GORENSTEIN SILTING MODULES AND GORENSTEIN PROJECTIVE MODULES

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Dedicated to Pu Zhang on the occasion of his 60th birthday

Abstract. (Partial) Gorenstein silting modules are introduced and investigated. It is shown that for finite dimensional algebras of finite CM-type, partial Gorenstein silting modules are in bijection with \( \tau_G \)-rigid modules; Gorenstein silting modules are the module-theoretic counterpart of 2-term Gorenstein silting complexes; and the relation between 2-term Gorenstein silting complexes, \( \tau \)-structures and torsion pairs in module categories. Furthermore, the corresponding version of the classical Brenner-Butler theorem in this setting are characterised; and the upper bound of the global dimension of endomorphism algebras of 2-term Gorenstein silting complexes over an algebra \( A \) are also characterised by terms of the Gorenstein global dimension of \( A \).

1. Introduction and Preliminaries

Silting complexes were first introduced by Keller and Vossieck [KV] to study \( \tau \)-structures in the bounded derived category of representations of Dynkin quivers. This kind of complexes is a generalization of tilting complexes and tilting modules. Later, 2-term silting complexes arouse the interest of many mathematicians. For examples, the connection with torsion pairs and support \( \tau \)-tilting modules are given([HKM, MM, AIR]); the counterpart of the classical tilting theorem and the global dimension of its endomorphism algebras are characterised([BZ1, BZ2]).

Silting modules introduced by Angeleri Hügel-Marks-Vitória [AMV1] over an arbitrary ring who are intended to generalize tilting modules in a similar fashion as 2-term silting complexes generalize 2-term tilting complexes and also, coincide with support \( \tau \)-tilting modules for finite dimensional algebras. Adachi-Iyama-Reiten [AIR] showed how 2-term silting complexes relate with silting modules, \( \tau \)-structures, and co-\( \tau \)-structures. Silting theory has been developed systematically, see for examples([MS, AMV2, A, LZ]).

The main idea of Gorenstein homological algebra is to replace projective modules by Gorenstein-projective modules. These modules were introduced by Enochs and Jenda
as a generalization of finitely generated modules of G-dimension zero over a two-sided Noetherian ring, in the sense of Auslander and Bridger [AB]. The subject has been developed to an advanced level, see for example [ABu, AR, Hap, EJ2, Ch, AM, Hol, CFH, BR, J, Chen, GZ, GK, RZ1, RZ2, RZ3, AS1, YLO, Z, CW, B1, G1].

Beligiannis [B2] introduced and studied the algebras of finite Cohen-Macaulay type (resp. finite CM-type for simply), which correspond to algebras of finite representation type in Gorenstein homological algebra. For this class of algebras, Gao [G2] introduced the relative transpose $\text{Tr}_G$ in terms of Gorenstein-projective modules and the corresponding Auslander-Reiten formula.

Based on these work, we draw a picture of all relevant modules as follows:

More precisely, there are the following natural questions:

**Question A:** Which class of modules coincides with the above $\tau_G$-rigid module?

**Question B:** What’s the correspondence in the bounded homotopy category with respect to the above modules?

The answers are given in the paper. We organize the paper as follows. In Section 2, we introduce the Gorenstein silting module and show that it has a bijective relationship with the relative rigid module over an algebra of finite CM-type. We also show the relations among Gorenstein tilting modules, Gorenstein star modules and Gorenstein silting modules. In Section 3, we show that the Gorenstein silting module is the module-theoretic counterpart of the 2-term Gorenstein silting complex. We also characterise it by the connection with the t-structure and torsion pair, and show the corresponding Brenner-Butler theorem, and characterise the global dimension of endomorphism algebras of 2-term Gorenstein silting complexes over an algebra $A$ by terms of the Gorenstein dimension of $A.$
First we give the main definitions in the paper.

For the algebra $A$, following [EJ2], an exact sequence $G_1 \xrightarrow{d_1} G_0 \rightarrow M \rightarrow 0$ is called a proper Gorenstein-projective presentation of $M$ if each $G_i$ is Gorenstein-projective and $\Hom_A(G, G_1) \rightarrow \Hom_A(G, G_0) \rightarrow \Hom_A(G, T) \rightarrow 0$ is exact for any Gorenstein-projective module $G$.

For a morphism $\theta : G_1 \rightarrow G_0$ with $G_1$ and $G_0$ being Gorenstein-projective modules.

We consider the class of $R$-modules $D_\theta := \{X \in \text{Mod}_R \mid \Hom_R(\theta, X) \text{ is epic} \}$.

**Definition 2.2** Let $R$ be a Noetherian ring. We say that an $R$-module $T$ is

- partial Gorenstein silting if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that
  
  $(Gs1)$ $D_\theta$ is a relative torsion class (i.e. closed for $G$-epimorphic images, $G$-extensions and coproducts);
  
  $(Gs2)$ $T$ lies in $D_\theta$.

- Gorenstein silting if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that $\text{Gen}_{G}(T) = D_\theta$.

Let $M$ be an $A$-module. Then there is a minimal Gorenstein-projective presentation $G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ of $M$. This induces the following exact sequences

$$0 \rightarrow \Hom_A(M, E) \rightarrow \Hom_A(G_0, E) \rightarrow \Hom_A(G_1, E) \rightarrow \text{Tr}_G M \rightarrow 0$$

and

$$0 \rightarrow \tau_G M \rightarrow D\Hom_A(G_1, E) \rightarrow D\Hom_A(G_0, E),$$

with $\text{Tr}_G M \in A(\text{Gproj})$-mod and $\tau_G M \in \text{mod}A(\text{Gproj})$. We denote by $A(\text{Gproj})$ the Cohen-Macaulay Auslander algebra of $A$.

**Definition 2.10** Let $A$ be a finite dimensional $k$-algebra of finite CM-type with the Gorenstein-projective generator $E$. We say that an $A$-module $M$ is $\tau_G$-rigid if $\Hom_{A(\text{Gproj})}((E, M), \tau_G M) = 0$ for the left $A(\text{Gproj})$-module $\Hom_A(E, M)$.

**Definition 3.1** Let $G^\bullet : G_1 \xrightarrow{d_1} G_0$ be a complex with $G_i \in \text{Gproj}A$ for $i = 0, 1$. We say that $G^\bullet$ is

- 2-term partial Gorenstein silting in $K^b(\text{Gproj}A)$ if it satisfies the following two conditions:
  
  (i) $G_1 \rightarrow \text{Im}d_1$ and $G_0 \rightarrow \text{Coker}d_1$ are right $\text{Gproj}A$-approximations;
  
  (ii) $\Hom_{D_{\text{gp}}(A)}(G^\bullet, G^\bullet[1]) = 0$.

- 2-term Gorenstein silting in $K^b(\text{Gproj}A)$ if it is a 2-term partial Gorenstein silting complex and $\text{thick}G^\bullet = D^b_{\text{gp}}(A)$.

Our main theorems are as follows:
**Theorem A** (Theorem 2.11 and 2.12) Let $A$ be a finite dimensional algebra of finite CM-type with the Gorenstein-projective generator $E$. Let $M$ be an $A$-module in $\text{mod} A$. Then the following statements are equivalent.

1. $M$ is $\tau_G$-rigid.
2. $\text{Hom}_A(E, M)$ is a $\tau$-rigid module, where $\tau$ is the Auslander-Reiten translation over $A(\text{Gproj})$.
3. $M$ is a partial Gorenstein silting $A$-module.

Moreover, $\tau_G M \cong \tau\text{Hom}_A(E, M)$.

Let $G^\bullet : G_1 \xrightarrow{d_1} G_0$ be a complex in $D^b_{\text{gpr}}(A)$ with $G_i \in \text{Gproj} A$ for $i = 0, 1$. Consider the subcategories of $\text{mod} A$:

$$T(G^\bullet) = \{ X \in \text{mod} A \mid \text{Hom}_{D^b_{\text{gpr}}(A)}(G^\bullet, X[1]) = 0 \}$$

and

$$F(G^\bullet) = \{ Y \in \text{mod} A \mid \text{Hom}_{D^b_{\text{gpr}}(A)}(G^\bullet, Y) = 0 \}.$$ 

Let $B = \text{End}_{D^b_{\text{gpr}}(A)}(G^\bullet)^{\text{op}}$. Consider the subcategories of $\text{mod} B$

$$\mathcal{X}(G^\bullet) = \text{Hom}_{D^b_{\text{gpr}}(A)}(G^\bullet, F(G^\bullet)[1]) \quad \text{and} \quad \mathcal{Y}(G^\bullet) = \text{Hom}_{D^b_{\text{gpr}}(A)}(G^\bullet, T(G^\bullet)).$$

**Theorem B** (Theorem 3.8, Theorem 3.12 and Theorem 3.16) Let $G^\bullet : G_1 \xrightarrow{d_1} G_0$ be a 2-term Gorenstein silting complex in $D^b_{\text{gpr}}(A)$, where $G_i \in \text{Gproj} A$. Then the following statements hold.

