Chiral Properties of Baryon Interpolating Fields

Keitaro Nagata *, Atsushi Hosaka,
Research Center for Nuclear Physics, Osaka University
Ibaraki 567-0047, Japan,
and
V. Dmitrašinović,
Institut J. Stefan, Dept. of Theoretical Physics F-1,
Jamova 39, 1000 Ljubljana, Slovenia;
Permanent address: Vinča Institute of Nuclear Sciences, lab 010
P.O.Box 522, 11001 Beograd, Serbia.

Abstract

We study the chiral transformation properties of all possible local (non-derivative) interpolating field operators for baryons consisting of three quarks with two flavors, assuming good isospin symmetry. We derive and use the relations/identities among the baryon operators with identical quantum numbers that follow from the combined colour, Dirac and isospin Fierz transformations. These relations reduce the number of independent baryon operators with any given spin and isospin. The Fierz identities also effectively restrict allowed baryon chiral multiplets. It turns out that the chiral multiplets of the baryons are equivalent to their Lorentz representation. For the two independent nucleon operators the only permissible chiral multiplet is the fundamental one \((1/2, 0) \oplus (0, 1/2)\). For the \(\Delta\), admissible Lorentz representations are \((1, 1/2) \oplus (1/2, 1)\) and \((0, 1/2) \oplus (1/2, 0)\). In the case of the \((1, 1/2) \oplus (1/2, 1)\) chiral multiplet the \(I(J) = 3/2(3/2)\) \(\Delta\) field has one \(I(J) = 1/2(3/2)\) chiral partner; otherwise it has none. We also consider the Abelian \((U_A(1))\) chiral transformation properties of fields and show that each baryon comes in two varieties: 1) with Abelian axial charge +3; and 2) with Abelian axial charge -1. In case of the nucleon these are the two Ioffe’s fields; in case of the \(\Delta\), the \((1, 1/2) \oplus (1/2, 1)\) multiplet has Abelian axial charge -1 and the \((3/2, 0) \oplus (0, 3/2)\) multiplet has Abelian axial charge +3.

1 Introduction

Chiral symmetry is one of the global symmetries of QCD and plays a key role in hadron physics. In the real world, chiral symmetry is spontaneously broken by the QCD ground state (“vacuum”), where a chiral condensate acquires a finite value. It is believed that the chiral symmetry

---

*Address after the 20th May 2007, Department of Physics, Chung-Yuan Christian University, Chung-Li 320, Taiwan
is restored at sufficiently high density/temperature. Properties of hadrons would be modified under such circumstances, and the chiral symmetry, as an almost exact symmetry, would become important in the classification of hadron spectra: in particular, one would observe a degeneracy of chiral partners.

We call chiral partners a set of hadrons that share a certain representation \( (I_R, I_L) \) of the chiral group \( SU(2)_R \times SU(2)_L \), where \( I_{R,L} \) label the representations of the right-, and left- isospin groups \( SU(2)_{R,L} \). In the Wigner-Weyl phase, all the hadrons that fall into one chiral multiplet, have their masses and axial-couplings determined by the chiral symmetry. Clear identification of chiral partners would greatly facilitate finding evidence for the chiral restoration, if there is any.

The best known example is the \( \sigma \) as the chiral partner of the \( \vec{\pi} \) meson, the quartet of fields \((\sigma, \vec{\pi})\) forming the chiral multiplet \((\frac{1}{2}, \frac{1}{2})\). Another example is the set of \((\vec{\rho}, \vec{a}_1)\) mesons, which are described by the interpolating fields \(\vec{\rho} \sim \bar{q} \gamma^\mu \vec{\tau} q, \vec{a}_1 \sim \bar{q} \gamma^\mu \gamma_5 \vec{\tau} q\) that form the chiral multiplet \((1, 0) \oplus (0, 1)\).

In contrast to the case of mesons, chiral partners of baryons are not well established. Even the mere existence of chiral partner(s) of the ground state baryon, the nucleon, is still an open question.

The main difficulty in finding the chiral partners of the nucleon is the lack of knowledge of chiral multiplets of baryons, because the observed baryon resonances do not manifestly belong to irreducible chiral representations in the spontaneously broken phase of chiral symmetry. Rather, a physical hadron is expected to be described by a linear combination of several chiral representations. At the level of the hadrons, one must assume/guess their chiral representations.

On the other hand, when one describes hadrons starting from the underlying quark substructure, the chiral multiplet (representation) for the hadrons can be determined uniquely, as shown in the case of the mesons, for instance \( \sigma, \vec{\pi} \) and \( \vec{\rho}, \vec{a}_1 \).

There is an infinite number of possibilities for baryon interpolating fields as, for instance one need not have only three quarks, but also five, seven or more. In addition, one may include (covariant) derivatives, or gluon fields into the baryon operators. Many lattice and QCD sum rule studies suggest, however, that the local, i.e. without derivatives, baryon operators consisting of three quarks successfully describe properties of the ground state baryons.

The construction of local operators for baryons was first studied by Ioffe \([1]\), Chung et al. \([2]\) and by Espriu et al. \([3]\). These authors showed that the Fierz transformations provide relations/identities among baryon operators with identical spin and isospin, and thus reduce the number of independent operators, e.g. they famously reduce the number of independent nucleon fields from five to two \([1, 3]\), and the number of independent \(\Delta\) fields from two to one (when one neglects fields belonging to the Lorentz group representation \(D(\frac{3}{2}, 0)\)).

The first systematic study of chiral multiplets of baryon operators was attempted by Cohen and Ji \([4]\). They did not explicitly discuss the Fierz identities among various fields, however, albeit they were aware of their existence. It is, in fact, necessary to consider the Fierz transformations, because they effectively restrict the allowed chiral multiplets of baryon operators.

