IS THE RIEMANN HYPOTHESIS A ONE- OR TWO-DIMENSIONAL PROBLEM?

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Abstract. We close, temporarily or not, a repeated attempt on the Riemann hypothesis, which ultimately proved inconclusive. There is however, in our opinion, much to be saved from it. It includes strong arguments for regarding R.H. as a two-dimensional problem.

0. Foreword

In a recent book [8], we gave a criterion for the validity of the Riemann hypothesis which expressed itself as a bound for a certain parameter-dependent hermitian form $\mathcal{H}$ on the space of complex-valued functions on some set of congruence classes. The study of $\mathcal{H}$, a seemingly purely algebraic problem benefitting from a great amount of structure, looked deceitfully easy: it was not, and we made repeatedly computational errors, for which we apologize, in a series of arXiv-prepublished texts. We wish not to let an erroneous paper be the last of this series. At the same time, we want to emphasize what should be retained from this approach, many elements of which are new and some, or so we hope, of remaining interest.

1. Introduction

The paper is based on the consideration of the distribution

$$\mathcal{S}_\infty(x, \xi) = \sum_{|j|+|k| \neq 0} a((j, k)) \delta(x - j) \delta(\xi - k)$$

in the plane, with $(j, k) = \text{g.c.d.}(j, k)$ and $a(r) = \prod_{p|r}(1 - p)$ for $r = 1, 2, \ldots$. There is a collection $(\mathcal{E}_\nu)_{\nu \neq \pm 1}$ of so-called Eisenstein distributions, $\mathcal{E}_\nu$ homogeneous of degree $-1 - \nu$, such that, as an analytic functional,

$$\mathcal{S}_\infty = \frac{1}{2i\pi} \int_{\text{Re}\nu = c > 1} \frac{\mathcal{E}_\nu}{\zeta(\nu)} d\nu = 12 \delta_0 + \sum_{\zeta(\rho) = 0} \text{Res}_{\nu = \rho} \left( \frac{\mathcal{E}_\nu}{\zeta(\nu)} \right),$$

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where $\delta_0$ is the unit mass at the origin of $\mathbb{R}^2$. This decomposition immediately points towards a possible link between $\zeta_\infty$ and the zeros of the Riemann zeta function: only, to obtain a full benefit of this distribution, one must appeal to pseudodifferential analysis, more precisely to the Weyl symbolic calculus of operators. This is the rule $\Psi$ that associates to any distribution $\mathcal{S} \in \mathcal{S}'(\mathbb{R}^2)$ the linear operator $\Psi(\mathcal{S})$ from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ weakly defined by the equation
\[
(\Psi(\mathcal{S})u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \mathcal{S}\left(\frac{x+y}{2}, \xi\right) e^{i\pi(x-y)\xi} u(y) \, dy \, d\xi.
\]
It defines pseudodifferential analysis, which has been for more than half a century one of the main tools in the study of partial differential equations. The methods used in the present context do not intersect the usual ones (and nothing would be gained here from previous acquaintance with the subject) and may call for the denomination of “pseudodifferential arithmetic” (Sections 5 and 6).

Eisenstein distributions will be described in detail in Section 3. They make up just a part of the domain of automorphic distribution theory, which relates to the classical one of modular form theory but is more precise. If one defines the Euler operator $2i\pi E = 1 + x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi}$ in the plane, the operator $\pi^2 E^2$ transfers under some map (the dual Radon transformation, in an arithmetic context) to the operator $\Delta - \frac{1}{4}$, where $\Delta$ is the automorphic Laplacian. While this is crucial in other applications [9], it is another feature of automorphic distribution theory that will be essential here: the way it can cooperate with the Weyl symbolic calculus. As will be seen in Section 8, the fact that $\mathcal{E}_\nu$ is automorphic for every $\nu$ is equivalent to the functional equation of the zeta function, providing it with a much richer aspect.

We shall make use especially of the collection of rescaling operators $Q^{2i\pi E}$, with
\[
(Q^{2i\pi E} \mathcal{S})(x, \xi) = Q \mathcal{S}(Qx, Q\xi), \quad Q > 0.
\]
In [8], attention was brought to the hermitian form $(w | \Psi (Q^{2i\pi E} \zeta_\infty) \, w)$. The following criterion for R.H. was obtained: that, given $\beta > 2$ and any function $w \in C^\infty(\mathbb{R})$ supported in $[0, \beta]$, one should have
\[
(w | \Psi (Q^{2i\pi E} \zeta_\infty) \, w) = O \left( Q^{\frac{3}{4} + \varepsilon} \right)
\]
as $Q \to \infty$ through squarefree integral values.
The next step consists in transforming the left-hand side of (1.5) into a finite-dimensional hermitian form. Given a positive integer $N$, one sets

$$
T_N(x, \xi) = \sum_{j,k \in \mathbb{Z}} a((j, k, N)) \delta(x - j) \delta(\xi - k).
$$

(1.6)

The distribution $T_\infty$ is obtained as the limit, as $N \to \infty$ (a notation meant to express that $N$ will be divisible by any given squarefree number, from a certain point on), of the distribution $T_N^\infty$ obtained from $T_N$ by dropping the term corresponding to $j = k = 0$. If $Q$ is squarefree, if the functions $v, u \in C^\infty(\mathbb{R})$ are supported in $[-\beta, \beta]$ and the supports of $v$ and $u$ do not intersect, finally if $N = RQ$ is a squarefree integer divisible by all primes less than $\beta Q$, one has

$$
(v \, | \, \Psi \left( Q^{2\pi i} T_\infty \right) u) = (v \, | \, \Psi \left( Q^{2\pi i} T_N \right) u).
$$

(1.7)

Now, the hermitian form on the right-hand side is amenable to an algebraic-arithmetic version. Indeed, transferring functions in $S(\mathbb{R})$ to functions on $\mathbb{Z}/(2 N^2) \mathbb{Z}$ under the linear map $\theta_N$ defined by the equation

$$
(\theta_N u) (n) = \sum_{n_1 \in \mathbb{Z}, \ n_1 \equiv n \mod 2 N^2} u \left( \frac{n_1}{N} \right), \quad n \mod 2 N^2,
$$

(1.8)

one obtains the identity

$$
(v \, | \, \Psi \left( Q^{2\pi i} T_N \right) u) = \sum_{m, n \mod 2 N^2} c(m, n) \, \theta_N v(m) \, (\theta_N u)(n).
$$

(1.9)

The coefficients $c(m, n)$ are fully explicit, and the symmetric matrix defining this hermitian form has a Eulerian structure. If $R \geq 2 \beta Q$, where we still assume that $v, u$ are supported in $[-\beta, \beta]$, $(\theta_N u)(n)$ depends only on the class of $n \mod R^2$.

Even though, under some support assumptions about $v, u$, one could reduce $(v \, | \, \Psi \left( Q^{2\pi i} T_N \right) u)$ to the expression

$$
\frac{1}{\pi} \sum_{Q_1, Q_2 = Q} \mu(Q_1) \sum_{T \mid R} \mu(T) \overline{v} \left( \frac{T}{Q_2} + \frac{Q_2}{T} \right) u \left( \frac{T}{Q_2} - \frac{Q_2}{T} \right),
$$

(1.10)

it would be a bad idea to proceed along such a line. Experience indeed shows that trying to analyze expressions involving the Möbius coefficients (here, $\mu(T)$, not $\mu(Q_1)$, is worrisome) always leaves us with some version of the
intractable Littlewood criterion. Also, the expression just given is reducible to a one-dimensional problem, in terms of the measure \( \sum_{k \neq 0} \mu(k) \delta(t - k) \), the decomposition of which into homogeneous components also involves the factor \( \zeta(\nu) \) in the denominator.

It is our opinion that there may still be some hope of finding the desired bound for the hermitian form \( (w \mid \Psi(Q^{2i\pi E} z_N) w) \) if one relies on the considerable amount of structure (all of which was not recalled in previous arXiv versions of the present manuscript) it benefits of. But this may be deceptive.

It has been suggested [2, p.735] that the true significance in R.H. may lie in its validity for families of \( L \)-functions rather than individual ones like the Riemann zeta function. Indeed, we shall show that the present approach would work (up to which point is yet unknown) equally well for Dirichlet \( L \)-functions or for the \( L \)-functions attached to Hecke eigenforms of the full unimodular group. It is in the second context only that the functional equation of the zeta function takes its significance as an automorphy property.

2. The Weyl calculus of operators

In space-momentum coordinates, the Weyl calculus, or pseudodifferential calculus, depends on one free parameter with the dimension of action, called Planck’s constant. In pure mathematics, even the more so when pseudodifferential analysis is applied to arithmetic, Planck’s constant becomes a pure number: there is no question that the good such constant in “pseudodifferential arithmetic” is 2, as especially put into evidence [8, Chapter 6] in the pseudodifferential calculus of operators with automorphic symbols. To avoid encumbering the text with unnecessary subscripts, we shall denote as \( \Psi \) the rule denoted as \( \text{Op}_2 \) in [8, (2.1.1)], to wit the rule that attaches to any distribution \( \mathcal{S} \in \mathcal{S}'(\mathbb{R}^2) \) the linear operator \( \Psi(\mathcal{S}) \) from \( \mathcal{S}(\mathbb{R}) \) to \( \mathcal{S}'(\mathbb{R}) \) weakly defined as

\[
(\Psi(\mathcal{S}) u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \mathcal{S} \left( \frac{x + y}{2}, \xi \right) e^{i\pi(x-y)\xi} u(y) \, d\xi \, dy, \quad (2.1)
\]

denominately a superposition of integrals (integrate with respect to \( y \) first). The operator \( \Psi(\mathcal{S}) \) is called the operator with symbol \( \mathcal{S} \). Its integral kernel is the function

\[
K(x, y) = (\mathcal{F}_2^{-1} \mathcal{S}) \left( \frac{x + y}{2}, \frac{x - y}{2} \right), \quad (2.2)
\]
where $\mathcal{F}^{-1}_2$ denotes the inverse Fourier transformation with respect to the second variable.

