar product with \( \partial_t R_j \) in (4.161) and \( \partial_t S_j \) in (4.162), summing up the resulting equations, integrating by parts, and using the decay of \( R_j, S_j \) and their first derivatives, we claim

\[
|\dot{\hat{x}}_i(t) - c_i| \leq O(\sqrt{\alpha}) + O(e^{-\delta t}). \tag{4.17}
\]
Indeed, according to the above, we obtain
\[
\dot{x}_i - c_i((\partial_x^2 R_i, \tilde{u})_{H^{-1}, H^1} + (\partial_x^2 S_i, \tilde{v})_{H^{-1}, H^1})
+ c_i((\partial_x^2 R_i, \tilde{u})_{H^{-1}, H^1} + (\partial_x^2 S_i, \tilde{v})_{H^{-1}, H^1}) - (\dot{x}_i - c_i) \int_{\mathbb{R}} (\partial_x R_i)^2 + (\partial_x S_i)^2 \, dx
= \sum_{i \neq j} (\dot{x}_j - c_j) \left( \int_{\mathbb{R}} (\partial_x R_i \partial_x R_j + \partial_x S_i \partial_x S_j) \, dx \right)
- \int_{\mathbb{R}} \left( \left( \tilde{u} + \sum_{j=1}^{N} R_j \right) \left( \tilde{v} + \sum_{j=1}^{N} S_j \right) \right) \left( \tilde{u}_x + \sum_{j=1}^{N} R_j \partial_x R_j \right) \partial_x R_i \, dx
- \int_{\mathbb{R}} \left( \left( \tilde{u} + \sum_{j=1}^{N} S_j \right) \left( \tilde{v} + \sum_{j=1}^{N} R_j \right) \right) \left( \tilde{v}_x + \sum_{j=1}^{N} S_j \partial_x S_j \right) \partial_x S_i \, dx
+ \int_{\mathbb{R}} \left( P * \frac{1}{2} \left( \tilde{u}_x + \sum_{j=1}^{N} R_j \right)^2 \right) \left( \tilde{v}_x + \sum_{j=1}^{N} S_j \right) \partial_x R_i \, dx
+ \int_{\mathbb{R}} \left( P * \frac{1}{2} \left( \tilde{v}_x + \sum_{j=1}^{N} S_j \right)^2 \right) \left( \tilde{u}_x + \sum_{j=1}^{N} R_j \right) \partial_x S_i \, dx
- \int_{\mathbb{R}} P * \left( \frac{1}{2} \left( \tilde{u}_x + \sum_{j=1}^{N} R_j \right)^2 \right) \left( \tilde{u}_x + \sum_{j=1}^{N} S_j \right) \partial_x R_i \, dx
- \int_{\mathbb{R}} \left( \frac{1}{2} \sum_{j=1}^{N} R_j^2 \right) \left( \tilde{u}_x + \sum_{j=1}^{N} S_j \right) \partial_x S_i \, dx.
\]

For every term, we have the following estimates:
\[
(\partial_x^2 R_i, \tilde{u})_{H^{-1}, H^1} = \int_{\mathbb{R}} \partial_x^2 R_i \tilde{u} \, dx = \int_{\mathbb{R}} [R_i - 2a_i \delta(x - \tilde{x}_i(t))] \tilde{u} \, dx
\]
\[
= \int_{\mathbb{R}} R_i \tilde{u} \, dx - 2a_i \| \tilde{u} \|_{L^\infty} \left( \left| \int_{\mathbb{R}} R_i \, dx \right| + 2a_i \right) \leq O(\sqrt{a}).
\]

Similarly,
\[
(\partial_x^2 S_i, \tilde{v})_{H^{-1}, H^1} \leq O(\sqrt{a}).
\]

For the term \( \int_{\mathbb{R}} \partial_x R_i \partial_x R_j \, dx \), we find
\[
\int_{\mathbb{R}} \partial_x R_i \partial_x R_j \, dx = - \int_{\mathbb{R}} \partial_x^2 R_i R_j \, dx = - \int_{\mathbb{R}} (R_i - 2a_i \delta(x - \tilde{x}_i(t))) R_j \, dx
= - \int_{\mathbb{R}} R_i R_j \, dx + 2a_i R_j (\tilde{x}_i(t)) \leq O(e^{-\frac{t}{4}}).
\]

Similarly,
\[
\int_{\mathbb{R}} \partial_x S_i \partial_x S_j \, dx \leq O(e^{-\frac{t}{4}}).
\]
Lemma 4.2. Assume that \((\tilde{u}, \tilde{v})\) satisfies (4.13). Then we have
\[
\int_{\mathbb{R}} \left[ \left( \tilde{u} + \sum_{j=1}^{N} R_j \right) \left( \tilde{v} + \sum_{j=1}^{N} S_j \right) \left( \tilde{u}_x + \sum_{j=1}^{N} R_j x \right) - \sum_{j=1}^{N} R_j S_j R_j x \right] R_{1z} dx \leq O(\sqrt{\alpha}) + O(e^{-\frac{z}{2}})
\]
and
\[
\int_{\mathbb{R}} \left[ \left( \tilde{u} + \sum_{j=1}^{N} R_j \right) \left( \tilde{v} + \sum_{j=1}^{N} S_j \right) \left( \tilde{v}_x + \sum_{j=1}^{N} S_j x \right) - \sum_{j=1}^{N} R_j S_j S_j x \right] S_{1z} dx \leq O(\sqrt{\alpha}) + O(e^{-\frac{z}{2}}).
\]

Proof. We calculate
\[
\int_{\mathbb{R}} \left[ \left( \tilde{u} + \sum_{j=1}^{N} R_j \right) \left( \tilde{v} + \sum_{j=1}^{N} S_j \right) \left( \tilde{u}_x + \sum_{j=1}^{N} R_j x \right) - \sum_{j=1}^{N} R_j S_j R_j x \right] R_{1z} dx \\
= \int_{\mathbb{R}} \left[ \left( \tilde{u} \left( \tilde{v} + \sum_{j=1}^{N} S_j \right) \left( \tilde{u}_x + \sum_{j=1}^{N} R_j x \right) + \sum_{j=1}^{N} R_j \tilde{v} \left( \tilde{u}_x + \sum_{j=1}^{N} R_j x \right) + \sum_{j=1}^{N} S_j \sum_{j=1}^{N} R_j \tilde{u}_x \\
+ \sum_{j=1}^{N} S_j \sum_{j=1}^{N} R_j \sum_{j=1}^{N} R_j S_j R_j x \right) R_{1z} dx \\
\leq \| \tilde{u} \|_{L^\infty} \| u \|_{L^\infty} \| v \|_{L^\infty} \| v \|_{L^\infty} \| R \|_{L^\infty} \| \tilde{v} \|_{L^\infty} \| ||u||_{L^\infty} \| \int_{\mathbb{R}} R_{1z} dx \\
+ \sum_{j=1}^{N} \| R_j \|_{L^\infty} \left( \sum_{j=1}^{N} \| S_j \|_{L^\infty} \right) \left( \int_{\mathbb{R}} \left( \tilde{u}_x \right)^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} R_{1z}^2 dx \right)^{\frac{1}{2}} \\
+ \sum_{i \neq j \text{ or } i \neq k \text{ or } j \neq k} R_i S_j R_k \| R_{1z} \| \leq O(\sqrt{\alpha}) + O(e^{-\frac{z}{2}}).
\]

Similarly, we have
\[
\int_{\mathbb{R}} \left[ \left( \tilde{u} + \sum_{j=1}^{N} R_j \right) \left( \tilde{v} + \sum_{j=1}^{N} S_j \right) \left( \tilde{v}_x + \sum_{j=1}^{N} S_j x \right) - \sum_{j=1}^{N} R_j S_j S_j x \right] S_{1z} dx \leq O(\sqrt{\alpha}) + O(e^{-\frac{z}{2}}).
\]

This completes the proof. \(\square\)

In order to estimate the next terms, we need the following lemma.