Baryon interpolating fields are constructed in such a way that they belong to the same (ir-
reducible) representations of the Lorentz and of the isospin $SU(2)$ group. It is not a priori obvious, however, that they also belong to the irreducible representations of the chiral group $SU(2)_R \times SU(2)_L$. We shall show that the Fierz identities/relations among baryon operators lead precisely to the (non-)vanishing of those linear combinations that form various chiral multiplets/representations. This should not be surprising as the Fierz identities form an implementation of the Pauli principle, and different permutation symmetry classes form distinct multiplets of composite particles. Hence it is necessary to take into account the Fierz identities among baryon operators.

The standard isospin formalism greatly facilitates derivation of the Fierz identities and chiral transformations of baryon operators, due to the fact that both the quarks and the nucleons belong to the iso-doublet representation. The composite Fierz identities (i.e. in both the Dirac and isospin space) and the chiral transformations of baryons are straightforwardly derived using the iso-doublet representation. We also consider the Abelian ($U_A(1)$) chiral transformations of baryons and show that most baryons come in two varieties: 1) with axial charge +3; and 2) with axial charge -1.

This paper is organized as follows. In section 2 we first define all possible quark bi-linear fields and summarize their chiral transformations. With quark bi-linear fields, all possible baryon operators can be systematically defined. We classify the baryon operators according to the representations of the Lorentz and the isospin groups. Then we derive the Fierz identities among the baryon operators for each representation of the Lorentz and isospin group. In section 3 we derive the Abelian and non-Abelian chiral transformations of the baryon operators as functions of the quarks’ chiral transformation parameters, using the iso-doublet representation. All possible chiral multiplets for the baryon operators are enumerated by taking into account the Fierz identities. The final section is a summary and an outlook to the future.

## 2 Baryon Operators

Local interpolating operators for baryons consisting of three quarks can be generally described as

$$B(x) \sim \epsilon_{abc} (q_a^T(x)\Gamma_1 q_b(x)) \Gamma_2 q_c(x),$$

(1)

where $q(x) = (u(x), d(x))^T$ is an iso-doublet quark field at location $x$, the superscript $T$ represents the transpose and the indices $a$, $b$ and $c$ represent the color. Hence the antisymmetric tensor in color space $\epsilon_{abc}$ ensures the baryons’ being color singlets. From now on, we shall omit the color indices and space-time coordinates. $\Gamma_{1,2}$ describe the tensor product of Dirac and isospin matrices. With a suitable choice of $\Gamma_{1,2}$, the baryon operators are defined so that they form an irreducible representation of the Lorentz and isospin groups, as we shall show in this section.

Note that we employ the iso-doublet form for the quark field $q$, although the explicit expressions in terms of up and down quarks are usually employed in lattice QCD and QCD sum rule studies. We shall see that the iso-doublet formulation leads to a simple classification of
baryons into isospin multiplets and to a straightforward derivation of Fierz identities and chiral transformations of baryon operators.

We begin with bi-linears of two quarks in Eq. (1). It is convenient to introduce a “tilde-transposed” quark field ˜\(q\) as follows

\[ ˜q = q^T C \gamma_5 (i \tau_2), \]

where \(C = i \gamma_2 \gamma_0\) is the Dirac field charge conjugation operator, \(\tau_2\) is the second isospin Pauli matrix, whose elements form the antisymmetric tensor in isodoublet space. Taking into account the Pauli principle, there are five non-vanishing possibilities for \(\Gamma_1\) (otherwise there would have been twice as many):

\[\begin{align*}
D_1 &= ˜qq, \\
D_2 &= ˜q \gamma^5 q, \\
D_3^\mu &= ˜q \gamma^\mu q, \\
D_4^\mu_i &= ˜q \gamma^\mu \gamma^5 \tau^i q, \\
D_5^{\mu \nu i} &= ˜q \sigma^{\mu \nu} \tau^i q.
\end{align*}\]

These quark bi-linear fields, \(D_1, D_2, D_3^\mu, D_4^\mu_i\) and \(D_5^{\mu \nu i}\), are referred to as the scalar, pseudo-scalar, vector, axial-vector and tensor diquarks, by their Lorentz transformation properties.

Firstly we investigate their Abelian chiral \((U(1)_A)\) transformation properties, which is given by \(U(1)_A\) transformation of the quark,

\[\begin{align*}
q &\rightarrow \exp(i \gamma_5 a)q, \\
\tilde{q} &\rightarrow \tilde{q} \exp(i \gamma_5 a),
\end{align*}\]

where \(a\) is an infinitesimal parameter for \(U(1)_A\). The scalar and pseudo-scalar diquarks transform as

\[\begin{align*}
\delta_5 D_1 &= 2iaD_2, \\
\delta_5 D_2 &= 2iaD_1,
\end{align*}\]

the vector and axial-vector diquarks,

\[\delta_5 D_{3,4} = 0,\]

the tensor diquark,

\[\delta_5 D_5^{\mu \nu i} = 2iaD_6^{\mu \nu i},\]

where \(D_6^{\mu \nu i} = ˜q \sigma^{\mu \nu} \gamma_5 \tau^i q\) is a dual-tensor diquark. Note that baryon operators constructed from the dual-tensor diquark can be expressed as functions of the tensor diquark by using the identity \(\sigma^{\mu \nu} \gamma_5 = -\frac{i}{2} \epsilon^{\mu \nu \alpha \beta} \sigma_{\alpha \beta}\).

