A concavity condition for existence of a negative Neumann-Poincaré eigenvalue in three dimensions

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Abstract

It is proved that if a bounded domain in three dimensions satisfies a certain concavity condition, then the Neumann-Poincaré operator on the boundary of the domain or its inversion in a sphere has at least one negative eigenvalue. The concavity condition is quite simple, and is satisfied if there is a point on the boundary at which the Gaussian curvature is negative.

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1 Introduction

The Neumann-Poincaré (abbreviated by NP) operator is a boundary integral operator which appears naturally when solving classical boundary value problems using layer potentials. Its study goes back to C. Neumann [17] and Poincaré [19] as the name of the operator suggests. Lately interest in the spectral properties of NP operator is growing rapidly, which is due to their relation to plasmonics [3, 5, 15], and significant progress is being made among which are work on continuous spectrum in two dimensions [6, 9, 11, 18] and Weyl asymptotic of eigenvalues in three dimensions [10] to name only a few.

However, there are still several puzzling questions on the NP spectrum (spectrum of the NP operator). A question on existence of negative NP eigenvalues in three dimensions is one of them. Unlike two-dimensional NP spectrum which is symmetric with respect to 0 and hence there are the same number of negative eigenvalues as positive ones (see, for example [9, 12]), not so many surfaces (boundaries of three-dimensional domains) are known to have negative NP eigenvalues. In fact, NP eigenvalues on spheres are all positive, and it is only in [1] which was published in 1994 that the NP operator on a very thin oblate spheroid is shown to have a negative eigenvalue. We emphasize that the NP eigenvalues on ellipsoids can be found explicitly using Lamé functions for which we also refer to [2, 7, 14, 20]. As far as we are aware of, there is no surface other than ellipsoids known to have a negative NP eigenvalue.

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It is the purpose of this paper to present a simple geometric condition which guarantees existence of a negative NP eigenvalue. To present the condition, let \( \Omega \) be a bounded domain with the \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). We say \( \partial \Omega \) is concave with respect to \( p \in \Omega \) if there is a point \( x \in \partial \Omega \) such that
\[
(x - p) \cdot \nu_x < 0,
\]
where \( \nu_x \) denotes the outward unit normal vector to \( \partial \Omega \). We emphasize that if \( \partial \Omega \) is \( C^2 \), then this condition is fulfilled if there is a point on \( \partial \Omega \) where the Gaussian curvature is negative. In fact, if \( (x - p) \cdot \nu_x \geq 0 \) for all \( p \in \Omega \) and \( x \in \partial \Omega \), then \( \Omega \) is convex and hence the Gaussian curvatures on \( \partial \Omega \) are non-negative. We prove in this paper that if the concavity condition (1.1) holds for some \( p \in \Omega \), then the NP operator defined either on \( \partial \Omega \) or the surface of inversion with respect \( p \) has at least one negative eigenvalue (see Theorem 2.1 and Corollary 2.2). We emphasize that (1.1) is not a necessary condition for existence of a negative NP eigenvalue; oblate spheroids have negative NP eigenvalues as mentioned before.

This paper is organized as follows. We review in section 2 the definition of the NP operator and state main results of this paper. Section 3 is to prove the transformation formula for the NP operator under the inversion in a sphere. We use this formula to prove the main results.

2 The NP operator and statements of main results

Let \( \Gamma(x) \) be the fundamental solution to the Laplacian, i.e.,
\[
\Gamma(x) = \begin{cases} 
\frac{1}{2\pi} \ln |x|, & d = 2, \\
-\frac{1}{4\pi|x|}, & d = 3.
\end{cases}
\]

As before, let \( \Omega \) be a bounded domain with the \( C^{1,\alpha} \) boundary for some \( \alpha > 0 \). The single layer potential \( S_{\partial \Omega}[^{\#}] \phi \) of a density function \( \phi \) is defined by
\[
S_{\partial \Omega}[^{\#}] \phi(x) := \int_{\partial \Omega} \Gamma(x - y)\phi(y) d\sigma(y), \quad x \in \mathbb{R}^d.
\]

