Quantifying Dip–Ramp–Plateau for the Laguerre Unitary Ensemble Structure Function

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Abstract: The ensemble average of $|\sum_{j=1}^{N} e^{ik\lambda_j}|^2$ is of interest as a probe of quantum chaos, as is its connected part, the structure function. Plotting this average for model systems of chaotic spectra reveals what has been termed a dip–ramp–plateau shape. Generalising earlier work of Brézin and Hikami for the Gaussian unitary ensemble, it is shown how the average in the case of the Laguerre unitary ensemble can be reduced to an expression involving the spectral density of the Jacobi unitary ensemble. This facilitates studying the large $N$ limit, and so quantifying the dip–ramp–plateau effect. When the parameter $a$ in the Laguerre weight $x^a e^{-x}$ scales with $N$, quantitative agreement is found with the characteristic features of this effect known for the Gaussian unitary ensemble. However, for the parameter $a$ fixed, the bulk scaled structure function is shown to have the simple functional form $\frac{2}{\pi} \arctan k$, and so there is no ramp-plateau transition.

1. Introduction

A prominent application of random matrix theory is to quantum chaos; see e.g. the text [22]. A basic postulate is that within blocks of the Hamiltonian corresponding to good quantum numbers (e.g. angular momentum etc.), and for large energy, the statistical properties of the rescaled energy levels coincides with the statistical properties of the bulk scaled eigenvalues of particular model Hamiltonians. The latter are random matrices: $N \times N$ real (complex) Hermitian matrices $H$ formed from matrices with standard real (complex) Gaussian entries $X$ according to $H = \frac{1}{2} (X + X^\dagger)$ in the case that the Hamiltonian admits (does not admit) a time reversal symmetry. In the real case, this class of random matrices is said to specify the Gaussian orthogonal ensemble (GOE), and in the complex case the Gaussian unitary ensemble (GUE). In a theoretical analysis, bulk scaling corresponds to first rescaling the eigenvalues $\lambda \mapsto \pi \lambda/\sqrt{2N}$ so that in the neighbourhood of the origin the mean spacing is unity, then taking the limit $N \to \infty$.

For spectral data, rescaling the energy levels is referred to as unfolding. In the case of the GOE or GUE, use can be made of the fact that to leading order the eigenvalue density
is given by the Wigner semi-circle functional form
\[ \rho_{(1)}(\lambda) = \left(\sqrt{2N}/\pi\right)(1 - \lambda^2/2N)^{1/2}, \]
supported on \( |\lambda| < \sqrt{2N}; \) see e.g. [17, Eq. (1.52)]. The edge eigenvalues in the data are discarded by restricting to \( |\lambda| < c\sqrt{2N} \) for a fixed \( 0 \ll c < 1, \) thus leaving only bulk eigenvalues. Unfolding of these bulk eigenvalues is carried out by rescaling \( \lambda \mapsto \lambda_j/\rho_{(1)}(\lambda_j), \) so that the new density is unity. For spectral data coming from less idealised circumstances, where the theoretical eigenvalue density is not known, or the data is noisy, poorly resolved or even incomplete, unfolding can no longer be precisely defined; see e.g. [33]. A fundamental question then arises: can an informative statistical quantity be found, providing at least a qualitative indicator of quantum chaos, without unfolding? In [29] it was proposed that the structure function, also known as the spectral form factor and first introduced into the study of quantum chaos by Berry [2], is well suited for this purpose.

The spectral form factor can be viewed as an example of the variance of a particular linear statistic. Before specifying the variance, let us first define the more general covariance. Thus, with \( A = \sum_{j=1}^{N} a(\lambda_j), B = \sum_{j=1}^{N} b(\lambda_j) \) two general linear statistics, so named since \( a(\lambda), b(\lambda) \) are functions of a single eigenvalue only, the corresponding covariance is defined by

\[ \text{Cov}(A, B) := \left( (A - \langle A \rangle)(B - \langle B \rangle) \right). \] (1.1)

Now choose

\[ A = \sum_{j=1}^{N} e^{ik_1\lambda_j}, \quad B = \sum_{j=1}^{N} e^{-ik_2\lambda_j}. \] (1.2)

In the special case \( k_1 = k_2 = k, \) the covariance (1.1) reduces to the variance

\[ \text{Var} A := S_N(k) = \left| \left( \sum_{j=1}^{N} e^{ik\lambda_j} \right) \right|^2 - \left| \left( \sum_{j=1}^{N} e^{ik\lambda_j} \right) \right|^2, \] (1.3)

and it is this quantity which is called the (unscaled) structure function.

The work [29] identified a qualitative property of the graph of the first average on the RHS of (1.3)—namely the existence of a minimum value separating the small and large \( k \) forms, and its neighbourhood—as an indicator of quantum chaos. There it was termed a correlation hole, and later as a dip–ramp–plateau, when it became prominent in the course of recent studies on the scrambling of information in black holes [7,11] and many body quantum chaos [6,8,9,41]. The term dip–ramp–plateau came about after the use of the GUE as a benchmark for the study of (1.3) [7], and in particular the first average, where the three behaviours inherent in the name are clearly visible in the corresponding graph. These more recent studies also identified an analogous effect for the first term in the rewrite of the covariance

\[ \text{Cov}(A, B) := \langle AB \rangle - \langle A \rangle \langle B \rangle, \] (1.4)

where \( A, B \) are given by (1.2) with

\[ k_1 = i\Gamma + k, \quad k_2 = -i\Gamma + k \quad (\Gamma > 0). \] (1.5)
Some insight into the dip–ramp–plateau effect is obtained upon relating the averages (1.3) and (1.4) to correlation functions. In relation to the latter, first recall that the joint eigenvalue probability density function (PDF) for the GOE and GUE is of the form

$$P_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{C_N} \prod_{i=1}^{N} w(\lambda_i) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta,$$

(1.6)

with \( w(x) = w^{(G)}(x) := e^{-\beta x^2/2} \) and \( \beta = 1 (\beta = 2) \) for the GOE (GUE). The corresponding \( k \)-point correlation function \( \rho(k) \) is specified in terms of \( P_N \) by

$$\rho(k)(\lambda_1, \ldots, \lambda_k) = \frac{N!}{(N-k)!} \int_{-\infty}^{\infty} d\lambda_{k+1} \cdots \int_{-\infty}^{\infty} d\lambda_N P_N(\lambda_1, \ldots, \lambda_N).$$

(1.7)

In the case \( k = 1 \), this corresponds to the spectral density. The ratio \( \rho(2)(\lambda_1, \lambda_2)/\rho(1)(\lambda_2) \) has the interpretation of the eigenvalue density at \( \lambda_1 \), given there is an eigenvalue at \( \lambda_2 \).