1. $(T(G^\bullet), F(G^\bullet))$ is a torsion pair for $\text{mod} A$.
2. $(\mathcal{X}(G^\bullet), \mathcal{Y}(G^\bullet))$ is a torsion pair in $\text{mod} B$ and there are equivalences

$$\text{Hom}_{D^b_{\text{gpr}}(A)}(G^\bullet, -) : T(G^\bullet) \xrightarrow{\cong} \mathcal{Y}(G^\bullet),$$

$$\text{Hom}_{D^b_{\text{gpr}}(A)}(G^\bullet, -[1]) : F(G^\bullet) \xrightarrow{\cong} \mathcal{X}(G^\bullet).$$

3. $\text{gldim} B \leq \text{Gdim} A + 1.$

Throughout $A$ is a finite dimensional $k$-algebra over a field $k$, and $\text{mod} A$ is the category of finitely generated left $A$-modules. A module $G$ of $\text{mod} A$ is Gorenstein-projective if there is an exact sequence

$$\cdots \xrightarrow{} P^{-1} \xrightarrow{} P^0 \xrightarrow{d_0} P^1 \xrightarrow{} \cdots$$

of projective modules of $\text{mod} A$, which stays exact after applying $\text{Hom}_A(-, P)$ for each projective module $P$, such that $G \cong \text{Ker} d_0$ (see [EJ2]). Denote by $\text{Gproj} A$ and $\text{proj} A$ the full subcategories of $\text{mod} A$ consisting of Gorenstein-projective modules and projective modules, respectively.

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}, \mathcal{Y}$ the full additive subcategories of $\mathcal{A}$. Let $M$ be an object of $\mathcal{A}$. Following [AS2], a morphism $f : X \xrightarrow{} M$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation of $M$ if any morphism from an object $\mathcal{X}$ to $M$ factors through $f$. 
$\mathcal{X}$ is called contravariantly finite if any object in $\mathcal{A}$ admits a right $\mathcal{X}$-approximation. A morphism $g : M \to Y$ with $Y \in \mathcal{Y}$ is called a left $\mathcal{Y}$-approximation of $M$ if any morphism from $M$ to an object $Y$ factors through $g$. $\mathcal{Y}$ is called covariantly finite if any object in $\mathcal{A}$ admits a left $\mathcal{Y}$-approximation.

Following [EJ2], an exact sequence $G_1 \xrightarrow{d_1} G_0 \to M \to 0$ (*) is called a proper Gorenstein-projective presentation of $M$ if each $G_i$ is Gorenstein-projective and $\text{Hom}_A(G, G_1) \to \text{Hom}_A(G, G_0) \to \text{Hom}_A(G, T) \to 0$ is exact for any Gorenstein-projective module $G$. (*) is minimal if $G_1 \to \text{Im}d_1$ and $G_0 \to M$ are right Gproj$A$-approximation. Moreover, the exact sequence $0 \to G_n \to \ldots \to G_1 \to G_0 \to M \to 0$ is called a proper Gorenstein-projective resolution of $M$ of length $n$ for some non-negative integer $n$, if each $G_i$ is all Gorenstein-projective and $0 \to \text{Hom}_A(G, G_n) \to \cdots \to \text{Hom}_A(G, G_0) \to \text{Hom}_A(G, M) \to 0$ is exact for any Gorenstein-projective module $G$. We say that $M$ has Gorenstein-projective dimension $d$, denoted by Gpd$M$, if $d$ is the least, and the Gorenstein dimension of $A$ is defined as follows:

$$\text{Gdim}A = \sup\{\text{Gpd}M | M \in \text{mod}A\}.$$ 

Now we write $C^b(A)$, $K^b(A)$ and $D^b(A)$ for the bounded complex category, bounded homotopy category and bounded derived category of $\text{mod}A$, respectively. Denote by $K^b(\text{Gproj}A)$ (resp. $K^b(\text{proj}A)$) the corresponding bounded homotopy category of Gorenstein-projective modules (resp. projective modules).

### 1.1. Gorenstein derived category

A complex $C^\bullet \in C^b(A)$ is $\mathcal{GP}$-acyclic, if $\text{Hom}_A(G, C^\bullet)$ is acyclic for each $G \in \text{Gproj}A$. A chain map $f^\bullet : X^\bullet \to Y^\bullet$ is a $\mathcal{GP}$-quasi-isomorphism, if $\text{Hom}_A(G, f^\bullet)$ is a quasi-isomorphism for each $G \in \text{Gproj}A$, i.e., there are isomorphisms of abelian groups for all $n \in \mathbb{Z}$,

$$H^n\text{Hom}_A(G, f^\bullet) : H^n\text{Hom}_A(G, X^\bullet) \cong H^n\text{Hom}_A(G, Y^\bullet).$$

Put

$$K^b_{\text{gpd}}(A) := \{X^\bullet \in K^b(A) | X^\bullet \text{ is } \mathcal{GP}\text{-acyclic}\}.$$ 

Then $K^b_{\text{gpd}}(A)$ is a thick triangulated subcategory of $K^b(A)$. Following [GZ], we have the following definition:

$$D^b_{\text{gp}}(A) := K^b(A)/K^b_{\text{gpd}}(A),$$

which is called the bounded Gorenstein derived category.

Following [AS1], an exact sequence $0 \to X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \to 0$ in $C^b(A)$ is $G$-exact if and only if $0 \to \text{Hom}_A(G, X^\bullet) \to \text{Hom}_A(G, Y^\bullet) \to \text{Hom}_A(G, Z^\bullet) \to 0$ is exact for all $G \in \text{Gproj}A$. In the module case, $g$ is called a $G$-epimorphism and $X$ is called a $G$-submodule of $Y$. From [GZ], if the exact sequence $0 \to X \to Y \to Z \to 0$ in $\text{mod}A$ is $G$-exact, then $X \to Y \to Z \to X[1]$ is a triangle in $D^b_{\text{gp}}(A)$.
1.2. CM-Auslander algebra. Recall from [B2] that $A$ is of finite Cohen-Macaulay type (resp. finite CM-type for simply), if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein-projective $A$-modules. Let $\{E_i\}_{i=1}^n$ be all non-isomorphic finitely generated Gorenstein-projective $A$-modules and $E = \bigoplus_{i=1}^n E_i$, and $A(Gproj) = \text{End}_A(E)^{op}$. Then $E$ is an $A$-$A(Gproj)$-bimodule, and $A(Gproj)$ is called the the Cohen-Macaulay Auslander (resp. CM-Auslander for simply) algebra of $A$.

1.3. torsion pairs and t-structures.

Definition 1.1. ([D]) A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\text{mod}A$ is called a torsion pair provided that:

1. $\mathcal{T} \cap \mathcal{F} = 0$;
2. If $T \rightarrow Y \rightarrow 0$ is exact with $T \in \mathcal{T}$ then $Y \in \mathcal{T}$;
3. If $0 \rightarrow Y \rightarrow F$ is exact with $F \in \mathcal{F}$ then $Y \in \mathcal{F}$;
4. For each $A$-module $X$ there is an exact sequence
   $$0 \rightarrow T \rightarrow X \rightarrow F \rightarrow 0$$
   with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Definition 1.2. ([BBD]) A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of the triangulated category $D^b_{gp}(A)$ is called a t-structure provided that:

1. $\mathcal{X}[1] \subset \mathcal{X}$ and $\mathcal{Y}[-1] \subset \mathcal{Y}$;
2. $\text{Hom}_{D^b_{gp}(A)}(X, Y[-1]) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
3. for any $C \in D^b_{gp}(A)$, there exists a distinguished triangle
   $$X \rightarrow C \rightarrow Y[-1] \rightarrow X[1]$$
   with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

2. Gorenstein silting modules

In this section, we introduce and study a class of modules, which we call the Gorenstein silting module, and show that it coincides with the relative rigid module over an algebra of finite CM-type introduced by Gao ([G2]). We also show the relations among Gorenstein tilting modules, Gorenstein star modules and Gorenstein silting modules.

2.1. Gorenstein silting modules. In this subsection, we introduce the Gorenstein silting module. Before this, we make some preparation.

Throughout this subsection, let $R$ be a Noetherian ring, and $\text{Mod}R$ the category of all left $R$-modules. We denote by $Gp(R)$ (resp. $Gi(R)$) the full subcategory of $\text{Mod}R$ consisting of Gorenstein-projective (resp. Gorenstein-injective) modules.

Now we assume that $Gp(R)$ is contravariantly finite in $\text{Mod}R$. For $R$-modules $M$ and $N$, we compute right derived functors of $\text{Hom}_R(M, N)$ using a Gorenstein-projective
resolution of $M$ ([EJ2], [Hol]). We will denote these derived functors by $\text{Gext}^1_R(M, N)$. A short exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is $G$-exact if and only if it is in $\text{Gext}^1_R(L, M)$.

Let $T$ be an $R$-module. Put

$$\text{Pres}_G(T) := \{ M \in \text{Mod}_R | \exists a G\text{-exact sequence } T_1 \rightarrow T_0 \rightarrow M \rightarrow 0 \text{ with } T_i \in \text{Add} T \} ,$$

$$\text{Gen}_G(T) := \{ M \in \text{Mod}_R | \exists a G\text{-exact sequence } T_0 \rightarrow M \rightarrow 0 \text{ with } T_0 \in \text{Add} T \} ,$$

and

$$T^{\perp} := \{ M \in \text{Mod}_R \text{Gext}^1_R(T, M) = 0 \} , \quad T_0^{\perp} := \{ M \in \text{Mod}_R \text{Hom}_R(T, M) = 0 \} ,$$

where Add$T$ denotes the subcategory of modules consisting of direct summands of direct sums of $T$.

