CAN A SINGLE PDE GOVERN WELL THE PROPAGATION OF THE ELECTRIC WAVE FIELD IN A HETEROGENEOUS MEDIUM IN 3D? *

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Abstract. It is demonstrated in this paper that the propagation of the electric wave field in a heterogeneous medium in 3D can sometimes be governed well by a single PDE, which is derived from the Maxwell’s equations. The corresponding component of the electric field dominates two other components. This justifies some past results of the second author with coauthors about numerical solutions of coefficient inverse problems with experimental electromagnetic data. In addition, since it is simpler to work in applications with a single PDE rather than with the complete Maxwell’s system, then the result of this paper might be useful to researchers working on applied issues of the propagation of electromagnetic waves in inhomogeneous media.

Key words. Maxwell’s equations, geodesic lines, domination of one component, experimental data for inverse problems

AMS subject classifications. 35Q61, 35R30

1. Introduction. In some previous works of the second author with coauthors coefficient inverse problems were solved for frequency dependent microwave experimental electromagnetic data using only the single Helmholtz equation, see, e.g. [6, 7, 8]. Reconstruction results were quite accurate ones. A similar observation took place in [1, 15], although for time dependent experimental data. Thus, a natural question to pose is: Given that the propagation of the electromagnetic wave field is governed by the Maxwell’s equations, why the use of only a single equation has provided accurate reconstruction results? A positive heuristic answer to this question can be found in the classical textbook of M. Born and E. Wolf [3, pages 695,696] for the frequency domain case. In addition, this question was positively addressed numerically in [2] for the time domain case and in [10] for the frequency domain case. It was demonstrated computationally in [2, 10] that if the incident electric wave field has only a single non zero component, then this component dominates two other components while propagating through the medium, and its propagation is well governed by a wave-like PDE. That PDE is either the Helmholtz equation in the frequency domain or the corresponding hyperbolic equation in the time domain.

The goal of this paper is to investigate the above question rigorously. We believe that the results of this paper might be useful not only for an analytical explanation of the accuracy of imaging results of [6, 7, 8] but also for applied mathematicians, physicists and engineers working on various topics of electromagnetic waves propagation. Indeed, it is clear that it is easier to work in applications with a single PDE rather than with the whole Maxwell’s system.

In section 2 we work in time domain. These results are used then in section

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3, where we derive our desired conclusion for the case of the frequency domain. In section 4, we link our main Theorem 2 of section 3 with the above cited results of [6, 7, 8]. In section 5 (Appendix) we prove a certain energy estimate.

2. Time Domain. Consider the Maxwell’s equations in a non magnetic medium

\[ \text{curl} \mathbf{H} = \varepsilon(x) \mathbf{E}_t, \quad \text{curl} \mathbf{E} = -\mathbf{H}_t, \quad \text{div} \mathbf{H} = 0, \quad x \in \mathbb{R}^3, t > 0, \]

where \( \varepsilon(x) \) is the spatially distributed dielectric constant. We work in this paper with dimensionless variables, since variables were made dimensionless in the above cited works about the experimental data. Thus, in vacuum \( \varepsilon(x) = 1 \), and we also assume that the magnetic permeability \( \mu \equiv 1 \). Let \( \mathbf{\nu} \in S^2 = \{ |\mathbf{\nu}| = 1 \} \) be a unit vector of the direction of propagation of the incident electric wave field. If the space \( \mathbb{R}^3 \) is vacuum, then equations (2.1) admit the following solution [13]:

\[ \mathbf{E}^0(x, t) = j \delta(t + t_0 - x \cdot \mathbf{\nu}), \quad \mathbf{H}^0(x, t) = (\mathbf{\nu} \times j) \delta(t + t_0 - x \cdot \mathbf{\nu}), \]

where \( \delta(t) \) is the Dirac delta function, \( t_0 \) is an arbitrary number, \( x \cdot \mathbf{\nu} \) denotes the scalar product of these two vectors and \( j \cdot \mathbf{\nu} = 0. \) The vector \( j \) defines the polarization of this wave,

\[ j = (j_1, j_2, j_3), \]

where \( j_1, j_2, j_3 \) are some constants. In the sequel we assume that \( j \in S^2 \). The orthogonality of vectors \( j \) and \( \mathbf{\nu} \) is necessary to satisfy the equation \( \text{div} \mathbf{H} = 0. \)

Below \( j \) and \( \mathbf{\nu} \) are assumed to be arbitrary but fixed vectors. Therefore we do not indicate dependence of the solution and some functions on these parameters for brevity, unless this is really necessary. Still, we use the parameter \( \mathbf{\nu} \) to indicate some domains and a plane wave for a more clear understanding. Below vectors \( j, \mathbf{\nu}, \mathbf{E}, \mathbf{H} \), etc. are row vectors, see, e.g. (2.3).

Let \( R > 0 \) be an arbitrary number. Consider the ball \( B \) with the center at \( \{0\} \) and the radius \( R \). Let the sphere \( S = \partial B \). Then

\[ B = \{ x \in \mathbb{R}^3 : |x| < R \}, \quad S = \{ x \in \mathbb{R}^3 : |x| = R \}. \]

We assume that

\[ 1 \leq \varepsilon(x) \leq \varepsilon_0 \quad \text{in} \quad B, \quad \varepsilon(x) = 1 \quad \text{in} \quad \mathbb{R}^3 \setminus B, \]

where \( \varepsilon_0 \geq 1 \) is a constant, and

\[ t_0 = \min_{x \in S} (x \cdot \mathbf{\nu}) = -R. \]

Let the incident plane wave propagates in the vacuum for \( t < 0 \) and meets the heterogeneous ball \( B \) at a moment of time \( t = 0 \). Then the propagation of the electromagnetic wave field is governed by the following Cauchy problem:

\[ \text{curl} \mathbf{H} = \varepsilon(x) \mathbf{E}_t, \quad \text{curl} \mathbf{E} = -\mathbf{H}_t, \quad (x, t) \in \mathbb{R}^4, \]

\[ \mathbf{E}|_{t<0} = \mathbf{E}^0(x, t), \quad \mathbf{H}|_{t<0} = \mathbf{H}^0(x, t). \]

For the sake of convenience, we reduce now problem (2.6), (2.7) to the case when only the vector function \( \mathbf{E}(x, t) \) is unknown. Since \( \text{div} (\text{curl} \mathbf{U}) = 0 \) for any appropriate
vector function $\mathbf{U}$, then applying the operator $\text{div}$ to both sides of the first equation (2.6), we obtain

\[
(2.8) \quad \text{div} (\varepsilon (x) \mathbf{E}_t) = 0.
\]

Integrating (2.8) with respect to $t$ and using (2.7), we obtain

\[
(2.9) \quad \text{div} (\varepsilon (x) \mathbf{E}(x,t)) - \text{div} (\varepsilon (x) \mathbf{E}^0 (x,\nu,0)) = 0.
\]

Note that $\text{div} \mathbf{E}^0 (x,t) = 0$, since $\mathbf{j} \cdot \mathbf{\nu} = 0$. Indeed,

\[
\text{div} \mathbf{E}^0 (x,t) = -(\mathbf{j} \cdot \mathbf{\nu}) \delta (t + t_0 - x \cdot \mathbf{\nu}) = 0.
\]

Hence,

\[
(2.10) \quad \text{div} (\varepsilon (x) \mathbf{E}^0 (x,0)) = \nabla \varepsilon (x) \cdot \mathbf{E}^0 (x,0) = (\nabla \varepsilon (x) \cdot \mathbf{j}) \delta (t_0 - x \cdot \mathbf{\nu}) = 0.
\]

This is because by (2.5) $\text{supp} \{\delta (t_0 - x \cdot \mathbf{\nu})\}$ is the tangent plane to $S$, namely $(x \cdot \mathbf{\nu}) = -R$, along which $\nabla \varepsilon (x) = 0$. Hence $\text{div} (\varepsilon (x) \mathbf{E}(x,t)) = 0$ for any $t$. Hence, problem (2.6), (2.7) is reduced to the following problem with $n(x) = \sqrt{\varepsilon (x)}$:

\[
(2.11) \quad n^2 (x) \mathbf{E}_{tt} - \Delta \mathbf{E} - \nabla (\mathbf{E} \cdot \nabla \ln n^2 (x)) = 0, \quad (x,t) \in \mathbb{R}^4,
\]

\[
(2.12) \quad \mathbf{E}|_{t<0} = \mathbf{E}^0 (x,t).
\]

Define two domains $D_- (\mathbf{\nu})$ and $D_+ (\mathbf{\nu})$ as

\[
D_- (\mathbf{\nu}) = \{ x \in \mathbb{R}^3 : x \cdot \mathbf{\nu} + R < 0 \},
\]

\[
D_+ (\mathbf{\nu}) = \{ x \in \mathbb{R}^3 : x \cdot \mathbf{\nu} + R \geq 0 \}.
\]

Note that the domain $D_- (\mathbf{\nu})$ is situated outside of $B$, while $B \subset D_+ (\mathbf{\nu})$.