Lemma 4.3. Under the same assumptions as in Lemma 4.2, we have
\[
\| P \left( \frac{1}{2} \left( \tilde{u}_x + \sum_{j=1}^{N} R_j x \right)^2 + \left( \tilde{u} + \sum_{j=1}^{N} S_j \right) + \left( \tilde{v} + \sum_{j=1}^{N} S_j \right) \left( \tilde{u}_x + \sum_{j=1}^{N} R_j x \right) + \left( \tilde{v}_x + \sum_{j=1}^{N} S_j x \right) \right) \|_{L^\infty} \\
\leq O(\sqrt{\alpha}) + O(e^{-\frac{z}{2}})
\]
and
\[
\left\| P \left( \frac{1}{2} (\bar{u}_x + \sum_{j=1}^N S_{jx})^2 (\bar{u} + \sum_{j=1}^N R_j) + (\bar{v} + \sum_{j=1}^N S_j) \right) \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right) + (\bar{v} + \sum_{j=1}^N S_j)^2 \right\|_{L^\infty} \leq O(\sqrt{\alpha}) + O(e^{-\frac{\alpha}{2}}).
\]

**Proof.** By using Hölder’s inequality and the triangle inequality, we get
\[
\left\| P \left( \frac{1}{2} (\bar{u}_x + \sum_{j=1}^N R_{jx})^2 (\bar{u} + \sum_{j=1}^N S_j) + (\bar{v} + \sum_{j=1}^N R_j) \right) \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right) + (\bar{v} + \sum_{j=1}^N S_j)^2 \right\|_{L^\infty} \leq \frac{1}{2} \int_{\mathbb{R}} \left( |\bar{u}_x + \sum_{j=1}^N R_{jx}|^2 - \sum_{j=1}^N R_{jx}^2 S_j \right) dx
\]
\[
+ \int_{\mathbb{R}} \left( |\bar{u} + \sum_{j=1}^N R_j|^2 - \sum_{j=1}^N R_j^2 S_j \right) dx
\]
\[
+ \int_{\mathbb{R}} \left( |\bar{v} + \sum_{j=1}^N S_j|^2 - \sum_{j=1}^N R_j^2 S_j \right) dx
\]
\[
= \frac{1}{2} I_{2,1} + I_{2,2} + I_{2,3}.
\]

For the term $I_{2,1}$, we have
\[
I_{2,1} = \int_{\mathbb{R}} \left( |\bar{u}_x + \sum_{j=1}^N R_{jx}|^2 - \sum_{j=1}^N R_{jx}^2 S_j \right) dx
\]
\[
\leq \int_{\mathbb{R}} \left( |\bar{u}_x + \sum_{j=1}^N R_{jx}|^2 - \sum_{j=1}^N R_{jx}^2 S_j \right) dx + \int_{\mathbb{R}} \left( \sum_{j=1}^N R_{jx}^2 \right) \bar{v} dx
\]
\[
+ \int_{\mathbb{R}} \left( \sum_{j=1}^N R_{jx}^2 \sum_{j=1}^N S_j - \sum_{j=1}^N R_{jx}^2 S_j \right) dx
\]
\[
\leq \||\bar{v}||_{L^\infty} \int_{\mathbb{R}} \left( |\bar{u}_x|^2 \right) dx + \int_{\mathbb{R}} \sum_{j=1}^N R_{jx}^2 dx + \int_{\mathbb{R}} \sum_{j=1}^N R_{jx}^2 S_j dx
\]
\[
\leq O(\sqrt{\alpha}) + O(e^{-\frac{\alpha}{2}}).
\]

For the term $I_{2,2}$, we have
\[
I_{2,2} = \int_{\mathbb{R}} \left( |\bar{u} + \sum_{j=1}^N R_j| \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right) + |\bar{v} + \sum_{j=1}^N S_j| \left( \bar{v}_x + \sum_{j=1}^N S_{jx} \right) - \sum_{j=1}^N R_j R_{jx} S_j \right) dx
\]
\[
\leq \int_{\mathbb{R}} \left( |\bar{u} + \sum_{j=1}^N R_j| \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right) \right) dx + \int_{\mathbb{R}} \left( |\sum_{j=1}^N R_j| \bar{u}_x + \sum_{j=1}^N S_{jx} \right) dx
\]
\[
+ \int_{\mathbb{R}} \left( \sum_{j=1}^N R_j \right) \left( \sum_{j=1}^N S_{jx} \right) \bar{u}_x dx + \int_{\mathbb{R}} \left( \sum_{j=1}^N R_j \right) \left( \sum_{j=1}^N S_{jx} \right) \left( \sum_{j=1}^N R_{jx} \right) - \sum_{j=1}^N R_j R_{jx} S_j \right) dx
\]
\[ \leq \| \bar{u} \|_{L^\infty} \int_R |u_x v_x| \, dx + \| v \|_{L^\infty} \left( \int_R \left( \sum_{j=1}^N R_j \right)^2 \, dx \right)^{\frac{1}{2}} \left( \int_R \bar{u}_x^2 \, dx \right)^{\frac{1}{2}} + \sum_{j=1}^N \| S_j \|_{L^\infty} \left( \int_R \left( \sum_{j=1}^N R_j \right)^2 \, dx \right)^{\frac{1}{2}} \left( \int_R \bar{u}_x^2 \, dx \right)^{\frac{1}{2}} + \int_R \left| \sum_{i \neq j \neq k \neq i} R_i S_{jx} R_{jx} \right| \, dx \]
\[ \leq O(\sqrt{\alpha}) + O(e^{\frac{\alpha}{4}}). \]

For this term \( I_{2.3} \), we obtain
\[ I_{2.3} = \int_R \left( \left( \bar{u} + \sum_{j=1}^N R_j \right)^2 \left( \bar{v} + \sum_{j=1}^N S_j \right) - \sum_{j=1}^N R_j S_j \right) \, dx \]
\[ \leq \int_R \left( \left( \bar{u} + \sum_{j=1}^N R_j \right)^2 \left( \bar{v} + \sum_{j=1}^N S_j \right) \right) \, dx + \int_R \left| \sum_{j=1}^N R_j S_j \right| \, dx \]
\[ \leq \| v \|_{L^\infty} \int_R \left| \bar{u} \right|^2 + \left| \sum_{j=1}^N R_j \right| + \left( \sum_{j=1}^N R_j \right)^2 - \sum_{j=1}^N R_j^2 \right) \, dx + \| \bar{v} \|_{L^\infty} \int_R \sum_{j=1}^N R_j^2 \, dx \]
\[ + \int_R \sum_{i \neq j} \sum_{j=1}^N R_j^2 S_i \, dx \leq O(\sqrt{\alpha}) + O(e^{\frac{\alpha}{4}}). \]

Accordingly, we get
\[ \left\| P \left( \frac{1}{2} \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right)^2 \left( \bar{v} + \sum_{j=1}^N S_j \right) + \left( \bar{u} + \sum_{j=1}^N R_j \right) \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right) \right) \right\|_{L^\infty} \]
\[ \leq O(\sqrt{\alpha}) + O(e^{\frac{\alpha}{4}}). \]

Similarly, we can prove
\[ \left\| P \left( \frac{1}{2} \left( \bar{v}_x + \sum_{j=1}^N S_{jx} \right)^2 \left( \bar{v} + \sum_{j=1}^N S_j \right) + \left( \bar{v} + \sum_{j=1}^N S_j \right) \right) \right\|_{L^\infty} \]
\[ \leq O(\sqrt{\alpha}) + O(e^{\frac{\alpha}{4}}). \]

This completes the proof of this lemma. \( \square \)

Thanks to Lemma 4.3, we have
\[ \int_R P \left( \frac{1}{2} \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right)^2 \left( \bar{v} + \sum_{j=1}^N S_j \right) + \left( \bar{u} + \sum_{j=1}^N R_j \right) \left( \bar{u}_x + \sum_{j=1}^N R_{jx} \right) \right) \]
\[ + \left( \bar{u} + \sum_{j=1}^N R_j \right)^2 \left( \bar{v} + \sum_{j=1}^N S_j \right) - \frac{1}{2} \sum_{j=1}^N S_{jx} R_j - \sum_{j=1}^N S_{jx} R_{jx} S_j - \sum_{j=1}^N S_{jx}^2 \right) \, dx \]
\[ \triangleq \int_R A(x) \partial_x^2 R \, dx = \int_R A(x) \left( R_i - 2a_i \delta(x - \tilde{x}_i(t)) \right) \, dx \]
Lemma 4.4. Under the same assumption as (4.2), we have

\[
\left\| P \* \left( \frac{1}{2} \left( \bar{u}_x + \sum_{j=1}^{N} S_{jx} \right) \right)^2 \left( \bar{v}_x + \sum_{j=1}^{N} R_j \right) + \left( \bar{v}_x + \sum_{j=1}^{N} S_{jx} \right) \left( \bar{u}_x + \sum_{j=1}^{N} R_j \right) \right\|_{L^\infty} \lesssim O(\sqrt{\alpha}) + O(e^{-\frac{\alpha}{2}}).
\]

