Measure and integration on Boolean algebras of regular open subsets in a topological space

Marcus Pivato† and Vassili Vergopoulos‡

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Abstract

The regular open subsets of a topological space form a Boolean algebra, where the \textit{join} of two regular open sets is the interior of the closure of their union. A \textit{credence} is a finitely additive probability measure on this Boolean algebra, or on one of its subalgebras. We develop a theory of integration for such credences. We then explain the relationship between credences, residual charges, and Borel probability measures. We show that a credence can be represented by a normal Borel measure, augmented with a \textit{liminal structure}, which specifies how two or more regular open sets share the probability mass of their common boundary. In particular, a credence on a locally compact Hausdorff space can be represented by a normal Borel measure and a liminal structure on the Stone-Čech compactification of that space. We also show how credences can be represented by Borel measures on the Stone space of the underlying Boolean algebra of regular open sets. Finally, we show that these constructions are functorial.

\textbf{Keywords:} regular open sets; Boolean algebra; Borel measure; compactification; Stone space; Gleason cover.

\textbf{MSC classification:} 60B05, 28C15, 28A60.

1 Introduction

Let \( S \) be a topological space, equipped with a Borel probability measure \( \mu \) having full support. It is well-known that the apparent “size” of a subset \( S \), as seen from a topological perspective, might greatly differ from its “size” from a measurable perspective. For example, suppose \( S \) is the unit interval \([0, 1]\) equipped with the Lebesgue measure. It is easy to construct a subset \( \mathcal{O}_n \subset S \) which is open and dense, but such that \( \mu[\mathcal{O}_n] < \frac{1}{n} \). Thus, if \( \mathcal{C}_n \)

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†THEMA, Université de Cergy-Pontoise.
‡Paris School of Economics, and University Paris 1 Panthéon-Sorbonne.
is the complement $O_n$, then $C_n$ is nowhere dense, but $\mu[C_n] > 1 - \frac{1}{n}$. If $C = \bigcup_{n=1}^{\infty} C_n$, then $C$ is meager in $[0, 1]$, but $\mu[C] = 1$. We can then construct a measurable function $f$ which is zero on the (co-meager) complement of $C$, but whose integral is arbitrarily large.

Real analysis is well-acquainted with these sorts of pathologies, and works around them. But they can still be inconvenient in some applications of probability theory — e.g. in the models of human decision-making employed in theoretical economics — where we would prefer our probability models to exhibit more intuitive behaviours. Note that the function $f$ constructed in the previous paragraph is discontinuous; it is not possible to obtain this sort of pathological behaviour with a continuous function. Likewise, the open sets $O_1, O_2, \ldots$ are not regular — they are not the interiors of their own closures. (Indeed, the only regular open dense subset of a topological space $S$ is $S$ itself.) This suggests that, by confining our attention to regular open sets and continuous functions, we can develop more “well-behaved” probability structures on topological spaces.

Unfortunately, the family $R(S)$ of regular open subsets of $S$ is not closed under unions or complementation, so we cannot even define a classical probability measure if we confine its domain to $R(S)$. But $R(S)$ does form a Boolean algebra under slightly different operations, and we can define a finitely additive, real-valued function on $R(S)$ with respect to this Boolean algebra structure. We will call such a structure a credence, because in a companion paper, we interpret it as representing the “beliefs” of a hypothetical agent about the likelihood of observing various regular subsets of $S$ [PV17].

In some situations, not all regular sets may be “observable” events for an agent, given her measurement technology. Thus, we might want to restrict the credence to some Boolean subalgebra $B$ within $R(S)$. For example, if $S = \mathbb{R}^N$, then $B$ could be the Boolean algebra of regular open sets with piecewise smooth boundaries. A credence plays the role of a (finitely additive) probability measure on $B$. However, in applications of probability theory, we often need to compute the integrals of real-valued functions. So such a credence would not be very useful, unless it came with a theory of integration. One goal of this paper is to develop such a theory. The other goal of this paper is to explicate the relationship between credences and Borel probability measures.

The remainder of this paper is organized as follows. Section 2 briefly reviews prior literature. In Section 3, we give some examples and basic results about credences. In Section 4, we develop a theory of integration for credences. In Section 5, we define the image of a credence under a measurable transformation, and use this to obtain a “change of variables” theorem for the integration theory of Section 4. In Section 6, we explain the relationship between credences, Borel probability measures, and finitely additive probability measures on topological spaces. These representations take an especially convenient form on compact spaces, or on compactifications of spaces, as explained in Section 7. In Section 8, we introduce a completely different representation of credences, in terms of Borel probability measures on the Stone space of the Boolean subalgebra $B$, using a construction which generalizes the Gleason cover of a topological space. Finally, in Section 9 we give a functorial formulation to the constructions in Section 8. The following diagram shows the logical dependencies between these sections.
2 Prior literature

The Boolean algebra of regular open subsets was introduced by Tarski [Tar37]. The first analysis of finitely additive measures on Boolean algebras was by Horn and Tarski [HT48], but this paper did not specifically consider the algebra of regular open sets. The literature inspired by [HT48] has focused mainly on identifying necessary and sufficient conditions for abstract Boolean algebras to support finitely additive measures with particular properties; see [DP08] and [BND13] for recent results and a review of this literature.

As we will soon see (Proposition 3.7), there is a close relationship between credences and residual measures—that is, measures on the sigma-algebra of Baire-property subsets of a topological space which vanish on all meager subsets of that space. Residual measures were studied by Armstrong and Prikry [AP78], Flachsmeyer and Lotz [FL78, FL80], and Zindulka [Zin00]. But these papers focused on countably additive measures, whereas we are interested in finitely additive ones (which we call residual charges). This is important, because countably additive residual measures are much harder to construct than finitely additive ones. Nevertheless, there are parallels between the finitely additive and countably additive cases. For example, Armstrong and Prikry [AP78, Proposition 7] observe that any residual measure on a space \( S \) can be represented by a measure on the Gleason cover of \( S \) (i.e. the Stone space of \( R^pS \)). Theorem 8.4 of the present paper makes a similar statement for credences defined on an arbitrary subalgebra \( \mathcal{B} \) of \( R^pS \). Thus, by combining Proposition 3.7 and Theorem 8.4 in the special case when \( \mathcal{B} = R^pS \), we obtain a version of Armstrong and Prikry’s Proposition 7 for residual charges.

Let \( \mathcal{D}(S) \) be the lattice of open subsets of a topological space \( S \). A credence on \( R(S) \) might seem superficially similar to a valuation defined on \( \mathcal{D}(S) \)\(^2\) But there are two important differences: first, credences are only defined on regular open sets; second, the additivity property for a credence is defined with respect to a special join operation \( \vee \) (see Section 3), whereas the additivity property of a valuation is defined with respect to the standard set-theoretic operations of union and intersection. These differences have two important consequences. First, there is now an integration theory for valuations [Eda95a].

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\(^1\) Flachsmeyer and Lotz called them hyperdiffusive measures.

\(^2\) A valuation is a function \( \nu : \mathcal{D}(S) \to \mathbb{R}_+ \) such that \( \nu(\emptyset) = 0 \) and for any \( \mathcal{O}, \mathcal{Q} \in \mathcal{D}(S) \), we have \( \nu(\mathcal{O} \cup \mathcal{Q}) = \nu(\mathcal{O}) + \nu(\mathcal{Q}) - \nu(\mathcal{O} \cap \mathcal{Q}) \), and also \( \nu(\mathcal{O}) \leq \nu(\mathcal{Q}) \) if \( \mathcal{O} \subseteq \mathcal{Q} \).
But as far as we know, there is no comparable theory of integration for credences on \( \mathcal{R}(S) \) or its subalgebras; we shall develop one in Section 4. Second, under fairly general conditions, a valuation on \( \mathcal{D}(S) \) can be uniquely extended to a charge defined on the Boolean algebra generated by \( \mathcal{D}(S) \) [Smi44, HT48], or even to a Borel measure [Law82, Eda95b, AMESD00, AM02, KL05]. But as we shall see in Section 6, the corresponding results for credences on \( \mathcal{R}(S) \) are much more subtle, because they involve not only a Borel measure but also a \textit{liminal structure}, which, roughly speaking, describes the way the credence deals with the \textit{boundaries} of regular open sets.

## 3 Credences

Throughout this paper, let \( S \) be a topological space. For any subset \( A \subseteq S \), let \( \text{int}(A) \) denote its interior, let \( \text{clos}(A) \) denote its closure, and let \( \partial A \) denote its boundary. An open subset \( O \subseteq S \) is \textit{regular} if \( O = \text{int}\left(\text{clos}(O)\right) \). For example, the interior of any closed subset of \( S \) is a regular set. For a concrete example, let \( S = \mathbb{R} \); then an open interval like \((0, 1)\) is a regular subset. However, a union like \((0, 1) \cup (1, 2)\) is a nonregular open subset. Clearly, the intersection of two regular subsets is another regular subset. Given any two regular open subsets \( Q, R \subseteq S \), we define \( Q \cup R := \text{int}\left(\text{clos}(Q \cup R)\right) \); this is the smallest regular open subset of \( S \) containing both \( Q \) and \( R \). For example, if \( S = \mathbb{R} \), then \((0, 1) \cup (1, 2) = (0, 2)\). Meanwhile, we define \( \neg D := \text{int}(S \setminus D) \), which is another regular open subset. The set \( \mathcal{R}(S) \) of all regular open subsets of \( S \) forms a Boolean algebra under the operations \( \cap, \cup, \text{ and } \neg \). [Fre04, Theorem 314P]. A subcollection \( \mathcal{B} \subseteq \mathcal{R}(S) \) is a \textit{Boolean subalgebra} if \( \mathcal{B} \) is closed under the operations \( \cap, \cup, \text{ and } \neg \).

**Example 3.1.** (a) A subset \( E \subseteq \mathbb{R} \) is \textit{elementary} if \( E := (a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_N, b_N) \) for some \(-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_N < b_N < \infty \). Any elementary subset is open and regular. Let \( \mathcal{E} \) be the collection of all elementary subsets of \( \mathbb{R} \); then \( \mathcal{E} \) is a Boolean subalgebra of the algebra of regular subsets of \( \mathbb{R} \). (Here, \( \cup \) indicates disjoint union.)

(b) Suppose \( S \) is a differentiable manifold. A subset \( \mathcal{H} \subseteq S \) is a \textit{smooth hypersurface} if there is a differentiable function \( \phi : S \rightarrow \mathbb{R} \) such that \( \mathcal{H} := \phi^{-1}\{r\} \) for some \( r \in \mathbb{R} \), and such that \( d\phi(h) \neq 0 \) for all \( h \in \mathcal{H} \). We will say that a regular open subset \( \mathcal{R} \subseteq S \) has a \textit{piecewise smooth boundary} if there is a finite collection \( \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N \) of smooth hypersurfaces such that \( \partial \mathcal{R} = (\mathcal{H}_1 \cap \partial \mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial \mathcal{R}) \). Let \( \mathcal{B}_{\text{smth}} \) be the set of regular open sets with piecewise smooth boundaries; then \( \mathcal{B}_{\text{smth}} \) is a Boolean subalgebra of \( \mathcal{R}(S) \).

(c) Suppose \( S \) is a topological vector space. A subset \( \mathcal{H} \subseteq S \) is a \textit{hyperplane} if there is a continuous linear function \( \phi : S \rightarrow \mathbb{R} \) such that \( \mathcal{H} := \phi^{-1}\{r\} \) for some \( r \in \mathbb{R} \). A regular open subset \( \mathcal{R} \subseteq S \) is a \textit{polyhedron} if there is a finite collection \( \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N \) of hyperplanes such that \( \partial \mathcal{R} = (\mathcal{H}_1 \cap \partial \mathcal{R}) \cup \cdots \cup (\mathcal{H}_N \cap \partial \mathcal{R}) \). (Heuristically, \( \mathcal{H}_n \cap \partial \mathcal{R} \) is a \textquotedblleft face\textquotedblright \ of \( \mathcal{R} \). Note that we do not require \( \mathcal{R} \) to be convex, or even connected.) Let \( \mathcal{B}_{\text{poly}} \) be the set of regular polyhedra; then \( \mathcal{B}_{\text{poly}} \) is a Boolean subalgebra of \( \mathcal{R}(S) \).

3 To be precise, this theory establishes a relationship between the Lebesgue integral with respect to a Borel measure on a space \( X \) and an integral with respect to a valuation on its \textit{upper space} \( \mathcal{U}(X) \).
Let \( S \subseteq \mathbb{R}^N \) be an open set. Let \( \mathcal{B}_{\text{jor}}(S) \) be the set of all regular open subsets of \( S \) whose boundaries have Lebesgue measure zero. This is a Boolean subalgebra of \( \mathcal{R}(S) \), which is sometimes called the **Jordan algebra**. (Note that \( \mathcal{B}_{\text{poly}}(\mathbb{R}^N) \subseteq \mathcal{B}_{\text{smth}}(\mathbb{R}^N) \subseteq \mathcal{B}_{\text{jor}}(\mathbb{R}^N) \).) 

A **credence** on \( \mathfrak{B} \) is a function \( \mu : \mathfrak{B} \rightarrow [0,1] \) with \( \mu[\mathcal{S}] = 1 \), such that for any finite collection \( \{B_n\}_{n=1}^N \) of disjoint elements of \( \mathfrak{B} \), we have

\[
\mu \left[ \bigvee_{n=1}^N B_n \right] = \sum_{n=1}^N \mu[B_n].
\]

Note an important difference from the usual definition of a measure: additivity is defined with respect to the operation \( \lor \), rather than ordinary union. We say that \( \mu \) has **full support** if \( \mu[B] > 0 \) for all nonempty \( B \in \mathfrak{B} \).

**Example 3.2.** Let \( S := (0,1) \), and let \( \mathcal{E} \) be the Boolean algebra of elementary regular open subsets of \((0,1)\), as defined in Example 3.1(a). For any \( \mathcal{E} \in \mathcal{E} \), if \( \mathcal{E} = (a,b) \) for some \( a < b \), then define \( \mu[\mathcal{E}] := b - a \). If \( \mathcal{E} = \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_N \) for some disjoint open intervals \( \mathcal{E}_1, \ldots, \mathcal{E}_N \), then define \( \mu[\mathcal{E}] := \mu[\mathcal{E}_1] + \cdots + \mu[\mathcal{E}_N] \). Then \( \mu \) is a credence on \( \mathcal{E} \). 

In effect, the credence in Example 3.2 is just the restriction of the Lebesgue measure \( \lambda \) to \( \mathcal{E} \). It is tempting to extrapolate from this example that we can obtain a credence on all of \( \mathcal{R}[0,1] \) by restricting \( \lambda \) to regular open sets. But this is not the case.

**Nonexample 3.3.** *The Lebesgue measure restricted to \( \mathcal{R}[0,1] \) is not a credence.*

**Proof sketch.**[4] Let \( \lambda \) be the Lebesgue measure, and let \( \mathcal{K} \) be a “fat” Cantor set — that is, a closed, nowhere dense subset of \([0,1] \) with \( \lambda[\mathcal{K}] > 0 \). Thus, \( \mathcal{U} := [0,1] \setminus \mathcal{K} \) is an open dense subset of \([0,1] \) with \( \lambda[\mathcal{U}] < 1 \). The set \( \mathcal{U} \) is a countable disjoint union of open intervals, and is not regular. However, it can be divided into two pieces, \( \mathcal{L} \) and \( \mathcal{R} \), which are defined by taking the “left half” and “right half” of each of open intervals comprising \( \mathcal{U} \). These are regular open sets, and \( \mathcal{R} = \neg \mathcal{L} \). Thus, \( \mathcal{L} \lor \mathcal{R} = [0,1] \). But clearly, \( \lambda[\mathcal{L}] + \lambda[\mathcal{R}] = \lambda[\mathcal{U}] < 1 = \lambda[0,1] \). Thus, \( \lambda \) is not finitely additive on \( \mathcal{R}[0,1] \).

(For a complete proof of a more general result, see Proposition 6.9 below.)

Nonexample 3.3 is rather disturbing. But the proof involves rather “exotic” sets, which are unlikely to arise in practical applications. Indeed, the next result says that, “for all practical purposes”, we can treat the Lebesgue measure as a credence. Let \( \mathfrak{B}_{\text{jor}}(\mathbb{R}^N) \) be the Jordan algebra defined in Example 3.1(d).

**Proposition 3.4** Let \( N \in \mathbb{N} \), let \( S \in \mathcal{R}(\mathbb{R}^N) \), and let \( \lambda \) be the normalized Lebesgue measure on \( S \). There exists a credence \( \mu \) on \( \mathcal{R}(S) \) such that \( \mu[B] = \lambda[B] \) for any \( B \in \mathfrak{B}_{\text{jor}}(S) \). Furthermore, \( \mu \) is invariant under all isometries. That is: if \( \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is an isometry, and \( \mathcal{R} \in \mathcal{R}(S) \) is such that \( \phi^{-1}(\mathcal{R}) \in \mathcal{R}(S) \) also, then \( \mu[\phi^{-1}(\mathcal{R})] = \mu[\mathcal{R}] \). Finally, for any \( \mathcal{R} \in \mathcal{R}(S) \), we have \( \mu[\mathcal{R}] \geq \lambda[\mathcal{R}] \).

[4] We are grateful to Joel David Hamkins for showing us this construction.
We will refer to the credence described in Proposition 3.4 as a *Lebesguesque* credence. This credence is not unique, but it is “unique enough” for practical purposes. Proposition 3.4 is actually a corollary of the following “extension theorem” for credences.

**Proposition 3.5** Let $S$ be a topological space, let $\mathcal{B} \subseteq \mathcal{R}(S)$ be a Boolean subalgebra, and let $\mu : \mathcal{B} \rightarrow [0, 1]$ be a credence on $\mathcal{B}$. Then there exists a credence $\nu : \mathcal{R}(S) \rightarrow [0, 1]$ such that $\nu(\mathcal{B}) = \mu(\mathcal{B})$ for all $\mathcal{B} \in \mathcal{B}$.

Furthermore, let $\sim$ be an equivalence relation on $\mathcal{R}(S)$, and suppose that $\mu$ is $\sim$-invariant on $\mathcal{B}$, meaning that $\mu[\mathcal{B}] = \mu[\mathcal{B}']$ for any $\mathcal{B}, \mathcal{B}' \in \mathcal{B}$ with $\mathcal{B} \sim \mathcal{B}'$. Then we can choose $\nu$ so that $\nu$ is $\sim$-invariant on $\mathcal{R}(S)$.

**Proof.** Let $\mathbb{R}^S$ be the vector space of all real-valued functions on $S$. Let $\mathcal{F}_{\mathcal{B}}$ be the linear subspace of $\mathbb{R}^S$ spanned by the characteristic functions of regular open sets. Thus, a typical element of $\mathcal{F}_{\mathcal{B}}$ is a finite linear combination of such characteristic functions; we will call this a *simple* function. Likewise, let $\mathcal{F}_{\mathcal{B}}$ be the linear subspace of $\mathbb{R}^S$ spanned by the characteristic functions of elements of $\mathcal{B}$; thus, $\mathcal{F}_{\mathcal{B}}$ is a subspace of $\mathcal{F}_{\mathcal{B}}$. Finally, let $\mathcal{Z}$ be the linear subspace of $\mathcal{F}_{\mathcal{B}}$ spanned by all simple functions of the form $(1_{\mathcal{R}} + 1_{\mathcal{Q}}) - (1_{\mathcal{R} \cap \mathcal{Q}} + 1_{\mathcal{R} \setminus \mathcal{Q}})$, for some $\mathcal{Q}, \mathcal{R} \in \mathcal{R}(S)$, as well as all simple functions of the form $1_{\mathcal{R}} - 1_{\mathcal{Q}}$ for some $\mathcal{Q}, \mathcal{R} \in \mathcal{R}(S)$ with $\mathcal{Q} \sim \mathcal{R}$.

Let $\mathcal{F}_{\mathcal{B}} := \mathcal{F}_{\mathcal{B}}/\mathcal{Z}$ be the quotient vector space. Thus, each element of $\mathcal{F}_{\mathcal{B}}$ is an equivalence class of simple functions. To understand these equivalence classes, suppose for simplicity that $\sim$ is trivial. If $f = \sum_{n=1}^{N} r_n 1_{\mathcal{R}_n}$ and $g = \sum_{m=1}^{M} q_m 1_{\mathcal{Q}_m}$ are simple functions (for some $\mathcal{R}_1, \ldots, \mathcal{R}_N, \mathcal{Q}_1, \ldots, \mathcal{Q}_M \in \mathcal{R}(S)$ and some constants $r_1, \ldots, r_N, q_1, \ldots, q_M \in \mathbb{R}$), then $f$ and $g$ are equivalent in $\mathcal{F}_{\mathcal{B}}$ if there is a collection of disjoint regular sets $\mathcal{P}_1, \ldots, \mathcal{P}_L \in \mathcal{R}(S)$ and constants $p_1, \ldots, p_L \in \mathbb{R}$ such that every $\mathcal{R}_n$ and every $\mathcal{Q}_m$ can be written as a join of some collection of elements from $\{\mathcal{P}_1, \ldots, \mathcal{P}_L\}$, and such that, for all $\ell \in [1 \ldots L]$, we have $p_\ell = \sum\{r_n; \ n \in \{1 \ldots N\}\} \cap \mathcal{P}_\ell \subseteq \mathcal{R}_n$ and also $p_\ell = \sum\{q_m; \ m \in \{1 \ldots M\} \cap \mathcal{P}_\ell \subseteq \mathcal{Q}_m\}$. In other words, if we define $h := \sum_{\ell=1}^{L} p_\ell 1_{\mathcal{P}_\ell}$, then we have $f = h = g$ everywhere except perhaps along the boundaries of $\mathcal{P}_1, \ldots, \mathcal{P}_L$.

For any $f \in \mathcal{F}_{\mathcal{B}}$, we will refer to its equivalence class by $\overline{f}$. Let $\mathcal{F}_{\mathcal{B}} := \{\overline{f}; \ f \in \mathcal{F}_{\mathcal{B}}\}$.

If $\overline{f} = \sum_{n=1}^{N} r_n \overline{1_{\mathcal{R}_n}}$, then we will say that $\overline{f}$ is *positive* if $r_n \geq 0$ for all $n \in \{1 \ldots N\}$, with $r_n > 0$ for at least one $n \in \{1 \ldots N\}$. For any $\overline{f}, \overline{g} \in \mathcal{F}_{\mathcal{B}}$, write $\overline{f} > \overline{g}$ if $\overline{f} - \overline{g}$ is positive. Then $\mathcal{F}_{\mathcal{B}}$ is an ordered vector space, and $\mathcal{F}_{\mathcal{B}}$ is a cofinal subspace (that is: for any $\overline{f} \in \mathcal{F}_{\mathcal{B}}$, there is some $\overline{g} \in \mathcal{F}_{\mathcal{B}}$ such that $g > f$). To see this, observe that the constant function $\overline{1}$ is in $\mathcal{F}_{\mathcal{B}}$ (because $S \in \mathcal{B}$), and we can always choose $r > 0$ large enough that $r\overline{1} \geq \overline{f}$.

Now, let $\mu : \mathcal{B} \rightarrow [0, 1]$ be a credence. Define $\phi : \mathcal{F}_{\mathcal{B}} \rightarrow \mathbb{R}$ by

$$\phi \left( \sum_{n=1}^{N} r_n \overline{1_{\mathcal{B}_n}} \right) := \sum_{n=1}^{N} r_n \mu[\mathcal{B}_n],$$

for any $\mathcal{B}_1, \ldots, \mathcal{B}_N \in \mathcal{B}$ and $r_1, \ldots, r_N \in \mathbb{R}$. This expression is well-defined on the equivalence classes of $\mathcal{F}_{\mathcal{B}}$ precisely because $\mu$ is a $\sim$-invariant credence on $\mathcal{B}$, so that
\[
\mu[B_1 \lor B_2] = \mu[B_1] + \mu[B_2] - \mu[B_1 \land B_2]
\]
for all \(B_1, B_2 \in \mathcal{B}\), and also \(\mu[B_1] = \mu[B_2]\) whenever \(B_1 \sim B_2\). Note that \(\phi\) is an order-preserving linear functional on \(\mathcal{F}_\mathcal{B}\).

A standard corollary of the Hahn-Banach Theorem yields an order-preserving linear functional \(\Phi : \mathcal{F}_\mathcal{R} \rightarrow \mathbb{R}\) which extends \(\phi\) — that is, \(\Phi(f) = \phi(f)\) for all \(f \in \mathcal{F}_\mathcal{B}\) [Con90, Theorem III.9.8, p.87]. Now define \(\nu : \mathcal{R}(S) \rightarrow \mathbb{R}\) by setting \(\nu[R] := \Phi(1_R)\) for all \(R \in \mathcal{R}(S)\). Note that \(\nu[R] \geq 0\) all \(R \in \mathcal{R}(S)\), because \(\Phi\) is order-preserving and \(\Phi(1_R) > 0\). To see that \(\nu\) is a credence, suppose \(R, Q \in \mathcal{R}(S)\) are disjoint. Then \(1_{R \cup Q} = 1_R + 1_Q\). Thus \(\nu[R \cup Q] = \Phi(1_{R \cup Q}) = \Phi(1_R) + \Phi(1_Q) = \nu[R] + \nu[Q]\), as desired.

To see that \(\nu\) is \(\sim\)-invariant, let \(R, Q \in \mathcal{R}(S)\), and suppose \(R \sim Q\). Then \(1_R = 1_Q\), so \(\nu[R] = \Phi(1_R) = \Phi(1_Q) = \nu[Q]\). Finally, to see that \(\nu\) extends \(\mu\), let \(B \in \mathcal{B}\). Then \(\nu[B] = \Phi(1_B) = \phi(1_B) = \mu(B)\), where each equality follows from the definition of the object on the left.

\[\Box\]

**Proof of Proposition 3.4.** Let \(\lambda\) be the normalized Lebesgue measure on \(S\), and for any \(B \in \mathcal{B}_{\text{jor}}(S)\), define \(\mu[B] := \lambda[B]\).

**Claim 1:** \(\mu\) is a credence on \(B \in \mathcal{B}_{\text{jor}}(S)\).

**Proof.** Let \(P, Q \in \mathcal{B}_{\text{jor}}(S)\). Suppose \(P\) and \(Q\) are disjoint. We must show that \(\lambda[P \lor Q] = \lambda[P] + \lambda[Q]\). Now, \(P \lor Q \subseteq P \cup Q \subseteq P \cup \mathcal{F} \cup Q\), where \(\mathcal{F} := (\partial P) \cap (\partial Q)\). Thus,

\[
\lambda[P] + \lambda[Q] = \lambda[P \lor Q] \leq \lambda[P \cup Q] \leq \lambda[P \cup \mathcal{F} \cup Q] = \lambda[P] + \lambda[\mathcal{F}] + \lambda[Q].
\]

But \(\lambda[\mathcal{F}] = 0\), because \(\mathcal{F} \subseteq \partial P\) and \(\lambda[\partial P] = 0\) because \(P \in \mathcal{B}_{\text{jor}}(S)\). Thus, \(\lambda[P \lor Q] = \lambda[P] + \lambda[Q]\), as claimed.

Claim 1

For any \(R, Q \in \mathcal{R}(S)\), write \(R \sim Q\) if there is some isometry \(\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N\) such that \(R = \phi^{-1}(Q)\). This defines an equivalence relation on \(\mathcal{R}(S)\). Observe that \(\mu\) is \(\sim\)-invariant on \(B \in \mathcal{B}_{\text{jor}}(S)\) (because the Lebesgue measure is invariant under all isometries). Now apply Proposition 3.5 to extend \(\mu\) to an isometry-invariant credence on all of \(\mathcal{R}(S)\). \[\Box\]

Some credences are very different from the ones in Examples 3.2 and Proposition 3.4.

**Example 3.6.** Let \(\mathcal{B} \subseteq \mathcal{R}(S)\) be a Boolean subalgebra. An **ultrafilter** is a collection \(\mathcal{U} \subseteq \mathcal{B}\) such that: (a) If \(U, V \in \mathcal{U}\), then \(U \cap V \in \mathcal{U}\}; (b) If \(\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{U}\); (c) \(\emptyset \notin \mathcal{U}\); and (d) For any \(B \in \mathcal{B}\), either \(B \in \mathcal{U}\) or \(\neg B \in \mathcal{U}\), but not both. Given any ultrafilter \(\mathcal{U}\), we can define a function \(\delta_{\mathcal{U}} : \mathcal{B} \rightarrow \{0, 1\}\) as follows: for any \(\mathcal{B} \in \mathcal{B}\), \(\delta_{\mathcal{U}}[B] = 1\) if \(B \in \mathcal{U}\), whereas \(\delta_{\mathcal{U}}[\emptyset] = 0\) if \(\emptyset \notin \mathcal{U}\). It is easy to verify that \(\delta_{\mathcal{U}}\) is a credence.

An ultrafilter \(\mathcal{U}\) is **fixed** if there is some point \(s \in S\) such that \(s \in \mathcal{U}\) for all \(U \in \mathcal{U}\). To obtain such an ultrafilter, let \(\mathcal{F}_s := \{R \in \mathcal{R}; s \in R\}\). Then \(\mathcal{F}_s\) is a **filter** (i.e. it satisfies properties (a), (b) and (c) above.) The Ultrafilter Theorem says that there is an ultrafilter \(\mathcal{U} \supseteq \mathcal{F}_s\) [Fre10, Theorem 2A10]. It is easy to see that \(\mathcal{U}\) is fixed at \(s\). In this case, \(\delta_{\mathcal{U}}\) can be interpreted as a “point mass” at \(s\). But there is a complication: \(\mathcal{U}\) is not unique.
There are many different ultrafilters fixed at \( s \), and each one defines a different credence—a different version of the “point mass” at \( s \). We will return to this in Example 6.2.

If \( \mathcal{U} \) is not fixed, then it is free. A free ultrafilter can behave like a point mass “at infinity” (see Example 6.3 below), or a point mass on the metaphorical “boundary” of the space \( S \) (Example 6.6). But it can also behave like a point mass in the interior of \( S \) “approached from one side” (Example 6.3).

Recall that a subset \( \mathcal{N} \subseteq S \) is nowhere dense if \( \text{int}[\text{clos}(\mathcal{N})] = \emptyset \). A subset \( \mathcal{M} \subseteq S \) is meager if it is a countable union of nowhere dense sets. Let \( \mathcal{M}(S) \) be the set of all meager subsets of \( S \); then \( \mathcal{M}(S) \) is an ideal under the standard set-theoretic operations. (That is: the union of two meager sets is meager, and the intersection of a meager set with any other set is meager.) A subset \( \mathcal{B} \subseteq S \) has the Baire property if \( \mathcal{B} = \mathcal{O} \triangle \mathcal{M} \) for some open \( \mathcal{O} \subseteq S \) and meager \( \mathcal{M} \subseteq S \). Let \( \mathfrak{A}(S) \) be the collection of all subsets with the Baire property; then \( \mathfrak{A}(S) \) is a Boolean algebra under the standard set-theoretic operations. Observe that \( \mathcal{M}(S) \subseteq \mathfrak{A}(S) \) as sets, but the Boolean algebra operations are different.

A probability charge on \( \mathfrak{A}(S) \) is a function \( \nu : \mathfrak{A}(S) \rightarrow [0,1] \) such that (1) \( \nu(S) = 1 \) and (2) \( \nu(A \cup B) = \nu(A) + \nu(B) \) for any disjoint \( A, B \in \mathfrak{A}(S) \). Let us say that \( \nu \) is a residual charge if, furthermore, \( \nu(\mathcal{M}) = 0 \) for all \( \mathcal{M} \in \mathcal{M}(S) \). Recall that a topological space \( S \) is a Baire space if the intersection of any countable family of open dense sets is dense. In particular, any locally compact Hausdorff space is Baire, and any completely metrizable space is Baire \([\text{Wil04}, \text{Corollary 25.4}]\). The next result says that credences have a particularly nice representation on Baire spaces.

**Proposition 3.7** Let \( S \) be a Baire space. Then there is a bijective correspondence between the credences on \( \mathfrak{A}(S) \) and the residual charges on \( S \). To be precise, if \( \nu \) is a residual charge on \( \mathfrak{A}(S) \), then we can obtain a credence by simply restricting \( \nu \) to \( \mathfrak{A}(S) \). Every credence arises in this fashion, and no two residual charges produce the same credence.

**Proof.** For any \( B_1, B_2 \in \mathfrak{A}(S) \), write \( B_1 \sim B_2 \) if \( B_1 \Delta B_2 \in \mathcal{M}(S) \); this is an equivalence relation on \( \mathfrak{A}(S) \). Let \( \mathfrak{A} := \mathfrak{A}(S)/\sim \); then the Boolean algebra operations on \( \mathfrak{A}(S) \) factor through to \( \mathfrak{A} \) (because \( \mathcal{M}(S) \) is an ideal), making \( \mathfrak{A} \) a Boolean algebra. Thus, a residual charge is equivalent to a finitely additive function on the Boolean algebra \( \mathfrak{A} \).

Recall that \( \mathfrak{A}(S) \) is a subset (but not a subalgebra) of \( \mathfrak{A}(S) \). Let \( \pi : \mathfrak{A}(S) \rightarrow \mathfrak{A} \) be the quotient map, and let \( \phi \) be the restriction of \( \pi \) to \( \mathfrak{A}(S) \). Then \( \phi \) is a Boolean algebra homomorphism from \( \mathfrak{A}(S) \) to \( \mathfrak{A} \) \([\text{Fre08}, \text{514I(b)}, \text{p.44}]\). Furthermore, if \( S \) is Baire, then \( \phi \) is bijective, and hence, an isomorphism \([\text{Fre08}, \text{514I(f)}]\). In other words, every \( \sim \)-equivalence class in \( \mathfrak{A}(S) \) contains a unique representative from \( \mathfrak{A}(S) \).

Thus, if \( \mu \) is any credence on \( \mathfrak{A}(S) \), then we obtain a finitely additive function \( \mu \circ \phi^{-1} : \mathfrak{A} \rightarrow [0,1] \), and from there, a residual charge \( \mu \circ \phi^{-1} \circ \pi : \mathfrak{A}(S) \rightarrow [0,1] \). Conversely, given any residual charge \( \nu \) on \( \mathfrak{A}(S) \), we obtain a finitely additive function \( \nu^* \) on \( \mathfrak{A} \), and from there, a credence \( \nu^* \circ \phi \) on \( \mathfrak{A}(S) \).

\[\text{Note that } \mathfrak{A}(S) \text{ is completely unrelated to the Baire sigma-algebra.}\]

\[\text{We are grateful to Robert Furber for pointing out this result to us.}\]

\[\text{\( \mathfrak{A} \) is called the category algebra of } S.\]

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Let \( s \in S \), and let \( \mathcal{U} \) be an ultrafilter fixed at \( s \). The “point mass” credence \( \delta_\mathcal{U} \) from Example 3.6 seems to contradict Proposition 3.7 since the singleton \( \{s\} \) is obviously meager. But this is misguided. If \( \mathcal{U} \) is an ultrafilter fixed at \( s \), then \( \delta_\mathcal{U} \) is equivalent to residual charge which gives probability 1 to every open neighbourhood of \( s \), but gives probability zero to \( \{s\} \) itself. This is possible because charges are only finitely (and not countably) additive.

This raises the question: why do we confine our attention to finitely additive credences? The set \( \mathcal{R}(S) \) is actually a \( \sigma \)-Boolean algebra under the following operations.\(^8\) Given any countable collection \( \{\mathcal{R}_n\}_{n=1}^\infty \) of regular subsets of \( S \), we define

\[
\bigvee_{n=1}^\infty \mathcal{R}_n = \text{int} \left[ \text{cl} \left( \bigcup_{n=1}^\infty \mathcal{R}_n \right) \right] \quad \text{and} \quad \bigwedge_{n=1}^\infty \mathcal{R}_n := \text{int} \left[ \bigcap_{n=1}^\infty \text{cl}(\mathcal{R}_n) \right].
\]  

However, if \( S \) satisfies mild topological conditions, then \( \mathcal{R}(S) \) can only support finitely additive credences. This fact is well-known, but for completeness we include a proof.

**Proposition 3.8** If \( S \) is a perfect, second-countable Hausdorff space, then there is no countably additive credence defined on the \( \sigma \)-algebra of regular subsets of \( S \).

**Proof.** (by contradiction) Suppose \( \mu \) is a countably additive credence on the \( \sigma \)-Boolean algebra \( \mathcal{R}(S) \) of regular subsets of \( S \). Let \( s_0 \in S \); say that \( s_0 \) is an atom of \( \mu \) if there is some \( \epsilon > 0 \) such that \( \mu(\mathcal{R}) \geq \epsilon \) for any \( \mathcal{R} \in \mathcal{R}(S) \) with \( s_0 \in \mathcal{R} \).

