Toric ideals with linear components: an algebraic interpretation of clustering the cells of a contingency table

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Abstract

In this paper we show that the agglomeration of rows or columns of a contingency table with a hierarchical clustering algorithm yields statistical models defined through toric ideals. In particular, starting from the classical independence model, the agglomeration process adds a linear part to the toric ideal generated by the $2 \times 2$ minors.

Keywords: independence model, toric ideals, toric statistical models

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1. Introduction

This paper aims to provide a geometric understanding of the clustering techniques for discrete data, and in particular for contingency tables. A two-way contingency table is an integer data table which collects the outcomes of two categorical random variables, i.e., it is a rectangular table of non-negative integer numbers. Contingency table are widely used in Statistics and the study of statistical models for this kind of data structures is an active research area. Especially when the contingency table is large, a natural problem is

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to ask whether there is some ways for simplifying the data structure losing as few information as possible. Therefore, a natural issue in that context is the problem of clustering the rows, respectively the column, of a contingency table, i.e. to find groups, respectively columns, with similar behavior.

In the literature, several techniques are studied in order to cluster the rows and columns of a contingency table. As the number of rows and columns is usually moderately small, most of the existing algorithms fall in the class of hierarchical algorithms. Among the most relevant references, we cite here the paper [1], where the clustering is defined in connection with Correspondence Analysis, and [2], where the clustering is endowed with suitable probability distributions, essentially derived from the chi-square and the Wishart densities. Such techniques are based on the spectral decomposition of a special matrix derived from the observed data. In terms of statistical models, they implicitly assume an underlying independence model, i.e., they first assume that the two random variables are independent, and then find discrepancies between the observed counts and the expected counts under independence.

In the papers cited above, the reader can find several real-data examples and a thorough description of the clustering process.

Since the definition of the algorithms is made in terms of the chi-squared distance inherited from the exploratory techniques, and in particular from Correspondence Analysis, there is a lack of study of the geometric description of the statistical models underlying the agglomerative process.

Here we use the language and tools from Algebraic Statistics in order to determine under which conditions the clustering has a natural algebraic and geometric counterpart. A survey of Algebraic Statistics can be found in the book [3]. The use of Algebraic Statistics for understanding the geometric structure of the statistical models has already been considered in other papers. Indeed, the availability of Computer Algebra systems and efficient packages for polynomial computation have led to the study of complex models for contingency tables. For instance, [4] defines weakened independence models using suitable sets of minors, while [5] adds a new condition on a subtable to the standard independence model. In [6] a modification of independence model to encode a special behavior of the diagonal cells of a square table is presented, while [7] defines statistical models to encode the notion of outliers and patterns of outliers in contingency tables through a model-based approach.

We collect some basic facts about the analysis of contingency tables within Algebraic Statistics in the next sections, focusing especially on the represen-
tation of a log-linear model in terms of toric ideals. We also prove that under the independence model, the action of merging two rows or two columns in a single cluster produce a new statistical model for smaller tables which falls again into the class of independence models. To do that, we prove and apply some results about toric ideals. In particular, we study how the ideal of the toric variety is affected by special properties of and operations on the design matrix. As a general reference to basic Algebraic Geometry we suggest [8] while for the basic ideas about toric ideals we refer to [9].

In this paper we show that: (i) the algebraic and geometric structure of the independence model is a natural framework for the clustering of the rows and columns of a contingency table; (ii) merging two columns or two rows of the table it is possible to characterize the induced toric model; (iii) merging a row with a column does not produce an easily interpretable statistical model.

The paper is structured as follows. In Section 2 we briefly recall the basic properties of toric ideals that we need. In Section 3 we describe how toric ideals appear in the study of statistical models. Section 4 is devoted to give the statistical motivation for our study which arise from the clustering algorithms of the rows or columns of a contingency table. In Section 5 we collect our main results while Section 6 contains pointer to further studies.

