MODULI SPACES OF LOW DIMENSIONAL LIE SUPERALGEBRAS

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ABSTRACT. In this paper, we study moduli spaces of low dimensional complex Lie superalgebras. We discover a similar pattern for the structure of these moduli spaces as we observed for ordinary Lie algebras, namely, that there is a stratification of the moduli space by projective orbifolds. The moduli spaces consist of some families as well as some singleton elements. The different strata are linked by jump deformations, which gives a unique manner of decomposing the moduli space which is consistent with deformation theory.

1. Introduction

In a series of papers, the authors and some collaborators have been studying moduli spaces of low dimensional Lie algebras, as well as moduli spaces of complex associative algebras, including algebras defined on a $\mathbb{Z}_2$-graded space. In the Lie algebra case, we have studied moduli spaces of complex Lie algebras of dimension up to 5, and real Lie algebras of dimension up to 4. In all of these cases, we found that the moduli space has a natural decomposition into strata which are parameterized by projective orbifolds of a very simple kind, which is a new point of view that had not appeared in the earlier literature.

Each stratum was of the form $\mathbb{P}^n/G$, where $G$ is a subgroup of the symmetric group $\Sigma_{n+1}$, which acts on $\mathbb{P}^n$ by permuting the projective coordinates. This led us to conjecture that every moduli space of finite dimensional Lie algebras has such a decomposition, and it also reasonable to guess that the same conjecture holds for Lie superalgebras. In this paper we prove that the conjecture holds for low dimensional complex Lie superalgebras.

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The classification of moduli spaces of superalgebras is complicated by the fact that a Levi decomposition of a superalgebra does not always exist, and the fact that for Lie superalgebras, a semisimple algebra may not be a direct sum of simple algebras. However, the definitions of solvable and nilpotent Lie superalgebras are the same as in the ordinary case. A semisimple Lie algebra is one whose solvable radical (maximal solvable ideal) is trivial. Moreover, if a Lie superalgebra is not solvable, then the quotient by the solvable radical is semisimple.

If a Lie superalgebra $L$ is solvable, then it has a codimension 1 ideal, so there is an exact sequence

$$0 \to M \to L \to W \to 0,$$

where $M$ is a $\mathbb{Z}_2$-graded ideal, and $W$ is a 1-dimensional algebra (which is necessarily trivial). However, $W$ may be $1|0$ or $0|1$-dimensional. As a consequence, solvable Lie superalgebras of a fixed dimension $m|n$ can be constructed from solvable Lie algebras of dimension $m-1|n$ or dimension $m|n-1$, so this method allows one to construct the solvable $m|n$-dimensional Lie superalgebras by a bootstrap analysis.

The situation with superalgebras which are not solvable is more complex, but in low dimensions this complication mostly disappears owing to the fact that there are not many examples of low dimensional complex semisimple Lie superalgebras. The paper [9] contains a description of the finite dimensional simple Lie superalgebras, as well as a prescription for constructing semisimple superalgebras. A more recent article [7], gives a more explicit description of the semisimple Lie superalgebras. The reader may also find the book [10] useful.

When a Lie superalgebra is not solvable, one has an exact sequence of the form

$$0 \to M \to L \to W \to 0,$$

where this time, $W$ is semisimple and $M$ is the solvable radical. Thus, for both solvable and nonsolvable Lie superalgebras, we can classify non semisimple algebras as extensions of either semisimple or trivial Lie algebras by solvable Lie algebras. There is a long history of the study of extensions of Lie algebras. In [6], a description of the process was given in the language of codifferentials, and the methods described in that article were used to construct the moduli spaces we are studying here.

The bidimension of a $\mathbb{Z}_2$-graded vector space is given in the form $m|n$, where $m$ is the dimension of the even part of the space and $n$ is the dimension of the odd part. An ordinary $m$-dimensional Lie algebra is simply a Lie superalgebra of dimension $m|0$, and it is necessary to
consider such algebras in studying the moduli spaces of superalgebras, because the dimension of the space $M$ or $W$ in the decomposition as an extension may be of the form $k|0$. In fact, in the study of $3|1$-dimensional superalgebras, one has to consider the extension of the $3|0$-dimensional simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ by a $0|1$-dimensional (trivial) algebra. However, other than this case, we won’t have to consider any semisimple superalgebras in the moduli spaces we construct, owing to the low dimensions of the spaces involved.

In this paper, we address the complete moduli spaces for Lie superalgebras in dimensions $1|1$, $1|2$, $2|1$, $1|3$, $2|2$ and $3|1$.

2. THE LANGUAGE OF CODIFFERENTIALS

Classically, the space of cochains $C(L, L)$ of a Lie algebra with coefficients in the adjoint representation is given by $C(L, L) = \text{Hom}(\wedge L, L)$, where $\wedge L$ is the exterior algebra of $L$. For an ordinary algebra, $\wedge L$ has dimension $n^{2^n}$, where $n = \dim(L)$, with components $C^k(L, L) = \text{Hom}(\wedge^k L, L)$ of dimension $\binom{n}{k}$. There is a natural Lie superalgebra structure on $C(L, L)$ and the Lie algebra structure itself is represented as an odd element $\ell$ of $C^2(L, L)$, which satisfies the codifferential condition $[\ell, \ell] = 0$. It is possible to extend this definition to the $\mathbb{Z}_2$-graded case, but there is a more fundamental approach, based on the fact that the exterior algebra of a $\mathbb{Z}_2$-graded space coincides in a natural manner with the symmetric algebra on the parity reversion of the $\mathbb{Z}_2$-graded space. Under this association, we obtain an equality between $C(L, L)$ and $C(\Pi L, \Pi L) = \text{Hom}(S(\Pi L), \Pi L)$. In fact, there is an isomorphism between $C(\Pi L, \Pi L)$ and the space of $\mathbb{Z}_2$-graded coderivations of the symmetric coalgebra $S(\Pi L)$. The difference in the expression of a Lie algebra structure on $L$ and a codifferential on $S(L)$ is easy to express. If $d$ is the codifferential on $S(W)$ corresponding to a Lie superalgebra structure $\ell$ on $L$, then

$$\ell(a \wedge b) = (-1)^a \pi(d(\pi a \cdot \pi b)).$$

Thus to convert from a Lie superalgebra expressed as a codifferential back to the standard form involves only multiplication by a sign. We will give our algebras in the form of codifferentials, but we will also indicate in some cases how to translate to the standard form.

3. CONSTRUCTION OF MODULI SPACES BY EXTENSIONS

Let us assume that $0 \to M \to L \to W \to 0$ gives an extension of the algebra structure on $W$ given by a codifferential $\delta$ by an algebra structure on $M$ given by the codifferential $\mu$. Then if $d$ is the corresponding
codifferential of the algebra structure on $L$, we have $d = \delta + \mu + \lambda + \psi$, where $\lambda \in \text{Hom}(M \otimes W, M)$ and $\psi \in \text{Hom}(S^2(W), M)$. The term $\lambda$ is traditional called the module structure on $M$ and the term $\psi$ is called the cocycle, although when $\mu \neq 0$, $\lambda$ is not precisely a module structure on $M$, nor is $\psi$ really a cocycle. However, we will use this terminology (even though it is not precisely correct). The condition that $d$ is a codifferential on $L$ is that $[d, d] = 0$, which is equivalent to the following three conditions:

1. $[\mu, \lambda] = 0$ (The compatibility condition)
2. $[\delta, \lambda] + \frac{1}{2} [\lambda, \lambda] + [\mu, \psi] = 0$ (The Maurer-Cartan Condition)
3. $[\delta + \lambda, \psi] = 0$ (The cocycle condition)

To construct an extension, we first fix $\delta$ and $\mu$, and then we solve the compatibility condition, which puts some constraints on the coefficients of $\lambda$. If $\beta \in \text{Hom}(W, M)$ is even, then replacing $\lambda$ with $\lambda + [\mu, \beta]$ generates an equivalent extension, so we use this to simplify the form of $\lambda$. Next, we consider the action of the group $G_{\delta, \mu}$ of transformations of $M \oplus W$, consisting of those block diagonal elements such that the action of the appropriate piece on $\delta$ or $\mu$ preserves this structure. This allows us to restrict the form of $\lambda$ even more.

Next, we apply the Maurer-Cartan (MC) condition to $\lambda$ and a generic $\psi$, which may place additional constraints on the coefficients of $\lambda$ and constraints on the coefficients of $\psi$. Finally, we apply the cocycle condition, to construct a $d$ which is a codifferential. Now, we can also apply a group $G_{\delta, \mu, \lambda}$ to restrict the coefficients of $\psi$ further, but in practice, we mostly did not do this, except possibly for a diagonal transformation.

After doing this, we find some codifferentials, and study their equivalence classes. Some of them naturally arise as families, and in this case, we check to see that they represent a projective family, in the sense that multiplying the coefficients by a nonzero complex number yields an isomorphic algebra.

This is not quite all the details involved, but we will illustrate the situation in our examples.

4. **Deformations of algebras and the versal deformation**

Given a 1-parameter family $d_t$ of algebras such that $d_0 = d$, then we say that this family determines a deformation of $d$. If $d_t \not\sim d$ for $t$ in some punctured nbd of $t = 0$, then we say that the deformation is nontrivial. If $d_t \sim d'$ for all nonzero $t$ in some punctured nbd of 0, then this deformation is called a jump deformation of $d$, while if $d_s \not\sim d_t$ for $s \neq t$ for small enough $s$ and $t$, then the deformation is called a smooth deformation. One can also have multiparameter deformations $d_t$ where
$t = (t_1, t_2, \cdots)$, in which case there may be 1-parameter curves in the $t$ space which determine jump deformations and other curves which determine smooth deformations.

There is a generalization of these multiparameter deformations called a deformation with a local base (see [1]), which is a commutative algebra $A$ such that there is an $A$-Lie algebra structure $d_A$ defined on $V \otimes A$, where $V$ is the underlying vector space on which the Lie algebra is defined, and $A$ is a local algebra, meaning it has a unique maximal ideal $m$. One requires that $A/m = \mathbb{K}$, the underlying field on which the Lie algebra is defined, so that there is a natural decomposition $A = \mathbb{K} \oplus m$. Then there is a natural map $V \otimes A \rightarrow V$, determined by the projection $A \rightarrow \mathbb{K}$, and $d_A$ is called a deformation with base $A$ if the induced map takes $d_A$ to $d$.

For super Lie algebras, it makes sense to work with $\mathbb{Z}_2$-graded commutative algebras, but we don’t take that point of view here, even though it would be interesting. If the reader is interested in seeing this type of analysis, we mention that in the study of low dimensional $L_\infty$ algebras in [5, 4] we did consider this more general perspective.

