Finding Large Monochromatic Diameter Two Subgraphs

Tom Fowler*
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332

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Abstract
Given a coloring of the edges of the complete graph on \( n \) vertices in \( k \) colors, by considering the neighbors of an arbitrary vertex it follows that there is a monochromatic diameter two subgraph on at least \( 1 + (n - 1)/k \) vertices. We show that for \( k \geq 3 \) this is asymptotically best possible, and that for \( k = 2 \) there is always a monochromatic diameter two subgraph on at least \( \lceil \frac{3}{4}n \rceil \) vertices, which again, is best possible.

1 Introduction

Ramsey’s Theorem implies that for every positive integer \( l \), there is an integer \( n \) such that for every two-coloring of the edges of a complete graph on at least \( n \) vertices, there is a monochromatic clique (or equivalently, a monochromatic diameter one subgraph) containing at least \( l \) vertices. Many variations of this classical problem have been considered, and there are several books devoted exclusively to the subject of Ramsey Theory ([2, 3]). I consider another variation that was proposed by Paul Erdős [1], which apparently has not been studied previously: Given a complete graph \( K_n \) on \( n \) vertices whose edges are colored red or blue, what is the largest (in terms of number of vertices) monochromatic diameter two subgraph that \( K_n \) is guaranteed to contain? Here monochromatic means that all of the edges have the same color and diameter two means that for every pair of vertices

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In the subgraph there is a path in the subgraph joining \( x \) and \( y \) and containing at most two edges.

More generally, given a complete graph \( K_n \) on \( n \) vertices whose edges are partitioned into \( k \) different sets which we will hereafter refer to as color classes, what is the largest monochromatic diameter two subgraph that \( K_n \) is guaranteed to have. Erdős [1] noticed that it is always possible to find a monochromatic diameter two subgraph of size \( 1 + \frac{(n-1)}{k} \) by considering at any vertex \( v \) the largest monochromatic star centered at \( v \). Is it possible to do better? We will show that the answer to this question is yes if and only if \( k = 2 \).

In section two of this paper we show that when \( k = 2 \), any edge two-coloring of \( K_n \) has a monochromatic diameter two subgraph containing at least \( \left\lceil \frac{3}{4}n \right\rceil \) vertices but there is a two-coloring of the edges of \( K_n \) such that the largest monochromatic diameter two subgraph contains at most \( \left\lceil \frac{3}{4}n \right\rceil \) vertices. In section 3, for \( k = 3 \) colors and for an integer \( s \) divisible by 3, we construct three-colorings of the edges of \( K_n \) on \( n = 2sk + 1 \) vertices whose largest monochromatic diameter two subgraph has size \( 2s + 3 \). In section 4, for any \( k \geq 4 \) and any positive integer \( s \), we induce a partition of the edges of the complete graph on \( n = 2sk + 1 \) vertices into \( k \) color classes from a construction of \( k \) circle graphs, each of which has no diameter two subgraph containing more than \( 2s + 7 \) vertices. Below we provide upper bounds when \( n \) does not have the form \( 2ks + 1 \).

When \( 2ks + 1 < n < 2k(s + 1) + 1 \), say \( n = 2ks + l \), the following construction along with the constructions above show that there is a \( k \) coloring of the edges of \( K_n \) whose largest monochromatic diameter two subgraph contains at most \( 2s + 6 + l \) vertices. Given an edge \( k \)-colored \( K_{n-1} \) on vertices \( \{1, \ldots, n-1\} \), create an edge \( k \)-coloring on \( K_n \) by introducing a new vertex labeled \( n \), coloring the edge \( \{n-1, n\} \) with any one of the \( k \) colors, and for \( 1 \leq x \leq n-2 \), coloring the edge \( \{x, n\} \) the same color as the edge \( \{x, n-1\} \). Then the largest monochromatic diameter two subgraph in \( K_n \) contains at most one more vertex than the largest monochromatic diameter two subgraph in \( K_{n-1} \) since the vertex \( n \) replicates the vertex \( n-1 \). When \( k = 3 \) colors and \( 6s + 1 < n < 6(s + 3) + 1 \) this gives an upper bound of \( 2s + 9 \).
2 Two Colors

Given a graph \( G = (V,E) \), and a subset \( A \subset V \), the graph \( G - A \), will denote the subgraph of \( G \) with vertex set \( V - A \) and edge set consisting of every edge in \( E \) which has both endpoints in \( V - A \).

First we exhibit edge colorings of \( K_n \) that have no monochromatic diameter two subgraphs on more than \( \lceil \frac{3}{4}n \rceil \) vertices. Partition \( V(K_n) \) into 4 sets \( R_1, R_2, B_1, B_2 \), each of size \( \lfloor \frac{n}{4} \rfloor \) or \( \lceil \frac{n}{4} \rceil \). Color an edge red if it has one endpoint in \( R_1 \) and the other in \( B_1 \) or if it has one endpoint in \( B_1 \) and the other in \( B_2 \) or one endpoint in \( B_2 \) and the other in \( R_2 \); otherwise color it blue. It is easily verified that the shortest path with all red edges between any vertex in \( R_1 \) and any vertex in \( R_2 \) has three edges. Similarly the shortest path with all blue edges between any vertex in \( B_1 \) and any vertex in \( B_2 \) has three edges. Therefore any monochromatic diameter two subgraph \( H \) must be disjoint from one of \( R_1, R_2, B_1, B_2 \), and hence contains at most \( \lceil \frac{3}{4}n \rceil \) vertices.

Now we show that any edge two-coloring of \( K_n \) will render a monochromatic diameter two subgraph of order at least \( \lceil \frac{3}{4}n \rceil \). Let \( R \cup B \) be a partition of the edges of a graph \( G \) into red edges, (those in \( R \)) and blue edges, (those in \( B \)). A path \( P \) in \( G \) is said to be a red (blue) path if all of \( P \)'s edges are red (blue). A vertex \( x \in V(G) \) is said to be a red (blue) violated vertex in \( G \) if there is a \( y \in V(G) \) such that there is no red (blue) \( x - y \) path in \( G \) with two or fewer edges. Similarly, a pair \( x,y \) of vertices is said to be a red (blue) violated pair in \( G \) if \( x \neq y \) and there is no red (blue) path in \( G \) joining \( x \) and \( y \) containing two or fewer edges.

Lemma 1 Let the edges of \( K_n \) be colored arbitrarily red or blue. A vertex \( x \in V(K_n) \) cannot be both red violated in \( K_n \) and blue violated in \( K_n \). Moreover if \( w,x \) is a red (blue) violated pair in \( K_n \) and \( y \) is a blue (red) violated vertex in \( K_n \) then the edges \( \{w,y\} \) and \( \{x,y\} \) have different colors.

Proof. First, suppose to the contrary that there are vertices \( r,b \) such that \( x,r \) is a red violated pair in \( K_n \) and \( x,b \) is a blue violated pair in \( K_n \). It follows that \( \{x,r\} \in B \) and \( \{x,b\} \in R \). If \( \{r,b\} \in R \) then \( xbr \) is a red path containing two edges joining \( x \) and \( r \) contradicting the assumption that \( x,r \) is a red violated pair in \( K_n \). A similar contradiction arises if \( \{x,b\} \in B \). Now suppose that \( w,x \) is a red violated pair in \( K_n \) and that there exists a vertex \( z \) such that \( y,z \) is a blue violated pair in \( K_n \). Clearly, the edges \( \{w,y\} \) and \( \{x,y\} \) can’t both be red. Now suppose that they are both blue.
This forces both of the edges \( \{w, z\} \) and \( \{x, z\} \) to be red since \( y, z \) is a blue violated pair in \( K_n \) which contradicts the assumption that \( w, x \) is a red violated pair. By symmetry, the same proof holds if the colors red and blue are interchanged. \( \square \)

Lemma 1 imposes a great deal of structure on the edges joining vertices in red and blue violated pairs. The key idea of the following proof is to use this structure to partition the red and blue violated vertices in an economical way.

Theorem 1 Let \( R \cup B \) be an arbitrary partition of the edges of \( K_n \). Then there exists a monochromatic diameter two subgraph \( H \) with \( |V(H)| \geq \lceil \frac{3n}{4} \rceil \). Moreover, there exists edge two-colorings for which equality is attained.

Proof. If there exists a spanning monochromatic diameter two subgraph \( H \), then we are done. Therefore, we may assume that there is a red violated pair \( r_1, r_2 \) in \( K_n \) and a blue violated pair \( b_1, b_2 \) in \( K_n \). By Lemma 1, \( \{r_1, r_2\} \cap \{b_1, b_2\} = \emptyset \). By Lemma 1 we may assume that \( \{r_1, b_1\}, \{r_2, b_2\} \in R \) and \( \{r_1, b_2\}, \{r_2, b_1\} \in B \). For \( \delta = 1, 2 \) and \( i = 1, 2, \ldots \) define \( R_{\delta,i} \) to be the set of all \( x \) that are red violated vertices in \( K_n - (R_{\delta,1} \cup \ldots \cup R_{\delta,i-1}) \) and for which the edge \( \{x, b_\delta\} \) is red and \( B_{\delta,i} \) to be the set of all \( y \) that are blue violated vertices in \( K_n - (B_{\delta,1} \cup \ldots \cup B_{\delta,i-1}) \) and for which the edge \( \{y, r_\delta\} \) is red. For \( i = 1 \) interpret the union to be empty and note that \( r_\delta \in R_{\delta,1} \) and \( b_\delta \in B_{\delta,1} \) for \( \delta = 1, 2 \). Since the number of vertices of \( K_n \) is finite there is a largest integer \( l \) such that at least one of \( R_{1,l}, R_{2,l}, B_{1,l}, B_{2,l} \) is nonempty.

