Asymptotic freedom in a scalar field theory on the lattice

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Abstract

An alternative model to the trivial $\phi^4$-theory of the standard model of weak interactions is suggested, which embodies the Higgs-mechanism, but is free of the conceptual problems of standard $\phi^4$-theory. We propose a $N$-component, $O(N)$-symmetric scalar field theory, which is originally defined on the lattice. The model can be motivated from SU(2) gauge theory. Thereby the scalar field arises as a gauge invariant degree of freedom. The scalar lattice model is analytically solved in the large $N$ limit. The continuum limit is approached via an asymptotically free scaling. The renormalized theory evades triviality, and furthermore gives rise to a dynamically formed mass of the scalar particle.

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In the standard model of weak interactions \([1]\), the main ingredient for the generation of the masses of \(W\)- and \(Z\)-bosons as well as fermions is a non-vanishing scalar condensate. In the usual Higgs mechanism, the non-vanishing scalar condensate shows up during a classical treatment of the scalar sector. However, it has been shown that a scalar theory with a \(\lambda \phi^4\) interaction of the scalar fields \(\phi\) is trivial at quantum level \([2]\). The quantum theory is only consistent with a zero renormalized coupling \(\lambda_R\), if the regulator (e.g. the momentum cutoff \(\Lambda\)) is removed. The situation can be most easily understood in the large \(N\)-limit of \(O(N)\)-symmetric \(\phi^4\)-theory \([3]\).

Renormalization enforces

\[
\frac{6}{\lambda} + \frac{1}{16\pi^2} \ln \frac{\Lambda^2}{\mu^2} = \frac{6}{\lambda_R(\mu)},
\]

where \(\mu\) is an arbitrary renormalization point. Since the stability of the bare action requires \(\lambda > 0\), one must require \(\lambda_R(\mu) \to 0\), if the cutoff \(\Lambda\) goes to infinity.

In order to evade this undesired situation in the Weinberg-Salam model, several proposals have been made: first, the scalar field can have a non-vanishing expectation value, and the residual interactions of the shifted field vanish \([4]\). Thus the conventional Higgs mechanism is possible. Second, the scalar theory is regarded as an effective theory, where the momentum cutoff \(\Lambda\) is finite \([5, 6]\). At energy scales near this cutoff, new physics comes into the game. The parameter range consistent with the phenomenology of weak interactions then impose an upper bound on the Higgs mass \(m_h\). Recent investigations find \(m_h \leq 710 \pm 60\) GeV. If the Higgs mass is close to the upper bound, the finite cutoff \(\Lambda\) is in the range \(2 \ldots 4 m_h\) \([6]\). It seems feasible that accelerator experiments in the near future will confirm or rule out the finite cutoff scenario.

Third, an analytical continuation of the coupling strength \(\lambda\) to negative values \([7]\) allows to remove the regulator in \([1]\). However, stability of the action then requires complex scalar fields. The complex nature of the fields induce imaginary parts into the effective potential of the scalar condensate \([8]\). The model most likely violates fundamental axioms of quantum field theory, e.g. the reflection positivity \([9]\).

The proposals above are not satisfactory from a field theoretical point of view. Searching for alternatives to trivial \(\phi^4\)-theory, Kuti et al. introduced higher derivative terms into the action of the scalar theory \([10, 11]\). They found ultraviolet stable fixed points and a Higgs mass in the TeV range, which escapes the triviality bounds. A different approach was proposed in \([12, 13, 14]\). There, the tree level scalar potential was generalized to contain arbitrary powers of the scalar field. If these powers are multiplied by appropriate inverse powers of the cutoff in order to restore the correct energy dimension, the models were observed to be renormalizable. Moreover, a continuum of renormalization group fixed points were found, some of which correspond to asymptotically free interactions \([13]\).

