Quasitoric stably normally split manifolds

Grigory Solomadin

Abstract

A smooth stably complex manifold is called a totally tangentially/normally split manifold (TTS/TNS-manifold, for short, resp.) if the respective complex tangential/normal vector bundle is stably isomorphic to a Whitney sum of complex linear bundles, resp. In this paper we construct manifolds $M$ s.t. any complex vector bundle over $M$ is stably equivalent to a Whitney sum of complex linear bundles. A quasitoric manifold shares this property iff it is a TNS-manifold. We establish a new criterion of the TNS-property for a quasitoric manifold $M$ via non-semidefiniteness of certain higher-degree forms in the respective cohomology ring of $M$. In the family of quasitoric manifolds, this generalises the theorem of J. Lannes about the signature of a simply connected stably complex TNS 4-manifold. We apply our criterion to show the flag property of the moment polytope for a nonsingular toric projective TNS-manifold of complex dimension 3.

1 Introduction

TTS- and TNS-manifolds appeared in the works of Conner and Floyd [9], Arthan and Bullet [1], Ochanine and Schwartz [13], Ray [17] and [19] related to a representation of a given complex cobordism class with a manifold from a prescribed family. A naturally arising problem here is to study TTS/TNS-manifolds in well-known families of manifolds, for example, quasitoric manifolds (see [8], [10]). Remind that any quasitoric manifold can be endowed with the natural stably complex structure. The stably complex manifold obtained in this way is a TTS-manifold. (See [8, Section 7.3] and formula (1) below.) In this way, the problem boils out to the study of quasitoric TNS-manifolds. (In other words, we want to describe TNS-manifolds in a special family of TTS-manifolds.)

Let us remind some facts about quasitoric TNS-manifolds (see [19]). The complex projective space $\mathbb{C}P^n$ is a TNS-manifold iff $n < 2$. Any invariant submanifold $Z \subset M$ of a quasitoric TNS-manifold $M^{2n}$ is again a TNS-manifold. These two facts together imply that the moment polytope of a quasitoric TNS-manifold has no triangular faces. (A quasitoric manifold over a polytope without triangular faces may be non-TNS, generally speaking, see Example 3.6.) If $M$ is a smooth projective toric TNS-manifold of complex dimension $n$ and $Z^{2(n−2)} \subset M^{2n}$ is any smooth closed subvariety of complex codimension 2, then the respective blow-up $Bl_Z M$ of the variety $M$ along $Z$ is a TNS-variety. Any Bott tower is a TNS-manifold. Successive blow-ups of any invariant submanifolds of complex codimension 2 starting from any Bott tower give different toric TNS-manifolds. Any polytope from famous families of simple polytopes such as flag nestohedra, graph-cubohedra and graph-associahedra admits a realisation as a 2-truncated cube with the canonical Delzant structure (see [6], [7]). The respective toric varieties are obtained from any Bott tower (of the necessary dimension) by consequent blow-ups of invariant subvarieties of complex dimension 2. Consequently, any combinatorial flag nestohedron, graph-cubohedron or graph-associahedron admits a toric TNS-variety over it. Another operations in the family of stably complex closed TNS-manifolds are cartesian product and connected sum of manifolds. In toric topology there are well-defined equivariant and box sum of any two quasitoric manifolds of the same dimension. We remark that equivariant and box sum of quasitoric TNS-manifolds are also TNS-manifolds.

The complete description of quasitoric TNS-manifolds of dimension 4 (toric surfaces, in particular) is given by the following Theorem in terms of the signature. (Remind that any quasitoric manifold is simply connected.)

*The work was done at the Steklov Institute of Mathematics RAS and supported by the Russian Science Foundation, grant 14-11-00414. E-mail: grigory.solomadin@gmail.com
Theorem 1.1 ([13], J. Lannes). Let $M^4$ be a stably complex simply connected closed manifold.

a) If the intersection form of two-dimensional cycles of $M^4$ is non-definite then the complex normal bundle of $M^4$ is stably equivalent to the sum $\xi_1 \oplus \xi_2$ for some complex linear bundles $\xi_1, \xi_2 \to M^4$.

b) If the intersection form of $M^4$ is definite, then $M^4$ is not a TNS-manifold.

One can use Theorem 1.1 to show that a smooth projective toric surface with the moment polygone $P^2 \subset \mathbb{R}^2$ is a TNS-manifold iff $P^2 \neq \Delta^2$, where $\Delta^2$ is the moment polygone of $\mathbb{CP}^2$, i.e. the Delzant triangle. (See Proposition 3.1.) However, the TNS-property for quasitoric manifolds is shown to depend not only on the combinatorial type of the moment polytope but also on the respective characteristic matrix in all dimensions greater than 2. (See Subsection 3.1.) Due to that reason, in this paper we focus on the study of toric TNS-manifolds.

For any element $a \in H^{2(n-k)}(M^{2n}; K)$, $a \neq 0$, $0 < k \leq n$, of the cohomology ring of a quasitoric manifold $M^{2n}$ we define a homogeneous (non-trivial) $k$-form $Q_a : H^2(M^{2n}; K) \to K$, by the formula $Q_a(x) = \langle a \cdot x^k, [M^{2n}] \rangle$ where $K$ is $\mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. This definition is $K$-linear w.r.t. $a$.

Definition 1.2. The form $Q_a$ is called admissible, if it is not semi-definite. In other words, the admissible form $Q_a$ has to take values of different signs.

The main result of this paper is as follows.

Theorem 1.3. Let $M^{2n}$ be a quasitoric manifold of dimension $2n$. Then $M^{2n}$ is a TNS-manifold iff the $2k$-form $Q_a$ is admissible for any integer $0 < k \leq n/2$ and any $a \in H^{2(n-2k)}(M^{2n}; \mathbb{Z})$, $a \neq 0$.

The conditions for odd $k$ are always satisfied so not included in the above Theorem. In the case of even $n$, among these homogeneous forms one has the $n$-form corresponding to the volume polynomial of the multifan of $M^{2n}$ [4]. There is a caveat: for any elements $a, b \in H^{2(n-2k)}(M^{2n}; \mathbb{Z})$ with admissible $2k$-forms $Q_a, Q_b$ the sum $Q_{a+b}$ is not admissible, generally speaking. That is why Theorem 1.3 requires admissibility of an infinite set of forms. In the case of $n = 3$ we reduce the condition of Theorem 1.3 to the study of a finite set of quadratic forms (see Theorem 3.4). Due to that reason one can check the TNS-property for a 6-dimensional quasitoric manifold using PC. We show that the TNS-property for a quasitoric manifold $M^{2n}$ is equivalent to the total splitting of any complex vector bundle over $M^{2n}$ after stabilisation (Theorem 2.10). We also study some operations on quasitoric TNS-manifolds, namely: the equivariant blow-up of an invariant submanifold of real codimension 4 of a given quasitoric TNS-manifold (Proposition 4.8) and equivariant connected sum of two given quasitoric TNS-manifolds at fixed points. We prove that the equivariant connected sum $M_1^{2n} \# M_2^{2n}$ of quasitoric manifolds of dimension $2n$, where $n$ is odd, is a TNS-manifold iff $M_1, M_2$ are TNS-manifolds (Proposition 4.3). This claim has an interesting generalisation to the case of even $n$ (ibid.). As an example, we show that the equivariant connected sum of any quasitoric 4-manifold $M^4$ with the fixed quasitoric 4-manifold $B^4$ at fixed points of opposite signs is a TNS-manifold (Proposition 4.6). (The moment polycone of $B^4$ is rectangular.) We also remark that this claim is easily generalised to the case of arbitrary even $n$.

It is natural to suppose that the TNS-property of a toric manifold depends only on the face lattice of the respective moment polytope, or is even equivalent to the flag property of the latter. These conjectures are discussed in Section 6. The cohomology algebra $H^*(M^{2n}; \mathbb{R})$ of a quasitoric manifold $M^{2n}$ is isomorphic to the quotient algebra $DO_{\mathbb{R}}(\mathbb{R}^n)/AnnV_\mathbb{F}$ of the algebra of differential operators with constant coefficients $DO_{\mathbb{R}}(\mathbb{R}^n)$ by the annihilator ideal $AnnV_\mathbb{F}$ of the volume polynomial $V_\mathbb{F}$ of the multifan $\mathcal{F}$ of $M^{2n}$ (see Theorem 5.3). This fact allows us to reformulate the TNS-criterion for a quasitoric manifold in terms of the respective volume polynomial, see Theorem 5.5. We also pose a 17 Hilbert’s problem-type question for some finite dimensional $\mathbb{R}$-algebras (see Theorem 5.6) and give a conjecture about real psd-forms. Finally, we remark that the TNS-criterion for a quasitoric manifold in terms of the respective volume polynomial is algorithmically verifiable, see Corollary 6.7.

2 TNS-criterion

The following Section contains some basic tools for the study of a TNS-property for quasitoric manifolds. The TNS-condition of a quasitoric manifold $M^{2n}$ is expressed in terms of the support functions of the cone $S(M^{2n}) \subseteq K^0(M^{2n})$ generated by the elements of the form $[\xi] - 1$, where $\xi \to M^{2n}$ is a
complex linear bundle, in Subsection 2.1 In the Subsection 2.3 we study this cone for products of complex projective spaces $\mathbb{C}P^n$, then for arbitrary quasitoric manifolds. The necessary definitions are given in Subsection 2.2. The name of Subsection 2.4 is self-explanatory.

2.1 TNS-property in terms of $K$-theory and cohomology ring

A quasitoric $2n$-dimensional smooth manifold $M^{2n}$ has the canonical stably complex structure

$$TM^{2n} \oplus \mathbb{C}^{m-n} \simeq \bigoplus_{i=1}^{m} \theta_i$$

for the complex line bundles $\theta_i \to M^{2n}, i = 1, \ldots, m$. (See [8].) Let $x_i := c_1(\theta_i)$.

**Theorem 2.1.** [3] Let $X$ be a finite CW-complex. Then the Chern character map $ch \otimes Q : K^*(X) \otimes Q \to H^*(X; Q)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded ring isomorphism.

Let $\Lambda = (\lambda^i_j)$ be the characteristic $(n \times m)$-matrix of the quasitoric manifold $M^{2n}$. Let $P^n \subset \mathbb{R}^n$ be the moment polytope of $M^{2n}$ with facets $F_1, \ldots, F_m$.

**Theorem 2.2.** [10] $H^*(M^{2n}; \mathbb{Z}) \simeq \mathbb{Z}[x_1, \ldots, x_m]/(I_H + J_H)$, $\deg x_i = 2$, $i = 1, \ldots, m$,

where $I_H = (x_i \cdots x_{i+k} | F_i \cap \cdots \cap F_{i+k} = \emptyset)$, $J_H = (\lambda^i_j x_1 + \cdots + \lambda^n_j x_m | i = 1, \ldots, n)$ are the ideals of the polynomial ring. The ring $H^*(M^{2n}; \mathbb{Z})$ is torsion-free and all respective odd-graded groups vanish.

Using the Atiyah and Hirzebruch spectral sequence and Theorem 2.2 one justifies the following

**Proposition 2.3.** $K(M^{2n}) := K^0(M^{2n})$ is a free abelian group (w.r.t. the sum) of rank equal to the Euler characteristic $\chi(M^{2n})$ of $M^{2n}$. Next, $K^1(M^{2n}) = 0$.

Remind that

$$K(\mathbb{C}P^n) \simeq \mathbb{Z}[x]/(x - 1)^{n+1},$$

where $x$ is a class of the complex conjugate $\bar{n} \to \mathbb{C}P^n$ to the tautological line bundle, and 1 is the class of the trivial linear vector bundle. (See [2].)

**Proposition 2.4.** Let $\xi \to M^{2n}$ be a complex linear vector bundle over $M^{2n}$. Then in $K(M^{2n})$ one has the relation $((\xi) - 1)^{n+1} = 0$.

