Probability Distribution Functions of Sunspot Magnetic Flux

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Abstract

We investigated the probability distributions of sunspot area and magnetic flux by using data from the Royal Greenwich Observatory and USAF/NOAA. We constructed a sample of 2995 regions with maximum-development areas >500 MSH (millionths of solar hemisphere), covering 146.7 yr (1874–2020). The data were fitted by a power-law distribution and four two-parameter distributions (tapered-power-law, gamma, lognormal, and Weibull distributions). The power-law model was unfavorable compared to the four models in terms of AIC, and was not acceptable according to the classical Kolmogorov–Smirnov test. The lognormal and Weibull distributions were excluded because their behavior extended to smaller regions (S ≪ 500 MSH) do not connect to previously published results. Therefore, our choices were tapered-power-law and gamma distributions. The power-law portion of the tapered-power-law and gamma distributions was found to have a power exponent of 1.35–1.9. Due to the exponential falloff of these distributions, the expected frequencies of large sunspots are low. The largest sunspot group observed had an area of 6132 MSH, and the frequency of sunspots larger than 10^3 MSH was estimated to be every 3–8 × 10^3 yr. We also estimated the distributions of the Sun-as-a-star total sunspot areas. The largest total area covered by sunspots on record was 1.67% of the visible disk, and can be up to 2.7% by artificially increasing the lifetimes of large sunspots in an area evolution model. These values are still smaller than those found on active Sun-like stars.

Unified Astronomy Thesaurus concepts: Solar activity (1475); Solar magnetic fields (1503); Solar photosphere (1518); Sunspots (1653); Starspots (1572)

Supporting material: machine-readable table

1. Introduction

Sunspots represent a variety of magnetic activities of the Sun (Solanki 2003; van Driel-Gesztelyi & Green 2015). It is thought that the dynamo mechanism in the solar interior intensifies and transports magnetic flux to the surface and eventually builds up active regions (ARs), including sunspots (Parker 1955). Even after four centuries of continuous observations, the sunspots still maintain important positions in the investigation of the dynamo processes in the Sun.

Another important aspect of sunspots is their close relationship with flare activity (Priest & Forbes 2002; Shibata & Magara 2011; Toriumi & Wang 2019). Statistical investigations have revealed that greater flares emanate from larger ARs (Sammis et al. 2000). This may be natural since larger ARs harbor more magnetic flux and thus have more magnetic free energy available. The largest observed sunspot group since the late-19th century was the one in 1947 April, the largest area of which was 6132 MSH (millionths of solar hemisphere; 1 MSH = 3.04 × 10^6 km^2) or about 1.2% of the visible solar disk (Figure 1). While this region was not flare active, another giant sunspot in 1946 July caused larger flares with geomagnetic disturbances (Toriumi et al. 2017). The formation mechanism of such great ARs is an interesting issue to be resolved.

The existence of spots is also known for other stars (Berdyugina 2005; Strassmeier 2009). One of the largest sunspots reported thus far was from a K0 giant XX Tri (HD 12545), which covered about 20% of the entire stellar surface (Strassmeier 1999). It was found that even solar-type stars producing the so-called superflares (Schaefer et al. 2000; Maehara et al. 2012) host star spots much larger than the solar ones (up to ~10% of the stellar hemisphere; Notsu et al. 2013, 2015). Therefore, the discussion of superflares on the Sun is closely related to the question of the production of super-large sunspots.

The key quantity we study in this paper is the probability distribution function (PDF) of the area S or magnetic flux Φ of sunspots. Bogdan et al. (1988) analyzed the Mt. Wilson white-light observations (1917–1982) of sunspots and showed that the sunspot umbral areas (1.5–141 MSH; the corresponding total sunspot areas would be about 5 times these; Solanki 2003) and follow a lognormal distribution. Hathaway & Choudhary (2008) obtained the same conclusion using the data from the Royal Greenwich Observatory (RGO) and the United States Air Force (USAF; data were compiled and distributed by the National Oceanic and Atmospheric Administration (NOAA)) covering the period of 1874–2007 (sunspot areas larger than 35 MSH). Baumann & Solanki (2005) studied the RGO data (1874–1976) of sunspots with areas >60 MSH, by making a distinction between a snapshot distribution and a maximum-area distribution; the former is derived from daily data (e.g., Bogdan et al. 1988; Hathaway & Choudhary 2008) while the latter is derived by following the time evolution of individual regions and by recording their maximum areas. They found that the two distributions are fitted by lognormal distributions that have similar parameter values. This property was also mentioned in Hathaway & Choudhary (2008).

The fall off of the lognormal distribution toward small sunspot areas may be because smaller magnetic concentrations...
the AR scales, with a power-law exponent of 2.69. All of these results are summarized and discussed later in Section 5 and shown in Figure 7. Balmaceda et al. (2009), Muñoz-Jaramillo et al. (2015), and Mandal et al. (2020) give detailed accounts of the issues with sunspot area calibration. Muñoz-Jaramillo et al. (2015) also tried several models (power law, lognormal, Weibull, exponential, and their combinations) to fit the area distributions.

In this paper, we use the data from RGO (1874–1976) and USAF/NOAA (1977–2020) and derive the maximum areas of individual regions (Section 2). Recurrent regions are counted only once when their areas reach the maximum. In order to compare these with magnetic field observations of ARs and smaller magnetic patches, we convert the sunspot areas to magnetic flux values (Section 2.2). Our primary interest here is whether the sunspot area or magnetic flux distribution extends to large values in the form of a power law or is tapered off, to address whether the Sun may have super-large sunspots like those in super-flare stars. Therefore, we limit the data of sunspots to areas 500 MSH or larger, and try to fit the data with five kinds of distribution functions; power-law, tapered-power-law, gamma, lognormal, and Weibull distributions (Section 3). Statistical examinations (Section 4) and a comparison with previously published results (Section 5) show a preference for the tapered-power-law and gamma distributions. Using the obtained results, we can predict the expected frequencies of super-large sunspots. In Section 6, we adopt a simple time-evolution model of sunspot areas and examine the effects of the assumptions we made in our analysis. Particularly, we estimate the snapshot (instantaneous) distribution of sunspot areas (Section 6.3) and also a distribution of Sun-as-a-star total sunspot areas (Section 6.4), and discuss their implications for super-large stellar spots (Section 7).

2. Data

2.1. Data Sources

The Greenwich Photoheliographic Results (GPR) in PDF files are available through the SAO/NASA Astrophysics Data System, NOAA, and the UK Solar System Data Center. The digitized data of sunspot areas recorded in GPR (1874 April–1976) are provided at several sites. Systematic observation and data collection of sunspots at RGO started on 1874 April 17 (Christie 1907). The correction for foreshortening was explained to have been applied to regions of angular distance within 80° from the disk center (Spencer Jones 1955), but regions beyond were occasionally recorded. From these data sets, we picked up regions whose foreshortening-corrected maximum-development areas (S_md) are 500 MSH or larger. If the corrected area S of a region was monotonically increasing or decreasing on the visible hemisphere, we defined the maximum value of S as the maximum-development area S_md, although the true maximum took place on the backside of the Sun. The effects of this assumption are estimated in Section 6. For recurrent regions, we only retained the maximum S over all

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4 ftp://ftp.ngdc.noaa.gov/STP/SOLAR_DATA/SOLAR_OBSERVATION/GREENWICH/
5 http://www.ukssdc.ac.uk/wdcc1/RGOPHR/
6 ftp://ftp.ngdc.noaa.gov/STP/SOLAR_DATA/SUNSPOT_REGIONS/Greenwich/
7 http://fenyi.solarobs.csfk.mta.hu/GPR/
8 http://solarcyclescience.com/activeregions.html
their disk passages because they were regarded as generated from the identical magnetic flux tubes generated by the dynamo. For this, we had to identify recurrent regions, and such lists are available in Maundr (1909) (1874–1906), the “Catalog of Recurrent Groups of Sun Spots” (1910–1955), “Ledger I: Recurrent Groups” (1916–1955), and “General Catalog of Groups of Sunspots” (1956–1976) sections of GPR.

The data after the cessation of RGO solar observations in 1976 were taken from USAF/NOAA (1977–2020)9. As no convenient lists are available to identify recurrent regions, we did this manually by relying on the following data: Hα synoptic charts in Solar Geophysical Data (SGD)9 (1977–1989), NOAA Report of Solar and Geophysical Activity (RSGA)10 (1990–2000), NOAA Weekly Preliminary Report and Forecast of SGD (Weekly PRF)7 (2001–2009), and Debrecen Photoheliographic Data (DPD)11 (2010–2020). We picked up regions with the foreshortening-corrected maximum-development area exceeding 500/1.20 = 415 MSH (see below). The distribution of angular distances from the disk center did not show a clear decline toward the limb and continued up to 909, implying that the foreshortening correction was applied to all the USAF/NOAA data.

In the end, we selected 2995 regions, 2175 from RGO and 820 from USAF/NOAA data, covering 1874 April to 2020 December, 146.7 yr (Table 1). Before obtaining the final list, we removed 363 and 101 regions from RGO and USAF/NOAA data as they were members of recurrent regions. The data fully cover 13 solar cycles, from Cycle 12 (1878 December to 1890 March) to Cycle 24 (2008 December to 2019 December).

### 2.2. Sunspot Area versus Magnetic Flux

In order to relate the sunspot area and the total radial unsigned magnetic flux (including the flux outside of the sunspots), we used the SHARP data series (Bobra et al. 2014) of the Helioseismic and Magnetic Imager (HMI; Scherrer et al. 2012; Schou et al. 2012) on board the Solar Dynamics Observatory (SDO; Pesnell et al. 2012). We picked up all available CEA (cylindrical equal area)-remapped definitive SHARP data from 2010–2015 that contained the sunspots of NOAA areas $\geq 500$ MSH within 45° from the disk center. In total, 137 patches were collected (one patch per region per day). From each patch, we carefully eliminated magnetic concentrations that were not related to the target ARs.

In order to evaluate the total unsigned magnetic flux in maxwell (Mx) from the list of the sunspot areas, we investigated the relation between the sunspot area, $S_{\text{HMI}}$, and the total unsigned radial flux, $\Phi_{\text{HMI}}$, which can also be calculated from the SHARP data. Note that we measured the total flux only within the smooth bounding curve of the SHARP data to minimize the possibility of including flux that was not related to the target AR and to reduce the noise effect. Figure 2(a) shows a scatterplot between $S_{\text{HMI}}$ and $\Phi_{\text{HMI}}$. From the least-squares fitting to this double logarithmic plot, we obtained the relation between the two parameters,

$$
\log(\Phi_{\text{HMI}} [\text{Mx}]) = (1.010 \pm 0.132) \times \log(S_{\text{HMI}} [\text{MSH}]) + (19.676 \pm 0.400),
$$

(1)

(\log means $\log_{10}$ for natural logarithm we use “ln” in this paper, where we assumed that both $\Phi_{\text{HMI}}$ and $S_{\text{HMI}}$ have errors (Deming 1943; Press et al. 1992). The obtained conversion equation shows that the total flux is almost linearly related to the sunspot area, $\Phi/S \approx 1660$ gauss (G) (AR flux/sunspot area for $S \gtrsim 500$ MSH). For comparison, Schrijver & Harvey (1994) derived AR flux/AR area $\approx 150$ G.