These observations have a tremendous impact on model building and phenomenol-
ogy, since only non-abelian gauge theories are believed so far to be asymptotically free and non-trivial. Such a scalar theory therefore would indeed serve as a possible candidate for the scalar sector of the Weinberg-Salam model. Unfortunately, it turned out \cite{12,14} that most of the UV-fixed points, found in \cite{12,13}, lead to singular effective potentials which are not useful in phenomenology. It was concluded in \cite{12,14} that the investigations in \cite{13} do not lead to new physics at the present stage of approximations. In the model proposed by Kuti et al. \cite{10,11}, complex ghost states appear. The discussions on the role of these states in phenomenology are still controversial.

In this letter, we propose a \(N\)-component, \(O(N)\)-symmetric scalar lattice model, which is a concrete realization of a renormalizable and asymptotically free scalar field theory, which, in addition, possesses well-behaved effective potentials, and is free of ghost states. Our model can be solved in the large \(N\)-limit (without resorting to further approximations as e.g. those used in \cite{12,13,14}).

The lattice scalar model which we wish to study is described by the following generating functional

\[
Z[j, \eta] = \int_{-\infty}^{+\infty} [d\phi_x] \prod_{\{x\}} \left[ 1 + \text{erf}(\Psi_x a^2) \right] \\
\times \exp \left\{ - \sum_x a^4 \left[ \frac{1}{2} \phi_x (-\partial^2) \phi_x - \Psi_x^2 - \eta_x \phi_x \right] \right\},
\]

\[
\Psi_x = \sqrt{\frac{3N}{2\lambda}} \left( m^2 + \frac{\lambda}{3N} j_x - \frac{\lambda}{6N} \phi_x^2 \right),
\]

where \(a\) is the lattice spacing, \(\partial^2\) is the lattice version of the Euclidean d’Alambert operator and \(\text{erf}\) is the error-function. The parameter \(m\) plays the role of a bare mass, and \(j\) is an external source, the meaning of which will be explained below. The functional derivative with respect to \(\eta_x\) generates scalar field \(\phi_x\) insertions in Green’s functions. Note that if we take the bare coupling strength \(\lambda\) to zero, we end up with a free scalar field theory.

Let us motivate the particular type \(2\) of the scalar field theory with the ”unusual” measure. Our basic idea is that the scalar degree of freedom naturally emerges from Yang-Mills theory as a gauge invariant degree of freedom, rather than being introduced by hand as in the electro-weak Lagrangian. In fact, such a scalar field theory with the ”non-trivial” integration measure naturally arises both in the lattice \(13\) and the continuum formulation \(10, 17, 18\). Let us illustrate this in the latter case. The gauge invariant partition function of Yang-Mills theory can be generically expressed in terms of the functional integral

\[
Z = \int \mathcal{D}\mu(\omega) \int \mathcal{D}A_i \exp\{-S_{YM}\},
\]
where $S_{YM}$ is the standard Yang-Mills action (see e.g. [1]) and $A_{i=1...3}$ denote the spatial components of the gauge field. $\omega$ is a time-independent, but space dependent element of the Cartan subgroup of the gauge group, and $\mu(\omega)$ denotes the corresponding Haar measure. The integration over the compact variable $\omega$ stems from the projection onto physical (i.e. gauge invariant) states [17, 18]. For a SU(2) gauge group, this integration can be parameterized by

$$\int D\mu(\omega) = \int_0^\pi D\chi \sin^2 \chi = \int_{-1}^{+1} da_0 \sqrt{1-a_0^2}, \quad a_0 = \cos \chi .$$  

(5)

The temporal component $A_0$ of the gauge field, which enters the action $S_{YM}$ in (4), is time independent and related to $\omega$ by

$$A_0 = -\frac{1}{T} \ln \omega , \quad T: \text{time period} ,$$  

(6)

which suggests to interpret $\omega$ as the (diagonal) Polyakov line. Let us emphasize that only the eigenvalues of the Polyakov line, i.e. $\exp\{\pm i\chi\}$, contribute to the integration (4) and that these eigenvalues (and therefore $\chi$) are gauge invariant.

The representation (4) also straightforwardly follows from continuum limit of the lattice formulation as well as from the standard continuum formulation [18] with Faddeev-Popov gauge fixing in the gauge

$$A_{0}^{ch} = 0 , \quad \partial_0 A_{n}^{n} = 0 ,$$  

(7)

where $A_0^n$ and $A_0^{ch}$ denote the "neutral" (diagonal) and the "charged" (off-diagonal) part of the gauge field. In this case the Haar measure arises from the Faddeev–Popov determinant.

