PERIODIC FREE RESOLUTIONS FROM TWISTED MATRIX FACTORIZATIONS

ABSTRACT. The notion of a matrix factorization was introduced by Eisenbud in the commutative case in his study of bounded (periodic) free resolutions over complete intersections. Since then, matrix factorizations have appeared in a number of applications. In this work, we extend the notion of (homogeneous) matrix factorizations to regular normal elements of connected graded algebras over a field.

Next, we relate the category of twisted matrix factorizations to an element over a ring and certain Zhang twists. We also show that in the AS-regular setting, every sufficiently high syzygy module is the cokernel of some twisted matrix factorization. Furthermore, we show that in this setting there is an equivalence of categories between the homotopy category of twisted matrix factorizations and the singularity category of the hypersurface, following work of Orlov.

Thomas Cassidy
Department of Mathematics
Bucknell University
Lewisburg, PA 17837

Andrew Conner
Ellen Kirkman
W. Frank Moore
Department of Mathematics
Wake Forest University
Winston-Salem, NC 27109

INTRODUCTION

The notion of a matrix factorization was introduced by Eisenbud in the commutative case in his study of bounded (periodic) free resolutions over complete intersections. Since then, matrix factorizations have appeared in a number of applications, including string theory, singularity categories, representation theory of Cohen-Macaulay modules, and other topics. Recently, Eisenbud and Peeva also extended the notion of matrix factorizations to higher codimension complete intersections.

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There are several candidates for the notion of a complete intersection in the case of a non-commutative algebra, based on their numerous characterizations in the commutative case \[9, 10, 13, 14\]. One possible approach is to first understand the hypersurface case by studying matrix factorizations in the non-commutative setting. In this work, we extend the notion of (homogeneous) matrix factorizations to regular normal elements of connected graded algebras over a field.

To state our results below, we assume that \(A\) is a connected, \(\mathbb{N}\)-graded, locally finite dimensional algebra over a field \(k\). We also fix a homogeneous normal regular element \(f \in A_+ = \bigoplus_{n>0} A_n\) and set \(B = A/(f)\). The regularity and normality of \(f\) provide us with a graded automorphism \(\sigma\) of \(A\) which we incorporate into the definition of matrix factorization. The use of \(\sigma\) to modify ring actions is the reason for our “twisted” terminology; see Definition 2.1.

Our first main result shows that just as in the commutative case, (reduced) twisted matrix factorizations give rise to (minimal) resolutions.

**Theorem A** (Proposition 2.2). A twisted left matrix factorization \((\varphi, \tau)\) of \(f\) gives rise to a complex \(\Omega(\varphi, \tau)\) of free left \(B\)-modules which is a graded free resolution of \(\text{coker } \varphi\) as a left \(B\)-module. If the twisted matrix factorization \((\varphi, \tau)\) is reduced (see Definition 2.5), then the graded free resolution is minimal. If the order of \(\sigma\) is finite, then the resolution is periodic of period at most twice the order of \(\sigma\).

As in the commutative case, one can consider the category of all twisted matrix factorizations of \(f\) over a ring \(A\), which we call \(\text{TMF}_A(f)\).

There is another context where twisting via an automorphism arises in the study of graded algebras: the Zhang twist \[27\]. The following theorem relates the category of twisted matrix factorizations of \(f\) over \(A\) to those over the Zhang twist of \(A\) with respect to a compatible twisting system \(\zeta\) (which we denote \(A^\zeta\)).

**Theorem B** (see Theorem 3.5). Let \(\zeta = \{\sigma^n | n \in \mathbb{Z}\}\) be the twisting system associated with the normalizing automorphism \(\sigma\). Then the categories \(\text{TMF}_A(f)\) and \(\text{TMF}_{A^\zeta}(f)\) are equivalent.

This result is somewhat surprising. If \(f\) is central in the Zhang twist \(A^\zeta\), then the complexes associated to matrix factorizations in \(\text{TMF}_{A^\zeta}(f)\) will be periodic of period at most two, while those coming from matrix factorizations in \(\text{TMF}_A(f)\) could have a longer period, depending on the order of \(\sigma\). It should be noted that \(f\) is not necessarily central in the Zhang twist. This peculiarity is illustrated in Example 6.2.

A major result in \[7\] that drives many of the applications of matrix factorizations is that, under appropriate hypotheses, every minimal graded free resolution is eventually given by a reduced matrix factorization. Using Jørgensen’s version of the Auslander-Buchsbaum formula for connected graded algebras\[15\], we are able to extend this result:

**Theorem C** (Theorem 4.7). Let \(A\) be a left noetherian Artin-Schelter regular algebra of dimension \(d\). Let \(f \in A_+\) be a homogeneous normal regular element and let \(B = A/(f)\). Then for every finitely generated graded left \(B\)-module \(M\), the \((d+1)^{st}\) left syzygy of \(M\) is the cokernel of some reduced twisted left matrix factorization of \(f\) over \(A\).

There is a suitable notion of homotopy in the category \(\text{TMF}_A(f)\), and we denote the associated homotopy category \(h\text{TMF}_A(f)\). Following Orlov’s lead \[23\], we provide a triangulated structure on \(h\text{TMF}_A(f)\) and prove the following Theorem.
**Theorem D** (Theorem 5.5). Let $A$ be a left noetherian Artin-Schelter regular algebra, and let $f \in A_+$ be a homogeneous normal regular element. Then the homotopy category of twisted matrix factorizations of $f$ over $A$ is equivalent to the bounded singularity category of $B$.

It should be noted that since the minimal resolution that comes from a twisted matrix factorization need not be periodic, some minor adjustments to Orlov’s original argument must be made.

The paper is organized as follows: Section 1 covers preliminaries, as well as sets up notation regarding various twists that will be used for the remainder of the paper. Section 2 covers the definition of matrix factorization, as well as the proof of Theorem A. Section 3 contains the background regarding the Zhang functor, as well as the proof of Theorem B. Section 4 includes the background on the Auslander-Buchsbaum theorem in our setting due to Jørgensen, as well as the precise statement and proof of Theorem C. Section 5 contains the categorical considerations for Theorem D, and Section 6 contains some examples.

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1. Preliminaries

The main results in this paper concern graded modules over graded rings, hence we work exclusively in that context. Let $A$ be a connected, $\mathbb{N}$-graded algebra over a field $k$. We assume $A$ is locally finite-dimensional: $\dim_k A_i < \infty$ for all $i \in \mathbb{N}$. Throughout, we work in the category $A$-GrMod of graded left $A$-modules with degree 0 morphisms, though our definitions and results have obvious analogs for graded right modules. Let $\sigma$ be a degree 0 graded algebra automorphism of $A$. For $M \in A$-GrMod, we write $M^\sigma$ for the graded left $A$-module with $M^\sigma = M$ as graded abelian groups and left $A$-action $a \cdot m = \sigma(a)m$. If $\varphi : M \to N$ is a degree 0 homomorphism of graded left $A$-modules, $\varphi^\sigma = \varphi : M^\sigma \to N^\sigma$ is also a graded module homomorphism. It is straightforward to check that the functor $(-)^\sigma$ is an autoequivalence of $A$-GrMod.

For any $n \in \mathbb{Z}$ and $M \in A$-GrMod, we write $M(n)$ for the shifted module whose degree $i$ component is $M(n)_i = M_{i+n}$. The degree shift functor $M \mapsto M(n)$ is also easily seen to be an autoequivalence of $A$-GrMod which commutes with $(-)^\sigma$; that is, $M(n)^\sigma = M^\sigma(n)$.

Let $f \in A_d$ be a normal, regular homogeneous element of degree $d$, and let $\sigma : A \to A$ be the graded automorphism of $A$ determined by the equation $af = f\sigma(a)$ for each $a \in A$. We call $\sigma$ the normalizing automorphism of $f$ and say $f$ is normalized by $\sigma$. Note that $f$ is normalized by $\sigma$ if and only if left multiplication by $f$ is a graded left module homomorphism $\lambda^f_M : M^\sigma(-d) \to M$ for all $M$. Moreover, $\lambda^f_M \varphi^\sigma(-d) = \varphi \lambda^f_M$ for any graded homomorphism $\varphi : M \to N$. The composite functor $M \mapsto M^\sigma(-d)$ will be used so frequently for fixed $\sigma$ and $d$ that we define $M^{tw} = M^\sigma(-d)$ and $M^{tw^{-1}} = M^{-\sigma^{-1}}(d)$.

In this paper we are especially interested in periodic resolutions. We say a degree 0 complex $P : \cdots \to P_2 \to P_1 \to P_0$ of graded left $A$-modules is periodic
of period $p$ if $p$ is the smallest positive integer such that there exists an integer $n$ and a morphism of complexes $t : P(n) \rightarrow P$ of (homological) degree $-p$ where $t : P_{i+p}(n) \rightarrow P_i$ is an isomorphism for all $i \geq 0$. Note the shift $(n)$ is applied to the internal grading of each free module in the complex. If such an integer $p$ exists, we say $P$ is periodic. If there exists an integer $m \geq 0$ such that the truncated complex $\cdots \rightarrow P_{m+2} \rightarrow P_{m+1} \rightarrow P_m$ is periodic, we say $P$ is periodic after $m$ steps.

We record a few straightforward facts about periodic complexes needed later.

**Lemma 1.1.** Let $P$ be a complex of graded free left $A$-modules.

1. If $P$ is periodic and there exists an integer $N > 0$ such that $\text{rank } P_j = \text{rank } P_N$ for all $j \geq N$, then $\text{rank } P_j = \text{rank } P_0$ for all $j \geq 0$.

2. Let $P$ be a complex of graded free left $A$-modules which is periodic of period $p$. If there exist an integer $n$ and an isomorphism of complexes $t : P(n) \rightarrow P$, then $P$ is also periodic of period $p$.

We say a resolution $(P_\bullet, d_\bullet)$ is minimal if $\text{im } d_i \subset A_+ P_{i-1}$ for all $i$, where $A_+ = \bigoplus_{i \geq 0} A_i$. Recall that every bounded below, graded module over a connected, $\mathbb{N}$-graded, locally finite-dimensional $k$-algebra has a minimal graded free resolution. This resolution is unique up to non-unique isomorphism of complexes (see, for example, [25]).

**Lemma 1.2.** Let $\sigma$ be a degree 0 graded automorphism of $A$. Let $P$ be a minimal graded free resolution of a bounded below, graded left $A$-module $M$. If $M^\sigma \cong M$ as graded modules, then the chain complexes $P^\sigma$ and $P$ are isomorphic.