### 2.3. Sunspot Area Calibration

It is known that the USAF/NOAA sunspot areas, $S_{\text{NOAA}}$, are systematically smaller than the RGO ones, $S_{\text{RGO}}$. In general, the multiplication of the USAF/NOAA values by $1.4–1.5$ gives better agreement with the RGO values (e.g., Fligge & Solanki 1997; Hathaway et al. 2002; Coumou et al. 2009; Hathaway 2015). However, Foukal (2014) showed that the inconsistency between the RGO and USAF/NOAA data sets was mainly due to very small sunspots ($\lesssim 2$ MSH) and that the areas for $>300$ MSH equalized. In order to examine if the NOAA records of larger ARs show systematically smaller values, we used the database developed by Mandal et al. (2020), which was calibrated to the same area scale as RGO and was extended to 2021.12 Among our

### Table 1

| No. | Date     | Data Source | Region Number | RGO Area (MSH) | Original Area10 | Magnetic Flux (Mx) |
|-----|----------|-------------|---------------|----------------|-----------------|-------------------|
| 1   | 1947-4-8 | RGO         | 14886         | 6132.0         | 6132            | 3.18E+23          |
| 2   | 1946-2-7 | RGO         | 14417         | 5202.0         | 5202            | 2.69E+23          |
| 3   | 1951-5-19| RGO         | 16763         | 4865.0         | 4865            | 2.51E+23          |
| 4   | 1946-7-29| RGO         | 14585         | 4720.0         | 4720            | 2.44E+23          |
| 5   | 1982-6-15| NOAA        | 3776          | 3720.0         | 3100            | 1.92E+23          |
| 6   | 1926-1-19| NOAA        | 9861          | 3716.0         | 3716            | 1.92E+23          |
| 7   | 1989-9-4 | NOAA        | 5395          | 4320.0         | 3600            | 2.23E+23          |
| 8   | 1990-11-19| NOAA       | 3776          | 3720.0         | 3100            | 1.92E+23          |
| 9   | 1938-1-21| NOAA        | 12673         | 3627.0         | 3627            | 1.87E+23          |
| 10  |          |             |               |                |                 |                   |

Notes. Table 1 is published in its entirety in machine-readable format. Only the top ten regions are shown here to explain the format of the table. NO. NOAA area before being converted to the RGO scale.

(This table is available in its entirety in machine-readable form.)
820 NOAA regions we excluded 58 regions on which we suspect that NOAA and Mandal et al. (2020) used different group definitions.

Figure 2(b) shows a comparison of the 762 regions. We found that the mean and standard deviation of the area ratios, $S_{\text{RGO}}/S_{\text{NOAA}}$, are $1.20 \pm 0.01$. Therefore, we simply assumed that the RGO data sets provide reliable values and the NOAA values are converted by

$$S = S_{\text{HMI}} = S_{\text{RGO}} = 1.20 \times S_{\text{NOAA}}. \quad (2)$$

Mandal et al. (2020) obtained a conversion factor of 1.48, but the method of comparison is different; they compared daily data while our data are region-wise, maximum-development areas. We also performed the analysis adopting a conversion factor of about 1.4 and basically found the same results, although detailed numerical values changed.

Under the assumption of Equation (2), we then applied Equation (1) to the maximum-development sunspot areas $S_{\text{md}}$ of both RGO and USAF/NOAA to generate the database on the maximum-development magnetic flux $\Phi_{\text{md}}$. A 500 MSH sunspot corresponds to a magnetic flux of $2.53 \times 10^{22}$ Mx. The largest sunspot so far, RGO 14886 in 1947 April (6132 MSH), is estimated to have a total flux of $3.18 \times 10^{23}$ Mx. Flux estimations of modern events such as NOAA 9169, 9393, 10486, and 12192 showed good agreements with the independent measurements by, e.g., Tian & Alexander (2008), Smyrli et al. (2010), Criscuoli et al. (2009), Zhang et al. (2010), and Sun et al. (2015).

3. Statistical Analysis

3.1. Definitions

Our data are made of $N_s = 2995$ values of sunspot magnetic flux at their maximum development, $\Phi_{\text{md},i}$ ($i = 1, 2, \ldots, N_s$) (subscript “s” denotes “sources” or “source flux tubes”). In the following, we simply designate $\Phi_{\text{md}}$ as $\Phi$ unless we have to distinguish between maximum-development values and other cases. The minimum value of $\Phi_s$ is $\Phi_1 = 2.53 \times 10^{22}$ Mx, corresponding to an area of 500 MSH. As a histogram representation of data depends on how one defines the data bins (Clauset et al. 2009), we will work on the complementary cumulative distribution function (CCDF; represented by $F(\Phi)$) as shown in Figure 3(a), which is uniquely defined in terms of a given observational data set (Equation (A4)). The CCDF is a decreasing function of its argument while the cumulative distribution function (CDF) is an increasing function; therefore, the usage of the CCDF is intuitively more straightforward when used in comparing data with a PDF, which is also a decreasing function of its argument in the present case.

The slope (or derivative) of the CCDF is the usual PDF (represented by $P(\Phi)$), but here we introduce dimensional parameters and define the flux emergence rate $f(\Phi)$ in units of regions $\text{Mx}^{-1} \text{Mm}^{-2} \text{d}^{-1}$ as (Figure 3(b))

$$f(\Phi) = P(\Phi) \frac{N_j}{A T}, \quad P(\Phi) = -\frac{dF(\Phi)}{d\Phi}, \quad (3)$$

where $A$ stands for the area of observation (the full Sun, e.g., $6.2 \times 10^6 \text{Mm}^2$ in Sections 4 and 5 and the hemisphere in Sections 6.3) and $T = 146.7 \text{yr} = 5.36 \times 10^4 \text{days}$. A histogram is generated showing $f(\Phi_j) = N_j/(A T \Delta \Phi_j)$, where $j$ stands for the bin number, and $N_j$ is the number of sunspot groups with flux values ranging from $\Phi_j$ to $\Phi_j + \Delta \Phi_j$. The flux bins are taken equidistant in log $\Phi$ ($\Delta \log \Phi = \text{constant} = 0.125$), so that $\Delta \Phi_j = \Phi_j \ln(10) \Delta(\log \Phi_j)$. The quantity $N_s = \sum_j N_j$ is explicitly used when we compare the observed histogram with a theoretical distribution function $P$.

In Figures 3(a) and (b), the solid curves represent the maximum-development areas while the dotted curves show the cases where we did not exclude the recurrent regions other than their maximum-area development. We can expect that a larger region may appear multiple times at smaller areas and increase

![Figure 2](image-url)
the counts at lower bins. The effect of not excluding recurrent regions is to slightly steepen the slope of the distribution.

3.2. Models and Fitting Procedures

In this paper, we try to fit the observed data set \( \Phi_t \) by

1. A power-law distribution (parameter: \( \alpha_1 \)),
2. A tapered-power-law distribution (parameters: \( \alpha_2, \beta_2 \)),
3. A truncated gamma distribution (parameters: \( \alpha_3, \beta_3 \)),
4. A truncated lognormal distribution (parameters: \( \mu, \sigma \)), and
5. A truncated Weibull distribution (parameters: \( k, \beta_3 \)).

A tapered-power-law distribution (also called the Pareto distribution of the third kind (Johnson et al. 1994) in contrast to the original Pareto distribution, which is a pure power law) has a CCDF that is a product of a power law and an exponential function. A gamma distribution has a PDF that is a product of a power law and an exponential function, and its CCDF is represented in terms of the gamma function. These have been used to describe the distribution functions of earthquake magnitudes (e.g., Kagan 2002; Serra & Corral 2017), in comparison with the power-law distribution, which is the Gutenberg–Richter relation in seismology (e.g., Utsu 1999).

The power-law distribution is a one-parameter model while the other four are two-parameter models. Models (1)–(3) have a power-law component with the exponents \( \alpha_1, \alpha_2, \) and \( \alpha_3, \) respectively. Models (2), (3), and (5) have an exponential component whose decay coefficients are represented by \( \beta_2, \beta_3, \) and \( \beta_5. \) The best-fit parameter values can be given systematically by the maximum-likelihood estimator (MLE), by maximizing the log-likelihood (LLH),

\[
\text{LLH} = \sum \ln(P(\Phi_t)).
\]

The definitions of these five distributions and their MLE solutions are given in Appendix A.

Whether one model is superior to the others can be assessed by comparing the AIC values (Akaike 1974),

\[
\text{AIC} = -2 \times \max(\text{LLH}) + 2K,
\]

where \( K \) is the number of parameters (\( K=1 \) or 2 in our analysis). (In another often-used criteria, Bayesian information criteria or BIC, \( K \ln n \) (\( n \) is the sample size) is used instead of \( 2K \); Burnham & Anderson 2002.) By increasing the number of parameters, the fitting becomes better and LLH increases. However, the introduction of more parameters is not justified if AIC does not decrease. If the AIC value of one model is smaller than the AIC of the other by 9–11 or more, the former model is regarded as better than the latter (Burnham et al. 2011).

AIC is a relative measure, and it is possible that the model preferred by AIC still gives a poor fit. Whether the fitting by one model is not satisfactory and should be rejected can be estimated by evaluating a statistical measure that quantifies the difference between the observed and modeled CCDFs and by comparing that measure with a theoretical threshold (Stephens 1970, 2016). Here, we use the Kolmogorov–Smirnov (K-S) statistic defined by Equation (B1). For a large-enough \( N_x \), the probability for the observed \( \sqrt{N_x} \) K-S value or larger to be obtained is theoretically given as a function (K-S function) that has a relatively weak dependence on \( N_x \). Alternatively, we can utilize simulation runs to estimate the probability (see below).

Once the parameter values are obtained, we were able to estimate their error ranges using the parametric bootstrap method (Efron & Tibshirani 1993; Burnham & Anderson 2002) as follows:

![Figure 3](https://example.com/figure3.png)

**Figure 3.** (a) The CCDF \( F(\Phi) \) of the magnetic flux \( \Phi_{\text{rad}} \) (Mx) of large sunspot regions (\( S_{\text{md}} \geq 500 \) MSH) at their maximum development, observed from 1874 April to the end of 2020 (146.7 yr). The top horizontal axis shows the corresponding sunspot areas. The dotted curve represents the distribution when recurrent regions not at their maximum development are all counted. (b) A histogram representing the distribution function \( f(\Phi_{\text{rad}}) \) of emergence rates of the magnetic flux \( \Phi_{\text{rad}} \) of large sunspot regions, per bin width, unit area in Mm², and day. The bin width is \( \Delta \log \Phi_{\text{rad}} = 0.125 \). Vertical bars indicate the \( \sqrt{N} \) uncertainty of \( N \) regions contained in each bin (the last bin only contains two regions). The dotted histogram represents the distribution when recurrent regions not at their maximum development are all counted.
Figure 4. (a) An MLE power-law fit (straight line) to the CCDF of magnetic flux $F(\Phi)$. (b) The power-law fit of panel (a) for $F(\Phi)$ is converted to the PDF $f(\Phi)$ and is overplotted on the histogram of the data.

