On the geometry of ghosts

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Abstract

An inspection of the precise geometric constructions underlying fundamental notions in quantum gauge field theories sheds light on various aspects which tend to be obscured in the usual formalisms. Revising the notions of mutually conjugated “internal” bundles we propose a general rule for the constructions of free quantum fields and their conjugates, naturally yielding the needed fundamental properties with regard to contractions, super-commutators, field momentum and Hamiltonian, and other quantities. This scheme applies to fields of all types; in particular we examine consequences in relation to ghosts and anti-ghosts.

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Introduction

Notions in quantum field theory are often introduced in a matrix language, which may obscure their geometric meaning and blur differences among distinct objects, while the most mathematically-oriented presentations tend to focus on functional analytical aspects in Hilbert spaces [15]. Here we consider a somewhat different point of view, believing that careful considerations in an explicitly geometric language can help to clarify the matter further. We can divide this task into various steps. The first step consists of describing the underlying finite dimensional bundles and their fiber structures, namely the theory’s “classical” (or “pre-quantum”) setting. While this aspect is widely treated in the literature, for a deeper insight there are fine points deserving special consideration, in particular with regard to the relation between mutually conjugate spaces, the geometry of spinors and its connection with spacetime geometry, and the geometry of pre-quantum ghost/anti-ghost fields. Details about these points, according to the view of this paper, can be found in previous work on a partly original presentation of gauge field theories [3, 4, 6, 8, 10, 12].

As for the functional spaces suitable for describing quantum states, we use an approach based on the distributional spaces of generalized semi-densities, whence one constructs multi-particle state spaces and a well-defined operator algebra generated by absorption and emission operators. If \( M \) is the spacetime manifold and \( P_m \rightarrow M \) is the sub-bundle of \( T^*M \) consisting of future mass-shells of mass \( m \), and if \( Z \rightarrow P_m \) is a vector bundle (so that we deal with a 2-fibered structure), then we introduce a corresponding “quantum bundle” whose fibers are spaces of generalized \( Z \)-valued semi-densities over mass-shells. Most basic concepts and results of standard differential geometry can be extended to such distributional bundles by a careful use of Frölicher’s notion of smoothness [11, 16, 17, 21, 2, 20].

If we aim at a quantum field theory formulated on curved spacetime, our path splits into two directions. On one hand, it can be shown that a timelike submanifold (representing a detector) carries a natural quantum formalism [5, 9], a sort of complicated clock that an observer can use for making predictions about the possible outcomes of measures. This formalism can be directly described in terms of momentum representation, but also determines a local quantum field setting in a spacetime neighbourhood of the timelike submanifold. So we can see quantum fields, at least locally, as sections of classical bundles tensorialized by a suitable \( \mathbb{Z}_2 \)-graded algebra \( \mathcal{O} \). Though this construction depends on the chosen detector, we may switch to a complementary view in which \( \mathcal{O} \)-valued fields over spacetime are considered as the fundamental objects; so we “forget” about detectors and observers and obtain a fully covariant, observer-independent field theory [11]. The drawback is that the notions of quantum states and transition probabilities become blurred. We may say that in curved spacetime the principle of correspondence has to be rethought. We may accept this difficulty as inherent in quantum field theory, while we recognize that a covariant formulation on a curved background is interesting under various respects.

The notion of a free quantum field stands at the junction between the above said approaches. Free quantum fields are of special interest also because, together with point interactions, they constitute (roughly speaking) the “building blocks” of field dynamics. A close inspection of the construction allows us to note details which are usually skipped. A free quantum field is best seen as a combination of particle absorption and anti-particle emission; this is true in all cases—not only for the Dirac field—except when the bundle of “internal degrees of freedom” is real (then particle and anti-particle coincide). Moreover, mutually conjugate fields are best seen as analogous constructions on mutually conjugate bundles, rather than equivalent objects obtained from one another by complex conjugation (possibly associ-
1 QUANTUM FIELDS: BASIC CONSTRUCTIONS

1.1 Generalized semi-densities and quantum states

Let $Z \rightarrow X$ be a finite-dimensional complex vector bundle over the real $m$-dimensional orientable manifold $X$, and choose a “positive” semi-vector bundle $(\wedge^m T^*X)^+ \subset \wedge^m T^*X$. Up to an isomorphism there is\footnote{For an account of positive semi-spaces and their rational powers, see [19, 9] and the bibliography therein.} a unique semi-vector bundle $U \rightarrow X$ such that $U \otimes U \cong (\wedge^m T^*X)^+$. A section $X \rightarrow U \otimes Z$ is called a $Z$-valued semi-density. We denote as $\mathcal{P}_0(X,Z)$ the vector space of all such sections which are smooth and have compact support. Its dual space in the standard test map topology \footnote{In the generalized sense $\delta[x]$ is valued into $\wedge^m T^*X$, so that $\delta[x] \otimes u$ is valued into $U \otimes U \otimes U^* \cong U$.} is assigned then one has the space $\mathcal{D}(X,Z)$ and called the space of $Z$-valued generalised semi-densities (so the word “generalized” is used here in the distributional sense).

In particular, a sufficiently regular ordinary section $\theta : X \rightarrow U \otimes Z$ is in $\mathcal{P}(X,Z)$ via the rule $\langle \theta, \sigma \rangle := \int_X \langle \theta(x), \sigma(x) \rangle, \sigma \in \mathcal{P}_0(X,Z^*)$.

Semi-densities have a special status among all kinds of generalised sections because of the natural inclusion $\mathcal{P}_0(X,Z) \subset \mathcal{P}(X,Z)$. Furthermore, if a fibered Hermitian structure of $Z \rightarrow X$ is assigned then one has the space $\mathcal{L}^2(X,Z)$ of all ordinary semi-densities $\theta$ such that $\langle \theta^1, \theta \rangle < \infty$. Let $0 \subset \mathcal{L}^2(X,Z)$ denote the subspace of all almost-everywhere vanishing sections; then the quotient $\mathcal{H}(X,Z) = \mathcal{L}^2(X,Z)/0$ is a Hilbert space, and we get a so-called rigged Hilbert space \footnote{An important result in the theory of distributions [25] then implies that $\mathcal{P}(X,Z)$ is dense in $\mathcal{D}(X,Z)$, namely any generalised semi-density can be approximated with arbitrary precision.}

$$\mathcal{P}_0(X,Z) \subset \mathcal{H}(X,Z) \subset \mathcal{D}(X,Z).$$

Elements in $\mathcal{D}(X,Z) \setminus \mathcal{H}(X,Z)$ can then be identified with the (non-normalizable) generalised states of the common physics terminology.

Let $\delta[x]$ be the Dirac density on $X$ with support $\{x\}, x \in X$. A generalised semi-density is said to be of Dirac type if it is of the form $\delta[x] \otimes u \in \mathcal{P}(X,Z)$ with\footnote{An important result in the theory of distributions [25] then implies that $\mathcal{P}(X,Z)$ is dense in $\mathcal{D}(X,Z)$, namely any generalised semi-density can be approximated with arbitrary precision.}$^2 u : X \rightarrow U^* \otimes Z$. We define $\mathcal{P}(X,Z)$ to be the space of all finite linear combinations of Dirac-type semi-densities. An important result in the theory of distributions [25] then implies that $\mathcal{P}(X,Z)$ is dense in $\mathcal{D}(X,Z)$, namely any generalised semi-density can be approximated with arbitrary precision.
1.2 Multi-particle states and elementary operators

(in the sense of the topology of distributional spaces) by a finite linear combination of Dirac-type semi-densities.

The assignment of a volume form \( \eta: X \to (\wedge^n T^*X)^+ \) and of a frame\(^3\) \((b_\alpha)\) of \(Z \to X\) determines the set \(\{B_{x\alpha}\} \subset \mathcal{P}(X, Z)\), called a generalised basis, where

\[ B_{x\alpha} \equiv \delta[x] \otimes \eta^{-1/2} \otimes b_\alpha(x). \]

Traditionally one would rather write \(B_{x\alpha}\) as \(|x, \alpha\rangle\) (say), but the point here is that we can introduce a handy “generalised index” notation. We write \(B^{x\alpha} \equiv \delta[x] \otimes \eta^{-1/2} \otimes b^\alpha(x)\), where \((b^\alpha)\) is the dual classical frame. Though contraction of any two distributions is not defined in the ordinary sense, a straightforward extension of the discrete-space operation yields

\[ \langle B^{x\alpha'}, B_{x\alpha} \rangle = \delta^{x'}_x \delta^\alpha_\alpha', \]

where \(\delta^{x'}_x\) is the generalised function usually indicated as \(\delta(x' - x)\). This is consistent with “index summation” in a generalised sense: if \(z \in \mathcal{P}_0(X, Z)\) and \(\zeta \in \mathcal{P}_0(X, Z^*)\) are test semi-densities, then we write

\[ z^{x\alpha} \equiv z^\alpha(x) \equiv \langle B^{x\alpha}, z \rangle , \quad \zeta_{x\alpha} \equiv \zeta_\alpha(x) \equiv \langle \zeta, B_{x\alpha} \rangle , \]

\[ \langle \zeta, z \rangle \equiv \zeta_{x\alpha'} \ z^{x\alpha} \langle B^{x\alpha'}, B_{x\alpha} \rangle \equiv \int_X \zeta_\alpha(x) \ z^\alpha(x) \ \eta(x) , \]

namely we interpret index summation with respect to the continuous variable \(x\) as integration, provided by the chosen volume form. This formalism can be extended to the contraction of two generalised semi-densities whenever it makes sense.

1.2 Multi-particle states and elementary operators

In order to deal with multi-particle states and different particle types, we introduce further notations in the context of §1.1. We set

\[ Z_0 \equiv \mathcal{P}_0(X, Z) , \quad Z^1 \equiv \mathcal{P}(X, Z) , \quad Z^1 \equiv \mathcal{P}(X, Z) , \]

\[ Z^n \equiv \diamondsuit^n Z^1 , \quad Z^n \equiv \diamondsuit^n Z^1 , \]

where \(\diamondsuit\) denotes either symmetrized or antisymmetrised tensor product (respectively for bosons and fermions). Then \(Z^n\) turns out to be dense in \(Z^n\), which in turn is dense either in the symmetrised or in the antisymmetrised subspace of \(\mathcal{P}(X^n, \otimes^n Z)\), \(X^n \equiv X \times \cdots \times X\). Next we set \(Z \equiv \bigoplus_{n=0}^\infty Z^n\), and assemble several particle types into one total state space

\[ \mathcal{V} := Z' \otimes Z'' \otimes \cdots \equiv \bigoplus_{n=0}^\infty \mathcal{V}^n , \]

where \(\mathcal{V}^n\), constituted of all elements of tensor rank \(n\), is the space of all states of \(n\) particles of any type.

We can also consider a dual construction, in an elementary sense, by replacing \(Z\) with its dual \(Z^*\), and obtain the “dual” space \(\mathcal{V}^*\). Moreover we note that using test semi-densities we obtain subspaces \(\mathcal{V}_0 \subset \mathcal{V}\) and \(\mathcal{V}_0^* \subset \mathcal{V}^*\).

