Charges from attractors

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Abstract
We describe how to recover the quantum numbers of extremal black holes from their near-horizon geometries. This is achieved by constructing the gravitational Noether–Wald charges which can be used for non-extremal black holes as well. These charges are shown to be equivalent to the $U(1)$ charges of appropriately dimensionally reduced solutions. Explicit derivations are provided for ten-dimensional type IIB supergravity and five-dimensional minimal gauged supergravity, with illustrative examples for various black hole solutions. We also discuss how to derive the thermodynamic quantities and their relations explicitly in the extremal limit, from the point of view of the near-horizon geometry. We relate our results to the entropy function formalism.

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1. Introduction

Studies of extremal black holes in string theory have regained importance with the advent of the attractor mechanism. In its simplest form the attractor mechanism states that the near-horizon geometry of an extremal black hole is fixed in terms of its charges. Further, it has been realized that there is a single function, called the entropy function, which determines the near-horizon geometry of extremal black holes \cite{1} (see also \cite{2}). Even though the entropy function provides the nonzero charges such as the electric, magnetic charges and angular momenta, for many extremal black holes, it does not always give the correct charges. For instance, there are apparent discrepancies when there are Chern–Simons terms for the gauge fields present in the Lagrangian. This is the case, for instance, in 5D minimal (and minimally gauged) supergravities. On the other hand, it has been believed \cite{4} that the near-horizon...
geometry of an extremal rotating black hole of 5D supergravities knows about only part of the full black hole angular momentum, called the horizon angular momentum. In [4] this has been argued to be the case for the BMPV black hole [16].

Given that finding the near-horizon geometries of the yet to be discovered extremal black hole solutions might be easier than finding the full black hole solutions, it will be useful to have a prescription to extract the quantum numbers of the full black hole from its near-horizon geometry. In this paper we show, by careful analysis of the near-horizon geometries of these black holes, that one can find the full set of asymptotic charges and angular momenta of extremal rotating black holes that satisfy certain assumptions.

For this, we first construct gravitational Noether charges following Wald [5] for several supergravity theories. These charges can be defined for Killing vectors of any given solution of the theory of interest. We mainly focus on type IIB in 10D, minimal and gauged supergravities in 5D. We present closed-form expressions for the Nother–Wald charges of these theories as integrals over compact submanifolds of co-dimension 2 of any given solution.

The 5D minimal gauged supergravity can be obtained by a consistent truncation of type IIB reduced on $S^5$ [22] (see also [23]). We show that the charges of the 5D theory can be obtained by the same dimensional reduction of the corresponding 10D charges. We further reduce the theory down to three dimensions and show that the Noether–Wald charges corresponding to Killing vectors that generate translations along compact directions are the same as the usual Noether charges for the corresponding Kaluza–Klein gauge fields in the dimensionally reduced theory. We use the understanding of the charges in the reduced theory to show how the entropy function may be modified to reproduce the charges of the 5D black holes.

We will argue that these Noether–Wald charges can be used to extract the charges of extremal black holes from their near-horizon geometries under certain assumptions which will be discussed later on. Thus, the formulae presented in this paper should prove useful in extracting the conserved charges of an extremal black hole from only its near-horizon geometry without having to know the full black hole solution. We exhibit the successes and limitations of our formulae by considering the examples of Gutowski–Reall black holes [12] and their generalizations [17] and BMPV [4, 16] black holes, black rings [18] and the 10D lift of Gutowski–Reall black holes [13].

The analysis of the conserved charges in this paper can be applied to many geometries other than the extremal black holes considered here and in particular to non-extremal black holes too.

In addition to the charges of a black hole, one is typically interested in the entropy, the mass, as well as the laws of black hole thermodynamics. Until now, the entropy has been defined in terms of a Noether charge only for non-extremal black holes [5]. To find these thermodynamic quantities and the laws of thermodynamics on the ‘extremal shell’, it was necessary to take the extremal limit of the relations defined for the non-extremal black holes (see, for instance, [1]). Furthermore, computations of quantities such as the mass, the Euclidean action and relations such as the first law and the Smarr formula relied on computing quantities in the asymptotic geometry. Hence, it would be desirable to derive appropriate relations intrinsically for extremal black holes, and with only minimal reference to the existence of an asymptotic geometry.

With this motivation, in the second part of the paper, we propose a definition of the entropy for extremal black holes in the near-horizon geometry that does not require taking the extremal limit of Wald’s entropy, but agrees with it. With a similar approach, we also derive the extremal limit of the first law from the extremal geometry, assuming only that the near-horizon geometry be connected to some asymptotic geometry. This definition of the entropy further allows us to derive a statistical version of the first law [6]. We also show that
this gives us the entropy function directly from a study of the appropriate Noether charge in the near-horizon geometry of extremal black holes. We will comment on the interpretation of the mass as well, from the point of view of the near-horizon solution.

The rest of the paper is organized as follows. In section 2, we review Wald’s construction of gravitational Noether charges and use it to derive the charges for type IIB supergravity (with the metric and the 5-form fields) and for the 5D minimal and gauged supergravity theories and show that they are related by dimensional reduction. In section 3, we show that the Noether–Wald charges are identical to the standard Noether charges for the Kaluza–Klein $U(1)$ gauge fields of the corresponding compact Killing vectors. We also discuss various assumptions under which these charges, when evaluated anywhere in the interior of the geometry, match with the standard Komar integrals evaluated in the asymptotes. Some issues of gauge (in)dependence of our charges are also addressed there. In section 4, we demonstrate how our formulae work on several examples of interest. The readers who are only interested in the formalism may skip this section. In section 5, we turn to modifying the entropy function formalism to include the Chern–Simons terms. In section 6, we discuss thermodynamics of the extremal black holes and define various physical quantities such as the entropy, chemical potentials for the charges and the mass. We end with conclusions in section 7. The example for black rings is given in the appendix.

2. Charges from Noether–Wald construction

Here, we derive expressions for the gravitational Noether charges corresponding to Killing isometries of the gravitational actions we are interested in following Wald [5, 7]. We review first the general formalism and point out some relevant subtleties. Then we construct these charges for 10D type IIB supergravity and for minimally gauged supergravity and Einstein–Maxwell–CS theory in 5D. Finally, we show how the 10D and 5D expressions can be related by dimensional reduction.

2.1. Review of Noether construction

Let us first review the construction of the charges and discuss some of the relevant properties. In [7], Lee and Wald described how to construct the Noether charges for diffeomorphism symmetries of a Lagrangian $L(\phi^i = g_{\mu\nu}, A_\mu, \ldots)$, a $d$-form in $d$ spacetime dimensions. For this, one first writes the variation of $L$ under arbitrary field variations $\delta \phi^i$ as

$$\delta L = E_i(\phi)\delta \phi^i + d\Theta(\delta \phi),$$  \hspace{1cm} (1)

where $E_i(\phi) = 0$ are the equations of motion and $\Theta$ is a $(d - 1)$-form. Secondly, one finds the variation of the Lagrangian under a diffeomorphism

$$\delta_\xi L = d(i_\xi L),$$  \hspace{1cm} (2)

where $\xi^a$ is the (infinitesimal) generator of a diffeomorphism. Then one defines the $(d - 1)$-form current $J_\xi$,

$$J_\xi = \Theta(\delta_\xi \phi) - i_\xi L,$$  \hspace{1cm} (3)

where $\delta_\xi \phi^i$ are the variations of the fields under the particular diffeomorphism. Then $J_\xi$ are conserved, i.e. $dJ_\xi = 0$, for any configuration satisfying the equations of motion. Since $J_\xi$ is closed, one can write (for trivial cohomology)

$$J_\xi = dQ_\xi$$  \hspace{1cm} (4)
for some \((d - 2)\)-form charge \(Q_\xi\). Now consider \(\xi\) to be a Killing vector and suppose that the field configurations on the given solution respect the symmetry generated by it, \(\mathcal{L}_\xi \phi^i = 0\). Since \(\Theta(\delta_\xi \phi^i)\) is linear in \(\mathcal{L}_\xi \phi^i\) we have \(\Theta(\delta_\xi \phi^i) = 0\) and so \(\mathcal{J}_\xi = -i_\xi L\). Next, let us illustrate that the charge defined as the integral \(\int_{\Sigma} Q_\xi\) over a compact \((d - 2)\)-surface \(\Sigma_r\) is conserved when (i) \(\xi\) is a Killing vector generating a periodic isometry \(\mathcal{J}_\xi\), (ii) \(\delta_\xi L = 0\) (as for Killing vectors in theories with \(L = 0\) on the solutions). Consider a \((d - 1)\)-hypersurface \(M_{12}\) which is foliated by compact \((d - 2)\)-hypersurfaces \(\Sigma_r\) over some interval \(\mathcal{R}_{12} \subset \mathbb{R}\). Using Gauss’ theorem one has

\[
\oint_{\partial M_{12}} Q_\xi = \int_{\mathcal{R}_{12}} \int_{\Sigma} \mathcal{J}_\xi
\]

for \(\partial M_{12} = \{\Sigma_1, \Sigma_2\}\). If \(\mathcal{J}_\xi = 0\), it follows that the charge \(\int_{\Sigma} Q_\xi\) does not depend on \(\Sigma_r\) and therefore is conserved along the direction \(r\). Next, let us assume that \(\xi\) generates translations along a periodic direction of \(\Sigma_r\). In general, \(\int_{\Sigma} \mathcal{J}_\xi\) receives contributions from terms in \(\mathcal{J}_\xi\) that contain the 1-form \(\xi\) dual to the Killing vector field \(\xi\) and terms that do not. The terms not involving \(\xi\) vanish by the periodicity of \(\xi\). Since \(\mathcal{J}_\xi = -i_\xi L\), there are no terms involving \(\xi\). Therefore \(\int_{\Sigma} Q_\xi\) is again independent of \(\Sigma_r\).

We will now discuss two important ambiguities in the above prescription. The first one is that the charge density defined by the equation \(\mathcal{J}_\xi = \delta_\xi L\) is ambiguous as \(Q_\xi = \delta_\xi L + d\Lambda_\xi\) does not change \(\mathcal{J}_\xi\) for some \((d - 3)\)-form \(\Lambda_\xi\). The extra term does not contribute to the integrated charge only if \(\Lambda_\xi\) is a globally defined \((d - 3)\)-form on \(\Sigma_r\), that is, it is periodic in the coordinates of \(\Sigma_r\) and non-singular. While this is the case for most of our examples, there may be situations in which, for instance, some gauge potentials that go into \(Q_\xi\) are only locally defined. Similarly, the conservation of \(Q_\xi\) is not guaranteed if any component of \(Q_\xi \in \Omega^{d-1}(M_{12})\) is not globally defined. To illustrate this, consider the \(\mathcal{J}_\xi = \delta_\xi L = 0\) case and let \(n\) be a normal to \(\Sigma_r\), such that \(dn = 0\). Then

\[
\frac{\partial}{\partial r} \oint_{\Sigma} Q_\xi = (i_n d) \oint_{\Sigma} Q_\xi = \oint_{\Sigma} i_n dQ_\xi + \oint_{\Sigma} d(i_n Q_\xi) = \oint_{\Sigma} d(i_n Q_\xi)
\]

which is only forced to vanish if \(i_n Q_\xi\) is globally defined on \(\Sigma_r\). The second, and a more important, ambiguity comes from possible boundary terms in the Lagrangian \(L\). For the boundary terms \(S_{\text{bdy}} = \int_{\partial M} L_{\text{bdy}} = \int_{M} dL_{\text{bdy}}\), the variation that gives the equations of motion is done on the boundary,

\[
\delta_\xi S_{\text{bdy}} = -i_\xi (dL_{\text{bdy}}) = \int_{\partial M} \delta_\xi L_{\text{bdy}} = \int_{M} d(\delta_\xi L_{\text{bdy}})
\]

(7)

Since \(\delta_\xi L_{\text{bdy}} = i_\xi (dL_{\text{bdy}}) + d(i_\xi L_{\text{bdy}})\), the current is just given by

\[
\mathcal{J}_\xi = -i_\xi L_{\text{bdy}} + d(i_\xi L_{\text{bdy}})
\]

(8)

and hence the charge is \(Q_\xi = i_\xi L_{\text{bdy}}\). This implies that boundary terms contribute only to conserved charges \(\int_{\Sigma} Q_\xi\) of (Killing) vectors that do not lie in \(\Sigma_r\).