**Proof.** Observe that $ch((\xi) - 1)^{n+1} = 0$, then use Theorem 2.1 $\square$

**Theorem 2.5.** [18] Proposition 3.2 One has

$$K(M^{2n}) \simeq \mathbb{Z}[\theta_1, \ldots, \theta_m]/(I_K + J_K),$$

where $\theta^\nu := \theta^\nu_1 \cdots \theta^\nu_m$ for $\nu = (v_1, \ldots, v_m)$. $I_K = ((\theta^\nu_1 - 1) \cdots (\theta^\nu_{i+k} - 1) | F_i \cap \cdots \cap F_{i+k} = \emptyset)$, $J_K = (\theta^\nu_1 \cdots \theta^\nu_m - 1 | i = 1, \ldots, n)$ are the ideals of the polynomial ring. In particular, the classes $[\theta_i]$ multiplicatively generate the ring $K(M^{2n})$, $i = 1, \ldots, m$.

**Remark 2.6.** Theorems 2.2 and 2.5 may also be deduced from the computation of the complex cobordism ring of a quasitoric manifold and further specialisation of the universal formal group law (for complex oriented generalised cohomology theories) to the cohomological or $K$-theoretical, respectively.

**Corollary 2.7.** For any $k = 1, \ldots, n$ one has

$$H^{2k}(M^{2n}; \mathbb{Q}) \simeq \mathbb{Q}[x^k | x \in H^2(M^{2n}; \mathbb{Z})]$$

**Proof.** Theorem 2.5 tells that $K(M^{2n}) \simeq \mathbb{Z}[\theta^\nu_\nu | \nu \in \mathbb{Z}^m]$. Theorem 2.1 implies that $ch_k(K(M^{2n}) \otimes \mathbb{Q}) = H^{2k}(M^{2n}; \mathbb{Q})$. It remains to observe that $ch_k(\theta^\nu_\nu) = (c_1(\theta^\nu_\nu))^k/k!$.
Consider the semigroup $C(M^{2n}) \subseteq K(M^{2n}) := K^0(M^{2n})$ generated by the elements of the form $[\xi] - 1$, where $\xi \to M^{2n}$ is any linear vector bundle over $M^{2n}$ (with respect to the Whitney sum operation).

**Corollary 2.8.** One has

$$C(M^{2n}) = \mathbb{Z}_{\geq}([\mathbb{E}_v] - 1) \mid v \in \mathbb{Z}^n),$$

where $\mathbb{Z}_{\geq}(\cdot)$ denotes the $\mathbb{Z}_{\geq}$-semigroup hull of a given abelian group.

**Proposition 2.9.** ([19] Lemma 2.3) Let $X, Y$ be finite CW-complexes. Suppose that $\xi, \eta \to X$ are linear vector bundles with totally split stably inverses. Let $f : Y \to X$ be a continuous map. Then the stably inverse vector bundles to $f^*\xi, f^*\xi \oplus \eta, \xi \eta \to X$ are totally split.

**Theorem 2.10.** Let $M^{2n}$ be a quasitoric manifold of dimension $2n$. Then the following conditions are equivalent:

(i) $M^{2n}$ is a TNS-manifold;

(ii) For any $i = 1, \ldots, m$, the stably inverse vector bundle to $\theta_i$ totally splits;

(iii) For any element $x \in K(M^{2n})$ there exists $N \in \mathbb{Z}$ s.t.

$$x = N + \sum_{v \in \mathbb{Z}^n_{\geq}} c_v[\mathbb{E}_v],$$

with all integers $c_v \geq 0$ being non-negative (the sum above is over the semigroup: $v = (v_1, \ldots, v_m) \in \mathbb{Z}^n_{\geq}$, only finite number of integers $c_v \in \mathbb{Z}$ are non-zero);

(iv) Any complex vector bundle $\xi \to M^{2n}$ stably totally splits.

**Proof.** (i) $\Rightarrow$ (ii). By the condition, there exists a totally split complex vector bundle $\alpha = \bigoplus_{i=1}^k \alpha_i \to M^{2n}$ and an integer $N \in \mathbb{N}$, s.t.

$$\theta + \alpha \simeq \mathbb{C}^N.$$

The claim now follows from the formula (1).

(ii) $\Rightarrow$ (iii). There exist complex vector bundles $\xi, \eta \to M^{2n}$, s.t. $x = [\xi] - [\eta]$ (see [2]). Next, for the stably inverse vector bundle $\zeta$ to $\eta$, i.e. $[\zeta] + [\eta] = k$, $k \in \mathbb{Z}$, one has

$$x = [\xi] + [\zeta] - k.$$

Hence, w.l.o.g. we may assume that $x = [\xi]$ is a class of a complex linear bundle. By Theorem 2.5, the equality (3) holds with ambient coefficients $c_v$ and $N = 0$ in the ring $K(M^{2n})$. It remains to eliminate the negative coefficients in this identity. Due to Proposition 2.9, for any $v \in \mathbb{Z}^n_{\geq}$, the linear vector bundle $[\mathbb{E}_v]$ has the totally split stably inverse. Hence, for any $v \in \mathbb{Z}^n_{\geq}$ s.t. $c_v < 0$ there exists $N_v \in \mathbb{Z}$ with $N_v + c_v[\mathbb{E}_v]$ represented by a class of some totally split vector bundle. Now the desired statement follows.

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) follow trivially.

The property (iv) from Theorem 2.10 is homotopy invariant for good topological spaces.

**Proposition 2.11.** Let $X, Y$ be homotopy equivalent finite CW-complexes. Suppose that every complex vector bundle over $X$ is stably totally split. Then every complex vector bundle over $Y$ is also stably totally split.

**Proof.** By the definition, there exist continuous maps $f : X \to Y$, $g : Y \to X$ s.t.

$$g \circ f \simeq_{\text{hot}} Id_X, f \circ g \simeq_{\text{hot}} Id_Y,$$

where $\simeq_{\text{hot}}$ denotes homotopy equivalence between maps. Let $\xi \to Y$ be a complex vector bundle over $Y$. By the hypothesis, there exist a totally split complex vector bundle $\alpha = \bigoplus_{i=1}^k \alpha_i \to X$ s.t.

$$f^*(\xi) \oplus \mathbb{C}^N = \alpha$$

for some $N \in \mathbb{N}$. Then one has:

$$g^*(f^*(\xi) \oplus g^*(\mathbb{C}^N)) = (f \circ g)^*(\xi) \oplus \mathbb{C}^N = g^*(\alpha).$$

The vector bundle $g^*(\alpha) = \bigoplus_{i=1}^k g^*(\alpha_i)$ is totally split. The vector bundles $(fg)^*(\xi)$ and $\xi$ are topologically equivalent, because they have the homotopy equivalent classifying maps. Hence,

$$\xi \oplus \mathbb{C}^N \simeq g^*(\alpha),$$

and $\xi$ is stably totally split vector bundle. Q.E.D.
2.2 Convex cones and operations on them

Recall some standard definitions of convex geometry. From now and on in this paper we consider convex cones only in finite-dimensional real linear spaces.

**Definition 2.12.** A convex cone \( \sigma \) in \( \mathbb{R}^n \) (with apex at the origin \( 0 \in \mathbb{R}^n \)) is any set of the form \( \text{con} \, X := \{ t_1 v_1 + \cdots + t_k v_k | k \in \mathbb{N}, t_1, \ldots, t_k \in \mathbb{R}_{\geq 0}; v_1, \ldots, v_k \in X \} \), where \( X \subset \mathbb{R}^n \). The maximal w.r.t. inclusion linear subset of the cone \( \sigma \) is called a lineality subspace \( \text{lin} \, \sigma \subset \mathbb{R}^n \) of \( \sigma \). The cone \( \sigma \) is closed, if it is closed as a subset of \( \mathbb{R}^n \), and salient, if \( \text{lin} \, \sigma = \emptyset \). The dimension \( \dim \sigma \) of the cone \( \sigma \) is the dimension of its linear hull \( \dim \mathbb{R}(\sigma) \). In case of \( \dim \sigma = n \), the cone \( \sigma \) is called full-dimensional.

**Proposition 2.13.** For any convex cone \( \sigma \subset \mathbb{R}^n \) there exists a salient cone \( \sigma' \subset \mathbb{R}^n \) s.t. one has \( \sigma = \sigma' + \text{lin} \, \sigma \).

Fix any basis \( e_1, \ldots, e_n \in \mathbb{R}^n \).

**Lemma 2.14.** Let \( \sigma \subset \mathbb{R}^n \) be a full-dimensional salient cone. Then \( \sigma \) has a supporting hyperplane having the normal with only rational coordinates in the basis \( e_1, \ldots, e_n \).

**Proof.** The normal of any supporting hyperplane to \( \sigma \) is an element of the dual cone \( \sigma^* \subset (\mathbb{R}^n)^* \). The cone \( \sigma \) is salient, hence, \( \text{int}(\sigma^*) \neq \emptyset \). The desired normal is any element of the interior \( \text{int}(\sigma^*) \) having all rational coordinates in the dual basis \( e^1, \ldots, e^n \in (\mathbb{R}^n)^* \).

**Definition 2.15.** A linear subspace \( U \subset \mathbb{R}^n \) is called rational w.r.t. the basis \( e_1, \ldots, e_n \in \mathbb{R}^n \) if \( U \) is generated by vectors \( u_1, \ldots, u_k \) of \( U \) having only rational coordinates in the basis \( e_1, \ldots, e_n \).

**Proposition 2.16.** Let \( \sigma \subset \mathbb{R}^n \) be a full-dimensional convex cone with rational lineality subspace \( \text{lin} \, \sigma \) w.r.t. the basis \( e_1, \ldots, e_n \in \mathbb{R}^n \). Then \( \sigma \) has a supporting hyperplane with the normal having only rational coordinates in the given basis of \( \mathbb{R}^n \).

The next step is to set up two different definitions of products for convex cones. The first one is the tensor product of cones (see [5]), and the second corresponds to the cartesian product of manifolds (see Corollary 2.30).

**Definition 2.17.** Let \( \sigma \subset U, \tau \subset V \) be convex cones in \( \mathbb{R} \)-linear spaces \( U, V \), respectively. The convex cone

\[
\sigma \otimes \tau := \{ u \otimes v | u \in U, v \in V \} \subseteq U \otimes V,
\]

is called a tensor product of cones \( \sigma, \tau \). For any \( u \in U, v \in V \) define

\[
x \ast y := u + v + u \otimes v.
\]

Also define the product of convex cones by the formula

\[
\sigma \ast \tau := \{ u \ast v | u \in U, v \in V \} \subseteq U \oplus V \oplus U \otimes V.
\]

By the definition, \( (\sigma \ast \tau, 1) = (\sigma, 1) \otimes (\tau, 1) \). Observe that the \( \ast \)-product is commutative and associative but not linear by either of the factors, generally speaking.

**Lemma 2.18.** Let \( \sigma = \text{lin} \, \sigma + \sigma' \subset U \), where \( \sigma, \sigma' \subset U \), \( \tau \subset V \) are convex cones. Then one has \( \sigma \otimes \tau = \text{lin} \sigma \otimes \mathbb{R}(\tau) + \sigma' \otimes \tau \).

**Proof.** An equality \( \sigma \otimes \tau = (\text{lin} \sigma \otimes \tau + \sigma' \otimes \tau \) clearly takes place. It remains to show that \( (\text{lin} \sigma) \otimes \tau = \text{lin} \sigma \otimes \mathbb{R}(\tau) \). The inclusion \( (\text{lin} \sigma) \otimes \tau \subseteq (\text{lin} \sigma) \otimes \mathbb{R}(\tau) \) clearly holds. The inverse inclusion \( (\text{lin} \sigma) \otimes \tau \supseteq (\text{lin} \sigma) \otimes \mathbb{R}(\tau) \) follows from the identities \( (\text{lin} \sigma) \otimes \tau = (- \text{lin} \sigma) \otimes (- \tau) = (\text{lin} \sigma) \otimes (- \tau), \mathbb{R}(\tau) = \tau - \tau \).