To define geodesic lines, we partially follow our paper [9]. The function $n(x)$ generates the Riemannian metric

\[
d\tau = n(x) |dx|, \quad |dx| = \sqrt{(dx_1)^2 + (dx_2)^2 + (dx_3)^2}.
\]

For each vector $\mathbf{\nu} \in \mathbb{S}^2$ define the plane $\Sigma (\mathbf{\nu})$ as

\[
(2.13) \quad \Sigma (\mathbf{\nu}) = \{ \xi \in \mathbb{R}^3 : \xi \cdot \mathbf{\nu} = -R \}.
\]

Observe that the plane $\Sigma (\mathbf{\nu})$ is tangent to $S$ at the point $\xi_{\text{tan}} = -R \mathbf{\nu}$. Hence, $\Sigma (\mathbf{\nu}) \cap B = \emptyset$ and the vector $\mathbf{\nu}$ is a normal vector to the plane $\Sigma (\mathbf{\nu})$. Consider an arbitrary point $y \in \Sigma (\mathbf{\nu})$. This point can be represented as

\[
(2.14) \quad y = y(a_2, a_3) = -R \mathbf{\nu} + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3, \quad (a_2, a_3) \in \mathbb{R}^2,
\]

where unit vectors $\mathbf{\nu}, \mathbf{e}_2 = \mathbf{j}, \mathbf{e}_3 = \mathbf{\nu} \times \mathbf{j}$ form an orthogonal triple. Note that vectors $\mathbf{e}_2, \mathbf{e}_3$ are parallel to the plane $\Sigma (\mathbf{\nu})$.

Let the function $\varphi (x, \mathbf{\nu})$ be the solution of the Cauchy problem for the eikonal equation,

\[
(2.15) \quad |\nabla \varphi (x, \mathbf{\nu})|^2 = n^2 (x), \quad \varphi (x, \mathbf{\nu})|_{x \in \Sigma (\mathbf{\nu})} = 0
\]
satisfying the following conditions:

\[
\varphi(x, \nu) \begin{cases} 
< 0 & \text{if } x \in D_-(\nu), \\
> 0 & \text{if } x \in D_+ (\nu).
\end{cases}
\]

The number \(|\varphi(x, \nu)|\) is the Riemannian distance between the point \(x\) and the plane \(\Sigma(\nu)\). From the Physics standpoint, \(|\varphi(x, \nu)|\) is the travel time between the point \(x\) and the plane \(\Sigma(\nu)\). For \(\xi \cdot \nu < -R\), i.e. in the domain \(D_-(\nu)\), the function \(\varphi(x, \nu)\) has the form \(\varphi(x, \nu) = x \cdot \nu + R\). To find the function \(\varphi(x, \nu)\) in the domain \(D_+(\nu)\), we need to solve problem (2.15), (2.16) in this domain. It is known that to do this, we need to solve the following Cauchy problem for a system of ordinary differential equations [14]:

\[
\begin{align*}
\frac{dx}{ds} &= \frac{p(x, \nu)}{n^2(x)}, & \frac{dp(x, \nu)}{ds} &= \nabla \ln n(x), & \frac{d\varphi(x, \nu)}{ds} &= 1, s > 0, \\
\end{align*}
\]

\[x|_{s=0} = y, \quad p|_{s=0} = \nu, \quad \varphi|_{s=0} = 0,
\]

where \(y \in \Sigma(\nu)\) is an arbitrary point of the plane \(\Sigma(\nu)\), see (2.14), \(s\) is the Riemannian arc length and

\[p(x, \nu) = \nabla \varphi(x, \nu).
\]

Note that equations (2.17), (2.18) imply that \(\varphi(\xi, \nu) = s\). In particular, this means that the second condition (2.15) is satisfied. Equations (2.17), (2.18) define a geodesic line \(\Gamma(x, \Sigma(\nu))\) in \(D_+(\nu)\) which connects the point \(x \in D_+(\nu)\) with the point \(y \in \Sigma(\nu)\) and orthogonal to \(\Sigma(\nu)\) at \(y\).

Cauchy problem (2.17), (2.18) has the unique solution

\[x = f(s, a_2, a_3), \quad p = g(s, a_2, a_3).
\]

These equations define the bundle of geodesic lines, which go out from different points \(y \in \Sigma(\nu)\) in direction \(\nu\). To find the geodesic line \(\Gamma(x, \Sigma(\nu))\), we need to invert the first equation (2.20) and calculate \(a_2\) and \(a_3\) and then to find \(y \in \Sigma(\nu)\), using formula (2.14). Set

\[D_+(\nu, R) = \{x : -R \leq x \cdot \nu \leq R\}, \quad T = \max_{x \in D_+(\nu, R)} \varphi(x, \nu).
\]

Hence, \(D_+(\nu, R) \subseteq D_+(\nu)\). Note that both sets \(D_+(\nu)\) and \(D_+(\nu, R)\) are closed ones, i.e. \(D_+(\nu) = \bar{D}_+(\nu)\) and \(D_+(\nu, R) = \bar{D}_+(\nu, R)\). Here \(T\) is a finite number. Indeed, let \(C(\nu, R)\) be the circular cylinder with the circle of the radius \(R\) and the axis \(x = (-R + s')\nu, s' \geq 0\), with generating lines orthogonal to the plane \(\Sigma(\nu)\). Consider the intersection \(D_0(\nu, R)\) of \(C(\nu, R)\) with \(D_+(\nu, R)\),

\[D_0(\nu, R) = C(\nu, R) \cap D_+(\nu, R).
\]

Then the function \(\varphi(x, \nu) = x \cdot \nu + R \leq 2R\) for \(x \in D_+(\nu, R) \setminus D_0(\nu, R)\), since \(x \cdot \nu \in [-R, R]\) in \(D_+(\nu, R)\). On the other hand the domain \(D_0(\nu, R)\) is finite. Therefore,

\[T = \max_{x \in D_0(\nu, R)} \left(\frac{2R}{x \cdot \nu} \varphi(x, \nu)\right).
\]
Consider the Jacobian
\begin{equation}
J(x) = \frac{\partial(x_1, x_2, x_3)}{\partial(s, a_2, a_3)}.
\end{equation}

From relations (2.17) and (2.18) follows that \( \partial x/\partial s = p = \nu \) and \( \partial x/\partial a_k = e_k \), \( k = 2, 3 \), at \( s = 0 \). Then the Jacobian is the determinant which rows are formed by components of three unite orthogonal vectors of the positive orientation. Hence,
\begin{equation}
J(x) = 1 \text{ for } x \in \Sigma(\nu).
\end{equation}

Note that \( J(x) = 1 \) for \( x \in D_0(\nu, R) \) as well since \( x = y + s\nu \) in \( D_0(\nu, R) \).

Denote by \( c(x) = 1/\sqrt{\varepsilon(x)} \) the speed of propagation of electromagnetic waves. By (2.4)
\begin{equation}
c_0 \leq c(x) \leq 1, \quad x \in \mathbb{R}^3,
\end{equation}
where \( c_0 = 1/\sqrt{\varepsilon_0} \).

