Efficient Bayesian estimation for GARCH-type models via Sequential Monte Carlo

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Abstract

This paper exploits the advantages of sequential Monte Carlo (SMC) to develop parameter estimation and model selection methods for GARCH (Generalized AutoRegressive Conditional Heteroskedasticity) style models. This approach provides an alternative method for quantifying estimation uncertainty relative to classical inference. We demonstrate that even with long time series, the posterior distribution of model parameters are non-normal, highlighting the need for a Bayesian approach and an efficient posterior sampling method. Efficient approaches for both constructing the sequence of distributions in SMC, and leave-one-out cross-validation, for long time series data are also proposed. Finally, we develop an unbiased estimator of the likelihood for the Bad Environment-Good Environment model, a complex GARCH-type model, which permits exact Bayesian inference not previously available in the literature.

Keywords: Markov chain Monte Carlo; Time series analysis; Volatility distribution; Cross-validation; Data annealing

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1 Introduction

In many financial applications, volatility, defined as the conditional standard deviation of asset returns, is an important quantity. Many financial applications are based on volatility forecasts, such as risk management, asset pricing and portfolio optimization (Engle, 2001). Therefore modelling of volatility has attracted a lot of research attention. One of the most influential models that describes the dynamics of volatility is the Autoregressive Conditional Heteroscedasticity (ARCH) model by Engle (1982) and the Generalized ARCH (GARCH) model by Bollerslev (1986), which is a more parsimonious extension of ARCH. The standard GARCH model of Bollerslev (1986) has successfully captured many characteristics of financial asset returns, and forms the basis for a wide range of expressions used to capture various empirical features of returns data. For instance, in order to capture the stylised fact of the leverage effect (Black, 1976), a range of models such as the exponential GARCH (EGARCH) model (Nelson, 1991), the Glosten-Jagannathan-Runkle GARCH model (GJR-GARCH) of Glosten et al. (1993), the absolute value GARCH (AGARCH) model of Hentschel et al. (1995), and the Bad Environment-Good Environment (BEGE) model (Bekaert et al., 2015) have been developed. Given the wide range of GARCH type models, it is crucial to develop an efficient statistical inference framework to select between these models.

Frequentist inference based on maximum likelihood estimation (MLE) is commonly employed with the GARCH class of models. A limitation of MLE is that it only produces point estimates of parameters, with the uncertainty of these parameter estimates based on asymptotic theory where parameter estimates are assumed to be normally distributed (Hamilton, 1994). With the development of new financial instruments based on volatility, such as volatility options, modelling the full predictive density of volatility is attracting attention (e.g. Corradi et al. (2009, 2011); and Griffin et al. (2016)). To produce such distributions, here we develop a Bayesian approach to statistical inference for the GARCH class of models. Under this approach, the unknown parameters are regarded as random variables, whose posterior distributions may be far from normal. In contrast to the confidence interval under classical statistics, the credible interval of Bayesian statistics is easily interpreted and gives a more direct and intuitive probability statement about uncertainty in the parameter estimates, (O’Hagan, 2004). Furthermore, our Bayesian approach can also provide full predictive distributions of the conditional volatility required for pricing volatility instruments.

An issue with Bayesian estimation is developing an efficient method for sampling from
the posterior distribution based on Monte Carlo methods (Robert and Casella, 2004). The most commonly used approach is the Markov Chain Monte Carlo (MCMC) method (Metropolis et al., 1953; Hastings, 1970), which has been utilised for Bayesian analysis of the GARCH class of models (see, for example, Kim et al. (1998), Vrontos et al. (2000), Takaishi (2009)). Despite the popularity of MCMC, it can be costly and difficult to tune the associated parameters when using MCMC (Roberts and Rosenthal, 2009) with conventional MCMC algorithms becoming less effective when the target distribution is far from normally distributed. In more recent times, the sequential Monte Carlo (SMC) method for static Bayesian models (Chopin, 2002; Del Moral et al., 2006) has been devised as a complementary method to MCMC. SMC traverses a set of particles through a sequence of probability distributions, which starts with one that is easy to sample from and ends with the ultimate target posterior distribution. The sequence of probability distributions can be attained either using a data annealing strategy (Chopin, 2002) or the method of likelihood annealing (Neal, 2001). SMC iteratively applies three main steps to traverse the population of particles, which include re-weighting, re-sampling, and moving (or mutation). SMC is parallelisable and easy to adapt to become more efficient compared with the MCMC method. It is possible to use parallelisation to save computation costs especially when the likelihood function is computationally intensive, which we demonstrate is easily achievable for the GARCH class of models. In addition, SMC is efficient at exploring irregular posterior distributions, often found with GARCH type models. The intermediate samples created by SMC embedded with data annealing strategy can be reused to perform posterior predictions. Also, the estimate of evidence, which is a by-product from SMC methods, can be used in Bayesian model selection.

In this paper, we utilise SMC methods to develop a novel fully Bayesian approach for the widely used GARCH class of models. This Bayesian framework can be used for estimation, prediction and model selection, and generating a full predictive density for conditional volatility. Our Bayesian SMC approach provides an attractive alternative to the MCMC or frequentist approaches. This paper also contributes to the SMC literature as we demonstrate that SMC data annealing is useful for leave-one-out cross-validation when dealing with time series data. Finally, we develop an unbiased likelihood estimator for the BEGE model, which permits exact Bayesian inference for this model. Bekaert et al. (2015) use an approximation to the likelihood, which as we show produces biased posterior distributions. South et al. (2019) use the BEGE model with biased likelihood as an example to illustrate a new SMC approach for parameter estimation. Our paper is a
comprehensive demonstration of the advantages that SMC can produce for GARCH-type models.

In Section 2, we provide an introduction to GARCH models and present specifically three models that are investigated in the paper. The methods for deriving an unbiased estimator for the likelihood of the BEGE model are also described. Section 3 provides a brief explanation of Bayesian statistics and the general SMC framework is given in detail. An efficient strategy to choose the sequence of intermediate probability distributions is also discussed. Section 4 provides a comparison of one-step forward predictions of volatility that are derived from the fully Bayesian approach and the classical approach, respectively. A leave-one-out cross-validation method for time series data is also described for evaluating the predictive performance of the forecasted volatility. In Section 5 a Bayesian model selection tool that is used to choose between candidate models is provided. Finally, a summary of the results is provided in Section 6 and concluding discussion is given in Section 7.

2 GARCH models

2.1 Introduction to GARCH family models

Research into GARCH type models remains a highly active area due to their success in modelling the volatility of financial time series data (see, for example, Giraitis et al. (2007), Alberg et al. (2008), Kristjanpoller and Minutolo (2015, 2016), and Dyhrberg (2016)). The standard GARCH model introduced by Bollerslev (1986) successfully established a dynamic model of time-varying volatility which captures some of the important characteristics of financial asset returns, numerous extreme values indicating distributions with heavy tails, and volatility clustering (Engle, 1982). The standard GARCH model has been shown to capture these features (Engle, 2004). Despite the long history of, and extensive literature on the classical GARCH model, the model is not free of limitations. The classical GARCH model does not account for the stylised feature of the leverage effect (Black, 1976), an asymmetric impact of past negative and positive returns on future volatility.

Numerous extensions have been proposed to address the problems and limitations of the standard GARCH model. One of the most popular models considered here is the GJR-GARCH (Glosten et al., 1993) which is an improvement over the classical GARCH model. Engle and Ng (1993) suggest that the GJR-GARCH model captures the leverage effect in stock returns best among other popular nonlinear models such as the EGARCH
(Nelson, 1991) model. The GJR-GARCH model originally assumed a Gaussian distribution of the return innovation, however, such a specification usually fails to generate a non-Gaussian distribution for returns which is one of the most important properties of financial time series (Francq and Zakoïan, 2011). Thus, we also consider an alternative model that accommodates conditional non-Gaussianity and was based on the GJR-GARCH model, which is called bad environment-good environment (BEGE) model (Bekaert et al., 2015).

2.2 Classical GARCH(1,1) model

GARCH models are observation driven, where the conditional variance is a function of not only its past values, but past innovations to returns. To begin, define the innovations in returns, $u_t$ as a shock to returns:

$$r_t = \mu + u_t,$$

$$u_t = \sigma_t \epsilon_t, \quad \epsilon_t \overset{i.i.d.}{\sim} N(0, 1)$$

where $r_t$ is the time series of asset returns with fixed mean $\mu$, with $u_t$ having a time varying conditional variance $\sigma_t^2$, with different GARCH models defining different processes for $\sigma_t^2$. In what follows we present three models within the GARCH family considered in this paper.

Under the classical GARCH($p$, $q$) model, the conditional variance is given by:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{q} \alpha_j u_{t-j}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-1}^2, \quad t \in \mathbb{Z},$$

where $\alpha_0 > 0$, $\alpha_j > 0$, $\beta_i > 0$, $p > 0$, $q > 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, and the stationary condition requires that $\sum_{j=1}^{q} \alpha_j + \sum_{i=1}^{p} \beta_i < 1$ (Bollerslev, 1986). The wide success of the parsimonious GARCH(1, 1) model in extensive empirical applications makes this model the workhorse of financial applications (see, for example, Engle (2001); Alexander and Lazar (2006); Francq and Zakoïan (2011)). The GARCH(1, 1) model considered here is given by:

$$\sigma_t^2 = \alpha_0 + \beta_1 \sigma_{t-1}^2 + \alpha_1 u_{t-1}^2,$$

where $\alpha_0$, $\alpha_1$, and $\beta_1$ are subject to the constraints discussed above in the general GARCH($p$, $q$)
model. The log-likelihood function of the GARCH(1,1) model is given by:

\[
L(r|Garch) = \sum_{t=1}^{T} \frac{1}{2} \left[ -\log \sigma_t^2 - \frac{u_t^2}{\sigma_t^2} - \log 2\pi \right],
\]

where \( r \) is the observed time series of asset returns, and \( \theta_{Garch} \) is a parameter set of the GARCH(1,1) model.