**Proposition 2.19.** For any convex cones \( \sigma \subset U, \tau \subset V \) one has

\[
\text{lin}(\sigma \otimes \tau) = (\text{lin} \sigma) \otimes \mathbb{R}(\tau) + \mathbb{R}(\sigma) \otimes \text{lin} \tau.
\]

In particular, the tensor product of two salient convex cones is salient.
Denote by $\sigma, \tau$ of the cones $2.3$ The cone

**Corollary 2.20.** Let $\sigma \subseteq U$, $\tau \subseteq V$ be full-dimensional convex cones with rational lineality subspaces $\text{lin} \sigma$, $\text{lin} \tau$ w.r.t. the bases $u_1, \ldots, u_k$ and $v_1, \ldots, v_l$ of linear spaces $U, V$, respectively. Then the subspace $\text{lin}(\sigma \otimes \tau) \subseteq U \otimes V$ is rational w.r.t. the basis $u_1 \otimes v_1, \ldots, u_k \otimes v_l$ of the linear space $U \otimes V$.

We need some facts about cyclic polytopes (see [22]). Let $x_n : \mathbb{R} \to \mathbb{R}^n, x_n(t) := (t, t^2, \ldots, t^n)$. The image of the real line $\mathbb{R}$ under the map $x_n$ is called a moment curve. For any $k > n$ the cyclic polytope $C_n(t_1, \ldots, t_k)$ is defined as a convex hull of $k$ distinct points $x_n(t_1), \ldots, x_n(t_k), t_1, \ldots, t_k \in \mathbb{R}$, of the moment curve.

**Theorem 2.21.** [22] (i) Cyclic polytope $C_n(t_1, \ldots, t_k)$ is a simplicial $n$-polytope;
(ii) $C_n(t_1, \ldots, t_k)$ has exactly $k$ vertices;
(iii) The combinatorial type of $C_n(t_1, \ldots, t_k)$ does not depend on the choice of $t_1, \ldots, t_k$.

For any $k$ let $C_k^n := C_n(-1, -1/2, \ldots, -1/k, 1/k, \ldots, 1/2, 1)$. Also let $C_k^n$ be the closure of the convex hull of the points $\{x_n(1/k) | k \in \mathbb{Z}\} \cup \{x_n(0)\}$ (see Fig. 1).

**Corollary 2.22.** $C_\infty^1 = [-1, 1]$. For $n \geq 2$, the vertices of $C_n^\infty$ are $\{x(1/k) | k \in \mathbb{Z}\} \cup \{x(0)\}$. $C_\infty^n = \bigcup_{k \leq 1} C_k^n$ is a compact convex body in $\mathbb{R}^n$.

**Proof.** The set $C_\infty^n$ is closed. It remains to notice that $C_k^n \subset C_{k+1}^n$ and $C_\infty^n \subset \mathbb{R}^n$, where $\mathbb{R}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid -1 \leq x_i \leq 1, i = 1, \ldots, n\}$ is a unitary $n$-hypercube. $\square$

### 2.3 The cone $S(M^{2n})$

Denote by $S(M^{2n}) \subseteq K(M^{2n}) \otimes \mathbb{R}$ the closure of the convex conical hull of the semigroup $C(M^{2n}) \subset K(M^{2n})$, i.e. $S(M^{2n}) = \text{con} C(M^{2n})$. In other words, $S(M^{2n})$ is the closure of the conical convex hull of elements of the form $[\xi] - 1$, where $\xi \to M^{2n}$ is a complex linear bundle. In this Subsection we give a description of $S(M^{2n})$ for an arbitrary quasitoric manifold $M^{2n}$.

The natural projection $K(CP^n) \to K(CP^{n-1})$ maps the cone $S(CP^n)$ onto the cone $S(CP^{n-1})$. Fix the basis $(x - 1), \ldots, (x - 1)^n$ of the linear space $K(CP^n) \otimes \mathbb{R} \simeq \mathbb{R}^n$ and let $e^1, \ldots, e^n$ be the
corresponding coordinates in $\tilde{K}(\mathbb{C}P^n) \otimes \mathbb{R}$. Let $A_n : \tilde{K}(\mathbb{C}P^n) \otimes \mathbb{R} \to \tilde{K}(\mathbb{C}P^n) \otimes \mathbb{R}$ be the matrix of the linear coordinate change defined uniquely by the conditions

$$A_n\left(\begin{array}{c} k \\ 1 \\ k \\ 2 \\ \vdots \\ k \\ n \end{array}\right) = (k, k^2, \ldots, k^n), \ k \in \mathbb{Z},$$

in the coordinates indicated above. From now and on in this paper, $(\binom{a}{b}) := \frac{a!}{(a-b)!(b)!}$ for $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{\geq 0}$; $(\binom{a}{b}) := 1$ for $a > 0$; $(\binom{a}{b}) := 0$ for $b \geq 0$. The matrix $A_n$ is well-defined for any $n \in \mathbb{N}$, since the polynomials $(\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{n})$ in $k$ span an $n$-dimensional linear subspace of the finite-dimensional real polynomial algebra $\mathbb{R}[k]$ ($n$ is fixed).

**Proposition 2.23.** $S(\mathbb{C}P^2) = \mathbb{R}_{\geq 0} \langle (x-1), (\overline{v}-1) \rangle$. If $n$ is odd, then $S(\mathbb{C}P^n) = \mathbb{R}_{\geq 0} \langle (x-1)^n, -(x-1)^n, S(\mathbb{C}P^{n-1}) \rangle$. If $n$ is even, then the image $A_n(S(\mathbb{C}P^n))$ is the cone over the compact convex body $P^{n-1}$ of $\mathbb{R}$-dimension $n-1$. Under the natural linear projection $\{e^n = 1\} \to \mathbb{R} \langle (x-1), \ldots, (x-1)^n \rangle$, the body $P^{n-1}$ maps bijectively onto $C^n_{\infty}$. In particular, the linear subspace $\text{lin} S(\mathbb{C}P^n) \subseteq \tilde{K}(\mathbb{C}P^n)$ is rational w.r.t. the above indicated basis of $\tilde{K}(\mathbb{C}P^n)$.

**Proof.** Due to the isomorphism [2] and Taylor expansion, one has:

$$x^k - 1 = \sum_{i=1}^{n} \binom{k}{i} (x-1)^i, \ k \in \mathbb{Z}. \quad (4)$$

By the Formula [4], the cone $A_n(S(\mathbb{C}P^n))$ is generated by the vectors $(k, k^2, \ldots, k^n) \in \mathbb{R}^n, \ k \in \mathbb{Z}$.

Let $n$ be odd. To prove the statement it is enough to check that any supporting function of the cone $S(\mathbb{C}P^n)$ vanishes on $(x-1)^n$. Consider the linear function $H: \tilde{K}(\mathbb{C}P^n) \otimes \mathbb{R} \to \mathbb{R}$ s.t. $H(S(\mathbb{C}P^n)) \geq 0$. Due to the identity [4], one has $0 \leq \lim_{k \to +\infty} H(x^k - 1) = \lim_{k \to +\infty} k^n H((x-1)^n)$. Hence, $H((x-1)^n) \geq 0$. Next, one has $\lim_{k \to -\infty} H(x^k - 1) = \lim_{k \to -\infty} k^n H((x-1)^n)$. We conclude that $H((x-1)^n) = 0$, as required.

Let $n$ be even. Dividing the generators $(k, k^2, \ldots, k^n)$, $k \in \mathbb{Z}$, by $k^n, k \neq 0$, we see that $A_n(S(\mathbb{C}P^n))$ is the cone over the convex body $C^n_{\infty}$ in the respective affine hyperplane. The explicit formula for $S(\mathbb{C}P^2)$ follows from Corollary 2.22

The following observation belongs to A. Ayzenberg.

**Remark 2.24.** For any integer $k$ the expansion coefficients of $x^k - 1 \in \tilde{K}(\mathbb{C}P^n)$ w.r.t. the basis $(x-1), \ldots, (x-1)^n$ are the binomial coefficients $(\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{n})$. A straightforward induction on $k$ shows that for any non-negative integer $l$ the inequality

$$\binom{k}{l}^2 \geq \binom{k}{l-1} \left(\binom{k}{l+1}\right)$$

holds. When $0 < l < k$, it is a well-known inequality on the binomial coefficients. On the other hand, this inequality means that the sequence $(\binom{1}{1}, \binom{2}{1}, \ldots, \binom{n}{1})$ is log-concave. This property of the projective space may be worth studying due to thesis [1].

**Lemma 2.25.** Let $x_1, \ldots, x_k \in K(M^k), k \geq 2$. Then one has

$$(\ldots((x_1 \ast x_2) \ast x_3) \ast \ldots) \ast x_k = \sum_{q=1}^{k} \sum_{1 \leq i_1 < \ldots < i_q \leq k} \sum x_{i_1} \cdots x_{i_q},$$

$$(x_1 - 1) \cdots (x_k - 1) = x_1 \cdots x_k - 1. \quad (5)$$

**Lemma 2.26.** The transition matrix from the basis $\{\langle y_1 - 1 \rangle^{v_1} \cdots \langle y_m - 1 \rangle^{v_m} \mid \sum_i v_i \leq n, v_i \in \mathbb{Z}_{\geq 0}\}$ to the basis $\{\langle y_1 \rangle^{v_1} \cdots \langle y_m \rangle^{v_m} - 1 \mid \sum_i v_i \leq n, v_i \in \mathbb{Z}_{\geq 0}\}$ of the linear space $\tilde{K}(\langle \mathbb{C}P^n \rangle^m) \otimes \mathbb{R}$ has only rational matrix elements.

**Proposition 2.27.** For any $n, m \in \mathbb{N}$ one has

$$S(\langle \mathbb{C}P^n \rangle^m) = S(\mathbb{C}P^n) \ast \cdots \ast S(\mathbb{C}P^n). \quad (6)$$

The subspace $\text{lin} S(\langle \mathbb{C}P^n \rangle^m) \subseteq \tilde{K}(\langle \mathbb{C}P^n \rangle^m)$ is rational w.r.t. the basis $\{y_1 \cdots y_m - 1 \mid \sum_i v_i \leq n, v_i \in \mathbb{Z}_{\geq 0}\}$ of the linear space $\tilde{K}(\langle \mathbb{C}P^n \rangle^m) \otimes \mathbb{R}$. 

7
Proof. The Formula \((6)\) is a straight consequence of the Formula \((5)\) and Künneth formula for \(K\)-theory (see [2]). The second claim follows from the Corollary 2.20, Proposition 2.23 and Lemma 2.26.

Let \(M^{2n}\) be a quasitoric manifold with the convex polytope with \(m\) facets. Consider the linear map \(R : K((\mathbb{C}P^n)^m) \to K(M^{2n})\) mapping the class \(y_i\) of the dual to the (pull-back of the) tautological line bundle over the \(i\)-th multiple in \((\mathbb{C}P^n)^m\) to \(\theta_i\). This map is well-defined due to Proposition 2.4 and Theorem 2.5. By the definition, \(R(y_1^1 \cdots y_m^n) = \theta^E\) holds for any \(v \in \mathbb{Z}^m\). Hence, \(S(M^{2n}) = R(S((\mathbb{C}P^n)^m))\) (see Corollary 2.8).

Lemma 2.28. There exist complex linear vector bundles \(\xi_i \to M^{2n}, i = 1, \ldots, \chi(M^{2n})\), such that \(e_1, \ldots, e_{\chi(M^{2n})} - 1\) constitute a basis of the free abelian group \(K(M^{2n})\), where \(e_i = [\xi_i], i = 1, \ldots, \chi(M^{2n})\). Given any complex linear vector bundle \(\xi \to M^{2n}\), the element \([\xi] - 1 \in K(M^{2n})\) has rational coordinates in the basis above.

Proof. Follows from Theorem 2.5 and multivariate Taylor formula for \(f(v_1, \ldots, v_m) = \theta^E - 1\).

