Intrinsic Ultracontractivity for Lévy Processes

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Abstract

We prove the intrinsic ultracontractivity for the semigroup generated by a large class of symmetric Lévy processes such that the Lévy measure satisfies some conditions in the neighborhood of 0, killed on exiting a bounded and connected Lipschitz domain.

1 Introduction

Intrinsic ultracontractivity has been studied extensively in recent years in the case of the symmetric diffusions (see e.g. [DS], [B]) and the symmetric α-stable process (see e.g. [CS], [K]). The concept of the intrinsic ultracontractivity for non-symmetric semigroups was introduced in [KS].

If the Lévy measure of symmetric Lévy processes $X_t$ is “uniformly separate” from 0 (see [I]) on truncated cone with vertex in the neighborhood of 0, we prove the intrinsic ultracontractivity for semigroup generated by the killed process on exiting a bounded and connected Lipschitz domain (Theorem [8]). In the case if the Lebesgue measure is absolutely continuous with respect to the Lévy measure then we show that the semigroup is intrinsic ultracontractive for any bounded open set (Remark [9]).

The paper is organized in the following way. In Section 2 we recall some definitions and prove facts about continuous and strictly positivity of a transition density of process killed on exiting a bounded open set. In Section 3 we prove the intrinsic ultracontractivity.

2 Preliminaries

In $\mathbb{R}^d$, $d \geq 1$, we consider a symmetric Lévy processes $X_t$. By $\nu$ we denote its (nonzero) Lévy measure and by $p(t, x, y) = p(t, x - y)$ the transition densities of $X_t$, which are assumed to be continuous for every $t > 0$ and defined for every $x, y \in \mathbb{R}^d$. In addition we assume that there exists a constant $c(\delta)$ such that $p(t, x) \leq c$ for $t > 0$ and $|x| \geq \delta$. 

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We use the notation $C = C(\alpha, \beta, \gamma, \ldots)$ to denote that the constant $C$ depends on $\alpha, \beta, \gamma, \ldots$. Usually values of constants may change from line to line, but they are always strictly positive and finite. Sometimes we skip in notation that constants depend on usual quantities (e.g. $d, D$). Next, we give some definitions. We denote

$$
\tau_D = \inf\{t > 0 : X_t \notin D\},
$$

$$
\eta_D = \inf\{t \geq 0 : X_t \notin D\}.
$$

Let $D$ be a bounded connected nonempty open set. In order to study the killed process on exiting of $D$ we construct its transition densities by the classical formula

$$
p_D(t, x, y) = p(t, x, y) - r_D(t, x, y),
$$

where

$$
r_D(t, x, y) = E_x[t > \tau_D; p(t - \tau_D, X_{\tau_D}, y)].
$$

The arguments used for Brownian motion (see e.g. [CZ]) will prevail in our case and one can easily show that $p_D(t, x, y)$, $t \geq 0$, satisfy the Chapman-Kolmogorov equation (semigroup property). Moreover the transition density $p_D(t, x, y)$ is a symmetric function $(x, y)$ a.s. With the above assumptions of the transition densities of the (free) process one can actually show that $p_D(t, x, \cdot)$ and $p_D(t, \cdot, x)$ can be chosen as continuous functions on $D$.

The semigroup given by the process $X_t$ killed on exiting of $D$ we denote by $P^D_t$. We set

$$
G_D(x, y) = \int_0^\infty p_D(t, x, y)dt
$$

and call the Green function for $D$.

$P^D_t$ is a strongly continuous semigroup of contractions on $L^2(D)$. Because $p_D(t, x, y)$ is symmetric a.e., we obtain that the operator $P^D_t$ is selfadjoint. For $D$ bounded we get from

continuity of $p(t, \cdot)$ that

$$
p_D(t, x, y) \leq p(t, x - y) \leq \sup_{x \in B(0, \text{diam}(D))} p(t, x) = C_1(t, D).
$$

Therefore $P^D_t$ is Hilbert-Schmidt operator, so it’s also compact. So, it’s well-known that there exists an orthonormal basis of real-valued eigenfunctions $\{\varphi_n\}_{n=0}^\infty$ with corresponding eigenvalues $\{e^{-\lambda_n t}\}_{n=0}^\infty$ satisfying $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, where all $\varphi_n$ are continuous.