If we now let the grade \([\phi]\) of a monomial element (a “decomposable tensor”) \(\phi \in \mathcal{V}\) to be the parity of the number of fermion factors it contains, then we obtain on \(\mathcal{V}\) a structure of

\(^3\)For notational simplicity we assume the frame’s domain to be the whole \(X\).
“super-algebra” (a $\mathbb{Z}_2$-graded algebra), products being performed in the appropriate tensor factors. Furthermore we can consider “interior products”, in the appropriate tensor factors, between elements in $\mathcal{V}^n$ and elements in $\mathcal{V}$, possibly to be intended in a generalized sense. These will be indicated by a vertical bar as (say) $\zeta | \psi$. We then obtain the rules
\[
\psi \circ \phi = (-1)^{[\phi][\psi]} \phi \circ \psi , \quad (\zeta \circ \xi) | \psi = \xi | (\zeta | \psi) ,
\]
\[
\zeta | (\phi \circ \psi) = (\zeta | \phi) \circ \psi + (-1)^{[z][\phi]} \phi \circ (\zeta | \psi) , \quad \phi, \psi \in \mathcal{V} , \zeta, \xi \in \mathcal{V}^1 ,
\]
valid whenever each of the involved factors has a definite grade. A linear map $X : \mathcal{V} \rightarrow \mathcal{V}$ is called a super-derivation (or anti-derivation) of grade $[X]$ if $[X \psi] = [X] + [\psi]$ and the graded Leibnitz rule
\[
X(\phi \circ \psi) = (X \phi) \circ \psi + (-1)^{[X][\phi]} \phi \circ X \psi
\]
is fulfilled.

The absorption operator associated with $\zeta \in \mathcal{V}^1$ and the emission operator associated with $z \in \mathcal{V}^1$ are the linear maps $\mathcal{V}_0 \rightarrow \mathcal{V}$ respectively defined as
\[
a[\zeta] \phi \equiv \zeta | \phi , \quad a[z] \phi \equiv z \circ \phi , \quad \phi \in \mathcal{V} .
\]
Similarly, we have operators $a[z], a^*[\zeta] : \mathcal{V}_0^* \rightarrow \mathcal{V}^*$, and one easily checks that $a[\zeta]$ and $a[z]$ are mutually transposed maps. Absorption and emission operators generate a vector space which turns out to be a $\mathbb{Z}_2$-graded algebra (the algebra product being the composition of endomorphisms) by letting the grades of $a[\zeta]$ and $a^*[\zeta]$ be $[\zeta]$ and $[z]$, respectively. The super-bracket of two operators $X, Y$ in this space is then defined by
\[
\{[X, Y] \} := XY - (-1)^{[X][Y']} YX .
\]
In particular, for $y, z \in \mathcal{V}^1$ and $\zeta, \xi \in \mathcal{V}^1$ we get
\[
\{[a[\zeta], a[\xi]]\} = \{(a^*[y], a^*[z])\} = 0 , \quad \{[a[\zeta], a^*[z]]\} = (\zeta, z) \mathbb{I} .
\]

The vector space $\mathcal{O}^1$ of all sums of the kind $a[\zeta] + a^*[z]$ has the subspace $\mathcal{O}_1$ of all finite linear combinations of absorption and emission operators associated with Dirac-type semi-densities. In particular we write $a^x \alpha \equiv a[B^x \alpha], a^*_{x \alpha} \equiv a^*[B_{x \alpha}]$, and obtain super-commutation rules
\[
\{[a^x \alpha, a^z \alpha']\} = \{(a^*_{x \alpha}, a^*_{x \alpha'})\} = 0 , \quad \{[a^x \alpha, a^*_{x \alpha'}]\} = \delta^x_{x'} \delta^{\alpha}_{\alpha'} ,
\]
where the latter is to be understood in a generalised sense: for $\zeta \in \mathcal{V}_0^1, z \in \mathcal{V}_0^1$, we write
\[
\{[a[\zeta], a^*[z]]\} = \{(\zeta x \alpha a^x \alpha, x \alpha' a^x \alpha')\} = \zeta x \alpha x \alpha' \{(a^x \alpha, a^*_{x \alpha'})\} = (\zeta, z) .
\]

Next we denote as $\mathcal{O}^n, n \in \mathbb{N}$, the vector space spanned by all compositions of $n$ emission and absorption operators ordered in such a way that all absorption operators stand on the right of any emission operator (normal order). A product $\mathcal{O}^n \times \mathcal{O}^p \rightarrow \mathcal{O}^{n+p}$ can be defined as composition together with normal reordering, obtained by imposing the modified rule
\[
\{[a^x \alpha, a^*_{x \alpha'}]\} = 0 .
\]
Setting $\mathcal{O}^0 \equiv \mathbb{C}$ we obtain a graded algebra $\mathcal{O} \equiv \bigoplus_{n=0}^{\infty} \mathcal{O}^n$ of linear maps $\mathcal{V}_0 \rightarrow \mathcal{V}$ (note that normal ordering is needed for obtaining an algebra of such maps). Moreover, $\mathcal{O}$ turns out to be a $\mathbb{Z}_2$-graded algebra, which can be identified with $\mathcal{V} \otimes \mathcal{V}^*$.

A suitable extension of $\mathcal{O}$ will be actually needed. Let $Z : \mathbb{R} \rightarrow \mathcal{O}$ be a local curve such that $\lim_{\lambda \rightarrow 0}[Z(\lambda)] \chi \in \mathcal{V}$ exists in the sense of distributions for all $\chi \in \mathcal{V}_0$. Then $\lim_{\lambda \rightarrow 0} Z(\lambda)$ is a well-defined linear map $\mathcal{V}_0 \rightarrow \mathcal{V}$ which belongs, in general, to an extended space $\mathcal{O}^* \supset \mathcal{O}$.
1.3 Conjugation and the role of Hermitian structure

In order to understand the precise relation between mutually conjugate fields, we need to keep in mind the notion of anti-dual space $\mathbf{V}^*$ and of conjugate space $\overline{\mathbf{V}}$ of a finite-dimensional complex vector space $\mathbf{V}$. In the finite-dimensional situation, the former can be simply defined as the complex vector space of all anti-linear functions $\mathbf{V} \to \mathbb{R}$, and the latter as its dual space. Complex conjugation yields then anti-isomorphisms $\mathbf{V} \leftrightarrow \overline{\mathbf{V}}$ and $\mathbf{V}^* \leftrightarrow \overline{\mathbf{V}}^*$. The “dotted-index” formalism is useful for dealing with component expressions related to $\overline{\mathbf{V}}$ and $\overline{\mathbf{V}}^*$.

The above notions can be seamlessly extended to complex vector bundles, and we observe that in most practical cases the fibers are assumed to be endowed with a Hermitian structure. This yields various isomorphisms and consequent possible simplifications of indexed expressions, specially with regard to conjugation. Nevertheless, a few preliminary hair-splitting observations may help us to handle the ensuing formalism better.

A Hermitian structure on $\mathbf{Z} \rightarrow X$ is a non-degenerate tensor field

$$h : X \to \overline{\mathbf{Z}}^* \otimes \mathbf{Z}^*$$

such that $\overline{h} = h^\dagger$. If $(b_\alpha)$ is a frame of $\mathbf{Z}$ then the conjugate frame of $\overline{\mathbf{Z}}$ is denoted as $(\overline{b_\alpha})$, and the anti-dual frame of $\overline{\mathbf{Z}}^*$ is denoted as $(\overline{b^\alpha})$. Accordingly we write

$$h = h^\alpha_\alpha \overline{b^\alpha} \otimes b_\alpha, \quad h^\# = h^\alpha_\alpha \overline{b_\alpha} \otimes b^\alpha, \quad h_\# = h^\alpha_\alpha b_\alpha \otimes \overline{b^\alpha},$$

where $h^\# : X \to \overline{\mathbf{Z}} \otimes_X \mathbf{Z}$ is the “inverse” of $h$. Then $h$ and $h^\#$ determine isomorphisms

$$\overline{b} : \mathbf{Z} \to \mathbf{Z}^* : \overline{z} \mapsto \overline{z^\alpha}, \quad \overline{\overline{b}} : \overline{\mathbf{Z}^*} \to \mathbf{Z} : z \mapsto z^\alpha,$$

$$\# : \mathbf{Z}^* \to \mathbf{Z} : \overline{\zeta} \mapsto \overline{\zeta^\#}, \quad \overline{\#} : \mathbf{Z} \to \overline{\mathbf{Z}} : \zeta \mapsto \overline{\zeta^\#},$$

over $X$, with $\overline{b}$ and $\overline{\overline{b}}$ being mutually inverse as well as $\overline{b}$ and $\#$.

If $z = z^\alpha b_\alpha \in \mathbf{Z}$, $\zeta = \zeta_\alpha b^\alpha \in \mathbf{Z}^*$, then we also write $\overline{z} = \overline{z^\alpha} \overline{b_\alpha} \in \overline{\mathbf{Z}}$, $\overline{\zeta} = \overline{\zeta_\alpha} \overline{b^\alpha} \in \overline{\mathbf{Z}}^*$, and

$$\overline{z^\alpha} = h^\alpha_\alpha \overline{z^\alpha} \overline{b^\alpha} \equiv \overline{z_\alpha} b^\alpha, \quad \overline{z^\alpha} = h^\alpha_\alpha z^\alpha \overline{b^\alpha} \equiv z_\alpha \overline{b^\alpha},$$

$$\overline{\zeta^\#} = h^\alpha_\alpha \zeta_\alpha \overline{b^\alpha} \equiv \overline{\zeta_\alpha} b^\alpha, \quad \overline{\zeta^\#} = h^\alpha_\alpha \overline{\zeta_\alpha} \overline{b^\alpha} \equiv \overline{\zeta_\alpha} b^\alpha.$$

Thus in many cases the Hermitian structure allows avoiding “dotted indices”, which are often used to distinguish components in conjugate spaces. In particular we may use

$$b^\alpha_\# = \overline{b_\alpha} \equiv h^\alpha_\alpha b^\alpha, \quad b^\alpha_\# = b^\alpha \equiv h^\alpha_\alpha \overline{b_\alpha}.$$

A Hermitian structure is specially relevant in relation to the fact that whenever a sector corresponding to a complex bundle $\mathbf{Z}$ is considered, then the theory also includes the sector corresponding to the conjugate bundle $\overline{\mathbf{Z}}$. These two classical bundles underlie the description of a couple particle-antiparticle.\footnote{Simplifications ensue if $\mathbf{Z} \equiv \mathbb{C} \otimes \mathbf{Z}_0$ where $\mathbf{Z}_0$ is a real vector bundle. Also note that a real metric $g$ of $\mathbf{Z}_0$ can be naturally extended to the a Hermitian structure of the complexified bundle.} Accordingly, from an ordinary section $\zeta : X \to \mathbf{Z}^*$ we get operators $a[\zeta]$ and $a^*[\zeta^\#]$, which can be respectively seen as the absorption of a particle and the emission of the related anti-particle. Similarly, an ordinary section $z : X \to \mathbf{Z}$ yields operators...
and its anti-particle, which are in general distinct. Thus an absorption operator with the same index type are related to internal states of a particle so that we get the generalized identity
\[ \bar{a}^{*\alpha} \equiv \bar{a}^*[\mathbb{B}^{\alpha}] , \quad a_{\alpha} \equiv a[\mathbb{B}_\alpha] . \]

We stress that, like the correspondences \( \zeta \to \zeta^\# \) and \( z \to z^\flat \) do not imply conjugation of \( \zeta \) and \( z \), so do \( \bar{a}^{*\alpha} \) and \( a_{\alpha} \). Actually, the conjugation relation is between the spaces on which these operators act, not between the operators themselves as generalized functions of \( x \). Hence we do not use the common notation \( \bar{a}^\dagger \) for emission operators, as the implied usual meaning for the “dagger” label is “transposition together with conjugation”. On the other hand, we could rightly set \( a^\dagger[\zeta] \equiv a^*[\zeta^\#] \), so that we see that if the Hermitian structure is positive-definite and only orthonormal frames are considered, then that notation doesn’t create difficulties essentially because one may identify high and low indices as well as dotted and non-dotted indices. Such identifications are routinely made in the literature [15, 18].