2. Toric varieties and toric ideals

In this paper we will work with toric varieties and with toric ideals, thus we recall some of the basic notions that we will need.

Given a $k \times (t+1)$ matrix $A = (a_{i,j})$ with non-negative integer entries, this can be used to define a map $T : \mathbb{R}^{t+1} \rightarrow \mathbb{R}^k$ in the following way

$$T(\zeta_0, \ldots, \zeta_t) =$$

$$(\zeta_0^{a_{1,1}} \zeta_1^{a_{1,2}} \cdots \zeta_t^{a_{1,t+1}}, \zeta_0^{a_{2,1}} \zeta_1^{a_{2,2}} \cdots \zeta_t^{a_{2,t+1}}, \ldots, \zeta_0^{a_{k,1}} \zeta_1^{a_{k,2}} \cdots \zeta_t^{a_{k,t+1}}).$$

The image of the map $T$, $X = T(\mathbb{R}^{t+1}) \subset \mathbb{R}^k$, is the toric variety associated to the matrix $A$. Thus, $T$ is a parametrization of $X$ and from this we see that $X$ is an irreducible algebraic variety.

As a toric variety $X$ is an algebraic variety, it is the zero locus of a set of polynomial equations. It is possible to associate to $X$ an ideal $I(X) \subset R = \mathbb{R}[p_1, \ldots, p_k]$ which is the ideal containing all polynomials vanishing on $X$.

As the matrix $A$ determines the toric variety via the map $T$, we will denote with $T(A)$ the toric variety $T(\mathbb{R}^{k+1})$. The matrix also determines
the ideal of the toric variety, and we will denote with \( I(A) \subset R \) the ideal \( I(T(R^{t+1})) \).

There are different ways of producing the toric ideal associated to a matrix. In particular, we will make use of the following algebraic approach. Consider the ring homomorphism

\[ \phi : R[p_1, \ldots, p_k] \rightarrow R[\zeta_0, \ldots, \zeta_t] \]

defined by setting

\[ \phi(p_i) = \zeta_0^{a_{i,1}} \zeta_1^{a_{i,2}} \cdots \zeta_t^{a_{i,t+1}} \quad (1) \]

for \( i = 1, \ldots, k \) and extending it by using the homomorphism properties. With these notations we get \( I(A) = \ker(\phi) \).

Special features of the matrix \( A \) reflect in geometric properties of the associated toric variety \( X \). In particular we examine here the special case when the matrix \( A \) has some repeated rows. In such case \( X \) is a hyperplane section of a cone over another toric variety, namely the toric variety defined by the matrix \( \bar{A} \) obtained from \( A \) by deleting each replicate of a given row. We can express this geometric property in algebraic terms as follows:

**Lemma 1.** Let \( A \) be an \( k \times (t + 1) \) matrix of non-negative integers having \( s \) rows \( i_1, \ldots, i_s \) equal to each other. Denote with \( \bar{A} \) the \((k - s + 1) \times (t + 1)\) matrix obtained by \( A \) deleting the \( s - 1 \) rows \( i_2, \ldots, i_s \). Then one has

\[ I(A) = I(\bar{A}) + J, \]

where \( J \) is the ideal generated by the degree one binomials \( p_{i_1} - p_{i_2}, \ldots, p_{i_1} - p_{i_s} \).

**Proof.** It is enough to provide a proof in the case \( s = 1 \), and we may assume that the first and the second rows are equal, i.e. \( i_1 = 1 \) and \( i_2 = 2 \). Then, \( T(A) = Y \cap H \) where \( H \) is the hyperplane of equation \( p_1 - p_2 = 0 \) and \( Y \) is the cone over \( T(\bar{A}) \) of vertex the point \( (1, 0, \ldots, 0) \). These remarks yields

\[ I(A) \supseteq I(\bar{A}) + \langle p_1 - p_2 \rangle. \]

