TIME-INCONSISTENT STOCHASTIC LINEAR–QUADRATIC CONTROL

YING HU†, HANQING JIN‡, AND XUN YU ZHOU§

Abstract. In this paper, we formulate a general time-inconsistent stochastic linear–quadratic (LQ) control problem. The time-inconsistency arises from the presence of a quadratic term of the expected state as well as a state-dependent term in the objective functional. We define an equilibrium, instead of optimal, solution within the class of open-loop controls, and derive a sufficient condition for equilibrium controls via a flow of forward–backward stochastic differential equations. When the state is one dimensional and the coefficients in the problem are all deterministic, we find an explicit equilibrium control. As an application, we then consider a mean–variance portfolio selection model in a complete financial market where the risk-free rate is a deterministic function of time but all the other market parameters are possibly stochastic processes. Applying the general sufficient condition, we obtain explicit equilibrium strategies when the risk premium is both deterministic and stochastic.

Key words. time-inconsistency, stochastic linear–quadratic control, equilibrium control, forward–backward stochastic differential equation, mean–variance portfolio selection

AMS subject classifications. 93E99, 60H10, 91B28

DOI. 10.1137/110853960

1. Introduction. Stochastic control is now a mature and well-established subject of study [10, 22]. Though not explicitly stated most of the time, a standing assumption in the study of stochastic control is the time-consistency, a fundamental property of conditional expectation with respect to a progressive filtration. As a result, an optimal control viewed from today will remain optimal when viewed from tomorrow. Time-consistency provides the theoretical foundation of the dynamic programming approach including the resulting Hamilton–Jacobi–Bellman (HJB) equation, which is in turn a pillar of modern stochastic control theory.

However, there are overwhelmingly more time-inconsistent problems than their time-consistent counterparts. Hyperbolic discounting [1, 17] and a continuous-time mean-variance portfolio selection model [23, 3] provide two well-known examples of time-inconsistency. Probability distortion, as in behavioral finance models [13], is yet another distinctive source of time-inconsistency.

One way to get around the time-inconsistency issue is to consider only pre-committed controls (i.e., the controls are optimal only when viewed at the initial

---

*Received by the editors November 4, 2011; accepted for publication (in revised form) May 9, 2012; published electronically June 19, 2012.

†IRMAR, Université Rennes 1, 35042 Rennes Cedex, France (ying.hu@univ-rennes1.fr). The research of this author was partially supported by the Marie Curie ITN grant, “Controlled Systems,” GA 213841/2008.

‡Mathematical Institute and Nomura Centre for Mathematical Finance, and Oxford–Man Institute of Quantitative Finance, The University of Oxford, Oxford OX1 3LB, UK (jin@maths.ox.ac.uk). The research of this author was partially supported by research grants from the Nomura Centre for Mathematical Finance and the Oxford–Man Institute of Quantitative Finance.

§Mathematical Institute and Nomura Centre for Mathematical Finance, and Oxford–Man Institute of Quantitative Finance, The University of Oxford, Oxford OX1 3LB, UK, and Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong (zhou@maths.ox.ac.uk). The research of this author was supported by a start-up fund of the University of Oxford, research grants from the Nomura Centre for Mathematical Finance and the Oxford–Man Institute of Quantitative Finance, and GRF grant CUHK419511.
time); see, e.g., [23] and nearly all of the follow-up work to date on the Markowitz problem, as well as [13], on the behavioral portfolio choice problem. While these controls are of practical and theoretical value, they have not really addressed the time-inconsistency nor provided solutions in a dynamic sense.

Motivated by practical applications, especially in mathematical finance, time-inconsistent control problems have recently attracted considerable research interest and efforts attempting to seek equilibrium, instead of optimal, controls. At a conceptual level, the idea is that a decision made by the controller at every instant of time is considered as a game against all the decisions made by future incarnations of the controller. An “equilibrium” control is therefore one such that any deviation from it at any time instant will be worse off. Taking this game perspective, Ekeland and Lazrak [8] approach the (deterministic) time-inconsistent optimal control, and Björk and Murgoci [5] and Björk, Murgoci, and Zhou [6] extend the idea to the stochastic setting, derive an (albeit very complicated) HJB equation, and apply the theory to a dynamic Markowitz problem. Yong [21] investigates a time-inconsistent deterministic linear–quadratic (LQ) control problem and derive equilibrium controls via some integral equations. However, study of time-inconsistent control is, in general, still in its infancy.

In this paper we formulate a general stochastic LQ control problem, where the objective functional includes both a quadratic term of the expected state and a state-dependent term. Each of these nonstandard terms introduces time-inconsistency into the problem in somewhat different ways. In contrast to most of the existing literature [8, 5, 6, 21], where an equilibrium control is defined within the class of feedback controls, we define our equilibrium via open-loop controls.\(^1\) Then we derive a general sufficient condition for equilibriums through a system of forward–backward stochastic differential equations (FBSDEs). An intriguing feature of these FBSDEs is that a time parameter is involved; thus these form a flow of FBSDEs. When the state process is scalar valued and all the coefficients are deterministic functions of time, we are able to reduce this flow of FBSDEs into several Riccati-like ordinary differential equations (ODEs), and hence obtain explicitly an equilibrium control, which turns out to be a linear feedback.

In the latter part of the paper, we study a continuous-time mean–variance portfolio selection model with state-dependent trade-off between mean and variance. A similar problem was first considered in [6] in the framework of feedback controls, and its solution was derived via a very complicated (generalized) HJB equation. Here we allow random market parameters (hence the model and approach of [6] will not work) and consider open-loop equilibriums. Applying the general sufficient condition and working through a delicate analysis, we will solve the corresponding FBSDEs and obtain equilibrium strategies. Again, these strategies happen to be linear feedbacks. We also compare our strategies with those ones in [6], when all the market coefficients are deterministic, and find that they are generally different. This suggests that how we define equilibrium controls is critical in studying time-inconsistent control problems.

The remainder of the paper is organized as follows. The next section is devoted to the formulation of our problem and the definition of equilibrium control. In section 3, we apply the spike variation technique to derive a flow of FBSDEs and a sufficient

\(^1\)Recall that the class of feedback controls is a subset of the class of open-loop controls. In standard (time-consistent) stochastic control theory, an optimal control is usually defined in the whole class of open-loops [10, 22]. Only under some conditions—foremost the system dynamic being Markovian—does an optimal control turn out to be a feedback control.
condition of equilibrium controls. Based on this general result, we solve in section 4 the case when the state is one dimensional and all the coefficients are deterministic.

In section 5, we formulate a continuous-time mean–variance portfolio selection model, which is a special case of the general LQ model investigated, and derive explicitly its solution. Finally, some concluding remarks are given in section 6.

2. Problem setting. Let $T > 0$ be the end of a finite time horizon, and let $(W_t)_{0 \leq t \leq T} = (W^1_t, \cdots, W^d_t)_{0 \leq t \leq T}$ be a $d$-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Denote by $(\mathcal{F}_t)$ the augmented filtration generated by $(W_t)$.

Throughout this paper, we use the following notation, with $l$ being a generic integer:

- $S^l$: the set of symmetric $l \times l$ real matrices.
- $L^2_G(\Omega; \mathbb{R}^l)$: the set of random variables $\xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$ with $\mathbb{E} \left[ |\xi|^2 \right] < +\infty$.
- $L^c_G(\Omega; \mathbb{R}^l)$: the set of essentially bounded random variables $\xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$.
- $L^2_G(t, T; \mathbb{R}^l)$: the set of $\{\mathcal{G}_s\}_{s \in [t, T]}$-adapted processes $f = \{f_s : t \leq s \leq T\}$ with $\mathbb{E} \left[ \int_t^T |f_s|^2 \, ds \right] < \infty$.
- $L^c_G(t, T; \mathbb{R}^l)$: the set of essentially bounded $\{\mathcal{G}_s\}_{s \in [t, T]}$-adapted processes.
- $L^2_G(\Omega; C(t, T; \mathbb{R}^l))$: the set of continuous $\{\mathcal{G}_t\}_{t \in [0, T]}$-adapted processes $f = \{f_s : t \leq s \leq T\}$ with $\mathbb{E} \left[ \sup_{s \in [t, T]} |f_s|^2 \right] < \infty$.

We will often use vectors and matrices in this paper, where all vectors are column vectors. For a matrix $M$, define

$$|M| = \sqrt{\sum_{i,j} m_{ij}^2} : \text{ Frobenius norm of a matrix } M.$$ 

For a square matrix $M$, we define $S(M) = \frac{1}{2}(M + M')$ as the symmetrization of $M$ and define $\text{tr}(M) = \sum_i M_{ii}$ as the trace of $M$. For a symmetric matrix $M$, we write $M \succeq 0$ if $M$ is positive semidefinite and write $M \succ 0$ if $M$ is positive definite.

We consider a continuous-time, $n$-dimensional nonhomogeneous linear controlled system

$$dX_s = [A_sX_s + B^u_su_s + b_s]ds + \sum_{j=1}^d [C_{s}^jX_s + D_j^su_s + \sigma_{s}^j]dW^j_s, \quad X_0 = x_0.$$  

(2.1)

Here $A$ is a bounded deterministic function on $[0, T]$ with value in $\mathbb{R}^{n \times n}$. The other parameters $B, C_j, D_j$ are all essentially bounded adapted processes on $[0, T]$ with values in $\mathbb{R}^{l \times n}$, $\mathbb{R}^{n \times l}$, respectively; $b$ and $\sigma_j$ are stochastic processes in $L^2_G(0, T; \mathbb{R}^n)$. The process $u \in L^2_G(0, T; \mathbb{R}^l)$ is the control, and $X$ is the state process valued in $\mathbb{R}^n$. Finally $x_0 \in \mathbb{R}^n$ is the initial state. It is obvious that for any control $u \in L^2_G(0, T; \mathbb{R}^l)$, there exists a unique solution $X \in L^2_G(\Omega; C(0, T; \mathbb{R}^n))$.

As time evolves, we need to consider the controlled system starting from time $t \in [0, T]$ and state $x_t \in L^2_G(\Omega; \mathbb{R}^n)$:

$$dX_s = [A_sX_s + B^u_su_s + b_s]ds + \sum_{j=1}^d [C_{s}^jX_s + D_j^su_s + \sigma_{s}^j]dW^j_s, \quad X_t = x_t.$$  

(2.2)

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
For any control $u \in L^2_F(t, T; \mathbb{R}^l)$, there exists a unique solution $X^{t, x, u} \in L^2_F(\Omega; C(t, T; \mathbb{R}^n))$.

At any time $t$ with the system state $X_t = x_t$, our aim is to minimize

$$J(t, x_t; u) \triangleq \frac{1}{2} \mathbb{E}_t \int_t^T ([Q_s X_s, X_s] + \langle R_s u_s, u_s \rangle) ds + \frac{1}{2} \mathbb{E}_t ([G X_T, X_T])$$

(2.3)

$$\frac{1}{2} \langle h \mathbb{E}_t [X_T], \mathbb{E}_t [X_T] \rangle - \langle \mu_1 x_t + \mu_2, \mathbb{E}_t [X_T] \rangle$$

over $u \in L^2_F(t, T; \mathbb{R}^l)$, where $X = X^{t, x, u}$, and $\mathbb{E}_t [\cdot] = \mathbb{E} [\cdot | \mathcal{F}_t]$. Here $Q$ and $R$ are both given essentially bounded adapted processes on $[0, T]$ with values in $\mathbb{S}^n$ and $\mathbb{S}^l$, respectively, $G, h, \mu_1, \mu_2$ are all constants in $\mathbb{S}^n, \mathbb{S}^l, \mathbb{R}^{n \times n}$, and $\mathbb{R}^n$, respectively. Throughout this paper, we assume that $-Q \succeq 0, R \succeq 0$ a.s., a.e., and $G \succeq 0$.