Let us show that the theory (4) of the gauge invariant variable $a_0$ has a similar structure as the model (2) of the scalar field. Assume that the spatial components $A_i$ in (4) are integrated out. This procedure would yield an effective theory of the scalar $a_0$, which is equivalent to the original Yang-Mills theory and which is hence
asymptotically free and non-trivial. This simple argument shows that asymptotically free and non-trivial effective scalar theories indeed exist.

In order to introduce the standard support of a scalar field integral, we define $a_0(x) = 2/\pi \arctan \phi(x)$, which maps the variable $a_0 \in [-1, +1]$ onto $\phi(x) \in (-\infty, \infty)$. The $a_0$-integral in (5) then results in

$$\int_{-\infty}^{\infty} D\phi \frac{2}{\pi} \sqrt{1 - \frac{4}{\pi^2} \arctan \frac{\phi}{1 + \phi^2}} \ldots$$

Figure 1 compares the measure of the $\phi$ functional integral (8) with that of our model (2), where we have set $j$ and $m$ to zero. For practical applications (e.g. a Monte-Carlo simulation), both curves are the same. The considerations above illustrate that the measure of the scalar functional integral (2) can be understood as a relict of the integration over the compact variable $\omega$, which ensures the proper projection onto physical states contributing to the Yang-Mills partition function. Thus the scalar field of our model can be interpreted as a gauge invariant degree of freedom of Yang-Mills theory. In this letter, we do not further pursue the rigorous construction of the scalar model from gauge theory, but rather study the coarse grained model (2) which is exactly solvable in the large $N$-limit.

In the naive continuum limit ($a \to 0$), the generating functional (2) takes the form

$$Z_{\text{naive}}[j = 0, \eta = 0] = \int_{-\infty}^{+\infty} [d\phi_x] \exp \{-S_{\text{naive}}\},$$

where (for illustrative purposes, we here choose $m = 0$ and $j = 0$)

$$S_{\text{naive}} = \int d^4x \left\{ \frac{1}{2} \phi_x (-\partial^2) \phi_x + \frac{\tilde{m}^2}{2} \phi_x^2 - \frac{\tilde{\lambda}}{N} \phi_x^4 \right\} + \mathcal{O}(a^2),$$

where

$$\tilde{m}^2 = \sqrt{\frac{2\lambda}{3N\pi}} \frac{1}{a^2}, \quad \tilde{\lambda} = \frac{\lambda}{24} \left(1 - \frac{2}{\pi}\right).$$

Thus the naive continuum limit corresponds to a continuum $\phi^4$-theory with negative quartic interaction. Nevertheless, the full lattice model (2) is well-defined, since the integrand in (2) behaves like $1/\phi_x^2 a^2$ for $\phi_x^2 a^2 \gg 1$. This implies that precisely the terms which vanish in the naive continuum limit stabilize the lattice action. Below, we will show that the above lattice model possesses at quantum level an interesting continuum limit, which is non-trivial in the sense that the scalar fields have non-vanishing vacuum expectation values. Furthermore, the continuum limit is approached via an asymptotically free scaling, i.e. $\lambda \to 0$ for $a \to 0$. One can already anticipate this remarkable scaling behavior, by starting from (3) and using standard perturbation theory. In leading order, our model essentially represents
standard \( \phi^4 \)-theory with, however, negative quartic coupling (see (11)). Due to this inverse sign of the coupling, the renormalization group \( \beta \)-function of our model is negative implying asymptotic freedom. In order to get access to the non-trivial vacuum properties, we do not further pursue perturbation theory, but will study the model in the large \( N \)-limit.