1. Generate $N_s$ sets of uniform random variables $y_i$, $i = 1, 2, \ldots N_s$ ($0 \leq y_i \leq 1$).
2. Find $\Phi_i$ such that $F(\Phi_i) = y_i$.
3. For the data set $\Phi_i$ ($i = 1, 2, \ldots N_s$), obtain the MLE solutions for the parameters and the K-S statistic.
4. Repeat (1)–(3) $m$ times; then the distributions of the parameter values and the K-S statistic are obtained.
5. Evaluate the $1\sigma$ width of these distributions and adopt them as the error ranges of the parameters.
6. For the K-S statistic, if the number of cases where the K-S values exceed the observed value $K-S_{obs}$ is obtained under the assumed model.

The $1\sigma$ values defined in step (5) scale basically as $1/\sqrt{N_s}$ and do not depend on $m$ if it is taken sufficiently large (we used $m = 10^3$).

The procedure described above has some points to consider. First, the MLE solution is not intended to geometrically fit a model CCDF to the observed CCDF. Particularly, it does not consider much about the fitting at the tail because the MLE solution is mostly determined by data points of the highest density, i.e., near to the lower end of the distribution. A more geometrically favorable solution may be obtained by, say, directly minimizing the K-S statistic. Second, the K-S statistic is not sensitive to misfitting at the lower and upper ends of the distribution because by definition both the observed and model CCDF match (taking the values of 0 or 1) at the ends. It is known that the contributions to K-S roughly scales as $1/\sqrt{F(1-F)}$ (Anderson & Darling 1952), and Clauset et al. (2009) suggested using a modified form of the K-S statistic by dividing its components by $1/\sqrt{F(1-F)}$ to enhance the sensitivity of the test at both ends. Here, we introduce a K-S statistic defined by Equation (B2), by only dividing by $1/\sqrt{F}$ to enhance the sensitivity at the tail.

In summary, we apply two methods.

1. Method 1: Seek the MLE solution, check the AIC values, and test the K-S statistic by the K-S function. Then by simulation runs, estimate the error ranges of the obtained parameters, and the $P$-value of the observed K-S statistic.
2. Method 2: Minimize K-S to obtain the solution, check the AIC values, and test the K-S statistic by the K-S function. Then by simulation runs, estimate the error ranges of the obtained parameters, and the $P$-value of the observed K-S statistic.

Method 2 was not applied to the power-law model because the deviations of the model at the tail are large and the introduction of K-S might not make sense.

4. Fitting Results

4.1. Power Law

Figure 4 shows the results of the power-law fitting. The MLE solution for the exponent is $\alpha = 2.91$, and the K-S statistic is large, $\Delta BIC = 4.1$. Therefore, the probability of obtaining such a value or larger is infinitesimally small, and the model is safely rejected.

4.2. Two-parameter Models

Figures 5(a)–(d) show the results of fitting by adopting MLE solutions (Method 1). Method 2 also gives similar plots. In Figure 6, both the CCDF and emergence rates $f(\Phi_{\text{mod}})$ together with the fitted functions are given by applying Method 2 to the gamma distribution, which is our most favored model.

Table 2 summarizes the results obtained from the MLE and K-S (except for the power-law) methods; the obtained parameter values (with error ranges), relative AIC values, K-S statistics and its theoretical probability, and $P$-values based on the K-S statistic. Here, $\Delta AIC (\Delta BIC = \Delta AIC$ among the two-parameter models) means the values of AIC with respect to its smallest value in the models (which happened to take place
for the MLE model applied to the Weibull distribution). Strictly speaking, AIC is defined when LLH is maximized (the MLE solution, Equation (5)), but we also applied the same formula by replacing max(LLH) with LLH from a particular solution not maximizing LLH. Therefore, for each model, ΔAIC is smaller for the MLE solution than for the solution with K-Sr = min. Likewise, the P-value derived from K-Sr is larger (better fitting) for the solution with K-Sr = min compared to the MLE solution.

The following properties can be found in this table.

1. The AIC values of the two-parameter models are much smaller than the case of the power law, so that all four models are better than the power law. The four models show ΔAIC values less than 5 and cannot be discriminated.
2. From the K-S probabilities, the MLE solutions show better performance but the K-Sr = min solutions are also acceptable.
3. From the P-values of the K-Sr metric, the solutions minimizing K-Sr show better performance but the MLE solutions are also acceptable.
4. The power-law indices of the tapered-power-law and the gamma distributions are $\alpha_2 = 1.8 - 1.9$ and $\alpha_3 = 1.35 - 1.45$. The reason why $\alpha_2 > \alpha_3$ is given in Appendix A.4; the tapered-power-law distribution
actually contains a mixture of two exponents $\alpha_2$ and $\alpha_2 - 1$, and its overall behavior is somewhere between them. If we extend the PDF toward smaller $\Phi_{md}$ values, the tapered-power-law distribution asymptotically approaches the power law with exponent $\alpha_2$. In any case, it is important to point out that both distributions show the power-law-like behavior with an exponent less than 2, namely, the overall contributions to the magnetic flux supply come mostly from large $\Phi_{md}$ regions.

5. As we discuss later (Figure 7), the behavior of the lognormal and Weibull distributions extended to smaller values of $\Phi_{md}$ is different from the tapered-power-law and gamma distributions (the latter two behave essentially like a power law with exponent $\approx 1.35$–1.9). The Weibull distributions approach a very flat power-law distribution $\propto \Phi_{md}^{-0.4}$, and the lognormal distributions decrease toward small $\Phi_{md}$. Therefore, our preferred models are the tapered-power-law and gamma distributions.

6. The tapered-power-law and gamma distributions show a steep falloff for large $\Phi_{md}$ values, steeper than power laws, indicating that the probability of having extremely large ARs is vanishingly small and there is a practical upper limit in the size and magnetic flux of emerging ARs.

Figure 6. (a) A fit to the CCDF $F(\Phi_{md})$ using the gamma-distribution model and by minimizing the K-Sr metric. (b) The gamma-distribution fit in panel (a) for $F(\Phi_{md})$ is converted to the PDF $f(\Phi_{md})$ and overplotted on the histogram of the data.

Table 2

| Model      | Method   | Parameter Values and Errors | $\Delta$AIC | $\sqrt{N_s}$ K-S | K-S Prob. | K-Sr P-value |
|------------|----------|----------------------------|-------------|----------------|-----------|-------------|
| Power      | MLE      | $\alpha_1 = 2.914 \pm 0.035$ | 184.6       | 4.14           | $<10^{-6}$ | 0.019       |
| Tapered    | MLE      | $\alpha_2 = 1.808 \pm 0.091$ | 2.79        | 1.059          | 0.210     | 0.290       |
|            | K-Sr = min $\alpha_2 = 1.886 \pm 0.153$ | $\beta_2 = 0.614 \pm 0.083$ | 3.77        | 1.211          | 0.105     | 0.444       |
| Gamma      | MLE      | $\alpha_3 = 1.358 \pm 0.138$ | 1.51        | 0.989          | 0.279     | 0.458       |
|            | K-Sr = min $\alpha_3 = 1.450 \pm 0.224$ | $\beta_3 = 0.547 \pm 0.083$ | 2.11        | 1.08           | 0.195     | 0.665       |
| Lognormal  | MLE      | $\mu = -0.300 \pm 0.099$ | 1.55        | 0.630          | 0.817     | 0.414       |
|            | K-Sr = min $\mu = -0.157 \pm 0.188$ | $\sigma = 0.728 \pm 0.061$ | 4.27        | 1.075          | 0.196     | 0.490       |
| Weibull    | MLE      | $k = 0.625 \pm 0.064$ | 0.00        | 0.827          | 0.497     | 0.911       |
|            | K-Sr = min $k = 0.629 \pm 0.068$ | $\beta_5 = 2.270 \pm 0.217$ | 0.06        | 0.868          | 0.434     | 0.850       |

Notes.

a Relative AIC values with respect to the minimum value of all the AIC values.

b K-S statistic derived from the observed data and the assumed model, multiplied by $\sqrt{N_s}$.

c Theoretical probability that the K-S metric shows values larger than the observed K-S metric.

d Probability based on the simulation runs showing that the simulated K-Sr values are larger than the observed K-Sr value.
5. Comparison with Published Results

In this section, we compare our results on the two-parameter models with the published observational data and fitting results. In Figure 7(a), which shows the flux emergence rates $f(\Phi_{md})$, the histogram in black represents our data (500 MSH $\leq S_{md} \leq 6132$ MSH, $2.53 \times 10^{22}$ Mx $\leq \Phi_{md} \leq 3.18 \times 10^{23}$ Mx). The four solid curves show our fits by minimizing the K-Sr metric; tapered-power-law (red), gamma (olive), Weibull (teal green), and lognormal (lime green) distributions. They are extended down to $\Phi_{md} = 10^{20}$ Mx ($S_{md} \approx 2$ MSH) and up to $\Phi_{md} = 10^{24}$ Mx ($S_{md} \approx 19,000$ MSH). Toward the smaller ends of $\Phi_{md}$, the tapered-power-law, gamma, and Weibull distributions approach power laws with exponents 1.89, 1.45, and 0.37, respectively.

The histogram in purple represents the data taken from Harvey & Zwaan (1993), who derived the emergence rates of bipolar ARs using the data obtained at NSO Kitt Peak (Livingston et al. 1976). Schrijver & Harvey (1994) reported that this distribution is fitted by a power law with an exponent of 2, and the amplitudes vary by roughly a factor of 10 between activity minimum and maximum, as shown by the two parallel lines in purple.

The thick dashed curve in lime green reproduces the lognormal fit to the RGO data ($S_{md} \geq 60$ MSH) by Baumann & Solanki (2005), extended down to $S_{md} \approx 2$ MSH (thin dashed curve). Below the peak at $S_{md} \approx 62.2$, the curve goes down to $f \rightarrow 0$. Our lognormal fit peaks at $S_{md} \approx 250$ MSH and then decreases to $f \rightarrow 0$. At least for our lognormal fitting, this decrease is not due to small flux tubes losing their darkness because even the smallest regions (500 MSH) in our sample are fairly large regions. Rather, this is a result of the shape of the observed distribution that bends down toward large $\Phi_{md}$, and the derived $\mu$ value may not represent any physical significance. Bogdan et al. (1988) gave the values 0.34–0.62 MSH for the peak of the instantaneous distribution function of sunspot umbral areas modeled by a lognormal distribution.

The thick dashed curve in teal green reproduces the Weibull function fit from Gopalswamy (2018), which covers all the data $S_{md} \geq 1$ MSH. The curve is very close to our Weibull fit.

