Spinning Equations for Objects of Some Classes in Finslerian Geometry

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Abstract—Spinning equations of Finslerian gravity, the counterpart of the Mathisson-Papapetrou spinning equations of motion are obtained. Two approaches of Finslerian geometries are formulated and discussed, the Cartan-Rund and Finsler-Cartan ones, as well as their corresponding spinning equations. The significance of the nonlinear connection and its relevance on spinning equations is noticed, and their deviations are examined.

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1. INTRODUCTION

The problem of motion of a test particle has been discussed by many authors in Finslerian geometry as an introductory point to express the behavior of particles defined within the framework of this geometry [1]. This type of achievement is always compared with its counterpart in Riemannian geometry. It is worth mentioning that the most important significance of Finslerian geometry is a description, at each point in its manifold, by coordinates and a direction, it leads to replacing manifolds with tangent bundles [2]. Accordingly, a new appearing quantities may be expected, able to interpret it from a physical point of view. This type of technicality is due to implementing the concept of geometrization of physics.

In our study, we are going to obtain the spinning equations for the Cartan-Rund approach for the orthodox Finsler space [3] and the Finsler-Cartan approach which is based on introducing a nonlinear connection [4], which plays a vital role in revisiting the falsifiability of geometrizing fundamental theories in physics and cosmology such as general relativity, brane, stings and gauge theories (see [5] and references therein).

The aim of this work is searching for a suitable Lagrangian function, inspired from the famous Bazanski Lagrangian [6] as expressed in the previous works [7–9]. Thus it is vital to derive their corresponding geodesic and geodesic deviation equations to obtain the version of the spinning object for short, using a specific type of transformation explained in our present work [9].

Consequently, this step will enable us to make sure the reliability of the obtained equation. This will guide us to determine its corresponding Lagrangian function, a step toward generalization, to describe the spinning equation of different objects. Also, we present a technique of commutation to obtain their deviation equation rather than the traditional method of Bazanski [6], which is based on variation with respect to the four-velocity, as well as using some identities to preserve the appearance of covariance in the system of equations [7–9]. The advantage of this method may give rise to examining the case of rate of change of the spinning object and its associated deviation equation.

2. FINSLERIAN GEOMETRY:
A BRIEF INTRODUCTION

Finslerian spaces are n-dimensional manifolds, in which the distance between two neighboring points \(P(x^i)\) and \(Q(x^i + dx^i)\) is given by the line element [1]

\[ ds = F(x^i, dx^i), \quad i = 1, 2, 3, \ldots n, \]

where \(F\) is a function of both the coordinates \(x^i\) and the directions \(\dot{x}^i = dx^i/ds\).

The metric of the Finsler space is defined as

\[ g_{\mu\nu} = \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}, \quad (1) \]

such that

\[ g_{\mu\nu} = g_{\mu\nu} dx^\mu \otimes dx^\nu. \quad (2) \]

A third-order tensor, the Cartan tensor is defined as follows:

\[ C_{(x, \dot{x})} = C_{\mu\nu\rho} dx^\mu \otimes dx^\nu \otimes dx^\rho, \quad (3) \]

such that

\[ C_{\mu\nu\rho} = \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial \dot{x}^\rho}. \quad (4) \]
The Cartan-Rund and Finsler-Cartan approaches. Riemannian geometry with slight modifications, the space becomes Finslerian geometry, there are two different types of connections: linear connections as expressed in the Finsler geometry, there are two different types of connections: linear connections as expressed in the Riemannian geometry with slight modifications, the Cartan-Rund and Finsler-Cartan approaches.

2.1. Finsler Geometry: The Cartan-Rund Approach

Chern defined the δ-derivative [3], in which

\[ \delta A^\mu \delta S = A^\mu \frac{d x^\nu}{d S}. \]  

(7)

The partial δ-derivative is defined as follows,

\[ A^\mu_{\nu} = \frac{\partial A^\mu}{\partial x^\nu} + ^* \Gamma^\mu_{\nu \rho} A^\nu \dot{x}^\rho, \]  

(8)

such that

\[ \delta A^\mu \delta S = A^\mu \frac{d x^\nu}{d S}. \]  