The first two terms in the cost functional (2.3) are standard in a classical LQ control problem, whereas the last two are unconventional. Specifically, the term $\frac{1}{2} \langle h \mathbb{E}_t [X_T], \mathbb{E}_t [X_T] \rangle$ is motivated by the variance term in a mean–variance portfolio choice model [11, 23], and the last term, $\langle \mu_1 x_t + \mu_2, \mathbb{E}_t [X_T] \rangle$, which depends on the state $x_t$ at time $t$, stems from a state-dependent utility function in economics [6].

Each of these two terms introduces time-inconsistency of the underlying model in somewhat different ways. With the time-inconsistency, the notion “optimality” needs to be defined in an appropriate way. Here we adopt the concept of equilibrium solution, which is, for any $t \in [0, T)$, optimal “infinitesimally” via spike variation.

Given a control $u^*$. For any $t \in (0, T)$, $\varepsilon > 0$, and $v \in L^2_F(\Omega; \mathbb{R}^l)$, define

$$u^{l, \varepsilon, v}_s = u^*_s + \varepsilon \mathbb{1}_{s \in [t, t + \varepsilon)}, \quad s \in [t, T].$$

**Definition 2.1.** Let $u^* \in L^2_F(0, T; \mathbb{R}^l)$ be a given control, and let $X^*$ be the state process corresponding to $u^*$. The control $u^*$ is called an equilibrium if

$$\lim_{\varepsilon \downarrow 0} \frac{J(t, X^*_t; u^{l, \varepsilon, v}_t) - J(t, X^*_t; u^*)}{\varepsilon} \geq 0,$$

where $u^{l, \varepsilon, v}$ is defined by (2.4), for any $t \in [0, T)$ and $v \in L^2_F(\Omega; \mathbb{R}^l)$.

The intuition behind this definition is similar to that in [8]. The controller at any time $t$ is playing a game against all his incarnations in the future. He can commit only for an infinitesimal time $\varepsilon$, so he can only hope to optimize in $[t, t + \varepsilon]$.

However, there is a critical difference between the definition here and those in [3, 5, 6, 8, 9]. An equilibrium control here is defined in the class of open-loop controls, whereas in the existing works only (Markovian) feedback controls are considered. In our definition, the perturbation of the control in $[t, t + \varepsilon)$ will not change the control process in $[t + \varepsilon, T]$, which is not the case with feedback controls. When the system is not Markovian, our definition applies well but the one by feedback control becomes unjustified.

In this paper, we will characterize equilibriums in the general case and identify them in some special cases including that of the mean–variance portfolio selection.

**3. Sufficient condition of equilibrium controls.** In this section we present a general sufficient condition for equilibriums. We derive this condition by the second-order expansion in the spike variation (we refer to [20] for the application of spike variation in optimal control theory), in the same spirit as that of proving the stochastic Pontryagin’s maximum principle [19, 22]. Again, since we do not assume the
Markovian property of the system, we are unable to follow the dynamic programming in the study of this problem as in the existing literature.

Let \( u^* \) be a fixed control, and let \( X^* \) be the corresponding state process. For any \( t \in [0, T] \), define in the time interval \([t, T]\) the processes \((p(\cdot; t), (k^j(\cdot; t))_{j=1, \ldots, d}) \in L^2_F(t, T; \mathbb{R}^n) \times (L^2_F(t, T; \mathbb{R}^n)^d) \) and \((P(\cdot; t), (K^j(\cdot; t))_{j=1, \ldots, d}) \in L^2_F(t, T; \mathbb{S}^n) \times (L^2_F(t, T; \mathbb{S}^n)^d)\) as the solutions to the following equations:

\[
\begin{align*}
\int_t^T dp(s; t) &= -[A^*_t p(s; t) + \sum_{j=1}^d (C^j_t)' k^j(s; t) + Q_s X^*_s] ds \\
&\quad + \sum_{j=1}^d k^j(s; t) dW^j_s, \quad s \in [t, T], \\
p(T; t) &= GX^*_T - \mu_1 X^*_T - \mu_2; \\
\int_t^T dP(s; t) &= -\left\{ A^*_t P(s; t) + P(s; t) A_s + Q_s \\
&\quad + \sum_{j=1}^d [(C^j_t)' P(s; t) C^j_t + (C^j_t)' K^j(s; t) + K^j(s; t) C^j_t] \right\} ds \\
&\quad + \sum_{j=1}^d K^j(s; t) dW^j_s, \quad s \in [t, T], \\
P(T; t) &= G.
\end{align*}
\]

Note that for each fixed \( t \in [0, T] \), the above equations are backward stochastic differential equations (BSDEs). So these essentially form a flow of BSDEs. From the assumption that \( Q \geq 0 \) and \( G \geq 0 \), it follows that \( P(s; t) \geq 0 \).

**Proposition 3.1.** For any \( t \in [0, T] \), \( \varepsilon > 0 \), and \( v \in L^2_F(\Omega; \mathbb{R}^d) \), define \( u^{t, \varepsilon, v} \) by \((2.4)\). Then

\[
\begin{align*}
J(t, X^*_t; u^{t, \varepsilon, v}) - J(t, X^*_t; u^*) &= \mathbb{E}_t \int_t^{t+\varepsilon} \left\{ \langle \Lambda(s; t), v \rangle + \frac{1}{2} \langle H(s; t) v, v \rangle \right\} ds \\
&\quad + o(\varepsilon),
\end{align*}
\]

where \( \Lambda(s; t) \triangleq B_s p(s; t) + \sum_{j=1}^d (D^j_s)' k^j(s; t) + R_s u^*_s \) and \( H(s; t) \triangleq R_s + \sum_{j=1}^d (D^j_s)' P(s; t) D^j_s \).

**Proof.** Let \( X^{t, \varepsilon, v} \) be the state process corresponding to \( u^{t, \varepsilon, v} \). Then by the standard perturbation approach (see, e.g., [22, section 4.2, pp. 126–128]), we have

\[
X^{t, \varepsilon, v}_s = X^*_s + Y^{t, \varepsilon, v}_s + Z^{t, \varepsilon, v}_s, \quad s \in [t, T],
\]

where \( Y \equiv Y^{t, \varepsilon, v} \) and \( Z \equiv Z^{t, \varepsilon, v} \) satisfy

\[
\begin{align*}
\left\{ dY_s &= A_s Y_s ds + \sum_{j=1}^d [C^j_s Y_s + D^j_s v 1_{s \in [t, t+\varepsilon]}] dW^j_s, \quad s \in [t, T], \\
Y_t &= 0; \\
\right. \\
\left\{ dZ_s &= [A_s Z_s + B_s v 1_{s \in [t, t+\varepsilon]}] ds + \sum_{j=1}^d C^j_s Z_s dW^j_s, \quad s \in [t, T], \\
Z_t &= 0.
\right.
\]

Moreover, by Theorem 4.4 in [22], we have

\[
\mathbb{E}_t \left[ \sup_{s \in [t, T]} |Y_s|^2 \right] = O(\varepsilon), \quad \mathbb{E}_t \left[ \sup_{s \in [t, T]} |Z_s|^2 \right] = O(\varepsilon^2).
\]

With \( A_s \) being deterministic, it follows from the dynamics of \( Y \) that \( \mathbb{E}_t [Y_s] = \int_t^s \mathbb{E}_u [A_s Y_u] d\nu = \int_t^s A_s \mathbb{E}_u [Y_u] d\nu \forall s \in [t, T] \). Hence we conclude that

\[
\mathbb{E}_t [Y_s] = 0 \forall s \in [t, T].
\]
By these estimates, we can calculate

$$2[J(t, X^*_t; u^{t,ε}, v) - J(t, X^*_t, u^*)]$$

$$= E_t \int_t^T \left[ (Q_s(2X^*_s + Y_s + Z_s), Y_s + Z_s) + (R_s(2u^*_s + v), v)1_{s \in [t, t+ε]} \right] ds$$

$$+ 2E_t [(G X^*_T, Y_T + Z_T)] + E_t [(G(Y_T + Z_T), Y_T + Z_T)]$$

$$- 2(hE_t [X^*_T] + μ_1 X^*_T + μ_2, E_t [Y_T + Z_T]) - \langle hE_t [Y_T + Z_T], E_t [Y_T + Z_T] \rangle$$

$$= E_t \int_t^T \left[ (Q_s(2X^*_s + Y_s + Z_s), Y_s + Z_s) + (R_s(2u^*_s + v), v)1_{s \in [t, t+ε]} \right] ds$$

$$+ E_t [2(GX^*_T - hE_t [X^*_T] - μ_1 X^*_T - μ_2, Y_T + Z_T) + (G(Y_T + Z_T), Y_T + Z_T)] + o(ε).$$

Recalling that $(p(·; t), k(·; t))$ and $(P(·; t), K(·; t))$ solve, respectively, (3.1) and (3.2), we have

$$E_t [(GX^*_T - hE_t [X^*_T] - μ_1 X^*_T - μ_2, Y_T + Z_T)]$$

$$= E_t \int_t^T \left\{ p(s; t), A_s(Y_s + Z_s) + B^*_s v1_{s \in [t, t+ε]} \right\}$$

$$- \left\langle A^*_s p(s; t) + \sum_{j=1}^d (C^*_s)^j k^j(s; t) + Q_s X^*_s, Y_s + Z_s \right\rangle$$

$$+ \sum_{j=1}^d (k^j(s; t), C^*_s(Y_s + Z_s) + D^*_s v1_{s \in [t, t+ε]} \right) ds$$

$$= E_t \int_t^T \left[ -(Q_s X^*_s, Y_s + Z_s) + \langle B_s p(s; t) + \sum_{j=1}^d (D^*_s)^j k^j(s; t), v1_{s \in [t, t+ε]} \right] ds$$

and

$$E_t [(G(Y_T + Z_T), Y_T + Z_T)]$$

$$= E_t \int_0^T \left[ -(Q_s(Y_s + Z_s), Y_s + Z_s) + \sum_{j=1}^d (D^*_s)^j P(s; t) D_s v, v1_{s \in [t, t+ε]} \right] ds + o(ε).$$

This proves (3.3). □

It follows from $R ≥ 0$ and $P(s; t) ≥ 0$ that $H(s; t) ≥ 0$. In view of (3.3), a sufficient condition for an equilibrium is

$$(3.4) \quad E_t \int_t^T |\Lambda(s; t)| ds < +∞, \quad \lim_{s \uparrow t} E_t [\Lambda(s; t)] = 0 \text{ a.s. } \forall t \in [0, T].$$

Under some condition, the second equality in (3.4) is ensured by

$$(3.5) \quad R_t u^*_t + B_t p(t; t) + \sum_{j=1}^d (D^*_t)^j k^j(t; t) = 0 \text{ a.s. } \forall t \in [0, T].$$

The following is the main general result for the time-inconsistent stochastic LQ control.

