Star-Shaped deviations

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Abstract

We propose the Star-Shaped deviation measures in the same vein as Star-Shaped risk measures and Star-Shaped acceptability indexes. We characterize Star-Shaped deviation measures through Star-Shaped acceptance sets and as the minimum of a family of Convex deviation measures. We also expose an interplay between Star-Shaped risk measures and deviation measures.

Keywords: Deviation measures, risk measures, Star-Shaped sets, acceptance sets.

1 Introduction

Since \cite{2} the axiomatization of risk measures has gained space in the literature. Their seminal paper argues that a “coherent” risk measure should satisfy four properties, among them, Positive Homogeneity. This property implies that the risk of a position is proportional to its size, i.e., for a risk measure $\rho$ and a positive real $\lambda$, it follows that $\rho(\lambda X) = \lambda \rho(X)$. However, Positive Homogeneity quickly came under criticism, mainly because the size of a financial position can affect the position risk due to liquidity risk, i.e., potential losses from difficulty into negotiating larger positions. In this sense, \cite{6}, \cite{8}, \cite{1} and \cite{12} argue against the Positive Homogeneity and sub-additivity assumptions adopted in the framework of coherent risk measures by focusing on Convexity, since, under $\rho(0) \leq 0$, it implies that for $\lambda \geq 1$, $\rho(\lambda X) \geq \lambda \rho(X)$.

Nonetheless, Convexity is actually stronger than purely demanding $\rho(\lambda X) \geq \lambda \rho(X)$ for $\lambda \geq 1$. The latter is called Star-Shapedness and is the focus of this study. In this sense, \cite{5} proposes the class of Star-Shaped risk measures. This nomenclature comes from the Star-Shaped property of the generated acceptance set. The reasoning for Star-Shapedness as sensible axiomatic requirement is that if a position is acceptable, any scaled reduction of it also is. The key-point in this theory is that a monetary risk measure is Star-Shaped if and only if it is the minimum of a family of Convex risk measures. This class gained some attention in the literature when \cite{13} explores allocations of Star-Shaped risk measures, \cite{17} relate them to the broader class of monetary risk measures, \cite{10} consider portfolio optimization and arbitrage, and \cite{24} explores the interplay with Star-Shaped acceptability indexes.

The arguments exposed in their work affect deviation measures in the same way as they affect monetary risk measures. Such concept of deviation is axiomatized for Convex functionals in \cite{27}, \cite{19} and \cite{9}. The main idea is to consider generalizations of the standard deviation and similar measures in an axiomatic fashion. See \cite{20} and \cite{26} for a comprehensive review. Recently, \cite{22}, \cite{21} and \cite{25} explore advantages of a more complete analysis that considers both risk and deviation measures.

Thus, it is reasonable to generalize both Positive Homogeneity and Convexity for deviation measures by Star-Shapedness. Therefore, in this paper, we define and explore the class of Star-Shaped deviation measures. In doing so, we obtain that a deviation measure is Star-Shaped if and only if it is the minimum of a family of Convex deviation measures. This result is obtained with distinct techniques from the one for risk measures since deviation measures do not fulfill the property of Monotonicity, which is crucial in the paper of \cite{5}. We develop a concept of acceptance set for deviation measures and show an interplay between Star-Shapedness for such sets and deviation measures. We also expose an interplay between Star-Shaped risk measures and deviation measures.

The intuitive reasoning is similar to the one for monetary risk measures since a reduction by scaling of a position must not produce larger deviations. Further, \cite{21} and \cite{18} study, respectively, compositions and decompositions of required capital and insurance premium into a risk measure and a deviation measure, which can be understood as the actuarial safety margin. Then, the representation of a Star-Shaped...
deviation as the minimum of Convex deviations suggests that using Star-Shaped deviation measures decreases the premium costs.

We consider a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). All equalities and inequalities are in the \(\mathbb{P}\)-a.s. sense. Let \(L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})\) be the spaces of (equivalence classes under \(\mathbb{P}\)-a.s. equality of) finite and essentially bounded random variables. We consider in \(L^\infty\) its strong topology. We say that a set \(A \subseteq L^\infty\) is Star-Shaped if \(\lambda A \subseteq A\) for all \(\lambda \in [0, 1]\), or equivalently, \(X \in A\) implies that \(\lambda X \in A\) for all \(\lambda \in [0, 1]\). We denote, by \(E[X], F_X\), and \(F_X^{-1}\), the expected value, the (increasing and right-continuous) cumulative probability function, and its left quantile for \(X \in L^\infty\) with respect to \(\mathbb{P}\). We denote that \(X\) and \(Y\) have the same distribution by \(X \sim Y\). The notation \(X \leq Y\), for \(X, Y \in L^\infty\), indicates stochastic dominance. For second-order it means \(E[f(X)] \leq E[f(Y)]\) for any increasing Convex function \(f: \mathbb{R} \to \mathbb{R}\), while for Convex order it means \(E[X] = E[Y]\) and \(E[f(X)] \leq E[f(Y)]\) for any Convex function. We define \(1_A\) as the indicator function for an event \(A \in \mathcal{F}\). We identify constant random variables with real numbers.

We begin by exposing the theoretical properties that appear in the literature regarding deviation measures, and we consider them in this paper.