(9)

and

\[ ^* \Gamma^\mu_{\nu \rho} = \{ \nu \rho \} - g^\alpha \beta [C^\rho_{\alpha \beta} \Gamma^\mu_{\nu \epsilon} + C^\epsilon_{\nu \alpha} \Gamma^\beta_{\epsilon \rho} - C^\epsilon_{\nu \beta} \Gamma^\alpha_{\epsilon \rho}] \dot{x}^\epsilon, \]

\[ \Gamma^\mu_{\nu \rho} = \gamma^\mu_{\nu \rho} \{ \nu \rho \} - C^\nu_{\delta \rho} \{ \nu \delta \} \dot{x}^\lambda, \]

\[ C^\nu_{\nu \delta} = g^\nu \delta C_{\nu \rho \delta} = g^\nu \delta \left( \frac{\partial g_{\nu \rho}}{\partial S} \right). \]

Equation (7) reduces to Riemannian geometry if \( C_{\nu \rho \delta} \equiv 0. \)

Moreover, the curvature tensor is defined with the commutation law as follows:

\[ A^\mu_{\nu \rho} - A^\mu_{\rho \nu} = K^\mu_{\nu \rho \delta} A^\delta, \]  

(10)

where \( K^\mu_{\nu \rho \delta} \) is the curvature tensor such that

\[ K^\mu_{\nu \rho \sigma} = ( ^* \Gamma^\mu_{\nu \rho, \sigma} - \frac{\partial ^* \Gamma^\mu_{\nu \rho}}{\partial x^\delta} \frac{\partial C^\delta}{\partial x^\sigma} ). \]

Thus, taking variation with respect to \( \Psi^\mu, \) we obtain

\[ \frac{\delta U^\beta}{\delta S} = 0. \]

(12)

Yet, their deviation equations may be obtained in a similar way as in [12] using the following commutation relation:

\[ \frac{\delta U^\mu}{\delta S} = \frac{\delta \Psi^\mu}{\delta \tau}. \]

(13)

Using (4) and (7), after some manipulations one obtains

\[ \frac{\delta^2 \Psi^\beta}{\delta S^2} = K^\alpha_{\mu \sigma} U^\mu U^\nu \Psi^\sigma. \]

(14)

3. SPINNING EQUATION IN FINSLERIAN GEOMETRY: THE CARTAN-RUND APPROACH

Also, using the following type of transformation [13], one finds,

\[ V^\mu = U^\mu + \frac{\beta \delta \Psi^\mu}{\delta S}, \]

(15)

where \( \beta \) is a parameter. We take the absolute \( \delta \) derivative on both sides.

\[ \frac{\delta V^\mu}{\delta \rho} = \frac{\delta}{\delta S} \left( U^\mu + \beta \frac{\delta \Psi^\mu}{\delta S} \right) \frac{d S}{d \rho}. \]

(16)

Next we substitute the geodesic and geodesic deviation equations and use the definition [10]

\[ S^\mu_{\nu} = \sigma \left[ (U^\mu \Psi^\nu) - (U^\nu \Psi^\mu) \right], \]

(17)
in which \( \beta = \sigma/m \), where \( S^{\mu\nu} \) is the spin tensor, \( \sigma \) is the magnitude of spin, and \( m \) is the mass of the object to get,

\[
\frac{\delta V^\alpha}{\delta \rho} = \frac{1}{2m} K^\alpha_{\beta\gamma\epsilon} S^{\gamma\epsilon} V^\beta,
\]

(18)

which is the analogous version to the Papapetrou equation [14] for a spinning objects, for short.

### 3.1. Spinning and Spinning Deviation Equation: The Cartan-Rund Approach

(i) The case \( P^\mu = mU^\mu \):

\[
L = g_{\mu\nu}(x, \dot{x}) U^\mu \frac{\delta \Phi^\nu}{\delta S} + S_{\mu\nu} \frac{\delta \Psi^{\mu\nu}}{\delta S} + \frac{1}{2m} K_{\mu\nu\rho\sigma} S^{\rho\sigma} U^\nu \Psi^\mu.
\]

(19)

Consequently, taking variation with respect to \( \Psi^\alpha \) and \( \Psi^{\alpha\beta} \), we get

\[
\frac{\delta U^\beta}{\delta S} = \frac{1}{2m} K^{\alpha}_{\mu\nu\sigma} S^{\nu\sigma} U^\mu,
\]

(20)

and

\[
\frac{\delta S^{\alpha\beta}}{\delta S} = 0.
\]

(21)

To obtain the corresponding deviation equation, we take the covariant delta derivative with respect to the path \( \tau \) on both sides and use the following conditions:

\[
\frac{\delta \Psi^\beta}{\delta S} = \frac{\delta U^\beta}{\delta \tau},
\]

