THE GINZBURG-LANDAU ORDER PARAMETER NEAR THE SECOND CRITICAL FIELD

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Abstract. In Ginzburg-Landau Theory of superconductivity, the density and location of the superconducting electrons is measured by a complex-valued wave function, the order parameter. In this paper, when the intensity of the applied magnetic field is close to the second critical field, and when the order parameter minimizes the Ginzburg-Landau functional defined over a two dimensional domain, the leading order approximation of its $L^2$-norm in 'small' squares is given as the Ginzburg-Landau parameter tends to infinity.

1. Introduction and main results

In this paper, we study the minimizers of the Ginzburg-Landau functional of superconductivity. In a two dimensional simply connected domain $\Omega$ with smooth boundary, the functional is defined over configurations $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ as follows,

$$
E(\psi, A) = \int_\Omega e_{\kappa, H}(\psi, A) \, dx
= \int_\Omega \left( |(\nabla - i\kappa H A)\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 + (\kappa H)^2 |\text{curl} A - 1)|^2 \right) \, dx. \quad (1.1)
$$

The modulus of the wave function $|\psi|$ measures the density of the superconducting electrons; the curl of the vector field $A$ measures the induced magnetic field; the parameter $H$ measures the intensity of the external magnetic field and the parameter $\kappa$ is a characteristic of the superconducting material. The ground state energy of the functional is,

$$
E_{\text{gs}}(\kappa, H) = \inf \{ E(\psi, A) \, : \, (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \}. \quad (1.2)
$$

The behavior of the ground state energy $E_{\text{gs}}(\kappa, H)$ and of the minimizers depend strongly on the intensity of the external field. Loosely speaking, there exist three critical values $H_{C_1}(\kappa)$, $H_{C_2}(\kappa)$ and $H_{C_3}(\kappa)$ such that, when the parameter $\kappa$ is sufficiently large and $(\psi_{\kappa, H}, A_{\kappa, H})$ is a minimizer of the functional in (1.1), the following is true:

- If the parameter $H$ satisfies $H < H_{C_1}$, then $|\psi_{\kappa, H}| > 0$ everywhere;
- if $H_{C_1} < H < H_{C_2}$, then $|\psi_{\kappa, H}|$ has isolated zeros; these zeros become evenly distributed in the domain $\Omega$ as $H$ becomes very close to $H_{C_2}$;
- if $H_{C_2} < H < H_{C_3}$, $|\psi_{\kappa, H}|$ is localized near the boundary of the domain $\Omega$;
- if $H > H_{C_3}$, $|\psi_{\kappa, H}| = 0$ everywhere.

In this paper, we investigate the behavior of the minimizers when $H$ is close to the critical value $H_{C_2}$. Existing mathematical results \cite{2, 3, 8, 11, 13} suggest that

$$
H_{C_2}(\kappa) = \kappa + o(\kappa) \quad \text{as} \ \kappa \to \infty.
$$

In \cite{8}, when the parameter $H$ satisfies

$$
H = \kappa + o(\kappa), \quad (\kappa \to \infty),
$$

it is obtained the following formula for the ground state energy,

$$
E_{\text{gs}}(\kappa, H) = E_{\text{surf}}|\partial \Omega|\kappa + E_{\text{Ab}}|\Omega| [\kappa - H]^2_+ + o \left( \max \left( \kappa, |\kappa - H|^2_+ \right) \right), \quad (\kappa \to \infty). \quad (1.3)
$$
Here $E_{\text{surf}}$ and $E_{\text{Ab}}$ are two universal constants, of which $E_{\text{Ab}}$ is related to the celebrated Abrikosov energy $[1]$. $|\partial \Omega|$ is the arc-length measure of the boundary and $|\Omega|$ is the Lebesgue (area) measure.

As a consequence of (1.3), if $(\psi, A)$ is a minimizer of the functional in (1.1), then,
\[
\int_{\Omega} |\psi|^4 \, dx = -\frac{2E_{\text{surf}}}{\kappa} |\partial \Omega| - 2E_{\text{Ab}} \left[ 1 - \frac{H}{\kappa} \right]^2 |\Omega| + o \left( \max \left( \frac{1}{\kappa}, \left[ 1 - \frac{H}{\kappa} \right]^2 \right) \right), \quad (\kappa \to \infty).
\]

Clearly, if the parameter $H$ satisfies $\kappa^{-1/2} \ll 1 - \frac{H}{\kappa} \ll 1$, then the bulk term in (1.4) is the dominant term. In this case, (1.4) is compatible with the following $L^\infty$-bound obtained in $[3]$,
\[
\|\psi\|_{L^\infty(\Omega_\kappa)} \leq C \left[ 1 - \frac{H}{\kappa} \right]^{1/2},
\]
where $\Omega_\kappa = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \kappa^{-1/4} \}$ and $C$ is a constant. In this paper we establish the additional asymptotics of $|\psi|^2$,
\[
\int_{\Omega} |\psi|^2 \, dx = -2E_{\text{Ab}} \left[ 1 - \frac{H}{\kappa} \right] |\Omega| + o \left( \left[ 1 - \frac{H}{\kappa} \right] \right), \quad (\kappa \to \infty).
\]

More precisely, the main result is:

**Theorem 1.1.** Suppose that $H$ is a function of $\kappa$, $H > \kappa$ and
\[
\kappa^{-1/2} \ll 1 - \frac{H}{\kappa} \ll 1, \quad (\kappa \to \infty).
\]

Let $Q_\ell$ be a square of side length $\ell$ such that,
\[
Q_\ell \subset \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \kappa^{-1/4} \},
\]
$\ell$ a function of $\kappa$, $\frac{\ell^2 \kappa H}{2\pi} \in \mathbb{N}$ and $1 - \frac{H}{\kappa} \ll \ell \ll 1$ as $\kappa \to \infty$.