**Theorem 3.2.** A control $u^* \in L^2_T (0, T; \mathbb{R}^l)$ is an equilibrium control if the following two conditions hold for any time $t$:
(i) The system of SDEs

\[
\begin{aligned}
    dX_t^* &= [A_s X_t^* + B^*_s u^*_s + b_s]ds + \sum_{j=1}^d [C^*_s X_t^* + D^*_s u^*_s + \sigma^*_s]dW^j_t, \quad s \in [0, T], \\
    X_0^* &= x_0, \\
    dp(s; t) &= -[A'_s p(s; t) + \sum_{j=1}^d (C'_s)^j k_j(s; t) + Q_s X^*_s]ds + \sum_{j=1}^d k_j(s; t)dW^j_t, \quad s \in [t, T], \\
    p(T; t) &= G X^*_T - h E_t[X^*_T] - \mu_1 X^*_T - \mu_2
\end{aligned}
\]

admits a solution \((X^*, p, k)\);

(ii) \(\Lambda(\cdot; t) \triangleq B p(\cdot; t) + \sum_{j=1}^d (D^j)' k_j(\cdot; t) j + R u^*\) satisfies condition (3.4).

Proof. Given a control \(u^* \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)\) satisfying (i) and (ii), for any \(v \in L^2_{\mathcal{F}}(\Omega; \mathbb{R}^l)\), define \(\Lambda\) and \(H\) as in Proposition 3.1. Then

\[
\lim_{\varepsilon \downarrow 0} \frac{J(t, X^*_t; u^{t, \varepsilon}) - J(t, X^*_t; u^*)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_t \int^t_{t+\varepsilon} \{\Lambda(s; t), v\} ds}{\varepsilon} \\
\geq \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}_t \int^t_{t+\varepsilon} \{\Lambda(s; t), v\} ds}{\varepsilon} \\
\geq 0,
\]

proving the result.  \(\square\)

Theorem 3.2 involves the existence of solutions to a flow of FBSDEs along with other conditions. Proving the general existence for this type of FBSDEs remains an outstanding open problem. In the rest of this paper we will focus on the case when \(n = 1\). This case is important especially in financial applications, as will be demonstrated by the mean–variance portfolio selection model.

When \(n = 1\), the state process \(X\) is a scalar-valued process evolving by the dynamics

\[
\begin{aligned}
    dX_t &= [A_s X_t + B^*_s u^*_s + b_s]ds + [C_s X_t + D_s u^*_s + \sigma_s]dW_t, \quad X_0 = x_0,
\end{aligned}
\]

where \(A\) is a bounded deterministic scalar function on \([0, T]\). The other parameters \(B, C, D\) are all essentially bounded and \(\mathcal{F}_t\)-adapted processes on \([0, T]\) with values in \(\mathbb{R}^l, \mathbb{R}^d, \mathbb{R}^{d \times d}\), respectively. Moreover, \(b \in L^2_{\mathcal{F}}(0, T; \mathbb{R})\) and \(\sigma \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)\).

In this case, the two adjoint equations for the equilibrium become

\[
\begin{aligned}
    dp(s; t) &= -[A_s p(s; t) + C^*_s k(s; t) + Q_s X^*_s]ds + k(s; t)'dW_s, \quad s \in [t, T], \\
    p(T; t) &= G X^*_T - h E_t[X^*_T] - \mu_1 X^*_T - \mu_2 \\
    dP(s; t) &= -[(2A_s + [C^*_s]^2)^2 P(s; t) + 2C'_s K(s; t) + Q_s]ds \\
    K(s; t)'dW_s, \quad s \in [t, T], \\
    P(T; t) &= G.
\end{aligned}
\]

For the reader’s convenience, we state here the \(n = 1\) version of Theorem 3.2.

**THEOREM 3.3.** An admissible control \(u^* \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^l)\) is an equilibrium control if, for any time \(t \in [0, T)\),

(i) the system of SDEs

\[
\begin{aligned}
    dX^*_t &= [A_s X^*_t + B^*_s u^*_s + b_s]ds + [C_s X^*_t + D_s u^*_s + \sigma_s]dW_t, \quad s \in [0, T], \\
    X^*_0 &= x_0, \\
    dp(s; t) &= -[A_s p(s; t) + C'_s k(s; t) + Q_s X^*_s]ds + k(s; t)'dW_s, \quad s \in [t, T], \\
    p(T; t) &= G X^*_T - h E_t[X^*_T] - \mu_1 X^*_T - \mu_2, \quad t \in [0, T],
\end{aligned}
\]
admits a solution \((X^*, p, k)\):

(ii) \(\Lambda(\cdot; t) \triangleq p(\cdot; t)B_\cdot + D'k(\cdot; t) + R. u^*\) satisfies the condition (3.4).

4. Equilibrium when coefficients are deterministic. Theorem 3.3 shows that one can obtain equilibrium controls by solving the system of FBSDEs (3.10). However, the FBSDEs in (3.10) are not standard since a “flow” of unknowns \((p(\cdot; t), k(\cdot; t))\) is involved. Moreover, there is an additional constraint (3.4) which, under some condition, boils down to an algebraic constraint (3.5) that acts on the “diagonal” (i.e., when \(s = t\)) of the flow. The unique solvability of this type of equations remains a challenging open problem even for the case \(n = 1\). However, we are able to solve this problem quite thoroughly when the parameters \(A, B, C, D, b, \sigma, Q,\) and \(R\) are all deterministic functions.

Throughout this section we assume all the parameters are deterministic functions of \(t\). In this case, since \(G\) has been also assumed to be deterministic, the BSDE (3.9) turns out to be an ODE with solution \(K \equiv 0\) and

\[
P(s; t) = Ge^{\int_s^t (2A_s + C_u^2) du + \int_s^T e^{\int_u^T (2A_v + C_v^2) dv} Q_v dv}.
\]

4.1. An ansatz. As in classical LQ control (see, e.g., [22]), we attempt to look for a linear feedback equilibrium. For this, given any \(t \in [0, T]\), we consider the following Ansatz:

\[
p(s; t) = M_sX_s^* - N_sE_t[X_s^*] - \Gamma_s^{(1)}X_s^* + \Phi_s, \quad 0 \leq t \leq s \leq T,
\]

where \(M, N, \Gamma^{(1)}, \Phi\) are deterministic differentiable functions with \(\dot{M} = m, \dot{N} = n, \dot{\Gamma}^{(1)} = \gamma^{(1)},\) and \(\Phi = \phi\). The advantage of this Ansatz is to separate the variables \(X_s^*, E_t[X_s^*]\) and \(X_s^*\) in the solution \(p(s; t)\), thereby reducing the complicated FBSDEs to four ODEs.

For any fixed \(t\), applying Ito’s formula to (4.1) in the time variable \(s\), we get

\[
dp(s; t) = \{M_s(A_s X_s^* + B'_s u^*_s + b_s) + m_s X_s^* - N_sE_t[A_s X_s^* + B'_s u^*_s + b_s]
- n_s E_t[X_s^*] - \gamma^{(1)} X_s^* + \phi_s\} ds + M_s(C_s X_s^* + D_s u^*_s + \sigma_s)'dW_s.
\]

Comparing the \(dW_s\) term with the \(dW_s\) term of \(dp(s; t)\) in (3.10), we obtain

\[
k(s; t) = M_s(C_s X_s^* + D_s u^*_s + \sigma_s), \quad s \in [t, T].
\]

Notice that \(k(s; t)\) turns out to be independent of \(t\).

Now we ignore the difference between the conditions (3.4) and (3.5) and put the above expressions of \(p(s; t)\) and \(k(s; t)\) into (3.5). Then we have

\[
[(M_s - N_s - \Gamma^{(1)}_s)X_s^* + \Phi_s]B_s + M_s D'_s[C_s X_s^* + D_s u^*_s + \sigma_s] + R_s u^*_s = 0, \quad s \in [0, T],
\]

from which we formally deduce

\[
u^*_s = \alpha_s X_s^* + \beta_s,
\]

where

\[
\alpha_s \triangleq -(R_s + M_s D'_s D_s)^{-1}[(M_s - N_s - \Gamma^{(1)}_s)B_s + M_s D'_s C_s],
\]

\[
\beta_s \triangleq -(R_s + M_s D'_s D_s)^{-1}(\Phi_s B_s + M_s D'_s \sigma_s).
\]
Next, comparing the $ds$ term in (4.2) with that in (3.10) (we suppress the argument $s$ here), we obtain

\[0 = mX^* + M(AX^* + B'u^* + b) - n\mathbb{E}_t[X^*] - N(AE_t[X^*] + B'E_t[u^*] + b) - \gamma^{(1)}X_t^* + \phi + AMX^* - AN\mathbb{E}_t[X^*] - A\Gamma^{(1)}X_t^* + A\Phi + MC'[CX^* + Du^* + \sigma] + QX^* \]

\[= [m + 2MA + M|C|^2 + Q + (MB' + MC'D)\alpha]X^* - [n + 2NA + NB'\alpha]\mathbb{E}_t[X^*] - (\gamma^{(1)} + A\Gamma^{(1)})X_t^* + [(M - N)(B'\beta + b) + \phi + A\Phi + MC'(D\beta + \sigma)].\]

Notice in the above that $X^* \equiv X_t^*$ and $\mathbb{E}_t[X^*] \equiv \mathbb{E}_t[X_t^*]$ due to the omission of $s$. This leads to the following equations for $M,N,\Gamma^{(1)},\Phi$ (again the argument $s$ is suppressed):

\[
\begin{aligned}
0 &= \dot{M} + (2A + |C|^2)M + Q \\
- M(B' + C'D)(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C], & s \in [0,T], \\
M_T &= G; \\
0 &= \dot{N} + 2AN \\
- NB'(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C], & s \in [0,T], \\
N_T &= h; \\
\Gamma^{(1)} &= -A\Gamma^{(1)}, & s \in [0,T], \\
\end{aligned}
\]

\[
\begin{aligned}
0 &= \dot{\Phi} + \{A - [(M - N)B' + MC'D](R + MD'D)^{-1}B\}\Phi + (M - N)b \\
+ C'M\sigma - [(M - N)B' + MC'D](R + MD'D)^{-1}MD'\sigma, & s \in [0,T], \\
\Phi_T &= -\mu_2.
\end{aligned}
\]

The solution to (4.7) is $\Gamma^{(1)}_s = \mu_1e^{\int_0^t A_{sd}dt}$. Equations (4.5) and (4.6) form a system of coupled Riccati equations for $(M,N)$

\[
\begin{aligned}
\dot{M} &= -[2A + |C|^2 + \Gamma^{(1)}B'(R + MD'D)^{-1}(B + D'C)]M - Q \\
+ (B + D'C)'(R + MD'D)^{-1}(B + D'C)M^2 \\
- B'(R + MD'D)^{-1}(B + D'C)MN, \\
M_T &= G; \\
\dot{N} &= -[2A + \Gamma^{(1)}B'(R + MD'D)^{-1}B]N \\
+ B'(R + MD'D)^{-1}(B + D'C)MN - B'(R + MD'D)^{-1}BN^2, \\
N_T &= h.
\end{aligned}
\]

Finally, once we get the solution for $(M,N)$, (4.8) is a simple ODE. Therefore, it is crucial to solve (4.9), which will be carried out in the next subsection.

\[^2\text{Strictly speaking, these are not Riccati equations in the usual sense as they are not symmetric. However, we still use the term so as to see the connection and difference between time-inconsistent and time-consistent LQ control problems.}\]
4.2. Solution to Riccati system (4.10). Formally, we define $J = \frac{M}{M}$ and study the following equation for $(M, J)$:

$$
\begin{aligned}
\dot{M} &= -\left[2A + |C|^2 + \Gamma(1) B'(R + M^+ D' D)^{-1}(B + D' C)\right] M - Q \\
&\quad + (B + D' C)'(R + M^+ D' D)^{-1}(B + D' C)M^2 \\
&\quad - B'(R + M^+ D' D)^{-1}(B + D' C)\frac{\dot{J}}{M}, \\
M_T &= G; \\
\dot{J} &= -|C|^2 - C' D(R + M^+ D' D)^{-1}(B + D' C)M \\
&\quad + \Gamma(1) B'(R + M^+ D' D)^{-1}D' C(M + \lambda), \\
J_T &= \frac{G}{\lambda},
\end{aligned}
$$

(4.10)

Proposition 4.1. If the system (4.10) admits a positive solution pair $(M, J)$, then the system (4.9) admits a positive solution pair $(M, J)$.