If one defines the Wigner function $\text{Wig}(v, u)$ of two functions in $\mathcal{S}(\mathbb{R})$ as the function in $\mathcal{S}(\mathbb{R}^2)$ such that

$$\text{Wig}(v, u)(x, \xi) = \int_{-\infty}^{\infty} \overline{v}(x + t) u(x - t) e^{2i\pi \xi t} dt,$$

one has

$$(v \mid \Psi(\mathcal{S}) u) = \langle \mathcal{S}, \text{Wig}(v, u) \rangle,$$

with $(v | u) = \int_{-\infty}^{\infty} \overline{v}(x) u(x) dx$, while straight brackets refer to the bilinear operation of testing a distribution on a function. Note that $\text{Wig}(v, u)(x, \xi) = 0$ unless $2x$ lies in the algebraic sum of the supports of $v$ and $u$. The function $\text{Wig}(v, u)$ has exactly the same role, in connection to the Weyl calculus $\Psi = \text{Op}_2$, as the function $W(v, u)$ [8, (2.1.3)] in connection to the rule $\text{Op}_1$ used in that reference and denoted as $\text{Op}$ there.

Another useful property of the calculus $\Psi$ is expressed by the following two equivalent identities, obtained with the help of elementary manipulations of the Fourier transformation or with that of (2.2),

$$\Psi(\mathcal{F}_{\text{symp}} \mathcal{S}) w = \Psi(\mathcal{S}) \overset{\vee}{w}, \quad \mathcal{F}_{\text{symp}} \text{Wig}(v, u) = \text{Wig}(v, \overset{\vee}{u}),$$

where $\overset{\vee}{w}(x) = w(-x)$ and the symplectic Fourier transformation in $\mathbb{R}^2$ is defined in $\mathcal{S}(\mathbb{R}^2)$ or $\mathcal{S}'(\mathbb{R}^2)$ by the equation

$$(\mathcal{F}_{\text{symp}} \mathcal{S})(x, \xi) = \int_{\mathbb{R}^2} \mathcal{S}(y, \eta) e^{2i\pi (xy - y\xi)} dyd\eta.$$

Introduce the Euler operator $2i\pi \mathcal{E} = 1 + x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi}$ and, for $t > 0$, the operator $t^{2i\pi \mathcal{E}}$ such that $(t^{2i\pi \mathcal{E}} \mathcal{S})(x, \xi) = t \mathcal{S}(tx, t\xi)$. Denoting as $U[2]$ the unitary rescaling operator such that $(U[2] u)(x) = 2^{\frac{1}{2}} u(x\sqrt{2})$, one can connect the Weyl calculi with Planck constants 1 and 2 by the equation [8, (2.1.14)]

$$U[2] \Psi(\mathcal{S}) U[2]^{-1} = \text{Op}_1(\text{Resc } \mathcal{S}),$$

with $\text{Resc} = 2^{-\frac{1}{2} + i\pi \mathcal{E}}$ or $(\text{Resc } \mathcal{S})(x, \xi) = \mathcal{S}(2^{\frac{1}{2}} x, 2^{\frac{1}{2}} \xi)$. This would enable us not to redo, in Section 5 and in Proposition 6.1, the proofs already written with another normalization in [8]; but, for self-containedness and
simplicity (specializing certain parameters), we shall rewrite all proofs dependent on the symbolic calculus.

The choice of the rule $\Psi$ makes it possible to avoid splitting into cases, according to the parity of the integers present there, the developments of pseudodifferential arithmetic in Section 5.

3. Eisenstein distributions

The objects in the present section are taken from the more detailed versions in [8, Section 2.2] or, especially, [7, Section 1.1]. Automorphic distributions are distributions in the Schwartz space $S'(\mathbb{R}^2)$, invariant under the linear changes of coordinates associated to matrices in $SL(2, \mathbb{Z})$. It is the theory of automorphic and modular distributions, developed over a 20-year span, that led to the definition of the basic distribution $T_\infty$ (4.3), and to that of Eisenstein distributions. It connects [7, Section 2.1] to the classical theory of automorphic functions and modular forms of the non-holomorphic type in the hyperbolic half-plane through a version of the dual Radon transformation dealing with $SL(2, \mathbb{Z})$-invariant objects. Our initial interest in developing these notions was stimulated [4, section 18] by the Lax-Phillips scattering theory [3]. We shall come back to this environment in Section 8.

**Definition 3.1.** If $\nu \in \mathbb{C}$, $\text{Re}\, \nu > 1$, the Eisenstein distribution $\mathcal{E}_{-\nu}$ is defined by the equation, valid for every $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle \mathcal{E}_{-\nu}, h \rangle = \sum_{|j|+|k|\neq 0} \int_0^\infty t^\nu h(jt, kt) \, dt = \frac{1}{2} \sum_{|j|+|k|\neq 0} \int_{-\infty}^\infty |t|^\nu h(jt, kt) \, dt.$$

(3.1)

The second equation is convenient when taking Fourier transforms with respect to $t$. It is immediate that the series of integrals converges if $\text{Re} \, \nu > 1$, in which case $\mathcal{E}_{-\nu}$ is well defined as a tempered distribution. Obviously, it is $SL(2, \mathbb{Z})$-invariant as a distribution, i.e., an automorphic distribution. It is homogeneous of degree $-1 + \nu$, i.e., $(2i\pi \mathcal{E}) \mathcal{E}_{-\nu} = \nu \mathcal{E}_{-\nu}$. Its name stems from its relation with the classical notion of non-holomorphic Eisenstein series, as made explicit in [8, p.93]: it is, however, a more precise concept.
Proposition 3.2. [7, p.13] As a tempered distribution, \( \mathcal{E}_{-\nu} \) extends as a
meromorphic function of \( \nu \in \mathbb{C} \), whose only poles are the points \( \nu = \pm 1 \):
these poles are simple, and the residues of \( \mathcal{E}_{\nu} \) there are
\[
\text{Res}_{\nu=1} \mathcal{E}_{-\nu} = 1 \quad \text{and} \quad \text{Res}_{\nu=-1} \mathcal{E}_{-\nu} = \delta_0,
\]
the unit mass at the origin of \( \mathbb{R}^2 \). Recalling the definition (2.6) of the symplectic Fourier transformation \( \mathcal{F}_{\text{symp}} \), one has, for \( \nu \neq \pm 1 \), \( \mathcal{F}_{\text{symp}} \mathcal{E}_{-\nu} = \mathcal{E}_{\nu} \).

Proof. We reproduce the proof given in [7, p.13-14]. Denote as \( (\mathcal{E}_{-\nu})_{\text{princ}} \) (resp. \( (\mathcal{E}_{-\nu})_{\text{res}} \)) the distribution defined in the same way as \( \mathcal{E}_{-\nu} \), except
for the fact that in the integral (3.1), the interval of integration \((0, \infty)\) is
replaced by the interval \((0, 1)\) (resp. \((1, \infty)\)), and observe that the distribution \( (\mathcal{E}_{-\nu})_{\text{res}} \) extends as an entire function of \( \nu \). As a consequence of
Poisson’s formula, one has when \( \text{Re} \nu > 1 \) the identity
\[
\int_1^{\infty} t^{-\nu} \sum_{(j,k) \in \mathbb{Z}^2} (\mathcal{F}_{\text{symp}} h)(tk, tj) \, dt = \int_1^{\infty} t^{-\nu} \sum_{(j,k) \in \mathbb{Z}^2} t^{-2} h(t^{-1}j, t^{-1}k) \, dt
\]
\[
= \int_0^1 t^{\nu} \sum_{(j,k) \in \mathbb{Z}^2} h(tj, tk) \, dt, \quad (3.3)
\]
from which one obtains that
\[
< \mathcal{F}_{\text{symp}} (\mathcal{E}_{\nu})_{\text{res}} , h > = < (\mathcal{E}_{-\nu})_{\text{princ}} , h > + \frac{h(0, 0)}{1 + \nu} + \frac{(\mathcal{F}_{\text{symp}} h)(0, 0)}{1 - \nu}. \quad (3.4)
\]
From this identity, one finds the meromorphic continuation of the function
\( \nu \mapsto \mathcal{E}_{\nu} \), including the residues at the two poles, as well as the fact that \( \mathcal{E}_{\nu} \)
and \( \mathcal{E}_{-\nu} \) are the images of each other under \( \mathcal{F}_{\text{symp}} \).

Note that if \( h = \text{Wig}(v, u) \), one has
\[
h(0, 0) = (v | \hat{u}), \quad (\mathcal{F}_{\text{symp}} h)(0, 0) = (v | u). \quad (3.5)
\]
We shall deal repeatedly with pairs of functions \( v, u \in \mathcal{S}(\mathbb{R}) \) such that the
supports of \( v \) and \( u \) are disjoint, and so are the supports of \( v \) and \( \hat{u} \). In such a case, we shall have, using (2.5), and without any condition on \( \nu \),
\[
< \mathcal{E}_{-\nu} , \text{Wig}(v, u) > = < (\mathcal{E}_{-\nu})_{\text{res}} , \text{Wig}(v, u) > + < (\mathcal{E}_{\nu})_{\text{res}} , \text{Wig}(v, \hat{u}) > \quad (3.6)
\]
□
Lemma 3.3. ([5, p.22], [7, p.15]) One has if $\nu \neq \pm 1$ the Fourier expansion

$$
\mathcal{E}_{-\nu}(x, \xi) = \zeta(\nu) |\xi|^{\nu-1} + \zeta(1+\nu) |x|^{\nu} \delta(\xi) + \sum_{r \neq 0} \sigma_{-\nu}(r) |\xi|^{\nu-1} \exp \left( 2i\pi \frac{r.x}{\xi} \right)
$$

(3.7)

where $\sigma_{-\nu}(r) = \sum_{1 \leq d|r} d^{-\nu}$: the first two terms must be grouped when $\nu = 0$.

Proof. Isolating the term for which $k = 0$ in (3.1), we write if $\text{Re} \nu > 1$, after a change of variable,

$$
\langle \mathcal{E}_{-\nu}, h \rangle = \zeta(1+\nu) \int_{-\infty}^{\infty} |t|^\nu h(t,0) dt + \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^\nu h(jt, kt) dt
$$

$$
= \zeta(1+\nu) \int_{-\infty}^{\infty} |t|^\nu h(t,0) dt + \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_{-1}^{-1} h) \left( \frac{j}{t}, kt \right) dt,
$$

(3.8)

where we have used Poisson’s formula at the end and denoted as $\mathcal{F}_{-1}^{-1} h$ the inverse Fourier transform of $h$ with respect to the first variable. Isolating now now the term such that $j = 0$, we obtain

$$
\frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_{-1}^{-1} h) \left( \frac{j}{t}, kt \right) dt = \zeta(\nu) \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_{-1}^{-1} h) (0, t) dt
$$

$$
+ \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_{-1}^{-1} h) \left( \frac{j}{t}, kt \right) dt,
$$

(3.9)

from which the main part of the lemma follows if $\text{Re} \nu > 1$ after we have made the change of variable $t \mapsto \frac{t}{k}$ in the main term. The continuation of the identity uses also the fact that the product $\zeta(\nu) |t|^{\nu-1}$ is regular at $\nu = -2, -4, \ldots$ and the product $\zeta(1+\nu) |t|^\nu$ is regular at $\nu = -3, -5, \ldots$, thanks to the trivial zeros of zeta. That the sum $\zeta(\nu) |\xi|^{\nu-1} + \zeta(1+\nu) |x|^\nu \delta(\xi)$ is regular at $\nu = 0$ follows from the facts that $\zeta(0) = -\frac{1}{2}$ and that the residue at $\nu = 0$ of the distribution $|\xi|^{\nu-1} = \frac{d}{d\xi} (|x|^{\nu} \text{sign} \xi)$ is $\frac{d}{d\xi} \text{sign} \xi = 2 \delta(\xi)$.