**Proof.** First note that $P^\sigma$ is a minimal graded free resolution of $M^\sigma$. Let $\psi : M^\sigma \rightarrow M$ be a graded isomorphism. Let $\Psi : P^\sigma \rightarrow P$ and $\Phi : P \rightarrow P^\sigma$ be the graded morphisms of complexes obtained by lifting $\psi$ and $\psi^{-1}$ respectively. By the Comparison Theorem, $\Psi \Phi$ and $\Phi \Psi$ are homotopic to the respective identity maps. Since the complexes $P$ and $P^\sigma$ are minimal, $\Psi \Phi$ and $\Phi \Psi$ are isomorphisms. Hence $\Psi$ and $\Phi$ are isomorphisms. \hfill $\Box$

### 2. Twisted matrix factorizations

The key to our study is the notion of a twisted matrix factorization. As will be evident, the notion can in fact be defined over any ring containing a normal, regular element. Indeed some results, such as Proposition 2.2, are readily seen to hold in this generality by forgetting the grading.

As above, let $A$ be a connected, $\mathbb{N}$-graded, locally finite-dimensional algebra over a field $k$. Let $f \in A_d$ be a normal, regular homogeneous element of degree $d$ and let $\sigma$ be its degree 0 normalizing automorphism. In this section we do not require $A$ to be Artin-Schelter regular.

**Definition 2.1.** A **twisted left matrix factorization** of $f$ over $A$ is an ordered pair of maps of finitely generated graded free left $A$-modules $(\varphi : F \rightarrow G, \tau : G^{tw} \rightarrow F)$ such that $\varphi \tau = \lambda_f^G$ and $\tau \varphi^{tw} = \lambda_f^F$.

Our definition is a naive generalization of the familiar notion from commutative algebra. We remark that the freeness of $G^{tw}$ requires $\sigma$ to be a graded automorphism. Also note that $(\varphi, \tau)$ is a twisted matrix factorization if and only if either $(\varphi^{tw}, \tau^{tw})$ or $(\tau, \varphi^{tw})$ is. It is easy to see that if $(\varphi, \tau)$ is a twisted matrix factorization, then both $\varphi$ and $\tau$ are injective since $f$ is regular.
Free modules over noncommutative rings need not have a well-defined notion of rank. Even among those that do, not all satisfy the rank conditions (a) \( f : A^n \to A^m \) an epimorphism \( \Rightarrow n \geq m \) and (b) \( f : A^n \to A^m \) a monomorphism \( \Rightarrow n \leq m \) (though (b) implies (a), see [20]). However, since we assume \( A \) is locally finite dimensional and morphisms preserve degree, the graded version of (b) clearly holds for graded free \( A \)-modules. Thus rank is well-defined for graded free \( A \)-modules.

As noted above, if \( \varphi : F \to G, \tau : G^{tw} \to F \) is a twisted left matrix factorization, then \( \varphi \) and \( \tau \) are injective. It follows that rank \( F = \text{rank} G \).

For categorical reasons (see below) we adopt the usual convention that the zero module is free on the empty set. We call the twisted factorization \((\varphi, \tau)\) where \( \varphi = \tau : 0 \to 0 \) the irrelevant factorization. We call a twisted factorization \((\varphi, \tau)\) trivial if \( \varphi = \lambda^A_f \) or \( \tau = \lambda^A_f \).

Paralleling the commutative case, twisted matrix factorizations provide a general construction of resolutions.

**Proposition 2.2.** Let \( (\varphi : F \to G, \tau : G^{tw} \to F) \) be a twisted left matrix factorization of a regular element \( f \in A \) with normalizing automorphism \( \sigma \). Set \( B = A/(f) \) and write \( \tau \) for reduction modulo \( f \). Then the complex

\[
\Omega(\varphi, \tau) : \cdots \to G^{tw} \overset{\varphi}{\to} F^{tw} \overset{\tau}{\to} G^{tw} \overset{\varphi}{\to} F \to G
\]

is a resolution of \( M = \text{coker} \varphi \) by free left \( B \)-modules.

Before giving the proof, we must address a potential source of confusion stemming from the fact that \( M \) is both a left \( A \)-module and a left \( B \)-module. Since \( \sigma(f) = f, \sigma \) induces an automorphism \( \bar{\sigma} : B \to B \). We will write \( M^{tw} \) to mean \( (AM)^\sigma(-d) \) and \( M^{tw} \) to mean \( (BM)^\sigma(-d) \). The reader should note that \( M^{tw} \cong B \otimes_A M^{tw} \) and likewise \( A(M^{tw}) \cong M^{tw} \). We denote repeated application of \( (-1)^{tw} \) by \( (-1)^{tw}, (-1)^{tw^2} \), etc. and likewise for \( (-1)^{tw} \).

The proof of Proposition 2.2 is a straightforward generalization of the commutative case.

**Proof.** Since \( \varphi \tau = \lambda^A_f \), we see that \( f(\text{coker} \varphi) = 0 \) so \( \text{coker} \varphi = \text{coker} \varphi \).

We prove exactness at \( F^{tw} \), exactness at \( G^{tw} \) being analogous. Let \( K \) be a graded free \( A \)-module and \( \kappa : K \to F^{tw} \) an \( A \)-module map such that \( \kappa \) is a \( B \)-module surjection onto \( \ker F^{tw} \). Then \( \text{im} \varphi^{tw} \kappa \subseteq fG^{tw} \) and we can define an \( A \)-module map \( h : K \to (G^{tw})^{tw+1} = G^{tw+1} \) by \( h(x) = g \) where \( g \in G^{tw} \) satisfies \( \varphi^{tw} \kappa(x) = f g \).

Any \( a \in A \) we have \( \varphi^{tw} \kappa(ax) = afg = f \sigma(a)g \) so \( h \) is \( A \)-linear.

Since \( f \) is regular, \( \kappa \) does not depend on the choice of \( g \). We have

\[
\lambda^F_{f^{tw}} \kappa = \tau^{tw-1} \varphi^{tw} \kappa = \tau^{tw-1} \lambda^G_{f^{tw}} \kappa = \lambda^F_{f^{tw}} \tau^{tw} h
\]

Again, since \( f \) is regular, \( \kappa = F^{tw} \kappa \), hence \( \ker F^{tw} = \text{im} \kappa \subseteq \text{im} F^{tw} \).

In light of Proposition 2.2, we make the following natural definition.

**Definition 2.3.** A morphism \( (\varphi, \tau) \to (\varphi', \tau') \) of twisted left matrix factorizations of \( f \) over \( A \) is a pair \( \Psi = (\Psi_G, \Psi_F) \) of degree 0 module homomorphisms \( \Psi_G : G \to
$G'$ and $\Psi_F : F \to F'$ such that the following diagram commutes.

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & G \\
\Psi_F & \downarrow & \Psi_G \\
F' & \xrightarrow{\varphi'} & G'
\end{array}
\]

A morphism $\Psi$ is an isomorphism if $\Psi_G$ and $\Psi_F$ are isomorphisms.

The regularity of $f$ guarantees that $(\Psi_G, \Psi_F) : (\varphi, \tau) \to (\varphi', \tau')$ is a morphism if and only if

$$(\Psi_F, \Psi^{tw}_G) : (\tau, \varphi^{tw}) \to (\tau', (\varphi')^{tw})$$

is. It is clear that $(\Psi_G, \Psi_F)$ is a morphism if and only if

$$(\Psi^{tw}_F, \Psi^{tw}_G) : ((\varphi)^{tw}, \tau^{tw}) \to ((\varphi')^{tw}, (\tau')^{tw})$$

is. We leave the straightforward proof of the next Proposition to the reader.

**Proposition 2.4.** A morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ of twisted matrix factorizations of $f$ over $A$ induces a chain morphism of complexes $\Omega(\Psi) : \Omega(\varphi, \tau) \to \Omega(\varphi', \tau')$. Twisted matrix factorizations $(\varphi, \tau)$ and $(\varphi', \tau')$ are isomorphic if and only if the complexes $\Omega(\varphi, \tau)$ and $\Omega(\varphi', \tau')$ are chain isomorphic.

**Definition 2.5.** We define the direct sum of twisted matrix factorizations $(\varphi, \tau)$ and $(\varphi', \tau')$ to be $(\varphi \oplus \varphi', \tau \oplus \tau')$. We call $(\varphi, \tau)$ reduced if it is not isomorphic to a factorization having a trivial direct summand.

We denote by $TMF_A(f)$ the category whose objects are all twisted left matrix factorizations of $f$ over $A$ and whose morphisms are defined as above. As previously mentioned, $TMF_A(f)$ has a zero object and all finite direct sums. Since morphisms are pairs of module maps, monomorphisms and epimorphisms are determined componentwise. Thus we have the following obvious fact.

**Proposition 2.6.** $TMF_A(f)$ is an abelian category.

Proposition 2.4 shows that forming the resolution $\Omega(\varphi, \tau)$ defines a functor from the abelian category of twisted matrix factorizations of $f$ over $A$ to the abelian category of complexes of finitely generated graded projective $B$-modules. We examine categories of twisted matrix factorizations more closely beginning in Section 3.

Next we show that a normal, regular homogeneous element gives rise to twisted matrix factorizations in many cases of interest. However, see Example 5.1. Let $B = A/\langle f \rangle$.

**Construction 2.7.** Let $M$ be a finitely generated graded left $B$-module with $pd_A M = 1$. Let $0 \to F \xrightarrow{\varphi} G \to 0$ be a minimal graded resolution of $M$ by graded free left $A$-modules. We have the commutative diagram

\[
\begin{array}{ccc}
F^{tw} & \xrightarrow{\varphi^{tw}} & G^{tw} \\
\lambda^{tw}_F & \downarrow & \lambda^{tw}_G \\
F & \xrightarrow{\varphi} & G
\end{array}
\]
Remark 2.8. The homomorphisms $\varphi^{tw} : F^{tw} \to G^{tw}$ and $\varphi : F \to G$ are identical on the underlying abelian groups. If we fix bases for $F$ and $G$, and keep the same bases for $F^{tw}$ and $G^{tw}$, the matrices of $\varphi$ and $\varphi^{tw}$ with respect to these bases are different. The matrix of $\varphi^{tw}$ is obtained by applying $\sigma^{-1}$ to each entry of the matrix of $\varphi$.

Since $M$ is a $B$-module, $f_A M = 0$, hence $\lambda_{G}^{\varphi^2} \subseteq \text{im} \varphi$. Thus by graded projectivity there exists a lift $\tau : G^{tw} \to F$ such that $\lambda_{G}^{\varphi^2} = \varphi \tau$. Note that since $f$ is regular, $\lambda_{G}^{\varphi^2}$ is injective, so $\tau$ is injective.

Next observe that $\varphi \tau \varphi^{tw} - \varphi \lambda_{G}^{\varphi^2} = 0$ and since $\varphi$ is injective, $\tau \varphi^{tw} = \lambda_{G}^{\varphi^2}$.

Iterating this process and applying $B \otimes_A -$ yields the complex $\Omega(\varphi, \tau)$ of Proposition 2.2

$$\cdots \to B \otimes_A F^{tw} \xrightarrow{1 \otimes \varphi^{tw}} B \otimes_A G^{tw} \xrightarrow{1 \otimes \tau} B \otimes_A F \xrightarrow{1 \otimes \varphi} B \otimes_A G \to 0$$

Proposition 2.9. The complex $\Omega(\varphi, \tau)$ is exact. It is a minimal graded free resolution of $B M$ if and only if $M$ has no $B$-free direct summand.

