0 Introduction.

We consider the set $\sigma_P$ of the power non-negative polynomials of several variables. By $Q_P$ we denote the class of the polynomials from $\sigma_1$ which can be represented as a sum of squares. It is shown in the classic work by D. Hilbert [3] that $Q_P$ does not coincide with $\sigma_P$. Step by step a number of polynomials belonging to $\sigma_P$ but not belonging to $Q_P$ was constructed (see [4]–[6]). It is interesting to note that many of these polynomials turn to be extremal in the class $\sigma_P$ [2].

In our paper we have made an attempt to work out a general approach to the investigation of the extremal elements of the convex sets $Q_P$ and $\sigma_P$. It seems to us that we have achieved a considerable progress in the case of $Q_P$. In the case of $\sigma_P$ we have made only the first steps. We also consider the class $\sigma_R$ of the non-negative rational functions. The article is based on the following methods:

1. We investigate non-negative trigonometrical polynomials and then, with the help of the Calderon transformation we proceed to the power polynomials.
2. The way of constructing support hyperplanes to the convex sets $Q_P$ and $\sigma_P$ is given in the paper.

Now we start with a more detailed description of the results of this article. Let us denote by $\sigma(N_1, N_2, N_3)$ the set of the trigonometrical polynomials

$$f(\alpha, \beta, \gamma) = \sum_{|k| \leq N_1} \sum_{|\ell| \leq N_2} \sum_{|m| \leq N_3} q(k, \ell, m) e^{i(k\alpha + \ell\beta + m\gamma)}$$

satisfying the condition

$$f(\alpha, \beta, \gamma) \geq 0, \quad (\alpha = \bar{\alpha}, \beta = \bar{\beta}, \gamma = \bar{\gamma}).$$

We denote by $Q(N_1, N_2, N_3)$ the set of trigonometrical polynomials of the class $\sigma(N_1, N_2, N_3)$ admitting representation

$$f(\alpha, \beta, \gamma) = \sum_{j=1}^{r} |F_j(\alpha, \beta, \gamma)|^2$$
where
\[ F_j(\alpha, \beta, \gamma) = \sum_{0 \leq k \leq N_1} \sum_{0 \leq \ell \leq N_2} \sum_{0 \leq m \leq N_3} q_j(k, \ell, m) e^{i(k\alpha + \ell\beta + m\gamma)} \quad (4) \]

It is clear that the set \( Q(N_1, N_2, N_3) \) is convex. In this article we give a method of constructing the support hyperplanes to the set \( Q(N_1, N_2, N_3) \). Hence we receive a number of general facts referring to the extremal points and faces of the set \( Q(N_1, N_2, N_3) \). Here we also introduce concrete examples of extremal points and faces.

Analogues results are received for the convex set \( \sigma(N_1, N_2, N_3) \) as well. In addition to the set of the non-negative trigonometrical polynomials we shall introduce the class \( \sigma_P(2N_1, 2N_2, 2N_3) \) of the power non-negative polynomials of the form
\[ f(x, y, z) = \sum_{0 \leq k \leq 2N_1} \sum_{0 \leq \ell \leq 2N_2} \sum_{0 \leq m \leq 2N_3} a_{k, \ell, m} x^k y^\ell z^m \quad (5) \]

By \( Q_P(2N_1, 2N_2, 2N_3) \) we denote the set of the power non-negative polynomials of the class \( \sigma_P(2N_1, 2N_2, 2N_3) \) admitting the representation
\[ f(x, y, z) = \sum_{j=1}^{r} |F_j(x, y, z)|^2 \quad (6) \]

where \( F_j(x, y, z) \) are polynomials of \( x, y, z \). With the help of the Calderon transformation the results obtained for the classes of the trigonometrical polynomials \( \sigma(N_1, N_2, N_3) \) and \( Q(N_1, N_2, N_3) \) we transfer onto the classes of the power polynomials \( \sigma_P(2N_1, 2N_2, 2N_3) \) and \( Q_P(2N_1, 2N_2, 2N_3) \). Let us note that a number of concrete examples of the extremal power polynomials is contained in important works \([2],[5]\).