□

Decompositions into homogeneous components of functions or distributions in the plane will be ever-present. Any function $h \in \mathcal{S}(\mathbb{R}^2)$ can be decomposed in $\mathbb{R}^2 \setminus \{0\}$ into homogeneous components according to the
equations, in which \( c > -1 \),

\[
h = \frac{i}{\nu} \int_{\text{Re}\nu=c} h_\nu d\nu, \quad h_\nu(x,\xi) = \frac{1}{2\pi} \int_0^\infty t^\nu h(tx,t\xi) dt. \tag{3.10}
\]

Indeed, the integral defining \( h_\nu(x,\xi) \) is convergent for \( |x| + |\xi| \neq 0, \text{Re}\nu > -1 \), and the function \( h_\nu \) so defined is \( C^\infty \) in \( \mathbb{R}^2 \setminus \{0\} \) and homogeneous of degree \(-1-\nu\); it is also analytic with respect to \( \nu \). Using twice the integration by parts associated to Euler’s equation

\[
-(1+\nu) h_\nu = \left(x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi}\right) h_\nu(x,\xi),
\]

one sees that the integral \( \frac{1}{i} \int_{\text{Re}\nu=c} h_\nu(x,\xi) d\nu \) is convergent for \( c > -1 \): its value does not depend on \( c \). Taking \( c = 0 \) and setting \( t = e^{2\pi t} \), one has for \( |x| + |\xi| \neq 0 \)

\[
h_{i\lambda}(x,\xi) = \int_{-\infty}^\infty e^{2\pi\tau \lambda} \cdot e^{2\pi\tau h(e^{2\pi\tau}x,e^{2\pi\tau}\xi)} d\tau, \tag{3.11}
\]

and the Fourier inversion formula shows that

\[
\int_{-\infty}^\infty h_{i\lambda}(x,\xi) d\lambda = h(x,\xi);
\]

this proves (3.10).

As a consequence, some automorphic distributions of interest (not all: Hecke distributions are needed too in general [9]) can be decomposed into Eisenstein distributions. A basic one is the “Dirac comb”

\[
\mathfrak{D}(x,\xi) = 2\pi \sum_{|j|+|k| \neq 0} \delta(x-j) \delta(\xi-k). \tag{3.12}
\]

Indeed, noting the inequality \(|\int_0^\infty t^\nu h(tx,t\xi) dt| \leq C (|x| + |\xi|)^{-\text{Re}\nu-1}\), one obtains if \( h \in S(\mathbb{R}^2) \) and \( c > 1 \), pairing (3.12) with (3.10), the identity

\[
\langle \mathfrak{D}, h \rangle = \frac{1}{i} \sum_{|j|+|k| \neq 0} \int_{\text{Re}\nu=c} d\nu \int_0^\infty t^\nu h(tj,tk) dt. \tag{3.13}
\]

Now, assuming that \( h \) is even as is possible since both \( \mathfrak{D} \) and \( \mathfrak{E}_{-\nu} \) are, one can rewrite (3.1) as

\[
\langle \mathfrak{E}_{-\nu}, h \rangle = \sum_{|m|+|n| \neq 0} \int_0^\infty t^{-\nu} h(mt,nt) dt, \quad c > 1. \tag{3.14}
\]

It follows that, for \( c > 1 \) [8, p.11],

\[
\mathfrak{D} = \frac{1}{i} \int_{\text{Re}\nu=c} \mathfrak{E}_{-\nu} d\nu = 2\pi + \frac{1}{i} \int_{\text{Re}\nu=0} \mathfrak{E}_{-\nu} d\nu, \tag{3.15}
\]

the second equation being a consequence of the first in view of (3.2).
In the reverse direction, Eisenstein distributions can be obtained from
the Dirac comb, making use of the resolvent of the Euler operator defined
[6, p.188] as
\[
(2i\pi E + \nu)^{-1} h (x, \xi) = \begin{cases} \int_0^1 t^{\nu} h(t x, t \xi) \, dt, & \text{Re} \, \nu > 0, \\ - \int_1^\infty t^{\nu} h(t x, t \xi) \, dt, & \text{Re} \, \nu < 0. \end{cases} \tag{3.16}
\]
That the two maps define bounded endomorphisms of \( L^2(\mathbb{R}^2) \) [6, (5.1.5)] is actually just Hardy’s inequality. The resolvent \((2i\pi E + \nu)^{-1}\) does not
preserve the space \( S(\mathbb{R}^2) \) but, if \( h \in S(\mathbb{R}^2) \), the two expressions
\[
\frac{1}{2\pi} \langle D, (2i\pi E + \nu)^{-1} \rangle = \sum_{|j|+|k|\neq 0} \int_0^1 t^{\nu} h(t j, t k) \, dt, \\
\frac{1}{2\pi} \langle D, (2i\pi E - \nu)^{-1} \rangle = - \sum_{|j|+|k|\neq 0} \int_1^\infty t^{-\nu} h(t j, t k) \, dt \tag{3.17}
\]
are convergent if \( c = \text{Re} \, \nu > 1 \). This is obvious so far as the second one
is concerned; for the first, choosing \( A > c + 1 \), we bound \( h(t j, t k) \) by
\( C \lfloor 1 + t(|j| + |k|) \rfloor^{-A} \) and make a change of variable.

Integral superpositions of Eisenstein distributions, such as the one in
(3.15), are to be interpreted in the weak sense in \( S'(\mathbb{R}^2) \), i.e., they make
sense when tested on arbitrary functions in \( S(\mathbb{R}^2) \). Of course, pole-chasing is
essential when changing contours of integration. But no difficulty concerning
the integrability with respect to \( \text{Im} \, \nu \) on the line ever occurs, because
of the identities
\[
(a - \nu)^A < \mathcal{E}_{-\nu}, W > = < (a - 2i\pi \mathcal{E})^A \mathcal{E}_{-\nu}, W > = < \mathcal{E}_{-\nu}, (a + 2i\pi \mathcal{E})^A W >, \tag{3.18}
\]
in which \( A = 0, 1, \ldots \) may be chosen arbitrarily large and \( a \) is arbitrary.

**Lemma 3.4.** Given \( \nu \in \mathbb{C}, \nu \neq \pm 1 \), and two functions \( v, u \in C^\infty(\mathbb{R}) \) with
supports contained in \( [a, b] \) with \( a \geq 0 \) and \( 0 < b^2 - a^2 < 8 \), one has
\[
(v \mid \Psi (\mathcal{E}_{-\nu}) u) = \zeta (\nu) \int_0^\infty x^{\nu-1} \mathcal{P}(x) u(x)^2 \, dx \\
+ 2 \sum_{r=\pm 1} \int_0^\infty x^{\nu-1} \mathcal{P} \left( x + \frac{r}{x} \right) u \left( x - \frac{r}{x} \right) \, dx. \tag{3.19}
\]
Proof. With a different normalization of the symbolic calculus, this was proved in [8, p.13]. Let us use the expansion (3.7), but only after we have substituted the pair \((\xi, -x)\) for \((x, \xi)\), which does not change \(E_{-\nu}(x, \xi)\) for any \(\nu \neq \pm 1\) because the Eisenstein distribution \(E_{-\nu}\) is automorphic: hence,

\[
E_{-\nu}(x, \xi) = \zeta(\nu) |x|^{\nu - 1} + \zeta(1 + \nu) \delta(x) |\xi|^{\nu} + \sum_{\nu \neq 0} \sigma_{-\nu}(r) |x|^{\nu - 1} \exp \left( -2i\pi \frac{r\xi}{x} \right).
\]

(3.20)

We immediately observe that, assuming only that \(v\) and \(u\) are supported in \([0, \infty[\), the first term of this expansion will contribute to \(\langle v | \Psi(E_{-\nu}) u \rangle\) the first term on the right-hand side of (3.19). As it follows from (2.2), if, as is the case here, the point 0 does not lie in the interior of the algebraic sum of the supports of the \(C^\infty\) functions \(v\) and \(u\), the second term of the expansion (3.20) will not contribute to \(\langle v | \Psi(E_{-\nu}) u \rangle\). Only the contribution of the series for \(k \neq 0\), to be designated as \(E_{\text{trunc}}^{\nu}(x, \xi)\), remains to be examined.

