THE NUMBER OF EDGES IN CRITICAL STRONGLY CONNECTED GRAPHS

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Abstract. We prove that the maximal number of directed edges in a vertex-critical strongly connected simple digraph on \( n \) vertices is \( \left( \frac{n}{2} \right) - n + 4 \).

1. Introduction

A directed graph (or digraph) without loops or multiple edges is called strongly connected if each vertex in it is reachable from every other vertex. It is called (vertex) critical strongly connected if, in addition to being strongly connected, it has the property that the removal of any vertex from it results in a non-strongly connected graph. We denote by \( M(n) \) the maximal number of edges in a critical strongly connected digraph on \( n \) vertices. Schwarz [3] conjectured (and proved for \( n \leq 5 \)) that \( M(n) \leq \left( \frac{n}{2} \right) \). This conjecture was proved by London in [2]. In this paper we determine the precise number of \( M(n) \), showing that it is \( \left( \frac{n}{2} \right) - n + 4 \). (The corresponding number for edge-critical strongly connected graphs is \( 2n - 2 \), see e.g. [1], pp 65-66.)

Here is some notation we shall use. Given a digraph \( D \) we denote by \( V(D) \) the set of its vertices, and by \( E(D) \) the set of edges. Throughout the paper the notation \( n \) will be reserved for the number of vertices in the digraph named \( D \). For a vertex \( v \) of \( D \) we write \( E_D^+(v) \) for the set of vertices \( u \) for which \( (v, u) \in E(D) \) and \( E_D^-(v) \) for the set of vertices \( u \) for which \( (u, v) \in E(D) \). We write \( d_D(v) \) for the degree of \( v \), namely \( |E_D^+(v)| + |E_D^-(v)| \). For a subset \( A \) of \( V(D) \) we write \( D - A \) for the graph obtained from \( D \) by removing all vertices in \( A \), together with all edges incident with them. If \( A \) consists of a single vertex \( a \), we write \( D - a \) for \( D - \{a\} \). By \( D/A \) we denote the digraph obtained from \( D \) by contracting \( A \), namely replacing all vertices of \( A \) by a single vertex \( a \), and defining \( E_D^+_{D/A}(a) = \bigcup \{ E_D^+(v) : v \in A \} \setminus A \) and \( E_D^-_{D/A}(a) = \bigcup \{ E_D^-(v) : v \in A \} \setminus A \).

2. The number of edges in vertex-critical graphs

Theorem 2.1. For \( n \geq 4 \)

\[ M(n) = \left( \frac{n}{2} \right) - n + 4 \]

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A vertex-critical graph with \(^\binom{n}{2} - n + 4\) edges is the following. Take a directed cycle \((v_1, v_2, \ldots, v_n)\), and add the directed edges \((v_i, v_j)\), \(3 \leq j < i \leq n\) and the edge \((v_2, v_1)\). Thus, what remains to be proved is that in a vertex-critical graph the number of edges does not exceed \(^\binom{n}{2} - n + 4\).

The proof will be based on two lemmas.

**Lemma 2.2.** Let \(D\) be a strongly connected digraph and \(v \in V(D)\) a vertex satisfying \(d(v) \geq n\). Then there exists a vertex \(z \in V(D) \setminus \{v\}\) such that \(D - z\) is strongly connected.

*Proof* The proof is by induction on \(n\). For \(n < 2\) the lemma is vacuously true, since its conditions are impossible to fulfill. For \(n = 2\) take \(z\) to be the vertex of the graph different from \(v\). Let now \(n > 2\) and suppose that the lemma is true for all graphs with fewer than \(n\) vertices. Let \(v\) be as in the lemma. There exists then a vertex \(u\) such that between \(u\) and \(v\) there is a double-arc (that is, two oppositely directed edges). Let \(C = D/\{u, v\}\), and name \(w\) the vertex of \(C\) replacing the shrunk pair \(\{u, v\}\). By a negation hypothesis, we may assume that \(D - u\) is not strongly connected. We claim then that \(d_C(w) \geq n - 1\). This will prove the lemma, since by the induction hypothesis it will follow that \(C\) has a vertex \(z\) different from \(w\) whose removal leaves \(C\) strongly connected. But then, clearly, also \(D - z\) is strongly connected.

