Positioning Precision Analysis of GNSS Multi-frequency Carrier Phase Combinations

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Abstract  GPS positioning precision is affected by various error sources, and traditional combinations of GPS carrier phase observations have their own limitations such as the wide-lane, the narrow-lane and the ionospheric-free combinations. To obtain the optimal positioning precision, a new linear combination method is addressed through the variance-covariance (VCV) of the GPS multi-frequency carrier phase combination equations, and the impact of the positioning precision is analyzed with the changing of the observation errors deduced by the law of error propagation. For the high precision positioning with only one carrier phase combination, the optimal combination method is deduced and further validated by an example of a baseline resolution with 60 km length. The result indicates that this method is the simplest, and the positioning precision is the best. Therefore, it is useful for long baseline quick positioning for different precision requirements in various distances.

Keywords  GPS; multi-frequency combination; propagation of errors; positioning precision

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Introduction

The positioning precision of the GPS carrier phase observations is affected by various error sources, such as atmospheric delay, satellite orbit errors, multipath, noise and etc. One effective method to reduce these errors is to use a linear combination of the GPS carrier phase observations. Traditional combinations (e.g. the widelane (WL), the narrowlane (NL) and the ionospheric-free (IF) combinations) suffer from their own limitations. For example, the positioning precision of the WL is degraded due to the amplification of the noise, even though the ambiguity in WL is easy to resolve. The NL cannot accurately determine the ambiguity because of its short wavelength, and the IF can eliminate the effect of ionospheric delay, but the ambiguity is not an integer. Therefore, these combination approaches for high precision positioning are mainly through mutually matching combinations to resolve ambiguities[1].

1  Equation and variance of carrier phase observation

For carrier phase observations from two receivers and two satellites in one epoch with frequency $L_5$ in GPS modernization[2], the double-difference carrier phase observation equations for each frequency can be expressed as:
\[ \nabla \Delta \varphi_i = - \frac{\nabla \Delta \varphi_i}{\lambda_i} + \nabla \Delta N_i = \nabla \Delta T + \frac{\nabla \Delta I + \nabla \Delta \delta r}{\lambda_i} + \nabla \Delta n_i + \nabla \Delta m_i \]  
(1)

\[ \nabla \Delta \varphi_2 = - \frac{\nabla \Delta \varphi_2}{\lambda_2} + \nabla \Delta N_2 = \nabla \Delta T + q_1 \frac{\nabla \Delta I + \nabla \Delta \delta r}{\lambda_2} + \nabla \Delta n_2 + \nabla \Delta m_2 \]  
(2)

\[ \nabla \Delta \varphi_3 = - \frac{\nabla \Delta \varphi_3}{\lambda_3} + \nabla \Delta N_3 = \nabla \Delta T + q_2 \frac{\nabla \Delta I + \nabla \Delta \delta r}{\lambda_3} + \nabla \Delta n_3 + \nabla \Delta m_3 \]  
(3)

then the variance-covariance (VCV) matrix can be deduced from Eqs.(1-3) as follows:

\[ \nabla \Delta D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \]  
(4)

where the diagonal elements are given as:

\[ D_{11} = (\nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3) / \lambda_1^2 + \lambda_2 \lambda_3 \nabla \Delta \sigma^2_1 / \lambda_1^2 + \nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3 \]

\[ D_{22} = (\nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3) / \lambda_2^2 + \lambda_1 \lambda_3 \nabla \Delta \sigma^2_1 / \lambda_2^2 + \nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3 \]

\[ D_{33} = (\nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3) / \lambda_3^2 + \lambda_1 \lambda_2 \nabla \Delta \sigma^2_1 / \lambda_3^2 + \nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3 \]

and the cross covariance terms are given as:

\[ D_{12} = D_{21} = (\nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3) / \lambda_1 \lambda_2 + \lambda_1 \lambda_3 \nabla \Delta \sigma^2_1 / \lambda_1 \lambda_2 + \nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3 \]

\[ D_{13} = D_{31} = (\nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3) / \lambda_1 \lambda_3 + \lambda_1 \lambda_2 \nabla \Delta \sigma^2_1 / \lambda_1 \lambda_3 + \nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3 \]

\[ D_{23} = D_{32} = (\nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3) / \lambda_2 \lambda_3 + \lambda_2 \lambda_1 \nabla \Delta \sigma^2_1 / \lambda_2 \lambda_3 + \nabla \Delta \sigma^2_1 + \nabla \Delta \sigma^2_2 + \nabla \Delta \sigma^2_3 \]

where the variances of the tropospheric delay \( \nabla \Delta \sigma^2_1 \), ionospheric delay \( \nabla \Delta \sigma^2_2 \) and orbit error \( \nabla \Delta \sigma^2_3 \) are m^2 in unit, and the noise and multipath variances are in unit of cycles^2. Here, in order to easily deduce equations, uncorrelation of noise and multipath are assumed in each frequency.