Proof. Exactness follows from Proposition 2.2. Since $F \xrightarrow{\varphi} G$ is a minimal resolution of $A M$, the complex $F^{tw^i} \xrightarrow{\varphi^{tw^i}} G^{tw^i}$ is a minimal resolution of $A M^{tw^i}$ for all $i \geq 0$. Thus we have $\text{im} \varphi^{tw^i} \subseteq A_+^{tw^i}$ for all $i \geq 0$. It follows that $\text{im}(1 \otimes \varphi^{tw^i}) \subseteq B_+(B \otimes_A G^{tw^i})$ for all $i \geq 0$. Thus it suffices to consider the maps $1 \otimes \tau^{tw^i}$.

Now, $B M$ has a $B$-free direct summand if and only if the twisted module $M^{tw^i}$ does.

Let $F = B \otimes_A F$. Since $\Omega(\varphi, \tau)$ is exact, for each $i \geq 0$ we have

$$\text{im}(1 \otimes \tau^{tw^i}) \cong \text{coker}(1 \otimes \varphi^{tw^i}) \cong B \otimes_A M^{tw^i} \cong M^{tw^i}$$

Since $\text{im}(1 \otimes \tau^{tw^i})$ is contained in the radical $B_+ F^{tw^i-1}$ if and only if no basis for the free $B$-module $F^{tw^i}$ intersects $\text{im}(1 \otimes \tau^{tw^i})$, the result follows. $\square$

Corollary 2.10. Under the hypotheses and notation of Proposition 2.9, for any integer $n$, the complex $\Omega(\varphi, \tau)^{tw^n}$ is a graded free resolution of $M^{tw^n}$

We can also express minimality of the resolution $\Omega(\varphi, \tau)$ in terms of the twisted matrix factorization.

Lemma 2.11. Let $A$ be a connected, $\mathbb{N}$-graded, locally finite-dimensional $k$-algebra and $f \in A_+$ a homogeneous normal, regular element. A twisted matrix factorization $(\varphi, \tau)$ of $f$ over $A$ is reduced if and only if $\Omega(\varphi, \tau)$ is a minimal graded free resolution.

Proof. The complex $\Omega(\varphi, \tau)$ is minimal if and only if $\text{coker}(1 \otimes \varphi^{tw^i})$ and $\text{coker}(1 \otimes \tau^{tw^i})$ have no $B$-free direct summands for all $i \geq 0$. Since the functor $(-)^{tw}$ preserves direct sums and free modules, the complex $\Omega(\varphi, \tau)$ is minimal if and
only if \( \text{coker}(1 \otimes \varphi) = \text{coker} \varphi \) and \( \text{coker}(1 \otimes \tau) = \text{coker} \tau \) have no \( B \)-free direct summands. The latter holds if and only if \((\varphi, \tau)\) is not isomorphic to a twisted factorization \((\varphi', \tau')\) where \( \varphi' \) or \( \tau' \) has \( \lambda_f^A \) as a summand. \( \square \)

Next we turn to periodicity. Clearly the complex \( \Omega(\varphi, \tau) \) is periodic of period at most \( 2n \) if \( \sigma \) has finite order \( n \). In practice, the period is often less than \( 2n \) (see Section 6 for some examples). Even when \( |\sigma| = \infty \), the complex may be periodic, as we show in the next proposition.

**Proposition 2.12.** The complex \( \Omega(\varphi, \tau) \) is periodic if and only if \( M \) has no \( B \)-free direct summands. The latter holds if and only if \((\varphi, \tau)\) is not isomorphic to a twisted factorization \((\varphi', \tau')\) where \( \varphi' \) or \( \tau' \) has \( \lambda_f^A \) as a summand. \( \square \)

**Proof.**

Suppose there exists an integer \( n \) such that \( M \) has no \( B \)-free direct summands. The latter holds if and only if \((\varphi, \tau)\) is not isomorphic to a twisted factorization \((\varphi', \tau')\) where \( \varphi' \) or \( \tau' \) has \( \lambda_f^A \) as a summand.

Conversely, suppose there exists an integer \( p \), an integer \( N \), and a degree \( n \)-morphism of complexes \( \Phi : \Omega(\varphi, \tau)(N) \rightarrow \Omega(\varphi, \tau) \) such that \( \Phi : \Omega_{i+p}(N) \rightarrow \Omega_i \) is an isomorphism for all \( i \geq 0 \). Since \( \Phi^2 \) also has this property, we may assume \( p = 2n \) is even.

By construction, minimal generators of \( \Omega_{i+p} \) can be taken to be minimal generators of \( \Omega_i \) with degrees shifted up by \( nd \). It follows that \( N = nd \). Thus the following diagram commutes, and

\[
\begin{array}{ccc}
M^\sigma & \cong & \text{coker}(1 \otimes \varphi^\sigma) \\
& \cong & \text{coker}(1 \otimes \varphi) \\
& \cong & M \\
\end{array}
\]

\[
\begin{array}{ccc}
B \otimes_A F & \xrightarrow{1 \otimes \varphi^\sigma} & B \otimes_A G \\
\Phi & & \Phi \\
B \otimes_A F & \xrightarrow{1 \otimes \varphi} & B \otimes_A G \\
\end{array}
\]

\( \square \)

### 3. Equivalent Categories of Twisted Matrix Factorizations

In [27], Zhang completely characterized pairs of graded \( k \)-algebras whose categories of graded modules are equivalent. With that characterization in mind, we consider the question of when categories of twisted matrix factorizations are equivalent. The following easy fact is useful later.

**Proposition 3.1.**

1. For any scalar \( \nu \in k^\times \), the categories \( \text{TMF}_A(f) \) and \( \text{TMF}_A(\nu f) \) are equivalent.
2. Let \( \phi : A \rightarrow A \) be a graded automorphism of \( A \). Then \( \text{TMF}_A(f) \cong \text{TMF}_A(\phi(f)) \).

**Proof.** For (1), the functors \((\varphi, \tau) \mapsto (\varphi, \nu \tau)\) and \((\varphi, \tau) \mapsto (\varphi, \nu^{-1} \tau)\) are easily seen to be inverse equivalences. For (2), first observe that \( \phi \sigma \phi^{-1} \) is the normalizing automorphism for \( \phi(f) \). Applying the functor \((-)^{\phi^{-1}}\) to any twisted matrix factorization of \( f \) over \( A \) produces the desired equivalence. \( \square \)

We briefly recall the basic definitions underlying Zhang’s graded Morita equivalence and encourage the interested reader to see [27] for more details.
A (left) twisting system for $A$ is a set $\zeta = \{\zeta_n \mid n \in \mathbb{Z}\}$ of graded $k$-linear automorphisms of $A$ such that $\zeta_n(\zeta_{m}(x)y) = \zeta_{m+n}(x)y$ for all $n, m, \ell \in \mathbb{Z}$ and $x \in A_{\ell}$, $y \in A_n$. For example, if $\phi$ is a graded $k$-linear automorphism of $A$, then setting $\zeta_n = \phi^n$ for all $n \in \mathbb{Z}$ gives a twisting system.

Given a twisting system $\zeta$, the Zhang twist of $A$ is the graded $k$-algebra $A^\zeta$ where $A^\zeta = A$ as graded $k$-vector spaces and for all $x \in A_{\ell}$ and $y \in A_n$, multiplication in $A^\zeta$ is given by $x \cdot y = \zeta_m(x)y$. Likewise, if $M$ is a graded left $A$-module, the twisted left $A^\zeta$-module $M^\zeta$ has the same underlying graded vector space as $M$, and for $m \in M_n$ and $z \in A_{\ell}$, $x \cdot m = \zeta_n(z)m$. Finally, we note that if $\varphi : M \to N$ is a degree 0 homomorphism of graded left $A$-modules, $\varphi : M^\zeta \to N^\zeta$ is also a degree 0 homomorphism of graded left $A^\zeta$-modules.

**Remark 3.2.** Aside from the use of the letter $\zeta$, the notation for the twisted module is identical to that used for the functor $(-)^{\sigma}$ above. However, the notions are not the same. One important difference is that for an integer $n$, the free left $A$-modules $A(n)^{\sigma}$ and $A^\zeta(n)^{\sigma}$ are identical, whereas the free left $A^\zeta$-modules $A(n)^{\zeta}$ and $A^\zeta(n)^{\zeta}$ - which have the same underlying graded vector space - are generally not identical, but are isomorphic via the map $\zeta_{-n}$. In light of this subtlety, the following simple lemma is not entirely trivial.

**Lemma 3.3.** Let $A$ be a connected, $\mathbb{N}$-graded, locally finite-dimensional $k$-algebra. Let $f \in A_d$ be a normal, regular homogeneous element with normalizing automorphism $\sigma$. Let $\zeta = \{\zeta_n \mid n \in \mathbb{Z}\}$ be a left twisting system.

1. If $\zeta_n(f) = c^n f$ for some $c \in k^\times$ and for all $n \in \mathbb{Z}$, then $f$ is normal and regular in $A^\zeta$ with normalizing automorphism $\hat{\sigma}(a) = c^{-\deg_a} \sigma \zeta_d(a)$.
2. If $\zeta$ further satisfies $\zeta_n \sigma \zeta_d = \sigma \zeta_{n+d}$ for all $n \in \mathbb{Z}$, we have $(A^w)^\zeta \cong (A^\zeta)^{tw} := (A^\zeta)^{\sigma}(-d)$ as free left $A^\zeta$-modules.

If the twisting system $\zeta$ is “algebraic,” meaning $\zeta_n \zeta_m = \zeta_{n+m}$ for all $n, m \in \mathbb{Z}$, the additional hypothesis of (2) becomes $\sigma \zeta_n = \zeta_n \sigma$ for all $n \in \mathbb{Z}$. In the common case where $\zeta_n = \phi^n$ for a $k$-linear automorphism $\phi : A \to A$, one needs only that $\sigma \phi = \phi \sigma$.

**Proof.** Let $a \in A_n$ be an arbitrary homogeneous element. To prove (1), we have

$$a \cdot f = \zeta_d(a)f = f \sigma(\zeta_d(a)) = \zeta_n^{-1}(f) \cdot \sigma \zeta_d(a) = c^{-n}f \cdot \sigma \zeta_d(a) = f \cdot \hat{\sigma}(a)$$

Thus $f$ is normal in $A^\zeta$. The equation also shows the regularity of $f$ in $A^\zeta$ follows from the regularity of $f$ in $A$, so $\hat{\sigma}$ is the normalizing automorphism.

For (2), first observe that $a \mapsto c^{\deg_a} \sigma$ defines a graded algebra automorphism $\lambda_c$ of $A^\zeta$. For any graded left $A^\zeta$-module $M$, $M \cong M^{\lambda_c}$ via the map $m \mapsto c^{\deg_m} m$ which we also denote $\lambda_c$.