Below we use the following assumptions:

**Assumptions:**
1. The function \( \varepsilon(x) \in C^\infty(\mathbb{R}^3) \), satisfies conditions (2.4).
2. There exists a positive constant \( J_0 \) such that \( J(x) > J_0 \) for \( D_+ (\nu, R) \).
3. Any point \( x \in D_+(\nu, R) \) can be connected with the plane \( \Sigma(\nu) \) by a single geodesic line \( \Gamma(x, \Sigma(\nu)) \) such that \( \Gamma(x, \Sigma(\nu)) \) is orthogonal to \( \Sigma(\nu) \) at a point \( y \in \Sigma(\nu) \).
4. Any two points \( x \) and \( y \) in \( \mathbb{R}^3 \) can be connected by a single geodesic line.

Under these Assumptions, the equality \( x = f(s, a_2, a_3) \) is invertible in \( D_+(\nu, R) \) and defines \( s = s(x, \nu) = \varphi(x, \nu) \) and parameters \( a_k = a_k(x, \nu) \), \( k = 2, 3 \), i.e. the point \( y \in \Sigma(\nu) \), see (2.14). By (2.17) and (2.18) if \( x = y \), then \( dx/ds = \nu \). The latter vector is directed along the geodesic line \( \Gamma(x, \Sigma(\nu)) \). Hence, \( \Gamma(x, \Sigma(\nu)) \) is orthogonal to \( \Sigma(\nu) \) at the point \( y \).

Define domains \( G_T(\nu) \) and \( G(T, \nu) \) in \( \mathbb{R}^4 \) as
\begin{equation}
G_T(\nu) = \{(x, t) : 0 \leq t < \min(|\varphi(x, \nu)|, T)\},
\end{equation}
\begin{equation}
G(T, \nu) = \{(x, t), |\varphi(x, \nu)| \leq t \leq T\}.
\end{equation}

To differentiate between notations of the Heaviside function \( H(t) \) and the magnetic wave field \( H(x, t) \), it is convenient to denote \( \theta_0(t) := H(t) \),
\begin{equation}
\theta_0(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}
\end{equation}

**Theorem 1.** Assume that the Assumption holds. Then for every vector \( \nu \in \mathbb{S}^2 \) the solution of problem (2.11), (2.12) can be represented in \( \mathbb{R}^4_T = \{(x, t) | 0 \leq t \leq T\} \) in the form
\begin{equation}
E(x, t) = \alpha^{-1}(x)\delta(t - \varphi(x)) + \tilde{E}(x, t)\theta_0(t - |\varphi(x)|),
\end{equation}
where \( \alpha^{-1}(x) \in C^\infty(D_+(\nu, R)) \), \( \alpha^{-1}(x) = 0 \) for \( x \in D_- \) and \( \tilde{E}(x, t) \in C^2(G(T, \nu)) \) and \( \tilde{E}(x, t) = 0 \) for \( t = |\varphi(x, \nu)| \).
\textbf{Proof.} Introduce functions \( \theta_k(t) \) as
\begin{equation}
\theta_{-3}(t) = \delta''(t), \quad \theta_{-2}(t) = \delta'(t), \quad \theta_{-1}(t) = \delta(t), \quad \theta_k(t) = \frac{t^k}{k!} \theta_0(t), \quad k = 1, 2, \ldots.
\end{equation}
Observe that \( \theta'_k(t) = \theta_{k-1}(t) \) for all \( k \geq -2 \). We seek the solution of problem (2.11), (2.12) in the form
\begin{equation}
E(x, t) = \sum_{k=-1}^{r} \alpha^k(x) \theta_k(t - \varphi(x)) + E^r(x, t),
\end{equation}
where the natural number \( r \) will be chosen later. Substituting representation (2.28) in (2.11), using the eikonal equation (2.15) and equating coefficients at \( \theta_k(t) \) for \( k = -2, -1, 0, 1, \ldots, r - 1 \), we obtain the following recursive formulas for finding coefficients \( \alpha^k(x) \):
\begin{equation}
2(\nabla \varphi(x) \cdot \nabla) \alpha^k(x) + \alpha^k(x) \Delta \varphi(x) + (\alpha^k(x) \cdot \nabla \ln n^2(x)) \nabla \varphi(x) = \Delta \alpha^{k-1}(x) + \nabla(\alpha^{k-1}(x) \cdot \nabla \ln n^2(x)), \quad k = -1, 0, 1, \ldots, r.
\end{equation}
Here we need to formally set
\begin{equation}
\alpha^{-2}(x) = 0.
\end{equation}
Since by (2.2) and (2.12) \( E = j \delta(t - (x \cdot \nu + R)) \) for \( t < 0 \), then we obtain
\begin{equation}
\alpha^{-1}(x) = 0, \quad \text{in} \quad D_-; \quad \alpha^{-1}|_{\Sigma(\nu)} = j,
\end{equation}
\begin{equation}
\alpha^k(x) = 0, \quad \text{in} \quad D_-; \quad \alpha^k|_{\Sigma(\nu)} = 0, \quad k = 0, 1, \ldots, r.
\end{equation}
Moreover, using (2.11), we obtain the following Cauchy problem for the residual \( E^r(x, t) \) of expansion (2.28)
\begin{equation}
n^2(x) \partial_t^2 E^r - \Delta E^r - \nabla(\nabla(\nabla \ln n^2(x))) = F^r(x, t), \quad E^r|_{t<0} = 0,
\end{equation}
where
\begin{equation}
F^r(x, t) = (\Delta \alpha^r(x) + \nabla(\alpha^r(x) \cdot \nabla \ln n^2(x))) \theta_r(t - \varphi(x)).
\end{equation}
We now construct solutions of equation (2.29) with the Cauchy data (2.31). We have along the geodesic line \( \Gamma(x, \Sigma(\nu)) : \)
\begin{equation}
2(\nabla \varphi(\xi) \cdot \nabla) \alpha^k(\xi) = 2(p(\xi) \cdot \nabla) \alpha^k(\xi) = 2n^2(\xi) \left( \frac{d\xi}{ds} \cdot \nabla \right) \alpha^k(\xi)
= 2n^2(\xi) \frac{d\alpha^k(\xi)}{ds},
\end{equation}
where \( \xi \in \Gamma(x, \Sigma(\nu)) \) is an arbitrary point. Moreover, it is stated in the paper [12] that the following formula valid along \( \Gamma(x, \Sigma(\nu)) \) (see Lemma 1, Equation (4.2)):
\begin{equation}
\frac{d \ln J(\xi)}{ds} = \text{div} \left( n^{-2}(\xi) \nabla \varphi(\xi) \right),
\end{equation}
where $J(\xi)$ is the Jacobian defined in (2.22), provided that $x = (x_1, x_2, x_3)$ is replaced with $\xi = (\xi_1, \xi_2, \xi_3)$. Calculating the right-hand-side of (2.35) and using (2.35), we find

$$\text{div} \left( n^{-2}(\xi) \nabla \varphi(\xi) \right) = n^{-2}(\xi) \Delta \varphi(\xi) + (\nabla n^{-2}(\xi) \cdot p(\xi)).$$

Hence, (2.35) implies

$$n^{-2}(\xi) \Delta \varphi(\xi) = \frac{d \ln J(\xi)}{ds} - (\nabla n^{-2}(\xi) \cdot p(\xi)) = \frac{d \ln(J(\xi)n^2(\xi))}{ds}.$$  

We have used here that, similarly (2.34),

$$-(\nabla n^{-2}(\xi) \cdot p(\xi)) = -n^2(\xi) \left(\nabla n^{-2}(\xi) \cdot \frac{d\xi}{ds}\right) = -n^2(\xi) \frac{d}{ds} n^{-2}(\xi) = \frac{d}{ds} \ln n^2(\xi).$$