In Figure 7(b), we extended the plot range to $10^{15}$ Mx $\leq \Phi_{md} \leq 10^{25}$ Mx ($2.3 \times 10^{-3}$ MSH $\leq S_{md} \leq 1.9 \times 10^{5}$ MSH), and added two more data sets. The thick solid curve in brown represents the data from Hagenaar et al. (2003), who investigated the emergence rates of small-scale bipolar magnetic patches (ephemeral regions) using the data from SOHO/MDI. The thick curves in navy blue (solid, dashed, dotted) represent data from Thornton & Parnell (2011, Figure 5), who analyzed the emergence rates of small-scale magnetic patches using the data from Hinode/SOT. The thin dashed line in teal green represents the downward extension of Gopalswamy (2018)’s Weibull distribution.

The thick dashed lines in black in Figures 7(a) and (b) show the power law of exponent 2.69 suggested by Thornton & Parnell (2011) to cover all the way from small-scale flux concentrations to large ARs. Our picture is different from theirs; the flux emergence rates of ARs are characterized by a power-law-like behavior of exponents between 1.45 (olive) and 1.89 (red) as shown in Figure 7. Our results are roughly consistent with the observations by Harvey & Zwaan (1993), who also showed that the distribution amplitudes varied by a factor of 10 between activity minima and maxima. Ephemeral regions and much smaller flux concentrations may have a...
power-law distribution with an exponent of 2.69, but they show little changes (or even anti-phase changes; Hagenaar et al. 2003) with the solar cycle, and they may give way to the AR component somewhere at around 2003. Many models have been proposed to represent the time evolution of sunspot areas (e.g., Kopecký 1956; Antalova & Macura 1986; Howard 1992; Hathaway & Choudhary 2008). Here, we use the model proposed by Kopecký (1956) because of its analytical simplicity and versatility. The time evolution of sunspot area \( S(t) \) is described by a differential equation

\[
\frac{dS}{dt} = -aS + K - bt \quad (t \geq 0),
\]

where \( a, b, \) and \( K \) are parameters; \( (a, b) \) control the shape of the time profile, and \( K \) controls the maximum size of the region. The solution to this equation is given as

\[
S = \frac{1}{a} \left[ K \left( 1 - e^{-at} \right) - bt \right],
\]

and \( S(t) \) takes the maximum value

\[
S_{\text{md}} = K - b t_{\text{md}} = \frac{b}{a} \ln \frac{aK + b}{b},
\]

at time

\[
t_{\text{md}} = \frac{1}{a} \ln \frac{aK + b}{b}.
\]

\( S(t) \) starts from \( S(0) = 0 \) and returns to \( S(t_{\text{life}}) = 0 \), where

\[
1 - \exp(-at_{\text{life}}) = \frac{b}{aK + b}.
\]

The solution for \( t_{\text{life}} \), if \( at_{\text{life}} \gg 1 \), is

\[
t_{\text{life}} \approx \frac{K}{b} + \frac{1}{a},
\]

and if \( at_{\text{life}} \ll 1 \), it is

\[
t_{\text{life}} \approx \frac{2K}{aK + b}.
\]

Kopecký (1956) used the empirical relations \( t_{\text{life}} \) (days) \( \approx 0.1 S_{\text{md}} \) (MSH) (Gnevyshev 1938) and \( t_{\text{life}}/t_{\text{md}} \approx 0.094 S_{\text{md}} \) (MSH) + 9.3 (Kopecký 1953) and adopted \( (a, b) = (0.3, 4.0) \), which roughly reproduces these empirical relations for \( S_{\text{md}} \leq 400 \) MSH. However, this setting makes the lifetimes of 500 MSH (minimum in our database) and 6132 MSH (maximum) regions as 50 days (1.8 solar rotations) and 480 days (1.3 yr, 17.6 solar rotations), respectively, which look too long. Nagovitsyn et al. (2019) suggested \( t_{\text{life}} \) (days) \( \approx 0.077 S_{\text{md}} \) (MSH), a slightly shorter lifetime for the specified \( S_{\text{md}} \) compared that in Gnevyshev (1938), but this also gives large values of lifetimes. According to Kopecký (1984), the

### Table 3

| \( S \) (MSH) | \( \Phi \) (Mx) | Interval (yr) |
|-------------|--------------|--------------|
| 500         | \( 2.5 \times 10^{22} \) | \( 5.242 \pm 0.001 \times 10^{-2} \) |
| 1000        | \( 5.1 \times 10^{22} \) | \( 1.82 \pm 0.4 \times 10^{-1} \) |
| 2000        | \( 1.0 \times 10^{23} \) | \( 1.20 \pm 0.4 \times 10^{0} \) |
| 3000        | \( 1.5 \times 10^{23} \) | \( 6.0 \pm 0.9 \times 10^{0} \) |
| 6132        | \( 3.2 \times 10^{23} \) | \( 5.2 \pm 0.2 \times 10^{2} \) |
| 10,000      | \( 5.2 \times 10^{23} \) | \( 8.2 \pm 0.3 \times 10^{4} \) |

Note. The ranges of values are based on the 1σ error ranges given in Table 2, namely, \( \alpha_3 = 1.450 \pm 0.224 \), \( \beta_3 = 0.547 \pm 0.083 \).
The time pro-

region size. The lifetimes of the 500 and 6132 MSH regions in

Hence, we roughly assigned a 7 day error bar. The regions whose

visible disk were assigned a 14 days error bar. The model with

sunspot lifetimes. The diamond signs denote the data points of the

longest lifetime of regions in the RGO observations was 8 solar

rotations (RGO recurrent series No. 2094, 1970 June 11 to

December 23, 195 days, maximum area = 1774 MSH). The region with the largest area (RGO region 14886, 6132 MSH)

had a lifetime of 95 days (1947 February 5 to May 11, observed

for four rotations).

As will be discussed in Section 6.3, the gamma function model for $F(\Phi_{\text{MD}})$ described in Section 5.1 gives the instantaneous distributions of sunspot area or magnetic flux, which are consistent with the observed distributions only if the region lifetimes are much shorter; a reasonable value we found is $(a, b) = (0.3, 20.0)$. The value of $a$ is fixed to 0.3 because the rise time of region growth ($\approx 1/a$) does not strongly depend on the region size. The lifetimes of the 500 and 6132 MSH regions in this model are 16 and 107 days, respectively. Figure 8(a) shows the time profiles $S(t)$ derived for $(a, b) = (0.3, 20.0)$ and for several values of $S_{\text{MD}}$. Figure 8(b) compares the models with $(a, b) = (0.3, 4.0)$ and $(0.3, 20.0)$, and other published results of sunspot lifetimes. The diamond signs denote the data points of the three longest-lived regions listed in Kopecky (1984). The asterisks denote the lifetimes of maximum sunspot areas $>3000$ MSH taken from Spencer Jones (1955). The regions whose emergence or decay (either of them) were not observed on the visible disk have uncertainties in their lifetimes between 1 and 13 days. Hence, we roughly assigned a 7 day error bar. The regions whose emergence and decay (both of them) were not observed on the visible disk were assigned a 14 days error bar. The model with $(a, b) = (0.3, 20.0)$ goes through the middle of the data points representing large ($>3000$ MSH) regions.

6.2. Effects of Time Evolution

The effects of the time evolution of sunspots were simulated as follows. By adopting a model of gamma distribution with $\alpha_3 = 1.450$ and $\beta_3 = 0.547$, we generated $N_s$ samples $(N_s = 2995)$ of $\Phi_{\text{MD}}$ by

$$F(\Phi_i) = i - 0.5 \frac{N_s}{N_s} (i = 1, 2, \ldots, N_s).$$

Then each model was placed at 27 equal-distant longitudes (27 is a rough number of the solar rotation period in days), and the magnetic flux was converted to the sunspot area by Equation (1), and evolved according to Equation (7). The observed maximum values of $S(t)$ in the longitude ranges of $\pm 80^\circ$ were recorded, converted to magnetic flux, and the CCDF was generated.

The solid curve in Figure 9 shows the result, compared with the true distribution designated by the dashed curve. After fitting the model, we found $\alpha_3 = 1.319$ and $\beta_3 = 0.590$. The flattening of the distribution from $\alpha_3 = 1.450$ to $\alpha_3 = 1.319$ (by about 0.13) is because we underestimated $S_{\text{MD}}$ and the data points on the original model (dashed) were shifted toward the left on the graph. From this, we can estimate that $\alpha_3 = 1.450$ derived from observations would actually be around $\alpha_3 = 1.68$, but still our conclusion will hold that the power-law exponent is less than 2.

6.3. Instantaneous Distribution

Once we have a model of the time evolution of sunspot areas, we can generate the instantaneous distribution function from the distribution of maximum-development areas. First, consider a simple case where the time evolution is represented by a step function (i.e., a spot appears with a maximum-development area and stays so until it suddenly disappears). If the lifetimes of regions do not depend on the areas and take a fixed value, the instantaneous distribution function is the same as the maximum-development...
distribution function. If the lifetimes are proportional to the areas, the instantaneous distribution function would be flatter than the maximum-development distribution function because larger regions live longer and have a higher probability of existence in snapshot data. In the general cases where the time evolution of sunspot areas is a function of time like in Equation (7), one must resort to numerical simulations to estimate the instantaneous distribution function of sunspot areas or flux emergence rates.

By adopting a model of gamma distribution with $\alpha_3 = 1.450$ and $\beta_3 = 0.547$, we generated $N_s$ samples of $\Phi_{md}$ by Equation (13) and converted them to $S_{md}$ by Equation (1). This time we extended the lower limit of the sunspot area and flux to 100 MSH and $4.97 \times 10^{21}$ Mx, so that $N_s = 1.8 \times 10^5$; the number of regions with $S_{md} \geq 500$ MSH was still 2995. Next, we used Equation (7) to evaluate and record daily values of $S(t)$ and $\Phi(t)$, leading to data samples $N_{rd} = 1.90 \times 10^4$ (region $\times$ day per hemisphere) for $S(t) \geq 500$ MSH. In this process we introduced a filter made of 13 consecutive 1s followed by 14 consecutive 0s, mimicking a 27 day modulation of visibility. This filter, replicated many times to cover the lifetimes of regions and its initial point randomly shifted between 0 and 26 points, was multiplied by daily values of $S(t)$ and $\Phi(t)$. Thus, we derived the histogram per unit area (Figure 10(a), the solid line in the histogram) by taking $A = \text{area of the solar hemisphere} = 3.1 \times 10^7$ Mm$^2$. Roughly this distribution is fitted by a gamma distribution with $\alpha_3 = 0.928$ and $\beta_3 = 0.654$. The power-law exponent decreased (1.450 → 0.928), giving a much flatter distribution. The observed value of $N_{rd}$ obtained by counting regions with $S(t) \geq 500$ MSH every day is $1.97 \times 10^4$ (region $\times$ day per hemisphere) and roughly agrees with the simulated results. On the other hand, if we used the evolution parameters $(a, b) = (0.3, 4.0)$, we ended up with a much larger value $N_{rd} = 6.10 \times 10^4$ (region $\times$ day per hemisphere), and the resulting data are shown as the dashed histogram in Figure 10(a). As a matter of fact, we selected the values $(a, b) = (0.3, 20.0)$ so that the value of $N_{rd}$ roughly matches the observation.