Proof. The proof is straightforward. □

In the following two subsections, we will study the system (4.10) for two cases. The main technique is the truncation method. This method involves “truncation functions” $\cdot \lor c$ for a small number $c > 0$ and $\cdot \land K$ for a big number $K$.

4.2.1. Standard case. We first consider the standard case where $R - \delta I \succeq 0$ for some $\delta > 0$.

Theorem 4.2. Assume that $R - \delta I \succeq 0$ for some $\delta > 0$ and $G \succeq h > 0$. Then (4.10) and (4.9) admit unique positive solution pairs if $Q + \Gamma(1) S(D' C B') \succeq 0$, and either (i) there exists a constant $\lambda \geq 0$ such that $B = \lambda D' C$, or (ii) $D' D - \delta I \succeq 0$ for some $\delta > 0$.

Proof. For fixed $c > 0$ and $K > 0$, consider the following truncated system of (4.10):

$$
\begin{aligned}
\dot{M} &= -\left[2A + |C|^2 + \Gamma(1) B'(R + M^+ D' D)^{-1}(B + D' C)\right] M - Q \\
&\quad + (B + D' C)'(R + M^+ D' D)^{-1}(B + D' C)M(M^+ \land K) \\
&\quad - B'(R + M^+ D' D)^{-1}(B + D' C)\frac{\dot{J}}{M^+ \land K}, \\
M_T &= G; \\
\dot{J} &= -\lambda(1) J - B'(R + M^+ D' D)^{-1}D' C(M^+ \land K), \\
J_T &= \frac{G}{\lambda},
\end{aligned}
$$

(4.11)

where $M^+ = \max\{M, 0\}$ and

$$
\lambda(1) \triangleq |C|^2 - C' D(R + M^+ D' D)^{-1}(B + D' C)(M^+ \land K) \\
&\quad + \Gamma(1) B'(R + M^+ D' D)^{-1}D' C + \frac{Q}{M^+ \lor c}.
$$

Since $R - \delta I \succeq 0$, the above system is locally Lipschitz with linear growth, and hence it admits a unique solution $(M_0, J_0)$. We omit the superscript $(c, K)$ when no confusion arises.

We are going to prove that $J \geq 1$ and that $M \in [\eta, L]$ for some $\eta > 0$ and $L > 0$ independent of $c$ and $K$ appearing in the truncation functions. To this end, denote

$$
\lambda(2) = (2A + |C|^2 + \Gamma(1) B'(R + M^+ D' D)^{-1}(B + D' C)) \\
&\quad - (B + D' C)'(R + M^+ D' D)^{-1}(B + D' C)(M^+ \land K) \\
&\quad + B'(R + M^+ D' D)^{-1}(B + D' C)\frac{M^+ \land K}{J \lor c}.
$$

Copyright © by SIAM. Unauthorized reproduction of this article is prohibited.
Then $\lambda^{(2)}$ is bounded, and $M$ satisfies

\begin{equation}
\dot{M} + \lambda^{(2)}M + Q = 0, \ M_T = G.
\end{equation}

Hence $M > 0$. As a result, the terms $R + M^+D'D$ and $M^+$ can be replaced by $\dot{R} + M'D'D$ and $M$, respectively, in (4.11) without changing their values.

Now we prove $J \geq 1$. Denote $\dot{J} \triangleq J - 1$; then $\dot{J}$ satisfies the ODE

\[
\dot{J} = -\lambda^{(1)}J - \left[\lambda^{(1)} + B'(R + MD'D)^{-1}D'C(M \wedge K)\right]
\]

where

\[
a^{(1)} = \lambda^{(1)} + B'(R + MD'D)^{-1}D'C(M \wedge K)
\]

\[
= |C|^2 - C'D(R + MD'D)^{-1}D'C(M \wedge K) + \Gamma^{(1)}B'(R + MD'D)^{-1}D'C + \frac{Q}{M \vee c}
\]

\[
\geq |C|^2 - C'D(R + MD'D)^{-1}D'C(M \wedge K) + \Gamma^{(1)}B'(R + MD'D)^{-1}D'C + \frac{Q}{M \vee c}
\]

\[
= \text{tr} \left\{ (R + MD'D)^{-1}|C|^2 + Q/(M \vee c)(R + MD'D) \right\} - \text{tr} \{ (R + MD'D)^{-1}D'C\} + \text{tr} \{ (R + MD'D)^{-1}\Gamma^{(1)}D'C\}
\]

\[
= \text{tr} \{ (R + MD'D)^{-1}H \}
\]

with $H \triangleq |C|^2 + Q/(M \vee c)(R + MD'D) - D'C\vee cDM + \Gamma^{(1)}S(D'C\vee c)$.

When $c$ is small enough such that $R - cD'D \succeq 0$, we have

\[
\frac{Q}{M \vee c}(R + MD'D) \succeq QD'D.
\]

Furthermore,

\[
\frac{|C|^2}{l}D'D - D'C\vee cD \succeq 0.
\]

Hence,

\[
H \geq \frac{QD'D + |C|^2R}{l} + \Gamma^{(1)}S(D'C\vee c) \succeq 0,
\]

and consequently $a^{(1)} \geq \text{tr} \{ (R + MD'D)^{-1}H \} \geq 0.$\footnote{Here we used the inequality that $\text{tr}(AB) \geq 0$ for any positive semidefinite matrices $A, B$.} We deduce that $\dot{J} \geq 0$, or equivalently, $J \geq 1$.

Next we prove $M$ is bounded above by a constant $L > 0$ independent of the truncation. Choosing $c$ small enough, the equation for $M$ turns out to be

\[
\begin{array}{l}
\dot{M} = (2A + |C|^2 + \Gamma^{(1)}B'(R + MD'D)^{-1}(B + D'C)M + Q - kM(M \wedge K), \\
M_T = G,
\end{array}
\]

\[
\geq \frac{QD'D + |C|^2R}{l} + \Gamma^{(1)}S(D'C\vee c) \succeq 0,
\]

\[
\begin{array}{l}
\dot{M} = (2A + |C|^2 + \Gamma^{(1)}B'(R + MD'D)^{-1}(B + D'C)M + Q - kM(M \wedge K), \\
M_T = G,
\end{array}
\]
where
\[
k = (B + D'C)'(R + MD'D)^{-1}(B + D'C) - B'(R + MD'D)^{-1}(B + D'C) \frac{1}{J}
\]
\[
= B'(R + MD'D)^{-1}B \left( 1 - \frac{1}{J} \right) + B'(R + MD'D)^{-1}D' \left( 2 - \frac{1}{J} \right) + C'D(R + MD'D)^{-1}D'C \\
\geq B'(R + MD'D)^{-1}D'C \left( 2 - \frac{1}{J} \right)
\]

If \( B = \lambda D'C \) for some \( \lambda \geq 0 \), then we have \( k \geq 0 \). Hence \( M \) admits an upper bound \( L \) independent of \( c \) and \( K \).

If \( D'D - \delta I \geq 0 \), then \( kM \) admits a bound independent of \( c \) and \( K \); hence once again \( M \) admits an upper bound \( L \) independent of \( c \) and \( K \).

Choosing \( K = L \) and examining again (4.12), we deduce that there exists \( \eta > 0 \) independent of \( c \) such that \( M \geq \eta \). It now suffices to take \( c = \eta \) to finish the proof. \( \square \)

4.2.2. Singular case. Let us now consider the singular case \( R \equiv 0 \). In this subsection we suppose that \( D'D - \delta I \geq 0 \) for some \( \delta > 0 \). Then the system of \((M, J)\) is

\[
\begin{align*}
\dot{M} &= -\left[ 2A + |C|^2 - (B + D'C)'(D'D)^{-1}(B + D'C) \\
&\quad + B'(D'D)^{-1}(B + D'C) \right] M - Q - \Gamma^{(1)} B'(D'D)^{-1}(B + D'C) \\
M_T &= G; \\
\dot{J} &= -\left[ |C|^2 - C'D(D'D)^{-1}(B + D'C) + (\Gamma^{(1)} B'(D'D)^{-1}D'C + Q) \frac{1}{\text{det}} \right] J \\
J_T &= \frac{G}{h}.
\end{align*}
\]

(4.13)

This system is even easier than the previous one. We will use the same truncation argument to prove the existence of a solution.

**Theorem 4.3.** Given \( G \geq h > 0, R \equiv 0 \), and \( D'D - \delta I \geq 0 \) for some \( \delta > 0 \), if \( Q + \Gamma^{(1)} B'(D'D)^{-1}(B + D'C) \geq 0 \) and \( Q + \Gamma^{(1)} B'(D'D)^{-1}D'C \geq 0 \), then (4.13) and (4.9) admit positive solution pairs.

**Proof.** For a fixed \( c > 0 \), consider the following truncated system:

\[
\begin{align*}
\dot{M} &= -\left[ 2A + |C|^2 - (B + D'C)'(D'D)^{-1}(B + D'C) \\
&\quad + B'(D'D)^{-1}(B + D'C) \right] M - Q - \Gamma^{(1)} B'(D'D)^{-1}(B + D'C), \\
M_T &= G; \\
\dot{J} &= -\left[ |C|^2 - C'D(D'D)^{-1}(B + D'C) + (\Gamma^{(1)} B'(D'D)^{-1}D'C + Q) \frac{1}{\text{det}} \right] J \\
J_T &= \frac{G}{h}.
\end{align*}
\]

(4.14)

This system is locally Lipschitz with linear growth, and hence it admits a unique solution pair \((M, J)\) depending on \( c \).

Define \( \bar{J} = J - 1 \). Then
\[
\dot{\bar{J}} = -\chi^{(3)} \bar{J} - a^{(3)}
\]
with $\lambda^{(3)} = |C|^2 - C'(D'D)^{-1}(B + D'C) + (\Gamma^{(1)}B'(D'D)^{-1}D'C + Q)\frac{1}{M \vee c}$ being bounded, and

$$a^{(3)} = \lambda^{(3)} + B'(D'D)^{-1}D'C$$

$$\geq (\Gamma^{(1)}B'(D'D)^{-1}D'C + Q)\frac{1}{M \vee c} \geq 0.$$  

Hence $J \geq 1$. Now we choose $c \leq 1$.

Denote $\lambda^{(4)} = 2A + |C|^2 - (B + D'C)'(D'D)^{-1}(B + D'C) + B'(D'D)^{-1}(B + D'C)\frac{1}{\sqrt{c}}, \quad \bar{Q} = Q + \Gamma^{(1)}B'(D'D)^{-1}B + D'C \geq 0$. Then $|\lambda^{(4)}|$ admits a bound independent of $c$, and

$$\bar{M} + \lambda^{(4)}M + \bar{Q} = 0, \quad M_T = G.$$  

Hence there exists some $\eta > 0$ (independent of $c$) such that $M \geq \eta$. Choosing $c = \eta$, we conclude the proof. \hfill $\square$

4.3. Equilibrium controls. We now present the main result of this section.

**Theorem 4.4.** Suppose $G \geq h > 0$. The system of the Riccati equations (4.9) admits a unique positive solution pair $(M, N)$ in the following three cases:

(i) $R - \delta I \geq 0$ for some $\delta > 0$, $\frac{QD'D + |C|^2R}{\sqrt{c}} + \Gamma^{(1)}S(D'C)B' \geq 0$, and $B = \lambda D'C$ for some $\lambda > 0$;

(ii) $R - \delta I \geq 0$ for some $\delta > 0$, $\frac{QD'D + |C|^2R}{\sqrt{c}} + \Gamma^{(1)}S(D'C)B' \geq 0$, and $D'D - \delta I \geq 0$ for some $\delta > 0$;

(iii) $R \equiv 0$, $D'D - \delta I \geq 0$ for some $\delta > 0$, $Q + \Gamma^{(1)}B'(D'D)^{-1}(B + D'C) \geq 0$, $Q + \Gamma^{(1)}B'(D'D)^{-1}D'C \geq 0$.