Similarly, we get

\[
\int_{\mathbb{R}} \left[ P \* \left( \frac{1}{2} \left( \bar{u}_x + \sum_{j=1}^{N} S_{jx} \right) \right)^2 \left( \bar{v}_x + \sum_{j=1}^{N} R_j \right) + \left( \bar{v}_x + \sum_{j=1}^{N} S_{jx} \right) \left( \bar{u}_x + \sum_{j=1}^{N} R_j \right) \right] dx \\
+ \left( \bar{v}_x + \sum_{j=1}^{N} S_{jx} \right) \left( \bar{u}_x + \sum_{j=1}^{N} R_j \right) \left( \bar{v}_x + \sum_{j=1}^{N} R_j \right) \frac{d^2}{dx^2} S_{0} dx \\
\leq O(\sqrt{\alpha}) + O(e^{-\frac{\alpha}{2}}).
\]
For the term $I_{3.3}$, we have
\[
\int_{\mathbb{R}} \left| \sum_{j=1}^{N} R_{jx}^2 \sum_{j=1}^{N} S_{jx} - \sum_{j=1}^{N} R_{jx}^2 S_{jx} \right| dx \leq \int_{\mathbb{R}} \sum_{i \neq j} \sum_{j=1}^{N} |R_{jx}^2 S_{ix}| dx \leq O(e^{-\frac{t}{4}}).
\]
Thus, we deduce that
\[
\left\| P \ast \left( \frac{1}{2} \left( \tilde{u}_x + \sum_{j=1}^{N} R_{jx} \right)^2 \left( \tilde{v}_x + \sum_{j=1}^{N} S_{jx} \right) - \frac{1}{2} \sum_{j=1}^{N} R_{jx}^2 S_{jx} \right) \right\|_{L^\infty} \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}})
\]
and
\[
\left\| P \ast \left( \frac{1}{2} \left( \tilde{v}_x + \sum_{j=1}^{N} S_{jx} \right)^2 \left( \tilde{u}_x + \sum_{j=1}^{N} R_{jx} \right) - \frac{1}{2} \sum_{j=1}^{N} S_{jx}^2 R_{jx} \right) \right\|_{L^\infty} \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}).
\]
This completes the proof of this lemma. \(\square\)

On account of Lemma 4.4, we obtain
\[
\int_{\mathbb{R}} P \ast \left( \frac{1}{2} \left( \tilde{u}_x + \sum_{j=1}^{N} R_{jx} \right)^2 \left( \tilde{v}_x + \sum_{j=1}^{N} S_{jx} \right) - \frac{1}{2} \sum_{j=1}^{N} R_{jx}^2 S_{jx} \right) \partial_x R_i \, dx \leq (O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}})) \int_{\mathbb{R}} |R_{ix}| \, dx \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}).
\]
Similarly, we find
\[
\int_{\mathbb{R}} P \ast \left( \frac{1}{2} \left( \tilde{v}_x + \sum_{j=1}^{N} S_{jx} \right)^2 \left( \tilde{u}_x + \sum_{j=1}^{N} R_{jx} \right) - \frac{1}{2} \sum_{j=1}^{N} S_{jx}^2 R_{jx} \right) \partial_x S_i \, dx \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}).
\]
Thanks to Lemmas 4.2–4.4, we arrive at
\[
|\dot{x}_i(t) - c_i||R_{ix}||_{L^2}^2 + ||S_{ix}||_{L^2}^2 + O(\sqrt{\alpha}) \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}).
\]
Since $||R_{ix}||_{L^2} > a_1$ and $||S_{ix}||_{L^2}^2 > b_1$, we have
\[
|\dot{x}_i(t) - c_i| \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}).
\]
Finally, we claim
\[
|x_i - \tilde{x}_i| \leq \frac{L}{12}.
\]
Indeed, if $x \notin [\tilde{x}_i - L/12, \tilde{x}_i + L/12]$, then
\[
u(x, t) \leq c_i e^{-\frac{t}{4}} + O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}) \leq c_i - O(\sqrt{\alpha}) - O(e^{-\frac{t}{4}})
\]
with $\alpha$ small enough and $L$ large enough. However,
\[
u(t, x) \geq \max_{x \in J(t)} \nu(x) \geq \nu(\tilde{x}_i) = c_i + O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}),
\]
which leads to a contradiction. Therefore, we have $|x_i - \tilde{x}_i| \leq L/12$. \(\square\)
4.2 The monotonicity property

Thanks to the preceding proposition, for $\epsilon_0 > 0$ small enough and $L_0 > 0$ large enough, one can construct $C^1$ functions $\tilde{x}_1, \ldots, \tilde{x}_N$ defined on $[0, t_0]$ such that (4.3)–(4.7) are satisfied. In this subsection, we investigate the almost monotonicity of functionals that are very close to the energy on the right of the $i$-th bump, $i = 1, \ldots, N - 1$ of $(u, v)$. Let $\Psi$ be a $C^\infty$-function such that

\[
\begin{cases}
0 < \Psi(x) < 1, & x \in \mathbb{R}, \\
|\Psi'''| \leq 10|\Psi'|, & x \in [-1, 1]
\end{cases}
\]

and

\[
\Psi(x) = \begin{cases}
e^{-|x|}, & x < -1, \\
1 - e^{-|x|}, & x > 1.
\end{cases}
\]

Setting $\Psi_K = \Psi(\cdot / K)$, for $j = 2, \ldots, N$, we introduce

\[
\begin{align*}
\mathcal{J}_{j,K}^u(t) &= \int_{\mathbb{R}} (u^2(t) + u_x^2(t))\Psi_{j,K}(t)dx,
\mathcal{J}_{j,K}^v(t) &= \int_{\mathbb{R}} (v^2(t) + v_x^2(t))\Psi_{j,K}(t)dx,
\mathcal{J}_{j,K}^{u,v}(t) &= \int_{\mathbb{R}} (u(t)v(t) + u_x(t)v_x(t))\Psi_{j,K}(t)dx,
\end{align*}
\]

where $\Psi_{j,K}(t, x) = \Psi_K(x - y_j(t))$ with $y_j(t) (j = 2, \ldots, N)$ defined in (4.6). Note that $\mathcal{J}_{j,K}^u(t)$ is close to $\|u(t)\|_{H^1(x>y_j(t))}$ and thus measures the energy on the right of the $(j-1)$-th bump of $u$, $\mathcal{J}_{j,K}^v(t)$ is close to $\|v(t)\|_{H^1(x>y_j(t))}$ and thus measures the energy on the right of the $(j-1)$-th bump of $v$, and $\mathcal{J}_{j,K}^{u,v}(t)$ is close to $\langle u(t), v(t) \rangle_{H^1 \times H^1(x>y_j(t))}$ and thus measures the energy on the right of the $(j-1)$-th bump of $(u, v)$. Finally, we set

\[
\sigma_0 = \frac{1}{4} \min(c_1, c_2 - c_1, \ldots, c_N - c_{N-1}).
\]

We have the following monotonicity result.

**Proposition 4.5** (Exponential decay of the functionals $\mathcal{J}_{j,K}^u(t)$, $\mathcal{J}_{j,K}^v(t)$ and $\mathcal{J}_{j,K}^{u,v}(t)$). Let $(u, v) \in Y([0, T])$ be a solution of two-component Novikov equations satisfying (4.4) on $[0, t_0]$. There exist $\alpha_0 > 0$ and $L_0 > 0$ only depending on $c_1$ such that if $0 < \alpha < \alpha_0$ and $L \geq L_0$, then for any $4 \leq K \leq \sqrt{T}$,

\[
\begin{align*}
\mathcal{J}_{j,K}^u(t) - \mathcal{J}_{j,K}^u(0) &\leq O(e^{-\frac{\alpha t}{\sqrt{T}}}), \\
\mathcal{J}_{j,K}^v(t) - \mathcal{J}_{j,K}^v(0) &\leq O(e^{-\frac{\alpha t}{\sqrt{T}}}), \\
\mathcal{J}_{j,K}^{u,v}(t) - \mathcal{J}_{j,K}^{u,v}(0) &\leq O(e^{-\frac{\alpha t}{\sqrt{T}}}), \quad \forall j \in \{2, \ldots, N\} \text{ and } t \in [0, t_0].
\end{align*}
\]

The proof of this proposition relies on the following Virial type identity.