**Definition 1.** A functional \(D: L^\infty \to \mathbb{R}_+ \cup \{\infty\}\) is a deviation measure. It may fulfill the following properties:

(i) **Non-Negativity:** For all \(X \in L^\infty\), \(D(X) = 0\) for constant \(X\) and \(D(X) > 0\) for non-constant \(X\);

(ii) **Translation Insensitivity:** \(D(X + C) = D(X), \quad \forall X \in L^\infty, \forall C \in \mathbb{R}\);

(iii) **Convexity:** \(\lambda D(X) + (1 - \lambda)D(Y) = D(\lambda X + (1 - \lambda)Y), \quad \forall X, Y \in L^\infty, \forall \lambda \in [0, 1]\);

(iv) **Positive Homogeneity:** \(\lambda D(X) = D(\lambda X), \quad \forall X \in L^\infty, \forall \lambda \geq 0\);

(v) **Star-Shapedness:** \(\lambda D(X) \geq D(\lambda X), \quad \forall X \in L^\infty, \forall \lambda \geq 1\).

(vi) **Lower Range Dominance:** \(D(X) \leq E[X] - \text{ess inf} X, \quad \forall X \in L^\infty\);

(vii) **Law Invariance:** If \(F_X = F_Y\), then \(D(X) = D(Y), \forall X, Y \in L^\infty\).

A deviation measure \(D\) is called proper if it fulfills (i) and (ii); Convex if it is proper and respects (iii); and Star-Shaped if it is proper and fulfills (iv); Lower Range Dominated if it satisfies (vi) and Law Invariant if it has (vii).

**Remark 2.** It is straightforward to prove that, for any \(X \in L^\infty\), the following is equivalent: (i) \(D\) is Star-Shaped; (ii) \(D(\lambda X) \leq \lambda D(X)\) for any \(0 \leq \lambda \leq 1\); (iii) \(\lambda \to \frac{D(\lambda X)}{D(X)}\) is non-decreasing. Moreover, for a proper sub-additive deviation measure, i.e. \(D(X + Y) \leq D(X) + D(Y), \forall X, Y \in L^\infty\), it is easy to see the equivalence between the following: (i) \(D\) is Star-Shaped; (ii) \(D\) is Positive Homogeneous; (iii) \(D\) is a general deviation measure.

**Example 3.** We now expose some examples of Star-Shaped deviation measures. Recall that for proper deviation measures, Star-Shapedness is implied by both Positive Homogeneity and Convexity. Further, some measures below do not satisfy \(D(X) > 0\) for non-constant \(X\), but only \(D(C) = 0\) for any \(C \in \mathbb{R}\). In some papers, such as [3], they are called variability measures or dispersion measures. Such maps can attend Non-Negativity when added to a proper deviation measures.

(i) **Standard Deviation (SD):** This is perhaps the most well-known measure of variability, being defined as \(SD(X) = E[(X - E[X])^2]^{1/2}\). It is a generalized deviation measure that represents the second moment around expectation and has been considered a proxy for risk in modern finance since the pioneering work of [15]. The SD inspires the whole conception of deviation measures, once the symmetry is dropped. This is important as dispersion from gains and losses have distinct impacts. The asymmetric forms of the SD are the lower and upper semi-deviations \((SD_-, SD_+)\). They consider dispersion only from values, respectively, below or above the expectation to avoid symmetry. This is necessary as not all dispersion in a financial position is undesirable, in fact, a result above its expected return is in general beneficial. They are defined as \(SD_-(X) = E[(X - E[X])^-]^2]^{1/2}\) and \(SD_+(X) = E[(X - E[X])^+]^2]^{1/2}\).

(ii) **Full Range (FR):** This extremely conservative generalized deviation measure is defined as \(FR(X) = \text{ess sup} X - \text{ess inf} X\) and represents the larger possible difference for two realizations of \(X\). Due to the conservatism of the FR, Lower and Upper Range (LR/UR) arise as adaptations to consider the range below or above the expectation, respectively. They are defined as \(LR(X) = E[X] - \text{ess inf} X\) and \(UR(X) = \text{ess sup} X - E[X]\). The idea is similar to that for \(SD_-\) and \(SD_+\).
(iii) Loss Value at Risk Deviation (LVaRD): This is a concept derived from the Loss VaR of [4], it is defined as $LVaRD_\alpha(X) = \sup_{u \geq 0} \{-F_{X-E[X]}^{-1}(\alpha(u)) - u\}$, where $\alpha : [0, \infty) \to (0, 1]$ is an increasing and right-continuous function which represents some benchmark loss. It is easy to check that, LVaRD is a variability measure not Convex and neither it is positively homogeneous unless $\alpha$ is constant. However, it is Star-Shaped and Translation Insensitive.

(iv) Regular based Deviation (RbD): Let $f : L^\infty \to \mathbb{R}$ be Monotone and Star-Shaped, with $f(X) \geq -E[X]$ and $f(X) = -E[X]$ if and only if $X$ is constant. Then we have that $D_f(X) = f(X - E[X])$ is the Star-Shaped deviation induced by $f$. See Theorem 12 for a concrete example. Under the same $f$, we can define the Lower and Upper Regular based Deviation (LD/UD) as $LD_f(X) = f((X - E[X])^-)$ and $UD_f(X) = f((X - E[X])^+)$, the intuition behind those is the same as for the semi-deviations and Upper/Lower Ranges. Note that all the previous examples of deviations are special cases of this approach. In the first example $f$ is the $L^2$ norm, where $D_f$ is the standard deviation, $SD_- = LD_f$ and $SD_+ = UP_f$. In the second example for the LR, we have $f(X) = -\text{ess inf} X$ and for UR, $f(X) = \text{ess sup} X$. Lastly, in the LVaRD, we have the risk measure LVaR doing the role of $f$. Further, we can define $D(X) = \min(UD_f(X), LD_f(X))$. This is a regularization of the ranges and is a Star-Shaped deviation measure.