In deriving the instantaneous distribution functions of sunspot magnetic flux from the observed data, we divided the data into three periods; activity maximum, intermediate state, and activity minimum. (The reason we did so is given in Section 6.4.) The dates (year and month) of activity maxima/minima are taken from SILSO (Sunspot Index and Long-term Solar Observations), and we define the maximum/minimum periods as being within 1.356 yr from the maximum/minimum dates. The mean lengths of the maximum, intermediate (between the maximum and minimum periods), and minimum states are 2.712, 5.424, and 2.712 yr, respectively. The mean length of the 13 cycles studied here is 10.85 yr, and 10.85/8 = 1.356 yr. The regions of $N_s = 2995$ with $S_{md} \geq 500$ MSH were divided into maximum (1466), minimum (114), and intermediate (1415) states, respectively (Table 4).

Figure 10(b) shows the observed instantaneous distribution functions of sunspot magnetic flux when all the data were used (the histogram in black) as well as the three activity phases treated separately (the histograms in red, green, and blue). We used all the data with $S \geq 100$ MSH. The histograms are all similar, meaning that they only change the magnitude and not the form of distribution. Particularly the histogram using all the data is reproduced well by the simulated results with $(a, b) = (0.3, 20.0)$ (Figure 10(a), the histogram in solid lines).

6.4. Sun-as-a-Star Distributions

From the instantaneous distribution of magnetic flux or sunspot areas, we can also derive the distribution function of total magnetic flux or total area of sunspot regions, summed over the hemisphere. This is a more fundamental quantity in considering solar irradiance modulation or luminosity variations of stars by starspots.

By adopting a model of gamma distribution with $\alpha_3 = 1.450$ and $\beta_3 = 0.547$, we generated $N_s$ samples of $\Phi_{hs}$ by Equation (13) and converted them to $S_{hs}$ by Equation (1). The distribution was extended down to $S_{hs} = 10$ MSH because in the process of the time evolution, regions with $S < 500$ MSH were generated, and these regions also contributed to the total area of sunspots. Then the models were distributed randomly over the time period of $0 \leq t \leq t_0 = 146.7$ yr, and were evolved according to Equation (7). If the time evolution of a region hit the end of the time domain $t = t_0$, the profile was folded and continued to $t = 0$, to make the statistical distribution stationary in time. The filter of 13 consecutive 1s and 14 consecutive 0s was multiplied to $S(t)$ as in Section 6.3. The daily summed values of magnetic flux over the solar hemisphere, $\Phi_{hs}$, were recorded, and a histogram of the PDF was generated (Figure 11(a)). In Figure 11, in addition to the abscissa in terms of $\Phi_{hs}$, the scale is also given for the sunspot areas summed over the hemisphere, $S_{hs}$.

Figure 11(a) shows the four histograms thus generated. The dotted histogram in black was made by simply using all the generated data values. The result is rather counterintuitive in that the distribution is very flat at the lower end; actually, it has a peak at 120 MSH. By looking at the histograms derived from the observations (Figure 11(b)), the histogram using all the data

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13 https://www.sidc.be/silso/cyclesminmax
is monotonically decreasing, and those derived from the activity maximum, intermediate, and minimum states show different behavior. In particular, the histogram of the activity maximum period shows a peak at around $3 \times 10^2$ Mx ($650$ MSH), but the sum of the three histograms (red, green, and blue) gives the solid histogram in black, which is monotonically decreasing.

The histograms in red, green, and blue in Figure 11(a) were obtained from simulations that mimic the activity-maximum, intermediate, and minimum states. If all are combined, we obtain the solid histogram in black, which roughly reproduces the observed histogram in Figure 11(b) (the histogram in black). Here, it must be stressed that the sum of the red, green, and blue histograms in Figure 11(a) does not lead to the dotted histogram in black (derived from using all the data) but to the solid histogram in black. In the simulated data the solar cycle modulation was not built in and had to be introduced manually. The dotted histogram in black (derived from using all simulated data) is similar in shape to the one derived from the intermediate activity state, meaning that the entire simulated data without cycle modulation represents an intermediate activity lasting the entire data period of 146.7 yr.

The corresponding histograms (in red, green, and blue) in Figures 11(a) and (b) do not match very well for various reasons. Our model for the time evolution of the sunspot areas is a very idealized one. The lifetime of a region was assumed to be a unique function of the maximum-development area, but in reality, there should be a statistical distribution of region lifetimes for a given maximum-development area. The division of data into three phases (activity maximum, intermediate, and minimum states) was made by a very simple procedure. Nevertheless, the maximum hemispheric summed area of sunspots expected from simulations is about 7420 MSH.

Table 4
Summary of Observed Quantities

| Data Source | Time (yr) | Regions per Full Sun | Daily Region Counts per Hemisphere | Number of Observations | Observed Maximum Values |
|-------------|----------|----------------------|-----------------------------------|------------------------|-------------------------|
|             | N_s      | N_rd                 | N_ad ($S_{ad} \geq 500$ MSH)      | N_d ($S \geq 500$ MSH) | Total Area (MSH) | Total Flux (Mx) | Region Lifetime (days) | Region Counts |
| All         | 146.71   | 2995                 | 1.97 x $10^4$                    | 4.42 x $10^3$          | 8382         | 4.36 x $10^{23}$ | 195           | 26           |
| Maximum     | 35.26    | 1466                 | 0.99 x $10^4$                    | 1.28 x $10^4$          | 8382         | 4.36 x $10^{23}$ |                |              |
| Minimum     | 37.65    | 114                  | 0.06 x $10^4$                    | 0.71 x $10^4$          | 2268         | 1.16 x $10^{23}$ |                |              |
| Intermediate| 73.80    | 1415                 | 0.92 x $10^4$                    | 2.43 x $10^3$          | 8080         | 3.54 x $10^{23}$ |                |              |

Notes.

a Recurrent regions were manually picked and counted only once when they showed the largest area.

b 1947 April 8

c RGO recurrent series No. 2094, 1970 June 11 to December 23 (Kopecky 1984).

d 1937 July 12
Figure 11. (a) Histograms of the PDF $h_{\Phi}(S_h)$ of the magnetic flux summed over the hemisphere, derived from a gamma distribution with $\alpha_1 = 1.450$ and $\beta_1 = 0.547$ combined with the sunspot area evolution model shown in Figure 8. The dotted histogram in black was derived by simply using all the generated data. The histograms in red, green, and blue were derived by mimicking the activity maximum, intermediate, and minimum states as defined in Table 4. The solid histogram in black was drawn by combining these three histograms. (b) The corresponding distributions derived from observations of RGO and NOAA. The histogram in black was derived by using all the data with areas larger than 10 MSH, while the histograms in red, green, and blue are based on data from the activity maximum, intermediate, and minimum phases of activity defined in Table 4.

Table 5
Simulation Runs for Sun-as-a-star Parameters

| Data Source | Simulation Parameters | Calculations | Simulated Maximum Values |
|-------------|-----------------------|--------------|--------------------------|
|             |                       | $N_s$| Total Area (MSH) | Total Flux (Mx) | Region Lifetime (days) | Region Counts |
| All         | (0.3, 20.0)           | 8.5 $\times$ 10^4 | 8270 | 4.30 $\times$ 10^23 | 101 | 15 |
| Maximum     | (0.3, 20.0)           | 4.2 $\times$ 10^4 | 7420 | 3.85 $\times$ 10^23 | 93 | 22 |
| Minimum     | (0.3, 20.0)           | 0.3 $\times$ 10^4 | 3680 | 1.90 $\times$ 10^23 | 65 | 5 |
| Intermediate| (0.3, 20.0)           | 4.0 $\times$ 10^4 | 6120 | 3.17 $\times$ 10^23 | 93 | 14 |
| Combined    | (0.3, 20.0)           | 8.5 $\times$ 10^4 | 7420 | 3.85 $\times$ 10^23 | 93 | 22 |
| Maximum$^c$ | (0.3, 4.0)            | 4.2 $\times$ 10^4 | 13770 | 7.19 $\times$ 10^23 | 413 | 44 |

Notes.

a The number of samples in the gamma function model with minimum values extended down to 4.86 $\times$ 10^20 Mx (10 MSH).

b The numbers do not have a significance of four or more digits because of the nature of the simulation. They are arbitrarily rounded off to the nearest ten.

c This is an artificial case of region lifetimes longer than the standard case of $(a, b) = (0.3, 20.0)$.

Table 5, which is not very far from the observed maximum value, 8382 MSH (Table 4).

Figure 12 compares two cases of the maximum activity phase; one is the same as in Figure 11 (except for the normalization parameter $T$; Appendix A.2) and the other adopts the parameter setting $(a, b) = (0.3, 4.0)$, i.e., the value originally suggested by Kopecký (1956) to reproduce the empirical relations $S_{\Phi_{\max}}$ (days) $\approx 0.1 S_{\Phi_{\min}}$ (MSH) by Gnevyshev (1938). The two histograms are not monotonic and have peaks at $\Phi_{\max} = 3.4 \times 10^{22}$ Mx ($S_{\Phi_{\max}} = 680$ MSH) for the former case and at $\Phi_{\max} = 1.8 \times 10^{23}$ Mx ($S_{\Phi_{\max}} = 3500$ MSH) for the latter case. The maximum value of the hemispheric summed area for the latter case is 13,770 MSH (Table 5).

The summed area of sunspots can be converted to the modulation in total solar irradiance (TSI; $\approx 1361$ W m$^{-2}$; Kopp 2021) by

$$\frac{\Delta \text{TSI}}{\text{TSI}} = c \frac{S_h [\text{MSH}]}{2 \times 10^6}$$

with $c \approx 0.22-0.31$ (Hudson et al. 1982). If we take the observed maximum value of $S_{\Phi_{\max}} = 8382$ MSH (covering 1.67% of the visible disk) and $c = 0.25$, we obtain $\Delta \text{TSI}/\text{TSI} = 1.05\%$. If we assume a hypothetical extreme case of $S_{\Phi_{\max}} = 13,770$ MSH discussed above (covering 2.7% of the visible disk), we obtain $\Delta \text{TSI}/\text{TSI} = 1.7\%$.

The reasons we revisited the case of $(a, b) = (0.3, 4.0)$ is, on the one hand, to show that this parameter setting leads to overestimated summed total areas of sunspots. On the other hand, it is important to remember that the two simulations
shown in Figure 12 are based on the same flux emergence rates, or the same strength of dynamo action. Sunspots live longer for the setting of \((a, b) = (0.3, 4.0)\) than the case of \((a, b) = (0.3, 20.0)\) by about a factor of 5, meaning that the former case represents a situation of lower diffusion of sunspot magnetic flux, or possibly weaker surface turbulent convection.

7. Summary and Discussion

We have investigated the probability distributions of sunspot areas \(S_{\text{ms}} \geq 500\) MSH by using the data from RGO (1987 April–1976) and USAF/NOAA (1977–2020). Recurrent regions were only counted once at their maximum-area development. The data scale of NOAA was adjusted to the scale of RGO by Equation (2), and the sunspot areas were converted to the magnetic flux contents of ARs by Equation (1). We obtained a sample of 2995 regions covering 146.7 yr.