Moreover, let $\Phi$ be a solution of ODE (4.8). Then $u^*(\cdot)$ given by (4.4) is an equilibrium.

**Proof.** Define $p(\cdot, \cdot)$ and $k(\cdot, \cdot)$ by (4.1) and (4.3), respectively. It is straightforward to check that $(u^*_t, X^*_t, p(\cdot, \cdot), k(\cdot, \cdot))$ satisfies the system of SDEs (3.10).

In all three cases, we can check that $\alpha_s$ and $\beta_s$ in (4.4) are both uniformly bounded, and hence $u^* \in L^2([0, T]; \mathbb{R})$ and $X^*_t \in L^2(\Omega; C([0, T]; \mathbb{R}))$.

Finally, denote $\Lambda(s; t) = R_su^*_s + p(s; t)B + D'_s(k(s; t))$. Plugging $p, k, u^*$, defined in (4.1), (4.3), and (4.4), into $\Lambda$, we have

$$\Lambda(s; t) = R_su^*_s + (M_sX^*_s - N_sE_t[X^*_s] - \Gamma^{(1)}sX^*_s + \Phi_s)B_s + M_sD'_s(C_sX^*_s) + D_su^*_s + \sigma_s$$

$$= (R_s + M_sD'_s(D_s))u^*_s + (B_s + D'_sC_s)M_sX^*_s - N_sE_t[X^*_s]B_s - \Gamma^{(1)}sX^*_sB_s$$

$$+ (\Phi_sB_s + M_sD'_s\sigma_s)$$

$$= -[(M_s - N_s - \Gamma^{(1)}s)B_s + M_sD'_s(C_s)X^*_s - \Phi_sB_s - M_sD'_s\sigma_s]$$

$$+ (B_s + D'_sC_s)M_sX^*_s - N_sE_t[X^*_s]B_s - \Gamma^{(1)}sX^*_sB_s + (\Phi_sB_s + M_sD'_s\sigma_s)$$

$$= (N_s + \Gamma^{(1)}sX^*_sB_s - N_sE_t[X^*_s]B_s) - \Gamma^{(1)}X^*_sB_s$$

$$= N_s[X^*_s - E_t[X^*_s]]B_s + \Gamma^{(1)}(X^*_s - X^*_t)B_s.$$  

Clearly $\Lambda$ satisfies the first condition in (3.4). Furthermore, we have

$$\lim_{s \downarrow t}E_t[X^*_s - E_t[X^*_s]] = 0, \quad \text{and} \quad \lim_{s \downarrow t}E_t[X^*_s - X^*_t] = 0;$$
hence \( \Lambda \) satisfies the second condition in (3.4).

By Theorem 3.3, \( u^* \) is an equilibrium. \( \square \)

Remark 4.5. If \( \mu_1 \geq 0 \) (e.g., in the mean–variance model to be studied subsequently), then \( \Gamma_t^{(1)} = \mu_1 e_f^T A ds \geq 0 \). With this condition, the first case and the third case in Theorem 4.4 can be simplified as follows:

(i') \( R - \delta I \geq 0 \) for some \( \delta > 0 \), and \( B = \lambda D'C \) for some \( \lambda \geq 0 \);

(ii') \( R \equiv 0 \), \( D'D - \delta I \geq 0 \) for some \( \delta > 0 \), and \( Q + \Gamma^{(1)}B'(D'D)^{-1}D'C \geq 0 \).

5. Mean-variance equilibrium strategies in complete market. In this section, we study the continuous-time Markowitz mean–variance portfolio selection model in a complete market. The problem is inherently time-inconsistent due to the variance term. Moreover, as in [6] we consider a state-dependent mean expectation. Hence there are two different sources of time-inconsistency. The definition of equilibrium strategies is in the sense of open-loop, which is different from the feedback sense in [5, 6].

The model is mathematically a special case of the general LQ problem formulated earlier in this paper, with \( n = 1 \) naturally. However, some coefficients are allowed to be random, so it is not a direct application of the previous section. Indeed the analysis in this section is much more involved due to the randomness of the coefficients.

For each \( t \in [0, T] \), consider a wealth-portfolio process \( (X_t, \pi_t) \) satisfying the wealth equation

\[
\begin{align*}
\begin{cases}
\frac{dX_s}{X_s} = r_s X_s ds + (\mu_s - r_s) \pi_s ds + \pi'_s \sigma_s dW_s, & s \in [t, T], \\
X_t = x_t,
\end{cases}
\end{align*}
\]

where \( r \in L^\infty(0, T; \mathbb{R}) \) is the interest rate process, and \( \mu \in L^2(0, T; \mathbb{R}^d) \) and \( \sigma \in L^2(0, T; \mathbb{R}^{d \times d}) \) are the drift rate vector and volatility processes of risky assets, respectively. We assume throughout that \( \sigma_s \sigma'_s - \varepsilon I \geq 0 \) for some \( \varepsilon > 0 \) to ensure the completeness of the market.

Denote \( \theta_t = \sigma_t^{-1} (\mu_t - r_t) \), \( u_t = \pi'_t \pi_t \). Then the wealth equation is equivalent to the equation of \( (X_t, u_t) \),

\[
\begin{align*}
\begin{cases}
\frac{dX_s}{X_s} = r_s X_s ds + \theta'_s u_s ds + u'_s dW_s, & s \in [t, T], \\
X_t = x_t.
\end{cases}
\end{align*}
\]

We interchangeably refer to \( \pi \) and \( u \) as (trading) strategies. It follows from our assumptions on \( \sigma \) that \( \pi \in L^2(0, T; \mathbb{R}) \) if and only if \( u \in L^2(0, T; \mathbb{R}) \). The objective of a mean–variance portfolio choice model at time \( t \in [0, T] \) is to achieve a balance between conditional variance and conditional expectation of terminal wealth, namely, to choose a strategy \( u \) so as to minimize

\[
J(t, x_t; u) \triangleq \frac{1}{2} \text{Var}_t (X_T) - (\mu_1 x_t + \mu_2) \mathbb{E}_t [X_T] = \frac{1}{2} \left( \mathbb{E}_t [X_T^2] - (\mathbb{E}_t [X_T])^2 \right) - (\mu_1 x_t + \mu_2) \mathbb{E}_t [X_T]
\]

with \( \mu_1 \geq 0 \). Here we insist that the weight between the conditional variance (as a risk measure) and the conditional expectation depend on the current wealth level, the reason having been elaborated in [6].

When the market parameters \( r \) and \( \theta \) are both deterministic, the problem is a special case that studied in section 4. In this section, we will find the equilibrium...
strategies for the model where the interest rate \( r \) is deterministic but \( \theta \) is allowed to be random.

The problem (5.1)–(5.3) is clearly a special case of LQ problem (2.2)–(2.3) with \( n = 1 \). The FBSDE (3.10) specializes to

\[
\begin{align*}
    dX^*_s &= [r_s X^*_s + \theta'_s u^*_s]ds + (u^*_s)'dW_s, \\
    dp(s; t) &= -r_s p(s; t)ds + k(s; t)'dW_s, \\
    p(T; t) &= X^*_T - \mathbb{E}_t[X^*_T] - \mu_1 X^*_T - \mu_2,
\end{align*}
\]

and the process \( \Lambda(s; t) \) in condition (3.4) is

\[ \Lambda(s; t) = p(s; t)\theta_s + k(s; t). \]

5.1. Formal derivation. As before, let us look for a solution in the form

\[ p(s; t) = M_s X^*_s - \Gamma_s^1 X^*_t + \Gamma_s^2 - \mathbb{E}_t[N_s X^*_s + \Gamma_s^3], \]

where \((M, U), (N, V), (\Gamma^1, \gamma^1), (\Gamma^2, \gamma^2), \) and \((\Gamma^3, \gamma^3)\) are solutions of the following BSDEs:

\[
\begin{align*}
    dM_s &= -F_{M,U}ds + U'_s dW_s, \\
    dN_s &= -F_{N,V}ds + V'_s dW_s, \\
    d\Gamma_s^1 &= -F^{(1)}(\gamma^1_s)'dW_s, \\
    d\Gamma_s^2 &= -F^{(2)}(\gamma^2_s)'dW_s, \\
    d\Gamma_s^3 &= -F^{(3)}(\gamma^3_s)'dW_s,
\end{align*}
\]

It is an easy exercise to obtain

\[
\begin{align*}
    d[N_s X^*_s] &= [r_s N_s X^*_s + N_s \theta'_s u^*_s - X^*_s F_{N,V} + V'_s u^*_s]'ds + [N_s u^*_s + X^*_s V]dW_s, \\
    d\mathbb{E}_t[N_s X^*_s] &= \mathbb{E}_t[r_s N_s X^*_s + N_s \theta'_s u^*_s - X^*_s F_{N,V} + V'_s u^*_s]'ds, \\
    d[M_s X^*_s] &= [r_s M_s X^*_s + M_s \theta'_s u^*_s - X^*_s F_{M,U} + U'_s u^*_s]'ds + [M_s u^*_s + X^*_s U]'dW_s.
\end{align*}
\]

Applying Ito’s formula to \( p(s; t) = M_s X^*_s - \Gamma_s^1 X^*_t + \Gamma_s^2 - \mathbb{E}_t[N_s X^*_s + \Gamma_s^3] \) and comparing the \( dW_s \) term in the second equation of (5.4), we get

\[ k(s; t) = X^*_s U_s + M_s u^*_s + \gamma^2_s - \gamma^1_s X^*_t. \]

Putting the expressions of \( p \) and \( k \) into the formal condition \( \Lambda(s; s) = 0 \), we obtain

\[
\begin{align*}
    u^*_s &= -M_s^{-1} \left[ \theta_s(M_s - N_s - \Gamma_s^1) + U_s - \gamma^1_s \right] X^*_s + \theta_s(\Gamma_s^2 - \Gamma_s^3) + \gamma^2_s \] \\
    &= \alpha_s X^*_s + \beta_s,
\end{align*}
\]

where

\[
\begin{align*}
    \alpha_s &\triangleq -M_s^{-1} \left[ \theta_s(M_s - N_s - \Gamma_s^1) + U_s - \gamma^1_s \right], \\
    \beta_s &\triangleq -M_s^{-1} \left[ \theta_s(\Gamma_s^2 - \Gamma_s^3) + \gamma^2_s \right].
\end{align*}
\]