In our paper we consider the case of three variables, but the obtained results can be easily transferred to any number of variables.

1 Main Notions

Let \( S \) be a set of points of the space \( R^3 \). Let us denote by \( \Delta = S - S \) the set of points \( x \in R^3 \) which can be represented in the form \( x = y - z, \quad y, z \in R^3 \).
The function $\Phi(x)$ is called Hermitian positive on $\Delta$ if for any points $x_1, x_2, \ldots, x_N \in S$ and numbers $\xi_1, \xi_2, \ldots, \xi_N$ the inequality
\[ \sum_{i,j} \xi_i \overline{\xi_j} \Phi(x_i - x_j) \geq 0 \] (7)
is true. We shall consider the lattice $S(N_1, N_2, N_3)$ consisting of the points $M(k, \ell, m)$ where $0 \leq k \leq N_1, 0 \leq \ell \leq N_2, 0 \leq m \leq N_3$. The set $\Delta(N_1, N_2, N_3)$ consists of the points $M(k, \ell, m)$ where $|k| \leq N_1, |\ell| \leq N_1, |m| \leq N_3$. By $P(N_1, N_2, N_3)$ we denote the class of functions which are Hermitian positive on $\Delta(N_1, N_2, N_3)$.

With each function $\Phi(k, \ell, m)$ from $P(N_1, N_2, N_3)$ we associate the Toeplitz matrices (see [8],[9]):
\[ B(l, m) = \begin{bmatrix}
\Phi(0, l, m) & \Phi(1, l, m) & \cdots & \Phi(N_1, l, m) \\
\Phi(-1, l, m) & \Phi(0, l, m) & \cdots & \Phi(N_1 - 1, l, m) \\
\vdots & \vdots & & \vdots \\
\Phi(-N, l, m) & \Phi(-N_1 + 1, l, m) & \cdots & \Phi(0, l, m)
\end{bmatrix} \] (8)

From the matrices $B(l, m)$ we construct the block Toeplitz matrices
\[ C_m = \begin{bmatrix}
B(0, m) & B(1, m) & \cdots & B(N_2, m) \\
B(-1, m) & B(0, m) & \cdots & B(N_2 - 1, m) \\
\vdots & \vdots & & \vdots \\
B(-N_2, m) & B(-N_2 + 1, m) & \cdots & B(0, m)
\end{bmatrix}, \quad |m| \leq N_3
\]

Finally from $C_k$ we make yet another block Toeplitz matrix
\[ A(N_1, N_2, N_3) = \begin{bmatrix}
C_0 & C_1 & \cdots & C_{N_3} \\
C_{-1} & C_0 & \cdots & C_{N_3 - 1} \\
\vdots & \vdots & & \vdots \\
C_{-N_3} & C_{-N_3 + 1} & \cdots & C_0
\end{bmatrix} \] (9)

**Proposition 1** (see [8],[9]). Inequality (7) is equivalent to the inequality $A(N_1, N_2, N_3) \geq 0$.

With the help of the function $\Phi(k, \ell, m)$ we introduce the linear functional (see[7]):
\[ L_\Phi(f) = \sum_{|k| \leq N_1} \sum_{|\ell| \leq N_2} \sum_{|m| \leq N_3} q(k, \ell, m) \Phi(k, \ell, m) \] (10)
Proposition 2 (see [8],[9]). If 
\[ \Phi(k, \ell, m) \in P(N_1, N_2, N_3) \] 
and \( f(\alpha, \beta, \gamma) = |F(\alpha, \beta, \gamma)|^2 \) then the relation 
\[ L_\Phi(f) = e^T A e \geq 0 \] (11) 
holds.