The data were fitted by a power-law distribution and four two-parameter distributions (tapered-power-law, gamma, log-normal, and Weibull distributions). The parameter values were obtained by the MLE method or by minimizing the revised K-S metric (K-Sr; Equation (B2)). The superiority of one model over the other was assessed by the AIC values. The acceptance or rejection of a specific model was assessed by the conventional K-S test, or by using K-Sr and its \(P\)-values (probability of realization).

The power-law model was unfavorable compared to the four models in terms of AIC, and was not acceptable by the classical K-S test. Among the two-parameter models, the lognormal and Weibull distributions performed well, but their behavior extended to smaller regions \((S \ll 500\) MSH) did not connect to the previously published results. Therefore, our choices were tapered-power-law and gamma distributions. The latter was more favorable considering its consistent sign of curvature of the PDF (Appendix A.4, Figure 13). We also preferred the model determined by K-Sr = min than the MLE solution; the former directly minimized the deviation of the model CCDF from the observed CCDF, with more weight on the tail of the distribution.

The power-law portion of the tapered-power-law and gamma distributions was found to be characterized by its power exponent 1.35–1.9. This was smoothly connected to the power-law distribution of exponent 2 for smaller ARs obtained by Harvey & Zwaan (1993). Modern observations on ephemeral regions and smaller flux concentrations by Hagenaar et al. (2003) and Thornton & Parnell (2011) implied an exponential decline or steeper power law. Although Thornton & Parnell (2011) suggested that all the magnetic structures were fitted by a single power law with an exponent \(\approx 2.7\), we argue that this is an extreme conclusion. Large sunspots and ordinary (or small) ARs show the power-law behavior with exponent 1.35–2, and their amplitude changes by a factor of 10 between activity maximum and minimum. Small flux concentrations, whether they follow a steeper power law or exponential decline, do not change significantly between activity maximum and minimum, and we suggest that they give way to the AR population at around \(\Phi_{\text{med}} \approx 10^{20}\) Mx.

The exponential falloff of our tapered-power-law and gamma distributions was significant, and the expected frequencies of large sunspots (Table 3) were low. The largest sunspot the Sun can generate was estimated to be around \(2 \times 10^4\) MSH. Such an upper limit to exist seems reasonable because the magnetic fields generating sunspots must be amplified within the convection zone, which has a finite size of \(2 \times 10^7\) km (30% of solar radius).

The effects of the time evolution of sunspot areas were estimated by introducing a model by Kopecký (1956). Our model assumed the lifetimes of sunspots to be roughly one-fifth of the empirical relation for the lifetimes proposed by Gnevyshev (1938), in order for the flux emergence rates we derived to be consistent with the instantaneous distribution of sunspot areas. Our assumption that the observed maximum area of a region on the visible hemisphere was a reasonable approximation to the true maximum area was confirmed.

By using the same evolution model, we converted the distribution of maximum-development areas to the distribution of instantaneous areas, and further derived the Sun-as-a-star distribution functions of total sunspot areas. In the current Sun, the largest total hemispheric area of sunspots recorded is 8382 MSH, i.e., covering 1.67% of the visible disk. By artificially increasing the lifetimes of large sunspots, the total area covered by sunspots could be increased, even up to 2.7% of the solar disk, leading to the modulation in TSI of about 1.7%.

Recently, Maehara et al. (2017) compared the appearance rates of starspots on slowly rotating solar-type stars and the rates of sunspots and found that the two distributions are not very dissimilar, speculating that the sunspots and larger starspots share a common physical origin. However, in the last 147 yr, we have not observed any sunspots greater than 6132 MSH in area. If the sunspot area distribution were to follow a power law and it was a matter of observation period, perhaps we could find some events of a relevant size by extracting historical records from a period even before that of the RGO data (Vaquero 2007). Our statistical analysis implies that the probability for the Sun to produce a much larger single AR is small; 10,000 MSH region every 3–8 \(\times 10^7\) yr. A similar argument on the hemispheric summed areas of sunspots was
not made in this paper because we do not have a model for its distribution function, namely, how the distribution decays at large values of summed areas. This will be a task performed in a future paper.

Our hypothetical simulation runs assuming an enhanced lifetime of sunspots (not enhanced generation of large sunspots) showed that the total area of sunspots can reach 2.7% of the visible disk. The emergence rates of sunspots reflect the process for the solar dynamo to generate magnetic flux, but the lifetime of magnetic structures is controlled by diffusion due to convective eddies. The latter may be an independent process from the dynamo. In the case of the Sun, the diffusive decay of sunspots leads to lifetimes of at most 7 months (Kopecky 1984). Under which conditions one may have larger starspots (stronger dynamo) or starspots living for many years (reduced diffusion) could be an important issue in understanding the nature of super-large starspots.

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Facilities: SDO(HMI), SOHO/MDI, Hinode(SOT), KPNO.

Appendix A

PDFs

A.1. Definitions

To be generic, we use $x$ instead of $\Phi$ for a statistical variable, and consider a semi-infinite range of $x_{\text{min}} \leq x$. The observed values of $x$ are indicated by $x_i$ ($i = 1, 2, \ldots, n$). The PDF and its CCDF for the flux emergence rates are denoted by $P(x)$ and $F(x)$, respectively, which are

$$P(x) = \frac{dF}{dx}, \tag{A1}$$

$$F(x) = \int_{x_{\text{min}}}^{x} P(x') \, dx', \tag{A2}$$

and

$$F(x_{\text{min}}) = \int_{x_{\text{min}}}^{x_{\text{min}}} P(x) \, dx = 1. \tag{A3}$$

The observed CCDF ($F_{\text{obs}}$) is defined as an aggregate of step functions,

$$F_{\text{obs}}(x) = (\text{number of data with } x_i \geq x) / n. \tag{A4}$$

At $x = x_i$, $F_{\text{obs}}(x)$ jumps from $i/n$ to $(i-1)/n$, and $F_{\text{obs}}(x_i) = i/n$ because of the “$\geq$” condition in Equation (A4).

A.2. Normalization

The emergence rate of regions with maximum-development magnetic flux $\Phi_{\text{md}}$ is given by Equation (3),

$$f(\Phi_{\text{md}}) = -\frac{N_d}{AT} \frac{dF(\Phi_{\text{md}})}{d\Phi_{\text{md}}} \tag{A5}$$

(Figures 3, 4, 5, 6, 7, and 9). Here, $T = 5.36 \times 10^4$ days, $A = 6.2 \times 10^6$ Mm$^2$ is the full-Sun area.

For instantaneous distributions of magnetic flux $\Phi$, we use $G(\Phi)$ for the CCDF, and the probability distribution $g(\Phi)$ is given by

$$g(\Phi) = -\frac{N_d}{AT} \frac{dG(\Phi)}{d\Phi} \tag{A6}$$

(Figure 10). Here, $T = 5.36 \times 10^4$ days, $A = 3.1 \times 10^6$ Mm$^2$ is the area of the solar hemisphere. $N_d$ is given in Table 4.

For the distributions of summed hemispheric magnetic flux $\Phi_{\text{hs}}$, we use $H(\Phi_{\text{hs}})$ for the CCDF and the PDF $h(\Phi_{\text{hs}})$ is given by

$$h(\Phi_{\text{hs}}) = -\frac{N_d}{T} \frac{dH(\Phi_{\text{hs}})}{d\Phi_{\text{hs}}} \tag{A7}$$

(Figures 11 and 12). Here, $T$ is actually the total number of observations (one observation per day), so that $T = 5.36 \times 10^4$ the observations shown in Figure 11 for all the cases (maximum, intermediate, minimum, all, and combined). In Figure 12, $T = 1.3 \times 10^4$ observations. $N_d$ is given in Table 4.

A.3. Power-law Distribution

First, we define the CCDF as (Kagan 2002)

$$F(x) = \left(\frac{x}{x_{\text{min}}}\right)^{-\alpha_1 + 1} \quad (\alpha_1 > 1), \tag{A8}$$

$$P(x) = \frac{1}{(\alpha_1 - 1)x_{\text{min}}} \left(\frac{x}{x_{\text{min}}}\right)^{-\alpha_1}. \tag{A9}$$

The MLE for the power exponent $\alpha_1$ is given by (e.g., Clauset et al. 2009)

$$\alpha_1 = 1 + \left[ \sum_{i=1}^{n} \ln \left( \frac{x_i}{x_{\text{min}}} \right) \right]^{-1.} \tag{A10}$$

A.4. Tapered-power-law Distribution

First, we define the CCDF as (Kagan 2002)

$$F(x) = \left(\frac{x}{x_{\text{min}}}\right)^{-(\alpha_2 - 1)} \exp \left[ -\beta_2 \frac{x - x_{\text{min}}}{x_{\text{min}}} \right] \quad (\alpha_2 > 1, \beta_2 \geq 0) \tag{A11}$$

($M$, $M_\ell$, $M_{\text{cm}}$, and $\beta$ in Kagan (2002) are $M = x$, $M_\ell = x_{\text{min}}$, $M_\ell / M_{\text{cm}} = \beta_2$, and $\beta = \alpha_2$ in our notation), which gives the PDF as follows:

$$P(x) = \frac{1}{x_{\text{min}}} \exp \left[ -\beta_2 \frac{x - x_{\text{min}}}{x_{\text{min}}} \right] \left[ \frac{1}{\alpha_2 - 1} \left( \frac{x}{x_{\text{min}}} \right)^{-\alpha_2} + \beta_2 \left( \frac{x}{x_{\text{min}}} \right)^{-(\alpha_2 - 1)} \right]. \tag{A12}$$
The MLE for parameters $\alpha_2$ and $\beta_2$ are given by Kagan & Schoenberg (2001) and Vere-Jones et al. (2001)

$$\beta_2 = \left[ n - (\alpha_2 - 1) \sum_{i=1}^{n} \ln \left( \frac{x_i}{x_{\min}} \right) \right]^{-1} \left[ \sum_{i=1}^{n} \frac{x_i - x_{\min}}{x_{\min}} \right]^{-1},$$

(A13)

$$\sum_{i=1}^{n} \left[ \alpha_2 - 1 + \beta_2 \frac{x_i}{x_{\min}} \right]^{-1} = \sum_{i=1}^{n} \frac{x_i}{x_{\min}}.$$  

(A14)

Equation (A14) must be solved numerically for $\alpha_2$.

Equation (A12) indicates that the PDF contains two power-law components with exponents $\alpha_2$ and $\alpha_2-1$, the former being the dominant component for small $x$. We can show that

$$\frac{d^2 \ln P}{d(\ln x)^2} = -\frac{\beta_2 x}{x_{\min}} \left[ 1 - \frac{\alpha_2 - 1}{(\alpha_2 - 1 + \beta_2 x/x_{\min})^2} \right].$$  

(A15)

The sign of this quantity is distributed as in Figure 13(a). For $\alpha_2 > 2$, $d^2 \ln P/d(\ln x)^2$ is always negative, and the slope of $\ln P$ as a function of $\ln x$ monotonically steepens as $x$ increases. If $1 < \alpha_2 < 2$, $d^2 \ln P/d(\ln x)^2$ changes sign at $\beta_2 x/x_{\min} = (\alpha_2 - 1)/(\alpha_2 - 1 - (\alpha_2 - 1))$, namely, the slope of $\ln P$ as a function of $\ln x$ first flattens and then steepens as $x$ increases. This reversed curvature is conspicuous if $\alpha_2$ is close to 1 (Figure 13(b)), and could be an undesirable feature of this distribution function if the inflection point appears in the fitting range ($x_{\min} < x$), or if we extend the distribution down below $x_{\min}$.