There is a special type of multiparameter deformation called a versal deformation, has the property that it induces every deformation with a local base in a natural manner. Moreover, there is a special type of versal deformation, called a miniversal deformation (see [1]) which can be constructed in a concrete fashion by beginning with an infinitesimal deformation $d_1 = d + t_i \delta_i$, where $\{\delta_i\}$ is a basis for $H^2(d)$, the second cohomology of the algebra $d$, see [2]. The deformation is called infinitesimal because it satisfies the Jacobi identity up to first order terms in the $t_i$. When studying Lie superalgebras, we only look at the even part of $H^2$, because we aren’t considering deformations with a graded commutative base.

In [3], a method of constructing a miniversal deformation for $L_\infty$ algebras was outlined, and we have developed tools using the Maple computer algebra system for carrying out the computations, which are mostly just applications of linear algebra, although the computation of the versal deformation involves solving systems of quadratic equations.

5. The Moduli Space of 1|1-dimensional Lie Superalgebras

For ordinary 2-dimensional Lie algebras, the moduli space consists of one nontrivial element, $\ell = \varphi_1 e_1 e_2$, which is solvable, but not nilpotent. Expressed as a codifferential, this solvable algebra has the form $d = \psi_2^{1,2}$.
For 1|1-dimensional Lie superalgebras, the situation is more interesting. There are 2 nonequivalent 1|1-dimensional Lie superalgebras. Let \( L = \langle f, e \rangle \) be a 1|1-dimensional vector space with \( e \) an even and \( f \) an odd basis element. The first algebra \( \ell_1 \) is given by \( \ell_1(e, f) = f \). This algebra is analogous to the ordinary Lie algebra \( \ell \) above. The second algebra \( \ell_2 \), is given by the formula \( \ell_2(f, f) = e \), with all other brackets vanishing. Because \( f \) is odd, \( f \wedge f \) is not equal to zero, a situation that cannot arise in the nongraded case.

The first algebra arises as an extension of the trivial algebra structure \( \delta = 0 \) on a 0|1-dimensional vector space \( W = \langle v_2 \rangle \) by the trivial algebra structure \( \mu = 0 \) on a 1|0-dimensional space \( M = \langle v_1 \rangle \). The structure \( \lambda \) is determined up to a constant multiple of \( \psi^{1,2}_1 \). The structure \( \psi \) must vanish as \( S^2(W) = 0 \), since \( W \) is a 1-dimensional odd vector space. Thus, up to isomorphism, we obtain that the only possible nontrivial structure is \( d_1 = \psi^{1,2}_1 \).

The second algebra arises as an extension of the trivial algebra structure \( \delta = 0 \) on a 1|0-dimensional vector space \( W = \langle v_1 \rangle \) by the trivial algebra structure \( \mu = 0 \) on a 0|1-dimensional space \( M = \langle v_2 \rangle \). In the language of codifferential, all of the maps \( \delta, \mu, \lambda \) and \( \psi \) must be odd, which forces \( \lambda = 0 \), and \( \psi \) to be a multiple of \( \psi^{1,1}_2 \). Thus the only nontrivial structure is given (up to isomorphism) by \( d_2 = \psi^{1,1}_2 \).

One can also proceed directly to construct the algebras by noting that the form of the algebra must be \( d = \psi^{1,2}_1 a + \psi^{1,1}_2 b \), and then checking that the condition \([d, d] = 0\) is equivalent to \( ab = 0\).

In the table below, we compute the cohomology of the two algebras. Here, \( h_n \) is the bi-dimension \( H^n \), the cohomology of the algebra in degree \( n \). Let us recall the meaning of the cohomology in low degrees. \( H^0 \) is the center of the algebra, \( H^1 \) is the space of nontrivial derivations of the algebra, \( H^2 \) classifies the infinitesimal deformations, and \( H^3 \) gives the obstructions to extending an infinitesimal deformation. An algebra for which the cohomology vanishes in all degrees is called totally rigid. In terms of actual deformations, only the odd part of \( H^2 \) counts, although one can interpret the even part in terms of deformations with a base given by a \( \mathbb{Z}_2 \)-graded algebra.

The algebra \( d_1 \) is totally rigid. We see that \( H^0 = \langle v_2 \rangle \) is the center of the algebra \( d_2 \). There is also a nontrivial even derivation of \( d_2 \), given by \( \varphi^1_1 + 2\varphi^2_2 \). This completely describes the moduli space of 1|1-dimensional Lie superalgebras. Since \( H^2 = 0 \) for both of these algebras, there are no nontrivial deformations for either one of them. Note that \( d_1 \) is solvable but not nilpotent, while \( d_2 \) is nilpotent.
Algebra | Codifferential | $h_0$ | $h_1$ | $h_2$ | $h_3$
---|---|---|---|---|---
$d_1$ | $\psi_1^{1,2}$ | $0|0$ | $0|0$ | $0|0$ | $0|0$
$d_2$ | $\psi_2^{1,1}$ | $0|1$ | $1|0$ | $0|0$ | $0|0$

**Table 1.** Cohomology of 1|1-Dimensional Complex Lie Algebras

6. **The moduli space of 2|1-dimensional Lie Superalgebras**

There are only three (families of) algebras on a 2|1-dimensional vector space. The corresponding dimension for the codifferentials is 1|2. One of these is a projective family $d_3(p : q)$, which means that $d_3(up : uq) \sim d_3(p : q)$ for all nonzero $u \in \mathbb{C}$. In this case, there are no isomorphisms between $d_3(p : q)$ and $d_3(x : y)$, except for the isomorphisms that give rise to the projective description in our notation. As is usually the case, when there is a family of algebras, there are some special values of the parameters $(p : q)$ such that the cohomology or even the deformation picture is different than generically. There is a special element $(0 : 0)$, which is called somewhat unfortunately the generic element of projective space by algebraic geometers, because the algebra corresponding to $(0 : 0)$ is never generic in its behavior. In this case, $d_3(0 : 0)$ is actually the trivial algebra, which has jump deformations to every nontrivial algebra in the moduli space. By a jump deformation, we mean a deformation $d_t$ of an algebra $d$, where $t$ is a (multi)-index such that $d_t \sim d'$ for some algebra $d'$ except when $t = 0$, in which case we obtain the original algebra.

The algebras $d_1$ and $d_3(p : q)$, except for $d_3(0 : 0)$, are solvable but not nilpotent, while the algebras $d_2$ and $d_3(0 : 0)$ are nilpotent. The algebra $d_2$ has a jump deformation to $d_1$, while $d_3(p : q)$ has a smooth deformation along the family. For the special cases of the parameters, only $d_3(1 : -2)$ does not behave generically in terms of its deformations, because it has a jump deformation to $d_1$ in addition to smooth deformations along the family.

All of the 2|1-dimensional algebras are given by extending the trivial algebra structure on either a 1|0 dimensional algebra by an algebra structure on a 1|1-dimensional space or by extending the trivial 0|1-dimensional algebra by an algebra structure on a 2|0-dimensional space. Because this case is fairly easy to describe, but shows some of the important features of the construction, we will give an explicit construction of the moduli space.

First, let us consider $W = \langle v_3 \rangle$, and $M = \langle v_1, v_2 \rangle$, so that $v_1$ is the only even basis element. The module structure $\lambda$ is of the form
Consider the case $G$. In fact, the group $\lambda$-stability condition is trivial, so we can express $G$ in the form $\beta$. The compatibility condition forces $\lambda$. Substituting $p$ for $\lambda$, we find we can reduce to $d_3(1:0)$. Next, let us assume that $\mu = \psi_1^{22}$. Then the compatibility condition gives $a_2 = -2a_1$. Since $[\mu, \beta] = 0$ for all $b$, we cannot simplify the expression for $\lambda$, but applying the group $g_{\delta,\mu}$ we find we can reduce to the case $a_1 = 1$ or $a_1 = 0$. The first case gives the algebra $d_1$, while the second gives the algebra $d_2$.

Finally, we have to consider the case $\mu = 0$. In this case, the compatibility condition is trivial, so we can express $\lambda = \psi_1^{1,3} p + \psi_2^{2,3} q$. Here we substituted $p = a_1$ and $q = a_2$, which is a notation we use when we suspect the relation between the $p$ and the $q$ gives a projective symmetry. In fact, the group $G_{\delta,\mu}$ acts on $\lambda$ by multiplying both coordinates by the same number, which is precisely what we expect if the structures form a projective family. We obtain $d = \lambda = d_3(p : q)$.

The reader may notice that we have already discovered all of the algebras by only looking at one of the possible decompositions. In fact, we suspect that if $V$ is a solvable $n|1$-dimensional algebra with $n > 1$, then there is a $(n-1)|1$-dimensional ideal. We already know from the study of $1|1$-dimensional algebras that the result does not hold for $n = 1$.

However, let us proceed with the case $W = \langle v_2, v_3 \rangle$ and $M = \langle v_1 \rangle$. First, we note that $\lambda$ and $\beta$ must both vanish, while $\psi = \psi_2^{1,1} c_1 + \psi_3^{1,1} c_2$.

| Algebra | Codifferential | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|---------|----------------|-------|-------|-------|-------|
| $d_1$   | $4\psi_2^{1,1} + \psi_1^{1,3} - 2\psi_2^{2,3}$ | 0|0|0|0|0 |
| $d_2$   | $4\psi_2^{1,1}$ | 0|2|3|1|1|0|0|
| $d_3(p : q)$ | $p\psi_1^{1,3} + q\psi_2^{2,3}$ | 0|0|1|0|0|0|
| $d_3(1 : -3)$ | $\psi_1^{1,3} - 3\psi_2^{2,3}$ | 0|0|1|0|0|1|
| $d_3(1 : 0)$ | $\psi_1^{1,3}$ | 0|1|2|0|0|1|0|
| $d_3(0 : 1)$ | $\psi_2^{2,3}$ | 1|0|1|1|1|1|1|
| $d_3(1 : -2)$ | $\psi_1^{1,3} - 2\psi_2^{2,3}$ | 0|0|1|0|2|1|0|
| $d_3(0 : 0)$ | $0$ | 1|2|5|4|6|6|6|6|
Let us first consider the case when $\mu = \psi_1^{1,2}$. In this case, we only need to consider the MC equation, and this forces $\psi = 0$. Thus $d = \mu$ which is isomorphic to $d_3(0 : 1)$.

Finally, consider the case when $\mu = 0$. Then there are no conditions on $\psi$, so we obtain $d = \psi_2^{1,1}c_1 + \psi_3^{1,1}c_2$. A little work with $G_{\delta,\mu}$ would show that we only have to consider 2 cases, where $c_1 = 1$ and $c_2 = 0$, or both of the coefficients vanish. In the first case, $d \sim d_2$ while the second case is given by $d = d_3(0 : 0)$, the trivial codifferential.

7. The moduli space of 1|2-dimensional Lie Superalgebras

There is one projective family $d_1(p : q)$, and three singletons in this moduli space. They correspond to codifferentials on a 2|1-dimensional space. Because the family $d_1(p : q)$ occurs first in the order of the description (we have ordered our algebras in such a manner that an algebra only deforms to one whose number is lower, except for the generic element), we don’t have any room for extra deformations for special values of the parameter $(p : q)$ in $d_1(p : q)$. There is an action of the symmetric group $\Sigma_2$ on the family by permuting the coordinates, which means that $d_1(p : q) \sim d_1(q : p)$ for all $(p : q)$. This means that the family $d_1(p : q)$ is parametrized by the projective orbifold $\mathbb{P}^1/\Sigma_2$.