Refer to the sets \( R_{1,1}, \ldots, R_{2,l} \) as red sets and the sets \( B_{1,1}, \ldots, B_{2,l} \) as blue sets.

(1) For every \( \delta \in \{1, 2\} \), \( \{b_\delta\} \) is not in any blue set (red set).

Proof of (1). We will prove by induction on \( i \) that \( r_\delta \notin B_{1,i} \cup B_{2,i} \) for every \( \delta \in \{1, 2\} \) and for every positive integer \( i \). Suppose that \( i = 1 \). Then \( r_\delta \) in either \( B_{1,1} \) or in \( B_{2,1} \) implies that \( r_\delta \) is a blue violated vertex in \( K_n \). This however is impossible by Lemma 1 since \( r_\delta \) in \( R_{\delta,1} \) implies that \( r_\delta \) is a red violated vertex in \( K_n \). Now suppose that \( i \geq 2 \). By the induction hypothesis, \( \{r_1, r_2\} \) is disjoint from \( B_{\alpha,1} \cup \ldots \cup B_{\alpha,i-1} \) for every \( \alpha \in \{1, 2\} \). Therefore, \( r_\delta \) is a red violated vertex (with \( r_{3-\delta} \)) in \( K_n - (B_{\alpha,1} \cup \ldots \cup B_{\alpha,i-1}) \) and hence cannot also be a blue violated vertex in \( K_n - (B_{\alpha,1} \cup \ldots \cup B_{\alpha,i-1}) \). It follows that \( r_\delta \) is not in \( B_{\alpha,i} \) for any \( \alpha \in \{1, 2\} \). Similar reasoning applies to \( b_\delta \). This proves (1).
We must show that a)

Suppose the assertion is false and let $K$ both red and blue violated in $w, x$ is a blue violated pair in $R$ is in is we prove: \[ \boxed{\text{propositions are false:} K} \]

because $\delta$ true. In particular, for every \[ \boxed{\text{any red set and any blue set are disjoint. To show a) suppose} \] and for every positive integer \[ \boxed{\text{we prove:}} \]

(2) The sets $R_{1,1}, R_{2,1}, R_{1,2}, \ldots, R_{2,t}, B_{1,1}, B_{2,1}, B_{1,2}, \ldots, B_{2,t}$ are pairwise disjoint.

Proof of (2). We must show that a) $R_{\delta,i}, R_{\alpha,j}$ are disjoint unless $\delta = \alpha$ and $i = j$, b) $B_{\delta,i}, B_{\alpha,j}$ are disjoint unless $\delta = \alpha$ and $i = j$ and c) that any red set and any blue set are disjoint. To show a) suppose $\delta \neq \alpha$ and $x \in R_{\delta,i}$. Clearly \{x, $b_\delta$\} is a red edge. Proposition (1) shows that $b_1, b_2$ is a blue violated pair in $K_n - (R_{\delta,1} \cup \ldots \cup R_{\delta,i-1})$ which by Lemma 1 implies that \{x, $b_\alpha$\} is a blue edge which means that $x$ cannot be contained in $R_{\alpha,j}$ for any $j$. So suppose $\delta = \alpha$ and $i < j$. In this case the definition of $R_{\delta,j}$ shows that $R_{\delta,i}$ and $R_{\delta,j}$ are disjoint. A similar argument shows that b) is true. In particular, for every $\delta \in \{1,2\}$ the only red (blue) set that $r_\delta$ \{b_\delta\} is in is $R_{\delta,1} (B_{\delta,1})$. To show that any red set and any blue set are disjoint we prove:

(3) For every $\alpha \in \{1,2\}$ and for every positive integer $i$ the following two propositions are false:

A) There is a vertex $w \in R_{\alpha,i}$ and a vertex $x$ in the union of the blue sets such that $w, x$ is a red violated pair in $K_n - (R_{\alpha,1} \cup \ldots \cup R_{\alpha,i-1})$.

B) There is a vertex $w \in B_{\alpha,i}$ and a vertex $x$ in the union of the red sets such that $w, x$ is a blue violated pair in $K_n - (B_{\alpha,1} \cup \ldots \cup B_{\alpha,i-1})$.

Proof of (3). Suppose the assertion is false and let $i$ be the smallest integer such that either A) or B) is true. Without loss of generality suppose that $w$ is in $R_{\alpha,i}$, $x$ is in $B_{\delta,j}$, $w, x$ is a red violated pair in $K_n - (R_{\alpha,1} \cup \ldots \cup R_{\alpha,i-1})$ and there is a vertex $y$ such that $x, y$ is a blue violated pair in $K_n - (B_{\delta,1} \cup \ldots \cup B_{\delta,j-1})$.

Claim: $y$ is not in $R_{\alpha,1} \cup \ldots \cup R_{\alpha,i-1}$. If it is then there is an integer $s < i$ and a vertex $z$ such that $y \in R_{\alpha,s}$ and $y, z$ is a red violated pair in $K_n - (R_{\alpha,1} \cup \ldots \cup R_{\alpha,s-1})$. By the choice of $i$, $z$ is not in $B_{\delta,1} \cup \ldots \cup B_{\delta,j-1}$. Therefore, $y, z$ is a red violated pair in the graph $K_n - (R_{\alpha,1} \cup \ldots \cup R_{\alpha,s-1} \cup B_{\delta,1} \cup \ldots \cup B_{\delta,j-1})$. Also, $x$ is not in $R_{\alpha,1} \cup \ldots \cup R_{\alpha,i-1}$ because $w, x$ is a red violated pair in $K_n - (R_{\alpha,1} \cup \ldots \cup R_{\alpha,i-1})$. Therefore $x, y$ is a blue violated pair in $K_n - (R_{\alpha,1} \cup \ldots \cup R_{\alpha,s-1} \cup B_{\delta,1} \cup B_{\delta,j-1})$ so $y$ is both red and blue violated in $K_n - (R_{\alpha,1} \cup \ldots \cup R_{\alpha,s-1} \cup B_{\delta,1} \cup \ldots \cup B_{\delta,j-1})$.
which contradicts Lemma 1. This proves the claim that \( y \) is not in \( R_{a,1} \cup \ldots \cup R_{a,i-1} \).

If \( w \) is not in \( B_{\delta,1} \cup \ldots \cup B_{\delta,j-1} \) then \( x \) is both red violated (with \( w \)) and blue violated (with \( y \)) in \( K_n - (R_{a,1} \cup \ldots \cup R_{a,i-1} \cup B_{\delta,1} \cup \ldots \cup B_{\delta,j-1}) \) which contradicts Lemma 1. So we may assume that there is a positive integer \( t \) with \( t < j \) and a vertex \( u \) such that \( w \in B_{\delta,t} \) and \( w, u \) is a blue violated pair in \( K_n - (B_{\delta,1} \cup \ldots \cup B_{\delta,t-1}) \). Remembering that \( x \in B_{\delta,j} \) we have that the edges \( \{w, r_\delta\}, \{x, r_\delta\} \) are both colored red. Since \( w, x \) is a red violated pair in \( K_n - (R_{a,1} \cup \ldots \cup R_{a,i-1}) \) it must be that \( r_\delta \in R_{a,1} \cup \ldots \cup R_{a,i-1} \). This implies that \( \delta = \alpha \) since the only red set that \( r_\delta \) is in is \( R_{\delta,1} \). Claim: \( u \) is not in \( R_{a,1} \cup \ldots \cup R_{a,i-1} \). Proof: Suppose by way of contradiction that the claim is false. Proposition (1) guarantees that \( b_1, b_2 \) is a blue violated pair in the graph \( K_n - (R_{a,1} \cup \ldots \cup R_{a,m-1}) \) for every positive integer \( m \) and allows us to apply the hypothesis of Lemma 1 to \( u, b_1, b_2 \) and to \( w, b_1, b_2 \). Hence, \( u \) in \( R_{a,1} \cup \ldots \cup R_{a,i-1} \) implies that \( \{u, b_3 - \alpha\} \) is a blue edge. Also, \( w \in R_{a,i} \) implies that \( \{w, b_3 - \alpha\} \) is a blue edge. Thus, \( wb_{3-\alpha}u \) is a blue path containing two edges. Since \( w, u \) is a blue violated pair in \( K_n - (B_{\delta,1} \cup \ldots \cup B_{\delta,t-1}) \) it must be that \( b_{3-\alpha} \) is in \( B_{\delta,1} \cup \ldots \cup B_{\delta,t-1} \). This implies that \( 3 - \alpha = \delta \) because the only blue set that \( b_{3-\alpha} \) is in is \( B_{3-\alpha,1} \). This contradicts the fact that \( \delta = \alpha \) and proves the claim that \( u \) is not in \( R_{a,1} \cup \ldots \cup R_{a,i-1} \).

Also, \( x \) is not in \( B_{\delta,1} \cup \ldots \cup B_{\delta,t-1} \) because \( t < j \) and \( x \in B_{\delta,j} \). Hence \( w \) is both red violated (with \( x \)) and blue violated (with \( u \)) in the graph \( K_n - (R_{a,1} \cup \ldots \cup R_{a,i-1} \cup B_{\delta,1} \cup \ldots \cup B_{\delta,t-1}) \) and this contradicts Lemma 1. Thus (3) holds.