Corollary 2.29. Suppose that \(S(M^{2n}) \neq \overline{K(M^{2n})} \otimes \mathbb{R}\). Then the cone \(S(M^{2n})\) has a supporting hyperplane with the normal vector having only rational coordinates in the basis from Lemma 2.28.

Proof. Theorem 2.5 and Lemma 2.28 imply that the matrix of the linear map \(R\) has only rational entries in the above bases of the linear spaces \(K((\mathbb{C}P^n)^m) \otimes \mathbb{R}, K(M^{2n}) \otimes \mathbb{R}\). It follows from Proposition 2.27 and Lemma 2.28 that the subspace \(\text{lin } S(M^{2n}) \subseteq \overline{K(M^{2n})} \otimes \mathbb{R}\) is rational w.r.t. the above basis. Now the claim follows from Proposition 2.16.

Corollary 2.30. For any quasitoric manifolds \(M_1^{2n_1}, M_2^{2n_2}\) an identity
\[
S(M_1^{2n_1} \times M_2^{2n_2}) = S(M_1^{2n_1}) \ast S(M_2^{2n_2})
\]
holds under the Künneth isomorphism \(K(M_1^{2n_1} \times M_2^{2n_2}) \simeq K(M_1^{2n_1}) \otimes K(M_2^{2n_2})\).

2.4 Proof of the main theorem

In order to prove the TNS-criterion we need an auxiliary

Lemma 2.31. Let \(S \subseteq L\) be a sub-semigroup of the free abelian group \(L \simeq \mathbb{Z}^d\) s.t. \(S\) contains a \(\mathbb{Z}\)-basis \(x_1, \ldots, x_d \in S\) of \(L\). Then the following conditions are equivalent:

(i) \(S = L\);

(ii) There exists an element \(v = \sum_i v^i x_i \in S\) s.t. \(v_i < 0, \ i = 1, \ldots, d\);

(iii) \(0 \in \text{int conv } S\), where \(\text{int conv}\) denote the interior and the convex hull of a set in \(\mathbb{R}^d\), resp.;

(iv) There is no such a linear function \(H : L \otimes \mathbb{R} \to \mathbb{R}, H \neq 0\) that \(S \subseteq \{H \geq 0\}\).

Proof. The implications (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (iv) are clear. (iv) \(\Rightarrow\) (iii) follows from the Supporting Hyperplane Theorem in \(\mathbb{R}^n\).

(iii) \(\Rightarrow\) (ii). The condition (iii) implies that there exists such an element \(u = \sum_i u^i x_i \in \text{int conv } S\) that \(u^i < 0, \ i = 1, \ldots, d\). Consider elements \(u_1, \ldots, u_d \in S\) and real positive numbers \(a_1, \ldots, a_d \in \mathbb{R}_{>0}\) s.t. \(u = \sum a^i u_i\). Consider a small variation \(b^i\) of \(a^i, i = 1, \ldots, d\), s.t. \(u' := \sum b^i u_i \in \text{int conv } S\), all coordinates (w.r.t. the basis \(x_1, \ldots, x_d\)) of \(u'\) are negative and all \(b^i \in \mathbb{Q}_{>0}\) are rational positive numbers. Let \(N \in \mathbb{N}\) be s.t. \(N b^i \in \mathbb{Z}_{\geq 0}, i = 1, \ldots, d\). Then all the coordinates of \(v := N u' \in S\) are negative (w.r.t. the basis \(x_1, \ldots, x_d\)), as required.

(ii) \(\Rightarrow\) (i). Let \(x \in L\). Consider a decomposition \(x = \sum_i a^i x_i, a_i \in \mathbb{Z}, i = 1, \ldots, d\). Let \(L_{\geq 0} \subseteq S\) be a sub-semigroup generated by \(x_1, \ldots, x_d\). Let \(N \in \mathbb{N}\) be a natural number s.t. \(N v < x\) (coordinate-wisely). Then \(x \in N v + L_{\geq 0}\). Hence \(x \in S\), as required.

Corollary 2.32. A quasitoric manifold \(M^{2n}\) of dimension \(2n\) is a TNS-manifold if \(S(M^{2n}) \neq \overline{K(M^{2n})} \otimes \mathbb{R}\). 

8
Proof. The TNS-property of $M^{2n}$ is equivalent to the condition $(iii)$ of Theorem 2.10. The latter is, in turn, equivalent to the condition

$$K(M^{2n}) = C(M^{2n}).$$

(7)

By Lemma 2.28, the semigroup $C(M^{2n}) = \mathbb{Z}_{\geq 0} \{ q^L - \epsilon \mid \epsilon \in \mathbb{Z}^m \}$ contains a $\mathbb{Z}$-basis of the free abelian group $K(M^{2n})$. Hence, by Lemma 2.31, the equality (7) is equivalent to the desired condition. \hfill \square

Define the homogeneous $\mathbb{Q}$-form $Q_a : H^2(M^{2n}; \mathbb{Q}) \to \mathbb{Q}$ of degree $k$ by the formula

$$Q_a(x) := \langle x^k a, [M^{2n}] \rangle, \quad x \in H^2(M^{2n}; \mathbb{Q}),$$

where $(\cdot, \cdot)$ is the canonical pairing.

Remark 2.33. For a quasitoric manifold $M^{2n}$, the volume polynomial of the multifan $\mathcal{F}$ corresponding to $M^{2n}$ is given by the formula

$$Vol_{\mathcal{F}}(c_1, \ldots, c_m) := \frac{1}{n!}((c_1x_1 + \cdots + c_mx_m)^n, [M^{2n}]), \quad c_i \in \mathbb{R}, \quad i = 1, \ldots, m.$$ 

An identity $Q_1(c_1x_1 + \cdots + c_mx_m) = \frac{1}{n!}Vol_{\mathcal{F}}(c_1, \ldots, c_m)$ clearly holds. Notice that the form $Q_a$ may be degenerate (see Example 2.36).

Proposition 2.34. Let $K = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. Then there is a bijective correspondence between $K$-linear functions $L : H^{2k}(M^{2n}; K) \to K$ and homogeneous $k$-forms $Q_a : H^2(M^{2n}; K) \to K$, $a \in H^{2(n-k)}(M^{2n}; K)$.

Proof. We give the proof for the case of $K = \mathbb{Q}$. The Poincaré duality implies that any $\mathbb{Q}$-linear function $L : H^{2k}(M^{2n}; \mathbb{Q}) \to \mathbb{Q}$ has the form $L(\cdot) = \langle \cdot, a, [M^{2n}] \rangle$ for some $a \in H^{2(n-k)}(M^{2n}; \mathbb{Q})$. Hence, Corollary 2.29 gives the required bijective correspondence. \hfill \square

Proof of Theorem 1.3 \Rightarrow. Assume the contrary. Then there exists $a \in H^{2(n-k)}(M^{2n}; \mathbb{Z})$ s.t. the form $Q_a$ is non-trivial and semidefinite. Let $x \in H^2(M^{2n}; \mathbb{Z})$ be s.t. $Q_a(x) > 0$. Consider a complex linear vector bundle $\xi \to M^{2n}$ with $c_1(\xi) = x$. The stably inverse vector bundle $a \to M^{2n}$ to $\xi$ is totally split by Theorem 2.10: $a = \bigoplus_{i=1}^n a_i$ for some $a_i \to M^{2n}$. Let $a_i := c_1(a_i)$. Apply the Chern character map to the identity $l + 1 - \xi = a \in K(M^{2n})$. The respective $4k$-component of the obtained identity is

$$-x^{2k} = \sum_{i=1}^n a_i^{2k}.$$ 

Now multiply the left and right parts of the last identity by $a$, then couple the obtained identity with $[M^{2n}]$ to get

$$-Q_a(x) = \sum_{i=1}^n Q_a(a_i).$$

But the latter contradicts the semidefiniteness of $Q_a$.

\hfill \Leftarrow. Assume the contrary. Then Corollary 2.29 and Lemma 2.28 imply that there exists a linear function $H : K(M^{2n}) \otimes \mathbb{R} \to \mathbb{R}$ s.t. $H(1) = 0$, $H(1) = 0 = H(\psi^k)$ for any $\psi \in \mathbb{Z}^m$. Define the linear function $L : H^*(M^{2n}; \mathbb{R}) \to \mathbb{R}$ by the formula

$$L(x) := H(ch^{-1}(x) + c^{−1}(x)), \quad x \in H^*(M^{2n}; \mathbb{R}),$$

where the bar denotes complex conjugation in $K$-theory. Notice, that $L(1) = 0$ and $L_{|H^{2(k+1)}(M^{2n}; \mathbb{R})} \equiv 0$ for any $k = 0, \ldots, [n/2]$. Suppose that $L \equiv 0$. Substituting $x = ch(\psi^k)$ one obtains

$$H(\psi^k) = -H(\psi^{-k}),$$

for any $\psi \in \mathbb{Z}^m$. Then $H \equiv 0$ — a contradiction. Hence, $L \not\equiv 0$.

Let $k$ be the greatest integer s.t. $L_{|H^{2k}(M^{2n}; \mathbb{R})} \not\equiv 0$. It follows from the definition of Chern character and $L \not\equiv 0$ that $k > 0$ is even. Due to Proposition 2.34, the linear function $L_{|H^{2k}(M^{2n}; \mathbb{Q})}$ gives a non-zero $k$-form $Q : H^2(M^{2n}; \mathbb{Q}) \to \mathbb{Q}$ of even degree. Using its homogeneity w.l.g. we may assume that $Q$ is integer. By the condition, this form is non-semidefinite. Hence, there exists an
element \( x \in H^2(M^{2n}; \mathbb{Z}) \) s.t. \( Q(x) = L(x^k) < 0 \). Let \( \xi \to M^{2n} \) be a complex linear vector bundle s.t. \( c_1(\xi) = x \). Then one has

\[
0 \leq H(\xi' + \xi^{'-1}) = L(ch(\xi')) = \sum_{i=1}^{k/2} \frac{a_i}{(2i)!} L(x^{2i}), \ a \in \mathbb{Z}.
\]

Hence,

\[
0 \leq \lim_{a \to +\infty} L(ch(\xi')) = L(x^k) \cdot (+\infty) = -\infty
\]
— a contradiction. Q.E.D.

Remark 2.35. \( \mathbb{Q} \)- and \( \mathbb{R} \)-analogues of Theorem 1.3 clearly take place.

Example 2.36. Let \( M^8 = \mathbb{CP}^2 \times \mathbb{CP}^2 \). The respective cohomology ring is \( H^*(M^8; \mathbb{Q}) \simeq \mathbb{Q}[x, y]/(x^3, y^3) \), where \( x, y \) are the first Chern classes of the pull-backs of the dual to the tautological bundles over the respective factors in \( \mathbb{CP}^2 \times \mathbb{CP}^2 \). We apply Theorem 1.3 to show that \( M^8 \) is not a TNS-manifold. The 4-form \( Q_1 \) is positive semidefinite: \( Q_1(ax + by) = 6a^2b^2 \), where \( a, b \in \mathbb{Q} \). (Notice that the intersection form of \( M^8 \) is non-definite: \( \sigma(M^8) = 1 < 3 = \dim H^2(M^8; \mathbb{R}) \).) Any element of \( H^4(M^8; \mathbb{Q}) \) has the form \( ax^2 + bxy + cy^2 \), where \( a, b, c \in \mathbb{Q} \). The quadratic form \( Q_{ax^2+bxy+cy^2} \) has matrix

\[
\begin{pmatrix}
  c & b \\
  b & a
\end{pmatrix}
\]

in the basis \( x, y \in H^2(M^8; \mathbb{Q}) \). Clearly, \( Q_{x^2+y^2} \) is a positive-definite form.

Corollary 2.37. Let \( M^{2n} \) be a quasitoric manifold of dimension \( 2n \). Then \( M^{2n} \) is a TNS-manifold iff for any \( 0 < k \leq n/2 \) \( H^{2k}(M^{2n}; \mathbb{R}) = ch_{2k}(S(M^{2n})) = \mathbb{R}_{\geq 0} \langle x^{2k} | x \in H^2(M^{2n}; \mathbb{Z}) \rangle \).