This is a typical pattern that has arisen in our studies of moduli spaces.

Note that the odd part of the dimension of $H^2$ is always 1, except for $d_1(0 : 0)$, where this number is 2. It is always the case that the generic element in a family has jump deformations to every other element in the family, and these are the only deformations of $d_1(0 : 0)$.

The element $d_2$ has a jump deformation to $d_1(1 : 1)$, while the other elements are rigid. The algebras $d_1(0 : 0)$, $d_3$ and $d_4$ are nilpotent, while $d_2$ and $d_1(p : q)$ are solvable but not nilpotent otherwise.
8. THE MODULI SPACE OF 3|1-DIMENSIONAL LIE SUPERALGEBRAS

This is an interesting moduli space because it has a non nilpotent element, given by an extension of the simple Lie algebra \( \mathfrak{sl}_2 \) by a 0|1-dimensional trivial algebra. This is the algebra \( d_1 \) in the list of algebras below. Other than this example, all 3|1-dimensional Lie algebras are solvable, so we next study how they are obtained by extensions.

8.1. Construction of the moduli space of 3|1-dimensional algebras. Consider a 0|1-dimensional vector space \( W = \langle v_4 \rangle \) and a 1|2-dimensional vector space \( M \). Note that the dimensions are reversed from the algebra picture because we are studying the algebras as codifferentials on a 1|3-dimensional space. There are three non-trivial 1|2-dimensional codifferentials. The generic \( \lambda \) is of the form 

\[
\lambda = a_{11} \psi_1^{1,4} + a_{22} \psi_2^{2,4} + a_{32} \psi_3^{2,4} + a_2 3 \psi_2^{3,4} + a_{33} \psi_3^{3,4},
\]

while \( \psi \) must vanish. The generic form of \( \beta \) is 

\[
\beta = \varphi^4_1 b_1 + \varphi^4_3 b_2.
\]

First we consider \( \mu = 2 \psi_2^{1,1} + \psi_1^{1,3} - 2 \psi_2^{2,3} \). The compatibility condition forces \( a_{32} \) and \( a_{33} \) to vanish and \( a_{21} = -2a_{11} \). Moreover, taking into account that we can add a term \([\mu, \beta]\) to \( \lambda \), it turns out that we can assume that \( \lambda = 0 \), so we obtain the codifferential \( d = \mu \), which is \( d_2(1 : 0) \) on our list of algebras.

Next, consider the case \( \mu = 2 \psi_2^{1,1} \). The compatibility condition forces \( a_{32} = 0 \) and \( a_{22} = -2a_{11} \). Moreover, \([\mu, \beta] = 0 \), so we are unable to eliminate any more terms in \( \lambda \). The MC equation is automatically satisfied, so all of the possible forms of \( \lambda \) above actually give codifferentials. The action of \( G_{8, \mu} \) on \( \lambda \) allows us to reduce to only three possible cases. The first is of the form 

\[
\lambda = p \psi_1^{1,4} - 2p \psi_2^{2,4} + \psi_2^{3,4} + q \psi_3^{3,4},
\]
\[ \lambda = \psi_1^{1,4} - 2\psi_2^{2,4} - 2\psi_3^{3,4}, \text{ or } \lambda = 0. \] The first choice of \( \lambda \) gives the family \( d_2(p : q) \), while the other 2 choices give \( d_3 \) and \( d_4 \), resp.

The last of the nonzero 1|2-dimensional codifferentials is \( \mu = p\psi_1^{1,3} + q\psi_2^{2,3} \). The condition \([\mu, \lambda] = 0\) has two possible solutions, the first holding for generic values of \((p : q)\), while the second holds only for \((0 : 0)\). In the first case, we have \( a_{32} = a_{33} = 0 \), and when \( q \neq 0 \), we can eliminate the \( a_{22} \) and \( a_{23} \) terms by adding an appropriate \([\mu, \beta]\) term, leaving only the coefficient \( a_{11} \), which can be further reduced to the cases \( a_{11} = 1 \) or \( a_{11} = 0 \). When \( a_{11} = 1 \), we obtain the codifferential \( d_5 \), while when \( a_{11} = 0 \), we obtain the subfamily \( d_6(p : q : r) \) of the family \( d_6(p : q : 0) \). When \( q = 0 \) and \( p = 1 \), we substitute these values into \( \mu \), and now we will introduce new \( p \) and \( q \) variables by setting \( a_{11} = p \) and \( a_{22} = q \). We still have a choice of \( a_{23} \) which can be reduced to being either 1 or 0. As it turns out, the new variables \( p \) and \( q \) play a significant role, because when \( a_{23} = 1 \) then if \( q \neq 0 \) we obtain the codifferential \( d_5 \), and if \( q = 0 \) we obtain \( d_6(1 : 0) \). Similarly, if \( a_{23} = 0 \), then when \( q \neq 0 \), we obtain the codifferential \( d_5 \) and when \( q = 0 \) we obtain the codifferential \( d_7(1 : 0) \).

Returning to the original subcases, when \((p : q) = (0 : 0)\) in \( \mu \), we obtain a more complicated solution because \( \mu = 0 \). Then \( \lambda \) is given by a matrix of the form

\[
A = \begin{bmatrix}
    a_{11} & 0 & 0 \\
    0 & a_{22} & a_{23} \\
    0 & a_{32} & a_{33}
\end{bmatrix}.
\]

Now, the compatibility and MC conditions are satisfied automatically, so any \( \lambda \) of this form gives a codifferential. However, the group \( G_{\delta,\mu} \) acts on the matrix of \( \lambda \) by conjugating the \( 2 \times 2 \) submatrix of \( \lambda \) by an element of \( \mathfrak{gl}(2, \mathbb{C}) \), and multiplying the matrix by a nonzero number. This is a familiar pattern which says that the submatrix can be reduced to a Jordan form, in which case, we get two possibilities, \( A = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 1 \\ 0 & 0 & r \end{bmatrix} \) or \( A = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{bmatrix} \), the latter corresponding to an eigenvalue of geometric multiplicity 2 in the \( 2 \times 2 \) submatrix. The first case gives \( d_6(p : q : r) \), the second \( d_7(p : q) \).

Finally, we have to consider the case \( W = \langle v_1 \rangle \) and \( M = \langle v_2, v_3, v_4 \rangle \), so that \( W \) is a 1|0-dimensional space and \( M \) is a 0|3-dimensional space. There are 3 distinct elements of the moduli space of 0|3-dimensional codifferentials, but the first one corresponds to the simple Lie algebra, so we only have to consider the other two cases. Now, for this space,
\( \lambda \) must vanish, but \( \psi \) has a more complicated form

\[
\psi = \psi_1^{1,1} c_1 + \psi_2^{1,1} c_2 + \psi_3^{1,1} c_3.
\]

The first case is given by \( \mu = \psi_2^{2,4} p + \psi_3^{3,4} q \). The MC condition gives three solutions, the first for generic values of \( p \) and \( q \), the second for \( p = 0 \) and the third for \( q = 0 \). It is interesting to note that the \( \mu \) is symmetric with respect to the interchange of \( p \) and \( q \), but this symmetry is broken by the interaction with the \( \psi \) term. Generically, the codifferential is isomorphic to \( d_6(0 : p : q) \), while if \( p = 0 \), then the codifferential is isomorphic to \( d_2(0 : q) \) if \( c_1 \neq 0 \) or \( d_6(0 : 0 : q) \) when \( c_1 = 0 \). On the other hand, if \( q = 0 \), then the codifferential is isomorphic to \( d_2(0 : p) \) when \( c_1 \neq 0 \) and \( d_6(0 : 0 : p) \) when \( c_1 = 0 \). Note that the symmetry in the isomorphism classes between \( p \) and \( q \) is restored in the isomorphism class, which has to be the case because \( \mu(p : q) \sim \mu(q : p) \), so extensions should not be different when \( p \) vanishes than when \( q \) vanishes.

The second case is given by \( \mu = \psi_2^{3,4} \). The MC equation forces \( \psi = 0 \), so \( d = \mu \), which is isomorphic to \( d_7(p : q) \).

Finally, the last case is given by \( \mu = 0 \), in which case the codifferential \( d \) is just \( \psi \) and there are no conditions. On the other hand, it is easy to see that whenever any of the terms in \( \psi \) is nonzero, we obtain an equivalent codifferential, which is isomorphic to \( d_4 \). Otherwise, we just obtain the zero codifferential, which is also given by \( d_7(0 : 0) \).

In the table of algebras below, we include some special subfamilies of \( d_6(p : q : r) \). Because this algebra is parametrized by \( \mathbb{P}^2/\Sigma_2 \), where the action of \( \Sigma_2 \) on \( \mathbb{P}^2 \) is given by interchanging the second two coordinates, there are special \( \mathbb{P}^1 \)s for which the cohomology does not follow the generic pattern. There are also some special points in the families for which the generic pattern does not hold. We include the special points of the \( d_2(p : q) \) family in the main table, but list the special points for \( d_6(p : q : r) \) and \( d_7(p : q) \) in separate tables.

### 8.2. Deformations of the 3|1-dimensional algebras.

The algebra \( d_1 \) is rigid. The family \( d_2(p : q) \) generically only deforms along the family. There are two special points \((1 : 1)\) and \((0 : 1)\) where the dimension of \( H^1 \) is 2, rather than the generic value 1, but this does not affect the deformations. On the other hand, \( d_2(0 : 0) \) has jump deformations to all the other elements in the family \( d_2(p : q) \), so the dimension of its \( H^2 \) is not generic. The algebra \( d_3 \) has a jump deformation to \( d_2(1 : -2) \) and also deforms smoothly in a nbd of \( d_2(1 : -2) \). In fact, if there is a jump deformation from an algebra to a member of a family, then there always smooth deformations in a nbd of this point. The algebra \( d_4 \) has
jump deformations to $d_2(x : y)$ for all $(x : y)$ as well as a jump to $d_3$. The algebra $d_5$ is completely rigid.