To prove that equality holds, let \( F(p_1, \ldots, p_k) \in I(A) \) and construct the polynomial \( G(p_2, \ldots, p_k) = F(p_2, p_2, \ldots, p_k) \) by setting \( p_1 = p_2 \). It is enough to show that \( G \in I(\bar{A}) \), i.e. it is enough to show that \( G \) vanishes on all the points of \( T(\bar{A}) \). Let \( (q_1, \ldots, q_k) \in T(\bar{A}) \) and notice that, as \( T(\bar{A}) \) is a cone, \( (x, q_2, \ldots, q_k) \in T(\bar{A}) \) for all values of \( x \in \mathbb{R} \). In particular, \( (q_2, q_2, \ldots, q_k) \in T(A) = T(\bar{A}) \cap H \) and hence \( G(q_2, \ldots, q_k) = 0 \) and the proof is completed. \( \square \)
3. Models and toric models

In Statistics, a two-way contingency table collects the outcomes of two categorical random variables, say $V_1$ and $V_2$, on a sample. If the first variable has $I$ levels, and the second variable has $J$ levels, we conventionally denote the sample space by the cartesian product $\{1, \ldots , I\} \times \{1, \ldots , J\}$ and the contingency table has integer nonnegative entries $(n_{i,j})_{i=1,\ldots , I, j=1,\ldots, J}$ where $n_{i,j}$ is the count of the individuals with $V_1 = i$ and $V_2 = j$.

A probability distribution for an $I \times J$ contingency table is an $I \times J$ table of probabilities $(p_{i,j})_{i=1,\ldots , I, j=1,\ldots, J}$ in the probability simplex

$$\Delta = \left\{(p_{i,j})_{i=1,\ldots , I, j=1,\ldots, J} : p_{i,j} \geq 0 , \sum_{i,j} p_{i,j} = 1 \right\}.$$

A statistical model for an $I \times J$ table is a subset of $\Delta$. A well known class of statistical models for contingency tables is the class of log-linear models, see e.g. [10], defined through linear conditions on the log of the probabilities, and therefore restricted to the interior of $\Delta$. More precisely, given an integer matrix $A$ with dimension $IJ \times (t+1)$, the log-linear model defined by $A$ is characterized by the linear system

$$\log(p) = A\beta.$$  \hspace{1cm} (2)

Here, $\log(p)$ is the vector of the log-probabilities (taken by ordering the cells lexicographically), $\beta = (\beta_0, \ldots , \beta_t) \in \mathbb{R}^{t+1}$ is the vector of model parameters. The number of free parameters is equal to $\text{rk}(A)$, the rank of $A$, and the degrees of freedom of the model are the difference $IJ - \text{rk}(A)$.

Exponentiating Eq. (2), we obtain a different expression of the same model, namely

$$p = \zeta^A,$$  \hspace{1cm} (3)

where $\zeta^A$ is written in vector notation, and $\zeta = \exp(\beta)$ is the vector of nonnegative parameters. This model is called toric model, as it expresses the probabilities in monomial form, and it allows to extend the model on the boundary of $\Delta$, see [11] and [3]. Notice that Eq. (3) is just Eq. (1) written in a shorter notation.

Eliminating the $\zeta$ parameters from Eq. (3) we obtain that the toric model is the variety defined by the toric ideal $\mathcal{I}(A)$ introduced above:

$$\mathcal{I}(A) = \langle p^a - p^b : A^ia = A^ib \rangle$$  \hspace{1cm} (4)
and therefore the model is:
\[ \mathcal{M}_A = T(A) \cap \Delta. \] (5)

Thus, toric models are defined by a finite set of binomial equations (up to the normalizing constant).

In this paper we will use especially the independence model, defined by the model matrix
\[ A = [1, r_1, \ldots, r_I, c_1, \ldots, c_J], \]
where

- \(1\) is a column vectors of 1s;
- \(r_i\) is the indicator vector of the cells in the \(i\)-th row of the table;
- \(c_j\) is the indicator vector of the cells in the \(j\)-th column of the table.