It is well known (see, for example, [4, 8]) that \( S_{\partial \Omega}[^{\#}] \phi \) satisfies the jump relation
\[
\frac{\partial}{\partial \nu} S_{\partial \Omega}[^{\#}] \phi \bigg|_{\pm}(x) = \left( \pm \frac{1}{2} I + K_{\partial \Omega}[^{*}] \right) \phi(x), \quad x \in \partial \Omega,
\]
where \( \frac{\partial}{\partial \nu} \) denotes the outward normal derivative, the subscripts \( \pm \) indicate the limit from outside and inside of \( \Omega \), respectively, and the operator \( K_{\partial \Omega}[^{*}] \) is defined by
\[
K_{\partial \Omega}[^{*}] \phi(x) := \int_{\partial \Omega} \nu_x \cdot \nabla_y \Gamma(x - y)\phi(y) d\sigma(y), \quad x \in \partial \Omega.
\]

The operator \( K_{\partial \Omega}[^{*}] \) is called the NP operator associated with the domain \( \Omega \) (or its boundary \( \partial \Omega \)). The operator \( S_{\partial \Omega}[^{\#}] \), as an operator on \( \partial \Omega \), maps \( H^{-1/2}(\partial \Omega) \) into \( H^{1/2}(\partial \Omega) \) continuously, and is invertible if \( d = 3 \). If \( d = 2 \), then there are domains where \( S_{\partial \Omega}[^{\#}] \) has
one-dimensional kernel, but by dilating the domain, it can be made to be invertible (see [22]). So, the bilinear form \( \langle \cdot, \cdot \rangle_{\partial \Omega} \), defined by

\[
\langle \varphi, \psi \rangle_{\partial \Omega} := -\langle \varphi, S_{\partial \Omega} \psi \rangle,
\]

for \( \varphi, \psi \in H^{-1/2}(\partial \Omega) \), is actually an inner product on \( H^{-1/2}(\partial \Omega) \) and it yields the norm equivalent to usual \( H^{-1/2} \)-norm (see, for example, [10]). Here, \( H^s \) denotes the usual \( L^2 \)-Sobolev space of order \( s \) and \( \langle \cdot, \cdot \rangle \) is the \( H^{-1/2}/H^{1/2} \) duality pairing. We denote the space \( H^{-1/2}(\partial \Omega) \) equipped with the inner product \( \langle \cdot, \cdot \rangle_{\partial \Omega} \) by \( H^* \).

It is proved in [12] that the NP operator \( K_{\partial \Omega}^* \) is self-adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{\partial \Omega} \). In fact, it is an immediate consequence of Plemelj’s symmetrization principle

\[
S_{\partial \Omega} K_{\partial \Omega}^* = K_{\partial \Omega} S_{\partial \Omega}.
\]

Here \( K_{\partial \Omega} \) is the adjoint of \( K_{\partial \Omega}^* \) with respect to the usual \( L^2 \)-inner product. Since \( K_{\partial \Omega}^* \) is compact on \( H^* \) if \( \partial \Omega \) is \( C^{1, \alpha} \), it has eigenvalues converging to 0.

For a fixed \( r > 0 \) and \( p \in \mathbb{R}^d \), let \( T_p : \mathbb{R}^d \setminus \{p\} \to \mathbb{R}^d \setminus \{p\}, d = 2, 3 \), be the inversion in a sphere, namely,

\[
T_p x := \frac{r^2}{|x - p|^2} (x - p) + p.
\]

For a given bounded domain \( \Omega \) in \( \mathbb{R}^d \), let \( \partial \Omega_p^* \) be the inversion of \( \partial \Omega \), i.e., \( \partial \Omega_p^* = T_p(\partial \Omega) \). If we invert \( \Omega \) in spheres of two different radii, then the resulting domains are dilations of each other. Since NP spectrum is invariant under dilation as one can see easily by a change of variables, we may fix the radius of the inversion sphere once for all.