(22)

and

\[
\frac{\delta \delta A^\alpha}{\delta S^2} = \frac{\delta \delta A^\alpha}{\delta \tau^2} = K^{\alpha}_{\beta\gamma\lambda} U^\beta U^\gamma \Psi^\lambda,
\]

(23)

to obtain the set of equations

\[
\frac{\delta^2 \Psi^\beta}{\delta S^2} = K^{\alpha}_{\mu\nu\sigma} U^{\mu\nu} \Psi^\sigma
\]

\[
+ \frac{1}{2m} (K^{\alpha}_{\mu\nu\sigma} S^{\nu\sigma} U^\mu)_\rho \Psi^\rho,
\]

(24)

and

\[
\frac{\delta^2 \Psi^{\alpha\beta}}{\delta S^2} = S^{\alpha\mu} K^{\beta}_{\mu\rho\sigma} U^\rho \Psi^\sigma.
\]

(25)

(ii) The case \( P^\mu \neq mU^\mu \). In this case we are going to derive the corresponding equations of motion and their deviation ones by proposing the following Lagrangian function:

\[
L = g_{\mu\nu}(x, \dot{x}) U^\mu \frac{\delta \Phi^\nu}{\delta S} + S_{\mu\nu} \frac{\delta \Psi^{\mu\nu}}{\delta S} + \frac{1}{2m} K_{\mu\nu\rho\sigma} S^{\rho\sigma} U^\nu \Psi^\mu + 2P_{\mu} U^\nu \Psi^{\mu\nu}.
\]

(26)

Taking the variation with respect to \( \Psi^\alpha \) and \( \Psi^{\alpha\beta} \), we get

\[
\frac{\delta U^\beta}{\delta S} = \frac{1}{2m} K^{\alpha}_{\mu\nu\sigma} S^{\nu\sigma} U^\mu,
\]

(27)

and

\[
\frac{\delta S^{\alpha\beta}}{\delta S} = 2P^{\alpha} U^\beta.
\]

(28)

Now, in order to obtain their corresponding deviation equation, one operates the covariant delta derivative with respect to the path \( \tau \) on both sides using the following conditions:

\[
\frac{\delta \Psi^\beta}{\delta S} = \frac{\delta U^\beta}{\delta \tau},
\]

(29)

and

\[
\frac{\delta \delta A^\alpha}{\delta S^2} = \frac{\delta \delta A^\alpha}{\delta \tau^2} = K^{\alpha}_{\beta\gamma\lambda} U^\beta U^\gamma \Psi^\lambda,
\]

(30)

to get

\[
\frac{\delta^2 \Psi^\beta}{\delta S^2} = K^{\alpha}_{\mu\nu\sigma} U^{\mu\nu} \Psi^\sigma
\]

\[
+ \frac{1}{2m} (K^{\alpha}_{\mu\nu\sigma} S^{\nu\sigma} U^\mu)_\rho \Psi^\rho,
\]

(31)

and

\[
\frac{\delta^2 \Psi^{\alpha\beta}}{\delta S^2} = S^{\alpha\mu} K^{\beta}_{\mu\rho\sigma} U^\rho \Psi^\sigma.
\]

(32)

### 4. FINSLER GEOMETRY: THE FINSLER-CARTAN APPROACH

It has been known that coordinates in different versions of Finsler space \( F(x, \dot{x}) \) are expressed within tangent manifolds (TM) rather than manifolds. This may give rise to generalizing the role of the directional derivative \( \dot{x} \), to become a mere coordinate system \( y^a \) viable for describing gauge potentials of an anisotropic gravitational field [5]. Such an issue was established by Cartan who kept existing the nonlinear connection, which in return may be feasible to express the metric-affine gravity theory (MAG) [15] in its Finslerian version. The virtue of implementing the nonlinear connection \( N^\mu^b \) becomes responsible for splitting the tangent bundle into two subbundles, one of the horizontal coordinates and the other assigned to vertical coordinates. Yet, in Finsler geometry the presence of the nonlinear connection is unique, while other linear connections are multiple. This is so valuable for describing an anisotropic gravitational field theory.
The building blocks of TM are centered on considering the adopted basis \((\delta_\mu, \partial_\alpha)\) for the horizontal coordinates, whose indices appear in Greek letters, while the vertical ones are expressed in Latin ones. In Finslerian geometry there are two different types of connections: linear connections as expressed in Riemannian geometry with slight modifications, and the nonlinear connection \(N^\alpha_k\). However, the latter cannot be expressed in Riemannian geometry due to expressing the tangent space not on a manifold but on a tangent bundle. Thus, this space is decomposed into two: a vertical space \(\frac{\partial}{\partial y^\alpha}\) and a horizontal space spanned by the elongated derivatives. This can be seen as describing the nonlinear connection as a gauge potential associated with its metric tensor \([5, 16]\).

