A CHARACTERIZATION OF SEMIAMPLENESS AND CONTRACTIONS OF RELATIVE CURVES

STEFAN SCHRÖER

Abstract. I give a cohomological characterization of semiample line bundles. The result is a generalization of both the Fujita–Zariski Theorem on semiampleness and the Grothendieck–Serre Criterion for ampleness. As an application of the Fujita–Zariski Theorem I characterize contractible curves in 1-dimensional families.

Introduction

The Fujita–Zariski Theorem asserts that a line bundle $\mathcal{L}$ that is ample on its base locus is semiample. Semiampleness means that a multiple $\mathcal{L}^\otimes n$, $n > 0$ is globally generated. For discrete base locus the result goes back to Zariski ([17], Thm. 6.2), and the general form is due to Fujita ([3], Thm. 1.10). This note contains two applications of the Fujita–Zariski Theorem.

The first section contains a generalization of both the Fujita–Zariski Theorem and the cohomological criterion for ampleness due to Grothendieck–Serre. The result is the following characterization: A line bundle $\mathcal{L}$ is semiample if and only if the modules $H^1(X, I \otimes \text{Sym}\mathcal{L})$ are finitely generated over the ring $\Gamma(X, \text{Sym}\mathcal{L})$ for every coherent ideal $I \subset \mathcal{O}_B$. Here $B \subset X$ is the stable base locus of $\mathcal{L}$. This gives a positive answer to Fujita’s question ([3], 1.16) whether it is possible to weaken the assumption in the Fujita–Zariski Theorem.

In the second section I generalize results of Piene [14] and Emsalem [2]. They used the Fujita–Zariski Theorem to obtain sufficient conditions for contractions in normal arithmetic surfaces. Our result is a characterization of contractible curves in 1-dimensional families over local noetherian rings in terms of complementary closed subsets. This also sheds some light on the noncontractible curve constructed by Bosch, Lütkebohmert, and Raynaud ([1], chap. 6.7). For proper normal algebraic surfaces, similar results appear in [15].

1. Characterization of semiampleness

Throughout this section, $R$ is a noetherian ring, $X$ is a proper $R$-scheme, and $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module. According to the Grothendieck–Serre Criterion ([3], Prop. 2.6.1) $\mathcal{L}$ is ample if and only if for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is an integer $n_0 > 0$ so that $H^1(X, \mathcal{F} \otimes \mathcal{L}^\otimes n) = 0$ for all $n > n_0$. Let me reformulate this in terms of graded modules. For a coherent $\mathcal{O}_X$-module $\mathcal{F}$, set

$$H^p_c(\mathcal{F}, \mathcal{L}) = H^p(X, \mathcal{F} \otimes \text{Sym}\mathcal{L}) = \bigoplus_{n \geq 0} H^p(X, \mathcal{F} \otimes \mathcal{L}^\otimes n).$$
This is a graded module over the graded ring $\Gamma_*(\mathcal{L}) = \Gamma(X, \text{Sym} \mathcal{L})$. The Grothendieck–Serre Criterion takes the form: $\mathcal{L}$ is ample if and only if the modules $H^1_i(\mathcal{F}, \mathcal{L})$ are finitely generated over the ring $\Gamma_0(\mathcal{L}) = \Gamma(\mathcal{O}_X)$ for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$. In this form it generalizes to the semiample case. Following Fujita, we define the stable base locus $B \subset X$ of $\mathcal{L}$ to be the intersection of the base loci of $\mathcal{L}^\otimes n$ for all $n > 0$. We regard it as a closed subscheme with reduced scheme structure.

\textbf{Theorem 1.1.} Let $B \subset X$ be the stable base locus of $\mathcal{L}$. Then the following are equivalent:

(i) The invertible sheaf $\mathcal{L}$ is semiample.

(ii) The modules $H^1_i(\mathcal{F}, \mathcal{L})$ are finitely generated over the ring $\Gamma_*(\mathcal{L})$ for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ and all integers $p \geq 0$.

(iii) The modules $H^1_i(\mathcal{I}, \mathcal{L})$ are finitely generated over the ring $\Gamma_*(\mathcal{L})$ for each coherent ideal $\mathcal{I} \subset \mathcal{O}_B$.