Now we will show that any red set and any blue set are disjoint. Suppose to the contrary that \( w \in R_{a,1} \cap B_{\delta,j} \) for some \( \alpha, \delta \in \{0, 1\} \) and for some positive integers \( i, j \). Let \( x, y \) be vertices such that \( w, x \) is a red violated pair in \( K_n - (R_{a,1} \cup \ldots \cup R_{a,i-1}) \) and \( w, y \) is a blue violated pair in \( K_n - (B_{\delta,1} \cup \ldots \cup B_{\delta,j-1}) \). By (3), \( x \notin B_{\delta,1} \cup \ldots \cup B_{\delta,j-1} \), \( y \notin R_{a,1} \cup \ldots \cup R_{a,i-1} \) and consequently \( w \) is both red and blue violated in \( K_n - (R_{a,1} \cup \ldots \cup R_{a,i-1} \cup B_{\delta,1} \cup \ldots \cup B_{\delta,j-1}) \) which contradicts Lemma 1. This proves (2).

From (2) it follows that at least one of the following four propositions holds:

(i) \( |R_{1,1} \cup R_{1,2} \cup \ldots \cup R_{1,l}| \leq \left\lfloor \frac{n}{4} \right\rfloor \).
(ii) \( |R_{2,1} \cup R_{2,2} \cup \ldots \cup R_{2,l}| \leq \left\lfloor \frac{n}{4} \right\rfloor \).
(iii) \( |B_{1,1} \cup B_{1,2} \cup \ldots \cup B_{1,l}| \leq \left\lfloor \frac{n}{4} \right\rfloor \).
(iv) \( |B_{2,1} \cup B_{2,2} \cup \ldots \cup B_{2,l}| \leq \left\lfloor \frac{n}{4} \right\rfloor \).

Without loss of generality assume proposition (i) holds. Let \( H' \) be the subgraph induced by the vertex set \( V(H') = V(K_n) - (R_{1,1} \cup \ldots \cup R_{1,l}) \). Claim:
For any two vertices $x$ and $y$ in $V(H')$, there exists a red path in $H'$ connecting $x$ to $y$ containing not more than two edges. Proof: Suppose by way of contradiction that $x, y$ is a red violated pair in $H'$. Since $b_1, b_2$ is a blue violated pair in $H'$, we can conclude from Lemma 1 (and if necessary, by relabelling) that the edge joining $x$ to $b_1$ is red. Now let an integer $s$ be the smallest integer such that $x, y$ is a red violated pair in $K_n - (R_{1,1} \cup \ldots \cup R_{1,s})$. Clearly such a $s$ exists because $x, y$ is a red violated pair in $K_n - (R_{1,1} \cup \ldots \cup R_{1,t})$. Now $x \in R_{1,s+1}$ by definition of $R_{1,s+1}$. Since $R_{1,t+1} = \emptyset$ we must have $s < l$ which implies that $x \in R_{1,1} \cup \ldots \cup R_{1,s+1} \subseteq R_{1,1} \cup \ldots \cup R_{1,l}$ and this contradicts $x \in V(H')$. This proves the claim that for any two vertices $x$ and $y$ in $H'$ there is a red path in $H'$ with not more than two edges joining $x$ to $y$. Define $H = (V(H'), E(H') \cap R)$. By this claim and the truth of proposition (i), $H$ is a monochromatic diameter two subgraph with $|V(H)| \geq \lceil \frac{3}{4} n \rceil$.

The coloring of the edges of $H_n$ described at the beginning of this paper shows that $\lceil \frac{3}{4} n \rceil$ is the best possible bound. This proves Theorem 1. Q.E.D.

3 Three Colors

Let $\{0, 1, \ldots, n - 1\}$ be a ground set. Addition will be modulo $n$ unless otherwise stated. For a positive integer $p$, and integers $a \leq b$ with $a, b \in \{0, \ldots, n - 1\}$ such that $a \equiv b (\text{mod } p)$ we will let $[a, b]_p$ denote the set of integers $x$ in $\{0, \ldots, n - 1\}$ such that $a \leq x \leq b$, and $x \equiv a (\text{mod } p)$. The notation $[a, b]$ will mean the set $[a, b]_1$ unless otherwise stated. For $a > b$ let $[a, b]_p$ be the set $[a, c]_p \cup [d, b]_p$ where $c \leq n - 1$ is the largest integer such that $a \equiv c (\text{mod } p)$ and $d \geq 0$ is the least integer such that $b \equiv d (\text{mod } n)$. If $p = 1$ the set $[a, b]$ will also be referred to as an interval. The quantity $|[a, b]|$, sometimes referred to as the length of the interval $[a, b]$, is defined to be $1 + b - a$ for $a \leq b$ and $|[a, n - 1]| + |[0, b]|$ otherwise. Given two sets $S, T \subseteq \{0, \ldots, n - 1\}$, $S^c$ will denote the set $\{0, \ldots, n - 1\} - S$, $S + T$ will denote the set of all elements of the form $s + t$ where $s \in S$ and $t \in T$ and $S - T$ will denote the set of all elements of the form $s - t$ where $s \in S$ and $t \in T$. If $x \in \{0, \ldots, n - 1\}$, $x + S$ will be short for $\{x\} + S$ and we will sometimes say that $x + S$ is a rotation of $S$. The notation $-S$ will be short for the set $\{0\} - S$. Given a graph $G = (V,E)$, the size of $G$ will equal $|V(G)|$. If $x \in V(G)$, the first neighborhood of $x$ will be the set of vertices $y$ for which $\{x, y\} \in E$, and the second neighborhood of $x$ will be the set of $z$ not in the first neighborhood of $x$ for which there is a $y$ in the first neighborhood of $x$, with $\{y, z\} \in E$. The set $N_x(G, 1)$ will denote the union of $\{x\}$ and the first
neighborhood of $x$, and the set $N_x(G, 2)$ will denote the union of $N_x(G, 1)$ and the second neighborhood of $x$. If $S \subseteq V(G)$ the subgraph of $G$ induced by $S$ is the subgraph consisting of the vertex set $S$ and every edge of $G$ that has both endpoints in $S$.

Given a positive integer $n \geq 2$ and a set $S \subseteq [1, \lfloor n/2 \rfloor]$, the circle graph on $n$ vertices determined by $S$ will denote the graph $C$ with vertex set $[0, n-1]$, having two vertices $x, y$ joined by an edge if and only if either $x - y \in S$ or $y - x \in S$. Note that $N_0(C, 2) = \{0\} \cup S \cup (-S) \cup (S + S) \cup (-S + S) \cup (-S - S)$. From this it follows that $x \in N_0(C, 2)$ if and only if $-x \in N_0(C, 2)$ and $x \in N^c_0(C, 2)$ if and only if $-x \in N^c_0(C, 2)$. If $C = ([0, n-1], E)$ is a circle graph, any rotation of $[0, n-1]$ defines a graph automorphism of $C$. Therefore, if $H$ is a largest size diameter two subgraph of $C$, we may assume $0 \in V(H)$ because if it is not a suitable rotation will show that there is a diameter two subgraph of $C$ with the same number of vertices as $H$ that contains 0. It also follows that $N_x(C, 2) = x + N_0(C, 2)$ and since $x + [0, n-1] = [0, n-1]$ that $N^c_x(C, 2) = x + N^c_0(C, 2)$. Defining $J = J(C)$ to be the circle graph on $n$ vertices determined by $N^c_0(C, 2) \cap [1, \lfloor n/2 \rfloor]$, we see that $\{x, y\} \in E(J)$ if and only if there is no $xy$ path in $C$ containing at most two edges. Thus $V(H)$ is an independent set in $J(C)$ (or equivalently $[0, n-1] - V(H)$ is a vertex cover in $J(C)$). Therefore lower bounds on the size of vertex covers of $J(C)$ imply upper bounds on the size of diameter two subgraphs of $C$. We rely heavily on the properties of circle graphs throughout the rest of the paper. In particular, we repeatedly use the fact that if $z \in N^c_0(C, 2)$ then for every $x \in [0, n-1]$, at most one of the two vertices $x$ and $x + z$ can be in any diameter two subgraph of $C$ because $x + z \in N^c_0(C, 2)$.

Let $n = 2sk + 1$ where $s$ is some positive integer and $k \geq 3$ is the number of colors used. Note that the circle graph determined by $[1, ks]$ is the complete graph on $n$ vertices. The strategy in our constructions will be to partition $[1, ks]$ into $k$ sets $N_1, \ldots, N_k$ and to prove for every $1 \leq i \leq k$ that the circle graph $C_i$ determined by $N_i$ has no diameter two subgraph on more than $2s + 7$ vertices.

Below are a series of Lemma’s used in proving the main result when $k = 3$ colors are used.

**Lemma 2** Let $s$ be a positive integer. The set $[2s + 1, 3s]$ determines a circle graph $C$ on $6s + 1$ vertices whose minimum cardinality vertex cover contains at least $4s$ vertices.

**Proof of Lemma 2.** Let $D$ be a minimum vertex cover of $C$. By the properties
of circle graphs we may assume $0 \notin D$. Since $N_0(C, 1) = \{0\} \cup [2s + 1, 4s]$, it follows that $[2s + 1, 4s] \subseteq D$. Also, $\{x, x + 4s\} \subseteq E(C)$ for every $x \in [1, 2s]$ and so at least one of $x$ and $x + 4s$ appears in $D$. This implies $|D| \geq 4s$ and proves Lemma 2.

**Lemma 3** Let $s$ be a multiple of 3. The set $[2s/3, s]$ determines a circle graph $C$ on $6s + 1$ vertices whose minimum vertex cover contains at least $4s$ vertices.

