An exact spin liquid state on the kagome lattice with topological order

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Abstract

We find an exact spin liquid state without time reversal symmetry on the kagome lattice with odd number of electrons per unit cell and explicit wave functions for all eigenstates. We also obtain that all spin-spin correlations are zero except trivial cases. We then show that there are anyonic excitations in our model. Finally, we demonstrate the existence of ground state degeneracy and gapless edge states indicating nontrivial topological order in our model. We also label all eigenstates on torus by local and global string operators.

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Introduction. –There has been enormous interest recently in topological quantum phases [1, 2] beyond the Landau paradigm [3], which is based on the idea of symmetry breaking and local order parameters. The newly discovered phases have many exotic phenomena such as electron fractionalization [4] and emergent gauge fields [5]. The traditional order parameter descriptions are either inapplicable or insufficient for these new phases.

Spin liquid states in dimension two (2D) are examples of such exotic phases. In the spin liquid states, strong quantum fluctuation forbids existence of nonzero spin order parameter. Although many numerical and analytical evidences indicate the existence of such exotic state in 2D, exact solution is absent until very recently [6, 7, 8, 9, 10, 11, 12]. These exact solutions provide us with a lot of intuition of the internal structure of spin liquid states.

Spin liquid states that spontaneously break time reversal symmetry (TRS) and spacial inversion symmetry are called chiral spin states [13]. An exact chiral spin state is proposed

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in Ref.[9] based on short-range anisotropic interaction [7], but this model have even number of electrons per unit cell, and thus not deserve to be called “Mott insulators” or “spin liquid” states in the traditional sense. We consider it an interesting question to find whether there exist other exact spin liquid states with arbitrarily given symmetry properties and odd number of electrons per unit cell.

In this letter we generalize the toric code model [8, 6] to the kagome lattice (Fig.1) and propose an exact spin liquid state that breaks TRS. This state is not chiral spin state in conventional sense because TRS is already broken in the Hamiltonian. The kagome lattice has odd number of spins per unit cell, and thus the proposed state is a true spin liquid state. Our model treat three spin components $\sigma^x, \sigma^y, \sigma^z$ democratically, which is more symmetrical than the $\sigma^x, \sigma^y$ model [8, 6]. We propose explicit wave functions for all the energy eigenstates and interpret these eigenstates as string-net [1] coherent states. We also compute spin-spin correlation functions and find that all of them are zero except the trivial cases of on-site correlations. After a short discussion of anyonic excitation in our model, we proceed to discuss the ground state degeneracy and gapless edge states indicating topological order. We also find all the eigenstates of the model on torus using string operators.

The model. —Our spin-$\frac{1}{2}$ Hamiltonian on the kagome lattice is given as:

$$H = -\sum_p V_p F_p$$

with summation over all elementary triangular and hexagonal plaquettes. We choose $F_p = \sigma_1^x \sigma_2^y \sigma_3^z$ on the triangles and $F_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^x \sigma_5^y \sigma_6^x$ on the hexagons (see Fig.1), where $\sigma_i^a$ are the well-known Pauli matrices. It follows from the property of Pauli matrices that $F_p^2 = 1$. Another crucial property of $F_p$ is that $[F_p, F_q] = 0$ for any pairs $p$ and $q$. So all $F_p$ can have eigenvalues$(\pm 1)$ simultaneously. Since $\sigma_i^a$ change sign under time reversal transformation, the Hamiltonian breaks TRS explicitly. The solubility of our model does not depend on signs of $V_p$, so we will choose all $V_p = V > 0$ for simplicity.
Firstly we consider a planar region with \( N \) sites, where \( N \) is very large. The number of elementary triangle and hexagon are shown to be \( 2N/3 \) and \( N/3 \), respectively. So there are \( N \) plaquettes, and \( F_p \) can label \( 2^N \) states. The number of physical spin states is also \( 2^N \). From this counting argument we know that eigenvalues of \( F_p \) label all the states on an infinite plane and the model is exactly solvable.