Now, $(A^{tw})^\zeta$ and $(A^\zeta)^{tw}$ have the same underlying graded vector space as $A$. We compute the left $A^\zeta$ action on both modules. With $a$ as above and $b \in A_m$, $A^\zeta$ acts on $(A^{tw})^\zeta$ by

$$a \cdot b = \hat{\sigma}(a) * b = \zeta_m(\hat{\sigma}(a))b = \zeta_m c^{-\deg_a} \sigma \zeta_d(a)b = c^{-\deg_a} \zeta_m \sigma \zeta_d(a)b$$

and on $(A^{tw})^{\zeta}$ by

$$a \cdot b = \zeta_m+\hat{\sigma}(a) * b = \sigma \zeta_{m+d}(a)b$$

since $b \in A_m = A^\sigma(-d)m+d$. Thus $((A^\zeta)^{tw})^{\lambda_c} = (A^{tw})^\zeta$ and the result follows. \qed
For completeness, we mention the left module version of Zhang’s theorem on graded Morita equivalence.

**Theorem 3.4** ([27]). Let \( k \) be a field and let \( A \) and \( A' \) be connected graded \( k \)-algebras with \( A_1 \neq 0 \). Then \( A \cong A'^\zeta \) for some twisting system \( \zeta \) if and only if the categories \( A\text{-GrMod} \) and \( A'\text{-GrMod} \) are equivalent.

The equivalence is given by \( M \mapsto M^\zeta \) for any graded \( A' \)-module \( M \) and is the identity on morphisms. We have the following.

**Theorem 3.5.** Let \( A \) be a connected, \( N \)-graded locally finite dimensional \( k \)-algebra. Let \( f \in A_d \) a normal, regular homogeneous element of degree \( d \) with normalizing automorphism \( \sigma \). Let \( \zeta = \{ \zeta_n \mid n \in \mathbb{Z} \} \) be a twisting system such that for all \( n \in \mathbb{Z} \), \( \zeta_n \sigma \zeta_d = \sigma \zeta_{n+d} \) and \( \zeta_n(f) = c^n f \) for some \( c \in k^\times \). Then the categories \( TMF_A(f) \) and \( TMF_{A^\zeta}(f) \) are equivalent.

**Proof.** By Proposition 3.1, it suffices to prove that \( TMF_A(f) \) is equivalent to \( TMF_{A^\zeta}(c^d f) \).

Let \((\varphi : F \to G, \tau : G^tw \to F)\) be a twisted left matrix factorization of \( f \) over \( A \). Let \( \lambda_c : (G^\zeta)^{tw} \to ((G^\zeta)^{tw})^{\lambda_c} \) be the graded isomorphism \( m \mapsto c^{\deg m} m \) as in the proof of Lemma 3.3. By Lemma 3.3(2) and the note preceding Remark 3.2, \((\varphi \circ \lambda_c : F^\zeta \to G^\zeta, \tau \lambda_c : (G^\zeta)^{tw} \to F^\zeta)\)

is a twisted matrix factorization of \( c^d f \) over \( A^\zeta \). The functoriality of Zhang’s category equivalence \((-)^\zeta\) ensures any morphism \((\alpha, \beta) : (\varphi, \tau) \to (\varphi', \tau')\) of twisted factorizations over \( A \) remains a morphism over \( A^\zeta \). This defines a functor \( TMF_A(f) \to TMF_{A^\zeta}(f) \).

The inverse equivalence is given by applying the inverse twisting system \( \zeta^{-1} = \{ \zeta_n^{-1} \mid n \in \mathbb{Z} \} \) to a twisted matrix factorization over \( A^\zeta \) and replacing \( \lambda_c \) by \( \lambda_{c^{-1}} \) in the above construction. \( \square \)

**Corollary 3.6.** The equivalence given in the preceding theorem completes a commutative diagram of functors

\[
\begin{array}{ccc}
TMF_A(f) & \longrightarrow & TMF_{A^\zeta}(f) \\
\text{coker} & & \text{coker} \\
\downarrow & & \downarrow \\
A\text{-GrMod} & \xrightarrow{Z} & A^\zeta\text{-GrMod}
\end{array}
\]

where \( Z \) denotes Zhang’s equivalence of categories, and \( \text{coker} \) sends the twisted matrix factorization \((\varphi, \tau)\) to \( \text{coker} \varphi \).

We do not know an example of a twisting system \( \zeta \) where \( f \) remains normal and regular in \( A^\zeta \) but \( TMF_A(f) \) and \( TMF_{A^\zeta}(f) \) are inequivalent.

In some cases, a normal, regular element can become central in an appropriate Zhang twist. By Proposition 2.12, twisted matrix factorizations of a central element produce resolutions with period at most 2. Example 6.2 below illustrates the following important subtlety.

**Corollary 3.7.** The period of a periodic minimal free resolution need not be invariant under a Zhang twist.
4. Noncommutative hypersurfaces

The bijection between periodic minimal free resolutions and reduced matrix factorizations over local rings hinges on the Auslander-Buchsbaum formula. Before giving a noncommutative version of this correspondence, we recall Jørgensen’s version of Auslander-Buchsbaum for connected graded $k$-algebras.

Throughout this section, let $A$ be a connected, $\mathbb{N}$-graded, locally finite dimensional $k$-algebra and additionally assume $A$ is left noetherian. Let $M$ be a finitely generated graded left $A$-module. The depth of $M$ is

$$\text{depth}_A(M) = \inf\{i \mid \text{Ext}_A^i(k,M) \neq 0\}$$

Note that $\text{depth}_A(M)$ is either an integer or $\infty$. We have the following special case of Jørgensen’s Auslander-Buchsbaum theorem.

**Theorem 4.1 ([13]).** With $A$ and $M$ as above, if the spaces $\text{Ext}_A^i(k,A)$ are finite dimensional $k$-vector spaces for all $0 \leq i \leq \text{depth}_A(A)$ and if $\text{pd}_A(M) < \infty$, then the Auslander-Buchsbaum formula

$$\text{pd}_A(M) + \text{depth}_A(M) = \text{depth}_A(A)$$

holds for $M$.

Theorem 4.1 says the Auslander-Buchsbaum formula holds for all finitely generated graded left modules over left noetherian Artin-Schelter regular algebras.

**Definition 4.2.** Let $A$ be a connected, $\mathbb{N}$-graded, locally finite-dimensional $k$-algebra. Then $A$ is Artin-Schelter regular (resp. Artin-Schelter Gorenstein) of dimension $d$ if

1. $\text{gl.dim}(A) = d < \infty$ (resp. $\text{inj.dim}(A) = d < \infty$ on both sides)
2. $\text{GKdim}(A) = d$
3. $\text{Ext}_A^i(k,A) = \delta_{i,d} k$

We frequently abbreviate these conditions AS-regular and AS-Gorenstein. We also note that the results below in which $A$ is a left noetherian AS-regular or AS-Gorenstein algebra do not require an explicit assumption that the Gelfand-Kirillov dimension is finite.

The following fact is a consequence of the long exact sequence in cohomology. We omit the straightforward proof.

**Lemma 4.3.** Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of graded left $A$ modules. If $\text{depth}_A(M) > \text{depth}_A(M'')$ then $\text{depth}_A(M') = \text{depth}_A(M'') + 1$.

Next, we need several facts relating the depth of $A$ to the depth of $A/(f)$ when $f$ is a normal, regular element. This result is classically known as Rees’ Lemma; we include a proof here for completeness.

**Lemma 4.4.** Let $A$ be a connected $\mathbb{N}$-graded $k$-algebra. Let $f \in A$ be a normal, regular homogeneous element and let $B = A/(f)$. Then

1. $\text{depth}_B(B) = \text{depth}_A(A) - 1$
2. $\text{depth}_A(M) = \text{depth}_B(M)$ for any finitely generated, graded left $B$-module $M$.
3. If $A$ is AS-Gorenstein, so is $B$. 

Proof. Let $N$ be a finitely generated, graded left $A$-module and consider the Cartan-Eilenberg change-of-rings spectral sequence

$$
\text{Ext}^p_B(k, \text{Hom}_A(B, N)) \Rightarrow \text{Ext}^{p+q}_A(k, N)
$$

Since $f \in A$ is regular, $\text{pd}_A B = 1$, hence the spectral sequence has only two nonzero rows.

$$
E^0_2 = \text{Ext}^p_B(k, \text{Hom}_A(B, N)) = \text{Ext}^p_B(k, N^f)
$$

$$
E^{p+q}_2 = \text{Ext}^p_B(k, \text{Ext}^q_A(B, N)) = \text{Ext}^p_B(k, N/fN)
$$

where $N^f = \{ n \in N \mid fn = 0 \}$. The associated long exact sequence is (\text{[5]} Theorem XV.5.11)

$$
\cdots \rightarrow E^{p,0}_2 \rightarrow \text{Ext}^p_A(k, N) \rightarrow E^{n-1,1}_2 \rightarrow E^{n+1,0}_2 \rightarrow \text{Ext}^{n+1}_A(k, N) \rightarrow E^{n,1}_2 \rightarrow \cdots
$$

To prove the first statement, set $N = A$. Since $f$ is regular, $N^f = 0$ and we have

$$
\text{Ext}^n_A(k, A) \cong \text{Ext}^{n-1}_B(k, B)
$$

The formula $\text{depth}_B(B) = \text{depth}_A(A) - 1$ follows.

For the second statement, let $M$ be a finitely generated, graded left $B$-module. Let $N = M$ viewed as an $A$-module via the natural quotient map. Then $N^f = M$ and $fN = 0$. For $i < d = \text{depth}_B(M)$

$$
\text{Ext}^i_B(k, M) \rightarrow \text{Ext}^i_A(k, M) \rightarrow \text{Ext}^{i-1}_B(k, M)
$$

is exact, hence $\text{Ext}^i_A(k, M) = 0$, and

$$
\text{Ext}^{d-2}_B(k, M) \rightarrow \text{Ext}^{d-1}_B(k, M) \rightarrow \text{Ext}^d_A(k, M) \rightarrow \text{Ext}^{d-1}_B(k, M)
$$

is exact, so $\text{Ext}^i_A(k, M) \neq 0$. This shows $\text{depth}_A(M) = \text{depth}_B(M)$.

For the last statement, assume $A$ is AS-Gorenstein of dimension $\mu = \text{inj.dim}(A)$. Then $\text{inj.dim}(B) = \mu - 1$ and $\text{GKdim}(B) = \text{GKdim}(A) - 1$ (\text{[22]} Theorem 3.6, Lemma 5.7). Finally, since $\text{Ext}^i_A(k, A) = \delta_{i,\mu}k$, equation (1) gives $\text{Ext}^i_B(k, B) = \delta_{i,\mu-1}k$. Thus $B$ is AS-Gorenstein of dimension $\mu - 1$. $\square$

Theorem \text{[4]} and Lemma \text{[4]} imply a graded $B$-module $M$ with $\text{pd}_A(M) = 1$ satisfies $\text{depth}_B(M) = \text{depth}_B(B)$. By analogy with the commutative case, it is tempting to call such a module “maximal Cohen-Macaulay.” In his notes \text{[4]}, Buchweitz defined the notion of a maximal Cohen-Macaulay module over any ring which is both left and right noetherian and has finite left and right injective dimension. We adopt a graded version of Buchweitz’s definition for left noetherian rings. A finitely generated graded module $M$ over a connected, $\mathbb{N}$-graded, locally finite dimensional left noetherian $k$-algebra $B$ of finite left and right injective dimension is called maximal Cohen-Macaulay if and only if $\text{Ext}^i_B(M, B) = 0$ for $i \neq 0$. The next Lemma shows that defining maximal Cohen-Macaulay modules in terms of depth is equivalent in our case.