This type of vision led Vacaru to express the MAG theory in Finsler space:

\[
\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^\alpha_k \frac{\partial}{\partial y^\alpha},
\]

such that

\[
N^\alpha_k = \frac{1}{2} g^{ab} \left(y^k \frac{\partial^2 F(x^i, y^j)}{\partial x^a \partial y^b} - \frac{\partial F(x^i, y^j)}{\partial x^b}\right).
\]

Due to the elongated derivatives, the Christoffel symbol of the horizontal space is defined as

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} (\delta_l g_{ki} + \delta_l g_{jk} - \delta_k g_{lj}).
\]

And its corresponding vertical affine connection remains the usual Christoffel symbol as given in Riemannian geometry, i.e.,

\[
C^i_{jk} = \frac{1}{2} g^{il} (\delta_l \delta g_{ki} + \delta_l \delta g_{jk} - \delta_k \delta g_{lj}),
\]

where \(\delta_\alpha = \delta / \partial y^\alpha\).

Accordingly, the equation of a geodesic has two branches, one is for horizontal components and the other for the vertical components as shown below:

\[
\nabla U^i = 0, \tag{33}
\]

where

\[
\nabla U^i = \frac{\delta U^i}{\delta s} + \Gamma^i_{jk} U^j U^k
\]

with \(U^i = dx^i / ds\), and

\[
\frac{DV^a}{Ds} = 0. \tag{34}
\]

Provided that

\[
\frac{DV^a}{Ds} = \frac{dV^a}{ds} + C^a_{bc} v^b V^c,
\]

and

\[
V^a = \frac{\delta y^a}{\delta s},
\]

this type of choice may be appropriate to express bigravity geometrically, which will be studied in our future work.

### 4.1. Metric Structure and Nonlinear Connection

It is well known that the metric structure for a combined space \(V^S\) of both vertical and horizontal coordinates is defined using the non-linear connection \([5]\). Such a connection may act as gauge potentials to express the case of a metric with anisotropy features in the following way:

\[
g_{MN} = \left(\begin{array}{ccc}
g_{\mu\nu} + h_{ab} N^a_\mu N^b_\nu & N^a_{\nu} & h_{ab} \\
N^a_\mu & h_{ab} & 0 \\
h_{ab} & 0 & 0
\end{array}\right), \tag{35}
\]

such that \(\mu, \nu = 0, 1, 2, 3\) and \(m, n = 4, 5, 6, 7\), where \(N^\beta_{\beta}\) acts as a set of gauge potentials. We define the 8-dimensional metric as

\[
dS^2 = G_{MN} dX^M \otimes dX^N, \tag{36}
\]

i.e.,

\[
ds^2 = g_{ij}(x, y) dx^i dx^j + h_{ab}(x, y) \delta y^a \delta y^b, \tag{37}
\]

where \(\delta y^a\) is the elongated derivative as defined in the adopted basis \((\delta, \delta)\) due to the existence of nonlinear connections.

Moreover, the spray \(G^\mu\) is connected with a nonlinear connection \(N^\alpha_\beta\), in which

\[
G^\mu = \frac{1}{4} N^\alpha_\beta \dot{x}^\beta, \tag{38}
\]

i.e., replacing \(y^\mu = \dot{x}^\mu\), one finds,

\[
N^\alpha_\beta = 2 \frac{\partial G^\alpha}{\partial \dot{x}^\beta}. \tag{39}
\]

The importance of the non-linear connection causes the ability to decompose the tangent space to the tangent bundle at point \((x, \dot{x})\) into a vertical span \(\frac{\partial}{\partial x}\) and a horizontal elongated derivative

\[
\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^\nu_{\mu} \frac{\partial}{\delta \dot{x}^\nu}. \tag{40}
\]

The nonlinear curvature is defined as

\[
R^\mu_{\nu\sigma} = \frac{\delta N^\nu_{\mu}}{\delta \dot{x}^\nu} - \frac{\delta N^\nu_{\nu}}{\delta \dot{x}^\sigma}. \tag{41}
\]

This may lead us to consider some of them as an illustration to this claim.
Such an explanation may be an advantage in describing the micro-internal behavior of particles in Finsler space [16]. The y-vector can also be regarded to measure the fluctuations of space-time. This can be seen in terms of $C_{ijk}$ or its associated curvature $S_{jkl}$.