The family of algebras $d_6(p : q : r)$ generically has $h_2 = 2$, which is precisely what one would expect as it deforms only along the family. For the special subfamilies, most of them deform generically, but $d_6(p : q : 0)$ also has a jump deformation to $d_5$, and $d_6(p : q : -2p)$ also has a jump to $d_2(p : q)$ as well as deforming in a nbd of $d_2(p : q)$. Now, for the special points, if they belong to a special subfamily, then they take part in any extra deformations which the subfamily has. However, one has to keep in mind that the symmetry of interchanging the last two coordinates must be taken into account, as was done when we listed the subfamilies. For example, the subfamily $d_6(p : q : 0)$ coincides (up to isomorphism) with the subfamily $d_6(p : 0 : q)$, so we only listed the first subfamily. Similarly, the element $d_6(1 : -2 : 0)$ is equivalent to $d_6(1 : 0 : -2)$, so it both jumps to $d_5$ and to $d_2(1 : 0)$. Other than these cases, there is only one additional special case, the element $d_6(0 : 1 : -1)$, which has a jump deformation to $d_1$. Finally, the

| Algebra | Codifferential | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|---------|----------------|-------|-------|-------|-------|
| $d_1$   | $\psi_4^{2,3} + \psi_3^{2,4} + \psi_2^{3,4}$ | 1 | 0 | 1 | 0 | 1 | 1 |
| $d_2(p : q)$ | $8\psi_2^{1,1} + p\psi_1^{1,4} - 2p\psi_2^{2,4} + \psi_2^{3,4} + q\psi_3^{3,4}$ | 0 | 0 | 1 | 0 | 0 | 1 |
| $d_2(0 : 1)$ | $8\psi_2^{3,1} + \psi_2^{2,4} + \psi_3^{3,4}$ | 0 | 1 | 2 | 0 | 0 | 1 |
| $d_2(1 : 1)$ | $8\psi_2^{3,1} + \psi_1^{1,4} - 2\psi_2^{2,4} + \psi_2^{3,4} + \psi_3^{3,4}$ | 0 | 1 | 0 | 1 | 1 | 0 | 1 |
| $d_2(0 : 0)$ | $8\psi_2^{3,1} + \psi_2^{2,4} + \psi_3^{3,4}$ | 0 | 1 | 4 | 2 | 4 | 1 | 2 |
| $d_3$ | $8\psi_2^{1,1} + \psi_1^{1,4} - 2\psi_2^{2,4} - 2\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 2 | 0 |
| $d_4$ | $8\psi_2^{1,1}$ | 0 | 3 | 7 | 2 | 4 | 5 | 1 | 2 |
| $d_5$ | $\psi_2^{2,3} + \psi_1^{1,4}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d_6(p : q : r)$ | $p\psi_1^{1,4} + q\psi_2^{2,4} + \psi_3^{3,4} + r\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 |
| $d_6(p : q : p + q)$ | $p\psi_1^{1,4} + q\psi_2^{2,4} + \psi_3^{3,4} + (p + q)\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 1 | 2 | 0 | 1 |
| $d_6(p : q : -p - q)$ | $p\psi_1^{1,4} + q\psi_2^{2,4} + \psi_3^{3,4} - (p + q)\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 1 | 2 | 0 | 1 |
| $d_6(p : q : 0)$ | $p\psi_1^{1,4} + q\psi_2^{2,4} + \psi_3^{3,4}$ | 0 | 1 | 3 | 0 | 0 | 3 | 1 | 0 |
| $d_6(p : q : -2p)$ | $p\psi_1^{1,4} + q\psi_2^{2,4} + \psi_3^{3,4} - 2p\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 3 | 1 | 0 |
| $d_6(0 : p : q)$ | $p\psi_2^{2,4} + \psi_3^{3,4}$ | 1 | 0 | 2 | 1 | 2 | 2 | 2 |
| $d_6(p : p : q)$ | $p\psi_1^{1,4} + p\psi_2^{2,4} + \psi_3^{3,4} + q\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 1 |
| $d_6(p : 2p : q)$ | $p\psi_1^{1,4} + p\psi_2^{2,4} + \psi_3^{3,4} + (2p + q)\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 1 | 0 |
| $d_6(p : q : -3p)$ | $p\psi_1^{1,4} + q\psi_2^{2,4} + \psi_3^{3,4} - 3p\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 1 |
| $d_7(p : q)$ | $p\psi_1^{1,4} + q\psi_2^{2,4} + q\psi_3^{3,4}$ | 0 | 0 | 4 | 0 | 0 | 4 | 0 | 0 |

Table 4. Cohomology of $3|1$-Dimensional Complex Lie Algebras
\[d_6(1 : -1 : 0) = \psi_1^4 - \psi_2^2 + \psi_3^4\]  
\[d_6(1 : -2 : 0) = \psi_1^4 - 2\psi_2^2 + \psi_3^4\]  
\[d_6(1 : -2 : -1) = \psi_1^4 - 2\psi_2^2 + \psi_3^4 - \psi_3^4\]  
\[d_6(1 : -3 : -2) = \psi_1^4 - 3\psi_2^2 + \psi_3^4 - 2\psi_3^4\]  
\[d_6(1 : 1 : 3) = \psi_1^4 + \psi_2^2 + \psi_3^4 + 3\psi_3^4\]  
\[d_6(2 : -1 : 2) = 2\psi_1^4 - \psi_2^2 + \psi_3^4 + 2\psi_3^4\]  
\[d_6(0 : 0 : 1) = \psi_1^3 + \psi_3^4\]  
\[d_6(0 : 1 : -1) = \psi_2^2 + \psi_3^4 - \psi_3^4\]  
\[d_6(1 : 0 : 1) = \psi_1^4 + \psi_3^4 + \psi_3^4\]  
\[d_6(1 : -2 : 1) = \psi_1^4 - 2\psi_2^2 + \psi_3^4 + \psi_3^4\]  
\[d_6(1 : -1 : 1) = \psi_1^4 - \psi_2^2 + \psi_3^4 + \psi_3^4\]  
\[d_6(1 : -3 : 1) = \psi_1^4 - 3\psi_2^2 + \psi_3^4 + \psi_3^4\]  
\[d_6(1 : -5 : -3) = \psi_1^4 - 5\psi_2^2 + \psi_3^4 - 3\psi_3^4\]  
\[d_6(0 : 1 : 1) = \psi_2^3 + \psi_3^4 + \psi_3^4\]  
\[d_6(1 : 1 : 2) = \psi_1^4 + \psi_2^2 + \psi_3^4 + 2\psi_3^4\]  
\[d_6(1 : -4 : -3) = \psi_1^4 - 4\psi_2^2 + \psi_3^4 - 3\psi_3^4\]  
\[d_6(2 : -3 : 1) = 2\psi_1^4 - 3\psi_2^2 + \psi_3^4 + \psi_3^4\]  
\[d_6(1 : -3 : 2) = \psi_1^4 - 3\psi_2^2 + \psi_3^4 + 2\psi_3^4\]  
\[d_6(1 : -4 : -2) = \psi_1^4 - 4\psi_2^2 + \psi_3^4 - 2\psi_3^4\]  
\[d_6(1 : 0 : 2) = \psi_1^4 + \psi_2^3 + 2\psi_3^4\]  
\[d_6(1 : -3 : 0) = \psi_1^4 - 3\psi_2^2 + \psi_3^4\]  
\[d_6(1 : 0 : 0) = \psi_1^4 + \psi_3^4\]  
\[d_6(0 : 0 : 0) = \psi_2^3\]

**Table 5.** Cohomology of special points in the family $d_6(p : q : r)$

A generic element $d_6(0 : 0 : 0)$ has jump deformations to $d_1$, $d_2(x : y)$ for all $(x : y)$, $d_5$ and $d_6(x : y : z)$ except $(0 : 0 : 0)$. Another important observation is that if any element of a family has a deformation to some algebra, then the generic element also has such a deformation. This is a case of the observation that if algebra $a$ deforms to algebra $b$ and algebra $b$ deforms to algebra $c$, then algebra $a$ also deforms to algebra $c$.

Generically, the algebra $d_7(p : q)$ has jump deformations to $d_6(p : q : q)$ and deforms in a nbd of $d_6(p : q : q)$ and $d_7(p : q)$. The special point $d_7(1 : 0)$ also has a jump deformation to $d_6$, while the special point $d_7(1 : -2)$ has jump deformations to $d_2(x : y)$ for all $(x : y)$ except
as to \( d_3 \). This family is parametrized by \( \mathbb{P}^1 \), with no action of a symmetric group. The generic element \( d_7(0 : 0) \) is just the trivial algebra, so it has jump deformations to every other element in the moduli space.

9. The Moduli Space of 2|2-dimensional Lie Superalgebras

9.1. Construction of the moduli space of 2|2-dimensional algebras. This is the most complicated of the moduli spaces in this paper to construct. We begin by considering extensions of the trivial algebra structure \( \delta = 0 \) on the 0|1-dimensional space \( W = \langle v_1 \rangle \) by an algebra structure \( \mu \) on a 2|1-dimensional space. The generic \( \lambda \) is of the form

\[
\lambda = \psi_1^{1,4} a_{11} + \psi_1^{2,4} a_{12} + \psi_2^{1,4} a_{21} + \psi_2^{2,4} a_{22} + \psi_3^{3,4} a_{33},
\]

which has a block diagonal matrix \( A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \). The \( \psi \) term must vanish, and \( \beta = \varphi_3^3 b \).

There are 4 nontrivial possibilities for \( \mu \). The first case is given by \( \mu = \psi_1^{1,3} p + \psi_1^{2,3} + \psi_2^{2,3} q \). There are two solutions to the compatibility condition, the first holding for generic values of \( p \) and \( q \), and the second holding only for \( p = q = 0 \). For the first solution, we have \( a_{21} = a_{31} = 0 \) and \( a_{11} = pa_{12} + a_{22} - a_{12} q \), so only two free variables remain, which we will denote by \( r = a_{22} \) and \( s = a_{12} \) for short. The MC condition is satisfied automatically, so \( d = \mu + \lambda \). When \( p \neq q \) and \( r \neq sq \), this gives the algebra \( d_1 \), when \( p = q \) and \( r \neq sq \), this gives the algebra \( d_5(p : 0) \), and finally when \( r = sq \), we obtain the algebra \( d_10(p : q : 0) \).
Next, consider the case when both \( p \) and \( q \) vanish. Then the compatibility condition yields \( a_{21} = 0 \), and adding a coboundary term allows us to eliminate the coefficient \( a_{12} \) as well, leaving two coefficients \( a_{11} = p \) and \( a_{22} = q \). The reader may be curious why we make this \( p \) and \( q \) substitution, and the explanation is that considering the action of \( G_{\delta,\mu} \) on \( \lambda \), we note that the two coefficients are scaled by a number, which suggests that they represent projective coefficients. Not all such projectively given coefficient relations survive the test of isomorphism, but in this case, they do, as the algebra we have constructed is isomorphic to \( d_5(p : q) \), except that when \( (p : q) = (0 : 0) \), we discover that \( d_5(0 : 0) \sim d_{10}(0 : 0 : 0) \), a type of occurrence which often happens in our construction of moduli spaces of algebras.

