Generalized Alignment of Gravitational Intencities and Electromagnetic Strengths in Kerr-Newman Space-Time

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Abstract

It is shown that, in the case of Kerr-Newman space-time, the complex electromagnetic strength \( \vec{E} + i \vec{H} \) and analogous complex intensity of the gravitational field \( \vec{F}_{\text{attr}} - \frac{i}{2} \vec{F}_{\text{gyr}} \) share the common complexified spatial direction.

1 Introduction

Considering interrelation of the Newtonian physics and the special relativity, it is instructive to mention that there arise no obstacles standing in the way of the relativistic generalizing the notion of a force in a fully coherent way, provided the basic theory is re-formulated in a Lorentz-invariant form. In mechanics, considering motion of a test particle, the force 4-vector proportional to the particle 4-acceleration is used to obtain the generalized relativistic definition of a force which exhibits a proper non-relativistic limit. Accordingly, turning to the field theory, a strength of a physical field, affecting the motion of particles which carry the corresponding ‘charge’, can be introduced on the base of the relevant specific forces. (Electromagnetic field is the obvious and most important instance where the above interpretation perfectly applies.)

A situation proves to be considerably more subtle when a similar transition from the flat space-time to a curved one is undertaken. Of course, considering motion of a classical test particle, both the 4-acceleration and the corresponding force 4-vector make sense and might be utilized for the extending of the force dynamics to the case of arbitrary curved space-time. However the manifestations of a 4-force alone does not exhaust the total effect the forces applied to a particle are responsible for. Specifically, admitting the 4-force to
represent the resulting extrinsic influence to motion of a particle (i.e. the action of all the relevant physical fields living in the space-time), the effect of the gravitational field proves to be missed — or, in the best case, characterized only indirectly and highly incompletely through the specific influence of the other fields affected by the gravitational distortion of the space-time geometry.

Indeed, whenever the particle is free, i.e. its worldline is geodetic and the 4-acceleration identically vanishes, it often should not be regarded to be unaffected by any force at all (as the obvious example of the sucking in a black hole demonstrates; less academic one is provided by the effect of accretion studied in astrophysics, see, e.g., Refs. [1]). Definitely, it seems more physically adequate to consider a freely falling particle to be, in a sense, ‘accelerated’ (‘dragged’?) in the gravitational field described by the curved space-time. Accordingly, an introduction of the corresponding gravitational force has to be regarded a meaningful problem.

It is worth noting here that, generally speaking, there still exist just opposite attitudes to the notion of a gravitational force. Specifically, the latter may be regarded as an obsolete speculative concept justified, in the best case, by the out-of-date habit alone. Indeed, working exclusively within framework of the 4-dimensional theory, a gravitational force is obviously a superfluous notion. However one may also interpret gravitational force as an inherent property of the corresponding physical field despite that the particular state of the motion of observers heavily affects such a characteristic and even may, in a sense, apparently eliminate it. The example of electromagnetism, where the energy and the momentum of the field are the functionals over the observable electric and magnetic strengths, may be regarded as an indirect argument in favor of the potential plausibility of such a relationship.

Not needing to predetermine a final judgement on the above dilemma, it is anyway worth analyzing the possibility of the consistent extending the concept of gravitational force to the case of a strong space-time curvature. It is especially important to follow a way strictly concordant with the weak field limit and to operate within physical framework (as opposed to formal mathematical considerations).

It is worth noting that investigations in the fields bearing a relation to the issue mentioned are of a comparatively long history. The latter is briefly reviewed in Ref. [2] where the extended list of references is given, see also Refs. [4], [3] and references therein.

Nevertheless the problem seems to be not properly cleared up. The goal pursued in the present work is to propound a consistent method of the description of the ‘true’ gravitational field which suits for arbitrarily strong gravitational field and simultaneously reveals the proper Newtonian limit. We shall see that it entails some new insight into the problem of description of observable manifestations of the gravitational field.

Generally speaking, the realization of the dynamical description of the gravitational field (i.e., here, the one referring to the concepts of forces and field intensities) is more intricate than the analogous treatments of the other classical fields, the electromagnetic one being a primary working object for a collating, of course. Concerning the physics, the two reasons should be mentioned in this respect.

The first intrinsic cause of troubles is the well known unavoidable co-existence of grav-
itutional forces in conjunction with inertial ones. Accordingly, attempting to describe the dynamical properties of the gravitational field itself, one has to be able to unambiguously distinguish the manifestations of the gravity from ones of the inertia. At the same time, within framework of the general relativity, the meaning of the separating the gravitational and inertial phenomena remains to be not satisfactorily understood. A conventional principle of the distinction of gravitational and inertial forces has not been clearly formulated; the formal analogues, mostly of mathematical origin (or even looking sometimes like terminological tricks), being used in practice instead. Sometimes, realizing the explicit splitting of forces into gravitational and inertial constituents, such a procedure is regarded as an artificial trick carried out after *ad hoc* fashion for the sake of convenience or a qualitative visualization of the covariant equations at most.

There is however a standpoint supposing an intrinsic character of the separation of the gravitational field and inertia manifestations within their combined ‘gravitational-inertial bundle’. Although the broadly interpreted principle of equivalence, see Ref. [6], states that the gravity and the inertia reveal themselves in local effects in indistinguishable ways (cf. Ref. [43]), it may nevertheless be supposed that there exist some their aspects which make them physically non-equivalent.

Specifically, there exist today no reasons (at least, for a theorist) to doubt that gravitational field possesses some own energy ‘separated’, in a sense, from the own energy of gravitating bodies. Moreover, this energy is able to be born across the space in a form of gravitational waves and, then, to be partially handed over to remote physical objects (*e.g.*, to a gravitational antenna). On the other hand, the inertia manifestations, heavily dependent on the motion of observers, surely cannot be associated with any such a long scale *energy transfer* detectable by an independent observer. It may be therefore argued that the own energy associated with the ‘gravitational-inertial bundle’ given is to be differently connected with its two ‘constituents’, no matter this relation is not clear yet.

It is worth noting that the only outcome of the above (perhaps not perfectly flawless) speculation which we would like to refer to is a certain evidence of a potential physical meaningfulness of attempts to separate the manifestations of the ‘active’ gravity from ‘superficial’ effects due to the ‘passive’ inertia. There are no evident reasons why one should *a priori* regard such a procedure as a mathematical exercise at most.

The next source of difficulties, standing in the way of description of the ‘pure’ gravity in terms of the measurable quantities, is the necessity to refer any observable force to some ‘observational platform’ — a frame of reference. Unfortunately, in the case of a curved space-time, the concept of a frame of reference, formulated in a way concordant with the flat space-time limit, cannot still now be regarded, on the whole, sufficiently definite and clear. In particular, the status of a successor of inertial frames of reference, constituting the foundation of the special relativity, remains to be not properly understood yet.

There exists however an opportunity to evade the surmounting the above problems in their generic posing. To that end, the present work focuses on a particular case *intermediate* between the flat and generic curved space-times where a sufficient symmetry (whose extreme degree characterizes the flat space-time) and arbitrarily strong curvature (associated with non-ignorable gravitational field) are combined. The subject of our analysis —
Kerr-Newman space-time \cite{4} — is one of the most important solvable models in the gravity theory. This case encompasses also Kerr, Reissner-Nordström and Schwarzschild fields (see Ref. \cite{8}) as particular limits, the results of the work holding true \textit{mutatis mutandis} for them as well.

The symmetry of the model picked on immediately gains us the well known natural way of the introduction of the reference platform (‘non-inertial frame of reference’) possessing a clear interpretation in physical terms. Accordingly, the theoretical tools used in the present work for the referring the spatial and temporal relations in Kerr-Newman space-time in terms of observable quantities (the so called apparent space and its derivatives, see section 2) are based just on a remnant the model inherits from the Minkowski space-time.

As opposed to the usual tendency, we do not strives to make the model maximally general. Instead the goal is pursued to limit ourselves with the only tools whose physical meaning is absolutely transparent, and to minimize the making use of any additional independent assumptions. Attaining it, there appears an opportunity to more or less directly apply the standard means used in similar situations within framework of the flat space-time theory. As a matter of fact, this element of the method possesses an unambiguous physical interpretation and proves to be a strictly unique one (in frame of the problem posing assumed).

Further, the standard way of the introducing of observable forces acting to test particles is followed. It allows one to define the corresponding static strengths (intensities) of physical fields living in Kerr-Newman space-time, yielding ultimately the completely plausible results. In particular, the observable intensities characterizing the gravitational-inertial manifestations and the observable strengths of the electromagnetic field immediately result.

Furthermore, by virtue of the model transparency, a straightforward speculation enables us to feel about for a method of the separating the gravity and inertia forces and to determine the true intensities of the gravitational field — attractive (gravitoelectric) and gyroscopic (gravitomagnetic) ones. The close connection of the gravitational field to the space-time curvature is the guiding relationship utilized.