Accordingly, the coordinate system of the tangent bundle has to be split into two categories, the horizontal and vertical ones. The causality of this performance is related to identifying Finsler geometry keeping the same behavior as Riemannian geometry to express bigravity theory using Finslerian geometry [17].

4.2. Geodesic and Geodesic Deviation in h-Derivative and v-Derivative

Paths of test particles in Finsler geometry may be determined by applying the action principle with the following Lagrangian function:

$$L = g_{\mu\nu}(x, y)U^\mu U^\nu + h_{ab}(x, y)V^a V^b.$$  \hspace{1cm} (42)

Taking the variation with respect to $U^\alpha$ and $V^c$, respectively, we obtain the set of geodesics (38) and (39).

Thus, if we apply (23) to both (33) and (34), we obtain their corresponding deviation equations. For h-derivative:

$$\frac{\nabla^2 \Psi^\alpha}{\nabla s^2} = \ddot{R}_{\beta\gamma\delta}^\alpha U^\beta U^\gamma \Psi^\delta,$$

where

$$\ddot{R}_{\beta\gamma\delta}^\alpha = \dddot{\Gamma}_{\beta\gamma\delta} - \ddot{\Gamma}_{\beta\gamma|\delta} + \dot{\Gamma}_{\beta\delta} \dot{\Gamma}_{\gamma\delta} - \dot{\Gamma}_{\beta\gamma} \dddot{\Gamma}_{\delta}.$$

And for v-derivative:

$$\frac{D^2 \Phi^\alpha}{D s^2} = C_{\beta\gamma\delta} V^\beta V^\gamma \Phi^\delta,$$

where

$$C_{\beta\gamma\delta} = \Gamma_{\beta\gamma|\delta} - \Gamma_{\beta\delta|\gamma} + \Gamma_{\beta\delta} \Gamma_{\gamma\delta} - \Gamma_{\beta\gamma} \Gamma_{\delta}.$$

5. SPINNING EQUATION OF FINSLER-CARTAN APPROACH

In a similar way, as presented in Section 3, we are going to derive the spinning equations, for short, using the following transformation.

Using the previous technique of deriving the spin equations, from geodesic and geodesic deviation equations, we suggest the following functions:

$$\bar{U} = U + \beta \frac{\nabla \Psi^\alpha}{\nabla s},$$  \hspace{1cm} (45)

and

$$\dot{V} = V + \beta \frac{D \Phi^\alpha}{D s}.$$  \hspace{1cm} (46)

Differentiating both sides of (45) and (46) with respect to $\nabla/\nabla s$ and $D/Ds$ and using (43) and (44) we obtain

$$\frac{\nabla \bar{U}^\beta}{\nabla s} = \frac{1}{2m} R_{\mu\sigma}^{\alpha} S^{\nu\sigma} U^\mu,$$  \hspace{1cm} (47)

and

$$\frac{D \bar{V}^\beta}{D s} = \frac{1}{2m} S_{\mu\sigma}^{\alpha} S^{\nu\sigma} V^\mu.$$  \hspace{1cm} (48)

5.1. Spinning and Spinning Deviation Equations Using the Finsler-Cartan Approach

(i) The case $P^\mu = mU^\mu$ and $\ddot{P}^a = mV^a$:

$$L = g_{\mu\nu}(x, y)U^\mu \frac{\nabla \Psi^\nu}{\nabla s} + S_{\mu\nu} \frac{\nabla \Psi^{\mu\nu}}{\nabla s}$$

$$+ h_{ab}(y)V^a \frac{D \Phi^b}{Ds} + \ddot{S}_{ab} \frac{D \Phi^b}{Ds}$$

$$\frac{1}{2m} R_{\mu\nu\rho\sigma} S^{\nu\sigma} U^\mu + \frac{1}{2m} S_{abcd} \dddot{S}^{\nu\sigma} U^\mu V^a.$$  \hspace{1cm} (49)