**Proof of Lemma 3.** Let $t = s/3$. We may assume that $0 \notin D$, where $D$ is a minimum cardinality vertex cover of $C$. Moreover, by using the fact that $C$ is a circle graph more carefully, we can by suitable rotation insure that there is a partition of $[0, n - 1]$ into maximal nonempty intervals such that:

1) $[0, n - 1] = [e_1, e_2] \cup [d_1, d_2] \cup \ldots \cup [e_{2l-1}, e_{2l}] \cup [d_{2l-1}, d_{2l}]$.

2) $0 = e_1 < e_2 + 1 = d_1 < d_2 + 1 = e_3 < \ldots < e_{2l-1} + 1 = d_{2l-1} \leq d_{2l} = n - 1$.

3) $[e_{2i-1}, e_{2i}] \cap D = \emptyset$ and $[d_{2i-1}, d_{2i}] \subseteq D$ for every $1 \leq i \leq l$.

We will denote the length $|[e_{2i-1}, e_{2i}]|$ of the interval $[e_{2i-1}, e_{2i}]$ by $x_i$ and the length of the interval $[d_{2i-1}, d_{2i}]$ by $y_i$.

Now $y_i \geq t + 1$ for every $1 \leq i \leq l$. To see this, let $[d_{2i-1}, d_{2i}]$ be an arbitrary maximal interval in $D$ and let $w$ be any vertex in this interval. Because $D$ is a minimum cardinality vertex cover there must be a vertex $z \in [0, n - 1] - D$ such that $\{w, z\} \subseteq E(C)$. But $z \in [0, n - 1] - D$ implies that $(z + [2t, 3t]) \cup (z - [2t, 3t]) \subseteq D$. Certainly $[d_{2i-1}, d_{2i}] \cap ((z + [2t, 3t]) \cup (z - [2t, 3t])) \neq \emptyset$. Since $[d_{2i-1}, d_{2i}]$ is a maximal interval in $D$, either $z + [2t, 3t] \subseteq [d_{2i-1}, d_{2i}]$ or $z - [2t, 3t] \subseteq [d_{2i-1}, d_{2i}]$ whence $y_i \geq t + 1$. Moreover, either $y_i \leq 2t - 2$ or $y_i \geq 3t$ for every $1 \leq i \leq l$. Otherwise, $2t - 1 \leq y_i \leq 3t - 1$ implies that $2t - 2 \leq d_{2i} - d_{2i-1} \leq 3t - 2$ which implies that $2t \leq (d_{2i-1} + 1) - (d_{2i-1} - 1) \leq 3t$ which is a contradiction since $d_{2i-1} - 1$ and $d_{2i} + 1$ are both in $[0, n - 1] - D$.

(1) We may assume $y_i \leq 2t - 2$ for every $1 \leq i \leq l$.

**Proof of (1).** First I will show that for every integer $q \in [0, n - 1]$ at least $3t$ vertices of the interval $[q, q + 5t - 1]$ must be in $D$. Fix $q$ and let $p, r$ be the least and greatest elements of $[q, q + 2t - 1]$ that are in $[0, n - 1] - D$. Now $p, r \in [0, n - 1] - D$ imply that $[p + 2t, p + 3t] \cup [r + 2t, r + 3t] \subseteq D$. Moreover, for every $1 \leq i \leq q - p - 1$, at least one of $p + i$ and $p + i + 3t$ is in $D$. It follows that every vertex in $[p + 2t, r + 3t]$ is either in $D$ or associated with a unique mate in $[p + 1, r - 1]$ that is in $D$. Moreover the definition of
Let $p$, $r$ implies that $[q, p - 1] \cup [r + 1, q + 2t - 1] \subset D$. Thus, at least $3t$ vertices of $[q, q + 5t - 1]$ are in $D$. Now suppose that some $y_i \geq 3t$. Let $[f_1, f_2]$ be a subinterval of $[d_{2i-1}, d_{2i}]$ with length exactly $3t$. By the result just proved, the three pairwise disjoint intervals $[f_2 + 1, f_2 + 5t]$, $[f_2 + 5t + 1, f_2 + 10t]$ and $[f_2 + 10t + 1, f_2 + 15t]$ each contribute at least $3t$ vertices to $D$. Since all three are disjoint from $[f_1, f_2]$, $|D| \geq 12t$ as desired. This proves (1).

Note that $y_i \leq 2t - 2$ implies $x_i + y_i + x_i + 1 \leq 2t$. To see this, note that for every integer $p$ where $y_i + 1 \leq p \leq x_i + y_i + x_i + 1 - 1$ there is an $x \in [e_{2i-1}, e_{2i}]$ and a $w \in [e_{2i+1}, e_{2i+2}]$ such that $p = w - x$. Since $y_i \leq 2t - 2$ for every $1 \leq i \leq l$, this implies that $2(x_1 + x_2 + \ldots + x_l) + (y_1 + \ldots + y_l) \leq 2lt$. Since $y_i \geq t + 1$ for every $1 \leq i \leq l$, we may assume $l \leq 11$ because $|D| = y_1 + \ldots + y_l$. Also, we have that $x_1 + \ldots + x_l \leq lt/2$. Since $y_1 + \ldots + y_l = 18t + 1 - (x_1 + \ldots + x_l)$, $|D| = y_1 + \ldots + y_l \geq (25t/2) + 1$ as desired. This proves Lemma 3.

**Lemma 4** $([5s/3] + 1, 8s/3])$ determines a circle graph $C$ on $6s + 1$ vertices whose largest diameter two subgraph contains at most $2s + 3$ vertices.

**Proof of Lemma 4.** Let $t = s/3$. It can be verified that $N_0^*(C, 2) = [8t + 1, 10t]$. Now $V(H) \subset N_0^*(C, 2) = \emptyset \cup N \cup S \cup \bar{N} \cup \bar{S}$ where $N = [5t + 1, 8t], S = [13t + 1, 18t], \bar{N} = [10t + 1, 13t]$ and $\bar{S} = [1, 5t]$. We claim that at most $s + 1$ vertices of the $8t$ vertices of $N \cup S$ can appear in $H$. To prove this, first assume that $N \cap V(H) = \emptyset$. The set $[13t + 1, 15t + 1]$ is not in the first neighborhood of $0$ nor is it in the first neighborhood of any vertex in $\bar{N}$. Since $0 \in V(H)$, this shows that $|N| + [[13t + 1, 15t + 1]] = 5t + 1$ vertices cannot appear in $H$. So suppose that $N \cap V(H) \neq \emptyset$ and let $x, y$ be the least and greatest elements of $N$ that are included in $H$. It suffices to show that at least $5t - 1$ of the vertices in $N \cup S$ don’t appear in $H$. Since $N_0^*(C, 2) = [8t + 1, 10t]$, including $x$ and $y$ in $H$ assures that $[x + 8t + 1, x + 10t] \cap V(H) = \emptyset = [y + 8t + 1, y + 10t] \cap V(H)$. Also, since $10t \in N_0^*(C, 2)$, at most half of the vertices in the set $[x + 1, y - 2t] \cup [x + 10t + 1, y + 8t]$ can appear in $V(H)$. Because $([5t + 1, x - 1] \cup [y + 1, 8t]) \cap V(H) = \emptyset$, the claim that at most $s + 1$ of the $8t$ vertices of $N \cup S$ is proved.

A similar claim holds for $\bar{N}$ and $\bar{S}$ whence $|V(H)| \leq 2s + 3$. This completes the proof of Lemma 4.

**Theorem 2** Let $s \geq 9$ be a multiple of $3$ and let $n = 6s + 1$. Then the complete graph on $n$ vertices can be partitioned into $3$ disjoint circle graphs $C_1$, $C_2$ and $C_3$ such that for each $i = 1, 2, 3$, the largest diameter two subgraph of $C_i$ contains at most $2s + 3$ vertices.
Proof. Let $N_1 = [1, s]$, $N_2 = [s + 1, 5s/3] \cup [(8s/3) + 1, 3s]$, and $N_3 = [(5s/3) + 1, (8s/3)]$ and let $C_i$ denote the circle graph determined by $N_i$ for $1 \leq i \leq 3$. Let $H_1$ be a largest size diameter two subgraph of $C_i$ $(i = 1, 2, 3)$ and let $s = 3t$.

It can be verified that $N_0^0(C_1, 2) = [2s + 1, 4s]$ so by Lemma 2, every vertex cover of $J(C_1)$ contains at least $4s$ vertices. Since $V(H_1)$ must be an independent set in $J(C_1)$ and complements of independent sets are vertex covers we have $|V(H_1)| \leq 2s + 1$.

As for $C_2$, note that $N_0^0(C_2, 1) = \{0\} \cup L \cup M \cup \bar{L}$ where $L = [3t + 1, 5t]$, $\bar{L} = [13t + 1, 15t]$ and $M = [8t + 1, 10t]$. Now $L + L = [6t + 2, 10t]$, $L + M = [11t + 2, 15t]$, $L + \bar{L} = [16t + 2, 18t] \cup [0, 2t - 1]$, $M + M = [16t + 2, 2t - 1]$, $M + \bar{L} = [3t + 1, 7t - 1]$ and $\bar{L} + \bar{L} = [8t + 1, 12t - 1]$. From this it can be verified that $N_0^0(C_2, 2) = [2t, 3t] \cup [15t + 1, 16t + 1]$. This shows that $J(C_2)$ is the circle graph of the hypothesis of Lemma 3. Applying the conclusion of Lemma 3 we get $|V(H_2)| \leq 2s + 1$.

Lemma 4 shows $|V(H_3)| \leq 2s + 3$ which completes the proof of Theorem 2.