2.2. The Noether–Wald charges for type IIB supergravity

Now we would like to find the Noether–Wald charges in 10D type IIB supergravity for configurations with just the metric and the 5-form turned on. As is standard, we work with the action

\[
\mathcal{L}_{\text{IIB}} = \frac{1}{16\pi G_{10}} \sqrt{-g} \left[ R - \frac{1}{4 \cdot 5!} F_5^2 \right]
\]

(9)
neglecting the self-duality of the 5-form and impose it only at the level of the equations of motion. We follow the procedure outlined in section 2.1 to find the Noether–Wald currents. Using the variations

\[ \delta(\sqrt{-g}R) = \sqrt{-g} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} + \sqrt{-g} g^{\mu\nu} \left[ \nabla_\alpha \delta \Gamma^\sigma_{\mu\nu} - \nabla_\nu \delta \Gamma^\sigma_{\mu\sigma} \right] \]

and

\[ \delta(\sqrt{-g} F_{\mu\nu}^2) = \sqrt{-g} \left[ 5 F_{\mu\nu}^{(5)} F_{\sigma\tau\alpha\lambda\kappa}^{(5)} - \frac{1}{2} g_{\mu\nu} F_{\sigma\tau\alpha\lambda\kappa}^{(5)} \right] \delta g^{\mu\nu} - 2 \cdot 5! \left[ \delta C^{(4)}_{\nu\sigma\alpha\lambda} \partial_\mu \left( \sqrt{-g} F_{\mu\nu\sigma\alpha\lambda}^{(5)} \right) - \partial_\mu \left( \delta C^{(4)}_{\nu\sigma\alpha\lambda} F_{\mu\nu\sigma\alpha\lambda}^{(5)} \right) \right], \]

(10)

where \( \delta \Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\alpha} \left[ \nabla_\nu \delta g_{\sigma\alpha} + \nabla_\sigma \delta g_{\mu\alpha} - \nabla_\alpha \delta g_{\mu\nu} \right] \). One can find the equations of motion

\[ R_{\mu\nu} - \frac{1}{16} F_{\mu\nu}^{(5)} F_{\sigma\tau\alpha\lambda\kappa}^{(5)} \epsilon^{\sigma\tau\alpha\lambda\kappa}_{\delta\epsilon\zeta\eta\kappa} = 0 \quad \text{and} \quad \partial_\mu \left( \sqrt{-g} F_{\sigma\tau\alpha\lambda\kappa}^{(5)} \right) = 0. \]

(11)

These are supplemented by the self-duality condition \( \star F_{\mu\nu}^{(5)} = F_{\mu\nu}^{(5)} \). The self-duality constraint \( F_{\mu\nu}^{(5)} = \star F_{\mu\nu}^{(5)} \) implies that \( F_{\mu\nu}^{(5)} = 0 \), and then the metric equation of motion in (11) implies \( R = 0 \) for any solution. Hence, the Lagrangian vanishes on the solutions and therefore the Noether–Wald current in (3) is given entirely by the 9-form \( \Theta \) (or equivalently by its dual vector field). This can be found from the total derivative terms in \( \delta \mathcal{L} \) by substituting \( \delta g_{\mu\nu} = \nabla_\mu \xi^\nu + \nabla_\nu \xi^\mu \) and \( \delta g_{\mu\nu}^{(2)} = 4 \partial_\mu \left( \xi^\sigma \epsilon^{\sigma\nu\alpha\lambda\kappa}_{\delta\epsilon\zeta\eta\kappa} \right) + \partial_\nu \left( \xi^\sigma \epsilon^{\sigma\mu\alpha\lambda\kappa}_{\delta\epsilon\zeta\eta\kappa} \right) \). This gives us the current

\[ J^\alpha = -2 \sqrt{-g} g^{\mu\nu} \left[ R_{\sigma\lambda} - \frac{1}{16} F_{\mu\nu}^{(5)} F_{\sigma\tau\alpha\lambda\kappa}^{(5)} \epsilon^{\sigma\tau\alpha\lambda\kappa}_{\delta\epsilon\zeta\eta\kappa} \right] \xi^\lambda \]

\[ + \partial_\mu \left[ -\sqrt{-g} g^{\mu\nu} g^{\sigma\tau} \left( \nabla_\sigma \xi_\tau - \nabla_\tau \xi_\sigma \right) + \frac{1}{2} \cdot 3! \sqrt{-g} \epsilon^{\mu\nu\alpha\lambda\kappa}_{\delta\epsilon\zeta\eta\kappa} \right], \]

(12)

where the first term vanishes by the equations of motion and the second term gives us the charge density

\[ Q^{\mu\nu} = - \frac{\sqrt{-g}}{16\pi G_{(10)}} \left[ \mathcal{A} \xi^\mu - \mathcal{A} \xi^\nu + \frac{1}{12} \xi^\nu \epsilon^{\alpha\beta\gamma}_{\delta\epsilon\zeta\eta\kappa} F_{\alpha\beta\gamma}^{(5)} \right]. \]

(13)

Noting that the self-duality constraint \( \sqrt{-g} F_{\mu\nu}^{(5)} = \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma\tau} F_{\rho\sigma\tau}^{(5)} \) implies

\[ \sqrt{-g} \frac{1}{3!} \epsilon_{\mu\nu}^{(5)} F_{\rho\sigma\tau}^{(5)} F_{\rho\sigma\tau}^{(5)} = \frac{1}{3!} \epsilon_{\mu\nu}^{(5)} \frac{1}{3!} \epsilon^{\alpha\beta\gamma}_{\delta\epsilon\zeta\eta\kappa} F_{\alpha\beta\gamma}^{(5)} \]

(14)

the Noether–Wald charge density (13) can be equivalently written as the 8-form

\[ Q^{(10)} = - \frac{1}{16\pi G_{(10)}} \left[ \mathcal{A} \xi^\nu - \frac{1}{2} i \xi^\nu C^{(4)} \wedge F^{(5)} \right]. \]

(15)

where \( \hat{\xi} \) is the dual 1-form of the vector field \( \xi^\mu \). This can be integrated over a compact 8D submanifold to get the corresponding conserved charge. A quick calculation verifies that the current for this charge vanishes identically as expected because of the vanishing Lagrangian. Hence, all charges that are computed from it are conserved as discussed in section 2.1. If we further assume that \( \mathcal{L}_\xi C^{(4)} = 0 \), we have \( i_\xi F^{(5)} = -d(i_\xi C^{(4)}) \). This can be used to rewrite (15) as

\[ Q^{(10)} = - \frac{1}{16\pi G_{(10)}} \left[ \mathcal{A} \xi^\nu + \frac{1}{2} C^{(4)} \wedge i_\xi F^{(5)} \right]. \]

(16)

up to an additional term proportional to \( d(C^{(4)} \wedge i_\xi C^{(4)}) \). This extra term does not contribute when integrated over a compact 8-manifold provided that \( C^{(4)} \wedge i_\xi C^{(4)} \) is a globally well-defined 7-form as we discussed in section 2.1. In such cases (16) can be used instead of (15).
In section 4, we will demonstrate that this formula reproduces conserved charges \[12\] of Gutowski–Reall black holes of type IIB in ten dimensions successfully. We hope this expression may be useful in obtaining the charges of the yet to be discovered black holes from their near-horizon geometries alone.

2.3. The Noether–Wald charges for 5D Einstein–Maxwell–CS

The action for 5D Einstein–Maxwell–Chern–Simons gravity is

\[ \mathcal{L} = \frac{1}{16\pi G_5} \left( \sqrt{-g} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) - 2 \left( F_5^{\mu} F_5^{\mu} - \frac{1}{4} g^{\mu\nu} F^2 \right) \right), \] (17)

which is the same as the action for the 5D minimal gauged supergravity up to the cosmological constant, which turns out not to contribute to the Noether charge. After a straightforward but slightly lengthy calculation it is easy to show that the Noether current for this action is

\[ J_\xi^\alpha = \frac{1}{16\pi G_5} \left[ 2\sqrt{-g} \left( R^{\alpha\lambda} - \frac{1}{2} g^{\alpha\lambda} R \right) - 2 \left( F_5^{\alpha} F_5^{\alpha} - \frac{1}{4} g^{\alpha\lambda} F^2 \right) \right] \xi_\lambda + 4(\xi \cdot A) \left[ \sqrt{-g} \left( \nabla_\alpha \xi_\mu + \nabla_\mu \xi_\alpha \right) \right]. \] (18)

The first two lines are simply proportional to the equations of motion and vanish on-shell and hence the Noether–Wald charges for this theory are

\[ Q_\xi^{\alpha\mu} = -\frac{1}{16\pi G_5} \left[ \sqrt{-g} \left( \nabla_\alpha \xi_\mu - \nabla_\mu \xi_\alpha \right) + 4(\xi \cdot A) \left( \sqrt{-g} F_5^{\alpha\mu} + \frac{2}{3\sqrt{3}} \epsilon^{\alpha\mu\lambda\sigma\rho} A_\sigma F_\rho \right) \right]. \] (19)

These expressions have also appeared recently in [8]. An alternative derivation of (19) in terms of KK charges will be presented in section 3.3. The charge density (19) can equivalently be written as the 3-form

\[ Q_\xi = -\frac{1}{16\pi G_5} \left[ \star d\xi + 4(\xi \cdot A) \left( \star F - \frac{4}{3\sqrt{3}} A \wedge F \right) \right]. \] (20)

As before, the charges can be obtained by integrating \( Q_\xi \) over a 3D compact sub-manifold. Note that if we set the gauge fields to zero we recover the standard Komar integral for the angular momentum.

2.4. Reduction from ten dimensions

Now, we will find the dimensional reduction of the 10D formula of conserved charges to the 5D formula to show that they are indeed identical, so let us first review the reduction formulae to obtain the equations of motion of 5D minimal gauged supergravity from 10D type IIB supergravity with only the metric and the self-dual 5-form \( F^{(5)} \) turned on [13, 14].