**Claim 1:** \( \mu \) cannot have any atoms.

**Proof.** (by contradiction) Suppose \( s_0 \in S \) was an atom. Thus, there is some \( \epsilon > 0 \) such that \( \mu(\mathcal{R}) \geq \epsilon \) for any \( \mathcal{R} \in \mathcal{R}(S) \) with \( s_0 \in \mathcal{R} \). Let \( \{\mathcal{N}_n\}_{n=1}^\infty \) be a countable neighbourhood base for \( s_0 \) (this exists because \( S \) is first-countable, because it is second-countable). By replacing each \( \mathcal{N}_n \) with \( \mathcal{N}_1 \cap \cdots \cap \mathcal{N}_n \) if necessary, we can assume without loss of generality that \( \mathcal{N}_1 \supseteq \mathcal{N}_2 \supseteq \cdots \). By then replacing each \( \mathcal{N}_n \) with the (larger) regular open set \( \text{int}[\text{cl}(\mathcal{N}_n)] \) if necessary, we can assume without loss of generality that \( \mathcal{N}_n \) is regular open. For all \( n \in \mathbb{N} \), let \( \mathcal{R}_n := -\mathcal{N}_n \). (Thus, \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \subseteq \cdots \)) Now define \( \mathcal{P}_1 := \mathcal{R}_1 \), and for all \( n \in \mathbb{N} \) with \( n \geq 2 \), define \( \mathcal{P}_n := \mathcal{R}_n \cap (-\mathcal{R}_{n-1}) \). Then \( \{\mathcal{P}_n\}_{n=1}^\infty \) is a collection of disjoint regular open sets, none of them containing \( s_0 \). By construction, for all \( n \in \mathbb{N} \) we have \( \mathcal{R}_n = \mathcal{P}_1 \lor \cdots \lor \mathcal{P}_n \).

**Claim 1A:** \( \bigvee_{n=1}^\infty \mathcal{P}_n = S \).

**Proof.** It suffices to show that \( \bigvee_{n=1}^\infty \mathcal{P}_n \) is dense in \( S \). Let \( s_1 \in S \setminus \{s_0\} \). Since \( S \) is Hausdorff, there exist disjoint open neighbourhoods \( \mathcal{B}_0 \) and \( \mathcal{B}_1 \) around \( s_0 \) and \( s_1 \), respectively. Since \( \{\mathcal{N}_n\}_{n=1}^\infty \) is a neighbourhood base of \( s_0 \), we can find some

\(^8\) Indeed, \( \mathcal{R}(S) \) is a complete Boolean algebra: it is closed even under the uncountably infinite versions of the operations \( \bigvee \) and \( \bigwedge \); see [Fre04, Theorem 314P] or [Wal74, Proposition 2.3, p.45].
\[ \mathcal{N}_n \subseteq \mathcal{B}_n. \] Thus, \( \mathcal{B}_1 \subseteq \mathcal{R}_n \). But \( s_1 \in \mathcal{B}_1 \) and \( \mathcal{R}_n = \mathcal{P}_1 \lor \cdots \lor \mathcal{P}_n \). Thus, we conclude that \( s_1 \in \bigvee_{n=1}^{\infty} \mathcal{P}_n \).

This holds for all \( s_1 \in \mathcal{S}\setminus\{s_0\} \). Thus, \( \mathcal{S}\setminus\{s_0\} \subseteq \bigvee_{n=1}^{\infty} \mathcal{P}_n \). But this is a dense subset of \( \mathcal{S} \) (because \( s_0 \) is not an isolated point, because \( \mathcal{S} \) is perfect). Thus, \( \bigvee_{n=1}^{\infty} \mathcal{P}_n = \mathcal{S} \).

Now, for all \( n \in \mathbb{N} \), we have \( \mu[\mathcal{N}_n] \geq \epsilon \) by the atomic property of \( s_0 \). Thus,

\[
\mu[\mathcal{P}_1] + \cdots + \mu[\mathcal{P}_n] = \mu[\mathcal{P}_1 \lor \cdots \lor \mathcal{P}_n] = \mu[\mathcal{R}_n] = 1 - \mu[\mathcal{N}_n] \leq 1 - \epsilon.
\]

Taking the limit as \( n \to \infty \), we conclude that \( \sum_{n=1}^{\infty} \mu[\mathcal{P}_n] \leq 1 - \epsilon \). But since \( \mu \) is sigma-additive, Claim 1A implies that \( \sum_{n=1}^{\infty} \mu[\mathcal{P}_n] = \mu[\mathcal{S}] = 1 \), which is a contradiction. To avoid the contradiction, \( s_0 \) cannot be an atom.  

\( \diamond \) Claim 1

Now, let \( \{\mathcal{B}_n\}_{n=1}^{\infty} \) be a countable base for the topology of \( \mathcal{S} \). For all \( n \in \mathbb{N} \), let \( s_n \in \mathcal{B}_n \); then \( \{s_n\}_{n=1}^{\infty} \) is a countable dense subset of \( \mathcal{S} \). Fix \( \epsilon \in (0, 1) \). For each \( n \in \mathbb{N} \), Claim 1 implies that \( s_n \) is not an atom of \( \mu \); thus, there exists \( \mathcal{R}_n \in \mathfrak{F}(\mathcal{S}) \) with \( s_n \subseteq \mathcal{R}_n \) such that

\[
\mu[\mathcal{R}_n] \leq \frac{\epsilon}{2^n}.
\]

(These sets are not necessarily disjoint.) Since \( \mu \) is sigma-additive,

\[
\mu\left[ \bigvee_{n=1}^{\infty} \mathcal{R}_n \right] \leq \sum_{n=1}^{\infty} \mu[\mathcal{R}_n] \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.
\]

But \( \bigvee_{n=1}^{\infty} \mathcal{R}_n = \mathcal{S} \), because \( \bigcup_{n=1}^{\infty} \mathcal{R}_n \) is dense, because it contains the dense set \( \{s_n\}_{n=1}^{\infty} \).

Thus, we conclude that \( \mu[\mathcal{S}] = \epsilon \). Since \( \epsilon < 1 \), this is a contradiction. \( \square \)

## 4 Integrators and conditional expectation

For any topological space \( \mathcal{S} \), let \( \mathcal{C}(\mathcal{S}, \mathbb{R}) \) denote the vector space of all continuous, real-valued functions on \( \mathcal{S} \). Let \( \mathcal{C}_b(\mathcal{S}, \mathbb{R}) \) be the Banach space of bounded, continuous, real-valued functions, with the uniform norm \( \| \cdot \|_\infty \). Let \( \mathfrak{E} \) be the Boolean algebra of elementary regular open subsets of \( \mathbb{R} \), as defined in Example 3.11(a). Let \( \mathfrak{B} \subseteq \mathfrak{F}(\mathcal{S}) \) be a Boolean subalgebra of \( \mathfrak{F}(\mathcal{S}) \). A function \( f : \mathcal{S} \to \mathbb{R} \) is \( \mathfrak{B} \)-comeasurable if \( \text{int} (f^{-1} [\text{clo} (\mathfrak{E})]) \in \mathfrak{B} \) for all \( \mathfrak{E} \in \mathfrak{E} \). Equivalently, \( f \) is \( \mathfrak{B} \)-comeasurable if \( \text{int} (f^{-1} (-\infty, r]) \in \mathfrak{B} \) and \( \text{int} (f^{-1} (r, \infty)) \in \mathfrak{B} \) for all \( r \in \mathbb{R} \). This section will develop an integration theory for comeasurable functions.

**Example 4.1.** (a) If \( f : \mathcal{S} \to \mathbb{R} \) is continuous, then \( f \) is \( \mathfrak{F}(\mathcal{S}) \)-comeasurable.

(b) Let \( \mathcal{S} \) be a differentiable manifold, and let \( \mathfrak{B}_{\text{smooth}} \) be the Boolean algebra of regular open sets with piecewise smooth boundaries, from Example 3.11(b). If \( f : \mathcal{S} \to \mathbb{R} \) is any differentiable function such that \( df(s) \neq 0 \) for all \( s \in \mathcal{S} \), then \( f \) is \( \mathfrak{B}_{\text{smooth}} \)-comeasurable.

Not every differentiable function is \( \mathfrak{B}_{\text{smooth}} \)-comeasurable. To see why some condition like \( df \neq 0 \) is required, suppose \( \mathcal{S} = \mathbb{R}^2 \), and define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f(x, y) := y^4 \sin(x/y) \) for
all \((x,y) \in \mathbb{R}^2\). Then \(f\) is differentiable everywhere on \(\mathbb{R}^2\) (with \(df(x,0) = 0\) for all \(x \in \mathbb{R}\)).

However, if \(\mathcal{R} := \text{int} \left( f^{-1}(\infty,0] \right)\), then \(\mathcal{R} \notin \mathcal{B}_{\text{smith}}\), because \(\partial \mathcal{R} = f^{-1}\{0\} = \{(x,y) \in \mathbb{R}^2; y = 0\text{ or } y = x/n\pi\text{ for some } n \in \mathbb{Z}\}\). This set is an infinite union of lines passing through the origin, which “converge” to the horizontal line \(y = 0\); hence it cannot be represented as a finite union of smooth curves.

(c) Let \(\mathcal{S}\) be a topological vector space, and let \(\mathcal{B}_{\text{poly}}\) be the Boolean algebra of regular open polyhedra, from Example 3.1(c). A function \(f : \mathcal{S} \to \mathbb{R}\) is affine if \(f = f_0 + r\) for some continuous linear function \(f_0 : \mathcal{S} \to \mathbb{R}\) and some constant \(r \in \mathbb{R}\). We say \(f\) is piecewise affine if there is a collection \(\mathcal{P}_1, \ldots, \mathcal{P}_N\) of disjoint regular open polyhedra such that \(\mathcal{S} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_N\), and a collection \(f_1, \ldots, f_N : \mathcal{S} \to \mathbb{R}\) of affine functions, such that \(f|_{\mathcal{P}_n} = f^n_{|\mathcal{P}_n}\) for all \(n \in \{1 \ldots N\}\). Any piecewise affine function is \(\mathcal{B}_{\text{poly}}\)-comeasurable.

(d) The sum of two \(\mathcal{B}\)-comeasurable functions is not necessarily \(\mathcal{B}\)-comeasurable. To see this, let \(\mathcal{S} = \mathbb{R}\), and let \(\mathcal{E}\) be the Boolean algebra of elementary sets from Example 3.1(a). Let \(f(x) := -2x\) and let \(g(x) := 2x + x^2 \sin(1/x)\), for all \(x \in \mathbb{R}\). Then both \(f\) and \(g\) are \(\mathcal{E}\)-comeasurable (because they are continuous and monotone). But if \(h := f + g\), then \(h(x) = x^2 \sin(1/x)\) for all \(x \in \mathbb{R}\). This function is not \(\mathcal{E}\)-comeasurable: if \(\mathcal{R} := \text{int} (h^{-1}(-\infty,0])\), then \(\mathcal{R}\) is an infinite union of open intervals, so \(\mathcal{R} \notin \mathcal{E}\). (Indeed, \(\partial \mathcal{R} = \{0\} \cup \{1/n\pi; n \in \mathbb{Z}\}\), which has a cluster point at 0.)

Let \(\mathcal{C}_B(\mathcal{S})\) be the set of all \(\mathcal{B}\)-comeasurable functions in \(\mathcal{C}_b(\mathcal{S}, \mathbb{R})\). This set is not necessarily closed under addition, as shown by Example 11(d). So, let \(\mathcal{G}_B(\mathcal{S})\) be the closed linear subspace of \(\mathcal{C}_b(\mathcal{S}, \mathbb{R})\) spanned by \(\mathcal{C}_B(\mathcal{S})\); then \(\mathcal{G}_B(\mathcal{S})\) is a Banach space under the uniform norm \(\|\cdot\|_\infty\). (If \(\mathcal{B} = \mathcal{A}(\mathcal{S})\), then \(\mathcal{G}_B(\mathcal{S}) = \mathcal{C}_B(\mathcal{S}) = \mathcal{C}_b(\mathcal{S}, \mathbb{R})\), because every continuous function is \(\mathcal{A}(\mathcal{S})\)-comeasurable.) For any subset \(\mathcal{B} \subseteq \mathcal{S}\), let \(\mathcal{G}_B(\mathcal{B}) := \{g|_{\mathcal{B}}; g \in \mathcal{G}_B(\mathcal{S})\}\). This is a linear subspace of \(\mathcal{C}_b(\mathcal{B}, \mathbb{R})\).

Now let \(\mathcal{R} \in \mathcal{B}\). A \(\mathcal{B}\)-partition of \(\mathcal{R}\) is a collection \(\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}\) (for some \(N \in \mathbb{N}\)) of disjoint elements of \(\mathcal{B}\) such that \(\mathcal{R} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_N\). For instance, if \(\mathcal{S} = \mathbb{R}\), then \(\{(0,1), (1,2)\}\) is a regular open partition of \((0,2)\).

An integrator on \(\mathcal{B}\) is a collection \(\mathbf{I} := \{I_B\}_{B \in \mathcal{B}}\), where, for all \(B \in \mathcal{B}\),

- \(I_B : \mathcal{G}_B(\mathcal{B}) \to \mathbb{R}\) is a bounded linear functional that is weakly monotonic —that is, for any \(f, g \in \mathcal{G}_B(\mathcal{B})\), if \(f(b) \leq g(b)\) for all \(b \in \mathcal{B}\), then \(I_B[f] \leq I_B[g]\);

- For any \(\mathcal{B}\)-partition \(\{\mathcal{B}_n\}_{n=1}^{N}\) of \(\mathcal{B}\), and for any \(g \in \mathcal{G}_B(\mathcal{B})\), we have

\[
I_B[g] = \sum_{n=1}^{N} I_{B_n}[g_{|B_n}].
\]

Even when \(\mathcal{B} = \mathcal{A}(\mathcal{S})\), it is not generally true that \(\mathcal{G}_B(\mathcal{B}) = \mathcal{C}_B(\mathcal{B}, \mathbb{R})\), for two reasons. First, some functions in \(\mathcal{C}_B(\mathcal{B}, \mathbb{R})\) cannot be extended to functions in \(\mathcal{C}_b(\mathcal{B}, \mathbb{R})\), where \(\overline{\mathcal{B}}\) is the closure of \(\mathcal{B}\). (For example, let \(\mathcal{S} := \mathbb{R}\), let \(\mathcal{B} := (0,1)\), and let \(g(x) := \sin(1/x)\) for all \(x \in \mathcal{B}\).) Second, not all functions in \(\mathcal{C}_b(\overline{\mathcal{B}}, \mathbb{R})\) can be extended to \(\mathcal{C}_b(\mathcal{S}, \mathbb{R})\), unless \(\mathcal{S}\) is a normal space, which we will not assume in general.
We say \( I \) is \textit{strictly monotonic} if, for all \( \mathcal{B} \in \mathfrak{B} \) and \( g \in \mathcal{C}_\mathfrak{B}(\mathcal{B}) \), if \( g(b) > 0 \) for all \( b \in \mathcal{B} \), then \( \mathbb{I}_\mathcal{B}[g] > 0 \). (Note that we only require strict monotonicity on \( \mathcal{C}_\mathfrak{B}(\mathcal{B}) \), not \( \mathcal{G}_\mathfrak{B}(\mathcal{B}) \).) If \( g \in \mathcal{G}_\mathfrak{B}(\mathcal{S}) \) and \( \mathcal{B} \in \mathfrak{B} \), we will abuse notation and write “\( \mathbb{I}_\mathcal{B}[g] \)” to mean \( \mathbb{I}_\mathcal{B}[g_{|\mathcal{B}}] \). Heuristically, \( \mathbb{I}_\mathcal{B}[g] \) should be interpreted as “the integral of \( g \) on \( \mathcal{B} \)”. (We will make this precise later.) Clearly, an integrator is equivalent to a single function \( I : \mathfrak{B} \times \mathcal{G}_\mathfrak{B}(\mathcal{S}) \to \mathbb{R} \) with the following properties:

- For every \( \mathcal{B} \in \mathfrak{B} \), the function \( \mathbb{I}_\mathcal{B} := I(\mathcal{B}, \bullet) : \mathcal{G}_\mathfrak{B}(\mathcal{S}) \to \mathbb{R} \) is a bounded linear functional which is weakly monotonic, such that \( I(\mathcal{B}, g) = 0 \) if \( g_{|\mathcal{B}} = 0 \).

- For every non-negative \( g \in \mathcal{G}_\mathfrak{B}(\mathcal{S}) \), if \( I(\mathcal{S}, g) \neq 0 \), then the function \( \mu_g : \mathfrak{B} \to [0, 1] \) defined by \( \mu_g[\mathcal{B}] := I(\mathcal{B}, g)/I(\mathcal{S}, g) \) is a credence on \( \mathfrak{B} \).

However, we find the earlier definition more illuminating. Now, let \( \mu \) be a credence on \( \mathfrak{B} \). We say that an integrator \( I \) is \textit{compatible} with \( \mu \) if we have \( \| \mathbb{I}_\mathcal{B} \|_\infty = \mu[\mathcal{B}] \), for all \( \mathcal{B} \in \mathfrak{B} \). If \( 1 \) is the constant function with value 1, then this implies that \( \mathbb{I}_\mathcal{B}[1] = \mu[\mathcal{B}] \). (This follows from the weak monotonicity of \( \mathbb{I}_\mathcal{B} \).) Furthermore, if \( \mu[\mathcal{B}] > 0 \), then we can then define \( \mathbb{E}_\mathcal{B}[f] := \mathbb{I}_\mathcal{B}[f]/\mu[\mathcal{B}] \) for all \( f \in \mathcal{G}_\mathfrak{B}(\mathcal{B}) \). Heuristically, \( \mathbb{E}_\mathcal{B}[f] \) is the \textit{conditional expectation} of \( f \) with respect to \( \mu \), given \( \mathcal{B} \). Observe that \( \mathbb{E}_\mathcal{B}[1] = 1 \). Furthermore, for any \( \mathfrak{B} \)-partition \( \{\mathcal{B}_n\}_{n=1}^N \) of \( \mathcal{B} \), formula (1) immediately yields the Bayesian formula:

\[
\mathbb{E}_\mathcal{B}[g] = \frac{1}{\mu[\mathcal{B}]} \sum_{n=1}^N \mu[\mathcal{B}_n] \mathbb{E}_\mathcal{B}_n [g_{|\mathcal{B}_n}] .
\]

\textbf{Example 4.2.} Let \( \mathcal{S} = (0, 1) \), let \( \mathfrak{C} \) be the Boolean algebra of elementary regular open subsets of \( (0, 1) \), and let \( \mu \) be the “Lebesgue” credence on \( \mathfrak{C} \) from Example 3.2. Then we obtain a \( \mu \)-compatible integrator as follows: for any \( \mathcal{E} = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \cdots \sqcup (a_N, b_N) \) in \( \mathfrak{C} \), and any \( g \in \mathcal{G}_\mathfrak{C}(0, 1) \), we define

\[
\mathbb{I}_\mathfrak{C}[g] := \int_{a_1}^{b_1} g(x) \, dx + \int_{a_2}^{b_2} g(x) \, dx + \cdots + \int_{a_N}^{b_N} g(x) \, dx ,
\]

where the integrals on the right-hand side can be read as Riemann integrals.

As we have already observed, every integrator defines a credence. Conversely, if \( \mathcal{S} \) is a finite set with the discrete topology, then every singleton is a regular set; in this case, for any credence \( \mu \), we can easily derive a unique \( \mu \)-compatible integrator \( \{\mathbb{I}_\mathcal{R}\}_{\mathcal{R} \in \mathfrak{R}(\mathcal{S})} \), by applying equation (2) when \( \mathcal{B}_1, \ldots, \mathcal{B}_N \) are singleton sets. But in general, the relationship between credences and integrators is more subtle. Typically, the Boolean algebra \( \mathfrak{R}(\mathcal{S}) \) is much smaller than the Borel sigma algebra on \( \mathcal{S} \), and the operation \( \lor \) is not the same as set-theoretic union. Thus, we cannot simply apply standard integration theory to credences on \( \mathfrak{R}(\mathcal{S}) \). Indeed, \textit{prima facie} it is not even clear how to \textit{define} integrals like the ones in formula (3). Nevertheless, in this section, we will prove the following result.
Theorem 4.3 Let $S$ be any topological space, let $\mathcal{B}$ be any Boolean subalgebra of $\mathcal{R}(S)$, and let $\mu$ be a credence on $\mathcal{B}$. There exists a unique integrator $I$ on $\mathcal{B}$ that is compatible with $\mu$. Furthermore, if $\mu$ has full support, then $I$ is strictly monotonic.

It might seem that there is not much left to prove here: can’t we just use the residual charge from Proposition 3.7 to compute a Lebesgue-type integral for any continuous real-valued function on $S$? Indeed, we will explore this strategy later, in Proposition 6.1. But this approach is not entirely satisfactory, for three reasons. First, it only works for Baire spaces. Second, it only works for credences defined on the full Boolean algebra $\mathcal{R}(S)$, whereas we want a theory which works for arbitrary Boolean subalgebras of $\mathcal{R}(S)$. Third, as we will discuss in Section 6, a representation in terms of residual charges is not always ideal, because residual charges can be somewhat pathological.

So, instead of relying on Proposition 3.7, we will construct the integrator in Theorem 4.3 from first principles. This is similar to standard constructions of the Lebesgue integral. But there are some subtle differences, because our Boolean algebras use $\lor$ and $\neg$ instead of set-theoretic union and complementation. So it is worth going through the details.

Notation. For the rest of this section, let $S$ be topological space, let $\mathcal{B}$ be a Boolean subalgebra of $\mathcal{R}(S)$, and let $\mu$ be a credence on $\mathcal{B}$. We will say a function $f : S \to \mathbb{R}$ is $\mathcal{B}$-simple if there is a finite $\mathcal{B}$-partition $\mathcal{P}$ of $S$ such that $f$ is constant on each cell of $\mathcal{P}$. We will say that $f$ is subordinate to $\mathcal{B}$. Let $\mathcal{F}$ denote the set of all $\mathcal{B}$-simple functions on $S$. We will first show how to integrate any $\mathcal{B}$-simple function relative to $\mu$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathcal{B}$-partitions of $S$. We define $\mathcal{P} \otimes \mathcal{Q} := \{ \mathcal{P} \cap \mathcal{Q}; \; \mathcal{P} \in \mathcal{P} \text{ and } \mathcal{Q} \in \mathcal{Q} \}$. This is another $\mathcal{B}$-partition, and is the “minimal common refinement” of $\mathcal{P}$ and $\mathcal{Q}$. If $f \in \mathcal{F}$ is subordinate to $\mathcal{P}$, then $f$ is also subordinate to $\mathcal{P} \otimes \mathcal{Q}$. If $f \in \mathcal{F}$ is subordinate to $\mathcal{P}$, and $g \in \mathcal{F}$ is subordinate to $\mathcal{Q}$, then the functions $f + g$ and $\min\{f, g\}$ are also $\mathcal{B}$-simple functions, and are subordinate to $\mathcal{P} \otimes \mathcal{Q}$.

Lemma 4.4 $\mathcal{F}$ is vector space under pointwise addition and scalar multiplication.

Proof. Clearly, if $f \in \mathcal{F}$ and $r \in \mathbb{R}$, then $rf \in \mathcal{F}$ (with the same subordinating partition as $f$). Now let $f_1, f_2 \in \mathcal{F}$. Suppose $f_1$ is subordinate to a $\mathcal{B}$-partition $\mathcal{P}$, and $f_2$ is subordinate to a $\mathcal{B}$-partition $\mathcal{Q}$. Then $f_1 + f_2$ is subordinate to $\mathcal{P} \otimes \mathcal{Q}$. □

For any $f \in \mathcal{F}$, if $f$ is subordinate to the $\mathcal{B}$-partition $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N\}$, and there are (possibly nondistinct) real numbers $r_1, r_2, \ldots, r_N$ such that $f(p) = r_n$ for all $p \in \mathcal{P}_n$ and all $n \in [1 \ldots N]$, then for any nonempty $\mathcal{B} \in \mathcal{B}$, we define

$$\int_\mathcal{B}^\circ f \ d\mu := \sum_{n=1}^{N} r_n \mu[\mathcal{P}_n \cap \mathcal{B}].$$

(4)

(The “$\circ$” is a reminder that this is not a standard Lebesgue integral.)

---

10 The behaviour of $f$ on the boundaries of the cells will be irrelevant for our purposes, so we allow it to be arbitrary. Note that this is slightly different from the definition of “simple function” in the proof of Proposition 3.5 but this should cause no confusion.
Lemma 4.5 Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\mathcal{B}$-partitions, and suppose $f$ is subordinate to both. The values of $\int_B f \, d\mu$ achieved by evaluating expression (4) relative to $\mathcal{P}$ and $\mathcal{Q}$ are the same.

Proof. (Case 1) Suppose $\mathcal{Q}$ is a refinement of $\mathcal{P}$. Then every cell in $\mathcal{P}$ is a finite join of cells in $\mathcal{Q}$. Furthermore, if $\mathcal{P} \in \mathcal{P}$ and $\mathcal{P} = \mathcal{Q}_1 \lor \cdots \lor \mathcal{Q}_n$ for some $\mathcal{Q}_1, \ldots, \mathcal{Q}_n \in \mathcal{Q}$, then $\mu(\mathcal{P}) = \mu(\mathcal{Q}_1) + \cdots + \mu(\mathcal{Q}_n)$, by (1). Thus, by grouping the terms in the expression (4) for $\int_B f \, d\mu$ with respect to $\mathcal{Q}$, and simplifying, we get the expression (4) for $\int_B f \, d\mu$ with respect to $\mathcal{P}$.

(Case 2) Suppose neither $\mathcal{Q}$ nor $\mathcal{P}$ refines the other. Let $\mathcal{R} := \mathcal{P} \otimes \mathcal{Q}$. Then $\mathcal{R}$ refines both $\mathcal{P}$ and $\mathcal{Q}$. Now apply Case 1. \qed

Corollary 4.6 Let $\mathcal{Q} = \{ \mathcal{Q}_n \}_{n=1}^N$ be a $\mathcal{B}$-partition of $\mathcal{S}$, and let $f \in \mathcal{F}$. Then for any nonempty $\mathcal{B} \in \mathcal{B}$, we have

$$\int_B f \, d\mu = \sum_{n=1}^N \int_{B \cap \mathcal{Q}_n} f \, d\mu.$$

Proof. If $f$ is subordinate to the $\mathcal{B}$-partition $\mathcal{P}$, then it is also subordinate to the $\mathcal{B}$-partition $\mathcal{P} \otimes \mathcal{Q}$. If we write the expression (4) for $\int_B f \, d\mu$ with respect to $\mathcal{P} \otimes \mathcal{Q}$, we get a sum of the expressions (4) for $\int_{B \cap \mathcal{Q}_n} f \, d\mu$ with respect to $\mathcal{P}$. \qed

Lemma 4.7 Let $f, g \in \mathcal{F}$, and let $r \in \mathbb{R}$. Let $\mathcal{B} \in \mathcal{B}$ be nonempty. Then:

(a) $\int_B r f \, d\mu = r \int_B f \, d\mu$, and $\int_B (f + g) \, d\mu = \int_B f \, d\mu + \int_B g \, d\mu$.

(b) If $f(b) \leq g(b)$ for all $b \in \mathcal{B}$, then $\int_B f \, d\mu \leq \int_B g \, d\mu$.

(c) For any $r \in \mathbb{R}$, if $f$ is the constant function with value $r$, then $\int_B f \, d\mu = r \mu(\mathcal{B})$.

(d) For any $f, g \in \mathcal{F}$ and $\epsilon > 0$, if $\|f - g\|_\infty \leq \epsilon$, then $|\int_B f \, d\mu - \int_B g \, d\mu| \leq \epsilon \mu(\mathcal{B})$.

Proof. Suppose $f$ is subordinate to the $\mathcal{B}$-partition $\mathcal{P}$ and $g$ is subordinate to the $\mathcal{B}$-partition $\mathcal{Q}$. Let $\mathcal{R} := \mathcal{P} \otimes \mathcal{Q}$. Then both $f$ and $g$ are subordinate to $\mathcal{R}$.

(a) The function $f + g$ is also subordinate to $\mathcal{R}$. If we evaluate $\int_B (f + g) \, d\mu$ via the expression (4) relative to $\mathcal{R}$, we see that it splits as the sum of the expressions (4) for $\int_B f \, d\mu$ and $\int_B g \, d\mu$ relative to $\mathcal{R}$. Thus, $\int_B (f + g) \, d\mu = \int_B f \, d\mu + \int_B g \, d\mu$.

(b) For every $b \in \mathcal{B}$, we have $f(b) \leq g(b)$. Thus, if we evaluate $\int_B f \, d\mu$ and $\int_B g \, d\mu$ via the relevant expressions (4) relative to $\mathcal{R}$, we see that every summand for $\int_B f \, d\mu$ is less than or equal to the corresponding summand for $\int_B g \, d\mu$. Thus, $\int_B f \, d\mu \leq \int_B g \, d\mu$. \hfill \qed
(c) If $f$ is a constant, then it is subordinate to the one-element partition $\{S\}$, and $S \cap B = B$. Now apply formula (4) to deduce that $\int_B f \, d\mu = r \mu[B]$.

(d) The function $(f - g)$ is $\mathcal{B}$-simple (it is subordinate to $\mathfrak{R}$) and $(f - g)(s) \leq \epsilon$ for all $s \in S$. Thus,

$$\int_B (f - g) \, d\mu \leq \int_B \epsilon \, d\mu = \epsilon \mu[B]. \quad (5)$$

Likewise, $\int_B (g - f) \, d\mu \leq \epsilon \mu[B]$. But then

$$\int_B (f - g) \, d\mu = \int_B (-(g - f)) \, d\mu = -\int_B (g - f) \, d\mu \geq -\epsilon \mu[B]. \quad (6)$$

Combining inequalities (5) and (6), we get that $|\int_B f \, d\mu - \int_B g \, d\mu| \leq \epsilon \mu[B]$. \qed

For any nonempty $B \in \mathcal{B}$ and $g \in G_B(\mathcal{B})$, let $F_g(\mathcal{B}) := \{f \in F; f(b) \leq g(b), \text{ for all } b \in B\}$. Then define

$$\mathbb{I}_B[g] := \sup_{f \in F_g(\mathcal{B})} \int_B f \, d\mu.$$ \hfill (7)

Theorem 4.3 is a consequence of the following result.

**Proposition 4.8** Let $S$ be any topological space, and let $\mathcal{B}$ be any Boolean subalgebra of $\mathfrak{R}(S)$. Let $\mu$ be any credence on $\mathfrak{B}$. Then the system $\mathbb{I} := \{\mathbb{I}_B\}_{B \in \mathcal{B}}$ of functionals defined by formula (7) is the unique integrator on $S$ that is compatible with $\mu$. Furthermore, if $\mu$ has full support, then $\mathbb{I}$ is strictly monotonic.

The proof of Proposition 4.8 involves a series of lemmas.

**Lemma 4.9** For any $B \in \mathcal{B}$, $g \in G_B(\mathcal{B})$ and $\epsilon > 0$, there exists $f \in F_g$ with $\|f - g\|_\infty < \epsilon$.

**Proof.** It suffices to show this in the case when $B = S$. Consider first the case where $g \in \mathcal{C}_B(\mathcal{S})$. Since $g$ is bounded, there is some $M \in \mathbb{N}$ that $|g(s)| < M$ for all $s \in S$. Fix $N \in \mathbb{N}$ such that $1/N < \epsilon$. For all $m \in [-MN \ldots MN)$, let $I_m := \left[\frac{m}{N}, \frac{m+1}{N}\right]$ (a closed interval in $\mathbb{R}$). Let $C_m := g^{-1}(I_m)$ and let $B_m := \text{int}(C_m)$. Then $B_m \in \mathcal{B}$, because $g$ is $\mathcal{B}$-comeasurable.

**Claim 1:** $\bigcup_{m=-MN}^{MN-1} B_m = S$.

**Proof.** Let $B^* := \bigcup_{m=-MN}^{MN-1} B_m$. We must show that $B^*$ is dense in $S$. Suppose not. Let $s \in S$ be a point not in the closure of $B^*$. Then there is some open neighbourhood $D$ of $s$ which does not intersect $B^*$. Now, for any $m$, the set $g^{-1}(\left[\frac{m}{N}, \frac{m+1}{N}\right])$ is a subset of $B_m$ (because it is an open subset of $C_m$). Thus, $D$ cannot intersect $g^{-1}(\left[\frac{m}{N}, \frac{m+1}{N}\right])$. Thus,
for all \( s' \in \mathcal{D} \) we must have \( g(s') = \frac{m}{N} \) for some \( m \in [-MN \ldots MN] \). In particular, \( g(s) = \frac{m_0}{N} \) for some \( m_0 \in [-MN \ldots MN] \). By making \( \mathcal{D} \) small enough, we can ensure that \( g(s') = \frac{m_0}{N} \) for all \( s' \in \mathcal{D} \) (because \( g \) is continuous). But then \( \mathcal{D} \) is an open subset of \( \mathcal{C}_{m_0} \); hence \( \mathcal{D} \subseteq \mathcal{B}_{m_0} \). Contradiction. \( \square \)

Let \( \mathcal{P}_{-MN} := \mathcal{B}_{-MN} \) and, for any \( m \in (-MN \ldots MN) \), define \( \mathcal{P}_m := \mathcal{B}_m \cap (-\mathcal{B}_{m-1}) \). Let \( \mathcal{M} := \{ m \in [-MN \ldots MN]; \mathcal{P}_m \neq \emptyset \} \), and define \( \mathfrak{P} := \{ \mathcal{P}_m \}_{m \in \mathcal{M}} \); then \( \mathfrak{P} \) is a \( \mathfrak{B} \)-partition of \( \mathcal{S} \). We define \( f \in \mathcal{F} \) as follows: for all \( m \in \mathcal{M} \), and all \( s \in \mathcal{P}_m \), define \( f(s) := \frac{m}{N} \). Meanwhile, for all \( s \in \mathcal{S} \) not in \( \bigcup_{m \in \mathcal{M}} \mathcal{P}_m \), define \( f(s) := g(s) \). Thus, \( f \in \mathcal{F} \), and \( f(s) \leq g(s) \) for all \( s \in \mathcal{S} \). Finally, for any \( s \in \mathcal{S} \), \( |f(s) - g(s)| \leq \frac{1}{N} < \epsilon \). Thus, \( \|f - g\|_{\infty} < \epsilon \), as desired.

Next, suppose \( g \) is a linear combination of functions in \( \mathcal{G}_B(\mathcal{S}) \). Since \( \mathcal{G}_B(\mathcal{S}) \) is closed under scalar multiplication, \( g \) can even be written in the form \( g = \sum_{n=1}^{N} g_n \) with \( g_n \in \mathcal{G}_B(\mathcal{S}) \) for any \( n \in [1 \ldots N] \), and some \( N \geq 1 \). Then, by the previous paragraph, there are functions \( f_n \in \mathcal{F}_{g_n} \) such that \( \|f_n - g_n\|_{\infty} < \epsilon/N \) for any \( n \in [1 \ldots N] \). Then, let \( f := \sum_{n=1}^{N} f_n \). Lemma 4.7 implies that \( f \in \mathcal{F} \). In fact, \( f \in \mathcal{F}_g(B) \). Finally, \( \|f - g\|_{\infty} < \epsilon \), by the Triangle Inequality.

Finally, let \( g \in \mathcal{G}_B(\mathcal{S}) \). Then there is some \( h_1 \in \mathcal{G}_b(\mathcal{S}, \mathbb{R}) \) with \( g - h_1 \|_{\infty} < \epsilon/3 \), such that \( h_1 \) is a linear combination of functions in \( \mathcal{G}_B(\mathcal{S}) \). Let \( h_2 := h_1 - \epsilon/3 \). Then \( h_2 \) is also a linear combination of functions in \( \mathcal{G}_B(\mathcal{S}) \), and \( g - h_2 \|_{\infty} < 2\epsilon/3 \), but also \( h_2 \leq g \). By the argument in the previous paragraph, there is some simple function \( f \in \mathcal{F} \) such that \( f \leq h_2 \) and \( \|f - h_2\|_{\infty} < \epsilon/3 \). Thus, \( f \leq g \) and and \( \|f - g\|_{\infty} < \epsilon \), as desired. \( \square \)

**Lemma 4.10** Let \( B \in \mathfrak{B} \) be nonempty. The function \( \mathbb{I}_B : \mathcal{G}_B(B) \rightarrow \mathbb{R} \) defined by formula (7) is linear and continuous with respect to the uniform norm on \( \mathcal{G}_B(B) \), and \( \|\mathbb{I}_B\|_{\infty} = \mu(B) \). It is also weakly monotone: if \( f(b) \leq g(b) \) for all \( b \in B \), then \( \mathbb{I}_B[f] \leq \mathbb{I}_B[g] \).

**Proof.** Linearity: scalar multiplication. For any \( g \in \mathcal{G}_B(B) \) and \( r > 0 \) it is easy to see that \( \mathcal{F}_{rg}(B) = \{ rf ; f \in \mathcal{F}_g(B) \} \). Thus,

\[
\mathbb{I}_B[r \ g] = \sup_{f \in \mathcal{F}_{rg}(B)} \int_B f \ d\mu = \sup_{f \in \mathcal{F}_g(B)} \int_B r \ f \ d\mu = r \ \mathbb{I}_B[g],
\]

where (*) is by Lemma 4.7(a). Now suppose \( r < 0 \). It suffices to consider the case \( r = -1 \). For any \( g \in \mathcal{G}_B(B) \), we define \( \mathcal{F}' = \left\{ f \in \mathcal{F} ; f(b) \geq g(b), \text{ for all } b \in B \right\} \).