To prove the claim note, first, that \(d_C(w) \geq n - 2\). This follows from the fact that each edge in \(D\) incident with \(v\) and different from the two edges joining \(v\) with \(u\), has its copy in \(C\). Since by our assumption \(D - u\) is not strongly connected, there are two edges in \(D\), say \((x, u)\) and \((u, y)\), such that \(y\) is not reachable from \(x\) in \(D - u\). If \(x = v\) then the edge \((w, y)\) is an edge in \(C\) not having a copy \((v, y)\) in \(D\), and thus can be added to the \(n - 2\) edges incident with \(v\) counted above, and thus \(d_C(w) \geq n - 1\), as desired. Similarly, if \(y = v\) then the edge \((x, w)\) shows that \(d_C(w) \geq n - 1\). If, on the other hand, \(x \neq v\) and \(y \neq v\), then one of \((x, w)\) or \((w, y)\) is an edge in \(C\) not counted above. \(\square\)

Note that the lemma proves the original conjecture of Schwarz, namely that \(M(n) \leq \binom{n}{2}\).

**Lemma 2.3.** Let \(D\) be a critical digraph and \(C\) a chordless cycle in it, such that \(V(C) \neq V(D)\). Then \(d(v) \leq n - |V(C)| + 2\) for all \(v \in V(C)\), with strict inequality holding for at least two vertices.

*Proof* Let \(J = D/V(C)\), and denote by \(c\) the vertex of \(J\) obtained from the contraction of \(C\). Write \(k\) for \(|V(C)|\). The graph \(J\) has \(n - k + 1\) vertices, and therefore, by Lemma 2.2, \(d_J(c) \leq n - k\). This implies that \(d(v) \leq n - k + 2\) for every \(v \in V(C)\).

Suppose now that, for some vertex \(w\) of \(C\), there obtains \(d(v) = n - k + 2\) for all vertices \(v \in V(C) \setminus \{w\}\). Then \(d_J(c) = n - k\), and all sets \(E_D^+(v) \setminus V(C)\)
(\(v \in V(C) \setminus \{w\}\)) are equal, and the same goes for the sets \(E_D^-(v) \setminus V(C)\). Moreover, \((E_D^-(w) \setminus V(C)) \subseteq E_D^-(v)\) and \((E_D^-(w) \setminus V(C)) \subseteq E_D^-(v)\) for all \(v \in V(C)\). But then \(D - w\) must be strongly connected, since if \((x, w)\) and 

\((w, y)\) are edges in \(D\), then \(y\) is reachable from \(x\) in \(D - w\) through vertices of \(V(C) \setminus \{w\}\).

\[\square\]

**Proof of Theorem 2.4.** The proof is by induction on \(n\). Write \(s_n\) for the value claimed by Theorem 2.1 for \(M(n)\), namely

\[s_n = \binom{n}{2} - n + 4\]

Since \(D\) is critically strongly connected, it contains a chordless cycle \(C\). Let \(|V(C)| = k\). If \(V(C) = V(D)\) then we are done because then \(|E(D)| = n \leq s_n\). Thus we may assume that \(V(C) \neq V(D)\), and since \(D\) is critical, this implies that \(n \geq k + 2\).

Let \(J = D/V(C)\), and denote by \(c\) the vertex of \(J\) obtained from the contraction of \(C\).