The following is the main result of this paper.

**Theorem 2.1.** Let \( \Omega \) be a bounded domain whose boundary is \( C^{1, \alpha} \) smooth for some \( \alpha > 0 \). If the concavity condition \((1.1)\) holds for some \( p \in \Omega \), then either \( K_{\partial \Omega}^* \) or \( K_{\partial \Omega_p^*}^* \) has a negative eigenvalue.

We have the following corollary as an immediate consequence of Theorem 2.1.

**Corollary 2.2.** Suppose \( \partial \Omega \) is \( C^2 \) smooth. If there is a point on \( \partial \Omega \) where the Gaussian curvature is negative, then either \( K_{\partial \Omega}^* \) or \( K_{\partial \Omega_p^*}^* \) for some \( p \in \Omega \) has a negative eigenvalue.

### 3 Inversion in a sphere

Just for simplicity we now assume the center \( p \) of the inversion sphere is 0, and denote \( \partial \Omega_p^* \) and \( T_p \) by \( \partial \Omega^* \) and \( T \), respectively.

Let \( x^* := Tx \). Since

\[
\frac{\partial x_i}{\partial x_j^*} = \frac{r^2 \delta_{ij}}{|x^*|^2} - 2r^2 \frac{x_i^* x_j^*}{|x^*|^4},
\]

the Jacobian matrix of \( T^{-1} \) is given by

\[
J_{T^{-1}} = \frac{r^2}{|x^*|^2} \left( I - 2 \frac{x^* (x^*)^t}{|x^*|^2} \right) = \frac{|x|^2}{r^2} \left( I - 2 \frac{x x^t}{|x||x|} \right).
\]
Here \( t \) denotes transpose. So, \( T \) is conformal. The change of variable formulas for line and surface are respectively given as follows:

\[
ds(x^*) = \frac{r^2}{|x|^2} ds(x),
\]

\[
dS(x^*) = \frac{r^4}{|x|^4} dS(x).
\]

It is known (see [13]) that

\[
|x^* - y^*| = \frac{r^2}{|x||y|} |x - y|.
\]

So we have

\[
\Gamma(x^* - y^*) = \begin{cases} 
\frac{\Gamma(x - y) - \Gamma(x) + \Gamma(y) + \Gamma(r^2)}{|x||y|} & \text{if } d = 2, \\
\frac{|x||y|}{r^2} \Gamma(x - y) & \text{if } d = 3.
\end{cases}
\]

For a function \( \varphi \) defined on \( \partial \Omega \), define \( \varphi^* \) on \( \partial \Omega^* \) by

\[
\varphi^*(y^*) := \varphi(y) \frac{|y|^d}{r^d}.
\]

Then, we can easily see using (3.5) that the following relation between the single layer potentials on domains \( \partial \Omega \) and \( \partial \Omega^* \) holds (see also [13]):

\[
S_{\partial \Omega^*}[\varphi^*](x^*) = \begin{cases} 
S_{\partial \Omega}[\varphi](x) - S_{\partial \Omega}[\varphi](0) + \left( \int_{\partial \Omega} \varphi ds \right) \left( \Gamma(r^2) - \Gamma(x) \right) & \text{if } d = 2, \\
\frac{|x|}{r} S_{\partial \Omega}[\varphi](x) & \text{if } d = 3.
\end{cases}
\]

We note that the map \( \varphi \mapsto \varphi^* \) is a conformal map from \( \mathcal{H}^*(\partial \Omega) \) to \( \mathcal{H}^*(\partial \Omega^*) \), in three dimensions, that is,

\[
\langle \varphi^*, \psi^* \rangle_{\partial \Omega^*} = \langle \varphi, \psi \rangle_{\partial \Omega}.
\]

This relation is also true in two dimensions if \( \varphi \) and \( \psi \) are of mean zero. The relationship between outward unit normal vectors \( \nu_{x^*} \) on \( \partial \Omega^* \) and \( \nu_x \) on \( \partial \Omega \) are given as follows:

\[
\nu_{x^*} = (-1)^m \left( I - \frac{x}{|x|} \frac{x'}{|x|} \right) \nu_x,
\]

where \( m = 1 \) if 0 is an exterior point of \( \Omega \) and \( m = 0 \) if 0 is an interior point of \( \Omega \). We emphasize that 0 is the inversion center.