1Throughout the present paper, Latin indices \(i, j\) etc. run over the isospin space 1, 2, 3, and Greek indices \(\alpha, \beta\) etc. run over the Lorentz space 0, 1, 2, 3.
We consider the chiral \((SU(2)_A)\) transformation
\[
q \rightarrow \exp(i\gamma_5 \tau \cdot a)q, \\
\tilde{q} \rightarrow \tilde{q} \exp(-i\tau \cdot a\gamma_5),
\]
(10a)
(10b)
where \(\vec{a}\) is the triplet of \(SU(2)_A\) group parameters. It is straightforward to obtain the chiral transformations of the diquarks: for the scalar and pseudo-scalar diquarks,
\[
\delta \vec{a} D = 0, \ (D = D_1, D_2),
\]
for the vector and axial-vector diquarks,
\[
\delta \vec{a} D_3^\mu = 2i a^i D_4^{\mu i}, \\
\delta \vec{a} D_4^{\mu i} = 2i a^i D_3^\mu,
\]
(12)
(13)
for the tensor diquark,
\[
\delta \vec{a} D_5^{\mu \nu i} = -2\epsilon^{ijk} a^j D_6^{\mu \nu k}.
\]
(14)
The scalar and pseudo-scalar diquarks \(D_{1,2}\) are chiral scalars (invariants) \((0, 0)\). The vector and axial-vector diquarks \(D_3^\mu, D_4^{\mu i}\) together belong to the chiral multiplet \((1, \frac{1}{2})\); therefore they are chiral partners, similar to the \((\sigma, \vec{\pi})\) case. The tensor diquark transforms into the dual-tensor diquark, and they together form the chiral multiplet \((1, 0) \oplus (0, 1)\).

Now we proceed to the baryon operators: There we consider the Pauli principle in two steps. The first step is the Pauli principle applied to the first and second quarks, i.e. to the diquarks, as already performed and discussed above. Second, additional constraint comes from the permutation of the second and the third quark, which corresponds to the Fierz transformation. Note that the Fierz transformation connects only baryon operators belonging to the same Lorentz and isospin group multiplets. Therefore, we may classify the baryon operators according to their Lorentz and isospin representations following Chung et al. It has been known that such baryon operators may couple either to the even or to the odd parity states. In the following discussions all the baryon operators will be defined as having even parity. We note, however, that the baryon operators belonging to the same chiral multiplet may have either parity.

Firstly, we consider the simplest case \(D(\frac{1}{2}, 0)_{I=\frac{1}{2}}\), where \(D(\frac{1}{2}, 0)\) denotes the representation of the Lorentz group and \(I = \frac{1}{2}\) denotes the isospin. There are five differently-looking operators,
\[
N_1 = (\tilde{q}q)q, \\
N_2 = (\tilde{q}\gamma_5q)\gamma_5q, \\
N_3 = (\tilde{q}\gamma_\mu q)\gamma^\mu q, \\
N_4 = (\tilde{q}\gamma_\mu \gamma_5 \tau^i q)\gamma^\mu \gamma_5 \tau^i q, \\
N_5 = (\tilde{q}\sigma_{\mu \nu} \tau^i q)\sigma^{\mu \nu} \tau^i q,
\]
(15)
(16)
(17)
(18)
(19)
where the subscripts 1, 2, ⋯, 5 describe their diquark components of Eq. (3). As a consequence of the Fierz transformations, see Appendix A, we obtain three identities:

\begin{align}
3N_3 &= -N_4, \\
N_4 &= 3(N_2 - N_1), \\
N_5 &= 6(N_1 + N_2).
\end{align}

Therefore, only two among the five operators are independent under the Pauli principle, as noted by Ioffe [1] and by Espriu et al. [3].

Next we consider \( D(\frac{1}{2}, 0)_{I=\frac{3}{2}} \). Baryon operators with \( I = \frac{3}{2} \) must contain either the axial-vector or the tensor diquark, so there are only two possibilities,

\begin{align}
\Delta_4^i &= (\tilde{q}\gamma_\mu\gamma_5\tau^i q)\gamma_\nu\tau^j P_{3/2}^{ij} q, \\
\Delta_5^i &= (\tilde{q}\sigma_{\mu\nu}\tau^i q)\sigma^{\mu\nu} P_{3/2}^{ij} q.
\end{align}

Here \( P_{3/2}^{ij} \) is the isospin-projection operator for \( I = \frac{3}{2} \), which is defined, together with an isospin-projection operator \( P_{1/2}^{ij} \) for \( I = \frac{1}{2} \), as

\begin{equation}
P_{3/2}^{ij} = \delta^{ij} - \frac{1}{3} \tau^i \tau^j, \quad P_{1/2}^{ij} = \frac{1}{3} \gamma^5 \tau^i \tau^j.
\end{equation}

The \( I = \frac{3}{2} \) projection operator satisfies \( \tau^i P_{3/2}^{ij} = 0 \), which ensures \( \tau^i \Delta_4^{i,5} = 0 \). Using the appropriate Fierz transformations, see Appendix A, we find that both these operators vanish,

\begin{align}
\Delta_4^i &= 0, \\
\Delta_5^i &= 0.
\end{align}

The vanishing of these operators implies that the nucleon operators belong only to the fundamental chiral representation, as we shall show in the next section. Moreover, the vanishing of the \( I = \frac{3}{2}(J^P = \frac{1}{2}^+) \) spatially symmetric states is in accord with the non-relativistic result.