As for the super-commutation rules for the two new elementary operators (besides the rules stated in §1.2), we note that for ordinary sections \( z \) and \( \zeta \) we have
\[ \{ [a^\beta], a^*[\zeta^\#] \} = \langle z^\flat, \zeta^\# \rangle I = \langle \zeta, z \rangle I , \]
so that we get the generalized identity
\[ \{ [a^\alpha(x), a^\beta(y)] \} = \{ [a_\beta(x), a^{*\alpha}(y)] \} = \delta^\alpha_\beta \delta(x - y) , \]
which is independent of the signature of the Hermitian structure \( h \).

Other super-commutators vanish. In particular, we note that an emission operator and an absorption operator with the same index type are related to internal states of a particle and its anti-particle, which are in general distinct. Thus
\[ \{ [a_\alpha(x), a^\beta(y)] \} = \{ [a^\alpha(x), a^{*\beta}(y)] \} = 0 . \]

Finally, we note that allowing normal ordering amounts to assuming the modified rules
\[ \{ [a^\alpha(x), a^\beta(y)] \} = \{ [a_\beta(x), a^{*\alpha}(y)] \} = 0 . \]

### 1.4 Distributional bundles and generalized frames

For a given particle type in Einstein’s spacetime \((M, g)\), the underlying “classical” geometric structure is that of a 2-fibered bundle \( \mathbb{Z} \to \mathcal{P}_m \to M \), where the top fibers describe the “internal degrees of freedom” and \( \mathcal{P}_m \subset \mathcal{P} \cong TM \) is the sub-bundle over \( M \) of future shells for the particle’s mass \( m \). At each \( x \in M \) we perform the constructions presented in the previous sections, with the generic manifold \( X \) now replaced by \((\mathcal{P}_m)_x \). In particular we get spaces \( \mathcal{Z}^1_x \equiv \mathcal{P}((\mathcal{P}_m)_x, Z_x) \), the fibered set \( \mathcal{Z}^1 := \bigcup_{x \in M} \mathcal{Z}^1_x \) and the multi-particle state bundle
\[ \mathcal{Z} := \bigoplus_{n=0}^\infty \mathcal{Z}^n \to M . \]

It turns out that \( \mathcal{Z} \to M \), as well as other similar or related bundles, is naturally a smooth vector bundle according to Frölicher’s notion of smoothness [11, 16, 17, 21, 2, 20].

Considering more particle types, one eventually gets the total quantum bundle\(^5\)
\[ \mathcal{V} := \mathcal{Z}' \otimes \mathcal{Z}'' \otimes \mathcal{Z}''' \otimes \cdots = \bigoplus_{n=0}^\infty \mathcal{V}^n \to M . \]

\(^5\)The quantum bundles for particle types of different mass are constructed over different mass-shell bundles.
1.5 Quantum configuration space

Similarly, one gets the Frölicher-smooth vector bundles $\mathcal{Z}_o \to M$ of all test fiber semi-densities and $\mathcal{Z} \to M$ of all finite sums fiber semi-densities of Dirac type.

Now consider an orthogonal splitting $T^*M \equiv P = P_\parallel \oplus P_\perp$ into “timelike” and “spacelike” $g$-orthogonal subbundles over $M$ (which can be seen as associated to the choice of an observer). Let $\eta_\perp$ be the volume form, associated with the metric, on the fibers of $P_\perp \to M$. The orthogonal projection $P \to P_\perp$ yields a distinguished diffeomorphism $P_m \leftrightarrow P_\perp$ for each $m$. The pull-back of $\eta_\perp$, denoted by the same symbol, is then a volume form on the fibers of $P_m$. The Leray form

$$\omega_m \equiv \omega[p_0 - e_m(\mu)] , \quad e_m(\mu) = (m^2 + |\mu|^2)^{1/2} ,$$

can now be then written as

$$\omega_m(p) = (2p_0)^{-1}\eta_\perp(p) , \quad p \in P_m , \quad p_0 \equiv e_m(\mu) .$$

This is a distinguished 3-form on each fiber of $P_m \to M$, and can also be regarded as a generalized density on each fiber of $P$.

It will be convenient to use the “spatial part” $\mu$ of the 4-momentum $p$ as a label, that is a generalised index for quantum states. If $(\alpha)$ is a frame of $Z \to P_m$ then we consider the generalised frame $\{B_{\alpha}(\mu)\} \equiv \{X_\mu \otimes \alpha\}$, where $X_\mu$ is defined as follows. For each $p \in P_m$ let $\delta_m(p)$ the Dirac density with support $\{p\}$ on the same fiber of $P_m \to M$, and let $\delta(y_\mu - \mu)$ be the generalised function characterised by $\delta_m[p](y_\mu) = \delta(y_\mu - \mu) d^3y_\mu$ in terms of linear coordinates $(y_\mu) \equiv (y_0, y_1, y_2, y_3) \equiv (y_0, y_\perp)$ in the fibers of $P$. Then for each $p \in P_m$ we regard $X_\mu$ as a generalised function of the variable $y_\perp$, with the expression

$$X_\mu(y_\perp) := l^{-3/2} \delta(y_\mu - \mu) \sqrt{d^3y_\perp} .$$

Here $l$ is a constant length needed in order to get an unscaled (“conformally invariant”) semi-density.

1.5 Quantum configuration space

In order to build a viable theory of quantum particles and their interactions one needs a time function, possibly associated with an observer of some kind. Having a global such structure in curved spacetime is a non-trivial requirement. However we may consider a somewhat weaker setting [5, 9], based on the assignment of a detector; that is a timelike submanifold $T \subset M$; indeed a momentum-space formalism for particle interactions, in terms of generalised semi-densities, can be exhibited as a sort of a complicated ‘clock’ carried by it. In the case of an inertial detector in flat spacetime, the Fourier transform relates the momentum-space and the position-space formalisms; this correspondence can be naturally extended to the curved spacetime case but, in general, only locally (in a sense to be made precise).

A generalised frame of free one-particle states along $T$ can be introduced by fixing any event $t_0 \in T \subset M$ and a classical frame $(\alpha)$ of the bundle $Z \to (P_m)_{t_0}$. The family of generalised semi-densities $\{B_{\alpha}(t_0)\}$ is then a generalised frame of $Z_{t_0} \to (P_m)_{t_0}$, which can be transported along $T$ by virtue of the underlying geometric structure. We obtain sections

$$B_{\alpha} : T \to \mathcal{D}(P_m, Z) : t \mapsto B_{\alpha}(t) = X_{p(t)} \otimes \alpha ,$$

\[\text{Let } M \text{ be a manifold with a chosen volume form } \eta, \text{ and } f \text{ a function on } M \text{ such that the submanifold } N \subset M \text{ is characterized by } f = 0 \text{ and } df \text{ nowhere vanishes on } N. \text{ Then the Leray form } \omega[f], \text{ often denoted as } \delta(f), \text{ is characterized [14] by the condition that } df \wedge \omega[f] = \eta \text{ holds on } N.\]

\[\text{This includes Fermi transport [7, 9] for the spacetime related factors, and a background connection of } Z \text{ which will have to be assumed [9].}\]
where \( p: T \to P_m : t \mapsto p(t) \) is Fermi-transported. This yields a trivialization
\[
\mathcal{P}(P_m, Z)_T \cong T \times \mathcal{P}(P_m, Z)_{t_0} ,
\]
which can be seen as determined by a suitable connection called the \textit{free-particle connection}. Eventually, the above arguments can be naturally extended to multi-particle bundles and states. When several particle types are considered, we get a trivialization \( \mathcal{V}_T \cong T \times \mathcal{Q} \) of the total quantum state bundle, where \( \mathcal{Q} \equiv \mathcal{V}_{t_0} \) can be seen as the “quantum configuration space”. The quantum interaction, an added term that modifies the free-field connection, can be constructed by assembling the classical interaction with a distinguished quantum ingredient \([5, 9]\). By construction, the free-particle transport preserves particle type and number. Accordingly, we also get the operator algebra
\[
\mathcal{O} \cong \mathcal{Q} \otimes J^* \equiv \mathcal{V}_{t_0} \otimes \mathcal{V}_{t_0}^* ,
\]
where the identification is determined via normal ordering.

The relation to position-space formalism can be summarized as follows. The restriction of the tangent bundle of \( M \) to base \( T \) splits as \( (TM)_T = (TM)^\parallel_T \oplus (TM)^\perp_T \) into “timelike” and “spacelike” \( g \)-orthogonal subbundles. Exponentiation determines, for each \( t \in T \), a diffeomorphism from a neighbourhood of 0 in \( (TM)^\perp_t \) to a spacelike submanifold \( M_t \subset M \), and so a 3-dimensional foliation of a neighbourhood \( N \equiv \bigcup_{t \in T} M_t \subset M \) of \( T \). A \textit{tempered} generalised semi-density on \( (P_m)_t \) yields, via Fourier transform, a generalised semi-density on \( (TM)^\perp_t \). A suitable restriction\(^a\) then yields, via exponentiation, a generalised semi-density on \( M_t \). This correspondence can be extended to \( Z \)-valued semi-densities by means of background linear connections of the various “internal” bundles. Eventually, the trivialisation \( \mathcal{V}_T \cong T \times \mathcal{Q} \) can be extended as \( \mathcal{V}_N \cong N \times \mathcal{Q} \). For an inertial detector in flat spacetime we essentially get the usual correspondence between momentum-space and position-space representation.