As usual, we express the metric in terms of the frame fields \( e^0, \ldots, e^9 \) and do the dimensional reduction along the compact 5-manifold \( \Sigma_c \) that is spanned by the 5-form \( e^5 \wedge e^6 \wedge e^7 \wedge e^8 \wedge e^9 = e^{56789} \). Then, the lift formula is [22] (see also [23])

\[ ds_{10}^2 = ds_5^2 + t^2 \sum_{i=1}^3 \left( d\mu_i \right)^2 + \mu_i^2 \left( d\xi_i + \frac{2}{t\sqrt{3}} A \right)^2, \] (21)

\[ F^{(5)} = (1 + *_{(10)}) \left[ -\frac{4}{t} \text{vol}_{(5)} + \frac{t^2}{\sqrt{3}} \sum_{i=1}^3 d\left( \mu_i \right) \wedge d\xi_i \wedge *_{(5)} F \right]. \]
where $\mu_1 = \sin \alpha$, $\mu_2 = \cos \alpha \sin \beta$, $\mu_3 = \cos \alpha \cos \beta$ with $0 \leq \alpha \leq \pi/2$, $0 \leq \beta \leq \pi/2$, $0 \leq \xi_i \leq 2\pi$ and together they parametrize $S^5$. Note that we define the Hodge star of a $p$-form $\omega$ in $n$ dimensions as $*(\omega)_{ij} = \frac{1}{p!} \epsilon_{ij_1 ... j_p} \alpha_{j_1 ... j_p}$, with $\epsilon_{0123456789} = 1$ and $\epsilon_{01234} = 1$ in an orthonormal frame. The 10D geometry is specified by $[e^0, \ldots, e^4]$, an orthonormal frame for the 5D metric $d\xi^2$, together with

\begin{align}
e^5 &= l \, dx, \quad e^6 = l \cos \alpha \, d\beta, \quad e^7 = l \sin \alpha \cos \beta [d\xi_1 - \sin^2 \beta \, d\xi_2 - \cos^2 \beta \, d\xi_3], \\
e^8 &= l \cos \alpha \sin \beta [d\xi_2 - d\xi_3], \quad e^9 = -\frac{2}{\sqrt{3}} A - l \sin^2 \alpha \, d\xi_1 - l \cos^2 \alpha [\sin^2 \beta \, d\xi_2 + \cos^2 \beta \, d\xi_3],
\end{align}

and the 5-form $[13, 22, 23]$

\begin{align}
F^{(5)} &= -\frac{4}{l} (e^{0-4} + e^{5-9}) + \frac{2}{\sqrt{3}} (e^{57} + e^{68}) \wedge (e^9 \wedge F) + (e^9 \wedge F).
\end{align}

One can write the 5-form RR field strength as $F^{(5)} = dC^{(4)}$ where

\begin{align}
C^{(4)} &= \Omega_4 + \cos \alpha e^{678} \wedge \left( e^9 + \frac{2}{\sqrt{3}} A \right) - \frac{2}{\sqrt{3}} \left[ A \wedge (e^{57} + e^{68}) \wedge \left( e^9 + \frac{2}{\sqrt{3}} A \right) \right] \\
&\quad + \frac{l}{2} \left( e^9 + \frac{2}{\sqrt{3}} A \right) \wedge (e^9 F + \frac{2}{\sqrt{3}} A \wedge F),
\end{align}

where $\Omega_4$ is a 4-form such that $e^{01234} = d\Omega_4$. Now we are ready to do the reduction of the 10D charge

\begin{align}
Q_{\chi} := -\frac{1}{16\pi G_{10}} \int_{\Sigma_5} (\ast d\hat{\chi} - \frac{1}{2} i_{\chi} C^{(4)} \wedge F^{(5)})
\end{align}

where $\Sigma_5$ is a compact 8D submanifold that is composed of a spacelike 3-surface $\Sigma$ in 5D and $\Sigma$. Hence, only $e^{5-9}$ will contribute to the integral. Let us consider $\chi$ to be a Killing vector of the 10D geometry which also reduces to a Killing vector of the 5D geometry and $\hat{\chi}$ be its dual 1-form. Then we find from the expression for the frame fields (21), (22),

\begin{align}
\hat{\chi} &= \hat{\chi}_5 + (i_{\chi} e^9) e^9 = \hat{\chi}_5 - \frac{2}{\sqrt{3}} (i_{\chi} A)e^9, \quad \text{so}
\end{align}

\begin{align}
\ast d\hat{\chi} = \ast d\hat{\chi}_5 - \frac{2}{\sqrt{3}} (i_{\chi} A) \ast d\hat{\chi}_5 + \frac{4}{3} (i_{\chi} A) \ast F + \ldots
\end{align}

where `$\ldots$' denotes terms that do not contribute to $Q_\chi$. Next, let us find the relevant terms in $C^{(4)}$ and $F^{(5)}$ (23), (24). Noting that $i_{\chi} (e^9 + \frac{2}{\sqrt{3}} A) = 0$, they are

\begin{align}
i_{\chi} C^{(4)} &= i_{\chi} \Omega_4 - \frac{2}{\sqrt{3}} (i_{\chi} A) (e^{57} + e^{68}) \wedge \left( e^9 + \frac{2}{\sqrt{3}} A \right) \\
&\quad + \frac{l}{\sqrt{3}} \left( e^9 + \frac{2}{\sqrt{3}} A \right) \wedge \left( i_{\chi} F + \frac{2}{\sqrt{3}} i_{\chi} (A \wedge F) \right) + \ldots
\end{align}

\begin{align}
F^{(5)} &= -\frac{4}{l} e^{6789} + \frac{2}{\sqrt{3}} (e^9 F - F \wedge A) + (e^{57} + e^{68}) + \ldots
\end{align}

\begin{align}
i_{\chi} C^{(4)} \wedge F^{(5)} &= -\frac{2}{l} i_{\chi} \Omega_4 + \frac{4}{3} (i_{\chi} A) (A \wedge F) \left( e^9 F + \frac{2}{\sqrt{3}} A \wedge F \right) + \ldots
\end{align}

After some algebra, the charge reads

\begin{align}
Q_{\chi} &= -\frac{1}{16\pi G_5} \int_{\Sigma} \left[ \ast d\hat{\chi}_5 + 4 (i_{\chi} A) F + \frac{16}{3\sqrt{3}} (i_{\chi} A) A \wedge F + \frac{2}{l} i_{\chi} \Omega_4 - \frac{4}{3} i_{\chi} (A \wedge F) \right].
\end{align}
We see immediately that for vectors in the directions of $\Sigma_1$ it just reproduces the 5D Noether charge (19). For vectors orthogonal to $\Sigma_1$, it is different, as is not unexpected, since typically in dimensional reduction the actions agree only up to boundary terms.

3. Charges from dimensional reduction

In this section, we will rederive the Noether–Wald charges for 5D supergravity of section 2.3 using further dimensional reduction. In particular, we will demonstrate that the 5D Noether–Wald charges can alternatively be obtained from Kaluza–Klein $U(1)$ charges. For this, we will first dimensionally reduce the 5D theory along the relevant Killing vectors and then find the Noether charges of the resulting gauge theory. Then we will lift the results back to 5D and show that they agree with the corresponding 5D Noether–Wald charges. Finally, we will discuss in which cases the charges obtained by our methods in the interior of the solution agree with the asymptotic ones.

3.1. Dimensional reduction

In five dimensions one can have two independent angular momenta, so we consider dimensional reduction over both compact Killing vector directions which generate translations along which we have the independent angular momenta. We will again assume that all fields obey the isometries and hence only need to consider zero modes in the compact directions.

We take lower case Greek letters $\alpha, \beta, \ldots \in \{t, r, \theta, \phi, \psi\}$ to be the 5D indices, upper case Latin $A, B, \ldots \in \{t, r, \theta\}$ to be the 3D indices and lower case Latin $a, b, \ldots, i, j, l, m, \ldots \in \{\phi, \psi\}$ to be the indices for the compactified directions in 5D or scalar fields in 3D. The appropriate reduction ansatz is

$$G_{\mu\nu} = \left( g_{MN} + h_{ij} B^i_M B^j_N, \frac{h_{in} B^i_M}{h_{mn}} \right), \quad A_m = \varphi_m \quad \text{and} \quad A_M = A^3D_M + \varphi_a B^a_M, \quad (31)$$

such that we get

$$F_{\mu\nu} = \left( \mathcal{F}_{MN} + (d\varphi_a \wedge B^a)^{MN}, \frac{\varphi_{n,M}}{\varphi_{m,N}} \right), \quad (32)$$

in terms of the 3D gauge fields $H^a = dB^a$ and $F^{3D} = dA^3D$, and we defined for simplicity $\mathcal{F} = F^{3D} + \varphi_a H^a$. The definition of $A^3D$ in (31) is needed to have the appropriate transformations of the KK and Maxwell $U(1)$ symmetries and arises naturally from the reduction using frame fields (see, for instance, [9] for details). Now, we find

$$F_{\mu\nu} F^{\mu\nu} = \mathcal{F}_{MN} \mathcal{F}^{MN} - 2h^{ab} \varphi_{a,M} \varphi_{b,M}, \quad \text{and} \quad \epsilon^{\mu_{\nu}\rho\sigma} A_{a,\mu} F_{\nu\rho} F_{\sigma\tau} = 4\epsilon^{LMN} \epsilon^{ab} (\varphi_{a,L} \mathcal{F}_{MN} \varphi_{b} - A^3D_{a,M} \varphi_{b,N}), \quad (33)$$

such that the 5D Lagrangian (17) can be rewritten as

$$\frac{16\pi}{\sqrt{g_5}} G_5 \times L^{3D} = \sqrt{-g} \sqrt{h} \left( R^{3D} - \frac{h_{ab}}{4} H^a_{MN} H^b_{MN} - \mathcal{F}_{MN} \mathcal{F}^{MN} + 2h^{ab} \varphi_{a,M} \varphi_{b,M} \right)$$

$$- \frac{8}{3\sqrt{3}} \epsilon^{LMN} \epsilon^{ab} (\varphi_{a,L} \mathcal{F}_{MN} \varphi_{b} - A^3D_{a,M} \varphi_{b,N}), \quad (34)$$

5 This dimensional reduction has been used recently in [10, 11] for defining the entropy functions for such theories.
where \( V_T \) is the ‘volume’ of the compact coordinates. One can now construct conserved currents using the Noether procedure for the gauge symmetries of the two \( U(1) \) gauge fields \( B_\mu \) and \( A_\mu^D \). We find the corresponding Noether charges for \( B_\mu \) to be

\[
J_\mu = -\frac{V_T}{16\pi G_5} \int_{S^4} \left( \sqrt{-g} \sqrt{h} (h_{ab} H^{ab} + 4 \partial_a \mathcal{F}^{\alpha \beta} + \frac{16 \partial_\alpha}{3} e L_i \epsilon^{\alpha \beta \gamma} \mathcal{A}_m \mathcal{A}_n \mathcal{A}_i) \right).
\]

which we identify as the two independent angular momenta. The Noether charge for \( A_\mu^D \) works out to be

\[
Q = -\frac{V_T}{4\pi G_5} \int_{S^4} \left( \sqrt{-g} \sqrt{h} \mathcal{F}^{\alpha \beta} + \frac{2}{3} e L_i \epsilon^{\alpha \beta \gamma} \mathcal{A}_m \mathcal{A}_n \mathcal{A}_i \right),
\]

which we identify with the 5D electric charge. Alternatively, these charges can be read off by writing the left-hand side of the equations of motion for the Lagrangian (34),

\[
-\partial_M \left( \sqrt{-g} \sqrt{h} (h_{ab} H^{ab} MN + 4 \partial_a \mathcal{F}^{MN}) + \frac{16 \partial_\alpha}{3} e L^M \epsilon^{\alpha \beta \gamma} \mathcal{A}_m \mathcal{A}_n \mathcal{A}_i \right) = 0
\]

\[
-4\partial_M \left( \sqrt{-g} h \mathcal{F}^{MN} + \frac{4}{3} \epsilon L^M \epsilon^{\alpha \beta \gamma} \mathcal{A}_m \mathcal{A}_n \mathcal{A}_i \right) = \frac{8}{3} \epsilon L^M \epsilon^{\alpha \beta \gamma} \mathcal{A}_m \mathcal{A}_n \mathcal{A}_i M,
\]

as a total derivative and interpreting the resulting total conserved quantities as the charges.