2.3 GJR-GARCH(1,1) model

The asymmetric GJR-GARCH(1,1) model is a GARCH-type model with a threshold indicator function, and has a different expression on conditional volatility \( \sigma_t^2 \) from the classical GARCH model, which is defined as:

\[
\sigma_t^2 = \alpha_0 + \beta_\sigma \sigma_{t-1}^2 + \phi u_{t-1}^2 + \phi^- u_{t-1}^2 (I_{u_{t-1} < 0}),
\]

where \( I \) is a dummy variable which takes the value of 1 if the innovation is negative, and 0 otherwise; and \( \alpha_0 > 0, \beta_\sigma > 0, \phi^- > 0 \) and \( \phi > 0 \) to ensure the conditional variance is positive and \( \beta_\sigma + \phi + 0.5\phi^- < 1 \) to ensure the conditional variance is stationary. In this model, the negative shocks increase the conditional variance relative to positive shocks due to the leverage effect in stock returns. The log-likelihood function of the GJR-GARCH(1,1) model has the same expression with the one of the GARCH(1,1) model provided in Equation (5).

2.4 Bad Environment-Good Environment (BEGE) model

The BEGE model is based on non-Gaussian innovations and is defined as:

\[
u_t = \sigma_p \omega_{p,t} - \sigma_n \omega_{n,t},\]

where

\[
\omega_{p,t} \sim \Gamma(p_t, 1), \quad \text{and} \quad \omega_{n,t} \sim \Gamma(n_t, 1),
\]

where \( \Gamma(k, \theta) \) is the so-called centered gamma distributions (Bekaert et al., 2015) with a zero mean. The innovation \( u_t \) is a linear combination of a bad environment component \( \omega_{n,t} \) with a shape parameter of \( n_t \), and a good environment component \( \omega_{p,t} \) with a shape parameter of \( p_t \). The conditional shock distribution of \( u_t \) is generated from two gamma-

\[1\]The probability density function of \( \Gamma(k, \theta) \) is \( f(x) = \frac{1}{\Gamma(k)\theta^k} (x + k\theta)^{k-1} \exp \left( - \frac{x}{\theta} \right) \) for \( x > -k\theta \).
distributed shocks. The time-varying shape parameters \( n_t \) and \( p_t \) are given by:

\[
p_t = p_0 + \rho_p p_t - 1 + \frac{\phi^+_p}{2\sigma_p^2} u_{t-1}^2 I_{u_{t-1} \geq 0} + \frac{\phi^-_p}{2\sigma_p^2} u_{t-1}^2 (1 - I_{u_{t-1} \geq 0}), \quad \text{and} \quad (8)
\]

\[
n_t = n_0 + \rho_n n_t - 1 + \frac{\phi^+_n}{2\sigma_n^2} u_{t-1}^2 I_{u_{t-1} \geq 0} + \frac{\phi^-_n}{2\sigma_n^2} u_{t-1}^2 (1 - I_{u_{t-1} \geq 0}). \quad (9)
\]

The overall conditional variance \( \sigma_t^2 \) of \( u_t \) in the BEGE model is:

\[
\sigma_t^2 \equiv \text{var}(r_t) = \sigma_p^2 p_t + \sigma_n^2 n_t, \quad (10)
\]

which is derived from the moment generating function of the centered gamma distribution (Bekaert et al., 2015). As with the conditional variance, moments of higher order such as conditional skewness and kurtosis can be easily derived with tractable expressions. These higher-order conditional moments are time-varying compared with traditional asymmetric volatility GARCH models. The log-likelihood function of the BEGE model is written as:

\[
L(r|\theta_{\text{BEGE}}) = \sum_{t=1}^{T} \log f_{\text{BEGE}}(r_t|r_{t-1}, \ldots, r_1; \theta_{\text{BEGE}}), \quad (11)
\]

where \( \theta_{\text{BEGE}} \) is a parameter set of the BEGE model, and the conditional likelihood of observation \( r_t \), \( f_{\text{BEGE}}(r_t|r_{t-1}, \ldots, r_1; \theta_{\text{BEGE}}) \) is approximated numerically according to Bekaert et al. (2015). The cumulative distribution function (CDF) of \( f_{\text{BEGE}}(r_t|r_{t-1}, \ldots, r_1; \theta_{\text{BEGE}}) \) at two points just above and below the observation \( r_t \) are numerically evaluated. Then a finite difference approximation method is used to find the derivative of the CDF, which approximates the probability density function of \( r_t \). As a result of the nature of approximation in this deterministic numerical method, the approximated likelihood of \( r_t \) is biased. Below, we propose an unbiased estimation for the likelihood of \( r_t \) of the BEGE model with importance sampling.

### 2.4.1 Unbiased estimation for the BEGE likelihood

According to Bekaert et al. (2015), the BEGE-distributed variable \( u_t \) can be rewritten as \( u_t = \omega_{p,t} - \omega_{n,t} \), where \( \omega_{p,t} \sim \tilde{\Gamma}(p_t, \sigma_p) \), \( \omega_{n,t} \sim \tilde{\Gamma}(n_t, \sigma_n) \), and \( u_t = r_t - \mu \). Then the BEGE density can take the form as

\[
f_{\text{BEGE}}(u_t) = \int_{\omega_{p,t}} f_{u_t}(u_t|\omega_{p,t}, \omega_{p,t}) f(\omega_{p,t}) d\omega_{p,t}
\]

\[
= \int_{\omega_{p,t}} f_{\omega_{p,t}}(u_t - \omega_{p,t}) f(\omega_{p,t}) d\omega_{p,t}. \quad (12)
\]
An unbiased estimator of $f_{BEGE}(u_t)$ can be constructed using Monte Carlo methods. Firstly, we sample $M$ independent samples $\omega_{p,t}^1, \ldots, \omega_{p,t}^M$ from $f(\omega_{p,t})$, then $f_{BEGE}(u_t)$ can be estimated as

$$f_{BEGE}(u_t) = \frac{1}{M} \sum_{i=1}^{M} f_{\omega_{n,t}}(\omega_{p,t}^i - u_t), \text{where } \omega_{p,t}^i \overset{i.i.d.}{\sim} f(\omega_{p,t}), i = 1, \ldots, M.$$ (13)

One main issue of the Monte Carlo integration method is that a large number of Monte Carlo draws, $M$, may be required to increase the accuracy of the Monte Carlo estimates, which makes it more computationally intensive. Therefore, we adopt importance sampling to reduce variance of Monte Carlo estimates and improve computational efficiency. When it is hard to sample from our target distribution $h(\cdot)$ directly, we can construct an importance distribution, $g(\cdot)$, which is easy to sample from. The following identity is the basis for importance sampling

$$\mathbb{E}_g \left[ \frac{h(x)}{g(x)} \right] = \int\frac{h(x)}{g(x)} g(x) dx = \int h(x) dx$$

$$\approx \frac{1}{M} \sum_{i=1}^{M} \frac{h(x_i)}{g(x_i)}, x_i \overset{i.i.d.}{\sim} g(\cdot).$$ (14)

In the case of the BEGE model, the target distribution is $h(\omega_{p,t}) = f_{\omega_{n,t}}(\omega_{p,t} - u_t)f(\omega_{p,t})$, and the importance sampling estimator of $f_{BEGE}(u_t)$ is given by

$$\hat{f}_{IS}(u_t) = \frac{1}{M} \sum_{i=1}^{M} \frac{h(\omega_{p,t}^i)}{g(\omega_{p,t}^i)}, \omega_{p,t}^i \overset{i.i.d.}{\sim} g(\omega_{p,t}),$$ (15)

where a sound proposal distribution $g(\omega_{p,t}^i)$ needs to be selected. The process of constructing an efficient importance distribution for estimating $f_{BEGE}(u_t)$ is provided in the Appendix. The general importance sampling estimator in (14) is unbiased. Andrieu and Roberts (2009) show that using an unbiased likelihood estimator within an MCMC produces a Markov chain that converges to the idealised posterior that assumes the likelihood can be computed exactly. Chopin et al. (2013) and Duan and Fulop (2015) show that using an unbiased likelihood estimator in an SMC algorithm for data annealing and likelihood annealing, respectively, produces an algorithm that also targets the exact posterior. We adopt a Bayesian statistical framework for parameter estimation, model selection and prediction for GARCH-type models that is described in the next section.
3 Bayesian Inference

3.1 Introduction to Bayesian Statistics

Bayes’ theorem captures learning from observed data and experience. Here, $y$ represents the observed data and $\theta$ represents the set of parameters of a model believed to have generated $y$. The likelihood function based on the chosen model is $f(y|\theta)$. Before seeing the observed data, the information regarding the parameters is reflected in a prior distribution $\pi(\theta)$. The posterior distribution of $\theta$ is given by:

$$\pi(\theta|y) = \frac{f(y|\theta)\pi(\theta)}{Z},$$

where

$$Z = f(y) = \int_{\theta} f(y|\theta)\pi(\theta)d\theta,$$

which is the normalising constant of the posterior that is independent of $\theta$.