Lemma 4.5. Let $A$ be a left noetherian, AS-regular algebra. Let $f \in A_+$ be a homogeneous normal, regular element and let $B = A/(f)$. Then for any finitely generated graded left $B$-module $M$, $\text{pd}_A(M) = 1$ if and only if $\text{Ext}^i_B(M, B) = 0$ for all $i \neq 0$. 

Proof. We have \(pd_A(M) = 1\) if and only if \(\text{Ext}_A^i(M,A) = 0\) for all \(i > 1\). One direction of this is clear. The other is Jørgensen’s Ext-vanishing theorem \([16]\).

Let \(d = \deg f\). Since \(0 \to A(-d) \xrightarrow{f} A \to B \to 0\) is a minimal graded free resolution of \(A B\), we see that \(\text{Ext}_A(B,A)\) is concentrated in homological degree 1 and \(\text{Ext}_A^1(B,A) \cong B(d)\) as graded left \(B\)-modules. Then the change of rings spectral sequence

\[
\text{Ext}_B^p(M, \text{Ext}_A^q(B,A)) \Rightarrow \text{Ext}_A^{p+q}(M,A)
\]

shows \(\text{Ext}_B^p(M,B) = 0\) for \(i \neq 0\) if and only if \(\text{Ext}_A^i(M,A) = 0\) for all \(i > 1\). \(\square\)

If \(\cdots \to P_2 \to P_1 \to P_0 \to M \to 0\) is an exact sequence of \(B\)-modules, we denote the \(j\)-th syzygy module \(\text{im}(P_j \to P_{j-1})\) by \(\Omega_j(M)\) where \(\Omega_0(M) = M\).

**Proposition 4.6.** Let \(A\) be a left noetherian Artin-Schelter regular algebra of dimension \(d\). Let \(f \in A_+\) be a normal, regular homogeneous element and let \(B = A/(f)\). Let \(M\) be a finitely generated, graded left \(B\)-module and \(P\) a graded projective \(B\)-module resolution of \(M\). Then \(pd_A(\Omega_i(M)) = 1\) for some \(0 \leq i \leq d\).

**Remark.** Recalling the construction of twisted matrix factorizations in Section 2, the Proposition shows their ubiquity over noetherian AS-regular algebras in the presence of a normal regular element.

**Proof.** First, we show \(\text{depth}_B(B) \geq \text{depth}_B(M)\). Indeed, by Theorem 4.1 and Lemma 4.4 we have

\[
\text{depth}_B(M) = \text{depth}_A(M) = \text{depth}_A(A) - pd_A(M) = \text{depth}_B(B) + 1 - pd_A(M)
\]

Since \(f M = 0\), we have \(pd_A(M) > 0\), hence \(\text{depth}_B(B) \geq \text{depth}_B(M)\). If \(\text{depth}_B(M) = \text{depth}_B(B)\), the equation above shows \(pd_A(M) = 1\), so assume \(\text{depth}_B(M) = i < \text{depth}_B(B)\). Graded projective modules are graded free so \(\text{depth}_B(P_j) = \text{depth}_B(B)\) for all \(j \geq 0\). Since

\[
0 \to \Omega_i(M) \to P_0 \to M \to 0
\]

is exact, \(\text{depth}_B(\Omega_i(M)) = i + 1\) by Lemma 4.3. Inductively applying Lemma 4.3 to the exact sequence

\[
0 \to \Omega_{i+1}(M) \to P_j \to \Omega_i(M) \to 0
\]

we obtain \(\text{depth}_B(\Omega_{d-i}(M)) = \text{depth}_B(B)\). It follows that

\[
pd_A(\Omega_{d-i}(M)) = 1
\]

as desired. \(\square\)

We are ready to prove our main theorem. For ease of notation, we will no longer specify the degree of the normal, regular homogeneous element \(f\) and instead reserve \(d\) for the dimension of the ambient AS-regular algebra. The functor \((-)^{tw}\) continues to denote the composition of \((-)^{tw}\) with an appropriate degree shift.

**Theorem 4.7.** Let \(A\) be a left noetherian Artin-Schelter regular algebra of dimension \(d\). Let \(f \in A_+\) be a homogeneous normal regular element and let \(\sigma\) be its normalizing automorphism. Let \(B = A/(f)\). If

\[
Q: \cdots \to Q_2 \to Q_1 \to Q_0
\]
is a minimal graded free left $B$-module resolution of a finitely generated graded left $B$-module $M$, then

1. The truncated complex $\cdots \to Q_{d+2} \to Q_{d+1} \to \cdots$ is chain isomorphic to $\Omega(\varphi, \tau)$ for some reduced twisted left matrix factorization $(\varphi, \tau)$.

Assuming further that $|\sigma| < \infty$, we have

2. $Q$ becomes periodic of period at most $2|\sigma|$ after $d+1$ steps.
3. $Q$ is periodic (of period at most $2|\sigma|$) if and only if $\pd_A(M) = 1$ and $M$ has no graded free $B$-module summand.
4. Every periodic minimal graded free left module resolution over $B$ has the form $\Omega(\varphi, \tau)$ for some reduced twisted left matrix factorization $(\varphi, \tau)$ of $f$ over $A$.

Proof. By Proposition 4.6, we have $\pd_A(\Omega_i(M)) = 1$ for some $0 \leq i \leq d$. If $\Omega_1(M) = \Omega_i(M) \oplus F$ where $F$ is a graded free $B$-module and $\Omega_i(M)$ has no free summand, then $\pd_A(\Omega_i(M)) = 1$. By Construction 2.7 and Proposition 2.9 there exists a twisted left matrix factorization $(\varphi, \tau)$ such that $\Omega(\varphi, \tau)$ is a periodic minimal graded free resolution of $\Omega_i(M)$. If $F[i]$ denotes the free module $F$ viewed as a complex concentrated in homological degree $i$, it follows that

$$\tilde{Q} : \Omega(\varphi, \tau) \oplus F[i] \to Q_{i-1} \to \cdots \to Q_0$$

is a minimal graded free resolution of $M$ (or, if $i = 0$, $\Omega(\varphi, \tau) \oplus F$ is a resolution). By uniqueness of minimal resolutions, $\tilde{Q} \cong Q$. Truncating each complex at homological degree $i+1$ and recalling that if $(\varphi, \tau)$ is a twisted matrix factorization, so are $(\varphi^{tw}, \tau^{tw})$ and $(\tau, \varphi^{tw})$, we have established (1).

If $|\sigma| < \infty$, the resolution $\tilde{Q}$ is periodic of period at most $2|\sigma|$ after $i + 2$ steps and rank $\tilde{Q}_j = \rank \tilde{Q}_{i+2}$ for all $j \geq i + 2$. This proves (2). Setting $i = 0$ and $\Omega_1(M) = \Omega_i(M)$, we also obtain the “if” direction of (3).

Now suppose that $Q$ is periodic of period $p$. If $\Omega_i(M)$ has a free summand, rank $Q_{p+i+1} = \rank Q_{i+1} > \rank Q_{i+2}$. But this is impossible, since $Q$ and $\tilde{Q}$ are isomorphic minimal free resolutions. Thus $\Omega_i(M)$ has no free direct summand and $\tilde{Q} : \Omega(\varphi, \tau) \to Q_0 \to \cdots \to Q_0$ is a minimal free resolution of $M$. By Lemma 1.1 rank $Q_j = \rank Q_0$ for all $j \geq 0$, so $M$ has no free direct summand.

By graded periodicity, $M = \coker(Q_1 \to Q_0)$ is isomorphic to $\coker(1 \otimes \varphi^{tw})$ or $\coker(1 \otimes \tau^{tw})$ for some $m$. Since both maps lift to injective maps of free $A$-modules, $\pd_A(M) = 1$. This completes the proof of (3). Since $(\varphi^{tw}, \tau^{tw})$ and $(\tau^{tw}, \varphi^{tw+1})$ are also twisted matrix factorizations, (4) follows as well. \qed

Taking $A$ to be the polynomial ring $k[x_1, \ldots, x_n]$, we recover a graded version of Theorem 6.1 of [2] as a special case of Theorem 4.5. We remark that the analogous theorem in [2] relies on the existence of regular sequences of length $\depth(A)$, whereas our proof necessarily avoids this assumption.

As a first corollary, we have the following useful fact.

**Corollary 4.8.** Let $A$, $f$, and $B$ as in the theorem. Assume $|\sigma| < \infty$. If $(Q, \partial)$ is a minimal graded free left $B$-module resolution of a finitely generated module, then $\im \partial_k$ has no free summands for $k \geq d + 1$.

We also see that resolutions of the trivial module $Bk$ have a very rigid structure. Note we do not need to assume $|\sigma| < \infty$. 

Corollary 4.9. Let $A$, $f$, and $B$ as in the theorem. There exists a minimal graded free resolution of the trivial $B$-module $Bk$ which becomes periodic of period at most 2 after $d+1$ steps.

Proof. Let $Q$ be a minimal graded free resolution of $Bk$. By Theorem 4.7, there exists a twisted left matrix factorization $(\varphi, \tau)$ of $f$ such that $\Omega(\varphi, \tau)$ is a minimal graded projective resolution of the $(d+1)$-st syzygy $\Omega_{d+1}(Bk)$.

Let $\bar{\sigma}: B \to B$ be the automorphism induced by $\sigma$. Clearly, $Bk^{\bar{\sigma}} \cong Bk$, so there exists a chain isomorphism $\Phi: Q^{\bar{\sigma}} \to Q$ by Lemma 1.2. By the 5-Lemma, $\Phi_{d+1}$ restricts to a graded $B$-module isomorphism $(\Omega_{d+1}(Bk))^{\bar{\sigma}} \cong \Omega_{d+1}(Bk)$.

The result follows from Proposition 2.12. \qed

Recall that a graded free resolution $(P_\bullet, d_\bullet)$ is called linear if $P_i$ is generated in degree $i$ for all $i \geq 0$.

Corollary 4.10. Let $A$, $f$, and $B$ be as in the theorem. Additionally assume $f$ is quadratic. Then $Q$ becomes a linear free resolution after $d+1$ steps.

Corollary 4.11. There exist bijections between isomorphism classes of reduced twisted left matrix factorizations of $f$ over $A$, isomorphism classes of nontrivial periodic minimal graded free resolutions of finitely generated graded left $B$-modules, and isomorphism classes of maximal Cohen-Macaulay left $B$-modules without free summands.