**Lemma 4.6** (Virial type identity). Let $(u, v) \in Y([0, T])$ with $0 < T \leq +\infty$ be a solution of (1.1) that satisfies (1.9). For any smooth space function $g : \mathbb{R} \to \mathbb{R}$, it holds that

\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2)g dx &= \int_{\mathbb{R}} u \left[u_x^2 v + 2P \ast \left(\frac{1}{2}u_x^2 v + uu_xv_x + u^2 v \right) + 2P_x \ast \left(\frac{1}{2}u_x^2 v_x \right) \right]g' dx + \int_{\mathbb{R}} (u^2 + u_x^2)g' dx, \\
\frac{d}{dt} \int_{\mathbb{R}} (v^2 + v_x^2)g dx &= \int_{\mathbb{R}} v \left[v_x^2 u + 2P \ast \left(\frac{1}{2}v_x^2 u + vv_x u_x + v^2 u \right) + 2P_x \ast \left(\frac{1}{2}v_x^2 u_x \right) \right]g' dx + \int_{\mathbb{R}} (v^2 + v_x^2)g' dx
\end{align*}
\]

and

\[
\frac{d}{dt} \int_{\mathbb{R}} (uv + u_xv_x)g dx
\]
Recall that the assumption ensures that this completes the proof.

Proof. By using the weak form of \( u \), and integrating by parts, we calculate
\[
\frac{d}{dt} \int_R (u^2 + u_x^2) \, dx = 2 \int_R (uu_t + u_x u_{xt}) \, dx + \int_R (u^2 + u_x^2) \, dx
\]
\[
= 2 \int_R (uu_t - uu_{xt} - uu_{xt}) \, dx + \int_R (u^2 + u_x^2) \, dx
\]
\[
= \int_R \left( 2u_x v - 2u \left( \frac{1}{2} u_x^2 v - uv u_{xx} - P * \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) - P_z * \left( \frac{1}{2} u_x^2 v_x \right) \right) \right) \, dx
\]
\[
+ \int_R (u^2 + u_x^2) \, dx
\]
\[
= \int_R \left( uu_x^2 + 2uP * \left( \frac{1}{2} u_x^2 u + vv_x u_x + v^2 u \right) + 2uP_z * \left( \frac{1}{2} u_x^2 v_x \right) \right) \, dx
\]
\[
+ \int_R (u^2 + u_x^2) \, dx.
\]
Similarly, we have
\[
\frac{d}{dt} \int_R (v^2 + v_x^2) \, dx = \int_R \left[ vv_x^2 u + 2vP * \left( \frac{1}{2} v_x^2 v + vv_x u_x + v^2 u \right) + 2vP_z * \left( \frac{1}{2} v_x^2 v_x \right) \right] \, dx
\]
\[
+ \int_R (v^2 + v_x^2) \, dx.
\]

Using the weak form of \((u, v)\) and its first derivative, and integrating by parts, we obtain
\[
\frac{d}{dt} \int_R (uv + u_x v_x) \, dx
\]
\[
= \int_R \left( uu_t + u_x v - u_{xt} v - uu_{xt} - uu_{xt} v - uv_{xx} \right) \, dx + \int_R (uv + u_x v_x) \, dx
\]
\[
= \int_R (uu_t + uv_{xt} - uu_{xt} + uu_{xt} v - uv_{xx}) \, dx + \int_R (uv + u_x v_x) \, dx
\]
\[
= \int_R \left( uu_{xt} + uv_{xx} + uP * \left( \frac{1}{2} v_x^2 u + vv_x u_x + v^2 u \right) + uP_z * \left( \frac{1}{2} v_x^2 v_x \right) \right) \, dx
\]
\[
+ \int_R \left( vP * \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) + vP_z * \left( \frac{1}{2} u_x^2 v_x \right) \right) \, dx + \int_R (uv + u_x v_x) \, dx.
\]
This completes the proof. \( \square \)

Proof of Proposition 4.5. Combining (4.3) and (4.6), we first note that for \( i = 2, \ldots, N \),
\[
\dot{y}_i(t) = \frac{\ddot{x}_{i-1}(t) + \ddot{x}_i(t)}{2} = \frac{c_{i-1} + c_i}{2} + O(\sqrt{\alpha}) + O(L^{-1}) \geq \frac{c_1}{2}.
\]
(4.18)

Recall that the assumption ensures that \( u \geq 0 \) and \( v \geq 0 \) on \( \mathbb{R} \). Now, applying the Virial type identity with \( g = \Psi_{i,K} \) and using (4.18), we get
\[
\frac{d}{dt} J_{i,K}^U(t) = -\dot{y}_i \int_R (u^2 + u_x^2) \Psi_{i,K}^U \, dx
\]
\[
+ \int_R \left[ uu_x^2 + 2uP * \left( \frac{1}{2} u_x^2 u + uu_x v_x + u^2 v \right) + 2uP_z * \left( \frac{1}{2} u_x^2 v_x \right) \right] \Psi_{i,K} \, dx
\]
\[
\leq -\frac{c_1}{2} \int_R (u^2 + u_x^2) \Psi_{i,K}^U \, dx + \int_R uu_x^2 v \Psi_{i,K} \, dx
\]
Next, the estimate of \( \tilde{I}_{1,i} \) (\( i = 1, 2, 3 \)), it holds that

\[
\tilde{I}_{1,i} \leq \frac{c_1}{20} \int_{\mathbb{R}} (u_x^2 + u_x^2) \Psi'_{i,K} dx + \frac{C}{K} \|u_0\|_{H^1(\mathbb{R})}^3 \|v_0\|_{H^1(\mathbb{R})} e^{-\frac{1}{2}(\sigma u + \frac{t}{4})}.
\]

Indeed, we divide \( \mathbb{R} \) into two regions \( D_i \) and \( D_i^c \) with

\[
D_i = \left[ \bar{x}_{i-1}(t) + \frac{L}{4}, \bar{x}_i(t) - \frac{L}{4} \right], \quad i = 2, \ldots, N.
\]

Combining (4.5) and (4.6), one can check that for \( x \in D_i^c \),

\[
|x - y_i(t)| \geq \frac{\bar{x}_i(t) - \bar{x}_{i-1}(t)}{2} - \frac{L}{4} \geq \frac{c_i - c_{i-1}}{4} t + \frac{L}{8} \geq \sigma_0 t + \frac{L}{8}.
\]

(4.19)

Let us begin by an estimate of \( \tilde{I}_{1,1} \). Using (4.19), the Sobolev imbedding and the exponential decay of \( \Psi'_{i,K} \) on \( D_i^c \), we get

\[
\tilde{I}_{1,1} = \int_{D_i} u u_x v \Psi'_{i,K} dx + \int_{D_i^c} u u_x v \Psi'_{i,K} dx
\leq \|u\|_{L^\infty(D_i)} \|v\|_{L^\infty(D_i)} \int_{D_i} (u_x^2 + u_x^2) \Psi'_{i,K} dx + \|\Psi'_{i,K}\|_{L^\infty(D_i)} \|u\|_{L^\infty(\mathbb{R})} \|v\|_{L^\infty(\mathbb{R})} \|u\|_{H^1}.
\]

Now, using the exponential decay of \( \varphi_c \) and \( \psi_c \) on \( D_i \), we have

\[
\|u\|_{L^\infty(D_i)} \leq \left\| u - \sum_{j=1}^{N} N \right\|_{L^\infty(D_i)} + \sum_{j=1}^{N} \|\varphi(\cdot - \bar{x}_j(t))\|_{L^\infty(\mathbb{R})} \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}),
\]

\[
\|v\|_{L^\infty(D_i)} \leq \left\| v - \sum_{j=1}^{N} v \right\|_{L^\infty(D_i)} + \sum_{j=1}^{N} \|\psi(\cdot - \bar{x}_j(t))\|_{L^\infty(\mathbb{R})} \leq O(\sqrt{\alpha}) + O(e^{-\frac{t}{4}}).
\]

(4.20)