Now introduce the microscopic density

$$n(1)(\lambda) = \sum_{j=1}^{N} \delta(\lambda - \lambda_j),$$

(1.8)

and use this to define the density-density correlation \( N(2) \),

$$N(2)(\lambda, \lambda') = \text{Cov} \left( n(1)(\lambda), n(1)(\lambda') \right).$$

(1.9)

The effect of the delta functions gives rise to integrals of the form (1.7) for \( k = 1 \) and \( k = 2 \), showing that

$$N(2)(\lambda, \lambda') = \rho(2)(\lambda, \lambda') + \delta(\lambda - \lambda') \rho(1)(\lambda') - \rho(1)(\lambda) \rho(1)(\lambda').$$

(1.10)

For general linear statistics \( A, B \) as defined above (1.1), the covariance (1.1) can be expressed in terms of \( N(2) \) as the double integral

$$\text{Cov} \left( A, B \right) = \int_{-\infty}^{\infty} d\lambda \ a(\lambda) \int_{-\infty}^{\infty} d\lambda' \ b(\lambda') \ N(2)(\lambda, \lambda')$$

$$= \int_{-\infty}^{\infty} d\lambda \ a(\lambda) \int_{-\infty}^{\infty} d\lambda' \ b(\lambda') \left( \rho(2)(\lambda, \lambda') + \delta(\lambda - \lambda') \rho(1)(\lambda') \right)$$

$$- \rho(1)(\lambda) \rho(1)(\lambda'),$$

(1.11)

where the second equality follows from (1.10). From this second expression, separating off the term in the integrand involving the product of densities corresponds to the form of the covariance (1.4), and specialising to \( A, B \) given by (1.2) with \( k_1, k_2 \) therein given by (1.5), we deduce

$$\langle AB \rangle = \int_{-\infty}^{\infty} d\lambda \ e^{(-\Gamma+ik)\lambda} \int_{-\infty}^{\infty} d\lambda' \ e^{(-\Gamma-ik)\lambda'} \left( \rho(2)(\lambda, \lambda') + \delta(\lambda - \lambda') \rho(1)(\lambda') \right)$$

$$+ \left| \int_{-\infty}^{\infty} e^{(-\Gamma-ik)\lambda'} \rho(1)(\lambda') \ d\lambda' \right|^2,$$

(1.12)
where $\rho^T_{(2)}$ denotes the truncated (also known as connected) two point correlation, obtained from $\rho_{(2)}$ by subtracting the product of the corresponding densities.

The significance of the decomposition (1.12) is that it distinguishes two distinct functional behaviours, both with respect to $N$, and with respect to $k$. With respect to $N$, the first term is proportional to $N$ while the second is proportional to $N^2$. That the first term is proportional to $N$ is a fundamental property of variances and covariances of smooth linear statistics in random matrix theory; see e.g. [38]. With respect to $k$, the first term increases linearly from zero (the ramp)—for an explanation in terms of screening in the underlying log-gas picture, see [17, Sect. 14.1], or for one in terms of universality see [14]—before asymptoting to a finite value (the plateau). In contrast, the second term decreases to zero (the dip) as $k$ increases.

In the case of the GUE (indicated by the use of the subscript $(G)$), the structure function (1.3) $S^G_N$ can be reduced to the single integral via the quite striking identity

$$S^G_N(k) = \int_0^k tK^{(L)}_N(t^2/2, t^2/2)\bigg|_{a=0} dt, \quad (1.13)$$

as found by Brézin and Hikami [3] (for recent alternative derivations see [18,36]). Here $K^{(L)}_N$ denotes the correlation kernel for the Laguerre unitary ensemble (LUE), the latter corresponding to the eigenvalue PDF (1.6) with $\beta = 2$ and weight

$$w(x) = w^{(L)}(x) = x^a e^{-x} \chi_{x > 0}, \quad (1.14)$$

where $\chi_A = 1$ for $A$ true and $\chi_A = 0$ otherwise; the specification of the correlation kernel is given in (2.7) below. Thus (1.13) is an example of an inter-relationship between different random matrix ensembles, each with unitary symmetry; for others (albeit of a different nature) see [12,16]. Moreover, with the linear statistics $A, B$ given by (1.2), as an extension of (1.13), it was derived in [18] that

$$\text{Cov}(A, B)^{(G)} = \int_0^{k_2} H^{(L)}(k_1 - k_2 + s, s) ds, \quad H^{(L)}(t_1, t_2) := \frac{t_1 + t_2}{2} K^{(L)}_N(t_1^2/2, t_2^2/2)\bigg|_{a=0} \quad (1.15)$$

(in the case that $k_1, k_2$ are given by (1.5) this identity was first given in [36]).

Our aim in this paper is to seek analogues of (1.13) and (1.15) for the LUE—it turns out that relative to the GUE the resulting structures are more complex, and we are restricted to extending (1.13). The quantity $S^{(L)}(k)$, in the special case $a = 0$, has been the subject of attention from the viewpoint of numerical plots [23] and approximate small $k$ analysis [31] in the context of recent studies on the supersymmetric Sachdev–Ye–Kitaev (SYK) model [23,25,30]. The latter in turn is of interest both from the viewpoint of information scrambling in black holes, and many body quantum chaos; see citations given above (1.4). In the case of $a$ proportional to $N$, an approximate analysis of $S^{(L)}(k)$ has been given in [4] in the context of a study of the reduced density matrix for a chaotic many body wave function. The LUE relates to supersymmetric models via the random chiral Hamiltonian structure

$$H = \begin{bmatrix} 0_{n \times n} & X \\ X^\dagger & 0_{N \times N} \end{bmatrix}, \quad (1.16)$$

where $X$ is an $n \times N$ ($n \geq N$) standard complex Gaussian matrix, with the square of the positive eigenvalues (which generally come in $\pm$ pairs) having joint distribution (1.6),
weight (1.14), \( a = n - N \); see [43], or [17, Sect. 3.1.1]. In relation to density matrices, which are positive definite matrices with unit trace, it is a fact that the eigenvalues of \( X^T X \) are the squared nonzero eigenvalues of (1.16) which gives relevance to the LUE; see e.g. [37] or [17, Sect. 3.3.4].

Whereas the identity (1.13) for \( S^{(G)}_N \) involves the correlation kernel for the LUE (specialised to \( a = 0 \)), it turns out that the analogous expression for \( S^{(L)}_N (k) \) involves the correlation kernel for the Jacobi unitary ensemble (JUE). The JUE corresponds to the eigenvalue PDF (1.6) with \( \beta = 2 \) and weight

\[
w(x) = w^{(J)}(x) = x^a (1 - x)^b \chi_{1 > x > 0}.
\]  

(1.17)

It appears specialised to the case \( b = 0 \).

**Theorem 1.1.** Let \( \rho^{(2)}_{(L)}(x, y) \) denote the truncated two-point correlation function for the LUE, and let \( \rho^{(J)}_{(1)}(x) \) denote the eigenvalue density for the JUE. We have

\[
- \int_{\mathbb{R}_+^2} e^{ik(x-y)} \rho^{(L)}_{(2)}(x, y) \, dx \, dy = \int_0^{1/(1+k^2)} \rho^{(J)}_{(1)}(x) \bigg|_{b=0} \, dx.
\]  

(1.18)

Equivalently

\[
S^{(L)}_N (k) = \int_{1/(1+k^2)}^1 \rho^{(J)}_{(1)}(x) \bigg|_{b=0} \, dx.
\]  

(1.19)

An application of (1.19) is to the calculation of the bulk scaled limit of \( S^{(L)}_N (k) \).

**Corollary 1.2.** Define

\[
S^{(L)}_\infty (k; \alpha) = \lim_{N \to \infty} \frac{1}{N} S^{(L)}_N (k) \bigg|_{a=\alpha N}.
\]  

(1.20)

Let \( 0 \leq c < 1 \) be specified by the equation

\[
c = \left( \frac{\alpha}{2 + \alpha} \right)^2.
\]  

(1.21)

define

\[
\rho^{(J), \text{global}}_{(1)}(x) := \frac{1}{\pi (1 - \sqrt{c})} \frac{1}{x} \left( x - c \right) \chi_{c < x < 1},
\]  

(1.22)

and specify \( k_c \geq 0 \) by the equation

\[
\frac{1}{1 + k_c^2} = c = \left( \frac{\alpha}{2 + \alpha} \right)^2.
\]  

(1.23)

We have
\[ S^{(L)}(k; \alpha) = \int_{1/(1+k^2)}^{1} \rho^{(J) \text{ global}}_N(x) \, dx \]

\[ = \frac{2}{\pi (1 - \sqrt{c})} \left( -\sqrt{c} \operatorname{Arctan} \sqrt{\frac{c(1-d)}{d-c}} + \operatorname{Arcsin} \sqrt{\frac{1-d}{1-c}} \right)_{d=1/(1+k^2)}, \]

valid for \( 0 \leq k \leq k_c \), and

\[ S^{(L)}_\infty(k; \alpha) = 1, \quad (1.25) \]

valid for \( k \geq k_c \).