Here the matrix \( A \) is defined by relations (8),(9), the function \( F(\alpha, \beta, \gamma) \) and the vector \( e \) have the forms
\[ F(\alpha, \beta, \gamma) = \sum_{0 \leq k \leq N_1} \sum_{0 \leq \ell \leq N_2} \sum_{0 \leq m \leq N_3} q(k, \ell, m)e^{\ell(ka+\ellb+m\gamma)} \] (12)
where
\[ e = \text{col}[h(0), h(1), ..., h(N_3)], \] (13)
\[ h(m) = \text{col}[g(o, m), g(1, m), ..., g(N_2, m)], \] (14)
\[ g(\ell, m) = \text{col}[d(0, \ell, m), d(1, \ell, m), ..., d(N_1, \ell, m)]. \] (15)

2 Support hyperplanes, extremal points and extremal faces of \( Q(N_1, N_2, N_3) \)

Let \( L \) be a linear a linear functional. The hyperplane \( H = [L, \alpha] \) is said to bound the set \( U \) if either
\[ L(f) \geq \alpha \] for all \( f \in U \) or \( L(f) \leq \alpha \) for all \( f \in U \).

A hyperplane \( H = [L, \alpha] \) is said to support a set \( U \) at a point \( f_0 \in U \) if
\[ L(f_0) = \alpha \] and if \( H \) bound \( U \).

Further we shall consider only such support hyperplanes \( H \) which have at least one common point with the corresponding convex set \( U \).

A point \( f_0 \) in the convex set \( U \) is called an extremal point of \( U \) if there exists no non-degenerate line segment in \( U \) that contains \( f_0 \) in its relative interior (see [10]). From Proposition 2 we obtain the following important assertion.

Corollary 1. The set of support hyperplanes for \( Q(N_1, N_2, N_3) \) coincides with the set of hyperplanes
\[ L_\Phi(f) = 0 \] (16)
where \( \Phi(k, \ell, m) \in P(N_1, N_2, N_3) \) and the corresponding matrix \( A(N_1, N_2, N_3) \) is such that
\[ \det A(N_1, N_2, N_3) = 0. \] (17)
Let us denote by $\nu_A$ the dimension of the kernel of the matrix $A(N_1, N_2, N_3)$. If vector $e \neq 0$ belongs to the kernel of $A(N_1, N_2, N_3)$ then the corresponding polynomial $f(\alpha, \beta, \gamma)$ (see (1) and (14)-(18)) belongs to $Q(N_1, N_2, N_3)$ and satisfies relation (19). The convex hull of such polynomials we denote by $D_A$. Using classical properties of a convex set we obtain the following assertions.

**Corollary 2.** If $\nu_A = 1$ and $f(\alpha, \beta, \gamma) \in D_A$ then the $f(\alpha, \beta, \gamma)$ is the extremal polynomial in the class $Q(N_1, N_2, N_3)$.

**Corollary 3.** If $\nu_A > 1$ then $D_A$ is the extremal face.

**Corollary 4.** If $f(\alpha, \beta, \gamma)$ is an extremal polynomial in the class $Q(N_1, N_2, N_3)$ then there exists a non trivial function $\Phi(k, \ell, m) \in P(N_1, N_2, N_3)$ such that $L_\Phi(f) = 0$.

**Example 1.** Let the relations

$$N_1 = N_2 = N_3 = 1$$

be valid. We set

$$B(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(0, 1) = B(1, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B(1, 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B(-1, 1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Then we have

$$C_0 = E_4, \quad C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} E_4 & C_1 \\ C_1^* & E_4 \end{bmatrix} \quad (18)$$

It is easy to see that the following linearly independent vectors

$$\begin{cases} e_1 = \text{col} \begin{bmatrix} 0, 0, 0, 1, 1, 0, 0, 0 \end{bmatrix} \\ e_2 = \text{col} \begin{bmatrix} 0, 0, -1, 0, 0, 1, 0, 0 \end{bmatrix} \\ e_3 = \text{col} \begin{bmatrix} 0, -1, 0, 0, 0, 0, 1, 0 \end{bmatrix} \\ e_4 = \text{col} \begin{bmatrix} -1, 0, 0, 0, 0, 0, 0, 1 \end{bmatrix} \end{cases} \quad (19)$$
form the basis of the kernel $A(1,1,1)$. The corresponding polynomials $F_k(\alpha, \beta, \gamma)$ have the forms

$$
\begin{align*}
F_1(\alpha, \beta, \gamma) &= e^{i(\alpha+\beta)} + e^{i\gamma}, \\
F_2(\alpha, \beta, \gamma) &= -e^{i\beta} + e^{i(\alpha+\gamma)}, \\
F_3(\alpha, \beta, \gamma) &= -e^{i\alpha} + e^{i(\beta+\gamma)}, \\
F_4(\alpha, \beta, \gamma) &= -1 + e^{i(\alpha+\beta+\gamma)}
\end{align*}
$$

(20)

It means that polynomials

$$
f_k(\alpha, \beta, \gamma) = |F_k(\alpha, \beta, \gamma)|^2, \quad (1 \leq k \leq 4)
$$

(21)

belong to the face $D_A$.