It is shown in particular that in the case of Kerr-Newman field the standard complex combination of the electric and magnetic strength vectors and analogous complex combination of the true gravitational intensities are \textit{proportional} (over \(\mathbb{C}\)) that may be referred to as the \textit{generalized alignment of the gravitational and electromagnetic fields}. To the best our knowledge, such a relationship is revealed for the first time. Another surprising implication is a somewhat unexpected form of the relationship of the intensities of the gravitational field and the curvature: we show that the former may not be regarded as a plain form (such as, \textit{e.g.}, a projection) of the latter. (For more details see section 8.)

As it was mentioned above, the related problems, concerning in particular the concepts of gravitational and inertial forces, have been treated, one way or another, in a number of works from the positions some of whose are of definite similarity to our one while others substantially differ. For the sake of easier comparison, we list in short the main approaches presented in the literature.

First of all it should be noted that the elements of the method used in the present work
can be found in Ref. [9] (see in particular therein the discussion of Problem 1 following §88). In the both cases the approaches are based on the analysis of the elementary measurement procedures and follow rather a physical intuition than mathematical likenesshood.

A semi-heuristic approach applied in Refs. [10]-[12] for the introduction of the true gravitational force is based on a plausible assumption concerning its ‘Lorentz-like’ transformation rule. The corresponding results, describing the attractive gravitational and centrifugal forces in Schwarzschild space-time and Reissner-Nordström space-time (see in particular Eqs. (12a), (12b) in [1]), agree with the particular cases of our formulae (see section 7). However the gyroscopic forces (gravitomagnetic and Coriolis) are not taken into account there.

The extended series of works [13]-[32] follows substantially different ideas, basing on the notion of the so called ‘optical geometry’ [15]. The physical background lying beyond the latter notion is exhibited in Ref. [20]. It relies first of all on a special regarding of some globally synchronized rescaled ‘universal time’ which is obtained from the proper time element by means of some conformal transformation, see [21, page 7]. This implies the regarding of the role of physical time distinct from one assumed in the present work, see section 4 statement (B). The possibility of global synchronization ensured by the conformal transformation is not a primary mandatory property of a time since the measuring of the latter is a local procedure. From our point of view the global ‘universal time’ introduced in [20] is not actually a time since it is not measured by any clock (showing proper time intervals along the clock’ worldline). The non-constant rescaling of readings of watches, used for explanation of the meaning of conformal transformation, means much more than a mere choice of the ‘local’ units of temporal duration. Further, the opt geometry treats trajectories of photons (light rays) as genuine straight lines. However it seems to be not a quite plausible conjecture. Specifically, by virtue of the equivalence principle, it is unreasonable to suppose that the light is not affected by the gravitational field. Indeed, the light is a ‘high frequency limit’ of the material object possessing the energy and, hence, the gravitational mass — periodic electromagnetic waves. The properties of light indeed allows one to obtain the absolute measure of a velocity magnitude (gravitational field is not able to force light signal to move faster or slower because its speed is maximal, anyway) but does not yield the standard of an invariable direction, the gravitational deflection of light and gravitational lensing being the well known relevant observable effects connected with variation of light ray direction. It is also worth noting that the approach in question involves a number of independent statements (claiming, for example, that (citing) “in static space-time gravitational force is velocity independent”, [24, page 943], etc.) which seem to be not convincingly motivated from the physical point of view. Resuming, it is stated in Ref. [19, page 734] that (citing) “the rotational effects <...> can be properly understood only after a fundamental revision of the very concept of centrifugal force”. The present work demonstrates however that the traditional regarding of the latter quite suffices and this is a nice sign since we should abstain from superfluos revisions of basic concepts until this is being allowed by the logic of problem.

Next, in Ref. [33] the method of interpretation of particle motion in Kerr space-time in terms of forces is suggested. The observers picked on are so called zero-angular-momentum
ones (ZAMOs, see also Ref. [33]) which are (citing) “standardly referred to as proper generalization of the Newtonian non-rotated rest ones”, some specific hypothesis on the properties of centrifugal force in such a frame being asserted. However the resulting description of the particle dynamics proves to entail improper non-relativistic limit. Indeed, Eqs. (64)-(67) of Ref. [34] (with $M = 0$) state that centrifugal inertial force may exist while Coriolis one identically vanishes. In the further works by the same author [35]-[37] the general covariant equations of the force balance in arbitrary space-time were suggested (including Kerr space-time as a particular example). The physical motivation of the method seems to be not quite sufficient however. For example, the gravitational force is chosen to coincide, up to some scalar factor, with the 4-acceleration of an observer. The only restriction imposed on the frame is the claim for observers’ worldlines to be hypersurface-orthogonal (‘hypersurface-orthogonal observers’, HOOs, see [35]). Thus the model suggested admits the existence of gravitational forces even in the case of the flat space-time, provided the observers’ worldlines are not geodetic. From a physical point of view such a development of the conventional concept seems to be not truly plausible.

The general approach widely used for the re-casting the equations of motion of test particles to the form of a force balance condition is based on the geometric projections of 4-dimensional characteristics to the directions parallel and orthogonal to observers’ 4-velocities. Alternative but similar method utilizes space-like space-time slicings within framework of the 3+1 splitting method. These are discussed in details in Ref. [3] and summarized recently in Ref. [4]. In Ref. [4] the relative gravitational force is defined via the relative 4-acceleration of the two families of observers. They play the role of the frames of reference and are allowed to be chosen without substantial restrictions. The gravitational force acting to test particles is defined in terms of the spatial projection of the Fermi-Walker derivative of the particle 4-momentum while the notion of inertial forces is not drawn in at all. In Ref. [5] Kerr-Newman metric is considered. The approach in question does not attempt to establish a relation of the mathematical formalism developed to the corresponding procedures of elementary measurement acts and observations which would confirm the interpretation of the formal relationships derived in physical terms. Similar method based on the space-time slicing is utilized also in Ref. [38]. The gravitational force is introduced in framework of the 3+1 splitting method. The projected equations of the test particle motion are re-interpreted basing on the analogies with the flat space-time dynamics. No inertial forces are taken into account.

In Ref. [39] the ‘pseudo-Newtonian’ gravitational force is defined by means of a sort of ‘integrating’ of the tidal gravitational force. The latter is expressed via the projection of the Riemann tensor. However the role of inertial forces is not allowed for.

Comparing with majority of the approaches mentioned, our one is distinctive by (i) its close connection to the physical origin of the problem, focusing on observable quantities and models of elementary measurement procedures, and (ii) the utilizing as far as possible the approaches and concepts developed within framework of the flat space-time theory with minimal drawing in additional postulates. Our results do not pretend at the current stage to any degree of generality (in particular, by virtue of a substantial role of the symmetries stipulated). Nevertheless a cogency of their interpretation is the important outcome which
could serve a ground for the further consistent extending of the concept of gravitational force to more general physical situations. A general theory has to reproduce them in the corresponding particular cases.

2 Spatial relationships in Kerr-Newman field from standpoint of uniformly rotated observers

Let us consider the line element of Kerr-Newman space-time (or Kerr space-time) in Carter’ representation (see Refs. [40],[41]):

\[-ds^2 = (p^2 + q^2) \left( \frac{dp^2}{\mathcal{P}} + \frac{dq^2}{\mathcal{Q}} \right) + \frac{1}{p^2 + q^2} \left( \mathcal{P}(d\tau + q^2d\sigma)^2 - \mathcal{Q}(d\tau - p^2d\sigma)^2 \right). \tag{1}\]

Here \(p, q, \tau, \sigma\) are the coordinates, \(\mathcal{P} = \mathcal{P}(p), \mathcal{Q} = \mathcal{Q}(q)\) are smooth functions. For simplicity, we require \(\mathcal{P}, \mathcal{Q}\) to be positive in the connected space-time region which will be considered. Since Kerr metric is recovered when

\[\mathcal{P} = \mathcal{P}_K(p) = a^2 - p^2, \quad \mathcal{Q} = \mathcal{Q}_K(q) = q^2 - 2mq + a^2, \tag{2}\]

while for the Kerr-Newman solution

\[\mathcal{P} = \mathcal{P}_{KN}(p) = \mathcal{P}_K(p), \quad \mathcal{Q} = \mathcal{Q}_{KN}(q) = \mathcal{Q}_K(q) + e^2, \tag{3}\]

(see Ref. [41]), \(\mathcal{P}\) and \(\mathcal{Q}\) are actually positive, provided \(m > a > 0, q > m + \sqrt{m^2 - a^2}, -a < p < a\). The symbols \(m, a\) and \(e\) denote real constant parameters: the mass, the specific angular momentum, and the electric charge of the ‘central’ source, respectively. (For the choice of the physical units assumed the light velocity \(c\) and the gravitational constant are equal to the unity.) The space-time region discriminated by the above conditions is adjacent to the asymptotically flat infinity. (The diverging observed whenever \(p \to \pm a\) can be shown to represent an apparent peculiarity reflecting the local fault of the coordinate system alone).