4 Four or more colors

Theorem 3 For all positive integers $k \geq 4$ and $s$, the complete graph $G$ on $n = 2sk + 1$ vertices can be decomposed into $k$ circle graphs $C_1, \ldots, C_k$ each on $n$ vertices such that $E(C_1), \ldots, E(C_k)$ forms a partition of the edges of $G$ and such that for every $1 \leq i \leq k$, the largest diameter two subgraph of $C_i$ contains at most $2s + 7$ vertices.

Proof of Theorem 3. For $j = 1, 2$ define $C_j$ to be the circle graph on $2sk + 1$ vertices determined by $[j, 2s - 2 + j]$. For integers $j$ satisfying $2 \leq j \leq k - 1$ and $j \notin \{(2k/3) - 2, (2k/3) - 1\}$ define $C_{j+1}$ to be the circle graph on $n$ vertices determined by $[js + 1, (j + 1)s]$. When $2k/3$ is an integer define $C_{2k/3-1}$ to be the circle graph on $n$ vertices determined by $[(2k - 6)s/3 + 1, 2ks/3]$, and $C_{2k/3}$ to be the circle graph on $n$ vertices determined by $[(2k - 6)s/3 + 1, 2ks/3 - 1]$. It can be verified that $E(C_1), E(C_2), \ldots, E(C_k)$ forms a partition of the edges of $G$. The proof will be broken up into a number of claims. In many cases the proofs that $C_i$ has no diameter two subgraph on more than $2s + 7$ vertices will actually imply the stronger result that every vertex cover of $J(C_i)$ contains at least $(2k - 2)s - 6$ vertices.

First we will prove the Lemma that covers a majority of the possibilities.
Lemma 5 Let \( k \geq 5 \) and \( s \) be positive integers and suppose that \( j \) is an integer such that \( j \notin \{2k-3)/3, (2k-2)/3, (2k-1)/3, 2k/3\} \) and \( 2 \leq j \leq k-2 \). Then the circle graph \( D \) on \( n = 2sk + 1 \) vertices determined by \([js + 1, (j + 1)s]\) has no diameter two subgraph containing more than \( 2s + 5 \) vertices.

Proof of Lemma 5. Let \( H \) be a largest diameter two subgraph of \( D \). Note that we may assume contains 0. Let \( A = [1, s-1], B = [js + 1, (j + 1)s], C = [2js + 2, (2j + 2)s], \bar{C} = [(2k-2j-2)s + 1, (2k-2j)s - 1], \bar{B} = [(2k-j-1)s + 1, (2k-j)s] \) and \( A = [(2k-1)s + 2, 2ks] \). It can be verified that \( N_0(D, 2) = \{0\} \cup A \cup B \cup \bar{C} \cup \bar{B} \cup \bar{A} \). If \( X, Y \) are sets of integers we write \( X < Y \) to mean \( X \subset \text{min} \ Y \). Let \( B^- = [(j-1)s + 1, js] \) and \( B^+ = [(j+1)s + 1, (j+2)s] \). If \( 2 \leq j \leq (k/2) - 1 \) then \( A < B^- < B < B^+ < C < \bar{C} < \bar{B} < \bar{A} \). If \( j = (k-1)/2 \) then \( A < B^- < B < B^+ < C = \bar{C} + 1 < \bar{B} < \bar{A} \). If \( k/2 \leq j \leq (2k-4)/3 - 1 \) then \( A < B^- < B < B^+ < \bar{C} < C < \bar{B} < \bar{A} \). Finally, if \( (2k+1)/3 \leq j \leq k - 2 \) then \( A < C < B^- < B < B^+ < \bar{B} < C < \bar{A} \). In any case, \( B^- \cup B^+ \) is disjoint from \( N_0(D, 2) \). If \( V(H) \cap (B \cup B) = \emptyset \) then \( V(H) = \{0\} \) and we would be done. From the symmetry we may assume that \( B \cap V(H) \neq \emptyset \).

Let \( x = js + 1 + l_x \leq y = (j + 1)s - l_y \) where \( (0 \leq l_x, l_y \leq s - 1) \) be the least and greatest elements of \( V(H) \cap [js + 1, (j+1)s] \) if they exist and let \( \bar{y} = n - js - s + l_y \leq \bar{x} = n - js - 1 - l_x \) where \( (0 \leq l_y, l_x \leq s - 1) \) be the least and greatest elements of \( V(H) \cap [n - js - s, n - js - 1] \) if they exist. Let \( t = \lceil 2js + 2, 2js + 2s \rceil \cap [n - 2js - 2s, n - 2js - 2s] \).

First it will be shown that we may assume \( B \cap V(H) \neq \emptyset \neq B \cap V(H) \). Suppose that \( B \cap V(H) = \emptyset \). The conditions on \( j \) imply that \( \bar{C} \) has no element in \( N_0(D, 1) \). Also, no element of \([2ks - 2js - 2s + 1, 2ks - 2js - 1] - [2js + 2, (2j + 2)s] \) is in the first neighborhood in \( D \) of any vertex in \([js + 1 + l_x, (j+1)s - l_y]\). Therefore \( (\bar{C} - C) \cap V(H) = \emptyset \). In addition, \( B^- \cup B^+ \subset N_0(D, 2) \) implies \( (\lceil 2js + 2, (2j + 1)s - l_y \rceil \cap [n - js - s, 2js + 2s - l_x, (2j + 2)s]) \cap V(H) = \emptyset \) and \( (\lceil (2k-1)s + 2, 2ks - l_y \rceil \cap [1 + l_x, s - 1]) \cap V(H) = \emptyset \). Remembering the definition of \( x \) and \( y \) and that \( l_x + l_y \leq s - 1 \) we get that at least \( 6s - 4 - t \) vertices in \( N_0(D, 2) \) are not in \( V(H) \), so \( |V(H)| \leq 2s + 1 \) as desired.

Therefore, assume that \( B \cap V(H) \neq \emptyset \neq B \cap V(H) \). Then the following subsets of \( N_0(D, 2) \) are not in \( V(H) \) for the reason stated:

(i) \([js + 1, js + l_x] \cup [(j+1)s + 1 - l_y, (j+1)s] \cup [n - js - s, n - js - s - 1 + l_y] \cup [n - js - l_x, n - js - 1] : \) definition of \( x, y, \bar{x}, \bar{y} \).

(ii) \([2js + 2, (2j+1)s - l_y] : [(j-1)s, js] \subset N_0(D, 2) \) and \( y = (j+1)s - l_y \in \)
$V(H)$.

(iii) $[(2j+1)s + 2 + l_x, (2j + 2)s] \subset \bar{N}^e_0(D, 2)$ and $x = js + 1 + l_x \in V(H)$.

(iv) $[n - 2js - 2s, n - (2j + 1)s - 2 - l_x] \cup [n - (2j + 1)s + l_y, n - 2js - 2]$; analogous to (ii) and (iii).

(v) $[(2k - 1)s + 2, 2ks - \min\{l_y, l_x\}] ; \bar{x}, y \in V(H)$ and $[(j - 1)s, js] \cap [(j + 1)s + 1, (j + 2)s] \subset \bar{N}^e_0(D, 2)$.

(vi) $[1 + \min\{l_x, l_y\}, s - 1] ; \bar{y}, x \in V(H)$ and $[(j - 1)s, js] \cap [(j + 1)s + 1, (j + 2)s]$. Suppose first that $j = (k - 1)/2$. Then $[2js + 2, 2js + 2s] = [n - 2js - 2s, n - 2js - 2] + 1$ so conditions (ii),(iii) and (iv) imply that $[(k - 1)s + 1, ks - \min\{1 + l_x, l_y\}] \cup [ks + 1 + \min\{l_y, 1 + l_x\}, (k + 1)s]$ is excluded from $H$. This and conditions (i), (v) and (vi) imply that at least 4s - 4 vertices are excluded from $H$. Since $|N_0(D, 2)| = 6s - 1$ it follows that $V(H) \leq 2s + 3$ as desired. So assume $j \neq (k - 1)/2$. Consider the quantity $q = |(n - 2js - 2s) - (2js + s)| = |2k - 4j - 2s|$. Then $q$ is always a positive even multiple of $s$ and $q < (2k - 1)s$.

I claim that $\{q, q + 1\} \cap \bar{N}^e_0(D, 2) \neq \emptyset$. For suppose $q \in \bar{N}^e_0(D, 2)$. Then because $q$ is an even multiple of $s$ it must be that $q = (j + 1)s$ or $q = (2j + 2)s$ or $q = (2k - j)s$. Suppose first that $q = (2k - j)s = \max \bar{B}$. If $q + 1 \in \bar{N}^e_0(D, 2)$ then $q + 1 = (2k - j - 1)s + 1$ or $q + 1 = (2ks - 2js - 2s + 1) + 1$ or $q + 1 = js + 1$, since $\{js + 1, 2ks - 2js - 2s + 1, 2ks - js - s + 1\}$ is the set of elements of $\bar{N}^e_0(D, 2)$ that could possibly equal an even multiple of $s$ plus one. The case $q + 1 = (2k - j - 1)s + 1$ is clearly impossible. The other cases are ruled out because $\bar{B} > \bar{C}$ and $\bar{B} > \bar{B}$ for every $2 \leq j \leq k - 2$ imply $q + 1 = 1 + \max \bar{B} > \max \{2ks - 2js - 2s + 1, js + 1\}$. Thus $q + 1 \in \bar{N}^e_0(D, 2)$.