The independence model contains all probability matrices such that the variables \(V_1\) and \(V_2\) are statistically independent. It is known, see [3] that the corresponding toric ideal \(\mathcal{I}(A)\) is generated by all \(2 \times 2\) minors of the probability table, that is
\[ \mathcal{I}(A) = \langle p_{i,j}p_{h,k} - p_{i,k}p_{h,j} : 1 \leq i < h \leq I, 1 \leq j < k \leq J \rangle. \] (6)

4. Clustering algorithms and their geometric counterpart

Let \(R_1, \ldots, R_I\) be the labels of the rows and \(C_1, \ldots, C_J\) be the labels of the columns of the contingency table under study. The classical agglomerative hierarchical clustering algorithms for a contingency table proceed as follows:

- At the initial step, each label is a separate cluster:
  \[ \{R_1\}, \ldots, \{R_I\}, \{C_1\}, \ldots, \{C_J\}. \]

- At step 1, we merge two elements of the partition above, based on some similarity criterion, and we obtain a partition with \((I + J - 1)\) clusters. There are different ways for defining the similarity in this step, and such different definitions yield different clustering algorithms. In our geometric study, we do not need to choose a specific similarity criterion, since our analysis is valid in general.
• At each step, we merge two elements of the partition, namely the two most similar elements, so that we lose a cluster at each step.

• At step \((I + J - 2)\) we define two clusters \(G_1\) and \(G_2\), and at the last step all row and column labels are merged together in a grand cluster.

Although in general an agglomerative clustering algorithm does not consider any restrictions on the intermediate steps, meaning that all pairs of clusters may be merged, in contingency tables analysis, the algorithms in the literature are usually defined to separately agglomerate the rows and the columns of the table, and we prove here that this prescription is fully motivated also under the geometric point of view. Therefore, we begin our analysis with the study of step 1 of the procedure above and, without loss of generality, we suppose that we have to merge the last two columns of the table.

Using the independence model, to merge the last two columns means to constrain the equality of the last two parameters: \(\beta_{t-1} = \beta_t\) in the log-linear representation (2) or \(\zeta_{t-1} = \zeta_t\) in the toric representation (3).

In a natural way, these equalities define a sub-model obtained from the matrix \(A\) of the independence model, with a new model matrix \(\tilde{A}\) obtained by summing the last two columns of \(A\).

The scheme is

\[
\begin{align*}
A &= [1, a_1, \ldots, a_t] & \longrightarrow \mathcal{I}(A) \\
\tilde{A} &= [1, a_1, \ldots, a_{t-2}, a_{t-1} + a_t] & \longrightarrow \mathcal{I}(\tilde{A})
\end{align*}
\]

The relation between these two ideals is given in Theorem 9, namely

\[
\mathcal{I}(\tilde{A}) = \mathcal{I}(A) + J,
\]

where \(J\) is a suitable ideal.

**Remark 2.** In this paper we assume that the first column of \(A\) is \(1 = (1, \ldots, 1)^T\). This a standard assumption for log-linear models and it does not affect the generality of our analysis. The algebraic counterpart of this assumption is the fact that the toric ideals involved are homogeneous, in particular it is possible to generate them using homogeneous binomials.
In terms of binomials, it is important to know whether the toric model associated to $\tilde{A}$ satisfies linear equations. Especially when these equations come from the ideal $\mathcal{J}$ in (7), this means that the linear binomials are generated by the clustering process. The relevance of degree one elements of the toric ideal rest on the identification of two cells of the table as follows.

**Remark 3.** The identification (or clustering) of two cells, say $(i, j)$ and $(h, k)$, has its algebraic counterpart in the linear binomial

$$p_{i,j} - p_{h,k}.$$ 

In fact, the equation $p_{i,j} - p_{h,k} = 0$ adds a constraint and the probabilities of the cells $k$ and $h$ must be equal.