Therefore, for \( 0 < \alpha < \alpha_0 \) and \( L > L_0 > 0 \) with \( \alpha_0 \ll 1 \) and \( L_0 \gg 1 \), we obtain

\[
\tilde{I}_{1,1} \leq \frac{c_1}{20} \int_{\mathbb{R}} (u_x^2 + u_x^2) \Psi'_{i,K} dx + \frac{C}{K} \|u_0\|_{H^1(\mathbb{R})}^3 \|v_0\|_{H^1(\mathbb{R})} e^{-\frac{1}{2}(\sigma u + \frac{t}{4})}.
\]

Before estimating the term \( \tilde{I}_{1,2} \), and using \( |\Psi''_{i,K}| \leq 10K^{-2} \Psi'_{i,K} \), we have

\[
(1 - \partial_x^2) \Psi'_{i,K}(x) = \Psi''_{i,K}(x) - \frac{1}{K^2} \Psi''_{i,K}(x) \geq \left( 1 - \frac{10}{K^2} \right) \Psi'_{i,K}(x), \quad \forall x \in \mathbb{R},
\]

and since \( K > 4 \), it holds that

\[
(1 - \partial_x^2)^{-1} \Psi'_{i,K}(x) \leq \left( 1 - \frac{10}{K^2} \right)^{-1} \Psi'_{i,K}(x), \quad \forall x \in \mathbb{R}.
\]

(4.21)

Next, the estimate of \( \tilde{I}_{1,2} \) gives us

\[
\tilde{I}_{1,2} = \int_{D_i} \left[ 2uP \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \right] \Psi'_{i,K} dx + \int_{D_i^c} \left[ 2uP \left( \frac{1}{2} u_x^2 v + uu_x v_x + u^2 v \right) \right] \Psi'_{i,K} dx
\leq \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} P \cdot |u_x^2 v + uu_x v_x + u^2 v| \Psi'_{i,K} dx
\]
+ \|\Psi_{i,K}\|_{L^\infty(D^*_i)} \|u\|_{L^\infty(R)} \int_R \left| u_x^2 v + 2uv u_x + 2u^2 v \right| dx \\
\leq 3 \|u\|_{L^\infty(D^*_i)} \|v\|_{L^\infty(D^*_i)} \int_R \left| u_x^2 + u^2 \Psi_{i,K} \right| dx + \frac{C}{K} \|u_0\|_{H^1(R)} \|v_0\|_{H^1(R)} e^{-\frac{1}{8}(\sigma \alpha t + \frac{\beta}{4})} \\
\leq \frac{c_1}{20} \int_R (u^2 + u_x^2) \Psi_{i,K} \, dx + \frac{C}{K} \|u_0\|_{H^1(R)} \|v_0\|_{H^1(R)} e^{-\frac{1}{8}(\sigma \alpha t + \frac{\beta}{4})},
\end{align*}

where Young’s inequality, the exponential decay of \( \Psi_{i,K} \) on \( D^*_i \), and the inequalities (4.20) and (4.21) are used. Let us tackle now the estimate of \( \tilde{I}_{1,3} \). On \( D^*_i \), we have
\begin{align*}
\int_{D^*_i} [uP_x * (u_x^2 v_x)] \Psi_{i,K}(t) \, dx &\leq \\|\Psi_{i,K}\|_{L^\infty(D^*_i)} \|v\|_{L^\infty(D^*_i)} \int_R u_x^2 |P \ast u| \, dx \\
&\leq \\|\Psi_{i,K}\|_{L^\infty(D^*_i)} \|v\|_{L^\infty(D^*_i)} \|P \ast u\|_{L^\infty(R)} \int_R u_x^2 \, dx.
\end{align*}

Applying Hölder’s inequality, we see that for all \( x \in \mathbb{R} \),
\begin{equation}
P \ast u = \frac{1}{2} \int_R e^{-|x-y|} u(y) \, dy \leq \frac{1}{2} \left( \int_R e^{-2|x-y|} \, dy \right)^{\frac{1}{2}} \left( \int_R u^2(y) \, dy \right)^{\frac{1}{2}} \leq \frac{1}{2} \|u\|_{L^2(R)},
\end{equation}

and then using (4.22) and the exponential decay of \( \Psi_{i,K} \) on \( D^*_i \), we have
\begin{align*}
\int_{D^*_i} [uP_x * (u_x^2 v_x)] \Psi_{i,K}(t) \, dx &\leq \frac{C}{K} \|u_0\|_{H^1(R)} \|v_0\|_{H^1(R)} e^{-\frac{1}{8}(\sigma \alpha t + \frac{\beta}{4})}.
\end{align*}

The estimate of \( \tilde{I}_{1,3} \) on \( D_i \) leads to
\begin{align*}
\int_{D_i} [uP_x * (u_x^2 v_x)] \Psi_{i,K}(t) \, dx &\leq \|u\|_{L^\infty(D_i)} \|v\|_{L^\infty(D_i)} \int_R |P \ast \Psi_{i,K}(t)| u_x^2 \, dx \\
&\leq \frac{c_1}{20} \int_R (u^2 + u_x^2) \Psi_{i,K} \, dx.
\end{align*}

Therefore, for \( 0 < \alpha < \alpha_0 \) and \( L > L_0 > 0 \) with \( \alpha_0 \ll 1 \) and \( L_0 \gg 1 \), it holds that
\begin{align*}
\frac{d}{dt} J^{u,K}_{j,K}(t) &\leq \frac{C}{K} \|u_0\|_{H^1(R)} \|v_0\|_{H^1(R)} e^{-\frac{1}{8}(\sigma \alpha t + \frac{\beta}{4})}.
\end{align*}

Integrating between 0 and \( t \), we obtain \( J^{u,K}_{j,K}(t) - J^{u,K}_{j,K}(0) \leq O(e^{-\frac{\sigma \alpha t}{8K}}) \). Similarly, we also obtain
\begin{align*}
J^{v,K}_{j,K}(t) - J^{v,K}_{j,K}(0) \leq O(e^{-\frac{\sigma \alpha t}{8K}}).
\end{align*}

Now, we need to prove the third inequality of this proposition. Applying the Virial type identity with \( g = \Psi_{i,K} \) and using (4.18), we get
\begin{align*}
\frac{d}{dt} J^{u,v}_{j,K}(t) &\leq \int_R \left[ u (uw u_x v_x + uP_x * \left( \frac{1}{2} u_x^2 u + uu_x u_x + u^2 u \right) + uP_x * \left( \frac{1}{2} u_x^2 u_x \right) \right] \Psi_{i,K} \, dx \\
+ \int_R \left[ vP_x * \left( \frac{1}{2} u_x^2 u + uu_x u_x + u^2 u \right) + vP_x * \left( \frac{1}{2} u_x^2 u_x \right) \right] \Psi_{i,K} \, dx \\
- \dot{y}_i \int_R (uv + u_x v_x) \Psi_{i,K} \, dx \\
= \int_R \left[ u (uw u_x v_x + uP_x * (v(uv + u_x v_x)) + vP_x * (u(uv + u_x v_x))) \right] \Psi_{i,K} \, dx \\
+ \int_R \left[ - \frac{1}{2} u_x^2 v^2 + vP_x * \left( \frac{1}{2} u_x^2 v \right) + vP_x * \left( \frac{1}{2} u_x^2 v_x \right) \right] \Psi_{i,K} \, dx \\
+ \int_R \left[ - \frac{1}{2} u_x^2 v^2 + uP_x * \left( \frac{1}{2} u_x^2 v \right) + uP_x * \left( \frac{1}{2} u_x^2 v_x \right) \right] \Psi_{i,K} \, dx \\
- \dot{y}_i \int_R (uv + u_x v_x) \Psi_{i,K} \, dx \\
\leq \tilde{I}_{2,1} + \tilde{I}_{2,2} + \tilde{I}_{2,3} - \dot{y}_i \int_R (uv + u_x v_x) \Psi_{i,K} \, dx.
\end{align*}
In the same way as the proof of the first inequality, we obtain
\[
\tilde{I}_{2.1} \leq \frac{c_1}{20} \int_{\mathbb{R}} (uv + u_x v_x) \Psi^{2}_{i,K} dx + \frac{C}{K} \|u_0\|_{\dot{H}^1(\mathbb{R})}^2 \|v_0\|_{\dot{H}^1(\mathbb{R})}^2 e^{-\frac{1}{4}(\sigma_0 t + \frac{1}{4})}.
\]