For a morphism $\theta : G_1 \rightarrow G_0$ with $G_1$ and $G_0$ being Gorenstein-projective modules, we consider the class of $R$-modules

$$D_\theta := \{ X \in \text{Mod}_R | \text{Hom}_R(\theta, X) \text{ is epic} \} .$$

We first collect some useful properties of $D_\theta$.

**Lemma 2.1.** Let $\theta : G_1 \rightarrow G_0$ with $G_1$ and $G_0$ being Gorenstein-projective modules, and $T$ the cokernel of $\theta$ such that $\theta$ is the proper Gorenstein-projective presentation of $T$.

(i) $D_\theta$ is closed under $G$-epimorphic images, $G$-extensions and direct products.

(ii) The class $D_\theta$ is contained in $T^{\perp}$. 

**Proof.** The statement (i) is easy to prove. Now we prove (ii). There exists the following diagram

$$\begin{array}{ccc}
G_1 & \xrightarrow{\theta} & G_0 & \xrightarrow{i} & T & \rightarrow & 0 \\
\pi & \downarrow & \downarrow & & \downarrow & & \\
\text{Im}\theta &ankle brace
\end{array}$$

Consider the $G$-exact sequence $0 \rightarrow \text{Im}\theta \rightarrow G_0 \rightarrow T \rightarrow 0$. Applying the functor $\text{Hom}_R(-, X)$ for any $X \in D_\theta$, then we get the $G$-exact sequence

$$\text{Hom}_R(G_0, X) \xrightarrow{i^*} \text{Hom}_R(\text{Im}\theta, X) \rightarrow \text{Gext}^1_R(T, X) \rightarrow 0 .$$

We show that $i^*$ is surjective. Let $f \in \text{Hom}_R(\text{Im}\theta, X)$. Since $X \in D_\theta$, there is a map $g : G_0 \rightarrow X$ such that $f = g\theta = gi\pi$. Since $\pi$ is an epimorphism, we have $f = gi$. Hence $\text{Gext}^1_R(T, X) = 0$, and so $X \in T^{\perp}$. □

**Definition 2.2.** We say that an $R$-module $T$ is

- **partial Gorenstein silting** if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that
(Gs1) $D_\theta$ is a relative torsion class (i.e. closed for $G$-epimorphic images, $G$-extensions and coproducts); 

(Gs2) $T$ lies in $D_\theta$.

• **Gorenstein silting** if there is a proper Gorenstein-projective presentation $\theta$ of $T$ such that $\text{Gen}_G(T) = D_\theta$.

**Proposition 2.3.** Let $T$ be an $R$-module with the proper Gorenstein-projective presentation $\theta : G_1 \to G_0$. If $T$ is a partial Gorenstein silting module with respect to $\theta$, and for each $E \in \text{Gp}(R)$, there exists a $G$-exact sequence $E \xrightarrow{\phi} T_0 \to T_{-1} \to 0$ with $T_0$ and $T_{-1}$ in $\text{Add}T$ such that $\phi$ is the left $D_\theta$-approximation, then $T$ is a Gorenstein silting module.

**Proof.** Since $T$ is a partial Gorenstein silting module with respect to $\theta$, it is clear that $\text{Gen}_G(T) \subseteq D_\theta$. Let $X$ be an object in $D_\theta$. Since $\text{Gp}(R)$ is contravariantly finite, there is an $R$-module $E \in \text{Gp}(R)$ and a $G$-epimorphism $E \xrightarrow{\phi} X$. By assumption this epimorphism factors through the left $D_\theta$-approximation $\phi : E \to T_0$ via a $G$-epimorphism $g : T_0 \to X$. Thus $X$ lies in $\text{Gen}_G(T)$. This means that $T$ is a Gorenstein silting module. \[\square\]

2.2. **Connection with Gorenstein tilting (resp. star) modules.** In this subsection, we characterise the relations among Gorenstein silting modules, Gorenstein tilting modules and Gorenstein star modules.

Throughout, let $R$ be a Noetherian ring such that $\text{Gp}(R)$ is the contravariantly finite subcategory of $\text{Mod}R$.

**Definition 2.4.** ([AS1,YLO,G1]) An $R$-module $T$ is called a Gorenstein tilting module if $T^G \subseteq \text{Pres}_G(T)$, or equivalently, it satisfies the following three conditions:

- **(T1)** $\text{Gpd}_R T \leq 1$.
- **(T2)** $\text{Gext}_R^1(T, T^{(I)}) = 0$ for all sets $I$.
- **(T3)** For any $E \in \text{Gp}(R)$, there exists a $G$-exact sequence $0 \to E \to T_0 \to T_{-1} \to 0$ with each $T_i \in \text{Add}T$.

**Definition 2.5.** ([Z]) An $R$-module $T$ is called a Gorenstein star module if

- (i) Any $G$-exact sequence $0 \to X \to T_0 \to Y \to 0$ with $T_0 \in \text{Add}T$ and $X \in \text{Gen}_G(T)$ is $\text{Hom}_R(T, -)$-exact.
- (ii) $\text{Gen}_G(T) = \text{Pres}_G(T)$.

**Lemma 2.6.** ([Z, Proposition 2.9]) The following statements are equivalent for an $R$-module $T$:

- (i) $T$ is a Gorenstein star module and $\text{Gen}_G(T)$ is closed under $G$-extension.
- (ii) $\text{Gen}_G(T) = \text{Pres}_G(T) \subseteq T^G \subseteq $.
Proposition 2.7. The following hold.

(1) Each Gorenstein tilting $R$-module is Gorenstein silting.

(2) Each Gorenstein silting $R$-module is a Gorenstein star module.

Proof. (1) Let $T$ be a Gorenstein tilting $R$-module. Then by definition there exists a $G$-exact sequence as follows:

$$0 \rightarrow G_1 \xrightarrow{\theta} G_0 \rightarrow T \rightarrow 0.$$ 

Let $X \in \text{Gen}_G(T)$. Then there exists a $G$-epimorphism $g : T(I) \rightarrow X$. Since $\text{Gpd}_R T \leq 1$ and $\text{Gext}_R^1(T, T(I)) = 0$, applying the functor $\text{Hom}_R(T, -)$ to $g$, it follows that $\text{Gext}_R^1(T, X) = 0$, and hence $X \in D_\theta$.

Let $X \in D_\theta$. Then by Lemma 2.1 $X \in T^{G\perp}$. It follows from the definition $T^{G\perp} = \text{Pres}_G(T)$ that $X \in \text{Pres}_G(T)$. This implies that $X \in \text{Gen}_G(T)$. Therefore, we prove that $\text{Gen}_G(T) = D_\theta$, and $T$ is a Gorenstein silting module with respect to $\theta$.

(2) Let $T$ be a Gorenstein silting $R$-module with the proper Gorenstein-projective presentation $\theta : G_1 \rightarrow G_0$. Let $X \in \text{Gen}_G(T)$. Then there is the $G$-epimorphic universal map $u : T(I) \rightarrow X$ for some index set $I$. We will show that $K := \ker u$ lies in $D_\theta = \text{Gen}_G(T)$. Pick $f : G_1 \rightarrow K$. Since $T(I) \in T^{G\perp}$, we get the following commutative diagram of exact rows

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\theta} & G_0 \\
\downarrow{f} & & \downarrow{\pi} \\
0 & \xrightarrow{\alpha} & K \\
\downarrow{g} & & \downarrow{h} \\
T & \xrightarrow{\alpha} & X \\
\downarrow{h} & & \downarrow{0} \\
0 & & 0
\end{array}
$$

By the universality of $u$, there is the morphism $h' : T \rightarrow T(I)$ such that $h = uh'$. Then by $ug = h\pi$ we have $u(g-h'\pi) = 0$. Hence there is a map $g' : G_0 \rightarrow K$ such that $g-h'\pi = vg'$, and moreover, we get from $vf = g'\theta$ that $f = g'\theta$. This implies that $K \in D_\theta = \text{Gen}_G(T)$, and so $\text{Gen}_G(T) \subseteq \text{Pres}_G(T)$. This means that $\text{Pres}_G(T) = \text{Gen}_G(T) \subseteq T^{G\perp}$. Thus Lemma 2.6 shows that $T$ is a Gorenstein star module.

Recall that a ring $R$ is Gorenstein if $R$ is two-sided Noetherian and has finite injective dimension, both as left and right $R$-module.

Theorem 2.8. Let $R$ be a 1-Gorenstein ring and $T$ an $R$-module. Then the following are equivalent.

(1) $T$ is a Gorenstein tilting module.

(2) $T$ is a Gorenstein silting module.

(3) $T$ is a Gorenstein star module and $\text{Gi}(R) \subseteq \text{Pres}_G(T)$.

Proof. First by [Z, Theorem 3.2] we know that $(1) \iff (3)$. The theorem immediately follows from Proposition 2.7.
Example 2.9. Let $A = k[x]/\langle x^2 \rangle$ with $k$ a field. Then $T_2(A) = \left( \begin{array}{c} A \\ 0 \\ A \end{array} \right)$ has 9 finite-dimensional indecomposable modules:

\[
G_1 = \left( \begin{array}{c} S \\ 0 \\ A \end{array} \right), \quad G_2 = \left( \begin{array}{c} A \\ 0 \\ A \end{array} \right), \quad G_3 = \left( \begin{array}{c} S \\ 0 \\ A \end{array} \right), \quad G_4 = \left( \begin{array}{c} A \\ 0 \\ A \end{array} \right),
\]

where $G_i$, $1 \leq i \leq 5$ are Gorenstein-projectives (see Beligiannis and Reiten [BR, p.101]). Note that $T_2(A)$ is 1-Gorenstein, and $G_8$ is a Gorenstein silting module.