One has

\[
\left( F_2^{-1} E_{-\nu}^{\text{trunc}} \right)(x, z) = \sum_{\nu \neq 0} \sigma_{-\nu}(r) |x|^{\nu - 1} \delta \left( z - \frac{r}{x} \right). \tag{3.21}
\]

Still using (2.2), the integral kernel of the operator \(\Psi(\mathcal{S}_r)\), with

\[
\mathcal{S}_r(x, \xi) := |x|^{\nu - 1} \exp \left( -2i\pi \frac{r\xi}{x} \right) = \mathcal{I}_r \left( x, \frac{\xi}{x} \right), \tag{3.22}
\]

is

\[
K_r(x, y) = \left( F_2^{-1} \mathcal{S}_r \right) \left( \frac{x + y}{2}, \frac{x - y}{2} \right) = \left| \frac{x + y}{2} \right|^{\nu} \delta \left( \frac{x^2 - y^2}{4} - r \right) = \left| \frac{x + y}{2} \right|^{\nu - 1} \delta \left( \frac{x - y}{2} - \frac{2r}{x+y} \right). \tag{3.23}
\]

If \(a \leq x, y \leq b\), the condition \(x^2 - y^2 = 4r \neq 0\) implies \(4|r| \leq b^2 - a^2\), so that only the terms with \(r = \pm 1\) remain. Making in the integral \(\int_{\Omega} K(x, y) \overline{v}(x) u(y) \, dx \, dy\) the change of variable which amounts to taking \(\frac{x+y}{2}\) and \(x - y\) as new variables, one obtains (3.19).
4. SOME DISTRIBUTIONS OF ARITHMETIC INTEREST

Set for \( j \neq 0 \)

\[ a(j) = \prod_{p \mid j} (1 - p), \quad (4.1) \]

where \( p \), in the role of defining the range of the subscript in a product, is always tacitly assumed to be prime. The distribution

\[ T_N(x, \xi) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \delta(x - j) \delta(\xi - k), \quad (4.2) \]

where the notation \( (j, k, N) \) refers to the g.c.d. of the three numbers, depends only on the “squarefree version” of \( N \), defined as \( N_\bullet = \prod_{p \mid N} p \). We also denote as \( T_N^\times \) the distribution obtained from \( T_N \) by discarding the term \( a(N) \delta(x) \delta(\xi) \), in other words by limiting the summation to all pairs of integers \( j, k \) such that \( |j| + |k| \neq 0 \). As \( N \to \infty \), a notation meant to convey that \( N \to \infty \) in such a way that any given finite set of primes constitutes eventually a set of divisors of \( N \), the distribution \( T_N^\times \) converges in the space \( S'(\mathbb{R}^2) \) towards the distribution

\[ T_\infty(x, \xi) = \sum_{|j| + |k| \neq 0} a((j, k)) \delta(x - j) \delta(\xi - k). \quad (4.3) \]

Indeed, if \( q \) denotes the least prime not dividing \( N \), one has

\[ T_\infty(x, \xi) - T_N^\times(x, \xi) = \sum_{|j| + |k| \neq 0} [a((j, k)) - a((j, k, N))] \delta(x - j) \delta(\xi - k), \quad (4.4) \]

where the sum may be restricted to the set of pairs \( j, k \) such that at least one prime factor of \( (j, k) \) does not divide \( N \), which implies \( (j, k) \geq q \). This convergence is very slow since, as a consequence of the theorem of prime numbers, in the case when \( N = \prod_{p < q} p \), \( q \) is roughly of the size of \( \log N \) only.

**Lemma 4.1.** [8, Lemma 3.1.1] For any squarefree integer \( N \geq 1 \), defining

\[ \zeta_N(s) = \prod_{p \mid N} (1 - p^{-s})^{-1}, \quad \text{so that} \quad \frac{1}{\zeta_N(s)} = \sum_{1 \leq T \mid N} \mu(T) T^{-s}, \quad (4.5) \]

where \( \mu \) is the Möbius indicator function, one has

\[ T_N^\times = \frac{1}{2i\pi} \int_{Re \nu = c} \frac{1}{\zeta_N(\nu)} \mathcal{E}_{-\nu} d\nu, \quad c > 1. \quad (4.6) \]

Then,

\[ T_\infty = \frac{1}{2i\pi} \int_{Re \nu = c} \mathcal{E}_{-\nu} d\nu, \quad c > 1. \quad (4.7) \]
Proof. Using the equation \( T^{-\frac{d}{4\pi}} \delta(x - j) = \delta(\frac{j}{T} - j) = T \delta(x - T j) \), one has with \( \mathcal{D} \) as introduced in (3.20)

\[
\frac{1}{2\pi} \prod_{p|N} (1 - p^{-2i\pi E}) \mathcal{D}(x, \xi) = \sum_{T|N} \mu(T) T^{-2i\pi E} \sum_{|j| + |k| \neq 0} \delta(x - j) \delta(\xi - k) \\
= \sum_{T|N} \mu(T) T \sum_{|j| + |k| \neq 0} \delta(x - T j) \delta(\xi - Tk) \\
= \sum_{T|N} \mu(T) T \sum_{|j| + |k| \neq 0} \delta(x - j) \delta(\xi - k) \\
= \sum_{|j| + |k| \neq 0} \mu(T) T \delta(x - j) \delta(\xi - k). 
\]

Since \( \sum_{T|(N,j,k)} \mu(T) T = \prod_{p|(N,j,k)} (1 - p) = a((N, j, k)) \), one obtains

\[
\frac{1}{2\pi} \prod_{p|N} (1 - p^{-2i\pi E}) \mathcal{D}(x, \xi) = \mathfrak{S}_N^x(x, \xi). 
\] (4.8)

Combining this equation with (3.15) and with \( (2i\pi E)_- = \nu \mathfrak{E}_{-\nu} \), one obtains if \( c > 1 \)

\[
\mathfrak{S}_N^x = \prod_{p|N} (1 - p^{-2i\pi E}) \left[ \frac{1}{2i\pi} \int_{\text{Re}\nu = c} \mathfrak{E}_{-\nu} d\nu \right] \\
= \frac{1}{2i\pi} \int_{\text{Re}\nu = c} \mathfrak{E}_{-\nu} \prod_{p|N} (1 - p^{-\nu}) d\nu, 
\] (4.10)

which is just (4.6). Equation (4.7) follows as well, taking the limit as \( N \to \infty \). Recall what was said immediately between (3.15) and (3.18) about integral superpositions of Eisenstein distributions such as the ones in (3.15) and (4.7).

Remark. In (4.7), introducing a sum of residues over all zeros of zeta with a real part above some large negative number, one can replace the line
Re \( \nu = c \) with \( c > 1 \) by a line \( \text{Re} \ \nu = c' \) with \( c' \) very negative: one cannot go further in the distribution sense. But [8, Theor. 3.2.2, Theor. 3.2.4], one can get rid of the integral if one agrees to interpret the identity in the sense of a certain analytic functional. Then, all zeros of zeta, non-trivial and trivial alike, enter the formula.

The following reduction of \( \mathcal{T}_\infty \) to \( \mathcal{T}_N \) is immediate, and fundamental for our purpose. Assume that \( v \) and \( u \in C^\infty(\mathbb{R}) \) are supported in \([0, \beta]\). Then, given a squarefree integer \( N = RQ \) (with \( R, Q \) integers) divisible by all primes \( < \beta Q \), one has

\[
(v \mid \Psi (Q^{2i\pi\mathcal{E}} \mathcal{T}_\infty) u) = (v \mid \Psi (Q^{2i\pi\mathcal{E}} \mathcal{T}_N) u) .
\]

Indeed, using (2.4) and noting that the transpose of \( 2i\pi\mathcal{E} \) is \(-2i\pi\mathcal{E}\), one can write the right-hand side as

\[
(v \mid \Psi (Q^{2i\pi\mathcal{E}} \mathcal{T}_N) u) = Q^{-1} \sum_{|j|+|k|\neq 0} a((j,k,N)) \text{Wig}(v,u) \left( \frac{j}{Q}, \frac{k}{Q} \right) .
\]

(4.12)

In view of the observation that follows (2.4), one has \( 0 < \frac{j}{Q} < \beta \) for all nonzero terms on the right-hand side, in which case all prime divisors of \( j \) divide \( N \).

We shall also need the distribution

\[
\mathcal{T}_\infty (x, \xi) = \sum_{|j|+|k|\neq 0} a \left( \left( j, k, \frac{x}{2} \right) \right) \delta(x-j)\delta(\xi-k),
\]

(4.13)

where \( a \left( \left( j, k, \frac{x}{2} \right) \right) \) is the product of all factors \( 1 - p \) with \( p \) prime \( \neq 2 \) dividing \( (j,k) \). One has

\[
\mathcal{T}_\infty (x, \xi) = \frac{1}{2i\pi} \int_{\text{Re} \ \nu = c} (1 - 2^{-\nu})^{-1} \frac{\mathcal{E}_{-\nu}(x, \xi)}{\zeta(\nu)} d\nu, \quad c > 1.
\]

(4.14)

In [8, Prop. 3.4.2 and 3.4.3], it was proved (with some differences essentially due to the present change of \( \text{Op}_1 \) to \( \Psi \)) that, if for some \( \beta > 2 \) and every function \( w \in C^\infty(\mathbb{R}) \) supported in \([0, \beta]\), one has

\[
(w \mid \Psi (Q^{2i\pi\mathcal{E}} \mathcal{T}_\infty) w) = O \left( Q^{\frac{1}{2} + \varepsilon} \right)
\]

(4.15)

as \( Q \to \infty \) through squarefree odd integral values, the Riemann hypothesis follows.
5. Pseudodifferential arithmetic

As noted in (4.11), one can substitute for the analysis of the hermitian form \( v \mid \Psi \left( Q^{2i\pi E} \Xi_N \right) u \) that of \( v \mid \Psi \left( Q^{2i\pi E} \Xi_N \right) u \), under the assumption that \( v, u \) are supported in \([0, \beta]\), provided that \( N \) is a squarefree odd integer divisible by all odd primes \(< \beta Q\). The new hermitian form should be amenable to an algebraic treatment. In this section, we make no support assumptions on \( v, u \), just taking them in \( S(\mathbb{R}) \).

We consider operators of the kind \( A = \Psi \left( Q^{2i\pi E} \mathcal{G} \right) \) with

\[
\mathcal{G}(x, \xi) = \sum_{j,k \in \mathbb{Z}} b(j, k) \delta(x - j) \delta(\xi - k), \tag{5.1}
\]

under the following assumptions: that \( N \) is an odd squarefree integer decomposing as the product \( N = QR \) of two positive integers, and that \( b \) satisfies the periodicity conditions

\[
b(j, k) = b(j + N, k) = b(j, k + N). \tag{5.2}
\]

A special case consists of course of the symbol \( \mathcal{G} = \Xi_N \). The aim is to transform the hermitian form associated to the operator \( \Psi \left( Q^{2i\pi E} \Xi_N \right) \) to an arithmetic version.