5. Homotopy category of twisted matrix factorizations

In [4], Buchweitz established an equivalence between the stable category of maximal Cohen-Macaulay modules over a noetherian ring $B$ with finite left and right injective dimensions and a quotient of the bounded derived category of modules over $B$ now called the singularity category. He noted the equivalence also holds in the graded case. In [23], Orlov proved that if $B$ is a graded factor algebra of a finitely generated, connected, $N$-graded noetherian $k$-algebra $A$ of finite global dimension by a central, regular element $W$, then the stable bounded derived category of graded $B$-modules is equivalent to a category Orlov called “the category of graded $D$-branes of type $B$ for the pair $(B, W)$.” In this section we extend Orlov’s result to factors of left noetherian AS-regular algebras by regular, normal elements. Much of Orlov’s work goes through with the obvious necessary changes. The key difference is that $|\sigma|$ need not be finite in our case, so we cannot appeal to periodicity of a resolution.

We continue to let $A$ be a connected, noetherian, $N$-graded, locally finite dimensional $k$-algebra and $f \in A_+$ a normal, regular homogeneous element with normalizing automorphism $\sigma$. Let $B = A/(f)$. We will also continue to consider left modules over $B$.

There is a natural functor of abelian categories $\mathcal{C} : TMFA(f) \to B-GrMod$ given on objects by $(\varphi, \tau) \mapsto \text{coker } \varphi$. (Recall from the proof of Proposition 2.2 that coker $\varphi$ is a $B$-module.) A morphism $\Psi: (\varphi, \tau) \to (\varphi', \tau')$ induces a well-defined map $\psi: \text{coker } \varphi \to \text{coker } \varphi'$ by $\pi'\Psi G \pi^{-1}$ where $\pi: G \to \text{coker } \varphi$ and $\pi': G' \to \text{coker } \varphi'$ are the canonical projections, and $\pi^{-1}$ is any section of $\pi$. 
The functor $C$ is not essentially surjective\footnote{A functor $F : C \to D$ is \textit{essentially surjective} if every object in $D$ is isomorphic to $F(c)$ for some object $c$ in $C$.} objects in the image of $C$ are finitely generated $B$-modules $M$ such that $\text{pd}_A M = 1$. By Lemma \ref{lem:pd}, if $A$ is left noetherian and AS-regular, the image of $C$ consists of maximal Cohen-Macaulay modules.

We denote the full subcategory of maximal Cohen-Macaulay modules in $B-$GrMod by $\text{MCM}(B)$. Following \cite{4}, we define the category of \textit{stable} maximal Cohen-Macaulay modules, which we denote $\text{MCM}^\text{st}(B)$, to have the same objects as $\text{MCM}(B)$, but for $M, N \in \text{MCM}^\text{st}(B)$,

$$\text{Hom}_{\text{MCM}^\text{st}(B)}(M, N) = \text{Hom}_B(M, N) / R$$

where $R$ is the subspace of morphisms which factor through a graded projective $B$-module.

As in \cite{4} and \cite{23}, let $D^b(B)$ be the bounded derived category of finitely generated graded left $B$-modules. A complex in $D^b(B)$ is called \textit{perfect} if it is isomorphic in $D^b(B)$ to a complex of finitely generated graded projective modules. Perfect complexes form a full, triangulated subcategory $D^b_{\text{perf}}(B)$ of $D^b(B)$. The \textit{singularity category of $B$} is defined to be the quotient category $D^b_{\text{sg}}(B) = D^b(B) / D^b_{\text{perf}}(B)$.

As noted in \cite{4}, the composition $\text{MCM}(B) \to D^b(B) \to D^b_{\text{sg}}(B)$, where the first functor takes a module to its trivial complex, factors uniquely through the quotient $\text{MCM}(B) \to \text{MCM}^\text{st}(B)$, yielding a functor $\mathcal{G} : \text{MCM}(B) \to D^b_{\text{sg}}(B)$. Buchweitz showed that $\mathcal{G}$ is an exact equivalence. The equivalence $\mathcal{G}$ induces a triangulated structure on $\text{MCM}^\text{st}(B)$. (It is possible to describe the triangulated structure independently, see \cite{4}, but as we will not need it, we omit any details.)

Thus it is natural to consider a “stable” version of the category $h\text{TMF}_A(f)$. To motivate the definition, we remark that the category $h\text{TMF}_A(f)$ is equivalent to the category of doubly-infinite sequences of graded free $A$-module homomorphisms of the form

$$\cdots \to F^{tw} \xrightarrow{\varphi^{tw}} G^{tw} \xrightarrow{\tau} F \xrightarrow{\varphi} G \xrightarrow{\tau^{-1}} F^{tw-1} \to \cdots$$

whose compositions are multiplication by $f$, and whose morphisms are maps of sequences

$$\Psi = (\ldots, \Psi^{tw}_F, \Psi^{tw}_G, \Psi_F, \Psi_G, \Psi^{tw-1}_F, \ldots)$$

satisfying the necessary commutation relations. We adopt the structure of the homotopy category of such sequences of graded projective $A$-modules.

\textbf{Definition 5.1.} A morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ is \textit{null homotopic} if there exists a pair $(s, t)$ of degree 0 module homomorphisms $s : G \to F'$ and $t : F \to G^{tw}$ such that $\Psi^{tw}_G = \varphi^{tw} \cdot s + t \tau$ and $\Psi_F = \tau' t + s \varphi$.

We denote by $h\text{TMF}_A(f)$ the quotient (homotopy) category of $\text{TMF}_A(f)$ with the same objects, and whose morphisms are equivalence classes of morphisms in $\text{TMF}_A(f)$ modulo null homotopic morphisms. Observe that taking $s : A \to A$ to be the identity map and $t : A \to A^{tw}$ to be zero shows the identity map $(\text{id}_A, \lambda_f^t) \to (\text{id}_A, \lambda_f^{tw})$ is null homotopic. Thus $(\text{id}_A, \lambda_f^t) \cong 0$ in $h\text{TMF}_A(f)$\footnote{Recall that objects $(\varphi, \tau), (\varphi', \tau')$ are isomorphic in $h\text{TMF}_A(f)$ if and only if there exist maps $\Phi$ and $\Psi$ between them such that $\text{id}_{(\varphi, \tau)} - \Phi \Psi$ and $\text{id}_{(\varphi', \tau')} - \Psi \Phi$ are null homotopic.}

A similar calculation shows $(\lambda_f^t, \text{id}_{A^{tw}}) \cong 0$. More generally we have the following.
Lemma 5.2. If \((\varphi, \tau) \in TMF_A(f)\) such that coker \(\varphi\) is a graded free \(B\)-module, then \((\varphi, \tau) \cong 0\) in \(hTMF_A(f)\).

Proof. The lemma is trivial if \(\varphi = 0\), so suppose \(\varphi \neq 0\) and \(M = \text{coker } \varphi = \text{coker } (B \otimes_A F \xrightarrow{1 \otimes \varphi} B \otimes_A G)\) is a graded free left \(B\)-module. Let \(\psi : M \to B \otimes_A G\) be a graded splitting of the canonical projection \(\pi\), viewed as a map of graded left \(A\)-modules. Since \(B \otimes_A G\) is isomorphic (as a left \(A\)-module) to the cokernel of \(\lambda_f^G : G^w \to G\), lifting \(\psi\) gives a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & F & \xrightarrow{\varphi} & G & \xrightarrow{\psi} & M & \rightarrow & 0 \\
\Psi_F & \downarrow & & \Psi_G & \downarrow & & \downarrow & & \\
0 & \rightarrow & G^w & \xrightarrow{\lambda_f^G} & G & \rightarrow & B \otimes_A G & \rightarrow & 0 \\
\tau & \downarrow & & \downarrow & & \downarrow & & \pi & \downarrow & \\
0 & \rightarrow & F & \xrightarrow{\varphi} & G & \rightarrow & M & \rightarrow & 0
\end{array}
\]

Since \(\pi \psi = \text{id}_M\), there exists \(s : G \to F\) such that \(\text{id}_F - \tau \Psi_F = s \varphi\) and \(\text{id}_G - \Psi_G = \varphi s\) by the comparison theorem. (The only \(A\)-module homomorphism \(M \to G\) is the zero map.) If we now set \(t = \Psi_F\), the morphism \((\Psi_G, t \Psi_F)\) of twisted matrix factorizations is chain homotopic to the identity map on \((\varphi, \tau)\) via the pair \((s, t)\).

Indeed, it is clear that \(\text{id}_F = \tau t + s \varphi\). To see that \(\text{id}^{tw}_{G} = \varphi^{tw}_{s+tw} + t \tau\), it suffices to show \(\Psi^{tw}_{G\tau} = t \tau\). This follows from the equalities

\[
0 = \Psi^{tw}_{G^{tw}} \varphi^{tw} - \lambda^{G^{tw}}_{f} \Psi^{tw}_{F} = \Psi^{tw}_{G^{tw}} \varphi^{tw} - \Psi^{tw}_{F} \lambda^{f}_{G} = \Psi^{tw}_{G^{tw}} \varphi^{tw} - \Psi^{tw}_{F} \varphi^{tw}
\]

and the injectivity of \(\varphi\).

Remark 5.3. Implicit in the previous proof is the fact that \(TMF_A(f) \to MCM(B)\), and hence the composite \(\underline{C} : TMF_A(f) \to MCM(B)\), is a full functor.

Our next objective is to establish the following fact.

Proposition 5.4. The category \(hTMF_A(f)\) is a triangulated category.

We begin with a few definitions. The translation functor on \(hTMF_A(f)\) is given by \((\varphi, \tau)[1] = (-\tau^{tw-1}, -\varphi)\) on objects and by \(\Psi[1] = (\Psi^{tw-1}_{F}, \Psi_G)\) on morphisms. For any morphism \(\Psi : (\varphi, \tau) \to (\varphi', \tau')\) the mapping cone of \(\Psi\) is the pair

\[
C(\Psi) = (\gamma : F' \oplus G \to G' \oplus F^{tw-1}, \delta : G^{tw} \oplus F \to F' \oplus G)
\]

where

\[
\gamma = \begin{pmatrix} \varphi' & 0 \\ \Psi_G & -\tau^{tw-1} \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} \tau' & 0 \\ \Psi_F & -\varphi \end{pmatrix}.
\]

By the above matrix notation, we mean that the maps \(\gamma\) and \(\delta\) are given as follows on ordered pairs:

\[
\gamma(x', y) = (\varphi'(x') + \Psi_G(y), -\tau^{tw-1}(y)) \quad \delta(y', x) = (\tau'(y') + \Psi_F(x), -\varphi(x)).
\]

It is straightforward to check that this pair is a twisted matrix factorization and there exist canonical inclusion and projection morphisms \(i : (\varphi', \tau') \to C(\Psi)\) and \(p :
Moreover, given a commutative square of twisted factorizations

\[
\begin{array}{c}
(\varphi, \tau) \xrightarrow{\Psi} (\varphi', \tau') \\
\downarrow \quad \downarrow \\
(\gamma, \delta) \xrightarrow{\Phi} (\gamma', \delta')
\end{array}
\]

an easy diagram chase shows \((\Pi G \oplus \Pi_{\text{tw}}^{-1} \oplus \Pi G', \Pi G')\) defines a morphism \(C(\Psi) \to C(\Phi)\). Note that the complex \(\Omega(C(\Psi))\) is the mapping cone of the induced morphism of complexes \(\Omega(\Psi) : \Omega(\varphi, \tau) \to \Omega(\varphi', \tau')\).