Let us first discuss the classical field, which is produced by the functional derivative of \( Z[j] \) with respect to the external source \( j \), i.e.

\[
C := - \frac{\delta}{\delta j(x)} \ln Z[j] |_{j=0}.
\]

In the naive continuum limit, this classical field basically measures \( \langle \phi^2 \rangle \) (up to a constant shift)

\[
C = \frac{\lambda}{6N} \left( 1 - \frac{2}{\pi} \right) \langle \phi^2 \rangle - \sqrt{\frac{2\lambda}{3\pi N}} \frac{1}{a^2} - \left( 1 - \frac{2}{\pi} \right) m^2 + \mathcal{O}(a^2).
\]

(13)

For \( C = 0 \), the scalar condensate is proportional to the mass parameter, as it is the case in a classical field theory with spontaneous symmetry breaking at tree level. Therefore, any non-vanishing value of \( C \) measures the non-trivial scalar condensate which occurs in the ground state of the quantum theory (for vanishing renormalized mass).

In the following, we will solve the lattice model (2) in the large \( N \) limit. For this purpose, we introduce the auxiliary field \( M_x \) by

\[
\prod_{\{x\}} \left[ 1 + \text{erf} \left( \Psi_x a^2 \right) \right] = \int_0^\infty [dM_x] \exp \left\{ - \sum_{\{x\}} a^4 \left( \sqrt{\frac{3N}{2\lambda}} M_x - \Psi_x \right)^2 \right\},
\]

(14)

where we have absorbed an unimportant factor into the measure of \( M_x \). Inserting (14) into the definition of the model (2), a Gaussian integral over the scalar fields is left. Performing this integration, one finds

\[
Z[j, \eta] = \int_0^\infty [dM_x] \exp \left\{ -S_M \right\},
\]

\[
S_M = \frac{1}{2} \text{tr} \ln(-\partial^2 + M_x) + \sum_x a^4 \left( \frac{3N}{2\lambda} M_x^2 - \frac{3N}{\lambda} m^2 M - j_x M_x \right) \]

- \frac{1}{2} \sum_{xy} a^4 \eta_x (-\partial^2 + M_x)^{-1}_{xy} \eta_y.
\]

This model is completely equivalent to that in (1). The latter version (15), however, is more suitable to perform the large \( N \)-expansion, since fluctuations of the field \( M_x \) are suppressed by powers of \( 1/N \). In order to study the continuum limit, we
carefully investigate the divergences of the trace term in (16) which will occur, if we shrink the lattice spacing to zero. Assuming a constant field $M$, this trace term is given by

$$N \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \ln \left( 2 \sum_{\mu=1}^{4} (1 - \cos k_\mu a) + Ma^2 \right) = -\frac{N}{a^4} \int_0^\infty \frac{ds}{s} e^{-sMa^2} e^{-8s} [I_0(2s)]^4 ,$$

(17)

where we have factored out the lattice volume and have dropped a $M$-independent constant. $I_0(x)$ is the Bessel function of the first kind. Inserting (17) in (16), one observes that the quadratic divergence, i.e. $\propto 1/a^2$, can be absorbed in the bare mass $m$. The logarithmic divergence of (17) turns out to be proportional to $M^2$ implying that (16) is renormalized by setting

$$\frac{3}{2\lambda(\Lambda^2)} - \frac{1}{64\pi^2} \ln \frac{\Lambda^2}{\mu^2} = \frac{3}{2\lambda_R(\mu)} , \quad \Lambda^2 := 1/a^2 ,$$

(18)

where $\mu$ is an arbitrary reference scale, and $\Lambda$ is the momentum cutoff provided by the lattice spacing. Equation (18) is of crucial importance. Firstly, it tells us, that the cutoff $\Lambda$ can be taken to infinity without enforcing a vanishing renormalized coupling strength $\lambda_R$. Hence, the model evades triviality, which is inherent in standard $\phi^4$-theory. Secondly, the theory (2) is asymptotically free, which can be easily verified by calculating the renormalization group $\beta$-function, which in the large $N$ limit is

$$\beta(\lambda) := \Lambda \frac{d\lambda(\Lambda)}{d\Lambda} = -\frac{\lambda^2}{48\pi^2} .$$

(19)

One might worry that the renormalized coupling diverges at a certain momentum at low energies. This behavior is generic for a large class of asymptotically free theories such as QCD and the Gross-Neveu model. In the latter model, it was observed that the divergence at low momenta is an artifact due to the assumption of a perturbative vacuum, and that the divergence is in fact screened by a mass which is dynamically formed in the true vacuum [19].