We finish this section with an important class of examples of (partial) Gorenstein silting modules: $\tau_G$-rigid modules over a finite dimensional $k$-algebra, where $\tau_G$ was introduced in [G2] for an algebra of finite CM-type.

2.3. Relative rigid modules over algebras of finite CM-type. Throughout this section, $A$ is a finite dimensional $k$-algebra of finite CM-type over a field $k$ and $\text{mod}A$ is the category of finitely generated left $A$-modules. All $A$-modules we consider are in $\text{mod}A$. Use the notation in the introduction. Denote by $A(\text{Gproj})$-mod the category of finitely generated right $A(\text{Gproj})$-modules and $A(\text{Gproj})$-mod the stable category of $A(\text{Gproj})$-mod modulo projective $A(\text{Gproj})$-modules. Let $D : A(\text{Gproj})$-mod $\rightarrow \text{mod}A(\text{Gproj})$ be the duality. For Simplicity, we denote the functors $\text{Hom}_A(\cdot, \cdot)$ by $(\cdot, \cdot)$ and $\text{Hom}_{A(\text{Gproj})}(\cdot, \cdot)$ by $(A(\text{Gproj})(\cdot, \cdot))$.

Let $M$ be an $A$-module. Then there is a minimal Gorenstein-projective presentation $G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ of $M$. This induces the following exact sequences

\[
0 \rightarrow (M, E) \rightarrow (G_0, E) \rightarrow (G_1, E) \rightarrow \text{Tr}_G M \rightarrow 0
\]

and

\[
0 \rightarrow \tau_G M \rightarrow D(G_1, E) \rightarrow D(G_0, E),
\]

with $\text{Tr}_G M \in A(\text{Gproj})$-mod and $\tau_G M \in \text{mod}A(\text{Gproj})$. Recall from [G2] that

\[
\text{Tr}_G : \text{mod}A/Gp(A) \rightarrow A(\text{Gproj})\text{-mod}
\]

is called the relative transpose of $A$, and moreover, $\text{Tr}_G$ is a faithful functor.

Definition 2.10. We say that an $A$-module $M$ is $\tau_G$-rigid if $\text{Hom}_{A(\text{Gproj})}((E, M), \tau_G M) = 0$ for the left $A(\text{Gproj})$-module $\text{Hom}_A(E, M)$.

Next we provide some properties for $\tau_G$-rigid module. For an $A$-module $M$, put

\[
\tau_G^0(M) := \{ X \in \text{mod}A(\text{Gproj}) | \text{Hom}_{A(\text{Gproj})}(X, \tau_G M) = 0 \}.
\]

Proposition 2.11. Let $M$ be an $A$-module and

\[
G_1 \xrightarrow{f} G_0 \rightarrow M \rightarrow 0
\]

the minimal proper Gorenstein-projective presentation of $M$. Then the following statements hold.

(1) $M$ is $\tau_G$-rigid if and only if $\text{Hom}_A(f, M)$ is an epimorphism.
(2) $M$ is $\tau_G$-rigid if and only if $\text{Hom}_A(E,M)$ is a $\tau$-rigid module, where $\tau$ is the Auslander-Reiten translation of $A(Gproj)$.

(3) $\tau_G M \cong \tau(E,M)$.

(4) Suppose that $M$ is $\tau_G$-rigid. Then $\text{Hom}_A(E,\text{Gen}_G M) \subseteq \perp^0(\tau_G M)$.

Proof. There is an exact sequence

$$0 \longrightarrow \tau_G M \longrightarrow D(G_1,E) \longrightarrow D(G_0,E). \quad (**)$$

Applying $\text{Hom}_{A(Gproj)}(N,-)$ for any $A(Gproj)$-module $N$, we have the following commutative diagram of exact sequences:

$$
\begin{array}{cccc}
0 & \longrightarrow & A(Gproj)(N,\tau_G M) & \longrightarrow & A(Gproj)(N,D(G_1,E)) & \longrightarrow & A(Gproj)(N,D(G_0,E)) \\
\end{array}
$$

$$
\begin{array}{ccc}
 & \cong & \cong \\
D_A(Gproj)((E,G_1),N) & \longrightarrow & D_A(Gproj)((E,G_0),N) & \longrightarrow & D_A(Gproj)((E,M),N) & \longrightarrow & 0.
\end{array}
$$

Then we get that $\text{Hom}_{A(Gproj)}((E,G_0),N) \longrightarrow \text{Hom}_{A(Gproj)}((E,G_1),N)$ is an epimorphism.

(1) and (2) Applying $\text{Hom}_A(E,-)$ the ($*$), we can get the projective resolution of $(E,M)$:

$$(E,G_1) \overset{\sigma}{\longrightarrow} (E,G_0) \longrightarrow (E,M) \longrightarrow 0.$$ Then we have from above arguments that $\text{Hom}_{A(Gproj)}((E,M),\tau_G M) = 0$ if and only if $\text{Hom}_{A(Gproj)}(\sigma, (E,M))$ is epic if and only if $\text{Hom}_A(f,M)$ is epic. Notice that the statement that $\text{Hom}_{A(Gproj)}(\sigma, (E,M))$ is epic means that $\text{Hom}_A(E,M)$ is a partial silting $A(Gproj)$-module, equivalently, $\text{Hom}_A(E,M)$ is $\tau$-rigid.

(3) Consider the exact sequence

$$0 \longrightarrow \tau(E,M) \longrightarrow D_{A(Gproj)}((E,G_1),A(Gproj)) \longrightarrow D_{A(Gproj)}((E,G_0),A(Gproj)).$$ Then we get from above arguments that $\tau_G M \cong \text{Hom}_{A(Gproj)}(A(Gproj),\tau_G M) \cong \tau(E,M)$.

(4) We have proved in (2) that $\text{Hom}_A(E,M)$ is a $\tau$-rigid module. If $Y \in \text{Gen}_G M$, then there is the $G$-exact sequence $M(I) \longrightarrow Y \longrightarrow 0$ for some index set $I$. Therefore we have that $(E,\text{Gen}_G M) \subseteq \text{Gen}(E,M) \subseteq (E,M)^{\perp I}$. Since there are the following isomorphisms

$$\text{Ext}^1_{A(Gproj)}((E,M), X) \cong D\text{Hom}^1_{A(Gproj)}(\tau^{-1}X, (E,M)) \cong D\text{Hom}^1_{A(Gproj)}(X, \tau(E,M)) \cong D\text{Hom}^1_{A(Gproj)}(X, \tau_G M),$$

it follows that $X \in (E,M)^{\perp I}$ if and only if $X \in \perp^0(\tau_G M)$. Hence $\text{Hom}_A(E,\text{Gen}_G M) \subseteq \perp^0(\tau_G M)$.

\[\Box\]

**Theorem 2.12.** Let $M$ be an $A$-module. Then $M$ is $\tau_G$-rigid if and only if $M$ is a partial Gorenstein silting $A$-module.
Proof. Let $G_1 \xrightarrow{\theta} G_0 \rightarrow M \rightarrow 0$ be the minimal proper Gorenstein-projective presentation of $M$. Since $G_0$ and $G_1$ are finitely generated, it follows that $D_{\theta}$ is closed under coproducts. Then we get from Lemma 2.1 that $D_{\theta}$ is a relative torsion class. On the other hand, we get from Proposition 2.11(1) that $M$ is $\tau_G$-rigid if and only if $\text{Hom}_A(\theta, M)$ is surjective. The latter implies that $M \in D_{\theta}$. Thus $M$ is $\tau_G$-rigid if and only if $M$ is partial Gorenstein silting. □

Recall from [DLJ] that an algebra $A$ is called $\tau$-tilting finite if it admits finite number of isomorphism classes of indecomposable $\tau$-rigid modules. Inspired on it, we introduce $\tau_G$-tilting finite algebras.

**Definition 2.13.** An algebra $A$ is called $\tau_G$-tilting finite if it admits finite number of isomorphism classes of indecomposable $\tau_G$-rigid modules.

**Corollary 2.14.** If $A(\text{Gproj})$ is $\tau$-tilting finite, then $A$ is $\tau_G$-tilting finite, and further has finite number of isomorphism classes of indecomposable partial Gorenstein silting modules.

*Proof.* By Proposition 2.11(1), we know that an $A$-module $M$ is $\tau_G$-rigid if and only if $\text{Hom}_A(E, M)$ is a $\tau$-rigid module. This implies the result by Theorem 2.12. □

### 3. 2-term Gorenstein silting complexes

In this section, we show that the 2-term Gorenstein silting complex is in bijection with the Gorenstein silting module. Then we characterise it by the connection with the $t$-structure and torsion pair. We also characterise the global dimension of endomorphism algebras of 2-term Gorenstein silting complexes over an algebra $A$ by terms of the Gorenstein dimension of $A$. Throughout, we denote by $A$ a finite dimensional $k$-algebra over a field $k$.

#### 3.1. 2-term Gorenstein silting complexes

In this subsection, we study 2-term Gorenstein silting complexes, and show the links with $t$-structures and torsion pairs. The Brenner-Butler theorem is given.