We define a **standard triangle** to be any sequence of maps in \(hTMF_A(f)\)

\[
(\varphi, \tau) \xrightarrow{\Psi} (\varphi', \tau') \xrightarrow{i} C(\Psi) \xrightarrow{p} (\varphi, \tau)[1].
\]

We define a **distinguished triangle** to be any triangle

\[
(\varphi, \tau) \xrightarrow{\Psi} (\varphi', \tau') \xrightarrow{\Psi'} (\varphi'', \tau'') \xrightarrow{\Psi''} (\varphi, \tau)[1]
\]

isomorphic to a standard triangle. For any twisted factorization \((\varphi, \tau)\), the triangle

\[
(\varphi, \tau) \xrightarrow{id} (\varphi, \tau) \to 0 \to (\varphi, \tau)[1]
\]

is distinguished. To see this, consider the diagram

\[
\begin{array}{ccccccccc}
(\varphi, \tau) & \xrightarrow{id} & (\varphi, \tau) & \xrightarrow{id} & 0 & \xrightarrow{id} & (\varphi, \tau)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\varphi, \tau) & \xrightarrow{id} & (\varphi, \tau) & \xrightarrow{i} & C(id) & \xrightarrow{p} & (\varphi, \tau)[1]
\end{array}
\]

and note that \(id_{C(id)}\) is null homotopic via the pair

\[
s : G \oplus F^{tw^{-1}} \to F \oplus G \quad t : F \oplus G \to G^{tw} \oplus F
\]

both given by \((x, y) \mapsto (0, x)\). Precomposing this homotopy with the canonical inclusion \((\varphi, \tau) \to C(id)\) shows that \(i\) is null homotopic. Thus the diagram commutes in \(hTMF_A(f)\), and is hence an isomorphism of triangles in \(hTMF_A(f)\).

To show \(hTMF_A(f)\) is triangulated, it remains to show distinguished triangles are closed under rotations and that the octahedral axiom holds. The argument very closely follows the proof of Theorem IV.1.9 in [11]. We discuss only rotations of distinguished triangles in detail, leaving the translation of the remainder of the proof from [11] to the interested reader.

To verify the class of distinguished triangles is closed under rotations, it suffices to consider standard triangles.

Let

\[
(\varphi, \tau) \xrightarrow{\Psi} (\varphi', \tau') \xrightarrow{i} C(\Psi) \xrightarrow{p} (\varphi, \tau)[1]
\]

be a standard triangle. To see the rotated triangle

\[
(\varphi', \tau') \xrightarrow{i} C(\Psi) \xrightarrow{p} (\varphi, \tau)[1] \xrightarrow{-\Psi[1]} (\varphi', \tau')[1]
\]
is distinguished, first observe that $C(i)$ is given by the pair

$$
(F' \oplus G) \oplus G' \xrightarrow{(\tau', 0, 0)} (G' \oplus F^{tw-1}) \oplus (F')^{tw-1}
$$

$$
(G' \oplus F) \oplus F' \xrightarrow{(\tau', 0, 0)} (F' \oplus G) \oplus G'.
$$

Let $\Theta : (\varphi, \tau)[1] \to C(i)$ be the morphism defined by the pair

$$
\Theta_{F^{tw-1}} : F^{tw-1} \xrightarrow{(0, \text{id}_{F^{tw-1}})} G' \oplus F^{tw-1} \oplus (F')^{tw-1}
$$

$$
\Theta_G : G \xrightarrow{(0, \text{id}_{G})} F' \oplus G \oplus G'.
$$

This gives a diagram

$$
\begin{array}{ccc}
(\varphi', \tau') & \xrightarrow{i} & C(\Psi) \\
\downarrow & & \downarrow p \\
(\varphi, \tau)[1] & \xrightarrow{\Psi[1]} & (\varphi', \tau')[1] \\
\downarrow & & \downarrow q \\
C(i) & \xrightarrow{\Theta} & C(i) \\
\downarrow & & \downarrow j \\
(\varphi', \tau') & \xrightarrow{j} & C(\Psi)
\end{array}
$$

where $j$ and $q$ are the canonical morphisms for the mapping cone $C(i)$. The first and last squares are easily seen to commute.

The middle square, however, commutes only up to homotopy. The morphism $j - \Theta p$ is seen to be null homotopic via the pair of maps

$$
s : G' \oplus F^{tw-1} \to F' \oplus G \oplus G'
$$

$$
t : F' \oplus G \to (G')^{tw} \oplus F \oplus F'
$$

both given by $(x, y) \mapsto (0, 0, x)$. To see that $\Theta$ is an isomorphism in $hTMF_A(f)$, let $\pi : C(i) \to (\varphi, \tau)[1]$ be the canonical projection. Then $\pi \Theta$ is the identity on $(\varphi, \tau)[1]$ and $\text{id}_{C(i)} - \Theta \pi$ is seen to be null homotopic by precomposing the pair $(s, t)$ above with the projection $C(i) \to C(\Psi)$. This shows the class of distinguished triangles is closed under rotations.

**Theorem 5.5.** Let $A$ be a left noetherian AS-regular algebra, $f \in A_+$ a homogeneous normal regular element, and $B = A/(f)$. Then the categories $hTMF_A(f)$, $MCM(B)$, and $D^b_{sg}(B)$ are equivalent.

**Proof.** Since $\mathcal{G}$ is known to be an exact equivalence, it suffices to show $hTMF_A(f) \cong MCM(B)$.

The functor $\mathcal{G} : TMF_A(f) \to MCM(B)$ factors through the projection to $hTMF_A(f)$ to complete the commutative diagram of functors

$$
\begin{array}{ccc}
TMF_A(f) & \xrightarrow{\mathcal{G}} & MCM(B) \\
\downarrow & & \downarrow \mathcal{G} \\
hTMF_A(f) & \xrightarrow{\mathcal{F}} & MCM(B) \\
\downarrow & & \downarrow \mathcal{G} \\
D^b(B) & & D^b_{sg}(B).
\end{array}
$$

To see this, it is enough to show that any null homotopic morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$ induces the zero map in $MCM(B)$. Specifically, we show the induced map
$\psi : \text{coker } \varphi \to \text{coker } \varphi'$ factors through the graded projective module $B \otimes_A G'$. The morphism $\Psi$ factors as

$$(\varphi, \tau) \xrightarrow{\Phi} (\gamma, \delta) \xrightarrow{\Pi} (\varphi', \tau')$$

through the twisted “horseshoe” factorization $(\gamma, \delta)$ where

$$\gamma : (G')_{tw} \oplus F' \xrightarrow{-\tau' \choose \text{id}} F' \oplus G',$$

$$\delta : (F')_{tw} \oplus (G')_{tw} \xrightarrow{-\varphi'_{tw} \choose \text{id}} (G')_{tw} \oplus F',$$

$\Phi_G = (s, \Psi_G), \Phi_F = (t, \Psi_F)$ and $\Pi$ is the canonical projection onto the second factor.

We claim $\text{coker } \gamma = B \otimes_A G'$. For any $x \in F'$ and $y \in G'$, $(x, y) = (0, y - \varphi'(x))$ in $\text{coker } \gamma$. Thus there is a surjection $G' \to \text{coker } \gamma$. The kernel of this surjection consists of $z \in G'$ such that $z = \varphi' \tau'(w) = fw$ for some $w \in (G')_{tw}$. So $\text{coker } \gamma = G'/fG' = B \otimes_A G'$.

Now the induced maps $\text{coker } \varphi \circ \delta : B \otimes_A G' \xrightarrow{\sim} \text{coker } \varphi'$ show the map $\psi$ induced by $\Psi$ factors through a graded projective module, hence is the zero map in $\text{MCM}(B)$. Thus the functor $\mathcal{F}$ is well-defined.

The triangulated structure on $\text{MCM}(B)$ is induced by $\mathcal{G}$, so to prove $\mathcal{F}$ is an exact functor, it suffices to check that $G\mathcal{F}$ is exact. By Proposition 2.9, $\Omega((\varphi, \tau)[1])$ is exact. Thus

$$0 \to \text{coker } (-\varphi) \to B \otimes_A F_{tw}^{-1} \to \text{coker } (-\tau_{tw}^{-1}) \to 0$$

is a short exact sequence in $B\text{-GrMod}$, and hence

$$\text{coker } (-\varphi) \to B \otimes_A F_{tw}^{-1} \to \text{coker } (-\tau_{tw}^{-1}) \to (\text{coker } (-\varphi))[1]$$

is a distinguished triangle in $D^b_{sg}(B)$. Since $B \otimes_A F_{tw}^{-1}$ is graded free, the first two morphisms are zero. Rotating the triangle yields

$$\text{coker } (-\tau_{tw}^{-1}) \cong (\text{coker } (-\varphi))[1] \cong (\text{coker } \varphi)[1] = (F(\varphi, \tau))[1]$$

in $D^b_{sg}(B)$. Thus we have a natural isomorphism $(\mathcal{F}(\varphi, \tau))[1] \cong \mathcal{F}((\varphi, \tau)[1])$. That $\mathcal{F}$ takes a standard triangle in $hTMF_A(f)$ to a distinguished triangle in $D^b_{sg}(B)$ follows from this natural isomorphism, the fact that for a morphism $\Psi : (\varphi, \tau) \to (\varphi', \tau')$, $\Omega(C(\Psi))$ is the mapping cone of $\Omega(\Psi) : \Omega(\varphi, \tau) \to \Omega(\varphi', \tau')$, and the usual property of mapping cones fitting into long exact sequences in homology.

By Construction 2.7, $\mathcal{C}$ is surjective on objects of $\text{MCM}(B)$, hence the same is true of $\mathcal{F}$. Since $\mathcal{C}$ is full by Remark 5.3, $\mathcal{F}$ is as well.

To see that $\mathcal{F}$ is injective on objects, we show $G\mathcal{F}$ is. Suppose $G\mathcal{F}(\varphi, \tau) \cong 0$ in $D^b_{sg}(B)$. Then $M = \text{coker } \varphi$ admits a finite length graded free $B$-module resolution, so $\text{Ext}^i_B(M, N) = 0$ for all $N$ and all $i \gg 0$. By Proposition 2.9, $\Omega(\varphi, \tau)$ is a graded free $B$-module resolution. Thus for some $n$, $\text{Ext}^i_B(\text{coker}(1 \otimes \varphi^{tw^n}), N) = 0$ for all $N$ and all $i > 0$. That is, $\text{coker}(1 \otimes \varphi^{tw^n})$ is graded free. As noted in the proof of Proposition 2.9, $\text{coker}(1 \otimes \varphi^{tw^n}) \cong M^{tw}$. Since $M$ is free if and only if $M^{tw}$ is, $M$ is graded free. By Lemma 5.2, $(\varphi, \tau) \not\cong 0$ in $hTMF_A(f)$.