3 TNS-manifolds in low dimensions

The following Section is devoted to the study of the quasitoric TNS-manifolds in dimensions 4, 6. A reduction of Theorem 1.3 to finitely many quadratic forms for 6-folds is given in Subsection 3.2.

3.1 Quasitoric TNS 4-folds

Here is a complete characterisation of smooth projective toric TNS-surfaces.

Proposition 3.1. Let \( M^4 \) be a smooth projective toric surface with the moment polygone \( P^2 \subset \mathbb{R}^2 \). Then \( M^4 \) is a TNS-manifold iff \( P^2 \) is distinct from the triangle \( \Delta^2 \).

Proof. Remind that the polygone \( P^2 \) has \( m \) edges. Due to Theorem 1.1 one has to check that the equality \( |\sigma(M^4)| = \dim H^2(M^4; \mathbb{R}) \) holds iff \( P^2 = \Delta^2 \). Due to the well-known formula of the signature and Euler characteristic of a toric manifold (e.g. \( \mathbb{R} \) Section 9.5), one has \( |\sigma(M^4)| = |4 - \dim H^2(M^4; \mathbb{R})| \leq \dim H^2(M^4; \mathbb{R}) \), \( \dim H^2(M^4; \mathbb{R}) = m - 2 \). Clearly, the equality holds here iff \( m = 3 \). It remains to notice that the only smooth projective toric surface over the triangle is \( \mathbb{CP}^2 \). Q.E.D.

Quasitoric non-TNS manifolds are more diverse starting from dimension 4 and so on. A straightforward computation implies the following

Proposition 3.2. Let \( M^4 \) be a quasitoric non-TNS 4-fold. Suppose that the moment polygone of \( M^4 \) is a 4-gon. Then the characteristic matrix of \( M^4 \) is \( GL_2(\mathbb{Z}) \)-equivalent to

\[
\begin{pmatrix}
  1 & 2 \\
  1 & 1
\end{pmatrix}, \begin{pmatrix}
  1 & -2 \\
  1 & -1
\end{pmatrix}, \begin{pmatrix}
  1 & -2 \\
  -1 & 1
\end{pmatrix}, \begin{pmatrix}
  1 & 2 \\
  -1 & -1
\end{pmatrix}.
\]

These manifolds have two different oriented diffeomorphism classes.
3.2 Quasitoric TNS 6-folds

Consider a quasitoric manifold $M^6$.

**Lemma 3.3.** For any classes $\alpha_1, \ldots, \alpha_k \in K(M^6)$ of complex linear vector bundles over $M^6$ the identity

$$ch_2((\alpha_1 - 1) \ast \cdots \ast (\alpha_k - 1)) = \frac{1}{2}(\sum_{i=1}^{k} c_1(\alpha_i))^2,$$

holds.

**Proof.** By Lemma 2.25 one has the chain of identities:

$$ch_2((\alpha_1 - 1) \ast \cdots \ast (\alpha_k - 1)) = ch_2(\sum_{q=1}^{k} \sum_{1 \leq i_1 < \cdots < i_q \leq k} (\alpha_{i_1} - 1) \cdots (\alpha_{i_q} - 1)) = \frac{1}{2} \sum_{i=1}^{k} (c_1(\alpha_i))^2 + \sum_{i<j} c_1(\alpha_i)c_1(\alpha_j) = \frac{1}{2}(\sum_{i=1}^{k} c_1(\alpha_i))^2.$$

**Theorem 3.4.** $M^6$ is a TNS-manifold iff

$$H^4(M^6; \mathbb{R}) = \mathbb{R}_{\geq 0}\langle (\sum_{i=1}^{m} a_ix_i)^2 | a_i = -1, 0, 1; \ i = 1, \ldots, m \rangle.$$

**Proof.** Due to Corollary 2.37 it is enough to prove that $ch_2(S(M^6)) = \mathbb{R}\langle (\sum_{i=1}^{m} a_ix_i)^2 | a_i = -1, 0, 1, \ i = 1, \ldots, m \rangle$. Corollary 2.20 and Proposition 2.23 imply that $S(M^6) = \mathbb{R}(t_{i_1} \ast \cdots \ast t_{i_k}) 1 \leq i_1 < \cdots < i_k \leq m$, $t_i = \theta_i^{\pm 1} - 1, \pm (\theta_i - 1)^3)$. Notice that for any $x \in K(M^6)$, $ch_2(x * (\pm(\theta_i - 1)^3)) = ch_2(x)$.

**Remark 3.5.** Let $v$ be a vertex of the polytope $P^3$. W.l.g. assume that $x_1, x_2, x_3$ correspond to the facets of $P^3$ meeting at $v$. The relations in $H^*(M^6; \mathbb{R})$ (see Theorem 2.22) imply that the right-hand side of the relation in Theorem 3.4 coincides identically with

$$\mathbb{R}_{\geq 0}\langle (\sum_{i=1}^{m} a_ix_i)^2 | a_i = -1, 0, 1; \ i = 4, \ldots, m \rangle.$$

This observation gives a further simplification of the TNS-criterion.

**Example 3.6.** One can see that the cones from Theorem 3.4 corresponding to toric manifolds with combinatorially equivalent moment polytopes are different. Let $P^3_1, P^3_2 \subset \mathbb{R}^3$ be the convex polytopes of the same combinatorial type as shown in Fig. 2 (namely, the connected sum of two cubes along vertices) with normal vectors

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix},$$

respectively. (Indices of the normal vectors of these polytopes are shown in Fig. 2). Both of these polytopes are obtained from the corresponding triangular prisms by truncation of two vertices and two edges. One can easily check that $P^3_1, P^3_2$ are Delzant polytopes. Consider the (smooth projective) toric manifolds $M^6_1, M^6_2$ of complex dimension 3 corresponding to the polytopes $P^3_1, P^3_2$, resp. Theorem 2.22 implies that $x_9, x_5, x_4, x_3, x_8, x_2, x_6, x_5, x_3$ and $x_2^2, x_8^2, x_5^2, x_6^2, x_3^2, x_2^2$ are the bases of $H^4(M_1; \mathbb{Q})$ and $H^4(M_2; \mathbb{Q})$, resp. One can calculate the cones from Theorem 3.4 using software programs (e.g. Sage, Singular). Namely, the cones $S(M_1), S(M_2)$ corresponding to $M_1, M_2$ have extreme rays (w.r.t. the above bases)

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
and have 4 and 7 facets, resp. In particular, $M_1, M_2$ are not TNS-manifolds.

**Corollary 3.7.** The cone $S(M^6)$ is polyhedral.

The analogue of Corollary 3.7 in higher dimensions does not hold, generally speaking. Due to Corollary 2.30 it is enough to give an example of a 8-quasitoric manifold $M^8$ with non-polyhedral cone $S(M^8)$.

**Example 3.8.** Let $M^8 = CP^2 \times CP^2$. In the denotations of Example 2.36 $ch_2(S(M^8)) = \mathbb{R}_{\geq 0}((ax + by)^2 | a, b \in \mathbb{R}) \subset H^4(M^8; \mathbb{R})$. The latter cone becomes the cone $\sigma = \mathbb{R}_{\geq 0}((1, 0, 0), (t^2, t, 1) | t \in \mathbb{R})$ after the suitable coordinate change in $H^4(M^8; \mathbb{R}) \simeq \mathbb{R}^3$. The cone $\sigma$ is not polyhedral. Hence, $S(M^8)$ is not polyhedral, as well.

# 4 Operations on quasitoric TNS-manifolds

The current Section contains some results on equivariant connected sum and TNS-property (Subsection 4.1). The blow-up along a complex codimension 2 invariant submanifold of a TNS toric manifold is generalised for quasitoric manifolds in Subsection 4.2.

## 4.1 Equivariant connected sum and quasitoric TNS-manifolds

We begin this Subsection with a recap on signs of fixed points of quasitoric manifolds. (See [8] Section 7.3.) Let $M^{2n}$ be a 2n-dimensional quasitoric manifold, and let $x \in M^{2n}$ be a fixed point under the natural action of the torus $T^n$. The fixed point $x$ is an intersection of pairwise different invariant submanifolds $M_{j_1}, \ldots, M_{j_n}$ of codimension 2 in $M^{2n}$. The sign $\sigma(x)$ of $M^{2n}$ at the fixed point $x$ is defined as 1, if the orientation of the real space $T_xM^{2n} \oplus \mathbb{R}^{2(m-n)}$ coincides with the orientation of the (realification) of the vector space $(\theta_{j_1} \oplus \cdots \oplus \theta_{j_n})_x$ determined by the orientation of the complex line bundles $\theta_{j_k}$, $k = 1, \ldots, n$, and is equal to $-1$, otherwise. One also has the formula

$$\sigma(x) = \det(\lambda_{j_1}, \ldots, \lambda_{j_n}),$$

provided that the respective inward-pointing normal vectors of the facets $F_{j_1}, \ldots, F_{j_n}$ of $P^n$ form a positive basis of $\mathbb{R}^n$.

For any two stably complex manifolds $M_1, M_2$ of dimension $2n$ there exists a natural stably complex structure on the connected sum $\tilde{M}_1 \sharp_{x_1,x_2} M_2$, $x_i \in M_i$, $i = 1, 2$ (see [8] Construction 9.1.11]). Choose a fixed point $x_i \in M_i$ of the quasitoric manifold $M_i^{2n}$ under the natural torus action, $i = 1, 2$. Then the respective connected sum is called an **equivariant connected sum** $\tilde{M}_1 \sharp_{x_1,x_2} M_2$ of the quasitoric manifolds $\tilde{M}_1 \sharp_{x_1,x_2} M_2$ along the fixed points $x_1, x_2$. It admits the action of the torus $(\mathbb{C}^\times)^n$ being a quasitoric manifold. The moment polytope of $\tilde{M}_1 \sharp_{x_1,x_2} M_2$ is equal to the connected sum of the moment polytopes corresponding to $M_1$ and $M_2$, at the vertices corresponding to the fixed points $x_1, x_2$. The characteristic matrix of $\tilde{M}_1 \sharp_{x_1,x_2} M_2$ is given explicitly in [8]. In order to define a (invariant) stably complex structure on $\tilde{M}_1 \sharp_{x_1,x_2} M_2$, one needs to endow this manifold with an orientation (for example,
given by the respective characteristic matrix). The manifold $M_1\overset{\sim}{\times} x_1, x_2 \overset{\sim}{\times} M_2$ is oriented diffeomorphic either to $M_1\overset{\sim}{\times} x_1, x_2 \overset{\sim}{\times} M_2$ or to $M_1\overset{\sim}{\times} x_1, x_2 \overset{\sim}{\times} M_2$. The restriction of the orientation on $M_1\overset{\sim}{\times} x_1, x_2 \overset{\sim}{\times} M_2$ to $M_1$ is equal to the orientation on $M_1$. Restricted to $M_2$, the orientation is equal or opposite to the orientation on $M_2$. In the first case, the introduced orientation on $M_1\overset{\sim}{\times} x_1, x_2 \overset{\sim}{\times} M_2$ is called compatible with the orientations on $M_i$, $i = 1, 2$.

**Proposition 4.1.** [8, Lemma 9.1.12] The equivariant connected sum $M_1\overset{\sim}{\times} x_1, x_2 \overset{\sim}{\times} M_2$ of omni-oriented quasitoric manifolds admits an orientation compatible with the orientations of $M_i$, $i = 1, 2$, if $\sigma(x_1) = -\sigma(x_2)$.

**Proposition 4.2.** Let $M_i^n$, $M_2^n$ be closed oriented manifolds of dimension $n$. Then the isomorphism of algebras

$$\tilde{H}^*(M_i^n X; \mathbb{R}) \cong (\tilde{H}^*(M_1^n; \mathbb{R}) \oplus \tilde{H}^*(X; \mathbb{R}))/I,$$

holds, where $X = M_2, \overline{M}_2$, $I = (D[*_{M_1}] - D[*X])$, and $D$ denotes the Poincaré duality operator. The multiplication between direct summands above is trivial in all dimensions.