\textit{Proof.} The implication (i)⇒(ii) is well known, and (ii)⇒(iii) is trivial. To prove (iii)⇒(i) we assume that $\mathcal{L}$ is not semiample. According to the Fujita–Zariski Theorem the restriction $\mathcal{L}|_B$ is not ample. By the Grothendieck–Serre Criterion there is a coherent ideal $\mathcal{I} \subset \mathcal{O}_B$ with $H^1(X, \mathcal{I} \otimes \mathcal{L}^\otimes n) \neq 0$ for infinitely many $n > 0$. Thus $H^1_*(\mathcal{I}, \mathcal{L})$ is not finitely generated over $\Gamma_0(\mathcal{L})$. Since $B \subset X$ is the stable base locus, the maps $\Gamma(X, \mathcal{L}^\otimes n) \to \Gamma(B, \mathcal{L}^\otimes n_B)$ vanish for all $n > 0$. Consequently, the irrelevant ideal $\Gamma_+(\mathcal{L}) \subset \Gamma_*(\mathcal{L})$ annihilates $H^1_*(\mathcal{I}, \mathcal{L})$, which is therefore not finitely generated over $\Gamma_*(\mathcal{L})$. \hfill\Box

Sommese introduced a quantitative version of semiampleness: Let $k \geq 0$ be an integer; a semiample invertible sheaf $\mathcal{L}$ is called $k$-ample if the fibers of the canonical morphism $f : X \to \text{Proj} \Gamma_*(\mathcal{L})$ have dimension $\leq k$. For example, 0-amplesness means ampleness.

\textbf{Theorem 1.2.} Let $\mathcal{L}$ be a semiample invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is $k$-ample if and only if the modules $H^{k+1}_*(\mathcal{F}, \mathcal{L})$ are finitely generated over the ground ring $R$ for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$.

\textit{Proof.} Set $Y = \text{Proj} \Gamma_*(\mathcal{L})$ and let $f : X \to Y$ be the corresponding contraction. Suppose $\mathcal{L}$ is $k$-ample. Choose $m_0 > 0$ so that $\mathcal{L}^\otimes m_0 = f^*(\mathcal{M})$ for some ample invertible $\mathcal{O}_Y$-module $\mathcal{M}$. Put $\mathcal{G} = \mathcal{F} \otimes (\mathcal{L} \oplus \mathcal{L}^\otimes 2 \oplus \ldots \oplus \mathcal{L}^\otimes m_0)$. Choose $m_0 > 0$ with $H^p(Y, R^q f_*(\mathcal{G} \otimes \mathcal{M}^\otimes m)) = 0$ for $p > 0$, $q \leq k+1$, and $m > m_0$. Consequently, the edge map $H^{k+1}_*(X, \mathcal{G} \otimes \mathcal{L}^\otimes m) \to H^0(Y, R^{k+1} f_*(\mathcal{G} \otimes \mathcal{M}^\otimes m))$ in the spectral sequence

$$H^p(Y, R^q f_*(\mathcal{G} \otimes \mathcal{M}^\otimes m)) \Rightarrow H^{p+q}(X, \mathcal{G} \otimes \mathcal{L}^\otimes m)$$

is injective for $m > m_0$. The fibers of $f : X \to Y$ are at most $k$-dimensional, so $R^{k+1} f_*(\mathcal{G}) = 0$. Thus $H^{k+1}_*(X, \mathcal{F} \otimes \mathcal{L}^\otimes m) = 0$ for all $n > n_0$. Conversely, assume that the condition holds. Seeking a contradiction we suppose that some fiber of $f : X \to Y$ has dimension $> k$. Using we find a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $R^{k+1} f_*(\mathcal{F}) \neq 0$. Replacing $\mathcal{L}$ by a suitable multiple, we have $\mathcal{L} = f^*(\mathcal{M})$ for some ample invertible $\mathcal{O}_Y$-module $\mathcal{M}$. Passing to a higher multiple if necessary, $H^p(Y, R^q f_*(\mathcal{F} \otimes \mathcal{M}^\otimes m)) = 0$ holds for $p > 0$, $q \leq k$, and $n > 0$. Then the edge map $H^{k+1}_*(X, \mathcal{F} \otimes \mathcal{L}^\otimes m) \to H^0(Y, R^{k+1} f_*(\mathcal{F} \otimes \mathcal{M}^\otimes m))$ is surjective for $n > 0$. Choose a global section $s \in \Gamma(Y, \mathcal{M}^\otimes n)$ for some $n > 0$ so that the open subset $Y_s \subset Y$ contains the set of associated points for $R^{k+1} f_*(\mathcal{F})$. Then $s \in \Gamma_*(\mathcal{M})$
is not a zero divisor for $H^0_*(R^{k+1}f_*(\mathcal{F}), \mathcal{M})$. It follows that $H^0_*(R^{k+1}f_*(\mathcal{F}), \mathcal{M})$ is nonzero for infinitely many degrees. Consequently, the same holds for $H^{k+1}_*(\mathcal{F}, \mathcal{L})$, which is therefore not finitely generated over $R$.