**Remark 1.3.** 1. The case \( a \) fixed is obtained by taking \( \alpha = 0 \) in the above formulas. From (1.23) this corresponds to \( k_c \to \infty \) so only the case (1.24) is required, which simplifies to

\[ S^{(L)}_\infty(k; 0) = \frac{2}{\pi} \operatorname{Arctan} k. \quad (1.26) \]

The absence of a ramp-plateau transition in the case, as distinct from the behaviour for \( \alpha > 0 \), was predicted in the work [4] relating to random density matrices.

2. In the Appendix, prompted by a referee, an approximate analysis leading to (1.26) is presented. To put this in context, we recall that with

\[ S^{(G)}_\infty(k) := \lim_{N \to \infty} \frac{1}{N} S^{(G)}_N(2\sqrt{2N} \tau), \quad (1.27) \]

Brézin and Hikami [3] proved as a consequence of (1.13) that

\[ S^{(G)}_\infty(k) = \begin{cases} \frac{2}{\pi} \left( \tau \sqrt{(1-\tau^2)} + \operatorname{Arctan} \tau \right), & 0 < \tau < 1, \\ 1, & \tau > 1. \end{cases} \quad (1.28) \]

In the same paper, it was shown how (1.28) can be also deduced by approximate working based on the universal form of the bulk truncated two-point correlation function (see e.g. [17, rewrite of (7.2)])

\[ \rho^{T \text{ bulk}}_2(x, y) = -\frac{(\sin[\pi \rho(x - y)])^2}{(\pi(x - y))^2}. \quad (1.29) \]

Here \( \rho \) is the local eigenvalue density, the value of which depends on the choice of units in the bulk scaling. The idea of the referee, developed in the Appendix, is to use this same starting point as a mechanism which gives an explanation of the result (1.26).

In Sect. 2 we revise how the correlation kernel determines the correlation functions for the LUE and JUE. For future use in the derivation of Theorem 1.1, we present differential identities for the correlation kernels in both cases, and also the evaluation of a key definite integral involving the Laguerre polynomials. An integral evaluation relating to the final term in (1.12) is derived in the first subsection of Sect. 3, while the proof of Theorem 1.1 is given in the second subsection. In Sect. 4 scaled limits relevant to the dip–ramp–plateau effect are calculated, with the proof of Corollary 1.2 given in the final subsection.
2. Preliminaries

Central to the study of the LUE are the Laguerre polynomials. These can be defined through the Rodrigues formula

\[ L_n^{(a)}(x) = \frac{x^{-a} e^x}{n!} \frac{d^n}{dx^n} \left( e^{-x} x^{n+a} \right) = \frac{(-1)^n}{n!} x^n + \frac{(-1)^{n-1}(a+n)}{(n-1)!} x^{n-1} + \cdots \tag{2.1} \]

A convenient normalisation is to introduce a proportionality constant so that the polynomials are monic (coefficient of leading monomial unity). Thus we define

\[ p_n^{(L)}(x) = n!(-1)^n L_n^{(a)}(x). \tag{2.2} \]

From standard properties of the Laguerre polynomials, the corresponding orthogonality relation is

\[ \int_{0}^{\infty} x^a e^{-x} p_m^{(L)}(x) p_n^{(L)}(x) \, dx = h_n^{(L)} \delta_{m,n}, \quad h_n^{(L)} = \Gamma(n+1) \Gamma(n+a+1). \tag{2.3} \]

The orthogonality (2.3) suggests introducing the orthogonal functions

\[ \psi_n^{(L)}(x) = \sqrt{w(x)} p_n^{(L)}(x), \quad w(x) = x^a e^{-x} \chi_{x>0}. \tag{2.4} \]

Considering (2.4) in squared variables, and so defining

\[ \hat{\psi}_n^{(L)}(X) = \sqrt{X} \psi_n^{(L)}(X^2), \]

one has that \( \hat{\psi}_n^{(L)}(X) \) form a complete set of eigenfunctions for the Schrödinger operator

\[ H^{(L)} = -\frac{d^2}{dX^2} + \frac{a^2 - 1/4}{X^2} + X^2, \quad X > 0; \]

see e.g. [10, Sect. 2.3]. This differential operator results as a specialisation to \( d = 1 \) of the radial part of the Schrödinger operator for the \( d \)-dimensional harmonic oscillator—there \( a - 1/2 \) relates to the quantum number for the corresponding angular part of the same Schrödinger operator; see e.g. [35]. The fact that \( H^{(L)} \) is self adjoint with respect to the inner product

\[ \langle f, g \rangle := \int_{0}^{\infty} f(x) g(x) \, dx, \tag{2.5} \]

gives an explanation for the orthogonality of the set of functions (2.4).

As previously remarked, the LUE corresponds to the eigenvalue PDF (1.3) with weight (1.14). Standard theory associated with the PDFs (1.6)—see e.g. [17, Ch. 5]—tells us that the \( k \)-point correlation functions (1.7) have the determinantal form

\[ \rho^{(k)}(x_1, \ldots, x_k) = \det \left[ K_N(x_j, x_l) \right]_{j,l=1}^{k}, \tag{2.6} \]

where \( K_N(x, y) \)—referred to as the correlation kernel—is specified by

\[ K_N(x, y) = \left( w(x) w(y) \right)^{1/2} \sum_{j=0}^{N-1} \frac{p_j(x) p_j(y)}{h_j}. \]
In (2.7) \( \{ p_j(x) \} \) are the set of monic orthogonal polynomials associated with the weight function \( w(x) \), normalisation \( h_j \),

\[
\int_{-\infty}^{\infty} w(x) p_j(x) p_k(x) \, dx = h_j \delta_{j,k}.
\] (2.8)

Important is the explicit form of the sum in (2.7), referred to as the Christoffel-Darboux formula (see e.g. [17, Prop. 5.1.3])

\[
\sum_{j=0}^{N-1} \frac{p_j(x) p_j(y)}{h_j} = \frac{1}{h_{N-1}} \frac{p_N(x) p_{N-1}(y) - p_N(y) p_{N-1}(x)}{x - y}.
\] (2.9)

Hence for the LUE we have

\[
K_N^{(L)}(x, y) = \frac{1}{h_N^{(L)}} \frac{\psi_N^{(L)}(x) \psi_N^{(L)}(y) - \psi_N^{(L)}(y) \psi_N^{(L)}(x)}{x - y}.
\] (2.10)

Crucial to our derivation of the results of Theorem 1.1 is an identity associated with the partial derivatives of (2.10) [42], [17, Proof of Prop. 5.4.2].