**Claim 1:** \( \mathbb{I}_B[g] = \inf_{f \in \mathcal{F}'} \int_B f \ d\mu \).

**Proof.** If \( f \in \mathcal{F}_g(B) \) and \( f' \in \mathcal{F}'(B) \), then \( f(b) \leq f'(b) \) for all \( b \in B \), and thus, \( \int_B f \leq \int_B f' \) by Lemma 4.7(b). It suffices to show that we can make this gap arbitrarily small.
Let $\epsilon > 0$. Lemma \ref{lem:4.9} yields some $f \in \mathcal{F}_g(\mathcal{B})$ such that $\|f - g\|_\infty < \epsilon/2$. Now let $f' := f + \epsilon$. Then $f' \in \mathcal{F}$, and for all $b \in \mathcal{B}$, we have $f'(b) = f(b) + \epsilon > g(b) - \epsilon/2 + \epsilon = g(b) + \epsilon/2 > g(b)$. Thus, $f' \in \mathcal{F}^g(\mathcal{B})$. However, $\|f - f'\|_\infty \leq \epsilon$. Thus, Lemma \ref{lem:1.7}(d) implies that $\left| \sup_{f \in \mathcal{F}_g(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu - \inf_{f \in \mathcal{F}^g(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu \right| \leq \epsilon$. It follows that

$$\left| \sup_{f \in \mathcal{F}_g(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu - \inf_{f \in \mathcal{F}^g(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu \right| \leq \epsilon.$$ 

Now the claim follows from defining formula \eqref{eq:7}, by letting $\epsilon \to 0$. \hfill \diamond \text{ Claim 1}

Clearly, $\mathcal{F}^{-g}(\mathcal{B}) = \{ -f ; \, f \in \mathcal{F}_g(\mathcal{B}) \}$. Thus,

$$\mathbb{I}_B[-g] \equiv \inf_{f \in \mathcal{F}^{-g}(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu \equiv \inf_{f \in \mathcal{F}_g(\mathcal{B})} \int_{\mathcal{B}} -f \, d\mu \equiv \inf_{f \in \mathcal{F}_g(\mathcal{B})} \left( -\int_{\mathcal{B}} f \, d\mu \right)$$

$$\quad = - \sup_{f \in \mathcal{F}_g(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu \equiv - \mathbb{I}_B[g],$$

as claimed. Here, $(\ast)$ is by Claim 1, $(\dagger)$ is by Lemma \ref{lem:1.7}(a), and $(\diamond)$ is by \eqref{eq:7}.

**Linearity: Addition.** Let $g_1, g_2 \in \mathcal{G}_B(\mathcal{B})$. For any $f_1 \in \mathcal{F}_{g_1}(\mathcal{B})$ and $f_2 \in \mathcal{F}_{g_2}(\mathcal{B})$, it is easy to see that $f_1 + f_2 \in \mathcal{F}_{g_1 + g_2}(\mathcal{B})$. Thus,

$$\mathbb{I}_B[g_1 + g_2] \equiv \sup_{f \in \mathcal{F}_{g_1 + g_2}(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu \equiv \sup_{f_1 \in \mathcal{F}_{g_1}(\mathcal{B})} \sup_{f_2 \in \mathcal{F}_{g_2}(\mathcal{B})} \int_{\mathcal{B}} (f_1 + f_2) \, d\mu$$

$$\quad \equiv \sup_{f_1 \in \mathcal{F}_{g_1}(\mathcal{B})} \sup_{f_2 \in \mathcal{F}_{g_2}(\mathcal{B})} \left( \int_{\mathcal{B}} f_1 \, d\mu + \int_{\mathcal{B}} f_2 \, d\mu \right) \equiv \sup_{f_1 \in \mathcal{F}_{g_1}(\mathcal{B})} \int_{\mathcal{B}} f_1 \, d\mu + \sup_{f_2 \in \mathcal{F}_{g_2}(\mathcal{B})} \int_{\mathcal{B}} f_2 \, d\mu$$

$$\equiv (\ast) \mathbb{I}_B[g_1] + \mathbb{I}_B[g_2],$$

(8)

where $(\ast)$ is by Lemma \ref{lem:1.7}(a) and both $(\diamond)$ are by formula \eqref{eq:7}. Meanwhile,

$$-\mathbb{I}_B[g_1 + g_2] \equiv \mathbb{I}_B[-(g_1 + g_2)] \geq \mathbb{I}_B[-g_1] + \mathbb{I}_B[-g_2] \equiv \mathbb{I}_B[g_1] - \mathbb{I}_B[g_2],$$

(9)

where $(\dagger)$ is by applying the derivation of inequality (8) to $-(g_1 + g_2) = (-g_1) + (-g_2)$, while both $(\ast)$ use the already-established scalar multiplication property of $\mathbb{I}_B$. Multiplying inequality (9) by $(-1)$, we get

$$\mathbb{I}_B[g_1 + g_2] \leq \mathbb{I}_B[g_1] + \mathbb{I}_B[g_2].$$

(10)

Combining inequalities (8) and (10), we obtain $\mathbb{I}_B[g_1 + g_2] = \mathbb{I}_B[g_1] + \mathbb{I}_B[g_2]$, as desired.

**Weak monotonicity:** Let $g_1, g_2 \in \mathcal{G}_B(\mathcal{B})$, and suppose $g_1(b) \leq g_2(b)$ for all $b \in \mathcal{B}$. Then clearly $\mathcal{F}_{g_1}(\mathcal{B}) \subseteq \mathcal{F}_{g_2}(\mathcal{B})$. Thus,

$$\mathbb{I}_B[g_1] \equiv \sup_{f \in \mathcal{F}_{g_1}(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu \leq \sup_{f \in \mathcal{F}_{g_2}(\mathcal{B})} \int_{\mathcal{B}} f \, d\mu = \mathbb{I}_B[g_1].$$
Continuity and norm. Let \( g \in \mathcal{G}_B(B) \). Since \( g \) is a bounded function, we have \( -\|g\|_\infty \leq g(s) \leq \|g\|_\infty \) for any \( s \in S \). Note that all constant real-valued functions belong to \( \mathcal{G}_B(B) \) and even to \( F \). Thus

\[
-\|g\|_\infty \mu(B) \leq \mathbb{I}_B[-\|g\|_\infty] \leq \mathbb{I}_B[g] \leq \mathbb{I}_B[\|g\|_\infty] \leq \|g\|_\infty \mu(B),
\]

where both (*) use Lemma 4.7(c), and both (†) use weak monotonicity. Thus, we obtain \( \|\mathbb{I}_B[g]\| \leq \|g\|_\infty \mu(B) \) for any \( g \in \mathcal{G}_B(B) \). This shows that \( \mathbb{I}_B \) is continuous and that \( \|\mathbb{I}_B\|_\infty \leq \mu(B) \). To see equality, let \( 1 \) be the constant 1-valued function on \( S \). Since \( 1 \) is itself a simple function, we have \( 1 \in \mathcal{F}_1 \). Indeed, \( 1 \) is the maximal element of \( \mathcal{F}_1 \). Thus,

\[
\mathbb{I}_B[1] \overset{(*)}{=} \int_B 1 \, d\mu = \mu[B] = \mu[B] \cdot \|1\|_\infty,
\]

where (*) is by (11), and (†) is by Lemma 4.7(c). It follows that \( \|\mathbb{I}_B\|_\infty = \mu(B) \).

Lemma 4.11. Let \( B \in \mathfrak{B} \) and let \( \mathfrak{P} = \{\mathcal{P}_n\}_{n=1}^N \) be a \( \mathfrak{B} \)-partition of \( B \). If \( g \in \mathcal{G}_B(B) \), then

\[
\mathbb{I}_B[g] = \sum_{n=1}^N \mathbb{I}_{\mathcal{P}_n}[g].
\]

Proof. For any \( n \in [1 \ldots N] \), if \( g_n := g|_{\mathcal{P}_n} \), then \( g_n \in \mathcal{G}_B(\mathcal{P}_n) \), and

\[
\sup_{f \in \mathcal{F}_g(B)} \int_{\mathcal{P}_n} f \, d\mu = \sup_{f \in \mathcal{F}_{g_n}(\mathcal{P}_n)} \int_{\mathcal{P}_n} f \, d\mu.
\]

To see this, note that if \( f \in \mathcal{F}_g(B) \), then \( f \in \mathcal{F}_{g_n}(\mathcal{P}_n) \). Conversely, for any \( f \in \mathcal{F}_{g_n}(\mathcal{P}_n) \), there exists \( f' \in \mathcal{F}_g(B) \) with \( f'|_{\mathcal{P}_n} = f|_{\mathcal{P}_n} \). Thus, both supremums in (12) have the same value. Now,

\[
\mathbb{I}_B[g] \overset{(*)}{=} \sup_{f \in \mathcal{F}_g(B)} \int_B f \, d\mu \overset{†}{=} \sup_{f \in \mathcal{F}_g(B)} \left( \sum_{n=1}^N \int_{\mathcal{P}_n} f \, d\mu \right) \leq \sum_{n=1}^N \sup_{f \in \mathcal{F}_{g_n}(\mathcal{P}_n)} \int_{\mathcal{P}_n} f \, d\mu \overset{(*)}{=} \sum_{n=1}^N \mathbb{I}_{\mathcal{P}_n}[g].
\]

Here, both (*) are by defining equation (11). Meanwhile, (†) is by applying Corollary 4.6 to each \( f \in \mathcal{F}_g(B) \), and (†) is by invoking equation (12) for each \( n \in [1 \ldots N] \).

It remains to show the reverse inequality. Let \( M := \|g\|_\infty \). Then \( M < \infty \). Fix \( \epsilon > 0 \). For all \( n \in [1 \ldots N] \), defining equation (7) yields some \( f_n \in \mathcal{F}_g(\mathcal{P}_n) \) such that

\[
\int_{\mathcal{P}_n} f_n \, d\mu \geq \mathbb{I}_{\mathcal{P}_n}[g] - \frac{\epsilon}{N}.
\]
Define $f: S \rightarrow \mathbb{R}$ by setting $f(s) := f_n(s)$ for all $s \in \mathcal{P}_n$ and $n \in [1 \ldots N]$, and define $f(s) := -M$ for all $s \in S \setminus (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_N)$. Then $f$ is also a simple function (subordinate to a refinement of $\mathfrak{P}$), and $f(b) \leq g(b)$ for all $b \in \mathcal{B}$; hence $f \in \mathcal{F}_g(\mathcal{B})$. Thus,

$$
\mathbb{I}_\mathcal{B}[g] \geq \left( a \right) \int_\mathcal{B} f \, d\mu = \left( b \right) \sum_{n=1}^{N} \int_{\mathcal{P}_n} f \, d\mu = \left( c \right) \sum_{n=1}^{N} \int_{\mathcal{P}_n} f_n \, d\mu \\
\geq \left( d \right) \sum_{n=1}^{N} \left( \mathbb{I}_{\mathcal{P}_n}[g] - \frac{\epsilon}{N} \right) = \left( e \right) \left( \sum_{n=1}^{N} \mathbb{I}_{\mathcal{P}_n}[g] \right) - \epsilon. \quad (15)
$$

Here (a) is by defining equation (7), (b) is by Corollary 4.6 (c) is by the definition of $f$, and (d) is by inequality (14). Inequality (15) holds for all $\epsilon > 0$, so we conclude

$$
\mathbb{I}_\mathcal{B}[g] \geq \sum_{n=1}^{N} \mathbb{I}_{\mathcal{P}_n}[g]. \quad (16)
$$

Combining inequalities (15) and (16) proves equation (11). \hfill \square

**Lemma 4.12** Suppose the credence $\mu$ has full support. Let $\mathcal{B} \in \mathfrak{B}$ be nonempty, and let $f \in \mathcal{C}_s(\mathcal{B})$. If $f(b) > 0$ for all $b \in \mathcal{B}$, then $\mathbb{I}_\mathcal{B}[f] > 0$. If $f(b) < 0$ for all $b \in \mathcal{B}$, then $\mathbb{I}_\mathcal{B}[f] < 0$.

**Proof.** By linearity, it is sufficient to consider the case where $f(b) > 0$ for all $b \in \mathcal{B}$. For all $n \in \mathbb{N}$, let $\mathcal{C}_n := f^{-1}(0, \frac{1}{n}]$. Note that $\bigcap_{n=1}^{\infty} (\mathcal{B} \cap \mathcal{C}_n) = \mathcal{B} \cap f^{-1}(0) = \emptyset$ (because $0 < f(b)$ for all $b \in \mathcal{B}$). There must then be some $n \in \mathbb{N}$ such that $\mathcal{B} \cap \mathcal{C}_n \neq \emptyset$. Let $\mathcal{Q} := \mathcal{B} \cap \text{int}(\mathcal{C}_n)$; then $\mathcal{Q} \subseteq \mathcal{B}$ and $\mathcal{Q} \in \mathfrak{B}$ (because $f$ is $\mathfrak{B}$-comeasurable). Thus, if $\mathcal{P} := \mathcal{B} \cap (-\mathcal{Q})$, then $\emptyset \neq \mathcal{P} \subseteq \mathcal{B}$ and $\mathcal{P} \in \mathfrak{B}$.

**Claim 1:** $f(p) \geq \frac{1}{n}$ for all $p \in \mathcal{P}$.

**Proof.** (by contradiction) Let $p \in \mathcal{P}$, and suppose $f(p) < \frac{1}{n}$. Then there is some open neighbourhood $\mathcal{O} \subseteq S$ containing $p$ such that $f(o) < \frac{1}{n}$ for all $o \in \mathcal{O}$ (because $f$ is continuous). Thus, $\mathcal{O} \subseteq \text{int}(\mathcal{C}_n)$. Thus, since $p \in \mathcal{P} \subseteq \mathcal{B}$ and $p \in \mathcal{O}$, we deduce that $p \in \mathcal{B} \cap \text{int}(\mathcal{C}_n) = \mathcal{Q}$. But $p \in -\mathcal{Q}$. Contradiction. \hfill \diamondsuit \text{Claim 1}

Let $\kappa: \mathcal{B} \rightarrow \mathbb{R}$ be the constant function with value $\frac{1}{n}$. Then

$$
\mathbb{I}_\mathcal{P}[f] \geq (s) \mathbb{I}_\mathcal{P}[\kappa] = \int_{\mathcal{P}} \kappa \, d\mu \overset{(t)}{=} \frac{1}{n} \cdot \mu[\mathcal{P}], \quad (17)
$$

where (s) is by Claim 1 and the “weak monotonicity” property of Lemma 4.10 and (t) is by Lemma 4.7(c). Meanwhile, $f(q) > 0$ for all $q \in \mathcal{Q}$. Thus, a similar argument yields

$$
\mathbb{I}_\mathcal{Q}[f] \geq 0. \quad (18)
$$
Thus,
\[
\begin{align*}
\mathbb{I}_B[f] & \equiv \mathbb{I}_Q[f] + \mathbb{I}_P[f] \geq \frac{1}{n} \cdot \mu[\mathcal{P}] > 0, \\
\mathbb{I}_B[f] & \equiv \mathbb{I}_Q[f] + \mathbb{I}_P[f] \geq \frac{1}{n} \cdot \mu[\mathcal{P}] > 0.
\end{align*}
\]

Here (\(\ast\)) is by Lemma 4.11 (because \(B = \mathcal{P} \vee Q\) and \(f \in \mathcal{C}_B(\mathcal{B})\)), while (\(^\dagger\)) is by inequalities (17) and (18). Finally, (\(\diamondsuit\)) is because \(\mathcal{P} \neq \emptyset\) and \(\mu\) has full support. \(\square\)

**Example 4.13.** The hypothesis of full support is needed for Lemma 4.12. To see this, let \(\mathcal{S} = \mathbb{N}\) with the discrete topology; then every subset of \(\mathcal{S}\) is regular, and \(\vee\) is just the union operation. Let \(\mathcal{B} = \wp(\mathbb{N})\) (the power set of \(\mathbb{N}\)), and let \(\mathcal{U}\) be a free ultrafilter in \(\wp(\mathbb{N})\), as defined in Example 3.6. In this case, conditions (c) and (d) take the form: (c) No finite subset of \(\mathbb{N}\) is in \(\mathcal{U}\); and (d) For any \(\mathcal{R} \subseteq \mathbb{N}\), either \(\mathcal{R} \in \mathcal{U}\), or \((\mathbb{N}\setminus\mathcal{R}) \in \mathcal{U}\), but not both.

Define \(\delta_\mathcal{U}\) as in Example 3.6. Then \(\delta_\mathcal{U}\) is a finitely additive probability measure on the power set of \(\mathbb{N}\), and thus, a credence. For any \(N \in \mathbb{N}\), property (c) implies that \([1 \ldots N] \notin \mathcal{U}\); thus, property (d) implies that \((N \ldots \infty) \in \mathcal{U}\). Thus, \(\delta_\mathcal{U}[1 \ldots N] = 0\) and \(\delta_\mathcal{U}(N \ldots \infty) = 1\).

Now let \(f(n) = \frac{1}{n}\) for all \(n \in \mathbb{N}\). Then \(f \in \mathcal{C}_\mathcal{U}(\mathbb{N}, \mathbb{R})\), and \(f\) is \(\mathcal{B}\)-comeasurable. For any \(N \in \mathbb{N}\), if \(\mathcal{R} := [1 \ldots N]\) and \(\mathcal{Q} := (N \ldots \infty)\), then \(\mathbb{I}_\mathcal{R}[f] = 0\) (because \(\delta_\mathcal{U}[\mathcal{R}] = 0\)) while \(\mathbb{I}_\mathcal{Q}[f] \leq 1/N\) by the “weak monotonicity” property of Lemma 4.10 (because \(f(q) \leq 1/N\) for all \(q \in \mathcal{Q}\)). Thus, Lemma 4.11 yields
\[
0 \leq \mathbb{I}_N[f] = \mathbb{I}_\mathcal{R}[f] + \mathbb{I}_\mathcal{Q}[f] \leq 0 + \frac{1}{N} = \frac{1}{N}.
\]

Letting \(N \to \infty\), we obtain \(\mathbb{I}_N[f] = 0\), despite the fact that \(f(n) > 0\) for all \(n \in \mathbb{N}\). \(\diamondsuit\)

Metaphorically speaking, the credence in Example 4.13 is like a “point mass at infinity”. Later, we will make this metaphor precise in Theorem 7.2.

**Proof of Proposition 4.8.** For any \(B \in \mathcal{B}\), Lemma 4.10 implies that \(\mathbb{I}_B\) is a bounded linear functional which is weakly monotonic and has \(\|\mathbb{I}_B\|_\infty = \mu[\mathcal{B}]\). Meanwhile, equation (11), follows from Lemma 4.11. Furthermore, if \(\mu\) has full support, then Lemma 4.12 implies that this conditional expectation system is strictly monotonic.

**Uniqueness:** Let \(\{\mathbb{I}_B\}_{B \in \mathcal{B}}\) be any integrator on \(\mathcal{S}\) that is compatible with \(\mu\). We must show that \(\mathbb{I}_B = \mathbb{I}_B^0\) for all \(B \in \mathcal{B}\). By linearity, it suffices to show \(\mathbb{I}_B[g] \leq \mathbb{I}_B^0[g]\) for any \(g \in \mathcal{G}_B(\mathcal{B})\) and any \(B \in \mathcal{B}\). By equation (11), it is sufficient to show \(\int_B f \, d\mu \leq \mu[\mathcal{B}] \cdot \mathbb{I}_B^0[g]\) for any \(B \in \mathcal{B}\) and \(g \in \mathcal{G}_B(\mathcal{B})\), and any \(f \in \mathcal{F}_g(\mathcal{B})\).

So let \(f \in \mathcal{F}_g(\mathcal{B})\), and let \(\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_N)\) be a \(\mathcal{B}\)-partition to which \(f\) is subordinate and, for any \(n \in [1 \ldots N]\), let \(r_n\) be the value of \(f\) on \(\mathcal{P}_n\). Then, for any \(n \in [1 \ldots N]\) and any \(s \in \mathcal{P}_n \cap\mathcal{B}\), we have \(r_n \leq g(s)\) (because \(f \in \mathcal{F}_g(\mathcal{B})\)). Since \(\mathbb{I}_{\mathcal{P}_n \cap\mathcal{B}}\) is monotonic, unimodular and linear, we obtain
\[
r_n \cdot \mu(\mathcal{P}_n \cap\mathcal{B}) = \mathbb{I}_{\mathcal{P}_n \cap\mathcal{B}}[r_n] \leq \mathbb{I}_{\mathcal{P}_n \cap\mathcal{B}}[g].
\]

(19)
Thus,
\[ \mathbb{I}_B^0[g] = \sum_{n=1}^N \mathbb{I}_{P_n \cap B}[g] \geq \sum_{n=1}^N r_n \cdot \mu(P_n \cap B) = \int_B f \, d\mu, \]
as claimed. Here, (a) is by equation (1), (b) is by inequality (19), and (c) is by equation (4).

5 Measurable functions and change of variables

In classical probability theory, a measurable function from a space \( X \) to a space \( Y \) can be used to “push forward” a probability measure from \( X \) to \( Y \). Conversely, it can be used to “pull back” a measurable real-valued function from \( Y \) to \( X \), and thereby convert an integral computation on \( Y \) into an integral computation on \( X \), via “change of variables”. We will now develop an analogous theory for credences and their associated integrators.

Let \( X \) and \( Y \) be two topological spaces. Let \( \mathcal{A} \subseteq \mathcal{R}(X) \) and \( \mathcal{B} \subseteq \mathcal{R}(Y) \) be Boolean subalgebras of the algebras of regular sets on \( X \) and \( Y \). A function \( \phi : X \to Y \) is measurable with respect to \( \mathcal{A} \) and \( \mathcal{B} \) if \( \phi^{-1}(B) \in \mathcal{A} \) for all \( B \in \mathcal{B} \). For example, if \( \phi \) is a continuous, open function, then \( \phi \) is measurable with respect to \( \mathcal{R}(X) \) and \( \mathcal{R}(Y) \) [Fre06, Appendix 4A2B, item (f)(iii), p.453]. Unfortunately, not all continuous functions are measurable with respect to the algebras of regular sets.\(^\text{11}\) Thus, we must introduce a weaker notion.

For any function \( \phi : X \to Y \) and subset \( B \subseteq Y \), we define \( \phi^{-1}(B) := \text{int} (\phi^{-1} \{ \text{clos}(B) \}) \). We say that \( \phi \) is comeasurable with respect to \( \mathcal{A} \) and \( \mathcal{B} \) if \( \phi^{-1}(B) \in \mathcal{A} \) for all \( B \in \mathcal{B} \). In particular, if \( Y = \mathbb{R} \) and \( \mathcal{B} = \mathcal{E} \) (the algebra of elementary sets), then this is the definition of “comeasurable” given for real-valued functions in Section \( 3 \).

Lemma 5.1 Let \( X \) and \( Y \) be topological spaces, and let \( \mathcal{A} \subseteq \mathcal{R}(X) \) and \( \mathcal{B} \subseteq \mathcal{R}(Y) \) be Boolean subalgebras of the algebras of regular sets.

(a) Any \((\mathcal{A}, \mathcal{B})\)-measurable function from \( X \) to \( Y \) is \((\mathcal{A}, \mathcal{B})\)-comeasurable.

(b) Suppose \( \phi : X \to Y \) is both continuous and open. Then for any \( B \in \mathcal{R}(Y) \), we have \( \text{int} (\phi^{-1} \{ \text{clos}(B) \}) = \phi^{-1}(B) \). Thus, \( \phi \) is \((\mathcal{A}, \mathcal{B})\)-measurable if and only if \( \phi \) is \((\mathcal{A}, \mathcal{B})\)-comeasurable. Furthermore, if \( \phi \) is \((\mathcal{A}, \mathcal{B})\)-(co)measurable, then the function \( \phi^{-1} : \mathcal{B} \to \mathcal{A} \) is a Boolean algebra homomorphism.

Proof. (a) Suppose \( \phi : X \to Y \) is measurable. Let \( B \in \mathcal{B} \). Let \( C := \neg B \); then \( C \in \mathcal{B} \) also. Thus, if we define \( D := \phi^{-1}(C) \), then \( D \in \mathcal{A} \), because \( \phi \) is measurable by hypothesis. However, \( \text{clos}(B) = C^c \). Thus,
\[ \phi^{-1} \{ \text{clos}(B) \} = \phi^{-1}(C^c) = \phi^{-1}(C)^c = D^c. \]

Thus, \( \text{int} (\phi^{-1} \{ \text{clos}(B) \}) = \text{int}(D^c) = \neg D \), which is an element of \( \mathcal{A} \), as desired.

\(^\text{11}\)For example, let \( X = Y = \mathbb{R} \) with the standard topology, let \( \phi(x) = x^2 \), and let \( B := (0, \infty) \). Then \( B \in \mathcal{R}(\mathbb{R}) \), but \( \phi^{-1}(B) = (-\infty, 0) \cup (0, \infty) \notin \mathcal{R}(\mathbb{R}) \).
(b) Suppose \( \phi \) is open and continuous. Then \( \phi^{-1}[\text{int}(B)] = \text{int}[\phi^{-1}(B)] \) and \( \phi^{-1}[\text{clos}(B)] = \text{clos}[\phi^{-1}(B)] \), for any \( B \subseteq \mathcal{Y} \). Thus, if \( B \in \mathfrak{H}(\mathcal{Y}) \), then

\[
\phi^{-1}(B) = \text{int} \left( \phi^{-1}[\text{clos}(B)] \right) = \phi^{-1} \left( \text{int}[\text{clos}(B)] \right) = \phi^{-1}(B),
\]

as claimed, where the last step is because \( \text{int}[\text{clos}(B)] = B \) because \( B \) is a regular open set. Given the identity (1), the conditions for \( \phi \) to be \((\mathfrak{A}, \mathfrak{B})\)-measurable and to be \((\mathfrak{A}, \mathfrak{B})\)-(co)measurable are logically equivalent.

Now suppose that \( \phi \) is \((\mathfrak{A}, \mathfrak{B})\)-(co)measurable. To see that \( \phi^{-1} : \mathfrak{B} \rightarrow \mathfrak{A} \) is a Boolean algebra homomorphism, first recall that

\[
\phi^{-1}(B_1 \cap B_2) = \phi^{-1}(B_1) \cap \phi^{-1}(B_2), \quad \text{for any } B_1, B_2 \subseteq \mathcal{Y}.
\]

Meanwhile, if \( B \in \mathfrak{B} \), then

\[
\phi^{-1}(\neg B) = \phi^{-1} \left( \text{int} \left( B^c \right) \right) = \text{int} \left( \left[ \phi^{-1}(B) \right]^c \right) = \neg \phi^{-1}(B).
\]

Finally, for any \( B_1, B_2 \in \mathfrak{B} \), de Morgan’s law yields \( B_1 \vee B_2 = \neg[(\neg B_1) \cap (\neg B_2)] \). Thus, an application of equations (2) and (3) yields \( \phi^{-1}(B_1 \vee B_2) = \phi^{-1}(B_1) \lor \phi^{-1}(B_2) \). \( \square \)

To illustrate Lemma 5.1, let \( \phi : \mathcal{X} \rightarrow \mathcal{Y} \) be any continuous function. Then it is easy to see that \( \phi \) is \((\mathfrak{H}(\mathcal{X}), \mathfrak{H}(\mathcal{Y}))\)-(co)measurable. Thus, if \( \phi \) is also open, then Lemma 5.1(b) says that \( \phi \) is \((\mathfrak{H}(\mathcal{X}), \mathfrak{H}(\mathcal{Y}))\)-measurable, and \( \phi^{-1} : \mathfrak{H}(\mathcal{Y}) \rightarrow \mathfrak{H}(\mathcal{X}) \) is a Boolean algebra homomorphism. The next example is more involved.

**Example 5.2.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be differentiable manifolds, and let \( \mathfrak{A} \subset \mathfrak{H}(\mathcal{X}) \) and \( \mathfrak{B} \subset \mathfrak{H}(\mathcal{Y}) \) be the Boolean algebras of regular open sets with piecewise smooth boundaries, as defined in Example 3.1(b). Let \( \phi : \mathcal{X} \rightarrow \mathcal{Y} \) be a submersion —that is, a differentiable function such that the derivative \( D_x \phi \) is a linear surjection from the tangent space \( T_x \mathcal{X} \) to the tangent space \( T_{\phi(x)} \mathcal{Y} \) for all for all \( x \in \mathcal{X} \). (This implies that \( \dim(\mathcal{X}) \geq \dim(\mathcal{Y}) \).) Then the Open Mapping Theorem implies that \( \phi \) is also open. Suppose \( B \in \mathfrak{B} \); then \( \partial B = (\mathcal{H}_1 \cap \partial B) \cup \cdots \cup (\mathcal{H}_N \cap \partial B) \) for some smooth hypersurfaces \( \mathcal{H}_1, \ldots, \mathcal{H}_N \subseteq \mathcal{Y} \). For all \( n \in [1 \ldots N] \), let \( \mathcal{H}'_n := \phi^{-1}(\mathcal{H}_n) \); then \( \mathcal{H}'_n \) is a smooth hypersurface in \( \mathcal{X} \). \( \Box \)

Meanwhile, if \( \mathcal{A} := \phi^{-1}(B) \), then \( \partial \mathcal{A} = \phi^{-1}(\partial B) \), because \( \phi \) is open and continuous. Thus,

\[
\partial \mathcal{A} = \phi^{-1}(\partial B) = \phi^{-1} \left( [\mathcal{H}_1 \cap \partial B] \cup \cdots \cup (\mathcal{H}_N \cap \partial B) \right) \\
= (\phi^{-1}[\mathcal{H}_1] \cap \phi^{-1}[\partial B]) \cup \cdots \cup (\phi^{-1}[\mathcal{H}_N] \cap \phi^{-1}[\partial B]) \\
= (\mathcal{H}'_1 \cap \partial \mathcal{A}) \cup \cdots \cup (\mathcal{H}'_N \cap \partial \mathcal{A}).
\]

Thus, \( \mathcal{A} \) has a piecewise smooth boundary; in other words, \( \mathcal{A} \in \mathfrak{A} \). This shows that \( \phi \) is \((\mathfrak{A}, \mathfrak{B})\)-measurable. Thus, Lemma 5.1(b) says that \( \phi^{-1} : \mathfrak{B} \rightarrow \mathfrak{A} \) is a Boolean algebra homomorphism. \( \Diamond \)

\( \Box \)

\( \Box \)

Suppose \( \mathcal{H}_n = \psi^{-1}_n \{0\} \) for some smooth function \( \psi_n : \mathcal{X} \rightarrow \mathbb{R} \). Then \( \mathcal{H}'_n := (\psi_n \circ \phi)^{-1} \{0\} \). If \( d\psi_n \) is never zero and \( \phi \) is a submersion, then the Chain Rule implies that \( d(\psi_n \circ \phi) \) is also never zero.
Continuing the notation of Lemma 5.1(b), suppose that $\mu$ is a credence on $\mathfrak{A}$. Let $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ be an open, continuous, $(\mathfrak{A}, \mathfrak{B})$-measurable function. The image (or “push-forward”) $\phi(\mu)$ is the function $\nu : \mathfrak{B} \rightarrow [0, 1]$ defined by setting $\nu[B] := \mu[\phi^{-1}(B)]$ for all $B \in \mathfrak{B}$. Since $\phi^{-1} : \mathfrak{B} \rightarrow \mathfrak{A}$ is a Boolean algebra homomorphism by Lemma 5.1(b), it is immediate that $\nu$ is a credence on $\mathcal{Y}$. The next result is the analog of the “Change of Variables” theorem for integration with respect to credences.

**Proposition 5.3** Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, and let $\mathfrak{A} \subseteq \mathfrak{R}(\mathcal{X})$ and $\mathfrak{B} \subseteq \mathfrak{R}(\mathcal{Y})$ be Boolean subalgebras of the algebras of regular sets. Let $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ be an open, continuous, $(\mathfrak{A}, \mathfrak{B})$-measurable function, and let $g \in \mathcal{G}_\mathfrak{B}(\mathcal{Y})$. Then:

(a) $g \circ \phi \in \mathcal{G}_\mathfrak{A}(\mathcal{X})$. Furthermore, if $g \in \mathcal{C}_\mathfrak{B}(\mathcal{Y})$, then $g \circ \phi \in \mathcal{C}_\mathfrak{A}(\mathcal{X})$.

(b) Let $\mu$ be a credence on $\mathfrak{A}$, and let $\nu := \phi(\mu)$. For any $B \in \mathfrak{B}$, if $A := \phi^{-1}(B)$, then $\int_A^\mu [g \circ \phi] = \int_B^\nu [g]$.

**Proof.** (a) **Case 1.** Suppose $g \in \mathcal{C}_\mathfrak{B}(\mathcal{Y})$. Let $r \in \mathbb{R}$, and let $B := [g^{-1}[r, \infty))$. Then $B \in \mathfrak{B}$, because $g$ is $\mathfrak{B}$-comeasurable. Thus,

\[
\int [g \circ \phi]^{-1}[r, \infty) = \int [\phi^{-1} (g^{-1}[r, \infty))] = \phi^{-1}(\int [g^{-1}[r, \infty]]) = \phi^{-1}(B) \in \mathfrak{A},
\]

where $(\ast)$ is because $\phi$ is open and continuous, and the last step is because $\phi$ is $(\mathfrak{A}, \mathfrak{B})$-measurable. By a similar argument, $\int [(g \circ \phi)^{-1}(-\infty, r)] \in \mathfrak{A}$ for all $r \in \mathbb{R}$. We conclude that $g \circ \phi$ is $\mathfrak{A}$-comeasurable. But also $g \circ \phi \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$, because $g \in \mathcal{C}_b(\mathcal{Y}, \mathbb{R})$. Thus, $g \circ \phi \in \mathcal{C}_\mathfrak{A}(\mathcal{X})$, as claimed.

**Case 2.** Now suppose $g = g_1 + \cdots + g_N$ for some $g_1, \ldots, g_N \in \mathcal{C}_\mathfrak{B}(\mathcal{Y})$. Then $g \circ \phi = g_1 \circ \phi + \cdots + g_N \circ \phi$. But for all $n \in \{1, \ldots, N\}$, $g_n \circ \phi \in \mathcal{C}_\mathfrak{A}(\mathcal{X})$, by Case 1. Thus, $g \circ \phi \in \mathcal{G}_\mathfrak{A}(\mathcal{X})$, as claimed.

**Case 3.** Now suppose $g \in \mathcal{G}_\mathfrak{B}(\mathcal{Y})$. Then $g$ is a limit (in the uniform norm) of a sequence $\{g_n\}_{n=1}^\infty$ where each $g_n$ is as in Case 2. The transformation $\mathcal{C}_b(\mathcal{Y}, \mathbb{R}) \ni h \mapsto h \circ \phi \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$ is continuous in the uniform norm. Thus, $g \circ \phi = \lim_{n \to \infty} g_n \circ \phi$ (in the uniform norm). By Case 2, $g_n \circ \phi \in \mathcal{G}_\mathfrak{A}(\mathcal{X})$ for all $n \in \mathbb{N}$. Thus, $g \circ \phi \in \mathcal{G}_\mathfrak{A}(\mathcal{X})$, because $\mathcal{G}_\mathfrak{A}(\mathcal{X})$ is closed in the uniform norm.

(b) Let $f : \mathcal{Y} \rightarrow \mathbb{R}$ be a $\mathfrak{B}$-simple function, subordinate to some $\mathfrak{B}$-partition $\{B_1, \ldots, B_N\}$ of $\mathfrak{B}$. For all $n \in \{1, \ldots, N\}$, let $r_n \in \mathbb{R}$ be the value of $f$ on $B_n$, and let $A_n := \phi^{-1}(B_n)$; then $A_n \in \mathfrak{A}$ because $\phi$ is $(\mathfrak{A}, \mathfrak{B})$-measurable. Furthermore, $\{A_1, \ldots, A_N\}$ is a $\mathfrak{A}$-partition of $\mathcal{X}$ (because $\phi^{-1} : \mathfrak{B} \rightarrow \mathfrak{A}$ is a Boolean algebra homomorphism, by Lemma 5.1(b)), and $f \circ \phi$ is an $\mathfrak{A}$-simple function subordinate to this partition. Thus, we have:

\[
\int_A^\mu (f \circ \phi) \, d\mu = \sum_{n=1}^N r_n \mu[A_n] = \sum_{n=1}^N r_n \mu[\phi^{-1}(B_n)] = \sum_{n=1}^N r_n \nu[B_n] = \int_B^\nu f \, dv.
\]
Here, both (*) are by defining formula (1), while (†) is by the definition of \( \nu = f(\mu) \).