**Proposition 3.** The polynomials $f_k(\alpha, \beta, \gamma)$ constructed by formulas (20) and (21) are extremal in the class $Q(1,1,1)$.

**Proof.** Let $B(0,0), B(0,1)$ and $B(1,0)$ be defined as above and let

$$
B(-1,1) = \begin{bmatrix} 0 & \gamma_1 \\ -\gamma_2 & 0 \end{bmatrix},
$$

(22)

$$
B(1,1) = \begin{bmatrix} 0 & \gamma_4 \\ \gamma_3 & 0 \end{bmatrix}.
$$

(23)

We consider the cases when $\gamma_j = 1, \ |\gamma_k| < 1, \ (k \neq j)$. In this cases vector $e_j$ (see (19)) belongs to the kernel of the corresponding matrix $A_j(1,1,1)$ and $\nu_j = 1$. Hence the polynomials $f_j(\alpha, \beta, \gamma)$ are extremal.

### 3 Support hyperplanes, extremal points and extremal faces of $\sigma(N_1, N_2, N_3)$

We will say that the function $\Phi(k, \ell, m) \in P(N_1, N_2, N_3)$ is extendible if $\Phi(k, \ell, m)$ admits an extension to the function of the class $P(\infty, \infty, \infty)$. We will use the following Rudin’s result [7].

**Proposition 4.** A function can be extended to a member of $P(\infty, \infty, \infty)$ if and only if $L_\Phi(f) \geq 0$ for every $f \in \sigma(N_1, N_2, N_3)$.

From Proposition 4 we deduce the following important assertion.

**Corollary 5.** The set of support hyperplanes of $\sigma(N_1, N_2, N_3)$ coincides with the set of hyperplanes $L_\Phi(f) = 0$ where $\Phi(k, \ell, m)$ is an extendible function from $P(N_1, N_2, N_3)$ and there exists a non-trivial polynomial $f_0 \in \sigma(N_1, N_2, N_3)$.
such that
\[ L_\Phi(f_0) = 0. \]
The convex hull of polynomials \( f_0 \in \sigma(N_1, N_2, N_3) \) satisfying relation \( L_\Phi(f_0) = 0 \) we denote by \( D_\Phi \). The number of linearly independent polynomials \( f_0 \) from \( D_\Phi \) we denote by \( \nu_\Phi \). From classical properties of a convex set we obtain the following assertions.

**Corollary 6.1.** If \( \nu_\Phi = 1 \) and \( f_0 \in D_\Phi \) then \( f_0 \) is the extremal polynomial in the class \( \sigma(N_1, N_2, N_3) \).

2. If \( \nu_\Phi > 1 \) then \( D_\Phi \) is the extremal face in the class \( \sigma(N_1, N_2, N_3) \).

3. If \( f_0 \) is an extremal polynomial in the class \( \sigma(N_1, N_2, N_3) \) then there exists a non-trivial extendible function \( \Phi(k, \ell, m) \in P(N_1, N_2, N_3) \) such that \( L_\Phi(f_0) = 0 \).