**Proof.** The desired map is induced by the contraction map of the connected sum of oriented manifolds $M_1\overset{\sim}{\times} X \to M_1 \vee X$. 

**Proposition 4.3.** Consider the equivariant connected sum $M_i^{2n} = M_1^{2n} \oplus M_2^{2n}$ of quasitoric manifolds $M_i^{2n}, M_2^{2n}$ with the orientation compatible with the orientations on the manifolds $M_1^{2n}, M_2^{2n}$.

a) Let $n$ be odd. Then $M_i^{2n}$ is a TNS-manifold iff $M_i^{2n}$ is a TNS-manifold, $i = 1, 2$;

b) Let $n$ be even. Then $M_i^{2n}$ is a TNS-manifold iff the homogeneous forms $Q_a$, $a \in H^{2(n-2k)}(M_i^{2n}; \mathbb{Z})$ of degree $0 < k < [n/2]$ are admissible for $M_i$, $i = 1, 2$, and the sum of the forms $Q + Q'$ is admissible, where $Q : H^2(M_i^{2n}; \mathbb{Z}) \to \mathbb{Z}$, $Q' : H^2(M_i^{2n}; \mathbb{Z}) \to \mathbb{Z}$ are the homogeneous forms of the top degree corresponding to the units $1_{M_i^{2n}} \in H^0(M_i^{2n}; \mathbb{Z})$, $1_{M_i^{2n}} \in H^0(M_i^{2n}; \mathbb{Z})$ in sense of Proposition 2.34 respectively.

**Proof.** Follows from Corollary 2.37 and Proposition 4.2.

Now let $n = 2$.

**Lemma 4.4.** Let $(a_1, b_1)$ be the index of the intersection form of a closed, compact, oriented, simply-connected 4-manifold $M_1, i = 1, 2$. Then the indices of the intersection forms of the manifolds $M_1 \sharp M_2, M_1 \sharp \overline{M}_2$, are equal to $(a_1 + a_2, b_1 + b_2), (a_1 + b_2, b_1 + a_2)$, resp.

**Proof.** Due to Proposition 4.2, the intersection form of the manifold $M_1\overset{\sim}{\times} X$ is equal to the direct sum of the respective intersection forms of $M_1, X$, where $X = M_2, \overline{M}_2$. It remains to notice that the intersection form of $\overline{M}_2$ is equal to the minus intersection form of $M_2$.

The indices of the intersection forms of quasitoric manifolds $\mathbb{C}P^2, \overline{\mathbb{C}P^2}$ are equal to $(1, 0), (0, 1)$, resp. The signs of all fixed points of $\mathbb{C}P^2 (\overline{\mathbb{C}P^2}$, resp.) are equal to $1 (-1$, resp.). Define the quasitoric manifold $B^4 := \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. It does not depend on the choice of the fixed points.

Let us explicitly check the following (without using Theorem 1.3).

**Proposition 4.5.** $B^4$ is a TNS-manifold.

**Proof.** By Theorem 2.5 it is enough to prove that for any linear bundle $\xi \to B^4$ the stably inverse to $\xi \oplus \xi$ is totally split. Proposition 4.2 implies that

$$H^*(B^4; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^3, y^3, xy, x^2 + y^2).$$

Let $t := c_1(\xi)$. One has $t^2 = ax^2$ for the generator $x^2 \in H^4(B^4; \mathbb{Z})$ and some $a \in \mathbb{Z}$. Let $\eta_1, \eta_2$ be the pullbacks of the tautological line bundles from $\mathbb{C}P^2, \overline{\mathbb{C}P^2}$ to $B^4$, resp. One has $c_1^1(\eta_1) = x^2$, $c_1^1(\eta_2) = -x^2$. Consider three cases.

1) $a = 0$. Then $\xi \oplus \xi$ is trivial, and there is nothing to prove.

2) $a < 0$. Then clearly one has

$$ch([\xi \oplus \xi \oplus a(\eta_2 + \eta_2)]) = 2 + 2|a|.$$

3) $a > 0$. Then similarly one has

$$ch([\xi \oplus \xi \oplus a(\eta_1 + \eta_1)]) = 2 + 2a.$$

The statement now follows from Theorem 2.1.
The manifold $B^4$ has exactly 2 fixed points of each sign. Choose fixed points $x_+, x_- \in B^4$ s.t. $\sigma(x_+) = 1, \sigma(x_-) = -1$.

**Proposition 4.6.** Let $x \in M^4$ be a fixed point of an arbitrary quasitoric manifold $M^4$. Then the equivariant connected sum $M^4_{x,x} \cup B^4$ ($M^4_{x,x} \cup B^4$, resp.) is a TNS-manifold, if $\sigma(x) = -1$ ($\sigma(x) = 1$, resp.). The respective orientation is compatible with the orientations on $M^4, B^4$.

### 4.2 Equivariant blow-ups and quasitoric TNS-manifolds

Let $M^{2n} = M(P, \Lambda)$ be the model of the quasitoric manifold given by a characteristic pair $(P, \Lambda)$. For any vertex $v = F_{i_1} \cap \cdots \cap F_{i_n} \in P$ the column vectors $\lambda_{i_1}, \ldots, \lambda_{i_n}$ of the characteristic $(n \times m)$-matrix

\[ \Lambda = (\lambda_i^j) \]

generate the cone $\sigma_v = \text{cone}(\lambda_{i_1}, \ldots, \lambda_{i_n}) \subset N \otimes \mathbb{R}$.

Consider two characteristic pairs $(P, \Lambda), (P', \Lambda')$. Let $N, N'$ be the lattices generated by the column vectors of $\Lambda, \Lambda'$, respectively. A weakly equivariant morphism $\varphi : M(P, \Lambda) \to M(P', \Lambda')$ between two quasitoric manifolds is by definition given by the lattice map $\psi : N \to N'$ s.t. for any vertex $v \in P$ there exists a vertex $v' \in P'$ with $\psi(\sigma_v) \subset \sigma_{v'}$.

Let $Z \subset M$ be a characteristic submanifold of $M$ of codimension 2$k$ corresponding to the face $G = F_{i_1} \cap \cdots \cap F_{i_k} \subset P$. Consider the Delzant polytope $\tilde{P} = \text{cut}_G P$ which is the truncation of the face $G$ of $P$, and a characteristic matrix $\Lambda$ obtained from $\Lambda$ by adding the vector $\lambda = \lambda_{i_1} + \cdots + \lambda_{i_k}$ corresponding to the truncation facet of $\tilde{P}$. The identity map on the lattices generated by the columns of $\Lambda, \Lambda'$ gives rise to a weakly equivariant morphism $\pi : M(\tilde{P}, \tilde{\Lambda}) \to M(P, \Lambda)$.

**Definition 4.7.** The weakly equivariant morphism $\pi : \text{Bl}_{Z(\alpha, -\beta)} M^{2n} = M(\tilde{P}, \tilde{\Lambda}) \to M(P, \Lambda)$ of quasitoric manifolds is called an equivariant blow-up of a quasitoric manifold $M$ at a characteristic submanifold $Z \subset M$.

**Proposition 4.8.** Let $Z \subset M$ be a characteristic submanifold of codimension 4 in a quasitoric TNS-manifold $M$. Then the equivariant blow-up $\text{Bl}_Z M^{2n}$ is a TNS-manifold.

**Proof.** Let $\tilde{\theta}, \tilde{\theta}_1, \ldots, \tilde{\theta}_m \to M^{2n}$ be the complex linear bundles corresponding to the truncation facet and all other facets of $\text{cut}_G P$ in the same order as in $P$, respectively. Due to Theorem 2.10 (ii), it is enough to check that any of the linear bundles $\tilde{\theta}, \tilde{\theta}_1, \ldots, \tilde{\theta}_m$ has a totally split stably inverse. One has $\tilde{\theta}_i = \pi^* \theta_i$ for any $i = 1, \ldots, m$. Hence, $\tilde{\theta}_i$ has a totally split stably inverse for any $i = 1, \ldots, m$ (see Proposition 2.9).

It remains to deal with the stably inverse to $\tilde{\theta}$. Let $y_i := c_1(\tilde{\theta}_i), y := c_1(\tilde{\theta})$. Consider the face $G = F_{i_1} \cap F_{i_2} \subset P^n$. W.l.g. we may assume that $\lambda_{i_1} = (1, 0, 0, \ldots, 0)^T, \lambda_{i_2} = (0, 1, 0, \ldots, 0)^T$ by using the necessary coordinate change. One has relations $y_1 y_2 = 0, y_1 = -y + a, y_2 = -y + b$ in the cohomology ring $H^*(\text{Bl}_Z M; \mathbb{Z})$, where $a = -\sum_{i=3}^m \lambda_i^1 y_i, b = -\sum_{i=3}^m \lambda_i^2 y_i$. Indeed, the first relation follows from the void intersection of the facets $F_{i_1}, F_{i_2}$ of $\tilde{P}$, and the other relations are corollaries of the linear relations in $H^*(\text{Bl}_Z M; \mathbb{Z})$. Let $\alpha, \beta \to \text{Bl}_Z M$ be the complex linear bundles s.t. $c_1(\alpha) = a, c_1(\beta) = b$. Observe that $c(\tilde{\theta}(\alpha \oplus \beta)) = (1 + y + a)(1 + y + b) = 1 + 2y + a + b = c(\tilde{\theta}^2 \alpha \beta)$, hence $\tilde{\theta}(\alpha \oplus \beta) \simeq \tilde{\theta}^2 \alpha \beta \oplus C^1$. We conclude that $\tilde{\theta}(\alpha \oplus \beta) \oplus \overline{\tilde{\theta}(\alpha \oplus \beta)} \simeq C^1$, i.e. the stably inverse to $\tilde{\theta} \alpha$ totally splits. By Proposition 2.9 the bundles $\alpha, \beta$ have totally split stably inverses. Applying Proposition 2.9 again to the tensor product of $\theta \alpha$ and $\pi$ we obtain the required.

## 5 Smooth projective toric TNS-manifolds

The main result of this Section is Theorem 5.12 describing the properties of the moment polytope for a smooth projective toric TNS-manifold.

Let $M^{2n} = M(P, \Lambda)$ be a non-singular projective toric variety of complex dimension $n > 2$. Let $P^n \subset \mathbb{R}^n$ be the respective moment polytope of $M^{2n}$. Let $F_i$ be the facets of $P^n$ with the respective normal vectors $\lambda_i = (\lambda_i^1, \lambda_i^2, \ldots, \lambda_i^n)^T \in \mathbb{R}^n, i = 1, \ldots, m$. Let $K$ be the simplicial complex corresponding to $P^n$ (i.e. the face lattice of the simplicial sphere $P^* \subset (\mathbb{R}^n)^*$). Consider a minimal missing face of $K$ of cardinality $k$. W.l.g. we may assume that vertices of that missing face correspond to the facets $F_1, \ldots, F_k \subset P$: $F_1 \cap \cdots \cap F_i \cap \cdots \cap F_k \neq \emptyset, 1 \leq i \leq k, F_1 \cap \cdots \cap F_k = \emptyset$. 