\textbf{Remark 1.3.} For a vector bundle $\mathcal{E}$, it might happen that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is semiample, whereas $\text{Sym}^n(\mathcal{E})$ fails to be globally generated for all $n > 0$. For example, let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ be a smooth proper curve of genus $g > p - 1$ so that the absolute Frobenius $\text{Fr}_X : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$ is zero. For an example see [11], p. 348, ex. 2.14. Let $D \subset X$ be a divisor of degree 1. According to the commutative diagram

\[ \begin{array}{cccc}
H^0(\mathcal{O}_X) & \longrightarrow & H^0(\mathcal{O}_D) & \longrightarrow & H^1(\mathcal{O}_X) \\
\text{Fr}_X & & \text{Fr}_X & & \text{Fr}_X \\
H^0(\mathcal{O}_X) & \longrightarrow & H^0(\mathcal{O}_{pD}) & \longrightarrow & H^1(\mathcal{O}_X(-pD)) \quad \longrightarrow \quad H^1(\mathcal{O}_X),
\end{array} \]

the $p$-linear map $\text{Fr}_X : H^1(\mathcal{O}_X(-D)) \to H^1(\mathcal{O}_X(-pD))$ is not injective. Hence there is a nontrivial extension

\[ 0 \to \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(D) \longrightarrow 0 \]

whose Frobenius pull back $\text{Fr}_X(\mathcal{E})$ splits. The surjection $\mathcal{E} \to \mathcal{O}_X(D)$ gives a section $A \subset \mathbb{P}(\mathcal{E})$ representing $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ with $A^2 = 1$ ([11], Prop. 2.6, p. 371). The Fujita–Zariski Theorem implies that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is semiample, and we obtain a birational contraction $\mathbb{P}(\mathcal{E}) \to Y$. It is easy to see that the exceptional set is an integral curve $R \subset \mathbb{P}(\mathcal{E})$ which has degree $p$ on the ruling. Hence $\mathbb{P}(\mathcal{E}) \to Y$ does not restrict to closed embeddings on the fibers of $\mathbb{P}(\mathcal{E}) \to X$. Consequently, $\text{Sym}^n(\mathcal{E})$ is not globally generated at any point $x \in X$.

2. Constructions of relative curves

Throughout this section, $R$ is a local noetherian ring, and $X$ is a proper $R$-scheme with 1-dimensional closed fiber $X_0 \subset X$. Then all fibers of the structure morphism $X \to \text{Spec}(R)$ are at most 1-dimensional. For example, $X$ could be a flat family of curves.

A Stein factor of $X$ is a proper $R$-scheme $Y$ together with a proper morphism $f : X \to Y$ so that $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is bijective (compare [13], sec. 5). Our objective is to describe the set of all Stein factors for a given $X$.

Let $C_i$, $i \in I$ be the finite collection of all 1-dimensional integral components of the closed fiber $X_0$. A subset $J \subset I$ yields a subcurve $C = \bigcup_{i \in J} C_i$. We call such a curve $C \subset X$ contractible if there is a Stein factor $f : X \to \tilde{Y}$ so that $f(C_i)$ is a closed point if and only if $i \in J$. According to [13], Theorem 5.4.1, a Stein factor is determined up to isomorphism by its restriction $f_0 : X_0 \to Y_0$. The task now is to determine the contractible curves $C \subset X$. It follows from [14] and [2] that all curves $C \subset X$ are contractible provided that the ground ring $R$ is henselian. In particular this holds if $R$ is complete. On the other hand, a noncontractible curve is discussed in [13], chapter 6.7.

