Support Vector Machines for Additive Models: Consistency and Robustness

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Abstract
Support vector machines (SVMs) are special kernel based methods and belong to the most successful learning methods since more than a decade. SVMs can informally be described as a kind of regularized M-estimators for functions and have demonstrated their usefulness in many complicated real-life problems. During the last years a great part of the statistical research on SVMs has concentrated on the question how to design SVMs such that they are universally consistent and statistically robust for nonparametric classification or nonparametric regression purposes. In many applications, some qualitative prior knowledge of the distribution P or of the unknown function $f$ to be estimated is present or the prediction function with a good interpretability is desired, such that a semiparametric model or an additive model is of interest.

In this paper we mainly address the question how to design SVMs by choosing the reproducing kernel Hilbert space (RKHS) or its corresponding kernel to obtain consistent and statistically robust estimators in additive models. We give an explicit construction of kernels — and thus of their RKHSs — which leads in combination with a Lipschitz continuous loss function to consistent and statistically robust SMVs for additive models. Examples are quantile regression based on the pinball loss function, regression based on the $\epsilon$-insensitive loss function, and classification based on the hinge loss function.

KEYWORDS: Support Vector Machine, SVM, additive model, consistency, robustness, kernel
1 Introduction

Kernel methods such as support vector machines belong to the most successful learning methods since more than a decade, see Schölkopf and Smola (2002). Examples are classification or regression models where we have an input space $\mathcal{X}$, an output space $\mathcal{Y}$, some unknown probability measure $P$ on $\mathcal{X} \times \mathcal{Y}$, and an unknown function $f : \mathcal{X} \to \mathbb{R}$ which describes the quantity of interest, e.g. the conditional quantile curve, of the conditional distribution of $P(\cdot|x)$, $x \in \mathcal{X}$. Support vector machines can informally be described as a kind of regularized M-estimators for functions and have demonstrated their usefulness in many complicated high-dimensional real-life problems. Besides several other nice features, one key argument for using SVMs has been the so-called “kernel trick” (Schölkopf et al., 1998), which decouples the SVM optimization problem from the domain of the samples, thus making it possible to use SVMs on virtually any input space $\mathcal{X}$. This flexibility is in strong contrast to more classical learning methods from both machine learning and non-parametric statistics, which almost always require input spaces $\mathcal{X} \subset \mathbb{R}^d$. As a result, kernel methods have been successfully used in various application areas that were previously infeasible for machine learning methods. As examples we refer to (i) SVMs where using probability measures, e.g. histograms, as input samples, have been used to analyze histogram data and coloured images (Hein and Bousquet, 2005, Sriperumbudur et al., 2009), (ii) SVMs for text classification and web mining (Joachims, 2002, Lafferty and Lebanon, 2005), and (iii) SVMs with kernels from computational biology, e.g. kernels for trees and graphs (Schölkopf et al., 2004).

For a data set $D_n = \{(x_1, y_1), \ldots, (x_n, y_n)\}$, the empirical SVM is defined as

$$f_{L,D_n,\lambda} := \arg \inf_{f \in H} \frac{1}{n} \sum_{i=1}^{n} L(x_i, y_i, f(x_i)) + \lambda \|f\|^2_H.$$  \hspace{1cm} (1)

That is, SVMs are based on three key components: (i) a convex loss function $L : \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \to [0, \infty)$ used to measure the quality of the prediction $f(x)$, (ii) a reproducing kernel Hilbert space (RKHS) $H$ of functions $f$ to specify the set of functions over which the expected loss is minimized, and (iii) the regularization term $\lambda \|f\|^2_H$ to reduce the danger of overfitting and to guarantee the existence of a unique SVM even if $L$ is not strictly convex. The RKHS is often implicitly defined by specifying a kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$. Details about the definition of SVMs and some examples will be given in
Section 2

During the last years a great part of the statistical research on SVMs has concentrated on the central question how to choose the loss function $L$, the RKHS $H$ or its kernel $k$, and sequences of regularization parameters $\lambda_n$ to guarantee that SVMs are universally consistent and statistically robust for classification and regression purposes. In a nutshell, it turned out in a purely non-parametric setup that SVMs based on the combination of a Lipschitz continuous loss function and a bounded continuous kernel with a dense and separable RKHS are universally consistent with desirable statistical robustness properties for any probability measure $P$ from which we observed the data set, see, e.g., Steinwart and Christmann (2008) and Christmann et al. (2009) for details. Examples are the combination of the Gaussian RBF-kernel with the pinball loss function for nonparametric quantile regression, with the $\epsilon$-insensitive loss function for nonparametric regression, or with the hinge loss function for nonparametric classification, see Section 2.

Although a nonparametric approach is often the best choice in practice due to the lack of prior knowledge on $P$, a semiparametric approach or an additive model (Friedman and Stuetzle, 1981; Hastie and Tibshirani, 1990) can also be valuable. For example, we may be interested due to practical reasons only in functions $f$ which offer a nice interpretation because an interpretable prediction function can be crucial if the prediction $f(x)$ has to be explainable to clients. This can be the case if the prediction is the expected claim amount of a client and these predictions are the basis for the construction of an insurance tariff. Here we will mainly consider additive models although models with a multiplicative can also be of interest. More precisely, for some $s \in \mathbb{N}$, the input space $\mathcal{X}$ is split up into $s \in \mathbb{N}$ non-empty spaces according to

$$\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_s$$

(2)

and only additive functions $f : \mathcal{X} \rightarrow \mathbb{R}$ of the form

$$f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_s(x_s), \quad x_j \in \mathcal{X}_j,$$

are considered, where $f_j : \mathcal{X}_j \rightarrow \mathbb{R}$ for $j \in \{1, \ldots, s\}$.

To our best knowledge, there are currently no results on consistency and statistical robustness published on SVMs based on kernels designed for additive models. Of course, one can use one of the purely nonparametric SVMs described above, but the hope is, that SVMs based on kernels especially designed for such situations may offer better results.
In this paper we address the question how to design specific SVMs for additive models. The main goal of this paper is that we give an explicit construction principle of kernels — and thus of their RKHSs — which leads in combination with a Lipschitz continuous loss function to consistent and statistically robust SMVs for additive models. Examples are SVMs in additive models for quantile regression based on the pinball loss function, for regression based on the \( \epsilon \)-insensitive loss function, and for classification based on the hinge loss function.

The rest of the paper is organized as follows. In Section 2 we collect some known results on loss functions, kernels and their RKHSs, and on support vector machines. These results are needed to state our results on consistency and statistical robustness of SVMs for additive models in Section 3. Although we have so far no result on the rates of convergence, our numerical examples given in Section 4 will demonstrate that SVMs based on kernels designed for additive models can easily outperform standard nonparametric SVMs if the assumption of an additive model is valid. Section 5 contains the discussion. All proofs are given in the Appendix.

2 Background on support vector machines

Let \( \mathcal{X} \) be a complete separable metric space and let \( \mathcal{Y} \) be a closed subset of \( \mathbb{R} \). We will always use the respective Borel-\( \sigma \)-algebras. The set of all probability measures on the Borel-\( \sigma \)-algebra of \( \mathcal{X} \times \mathcal{Y} \) is denoted by \( \mathcal{M}_1(\mathcal{X} \times \mathcal{Y}) \). The random input variables \( X_1, \ldots, X_n \) take their values in \( \mathcal{X} \) and the random output variables \( Y_1, \ldots, Y_n \) take their values in \( \mathcal{Y} \). It is assumed that \( (X_1, Y_1), \ldots, (X_n, Y_n) \) are independent and identically distributed according to some unknown probability measure \( P \in \mathcal{M}_1(\mathcal{X} \times \mathcal{Y}) \). Since \( \mathcal{Y} \subset \mathbb{R} \) is closed, \( P \) can be split into the marginal distribution \( P_\mathcal{X} \) on \( \mathcal{X} \) and the conditional distribution \( P(\cdot | x) \) of \( \mathcal{Y} \) given \( x \).