**Definition 3.1.** ([CW]) Let $G^\bullet : G_1 \xrightarrow{d_1} G_0 \rightarrow 0$ be a complex in $D_{gp}^b(A)$ with $G_i \in \text{Gproj}A$ for $i = 0, 1$. We say that $G^\bullet$ is

- **2-term partial Gorenstein silting** if it satisfies the following two conditions:
  
  (i) $G_1 \rightarrow \text{Im}d_1$ and $G_0 \rightarrow \text{Coker}d_1$ are right $\text{Gproj}A$-approximations;

  (ii) $\text{Hom}_{D_{gp}^b(A)}(G^\bullet, G^\bullet[1]) = 0$;

- **2-term Gorenstein silting** in $K^b(\text{Gproj}A)$ if it is a 2-term partial Gorenstein silting complex and thick$G^\bullet = K^b(\text{Gproj}A)$. 

\[\]
Let $G^\bullet : G_1 \xrightarrow{d_1} G_0$ be a complex in $D_{gp}^b(A)$ with $G_i \in \text{Gproj} A$ for $i = 0, 1$. Consider the subcategories of $\text{mod} A$:

$$\mathcal{T}(G^\bullet) = \{ X \in \text{mod} A \mid \text{Hom}_{D_{gp}^b(A)}(G^\bullet, X[1]) = 0 \}$$

and

$$\mathcal{F}(G^\bullet) = \{ Y \in \text{mod} A \mid \text{Hom}_{D_{gp}^b(A)}(G^\bullet, Y) = 0 \}.$$

Assume that $G^\bullet : G_1 \rightarrow G_0$ is a 2-term Gorenstein silting complex in $K^b(\text{Gproj} A)$. Consider the subcategories of $D_{gp}^b(A)$:

$$D_{gp}^{\leq 0}(G^\bullet) = \{ X^\bullet \in D_{gp}^b(A) \mid \text{Hom}_{D_{gp}^b(A)}(G^\bullet, X^\bullet[i]) = 0, \text{ for } i > 0 \}$$

and

$$D_{gp}^{\geq 0}(G^\bullet) = \{ X^\bullet \in D_{gp}^b(A) \mid \text{Hom}_{D_{gp}^b(A)}(G^\bullet, X^\bullet[i]) = 0, \text{ for } i < 0 \}$$

Then we have the following facts.

**Lemma 3.2.** Let $G^\bullet$ be a 2-term Gorenstein silting complex in $K^b(\text{Gproj} A)$. Then

1. $(D_{gp}^{\leq 0}(G^\bullet), D_{gp}^{\geq 0}(G^\bullet))$ is a t-structure in $D_{gp}^b(A)$.
2. $\mathcal{T}(G^\bullet) = D_{gp}^{\leq 0}(G^\bullet) \cap \text{mod} A$ and $\mathcal{F}(G^\bullet) = D_{gp}^{\geq 1}(G^\bullet) \cap \text{mod} A$.

**Proof.** Conclusion (1) can be obtained from [KY, Section 3.3]. (2) can be obtained by definitions. \[\square\]

Next we consider the relations between 2-term Gorenstein silting complexes, t-structures and torsion pairs in module categories. A key lemma is given below.

**Lemma 3.3.** For any $X^\bullet \in D_{gp}^b(A)$ and $n \in \mathbb{Z}$, we have a functorial exact sequence

$$0 \rightarrow \text{Hom}_{D_{gp}^b(A)}(G^\bullet, H^n(X^\bullet)[1]) \rightarrow \text{Hom}_{D_{gp}^b(A)}(G^\bullet, X^\bullet[n])$$

$$\rightarrow \text{Hom}_{D_{gp}^b(A)}(G^\bullet, H^n(X^\bullet)) \rightarrow 0.$$ 

**Proof.** For $X^\bullet[n] \in D_{gp}^b(A)$, applying $\text{Hom}_{D_{gp}^b(A)}(\cdot, X^\bullet[n])$ to a distinguished triangle

$$G_1 \xrightarrow{d_1} G_0 \rightarrow G^\bullet \rightarrow G_1[1],$$

we have a short exact sequence

$$0 \rightarrow \text{Coker} \left( \text{Hom}_{D_{gp}^b(A)}(d^1, X^\bullet[n - 1]) \right) \rightarrow \text{Hom}_{D_{gp}^b(A)}(G^\bullet, X^\bullet[n])$$

$$\rightarrow \text{Ker} \left( \text{Hom}_{D_{gp}^b(A)}(d^1, X^\bullet[n]) \right) \rightarrow 0.$$

Since

$$\text{Ker} \left( \text{Hom}_{D_{gp}^b(A)}(d^1, X^\bullet[n]) \right) \cong \text{Ker} \left( \text{Hom}_A(d^1, H^n(X^\bullet)) \right) \cong \text{Hom}_{D_{gp}^b(A)}(G^\bullet, H^n(X^\bullet)),$$

and

$$\text{Coker} \left( \text{Hom}_{D_{gp}^b(A)}(d^1, X^\bullet[n - 1]) \right) \cong \text{Coker} \left( \text{Hom}_A(d^1, H^{n-1}(X^\bullet)) \right) \cong \text{Hom}_{D_{gp}^b(A)}(G^\bullet, H^{n-1}(X^\bullet)[1]),$$

we get the desired exact sequence. \[\square\]
**Proposition 3.4.** Let $G^\bullet$ be a 2-term Gorenstein silting complex in $K^b(\text{Gproj}A)$, and $C_{gp}(G^\bullet) := D_{gp}^0(G^\bullet) \cap D_{gp}^0(G^\bullet)$ the heart of the induced t-structure $(D_{gp}^{\leq 0}(G^\bullet), D_{gp}^{\geq 0}(G^\bullet))$. Let $B = \text{End}_{D_{gp}(A)}(G^\bullet)^{op}$.

1. $C_{gp}(G^\bullet)$ is an abelian category and the short exact sequences in $C_{gp}(G^\bullet)$ are precisely the triangles in $D_{gp}(A)$ all of whose vertices are objects in $C_{gp}(G^\bullet)$.

2. For a complex $X^\bullet$ in $D_{gp}(A)$, we have that $X^\bullet$ is in $C_{gp}(G^\bullet)$ if and only if $H^i(X^\bullet)$ is in $\mathcal{T}(G^\bullet)$, $H^{-i}(X^\bullet)$ is in $\mathcal{F}(G^\bullet)$ and $H^i(X^\bullet) = 0$ for $i \neq -1,0$.

3. The functor $\text{Hom}_{D_{gp}(A)}(G^\bullet, \underline{-}) : C_{gp}(G^\bullet) \to \text{mod}B$ is an equivalence of abelian categories.

4. For any $E \in \text{Gproj}A$, there is a triangle in $D_{gp}(A)$

$$E \xrightarrow{e} G^\bullet \xrightarrow{f} G'^\bullet \xrightarrow{g} E[1] \quad (\triangle_{G^\bullet})$$

with $G^\bullet, G'^\bullet \in \text{add}G^\bullet$.

5. Suppose that $A$ is of finite CM-type with the Gorenstein-projective generator $E$. Then

$$Q^\bullet : \text{Hom}_{D_{gp}(A)}(G^\bullet, G^\bullet) \xrightarrow{\text{Hom}_{D_{gp}(A)}(G^\bullet, \underline{f})} \text{Hom}_{D_{gp}(A)}(G^\bullet, G'^\bullet)$$

is a 2-term partial silting complex in $K^b(\text{proj}B)$, where $f$ is the map from the triangle $\triangle_{G^\bullet}$ of the Gorenstein-projective generator $E$.

**Proof.** (1) The pair $(D_{gp}^{\leq 0}(G^\bullet), D_{gp}^{\geq 0}(G^\bullet))$ is a t-structure in $D_{gp}(A)$. Then we come to the conclusion.

(2) From Lemma 3.3, we have that

$$D_{gp}^{\leq 0}(G^\bullet) = \{ X^\bullet \in D_{gp}(A) \mid \text{Hom}_{D_{gp}(A)}(G^\bullet, H^i(X^\bullet)) = 0 \text{ for } i > 0, \text{Hom}_{D_{gp}(A)}(G^\bullet, H^j(X^\bullet)[1]) = 0 \text{ for } j \geq 0 \}$$

$$= \{ X^\bullet \in D_{gp}(A) \mid H^i(X^\bullet) = 0 \text{ for } i > 0 \text{ and } \text{Hom}_{D_{gp}(A)}(G^\bullet, H^0(X^\bullet)[1]) = 0 \}$$

$$= \{ X^\bullet \in D_{gp}(A) \mid H^i(X^\bullet) = 0 \text{ for } i > 0 \text{ and } H^0(X^\bullet) \in \mathcal{T}(G^\bullet) \},$$

and

$$D_{gp}^{\geq 0}(G^\bullet) = \{ X^\bullet \in D_{gp}(A) \mid \text{Hom}_{D_{gp}(A)}(G^\bullet, H^i(X^\bullet)) = 0 \text{ for } i < 0, \text{Hom}_{D_{gp}(A)}(G^\bullet, H^j(X^\bullet)[1]) = 0 \text{ for } j < -1 \}$$

$$= \{ X^\bullet \in D_{gp}(A) \mid H^i(X^\bullet) = 0 \text{ for } i < -1 \text{ and } \text{Hom}_{D_{gp}(A)}(G^\bullet, H^{-1}(X^\bullet)) = 0 \}$$

$$= \{ X^\bullet \in D_{gp}(A) \mid H^i(X^\bullet) = 0 \text{ for } i < -1 \text{ and } H^{-1}(X^\bullet) \in \mathcal{F}(G^\bullet) \}.$$ 

Therefore, we get that $X^\bullet \in C_{gp}(G^\bullet)$ if and only if $H^0(X^\bullet) \in \mathcal{T}(G^\bullet)$, $H^{-1}(X^\bullet) \in \mathcal{F}(G^\bullet)$ and $H^i(X^\bullet) = 0$ for $i \neq -1,0$.