If $(\psi, A)$ is a minimizer of the functional in (1.1), then,
\[
\frac{1}{|Q_\ell|} \int_{Q_\ell} |\psi|^2 \, dx = -2E_{\text{Ab}} \left[ 1 - \frac{H}{\kappa} \right] + o \left( 1 - \frac{H}{\kappa} \right), \quad (\kappa \to \infty).
\]

Here $E_{\text{Ab}} \in [-\frac{1}{2}, 0]$ is a universal constant defined in (3.10).

A key step to prove Theorem 1.1 is the approximation of the order parameter $\psi$ by a periodic eigenfunction of the Landau Hamiltonian (see Theorem 4.5). In [3], such an approximation is given when $\kappa$ and $H$ satisfy,
\[
\kappa^{-2/5} \ll 1 - \frac{H}{\kappa} \ll \frac{1}{(\ln \kappa)^2}, \quad (\kappa \to \infty).
\]

This assumption is restrictive compared to that of Theorem 1.1. Furthermore, the result of Theorem 1.1 goes beyond the result of [3] as the formula (1.7) is new.

Notice that the result of Theorem 1.1 is particularly useful if we know that $E_{\text{Ab}} = -\frac{1}{2}$. If this is the case, (1.4) and (1.6) together yield,
\[
\lim_{\kappa \to \infty} \int_{\Omega} \left( A - |u|^2 \right) \, dx = 0, \quad A = -2E_{\text{Ab}}, \quad u = \left[ 1 - \frac{H}{\kappa} \right]^{-1/2} \psi.
\]

Such a bound is helpful to construct a vortex structure of $u$ [12]. However, if $E_{\text{Ab}} < -\frac{1}{2}$, a convergence such as the one in (1.8) does not hold and the profile of $|\psi|^2$ is not homogeneous a.e. in $\Omega$.

We conclude this introduction by clarifying some notation that will be used throughout this paper. If $a(\kappa)$ and $b(\kappa)$ are two positive functions, we write $a(\kappa) \approx b(\kappa)$ to mean that there exist

\footnote{The notation $a(\kappa) \ll b(\kappa)$ means that $a$ and $b$ are positive functions and $a/b \to 0$ as $\kappa \to \infty$}
positive constants $\kappa_0, c_1$ and $c_2$ such that $c_1 b(\kappa) \leq a(\kappa) \leq c_2 b(\kappa)$ for all $\kappa \geq \kappa_0$. The notation $a(\kappa) \sim b(\kappa)$ means that $a(\kappa) = b(\kappa)(1 + o(1))$ as $\kappa \to \infty$. The notation $a(\kappa) \ll b(\kappa)$ means $a(\kappa) = o(1) b(\kappa)$ as $\kappa \to \infty$. Constants in the remainder of inequalities are all denoted by the letter $C$, whose value might change from a line to another.

Finally, notice that in the parameter regime of Theorem 2.2

$$\left[1 - \frac{H}{\kappa}\right] = \left[\frac{\kappa}{H} - 1\right] (1 + o(1)), \quad (\kappa \to \infty).$$

This remark will be often used throughout the paper.

2. USEFUL ESTIMATES

In this section, we collect a priori estimates satisfied by critical points of the functional in (1.1). Notice that critical points of the functional in (1.1) satisfy the Ginzburg-Landau equations:

$$\begin{cases} 
-\nabla \cdot \text{curl } A = (\kappa H)^{-1} \text{Im}(\psi (\nabla - i\kappa H A)\psi), \quad \text{in } \Omega, \\
\nu \cdot (\nabla - i\kappa H A)\psi = 0, \quad \text{curl } A = 1, \quad \text{on } \partial \Omega.
\end{cases}$$

Here $\nu$ is the unit inward normal vector of $\partial \Omega$. The set of estimates in Theorem 2.1 appeared first in [10] (for a more particular regime) and then proved for a wider regime in [6].

**Lemma 2.1.** There exist positive constants $\kappa_0$ and $C$ such that, if $\kappa \geq \kappa_0$, $H \geq \frac{\kappa}{2}$ and $(\psi, A)$ is a solution of (2.7), then,

$$\|\text{curl } A - 1\|_{C^1(\Omega)} + \kappa^{-1} \|\text{curl } A - 1\|_{C^2(\Omega)} \leq C \kappa^{-1},$$

$$\|\nabla - i\kappa H A\psi\|_{L^\infty(\Omega)} \leq C \kappa, \quad e_{\kappa, H}(\psi, A) \leq C \kappa^2.$$ (2.3)

The sharp $L^\infty$ bound in the next theorem is established in [8]. It plays a key role in the proof of Theorem 1.1.