Accordingly, taking variation with respect to $\Psi^\alpha$ and $\Psi^{\alpha\beta}$, one obtains

$$\frac{\nabla \bar{U}^\beta}{\nabla s} = \frac{1}{2m} R_{\mu\sigma}^{\alpha} S^{\nu\sigma} U^\mu.$$  \hspace{1cm} (50)

In a similar way we obtain their corresponding deviation equations. For h-components:

$$\frac{\nabla^2 \Psi^\beta}{\nabla s^2} = R_{\mu\nu\rho}^{\alpha} U^\mu U^\nu \Psi^\rho$$

$$\frac{1}{2m} (R_{\mu\nu\rho}^{\alpha} S^{\nu\sigma} U^\mu)_{||\rho} \Psi^\rho,$$  \hspace{1cm} (51)

and

$$\frac{\nabla^2 \Psi^{\alpha\beta}}{\nabla s^2} = S_{\alpha\mu}^{\beta\nu} R_{\mu\nu\rho}^{\beta} U^\mu U^\nu \Psi^\rho.$$  \hspace{1cm} (52)

For v-components, taking variation with respect to $\Phi^a$ and $\Phi^{ab}$, we get

$$\frac{D \bar{V}^a}{Ds} = \frac{1}{2m} S_{bcd}^{\alpha} \dddot{S}^{\nu\sigma} V^b,$$  \hspace{1cm} (53)

and

$$\frac{DS^{\alpha\beta}}{Ds^2} = 0.$$  \hspace{1cm} (54)

Using the similar technique for obtaining spinning deviation equations, we obtain

$$\frac{D \Phi^{ab}}{Ds} = \frac{1}{2m} S_{\mu\sigma}^{\alpha} S^{\nu\sigma} V^\mu,$$  \hspace{1cm} (55)

and

$$\frac{D^2 \Phi^a}{Ds^2} = S_{bcd}^{\alpha} V^b V^c \Phi^d + \frac{1}{2m} (S_{bcd}^{\alpha} \dddot{S}^{\nu\sigma} V^b)_{||c} \Phi^e.$$  \hspace{1cm} (56)
and

$$\frac{D^2 \Phi_{ab}}{Ds^2} = S^a_{\ cde} V^d \Psi^e. \quad (57)$$

(ii) The case $P^\mu \neq mU^\mu$ and $\tilde{P}^a \neq mV^a$. In this case we obtain the spinning and spinning deviation equations for objects having some intrinsic properties due to regarding that the momentum is not solely a product of mass times four velocity vector $[14]$:

$$L = g_{\mu\nu}(x) P^\mu \frac{\nabla \Phi^\nu}{\nabla s} + S_{\mu\nu} \frac{\nabla \Psi_{\mu\nu}}{\nabla s} + h_{ab}(y) \tilde{P}^a D\Phi^b + \tilde{S}_{ab} D\Phi^{ab}$$

$$+ \frac{1}{2} L_{\mu\nu\rho\sigma} S^{\rho\sigma} U^\nu \Psi^\sigma + \frac{1}{2} S_{\alpha\beta\gamma\delta} S^{\alpha\beta\gamma\delta} U^b \Psi^a$$

$$+ 2 P^b U^\nu \Psi^\mu + 2 \tilde{P}^b V^\nu \Psi^\mu. \quad (58)$$

Thus, for h-components, we take variation with respect to $\Psi^\alpha$ and $\Psi^\beta$:

$$\frac{\nabla P^\alpha}{\nabla s} = \frac{1}{2} P^\alpha_{\mu\sigma} S^{\nu\sigma} U^\mu, \quad (59)$$

and

$$\frac{\nabla S^{\alpha\beta}}{\nabla s} = 2 P^\alpha U^\beta. \quad (60)$$

In a similar way we obtain their corresponding deviation equations:

$$\frac{\nabla^2 \Psi^\beta}{\nabla s^2} = R^\alpha_{\mu\sigma} P^\mu U^\nu \Psi^\sigma$$

$$+ \frac{1}{2} (P^\alpha_{\mu\sigma} S^{\nu\sigma} U^\mu)|\rho \Psi^\rho, \quad (61)$$

and

$$\frac{\nabla^2 \Psi^{\alpha\beta}}{\nabla s^2} = S^{\alpha\mu} R^\beta_{\mu\sigma} U^\nu \Psi^\sigma + 2 (P^\alpha U^\beta)|\rho \Psi^\rho. \quad (62)$$