In order to demonstrate that the model (2) possesses non-trivial ground state properties, we present the effective potential $U$ for the composite field $C$. This potential can be obtained by a Legendre transformation of $Z[j, \eta = 0]$ with respect to $j$ thereby assuming constant classical fields $C$. In the large $N$ limit, the final result is for the case of a vanishing renormalized mass (details will be presented elsewhere)

$$U(C) = \frac{N}{64\pi^2} C^2 \left( \ln \frac{C}{M_0} - \frac{1}{2} \right) ,$$

(20)

where $M_0$ is a renormalization group invariant scale, which arises from dimensional transmuting the coupling strength $\lambda_R$. Functional derivatives of the effective action
with respect to the classical field provide vertex functions which contain insertions of the composite field $C$. Differentiations of the effective potential with respect to constant fields yield vertex functions at zero momentum transfer. From (20), we learn that our model possesses a tower of non-vanishing vertex functions, the energy scale of which is set by the renormalization group invariant scale $M_0$. Our model therefore exhibits non-vanishing interactions in the continuum limit (non-triviality). Stationary points $C_0$ of the potential (20) immediately get a physical interpretation, if we calculate the propagator of the scalar particle $\phi_x$. Taking twice the functional derivative of $Z[j=0,\eta]$ with respect to $\eta$ (and setting $\eta = 0$ afterwards), this propagator is in leading order of the large $N$-expansion

$$S_{xy} = \frac{1}{-\partial^2 + C_0}.$$  \hfill (21)

We therefore identify $C_0$ as the mass squared of the scalar particle.

On the other hand, stationary points of the effective potential correspond to possible candidates for the vacuum. Each stationary point describes a phase of the model. The phase with minimal potential $U$ constitutes the ground state at zero temperature. In the case of the potential (20), two phases are present. In the phase characterized by $C_0 = 0$, the scalar particle is massless. However, the second phase (provided by the stationary point at $C_0 = M_0$) describes the true ground state, since this state has a lower vacuum energy density, i.e. $-M_0^2/128\pi^2$ (note that the perturbative state has by definition zero energy density). We call this phase non-trivial, since it exhibits a dynamically formed mass of the scalar particle. In the latter phase, the ratio of the dynamical mass squared and the vacuum energy density is in leading order of the large $N$-expansion

$$\sqrt{N} \frac{M_0}{-U(C_0 = M_0)} = 8\sqrt{2}\pi.$$  \hfill (22)

To estimate the order of magnitude of the scalar mass, we set $N = 1$ and choose a generic value of the vacuum energy density of the theory of weak interactions, i.e. $U(C_0 = M_0) \approx -(200 \text{ GeV})^4$. From (22), we then find a mass of the scalar particle quite above 1 TeV. This result supplements those obtained in [11], where a Higgs mass in the TeV range was also reported in a different type of scalar field theory. It was first shown by Kuti et al. [10, 11] that certain scalar field theories possess ultraviolet fixed points, and that the Higgs mass predicted by these theories evade the triviality bounds. It was argued by renormalization group techniques first by Morris [12] and independently by Halpern and Huang [13] that certain non-polynomial scalar field theories which contain the momentum cutoff at tree level are non-trivial.

\footnote{This ensures that $U(C)$ is stationary at the corresponding $C$ value.}
and asymptotically free. In this letter, we have provided for the first time an explicit example for such a theory. We have proposed an O(N)-symmetric scalar field theory on the lattice which is solvable in the large N-limit. The key point is that the functional integration over the scalar field is supplemented with a non-trivial measure. We have argued that such a measure naturally arises, if the scalar field is interpreted as effective gauge invariant degree of freedom of SU(2) gauge theory. The quantum model of our toy theory has a continuum limit and is asymptotically free. The model exhibits two phases. In the vacuum phase, the scalar particle possesses a dynamically generated mass in the TeV range. The results therefore signal non-triviality.

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