We can write down the following wave functions for all eigenstates:

\[
|\psi_\lambda\rangle = \prod_p (1 + \lambda_p F_p) |\psi_0\rangle
\]  

(2)

where \( p \) exhaust all elementary triangular and hexagonal plaquettes. \(|\psi_0\rangle\) is an arbitrary spin state which we choose as the state satisfying \( \sigma_i^z = +1 \) for all sites. The irrelevant normalization factors have been omitted. Different choices of \( \lambda_p = \pm 1 \) label different states. The \( F_p \) expectation in \(|\psi_\lambda\rangle\) is:

\[
\frac{\langle \psi_\lambda | F_p | \psi_\lambda \rangle}{\langle \psi_\lambda | \psi_\lambda \rangle} = \lambda_p
\]

(3)

so \(|\psi_\lambda\rangle\) is an eigenstate of \( H \) with eigenvalue \( E = -\sum_p \lambda_p V_p \). The ground state is the state with all \( \lambda_p = 1 \). The above simple wave functions have transparent explanation in the physical language of string-net condensation \([1]\). We will demonstrate this in our model.

A caution in order is that this wave function can completely determine the state only on an infinite plane. On systems with boundary like disk or closed like torus, there are other degrees of freedom missing. We will return to this question later.

We define two types of strings \( T_1 \) and \( T_2 \) \([6,1]\). \( T_1 \) strings walk on triangles while \( T_2 \) strings walk on hexagon (see Fig.2). We define the string operators as \( P_t = \prod_{i \in t} \sigma_i^a \) for each string \( t \). The choice of \( \sigma_i^a \) depends on both the site \( i \) and the string type. For \( T_1 \) strings, \( \sigma_i^a = \sigma^y, \sigma^z, \sigma^x \) at the three vertexes 1,2,3 of each triangle, respectively. For \( T_2 \) strings, \( \sigma_i^a = \sigma^z, \sigma^y, \sigma^x, \sigma^z, \sigma^y, \sigma^x \) at the six vertexes 1,2,3,4,5,6 of the hexagon, respectively. An important property of this definition is that \( T_1 \) and \( T_2 \) string operators involve different Pauli matrices at any sites \( i \), from which we can readily obtain the simple commutative relations among \( P_t \). For arbitrary \( T_1 \) strings \( s, t \) and \( T_2 \) strings \( u, v \), we have \([P_s, P_t] = 0 \), \([P_u, P_v] = 0 \); \([P_s, P_u] = 0 \) if \( s \) and \( u \) intersect each other by even times, while \([P_s, P_u] = 0 \) if \( s \) and \( u \) intersect each other by odd times.

These strings are divided into two classes by their topology. A string is called open string if it has endpoints, otherwise is called closed string \([1]\). From the property of Pauli matrices we can find \([P_t, F_p] = 0 \) for any \( p \) for a closed string \( t \), and it follows from this that \([P_t, H] = 0 \). For open string \( s \) with two endpoints on plaquettes \( p, q \), we have \([P_s, F_a] = 0 \) if \( a = p \) or \( q \), and \([P_s, F_a] = 0 \) otherwise. So \( P_s \) changes the sign of \( F_p \) at its two endpoints. If we regard negative \( F_p \) as vortex at \( p \), then \( P_s \) applied to the ground state will creates two vortexes at its endpoints and increase the energy by \( 2V \).

It is interesting that \( F_p \) are also closed strings operators defined above. Actually, for a \( T_1 \) string \( t_1 \) surrounding a single hexagon \( p \) (such as string \( c \) in fig. 2), we have \( P_{t_1} = F_p \). Similar relations also hold for any \( T_2 \) string surrounding a single triangle. So we conclude
Figure 2: Strings on the kagome lattice. \(a, b\) are open strings, while \(c, d\) are closed strings. \(a, c\) are \(T_1\) strings, while \(b, d\) are \(T_2\) strings. The letters \(x, y, z\) denote the specific Pauli matrices appearing in the string operators.

that \(F_p\) are special cases of closed string operators. This observation gives physical meaning to the above wave function. We expand the wave function:

\[
|\psi_\lambda\rangle = (1 + \sum_p \lambda_p F_p + \sum_{p_1, p_2} \lambda_{p_1} \lambda_{p_2} F_{p_1} F_{p_2} + \cdots)|\psi_0\rangle
\]  

(4)

each term is a closed string state. So \(|\psi_\lambda\rangle\) are coherent states of closed strings.

We mention that there are also two types of strings in the models of Kitaev and Wen [6, 13, 18], but strings in our model have the additional feature that \(T_1\) and \(T_2\) strings walk on triangles and hexagons, respectively. So their properties are different. One of the differences is that there are twice as many triangles as hexagons and thus more room for open \(T_1\) string endpoints than for \(T_2\) string endpoints.