**Proposition 5.1.** Define the operator \( B \) by the identity

\[
(Bu)(Qx) = Q^{-1} \left[ \Psi \left( Q^{2i\pi E} \mathcal{G} \right) (y \mapsto u(Qy)) \right](x), \tag{5.3}
\]

in other words

\[
(Bu)(x) = \frac{1}{2Q^2} \int_{\mathbb{R}^2} \mathcal{G} \left( \frac{x + y}{2}, \xi \right) u(y) \exp \left( \frac{i\pi}{Q^2} (x - y)\xi \right) dy d\xi. \tag{5.4}
\]

Setting \( v_Q(x) = v(Qx) \) and \( u_Q(x) = u(Qx) \), one has the identity

\[
(v \mid Bu) = (v_Q \mid A u_Q). \tag{5.5}
\]

**Proof.** Starting from

\[
(Bu)(Qx) = \frac{1}{2} \int_{\mathbb{R}^2} \mathcal{G} \left( \frac{Q(x + y)}{2}, Q\xi \right) e^{i\pi (x - y)\xi} u(Qy) dy d\xi, \tag{5.6}
\]
one obtains (5.4). From (5.3),

\[
(v \mid B u) = Q \int_{-\infty}^{\infty} \overline{\tau}(Qx) (Bu)(Qx) \, dx = Q \int_{-\infty}^{\infty} \overline{\tau}(Qx) \cdot (Q^{-1}Au_Q)(x) \, dx = \int_{-\infty}^{\infty} \overline{\tau}(Qx) \cdot (Au_Q)(x) \, dx = (v_Q \mid Au_Q).
\]

(5.7)

Lemma 5.2. Set, for \( u \in S(\mathbb{R}) \),

\[
(\theta_N u)(n) = \sum_{\ell \in \mathbb{Z}} u \left( \frac{n}{N} + 2\ell N \right), \quad n \mod 2N^2.
\]

(5.8)

On the other hand, set

\[
(\kappa u)(n) = \sum_{\ell_1 \in \mathbb{Z}} u \left( \frac{n}{R} + 2QN\ell_1 \right), \quad n \in \mathbb{Z}/(2N^2)\mathbb{Z}.
\]

(5.9)

One has

\[
(\kappa u)(n) = (\theta_N u_Q)(n).
\]

(5.10)

Proof. It is immediate. \qed

The following proposition reproduces [8, Prop. 4.1.2], with the simplification brought by the fact, made possible by the choice of the Weyl calculus \( \Psi \) rather than \( \text{Op}_1 \), that we may take \( N \) odd.

Proposition 5.3. With \( N = RQ \) and \( b(j,k) \) satisfying the properties in the beginning of this section, define the function

\[
f(j, s) = \frac{1}{N} \sum_{k \mod N} b(j,k) \exp \left( \frac{2\pi i ks}{N} \right), \quad j, s \in \mathbb{Z}/N\mathbb{Z}.
\]

(5.11)

Set, noting that the condition \( m - n \equiv 0 \mod 2Q \) implies that \( m + n \) too is even,

\[
c_{R,Q} (\mathcal{G}; m, n) = \text{char}(m+n \equiv 0 \mod R, m-n \equiv 0 \mod 2Q) f \left( \frac{m+n}{2R}, \frac{m-n}{2Q} \right).
\]

(5.12)
Then, one has
\[
(v \mid \Psi (Q^{2i\pi} \mathcal{G}) \ u) = \sum_{m,n \in \mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q} (\mathcal{G}; m, n) \overline{\theta_N v(m)} (\theta_N u)(n).
\] (5.13)

**Proof.** There is no restriction here on the supports of \(v, u\), and one can replace these two functions by \(v_Q, u_Q\). In view of (5.5) and (5.10), the identity (5.13) is equivalent to
\[
(v \mid B u) = \sum_{m,n \in \mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q} (\mathcal{G}; m, n) \overline{\kappa v(m)} (\kappa u)(n).
\] (5.14)

From (5.4), one has
\[
(v \mid B u) = \frac{1}{2Q^2} \int_{-\infty}^{\infty} \overline{\mathcal{G}(x, \xi)} \ u(y) \ \overline{\theta_N v(m) (\theta_N u)(n)} \ dy \ d\xi
= \frac{1}{Q^2} \int_{-\infty}^{\infty} \overline{\mathcal{G}(y, \xi)} \ u(2y - x) \ \exp \left( \frac{2i\pi}{Q^2} (x - y) \right) \ dy \ d\xi
= \frac{1}{Q^2} \int_{-\infty}^{\infty} \overline{\mathcal{G}(x, \xi)} \ \sum_{j,k \in \mathbb{Z}} b(j, k) \ u(2j - x) \ \exp \left( \frac{2i\pi}{Q^2} (x - j)k \right).
\] (5.15)

Since \(b(j,k) = b(j,k + N)\), one replaces \(k\) by \(k + N\ell\), the new \(k\) lying in the interval \([0, N - 1]\) of integers. One has (Poisson’s formula)
\[
\sum_{\ell \in \mathbb{Z}} \exp \left( \frac{2i\pi}{Q^2} (x - j)\ell N \right) = \sum_{\ell \in \mathbb{Z}} \exp \left( \frac{2i\pi}{Q} (x - j)\ell R \right) = \frac{Q}{R} \sum_{\ell \in \mathbb{Z}} \delta \left( x - j - \frac{\ell Q}{R} \right),
\] (5.16)

and
\[
(B u)(x) = \frac{1}{N} \sum_{j \in \mathbb{Z}} b(j,k) \sum_{\ell \in \mathbb{Z}} u \left( j - \frac{\ell Q}{R} \right) \ \exp \left( \frac{2i\pi(x - j)k}{Q^2} \right) \ \delta \left( x - j - \frac{\ell Q}{R} \right)
= \sum_{m \in \mathbb{Z}} t_m \delta \left( x - \frac{m}{R} \right),
\] (5.17)

with \(m = Rj + \ell Q\) and \(t_m\) to be made explicit: we shall drop the summation with respect to \(\ell\) for the benefit of a summation with respect to \(m\). Since,
when \( x = j + \frac{4Q}{R} = m \) , one has \( \frac{x}{Q} = \frac{m - Rj}{NQ} \) and \( j - \frac{Q}{R} = 2j - x = 2j - \frac{m}{R} \), one has
\[
\begin{align*}
t_m &= \frac{1}{N} \sum_{j \in \mathbb{Z}} b(j, k) \chi(m \equiv Rj \mod Q) u \left( 2j - \frac{m}{R} \right) \exp \left( \frac{2i\pi k(m - Rj)}{NQ} \right) \\
&= \frac{1}{N} \sum_{0 \leq j < QN} b(j, k) \chi(m \equiv Rj \mod Q) \\
&\quad \times \left( \sum_{\ell_1 \in \mathbb{Z}} u \left( 2(j + \ell_1QN) - \frac{m}{R} \right) \exp \left( \frac{2i\pi k(m - Rj)}{QN} \right) \right). 
\end{align*}
\]
(5.18)

Recalling the definition (5.9) of \( \kappa_v \), one obtains
\[
\begin{align*}
t_m &= \frac{1}{N} \sum_{0 \leq j < QN} b(j, k) \chi(m \equiv Rj \mod Q) \\
&\quad \times (\kappa_u)(2Rj - m) \exp \left( \frac{2i\pi k(m - Rj)}{QN} \right). 
\end{align*}
\]
(5.19)

The function \( \kappa_v \) is \((2N^2)\)-periodic, so one can replace the subscript \( 0 \leq j < RQ^2 \) by \( j \mod RQ^2 \). Using (5.17), we obtain
\[
\begin{align*}
(v | B_u) &= \frac{1}{N} \sum_{j \mod RQ^2} \sum_{0 \leq k < N} b(j, k) \sum_{m_1 \in \mathbb{Z}} \chi(m_1 \equiv Rj \mod Q) \\
&\quad \times \varphi \left( \frac{m_1}{R} \right) (\kappa_u)(2Rj - m_1) \exp \left( \frac{2i\pi k(m_1 - Rj)}{QN} \right). 
\end{align*}
\]
(5.20)

The change of \( m \) to \( m_1 \) is just a change of notation.

Fixing \( k \), we trade the set of pairs \( m_1, j \) with \( m_1 \equiv Rj \mod Q \) for the set of pairs \( m, n \in (\mathbb{Z}/(2N^2)\mathbb{Z}) \times (\mathbb{Z}/(2N^2)\mathbb{Z}) \), where \( m \) is the class mod \( 2N^2 \) of \( m_1 \) and \( n \) is the class mod \( 2N^2 \) of \( 2Rj - m_1 \). Of necessity, \( m + n \equiv 0 \mod 2R \) and \( m - n \equiv 2(m - Rj) \equiv 0 \mod 2Q \). Conversely, given a pair of classes \( m, n \mod 2N^2 \) satisfying these conditions, the equation \( 2Rj - m = n \) uniquely determines \( j \mod \frac{2N^2}{2R} = RQ^2 \), as it should.
The sum $\sum_{m \equiv m \mod 2N^2} \nu \left( \frac{m}{R} \right)$ is just $(\kappa v)(m)$, and we have obtained the identity
\[(v | Bu) = \sum_{m,n \mod 2N^2} c_{R,Q} (\mathcal{G}; m, n) (\kappa v)(m)(\kappa u)(n), \quad (5.21)\]
provided we define
\[c_{R,Q} (\mathcal{G}; m, n) = \frac{1}{N} \text{char}(m + n \equiv 0 \mod R, m - n \equiv 0 \mod 2Q) \sum_{k \mod N} b \left( \frac{m + n}{2R}, k \right) \exp \left( \frac{2i\pi k \cdot m - n}{N \cdot 2Q} \right), \quad (5.22)\]
which is just the way indicated in (5.11), (5.12).

\[\square\]

6. The $(R, Q)$-decomposition of the main hermitian form

We assume that $N = RQ$ is a squarefree odd integer and we now make the coefficients $c_{R,Q} (\mathcal{E}_N; m, n)$ fully explicit. The following reproduces [8, Prop. 4.2.8] with the necessary modifications necessitated by the choice of the symbolic calculus $\Psi$. Note that this choice saved us much irritation with the prime 2.