**Proposition 2.1.** Let \( \psi_n^{(L)}(x) \) be given by (2.4) and \( K_N^{(L)}(x, y) \) by (2.10). We have

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) (xy)^{1/2} K_N^{(L)}(x, y) = -\frac{(xy)^{1/2}}{2h_{N-1}^{(L)}} \left( \psi_N^{(L)}(x) \psi_N^{(L)}(y) + \psi_N^{(L)}(x) \psi_N^{(L)}(y) \right).
\] (2.11)

**Proof.** We proceed as in the derivation outlined in [17, Proof of Prop. 5.4.2]. The orthogonal functions (2.4) satisfy the matrix differential recurrence

\[
x \frac{d}{dx} \begin{bmatrix} \psi_n^{(L)}(x) \\ \psi_{n-1}^{(L)}(x) \end{bmatrix} = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix} \begin{bmatrix} \psi_n^{(L)}(x) \\ \psi_{n-1}^{(L)}(x) \end{bmatrix},
\] (2.12)

where

\[
A_{11}(x) = -A_{22}(x) = -\frac{1}{2} (x - 2n - a), \quad A_{12}(x) = n(a + n), \quad A_{21}(x) = 1.
\] (2.13)

For general differentiable \( f = f(x, y) \) we can check

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) (xy)^{1/2} f = \frac{(xy)^{1/2}}{x - y} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) f.
\]

Choosing \( f = (\psi_N^{(L)}(x) \psi_N^{(L)}(y) - \psi_N^{(L)}(y) \psi_N^{(L)}(x))/h_N^{(L)} \), it follows that

\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) (xy)^{1/2} K_N^{(L)}(x, y)
= \frac{1}{h_N^{(L)}(x - y)} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \begin{bmatrix} \psi_N^{(L)}(x) \\ \psi_N^{(L)}(y) \end{bmatrix}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \psi_N^{(L)}(x) \\ \psi_N^{(L)}(y) \end{bmatrix}.
\]
For the partial derivatives on the RHS, use of (2.12) shows they can be carried out to obtain
\[
\begin{bmatrix}
\psi_N^{(L)}(x) & \psi_{N-1}^{(L)}(x)
\end{bmatrix}
\begin{bmatrix}
-A_{21}(x) - A_{21}(y) & A_{11}(x) + A_{22}(y) \\
-A_{22}(x) + A_{11}(y) & A_{12}(x) - A_{12}(y)
\end{bmatrix}
\begin{bmatrix}
\psi_N^{(L)}(y) \\
\psi_{N-1}^{(L)}(y)
\end{bmatrix}
\]
\[
\begin{bmatrix}
\psi_N^{(L)}(x) & \psi_{N-1}^{(L)}(x)
\end{bmatrix}
\begin{bmatrix}
0 & -\frac{1}{2} \\
-\frac{1}{2} & 0
\end{bmatrix}
\begin{bmatrix}
\psi_N^{(L)}(y) \\
\psi_{N-1}^{(L)}(y)
\end{bmatrix}
\]
\[
= -\frac{1}{2} \left( \psi_N^{(L)}(x)\psi_{N-1}^{(L)}(y) + \psi_N^{(L)}(x)\psi_{N-1}^{(L)}(y) \right),
\]
and (2.11) follows.

Also of importance is the closed form evaluation of the integral
\[
I_{jk}^{(L)}(s) := \int_0^\infty L_j^{(a)}(x)L_k^{(a)}(x)x^a e^{(s-1)x} \, dx, \quad \text{Re } s < 1, \tag{2.14}
\]
which we interpret as the Laplace-Fourier transform of \( L_j^{(a)}(x)L_k^{(a)}(x)w_i^{(L)}(x) \). In fact its value can be read off by specialising formulas given in standard compendiums of integral evaluations [39, Entries 2.19.14.6], [20, Entry 7.414.4], [13, Entry 4.11 (35)]. We owe our knowledge of these references due to them appearing in the paper [28, Sect. 4], which considers further generalisations of the integrals of products of Laguerre polynomials. In a random matrix context, (2.14) first appeared in the work of Haagerup and Thorbjørnsen [21], where its evaluation was stated as a known result (with reference to the early work [32] also referenced in [28]), and a verification type proof was given.

A companion Fourier-Laplace transform to (2.14) is
\[
I_{jk}^{(G)}(s) := \int_{-\infty}^\infty e^{sx} H_j(x)H_k(x)e^{-x^2} \, dx, \tag{2.15}
\]
where \( H_n(x) \) denotes the Hermite polynomial of degree \( n \). This features prominently in the derivation of (1.13) given in [18,36], the general strategy of which underpins our derivation of the identities of Theorem 1.1. The evaluation of (2.15) can be found in a number of references—many are listed in [18, statement of Prop. 13]. The most structurally revealing make use of generating functions. This motivates us to give a self contained generating function approach to compute (2.14). For this we take as background knowledge the generating function formulas [20, Entry 8.975.1]
\[
\sum_{n=0}^\infty t^n L_n^{(a)}(x) = (1 - t)^{-a+1}e^{-tx/(1-t)}, \quad |t| < 1, \tag{2.16}
\]
and [40, Eq. 5.2 (12)]
\[
\sum_{n=0}^\infty \frac{(c)_n}{n!} F_1(-n, b, c; x)t^n = (1 - t)^{b-c}(1 - t + xt)^{-b}, \quad |t| < 1, \tag{2.17}
\]
where
\[
(c)_n := \frac{\Gamma(c+n)}{\Gamma(c)}. \tag{2.18}
\]
Proposition 2.2. Let $\HyperTwo{2}{1}{}$ denote the Gauss hypergeometric function. Define $I_{jk}^{(L)}(s)$ by (2.14). We have

$$I_{jk}^{(L)}(s) = \Gamma(a + 1) \frac{(a + 1)_j (a + 1)_k}{j! k!} (1 - s)^{-(a + 1)} \left( -\frac{s}{1 - s} \right)^{j+k} \times \HyperTwo{2}{1}{-k, -j, a + 1; 1/s^2}. \quad (2.19)$$

Proof. Use of the generating function (2.16) shows that

$$\sum_{j, k=0}^{\infty} t_1^j t_2^k I_{jk}^{(L)}(s) = (1 - t_1)^{-(a + 1)} (1 - t_2)^{-(a + 1)}$$

$$\times \int_0^{\infty} e^{-t_1 x/(1-t_1) - t_2 x/(1-t_2)} x^a e^{s x} dx. \quad (2.20)$$

The integral in (2.20) reduces to the integral definition of the gamma function after a simple change of variables. Introducing the notation

$$Y = 1 - s + t_2/(1 - t_2), \quad \tilde{Y} = (1 - t_2)Y = 1 - s + st_2, \quad (2.21)$$

and upon some simple manipulation, this shows

$$\sum_{j, k=0}^{\infty} t_1^j t_2^k I_{jk}^{(L)}(s) = \Gamma(a + 1) \tilde{Y}^{-(a + 1)} \left( 1 - t_1 \left( 1 - \frac{1}{\tilde{Y}} \right) \right)^{-(a + 1)}. \quad (2.22)$$

Using the binomial theorem, we read off from (2.22) that the coefficient of $t_1^j$ is

$$\frac{(a + 1)_j}{j!} \left( 1 - \frac{1}{\tilde{Y}} \right)^j = \frac{(a + 1)_j}{j!} \left( \frac{\tilde{Y} - (1 - t_2)}{\tilde{Y}} \right)^j,$$

where the equality follows from the definition (2.21). Hence we have

$$\sum_{k=0}^{\infty} t_2^k I_{jk}^{(L)}(s) = \Gamma(a + 1) \frac{(a + 1)_j}{j!} \tilde{Y}^{-(a + 1 + j)} \left( \tilde{Y} - (1 - t_2) \right)^j$$

$$= \Gamma(a + 1) \frac{(a + 1)_j}{j!} (1 - s)^{-(a + 1 + j)} (-s)^j J(t_2; s), \quad (2.23)$$

where

$$J(t_2; s) := (1 - \mu t_2)^{b-c} (1 - \mu t_2 + x \mu t_2)^{-b}, \quad (2.24)$$

with

$$b = -j, \quad c = a + 1, \quad \mu = \frac{s}{s - 1}, \quad x = \frac{1}{s^2}. \quad (2.25)$$

The coefficient of $t_2^k$ in the power series expansion of (2.24) can be read off from (2.17). Using this in (2.23) gives (2.19). \qed
The identity (1.18) involves the correlation kernel for the JUE, with the latter in turn relating to the weight function (1.17). The corresponding (monic) polynomials as specified by the requirement (2.8) are simply related to the Jacobi polynomials. However for present purposes it is preferable to write them in hypergeometric form (see e.g. [5])

\[ p_n^{(J)}(x) = (-1)^n \frac{(a + 1)_n}{(a + b + n + 1)_n} \frac{2}{n} F_1(-n, a + b + n + 1, a + 1; x), \]

(2.26)

and we read off from the same reference the explicit value of the norm

\[ h_n^{(J)} = \frac{n! \Gamma(a + n + 1) \Gamma(b + n + 1)}{(a + b + 2n + 1) \Gamma(a + b + n + 1)}. \]

(2.27)

We have use for a differential identity satisfied by \( K_N^{(J)}(x, x) \) (note that according to (2.6) this is equal to \( \rho_n^{(J)}(x) \)). It is a minor linear change of variables of a result in [27, Lemma 5.6]. We will give a different derivation, as a special case of a more general differential identity, of the type given in Proposition 2.1, but now in relation to the Jacobi correlation kernel.