For the term \(\tilde{I}_{2.2}\), we note
\[
|u_x| \leq u \quad \text{and} \quad |v_x| \leq v.
\]
Thus,
\[
(u + u_x)(v + v_x) \geq 0 \quad \text{and} \quad (u - u_x)(v - v_x) \geq 0,
\]
which is
\[
|uv_x + u_x v| \leq |uv + u_x v_x|.
\]

Thanks to this estimate (4.23), we have
\[
\tilde{I}_{2.2} = \frac{1}{2} \int_{\mathbb{R}} \left( (P_{xx} - P) \ast u^2 v + P \ast (u_x^2 v) + P \ast (u_x v_x) \right) v \Psi^{2}_{i,K} dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}} \left( P \ast ((u_x^2 - u^2) v) + P \ast (2u u_x v + u^2 v_x + u_x^2 v) \right) v \Psi^{2}_{i,K} dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}} \left( P \ast ((u + u_x v_x) u_x) + P \ast ((u_x + u_x v) u) \right) v \Psi^{2}_{i,K} dx
\]
\[
\leq \frac{1}{2} \int_{\mathbb{R}} \left( P \ast ((u + u_x v_x) u_x) + P \ast ((u + u_x v_x) u) \right) v \Psi^{2}_{i,K} dx.
\]

In the same way as the proof of the first inequality, we find
\[
\tilde{I}_{2.2} \leq \frac{c_1}{20} \int_{\mathbb{R}} (uv + u_x v_x) \Psi^{2}_{i,K} dx + \frac{C}{K} \|u_0\|_{\dot{H}^1(\mathbb{R})}^2 \|v_0\|_{\dot{H}^1(\mathbb{R})}^2 e^{-\frac{1}{4}(\sigma_0 t + \frac{1}{4})}.
\]

Similarly, we have
\[
\tilde{I}_{2.3} \leq \frac{c_1}{20} \int_{\mathbb{R}} (uv + u_x v_x) \Psi^{2}_{i,K} dx + \frac{C}{K} \|u_0\|_{\dot{H}^1(\mathbb{R})}^2 \|v_0\|_{\dot{H}^1(\mathbb{R})}^2 e^{-\frac{1}{4}(\sigma_0 t + \frac{1}{4})}.
\]

Therefore, for \(0 < \alpha < \alpha_0\) and \(L > L_0 > 0\) with \(\alpha_0 \ll 1\) and \(L_0 \gg 1\), it holds that
\[
\frac{d}{dt} J^{u,v}_{j,K}(t) \leq \frac{C}{K} \|u_0\|_{\dot{H}^1(\mathbb{R})}^2 \|v_0\|_{\dot{H}^1(\mathbb{R})}^2 e^{-\frac{1}{4}(\sigma_0 t + \frac{1}{4})}.
\]

Integrating between 0 and \(t\), we obtain
\[
J^{u,v}_{j,K}(t) - J^{u,v}_{j,K}(0) \leq O(e^{-\frac{\alpha_0 t}{K}}).
\]

This completes the proof of this proposition. \(\square\)

### 4.3 Localized estimates and global identities

We define the function \(\Phi_1 = \Phi_1(t, x)\) by \(\Phi_1 = 1 - \Psi_{2,K} = 1 - \Psi_{K}(\cdot - y_2(t))\), \(\Phi_N = \Psi_{N,K} - \Psi_{K}(\cdot - y_N(t))\) and for \(i = 2, \ldots, N - 1,\)
\[
\Phi_i = \Psi_{i,K} - \Psi_{i+1,K} = \Psi_{K}(\cdot - y_i(t)) - \Psi_{K}(\cdot - y_{i+1}(t)),
\]
where \(\Psi_{K}\) and \(y_i\) are defined in the previous subsection. It is easy to check that \(\sum_{i=1}^{N} \Phi_{i,K} \equiv 1\). We take \(L > 0\) and \(L/K > 0\) large enough so that \(\Phi_i\) satisfies
\[
|1 - \Phi_{i,K}| \leq 4e^{-\frac{i}{4K}} \quad \text{on} \quad \left[\tilde{x}_i - \frac{L}{4}, \tilde{x}_i + \frac{L}{4}\right]
\]

(4.24)
and
\[ |\Phi_i| \leq 4e^{-\frac{3}{2}x} \quad \text{on} \quad \left[ \tilde{x}_j - \frac{L}{4}, \tilde{x}_j + \frac{L}{4} \right] \quad \text{whenever} \quad j \neq i. \]  

(4.25)

We now use the following localized versions of \( E_u, E_v, H \) and \( F \) defined for \( i \in \{1, \ldots, N\} \):

\[
\begin{align*}
E_{ui}(u) &= \int_{\mathbb{R}} (u^2 + u_x^2) \Phi_i \, dx, \\
E_{vi}(v) &= \int_{\mathbb{R}} (v^2 + v_x^2) \Phi_i \, dx, \\
H_i(u, v) &= \int_{\mathbb{R}} (uv + u_x v_x) \Phi_i \, dx, \\
F_i(u, v) &= \int_{\mathbb{R}} (u^2 v^2 + \frac{1}{3} u^2 v_x^2 + \frac{1}{3} v^2 u_x^2 + \frac{4}{3} u v u_x v_x - \frac{1}{3} u_x^2 v_x^2) \Phi_i \, dx.
\end{align*}
\]

(4.26)

Please note that henceforth we take \( K = \sqrt{L/8} \).

The following lemma gives a localized version of (3.8). Note that the functionals \( H_i \) and \( F_i \) do not depend on time in the statement below since we fix \( \tilde{x}_i \).

**Lemma 4.7.** Given \( N \) real numbers \( \tilde{x}_1 < \cdots < \tilde{x}_N \) with \( \tilde{x}_i - \tilde{x}_{i-1} \geq 2L/3 \), we define \( J_i \) as in (4.6) and assume that for \( i = 1, \ldots, N \), there exists an \( x_i \in J_i \) such that \( |x_i - \tilde{x}_i| \leq L/12 \) and \( u(x_i)v(x_i) \) is defined for \( i \in {1, \ldots, N} \).

(4.27)

**Proof.** Let \( i \in \{1, \ldots, N\} \) be fixed. We introduce the functions \( g_u, g_v \) and \( h \) defined by

\[
\begin{align*}
g_u(x) &= \begin{cases} 
  u - u_x, & x < x_i, \\
  u + u_x, & x > x_i,
\end{cases} \\
g_v(x) &= \begin{cases} 
  v - v_x, & x < x_i, \\
  v + v_x, & x > x_i
\end{cases}
\end{align*}
\]

and

\[
h(x) = \begin{cases} 
  uv - \frac{1}{3} (uv)_x - \frac{1}{3} u_x v_x, & x < x_i, \\
  uv + \frac{1}{3} (uv)_x + \frac{1}{3} u_x v_x, & x > x_i
\end{cases}
\]

Integrating by parts, we compute

\[
\begin{align*}
\int_{\mathbb{R}} h(x) g_u(x) g_v(x) \Phi_i \, dx &= \int_{-\infty}^{x_i} \left( uv - \frac{1}{3} (uv)_x - \frac{1}{3} u_x v_x \right)(uv - (uv)_x + u_x v_x) \Phi_i \, dx \\
&\quad + \int_{x_i}^{\infty} \left( uv + \frac{1}{3} (uv)_x - \frac{1}{3} u_x v_x \right)(uv + (uv)_x + u_x v_x) \Phi_i \, dx \\
&= \int_{-\infty}^{x_i} \left( u^2 v^2 + \frac{1}{3} u^2 v_x^2 + \frac{1}{3} v^2 u_x^2 + \frac{4}{3} u v u_x v_x - \frac{1}{3} u_x^2 v_x^2 \right) dx - \frac{4}{3} M_i^2 \Phi_i(x_i) \\
&\quad + \frac{2}{3} \int_{x_i}^{\infty} u^2 v^2 \Phi_i \, dx - \frac{2}{3} \int_{-\infty}^{x_i} u^2 v^2 \Phi_i \, dx.
\end{align*}
\]

(4.28)

Recall that we take \( K = \sqrt{L/8} \) and thus \( |\Phi_i| \leq C/K = O(L^{-1/2}) \). Moreover, since \( |x_i - \tilde{x}_i| \leq L/12 \), it follows from (4.24) that \( \Phi_i(x_i) = 1 + O(e^{-\sqrt{T}}) \) and thus

\[
\int_{\mathbb{R}} h(x) g_u(x) g_v(x) \Phi_i \, dx = F_i(u, v) - \frac{4}{3} M_i^2 + \| u_0 \|^2_{H^1(\mathbb{R})} \| v_0 \|^2_{H^1(\mathbb{R})} O(L^{-1/2}) + O(e^{-\sqrt{T}}).
\]

(4.29)
On the other hand, we firstly claim that \( x_1 \) is the maximal point of the function \( u(x,t)v(x,t) \) on \( (-\infty,y_2(t)+aL) \), \( x_N \) is the maximal point of the function \( u(x,t)v(x,t) \) on \([y_N(t)-aL,\infty)\) and \( x_i \) is the maximal point of the function \( u(x,t)v(x,t) \) on \([y_i(t)-aL,y_{i+1}(t)+aL]\), where \( a \) is a constant chosen later and \( i \in \{2,\ldots,N-1\} \).