For geometries with just one independent angular momentum, one can apply the above formulae in a straightforward way or do a reduction only down to 4D as in such cases only one \( U(1) \) isometry is expected in the geometry. The computations for the latter are identical to those here, so we just state the expressions for the angular momentum along \( \delta_\xi \) and the charge:

\[
J = -\frac{V_T}{16\pi G_5} \int_{S^4} \left( \sqrt{-g} e^{2\xi} (e^{2\psi} H^{\alpha \beta} + 4 \partial_\alpha \mathcal{F}^{\alpha \beta} + \frac{8 \partial_\alpha}{3} e L_i \epsilon^{\alpha \beta \gamma} \mathcal{A}_m \mathcal{A}_n \mathcal{A}_i) \right),
\]

\[
Q = -\frac{V_T}{4\pi G_5} \int_{S^4} \left( \sqrt{-g} e^{2\psi} \mathcal{F}^{\alpha \beta} + \frac{1}{3} e L_i \epsilon^{\alpha \beta \gamma} \mathcal{A}_m \mathcal{A}_n \mathcal{A}_i \right),
\]

where \( e^{2\psi} = g_{\psi \psi} \), \( V_T \) is the periodicity of \( \psi \) and the conservation follows by the equations of motion

\[
-\partial_M \left( \sqrt{-g} e^{2\psi} (e^{2\psi} H^{MN} + 4 \partial_a \mathcal{F}^{MN}) + \frac{8 \partial_\alpha}{3} e^{ABMN} (\partial_\alpha \mathcal{F}^{AB} - 2 \partial_\alpha \mathcal{A}_B^{(4)}) \right) = 0,
\]

\[
-4\partial_M \left( \sqrt{-g} e^{2\psi} \mathcal{F}^{MN} + \frac{2}{3} e^{ABMN} \epsilon^{\alpha \beta \gamma} (\partial_\alpha \mathcal{F}^{AB} - 2 \partial_\alpha \mathcal{A}_B^{(4)}) \right) = \frac{8}{3} e^{ABMN} \mathcal{F}^{AB} \mathcal{A}_M.
\]

### 3.2. Oxidation of the angular momentum

Now we would like to demonstrate that the lower dimensional Noether charges above, when lifted back to 5D, give the Noether–Wald charges for the compactified Killing vectors. For simplicity, we look at the expression with only one independent angular momentum and only one dimension (along \( \psi \)) reduced. Our results will hold in general though, as the gauge theory corresponding to the angular momentum is Abelian, so we can examine different Killing vectors independently. First, we note that the dimensional reduction ansatz can be obtained with the following triangular form of the frame fields [9]:

\[
V^I_\mu = \begin{pmatrix} v_M^I & e^\sigma B_M^I \\ 0 & e^\sigma \end{pmatrix} \text{ and the inverse } V^M_1 = \begin{pmatrix} v_M^1 & -v_N^1 B_N^1 \\ 0 & e^{-\sigma} \end{pmatrix}.
\]
with (bold Latin) tangent space indices $A, B, \ldots \in \{0, \ldots, 4\}$ and $a, b, \ldots \in \{0, \ldots, 3\}$ such that we can write the 4D fields in terms of the 5D fields (but still in 4D coordinates):

$$B_M = e^{-\sigma} V^4_M, \quad H_{MN} = e^{-\sigma} (dV^4)_{MN} - 2e^{-\sigma} ((de^\sigma) \wedge B)_{MN},$$

$$\delta A^\sigma = \xi^\mu V^I_\mu \quad \text{and} \quad \sigma' = \xi^\mu A_\mu.$$ (44)

Now the conservation equation (41) for the angular momentum hypersurface $\Sigma_1$ outside the horizon). That is, provided there exists a spacelike necessarily extremal) black hole (or in general any spacetime with a suitable asymptotic position of the surface on which they are computed,

$$\frac{Q}{\partial M} = \frac{Q}{m} \quad \text{where} \quad \frac{Q}{m} = \int_M dM \partial N Q^{MN} = 0$$

where $\Sigma_1$ and $\Sigma_\infty$ are the boundaries of the volume $M$—provided that the $U(1)$ theory is defined throughout the bulk volume and we can consistently compactify the manifold (at least outside the horizon).

Hence, the black hole charge and angular momentum as defined on a spacelike $(d - 2)$ hypersurface $\Sigma_\infty$ at the asymptotes are given by the corresponding KK or Noether–Wald charge, computed over any spacelike $(d - 2)$ hypersurface $\Sigma_\infty$ in the spacetime for any (not necessarily extremal) black hole (or in general any spacetime with a suitable asymptotic boundary). That is, provided there exists a spacelike $(d - 1)$ hypersurface $M$ with $\partial M = \{\Sigma_1, \Sigma_\infty\}$ on which the following sufficient conditions are satisfied:

1. The relevant compact Killing vector is a restriction to $\Sigma_1$ of a Killing vector field that is globally defined on $M$ and generates a constant periodicity.
(2) There are no sources, i.e. the vacuum equations of motion for the gauge fields are satisfied.

(3) There exists a smooth fibration of surfaces \((\Sigma \rightarrow [r_0, \infty]) = \mathcal{M}\) such that \(\pi^{-1}r_0 = \Sigma_0, \lim_{r \to \infty} \pi^{-1}r = \Sigma_\infty\).

An example where these conditions are satisfied is the region outside the (outer) horizon of a stationary black hole solution with an \(S^{d-2}\) horizon topology, embedded in a geodesically complete spacetime with an asymptotic \(S^{d-2}\) boundary. One example where these conditions are violated is that of black rings [18] which will be considered separately in an appendix.

3.3.2. Gauge issues. The contributions of the CS term in the conserved quantities in (3.1) depend explicitly on the gauge potentials. This does not however make them gauge dependent. To see this in 5D, let us consider the electric charge computed by the Noether procedure which is given in [4] as \(\frac{1}{4\pi G_5} \int_{\Sigma} (\ast F + \frac{2}{\sqrt{3}} A \wedge F)\). We note that the charges get contributions of the form \(\int_{\Sigma} dA \wedge F\) that change under a transformation \(\delta A = d\Lambda\) as \(\int_{\Sigma} dA \wedge F = \int_{\Sigma} d(\Lambda F) = 0\) because \(\Sigma\) is compact. From the 3D point of view the KK scalars \(A\) may depend on a 5D gauge transformation. However, \(\Lambda\) must be periodic in the angular coordinates so that the contributions from \(d\Lambda\) vanish after integration. This is also the reason why the term containing \(\xi \cdot A\) in equation (19) is gauge independent for compact Killing vectors. On the other hand, the Noether charge for a non-compact Killing vector is gauge dependent and hence is only physically relevant when measured with respect to some boundary condition or as a difference of charges.

4. Examples

So far we have derived Noether charges for various supergravity theories that may be used to calculate the electric charges and angular momenta of the solutions. In particular, they can be used in the near-horizon geometries to calculate the conserved charges of the corresponding black holes. In this section, we will demonstrate with several examples how our charges successfully reproduce the known black hole charges in different dimensions, for equal or unequal angular momenta and independent of the asymptotic geometries. We will start with a 10D example and then cover 5D examples, first with one angular momentum in AdS and flat asymptotics, and then with unequal angular momenta in asymptotic AdS.

4.1. The 10D Gutowski–Reall black hole

In [12], Gutowski and Reall found the first example of a supersymmetric black hole which asymptotes to AdS5 as a solution to minimal gauged supergravity in 5D (see also [17, 34–36]). Their solution was lifted to a solution to 10D type IIB supergravity in [13] and shown to admit two supersymmetries. In [14] (see also [15]), the near-horizon geometry of this 10D black hole was studied. Here, we use the formulae found in section 2.2 to calculate the Noether–Wald charges in the near-horizon geometry and show that they agree with the charges of the black hole measured from the asymptotes. The 10D metric of this near-horizon geometry is \(ds^2_{10} = \eta_{ab} e^a e^b\) with the orthonormal frame

\[
\begin{align*}
e^0 &= \frac{2r}{\omega} dt - \frac{3\omega^2}{4l} \sigma^1, & e^1 &= \frac{\omega l}{2\lambda} dr, & e^2 &= \frac{\omega}{2} \sigma^1, \\
e^3 &= \frac{\omega}{2} \sigma^2, & e^4 &= \frac{\omega}{2l} \sigma^1.
\end{align*}
\]
and the 5-form is
\[ F^{(5)} = \frac{-4}{l} (e^{0-4} + e^{5-9}) - \frac{1}{l} (e^{57} + e^{68}) \wedge \left[ -3e^{023} + e^{014} + \frac{2\lambda}{\omega} e^{234} + e^9 \wedge \left( 3e^{14} - e^{23} - \frac{2\lambda}{\omega} e^{01} \right) \right] \]

(48)

where \( e^5 \cdots e^9 \) are given in (22) and
\[ A = \frac{\sqrt{3}}{2} \left( \frac{2e}{\omega} dt + \frac{\omega^2}{4l^2} \sigma^L \right) = \frac{\sqrt{3}}{2} \left( e^9 + \frac{2\omega}{\lambda} e^4 \right), \quad \lambda = \sqrt{l^2 + 3\omega^2} \]
and
\[ \sigma^L_1 = \sin \phi d\theta - \sin \theta \cos \phi d\psi, \quad \sigma^L_2 = \cos \phi d\theta + \sin \theta \sin \phi d\psi, \]
(49)

\[ \sigma^L_1 = \frac{\lambda}{\sqrt{l^2 + 3\omega^2}} = \frac{\lambda}{\sqrt{l^2 + 3\omega^2}} \]
and hence the relevant terms in \( \star d \hat{\chi} \) are \( \frac{\omega}{\sqrt{l^2 + 3\omega^2}} (4l^2 + 3\omega^2)e^{2-9} \). Similarly, we find
\[ C^{(4)} \wedge i_x F^{(5)} = \frac{\omega}{l^3} (2l^2 + \omega^2) \frac{1}{8} \sigma^L_1 \wedge \sigma^L_2 \wedge \sigma^L_5 \wedge e^{56789}. \]
(51)

After noting that the integral over \( \frac{1}{8} \sigma_{123} \wedge e^{56789} \) gives a factor of \( 2\pi l^5 \), we find
\[ Q_{\bar{\partial}_\phi} = -\frac{1}{16\pi l^2 G_5} \int_{S_l^{7,9}} \left[ \star d \hat{\chi} + \frac{1}{2} C^{(4)} \wedge i_x F^{(5)} \right] = -\frac{3\pi \omega^4}{8l G_5} \left( 1 + \frac{2\omega^2}{3l^2} \right), \]
(52)

which agrees with the angular momentum, up to a minus sign, that comes from the definition of the angular momentum as minus the Noether charge [12]. For \( \chi = \bar{\partial}_c + \bar{\partial}_e + \bar{\partial}_f \), we have \( i_x e^0 = -l^\perp \). One can calculate the 10D current and find that
\[ \star d \hat{\chi} + \frac{1}{2} C^{(4)} \wedge i_x F^{(5)} = \frac{4l}{\sqrt{3}} \left( \star_5 F + \frac{2}{\sqrt{3}} A \wedge F \right) \wedge e^{5678} \wedge \left( e^9 + \frac{2}{\sqrt{3}} A \right) + \cdots. \]
(53)

Therefore, the corresponding charge is
\[ Q_{\bar{\partial}_1 + \bar{\partial}_2 + \bar{\partial}_3} = -\frac{\pi l \omega^2}{4 G_5} \left( 1 + \frac{2\omega^2}{3l^2} \right). \]
(54)

This differs from the answer \( Q^{(GR)} = \frac{\pi \omega^2}{2G_5} \left( 1 + \frac{\omega^2}{l^2} \right) \) [12] by a factor of \(-l/\sqrt{2} l\). The minus sign is because of a difference in our conventions from those of [12] and the factor of \( l \) is there to make the charge \( Q^{(GR)} \) dimensionless. The Killing vector \( \bar{\partial}_c + \bar{\partial}_e + \bar{\partial}_f \) has a period of \( 6\pi \) and to normalize it to have a period of \( 2\pi \) we have to multiply it by a factor of \( 3 \). If we take this into account the extra factor reduces to \( \sqrt{3}/2 \). This is precisely the factor required to define the 5D gauge field in the conventions of dimensional reduction from 10D to 5D [22]. Thus, we find complete agreement between our 10D computation of charges from the NHG and the asymptotic black hole charges of [12].

