A duality of fields

Wen-Du Li\textsuperscript{a,b} and Wu-Sheng Dai\textsuperscript{b*}

\textsuperscript{a}Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin, 300071, P. R. China
\textsuperscript{b}Department of Physics, Tianjin University, Tianjin 300350, P. R. China

Abstract: It is shown that there exists a duality among fields. If a field is dual to another field, the solution of the field can be obtained from the dual field by the duality transformation. We give a general result on the dual fields. Different fields may have different numbers of dual fields, e.g., the free field and the $\phi^4$-field are self-dual, the $\phi^n$-field has one dual field, a field with an $n$-term polynomial potential has $n + 1$ dual fields, and a field with a nonpolynomial potential may have infinite number of dual fields. All fields which are dual to each other form a duality family. This implies that the field can be classified in the sense of duality, or, the duality family defines a duality class. Based on the duality relation, we can construct a high-efficiency approach for seeking the solution of field equations: solving one field in the duality family, all solutions of other fields in the family are obtained immediately by the duality transformation. As examples, we consider some $\phi^n$-fields, general polynomial-potential fields, and the sine-Gordon field.

\textsuperscript{1}daiwusheng@tju.edu.cn.
## Contents

1 Introduction 2

2 The $\phi^n$-field 3
   2.1 The duality 3
   2.2 Solving field equations by means of the duality 4

3 The general polynomial potential 4
   3.1 The duality 4
   3.2 Solving field equations by means of the duality 6

4 The sine-Gordon equation 6
   4.1 The duality 6
      4.1.1 The $1 + n$-dimensional solution 7
      4.1.2 The $1 + 1$-dimensional solution 8
   4.2 The duality family of the sine-Gordon field 9

5 Fields with general potentials 10

6 Constructing the dual fields from the solution 11

7 The duality of polynomial fields: examples 11
   7.1 The self-duality: free fields and $\phi^4$-fields 11
   7.2 The $\phi^1$-field and the $\phi^{-2}$-field 12
      7.2.1 The $1 + 3$-dimensional solution 13
      7.2.2 The $1 + 1$-dimensional solution 13
   7.3 The $\phi^2$-field and the $\phi^6$-field 14
   7.4 $V(\phi) = \lambda\phi^3 + \kappa\phi$ and its duality 14
      7.4.1 $U(\phi) = \eta\phi^6 + \sigma\phi^4$ 15
      7.4.2 $U(\phi) = \eta\phi^{-2} + \sigma\phi^4$ 15
   7.5 $V(\phi) = \lambda\phi^n + \kappa_1\phi^{2n-2} + \kappa_2$ and its duality 16
      7.5.1 $U(\phi) = \kappa_1\phi^{-2}$ 16
      7.5.2 $U(\phi) = \lambda\phi$ 17

8 Conclusion 17

A A solution of the scalar field equation 17
1 Introduction

In this paper, we show that there exists a duality between fields. Furthermore, we show that the duality can serve as a method of solving field equations.

Consider a scalar field with the potential $V(\phi)$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi)$$

(1.1)

and the field equation is

$$\Box \phi + m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} = 0.$$  \hspace{1cm} (1.2)

It will be shown that the field with the potential $V(\phi)$ may have dual fields determined by some other potentials.

A field may have different numbers of dual fields. A $\phi^n$-field has one dual field, a field with an $n$-term polynomial potential has $n + 1$ dual fields, a field with a nonpolynomial potential may have infinite number of dual fields, and there are also self-dual fields. Dual fields form a duality family; all fields in the family are dual to each other. This allows us to classify fields based on the duality: a duality family forms a duality class.

The duality can serve as a high-efficiency approach for seeking the solution of field equations. The duality relation relates a field and its dual field, so the duality relation can be used to find the solution of a field equation from the solution of the field equation of its dual field. In a duality family, if one field in the family is solved, then the solution of all other fields in the family can be obtained immediately by the duality transformation. Concretely, we construct the dual field of the $\phi^n$-field, the general-polynomial-field ($V(\phi)$ is a general polynomial of $\phi$), and the sine-Gordon field.

In physics, the duality plays an important role. The duality bridges two different physical systems and reveals inherent relations in physics. Various dualities are found in many physical problems. The S-duality (strong–weak duality) relates a strongly coupled theory to an equivalent theory with a small coupling constant, such as the duality between two perturbed quantum many-body systems [1], the (2+1)-dimensional duality of free Dirac or Majorana fermions and strongly-interacting bosonic Chern-Simons-matter theories [2], the electric-magnetic duality [3–5], the duality of higher spin free massless gauge fields [6], the duality in the couple of gauge field to gravity [7], and in string theory [8, 9]. The AdS/CFT duality has been found in many physical areas [10–14]. The fluid/gravity duality bridges fluid systems which is described by the Navier-Stokes equation and spacetime which is described by the Einstein equation [15–27]. The gravoelectric duality is another duality of spacetime [28–32]. In string theory there is an T-duality [33, 34].