**Theorem 2.2.** Suppose that $\kappa$ and $H$ satisfy,

$$\kappa^{-1/2} \ll 1 - \frac{H}{\kappa} \ll 1, \quad (\kappa \to \infty).$$

There exist constants $C > 0$ and $\kappa_0$ such that, if $\kappa \geq \kappa_0$ and $(\psi, A)$ is a solution of (2.7), then,

$$\|\psi\|_{L^\infty(\Omega_\kappa)} \leq C \left[1 - \frac{H}{\kappa}\right]^{1/2},$$

where

$$\Omega_\kappa := \{x \in \Omega \mid \text{dist}(x, \partial \Omega) \geq \kappa^{-1/4}\}.$$

3. THE LIMITING PROBLEM

3.1. **Reduced Ginzburg-Landau functional.** Given a constant $b \geq 0$ and an open set $D \subset \mathbb{R}^2$, we define the following Ginzburg-Landau energy,

$$G_{b,D}(u) = \int_D \left(b(|\nabla - iA_0|u|^2 - |u|^2 + \frac{1}{2} |u|^4\right) dx.$$ (3.1)

Here $A_0$ is the canonical magnetic potential,

$$A_0(x_1, x_2) = \frac{1}{2} (-x_2, x_1), \quad (x = (x_1, x_2) \in \mathbb{R}^2).$$ (3.2)

We will consider the functional $G_{b,D}$ first with Dirichlet and later with (magnetic) periodic boundary conditions. It will be clear from the context what is meant.

Consider the functional with Dirichlet boundary conditions and for $b > 0$. If the domain $D$ is bounded, completing the square in the expression of $G_{b,D}$ shows that $G_{b,D}$ is bounded from
below. Thus, starting from a minimizing sequence, it is easy to check that $G_{b,D}$ has a minimizer. A standard application of the maximum principle shows that, if $u$ is any minimizer of $G_{b,D}$, then

$$|u| \leq 1, \quad \text{in } D,$$

see e.g. [72].

Given $R > 0$, we denote by $K_R = (-R/2, R/2) \times (-R/2, R/2)$ a square of side length $R$. Let,

$$m_0(b, R) = \inf_{u \in H^1_0(K_R; \mathbb{C})} G_{b,K_R}(u).$$  (3.4)

The following remark will be useful. If $u_R$ is a minimizer of $\mathcal{E}$, then $u_R$ satisfies the Ginzburg-Landau equation,

$$-(\nabla - iA_0)^2 u_R = b(1 - |u_R|^2) u_R, \quad \text{in } Q_R.$$  (3.5)

Recall that $u_R \in H_0^1(Q_R)$. We extend $u_R$ by magnetic periodicity to all $\mathbb{R}^2$, i.e. to a function in the space $E_R$ introduced below. That way, $u_R$ satisfies

$$-(\nabla - iA_0)^2 u = (1 - |u|^2)u,$$

in all $\mathbb{R}^2$. We can apply Theorem 3.1 in [4] to get that,

$$\forall b \in [1 - b_0, 1], \quad |u| \leq C_{\max} \sqrt{1 - b},$$  (3.6)

where $b_0$ and $C_{\max}$ are universal constants.

3.2. Periodic minimizers. We introduce the following space,

$$E_R = \left\{ u \in H^1_{0,loc}(\mathbb{R}^2; \mathbb{C}) : u(x_1 + R, x_2) = e^{iRx_2/2} u(x_1, x_2), \right. \left. u(x_1, x_2 + R) = e^{-iRx_1/2} u(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2 \right\}.  \quad (3.7)$$

Notice that the periodicity conditions in (3.6) are constructed in such a manner that, for any function $u \in E_R$, the functions $|u|$, $|\nabla A_0 u|$ and the vector field $\nabla A_0 u$ are periodic with respect to the lattice generated by $K_R$.

Recall the functional $G_{b,D}$ in (3.1) above. We introduce the ground state energy,

$$m_p(b, R) = \inf_{u \in E_R} G_{b,K_R}(u).$$  (3.8)

The next proposition exhibits a relation between the ground state energies $m_0(b, R)$ and $m_p(b, R)$. It is proved in [4].

**Proposition 3.1.** Let $m_0(b, R)$ and $m_p(b, R)$ be as introduced in (3.4) and (3.7) respectively. For all $b > 0$ and $R > 0$, we have,

$$m_0(b, R) \geq m_p(b, R).$$

Furthermore, there exist universal constants $\epsilon_0 \in (0, 1)$ and $C > 0$ such that, if $b \geq 1 - \epsilon_0$ and $R \geq 2$, then,

$$m_0(b, R) \leq m_p(b, R) + C[1 - b_0] + R.$$  (3.9)

3.3. The periodic Schrödinger operator with constant magnetic field. In this section, we assume the quantization condition that $|K_R|/(2\pi)$ is an integer, i.e. there exists $N \in \mathbb{N}$ such that,

$$R^2 = 2\pi N.$$  (3.10)

Recall the magnetic potential $A_0$ introduced in (3.2) above. Consider the operator,

$$P_R = -(\nabla - iA_0)^2$$ in $L^2(K_R),$  (3.11)
Theorem 3.4. Let $E_R$ be the self-adjoint realization associated with the closed quadratic form

$$E_R \ni f \mapsto Q_R(f) = \| (\nabla - i A_0) f \|_{L^2(K_R)}^2.$$  \hfill (3.11)

The operator $P_R$ is with compact resolvent. Denote by $\{\mu_j(P_R)\}_{j \geq 1}$ the increasing sequence of its distinct eigenvalues (i.e. without counting multiplicity).

The following proposition may be classical in the spectral theory of Schrödinger operators, but we refer to [2] or [3] for a simple proof.

Proposition 3.2. Assuming $R$ is such that $|K_R| \in 2\pi \mathbb{N}$, the operator $P_R$ enjoys the following spectral properties:

1. $\mu_1(P_R) = 1$ and $\mu_2(P_R) \geq 3$.
2. The space $L_R = \text{Ker}(P_R - 1)$ is infinite dimensional and $\dim L_R = |K_R|/(2\pi)$.