1.6 Quantum fields

If a fibered Hermitian structure of \( Z \hookrightarrow P_m \) is assumed, then any \( \zeta \in \mathcal{P}(P_m, Z^*) \) yields an absorption operator \( a[\zeta] \) and an emission operator \( a^*[\zeta^\#] \) as well. Proceeding as in §1.3 we can now see \( a^\alpha \) and \( a^\alpha \) as generalised functions of momentum, which in terms of the previously described generalised frames can be written as
\[
a^\alpha(p_\perp) \equiv a^\alpha = a[B^\alpha\#] \equiv a[X^p \otimes b^\alpha] , \quad a^\alpha(p_\perp) \equiv a^{\rho\alpha} := a^\alpha[(B^{\rho\alpha})\#] \equiv a^\alpha[X^p \otimes \bar{b}^\alpha] .
\]
Consistently with the generalised index notation we also write \( a[\zeta] = \zeta_{\rho\alpha} a^{\rho\alpha} , a^*[\zeta^\#] = \zeta_{\rho\alpha} a^{\star\rho\alpha} \), and eventually
\[
a[\zeta] = a^{\rho\alpha} B_{\rho\alpha} , \quad a^*[\zeta^\#] = a^{\star\rho\alpha} B_{\rho\alpha} .
\]

Essentially, free quantum fields are introduced as combinations of Fourier transforms and anti-transforms of the above objects. However, the fact that \( Z \) is in general a vector bundle over \( P_m \) may stand in the way of expressing a quantum field as a section of some bundle over \( M \). In order to overcome this difficulty we first note that in the situations of interest \( Z \) is a subbundle of a “semi-trivial” bundle, namely
\[
Z \subseteq P_m \times Z' \]
\(^a\)A distribution can be restricted to an open set \([25]\).
where $Z' \to M$ is a vector bundle. For each $p \in P_m$, the fiber’s algebraic structure determines a projection $\Pi_Z(p) : Z' \to Z_p$, which can be expressed as

$$\Pi_Z(p) = b_\alpha(p) \otimes b^\alpha(p),$$

in a suitable frame adapted to $Z_p$; if $Z = P_m \times_M Z'$ then $\Pi_Z(p)$ is just the identity. Analogously, the map

$$\Pi_{Z^*}(p) = \tilde{b}_\alpha(p) \otimes \tilde{b}^\alpha(p)$$

is either the identity or the projection onto $Z_p^* \cong Z_p$. Now, by composing the second tensor factors with absorption and emission operators, and doing transpositions for formal purposes, we obtain the maps

$$\Phi^+ : P_m \to \mathcal{O} \otimes Z : p \mapsto a^\alpha(p) \otimes b_\alpha(p),$$

$$\Phi^- : P_m \to \mathcal{O} \otimes Z^* : p \mapsto a^{*\alpha}(p) \otimes \tilde{b}_\alpha(p).$$

Working with a chosen observer we label momenta $p \in P_m$ by their “spatial” part $p_L$ ($§1.4$). We may then select an orthonormal frame $(b_\alpha(0))$ corresponding to $p_L = 0$. In the situations of our interest one finds that, for each $p_L \in P_L$, a natural and essentially unique unitary transformation $K(p_L) : Z_0 \to Z_{p_L}$, which yields the orthonormal frames

$$(b_\alpha(p_L)) \equiv (K(p_L)b_\alpha(0)) = (K^\beta_\alpha(p_L) b_\beta(0)), \quad p_L \in P_m.$$  

We then note that, because of unitarity, the conjugate frames $(\tilde{b}_\alpha(p_L))$ transform with the same rule, while both the dual frame $(b^\alpha(p_L))$ and the anti-dual frame $(\tilde{b}^\alpha(p_L))$ transform according to the inverse matrix $(K_{\alpha}^\beta(p_L))$. We now express $\Phi^+$ and $\Phi^-$ as

$$\Phi^+(p_L) = \Phi^{*\alpha}(p_L) \otimes b_\alpha(0) \equiv (K_\beta^\alpha(p_L) a^\beta(p_L)) \otimes b_\alpha(0),$$

$$\Phi^-(p_L) = \Phi^{-\alpha}(p_L) \otimes \tilde{b}_\alpha(0) \equiv (K_{\alpha}^\beta(p_L) a^{*\beta}(p_L)) \otimes \tilde{b}_\alpha(0).$$

The above components $\Phi^{\alpha\alpha}$ and $\Phi^{*-\alpha}$ are written in the frame $(b_\alpha(0))$, which is independent of momentum. Next we consider again the setting described in $§1.5$, and realize that for all $t \in T$ we can perform spatial Fourier transforms and anti-transforms of $\Phi^{\alpha\alpha}$ and $\Phi^{-\alpha}$, obtaining $\mathcal{O}$-valued distributions on $(TM)_T^\bot$. We then get the generalized map

$$\phi = \phi^\alpha b_\alpha(0) : (TM)_T^\bot \to \mathcal{O} \otimes Z$$

whose components have the expression

$$\phi^\alpha(x) \equiv \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} K_\beta^\alpha(p_L)(e^{-i(p,x)} a^\beta(p_L) + e^{i(p,x)} a^{*\beta}(p_L)), \quad p_0 \equiv (m^2 + p_L^2)^{1/2},$$

and are easily seen to fulfill the Klein-Gordon equation. We remark that the above free field can always be seen as a combination of \textit{particle absorption and anti-particle emission}, where the terms “particle” and “antiparticle” refer to the internal bundles $Z$ and $Z'$; if $Z$ is real then these coincide, and we lose the distinction.

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9 Most notably, the inclusion is proper in the case of the electron and positron bundles ($§2.4.1$).

10 The factor $(2p_0)^{-1/2}$ is related to the Leray form of the mass shell.
In comparison with the notion of a field defined on \( M \), the above scheme can be seen as yielding a kind of linearized construction. If \((M, g)\) is Minkovski spacetime and we have an inertial orthogonal decomposition \( M = T \times X \), then by obvious identifications we also obtain a true field over \( M \); but note that the affine space \( X \) (the space of “positions” of the chosen observer) has here a distinguished point, namely the detector’s position, so that it can be identified with a vector space. In curved spacetime a section of a vector bundle over \( M \) can be obtained too, but possibly only in a neighbourhood of the detector \( T \). Without entering details, the construction uses the local isomorphism (§1.5) of a neighbourhood of \( t \) in \( M \) with a neighbourhood of 0 in \( (TM)_t^\perp \), together with parallel transport of \( b_\gamma(0) \) along the spacelike geodesic from \( t \) to \( x \in M_t \) relatively to a fixed background connection (possibly related to gauge-fixing). We stress that the components \( \phi^\alpha(x) \) are valued in a fixed algebra of linear operators on the space \( Q \) of quantum states. This is actually the extended operator space \( \mathcal{O}^* \) (§1.2), but we’ll indicate it as \( \mathcal{O} \) for notational simplicity.

The above constructions yield a so-called free quantum field, which is an essentially unique, well-defined object, fulfilling the Klein-Gordon equation and determined by the underlying classical geometry. More generally we can consider arbitrary generalised sections \( M \to \mathcal{O} \otimes_M Z \). The quantum fields of a theory can be described as generalised sections \( M \to \mathcal{E} = \mathcal{O} \otimes E \), where \( E \to M \) is the classical “configuration bundle” (this is the finite-dimensional vector bundle whose sections are the “pre-quantum” fields) and \( E \to M \) is the corresponding “quantum bundle”.

For simplicity of notation and exposition, in the rest of this paper we’ll work in flat spacetime with a given inertial decomposition \( M = T \times X \) (and \( X \) is identified with a vector space as remarked above), but we stress that most constructions and results, with proper caveats, can be recast in a more general scenario.

### 1.7 Conjugate fields

The scheme sketched in §1.6 is suitable for describing bosonic and fermionic free quantum fields, as the differences between these two cases are dealt with by the super-commutation rules among absorption and emission operators. However there is a complication, related to conjugate fields, which deserves a thorough discussion.

We begin by clarifying a notational issue. In the standard theoretical physics literature, complex conjugation is usually indicated by an asterisk. In mathematics, complex conjugation is usually indicated by an overbar, while an asterisk labels transposition. We’ll stick to the mathematics usage, and note that there is one situation of apparent conflict: the “Dirac adjoint” \( \bar{\psi} \) of a Dirac spinor \( \psi \). Actually it turns out that this is easily adjusted, as the space \( W \) of 4-spinors has a natural Hermitian structure of signature \((2,2)\), and \( \bar{\psi} \in W^* \) is exactly the element corresponding to the complex conjugate of \( \psi \) via the induced isomorphism \( W \leftrightarrow W^* \). In general, no issue arises about denoting the Hermitian transpose of \( \phi \) as \( \phi^\dagger \), though that is somewhat pleonastic as one could just write \( \bar{\phi} \) implying the isomorphism \( Z \leftrightarrow Z^* \) determined by Hermitian structure. But note that \( \psi^\dagger \equiv \bar{\psi} \gamma^0 \), in the Dirac context, is the transpose of \( \bar{\psi} \) with respect to a different, positive definite Hermitian structure which is associated with the chosen observer.

The above considerations are valid in the classical field context, but in the quantum context there are further complications. With regard to conjugation we have two possible

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11 One finds further notational variations, however. For example in [15] an asterisk stands for Hermitian transposition while dual spaces are labeled by the symbol #.

12 These issues were thoroughly examined in previous papers [3, 4, 6].
constructions: we can take the complex conjugate of $\phi$, and also make the same construction used for $\phi$ but replacing the internal bundle $Z$ with the “anti-particle” bundle $\overline{Z}$. Moreover we can apply transposition in the operator algebra $\mathcal{O}$, indicated by an asterisk. Hence in connection with the free field $\phi = \phi^\alpha b_\alpha(0)$ introduced in §1.6 we also obtain the fields

$$\phi^* = \phi^\alpha^* b_\alpha(0), \quad \bar{\phi} = \bar{\phi}^\alpha b_\alpha(0), \quad \bar{\phi}^* = (\bar{\phi}^\alpha)^* \bar{b}_\alpha(0),$$

where\(^\text{13}\)

$$\phi^\alpha^*(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} K^\beta_\alpha(p_\perp) \left( e^{-i \langle p, x \rangle} a_\beta(p_\perp) + e^{i \langle p, x \rangle} a^\beta(p_\perp) \right),$$

$$\bar{\phi}_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} \bar{K}_\alpha^\beta(p_\perp) \left( e^{i \langle p, x \rangle} a_\beta(p_\perp) + e^{-i \langle p, x \rangle} a^\beta(p_\perp) \right),$$

$$\bar{\phi}_\alpha^*(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} \bar{K}_\alpha^\beta(p_\perp) \left( e^{i \langle p, x \rangle} a_\beta(p_\perp) + e^{-i \langle p, x \rangle} a^\beta(p_\perp) \right).$$

We then see that $\bar{\phi}^*$ is exactly the “free field of the conjugate bundle”, namely it can be obtained by the same construction as $\phi$, after replacing $Z$ with $\overline{Z}$, as a combination of anti-particle absorption and particle emission operators. Up to identifications which seem obvious in the matrix formalism, $\bar{\phi}^*$ is essentially the field which in a generic setting is usually denoted as $\phi^\dagger$.

In a classical field theory one deals with fields and their complex conjugates, which are to be replaced with $\mathcal{O}$-valued fields upon quantization. Free fields play a specially important role as basic “building blocks” of field dynamics. When we evaluate any functional of the fields in terms of free fields, we are to replace the classical components $\phi^\alpha(x)$ with the expression written in §1.6. Which is the correct replacement for $\bar{\phi}_\alpha(x)$? The answer depends on certain properties that field super-commutators must obey. We claim that it is

$$\bar{\phi}_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} \bar{K}_\alpha^\beta(p_\perp) \left( \pm e^{-i \langle p, x \rangle} a_\beta(p_\perp) + e^{i \langle p, x \rangle} a^\beta(p_\perp) \right),$$

where the upper sign in the first term in the integrand holds for boson fields, while the lower sign holds for fermion fields. Thus $\bar{\phi}_\alpha$ coincides with $\bar{\phi}_\alpha^*$ for bosons but not for fermions. This seems to be in contrast with standard presentations, as far as fermion fields are concerned; however we’ll argue that the minus sign is actually present in the usual expression for the Dirac-adjoint quantum field $\tilde{\psi}$, though somewhat hidden in the intricacies of the matrix formalism. Moreover we’ll check that the required identities, and the free-field expressions of the most important functionals, do follow from the above prescription.\(^\text{14}\)

\textbf{Remark.} Because of the isomorphism $\overline{Z} \leftrightarrow Z^*$, we can equivalently view $Z^*$ as the internal anti-particle bundle. Both views require the Hermitian structure unless we deal with real bundles (which by the way is exactly the case of the ghost and anti-ghost fields, see §2.4.3).