Now, let \( g \in \mathcal{G}_B(Y) \); then \( g \circ \phi \in \mathcal{G}_B(X) \), by part (a). Let \( \mathcal{F}_Y \) be the set of all \( \mathcal{B} \)-simple functions on \( Y \), and let \( \mathcal{F}_g(\mathcal{B}) := \{ f \in \mathcal{F}_Y; \; f(b) \leq g(b), \text{ for all } b \in \mathcal{B} \} \). Let \( \mathcal{F}_X \) be the set of all \( \mathcal{A} \)-simple functions on \( X \), and let \( \mathcal{F}_{g \circ \phi}(\mathcal{A}) := \{ f \in \mathcal{F}_X; \; f(a) \leq g \circ \phi(a), \text{ for all } a \in \mathcal{A} \} \). Then \( f \circ \phi \in \mathcal{F}_{g \circ \phi}(\mathcal{A}) \) for every \( f \in \mathcal{F}_g(\mathcal{B}) \). Thus,

\[
\mathbb{I}^{\mu}_A[g \circ \phi] = \sup_{f \in \mathcal{F}_{g \circ \phi}(\mathcal{A})} \int_A f \; d\mu \geq \sup_{f \in \mathcal{F}_g(\mathcal{B})} \int_A (f \circ \phi) \; d\mu \geq \mathbb{I}^{\nu}_B[g], \tag{5}
\]

Here, both (*) are by defining formula (1), (†) is because we have just observed that \( \{ f \circ \phi; \; f \in \mathcal{F}_g(\mathcal{B}) \} \subseteq \mathcal{F}_{g \circ \phi}(\mathcal{A}) \), and (🌹) is by applying equation (1) to each \( f \in \mathcal{F}_g(\mathcal{B}) \). Meanwhile

\[
-\mathbb{I}^{\mu}_A[g \circ \phi] = \mathbb{I}^{\mu}_A[-g \circ \phi] \geq \mathbb{I}^{\nu}_B[-g] = -\mathbb{I}^{\nu}_B[g], \tag{6}
\]

where (*) is obtained like inequality (5). Multiplying both sides of (6) by \(-1\), we get

\[
\mathbb{I}^{\mu}_A[g \circ \phi] \leq \mathbb{I}^{\nu}_B[g]. \tag{7}
\]

Combining inequalities (5) and (7), we get \( \mathbb{I}^{\mu}_A[g \circ \phi] = \mathbb{I}^{\nu}_B[g] \), as desired. \( \Box \)

Clearly, the composition of any two measurable functions is measurable. Likewise, part (b) of the next result shows that the composition of two comeasurable functions is comeasurable, as long as it satisfies an auxiliary condition; roughly speaking, the composite function must “preserve negations”. Meanwhile, part (a) says that the composition of a comeasurable function with a measurable function is comeasurable.

**Proposition 5.4** Let \( X, Y, Z \) be three topological spaces, and let \( \mathcal{A} \subseteq \mathcal{R}(X), \; \mathcal{B} \subseteq \mathcal{R}(Y) \) and \( \mathcal{C} \subseteq \mathcal{R}(Z) \) be Boolean subalgebras of the algebras of regular sets. Let \( \phi : X \rightarrow Y \) and \( \psi : Y \rightarrow Z \) be functions.

(a) If \( \phi \) is (\( \mathcal{A}, \mathcal{B} \))-comeasurable, and \( \psi \) is (\( \mathcal{B}, \mathcal{C} \))-measurable, then \( \psi \circ \phi \) is (\( \mathcal{A}, \mathcal{C} \))-comeasurable.

(b) Suppose \( \phi \) is continuous and (\( \mathcal{A}, \mathcal{B} \))-comeasurable, while \( \psi \) is continuous and (\( \mathcal{B}, \mathcal{C} \))-comeasurable. If \( (\psi \circ \phi)^{-}(C) \subseteq -((\psi \circ \phi)^{-}(C)) \) for all \( C \in \mathcal{C} \), then \( \psi \circ \phi \) is (\( \mathcal{A}, \mathcal{C} \))-comeasurable.

**Proof.** (a) Let \( C \in \mathcal{C} \). Let \( D := -C \); then \( D \in \mathcal{C} \) also. Thus, if we define \( \mathcal{B} := \psi^{-1}(D) \), then \( \mathcal{B} \in \mathcal{B} \), because \( \psi \) is (\( \mathcal{B}, \mathcal{C} \))-measurable, by hypothesis. But

\[
D = \operatorname{clos}(C)^C.
\]

Thus, \( \mathcal{B} = \psi^{-1}(D) = \psi^{-1}\left[\operatorname{clos}(C)^C\right] = (\psi^{-1}\left[\operatorname{clos}(C)\right])^C. \)

Thus, \( \psi^{-1}\left[\operatorname{clos}(C)\right] = \mathcal{B}^C = \operatorname{clos}(E) \), where \( E := -\mathcal{B} \) so that \( E \in \mathcal{B} \).

Thus, \( (\psi \circ \phi)^{-}\left[\operatorname{clos}(C)\right] = \phi^{-1}(\psi^{-1}\left[\operatorname{clos}(C)\right]) = \phi^{-1}\left[\operatorname{clos}(E)\right]. \)

Thus, \( \operatorname{int}\left((\psi \circ \phi)^{-}\left[\operatorname{clos}(C)\right]\right) = \operatorname{int}\left(\phi^{-1}\left[\operatorname{clos}(E)\right]\right) \),
which is an element of $\mathfrak{A}$, because $\phi$ is $(\mathfrak{A}, \mathfrak{B})$-comeasurable, by hypothesis.

(b) The proof uses the following claim.

**Claim 1:** For any $D \in \mathfrak{C}$, $\phi^- \circ \psi^{-}(D) \subseteq (\psi \circ \phi)^-(D)$.

**Proof.** Let $B := \psi^{-}(D)$. Then $B \in \mathfrak{B}$ because $\psi$ is comeasurable. But $B = \operatorname{int} (\psi^{-1} [\operatorname{clos}(D)]) \subseteq \psi^{-1} [\operatorname{clos}(D)]$, which is a closed set (because $\psi$ is continuous). Thus, $\operatorname{clos}(B) \subseteq \psi^{-1} [\operatorname{clos}(D)]$. Thus,

$$
\phi^-(B) = \operatorname{int} (\phi^{-1} [\operatorname{clos}(B)]) \subseteq \operatorname{int} (\phi^{-1} [\psi^{-1} (\operatorname{clos}[D])]) = \operatorname{int} ((\psi \circ \phi)^{-1} [\operatorname{clos}(D)]) = (\psi \circ \phi)^-(D).
$$

In other words, $\phi^- \circ \psi^{-}(D) \subseteq (\psi \circ \phi)^-(D)$. \hfill $\Box$ Claim 1

Now, let $C \in \mathfrak{C}$, and let $D := \neg C$. Then $D \in \mathfrak{C}$ also, and

$$
\phi^- \circ \psi^{-}(D) \subseteq (\psi \circ \phi)^-(D) = (\psi \circ \phi)^-(\neg C) \subseteq (\psi \circ \phi)^-(C),
$$

where $(\ast)$ is by Claim 1 and $(\dag)$ is by the hypothesis on $\psi \circ \phi$. Thus,

$$
(\psi \circ \phi)^-(\neg C) \equiv (\psi \circ \phi)^-(C) \subseteq (\psi \circ \phi)^-(\neg D) + (\psi \circ \phi)^-(D).
$$

Here, $(\ast)$ is because $(\psi \circ \phi)^-(C) \in \mathfrak{R}(\mathcal{X})$ (because $\psi \circ \phi$ is continuous), and the negation operator $\neg$ is an involution on $\mathfrak{R}(\mathcal{X})$. Meanwhile, $(\dag)$ is by negating both sides of $(\ast)$ (thereby reversing the direction of inclusion). But $\psi^{-}(D) \in \mathfrak{B}$ because $\psi$ is $(\mathfrak{B}, \mathfrak{C})$-measurable; thus, $\phi^- \circ \psi^{-}(D) \in \mathfrak{A}$ because $\phi$ is $(\mathfrak{A}, \mathfrak{B})$-comeasurable. Thus, equation $(\ast)$ implies that $(\psi \circ \phi)^-(C) \in \mathfrak{A}$, because $\mathfrak{A}$ is closed under negation.

This argument holds for all $C \in \mathfrak{C}$; thus, $\psi \circ \phi$ is $(\mathfrak{A}, \mathfrak{C})$-comeasurable.

\[\square\]

To see how the hypothesis of Proposition 5.4(b) could fail, suppose $(\psi \circ \phi)^{-1}(\partial C)$ contained an open subset $\mathcal{O}$. (Clearly, this could only happen if $\psi \circ \phi$ was not an open function.) Then $\mathcal{O} \subseteq (\psi \circ \phi)^{+}(C)$ and $\mathcal{O} \subseteq (\psi \circ \phi)^{+}(-C)$; so that $(\psi \circ \phi)^{+}(\neg C)$ and $(\psi \circ \phi)^{+}(C)$ would be non-disjoint, and hence $(\psi \circ \phi)^{+}(\neg C) \nsubseteq (\psi \circ \phi)^{+}(C)$.

If $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is merely comeasurable, but not measurable, then we cannot use $\phi$ to “push forward” a credence $\mu$ from $\mathcal{X}$ to $\mathcal{Y}$ as in Proposition 5.3. Nevertheless, Proposition 5.4(a) still allows us to “push forward” integration with respect $\mu$. To see this, suppose $\mathcal{Z} = \mathbb{R}$ and $\mathfrak{E} = \mathfrak{C}$ (the Boolean algebra of elementary functions from Example 3.1(a)). If $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ is $(\mathfrak{A}, \mathfrak{B})$-comeasurable, and $g : \mathcal{Y} \rightarrow \mathbb{R}$ is $(\mathfrak{B}, \mathfrak{E})$-measurable, then Proposition 5.4(a) says that $g \circ \phi$ is $(\mathfrak{A}, \mathfrak{E})$-comeasurable; hence $\mathbb{P}_{\mathfrak{A}}[g \circ \phi]$ is well-defined for any $\mathfrak{A} \in \mathfrak{A}$. This sort of computation plays a key role in [PV17], where $g$ is interpreted as a “utility function”, and $\mathbb{P}_{\mathfrak{A}}[g \circ \phi]$ is interpreted as “expected utility”.

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Remark 5.5. The fact that the function $\phi$ is both continuous and open is crucial in Lemma 5.1 and Proposition 5.3. But a continuous function $\phi$ need not be open to induce a Boolean algebra homomorphism. To see this, note that any clopen subset of a topological space $X$ is a regular open set. The collection of all clopen sets of $X$ is a Boolean subalgebra of $\mathcal{R}(X)$, in which the Boolean operations of $\mathcal{R}(X)$ agree with the standard set-theoretic operations of union, intersection, and complementation. Let $X$ and $Y$ be two topological spaces, and let $\mathcal{Cl}(X)$ and $\mathcal{Cl}(Y)$ be their Boolean algebras of clopen sets. If $\phi : X \rightarrow Y$ is any continuous function, then it easy to verify that $\phi$ is measurable with respect to $\mathcal{Cl}(X)$ and $\mathcal{Cl}(Y)$, and $\phi^{-1} : \mathcal{Cl}(Y) \rightarrow \mathcal{Cl}(X)$ is a Boolean algebra homomorphism.

6 Liminal structures

Sections 3, 4 and 5 considered arbitrary Boolean subalgebras of $\mathcal{R}(S)$. But this section and Section 7 only consider credences defined on the full Boolean algebra $\mathcal{R}(S)$. We are interested in whether such credences, and their associated integrators, can be represented in terms of a traditional measure (either finitely additive or countably additive) defined on some Boolean algebra or sigma-algebra of subsets of $S$. Proposition 3.7 already suggests one answer to this question.

Proposition 6.1 Suppose $S$ is a Baire space. Let $\mu$ be a credence on $\mathcal{R}(S)$, with integrator $\{I_R\}_{R \in \mathcal{R}(S)}$. Let $\nu$ be the unique residual charge associated with $\mu$ by Proposition 3.7. Then for any $B \in \mathcal{R}(S)$, and any $g \in C_b(S, \mathbb{R})$, we have

$$I_B[g] = \int_B g \, d\nu.$$

Proof. Let $\mathcal{B}(S)$ and $\mathcal{M}(S)$ be as defined prior to Proposition 3.7. Recall that $\nu$ is a charge on $\mathcal{B}(S)$ such that $\nu(\mathcal{M}) = 0$ for all $\mathcal{M} \in \mathcal{M}(S)$, and such that $\nu[R] = \mu[R]$ for every $R \in \mathcal{R}(S)$. Let $f : B \rightarrow \mathbb{R}$ be a simple function, subordinate to a regular open partition $\mathfrak{P} = \{P_1, P_2, \ldots, P_N\}$ of $B$. For all $n \in [1 \ldots N]$, suppose $f(p) = r_n$ for all $p \in P_n$. Then

$$\int_B f \, d\nu \equiv \sum_{n=1}^N r_n \nu[P_n] \equiv \sum_{n=1}^N r_n \mu[P_n] \equiv \int_B f \, d\mu. \quad (1)$$

Here, $(\ast)$ is because $\nu[\partial P_n] = 0$ for every $n \in [1 \ldots N]$ (because $\partial P_n \in \mathcal{M}(S)$). Next, $(\dagger)$ is because $\nu[P_n] = \mu[P_n]$ for all $n \in [1 \ldots N]$, by Proposition 3.7. Finally, $(\circ)$ is by defining formula (4).

Now let $g \in C_b(S, \mathbb{R})$, and let $\mathcal{F}_g(B)$ be the set of simple functions $f$ such that $f(b) \leq g(b)$ for all $b \in B$. Then

$$\int_B g \, d\nu \equiv \sup_{f \in \mathcal{F}_g(B)} \int_B f \, d\nu \equiv \sup_{f \in \mathcal{F}_g(B)} \int_B f \, d\mu \equiv I_B[g],$$

as desired. Here, $(\ast)$ is because $g$ can be uniformly approximated from below by elements of $\mathcal{F}_g(B)$, $(\dagger)$ is by equation (1), and $(\circ)$ is by defining formula (1).

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The representation in Proposition 6.1 is not entirely satisfactory, because finitely additive charges can exhibit somewhat pathological behaviour. If possible, we would like to represent a credence using a \textit{countably additive} measure—ideally, a Borel measure. If we \textit{must} use a charge, then we would like it to be well-behaved. But residual charges can be badly behaved. To see this, let $\mathcal{U}$ be an ultrafilter fixed at some point $s \in \mathcal{S}$, and let $\delta_{\mathcal{U}}$ be the credence in Example 3.6. Let $\nu$ be the residual charge associated to $\delta_{\mathcal{U}}$ by Proposition 3.7. Then for any regular open set $\mathcal{R}$ containing $s$, we must have $\nu[\mathcal{R}] = \delta_{\mathcal{U}}[\mathcal{R}] = 1$. Since any open set is equal to a regular open set modulo some meager set, this implies that $\nu[\mathcal{O}] = 1$ for any open neighbourhood of $s$. But $\nu\{s\} = 0$, because the singleton $\{s\}$ is meager. Thus, $\nu$ violates \textit{normality}—a basic “continuity” condition for measures, which requires the measure of any set to be well-approximated from above by open sets and well-approximated from below by closed sets. It would be better to represent credences using \textit{normal} charges or Borel measures. We will now construct such representations.

Let $\mathcal{B}_c(\mathcal{S})$ be the Borel sigma-algebra on $\mathcal{S}$—that is, the smallest sigma-algebra containing all open and closed subsets of $\mathcal{S}$. A \textit{Borel probability measure} is a (countably additive) probability measure on $\mathcal{B}_c(\mathcal{S})$. Since $\mathcal{A}(\mathcal{S})$ is a subset of $\mathcal{B}_c(\mathcal{S})$, it is tempting to think that every Borel probability measure on $\mathcal{S}$ defines a credence on $\mathcal{A}(\mathcal{S})$. Nonexample 3.3 already showed that this is not the case. Nevertheless, we might conversely hope that every credence $\mu$ on $\mathcal{A}(\mathcal{S})$ could be represented by a Borel measure $\nu$ in an essentially unique way, such that integration with respect to $\nu$ (in the classical sense) will be the same as integration with respect to $\mu$ (in the sense defined in Section 4). But as we shall now see, this is not the case either.

Let $\mu$ be a credence on $\mathcal{A}(\mathcal{S})$, and let $I := \{\mathcal{I}_\mathcal{R}\}_{\mathcal{R} \in \mathcal{A}(\mathcal{S})}$ be the associated integrator from Theorem 4.3. Let $\nu$ be a Borel probability measure on $\mathcal{B}(\mathcal{S})$. Let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{S}, \mathbb{R})$ be some collection of continuous functions. We will say that $\nu$ satisfies the \textit{Riesz representation property} for $\mathcal{I}_\mathcal{S}$ on $\mathcal{F}$ if

$$\mathcal{I}_\mathcal{S}[f] = \int_\mathcal{S} f \, d\nu, \quad \text{for all } f \in \mathcal{F}. \quad (2)$$

In this case, what is the relationship between $\nu$ and $\mu$? What is the relationship between $\nu$ and $I$? As the next examples show, this question does not have a simple answer.

**Example 6.2.** Let $\mathcal{S} := [-1, 1]$. Let $\mathcal{U}_0 \subseteq \mathcal{A}[-1, 1]$ be an ultrafilter fixed at 0, and define the credence $\delta_0 := \delta_{\mathcal{U}_0}$ as in Example 3.6. The associated integrator is defined as follows: for any $\mathcal{R} \in \mathcal{A}(\mathcal{S})$, and any $f \in \mathcal{C}_b(\mathcal{S}, \mathbb{R})$,

$$\mathcal{I}_\mathcal{R}[f] = \begin{cases} f(0) & \text{if } \mathcal{R} \in \mathcal{U}_0 \text{ (in particular, if } 0 \in \mathcal{R}); \\ 0 & \text{if } \mathcal{R} \notin \mathcal{U}_0 \text{ (in particular, if } 0 \in -\mathcal{R}). \end{cases}$$

Heuristically, $\delta_0$ is like a “point mass” at zero, but with an additional feature: if the point 0 lies on the boundary between a regular set $\mathcal{R}$ and its negation $-\mathcal{R}$, then exactly one of $\mathcal{R}$ or $-\mathcal{R}$ can “claim ownership” of 0; this decision is made by the ultrafilter $\mathcal{U}_0$. For example, exactly one of the following two statements is true:

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• For all \( \epsilon > 0 \), \( \delta_0((0, \epsilon]) = 1 \) while \( \delta_0((-\epsilon, 0]) = 0 \).
• For all \( \epsilon > 0 \), \( \delta_0((0, \epsilon]) = 0 \) while \( \delta_0((-\epsilon, 0]) = 1 \).

The ultrafilter \( \mathcal{U}_0 \) also decides the “ownership” of zero in more complicated cases. For example, let

\[
\mathcal{E}_+ := \bigcup_{n=1}^{\infty} \left( \frac{1}{2n+1}, \frac{1}{2n} \right) \quad \text{and} \quad \mathcal{O}_+ := \bigcup_{n=1}^{\infty} \left( \frac{1}{2n}, \frac{1}{2n-1} \right)
\]

while \( \mathcal{E}_- := \bigcup_{n=1}^{\infty} \left( -\frac{1}{2n+1}, -\frac{1}{2n} \right) \quad \text{and} \quad \mathcal{O}_- := \bigcup_{n=1}^{\infty} \left( -\frac{1}{2n-1}, -\frac{1}{2n} \right). \]

These are four disjoint regular open sets, with \( \mathcal{E}_+ \cup \mathcal{O}_+ \cup \mathcal{E}_- \cup \mathcal{O}_- = [-1, 1] \). Thus, one of the four sets \( \mathcal{E}_+ \), \( \mathcal{O}_+ \), \( \mathcal{E}_- \), and \( \mathcal{O}_- \) gets \( \delta_0 \)-measure 1 (i.e. claims “ownership” of 0), while the other three get \( \delta_0 \)-measure 0 — the ultrafilter \( \mathcal{U}_0 \) decides which one. \( \diamond \)

**Example 6.3.** We will now refine Example 6.2. Let \( \mathcal{F}_+ := \{ \mathcal{R} \in \mathcal{A}[-1, 1]; \ (0, \epsilon) \subseteq \mathcal{R} \text{ for some } \epsilon > 0 \} \). Let \( \mathcal{F}_- := \{ \mathcal{R} \in \mathcal{A}[-1, 1]; \ (-\epsilon, 0) \subseteq \mathcal{R} \text{ for some } \epsilon > 0 \} \). Then \( \mathcal{F}_+ \) and \( \mathcal{F}_- \) are both free filters. Let \( \mathcal{U}_+ \) be a free ultrafilter containing \( \mathcal{F}_+ \) and let \( \mathcal{U}_- \) be a free ultrafilter containing \( \mathcal{F}_- \). Define the credences \( \delta_\pm : \mathcal{A}[-1, 1] \to \mathbb{R} \) using \( \mathcal{U}_\pm \) as in Example 3.6. Fix some constants \( \varphi_\pm \in [0, 1] \) such that \( \varphi_- + \varphi_+ = 1 \). Finally, let \( \lambda \) be a Lebesegesque credence on \([-1, 1]\), as defined by Proposition 3.4. For any \( \mathcal{R} \in \mathcal{A}[-1, 1] \), define \( \nu[\mathcal{R}] := \frac{1}{2} (\lambda[\mathcal{R}] + \varphi_- \delta_-[\mathcal{R}] + \varphi_+ \delta_+[\mathcal{R}]) \). Then \( \nu \) is a credence on \( \mathcal{A}[-1, 1] \).

If \( 0 \in \mathcal{R} \) then \( \nu[\mathcal{R}] = \lambda[\mathcal{R}]/2 \), whereas if \( 0 \notin \mathcal{R} \), then \( \nu[\mathcal{R}] = (\lambda[\mathcal{R}] + 1)/2 \). But if 0 lies on the boundary between \( \mathcal{R} \) and \( -\mathcal{R} \), then \( \nu \)'s behaviour is determined by \( \mathcal{U}_\pm \), and \( \varphi_\pm \). In particular, for any \( \epsilon > 0 \), we have \( \nu([-\epsilon, 0]) = (\epsilon + \varphi_-)/2 \) and \( \nu([0, \epsilon]) = (\epsilon + \varphi_+)/2 \).

Let \( \lambda_* \) be the Lebesgue measure (not to be confused with the Lebesguesque credence \( \lambda \)), and let \( \delta_* \) be the Borel probability measure which assigns probability 1 to the singleton \( \{0\} \) (not to be confused with the credences \( \delta_\pm \) defined above). Let \( \nu := \frac{1}{2} (\lambda + \delta_*) \). Then \( \nu \) is a Radon measure on \([-1, 1]\), and satisfies the Riesz representation property (2) for \( \mathcal{L} \) on \( \mathcal{C}(\mathcal{S}, \mathbb{R}) \) for any choice of ultrafilters \( \mathcal{U}_\pm \) and any choice of constants \( \varphi_\pm \). In other words, there are many different credences which “look like” the same Radon measure on \( \mathcal{S} \), in the sense of equation (2). \( \diamond \)

We will now develop a theory that explains these examples. Let \( \mathfrak{A}(\mathcal{S}) \) be the Boolean algebra generated by the open subsets of \( \mathcal{S} \). Thus, \( \mathfrak{A}(\mathcal{S}) \) contains all open subsets, all closed subsets, and all finite unions and intersections of such sets. For any \( \mathcal{T} \subseteq \mathcal{S} \), let \( \mathfrak{A}(\mathcal{T}) := \{ \mathcal{A} \cap \mathcal{T}; \ \mathcal{A} \in \mathfrak{A}(\mathcal{S}) \} \); this is the Boolean algebra generated by all relatively open and relatively closed subsets of \( \mathcal{T} \). A **charge** on \( \mathfrak{A}(\mathcal{T}) \) is a function \( \nu : \mathfrak{A}(\mathcal{T}) \to [0, 1] \) which is finitely additive — i.e. \( \nu[\mathcal{A} \cup \mathcal{B}] = \nu[\mathcal{A}] + \nu[\mathcal{B}] \) for any disjoint \( \mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{T}) \). In particular, any Borel measure on \( \mathcal{T} \) restricts to a charge on \( \mathfrak{A}(\mathcal{T}) \) in the obvious way. We will say that \( \nu \) is a **probability charge** on \( \mathcal{T} \) if \( \nu[\mathcal{T}] = 1 \).

\( \mathfrak{R}(\mathcal{S}) \) is a subset of \( \mathfrak{A}(\mathcal{S}) \), but it is not a sub-algebra, because the join operator on \( \mathfrak{R}(\mathcal{S}) \) is not union. Furthermore, Examples 6.2 and 6.3 show that not every probability charge
on $\mathfrak{A}(S)$ determines a unique credence on $\mathfrak{H}(S)$. Nevertheless, if $S$ is a $T_4$ space, we will see that any credence can be represented by a probability charge on $\mathfrak{A}(S)$, enriched with an auxiliary, “liminal” structure.

Let $\nu$ be a probability charge on $\mathfrak{A}(S)$. For any $B \in \mathfrak{A}(S)$, let $\nu_B$ be the restriction of $\nu$ to a charge on $\mathfrak{A}(B)$. A **liminal charge structure subordinate to** $\nu$ is a collection $\{\rho_R\}_{R \in \mathfrak{H}(S)}$, where, for all $R \in \mathfrak{H}(S)$, $\rho_R$ is a charge on $\mathfrak{H}(\partial R)$ which is absolutely continuous with respect to $\nu$, such that for any regular partition $\{R_1, \ldots, R_N\}$ of $S$, we have

$$\rho_{R_1} + \cdots + \rho_{R_N} = \nu_{\partial R_1 \cup \cdots \cup \partial R_N}. \quad (3)$$

In particular, for any $R \in \mathfrak{H}(S)$, if $Q := -R$ (so that $\partial Q = \partial R$), then $\rho_R + \rho_Q = \nu_{\partial R}$. Heuristically, $\rho_R$ and $\rho_Q$ describe the way that $R$ and $Q$ “share” the $\nu$-mass of their common boundary. Given such a structure, we can define a function $\mu : \mathfrak{H}(S) \rightarrow [0, 1]$ by

$$\mu[R] := \nu(R) + \rho_R(\partial R), \quad \text{for all } R \in \mathfrak{H}(S). \quad (4)$$

It is easy to verify that $\mu$ is a credence.\(^\text{13}\) The first main result of this section establishes that, on any $T_4$ topological space, every credence arises in this fashion.\(^\text{14}\)

A charge $\nu$ is **normal** if, for every $B \in \mathfrak{A}(S)$, we have $\nu[B] = \sup\{\nu[C]; \ C \subseteq B$ and $C$ closed in $S\}$ and $\nu[B] = \inf\{\nu[O]; \ B \subseteq O \subseteq S$ and $O$ open in $S\}$. A liminal charge structure $\{\rho_R\}_{R \in \mathfrak{H}(S)}$ is normal if $\rho_R$ is a normal charge on $\partial R$ for all $R \in \mathfrak{H}(S)$.

**Theorem 6.4** Let $S$ be a $T_4$ space, let $\mu$ be a credence on $\mathfrak{H}(S)$, and let $I := \{I_R\}_{R \in \mathfrak{H}(S)}$ be the $\mu$-compatible integrator. There is a unique normal probability charge $\nu$ on $\mathfrak{A}(S)$ that satisfies the Riesz representation property (2) for $I_S$ on $C_b(S, \mathbb{R})$. Furthermore, there is a unique normal liminal charge structure $\{\rho_R\}_{R \in \mathfrak{H}(S)}$ which is subordinate to $\nu$, such that for any $R \in \mathfrak{H}(S)$, $\mu$ satisfies equation (4), and also

$$I_R[f] = \int_R f \, d\nu + \int_{\partial R} f \, d\rho_R, \quad \text{for all } f \in C_b(S, \mathbb{R}). \quad (5)$$

In particular, if $\nu[\partial R] = 0$, then (4) and (5) say that $\mu[R] = \nu(R)$ and $I_R[f] = \int_R f \, d\nu$.

**Proof of Theorem 6.4.** Let $I := \{I_R\}_{R \in \mathfrak{H}(S)}$ be the $\mu$-compatible integrator from Theorem 4.3. Then $I_S$ is a continuous, positive linear functional on the Banach space $C_b(S, \mathbb{R})$. Since $S$ is a normal Hausdorff space, a version of the Riesz Representation Theorem says that there is a unique normal probability charge $\nu$ on $\mathfrak{A}(S)$ that satisfies the Riesz representation property (2) for $I_S$ on $C_b(S, \mathbb{R})$ [AB06, Theorem 14.9].

Let $R \in \mathfrak{H}(S)$. Since $\nu$ is normal, for any $\epsilon > 0$, there is a closed set $K_\epsilon \subseteq R$ with

$$\nu[K_\epsilon] > \nu[R] - \epsilon. \quad (6)$$

\(^{13}\)Proof sketch. For any regular partition $\{R_1, \ldots, R_N\}$ of $S$, equations (3) and (4) together yield $\mu[R_1] + \cdots + \mu[R_N] = 1$.

\(^{14}\)Recall that a Hausdorff topological space $S$ is $T_4$ if, for any disjoint closed subsets $C_1, C_2 \subseteq S$, there exist disjoint open sets $O_1, O_2 \subseteq S$ with $C_1 \subseteq O_1$ and $C_2 \subseteq O_2$. For example, any metrizable space is $T_4$. 29
Proof. We have

\[ \alpha_\epsilon(K_\epsilon) = 1 \quad \text{and} \quad \alpha_\epsilon(R^\epsilon) = 0. \]  

(7)

Let \( \beta_\epsilon := 1 - \alpha_\epsilon \). Then for any \( f \in C_b(S, \mathbb{R}) \), we have \( f = \alpha_\epsilon f + \beta_\epsilon f \); thus

\[ \mathbb{I}_R[f] = \mathbb{I}_R[\alpha_\epsilon f + \beta_\epsilon f] = \mathbb{I}_R[\alpha_\epsilon f] + \mathbb{I}_R[\beta_\epsilon f]. \]  

(8)

Claim 1: For any \( f \in C_b(S, \mathbb{R}) \), we have \( \mathbb{I}_R[\alpha_\epsilon f] = \int_R \alpha_\epsilon f \, d\nu \).

Proof. We have

\[ \mathbb{I}_S[\alpha_\epsilon f] = (i) \mathbb{I}_R[\alpha_\epsilon f] + \mathbb{I}_R[\alpha_\epsilon f] \quad \text{and} \quad (\odot) \quad \mathbb{I}_S[\alpha_\epsilon f] = (i) \int_S \alpha_\epsilon f \, d\nu = \int_S \alpha_\epsilon f \, d\nu + \int_{R^\epsilon} \alpha_\epsilon f \, d\nu \]

and

\[ = (\odot) \quad \int_{R^\epsilon} \alpha_\epsilon f \, d\nu + \int_{R^\epsilon} 0 \, d\nu = \int_{R^\epsilon} \alpha_\epsilon f \, d\nu + 0. \]

Combining these observations yields the claim. Here (i) is by equation (1), (\odot) is by the Riesz representation property (2), and both (\dagger) and (\ddagger) are by (7). \( \diamond \) Claim 1

Claim 2: For any \( f \in C_b(S, \mathbb{R}) \), we have \( \lim_{\epsilon \to 0} \int_R \alpha_\epsilon f \, d\nu = \int_R f \, d\nu \).

Proof. \[
\left| \int_R f \, d\nu - \int_R \alpha_\epsilon f \, d\nu \right| = \left| \int_R (1 - \alpha_\epsilon) f \, d\nu \right| = \left| \int_R \beta_\epsilon f \, d\nu \right|
\]

\[ \leq \|f\|_{\infty} \int_R |\beta_\epsilon| \, d\nu \leq \|f\|_{\infty} \cdot \nu[R \setminus \text{supp}(\beta_\epsilon)] \]

\[ \leq (\ast) \|f\|_{\infty} \cdot \nu[R \setminus K_\epsilon] \leq (\odot) \|f\|_{\infty} \cdot \epsilon \xrightarrow{\epsilon \to 0} 0, \]

as desired. Here, (i) is by the defining properties (4) of the function \( \alpha_\epsilon \) (since \( \beta_\epsilon = 1 - \alpha_\epsilon \)), while (\odot) is by inequality (6). \( \diamond \) Claim 2

Taking the limit as \( \epsilon \to 0 \) in equation (5), and combining Claims 1 and 2 we obtain:

\[ \mathbb{I}_R[f] = \int_R f \, d\nu + \lim_{\epsilon \to 0} \mathbb{I}_R[\beta_\epsilon f], \quad \text{for all} \ f \in C_b(S, \mathbb{R}). \]  

(9)

Claim 3: There is a bounded positive linear functional \( \Phi_R : C_b(\partial R^\epsilon, \mathbb{R}) \to \mathbb{R} \) with \( \|\Phi_R\|_{\infty} \leq 1 \), such that

\[ \lim_{\epsilon \to 0} \mathbb{I}_R[\beta_\epsilon f] = \Phi_R(f|_{\partial R}), \quad \text{for all} \ f \in C_b(S, \mathbb{R}). \]  

(10)
Proof. The proof involves two subclaims.

**Claim 3A:** \( \lim_{\epsilon \to 0} \mathbb{I}_R[\beta, f] \) does not depend upon the particular system of sets \( \{\mathcal{K}_\epsilon\}_{\epsilon > 0} \) and functions \( \{\alpha_\epsilon\}_{\epsilon > 0} \) that we use in the above construction, as long as they satisfy the defining conditions (9) and (7).

Proof. Observe that equation (9) can be rewritten:

\[
\lim_{\epsilon \to 0} \mathbb{I}_R[\beta, f] = \mathbb{I}_R[f] - \int_R f \, d\nu,
\]

and the right hand side clearly does not depend upon \( \{\mathcal{K}_\epsilon\}_{\epsilon > 0} \) and \( \{\alpha_\epsilon\}_{\epsilon > 0} \). \( \Box \) Claim 3A

In the following argument, Claim 3A means that we can assume without loss of generality that \( \{\mathcal{K}_\epsilon\}_{\epsilon > 0} \) and \( \{\alpha_\epsilon\}_{\epsilon > 0} \) have whatever additional properties we require.

**Claim 3B:** Let \( f, g \in \mathcal{C}(\mathcal{S}, \mathbb{R}) \). If \( f|_{\partial \mathcal{R}} = g|_{\partial \mathcal{R}} \), then \( \lim_{\epsilon \to 0} \mathbb{I}_R[\beta, f] = \lim_{\epsilon \to 0} \mathbb{I}_R[\beta, g] \).

Proof. Fix \( \epsilon > 0 \). For all \( r \in \partial \mathcal{R} \), there exists an open neighbourhood \( \mathcal{B}_r^\prime \subseteq \mathcal{S} \) around \( r \) such that \( |f(b) - f(r)| < \frac{\epsilon}{2} \) for all \( b \in \mathcal{B}_r^\prime \) (because \( f \) is continuous). Likewise, there exists an open neighbourhood \( \mathcal{B}_r'' \subseteq \mathcal{S} \) around \( r \) such that \( |g(b) - g(r)| < \frac{\epsilon}{2} \) for all \( b \in \mathcal{B}_r'' \) (because \( g \) is continuous). Let \( \mathcal{B}_r^\epsilon := \mathcal{B}_r^\prime \cap \mathcal{B}_r'' \). Then for all \( b \in \mathcal{B}_r^\epsilon \),

\[
|f(b) - g(b)| = |f(b) - f(r) + f(r) - g(b)| \\
\leq |f(b) - f(r)| + |g(r) - g(b)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Now let \( \mathcal{B}_\epsilon := \bigcup_{r \in \partial \mathcal{R}} \mathcal{B}_r^\epsilon \). Then \( \mathcal{B}_\epsilon \) is an open neighbourhood of \( \partial \mathcal{R} \), and by combining the inequalities (11) for all \( r \in \partial \mathcal{R} \), we obtain

\[
|f(b) - g(b)| < \epsilon, \quad \text{for all } b \in \mathcal{B}_\epsilon.
\] (12)

Let \( \overline{\mathcal{R}} \) denote the closure of \( \mathcal{R} \) in \( \mathcal{S} \). Then \( \overline{\mathcal{R}} \setminus \mathcal{B}_\epsilon \) is a closed subset of \( \mathcal{S} \). But \( \mathcal{R} = \mathcal{R} \cup \partial \mathcal{R} \), and \( \partial \mathcal{R} \subseteq \mathcal{B}_\epsilon \) by construction; thus, \( (\overline{\mathcal{R}} \setminus \mathcal{B}_\epsilon) \subseteq \mathcal{R} \). By replacing \( \mathcal{K}_\epsilon \) with \( \mathcal{K}_\epsilon \cup \mathcal{B} \setminus \mathcal{B}_\epsilon \) if necessary, we can assume without loss of generality that \( (\mathcal{R} \setminus \mathcal{K}_\epsilon) \subseteq \mathcal{B}_\epsilon \), for each \( \epsilon > 0 \). Thus,

\[
\left| \mathbb{I}_R[\beta, f] - \mathbb{I}_R[\beta, g] \right| = \left| \mathbb{I}_R[\beta, f - \beta, g] \right| = \left| \mathbb{I}_R[\beta, f - g] \right| \\
\leq \| \beta, f - g \|_\infty \leq \sup_{s \in \text{supp}(\beta)} |f(s) - g(s)| \\
\leq \sup_{b \in \mathcal{B}_\epsilon} |f(b) - g(b)| \leq \epsilon.
\] (13)

Here, \( (\diamond) \) is because \( \| \beta \|_\infty = 1 \), \( (\ast) \) is because \( \text{supp}(\beta) \subseteq \mathcal{R} \setminus \mathcal{K}_\epsilon \subseteq \mathcal{B}_\epsilon \) (by the defining conditions (7)), and \( (\dagger) \) is by inequality (12).