Using formulae (2.34) and (2.36) and replacing in (2.29) $x$ with $\xi$, we obtain

$$2n^2(\xi) \left[ \frac{d\alpha_k(\xi)}{ds} + \alpha_k(\xi) \frac{d \ln(\sqrt{J(\xi)n(\xi)})}{ds} \right] + (\alpha_k(\xi) \cdot \nabla \ln n^2(\xi)) \nabla \varphi(\xi) = \Delta \alpha^{k-1}(\xi) + \nabla(\alpha^{k-1}(\xi) \cdot \nabla \ln n^2(\xi)), \quad k = -1, 0, 1, \ldots, r.$$  

Multiplying equation (2.37) by $\sqrt{J(\xi)}/(2n(\xi))$, we transform equation (2.29) along $\Gamma(x, \Sigma(\nu))$ to the recursive form

$$\frac{d}{ds} \left( \alpha^k(\xi)n(\xi)\sqrt{J(\xi)} \right) - \frac{n(\xi)\sqrt{J(\xi)}}{2} (\alpha^k(\xi) \cdot \nabla n^{-2}(\xi)) p(\xi) = Q^k(\xi), \quad k = -1, 0, 1, \ldots, r;$$

were

$$Q^k(\xi) = \frac{\sqrt{J(\xi)}}{2n(\xi)} \left[ \Delta \alpha^{k-1}(\xi) + \nabla(\alpha^{k-1}(\xi) \cdot \nabla \ln n^2(\xi)) \right].$$

When we solve equations (2.38) going from $k$ to $k + 1$, the function $Q^k(\xi)$ is always known from the previous step. It follows from (2.30) and (2.39) that $Q^{-1}(\xi) = 0$. Functions $\alpha^k(\xi)$ satisfy on $\Sigma(\nu)$ conditions (2.31). We also recall that by (2.4) and (2.21) $n(\xi)|_{\Sigma(\nu)} = 1$, $J|_{\Sigma(\nu)} = 1$.

Hence, integrating (2.38) with respect to $s$, we obtain a recursive integral equation along the geodesic line $\Gamma(x, \Sigma(\nu))$

$$\alpha^k(x) = \frac{1}{n(x)\sqrt{J(x)}} \left( A^k + \int_{\Gamma(x, \Sigma(\nu))} \left[ Q^k(\xi) \right. \right.$$  

$$\left. + \frac{n(\xi)\sqrt{J(\xi)}}{2} (\alpha^k(\xi) \cdot \nabla n^{-2}(\xi)) p(\xi) \right] ds, \quad k = -1, 0, 1, \ldots, r,$$

where $\xi = f(s, a_2, a_3)$, the vector function $f(s, a_2, a_3)$ is defined in (2.20) and

$$A^{-1} = j, \quad A^0 = 0, \quad k = 0, 1, \ldots, r.$$
Recall that \( s \) is the Riemannian arc length of \( \Gamma(\xi, \Sigma(\nu)) \). Equation (2.40) is a Volterra-type integral equation of the second kind along the curve \( \Gamma(x, \Sigma(\nu)) \). Therefore, this equation can be solved by the method of successive approximations which is rapidly converging. We can solve equation (2.40) for different \( k = -1, ..., r \) step-by-step, starting from \( k = -1 \).

As it will be clear in the sequel, the most important role plays the function \( \alpha^{-1}(x) \). We now represent this function through the resolvent \( R(x, \xi) \) of equation (2.40) for \( k = -1 \). First, we introduce the vector function \( \beta(x) \)

\[
\beta(x) = n(x) \sqrt{J(x)} \alpha^{-1}(x).
\]

Then equation for this function has the form

\[
\beta(x) = j + \frac{1}{2} \int_{\Gamma(x, \Sigma(\nu))} \left( \beta(\xi) \cdot \nabla n^{-2}(\xi) \right) p(\xi) \, ds.
\]

The more convenient form of this equation is:

\[
\beta(x) = j + \int_{\Gamma(x, \Sigma(\nu))} \beta(\xi) K_0(\xi) \, ds,
\]

where \( K_0(\xi) \) is the \( 3 \times 3 \) matrix

\[
K_0(\xi) = \frac{1}{2} (\nabla n^{-2}(\xi))^* p(\xi)
\]

and \( (\nabla n^{-2}(\xi))^* \) is the transposed vector \( (\nabla n^{-2}(\xi)) \), i.e. column vector, while \( p(\xi) \) is row vector.

Represent \( \beta(x) \) as

\[
\beta(x) = \sum_{n=0}^{\infty} \beta^n(x),
\]

where

\[
\beta^0(x) = j, \quad \beta^n(x) = \int_{\Gamma(x, \Sigma(\nu))} \beta^{n-1}(\xi) K_0(\xi) \, ds, \quad n = 1, 2, \ldots.
\]

Then

\[
\beta^1(x) = j \int_{\Gamma(x, \Sigma(\nu))} K_0(\xi) \, ds,
\]

Next,

\[
\beta^2(x) = j \int_{\Gamma(x, \Sigma(\nu))} \int_{\Gamma(\xi, \Sigma(\nu))} K_0(\xi') \, ds' K_0(\xi) \, ds.
\]

Changing here the repeated integration by place and then replacing \( s' \) with \( s \) and vice
versa, we obtain

$$\beta^2(x) = j \int_{\Gamma(x, \Sigma(\nu))} K_1(x, \xi) \, ds,$$

where

$$K_1(x, \xi) = \int_{\Gamma(x, \Sigma(\nu)) \setminus \Gamma(\xi, \Sigma(\nu))} K_0(\xi) K_0(\xi') \, ds'. $$

Similarly,

$$\beta^n(x) = j \int_{\Gamma(x, \Sigma(\nu))} \int_{\Gamma(\xi, \Sigma(\nu))} K_{n-2}(\xi, \xi') \, ds' \, K_0(\xi) \, ds$$

$$= j \int_{\Gamma(x, \Sigma(\nu))} K_{n-1}(x, \xi) \, ds, \quad n \geq 2,$$

where

$$K_{n-1}(x, \xi) = \int_{\Gamma(x, \Sigma(\nu)) \setminus \Gamma(\xi, \Sigma(\nu))} K_{n-2}(\xi', \xi') \, ds' \, K_0(\xi).$$

and $K_0(x, \xi) = K_0(\xi)$. Thus, we obtain

$$\beta(x) = j \left( I + \int_{\Gamma(x, \Sigma(\nu))} R(x, \xi) \, ds \right).$$

Here $I$ is the identity matrix and $R(x, \xi)$ is defined as

$$(2.41) \quad R(x, \xi) = \sum_{n=0}^{\infty} K_n(x, \xi).$$

Finally we obtain for $\alpha^{-1}(x)$ the following formula

$$\alpha^{-1}(x) = \frac{j}{n(x) \sqrt{J(x)}} \left( I + \int_{\Gamma(x, \Sigma(\nu))} R(x, \xi) \, ds \right).$$

Applying the above technique, similar formulae can be easily derived for $\alpha^k(x)$, $k = 0, 1, \ldots, r$.

Denote

$$P_k(x) = \int_{\Gamma(x, \Sigma(\nu))} Q_k(\xi) \, ds.$$

Then

$$(2.42) \quad \alpha^k(x) = \frac{1}{n(x) \sqrt{J(x)}} \left( P_k(x) + \int_{\Gamma(x, \Sigma(\nu))} P_k(\xi) R(x, \xi) \, ds \right), \quad k = 0, 1, \ldots, r.$$
We estimate \( R(x, \xi) \) later in the proof of Theorem 2. The uniform convergence of series (2.41) in \( D_+(\nu, R) \) follows from that estimate.

Since the function \( n(x) \in C^\infty(\mathbb{R}^3) \), then functions \( f(s, a_2, a_3) \) and \( J(x) \) belong to \( C^\infty(D_+(\nu, R)) \). Hence, all functions \( \alpha^k(x) \in C^\infty(D_+(\nu, R)) \). Moreover, \( \alpha^{-1}(x) = 0 \) for \( k = 0, 1, \ldots, r \), if \( x \) lies outside \( D_0(\nu, R) \) since \( \text{supp} \nabla \varepsilon(x) \subset B \). Therefore, functions \( \alpha^k(x) \) for \( k = 0, 1, \ldots, r \) are compactly supported in \( D_+(\nu, R) \). Hence, \( F^r(x, t) = 0 \) for \((x, t) \in D_0(\nu, R), t \geq 0\) (see notation (2.21)). Moreover, \( F^r(x, t) = 0 \) for \((x, t) \in G_T(\nu)\). These two facts imply that function \( F^r(x, t) \) is compactly supported in domain \( \mathbb{R}^*_T \supseteq G(T, \nu) \). Hence, it follows from (2.32) that the vector function \( E^r(x, t) \) vanishes in \( G_T(\nu) \) and it is compactly supported in \( \mathbb{R}^*_T \) because the speed of electromagnetic waves is finite.