For \( D(1, \frac{1}{2})_{I=\frac{1}{2}} \), baryon operators may contain the vector and the axial-vector, or the tensor diquark. Hence there are three operators

\begin{align}
N_3^\mu &= (\tilde{q}\gamma_\mu q)\Gamma_{3/2}^{\mu} q, \\
N_4^\mu &= (\tilde{q}\gamma_\mu\gamma_5\tau^i q)\Gamma_{3/2}^{\mu} \gamma^5 \tau^i q, \\
N_5^\mu &= i(\tilde{q}\sigma_{\alpha\beta}\gamma_\mu q)\Gamma_{3/2}^{\mu} \gamma^\alpha \gamma^\beta \tau^i q,
\end{align}

where the imaginary identity \( i \) is introduced into the definition Eq. (30) in order to maintain the reality of all coefficients in the Fierz identities, see Appendix A. Similarly to the isospin projection operators, \( \Gamma_{3/2}^{\mu\nu} \) is the spin-projection operator for \( J = \frac{3}{2} \) states, which is defined, together with the \( J = \frac{1}{2} \) projection operator \( \Gamma_{1/2}^{\mu\nu} \), by

\begin{equation}
\Gamma_{3/2}^{\mu\nu} = g^{\mu\nu} - \frac{1}{4} \gamma^\mu \gamma^\nu, \quad \Gamma_{1/2}^{\mu\nu} = \frac{1}{4} \gamma^\mu \gamma^\nu.
\end{equation}
Owing to this projection operator, the $J = \frac{3}{2}$ baryon operators satisfy the Rarita-Schwinger condition $\gamma_\mu N^\mu_{3,4} = 0$. Here we comment that a vector-spinor (Rarita-Schwinger) field must satisfy another condition, $\partial_\mu N^\mu = 0$, that can be satisfied by employing another spin projection operator that contains derivatives \[3], which we do not use here, in compliance with the standard usage in lattice QCD and QCD sum rules. The Fierz transformation provides two relations

\[ N^\mu_3 = N^\mu_4, \quad (32) \]
\[ N^\mu_5 = -2N^\mu_3. \quad (33) \]

There is therefore one independent $J = \frac{3}{2} (I = \frac{1}{2})$ operator.

For $D(1, \frac{1}{2})_{I=\frac{3}{2}}$, there are two operators

\[ \Delta^{\mu i}_4 = (\bar{q}\gamma_\mu \gamma_5 \tau^i q)\Gamma^{\mu\nu\alpha\beta} P^{ij}_{3/2} q, \quad (34) \]
\[ \Delta^{\mu i}_5 = i(\bar{q}\sigma_{\alpha\beta} \tau^i q)\Gamma^{\mu\alpha\gamma\beta} P^{ij}_{3/2} q. \quad (35) \]

We obtain the Fierz identity

\[ \Delta^{\mu i}_4 = \Delta^{\mu i}_5. \quad (36) \]

Therefore, there is only one independent $J = \frac{3}{2} (I = \frac{3}{2})$, $D(1, \frac{1}{2})_{I=\frac{3}{2}}$ operator, in accord with Ioffe’s claim \[1\]. This is not the only possible $J = \frac{3}{2}, I = \frac{3}{2}$ operator, however.

There is another $J = \frac{3}{2}$ operator in the $D(\frac{3}{2}, 0)_{I=\frac{3}{2}}$ Lorentz representation, that is a Lorentz tensor

\[ N^{\mu\nu}_{5} = (\bar{q}\sigma_{\alpha\beta} \tau^i q)\Gamma^{\mu\nu\alpha\beta} \tau^i q, \quad (37) \]

where $\Gamma^{\mu\nu\alpha\beta}$ is another $J = \frac{3}{2}$ projection operator defined as

\[ \Gamma^{\mu\nu\alpha\beta} = \left( g^{\mu\alpha} g^{\nu\beta} - \frac{1}{2} g^{\nu\beta} \gamma^\mu \gamma^\alpha + \frac{1}{2} g^{\mu\beta} \gamma^\nu \gamma^\alpha + \frac{1}{6} \sigma^{\mu\nu\sigma} \sigma_{\alpha\beta} \right), \quad (38) \]

which ensures that $N^{\mu\nu}_{5} = -N^{\nu\mu}_{5}$, $\gamma_\mu N^{\mu}_{5} = 0$. Using the Fierz transformation we obtain $N^{\mu\nu}_{5} = 0$, i.e., this field vanishes due to the Pauli principle.

Finally in the $D(\frac{3}{2}, 0)_{I=\frac{3}{2}}$ Lorentz representation, there is one $\Delta$ operator

\[ \Delta^{\mu i}_5 = (\bar{q}\sigma_{\alpha\beta} \tau^i q)\Gamma^{\mu\nu\alpha\beta} P^{ij}_{3/2} q. \quad (39) \]

The Fierz transformation generates only the trivial relation $\Delta^{\mu i}_5 = \Delta^{\mu i}_5$, so this operator survives the total anti-symmetrization and provides a rarely considered alternative for the $\Delta(1232)$ interpolating field.

We have defined various local operators for three-quark baryons by classifying them according to their Lorentz and isospin group representations. We have explicitly shown that the Fierz transformation connects the baryon operators with identical Lorentz and isospin properties. In the next section, we shall show that the Fierz identities restrict the Pauli-allowed baryons and hence also restrict their chiral multiplets. In this sense, the chiral $SU(2)_R \times SU(2)_L$ symmetry plays a similar role in the relativistic treatment of baryons, to the one of $SU_{FS}(6)$ symmetry in the nonrelativistic treatment. Presumably, one can apply this form of the Pauli principle also to other, non-local three-quark fields that correspond to the orbitally excited states, but we leave that problem for another occasion.
3 Chiral Transformations

In this section, we investigate the chiral transformations of three-quark baryon operators. The chiral mixing of baryon operators is caused by their diquark components, so it is convenient to classify the baryon operators according to their diquark chiral multiplets:

\[ D_1 \]
\[ D_2 \] \( (0, 0) \),

\[ D_3 \]
\[ D_{4 \mu} \] and \( D_{5 \mu \nu} \) \( (\frac{1}{2}, \frac{1}{2}) \) and \( D_{5 \mu \nu} \) \( (1, 0) + (0, 1) \).