(v) Loss-Deviation (LD): This measure is linked to the dispersion of results worse than a benchmark, typically a risk measure, measured by usual $p$-norms. This concept is explored by [22] and [23]. Let $f : L^\infty \to \mathbb{R}$ be Monotone and Positive Homogeneous such that $f(X + c) = f(X) + c$ for any $X \in L^\infty$ and any $c \in \mathbb{R}$. Then, and its loss-deviation is $LD(X) = ||(X - f(X))^-||_p, p \in [1, \infty]$. This deviation is a generalization of the lower semi-deviation, and it is not Convex for any concave $f$, except for the negative expectation. However, it is Star-Shaped since it is Positive Homogeneous.

(vi) Minkowski Deviation ($MD_A$): Given an acceptance set $A$ that is Star-Shaped, radially bounded for non-constants and stable under scalar addition (see [16] for precise definition and financial intuition of those properties), the Minkowski deviation is defined as $MD_A(X) = \inf\{m > 0 : \frac{1}{m} \in A\}$. Any Positive Homogeneous deviation measure $D$ is a Minkowski Deviation by taking $A = \{X \in L^\infty : D(X) \leq 1\}$. MD is Convex if and only if the acceptance set also is Convex. Related to our main result in Theorem 5, by letting $A_Y = \{\lambda Y + c : \lambda \in [0, 1], c \in \mathbb{R}\}$, Proposition 3.1 of [16] gives $MD_A(X) = MD_{\cup_{Y \in A} A_Y}(X) = \inf_{\lambda \in A} MD_{A_Y}(X)$. Nonetheless, their framework can not embrace deviation measures that are not Positive Homogeneous, in particular those that are the focus of this study.

(vii) Interquantile Deviation (IQD): Based on Value at Risk, defined as $VaR^\alpha(X) = -F_{-X}^{-1}(\alpha), \alpha \in (0, 1)$, the IQD is a commonly used measure of dispersion in statistics, it measures the distance between two quantiles, for $\alpha \in (0, 0.5)$ we have that $IQD^\alpha(X) = VaR^\alpha(X) - VaR^{1-\alpha}(X)$. This measure is not Convex, while it is Positive Homogeneous. Hence, it is Star-Shaped. Furthermore, while it is Translation Insensitive, it is only Non-Negative under a mixture with another proper deviation measure. This measure is studied in [28] and [3]. We have that $D^\alpha(X) = (IQD^\alpha(X))^2 + SD(X)$ is a non-trivial, Star-Shaped deviation measure that is neither Convex nor Positive Homogeneous.

(viii) Inter-ES Deviation (IED): This deviation is similar to the IQD, however, here the Expected Shortfall (ES) does the same role as the VaR in the IQD. For $\alpha \in (0, 1)$, the ES is defined as $ES^\alpha(X) = \frac{1}{\alpha} \int_0^\alpha VaR^\alpha(X)ds$. Then, we have that $IED^\alpha(X) = ES^\alpha(X) - ES^{1-\alpha}(X)$. This deviation measure in convex-order consistent, Law Invariant and has all properties of a generalized deviation measure, with exception for Non-Negativity. Again, this can be easily solved by adding it to a proper deviation measure. Furthermore, $IED^\alpha$ is the smallest Law Invariant, Translation Insensitive Convex functional dominating $IQD^\alpha$, see [28] Theorem 5 and Example 7. Again, we can easily derive a Star-Shaped deviation measure that is neither Convex nor Positive Homogeneous by squaring the IED and adding it to a Star-Shaped deviation measure.

2 Results

Proposition 4 below is a direction on how to extend the concept of acceptance set for monetary risk measures to the framework of deviation measures. A more extensive study on such acceptance sets is beyond the scope of this paper and will be postponed for future research. Regarding an alternative approach based on Positive Homogeneity, see [16].
**Proposition 4.** The following is equivalent for a deviation measure $D : L^\infty \to \mathbb{R}_+ \cup \{\infty\}$:

(i) $D$ is Star-Shaped.

(ii) $A_D := \{X \in L^\infty : D(X) \leq E[X]\}$ is Star-Shaped.

(iii) There is a Star-Shaped set $A$ such that $D(X) = D_A(X) := \inf\{m \in \mathbb{R} : X + m \in A\} + E[X]$, for all $X \in L^\infty$.

In this case we have that $D_{A_D} = D$ and $A \subseteq A_{D_A}$.

**Proof.** (i) $\implies$ (ii). Note that $X \in A_D$ if and only if $0 \leq D(X) \leq E[X]$. Then, for any $X \in A$ and $\lambda \in [0, 1]$ we have that $D(\lambda X) \leq \lambda D(X) \leq \lambda E[X]$. Thus, $\lambda X \in A$ which implies that $A_D$ is Star-Shaped.

(ii) $\implies$ (iii). Let $A = A_D$. Then

$$D_A(X) - E[X] = \inf\{m \in \mathbb{R} : X + m \in A_D\}$$

$$= \inf\{m \in \mathbb{R} : -E[X + m] + D(X + m) \leq 0\}$$

$$= \inf\{m \in \mathbb{R} : -E[X] + D(X) \leq m\}$$

$$= D(X) - E[X].$$

(iii) $\implies$ (i). Let $\lambda \in (0, 1]$ the case where $\lambda = 0$ is trivial. Note that if $A \subseteq \frac{1}{\lambda} A$, then for any $X \in L^\infty$ we have that

$$D(\lambda X) = \inf\{m \in \mathbb{R} : \lambda X + m \in A\} + E[\lambda X]$$

$$\leq \lambda \inf\{m \in \mathbb{R} : X + m \in A\} + E[X] = \lambda D(X).$$

Moreover, $D_{A_D} = D$ is trivial from previous items. Further, let $X \in A$. Then

$$0 \geq \inf\{m \in \mathbb{R} : X + m \in A\} = D_A(X) - E[X].$$

Hence, $X \in A_{D_A}$.  