**Proposition 6.1.** One has
\[c_{R,Q} (\mathcal{E}_N; m, n) = \text{char}(m + n \equiv 0 \mod R) \text{char}(m - n \equiv 0 \mod 2Q) \times \mu \left( \left( \frac{m + n}{2R}, N \right) \right) \text{char} \left( N = \left( \frac{m + n}{2R}, N \right) \left( \frac{m - n}{2Q}, N \right) \right). \quad (6.1)\]

**Proof.** We shall give two proofs of this formula, fundamental for our aim. One has the identity, most certainly found many times,
\[S(N, k) = \sum_{r \mod N} \exp \left( -\frac{2i\pi rk}{N} \right) = \mu(N) a((k, N)). \quad (6.2)\]
For completeness’ sake, we reproduce the proof given in [8, (4.2.53)]. Set $N = pN_p$ for every $p | N$ and, choosing integral coefficients $\alpha_p$ such that $\sum_{p | N} \alpha_p N_p = 1$, write $\frac{1}{N} = \sum_{p | N} \frac{\alpha_p}{p}$. The coefficient $\alpha_p$, which is unique
mod $p$, is not divisible by $p$. Under the identification of $\mathbb{Z}/N\mathbb{Z}$ with $\prod_{p|N} \mathbb{Z}/p\mathbb{Z}$, one writes $r = (r^p)_{p|N}$. Then,

$$\exp\left(-2i\pi \frac{kr}{N}\right) = \prod_{p|N} \exp\left(-2i\pi kr \frac{\alpha_p}{p}\right) = \prod_{p|N} \exp\left(-2i\pi k \frac{r^p}{p}\right)$$  \hspace{1cm} (6.3)

and

$$S(N; k) = \prod_{p|N} \sum_{r^p \in (\mathbb{Z}/p\mathbb{Z})^\times} \exp\left(-2i\pi k \sum_{p|N} \frac{\alpha_p r^p}{p}\right).$$  \hspace{1cm} (6.4)

Since $p \nmid \alpha_p$, the value of the $p$th factor is $p - 1$ if $p|k$, $-1$ if $p \nmid k$. Hence,

$$S(N; k) = \prod_{p|N} \left[(-1)(1 - p \text{ char}(p|k))\right] = \mu(N) a((k, N)).$$  \hspace{1cm} (6.5)

Then,

$$\mu((j, N)) a((j, k, N)) = \mu((j, N)) a((k, (j, N))) = \sum_{r \mod (j, N)} \exp\left(-\frac{2i\pi r k}{(j, N)}\right).$$  \hspace{1cm} (6.6)

With $b(j, k) = a((j, k, N))$, one has

$$f(j, s) = \frac{1}{N} \sum_{k \mod N} a((j, k, N)) \exp\left(\frac{2i\pi ks}{N}\right) = \frac{\mu((j, N))}{N} \sum_{r \mod (j, N)} \sum_{k \mod N} \exp\left(\frac{2i\pi k}{N} \left(s - \frac{rN}{(j, N)}\right)\right).$$  \hspace{1cm} (6.7)

The inside sum coincides with $N \text{ char} \left(s \equiv \frac{rN}{(j, N)} \mod N\right)$. This requires first that $s \equiv 0 \mod \frac{N}{(j, N)}$, in other words $js \equiv 0 \mod N$, and we set $s = S \equiv \frac{Ns'}{(j, N)}$. Then,

$$N \text{ char} \left(s - \frac{rN}{(j, N)} \equiv 0 \mod N\right) = N \text{ char} \left(\frac{N}{(j, N)} (s' - r) \equiv 0 \mod N\right)$$

$$= N \text{ char} \left(s' - r \equiv 0 \mod (j, N)\right).$$  \hspace{1cm} (6.8)
Since
\[
\sum_{r \mod (j, N)} N \text{char} (s' - r \equiv 0 \mod (j, N)) = N \text{char} ((s', (j, N)) = 1),
\]
one obtains
\[
f(j, s) = \mu((j, N)) \text{char} (js \equiv 0 \mod N) \times \text{char} \left[ \frac{(j, N)s}{N}, (j, N) \right] = 1
\]
\[
= \mu((j, N)) \text{char} (js \equiv 0 \mod N) \times \text{char} \left[ ((j, N)s, (j, N)N) = N \right].
\]
(6.9)
The condition \(((j, N)s, (j, N)N) = N\), or \((j, N)(s, N) = N\), implies the condition \(js \equiv 0 \mod N\). Hence,
\[
f(j, s) = \mu((j, N)) \text{char} \left[ ((j, N), (s, N)) = N \right].
\]
(6.10)
The equation (6.1) follows then from (5.12).
\[\square\]

In view of the importance of the formula (6.1), let us give a short alternative proof of it, based on using Proposition 5.3 “in reverse”. So that (5.12) should be satisfied with \(c_{R,Q} (\mathfrak{T}_N; m, n)\) as it appears in (6.1), we must take
\[
f(j, s) = \text{char}(j, s \in \mathbb{Z}) \mu((j, N)) \text{char}(N = (j, N)(s, N)).
\]
(6.12)
A function \(b(j, k)\) leading to (5.11) can then be obtained by an inversion of that formula, to wit
\[
b(j, k) = \sum_{s \mod N} f(j, s) \exp \left(-\frac{2i\pi ks}{N}\right).
\]
(6.13)
If \(N = N_1 N_2\), one writes, choosing \(a, d\) such that \(aN_1 + dN_2 = 1\), \(\frac{1}{N} = \frac{d}{N_1} + \frac{a}{N_2}\), so that
\[
\exp \left(-\frac{2i\pi ks}{N}\right) = \exp \left(-\frac{2i\pi ds}{N_1}\right) \exp \left(-\frac{2i\pi ak}{N_2}\right)
\]
(6.14)
This leads to the equation (with an obvious notation) \(b(j, k) = b_{N_1}(j, k)b_{N_2}(j, k)\), and the proof may be reduced to the case when \(N\) is a prime \(p\). Then, in the formula (6.13), the term obtained for \(s \equiv 0\) is \(\text{char}(j \not\equiv 0 \mod p)\), and the sum of terms obtained for \(s \not\equiv 0\) is \(-\text{char}(j \equiv 0 \mod p) \times [p \text{ char}(k \equiv 0 \mod p)\)].
0 \mod p) - 1]. Overall, the $p$-factor under investigation is thus

$$
\text{char}(j \not\equiv 0) + \text{char}(j \equiv 0) \left[ 1 - p \text{ char}(k \equiv 0) \right] = 1 - p \text{ char}(j \equiv k \equiv 0).
$$

(6.15)

This concludes this verification.

**Lemma 6.2.** The coefficients introduced in (5.12) are given as

$$
c_{R, Q} (T_N; m, n) = \text{char}(m + n \equiv 0 \mod 2R) \mu \left( \left( \frac{m + n}{2R}, R \right) \right) \text{char} \left( R = \left( \frac{m + n}{2R}, R \right) (m - n, R) \right) \\
\times \sum_{Q_1 Q_2 = Q} \mu(Q_1) \text{char}(m \equiv n \equiv 0 \mod Q_1) \text{char}(m - n \equiv 0 \mod 2Q_2^2).
$$

(6.16)

**Proof.** We have recopied on the first line of the right-hand side the factor of (6.1) depending on $R$. Making use of $p$-adic absolute values to trace the divisibility by $p$ or $p^2$, the factor depending on $Q$ is

$$
\prod_{p|Q} \text{char}(m - n \equiv 0 \mod p) \mu((m + n, p)) \text{char} \left( p = (m + n, p) \left( \frac{m - n}{p}, p \right) \right)
= \prod_{p|Q} \text{char}(m - n \equiv 0 \mod p) \left[ \text{char}(m + n \not\equiv 0 \mod p) \text{char}\left( |m - n|_p \leq \frac{1}{p^2} \right) \\
- \text{char}(m + n \equiv 0 \mod p) \text{char}\left( |m - n|_p = \frac{1}{p} \right) \right].
$$

(6.17)

If one adds $\text{char}(m + n \equiv 0 \mod p) \text{char}\left( |m - n|_p \leq \frac{1}{p^2} \right)$ to either term within the last brackets, one rewrites the $Q$-factor under examination as

$$
\prod_{p|Q} \text{char}(m - n \equiv 0 \mod p) \left[ \text{char}\left( |m - n|_p \leq \frac{1}{p^2} \right) \\
- \text{char}(m + n \equiv 0 \mod p) \text{char}\left( |m - n|_p \leq \frac{1}{p} \right) \right] \\
= \prod_{p|Q} \left[ \text{char}(m - n \equiv 0 \mod p^2) - \text{char}(m \equiv n \equiv 0 \mod p) \right].
$$

(6.18)
The $Q$-factor under examination is thus
\[ \sum_{Q_1, Q_2 = Q} \mu(Q_1) \text{char}(m \equiv n \equiv 0 \mod Q_1) \text{char}(m - n \equiv 0 \mod 2Q_2^2) \quad (6.19) \]
and the lemma follows.

The point of the next lemma is that $(\theta_N v)(m)$, which depends in principle on the class of $m \mod 2N^2$, depends only on the class of $m \mod R^2$ under a certain support condition.

**Lemma 6.3.** Let $N = RQ$ be a squarefree odd integer, let $\beta > 0$ be given and assume that $R \geq 2\beta Q$. If two integers $m_0$ and $m'$ are such that $|m'| < \frac{R^2}{2}$, finally $|m_0| < \beta N$, one has of necessity $m_0 = m'$: in particular, such an $m_0$ does not exist unless $|m'| < \beta N$. If a function $v \in C^\infty(\mathbb{R})$ is supported in $[-\beta, \beta]$ and $m \in \mathbb{Z}/(2N^2)\mathbb{Z}$, one has $(\theta_N v)(m) = v\left(\frac{m'}{N}\right)$, where $m'$ is given by the conditions that $|m'| < \frac{R^2}{2}$ and that the class of $m'$ mod $R^2$ agrees with the image of $m$ under the canonical homomorphism $\mathbb{Z}/(2N^2)\mathbb{Z} \to \mathbb{Z}/R^2\mathbb{Z}$.

**Proof.** If $|m'| < \frac{R^2}{2}$ and $\ell = 1, 2, \ldots$, one has $|m' + \ell R^2| > \frac{R^2}{2} > \beta N$: hence, in the class of $m'$ mod $R^2$, only $m'$ can lie in $[\beta N, \beta N]$. To compute $(\theta_N v)(m)$, define $m'$ according to the last sentence of the lemma. If $|m'| \geq \beta N$, there is no integer $m_0$ such that $|m_0| < \beta N$, the class of which mod $R^2$ would lie in the image of $m$ under the natural homomorphism $\mathbb{Z}/(2N^2)\mathbb{Z} \to \mathbb{Z}/R^2\mathbb{Z}$: as a consequence of (5.8), one thus has $(\theta_N v)(m) = 0$, while $v\left(\frac{m'}{N}\right) = 0$ too. If $|m'| < \beta N$, the only such $m_0$ is $m'$, and the series (5.8) for $(\theta_N v)(m)$ reduces to $v\left(\frac{m'}{N}\right)$.