And to v-coordinates, we can obtain

$$\frac{D P^a}{Ds} = \frac{1}{2} S^a_{\ cde} S^{\alpha\beta} V^b, \quad (63)$$

and

$$\frac{D S^{ab}}{Ds^2} = 2 P^a U^b. \quad (64)$$

Using the similar technique for obtaining spinning deviation equations, we get

$$\frac{D^2 \Phi^a}{Ds^2} = S^a_{\ cde} \tilde{P}^b V^c \Phi^d + \frac{1}{2} (S^a_{\ cde} S^{\alpha\beta} V^b)|\rho \Psi^\rho, \quad (65)$$

and

$$\frac{D^2 \Phi^{ab}}{Ds^2} = S^{ac} S^{b}_{\ cde} V^d \Psi^e + 2 (P^a V^b)|\rho \Psi^d. \quad (66)$$

Due to the role of the adopted basis $(\delta_\nu)$, the two equations of spinning motion look independently. From this perspective, there is no interacting terms between $U^\alpha$ and $V^\alpha$, considering $v^\alpha = dx^\alpha/ds$. This means that we obtain the associated rate of change of spinning acceleration i.e., the spinning jerk. This may have an impact on adjusting trajectories of space navigation.

6. APPLICATIONS OF SPINNING EQUATIONS IN FINSLER SPACES

6.1. Finslerian Space: the Brandt Approach

Brandt [18] has suggested the following square line element to obtain the geodesic in Finslerian fields, provided that he discarded the effect of nonlinear connection, to express the Lagrangian function as combined between velocity and acceleration. In our present work we are going to extend his vision to obtain the associated equations of spinning and rate of change of spinning (spinning jerk) related to proper maximal acceleration,

$$d\sigma^2 = g_{\mu\nu} U^\mu U^\nu + \rho_0 D\mu D\nu, \quad (67)$$

where $DU^\mu = dU^\mu + \{\mu_{\lambda\nu}\} U^\lambda U^\nu$. Thus the corresponding Lagrangian function becomes

$$L_\sigma = g_{\mu\nu} U^\mu U^\nu + \rho_0 g_{\mu\nu} \frac{DU^\mu}{D\sigma} \frac{DU^\nu}{D\sigma}, \quad (68)$$

where $\rho_0$ is the proper maximum acceleration [19]. Taking variation with respect to $x^\mu$ and $v^\alpha$, one obtains

$$\frac{DU^\alpha}{D\sigma} = 0, \quad (69)$$

and

$$\frac{D^2 U^\alpha}{D\sigma^2} = 0. \quad (70)$$

Accordingly, we suggest that the Bazanski Lagrangian becomes

$$L_\sigma = g_{\mu\nu} U^\mu \frac{D \Psi^\nu}{D\sigma} + \rho_0 g_{\mu\nu} \frac{DU^\mu}{D\sigma} \frac{D \Phi^\mu}{D\sigma}, \quad (71)$$

where $\Phi^\mu$ is the corresponding deviation vector.

Consequently, taking variation with respect to $\Psi^\alpha$ and $\Phi^\alpha$ with $\Phi^\beta = D \Psi^\beta / D\sigma$ simultaneously, we obtain the equations expressed in $(33)$ and $(34)$, respectively. Following the same technique as mentioned in the previous sections, we obtain their corresponding deviation equations:

$$\frac{D^2 \Psi^\mu}{D\sigma^2} = R^\mu_{\nu\rho\sigma} U^\nu U^\rho \Psi^\delta, \quad (72)$$

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such that the metric of the tangent bundle of space-

Also, in a similar way as in the previous sections, we
derive their corresponding spinning equations in the
following way: we assume the Lagrangian

\[ L = g_{\mu\nu} P^\mu \frac{D \Psi^\nu}{D \sigma} + S_{\mu} \frac{D \Psi^\mu}{D \sigma} + \frac{1}{2} R_{\mu\nu\rho\sigma} U^\nu S^\rho S^\sigma \Psi^\mu + 2 P_{\mu} U^\nu \Psi^\mu + \rho_0 g_{\mu\nu} \frac{D P^\mu}{D \sigma} \frac{D \Psi^\nu}{D \sigma} + 2 \frac{dP^\mu}{d\sigma} \frac{dU^\nu}{d\sigma} \frac{D \Psi^\mu}{D \sigma} + 2 \frac{d^2P^\mu}{d\sigma^2} \frac{dU^\nu}{d\sigma} \frac{D \Psi^\mu}{D \sigma}. \]  