We now show our main result below in Theorem 5 by showing a similar result to Theorem 5 in [5]. However, note that beyond Star-Shapedness and Convexity, which play a key role in the Theorem, we ask nothing but the most basic axioms of deviation measures, namely Non-Negativity and Translation Insensitivity. While in [5], they also demand Normalization, the role of Normalization was explored in [17].

**Theorem 5.** $D : L^\infty \to \mathbb{R}_+ \cup \{\infty\}$ is a Star-Shaped deviation measure if and only if there is a family $\{D_i\}_{i \in \mathcal{I}}$ of Convex deviation measures such that the representation

$$D(X) = \min_{i \in \mathcal{I}} D_i(X), \ \forall \ X \in L^\infty,$$

(1)

holds. Moreover, such family can be chosen as the one composed by the Convex deviation measures that dominate $D$, i.e.

$$\mathcal{I} = \{\beta : L^\infty \to \mathbb{R}_+ \cup \{\infty\} : \beta \text{ Convex deviation measure and } \beta \geq D\}.$$

**Proof.** Let $D$ be given as in (1), we shall show that it is a Star-Shaped deviation measure. For Non-Negativity take nonconstant $X \in L^\infty$ and $i^* \in \mathcal{I}$ such that $D(X) = D_{i^*}(X)$. By Non-Negativity of $D_{i^*}$, $0 < D_{i^*}(X) = D(X)$. One argues similarly for $X \in \mathbb{R}$. For Translation Insensitivity we have for any $m \in \mathbb{R}$ and any $X \in L^\infty$ that

$$D(X + m) = \min_{i \in \mathcal{I}} D_i(X + m) = \min_{i \in \mathcal{I}} D_i(X) = D(X).$$

For Star-Shapedness, let $\lambda \leq 1$. Then, as each $D_i$ is Convex,

$$D(\lambda X) = \min_{i \in \mathcal{I}} D_i(\lambda X) \leq \lambda \min_{i \in \mathcal{I}} D_i(X) = \lambda D(X).$$
For the converse, let \( D \) be Star-Shaped deviation measure, \( \mathcal{A} = \mathcal{A}_D \) and \( \mathcal{D}_A \) be as defined in Proposition 4.

We then have that \( \mathcal{D}_A(X) = D(X) \).

Now we will find a family of Convex deviation measure such that equation (1) holds. For any \( Y \in L^\infty \) we define

\[
\mathcal{A}_Y = \text{conv} \{(Y - E[Y] + D(Y)) \cup \{0\}) + \mathbb{R}_+ = \{\lambda(Y - E[Y] + D(Y)) + m : \lambda \in [0, 1], m \geq 0\},
\]

and let \( D_Y(X) = D_{\mathcal{A}_Y}(X) \). We have that each \( \mathcal{A}_Y \) is Convex since \( \mathbb{R}_+ \) is a Convex cone. It is easy to see that \( D_Y(X) = D(X) = 0 \) for any \( X \) constant. For non-constant \( X \), if \( X \in \mathcal{A}_Y \), then \( X = \lambda(Y - E[Y] + D(Y) + k) \), where \( k \in \mathbb{R}_+ \) and \( \lambda \in [0, 1] \). In this case

\[
E[-X] + D(X) = E[-\lambda(Y - E[Y] + D(Y) + k)] + D(\lambda(Y - E[Y] + D(Y) + k))
\]

\[
= -\lambda(D(Y) + k) + D(\lambda Y)
\]

\[
\leq \lambda(D(Y) - D(Y) - k) = -\lambda k \leq 0.
\]

Thus, \( X \in \mathcal{A} \), which implies \( \mathcal{A}_Y \subseteq \mathcal{A} \) and, consequently,

\[
D_Y(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\} + E[X] \geq \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\} + E[X] = D(X).
\]

Thus, \( D(X) \leq \inf\{D_Y(X) : Y \in L^\infty\} \). Furthermore, since \( X + E[-X] + D(X) \in \mathcal{A}_X \), we have that

\[
D_X(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_X\} + E[X] \leq E[-X] + D(X) + E[X] = D(X).
\]

Hence, we have that \( D(X) = D_X(X) \) and \( D(X) = \min\{D_Y(X) : Y \in L^\infty\} \).

Now, we need to show that each \( D_Y \) defines a Convex deviation measure. When \( Y \) is constant \( \mathcal{A}_Y = \mathbb{R}_+ \) and we have that \( D_Y(X) = 0 \) if \( X \) is constant and \( D_Y(X) = \infty \) otherwise. This obviously defines a trivial generalized deviation measure. Therefore, we only have to show the deviation properties in case \( Y \) is non-constant. For Translation Insensitivity, let \( c \in \mathbb{R} \) and \( X \in L^\infty \). Then

\[
D_Y(X + c) = \inf\{m \in \mathbb{R} : X + m + c \in \mathcal{A}_Y\} + E[X + c]
\]

\[
= \inf\{m - c \in \mathbb{R} : X + m \in \mathcal{A}_Y\} + E[X] + c
\]

\[
= \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\} + E[X] = D_Y(X).
\]