\(^{13}\)We used $K^\dagger = \bar{K}$.

\(^{14}\)The notion of anti-particle is often introduced, according to an historical presentation, in the discussion of the Dirac spinor field, in relation to the equal-time commutation rules which quantum fields and their conjugates are required to obey as an implementation of the principle of correspondence. The same discussion is also offered as a justification for the introduction of anti-commuting absorption and emission operators.
1.8 Recalls about propagators

In terms of the decomposition $M = T \times X$ discussed in §1.6 we write $x \equiv (t, x_\perp) \in M$. We also have (§1.4) the splitting $T^*M \equiv P = P_0 \oplus P$. This is the bundle of momenta, which is trivial in the flat case. We write $p \equiv (p_0, p_\perp) \in P$. If $p \in P_m \subset P$ then

$$p_0 = \nu_m(p_\perp) \equiv \sqrt{m^2 + |p_\perp|^2}.$$  

The evaluation of field super-commutators yields the integrals

$$D^\pm(x) \equiv \frac{\pm 1}{(2\pi)^3} \int \frac{d^3p_\perp}{2p_0} e^{\mp i\langle p, x \rangle}, \quad p \in P_m,$$

which are well-defined distributions. The convention of using the symbol $p_0$ as a positive “on-shell” function of $p_\perp$ is common and here we’ll use it, though it could be confusing if one aims at a systematical understanding of the relations among special generalized densities and propagators.\(^\text{15}\) Despite appearance, these are full Fourier transforms, since they can also be written in the form

$$D^\pm(x) = \frac{1}{(2\pi)^3} \int \frac{d^4p}{2p_0} e^{\mp i\langle p, x \rangle} \delta(p_0 \mp \nu_m(p)) .$$

As such, they are recognized as the Fourier transform and minus the Fourier anti-transforms of the Leray density $\omega_m$ of $P_m$ divided by $2\pi$ and seen as a generalized density on $P$ (§1.4).

We’ll be also involved with the partial derivatives

$$D^\pm_\lambda(x) \equiv \frac{\partial}{\partial x^\lambda} D(x) = \frac{-i}{(2\pi)^3} \int \frac{d^3p_\perp}{2\nu_m(p)} p_\lambda e^{\mp i\langle p, x \rangle}, \quad \lambda = 0, 1, 2, 3.$$  

Moreover we set $\mathcal{D} \equiv D^+ + D^-$, and find

$$D^+(x) = -D^-(x) \quad \Rightarrow \quad \mathcal{D}(-x) = -\mathcal{D}(x) .$$

Finally we obtain the “zero-time” relations

$$D^+(0, x_\perp) = -D^-(0, x_\perp) \quad \Rightarrow \quad D(0, x_\perp) = 0 ,$$  

$$D^\pm_0(0, x_\perp) = -\frac{i}{2} \delta(x_\perp) \quad \Rightarrow \quad -i \delta(x_\perp) = D_0(0, x_\perp) \equiv D^+_0(0, x_\perp) + D^-_0(0, x_\perp) .$$

While the generalized density $\omega_m$ is observer-dependent, the combination $\mathcal{D} \equiv D^+ + D^-$ turns out to be a geometrically well-defined object, as it is the Fourier transform of the observer-independent Leray form\(^\text{16}\) $\omega[g - m^2]$ where $g(p) \equiv p^2$. Hence the above identities imply that $\mathcal{D}$ and its derivatives vanish outside the causal cone.

1.9 Field super-commutators

The basic super-commutation rules of emission and absorption operators (§1.3) can be rewritten in the present context as follows. We have

$$\{[a^\alpha(p_\perp), a^\beta_0(q_\perp)] = \{[a^\beta_0(p_\perp), a^\alpha(q_\perp)] = \delta^\alpha_\beta \delta(p_\perp - q_\perp) ,$$

\(^{15}\)In the physics literature one tries to avoid such issues, possibly by \textit{ad hoc} spatial variable changes.\(^{16}\) Usually denoted as $\delta(p^2 - m^2)$.
while other super-commutators vanish, namely
\[
0 = \{[a^\alpha(p_\perp), a^\beta(q_\perp)] = \{[a^\alpha(p_\perp), a_\beta(q_\perp)] = \{[a_\alpha(p_\perp), a^\beta(q_\perp)] = \{[a_\alpha(p_\perp), a_\beta(q_\perp)] = \{a_\alpha(p_\perp), a_\beta(q_\perp)] = \{a_\alpha(p_\perp), a_\beta(q_\perp)] = \{[a^\alpha(p_\perp), a^\beta(q_\perp)] = \{a^\alpha(p_\perp), a^\beta(q_\perp)] = \{[a^\alpha(p_\perp), a_\beta(q_\perp)] = \{[a^\alpha(p_\perp), a_\beta(q_\perp)] = \{a^\alpha(p_\perp), a^\beta(q_\perp)] = \{a^\alpha(p_\perp), a^\beta(q_\perp)] = \{[a^\alpha(p_\perp), a^\beta(q_\perp)] = \{a^\alpha(p_\perp), a^\beta(q_\perp)] = \{[a^\alpha(p_\perp), a_\beta(q_\perp)] = \{[a^\alpha(p_\perp), a_\beta(q_\perp)] = \{a^\alpha(p_\perp), a_\beta(q_\perp)] .
\]

Then, for any two events \(x, x' \in \mathbf{M}\) we have the vanishing super-commutators\(^1\)
\[
\{[\phi^\alpha(x), \phi^\beta(x')] = \{[\phi^\alpha(x), \phi^\beta(x')] = \{[\phi^\alpha(x), \phi^\beta(x')] = 0 ,
\]
Moreover we find the super-commutators
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta(x')] = \delta^\alpha_\beta (D^+(x + x') \pm D^-(x + x')) ,
\]
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta(x')] = \delta^\alpha_\beta (D^+(x - x') \pm D^-(x - x')) ,
\]
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta(x')] = \delta^\alpha_\beta (D^+(x - x') + D^-(x - x')) = \delta^\alpha_\beta D(x - x') ,
\]
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta,\lambda(x')] = \delta^\alpha_\beta (D^+_\lambda(x + x') \mp D^-_\lambda(x + x')) ,
\]
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta,\lambda(x')] = \delta^\alpha_\beta (-D^-_\lambda(x - x') \mp D^+_\lambda(x - x')) ,
\]
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta,\lambda(x')] = -\delta^\alpha_\beta (D^-_\lambda(x - x') + D^+_\lambda(x - x')) = -\delta^\alpha_\beta D_\lambda(x - x') ,
\]
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta,\lambda(x')] = \delta^\alpha_\beta (D^+_\lambda(x - x') + D^-_\lambda(x - x')) = \delta^\alpha_\beta D_\lambda(x - x') ,
\]
where \(\tilde{\phi}_\alpha,\lambda \equiv \partial \tilde{\phi}_\alpha / \partial x^\lambda\) and the like, and double signs apply to the alternative boson/fermion.

We now observe that, out of the above non-vanishing super-commutators, those which involve \(\tilde{\psi}\) and its derivatives depend on the difference \(x - x'\) and are expressed in terms of the observer-independent distribution \(\mathcal{D}\). This fact endorses our prescription of \(\tilde{\phi}\) as the right free quantum field replacement for a classical field \(\phi\). In the bosonic case \(\tilde{\phi}\) coincides with \(\phi^\dagger\) and \(\phi^\ast\), which in a generic context is usually indicated as \(\phi^\dagger\) (§1.7).

At equal times \((x^0 = x'^0)\) we obtain
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta(x')] = 0 ,
\]
\[
\{[\phi^\alpha(x), \tilde{\phi}_\beta,0(x')] = -\{[\phi^\alpha(x), \tilde{\phi}_\beta(x')] = -i \delta^\alpha_\beta \delta(x_\perp - x'_\perp) .
\]

1.10 Conjugate momenta and the Hamiltonian

In the context of Lagrangian field theory one sets \(\pi_\alpha := \partial \ell / \partial \phi_0^\alpha\) where \(\ell d^4x\) is the total Lagrangian density. In a Hamiltonian setting, \(\pi_\alpha\) plays the role of the “conjugate momentum” associated with \(\phi^\alpha\). The required equal-time super-commutation rules are of the type
\[
\{[\phi^\alpha(x), \pi_\beta(x')] = \pm i \delta^\alpha_\beta \delta(x_\perp - x'_\perp) \sqrt{g} ,
\]
\[
\{[\phi^\alpha(x), \phi^\beta(x')] = \{[\pi_\alpha(x), \pi_\beta(x')] = 0 .
\]
with \( x \equiv (t, x) \), \( x' \equiv (t, x') \), \( \sqrt{|g|} \equiv \sqrt{|\det g|} \). These rules are to be directly checked to hold true for free fields; their validity for critical sections\(^{18}\) can then be inferred by general arguments based on the form of the dynamics. Note that, in standard expressions written in terms of field components, the product of field components valued at the same spacetime point is defined by normal ordering (§1.2), in order to obtain \( \mathcal{O} \)-valued quantities. Instead, normal ordering is not assumed in the above rules, which must be intended in a generalized distributional sense.

The Hamiltonian density and the Hamiltonian of a general field theory are the functionals

\[
\phi \mapsto \mathcal{H}[\phi] = \Pi_\alpha[\phi] \phi_\alpha^\dagger - \ell[\phi], \quad H[\phi](t) = \int d^3 x_\perp \mathcal{H}[\phi](t, x_\perp).
\]

In particular one is interested in the free Hamiltonian in each sector of the theory, obtained by dropping all interactions with other sectors and then evaluating through free fields. A basic example is obtained from the sector Lagrangian

\[
\ell_{\text{free}}[\phi, \bar{\phi}] = \left( \frac{1}{2} g^{\lambda \mu} \bar{\phi}_{\alpha, \lambda} \phi_{\alpha, \mu} - \frac{1}{2} m^2 \bar{\phi}_{\alpha} \phi_{\alpha} \right) \sqrt{|g|}, \quad \lambda, \mu = 0, 1, 2, 3,
\]

whence

\[
\Pi_\alpha = \frac{1}{2} g^{0 \lambda} \bar{\phi}_{\alpha, \lambda} \sqrt{|g|}, \quad \Pi^\alpha = \frac{1}{2} g_{\mu}^{0 \lambda} \phi_{\alpha, \mu} \sqrt{|g|},
\]

\[
-\frac{1}{\sqrt{|g|}} \mathcal{H}_{\text{free}}[\phi, \bar{\phi}] = \bar{\phi}_{\alpha, 0} \phi_{\alpha}^0 - \frac{1}{2} g^{\lambda \mu} \bar{\phi}_{\alpha, \lambda} \phi_{\alpha, \mu} + \frac{1}{2} m^2 \bar{\phi}_{\alpha} \phi_{\alpha} =
\]

\[
= \frac{1}{2} \bar{\phi}_{\alpha, 0} \phi_{\alpha}^0 - \frac{1}{2} g^{ij} \bar{\phi}_{\alpha, i} \phi_{\alpha, j} + \frac{1}{2} m^2 \bar{\phi}_{\alpha} \phi_{\alpha}, \quad i, j = 1, 2, 3.
\]

By evaluation through quantum free fields we then see, keeping the results of §1.9 into account, that the above written equal-time super-commutation rules are indeed fulfilled. Moreover, allowing normal ordering we obtain\(^{19}\)

\[
H_{\text{free}}[\phi, \bar{\phi}] = \frac{1}{2} \int d^3 x_\perp \left( \bar{\phi}_{\alpha, 0} \phi_{\alpha}^0 - g^{ij} \bar{\phi}_{\alpha, i} \phi_{\alpha, j} + m^2 \bar{\phi}_{\alpha} \phi_{\alpha} \right)(t, x_\perp) =
\]

\[
= \frac{1}{2} \int d^3 p_\perp p_0 \left( a^\alpha_\beta(p_\perp) a^\dagger_\beta(p_\perp) + a^\dagger_\beta(p_\perp) a^\alpha_\beta(p_\perp) \right),
\]

which holds for boson and fermion fields alike.