When our interest is in generating samples of $\theta$ from the posterior, the normalising constant $Z$ can be ignored and then the posterior function of $\theta$ can be expressed as:

$$\pi(\theta|y) \propto f(y|\theta)\pi(\theta).$$

Given that directly sampling from the posterior distribution $\pi(\theta|y)$ is generally infeasible, the Markov chain Monte Carlo (MCMC) method [Metropolis et al., 1953; Hastings, 1970] is the most widely used sampling method. One standard MCMC algorithm is the Metropolis-Hastings (MH) algorithm, which is straightforward to implement. The MH algorithm is presented in Algorithm 3.1. Under the MH algorithm, a proposal distribution $q(\theta^*|\theta)$ shall be carefully chosen, where the new sample candidate value $\theta^*$ depends on the current state $\theta$. The decision of accepting or rejecting $\theta^*$ depends on the following acceptance probability:

$$\alpha(\theta \rightarrow \theta^*) = \min \left(1, \frac{f(y|\theta^*)\pi(\theta^*)q(\theta|\theta^*)}{f(y|\theta)\pi(\theta)q(\theta^*|\theta)} \right).$$

The reader is referred to [Chib and Greenberg, 1995] for the theoretical details of MH algorithms. According to [Andrieu and Roberts, 2009], the exact likelihood $f(y|\theta)$ can be replaced by its unbiased estimator $\hat{f}(y|\theta)$, with the corresponding MCMC algorithm still converging to the posterior distribution $\pi(\theta|y)$.

However, the popular MCMC algorithm is not free from limitations. For the MH al-
Algorithm 3.1 Metropolis-Hastings Algorithm

1: Initialise $\theta^0$
2: for all iterations $t \in 1...T$ do
3: Given the current parameter value $\theta = \theta^{t-1}$, draw proposed $\theta^* \sim q(\theta^* | \theta)$
4: Calculate the acceptance probability:
   $$\alpha(\theta \rightarrow \theta^*) = \min \left(1, \frac{f(y|\theta^*)\pi(\theta^*)q(\theta^*|\theta)}{f(y|\theta)\pi(\theta)q(\theta|\theta^*)} \right),$$
5: Set $\theta^t \leftarrow \theta^*$ with probability $\alpha(\theta \rightarrow \theta^*)$, otherwise set $\theta^t \leftarrow \theta^{t-1}$.
6: end for

algorithm, an optimal proposal distribution $q(\theta^* | \theta)$ can lead to better Markov chains that converge more quickly and efficiently explore the parameter space. But it may be costly to tune the proposal distribution. In order to improve the efficiency of MCMC algorithms, adaptive MCMC methods have been studied in the literature (e.g. Haario et al., 2001, 2006; Roberts and Rosenthal, 2009). However, the Markov property of the process may be destroyed with many adaptive strategies and optimal acceptance rates targeted by some adaptive methods are based on strong posterior distribution assumptions, which are unlikely to hold in many applications. Moreover, when the target distribution is multimodal, the conventional MCMC algorithm may become stuck in a local mode for many iterations. An alternative method named sequential Monte Carlo (SMC) was designed, which could be viewed as a complementary method to MCMC. SMC has several advantages over MCMC and a more detailed description of this method is given in the next section.

3.2 Sequential Monte Carlo

3.2.1 Sequential Monte Carlo framework

Here, we are focusing on the use of SMC for static Bayesian models (Chopin, 2002). In SMC, a set of $N$ weighted particles, $\{W_t^i, \theta_t^i\}_{i=1}^N$, are traversed through a sequence of posterior distributions. Therefore, we need to build a sequence of distributions $\pi_t(\theta|y)$ for $t = 0, \ldots, T$, which explore the ultimate target distribution gradually and where $\pi_T(\theta|y)$ equals $\pi(\theta|y)$. One method to construct the sequence of distributions is the data annealing strategy (Chopin, 2002). Under this method the target sequence is created as $\pi_t(\theta|y_{1:t}) \propto f(y_{1:t}|\theta)\pi(\theta)$, where $y_{1:t}$ represents the observation data up to time $t$. Another method called likelihood annealing (Neal, 2001) can also be used to generate the sequence when the whole observed dataset is already available. The sequence formed by the likelihood
The annealing method is given by:

\[
\pi_t(\theta|y) = \frac{f(y|\theta)\gamma_t \pi(\theta)}{Z_t} \propto f(y|\theta)^{\gamma_t} \pi(\theta),
\]

where \(\gamma_t\) is referred to as the temperature and \(0 = \gamma_0 \leq \cdots \leq \gamma_t \leq \cdots \leq \gamma_T = 1\). We use both of the methods in this study, however, the data annealing approach is preferred when data are collected sequentially. A more detailed analysis and comparison between these two sequence constructions is demonstrated in Section 3.2.2. As with MCMC, unbiased likelihood estimators in SMC and exact Bayesian inference (up to Monte Carlo error) can be produced (Chopin et al., 2013; Duan and Fulop, 2015).

In order to generate a properly weighted sample for target \(t\), we need to reweight the particle set at target \(t - 1\), \(\{W_{t-1}^i, \theta_{t-1}^i\}_{i=1}^N\). The reweighting step can be achieved by importance sampling using:

\[
w_t^i = W_{t-1}^i \tilde{w}_t^i = W_{t-1}^i \frac{\eta_t(\theta_{t-1}^i|y)}{\eta_{t-1}(\theta_{t-1}^i|y)},
\]

(Chopin, 2002), for \(i = 1, \ldots, N\) where \(w_t^i\) needs to be normalised, and \(\eta_t(\cdot)\) is the unnormalised version of \(\pi_t(\cdot)\). The so-called incremental weight \(\tilde{w}_t^i\) is calculated by dividing the unnormalised target density at \(t\), \(\eta_t(\theta_{t-1}^i|y)\), by the unnormalised target density at \(t - 1\), \(\eta_{t-1}(\theta_{t-1}^i|y)\). Unfortunately, the reweighting step generally leads to reductions in the weighted samples efficiency, which can be measured by the effective sample size (ESS, Liu (2008)). ESS refers to the number of perfect samples among all the \(N\) generated samples and is computed by:

\[
\text{ESS}_t = \frac{N}{1 + \text{CV}[w_t(\theta)]},
\]

where \(\text{CV}[w_t(\theta)]\) is the coefficient of variation of the unnormalised weights of target \(t\). The ESS can be approximated using the normalized weights as \(1/\sum_{i=1}^N(W_t^i)^2\). Therefore, when the ESS becomes too small, we resample the particle values proportional to their weights. This resets the approximate ESS back to \(N\).

However, the resampling strategy will result in duplication, especially for particle values with relatively large weight. A moving step is required to increase the number of unique particles. The moving step is usually the most computationally expensive part of the SMC process, and the MCMC kernels of invariant distribution \(\pi_t\) are commonly employed in the literature. Generally, in static Bayesian models \(R_t\) iterations of a multivariate normal random walk MCMC kernel (Chopin, 2002) are taken, the approach adopted here is designed to diversify the particles. We propose to use the MH MCMC algorithm in
Algorithm 3.1. It is important to note that the population of particles can be used to automatically tune the covariance matrix of the multivariate normal random walk. As the proposals of particles can be rejected by the MCMC kernel, it is then necessary to repeat the MCMC kernel $R_t$ times to maintain a target level of diversity.

According to Salomone et al. (2018), $R_t$ can be determined adaptively by choosing an optimal tuning scale for the covariance matrix of the MCMC kernel. The optimal scale is selected based on expected square jumping distance (ESJD; Sherlock and Roberts, 2009). We randomly assign each particle one of several tuning scales and we estimate the ESJD for each particle using a single MCMC iteration. The tuning scale which resulted in the highest median ESJD value is selected as optimal. The ESJD is given by:

$$\text{ESJD} = \alpha(\theta^i_t \rightarrow \theta^*) \Lambda^i_t = \alpha(\theta^i_t \rightarrow \theta^*)(\theta^i_t \rightarrow \theta^*)^T \Sigma_t^{-1}(\theta^i_t \rightarrow \theta^*),$$

which is the product of the MH acceptance ratio and the Mahalanobis square jumping distance (Salomone et al., 2018), $i = 1, \ldots, N$, and $t = 0, \ldots, T$. After obtaining the optimal scale, we keep applying an MCMC kernel to all the particles until a target number of particles’ ESJD values are higher than a threshold. We set this threshold at the median value of the Mahalanobis distances between each pair of particles before the move step. As can be seen, the total number of MCMC iterations can be adaptively attained after meeting the threshold. The SMC sampling method is summarized in Algorithm 3.2.

**Algorithm 3.2 SMC Sampler**

1. Sample $\theta^i_0 \sim \pi(\cdot)$ and set $W^i_0 = 1/N$ for $i = 1, \ldots, N$
2. for $t = 1, \ldots, T$ do
3. 
4. 
5. 
6. 
7. if $\text{ESS} < \alpha N$, with $\alpha \in (0, 1]$ then
8. 
9. end if
10. end for
3.2.2 Efficient sequence construction method

Under the SMC approach, the sequence of target distributions can be defined through both likelihood annealing and data annealing. These two annealing methods provide similar estimates of posterior distributions for the parameters of the three GARCH-type models, which are shown in Section 6.1. Under the data annealing strategy, the sequence of target distributions are obtained smoothly when the observations arrive one at a time. The sequence of posterior distributions constructed by using data annealing method are given by:

$$\pi_t(\theta|y_{1:t}) \propto f(y_{1:t}|\theta)\pi(\theta),$$

where $t = 0, \ldots, T$, and $y_{1:t}$ represents the data available up to current time $t$. The corresponding unnormalised weight of the particle $i$ at time $t$ is thus:

$$w_t^i = W_t^{i-1} \frac{f(y_{1:t}|\theta_{t-1}^i)}{f(y_{1:t-1}|\theta_{t-1}^i)} = W_t^{i-1} f(y_t|y_{1:t-1}, \theta_{t-1}^i),$$

which needs to be normalised. It is important to note that to update the weights only the conditional likelihood of the introduced data point needs to be computed. As more returns become available, the re-weighting step in data annealing does not need to re-process previously collected data. This provides extensive computational savings, especially when the time series is long.