Consequently, denoting by $\Pi_1$ the orthogonal projection on the space $L_R$ (in $L^2(K_R)$) and by $\Pi_2 = \text{Id} - \Pi_1$, we have for all $f \in D(P_R)$,

$$\langle P_R \Pi_2 f, \Pi_2 f \rangle_{L^2(K_R)} \geq 3\| f \|_{L^2(K_R)}^2.$$  \hfill (3.12)

The next lemma is a consequence of the existence of a spectral gap between the first two eigenvalues of $P_R$. It is proved in [5].

Lemma 3.3. Given $p \geq 2$, there exists a constant $C_p > 0$ such that, for any $\gamma \in (0, \frac{1}{2})$, $R \geq 1$ and $f \in D(P_R)$ satisfying

$$Q_R(f) - (1 + \gamma)\| f \|_{L^2(K_R)}^2 \leq 0,$$  \hfill (3.13)

the following estimate holds,

$$\| f - \Pi_1 f \|_{L^p(K_R)} \leq C_p \sqrt{\gamma} \| f \|_{L^2(K_R)}.$$  \hfill (3.14)

Here $\Pi_1$ is the projection on the space $L_R$.

3.4. The Abrikosov energy. We introduce the following energy functional (the Abrikosov energy),

$$F_R(v) = \int_{K_R} \left( \frac{1}{2} |v|^4 - |v|^2 \right) dx.$$  \hfill (3.15)

The energy $F_R$ will be minimized on the space $L_R$, the eigenspace of the first eigenvalue of the periodic operator $P_R$,

$$L_R = \{ u \in E_R : P_R u = u \}.$$  \hfill (3.16)

We need the following theorem which we take from [2] [7].

Theorem 3.4. Let

$$\forall R > 0 , \quad c(R) = \min \{ F_R(u) : u \in L_R \}.$$  \hfill (3.17)

There exists a constant $E_{\text{Ab}} \in \left[ -\frac{1}{2}, 0 \right]$ such that,

$$E_{\text{Ab}} = \lim_{R \to \infty} \frac{c(R)}{R^2}.$$  \hfill (3.18)

It is observed in [7] that there is a relationship between the ground state energies $m_p(b, R)$ and $c(R)$. This is recalled in the next theorem.

Theorem 3.5. Let $m_p(b, R)$ and $c(R)$ be as introduced in (3.7) and (3.16) respectively. For all $b > 0$ and $R > 0$, we have,

$$m_p(b, R) \leq |1 - b|^2 c(R).$$  \hfill (3.19)

Furthermore, there exist universal constants $\epsilon_0 \in (0, 1)$ and $C > 0$ such that, if $R \geq 2$, $b \geq 1 - \epsilon_0$, and $0 < \sigma < 1/2$, then,

$$m_p(b, R) \geq |1 - b|^2 \left( (1 + 2\sigma)c(R) - C\sigma^{-3}(1 - b)^2 R^4 \right).$$  \hfill (3.20)
4. Energy in small squares

In this section, the notation $Q_\ell$ stands for a square in $\mathbb{R}^2$ of side length $\ell > 0$

$$Q_\ell = (-\ell/2 + a_1, a_1 + \ell/2) \times (-\ell/2 + a_2, a_2 + \ell/2),$$

where $a = (a_1, a_2) \in \mathbb{R}^2$.

If $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times \dot{H}^1(\Omega; \mathbb{R}^2)$, we denote by $e(\psi, A) = |(\nabla - i\kappa H A)\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2} |\psi|^4$. Furthermore, we define the Ginzburg-Landau energy of $(\psi, A)$ in a domain $\mathcal{D} \subset \Omega$ as follows,

$$\mathcal{E}(\psi, A; \mathcal{D}) = \int_{\mathcal{D}} e(\psi, A) \, dx + (\kappa H)^2 \int_{\Omega} |\text{curl}(A - F)|^2 \, dx. \quad (4.1)$$

Also we introduce the functional,

$$\mathcal{E}_0(\psi, A; \mathcal{D}) = \int_{\mathcal{D}} \left( |(\nabla - i\kappa H A)\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) \, dx. \quad (4.2)$$

If $\mathcal{D} = \Omega$, we sometimes omit the dependence on the domain and write $\mathcal{E}_0(\psi, A)$ for $\mathcal{E}_0(\psi, A; \Omega)$.

The results of this section will be derived under the assumption that the magnetic field $H$ satisfies,

$$H = \kappa - \mu(\kappa) \sqrt{\kappa}, \quad (4.3)$$

where the function $\mu(\kappa)$ satisfies,

$$\liminf_{\kappa \to \infty} \mu(\kappa) = \infty \quad \text{and} \quad \limsup_{\kappa \to \infty} \frac{\mu(\kappa)}{\sqrt{\kappa}} = 0. \quad (4.4)$$

The assumptions (4.3)-(4.4) are equivalent to those in Theorem 1.1. As mentioned in the introduction, (4.3)-(4.4) cover a range of the parameter $H$ wider than the one assumed in [3]. As a consequence, the results here are stronger than those of [3].