The goal is to find a good predictor \( f : \mathcal{X} \rightarrow \mathbb{R} \) which predicts the value \( y \) of an output variable after observing the value \( x \) of the corresponding input variable. The quality of a prediction \( t = f(x) \) is measured by a loss function

\[
L : \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \rightarrow [0, \infty), \quad (x, y, t) \mapsto L(x, y, t).
\]

It is assumed that \( L \) is measurable and \( L(x, y, y) = 0 \) for every \( (x, y) \in \mathcal{X} \times \mathcal{Y} \) — that is, the loss is zero if the prediction \( t \) equals the actual value \( y \) of the
output variable. In addition, we make the standard assumption that

\[ L(x, y, \cdot) : \mathbb{R} \to [0, \infty), \quad t \mapsto L(x, y, t) \]

is convex for every \((x, y) \in \mathcal{X} \times \mathcal{Y}\) and that additionally the following uniform Lipschitz property is fulfilled for some real number \( |L|_1 \in (0, \infty) \):

\[
\sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} |L(x, y, t) - L(x, y, t')| \leq |L|_1 \cdot |t - t'| \quad \forall t, t' \in \mathbb{R}.
\] (3)

We restrict our attention to Lipschitz continuous loss functions because the use of loss functions which are not Lipschitz continuous (such as the least squares loss which is only locally Lipschitz continuous on unbounded domains) usually conflicts with robustness; see, e.g., Steinwart and Christmann (2008, §10.4).

The quality of a (measurable) predictor \( f : \mathcal{X} \to \mathbb{R} \) is measured by the risk

\[
\mathcal{R}_{L,P}(f) = \int_{\mathcal{X} \times \mathcal{Y}} L(x, y, f(x)) P(d(x,y)).
\]

By different choices of \( \mathcal{Y} \) and the loss function \( L \), different purposes are covered by this setup – e.g. binary classification for \( \mathcal{Y} = \{-1; +1\} \) and the hinge loss

\[ L_{\text{hinge}}(x, y, t) := \max\{0, 1 - yt\}, \]

regression for \( \mathcal{Y} = \mathbb{R} \) and the \( \epsilon \)-insensitive loss

\[ L_{\epsilon}(x, y, t) := \max\{0, |y - t| - \epsilon\} \]

where \( \epsilon > 0 \), and quantile regression for \( \mathcal{Y} = \mathbb{R} \) and the pinball loss

\[ L_{\tau}(x, y, t) := \begin{cases} (\tau - 1)(y - t), & \text{if } y - t < 0, \\ \tau(y - t), & \text{if } y - t \geq 0, \end{cases} \] (4)

where \( \tau \in (0, 1) \).

An optimal predictor is a measurable function \( f^* : \mathcal{X} \to \mathbb{R} \) which attains the minimal risk, called Bayes-risk,

\[
\mathcal{R}_{L,P}^* = \inf_{f : \mathcal{X} \to \mathbb{R}} \text{measurable} \mathcal{R}_{L,P}(f).
\]
The optimal predictor in a set $\mathcal{F}$ of measurable functions $f : \mathcal{X} \to \mathbb{R}$ is an $f^* \in \mathcal{F}$ which attains the minimal risk

$$R_{L,P,F}^* = \inf_{f \in \mathcal{F}} R_{L,P}(f).$$

For example, the goal of quantile regression is to estimate a conditional quantile function, i.e., a function $f^*_{\tau,P} : \mathcal{X} \to \mathbb{R}$ such that

$$P\left( (-\infty, f^*_{\tau,P}(x)) \mid x \right) \geq \tau \quad \text{and} \quad P\left( [f^*_{\tau,P}(x), \infty) \mid x \right) \geq 1 - \tau$$

for the quantile $\tau \in (0, 1)$. If $f^*_{\tau,P} \in \mathcal{F}$, then the conditional quantile function $f^*_{\tau,P}$ attains the minimal risk $R^*_{L_{\tau,P},F}$ for the pinball loss $L_{\tau}$ (with parameter $\tau$) so that quantile regression can be done by trying to minimize the risk $R^*_{L_{\tau,P}}$ in $\mathcal{F}$.

One way to build a non-parametric predictor $f$ is to use a support vector machine

$$f_{L,P,\lambda} := \arg \inf_{f \in H} R_{L,P}(f) + \lambda \|f\|_H^2,$$  \hspace{1cm} (5)

where $H$ is a reproducing kernel Hilbert space (RKHS) of a measurable kernel $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, and $\lambda > 0$ is a regularization parameter to reduce the danger of overfitting, see e.g., Vapnik (1998), Schölkopf and Smola (2002) or Steinwart and Christmann (2008) for details. The reproducing property of $k$ states that, for all $f \in H$ and all $x \in \mathcal{X}$,

$$f(x) = \langle f, \Phi(x) \rangle_H$$

where $\Phi : \mathcal{X} \to H$, $x \mapsto k(\cdot, x)$ denotes the canonical feature map. A kernel $k$ is called bounded, if

$$\|k\|_{\infty} := \sup_{x \in \mathcal{X}} \sqrt{k(x,x)} < \infty.$$  \hspace{1cm} (6)

Using the reproducing property and $\|\Phi(x)\|_H = \sqrt{k(x,x)}$, we obtain the well-known inequalities

$$\|f\|_{\infty} \leq \|k\|_{\infty} \|f\|_H$$

and

$$\|\Phi(x)\|_{\infty} \leq \|k\|_{\infty} \|\Phi(x)\|_H \leq \|k\|_{\infty}^2$$  \hspace{1cm} (7)

for all $f \in H$ and all $x \in \mathcal{X}$. As an example of a bounded kernel, we mention the popular Gaussian radial basis function (GRBF) kernel defined by

$$k_{\gamma}(x, x') = \exp(-\gamma^{-2} \|x - x'\|_{\mathbb{R}^d}^2), \quad x, x' \in \mathcal{X},$$  \hspace{1cm} (8)
where $\gamma$ is some positive constant and $\mathcal{X} \subset \mathbb{R}^d$. This kernel leads to a large RKHS which is dense in $L_1(\mu)$ for all probability measures $\mu$ on $\mathbb{R}^d$. We will also consider the polynomial kernel

$$k_{m,c}(x, x') = (\langle x, x' \rangle_{\mathbb{R}^d} + c)^m, \quad x, x' \in \mathcal{X},$$

where $m \in (0, \infty)$, $c \in (0, \infty)$ and $\mathcal{X} \subset \mathbb{R}^d$. The dot kernel is a special polynomial kernel with $c = 0$ and $m = 1$. The polynomial kernel is bounded if and only if $\mathcal{X}$ is bounded.

Of course, the regularized risk

$$R_{L,P,\lambda}^\text{reg}(f) := R_{L,P}(f) + \lambda \|f\|_H^2$$

is in general not computable, because $P$ is unknown. However, the empirical distribution

$$D_n = \frac{1}{n} \sum_{i=1}^n \delta(x_i, y_i)$$

corresponding to the data set $D_n = ((x_1, y_1), \ldots, (x_n, y_n))$ can be used as an estimator of $P$. Here $\delta(x_i, y_i)$ denotes the Dirac distribution in $(x_i, y_i)$. If we replace $P$ by $D_n$ in (5), we obtain the regularized empirical risk $R_{L,D_n,\lambda}^\text{reg}(f)$ and the empirical SVM $f_{L,D_n,\lambda}$. Furthermore, we need analogous notions where $(x_i, y_i)$ is replaced by random variables $(X_i, Y_i)$. Thus, we define

$$D_n = \frac{1}{n} \sum_{i=1}^n \delta(X_i, Y_i).$$

Then, for every $\omega \in \Omega$, $D_n(\omega)$ is the empirical distribution corresponding to the data set $((X_1(\omega), Y_1(\omega)), \ldots, (X_n(\omega), Y_n(\omega)))$ and, accordingly, $R_{L,D_n,\lambda}^\text{reg}(f)$ denotes the mapping $\Omega \to \mathbb{R}$, $\omega \mapsto R_{L,D_n(\omega),\lambda}^\text{reg}(f)$, and $f_{L,D_n,\lambda}$ denotes the mapping $\Omega \to H$, $\omega \mapsto f_{L,D_n(\omega),\lambda}$.