We have that $p_D(t, x, \cdot) \in L^2(D)$ so we can represent this function as

$$
p_D(t, x, \cdot) = \sum_{n=0}^\infty <p_D(t, x, \cdot), \varphi_n > \varphi_n.
$$

But $<p_D(t, x, \cdot), \varphi_n > = P^D_t \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x)$, so

$$
p_D(t, x, y) = \sum_{n=0}^\infty e^{-\lambda_n t} \varphi_n(x) \varphi_n(y).
$$
Now, let us observe that the above series are uniformly convergent, it follows from \( |\varphi_n| \leq e^{\lambda_n t/3} C_1(t/3, D) \) and

\[
\sum_{n=0}^{\infty} e^{-\lambda_n t/3} = \int_D p_D(t/3, x, x)dx \leq C_1(t/3, D)|D|.
\]

Hence, we get that \( p_D(t, \cdot, \cdot) \in C(D \times D) \). Therefore \( p_D(t, x, y) = p_D(t, y, x) \) for any \( t > 0 \) and \( x, y \in D \).

Next, we show that \( p_D(t, \cdot, \cdot) \) is strictly positive on \( D \times D \). First, let us observe that for any \( x \in D \) we have

\[
p_D(t, x, x) = \int_D p_D(t/2, x, z)p_D(t/2, z, x)dz \geq (P^x(\tau_D > t/2) )^2/|D| > 0.
\]

Let \( K \subset D \) be a compact and connected set. By continuity of \( p_D(t, \cdot, \cdot) \) we obtain that for any \( x \in K \) there is a radius \( r_x \) such that

\[
p_D(t, x, y) > 0 \quad \text{for} \quad x, y \in B(x, 2r_x).
\]

Because \( K \) is compact, there are \( x_1, \ldots, x_k \in K \) such that \( K \subset \bigcup_{i=1}^k B(x_i, r_{x_i}) \). Now, we use a fact that \( K \) is connected to get from the Chapman-Kolmogorov equation that

\[
p_D(kt, x, y) > 0 \quad \text{for} \quad x, y \in K.
\]

Hence we have that \( p_D(s, x, y) > 0 \) for \( s \geq kt \) and \( x, y \in K \). Therefore \( G_D(x, y) > 0 \), first for \( x, y \in K \) and next for any \( x, y \in D \). This gives us that \( p_D(t, x, y) \) is strictly positive on \( D \) for any \( t > 0 \). So we obtain that \( \varphi_0 \) is strictly positive on \( D \) too.

**Lemma 1.** For any \( x \in D \) and \( t > 0 \) we have

\[
p_D(t, x, y) \leq C(t, D)E^x\tau_D E^y\tau_D.
\]

**Proof.** By the Chapman-Kolmogorov equation we obtain for \( t > 0 \)

\[
p_D(t, x, y) = \int_D p_D(t/2, x, z)p_D(t/2, z, y)dz \leq C_1(t/2, D)P^x(\tau_D > t/2).
\]

Applying again the Chapman-Kolmogorov equation together with the above inequality we get

\[
p_D(t, x, y) \leq C_1P^x(\tau_D > t/4)\int_D p_D(t/2, z, y)dz = C_1P^x(\tau_D > t/4)P^y(\tau_D > t/2).
\]

The application of Chebyshev’s inequality completes the proof.

**Definition 2.** The semigroup \( \{P^D_t\} \) is said to be **intrinsic ultracontractive** if, for any \( t > 0 \), there exists a constant \( c_t \) such that

\[
p_D(t, x, y) \leq c_t\varphi_0(x)\varphi_0(y), \quad x, y \in D.
\]
Proposition 3. Let $D$ be a bounded connected nonempty open set. Then $\{P_t^D\}$ is intrinsic ultracontractive if and only if there is a constant $C$ such that $E^x\tau_D \leq C\varphi_0(x)$.