First, we consider the baryon operators \( N_1 \) and \( N_2 \) that are constructed from the scalar and pseudo-scalar diquarks. As shown in Eq. (11), the scalar and pseudo-scalar diquarks are chiral scalars \((0, 0)\), therefore \( N_1 \) and \( N_2 \) belong to the fundamental representation, whose transformations are simply,

\[ \delta_5 a \overset{5}{N} = i \gamma_5 \tau \cdot a N, \quad (N = N_1, N_2). \]  

(40)

Under the Abelian chiral transformation the rule is also linear, but more complicated as it mixes the two nucleon fields:

\[ \delta_5 N_1 = ia \gamma_5 (N_1 + 2N_2) \]  

(41)

\[ \delta_5 N_2 = ia \gamma_5 (N_2 + 2N_1). \]  

(42)

In other words these two nucleon fields appear to form a two-dimensional representation of the Abelian chiral symmetry \( U_L(1) \times U_R(1) \), or an \( U_A(1) \) doublet. Of course, all irreducible representations of any Abelian Lie group are one-dimensional, so the “chiral doublet” two-dimensional representation of \( U_A(1) \) furnished by the fields \( N_{1,2} \) and defined by Eqs. (41) and (42) must be a reducible one. We may perform the reduction by taking the symmetric and antisymmetric linear combinations of two nucleon fields \( N_{1,2} \):

\[ N_n = (N_1 + N_2) \]  

(43)

\[ N_m = (N_1 - N_2). \]  

(44)

Then their Abelian chiral transformation properties are

\[ \delta_5 N_n = 3ia \gamma_5 N_n \]  

(45)

\[ \delta_5 N_m = -ia \gamma_5 N_m. \]  

(46)

Note the factor 3 in front of the r.h.s. of Eq. (45), i.e., it is the “triply naive” Abelian axial baryon charge transformation law, as it should be for an object consisting of three quarks, and the negative sign in front of the r.h.s. of Eq. (46), as it should for an Abelian “mirror” nucleon.

With the vector and axial-vector diquarks, we can construct six kinds of baryon operators, four of them are \( I = \frac{1}{2} \): \( N_3 \), \( N_{3 \mu} \), \( N_4 \) and \( N_{4 \mu} \), and two of them \( I = \frac{3}{2} \): \( \Delta_4 \) and \( \Delta_{4 \mu} \). The \( SU(2)_A \) transformation defined by Eq. (10a) can change the isospin of the baryon operators, but it cannot change its spin \( J \), as familiar from the case of mesons e.g. \((\sigma, \vec{\pi})\). Therefore, we may

\[ ^2 \text{We refer to such fields that transform with the positive sign on the r.h.s. as “naive” or covariant, and to those that transform with the negative sign as “mirror” or contra-variant.} \]
divide chiral transformations of these fields into two sets: one is the set of $J = \frac{1}{2} (N_3, N_4, \Delta^i_4)$ fields and the other is the set of $J = \frac{3}{2} (N^\mu_3, N^\mu_4, \Delta^{\mu i}_4)$.

The $SU(2)_A$ transformations of the first set ($J = \frac{1}{2}$) are given by

$$
\delta^g_5 N_3 = -i a \cdot \tau_5 N_3 - \frac{2}{3} i a \cdot \tau_5 N_4 - 2i\gamma_5 a \cdot \Delta_4,
$$

$$
\delta^g_5 N_4 = -2i a \cdot \tau_5 N_3 + \frac{1}{3} i a \cdot \tau_5 N_4 - 2i\gamma_5 a \cdot \Delta_4,
$$

$$
\delta^g_5 \Delta^i_4 = -2i\gamma_5 a^j P^{ij}_{3/2} N_3 - \frac{2}{3} i\gamma_5 a^j P^{ij}_{3/2} N_4 + \frac{2}{3} i\tau^i\gamma_5 a \cdot \Delta_4 - i a \cdot \tau_5 \Delta^i_4.
$$

As mentioned earlier, these baryon operators transform only one $J = \frac{1}{2}$ field into another, i.e., the $J = \frac{1}{2}$ operators close this algebra and there is no mixing of the $J = \frac{1}{2}$ and $J = \frac{3}{2}$ operators.

We find that Eqs. (47) can be reduced to irreducible components by taking the antisymmetric linear combination of the two nucleon fields:

$$
\delta^g_5 (N_3 - N_4) = i a \cdot \tau_5 (N_3 - N_4),
$$

$$
\delta^g_5 (3N_3 + N_4) = -\frac{5}{3} i a \cdot \tau_5 (3N_3 + N_4) - 8i\gamma_5 a \cdot \Delta_A,
$$

$$
\delta^g_5 \Delta^i_4 = -\frac{2}{3} i\gamma_5 a^j P^{ij}_{3/2} (3N_3 + N_4) + \frac{2}{3} i\tau^i\gamma_5 a \cdot \Delta_A - i a \cdot \tau_5 \Delta^i_4,
$$

i.e., $(N_3 - N_4)$ forms the fundamental representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, whereas $(3N_3 + N_4)$ together with $\Delta^i_4$ form the higher dimensional representation $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$: $3N_3 + N_4$ and $\Delta^i_4$ appear to be chiral partners. The Fierz identities Eqs. (20), (21), and (26), however, forbid exactly the appearance of the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ baryon fields/chiral partners, as the said identities make precisely these (linear combinations of) fields vanish identically.

Hence the only possible chiral multiplet containing the nucleon field is the fundamental one, i.e.,

$$
\delta^g_5 N = i a \cdot \tau_5 N, \quad (N = N_3, N_4),
$$

while the Abelian chiral transformation is given by

$$
\delta_5 N = -i\gamma_5 a N, \quad (N = N_3, N_4),
$$

hence both of these nucleon fields have the single-mirror property under the Abelian chiral transformation.