We now describe what is the effect on the toric ideals of summing up two columns of the matrix $A$. In the examples below the computations are carried out with the software 4ti2, see [12].

**Example 4.** For instance the following matrix is the model matrix of the independence model for $3 \times 3$ contingency tables, and represents the most simple example for our theory.

$$A = \begin{pmatrix}
  1 & 1 & 0 & 0 & 1 & 0 & 0 \\
  1 & 1 & 0 & 0 & 1 & 0 & 0 \\
  1 & 1 & 0 & 0 & 0 & 0 & 1 \\
  1 & 0 & 1 & 0 & 1 & 0 & 0 \\
  1 & 0 & 1 & 0 & 0 & 1 & 0 \\
  1 & 0 & 1 & 0 & 0 & 0 & 1 \\
  1 & 0 & 0 & 1 & 1 & 0 & 0 \\
  1 & 0 & 0 & 1 & 0 & 1 & 0 \\
  1 & 0 & 0 & 1 & 0 & 0 & 1 
\end{pmatrix}$$  

and the corresponding toric ideal is

$$\mathcal{I}(A) = \langle -p_{1,3}p_{2,1} + p_{1,1}p_{2,3}, p_{1,3}p_{3,2} - p_{1,2}p_{3,3}, p_{2,3}p_{3,2} - p_{2,2}p_{3,3}, $$

$$p_{2,3}p_{3,1} - p_{2,1}p_{3,3}, -p_{1,2}p_{2,1} + p_{1,1}p_{2,2}, p_{1,3}p_{2,2} - p_{1,2}p_{2,3}, p_{2,2}p_{3,1} = p_{2,1}p_{3,2}, $$

$$p_{1,3}p_{3,1} - p_{1,1}p_{3,3}, -p_{1,2}p_{3,1} + p_{1,1}p_{3,2} \rangle.$$  

(9)
If we merge the last two columns of the contingency table, we obtain a model matrix $\tilde{A}$ with some repeated rows, and the corresponding toric ideal is

$$I(\tilde{A}) = \langle p_{3,2} - p_{3,3}, -p_{1,2} + p_{1,3}, -p_{2,2} + p_{2,3}, p_{1,2}p_{3,1} - p_{1,1}p_{3,3}, p_{2,2}p_{3,1} - p_{2,1}p_{3,3}, -p_{1,2}p_{2,1} + p_{1,1}p_{2,2} \rangle. \quad (10)$$

The new ideal $I(\tilde{A})$ contains $I(A)$ and has three additional generators: $p_{3,2} - p_{3,3}$, $-p_{1,2} + p_{1,3}$, and $-p_{2,2} + p_{2,3}$. This means that the analytical condition $\beta_5 = \beta_6$ in the log-linear formulation of the independence model translates into the identification of the cells in the last two columns of the $3 \times 3$ table: $(1,2)$ with $(1,3)$ in the first row, $(2,2)$ with $(2,3)$ in the second row, and $(3,2)$ with $(3,3)$ in the third row. A similar argument can be repeated when summing any two columns which are both rows indicators, or columns indicators of the table.

**Remark 5.** Through a quick inspection of the generators of $I(A)$ in the example above, one sees that the ideal is generated by:

(a) three binomials of degree two: $p_{1,2}p_{3,1} - p_{1,1}p_{3,3}$, $p_{2,2}p_{3,1} - p_{2,1}p_{3,3}$, $-p_{1,2}p_{2,1} + p_{1,1}p_{2,2}$. Such binomials are the generators of the toric ideal of the independence model for a $3 \times 2$ table;

(b) three binomials of degree one: $p_{3,2} - p_{3,3}$, $-p_{1,2} + p_{1,3}$, $-p_{2,2} + p_{2,3}$. Such binomial yields the identification of the two merged columns, as discussed above.

The agglomeration of the last two columns of the table is sketched in Figure 1. In the next section, we will prove that this fact is valid in general for the independence model.