**Proposition 4.1.** Suppose that the magnetic field $H$ satisfies (4.3) and (4.4). There exist positive constants $C$, $R_0$ and $\kappa_0$ such that the following is true. Let $\kappa$, $\ell$, and $\delta$ satisfy $\kappa \geq \kappa_0$, $R_0 \kappa^{-1} \leq \ell \leq 1/2$ and $\delta \in [0, 1[$. If $(\psi, A) \in H^1(\Omega; \mathbb{C}) \times \dot{H}^1(\Omega; \mathbb{R}^2)$ is a minimizer of (1.1), and $Q_\ell \subset \Omega$ is a square of side length $\ell$ satisfying,

$$Q_\ell \subset \Omega_\kappa = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \kappa^{-1/4} \},$$

then,

$$\frac{1}{|Q_\ell|} \mathcal{E}_0(\psi, A; Q_\ell) \leq (1 + \delta) [\kappa - H]^2 \frac{c(R)}{R^2} + C \left( \delta^{-1} \ell^2 \kappa + \delta \kappa + \ell^{-1} \right) [\kappa - H].$$

Here $R = \sqrt{\kappa} H \ell$, $c(R)$ is the function introduced in (3.16) and $\mathcal{E}_0$ is the functional introduced in (4.2).

**Proof.** After performing a gauge transformation, we may suppose that the magnetic potential $A$ satisfies (see [3, (5.31)]),

$$|A(x) - A_0(x)| \leq \frac{C \ell}{\sqrt{\kappa} H}, \quad \forall \ x \in Q_\ell, \quad (4.5)$$

where $A_0$ is the magnetic potential introduced in (3.2).

Without loss of generality, we may assume that,

$$Q_\ell = (-\ell/2, \ell/2) \times (-\ell/2, \ell/2) \subset \Omega_\kappa.$$

Let $b = H/\kappa$, $R = \ell \sqrt{\kappa} H$ and $u_R \in H_0^1(Q_R)$ be a minimizer of the functional $\mathcal{G}_{b, Q_R}$ introduced in (3.1), i.e. $\mathcal{G}_{b, Q_R}(u_R) = m_0(b, R)$ where $m_0(b, R)$ is introduced in (3.2).

Let $\chi_R \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function such that,

$$0 \leq \chi_R \leq 1 \quad \text{in} \ \mathbb{R}^2, \quad \text{supp} \chi_R \subset Q_{R+L}, \quad \chi_R = 1 \quad \text{in} \ Q_R.$$
and \(|\nabla \chi_R| \leq C/L\) for some universal constant \(C\). Let 
\(\eta_R(x) = 1 - \chi_R(x \sqrt{\kappa H})\) for all \(x \in \mathbb{R}^2\).
We introduce the function (whose construction is inspired from \([13]\))
\[
\varphi(x) = 1_{Q_\ell}(x)u_R(x \sqrt{\kappa H}) + \eta_R(x)\psi(x), \quad \forall \ x \in \Omega.
\]
(4.6)
Notice that by construction, \(\varphi\) satisfies,
\[
\varphi(x) = \begin{cases} 
   u_R(x \sqrt{\kappa H}) & \text{if } x \in Q_\ell, \\
   \eta_R(x \sqrt{\kappa H})\psi(x) & \text{if } x \in Q_{\ell + \frac{1}{\sqrt{\kappa H}} \setminus Q_\ell}, \\
   \psi(x) & \text{if } x \in \Omega \setminus Q_{\ell + \frac{1}{\sqrt{\kappa H}} \setminus Q_\ell}.
\end{cases}
\]
This allows us to get that, for all \(\delta \in (0, 1)\) (see \([7, (4.13)]\)),
\[
\mathcal{E}(\varphi, A; \Omega) \leq \mathcal{E}(\psi, A; \Omega \setminus Q_{\ell}) + \left(\frac{1 + \delta}{b}\right)m_0(b, R) + r_0(\kappa),
\]
(4.7)
where \(m_0(b, R)\) is defined in \([3,4]\), and for some constant \(C\), \(r_0(\kappa)\) is given as follows,
\[
r_0(\kappa) = C\left[\delta^{-1}(\kappa H)^2\|A - A_0\|^2_{L^2(Q_{\ell})} + \delta \ell^2 \kappa^2 \|u_R\|^2\right]
\]
\[
+ \left[\mathcal{E}_0(\eta_R(x \sqrt{\kappa H})\psi, A; Q_{\ell + \frac{1}{\sqrt{\kappa H}} \setminus Q_\ell}) - \mathcal{E}_0(\psi, A; Q_{\ell + \frac{1}{\sqrt{\kappa H}} \setminus Q_\ell})\right].
\]
(4.8)
Since \((\psi, A)\) is a minimizer, we have,
\[
\mathcal{E}(\psi, A) \leq \mathcal{E}(\varphi, A; \Omega).
\]
Since \(\mathcal{E}(\psi, A; \Omega) = \mathcal{E}(\psi, A; \Omega \setminus Q_{\ell}) + \mathcal{E}_0(\psi, A; Q_{\ell})\), the estimate \([17]\) gives us,
\[
\mathcal{E}_0(\psi, A; Q_{\ell}) \leq \left(\frac{1 + \delta}{b}\right)m_0(b, R) + r_0(\kappa).
\]
We use the estimates in \([3,8]\) and Theorem \([3,5]\) to write,
\[
\mathcal{E}_0(\psi, A, Q_{\ell}) \leq \left(\frac{1 + \delta}{b}\right)\left(1 - b^3 + c(R) + C[1 - b]_+ R\right) + r_0(\kappa).
\]
(4.9)
Next we control the error term \(r_0(\kappa)\). The first term in \(r_0(\kappa)\) is controlled by using \([2,1, 4.5]\)
and \([3, 5]\). That way we write,
\[
\delta^{-1}(\kappa H)^2\|A - A_0\|^2_{L^2(Q_{\ell})} + \delta \ell^2 \kappa^2 \|u_R\|^2 \leq C\left(\delta^{-1} \ell^4 \kappa^2 + \delta \ell^2 \kappa^2\right)[1 - b].
\]
The second term in \(r_0(\kappa)\) is controlled as follows. An integration by parts allows us to write,
\[
\mathcal{E}_0(\eta_R(x \sqrt{\kappa H})\psi, A; Q_{\ell + \frac{1}{\sqrt{\kappa H}} \setminus Q_\ell}) = \frac{\kappa^2}{2} \int_{Q_{\ell + \frac{1}{\sqrt{\kappa H}} \setminus Q_\ell}} \left(\eta_R^4(x \sqrt{\kappa H}) - 2\eta_R^2(x \sqrt{\kappa H})\right)|\psi|^4 \, dx
\]
\[
+ \int_{Q_{\ell + \frac{1}{\sqrt{\kappa H}} \setminus Q_\ell}} |\nabla \eta_R(x \sqrt{\kappa H})|^2 |\psi|^2 \, dx.
\]
Proposition 4.2. Suppose that the assumptions in Proposition 4.1 are true. Let be a cut-off function satisfying,