Next, consider \( \mu = \psi_1^{1,3} + \psi_2^{2,3} \). The compatibility condition forces \( a_{33} = 0 \) and by adding a coboundary, we could eliminate either the \( a_{11} \) or the \( a_{22} \) term, but this time, we found it convenient not to use this simplification, for reasons we will explain. The matrix of \( \lambda \) is essentially a \( 2 \times 2 \) block matrix, and the action of \( G_{\delta,\mu} \) on \( \lambda \) is essentially given by conjugation of the \( 2 \times 2 \) submatrix of \( \lambda \), up to a constant multiple. This action we know well, and it gives isomorphism classes of Jordan decompositions, so we know the decomposition can be reduced to some simple cases. The first is when the submatrix is of the form

\[
A = \begin{bmatrix} p & 1 \\ 0 & q \end{bmatrix}.
\]

In this case, whenever \( p \neq q \), we obtain \( d_1 \), and when \( p = q \), we obtain \( d_5(p : 0) \). The second is given by the diagonal matrix \( \text{diag}(p, p) \), and in this case, independently of \( p \), we obtain \( d_{11}(1 : 0) \).

Now, let \( \mu = \psi_3^{1,2} + 2\psi_3^{1,1} \). The compatibility condition forces \( a_{12} = 0 \) and \( a_{11} = -a_{22} - a_{33} \), but there is no additional simplification by adding a coboundary term, since \( [\mu, \beta] \) vanishes. The matrix of \( \lambda \) is determined by two coefficients. This time we found it convenient to set \( a_{22} = p \) and \( a_{33} = -p - q \). The MC condition is automatically satisfied, and we obtain that the algebra is isomorphic to \( d_6(p : q) \).

The last nontrivial \( \mu \) is \( \mu = \psi_3^{1,1} \). The compatibility conditions force \( a_{12} = 0 \) and \( a_{33} = -a_{11} \). Adding a coboundary term does not change anything, but applying \( G_{\delta,\mu} \), we discover that unless \( a_{11} = a_{22} \), we can eliminate the \( a_{21} \) term, and in this case, we obtain \( d_{11}(p : q) \), except when \( p = q \). When \( p = q \neq 0 \), we obtain \( d_8 \), and when \( p = q = 0 \), we obtain \( d_9 \). When \( a_{11} = a_{22} \), and \( a_{21} \neq 0 \), then when \( a_{21} \neq 0 \), we obtain \( d_7(a_{11}, a_{11}) \), and otherwise if \( a_{11} \neq 0 \), we get \( d_8 \) and when \( a_{11} = 0 \) we obtain \( d_9 \).

Finally, we need to analyze the case \( \mu = 0 \). Then \( \lambda \) is given by the generic value of \( \lambda \), whose matrix is the block diagonal \( A \) given
in the beginning of this section, and the group $G_{\delta,\mu}$ acts by reducing to Jordan form. Thus there are two cases to consider, given by the matrices below.

$$
\begin{bmatrix}
p & 1 & 0 \\
0 & q & 0 \\
0 & 0 & r
\end{bmatrix}, \quad \begin{bmatrix}
p & 0 & 0 \\
0 & p & 0 \\
0 & 0 & q
\end{bmatrix}.
$$

The first matrix gives the algebra $d_{10}(p : q : r)$, while the second gives $d_{11}(p : q)$.

Next, we extensions of the trivial algebra structure $\delta = 0$ on the 1|0-dimensional space $W = \langle v_2 \rangle$ by an algebra structure $\mu$ on a 1|2-dimensional space $M = \langle v_1, v_3, v_4 \rangle$. The generic $\lambda$ is of the form

$$
\lambda = \psi_3^{1,2} a_{31} + \psi_4^{1,2} a_{31} + \psi_1^{2,3} a_{12} + \psi_1^{2,4} a_{13}.
$$

Generically, $\psi = \psi_3^{2,2} c_1 + \psi_4^{2,2} c_2$ and $\beta = \varphi_1^{2} b$. There are three nontrivial possibilities for $\mu$.

The first case is $\mu = 4\psi_3^{1,1} + \psi_4^{1,4} - 2\psi_1^{3,3}$. The compatibility condition gives $a_{12} = a_{31} = 0$ and $a_{21} = 8 a_{13}$, but adding a coboundary term allows us to eliminate $\lambda$. The MC condition forces $\psi = 0$, so the resulting algebra is just $\mu$, which is isomorphic to $d_7(1 : 0)$.

The second case is $\mu = 4\psi_3^{1,1}$. The compatibility condition gives $a_{12} = a_{13} = 0$, and adding a coboundary term allows us to assume $a_{12} = 0$, leaving only the coefficient $a_{31}$ in $\lambda$, which can be taken to be 1 or 0. The MC condition and the cocycle condition are automatically satisfied, and thus we have no conditions on the coefficients $c_1$ or $c_2$. Assuming $a_{31} = 1$, then when $c_1 = -4 c_2^2$, the algebra is isomorphic to $d_2$, and otherwise, it is isomorphic to $d_4$. When $a_{31} = 0$, if $c_2 \neq 0$, it is $d_2$, if $c_2 = 0$ and $c_1 \neq 0$, then we obtain $d_6(0 : 0)$, while if $c_1 = c_2 = 0$, we obtain $d_9$.

Finally, when $\mu = \psi_1^{1,4} p + \psi_3^{3,4} q$ (note this includes the case $\mu = 0$), the compatibility condition gives three solutions, a generic one, one for $q = -p$, and one where $p = q = 0$. Let us consider the generic case first, which forces $a_{12} = a_{21} = a_{31} = 0$.

The MC condition gives three solutions, depending on $p$ and $q$. The first case is generic in terms of $p$ and $q$, and $\psi$ must vanish. The resulting algebra is isomorphic to $d_{10}(p : 0 : q)$ except that if $p = 0$, we must have $a_{1,3,1} \neq 0$. When $p = a_{1,3} = 0$, we obtain $d_{11}(0 : q)$, in other words, we get $d_{11}(0 : 1)$ and $d_{11}(0 : 0)$.

The second solution to the MC condition has $q = 0$ and $c_2 = 0$. This one gives rise to several different algebras, which is not surprising, since we have the variables $c_1, a_{1,3,1}$ and $p$ to consider. Depending on the
values of these coefficients, we get $d_7(0 : 1)$, $d_7(0 : 0)$, $d_9$, $d_{10}(1 : 0 : 0)$, $d_{10}(0 : 0 : 0)$ and $d_{11}(0 : 0)$.

The final solution to the MC condition occurs when both $p$ and $q$ vanish, and this time, the cocycle condition forces either $c_2$ or $a_{1,3}$ to vanish. In the first case, we have two potentially nonzero coefficients, $c_1$ and $a_{1,3}$, and four algebras arise, $d_7(0 : 0)$, $d_9$, $d_{10}(0 : 0 : 0)$, and $d_{11}(0 : 0)$.

The second solution of the compatibility condition has $q = -p$ and $a_{31} = 0$, leaving $a_{21}$, $a_{12}$ and $a_{1,3}$ undetermined. When $p \neq 0$ we can assume it is 1, and in this case, we can eliminate the coefficient $a_{13}$ by adding a coboundary term. By applying an element of $G_{\delta, \mu}$, we can also reduce to the cases where $a_{21}$ and $a_{12}$ are either 1 or 0. When they both equal 1, the MC condition forces $c_1 = 0$ and $c_2 = -1$, completely determining the algebra, which is $d_3$. When $a_{21} = 0$ and $a_{12} = 1$, the MC condition forces $c_1 = c_2 = 0$, giving the algebra $d_5(0 : 1)$. When $a_{21} = 1$ and $a_{12} = 0$, the MC condition again forces $c_1 = c_2 = 0$, giving the algebra $d_6(1 : 0)$. Finally, when $a_{21} = a_{12} = 0$, the MC condition also forces $c_1 = c_2 = 0$, giving the algebra $d_{10}(1 : 0 : -1)$.

Now, we study the case when $p = 0$. The MC condition has two solutions, $a_{21} = 0$ or both $a_{12}$ and $a_{13}$ vanish. When $a_{21} = 0$, the cocycle condition gives $c_1a_{12} + c_2a_{13} = 0$. We can break the solution to the cocycle condition into 3 parts. After some analysis, we obtain the algebras $d_6$, $d_7(0 : 0)$, $d_9$, $d_{10}(0 : 0 : 0)$, and $d_{11}(0 : 0)$. The other solution to the MC condition gives $a_{12} = a_{13} = 0$, and the cocycle condition puts no restriction on $\psi$. Depending on the values of $a_{21}$, $c_1$ and $c_2$, we obtain the algebras $d_4$, $d_6$, $d_9$ and $d_{11}(0 : 0)$.

The third and last solution of the compatibility condition gives $p = q = 0$, in other words, $\mu = 0$. There are 2 solutions to the MC condition, $a_{21} = a_{31} = 0$ or $a_{12} = a_{13} = 0$. In the first case, the cocycle condition gives $c_1a_{12} + c_2a_{13} = 0$, which can be divided into 3 cases, when $c_1 \neq 0$, when $c_1 = c_2 = 0$ and when $c_1 = a_{13} = 0$. The first gives $d_7(0 : 0)$ when $a_{13} \neq 0$ and $d_9$ when $a_{13} = 0$. The second gives $d_{10}(0 : 0 : 0)$ as long as not both $a_{12}$ and $a_{13}$ vanish, and $d_{11}(0 : 0)$ otherwise. The third solution gives $d_7(0 : 0)$ when neither $a_{12}$ nor $c_2$ vanish, $d_{10}(0 : 0 : 0)$ when $a_{12} \neq 0$ and $c_2 = 0$, $d_9$ when $a_{12} = 0$ and $c_2 \neq 0$ and $d_{11}(0 : 0)$ when both $a_{12}$ and $c_2$ vanish. For the second solution to the MC condition, the cocycle condition is trivial. The solutions, depending on the values of the coefficients $a_{21}$, $a_{31}$, $c_1$ and $c_2$, are $d_4$, $d_6(0 : 0)$, $d_9$, and $d_{11}(0 : 0)$.

This completes the construction of the moduli space of 2|2-dimensional algebras.
Deformations of the symmetric group. Generically, deformations are only along the family.