Convexity follows because \( \mathcal{A}_Y \) is Convex. In fact, for any \( \lambda \in [0, 1] \) and any \( X, Z \in L^\infty \) we have

\[
\lambda D_Y(X) + (1 - \lambda)D_Y(Z)
\]

\[
= \inf\{\lambda m_1 + (1 - \lambda)m_2 \in \mathbb{R} : X + m_1 \in \mathcal{A}_Y, Y + m_2 \in \mathcal{A}_Y\} + E[\lambda X] + E[(1 - \lambda)Z]
\]

\[
\geq \inf\{\lambda m_1 + (1 - \lambda)m_2 \in \mathbb{R} : \lambda(X + m_1) + (1 - \lambda)(Z + m_2) \in \mathcal{A}_Y\} + E[AX + (1 - \lambda)Z]
\]

\[
= \inf\{m \in \mathbb{R} : \lambda X + (1 - \lambda)Z + m \in \mathcal{A}_Y\} + E[\lambda X + (1 - \lambda)Z] = D_Y(\lambda X + (1 - \lambda)Z).
\]

For Non-Negativity, we already showed that if \( c \in \mathbb{R} \) then \( D_Y(c) = D(c) = 0 \). For \( X \in L^\infty \) non-constant, we have that \( D_Y(X) \geq D(X) > 0 \). Thus, each \( D_Y \) is a Convex deviation measure. Moreover, let \( \mathcal{I} = \{\beta : L^\infty \rightarrow \mathbb{R}_+ \cup \{\infty\}, \beta \text{ is Convex deviation measure and } \beta \geq D\} \). We have that \( D(X) \leq \inf_{I \in \mathcal{I}} \beta(X) \). Since \( D_X \in \mathcal{I} \), we have that \( D(X) = \min_{I \in \mathcal{I}} \beta(X), \forall X \in L^\infty \).

\[\square\]

Remark 6. The minimum in equation (1) cannot be replaced by an infimum. In order to verify it, note that it could easily lead to a situation where for a non-constant random variable \( X \in L^\infty \) we may have \( \inf_{\beta \in \mathcal{I}} D_\beta(X) = 0 \). This directly conflicts to Non-Negativity, even if all \( D \) are deviation measures.

Remark 7. The set \( \mathcal{I} \) in the representation of last Theorem is not unique. Nonetheless, under a relaxation we have some uniqueness result. For any set \( \mathcal{I} \) of Convex deviation measures, define its relaxation as

\[
\mathcal{I}^* = \left\{ \beta : L^\infty \rightarrow \mathbb{R}_+ \cup \{\infty\} : \beta \text{ is Convex deviation measure and } \beta \geq \min_{I \in \mathcal{I}} \beta \right\}.
\]

Note that \( \min \beta \) is not necessarily well-defined. However, if it is well-defined and \( D = \min_{I \in \mathcal{I}_1} \beta_i = \min_{I \in \mathcal{I}_2} \beta_i \), then we directly have that

\[
\mathcal{I}^* = \mathcal{I}_1^* = \mathcal{I}_2^* = \left\{ \beta : L^\infty \rightarrow [0, \infty] : \beta \text{ is Convex deviation measure and } \beta \geq D \right\}.
\]

The Corollary below provides the same result as Theorem 5 but for different classes of deviation measures.
Corollary 8. Let $\mathcal{D} : L^\infty \to \mathbb{R}_+ \cup \{\infty\}$ be a Star-Shaped deviation measure represented under $\mathcal{I}$ in the context of Theorem 5. Then:

(i) $\mathcal{D}$ is Positive Homogeneous if and only if there exists some $\mathcal{I}$ composed by generalized deviation measures such that equation (1) holds.

(ii) $\mathcal{D}$ is Lower Range Dominated if and only if there is some $\mathcal{I}$ composed by lower ranged deviation measures such that equation (1) holds.

In any case, such families can be chosen, respectively, as the one composed by law-invariant and star-shaped measures. But not all law-invariant and star-shaped deviation measures respect convex order. Note that not all law-invariant star-shaped deviation measures such that equation (1) holds.

Proof. For (i), if $\mathcal{I}$ is composed by generalized deviation measures, then for any $\lambda \geq 0$ and $X \in L^\infty$ we have that

$$D(\lambda X) = \min_{i \in \mathcal{I}} \lambda \mathcal{D}_i(X) = \lambda \min_{i \in \mathcal{I}} \mathcal{D}_i(X) = \lambda D(X).$$

Thus, $\mathcal{D}$ is Positive Homogeneous. For the converse, let $\mathcal{D}'$ be defined as in Theorem 5, but now with $\mathcal{A}_Y = \text{convco}(\{Y - E(Y) + D(Y)\} + \mathbb{R}_+ \cup \{0\}) + \mathbb{R}_+$, where convco means the Convex conic hull i.e. $\text{convco}(A) = \{k(X + Y) : X, Y \in A, k \geq 0\}$. Clearly $\mathcal{A}_Y$ is a cone. Then, for any $X \in L^\infty$ and $\lambda \geq 0$ we get that

$$D_Y(\lambda X) = \inf \{m \in \mathbb{R} : \lambda X + m \in \mathcal{A}_Y\} + E[\lambda X]$$

$$= \inf \{\lambda n \in \mathbb{R} : \lambda(X + n) \in \mathcal{A}_Y\} + E[\lambda X]$$

$$= \lambda \left( \inf \{n \in \mathbb{R} : X + n \in \mathcal{A}_Y\} + E[X] \right) = \lambda D_Y(X).$$

Then, each $\mathcal{D}'$ is Positive Homogeneous. The facts that $D(X) = \min \{D_Y(X) : Y \in L^\infty\}$ and $\mathcal{I} = \{\beta : L^\infty \to \mathbb{R}_+ \cup \{\infty\} : \beta$ generalized deviation measure and $\beta \geq D\}$ follow as in Theorem 5.