The behavior exhibited in Example 4 is very peculiar and, in general, the toric ideal is not so easily described. The next example is not directly related to the independence model, but it shows that linear polynomials are not enough in general for describing the toric ideal $I(\tilde{A})$. Two more examples shows that linear polynomials does not appear when merging a row with a column of the table.
Example 6. In general, it is not enough to add linear binomials to $\mathcal{I}(A)$ to obtain $\mathcal{I}(\tilde{A})$. For instance, if we compute the toric ideal associated to the following matrix

$$
A = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}
$$

in the ring $\mathbb{R}[p_1, p_2, p_3, p_4]$, we obtain the zero ideal, i.e. $T(A) = \mathbb{R}^4$ but if we sum the last two columns of $A$, we get the matrix $\tilde{A}$ and $\mathcal{I}(\tilde{A}) = \mathcal{I}(A) + (p_1 p_2 - p_3 p_4)$. Thus a generator of degree two appears in $\mathcal{I}(\tilde{A})$.

Example 7. Now we sum two columns of the model matrix of the independence model pertaining to different objects, i.e., one indicator vector of a row and one indicator vector of a column. Let us consider Example on the $3 \times 3$ independence model, but now we sum the indicator vectors of the last row and of the last column. The new matrix is now

$$
\tilde{B} = \begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 2 & 0 & 0
\end{pmatrix}.
$$

(11)
As the matrices $A$ and $\tilde{B}$ have the same kernel, the toric ideals must be equal: $\mathcal{I}(A) = \mathcal{I}(\tilde{B})$. From the point of view of Statistics, this behavior is natural, since no degrees of freedom are added when moving from $A$ to $\tilde{B}$.

Example 8. When the agglomeration process is iterated twice the situation is more complicated. Let us consider the independence model for $4 \times 4$ contingency tables, whose model matrix $A$ written as in Eq. (8) with the suitable number of rows and columns. The matrix $A$ has 16 rows and 9 columns: 1 column of 1s, 4 columns with the indicator of the rows, and 4 columns with the indicators of the columns. The toric ideal $\mathcal{I}(A)$ is generated by the 36 $2 \times 2$ minors of the probability table, as prescribed from Eq. (6). When we merge the last row with the last column of the table (i.e., we sum the fifth and the ninth column of $A$), we obtain a matrix $\tilde{B}$ in analogy with the matrix in Eq. (11) and the corresponding ideal is $\mathcal{I}(\tilde{B}) = \mathcal{I}(A)$. If we merge two further objects, namely the third row with the third column of the table, we define a new matrix $\tilde{C}$, and the ideal is

$$\mathcal{I}(\tilde{C}) = \mathcal{I}(A) + \langle p_{4,3} - p_{3,4}, p_{3,4}^2 - p_{3,3}p_{4,4} \rangle$$

and the two added binomials do not have a simple statistical counterpart.

5. General facts and application to the independence model

There is a quite general relation between the toric ideal of $A$ and the one of $\tilde{A}$.

Theorem 9. Consider the diagram of commutative rings

$$\begin{array}{ccc}
n[p_1, \ldots, p_{IJ}] & \mathcal{I} & \tilde{A} \\
\phi \downarrow & & \phi^{-1} \\
\kappa[\zeta_0, \ldots, \zeta_{t-1}, \zeta_t] & \alpha \uparrow & \kappa[\zeta_0, \ldots, \zeta_{t-1}]
\end{array}$$

where the maps are defined as follows:

$$\phi(p_i) = \zeta_0^{a_{i,1}} \cdots \zeta_t^{a_{i,t}},$$

$$\psi(p_i) = \zeta_0^{a_{i,1}} \cdots \zeta_t^{a_{t-1} + a_{i,t}},$$

and $\alpha(\zeta_i) = \zeta_i$ for $i \neq t$ and $\alpha(\zeta_t) = \zeta_{t-1}$. Then, with the notation of the previous sections, we have

$$\mathcal{I}(\tilde{A}) = \mathcal{I}(A) + \alpha^{-1}(\zeta_{t-1} - \zeta_t).$$
Proof. The proof follows immediately noticing that $\phi = \alpha \circ \psi$. \qed

We now derive some interesting facts about the degree one elements of $\mathcal{I}(\tilde{A})$.