**Proposition 2.3.** We have

\[ \frac{d}{ds}(s(1-s)K_N^{(J)}(s, s)) = -\frac{(2N + a + b)}{h_{N-1}^{(J)}}w^{(J)}(s)p_n^{(J)}(s)p_{n-1}^{(J)}(s). \]

(2.28)

**Proof.** The weight function for the Jacobi unitary ensemble supported on \((-1, 1)\) is

\[ \tilde{w}^{(J)}(x) = (1 - x)^a(1 + x)^b \chi_{-1 < x < 1}. \]

Denote the corresponding correlation kernel by \( \tilde{K}_N^{(J)}(x, y) \), the corresponding monic orthogonal polynomials by \( \tilde{p}_n^{(J)}(x) \), their norm by \( h_n^{(J)} \), and corresponding orthogonal functions \( \tilde{\psi}_n^{(J)}(x) \) (see \( \sqrt{\tilde{w}^{(J)}(x)}\tilde{p}_n^{(J)}(x) \)).

For this variant of the JUE, the Jacobi analogue of (2.11) is given in [17, 2nd last displayed equation in proof of Prop. 5.4.2],

\[
\left\{ (1 - x^2) \frac{\partial}{\partial x} + (1 - y^2) \frac{\partial}{\partial y} \right\} (1 - x^2)^{1/2}(1 - y^2)^{1/2} \tilde{K}_N^{(J)}(x, y) = -\frac{(2N + a + b)}{2h_{N-1}^{(J)}} \left( \tilde{\psi}_N^{(J)}(x)\tilde{\psi}_N^{(J)}(y) + \tilde{\psi}_{N-1}^{(J)}(x)\tilde{\psi}_{N-1}^{(J)}(y) \right).
\]

After simple manipulation, and the linear change of variables \( x = 1 - 2u, y = 1 - 2v \), this is seen to be equivalent to a differential identity for the correlation kernel of the JUE as originally defined in terms of the weight (1.17),

\[
\left\{ 1 - (u + v) + u(1 - u) \frac{\partial}{\partial u} + v(1 - v) \frac{\partial}{\partial v} \right\} K_N^{(J)}(u, v) = -\frac{(2N + a + b)}{2h_{N-1}^{(J)}} \left( \psi_N^{(J)}(u)\psi_N^{(J)}(v) + \psi_{N-1}^{(J)}(u)\psi_{N-1}^{(J)}(v) \right).
\]
Taking the limit $u, v \to s$, the LHS reduces to

$$
\left(1 - 2s + s(1 - s) \frac{d}{ds}\right) K_N^{(J)}(s, s) = \frac{d}{ds} s(1 - s) K_N^{(J)}(s, s),
$$

which is the LHS of (2.28), and taking the same limit on the RHS, we see the RHS of (2.28) results.

\[\Box\]

3. Calculation of the Structure Function and Related Averages

3.1. Fourier-Laplace transform of the density. The Fourier-Laplace transform of the density appears in the expression (1.12), which in turn relates to the form of the covariance (1.4) as implied by (1.11). The evaluation of its derivative has been given in [21, Th. 6.4]. Revising its proof is an instructive preparation for the proof of Theorem 1.1.

**Proposition 3.1.** We have

$$
\int_0^\infty t\rho_{(1)}(t)e^{st} dt = \frac{N(N + a)}{(1 - s)^{2N+a}} 2F_1\left(\begin{array}{c} -N + 1 - a, -N + 1, 2; \frac{s^2}{2} \end{array}\right). \tag{3.1}
$$

**Proof.** Noting that

$$
\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right) (x y)^{1/2} K_N^{(L)}(tx, ty) = (x y)^{1/2} \frac{d}{dt} t K_N^{(L)}(tx, ty),
$$

we see upon replacing $x, y$ by $xt, yt$ in (2.11), then setting $x = y = 1$, that

$$
\frac{d}{dt} t K_N^{(L)}(t, t) = -\frac{1}{h_N^{(L)}} \psi_N^{(L)}(t) \psi_N^{(L)}(t). \tag{3.2}
$$

We know from (2.6) that $K_N^{(L)}(t, t) = \rho_{(1)}(t)$ and so after multiplying both sides of (3.2) by $e^{st}$ and integrating we deduce

$$
s \int_0^\infty t K_N^{(L)}(t, t)e^{st} dt = \frac{1}{h_N^{(L)}} \int_0^\infty \psi_N^{(L)}(t) \psi_N^{(L)}(t)e^{st} dt = -\frac{N!}{\Gamma(N + a)} \int_0^\infty t^a e^{-t(1-s)} L_N^{(a)}(t)L_N^{(a)}(t) dt.
$$

The integrand is an example of (2.14), and so application of Proposition 2.2 gives

$$
\int_0^\infty t K_N^{(L)}(t, t)e^{st} dt = \frac{(a + 1)_N}{(N - 1)!} \frac{(1 - s)^{2N-2}}{(1 - s)^{2N+a}} 2F_1\left(\begin{array}{c} -N + 1, -N, a + 1; 1/s^2 \end{array}\right).
$$

Use of the polynomial identity [21, Eq. (6.17)]

$$
2F_1\left(\begin{array}{c} -j, -k, a + 1; 1/s^2 \end{array}\right) = \frac{k!}{(k-j)!(a+1)_j} \frac{1}{s^2} 2F_1\left(\begin{array}{c} -j - a, -j, 1+k-j; s^2 \end{array}\right), \tag{3.3}
$$

valid for $j, k$ non-negative integers with $j \leq k$, reduces this to (3.1). \[\Box\]
3.2. Proof of Theorem 1.1. Our proof of Theorem 1.1 makes use of the operator

\[ B_{x,y} = 1 + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \]

In light of the operator identity

\[ (xy)^{-1/2} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) (xy)^{1/2} = B_{x,y} \]

we see from Proposition 2.1 that

\[ B_{x,y} K_N^{(L)}(x, y) = -\frac{1}{2h_{N-1}^{(L)}} \left( \psi_N^{(L)}(x) \psi_{N-1}^{(L)}(y) + \psi_{N-1}^{(L)}(x) \psi_N^{(L)}(y) \right). \]  

We observe too the skew self-adjoint property

\[ \langle f, B_{x,y} g \rangle_2 = -\langle B_{x,y} f, g \rangle_2, \quad \langle f, g \rangle_2 := \int_{\mathbb{R}_+^2} f(x, y) g(x, y) \, dx \, dy, \]  

as well as the identity

\[ (B_{z_1, z_2} - 1)e^{z_1 x + z_2 y} = (B_{x,y} - 1)e^{z_1 x + z_2 y}. \]  