The data annealing strategy is also useful for performing posterior prediction. One way to consider if a GARCH-type model is appropriate would be to focus on the model’s ability to forecast. Our Bayesian approach provides a clear method to deduce the posterior predictive distribution conditional on currently available data. The intermediate posterior distribution $\pi_{t-1}(\theta|y_{1:t-1})$ from data annealing is utilised when forecasting one-step ahead posterior prediction for data at time $t$. Meanwhile, the corresponding posterior predictive density that is used to measure the prediction accuracy can be estimated. Section 4 details the procedure of estimating the posterior predictive distribution.

4 Bayesian posterior predictive distribution

In this section, the fully Bayesian and traditional classical approaches are compared in the context of generating one-step ahead forecasts of conditional variances. The distribution of the one-step ahead conditional variances together with 95% prediction intervals are offered by the fully Bayesian approach, which cannot be provided by the classical approach.
We also provide a fully Bayesian approach for assessing time series models’ predictive performance, which can be achieved easily with SMC data annealing.

4.1 Bayesian posterior predictive distribution

Under the fully Bayesian approach, the posterior predictive distribution for the unknown observations can be produced. New observations \( \hat{y}_{t+1} \) can be drawn from the posterior predictive distribution given the original data \( y_{1:t} \) up to date \( t \):

\[
P(\hat{y}_{t+1}|y_{1:t}) = \int_{\theta} P(\hat{y}_{t+1}|\theta, y_{1:t}) \pi_{t}(\theta|y_{1:t}) \, d\theta.
\]

The posterior predictive distribution accounts for the uncertainty associated with the parameters without making asymptotic assumptions. Here, we consider one-step ahead forecasts for the posterior predictions of the conditional variance \( \hat{\sigma}^2_{t+1} \) given its past values and the real data up to time \( t \), \( \sigma^2_{1:t} \) and \( y_{1:t} \).

The posterior predictive distribution is particularly easy to estimate via SMC with data annealing. Denote the SMC weighted sample from the posterior \( \pi_{t}(\theta|y_{1:t}) \) as \( \{W_{i}^{t}, \theta_{i}^{t}\}_{i=1}^{N} \). Then, the posterior predictive distribution can be approximated by \( \{W_{i}^{t}, (\hat{\sigma}^2_{t+1})_{i}^{t}\}_{i=1}^{N} \) where \( (\hat{\sigma}^2_{t+1})_{i}^{t} \sim p(\sigma^2_{t+1}|y_{1:t}, \sigma^2_{1:t}, \theta_{i}^{t}) \) for \( i = 1, \ldots, N \). From the weighted sample, a point estimate (e.g. mean or median) and predictive interval can be easily estimated. The computation associated with constructing the approximation to the predictive posterior can be easily vectorised and parallelised.

4.2 Forecast through maximum likelihood estimation (MLE) approach

We compare the Bayesian SMC approach for forecasting with MLE. In order to form one-step ahead forecasts with MLE, at time \( t \) we firstly find the maximum likelihood estimate \( \theta_{t}^{\text{MLE}} \) based on the observations up to time \( t \). MLE parameter estimates are obtained by maximising each model’s log-likelihood provided in Section 2 via numerical optimisation. After obtaining the MLE parameter estimates at time \( t \), a point estimate of the prediction of the conditional variance from the three GARCH-type models at time \( t + 1 \) can be obtained by utilizing one of Equations (4), (6) or (10) in Section 2.

4.3 Forecast evaluation

In order to assess which model provides more accurate forecasts, we adopt the leave-future-out cross-validation (LFO-CV) method (Bürkner et al., 2019).
state that the common leave-one-out cross-validation (LOO-CV) is not suitable for assessing predictive performance when the data is sequentially ordered in time since future observations may provide some information that affects in-sample model fit, and thus the predictions may be affected if only leaving one observation out at a time. Therefore, when making one-step ahead forecasts at time $t+1$ we use LFO-CV that leaves out all the future values after time $t$ to assess predictive performance for time series models. To measure the predictive accuracy, the expected log posterior density (elpd; Vehtari et al., 2017) is approximated by:

$$\text{elpd}_{\text{LFO}} = \sum_{t=\tau-1}^{T-1} \log p(y_{t+1}|y_{1:t}),$$

where $\tau$ is the time point that we decide to start assessing the predictions, $T$ is the number of observations in the dataset, and the density $p(y_{t+1}|y_{1:t})$ can be approximated by evaluating an estimated density, which is constructed using kernel density estimation based on the weighted sample $\{w_{it}^i, \hat{y}_{it+1}^i\}_{i=1}^N$ at the true observation $y_{t+1}$. The standard approach to LFO-CV would be to re-compute the posterior distribution (referred to here as a refit) as each new data point is introduced, however, this would be computationally intensive. Bürkner et al. (2019) propose an algorithm to approximate LFO-CV and reduce the number of refits reducing computation time. However, our data annealing SMC naturally re-estimates the posterior distribution in a computationally efficient manner as more data are introduced, and hence, we demonstrate it is valuable for performing LFO-CV. The weighted sample from the posterior $\{w_{it}^i, \theta_t^i\}_{i=1}^N$ based on different subsets of observations $y_{1:t}$ is a by-product of posterior simulation via data annealing SMC, which means there is no need to refit the time series model again to perform and test predictions.

A simulation study is performed in Section 6.2 to evaluate the predictive performance for conditional variance and explore the relative merits of the fully Bayesian approach relative to the MLE approach. To assess predictive accuracy we provide an estimated $\text{elpd}_{\text{LFO}}$ and a 95% one-step forward prediction interval for each candidate model with different simulated datasets. We also illustrate that the predictive distribution of the conditional volatility can provide us with different levels of confidence about the magnitude of the volatility at the next time point. We count and record the true conditional variances from the simulated data that fall inside the 2.5% and 97.5% quantiles. We show in Section 6.2 that the posterior predictive interval from each true model, which accounts for all sources of variability in the model, provides a satisfactory coverage of the true values.
5 Model choice

Practitioners are not only interested in single model analysis, but also the relative performance of competing models. Under a fully Bayesian framework, the relative performance of one model against another candidate model is assessed using the Bayes factor (Jeffreys, 1933; Kass and Raftery, 1995). The Bayes factor compares two different models, $m_1$ and $m_2$, with parameter vectors $\theta_{m_1}$ and $\theta_{m_2}$ is given by:

$$BF_{m_1,m_2} = \frac{P(y|m_1)}{P(y|m_2)} = \frac{\int P(y|\theta_{m_1},m_1)P(\theta_{m_1}|m_1)d\theta_{m_1}}{\int P(y|\theta_{m_2},m_2)P(\theta_{m_2}|m_2)d\theta_{m_2}},$$

which is just the ratio of normalising constants under each model. The normalising constant is also referred to as the marginal likelihood or evidence. The larger the evidence in favour of model $m_1$ relative to $m_2$ is, the greater the Bayes factor $BF_{m_1,m_2}$. One challenge of implementing Bayes factor is the intensive computation of the evidence. A review of some commonly used methods of estimating the model evidence is given in Friel and Wyse (2012).

Fortunately, an estimate of the evidence can be obtained as a by-product from SMC methods. The estimate of evidence $Z$ can be computed as $Z = Z_T/Z_0 = \prod_{t=1}^{T} Z_t/Z_{t-1}$ with $Z_0 = 1$ (Del Moral et al., 2006). Under likelihood annealing SMC, the ratio of normalizing constants $Z_t/Z_{t-1}$ is given by:

$$\frac{Z_t}{Z_{t-1}} = \int_\theta f(y|\theta)^{\gamma_t-\gamma_{t-1}} \pi_{t-1}(\theta|y) d\theta.$$

Then the estimate of log evidence $\log Z$ is given by:

$$\log Z = \sum_{t=1}^{T} \log E_{\pi_{t-1}}[f(y|\theta)^{\gamma_t-\gamma_{t-1}}]$$

$$\approx \sum_{t=1}^{T} \log \left( \frac{N}{\sum_{i=1}^{N} w^t_i} \right),$$

where $w^t_i$ is the unnormalised weight from Algorithm 3.2. A similar estimator under data annealing can be easily derived. The estimates of log evidence of the three GARCH models obtained through both likelihood annealing and data annealing SMC are provided in Section 6.3.

Other widely used model selection tools such as the Bayesian information criterion (BIC) (Schwarz et al., 1978), which provides a crude approximation to the logarithm of Bayes factor (Kass and Raftery, 1995), and deviance information criterion (DIC) (Spiegelhalter et al.,
have restrictive assumptions and limitations. For instance, the BIC will behave poorly when the implicit prior assumed by BIC is not a multivariate normal distribution (Kuha, 2004), and DIC is only valid when the posterior can be approximated well with a multivariate normal distribution (Gelman et al., 2014). However, the fully Bayesian approach based on the Bayes factor does not rely on the restrictive approximations. A simulation study provided in Section 6.3 demonstrates the Bayesian model evidence’s ability to select the true model for the motivating GARCH-type models of interest in this paper.

6 Results

The plan of this section is as follows. In Section 6.1 we present results for the posterior distributions obtained by SMC algorithms for the GARCH-type models presented in Section 2. The one-step ahead forecasts of conditional variances by using both classical MLE and Bayesian approaches are illustrated in Section 6.2. The model selection results obtained from the fully Bayesian approach are presented in Section 6.3. Computer code for implementing our methods is available at https://github.com/DanLAct/GARCH-SMC.