**Proposition 10.** A linear binomial $p_k - p_h$ belongs to $\mathcal{I}(\tilde{A})$ if and only if $a_r(k) = a_r(h)$ for $r = 0, \ldots, t - 2$ and $a_{t-1}(k) + a_t(k) = a_{t-1}(h) + a_t(h)$.

**Proof.** We directly use the parametrization to compute

$$p_h - p_k = \tilde{\zeta}_0 \tilde{\zeta}_{a_1}(h) \cdots \tilde{\zeta}_{a_{t-2}}(h) \tilde{\zeta}_{a_{t-1}}(h) + a_t(h) - \tilde{\zeta}_0 \tilde{\zeta}_{a_1}(k) \cdots \tilde{\zeta}_{a_{t-2}}(k) \tilde{\zeta}_{a_{t-1}}(k) + a_t(k)$$

and this is the zero polynomial if and only if the statement holds. \qed

In several situations of practical interest, the model matrix is a binary matrix, i.e., it only involves zeros and ones. This is the case, for example, for the independence model.

**Corollary 11.** Let $A$ and $\tilde{A}$ as above be two binary matrices. A linear polynomial $p_k - p_h$ belongs to $\mathcal{I}(\tilde{A})$ if and only if it belongs to $\mathcal{I}(A)$ or $a_r(k) = a_r(h)$ for $r = 0, \ldots, t - 2$ and $a_{t-1}(k) \neq a_{t-1}(h)$, $a_t(k) \neq a_t(h)$.

**Proof.** Let $k$ and $h$ be two rows of $A$ such that $a_r(k) = a_r(h)$ for $r = 0, \ldots, t - 2$. Let us consider the $2 \times 2$ subtable of $A$ formed by the rows $k$ and $h$, and by the columns $t - 1$ and $t$. It is easy to check that there are only four possible configurations, up to permutations of rows and columns:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Among such configurations, only the first one satisfies the last condition of the statement. In configuration $(a)$, the new binomial $p_k - p_h$ is added to $\mathcal{I}(\tilde{A})$ as a consequence of Lemma 1, while in configurations $(b)$ and $(d)$ the binomial $p_k - p_h$ is already in $\mathcal{I}(A)$ because the rows $k$-th and $h$-th of $A$ are equal. Configuration $(c)$ does not produce any linear binomial since the two rows in $\tilde{A}$ are not equal. \qed

We conclude with a nice result about the independence model. This explain why Example 4 is so well-behaved.
Proposition 12. Let $A$ be the model matrix of an independence model. Produce a matrix $\tilde{A}$ by summing up two columns of $A$ both rows indicators, or both columns indicators, of the table. Then,

$$I(\tilde{A}) = I(A) + J$$

where the ideal $J$ is generated by linear binomials. More precisely, if we sum the indicator functions of columns $a$ and $b$, then $J$ contains all the binomials $p_{i,a} - p_{i,b}$ for all values of $i$.

Proof. Without loss of generality let us assume that $A$ is the model matrix of an independence model over an $I \times J$ table $T_1$. Also, we assume to sum the last two columns of $A$, say column $t - 1$ and columns $t$, coming from the indicator functions of the last two columns of the table, say column $J - 1$ and column $J$. Notice that $\tilde{A}$ has rows which are equal in pairs, namely the rows indexed by

$$p_{1,J - 1}, \ldots, p_{I,J - 1}$$

coincide, respectively, with the rows indexed by,

$$p_{1,J}, \ldots, p_{I,J}.$$  \hspace{1cm} (12)