4.2. 5D black holes

Now we turn to black hole solutions in 5D Einstein–Maxwell–CS and minimal gauged supergravity.
4.2.1. Equal angular momenta: BMPV and GR. Let us consider two examples that are similar in the near-horizon geometry, with a squashed $S^3$ horizon, but differ by their asymptotic behaviour: the BMPV black hole \cite{4,16} with asymptotically flat geometry and the Gutowski–Reall (GR) black hole \cite{12} with asymptotically $AdS_5$ geometry.

Their near-horizon solutions can be put into the form

$$ds^2 = v_1 \left( -r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (\sigma_1^2 + \sigma_2^2 + \eta (\sigma_1 + \sigma_2 \, dt)^2),$$

$$A = -er dt + p (\sigma_1 - ar \, dt),$$

which, when dimensionally reduced along the $\psi$-direction, gives $ds^2 = v_1 ( -r^2 dt^2 + \frac{dr^2}{r^2} ) + v_2 (d\theta^2 + \sin^2 \theta \, d\phi^2 )$. This has $AdS_2 \times S^3$ symmetry as expected. The fields take the form $B = -r \alpha dt + \cos \theta \, d\phi$, $e^{2\alpha} = v_2 \eta$, $\omega = p$ and $A^{(4)} = -er \, dt$. For the BMPV case, we find

$$v_1 = v_2 = \frac{\mu}{4}, \quad \eta = 1 - \frac{j^2}{\mu^2}, \quad \alpha = \frac{j}{\sqrt{\mu^2 - j^2}},$$

$$e = -\frac{\sqrt{3} \mu^2}{4 \sqrt{\mu^2 - j^2}} \quad \text{and} \quad p = \frac{\sqrt{3} j}{4 \mu}.$$  

Evaluating the 4D quantities and noting that $\epsilon^{tr\phi\theta} = 1$ and $V_{\gamma^r} = 4 \pi$, \eqref{55}, \eqref{56} gives us $J = \frac{\pi j}{4 \alpha s}$, which is equal in magnitude to the angular momentum in \cite{4} up to a factor of 2, which arises from the canonical normalization of the Killing vector $\xi = 2 \partial_\psi$ and $Q = \frac{\sqrt{3} \pi \mu}{2 \alpha r}$. 

For the GR case, we have

$$v_1 = \frac{\omega l}{2 \lambda}, \quad v_2 = \frac{\omega^2}{4}, \quad \eta = 1 + \frac{3 \omega^2}{4 l^2}, \quad \alpha = -\frac{3 \omega l^2}{\lambda^2 l^2 + 3 \omega^2},$$

$$e = \frac{l}{2 \sqrt{3}} \alpha, \quad p = \frac{\sqrt{3} \omega^2}{8 l^2}.$$ 

Note that we have defined $A$ with an overall factor of $-1$ compared to \cite{14} to account for a different convention for the CS term. This gives the results $J = -\frac{\pi \omega^2}{2 \alpha r} (1 + \frac{\omega^2}{2 l^2})$ and $Q = \frac{\sqrt{3} \pi \omega^2}{2 \alpha r} (1 + \frac{\omega^2}{2 l^2})$ as expected. Note that \cite{12} does not use the canonical normalization for $\partial_\psi$ of \cite{4}.

4.2.2. Non-equal angular momenta: supersymmetric black holes. Here, we present as the most simple example the $N = 2$ supersymmetric black holes with non-equal angular momenta of \cite{17}, which are asymptotically $AdS_5$, just as the GR case. We start off with the metric in the form \cite{17}

$$g_{rr} = \frac{-\Delta_t}{(\rho^2 \Xi_a \Xi_b)(\rho^2 \Xi_a \Xi_b) (1 + r^2) - \Delta_t (2m \rho^2 - q^2 + 2abr \rho^2)},$$

$$g_{rr} = \rho^2 \Delta_r, \quad g_{\theta \phi} = \frac{\rho^2}{\Delta_r}, \quad g_{\psi \psi} = \frac{\rho^2}{\Delta_r} \sin^2 \theta,$$

$$g_{\phi \phi} = \frac{-\Delta_t \rho^2 \Xi_a \Xi_b}{\rho^2 \Xi_a \Xi_b} (a(2m \rho^2 - q^2) + b q \rho^2 (1 + a^2)),$$

$$g_{\phi \psi} = g_{\psi \phi} (a \leftrightarrow b, \sin \theta \leftrightarrow \cos \theta),$$

$$g_{\phi \phi} = \frac{\sin^2 \theta}{\rho^2 \Xi_a \Xi_b} ((r^2 + a^2) \rho^4 \Xi_a + a \sin^2 \theta (a(2m \rho^2 - q^2) + 2b q \rho^2)),$$

$$g_{\psi \psi} = g_{\phi \phi} (a \leftrightarrow b, \sin \theta \leftrightarrow \cos \theta),$$

$$g_{\phi \psi} = \frac{\sin^2 \theta \cos^2 \theta}{\rho^2 \Xi_a \Xi_b} (ab(2m \rho^2 - q^2) + (a^2 + b^2) q \rho^2).$$
with the gauge field
\[ A = \frac{\sqrt{3}q}{2\rho^2} \left( \Delta_t \Xi_a \Xi_b \, dt - \frac{a \sin^2 \theta}{\Xi_a} \, d\phi - \frac{b \cos^2(\theta)}{\Xi_b} \, d\psi \right), \] (59)
where
\[ \rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Delta_t = 1 - a^2 \cos^2 \theta b^2 \sin^2 \theta, \]
\[ \Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 + r^2) + q^2 + 2abq}{r^2 - 2m}, \]
\[ \Xi_a = 1 - a^2 \quad \text{and} \quad \Xi_b = 1 - b^2. \]
We consider the case with saturated BPS limit and no CTCs, which requires
\[ q = \frac{m}{1 + a + b}, \quad m = (a + b)(1 + a)(1 + b)(1 + a + b). \] (61)

Now we can find the near-horizon geometry with explicit AdS\(_2\) symmetry as in \([18]\), by redefining
\[ \tilde{t} = \epsilon t, \quad \tilde{r} = \frac{4(1 + 3a + a^2 + 3b + b^2 + 3ab) r - \sqrt{a + b + ab}}{(1 + a)(1 + b)(a + b) \epsilon}, \]
\[ d\tilde{\phi} = dt + d\phi, \quad d\tilde{\psi} = dt + d\psi, \] (62)
then taking the limit of \( \epsilon \to 0 \) and applying a gauge transformation to get rid of a constant term in \( A_t \). We can read off the 3D scalar fields \( h_{mn} \) and \( \phi \) and find
\[ B_{MN}^a = h^{ab} G_{aN}, \quad g_{MN} = G_{MN} - B_{M}^{ab} B_{N}^{ab} \quad \text{and} \quad A_{MN}^a = A_M - \phi \delta_a B_{MN}. \] (63)

Noting that \( Y_{72} = 4\pi^2 \), equations (35) give us the angular momenta \( J_{\tilde{\phi}} = \frac{\pi a^2 + b^2 + 3ab + b^2 - 3ab}{4G_{55}(1 - a)(1 - b)} \) and \( J_{\tilde{\psi}} = \frac{\pi b^2 + a^2 + 3ab + b^2 - 3ab}{4G_{55}(1 - b)(1 - a)} \). These agree precisely with the corresponding asymptotic angular momenta of \([18]\).

5. Charges from the entropy function

The original incarnation of the entropy function formalism \([1, 3]\) was not only a useful tool for finding near-horizon solutions, but also for extracting the conserved charges from a given solution. However, in the presence of Chern–Simons terms, the entropy function formalism captures only part of the conserved charges. We demonstrate here two equivalent ways to cure this problem. Let us first recall the entropy function formalism \([1, 3]\).

One considers a general theory of gravity described by the Lagrangian density \( \mathcal{L} \) with Abelian gauge fields \( F^i(\chi) \) and scalar fields \( \Phi^i(\chi) \). Then one writes the most general ansatz for the near-horizon geometry assuming the isometries of AdS\(_2 \times S^1 \) (for simplicity, we consider here \( d = 4 \) as in \([1, 3]\)):
\[ ds^2 = v_1(\theta) \left( -r^2 \, dt^2 + \frac{dr^2}{r^2} \right) + \beta^2 (d\theta^2 + v_2(\theta)(d\phi^2 - ar \, dt)^2), \]
\[ F^i = (\epsilon^i - ab(\theta)) \, dr \wedge dt + \partial_\beta b(\theta) \, d\theta \wedge (d\phi - ar \, dt) \quad \text{and} \quad \Phi^i = u^i(\theta), \] (64)
in terms of the parameters \( \{a, \epsilon^i, \beta\} \) and \( \theta \)-dependent scalars \( \{v_1(\theta), b(\theta), u^i(\theta)\} \). Then, one defines the ‘reduced action’ \( f(\alpha, \tilde{\epsilon}, \tilde{\beta}, \tilde{v}(\theta), \tilde{b}(\theta), \tilde{u}(\theta)) = \int d\phi \, d\psi \mathcal{L} \) —a functional that generates the equations of motion:
\[ \frac{\delta f}{\delta \epsilon} = \frac{\delta f}{\delta \beta(\theta)} = \frac{\delta f}{\delta v(\theta)} = \frac{\delta f}{\delta \alpha(\theta)} = 0, \]
where the functional derivatives can be understood in terms of the Fourier coefficients in the expansion along \( \theta \), and
\[ \frac{\delta f}{\delta \alpha} = q_i, \quad \frac{\delta f}{\delta \epsilon} = j, \] (65)
where \( q_i \) and \( j \) are supposed to give the charges of the black hole. Then, the entropy function is defined to be the Legendre transform of the reduced action
\[
E(j, q_i, \beta, \tilde{v}(\theta), \tilde{b}(\theta), \tilde{u}(\theta)) = 2\pi (e' q_i + \alpha j - f).
\]
Finally, the entropy of the black hole is \( S = E \), evaluated on the solution.

5.1. Completing the equations of motion

In section 3.1, we learned how to find the conserved charges in the presence of Chern–Simons by writing the KK gauge field equations of motion in a conserved form. Since we now know the right reduction ansatz, we just need to find a mechanism to parametrize both the variation with respect to \( A_t \) and \( B_t \) and the integration of the right-hand side of the equations of motion to obtain the closed form. One such mechanism is a modification of the ansatz with the pure gauge terms \{\( \epsilon_i \), \( \omega^a \}\) to do the variations \( \delta L / \delta A_t \) and \( \delta L / \delta B_t \), and with a dummy function \( c(r) \), that introduces an artificial and unphysical \( r \)-dependence into fields that are constant by the symmetries. \( c(r) \) then allows us to keep track of their, otherwise vanishing, derivatives and to do their integration on the right-hand side of the equations of motion. Hence, we write
\[
A_t = -(\epsilon_i + \epsilon' r) dr + c(r) p_i^j(\theta)(d\phi^a - (\omega^a + \alpha^a r) dr),
\]
\[
ds^2 = v(\theta) \left( -r^2 dr^2 + \frac{dr^2}{r^2} \right) + \beta^2 (d\theta^2 + \eta_{ab}(\theta)(d\phi_a - (\omega^a + \alpha^a r) dr)(d\phi_b - (\omega^b + \alpha^b r) dr))
\]
and we also wrap all scalar fields that appear in the Chern–Simons terms with a factor of \( c(r), u_i(\theta, r) = c(r) u_i(\theta) \). The solution corresponds to setting \( c(r) = 1 \) and \( c'(r) = 0 \), which we can either implement by furnishing \( c(r) \) with a control parameter or by choosing \( c(r) \), s.t. \( c(r_0) = 1 \) and \( c'(r_0) = 0 \) for some \( r_0 \), but \( c'(r) \neq 0 \) for \( r \neq r_0 \). The equations of motion for the gauge fields are then \( \partial_r \frac{\delta L}{\delta \epsilon} = \frac{\partial f}{\partial \epsilon} \) and give rise to the conserved charges
\[
Q_i = \frac{\delta f}{\delta \epsilon_i} - \int dr \frac{\delta f}{\delta \epsilon_i} \quad \text{and} \quad J_a = \frac{\delta f}{\delta \alpha^a} - \int dr \frac{\delta f}{\delta \alpha^a},
\]
evaluated on the solution. A simple variation of this is \( c(r) = 1 + \frac{1}{r} \), with \( n \) being the number of 3D scalar fields in the CS term, which automatically takes care of the integration of the second term and ensures that all remnant dummy terms will disappear in the first term at \( r = 0 \).