Support vector machines $f_{L,P,\lambda}$ need not exist for every probability measure $P \in \mathcal{M}_1(\mathcal{X} \times \mathcal{Y})$; for Lipschitz continuous loss functions it is sufficient for the existence of $f_{L,P,\lambda}$ that $\int L(x, y, 0) P(d(x, y)) < \infty$. This condition may be violated by heavy-tailed distributions $P$ and, in this case, it is possible that $R_{L,P}(f) = \infty$ for every $f \in H$.

In order to enlarge the applicability of support vector machines to heavy-tailed distributions, the following extension has been developed in [Christmann et al. (2009)]. Following an idea already used by [Huber (1967)] for
M-estimates in parametric models, a *shifted loss function* $L^*: \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$L^*(x, y, t) = L(x, y, t) - L(x, y, 0) \quad \forall (x, y, t) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}.$$  

Then, similar to the original loss function $L$, define the $L^*$-risk by

$$\mathcal{R}_{L^*, P}(f) = \int L^*(x, y, f(x)) P(dx, dy)$$
and the regularized $L^*$-risk by

$$\mathcal{R}_{L^*, P, \lambda}(f) = \mathcal{R}_{L^*, P}(f) + \lambda \|f\|^2_H$$

for every $f \in H$. In complete analogy to (5), we define the support vector machine based on the shifted loss function $L^*$ by

$$f_{L, P, \lambda} := \arg \inf_{f \in H} \mathcal{R}_{L^*, P}(f) + \lambda \|f\|^2_H. \quad (9)$$

If the support vector machine $f_{L, P, \lambda}$ defined by (9) exists, we have seemingly defined $f_{L, P, \lambda}$ in two different ways now. However, the two definitions coincide in this case and the following theorem summarizes some basic results of Christmann et al. (2009).

**Theorem 1.** Let $L$ be a convex and Lipschitz continuous loss function and let $k$ be a bounded kernel. Then, for every $P \in \mathcal{M}_1(\mathcal{X} \times \mathcal{Y})$ and every $\lambda \in (0, \infty)$, there exists a unique SVM $f_{L, P, \lambda} \in H$ which minimizes $\mathcal{R}_{L^*, P, \lambda}$, i.e.

$$\mathcal{R}_{L^*, P}(f_{L, P, \lambda}) + \lambda \|f_{L, P, \lambda}\|^2_H = \inf_{f \in H} \mathcal{R}_{L^*, P}(f) + \lambda \|f\|^2_H.$$

If the support vector machine $f_{L, P, \lambda}$ defined by (9) exists, then the two definitions (5) and (9) coincide.

### 3 Support vector machines for additive models

#### 3.1 Model and assumptions

As described in the previous section, the goal is to minimize the risk $f \mapsto \mathcal{R}_{L, P}(f)$ in a set $\mathcal{F}$ of functions $f : \mathcal{X} \to \mathbb{R}$. In this article, we assume an *additive model*. Accordingly, let

$$\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_s$$
where $X_1, \ldots, X_s$ are non-empty sets. For every $j \in \{1, \ldots, s\}$, let $F_j$ be a set of functions $f_j : X_j \to \mathbb{R}$. Then, we only consider functions $f : X \to \mathbb{R}$ of the form

$$f(x_1, \ldots, x_s) = f_1(x_1) + \cdots + f_s(x_s) \quad \forall (x_1, \ldots, x_s) \in X_1 \times \cdots \times X_s$$

for $f_1 \in F_1, \ldots, f_s \in F_s$. Thus,

$$F := \{f_1 + \cdots + f_s : f_j \in F_j, \ 1 \leq j \leq s\}.$$  \hspace{1cm} (10)

In (10), we have identified $f_j$ with the map $X \to \mathbb{R}$, $(x_1, \ldots, x_s) \mapsto f_j(x_j)$. Such additive models can be treated by support vector machines in a very natural way. For every $j \in \{1, \ldots, s\}$, choose a kernel $k_j$ on $X_j$ with RKHS $H_j$. Then, the space of functions

$$H := \{f_1 + \cdots + f_s : f_j \in H, \ 1 \leq j \leq s\}$$

is an RKHS on $X = X_1 \times \cdots \times X_s$ with kernel $k = k_1 + \cdots + k_s$; see Theorem 2 below. In this way, SVMs can be used to fit additive models and SVMs enjoy at least three appealing features: First, it is guaranteed that the predictor has the assumed additive structure $(x_1, \ldots, x_s) \mapsto f_1(x_1) + \cdots + f_s(x_s)$. Second, it is possible to still use the standard SVM machinery including the kernel trick \cite[§2]{scholkopf2002learning} and implementations of SVMs – just by selecting a kernel $k = k_1 + \cdots + k_s$. Third, the possibility to choose different kernels $k_1, \ldots, k_s$ offers a great flexibility. For example, take $s = 2$ and let $k_1$ be a GRBF kernel on $\mathbb{R}^{d_1}$ and $k_2$ be a GRBF kernel on $\mathbb{R}^{d_2}$. Since the RKHS of a Gaussian kernel is an infinite dimensional function space, we get non-parametric estimates of $f_1$ and $f_2$. As a second example, consider a semiparametric model with $f = f_1 + f_2$ where $f_1 : x_1 \mapsto f_1(x_1)$ is assumed to be a polynomial function of order at most $m$ and $f_2 : x_2 \mapsto f_2(x_2)$ may be some complicated function. Then, this semiparametric model can be treated by simply taking a polynomial kernel on $\mathbb{R}^{d_1}$ for $k_1$ and a GRBF kernel on $\mathbb{R}^{d_2}$ for $k_2$. This can be used, for example, in order to model changes in space (for $d_1 \leq 3$ and $x_1$ specifying the location) or in time (for $d_1 = 1$ and $x_1$ specifying the point in time).

**Theorem 2.** For every $j \in \{1, \ldots, s\}$, let $X_j$ be a non-empty set and

$$k_j : X_j \times X_j \to \mathbb{R}, \quad (x_j, x_j') \mapsto k_j(x_j, x_j'),$$
be a kernel with corresponding RKHS $H_j$. Define $k = k_1 + \cdots + k_s$. That is, 
\[ k((x_1, \ldots, x_s), (x'_1, \ldots, x'_s)) = k_1(x_1, x'_1) + \cdots + k_s(x_s, x'_s) \]
for every $x_j, x'_j \in \mathcal{X}_j$, $j \in \{1, \ldots, s\}$. Then, $k$ is a kernel on $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_s$ with RKHS 
\[ H := \{ f_1 + \cdots + f_s : f_j \in H_j, \ 1 \leq j \leq s \} \]
and the norm of $H$, given in (2), fulfills
\[ \|f_1 + \cdots + f_s\|_H^2 \leq \|f_1\|_{H_1}^2 + \cdots + \|f_s\|_{H_s}^2 \quad \forall f_1 \in H_1, \ldots, f_s \in H_s. \] (11)

If not otherwise stated, we make the following assumptions throughout the rest of the paper although some of the results are also valid under more general conditions.

**Main assumptions**

(i) For every $j \in \{1, \ldots, s\}$, the set $\mathcal{X}_j$ is a complete, separable metric space; $k_j$ is a continuous and bounded kernel on $\mathcal{X}_j$ with RKHS $H_j$. Furthermore, $k = k_1 + \cdots + k_s$ denotes the kernel on $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_s$ defined in Theorem 2 and $H$ denotes its RKHS.

(ii) The subset $\mathcal{Y} \subset \mathbb{R}$ is closed.

(iii) The loss function $L$ is convex and fulfills the uniform Lipschitz continuity (3) with Lipschitz constant $|L|_1 \in (0, \infty)$. In addition, $L(x, y, y) = 0$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Note that every closed subset of $\mathbb{R}^d$ is a complete, separable metric space. We restrict ourselves to Lipschitz continuous loss functions and continuous and bounded kernels because it has been shown earlier that these assumptions are necessary in order to ensure good robustness properties; see e.g. [Steinwart and Christmann, 2008, § 10]. The condition $L(x, y, y) = 0$ is quite natural and practically always fulfilled – it means that the loss of a correct prediction is 0. Our assumptions cover many of the most interesting cases. In particular, the hinge loss (classification), the $\epsilon$-insensitive loss (regression) and the pinball loss (quantile regression) fulfill all assumptions. Many commonly used kernels are continuous. In addition, the Gaussian kernel is always bounded, the linear kernel and all polynomial kernels are bounded if and only if $\mathcal{X}_j$ is bounded. From the assumption that the kernels $k_j$ are continuous and bounded on $\mathcal{X}_j$, it follows that the kernel $k = k_1 + \ldots k_s$ is continuous and bounded on $\mathcal{X}$. 