The NP operators \( K_{\partial \Omega}^* \) and \( K_{\partial \Omega^*}^* \) are related in the following way:

**Lemma 3.1.** Suppose that 0 is the center of the inversion sphere.

(i) If \( 0 \in \Omega \), then

\[
K_{\partial \Omega^*}^*[\varphi^*](x^*) = \begin{cases} 
-\frac{|x|^2}{r^2} K_{\partial \Omega}[\varphi](x) + \left( \int_{\partial \Omega} \varphi ds \right) \frac{x \cdot \nu_x}{2\pi r^2} & \text{if } d = 2, \\
-\frac{|x|^3}{r^3} K_{\partial \Omega}[\varphi](x) - \frac{|x|}{r^3} (x \cdot \nu_x) S_{\partial \Omega}[\varphi](x) & \text{if } d = 3.
\end{cases}
\]
Proof. Since the difference of proofs for (3.10) and (3.11) is just the sign of the normal vector, we only prove the first one.

If \( d = 2 \), we use (3.1), (3.2), (3.5), (3.6), and (3.9) to have

\[
K_{\partial \Omega^*}[\varphi^*](x^*) = \int_{\partial \Omega^*} \nu_x \cdot \nabla_x \Gamma(x^* - y^*) \varphi^*(y^*) \, ds(y^*)
\]

\[
= \int_{\partial \Omega} \left( I - 2 \frac{x}{|x|} \frac{x^t}{|x|} \right) \nu_x \cdot J_{I - 1} \nabla_x \left( \Gamma(x - y) - \Gamma(x) \right) \varphi(y) \, ds(y)
\]

\[
= -\frac{|x|^2}{r^2} \int_{\partial \Omega} \nu_x \cdot \left( \nabla_x \Gamma(x - y) - \nabla_x \Gamma(x) \right) \varphi(y) \, ds(y)
\]

\[
= -\frac{|x|^2}{r^2} K_{\partial \Omega^*}[\varphi](x) + \left( \int_{\partial \Omega} \varphi ds \right) \frac{x \cdot \nu_x}{2 \pi r^2}.
\]

The three-dimensional case can be proved similarly. In fact, we have

\[
K_{\partial \Omega^*}[\varphi^*](x^*) = \int_{\partial \Omega^*} \nu_x \cdot \nabla_x \Gamma(x^* - y^*) \varphi^*(y^*) \, dS(y^*)
\]

\[
= \int_{\partial \Omega} \left( I - 2 \frac{x}{|x|} \frac{x^t}{|x|} \right) \nu_x \cdot J_{I - 1} \nabla_x \left( \frac{|y|^2}{r^2} \Gamma(x - y) \right) \varphi(y) \, dS(y)
\]

\[
= -\frac{|x|^2}{r} \int_{\partial \Omega} \nu_x \cdot \left( \nabla_x \Gamma(x - y) + \frac{x}{r^2|x|} \Gamma(x - y) \right) \varphi(y) \, dS(y)
\]

\[
= -\frac{|x|^3}{r^3} K_{\partial \Omega^*}[\varphi](x) - \frac{|x| (x \cdot \nu_x)}{r^3} S_{\partial \Omega}[\varphi](x).
\]

This completes the proof. \( \square \)

If \( \varphi \) is an eigenvector of \( K_{\partial \Omega^*} \) with corresponding eigenvalue \( \lambda \neq 1/2 \), then \( \int_{\partial \Omega} \varphi ds = 0 \).