The other computations follow just as in the original form of the entropy function, using \( c = 1, c' = 0 \) throughout. Note that the entropy function is still computed as originally defined, \( E = 2\pi (\frac{\partial f}{\partial \epsilon} \omega^a + \frac{\partial f}{\partial \alpha^a} \epsilon' - f) \), i.e. not using the conserved charges.

One can easily see that this gives the equations of motion, and it also gives the correct value for the entropy as the original derivation \([1, 3]\) is independent of what the conserved charges are. This can also be seen by repeating the derivation in section 6.4 with the original action (34). As a simple example we have already written the 4D ansatz (55) in section 4.2.1 in a suggestive form, such that the coefficients can be read off from (56) and (57) with \( \beta = \sqrt{v} \). We note that the \( \omega^a \) parameters do not appear here in the action. A simple computation reveals that this indeed gives the results in section 4.2.1.

5.2. Gauge invariance from boundary terms

In section 3.3, we found that the charges are gauge invariant. However, it would be desirable if we could impose gauge invariance at the level of the Lagrangian of the 3D action (34).
The result can, in principle, be oxidized back to 5D, but we will stick for simplicity to 3D. The only term of concern is $A^{3D} \wedge dA$ which, after integration, gives a boundary term $\Lambda dA^{3D} \wedge dA$. This can be re-expressed as $d(\Lambda A^{3D} \wedge dA) - A^{3D} d\Lambda \wedge dA$, where the first term vanishes if we consider a stationary boundary. The second term is suitably cancelled by adding a boundary term $A^{bdy} A^{3D} \wedge dA^{bdy}$, which is identical to a bulk term $d(\Lambda A^{3D} \wedge dA)$. Expressed in index notation, and furnished with appropriate factors, the boundary term that we need to add corresponds to the bulk term is

$$\delta L^{3D} = -\frac{\mathcal{V}}{16\pi} \frac{1}{3} \epsilon_{LMN} \epsilon^{ab} \left( A^{a,LF}_{MN} A^{b} + 2 A^{a,LF}_{MN} A^{b,N} \right),$$

eliminating the gauge-dependent term. A quick calculation shows that this does not affect the value of the charges (35), (36). Effectively, what we have done is to differentiate the components of the 5D gauge field in the CS term whose gauge transformations do not vanish automatically by periodicity constraints, and remove the derivative from other components by an integration by parts. Hence, the right-hand side of each of the 3D gauge field equations of motion does vanish, and the charges are just the conjugate momenta of the gauge fields $A^{3D}$.

6. Thermodynamic charges

Having computed the charges of the $S^{d-2}$ isometries, we now turn to the charges of the AdS$_2$ isometries. In particular, we will concentrate on the charge of $\partial_t$, as this will be related to the thermodynamic quantities’ entropy $S$ and mass $M$. First, we will compute the Poincaré time Noether charge from the Hamiltonian in the NHG and propose a new definition of the black hole entropy for extremal black holes in the NHG in terms of this charge—similar to Wald’s definition for non-extremal black holes. Then we (i) justify this definition by showing that it gives the right extremal limit of the first law, (ii) derive from the Noether charge a statistical version of the first law suitable for extremal black holes and (iii) rederive the entropy function directly from the definition of the entropy. Finally, we discuss the notion of mass as seen from the NHG by deriving a Smarr-like formula.
6.1. Poincaré time Hamiltonian

For the Poincaré time Killing vector $\partial_t$, one expects the Noether charge to be related to the Hamiltonian, which we will explore now.

Since the theory is generally diffeomorphism invariant, we expect the bulk contribution to vanish. So we concentrate on boundary terms $S_{\text{bdy}} = \int_{\Sigma} L_{\text{bdy}}$ that are necessary to cancel total derivatives $d\Theta$ in the variation of the bulk action $\delta S = \int (E^i \delta \phi^i + d\Theta(\delta \phi))$. In our example, we have to consider both the variations of the metric and of the 3D gauge fields. For the gauge fields, the term that we ignored in the derivation of the equations of motion was

$$\partial_{\mu} \Theta^\mu = \partial_{\mu} \left( \frac{\delta L}{\delta A_{\nu,\mu}} \delta A_{\nu} + \frac{\delta L}{\delta B_{\nu,\mu}} \delta B_{\nu} \right).$$

For a complete spacetime, the textbook answer is to place the usual restriction $\delta A_{\text{bdy}} = \delta B_{\text{bdy}} = 0$. Then, the only boundary term that one needs to add in order to make the variational principle consistent is a Gibbons–Hawking-like term that compensates for a total derivative $d\Theta$. The term that we ignored in the derivation of the equations of motion was $\frac{\delta L}{\delta A_{\nu,\mu}} \delta A_{\nu} + \frac{\delta L}{\delta B_{\nu,\mu}} \delta B_{\nu}$.

Now, we can read off the Hamiltonian of the NHG if it were an isolated solution. By definition, $L_{\delta} g_{\mu \nu} = 0$, such that the canonical Hamiltonian is just $H = -\int_{\Sigma_{\text{bdy}}} i_{\partial_t} L_{\text{GH}}$ with the time slice of $B$ being $\Sigma_{\text{bdy}} = S^1$. Since $\partial_t$ is a Killing vector, a quick calculation shows $i_{\partial_t} \sqrt{-\gamma} K = \sqrt{-\gamma} N_{M N} (d\delta \gamma)^{MN}$, and hence the Hamiltonian is just

$$H = -\int_{\Sigma_{\text{bdy}}} i_{\partial_t} L_{\text{GH}} = \frac{\sqrt{-\gamma}}{16\pi G_5} \int_{S^1} d\Theta \sqrt{-g} N_{M N} (d\delta \gamma)^{MN}. \quad (74)$$

Now, if we consider the near-horizon geometry being embedded in the full black hole solution, we cannot put $\delta A_{\text{bdy}} = \delta B_{\text{bdy}} = 0$, but we need to satisfy the variational principle by adding a Hawking–Ross-like boundary term as in [28]:

$$\mathcal{L}_{\text{HR}} = n_M \left( \frac{\delta L}{\delta A_{N,M}} A_N + \frac{\delta L}{\delta B^a_{N,M}} B^a_N \right) = -n_N \left( \tilde{Q}^{MN} A_N + J^a_{MN} B^a_N \right), \quad (75)$$

and impose the condition to keep the charges fixed under variations of the boundary fields. Now, the boundary action varies as

$$\delta S_{\text{HR}} = -\int_{\partial M} d^2 \sigma n_M \left( (\delta \tilde{Q}^{MN}) A_N + (\delta J^a_{MN}) B^a_N \right) - \int_{\partial M} d^2 \sigma n_M \left( \tilde{Q}^{MN} \delta A_N + J^a_{MN} \delta B^a_N \right). \quad (76)$$

where the second term cancels the total derivative in the variation of the bulk action (note the inward-pointing $n$) and the first term vanishes as the charges are fixed. A little caveat occurs if we use the gauge-dependent form of the action (34), when $\tilde{Q} \neq Q$, however the missing bit does not depend on the 3D gauge fields, but only on the scalar fields, and hence it is invariant under variations of the gauge fields. If we consider the gauge-independent form of the action (70), then $\tilde{Q} = Q$. Again, by definition we have $L_{\delta} B^i = 0$, and we will choose a gauge such that $L_{\partial_t} A^i = 0$, and the canonical Hamiltonian is just

$$H = -\int_{S^1} i_{\partial_t} (L_{\text{HR}} + L_{\text{GH}}). \quad (77)$$
Due to the AdS\(_2\) symmetries, we have \(\int_{\Sigma^1} \partial_t (Q \wedge A)\) and similar for \(J_i \wedge B^i\). This puts the Hawking–Ross contribution to the boundary Hamiltonian to 

\[\int_{\Sigma^1} \partial_t (Q \wedge A) + 4 \mathcal{F}^MN i_{\beta} (\partial_{\alpha} B^\alpha + A) + 16 \frac{\sqrt{3}}{3} \epsilon_{MNL} \epsilon_{ij} A^i_{L} A^j\]

(78)

We now compare (78) with the Noether charge obtained by dimensional reduction of the 5D expression (20). For this, we work out how the individual terms look like in 3D with the notation of section 3.1. We consider only the components \(Q^MN\) in the non-compact directions and only zero modes of the fields in the compact directions. Hence, we get from the reduction formulae (31)–(34)

\[Q^MN_{\xi^3D} = -\frac{\sqrt{-g} \mathcal{H}}{2\pi G_5} \int_{B^2} Q^{3D} (d^\xi^3D)_{M}^N + (d^\xi^3D)_{M}^N \cdot \mathbf{B}^i h^i_{HH} (\mathcal{F}^MN) + 4 (d^\xi^3D)_{M}^N \cdot \mathbf{B}^i h^i_{HH} (\mathcal{F}^MN) + 4 (d^\xi^3D)_{M}^N \cdot \mathbf{B}^i h^i_{HH} (\mathcal{F}^MN)\]

(79)

Now, we can write the charges of \(\xi^3D\), the non-compact components of \(\xi\), and \(\chi\), its compact components, separately:

\[Q^MN_{\xi^3D} = -\frac{\sqrt{-g} \mathcal{H}}{2\pi G_5} \int_{B^2} Q^{3D} (d^\xi^3D)_{M}^N + (\xi^3D)_{M}^N \cdot \mathbf{B}^i h^i_{HH} (\mathcal{F}^MN) + 4 (\xi^3D)_{M}^N \cdot \mathbf{B}^i h^i_{HH} (\mathcal{F}^MN)\]

(80)

where we have implicitly done an integration over the compact coordinates. Thus we see that (78) is just the Noether charge \(Q^3D_{\partial_t}\) in 3D (80) as expected, and we have yet another confirmation of the KK charge (35), as it matches with (80).

6.2. Entropy

The entropy \(S\) of non-extremal black holes was shown by Wald [5] to be given by the Noether charge \(\kappa S = \int_{\partial \mathcal{B}} \mathcal{Q}_\xi\) of the timelike Killing vector \(\xi\) that generates the horizon, evaluated on the bifurcate \((d - 2)\) surface \(\mathcal{B}\) of the horizon, and \(\kappa\) is the surface gravity of the horizon. Jacobsen, Myers and Kang [19] later showed that the charge can be evaluated anywhere on the horizon, provided all fields are regular at the bifurcation surface. After a coordinate transformation, one sees that this requires all gauge fields to vanish on the horizon, such that the gauge is fixed to \(\xi \cdot A = 0\) at the horizon, and hence eliminates the ambiguity of the gauge dependence of the Noether charge.