If \( \Phi(k, \ell, m) \) is an extendible function then there exists a positive measure \( \mu(\alpha, \beta, \gamma) \) such that (Bochner theorem)

\[ \Phi(k, \ell, m) = \frac{1}{(2\pi)^3} \int_G e^{i(k\alpha + \ell\beta + m\gamma)} d\mu \tag{24} \]

where Domain \( G \) is defined by the inequalities \(-\pi \leq \alpha, \beta, \gamma \leq \pi\). From relations (1), (10) and (24) we deduce the following well-known representation (see[7])

\[ L_\Phi(f) = \frac{1}{(2\pi)^3} \int_G f(\alpha, \beta, \gamma) d\mu. \tag{25} \]

**Corollary 7.** If \( f_0(\alpha, \beta, \gamma) \) belongs to the class \( \sigma(N_1, N_2, N_3) \) and in a certain point \( f_0(\alpha_0, \beta_0, \gamma_0) = 0 \) then \( f_0(\alpha, \beta, \gamma) \) is either extremal or belongs to the extremal face.

### 4 The power non-negative polynomials

We shall use the linear Calderon transformation

\[ (Cf)(\alpha, \beta, \gamma) = f_1(x, y, z) \tag{26} \]

which is defined by formulas

\[ e^{i\alpha} = \frac{x+i}{x-i}, \quad e^{i\beta} = \frac{y+i}{y-i}, \quad e^{i\gamma} = \frac{z+i}{z-i}, \]

\[ f_1(x, y, z) = f\left(\frac{x+i}{x-i}, \frac{y+i}{y-i}, \frac{z+i}{z-i}\right) \left(\frac{x^2 + 1}{N_1} + 1\right) \left(\frac{y^2 + 1}{N_2} + 1\right) \left(\frac{z^2 + 1}{N_3}\right)\]. \tag{27}
Proposition 5 (see [9], Ch. 3). The Calderon transformation $C$ maps $\sigma(N_1, N_2, N_3)$ onto $\sigma_P(2N_1, 2N_2, 2N_3)$ and $Q(N_1, N_2, N_3)$ onto $Q_P(2N_1, 2N_2, 2N_3)$.

Using linearity of the operator $C$ we deduce from Proposition 5 the following assertion.

Corollary 8. The Calderon transformation $C$ maps the extremal points and faces of $\sigma(N_1, N_2, N_3)$ and $Q(N_1, N_2, N_3)$ onto extremal points and faces of $\sigma_P(2N_1, 2N_2, 2N_3)$ and $Q_P(2N_1, 2N_2, 2N_3)$ respectively.

Example 2. Let us consider the polynomials

\[ P_1(x, y, z) = (xyz - z + y + x)^2, \]  
\[ P_2(x, y, z) = (yz - xz + xy + 1)^2, \]  
\[ P_3(x, y, z) = (yz - xz - xy - 1)^2, \]  
\[ P_4(x, y, z) = (xz + yz + xy - 1)^2. \]

Proposition 6. Polynomials $P_k(x, y, z)$ ($k=1, 2, 3, 4$) are extremal polynomials in the classes $Q_P(2, 2, 2)$ and $\sigma_P(2, 2, 2)$.

Proof. Using Example 1 and Corollary 8 we deduce that polynomials $P_k(x, y, z)$ ($k=1, 2, 3, 4$) are extremal in the class $Q_P(2, 2, 2)$. We remark that $\deg P_k(x, y, z) \leq 6$. It is proved for this case (see [1]) that the extremal polynomials in the class $Q_P$ are extremal in the class $\sigma_P(2, 2, 2)$ as well. The proposition is proved.

We remark that an extremal polynomial in the class $Q_P$ is given in the paper [1].

5 Extremal trigonometrical and power polynomials in the classes $\sigma$ and $\sigma_P$

In my book [9] I give the method of constructing trigonometrical polynomials belonging to $\sigma(N_1, N_2, N_3)$ but not belonging to $Q(N_1, N_2, N_3)$. This method can also be used for constructing extremal polynomials in the classes $\sigma(N_1, N_2, N_3)$ and $\sigma_P(N_1, N_2, N_3)$. For illustrating this fact we consider the following example.

Example 3. Let us introduce the polynomial

\[ f_0(\alpha, \beta, \gamma) = 4 \cos(\alpha + \beta + \gamma) - \cos(-\alpha + \beta + \gamma) - \cos(\alpha - \beta + \gamma) + \cos(\alpha + \beta - \gamma) \]
It is shown in the book ([9]Ch.3) that $f_0(\alpha, \beta, \gamma) - m(0 < m \leq 4 - 2^{3/2})$ belongs to $\sigma(1, 1, 1)$ but does not belong to $Q(1, 1, 1)$.