Application of first (3.6), then (3.5), then a direct calculation, and finally (3.4) shows

\begin{align*}
(B_{z_1, z_2} - 1) & \int_{\mathbb{R}_+^2} e^{z_1 x + z_2 y} \left( K_N^{(L)}(x, y) \right)^2 \, dx \, dy \\
& = \int_{\mathbb{R}_+^2} (B_{x,y} - 1)e^{z_1 x + z_2 y} \left( K_N^{(L)}(x, y) \right)^2 \, dx \, dy \\
& = -\int_{\mathbb{R}_+^2} e^{z_1 x + z_2 y} \left( B_{x,y} + 1 \right) \left( K_N^{(L)}(x, y) \right)^2 \, dx \, dy \\
& = -2 \int_{\mathbb{R}_+^2} e^{z_1 x + z_2 y} \left( B_{x,y} \right) K_N^{(L)}(x, y) \, dx \, dy \\
& = \frac{1}{h_{N-1}^{(L)}} \int_{\mathbb{R}_+^2} e^{z_1 x + z_2 y} K_N^{(L)}(x, y) \left( \psi_N^{(L)}(x) \psi_{N-1}^{(L)}(y) + \psi_{N-1}^{(L)}(x) \psi_N^{(L)}(y) \right) \, dx \, dy.
\end{align*}

Now apply the operator

\[ \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \]

to the expression in the first line of (3.7). In the form given in the final line of (3.7) this has the effect of creating a factor \((x - y)\) inside the integrand. But from (2.10)

\[ (x - y) K_N^{(L)}(x, y) = \frac{1}{h_{N-1}^{(L)}} \left( \psi_N^{(L)}(x) \psi_{N-1}^{(L)}(y) - \psi_{N-1}^{(L)}(x) \psi_N^{(L)}(y) \right). \]
Hence we deduce from (3.7) that

\[
\left( \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right) (B_{z_1, z_2} - 1) \int_{\mathbb{R}^2_+} e^{z_1 x + z_2 y} \left( K_N^{(L)}(x, y) \right)^2 dxdy
= \frac{1}{(h_{N-1}^{(L)})^2} \int_{\mathbb{R}^2_+} e^{z_1 x + z_2 y} \left( (\psi_N^{(L)}(x)\psi_{N-1}^{(L)}(y))^2 - (\psi_N^{(L)}(x)\psi_N^{(L)}(y))^2 \right) dxdy
= \left( \frac{N!(N-1)!}{h_N^{(L)}} \right)^2 \left( I_{N,N}^{(L)}(z_1)I_{N-1,N-1}^{(L)}(z_2) - I_{N,N}^{(L)}(z_2)I_{N-1,N-1}^{(L)}(z_1) \right),
\]

where the second equality follows from the definitions (2.4) and (2.14). The significance of this expression is that according to Proposition 2.2 all terms on the RHS can be evaluated explicitly, reducing it to

\[
2F1 \cdot J. Forrester
\]

The Gaussian analogue of the equality between the first line of (3.8) and (3.9) is given by [18, Equality between LHS of (3.18) and final expression in (3.19)]. Comparison between the two shows that the present Laguerre case is more complicated as the first line of (3.8) involves second order partial derivatives, whereas its Gaussian analogue only involves first order partial derivatives. Due to this complication, we have not been able to deduce a Laguerre analogue of (1.15). However, if we consider instead the special case of the covariance corresponding to the structure function (1.3), further progress is possible.

Thus set \( z_1 = -z_2 = it \) in the equality between the LHS of (3.8) and (3.9). This gives the simplified identity

\[
\frac{t}{4} \int_{\mathbb{R}^2_+} e^{it(x-y)} \left( K_N^{(L)}(x, y) \right)^2 dxdy = \left( \frac{(a+1)_N}{(N-1)!} \right)^2 \times (1 + t^2)^{-a-1} \left( \frac{t^2}{1 + t^2} \right)^{2N} 2F1 \left( -N, -N, a + 1; -1/t^2 \right)
\times 2F1 \left( -N + 1, -N + 1, a + 1; -1/t^2 \right).
\]

In terms of the variable \( u = 1/(1 + t^2) \) (3.10) reads

\[
\frac{d}{du} \left( u(1-u) \frac{d}{du} \right) \int_{\mathbb{R}^2_+} e^{i\sqrt{1-u}u(x-y)} \left( K_N^{(L)}(x, y) \right)^2 dxdy = \left( \frac{(a+1)_N}{(N-1)!} \right)^2 \times u^a \left( 1 - u \right)^{2N-1} 2F1 \left( -N, -N, a + 1; u/(u - 1) \right)
\times 2F1 \left( -N + 1, -N + 1, a + 1; u/(u - 1) \right).
\]

Recalling now the Pfaff–Kummer transformation for the Gauss hypergeometric function

\[
2F1(\alpha, \beta, \gamma; z) = (1 - z)^{-\alpha} 2F1(\alpha, \gamma - \beta, \gamma; z/(z - 1))
\]
allows the RHS of (3.11) to be simplified, reducing it to
\[
\left(\frac{(a + 1)N}{(N - 1)!}\right)^2 u^a_2 F_1(-N, N + a + 1, a + 1; u) _2 F_1(-N + 1, N + a, a + 1; u) = -\frac{2N + a}{h^{(J)}_{N-1}} u^a p_N^{(J)}(u) p_{N-1}^{(J)}(u) \bigg|_{b=0} = \frac{d}{du} \left( u(1 - u)K_N^{(J)}(u, u) \right) \bigg|_{b=0},
\]
where the first equality follows from (2.26) and (2.27) and the second from Proposition 2.3. Equating this to the LHS of (3.11) and taking the indefinite integral of both sides shows
\[
\frac{d}{du} \int \mathbb{R}_+^2 e^{i\sqrt{(1-u)/u}(x-y)} \left( K_N^{(L)}(x, y) \right)^2 \, dx \, dy = K_N^{(J)}(u, u) \bigg|_{b=0};
\]
Integrating both sides from 0 to s and setting \(\sqrt{(1-s)/s} = k\) gives the sought identity (1.18), upon identifying \((K_N^{(L)}(x, y))^2 = -\rho_2^{(T)}(x, y)\) and \(K_N^{(J)}(u, u) = \rho_1^{(J)}(u)\).

The sum rules
\[
\int_{\mathbb{R}_+^2} \delta(x - y)\rho_1^{(L)}(y) \, dx \, dy = N, \quad \int_0^1 \rho_1^{(J)}(u) \, du = N,
\]
which are simply normalisation conditions, show that (1.19) is equivalent to (1.18), after recalling too (1.11).