We shall need also the potential of the electromagnetic field filling Kerr-Newman space-time. Its covector equals

\[\mathcal{A} = \frac{e \cdot q}{p^2 + q^2} (d\tau - p^2d\sigma). \tag{4}\]

It is worth noting that Boyer-Lindquist’ (BL) representation \[12\] of Kerr-Newman metric is derived from Eqs. (1)-(4) by means of the local coordinate transformation

\[p = a \cos \theta, \quad q = r, \quad \sigma = -a^{-1}\varphi, \quad \tau = t - a\varphi. \tag{5}\]

It proves to be more convenient for us to carry out all the intermediate work making use of Carter’ coordinates since they enable one to gain advantage of a remarkable symmetry of the ‘radial’ and ‘angular azimuthal’ coordinates, \(q\) and \(p\), respectively, which Kerr-Newman
metric possesses and which is manifest just when using Carter’s representation. On the other hand the interpretation of the physical relationships implied is usually more manifest in terms of BL coordinates. In particular Carter’s coordinates obviously fail in the case of the vanishing of the angular momentum parameter \( a \to 0 \).

We shall consider the family of (formally structureless point-like) observers whose world-lines are described by equations

\[
p, q \text{ are constant, } \sigma = -a^{-1}(\phi + \omega t), \quad \tau = (1 - a\omega)t - a\phi, \quad \phi \text{ being a constant.} \quad (6)
\]

In Boyer-Lindquist coordinates these read \( \varphi = \omega t + \text{const}, \quad r, \theta \text{ are constant (cf. Ref. [43]).} \)

In the flat space-time limit the observers reduce to uniformly rotated ones. In Eq. (6) the BL coordinate \( t \) may be regarded as a free parameter on the curves of the congruence. The parameter \( \phi \) (identified modulo \( 2\pi \)) together with (constant) \( p \) and \( q \) label every individual observer. \( \omega \) is the predefined constant (the formal frequency) associated with the whole congruence and characterizing the rate of the steady rotation of the family of observers as a whole. (One should not attach \textit{a priori} a direct physical meaning to the numeric values of \( \omega \), noting however that in the case \( \omega = 0 \) the observers may be regarded non-rotating with respect to the static flat asymptotic.)

Let us mention that, referring to a congruence of worldlines of point-like observers, we deal with the realization of the ‘observational platform’ which was named in Ref. [2] a “threading point of view”.

It is worth noting that the norm of the vector field

\[
U = (1 - a\omega)\frac{\partial}{\partial \tau} - \frac{\omega}{a}\frac{\partial}{\partial \sigma}
\]

tangent to the congruence (6) amounts to

\[
||U||^2 = \frac{P Q}{a^2(p^2 + q^2)}[P - Q],
\]

where \( P = P(p) = \frac{(a - \omega(a^2 - p^2))^2}{p}, \quad Q = Q(q) = \frac{(a - \omega(a^2 + q^2))^2}{q} \).

Thus the congruence (6) is timelike, provided

\[
P - Q > 0. \quad (8)
\]

Giving suitable \( \omega \), we restrict consideration to the space-time region (assuming it to be non-empty) where the above timelikeness condition is fulfilled.

Having made these preliminary remarks, let us consider a pair of close observers, associated, say, with the (fixed) parameter values \( \phi, p, q \) and \( \phi + d\phi, p + dp, q + dq \), respectively. Analyzing the exchange by light signals, it is straightforward to show that the distance \( dl \) separating these two infinitesimally close observers amounts to the square root of the quadratic form

\[
dl^2 = (p^2 + q^2) \left\{ \frac{dp^2}{\mathcal{P}} + \frac{dq^2}{\mathcal{Q}} + \frac{d\phi^2}{P - Q} \right\}. \quad (9)
\]
It is positively defined by virtue of the condition (8).

From a mathematical standpoint the symmetric tensor determining the quadratic form
\( (9) \) can be obtained as a pull-back of the space-time tensor
\( h_{\mu \nu} = (-g_{\mu \nu} + (U_{\lambda} U^{\lambda})^{-1} U_{\mu} U_{\nu}) \)
with respect to the map \( (p, q, \tau, \sigma) \rightarrow (p, q, \phi = -\omega \tau - a(1 - a \omega) \sigma) \).
Here \( g_{\mu \nu} \) denotes the components of the metric tensor determining the line element
\( ds^2 \), Eq. (1), \( U_{\mu} \) denotes the covariant components of the vector field (7).
(Cf. the derivation of Eqs. (84,6), (84,7) in Ref. (1).) Thus the metric (9) may be also regarded as minus the projection of the space-time metric onto the infinitesimal 3-spaces orthogonal to the observers’ worldlines.

It has to be noted that Eq. (9) is in fact a quantitative realization of the following statements which are not usually explicitly formulated:

(A) The speed of light detected by an arbitrary observer in his/her sufficiently small neighborhood coincides with the universal constant \( c \) independently of the features of observer motion and the state of the ambient gravitation field.

(B) The time measured by any observer coincides with the interval (the proper time)
\[ c^{-1} \int \sqrt{ds^2} \]
accumulated along his/her worldline.

It is also implied that observers determine their mutual distance by means of a local light ranging. This method is interpreted within framework of the geometric optic, light rays being represented by (infinitesimal) segments of null geodesics.

It is important that, referring to an arbitrary small neighbourhood of an observer, the statement A stipulates the possibility to make a path of the light signal arbitrarily short that ensures the necessary relaxing of the role of the deflection of the probing ray due to all the causes (except of a corner in the point of reflection from a ‘target’). Essentially, the claim (A) expresses the exclusive role of light signals for the ensuring the relativistic consistent measure of a length. This is a commonplace under the conditions corresponding to the special relativity. Moreover, in practice, the standard of length is established as a fixed number of wavelengths of a definite transition line in cesium. All the more, in the case of essential role of the acceleration or the gravitational field, there may exist no relativistic means to introduce the notion of a length other than to assume the invariance of the local velocity of light. In principle, such an assumption might lead to physical inconsistencies (whose actual existence is to be verified with the help of observations analogous to Michelson’ experiment) but in such a case this is the notion of a local infinitesimal length which would fail to be physically meaningful.

The important property of the length element (9) is its independence on a particular moment of measurement. The latter fact is a trivial consequence of the obvious property of the congruence (7): it consists exactly of the orbits of a space-time isometry group. Indeed, the vector field (7) is evidently the Killing field for the metric (1) whose coefficients depend on neither \( \tau \) nor \( \sigma \). Hence the ‘development towards the future’ of the observers is geometrically the space-time isometry motion. Thus, from their standpoint any invariant characteristic of the spatial geometry observed is necessarily invariant with time. The infinitesimal distance is just an instance of such a characteristic.
Thus it can be stated that any two nearby observers from the family chosen may be regarded \textit{mutually immobile}. On an equal footing it may be also said that they are in fact ‘immersed’ in the \textit{static Riemannian space} whose metric is represented in the coordinate chart \( \{ p, q, \phi \} \) by the expression (9).

Hereinafter, for the sake of convenience of references, this 3-dimensional space will be named the \textit{apparent space}. By virtue of the universality and the very meaning of the above claims (A) and (B), the apparent space encodes all the local spatial relationships leaned on the notion of a \textit{distance} which are available for the observers under consideration.

It is worth emphasizing that the latter claim is based on a theoretical model of the physical observation carried out by observers and involves no additional conjections.

It should exist no serious objections against the statement that the apparent space is a really existing physical-geometrical entity and that particular local observations may (and have to) refer to it for the \textit{concordant joining} local results into a unified picture.

This way, ‘solidifying’ the observational platform, we come at a qualitative situation which is characteristic of the special relativity and which can be consistently treated by the straightforward methods analogous to ones applied in the case of the flat space-time.

\section{Kinematic of a test particle in apparent space}

Let us consider a test particle moving in accordance with the equations \( x^\alpha = x^\alpha(\nu) \), where \( \nu \) denotes the proper time along the particle worldline. Given the notion of 3-dimensional apparent space, the vector of velocity \( \vec{v} \), the vector of momentum \( \vec{P} \) (whose symbol should not be mixed up with the function \( \mathcal{P} = \mathcal{P}(p) \)) and the energy \( \mathcal{E} \) of the particle can be determined. The above characteristics of particle motion are expressed in terms of invariant geometrical objects (scalars and vectors) \textit{over the apparent space}.

Specifically, in physical terms, the vector of speed is to be defined as an infinitesimal displacement of the particle divided to the (also infinitesimal) duration of the corresponding motion which is measured by the observer being in the point occupied initially by the particle. It is worth mentioning that the time interval of the particle motion is not \textit{a priori} referred to any prescribed ‘timelike coordinate’ or the 3+1 space-time splitting. Rather it is defined by means of the modeling of the exchange of light signals (\textit{i.e.} the local light ranging) and is based on the above claims (A) and (B) which just suffice for this purpose, provided the observers carrying out the measurements are properly specified.