2 Quantum fields in a gauge theory

2.1 Introductory remarks

We can view the notion of a quantum field in two different ways. One way, explored in §1, uses a very restricted geometric setting, requiring an inertial observer in flat spacetime, in order to introduce the field in terms of the elementary operators on particle state space. Those constructions can be generalized, at least locally, to a curved background, but the need for an observer of some kind remains. Actually, effective arguments in QFT need that too. However one could consider a different point of view by studying a field theory in a “quantum bundle”\(^{18}\)

\(^{18}\)That is solutions of the full field equations with interactions.

\(^{19}\)We are not explicitly writing this calculation, which turns out to be longer than one would expect at first sight.
2.2 Remarks about fiber endomorphisms

\[ \mathcal{E} \cong \mathcal{O} \otimes E \to M, \] corresponding to the classical “configuration bundle” \( E \to M \) (necessarily a vector bundle) via fiber tensorialization by a suitable, fixed \( \mathbb{Z}_2 \)-graded algebra \( \mathcal{O} \).

We may argue that, in general, the compatibility between these two points of view is local and partial, as well as the complementarity between particle and field representations.

The fundamental differential geometric notions for quantum bundles, and a related jet bundle approach to Lagrangian field theory and symmetries, were studied in a previous paper [11] by exploiting Frölicher’s notion of smoothness [16, 17, 21, 2, 20]. Eventually, most aspects work quite similarly to their finite-dimensional counterparts, but there are delicate points. In particular, fiber coordinates (different from base coordinates) are now \( \mathcal{O} \)-valued, so that the meaning of partial derivatives with respect to these is not obvious at first sight. However it can be shown that geometrically meaningful classical expressions containing such derivatives do have a well-defined quantum counterpart if the expressions to be derived are polynomials in the fiber coordinates (which is certainly the case in most physically relevant situations). The derivation rules are then essentially the same, though with additional conditions regarding the ordering of factors. The above cited paper [11] could be seen, in principle, as a prerequisite for a Lagrangian quantum field theory, but it is not strictly needed for a first reading of what follows.

A further remark regards the relations between mutually conjugate fields. It is often stressed that the ghost and anti-ghost fields are independent of each other. Indeed, they appear asymmetrically in the Lagrangian. By contrast, the Dirac fields \( \psi \) and \( \bar{\psi} \) are formally exchanged by conjugation in the Lagrangian and in the field equations. Whatever the form of the field equations, however, any fields \( \phi \) and \( \bar{\phi} \) (§1.7) could be seen as mutually independent even if they admit conjugate solutions. This point of view is strengthened by the observation that one obtains the field equations by varying them independently, as well as from other considerations such as the derivation of the free Hamiltonian (§2.5).
on $\text{End} F$ given by

$$H : \text{End} F \times \text{End} F \to \mathbb{C} : (X, Y) \mapsto \text{Tr}(X^\dagger \circ Y).$$

Moreover every endomorphism can be uniquely written as the sum of a anti-Hermitian and a Hermitian endomorphism, namely one obtains\(^{20}\) the real splitting $\text{End} F = \mathcal{L} \oplus i \mathcal{L}$. Now it’s easy to check that the restrictions of $H$ to these real $n^2$-dimensional subbundles are real Euclidean (i.e. positive) scalar products. The above statement about the signature of the real 2-form $G$ then follows from the observation that $\mathcal{L}$ and $i \mathcal{L}$ are respectively characterized by the properties $X^\dagger = -X$ and $X^\dagger = X$ for any element $X$. Note how the assignment of a Hermitian structure on $F$ determines a splitting of the real vector space underlying $\text{End} F$ into the direct sum of two subspaces of opposite signatures.

If $(b_i)$ is an orthonormal frame of $F$ then the matrix of a section $X : M \to \mathcal{L}$ is anti-Hermitian. In particular, one can always find an orthonormal frame $(l_i)$ of $\mathcal{L}$ related to $(b_i)$ by the relations $l_i = l_{ij} b_j \otimes b^j$, where the matrices $(l_{ij})$ are constant. Then we obtain the constant coefficients (structure constants)

$$c_{JH}^I \equiv \langle l^I, [l_J, l_H] \rangle,$$

where $(l^I)$ is the dual frame.

### 2.3 Pre-quantum fields of an essential gauge theory

A gauge field theory with one fermion type can be formulated by assuming, as the fundamental geometric data, two complex bundles over a 4-dimensional manifold $M$:

- the two-spinor bundle (or Weyl bundle) $U \rightarrow M$ has 2-dimensional fibers, and the fibers of $\Lambda^2 U \rightarrow M$ are endowed with a Hermitian structure (but not the fibers of $U$ itself);
- the Hermitian bundle $F \rightarrow M$, whose fibers describe the internal degrees of freedom of fermions besides spin.

Then it turns out\(^{[3, 4, 6]}\) that that the fibers of the Hermitian subbundle $H \subset U \otimes \overline{U}$ are naturally endowed with a Lorentz structure, and there is a natural Clifford morphism $\gamma : H \rightarrow \text{End} W$, where $W := U \oplus_M \overline{U}^*$ can be identified as the Dirac bundle. The gravitational structure is jointly described by a scaled tetrad\(^{21}\) $\Theta : TM \rightarrow \mathbb{L} \otimes H$ and by a linear connection $\Gamma$ of $U \rightarrow M$ (2-spinor connection), which can be included among the variables of a comprehensive Lagrangian theory. In this article, however, we’ll assume a fixed gravitational background, represented by an assigned couple $(\Theta, \Gamma)$.

Consider the following pre-quantum fields:

- a “matter” field $\psi : M \rightarrow W \otimes_M F$;
- a gauge field, namely a linear Hermitian connection of $F \rightarrow M$.

The latter can be seen as a section $\alpha : M \rightarrow \Gamma$, where $\Gamma \rightarrow M$ is an affine bundle whose “derived” vector bundle (the bundle of differences of linear Hermitian connections) is $TM \otimes_M \mathcal{L} \rightarrow M$. Now the quantum theory requires the fields to be sections of vector bundles, whose fibers are tensorialized by a suitable operator algebra $\mathcal{O}$ (§1.5, 1.6). For gauge fields, this requirement is met by the choice of a local curvature-free connection $\alpha_0$. The field $\alpha$ is then represented by the difference $A \equiv \alpha - \alpha_0$.

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\(^{20}\)Then $\mathcal{L}$ is the Lie-algebra bundle of the group bundle of all unitary fiber automorphism. More generally, a fiber’s symmetry may be described by a different group bundle and its derived Lie-algebra bundle.

\(^{21}\)Here $\mathbb{L}$ is the space of length units. See\(^{[9, 19]}\) for a thorough account of unit spaces.
2.4 Gauge theory’s quantum free fields

2.4.1 Dirac field

We now inspect the case of the electron field, namely the internal bundle is the bundle $\mathbf{W}$ of Dirac spinors (these results straightforwardly carry over to $\mathbf{W} \otimes F$ for fermions with larger internal structure). Then we’ll briefly comment about the formal differences with usual presentations.

We first recall [4, 6] that the “semi-trivial” bundle $\mathbf{P}_m \times_M \mathbf{W} \rightarrow \mathbf{P}_m$ has the distinguished decomposition $\mathbf{W}^+ \oplus_{\mathfrak{p}_m} \mathbf{W}^-$, where $\mathbf{W}_p^\pm := \ker(m \mp \gamma_p)$. The bundles $\mathbf{W}^\pm \rightarrow \mathbf{P}_m$ are mutually orthogonal in the Hermitian metric associated with Dirac conjugation, which has the signature $(+−−)$; the sign of its restriction to $\mathbf{W}^\pm$ is the same as the label. A Dirac frame

$$(\zeta_\alpha(p)) \equiv (u_\alpha(p); v_\beta(p)) \ , \ \alpha = 1, 2, 3, 4 \ , \ A, B = 1, 2 \ ,$$

is adapted to the above decomposition at $p \in \mathbf{P}_m$. We have a distinguished transformation $K(p_\perp) : \mathbf{W} \rightarrow \mathbf{W}$ expressing it in terms of a frame independent of $p$, e.g. the Dirac frame $(\zeta_\alpha(0))$ associated with the chosen observer. Namely $\zeta_\alpha(p_\perp) = K^\beta_\alpha(p_\perp) \zeta_\beta(0)$, where the $4 \times 4$ matrix of $K(p_\perp)$ in the frame $(\zeta_\alpha(0))$ can be expressed as

$$K(p_\perp) = \sqrt{\frac{m}{2(E_m(p_\perp) + m)}} (\mathbb{I} + \frac{i}{m} p_\perp \gamma^\lambda \gamma_0) \ , \ \ (m^2 + |p_\perp|^2)^{1/2} \equiv E_m(p_\perp) \equiv p_0 > 0 \ .$$

This is essentially the transformation $K$ appearing in the definition of the components of the free quantum field (§1.6), but there is a slight complication: the particle (electron) and

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22We are also ignoring the Higgs field and related issues. An account of these aspects in the pre-quantum geometric context presented here can be found in previous papers [8, 12].
anti-particle (positron) bundles are now $W^+$ and $W^-$, so they are not mutually conjugate bundles. Accordingly, we introduce the absorption and emission operators
\[
\begin{align*}
    a^A(p_\perp) &\equiv a^A[X^p \otimes u^A(p)] , \\
    c_\alpha(p_\perp) &\equiv a^A[X^p \otimes \bar{v}_A(p)] , \\
    c_A(p_\perp) &\equiv a^A[X^p \otimes \bar{v}_A(p)] , \\
    a_\alpha^*(p_\perp) &\equiv a^A[X^p \otimes u_A(p)] ,
\end{align*}
\]
and obtain the non-vanishing anti-commutators
\[
\{a^A(p_\perp), a^\beta_B(q_\perp)\} = \{c_B(p_\perp), c^A(q_\perp)\} = \delta^A_B \delta(p_\perp - q_\perp) .
\]
Repeating the construction seen in §1.6 we now express the elementary operators in the frame $(\zeta_\alpha(0))$, namely
\[
\begin{align*}
    a^\alpha(p) &\equiv K^\alpha_\alpha(p) a^A(p) , \\
    c^\alpha(p) &\equiv K^\alpha_\alpha(p) c^A(p) , \\
    c_\alpha(p) &\equiv K^\alpha_\alpha(p) c_A(p) , \\
    a_\alpha^*(p) &\equiv K^\alpha_\alpha(p) a_\alpha^*(p) ,
\end{align*}
\]
whence by straightforward calculations we get the anti-commutators
\[
\begin{align*}
    \{a_\alpha^*(p), a^\beta(q)\} &= \frac{1}{2m} (m \ I_p + p_\lambda \gamma^\lambda) \delta(p_\perp - q_\perp) , \\
    \{c_\alpha(p), c^\beta(q)\} &= \frac{1}{2m} (m \ I_p - p_\lambda \gamma^\lambda) \delta(p_\perp - q_\perp) , \\
    \{a_\alpha(p), c^\beta(q)\} &= \{c_\alpha(p), a^\beta(q)\} = 0 .
\end{align*}
\]
Now, according to the general prescription introduced in §1.6, we consider the free fields $\psi$ and $\bar{\psi}$ whose components in the frame $(\zeta_\alpha(0))$ are
\[
\begin{align*}
    \psi^\alpha(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p_\perp}{\sqrt{2p_0}} \left( e^{-i p_\perp \cdot x} a^\alpha(p_\perp) + e^{i p_\perp \cdot x} c^\alpha(p_\perp) \right) , \\
    \bar{\psi}_\alpha(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p_\perp}{\sqrt{2p_0}} \left( -e^{-i p_\perp \cdot x} c_\alpha(p_\perp) + e^{i p_\perp \cdot x} a_\alpha^*(p_\perp) \right) .
\end{align*}
\]
It's not difficult to check that these fulfil the Dirac equation and the conjugate Dirac equation, respectively.