Letting \( \epsilon \to 0 \) in inequality (13), we obtain \( \lim_{\epsilon \to 0} \mathbb{I}_R[\beta, f] = \lim_{\epsilon \to 0} \mathbb{I}_R[\beta, g] \). \( \Box \) Claim 3B

Let \( \mathcal{C}_1(\partial \mathcal{R}, \mathbb{R}) := \{ f|_{\partial \mathcal{R}} : f \in \mathcal{C}_0(\mathcal{S}, \mathbb{R}) \} \). Claim 3B implies that we can define a function \( \Phi_\mathcal{R} : \mathcal{C}_1(\partial \mathcal{R}, \mathbb{R}) \to \mathbb{R} \) by equation (10). The linearity of \( \Phi_\mathcal{R} \) follows automatically from
the linearity of $\mathbb{I}_R$ and the fact that $(r f + g)|_{\partial R} = r f|_{\partial R} + g|_{\partial R}$ for any $f, g \in C_b(R, \mathbb{R})$ and $r \in \mathbb{R}$. Likewise, $\Phi_R$ is positive because $\mathbb{I}_R$ is weakly monotone.

Next, recall that $\partial R$ is a closed subset of $S$, and $S$ is $T_4$; thus, the Tietze Extension Theorem implies that $C_1(\partial R, \mathbb{R}) = C_b(\partial R, \mathbb{R})$. Thus, the function $\Phi_R$ is well-defined on all of $C_0(\partial R, \mathbb{R})$.

Finally, to show that $\|\Phi_R\|_\infty \leq 1$, let $f \in C_b(\partial R, \mathbb{R})$. Suppose $\|f\|_\infty = M$, so we can think of $f$ as a function $f : \partial R \rightarrow [0, M]$. The Tietze Extension Theorem yields a continuous function $F : S \rightarrow [-M, M]$ such that $F|_{\partial R} = f$. Thus, for all $\epsilon > 0$, we have $\|\beta_\epsilon F\|_\infty \leq M$, and thus, $\|\mathbb{I}_R[\beta_\epsilon F]\| \leq M \cdot \mu(R) \leq M$ (because $I$ is compatible with $\mu$).

Thus,

$$|\Phi_R(f)| = \lim_{\epsilon \to 0} \|\mathbb{I}_R[\beta_\epsilon F]\| \leq M = \|f\|_\infty.$$

This holds for all $f \in C_b(\partial R, \mathbb{R})$, so $\|\Phi_R\|_\infty \leq 1$. \hfill $\Diamond$ \textbf{Claim 3}

Now, $\partial R$ is a closed subset of the $T_4$ space $S$; thus, $\partial R$ is also $T_4$ [Wil04, Theorem 15.4(a)]. Thus, a version of the Riesz Representation Theorem [AB06, Theorem 14.9] yields a unique normal charge $\rho_R$ on $\mathcal{A}(\partial R)$ such that

$$\Phi_R(f) = \int_{\partial R} f \, d\rho_R, \quad \text{for all } f \in C_b(\partial R, \mathbb{R}).$$

Combining this with equations (9) and (10), we obtain equation (5).

Now, let $Q := -\partial R$. Note that $\partial Q = \partial R$. By repeating the above argument for $Q$, we obtain another normal charge $\rho_Q$ on $\mathcal{A}(\partial Q) = \mathcal{A}(\partial R)$, such that

$$\mathbb{I}_Q[f] = \int_Q f \, d\nu + \int_{\partial R} f \, d\rho_Q, \quad \text{for all } f \in C_b(S, \mathbb{R}).$$

Thus, for any $f \in C_b(S, \mathbb{R})$, we have

$$\int_S f \, d\nu \overset{(\ast)}{=} \int_R f \, d\nu + \int_{\partial R} f \, d\nu + \int_Q f \, d\nu, \quad \text{but also},$$

$$\int_S f \, d\nu \overset{(\ast)}{=} \mathbb{I}_S[f] \overset{(\dagger)}{=} \mathbb{I}_R[f] + \mathbb{I}_Q[f]$$

$$= \int_R f \, d\nu + \int_{\partial R} f \, d\rho_R + \int_Q f \, d\nu + \int_{\partial R} f \, d\rho_Q,$$

where $(\ast)$ is by the equation (9), $(\dagger)$ is by equation (11), and $(\ast)$ is by the equations (5) and (14).

Subtracting (15) from (16) and rearranging, we obtain:

$$\int_{\partial R} f \, d\nu_{\partial R} = \int_{\partial R} f \, d\rho_R + \int_{\partial R} f \, d\rho_Q = \int_{\partial R} f \, d(\rho_R + \rho_Q), \quad \text{for all } f \in C_b(S, \mathbb{R}).$$
However, as we earlier noted, the Tietze Extension Theorem implies that $C_b(\partial \mathcal{R}, \mathbb{R}) := \{ f|_{\partial \mathcal{R}}; \ f \in C_b(\mathcal{S}, \mathbb{R}) \}$. Thus, we obtain

$$\int_{\partial \mathcal{R}} f \ d\nu_{\partial \mathcal{R}} = \int_{\partial \mathcal{R}} f \ d(\rho_\mathcal{R} + \rho_\mathcal{Q}), \ \text{for all } f \in C_b(\partial \mathcal{R}, \mathbb{R}).$$

(17)

Now, $\nu_{\partial \mathcal{R}}$ is normal because it is a restriction of the normal charge $\nu$ to $\mathfrak{A}(\partial \mathcal{R})$, while $(\rho_\mathcal{R} + \rho_\mathcal{Q})$ is normal because it is a sum of two normal charges on $\mathfrak{A}(\partial \mathcal{R})$. Thus, statement (17) and the uniqueness part of the Riesz Representation Theorem yield

$$\rho_\mathcal{R} + \rho_\mathcal{Q} = \nu_{\partial \mathcal{R}}.$$ 

(18)

**Claim 4:** $\rho_\mathcal{R}$ and $\rho_\mathcal{Q}$ are absolutely continuous relative to $\nu_{\partial \mathcal{R}}$.

**Proof.** Let $\mathcal{U} \subseteq \partial \mathcal{R}$, and suppose that $\nu_{\partial \mathcal{R}}[\mathcal{U}] = 0$. Then equation (18) implies that $\rho_\mathcal{R}[\mathcal{U}] + \rho_\mathcal{Q}[\mathcal{U}] = 0$. Since these are both positive measures, this means that $\rho_\mathcal{R}[\mathcal{U}] = \rho_\mathcal{Q}[\mathcal{U}] = 0$. This conclusion holds whenever $\nu_{\partial \mathcal{R}}[\mathcal{U}] = 0$. Thus, $\rho_\mathcal{R}$ and $\rho_\mathcal{Q}$ are absolutely continuous relative to $\nu_{\partial \mathcal{R}}$. \hfill \Box

Equation (18) is obviously a special case of the equation (3) for the two-element partition $\{\mathcal{R}, \mathcal{Q}\}$. To prove equation (3) in general, let $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ be any regular open partition of $\mathcal{S}$, and generalize equations (15), (16) and (17) in the obvious way.

It remains to establish formula (4). Let $\mathcal{R} \in \mathfrak{A}(\mathcal{S})$. Let $1$ be the constant 1-valued function; then $1 \in C_b(\mathcal{S}, \mathbb{R})$, and we have

$$\mu[\mathcal{R}] = \mathbb{1}_{\mathcal{R}}[1] = \int_{\mathcal{S}} 1 \ d\nu + \int_{\partial \mathcal{R}} 1 \ d\rho_\mathcal{R} = \nu[\mathcal{R}] + \rho_\mathcal{R}[\partial \mathcal{R}],$$

as desired. Here $(*)$ is by equation (4). \hfill \Box

If $\mathcal{S}$ is locally compact, then there is a nicer representation of credences. Let $C_b(\mathcal{S}, \mathbb{R})$ be the Banach space of all functions in $C_b(\mathcal{S}, \mathbb{R})$ which “vanish at infinity”, meaning that for any $\epsilon > 0$, there is some compact subset $\mathcal{K} \subseteq \mathcal{S}$ such that $|f(s)| \leq \epsilon$ for all $s \in \mathcal{S}\setminus\mathcal{K}$. Likewise, let $\mathfrak{N}_b(\mathcal{S})$ be the set of all regular open subsets of $\mathcal{S}$ with compact closures.

Let $\nu$ be a Borel measure on $\mathcal{S}$. For any $\mathcal{B} \in \mathfrak{B}_b(\mathcal{S})$, let $L^1(\mathcal{B}, \nu)$ denote the Banach space of real-valued, $\nu$-integrable functions on $\mathcal{B}$, modulo equality $\nu$-almost everywhere. Let $L^1(\mathcal{B}, \nu; [0, 1])$ be the set of $[0, 1]$-valued functions in $L^1(\mathcal{B}, \nu)$. A **liminal density structure subordinate to $\nu$** is a collection $\{\phi_{\mathcal{R}}\}_{\mathcal{R} \in \mathfrak{N}(\mathcal{S})}$, where, for all $\mathcal{R} \in \mathfrak{A}(\mathcal{R})$, $\phi_{\mathcal{R}} \in L^1(\partial \mathcal{R}, \nu; [0, 1])$ is a function such that, for any regular partition $\{\mathcal{R}_1, \ldots, \mathcal{R}_N\}$ of $\mathcal{S}$, we have

$$\phi_{\mathcal{R}_1} + \cdots + \phi_{\mathcal{R}_N} = 1, \ \text{\nu-almost everywhere on } \partial \mathcal{R}_1 \cup \cdots \cup \partial \mathcal{R}_N.$$ 

(19)

In particular, for any $\mathcal{R} \in \mathfrak{A}(\mathcal{S})$, if $\mathcal{Q} := \neg \mathcal{R}$ (so that $\partial \mathcal{Q} = \partial \mathcal{R}$), then (19) implies that $\phi_{\mathcal{Q}} = 1 - \phi_{\mathcal{R}}$, $\nu$-a.e. on $\partial \mathcal{R}$. Heuristically, $\phi_{\mathcal{R}}$ and $\phi_{\mathcal{Q}}$ describe the way in which $\mathcal{R}$ and

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15If $\nu[\mathcal{B}] = 0$, then $L^1(\mathcal{B}, \nu)$ is trivial.
Proof. Let prove a slightly weaker result, for locally compact 

\text{compact Hausdorff space, then } every \text{ credence arises in this fashion. But first we will prove a slightly weaker result, for locally compact } T_4 \text{ spaces. Recall that a Borel measure } \nu \text{ is } \text{Radon} \text{ if, for all } B \in \mathcal{B}(\mathcal{S}), \text{ we have } \nu[B] = \sup\{\nu[K] \mid K \subseteq B \text{ and } K \text{ compact in } \mathcal{S}\}, \text{ and } \nu[B] = \inf\{\nu[O] \mid B \subseteq O \subseteq \mathcal{S} \text{ and } O \text{ open in } \mathcal{S}\}.

\textbf{Theorem 6.5} Let } \mathcal{S} \text{ be a locally compact } T_4 \text{ space, let } \mu \text{ be a credence on } \mathcal{A}(\mathcal{S}), \text{ and let } I := \{I_R\}_{R \in \mathcal{A}(\mathcal{S})} \text{ be the } \mu\text{-compatible integrator. There is a unique Radon measure } \nu \text{ on } \mathcal{S} \text{ that satisfies the Riesz representation property (2) for } \mathbb{I}_S \text{ on } C_0(\mathcal{S}, \mathbb{R}). \text{ Furthermore, there is a unique liminal density structure } \{\phi_R\}_{R \in \mathcal{A}(\mathcal{S})} \text{ such that } \mu \text{ satisfies equation (20) for all } R \in \mathcal{A}_0(\mathcal{S}), \text{ while }

I_R[f] = \int f \, d\nu + \int_{\partial R} f \, dp_{\partial R}, \quad (21)

\text{for all } f \in C_0(\mathcal{S}, \mathbb{R}) \text{ and } R \in \mathcal{A}(\mathcal{S}). \text{ Also, (21) holds for all } f \in C(\mathcal{S}, \mathbb{R}) \text{ and } R \in \mathcal{A}_0(\mathcal{S}).

\textbf{Proof.} \text{ Let } I := \{I_R\}_{R \in \mathcal{A}(\mathcal{S})} \text{ be the } \mu\text{-compatible integrator. Then } \mathbb{I}_S \text{ is a continuous, positive linear functional on the Banach space } C_0(\mathcal{S}, \mathbb{R}), \text{ with } \|\mathbb{I}_S\|_\infty = 1. \text{ Since } \mathcal{S} \text{ is a locally compact Hausdorff space, the Riesz Representation Theorem says that there is a unique Radon probability measure } \nu \text{ on } \mathcal{B}(\mathcal{S}) \text{ that satisfies the Riesz representation property (2) for } \mathbb{I}_S \text{ on } C_0(\mathcal{S}, \mathbb{R}) \text{ [Fol84, Theorem 7.17].}

At this point, the proof is very similar to the proof of Theorem 6.5, except with the words "(normal) charge" everywhere replaced by "(Radon) measure", and with } C_b(\mathcal{S}, \mathbb{R}) \text{ replaced by } C_0(\mathcal{S}, \mathbb{R}). \text{ However, there is a subtle change in the argument immediately after the proof of Claim 3. Now, for any } R \in \mathcal{A}(\mathcal{S}), \text{ } \partial R \text{ is a closed subset of the locally compact Hausdorff space } \mathcal{S}; \text{ thus, } \partial R \text{ is also a locally compact Hausdorff space [Wil04, Theorem 18.4]. Thus, the Riesz Representation Theorem yields a unique Radon measure } \rho_R \text{ on } \mathcal{B}(\partial R) \text{ such that }

\Phi_R(f) = \int_{\partial R} f \, d\rho_R, \quad \text{for all } f \in C_0(\partial R, \mathbb{R}).

Combining this with the relevant versions of equations (9) and (10), we obtain equation (5) for all } f \in C_0(\mathcal{S}, \mathbb{R}).

The conclusion of Claim 4 is still true. But now, since we are now dealing with (sigma-additive) measures, we can apply the Radon-Nikodym Theorem to obtain non-negative functions } \phi_R \text{ and } \phi_Q \in L^1(\partial R, \nu) \text{ such that } dp_R = \phi_R \, d\nu_{\partial R} \text{ and } dp_Q = \phi_Q \, d\nu_{\partial R} \text{ (where } \nu_{\partial R} \text{ is the restriction of } \nu \text{ to } \mathcal{B}(\partial R)). \text{ Substituting this into equation (5) yields (21) for all } f \in C_0(\mathcal{S}, \mathbb{R}). \text{ Finally, equation (18) implies that } \phi_R(s) + \phi_Q(s) = 1 \text{ for } \nu\text{-almost all } s \in \partial R, \text{ as claimed. Since } \phi_R \text{ and } \phi_Q \text{ are non-negative, this implies that } 0 \leq
\( \phi_R(s), \phi_Q(s) \leq 1 \) for \((\nu)\)-almost all \(s \in \partial R\). In other words, \(\phi_R, \phi_Q \in \mathbb{L}^1(\partial R, \nu; [0, 1])\), as claimed.

To prove equation (19) in general, let \(\{R_1, \ldots, R_N\}\) be any regular open partition of \(S\), and generalize equations (15, 16) and (17) in the obvious way.

Next we establish formula (21) for any \(R \in \mathcal{R}_0(S)\) and \(f \in \mathcal{C}(S, \mathbb{R})\). If \(R \in \mathcal{R}_0(S)\), then its closure \(\overline{R}\) is compact. For all \(r \in \overline{R}\), let \(\mathcal{O}_r\) be an open neighbourhood of \(r\) whose closure \(\overline{\mathcal{O}_r}\) is compact (this exists because \(S\) is locally compact and Hausdorff). The collection \(\{\mathcal{O}_r; r \in \overline{R}\}\) is an open cover of the compact set \(\overline{R}\), so it admits a finite subcover, say \(\{\mathcal{O}_{r_1}, \ldots, \mathcal{O}_{r_N}\}\). Let \(\mathcal{U} := \mathcal{O}_{r_1} \cup \cdots \cup \mathcal{O}_{r_N}\); this is an open set containing \(\overline{R}\), and \(\overline{\mathcal{U}} = \overline{\mathcal{O}_{r_1}} \cup \cdots \cup \overline{\mathcal{O}_{r_N}}\) is a finite union of compact sets, hence compact. Now, \(\overline{R}\) and \(\mathcal{U}\) are disjoint closed subsets of the \(T_4\) space \(S\); thus, Urysohn’s Lemma yields a function \(h \in \mathcal{C}_0(S, \mathbb{R})\) such that \(h(r) = 1\) for all \(r \in \overline{R}\), while \(h(s) = 0\) for all \(s \in \mathcal{U}\).

Let \(f \in \mathcal{C}(S, \mathbb{R})\). Then \(\text{supp}(hf) \subseteq \overline{\mathcal{U}}\), a compact set; thus, \(hf \in \mathcal{C}_0(S, \mathbb{R})\). Thus,

\[
\mathbb{I}_R[f] = \mathbb{I}_R[hf] = \int_{\partial R} h f \, d\nu + \int_{\partial R} h f \cdot \phi_R \, d\nu_{\partial R},
\]

which yields the desired equation (21) for \(f\). Here, both \((\ast)\) are because \(h(r) = 1\) for all \(r \in \overline{R}\), by construction, while \((\dagger)\) is by equation (21), which we can already apply to \(hf\) because it is an element of \(\mathcal{C}_0(S, \mathbb{R})\).

Finally, setting \(f = 1\) in equation (21), we obtain:

\[
\mu[R] = \mathbb{I}_R[1] = \int_{\mathcal{R}} 1 \, d\nu + \int_{\partial R} 1 \cdot \phi_R \, d\nu_{\partial R} = \nu[R] + \int_{\partial R} \phi_R \, d\nu_{\partial R},
\]

which yields (20) for any \(R \in \mathcal{R}_0(S)\).

Note that \(\nu\) is not necessarily a probability measure in Theorem 6.3. The set \(S\) itself is not necessarily an element of \(\mathcal{R}_0(S)\) (unless \(S\) is compact), so we cannot apply formula (20) to obtain \(\nu[S] = \mu[S] = 1\). The next example illustrates this, and also shows why we restrict formulae (20) and (21) to \(\mathcal{R}_0(S)\) and \(\mathcal{C}_0(S, \mathbb{R})\). It also demonstrates that the representation given by Theorem 6.3 is not always very informative.

**Example 6.6.** Let \(S = (0, 1)\); this space is locally compact and \(T_4\). Let \(\mathfrak{F} := \{R \in \mathcal{R}(0, 1); (0, \epsilon) \subseteq R\) for some \(\epsilon > 0\}\). Then \(\mathfrak{F}\) is a free filter. Use the Ultrafilter Theorem to extend \(\mathfrak{F}\) to a free ultrafilter \(\mathcal{U} \subseteq \mathcal{R}(0, 1)\), and then define the credence \(\delta_\mathcal{U}\) as in Example 3.6.

It is easy to see that \(\delta_\mathcal{U}\) can be represented by a (finitely additive) charge on \(\mathcal{R}(0, 1)\), as described in Theorem 6.3. However, there is clearly no (countably additive) Borel measure on \((0, 1)\) which satisfies (20) for all \(R \in \mathcal{R}(0, 1)\). Indeed, any set in \(\mathcal{R}_0(0, 1)\) must be bounded away from zero. Likewise, any function \(f \in \mathcal{C}_0((0, 1]; \mathbb{R})\) must have the property that \(\lim_{s \to 0} f(s) = 0\). Thus, there is only one Borel measure which satisfies (20) for all \(R \in \mathcal{R}_0(0, 1)\) and satisfies (21) for all \(f \in \mathcal{C}_0((0, 1]; \mathbb{R})\) —namely, the zero measure.
Example 4.13 has a credence similar to Example 6.6. But if $\mathcal{S}$ is a compact space, then pathological examples like these cannot exist.

**Corollary 6.7** Let $\mathcal{S}$ be a compact Hausdorff space, let $\mu$ be a credence on $\mathcal{R}(\mathcal{S})$, and let $I := \{I_R\}_{R \in \mathcal{R}(\mathcal{S})}$ be the $\mu$-compatible integrator. There is a unique normal Borel probability measure $\nu$ on $\mathcal{S}$ that satisfies the Riesz representation property (2) for $I_S$ on $C(\mathcal{S}, \mathbb{R})$. Furthermore, there is a unique liminal density structure $\{\phi_R\}_{R \in \mathcal{R}(\mathcal{S})}$ which is subordinate to $\nu$ and which satisfies equations (20) and (21) for all $R \in \mathcal{R}(\mathcal{S})$ and all $f \in C(\mathcal{S}, \mathbb{R})$.

**Proof.** If $\mathcal{S}$ is compact, then $\mathcal{R}_0(\mathcal{S}) = \mathcal{R}(\mathcal{S})$ and $C_0(\mathcal{S}, \mathbb{R}) = C(\mathcal{S}, \mathbb{R})$, while a Borel measure $\nu$ is Radon if and only if it is normal. Now apply Theorem 6.5. □

**Example 6.8.** (a) Let $\mathcal{U} \subset \mathcal{R}(\mathcal{S})$ be an ultrafilter fixed at some point $s$ in $\mathcal{S}$, and define $\delta_{\mathcal{U}}$ as in Example 3.6. To satisfy equations (20) and (21), let $\nu$ be the Borel probability measure which assigns probability 1 to $\{s\}$ (i.e. the “point mass” at $s$). For any $R \in \mathcal{R}(\mathcal{S})$ with $s \in \partial R$, define $\phi_R := 1$ if $R \in \mathcal{U}$, and $\phi_R := 0$ if $R \notin \mathcal{U}$. (If $s \notin \partial R$, then $\nu(\partial R) = 0$, so the values of $\phi_R$ and $\phi_{\neg R}$ are irrelevant.)

(b) Let $\mathcal{S} \in \mathcal{R}(\mathbb{R}^N)$, and let $\mu$ be a Lebesguesque credence on $\mathcal{B}_{\text{for}}(\mathcal{S})$, as in Proposition 3.1. To satisfy equations (20) and (21), let $\nu$ be the (normalized) Lebesgue measure on $\mathcal{S}$. If $R \in \mathcal{B}_{\text{for}}(\mathcal{S})$, then $\nu(\partial R) = 0$, so the values of $\phi_R$ and $\phi_{\neg R}$ are irrelevant. But if $R \notin \mathcal{B}_{\text{for}}(\mathcal{S})$, then $\nu(\partial R) > 0$. In this case, the functions $\phi_R$ and $\phi_{\neg R}$ describe how the nonzero Lebesgue measure of $\partial R$ is “shared” between $R$ and $\neg R$. Thus, $\mu[R] \geq \nu[R]$ and $\mu[\neg R] \geq \nu[\neg R]$, with at least one of these inequalities being strict. □

Can we eliminate the “liminal” terms in equations (20) and (21)? Perhaps if the credence $\mu$ was particularly nice, or if the Borel probability measure $\nu$ was “smooth enough”, then these terms would vanish. In this case, equation (21), for example, would reduce to the more familiar expression:

$$I_R[f] = \int_R f \, d\nu.$$

Meanwhile, equation (20) would say that $\mu[R] = \nu[R]$ for all $R \in \mathcal{R}(\mathcal{S})$—in other words, $\nu$ would define a credence when restricted to $\mathcal{R}(\mathcal{S})$. Nonexample 3.3 already shows that this is not the case for the Lebesgue measure on $[0, 1]$. But the next result goes much further: it is never the case, for any Borel measure on a broad class of topological spaces.

Say that a topological space $\mathcal{S}$ is **projectible** if there is an open, continuous function from $\mathcal{S}$ to $[0, 1]$. For example, any open subset of a topological vector space is projectible. Also, any fibre bundle over any open subset of $[0, 1]$ (with any fibre space) is projectible. We will say that $\mathcal{S}$ is **locally projectible** if every point in $\mathcal{S}$ has a regular open neighbourhood which is projectible. For example, any topological manifold is locally projectible.

**Proposition 6.9** If $\mathcal{S}$ is locally projectible, then there is no Borel probability measure on $\mathcal{S}$ which defines a credence when restricted to $\mathcal{R}(\mathcal{S})$. Thus, if $\mu$ is any credence on $\mathcal{R}(\mathcal{S})$ admitting a liminal density representation (20), then the liminal density structure $\{\phi_R\}_{R \in \mathcal{R}(\mathcal{S})}$ is nontrivial.
In particular, if $S$ is any bounded open subset of $\mathbb{R}^N$, then Proposition 6.9 says that the Lebesgue measure on $S$ cannot define a credence on $\mathcal{R}(S)$. Another consequence of this result is that the Borel measure which appears in Theorem 6.5 and Corollary 6.7 must be different than the residual charge which appears in Propositions 3.7 and 6.1. If $S$ satisfies the hypothesis of Proposition 6.9, then no Borel probability measure on $S$ can be a residual charge, even if they represent the same credence.

Proof. (Case 1) Suppose $S$ is projectible. Let $\phi : S \to [0,1]$ be an open, continuous function. Let $\nu'$ be a Borel probability measure on $S$, and let $\mu'$ be the restriction of $\nu'$ to $\mathcal{R}(S)$. Let $\nu := \phi(\nu')$; this is a Borel probability measure on $[0,1]$. Let $\mu := \phi(\mu')$; this is the restriction of $\nu$ to $\mathcal{R}[0,1]$. Now, $\phi^{-1} : \mathcal{R}[0,1] \to \mathcal{R}(S)$ is a Boolean algebra homomorphism [Pre06, 4A2B (f)(iii)] (or Lemma 5.1(b)). Thus, if $\mu'$ is a credence on $\mathcal{R}(S)$, then $\mu$ is a credence on $\mathcal{R}[0,1]$. So to prove the theorem, it suffices to show:

*There is no Borel measure $\nu$ on $[0,1]$ such that $\mu$ is a credence on $\mathcal{R}[0,1]$.*

Our proof strategy is somewhat similar to the strategy sketched for Nonexample 3.3, but more general, since it must work for any Borel measure on $[0,1]$.

**Claim 1:** If $\nu$ has an atom in $(0,1)$, then $\mu$ is not a credence.

Proof. Suppose $\nu$ has an atom at some point $x \in (0,1)$. Let $\mathcal{L} := [0,x)$ and $\mathcal{R} := (x,1]$. Then clearly, $\mathcal{L}$ and $\mathcal{R}$ are regular open subsets of $[0,1]$ with $\mathcal{L} \lor \mathcal{R} = [0,1]$. But

$$\mu[\mathcal{L}] + \mu[\mathcal{R}] = \nu[\mathcal{L}] + \nu[\mathcal{R}] = \nu[\mathcal{L} \cup \mathcal{R}] = 1 - \nu\{x\} < 1 = \mu[0,1].$$

Thus, $\mu$ violates the finite additivity equation (1), so it is not a credence. \(\diamondsuit\) Claim 1

So, without loss of generality, we assume that $\nu$ has no atoms in $(0,1)$. Thus, for any $r \in (0,1)$, and any $\epsilon > 0$, there exists $\delta > 0$ such that $\nu(r - \delta, r + \delta) < \epsilon$.

Now, let $\{r_n\}_{n=1}^{\infty}$ be a countable dense subset of $(0,1)$ (for example, the set of all rationals in $(0,1)$). For all $n \in \mathbb{N}$, since $r_n$ is not an atom, there exists $\delta'_n > 0$ such that $\nu(r_n - \delta'_n, r_n + \delta'_n) < 1/2^n$. For all $n \in \mathbb{N}$, define the open set $O_n$ as follows:

- Let $O_1 := (r_1 - \delta_1, r_1 + \delta_1)$.
- Let $n \geq 2$. By induction, suppose that $O_n$ has already been defined, and contains $\{r_1, \ldots, r_n\}$. Let $m(n) := \min\{n \in \mathbb{N} : r_n \notin \text{clos}(O_n)\}$. (Thus, $m(n) \geq n + 1$.) Define $q_n := t_{m(n)}$, and let $\delta''_n := \inf\{|q_n - u| : u \in O_n\}$; then $\delta''_n > 0$. Let $\delta_n := \min\{\delta'_m(n), \delta''_n\}$, and define $O_{n+1} := O_n \cup (q_n - \delta_n, q_n + \delta_n)$.

In this way, we obtain an increasing sequence $(O_1 \subseteq O_2 \subseteq \cdots)$ of open sets. Let $U := \bigcup_{n=1}^{\infty} O_n$; then $U$ is an open subset of $[0,1]$. Furthermore, $U$ is dense in $[0,1]$. To see this, note that, for any $N \in \mathbb{N}$, we must have $\{r_1, \ldots, r_N\} \subset \text{clos}(O_N)$. Thus, $\{r_n\}_{n=1}^{\infty} \subset \text{clos}(U)$, so since $\{r_n\}_{n=1}^{\infty}$ is dense in $(0,1)$, it follows that $\text{clos}(U) = [0,1]$. 37
By construction, we can write $\mathcal{U}$ as a disjoint union of open intervals:

$$
\mathcal{U} = \bigcup_{n=1}^{\infty} (q_n - \delta_n, q_n + \delta_n) = \bigcup_{n=1}^{\infty} (r_m(n) - \delta_n, r_m(n) + \delta_n).
$$

Thus, $\nu(\mathcal{U}) = \sum_{n=1}^{\infty} \nu(r_m(n) - \delta_n, r_m(n) + \delta_n) \leq \sum_{n=1}^{\infty} \nu(r_m(n) - \delta_m(n), r_m(n) + \delta_m(n)) < \sum_{n=1}^{\infty} \frac{1}{2^{m(n)}} \leq \sum_{m=1}^{\infty} \frac{1}{2^m} = 1. \tag{22}$

For all $n \in \mathbb{N}$, let $\mathcal{U}_n := (q_n - \delta_n, q_n + \delta_n)$, and let $\mathcal{L}_n := (q_n - \delta_n, q_n)$ and $\mathcal{R}_n := (q_n, q_n + \delta_n)$ be the left and right “halves” of $\mathcal{U}_n$. Let $\mathcal{Q} := \{q_n\}_{n=1}^{\infty}$. Then $\mathcal{U} = \mathcal{L} \sqcup \mathcal{Q} \sqcup \mathcal{R}$, so $\nu(\mathcal{U}) = \nu(\mathcal{L}) + \nu(\mathcal{Q}) + \nu(\mathcal{R}) = \nu(\mathcal{L}) + \nu(\mathcal{R})$ (because $\nu(\mathcal{Q}) = 0$ because $\mathcal{Q}$ is a countable set and $\nu$ has no atoms).

**Claim 2:** $\mathcal{L}$ and $\mathcal{R}$ are disjoint regular open sets.

**Proof.** Clearly, $\mathcal{L}$ and $\mathcal{R}$ are open and disjoint; it remains to show regularity. Let $\mathcal{I} := \text{int}[\text{clos}(\mathcal{L})]$; we must show that $\mathcal{I} = \mathcal{L}$. Since $\mathcal{L}$ is an open subset of clos($\mathcal{L}$), we have $\mathcal{L} \subseteq \mathcal{I}$; we must show that $\mathcal{I} \subseteq \mathcal{L}$.

First, note that clos($\mathcal{L}$) $\subseteq [0,1]\setminus \mathcal{R}$; thus, $\mathcal{I} \subseteq [0,1]\setminus \mathcal{R}$. Since $\mathcal{I}$ is open, we have $\mathcal{I} = \bigcup_{j=1}^{\infty} \mathcal{I}_j$ for some countable collection $\{\mathcal{I}_j\}_{j=1}^{\infty}$ of disjoint open intervals. For all $j \in \mathbb{N}$, let $\mathcal{I}_j = (a_j, b_j)$.

**Claim 2A:** For all $j \in \mathbb{N}$, $a_j \notin \mathcal{L}$ and $b_j \notin \mathcal{L}$.

**Proof.** Since $\{\mathcal{I}_n\}_{n=1}^{\infty}$ are disjoint, we must have $a_j, b_j \notin \mathcal{I}_n$ for all $n \in \mathbb{N}$. Thus, $a_j, b_j \notin \bigcap_{n=1}^{\infty} \mathcal{I}_n = \mathcal{I}$. But $\mathcal{L} \subseteq \mathcal{I}$. Thus, $a_j, b_j \notin \mathcal{L}$.

Now, fix $j \in \mathbb{N}$. We must have $\mathcal{I}_j \cap \mathcal{L} \neq \emptyset$, because $\mathcal{L}$ is dense in clos($\mathcal{L}$), and $\mathcal{I}_j$ is an open subset of clos($\mathcal{L}$). Thus, there is some $n \in \mathbb{N}$ such that $\mathcal{I}_j \cap \mathcal{L}_n \neq \emptyset$.

**Claim 2B:** $b_j = q_n$.

**Proof.** (by contradiction) Recall that $\mathcal{I}_j = (a_j, b_j)$ and $\mathcal{L}_n := (q_n - \delta_n, q_n)$. Thus, if $\mathcal{I}_j \cap \mathcal{L}_n \neq \emptyset$, then $q_n - \delta_n < b_j$. If $q_n < b_j$, then $\mathcal{I}_j$ would overlap $\mathcal{R}_n$, and hence $\mathcal{R}$, contradicting the fact that $\mathcal{I} \subseteq [0,1]\setminus \mathcal{R}$. Thus, $q_n - \delta_n < b_j \leq q_n$. But then we must have $b_j = q_n$, by Claim 2A.

**Claim 2C:** $a_j = q_n - \delta_n$.

**Proof.** (by contradiction) If $a_j > q_n - \delta_n$, then $a_j \in \mathcal{L}_n$ (by Claim 2B), contradicting Claim 2A. Thus, $a_j \leq q_n - \delta_n$. On the other hand, if $a_j < q_n - \delta_n$, then the open interval $(a_j, q_n - \delta_n)$ must intersect $\mathcal{U}$ (because $\mathcal{U}$ is dense in $[0,1]$), which means it must intersect $\mathcal{U}_m = (q_m - \delta_m, q_m + \delta_m)$ for some $m \in \mathbb{N}\setminus\{n\}$. This means that $a_j < q_m + \delta_m$.

Recall that $\mathcal{U}_m = \mathcal{L}_m \sqcup \{q_m\} \sqcup \mathcal{R}_m$, where $\mathcal{R}_m = (q_m, q_m + \delta_m)$. Also recall that $\mathcal{I}_j = (a_j, b_j)$. If $q_m \leq a_j < q_m + \delta_m$, then clearly $\mathcal{I}_j$ overlaps $\mathcal{R}_m$, which contradicts the fact that $\mathcal{I} \subseteq [0,1]\setminus \mathcal{R}$. Thus, we must have $a_j < q_m$.
Meanwhile, if \( q_n - \delta_n < q_m - \delta_m \), then \((a_j, q_n - \delta_n)\) is disjoint from \( U_m \), contradicting our assumption that they overlap. Thus, we must have \( q_m - \delta_m \leq q_n - \delta_n \). But if \( q_n - \delta_m \leq q_n - \delta_n < q_m + \delta \), then \( L_n \) and \( U_m \) would overlap, contradicting the fact that \( U_n \) and \( U_m \) are disjoint by definition (because \( n \neq m \)). Thus, we must also have \( q_m + \delta_m \leq q_n - \delta_n \). Putting it all together, we have \( a_j < q_m < q_m + \delta_m \leq q_n - \delta_n < q_n = b_j \), where the last equality is by Claim 2B. This means that \((q_m, q_m + \delta_m) \subset (a_j, b_j)\); in other words, \( R_m \subset I_j \). But again, this contradicts the fact that \( I \subseteq [0,1] \setminus R \).

To avoid these contradictions, we must have \( a_j = q_n - \delta_n \). \( \nabla \) Claim 2C

Claims 2B and 2C together imply that \( I_j = (q_n - \delta_n, q_n) \); in other words, \( I_j = L_n \). This argument works for all \( j \in \mathbb{N} \). Thus, every open interval of \( I \) is actually one of the intervals of \( L \). Thus, \( I \subseteq L \). But we have already noted that \( L \subseteq I \). Thus, \( L = I \), as desired. Thus, \( L \) is regular. The proof for \( R \) is similar. \( \diamond \) Claim 2

Now, \( L \sqcup R = U \setminus Q \), which is dense in \( U \) (because \( Q \) is countable and \( U \) is open). But \( U \) is dense in \([0,1]\). Thus, \( L \sqcup R \) is dense in \([0,1]\). Thus, \( L \sqcup R = [0,1] \). But

\[
\mu[L] + \mu[R] = \nu[L] + \nu[R] = \nu[L \sqcup R] \leq \nu[U] < 1 = \mu[0,1] = \mu[L \sqcup R],
\]

where \((*)\) is by inequality (22). Thus, \( \mu \) violates the finite additivity equation (1), so it is not a credence.