In this study we use $N = 10,000$ particles in SMC to generate posterior distributions. The time series data we use are monthly log stock returns for the S&P 500 Composite Index from July 1926 to January 2018 from the Center of Research in Securities Prices (CRSP). The parameters in the GARCH-type models we aim to estimate respectively are $\theta_{\text{Garch}(1,1)} = (\mu^G, \alpha_0^G, \alpha_1, \beta_1)$, $\theta_{\text{GJR}} = (\mu^\text{GJR}, \alpha_0^\text{GJR}, \beta_\sigma, \phi, \phi^-)$, and $\theta_{\text{BEGE}} = (\mu^B, p_0, n_0, \rho_p, \rho_n, \phi^+_p, \phi^-_p, \phi^+_n, \phi^-_n, \sigma_p, \sigma_n)$. Some constraints on the parameters are imposed initially, so that the priors are set as below. The priors on the parameters of the GARCH(1,1) model are given by:

$$
\alpha_0^G \sim U(0, 0.3), \quad \alpha_1 \sim U(0, 0.5),
\beta_1 \sim U(0, 0.99), \quad \mu^G \sim U(-0.9, 0.9),
$$

where $U(\cdot)$ denotes a uniform probability density function. To satisfy stationarity it is necessary to enforce $\alpha_1 + \beta_1 \leq 0.9999$. The priors on the parameters of the GJR-GARCH(1,1) model are given by:

$$
\alpha_0^{\text{GJR}}, \phi, \phi^- \sim U(0, 0.3), \quad \beta_\sigma \sim U(0, 0.99), \quad \mu^{\text{GJR}} \sim U(-0.9, 0.9).
$$
To satisfy stationarity we need $\phi + \frac{1}{2}\phi^{-} + \beta_{\sigma} \leq 0.9999$. The priors on the parameters of the BEGE model are given by:

$$p_{0}, \phi_{p}^{+}, \phi_{p}^{-} \sim U(0, 0.5),$$
$$\sigma_{p}, \sigma_{n} \sim U(0, 0.3), \quad \rho_{p}, \rho_{n} \sim U(0, 0.99),$$
$$n_{0} \sim U(0, 1), \quad \phi_{n}^{+} \sim U(-0.2, 0.1),$$
$$\phi_{n}^{-} \sim U(0, 0.75), \quad \mu^{B} \sim U(-0.9, 0.9).$$

Additionally, to satisfy stationarity we require that $\rho_{p} + \frac{1}{2}\phi_{p}^{+} + \frac{1}{2}\phi_{p}^{-} \leq 0.995$ and $\rho_{n} + \frac{1}{2}\phi_{n}^{+} + \frac{1}{2}\phi_{n}^{-} \leq 0.995$. All parameters are assumed to be independent in the prior.

### 6.1 Posterior Distributions

The univariate posterior distributions obtained by our SMC algorithms for the parameters of the GARCH(1, 1), GJR-GARCH(1, 1) and BEGE models are shown in Figure 1, 2 and 3, respectively. As can be seen in the figures, each parameter’s posterior distribution derived by data annealing SMC is very close to that obtained under likelihood annealing SMC. Except for the parameters of the fixed mean value $\mu = (\mu^{G}, \mu^{GJR})$ under GARCH(1, 1) and GJR-GARCH(1, 1) model, all other posterior distributions demonstrate that they are not symmetric and, in the BEGE case, are far from normally distributed, which contradicts the assumption of normally distributed parameter estimates under standard asymptotic inference.

Figure 1: Marginal posterior distributions of the parameters of the GARCH(1,1) model by using SMC.

Figure 3 demonstrates that the approximate likelihood of $f_{BEGE}(u_{t})$ can lead to biased posteriors, especially for $\sigma_{p}$ and $\mu^{B}$. Even though the unbiased estimators of the likelihood produced by Monte Carlo integration and importance sampling produce similar posterior
Figure 2: Marginal posterior distributions of the parameters of the GJR-GARCH(1,1) model by using SMC.

Figure 3: Marginal posterior distributions of the parameters of the BEGE model by using SMC.
distributions, the importance sampling estimating method is more efficient. To test the
efficiency of these two estimation methods, we choose three sets of parameters and 100
independent likelihood estimates for the two estimators with different Monte Carlo sample
sizes $M$. As shown in Table 1, the Monte Carlo integration requires to draw at least
5,000 times from a density to guarantee the standard deviation is lower than 1, while
the importance sampling only needs 1,000 samples to achieve a similar level of variance.
Therefore, we adopt importance sampling estimator when using SMC to sample from the
posterior for the BEGE model below.

Table 1: Comparison of unbiased Monte Carlo integration estimator and importance sampling
estimator

| Parameters | Number of Monte Carlo Iterations | Standard Deviation |
|------------|---------------------------------|-------------------|
|            | Monte Carlo Integration         | Importance Sampling |
| Parameter Set 1 | 100 | 4.86 | 1.63 |
|             | 1000 | 3.20 | 0.74 |
|             | 5000 | 0.96 | 0.50 |
|             | 10000 | 0.64 | 0.39 |
| Parameter Set 2 | 100 | 6.09 | 1.63 |
|             | 1000 | 1.30 | 0.97 |
|             | 5000 | 0.59 | 0.77 |
|             | 10000 | 0.42 | 0.69 |
| Parameter Set 3 | 100 | 7.4 | 1.45 |
|             | 1000 | 1.47 | 0.84 |
|             | 5000 | 0.59 | 0.62 |
|             | 10000 | 0.41 | 0.55 |

Parameter Set 1: $p_0 = 0.081, \sigma_p = 0.004, \rho_p = 0.857, \phi_p^+ = 0.084, \phi_p^- = 0.151, n_0 = 0.817, \sigma_n = 0.023, \rho_n = 0.466, \phi_n^+ = -0.131, \phi_n^- = 0.323, \mu_B = 0.007$;
Parameter Set 2: $p_0 = 0.049, \sigma_p = 0.006, \rho_p = 0.845, \phi_p^+ = 0.120, \phi_p^- = 0.155, n_0 = 0.700, \sigma_n = 0.022, \rho_n = 0.571, \phi_n^+ = -0.180, \phi_n^- = 0.475, \mu_B = 0.009$;
Parameter Set 3: $p_0 = 0.122, \sigma_p = 0.006, \rho_p = 0.837, \phi_p^+ = 0.108, \phi_p^- = 0.155, n_0 = 0.792, \sigma_n = 0.023, \rho_n = 0.557, \phi_n^+ = -0.183, \phi_n^- = 0.433, \mu_B = 0.008$.

The likelihood function for the BEGE model is computationally expensive. We signifi-
cantly accelerate the likelihood calculations for the proposed parameters of all particles
in one iteration of the MCMC step of the SMC algorithm by taking advantage of parallel
computing. More specifically, we allocate a set of these proposed parameters to each core
on a multi-core machine. Using the importance sampling estimator of the likelihood
and 12 cores, the SMC algorithm with data annealing applied to the BEGE model runs 6.7
times faster than a completely serial SMC algorithm. Furthermore, the likelihood anneal-
ing SMC applied to the BEGE model runs 6.5 times faster than a completely serial SMC
algorithm.

Figure 4 shows the posterior distributions for the BEGE model explored by using MH MCMC and SMC methods. The MH MCMC requires tuning of the proposal distribution. To make implementing the MH MCMC easier, the final population of SMC particles are used to form the proposal distribution. More specifically, we use a multivariate normal random walk with a covariance estimated from the SMC particles. However, even though we give MH MCMC an advantage by avoiding the tuning step, the posteriors obtained by MCMC still perform poorly compared with the ones from SMC methods in Figure 4. For instance, the posterior distribution for parameter $\sigma_p$ obtained by MCMC has an obvious unexpected bump, and the trace plot in Figure 5 shows that MCMC can get stuck and the Markov chain explores the parameter space slowly. The runtime of the MH MCMC algorithm with 1 million iterations is more than 2.2 times slower than the data annealing SMC that exploits parallel computing.

![Marginal posterior distributions of the parameters of the BEGE model by using SMC and MCMC methods for real data.](image)

Figure 4: Marginal posterior distributions of the parameters of the BEGE model by using SMC and MCMC methods for real data.

Then both the MH MCMC and SMC methods are used to sample the posterior distributions for the BEGE model with simulated data, and the results are shown in Figure 6. We choose the posterior point estimates with the highest likelihood from the results of the SMC method with real data to be the initial values for the MCMC, which are the red dashed lines in Figure 6. As can be seen, the estimates of posterior distribution produced by MH MCMC are less smooth than SMC and the point estimates used to generate sim-
Figure 5: Trace plots and marginal posterior distributions of the parameters of the BEGE model by using MCMC method for real data.

Simulated data are also contained by the posterior distributions. This simulation study can be viewed as a validation of the effectiveness of the proposed SMC methods.