Proof. Suppose that $\{P_t^D\}$ is intrinsic ultracontractive that is

$$p_D(t, x, y) \leq c_t \varphi_0(x) \varphi_0(y).$$

Because $p_D(t, \cdot, \cdot)$ and $\varphi_0(\cdot)$ are continuous and strictly positive, we have (see Theorem 3.2 in [DS]) that there is $\tilde{c}_t$ such that

$$\tilde{c}_t \varphi_0(x) \varphi_0(y) \leq p_D(t, x, y).$$

If we integrate the above inequality with respect to $dt$ we get

$$C\varphi_0(x)\varphi_0(y) \leq G_D(x, y).$$

And by integrating with respect to $dy$

$$\tilde{C}\varphi(x) \leq E^x\tau_D.$$ 

Now, suppose that $E^x\tau_D \leq C\varphi_0(x)$. From Lemma [1] we have

$$p_D(t, x, y) \leq C_t E^x\tau_D E^y\tau_D,$$

what ends the proof. \qed

3 Main results

We prove intrinsic ultracontractivity for the semigroup $P_t^D$ generated by the symmetric Lévy process, whose a Lévy measure satisfies

$$\forall r > 0, \gamma \in (0, \pi) \exists \rho > 0 \inf_{|y| = \rho, \Gamma_\gamma(y)} \nu(\Gamma_\gamma(y) \cap B(0, r)) > 0, \quad (1)$$

where $\Gamma_\gamma(y)$ is a right circular cone of angle $\gamma$ at the vertex in $y$.

Notation and the proof of following theorem is similar as in paper [K]. We assume that $D$ is a bounded and connected Lipschitz domain. That is there exist $\gamma_0$ and $R_0 > 0$ and a cone $\Gamma_{\gamma_0} = \{(y, x) : 0 < x, y \in \mathbb{R}^{d-1}, \gamma_0|y| < x\}$ such that for every $Q \in \partial D$, there is a cone $\Gamma_{\gamma_0}(Q)$ with vertex $Q$, isometric with $\Gamma_{\gamma_0}$ and satisfying $\Gamma_{\gamma_0}(Q) \cap B(Q, R_0) \subset D$. Denote $U(\sigma) = \{x \in D : \delta_D(x) < \sigma\}$, where $\sigma \leq \frac{R_0}{4\sqrt{1+\gamma_0^2}}$. Then for any $x \in U(\sigma)$ there are a point $y$ and a cone $\Gamma_{\gamma_0}(y)$ such that $|y - x| < \sigma(1 + \sqrt{1+\gamma_0^2}) \leq \frac{R_0}{2}$ and $\Gamma_{\gamma_0}(y) \cap B(x, R_0/2) \subset D \cap U(\sigma)^c$. 

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We fix \( x_0 \in D \) and let \( r > 0 \) be such that \( \overline{B(x_0,2r)} \subset D \). Denote \( K = \overline{B(x_0,r)} \), \( L = B(x_0,2r) \), \( M = D \setminus K \) and \( N = D \setminus L \). We deal that \( r \leq \rho_0 \). Define stopping time \( S_n \) and \( T_n \)

\[
\begin{align*}
S_1 &= 0, \\
T_n &= S_n + \eta_M \circ \theta_{S_n}, \\
S_n &= T_{n-1} + \eta_L \circ \theta_{T_{n-1}}.
\end{align*}
\]

Now, we prove the following lemma.

**Lemma 4.** There exists a constant \( c = c(D, x_0) \) such that

\[
P^x(X(\eta_M) \in K) \geq cE^x \eta_M
\]

for all \( x \in \mathbb{R}^d \).

**Proof.** From (1) we get existing a constant \( \sigma_0 \leq r \) such that

\[
\inf_{|y|=\sigma_0(1+\sqrt{1+\gamma_0})} \nu(\Gamma_\gamma_0(y) \cap B(0,R_0/2)) = C_1.
\]

Denote \( W = \{ x \in D : \delta_D(x) \geq \sigma_0/2 \} \setminus B(x_0, r) \).