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3.2 Consistency

SVMs are called universally consistent, if the risk of the SVM estimator $f_{L,D_n,\lambda_n}$ converges, for all probability measures $P$, in probability to the Bayes-risk, i.e.

$$\mathcal{R}_{L^*, P}(f_{L,D_n,\lambda_n}) \overset{P}{\longrightarrow} \mathcal{R}^*_{L^*, P} \quad (n \to \infty).$$

(12)

In order to obtain universal consistency of SVMs, it is necessary to choose a kernel with a large RKHS. Accordingly most known results about universal consistency of SVMs assume that the RKHS is dense in $C(X)$ where $X$ is a compact metric space (see e.g. Steinwart (2001)) or, at least, that the RKHS is dense in $L_q(P_X)$ for some $q \in [1, \infty)$. In this paper, we consider an additive model where the goal is to minimize the risk $f \mapsto \mathcal{R}_{L,P}(f)$ in the set

$$\mathcal{F} = \{f_1 + \cdots + f_s : f_j \in \mathcal{F}_j, \ 1 \leq j \leq s\}.$$

For the consistency of SVMs in an additive model, we do not need that the RKHS $H = H_1 + \cdots + H_s$ is dense in the whole space $L_q(P_X)$; instead, we only assume that each $H_j$ is dense in $\mathcal{F}_j$. As usual, $\mathcal{L}_q(\mu)$ denotes the set of all $q$-integrable real-valued functions with respect to some measure $\mu$ and $L_q(\mu)$ denotes the set of all equivalence classes in $\mathcal{L}_q(\mu)$. Theorem 3 shows consistency of SVMs in additive models. That is, the $L^*$-risk of $f_{L,D_n,\lambda_n}$ converges in probability to the smallest possible risk in $\mathcal{F}$.

**Theorem 3.** Let the main assumptions (p. 10) be valid. Let $P \in \mathcal{M}_1(X \times Y)$ such that

$$H_j \subset \mathcal{F}_j \subset \mathcal{L}_1(P_{X_j}), \quad 1 \leq j \leq s,$$

and let $H_j$ be dense in $\mathcal{F}_j$ with respect to $\| \cdot \|_{L_1(P_{X_j})}$. Then, for every sequence $(\lambda_n)_{n \in \mathbb{N}} \subset (0, \infty)$ such that $\lim_{n \to \infty} \lambda_n = 0$ and $\lim_{n \to \infty} \lambda_n^2 n = \infty$,

$$\mathcal{R}_{L^*, P}(f_{L,D_n,\lambda_n}) \longrightarrow \mathcal{R}^*_{L^*, P, \mathcal{F}} \quad (n \to \infty)$$

in probability.

In general, it is not clear whether convergence of the risks implies convergence of the SVM $f_{L,D_n,\lambda_n}$. However, the following theorem will show such a convergence for quantile regression in an additive model - under the condition that the quantile function $f^*_{\tau,P}$ actually lies in $\mathcal{F} = \mathcal{F}_1 + \cdots + \mathcal{F}_s$. In order to formulate this result, we define

$$d_0(f,g) = \int \min \{1, |f - g|\} \, dP_X$$

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where \(f, g : \mathcal{X} \to \mathbb{R}\) are arbitrary measurable functions. It is known that \(d_0\) is a metric describing convergence in probability.

**Theorem 4.** Let the main assumptions (p. [10]) be valid. Let \(P \in \mathcal{M}_1(\mathcal{X} \times \mathcal{Y})\) such that

\[
H_j \subset F_j \subset L_1(P_{X_j}) \quad \forall j \in \{1, \ldots, s\}
\]

and \(H_j\) is dense in \(F_j\) with respect to \(\| \cdot \|_{L_1(P_{X_j})}\). Let \(\tau \in (0, 1)\) and assume that the quantile function \(f_{\tau, P}^*\) is \(P_X\)-almost surely unique and that

\[
f_{\tau, P}^* \in F.
\]

Then, for the pinball loss function \(L = L_\tau\) and for every sequence \((\lambda_n)_{n \in \mathbb{N}} \subset (0, \infty)\) such that \(\lim_{n \to \infty} \lambda_n = 0\) and \(\lim_{n \to \infty} \lambda_n^2 n = \infty\),

\[
d_0(f_{L, P, \lambda_n}, f_{\tau, P}^*) \longrightarrow 0 \quad (n \to \infty)
\]

in probability.

### 3.3 Robustness

During the last years some general results on the statistical robustness properties of SVMs have been shown. Many of these results are directly applicable to SVMs for additive models if the kernel is bounded and continuous (or at least measurable) and the loss function is Lipschitz continuous. For brevity we only give upper bounds for the bias and the Bouligand influence function for SVMs, which are both even applicable for non-smooth loss functions like the pinball loss for quantile regression, and refer to Christmann et al. (2009) and Steinwart and Christmann (2008, Chap. 10) for results on the classical influence function proposed by (Hampel, 1968, 1974) and to Hable and Christmann (2009) for qualitative robustness of SVMs.

Define the function

\[
T : \mathcal{M}_1(\mathcal{X} \times \mathcal{Y}) \to H, \quad T(P) := f_{L, P, \lambda},
\]

which maps each probability distribution to its SVM. In robust statistics we are interested in smooth and bounded functions \(T\), because this will give us stable SVMs within small neighborhoods of \(P\). If an appropriately chosen derivative of \(T(P)\) is bounded, then we expect the value of \(T(Q)\) to be close to the value of \(T(P)\) for distributions \(Q\) in a small neighborhood of \(P\).
The next result shows that the $H$-norm of the difference of two SVMs increases with respect to the mixture proportion $\varepsilon \in (0, 1)$ at most linearly in gross-error neighborhoods. The norm of total variation of a signed measure $\mu$ is denoted by $\|\mu\|_M$.

**Theorem 5** (Bounds for bias). *If the main assumptions (p. 10) are valid, then we have, for all $\lambda > 0$, all $\varepsilon \in [0, 1]$, and all probability measures $P$ and $Q$ on $\mathcal{X} \times \mathcal{Y}$, that*

$$
\|T(Q) - T(P)\|_\infty \leq c \|P - Q\|_M, \quad (14)
\|T((1 - \varepsilon)P + \varepsilon Q) - T(P)\|_\infty \leq c \|P - Q\|_M \varepsilon, \quad (15)
$$

*where $c = \frac{1}{\lambda} \|k\|^2_\infty |L|_1$.*

Because of (7), there are analogous bias bounds of SVMs with respect to the norm in $H$, if we replace $c$ by $\tilde{c} := \frac{1}{\lambda} \|k\|_\infty |L|_1$.

While F.R. Hampel’s influence function is related to a Gâteaux-derivative which is linear, the Bouligand influence function is related to the Bouligand derivative which needs only to be *positive homogeneous*. Because this weak derivative is less known in statistics, we like to recall its definition. Let $E_1$ and $E_2$ be normed linear spaces. A function $f : E_1 \to E_2$ is called *positive homogeneous* if $f(\alpha x) = \alpha f(x)$ for all $\alpha \geq 0$ and for all $x \in E_1$. If $U$ is an open subset of $E_1$, then a function $f : U \to E_2$ is called *Bouligand-differentiable* at a point $x_0 \in U$, if there exists a positive homogeneous function $\nabla^B f(x_0) : U \to E_2$ such that

$$
\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - \nabla^B f(x_0)(h)\|_{E_2}}{\|h\|_{E_1}} = 0,
$$

see Robinson (1991).