We now apply the method of energy estimates to the problem (2.32) in the domain \( \mathbb{R}^*_T \supseteq G(T, \nu) \) to estimate function \( E^r(x, t) \). This method is a powerful tool for investigations of various problems of mathematical physics. Applications of this methods for boundary value problems are given in [4], [11] and many other books related to partial differential equations. So, main ideas of this method are well known. At the same time, we can not give a reference of the exact result that we need for our goal. Therefore, for the completeness of the proof and for the reader’s convenience we formulate below a lemma related to the estimate of solution to problem (2.32). Let \( Y(t, T) = \mathbb{R}^*_T \cap \{ t = \text{const} \} \).

**Lemma.** Let \( \varepsilon(x) \) satisfy the conditions

\[
\|\varepsilon\|_{C^r(\mathbb{R}^3)} \leq \mu, \quad \|\ln \varepsilon\|_{C^{r+1}(\mathbb{R}^3)} \leq \mu,
\]

with a positive constant \( \mu \) and function \( F^r(x, t) \) belongs to \( H^r(\mathbb{R}^*_T) \) and satisfies the inequality

\[
\|F^r\|_{H^r(\mathbb{R}^*_T)} \leq M.
\]

Then the solution of problem (2.32) \( E^r \in H^{r+1}(Y(t, T)) \) for all \( t \in (0, T) \) and there exists a positive constant \( C_1 = C_1(\mu, T) \) such that the following estimates hold:

\[
\|E^r\|_{H^{r+1}(Y(t, T))} \leq C_1 M, \quad \|\partial t E^r\|_{H^{r}(Y(t, T))} \leq C_1 M.
\]

The proof of this Lemma is given in the Appendix.

In our case \( E^r \in H^r(\mathbb{R}^*_T) \) and \( \varepsilon \in C^\infty(\mathbb{R}^3) \). Hence, conditions (2.43) and (2.44) valid with some positive \( \mu \) and \( M \). Applying Lemma, we obtain that the solution of problem (2.32) is such that \( E^r \in H^{r+1}(Y(t, T)) \) and \( \partial t E^r \in H^r(Y(t, T)) \) for all \( t \in [0, T] \). Choosing \( r = 4 \) and applying Lemma, we obtain \( E^4 \in H^5(Y(t, T)) \) for all \( t \in [0, T] \) and, hence, \( E^4 \in H^5(\mathbb{R}^*_T) \). Therefore the embedding theorem implies that \( E^4 \in C^2(\mathbb{R}^*_T) \). Hence, the vector function \( E^4 \in C^2 \left( \mathbb{R}^*_T \right) \) and is continuous together with space derivatives up to the second order across the characteristic wedge \( t = |\varphi(x)| \). In particular, \( E^4(x, t) = 0 \) for \( t = |\varphi(x)| \).

Setting

\[
\hat{E}(x, t) = \sum_{k=0}^{4} \alpha^k(x) \frac{(t - \varphi(x))^k}{k!} + E^4(x, t),
\]
we obtain (2.26) as well as the required smoothness $\mathbf{E} \in C^2 \left( G(\nu, T) \right)$. \hfill \Box

**Remark 1.** The equality (2.46) implies the following formula, which we use below:

$$\lim_{t \to \varphi(x)^+} \mathbf{E}(x, t) = \alpha^0(x).$$

3. **Frequency domain.** Consider the Fourier transform $\tilde{\mathbf{E}}(x, k)$ of the function $\mathbf{E}(x, t)$,

$$\tilde{\mathbf{E}}(x, k) = \int_{-\infty}^{\infty} \mathbf{E}(x, t) \exp(-ikt) \, dt$$

$$= \tilde{\mathbf{E}}^0(x, k) + \int_{0}^{\infty} \mathbf{E}(x, t) \exp(-ikt) \, dt,$$

where $k = 2\pi/\lambda$ is the wave number, $\lambda$ is the dimensionless wavelength and

$$\tilde{\mathbf{E}}^0(x, k) = j \exp(i(x \cdot \nu + R))\theta_0(-x \cdot \nu - R)$$

is the Fourier transform of $\mathbf{E}^0(x, t)$. The existence of the integral in (3.1) follows from results of Vainberg [17] which claim that the vector function $\mathbf{E}(x, t)$ decays exponentially together with its appropriate derivatives as $t \to \infty$ while $x$ runs over any bounded domain $\Omega \subset \mathbb{R}^3$. Next, theorem 3.3 of [16] and theorem 6 of Chapter 9 of [17] guarantee that $\tilde{\mathbf{E}}(x, k)$ is the solution to the equation

$$(\Delta + k^2 n^2(x)) \tilde{\mathbf{E}} + \nabla(\tilde{\mathbf{E}} \cdot \nabla \ln n^2(x)) = 0, \quad x \in \mathbb{R}^3,$$

where the scattering field

$$\tilde{\mathbf{E}}^{sc}(x, k) = \tilde{\mathbf{E}}(x, k) - \tilde{\mathbf{E}}^0(x, k)$$

satisfies the radiation condition as $|x| \to \infty$.

We now consider the vector function $\tilde{\mathbf{E}}(x, k)$ in (3.1) for $x \in D_+(\nu, R)$. Using representation (2.26), we obtain

$$\tilde{\mathbf{E}}(x, k) = \alpha^{-1}(x) \exp(-ik\varphi(x)) + \int_{\varphi(x)}^{\infty} \tilde{\mathbf{E}}(x, t) \exp(-ikt) \, dt.$$

Integrating by parts in (3.3) and using formula, we obtain

$$\tilde{\mathbf{E}}(x, k) = \alpha^{-1}(x) \exp(-ik\varphi(x)) + \frac{\exp(-ik\varphi(x))}{ik} \alpha^0(x)$$

$$+ \frac{1}{ik} \int_{\varphi(x)}^{\infty} \tilde{\mathbf{E}}_t(x, t) \exp(-ikt) \, dt,$$
Thus,
\[
\tilde{E}(x, k) = \alpha^{-1}(x) \exp(-ik\varphi(x)) + O\left(\frac{1}{k}\right), \quad k \to \infty, \quad \forall x \in D_+(\nu, R).
\]

Consider now the equation
\[
(\Delta + k^2 n^2(x)E) = 0, \quad x \in \mathbb{R}^3,
\]
with the incident plane wave \(\tilde{E}^0(x, k) = j \exp(i k(x \cdot \nu + R))\theta_0(-x \cdot \nu - R)\) and the radiation condition for \(\left(\tilde{E} - \tilde{E}^0\right)\). Note that \(\tilde{E} \cdot \nu = 0\) and \(\tilde{E} \cdot (\nu \times j) = 0\) since \(\tilde{E}^0 \cdot \nu = 0\) and \(\tilde{E}^0 \cdot (\nu \times j) = 0\) because \(\tilde{E}^0\) is parallel to \(j\) and also \(j \cdot \nu = 0\).

Consider the function \(\tilde{\pi}(x, t) = \tilde{E} \cdot j\). Then
\[
(\Delta + k^2 n^2(x)\tilde{\pi}(x, k)) = 0, \quad x \in \mathbb{R}^3,
\]
with the incident plane wave \(\tilde{\pi}^0(x, k) = \exp(i k(x \cdot \nu + R))\theta_0(-x \cdot \nu - R)\) and the radiation condition for \(\left(\tilde{\pi} - \tilde{\pi}^0\right)\).