Figure 6: Marginal posterior distributions of the parameters of the BEGE model by using SMC and MCMC methods for simulated data.
6.2 Posterior Predictive Distribution

In this section we perform a simulation study which shows the relative merits of the Bayesian approach over the classical approach for forecasting the conditional variance. Firstly, we simulate the conditional variance of stock returns $\sigma^2_{1:1099}$ from the GARCH(1, 1) model, the GJR-GARCH(1, 1) model and the BEGE model, respectively. The parameters used to generate the three simulated data sets are randomly selected from the final population of SMC particles when modelling real time series data with the three GARCH-type models. The initial values of conditional variance at time $t = 1$ of the three GARCH-type models are $(\sigma_1^2)_{\text{GARCH}} = 0.0026$, $(\sigma_1^2)_{\text{GJR-GARCH}} = 0.0022$ and

$$(\sigma_1^2)_{\text{BEGE}} = \sigma_p^2 + \sigma_n^2 = 0.0013.$$  

The total number of simulated data in our study is 1099. We consider the performance of one-step ahead prediction based on simulated conditional variances $t = 201$ to $t = 1099$. Besides using the true model to fit the simulated data, we also perform a study of model misspecification by using misspecified models to fit the simulated data and make predictions.

Here, we present the results for the one-step ahead forecasts of conditional variances of the three GARCH-type models obtained via data annealing SMC and MLE approaches.
In order to provide a clearer visual interpretation, in Figure 7 we illustrate the results for forecasted conditional volatilities, which are the square roots of the forecasted conditional variances. Figure 7 shows the 95% prediction interval bounds for the last 100 predicted volatilities \( \hat{\sigma}_{1000:1099} \) simulated from the three GARCH-type models by using our Bayesian approach. The 95% prediction interval is built simply by computing the 0.025 and 0.975 quantiles of the square roots of the posterior predictive sample of the conditional variances. However, the classical approach cannot produce such prediction intervals for the conditional volatility. It is obvious that these prediction intervals capture the simulated data successfully. We take posterior medians as point estimators for the Bayesian approach. After illustrating these posterior medians together with the classical point estimators of three GARCH-type models in Figures 7, we can see that the differences between the Bayesian and classical point estimators of predictions are small.

Table 2: Frequency of simulated conditional variances falling outside 95% prediction interval

| Model               | Data          | Simulated Data from GARCH(1, 1) | Simulated Data from GJR-GARCH(1, 1) | Simulated Data from BEGE |
|---------------------|---------------|---------------------------------|-------------------------------------|--------------------------|
| GARCH(1, 1)         | 0.9%          | 34.9%                           | 47.8%                               |
| GJR-GARCH(1, 1)     | 8.8%          | 0.8%                            | 28.4%                               |
| BEGE                | 2.8%          | 0.1%                            | 1.2%                                |

The 95% Bayesian prediction intervals for the one-step ahead forecasted conditional variance of the three GARCH-type models contain true conditional variances with a satisfactory coverage level. Besides checking the accuracy of Bayesian predictive distributions for conditional volatility by using the true model, we also use misspecified models to fit the simulated data and compare their performance with the true model. Table 2 summarises the frequency of true values of simulated conditional variances \( t = 201 \) to \( t = 1099 \) falling outside each model’s 95% prediction interval. For the data simulated from the BEGE model, there are 1.2% of true squared volatilities falling outside the predicted interval for the true BEGE model, but the other two misspecified models provide poor predictive performance. For the data simulated from the GARCH(1, 1) model and GJR-GARCH(1, 1) model, there are less than 1% of the true conditional variances outside the associated 95% prediction interval for these true models. Overall, the Bayesian prediction intervals demonstrate reasonable coverage rates of true conditional volatilities for the three GARCH-type models in the simulation study, and such a conclusion is reinforced by considering each model’s estimated \( \text{elpd}_{\text{LFO}} \) in Table 3. The model with the highest value of \( \text{elpd}_{\text{LFO}} \) is the one that has the best predictive performance. We can see that there is an agreement
between the results of elpd_{LFO} and coverage rates of the 95% prediction intervals used to assess candidate models’ predictive performance. It is worth noting that the BEGE model performs well for forecasting conditional variances under the dataset simulated from the GJR-GARCH(1, 1) model, which indicates that the performance of the BEGE model is robust to model misspecification in this respect. Therefore, even though the real volatility of the real stock returns is unobservable, this simulation study provides some confidence that the real volatility can be adequately captured and covered by the prediction interval with adoption of the Bayesian approach.

Table 3: Approximated elpd_{LFO} for one-step ahead forecasts of simulated conditional variances from \( t = 201 \) to \( t = 1099 \)

| Model        | Data from GARCH(1,1) | Simulated Data from GJR-GARCH(1,1) | Simulated Data from BEGE |
|--------------|----------------------|-----------------------------------|--------------------------|
| GARCH(1, 1)  | 6531.4               | 3895.3                            | -847.9                   |
| GJR-GARCH(1, 1) | 6278.8             | 6364.0                            | 5492.9                   |
| BEGE         | 6364.4               | 6393.9                            | 6188.1                   |

Figure 8 shows the time series for real stock returns and one-step ahead forecasts for conditional variances obtained via the Bayesian approach. The Bayesian posterior median value of conditional variance is set as the point estimate of one-step ahead prediction. It is immediately apparent that the Bayesian approach can provide efficient predictions for the three GARCH-type models, which successfully capture the features of the data such as the evident growth of volatility following the negative shock of 1987 crash.

Another promising aspect of the Bayesian approach is reflected in the provision of the predictive distribution of conditional volatility, which quantifies the uncertainty in the one-step ahead predicted volatility. However, this predictive distribution cannot be analytically derived. From the predictive distribution, more information about the one-step forward unobserved volatility can be extracted. For instance, the probability of the conditional volatility at the next time point being within any particular range can be computed. Figure 9 (a) shows a distribution of the predicted conditional volatility of the GJR-GARCH(1, 1) model at time \( t = 586 \) conditional on real data \( t = 1 \) to \( t = 585 \). From this distribution, we can see that the conditional volatilities with values around 0.076 have a high probability of being observed. In contrast, the predictive distribution in Figure 9 (b) indicates that there is a very low chance of the conditional volatility being greater than 0.076, but it shows a high chance of a relatively low conditional volatility being observed at time \( t = 743 \) from the BEGE model. Figure 9 (c) shows that there is a lot of uncertainty
Figure 8: One-step ahead Bayesian forecasts for conditional variance of GARCH(1, 1), GJR-GARCH(1, 1) and BEGE with real time series data by using the Bayesian approach. The prediction period is from March 1943 to January 2018. The top panel shows monthly log-return series, and the bottom panel shows the results of one-step ahead forecasted conditional variances from the three GARCH-type models.

Figure 9: Predictive distribution for real data. (a) One-step ahead predicted conditional volatility at time $t = 586$ of GJR-GARCH(1, 1) model; (b) One-step ahead predicted conditional volatility at time $t = 743$ of BEGE model; (c) One-step ahead predicted conditional volatility at time $t = 580$ of BEGE model.
about the relatively high volatility at time $t = 580$.

### 6.3 Model Choice

In this section we perform a simulation study to assess the performance of the SMC estimator of the Bayesian model evidence for model selection for the GARCH-type models. We first simulate five sets of data from each competing model of interest, producing 15 datasets in total. The associated parameter values used to generate simulated data are randomly selected from the SMC posterior samples of the three GARCH-type models in Section 6.1. After obtaining simulated data, we use SMC to estimate the log evidence for each candidate model given different sets of simulated data.

#### Table 4: Estimation of Log Evidence

| Model          | Data | 1    | 2    | 3    | 4    | 5    |
|----------------|------|------|------|------|------|------|
| GARCH(1, 1)    |      | 1811.4 | 1578.5 | 1843.6 | 1936.2 | 1728.8 |
| GJR-GARCH(1, 1)|      | 1810.2 | 1576.6 | 1842.1 | 1934.2 | 1727.9 |
| BEGE           |      | 1805.1 | 1572.5 | 1837.8 | 1930.1 | 1719.9 |

| Model          | Data | 1    | 2    | 3    | 4    | 5    |
|----------------|------|------|------|------|------|------|
| GARCH(1, 1)    |      | 1716.0 | 1804.1 | 1941.5 | 2017.7 | 1778.5 |
| GJR-GARCH(1, 1)|      | 1730.1 | 1811.1 | 1952.6 | 2021.2 | 1784.0 |
| BEGE           |      | 1723.8 | 1805.4 | 1943.7 | 2014.3 | 1777.0 |

| Model          | Data | 1    | 2    | 3    | 4    | 5    |
|----------------|------|------|------|------|------|------|
| GARCH(1, 1)    |      | 1931.0 | 1856.1 | 1819.2 | 1876.5 | 1786.4 |
| GJR-GARCH(1, 1)|      | 1945.2 | 1865.2 | 1831.0 | 1886.9 | 1803.9 |
| BEGE           |      | 1978.8 | 1881.4 | 1873.1 | 1915.7 | 1862.6 |

| Model          | Data Annealing SMC | Likelihood Annealing SMC |
|----------------|--------------------|--------------------------|
| GARCH(1, 1)    | 1811.5             | 1811.4                   |
| GJR-GARCH(1, 1)| 1817.5             | 1818.0                   |
| BEGE           | 1845.4             | 1844.9                   |

The results of each model’s estimation of log evidence given the simulated data are presented in Table 4. As the data generating process (DGP) is known, the performance of the evidence can be assessed by checking if the true model is correctly selected according to the estimated value of the log evidence. As can be seen from Table 4, for
each simulated dataset the model with the highest estimated log evidence corresponds to
the DGP. This demonstrates the effectiveness of the fully Bayesian approach for model
choice and the SMC evidence estimator for the GARCH-type models and length of time
series investigated here. It is worth noting that for simulated data from the GARCH(1, 1)
model, the GJR-GARCH(1, 1) model has only a slightly smaller log evidence value com-
pared to the GARCH(1, 1) model. However, when data is generated from the GJR-
GARCH(1, 1) model, the log evidence of the GARCH(1, 1) model is much smaller than the
GJR-GARCH(1, 1) model. When the GARCH(1, 1) model is true, there is a possibility
that the GJR-GARCH(1, 1) model can mimic the behaviour of the GARCH(1, 1) model
since the GARCH(1, 1) model is nested within the GJR-GARCH(1, 1) model.