For extremal black holes, \(\kappa = 0\) on the horizon \((r = 0)\), so Wald does not give a suitable definition of \(S\), and furthermore there is no bifurcation surface—putting the gauge fixing in doubt. In the AdS NHG, there should be no special point where to compute physical quantities. Using the concept that the entropy is intrinsic to the horizon, and hence does not require embedding the NHG into an asymptotic geometry, those problems are cured by defining the entropy as

\[S = \frac{2\pi}{\kappa (r_{\text{bdy}})} \int_{\partial \mathcal{B}} H_1 (r_{\text{bdy}})\]

(82)
in the dimensionally reduced theory with the boundary placed at any radius $r_{\text{bdy}} \neq 0$. The fact that the 3D theory is static allows us to use \[ \kappa = - \frac{g_{tt,r}}{2\sqrt{-g_{tt}g_{rr}}} \] (83)

that is well defined and physically motivated as the acceleration of a probe at any radius $r$ with respect to an asymptotic observer and hence related to the temperature of the Unruh radiation. It also ensures that the entropy is independent of $r_{\text{bdy}}$ with well-defined limits $r_{\text{bdy}} \to 0$ and $r_{\text{bdy}} \to \infty$. Now, in terms of the Noether charge (80), the entropy is just as expected

\[
S = \frac{2\pi}{\kappa(r)} \int_{\Sigma_1} Q_\delta(r) \tag{84}
\]

in the gauge $\xi \cdot A(r) = \xi \cdot B(r) = 0$; but evaluated at $r \neq 0$, rather than $r = 0$ that one would naively expect. We will see in the following three subsections that this definition of the entropy naturally arises from black hole thermodynamics.

6.3. First law

Since we now have an expression for the entropy intrinsic to the extremal limit, let us see whether we can also find an expression for its variation as derived for non-extremal black holes by Wald in [5]. First let us write the Noether charge for the gauge-invariant action (70) in 3D for $\xi^2 = 0$, as

\[
Q_{\xi^{3D}}(r) = \frac{\kappa(r)}{2\pi} S - \xi^{3D} \cdot A(r) Q_{\text{el}} - \xi^{3D} \cdot B^0(r) J_0. \tag{85}
\]

Then, we consider variations of the dynamical fields $\delta \phi^i$ that keep the solution on-shell and use the identity $\delta dQ^{\xi^{3D}} = d(\xi^{3D} \cdot \Theta)$ [5], with $\Theta$ defined in section 2, such that we can relate the variation of the charge evaluated over two boundaries $\Sigma_1$ and $\Sigma_2$ of a spacelike $(d-1)$ surface:

\[
\int_{\Sigma_1} (\delta Q^{\xi^{3D}} - \xi^{3D} \cdot \Theta) = \int_{\Sigma_2} (\delta Q^{\xi^{3D}} - \xi^{3D} \cdot \Theta). \tag{86}
\]

Now, let us move the boundaries into the near-horizon geometry ($\to \Sigma_H$) and into some asymptotic limit ($\to \Sigma_\infty$). On $\Sigma_H$, we have

\[
\int_{\Sigma_H} \xi^{3D} \cdot \Theta = \int_{\Sigma_H} \xi^{2L} d\phi^M e_{LMN} \left(-g^{OP} \delta_{OP}^N + g_{OP}^{\gamma} \delta_{OP}^\gamma - \frac{\delta L}{\delta A_{O,N}} \delta A_O + \frac{\delta L}{\delta B_{O,N}} \delta B_O \right)
\]

\[
= \frac{S}{2\pi} \delta \kappa - Q_{\text{el}} \delta (\xi^{3D} \cdot A) - J_i \delta (\xi^{3D} \cdot B^i). \tag{87}
\]

where we used for the second equality the AdS$_2$ isometries, and assumed an Einstein–Hilbert term for the gravitational action, and any gauge field term that can be written with only first derivatives of $A$, such as (70). The right-hand side of (86) can be interpreted by following Wald and defining the canonical energy, i.e. the Hamiltonian measured by an asymptotic observer at $\Sigma_\infty$, $E = \int_{\Sigma_\infty} (Q^{\xi^{3D}} - \xi^{3D} \cdot V)$ with some $(d-1)$-form $V: \delta \int_{\Sigma_\infty} \xi^{3D} \cdot V = \int_{\Sigma_\infty} \xi^{3D} \cdot \Theta$. This corresponds, for the asymptotic boundary conditions $A = B = 0$ and suitable normalization of $\xi^{3D}$, to the mass. Altogether, (87) now gives us an expression similar to the first law

\[
\frac{\kappa(r)}{2\pi} \delta S + \Phi(r) \delta Q_{\text{el}} + \Omega^i(r) \delta J_i = \delta E \tag{88}
\]
at some \( r \neq 0 \), where \( \Phi(r) = -\xi_3^D \cdot A(r) \) and \( \Omega(r) = -\xi_3^D \cdot B^i(r) \) measure the co-rotating electric potential and angular frequency\(^6\) at \( r \) in the NHG with respect to the definition of \( E \). This, however, is not yet a relation for the full black hole, but captures only physics outside \( \Sigma_r \). The extremal limit of the non-extremal first law of the full black hole solution is reproduced by taking the limit \( r \to 0 \):

\[
\Phi_H \delta Q_{el} + \Omega_H \delta J_b = \delta E,
\]

where \( \Phi_H = -\xi_3^D \cdot A(0) \) and \( \Omega_H = -\xi_3^D \cdot B(0) \) are the horizon co-rotating electric potential and angular frequency. It is interesting to observe though that (88) and the corresponding expressions for the Smarr formula resemble the first law of a finite temperature black hole, even though its physical significance is limited, as \( \Sigma_r \) for \( r \neq 0 \) is not a horizon.

An interesting observation and lesson is that when embedding the near-horizon solution into an asymptotic solution, but computing Noether charges in the NHG, we need to use the gauge-invariant action (70) and the full Noether charge, because there is no boundary of the NHG on which we were allowed to fix the gauge fields and its gauge variations.

We see that our version of the first law also holds for perturbations away from extremality, which connects it smoothly (in a thermodynamic sense) to the near-extremal limit of the non-extremal black hole, again supporting our definition of the entropy.

### 6.4. Entropy function and the Euclidean action

Now, let us continue following Wald [5] and relate the (integrated) mass (or energy \( E \)) to the entropy. Starting with (85), we apply Gauss’ law to find

\[
\frac{\kappa(r)}{2\pi} S - \xi_3^D \cdot A(r) Q_{el} - \xi_3^D \cdot B^i(r) J_b = E - \int_{\mathcal{M}} \mathcal{J}_\xi + \int_{\Sigma_{\infty}} \xi_3^D \cdot V =: E - \frac{\kappa(r)}{2\pi} I(r),
\]

where the Euclidean action\(^7\) \( I \) is now, in principle, a function of the radial position of \( \Sigma_H \), since \( \delta \mathcal{M} = \{ \Sigma_H, \Sigma_{\infty} \} \). Even though \( I \) is defined only for \( \kappa \neq 0 \) as the integral of the analytically continued Lagrangian, with \( \tau = it \) having period \( \frac{2\pi}{\kappa} \), one would like to find a well-defined limit as \( \kappa \to 0 \), i.e. \( r \to 0 \), representing the full extremal black hole solution. This requires

\[
\Phi_H Q_{el} + \Omega_H J_b = E.
\]

This relation can be taken as a (gauge-dependent) definition of the mass of the black hole in the near-horizon geometry. We note that since the action is gauge-invariant, (91) is gauge-independent in the sense that a gauge transformation that changes \( \Phi_H \) and \( \Omega_H \) on \( \Sigma_0 \) changes \( E \) at \( \Sigma_\infty \) accordingly. In the appropriate gauge in which \( E = M \), it should agree with the BPS (or extremality) condition—as we verified for BMPV and GR—and with an applicable Smarr-like formula, supposed one has a full solution at hand. Now, let us study the remaining terms of (90). Again, we make use of the AdS\(_2\) geometry to find that \( \xi_3^D \cdot (A(r) - A(0)) / \kappa(r) = F_{\tau r}^3 = -E_H \) is the constant co-rotating electric field strength in the NHG, as is \( \xi_3^D \cdot (B'(r) - B'(0)) / \kappa(r) = H_{\tau r} = -H_H \) the field strength of the KK gauge field. Now, (90) reads

\[
S = -2\pi (E_H Q_{el} + H_H J_b) - I,
\]

with all terms, including \( I \), being independent of the position \( r \neq 0 \) of \( \Sigma_H \) in the NHG. (92) also holds in the limit as \( r \to 0 \). A similar expression was proposed and discussed in

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\(^{6}\) To illustrate that this definition of \( \Omega \) corresponds to that in [5], consider a vector \( \xi = \partial_t - \Omega \partial_\phi \) in static coordinates with a diagonal metric \( g \) and \( \xi = \partial_t \) in co-rotating coordinates with a non-diagonal metric \( g' \). Then \( \xi = g_{\tau r} \partial_\tau - \Omega g_{\tau \phi} \partial_\phi = g_{\tau r} \partial_\tau + B_{\tau r} \partial_\phi \). A similar argument follows from requiring constant normalization of \( \xi \) and considering \( g_{\tau r} + \Omega g_{\tau \phi} = g_{\tau r} \) in the explicit coordinate transformation.

\(^{7}\) \( I \) equals the Euclidean action only for stationary spacetimes, see [5].
a statistical context by Silva in [6], where it was motivated by taking the extremal limit of non-extremal black holes, assuming an appropriate expansion of $\Phi_H$ and $\Omega_H$ in terms of the inverse temperature. This is identical to (92), provided one identifies the NHG field strengths with the appropriate expansion coefficients in [6]. Note that this relation is particular for extremal black holes and profoundly different from the relation of the entropy to the Euclidean action for non-extremal black holes [29, 30].

Let us now show how this relates to the entropy function formalism. Given $I = \frac{-2\pi}{k(r)} \left( \int_M i_{\xi} L + \int_{\Sigma_{\infty}} i_{\xi} V \right) [5]$, we use the fact that the spacetime in the NHG can be trivially foliated with spheres to rewrite this as

$$I = \frac{-2\pi}{k(r)} \left[ \int_{\Sigma_{\infty}} i_{\xi} L + \int_{\Sigma_{\infty}} i_{\xi} V - \int_0^r \int_{S^2} i_{\xi} L \right] = : I_0 + \frac{2\pi}{k(r)} \int_0^r i_{\xi} \int_{\Sigma_{\infty}} L, \quad (93)$$

where $\partial M_0 = \{\Sigma_{r=0}, \Sigma_{\infty}\}$. Since $\int_{S^2} L$ is supposed to be invariant under the AdS$_2$ isometries, it is proportional to the volume form on AdS$_2$ and $(\int_0^r i_{\xi} \int_{S^2} L)/k(r) = \Phi_{S^2} L = \text{const}$. Now, the fact that $I = \text{const}$ implies that $I_0 = 0$ and we are left with

$$S = -2\pi \left( E_H Q_{\text{el}} + H_H J_I + \Phi_{S^2} \right). \quad (94)$$

This is just the entropy function for the gauge-invariant action (70). The same derivation can be applied to the original action (34) to give its corresponding entropy function. In that case $\mathcal{E}$ in (91) will have a different value, because of the boundary terms in the action, stressing again the need to work with (70) when relating the NHG to the asymptotic geometry.

6.5. Mass

Even though the mass of extremal black holes is fixed by the extremality (or BPS) relation (91), let us now study its physical interpretation from the point of view of the NHG by deriving a Smarr-like formula for the 5D Einstein–Maxwell–CS case.