Further, knowing the speed, the energy and the momentum of a particle follows from the standard local relativistic equations generalized to the case of non-flat spatial ground (the apparent space). This way, it is straightforward to show that

\begin{equation}
\mathcal{E} = \sqrt{\mathcal{P}^2 + \mathcal{Q}^2} \cdot \frac{\omega C(\sigma) - a(1 - a\omega)C(\tau)}{\sqrt{\mathcal{P}^2 - \mathcal{Q}^2}}, \tag{10}
\end{equation}

\begin{equation}
\mathcal{P} = \frac{1}{p^2 + q^2} \left\{ \epsilon(p) R(p) \frac{\partial}{\partial p} + \epsilon(q) R(q) \frac{\partial}{\partial q} \right\},
\end{equation}

\begin{equation}
\vec{P} = \epsilon(p) R(p) \frac{\partial}{\partial p} + \epsilon(q) R(q) \frac{\partial}{\partial q}.
\end{equation}
\[- \left[ (C(τ)p^2 + C(σ)) \frac{a - \omega(a^2 - p^2)}{\mathcal{P}} \right. + \left. (C(τ)q^2 - C(σ)) \frac{a - \omega(a^2 + q^2)}{\mathcal{Q}} \right] \frac{\partial}{\partial φ}, \right) \]

(11)

where
\[
R(p) = \sqrt{\mathcal{P}(C(0) - µ^2p^2) - (C(τ)p^2 + C(σ))^2},
\]
\[
R(q) = \sqrt{(C(τ)q^2 - C(σ))^2 - \mathcal{Q}(C(0) + µ^2q^2)).}
\]

(12)

Here \(µ\) is the (constant) mass of the particle at rest. The symbols \(C(τ), C(σ), C(0)\) may be interpreted as the notations of the following expressions:
\[
C(τ) = \frac{µ}{p^2 + q^2} \left[ \mathcal{P} \left( \frac{dτ}{dν} + q^2 \frac{dσ}{dν} \right) - \mathcal{Q} \left( \frac{dτ}{dν} - p^2 \frac{dσ}{dν} \right) \right],
\]
\[
C(σ) = \frac{µ}{p^2 + q^2} \left[ q^2 \mathcal{P} \left( \frac{dτ}{dν} + q^2 \frac{dσ}{dν} \right) + p^2 \mathcal{Q} \left( \frac{dτ}{dν} - p^2 \frac{dσ}{dν} \right) \right],
\]
\[
C(0) = \frac{1}{2} µ^2 \left[ (p^2 - q^2) + (p^2 + q^2)^2 \left( \frac{1}{\mathcal{P}} \left( \frac{dp}{dν} \right)^2 - \frac{1}{\mathcal{Q}} \left( \frac{dq}{dν} \right)^2 \right) \right.
\]
\[
\left. + \mathcal{P} \left( \frac{dτ}{dν} + q^2 \frac{dσ}{dν} \right)^2 + \mathcal{Q} \left( \frac{dτ}{dν} - p^2 \frac{dσ}{dν} \right)^2 \right].
\]

(13)

At the same time these are none other than the \textit{geodesic integrals} for the metric (1), cf. [11]. Thus, claiming \(C(τ), C(σ), C(0)\) to be constant (together with the condition specifying \(ν\) as the proper time), one obtains the complete set of the first integrals of the geodesic equations for the metric (1). The symbols \(ε(p) = \text{sign}(dp/dν), ε(q) = \text{sign}(dq/dν),\) involved in Eq. (11), equal to +1 or −1. They however are not the true constants because the sign of each of them is reversed in the points where the argument of the corresponding square root, either \(R(p)\) or \(R(q)\), vanishes (and where the value of \(ε(∗)\) does not matter), see Eqs. (12).

It is useful to notice in conclusion that the momentum vector admits the alternative representation
\[
\tilde{\mathcal{P}} = µ \left( \frac{dp}{dν} \frac{∂}{∂p} + \frac{dq}{dν} \frac{∂}{∂q} + \frac{dφ}{dν} \frac{∂}{∂φ} \right),
\]
\textit{i.e.}, it is a pull forward of the vector \(µ \cdot ∂/∂ν\) tangent to the particle trajectory. This is not surprising since by virtue of definitions the direction of momentum coincides with one of infinitesimal displacement of the particle in the apparent space.

4 The observable dynamics of test particles

Given the particle momentum, the problem of the determination of its \textit{physical variation rate} can be posed. For definiteness we denote the latter as \(D\tilde{\mathcal{P}}/δτ\ (δτ\ means the
infinitesimal physical time interval and should not be mixed up with the variation of the coordinate $\tau$). The very realization of the operator $D/\delta\tau$ has to be based on physical arguments which are discussed below.

Let us consider the generalized relativistic version of the Newton law

$$\vec{F} = \frac{D}{\delta\tau} \vec{P}.$$  

Essentially, this equation simply allows one to introduce the notion of observable force acting to a test particle. The latter may be regarded as the ‘cause’ of observable acceleration of the particle in the apparent space. This line may be further continued pursuing the goal of the rigorous introduction of observable strengths (intencities) of physical fields including gravitational one. Accordingly, one may interpret the forces as the result of the action of the fields to the corresponding ‘charges’ the particle possesses.

The basic relativistic recipe of the determination of the rate of vector variation is of course well known and may sound as follows: Let the two nearby points on the particle trajectory be chosen, the first of them coinciding with position of the observer carrying out the measurement. Further let the vector ‘dragged’ with the particle (speed, momentum, etc.) be registered in the two points mentioned, and the corresponding difference be found. Dividing it to the duration of the particle motion, the desirable ‘rate of vector variation’ follows.

Concerning the time of the particle motion, it is determined by the observer, occupying the point of the initial particle position. The basic setup of its measurement is standard: the observer emits a series of light signals towards the second observer which is situated in the point of the future final particle position and registers the light signal reflected by the second observer at the moment when the particle arrives at the his/her point. Then, from the standpoint of the first observer, the middle between the events of the emission and the reception for that light signal is just the moment of the particle arrival at the final point (precisely by virtue of the statement (A) formulated in section 2). Then the corresponding time interval of the particle motion immediately follows.

The determination of the difference of the two vectors ‘attached’ to the nearby but distinct points is less trivial. On one hand, this is a classical problem of the non-euclidean geometry which is exhaustively treated in terms of the parallel transport mathematically equivalent to the notion of connection. On the other hand, such a solution would be, in principle, a formal-mathematical one, while we need strictly ‘physically meaningful’ interpretations in terms of ‘observable’ quantities. However one may also argue that the practice of local measurements of spatial relationships has in fact no sufficient experience in the dealing with the substantially curved spatial background where the spatial curvature affects the essence of the physical process realizing the measurement procedure. Accordingly, it cannot be a priori stated quite definitely what mathematical tools of Riemannian geometry might be qualified as ‘physically realizable’. Nevertheless the parallel vector transport along infinitesimal paths obviously may. Indeed, the observers are certainly able to obtain the components of the apparent metric tensor, measuring their mutual distances by means of the light ranging. Then they
may obtain the corresponding connection (calculating the Christoffel symbols, for example) in the point of measurement. This allows them further to determine the corrections necessary for the re-casting the directly observable variations of components of a vector under consideration into the components of vector variation (represented, mathematically, by the absolute differential contracted with the displacement vector). Thus one may conjecture the following:

(C) In the static apparent space, the observable variation of a vector, dragged from a point to another nearby one, is determined by means of the local parallel transport with respect to the torsion-free connection compatible with the apparent metric.

Said another way, living in the static curved Riemannian space, one is to apply the corresponding standard geometric tools for the description of the spatial relations involved in physical processes. Anyway, such a position is the most natural and, simultaneously, straightforward one within the framework considered.

Indeed, it would be more than strange to suppose the connection determining ‘physical’ parallel transport to be incompatible with the spatial metric. Thus the only pertinent alternative could be the introducing of a non-zero torsion. However no compelling reasons in favor of such a modification of the statement (C) are immediately perceptible and we limit ourselves here with these remarks.

Nevertheless, strictly speaking, (C) is a separate additional postulate besides the statements (A) and (B). However, one way or another, some guiding concept has to be introduced to properly formalize the observations over a substantially curved spatial background. Essentially, we deal here with a sort of basic definitions ensuring the consistent regarding of one of ‘primitive’ notions. Specifically, it clarifies the meaning (the method of measurement) of a variation of a vector ‘dragged’ in the curved space where the existence of the preferable Cartesian coordinates yielding trivial connection may not be stipulated. It is worth mentioning that the conjecture (C) is the first and, in fact, the only statement concerning the principles of measurements and asserting something new which we use in the present work. Indeed, the meaning of the claims (A) and (B) (see section 2) is, in a sense, just opposite: essentially, they state that the space-time curvature and acceleration of observers does not locally affect the role of light signals and the basic model of time measurements adopted in the flat space-time physics.