The *Bouligand influence function* (BIF) of the map $T : M_1(\mathcal{X} \times \mathcal{Y}) \to H$ for a distribution $P$ in the direction of a distribution $Q \neq P$ was defined by Christmann and Van Messem (2008) as

$$
\lim_{\varepsilon \downarrow 0} \frac{\|T((1 - \varepsilon)P + \varepsilon Q) - T(P) - \text{BIF}(Q; T, P)\|_H}{\varepsilon} = 0. \quad (16)
$$

Note that the BIF is a special Bouligand-derivative

$$
\lim_{\|\varepsilon(Q - P)\| \to 0} \frac{\|T((P + \varepsilon(Q - P)) - T(P) - \text{BIF}(Q; T, P))\|_H}{\|\varepsilon(Q - P)\|} = 0
$$
due to the fact that $Q$ and $P$ are fixed, and it is independent of the norm on $M_1(X \times Y)$. The partial Bouligand derivative with respect to the third argument of $L^\star$ is denoted by $\nabla_3^B L^\star(x, y, t)$. The BIF shares with F.R. Hampel’s influence function the interpretation that it measures the impact of an infinitesimal small amount of contamination of the original distribution $P$ in the direction of $Q$ on the quantity of interest $T(P)$. It is thus desirable that the function $T$ has a bounded BIF. It is known that existence of the BIF implies existence of the IF and in this case they are equal. The next result shows that, under some conditions, the Bouligand influence function of SVMs exists and is bounded, see Christmann et al. (2009) for more related results.

**Theorem 6** (Bouligand influence function). *Let the main assumptions (p. 14) be valid, but assume that $\mathcal{X}$ is a complete separable normed linear space. Let $P, Q \in M_1(X \times Y)$. Let $L$ be the pinball loss function $L^\tau$ with $\tau \in (0, 1)$ or let $L$ be the $\epsilon$-insensitive loss function $L_\epsilon$ with $\epsilon > 0$. Assume that for all $\delta > 0$ there exist positive constants $\xi_P, \xi_Q, c_P, c_Q$ such that for all $t \in \mathbb{R}$ with $|t - f_{L,P,\lambda}(x)| \leq \delta \|k\|_\infty$ the following inequalities hold for all $a \in [0, 2\delta \|k\|_\infty]$ and $x \in \mathcal{X}$:

$$
P([t, t + a] \mid x) \leq c_P a^{1 + \xi_P} \quad \text{and} \quad Q([t, t + a] \mid x) \leq c_Q a^{1 + \xi_Q}.$$  

(17)

Then the Bouligand influence function $\text{BIF}(Q; T, P)$ of $T(P) := f_{L,P,\lambda}$ exists, is bounded, and equals

$$
\frac{1}{2} \left( \mathbb{E}_P \nabla_3^B L^\star(X, Y, f_{L,P,\lambda}(X)) \Phi(X) - \mathbb{E}_Q \nabla_3^B L^\star(X, Y, f_{L,P,\lambda}(X)) \Phi(X) \right). 
$$

(18)

Note that the Bouligand influence function of the SVM only depends on $Q$ via the second term in (18). The interpretation of the condition (17) is that the probability that $Y$ given $x$ is in some small interval around the SVM is essentially at most proportional to the length of the interval to some power greater than one.

For the pinball loss function, the BIF given in (18) simplifies to

$$
\frac{1}{2\lambda} \int_{\mathcal{X}} (P((-\infty, f_{L,P,\lambda}(x)] \mid x) - \tau) \Phi(x) P_{\mathcal{X}}(dx)
$$

$$
- \frac{1}{2\lambda} \int_{\mathcal{X}} (Q((-\infty, f_{L,P,\lambda}(x)] \mid x) - \tau) \Phi(x) Q_{\mathcal{X}}(dx).
$$

(19)

---

1Bouligand derivatives are only defined in normed linear spaces. E.g., $\mathcal{X} \subset \mathbb{R}^d$ a linear subspace.
The BIF of the SVM based on the pinball loss function can hence be interpreted as the difference of the integrated and with \( \frac{1}{2\lambda} \Phi(x) \) weighted difference between the estimated quantile level and the desired quantile level \( \tau \).

Recall that the BIF is a special Bouligand derivative and thus positive homogeneous in \( h = \varepsilon(Q - P) \). If the BIF exists, we then immediately obtain

\[
f_{L, (1-\alpha \varepsilon)P + \alpha \varepsilon Q, \lambda} - f_{L, P, \lambda} = T(P + \alpha h) - T(P) = \alpha \text{BIF}(Q; T, P) + o(\alpha h) \tag{20}
\]

for all \( \alpha \geq 0 \). This equation gives us a nice approximation of the asymptotic bias term \( f_{L, (1-\varepsilon)P + \varepsilon Q, \lambda} - f_{L, P, \lambda} \), if we consider the amount \( \alpha \varepsilon \) of contamination instead of \( \varepsilon \).

4 Examples

In this section we would like to illustrate our theoretical results on SVMs for additive models with a few finite sample examples. The goals of this short section are twofold. We like to get some preliminary insight how SVMs based on kernels designed for additive models work for finite sample sizes when compared to the standard GRBF kernel defined on the whole input space and to get some ideas for further research on this topic. We also like to apply support vector machines based on the additive kernels treated in this paper to a real-life data set.

4.1 Simulated example

Let us consider the following situation of median regression. We have two independent input variables \( X_1 \) and \( X_2 \) each with a uniform distribution on the interval \([0, 1]\) and the output variable \( Y \) given \( x = (x_1, x_2) \) has a Cauchy distribution (and thus not even the first moment does exist) with center \( f(x_1, x_2) := f_1(x_1) + f_2(x_2) \), where \( f_1(x_1) := 7 + 5x_1^2 \) and \( f_2(x_2) := \sin(5x_2)\cos(17x_2) \). Hence the true function \( f \) we like to estimate with SVMs has an additive structure, where the first function is a polynomial of order two and the second function is a smooth and bounded function but no polynomial. Please note, that here \( \mathcal{X} = [0, 1]^2 \) is bounded whereas \( \mathcal{Y} = \mathbb{R} \) is unbounded.
As $\mathcal{X}$ is bounded, even a polynomial kernel on $\mathcal{X}$ is bounded which is not true for unbounded input spaces. We simulated three data sets of this type with sample sizes $n = 500$, $n = 2,000$, and $n = 10,000$. We compare the exact function $f$ with three SVMs $f_{L,D,\lambda_n}$ fitted by the three data sets, where we use the pinball loss function with $\tau = 0.5$ because we are interested in median regression.

- **Nonparametric SVM.** We use an SVM based on the 2-dimensional GRBF kernel $k$ defined in (8) to fit $f$ in a totally nonparametric manner.

- **Nonparametric additive SVM.** We use an SVM based on the kernel $k = k_1 + k_2$ where $k_1$ and $k_2$ are 1-dimensional GRBF kernels.

- **Semiparametric additive SVM.** We use an SVM based on the kernel $k = k_1 + k_2$ where $k_1$ is a polynomial kernel of order 2 to fit the function $f_1$ and $k_2$ is a 1-dimensional GRBF kernel to fit the function $f_2$.

Our interest in these examples is to check how well SVMs using kernels designed for additive models perform in these situations. No attempt was made to find optimal values of the regularization parameters $\lambda$ and the kernel parameter $\gamma$ by using a grid search or cross-validation, because we did not want to mix the quality of such optimization strategies with the choice of the kernels. We therefore fixed $\gamma = 2$ and used the simple non-stochastic specification $\lambda_n = 0.05n^{-0.45}$ for the regularization parameter which guarantees that our consistency result from Section 3 is applicable.

From Figures 1 to 3 we can draw the following conclusions for this special situation.

i) If the additive model is valid, all three SVMs give comparable and reasonable results if the sample size $n$ is large enough even for Cauchy distributed error terms, see Figure 1. This is in good agreement with the theoretical results derived in Section 3.

ii) If the sample size is small to moderate and if the assumed additive model is valid, then both SVMs based on kernels especially designed for additive models show better results than the standard 2-dimensional GRBF kernel, see Figures 2 and Figure 3.

iii) The difference between the nonparametric additive SVM and semiparametric additive SVM was somewhat surprisingly small for all three
sample sizes, although the true function had the very special structure which is in favour for the semiparametric additive SVM.