In section 2, we consider the duality of the $\phi^n$-field. In section 3, we consider the duality of the general polynomial potential. In section 4, we consider the duality of the sine-Gordon field. In section 6, we discuss an approach of the construction of dual fields. In section [7 ?], we give a general result on dual fields. In section 7, we provide some examples. The conclusion is given in section 8. Moreover, we add an appendix on the solution of field equations.
2 The $\phi^n$-field

In this section, we consider the duality of the $\phi^n$-field.

2.1 The duality

Two scalar fields $\phi(x^\mu)$ and $\varphi(y^\mu)$ with the potentials

\[ V(\phi) = \lambda \phi^a, \]  
\[ U(\varphi) = \eta \varphi^A, \]  

are dual to each other if

\[ \frac{2}{2 - a} = \frac{2 - A}{2}. \]  

The dual fields are related by the following duality relations:

\[ \phi \rightarrow \varphi^{\frac{2 - a}{2}}, \]  
\[ x^\mu \rightarrow \frac{2}{2 - a} y^\mu, \quad \mu = 0, 1, \ldots \]  

and

\[ \lambda \rightarrow -G, \]  
\[ G \rightarrow -\eta, \]  

where

\[ G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi^a, \]  
\[ G = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2 + \eta \varphi^A. \]  

are two Lorentz scalars corresponding to $\phi$ and $\varphi$, respectively.

The duality relation can be verified directly.

The field equation of the potential (2.1) is

\[ \Box \phi + m^2 \phi + a \lambda \phi^{a-1} = 0. \]  

The duality transformations (2.4) and (2.5) give

\[ \Box \phi \rightarrow \frac{2 - a}{2} \varphi^{\frac{2 - a}{2}} \Box \varphi + \frac{a}{2} \varphi^{\frac{2 - a}{2}} \partial_\mu \varphi \partial^\mu \varphi. \]  

Substituting the transformation (2.11) into the field equation (2.10) gives

\[ \frac{2 - a}{2} \varphi^{\frac{2 - a}{2}} \Box \varphi + \frac{a}{2} \varphi^{\frac{2 - a}{2}} \partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^{\frac{2 - a}{2}} + a \lambda \varphi^{\frac{2(a-1)}{2}} = 0. \]  

The coupling constant $\lambda$ in the potential $V(\phi)$, by the duality transformation (2.6), is replaced by:

\[ \lambda \rightarrow -\left( \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} m^2 \varphi^2 + \eta \varphi^A \right). \]  

Using the duality relation (2.3), we arrive at the field equation of $\varphi$,

\[ \Box \varphi + m^2 \varphi + A \eta \varphi^A-1 = 0. \]
2.2 Solving field equations by means of the duality

The duality relation bridges various fields and can serve as a method of solving field equations. Once a field equation is solved, its dual field is also solved.

It can be checked that the field equation with the potential \( (2.10) \) has an implicit solution (Appendix A):

\[
\beta_\mu x^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left( \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 - G \right)}} d\phi = 0,
\]

(2.15)

where \( \beta_\mu \) is a constant.

Substituting the duality transformations \( (2.4) \) and \( (2.5) \) into the solution \( (2.15) \) gives

\[
\beta_\mu y^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 \varphi^2 + \left( - G \varphi - \frac{2a}{2-a} \right) - (-\lambda) \right]}} d\varphi = 0.
\]

(2.16)

This is just a solution of the field equation with \( U(\varphi) = -G\varphi^{-\frac{2a}{2-a}} \). By the duality relations \( (2.3), (2.6), (2.7), \) and Eq. \( (2.9) \), we can see that this is the solution of the field equation \( (2.14) \):

\[
\beta_\mu y^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left( \frac{1}{2} m^2 \varphi^2 + \eta \varphi^A - G \right)}} d\varphi = 0.
\]

(2.17)

3 The general polynomial potential

In this section, we consider the duality between the field with general polynomial potentials. The general polynomial is a superposition power series containing arbitrary real-number powers.