If $q = (j + 1)s$ then the restrictions on $j$ in the hypothesis of the Lemma force $j = (2k - 3)/5$ which implies that $q = (2k + 2)s/5 \neq (6k - 4)s/5 = (2ks - 2js - 2s + 1) - 1$. By similar considerations as above this means that $q + 1 \in \bar{N}^e_0(D, 2)$ as desired. If $q = (2j + 2)s$ then the restrictions on $j$ force $j = (k - 2)/3$ whence $q + 1 = 1 + (2k + 2)s/3 \neq 2ks - 2js - 2s + 1$ and $q + 1 \neq 2ks - js - s + 1$. Again, this shows that $q + 1 \in \bar{N}^e_0(D, 2)$ and proves the claim that $\{q, q + 1\} \cap \bar{N}^e_0(D, 2) \neq \emptyset$.

Let $q' = \min (\{q, q + 1\} \cap \bar{N}^e_0(D, 2))$. For every $z \in [0, 2sk]$ at most one of the vertices $z$ and $z + q'$ can appear in $H$. Therefore,

(vii) "Almost" half of $[(2j+1)s - \min\{l_y, l_x\} + 1, (2j + 1)s + \min\{l_x, l_y\}] \cup [n - (2j + 1)s - \min\{l_y, l_x\}, n - (2j + 1)s - 1 + \min\{l_y, l_x\}]$ is not in $V(H)$.
In (vii) we say "almost" half because if $q \notin N_0(D, 2)$ and $q+1 \in N_0(D, 2)$ then we lose 1 vertex in the count. Thus at least $\min\{l_x, l_y\}+\min\{l_x, l_y\} - 1$ vertices from the set in (vii) are not in $V(H)$. Using (i)-(vii) we see that at least $6s - 8 - t$ vertices are excluded from $H$ which implies $|V(H)| \leq 2s + 5$ as desired. This completes the proof of Lemma 5.

**Lemma 6** Suppose $s$ and $k \geq 4$ are given positive integers and let $n = 2sk+1$. Then the circle graph $D$ on $n$ vertices determined by $N = [1, 2s-1]_2$ has no diameter two subgraph $H$ on more than $2s + 3$ vertices.

**Proof of Lemma 6.** Throughout this proof we will abbreviate $[a, b]_2$ by $[a,b]$. Let $H$ be a largest diameter two subgraph of $D$. Without loss of generality $H$ contains 0. Now $N_0(D, 2) = [(2k-4)s + 3, 2ks - 1] \cup [0, 4s - 2] \cup [1, 2s-1] \cup [(2k-2)s+2, 2ks]$ so $2s+1, (2k-4)s-1, (2k-2)s \in N_0(D, 2).$ Thus at most one third of the vertices from each of the following two subsets of $N_0(D, 2)$ can appear in $H$:

1) $[1, 2s-3] \cup [(1, 2s-3) + 2s + 1] \cup [(1, 2s-3) + (2k-2)s]$

2) $[4, 2s] \cup [(4, 2s) + (2k-4)s - 1] \cup [(4, 2s) + (2k-2)s]$

Therefore at least $4s - 4$ vertices are excluded from $H$ so $|V(H)| \leq |N_0(D, 2)| - (4s - 4) = 2s + 3$ as desired. This proves Lemma 6.

**Lemma 7** Suppose $s$ and $k \geq 4$ are given positive integers and that $n = 2sk+1$. Then the circle graph $D$ on $n$ vertices determined by $N = [2, 2s]_2$ has no diameter two subgraph $H$ on more than $2s + 1$ vertices.

**Proof of Lemma 7.** Let $H$ be a largest diameter two subgraph of $D$. We may assume $H$ contains 0. It is easily verified that $N_0(D, 2) = [(2k-4)s + 1, 2ks - 1]_2 \cup [0, 4s]_2$. Note that $|N_0(D, 2)| = 4s + 1$. Since $(2k-4)s - 1 \in N_0(D, 2)$, at most half of the set $[2, 4s]_2 \cup [(2k-4)s + 1, 2ks - 1]_2$ is in $V(H)$. Thus, $H$ contains at most $2s + 1$ vertices. This proves Lemma 7.

**Lemma 8** Suppose $s$ and $k \geq 4$ are given positive integers and that $n = 2sk+1$. Then the circle graph $D$ on $n$ vertices determined by $N = [(k - 1)s + 1, ks]$ has no diameter two subgraph on more than $2s + 3$ vertices.

**Proof of Lemma 8.** Let $H$ be a largest monochromatic diameter two subgraph in $D$. Without loss of generality $H$ contains 0. It is easily verified that $N_0(D, 2) = [(2k-2)s + 2, 2sk] \cup [0, 2s - 1] \cup [(k-1)s + 1, (k+1)s]$. Since $(k-1)s$ and $2(k-1)s$ are elements of $N_0(D, 2)$ at most one third of the set $[2, 2s - 1] \cup [(k-1)s + [2, 2s - 1]) \cup [(2k-2)s + [2, 2s - 1])$ can appear in $V(H)$. Hence $|V(H)| \leq 2s + 3$. This proves Lemma 8.
Lemma 9 Suppose $k \geq 6$ is an integer divisible by 3 and $s$ is a positive integer. Then the circle graph $D = C_{2k/3+1}$ on $2sk + 1$ vertices determined by $[(2ks/3) + 1, ((2k + 3)s/3)$ contains no diameter two subgraph on more than $2s + 7$ vertices.

Proof of Lemma 9. Let $j = 2k/3$ and let $H$ be a largest diameter two subgraph of $D$. Without loss of generality $H$ contains 0. Clearly $N_0(D, 2) = [(3j - 1)s + 2, 3js] \cup [0, s - 1] \cup [(j - 2)s + 1, js - 1] \cup [js + 1, (j + 1)s] \cup [(2j - 1)s + 1, 2js] \cup [2js + 2, (2j + 2)s]$. Define for $1 \leq i \leq s - 1$, $v_{1,i} = i$, $v_{2,i} = (j - 2)s + i$, $v_{3,i} = js + i$, $v_{4,i} = 2js + 1 + i$ and $\bar{v}_{1,i} = (3j - 1)s + 1 + i$, $\bar{v}_{2,i} = (2j + 1)s + 1 + i$, $\bar{v}_{3,i} = (2j - 1)s + 1 + i$ and $\bar{v}_{4,i} = (j - 1)s + i$. Let $K_i$ be the subgraph of $J(D)$ induced by $\{v_{1,i}, v_{2,i}, v_{3,i}, v_{4,i}\}$ and let $\bar{K}_i$ be the subgraph of $J(D)$ induced by $\{\bar{v}_{1,i}, \bar{v}_{2,i}, \bar{v}_{3,i}, \bar{v}_{4,i}\}$.

It is easy to verify that $\{2s, (j - 2)s, js, (j + 2)s + 1, 2js + 1\} \subset N_0(D, 2)$. Therefore, for every $1 \leq i \leq s - 1$, $K_i$ ($\bar{K}_i$) consists of a complete graph on 4 vertices with the edge between $v_{3,i}$ and $v_{4,i}$ ($\bar{v}_{3,i}$ and $\bar{v}_{4,i}$) deleted. For every $1 \leq i \leq s - 1$ let $B_i = V(K_i) - V(H)$ and $\bar{B}_i = V(\bar{K}_i) - V(H)$. By definition of $J(D)$, $B_i$ ($\bar{B}_i$) is a vertex cover of $K_i$ ($\bar{K}_i$). Note that $|B_i| \geq 2$ ($|\bar{B}_i| \geq 2$) with equality if and only if $v_{3,i}, v_{4,i} \in V(H)$ ($\bar{v}_{3,i}, \bar{v}_{4,i} \in V(H)$).

This fact is used repeatedly in proving the following propositions.

Let $i' \in [1, s - 1]$. Then

(1). $v_{1,i'} \in V(H)$ implies
   (1a). $\bar{v}_{1,i} \in \bar{B}_i$ for $1 \leq i \leq i'$.
   (1b). $\bar{v}_{3,i} \in \bar{B}_i$ for $1 \leq i \leq i' - 1$.

(2). $v_{2,i'} \in V(H)$ implies
   (2a). $v_{4,i} \in B_i$ for $1 \leq i \leq s - 1$.
   (2b). $|B_i| \geq 3$ for $1 \leq i \leq s - 1$.
   (2c). $|B_{i'}| + |B_{i'}| + \ldots + |B_{s-1}| + |B_{s-1}| \geq 6(s - i')$.
   (2d). $v_{3,i} \in B_i$ for $1 \leq i \leq i'$.
   (2e). $\bar{v}_{4,i} \in \bar{B}_i$ for $i' \leq i \leq s - 1$.

(3). $\bar{v}_{2,i'} \in V(H)$ implies
   (3a). $\bar{v}_{4,i} \in \bar{B}_i$ for $1 \leq i \leq s - 1$.
   (3b). $|\bar{B}_i| \geq 3$ for $1 \leq i \leq s - 1$.
   (3c). $|B_{i'}| + |\bar{B}_{i'}| + \ldots + |B_{s'}| + |\bar{B}_{s'}| \geq 6i'$.
   (3d). $\bar{v}_{3,i} \in \bar{B}_i$ for $i' \leq i \leq s - 1$.
   (3e). $v_{4,i} \in B_i$ for $1 \leq i \leq i'$.

(4). $v_{4,i'} \in V(H)$ implies
   (4a). $v_{3,i'+1} \in B_{i'+1}$.
(4b). $\bar{v}_{4,i} \in B_i$ for $1 \leq i \leq i'$.

(5). $\bar{v}_{4,i'} \in V(H)$ implies

(5a). $\bar{v}_{3,i'-1} \in B_{i'-1}$.

Proof of (1)-(5) Following each assertion will be the integers in $N_0^c(D, 2)$ that prove the assertion. (1a),(2d),(2e),(3d) and (3e), $[s, 2s]$; (2b), $\lfloor (2j - 2)s + 1, (2j - 1)s \rfloor$; (2a) and (3a), $\lfloor (j + 1)s + 1, (j + 3)s \rfloor$; (4a) and (5a), $js$; (4b), $\lfloor (j + 1)s + 1, (j + 2)s \rfloor$.