(3) The proof can be found in [HKM, Theorem 1.3].

(4) Let $G'^{\bullet} \to E[1]$ be a right add$G^\bullet$-approximation of $E[1]$. Extend it to a triangle

$$E \to H^\bullet \to G'^{\bullet} \to E[1], \quad (*)$$
Lemma 3.5. The following hold:

where $H^\bullet$ is a 2-term complex in $K^b(\operatorname{Gproj} A)$. By applying the functors $\operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, -)$ and $\operatorname{Hom}_{D^b_{sp}(A)}(-, G^\bullet)$ to the triangle $(\ast)$, we have that

$$\operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, H^\bullet[1]) = 0 \quad \text{and} \quad \operatorname{Hom}_{D^b_{sp}(A)}(H^\bullet, G^\bullet[1]) = 0.$$  

Applying $\operatorname{Hom}_{D^b_{sp}(A)}(-, H^\bullet)$ yields $\operatorname{Hom}_{D^b_{sp}(A)}(H^\bullet, H^\bullet[1]) = 0$. Hence $G^\bullet \oplus H^\bullet$ is a 2-term partial Gorenstein silting complex in $K^b(\operatorname{Gproj} A)$. The triangle $(\ast)$ shows that $E \in \operatorname{thick}(G^\bullet \oplus H^\bullet)$ and so $G^\bullet \oplus H^\bullet$ is a 2-term Gorenstein silting complex. Therefore, we get the desired triangle $\Delta_{G^\bullet}$.

(5) Let $\alpha$ be a morphism in $\operatorname{Hom}_{K^b(\operatorname{proj} B)}(Q^\bullet, Q^\bullet[1])$. Then it has the following form

$$\operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, G'^\bullet) \xrightarrow{\operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, f)} \operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, G''^\bullet),$$

and there is a morphism $h : G'^\bullet \to G''^\bullet$ such that $\alpha = \operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, h)$. Since $\operatorname{Hom}_{D^b_{sp}(A)}(E, E[1]) = 0$, there are unique morphisms $h_1$, $h_2$ such that the following diagram

$$\begin{array}{cccccccc}
E & \xrightarrow{e} & G'^\bullet & \xrightarrow{f} & G''^\bullet & \xrightarrow{g} & E[1] \\
\downarrow h_1 & & \downarrow h & & \downarrow h_2 & & \downarrow h_1[1] \\
G'^\bullet & \xrightarrow{f} & G''^\bullet & \xrightarrow{g} & E[1] & \xrightarrow{-e[1]} & G''^\bullet[1].
\end{array}$$

is commutative. So there is a morphism $h_3 : G''^\bullet \to G''^\bullet$ such that $h_2 = gh_3$, and also,

$$g(h - h_3 f) = gh - gh_3 f = gh - h_2 f = 0.$$  

Hence there is a morphism $h_4$ such that $h - h_3 f = f h_4$. Applying $\operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, -)$ to $h - h_3 f = f h_4$ yields

$$\alpha = \operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, h_3) \operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, f) + \operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, f) \operatorname{Hom}_{D^b_{sp}(A)}(G^\bullet, h_4),$$

which implies that $\alpha$ regarded as a map in $\operatorname{Hom}_{K^b(\operatorname{proj} B)}(Q^\bullet, Q^\bullet[1])$ is null-homotopic. Thus, $Q^\bullet$ is a 2-term partial silting complex in $K^b(\operatorname{proj} B)$.  

Let $X, Y \in \operatorname{mod} A$ and $f : X \to Y$. We denote by $\operatorname{GIm} f$ the $G$-image of $f$, this is, $\operatorname{Hom}_A(G, X) \to \operatorname{Hom}_A(G, \operatorname{Im} f)$ remains to be epic for any $G \in \operatorname{Gproj} A$. Let $X \in \operatorname{mod} A$. Consider the canonical sequence of $X$:

$$0 \to tX \xrightarrow{i_X} X \to X/tX \to 0,$$

where $tX = \sum \operatorname{GIm} f$ with $f \in \operatorname{Hom}_A(H^0(G^\bullet), X)$ such that any $g : E \to X$ factors through $f$ for any $E \in \operatorname{Gproj} A$. We collect some properties of $\mathcal{T}(G^\bullet)$ and $\mathcal{F}(G^\bullet)$.

Lemma 3.5. The following hold:

1. $\mathcal{T}(G^\bullet)$ is closed under $G$-epimorphic images.
2. $\mathcal{F}(G^\bullet)$ is closed under $G$-submodules.
(3) For any \(X \in \text{mod} A\), \(\text{Hom}_A(H^0(G^\bullet), i_X)\) is an isomorphism.

Proof. (1) Let \(0 \to X \to Y \to Z \to 0\) be a \(G\)-exact sequence in \(\text{mod} A\). Applying \(\text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, -)\), we get the following exact sequence

\[
\text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, Y[1]) \to \text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, Z[1]) \to \text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, X[2]).
\]

Since \(\text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, X[2]) = 0\), then we have that \(Y \in \mathcal{T}(G^\bullet)\) implies \(Z \in \mathcal{T}(G^\bullet)\).

(2) Let \(0 \to X \to Y \to Z \to 0\) be a \(G\)-exact sequence in \(\text{mod} A\). Applying \(\text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, -)\), we get the following exact sequence

\[
0 \to \text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, X) \to \text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, Y) \to \text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, Z).
\]

Hence \(Y \in \mathcal{F}(G^\bullet)\) implies \(X \in \mathcal{F}(G^\bullet)\).

(3) By the definition of \(tX\), we immediately obtain the desired isomorphism. \(\square\)

Lemma 3.6. There exists a triangle in \(D^b_{\text{sp}}(A)\) for a \(2\)-term Gorenstein silting complex \(G^\bullet\) of the form

\[
H^{-1}(G^\bullet)[1] \to G^\bullet \to H^0(G^\bullet) \to H^{-1}(G^\bullet)[2].
\]

Proof. Let \(G^\bullet := 0 \to \text{Im} d^1 \to G_0 \to 0\). We have a \(G\)-exact sequence

\[
0 \to \text{Ker} d^1[1] \to G^\bullet \to G^\bullet \to 0
\]

in \(C^b(A)\). Since \(G^\bullet\) is \(GP\)-quasi-isomorphic to \(H^0(G^\bullet)\), then \(G^\bullet \cong H^0(G^\bullet)\) in \(D^b_{\text{sp}}(A)\), we get the desired triangle in \(D^b_{\text{sp}}(A)\) of the form

\[
H^{-1}(G^\bullet)[1] \to G^\bullet \to H^0(G^\bullet) \to H^{-1}(G^\bullet)[2].
\]

\(\square\)

Lemma 3.7. For any \(X \in \text{mod} A\), we have a functorial isomorphism

\[
\text{Hom}_{D^b_{\text{sp}}(A)}(G^\bullet, X) \cong \text{Hom}_A(H^0(G^\bullet), X)
\]

and a monomorphism

\[
\text{Hom}_{D^b_{\text{sp}}(A)}(H^0(G^\bullet), X[1]) \to \text{Hom}_A(G^\bullet, X[1]).
\]

Proof. Applying \(\text{Hom}_{D^b_{\text{sp}}(A)}(-, X)\) to the triangle in Lemma 3.6

\[
H^{-1}(G^\bullet)[1] \to G^\bullet \to H^0(G^\bullet) \to H^{-1}(G^\bullet)[2]
\]

and using that there is no non-zero negative extensions between modules, we get the required isomorphism and monomorphism. \(\square\)

Theorem 3.8. The following are equivalent for a complex \(G^\bullet : G_1 \to G_0\) with \(G_i \in \text{Gproj} A\).

(1) \(G^\bullet\) is a \(2\)-term Gorenstein silting complex in \(D^b_{\text{sp}}(A)\).

(2) \(\mathcal{T}(G^\bullet) \cap \mathcal{F}(G^\bullet) = 0\) and \(H^0(G^\bullet) \in \mathcal{T}(G^\bullet)\).

(3) \(\mathcal{T}(G^\bullet) \cap \mathcal{F}(G^\bullet) = 0\) and \(t(X) \in \mathcal{T}(G^\bullet), X/tX \in \mathcal{F}(G^\bullet)\) for all \(X \in \text{mod} A\).
Proposition 3.10. \((\mathcal{T}(G^*), \mathcal{F}(G^*))\) is a torsion pair for \(\text{mod} A\).

Proof. \((1) \iff (2)\) \(\text{Hom}_{D^b_{gp}(A)}(G^*, G^*[i]) = 0\) for all \(i > 0\) if and only if \(H^0(G^*) \in \mathcal{T}(G^*)\) by Lemma 3.3. For any \(X \in \mathcal{T}(G^*) \cap \mathcal{F}(G^*)\), \(\text{Hom}_{D^b_{gp}(A)}(G^*, X[n]) = 0\) for all \(n \in \mathbb{Z}\) and hence \(X = 0\). Conversely, let \(X^* \in D^b_{gp}(A)\) with \(\text{Hom}_{D^b_{gp}(A)}(G^*, X[n]) = 0\) for all \(n \in \mathbb{Z}\). Then by Lemma 3.3, \(H^n(X^*) \in \mathcal{T}(G^*) \cap \mathcal{F}(G^*) = 0\).