Concerning (ii), if $\mathcal{I}$ is composed by Lower Range Dominated deviation measures, then for any $X \in L^\infty$ we have that

$$D(X) = \min_{i \in \mathcal{I}} \lambda \mathcal{D}_i(X) \leq E[X] - \text{ess inf} X.$$

Thus, $\mathcal{D}$ is Lower Range Dominated. For the converse, note that the singleton containing only $\mathcal{D}$ itself satisfies equation (1). Hence, if $\mathcal{D}$ is Lower Range Dominated then there is some $\mathcal{I}$ composed by lower ranged deviation measures such that equation (1) holds. Lastly, let $\mathcal{D}'$ be defined as in Theorem 5, but now with $\mathcal{A}_Y = \text{convco}(\{Y - E(Y) + D(Y)\} + L^\infty_+ \cup \{0\}) + L^\infty_+$. Clearly $\mathcal{A}_Y$ is a Monotone set that contains $L^\infty_+$, which implies $X - \text{ess inf} X \in \mathcal{A}_Y$. Then, for any $X \in L^\infty$ we get that

$$D_Y(X) = \inf \{m \in \mathbb{R} : X + m \in \mathcal{A}_Y\} + E[X] \leq E[\lambda X] - \text{ess inf} X.$$

Then, each $\mathcal{D}'$ is Lower Range Dominated. The facts that $D(X) = \min \{D_Y(X) : Y \in L^\infty\}$ and $\mathcal{I} = \{\beta : L^\infty \to \mathbb{R}_+ \cup \{\infty\} : \beta$ Lower Range Dominated Convex deviation measure and $\beta \geq D\}$ follow as in Theorem 5.

Remark 9. The same reasoning for preservation of properties in Corollary 8 is not true for Law Invariance. We thank an anonymous reviewer for raising this point and providing the concrete example. If $D$ can be written as the minimum of law-invariant convex deviation measures, then $D$ is law-invariant and star-shaped. But not all law-invariant and star-shaped $D$ can be written as the minimum of law-invariant convex deviation measures. The intuitive reason is that law-invariant convex deviation measures respect convex order (under some continuity), so by taking a minimum one arrives at a law-invariant star-shaped deviation measure which is consistent with convex order. Note that not all law-invariant star-shaped deviation measures respect convex order. To see a concrete example without imposing any continuity, let $X$ be uniform on $[-2, 2]$, and $Y = (2 - X)1_{(X > 0)} - (2 + X)1_{(X < 0)}$, which is also uniform on $[-2, 2]$; $Z : = X/2 + Y/2$ is distributed on $[-1, 1]$ with equal probability. Let $D$ be IQD at level 0.4 (i.e., 0.4-quantile minus 0.6-quantile) plus SD (this is a positively homogeneous and proper deviation measure). We can easily compute $D(X) = D(Y) = 4/5 + (4/3)^{1/2} < 2$ and $D(Z) = 2 + (2/3)^{1/2} > D(X)$. If $D$ is the minimum of some law-invariant convex deviation measures $\{D_i\}_{i \in \mathcal{I}}$, then we must have $D(Z) \leq D(X)$, since $D_i(Z) \leq D_i(X)/2 + D_i(Y)/2 = D_i(X)$ for each $i \in \mathcal{I}$. This is a contradiction.
[17] show that the Star-Shapedness of the minimum of a family of Convex risk measures is closely related to the behavior of each measure around 0. The same holds for deviation measures. However, a proper deviation measure is automatically Normalized as Non-Negativity implies $D(0) = 0$. Hence our result that the minimum of Convex deviation measures is a Star-Shaped deviation measure, with no need for the extra assumption of Normalization found in [5]. It is possible to write any proper deviation measure as the minimum of a family of deviation measures that are Convex. However, to do so, we must drop Non-Negativity, just like one needs to drop Normalization to write a monetary risk measure as the minimum of Convex risk measures. See [11]. The next Proposition shows this fact. To that, let $\chi_A$ be the characteristic function of $A$, i.e. $\chi_A(X) = 0$ if $X \in A$ and $\infty$ otherwise. Note that $\chi_R$ defines a trivial proper deviation measure.

**Proposition 10.** $D : L^\infty \to \mathbb{R}_+ \cup \{\infty\}$ is a proper deviation measure if and only if there is a family of Translation Insensitive deviation measures $\{D_i\}_{i \in I}$ which respects Convexity and such that $\chi_R \in \{D_i\}_{i \in I}$, $D_i(X) = 0$ holds only if $X \in \mathbb{R}$ and such that the representation holds

$$D(X) = \min_{i \in I} D_i(X), \quad \forall X \in L^\infty.$$ 

Moreover, such a family can be chosen as the one composed by the Translation Insensitive deviation measures that fulfill Convexity and dominate $D$.

**Proof.** To see that $\min_{i \in I} D_i(X)$ defines a proper deviation measure note that Non-Negativity follows from the presence of $\chi_R$ in $I$, i.e. for constant $X$ we have that $0 \leq \min_{i \in I} D_i(X) = \chi_R(X) = 0$. Translation Insensitivity follow the same reasoning as Theorem 5. The converse follows the same steps as in the proof of Theorem 5, but defining $A_Y$ as

$$A_Y = \{Y - E[Y] + D(Y)\} + \mathbb{R}_+.$$ 

Translation Insensitivity, Convexity of each $D_Y$ and that $D_X(X) = D(X)$ will follow from the same reasoning. However, we will lose the Star-Shapedness of each $D_Y$, and consequently, of $D$, as not all $A_Y$ may be Star-Shaped. Nevertheless, more importantly, we will also lose Non-Negativity, as for any non-constant $Y$ and constant $c$ we will have that $D_Y(c) = \infty$. While it follows that $D_c = \chi_R$ for any constant $c$.