Moreover, each model’s estimated log evidence for real time series data is also provided
in Table 4. There is strong evidence that the BEGE model is pre ferred over the other two
GARCH-type models for this dataset. This is consistent with the result from Bekaert et al.
(2015) and reflects the fact that the BEGE model endowed with non-Gaussian innovations
outperforms the traditional GARCH model and the standard asymmetric GJR-GARCH
model when applied to equity market returns.

7 Discussion

In this article, a novel Bayesian framework for the analysis of GARCH-type models
has been developed, that allows for parameter estimation, model selection and prediction.
To this end, an efficient fully Bayesian approach has been developed using SMC. To obtain
exact Bayesian inference for the BEGE model, we proposed to use an unbiased likelihood
estimator in place of the biased approximation adopted in Bekaert et al. (2013). This
approach, in comparison to traditional estimation inference and forecasting techniques,
ofers a number of important advantages. Overall, in contrast to traditional methods, the
Bayesian framework offers a direct method for quantifying the uncertainty surrounding
both model parameter estimates and volatility forecasts. It is shown that SMC offers a
number of advantages over standard Monte-Carlo methods. For such applications on long
time series data, the SMC approach proposed here offers an efficient method generating
the full predictive distribution of volatility, computation of the evidence for model com-
parison and undertaking leave-one-out cross-validation. Although this study has focused
on GARCH-type models, the benefits of the proposed approach would translate to other
time series models for alternative economic or financial problems. We have demonstrated
here that SMC data annealing can be an efficient approach for comparing the predictive
performance of time series models. An interesting avenue for future research would be to explore this idea more comprehensively across different time series models and to draw comparisons with the approximate approach of Bürkner et al. (2019).

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Appendix: Importance sampling estimator of likelihood of the BEGE model

As described in Section 2.4.1, we need an importance distribution that will produce a low variance estimator of the likelihood with a small Monte Carlo sample size. This appendix shows that the likelihood requires to be estimated only when both $p_t > 1$ and $n_t > 1$, and in other cases an exact likelihood can be computed. For simplicity, we only consider here the likelihood for a single observation. Four cases are considered to compute the likelihood $f_{\text{BEGE}}(u)$ with the following expression

$$f_{\text{BEGE}}(u) = \int_{\omega_p} f_{\omega_n}(\omega_p - u) f(\omega_p) \, d\omega_p,$$

where

$$f_{\omega_p}(\omega_p) = \frac{1}{\Gamma(p)\sigma_p^p} (\omega_p - \overline{\omega}_p)^{p-1} \exp \left( -\frac{1}{\sigma_p} (\omega_p - \overline{\omega}_p) \right), \text{ for } \omega_p > \overline{\omega}_p$$

$$f_{\omega_n}(\omega_p - u) = \frac{1}{\Gamma(n)\sigma_n^n} (\omega_p - u - \overline{\omega}_n)^{n-1} \exp \left( -\frac{1}{\sigma_n} (\omega_p - u - \overline{\omega}_n) \right), \text{ for } \omega_p - u > \overline{\omega}_n$$

where $p \geq 1$, $n \geq 1$, $\overline{\omega}_p = -p\sigma_p$, and $\overline{\omega}_n = -n\sigma_n$. The lower limit of integration in the expression for $f_{\text{BEGE}}(u)$ is $\omega_p$, which must be greater than $\overline{\omega}_p$ and $\overline{\omega}_n + u$. Therefore, we define the lower limit of integration as $\delta = \max(\overline{\omega}_p, \overline{\omega}_n + u)$.

Firstly, define $\sigma = \frac{1}{\sigma_p} + \frac{1}{\sigma_n}$. When $p = n = 1$ an exact expression for $f_{\text{BEGE}}(u)$ can
be found by

\[
f_{\text{BEGE}}(u) = \int_{\delta}^{\infty} \frac{1}{\sigma_p} \exp \left( -\frac{1}{\sigma_p} (\omega_p - \bar{\omega}_p) \right) \frac{1}{\sigma_n} \exp \left( -\frac{1}{\sigma_n} (\omega_n - (u + \bar{\omega}_n)) \right) d\omega_p
\]

\[
= k_1 \int_{\delta}^{\infty} \exp (-\sigma \omega_p) d\omega_p
\]

\[
= k_1 \frac{1}{\sigma} \exp (-\sigma \delta),
\]

where \( k_1 = \frac{1}{\sigma_p \sigma_n} \exp \left( \frac{\bar{\omega}_p}{\sigma_p} \right) \exp \left( \frac{u + \bar{\omega}_n}{\sigma_n} \right) \).

Then, the log-likelihood of the BEGE model in this case is

\[
\log f_{\text{BEGE}}(u) = \log k_1 - \log \sigma - \sigma \delta.\]

When \( p = 1 \) and \( n > 1 \), an exact expression for the likelihood can be found as

\[
f_{\text{BEGE}}(u) = \int_{\delta}^{\infty} \frac{1}{\sigma_p} \exp \left( -\frac{1}{\sigma_p} (\omega_p - \bar{\omega}_p) \right) \frac{1}{\Gamma(n)\sigma_n} (\omega_p - u - \bar{\omega}_n)^{n-1} \exp \left( -\frac{1}{\sigma_n} (\omega_p - u - \bar{\omega}_n) \right) d\omega_p,
\]

define \( k_2 = \frac{1}{\sigma_p \Gamma(n)\sigma_n} \exp \left( \frac{\bar{\omega}_p}{\sigma_p} \right) \), and \( c_1 = k_2 \exp \left( -\frac{1}{\sigma_p} (u + \bar{\omega}_n) \right) \), then the likelihood can be written as

\[
f_{\text{BEGE}}(u)
\]

\[
= \int_{\delta}^{\infty} k_2 \exp \left( -\frac{1}{\sigma_n} (\omega_p - u - \bar{\omega}_n) \right) \exp \left( -\frac{\omega_p}{\sigma_p} + \frac{u + \bar{\omega}_n}{\sigma_p} - \frac{u + \bar{\omega}_n}{\sigma_p} \right) (\omega_p - u - \bar{\omega}_n)^{n-1} d\omega_p
\]

\[
= \int_{\delta}^{\infty} c_1 \exp (-\sigma (\omega_p - u - \bar{\omega}_n)) (\omega_p - u - \bar{\omega}_n)^{n-1} d\omega_p
\]

\[
= c_1 \frac{\Gamma(n)}{\sigma p} \int_{\delta}^{\infty} \frac{\sigma_n}{\Gamma(n)} (\omega_p - u - \bar{\omega}_n)^{n-1} \exp (-\sigma (\omega_p - u - \bar{\omega}_n)) d\omega_p.
\]

We set \( x_p = \omega_p - u - \bar{\omega}_n = \omega_p - (u + \bar{\omega}_n) \), which gives \( \omega_p = x_p + u + \bar{\omega}_n \) and \( d\omega_p = dx_p \).

The likelihood then can be written as

\[
f_{\text{BEGE}}(u) = c_1 \frac{\Gamma(n)}{\sigma p} \int_{\delta - (u + \bar{\omega}_n)}^{\infty} \frac{\sigma_n}{\Gamma(n)} x_p^{n-1} \exp (-\sigma x_p) dx_p
\]

\[
= \begin{cases} 
\frac{\Gamma(n)}{\sigma p}, & \text{if } \delta = u + \bar{\omega}_n \\
\frac{\Gamma(n)}{\sigma p} \left[ 1 - F(\bar{\omega}_p - (u + \bar{\omega}_n), n, \sigma) \right], & \text{if } \delta = \bar{\omega}_p,
\end{cases}
\]

where \( F \) is the CDF of a gamma distribution with parameters \( n \) and \( \sigma \) over the interval of \((0, \bar{\omega}_p - (u + \bar{\omega}_n))\). Similarly, in the case of \( n = 1 \) and \( p > 1 \) an exact expression for the likelihood can be found as

\[
f_{\text{BEGE}}(u) = \int_{\delta}^{\infty} \frac{1}{\sigma_n} \exp \left( -\frac{1}{\sigma_n} (\omega_p - (u + \bar{\omega}_n)) \right) \frac{1}{\Gamma(p)\sigma_p} (\omega_p - \bar{\omega}_p)^{p-1} \exp \left( -\frac{1}{\sigma_p} (\omega_p - \bar{\omega}_p) \right) d\omega_p,
\]

define \( k_3 = \frac{1}{\sigma_p \Gamma(p)\sigma_p} \exp \left( \frac{u + \bar{\omega}_n}{\sigma_p} \right) \) and \( c_2 = k_3 \exp \left( -\frac{\bar{\omega}_n}{\sigma_p} \right) \), then the likelihood can be
written as

\[ f_{\text{BEGE}}(u) = \int_{\delta}^{\infty} k_3 \exp \left( -\frac{1}{\sigma_p} (\omega_p - \overline{\omega}_p) \right) \exp \left( -\frac{\omega_p}{\sigma_n} + \frac{\overline{\omega}_p}{\sigma_n} - \frac{\overline{\omega}_n}{\sigma_n} \right) (\omega_p - \overline{\omega}_n)^{p-1} d\omega_p \]

\[ = \int_{\delta}^{\infty} c_2 (\omega_p - \overline{\omega}_n)^{p-1} \exp (-\sigma (\omega_p - \overline{\omega}_p)) d\omega_p \]

\[ = c_2 \frac{\Gamma(p)}{\sigma_p} \int_{\delta}^{\infty} \frac{\sigma_p}{\Gamma(p)} (\omega_p - \overline{\omega}_n)^{p-1} \exp (-\sigma (\omega_p - \overline{\omega}_p)) d\omega_p. \]

We set \( x_p = \omega_p - \overline{\omega}_p \), which gives \( \omega_p = x_p + \overline{\omega}_p \) and \( d\omega_p = dx_p \). The likelihood then can be written as

\[ f_{\text{BEGE}}(u) = c_2 \frac{\Gamma(p)}{\sigma_p} \int_{\delta-x_p}^{\infty} \frac{\sigma_p}{\Gamma(p)} x_p^{p-1} \exp (-\sigma x_p) dx_p \]

\[ = \begin{cases} 
    c_2 \frac{\Gamma(p)}{\sigma_p}, & \text{if } \delta = \overline{\omega}_p \\
    c_2 \frac{\Gamma(p)}{\sigma_p} [1 - F(u + \overline{\omega}_n - \overline{\omega}_p, p, \sigma)], & \text{if } \delta = u + \overline{\omega}_n,
\end{cases} \]

where \( F \) is the CDF of a gamma distribution with parameters \( p \) and \( \sigma \) over the interval of \((0, u + \overline{\omega}_n - \overline{\omega}_p)\).