3.1 The duality

Two scalar fields \( \phi(x^\mu) \) and \( \varphi(y^\mu) \) of the potentials

\[
V(\phi) = \lambda \phi^a + \sum_n \kappa_n \phi^{b_n}, \quad (3.1)
\]

\[
U(\varphi) = \eta \varphi^A + \sum_n \sigma_n \varphi^{B_n} \quad (3.2)
\]

are dual to each other if

\[
\frac{2 - A}{2} = \frac{2 - a}{2}, \quad (3.3)
\]

\[
2 - b_n = \frac{2}{2 - A} (2 - B_n) \quad \text{or} \quad \frac{2}{2 - a} (2 - b_n) = 2 - B_n. \quad (3.4)
\]

The dual fields are related by the following duality relations:

\[
\phi \rightarrow \varphi^{\frac{2}{2-a}}, \quad (3.5)
\]

\[
x^\mu \rightarrow \frac{2}{2-a} y^\mu, \quad \mu = 0, 1, \ldots \quad (3.6)
\]
and
\begin{align}
\lambda &\to -\mathcal{G}, \\
G &\to -\eta, \\
\kappa_n &\to \sigma_n,
\end{align}
(3.7) (3.8) (3.9)

where
\begin{align}
G &= \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{2} m^2 \phi^2 + V (\phi), \\
\mathcal{G} &= \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi + \frac{1}{2} m^2 \varphi^2 + U (\varphi)
\end{align}
(3.10) (3.11)

are two Lorentz scalars corresponding to \( \phi \) and \( \varphi \), respectively.

The field equation with the potential (3.1) is
\[ \Box \phi + m^2 \phi + a \lambda \phi^{a-1} + \sum_n b_n \kappa_n \phi^{b_n-1} = 0. \] (3.12)

The duality transformations (3.5) and (3.6) give
\[ \Box \phi \to \frac{2 - a}{2} \varphi^{2-a} \Box \varphi + \frac{a}{2} \varphi^{2-a} \partial_{\mu} \varphi \partial^{\mu} \varphi. \] (3.13)

Substituting the transformation into the field equation (3.12) gives an equation of \( \varphi \):
\[ \frac{2 - a}{2} \varphi^{2-a} \Box \varphi + \frac{a}{2} \varphi^{2-a} \partial_{\mu} \varphi \partial^{\mu} \varphi + m^2 \varphi^{2-a} + a \lambda \varphi^{2(a-1)/(2-a)} + \sum_n b_n \kappa_n \varphi^{2(b_n-1)/(2-a)} = 0. \] (3.14)

The transformation of the coupling constant is given by the duality transformation (3.7):
\[ \lambda \to - \left( \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi + \frac{1}{2} m^2 \varphi^2 + \eta \varphi^A + \sum_n \sigma_n \varphi^{B_n} \right). \] (3.15)

Then we arrive at the field equation of \( U (\varphi) = \eta \varphi^A + \sum_n \sigma_n \varphi^{B_n} \) with the duality relations (3.3), (3.4), and (3.9):
\[ \Box \varphi + m^2 \varphi + A \eta \varphi^{A-1} + \sum_n B_n \sigma_n \varphi^{B_n-1} = 0. \] (3.16)

It can be seen from the potential (3.1) that in a general polynomial potential \( V (\phi) = \lambda \phi^a + \sum_n \kappa_n \phi^b, \) every term in the potential can be chosen as the first term in \( V (\phi) \) to play the role of \( \lambda \phi^a \). Different choices give different dual fields. For an \( n \)-term polynomial potential, there are \( n \) choices. Therefore, all potentials who are dual to each other form a duality family with \( n + 1 \) members.
3.2 Solving field equations by means of the duality

Similarly, the field equation (3.12) has an implicit solution:

$$\beta_\mu x^\mu + \int \frac{\sqrt{-a^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 \varphi^2 + (\lambda \varphi^a + \sum_n \kappa_n \varphi^{bn}) - G \right]}} d\varphi = 0. \quad (3.17)$$

Substituting the duality transformations (3.5) and (3.6) into the solution (3.17) and using the duality relations (3.3) and (3.4), we arrive at

$$\beta_\mu y^\mu + \int \frac{\sqrt{-a^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 \varphi^2 + (-G \varphi^{-\frac{a}{2}} + \sum_n \kappa_n \varphi^{\frac{2bn-2a}{2-a}}) - (-\lambda) \right]}} d\varphi = 0. \quad (3.18)$$

This is just a solution of the field equation with $U(\varphi) = -G \varphi^{-\frac{a}{2}} + \sum_n \kappa_n \varphi^{\frac{2bn-2a}{2-a}}$. By the duality relations (3.3), (3.4), (3.7), (3.8), (3.9), and Eq. (3.11), we can see that this is the solution of the field equation with the potential (2.14):

$$\beta_\mu y^\mu + \int \frac{\sqrt{-a^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 \varphi^2 + U(\varphi) - G \right]}} d\varphi = 0. \quad (3.19)$$

Obviously, for an $n$-term polynomial potential, once a potential in the duality family is solved, the other $n$ potentials are immediately solved.