We seek to describe contractible curves $C \subset X$ in terms of complementary closed subsets $D \subset X$. We need a definition: Suppose $D \subset X$ is a closed subset of codimension $\leq 1$. Let $R \subset R^\prime$ be the completion with respect to the maximal ideal, $X'$ the normalization of $X \otimes_R R^\prime$, and $C'_i, C'', D' \subset X'$ the preimages of
$C_i, D \subset X$, respectively. Let $h : X' \to Z'$ be the contraction of all $C'_i \subset X'_0$ disjoint from $C'$. We call $D$ persistent if $h(D') \subset Z'$ has codimension $\leq 1$.

**Example 2.1.** Suppose $R$ is a discrete valuation ring with residue field $k$ and fraction field $K$. Let $X$ be the proper $R$-scheme obtained from $X' = \mathbb{P}_R^1$ by identifying the closed points $0, \infty \in \mathbb{P}_k^1$. Then the closure $D \subset X$ of the point $0 \in \mathbb{P}_K^1$ is not persistent.

**Theorem 2.2.** Suppose $J \subset I$ is a subset so that the curve $C = \bigcup_{i \in J} C_i$ is connected. Then $C \subset X_0$ is contractible if and only if there is a persistent closed subset $D \subset X$ of codimension $\leq 1$ disjoint from $C$ and intersecting each irreducible component $C_i \subset X_0$ with $i \notin J$.

**Proof:** Assume that $C$ is contractible. The corresponding contraction $f : X \to Y$ maps $C$ to a single point. Let $V \subset Y$ be an affine open neighborhood of $f(C)$. Set $U = f^{-1}(V)$ and $D = X - U$. Clearly $D \cap C = \emptyset$. Furthermore, $D \cap C_i \neq \emptyset$ for $i \notin J$; otherwise $f(C_i)$ would be a proper curve contained in the affine scheme $V$, which is absurd. Let $X', Y'$ be the normalizations of $X \otimes_R R^\wedge, Y \otimes_R R^\wedge$, respectively. The induced morphism $f' : X' \to Y'$ is the contraction of the preimage $C' \subset X'$ of $C$. The preimage $V' \subset Y'$ of $V$ is affine, so $Y - V$ is of codimension $\leq 1$ (II, 2.2.6). Hence the preimage $D' \subset X'$ of $D$ is of codimension $\leq 1$. Obviously, the same holds if we contract the preimages $C'_i \subset X'$ of $C_i$ disjoint from $C'$. Thus $D \subset X$ is of codimension $\leq 1$ and persistent.

Conversely, assume the existence of such a subset $D \subset X$. Set $U = X - D$. We claim that the affine hull $U^{\text{aff}} = \text{Spec } \Gamma(U, \mathcal{O}_X)$ is of finite type over $R$ and that the canonical morphism $U \to U^{\text{aff}}$ is proper.

Suppose this for a moment. Then $U \to U^{\text{aff}}$ contracts $C$ and is a local isomorphism near each $x \in U_0 - C$. Choose for each $x \in X_0 - C$ an affine open neighborhood $U_x \subset X$ of $x$ disjoint to the exceptional set of $U \to U^{\text{aff}}$. Then $U_x \cap U \to U^{\text{aff}}$ is an open embedding. It is easy to see that the schemes $U_x \bigcup_{U_x \cap U} U^{\text{aff}}, x \in X_0 - C$ and $U^{\text{aff}}$ form an open cover of a proper $R$-scheme $Y$. The induced morphism $f : X \to Y$ is the desired contraction.

It remains to verify the claim. Let $R \subset R^\wedge$ be the completion. According to [2], VIII Corollary 3.4, the scheme $U^{\text{aff}}$ is of finite type if and only if $U^{\text{aff}} \otimes_R R^\wedge$ is of finite type. Furthermore, $U \to U^{\text{aff}}$ is proper if and only if $U^{\text{aff}} \otimes_R R^\wedge$ is proper if and only if $U^{\text{aff}}$ is proper after tensoring with $R^\wedge$ ([2], VIII Cor. 4.8). Since $U^{\text{aff}} \otimes_R R^\wedge = (U \otimes_R R^\wedge)^{\text{aff}}$ by [4], Proposition 21.12.2, it suffices to prove the claim under the additional assumption that $R$ is complete.