The special point \(d_1\) while the special point \(d_2\) points that do not fit the generic picture, but deform generically. The generic deformations are given in the table below.

| Algebra | Codifferential | \(h_0\) | \(h_1\) | \(h_2\) | \(h_3\) |
|---------|----------------|--------|--------|--------|--------|
| \(d_1\) | \(\psi_1^{1,3} + \psi_1^{2,3} + 2\psi_2^{2,3} + \psi_1^{2,4} + \psi_2^{2,4}\) | 0|0 | 0|0 | 0|0 |
| \(d_2\) | \(\psi_1^{1,2} + 4\psi_1^{1,1} + \psi_1^{2,2} + 2\psi_2^{2,2}\) | 0|2 | 2|2 | 2|0 | 0|0 |
| \(d_3\) | \(\psi_3^{1,2} + 4\psi_3^{1,1} + 2\psi_2^{1,4} + \psi_2^{2,4}\) | \(-\psi_3^{3,4} + \psi_1^{1,2} - \psi_2^{3,4} - \psi_4^{2,4}\) | 0|0 | 0|1 | 1|0 | 0|0 |
| \(d_4\) | \(8\psi_3^{1,1} + \psi_1^{1,2} + 2\psi_4^{1,1}\) | 0|2 | 3|2 | 3|2 | 2|2 |
| \(d_5(p : q)\) | \(\psi_2^{2,3} + p - q\psi_1^{1,4} + p\psi_2^{2,4} + q\psi_3^{3,4}\) | 0|0 | 1|0 | 0|1 | 0|0 |
| \(d_6(p : q)\) | \(\psi_3^{1,2} + 4\psi_3^{1,1} + \psi_1^{1,4} + 2(p - q)\psi_2^{1,4}\) | \(+p\psi_2^{2,4} - (p + q)\psi_3^{3,4}\) | 0|0 | 1|0 | 0|1 | 0|0 |
| \(d_7(p : q)\) | \(4\psi_3^{1,1} + p\psi_1^{1,4} + \psi_2^{1,4} + q\psi_2^{2,4} + 2p\psi_3^{3,4}\) | 0|0 | 1|0 | 0|1 | 0|0 |
| \(d_8\) | \(4\psi_3^{1,1} + \psi_1^{1,4} + \psi_2^{2,4} - 2\psi_3^{3,4}\) | 0|0 | 2|0 | 0|3 | 1|0 |
| \(d_9\) | \(4\psi_3^{1,1}\) | 1|2 | 5|4 | 6|6 | 6|6 |
| \(d_{10}(p : q : r)\) | \(p\psi_1^{1,4} + \psi_2^{1,4} + q\psi_2^{2,4} + r\psi_3^{3,4}\) | 0|0 | 2|0 | 0|2 | 0|0 |
| \(d_{11}(p : q)\) | \(p\psi_1^{1,4} + p\psi_2^{2,4} + q\psi_3^{3,4}\) | 0|0 | 4|0 | 0|4 | 0|0 |

Table 7. Cohomology of 2|2-Dimensional Complex Lie Algebras

9.2. Deformations of the 2|2-dimensional algebras. The algebras \(d_1, d_2\) and \(d_3\) are rigid, although only \(d_1\) is totally rigid (cohomology vanishes identically). The algebra \(d_4\) has jump deformations to both \(d_2\) and \(d_3\).

The family \(d_5(p : q)\) is parameterized by \(\mathbb{P}^1\), with no action of a symmetric group. Generically, deformations are only along the family. The special point \(d_5(1 : 0)\) has an additional jump deformation to \(d_3\), while the special point \(d_6(0 : 1)\) has a jump deformation to \(d_1\). The points \(d_5(1 : 1), d_5(2 : 1)\) and \(d_5(3 : 1)\) have cohomology dimensions that do not fit the generic picture, but deform generically. The generic point \(d_5(0 : 0)\) is quite special. It is isomorphic to \(d_{10}(0 : 0 : 0)\), and it has jump deformations to \(d_1, d_3, d_5(x : y)\) except \(0 : 0\), \(d_6(x : y)\) except \(1 : 1\) and \(0 : 0\), \(d_7(x : y)\) and \(d_{10}(x : y : z)\) except \(0 : 0 : 0\).

The family \(d_6(p : q)\) is parameterized by \(\mathbb{P}^1/\Sigma_2\), where \(\Sigma_2\) acts by permuting the coordinates \((p : q)\). Generically, an point in this family only has deformations in a nbd of the point. The special point \(d_6(1 : 0)\) also has a jump deformation to \(d_3\), while the points \(d_6(1 : 1)\) and \(d_6(1 : -1)\) deform generically. The generic point \(d_6(0 : 0)\) has jump deformations to \(d_2, d_3, d_4, \) and \(d_6(x : y)\) except \(0 : 0\). The cohomology of the elements in this family is given in the table below.

The family \(d_7(p : q)\) is parameterized by \(\mathbb{P}^1\), with no action of a symmetric group. Generically, elements in the family deform only along
the family. The special point \( d_7(1 : 1) \) deforms in a nbd of \( d_6(1 : 1) \) but does not have a jump deformation to \( d_6(1 : 1) \). The special points \( d_7(1 : 0) \), \( d_7(0 : 1) \), \( d_7(1 : -1) \) and \( d_7(3 : 2) \) deform generically. The generic element \( d_7(0 : 0) \) has jump deformations to \( d_3 \), \( d_6(x : y) \) except \((0 : 0)\), and \( d_7(x : y) \) except \((0 : 0)\).

The algebra \( d_8 \) has jump deformations to \( d_6(1 : 1) \), \( d_7(1 : 1) \) and deforms in a nbd of each of these points.

The algebra \( d_9 \) has jump deformations to \( d_3 \), \( d_4 \), and \( d_6(x : y) \) and \( d_7(x : y) \) for all \((x : y)\) as well as \( d_8 \).

The family \( d_{10}(p : q : r) \) is parametrized by \( \mathbb{P}^2/\Sigma_2 \), where \( \Sigma_2 \) acts by permuting the first two coordinates. Generically, elements deform only along the family. There are some special subfamilies parametrized by \( \mathbb{P}^1 \), for which the cohomology or deformation theory is not generic, and also some special points, each of which belongs to one or more of the special subfamilies, for which the cohomology or deformation theory is even more unusual.
The members of the subfamily $d_{10}(p : q : p - q)$ have deformations in a nbd of $d_5(p : p - q)$, but don’t have jump deformations to this point. The members of the subfamily $d_{10}(p : q : 0)$ all have jump deformations to $d_1$. The members of the subfamily $d_{10}(p : q : -2p)$ have jump deformations to $d_7(p : q)$ and deform in a nbd of this point, while the members of the subfamily $d_{10}(p : q : -p - q)$ jump to $d_6(p : q)$ and deform in a nbd of this point. The other special subfamilies have non generic cohomology, but generic deformations.

The special point $d_{10}(1 : 0 : -1)$ has additional jump deformations to $d_3$, $d_5(0 : 1)$, $d_6(1 : 0)$ and deforms in a nbd of $d_5(0 : 1)$ and $d_6(1 : 0)$. The special point $d_{10}(1 : 2 : -1)$ has a jump deformation to $d_5(1 : -1)$, $d_{10}(1 : 0 : 1)$ has a jump to $d_5(1 : 1)$, $d_{10}(1 : 2 : 1)$ has a jump to $d_5(2 : 1)$, as well as deformations in a nbd of these points. The special point $d_{10}(1 : -1 : 2)$ has jumps to $d_5(1 : 2)$ and $d_7(1 : -1)$, while $d_{10}(1 : 3 : -2)$ has jumps to $d_5(1 : -2)$ and $d_7(1 : 3)$. The algebra $d_{10}(1 : 0 : 0)$ has jumps to $d_1$ and $d_7(0 : 1)$, and $d_{10}(1 : 0 : -2)$ has a jump to $d_7(1 : 0)$, $d_{10}(1 : 2 : -2)$ and $d_{10}(1 : 2 : -4)$ have jumps to $d_7(1 : 2)$. The algebra $d_{10}(1 : 2 : 0)$ jumps to $d_1$, $d_{10}(1 : 2 : -3)$ has a jump to $d_6(1 : 2)$, while $d_{10}(1 : -1 : 0)$ has jumps to $d_1$ and $d_6(1 : -1)$ and $d_{10}(1 : 1 : 0)$ has jumps to $d_1$ and $d_5(1 : 0)$.

The rest of the special points only have non generic cohomology, except $d_{10}(0 : 0 : 0)$ which jumps to $d_1$, $d_3$, $d_5(x : y)$ except $(0 : 0)$, $d_6(x : y)$ except $(1 : 1)$ and $(0 : 0)$, $d_7(x : y)$ and $d_{10}(x : y : z)$ except $(0 : 0 : 0)$. Note this is exactly the deformation pattern for $d_5(0 : 0)$, which is necessary as $d_5(0 : 0)$ and $d_{10}(0 : 0 : 0)$ are isomorphic algebras.

| Algebra $d_7(p : q)$ | Codifferential $4\psi_3^{1,1} + p\psi_{1,4} + \psi_{2,4} + q\psi_{3,4} - 2p\psi_{3,4}$ | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|----------------------|-------------------------------------------------|-------|-------|-------|-------|
| $d_7(1 : 0)$         | $4\psi_3^{1,1} + \psi_{1,4} + \psi_{2,4} - 2\psi_{3,4}$ | 1 | 0 | 1 | 1 |
| $d_7(0 : 1)$         | $4\psi_3^{1,1} + \psi_{1,4} + \psi_{2,4} - 2\psi_{3,4}$ | 0 | 1 | 2 | 0 |
| $d_7(1 : 1)$         | $4\psi_3^{1,1} + \psi_{1,4} + \psi_{2,4} + \psi_{3,4} - 2\psi_{3,4}$ | 0 | 0 | 1 | 0 |
| $d_7(1 : -1)$        | $4\psi_3^{1,1} + \psi_{1,4} + \psi_{2,4} - 2\psi_{3,4}$ | 0 | 0 | 1 | 0 |
| $d_7(3 : 2)$         | $4\psi_3^{1,1} + 3\psi_{1,4} + \psi_{2,4} + 2\psi_{3,4} - 6\psi_{3,4}$ | 0 | 0 | 1 | 0 |
| $d_7(0 : 0)$         | $4\psi_3^{1,1} + \psi_{2,4}$ | 1 | 1 | 3 | 3 | 4 | 4 | 4 | 4 | 4 | 4 |
The Moduli Space of Complex 1|3-dimensional Lie Superalgebras

10.1. Construction of the moduli space of 1|3-dimensional algebras. Consider a 0|1-dimensional vector space $W = \langle v_4 \rangle$ and a 3|0-dimensional vector space $M$. There is no nontrivial 3|0-dimensional codifferential, so $\mu u$ must vanish. We have $\lambda = \sum_{i,j} \psi^{j,4}_{i} a_{i,j}$, which is given by a $3 \times 3$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Moreover, $\psi$ also must vanish. Thus we easily see that the algebras arising in this manner are given by the Jordan decomposition of the matrices $A$. This gives us three cases to consider, given by the three matrices below:

$$
\begin{align*}
\begin{bmatrix} p & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & r \end{bmatrix}, & \quad \begin{bmatrix} p & 0 & 0 \\ 0 & p & 1 \\ 0 & 0 & q \end{bmatrix}, & \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{align*}
$$

The first matrix corresponds to the codifferential $d_1(p : q : r)$, the second to $d_2(p : q)$ and the third to $d_3$. There is a fourth case of Jordan decomposition, given by the zero matrix, but that gives the trivial algebra, which we don’t list in our table explicitly.