Applying again Ito’s formula to \( p \) and using the above expression of \( u \), we deduce

\[
\begin{align*}
    dp(s; t) &= [-F_{M,U} X^*_s + r_s M_s X^*_s + (\theta_s M_s + U_s)(\alpha X^*_s + \beta_s) - F^{(2)} + X^*_s F^{(1)}]'ds \\
    &\quad + \mathbb{E}_t[F_{N,V} X^*_s - r_s N_s X^*_s - (\theta_s N_s + V_s)(\alpha X^*_s + \beta_s) + F^{(3)}]'ds + k(s, t)'dW_s,
\end{align*}
\]
while the second equation in (5.4) gives
\[
dp(s; t) = \{-r_s M_s X_s^* + r_s \Gamma_s^{(1)} X_s^* - r_s \Gamma_s^{(2)} + r_s \mathbb{E}_t[N_s X_s^* + \Gamma_s^{(3)}]\} ds + k(s; t) dW_s.
\]
Comparing the corresponding terms, we obtain (again we suppress the subscripts \(s \in [t, T]\))
\[
F_{M,U} = 2r_M + (\theta M + U)' \alpha,
\]
\[
F_{N,V} = 2r_N + (\theta N + V)' \alpha,
\]
\[
F^{(1)} = r \Gamma^{(1)},
\]
\[
F^{(2)} = r \Gamma^{(2)} + (\theta M + U)' \beta,
\]
\[
F^{(3)} = r \Gamma^{(3)} + (\theta N + V)' \beta.
\]

**5.2. Solution to the BSDEs (5.6).** It now suffices to solve the BSDEs (5.6). Its third equation can be easily solved, whose solution is
\[
\Gamma^{(1)}_t = \mu_1 \int_t^T r_s ds, \quad \gamma^{(1)}_t = 0.
\]
Noting that the first two equations are identical, we conclude that
\[
M = N, \quad U = V.
\]
Then
\[
F^{(2)} - F^{(3)} = r(\Gamma^{(2)} - \Gamma^{(3)}).
\]
By the last two equations in (5.6), we have
\[
\Gamma^{(2)}_s - \Gamma^{(3)}_s = -\mu_2 \int_t^s r_s ds \triangleq \Delta_s.
\]
To proceed, let us recall some facts about bounded-mean-oscillation (BMO) martingales; see Kazamaki [15]. The process \(Z \cdot W \triangleq \int_0^s Z'_r dW_r\) is a BMO martingale if and only if there exists a constant \(C > 0\) such that
\[
\mathbb{E} \left[ \int_T^\tau |Z_s|^2 ds \right] \leq C
\]
for all stopping times \(\tau \leq T\). For every such \(Z\), the stochastic exponential of \(Z \cdot W\) denoted by \(\mathcal{E}(Z \cdot W)\) is a positive martingale, and for any \(p > 1\), there exists a constant \(C_p > 0\) such that \(\mathbb{E} \left[ \left( \int_T^\tau |Z_s|^2 ds \right)^p \right] \leq C_p\) for any stopping time \(\tau \leq T\). Moreover, if \(Z \cdot W\) and \(V \cdot W\) are both BMO martingales, then under the probability measure \(Q\) defined by \(\frac{dQ}{dP} = \mathcal{E}_T(V \cdot W)\), \(W^Q_t \triangleq W_t - \int_0^t V_r dr\) is a standard Brownian motion, and \(Z \cdot W^Q\) is a BMO martingale.

Now, plugging the definition of \(\alpha\) into the first equation in (5.6), we get the BSDE satisfied by \((M, U)\):
\[
(5.8) \quad \begin{cases} 
    dM_s = -(2r_s M_s - U'_s \theta_s + \Gamma_s^{(1)} |\theta_s|^2 - M_s^{-1} |U_s|^2 + \Gamma_s^{(1)} M_s^{-1} U'_s \theta_s) ds + U'_s dW_s, \\
    M_T = 1.
\end{cases}
\]
This is a type of indefinite stochastic Riccati equation due to the presence of \(M^{-1}\) in the driver; however, it is different from the one studied in [12].
PROPOSITION 5.1. BSDE (5.8) admits a unique solution \((M, U) \in L^\infty_T(0, T; \mathbb{R}) \times L^2_T(0, T; \mathbb{R}^d)\) satisfying \(M \geq c\) for some constant \(c > 0\). Moreover, \(U \cdot W\) is a BMO martingale.

Proof. Once again, we will prove the existence by a truncation argument. Let \(c > 0\) be a given number to be chosen later. Consider the following quadratic BSDE:

\[
\left\{ \begin{array}{l}
dM_s = -\left[ 2r_s M_s - U'_s \theta_s + \Gamma_s^{(1)} |\theta_s|^2 - \frac{|U'_s|^2}{M_s \vee c} + \Gamma_s^{(1)} \frac{U'_s \theta_s}{M_s \vee c} \right] ds + U'_s dW_s, \\
M_T = 1.
\end{array} \right.
\]

This BSDE is a standard quadratic BSDE. Hence there exists a solution \((M^c, U^c) \in L^\infty_T(0, T; \mathbb{R}) \times L^2_T(0, T; \mathbb{R}^d)\), and \(U^c : W\) is a BMO martingale; see [16] and [18].

We can rewrite the above BSDE as

\[
\left\{ \begin{array}{l}
dM_s = -(2r_s M_s + \Gamma_s^{(1)} |\theta_s|^2) ds \\
+ U'_s [dW_s - (\Gamma_s^{(1)} \frac{1}{M_s \vee c} \theta_s - \theta_s - \frac{1}{M_s \vee c} U'_s) ds], \\
M_T = 1.
\end{array} \right.
\]

As \((\Gamma_s^{(1)} \frac{1}{M_s \vee c} \theta - \theta - \frac{1}{M_s \vee c} U^c) \cdot W\) is a BMO martingale, there exists a new probability measure \(Q\) such that

\[
W^Q_t = W_t - \int_0^t \left( \Gamma_s^{(1)} \frac{1}{M_s \vee c} \theta_s - \theta_s - \frac{1}{M_s \vee c} U^c_s \right) ds
\]

is a Brownian motion under \(Q\).

Hence,

\[
M^c_t = \mathbb{E}_s^Q \left[ e^{2 \int_s^T \Gamma_v^{(1)} \frac{1}{M_v \vee c} \theta_v - \theta_v - \frac{1}{M_v \vee c} U^c_v dv} \int_s^T \Gamma_v^{(1)} e^{2 \int_s^v \Gamma_u^{(1)} \frac{1}{M_u \vee c} \theta_u - \theta_u - \frac{1}{M_u \vee c} U^c_u} du \right],
\]

from which we deduce that there exists a constant \(\eta > 0\) independent of \(c\) such that \(M \geq \eta\). Taking \(c = \eta\), we obtain a solution.

Let us now prove the uniqueness. First, we note that if \((M, U) \in L^\infty_T(0, T; \mathbb{R}) \times L^2_T(0, T; \mathbb{R}^d)\) is a solution and there exists \(c > 0\) such that \(M \geq c\), then \(U \cdot W\) is a BMO martingale. Let us define

\[
Y_s = M_s^{-1}, \quad Z_s = -M_s^{-2} U_s.
\]

Then \((Y, Z)\) is a solution in \(L^\infty_T(0, T; \mathbb{R}) \times L^2_T(0, T; \mathbb{R}^d)\) of the BSDE

\[
\left\{ \begin{array}{l}
dY_s = [-2r_s Y_s - Z'_s \theta_s - \Gamma_s^{(1)} |\theta_s|^2 Y^2_s + \Gamma_s^{(1)} Y_s Z'_s] ds + Z'_s dW_s, \\
Y_T = 1.
\end{array} \right.
\]

Moreover, \(Z \cdot W\) is a BMO martingale.

It suffices to prove the uniqueness of the solution to BSDE (5.11). For this, let \((Y^{(1)}, Z^{(1)})\) and \((Y^{(2)}, Z^{(2)})\) be two solutions in \(L^\infty_T(0, T; \mathbb{R}) \times L^2_T(0, T; \mathbb{R}^d)\) such that \(Z^{(1)} \cdot W\) and \(Z^{(2)} \cdot W\) are BMO martingales. Set

\[
\tilde{Y} = Y^{(1)} - Y^{(2)}, \quad \tilde{Z} = Z^{(1)} - Z^{(2)}.
\]

Then

\[
\left\{ \begin{array}{l}
d\tilde{Y}_s = [-2r_s \tilde{Y}_s - \tilde{Z}'_s \theta_s - \Gamma^{(1)} |\theta_s|^2 (Y^{(1)}_s + Y^{(2)}_s) \tilde{Y}_s \\
+ \Gamma^{(1)} \theta'_s (Y^{(1)}_s Z^{(1)}_s + Y^{(2)}_s Z^{(2)}_s)] ds + \tilde{Z}'_s dW_s, \\
\tilde{Y}_T = 0.
\end{array} \right.
\]
Applying Ito’s formula to $|\bar{Y}_s|^2$ and taking conditional expectation, we deduce (where $C > 0$ is a constant which may change from line to line)

$$
|\bar{Y}_s|^2 + \mathbb{E}_s \left[ \int_s^T |\bar{Z}_r|^2 dr \right] \\
\leq C \mathbb{E}_s \left[ \int_s^T (|\bar{Y}_r| + |\bar{Z}_r| + |Z_r^{(1)}||\bar{Y}_r|)dr \right] \\
\leq C \mathbb{E}_s \left[ \int_s^T |\bar{Y}_r|^2 dr \right] + \frac{1}{2} \mathbb{E}_s \left[ \int_s^T |\bar{Z}_r|^2 dr \right] + C \sqrt{\mathbb{E}_s \left[ \int_s^T |Z_r^{(1)}|^2 dr \right]} \mathbb{E}_s \left[ \int_s^T |\bar{Y}_r|^4 dr \right].
$$

Let us assume that $s \in [T - \delta, T]$. Then by setting

$$
\bar{Y}_{T-\delta,T}^s = \|\bar{Y}\|_{L^2(T-\delta,T;\mathbb{R})},
$$

we obtain

$$
|\bar{Y}_s|^2 \leq C(\delta + \delta^{1/2})|\bar{Y}_{T-\delta,T}^s|^2.
$$

Hence,

$$
|\bar{Y}_{T-\delta,T}^s|^2 \leq C\delta^{1/2}|\bar{Y}_{T-\delta,T}^s|^2.
$$

By taking $\delta$ sufficiently small, we deduce that $\bar{Y}_{T-\delta,T}^s = 0$. We conclude the proof of uniqueness by continuing on $[T - 2\delta, T - \delta], \ldots$, until time 0 is reached.

Then we consider the BSDE satisfied by $(\Gamma^{(2)}, \gamma^{(2)})$:

$$
\begin{cases}
    d\Gamma_t^{(2)} = - \left[ r_t \Gamma_t^{(2)} - \left( \theta_t + \frac{U_t}{M_t} \right) \gamma_t^{(2)} - \left( |\theta_t|^2 + \frac{U_t^2}{M_t} \right) \Gamma_t \right] dt + (\gamma_t^{(2)})' dW_t, \\
    \Gamma_T^{(2)} = -\mu_2.
\end{cases}

\tag{5.13}
$$

**Proposition 5.2.** BSDE (5.13) admits a unique solution $(\Gamma^{(2)}, \gamma^{(2)}) \in L^\infty(0, T; \mathbb{R}) \times L^2(0, T; \mathbb{R}^d)$. Moreover, $\gamma^{(2)} \cdot W$ is a BMO martingale.