Exact spin correlation. –The spin-spin correlation functions can be extracted from the exact wave functions. Since wave functions of all eigenstates are known, we can obtained all the spin-spin correlations by direct computation:

\[
\sigma_i^a \sigma_j^b |\psi_\lambda\rangle = \prod_s (1 - \lambda_s F_s) \prod_t (1 + \lambda_t F_t) \sigma_i^a \sigma_j^b |\psi_0\rangle
\]  

(5)

Where \(s\) exhausts elementary plaquettes with the property \(\{P_s, \sigma_i^a \sigma_j^b\} = 0\), and \(t\) exhausts elementary plaquettes with the property \(\{P_s, \sigma_i^a \sigma_j^b\} = 0\). We can check that such \(s\) exist except in the trivial cases when \(i = j\) and \(a = b\). Since \((1 + \lambda_s F_s)(1 - \lambda_s F_s) = 0\), we obtain the exact result:

\[
\langle \psi_\lambda | \sigma_i^a \sigma_j^b | \psi_\lambda \rangle = 0
\]  

(6)

except the trivial cases \(\langle \psi_\lambda | \sigma_i^a \sigma_j^b | \psi_\lambda \rangle = \langle \psi_\lambda | 1 | \psi_\lambda \rangle = 1\). This computation is analogous to the computation of spin-spin correlations in the Kitaev model [14], where they are found
to be exactly zero beyond the nearest neighbor. The spin-spin correlations in our model are even more exotic. They are identically zero for two different sites including the case of nearest neighbors. We also mention that the spin correlations in the earlier models \[1, 6\] can also be computed using explicit wavefunctions. Our method can also be generalized to computation of multi-spin correlations. Another feature is that the result is valid for all energy eigenstates, besides the ground state. This remarkable feature is also shared by the Kitaev’s model on honeycomb lattice \[7, 14\].

The zero spin correlations indicate very strong quantum fluctuation in our model. So the conventional idea of using spins as local order parameter will not give meaningful result here. But there is topological order \[1\] in this model manifesting in ground state degeneracy on closed manifold and gapless edge states at the boundary of open systems.

**Anyonic excitation.** – The anyonic excitation in our model has similar properties to earlier models \[6, 7, 8\]. The energy of an open string is not changed if we move the two endpoints far apart, i.e. the endpoints are deconfined. We call these endpoints \(T_1\) and \(T_2\) vortexes. We also mention that these vortexes can be interpreted as \(Z_2\) vortex excitations of an emergent \(Z_2\) gauge field \[1, 15\].

The vortex hopping operators is very simple, they are the same matrices appearing in the definition of \(P_t\). The statistical angles of \(T_1\) and \(T_2\) vortexes can be shown to be both zero by the statistical algebra method \[16\]. So the same type of vortexes see each other as bosons. The mutual statistical angle of \(T_1\) and \(T_2\) vortexes can be computed in a different way. Following Kitaev’s work \[6\], we consider the physical process of moving \(T_2\) vortex \(a\) around \(T_1\) vortex \(b\) along the path of string \(u\) (shown in Fig.3). The initial state is \(|\psi_i\rangle = P_s P_t |\psi\rangle\), where \(|\psi\rangle\) is a state with all vortexes far away from \(a, b\). The final state is:

\[
|\psi_f\rangle = P_u |\psi_i\rangle = P_u P_s P_t |\psi\rangle = -P_s P_t P_u |\psi\rangle = -|\psi_i\rangle 
\]

the result does not depends on detail shape of \(u\). The only important thing is that \(b\) is enclosed by \(u\). So the phase factor \(-1\) is purely topological. The mutual statistical angle \(\theta\) is defined by \(e^{2i\theta} = -1\). So we have \(\theta = \pi/2\) which is half of fermion’s. This anyonic mutual statistic is a signal of topological order in our model.

**Ground state degeneracy and gapless edge states.** – The ground state degeneracy on a closed manifold often indicate topological order hidden in a phase \[17, 11, 13\]. To study the topological order of our model, we consider a periodic system, i.e. we put the system on a torus. For a \(m \times n\) system (as Fig.4), there are \(2mn\) triangles, \(mn\) hexagons, and \(3mn\) sites (the sites at opposite boundaries are actually the same site and should not be re-counted). Now the mutual independency of \(F_p\) is destroyed by the periodic boundary condition. Actually, one can readily check that \(\prod_{p \in \text{triangle}} F_p = 1\) and \(\prod_{p \in \text{hexagon}} F_p = 1\), so the number of independent \(F_p\) are \(3mn - 2\).