Ultimately, basing on the above speculation, a straightforward calculation, whose technical details are omitted here for brevity, yields the following simple explicit representation of the operator of the rate of vector variation:

\[
\frac{D}{\delta \tau} = \bar{v} \cdot \mathcal{D}, \quad (\bar{v} = \mathcal{E}^{-1} \mathcal{\bar{P}} \text{ is the observable velocity of a particle}).
\]

Here \( \mathcal{D} \) denotes the covariant (absolute) differential in the apparent space, the symbol ‘\( \cdot \)’ denotes the contraction of a vector (at left) with an 1-form (at right).

It is worth emphasizing that this plausible equation is not a conjectural definition of its l.h.s., i.e., a ‘physical derivative’ of a vector (although it would be a rather natural one). Rather it is a straightforward consequence and follows from the statements (A)–(C).
5 The forces affecting the motion of test particles

It was shown that, utilizing the concept of the apparent space and tracing the particle’s motion, the observers reveal the following force applied to particle:

$$\vec{F} = \varepsilon^{-1}(\vec{\mathcal{P}} \cdot \mathcal{D}) \vec{\mathcal{F}}.$$  \hspace{1cm} (14)

Using Eqs. (11), (9), the r.h.s. expression can be explicitly calculated in a general case. Specifically, dealing with the most general case of the arbitrary prescribed motion of a test particle (caused by the unspecified influence on it) with arbitrary timelike worldline, the following identity takes place:

$$(\vec{\mathcal{P}} \cdot \mathcal{D}) \vec{\mathcal{F}} \equiv \varepsilon^{2} \mathcal{F}_{\text{mass}} + \varepsilon[\vec{\mathcal{P}} \times \mathcal{F}_{\text{mom}}] + \varepsilon \mathcal{F}_{\text{ext}}.$$  \hspace{1cm} (15)

Here \(\cdot \times \cdot\) denote the antisymmetric vector cross-product in the apparent metric (9). The following series of vector fields living in the apparent space are introduced:

$$\mathcal{F}_{\text{mass}} = \left[ \frac{1}{2} \frac{2p\mathcal{P} - (p^2 + q^2)\mathcal{P}'}{(p^2 + q^2)^2} + \frac{a - \omega(a^2 - p^2)}{(p^2 + q^2)(\mathcal{P} - \mathcal{Q})} \left( \frac{a - \omega(a^2 - p^2)}{2p} \frac{\mathcal{P}'}{2p} + 2\omega q \right) \right] \frac{\partial}{\partial p},$$  \hspace{1cm} (16)

$$\mathcal{F}_{\text{mom}} = \left[ \frac{2q\mathcal{P}}{(p^2 + q^2)^2} + \frac{a - \omega(a^2 - p^2)}{(p^2 + q^2)(\mathcal{P} - \mathcal{Q})} \left( \frac{2q\mathcal{P}'}{(p^2 + q^2)(\mathcal{P} - \mathcal{Q})} \right) \right] \frac{\partial}{\partial p} + \left[ \frac{-2p\mathcal{Q}}{(p^2 + q^2)^2} + \frac{a - \omega(a^2 + q^2)}{(p^2 + q^2)(\mathcal{P} - \mathcal{Q})} \left( \frac{\mathcal{Q}'\mathcal{P}'}{(p^2 + q^2)(\mathcal{P} - \mathcal{Q})} \right) \right] \frac{\partial}{\partial q},$$  \hspace{1cm} (17)

$$\varepsilon \mathcal{F}_{\text{ext}} = \frac{1}{2(p^2 + q^2)(\vec{\mathcal{P}} \cdot d\mathcal{P}(0))} \vec{\mathcal{P}} \left[ \mathcal{P} dC(0) \right] - 2 \left( p^2 C(\tau) + C(\sigma) \right) \left( p^2 dC(\tau) + dC(\sigma) \right) \frac{\partial}{\partial p} + \frac{1}{2(p^2 + q^2)(\vec{\mathcal{P}} \cdot d\mathcal{Q}(0))} \vec{\mathcal{P}} \left[ 2dC(0) \right] + 2 \left( q^2 C(\tau) - C(\sigma) \right) \left( q^2 dC(\tau) - dC(\sigma) \right) \frac{\partial}{\partial q} - \frac{1}{(p^2 + q^2)} \vec{\mathcal{P}} \left[ \frac{a - \omega(a^2 - p^2)}{\mathcal{P}} \left( p^2 dC(\tau) + dC(\sigma) \right) + \frac{a - \omega(a^2 + q^2)}{2q} \left( q^2 dC(\tau) - dC(\sigma) \right) \right] \frac{\partial}{\partial \phi}.$$  \hspace{1cm} (18)
It is assumed here that the worldline of the particle is timelike and, besides, the momentum components \( \vec{P} \, dp, = \vec{P} \, dq \) are nonzero. The latter limitation is not a substantial one and there exists an equivalent representation of \( \vec{F}_{\text{ext}} \):

\[
\vec{F}_{\text{ext}} = \mu \sqrt{ p^2 + q^2 } \sqrt{ \mathcal{D} } \, \sqrt{ P - Q } \times \\
\left[ (a - \omega (a^2 - p^2)) \, \mathcal{D} \left( \frac{d\tau}{dv} - p^2 \frac{d\sigma}{dv} \right) \\
- (a - \omega (a^2 + q^2)) \, \mathcal{D} \left( \frac{d\tau}{dv} + q^2 \frac{d\sigma}{dv} \right) \right]^{-1} \times \\
\left\{ \frac{D}{dv} \frac{d}{dp} - \frac{D}{dv} \frac{d}{dq} + \frac{p(1 - a \omega)}{D} \frac{d}{dv} - \frac{\omega}{D} \frac{d}{dv} \right\}
\]

(19)

which evidently does not assert it. It is worth pointing out an additional form of Eq. (18) which manifests the meaning of the variation of non-conservation of geodesic integrals (13), if any:

\[
\vec{F}_{\text{ext}} = \frac{\sqrt{ \mathcal{D} } \, \sqrt{ P - Q } \, \left( \frac{1}{2} \left( \frac{\partial}{\partial p} \frac{d}{dp} + \frac{\partial}{\partial q} \frac{d}{dq} \right) + \frac{p(1 - a \omega)}{D} \frac{d}{dv} - \frac{\omega}{D} \frac{d}{dv} \right)}{(p^2 + q^2)^{3/2} \left( \omega C_{(\sigma)} - a (1 - a \omega) C_{(\tau)} \right)} \\
+ \frac{1}{2} \left( \frac{\partial}{\partial p} \frac{d}{dp} + \frac{\partial}{\partial q} \frac{d}{dq} \right) \left[ \left( a - \omega (a^2 - p^2) \right) \partial \frac{dC_{(\sigma)}}{dv} + \epsilon(1) \frac{d}{dv} \left( a - \omega (a^2 - p^2) \right) \right] \\
+ \frac{1}{2} \left( \frac{\partial}{\partial p} \frac{d}{dp} + \frac{\partial}{\partial q} \frac{d}{dq} \right) \left[ \left( a - \omega (a^2 + q^2) \right) \partial \frac{dC_{(\sigma)}}{dv} + \epsilon(1) \frac{d}{dv} \right]
\]

(20)

where

\[
\vec{v}_{(pq)} = \epsilon(p) \frac{\partial}{\partial p} + \epsilon(q) \frac{\partial}{\partial q}
\]

\[
\vec{v}_{(p\phi)} = \frac{a - \omega (a^2 - p^2)}{\mathcal{D}} \partial + \epsilon(p) \frac{C_{(\sigma)} + p^2 C_{(\tau)}}{R_{(p)}} \partial
\]

\[
\vec{v}_{(q\phi)} = \frac{a - \omega (a^2 + q^2)}{\mathcal{D}} \partial + \epsilon(q) \frac{C_{(\sigma)} - q^2 C_{(\tau)}}{R_{(q)}} \partial
\]

\( (R_{(p)}, R_{(q)}) \) are defined by Eqs. (12). Here \( \vec{F} \) has to be interpreted as r.h.s. of Eq. (11).

The decomposition (13) is derived as follows.

The contribution involving \( \vec{F}_{\text{ext}} \) is defined in such a way to accumulate all the terms including the second order derivatives of the coordinate functions specifying the particle worldline. They are incorporated with the corresponding expressions quadratic in the first order derivatives in order to yield in total the (components of) covariant derivative of the 4-vector tangent to the particle worldline (cf. Eq. (13)). Said another way, \( \vec{F}_{\text{ext}} \) is chosen to represent the constituent of the force (14) causing the 4-acceleration of the particle. It is a priori clear that \( \vec{F}_{\text{ext}} \) has to be a linear function of the components of 4-acceleration.