Proof. As $-(\theta + \frac{U}{M} \cdot W$ is a BMO martingale, it suffices to apply the result of section 3 in [4] to deduce that BSDE (5.13) admits a unique solution $(\Gamma^{(2)}, \gamma^{(2)}) \in L^2(0, T; \mathbb{R}) \times L^2(0, T; \mathbb{R}^d)$. Let $\mathbb{Q}$ be the probability measure defined by $d\mathbb{Q} = \mathcal{E}_T(-\theta + \frac{U}{M} \cdot W)$. Then under $\mathbb{Q}$,

$$
W_t^\mathbb{Q} = W_t + \int_0^t (\theta_s + M_s^{-1} U_s) ds
$$

is a Brownian motion and $U \cdot W^\mathbb{Q}$ is a BMO martingale. Furthermore,

$$
d\Gamma_t^{(2)} = - \left[ r_t \Gamma_t^{(2)} - \left( |\theta_t|^2 + \frac{U_t^2}{M_t} \right) \Gamma_t \right] dt + (\gamma_t^{(2)})' dW_t^\mathbb{Q}, \quad \Gamma_T^{(2)} = -\mu_2.
$$
Hence

\[ \Gamma_t^{(2)} = \mathbb{E}_t^Q \left[ -e_t^\tau r_\ast d\mu_2 - \int_t^T e_s^{\tau'} r_\ast d\Gamma_s \left( |\theta_s|^2 + \frac{U_s' \theta_s}{M_s} \right) ds \right]. \]

From this we deduce that \( \Gamma^{(2)} \) is a bounded process. Moreover, from (5.13),

\[
\mathbb{E}_t^Q \left[ \int_t^T |\gamma_s^{(2)}|^2 ds \right] = \mathbb{E}_t^Q \left[ \int_t^T (\gamma_s^{(2)}')^2 dW_s^Q \right]^2
\[ = \mathbb{E}_t^Q \left[ \Gamma_t^{(2)} - \Gamma_t^{(2)} + \int_t^T r_s \Gamma_s^{(2)} - \Gamma_s \left( |\theta_s|^2 + \frac{U_s' \theta_s}{M_s} \right) ds \right]^2. \]

Hence from the last equality, \( \gamma^{(2)} \cdot W^Q \) is a BMO martingale under \( \mathbb{Q} \), and then \( \gamma^{(2)} \cdot W \) is a BMO martingale under \( \mathbb{P} \).

With \( M, U, \gamma^{(2)} \) obtained, we can construct a (feedback) strategy

\[ u_s^* = \alpha_s X_s^* + \beta_s, \]

where

\[ \alpha_s \triangleq \frac{\Gamma_s^{(1)} \theta_s - U_s}{M_s}, \quad \beta_s \triangleq -\frac{\Gamma_s \theta_s + \gamma_s^{(2)} - U_s}{M_s}. \]

In order to confirm that the above is indeed an admissible feedback strategy, we need to prove the following technical result. Its proof is intriguing in its own right.

**Proposition 5.3.** Let \( X^* \) be the solution to the first equation of (5.4) where \( u^* \) is substituted by (5.14). Then \( X^* \in L^2_T(0, T; C(0, T; \mathbb{R})) \) and \( u^* \in L^2_T(0, T; \mathbb{R}^d) \).

**Proof.** Plugging the feedback strategy \( u^* \) into the wealth equation (5.2), we get

\[ X_t^* = \rho_t \left( x_0 - \int_0^t \rho_s^{-1} \alpha_s \beta_s ds + \int_0^t \rho_s^{-1} \beta_s dW_s^\theta \right), \]

with \( W_s^\theta = W_t + \int_t^s \theta_s ds \) and \( \rho_t = e^{\int_0^t r_\ast ds} \mathcal{E}_t(\alpha \cdot W^\theta) \).

On the one hand,

\[
\mathcal{E}_t(\alpha \cdot W^\theta) = e^{-\int_0^t \frac{|\alpha_s|^2}{2} ds - \int_0^t \frac{U_s' \theta_s}{M_s} dW_s + \theta_s ds}
\[ = e^{-\int_0^t \frac{|\alpha_s|^2}{2} ds - \int_0^t \frac{U_s' \theta_s}{M_s} dW_s + \theta_s ds} \left( \frac{1}{M_s} \right)^{2 - \frac{U_s' \theta_s}{M_s} dW_s}
\[ = e^{-\int_0^t \frac{|\alpha_s|^2}{2} ds - \frac{U_s' \theta_s}{M_s} dW_s} \left( \frac{M_s}{2} \right)^{2 - \frac{U_s' \theta_s}{M_s} dW_s} \mathcal{E}_t \left( \frac{\Gamma^{(1)} \theta - U_s}{M_s} \cdot W \right). \]

Applying Itô’s formula to \( \ln(M) \), we get

\[
d\ln(M_s) = \left[ -2r_s + \frac{U_s' \theta_s}{M_s} - \Gamma_s^{(1)} \frac{|\theta_s|^2}{M_s} + \frac{1}{2} \frac{|U_s|^2}{M_s^2} - \Gamma_s^{(1)} \frac{U_s' \theta_s}{M_s^2} \right] ds + \frac{U_s'}{M_s} dW_s
\[ = \left[ -2r_s + \frac{|\alpha_s|^2}{2} - \frac{1}{2} \frac{\Gamma_s^{(1)} \theta_s}{M_s} - \Gamma_s^{(1)} \frac{|\theta_s|^2}{M_s} \right] ds + \frac{U_s'}{M_s} dW_s.\]
Finally, regarding (5.16) we denote by $X^*$ the solution to BSDE (5.15). For any $p > 1$, 
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \rho_s^{-1} \alpha_s \beta_s ds \right|^p \right] \leq c_p \left( \mathbb{E} \left[ \left( \int_0^T |\alpha_s|^2 ds \right)^{2p} \right] + \mathbb{E} \left[ \left( \int_0^T |\beta_s|^2 ds \right)^{2p} \right] \right)^{1/2p} < +\infty.
\]
Similarly we have $\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \rho_s^{-1} \theta_s \beta_s ds \right|^p \right] < +\infty$. Also we have 
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \rho_s^{-1} \beta_s dW_s \right|^{2p} \right] \leq \tilde{c}_p \mathbb{E} \left[ \left( \int_0^T \rho_s^{-2} |\beta_s|^2 ds \right)^p \right] \leq \tilde{c}_p \mathbb{E} \left[ \sup_{t \in [0,T]} \rho_t^{-2p} \left( \int_0^T |\beta_s|^2 ds \right)^p \right] < +\infty,
\]
where $c_p, \tilde{c}_p$ are both constants depending only on $p$. These two inequalities lead to $X^* \in L^2_T(\Omega, C(0, T; \mathbb{R}))$.

Finally, regarding $(X^*, u^*)$ as the solution to the BSDE, we get 
\[
\{ dX_s = r_s X_s ds + \theta'_s u_s ds + u'_s dW_s, \quad s \in [0, T], \}
\]
\[
X_T = X^*_T.
\]
By the standard estimates for the Lipschitz BSDE, $u^* \in L^2_T(0, T; \mathbb{R}^d)$ as soon as $X^* \in L^2_T(\Omega, C(0, T; \mathbb{R}))$.

5.3. Equilibrium strategy. Summarizing the preceding analysis, we obtain finally the main result of this section.

THEOREM 5.4. Let $(M, U)$ and $(\Gamma^{(2)}, \gamma^{(2)})$ be the solutions to BSDEs (5.8) and (5.13), respectively, and $\Gamma_s = -\mu_2 e^{\int_0^s r_u du}$. Then 
\[
u^* = -M_s^{-1} \left[ (U_s - \theta_s \mu_1 e^{\int_0^s r_u du}) X^*_s + \Gamma_s \theta_s + \gamma_s^{(2)} \right]
\]
is an equilibrium strategy.
Proof. Define \( p, k \) by (5.5) and (5.7) (recall that \( N = M \) and \( V = U \)). It is easy to check that \( u^*, X^*, p, k \) satisfies (5.4). Furthermore, \( \Lambda \) in the condition (3.4) is

\[
\Lambda(s;t) = p(s;t)\theta_s + k(s;t) \\
= \left( M_sX^*_s - \Gamma^{(1)}_sX^*_t + \Gamma^{(2)}_s - E_t \left[ M_sX^*_s + \Gamma^{(3)}_s \right] \right) \theta_s \\
+ X^*_sU_s + M_su^*_s + \gamma^{(2)}_s - \gamma^{(1)}_sX^*_t \\
= \left( M_sX^*_s + \Gamma^{(3)}_s - E_t \left[ M_sX^*_s + \Gamma^{(3)}_s \right] \right) \theta_s + \Gamma^{(1)}_s(X^*_s - X^*_t)\theta_s.
\]

Since \( M, \theta, \Gamma^{(3)}, \Gamma^{(1)} \) are all essentially bounded, \( E_t[\sup_{s \in [t,T]}(X^*_s)^2] < +\infty \), we deduce that \( \Lambda \) meets condition (3.4). It follows from Theorem 3.3 that \( u^* \) is an equilibrium.

5.4. Examples. Equilibrium strategies for mean–variance models have been studied in [3, 5, 6], among others, in different frameworks. In this subsection, we will compare our results with some existing ones in the literature.

5.4.1. Deterministic risk premium. Let us first consider the case when the risk premium is a deterministic function of time. Then \( U = 0, \gamma^{(2)} = 0, \) and

\[
M_t = e^{2\int_t^T r_v dv}\left(1 + \mu_1 \int_t^T e^{-\int_t^s r_v dv} |\theta_s|^2 ds \right).
\]

The equilibrium strategy is given by

\[
u^*_t = \frac{\mu_1 e^{\int_t^T r_v dv} \theta_t X^*_t + \mu_2 e^{\int_t^T r_v dv}}{M_t} \theta_t.
\]

In the appendix, we obtain that the precommitted optimal control for the problem starting at \( t = 0 \) is also in an affine feedback form

\[
u^{pre}(t,x) = -x\theta_t + e^{\int_0^t r_v dv}(x_0 + (\mu_1 x_0 + \mu_2)e^{\int_0^t |\theta_s|^2 ds + \mu_2 \int_0^t \theta_s dW_s}) \theta_t.
\]

Case 1: \( \mu_1 = 0 \).

When \( \mu_1 = 0 \), the objective is exactly the same as in [3] and [5]; however, in those papers the equilibrium is defined within the class of (deterministic) feedback controls.

By Theorem 5.4,

\[
u^*_t = e^{\int_t^T r_v dv} \theta_t
\]

is a mean–variance equilibrium strategy. This equilibrium coincides with the one obtained in [3] and [5], although the definitions of equilibrium are different. The ex-post reason is that the feedback part of our equilibrium is absent, and thus so is the gap between the two definitions.

To compare the performance of this equilibrium control with the precommitted optimal control at time \( t = 0 \), we calculate the objective function value \( J(0,x_0;u^*) \).

It is not hard to get that \( \mathbb{E}[X_T^*] = e^{\int_0^T r_v dv} x_0 + \mu_2 \int_0^T |\theta_s|^2 ds + \mu_2 \int_0^T \theta_s dW_s; \) so,

\[
\mathbb{E}[X_T^*] = e^{\int_0^T r_v dv} x_0 + \mu_2 \int_0^T |\theta_s|^2 ds, \quad \text{Var}(X_T^*) = \mu_2^2 \int_0^T |\theta_s|^2 ds.
\]
Hence the objective value of \( u^* \) at time \( t = 0 \) is
\[
J(0, x_0; u^*) = -\frac{\mu_2^2}{2} \int_0^T |\theta_s|^2 ds - \mu_2 e^{f_0^\top r_s} dx_0.
\]
It is derived in the appendix that the precommitted optimal portfolio \( u^{pre} \) at time \( t = 0 \) has the objective value
\[
J(0, x_0; u^{pre}) = -\frac{\mu_2^2}{2} (e^{f_0^\top |\theta_0|^2} - 1) - \mu_2 e^{f_0^\top r_s} dx_0.
\]
Clearly, \( J(0, x_0; u^*) > J(0, x_0; u^{pre}) \). Moreover, we can easily see the difference between the two objective values.