Also, (2b) follows from (2a). By (2b) and (2e) we get (2c). The assertion (3c) follows similarly. This proves (1)-(5).

(6) Suppose that $v_{1,i}, v_{2,i} \in B_i$ for every $i \in [i', i'')$. Then $|B_{i'}| + |B_{i'+1}| + \ldots + |B_{i''}| \geq 3(1 + i'' - i') - 1$. Similarly, if $\bar{v}_{1,i}, \bar{v}_{2,i} \in \bar{B}_i$ for every $i \in [i', i'')$ then $|B_{i'}| + \ldots + |B_{i''}| \geq 3(1 + i'' - i') - 1$.

Proof of (6). The hypothesis implies that $V(H) \cap V(K_i) \subset \{v_{3,i}, v_{4,i}\}$ for $i' \leq i \leq i''$. Consider all of the indices $p_i$, $i' \leq p_1 < p_2 < \ldots < p_q \leq i''$ such that $|B_{p_i}| = 2$ for $i \in [1, q]$. If $q \leq 1$ then we are done. Otherwise, let $r \in [1, q - 1]$. Claim: there is an index $u_r$ such that $p_r < u_r < p_{r+1}$ and $|B_{u_r}| = 4$. Suppose not, then (4a) implies that $V(H) \cap \bigcup V(K_{p_r+1}) = v_{4,p_r+1}$ and $V(H) \cap V(K_{p_r+1}) = v_{3,p_r+1} - 1$. This implies that there is an index $u'$ with $p_r \leq u' < p_{r+1}$ such that $v_{4,u'}, v_{3,u'+1} \in V(H)$ which is a contradiction to (4a). Then for $r \in [1, q - 1]$ we have $|B_{p_r}| + |B_{u_r}| = 6$ and the result follows. A similar argument proves the other part of (6).

(7) We may assume that for every $1 \leq i \leq s - 1$, $v_{2,i} \in B_i$ and $\bar{v}_{2,i} \in \bar{B}_i$.

Proof of (7) By (2b) and (3b) we may assume there is an integer $i' \in [1, s-1]$ such that exactly one of the following two propositions is true:

a) $v_{2,i'} \in V(H)$ and $\{\bar{v}_{2,1}, \ldots, \bar{v}_{2,s-1}\} \cap V(H) = \emptyset$

b) $\bar{v}_{2,i'} \in V(H)$ and $\{v_{2,1}, \ldots, v_{2,s-1}\} \cap V(H) = \emptyset$.

Otherwise $|B_1| + |B_1| + \ldots + |B_{s-1}| \geq 6s - 6$ or $\{v_{2,1}, \bar{v}_{2,1}, \ldots, \bar{v}_{2,s-1}\} \cap V(H) = \emptyset$ and either way we would be done.

Assume first that a) is true and let $i'$ be the least integer in $[1, s-1]$ such that $v_{2,i'} \in V(H)$. By (2a),(2b) and (2c) we have that $v_{4,i} \in B_i$ for every $i \in [1, s-1]|B_i| \geq 3$ for $i \in [1, s-1]$, $|B_{i'}| + |B_{i'}| + \ldots + |B_{s-1}| \geq 6(s-i')$, and hence we may assume that $i' > 1$. By (2d) we have $v_{3,i} \in B_i$ for every $1 \leq i \leq i'$. Thus, for $1 \leq i < i'$, $B_i = \{v_{1,i}, v_{2,i}, v_{3,i}, v_{4,i}\}$ or $\bar{B}_i = \{v_{2,i}, v_{3,i}, v_{4,i}\}$. For indices $i$ in which the first possibility holds, $|B_i| + |\bar{B}_i| \geq 6$. So we may
assume there is an integer \( i'' \in [1, i' - 1] \) that is the greatest index such that the latter holds. By (1a) we know that \( \bar{v}_{1,i} \notin V(H) \) for every \( 1 \leq i \leq i'' \). From (6), this implies that \(|B_1| + |\bar{B}_1| + |B_2| + \ldots + |\bar{B}_{i''}| \geq 6i'' - 1\) so \(|B_1| + |\bar{B}_1| + \ldots + |\bar{B}_{s-1}| \geq 6s - 7\) whence \(|V(H)| \leq 2s + 4\). A small modification of this approach works if b) is assumed to hold. This proves (7).

By (7), we may assume that \( v_{2,i} \in B_i \) and \( \bar{v}_{2,i} \in \bar{B}_i \) for \( 1 \leq i \leq s - 1 \). Let \( l \) denote the greatest integer in \([1, s - 1]\) (if it exists) such that \( v_{3,l}, v_{4,l} \in V(H) \) and let \( \bar{l} \) denote the least integer in \([1, s - 1]\) (if it exists) such that \( \bar{v}_{3,l}, \bar{v}_{4,l} \in V(H) \). Assume first that they both exist. Note that by (4b), \( l < \bar{l} \). Let \( l' < l \) be the greatest integer (if it exists) such that \( v_{1,l'} \in V(H) \). By (1a), (1b) and (4b), we have \(|B_1| = 4\) for \( 1 \leq i \leq l' - 1 \) and consequently that \(|B_1| + |\bar{B}_1| + \ldots + |B_{l'-1}| + |\bar{B}_{l'-1}| \geq 6(l' - 1) - 6\). By (6) and the fact that \( l < \bar{l} \) we have that \(|B_1| + |\bar{B}_1| + \ldots + |B_{l-1}| + |\bar{B}_{l-1}| \geq 6(l' - 1) - 1\) and so \(|B_1| + |\bar{B}_1| + \ldots + |B_{s-1}| + |\bar{B}_{s-1}| \geq 6s - 7\). If \( l' \) did not exist (6) would lead to the same conclusion. Similar reasoning shows that \(|B_{l'+1}| + |\bar{B}_{l'+1}| + \ldots + |B_{s-1}| + |\bar{B}_{s-1}| \geq 6s - 10\) and so \(|V(H)| \leq 2s + 7\). If one or both of \( l \) or \( \bar{l} \) does not exist the above technique can be modified. This proves Lemma 9.

**Lemma 10** Let \( k \geq 6 \) be an integer divisible by 3, let \( s \) be a positive integer and suppose \( j = 2k/3 \). Then the circle graph \( D = C_{2k/3-1} \) on \( n = 2sk + 1 \) vertices determined by \( [(j - 2)s + 2, js]_2 \) has no diameter two subgraph containing more than \( 2s + 5 \) vertices.

**Proof of Lemma 10.** Let \( H \) be a largest diameter two subgraph of \( D \). Without loss of generality \( H \) contains 0. It is easily verified that \( N_0(D, 2) = [(3j - 2)s + 3, 2sk - 1]_2 \cup [0, 2s - 2]_2 \cup [(j - 2)s + 2, js]_2 \cup [js + 1, js + 4s - 3]_2 \cup [(2j - 4)s + 4, 2js]_2 \cup [2js + 1, (2j + 2)s - 1]_2 \). Let \( H \) be a largest diameter two subgraph of \( D \) containing 0 and for an integer \( i \) satisfying \( 1 \leq i \leq s - 1 \) define \( K_i \) to be the subgraph of \( J(D) \) induced by \( \{v_{1,i}, v_{2,i}, v_{3,i}, v_{4,i}\} \), where \( v_{1,i} = 2i, v_{2,i} = (j - 2)s + 2i, v_{3,i} = (2j - 4)s + 2i + 2, v_{4,i} = js - 1 + 2i, \) and let \( K_i \) be the subgraph of \( J(D) \) induced by \( \{\bar{v}_{1,i}, \bar{v}_{2,i}, \bar{v}_{3,i}, \bar{v}_{4,i}\} \), where \( \bar{v}_{1,i} = (3j - 2)s + 1 + 2i, \bar{v}_{2,i} = 2js + 1 + 2i, \bar{v}_{3,i} = (j + 2)s + 1 + 2i \) and \( \bar{v}_{4,i} = (2j - 2)s + 2 + 2i \). Since \( \{2s - 1, (j - 4)s + 3, (j - 2)s, js - 1, (2j - 4)s + 2\} \subset N_0(D, 2) \) the graph \( K_i \) (\( K_i \)) consists of a complete graph on 4 vertices with the edge \( \{v_{2,i}, v_{3,i}\} \) (\( \{\bar{v}_{2,i}, \bar{v}_{3,i}\} \)) deleted. For every \( 1 \leq i \leq s - 1 \) let \( B_i = V(K_i) - V(H) \) and \( \bar{B}_i = V(K_i) - V(H) \). By definition of \( J(D) \), \( B_i \)
Following each assertion will be the integers in and suppose \( j = 2k/3 \). Then the circle graph \( D = C_{(2k/3)} \) on \( 2sk + 1 \) vertices

\((\bar{B}_i)\) is a vertex cover of \( K_i \) \((K_i)\) in \( J(D) \). Therefore |\( B_i | \geq 2 \ (|\bar{B}_i | \geq 2)\) with equality if and only if \( v_{2,i}, v_{3,i} \in V(H) \) \((\bar{v}_{2,i}, \bar{v}_{3,i} \in V(H)\)) This fact is used repeatedly below.