After subtracting \( \vec{F}_{\text{ext}} \), the residual part of l.h.s. of Eq. (13) does not involve second order derivatives of the coordinate functions. It depends on their first order derivatives alone. Accordingly, the problem may be posed to expand it with respect to the ‘physical basis’ (11). Such a representation proves to be linear (cancelling out the overall factor \( \mathcal{D} \)).
with respect to the components of momentum and the energy which are now the only terms characterizing the details of the particle motion. The vectors $\vec{F}_{\text{mass}}, \vec{F}_{\text{mom}}$ are just the corresponding ‘coefficients’ coupled with energy and momentum, respectively, independent on any such a characteristic.

Summarizing, Eq. (15)

- singles out the part of $\mathcal{E}^{-1}(\vec{\mathcal{J}} \cdot \mathcal{D})\vec{\mathcal{J}}$ linear with respect to the covariant derivatives of the 4-vector tangent to particle worldline ($\vec{F}_{\text{ext}}$) and, then,

- decomposes the residual part of $\mathcal{E}^{-1}(\vec{\mathcal{J}} \cdot \mathcal{D})\vec{\mathcal{J}}$ into the linear expansion with the vector coefficients ($\vec{F}_{\text{mass}}, \vec{F}_{\text{mom}}$) with respect to $\mathcal{E}$ and $\vec{\mathcal{J}}$.

The result of the above procedure is obviously unique.

The vectors $\vec{F}_{\text{ext}}, \mathcal{E}\vec{F}_{\text{mass}}, \left[ \mathcal{J} \times \vec{F}_{\text{mom}} \right]$ are the forces acting to the test particle. Eq. (15) (divided to $\mathcal{E}$) is just the equation of their balance. To better substantiate this statement, let us discuss the physical aspect of the above result.

The vector $\vec{F}_{\text{ext}}$ admits a straightforward and univocal interpretation: it represents precisely the external (i.e. neither gravitational nor inertial) force applied to the massive particle since it is immediately associated with the 4-acceleration of particle, being a linear function of the components of the latter, see Eq. (19). In particular, the 4-acceleration and $\vec{F}_{\text{ext}}$ vanish strictly simultaneously. (The direct implication trivially follows from Eq. (19) while the inverse one requires additionally the taking into account the identity $g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{D}{dt} \frac{dx^\beta}{dt} = 0$.) Thus $\vec{F}_{\text{ext}}$ vanishes if and only if the worldline is geodesic which describes just the motion free of external non-gravitational influence.

The physical meaning of the force $\vec{F}_{\text{ext}}$ is clearly manifested in the most simple case when the ‘external’ influence to test particle is caused by the electromagnetic field inherent to Kerr-Newman space-time. To that end, let us assume that the particle carries a small electric charge $e$ whose influence to the space-time geometry, as well as the corresponding radiative braking, may be neglected. It is convenient to describe the motion of such a particle in terms of ‘geodesic integrals’ defined by Eqs. (13). Specifically, for the free charged particle they are not constant but

$$C(0) = c(0), \quad C(\tau) = c(\tau) - (e e) \frac{q}{p^2 + q^2}, \quad C(\sigma) = c(\sigma) + (e e) \frac{q p^2}{p^2 + q^2}. \quad (21)$$

Here $c(0), c(\tau), c(\sigma)$ are the actual constants of motion for the configuration in question.

Substituting expressions (21) into Eq. (20), the differentials of geodesic integrals are calculated in explicit form. Further, making use of Eqs. (10), (11), the geodesic integrals are eliminated in favor of particle energy and momentum $\mathcal{E}, \vec{\mathcal{J}}$. Finally, the well known Lorentz’ equation follows

$$\vec{F}_{\text{ext}} = e(\vec{E} + [\vec{v} \times \vec{H}]). \quad (22)$$

(it is worth reminding that the speed $\vec{v}$ equals $\mathcal{E}^{-1}\vec{\mathcal{J}}$). Here the electric and magnetic field strengths $\vec{E}, \vec{H}$, represented in the form of a complex sum $\vec{E} + i\vec{H}$, are defined as
follows:

$$\vec{E} + i \vec{H} = -\frac{e}{(q+ip)^2} \sqrt{\frac{\mathcal{P}^2}{(p^2+q^2)(P-Q)}} \cdot \vec{X},$$  \hspace{1cm} (23)

where $\vec{X} = a - \omega (a^2 - p^2) \frac{\partial}{\partial q} + i \frac{a - \omega (a^2 + q^2)}{2} \frac{\partial}{\partial p}$  \hspace{1cm} (24)

($e$ is the charge inducing the electromagnetic field in Kerr–Newman space-time, see Eqs. (3), (4)).

It is worth noting that the vectors $\vec{E}$ and $\vec{H}$ do not depend on any characteristic of the test particle or the features of its motion. They are determined by the properties of the space-time, the electromagnetic field, and the rate of observers rotation (characterized by the formal frequency $\omega$). Thus $\vec{E}$ and $\vec{H}$ are just the observable electromagnetic strengths detected by uniformly rotated observers.

The above result concerning the strengths of electromagnetic field measured by means of the analysis of motion of charged test particles is completely concordant with what one expects basing on the experience gained from the flat space-time electrodynamics.

Now let us consider the two further contributions to the force balance equation (15). They are associated with the gravitational and inertial forces, the physical essence of the configuration considered ensuring the presence of the both of them.

Indeed, let the motion of a test particle be such that the ‘geodesic integrals’ are constant. We have mentioned that the latter claim corresponds to a free character of the particle motion. According to the basic assertion of the general relativity, the free geodesic motion of test particles is just a realization of the gravitational influence to it which we, in turn, associate with some gravitational force. Additionally, one have to take into account the effect of inertial force depending on the frame of reference chosen, the latter force being locally inseparable from the former one as far as one concerns their influence to particle motion. The equation

$$\vec{F} = \mathcal{E} \vec{X}_{mass} + [\mathcal{P} \times \vec{X}_{mon}].$$  \hspace{1cm} (25)

taking place for free particles describes just the influence of the gravitational and inertial forces to them. These force a ‘free’ particle to deviate from a ‘straight line’ — a geodesic in the apparent space which would be the trajectory of the particle motion, provided the resulting force (25) vanishes. Eq. (23) closely resembles the equation (22) describing the action of electric and magnetic fields. It is worth noting that the ‘massive’ (gravitational and inertial, simultaneously) ‘charge’ coincides with the relativistic (velocity dependent) mass of the particle $\mathcal{E}$, not merely the rest mass $\mu$.

An intriguing problem is now the separating of the inertial and ‘true gravitational’ constituents of the combined gravitational-inertial bundle of observable intensities (25) which are, in accordance with the principle of equivalence, linearly superposed.

\footnote{Eq. (25) holds true for massless test particles as well. In particular, the photons are also subjected to the action of the gravitational and inertial forces. This point exhibits a substantial distinction of the present method and the approach exploited in Ref. [13] (and provides us with the concise explanation of the meaning of the well known light deflection effect).}
6 Separating gravitational and inertial forces

Dealing with forces characterizing the action of the gravitational field, it would be important to be able to elicit their ‘pure gravitational’ constituents. In this case one could establish a bridge connecting the hardly caught notion of a true gravitational field (setting it off against inertia manifestations not associated with ‘physical fields’) and the quantities which can be measured in an observation, cf. [6]. A natural methodological condition to be a priori stipulated is the operating with primordial relativistic instruments — watches and light rays — in accordance with a strictly definite relativistic measurement procedure.

There is no satisfactory universal regarding of a gravitational force in framework of the general relativity yet. A number of different approaches have been suggested and they lead to distinct conclusions when treating equivalent problems (cf. Ref. [2]). Unfortunately, no entirely convincing criterium is now known which would enable one to distinguish a single concept of gravitational force among suggested ones.

Nevertheless, it should be no objections against the following simple observation: the gravitational field is closely connected with the space-time curvature; in particular, it may exist (and does exist) only in the case of a curved space-time.

A crucial role of the curvature in the qualitative recognizing of the true gravitational field is generally accepted although the relevant quantitative outcome obtained so far seems not so universal and perfectly convincing. Mostly, the curvature is called to account for the tidal effects caused by the inhomogeneous gravity.

We would like to suggest here a different approach which does not pretend (in the form presented, at least) to any degree of generality and is currently limited to the case of Kerr-Newman space-time (and, maybe, some its nearby generalizations). Nevertheless it allows one to describe a full scope of expectable local manifestations of the gravitational field and reveals some its surprising properties.

The main idea realized below is straightforward: having obtained the combined gravitational-inertial bundle of specific force intensities \( \{ \hat{F}_{\text{mass}}, \hat{F}_{\text{mom}} \} \) determined by Eqs. (16), (17), one may attempt to separate in them a constituent which is intimately connected with the curvature. One may hope that such a constituent should describe just the a true gravitational intensities while the residue part of \( \{ \hat{F}_{\text{mass}}, \hat{F}_{\text{mom}} \} \) is interpreted as responsible for the effects of inertia.