4 The sine-Gordon equation

The sine-Gordon potential is not a polynomial potential. We first expand the sine-Gordon potential as a power series. Each term in the expansion is a power function, which has a duality discussed in section 2. Calculating the duality of each term of the series and then summing up the series give the dual potential of the sine-Gordon potential.

4.1 The duality

The sine-Gordon equation is [35]

$$\Box \phi + \frac{m^3}{\sqrt{\lambda}} \sin \left( \frac{\sqrt{\lambda}}{m} \phi \right) = 0. \quad (4.1)$$

First rewrite the sine-Gordon Lagrangian, $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^4}{\lambda} \left[ 1 - \cos \left( \frac{\sqrt{\lambda}}{m} \phi \right) \right]$, as

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^4 \phi^2 - V(\phi) \quad (4.2)$$

with

$$V(\phi) = \frac{m^4}{\lambda} \left[ 1 - \cos \left( \frac{\sqrt{\lambda}}{m} \phi \right) \right] - \frac{1}{2} m^2 \phi^2. \quad (4.3)$$
Expanding $V(\phi)$ gives

$$V(\phi) = -\frac{\lambda}{24}\phi^4 - \sum_{n=3}^{\infty} \frac{(-1)^n \lambda^{n-1}}{(2n)! m^{2(n-2)}} \phi^{2n}. \quad (4.4)$$

The leading term in Eq. (4.4) is the $\phi^4$ term.

First regarding the expansion (4.4) as a polynomial (though it is indeed a series), we can obtain the dual potential by the duality transformation of the general polynomial potential, Eqs. (3.3), (3.4), (3.7), (3.8), and (3.9):

$$A = 4, \quad B_n = 4 - 2n, \quad n = 3, 4, 5, \ldots,$$

$$\eta = -G, \quad \sigma_n = \frac{(-1)^n \lambda^{n-1}}{(2n)! m^{2(n-2)}}. \quad (4.5)$$

The duality transformations here, by Eqs. (3.5) and (3.6), are

$$\phi \rightarrow \varphi^{-1}, \quad (4.6)$$

$$x^\mu \rightarrow -y^\mu. \quad (4.7)$$

The potential of the dual field can be obtained by substituting the duality relation (4.5) into Eq. (3.2) and then summing up the series:

$$U(\varphi) = -G\varphi^4 - \sum_{n=3}^{\infty} \frac{(-1)^n \lambda^{n-1}}{(2n)! m^{2(n-2)}} \varphi^{2n-2}$$

$$= -G\varphi^4 + \frac{m^4}{\lambda} \varphi^4 \left(1 + \cos \frac{\sqrt{\lambda}}{m\varphi}\right) - \frac{1}{2} m^2 \varphi^2. \quad (4.8)$$

The field equation of the dual field then reads

$$\Box \phi - 4G\varphi^3 + \frac{4m^4}{\lambda} \varphi^3 \left(1 - \cos \frac{\sqrt{\lambda}}{m\varphi}\right) - \frac{m^3}{\sqrt{\lambda}} \varphi^3 \sin \frac{\sqrt{\lambda}}{m\varphi} = 0. \quad (4.9)$$

Different solution of $\phi$ gives different $G$ and different $G$ gives different dual potentials.

The expansion (4.4) has infinite terms, so the duality family has infinite members.

### 4.1.1 The $1+n$-dimensional solution

The sine-Gordon equation (4.1) has a $1+n$-dimensional solution [35]

$$\phi = \frac{4m}{\sqrt{\lambda}} \arccot \left(i \cot \left(\frac{\beta_{\mu} x^\mu}{2 \sqrt{-\beta^2 m}}\right)\right). \quad (4.10)$$

Substituting the solution (4.10) into Eq. (3.10) gives

$$G = \frac{2m^4}{\lambda}. \quad (4.11)$$
Then the potential of the dual field, by Eq. (4.8), reads
\[ U(\phi) = -\frac{m^4}{\lambda} \phi^4 \left( 1 + \cos \frac{\sqrt{\lambda}}{m\phi} \right) - \frac{1}{2} m^2 \phi^2. \] (4.12)