4. Scaled Limits

4.1. Global scaling. Generally a global scaling limit in random matrix theory is when the entirety of the spectrum plays a role. In the LUE this takes effect when eigenvalues are scaled according to \(\lambda_j = 4Nx_j\), the point being that in the variables \(\{x_j\}\) the limiting support is compact. Note that this latter feature is not dependent on the specific choice of (positive) proportionality—the choice of 4 is for convenience. There are two distinct cases: either the Laguerre parameter \(a\) is held fixed, or the Laguerre parameter is scaled with \(N\). In fact the former is the case \(\alpha = 0\) in the second scenario, for which
\[
\lim_{N \to \infty} 4\rho_1^{(L)}(4Nx) \bigg|_{a=Na} = \frac{2}{\pi} \sqrt{(c_+^2 - x)(x - c_-^2)} x_{c_+^2 < x < c_-^2}, \quad c_\pm := \frac{1}{2} \left(\sqrt{\alpha + 1} \pm 1\right),
\]
where this functional form is known as the Marchenko–Pastur density [38]. Hence for a general linear statistic \(A = \sum_{j=1}^N a(\lambda_j/4N)\)
\[
\lim_{N \to \infty} \frac{1}{N} \langle A \rangle^{(L)} = \frac{2}{\pi} \int c_-^2 \frac{a(x)}{x} \sqrt{(c_+^2 - x)(x - c_-^2)} \, dx.
\]
In the special case \(a(x) := a_s(x) = xe^{sx}\), and with \(A_s := \sum_{j=1}^N a_s(\lambda_j/4N)\), it follows from Proposition 3.1 that (4.2) reduces to
\[
\lim_{N \to \infty} \frac{1}{N} \langle A_s \rangle^{(L)} = (1 + \alpha) e^{(s/2)(1+\alpha/2)}_0 F_1\left(2; (1 + \alpha)(s/2)^2\right),
\]
which can also be obtained directly from (4.2); see [21, Sect. 6.6].
It is fundamental in random matrix theory that the variance of a smooth linear statistic in the global scaling limit is of $O(1)$. From the definition (1.3) of $S_N(k)$ we have that
\[
\text{Var}\left(\sum_{j=1}^{N} e^{ik\lambda_j/\sqrt{N}}\right)^{(L)} = S_N^{(L)}(k/\sqrt{2N}) = \int_{1/(1+(k/\sqrt{2N})^2)}^{1} \rho^{(J)}_{(1)}(x) \bigg|_{b=0} \ dx \quad (4.4)
\]
(here the factor of $\sqrt{2}$ in the global scaling is for later convenience; recall the second sentence of the first paragraph above), where the second equality follows from (1.19). Recalling the symmetry of the Jacobi ensemble under the mappings $a \leftrightarrow b$, we have that $\rho^{(J)}_{(1)}(x)$ can be evaluated explicitly; see [18, Eq. (3.28)].

\[
\int_{1/(1+(k/\sqrt{2N})^2)}^{1} \rho^{(J)}_{(1)}(x) \bigg|_{b=0} \ dx = \int_{0}^{(k/\sqrt{2N})^2/(1+(k/\sqrt{2N})^2)} \rho^{(J)}_{(1)}(x) \bigg|_{a=0} \ dx = \frac{1}{2N^2} \int_{0}^{k^2/(1+(k/\sqrt{2N})^2)} \rho^{(J)}_{(1)}(x/2N^2) \bigg|_{a=0} \ dx, \quad (4.5)
\]
where the second equality follows by a simple change of variables.

The utility of (4.5) follows from the standard limit theorem in random matrix theory (see e.g. [17, Sect. 7.2.5]) that
\[
\lim_{N \to \infty} \frac{1}{2N^2} \rho^{(J)}_{(1)}(x/2N^2) \bigg|_{a=0} = \rho^{\text{hard}}_{(1)}(x) \bigg|_{a=0} = \frac{1}{4} \left( (J_0(x^{1/2}))^2 + (J_1(x^{1/2}))^2 \right), \quad (4.6)
\]
where $J_n(v)$ denotes the Bessel function of order $n$ and $\rho^{\text{hard}}_{(1)}(x)$ denotes the scaled hard edge state with unitary symmetry [15], with a remainder term that can readily be checked to be uniform for $x$ on a compact set of the positive half line. Hence
\[
S_{\infty}^{(L)\text{global}}(k) := \lim_{N \to \infty} S_N^{(L)}(k/\sqrt{2N}) = \int_{0}^{k^2} \rho^{\text{hard}}_{(1)}(x) \bigg|_{a=0} \ dx = \frac{1}{4} \int_{0}^{k^2} \left( (J_0(x^{1/2}))^2 + (J_1(x^{1/2}))^2 \right) \ dx. \quad (4.7)
\]

We remark that since for $x \to \infty$, $\rho^{\text{hard}}_{(1)}(x) \sim 1/(2\pi x^{1/2})$ (this holds independent of the parameter $a$; see [17, Eq. (7.74)]), for $k \to \infty$
\[
S_{\infty}^{(L)\text{global}}(k) \sim \frac{k}{\pi}. \quad (4.8)
\]
This corresponds to the ‘ramp’ in the dip–ramp–plateau effect discussed in the Introduction. We remark too that the global scaling limit of the structure function for the GUE as implied by (1.13) is also given by the same functional form (4.7) (note that this is dependent on the precise choice of the proportionality in the global scaling—more generally this statement would hold after appropriately identifying $k$) [18]. In the latter reference it is noted that the integral in (4.7) can be evaluated explicitly; see [18, Eq. (3.28)].

In addition to the variance of a smooth linear statistic in the global scaling limit being $O(1)$, another generic feature is that their limiting distribution satisfies a central limit theorem; see [38]. Recently the question of the rate of convergence to the central limit theorem has attracted attention from a number of different viewpoints [1,24,26]. The
formula (4.4) allows the convergence rate question to be addressed for the variance of the specific linear statistic relating to the structure function in the LUE. According to (4.5), this is determined in turn by the rate of convergence of the hard edge scaled density for the JUE. On this, we have the recent large $N$ expansion [34, Prop. 2] (see also the related work [19])

$$\rho^{(J)}(x/2N^2)\bigg|_{a=0} = \rho^{\text{hard}}_{(1)}(x)\bigg|_{a=0} + \frac{b}{N^2} \frac{dx}{dx} \rho^{\text{hard}}_{(1)}(x)\bigg|_{a=0} + O\left(\frac{1}{N^2}\right), \quad (4.9)$$

telling us that the rate is $O(1/N)$.

4.2. Bulk scaling of the linear statistic $\sum_{j=1}^{N} e^{i k j}$. Bulk scaling refers to using a linear change of variables so that the eigenvalues away from the edges have nearest neighbour of order unity for $N$ large. For the Laguerre ensemble, the support of the eigenvalue density is an interval of length proportional to $N$, in both the cases of $a$ fixed or proportional to $N$.

Before considering the corresponding limiting form of $S_{N}^{(L)}$, in view of the interest in (1.12) for $A, B$ given by (1.2) with $k_1 = k_2 = k$, we first make some remarks in relation to the average of the linear statistic $\sum_{j=1}^{N} e^{i k j}$. For $N, k$ large, it follows from (4.2) that

$$\left\langle \sum_{j=1}^{N} e^{i k j} \right\rangle^{(L)} \sim \frac{2N}{\pi} \int_{c_-}^{c_+} \frac{e^{i4Nkx}}{x} \sqrt{(c_+ - x)(x - c_-)} \, dx. \quad (4.10)$$

There are two distinct behaviours, depending on $\alpha = 0$ (and thus $c_- = 0$) or $\alpha > 0$. In the former, expanding the integrand in the neighbourhood of $x = 0$ and changing variables shows

$$\left\langle \sum_{j=1}^{N} e^{i k j} \right\rangle^{(L)} \sim \sqrt{\frac{N}{i \pi k}}. \quad (4.11)$$

As noted in [23], the absolute value squared of (4.11) being of order $N$, and its slow (relative to the Gaussian case [7,18]) $O(1/k)$ ‘dip’ obscures the ‘ramp’ in the dip–ramp–plateau effect. In contrast, for $\alpha > 0$, expanding the integrand in the neighbourhoods of both endpoints $c_+^2 > c_-^2 > 0$ shows that for some $s_{\pm}(\alpha)$ independent of $k, N$

$$\left\langle \sum_{j=1}^{N} e^{i k j} \right\rangle^{(L)} \sim \frac{1}{N^{1/2}} \frac{1}{k^{3/2}} \left( s_{+}(\alpha)e^{i e_+^2 4Nk} + s_{-}(\alpha)e^{i e_-^2 4Nk} \right), \quad (4.12)$$

which exhibits the same rate of decay in both $N$ and $k$ as the Gaussian case, indicating that the dip and ramp are distinct effects (see also [4, Sect. IV.B with the identification $k = \tau/N$]).
4.3. Bulk scaling of the structure function $S_N^{(L)}$ and proof of Corollary 1.2. The limiting form of $S_N^{(L)}$ is easy to compute from (1.19). The latter reduces the task to computing the global limiting form of the density of the JUE—it is a global scaling since there are of order $N$ eigenvalues in the interval $(c, 1)$ for any $0 < c < 1$. This is known in random matrix theory from a result of Wachter [44],

$$\lim_{N \to \infty} \frac{1}{N} \rho^{(1)}_{\text{JUE}}(x) \big|_{b=0, a=\alpha N} = \frac{1}{\pi (1 - \sqrt{c})} \frac{1}{\sqrt{x - c}} \chi_{c < x < 1}, \quad \frac{1}{(1 - \sqrt{c})} = 1 + \alpha/2. \quad (4.13)$$

The statement of Corollary 1.2 now follows, where the integral in (1.24) has been evaluated with the help of computer algebra.