At first glance such an attempt cannot allow a satisfactory non-artificial solution. Indeed, the curvature is characterized by mathematical expressions which necessarily involves the second order derivatives of the coordinate components of the space-time metric. On the contrary, the vector fields \( \{ \hat{F}_{\text{mass}}, \hat{F}_{\text{mom}} \} \) are constructed from the first their derivatives at most.

However, the spinor components of the curvature of the metric (11) can be represented as some linear homogeneous operators applied to the functions

\[
\Omega_{(p)} = \mathcal{P}'(p) - \frac{2p}{p^2 + q^2}(\mathcal{P} - \mathcal{Q}), \quad \Omega_{(q)} = \mathcal{Q}'(q) + \frac{2q}{p^2 + q^2}(\mathcal{P} - \mathcal{Q})
\]  (26)
Moreover, the metric (1) is (locally) flat if and only if
\[ \Omega_{(p)} = 0 = \Omega_{(q)}. \]

For the sake of convenience of references, we shall call hereinafter the expressions (26) the elements of the \textit{pre-curvature} of the Kerr-Newman metric (1).

Thus although the curvature involves the second order derivatives of the metric components (in our case the second order derivatives of the functions \( \mathcal{P} \) and \( \mathcal{Q} \)), it also may be interpreted as a subsidiary object derived from pre-curvatures (26). In a sense, \( \Omega_{(p)}, \Omega_{(q)} \) may be used as the equivalent representation of the curvature.

The above property of the curvature of the Kerr-Newman space-time and the explicit mathematical structure of the expressions of \( \vec{F}_{\text{mass}}, \vec{F}_{\text{mom}} \), see Eqs. (16), suggest the following natural statement realizing the ‘minimal coupling’ of the ‘true gravity’ and the curvature:

\textbf{Conjecture:} In the case of metric (1) the true intensities of the gravitational field are linear homogeneous functions of the pre-curvature elements (26).

Indeed, it is impossible to separate in r.h.s.’s of Eqs. (16) the other additive sub-expressions which would vanish \textit{strictly simultaneously} with the pre-curvatures (and the curvature itself) for arbitrary functions \( \mathcal{P} \) and \( \mathcal{Q} \).

Then, adopting the validity of the above conjecture, a straightforward inspection of Eqs. (16), (17) immediately enables one to find the desirable \textit{gravitational} (gravitoelectric) and \textit{gyroscopic} (gravitomagnetic) intensities \( \vec{F}_{\text{attr}} \) and \( \vec{F}_{\text{gyr}} \). The most advantageous way of their representation turns out to be the introducing of the complexified vector sum \( \vec{F}_{\text{attr}} - \frac{1}{2} \vec{F}_{\text{gyr}} \) (similar to the complex vector \( \vec{E} + i \vec{H} \) utilized in electrodynamics) which proves to be factorizable as follows:

\[ \vec{F}_{\text{attr}} - \frac{1}{2} \vec{F}_{\text{gyr}} = - \frac{1}{2} \left( a - \omega \left( a^2 + q^2 \right) \right) \Omega_{(q)} - i \left( a - \omega \left( a^2 - p^2 \right) \right) \Omega_{(p)} \quad \vec{X}. \] (27)

The complex-valued vector field \( \vec{F} \) has been defined in Eq. (24).

The residue parts of the specific forces \( \vec{F}_{\text{mass}} \) and \( \vec{F}_{\text{mom}} \) which leave after the subtracting of \( \vec{F}_{\text{attr}} \) and \( \vec{F}_{\text{gyr}} \), respectively, provide obviously the \textit{inertial centrifugal} \( \vec{F}_{\text{cf}} \) and \textit{Coriolis} \( \vec{F}_{\text{Cor}} \) intensities. They are equal to

\[ \vec{F}_{\text{cf}} = \frac{\omega^2}{P - Q} \left[ q \frac{\partial}{\partial q} - p \frac{\partial}{\partial p} \right], \] (28)

\[ \vec{F}_{\text{Cor}} = \frac{2 \omega}{(p^2 + q^2)(P - Q)} \left( a - \omega(a^2 - p^2) \right) \left( \frac{a}{P} - \frac{a - \omega(a^2 + q^2)}{Q} \right) \times \left[ q \frac{\partial}{\partial p} + p \frac{\partial}{\partial q} \right]. \] (29)

One may suppose that pre-curvature should be connected with Lanczos potential (see Ref. [44]) which also gives rise to the curvature (specifically, Weyl spinor) as a result of application of the first order differential operator. A straightforward examination does not confirm it however.

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\[ \text{One may suppose that pre-curvature should be connected with Lanczos potential (see Ref. [44]) which also gives rise to the curvature (specifically, Weyl spinor) as a result of application of the first order differential operator.} \]
and determine jointly the inertial force
\[
\vec{F}_{\text{in}} = \mathcal{E} \vec{F}_{\text{cf}} + [\vec{\mathcal{P}} \times \vec{F}_{\text{Cor}}].
\]

Examining the results obtained, it is worth stressing that the remarkable proportionality (generalized alignment) of the ‘complexified gravitational intensity’ (27) and the ‘complex electromagnetic strength’ (23) was in no way assumed initially. It is therefore a surprising and highly promising fact. Although the implications of the coincidence of ‘complexified directions’ of the gravitational and electromagnetic field strengths in the Kerr-Newman space-time are not manifest yet, this relationship cannot be a mere occasion. In any case, it should be regarded as a strong indirect argument in favor of the plausibility of the expressions for the true gravitational intensities suggested, as well as gives evidence for the validity of the approach applied at whole.

Another curious fact worth mentioning here is the orthogonality of the centrifugal and Coriolis intensities:
\[
\vec{F}_{\text{cf}} \cdot \vec{F}_{\text{Cor}} = 0.
\]
It also was not assumed beforehand, gaining therefore some new insight into the properties of the inertia manifestations.

7 Boyer-Lindquist picture

It is worth transforming the main relationships derived above to the Boyer-Lindquist co-ordinates [42] which are usually utilized when considering the physics-related properties of Kerr and Kerr-Newman metrics, cf. Ref. [43].

Reminding the standard abbreviating notations
\[
\Delta = \Delta(r) = r^2 - 2mr + a^2 + e^2, \quad \rho^2 = \rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta
\]
and introducing the following additional ones
\[
\Upsilon_{(p)} = \Upsilon_{(p)}(\theta) = 1 - \omega a \sin^2 \theta, \quad \Upsilon_{(q)} = \Upsilon_{(q)}(r) = a - \omega (a^2 + r^2), \quad \Gamma = \Gamma(r, \theta) = \Upsilon_{(p)}^2 \Delta - \Upsilon_{(q)}^2 \sin^2 \theta > 0,
\]
\[
\vec{\mathcal{Y}} = \Upsilon_{(p)} \Delta \frac{\partial}{\partial r} - i \Upsilon_{(q)} \sin^2 \theta \frac{\partial}{\partial \theta},
\]

it can be shown that the complexified strength of the electromagnetic field \( \vec{E} + i \vec{H} \), the complexified intensity of gravitational field \( \vec{F}_{\text{attr}} - \frac{i}{2} \vec{F}_{\text{gyr}} \), and and intensities of inertia manifestations \( \vec{F}_{\text{cf}}, \vec{F}_{\text{Cor}} \) equal
\[
\vec{E} + i \vec{H} = -\frac{e}{(r + ia \cos \theta)^2} \sqrt{\rho^2 \Gamma} \vec{\mathcal{Y}}, \quad (34)
\]
\[
\vec{F}_{\text{attr}} - \frac{i}{2} \vec{F}_{\text{gyr}} = -\left\{ \left( m \left( r^2 - a^2 \cos^2 \theta \right) - e^2 r \right) \Upsilon_{(p)} + i \left( 2mr - e^2 \right) \Upsilon_{(q)} \cos \theta \right\} \frac{\vec{\mathcal{Y}}}{\rho^2 \Gamma}, \quad (35)
\]
\[ \mathbf{F}_{\text{cl}} = \omega^2 \frac{\Delta \sin \theta}{\Gamma} \left[ r \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} \right], \]  
\[ \mathbf{F}_{\text{Cor}} = 2\omega \frac{\Upsilon_{(p)} \Delta - a \Upsilon_{(q)} \sin^2 \theta}{\rho^2 \Gamma} \left[ \Delta \cos \theta \frac{\partial}{\partial r} - r \sin \theta \frac{\partial}{\partial \theta} \right]. \]

It is worth noting that except of the \( \Gamma \)-dependent factors involved in denominators (which play the role of the special relativistic \( \gamma \)-factor) all the dependencies on the formal frequency of observers rotation \( \omega \) are either linear (for electromagnetic field) or quadratic (for gravitational field and inertia).