\((2) \Rightarrow (3)\) Let \(X \in \text{mod} A\). Since \(H^0(G^*) \in \mathcal{T}(G^*)\), it follows that \(tX \in \mathcal{T}(G^*)\). Next, since there is an isomorphism by Lemma 3.7

\[
\text{Hom}_{D^b_{gp}(A)}(G^*, X/tX) \cong \text{Hom}_A(H^0(G^*), X/tX),
\]

and \(\text{Hom}_A(H^0(G^*), i_X)\) is an isomorphism, it follows that \(\text{Hom}_{D^b_{gp}(A)}(G^*, X/tX) = 0\) and hence \(X/tX \in \mathcal{F}(G^*)\).

\((3) \Rightarrow (4)\) It can be obtained by the definition.

\((4) \Rightarrow (2)\) We just need to prove that \(H^0(G^*) \in \mathcal{T}(G^*)\). By Lemma 3.7

\[
0 = \text{Hom}_{D^b_{gp}(A)}(G^*, \mathcal{F}(G^*)) \cong \text{Hom}_A(H^0(G^*), \mathcal{F}(G^*)),
\]

it follows from \((\mathcal{T}(G^*), \mathcal{F}(G^*))\) is a torsion pair that \(H^0(G^*) \in \mathcal{T}(G^*)\). \(\square\)

Remark 3.9. Note that the torsion pair \((\mathcal{T}(G^*), \mathcal{F}(G^*))\) coincides with \((D_\theta, T^0)\) defined in the subsection 2.1.

Proof. Let \(G^* : G_1 \xrightarrow{\theta} G_0\) be a 2-term Gorenstein silting complex in \(D^b_{gp}(A)\), and \(T = H^0(G^*) = \text{Coker} \theta\). On one hand, consider the distinguished triangle in \(D^b_{gp}(A)\)

\[
G_1 \xrightarrow{\theta} G_0 \rightarrow G^* \rightarrow G_1[1].
\]

Applying the functor \(\text{Hom}_{D^b_{gp}(A)}(-, X)\) for any \(A\)-module \(X\), there is the induced exact sequence

\[
\text{Hom}_{D^b_{gp}(A)}(G_0, X) \longrightarrow \text{Hom}_{D^b_{gp}(A)}(G_1, X) \longrightarrow \text{Hom}_{D^b_{gp}(A)}(G^*[1], X) \longrightarrow 0.
\]

Since \(\text{Hom}_{D^b_{gp}(A)}(G_i, X) \cong \text{Hom}_A(G_i, X)\) with \(i = 0, 1\), we get that \(X \in D_\theta\) if and only if \(\text{Hom}_{D^b_{gp}(A)}(G^*, X[1]) = 0\) if and only if \(X \in \mathcal{T}(G^*)\).

On the other hand, since

\[
\text{Hom}_A(T, X) \cong \text{Hom}_{D^b_{gp}(A)}(T, X) \cong \text{Hom}_{D^b_{gp}(A)}(G^*, X),
\]

we get that \(X \in T^0\) if and only if \(X \in \mathcal{F}(G^*)\). \(\square\)

Proposition 3.10. Let \(G^*\) be a 2-term Gorenstein silting complex in \(D^b_{gp}(A)\) and \((\mathcal{T}(G^*), \mathcal{F}(G^*))\) the torsion pair induced by \(G^*\).

1. For any \(X \in \text{mod} A\), \(X \in \text{add} H^0(G^*)\) if and only if \(X\) is Ext-projective in \(\mathcal{T}(G^*)\).

2. For any \(X \in \mathcal{T}(G^*)\), there is a \(G\)-exact sequence \(0 \rightarrow L \rightarrow T_0 \rightarrow X \rightarrow 0\) with \(T_0 \in \text{add} H^0(G^*)\) and \(L \in \mathcal{T}(G^*)\).
Proof. Assume that $X$ is Ext-projective in $T(G^*)$. Since $T(G^*) = \text{Fac}H^0(G^*)$, there is a $G$-exact sequence

$$0 \to L \to T_0 \overset{\alpha}{\to} X \to 0,$$

where $T_0 \overset{\alpha}{\to} X$ is a right add$H^0(G^*)$-approximation. Since $\text{Hom}_A(H^0(G^*), \alpha)$ is an epimorphism, we have that $\text{Hom}_{D^b_{gpa}(A)}(G^*, \alpha)$ is an epimorphism. Applying $\text{Hom}_{D^b_{gpa}(A)}(G^*, -)$ to (**) we have an exact sequence

$$\text{Hom}_{D^b_{gpa}(A)}(G^*, T_0) \overset{\text{Hom}_{D^b_{gpa}(A)}(G^*, \alpha)}{\to} \text{Hom}_{D^b_{gpa}(A)}(G^*, X) \to \text{Hom}_{D^b_{gpa}(A)}(G^*, L[1]) \to 0.$$ 

Then $\text{Hom}_{D^b_{gpa}(A)}(G^*, L[1]) = 0$ which implies that $L$ is in $T(G^*)$. Thus, by assumption, the sequence (**) splits, and hence $X$ is in add$H^0(G^*)$.

By the monomorphism in Lemma 3.7, we have that add$H^0(G^*)$ is Ext-projective in $T(G^*)$. □

**Theorem 3.11.** Suppose that $A$ is a Gorenstein algebra of finite CM-type with the Gorenstein projective generator $G$, and $A(Gproj) = \text{End}_A(E)\text{op}$. Let $G^* : G_1 \overset{\theta}{\to} G_0$ be a 2-term complex in $D^b(A)$, and $T = H^0(G^*) = \text{Coker}\theta$. Then the following statements are equivalent.

1. $T$ is a Gorenstein silting module with respect to $\theta$ in mod$A$.
2. $G^* : G_1 \overset{\theta}{\to} G_0$ is a 2-term Gorenstein silting complex in $D^b_{gpa}(A)$.

**Proof.** First, we claim that $T$ is a partial Gorenstein silting module with respect to $\theta$ if and only if $G^*$ is a 2-term partial Gorenstein silting complex.

Assume that $T$ is a partial Gorenstein silting module. By definition we know that $\text{Hom}_A(\theta, T)$ is an epimorphism. Then we get that $\text{Hom}_A(Gproj)((E, \theta), (E, T))$ is an epimorphism from the proof of Proposition 2.11(1). Let $\sigma := \text{Hom}_A(E, \theta)$ and $f : (E, G_1) \to (E, T)$. By the following diagram,

$$\begin{array}{ccc}
\text{Hom}_A(E, G_1) & \overset{\sigma}{\longrightarrow} & \text{Hom}_A(E, G_0) \\
\downarrow h & & \downarrow g \\
\text{Hom}_A(E, G_1) & \overset{\sigma}{\longrightarrow} & \text{Hom}_A(E, G_0) \\
\downarrow & & \downarrow \pi \\
\text{Hom}_A(E, T) & \overset{\pi}{\longrightarrow} & \text{Hom}_A(E, T)
\end{array}$$

there exists a morphism $g : \text{Hom}_A(E, G_0) \to \text{Hom}_A(E, T)$, such that $f = g\sigma$. Since $\text{Hom}_A(E, G_1)$ is projective, there exists a morphism $h : \text{Hom}_A(E, G_1) \to \text{Hom}_A(E, G_0)$, such that $f = \pi h$. Similarly since $\text{Hom}_A(E, G_0)$ is projective, there exists a morphism $s_0 : \text{Hom}_A(E, G_0) \to \text{Hom}_A(E, G_0)$, such that $g = \pi s_0$. Therefore $\pi h = \pi s_0 \sigma$, i.e., $\pi(h - s_0 \sigma) = 0$. It follows that there is $s_1 : \text{Hom}_A(E, G_1) \to \text{Hom}_A(E, G_1)$,
such that $h - s_0\sigma = \sigma s_1$, which shows that $h$ is null-homotopic. This implies that
\[ \text{Hom}_{A(G_{proj})}((E, G^\bullet), (E, G^\bullet)[1]) = 0. \]
Therefore, we get that $\text{Hom}_{\text{D}_{gp}(A)}(G^\bullet, G^\bullet[1]) = 0$. This implies that $G^\bullet$ is a 2-term partial Gorenstein silting complex.

Conversely, if $G^\bullet$ is a 2-term partial Gorenstein silting complex, then by the diagram above, we have that $h = s_0\sigma + \sigma s_1$. Then $f = \pi h = \pi s_0\sigma + \pi \sigma s_1 = (\pi s_0)\sigma$, which means that $\text{Hom}_{A(G_{proj})}((E, \theta), (E, T))$ is an epimorphism. Therefore $\text{Hom}_{A}(\theta, T)$ is an epimorphism, and so $T \in D_{\theta}$. This implies that $T$ is a partial Gorenstein silting module with respect to $\theta$.