### 7. On the Structure of the Main Hermitian Form

**Lemma 7.1.** Let $N$ be a squarefree odd integer, and let $v, u$ be two functions in $\mathcal{S}(\mathbb{R})$: assume that the supports of $v$ and $u$ do not intersect. One has for every squarefree odd integer $N$ the identity
\[ (v \mid \Psi(\Sigma_N) u) = \frac{1}{\pi} \sum_{T | N} \mu(T) \sum_{j \neq 0, k \in \mathbb{Z}} \overline{v}\left(T j + \frac{k}{T}\right) u\left(T j - \frac{k}{T}\right). \quad (7.1) \]
Proof. We start from the definition (3.12) of $\mathcal{D}$ and the fact that the distribution $(x, \xi) \mapsto \mathcal{F}_2^{-1}(\mathcal{D}(x, \xi) + \delta(x)\delta(\xi))$ is invariant under $\mathcal{F}_2^{-1}$: hence,

$$\left(\mathcal{F}_2^{-1}\mathcal{D}\right)(r, s) = \mathcal{D}(r, s) + 2\pi \delta(r) (\delta(r) - 1).$$

(7.2)

The two extra terms do not matter for our purpose since, evaluated at the pair $(r, s) = (\frac{x+y}{2}, \frac{x-y}{2})$ as demanded by the recipe (2.2) for computing the integral kernel of an operator characterized by its symbol, their sum involves the factor $\delta(\frac{x+y}{2})$ and, under the assumption that the supports of $v$ and $\check{u}$ do not intersect, one has $v(x)u(y) = 0$ when $x + y = 0$. In order not to bother about the extra terms, let us use the sign $\equiv$, between two distributions in the arguments $r, s$, to indicate that their difference is “divisible” by $\delta(r)$. Hence,

$$\left(\mathcal{F}_2^{-1}\mathcal{D}\right)(r, s) \equiv \mathcal{D}(r, s).$$

(7.3)

Together with the operator $2i\pi\mathcal{E}$, let us introduce the operator $2i\pi\mathcal{E}^\blacklozenge = r \frac{\partial}{\partial r} - s \frac{\partial}{\partial s}$ when the coordinates $(r, s)$ are used on $\mathbb{R}^2$. One has $\mathcal{F}_2^{-1}[(2i\pi\mathcal{E})\mathcal{S}] = (2i\pi\mathcal{E}^\blacklozenge)\mathcal{F}_2^{-1}\mathcal{S}$ for every tempered distribution $\mathcal{S}$.

From (4.9), one obtains

$$\mathcal{F}_2^{-1}\mathcal{T}^\blacklozenge = \frac{1}{2\pi} \prod_{p|N} \left(1 - p^{-2i\pi\mathcal{E}^\blacklozenge}\right) \mathcal{D} = \frac{1}{2\pi} \sum_{T|N} \mu(T) T^{-2i\pi\mathcal{E}^\blacklozenge} \mathcal{D},$$

(7.4)

explicitly

$$\left(\mathcal{F}_2^{-1}\mathcal{T}^\blacklozenge\right)(r, s) \equiv \frac{1}{2\pi} \sum_{T|N} \mu(T) \sum_{|j| + |k| \neq 0} \delta \left( \frac{r}{T} - j \right) \delta(Ts - k).$$

(7.5)

The integral kernel of the operator $\Psi (\mathcal{T}^\blacklozenge_N)$ is

$$K(x, y) = \left(\mathcal{F}_2^{-1}\mathcal{T}^\blacklozenge_N\right) \left( \frac{x+y}{2}, \frac{x-y}{2} \right)$$

$$= \frac{1}{2\pi} \sum_{T|N} \mu(T) \sum_{|j| + |k| \neq 0} \delta \left( \frac{x+y}{2T} - j \right) \delta \left( \frac{T(x-y)}{2} - k \right)$$

$$= \frac{1}{\pi} \sum_{T|N} \mu(T) \sum_{|j| + |k| \neq 0} \delta \left( x - Tj - \frac{k}{T} \right) \delta \left( y - Tj + \frac{k}{T} \right).$$

(7.6)

The equation (7.1) follows.

□
That the triple series on the right-hand side of (7.1) is absolutely convergent follows immediately: it suffices, with some \( A > 3 \), to write
\[
\left| v \left( T_j + \frac{k}{T} \right) u \left( T_j - \frac{k}{T} \right) \right| \leq C (1 + |T_j|)^{-A} \left( 1 + \frac{|k|}{T} \right)^{-A}
\]
\[
\leq C (1 + T)^{-2} (1 + |T_j|)^{-A+2} \left( 1 + \frac{|k|}{T} \right)^{-A}.
\]
(7.7)

The sole difficulty in the search for the estimate that would yield R.H. consists in the dependence on \( Q \): finding a \( O \left( Q^{1+\varepsilon} \right) \) is easy, but we need a \( O \left( Q^{\frac{1}{2}+\varepsilon} \right) \). The following theorem exhibits the link between the operators with symbols \( T_N \) and \( Q \).

**Theorem 7.2.** Let \( N = RQ \) be a squarefree odd integer. Given a complex-valued function \( \psi \) on \( \mathbb{Z}/(2N^2)\mathbb{Z} \), identified with \( (\mathbb{Z}/R^2\mathbb{Z}) \times (\mathbb{Z}/(2Q^2)\mathbb{Z}) \), a product in which the coordinates are denoted as \( n', n'' \), define the function \( \Lambda_{R,Q} \psi \) by the equation
\[
(\Lambda_{R,Q} \psi) (n', n'') = \psi (n', -n'').
\]
(7.8)

Introduce the automorphism \( \Lambda_{R,Q}^\sharp \) of \( S(\mathbb{R}) \) defined by the equation
\[
\Lambda_{R,Q}^\sharp = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} \exp \left[ \frac{2R}{Q} \left( \tau \frac{d}{dx} + i\pi \sigma x \right) \right],
\]
(7.9)
in other words
\[
\left( \Lambda_{R,Q}^\sharp \right) (x) = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} w \left( x + \frac{2R\tau}{Q} \right) \exp \left( \frac{2i\pi \sigma (Nx + R^2\tau)}{Q^2} \right),
\]
(7.10)
so that, for every \( w \),
\[
\theta_N \Lambda_{R,Q}^\sharp w = \Lambda_{R,Q} \theta_N w.
\]
(7.11)

Then, for every pair of functions \( v, u \in S(\mathbb{R}) \), one has
\[
(v \mid \Psi (Q^{2i\pi \varepsilon} \varpi_N) u) = \mu(Q) \left( v \mid \Psi (\varpi_N) \Lambda_{R,Q}^\sharp u \right).
\]
(7.12)

**Proof.** With another normalization of the operator calculus, this was given in [8, Cor. 4.2.7]. The main part was given two different proofs, one in [8,
Prop. 4.2.3] and the other in [8, p.62-64].

What Theorem 7.2 says is that (up to the factor $\mu(Q)$), applying $Q^{2\pi i}E$ to $T_{N}$ has the same effect as transforming the $\theta_{N}$-transform of the second function $u$ of the pair under the reflection $(n', n'') \mapsto (n', -n'')$.

□

In [8, Section 3.4], besides the operator with symbol $Q^{2\pi i}E_{\infty}$ or $Q^{2\pi i}E_{T_{N}}$, we considered the operator with symbol $Q^{\pi i}E_{\infty}$ or $Q^{\pi i}E_{T_{N}}$. It is not surprising that the criterion given in (1.5) should extend in this case, only replacing $Q^{\frac{1}{2}+\varepsilon}$ on the right-hand side by $Q^{\frac{1}{4}+\varepsilon}$: however, one cannot truly reduce this case to the preceding one, since $Q$ is not, in the criterion, just a large number, but a squarefree integer. We quote this extension because of the following special case, which suggests that Hilbert-space methods might play a role in the problem.

**Theorem 7.3.** [8, Theor. 4.3.4] Let $Q$ be a squarefree odd integer. Let $\Lambda_{Q}$ be the set $\{\{\lambda \mod Q : \lambda^2 \equiv 1, Q\}\}$, in other words the set of $\lambda \in (\mathbb{Z}/Q\mathbb{Z})^\times$ such that $\lambda \equiv \pm p$ for every $p|Q$. Let $\chi_{Q} : \Lambda_{Q} \to \{\pm 1\}$ be the character defined by setting $\chi_{Q}(\lambda) = -1$ if one has $\lambda \equiv 1 \mod p$ except for an odd number of $p$'s. Given $\tau \in (\mathbb{Z}/Q\mathbb{Z})^\times$, introduce the measure on the line

$$d_{Q,\tau}(x) = \sum_{\lambda \in \Lambda_{Q}} \chi_{Q}(\lambda) \sum_{\ell \in \mathbb{Z}} \delta\left(x - Q\ell + \tau \lambda \sqrt{Q}\right).$$

(7.13)

Then, given $w \in \mathcal{S}(\mathbb{R})$, one has

$$\langle w | \Psi(Q^{\pi i}E_{Q}) w \rangle = 2^{-\iota(Q)} \sum_{\tau \in (\mathbb{Z}/Q\mathbb{Z})^\times} |(d_{Q,\tau} | w) |^2,$$

(7.14)

where $\iota(Q)$ is the number of distinct prime factors of $Q$.

Under the transfer by $\theta_{Q^2}$, a operator defined just like (5.8), just replacing $N$ by $Q^2$, the operator $\Psi(Q^{\pi i}E_{Q})$ transfers to the operator of orthogonal projection on the subspace of $\mathbb{Z}/(2Q)\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \prod_{p|Q}\mathbb{Z}/p\mathbb{Z}$ consisting of functions which are “totally odd”, meaning by this they are separately odd with respect to each of the coordinates of this product (with $p \neq 2$). Needless to say, there is no understanding of this space, in $\mathcal{S}(\mathbb{R})$, in classical terms.
The operator with symbol $Q^{i\pi E}E_T$ is a nonnegative symmetric operator on $\mathcal{S}(\mathbb{R})$, unbounded in any classical sense. Also, one has for it quite precise $Q$-estimates since [8, Prop. 4.5.1] one has for every $\varepsilon > 0$, if $v, u \in \mathcal{S}(\mathbb{R})$ and $Q = \prod_{p \leq q} p$, the estimate

$$
(v | \Psi (Q^{i\pi E}E_T) u) = e^{-\gamma Q^{1/2} \log q} \left[ (v | u) + \mu(Q) (v | \hat{u}) \right] + O \left( Q^\varepsilon \right).
$$

Yes, it is easy to replace in the criterion the pair $w, w$ by a more general pair $v, u$: it suffices to make support assumptions (compatible if so desired with the conditions $(v | u) = (v | \hat{u}) = 0$) so that the only term of (3.18) that will remain is the one corresponding to $r = 1$: for, in this case, ensuring that the expression (3.18) is nonzero for some choice of $v, u$ is immediate. But, in order to prove or disprove R.H., it is the operator with symbol $Q^{i\pi E}E_N$, with $N$ very large with respect to $N$, that we would need to consider. Analyzing what can remain in this case, if anything, of the Hilbert space structure put into evidence when $N = Q$, might help.