The representation (30)-(37) is convenient in particular for the analysis of the physically important limiting cases. Choosing \( a = 0 \), the field strengths in the Reissner-Nordström space-time follow. If \( e = 0 \) the characteristics of the Kerr field and, with \( a = 0 \), Schwarzschild field immediately result.

A weak field limit can be easily examined as well. It can be shown that it completely conforms to the corresponding non-relativistic (Newtonian) values. Additionally, the following expression for the gyroscopic (gravitomagnetic) gravitational intensity detected by asymptotically non-rotated observers (\( \omega = 0 \)) in the case of Kerr space-time is of interest:

\[ \mathbf{F}_{\text{gyr}} \simeq \frac{2ma}{r^3} \left( 2 \cos \theta \frac{\partial}{\partial r} - \frac{\sin^2 \theta \, \partial}{r \, \partial \theta} \right). \]

It possesses no immediate Newtonian analogue, of course.

8 Summary and discussion

Summarizing, we started with a simple observation that in the case of the stationary Kerr-Newman space-time the natural assumption on the local constancy of the light speed (statement (A) of section 2) and the well known description of ideal clocks (statement (B) therein) imply the unique manifest representation of the spatial relations referring to the notion of a distance. Specifically, the uniformly rotated observers are in fact immersed in the ‘constant’ (static) curved Riemannian space endowed with the metric (9) — the ‘apparent space’.

Accordingly, it immediately follows from the above statement that, tracing the motion of a test particle, the observers find that its vector velocity, vector momentum and energy are described by the equations (11) which are deduced by means of the standard relativistic mechanics applied in vicinity of the observation point.

Next, given the energy and momentum of a particle, their rate of variation can be determined, bearing in mind the subsequent introducing the notion of observable force acting to a particle. At this stage an additional guiding assumption is to be postulated: it asserts that the infinitesimal variation of a vector ‘attached’ to the moving particle is determined with the help of the torsion-free connection compatible with the apparent metric which yields the ‘physical measure’ of a distance (see the claim (C) in section 4).
After some technical work we obtain the explicit expression $E_{\vec{F}} = \vec{F}_{\text{mass}} + [\vec{P} \times \vec{F}_{\text{mom}}] + \vec{F}_{\text{ext}}$ for the observable force $\vec{F}$ applied to the particle (for the meaning of the notations see Eqs. (16), (17), (18)–(20)).

The principle of the separating the above constituents of the total force $\vec{F}$ determines their meaning. Specifically, $\vec{F}_{\text{ext}}$ is the external (non-gravitational and non-inertial) force closely connected with the standard 4-force (its components are linear homogeneous functions of the components of the latter). In the case of a free charged particle it coincides with the Lorentz force $e(\vec{E} + [\vec{v} \times \vec{H}])$, where $\vec{E}$ and $\vec{H}$ are the observable electric and magnetic strengths, respectively, determined by Eq. (23). Further, $E_{\vec{F}_{\text{mass}}}$ is the attractive ‘massive’ mixed gravitational-inertial force, $\vec{F}_{\text{mass}}$ being characteristic of its intensity. Similarly, $[\vec{P} \times \vec{F}_{\text{mom}}]$ is the mixed gravitomagnetic-Coriolis force possessing the own intensity $\vec{F}_{\text{mom}}$. It is important to note that both $\vec{F}_{\text{mass}}$ and $\vec{F}_{\text{mom}}$ do not involve any specific characteristic of a test particle or its motion and are determined by the space-time geometry (and depend on the observers).

The final crucial step enabling one to describe in physical terms the gravitational field associated with the model considered is a plausible separation of the combined gravitational-inertial forces (and the corresponding intensities) into their purely gravitational and purely inertial constituents. The trick utilized for this purpose is based on the inspection of the variation of the combined gravitational-inertial force ‘in response’ to the variation of the space-time geometry running over the class of metrics described by Eq. (1) which involves two arbitrary functions $\mathcal{P}(p)$ and $\mathcal{Q}(q)$. (Strictly speaking, this procedure requires the leaving the class of Kerr-Newman metrics. Said another way, it would be insufficient to take the metric (1) with the functions $\mathcal{P}(p)$, $\mathcal{Q}(q)$ fixed to the form (3) or (2) alone.)

More definitely, the common terms in the gravitational-inertial intensities and the space-time curvature components which vanish strictly simultaneously for arbitrary functions $\mathcal{P}$, $\mathcal{Q}$ are singled out. (In accordance with their role these terms were named pre-curvatures, see section 6). There exist all the reasons to identify the corresponding constituents of the intensities with the pure gravitational contribution. Taking them off from the combined gravitational-inertial forces, it is natural to interpret the residual forces as pure inertial in a nature.

It is worth noting that the method of the separating the inertial and gravitational intensities applied — a ‘variation’ of the space-time geometry — cannot be associated with any conceivable ‘local experiment’ and thus our result does not conflict with the equivalence principle (cf. Ref. [11]).

Ultimately, we obtain the attractive gravitational (gravitoelectric) intensity $\vec{F}_{\text{attr}}$ and the gyroscopic (gravitomagnetic) intensity $\vec{F}_{\text{gyr}}$ determined by Eq. (27) and Eq. (35), the centrifugal inertial intensity (Eqs. (28), (36)), and the Coriolis intensity (Eqs. (29), (37)), respectively.

These equations reveal a proper behavior in the weak field limit. Moreover, the tools which we used for their derivations are based on a remnant of the model which is directly inherited from the physics of Minkowski space-time. Accordingly, the method applied proves to be strictly unique and gains advantage of an unambiguous physical interpretation.
It is instructive to mention that the specific role of light signals (expressed by the statement (A) in section (2)) does not attribute to the trajectories of light rays the role of a ‘standard’ of a constant direction. Specifically, we used the statement (A) for the deriving the formula (9) determining local distance. Here the consideration is local and the geodesic property of worldlines of photons (second order effect) is not actually used. All what we need here is the vanishing on them of the interval (1), i.e. the fact that worldlines of light signals are null. Accordingly, there is no reason for finite segments of light rays to be straight (geodetic) with respect to the observable geometry of the apparent space. It is indeed the case and, generically, the observable acceleration of photons does not vanish. In physical terms this means just the varying of the ray direction, providing the theoretical base for invariant interpretation of the light deflection effect.

Photons feel the same gravitational and inertial forces as the particles possessing a rest mass do. However the extrinsic force $\vec{F}_{\text{ext}}$ is here identically zero (precisely because photon worldlines are geodesic). In this respect our approach differs from one by Abramowicz et al where the rays are regarded as ‘dynamically straight’ [13].

Another point worth mentioning is the problem of a kind of mass involved in gravitational interaction. Here we shall speak about the mass of test particles. In according with the above derivation their ‘gravitational charge’ is $\mathcal{E} = \mu/\sqrt{1 - \vec{v}^2}$ (let us remind that the scalar product and, in particular, $\vec{v}^2$ is to be determined with respect to the metric of the apparent space). Thus we have an instance of manifestation of the equivalence principle claiming in our case that the inertial mass $\mu/\sqrt{1 - \vec{v}^2}$ characterizes also the gravitational properties of a particle.

Now we would like to mention some simple consequences which are more or less automatically inferred from the results presented.

First, it is worth noting that the formulae (34), (35) yield certain characteristic of the ‘classical spin’ of the electromagnetic and gravitational fields. Obviously, it can be associated with the power of the factor $\Gamma^{-1/2}$, cf. [14], Eq. (12a)]. Hence, measuring strengths of the electromagnetic field for the various angular velocities of observers, one finds the spin equal to 1 while for the gravitational field it amounts to 2. One thus has to be careful when resorting to too direct analogues between the gravitation and the electromagnetism.

Further, although it could seem surprising, one has to doubt of the widespread regarding of the gravitational field as a source of tidal effects, assuming it to be proportional to (the proper projections of) the curvature tensor. Indeed, we see that in the case of Kerr-Newman field at least both the curvature and field intensities are subsidiary objects derived from another one named pre-curvature which depends on the first order derivatives of the metric. (Unfortunately, the covariant meaning of pre-curvature is not known yet.) It is essential that the curvature involves the derivatives of the pre-curvature elements while the gravitational field intensities are simply their linear functions. Thus the intensity of gravitational field is not a derivative of the space-time curvature. They are not algebraically connected (but nevertheless vanish strictly simultaneously).

Finally, one may express a hope that the surprising remarkable ‘complex proportionality’ of ‘the complexified strengths’ $\vec{E} + i\vec{H}$ and $\vec{F}_{\text{attr}} - i\vec{F}_{\text{gyr}}$ for the electromagnetic and
gravitational fields, respectively (see Eqs. (24), (33)), as well as the very singling out the purely gravitational field intensities, could yield a new insight in our concepts concerning the physical manifestations of the gravitational field.

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