(Case 2) Now suppose \( S \) is locally projectible. Let \( \nu \) be a Borel probability measure on \( S \). Since \( S \) is locally projectible, there exists some regular open subset \( S_0 \subseteq S \) such that \( S_0 \) is projectible and \( \nu[S_0] > 0 \). For all Borel subsets \( B \subseteq S_0 \), define \( \nu_0[B] := \nu[B]/\nu[S_0] \); then \( \nu_0 \) is a Borel probability measure on \( S_0 \). By contradiction, suppose \( \nu \) defines a credence \( \mu \) when restricted to \( R(S) \). Since \( S_0 \in R(S) \), we have \( \mu[S_0] = \nu[S_0] > 0 \), and \( R(S_0) = \{ R \in R(S); \ R \subseteq R_0 \} \). Thus, for all \( R \in R(S_0) \), we can define \( \mu_0[R] := \mu[R]/\mu[S_0] \), to obtain a credence on \( R(S_0) \). Equivalently, \( \mu_0[R] = \nu_0[R] \) for all \( R \in R(S_0) \); in other words, \( \nu_0 \) defines a credence when restricted to \( R(S_0) \). But this contradicts Case 1, because \( S_0 \) is projectible. \( \square \)

7 Compactification representations

Corollary 6.7 shows that liminal representations are especially useful on compact spaces. This suggests that we could greatly improve the representation in Theorem 6.5 by compactifying \( S \). Let \( \overline{S} \) be a compactification of \( S \) (i.e. a compact Hausdorff space containing \( S \) as a dense subspace). Let \( C_{\overline{S}}(S, \mathbb{R}) := \{ f|_S; \ f \in C(\overline{S}, \mathbb{R}) \} \) — this is the set of all continuous functions in \( C(S, \mathbb{R}) \) which can be continuously extended to \( \overline{S} \). If such an extension exists, then it is unique, because \( S \) is dense in \( \overline{S} \). For any \( f \in C_{\overline{S}}(S, \mathbb{R}) \), let \( \overline{f} \) denote its unique extension to \( C(\overline{S}, \mathbb{R}) \).

Example 7.1. Let \( S \) be a locally compact Hausdorff space.
(a) Suppose $S$ is not compact, and let $S^*$ be its Alexandroff compactification. For any $f : S \to \mathbb{R}$ and $L \in \mathbb{R}$, we will write “$L = \lim_{s \to \infty} f(s)$” if, for any open neighbourhood $O$ around $L$, there is some compact subset $K \subset S$ such that $f(S \setminus K) \subseteq O$. Then $C_{S^*}(S, \mathbb{R}) = \{ f \in \mathcal{C}(S, \mathbb{R}) : \lim_{s \to \infty} f(s) \text{ is well-defined} \}$. If $f \in C_{S^*}(S, \mathbb{R})$, then $\overline{f}(\infty) = \lim_{s \to \infty} f(s)$.

(b) Let $S$ be a totally bounded metric space. Then $S$ is locally compact if and only if it is locally complete — i.e., every point has a neighbourhood within which every Cauchy sequence converges. Let $\overline{S}$ be the (metric) completion of $S$. Then $\overline{S}$ is a compactification of $S$ (as a topological space), and $\mathcal{C}(\overline{S}, \mathbb{R})$ is the set of uniformly continuous real-valued functions on $S$. (The same is true if $S$ is a totally bounded uniform space, and $\overline{S}$ is its (uniform) completion $\text{[Wil04, Theorems 39.10 and 39.13].}$

(c) Let $\beta S$ be the Stone-Čech compactification of $S$. The Stone-Čech Extension Theorem implies that every bounded continuous real-valued function on $S$ has a continuous extension to $\beta S$. Thus, $C_{\beta S}(S, \mathbb{R}) = C_b(S, \mathbb{R})$.

(d) Let $\mathbb{R} := [-\infty, \infty]$, with the obvious topology. Then $\mathbb{R}$ is a compactification of $\mathbb{R}$, and $\mathcal{C}(\mathbb{R}, \mathbb{R}) = \{ f \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \lim_{s \to \infty} f(s) \text{ and } \lim_{s \to -\infty} f(s) \text{ are well-defined} \}$.

(e) Example (d) can be generalized as follows. Let $(K_1 \subseteq K_2 \subseteq \cdots)$ be a compact exhaustion of $S$ — that is, an increasing sequence of compact subsets of $S$, such that every compact subset of $S$ is contained in some $K_n$. An end of $S$ is a decreasing sequence $\epsilon := (O_1 \supseteq O_2 \supseteq \cdots)$, where for all $n \in \mathbb{N}$, $O_n$ is a connected component of $S \setminus K_n$. Let $\mathcal{E}(S)$ be the set of ends of $S$. (The definition of $\mathcal{E}(S)$ is independent of the exact choice of compact exhaustion.) The Freudenthal compactification of $S$ is the set $\mathcal{F} := S \sqcup \mathcal{E}(S)$, where every open subset of $\mathcal{F}$ remains open in $\mathcal{F}$, and where, for each $\epsilon \in \mathcal{E}(S)$, if $\epsilon := (O_1 \supseteq O_2 \supseteq \cdots)$, then the sets $\{Q_n \cup \{\epsilon\}\}_{n=1}^{\infty}$ form a neighbourhood base for $\epsilon$ in $\mathcal{F}$. See [Pes90] for more information. For example, $\mathcal{E}(\mathbb{R}) = \{\pm\infty\}$, and the Freudenthal compactification of $\mathbb{R}$ is $\mathbb{R}$, as defined in Example (d). (However, if $N \geq 2$, then $\mathcal{E}(\mathbb{R}^N)$ is a singleton, so the Freudenthal compactification of $\mathbb{R}^N$ is the same as its Alexandroff compactification.)

For any $f \in \mathcal{C}(S, \mathbb{R})$, $\epsilon \in \mathcal{E}(S)$, and $L \in \mathbb{R}$, write “$\lim_{s \to \epsilon} f(s) = L$” if, for any neighbourhood $U$ of $L$, there is some $n \in \mathbb{N}$ such that $f(O_n) \subseteq U$ (where $\epsilon := (O_1 \supseteq O_2 \supseteq \cdots)$). Then $C_{\mathcal{F}}(S, \mathbb{R}) = \{ f \in \mathcal{C}(S, \mathbb{R}) : \lim_{s \to \epsilon} f(s) \text{ is well-defined for every } \epsilon \in \mathcal{E}(S) \}$. If $f \in C_{\mathcal{F}}(S, \mathbb{R})$, then $\overline{f}(\epsilon) = \lim_{s \to \epsilon} f(s)$ for each $\epsilon \in \mathcal{E}(S)$.

(f) Let $\varphi(S)$ be the power set of $S$. A proximity on $S$ is a symmetric, reflexive binary relation $\sim$ on $\varphi(S)$ such that, for all nonempty $A, B, C \subseteq S$, we have: (i) $\emptyset \not\sim A$; (ii) $A \sim (B \cup C)$ if and only if $A \sim B$ or $A \sim C$; and (iii) If $A \not\sim B$, then there exist disjoint $D, E \subseteq X$ with $A \not\sim (S \setminus D)$ and $B \not\sim (S \setminus E)$ [Wil04, §40]. A proximity $\sim$ is compatible with the topology of $S$ if, for any $A \subseteq S$, we have $\text{clos} (A) := \{ s \in S : \{ s \} \sim A \}$. If $\overline{S}$ is any Hausdorff compactification of $S$, then we get a compatible proximity on $S$ by stipulating that $A \sim B$ if and only if $\text{clos}_{\overline{S}}(A) \cap \text{clos}_{\overline{S}}(B) \neq \emptyset$ (where $\text{clos}_{\overline{S}}(A)$ is the closure of $A$ in $\overline{S}$). In fact, every compatible proximity arises from a compactification in this fashion; thus,
there is a bijective correspondence between the compactifications of $S$ and the compatible proximities [Wil04] Definition 41.2.

The **elementary** proximity $\simeq$ on $\mathbb{R}$ is defined by stipulating that $A \simeq B$ if and only if $\text{clos}(A) \cap \text{clos}(B) \neq \emptyset$ (for any $A, B \subseteq \mathbb{R}$). If $\sim$ is a compatible proximity on $S$, then a function $f : S \to \mathbb{R}$ is **proximity-preserving** if, for all $A, B \subseteq S$ such that $A \sim B$, we have $f(A) \approx f(B)$. Every proximity-preserving function is continuous, but not every continuous function is proximity preserving. However, if $\sim$ arises from the compactification $\overline{S}$, then $C_\sim(S, \mathbb{R})$ is the precisely set of proximity-preserving functions from $S$ to $\mathbb{R}$. (This follows by combining Corollary 36.20 with Theorems 36.19, 40.10(b) and 41.1 of [Wil04].) $\diamond$

Let $\overline{S}$ be a compactification of $S$. For any $R \in \mathcal{R}(S)$, there is a unique $\overline{R} \in \mathcal{R}(\overline{S})$ such that $\overline{R} \cap S = R$ (see Lemma 7.4 below). We will refer to $\overline{R}$ as the **extension** of $R$.

**Theorem 7.2** Let $S$ be a locally compact Hausdorff space. Let $\mu$ be a credence on $\mathcal{R}(S)$, and let $I := \{I_R\}_{R \in \mathcal{R}(S)}$ be the $\mu$-compatible integrator. Let $\overline{S}$ be a compactification of $S$. There is a unique normal Borel measure $\overline{\nu}$ on $\overline{S}$, and a unique liminal density structure $(\overline{\phi}_R)_{R \in \mathcal{R}(\overline{S})}$ which is subordinate to $\overline{\nu}$, such that for any $R \in \mathcal{R}(S)$, we have

$$\mu[R] = \overline{\nu}(\overline{R}) + \int_{\overline{R}} \overline{\phi}_R \ d\overline{\nu},$$

where $\overline{R}$ is the unique extension of $R$ to $\overline{S}$. Furthermore, for any $f \in C_\sim(S, \mathbb{R})$, we have

$$I_R[f] = \int_{\overline{R}} \overline{f} \ d\overline{\nu} + \int_{\overline{R}} \overline{f} \overline{\phi}_R \ d\overline{\nu},$$

where $\overline{f}$ is the unique extension of $f$ to $C(\overline{S}, \mathbb{R})$.

**Example 7.3.** (a) If $\overline{S}$ is the Alexandroff compactification of $S$, then (2) holds for any $f \in C(S, \mathbb{R})$ such that $\lim_{s \to \infty} f(s)$ is well-defined.

(b) If $S$ is a totally bounded, locally complete metric space, and $\overline{S}$ is its completion, then (2) holds for all uniformly continuous $f \in C(S, \mathbb{R})$.

(c) If $\overline{S}$ is the Stone-Čech compactification of $S$, then (2) holds for all $f \in C_b(S, \mathbb{R})$. $\diamond$

The proof of Theorem 7.2 depends on the following lemma, which establishes that each element of $\mathcal{R}(S)$ has a unique “extension” to an element of $\mathcal{R}(\overline{S})$.

**Lemma 7.4** Let $\overline{S}$ be a compactification of a locally compact Hausdorff space $S$.

(a) If $\overline{R} \in \mathcal{R}(\overline{S})$, then $\overline{R} \cap S \in \mathcal{R}(S)$. Furthermore, the function $\overline{R} \mapsto \overline{R} \cap S$ is a Boolean algebra isomorphism from $\mathcal{R}(\overline{S})$ to $\mathcal{R}(S)$.

(b) Let $\overline{R} \in \mathcal{R}(\overline{S})$ and let $R := S \cap \overline{R}$. Let $C_1(\overline{R}, \mathbb{R}) := \{f_{\overline{R}}; \ f \in C(\overline{S}, \mathbb{R})\}$ and let $C_1(R, \mathbb{R}) := \{f_R; \ f \in C(S, \mathbb{R})\}$. Define $\Phi : C_1(\overline{R}, \mathbb{R}) \to C_1(R, \mathbb{R})$ by setting $\Phi(f) := f_{\overline{R}}$ for all $f \in C_1(\overline{R}, \mathbb{R})$. Then $\Phi$ is an order-preserving, continuous linear bijection, and $\|f_{\overline{R}}\|_\infty = \|f\|_\infty$ for all $f \in C_1(\overline{R}, \mathbb{R})$. 

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Proof. (a) Homomorphism. Let \( \iota : \mathcal{S} \rightarrow \mathcal{S} \) be the inclusion map. This is a continuous function, because \( \mathcal{S} \) has the subspace topology it inherits from \( \mathcal{S} \). It is also an open function, because \( \mathcal{S} \) is an open subset of \( \mathcal{S} \) [Eng89, Theorem 3.5.8]. Thus \( \iota^{-1} : \mathcal{A}(\mathcal{S}) \rightarrow \mathcal{A}(\mathcal{S}) \) is a Boolean algebra homomorphism [Fre06, 4A2B (f)(iii)] (or Lemma 5.1(b)). But for any \( \mathcal{R} \in \mathcal{A}(\mathcal{S}) \), we have \( \iota^{-1}(\mathcal{R}) = \mathcal{S} \cap \mathcal{R} \).

Injective. Let \( \mathcal{R}, \mathcal{O} \in \mathcal{A}(\mathcal{S}) \). Suppose \( \mathcal{R} \neq \mathcal{O} \), then either \( \mathcal{R} \cap (-\mathcal{O}) \neq \emptyset \) or \( \mathcal{O} \cap (-\mathcal{R}) \neq \emptyset \). Suppose the former. Then \( \mathcal{R} \cap (-\mathcal{O}) \) is a nonempty open subset of \( \mathcal{S} \). But \( \mathcal{S} \) is dense in \( \mathcal{S} \). Thus, \( \mathcal{S} \cap \mathcal{R} \cap (-\mathcal{O}) \neq \emptyset \), which implies that \( \mathcal{S} \cap \mathcal{R} \neq \mathcal{S} \cap \mathcal{O} \).

Surjective. Let \( \mathcal{R} \in \mathcal{A}(\mathcal{S}) \). Define \( \mathcal{R} := \operatorname{int}_{\mathcal{S}}[\operatorname{clos}_{\mathcal{S}}(\mathcal{R})] \). Let \( \mathcal{O} \) be the topology of \( \mathcal{S} \), and let \( \mathcal{S} \) be the topology of \( \mathcal{S} \). Now, \( \mathcal{R} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \), and \( \mathcal{R} \) is open in \( \mathcal{S} \), so \( \mathcal{R} \subseteq \operatorname{int}_{\mathcal{S}}[\operatorname{clos}_{\mathcal{S}}(\mathcal{R})] = \mathcal{R} \). Thus, \( \mathcal{R} \subseteq \mathcal{S} \cap \mathcal{R} \). Conversely,

\[
\mathcal{R} = \bigcup \{ \mathcal{O} \in \mathcal{O} ; \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \}.
\]

Thus, \( \mathcal{R} \cap \mathcal{S} = \bigcup \{ \mathcal{S} \cap \mathcal{O} ; \mathcal{O} \in \mathcal{O} \text{ and } \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \} \subseteq \bigcup \{ \mathcal{O} \cap \mathcal{S} ; \mathcal{O} \in \mathcal{O} \text{ and } \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \} = \operatorname{int}_{\mathcal{S}}[\operatorname{clos}_{\mathcal{S}}(\mathcal{R})] \] \( \{ \mathcal{O} \in \mathcal{O} \text{ and } \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \} \)

Here, \( \{ \mathcal{O} \cap \mathcal{S} ; \mathcal{O} \in \mathcal{O} \text{ and } \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \} \) because \( \mathcal{R} \subseteq \mathcal{S} \) and \( \mathcal{S} \) has the subspace topology. Finally, \( \{ \mathcal{O} \cap \mathcal{S} ; \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \} \)

(b) The function \( \Phi \) is clearly order-preserving and linear. It is also continuous since we clearly have \( \| f \|_{\infty} \leq \| f \|_{\infty} \) for all \( f \in C_{1}(\mathcal{S}) \). In fact, we even have the equality of the two norms, because \( f \) is continuous and \( \mathcal{R} \) is dense in \( \mathcal{S} \). To see that \( \Phi \) is surjective, let \( f \in C_{1}(\mathcal{S}) \). Then there is some \( g \in C_{1}(\mathcal{S}) \) such that \( f = g \). Let \( g \in C_{1}(\mathcal{S}) \) be the (unique) continuous extension of \( g \) to \( \mathcal{S} \). Then define \( \overline{f} := \frac{f}{\| f \|_{\infty}} \). Then clearly, \( \overline{f} \) is because \( \mathcal{R} \subseteq \mathcal{S} \) and \( \mathcal{S} \) has the subspace topology. Finally, \( \{ \mathcal{O} \cap \mathcal{S} ; \mathcal{O} \subseteq \operatorname{clos}_{\mathcal{S}}(\mathcal{R}) \} \)

Remark. In the proof of Lemma 7.3(a), it is crucially important that \( \mathcal{S} \) be an open dense subset of \( \mathcal{S} \). Let’s say that a compactification is proper if it has this property. If \( \mathcal{S} \) is a Hausdorff space, then the following are equivalent: (1) \( \mathcal{S} \) is locally compact; (2) \( \mathcal{S} \) has a proper compactification; (3) every compactification of \( \mathcal{S} \) is proper [Eng89, Theorem 3.5.8]. For this reason, Theorem 7.2 only applies to locally compact Hausdorff spaces.

Proof of Theorem 7.2. Let \( \mu \) be a credence on \( \mathcal{A}(\mathcal{S}) \). Define the credence \( \overline{\mu} \) on \( \mathcal{A}(\mathcal{S}) \) by setting \( \overline{\mu}(\mathcal{R}) := \mu[\mathcal{S} \cap \mathcal{R}] \) for all \( \mathcal{R} \in \mathcal{A}(\mathcal{S}) \). Lemma 7.3(a) implies that this is a well-defined credence. For any \( \mathcal{R} \in \mathcal{A}(\mathcal{S}) \), if \( \mathcal{R} \) is the (unique) element of \( \mathcal{A}(\mathcal{S}) \) such that \( \mathcal{S} \cap \mathcal{R} = \mathcal{R} \), then we have \( \mu[\mathcal{R}] = \overline{\mu}[\mathcal{R}] \).
Let \( \mathbf{I} := \{ \mathbb{I}_R \}_{R \in \mathfrak{R}(S)} \) be the unique \( \pi \)-compatible integrator from Theorem 4.3. Corollary 6.7 yields a unique normal Borel probability measure \( \nu \) on \( S \) and liminal density structure \( \phi \). For any \( R \in \mathfrak{R}(S) \) and \( f \in C_1(R, \mathbb{R}) \), define

\[
\mathbb{I}_R[f] := \mathbb{I}_R(\overline{f}),
\]

(3)

where \( \overline{f} \) is the unique element of \( C_1(\overline{R}, \mathbb{R}) \) such that \( \overline{f}_R = f \), as given by Lemma 7.4(b).

**Claim 1:** \( \mathbf{I} := \{ \mathbb{I}_R \}_{R \in \mathfrak{R}(S)} \) is a \( \mu \)-compatible integrator on \( S \).

**Proof.** Let \( R \in \mathfrak{R}(S) \). Lemma 7.4(b) says the transformation \( f \mapsto \overline{f} \) is an order-preserving, norm-preserving, linear isomorphism from \( C_1(R, \mathbb{R}) \) to \( C_1(\overline{R}, \mathbb{R}) \). Thus, \( \mathbb{I}_R \) is a weakly monotone, bounded linear functional on \( C_1(R, \mathbb{R}) \), with

\[
\| \mathbb{I}_R \|_\infty \equiv (\ast) \quad \| \mathbb{I}_\pi \|_\infty \equiv (\dagger) \quad \pi[R] \equiv (\circ) \quad \mu[R].
\]

Here, \( \ast \) is because the function \( f \mapsto \overline{f} \) is norm-preserving, \( \dagger \) is because the integrator \( \mathbf{I} \) is compatible with \( \pi \), and \( \circ \) is by the definition of \( \pi \).

It remains to verify formula (1). Let \( \{ R_1, \ldots, R_N \} \) be a regular open partition of \( R \). Then \( \{ \overline{R}_1, \ldots, \overline{R}_N \} \) is a regular open partition of \( \overline{R} \) (because of the Boolean algebra isomorphism from Lemma 7.4(a)). For any \( f \in C_1(R, \mathbb{R}) \), we then have

\[
\mathbb{I}_R[f] \equiv (\ast) \quad \mathbb{I}_\pi[\overline{f}] \equiv (\dagger) \quad \sum_{n=1}^N \mathbb{I}_{\overline{R}_n}[f_{\overline{R}_n}] \equiv (\circ) \quad \sum_{n=1}^N \mathbb{I}_{R_n}[f_{R_n}],
\]

as desired. Here, \( \ast \) is by the definition of the functional \( \mathbb{I}_R \) via (3), and \( \circ \) is by the definition of the functionals \( \mathbb{I}_{R_1}, \ldots, \mathbb{I}_{R_N} \) via (3). Meanwhile, \( \dagger \) is by equation (1), because the system \( \{ \mathbb{I}_\pi \}_{R \in \mathfrak{R}(S)} \) is an integrator by hypothesis. \( \diamond \) Claim 1

Now, Theorem 4.3 says that there is a unique \( \mu \)-compatible integrator on \( \mathfrak{R}(S) \). Combining this with Claim 1, we conclude that this integrator must be the one defined by formula (3). Combining formula (3) with the versions of equations (20) and (21) in Corollary 6.7 we obtain equations (1) and (2).

### 8 Integration via Stone spaces

In this section, we will introduce an entirely different representation of integrators, which is applicable to a credence defined on a Boolean subalgebra of regular open sets. For any topological space \( T \), let \( \mathfrak{C}(T) \) be the collection of all clopen subsets of \( T \). This collection is a Boolean algebra under the standard set-theoretic operations of union, intersection, and complementation. A **Stonean space** is a compact, totally disconnected Hausdorff space. For any Boolean algebra \( \mathfrak{B} \), let \( \sigma(\mathfrak{B}) \) be the set of all ultrafilters of \( \mathfrak{B} \). For any \( B \in \mathfrak{B} \), let
The collection \( \mathcal{B} := \{ \mathcal{U} \in \sigma(\mathcal{B}); \mathcal{B} \in \mathcal{U} \} \). The collection \( \{ \mathcal{B}^* \}_{\mathcal{B} \in \mathcal{B}} \) is a base of clopen sets for a topology on \( \sigma(\mathcal{B}) \), making \( \sigma(\mathcal{B}) \) into Stonean space. This is called the Stone space of \( \mathcal{B} \). The Stone Representation Theorem says that there is a Boolean algebra isomorphism from \( \mathcal{B} \) to \( \text{Clp}[\sigma(\mathcal{B})] \) given by \( \mathcal{B} \ni \mathcal{B} \mapsto \mathcal{B}^* \in \text{Clp}[\sigma(\mathcal{B})] \).

If \( \mathcal{A} \) is another Boolean algebra, and \( h : \mathcal{A} \longrightarrow \mathcal{B} \) is a Boolean algebra homomorphism, then we obtain a continuous function \( H : \sigma(\mathcal{B}) \longrightarrow \sigma(\mathcal{A}) \) which maps each ultrafilter in \( \mathcal{B} \) to its \( h \)-preimage ultrafilter in \( \mathcal{A} \). This yields a contravariant functor \( \sigma \) from the category of Boolean algebras to the category of Stonean spaces and continuous functions. In fact, the Stone Duality Theorem says that \( \sigma \) is a functorial isomorphism between these two categories.

Now, let \( \mathcal{S} \) be a locally compact Hausdorff space, and let \( \hat{\mathcal{S}} \) be its Stone-Čech compactification. Recall from Lemma 7.4(a) that there is a Boolean algebra isomorphism \( h : \mathcal{R}(\hat{\mathcal{S}}) \rightleftharpoons \mathcal{R}(\mathcal{S}) \) given by \( h(\hat{\mathcal{R}}) := \hat{\mathcal{R}} \cap \mathcal{S} \), for all \( \hat{\mathcal{R}} \in \mathcal{R}(\hat{\mathcal{S}}) \). For any \( \mathcal{R} \in \mathcal{R}(\mathcal{S}) \), let \( \hat{\mathcal{R}} := h^{-1}(\mathcal{R}) \) — the unique element of \( \mathcal{R}(\hat{\mathcal{S}}) \) such that \( \hat{\mathcal{R}} \cap \mathcal{S} = \mathcal{R} \). Let \( \mathcal{B} \subseteq \mathcal{R}(\mathcal{S}) \) be any Boolean subalgebra, and let \( \hat{\mathcal{B}} := \{ \hat{\mathcal{B}}; \mathcal{B} \in \mathcal{B} \} \); this is a Boolean subalgebra of \( \mathcal{R}(\hat{\mathcal{S}}) \), and is isomorphic to \( \mathcal{B} \) via \( h \). We say that \( \mathcal{B} \) is generative if \( \hat{\mathcal{B}} \) is a base for the topology of \( \hat{\mathcal{S}} \).

**Lemma 8.1** If \( \mathcal{B} \) is generative, then it is a base for the topology of \( \mathcal{S} \).

**Proof.** It suffices to show that every open subset of \( \mathcal{S} \) contains a \( \mathcal{B} \)-neighbourhood around each of its points. So, let \( \mathcal{O} \subseteq \mathcal{S} \) be open, and let \( s \in \mathcal{O} \). Then \( \mathcal{O} \) is also open in \( \hat{\mathcal{S}} \) (because \( \mathcal{S} \) is an open subset of \( \hat{\mathcal{S}} \)). Thus, there exists \( \hat{\mathcal{B}} \subseteq \hat{\mathcal{B}} \) with \( s \in \hat{\mathcal{B}} \subseteq \mathcal{O} \) (because \( \hat{\mathcal{B}} \) is a base for the topology of \( \hat{\mathcal{S}} \)). Note that \( \hat{\mathcal{B}} = \hat{\mathcal{B}} \cap \mathcal{S} \) (because \( \hat{\mathcal{B}} \subseteq \mathcal{S} \)); thus \( \hat{\mathcal{B}} \subseteq \mathcal{B} \). This works for any \( s \) and \( \mathcal{O} \); thus, \( \mathcal{B} \) is a base for the topology of \( \mathcal{S} \). \( \square \)

The converse of Lemma 8.1 is false: for \( \mathcal{B} \) to be generative, it is *not sufficient* that \( \mathcal{B} \) be a base for the topology of \( \mathcal{S} \). For example, let \( \mathcal{S} = \mathbb{N} \), with the discrete topology; then \( \mathcal{R}(\mathbb{N}) = \varnothing(\mathbb{N}) \). Let \( \mathfrak{F} \) be the set of all finite subsets of \( \mathbb{N} \), and let \( \mathfrak{E} := \{ \mathbb{N} \setminus \mathcal{F}; \mathcal{F} \in \mathfrak{F} \} \); then \( \mathcal{B} := \mathfrak{F} \cup \mathfrak{E} \) is a Boolean subalgebra of \( \mathcal{R}(\mathbb{N}) \) which generates the topology of \( \mathbb{N} \), because it contains all singleton sets. Let \( \hat{\mathbb{N}} \) be the Stone-Čech compactification of \( \mathbb{N} \), and let \( \hat{\mathfrak{F}} := \{ \hat{\mathbb{N}} \setminus \mathcal{F}; \mathcal{F} \in \mathfrak{F} \} \). Then \( \hat{\mathcal{B}} = \mathfrak{F} \cup \hat{\mathcal{E}} \), which does *not* generate the topology of \( \hat{\mathbb{N}} \). Nevertheless, the full Boolean algebra \( \mathcal{R}(\hat{\mathcal{S}}) \) itself is always generative, because of the next lemma (using the fact that \( \mathcal{R}(\hat{\mathcal{S}}) = \mathcal{R}(\hat{\mathcal{S}}) \)).

**Lemma 8.2** Let \( \mathcal{S} \) be a locally compact Hausdorff space. Then \( \mathcal{R}(\mathcal{S}) \) is a base for the topology on \( \mathcal{S} \).

**Proof.** Let \( s \in \mathcal{S} \), and let \( \mathcal{O} \subseteq \mathcal{S} \) be any open neighbourhood of \( s \). Since \( \mathcal{S} \) is locally compact, there is a compact subset \( \mathcal{K} \subseteq \mathcal{O} \) which is also a neighbourhood of \( s \). Let \( \mathcal{R} := \text{int}(\mathcal{K}) \); then \( \mathcal{R} \) is a regular open subset of \( \mathcal{S} \), and \( s \in \mathcal{R} \subseteq \mathcal{O} \), as desired. \( \square \)

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16See e.g. [Wal74, Theorem 2.10, p.51], [PW88, §3.2], or [Fre04, Sections 311E, 311F and 311I].
17See e.g. [Wal74, Example 10.8(a), p. 247], [Joh86, §4.1, p.71], or [Fre04, Sections 312Q and 312R].
Let $S^* := \sigma(\mathcal{B})$. Via the Stone Duality Theorem, the isomorphism $h : \mathcal{R}(\hat{S}) \cong \mathcal{R}(S)$ induces a homeomorphism $H : S^* \rightarrow \sigma(\mathcal{B})$. To be precise, for any $s^* \in S^*$ (an ultrafilter in $\mathcal{B}$), we have

$$H(s^*) = \left\{ \hat{B} ; B \in s^* \right\}. \quad (1)$$

**Proposition 8.3** Let $S$ be a locally compact Hausdorff space, and suppose $\mathcal{B}$ is a generative subalgebra of $\mathcal{R}(S)$. For any $s^* \in S^*$, the intersection

$$\bigcap_{B \in s^*} \text{clos}(\hat{B}). \quad (2)$$

contains only a single element, which we will denote by $p(s^*)$. This determines a continuous surjective function $p : S^* \rightarrow \hat{S}$.

If $S$ itself is compact, then $\hat{S} = S$, so that $p$ is a continuous surjection from $S^*$ into $S$. In the special case when $\mathcal{B} = \mathcal{R}(S)$, the pair $(S^*, p)$ is called the **Gleason cover** (or the projective cover, or the absolute) of the space $S$.

**Proof of Proposition 8.3** Recall that each $s^*$ in $S^*$ is a filter in $\mathcal{B}$. Thus, by defining formula (1) and Lemma 7.4(a), $H(s^*)$ is a filter in $\hat{\mathcal{B}}$—hence, a filter of subsets of $\hat{S}$. Thus, the collection $\{ \text{clos}(\hat{B}) ; \hat{B} \in H(s^*) \}$ is a filter of closed subsets of $\hat{S}$, so it satisfies the Finite Intersection Property. Thus, the intersection (2) is nonempty, because $\hat{S}$ is compact.

To see that (2) is a singleton, let $\hat{s}_1$ be some element of (2), and let $\hat{s}_2$ be any other element of $\hat{S}$. There exists a disjoint open sets $\hat{O}_1, \hat{O}_2 \subseteq \hat{S}$ with $\hat{s}_1 \in \hat{O}_1$ and $\hat{s}_2 \in \hat{O}_2$ (because $\hat{S}$ is Hausdorff). Then there exists $\hat{B} \in \hat{\mathcal{B}}$ such that $\hat{s}_1 \in \hat{B} \subseteq \hat{O}_1$ (because $\mathcal{B}$ generates the topology of $\hat{S}$, because $\mathcal{B}$ is generative). Let $B := \hat{B} \cap S$; then $B \in \mathcal{B}$.

**Claim 1**: $B \in s^*$.

**Proof.** (by contradiction) Suppose $B \notin s^*$, then $\neg B \in s^*$, because $s^*$ is an ultrafilter.

But $\hat{s}_1 \in \hat{B}$, so $\hat{s}_1 \notin \text{clos}(\neg \hat{B}) = \text{clos}(\neg B)$; thus, $\hat{s}_1$ is *not* the intersection (2), which is a contradiction. To avoid this contradiction, we must have $B \in s^*$.

Now, $\hat{B} \subseteq \hat{O}_1$, so $\hat{B}$ is disjoint from $\hat{O}_2$. Thus, $\text{clos}(\hat{B})$ is also disjoint from $\hat{O}_2$ (because $\hat{O}_2$ is open). In particular, $\hat{s}_2 \notin \text{clos}(\hat{B})$. Thus, $\hat{s}_2$ is *not* an element of the intersection (2). This holds for all $\hat{s}_2 \neq \hat{s}_1$, so we conclude that $\hat{s}_1$ is the *only* element of (2). Thus, the function $p$ is well-defined.

$p$ is surjective. Let $\hat{s} \in \hat{S}$. Let $\mathcal{B}_\hat{s}$ be the set of all elements in $\hat{\mathcal{B}}$ containing $\hat{s}$. This is clearly a filter. Thus, by the Ultrafilter Theorem, it can be completed to an ultrafilter $\mathcal{U}$—i.e. an element of $\sigma(\hat{\mathcal{B}})$. Let $s^* \in S^*$ be the (unique) element such that $H(s^*) = \mathcal{U}$.

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18See e.g. [Wal74 §10.54, p.288], [John6 §3.10, p.107], or [PW88 Chap. 6]. We thank Vincenzo Marra for introducing us to Gleason covers.
For any \( s' \in \hat{S} \setminus \{\hat{s}\} \), there are disjoint open sets \( \hat{O}, \hat{O}' \subseteq \hat{S} \) such that \( \hat{s} \in \hat{O} \) and \( s' \in \hat{O}' \) (because \( \hat{S} \) is Hausdorff). Since \( \hat{B} \) is a base for the topology of \( \hat{S} \), there exists \( \hat{B} \in \hat{B}_s \) such that \( \hat{s} \in \hat{B} \subseteq \hat{O} \). Now, \( \text{clo}(\hat{B}) \) is disjoint from \( \hat{O}' \) (because \( \hat{O}' \) is open); thus \( s' \notin \text{clo}(\hat{B}) \). Thus, \( \hat{s} \) is not an element of \( \{2\} \). This holds for all \( s' \in \hat{S} \setminus \{\hat{s}\} \). But we have already established that \( \{2\} \) is nonempty; thus, we must have \( p(s^*) = \hat{s} \).

\( p \) is continuous. Let \( \hat{s} \in \hat{S} \) and let \( \hat{O} \subseteq \hat{S} \) be an open neighbourhood of \( \hat{s} \). Now \( \hat{S} \) is compact Hausdorff, hence locally compact; thus, there exists a compact subset \( \hat{K} \subseteq \hat{S} \) such that \( \hat{s} \in \hat{K} \subseteq \hat{O} \), where \( \hat{O}_1 := \text{int}(\hat{K}) \). There exists \( \hat{B} \in \hat{B}_s \) such that \( \hat{s} \in \hat{B} \subseteq \hat{O}_1 \) (because \( \hat{B} \) is a base for the topology of \( \hat{S} \)). Let \( \hat{B}^0 := \{ U \in \sigma(\hat{B}) \}; \hat{B} \in \{ U \} \}; \) as noted above, this is one of the elements in the clopen basis for the topology on \( \sigma(\hat{B}) \). Thus, if we define \( \hat{B}^1 := H^{-1}(\hat{B}^0) \), then \( \hat{B}^1 \) is an open subset of \( \hat{S}^* \) (because \( H \) is continuous).

Claim 2: \( p^{-1}(\{\hat{s}\}) \subseteq \hat{B}^1 \).

Proof. Let \( s^* \in p^{-1}(\{\hat{s}\}) \). Then \( p(s^*) = \hat{s} \), which, by defining formulae \( \{1\} \) and \( \{2\} \), means that \( \hat{s} \) is contained in \( \text{clo}(\hat{K}) \) for all \( \hat{K} \in H(s^*) \). Now, if \( \hat{B} \notin H(s^*) \), then we must have \( \neg \hat{B} \in H(s^*) \) (because \( H(s^*) \) is an ultrafilter in \( \hat{B} \)). But clearly \( \hat{s} \notin \text{clo}(\neg \hat{B}) \) (because \( \hat{s} \notin \hat{B} \)), so this is a contradiction. Thus, we must have \( \hat{B} \in H(s^*) \). Therefore, \( H(s^*) \in \hat{B}^0 \), so \( s^* \in \hat{B}^1 \).

Claim 3: \( p(\hat{B}^1) \subseteq \hat{O} \).

Proof. For any \( s^* \in \hat{B}^1 \), we have \( H(s^*) \in \hat{B}^0 \), which means \( \hat{B} \in H(s^*) \). Thus, equations \( \{1\} \) and \( \{2\} \) together imply that \( p(s^*) \in \text{clo}(\hat{B}) \). But \( \hat{B} \subseteq \hat{O}_1 \), so \( \text{clo}(\hat{B}) \subseteq \text{clo}(\hat{O}_1) \subseteq \hat{K} \subseteq \hat{O} \). Thus, \( p(s^*) \in \hat{O} \) for all \( s^* \in \hat{B}^1 \).

Claims 2 and 3 show that, for any open neighbourhood \( \hat{O} \subseteq \hat{S} \) around \( \hat{s} \), there is some open neighbourhood \( \hat{B}^1 \) around \( p^{-1}(\{\hat{s}\}) \) such that \( p(\hat{B}^1) \subseteq \hat{O} \). Thus, \( p \) is continuous at each point in \( p^{-1}(\{\hat{s}\}) \). This argument holds for all \( \hat{s} \in \hat{S} \); thus, \( p \) is continuous on \( \hat{S}^* \).