### 8. A one- or a two-dimensional problem?

The measure

$$
\sum_{k \neq 0} \mu(k) \delta(x - k) = \frac{1}{2i\pi} \int_{\Re \nu = c} \frac{|x|^{\nu - 1}}{\zeta(\nu)} d\nu
$$

also leads to a criterion for R.H., just replacing the two-dimensional distribution $E_{-\nu}$ by the power $|x|^{\nu - 1}$: but this criterion always ends up with some version of Littlewood’s, in terms of sums of values of the Möbius indicator. The two-dimensional problem, though inconclusive too, seems to lie deeper. First, the consideration of hermitian forms is well-documented as lying at the crux of many mathematical problems. Next, Theorem 7.2 gives the hermitian form of interest a large amount of structure, possibly preparing (Theorem 7.3) for the use of Hilbert space methods.

Finally, we shall observe here that, with this point of view, the Riemann zeta function fits naturally as an element of two distinct families of $L$-functions: that of Dirichlet $L$-functions and, perhaps more surprisingly, those associated to Hecke eigenforms of the full unimodular group. In each case, the appropriate distribution taking the place of $E_{-\nu}$ seems to be a possible starting point for the study of R.H.
Let $\chi$ be a Dirichlet character mod $M$ and let $L(s, \chi)$ be the associated $L$-function. Setting $a_{\chi}(r) = \prod_{p | r} (1 - p \chi(p))$, one defines with $c > 1$

$$T_{\chi}(x, \xi) = \sum_{|j| + |k| \neq 0} a_{\chi}((j, k)) \delta(x - j) \delta(\xi - k) = \frac{1}{2i\pi} \int_{\text{Re} \nu = c} \frac{E_{-\nu}(x, \xi)}{L(\nu, \chi)} d\nu. \quad (8.2)$$

One uses also the obvious definition of $T_{\chi}(N)$ and the partial product

$$L_N(s, \chi) = \prod_{p | N} (1 - \chi(p) p^{-s})^{-1} = \sum_{T | N} \mu(T) \chi(T) T^{-s}. \quad (8.3)$$

As a first simplification, we may, when taking up this case, consider only the squarefree odd integers $Q$ relatively prime to $M$: this does not harm the criterion. Then, it is only in the $R$-factors that the fact that $\chi$ is not the trivial character will be felt. All the computations in Section 5, which did not depend on the function $b$ satisfying the required periodicity conditions, extend. The first difference with the Riemann case occurs in Proposition 6.1. Applying Proposition 5.3, one has now

$$f(j, s) = \frac{1}{N} \sum_{k \mod N} a_{\chi}((j, k, N)) \exp\left(\frac{2i\pi ks}{N}\right), \quad j, s \in \mathbb{Z}/N\mathbb{Z}. \quad (8.4)$$

Again,

$$a_{\chi}((j, k, N)) = a_{\chi}((j, k, N_1)) a_{\chi}((j, k, N_2)) \quad (8.5)$$

if $N = N_1 N_2$. If $aN_1 + dN_2 = 1$, one has (6.14)

$$\exp\left(\frac{2i\pi ks}{N}\right) = \exp\left(\frac{2i\pi ds}{N_1}\right) \exp\left(\frac{2i\pi ak}{N_2}\right). \quad (8.6)$$

Since $(d, N_1) = 1$, the prime divisors of $(j, k, N_1)$ are the same as prime divisors of $(j, dk, N_1)$. It follows that $f(j, s)$ has a Eulerian structure, the factor of which corresponding to the prime divisor $p$ of $N$ being

$$f_p(j, s) = \frac{1}{p} \sum_{k \mod p} a_{\chi}((j, k, p)) \exp\left(\frac{2i\pi ks}{p}\right), \quad j, s \in \mathbb{Z}/p\mathbb{Z}. \quad (8.7)$$

The contribution of the term for which $k = 0$ is

$$\frac{1}{p} a_{\chi}((j, p)) = \frac{1}{p} [1 - p \chi(p) \text{char}(j \equiv 0 \mod p)]. \quad (8.8)$$

The remaining sum is

$$\frac{1}{p} \sum_{k \in (\mathbb{Z}/p\mathbb{Z})^\times} \exp\left(\frac{2i\pi ks}{p}\right) = \frac{1}{p} [p \text{char}(s \equiv 0 \mod p) - 1]. \quad (8.9)$$
Hence,
\[ f(j, s) = \prod_{p \mid N} \left[ \text{char}(s \equiv 0 \, \text{mod} \, p) - \chi(p) \, \text{char}(j \equiv 0 \, \text{mod} \, p) \right]. \]  
(8.10)

It follows that
\[
c_{R,Q} \left( \Sigma_N^X; m, n \right) = \text{char}(m + n \equiv 0 \, \text{mod} \, 2R) \, \text{char}(m - n \equiv 0 \, \text{mod} \, 2Q) \\
\quad \times \prod_{p \mid N} \left[ \text{char} \left( \frac{m - n}{Q} \equiv 0 \, \text{mod} \, p \right) - \chi(p) \, \text{char} \left( \frac{m + n}{R} \equiv 0 \, \text{mod} \, p \right) \right],
\]  
(8.11)

which reduces when \((Q, M) = 1\) to
\[
c_{R,Q} \left( \Sigma_N^X; m, n \right) = \text{char}(m + n \equiv 0 \, \text{mod} \, 2R) \, \text{char}(m - n \equiv 0 \, \text{mod} \, 2Q) \\
\quad \times \prod_{p \mid R} \left[ \text{char}(m - n \equiv 0 \, \text{mod} \, p) - \chi(p) \, \text{char} \left( \frac{m + n}{R} \equiv 0 \, \text{mod} \, p \right) \right]. \]  
(8.12)

The only difference with the case of the Riemann zeta function is the necessity, as expected, to replace the Möbius indicator \(\mu\) by the product \(\mu \chi\).

The original Riemann problem fits also as a particular case of a question involving modular form theory. Given a non-trivial character \(\chi: \mathbb{Q}^\times \to \mathbb{C}^\times\) and a real number \(\lambda\), consider [7, p.20-25] the distribution
\[
N_{\chi,i\lambda}(x, \xi) = \frac{1}{4} \sum_{mn \neq 0} \chi \left( \frac{m}{n} \right) \, |n|^{i\lambda} \, |\xi|^{-1-i\lambda} \, \exp \left( \frac{2i\pi mn \, x}{\xi} \right),
\]  
(8.13)

and, with \(\varepsilon = 0\) or \(1\) such that \(\chi(-1) = (-1)^\varepsilon\), define the functions
\[
L(s, N_{\chi,i\lambda}) = \sum_{m,n \geq 1} \chi \left( \frac{m}{n} \right) \, m^{-s-i\lambda} \, n^{-s+i\lambda}, \\
L^*(s, N) = \pi^{-s} \Gamma \left( \frac{s + \varepsilon}{2} + \frac{i\lambda}{4} \right) \Gamma \left( \frac{s + \varepsilon - i\lambda}{2} - \frac{i\lambda}{4} \right) \, L(s, N_{\chi,i\lambda}).
\]  
(8.14)

The distribution \(N_{\chi,i\lambda}\) is automorphic (i.e., invariant under the linear transformations of \(\mathbb{R}^2\) associated to matrices \(g\) in \(\text{SL}(2, \mathbb{Z})\): it is so when \(g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\) with \(b \in \mathbb{Z}\) but the case when \(g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) remains to be examined) if and only if the function \(s \mapsto L(s, N_{\chi,i\lambda})\) extends as an entire function, polynomially bounded in vertical strips, satisfying the functional equation
\[
L^*(s, N_{\chi,i\lambda}) = (-1)^\varepsilon L^*(1 - s, N_{\chi,i\lambda}).
\]  
(8.15)
In such a case, the character $\chi$ must be unitary (Ramanujan’s theorem for Maass forms [9]). The knowledge of the distribution $\mathfrak{N}_{\chi,i\lambda}$ certainly determines $-1 - i\lambda$, its degree of homogeneity, but does not determine $\chi(p)$ for every $p$: only up to the possible change of $\chi(p)p^{-\frac{i\lambda}{2}}$ to its inverse.

If $\chi$ is the trivial (constant) character and one replaces in (8.13) the subscript $mn \neq 0$ by $|m| + |n| \neq 0$, one obtains [7, p.15] a definition of the Eisenstein distribution $\frac{1}{2} E_{i\lambda}$ and, generalizing (8.14), one has

$$L \left( s, \frac{1}{2} E_{i\lambda} \right) = \zeta \left( s - \frac{i\lambda}{2} \right) \zeta \left( s + \frac{i\lambda}{2} \right). \quad (8.16)$$

The fact that $E_{i\lambda}$ is automorphic for every $\lambda$ is thus equivalent to the functional equation of zeta: more precisely (since the two factors on the right-hand side of (8.16) must be exchanged), it becomes equivalent to it if completed by the fact that the distribution $\frac{1}{2} E_{i\lambda}$ and its $F_{\text{sym}}$-transform have the same $L$-function. One may thus consider the functional equation of zeta as a property of automorphy.

The theory of Eisenstein distributions and distributions $\mathfrak{N}_{\chi,i\lambda}$ relates (it is more precise) to that of Eisenstein series and modular forms of the non-holomorphic species. This has been made explicit in [7, Section 2.1]: let us just mention here that $\mathfrak{N}_{\chi,i\lambda}$ then deserves to be called a Hecke eigenform. One might attack the Riemann hypothesis for the function

$$L \left( s, \frac{1}{2} \mathfrak{N}_{\chi,i\lambda} \right) = \zeta \left( s + \frac{i\lambda}{2}, \chi \right) \zeta \left( s - \frac{i\lambda}{2}, \chi \right), \quad (8.17)$$

with $\zeta(\nu, \chi) = \sum_{n \geq 1} n^{-\nu} \chi(n)$, by the consideration of the distribution

$$\mathcal{T}_{\chi}(x, \xi) = \sum_{|j| + |k| \neq 0} a^\chi((j,k)) \delta(x - j) \delta(\xi - k), \quad (8.18)$$

where $a^\chi(r) = \prod_{p|r} (1 - p \chi(p))$.

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