**Remark.** As in the generic fermion case examined in §1.7, the minus sign in the expression of $\bar{\psi}_\alpha(x)$ above is needed in order to obtain the correct supercommutator identities and expressions of field functionals in terms of basic operators. In order to make a thorough comparison with the matrix formulas found in usual presentations, we could adjust some conventions and absorb that sign into the definition of $c_\alpha(p_\perp)$, and also relate this to the *negative* Hermitian metric of the positron sector.

### 2.4.2 Gauge field

The free gauge field is defined by
\[
\begin{align*}
    A_\lambda^i (x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p_0}} \left( e^{-i(p \cdot x)} b_\lambda^i(p_\perp) + e^{i(p \cdot x)} b_\lambda^{*i}(p_\perp) \right) , \\
    b_\lambda^i (p_\perp) &\equiv a[X^p \otimes e_\lambda \otimes t^i] , \\
    b_\lambda^{*i}(p_\perp) &\equiv a^*[X^p \otimes (e_\lambda)^{\#} \otimes (t^i)^{\#}] ,
\end{align*}
\]
where $(e_\lambda)$ is a possibly orthonormal spacetime frame and $(e_\lambda)^{\#}$ is the co-vector frame associated to it via the spacetime metric; $((t^i)^{\#})$ is the frame of $\mathcal{L}$ associated to the frame $(t^i)$ of $\mathcal{L}^*$ via the metric $G$ (§2.2). Here we are not discussing frames adapted to gauge symmetry.
2.4.3 Ghost and anti-ghost fields

The ghost and anti-ghost fields (§2.3) are distinct, independent fields, the isomorphism $\mathcal{L} \cong \mathcal{L}^*$ notwithstanding. One could view the couple $(\omega, \bar{\omega})$ as a unique field $M \rightarrow \text{End} F \equiv \mathcal{L} \oplus i \mathcal{L}$, namely a section of the complexified bundle of $\mathcal{L} \rightarrow M$. The situation is then somewhat similar to that of the Dirac field, but simpler as we do not have to deal with frames dependent on momenta. Also note that seeing mutually conjugate fields as valued in mutually dual internal bundles is indeed consistent with a general view, valid both in the real case and in the complex case with a Hermitian structure (see the remark concluding §1.7).

According to the scheme presented in §1.6, the quantum free fields $\omega$ and $\bar{\omega}$ are defined to have the components

$$\omega_I(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 p}{\sqrt{2} p_0} \left( e^{-i(p,x)} g^I(p_\perp) + e^{i(p,x)} k^I(p_\perp) \right),$$

$$\bar{\omega}_I(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 p}{\sqrt{2} p_0} \left( -e^{-i(p,x)} k_I(p_\perp) + e^{i(p,x)} g^*_I(p_\perp) \right),$$

where $p_0 = |p_\perp| (m = 0)$ and

$$g^I(p_\perp) := a[X^p \otimes \ell]\, , \quad g^*_I(p_\perp) := a^*[X^p \otimes \ell]\, ,$$

$$k_I(p_\perp) := a[X^p \otimes I]\, , \quad k^*_I(p_\perp) := a^*[X^p \otimes I]\, .$$

The minus sign in $\bar{\omega}_I$ is related to the fact that these are assumed to be fermion fields.

2.5 Special functionals

Certain density functionals of the fields have special roles, and their spatially integrated evaluation through free fields yields remarkably simple expressions, independent of time. We are not going to write detailed calculations, but we stress that the following reported results for the fermion fields critically depend on our assumptions on the form of the free conjugate fields. In particular, note the expression of free 4-momentum for ghosts.

All expressions are written by allowing normal ordering.

Dirac charge

The Dirac current is the 3-form $\langle \bar{\psi} \gamma^\lambda \psi \rangle \, dx_\lambda$, where $dx_\lambda \equiv \partial x_\lambda | d^4x$. Its restriction to constant-time hypersurfaces is the scalar density $\langle \bar{\psi} \gamma^0 \psi \rangle \, dx_0 \equiv \langle \bar{\psi} \gamma^0 \psi \rangle \, d^3x_\perp$, whose global value is the Dirac charge

$$Q_{\text{Dir}} = \int d^3x_\perp \bar{\psi}(x) \gamma^\lambda \psi(x) = \int \frac{d^3 p_\perp}{2m} \left( a^*_\lambda(p_\perp) a^\lambda(p_\perp) - c^*_\lambda(p_\perp) c_\lambda(p_\perp) \right).$$

Dirac momentum

We recall that in flat spacetime the canonical energy-momentum tensor evaluated through a field $\phi$ is a section $T[\phi] : M \rightarrow T^*M \otimes \wedge^3 T^*M$, with the coordinate expression

$$T[\phi] = (\phi^\alpha, \partial^\alpha \ell[\phi] - \ell[\phi] \delta^\alpha_\lambda) \, dx^\lambda \otimes dx_\mu.$$

\textsuperscript{23}See [13] for an extension to curved spacetimes.
In the case of the free Dirac field we obtain
\[ \mathcal{T}[\psi, \bar{\psi}] = \frac{i}{2} \left( -\bar{\psi}_\lambda \gamma^\mu \psi + \bar{\psi} \gamma^\mu \psi_\lambda \right) dx^\lambda \otimes dx_\mu. \]

The corresponding 4-momentum density
\[ \mathcal{P}_\lambda dx^\lambda \otimes dx_0 = \frac{i}{2} \left( -\bar{\psi}_\lambda \gamma^0 \psi + \bar{\psi} \gamma^0 \psi_\lambda \right) dx^\lambda \otimes dx_0 \]
can be defined by a suitable pull-back via the inclusions of the constant-time hyper-planes into \( M \), and 4-momentum is defined to be the 1-form \( P_\lambda dx^\lambda \). We obtain
\[ P_\lambda \equiv \frac{i}{2} \int dx_0 \left( -\bar{\psi}_\lambda \gamma^0 \psi + \bar{\psi} \gamma^0 \psi_\lambda \right) \equiv \frac{i}{2} \int d^3x_\perp \left( -\bar{\psi}_\lambda \gamma^0 \psi + \bar{\psi} \gamma^0 \psi_\lambda \right) = \int d^3p_\perp \frac{p_\lambda}{2m} \left( a^*_\lambda(p_\perp) a(p_\perp) + c^*_\lambda(p_\perp) c(p_\perp) \right). \]

**Dirac Hamiltonian**

The Hamiltonian density \( \mathcal{H} \) of a Lagrangian field theory has the expression
\[ \mathcal{H}[\phi] = \Pi_\alpha[\phi] \phi^*_0 - \ell[\phi]. \]
The free field Hamiltonian density, for each sector of any theory, is obtained by keeping only those terms in which no contribution from other sectors appears, and evaluating it through free fields. For the Dirac sector we obtain
\[ \mathcal{H}_{\text{free}}[\psi, \bar{\psi}] = \frac{i}{2} \left( \bar{\psi}_i \gamma^i \psi - \bar{\psi} \gamma^i \psi_i \right) + m \bar{\psi} \psi = \frac{i}{2} \left( \bar{\psi} \gamma^0 \psi_0 - \bar{\psi}_0 \gamma^0 \psi \right), \]
which is just the 0-component of 4-momentum (the latter equality was written by taking the Dirac equation into account).

**Ghost momentum and Hamiltonian**

In the ghost-antighost sector, components of the canonical energy-momentum tensor have the expression
\[ T^\mu_\lambda[\omega, \bar{\omega}] = g^{\mu\nu} (\bar{\omega}_\lambda, \omega^\nu_\lambda + \bar{\omega}_1, \lambda \omega^\nu_\mu) - g^{\nu\rho} \bar{\omega}_1, \nu \omega^\nu_\rho \delta^\mu_\lambda, \]
where \( \omega^\nu_\mu \equiv \nabla_\mu \omega^\nu - \omega^\nu_\mu + c^\nu_M \omega^M A^\mu_\nu \). Then the components of the ghost 4-momentum density, evaluated through free fields, are
\[ T^0_\lambda[\omega, \bar{\omega}] = \bar{\omega}_{1,0} \omega^\lambda_0 + \bar{\omega}_1, \lambda \omega^\lambda_0 - g^{\nu\rho} \bar{\omega}_1, \nu \omega^\nu_\rho \delta^0_\lambda. \]
By spatial integration we then obtain the free ghost 4-momentum, with components
\[ P_\lambda[\omega, \bar{\omega}] = \int d^3p_\perp p_\lambda \left( k^\nu(p_\perp) k^\nu(p_\perp) + g^\nu(p_\perp) g^\nu(p_\perp) \right), \quad p_0 = |p_\perp|, \quad m = 0. \]
Again, we easily check that the free Hamiltonian density is \( \mathcal{H}_{\text{free}}[\omega, \bar{\omega}] = T^0_0[\omega, \bar{\omega}] \), so that the 0-component of \( P \) coincides with the free Hamiltonian \( \mathcal{H}_{\text{free}}[\omega, \bar{\omega}] \equiv \int d^3x_\perp \mathcal{H}_{\text{free}}[\omega, \bar{\omega}]. \)
2.6 Canonical supercommutation rules

Faddeev-Popov current

The one-parameter transformation \( \omega \rightarrow e^{\tau} \omega, \tilde{\omega} \rightarrow e^{-\tau} \tilde{\omega}, \tau \in \mathbb{R} \), obviously preserves the Lagrangian. The corresponding infinitesimal symmetry determines (§2.7) the Faddeev-Popov current \( \mathcal{J}_{\text{FP}} = \mathcal{J}_{\text{FP}}^\lambda \, dx_\lambda \) where (in orthonormal spacetime coordinates)

\[
\mathcal{J}_{\text{FP}}^\lambda = g^{\lambda \mu} (\tilde{\omega}_{i,\mu} \omega^i - \tilde{\omega}_i \omega^i_\mu) .
\]