| Algebra | Codifferential | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|---------|----------------|-------|-------|-------|-------|
| $d_{10}(p : q : r)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} + r\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : q : p - q)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} + (p - q)\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : 0 : q)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{3,4}_{3}$ | 1 | 0 | 2 | 1 |
| $d_{10}(p : 2p : q)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + 2p\psi^{2,4}_{2} + q\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : q : 0)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} - 2p\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : q : -p - q)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} + (p + q)\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : 3p : q)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + 3p\psi^{2,4}_{2} + q\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : q : 2p - q)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} + (2p - q)\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : q : p)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} + p\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : -p : q)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} - p\psi^{2,4}_{2} + q\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : q : -3p)$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} - 3p\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |
| $d_{10}(p : q : -(2p + q))$ | $p\psi^{1,4}_{1} + \psi^{2,4}_{1} + q\psi^{2,4}_{2} - (2p + q)\psi^{3,4}_{3}$ | 0 | 0 | 2 | 0 |

Table 11. Cohomology of subfamilies of $d_{10}(p : q : r)$
### Table 12.

| Algebra      | Codifferential                                                                 | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|--------------|---------------------------------------------------------------------------------|-------|-------|-------|-------|
| $d_{10}(p : q : r)$ | $p\psi_1^{1,4} + \psi_1^{2,4} + q\psi_2^{2,4} + r\psi_3^{3,4}$             | 0|0 2|0 2|0 0 |
| $d_{10}(1 : 0 : -1)$ | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_3^{3,4}$                | 1|0 2|3 4|4 4 |
| $d_{10}(1 : 2 : -1)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} - \psi_3^{3,4}$           | 0|0 2|2|3 1 |1 |
| $d_{10}(1 : 0 : 1)$   | $\psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4}$                | 1|0 2|1|2 3 3 |
| $d_{10}(1 : 2 : 1)$   | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} + \psi_3^{3,4}$           | 0|0 2|0|1 3 |2 |
| $d_{10}(1 : -1 : 2)$  | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_2^{2,4} + 2\psi_3^{3,4}$           | 0|0 2|0|0 4 0 |
| $d_{10}(1 : 3 : -2)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} - 2\psi_3^{3,4}$          | 0|0 2|0|3 0 |
| $d_{10}(1 : 0 : 0)$   | $\psi_1^{1,4} + \psi_2^{2,4}$                                         | 1|1|3 4|4 4 |
| $d_{10}(1 : 0 : -2)$  | $\psi_1^{1,4} + \psi_2^{2,4} - 2\psi_3^{3,4}$                 | 1|0 2|1|2 3 |3 |
| $d_{10}(1 : 2 : 0)$   | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4}$                        | 0|1 3|0|1 3 |2 |
| $d_{10}(1 : 2 : -2)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} - 2\psi_3^{3,4}$          | 0|0 2|2|3 3 |1 |
| $d_{10}(1 : 2 : -3)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} - 3\psi_3^{3,4}$          | 0|0 2|0|1 3 |2 |
| $d_{10}(1 : 2 : -4)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} - 4\psi_3^{3,4}$          | 0|0 2|0|1 3 |2 |
| $d_{10}(1 : -1 : 0)$  | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_3^{3,4}$                         | 0|1 3|0|0 4 |0 |
| $d_{10}(1 : 1 : 0)$   | $\psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4}$                         | 0|1 3|0|0 3 |0 |
| $d_{10}(1 : 0 : 2)$   | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_3^{3,4}$                        | 1|0 2|1|2 2 |3 |
| $d_{10}(1 : 0 : -3)$  | $\psi_1^{1,4} + \psi_2^{2,4} - 3\psi_3^{3,4}$                 | 1|0 2|1|2 2 |3 |
| $d_{10}(1 : 3 : -1)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} - \psi_3^{3,4}$          | 0|0 2|2|2 1 |1 |
| $d_{10}(1 : 3 : 5)$   | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} + 5\psi_3^{3,4}$         | 0|0 2|0|2 1 |1 |
| $d_{10}(1 : 3 : 1)$   | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} + \psi_3^{3,4}$          | 0|0 2|0|2 1 |1 |
| $d_{10}(1 : 3 : 3)$   | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} + 3\psi_3^{3,4}$         | 0|0 2|0|2 1 |1 |
| $d_{10}(1 : 3 : -3)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} - 3\psi_3^{3,4}$          | 0|0 2|2|2 1 |1 |
| $d_{10}(1 : 3 : -5)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} - 5\psi_3^{3,4}$          | 0|0 2|0|2 1 |1 |
| $d_{10}(1 : 3 : -7)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} - 7\psi_3^{3,4}$         | 0|0 2|0|2 1 |1 |
| $d_{10}(1 : 3 : -9)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{2,4} - 9\psi_3^{3,4}$         | 0|0 2|0|2 1 |1 |
| $d_{10}(1 : -1 : 3)$  | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_2^{2,4} + 3\psi_3^{3,4}$         | 0|0 2|0|2 2 |2 |
| $d_{10}(1 : -3 : 5)$  | $\psi_1^{1,4} + \psi_2^{2,4} - 3\psi_2^{2,4} + 5\psi_3^{3,4}$         | 0|0 2|0|2 0 |2 |
| $d_{10}(1 : 5 : -3)$  | $\psi_1^{1,4} + \psi_2^{2,4} + 5\psi_2^{2,4} - 3\psi_3^{3,4}$          | 0|0 2|0|2 0 |2 |
| $d_{10}(1 : -1 : 1)$  | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_2^{2,4} + 3\psi_3^{3,4}$         | 0|0 2|2|2 2 |2 |
| $d_{10}(1 : -3 : 3)$  | $\psi_1^{1,4} + \psi_2^{2,4} - 3\psi_2^{2,4} - 3\psi_3^{3,4}$         | 0|0 2|0|2 0 |2 |
| $d_{10}(1 : -3 : 1)$  | $\psi_1^{1,4} + \psi_2^{2,4} - 3\psi_2^{2,4} + \psi_3^{3,4}$          | 0|0 2|0|2 0 |2 |
| $d_{10}(0 : 0 : 0)$   | $\psi_1^{2,4}$                                                        | 1|1|4 3|5 6 |6 6 |
The other possible decomposition is given by the 1|0-dimensional space \( W = \langle v_3 \rangle \) and the 2|1-dimensional space \( M = \langle v_1, v_2, v_4 \rangle \). There are 4 nontrivial possibilities for \( \mu \), given by the elements in the 2|1-dimensional moduli space already described in this paper. The \( \lambda \) term is of the form \( \lambda = \psi_1^{3,4} a_{31} + \psi_4^{2,3} a_{32} + \psi_1^{3,4} a_{1,3} + \psi_2^{3,4} a_{23} \), while \( \psi = \psi_4^{3,3} a \) and \( \beta = \varphi_1^3 b_1 + \varphi_2^3 b_2 \).

The first case is \( \mu = \psi_1^{1,4} p + \psi_1^{2,4} + \psi_2^{2,4} q \). The compatibility condition forces \( a_{31} = a_{32} = 0 \). If neither \( p \) nor \( q \) vanish, then by adding an appropriate \([\mu, \beta]\) term, we could eliminate \( \lambda \), but in general, we can at least eliminate the \( a_{13} \) term. Thus we reduce to the case \( \lambda = \psi_2^{3,4} a_{23} \). The MC equation forces \( \psi = 0 \). By applying an element of \( G_{5,\mu} \) we can reduce to the case where \( a_{23} \) is either 1 or 0. When \( a_{23} = 1 \), we obtain the algebra \( d_1(p : q : 0) \). The case when \( a_{23} = 0 \) is a bit more complex. When neither \( p \) nor \( q \) vanishes, we obtain \( d_1(p : q : 0) \), but when \( q = 0 \) we obtain \( d_2(0 : p) \) and similarly when \( p = 0 \) we obtain \( d_2(0 : q) \).

For \( \mu = \psi_1^{1,4} + \psi_2^{2,4} \), the compatibility condition forces \( a_{31} = a_{32} = 0 \), and then by adding a \([\mu, \beta]\) term, we can make \( \lambda \) vanish. The MC equation also forces \( \psi \) to vanish. Thus we get the codifferential \( \mu \) which is \( d_2(1 : 0) \) in our list.

For \( \mu = \psi_4^{1,2} + 2\psi_4^{1,1} \), applying the compatibility condition and taking into account the addition of a coboundary term, \( \lambda \) can be made to vanish. This time, the MC condition does not force \( \psi = 0 \), but we can assume \( c = 1 \), which gives \( d_4 \), or \( c = 0 \), which gives \( d_5 \).

| Algebra | Codifferential | \( h_0 \) | \( h_1 \) | \( h_2 \) | \( h_3 \) |
|---------|----------------|---------|---------|---------|---------|
| \( d_1(p : q : r) \) | \( p\psi_1^{1,4} + \psi_1^{2,4} + q\psi_2^{2,4} + \psi_3^{3,4} + r\psi_3^{3,4} \) | 0 | 0 | 0 | 0 |
| \( d_2(p : q) \) | \( p\psi_1^{1,4} + p\psi_2^{2,4} + q\psi_3^{3,4} \) | 0 | 0 | 4 | 0 |
| \( d_3 \) | \( \psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4} \) | 0 | 0 | 8 | 0 |
| \( d_4 \) | \( 4\psi_1^{1,1} + \psi_4^{2,3} \) | 0 | 1 | 4 | 0 |
| \( d_5 \) | \( \psi_4^{1,2} + 4\psi_4^{1,1} \) | 1 | 1 | 5 | 1 |
| \( d_6 \) | \( 4\psi_4^{1,1} \) | 2 | 1 | 7 | 2 |

Table 13. Cohomology of 1|3-Dimensional Complex Lie Algebras

For \( \mu = 2\psi_4^{1,1} \), we reduce to the case where only \( a_{32} \) does not vanish, so \( \lambda = \psi_2^{3,4} a_{32} \). We can assume that \( a_{32} \) is either 1 or 0. When \( a_{32} = 1 \), the MC condition does not impose any restriction on \( \psi \), nor does the cocycle condition. Nevertheless, independently of the value of \( c \), the algebra is isomorphic to \( d_4 \). When \( a_{32} = 0 \), we again don’t get any
restriction on $\psi$, but can take $c = 1$ or $c = 0$. The first gives $d_5$, while
the second gives $d_6$.