In the case of \( p > 1 \) and \( n > 1 \), we use importance sampling method to estimate likelihood of the BEGE model. Here we attempt to find a parametric importance distribution \( g(\omega_p) \) that matched closely the target distribution \( h(\omega_p) \). Firstly, we determine the mode and curvature of \( h(\omega_p) \). The expression for \( h(\omega_p) \) is given by

\[ h(\omega_p) = \frac{1}{\Gamma(p)\sigma_p^p} (\omega_p - \overline{\omega}_p)^{p-1} \exp \left( -\frac{1}{\sigma_p} (\omega_p - \overline{\omega}_p) \right) \frac{1}{\Gamma(n)\sigma_n^n} (\omega_p - u - \overline{\omega}_n)^{n-1} \exp \left( -\frac{1}{\sigma_n} (\omega_p - u - \overline{\omega}_n) \right). \]

We set \( m(\omega_p) = \log [h(\omega_p)] \), which is given by

\[ m(\omega_p) = -\log \Gamma(p) - p \log \sigma_p + (p-1) \log (\omega_p - \overline{\omega}_p) - \frac{1}{\sigma_p} (\omega_p - \overline{\omega}_p) \]

\[ - \log \Gamma(n) - n \log \sigma_n + (n-1) \log (\omega_p - u - \overline{\omega}_n) - \frac{1}{\sigma_n} (\omega_p - u - \overline{\omega}_n). \]

The first and second derivatives of \( m(\omega_p) \) are given by

\[ m'(\omega_p) = \frac{p - 1}{\omega_p - \overline{\omega}_p} + \frac{n - 1}{\omega_p - u - \overline{\omega}_n} - \sigma, \]

\[ m''(\omega_p) = \frac{1 - p}{(\omega_p - \overline{\omega}_p)^2} + \frac{1 - n}{(\omega_p - u - \overline{\omega}_n)^2}. \]
The mode $\hat{\omega}_p$ of $m(\omega_p)$ can be found by setting $m'(\omega_p) = 0$, then we get

$$(p - 1)(\omega_p - u) + (n - 1)(\omega_p - \bar{\omega}_n) - \sigma(\omega_p - \bar{\omega}_p)(\omega_p - u - \bar{\omega}_n) = 0,$$

$$\sigma\omega_p^2 + [-\sigma(u + \bar{\omega}_n + \bar{\omega}_p) + (2 - n - p)]\omega_p + [\sigma\bar{\omega}_p(u + \bar{\omega}_n) + (p - 1)(u + \bar{\omega}_n) + (n - 1)\bar{\omega}_n] = 0.$$

By setting $a = \sigma$, $b = -\sigma(u + \bar{\omega}_n + \bar{\omega}_p) + (2 - n - p)$, and $c = \sigma\bar{\omega}_p(u + \bar{\omega}_n) + (p - 1)(u + \bar{\omega}_n) + (n - 1)\bar{\omega}_n$, then we can write $a\omega_p^2 + b\omega_p + c = 0$ and the mode $\hat{\omega}_p = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ can be easily derived. The variance of $m(\omega_p)$ is estimated by using the second-derivative of $m(\omega_p)$ around the mode $\hat{\omega}_p$, which is given by

$$\hat{V} = -\left[\frac{1 - p}{(\hat{\omega}_p - \bar{\omega}_p)^2} + \frac{1 - n}{(\hat{\omega}_p - u - \bar{\omega}_n)^2}\right]^{-1}.$$

We propose a gamma distribution, $X \sim \Gamma(i, j)$ for $x > 0$, with mode $\hat{\omega}_p$ and variance $\hat{V}$ to be the importance distribution. The mode of this gamma distribution is $(i - 1)j$ and the variance is $ij^2$. As $\omega_p > \delta$, we need to make the importance distribution greater than $\delta$ and a shift gamma distribution, $Y = X + \delta$ for $y > \delta$, is selected. As $x = y - \delta$, we can show that $|\frac{dx}{dy}| = 1$. Then an expression for the density of this importance distribution $g(y)$ is given by

$$g(y) = f_X(x(y))\left|\frac{dx}{dy}\right| = \frac{1}{j^\Gamma(i)}(y - \delta)^{i-1}\exp\left(-\frac{y - \delta}{j}\right),$$

where the mode of this distribution is $(i - 1)j + \delta$ and the variance is $ij^2$. The value of parameter $i$ and $j$ can be derived by solving $(i - 1)j + \delta = \hat{\omega}_p$ and $ij^2 = \hat{V}$, and the results are given by

$$i = \frac{-b_i + \sqrt{b_i^2 - 4\hat{V}^2}}{2\hat{V}}, \text{ where } b_i = -2\hat{V} - (\hat{\omega}_p - \delta)^2,$$

$$j = \frac{\hat{\omega}_p - \delta}{i - 1}.$$

After obtaining the importance distribution $g(\cdot)$, we can unbiasedly estimate $f_{BEGE}(u)$ by following Equation 15. It is worth noting that this importance distribution can be used only when the mode $\hat{\omega}_p$ is greater than $\delta$, which is the lower limit of integration in the expression for $f_{BEGE}(u)$. For this case, it can be proved that $\hat{\omega}_p$ is always greater than $\delta$. 

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and the details of proof are shown below.

(1) when $\delta = \overline{w}_p > (u + \overline{w}_n),$

$$(2a\delta + b)^2 = (2\sigma\overline{w}_p + b)^2$$

$$= b^2 + 4\sigma^2\overline{w}_p^2 + 4\sigma\overline{w}_p[-\sigma(u + \overline{w}_n + \overline{w}_p) + (2 - n - p)]$$

$$= b^2 + 4\sigma^2\overline{w}_p^2 - 4\sigma^2\overline{w}_p^2 - 4\sigma^2\overline{w}_p(u + \overline{w}_n) + 4\sigma\overline{w}_p(2 - n - p)$$

$$= b^2 - [4\sigma^2(u + \overline{w}_n)\overline{w}_p + 4\sigma(n - 1)\overline{w}_p + 4\sigma(p - 1)\overline{w}_p]$$

$$= b^2 - 4ad, \text{ where } d = \sigma(u + \overline{w}_n)\overline{w}_p + (n - 1)\overline{w}_p + (p - 1)\overline{w}_p,$$

∴ $c = \sigma(u + \overline{w}_n)\overline{w}_p + (n - 1)\overline{w}_p + (p - 1)(u + \overline{w}_n), \text{ and } \overline{w}_p > (u + \overline{w}_n)$

∴ $d > c, 4ad > 4ac, b^2 - 4ad < b^2 - 4ac, (2a\delta + b)^2 < b^2 - 4ac, 2a\delta + b < \sqrt{b^2 - 4ac},$

$$\delta < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\delta < \hat{\omega}_p;$$

(2) when $\delta = (u + \overline{w}_n) > \overline{w}_p,$

$$(2a\delta + b)^2 = (2\sigma(u + \overline{w}_n) + b)^2$$

$$= b^2 + 4\sigma^2(u + \overline{w}_n)^2 + 4\sigma(u + \overline{w}_n)[-\sigma(u + \overline{w}_n + \overline{w}_p) + (2 - n - p)]$$

$$= b^2 + 4\sigma^2(u + \overline{w}_n)^2 - 4\sigma^2(u + \overline{w}_n)^2 - 4\sigma^2\overline{w}_p(u + \overline{w}_n) - 4\sigma(u + \overline{w}_n)(n + p - 2)$$

$$= b^2 - 4\sigma[(u + \overline{w}_n)\overline{w}_p + (n - 1)(u + \overline{w}_n) + (p - 1)(u + \overline{w}_n)]$$

$$= b^2 - 4ae, \text{ where } e = \sigma(u + \overline{w}_n)\overline{w}_p + (n - 1)(u + \overline{w}_n) + (p - 1)(u + \overline{w}_n),$$

∴ $c = \sigma(u + \overline{w}_n)\overline{w}_p + (n - 1)\overline{w}_p + (p - 1)(u + \overline{w}_n), \text{ and } (u + \overline{w}_n) > \overline{w}_p$

∴ $e > c, 4ae > 4ac, b^2 - 4ae < b^2 - 4ac, (2a\delta + b)^2 < b^2 - 4ac, 2a\delta + b < \sqrt{b^2 - 4ac},$

$$\delta < \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\delta < \hat{\omega}_p.$$

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