Evaluating \( \mathcal{J}_{\text{FP}}^\lambda \) through free fields, integrating on constant-time hyperplanes and allowing normal ordering we find

\[
\int d^3 x_\perp \mathcal{J}_{\text{FP}}^\lambda (x) = g^{\lambda \mu} \int d^3 x_\perp (\tilde{\omega}_{i,\mu} \omega^i - \tilde{\omega}_i \omega^i_\mu)(x) =
\]

\[
i g^{\lambda \mu} \int \frac{d^3 p}{p_0} p_\mu \left( k^i (p_\perp) k_i (p_\perp) + g^i (p_\perp) g^i (p_\perp) \right) .
\]

2.6 Canonical supercommutation rules

In order to be convinced of the consistency of our setting we should recover the basic supercommutators among the free fields evaluated at different events \( x, x' \in M \). We already did that in the generic setting, which also includes ghost fields and unconstrained gauge fields. As for the Dirac field we obtain

\[
\{ \tilde{\psi}_\alpha (x), \psi^\beta (x') \} = \frac{1}{2m} \left( (-m \mathbb{1} + i \gamma^\lambda \partial_\lambda) \mathcal{D}(x-x') \right)^\beta_\alpha .
\]

Moreover for equal-time events \( (x^0 = x'^0) \) we obtain

\[
\{ \left( \gamma^0 \right)_\alpha (x), \psi^\beta (x') \} = \{ \tilde{\psi}_\alpha (x), (\gamma^0 \psi)^\beta (x') \} = \frac{1}{2m} \delta^\beta_\alpha \delta (x_\perp - x'_\perp) .
\]

Next we want to check the equal-time super-commutation rules between field components and conjugate momenta, in each sector, to be of the general form written in §1.10. We work in flat spacetime and set \( \sqrt{| g |} = 1 \). Considering the above identity we’d rather write the canonical momentum conjugate to the Dirac field \( \psi \) as \( \pi_\alpha = 2i m (\tilde{\psi} \gamma^0)_\alpha \). However the factor \( (2m)^{-1} \) in the anti-commutator can be absorbed by inserting factors \( \sqrt{2m} \) in the definitions of \( \psi \) and \( \tilde{\psi} \) (this is indeed found in the literature), and we obtain

\[
\{ \pi_\alpha (x), \psi^\beta (x') \} = i \delta^\beta_\alpha \delta (x_\perp - x'_\perp) ,
\]

\[
\pi_\alpha = i (\tilde{\psi} \gamma^0)_\alpha .
\]

The expression for \( \pi_\alpha \) derived from the Lagrangian (§2.3) has a further factor \( \frac{1}{2} \), which can be made to disappear by changing the Lagrangian via the addition of a suitable divergence term. Similar results hold for the conjugate sector, with \( \pi^\alpha = i (\gamma^0 \psi)^\alpha \).

The expression \( \pi^\alpha = (F^{0\lambda} + g^{00} n_i) \) for the canonical momentum conjugate to the gauge field (§2.3) contains the term \( n_i \) which commutes with everything, so it may seem that it could be just dropped.\(^{24}\) However the remaining term \( F^{0\lambda} \) does not possess the required property, so that instead one keeps both terms and uses the replacement \( n^i \rightarrow -\frac{1}{i} g^{\lambda \mu} A^\lambda_{i,\mu} \), which is just the Euler-Lagrange field equation for \( n \) (in orthonormal coordinates). For \( \xi = 1 \) (the “Feynman gauge”) we get

\[
\pi^\lambda = g^{\lambda \mu} (- A_{i,\mu,0} + A_{i,0,\mu} - c_{i,j,H} A^j_{i,\mu} A^H_0) - g^{\lambda \mu} g^{\mu \nu} A^i_{\nu,\mu} .
\]

\(^{24}\)Indices related to \( \Sigma \) are raised and lowered via the Euclidean metric introduced in §2.2.
Since $A$ is a boson field, it obeys the standard commutation rules. In particular, one easily checks that the spatial derivatives of $A$, and the components of $A$ itself, do not contribute to the equal-time commutator with $A$. The part of $\pi^\lambda$ which contains time derivatives of $A$ is just $-g^{\lambda\mu} A_{\mu,0} \equiv -A^\mu_{,0}$, so that at equal times we eventually have

$$[A^\mu_{,0}(t,x_\perp), A^\nu_{,0}(t,x'_\perp)] = [A^\rho_{,0}(t,x_\perp), A^\lambda_{,0}(t,x'_\perp)] = -i \delta_\lambda^\mu \delta_\rho^\nu \delta(x_\perp - x'_\perp).$$

Finally we consider ghosts and anti-ghosts. At equal times we have

$$\{\tilde{\omega}_j(x), \omega^j(x')\} = \{\omega^j_0(x), \tilde{\omega}_j(x')\} = i \delta^j_0 \delta(x_\perp - x'_\perp),$$

whence also $\{\tilde{\omega}_j(x), \omega^j_0(x')\} = 0$. From the expressions of the canonical momenta conjugate to $\omega^j$ and $\tilde{\omega}_j$ (§2.3), using the shorthand $\omega^j_\mu \equiv \omega^j \mu + c^j_{j\mu} \omega^\mu A^H_\mu$, we then get

$$\{\omega^j(x), \pi_j(x')\} = \{\tilde{\omega}_j(x'), \omega^j(x)\} = i \delta^j_0 \delta(x_\perp - x'_\perp),$$

$$\{\tilde{\omega}_j(x), \pi_j(x')\} = -\{\omega^j_0(x'), \tilde{\omega}_j(x)\} = i \delta^j_0 \delta(x_\perp - x'_\perp).$$

### 2.7 Remarks about BRST symmetry

There exists a large literature [24, 23, 22] about the jet bundle formulation of Lagrangian field theories and their symmetries. An approach consistent with the notion of quantum bundle presented here was proposed in a previous paper [11]; we refer to it for further citations and mathematical details.

Let $\mathcal{E} \equiv \mathcal{O} \otimes E \to M$ be a quantum bundle over the spacetime manifold, and consider a first-order Lagrangian density $\mathcal{L} = \ell \, d^4x : J\mathcal{E} \to \mathcal{O} \otimes \wedge^4T^*M$. A morphism $v : J\mathcal{E} \to V\mathcal{E}$ over $\mathcal{E}$, from the first-jet bundle to the vertical bundle, determines a geometrically well-defined “generalized Lie derivative” with the coordinate expression

$$\delta[v]\mathcal{L} = \left(\frac{\partial \ell}{\partial y^\alpha} v^\alpha + \frac{\partial \ell}{\partial y^\lambda} \frac{dv^\alpha}{dx^\lambda}\right)d^4x,$$

$$v = v^\alpha \frac{\partial}{\partial y^\alpha},$$

where $y^\alpha$ are generic fiber coordinates and $dv^\alpha/dx^\lambda$ indicates the components of the “horizontal differential” $dv^\alpha$ (yielding the total differential of $v^\alpha$ under evaluation through a field).

A generalized version of the Noether theorem can be now stated as follows. If $\delta[v]\mathcal{L} = d_{\mathcal{N}}\mathcal{N}$ with $\mathcal{N} : J\mathcal{E} \to \mathcal{O} \otimes \wedge^3T^*M$ then

$$\mathcal{J} = J^\lambda dx_\lambda = \frac{\partial \ell}{\partial y^\lambda} v^\alpha \, dx_\lambda - \mathcal{N} : J\mathcal{E} \to \mathcal{O} \otimes \wedge^3T^*M$$

is a conserved current, namely its evaluation through a ”critical field” (a solution of the Euler-Lagrange equations) yields a closed 3-form. We say that $v$ is a first-order infinitesimal vertical symmetry (i.v.s.). The Faddeev-Popov current considered in §2.5 is the special case in which $v^j = \omega^j$, $v_j = -\tilde{\omega}_j$.

If $v$ is an i.v.s. then

$$Q \equiv \int d^3x_\perp J^0(x) = \int d^3x_\perp \pi_\alpha(x) v^\alpha \circ j\phi(x)$$
is constant when evaluated through critical sections. Now suppose $J^0$ is of even grade in $\mathcal{O}$ and $\{\phi^\alpha(x) , v^\beta[\phi](x')\} = 0$ at equal times. Then it’s not difficult to see that

$$\{Q, \phi^\alpha(x)\} = [Q, \phi^\alpha(x)] = \int \mathrm{d}^3 x' \left[ \Pi^\beta(x') v^\beta[\phi](x') , \phi^\alpha(x) \right] = -i v^\alpha[\phi](x) \sqrt{|g|(x)} .$$

Since $v^\alpha[\phi] \equiv \delta[v] \phi^\alpha$, we can rewrite the above result as

$$\delta[v] \phi^\alpha = i [Q, \phi^\alpha] |g|^{-1/2} .$$

One also says that the i.v.s. $v$ is generated by the corresponding charge $Q$.

In the context of the gauge theory introduced in §2.3 we now consider

$$v = v^{\alpha i} \frac{\partial}{\partial \psi^{\alpha i}} + v_{\alpha i} \frac{\partial}{\partial \bar{\psi}^{\alpha i}} + v^I_\lambda \frac{\partial}{\partial A^I_\lambda} + v^I \frac{\partial}{\partial \omega^I} + v_i \frac{\partial}{\partial \bar{\omega}_i} ,$$

$$v^{\alpha i} = \theta I^j_\lambda \omega^j \psi^{\alpha i} , \quad v_{\alpha i} = \theta I^j_1 \bar{\psi}^{\alpha j} \omega^i , \quad v^I_\lambda = \theta \omega^I_\lambda , \quad v^I = \frac{1}{2} \theta c^I_\mu \omega^j \omega^H , \quad v_i = \theta n_i ,$$

which turns out to be an i.v.s. for any choice of an odd-graded $\theta \in \mathcal{O}$. The current turns out to be the sum $\theta J(\psi, A) + \theta J_{\text{ghost}}$ of two separately conserved terms, which in orthonormal coordinates have the expressions

$$J(\psi, A) = \left( -i \langle \bar{\psi} \gamma^\lambda \omega \psi \rangle + \langle F^{\lambda \mu}, \omega; \mu \rangle \right) \mathrm{d}x_\lambda ,$$

$$J_{\text{ghost}} = g^{\lambda \mu} \left( n_I \omega^I_{\mu} - \frac{1}{2} \bar{\omega}_{I, \mu} c^I_{j H} \omega^j \omega^H \right) \mathrm{d}x_\lambda .$$

We can get rid of the arbitrary constant $\theta$ by setting $\delta[v] \equiv \theta S$ and defining the charge $Q$ so that the charge in the above general sense is actually $\theta Q$. That argument applies because $\theta Q$ is of even grade, and it can be checked that the identity $\{\phi^\alpha(x) , v^\beta[\phi](x')\} = 0$ is fulfilled in each sector. Hence the transformation $S$ acts on field components as

$$S \phi^\alpha = i \{Q, \phi^\alpha\} |g|^{-1/2} .$$

It’s not difficult to check the above relation in each sector by using the supercommutation identities found for free fields. Since the charge is constant when evaluated through fully interacting critical fields, one must assume those identities remain valid; as we already observed, that is usually inferred by general arguments based on the form of the dynamics.
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