Similarly to the first set, at first sight the $J = \frac{3}{2}$ fields may be in both the $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$ and in the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ multiplet. In this case, however, the Fierz identities Eq. (32) forbid the fundamental representation, leaving the $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ as the only possible chiral multiplet for these $J = \frac{3}{2}$ fields, with the transformation law

$$
\delta^g_5 N^\mu = \frac{5}{3} i a \cdot \tau_5 N^\mu + 2i\gamma_5 a \cdot \Delta^\mu_4,
$$

$$
\delta^g_5 \Delta^{\mu i}_4 = \frac{8}{3} i\gamma_5 a^j P^{ij}_{3/2} N^\mu - \frac{2}{3} i\tau^i\gamma_5 a \cdot \Delta^{\mu i}_4 + i a \cdot \tau_5 \Delta^{\mu i}_4,
$$

hence the other is the set of $J = \frac{3}{2} (N^\mu_3, N^\mu_4, \Delta^{\mu i}_4)$.
where \( N^\mu = N_3^\mu, N_4^\mu \), and \( \Delta_4^{\mu i} \) are chiral partners belonging to the chiral multiplet \((1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)\). Their Abelian chiral transformations are given by

\[
\delta_5 N^\mu = i\gamma_5 a N^\mu, \quad (N^\mu = N_3^\mu, N_4^\mu),
\]

hence all of these (linear combinations of) fields belong to the single-naive Abelian representation.

Next we proceed to baryons constructed from the tensor diquark. As mentioned earlier, the chiral transformation does not change the spin. Due to the Fierz identity Eq. (27), \( \Delta_5^{\mu i} \) the chiral partner of the \( N_5 \) vanishes identically and \( N_5 \) can only belong to the fundamental representation; thus we obtain

\[
\delta_5^{\vec{a}} N_5 = i\vec{\tau} \cdot \vec{a} \gamma_5 N_5.
\]

For the Abelian chiral transformation

\[
\delta_5 N_5 = 3i\gamma_5 a N_5,
\]

they transform as triple-naive.

Thus we have considered all possible nucleon operators: \( N_1, N_2, \cdots, N_5 \). We found two sets of two non-vanishing operators: \((N_1, N_2)\) and \((N_4, N_5)\); each pair belonging to the fundamental representation \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\). Hence the nucleon operators cannot have any chiral partners. Moreover, the Fierz identities Eqs. (21) and (22), impose the equivalence of the two pairs. Thus, they restrict the allowed fields to only two, e.g. \((N_1 \pm N_2) \equiv (N_4, N_5)\). The Abelian chiral transformation properties distinguish between the two remaining nucleon operators \((N_1 \pm N_2)\).

Cohen and Ji [4] have also pointed out the fact that the \( I(J) = \frac{3}{2} \) nucleon operators belong only to the fundamental representation. It is important to note that it is the Pauli principle that forbids the higher dimensional chiral representation for the nucleon operators.

For the spin 3/2 fields in the Lorentz representation \( D((\frac{1}{2}, \frac{3}{2}) I = \frac{3}{2}, \frac{1}{2}) \), we obtain

\[
\delta_5^\vec{a} N_5^\mu = \frac{5}{3} i\vec{\tau} \cdot \vec{a} \gamma_5 N_5^\mu - 4i\gamma_5 \vec{a} \cdot \Delta_5^\mu,
\]

\[
\delta_5^{\vec{a} i\gamma_5} \Delta_5^{\mu i} = \frac{4}{3} i\gamma_5 a^j P_{ij/2} N_5^\mu - \frac{2}{3} i\tau^j \gamma_5 a \cdot \Delta_5^\mu + i\vec{a} \cdot \tau_5 \Delta_5^{\mu i}.
\]

Therefore these fields also belong to the \((1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)\) representation. The Fierz identities Eqs. (33) and (36), however, ensure that the two sets of \((1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)\) baryon operators, \((N_4^\mu, \Delta_4^{\mu i})\) and \((N_5^\mu, \Delta_5^{\mu i})\), are actually equivalent. Their Abelian chiral transformations are given as

\[
\delta_5 N = i\gamma_5 a N, (N = N_4^\mu, \Delta_5^{\mu i}),
\]

hence they have the single-naive Abelian properties. So, it turns out that the only allowed chiral multiplet for the \( D(1, \frac{1}{2}) \) baryons is the \((1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)\) one, in which case the \( I = \frac{3}{2} \) and \( I = \frac{1}{2} \) baryons are chiral partners. It has been shown in QCD sum rules [6, 7] that the \( \Delta_5^{\mu} \) operator gives a good description of the \( \Delta(1232) \), but its chiral partners, the baryons with \( I(J) = \frac{3}{2}(\frac{3}{2}), \frac{3}{2}(\frac{3}{2}) \)
were not discussed there. On the other hand, lattice QCD studies \[^{8,9}\] show that the $\Delta^{\mu}_4$ and $N^{\mu}_4$ operator describe the $\Delta(1232)$ and the $N(1520)$ resonance.

There is another candidate \[^{2}\] for the $\Delta(1232)$ interpolating field operator: the $D(\frac{3}{2}, 0)_{I=\frac{3}{2}}$ representation Eq. \[(53)\]; here we obtain

$$
\delta_5^{\mu\nu}\Delta_5^{\mu\nu} = i\tau \cdot a_\gamma_5\Delta_5^{\mu\nu} - 2\epsilon_{ijk}a^j\gamma_5\Delta_5^{\mu\nu} + \frac{2}{3}i\tau\gamma_5 a \cdot \Delta_5^{\mu\nu}.
$$

(61)

Therefore the $\Delta_5^{\mu\nu}$ belongs to the $(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$ chiral multiplet, and it has no chiral partner. The Abelian chiral transformation is given by

$$
\delta_5 \Delta_5^{\mu\nu} = 3i\gamma_5 a\Delta_5^{\mu\nu},
$$

(62)

hence it is triple-naive Abelian field.