We need to show that \( x_1 \) is the maximal point of the function \( u(x,t)v(x,t) \) on \([y_i(t)-aL,y_{i+1}(t)+aL]\), where \( i \in \{2,\ldots,N-1\} \). If \( x \in [y_i(t)-aL,y_{i+1}(t)+aL] \), then

\[
 u(x,t)v(x,t) \leq c_i e^{-\frac{1}{2}L} + c_i e^{\frac{1}{2}L} + O(\sqrt{t}) + O(e^{-\frac{1}{2}}).
\]

Choosing \( a \) with

\[
c_i e^{-\frac{1}{2}L} + c_i e^{\frac{1}{2}L} + O(\sqrt{t}) + O(e^{-\frac{1}{2}}) \leq c_i - O(\sqrt{t}) - O(e^{-\frac{1}{2}}),
\]

and using the same estimate as above, we can prove the similar conclusions for \( x_1 \) and \( x_N \).

Next, we define the intervals \( \bar{J}_1 = (-\infty,y_2(t)+aL) \) and \( \bar{J}_N = [y_N(t)-aL,\infty) \) and for \( i = 2,\ldots,N-1 \), \( \bar{J}_i = [y_i(t)-aL,y_{i+1}(t)+aL] \), where \( a \) is chosen above. It holds that

\[
\int_{\bar{J}_i} h(x)g_u(x)g_v(x)\Phi_i dx = \int_{\bar{J}_i} h(x)g_u(x)g_v(x)\Phi_i dx + \int_{\bar{J}_i} h(x)g_u(x)g_v(x)\Phi_i dx 
\leq \frac{4}{3} M_i \int_{\bar{J}_i} g_u(x)g_v(x)\Phi_i dx
\]

where we have used the fact that \( h(x) \leq 4u(x)v(x)/3 \leq 4M_i/3 \) on the \( \bar{J}_i \). One chooses \( L \) large enough satisfying \( aL > 10\sqrt{L} \), which leads to

\[
\int_{\bar{J}_i} h(x)g_u(x)g_v(x)\Phi_i dx \leq C\|u_0\|_{H^1(\mathbb{R})}^2\|u_0\|_{H^1(\mathbb{R})} e^{-\tilde{a}L} \leq O(L^{-1}),
\]

where \( C \) and \( \tilde{a} \) are constants. Therefore,

\[
\frac{4}{3} M_i^2 - \frac{4}{3} M_i H_i(u,v) + F_i(u,v) \leq O(L^{-\frac{1}{2}}), \quad i \in \{1,\ldots,N\}.
\]

This completes the proof. \( \square \)

**Lemma 4.8.** For any \( Z \in \mathbb{R}^N \) satisfying \( |z_i-z_{i-1}| \geq L/2 \) and any \( (u,v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \), it holds that

\[
E_u(u) - \sum_{i=1}^N E_u(\varphi_{e_i}) = \|u - R_Z\|_{H^1}^2 + 4 \sum_{i=1}^N a_i(u(z_i) - a_i) + O(e^{-\frac{1}{2}}),
\]

\[
E_v(v) - \sum_{i=1}^N E_v(\psi_{e_i}) = \|v - S_Z\|_{H^1}^2 + 4 \sum_{i=1}^N b_i(v(z_i) - b_i) + O(e^{-\frac{1}{2}}).
\]

**Proof.** Using the relation between \( \varphi \) and its derivative, and integrating by parts, we get

\[
E_u(u - R_Z) = E_u(u) + E_u(R_Z) - 2 \sum_{i=1}^N \int_{\bar{J}_i} \left( u \varphi_{e_i}(-z_i) + u_x \partial_x \varphi_{e_i}(-z_i) \right) dx
\]

\[
= E_u(u) + E_u(R_Z) - 2 \sum_{i=1}^N \int_{\bar{J}_i} u \varphi_{e_i}(-z_i) dx + 2 \sum_{i=1}^N \int_{z_i}^\infty u_x \varphi_{e_i}(-z_i) dx
\]

\[
- 2 \sum_{i=1}^N \int_{-\infty}^{z_i} u_x \varphi_{e_i}(-z_i) dx
\]

\[
= E_u(u) + E_u(R_Z) - 4 \sum_{i=1}^N a_i u(z_i).
\]
On the other hand, since \(|z_i - z_{i-1}| \geq L/2\), it is easy to check that
\[
E(RZ) = \sum_{i=1}^{N} E(\varphi_{ci}) + O(e^{-\frac{7}{8}L}).
\]
Combining these two identities, we see that the desired result follows; similarly, we obtain the second equality of this lemma.

\[\square\]

4.4 End of the proof of Theorem 1.3

Before we give the final proof, we need to prove the following lemmas.

**Lemma 4.9.** Assume \(\|u_0 - R_{Z_0}\|_{H^1} + \|v_0 - S_{Z_0}\|_{H^1} < \epsilon\). Then for any \(i = \{1, \ldots, N\}\), we have
\[
\begin{align*}
|H_i(u_0, v_0) - H_i(\varphi_{ci}, \psi_{ci})| &< O(\epsilon) + O(e^{-\sqrt{L}}), \\
|E_{ui}(u_0) - E_{ui}(\varphi_{ci})| &< O(\epsilon) + O(e^{-\sqrt{L}}), \\
|E_{vi}(v_0) - E_{vi}(\psi_{ci})| &< O(\epsilon) + O(e^{-\sqrt{L}}).
\end{align*}
\]

**Proof.** Define the interval \(J_i = [z_i^0 - L/4, z_i^0 + L/4]\). Using (4.24), (4.25) and the exponential decay of \(\varphi_{ci}'s\) and \(\psi_{ci}'s\), we have
\[
|H_i(\varphi_{ci}, \psi_{ci}) - H_i(u_0, v_0)| = \left| \int_{\mathbb{R}} (\varphi_{ci} \psi_{ci} + \varphi_{ci}' \psi_{ci}') dx - \int_{\mathbb{R}} (u_0 v_0 + u_0 v_0) \Phi_i dx \right|
\leq \left| \int_{J_i} (\varphi_{ci} \psi_{ci} + \varphi_{ci}' \psi_{ci}') dx - \int_{J_i} (u_0 v_0 + u_0 v_0) dx \right| + O(e^{-\sqrt{L}}) + O(e^{-\frac{7}{8}L}).
\]

Since \(\|u_0 - R_{Z}\|_{H^1} + \|v_0 - S_{Z}\|_{H^1} < \epsilon\), we obtain
\[
\begin{align*}
\|u_0 - \varphi_{ci}\|_{H^1(J_i)} + \|v_0 - \psi_{ci}\|_{H^1(J_i)} &\leq \|u_0 - R_{Z}\|_{H^1(J_i)} + \|v_0 - S_{Z}\|_{H^1(J_i)} + \sum_{i \neq j} (\|\varphi_{ci}\|_{H^1(J_i)} + \|\psi_{ci}\|_{H^1(J_i)}) \\
&\leq \|u_0 - R_{Z}\|_{H^1(R)} + \|v_0 - S_{Z}\|_{H^1(R)} + O(e^{-\frac{7}{8}L}) < \epsilon + O(e^{-\frac{7}{8}L}).
\end{align*}
\]

Using the same way in proving the third inequality in Lemma 3.5, we obtain
\[
|H_i(\varphi_{ci}, \psi_{ci}) - H_i(u_0, v_0)| < O(\epsilon) + O(e^{-\sqrt{L}}).
\]

Similarly,
\[
|E_{ui}(\varphi_{ci}) - E_{ui}(u_0)| < O(\epsilon) + O(e^{-\sqrt{L}}) \quad \text{and} \quad |E_{vi}(\psi_{ci}) - E_{vi}(v_0)| < O(\epsilon) + O(e^{-\sqrt{L}}).
\]

This completes the proof.