We will use the construction from Proposition \[8.3\] to give a new representation of integration with respect to a credence. For any \( g \in C_b(\hat{S}, \mathbb{R}) \), the Stone-Čech Extension Theorem yields a unique extension \( \hat{g} \in C(\hat{S}, \mathbb{R}) \) such that \( \hat{g}_{|S} = g \). Let \( \mathcal{B} \) be a Boolean subalgebra of \( \mathcal{P}(\hat{S}) \), and let \( S^* \) and \( p : S^* \longrightarrow \hat{S} \) be as in Proposition \[8.3\]. We then define \( g^* := \hat{g} \circ p : S^* \longrightarrow \mathbb{R} \). The transformation \( g \rightarrow g^* \) is a bounded linear function from \( C_b(\hat{S}, \mathbb{R}) \) into \( C(S^*, \mathbb{R}) \). Now let \( \mu \) be a credence on \( \mathcal{B} \). We can then define a probability charge \( \mu^* \) on the Boolean algebra \( \mathcal{C}_{lp}(S^*) \) as follows:

\[
\text{for all } B \in \mathcal{B}, \quad \mu^*[B^*] := \mu[B], \quad \text{where } B^* := \{ s^* \in S^*; B \in s^* \}. \quad (3)
\]

(Recall that the Stone Representation Theorem says that the map \( \mathcal{B} \rightarrow B^* \) is a Boolean algebra isomorphism from \( \mathcal{B} \) to \( \mathcal{C}_{lp}(S^*) \).) We now have everything we need for the next result. Recall the definitions of \( G_{\mathcal{B}}(S) \) and \( I = \{1\}_{\mathcal{B} \subseteq \mathcal{B}} \) from Section \[4\].

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Theorem 8.4 Let $\mathcal{S}$ be a locally compact Hausdorff space, let $\mathcal{B}$ be a generative Boolean subalgebra of $\mathcal{R}(\mathcal{S})$, let $\mu$ be a credence on $\mathcal{B}$, and let $\mu^*$ be the corresponding probability charge on $\mathcal{Clp}(\mathcal{S}^*)$. Then $\mu^*$ can be extended to a unique Borel probability measure on $\mathcal{S}^*$. Furthermore, for any $g \in \mathcal{G}_\mathcal{B}(\mathcal{S})$ and any $\mathcal{D} \in \mathcal{B}$, we have

$$\mathbb{I}_\mathcal{D}[g] = \int_{\mathcal{D}^*} g^* \, d\mu^*. \quad (4)$$

This representation theorem has two advantages over the representations from Sections 6 and 7. First, it applies to a credence defined on any generative Boolean subalgebra $\mathcal{B}$ of $\mathcal{R}(\mathcal{S})$. Second, the representation only requires a single Borel measure $\mu^*$, not an entire liminal structure. The disadvantage is that $\mu^*$ is defined on $\mathcal{S}^*$, a topological space even larger and more exotic than the Stone-Čech compactification $\hat{\mathcal{S}}$ (as shown by Proposition 8.3). The proof of Theorem 8.4 depends on two lemmas. The first one is a straightforward variant of the Kolmogorov Consistency Theorem, but for completeness, we provide a proof.

Lemma 8.5 Let $\mathcal{T}$ be any Stonean space, and let $\mu$ be a probability charge on $\mathcal{Clp}(\mathcal{T})$. Then there is a unique Borel probability measure $\nu$ on $\mathcal{T}$ which extends $\mu$.

Proof. Let $\mathcal{F}_0 := \{1_B; \ B \in \mathcal{Clp}(\mathcal{T})\}$, and let $\mathcal{F}$ be the set of finite linear combinations of elements in $\mathcal{F}_0$. Then $\mathcal{F}$ is an algebra of continuous functions on $\mathcal{T}$, because the sum or product of any two elements of $\mathcal{F}$ is also an element of $\mathcal{F}$ (because $\mathcal{Clp}(\mathcal{T})$ is a Boolean algebra under standard set-theoretic operations).

Claim 1: $\mathcal{F}$ separates points.

Proof. For any distinct $s, t \in \mathcal{T}$, there is an open neighbourhood around $s$ which excludes $t$ (because $\mathcal{T}$ is Hausdorff). Thus, there is a clopen neighbourhood $\mathcal{B}$ around $s$ which excludes $t$ (because $\mathcal{T}$ is totally disconnected). Thus, $1_B$ separates $s$ from $t$. $\diamondsuit$ Claim 1

Combining Claim 1 with the Stone-Weierstrass theorem, we deduce that $\mathcal{F}$ is dense in $\mathcal{C}(\mathcal{T}, \mathbb{R})$ in the uniform topology. Define $\mathbb{I}^\mu : \mathcal{F} \to \mathbb{R}$ by

$$\mathbb{I}^\mu \left( \sum_{n=1}^N r_n 1_{B_n} \right) := \sum_{n=1}^N r_n \mu[B_n], \quad \text{for any } \sum_{n=1}^N r_n 1_{B_n} \in \mathcal{F}. $$

Then $\mathbb{I}^\mu$ is a bounded linear functional on $\mathcal{F}$. But $\mathcal{F}$ is dense in $\mathcal{C}(\mathcal{T}, \mathbb{R})$, so $\mathbb{I}^\mu$ extends to a unique bounded linear functional $\mathbb{I}^\mu : \mathcal{C}(\mathcal{T}, \mathbb{R}) \to \mathbb{R}$. Since $\mathcal{T}$ is compact Hausdorff, the Riesz Representation Theorem yields a unique Borel probability measure $\nu$ such that $\mathbb{I}^\mu(f) = \int_{\mathcal{T}} f \, d\nu$ for all $f \in \mathcal{C}(\mathcal{T}, \mathbb{R})$. In particular, for any $\mathcal{B} \in \mathcal{Clp}(\mathcal{T})$, we have $\mu[\mathcal{B}] = \mathbb{I}^\mu[1_B] = \int_{\mathcal{T}} 1_B \, d\nu = \nu[\mathcal{B}]$, so $\nu$ extends $\mu$, as claimed. $\square$

Lemma 8.6 Let $s^* \in \mathcal{S}^*$ and let $\hat{s} \in \hat{\mathcal{S}}$. Then $p(s^*) = \hat{s}$ if and only if, for every $\hat{\mathcal{B}} \in \hat{\mathcal{B}}$ with $\hat{s} \in \hat{\mathcal{B}}$, we have $(\mathcal{S} \cap \hat{\mathcal{B}}) \in s^*$. 47
Proof. For any $\mathcal{B} \in \mathfrak{B}$, let $\hat{\mathcal{B}}$ denote the (unique) element of $\hat{\mathfrak{B}}$ such that $S \cap \hat{\mathcal{B}} = \mathcal{B}$.

"$\implies$" By defining formula (2), $p(s^*) = \hat{s}$ if $\hat{s} \in \text{clo}(\hat{Q})$ for all $Q \in s^*$ —or equivalently, if every open neighbourhood of $\hat{s}$ overlaps $\hat{Q}$ for every $Q \in s^*$. In particular, if $\hat{\mathcal{B}} \in \hat{\mathfrak{B}}$ is a neighbourhood of $\hat{s}$, then $\hat{\mathcal{B}}$ overlaps $\hat{Q}$ for every $Q \in s^*$. If $\mathcal{B} := S \cap \hat{\mathcal{B}}$, then $\neg\mathcal{B} = S \setminus (\neg\hat{\mathcal{B}})$; thus, $\neg\mathcal{B}$ is not an element of $s^*$ (because $(\neg\hat{\mathcal{B}}) \cap \hat{\mathcal{B}} = \emptyset$). Since $s^*$ is an ultrafilter, this means that $\mathcal{B}$ itself must be an element of $s^*$, as claimed. This argument holds for any $\hat{\mathcal{B}} \in \hat{\mathfrak{B}}$ with $\hat{s} \in \hat{\mathcal{B}}$.

"$\impliedby$" Let $\hat{\mathcal{O}} \subseteq \hat{S}$ be any open neighbourhood of $\hat{s}$. Since $\hat{\mathfrak{B}}$ generates the topology on $\hat{S}$, there is some $\hat{\mathcal{B}} \in \hat{\mathfrak{B}}$ such that $\hat{s} \in \hat{\mathcal{B}} \subseteq \hat{\mathcal{O}}$; thus, if $\mathcal{B} := S \cap \hat{\mathcal{B}}$, then $\mathcal{B} \in s^*$. Now, let $Q \in s^*$ be arbitrary. Then $Q \cap \mathcal{B} \neq \emptyset$ (because $s^*$ is a filter). Thus, $Q \cap \hat{\mathcal{O}} \neq \emptyset$ (because $\mathcal{B} \in \hat{\mathcal{B}} \cap \hat{\mathcal{O}}$). This shows that every element of $s^*$ intersects $\hat{\mathcal{O}}$. This argument works for any open neighbourhood $\hat{\mathcal{O}}$ of $\hat{s}$; thus, every open neighbourhood of $\hat{s}$ overlaps every element of $s^*$; hence (2) yields $p(s^*) = \hat{s}$.

\[ \square \]

Proof of Theorem 8.4. The fact that $\mu^*$ can be extended to a unique Borel probability measure on $S^*$ follows immediately from Lemma 8.3. It remains to prove equation (4).

Invoking the canonical homeomorphism between $S^*$ and $\sigma(\hat{\mathfrak{B}})$ defined by formula (1), we will assume for simplicity that $S^* = \sigma(\hat{\mathfrak{B}})$. Thus, each point in $S^*$ is identified with an ultrafilter in the Boolean algebra $\hat{\mathfrak{B}}$, so Lemma 8.6 takes the following simpler form:

Claim 1: For any $s^* \in S^*$ and $\hat{s} \in \hat{S}$, we have $p(s^*) = \hat{s}$ if and only if every $\hat{\mathcal{B}}$-neighbourhood of $\hat{s}$ is an element of $s^*$.

For any $g \in C_0(S, \mathbb{R})$, let $\hat{g} \in C(\hat{S}, \mathbb{R})$ be the (unique) Stone-Cech extension of $g$.

Claim 2: If $g \in G_{\mathfrak{B}}(S)$, then $\hat{g} \in G_{\hat{\mathfrak{B}}}(\hat{S})$.

Proof. Every function in $G_{\mathfrak{B}}(S)$ is a uniform limit of linear combinations of functions in $C_{\mathfrak{B}}(S)$, and the transformation $C(S, \mathbb{R}) \ni f \to \hat{f} \in C(\hat{S}, \mathbb{R})$ is linear and continuous. So it suffices to show that $\hat{g} \in C_{\hat{\mathfrak{B}}}(\hat{S})$ whenever $g \in C_{\mathfrak{B}}(S)$. So, suppose $g \in C_{\mathfrak{B}}(S)$. Let $r \in \mathbb{R}$, and let $\hat{\mathcal{B}} := \text{int}(\hat{g}^{-1}(-\infty, r])$. Then $\hat{\mathcal{B}}$ is a regular open subset of $\hat{S}$ (because $\hat{g}$ is continuous). Let $\mathcal{B} := S \cap \hat{\mathcal{B}}$. Then

\[ \mathcal{B} = S \cap \text{int}(\hat{g}^{-1}(-\infty, r]) = \text{int}(S \cap \hat{g}^{-1}(-\infty, r]) = \text{int}(g^{-1}(-\infty, r]) \in \mathfrak{B}. \tag{5} \]

Here, (*) is because $S$ is an open subset of $\hat{S}$ (because $S$ is locally compact), (†) is because $g := \hat{g}_S$ by the definition of $\hat{g}$, and the last step is because $g \in C_{\mathfrak{B}}(S)$.

But $\hat{\mathfrak{B}}$ was defined using the isomorphism from Lemma 7.4(a); in other words, $\hat{\mathfrak{B}} = \{\hat{\mathcal{R}} \in \mathfrak{R}(\hat{S}); S \cap \hat{\mathcal{R}} \in \mathfrak{B}\}$. Thus, equation (5) establishes that $\hat{\mathcal{B}} \in \hat{\mathfrak{B}}$.

By a very similar argument, $\text{int}(\hat{g}^{-1}[r, \infty)) \in \hat{\mathfrak{B}}$. This argument works for all $r \in \mathbb{R}$; thus, $\hat{g} \in C_{\hat{\mathfrak{B}}}(\hat{S})$, as desired.

\[ \Diamond \text{ Claim 2} \]
For any $\mathring{B} \in \mathfrak{B}$, define $B^* := \{s^* \in S^*; \mathring{B} \in s^*\}$. This is a clopen subset of $S^*$. The function $\mathring{B} \mapsto B^*$ is a Boolean algebra isomorphism from $\mathfrak{B}$ to $\mathfrak{Cl}(S^*)$ (by the Stone Representation Theorem).

**Claim 3:** If $\mathring{B} \in \mathfrak{B}$ then $p^{-1}(\mathring{B}) \subseteq B^* \subseteq p^{-1}[\text{clos}(\mathring{B})]$.\footnote{It is tempting to think that $p^{-1}(\mathring{B}) = B^*$ for all $\mathring{B} \in \mathfrak{B}$. But this cannot be true. To see this, note that $(-\mathring{B})^* = (B^*)^c$ (by the Stone Representation Theorem). Thus, if $p^{-1}(\mathring{B}) = B^*$ and $p^{-1}(-\mathring{B}) = (-\mathring{B})^* = (B^*)^c$, then we would have $p^{-1}(\partial B) = \emptyset$, contradicting the surjectivity of $p$.}

**Proof.** Let $s^* \in S^*$, and let $\mathring{s} := p(s^*)$. Then $(s^* \in p^{-1}(B)) \iff (\mathring{s} \in \mathring{B}) \iff (B \in s^*) \iff (s^* \in B^*)$, where "\iff" is by Claim 1. Thus, $p^{-1}(\mathring{B}) \subseteq B^*$.

Now let $s^* \in B^*$; then $\mathring{B} \in s^*$. Let $\mathring{\mathcal{O}} \subseteq \mathring{S}$ be any open neighbourhood of $\mathring{s}$. Then $\mathring{\mathcal{O}}$ contains a $\mathfrak{B}$-neighbourhood $\mathring{\mathcal{Q}}$ of $\mathring{s}$, because $\mathfrak{B}$ is a base for the topology of $\mathring{S}$. Now, $\mathring{\mathcal{Q}} \in s^*$ by Claim 1 and thus, $\mathring{\mathcal{Q}} \cap \mathring{B} \neq \emptyset$, because $s^*$ is a filter. Thus, $\mathring{\mathcal{O}} \cap \mathring{B} \neq \emptyset$, because $\mathring{\mathcal{Q}} \subseteq \mathring{\mathcal{O}}$. Thus, $\mathring{B}$ overlaps every neighbourhood of $\mathring{s}$; thus, $\mathring{s} \in \text{clos}(\mathring{B})$. Thus, $s^* \in p^{-1}[\text{clos}(\mathring{B})]$. This shows that $B^* \subseteq p^{-1}[\text{clos}(\mathring{B})]$. \hfill \diamond Claim 3

Recall from Section 4 that we compute $\mathbb{I}^\mu_S(g)$ by approximating $g$ from below by $\mathfrak{B}$-simple functions. Thus we need to translate $\mathfrak{B}$-simple functions over $S$ into $\mathfrak{B}$-simple functions on $\mathring{S}$ and into $\mathfrak{Cl}(S^*)$-simple functions on $S^*$. It will be convenient to work with a particular class of simple function. A function $\mathring{f} : \mathring{S} \rightarrow \mathbb{R}$ is a $\mathfrak{B}$-pyramidal function if there exists a nested sequence of subsets $\mathring{S} = \mathring{B}_0 \supseteq \mathring{B}_1 \supseteq \cdots \supseteq \mathring{B}_N$, with $\mathring{B}_1, \ldots, \mathring{B}_N \in \mathfrak{B}$, and real numbers $r_0 \in \mathbb{R}$ and $r_1, \ldots, r_N \in \mathbb{R}_+$ such that

$$\mathring{f} = \sum_{n=0}^{N} r_n 1_{\mathring{B}_n}.$$  \hfill (6)

For notational convenience, we define $\mathring{B}_{N+1} := \emptyset$. For all $n \in \{0\ldots N\}$, let $R_n := r_0 + \cdots + r_n$; then $\mathring{f}(\mathring{b}) = R_n$ for all $\mathring{b} \in \mathring{B}_n \cap (-\mathring{B}_{n+1})$. Observe that $R_0 < R_1 < \cdots < R_N$ (hence the term “pyramidal”). Let $\mathring{F}$ be the set of all $\mathfrak{B}$-pyramidal functions on $\mathring{S}$. Likewise, let $F'$ be the set of all $\mathfrak{B}$-pyramidal functions on $S$. Finally, let $F^*$ be the set of all $\mathfrak{Cl}(S^*)$-pyramidal functions on $S^*$.

Let $\mathring{f} \in \mathring{F}$ be as in formula (6), where $\mathring{B}_0 = \mathring{S}$ and $\mathring{B}_1, \ldots, \mathring{B}_N \in \mathfrak{B}$. For all $n \in \{1\ldots N\}$, let $B^*_n := \{s^* \in S^*; \mathring{B}_n \in s^*\}$. Define

$$f^* := \sum_{n=0}^{N} r_n 1_{B^*_n} \text{ and } \mathring{f} := \sum_{n=0}^{N} r_n 1_{\text{clos}(\mathring{B}_n)}.$$  \hfill (7)

where of course, $B^*_0 = S^*$ and $\text{clos}(\mathring{B}_0) = \mathring{S}$.

**Claim 4:**

(a) The function $\mathring{f} \mapsto \mathring{f}_{\mid \mathring{S}}$ is a bijection from $\mathring{F}$ to $F'$.

(b) The function $\mathring{f} \mapsto f^*$ is a bijection from $\mathring{F}$ to $F^*$.

(c) For any $\mathring{f} \in \mathring{F}$, we have $\mathring{f} \circ p \leq f^* \leq \mathring{f} \circ p$.
Proof. Part (a) follows from the fact that the map \( \hat{B} \mapsto \hat{B} \cap \mathcal{S} \) is a bijection from \( \hat{\mathfrak{B}} \) to \( \mathfrak{B} \), as shown by Lemma 4.1(a). Part (b) follows from the Stone Representation Theorem. Part (c) follows from Claim 3.

Claim 5: Let \( \hat{f} \in \hat{\mathcal{F}} \) and let \( \hat{g} \in \mathcal{C}(\hat{\mathcal{S}}, \mathbb{R}) \). If \( \hat{f} \leq \hat{g} \), then \( \mathcal{F} \leq \mathcal{G} \).

Proof. Suppose \( \hat{f} \in \hat{\mathcal{F}} \) is as in formula (6), where \( \hat{B}_0 = \hat{\mathcal{S}} \) and \( \hat{B}_1, \ldots, \hat{B}_N \in \hat{\mathfrak{B}} \); thus, \( \mathcal{F} \) is as in formula (7) (right). Let \( \hat{s} \in \hat{\mathcal{S}} \); we must show that \( \mathcal{F}(\hat{s}) \leq \mathcal{G}(\hat{s}) \). Let \( \hat{N}+1 := \emptyset \).

Observe that \( \hat{\mathcal{S}} = \text{clos}(\hat{B}_0) \supseteq \text{clos}(\hat{B}_1) \supseteq \cdots \supseteq \text{clos}(\hat{B}_N) \). Thus, if \( \hat{s} \in \text{clos}(\hat{B}_m) \), then \( \hat{s} \in \text{clos}(\hat{B}_n) \) for all \( n \leq m \). Let \( m := \max\{n \in [0 \ldots N] : \hat{s} \in \text{clos}(\hat{B}_n)\} \), and let \( R := r_0 + \cdots + r_m \). Then there is a net \( (\hat{b}_j)_{j \in \mathcal{J}} \) in \( \hat{B}_m \) converging to \( \hat{s} \) (where \( \mathcal{J} \) is some directed set). Now, \( \hat{s} \notin \text{clos}(\hat{B}_{m+1}) \), so \( -\hat{B}_{m+1} \) is an open neighbourhood of \( \hat{s} \), so there is some \( j_0 \in \mathcal{J} \) such that for all \( j > j_0 \), we have \( \hat{b}_j \in \hat{B}_m \cap (-\hat{B}_{m+1}) \), and thus, \( \hat{f}(\hat{b}_j) = R \), which means \( \hat{R} \leq \hat{g}(\hat{b}_j) \) (because \( \hat{f} \leq \hat{g} \)). Thus, \( \hat{R} \leq \hat{g}(\hat{s}) \), because \( \hat{g} \) is continuous and \( (\hat{b}_j)_{j \in \mathcal{J}} \) converges to \( \hat{s} \). But \( \hat{f}(\hat{s}) = R \), because \( \hat{s} \in \text{clos}(\hat{B}_m) \backslash \text{clos}(\hat{B}_{m+1}) \). Thus, \( \mathcal{F}(\hat{s}) \leq \mathcal{G}(\hat{s}) \). This argument holds for all \( \hat{s} \in \hat{\mathcal{S}} \); thus, \( \mathcal{F} \leq \mathcal{G} \), as claimed.

Let \( \mathcal{F} \) be the set of all \( \mathfrak{B} \)-simple functions on \( \mathcal{S} \). For any \( g \in \mathcal{G}_\mathfrak{B}(\mathcal{S}) \), define \( \mathcal{F}_g := \{f \in \mathcal{F} : f \leq g\} \), as in Section 4. Meanwhile, define \( \mathcal{F}_g' := \{f \in \mathcal{F}' : f \leq g\} \). Note that \( \mathcal{F}_g' \subseteq \mathcal{F}_g \).

For any \( f \in \mathcal{F} \) and any \( D \in \mathfrak{B} \), define \( \int_D f \, d\mu = \int_D f \, d\mu \) as in formula (4).

Claim 6: For any \( f \in \mathcal{F}_g \), there exists \( f' \in \mathcal{F}_g' \) such that \( \int_D f' \, d\mu = \int_D f \, d\mu \) for all \( D \in \mathfrak{B} \).

Proof. By definition, there is a \( \mathfrak{B} \)-partition \( \{\mathcal{P}_1, \ldots, \mathcal{P}_L\} \) of \( \mathcal{S} \) such that \( f \) is constant on each cell of the partition. Let \( R_0 < R_1 < \cdots < R_N \) be the (finite) set of values which \( f \) takes on these cells. For all \( n \in [0 \ldots N] \), let \( \mathcal{L}_n := \{\ell \in [1 \ldots L] : \mathcal{P}_\ell \) takes the value \( R_n \) on \( \mathcal{P}_\ell\} \), and then define \( \mathcal{P}_n' := \bigvee_{\ell \in \mathcal{L}_n} \mathcal{P}_\ell \); thus, \( \mathcal{P}_1', \ldots, \mathcal{P}_N' \in \mathfrak{B} \) are disjoint. Finally, for all \( n \in [0 \ldots N] \), define \( \mathcal{B}_n := \mathcal{P}_n' \vee \mathcal{P}_{n+1}' \vee \cdots \vee \mathcal{P}_N' \in \mathfrak{B} \). Then \( \mathcal{S} = \mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \cdots \supseteq \mathcal{B}_N \). Meanwhile, define \( r_0 := R_0 \), and for all \( n \in [1 \ldots N] \), let \( r_n := R_n - R_{n-1} \). Thus, for any \( n \in [1 \ldots N] \), \( r_n > 0 \), and \( R_n = r_0 + \cdots + r_n \). Finally, define
\[
f' := \sum_{n=0}^N r_n 1_{\mathcal{B}_n}.
\]

Then \( f' \in \mathcal{F}' \), and for any \( D \in \mathfrak{B} \), we have
\[
\int_D f' \, d\mu = \sum_{n=0}^N \int_D r_n 1_{\mathcal{B}_n} \, d\mu = \sum_{n=0}^N r_n \mu[D \cap \mathcal{B}_n] = \sum_{n=0}^N r_n \mu[D \cap \bigvee_{m=n}^N \mathcal{P}_m']
\]
\[
= \sum_{n=0}^N r_n \left( \sum_{m=n}^N \mu[D \cap \mathcal{P}_m'] \right) = \sum_{n=0}^N r_n \mu[D \cap \mathcal{P}_m'] 
\]
\[
= \sum_{m=0}^N \sum_{n \leq m} r_n \mu[D \cap \mathcal{P}_m'] = \sum_{m=0}^N R_m \mu[D \cap \mathcal{P}_m']
\]
Proof. (a) If $\text{Claim 7:}$

For any $p$, must show that transformation $\ell$ is injective by Claim 4(a). To see that it is surjective, let $\hat{f} \mapsto f^*$ is a bijection from $\hat{F}_g$ to $F'_g$. By dropping to a subnet if necessary, we can find some $\ell \in [1 \ldots L]$ such that $q_j \in P_\ell$ for all $j \in J$. Find the unique $n \in [1 \ldots N]$ such that $\ell \in \mathcal{L}_n$. Then $P_\ell \subseteq P_n$, so for all $j \in J$, we have $q_j \in P'_n$ and $f(q_j) = R_n$. Thus, $R_n \leq g(q_j)$ for all $j \in J$ (because $f \leq g$), and thus $R_n \leq g(s)$, because $g$ is continuous, and $(q_j)_{j \in J}$ converges to $s$.

Now, the net $(q_j)_{j \in J}$ is a subset of $P'_n$, which is disjoint from the open set $B_{n+1}$. Thus, $s \notin B_{n+1}$. So, let $f' \leq g$, and thus, $f' \in F'_g$. Thus, $f'(s) \leq g(s)$. But we have already established that $R_n \leq g(s)$; thus, $f'(s) \leq g(s)$, as desired.

This argument holds for all $s \in S$; thus, $f' \leq g$, and thus, $f' \in F'_g$. \hfill $\Box$ Claim 6

For any $\hat{g} \in C(\hat{S}, \mathbb{R})$, let $\hat{F}_g := \{ \hat{f} \in \hat{F}; \hat{f} \leq \hat{g} \}$. Likewise, for any $g \in C(S^*, \mathbb{R})$, let $\hat{F}_g^* := \{ f \in F^*; f \leq g \}$.

Claim 7: Let $g \in G_B(S)$, and let $\hat{g}$ be the unique extension of $g$ to $G_B(S)$, and let $g^* := \hat{g} \circ p$.

(a) The function $\hat{f} \mapsto \hat{f}|_S$ is a bijection from $\hat{F}_g$ to $F'_g$.

(b) The function $\hat{f} \mapsto f^*$ is a bijection from $\hat{F}_g$ to $F^*_g$.

Proof. (a) If $\hat{f} \in \hat{F}_g$, then $\hat{f} \leq \hat{g}$, and hence $\hat{f}|_S \leq \hat{g}|_S = g$. Thus, $\hat{f}|_S \in F'_g$. The transformation $\hat{f} \mapsto \hat{f}|_S$ is injective by Claim 4(a). To see that it is surjective, let $f \in F'_g$. The surjectivity in Claim 4(a) yields a unique $\hat{f} \in \hat{F}$ such that $(\hat{f})|_S = f$. We must show that $\hat{f} \in \hat{F}_g$. So, let $\hat{s} \in \hat{S}$; we must show that $\hat{f}(\hat{s}) \leq \hat{g}(\hat{s})$.

Suppose $\hat{f}$ is as in formula (8), where $\hat{B}_0 = \hat{S}$ and $\hat{B}_1, \ldots, \hat{B}_N \in \hat{B}$. Let $\hat{B}_{N+1} := \emptyset$. Let $m := \max \{ n \in [0 \ldots N]; \hat{s} \in \text{clos} (\hat{B}_n) \}$, and let $R := r_0 + \cdots + r_m$. Let $B_m := S \cap \hat{B}_m$. Then $B_m$ is a dense subset of $\hat{B}_m$, because $S$ is a dense subset of $\hat{S}$, and $\hat{B}_m$ is an open subset of $\hat{S}$. Thus, $\text{clos}(B_m) = \text{clos}(\hat{B}_m)$. But $\hat{s} \in \text{clos}(\hat{B}_m)$, so there is a net $(b_j)_{j \in J}$ in $B_m$ converging to $\hat{s}$ (where $J$ is some directed set). We have $f(b_j) \leq g(b_j)$ for all $j \in J$, because $f \leq g$ (because $f \in F'_g$). Now, $\hat{s} \notin \text{clos}(\hat{B}_{m+1})$, so $-\hat{B}_{m+1}$ is an open neighbourhood of $\hat{s}$. Thus, there is some $j_0 \in J$ such that for all $j > j_0$, we have $b_j \in B_m \cap (\hat{B}_{m+1})$, and thus, $f(b_j) = R$, which means $R \leq g(b_j)$ (because $f \leq g$), and hence $R \leq \hat{g}(b_j)$ (because $\hat{g}|_S = g$). Thus, $R \leq \hat{g}(\hat{s})$, because $\hat{g}$ is continuous and $(b_j)_{j \in J}$ converges to $\hat{s}$. But $\hat{f}(\hat{s}) \leq R$ (because $\hat{s} \in -\hat{B}_{m+1}$), so this means that $\hat{f}(\hat{s}) \leq \hat{g}(\hat{s})$, as desired.

This argument works for all $\hat{s} \in \hat{S}$; thus $\hat{f} \leq \hat{g}$, and thus, $\hat{f} \in \hat{F}_g$, as desired.
(b) If \( \hat{f} \in \hat{F}_g \), then \( f \leq \hat{g} \). Thus, \( \hat{f} \leq \hat{g} \), by Claim 4. Thus, \( \hat{f} \circ p \leq \hat{g} \circ p \). But \( f^* \leq f \circ p \) by Claim 4(c), while \( \hat{g} \circ p = g^* \) by definition. Thus, \( f^* \leq g^* \). Thus, \( f^* \in F_g^* \). The function \( \hat{f} \mapsto f^* \) is injective by Claim 4(b). It remains to show that it is surjective.

Let \( f_0 \in F_g^* \). The surjectivity in Claim 4(b) yields \( \hat{f} \in \hat{F} \) such that \( f_0 = f^* \). We have

\[
\hat{f} \circ p \leq f^* = f_0 \leq g^* = \hat{g} \circ p,
\]

where \((*)\) is by Claim 4(c), \((\dagger)\) is because \( f_0 \in F_g^* \), and the equalities are true by the definitions of the functions in question. But \( p \) is surjective, so inequality \((\ddagger)\) implies that \( \hat{f} \leq \hat{g} \). Thus, \( \hat{f} \in \hat{F}_g \), as desired.

For any \( f \in F \), define \( \int_S f \, d\mu \) as in formula (4). Meanwhile, for any \( f \in F^* \), let \( \int_{S^*} f \, d\mu^* \) be the standard Lebesgue integral with respect to the Borel measure \( \mu^* \).

**Claim 8:** For any \( \hat{f} \in \hat{F} \) and \( D \in \mathfrak{B} \), we have

\[
\int_D \hat{f}_S \, d\mu = \int_{D^*} f^* \, d\mu^*.
\]

**Proof.** Suppose \( \hat{f} \in \hat{F} \) is as in formula (4), where \( \hat{B}_0 = \hat{S} \) and \( \hat{B}_1, \ldots, \hat{B}_N \in \hat{B} \). For all \( n \in [1 \ldots N] \), let \( B_n := \hat{B}_n \cap S \). Then \( B_n \in \mathfrak{B} \), and for any \( \mathcal{D} \in \mathfrak{B} \), we have

\[
\int_D \hat{f}_S \, d\mu \equiv \sum_{n=0}^N r_n \mu[\mathcal{D} \cap B_n] \equiv \sum_{n=0}^N r_n \mu^*[(\mathcal{D} \cap B_n)^*]
\]

as claimed. Here, \((*)\) is by the linearity of \( \hat{f}_S \) from Lemma 4.7(a), because \( \hat{f}_S = \sum_{n=0}^N r_n 1_{B_n} \). Meanwhile, \((\dagger)\) is by defining formula (3), and \((\ddagger)\) is because the transformation \( B \mapsto B^* \) is a Boolean algebra homomorphism from \( \mathfrak{B} \) to \( \mathfrak{C}(S^*) \). Finally, \((\circ)\) is by equation (7) (left) and the linearity of the Lebesgue integral.

Now let \( g \in \mathcal{G}(S) \) and let \( \mathcal{D} \in \mathfrak{B} \). Then

\[
\Pi^\mu_D(g) \equiv \sup_{f \in F_g} \int_D f \, d\mu = \sup_{h \in \hat{F}_g} \int_D h \, d\mu = \sup_{\hat{f} \in \hat{F}_g} \int_D \hat{f}_S \, d\mu
\]

\[
\equiv \sup_{\hat{f} \in \hat{F}_g} \int_{D^*} f^* \, d\mu^* = \sup_{h \in \hat{F}_{g^*}} \int_{D^*} h \, d\mu^* = \int_{D^*} g^* \, d\mu^*;
\]

which proves equation (4). Here, \((*)\) is by defining formula (7), and \((\#)\) is by Claim 6. Next, \((\dagger)\) is by Claim 7(a), and \((\circ)\) is by Claim 8. Next \((\ddagger)\) is by Claim 7(b). Finally, to see \((\circ)\), recall that we showed in the proof of Lemma 8.5 that \( F^* \) is uniformly dense in \( \mathcal{C}(S^*, \mathbb{R}) \), by invoking the Stone-Weierstrass Theorem. Thus, we can uniformly approximate \( g^* \) from below with elements of \( F_g^* \). Thus, \((\circ)\) follows from the fact that \( \mu^* \)-integration is continuous with respect to the uniform norm on \( \mathcal{C}(S^*, \mathbb{R}) \). \( \square \)
9 A categorical perspective

A \textit{credence space} is an ordered triple \((S, \mathcal{B}, \mu)\), where \(S\) is a locally compact Hausdorff space, \(\mathcal{B}\) is a generative Boolean subalgebra of \(\mathcal{R}(S)\), and \(\mu\) is a credence on \(\mathcal{B}\). If \(C_1 = (S_1, \mathcal{B}_1, \mu_1)\) and \(C_2 = (S_2, \mathcal{B}_2, \mu_2)\) are two credence spaces, then a \textit{morphism} from \(C_1\) to \(C_2\) is a continuous function \(\phi : S_1 \rightarrow S_2\) such that \(\phi^{-1} : \mathcal{B}_2 \rightarrow \mathcal{B}_1\) is a Boolean algebra homomorphism and \(\mu(\mu_1) = \mu_2\). It is easily verified that the composition of two such morphisms is also a credence space morphism. Let \(\text{Cred}\) be the category of credence spaces and their morphisms.

Let \(\text{CmpCrd}\) be the full subcategory of \(\text{Cred}\) consisting of all \textit{compact} credence spaces and their morphisms. Via Stone-Čech compactification, we can define a functor from \(\text{Cred}\) into \(\text{CmpCrd}\) as follows. Let \(C = (S, \mathcal{B}, \mu)\) be a credence space. Lemma 7.4(a) yields a Boolean algebra isomorphism \(h : \mathcal{R}(\hat{S}) \cong \mathcal{R}(S)\) given by \(h(\hat{R}) = \hat{R} \cap S\). Let \(\hat{\mathcal{B}} := h^{-1}(\mathcal{B})\); then \(\hat{\mathcal{B}}\) is a Boolean subalgebra of \(\mathcal{R}(\hat{S})\), and is isomorphic to \(\mathcal{B}\) via \(h\). For any \(\hat{B} \in \hat{\mathcal{B}}\), define \(\hat{\mu}[\hat{B}] := \mu[\hat{B} \cap S]\); then \(\hat{\mu}\) is a credence on \(\hat{\mathcal{B}}\). The triple \(\hat{C} := (\hat{S}, \hat{\mathcal{B}}, \hat{\mu})\) is a compact credence space, which we will call the \textit{Stone-Čech compactification} of \(C\).

Now let \(C_1 = (S_1, \mathcal{B}_1, \mu_1)\) and \(C_2 = (S_2, \mathcal{B}_2, \mu_2)\) be two credence spaces, and let \(\phi : S_1 \rightarrow S_2\) be a credence space morphism. Since \(S_2 \subseteq \hat{S}_2\), we can regard \(\phi\) as a continuous function from \(S_1\) into the compact space \(\hat{S}_2\). The Stone-Čech Extension Theorem yields a unique extension to a continuous function \(\hat{\phi} : \hat{S}_1 \rightarrow \hat{S}_2\).

\textbf{Proposition 9.1} \(\hat{\phi}\) is a credence space morphism from \(\hat{C}_1\) to \(\hat{C}_2\). The transformation \(\beta\) defined by \(C \mapsto \hat{C}\) and \(\phi \mapsto \hat{\phi}\) is a faithful functor from \(\text{Cred}\) to \(\text{CmpCrd}\).

\textit{Proof.} We already know that \(\hat{\phi}\) is continuous. We must show that \(\hat{\phi}^{-1} : \hat{\mathcal{B}}_2 \rightarrow \hat{\mathcal{B}}_1\) is a Boolean algebra homomorphism and that \(\hat{\phi}[\hat{\mu}_1] = \hat{\mu}_2\).