14
Proposition 5.1. Let $k > 2$. Then the matrix $(\lambda_1, \ldots, \lambda_k)$ is $GL_n(\mathbb{Z})$-equivalent either to

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}
or to
\begin{pmatrix}
1 & 0 & \ldots & 0 & -1 \\
0 & 1 & \ldots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & -1 \\
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
\]

Proof. Let $i > k$ be a facet index s.t. $F_1 \cap \cdots \cap F_{k-1} \cap F_i$ contains a vertex of $P$. Consider the standard unitary basis $e_1, \ldots, e_n \in \mathbb{R}^n$, $e_i = \delta_i^j$. Since $P^n$ is a Delzant polytope, there is a $GL_n(\mathbb{Z})$-transform taking $\lambda_1, \ldots, \lambda_{k-1}$ to the vectors $e_1, \ldots, e_{k-1}$, resp. Clearly, the matrix $(\lambda_1, \ldots, \lambda_k)$ has rank $k - 1$ or $k$. In the first case, one has $\lambda_k = (a_1, \ldots, a_{k-1}, 0, \ldots, 0)^T$ for some $a_i \in \mathbb{Z}$, $i = 1, \ldots, k - 1$. For any $i = 1, \ldots, k - 1$ there is $j > k$, s.t. $F_1 \cap \cdots \cap F_i \cap \cdots \cap F_k \cap F_j$ has a vertex. Then the Delzant condition implies $a_ic_i = 1$ for some $c_i \in \mathbb{Z}$. Hence, $a_i = \pm 1$ for $i = 1, \ldots, k - 1$. Since $\eta_k$ is an inward-pointing normal vector to $F_i$, one has $a_1 = \cdots = a_k = -1$.

Now suppose that $rk(\lambda_1, \ldots, \lambda_k) = k$. Applying the corresponding $GL_n(\mathbb{Z})$-transform to $P^n$ one obtains $\eta_k = ce_k$ for some $c \in \mathbb{Z}$. The Delzant condition implies $cd = -1$ for some $d \in \mathbb{Z}$. Hence $c = \pm 1$. Applying the corresponding $GL_n(\mathbb{Z})$-transform one reduces the matrix to the desired form. \qed

Let $n = k = 3$. Clearly, $\{1, \ldots, m\} = \{1, 2, 3\} \cup S_1 \cup S_2$, where $S_1, S_2$ are s.t. $\cup_{i \in S_1} F_i$ and $\cup_{j \in S_2} F_j$ have empty intersection.

Consider the case $(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Proposition 5.2. There exist $i \in \{1, 2\}$ s.t. for any $p \in S_i$, $k = 1, 2, 3$ one has $\lambda_p^k \geq 0$.

Proof. Denote by $x \in \mathbb{R}^3$ the common vertex belonging to planes containing the facets $F_1, F_2, F_3$. Consider the minimal convex polyhedral cone $\sigma \subset \mathbb{R}^3$ containing $P^3$ with origin in $x$. Choose indices $\{i, j\} = \{1, 2\}$ s.t. $\cup_{p \in S_i} F_p$ is the closest part to $x$. (More precisely, there exists a dividing plane $\{H = c\}$ s.t. $\cup_{p \in S_i} F_p \cap \{H < c\} = \cup_{p \in S_i} F_p$ and $\cup_{p \in S_i} F_p \cap \{H > c\} = \emptyset$.) For any facet $F_p, p \in S_i$, the corresponding plane intersects the interiors of all 3 facets of $\sigma$. Hence, the normal vector $\eta_p$ belongs to the cone generated by $(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T$, and has only non-negative coordinates. \qed

Denote by $S_+, S_\pm$ the sets $S_i, S_j$ from the Proposition 5.1 resp. (By abuse of the notation, the values $i, j$ are now undefined.) The characteristic matrix of $M^6$ has form $(Id_3 | A | B)$, where $A = (a_{ij})$, $a_{ij} \geq 0$, $B = (b_{ij})$. By the definition, $M^6$ is the equivariant connected sum of quasitoric manifolds $M^6_+, M^6_\pm$, where $M^6_+ = M(P_+, (Id_3 | A))$, $M^6_\pm = M(P_\pm, (Id_3 | B))$, and $P_+, P_\pm \subset \mathbb{R}^3$ are the respective combinatorial convex polytopes. Let $F_{+,i}$ be the facets of $P_+$ corresponding to $F_i \in P^3$, $i \in \{1, 2, 3\} \cup S_+$. Let $l_i := \sum_{j \in S_+} a_{ij}^2 x_j \in H^2(M^6; \mathbb{Z}), i = 1, 2, 3$.

Proposition 5.3. The quadratic forms $Q_{\ell_i}$ are non-trivial and positive-semidefinite for $i = 1, 2, 3$.

The proof is given below. Consider the characteristic submanifold $M^4_{+,i}$ of $M^6_+$ corresponding to the facet $F_{+,i}$, $i = 1, 2, 3$. The corresponding moment polygone and characteristic matrix are denoted by $P_{+,i}^4, \Lambda_{+,i}$ resp. The matrix $\Lambda_{+,i}$ is equal to the submatrix of $\Lambda$ consisting of the rows complement to the $i$-th and columns with indices $(\{1, 2, 3\} \setminus \{i\}) \cup \{k \in S_\pm | F_i \cap F_k \neq \emptyset\}$ (see [5]).

Proposition 5.4. [8] Lemma 7.3.19 For any non-singular projective toric variety $X$ and any fixed point $x \in X$ one has $\sigma(x) = 1$.

Lemma 5.5. For any index $i = 1, 2, 3$ and a vertex $v \in P_{+,i}^4$, one has

$$\sigma(v) = \begin{cases} -1, \text{ if } v \text{ is different from } F_{+,1} \cap F_{+,2} \cap F_{+,3} \text{ and incident to it,} \\
1, \text{ otherwise.} \end{cases}$$

15
Proof. \( \) W.l.g. let \( i = 1 \). Proposition 5.4 and Formula (8) imply that for any vertex \( v \in P_{i=1}^4 \) different from \( F_{i=1} \cap F_{i=2} \cap F_{i=3} \) and not incident with the latter, one has \( \sigma(v) = 1 \). The characteristic vectors of the edges \( F_{i=1} \cap F_{i=2} \cap F_{i=3} \) are outward-pointing, whereas the characteristic vectors of all other edges in \( P_{i=1}^4 \) are inward-pointing. This implies the desired identities on the signs of the remaining 3 vertices of \( P_{i=1}^4 \).

Remind that the index \( \text{ind}_v(v) \) of a vertex \( v = F_{i_1} \cap \cdots \cap F_{i_n} \in P^n \) of the moment polytope w.r.t. the generic vector \( \nu \in \mathbb{N} \otimes \mathbb{R}^n \) is by definition equal to the number of negative scalar products of \( \nu \) with the conjugate basis to \( \lambda_1, \ldots, \lambda_n \) (see [8] Section 7.3).

**Proposition 5.6.** [8] Theorem 9.4.8]

\[
\sigma(M^{4n}) = \sum_{v \in M^F} (-1)^{\text{ind}_v(v)} \sigma(v).
\]

**Lemma 5.7.** Let \( i = 1, 2, 3 \). Then there exists \( \nu \in \mathbb{R}^2 \) (in general position) and a vertex \( w \in P_{i=1}^4 \), \( w \neq F_{i=1} \cap F_{i=2} \cap F_{i=3} \), s.t. for any vertex \( v \in P_{i=1}^4 \), one has \( \text{ind}_v(v) = 0 \), if \( v = F_{i=1} \cap F_{i=2} \cap F_{i=3} \) or \( v = w \), and \( \text{ind}_v(v) = 1 \), otherwise.

**Proposition 5.8.** For any \( i = 1, 2, 3 \), \( M^{i=1} \) is not a TNS-manifold.

**Proof.** It follows from Lemmas 5.3, 5.7 and Proposition 5.6 that \( \sigma(M^{i=1}) = (m - 2)(-1)^1 \cdot 1 + (-1)^0 \cdot 1 + (-1)^0 \cdot (-1) = 2 - m = - \dim H^2(M^{i=1}; \mathbb{R}) \). It remains to use Theorem 1.3.

Now suppose that \( (\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \). We return to the notation \( S_1, S_2 \subset \{1, \ldots, m\} \).

Let \( l := \sum_{i \in S_1} \lambda_i^3 x_i, l' := \sum_{i \in S_2} \lambda_i^3 x_i \).

**Proposition 5.9.** The quadratic form \( Q_l \) has rank 1. In particular, it is semidefinite.

**Proof.** Theorem 2.2 implies that one has \( l = -l' \) in the cohomology ring of \( M^6 \) and \( \dim(x_3, \ldots, x_m) = m - 3 \). Hence, it is the only linear dependence between the elements \( x_3, \ldots, x_m \) in \( H^2(M^6; \mathbb{R}) \) up to multiplication by scalars. Then for any \( x \in H^2(M^6; \mathbb{Z}) \), \( Q_l(x) = -Q_l(x) \) holds. For any \( i \in S_1, j \in S_2 \) one has \( F_i \cap F_j = \emptyset \), thus \( x_i x_j = 0 \) (see Theorem 2.2). Now it follows that for any \( x \in \mathbb{R}(x_i \in S_1 \cup S_2) \) : \( Q_l(x) = 0 \). The Poincaré duality implies that \( Q_l \neq 0 \). We conclude that \( Q_l \) has rank 1, taking a non-zero value on \( x_3 \).

**Lemma 5.10.** Let \( n, k > 2 \). Suppose that \( P^n \) has facets \( F_1, \ldots, F_k \) corresponding to a minimal missing face of the corresponding face lattice on \( P^n \). Then for any \( i = 1, \ldots, k \) the face lattice on the polytope \( F_i \) has a minimal missing face \( F_1 \cap F_i, \ldots, F_i \cap F_k, F_k \cap F_i \) of cardinality \( k - 1 \).

**Proof.** Follows from the fact that the intersection of any two facets of a simple convex \( n \)-polytope (\( n > 2 \)) is either empty or has codimension 2.

**Lemma 5.11.** [22] A simple convex polyhedron \( P^3 \) is flag iff \( P^3 \) has no 3-belts and \( P \neq \Delta^3 \).

The results of this Subsection imply the following

**Theorem 5.12.** Let \( M^{2n} \) be a nonsingular projective toric variety of complex dimension \( n \). Suppose that \( M^{2n} \) is a TNS-manifold. Then the face lattice of the moment polytope \( P^n \) has no minimal missing faces of cardinality \( n \). In particular, the moment polytope \( P^3 \) of a toric 3-dimensional TNS-manifold is a flag polytope.

**Example 5.13.** Consider the toric manifolds \( M^6_1, M^6_2 \) from Example 3.6. Then the indices of the quadratic forms \( Q_{x_3+x_5+x_6}, Q_{x_4+2x_5+x_6}, Q_{2x_4+x_5+x_6} \) are equal to \( (2, 0), (2, 0), (2, 0) \) and \( (1, 0), (2, 1), (2, 1) \), resp.
6 Concluding remarks

Proposition 3.1 and Theorem 5.12 hint that there is a possible connection between the combinatorial type of the moment polytope of a smooth projective toric variety and the respective TNS-property.

**Conjecture 6.1.** Let \( X, Y \) be non-singular projective toric varieties of complex dimension \( n \) with combinatorially equivalent moment polytopes. Then \( X \) is a TNS-manifold iff \( Y \) is a TNS-manifold.

Proposition 3.2 shows that the analogue of Conjecture 6.1 in the category of quasitoric manifolds is, generally speaking, false in any dimension greater than two. Indeed, one has to consider the product of one of the manifolds \( M \) is, generally speaking, false in any dimension greater than two. Indeed, one has to consider the product of one of the manifolds \( M \) with \((\mathbb{C}P^1)^{n-2}\). The obtained manifold then is a quasitoric non-TNS manifold over the cube \( I^n \). But \((\mathbb{C}P^1)^n\) is also a (quasi)toric manifold over \( I^n \) being a TNS-manifold. Theorem 5.12 also allows to pose the following

**Conjecture 6.2.** Let \( M^{2n} \) be a nonsingular projective toric variety of complex dimension \( n \). Then \( M^{2n} \) is a TNS-manifold iff the moment polytope \( P^n \) of \( M^{2n} \) is a flag polytope.