Finally, when $\mu = 0$, the first restriction on $\lambda$ comes from the $MC$
equation, which forces either $a_{13} = a_{23} = 0$, or $a_{31} = a_{32} = 0$. For
the first case, we consider the action of $G_{\delta,\mu}$ on $\lambda$, which turns out
to be equivalent to the action of $\text{GL}(2, \mathbb{C})$ on the vector $(a_{31}, a_{32})$.
This action gives us only two equivalence classes, that of $(1, 0)$ and
$(0, 0)$. The first class corresponds to $a_{31} = 1$ and $a_{32} = 0$. There is
no restriction on $\psi$, but independently of the value of $c$, we obtain $d_5$.
Similarly, when both $a_{31}$ and $a_{32}$ vanish, then we obtain $d_6$. Finally, if
$a_{31} = a_{32} = 0$, then we get a similar action of $\text{GL}(2, \mathbb{C})$ on the vector
$(a_{13}, a_{23})$, again with two equivalence classes. The nonvanishing class
$a_{13} = 1$ and $a_{23} = 0$ forces $\psi = 0$ by the cocycle condition, and we
obtain $d_2(0 : 0)$. The vanishing class gives $\lambda = 0$, and there is no
restriction on $\psi$. When $c \neq 0$ we obtain $d_6$. The case $c = 0$ gives us the
trivial algebra. This completes the description of the moduli space.

| Algebra          | Codifferential                              | $h_0$ | $h_1$ | $h_2$ | $h_3$
|------------------|---------------------------------------------|-------|-------|-------|-------|
| $d_1(p : q : r)$ | $p\psi_1^{1.4} + \psi_1^{2.4} + q\psi_2^{2.4}$ | 0|0 | 2|0 | 0|2 | 0|0 |
| $d_1(p : q : 0)$ | $p\psi_1^{1.4} + \psi_1^{2.4} + q\psi_2^{2.4} + \psi_3^{3.4}$ | 1|0 | 2|1 | 2|2 | 2|2 |
| $d_1(p : 2p : q)$ | $p\psi_1^{1.4} + \psi_1^{2.4} + 2p\psi_2^{2.4}$ | 0|0 | 2|0 | 1|2 | 0|1 |
| $d_1(p : q : p + q)$ | $p\psi_1^{1.4} + \psi_1^{2.4} + q\psi_2^{2.4}$ | 0|0 | 2|0 | 1|2 | 0|1 |
| $d_1(p : 3p : q)$ | $p\psi_1^{1.4} + \psi_1^{2.4} + 3p\psi_2^{2.4}$ | 0|0 | 2|0 | 0|2 | 1|0 |
| $d_1(p : -p : q)$ | $p\psi_1^{1.4} + \psi_1^{2.4} - p\psi_2^{2.4}$ | 0|0 | 2|0 | 0|2 | 2|0 |
| $d_1(p : q : -2p + q)$ | $p\psi_1^{1.4} + \psi_1^{2.4} + q\psi_2^{2.4}$ | 0|0 | 2|0 | 0|2 | 2|0 |
| $d_1(p : q : 2p + q)$ | $p\psi_1^{1.4} + \psi_1^{2.4} + q\psi_2^{2.4}$ | 0|0 | 2|0 | 0|2 | 1|0 |

Table 14. Cohomology of special subfamilies of the
family $d_1(p : q : r)$
10.2. **Deformations of the $1|3$-dimensional algebras.** The family of algebras $d_1(p : q : r)$ is parametrized by $\mathbb{P}^2/\Sigma_3$, where $\Sigma_3$ acts by permuting the coordinates. There are a lot of special subfamilies and

| Algebra     | Codifferential | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|-------------|----------------|-------|-------|-------|-------|
| $d_1(1 : -1 : 0)$ | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_2^{2,4} + \psi_2^{3,4}$ | 1 | 0 | 2 | 1 | 3 | 2 | 4 | 3 |
| $d_1(1 : 2 : 0)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} + \psi_2^{3,4}$ | 1 | 0 | 2 | 1 | 3 | 2 | 3 | 3 |
| $d_1(1 : 2 : 4)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} + \psi_2^{3,4} + 4\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 2 |
| $d_1(1 : 2 : 3)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 2\psi_2^{2,4} + \psi_2^{3,4} + 3\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 2 | 1 | 2 |
| $d_1(1 : -1 : 2)$ | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_2^{2,4} + \psi_2^{3,4} + 2\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 2 | 2 | 1 |
| $d_1(1 : -3 : 4)$ | $\psi_1^{1,4} + \psi_2^{2,4} - 3\psi_2^{3,4} + \psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 1 | 2 | 0 | 1 |
| $d_1(3 : -4 : -8)$ | $3\psi_1^{1,4} + \psi_2^{2,4} - 4\psi_2^{3,4} + \psi_3^{3,4} - 8\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 1 | 2 | 0 | 1 |
| $d_1(3 : -4 : 7)$ | $3\psi_1^{1,4} + \psi_2^{2,4} - 4\psi_2^{3,4} + \psi_3^{3,4} + 7\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 1 | 2 | 0 | 1 |
| $d_1(1 : -2 : -4)$ | $\psi_1^{1,4} + \psi_1^{2,4} - 2\psi_2^{2,4} + \psi_3^{3,4} - 4\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 1 | 2 | 1 | 1 |
| $d_1(1 : 0 : 0)$ | $\psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4}$ | 1 | 0 | 2 | 1 | 2 | 2 | 2 | 2 |
| $d_1(1 : 3 : 0)$ | $\psi_1^{1,4} + \psi_1^{2,4} + 3\psi_2^{2,4} + \psi_2^{3,4}$ | 1 | 0 | 2 | 1 | 2 | 2 | 3 | 2 |
| $d_1(1 : -1 : 3)$ | $\psi_1^{1,4} + \psi_2^{2,4} - \psi_2^{2,4} + \psi_2^{3,4} + 3\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 2 | 4 | 0 |
| $d_1(1 : -3 : 3)$ | $\psi_1^{1,4} + \psi_2^{2,4} - 3\psi_2^{3,4} + \psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 2 | 3 | 0 |
| $d_1(1 : 3 : 7)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{3,4} + \psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 0 |
| $d_1(1 : 3 : 9)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{3,4} + \psi_3^{3,4} + 9\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 0 |
| $d_1(1 : 3 : 5)$ | $\psi_1^{1,4} + \psi_1^{2,4} + 3\psi_2^{2,4} + \psi_2^{3,4} + 5\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 2 |
| $d_1(1 : 3 : -5)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{3,4} + \psi_3^{3,4} - 5\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 2 |
| $d_1(1 : 3 : -7)$ | $\psi_1^{1,4} + \psi_1^{2,4} + 3\psi_2^{2,4} + \psi_3^{3,4} - 7\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 1 |
| $d_1(1 : -3 : -5)$ | $\psi_1^{1,4} + \psi_1^{2,4} + 3\psi_2^{2,4} + \psi_3^{3,4} - 5\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 2 |
| $d_1(1 : -2 : 0)$ | $\psi_1^{1,4} + \psi_2^{2,4} - 2\psi_2^{3,4}$ | 1 | 0 | 2 | 1 | 2 | 2 | 3 | 2 |
| $d_1(1 : -7 : 0)$ | $\psi_1^{1,4} + \psi_2^{2,4} - 7\psi_2^{3,4} + \psi_3^{3,4}$ | 1 | 0 | 2 | 1 | 2 | 2 | 2 | 2 |
| $d_1(3 : -5 : -7)$ | $3\psi_1^{1,4} + \psi_2^{2,4} - 5\psi_2^{3,4} + \psi_3^{3,4} - 7\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 0 | 2 | 1 | 0 |
| $d_1(1 : 5 : -9)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 5\psi_2^{3,4} + \psi_3^{3,4} - 9\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 1 | 0 |
| $d_1(1 : 3 : -13)$ | $\psi_1^{1,4} + \psi_2^{2,4} + 3\psi_2^{3,4} + \psi_3^{3,4} - 13\psi_3^{3,4}$ | 0 | 0 | 2 | 0 | 2 | 1 | 0 |
| $d_1(0 : 0 : 0)$ | $\psi_1^{2,4} + \psi_2^{3,4}$ | 1 | 0 | 3 | 1 | 4 | 3 | 5 | 4 |

**Table 15.** Cohomology of special points in the family $d_1(p : q : r)$

special points, for which the cohomology is not generic. However, none of these special cases, except the generic point $d_1(0 : 0 : 0)$ give rise to any extra deformations, which makes sense, since $d_1(p : q : r)$ is the first element in our list, so shouldn’t have any extra deformations. We
do have a 2-parameter family of deformations, as elements deform along the family. The generic element $d_1(0:0:0)$ has jump deformations to element in the family except itself.

The family $d_2(p:q)$ is parametrized by $\mathbb{P}^1$, with no action of a symmetric group. Generically, an element $d_2(p:q)$ has a jump deformation to $d_1(p:p:q)$ and smooth deformations in nbds of $d_1(p:p:q)$ as well as $d_2(p:q)$. Again, there are special points, but no extra deformations, because the elements already deform to everything they could. The exception is $d_2(0:0)$, which has jump deformations to $d_1(x:y:z)$ for all $(x:y:z)$ and $d_2(x:y)$ for all $(x:y)$ except $(0:0)$.

The algebra $d_3$ has jump deformations to $d_1(1:1:1)$ and $d_2(1:1)$ as well as smooth deformations in nbds of these points. The description of the deformation picture of the first 3 algebras corresponds exactly to the description of the moduli space of $3 \times 3$ matrices with the action given by conjugation by an element in $\text{GL}(3, \mathbb{C})$ and multiplication by a nonzero complex number. This picture arises in many of the moduli spaces of different algebraic objects.

The algebra $d_4$ is rigid, even though the dimension of $H^2$ is 8, because it is really $8|0$-dimensional, and only odd elements of $H^2$ contribute to deformations. The algebra $d_5$ has a jump deformation to $d_4$, and $d_6$ has jump deformations to $d_4$ and $d_5$. This completes the description of the deformations of the elements in the moduli space of 1|3-dimensional complex Lie superalgebras.

| Algebra       | Codifferential                           | $h_0$ | $h_1$ | $h_2$ | $h_3$ |
|---------------|-----------------------------------------|-------|-------|-------|-------|
| $d_2(p:q)$    | $p\psi_1^{1,4} + p\psi_2^{2,4} + \psi_3^{3,4} + q\psi_3^{3,4}$ | 0|0|4|0|0|0|0 |
| $d_2(0:1)$    | $\psi_3^{3,4} + \psi_3^{3,4}$           | 2|0|2|6|4|8|6 |
| $d_2(1:0)$    | $\psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4}$ | 1|0|4|1|4|4|4 |
| $d_2(1:2)$    | $\psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4} + 2\psi_3^{3,4}$ | 0|0|0|0|3|0|3 |
| $d_2(2:1)$    | $2\psi_1^{1,4} + 2\psi_2^{2,4} + \psi_3^{3,4} + \psi_3^{3,4}$ | 0|0|4|0|2|4|0 |2 |
| $d_2(1:1)$    | $\psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4} + 3\psi_3^{3,4}$ | 0|0|4|0|0|4|4|0 |
| $d_2(3:1)$    | $3\psi_1^{1,4} + 3\psi_2^{2,4} + \psi_3^{3,4} + \psi_3^{3,4}$ | 0|0|4|0|0|4|2|0 |
| $d_2(1:-1)$   | $\psi_1^{1,4} + \psi_2^{2,4} + \psi_3^{3,4} - \psi_3^{3,4}$ | 0|0|4|0|0|4|6|0 |
| $d_2(0:0)$    | $\psi_3^{3,4}$                           | 2|0|5|2|8|5|11|8 |

Table 16. Cohomology of the special points in the family $d_2(p:q)$
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