We impose conditions below, which guarantee that the electric wave field \(\tilde{E}_j = j \tilde{\pi}(x, k)\) is close to \(\tilde{E}\) at the high values of the wave number \(k\), which is equivalent to small wavelengths \(\lambda\). First, suppose that these two electric wave fields are indeed close to each other in the norm of the space \(C\left(D_+(\nu, R)\right)\). This means that
\[
\tilde{E}(x, k) = \tilde{E}(x, k) + V(x, k), \quad x \in D_+(\nu, R),
\]
\[
\|V(x, k)\|_{C(D_+(\nu, R))} \leq \sigma,
\]
where \(\sigma > 0\) is a small number. Consider the component \(\tilde{u}(x, k) = \tilde{E}(x, k) \cdot j\). Then
\[
\tilde{u}(x, k) = \tilde{\pi}(x, k) + V(x, k) \cdot j,
\]
\[
\|V(x, k) \cdot j\|_{C(D_+(\nu, R))} \leq \sigma,
\]
which means that the function \(\tilde{u}(x, k)\) approximates well the solution of the Helmholtz equation \(\left(\tilde{E}^0\right)\) with the above incident plane wave and corresponding radiation conditions.

Next, since \(j \cdot \nu = j \cdot (\nu \times j) = 0\), then \(\left(\tilde{E} \cdot \nu = V(x, k) \cdot \nu\right)\) implies that
\[
\tilde{E}(x, k) \cdot \nu = \tilde{E}(x, k) \cdot \nu,
\]
\[
\tilde{E}(x, k) \cdot (\nu \times j) = V(x, k) \cdot (\nu \times j).
\]
Hence, components \(\tilde{E}(x, k) \cdot \nu\) and \(\tilde{E}(x, k) \cdot (\nu \times j)\) are sufficiently small,
\[
\|\tilde{E}(x, k) \cdot \nu\|_{C(D_+(\nu, R))} \leq \sigma, \quad \|\tilde{E}(x, k) \cdot \nu\|_{C(D_+(\nu, R))} \leq \sigma.
\]

Thus, it follows from \(\left(3.7\right)-\left(3.11\right)\) that the component \(\tilde{u}(x, k) = \tilde{E}(x, k) \cdot j\) of the electric wave field dominates two other components and it is close to the solution.
$$\pi(x, k)$$ of the Helmholtz equation supplied by the above incident plane wave and radiation conditions.

**Remark 2.** Since experimental data have noise, then it is sufficient to obtain a good approximation $$\pi(x, k)$$ for the component $$\tilde{u}(x, k) = \mathbf{E}(x, k) \cdot \mathbf{j}$$ of the electric wave field in the $$C(D_+(\mathbf{\nu}, R))$$-norm.

What is left to do is to prove (3.7), (3.8). And this is what is done in the rest of this section.

It is easy to derive a complete analog of formula (3.4) for the function $$\mathbf{E}$$. To do this, one should consider the Cauchy problem for the time dependent analog of (3.5) and repeat arguments of Theorem 1 for the case when the term $$\nabla (\mathbf{E} \cdot \nabla \ln n^2(x))$$ is neglected in (2.11). Hence,

$$\mathbf{E}(x, k) = \hat{\alpha}^{-1}(x) \exp(-ik\varphi(x)) + O \left( \frac{1}{k} \right), \quad k \to \infty, \quad \forall x \in D_+(\mathbf{\nu}, R),$$

where

$$\hat{\alpha}^{-1}(x) = \frac{\mathbf{j}}{n(x) \sqrt{J(x)}},$$

**Theorem 2.** Suppose that the Assumptions hold. Let $$\eta > 0$$ be such a constant that

$$\|\nabla n(x)\|_{C(\mathbf{\Pi})} \leq \eta.$$  

Then

$$\bar{\mathbf{E}}(x, k) - \mathbf{E}(x, k) = \mathbf{V}(x, k) + O \left( \frac{1}{k} \right), \quad k \to \infty, \quad x \in D_+(\mathbf{\nu}, R)$$

and

$$\bar{\mathbf{E}}(x, k) \cdot \mathbf{j} = \frac{\exp(-ik\varphi(x))}{n(x) \sqrt{J(x)}} + \mathbf{V}(x, k) \cdot \mathbf{j} + O \left( \frac{1}{k} \right),$$

$$\bar{\mathbf{E}}(x, k) \cdot \mathbf{\nu} = \mathbf{V}(x, k) \cdot \mathbf{\nu} + O \left( \frac{1}{k} \right),$$

$$\bar{\mathbf{E}}(x, k) \cdot (\mathbf{\nu} \times \mathbf{j}) = \mathbf{V}(x, k) \cdot (\mathbf{\nu} \times \mathbf{j}) + O \left( \frac{1}{k} \right), \quad k \to \infty, \quad x \in D_+(\mathbf{\nu}, R),$$

where

$$\mathbf{V}(x, k) = (\alpha^{-1}(x) - \hat{\alpha}^{-1}(x)) \exp(-ik(x \cdot \mathbf{\nu})), \quad \mathbf{V}(x, k) \bigg| \leq \frac{1}{\sqrt{J_0}} \exp(\eta T) - 1, \quad x \in D_+(\mathbf{\nu}, R).$$

Thus, if the number $$\eta$$ in (3.14) is sufficiently small, then (3.7) and (3.8) hold.

**Proof.** Using formulae (2.26) and (3.4) we obtain the first relation (3.16). Formulae (3.15) follow from (3.16) as well as from (3.4). Then, using (3.4) and (3.13), we obtain

$$|\mathbf{V}(x, k)| = |\alpha^{-1}(x) - \hat{\alpha}^{-1}(x)| = \frac{1}{n(x) \sqrt{J(x)}} \left| \int_{\Gamma(x, \xi)} R(x, \xi) ds \right|,$$
We now estimate $R(x, \xi)$ for $x \in D_+(\nu, R)$ and $\xi \in \Gamma(x, \Sigma(\nu))$. Note that $R(x, \xi) = 0$ for $x \in D_+(\nu, R) \setminus D_0(\nu, R)$ since $\nabla n(x) = 0$ and, hence, all $K_n(x, \xi) = 0$ in this domain. So, we need estimate $R(x, \xi)$ for $x \in D_0(\nu, R)$ only.

Introduce the matrix norm for a matrix $K(x, \xi) = (k_{ij}(x, \xi))_{i,j=1}^3$ as

$$\|K(x, \xi)\| = \max_{i,j=1,2,3} |k_{ij}(x, \xi)|.$$ 

Let $x \in D_0(\nu, R)$ and $\varphi(x) = s_0$ and $\varphi(\xi) = s$. Then using (2.41) and formulae for $K_n(x, \xi)$, $n = 0, 1, 2, \ldots$, we obtain

$$\|K_0(x, \xi)\| \leq \eta, \quad x \in D_0(\nu, R), \quad \xi \in \Gamma(x, \Sigma(\nu)),$$

$$\|K_1(x, \xi)\| \leq \eta^2(s_0 - s), \quad x \in D_0(\nu, R), \quad \xi \in \Gamma(x, \Sigma(\nu)),$$

$$\|K_n(x, \xi)\| \leq \eta^{n+1}(s_0 - s)^n/n!, \quad x \in D_0(\nu, R), \quad \xi \in \Gamma(x, \Sigma(\nu)), \quad n = 2, 3, \ldots$$

Hence

$$\|R(x, \xi)\| \leq \eta \exp(\eta(s_0 - s)), \quad x \in D_0(\nu, R),$$

Since $n^{-1}(x) \leq 1$ and $J(x) \geq J_0$, then we arrive at the estimate:

$$|\mathbf{V}(x, k)| \leq \frac{1}{\sqrt{J_0}}[\exp(\eta s_0) - 1] \leq \frac{1}{\sqrt{J_0}}[\exp(\eta T) - 1], \quad x \in D_0(\nu, R).$$

Thus, we obtain the estimate in the second line of (3.16). This estimate concludes the proof. □

4. Relevance to Experimental Results of [6, 7, 8]. We now explain why Theorem 2 at least partially justifies the validity of modeling of the propagation of electromagnetic waves in the frequency domain by the single Helmholtz equation (3.6) in the works of the second author with coauthors on experimental data [6, 7, 8]. We say “at least partially” because a completely precise explanation is unlikely possible since we deal here with a sort of a “mathematics-to-physics bridge”.