It has been mentioned that closed string operators \(P_t\) commute with \(H\) and thus do not change the energy eigenvalue. But does \(P_t\) have chance to transform a given energy eigenstate to another eigenstate that degenerate with it? For the cases when \(t\) is homotopically trivial, i.e. \(t\) is able to be deformed to a point, the answer is no. Actually, when \(t\) is a homotopically trivial \(T_1\) string, it can be checked that \(P_t = \prod_{p} F_{p}\), with \(p\) exhausting the hexagons enclosed by \(t\). So \(P_t\) cannot transform a given energy eigenstate to another. Similar result holds for \(T_2\) strings.
Figure 3: Statistic of open string endpoints. *s* is an open $T_2$ string with one endpoint *a* and the other endpoint far away, while *t* is an open $T_1$ string with one endpoint *b* and the other endpoint also far away. *u* is a closed string enclosing *b*.

But there are also homotopically nontrivial strings called global strings on the torus. Consider four global strings $t_1$, $t_2$, $t_3$ and $t_4$ shown in Fig.4. We have $\{P_{t_1}, P_{t_3}\} = 0$ and $\{P_{t_2}, P_{t_4}\} = 0$ by their intersecting times. All other pairs of string operators are commutative. Since all these operators commute with $F_p$, we can choose the common eigenstates of $P_{t_1}$, $P_{t_2}$ and $F_p$ as the basis of the Hilbert space. To find whether eigenvalues of $P_{t_1}$, $P_{t_2}$ and $F_p$ can label all states completely, we just need to do a simple counting: All of $P_{t_1}$, $P_{t_2}$ and $F_p$ have two eigenvalues $\pm 1$, so they can label $2^{3mn-2} \times 2 \times 2 = 2^{3mn}$ states. This is just the dimension of physical Hilbert space. So $P_{t_1}$, $P_{t_2}$ and $F_p$ are complete labels and we have found all energy eigenstates on the torus. From the above discussion we conclude that states with given $F_p$ has additional 4-fold degeneracy coming from two global string operators $P_{t_1}$ and $P_{t_2}$. It follows from this that states with all $F_p = 1$ are 4-fold degenerate, i.e. the ground state degeneracy is 4.

Furthermore, we can figure out how these eigenstates transform under $P_{t_3}$ and $P_{t_4}$. Since $\{P_{t_3}, P_{t_1}\} = 0$ and $[P_{t_3}, P_{t_2}] = 0$, we conclude that $P_{t_3}$ change the sign of $P_{t_1}$ but not of $P_{t_2}$. By similar argument, we can also show that $P_{t_4}$ change the sign of $P_{t_2}$ but not of $P_{t_1}$. This argument also shows that both $\pm 1$ eigenvalues of $P_{t_1}$ and $P_{t_2}$ are accessible by some physical states.

We mention that $t_1$ and $t_2$ can be chosen rather arbitrarily, with only requirements that they circle the torus in $m$ and $n$ directions respectively, and they are of the same string type. The physics here will not change with our choices because it is topological.

We also find the existence of gapless edge states in our model, which is another manifestation of topological order. If we cut the torus open along the bold lines in Fig.4, the dimension of Hilbert space is larger by a factor $2^L$ than the case of torus, where $L \approx m + n$ when $m, n$ are large. But the number of $F_p$ labeling energies is unchanged when we cut the torus open. So there is a large degeneracy proportional to $2^L$, which we can interpret as various edge excitations with zero energy [1, 8].

**Conclusion.** –An exact spin liquid state has been given on the kagome lattice based on
Figure 4: Kagome lattice on torus. $m = n = 3$ in this figure. The effective region is within the dark bold lines. The four copies of the point “A” should be regarded as one and the same point because of periodic boundary condition. $t_1, t_2, t_3$ and $t_4$ are all closed strings on the torus. $t_1, t_2$ are $T_1$ strings, while $t_3, t_4$ are $T_2$ strings. None of them can be deformed to a point and thus they are called global strings.

multi-spin interactions. All spin-spin correlations are found to be zero except trivial cases. There are also anyonic excitations and ground state degeneracy indicating nontrivial topological order. We think that this exact spin liquid state with odd number of spins per unit cell might be helpful to improve our understanding of 2D spin liquid states. Furthermore, the existence of such exact states suggests that there maybe exist other interesting states based on multi-spin interactions.

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