Also, taking variation with respect to \( \Psi^\alpha \) and \( \Psi^{\alpha\beta} \), we get for \( h \)-coordinates

\[ \frac{D P^\mu}{D \sigma} \frac{D \Psi^\nu}{D \sigma} = \frac{1}{2} R_{\nu\rho\delta} U^\rho S^\delta, \]  

and

\[ \frac{D S^{\mu\nu}}{D \sigma} = 2 P^\mu U^\nu. \]  

For \( v \)-coordinates we take variation with respect to \( \Phi^\alpha \) and \( \Phi^{\alpha\beta} \) to obtain

\[ \frac{D^2 P^\mu}{D \sigma^2} = \frac{1}{2} R_{\nu\rho\delta} U^\rho S^\delta, \]  

and

\[ \frac{D^2 S^{\mu\nu}}{D \sigma^2} = 2 \frac{D^2 P}{D \sigma^2} \frac{D U^\nu}{D \sigma}. \]  

Nevertheless, the above tangent bundle space can
be expressed in Kaluza–Klein space as one geodesic equation, where the first four components are expressing the acceleration vector, while the last ones are defining the jerk vector.

6.2. On the Relation Between the Spin Tensor
and the Proper Maximum Acceleration

Brandt [20] showed a relationship between the
proper maximal acceleration \( \rho_0 \) and the Riemann–Christoffel Tensor in the following way:

\[ \rho_0^2 = R_{\alpha\beta\gamma\sigma} U^{\beta} U^{\gamma} N^{\alpha} N^{\sigma}, \]  

in which the corresponding unit vector is defined as \( N^\alpha N_\alpha = -1 \). If we assume the N-vector is connected with the deviation vector as

\[ N^\alpha = \frac{s}{m} \Psi^\alpha, \]  

then we find out that

\[ \rho_0^2 = \left( \frac{s}{m} \right)^2 R_{\alpha\beta\gamma\sigma} U^{\beta} U^{\gamma} \Psi^\alpha \Psi^\sigma. \]  

Consequently, we obtain the following relation:

\[ \rho_0^2 = \frac{1}{2m} R_{\alpha\beta\gamma\sigma} U^\beta S^{\gamma\sigma} N^{\alpha}. \]  

Meanwhile, using the Papapetrou equation [14], we find that

\[ \rho_0^2 = \frac{DU_\alpha}{DS} N^{\alpha}. \]  

Yet, multiplying both sides by \( N_\epsilon \), provided that
\( N^\alpha N_\epsilon = \delta_\epsilon^\alpha \), we obtain the relationship between the covariant acceleration and the proper maximum acceleration [20]

\[ \frac{DU_\alpha}{DS} = \rho_0^2 N^{\alpha}. \]
Using Eqs. (29) and (88) in (89), we get
\[ \frac{D\Psi_{\alpha}}{D\tau} = \frac{s}{m} \rho_0^2 \Psi_{\alpha}. \]  
(92)

If we apply covariant derivatives to both sides of (92), we find that
\[ \frac{D^2\Psi_{\alpha}}{D\tau^2} = \left( \frac{s\rho_0}{m} \right)^2 \rho_0^2 \Psi_{\alpha}. \]  
(93)

The latter shows the significance of a wave equation regulated by the quantity \( (s\rho_0/m)^2 \), from which we can relate the deviation vector and the proper maximum acceleration with the ratio between the magnitude of spinning object and its mass.

7. CONCLUSIONS

In this study, we have obtained the set of spinning and spinning deviation equations for the Riemannian analog in Finslerian geometry, as clarified in the Cartan–Rund and Finsler–Cartan approaches. In this study, we have obtained sets of spinning and spinning deviation equations to examine the stability of any rotating object on its orbit. This study is in favor of the argument of Aringazin and Dzhunushaliev \[21\] who emphasized that the gyroscopic precession in Finsler is different from the usual Riemannian treatment. Meanwhile, it is quite evident to note that the behavior of the spin connection as a gauge potential for a covariant derivative under general coordinate transformation and local Lorentz transformation in Riemannian geometry \[22\] may be equivalent to that of the nonlinear connection in Finsler geometry. Owing to this analogy, we regard the behavior of the spin connection in non-Riemannian geometries as an untested effect of nonlinear connection in these types of geometries. Such a result will be extended to examine its role in torsion and curvature, having a Finslerian flavor \[23\] on the spinning motion and the effect of their corresponding spinning jerk in our future work.

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