## 2 Positioning precision analysis of carrier phase combinations

Given the integer of \( i, j \) and \( k \), a random linear multi-frequency phase combination of double-difference observations can be expressed as:

\[ \nabla \Delta \varphi = [i \ j \ k] \begin{bmatrix} \nabla \Delta \varphi_1 \\ \nabla \Delta \varphi_2 \\ \nabla \Delta \varphi_3 \end{bmatrix} \quad (i, j, k \in Z) \]  
(5)

with the combination wavelength as:

\[ \lambda_c = \frac{\lambda_i \lambda_j \lambda_k}{i \lambda_i + j \lambda_j + k \lambda_k} \]  
(6)

Through Eqs.(4) and (5), the variance of combination observations, \( \nabla \Delta \sigma^2_c \), can be written as:

\[ \nabla \Delta \sigma^2_c = [i \ j \ k] \nabla \Delta D [i \ j \ k]^T = i^2 \left( \frac{\nabla \Delta \sigma^2_{1r}}{\lambda_1^2} + \lambda_1 \lambda_2 \nabla \Delta \sigma^2_{1r} / \lambda_1^2 \right) + \lambda_1 \lambda_2 \lambda_3 \nabla \Delta \sigma^2_{1r} / \lambda_1^2 \lambda_2 \lambda_3 + 2j(i \ \nabla \Delta \sigma^2_{2r} / \lambda_2^2 + \lambda_2 \lambda_3 \nabla \Delta \sigma^2_{2r} / \lambda_2^2 \lambda_3 + \nabla \Delta \sigma^2_{2r}) + 2jk(i \ \nabla \Delta \sigma^2_{3r} / \lambda_3^2 + \lambda_3 \lambda_1 \nabla \Delta \sigma^2_{3r} / \lambda_3^2 \lambda_1 + \nabla \Delta \sigma^2_{3r}) + 2jk(i \ \nabla \Delta \sigma^2_{1r} + \lambda_1 \lambda_2 \nabla \Delta \sigma^2_{1r} / \lambda_1^2 \lambda_2 + \lambda_2 \lambda_3 \nabla \Delta \sigma^2_{1r} / \lambda_2^2 \lambda_3 + \nabla \Delta \sigma^2_{1r}) \]  
(7)

where the unit of \( \nabla \Delta \sigma^2_c \) is in cycle^2; \( \nabla \Delta \sigma^2_{1r} = \nabla \Delta \sigma^2_{1r} / \lambda_1^2 \) and \( \nabla \Delta \sigma^2_{2r} = \nabla \Delta \sigma^2_{2r} / \lambda_2^2 \). To simply express, we still use the expression of \( \nabla \Delta \sigma^2_{1r} \) and \( \nabla \Delta \sigma^2_{2r} \) throughout this paper. From Eq.(7), we can see that the value of \( \nabla \Delta \sigma^2_c \) is changing with \( i, j \) and \( k \). Therefore, it is possible to find a group value of \( (i, j, k) \) to improve the positioning precision.

### 2.1 Positioning precision analysis in special conditions

The variance \( \nabla \Delta \sigma^2_c \) in Eq.(7) can be divided into the following several items.

1) Variance of noise and multipath error \( \nabla \Delta \sigma^2_{c n} \):

\[ \nabla \Delta \sigma^2_{c n} = \nabla \Delta \sigma^2_{c n} (i^2 + j^2 + k^2) \]  
(8)

where the unit of \( \nabla \Delta \sigma^2_{c n} \) is the cycles^2. If the \( \nabla \Delta \sigma^2_{c n} \) is the minimum, the combination value of \( (i, j, k) \) should be the (1,0,0), (0,1,0) or (0,0,1). It can be seen that the \( L_1 \) observation is better than other combination observations after the atmospheric error is eliminated.