We have studied the chiral transformations of the spin-$\frac{1}{2}$ and $\frac{3}{2}$ baryon operators. Due to our restriction to two light flavors, it turns out that the allowed chiral multiplets of three-quark baryon operators have the same labels/dimensions as the Lorentz representations that they belong to: $D(\frac{1}{2}, 0)$ baryons belong to the $(\frac{1}{2}, 0)$ chiral multiplet, the $D(1, \frac{1}{2})$ baryons belong to the $(1, \frac{1}{2})$ and the $D(\frac{3}{2}, 0)$ baryon operators belong to the $(\frac{3}{2}, 0)$. On the other hand, their Abelian chiral transformation properties depend on other properties of the operators, namely the number of right and left components of the quark field, as explained below.

The $D(1, \frac{1}{2})_{I=\frac{1}{2}, 3}=\frac{1}{2}$ chiral partners of the $\Delta$ are of special interest. Equations \[(53)\] and \[(54)\] determine, to lowest order in perturbation theory, their (bare) axial-coupling constant matrix. In order to see this explicitly, we consider their flavor components, which are obtained as

$$
N^{\mu}_4 = \begin{pmatrix}
\sqrt{6}\phi_{\frac{1}{2}, \frac{1}{2}}

\sqrt{6}\phi_{\frac{1}{2}, -\frac{1}{2}}
\end{pmatrix},
$$

(63)

$$
\Delta^{\mu+}_4 = \begin{pmatrix}
-\sqrt{2}\phi_{\frac{3}{2}, -\frac{1}{2}}

-\sqrt{2}\phi_{\frac{3}{2}, -\frac{3}{2}}
\end{pmatrix}, \quad \Delta^{\mu-}_4 = \begin{pmatrix}
\sqrt{2}\phi_{\frac{3}{2}, \frac{1}{2}}

\sqrt{2}\phi_{\frac{3}{2}, \frac{3}{2}}
\end{pmatrix},
$$

(64)

where $\Delta^{\mu\pm}_4 = (\bar{q}\gamma_\mu\gamma_5q)\Gamma^{\mu\nu}_{3/2}\gamma_5 P^{\pm}_{3/2}q$, and the third component is eliminated by the subsidiary condition $\tau^3\Delta^{\mu3}_4 = -\tau^1\Delta^{\mu1}_4 - \tau^2\Delta^{\mu2}_4$. $\phi_{I, I_z}$ are the properly normalized flavor wave-functions with the given isospin. In terms of the axial rotation about the third axis $\bar{a} = (0, 0, a_3)$, we obtain the axial-coupling constant matrix from Eqs. \[(55)\] and \[(54)\], e.g. for $\phi_{\frac{1}{2}, \frac{1}{2}}$, $\phi_{\frac{1}{2}, -\frac{1}{2}}$ and $\phi_{\frac{3}{2}, \frac{3}{2}}$:

$$
\delta_5^{3\phi_{\frac{3}{2}, \frac{3}{2}}} = i\gamma_5 a_3 \phi_{\frac{3}{2}, \frac{3}{2}},
$$

(65)

$$
\delta_5^{3\phi_{\frac{3}{2}, \frac{1}{2}}} = i\gamma_5 a_3 \phi_{\frac{3}{2}, \frac{1}{2}},
$$

(66)

with the familiar (“$SU_{FS}(6)$” value $\frac{2}{3}$ for its “nucleon” component. Manifestly, this value was not obtained using the $SU_{FS}(6)$ symmetry, but from the chiral $SU(2)_R \times SU(2)_L$ symmetry.
4 Summary and Conclusions

We have investigated the chiral multiplets consisting of local three-quark baryon operators, where we took into account the Pauli principle by way of the Fierz transformation. All spin 1/2 and 3/2 baryon operators were classified according to their Lorentz and isospin group representations, where spin and isospin projection operators were employed.

We derived all non-trivial relations between various baryon operators due to the Fierz transformations, and thus found the independent baryon fields. We showed that the Fierz transformation connects only the baryon operators with identical group-theoretical properties, i.e., belonging to the same chiral multiplet. Then we studied chiral transformations of the baryon operators, where the Fierz identities restrict the permissible baryons’ chiral multiplets.

We also found that baryons with different isospins may mix under the chiral transformations, i.e., they may belong to the same chiral multiplet, whereas baryons with different spins can not. The parity does not play an apparent role in the chiral properties of the baryon operators. Thus, naively, the nucleon could have a chiral partner - a baryon operator with \( I(J^P) = \frac{3}{2}(\frac{1}{2} \pm) \). This chiral partner vanishes identically, however, due to the Pauli principle/Fierz identities.

There are two possible choices for the baryon operator with \( J = \frac{3}{2} \); one is \( D(\frac{1}{2}, \frac{3}{2}) \) and the other is \( D(\frac{3}{2}, 0) \). In the former choice, the baryon isodoublet with \( I(J) = \frac{1}{2}(\frac{3}{2}, \frac{3}{2}) \) are the \( \Delta \)'s chiral partners. In the latter choice, the \( \Delta \) does not have a chiral partner. The two \( \Delta \) fields also differ in their Abelian chiral properties: the \((1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)\) multiplet has Abelian axial charge -1 and the \((\frac{3}{2}, 0) \oplus (0, \frac{3}{2})\) multiplet has Abelian axial charge +3. The Abelian \((U_A(1))\) chiral properties also resolve the nucleon field ambiguity (Ioffe [1]): one nucleon has Abelian axial charge +3; another has Abelian axial charge -1. Physical consequences of this \( U_A(1) \) ambiguity have to some extent already been explored in Ref. [10] in the case of nucleons, whereas the question remains completely unexplored in the case of the \( \Delta \).