Now each curve in $X_0$ is contractible. Observe that the contraction of $C$ does not change $U^{\text{aff}}$, so we can as well assume that $C$ is empty. Now our goal is to prove that $U$ is affine. Since $R$ is complete, hence universally Japanese, the normalization $X' \to X$ is finite. Using Chevalley’s Theorem ([2], Thm. 6.7.1), we reduce the problem to the case that $X$ is normal. Now the irreducible components of $X$ are the connected components. Treating them separately we may assume that $X$ is connected. Contracting the curves $C_i$ contained in $D$ we can assume that $D_0$ is finite and intersects each $C_i$. If $D = X$ or $D = \emptyset$ there is nothing to prove. Assume that $D \subset X$ is of codimension 1, in other words a Weil divisor. The problem is that it might not be Cartier. To overcome this, consider the graded quasicoherent $\mathcal{O}_X$-algebra $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{O}_X(nD)$. The graded subalgebra $\mathcal{R}' \subset \mathcal{R}$ generated by $\mathcal{R}_1 = \mathcal{O}_X(D)$ is of finite type over $\mathcal{O}_X$. Set $X' = \text{Proj } (\mathcal{R}')$ and let $g : X' \to X$
be the structure morphism. Then $g$ is projective and $\mathcal{O}_{X'}(1)$ is a $g$-very ample invertible $\mathcal{O}_{X}$-module. The canonical maps $D: \mathcal{O}_{X}(nD) \to \mathcal{O}_{X}((n+1)D)$ induce a homomorphism $\mathcal{R}' \to \mathcal{R}'$ of degree one, hence a section $s: \mathcal{O}_{X'} \to \mathcal{O}_{X'}(1)$. It follows from the definition of homogeneous spectra that $s$ is bijective over $U$ and vanishes on $g^{-1}(D)$. Thus the corresponding Cartier divisor $D' \subset X'$ representing $\mathcal{O}_{X'}(1)$ has support $g^{-1}(D)$.

Let $A \subset X'_0$ be a closed integral subscheme of dimension $n > 0$. If $g(A) \subset X_0$ is a curve, then $A$ is not contained in $D'$ but intersects $D'$. Hence $D' \cdot A > 0$. If $g(A) \subset X$ is a point, then $\mathcal{O}_{A}(1)$ is ample, so $(D')^n \cdot A > 0$. By the Nakai criterion for ampleness we conclude that $\mathcal{O}_{X'}(1)$ is ample on its base locus. Now the Fujita–Zariski Theorem tells us that $\mathcal{O}_{X'}(1)$ is semiample. It follows that $U \simeq X' - D'$ is affine. This finishes the proof.

Let us consider the special case that the total space $X$ is a normal surface. Replacing $R$ by $\Gamma(X, \mathcal{O}_X)$, we are in the following situation: Either $R$ is a discrete valuation ring, such that $X \to \text{Spec}(R)$ is a flat deformation of $X_0$, or $R$ is a local normal 2-dimensional ring, hence $X \to \text{Spec}(R)$ is the birational contraction of $X_0$. In either case we call a Weil divisor $H \in Z^1(X)$ horizontal if it is a sum of prime divisors not supported by $X_0$.

Suppose $J \subset I$ is a subset with $C = \bigcup_{i \in J} C_i$ connected. Let $V \subset X_0$ be the union of all $C_i$ disjoint from $C$.

**Corollary 2.3.** Notation as above. Then $C \subset X_0$ is contractible if and only if there is a horizontal Weil divisor $H \subset X$ disjoint from $C$ with the following property: For each $C_i$, $i \not\in J$, either $H$ intersects $C_i$, or $H$ intersects a connected component $V' \subset V$ with $V' \cap C_i \neq \emptyset$.

**Proof.** Suppose $C \subset X_0$ is contractible. Let $D \subset X$ be a persistent Weil divisor as in Theorem 2.2. Then its horizontal part $H \subset D$ satisfies the above conditions. Conversely, assume there is a horizontal Weil divisor $H \subset X$ as above. It follows that $D = H + V$ is a persistent Weil divisor disjoint from $C$ intersecting each $C_i$ with $i \not\in J$. Thus $C \subset X_0$ is contractible.

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Mathematisches Institut, Ruhr-Universität, 44780 Bochum, Germany

Current address: M.I.T. Department of Mathematics, 77 Massachusetts Avenue, Cambridge MA 02139-4307, USA

E-mail address: s.schroeer@ruhr-uni-bochum.de