**Assertion 2.4.**

\[|E(D)| - |E(J)| \leq s_n - s_{n-k+1}\]

Consider first the case \(k = 2\). Let \(v\) be one of the two vertices of \(C\). The graph \(J\), being the contraction of a strongly connected graph, is itself strongly connected, and since \(D - v\) is not strongly connected, we have \(J \neq D - v\). This implies that \(|E(J)| > |E(D - v)|\), and hence

\[|E(D)| - |E(J)| \leq d_D(v) - 1 \leq n - 2 = s_n - s_{n-1}\]

Assume now that \(k \geq 3\). Let \(v\) be a vertex of \(C\) having maximal degree, namely \(d_D(v) \geq d_D(u)\) for all \(u \in V(C)\). Let \(r\) be the number of edges in \(D\) not incident with any vertex of \(C\). Then

\[|E(D)| = r - k + \sum_{u \in V(C)} d_D(u)\]

and

\[|E(J)| = d_J(c) + r \geq d_D(v) - 2 + r\]

and therefore

\[|E(D)| - |E(J)| \leq 2 - k + \sum_{u \in V(C) \setminus \{v\}} d_D(u)\]

\[\leq 2 - k + 2(n - k + 1) + (k - 3)(n - k + 2)\]

\[= (k - 1)n - (k^2 - 2k + 2) \leq s_n - s_{n-k+1}\]

which proves the assertion.

If \(J\) is critical, then the theorem follows from Assertion 2.4 and the induction hypothesis. So, we may assume that \(J\) is not critical. But, for every vertex \(u\) different from \(c\), the graph \(J - u\) is not strongly connected, since the graph \(D - u\) is not strongly connected. Hence, by Lemma 2.2, we have
On the other hand, the fact that \( J \) is not critical means that \( J - c = D - V(C) \) is strongly connected.

We next show:

**Assertion 2.5.**

\[
\sum_{v \in V(C)} d_D(v) \leq (n - 1)k - n + 4
\]

**Proof of the assertion** By Lemma 2.2, \( d(v) < n \) for all vertices \( v \). Hence, if \( d_D(v) = 2 \) for some \( v \in V(C) \) then the assertion is true. So, we may assume that \( d_D(v) \neq 2 \) for all \( v \in V(C) \). This means that \( (E^+_D(v) \cup E^-_D(v)) \setminus V(C) \neq \emptyset \) for every \( v \in V(C) \). Let \( V(C) = \{v_1, v_2 \ldots v_k\} \) and \( E(C) = \{(v_i, v_{i+1}) : 1 \leq i \leq k\} \) (where, as usual, the indices are taken modulo \( k \)). Without loss of generality we may assume that \( E^+_D(v_1) \setminus V(C) \neq \emptyset \). If \( E^-_D(v_3) \setminus V(C) \neq \emptyset \) then \( D - v_2 \) is strongly connected. Thus we may assume that \( E^-_D(v_3) \setminus V(C) = \emptyset \) and \( E^+_D(v_3) \setminus V(C) \neq \emptyset \). Applying this argument again and again, we conclude that \( k \) is even and that \( E^-_D(v_i) \setminus V(C) = \emptyset \) for all odd \( i \) and \( E^+_D(v_i) \setminus V(C) = \emptyset \) for all even \( i \). By (1) it follows that for every two adjacent vertices on \( C \) the total number of edges incident with them and not belonging to \( C \) does not exceed \( n - k \). This implies that:

\[
\sum_{v \in V(C)} d_D(v) \leq \frac{k}{2}(n - k) + 2k \leq (n - 1)k - n + 4
\]

proving the assertion.

Recall now that \( n \geq k + 2 \) and that \( D - V(C) \) is strongly connected. Hence \( D - V(C) \) contains a chordless cycle \( C' \). Let \( k' = |V(C')| \). The same arguments as above hold when \( C \) is replaced by \( C' \), and thus we may assume that

\[
\sum_{v \in V(C')} d_D(v) \leq (n - 1)k' - n + 4
\]

This, together with Lemma 2.3, yields:

\[
\sum_{v \in V(D)} d_D(v) \leq (n - 1)k - n + 4 + (n - 1)k' - n + 4 + (n - 1)(n - k - k') = 2s_n
\]

which means that

\[
|E(D)| \leq s_n
\]

\[\square\]
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