Random integral currents.

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Abstract

For nice functions, invariant means over integral currents (certain generalized surfaces), can be uniquely defined. That may have applications to define Nambu-like string theory.

1 Currents (generalized surfaces).

Let $K$ be a compact set in $\mathbb{R}^n$. Let $\Omega^m$ be a space of $C^\infty$ differential forms on $K$, with $C^\infty$ norm. Space of $m$ currents $T_m$ is the space of continuous linear functionals on $\Omega^m$. Polyhedral chains are particular cases of currents, linear functionals on forms they define are integrals over polyhedral chains.

For $\omega \in \Omega^m$,

$$M(\omega) = \sup_{x \in K} \| \omega(x) \|,$$

where for $\xi \in \Lambda^m$, $\| \xi \| = \sup_{|\gamma| \leq 1} (\xi \cdot \gamma)$, $\gamma$ a simple $m$-vector.

Let $M(T)$ be the dual of $M(\omega)$,

$$M(T) = \sup_{\omega \in \Omega^m, M(\omega) \leq 1} (T(\omega))$$

Space of normal currents $N$ is a linear space of currents with $M(T) + M(\partial T) < \infty$.

Rectifiable currents $R$ are currents which may be approximated in $M$ semi-norm by integer Lipschitz chains, images under Lipschitz maps of polyhedral chains with integer coefficients.

Integral currents $I$ are normal currents such that $T$ and $\partial T$ are rectifiable currents. Integral currents form an abelian group. We equip integral currents with flat semi-norm $\| \|_F$

$$\| T \|_F = \inf (M(R) + M(S)), \quad T = R + \partial S, \quad R, S \text{ are rectifiable}$$

It is clear that $\| T \|_F \leq M(T)$.
Addition-invariant measure on integral currents $I_m$

Let $f(X), X \in I_m$ be a bounded uniformly continuous function on space of integral $m-$currents $I_m$ with flat semi-norm. Let

$$O_f = \{ f(X + Y) | Y \in I_m \}$$

be its $I_m$ orbit.

**Lemma 1.** A sequence in $O_f$ has a subsequence convergent point-wise in $I_m$ to a bounded continuous function on $I_m$.

Let $B_m \Lambda = \{ T \in I_m | M(T) + M(\partial T) \leq \Lambda \}$. $B_m \Lambda$ is compact in the flat semi-norm $\| \cdot \|_F$. Using diagonal argument, it follows that a sequence in $O_f$ has a point-wise convergent subsequence on $I_m$. Indeed, let $n_1(k)$ be a subsequence uniformly convergent on $B_m \Lambda$, $n_2(k)$ a subsequence of $n_1$ uniformly convergent on $B_{2\Lambda}$, ... $n_p(k)$ a subsequence of $n_{p-1}(k)$ uniformly convergent on $B_{p\Lambda}$. Let $\hat{k} = n_k(k)$. Then $f_{\hat{k}}(X) \to h(X)$ point-wise, $X \in I_m$.

**Lemma 2.**
1) The orbit $O_f$ is relatively compact in the weak topology.
2) Weakly closed convex hull of $O_f$ is weakly compact.

A space dual to the space of continuous bounded functions on a normal topological space $S$ is the space $B$ of regular Borel measures on the field of closed sets, and with norm being total variation. $I_m$ is a space with a semi-norm $\| \cdot \|_F$, and it is normal. A sequence in $O_f$ has a point-wise convergent subsequence $f_{\hat{k}}(X)$, and $f_{\hat{k}}(X)$ is uniformly bounded. By dominated convergence theorem, for any measure $\mu \in B$, $\int f_{\hat{k}}(X)d\mu$ is convergent. Therefore the orbit $O_f$ is relatively sequentially compact in the weak topology.

**Theorem 1.** Let $f(X), X \in I_m$ be a bounded uniformly continuous function on space of integral $m-$currents $I_m$ with flat semi-norm. There is unique mean of $f(X)$ over $X \in I_m$, invariant under addition in $I_m$. That is, there is uniquely defined constant $<f>$, $<f(\cdot + Y)> = <f(\cdot)>$, such that for any $\epsilon > 0$ there exists $\{ \lambda_i \in \mathbb{R}, Y_i \in I_m | \lambda_i \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \}$, such that

$$\sup_X \left| \sum_{i=1}^{N} \lambda_i f(X + Y_i) - <f> \right| < \epsilon$$

$I_m$ is an abelian group and acts on continuous bounded functions by shifts; such action is distal. From Markov-Kakutani theorem [4], [5], there is a unique fixed point of the action of $I_m$ on weakly compact convex hull of the orbit $O_f$. 

2
An easy modification of the above argument can be used to compute mean over currents with prescribed boundary, by averaging over currents with zero boundary:

**Theorem 2.** Let \( I^0_m \) be space of integral \( m \)-currents \( T \) with zero boundary, \( \partial T = 0 \). Let \( f(X), X \in I^0_m \) be a bounded uniformly continuous function. There is unique mean of \( f(X) \) over currents in \( I^0_m \), invariant under addition in \( I^0_m \).

Motivated by applications, we give an example of a family of functions for which an invariant mean can be defined:

**Proposition 1.**

Let \( k \) be a \( C^\infty \) 2-form on \( \mathbb{R}^n \) with compact support, and with \( \max \{ ||k||, ||dk|| \} < \infty \). Let

\[
g_k(X) = \exp \left( i \int k|X\right) \exp \left( i||X||_F\right), X \in I_2. \tag{3} \]

(where \( \int k|X \) is an integral of a 2-form \( k \) over integral current \( X \in I_2 \)). Let \( G_k \) be the \( I_2 \) orbit of \( g_k(X) \),

\[
G_k = \{ g_k(X + Y)|X, Y \in I_2 \}. \tag{4} \]

Functions in \( G_k \) are uniformly bounded, and equicontinuous, therefore the \( I_2 \) mean \( <g_k> \) can be uniquely defined.

Indeed, \( |g_k| \leq 1 \), and

\[
|g_k(X + Y) - g_k(\bar{X} + Y)| = \\
\left| \exp \left( i \int k|(X + Y)\right) \exp \left( i||X + Y||_F\right) \left( 1 - \exp \left( i \int k|(X - \bar{X})\right) \exp \left( i||\bar{X} + Y||_F - i||X + Y||_F\right) \right) \right| \\
\leq | \int k|\left( X - \bar{X} \right) | + ||X - \bar{X}||_F \\
\leq (1 + \max \{ ||k||, ||dk|| \}) ||X - \bar{X}||_F
\]

(we used that \( |1 - e^{i\alpha}| \leq |\alpha|, \alpha \in \mathbb{R} \).

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