We recall that accurate reconstruction results were obtained in [6, 7, 8] when solving coefficient inverse problems. Experimental data in these references were collected for the cases when rather small inclusions mimicking land mines and improvised explosive devices were embedded in an otherwise uniform background (dry sand). The dielectric constant was not changing within such an inclusion, although this was not an assumption in reconstruction algorithms. Therefore,

$$\nabla n(x) = \begin{cases} 0 \text{ within such an inclusion,} \\ 0 \text{ in the background.} \end{cases}$$

The question remains now about the discontinuity of the function $n(x)$ at the inclusion/background interface. Since any solution of an elliptic equation, such as, e.g. (3.6), is sufficiently smooth outside of discontinuities of its coefficients [5], then we conjecture that the medium “percepts” the functions $n(x)$ in those inclusions as a smooth function with rather non-small values of $|\nabla n(x)|$ near those interfaces. In fact, this has been observed in computed images of [6, 7, 8]. Now, since values of $|\nabla n(x)|$ were not small only in close proximities of those interfaces and volumes of those proximities were small, then this means that norms $||\nabla n(x)||_{L^2(B)}$ were actually small. On the other hand, since finite differences with relatively small numbers
of grid points were used in [6, 7, 8] to solve inverse problems and since all norms in a finite dimensional space are equivalent, then the smallness of the discrete norm \( \|\nabla n(x)\|_{L_2(B)} \) is equivalent to the smallness of the discrete norm \( \|\nabla n(x)\|_{C(B)} \), which is close to the smallness assumption imposed in Theorem 2 on the number \( \eta \) in (3.14). Note that smallness assumptions were not used in algorithms of [6, 7, 8].

Thus, Theorem 2 explains, at least partially, accurate reconstructions in [6, 7, 8].

5. Appendix. Proof of the Lemma.

Denote \( E_r^j, j = 1, 2, 3 \), components of vector function \( E^r \). Calculating scalar product of both sides of equation (2.32), \( 2\partial_t E^r \), using \( n^2(x) = \varepsilon(x) \) and applying the identity

\[
2\partial_t E^r \cdot \Delta E^r = 2\text{div} \left( \sum_{j=1}^{3} (\partial_t E^r_j) \nabla E^r_j \right) - \partial_t \left( \sum_{j=1}^{3} |\nabla E^r_j|^2 \right)
\]

we obtain:

\[
\partial_t \left( \varepsilon(x)|\partial_t E^r|^2 + \sum_{j=1}^{3} |\nabla E^r_j|^2 \right) - 2\text{div} \left( \sum_{j=1}^{3} (\partial_t E^r_j) \nabla E^r_j \right) = -2 \sum_{i,j=1}^{3} \partial_t E^r_i \left[ (\partial_x E^r_j) \partial_x \varepsilon(x) + E^r_j \partial_x \ln \varepsilon(x) \right] = 2(\partial_t E^r) \cdot F^r.
\]

Integrating identity (5.1) over the domain \( \mathbb{R}_t^4 \), \( t \in (0, T] \) and taking into account that \( E^r \) and \( F^r \) are compactly supported in \( \mathbb{R}_t^4 \) and the initial zero data, we arrive to the equality

\[
\int_{\mathbb{R}_t^4} \left( \varepsilon(x)|\partial_t E^r(x, t)|^2 + \sum_{j=1}^{3} |\nabla E^r_j(x, t)|^2 \right) dx
\]

\[
= 2 \int_{\mathbb{R}_t^4} \sum_{i,j=1}^{3} (\partial_x E^r_i(x, \tau)) (\partial_{x_j} E^r_j(x, \tau)) \partial_{x_i} \ln \varepsilon(x) dxd\tau
\]

\[
+ 2 \int_{\mathbb{R}_t^4} \sum_{i,j=1}^{3} (\partial_x E^r_i(x, \tau)) (E^r_j(x, \tau)) \partial_{x_i} \ln \varepsilon(x) dxd\tau
\]

\[
+ 2 \int_{\mathbb{R}_t^4} \partial_t E^r(x, \tau) \cdot F^r(x, \tau) dxd\tau.
\]

Transform this equality using assumption (2.4), (2.43), the algebraic inequalities

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we obtain the more general inequality

\[
\varepsilon_0 \int_{Y(t,T)} \left( |\partial_t E^r(x, t)|^2 + \sum_{j=1}^{3} |\nabla E_j^r(x, t)|^2 \right) \, dx 
\leq C_1 \int_{\mathbb{R}_+^4} \left[ |\partial_t E^r(x, \tau)|^2 + \sum_{j=1}^{3} |\nabla E_j^r(x, \tau)|^2 + |E^r(x, \tau)|^2 \right] \, dx \, d\tau
\]

(5.3) 

+ \int_{\mathbb{R}_+^4} |F^r(x, \tau)| \, dx \, d\tau.

Using the inequality

\[
|F^r(x, t)|^2 = \left( \int_0^t \partial_t E^r(x, \tau) \, d\tau \right)^2 \leq T \int_0^t |\partial_t E^r(x, \tau)|^2 \, d\tau,
\]

we obtain the more general inequality

\[
\int_{Y(t,T)} \left( |\partial_t E^r(x, t)|^2 + \sum_{j=1}^{3} |\nabla E_j^r(x, t)|^2 + |E^r(x, t)|^2 \right) \, dx 
\leq C_1 \int_{\mathbb{R}_+^4} \left[ |\partial_t E^r(x, \tau)|^2 + \sum_{j=1}^{3} |\nabla E_j^r(x, \tau)|^2 + |E^r(x, \tau)|^2 \right] \, dx \, d\tau
\]

(5.4) 

+ \int_{\mathbb{R}_+^4} |F^r(x, \tau)| \, dx \, d\tau.

Applying Gronwall-Bellman to inequality (5.4), we find

\[
\int_{Y(t,T)} \left( |\partial_t E^r(x, t)|^2 + \sum_{j=1}^{3} |\nabla E_j^r(x, t)|^2 + |E^r(x, t)|^2 \right) \, dx \leq \|F^r\|_{L^2_{\mathbb{R}_+^4}}^2 \exp(C_1T).
\]

Thus, we have obtained the inequalities:

(5.5) \quad \|E^r\|_{H^1(Y(t,T))} \leq C_1 M, \quad \|\partial_t E^r\|_{L^2(Y(t,T))} \leq C_1 M.

where \( M \) is defined in (2.44).

Differentiating equation (2.32) \( k \leq r \) times with respect to \( t \) and then calculating scalar product of both sides of the resulting equation with the vector function \( 2\partial_t^k E^r \), we obtain relations (5.1)-(5.5) with \( \partial_t^k E^r \) instead \( E^r \). Therefore, the following estimates hold

(5.6) \quad \|\partial_t^k E^r\|_{H^1(Y(t,T))} \leq C_1 M, \quad \|\partial_t^{k+1} E^r\|_{L^2(Y(t,T))} \leq C_1 M, \; k \leq r.