\[ \psi \in C^\infty(\mathbb{R}^2) \]

Remembering the definition of \( b \), we finish the proof of Proposition 4.1.

We use Theorem 2.2 to write,

\[ \mathcal{E}_0(\eta_R(x\sqrt{\kappa H})\psi, A; Q_{\ell,\kappa} \backslash Q_{\ell}) - \mathcal{E}_0(\psi, A; Q_{\ell,\kappa} \backslash Q_{\ell}) \leq C\ell|\kappa - 1| \]

Therefore, the term \( r_0(\kappa) \) satisfies,

\[ r_0(\kappa) \leq C\left(\delta^{-1}\ell^2\kappa^2 + \delta\kappa^2 + \ell^{-1}\kappa\right)[1 - b]\]

Remembering the definition of \( b = H/\kappa \) and the assumption in (4.3)-(4.4) on \( H \), we finish the proof of Proposition 4.1.

The proof of the next proposition is similar to that of Proposition 4.1.

Proposition 4.2. Suppose that the assumptions in Proposition 4.1 are true. Let \( \chi_\ell \in C^\infty_c(Q_{\ell}) \) be a cut-off function satisfying,

\[ \chi_\ell = 1 \text{ in } Q_{\ell,\kappa - 1}, \quad 0 \leq \chi_\ell \leq 1, \quad |\nabla \chi_\ell| \leq c\sqrt{\kappa H} \text{ in } Q_{\ell}, \]

where \( c \) is a universal constant.

There exists a constant \( C \) such that, if \( (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \) is a minimizer of (1.1), then,

\[ \frac{1}{|Q_{\ell}|}\mathcal{E}_0(\chi_\ell \psi, A; Q_{\ell}) \leq \frac{1 + \delta}{|Q_{\ell}|}\mathcal{E}_0(\psi, A; Q_{\ell}) + C\left(\delta^{-1}\ell^2\kappa + \delta\kappa + \ell^{-1}\kappa\right)[\kappa - H]. \]

Remark 4.3. Let \( A \) and \( B \) be two functions of \( \kappa \) such that,

\[ \mu \ll A \ll \kappa^{1/2} \quad \text{and} \quad 1 \ll B \ll \mu, \]

where \( \mu \) is as in (4.3)-(4.4).

The choice \( \delta = B\kappa^{-1/2} \) and \( \ell = \frac{[A\kappa^{-1/2}\sqrt{\kappa H}]}{\sqrt{2\pi \kappa H}} \approx A\kappa^{-1/2} \) makes the error terms in Propositions 4.1 and 4.2 of order \( o\left([\kappa - H]^2\right) \). Here \([\cdot]\) is the floor function (integer part). The choice of \( \ell \) forces \( R = \ell\sqrt{\kappa H} \) to satisfy \( R^2 \in 2\pi N \). This condition is needed to use the results of Section 3.2.

The above choice explains the assumption made on \( \ell \) in Theorem 1.1.

In the sequel, we suppose that the parameters \( \delta \) and \( \ell \) are selected as in Remark 4.3.

Theorem 4.4. Suppose that the magnetic field \( H \) satisfies (4.3) and (4.4). Let \( (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \) be a minimizer of (1.1) and \( Q_{\ell} \subset \Omega \) a square of side length \( \ell \) such that,

\[ Q_{\ell} \subset \Omega_\kappa = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \kappa^{-1/4} \}, \]

and \( \ell \) given in Remark 4.3. As \( \kappa \to \infty \), there holds:
Let $\psi$ be the function in Proposition 4.2. Define the function $\rho(x;\kappa;\ell) = \mathcal{E}_0(\psi;A;Q_\ell) + o(|\kappa-H|^2)$. Thus, $\rho(x;\kappa;\ell)$ satisfies $\frac{\kappa^2}{2} \int_{Q_\ell} |\psi|^4 \, dx = \mathcal{E}_0(\psi;A;Q_\ell) + \int_{\partial Q_\ell} \nu \cdot (\nabla - i \kappa H A) \psi \overline{\psi} \, dx.$ Thanks to the estimates in (4.1), (4.3) and the choice of $\ell$ in Remark 4.3, the boundary term is, $\int_{\partial Q_\ell} \nu \cdot (\nabla - i \kappa H A) \psi \overline{\psi} \, dx = O\left( \ell \kappa \left[ 1 - \frac{H}{\kappa} \right]^{1/2} \right) = o\left( \ell^2 |\kappa-H|^2 \right).$ Thus, $-\frac{1}{|Q_\ell|} \frac{\kappa^2}{2} \int_{Q_\ell} |\psi|^4 \, dx = \frac{1}{|Q_\ell|} \mathcal{E}_0(\psi;A;Q_\ell) + o(|\kappa-H|^2) = [\kappa-H]^2 E_{Ab} + o(|\kappa-H|^2).$