4.2 Example: additive model using SVMs for rent standard

Let us now consider a real-life example of the rent standard for dwellings in a large city in Germany. Many German cities compose so-called rent standards to make a decision making instrument available to tenants, landlords, renting advisory boards, and experts. Such rent standards can in particular be used for the determination of the local comparative rent, i.e. the net rent as a function of the dwelling size, year of construction of the house, geographical information etc. For the construction of a rent standard, a representative random sample is drawn from all households and questionnaires are used to determine the relevant information by trained interviewers. Fahrmeir et al. (2007) described such a data set consisting of \( n = 3,082 \) rent prizes in Munich, which is one of the largest cities in Germany. They fitted the following additive model

\[
\text{price} = f_1(\text{size}) + f_2(\text{year}) + \beta_0 + \beta_1 \text{region}_1 + \beta_2 \text{region}_2 + \text{error},
\]

where the following variables were used:

- price : net rent price per square meter in DM \([1 \text{ €} \approx 1.96 \text{ DM}]\)
- size : size in square meter of the dwelling \([\text{between 20 and 160}]\)
- year : year \([\text{between 1918 and 1997}]\)
- region_1 : good residential area \([0=\text{no}, 1=\text{yes}]\)
- region_2 : best residential area \([0=\text{no}, 1=\text{yes}]\).

Hence region_1 and region_2 are dummy variables with respect to a standard residential area. Fahrmeir et al. (2007) used a special spline method for estimating the functions \(f_1\) and \(f_2\).

For illustration purposes of SVMs with additive kernels investigated in the present paper, we used a nonparametric additive SVM for median regression. More precisely, we used the pinball loss function with \(\tau = 0.5\) and the kernel

\[
k(x, x') = \sum_{j=1}^{4} k_j(x_j, x'_j), \quad x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \quad x' = (x'_1, x'_2, x'_3, x'_4) \in \mathbb{R}^4,
\]
where

\( k_1 : \mathbb{R} \rightarrow \mathbb{R} \) Gaussian RBF kernel with \( \gamma = 1 \) for size
\( k_2 : \mathbb{R} \rightarrow \mathbb{R} \) Gaussian RBF kernel with \( \gamma = 1 \) for year
\( k_3 : \mathbb{R} \rightarrow \mathbb{R} \) dot kernel for region_1
\( k_4 : \mathbb{R} \rightarrow \mathbb{R} \) dot kernel for region_2.

In analogy to the simulated examples given above, the regularizing parameter was again set to \( \lambda_n = 0.05n^{-0.45} = 0.00135 \) such that our theoretical results are applicable.

The left plot in Figure 4 shows the estimated median net rent price of one square meter depending on the size of the dwelling and the year of the construction for a dwelling in a standard residential area. The plot shows that the median of the net rent prices per square meter is fairly stable for construction years up to 1960, but a more or less linear increase is visible for newer buildings. The plot also shows that the median of the net rent prices per square meter is especially high for dwellings of size less than 80 square meter, that the price is nearly constant for sizes between 80 and 140 square meter, and then a slight increase of the square meter prize seems to occur for even larger dwellings. The slope parameters were estimated by \( \hat{\beta}_1 = 1.38 \) for good residential area (region_1 = 1) and \( \hat{\beta}_1 = 3.46 \) for best residential area (region_2 = 1). Hence, we obtain apart from these level shifts the same surfaces for dwellings located in good or in best residential areas. We would like to mention that we used this real-life example just for illustration purposes, but nevertheless our results are in good agreement with the more detailed statistical analysis of this data set made by Fahrmeir et al. (2007) who used different statistical techniques.

From an applied point of view, one may also be interested in the 10% highest net rent prices depending on the four explanatory variables. We therefore repeated our computations using the same kernel but instead of \( \tau = 0.50 \) for median regression we used \( \tau = 0.90 \) to obtain estimates for the 90\% quantiles of the net rent prizes depending on the four explanatory variables. The right plot in Figure 4 shows the estimated 90\% quantile net rent prices of one square meter depending on the size of the dwelling and the year of the construction for a dwelling in a standard residential area. The slope parameters were estimated by \( \hat{\beta}_1 = 1.59 \) for good residential area (region_1 = 1) and \( \hat{\beta}_1 = 4.24 \) for best residential area (region_2 = 1). The

Some numerical computations showed that the SVM results were fairly stable with respect to other choice of \( \lambda_n \), e.g. \( 4\lambda_n \), for this particular data set and we will hence only show the results \( \lambda_n = 0.05n^{-0.45} \).
shape of the surface is quite similar to the shape of the surface in the previous plot for the estimated median net rent prices. However, the plot may give an indication for a moderate peak for the 90% quantile net rent prices for dwellings of size 100 square meter.

5 Discussion

Support vector machines belong to the class of modern statistical machine learning methods based on kernels. The success of SVMs is partly based on the kernel trick which makes SVMs usable even for abstract input spaces, their universal consistency, their statistical robustness with respect to small model violations, and on the existence of fast numerical algorithms. During the last decade there has been considerable research on these three topics. To obtain universal consistency one needs a sufficiently large reproducing kernel Hilbert space $H$, such that many interesting SVMs are based on Hilbert spaces with infinite dimension. Due to the no-free-lunch theorem (Devroye, 1982), there exists in general no uniform rate of convergence of SVMs on the set of all probability measures.

Although such a nonparametric approach is often the best choice in practice due to the lack of prior knowledge on the unknown probability measure $P$, a semiparametric approach or an additive model (Friedman and Stuetzle, 1981, Hastie and Tibshirani, 1990) can also be valuable for at least two reasons: (i) In some applications some weak knowledge on $P$ or on the unknown function $f$ to be estimated, say the conditional quantile curve, is known, e.g. $f$ is known to be bounded or at least integrable. (ii) Due to practical reasons, we may be interested only in functions $f$ which offer a nice interpretation although there might be a measurable function with a smaller risk, because an interpretable prediction function can be crucial in some applications. An important class of statistical models whose predictions are relatively easily to interpret are additive models.

Therefore, support vector machines for additive models were treated in this paper and some results on their consistency and statistical robustness properties were derived.

Some simple numerical examples showed that SVMs based on kernels especially designed for an additive model can yield better predictions than the standard SVM based on the classical Gaussian RBF kernel, if an additive model is indeed valid.
It may be worthwhile to investigate the rates of convergence of SVMs based on kernels designed for additive models with SVMs based on standard kernels, because our simple numerical examples seem to indicate that there might be some gain with respect to the rate of convergence. However, this is beyond the scope of this paper.

We would like to mention the well-known fact that not only the sum of $s$ kernels is a kernel but also the product of $s$ kernels is a kernel. Hence it seems to be possible to derive similar results than those given here for additive models also for multiplicative models.

Finally, we would like to mention that there are of course many other statistical estimation techniques for additive models, e.g. splines and boosting, but a comparison of these methods with SVMs based on additive kernels is beyond the scope of this paper.

Appendix: Proofs

Proof of Theorem 2. First fix any $j \in \{1, \ldots, s\}$ and define the mapping 

$$\tilde{k}_j : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \quad \text{via} \quad \tilde{k}_j((x_1, \ldots, x_s), (x'_1, \ldots, x'_s)) = k_j(x_j, x'_j)$$

for every $(x_1, \ldots, x_s) \in \mathcal{X}$ and $(x'_1, \ldots, x'_s) \in \mathcal{X}$. Accordingly, for every $f_j \in H_j$, define $\tilde{f}_j : \mathcal{X} \rightarrow \mathbb{R}$ via 

$$\tilde{f}_j(x_1, \ldots, x_s) = f_j(x_j) \quad \forall (x_1, \ldots, x_s) \in \mathcal{X}.$$ 

Then, it is easy to see that 

$$\tilde{H}_j = \{ \tilde{f}_j : \mathcal{X} \rightarrow \mathbb{R} : f_j \in H_j \}$$

is a Hilbert space with inner product and norm given by 

$$\langle \tilde{f}_j, \tilde{h}_j \rangle_{\tilde{H}_j} = \langle f_j, h_j \rangle_{H_j} \quad \text{and} \quad \|\tilde{f}_j\|_{\tilde{H}_j} = \|f_j\|_{H_j}$$ (1)

for every $f_j \in H_j$ and $g_j \in H_j$. Hence, for every $x = (x_1, \ldots, x_s) \in \mathcal{X}$, we get $\tilde{k}_j(\cdot, x) \in \tilde{H}_j$ and 

$$\tilde{f}_j(x) = f_j(x_j) = \langle f_j, k_j(\cdot, x_j) \rangle_{H_j} = \langle \tilde{f}_j, \tilde{k}_j(\cdot, x) \rangle_{\tilde{H}_j} \quad \forall f_j \in H_j$$
where the last equality follows from (1) and the definition of $\tilde{k}_j$. That is, $\tilde{k}_j$ is a reproducing kernel and $\tilde{H}_j$ is its RKHS.