Apply now the mathematical induction method to prove estimate (2.45). Suppose that for some \( n, 1 < n - 1 < r - 1, \) the estimates similar (5.5), (5.6) hold:

(5.7) \quad \|E^r\|_{H^{n-1}(Y(t,T))} \leq C_1 M, \quad \|\partial_t E^r\|_{H^{n-2}(Y(t,T))} \leq C_1 M,
\quad \|\partial_t^k E^r\|_{H^{n-1}(Y(t,T))} \leq C_1 M, \quad \|\partial_t^{k+1} E^r\|_{H^{n-2}(Y(t,T))} \leq C_1 M, \; k \leq r - (n - 2),
and prove that the similar estimates are valid when \( n - 1 \) replaced with \( n \). Denote

\[
D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}},
\]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is a multi index, \( \alpha_1, \alpha_2, \alpha_3 \) are integer nonnegative numbers and \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \). We shall use the Leibnitz formula for a product of two functions

\[
D^\alpha (uv) = \sum_{\beta \leq \alpha} C^\beta_\alpha (D^\beta u)(D^{\alpha - \beta} v),
\]

where \( \beta = (\beta_1, \beta_2, \beta_3) \), \( C^\beta_\alpha = C_\alpha^{\beta_1} C_\alpha^{\beta_2} C_\alpha^{\beta_3} \) is product of the binomial coefficients and \( \beta \leq \alpha \) means that \( \beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2, \beta_3 \leq \alpha_3 \). Applying the differential operator \( D^\alpha \) with \( |\alpha| = n \) to equation (2.32) and using the given above formula, we obtain

\[
\varepsilon(x) \partial_t^2 D^\alpha E^r - \Delta D^\alpha E^r + \sum_{\beta \leq \alpha, \beta \neq \alpha} C^\beta_\alpha (\partial_t^2 D^\beta E^r) (D^{\alpha - \beta} \varepsilon(x)) \\
+ \sum_{j=1}^3 \sum_{\beta \leq \alpha} C^\beta_\alpha [(\nabla D^\beta E^r_j)(\partial_{x_j} D^{\alpha - \beta} \varepsilon(x)) + (D^\beta E^r_j) \nabla (\partial_{x_j} D^{\alpha - \beta} \ln \varepsilon(x))] \\
= D^\alpha F^r(x, t).
\]

(5.8)

Calculating scalar product of both sides of equation (5.8) and \( 2 \partial_t D^\alpha E^r \), we obtain the relation similar in the main part to (5.1), namely:

\[
\partial_t \left( \varepsilon |\partial_t D^\alpha E^r|^2 + \sum_{j=1}^3 \left| \nabla D^\alpha E^r_j \right|^2 \right) - 2 \text{div} \left( \sum_{j=1}^3 (\partial_t D^\alpha E^r_j) \nabla E^r_j \right) \\
+ 2 \sum_{\beta \leq \alpha, \beta \neq \alpha} C^\beta_\alpha \left[ (\partial_t D^\alpha E^r) \cdot \partial_t^2 D^\beta E^r \right] (D^{\alpha - \beta} \varepsilon(x)) \\
- 2 \sum_{j=1}^3 \sum_{i \neq j} C^\beta_\alpha [(\partial_t D^\alpha E^r_i) (\partial_t D^\alpha E^r_j) \partial_{x_j} \ln \varepsilon(x) + (D^\beta E^r_j) \partial_{x_i, x_j} D^{\alpha - \beta} \ln \varepsilon(x)] \\
= 2 (\partial_t D^\alpha E^r) \cdot F^r.
\]

(5.9)

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Integrating this identity over domain \( \mathbb{R}_4^1, t \in (0, T] \), we arrive to the equality

\[
\int_{Y(t,T)} \left( \varepsilon(x)|\partial_t D^\alpha E^r(x, t)|^2 + \sum_{j=1}^3 |\nabla D^\alpha E_j^r(x, t)|^2 \right) \, dx \\
= -2 \sum_{\beta \leq \alpha, \beta \neq \alpha} C_\alpha^\beta \int_{\mathbb{R}_4^1} \left( (\partial_t D^\alpha E^r(x, \tau)) \cdot \partial_t^2 D^\beta E^r(x, \tau) \right) (D^{\alpha - \beta} \varepsilon(x)) \, dx d\tau \\
+ 2 \sum_{i,j=1}^3 \sum_{\beta \leq \alpha} C_\alpha^\beta \int_{\mathbb{R}_4^1} (\partial_t D^\alpha E_i^r(x, \tau)) (\partial_{x_i} D^\beta E_j^r(x, \tau)) \partial_{x_j} D^{\alpha - \beta} \ln \varepsilon(x) \, dx d\tau \\
+ 2 \sum_{i,j=1}^3 \sum_{\beta \leq \alpha} C_\alpha^\beta \int_{\mathbb{R}_4^1} (\partial_t D^\alpha E_i^r(x, \tau)) (D^\beta E_j^r(x, \tau)) \partial_{x_i x_j} D^{\alpha - \beta} \ln \varepsilon(x) \, dx d\tau \\
(5.10) + 2 \int_{\mathbb{R}_4^1} \partial_r D^\alpha E^r(x, \tau) \cdot D^\alpha F^r(x, \tau) \, dx d\tau.
\]

Use now assumption (2.4), (2.43) and the inequality \( 2a \cdot b \leq |a|^2 + |b|^2 \). Then, taking into account that \( \sum_{\beta \leq \alpha} C_\alpha^\beta \leq 2^{3n} \) and \( C_\alpha^\beta \leq 2^m \), we obtain

\[
\varepsilon_0 \int_{Y(t,T)} \left( |\partial_t D^\alpha E^r(x, t)|^2 + \sum_{j=1}^3 |\nabla D^\alpha E_j^r(x, t)|^2 \right) \, dx \\
\leq C_2 \int_{\mathbb{R}_4^1} \left[ |\partial_t D^\alpha E^r(x, \tau)|^2 + \sum_{j=1}^3 |\nabla D^\alpha E_j^r(x, \tau)|^2 \right] \, dx d\tau \\
+ 2^n \mu \sum_{\beta \leq \alpha, \beta \neq \alpha} \int_{\mathbb{R}_4^1} \left( |\partial_t^2 D^\beta E^r(x, \tau)|^2 \, dx d\tau + \sum_{j=1}^3 |\nabla D^\beta E_j^r(x, \tau)|^2 \right) \\
(5.11) + \int_{\mathbb{R}_4^1} |D^\alpha F^r(x, \tau)|^2 \, dx d\tau,
\]

where \( C_2 = 2^{3n+1} \mu \).

Since the relations \( \beta \leq \alpha, \beta \neq \alpha \) mean that \( |\beta| \leq |\alpha| - 1 = n - 1 \), then by the induction assumption (5.7), there exists a positive constant \( C_1 \) such that

\[
2^n \mu \sum_{\beta \leq \alpha, \beta \neq \alpha} \int_{\mathbb{R}_4^1} \left( |\partial_t^2 D^\beta E^r(x, \tau)|^2 \, dx d\tau + \sum_{j=1}^3 |\nabla D^\beta E_j^r(x, \tau)|^2 \right) \leq C_1 M^2.
\]

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Then we derive from (5.11) that
\[
\int_{Y(t,T)} \left( |\partial_t^k D^\alpha E'(x,t)|^2 + \sum_{j=1}^{3} |\nabla D^\alpha E''_j(x,t)|^2 \right) \, dx \leq C_1 \int_{\mathbb{R}^4_t} \left[ |\partial_t D^\alpha E'(x,\tau)|^2 + \sum_{j=1}^{3} |\nabla D^\alpha E''_j(x,\tau)|^2 \right] \, dx \, d\tau
\]
(5.12)
\[+ C_1 M^2.\]

Applying the Gronwall’s inequality, we obtain
\[
\int_{Y(t,T)} \left( |\partial_t D^\alpha E'(x,t)|^2 + \sum_{j=1}^{3} |\nabla D^\alpha E''_j(x,t)|^2 \right) \, dx \leq C_1 M^2, \quad |\alpha| = n.
\]
(5.13)

Differentiating equation (5.8) \( k \leq r - (n - 1) \) times with respect to \( t \) and then calculating scalar product of both sides of the obtained equation and \( 2\partial_t^k E' \), we obtain relations (5.8)-(5.13) with \( \partial_t^k D^\alpha E' \) instead \( D^\alpha E' \). Therefore, the following estimates hold
\[
\int_{Y(t,T)} \left( |\partial_t^{k+1} D^\alpha E'(x,t)|^2 + \sum_{j=1}^{3} |\nabla \partial_t^k D^\alpha E''_j(x,t)|^2 \right) \, dx \leq C_1 M^2, \quad k \leq r - (n - 1).
\]

Thus, inequalities (5.7) hold with \( n - 1 \) replaced by \( n \). This justifies the mathematical induction method and we can set \( n = r + 2 \) in (5.7). The latter proves the required inequalities (2.45). \( \square \)

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