First, we prove that for \( x \in W \), we have

\[
P^x(X(\tau_M) \in K) \geq c_1,
\]

for some constant \( c = c(r, D) \). Let \( \rho_1 \) be such that

\[
\inf_{|y|=\rho_1; \Gamma_1(y)} \nu(\Gamma_1(y) \cap B(0,r)) = C_2 > 0.
\]

Denote \( J = D \setminus B(x_0, r - \rho_1/4) \). Indeed, from the Ikeda-Watanabe formula we have

\[
P^x(X(\tau_M) \in K) \geq P^x(X(\tau_J) \in B(x_0, r - \rho_1/4))
\]

\[
\geq P^x(X(\tau_J) \in B(x_0, r - \rho_1/2))
\]

\[
= \int_J G_J(x,y) \nu(B(x_0, r - \rho_1/2) - y) \, dy
\]

\[
\geq \int_W G_J(x,y) \nu(B(x_0, r - \rho_1/2) - y) \, dy.
\]

Because \( p_J(t, \cdot, \cdot) \) is continuous and positive function on \( J \times J \) and \( W \times W \) is compact subset of \( J \times J \), we get \( \inf_{x,y \in W} p_J(t, x, y) > 0 \). So,

\[
\inf_{x,y \in W} G_J(x,y) \geq \int_0^\infty \inf_{x,y \in W} p_J(t, x, y) \, dt = c > 0.
\]
Besides, we have
\[ \inf_{y \in B(x_0, r + \rho_1/2)} \nu((B(x_0, r - \rho_1/2) - y) \geq \inf_{|y| = \rho_1; \Gamma_1(y)} \nu(\Gamma_1(y) \cap B(0, r)) > 0. \]

Therefore
\[ P^x(X(\tau_M) \in K) \geq \epsilon \int_{B(x_0, r + \rho_1/2) \setminus B(x_0, r)} \nu(B(x_0, r - \rho_1/2) - y) dy = C > 0. \]

From (2) and the fact that \( E^x \tau_M \leq \tilde{C} \) we obtain the claim of the lemma for \( x \in W \).

Now, let \( x \in D \setminus (W \cup K) = U(\sigma_0/2) \). Then from Strong Markov Property we get
\[
P^x(X(\tau_M) \in K) = E^x(P^{X(\tau_U(\sigma_0/2))}(X(\tau_M) \in K)) \geq c_2 E^x(E^{X(\tau_U(\sigma_0/2))}\tau_M) \\
= c_2(E^x\tau_M - E^x\tau_U(\sigma_0/2)).
\]

And from (2) we obtain
\[
P^x(X(\tau_M) \in K) = E^x(X(\tau_U(\sigma_0/2)) \in W \cup K, P^{X(\tau_U(\sigma_0/2))}(X(\tau_M) \in K)) \geq c_1 P^x(X(\tau_U(\sigma_0/2)) \in W \cup K).
\]

But
\[
P^x(X(\tau_U(\sigma_0/2) \in W \cup K) \geq P^x(X(\tau_U(\sigma_0/2)) \in D \cap U(\sigma_0)) \\
= \int_{U(\sigma_0/2)} G_U(\sigma_0/2)(x, y)\nu(D \cap U(\sigma_0) - y) dy \\
\geq C_1 \int_{U(\sigma_0/2)} G_U(\sigma_0/2)(x, y) dy = C_1 E^x\tau_U(\sigma_0/2)
\]

Hence
\[
P^x(X(\tau_M) \in K) = (\frac{1}{2} + \frac{1}{2}) P^x(X(\tau_M) \in K) \\
\geq \frac{c_2}{2} (E^x\tau_M - E^x\tau_{D \setminus (W \cup K)}) + \frac{C_1}{2} (E^x\tau_{D \setminus (W \cup K)}) \\
\geq \frac{c_2 \wedge C_1}{2} E^x\tau_M.
\]

For \( x \in D \) we have \( E^x\tau_M = E^x\eta_M \), and the claim of the lemma for \( x \in D \) of course is obvious, so it ends the proof.

**Lemma 5.** For all \( x \in \mathbb{R}^d \) there exists a random variable \( Z \) such that for all \( n \geq Z \) we have \( T_n = \eta_D \) almost surely \( P^x \).
Proposition 6. Let $C$ be an nonempty open subset of $D$. Then there is $c$ such that

$$E^x \int_0^{\tau_D} 1_C(X_t) dt \geq cE^x \tau_D.$$ 

Theorem 7. There exists a constant $C$ such that

$$E^x \tau_D \leq C \varphi_0(x),$$

for all $x \in D$. 

Proof. We will show that there exists a constant $\beta < 1$ such that $P^x(T_n < \eta_D) \leq \beta^n$ for all $x \in \mathbb{R}^d$ and $n \geq 1$.