Next, it follows from (Berlinet and Thomas-Agnan, 2004, § 4.1) that $k = \tilde{k}_1 + \cdots + \tilde{k}_s$ is a reproducing kernel on $\mathcal{X}$ with RKHS $H = \tilde{H}_1 + \cdots + \tilde{H}_s$ and norm

$$||f||^2_H = \min_{f = f_1 + \cdots + f_s \in \tilde{H}_1 + \cdots + \tilde{H}_s} ||\tilde{f}_1||^2_{\tilde{H}_1} + \cdots + ||\tilde{f}_s||^2_{\tilde{H}_s} = \min_{f = f_1 + \cdots + f_s \in H_1 + \cdots + H_s} ||f||^2_{H_1} + \cdots + ||f||^2_{H_s}$$  \hspace{1cm} (2)

Using the reduced notation $f_1 + \cdots + f_s$ instead of $\tilde{f}_1 + \cdots + \tilde{f}_s$, inequality (11) follows. \hfill \Box

In order to prove Theorem 3, the following proposition is needed. It provides conditions on $H_j$ and $F_j$ under which the minimal risk over $H = H_1 + \cdots + H_s$ is equal to the minimal risk over the larger $F = F_1 + \cdots + F_s$.

**Proposition 7.** Let the main assumptions (p. 10) be valid. Let $P \in \mathcal{M}_1(\mathcal{X} \times \mathcal{Y})$ such that

$$H_j \subset F_j \subset L_1(P_{X_j}) \quad \forall j \in \{1, \ldots, s\}$$

and $H_j$ is dense in $F_j$ with respect to $\| \cdot \|_{L_1(P_{X_j})}$. Then,

$$\mathcal{R}_{L^*, P, H}^* := \inf_{f \in H} \mathcal{R}_{L^*, P}^*(f) = \mathcal{R}_{L^*, P, F}^*.$$ \hspace{1cm} (3)

**Proof of Proposition 7.** According to the definitions, it only remains to prove $\mathcal{R}_{L^*, P, H}^* \leq \mathcal{R}_{L^*, P, F}^*$. To this end, take any $f \in F$ and any $\varepsilon > 0$. Then, by assumption there are functions

$$f_j \in F_j, \quad j \in \{1, \ldots, n\},$$

such that $f = f_1 + \cdots + f_s$ and, for every $j \in \{1, \ldots, s\}$, there is an $h_j \in H_j$ such that

$$\|h_j - f_j\|_{L_1(P_{X_j})} < \frac{\varepsilon}{s \cdot |L|_1}.$$ \hspace{1cm} (4)
Hence, for \( h = h_1 + \cdots + h_s \in H \),
\[
| \mathcal{R}_{L^*,P}(h) - \mathcal{R}_{L^*,P}(f) | \leq \int | L(x, y, h(x)) - L(x, y, f(x)) | P(d(x, y)) \\
\leq | L | \int | h(x) - f(x) | P_\mathcal{X}(dx) \leq | L | \sum_{j=1}^{s} \int | h_j(x_j) - f_j(x_j) | P_{\mathcal{X}_j}(dx_j) \\
< \varepsilon
\]

\( \square \)

**Proof of Theorem 3.** To avoid handling too many constants, let us assume \( \| k \|_{\infty} = 1 \). According to (6), this implies \( \| f \|_{\infty} \leq \| f \|_H \) for all \( f \in H \). Now we use the Lipschitz continuity of \( L \) to obtain, for all \( g \in H \),
\[
| \mathcal{R}_{L^*,P}(f_{L,P,\lambda_n}) - \mathcal{R}_{L^*,P}(g) | \leq \\
\leq \int | L(x, y, f_{L,P,\lambda_n}(x)) - L(x, y, g(x)) | P(d(x, y)) \\
\leq | L | \int | f_{L,P,\lambda_n}(x) - g(x) | P_\mathcal{X}(dx) \leq | L | \int \| f_{L,P,\lambda_n} - g \|_\infty P_\mathcal{X}(dx) \\
\leq | L | \| f_{L,P,\lambda_n} - g \|_H . \tag{5}
\]

Let \( \Phi \) denote the canonical feature map which corresponds to the kernel \( k \). According to Christmann et al. (2009, Theorem 7), for every \( n \in \mathbb{N} \), there is a bounded, measurable function \( h_n : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) such that
\[
\| h_n \|_{\infty} \leq | L | \tag{6}
\]
and, for every \( Q \in \mathcal{M}_1(\mathcal{X} \times \mathcal{Y}) \),
\[
\| f_{L,P,\lambda_n} - f_{L,Q,\lambda_n} \|_H \leq \lambda_n^{-1} \| \mathbb{E}_P h_n \Phi - \mathbb{E}_Q h_n \Phi \|_H . \tag{7}
\]
Fix any \( \varepsilon \in (0, 1) \) and define
\[
B_n := \left\{ D_n \in (\mathcal{X} \times \mathcal{Y})^n : \| \mathbb{E}_P h_n \Phi - \mathbb{E}_{D_n} h_n \Phi \|_H \leq \frac{\lambda_n \varepsilon}{| L |} \right\} \tag{8}
\]
where \( D_n \) denotes the empirical distribution of the data set \( D_n \). Then, (6), (7) and (8) yield
\[
| \mathcal{R}_{L^*,P}(f_{L,P,\lambda_n}) - \mathcal{R}_{L^*,P}(f_{L,D_n,\lambda_n}) | \leq \varepsilon \quad \forall D_n \in B_n . \tag{9}
\]
Now let us turn over to the probability $P^n(B_n)$. By use of Hoeffding’s inequality, we will show that
\[
\lim_{n \to \infty} P^n(B_n) = 1 .
\] (10)

To this end, we first observe that $\lambda_n n^{1/2} \to \infty$ implies that $\lambda_n \varepsilon \geq n^{-1/2}$ for all sufficiently large $n \in \mathbb{N}$. Moreover, (9) and our assumption $\|k\|_{\infty} = 1$ yield $\|h_n \Phi\|_{\infty} \leq |L|_1$. Define
\[
a_n := |L|_1^{-1} \varepsilon \lambda_n \quad \text{and} \quad \xi_n := 3 \frac{|L|_1^{-2} \varepsilon^2 \lambda_n^2 n}{8 |L|_1^{-1} \varepsilon \lambda_n + 3} = \frac{3}{8} \frac{a_n^2 n}{a_n + 3}
\]
and note that, for sufficiently large $n$,
\[
\frac{\sqrt{2 \xi_n} + 1}{\sqrt{n}} + \frac{4 \xi_n}{3n} = \frac{a_n}{2} \cdot \frac{\sqrt{3}}{\sqrt{a_n + 3}} + \frac{1}{\sqrt{n}} + \frac{a_n}{2} \cdot \frac{a_n}{a_n + 3} < \frac{a_n}{2} + \frac{1}{\sqrt{n}} + \frac{a_n}{2} \cdot \frac{1}{3} < a_n = |L|_1^{-1} \varepsilon \lambda_n . \quad (11)
\]