The case of fixed $a$ corresponds to the case $\alpha = 0$ $(c = 0)$ in this formula and so

$$\lim_{N \to \infty} \frac{1}{N} \rho^{(1)}_{\text{JUE}}(x) \big|_{b=0, a \text{ fixed}} = \frac{1}{\pi} \frac{1}{\sqrt{x (1 - x)}} \chi_{0 < x < 1}. \quad (4.14)$$

Use of this in (1.19) validates (1.26).

Contrary to the results of an approximate analysis [23, Eq. (3.15), Fig. 3], [31, Eq. (4.17), Figure 3], our exact result (1.26) shows that for the LUE with $a$ fixed there is no transition from a ramp to plateau in the graphical shape of $S^{(L)}_{\infty}(k; 0)$. The exact result exhibits the limiting forms

$$S^{(L)}_{\infty}(k; 0) \sim \frac{2k}{\pi} - \frac{2k^3}{3\pi} + O(k^5) \quad k \to 0^+$$

and

$$S^{(L)}_{\infty}(k; 0) \sim 1 - \frac{2}{\pi k} + \frac{2}{3\pi k^3} + O(k^{-5}), \quad (4.15)$$

and $S^{(L)}_{\infty}(k; 0)$ is real analytic for $k > 0$.

In contrast to the behaviour of $S^{(L)}_{\infty}(k; 0)$, (1.24) and (1.25) show that with $a = \alpha N$ there is a transition to a plateau $S^{(L)}_{\infty}(k; \alpha) = 1$, occurring at the value of $k$ specified by (1.23). Like in the Gaussian case [3] the ramp portion of the graph is curved, although the leading small $k$ form is linear

$$S^{(L)}_{\infty}(k; \alpha) \sim \frac{2\sqrt{1 + \alpha}}{\pi} k + O(k^2). \quad (4.16)$$

Graphical plots indicate that for $0 < k < k_c$, $S^{(L)}_{\infty}(k; \alpha)$ is concave, with curvature increasing as $\alpha$ decreases. As $k \to k_c^-$, use of the first of the expressions in (1.24) shows

$$S^{(L)}_{\infty}(k; \alpha) \sim 1 - \frac{2}{3\pi (1 - \sqrt{c})} \frac{1}{c} \left( \frac{k_c^2 - k^2}{(1 + k^2)(1 + k_c^2)} \right)^{3/2} + O\left( \frac{k_c^2 - k^2}{(1 + k^2)(1 + k_c^2)} \right)^{5/2}, \quad (4.17)$$

Hence both the function value (which is equal to 1), and the value of its first derivative (which is equal to 0) agree at the transition to the plateau.
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Appendix

In Remark 1.3.2 it was noted that a referee outlined to the author an approximate analysis which reproduces (1.26). This analysis is based on the universal form (1.29), or equivalently its Fourier transform

$$
\int_{-\infty}^{\infty} \frac{\sin^2(\pi \rho w)}{w^2} e^{i wt} dw = \begin{cases} 
\pi^2 \left( \rho - \tau/(2\pi) \right), & \tau/(2\pi) < \rho \\
0, & \tau/(2\pi) > \rho,
\end{cases}
$$

(A.1)

where it is assumed $\tau > 0$. In the same remark we commented that this idea can be also found in the original paper of Brézin and Hikami [3], where it was used to anticipate the findings of their exact analysis giving the functional form (1.28) for the GUE; see also [36, Sect. 3].

To present the argument, first note that combining (1.3) with the first line of (1.12) in the case $\Gamma_1 = 0$ shows

$$
S_N(k) = \int_{-\infty}^{\infty} d\lambda e^{ik\lambda} \int_{-\infty}^{\infty} d\lambda' e^{ik\lambda'} \left( \rho^T_{(2)}(\lambda, \lambda') + \delta(\lambda - \lambda')\rho_{(1)}(\lambda') \right).
$$

(A.2)

Next change variables $\lambda, \lambda' \mapsto 4N\lambda, 4N\lambda'$ so that the density for large $N$ has the leading form

$$
4\rho_{(1)}(4N\lambda) \sim N\rho^{MP}(\lambda), \quad \rho^{MP}(\lambda) := \frac{2}{\pi} \sqrt{(1 - \lambda)/\lambda} \chi_{0 < \lambda < 1}
$$

(A.3)

(this is (4.1) with $\alpha = 0$). Following [3], or the independent working of the referee, the key hypothesis is the approximation

$$
4^2 \rho^T_{(2)}(4N\lambda, 4N\lambda') \approx -\frac{\left( \sin[\pi N\rho^{MP}((\lambda + \lambda')/2)(\lambda - \lambda')] \right)^2}{(\pi(\lambda - \lambda'))^2} \chi_{0 < \lambda, \lambda' < 1}
$$

(A.4)

which is based on the universal bulk scaling form (1.29).

Noting that the double integral of the second term in (A.2) equals $N$ as a normalisation, and changing variables

$$
w = N(\lambda - \lambda'), \quad u = (\lambda + \lambda')/2
$$

then gives as a large $N$ approximation

$$
\frac{1}{N} S_N^{(L)}(k) \approx 1 - \frac{1}{\pi^2} \int_{-\infty}^{\infty} dw \int_{0}^{1} du \frac{\sin^2(\pi \rho^{MP}(u) w)}{w^2} e^{4i w k}.
$$

(A.5)
Changing the order of integration, and using \((A.1)\) with \(\tau = 4k > 0\) then shows

\[
\frac{1}{N} S_N^{(L)}(k) \approx 1 - \int_{0}^{u^*} \left( \rho_{MP}(u) - \frac{2k}{\pi} \right) du, \tag{A.6}
\]

where \(u^* = u^*(k)\) is such that

\[
\frac{\pi}{2} \rho_{MP}(u^*) = k, \tag{A.7}
\]

and thus from \((A.3)\) has the explicit value

\[
u^* = \frac{1}{1 + k^2}. \tag{A.8}
\]

Differentiating \((A.6)\) with respect to \(k\), and making use of \((A.7)\) and \((A.8)\) shows

\[
\frac{d}{dk} \left( \frac{1}{N} S_N(k) \right) \approx \frac{2}{\pi} u^* = \frac{2}{\pi} \frac{1}{1 + k^2}, \tag{A.9}
\]

which we see is in precise agreement with \((1.26)\). As emphasised by the referee, a further prediction of this working is that for a random matrix ensemble in the unitary symmetry class, and thus possessing a bulk scaled two-point function \((1.29)\), the ramp-plateau transition will be absent whenever the corresponding spectral density is unbounded. Thus the distinction of the behaviours \((1.26)\) when the limiting spectral density is given by \((A.3)\), and \((1.24), (1.25)\) when the limiting spectral density is given by \((4.1)\).

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