Let $R = B(x_0, \text{diam}(D)) \setminus K$ and $\varepsilon = \inf_{x, y \in B(x_0, \text{diam}(D) - \rho_1/2) \setminus B(x_0, 2r)} G_R(x, y)$ then from the Ikeda-Watanabe formula for $x \in N$ we get

$$P^x(X(\eta_M) \in D^c) \geq P^x(X(\eta_R) \in B^c(x_0, \text{diam}(D)))$$

$$\geq \int_{B(x_0, \text{diam}(D) - \rho_1/2) \setminus B(x_0, 2r)} G_R(x, y) \nu(B^c(x_0 - y, \text{diam}(D)))dy$$

$$\geq \varepsilon \int_{B(0, \text{diam}(D) - \rho_1/2) \setminus B(0, \text{diam}(D) - \rho_1)} \nu(B^c(y, \text{diam}(D)))dy \geq \varepsilon C_2 c = 1 - \beta.$$ 

Consequently, for any $x \in \mathbb{R}^d$ and $n \geq 1$ we get

$$P^x(T_n < \eta_D, T_{n+1} = \eta_D) = P^x(T_n < \eta_D, S_{n+1} = \eta_D) + P^x(T_n < \eta_D, S_{n+1} < \eta_D, X(T_{n+1}) \in D^c)$$

$$= P^x(T_n < \eta_D, S_{n+1} = \eta_D) + P^x(T_n < \eta_D, X(S_{n+1}) \in N, X(\eta_M) \circ \theta_{S_{n+1}} \in D^c)$$

$$= P^x(T_n < \eta_D, S_{n+1} = \eta_D) + E^x(T_n < \eta_D, X(S_{n+1}) \in N, P^x(S_{n+1})(X(\eta_M) \in D^c))$$

$$\geq (1 - \beta)P^x(T_n < \eta_D, S_{n+1} = \eta_D) + (1 - \beta)P^x(T_n < \eta_D, S_{n+1} < \eta_D)$$

$$= (1 - \beta)P^x(T_n < \eta_D).$$ 

Hence, we obtain

$$P^x(T_{n+1} < \eta_D) = P^x(T_n < \eta_D) - P^x(T_n < \eta_D, T_{n+1} = \eta_D)$$

$$\leq P^x(T_n < \eta_D) - (1 - \beta)P^x(T_n < \eta_D) = \beta P^x(T_n < \eta_D).$$ 

Applying the Borel-Cantelli Lemma ends the proof of lemma. 

The above lemma allow us to prove similarly as Theorem 8 in [K] the following proposition.
Proof. We have, for all $t > 0$,
\[ e^{-\lambda_0 t} \varphi_0(x) = \int_D p_D(t, x, y) \varphi_0(y) dy. \]
By integration this with respect $dt$ we get
\[ \varphi_0(x) = \lambda_0 \int_D G_D(x, y) \varphi_0(y) dy. \]
Because $\varphi_0$ is continuous and positive, we obtain that there is a constant $\varepsilon > 0$ such that a set
\[ C = \{ x : \varphi_0(x) > \varepsilon \} \]
is nonempty. By Proposition 6 we have
\[
E^x \tau_D \leq c^{-1} \int_C G_D(x, y) dy \leq (c\varepsilon)^{-1} \int_C G_D(x, y) \varphi_0(y) dy \\
\leq (c\varepsilon)^{-1} \int_D G_D(x, y) \varphi_0(y) dy = (c\varepsilon \lambda_0)^{-1} \varphi_0(x).
\]
Applying Lemma 3 give us the theorem below.

**Theorem 8.** Let $D$ be an bounded and connected Lipschitz domain. If the Lévy measure of symmetric Lévy process $X_t$ satisfies (1), then the semigroup \( \{P^D_t\} \) is intrinsic ultracontractive.

**Remark 9.** Suppose that the symmetric Lévy process $X_t$ has the Lévy measure such that the Lebesgue measure is absolutely continuous with respect to it. Then the semigroup $P^D_t$ is intrinsic ultracontractive for any bounded open set.

**Proof.** Proof of this remark is the same as the proof of Theorem 1 in [K].

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