So (3.6) and Lemma 3.1 for \( d = 2 \) show that \( \varphi^* \) is an eigenvector of \( K_{\partial \Omega^*} \) and the corresponding eigenvalue is \( -\lambda \) if \( 0 \in \Omega \) and \( \lambda \) if \( 0 \in \Omega^* \). Since the spectrum \( \sigma(K_{\partial \Omega^*}) \) of the NP operator in two dimensions is symmetric with respect to 0, we infer that \( \sigma(K_{\partial \Omega^*}) = \sigma(K_{\partial \Omega}) \), namely, the spectrum is invariant under inversion. This fact is known \([21]\), but the above yields an alternative proof.

In three dimensions, we obtain the following identities.

**Proposition 3.2.** Suppose \( d = 3 \) and 0 is the center of the inversion sphere.

(i) If \( 0 \in \Omega \), then

\[
\langle K_{\partial \Omega^*}[\varphi^*], \varphi^* \rangle_{\partial \Omega^*} + \langle K_{\partial \Omega^*}[\varphi^*], \varphi \rangle_{\partial \Omega} = \int_{\partial \Omega} \frac{x \cdot \nu_x}{|x|^2} |S_{\partial \Omega}[\varphi](x)|^2 \, dS.
\]
If \(0 \in \overline{\Omega}^c\), then
\[
\langle K_{\partial \Omega}^* \varphi^*, \varphi^* \rangle_{\partial \Omega^*} - \langle K_{\partial \Omega} \varphi, \varphi \rangle_{\partial \Omega} = -\int_{\partial \Omega} \frac{x \cdot \nu_x}{|x|^2} |S_{\partial \Omega}[\varphi](x)|^2 dS. \tag{3.13}
\]

**Proof.** The identity follows from (3.7) and (3.10) immediately. In fact, we have
\[
\langle K_{\partial \Omega}^* \varphi^*, \varphi^* \rangle_{\partial \Omega^*} = \int_{\partial \Omega} \frac{|x|^3}{r^3} K_{\partial \Omega}^*[\varphi^*](x) \overline{S_{\partial \Omega}[\varphi^*](x)} S_{\partial \Omega}[\varphi](x) \frac{x^3}{|x|^3} dS(x) = -\langle K_{\partial \Omega}[\varphi], \varphi \rangle_{\partial \Omega} + \int_{\partial \Omega} \frac{x \cdot \nu_x}{|x|^2} |S_{\partial \Omega}[\varphi](x)|^2 dS(x),
\]
which proves (3.12). (3.13) can be proved similarly.

We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1** Suppose \(\Omega\) satisfies (1.1) at \(p \in \Omega\). Without loss of generality we assume that \(p = 0\). Then there is \(x_0 \in \partial \Omega\) such that \(x_0 \cdot \nu_{x_0} < 0\). Choose an open neighborhood \(U\) of \(x_0\) in \(\partial \Omega\) so that \(x \cdot \nu_x < 0\) for all \(x \in U\). Since \(S_{\partial \Omega} : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)\) is invertible in three dimensions (see [22]), we may choose \(\varphi \neq 0\) so that \(S_{\partial \Omega}[\varphi]\) is supported in \(U\). We then infer from (3.12) that
\[
\langle K_{\partial \Omega}^* \varphi^*, \varphi^* \rangle_{\partial \Omega^*} + \langle K_{\partial \Omega}^*[\varphi], \varphi \rangle_{\partial \Omega} < 0.
\]
Therefore, the numerical range of either \(K_{\partial \Omega}^*\) or \(K_{\partial \Omega}^*[\varphi]\) has a negative element. It implies that either \(K_{\partial \Omega}^*\) or \(K_{\partial \Omega}^*[\varphi]\) has at least one negative eigenvalue. This completes the proof.

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