On the analytic zero divisor conjecture of Linnell

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Abstract. In this note we prove that in the case of finitely generated amenable groups the classical zero divisor conjecture implies the analytic zero divisor conjecture of Linnell.
1 Introduction

Let $G$ be a discrete group. Denote by $\mathbb{C}G$ the complex group algebra of $G$. The following conjecture is called the zero divisor conjecture.

**Conjecture 1** Let $G$ be a torsion free group. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in \mathbb{C}G$, then $\alpha \beta \neq 0$.

Now let

$$L^2(G) = \{ \alpha : G \to \mathbb{C} \mid \sum_{g \in G} |\alpha(g)|^2 < \infty \}.$$ 

Then the Hilbert space $L^2(G)$ is a two-sided $\mathbb{C}G$-module. In [3], Linnell formulated an analytic version of the zero divisor conjecture.

**Conjecture 2** Let $G$ be a torsion free group. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^2(G)$, then $\alpha \beta \neq 0$.

Since $\mathbb{C}G \subseteq L^2(G)$, the second conjecture implies the first one. The goal of this paper is to prove that for finitely generated amenable groups the two conjectures are actually equivalent.

**Theorem 1** Let $G$ be a finitely generated amenable group. If $0 \neq \alpha \in \mathbb{C}G$, $0 \neq \beta \in L^2(G)$ and $\alpha \beta = 0$, then there exists $0 \neq \gamma \in \mathbb{C}G$ such that $\alpha \gamma = 0$.

2 Invariant subspaces and the von Neumann dimension

A linear subspace $V \subseteq L^2(G)$ is called an invariant subspace it is a right $\mathbb{C}G$-submodule of $L^2(G)$, that is if $\beta \in V$ and $\gamma \in \mathbb{C}G$, then $\beta \gamma \in V$. The von Neumann dimension of an invariant subspace $V$ is a real number defined the following way [4],

$$\dim_G V = \langle P_V 1_e, 1_e \rangle$$

where $P_V$ denotes the orthogonal projection onto the closure of $V$ and $1_e \in L^2(G)$ takes the value 1 on the unit of $G$ and vanishes everywhere else. We list some properties of the von Neumann dimension [4].

1. If $V \neq 0$, then $\dim_G(V) > 0$.
2. $\dim_G(L^2(G)) = 1$.
3. If $V \subseteq W$ are invariant subspaces, then $\dim_G(V) \leq \dim_G(W)$.
4. If $T : L^2(G) \to L^2(G)$ is a bounded linear transformation and a right $\mathbb{C}G$-module homomorphism, then

$$\dim_G(\text{Ker} \ T) + \dim_G(\text{Ran} \ T) = 1.$$
For an arbitrary linear subspace $W \subseteq L^2(G)$ and a finite subset $A \subseteq G$ one can consider
\[
\dim_A(W) = \frac{\sum_{g \in A} \langle P_W 1_g, 1_g \rangle}{|A|},
\]
where $1_g \in L^2(G)$ takes the value 1 on $g$ and vanishes everywhere else. [2]. Obviously, $\dim_A(W)$ is a non-negative real number, and if $W \subseteq Z$, then $\dim_A(W) \leq \dim_A(Z)$. If $V$ is an invariant subspace, then for any finite subset $A \subseteq G$, $\dim_A(V) = \dim_G(V)$. Note that if $W$ is a finite dimensional subspace consisting of elements supported on $A$, then $\dim_A(W) = \dim C W | A |$.

3 Approximation of the von Neumann dimension

The goal of this section is to prove a variant of a result of Dodziuk and Mathai [1]. First let us recall the notion of amenability. Let $G$ be a finitely generated group with symmetric generator set $\{g_1, g_2, \ldots, g_k\}$. The generators determine a word-metric $d$ on $G$ such that the right multiplications by the elements of $G$ are isometries. The group $G$ is called amenable if it has an exhaustion $F_1 \subseteq F_2 \subseteq \cdots, \bigcup_{k=1}^\infty F_k = G$ by finite subsets such that for any fixed $r \in \mathbb{N}$,
\[
\lim_{k \to \infty} \frac{\partial_r F_k}{|F_k|} = 0,
\]
where $\partial_r F_k = \{g \in F_k \mid d(g, G \setminus F_k) \leq r \}$. Before stating our proposition we list some notations and definitions. Let $\alpha \in \mathbb{C}G$, then $M_\alpha : L^2(G) \to L^2(G)$ is the left multiplication by $\alpha$. Then $\text{Ker} M_\alpha$ is an invariant subspace. The width of $\alpha$, $w(\alpha)$ is defined as $\max_{g \in G, \alpha(g) \neq 0} d(e, g)$, where $e$ is the unit of $G$. Let $G_k = F_k \setminus \partial_w(\alpha) F_k$. Note that if $\beta$ is supported on $G_k$, then $\alpha \beta$ will be supported on $F_k$. We consider the following spaces,
\[
V_k = \{\beta \in L^2(G) \mid \text{supp}(\beta) \subseteq F_k\}
\]
\[
W_k = \{\beta \in L^2(G) \mid \text{supp}(\beta) \subseteq G_k\}
\]

Proposition 1 $\dim_G(\text{Ker} M_\alpha) = \lim_{k \to \infty} \dim F_k (V_k \cap \text{Ker} M_\alpha)$.

Proof: Let $M_\alpha^k : W_k \to V_k$ the restriction of $M_\alpha$ onto $W_k$. Then $\text{Ker} M_\alpha^k \subseteq V_k \cap \text{Ker} M_\alpha$, $\text{Ran} M_\alpha^k \subseteq \text{Ran} M_\alpha$ and $\dim C \text{Ker} M_\alpha^k + \dim C \text{Ran} M_\alpha^k = |G_k|$. Therefore,
\[
\dim F_k (\text{Ker} M_\alpha^k) \leq \dim F_k (V_k \cap \text{Ker} M_\alpha) \leq \dim_G(\text{Ker} M_\alpha) \tag{1}
\]
\[
\dim F_k (\text{Ran} M_\alpha^k) \leq \dim_G(\text{Ran} M_\alpha) \tag{2}
\]
\[ \dim \mathcal{F}_k(\text{Ker} \, M^k_\alpha) + \dim \mathcal{F}_k(\text{Ran} \, M^k_\alpha) = \frac{|G^k|}{|\mathcal{F}_k|} \]  

(3)

By (3),

\[ \lim_{k \to \infty} (\dim \mathcal{F}_k(\text{Ker} \, M^k_\alpha) + \dim \mathcal{F}_k(\text{Ran} \, M^k_\alpha)) = 1 = \dim_G(\text{Ker} \, M_\alpha) + \dim_G(\text{Ran} \, M_\alpha) \]  

(4)

Hence, by (2) and (3), \( \lim_{k \to \infty} \dim \mathcal{F}_k \text{Ker} \, M^k_\alpha = \dim_G(\text{Ker} \, M_\alpha) \), thus the Proposition follows.

4 The proof of the Theorem

Let \( 0 \neq \alpha \in \mathbb{C}G \), \( 0 \neq \beta \in L^2(G) \) and \( \alpha \beta = 0 \). Then \( \text{Ker} \, M_\alpha \neq 0 \), hence \( \dim_G(\text{Ker} \, M_\alpha) > 0 \). By the Proposition, for sufficiently large \( k \), there exists \( 0 \neq \gamma \in \mathbb{V}_k \cap \text{Ker} \, M_\alpha \). Then \( \gamma \in \mathbb{C}G \) and \( \alpha \gamma = 0 \).

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