\textit{Homomorphism.} Recall that the Boolean algebra isomorphisms \(h_1 : \mathcal{R}(\hat{S}_1) \cong \mathcal{R}(S_1)\) and \(h_2 : \mathcal{R}(\hat{S}_1) \cong \mathcal{R}(S_2)\) from Lemma 7.4(a) are defined by \(h_1(\hat{R}_1) := \hat{R}_1 \cap S_1\) and \(h_2(\hat{R}_2) := \hat{R}_2 \cap S_2\) for all \(\hat{R}_1 \in \mathcal{R}(\hat{S}_1)\) and \(\hat{R}_2 \in \mathcal{R}(\hat{S}_2)\). Furthermore, \(\hat{\mathcal{B}}_1 := h_1^{-1}(\mathcal{B}_1)\) and \(\hat{\mathcal{B}}_2 := h_2^{-1}(\mathcal{B}_2)\), so by restricting \(h_1\) and \(h_2\), we get isomorphisms \(h_1 : \hat{\mathcal{B}}_1 \cong \mathcal{B}_1\) and \(h_2 : \hat{\mathcal{B}}_2 \cong \mathcal{B}_2\). Let \(h_1 : \hat{\mathcal{B}}_1 \cong \mathcal{B}_1\) be the inverse of \(h_1\) — this is also an isomorphism. Finally, \(\phi^{-1} : \mathcal{B}_2 \rightarrow \mathcal{B}_1\) is a Boolean algebra homomorphism because \(\phi\) is a credence space morphism. Thus, the fact that \(\hat{\phi}^{-1} : \hat{\mathcal{B}}_2 \rightarrow \hat{\mathcal{B}}_1\) is a Boolean algebra homomorphism is an immediate consequence of the following commuting diagram

\[
\begin{array}{ccc}
\hat{\mathcal{B}}_2 & \xrightarrow{h_2} & \mathcal{B}_2 \\
\hat{\phi}^{-1} \downarrow & & \downarrow \phi^{-1} \\
\hat{\mathcal{B}}_1 & \xleftarrow{h_1} & \mathcal{B}_1
\end{array}
\]

\footnote{Lemma 5.1(b) and Remark 5.5 provide sufficient conditions for \(\phi^{-1}\) to be a Boolean algebra homomorphism from \(\mathcal{B}_2\) to \(\mathcal{B}_1\).}
Measure-preserving. Let \( \hat{B}_2 \in \hat{B} \). If \( B_2 := h_2(\hat{B}_2) \), then \( \hat{\mu}_2[\hat{B}_2] = \mu_2[B_2] \). Likewise, if \( \hat{B}_1 := \hat{\phi}^{-1}(\hat{B}_2) \) and \( B_1 := h_1(\hat{B}_1) \), then \( \hat{\mu}_1[\hat{B}_1] = \mu_1[B_1] \). But from the above commuting diagram, we see that \( \hat{\phi}^{-1}(B_2) = B_1 \). Thus, \( \mu_1[B_1] = \mu_2[B_2] \), because \( \phi \) is a credence space morphism. It follows that \( \hat{\mu}_1[\hat{B}_1] = \hat{\mu}_2[\hat{B}_2] \), as desired.

The proof that \( \beta \) is a faithful functor is identical to the proof of the corresponding properties for the standard Stone-Čech compactification functor in the category of topological spaces.

A **Stonean probability space** is an ordered pair \((S, \mu)\), where \( S \) is a Stonean topological space, and \( \mu \) is a Borel measure on \( S \). Let \( \text{StPr} \) be the category of Stonean probability spaces and continuous, measure-preserving functions. Via Theorem 8.4, we can define a functor from \( \text{Cred} \) to \( \text{StPr} \) as follows. Let \( C = (S, \mathcal{B}, \mu) \) be a credence space. Let \( S^* := \sigma(\mathcal{B}) \), and let \( \mathcal{B}^* := \mathcal{C}lP(S^*) \); thus, \( \mathcal{B}^* \) is isomorphic to \( \mathcal{B} \) by the Stone Representation Theorem. Let \( \mu^* \) be the Borel probability measure defined on \( S^* \) by formula (3) and Lemma 8.5. Thus, \( C^* := (S^*, \mu^*) \) is a Stonean probability space, which will call the **Stone representation** of \( C \).

Now let \( C_1 = (S_1, B_1, \mu_1) \) and \( C_2 = (S_2, B_2, \mu_2) \) be two credence spaces, and let \( \phi : S_1 \rightarrow S_2 \) be a credence space morphism. Since \( \phi^{-1} : B_2 \rightarrow B_1 \) is Boolean algebra homomorphism, the Stone Duality Theorem yields a continuous function \( \phi^* : S_1^* \rightarrow S_2^* \).

**Proposition 9.2** \( \phi^* \) is continuous, measure-preserving function from \( C_1^* \) to \( C_2^* \). The transformation \( \sigma \) defined by \( C \mapsto C^* \) and \( \phi \mapsto \phi^* \) is a faithful functor from \( \text{Cred} \) to \( \text{StPr} \).

**Proof.** Suppose \( C_1^* := (S_1^*, \mu_1^*) \) and \( C_2^* := (S_2^*, \mu_2^*) \), where \( S_1 = \sigma(B_1) \) and \( S_2 = \sigma(B_2) \). The Stone Duality Theorem says \( \phi^* : S_1^* \rightarrow S_2^* \) is continuous. To show that \( \phi^* \) is measure-preserving, it suffices to show that it preserves the measures of clopen sets, because Lemma 8.3 says that a Borel measure on a Stonean space is entirely determined by its values on clopen sets. So, let \( Q_2 \in \mathcal{C}lP[S_2^*] \) and let \( Q_1 := (\phi^*)^{-1}(Q_2) \); we must show that \( \mu_1^*[Q_1] = \mu_2^*[Q_2] \).

Recall that \( S_2^* \) is the set of all ultrafilters in \( B_2 \), and the Stone Representation Theorem says that there exists some \( B_2 \in \mathcal{B}_2 \) such that \( Q_2 = B_2^* \), where \( B_2^* := \{ s_2^* \in S_2^* : B_2 \in s_2^* \} \). Thus, defining formula (3) says that \( \mu_2^*[Q_2] = \mu_2[B_2] \). Let \( B_1 := \phi^{-1}(B_2) \); then \( B_1 \in B_1 \) (because \( \phi \) is a credence space morphism), and for all \( s_1^* \in S_1^* \), we have

\[
\left( s_1^* \in Q_1 \right) \iff \left( \phi^*(s_1^*) \in Q_2 \right) \iff \left( B_2 \in \phi^*(s_1^*) \right) \iff \left( \phi^{-1}(B_2) \in s_1^* \right) \iff \left( B_1 \in s_1^* \right),
\]

where \((*)\) is because \( \phi^*(s_1^*) = \{ B \in B_2 : \phi^{-1}(B) \in s_1^* \} \). Thus, we see that \( Q_1 = B_1^* \), where \( B_1^* := \{ s_1^* \in S_1^* : B_1 \in s_1^* \} \). Thus, defining formula (3) says that \( \mu_1^*[Q_1] = \mu_1[B_1] \). But \( \mu_1[B_1] = \mu_2[B_2] \) because \( \phi(\mu_1) = \mu_2 \) and \( \phi^{-1}(B_2) = B_1 \). Thus, we conclude that \( \mu_1^*[Q_1] = \mu_2^*[Q_2] \), as desired.

**Functor.** Let \( C_3 = (S_3, B_3, \mu_3) \), let \( \sigma(C_3) := (S_3^*, \mu_3^*) \), and let \( \psi : S_2 \rightarrow S_3 \) be another credence space homomorphism. We must show that \( \psi \circ \phi^* = \psi^* \circ \phi^* \). To see this, recall that \( (\psi \circ \phi)^* = \psi^* \circ \phi^* \), and \( \phi^* \) and \( \psi^* \) are the results of applying the Stone Duality functor to the Boolean algebra homomorphisms \( (\psi \circ \phi)^{-1} : B_3 \rightarrow B_1 \), \( \psi^{-1} : B_3 \rightarrow B_2 \), and \( \phi^{-1} : B_2 \rightarrow B_1 \), respectively. Furthermore, \((\psi \circ \phi)^{-1} = \phi^{-1} \circ \psi^{-1} \). Thus, \((\psi \circ \phi)^* = \psi^* \circ \phi^* \).
Faithful. Let $\xi : S_1 \rightarrow S_2$ be another credence space homomorphism, and suppose $\xi^* = \phi^*$. By Stone Duality, this means that $\xi^{-1} : B_2 \rightarrow B_1$ and $\phi^{-1} : B_2 \rightarrow B_1$ are the same Boolean algebra homomorphism; in other words, $\xi^{-1}(B_2) = \phi^{-1}(B_2)$ for all $B_2 \in B_2$. Let $s \in S_1$ and suppose $\phi(s) \neq \xi(s)$. Then there exists $B_2 \in B_2$ such that $\phi(s) \in B_2$ but $\xi(s) \notin B_2$ (because by Lemma 4.3, $B_2$ is a base for the topology of $S_2$, which is Hausdorff). Then $s \in \phi^{-1}(B)$ but $s \notin \xi^{-1}(B)$, which contradicts the fact that $\xi^{-1}(B_2) = \phi^{-1}(B_2)$. By contradiction, we must have $\phi(s) = \xi(s)$ for all $s \in S$ — in other words, $\phi = \xi$.

An integration space is a triple $I = (S, G, \mathbb{I})$, where $S$ is a locally compact Hausdorff space, $G$ is a linear subspace of $C(S, \mathbb{R})$, and $\mathbb{I} : G \rightarrow \mathbb{R}$ is a weakly monotonic, bounded linear functional with norm 1. Heuristically, for any $g \in G$, we can think $\mathbb{I}[g]$ as the “integral” of $g$ with respect to some hypothetical probability measure. For example, if $C = (S, B, \mu)$ is a credence space, then we obtain an integration space $\Upsilon(C) := (S, G, \mathbb{I}^\mu_S)$, where $G := G_B(S)$ and $\mathbb{I}^\mu_S$ is the $\mu$-compatible integrator from Theorem 4.3.

If $I_1 = (S_1, G_1, \mathbb{I}_1)$ and $I_2 = (S_2, G_2, \mathbb{I}_2)$ are two integration spaces, then a morphism from $I_1$ to $I_2$ is a continuous function $\phi : S_1 \rightarrow S_2$ such that, for all $g \in G_2$, we have $g \circ \phi \in G_1$ and $\mathbb{I}_1[g \circ \phi] = \mathbb{I}_2[g]$. For example, if $C_1$ and $C_2$ are two credence spaces, and $\phi$ is a credence space morphism from $C_1$ to $C_2$, then Proposition 5.3 says that $\phi$ is also an integration space morphism from $\Upsilon(C_1)$ to $\Upsilon(C_2)$. Thus, if $Int$ is the category of integration spaces and their morphisms, then the transformation $\Upsilon$ defined by $C \mapsto \Upsilon(C)$ and $\phi \mapsto \phi$ is a functor from $Cred$ to $Int$. If $CmpInt$ is the subcategory of compact integration spaces, then $\Upsilon$ restricts to a functor from $CmpCrd$ into $CmpInt$. Let $\Upsilon \circ \beta : Cred \rightarrow CmpInt$ be the functor obtained by composing $\Upsilon$ with the functor $\beta$ from Proposition 9.1.

Proposition 9.3
(a) For any credence space $C = (S, B, \mu)$, the inclusion map $\iota_C : S \hookrightarrow \widehat{S}$ is an integration space morphism from $\Upsilon(C)$ to $\Upsilon(\widehat{C})$.

(b) The collection $\{\iota_C ; C \in Cred\}$ is a natural transformation from $\Upsilon$ to $\Upsilon \circ \beta$.

Proof. (a) Let $\widehat{C} = (\widehat{S}, \widehat{B}, \widehat{\mu})$. Then $\Upsilon(\widehat{C}) = (\widehat{S}, \widehat{G}, \mathbb{I}_S^\widehat{\mu})$, where $\widehat{G} = G_{\widehat{B}}(\widehat{S})$, and $\mathbb{I}_S^\widehat{\mu}$ is obtained from Theorem 4.3. The inclusion map $\iota_C : S \hookrightarrow \widehat{S}$ is continuous because $\widehat{S}$ is a compactification of $S$. Let $\widehat{g} \in G_{\widehat{B}}(\widehat{S})$, and let $g := \widehat{g} \circ \iota_C$; then $g = \widehat{g}_{\iota_S}$.

Claim 1: $g \in G_B(S)$.

Proof. Every function in $G_B(\widehat{S})$ is a uniform limit of linear combinations of functions in $C_B(\widehat{S})$, and the transformation $C(\widehat{S}, \mathbb{R}) \ni \widehat{f} \mapsto \widehat{f}_{\iota_S} \in C(S, \mathbb{R})$ is linear and continuous, by Lemma 7.4(b). So it suffices to show that $g \in C_B(\widehat{S})$ whenever $\widehat{g} \in C_B(\widehat{S})$. So, suppose $\widehat{g} \in C_B(\widehat{S})$. Let $r \in \mathbb{R}$, and let $B := \text{int}(g^{-1}(-\infty, r])$; we must show that $B \in B$. To see this, let $\widehat{B} := \text{int}(\widehat{g}^{-1}(-\infty, r])$; then we know that $\widehat{B} \in \widehat{B}$ because $\widehat{g} \in C_B(\widehat{S})$. Now, as shown in equation (5) in the proof of Theorem 8.4, $B = S \cap \widehat{B}$. But Lemma 7.4(a) says that $\widehat{B} = \{\widehat{R} \in \mathcal{R}(\widehat{S}); \ S \cap \widehat{R} \in B\}$. Thus $B \in B$. By a
very similar argument, \( \text{int} (g^{-1}[r, \infty)) \in \mathcal{B} \). This argument works for all \( r \in \mathbb{R} \); thus, \( g \in \mathcal{C}_\mathcal{B}(\mathcal{S}) \), as desired.

\( \diamond \) Claim 1

It remains to show that \( \mathbb{I}_s^n g = \mathbb{I}_{\mathcal{S}}^n \hat{g} \). Let \( \mathcal{F} \) be the set of \( \mathcal{B} \)-simple functions of \( \mathcal{S} \), and let \( \hat{\mathcal{F}} \) be the set of \( \hat{\mathcal{B}} \)-simple functions on \( \hat{\mathcal{S}} \).

**Claim 2:** For any \( \hat{f} \in \hat{\mathcal{F}} \), if \( f = \hat{f}_{1, \mathcal{S}} \), then \( f \in \mathcal{F} \), and \( \int_{\mathcal{S}} f \, d\mu = \int_{\hat{\mathcal{S}}} \hat{f} \, d\hat{\mu} \).

**Proof.** Let \( \hat{f} \in \hat{\mathcal{F}} \) be subordinate to the \( \hat{\mathcal{B}} \)-partition \( \{ \hat{\mathcal{B}}_1, \ldots, \hat{\mathcal{B}}_N \} \). Then \( \hat{f}_{1, \mathcal{S}} \) is a simple function on \( \mathcal{S} \), subordinate to the \( \mathcal{B} \)-partition \( \{ \mathcal{B}_1, \ldots, \mathcal{B}_N \} \), where for all \( n \in [1 \ldots N] \), \( \mathcal{B}_n := \mathcal{S} \cap \hat{\mathcal{B}}_n \), and the value that \( \hat{f}_{1, \mathcal{S}} \) takes on \( \mathcal{B}_n \) is the same as the value that \( f \) takes on \( \hat{\mathcal{B}}_n \)—call this value \( r_n \). Thus,

\[
\int_{\mathcal{S}} f \, d\mu = \sum_{n=1}^N r_n \mu[\mathcal{B}_n] = \sum_{n=1}^N r_n \hat{\mu}[\hat{\mathcal{B}}_n] = \int_{\hat{\mathcal{S}}} \hat{f} \, d\hat{\mu},
\]

where both (\( * \)) are by defining formula 1, and (\( \dagger \)) is by the definition of \( \hat{\mu} \). \( \diamond \) Claim 2

Let \( \mathcal{F}_g := \{ f \in \mathcal{F}; \, f \leq g \} \), and let \( \hat{\mathcal{F}}_g := \{ \hat{f} \in \hat{\mathcal{F}}; \, \hat{f} \leq \hat{g} \} \).

**Claim 3:** \( \mathcal{F}_g = \{ \hat{f}_{1, \mathcal{S}}; \, \hat{f} \in \hat{\mathcal{F}}_g \} \).

**Proof.** “\( \subseteq \)” If \( \hat{f} \in \hat{\mathcal{F}}_g \), then \( \hat{f}_{1, \mathcal{S}} \in \mathcal{F}_g \) by Claim 2. Furthermore, if \( \hat{f} \in \hat{\mathcal{F}}_g \), then \( \hat{f}(\hat{s}) \leq \hat{g}(\hat{s}) \) for all \( \hat{s} \in \hat{\mathcal{S}} \), which means that \( \hat{f}_{1, \mathcal{S}}(s) \leq g(s) \) for all \( s \in \mathcal{S} \), and hence, \( \hat{f}_{1, \mathcal{S}} \in \mathcal{F}_g \).

“\( \supseteq \)” Let \( f \in \mathcal{F}_g \) be subordinate to the \( \mathcal{B} \)-partition \( \{ \mathcal{B}_1, \ldots, \mathcal{B}_N \} \). For all \( n \in [1 \ldots N] \), let \( \hat{\mathcal{B}}_n \in \hat{\mathcal{B}} \) be the unique element such that \( \mathcal{B}_n = \mathcal{S} \cap \hat{\mathcal{B}}_n \). Then \( \{ \hat{\mathcal{B}}_1, \ldots, \hat{\mathcal{B}}_N \} \) is a \( \hat{\mathcal{B}} \)-partition of \( \hat{\mathcal{S}} \) (by Lemma 7.4(a)). Let \( \hat{f} \in \hat{\mathcal{F}} \) be the unique simple function on \( \hat{\mathcal{S}} \) subordinate to this partition, such that for all \( n \in [1 \ldots N] \), the value that \( \hat{f} \) takes on \( \hat{\mathcal{B}}_n \) is the same as the value that \( f \) takes on \( \mathcal{B}_n \), and \( \hat{f}_{1, \mathcal{B}_n} = f_{1, \mathcal{B}_n} \) \( \dagger \). Meanwhile, define \( \hat{f}(\hat{s}) = \hat{g}(\hat{s}) \) for all \( \hat{s} \in (\partial \hat{\mathcal{B}}_n) \setminus \mathcal{S} \), for all \( n \in [1 \ldots N] \). Then clearly \( \hat{f}_{1, \mathcal{S}} = f \). It remains to show that \( \hat{f} \in \hat{\mathcal{F}}_g \).

For any \( \hat{s} \in \hat{\mathcal{S}} \), we must show that \( \hat{f}(\hat{s}) \leq \hat{g}(\hat{s}) \). If \( \hat{s} \in \mathcal{S} \), then \( \hat{f}(\hat{s}) = f(\hat{s}) \leq g(\hat{s}) = \hat{g}(\hat{s}) \), so we’re done. So suppose \( \hat{s} \in \hat{\mathcal{S}} \setminus \mathcal{S} \). If \( \hat{s} \in \partial \hat{\mathcal{B}}_n \) for some \( n \in [1 \ldots N] \), then \( \hat{f}(\hat{s}) = \hat{g}(\hat{s}) \) by definition, so we’re done. So suppose that \( \hat{s} \in \hat{\mathcal{B}}_n \) for some \( n \in [1 \ldots N] \). Let \( r_n \) be the (constant) value of \( f \) on \( \hat{\mathcal{B}}_n \).

Since \( \mathcal{B}_n \) is dense in \( \hat{\mathcal{B}}_n \), there is a net \( (b_j)_{j \in J} \) in \( \mathcal{B}_n \) converging to \( \hat{s} \) (for some directed set \( J \)). For all \( j \in J \), we have \( f(b_j) = \hat{f}(b_j) = r_n \) (because \( b_j \in \hat{\mathcal{B}}_j \)) while \( f(b_j) \leq g(b_j) \) (because \( f \in \mathcal{F}_g \)), so that \( r \leq g(b_j) = \hat{g}(b_j) \). Thus, \( r_n \leq \hat{g}(\hat{s}) \), because \( \hat{g} \) is continuous and \( (b_j)_{j \in J} \) converges to \( \hat{s} \). But \( \hat{f}(\hat{s}) = r_n \) also, because \( \hat{s} \in \hat{\mathcal{B}}_n \). Thus, \( \hat{f}(\hat{s}) \leq \hat{g}(\hat{s}) \), as desired. \( \diamond \) Claim 3

\( ^{21} \)Recall that the value of a simple function on the boundaries of its subordinating partition is arbitrary, and has no effect on the integral; the only important thing is that \( \hat{f} \) is dominated by \( \hat{g} \).
We now have
\[
\mathbb{P}^\phi[g] \equiv \sup_{f \in F_g} \int_S f \, d\mu = \sup_{f \in \hat{F}_g} \int_{\hat{S}} \hat{f}_1 \, d\mu = \sup_{f \in \hat{F}_g} \int_{\hat{S}} \hat{f} \, d\hat{\mu} \equiv \mathbb{P}_g^\phi,
\]
as desired. Here, both \((*)\) equalities are by defining formula (7), \((†)\) is by Claim 3, and \((⊙)\) is by Claim 2.

(b) Let \(C_1 = (S_1, \mathcal{B}_1, \mu_1)\) and \(C_2 = (S_2, \mathcal{B}_2, \mu_2)\) be two credence spaces, and let \(\phi : \mathcal{S}_1 \rightarrow \mathcal{S}_2\) be a credence space homomorphism. Suppose \(\hat{C}_1 = (\hat{S}_1, \hat{\mathcal{B}}_1, \hat{\mu}_1)\) and \(\hat{C}_2 = (\hat{S}_2, \hat{\mathcal{B}}_2, \hat{\mu}_2)\), and let \(\hat{\phi} := \beta(\phi) : \hat{S}_1 \rightarrow \hat{S}_2\). Recall that the underlying topological spaces of \(\Upsilon(C_1)\) and \(\Upsilon(C_2)\) are still \(S_1\) and \(S_2\), while \(\Upsilon(\phi) = \phi\). Likewise, the underlying topological spaces of \(\Upsilon \circ \beta(C_1)\) and \(\Upsilon \circ \beta(C_2)\) are \(\hat{S}_1\) and \(\hat{S}_2\), while \(\Upsilon \circ \beta(\phi) = \hat{\phi}\). Let \(\iota_1 : S_1 \hookrightarrow \hat{S}_1\) and \(\iota_2 : S_2 \hookrightarrow \hat{S}_2\) be the inclusion maps. Then for any \(s \in S_1\), we have \(\hat{\phi} \circ \iota_1(s) = \hat{\phi}_{1\mathcal{S}_1}(s) = \phi(s) = \iota_2 \circ \phi(s)\). Thus, the following diagram commutes:

\[
\begin{array}{ccc}
S_1 & \longrightarrow & S_2 \\
\phi \downarrow & & \downarrow \iota_2 \\
\hat{S}_1 & \longrightarrow & \hat{S}_2
\end{array}
\]

Such a diagram commutes for any choice of \(C_1, C_2\) and \(\phi\). Thus, \(\{\iota_C ; C \in \text{Cred}\}\) is a natural transformation from \(\Upsilon\) to \(\Upsilon \circ \beta\). \(\square\)

A \textit{locally compact probability space} is an ordered pair \((S, \mu)\), where \(S\) is a locally compact Hausdorff space and \(\mu\) is a Borel probability measure on \(S\). For any such space \((S, \mu)\), let \(C_0(S, \mathbb{R})\) be the set of all continuous, real-valued functions on \(S\) which converge to zero at infinity; then we obtain an integration space \(\Xi(S, \mu) := (S, \mathcal{G}, \lambda)\) where \(\mathcal{G} := C_0(S, \mathbb{R})\) and \(\lambda[g] := \int_S g \, d\mu\) for all \(g \in \mathcal{G}\).

\[\text{If} \ (S_1, \mu_1) \ \text{and} \ (S_2, \mu_2) \ \text{are locally compact probability spaces, and} \ \phi : S_1 \rightarrow S_2 \ \text{is a continuous, measure-preserving function, then} \ \hat{\phi} \ \text{is also an integration space isomorphism from} \ \Xi(S_1, \mu_1) \ \text{to} \ \Xi(S_2, \mu_2). \ \text{Thus, if} \ \text{LCPr} \ \text{is the category of locally compact probability spaces and continuous, measure-preserving functions, then the transformation} \ \Xi \ \text{defined by} \ (S, \mu) \mapsto \Xi(S, \mu) \ \text{and} \ \phi \mapsto \hat{\phi} \ \text{is a functor from} \ \text{LCPr} \ \text{into} \ \text{Int}. \ \text{Furthermore,} \ \text{StPr} \ \text{is a subcategory of} \ \text{LCPr}, \ \text{and} \ \Xi \ \text{restricts to a functor from} \ \text{StPr} \ \text{into} \ \text{CmpInt}. \ \text{Let} \ \Xi \circ \sigma : \text{Cred} \longrightarrow \text{CmpInt} \ \text{be the functor obtained by composing} \ \Xi \ \text{with the functor} \ \sigma \ \text{from Proposition 9.2.}

\]

For any credence space \(C = (S, \mathcal{B}, \mu)\), Proposition \[8.3\] yields a continuous surjection \(p_C : S^* \rightarrow \mathcal{S}\), defined by formula (2). We will refer to \(p_C\) as the \textit{Gleason map} for \(C\).

\[\text{Proposition 9.4} \quad (a) \ \text{For any credence space} \ C, \ \text{the map} \ p_C \ \text{is an integration space morphism from} \ \Xi(C^*) \ \text{to} \ \Upsilon(\hat{C}).\]

\[\text{22Inden, the Riesz Representation Theorem says that every integration space with} \ \mathcal{G} = C_0(S, \mathbb{R}) \ \text{arises from a locally compact probability space in this way.}\]
Any Stonean probability space \((S^*, \mu^*)\) can be seen as a credence space \((S^*, \mathcal{C}(S^*), \mu^*)_p\) (because \(\mathcal{C}(S^*)\) is itself a Boolean subalgebra of \(\mathcal{H}(S^*)_p\).) Any continuous function between two Stonean spaces automatically induces a Boolean algebra homomorphism between their algebras of clopen sets (see Remark 5.5). Thus, if this function is also measure-preserving (i.e. if it is a morphism in the category \(\text{StPr}\)), then it is a credence space morphism. Thus, through a slight abuse of notation, we can regard \(\text{StPr}\) as a subcategory of \(\text{Cred}\), so that the system \(\{p_C; \ C \in \text{Cred}\}\) is a natural transformation from \(\Xi \circ \sigma \) to \(\Upsilon \circ \beta\).

Proof of Proposition 9.4. (a) Let \(\hat{C} = (\hat{S}, \hat{B}, \hat{\mu})\); then \(\Upsilon(\hat{C}) = (\hat{S}, \hat{\sigma}, \hat{\mu\|}_\hat{S})\), where \(\hat{\sigma} = \mathcal{G}_\|_\hat{S}(\hat{S})\), and \(\hat{\mu\|}_\hat{S}\) is obtained from Theorem 4.3. Meanwhile, \(C^* = (S^*, \mu^*)\), where \(\mu^*\) is the Borel probability measure defined on \(S^*\) by formula (3) and Lemma 8.3. Thus, \(\Xi(C^*) = (S^*, G^*, \|\|)^*\), where \(G^* = C(S^*, \mathbb{R})\) and \(\|\|_*[g] = \int_{S^*} g \, d\mu^*\) for all \(g \in G^*\).

The function \(p_C : S^* \rightarrow \hat{S}\) is continuous by Proposition 8.3. For any \(\hat{g} \in \hat{G}\), we automatically have \(\hat{g} \circ p_C \in G^*\), because \(G^* = C(S^*, \mathbb{R})\). In the notation of Theorem 8.4, \(\hat{g} \circ p_C = g^*.\) Thus,

\[
\|\|_*[\hat{g} \circ p_C] = \|\|_*[g^*] \quad \overset{(*)}{=} \quad \|\|_*[\hat{g}] \quad \overset{(**)}{=} \quad \|\|_*[\hat{g}],
\]

as desired. Here, \((*)\) is by Theorem 8.4 and \((**\)) is by Proposition 9.3(a).

(b) Let \(C_1 = (S_1, \mathcal{B}_1, \mu_1)\) and \(C_2 = (S_2, \mathcal{B}_2, \mu_2)\) be two credence spaces, and let \(\phi : S_1 \rightarrow S_2\) be a credence space homomorphism. Suppose \(\hat{C}_1 = (\hat{S}_1, \hat{B}_1, \hat{\mu}_1)\) and \(\hat{C}_2 = (\hat{S}_2, \hat{B}_2, \hat{\mu}_2)\), and let \(\hat{\phi} := \beta(\phi) : \hat{S}_1 \rightarrow \hat{S}_2\). Let \(\mathcal{C}_1^* = (S_1^*, \mu_1^*)\) and \(\mathcal{C}_2^* = (S_2^*, \mu_2^*)\), and let \(\phi^* := \sigma(\phi) : S_1^* \rightarrow S_2^*\). Let \(p_1 := p_{C_1} : S_1^* \rightarrow \hat{S}_1\) and \(p_2 := p_{C_2} : S_2^* \rightarrow \hat{S}_2\) be the Gleason maps. We must show that the following diagram commutes:

\[
\begin{array}{ccc}
S_1^* & \xrightarrow{\phi^*} & S_2^* \\
p_1 \downarrow & & \downarrow p_2 \\
\hat{S}_1 & \xrightarrow{\phi} & \hat{S}_2 \\
\end{array}
\]  
(1)

Let \(s_1^* \in S_1^*\). Let \(\hat{s}_2 := \hat{\phi} \circ p_1(s_1^*)\) and let \(\hat{s}_2^* := p_2 \circ \phi^*(s_1^*).\) We must show that \(\hat{s}_2 = \hat{s}_2^*\).

By contradiction, suppose not. Since \(\hat{S}_2\) is Hausdorff, there exist disjoint open sets \(\hat{O}_2\)

---

23For \(p_C\) to be a credence space morphism, \(p_C^{-1} : \hat{S} \rightarrow \mathcal{C}(\hat{S})*\) must be a Boolean algebra homomorphism. But this is false in general, as explained in Footnote 19.
and \( \hat{\mathcal{O}}_2 \) with \( \hat{s}_2 \in \hat{\mathcal{O}}_2 \) and \( \hat{s}_2' \in \hat{\mathcal{O}}_2' \). Since \( \hat{\mathcal{B}}_2 \) generates the topology of \( \hat{\mathcal{S}}_2 \), we can find \( \hat{\mathcal{B}}_2, B_2 \in \hat{\mathcal{B}}_2 \) with \( \hat{s}_2 \in \hat{\mathcal{B}}_2 \subseteq \hat{\mathcal{O}}_2 \) and \( \hat{s}_2' \in B_2 \subseteq \hat{\mathcal{O}}_2' \).

Let \( \hat{s}_1 := p_1(s_1^*) \). Then \( \hat{\phi}(\hat{s}_1) = \hat{s}_2 \). Thus, if \( \hat{\mathcal{B}}_1 := \hat{\phi}^{-1}(\hat{B}_2) \), then \( \hat{\mathcal{B}}_1 \) is an open neighbourhood around \( \hat{s}_1 \), and \( \hat{B}_1 \in \hat{\mathcal{B}}_1 \), because \( \hat{\phi} \) is a credence space morphism. Let \( B_1 := S_1 \cap \hat{B}_1 \). Then Lemma 8.6 says that \( B_1 \in \sigma_{\phi}^* \) (because \( p_1(s_1^*) = \hat{s}_1 \)).

Meanwhile, let \( s_2^* := \phi^*(s_1^*) \); then \( p_2(s_2^*) = \hat{s}_2' \). Let \( \mathcal{B}_2' := \mathcal{S}_2 \cap \mathcal{B}_2 \). Then Lemma 8.6 says that \( \mathcal{B}_2' \in s_2^* \) (because \( p_2(s_2^*) = \hat{s}_2' \)). But \( s_2^* = \phi^*(s_1^*) = \{ B \in \mathcal{B}_2; \ \phi^{-1}(B) \in \sigma_{\phi}^* \} \). Thus, if \( \mathcal{B}_2' \in s_2^* \), then \( \mathcal{B}_1' := \phi^{-1}(\mathcal{B}_2') \in \sigma_{\phi}^* \).

At this point, we have \( \mathcal{B}_1 \in \sigma_{\phi}^* \) and \( \mathcal{B}_1' \in s_1^* \). Thus, \( \mathcal{B}_1 \cap \mathcal{B}_1' \neq \emptyset \), because \( s_1^* \) is a filter. Let \( b_1 \in \mathcal{B}_1 \cap \mathcal{B}_1' \). Then \( b_1 \in \mathcal{B}_1 \subseteq \mathcal{B}_1' \), and \( \phi_{\mathcal{B}_1} = \phi \), so we get

\[
\phi(b_1) = \hat{\phi}(b_1) \in \hat{\phi}(\mathcal{B}_1) \subseteq \hat{\mathcal{B}}_2 \subseteq \hat{\mathcal{O}}_2. \tag{2}
\]

Meanwhile, \( b_1 \in \mathcal{B}_1' \), so we get

\[
\phi(b_1) \in \phi(\mathcal{B}_1') \subseteq \mathcal{B}_2' \subseteq \mathcal{B}_2 \subseteq \hat{\mathcal{O}}_2'. \tag{3}
\]

Combining equations (2) and (3), we see that \( \phi(b_1) \in \hat{\mathcal{O}}_2 \cap \hat{\mathcal{O}}_2' \), which contradicts the fact that \( \hat{\mathcal{O}}_2 \) and \( \hat{\mathcal{O}}_2' \) are disjoint by construction.

To avoid the contradiction, we must have \( \hat{s}_2 = \hat{s}_2' \) — i.e. \( \hat{\phi} \circ p_1(s_1^*) = p_2 \circ \phi^*(s_1^*) \). Since this holds for all \( s_1^* \in \sigma_{\phi}^* \), we conclude that \( \hat{\phi} \circ p_1 = p_2 \circ \phi^* \), hence the diagram (1) commutes. \( \square \)

If we restrict attention to the Boolean algebra of all regular sets, then Propositions 9.1 and 9.2 admit a simpler formulation. A full credence space is an ordered pair \((\mathcal{S}, \mu)\), where \( \mathcal{S} \) is a locally compact Hausdorff space and \( \mu \) is a credence on \( \mathcal{H}(\mathcal{S}) \). Let \( \text{Cred}_0 \) be the category of full credence spaces — this is a full subcategory of \( \text{Cred} \). Likewise, let \( \text{CmpCrd}_0 \) be the full category compact full credence spaces — this is a full subcategory of both \( \text{Cred} \) and \( \text{CmpCrd} \).

A topological space is extremally disconnected if the closure of every open subset is itself an open subset. Any extremally disconnected space is totally disconnected. A Gleason space is a compact, Hausdorff, extremally disconnected space (i.e. an extremally disconnected Stonean space). A Gleason probability space is an ordered pair \((\mathcal{S}, \mu)\), where \( \mathcal{S} \) is a Gleason space and \( \mu \) is a Borel probability measure on \( \mathcal{S} \). Let \( \text{GlPr} \) be the category of Gleason probability spaces and continuous, measure-preserving functions — this is a full subcategory of \( \text{StPr} \).

Proposition 9.5  \( \textbf{(a)} \) If \( \mathcal{C} \) is a full credence space, then \( \check{\mathcal{C}} \) is also a full credence space. Thus, \( \beta \) is a faithful functor from \( \text{Cred}_0 \) to \( \text{CmpCrd}_0 \).

\( \textbf{(b)} \) If \( \mathcal{C} \) is a full credence space, then \( \mathcal{C}^* \) is a Gleason probability space. Thus, \( \sigma \) is a faithful functor from \( \text{Cred}_0 \) to \( \text{GlPr} \).
Proof. If $C = (S, \mu)$ is a full credence space, then $\beta(C) = (\hat{S}, \hat{\mu})$ is obviously a compact full credence space, because $\hat{S}$ is compact. Meanwhile, $\sigma(C)$ is a Gleason probability space, because $R(S)$ is a complete Boolean algebra (see Footnote 3), and the Stone space of any complete Boolean algebra is totally disconnected. Finally, the functorial claims follow immediately from Propositions 9.1 and 9.2.

The natural transformation claims in Propositions 9.3(b) and 9.4(b) clearly continue to hold for the restricted functors in Proposition 9.5.

Conclusion

This paper has developed both an integration theory and a representation theory for credences on Boolean algebras of regular open sets. But many intriguing questions remain unanswered. How much of classical measure theory can be extended to credences? For example, is there a notion of “Cartesian product” for two credences, which satisfies a version of the Fubini-Tonelli Theorem? There is a natural way to define signed and complex-valued credences; do these admit a Hahn-Jordan Decomposition Theorem? The most obvious formulation of the Radon-Nikodym Theorem is false for credences, in general. But is there a version of this theorem for some suitably modified notion of “absolute continuity”? If $S$ is a topological group, then there is a natural notion of an invariant ("Haar") credence on $S$. Does such a credence always exist? When is it unique? If $S$ is a locally compact abelian group, then can we develop a version of harmonic analysis using this credence? Likewise, if $\phi : S \rightarrow S$ is a homeomorphism (i.e. a dynamical system), then there is a natural notion of “$\phi$-invariant credence”. How much of classical ergodic theory can be extended to such credences? In particular, do $\phi$-invariant credences always exist? Since credences interact nicely with the topology of $S$, would the ergodic theory of $\phi$-invariant credences provide insights into $S$ as a topological dynamical system?

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24See [Wal74, Proposition 2.5, p.47] or [Fre04, Theorem 314S].

25We are grateful to David Fremlin for this observation.
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