Finally, the assumption on the support of the function $\chi_\ell$, the bound (2.4) and the choice of $\ell$ together give us, $\int_{Q_\ell} |\chi_\ell \psi|^4 \, dx = \int_{Q_\ell} |\psi|^4 + O\left( \frac{\ell}{\sqrt{\kappa H}} \left[ 1 - \frac{H}{\kappa} \right]^{2} \right) = o\left( \left[ 1 - \frac{H}{\kappa} \right]^2 \right).$
There holds,
\[ \|v - \Pi_1(v)\|_{L^p(K_R)} \leq C \sqrt{1 - \frac{H}{\kappa}} \|v\|_{L^2(K_R)}, \quad (p \in \{2, 4\}), \]
and
\[ E_0(\psi, A_0; Q_\ell) \geq \int_{K_R} \left( \frac{1 - \frac{\kappa}{H}}{4} |\Pi_1(v)|^2 - \frac{\kappa}{2H} |v|^4 \right) dx \]
\[ \geq \left[ 1 - \frac{\kappa}{H} \right]^2 \left( \left( 1 + 2 \left( \sigma + 1 - \frac{\kappa}{H} \right) \right) c(R) - C\sigma^{-3} \left( 1 - \frac{H}{\kappa} \right)^2 R^4 \right), \quad (\sigma \in (0, 1/2)). \]

Here \( \Pi_1 \) is the projection introduced in Proposition 3.2.

Proof. Applying a translation, we may suppose that the center of \( Q_\ell \) is \( a_j = 0 \) (this amounts to a gauge transformation). We may select \( \kappa \) sufficiently large so that \( R = \ell\sqrt{\kappa H} \) lives in any preassigned neighborhood of infinity. That way, we have
\[ \frac{c(R)}{R^2} = E_{\text{Ab}}(1 + o(1)) \leq \frac{E_{\text{Ab}}}{2} < 0. \]

As a consequence, we get from Propositions 4.1 and 4.2 that,
\[ \int_{Q_\ell} \left( |\nabla - i\kappa H A_0|^{2|\chi(\sigma)| - \kappa^2|\chi(\sigma)|^2} \right) dx < 0. \]

The change of variable \( x \mapsto \sqrt{\kappa H} (x - a_j) \) yields,
\[ \int_{K_R} \left( |\nabla - iA_0|^{2|\chi(\sigma)| - (1 + \gamma)|\chi(\sigma)|^2} \right) dx < 0, \]
with \( \gamma = \frac{\kappa}{H} - 1 \approx 1 - \frac{\kappa}{H} \). The first estimate of Theorem 4.5 follows by applying Lemma 3.3.

Next we prove the remaining estimates of Theorem 4.5. Notice that the change of variable \( x \mapsto \sqrt{\kappa H} (x - a_j) \) and Theorem 3.2 together tell us,
\[ E_0(\psi, A_0; Q_\ell) = \int_{K_R} \left( |\nabla - iA_0|^{2|\chi(\sigma)| - \frac{\kappa}{H}|\chi(\sigma)|^2 |\chi(\sigma)|^2 + \frac{\kappa}{2H} |v|^4} \right) dx \]
\[ \geq \int_{K_R} \left( 1 - \frac{\kappa}{H} \right) |\Pi_1(v)|^2 + \frac{\kappa}{2H} |v|^4 \right) dx. \]  
(4.10)

Let \( b = \kappa/H \). Recall that \( \psi \) satisfies in \( Q_\ell \) the pointwise bound
\[ |\psi| \leq C \left[ 1 - \frac{H}{\kappa} \right]^{1/4} \approx [b - 1]^{1/4}. \]
Consequently, \(|v| \leq C[b - 1]^{1/2}\). This is the key estimate to finish the proof of Theorem 4.5. The method used is the same as that of [7, Theorem 2.11].

We established that \( \|v - \Pi_1(v)\|_{L^4(K_R)} \leq C \sqrt{1 - \frac{\kappa}{H}} \|v\|_{L^2(K_R)}. \) This inequality gives us that,
\[ \|v\|_{L^4(K_R)} \geq \|\Pi_1(v)\|_{L^4(K_R)} - C\sqrt{b - 1}\|v\|_{L^2(K_R)}. \]

As a consequence, we get with a new constant \( C \) and for all \( \sigma \in (0, 1/2) \),
\[ \|v\|_{L^4(K_R)}^4 \geq (1 - \sigma)\|\Pi_1(v)\|_{L^4(K_R)}^4 - C\sigma^{-3}(b - 1)^2\|v\|_{L^2(K_R)}^4. \]  
(4.11)