Consequently, Hoeffding’s inequality in Hilbert spaces (see Steinwart and Christmann (2008, Corollary 6.15)) yields for $B = 1$ the bound
\[
P^n(B_n) = P^n \left( \left\{ D \in (\mathcal{X} \times \mathcal{Y})^n : \|E_P h_n \Phi - E_D h_n \Phi\|_H \leq \frac{\lambda_n \varepsilon}{|L|_1} \right\} \right) \geq P^n \left( \left\{ D \in (\mathcal{X} \times \mathcal{Y})^n : \|E_P h_n \Phi - E_D h_n \Phi\|_H \leq \frac{\sqrt{2 \xi_n} + 1}{\sqrt{n}} + \frac{4 \xi_n}{3n} \right\} \right) \geq 1 - \exp \left( -\frac{3}{8} \cdot \frac{\varepsilon^2 \lambda_n^2 n/|L|_1^2}{|L|_1 + 3} \right) = 1 - \exp \left( -\frac{3}{8} \cdot \frac{\varepsilon^2 \lambda_n^2 n}{|L|_1^2 (|L|_1 + 3)} \right)
\]
for all sufficiently large values of $n$. Now (10) follows from $\lambda_n \to 0$ and $\lambda_n n^{1/2} \to \infty$. According to (9) and (10),
\[
\mathcal{R}_{L^*, P}(f_{L, P, \lambda_n}) - \mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}) \to 0 \quad (n \to \infty)
\]
in probability. Note that,
\[
\left| \mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}) - \mathcal{R}_{L^*, P, F}^* \right| \leq \left| \mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}) - \mathcal{R}_{L^*, P, H}^* \right| + \left| \mathcal{R}_{L^*, P, H}^* - \mathcal{R}_{L^*, P, F}^* \right|
\]
\[
\leq \left| \mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}) - \mathcal{R}_{L^*, P}(f_{L, P, \lambda_n}) \right| + \left| \mathcal{R}_{L^*, P}(f_{L, P, \lambda_n}) - \mathcal{R}_{L^*, P, H}^* \right| \quad (12)
\]
As shown above, the first term in (12) converges in probability to 0. Therefore, it only remains to prove that the second term converges to 0. To this end, define, for every \( f \in H \), the affine linear function
\[
A_f^* : \mathbb{R} \to \mathbb{R}, \quad \lambda \mapsto \mathcal{R}_{L^*, P}(f) + \lambda\|f\|^2_H - \mathcal{R}_{L^*, P, H}^*.
\]
Then, a continuity result for the pointwise infimum of a family of affine functions (see e.g. [Steinwart and Christmann, 2008, A.6.4]) yields
\[
\lim_{n \to \infty} \inf_{f \in H} A_f^*(\lambda_n) = \inf_{f \in H} A_f^*(0).
\]
However, according to the definitions,
\[
\inf_{f \in H} A_f^*(\lambda_n) = \mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}) + \lambda_n\|f_{L, D_n, \lambda_n}\|^2_H - \mathcal{R}_{L^*, P, H}^* \quad \forall \ n \in \mathbb{N}
\]
and \( \inf_{f \in H} A_f^*(0) = 0 \). Hence,
\[
0 \leq \limsup_{n \to \infty} \left( \mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}) - \mathcal{R}_{L^*, P, H}^* \right) \leq 0.
\]

Proof of Theorem 4. Since the quantile function \( f_{\tau, P}^* \) attains the minimal risk \( \mathcal{R}_{L^*, P}^* \) for the pinball loss \( L = L_\tau \) [Koenker, 2005, § 1.3], the assumption \( f_{\tau, P}^* \in F \) implies \( \mathcal{R}_{L^*, P, F}^* = \mathcal{R}_{L^*, P}^* \). Hence, an application of Theorem 3 yields
\[
\mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}) \to \mathcal{R}_{L^*, P}^* \quad (n \to \infty) \tag{13}
\]
in probability. It is shown in [Christmann et al., 2009, Corollary 31] that, for all sequences \( (f_n)_{n \in \mathbb{N}} \) of measurable functions \( f_n : \mathcal{X} \to \mathbb{R} \),
\[
\mathcal{R}_{L^*, P}(f_n) \to \mathcal{R}_{L^*, P}^* \quad \text{implies} \quad d_0(f_n, f_{\tau, P}^*) \to 0.
\]
This proves Theorem 4 in the following way: According to the characterization of convergence in probability by means of almost surely convergent subsequences [Dudley, 2002, Theorem 9.2.1], it follows from (13) that, for every subsequence of \( \mathcal{R}_{L^*, P}(f_{L, D_n, \lambda_n}), \ n \in \mathbb{N} \), there is a further subsequence which converges almost surely to \( \mathcal{R}_{L^*, P}^* \). Hence, according to the cited result [Christmann et al., 2009, Corollary 31], for every subsequence of
\[
d_0(f_{L, D_n, \lambda_n}, f_{\tau, P}^*), \quad n \in \mathbb{N},
\]
there is a further subsequence which converges almost surely to 0. That is,
\[
d_0(f_{L, D_n, \lambda_n}, f_{\tau, P}^*) \to 0 \text{ in probability.} \quad \square
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Figure 1: Quantile regression using SVMs and pinball loss function with \( \tau = 0.5 \). Model: \( Y | (x_1, x_2) \sim f_1(x_1) + f_2(x_2) + \text{Cauchy-errors} \), where \( f_1(x_1) := 7 + 5x_1^2 \) and \( f_2(x_2) = \sin(5x_2) \cos(17x_2) \) and \( x_1 \) and \( x_2 \) are observations of independent and identically uniform distributed random variables on the interval \([0, 1]\). The regularization parameter is \( \lambda_n = 0.05n^{-0.45} \), and the kernel parameter of the Gaussian RBF kernel is \( \gamma = 2 \). Upper left subplot: true function \( f(x_1, x_2) = f_1(x_1) + f_2(x_2) \). Upper right subplot: SVM fit based on GRBF kernel \( k \) on \( \mathcal{X} = \mathbb{R}^2 \). Lower left subplot: SVM fit based on the sum of two 1-dimensional GRBF kernels. Lower right subplot: SVM fit based on the sum of a 1-dimensional polynomial kernel on \( \mathbb{R} \) and a 1-dimensional GRBF kernel.
Figure 2: Quantile regression using SVMs and pinball loss function with $\tau = 0.5$. Model: $Y|(x_1, x_2) \sim f_1(x_1) + f_2(x_2) + \text{Cauchy-errors}$, where $f_1(x_1) := 7 + 5x_1^2$ and $f_2(x_2) = \sin(5x_2)\cos(17x_2)$ and $x_1$ and $x_2$ are observations of independent and identically uniform distributed random variables on the interval $[0, 1]$. The regularization parameter is $\lambda_n = 0.05n^{-0.45}$, and the kernel parameter of the Gaussian RBF kernel is $\gamma = 2$. Upper left subplot: true function $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$. Upper right subplot: SVM fit based on GRBF kernel $k$ on $\mathcal{X} = \mathbb{R}^2$. Lower left subplot: SVM fit based on the sum of two 1-dimensional GRBF kernels. Lower right subplot: SVM fit based on the sum of a 1-dimensional polynomial kernel on $\mathbb{R}$ and a 1-dimensional GRBF kernel.
Figure 3: Quantile regression using SVMs and pinball loss function with \( \tau = 0.5 \). Model: \( Y | (x_1, x_2) \sim f_1(x_1) + f_2(x_2) + \text{Cauchy-errors} \), where 
\[ f_1(x_1) := 7 + 5x_1^2 \] and 
\[ f_2(x_2) = \sin(5x_2)\cos(17x_2) \] and \( x_1 \) and \( x_2 \) are observations of independent and identically uniform distributed random variables on the interval \([0, 1]\). The regularization parameter is \( \lambda_n = 0.05n^{-0.45} \), and the kernel parameter of the Gaussian RBF kernel is \( \gamma = 2. \) Upper left subplot: true function \( f(x_1, x_2) = f_1(x_1) + f_2(x_2) \). Upper right subplot: SVM fit based on GRBF kernel \( k \) on \( \mathcal{X} = \mathbb{R}^2 \). Lower left subplot: SVM fit based on the sum of two 1-dimensional GRBF kernels. Lower right subplot: SVM fit based on the sum of a 1-dimensional polynomial kernel on \( \mathbb{R} \) and a 1-dimensional GRBF kernel.

\[ n = 500 \]
Figure 4: Plot for the fitted additive model for the rent standard data set based on a nonparametric additive SVM for quantile regression, i.e., pinball loss function with $\tau = 0.50$ (left) and $\tau = 0.90$ (right). The surface gives the estimated median (left) or 90% quantile (right) net rent price of one square meter depending on the size of the dwelling and the year of the construction for a standard residential area, i.e., region_1 = region_2 = 0.

$\tau = 0.5$  

$\tau = 0.9$