2) Variance of tropospheric and orbital errors \( \nabla \Delta \sigma^2_{c T} \):

\[ \nabla \Delta \sigma^2_{c T} = \nabla \Delta \sigma^2_{c T} (i^2 + j^2 + k^2) \]  
(9)

When the combination \( (i, j, k) \) satisfies the following equation:

\[ (i / \lambda_i + j / \lambda_j + k / \lambda_k) = 0 \]  
(10)
the tropospheric and orbital errors could reach the minimum in unit of cycle^2. Although Eq.(10) has no physical sense (the wavelength is infinite), the errors can reach the minimum with the combination of i, j and k.

3) Variance of ionospheric error V Δσ^2(i):

\[ V Δσ^2(i) = V Δσ^2(i^2λ_i^2 + j^2λ_j^2 + k^2λ_k^2 + 2jλ_jλ_y + 2kλ_kλ_y + iλ_iλ_y)\]

When the combination (i, j, k) satisfies the following equation:

\[ (iλ_i + jλ_j + kλ_k)^2 = 0 \]

the ionospheric error could reach the minimum in unit of cycle^2. While \( k = 0, i/j = -λ_i/λ_j \), Eq.(12) is just the dual-frequency ionospheric-free combination[3].

It is important to note that the variance in units of cycles^2 may not have the minimum in unit of m^2. For example, the variance of tropospheric and orbital errors in unit of m^2 is:

\[ V Δσ^2(T_{m}) = V Δσ^2(i^2λ_i^2 + j^2λ_j^2 + k^2λ_k^2)\]

It can be seen that the errors of troposphere and orbit cannot be eliminated in distance through combinations.

2.2 A new linear combination

Above, it has been shown that the variance in units of cycle^2 may not have the minimum in unit of m^2. Similarly, short-wave combinations may reduce the errors in distance, but it will sometimes bring much difficulty for ambiguity resolution. Therefore, we defined the optimal combination[4] which not only has the minimum value of variance in unit of m^2, but also makes the variance in units of cycle^2 less than or closer to the variance of L1 frequency observation. In this way, the high positioning precision can be obtained. Meanwhile, the difficulty of ambiguity resolution can reach the level that is less than or closer to L1 observation.

When the (i, j, k) satisfies Eqs.(10) and (12), the ionospheric and tropospheric errors in units of cycle^2 can reach the minimum, and the ionospheric error in distance is almost zero. Fig.1 shows the geometrical planes of Eqs.(10) and (12). When the value of (i, j, k) is located between two planes, the variance can reach optimization. Therefore, the expression satisfying the above conditions can be written as \( \frac{\partial V Δσ}{\partial i} = 0 \), namely:

\[ i(λ_i^2V Δσ^2_i + λ_j^2V Δσ^2_j + λ_k^2V Δσ^2_k) + j(\lambda_i^2V Δσ^2_i + λ_j^2V Δσ^2_j + λ_k^2V Δσ^2_k) + k(\lambda_i^2V Δσ^2_i + λ_j^2V Δσ^2_j + λ_k^2V Δσ^2_k) = 0 \]

(13)

Fig.1 Selection of optimal precision for three-carrier phase

Then, the optimal integer combination should be on the plane, as shown in Eq.(13), or on both sides, and the combination still satisfies the definition of the optimal combination. So, we can obtain the optimal combination with the least variance using Eqs.(7) and (13).

3 Test of determining optimal phase combination

3.1 Determination of variance parameters and linear combination

Because current GPS has no L5 observation, we use dual-frequency observations to test the conclusion in Section 2.2.

When \( k = 0 \), Eq.(13) can be written as:

\[ i/j = \frac{λ_j}{λ_i}V Δσ^2_i + \frac{λ_i}{λ_j}V Δσ^2_j \]

(14)

namely, \( \frac{λ_j}{λ_i} < i/j < \frac{λ_i}{λ_j} \) and the value range of combination (i, j) is shown in Fig.2:
The standard deviation of each observation error for > 50 km long baseline was described in References [3-5], and is shown in Table 1. When the $V^2_{\Delta \sigma_i^2}$ and $V^2_{\Delta \sigma_j^2}$ are known, the geometrical line defined by Eq.(14) can be determined. The more excellent combinations can be obtained through calculations when the integer combinations near the line are put into Eq.(7). Removing the combinations with shorter wavelengths and larger variances, the candidate combinations are shown in Table 2.