Using the pointwise bound of \( v \), \(|v| \leq [b - 1]^{1/2}\), we get that,
\[ \|v\|_{L^4(K_R)}^4 \geq (1 - \sigma)\|\Pi_1(v)\|_{L^4(K_R)}^4 - C\sigma^{-3}[1 - b]^2 R^4. \]
We use this bound to get a lower bound of the term in (4.10). That way we get that,

\[
\int_{K_R} \left( 1 - \frac{\kappa}{H} \right) |\Pi_1(v)|^2 + \frac{\kappa}{2H} |v|^4 \, dx 
\geq \int_{K_R} \left( -(b-1)|\Pi_1(v)|^2 + \frac{1}{2}(1 - \sigma + \gamma)|\Pi_1(v)|^4 \right) \, dx - C\sigma^{-3}[1 - b]^2 R^4,
\]

where \( \gamma = \frac{\kappa}{H} - 1 \). By introducing the new function \( u \in L_R \) as follows,

\[
\Pi_1 v = \left( \frac{[b-1]}{1 - \sigma + \gamma} \right)^{1/2} u,
\]

we get that

\[
\int_{K_R} \left( -(b-1)|\Pi_1(v)|^2 + \frac{1}{2}(1 - \sigma + \gamma)|\Pi_1(v)|^4 \right) \, dx = \frac{[1 - b]^2}{1 - \sigma + \gamma} F_R(u).
\]

Using the lower bound \( F_R(u) \geq c(R) \) finishes the proof of Theorem 4.5.

\[\square\]

**Remark 4.6.** Recall the functions \( A \) and \( B \) from Remark 4.3 together with the choice of the parameters \( \delta \) and \( \ell \). The assumptions (4.3)-(4.4) on \( H \) and the properties of \( A \) and \( B \) give us,

\[
\left[ 1 - \frac{H}{\kappa} \right] R^2 = o(1), \quad \text{as } \kappa \to \infty.
\]

We select the parameter \( \sigma \) in Theorem 4.5 as a function of \( \kappa \) such that,

\[
\left( 1 - \frac{H}{\kappa} \right) R^{2} \right)^{-1/3} \leq \sigma \ll 1 \quad \text{as } \kappa \to \infty.
\]

This forces the error term

\[
\sigma^{-3} \left[ 1 - \frac{H}{\kappa} \right] R^4
\]

to be of order \( o(R^2) \).

The choice of \( \sigma \) as done in Remark 4.6 allows us to prove Theorem 1.1. This choice, together with the choice of the parameters \( \delta \) and \( \ell \) in Remark 4.3 force all error terms in Propositions 4.1-4.2 and Theorem 4.5 to be of lower order compared to the principal terms.

**Proof of Theorem 1.1.** Notice that (4.3)-(4.4) ensure that

\[
\left[ \frac{\kappa}{H} - 1 \right] = \left[ 1 - \frac{H}{\kappa} \right] (1 + o(1)), \quad \text{as } \kappa \to \infty.
\]

Combining the results of Propositions 4.1-4.2 and Theorem 4.5 we get that,

\[
\frac{1}{|Q_\ell|} \int_{K_R} \left( 1 - \frac{\kappa}{H} \right) |\Pi_1(v)|^2 + \frac{\kappa}{2H} |v|^4 \, dx \leq [\kappa - H]^2 \left( E_{\Lambda B} + o(1) \right).
\]

Using the estimate \( \|v - \Pi_1(v)\|_{L^2(K_R)} \leq C\sqrt{1 - \frac{H}{\kappa}} \|v\|_{L^2(K_R)} \), we can replace \( \|\Pi_1(v)\|_{L^2(K_R)} \) by \( \|v\|_{L^2(K_R)} \) to leading order. That way we get,

\[
\frac{1}{|Q_\ell|} \int_{K_R} \left( 1 - \frac{\kappa}{H} \right) (1 + o(1)) |v|^2 + \frac{\kappa}{2H} |v|^4 \, dx \leq [\kappa - H]^2 \left( E_{\Lambda B} + o(1) \right).
\]

Applying the change of variable \( x \to a_j + \frac{v}{\sqrt{\kappa H}} \) and remembering the definition of \( v \) in Theorem 4.5 we get,

\[
\frac{\kappa H}{|Q_\ell|} \int_{Q_\ell} \left( 1 - \frac{\kappa}{H} \right) (1 + o(1)) |\chi v|^2 + \frac{\kappa}{2H} |\chi v|^4 \, dx \leq [\kappa - H]^2 \left( E_{\Lambda B} + o(1) \right).
\]
Theorem 4.4 tells us that
\[ \int_{Q_{\ell}} |\chi_{\ell}\psi|^4 \, dx = -2E_{Ab} \left[ 1 - \frac{H}{\kappa} \right]^2 + o \left( \left[ 1 - \frac{H}{\kappa} \right]^2 \right). \]
Consequently, we get that,
\[ \kappa H \int_{Q_{\ell}} \left( 1 - \frac{\kappa}{H} \right) (1 + o(1)) |\chi_{\ell}\psi|^2 = E_{Ab} [\kappa - H]^2 + |\kappa - H|^2 \left( E_{Ab} + o(1) \right). \]
Remembering the assumptions (4.3)-(4.4) on $H$, we deduce that,
\[ \frac{1}{|Q_{\ell}|} \int_{Q_{\ell}} (1 + o(1)) |\chi_{\ell}\psi|^2 = -2E_{Ab} \left[ 1 - \frac{H}{\kappa} \right] + o \left( \left[ 1 - \frac{H}{\kappa} \right] \right). \]
The assumption on the support of the function $\chi_{\ell}$, the bound (2.4) and the choice of $\ell$ together give us,
\[ \int_{Q_{\ell}} |\chi_{\ell}\psi|^2 \, dx = \int_{Q_{\ell}} |\psi|^2 \, dx + O \left( \frac{\ell}{\sqrt{\kappa H}} \left[ 1 - \frac{H}{\kappa} \right] \right) = o \left( \left[ 1 - \frac{H}{\kappa} \right] \right). \]
This finishes the proof of Theorem 1.1. \qed

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