Case 2: \( \mu_2 = 0 \).
When \( \mu_2 = 0 \), the objective is equivalent to the one in [6]. In this case, our equilibrium is, explicitly,
\[
u^*_t = \frac{\mu_1 e^{f_0^\top r_s} \theta_t}{M_t} X^*_t.
\]
In [6], the equilibrium is defined for the class of feedback controls as in [5]. Therein the equilibrium strategy is derived in a linear feedback form \( u^{fbc}_t = c^{fbc}_t X^*_t \) with \( c^{fbc}_t \) uniquely determined by an integral equation (whose unique solvability is established). We can easily show that the linear coefficient of our equilibrium above does not satisfy the integral equation in [6]. Thus, in turn, indicates the difference between the two definitions of equilibrium (open-loop and feedback).

To compare the performance of these two different equilibrium controls, together with the precommitted optimal control at time \( t = 0 \), we calculate \( J(0, x_0; u) \) for \( u = u^*, u^{fbc}, \) and \( u^{pre} \), respectively. Denote \( c_t = \frac{\mu_1 e^{f_0^\top r_s} \theta_t}{M_t} \); then it is an easy exercise to get \( X^*_t = x_0 e^{f_0^\top (r_t + c_t \theta_t)} dt \), the variance of \( X^*_t = x_0^2 e^{2 f_0^\top (r_t + c_t \theta_t)} dt (e^{f_0^\top |c_t|^2} dt - 1) \), leading to
\[
J(0, x_0; u^*) = \frac{x_0^2}{2} e^{2 f_0^\top (r_t + c_t \theta_t)} dt (e^{f_0^\top |c_t|^2} dt - 1) - x_0^2 \mu_1 e^{f_0^\top (r_t + c_t \theta_t)} dt.
\]
Similarly,
\[
J(0, x_0; u^{fbc}) = \frac{x_0^2}{2} e^{2 f_0^\top (r_t + (c^{fbc})_t \theta_t)} dt (e^{f_0^\top |c^{fbc}|^2} dt - 1) - x_0^2 \mu_1 e^{f_0^\top (r_t + (c^{fbc})_t \theta_t)} dt.
\]
By the calculation in the appendix, we have
\[
J(0, x_0; u^{pre}) = -\frac{x_0^2}{2} \mu_1^2 (e^{f_0^\top |\theta_0|^2} dt - 1) - x_0^2 \mu_1 e^{f_0^\top r_s} ds.
\]
Clearly,
\[
J(0, x_0; u^*) > J(0, x_0; u^{pre}), \quad J(0, x_0; u^{fbc}) > J(0, x_0; u^{pre}).
\]
Moreover, we can easily compare \( J(0, x_0; u^*) \) and \( J(0, x_0; u^{fbc}) \) due to their explicit expressions.
5.4.2. Stochastic risk premium. When the risk premium of the market is a stochastic process, the PDE (HJB equation) approach employed by [5] and [6], where the definition of equilibrium is in the class of feedback controls, no longer works. Czichowsky [7] studied this problem (albeit with a different definition of mean–variance efficiency) in the case when \( \mu_1 = 0 \) by a discretization method. To the best of our knowledge, our result is the first attempt to formulate and find equilibrium with random market parameters in the general case.

Case 1: \( \mu_1 = 0 \).
When \( \mu_1 = 0 \), we have \( U = 0 \), and \( M_t = e^{2 \int_t^T r_s ds} \), and our equilibrium is
\[
\begin{align*}
 u^*_t &= e^{-\int_t^T r_s ds} \mu_2 \theta_t - e^{-2 \int_t^T r_s ds} \gamma^{(2)} (2).
\end{align*}
\]
This strategy consists of two parts. The first is in the same form as that in the deterministic risk premium case, and the second is to hedge the uncertainty arising from the randomness of \( \theta \).

Case 2: \( \mu_2 = 0 \).
When \( \mu_2 = 0 \), we have \( \gamma^{(2)} = 0 \), and our equilibrium is
\[
\begin{align*}
 u^*_t = \left( \frac{\mu_1 e^{\int_t^T r_s ds}}{M_t} \theta_t - U_t}{M_t} \right) X^*_t.
\end{align*}
\]
The linear feedback coefficient in this equilibrium also consists of two parts. The first part is formally the same as its deterministic counterpart, whereas the second part is for the randomness of the parameter \( \theta \).

6. Concluding remarks. This paper, we believe, has posed more questions than answers. The flow of FBSDEs (3.6) is an interesting class of equations, whose general solvability begs for systematic investigations. How to adapt the generalized HJB approach of [5, 6] to our open-loop control framework, even when all the coefficients are deterministic, warrants a careful study (but notice the fundamental difference in the definitions of equilibrium). Extension beyond the realm of LQ control may open up an entirely new avenue for stochastic control. Finally, how our game theoretic formulation may be extended to other types of time-inconsistency, e.g., that caused by probability distortion, promises to be an equally exciting research topic. The research on the last problem is in progress and will appear in a forthcoming paper.

Appendix. Precommitted optimal mean–variance control. We consider the precommitted optimal control problem at time \( t = 0 \),
\[
\begin{align*}
 &\min J(0, x_0; u), \\
 &dX_t = (r_t X_t + \pi_t' \theta_t) dt + \pi_t' dW_t, \\
 &X_0 = x_0.
\end{align*}
\]

From the existing study on precommitted mean–variance problems such as [2], it follows that the optimal terminal state must be in the form \( X_T = \lambda - \mu \rho \) with constants \( \lambda \) and \( \mu \), random variable \( \rho = e^{-\int_0^T (r_s + |\theta_s|^2/2) ds - \int_0^T \theta_s dW_s} \), and the constraint
\[
\lambda \mathbb{E} [\rho] - \mu \mathbb{E} [\rho^2] = x_0.
\]
With this form of the terminal state, the objective value can be written as a function of $\mu$:

$$f(\mu) := \frac{\mu^2}{2} \text{Var}(\rho) - \frac{\mu_1 x_0 + \mu_2}{\mathbb{E}[\rho]} (x_0 + \mu \mathbb{E}[\rho^2] - \mu(\mathbb{E}[\rho])^2)$$

$$= \left( \frac{\mu^2}{2} - \frac{\mu_1 x_0 + \mu_2}{\mathbb{E}[\rho]} \right) \text{Var}(\rho) - \frac{\mu_1 x_0 + \mu_2}{\mathbb{E}[\rho]} x_0.$$ 

So the minima of $f(\mu)$, given $\text{Var}(\rho) \neq 0$, is achieved at $\mu^* = \frac{\mu_1 x_0 + \mu_2}{\mathbb{E}[\rho]}$ with the minimal objective value

$$V^{pre}(x_0) = f(\mu^*) = -\frac{1}{2} \left( \frac{\mu_1 x_0 + \mu_2}{\mathbb{E}[\rho]} \right)^2 \text{Var}(\rho) - \frac{\mu_1 x_0 + \mu_2}{\mathbb{E}[\rho]} x_0.$$ 

When the parameters $r$ and $\theta$ are deterministic, we can get the explicit optimal value for the precommitted problem (A.1):

$$V^{pre}(x_0) = -\frac{1}{2} (\mu_1 x_0 + \mu_2)^2 (e^{T} [\theta s^2] - 1) - (\mu_1 x_0 + \mu_2)e^{T} r^T s \theta.$$ 

Furthermore, the corresponding optimal control can be written as the affine feedback control

$$u_{pre}(t, x) = -\theta_t x + e^{T} r^T s (x_0 + (\mu_1 x_0 + \mu_2)e^{T} (\theta s^2 - r) s) \theta_t.$$ 

**Acknowledgment.** The authors thank the three referees and the associate editor for their helpful comments and suggestions.

**REFERENCES**

[1] G. Ainslie, Derivation of “rational” economic behavior from hyperbolic discount curves, The Amer. Economic Rev., 81 (1991), pp. 334–340.

[2] T. R. Bielecki, S. R. Pliska, H. Jin, and X. Y. Zhou, Continuous-time mean–variance portfolio selection with bankruptcy prohibition, Math. Finance, 15 (2005), pp. 213–244.

[3] S. Basak and G. Chabakauri, Dynamic mean–variance asset allocation, Rev. Financial Stud., 23 (2010), pp. 2970–3016.

[4] P. Briand and F. Confortola, BSDEs with stochastic Lipschitz condition and quadratic PDEs in Hilbert spaces, Stochastic Process. Appl., 118 (2008), pp. 818–838.

[5] T. Björk and A. Murgoci, A General Theory of Markovian Time Inconsistent Stochastic Control problems, Working paper 1694759, Social Science Research Network (SSRN), 2010. Available online at http://papers.ssrn.com/sol3/papers.cfm?abstract_idx, 1694759.

[6] T. Björk, A. Murgoci, and X. Y. Zhou, Mean-variance portfolio optimization with state dependent risk aversion, Math. Finance, to appear. Published online Feb. 3, 2012 DOI:10.111/j.1467-9965.2011.005.15.x.

[7] C. Czichowsky, Time-Consistent Mean-Variance Portfolio Selection in Discrete and Continuous Time, Working paper 661, National Centre of Competence in Research, Financial Valuation and Risk Management, Berne, Switzerland, 2010.

[8] I. Ekeland and A. Lazriah, Being Serious about Non-Commitment: Subgame Perfect Equilibrium in Continuous Time, preprint, arXiv:math/0604264, 2006.

[9] I. Ekeland and T. A. Pirvu, Investment and consumption without commitment, Math. Financial Economics, 2 (2008), pp. 57–86.

[10] W. Fleming and M. Soner, Controlled Markov Processes and Viscosity Solutions, 2nd ed., Springer-Verlag, New York, 2006.

[11] Y. Hu and X. Y. Zhou, Constrained stochastic LQ control with random coefficients, and application to portfolio selection, SIAM J. Control Optim., 44 (2005), pp. 444–466.

[12] Y. Hu and X. Y. Zhou, Indefinite stochastic Riccati equations, SIAM J. Control Optim., 42 (2003), pp. 123–137.
[13] H. Jin and X. Y. Zhou, Behavioral portfolio selection in continuous time, Math. Finance, 18 (2008), pp. 385–426.
[14] H. Jin and X. Y. Zhou, Erratum to Behavioral portfolio selection in continuous time, Math. Finance, 20 (2010), pp. 521–525.
[15] N. Kazamaki, Continuous Exponential Martingales and BMO, Lecture Notes in Math. 1579, Springer, Berlin, 1994.
[16] M. Kobylanski, Backward stochastic differential equations and partial differential equations with quadratic growth, Ann. Probab., 28 (2000), pp. 558–602.
[17] G. Loewenstein and D. Prelec, Anomalies in intertemporal choice: Evidence and an interpretation, Quart. J. Econom., 107 (1992), pp. 573–597.
[18] M. A. Morlais, Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem, Finance Stoch., 13 (2009), pp. 121–150.
[19] S. Peng, A general stochastic maximum principle for optimal control problems, SIAM J. Control Optim., 28 (1990), pp. 966–979.
[20] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, John Wiley, New York, 1962.
[21] J. Yong, A deterministic linear quadratic time-inconsistent optimal control problem, Math. Control Related Fields, 1 (2011), pp. 83–118.
[22] J. Yong and X. Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, 1999.
[23] X. Y. Zhou and D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, Appl. Math. Optim., 42 (2000), pp. 19–33.