A.5. Truncated Gamma Distribution

We first define the PDF as (Kagan 2002)

$$P(x) = C \left( \frac{x}{x_{\min}} \right)^{\alpha_3} \exp \left[ -\beta_3 \frac{x}{x_{\min}} \right] \text{ for } \alpha_3 > 1, \beta_3 > 0,$$

(A16)

where $C$ is fixed from the normalization condition as

$$C = \frac{\beta_3^{1-\alpha_3}}{x_{\min} \Gamma(1-\alpha_3, \beta_3)}$$  

(A17)

and $\Gamma$ stands for the incomplete gamma function defined by

$$\Gamma(a, y) = \int_{y}^{\infty} t^{a-1}e^{-t}dt.$$  

(A18)

If $\alpha_3 < 0$ (which is more common in cases in application), $P(x)$ can be defined down to $x = 0$. Here, we consider the case $\alpha_3 > 1$ so that $x$ must be bounded from below as $x \geq x_{\min}$, hence, the name truncated gamma distribution. The CCDF is given by

$$F(x) = \frac{\Gamma(1-\alpha_3, \beta_3 x/x_{\min})}{\Gamma(1-\alpha_3, \beta_3)}.$$  

(A19)

In order for the $\Gamma$-function not to diverge at $\beta_3 \rightarrow 0$ and $\alpha_3 > 1$, a recurrence formula

$$\Gamma(a + 1, y) = a \Gamma(a, y) + y^a e^{-y}$$  

(A20)

is used to rewrite $F(x)$ as

$$F(x) = \frac{\Gamma(3-\alpha_3, \eta) - \eta^{2-\alpha_3} e^{-\eta} - (2-\alpha_3) \eta^{1-\alpha_3} e^{-\eta}}{\Gamma(3-\alpha_3, \beta_3) - \beta_3^{2-\alpha_3} e^{-\beta_3} - (2-\alpha_3) \beta_3^{1-\alpha_3} e^{-\beta_3}}$$  

(A21)

which can be safely used for $1 < \alpha_3 < 3$. 

Figure 13. (a) The locus of $d^2 \ln P/d(\ln x)^2 = 0$. In the area above this curve, the slope of $\ln P$ as a function of $\ln x$ monotonically steepens as $x$ increases. (b) An example of $P(x)$ for $\alpha_2 = 1.1$ and $\beta_2 = 0.01$. The slope is not monotonic, and an inflection point appears.
To obtain the MLE solutions for $\alpha_3$ and $\beta_3$, the LLH, 

$$\text{LLH} = n \ln C - \alpha_3 \sum_{i=1}^{n} \frac{x_i}{x_{\text{min}}} - \beta_3 \sum_{i=1}^{n} \frac{x_i - x_{\text{min}}}{x_{\text{min}}}$$  \hspace{1cm} (A22) 

is usually maximized for both $\alpha_3$ and $\beta_3$ numerically (Johnson et al. 2011). Here, we use the condition $\partial \text{LLH}/\partial \beta_3 = 0$ explicitly, i.e., 

$$\ln \Gamma(1 - \alpha_3, \beta_3) = -\alpha_3 \ln \beta_3 - \beta_3 - \ln \left[ \frac{\alpha_3 - 1}{\beta_3} + \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_{\text{min}}} \right],$$  \hspace{1cm} (A23) 

which gives $\beta_3$ if $\alpha_3$ is specified. Therefore, LLH is maximized for $\alpha_3$ to obtain the MLE solution.

### A.6. Truncated Lognormal Distribution

We first define the PDF as 

$$P(x) = \frac{C}{\sqrt{2\pi}\sigma x} \exp \left[ -\frac{[\ln(x/x_{\text{min}}) - \mu]^2}{2\sigma^2} \right],$$  \hspace{1cm} (A24) 

where $C$ is fixed from the normalization condition as 

$$C = 2 \left[ 1 - \text{erf} \left( -\frac{\mu}{\sqrt{2}\sigma} \right) \right]^{-1}.$$  \hspace{1cm} (A25) 

and erf stands for the error function defined by 

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_{0}^{y} e^{-t^2} \, dt.$$  \hspace{1cm} (A26) 

The peak of $P$ is at $x_{\text{min}} \exp(\mu - \sigma^2)$. The lognormal distributions can generally be defined down to $x = 0$, and under such cases, the mean and the dispersion are given by $x_{\text{min}} \exp(\mu + \sigma^2/2)$ and $x_{\text{min}}^2 \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$. In our case, we truncate the distribution at $x = x_{\text{min}}$, so that the mean and the dispersion are not given by these formulas. The CCDF is given by 

$$F(x) = \left[ 1 - \text{erf} \left( \frac{\ln(x/x_{\text{min}}) - \mu}{\sqrt{2}\sigma} \right) \right] \times \left[ 1 - \text{erf} \left( -\frac{\mu}{\sqrt{2}\sigma} \right) \right]^{-1}.$$  \hspace{1cm} (A27) 

By introducing $\zeta = \mu/\sigma$, the MLE solution is given by Crow & Shimizu (2020) as 

$$\sigma = \frac{1}{n} \sum_{i=1}^{n} \ln \frac{x_i}{x_{\text{min}}} \left[ \zeta + \sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{1 - \text{erf}(-\zeta/\sqrt{2})} \right]^{-1},$$  \hspace{1cm} (A28) 

and 

$$1 - \sqrt{\frac{2}{\pi}} \frac{e^{-\zeta^2/2}}{1 - \text{erf}(-\zeta/\sqrt{2})} = \frac{1}{n} \sum_{i=1}^{n} \left[ \frac{1}{\sigma} \ln \frac{x_i}{x_{\text{min}}} \right] - \zeta^2.$$  \hspace{1cm} (A29) 

Equation (A29) must be solved numerically for $\zeta$.

### A.7. Truncated Weibull Distribution

The CCDF and PDF are given by Weibull (1939) 

$$F(x) = \exp \left[ -\beta_3 \left( \frac{x}{x_{\text{min}}} \right)^k \right]$$  \hspace{1cm} (k > 0, $\beta_3 > 0$) \hspace{1cm} (A30) 

and 

$$P(x) = \frac{\beta_3 k}{x_{\text{min}} \beta_3} \left( \frac{x}{x_{\text{min}}} \right)^{k-1} \exp \left[ -\beta_3 \left( \frac{x}{x_{\text{min}}} \right)^k \right] + \beta_3.$$  \hspace{1cm} (A31) 

If $k \geq 1$, $P(x)$ can be defined down to $x = 0$. Here, we consider the case $k < 1$, so that the equations above are here modified to include the truncation at $x = x_{\text{min}}$. The MLE solutions are given by Wingo (1989) 

$$\beta_3 = \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{x_i}{x_{\text{min}}} \right]^k - 1,$$  \hspace{1cm} (A32) 

and 

$$\frac{n}{k} + \sum_{i=1}^{n} \ln \frac{x_i}{x_{\text{min}}} \left[ 1 - \beta_3 \left( \frac{x_i}{x_{\text{min}}} \right)^k \right] = 0.$$  \hspace{1cm} (A33) 

Equation (A33) must be solved numerically for $k$.

The behavior of $P$ near $x \approx x_{\text{min}}$ is 

$$P(x) \approx \frac{\beta_3 k}{x_{\text{min}}} \left( \frac{x}{x_{\text{min}}} \right)^{-(1-k)} \exp(\beta_3 k)$$  \hspace{1cm} (A35) 

so that the distribution approaches a power law with exponent $1-k$.

### Appendix B

#### Evaluation Metrics

The goodness-of-fit metrics measure the distance between the theoretical and observed CCDFs. In this paper, we use the K-S metric and its variant, K-Sr.

#### B.1. K-S Metric

The K-S metric is defined as (Stephens 1970, 2016) 

$$\text{K-S} = \max_{1 \leq i \leq n} \left( \frac{n + 1 - i}{n} - F(x_i), F(x_i) - \frac{n - i}{n} \right).$$  \hspace{1cm} (B1) 

For large-enough $n$, the probability of obtaining $\sqrt{n}$ K-S larger than a specified value is given by the so-called K-S function with a relatively weak dependence on $n$, regardless of the form of CCDF. We used the method proposed by Simard & L’Ecuyer (2011) to calculate the K-S function.

#### B.2. Modified K-S Metric

Since the observed and theoretical CCDFs match at $x \approx x_{\text{min}}$ and $x \rightarrow \infty$ by definition, the data points in these regions make fewer contributions to the K-S metric compared to mid-data.
points. As a matter of fact, the contributions scale as \( \sqrt{F(1 - F)} \) (Anderson & Darling 1952). Therefore, one can obtain more uniform contributions from all the data by dividing the difference by \( 1/\sqrt{F(1 - F)} \) (Anderson & Darling 1952; Clauset et al. 2009). In this paper we adopt

\[
K_{-SR} = \max_{1 \leq i \leq n} \frac{n + 1 - i - F(x_i)}{F(x_i) - \frac{n - i}{n}} / \sqrt{F(x_i)}, \tag{B2}
\]

to emphasize the tail portion only. A mismatching near \( x \approx x_{\text{min}} \) may occur from physical (e.g., limitation in detecting small sunspots, which is very unlikely in the present study because 500 MSH sunspots are big) or technical reasons. Therefore, we do not want to amplify it.

### Appendix C

**Comparison with Published Data**

Here, we summarize the flux emergence rates (Equation (3)) given in the literature and how they were converted and plotted in Figure 7 in units of \( \text{Mx}^{-1} \text{Mm}^{-2} \text{d}^{-1} \). The values of \( n, A, \) and \( T \) differ in individual data sources.

#### C.1. Thornton & Parnell (2011)

Thornton & Parnell (2011) analyzed the emergence rates of small-scale magnetic field patches using the data from Hinode/SOT and obtained a formula extending all the way up to the AR scales,

\[
f_{\text{thp}} = \frac{n_0}{F_{\Phi}} \left( \frac{\Phi}{\Phi_0} \right)^{-\alpha} \text{[Mx}^{-1} \text{cm}^{-2} \text{day}^{-1}], \tag{C1}
\]

where \( n_0 = 3.14 \times 10^{-14} \text{cm}^{-2} \text{day}^{-1} \), \( \Phi_0 = 1.0 \times 10^{16} \text{Mx} \), and \( \alpha = 2.69 \). In Figure 7, the magnetic flux emergence rate is plotted as \( f_{\text{thp}} = f_{\text{thp}} \times 10^{16} \), and the last factor is \( \text{(Mm/cm)}^2 \).