**Lemma 4.10.** Assume that \(u \in H^s, v \in H^s, s > 3\) and \(u \geq 0, v \geq 0\). Let
\[
M_i = u(x_i) v(x_i) = \max_{x \in J_i} u(x) v(x),
\]
and set \(\alpha_0 = A(e_0^{1/4} + L_0^{-1/8})\). Then we have
\[
|M_i - c_i| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^{\frac{1}{2}}), \quad i = 1, \ldots, N.
\]

**Proof.** Define the second-order polynomials \(\hat{P}^i\) and \(\tilde{P}^i\) respectively by
\[
\hat{P}^i(y) = y^2 - H_i(u, v)y + \frac{3}{4} F_i(u, v) \quad \text{and} \quad \tilde{P}^i(y) = y^2 - H_i(u_0, v_0)y + \frac{3}{4} F_i(u_0, v_0).
\]
For the peaked solution, $H(\varphi_{c_i}, \psi_{c_i}) = 2c_i$ and $F(\varphi_{c_i}, \psi_{c_i}) = 4c_i^2/3$, and the above polynomial becomes
\[
\hat{P}_0(y) = y^2 - 2c_i y + c_i^2 = (y - c_i)^2.
\]

Since
\[
\hat{P}_i(y) = \hat{P}_0(y) + (H(\varphi_{c_i}, \psi_{c_i}) - H_i(u_0, v_0))y + (H_i(u_0, v_0) - H_i(u, v))y
\]
\[
+ \frac{3}{4} (F_i(u_0, v_0) - F(\varphi_{c_i}, \psi_{c_i})) + (F_i(u, v) - F(u_0, v_0)),
\]
it follows that
\[
\sum_{i=1}^N \hat{P}_i(M_i) = \sum_{i=1}^N \hat{P}_0(M_i) + \sum_{i=1}^N M_i (H(\varphi_{c_i}, \psi_{c_i}) - H_i(u_0, v_0))
\]
\[
+ \frac{4}{3} \left( F(u_0, v_0) - \sum_{i=1}^N F(\varphi_{c_i}, \psi_{c_i}) \right) + \sum_{i=1}^N M_i (H_i(u_0, v_0) - H_i(u, v)) \lesssim O(L^{-\frac{1}{4}}).
\]

Using the Abel transformation of the last term and the monotonicity property and combining Lemma 4.9, we thus get
\[
\sum_{i=1}^N (M_i - c_i)^2 \lesssim \sum_{i=1}^N M_i |H(\varphi_{c_i}, \psi_{c_i}) - H_i(u_0, v_0)| + \frac{3}{4} |F(u_0, v_0) - F(R, S)| + O(L^{-\frac{1}{4}}) + O(\epsilon)
\]
\[
\lesssim O(L^{-\frac{1}{2}}) + O(\epsilon),
\]
which completes the proof of the lemma.

**Lemma 4.11.** Assume $|M_i - c_i| \lesssim O(L^{-1/4}) + O(\epsilon^{1/2})$, $i = 1, \ldots, N$. Then for any $i \in \{1, \ldots, N\}$, we have
\[
|u(x_i) - a_i| \lesssim O(L^{-1/4}) + O(\epsilon^{1/2}) \quad \text{and} \quad |v(x_i) - b_i| \lesssim O(L^{-1/4}) + O(\epsilon^{1/2}).
\]

Note the monotonicity property that $J^u_{x_i}(t)$ is close to $\|u(t)\|_{H^1(x > y_i(t))}$ and thus measures the energy on the right of the $(j - 1)$-th bump of $u$ and $J^v_{x_i}(t)$ is close to $\|v(t)\|_{H^1(x > y_i(t))}$ and thus measures the energy on the right of the $(j - 1)$-th bump of $v$. We thus use the induction argument to prove this lemma.

**Proof of Lemma 4.11. Step 1.** Define
\[
g_{uN}(x) = \begin{cases} 
\frac{u(x) - u_N(x)}{x < x_N}, \\
\frac{u(x) + u_N(x)}{x > x_N}.
\end{cases}
\]

Then we have
\[
0 \lesssim \int_R g_{uN}^2(x) \Phi_N(x) dx
\]
\[
= \int_R (u^2 + u_N^2) \Phi_N(x) dx - 2u^2(x_N) \Phi_N(x_N)
\]
\[
+ \int_{-\infty}^{x_N} u^2 \Phi'_N dx - \int_{x_N}^{+\infty} u^2 \Phi'_N dx.
\]

Using $|x_i - \tilde{x}_i| \lesssim L/12$, we see that from (4.24), $\Phi_i(x_i) = 1 + O(\epsilon \sqrt{T})$, and thus,
\[
E_{uN}(u(t)) - 2u^2(x_N) + O(L^{-\frac{1}{2}}) \geq 0.
\]

Note that
\[
E_{uN}(u(t)) = J^u_{x_i}(t) \quad \text{and} \quad E_{uN}(u(t)) - E_{uN}(u(0)) \lesssim O(e^{-\sigma_N \sqrt{T}}).
\]
We thus get
\[ u^2(x_N) \leq O(L^{-\frac{1}{2}}) + O(\epsilon) + a_N^2. \]
Similarly,
\[ v^2(x_N) \leq O(L^{-\frac{1}{2}}) + O(\epsilon) + b_N^2. \]
Combining the identity \(|M_N - c_N| \leq O(L^{-1/4}) + O(\epsilon^{1/2})\), we arrive at
\[ |u(x_N) - a_N| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2) \quad \text{and} \quad |v(x_N) - b_N| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2). \]

**Step 2.** Assume that for any \(k \leq i < N\), we have
\[ |u(x_i) - a_i| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2) \quad \text{and} \quad |v(x_i) - b_i| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2). \]
We need to prove
\[ |u(x_{i-1}) - a_{i-1}| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2) \quad \text{and} \quad |v(x_{i-1}) - b_{i-1}| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2). \]

Firstly, we have
\[
\sum_{i=k-1}^{N} u^2(x_i) \leq \frac{1}{2} \sum_{i=k-1}^{N} E_{u}(u(t)) + O(L^{-\frac{1}{2}})
\leq \frac{1}{2} \left( J_{k-1,K}^{u} + 1 \right) + \frac{1}{2} \left( J_{k-1,K}^{u} + 1 \right) + \frac{1}{2} \sum_{i=k-1}^{N} E_{u}(\varphi_{c_i})
+ \frac{1}{2} \sum_{i=k-1}^{N} E_{u}(\varphi_{c_i}) + O(L^{-\frac{1}{2}}).
\]
It follows that
\[
\sum_{i=k-1}^{N} u^2(x_i) \leq \frac{1}{2} \sum_{i=k-1}^{N} E_{u}(\varphi_{c_i}) + O(L^{-\frac{1}{2}}) + O(\epsilon).
\]
Using the assumption, we find \(u^2(x_{k-1}) \leq a_{k-1}^2 + O(L^{-\frac{1}{2}}) + O(\sqrt{\epsilon})\). Similarly,
\[ v^2(x_{k-1}) \leq b_{k-1}^2 + O(L^{-\frac{1}{2}}) + O(\sqrt{\epsilon}). \]
Those together with the identity (4.10) imply that for any \(i = 1, \ldots, N\),
\[ |u(x_i) - a_i| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2) \quad \text{and} \quad |v(x_i) - b_i| \leq O(L^{-\frac{1}{2}}) + O(\epsilon^2). \]
This completes the proof. \(\square\)

**End of the proof of Theorem 1.3.** To conclude the proof, from (4.8), it thus suffices to prove that there exists a \(C > 0\), which does not depend on \(A\) such that
\[ |u(x_N) - a_N| \leq C(L^{-\frac{1}{2}} + \epsilon^2) \quad \text{and} \quad |v(x_N) - b_N| \leq C(L^{-\frac{1}{2}} + \epsilon^2), \]
which has been verified by (4.11). From (4.5) and (4.7), we know that for \(i = 2, \ldots, N\),
\[ x_i - x_{i-1} \geq \frac{2}{3} L. \]
This completes the proof of the theorem. \(\square\)

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