That $\mathcal{F}$ is faithful now follows from the triangulated structure (see Theorem 3.9 of [24]).
Zhang proves in [27] that, among other properties, being noetherian, AS regular or AS-Gorenstein is invariant under graded Morita equivalence. Thus, in the case where \( A \) is left noetherian and AS-regular, the equivalence theorems of Section 3 imply equivalences of the corresponding categories of singularities.

6. Examples

**Example 6.1.** Let \( V \) be a finite-dimensional vector space over a field \( k \) with skew-symmetric, nondegenerate form \( \omega \). Assume \( \dim V = 2n \geq 4 \) and let \( \mathfrak{h} \) be the corresponding Heisenberg Lie algebra and \( U(\mathfrak{h}) \) its universal enveloping algebra. Then \( U(\mathfrak{h}) \) can be presented by generators \( x_1, \ldots, x_n, y_1, \ldots, y_n \) subject to the relations

\[
[x_i, x_j] = [y_i, y_j] = 0 \\
[x_i, y_j] = 0 \text{ for } i \neq j \\
[x_1, y_1] = [x_2, y_2] = \cdots = [x_n, y_n]
\]

Since \( \mathfrak{h} \) is a finite-dimensional Lie algebra, \( U(\mathfrak{h}) \) is Artin-Schelter regular of dimension \( 2n + 1 \). The element \( f = \{x_1, y_1\} \) is central and regular, and \( B = U(\mathfrak{h})/(f) \cong k[x_1, \ldots, x_n, y_1, \ldots, y_n] \) is a commutative polynomial ring. By Hilbert’s Syzygy Theorem, every finitely generated left \( B \)-module has a finite minimal graded free resolution. Thus there exist no nontrivial reduced twisted left matrix factorizations of \( f \).

**Example 6.2.** Let \( A = k[x, y][w; \zeta] \) be a graded Ore extension of a commutative polynomial ring in two variables by a graded automorphism \( \zeta \), where \( wg = \zeta(g)w \) for all \( g \in k[x, y] \). Then \( w^2 \) is regular and its normalizing automorphism is \( \sigma = \zeta^{-2} \). After choosing bases, we define homomorphisms \( \varphi : F \to G \) and \( \tau : G^\sigma \to F \) of graded free left \( A \)-modules via right multiplication by the matrices

\[
[\varphi] = \begin{pmatrix} w & -\zeta(x) \\ 0 & w \end{pmatrix} \quad \text{and} \quad [\tau] = \begin{pmatrix} w & \zeta^2(x) \\ 0 & w \end{pmatrix}
\]

Note \([\varphi^\sigma] = \begin{pmatrix} w & -\zeta^3(x) \\ 0 & w \end{pmatrix}\). A straightforward verification shows that \( \varphi \tau = \lambda_w^2 \) and \( \tau \varphi^\sigma = \lambda_{w^2} \). (We remind the reader that since we work with left modules, the composition is computed by multiplying matrices in the opposite order.) Examining for periodicity, we see that the minimal resolution \( \Omega(\varphi, \tau) \) is periodic of period \( p \) if and only if \( \zeta^p(x) = cx \) for some integer \( p \) and scalar \( c \). This example suggests a useful method for constructing twisted factorizations with desired properties. For example, if \( \zeta(x) = x + y \) and \( \zeta(y) = qy \) where \( q \) is a primitive \( n \)-th root of unity, then the resolution is periodic of period \( n \).

As another example, taking \( \zeta(x) = (x + y)/2 \) and \( \zeta(y) = y/2 \) we obtain a resolution which is not periodic. But it is interesting to note that since \( \zeta^n(x) \to 0 \) as \( n \to \infty \), the limiting matrix \( \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix} \) defines a minimal resolution which is periodic of period 1. In this sense, the resolution becomes periodic after infinitely many steps.

In any case, extend \( \zeta \) to a graded automorphism of \( A \) by \( \zeta(w) = w \). Let \( Z = \{\zeta^n \mid n \in \mathbb{Z}\} \) be the associated twisting system. The twisted multiplication in \( A^Z \) gives

\[
g \ast w^2 = \zeta^2(g)w^2 = w^2 g = w^2 \ast g
\]
for all $g \in A$ so $w$ is central in $A^2$. By Proposition 2.12, every twisted matrix factorization of $w^2$ over $A^2$ gives rise to a minimal graded free resolution of period at most 2.

**Example 6.3.** Let $A = k\langle x, y, z \rangle / \langle r_1, r_2, r_3 \rangle$ where

$$r_1 = yz + yx - x^2$$
$$r_2 = zx + zx - y^2$$
$$r_3 = xy + yx - z^2$$

The algebra $A$ is a nondegenerate 3-dimensional Sklyanin algebra. The element $g = 2 \ast (y^3 + xyz - yzx - x^3)$ (the factor of 2 is only to clean up the twisted matrix factorization) is central and regular in $A$, so $\sigma = \text{id}_A$. Let

$$\varphi = \begin{pmatrix} x & y & z \\ -yz - 2x^2 - yx & xx - xz & x \\ xy - 2yx & xz - x^2 & y \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} -zy & -x & -z \\ zz - xx & z & y \\ xy & -y & x \\ 2xyz - 4x^3 - 2y^2 - 2(xz - yx) & -z & -z \end{pmatrix}$$

be matrices with entries in $A$. One can check (it is not trivial) that $\tau \varphi = \varphi \tau = g \text{id}_A$. This matrix factorization produces a minimal resolution of the second syzygy module in a minimal resolution of the trivial module $Bk$. Indeed, if we put

$$M_2 = \begin{pmatrix} -x & -z & y \\ z & y & x \\ -2x^2 - 2y^2 - 2(xy - yx) & -z \end{pmatrix}$$

then

$$\cdots \xrightarrow{\varphi} B(-5)^3 \oplus B(-6) \xrightarrow{\tau} B(-3) \oplus B(-4)^3 \xrightarrow{\varphi}$$
$$B(-2)^3 \oplus B(-3) \xrightarrow{M_2} B(-1)^3 \xrightarrow{M_1} B$$

is a minimal graded free left $B$-module resolution of $Bk$.

**Example 6.4.** Let $A = k_q[x, y]$ be the skew polynomial ring where $yx = qxy$ for some fixed $q \in k^\times$. Let $g$ be the graded automorphism of $A$ given by $g(x) = \lambda x$ and $g(y) = \lambda^{-1} y$ where $\lambda$ is a primitive $n$-th root of unity. Let $G = \langle g \rangle$, the cyclic group of order $n$, act on $A$ with invariant subring $A^G$. Classically (when $q = 1$), this is an $A_n$ Kleinian singularity. It is not hard to check that $A^G$ is generated by $X := x^n$, $Y := xy$, and $Z := y^n$, and $A^G \cong C/(\omega)$, where

$$C = k\langle X, Y, Z \rangle / \langle YX - q^nXY, ZX - q^nZX, ZY - q^nYZ \rangle$$

is a skew polynomial ring and $\omega := XZ - q^{-1/n}Y^n$ is a regular normal element of $C$. [13] Case 2.2]. Let $C$ be graded by setting $\deg X = \deg Z = n$, and $\deg Y = 2$. Note that $C$ is noetherian and AS-regular of dimension 3 and one has relations

$$\omega X = q^n X \omega, \quad \omega Y = Y \omega, \quad \omega Z = q^{-n^2} Z \omega.$$

The sets $M_j = \{ a \in A \mid g(a) = \lambda^j a \}$ for $0 \leq j < n$ are graded left $R = A^G$ modules, generated by $x^j$ and $y^{n-j}$. Note that $M_0 = R$, and henceforth assume $j \neq 0$.

As a module over $C$, a minimal resolution of $M_j$ has the form

$$0 \rightarrow C(-2n + j) \oplus C(-n - j) \xrightarrow{G_j} C(-j) \oplus C(-n + j) \xrightarrow{D} M_j \rightarrow 0,$$
where the maps are given by right multiplication by $G_j$:

$$D := \begin{pmatrix} x_j \\ y^{n-j} \end{pmatrix} \quad \text{and} \quad G_j := \begin{pmatrix} -q^{-(n-j)} Y^{n-j} & q^{(n-j)j} X \\ -Z & q^{nj-(\frac{j}{2})} Y_j \end{pmatrix}$$

Thus $pd_C M_j = 1$, and hence $M_j$ is a maximal Cohen-Macaulay $R$-module. It is worth noting that when $q = 1$, the $M_j$ form a complete set of maximal Cohen-Macaulay $R$-modules [21, Example 5.25].

Next, observe that $G_{n-j} G_j = \begin{pmatrix} -q^{-(n-j)}j \omega & 0 \\ 0 & -q^{(n-j)j + n^2} \omega \end{pmatrix} = G_j G_{n-j}$

This shows $0 \to M_{n-j}(-n) \xrightarrow{G_j} R(-j) \oplus R(-n+j) \xrightarrow{D_j} M_j \to 0$, where $G_j$ is the $R$-module map induced on coker $G_{n-j}$ by $G_j$, is an exact sequence of $R$-modules. So a minimal graded $R$-module resolution of $M_j$ is periodic of period at most 2 for every $0 < j < n$. (When $n = 2$, the resolution has period 1.) With a small adjustment, we obtain a complex arising from a twisted matrix factorization of $\omega$. Let

$$\Delta := \begin{pmatrix} -1 & 0 \\ 0 & -q^{n^2} \end{pmatrix}, \quad N_j := G_j \Delta^{-1} = \begin{pmatrix} -q^{-(n-j)} Y^{n-j} & -q^{(n-j)j + n^2} X \\ Z & -q^{nj-(\frac{j}{2})n^2} Y_j \end{pmatrix},$$

and $P_{n-j} := q^{-(n-j)} G_{n-j}$. Then we have

$$P_{n-j} := \begin{pmatrix} -q^{-(\frac{j}{2})j-(n-j)} Y_j \\ -q^{-(n-j)j} Z \\ q^{(n-j)^2-(n-j)} Y^{n-j} \end{pmatrix},$$

and

$$N_j^\sigma = \begin{pmatrix} -q^{-(n-j)} Y^{n-j} & -q^{(n-j)j} X \\ q^{n^2} Z & -q^{nj-(\frac{j}{2})n^2} Y_j \end{pmatrix}.$$

Finally we have $P_{n-j} N_j = \omega I = N_j^\sigma P_{n-j}$ as desired. We note that $|\sigma| = |q|$, which can be an arbitrary positive integer or infinite.

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