Based on the Lower Range Dominance, it is possible to obtain an interplay between risk and deviation measures. Theorem 2 of [27] ensures this for generalized and Convex deviations. Moreover, in Theorem 12, we do the same for Star-Shaped deviations. This claim is a generalization of [27] Theorem 2.

**Definition 11.** A functional $\rho : L^\infty \to \mathbb{R}$ is a Star-Shaped risk measure if it has the following properties:

(i) **Monotonicity:** If $X \leq Y$, then $\rho(X) \geq \rho(Y)$, $\forall X, Y \in L^\infty$.

(ii) **Translation Invariance:** $\rho(X + C) = \rho(X) - C$, $\forall X, Y \in L^\infty$, $\forall C \in \mathbb{R}$.

(iii) **Normalization:** $\rho(0) = 0$.

(iv) **Star-Shapedness:** $\rho(\lambda X) \geq \lambda \rho(X)$, $\forall X \in L^\infty$, $\forall \lambda \geq 1$.

**Theorem 12.** We have that:

(i) if $\rho : L^\infty \to \mathbb{R}$ is a Star-Shaped risk measure such that $\rho(X) > -E[X]$ for any non-constant $X$, then $D(X) = \rho(X - E[X])$ is a Star-Shaped deviation measure.

(ii) if $D : L^\infty \to \mathbb{R}_+ \cup \{\infty\}$ is a Star-Shaped deviation measure such that $D(X) \leq E[X] - \text{ess inf} X$ for any $X$, then $\rho(X) = -E[X] + D(X)$ is a Star-Shaped risk measure.

**Proof.** For (i), Translation Insensitivity follows by $D(X + c) = \rho(X + c - E[X] - c) = D(X)$. For Star-Shapedness, let $\lambda \geq 1$, then $D(\lambda X) = \rho(\lambda(X - E[X])) \geq \lambda(\rho(X) - E[X]) = \lambda D(X)$. For Non-Negativity, if $c \in \mathbb{R}$ we have that $D(c) = \rho(c - E(c)) = \rho(0) = 0$. If $X$ is non-constant, then $D(X) = \rho(X) - (-E[X]) > 0$.

Regarding (ii), Translation Invariance follows as $\rho(X + c) = -c - E[X] + D(X + c) = \rho(X) - c$. Normalization is as $\rho(0) = -E[0] + D(0) = 0$. For Star-Shapedness, let $\lambda \geq 1$. Then $\rho(\lambda X) = -E[\lambda X] + D(\lambda X) \leq \lambda(-E[X] + D(X)) = \lambda \rho(X)$. For Monotonicity, let $X \geq Y$. Then for any $\lambda \in (0, 1)$
there is $Z \geq X$ such that $X = \lambda Y + (1 - \lambda)Z$. By Theorem 5 we have that $\mathcal{D}$ is the point-wise minimum of a family of Convex deviation measures. Thus, we get
\[
\rho(X) \leq -E(\lambda Y + (1 - \lambda)Z) + \min_{\mathcal{D}} \{\lambda \mathcal{D}_i(Y) + (1 - \lambda)\mathcal{D}_i(Z)\}
\]
\[
\leq -\lambda E[Y] - (1 - \lambda)E[Z] + (1 - \lambda)(E[Z] - \text{ess inf } Z) + \lambda \min_{\mathcal{D}} \mathcal{D}_i(Y)
\]
\[
\leq \lambda (E[-Y] + \min_{\mathcal{D}} \mathcal{D}_i(Y)) - (1 - \lambda) \text{ ess inf } X = \lambda \rho(Y) - (1 - \lambda) \text{ ess inf } X.
\]
Since for any $\lambda \in (0, 1)$ there is $Z \geq X$ that satisfies the inequality, we then get that
\[
\rho(X) \leq \lim_{\lambda \uparrow 1} (\lambda \rho(Y) - (1 - \lambda) \text{ ess inf } X) = \rho(Y).
\]

\[\Box\]

Remark 13. In the conditions of the last Theorem, both $\rho$ and $\mathcal{D}$ also inherit some properties such as lower semicontinuity and Law Invariance. Furthermore, one can replace the expectation in those formulations for another Star-Shaped risk measure $\mu$ as long as $\mu(X) + \mathcal{D}(X) \leq -\text{ess inf } X$ for any $X \in L^\infty$. This property is called Limitedness, and such composition is studied in [21]. Furthermore, we have in the context of Theorem 5 and Theorem 12 an interplay of acceptance sets. Let $\mathcal{D}$ be some Star-Shaped deviation measure represented under $\mathcal{I}$ and generating the Star-Shaped risk measure $\rho$. The acceptance set for monetary risk measures is traditionally defined as $A_\rho = \{X \in L^\infty : \rho(X) \leq 0\}$. Then it is easy to observe that $A_{\mathcal{D}} = \cup_{\mathcal{I}} \mathcal{A}_{\mathcal{D}_i} = A_\rho$.

We have as a direct corollary that under Lower Range Dominance it is possible to extend the last claim in Proposition 4.

**Corollary 14.** Let $\mathcal{D} : L^\infty \to \mathbb{R}_+ \cup \{\infty\}$ be a Lower Range Dominated Star-Shaped deviation measure and consider the notation in Proposition 4. Then $cl(\mathcal{A}) = A_{\mathcal{D}_A}$.