Figure 5 of Thornton & Parnell (2011) was also copied in Figure 7 using geometrical approximations to their curves.

#### C.2. Harvey & Zwaan (1993) and Schrijver & Harvey (1994)

Harvey & Zwaan (1993) derived the emergence rates of bipolar ARs using the data obtained at NSO Kitt Peak (Livingston et al. 1976). The data were taken on 739 days nearly uniformly distributed from 1975–1986. Table 6 reproduces the data given in their Table 1 before various corrections (data gaps, etc.) were applied. We converted the area data \( A_j \) and \( \Delta A_j \) in square degrees (1 \text{deg}^2 = 1.48 \times 10^{18} \text{cm}^2) \) to magnetic flux as

\[
\Phi_j[\text{Mx}] = 150 \times A_j \times 1.48 \times 10^{18}, \tag{C2}
\]

\[
\Delta \Phi_j[\text{Mx}] = 150 \times \Delta A_j \times 1.48 \times 10^{18}. \tag{C3}
\]

The conversion factor 150 Mx cm\(^{-2}\) (or mean field strength) was taken from Schrijver & Harvey (1994), who suggested 136 and 153 derived from different methods. The histogram in Figure 7 shows the counts \( N_j/\tau_{\text{obs}} \Delta \Phi_j S_{\text{hm}} \), where \( \tau_{\text{obs}} = 739 \text{days} \) and \( S_{\text{hm}} = 3.04 \times 10^6 \text{(Mm}^2\text{)} \) is the area of the solar hemisphere.

Schrijver & Harvey (1994) reported that the area distribution \( N_j(A_j) \) is fitted by a power law with exponent \( p = 2 \), namely,

\[
\tilde{N}(A_j) = a^* A_j^{-p}, \tag{C4}
\]

with amplitudes \( a^* = 1.23 \) for activity minima and \( a^* = 10 \) for activity maxima. This range of values is shown in Figure 7 as two parallel lines, after the following conversion:

\[
\tilde{N}(\Phi_j) = a^* A_j^{-p} / [150 \times (1.48 \times 10^{18}) \times S_{\text{hm}}]. \tag{C5}
\]

#### C.3. Hagenaar et al. (2003)

Hagenaar et al. (2003) investigated the emergence rates of small-scale bipolar magnetic field patches (ephemeral regions) using data from SOHO/MDI. The data were taken between 1996 and 2001. In their Figure 11, they presented the flux emergence rates in number of regions per day per \( 10^{18} \text{Mx}^2 \) bin over the whole solar surface for the data of 1997 October and 2000 August as

\[
f_{\text{h}} = S_{\text{cm}} (28.4 \times 10^{-20}) \exp(-\Phi/(10^{18}[\text{Mx}] \times 5.5 f_{\text{c}})) \tag{C6}
\]

(1997 October),

\[
f_{\text{h}} = S_{\text{cm}} (21.1 \times 10^{-20}) \exp(-\Phi/(10^{18}[\text{Mx}] \times 5.2 f_{\text{c}})) \tag{C7}
\]

(2000 August).

Here, \( 1 \times 10^{18} \text{Mx} < \Phi \lesssim 6 \times 10^{18} \text{Mx} \), \( S_{\text{cm}} = 6.1 \times 10^{22} \text{Mx}^2 \) is the whole solar area in cm\(^2\), and \( f_{\text{c}} = 1.6 \) is a conversion factor to put the MDI field strength to the scale of NSO Kitt Peak. In Figure 7, these were converted to

\[
\tilde{f}_{\text{h}} = f_{\text{h}} / (1.0 \times 10^{18}[\text{Mx}] \times S_{\text{mm}}), \tag{C8}
\]

where \( S_{\text{mm}} = 6.1 \times 10^6 \text{Mm}^2 \) is the whole solar area in Mm\(^2\).

### Table 6

Data reproduced from Harvey & Zwaan (1993)

| \( j \) | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    | 11    |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( A_j \) | 2.90  | 3.93  | 4.95  | 5.95  | 6.99  | 8.00  | 8.91  | 9.92  | 11.34 | 13.13 | 15.47 |
| \( \Delta A_j \) | 1.0   | 1.0   | 1.0   | 1.0   | 1.0   | 1.0   | 1.0   | 1.0   | 2.0   | 2.0   | 2.0   |
| \( N_j \) | 313   | 155   | 77    | 65    | 44    | 32    | 30    | 24    | 56    | 26    | 27    |

| \( j \) | 12    | 13    | 14    | 15    | 16    | 17    | 18    | 19    | 20    | 21    | 22    |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( A_j \) | 17.29 | 19.32 | 22.63 | 26.15 | 30.10 | 35.89 | 44.07 | 51.66 | 60.70 | 68.50 | 73.30 |
| \( \Delta A_j \) | 2.0   | 2.0   | 4.0   | 4.0   | 4.0   | 8.0   | 8.0   | 8.0   | 8.0   | 8.0   | 8.0   |
| \( N_j \) | 30    | 17    | 23    | 17    | 9     | 23    | 3     | 5     | 1     | 0     | 1     |

**Note.** Here \( j \) represents the bin number, \( A_j \) and \( \Delta A_j \) represent the area and bin in square degrees, and \( N_j \) represents the number of regions of the \( j \)th bin.
Baumann & Solanki (2005) presented the lognormal distributions fitted to the RGO sunspot area data. They adopted the form
\[
\ln \left( \frac{dN}{dA} \right) = -\frac{(\ln A - \ln (A_{\text{min}}))^2}{2\ln \sigma_A} + \ln \left( \frac{dN}{dA} \right)_{\text{max}} 
\]
(C9)
with the normalization
\[
\int_{\ln A_{\text{min}}}^{\infty} \frac{dN}{dA} dA = 1 
\]
(C10)
and parameter values \(A_{\text{min}} = 60\) MSH, \(\langle A \rangle = 62.2\) MSH, and \(\sigma_A = 2.45\) MSH. These are converted to our standard form (e.g., Equation (A24))
\[
P_B(A) = \frac{C}{\sqrt{2\pi} \sigma A} \exp \left[ -\frac{(\ln (A/A_{\text{min}}) - \bar{\mu})^2}{2\sigma^2} \right] 
\]
(C11)
by taking \(\bar{\mu} = \ln (A) + \ln \sigma_A, \bar{\sigma} = \sqrt{\ln \sigma_A}\). Then, by applying Equation (1), we obtain Equation (A24) with \(\mu = \bar{\mu}, \sigma = \sigma_B (p = 1.02)\).

\section{5. Gopalswamy (2018)}

Gopalswamy (2018) made a general study of the applicability of the Weibull distribution to various indices of solar activity. Gopalswamy used the formula for the CCDF, represented by \(y\), as
\[
\log y = a \left[ 1 - \exp \left( -\frac{\gamma - \log x}{\eta} \right) \right] 
\]
(C12)
When \(x\) is the sunspot area in MSH observed at RGO and NOAA (1874 May to 2016 December) and \(y\) is the number of regions with area >\(x\), Gopalswamy (2018) gave the parameter values \(a = 2.5, \gamma = 3.3, \text{ and } \eta = 0.8\). This should match our representation of the CCDF,
\[
F_G = n \exp \left[ -\beta \left( \frac{x}{x_{\text{min}}} \right)^k \right] (x \geq x_{\text{min}}). 
\]
(C13)
Here, we adopt the un-normalized CCDF with a factor \(n = 41,433\) representing the total number of data. The parameters are related as follows:
\[
k = \frac{\eta \ln 10}{\ln 10 - \ln n}, 
\]
(C14)
\[
\beta = a \ln 10 - \ln n, 
\]
(C15)
\[
x_{\text{min}} = 10^\left( 1 - \frac{1}{a} \ln n \right)^{1/k}. 
\]
(C16)
After converting sunspot area \(x\) to magnetic flux \(\Phi\) by Equation (1), and using \(t_{\text{obs}} = 5.21 \times 10^4 \) days (from 1874 May to 2016 December) and \(S_{\text{obs}} = 6.1 \times 10^6\) (the whole solar area in Mm²), a Weibull PDF multiplied by \(n/(t_{\text{obs}}S_{\text{obs}})\) is plotted in Figure 7. We assume that Gopalswamy (2018) did not make particular considerations on recurrent regions, so some mismatching is anticipated.

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Smyrli, A., Zuccarello, F., Romano, P., et al. 2010, A&A, 521, A56
Solanki, S. K. 2003, A&ARv, 11, 153
Spencer Jones, H. 1955, Sunspot and Geomagnetic-Storm Data Derived from Greenwich Observations (London: Her Majesty’s Stationery Office), 1874
Spruit, H. C. 1977, SoPh, 55, 3
Stephens, M. A. 1970, J. R. Stat. Soc. B, 32, 115
Stephens, M. A. 2016, Wiley StatsRef: Statistics Reference Online (Hoboken, NJ: Wiley)
Strassmeier, K. G. 1999, A&A, 347, 225
Strassmeier, K. G. 2009, A&ARv, 17, 251
Sun, X., Bobra, M. G., Hoeksema, J. T., et al. 2015, ApJL, 804, L28
Thornton, L. M., & Parnell, C. E. 2011, SoPh, 269, 13
Tian, L., & Alexander, D. 2008, ApJ, 673, 532
Tsuneta, S., Ichimoto, K., Katsukawa, Y., et al. 2008, SoPh, 249, 167
Utsu, T. 1999, PApGe, 155, 509
van Driel-Gesztelyi, L., & Green, L. M. 2015, LRSP, 12, 1
Vaquero, J. M. 2007, AdSpR, 40, 929
Vere-Jones, D., Robinson, R., & Yang, W. 2001, GeoJL, 144, 517
Weibull, W. 1939, Proc. No.153, The Phenomenon of Rupture in Solids (Stockholm: Royal Swedish Institute for Engineering Research)
Wingo, D. R. 1989, Stat. Pap., 30, 39
Zhang, J., Wang, Y., & Liu, Y. 2010, ApJ, 723, 1006
Zwaan, C. 1987, ARA&A, 25, 83

Toriumi, S., Schrijver, C. J., Harra, L. K., Hudson, H., & Nagashima, K. 2017, ApJ, 834, 56
Toriumi, S., & Wang. H. 2019, LRSP, 16, 3
Tsuneta, S., Ichimoto, K., Katsukawa, Y., et al. 2008, SoPh, 249, 167
Utus, T. 1999, PApGe, 155, 509
van Driel-Gesztelyi, L., & Green, L. M. 2015, LRSP, 12, 1
Vaquero, J. M. 2007, AdSpR, 40, 929
Vere-Jones, D., Robinson, R., & Yang, W. 2001, GeoJL, 144, 517
Weibull, W. 1939, Proc. No.153, The Phenomenon of Rupture in Solids (Stockholm: Royal Swedish Institute for Engineering Research)
Wingo, D. R. 1989, Stat. Pap., 30, 39
Zhang, J., Wang, Y., & Liu, Y. 2010, ApJ, 723, 1006
Zwaan, C. 1987, ARA&A, 25, 83

The Astrophysical Journal, 943:10 (21pp), 2023 January 20
Sakurai & Toriumi