Let us suppose there is some asymptotic geometry attached to the near-horizon geometry in such a way that the conditions in section 3.3 are satisfied, and follow closely the derivation by Gauntlett, Myers and Townsend in [4] for a few steps. The mass, $\mathcal{E}$ in a gauge in which $A = B = 0$ at $\Sigma_{\infty}$, can be rewritten using Gauss’s law in 5D as

$$M = -\frac{d - 2}{d - 3} \frac{1}{16\pi G_5} \int_{\Sigma_{\infty}} \star d\hat{k} = \frac{3}{2} \frac{1}{16\pi G_5} \left[ -\int_{\Sigma} \star d\hat{k} + \int_M \star \Box \hat{k} \right], \quad (95)$$

for some $\partial M = \{\Sigma, \Sigma_{\infty}\}$ and $k$ being the asymptotic unit norm timelike Killing vector. Assuming we work in a gauge in which $\mathcal{L}_{\hat{k}} A = 0$, and using the relations $\Box k_\mu = -R_{\mu\nu} k^\nu, \mathcal{L}_{\hat{k}} \Omega = i_{\hat{k}} (d\Omega) + d(i_{\hat{k}} \Omega)$ for any form $\Omega$ and the equations of motion for $g$ and $A$, the result is

$$M = \frac{3}{2} \frac{1}{16\pi G_5} \int_{\Sigma} \left[ \star d\hat{k} + 4(k \cdot A) \star F - \frac{4}{9} \star (\hat{k} \wedge (\hat{A} \cdot F)) + \frac{16}{3\sqrt{3}} (k \cdot A) A \wedge F \right], \quad (96)$$

plus a term at $\Sigma_{\infty}$ that vanishes as $A \to 0$. In dimensions other than $d = 5$, there will be an extra term that cannot be expressed as a surface integral at $\Sigma_H$. For details see [4]. Now, we see that the first, second and last terms combine to give the Noether charge (19). Decomposing $k$ into its compact and non-compact components, $k = \partial t + \Omega \chi$, and choosing $\Sigma$ to be an $r = \text{const}$ surface in the NHG, we find from the 3D expressions (80), (81) that this gives us

$$M = \frac{3}{2} \left[ \frac{k(r)}{2\pi} S + \Omega \chi J_I \right] + \Phi(r) Q_{\text{el}},$$

$$-\frac{1}{8\pi G_5} \left[ V_f \int_{S^2} (\partial_t \cdot A) \star F - \int_{\Sigma} \star ((\partial_t + \Omega \chi) \wedge (\hat{A} \cdot F)) \right]. \quad (97)$$
In $(\hat{\partial}_i + \Omega^i \hat{J}_i) \wedge (\hat{A} \cdot F)$, we find that in terms of frame fields the relevant components are $(\hat{\partial}_i + \Omega^i \hat{J}_i)_{00}$, $A_0$ and $F_{01}$, since the AdS$_2$ symmetries restrict non-vanishing $F_{M1}$ to $M = 0$. This makes the last term vanishing, such that we get in the limit $r \to 0$ the Smarr formula,

$$M = \frac{1}{2} \Omega^2 H + \Phi H Q_{el}.$$  

(98)

that agrees with the near-horizon limit of the non-extremal one. From the point of view of the near-horizon solution, we find that the mass is now a gauge-dependent expression, with the gauge given by the embedding of the near-horizon solution in the asymptotic solution. We find that (98) looks different from (91), however they are in agreement since $\Omega_H$ vanishes for BMPV black holes [4].

7. Conclusions

In this paper, we presented expressions for conserved currents and charges of 10D type IIB supergravity (with the metric and 5-form) and minimal (gauged) supergravity theories in five dimensions. These have been obtained following Wald’s construction of gravitational Noether charges. Those of the 5D gauged supergravity can also be obtained by the dimensional reduction of the 10D formulae. We further showed that the Noether charges of the higher dimensional theories, after the dimensional reduction, match precisely with the Noether charges of gauge fields obtained by Kaluza–Klein reduction over the compact Killing vector directions of interest. Our expressions for the charges should be valid generally for both extremal and non-extremal geometries. We then turned their applications to extremal black holes and demonstrated that, when evaluated in the near-horizon geometries, our charges reproduce the conserved charges of the corresponding extremal black holes under certain assumptions. In particular, we exhibited that our methods give the correct electric charges and angular momenta for the BMPV and Gutowski–Reall black holes.

A host of new solutions to supergravity theories with AdS$_2$ isometries have been found recently [20] and many more such solutions are expected to be found in the future. These solutions may be interpreted as the near-horizon geometries of some yet to be found black holes. In such cases, our results should be useful in extracting the black hole charges without knowing the full black hole solutions but just the near-horizon geometries. On the other hand, the holographic duals of string theories in the NHG are expected to be supersymmetric conformal quantum mechanics. Our conserved charges should be part of the characterizing data of these conformal quantum mechanics.

We argued that the black holes with AdS$_3$ near horizons do not satisfy our assumptions when embedded in the black hole asymptotes with $S^{d-2}$ isometries (rather than the black string asymptotes). Supersymmetric black rings are the main examples for which our formulae do not seem to apply. More generally for black holes with AdS$_3$ one has to find the correct way to extract the conserved charges separately which we would like to return to in future.

We then presented a new entropy function valid for rotating black holes in 5D with CS terms which gives the correct electric charges as well as the entropy. This is an improvement over [21]. We used appropriate boundary terms that make the action fully gauge independent which turns out to be relevant to obtain the thermodynamics in the second part of the paper.

In the second part of the paper we exhibited a new definition of the entropy as a Noether charge, and a derivation of the first law, which are applicable for extremal black holes directly. We used this definition to produce the statistical version of the first law and moved on to rederive the entropy function from a more physical perspective. Finally, we commented on the physical interpretation of the mass in the near-horizon solution. The relevant calculations were done in the near-horizon geometry, only assuming an embedding into some asymptotic
solution for the purpose of formally defining the mass. We did not, however, produce a conserved charge corresponding to the level number. In terms of the 5D fields, the expression in [27] is just proportional to \( \int_{\Sigma_0} \ast F \), which is conserved in the NHG by the symmetries, but not by the equations of motion in a general geometry. Various potentially interesting candidates, such as the R-charge and global AdS2 time Noether–Wald charge, did not produce an interesting result.

We find that the gauge-independent thermodynamic quantities can be evaluated everywhere in the near-horizon geometry, as they are a statement about the near-horizon geometry. In particular, they are the entropy, Euclidean action and charges and their chemical potentials, as well as the statistical version of the first law (92). Relations and quantities related to the asymptotic geometry and to thermodynamics of non-extremal black holes (the mass, horizon electric potential and angular frequency, as well as the first law and Smarr formula) however are gauge-dependent from the point of view of the near-horizon geometry. This means that the former ones may be more relevant for characterizing attractors.

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Appendix. Black rings

The non-equal angular momentum generalization of the BMPV case is the supersymmetric black ring [18]. It is an excellent counterexample in which the conditions in section 3.3 are not satisfied. To demonstrate this, we sketch out the derivation of the asymptotic and near-horizon limits as given in [18]. The general form of the solution is given by

\[
\begin{aligned}
\mathrm{d}s^2 &= -f^2(dt + \omega_\theta \, d\phi + \omega_\psi \, d\psi)^2 + \frac{f^{-1}R^2}{(x - y)^2} \times \left( \frac{\mathrm{d}y^2}{y^2 - 1} + \frac{\mathrm{d}x^2}{1 - x^2} + (1 - x^2) \, d\phi^2 + (y^2 - 1) \, d\psi^2 \right) \\
A &= \frac{\sqrt{3}}{2} \left( f (dt + \omega) - \frac{q}{2} (1 + x) \, d\phi + (1 + y) \, d\psi \right),
\end{aligned}
\]

where \( y \in [\infty, -1], x \in [-1, 1], \phi, \psi \in \mathbb{R}/2\pi \mathbb{Z} \) and \( f^{-1} = 1 + \frac{q^2}{2R^2} (x - y) - \frac{q^2}{2R^2} (x^2 - y^2), \omega_\phi = -\frac{q}{2R^2} (1-x^2)(3Q-q^2(3+x+y)) \) and \( \omega_\psi = \frac{3q}{2R^2}(1+y) + \frac{q^2}{2R^2} (1 - y^2)(3Q - q^2(3+x+y)) \).

The asymptotic limit is given by \((x + 1) \to +0 \) and \((y + 1) \to -0 \), and its geometry of a squashed sphere with broken isometry \( SO(4) \to U(1)^2 \) can be made manifest by combining \((x, y)\) into a radial coordinate \( \rho \in \mathbb{R}_+ \) and an angular coordinate \( \Theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \):

\[
\begin{aligned}
\rho \sin \Theta &= \frac{R \sqrt{y^2 - 1}}{x - y} \quad \text{and} \quad \rho \cos \Theta = \frac{R \sqrt{1 - x^2}}{x - y}.
\end{aligned}
\]
The near-horizon limit, on the other hand, is given by $y \to -\infty$, such that the appropriate radial and angular coordinates are $r = -\frac{\epsilon}{y}$ and $\cos \theta = x$. A first observation is that the two limits are just points in the ‘opposite’ coordinates, $(\rho, \theta) \to (R, \frac{\pi}{2})$ and $(r, \theta) \to (R, \pi)$. To obtain the near-horizon geometry in a suitable form, we define $\chi = \phi - \psi$, take the limit $r = \epsilon \tilde{r} R^{-1}$, $t = \epsilon^{-1} \tilde{t}$, $\epsilon \to 0$ and get

$$\text{d}s^2 = \frac{q^2}{4\tilde{r}^2} \text{d}\tilde{r}^2 - \frac{\tilde{r}}{q} \text{d}t \text{d}\psi + \frac{3((q^2 - Q)^2 - 4q^2 R^2)}{4q^2} \text{d}\psi^2 + \frac{9^2}{4}(\text{d}\theta^2 + \sin^2 \theta \text{d}\chi^2)$$

and

$$A = -\frac{\sqrt{3}}{4q}((q^2 + Q) \text{d}\psi + q^2 (1 + \cos \theta) \text{d}\chi).$$

Now, we also see that the topology of the horizon is $S^1 \times S^2$ with $U(1) \times SO(3) \ni U(1)^2$ isometry and whose subgroup $U(1)^2$ is not guaranteed to agree with the $U(1)^2$ of the asymptotic geometry. The $\text{AdS}_2$ geometry is more apparent after dimensional reduction, when $g_{\mu\nu} \propto \tilde{r}^2$ is restored, and after suitably rescaling $\tilde{t}$. Furthermore it is shown in [18] that the $\text{AdS}_2$ and $S^1$ combine into a local $\text{AdS}^3$. The conserved charges are now $J_\phi = \frac{\pi}{16 G_5} q^{-1}((q^2 - Q)^2 - 12q^2 R^2)$, $J_\chi = -\frac{\pi}{16 G_5} q(q^2 + Q)$ and $Q_{A1} = \frac{\sqrt{3}}{2G_5} Q$, or in the old coordinates $J_\phi = \frac{\pi}{16 G_5} q^{-1}((q^2 - Q)^2 + 2q^2(q^2 - 2Q - 6R^2))$, $J_\chi = \frac{\pi}{16 G_5} q Q$. They compare to the asymptotic quantities computed in [18] $J_\phi = \frac{\pi}{8 G_5} q(3Q - q^2)$, $J_\chi = \frac{\pi}{8 G_5} q(6R^2 + 3Q - q^2)$ and $Q_{A1} = \frac{\sqrt{3}}{2G_5} Q$.

The distinguishing feature here is that black rings have an $\text{AdS}_3 \times S^2$ near-horizon geometry. Thus the $S^1 \times S^2$ of the horizon and the $S^3$ of the asymptotic hypersurface are topologically distinct, such that there is no continuous fibration of hypersurfaces over $r$ between them. In particular, the coordinates that describe the asymptotic $S^3$ shrink the horizon and the area bounded by the black ring to a point in 3D (or an $S^3 \times S^1$ in 5D), and are missing part of the boundary of the full solution because of the difference in topology. This missing part shrinks into the coordinate singularity that also contains the horizon, so the flux that passes through that part of the boundary will not be seen from the asymptotic geometry.

It is not inconceivable that if we consider the black rings on Taub–Nut spaces like in [31–33] and obtain a 4D black hole which satisfies our criteria one may yet be able to recover the charges of such black rings.

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