Now we consider the polynomial
\[ f(\alpha, \beta, \gamma) = 2^{3/2} - \cos(\alpha + \beta + \gamma) - \cos(-\alpha + \beta + \gamma) - \cos(\alpha - \beta + \gamma) + \cos(\alpha + \beta - \gamma) \]
(33)

The last equality we rewrite in the form
\[ f(\alpha, \beta, \gamma) = 2[2^{1/2} - \cos \alpha \cos(\beta + \gamma) + \sin \alpha \sin(\beta - \gamma)] \]
(34)

**Proposition 7.** The polynomial $f(\alpha, \beta, \gamma)$ is an extremal one in the class $\sigma(1, 1, 1)$.

**Proof.** It follows from formula (34) that the polynomial $f(\alpha, \beta, \gamma)$ has the following zeroes:
1. $\alpha_1 = \pi/4, \beta(0, 1), \gamma(0, 1)$.
2. $\alpha_2 = -\pi/4, \beta(0, 0), \gamma(0, 0)$.
3. $\alpha_3 = \pi/4, \beta(0, -1), \gamma(0, -1)$.
4. $\alpha_4 = -\pi/4, \beta(0, -2), \gamma(0, -2)$.
5. $\alpha_5 = 3\pi/4, \beta(-1, -1), \gamma(-1, -1)$.
6. $\alpha_6 = -3\pi/4, \beta(-1, 0), \gamma(-1, 0)$.
7. $\alpha_7 = 3\pi/4, \beta(1, -1), \gamma(1, -1)$.
8. $\alpha_8 = -3\pi/4, \beta(1, 0), \gamma(1, 0)$.

Since the polynomial $f(\alpha, \beta, \gamma)$ is non-negative all its first derivatives in the points of zeroes are equal to zero. This is also true for the polynomials $g(\alpha, \beta, \gamma) \in \sigma(1, 1, 1)$ and such that $g(\alpha, \beta, \gamma) \leq f(\alpha, \beta, \gamma)$. Thus we obtain 32 linear equations on 27 coefficients of the polynomial $g(\alpha, \beta, \gamma)$. This linear system is of 26th rank, that is only $cf(c = const)$ satisfies this system. The proposition is proved.

From Proposition 7 using the Calderon transformation we obtain the following assertion.

**Proposition 8.** The polynomial
\[ f(x, y, z) = 2^{3/2}(1+x^2)(1+y^2)(1+z^2) + 8z(y+x)(yx-1) - 2(z^2-1)((yx+1)^2 - (x-y)^2) \]
(35)

is an extremal in the class $\sigma_P(2, 2, 2)$.

As it was mentioned in the introductory part some other extremal polynomials in the class $\sigma_P$ were known earlier [2].
6 Non-negative rational functions

Let us consider the class $\sigma_{R}(N, M)$ of the non-negative rational functions of the form

$$R(x, y) = \frac{p(x, y)}{q(x, y)}$$

where $p(x, y)$ and $q(x, y)$ are polynomials with real coefficients and $deg p(x, y, z) \leq N$, and $deg q(x, y, z) \leq M$.

As Artin’s result shows the function $R(x, y, z)$ can be represented in the form

$$R(x, y) = \sum_{k=1}^{r} \frac{p_{k}^2(x, y)}{q_{k}^2(x, y)}$$

(36)

where $p_k(x, y, z)$ and $q_k(x, y, z)$ are polynomials. It is clear that $\sigma_{R}(N, M)$ is a convex set.

**Proposition 9.** If $R(x, y, z)$ is an extremal function of the convex set $\sigma(N, M)$ then $R(x, y, z)$ admits the representation

$$R(x, y) = \frac{p^2(x, y)}{q^2(x, y)}$$

(37)

where $p(x, y, z)$ and $q(x, y, z)$ are polynomials and $p(x, y)^2$ is extremal in the class of the polynomials $Q_{P}$.

Thus the results concerning the class $Q_{P}$(section 4) can be useful for the investigation of the non-negative rational functions.

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