Some partial results have already been pointed out by Ioffe [1], by Chung, Dosch, Kremmer and Schall [2], and by Cohen and Ji [4], but they were limited in their scope. For example, Ioffe [1] and Chung, Dosch, Kremmer and Schall [2], were concerned only with the Fierz relations and the number of independent fields, not with their chiral properties. Moreover, Ioffe did not mention the \( D(\frac{3}{2}, 0)_{I=\frac{3}{2}} \) spin 3/2 fields - he only considered the Lorentz representation \( D(\frac{1}{2}, 1)_{I=\frac{3}{2}} \) spin 3/2 fields; only Chung et al. [2] considered the \( D(\frac{3}{2}, 0)_{I=\frac{3}{2}} \) spin 3/2 fields. Cohen and Ji [4], on the other hand, were concerned primarily with the chiral properties. We have unified these two questions and showed that only a complete analysis leads to a meaningful answer to both. Our results answer a specific question raised recently in the study of baryon chiral multiplets, [11], and thus open the door to further explicit studies in this field.

We have employed the standard isospin formalism instead of the explicit expressions in terms of different flavored quarks in the flavor components of the baryon fields that are commonplace in this line of work. By using the isospin formalism, we have been able to derive all Fierz identities and chiral transformations of the baryons systematically. The extension to SU(3) is not as straightforward as one might have imagined, however, so we leave it for another occasion.
Acknowledgments

We wish to thank Dr. N. Ishii and Prof. W. Bentz, for valuable conversations about the Fierz transformation of triquark fields and (in)dependence of the nucleon interpolating fields. We also thank Prof. D. Jido, for the fruitful discussions on the Fierz transformation in the $L-R$ representation. One of us (V.D.) wishes to thank Prof. S. Fajfer for hospitality at the Institute Jožef Stefan, where this work was started and to a large extent completed, and to Prof. H. Toki for hospitality at RCNP, where it was finalized.

A Fierz Transformation

We summarize the detailed results of the Fierz transformation of baryons. After the Fierz transformation of the isospin, Dirac and color, the Fierz transformed field $\mathcal{F}[N]$ satisfy the relation $N = -\mathcal{F}[N]$, namely the Pauli principle.

For $D(\frac{1}{2}, 0)$ and $I = \frac{1}{2}$,

$$\mathcal{F} \left[ \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{pmatrix} \right] = -\frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & -1 & \frac{1}{2} \\ 1 & 1 & -1 & 1 & \frac{1}{2} \\ 4 & -4 & -2 & -2 & 0 \\ -12 & 12 & -6 & 2 & 0 \\ 36 & 36 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{pmatrix}.$$ \hspace{1cm} (67)

For the $D(\frac{1}{2}, 0)$ and $I = \frac{3}{2}$ operators,

$$\mathcal{F}[\Delta^i_4] = \frac{1}{2} \Delta^i_4,$$ \hspace{1cm} (68)

$$\mathcal{F}[\Delta^i_5] = -\frac{1}{2} \Delta^i_5.$$ \hspace{1cm} (69)

For the $D(\frac{1}{2}, 1)$ and $I = \frac{1}{2}$ operators,

$$\mathcal{F} \left[ \begin{pmatrix} N_{3\mu} \\ N_{4\mu} \\ N_{5\mu} \end{pmatrix} \right] = -\frac{1}{8} \begin{pmatrix} 2 & 2 & -2 \\ 6 & -2 & -2 \\ -12 & -4 & 0 \end{pmatrix} \begin{pmatrix} N_{3\mu} \\ N_{4\mu} \\ N_{5\mu} \end{pmatrix}.$$ \hspace{1cm} (70)

For the $D(\frac{1}{2}, 1)$ and $I = \frac{3}{2}$ operators,

$$\mathcal{F} \left[ \begin{pmatrix} \Delta^{i\mu}_4 \\ \Delta^{i\mu}_5 \end{pmatrix} \right] = -\frac{1}{4} \begin{pmatrix} 2 & 2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} \Delta^{i\mu}_4 \\ \Delta^{i\mu}_5 \end{pmatrix}.$$ \hspace{1cm} (71)

For the $D(\frac{3}{2}, 0)$ operators,

$$\mathcal{F} [N^{\mu\nu}_5] = \frac{1}{2} N^{\mu\nu}_5,$$ \hspace{1cm} (72)

$$\mathcal{F} [\Delta^{\mu\nu}_5] = -\Delta^{\mu\nu}_5.$$ \hspace{1cm} (73)
References

[1] B.L. Ioffe, Nucl. Phys. B 188, 317 (1981); ibid. B 191, 591 (1981) (E); Z. Phys. C 18, 67 (1983).
[2] Y. Chung, H. G. Dosch, M. Kremer and D. Schall, Nucl. Phys. B 197 (1982) 55.
[3] D. Espriu, P. Pascual and R. Tarrach, Nucl. Phys. B 214 (1983) 285.
[4] T.D. Cohen and X. Ji, Phys. Rev. D 55, 6870 (1997).
[5] M. Benmerrouche, R. M. Davidson and N. C. Mukhopadhyay, Phys. Rev. C 39, 2339 (1989).
[6] F. X. Lee and X. Y. Liu, Phys. Rev. D 66, 014014 (2002) arXiv:nucl-th/0203051.
[7] F. X. Lee, arXiv:nucl-th/0605065.
[8] J. M. Zanotti, D. B. Leinweber, A. G. Williams, J. B. Zhang, W. Melnitchouk and S. Choe [CSSM Lattice collaboration], Phys. Rev. D 68, 054506 (2003) arXiv:hep-lat/0304001.
[9] D. B. Leinweber, W. Melnitchouk, D. G. Richards, A. G. Williams and J. M. Zanotti, Lect. Notes Phys. 663, 71 (2005) arXiv:nucl-th/0406032.
[10] K. Nagata, A. Hosaka and V. Dmitrašinović, “$U_A(1)$ symmetry restoration does not imply nucleon parity doubling”, in preparation.
[11] S. R. Beane and M. J. Savage, Phys. Lett. B 556, 142 (2003)