### Table 1 Standard deviations of each observation error

| Observation errors    | Standard deviations |
|-----------------------|---------------------|
| Noise/multipath       | 2 mm                |
| Troposphere/orbit     | 10 cm               |
| Ionosphere            | 28 cm               |

### Table 2 Variance value of each linear phase combination observation

| $i$ | $j$ | Combination Wavelength $\lambda$ (m) | $V^2_{\Delta \sigma_i^2}$ /cycles$^2$ | $V^2_{\Delta \sigma_j^2}$ /m$^2$ | Note                        |
|-----|-----|--------------------------------------|-------------------------------------|--------------------------------|-----------------------------|
| 0   | 1   | 0.244 2                             | 3.732 7                            | 0.222 6                           | $L_2$ only                  |
| 1   | 0   | 0.190 3                             | 2.441 1                            | 0.088 4                           | $L_1$ only                  |
| 1   | -1  | 0.862 2                             | 0.187 3                            | 0.139 2                           | WL                          |
| 1   | 1   | 0.107 0                             | 12.160 4                           | 0.139 1                           | NL                          |
| 2   | -2  | 0.431 1                             | 0.749 3                            | 0.139 2                           |                             |
| 2   | -1  | 0.155 9                             | 1.524 2                            | 0.037 0                           |                             |
| 3   | -3  | 0.287 4                             | 1.686 0                            | 0.139 2                           |                             |
| 3   | -2  | 0.132 0                             | 0.981 9                            | 0.017 1                           |                             |
| 4   | -4  | 0.215 5                             | 2.997 2                            | 0.139 2                           |                             |
| 4   | -3  | 0.114 5                             | 0.811 9                            | 0.010 6                           |                             |
| 6   | -5  | 0.090 5                             | 1.603 0                            | 0.013 1                           |                             |
| 77  | -60 | 0.006 3                             | 252.521 4                          | 0.010 0                           | IF                          |

### Table 3 Deviations of positioning using each method

| Observation type | $N$ / m | $E$ / m | $H$ / m |
|------------------|---------|---------|---------|
| $L_1$ -only      | 0.236   | 0.365   | 0.605   |
| $L_1/L_2$        | 0.000   | -0.031  | 0.039   |
| WL               | 0.019   | -0.131  | 0.205   |
| IF               | 0.000   | -0.031  | 0.039   |
| (3, -2)          | -0.017  | -0.044  | 0.057   |

### 3.2 Test results

From Table 2, the combinations of (4, -3) and (3, -2) satisfies the definition of optimal combination, but the ambiguity resolution of combination (4, -3) is very difficult to fix because of its shorter wavelength, so (3, -2) is chosen in the test. This combination has a moderate wavelength (0.132 m) and its variance (0.981 9) in unit of cycles$^2$ is less than the $L_1$, like in unit of m$^2$.

In order to only compare with the $L_1$, 30 min observations of the $L_1/L_2$, WL and IF combinations with a 60 km baseline, 30 s sample rate, and 15° altitude angle were simulated with 7 visible satellites. For the resolution of $L_1$-only and (3, -2), the ionospheric error is corrected by the Klobuchar model, tropospheric error is corrected by the Hopfield model, and the Lambda method is used to search the ambiguity. The Gpsurvey2.35 software is used for the resolution of $L_1/L_2$, WL and IF combinations. Deviations of positioning using different observation types are shown in Table 3.

### 4 Conclusions

A new method for determining optimal multi-frequency phase combinations is presented in this study. Test results show that the ionospheric error is the dominant influence on the $L_1$ -only observation (in Table 3). When the $L_1/L_2$ and IF observations were used, the positioning precision are the best because the ionospheric error is eliminated, while the result of WL observation is terrible for the long wavelength and large noise. When the optimal combination (3, -2) is used, the positioning precision is slightly worse than the $L_1/L_2$ and IF combinations, but the ambiguity search of the $L_2$ is not required in the baseline resolution. It has been indicated that this method is the simplest and the precision is the best, and we can
use it for long baseline quick positioning as different precision requests in various distances. In addition, according to the Section 2.2, for the triple-frequency observation, the same way can be developed to obtain the optimal combination in various distance GPS baselines.

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Notes to Contributors

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- GIS
- GPS
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