Let \( i' \in [1, s - 1]. \) Then

1. \( v_{3,i'} \in V(H) \) \((\bar{v}_{3,i'} \in V(H))\) implies:
   1a. \( \{v_{4,i}, v_{3,i}\} \cap V(H) = \emptyset \ (\{v_{3,i}, \bar{v}_{4,i}\} \cap V(H) = \emptyset \) for \( 1 \leq i \leq s - 1\).
   1b. \( \bar{v}_{2,i} \in \bar{B}_i \) for \( 1 \leq i \leq i'\). \((v_{2,i} \in B_i \) for \( i' < i \leq s - 1).\)
   1c. \( v_{2,i'+1} \in B_{i'+1} \ (\bar{v}_{2,i'}-1 \in \bar{B}_{i'-1}).\)
   1d. \( \bar{v}_{4,i'} \in \bar{B}_{i'} \ (v_{4,i'} \in B_{i'}).\)

2. \( v_{2,i'} \in V(H) \) \((\bar{v}_2,i' \in V(H))\) implies:
   2a. \( \bar{v}_{1,i'-1} \in \bar{B}_{i'-1} \ (v_{1,i'+1} \in B_{i}).\)
   2b. \( \bar{v}_{4,i'} \in \bar{B}_{i'} \ (v_{4,i'} \in B_{i'}).\)

3. \( \bar{v}_{1,i'} \in V(H) \) \((v_{1,i'} \in V(H))\) implies
   3a. \( \bar{v}_{4,i} \in \bar{B}_i \) for \( i' < i \leq s - 1 \ (v_{4,i} \in B_i \) for every \( 1 \leq i \leq i').\)

Proof of (1)-(3). Following each assertion will be the integers in \( N_0(D, 2) \) that prove the assertion. \( 1a \) \([1, js - 1]_1; \ 1b \) \([1, js - 1]_2; \ 1c \) \((j - 2)s; \ 1d \) \( 2s; \ 2a \) \( 2js - 1; \ 2b \) \( js + 2; \ 3a \) \([1, js - 1]_2. \) This proves (1)-(3).

Consider the two propositions i) There is an \( i \in [1, s - 1] \) such that \( \{v_{2,i}, v_{3,i}\} \subset V(H) \) and ii) There is an \( i \in [1, s - 1] \) such that \( \{\bar{v}_{2,i}, \bar{v}_{3,i}\} \subset V(H). \) By \( 1a \) at most 1 of the propositions is true. Also, we may assume at least 1 is true, otherwise \( |B_i| + |\bar{B}_i| \geq 6 \) for every \( i \in [1, s - 1] \) and we would be done. Suppose first that proposition i) is true and let \( 1 \leq p_1 < p_2 < \ldots < p_r \leq s - 1 \) be the indices for which \( \{v_{2,p_i}, v_{3,p_i}\} \subset V(H) \). Clearly \( |B_{i}| \geq 3 \) for every \( i \in [1, s - 1] - \{p_1, \ldots, p_r\} \) and since proposition ii) is false, \( |\bar{B}_i| \geq 3 \) for every \( i \in [1, s - 1] \). By \( 1a, 1b \) and \( 1d \), we must have \( V(K_{p_i}) \cap V(H) \subset \{v_{1,p_i}\} \) for \( i \in [1, r]. \) If \( \bar{v}_{1,i} \notin V(H) \) then \( |\bar{B}_{p_i}| + |B_{p_i}| \geq 6. \) So let \( i' \in [1, r] \) be the least subindex such that \( \bar{v}_{1,i'} \notin V(H). \) By \( 3a, \bar{v}_{4,i'} \in B_{p_i} \) for all \( p'_r < i' \leq p_r. \) This and \( 1a, 1b, 2a \) imply that \( |\bar{B}_{p_i}| = 4 \) for \( i \in [i' + 1, r]. \) By \( 1c \) \( p_{i+1} > p_i + 1 \) for every \( i \in [1, r - 1] \) and hence \( |\bar{B}_{p_i}| \geq 3 \) for all \( i \in [1, r]. \) Therefore, for every \( i \in [i' + 1, r], \) we have \( |\bar{B}_{p_i}| + |\bar{B}_{p_i}| + |B_{p_i}| + |B_{p_i}| \geq 12. \) This implies that \( |B_1| + |\bar{B}_1| + \ldots + |\bar{B}_{s-1}| \geq 6s - 7 \) whence \( |V(H)| \leq 2s + 4. \) If proposition ii) is true this technique can be modified slightly by letting \( i' \) be the greatest subindex for which \( v_{1,p_{i'}} \in V(H). \) This proves Lemma 10.

**Lemma 11** Let \( k \geq 6 \) be an integer divisible by \( 3, \) let \( s \) be a positive integer and suppose \( j = 2k/3. \) Then the circle graph \( D = C_{(2k/3)} \) on \( 2sk + 1 \) vertices
Suppose first that \([j - 2)s + 1, js - 1]_2\) has no diameter two subgraph containing more than \(2s + 5\) vertices.

Proof of Lemma 11. Let \(H\) be a largest size diameter two subgraph of \(D\). Without loss of generality, \(H\) contains 0. It is easily verified that \(N_0(D, 2) = [(3j - 2)s + 3, 2sk - 1]_2 \cup [0, 2s - 2]_2 \cup [(j - 2)s + 1, js - 1]_2 \cup [js + 3, (j + 4)s - 1]_2 \cup [(2j - 4)s + 2, 2js - 2]_2 \cup [2js + 2, (2j + 2)s]_2\). For \(1 \leq i \leq s - 1\) define \(v_{1,i} = 2i, v_{2,i} = (j - 2)s + 1 + 2i, v_{3,i} = js + 1 + 2i, v_{4,i} = (2j - 4)s + 2i, v_{5,i} = (3j - 2)s + 1 + 2i, v_{6,i} = 2js + 2i, v_{7,i} = (2j - 2)s + 2i,\) and \(v_{8,i} = (j + 2)s + 1 + 2i\). For the same set of \(i\) define \(K_i\) to be the subgraph of \(J = J(D)\) induced by the vertex set \(\{v_{1,i}, v_{2,i}, v_{3,i}, v_{4,i}\}\) (\(\{v_{5,i}, v_{6,i}, v_{7,i}, v_{8,i}\}\)) deleted. For every \(i \in [1, s - 1]\), let \(B_i = V(K_i) - V(H)\) and \(\bar{B}_i = V(K_i) - V(H)\). By the nature of \(J(D)\), \(B_i (\bar{B}_i)\) must be a vertex cover of \(K_i (\bar{K}_i)\) in \(J\). Note that \(|B_i| \geq 2 (|\bar{B}_i| \geq 2)\) with equality if and only if \(v_{2,i}, v_{4,i} \in V(H) (v_{5,i}, v_{6,i} \in V(H))\).

There can be at most one index \(i \in [1, s - 1]\) such that \(v_{2,i}, v_{4,i} \in V(H)\) because \(1, (j - 2)s - 1]_2 \subset N_0(D, 2)\) and \(v_{2,i} \in V(H)\) implies that \(v_{4,i'} \in B_{i'}\) for every \(i' \in [1, i - 1]\). Similar reasoning implies that \(v_{5,i}, v_{6,i} \in V(H)\) for at most one \(i \in [1, s - 1]\). Therefore \(|B_1| + |\bar{B}_1| + |B_2| + \ldots + |B_{s-1}| \geq 6s - 8\) which implies \(|V(H)| \leq 2s + 5\), as desired. This proves Lemma 11.

**Lemma 12** Let \(k \geq 4\) and \(s\) be a positive integers. If \(j \in \{(2k - 2)/3, (2k - 1)/3\}\) and \(j\) is an integer then the circle graph \(D\) on \(n = 2sk + 1\) vertices determined by \([js + 1, (j + 1)s]\) has no diameter two subgraph on more than \(2s + 7\) vertices.

**Proof of Lemma 12.** Suppose first that \(j = (2k - 2)/3\). As usual, let \(H\) be a largest diameter 2 subgraph of \(D\) containing 0. It is easily verified that \(N_0(D, 2) = [(3j + 1)s + 2, (3j + 2)s] \cup [0, s - 1] \cup [js + 1, (j + 2)s - 1] \cup [2js + 2, (2j + 2)s]\). Now \(js, 2js \in N_0(D, 2)\), so at most one third of each of the following two subsets of \(N_0(D, 2)\) can occur in \(H\):

1. \([2, s - 1] \cup [js + 2, s - 1] \cup [2js + 2, s - 1]\)
2. \([(3j + 1)s + 2, (3j + 2)s] \cup [(3j + 1)s + 2, (3j + 2)s - 1 - j] \cup [(3j + 1)s + 2, (3j + 2)s - 1 - 2js]\).

Thus at least \(4s - 8\) vertices are excluded from \(H\) so \(|V(H)| \leq 2s + 5\).

Now suppose \(j = (2k - 1)/3\). Let \(H\) be a largest diameter two subgraph containing 0. Since \(N_0(D, 2) = [3js + 2, (3j + 1)s] \cup [0, s - 1] \cup [(j - 1)s + 1, js - 1] \cup [js + 3, (j + 4)s - 1] \cup [(2j - 4)s + 2, 2js - 2] \cup [2js + 2, (2j + 2)s]\).
Therefore at most one third of each of the following two subsets of $N_0(D, 2)$ can appear in $H$:

1) $[2, s - 1] \cup [(2, s - 1) + (j - 1)s - 1] \cup ([2, s - 1] + 2js)$.
2) $[3js + 2, (3j + 1)s - 1] \cup ([3js + 2, (3j + 1)s - 1] - (j - 1)s + 1) \cup ([3js + 2, (3j + 1)s - 1] - 2js)$.

Thus $|V(H)| \leq 2s + 7$ as desired. This proves Lemma 12.

Lemma’s 5-12 complete the proof of Theorem 3. Q.E.D.

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