A convex \( n \)-polytope \( P^n \subset \mathbb{R}^n \) is flag iff the corresponding face lattice of the moment polytope \( P^n \) has minimal missing faces only of cardinality 2. So, in order to study the above conjectures in real dimension 8, one has to find a Delzant 4-polytope \( P^4 \) having only facets with no 3-belts or triangles, and \( P^4 \) having a minimal missing face of cardinality 3. Such an example is not known to the author. It is also plausible to expect the future proofs of the above conjectures to rely on the existence of the complex/algebraic structure on the respective toric variety. In connection with this we mention the different well-known descriptions of the K-theory ring of a toric variety obtained by Pukhlikov and Khovanskii [15], Morelli [12] and Klyachko.

The relation between the top-degree form in the TNS-criterion (Theorem 1.3) and the volume \( V(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m] \) of degree \( d \geq 0 \) the algebra

\[
A(V) := DOp_\mathbb{R}(\mathbb{R}^m)/AnnV
\]

is a Poincaré algebra of virtual rank \( d \) (see [21]), where \( DOp_\mathbb{R}(\mathbb{R}^m) \) is the algebra of differential operators in \( m \) variables with real constant coefficients and \( AnnV \) is the annihilator ideal of \( V \). The natural grading on the algebra \( DOp_\mathbb{R}(\mathbb{R}^m) \) induces the grading on \( A(V) = A_*(V) = \bigoplus_{d=0}^{\infty} A_d(V) \). The following theorem was formulated by Pukhlikov and Khovanskii in case of a smooth projective toric variety \( M^{2n} \) and its respective fan \( \mathcal{F} \) (in terms of the Chow ring, [16]). It was proved by Timorin [21]. In case of a quasitoric manifold \( M^{2n} \) it follows from the results of Ayzenberg and Matsuda [4] about the dual algebra of a multifan, in the particular case of a simplicial complex on a sphere.

**Theorem 6.3.** [4, Theorem 8.2] Let \( M^{2n} \) be a quasitoric manifold with the multifan \( \mathcal{F} \subset \mathbb{R}^n \) having \( m \) rays. Then the isomorphism of algebras

\[
H^*(M^{2n}; \mathbb{R}) \cong A_*(V_\mathcal{F}), \quad a \mapsto D_a,
\]

holds. The canonical pairing \( \langle a, [M^{2n}] \rangle, \quad a \in H^{2n}(M^{2n}; \mathbb{R}) \), coincides with the evaluating of \( D_aV_\mathcal{F} \).

One has

**Lemma 6.4.** [4, p.19] Consider a homogeneous polynomial \( V(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m] \) of degree \( d \geq 0 \). Let \( c_1, \ldots, c_m \in \mathbb{R} \). Then for the linear differential operator \( D^d_\xi := c_1 \partial_{x_1} + \cdots + c_m \partial_{x_m}, \quad \xi = (c_1, \ldots, c_m) \), the formula

\[
D^d_\xi V = d!V(c_1, \ldots, c_m),
\]

holds.

Define the homogeneous form \( Q_\alpha : A_1(V_\mathcal{F}) \rightarrow \mathbb{R} \) of degree \( k \) by the formula \( Q_\alpha(x) := \alpha x^kV_\mathcal{F} \), where \( k = 1, \ldots, n \) and \( \alpha \in A_{d-k}(V_\mathcal{F}) \). Theorem 6.3 and Lemma 6.4 allow us to reformulate the TNS-criterion (Theorem 1.3) in the following way.
Theorem 6.5. Let $M^{2n}$ be a quasitoric manifold with the multifan $\mathcal{T} \subset \mathbb{R}^n$ having $m$ fans. Then the following conditions are equivalent:

(i) $M^{2n}$ is a TNS-manifold;

(ii) One has
\[ \mathbb{R}_{\geq 0} \langle x^k \mid x \in A_1(V) \rangle = A_k(V), \]
where $k = 1, \ldots, n$;

(iii) The homogeneous $k$-form $Q_\alpha$ is admissible for any $k = 1, \ldots, n$ and $\alpha \in A_{n-k}(V)$;

(iv) For any homogeneous differential operator $D \in DOP_{\mathbb{R}}(\mathbb{R}^m)$, $\deg D = k$, if the polynomial $DV_{\alpha}$ is non-zero, then $DV_{\alpha}$ takes values of opposite signs, where $k = 0, \ldots, n - 1$.

We remark that an analogue of some equivalences in the above Theorem 6.5 takes place for any homogeneous polynomial $V(x_1, \ldots, x_m)$ of degree $d$ with real coefficients.

Theorem 6.6. Let $V(x_1, \ldots, x_m)$ be a homogeneous polynomial of degree $d$ with real coefficients. Then the following conditions are equivalent:

(i) One has
\[ \mathbb{R}_{\geq 0} \langle x^k \mid x \in A_1(V) \rangle = A_k(V), \]
where $k = 1, \ldots, d$;

(ii) The homogeneous form $Q_\alpha : A_1(V) \to \mathbb{R}$ is admissible for any $k = 1, \ldots, d$ and $\alpha \in A_{d-k}(V)$;

(iii) For any homogeneous differential operator $D \in DOP_{\mathbb{R}}(\mathbb{R}^m)$, $\deg D = k$, if the polynomial $DV_{\alpha}$ is non-zero, then $DV_{\alpha}$ takes values of opposite signs, where $k = 0, \ldots, d - 1$.

Proof. The equivalence $(ii) \iff (iii)$ follows from Lemma 6.4. The equivalence $(i) \iff (ii)$ is straightforward to show using the arguments of Subsection 2.4. In order to do that, one has to use two facts about the algebra $A(V)$. First, $A(V)$ is a Poincaré algebra (see [21]). Second, for any $k = 1, \ldots, d$ one has $A_k(V) = \mathbb{R} \langle x^k \mid x \in A_1(V) \rangle$. The last identity is a consequence of the easily shown formula $(-1)^r \sum_{i \in \{1, \ldots, r\}} (-1)^{|i|} \sum_{i \in \{1, \ldots, r\}}$, taking place in the polynomial algebra $\mathbb{R}[y_1, \ldots, y_r]$, $r \in \mathbb{N}$.

The next observation was suggested by A. Ayzenberg.

Corollary 6.7. The condition $(iii)$ of Theorem 6.6 (and the condition $(iv)$ of Theorem 6.5) is algorithmically verifiable.

Proof. It is clear that this condition could be written as a closed arithmetic formula of first order on the coefficients $d_{i_1, \ldots, i_m}$ of the differential operators:

\[ \forall k \in \{0, \ldots, n - 1\} \forall \{d_{i_1, \ldots, i_m} \mid i_1 + \cdots + i_m = k\} : \]

\[ \sum_{i_1 + \cdots + i_m = k} d_{i_1, \ldots, i_m} \partial_1^{i_1} \cdots \partial_m^{i_m} V_{\alpha} \neq 0 \Rightarrow \]

\[ \left( - \sum_{i_1 + \cdots + i_m = k} d_{i_1, \ldots, i_m} \partial_1^{i_1} \cdots \partial_m^{i_m} V_{\alpha} \geq 0 \right) \land \left( - \sum_{i_1 + \cdots + i_m = k} d_{i_1, \ldots, i_m} \partial_1^{i_1} \cdots \partial_m^{i_m} V_{\alpha} \leq 0 \right). \]

Hence, the claim follows now from Tarski algorithm [20].

A 17-th Hilbert’s problem-type question for finite-dimensional algebras rises (see [14], Chapter 7).

Problem. Describe explicitly the family of homogeneous polynomials $V(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m]$ of degree $d \geq 2$, $m \geq 1$, satisfying the conditions of Theorem 6.6.

It is natural to conjecture that the condition $(iii)$ of Theorem 6.6 is equivalent to the condition on the different signs of non-zero quadratic forms $DV_{\alpha}$, $D \in DOP_{\mathbb{R}}(\mathbb{R}^m)$. This follows from the following Conjecture about real psd-forms.

Conjecture 6.8. Consider a homogeneous polynomial $V(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m]$ of degree $d \geq 3$, $m \geq 1$. Let $V$ be a psd-form, i.e. for any $x_1, \ldots, x_m \in \mathbb{R}$, $V(x_1, \ldots, x_m) \geq 0$. Then there exists a homogeneous differential operator $D \in DOP_{\mathbb{R}}(\mathbb{R}^m)$ with constant real coefficients, $0 < \deg D < d$, s.t. the polynomial $DV$ is a non-zero psd-form.
Acknowledgements

V.M. Buchstaber provided an invaluable assistance on all stages of the conducted research and preparation of the present paper. I am grateful to A.A. Ayzenberg and T.E. Panov for different valuable remarks, N.Y. Erokhovets for the communication on the combinatorics of convex polytopes and V.A. Kirichenko for the remarks on the volume polynomial. Being the “Young Russian Mathematics” award winner, I would like to thank its sponsors and jury.

References

[1] Rob Arthan and Shaun Bullet, The homology of $MO(1)^{\infty}$ and $MU(1)^{\infty}$, Journal of Pure and Applied Algebra 26 (1982), 229–234.

[2] Michael Atiyah, Lectures on $K$-theory, W.A. Benjamin Inc., 1967.

[3] Michael Atiyah and Fridriech Hirzebruch, Vector Bundles and Homogeneous Spaces, AMS Symposium in Pure Math. III (1960), 197–222.

[4] Anton Ayzenberg and Mikiya Masuda, Volume polynomials and duality algebras of multi-fans, Arnold Math. J. 2 (2016), 329–381, available at arXiv:1509.03008[math.CO].

[5] Tristram Bogart, Mark Contois, and Joseph Gubeladze, Hom-polytopes, Mathematische Zeitschrift 273 (2013), no. 3-4, 1267–1296.

[6] Victor M. Buchstaber and Vadim D. Volodin, Sharp upper and lower bounds for nestohedra, Izv. Math. 75 (2011), 1107–1133.

[7] ______, Combinatorial 2-truncated Cubes and Applications, Associahedra, Tamari Lattices and Related Structures, Springer, Basel, 2012, pp. 161–186.

[8] Victor M. Buchstaber and Taras E. Panov, Toric topology, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015.

[9] Pierre Conner and Edwin Floyd, The relation of Cobordism to $K$-theories, Lecture Notes in Math., vol. 28, Springer, Berlin, 1966.

[10] Michael W. Davis and Tadeusz Januszkiewicz, Convex polytopes, Coxeter orbifolds and torus actions, Duke Math. J. 62 (1991), no. 2, 417–451.

[11] June Huh, Rota’s conjecture and positivity of algebraic cycles in permutohedral varieties, Ph.D. Thesis, The University of Michigan, 2014.

[12] Robert Morelli, The $K$ theory of a toric variety, Adv. Math. 100 (1993), 154–182.

[13] S. Ochanine and L. Schwartz, Une remarque sur les générateurs du cobordisme complexe, Mathematische Zeitschrift 190 (1985), 543–557.

[14] Victor Prasolov, Polynomials, Algorithms and Computation in Mathematics, vol. 11, Springer-Verlag Berlin Heidelberg, 2004.

[15] A.V. Pukhlikov and A.G. Khovanskii, Finitely additive measures of virtual polyhedra, Algebra i Analiz 4 (1992), 337–356.

[16] ______, The Riemann–Roch theorem for integrals and sums of quasipolynomials on virtual polytopes, St. Petersburg Mathematical Journal 4 (1993), no. 2, 337–356.

[17] Nigel Ray, On a construction in bordism theory, Proc. Edinburgh Mat. Soc. 29 (1986), 413–422.

[18] P. Sankaran and V. Uma, K-theory of quasi-toric manifolds, Osaka J. Math. 44 (2007), 71–89.

[19] G. Solomadin, Quasitoric totally normally split representatives in unitary cobordism ring, Preprint (2018), available at arXiv:1704.07403[math.AT]. To appear in: Math. Notes.

[20] A. Tarski, A decision method for elementary algebra and geometry, The RAND Corp., 1948. 2-nd ed.

[21] Vladlen Timorin, An analogue of the Hodge–Riemann relations for simple convex polytopes, Russian Mathematical Surveys 54 (1999), 381–426.

[22] Gunther Ziegler, Lectures on Polytopes, Graduate Texts in Math., V.152, New York: Springer-Verlag, 1995.