(1)$\Rightarrow$(2) By Theorem 3.8, we prove that $T(G^\bullet) \cap F(G^\bullet) = 0$ and $H^0(G^\bullet) \in T(G^\bullet)$. From Remark 3.9, we have $T = H^0(G^\bullet) \in D_{\theta} = T(G^\bullet)$. Let $X \in T(G^\bullet) \cap F(G^\bullet) = D_{\theta} \cap T^{\perp,0} = \text{Gen}_G(T) \cap T^{\perp,0}$. Then there is a $G$-epimorphism $T_0 \longrightarrow X \longrightarrow 0$ with $T_0 \in \text{Add} T$, and $\text{Hom}_{A}(T, X) = 0$. Therefore we can get from the induced exact sequence $0 \longrightarrow (X, X) \longrightarrow (T_0, X)$ that $X = 0$. Thus $G^\bullet : G_{1, \theta} G_0$ is a 2-term Gorenstein silting complex in $\text{D}_{gp}(A)$.

(2)$\Rightarrow$(1) By the above claim, we see that $T$ is a partial Gorenstein silting module, and so $\text{Gen}_G(T) \subseteq D_{\theta}$. From Proposition 3.9, for any $X \in D_{\theta} = T(G^\bullet)$, there is a $G$-exact sequence
\[ 0 \longrightarrow L \longrightarrow T_0 \longrightarrow X \longrightarrow 0 \]
with $T_0 \in \text{add} H^0(G^\bullet) = \text{add} T$ and $L \in T(G^\bullet)$. Then we get that $X \in \text{Gen}_G(T)$. Therefore $T$ is a Gorenstein silting module with respect to $\theta$ in $\text{mod} A$. \hfill $\square$

Let $B = \text{End}_{\text{D}_{gp}(A)^{\text{op}}}(G^\bullet)^{\text{op}}$. Consider the subcategories of $\text{mod} B$
\[ \mathcal{X}(G^\bullet) = \text{Hom}_{\text{D}_{gp}(A)}(G^\bullet, F(G^\bullet)[1]) \quad \text{and} \quad \mathcal{Y}(G^\bullet) = \text{Hom}_{\text{D}_{gp}(A)}(G^\bullet, T(G^\bullet)). \]

Then we can draw the Brenner-Butler theorem in this setting.

**Theorem 3.12.** Let $G^\bullet$ be a 2-term Gorenstein silting complex in $\text{D}_{gp}(A)$. Then $(\mathcal{X}(G^\bullet), \mathcal{Y}(G^\bullet))$ is a torsion pair in $\text{mod} B$ and there are equivalences
\[ \text{Hom}_{\text{D}_{gp}(A)}(G^\bullet, -) : T(G^\bullet) \longrightarrow \mathcal{Y}(G^\bullet), \]
and
\[ \text{Hom}_{\text{D}_{gp}(A)}(G^\bullet, [1]) : F(G^\bullet) \longrightarrow \mathcal{X}(G^\bullet). \]

The equivalences send $G$-exact sequences with terms in $T(G^\bullet)$ (resp. $F(G^\bullet)$) to short exact sequences in $\text{mod} B$.

**Proof.** This follows from Proposition 3.4 (1) and (3), using that $T(G^\bullet) \cup F(G^\bullet) \subseteq C_{gp}(G^\bullet)$.

We finish this section with an interesting property of the 2-term Gorenstein silting complex over a finite dimensional Gorenstein algebra $A$ of finite CM-type with the Gorenstein-projective generator $E$. 
Let $G^\bullet : G_1 \rightarrow G_0$ be the 2-term complex over $\text{Gproj} A$, and set
\[ P^\bullet : \text{Hom}_A(E, G_1) \rightarrow \text{Hom}_A(E, G_0). \]

**Proposition 3.13.** Suppose that $A$ is a Gorenstein algebra. Then $G^\bullet$ is a 2-term Gorenstein silting complex in $D_{sp}^b(A)$ if and only if $P^\bullet$ is a 2-term silting complex in $D_{sp}^b(\text{A(Gproj)})$.

**Proof.** Since $\text{Hom}_A(E, -)$ is a fully faithful functor, we have that
\[ \text{Hom}_{D_{sp}^b(\text{A(Gproj)})}(P^\bullet, P^\bullet[1]) \cong \text{Hom}_{D_{sp}^b(A)}(G^\bullet, G^\bullet[1]). \]
On the other hand, from [GZ], there is a triangle-equivalence $D_{sp}^b(A) \cong D^b(\text{A(Gproj)})$ induced by $\text{Hom}_A(E, -)$. This completes the proof. \hfill $\square$

### 3.2. On global dimension.

In this subsection, we compare the global dimension between $A$ and $B$, where $G^\bullet$ is a 2-term Gorenstein silting complex in $D_{sp}^b(A)$, and $B = \text{End}_{D_{sp}^b(A)}(G^\bullet)^{op}$.

Recall from [IY], for full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $D_{sp}^b(A)$, denote
\[ \mathcal{X} \ast \mathcal{Y} := \{ Z \in D_{sp}^b(A) \mid \text{there exists a triangle } X \rightarrow Z \rightarrow Y \rightarrow X[1] \text{ in } D_{sp}^b(A) \text{ with } X \in \mathcal{X} \text{ and } Y \in \mathcal{Y} \}. \]
By the octahedral axiom, we have that $(\mathcal{X} \ast \mathcal{Y}) \ast Z = \mathcal{X} \ast (\mathcal{Y} \ast Z)$. Call $\mathcal{X}$ extension closed if $\mathcal{X} \ast \mathcal{X} = \mathcal{X}$. Now fix $\text{GP} = \text{add} G^\bullet$ and $G_c = \text{GP} \cap C_{sp}(G^\bullet)$.

**Lemma 3.14.** $C_{sp}(G^\bullet) \subset \text{GP} \ast \text{GP}[1] \ast \cdots \ast \text{GP}[d+1]$ for some non-negative integer $d$.

**Proof.** Note that
\[ C_{sp}(G^\bullet) \subset D_{sp}^{\leq l}(G^\bullet) \subset \text{GP} \ast \text{GP}[1] \ast \cdots \ast \text{GP}[l-1] \ast \text{GP}[l] \]
for some $l > 0$. For any $X^\bullet \in C_{sp}(G^\bullet)$, we have $H^i(X^\bullet) = 0$ for $i \neq -1, 0$. Taking a projective resolution $P^\bullet$ of $X^\bullet$, then there exists some non-negative integer $d$ such that $H^i(P^\bullet) = 0$ for $i > 0$ or $i < -d - 1$. Therefore
\[ \text{Hom}_{D_{sp}^b(A)}(X^\bullet, G^\bullet[i]) \cong \text{Hom}_{D_{sp}^b(A)}(P^\bullet, G^\bullet[i]) = 0, \ i \geq d+2, \]
which implies that $X^\bullet \in \text{GP} \ast \text{GP}[1] \ast \cdots \ast \text{GP}[d+1]$. \hfill $\square$

**Lemma 3.15.** For the complex $X^\bullet \in C_{sp}(G^\bullet) \cap (\text{GP}_c \ast \text{GP}_c[1] \ast \cdots \ast \text{GP}_c[m])$ for some $m \geq 0$, we have $\text{pdHom}_{D_{sp}^b(A)}(G^\bullet, X^\bullet)_B \leq m$.

**Proof.** Let $X^\bullet_0 = X^\bullet$. There are triangles
\[ X_{i+1} \rightarrow O_i \xrightarrow{g_i} X_i \rightarrow X_{i+1}[1], \ 0 \leq i \leq m - 1 \]
where $O_i \in \text{GP}_c$ and $X_i \in \text{GP}_c \ast \text{GP}_c[1] \ast \cdots \ast \text{GP}_c[m - i]$. Since $\text{Hom}_{D_{sp}^b(A)}(G^\bullet, G^\bullet[i]) = 0$ for all $i > 0$, we have that $g_i$ is a right GP-approximation of $X_i^\bullet$. Then we get the following induced exact sequence
\[ \text{Hom}_{D_{sp}^b(A)}(G^\bullet, X^\bullet_{i+1}) \rightarrow \text{Hom}_{D_{sp}^b(A)}(G^\bullet, O_i^\bullet) \xrightarrow{\text{Hom}_{D_{sp}^b(A)}(G^\bullet, g_i)} \text{Hom}_{D_{sp}^b(A)}(G^\bullet, X_i^\bullet) \rightarrow 0. \]
Then we get that
\[ \text{pdHom}_{D^b_{\text{gp}}(A)}(G^\bullet, X^\bullet_i)_B \leq \text{pdHom}_{D^b_{\text{gp}}(A)}(G^\bullet, X^\bullet_{i+1})_B + 1. \]
Therefore \( \text{pdHom}_{D^b_{\text{gp}}(A)}(G^\bullet, X^\bullet)_B \leq \text{pdHom}_{D^b_{\text{gp}}(A)}(G^\bullet, X^\bullet_m)_B + m = m. \)

\[ \square \]

**Theorem 3.16.** Assume that \( A \) has Gorenstein dimension \( d \) for some positive integer \( d \). Then \( \text{gldim} B \leq d + 1 \).

**Proof.** If \( G^\bullet \) is Gorenstein tilting, then \( G^\bullet \in \mathcal{C}_{\text{gp}}(G^\bullet) \). Therefore we have that \( \text{GP} = \text{GP}_{\mathcal{C}} \).
It follows from Lemma 3.14 and 3.15 that \( \text{gldim} B \leq \text{Gdim} A + 1 \). \[ \square \]

**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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