Thus, we construct the matrix $\tilde{B}$ obtained by $\tilde{A}$ deleting all the rows in (12). Using Lemma \[1\] we know that

$$\mathcal{J}(\tilde{B}) + \langle p_{1,J - 1} - p_{1,J}, \ldots, p_{I,J - 1} - p_{I,J}\rangle = \mathcal{J}(\tilde{A}).$$

Now, we notice that $\tilde{B}$ is the model matrix of an independence model on the table $T_2$ obtained by removing column $J$ by table $T_1$. In particular, $\mathcal{J}(\tilde{B})$ is generated by the $2 \times 2$ minors of $T_2$, i.e. by the $2 \times 2$ minors of $T_1$ not involving the last column, i.e. by the $2 \times 2$ minors not involving any of the variables $p_{1,J}, \ldots, p_{I,J}$. Thus, it is immediate to check that

$$\mathcal{J}(A) + \langle p_{1,J - 1} - p_{1,J}, \ldots, p_{I,J - 1} - p_{I,J}\rangle = \mathcal{J}(\tilde{B}) + \langle p_{1,J - 1} - p_{1,J}, \ldots, p_{I,J - 1} - p_{I,J}\rangle,$$

and the conclusion follows. \[\square\]

Remark 13. Proposition \[12\] is the key for iterating the clustering process within the independence model. Identifying two columns of the table produces linear equations given by the identified cells and a smaller table. If we now identify columns, or rows, of this new table, we have to add new linear equations and we have to deal with new, smaller tables.
6. Further questions

In this paper we addressed the following question:

\( Q_1 \) What does it happen when we identify two parameters of the log-linear (or toric) model? Recall that the first parameter \( \beta_0 \) is fixed, so that the question only concerns \( \beta_1, \ldots, \beta_t \)?

We provided examples and general results, see Theorem 9, about this question. Also, we answered question \( Q_1 \) in some special cases, e.g. for the independence model we produced a complete answer, see Proposition 12. Question \( Q_1 \) leads to new interesting problems to be addressed, both from the point of view of Algebra and from the point of view of applications to Statistics.

First, it is now natural to consider new questions, which is in some sense the converse of \( Q_1 \), namely:

\( Q_2 \) Given a toric model with model matrix \( A \), what does it happen to \( I(A) \) (and to the model matrix \( A \)) when we add a linear binomial of the form \( p_k - p_h \) to the generators of \( I(A) \)?

The treatment of this question will be subject to further analysis. We think that the following is a promising approach: first we form the toric ideal associated to the matrix \( A \).

\[ A = [a_0, \ldots, a_t] \mapsto I(A). \]

Then we form the ideal \( J = I(A) + \langle p_k - p_h \rangle \) and we consider the following questions: under which conditions on the linear binomial \( p_h - p_k \) is \( J = I(B) \) for some matrix \( B \)? If, for a given choice of a linear binomial, \( J \) is not a toric ideal, how can we find a matrix \( B \) such that \( J \subset I(B) \)?

From the perspective of the analysis of statistical models, it is interesting to study other approaches to cluster analysis for contingency tables. In this paper, we focused our attention to the hierarchical agglomerative algorithms, but there are also different approaches to the problem. For instance, in [13] the problem of clustering is considered in the framework of mixture models.
Moreover, also with the names of bi-clustering or co-clustering, several algorithms are now available for a wide range of applications, from Psychometry to Molecular Biology. For more details on such techniques see for instance [14], [15], and [16]. Such procedures for finding patterns in data matrices differ in several aspects (the patterns they seek, the assumptions on the underlying probability models, and so on), but all of them are essentially based on mixtures of independence models or other log-linear models with support on suitable regions of the contingency table to be determined by minimizing some cost functions. The geometry of such statistical models is strictly related with the notion of nonnegative rank of a matrix, and some preliminary results can be found in [17], [18] and [19], where some applications to mixture models in statistics are introduced.

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