**Proof.** Under Lower Range Dominance, we get by Theorem 12 that $\rho(X) = -E[X] + \mathcal{D}(X)$ defines a Star-Shaped risk measure for any Star-Shaped deviation $\mathcal{D}$. In particular, $\mathcal{D}_A(X) = \rho_A(X) + E[X]$. Then, Proposition 4.3 in [7] assures that $cl(\mathcal{A}) = \{X \in L^\infty : \rho_A(X) \leq 0\}$. Hence, $cl(\mathcal{A}) = A_{\mathcal{D}_A}$.

In the context of last Theorem there is also an interplay between increasing Convex order for $\rho$ and Convex order for $\mathcal{D}$, that allows us to prove results that mimic Theorems 11 and 12 in [5].

**Corollary 15.** Let the probability space be atomless and $\mathcal{D} : L^\infty \to \mathbb{R}_+ \cup \{\infty\}$ a Lower Range Dominated Star-Shaped deviation measure. Then:

(i) $\mathcal{D}$ is Law Invariant if and only if there is a Star-Shaped set $G$ of non-increasing functions $g : (0, 1) \to \mathbb{R}$ with $g(1^-) \geq 0$ such that
\[
\mathcal{D}(X) = \inf_{g \in G \alpha \in (0, 1)} \sup \{\text{VaR}^\alpha(X - E[X]) - g(\alpha)\}, \forall X \in L^\infty.
\]

(ii) $\mathcal{D}$ is Convex order consistent if and only if there is a Star-Shaped set $G$ of non-increasing functions $g : (0, 1) \to \mathbb{R}$ with $g(1^-) \geq 0$ such that
\[
\mathcal{D}(X) = \sup_{g \in G \alpha \in (0, 1)} \{\text{ES}^\alpha(X - E[X]) - g(\alpha)\}, \forall X \in L^\infty.
\]

**Proof.** (i) It is straightforward to verify that (2) defines a law-invariant Star-Shaped deviation measure. Lower Range Dominance follows from the Monotone behavior in $\alpha$ for VaR and the functions in $G$ as
\[
\mathcal{D}(X) \leq \text{VaR}^\alpha(X - E[X]) - \sup_{g \in G} g(1^-) \leq E[X] - \text{ess inf } X, \forall X \in L^\infty.
\]

For the converse, we have by Theorem 12 that $\rho(X) = -E[X] + \mathcal{D}(X)$, $X \in L^\infty$ is a Law Invariant Star-Shaped risk measure, which implies it is Monotone regarding to increasing order. Thus, by Theorem 12 of [5] there is a Star-Shaped set $G$ of non-increasing functions $g : (0, 1) \to \mathbb{R}$ with $g(1^-) \geq 0$ such that for any $X \in L^\infty$ that
\[
\rho(X) = \inf_{g \in G \alpha \in (0, 1)} \sup \{\text{VaR}^\alpha(X) - g(\alpha)\}.
\]

By adding $E[X]$ in both sides of the last equation we get the claim.
(ii) That (2) defines a Lower Range Dominated Law Invariant Star-Shaped deviation measure it is similar to item (i) since \( \alpha \to ES^\alpha(X) \) is Monotone for any \( X \in L^\infty \). Let \( X, Y \in L^\infty \) such that \( X \succeq Y \) in Convex order. Then we have that \( E[X] = E[Y] \) and \( ES^\alpha(X) \leq ES^\alpha(Y) \) for all \( \alpha \in (0,1) \). This directly implies \( D(X) \leq D(Y) \). For the converse, we have by Theorem 12 that \( \rho(X) = -E[X] + D(X) \), \( X \in L^\infty \) is a Law Invariant Star-Shaped risk measure. We claim that \( \rho \) is consistent with respect to increasing Convex order. Let \( X \succeq Y \) in such order. Then \( X - E[X] \succeq Y - E[Y] \) in Convex order. Thus, \( E[X] \geq E[Y] \) and \( D(X) = D(X - E[X]) \leq D(Y - E[Y]) = D(Y) \). This directly implies \( \rho(X) \leq \rho(Y) \). Now, by Theorem 12 of [5] there is a Star-Shaped set \( G \) of non-increasing functions \( g: (0,1) \to \mathbb{R} \) with \( g(1^-) \geq 0 \) such that for any \( X \in L^\infty \),

\[
\rho(X) = \inf_{g \in G} \sup_{\alpha \in (0,1)} \{ ES^\alpha(X) - g(\alpha) \}.
\]

By adding \( E[X] \) in both sides of the last equation we get the claim.

It is shown in [14] that minimum of Law Invariant Convex risk measures are precisely the Second Stochastic Dominance (SSD) consistent risk measures. We now show that a similar result holds for Lower Range Dominated Star-Shaped deviation measures where SSD should be replaced by Convex order.

**Corollary 16.** Let the probability space be atomless. \( D: L^\infty \to \mathbb{R}_+ \cup \{\infty\} \) is a Convex order consistent Lower Range Dominated Star-Shaped deviation measure there exists some \( I \) composed by Law Invariant convex deviation measures such that equation (1) holds.

**Proof.** On atomless probability spaces any Law Invariant Convex Deviation is consistent to Convex order, see [9] for instance. It is straightforward to verify that the same is true for the supremum, the infimum, or any convex combination of Law Invariant Convex Deviations. Moreover, the infimum in representation (3) of item (ii) of Corollary 15 is attained by taking the star-shaped set \( G = \{ g_Y(\alpha) = ES^\alpha(Y): E[-Y] + D(Y) \leq 0 \} \), see Theorem 3.1 in [14] and [5] for details. Hence, under such reasoning, the claim is directly obtained.

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