The field equation of the dual field is then
\[ \Box \phi + \frac{4m^4}{\lambda} \phi^3 \left( 1 + \cos \frac{\sqrt{\lambda}}{m\phi} \right) + \frac{m^3}{\sqrt{\lambda}} \phi^2 \sin \frac{\sqrt{\lambda}}{m\phi} = 0. \] (4.13)

The solution of the dual field can be obtained by substituting the duality transformations (3.5) and (3.6) into the solution (4.10):
\[ \phi = \left[ \frac{4m}{\sqrt{\lambda}} \arccot \left( \frac{\cos \left( \frac{\sqrt{2} \mu}{2\sqrt{-\beta^2}} m \right)}{\cosh \left( \frac{\sqrt{2} \beta}{2\sqrt{-\beta^2}} mx \right)} \right) \right]^{-1}. \] (4.14)

### 4.1.2 The 1 + 1-dimensional solution

The sine-Gordon equation (4.1) also has a 1 + 1-dimensional solution [36, 37],
\[ \phi = \frac{4m}{\sqrt{\lambda}} \arctan \left( \frac{\cos \left( \frac{\sqrt{2} \mu}{2\sqrt{-\beta^2}} m \right)}{\cosh \left( \frac{\sqrt{2} \beta}{2\sqrt{-\beta^2}} mx \right)} \right). \] (4.15)

Substituting the solution (4.10) into (3.10) gives
\[ G = \frac{8m^4}{\lambda} \left[ \cos \left( \sqrt{2}mt \right) + \cosh \left( \sqrt{2}mx \right) + 2 \right]^{-1}. \] (4.16)

The potential of the dual field, by Eq. (4.8), reads
\[ U(\varphi) = -\frac{8m^4}{\lambda} \left[ \cos \left( \sqrt{2}mt \right) + \cosh \left( \sqrt{2}mx \right) + 2 \right]^{-1} \varphi^4 + \frac{m^4}{\lambda} \varphi^4 \left( 1 - \cos \left( \frac{\sqrt{\lambda}}{m\varphi} \right) \right) - \frac{1}{2} m^2 \varphi^2. \] (4.17)

The field equation of the dual field is
\[ \Box \varphi - \frac{32m^4}{\lambda} \left[ \cos \left( \sqrt{2}mt \right) + \cosh \left( \sqrt{2}mx \right) + 2 \right]^{-1} \varphi^3 + \frac{4m^4}{\lambda} \varphi^3 \left( 1 - \cos \frac{\sqrt{\lambda}}{m\varphi} \right) - \frac{m^3}{\sqrt{\lambda}} \varphi^3 \sin \frac{\sqrt{\lambda}}{m\varphi} = 0. \] (4.18)

The solution of the dual field can be obtained by substituting the duality transformations (4.6) and (4.7) into the solution (4.15):
\[ \varphi = \left[ \frac{4m}{\sqrt{\lambda}} \arctan \left( \frac{\cos \left( \frac{\sqrt{2} \mu}{2\sqrt{-\beta^2}} m \right)}{\cosh \left( \frac{\sqrt{2} \beta}{2\sqrt{-\beta^2}} mx \right)} \right) \right]^{-1}. \] (4.19)

This is the solution of the field equation (4.18).
4.2 The duality family of the sine-Gordon field

In the above, by regarding the expansion of the sine-Gordon potential as a "polynomial potential", we obtain the dual potential of the sine-Gordon potential. In section 3, we write a polynomial potential in the form of \( V(\phi) = \lambda \phi^a + \sum_n \kappa_n \phi^{bn} \), in which the first term \( \lambda \phi^a \) is arbitrarily chosen from the polynomial and the other terms in the polynomial are incorporated into the sum. However, every term in \( V(\phi) \) can be chosen as the first term and different choices lead to different dual potentials. In the following, we give a general discussion on the dual potential of the sine-Gordon potential by choosing different terms in the expansion as the first term.

Rewriting the expansion of \( V(\phi) \), Eq. (4.3), as \( V(\phi) = - \frac{(-1)^\chi}{(2\chi)! \; m^{2(\chi-2)}} \phi^{2\chi} - \sum_{n=2(n \neq \chi)}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\lambda^{n-1}}{m^{2(n-2)}} \phi^{2n} \), \( \chi = 2, 3, 4 \cdots \). (4.20)

It can be seen that the expansion (4.3) is just the special case of Eq. (4.20) with \( \chi = 2 \).

Similarly, regarding the expansion (4.20) as a "polynomial", we obtain the dual potential by the duality transformation of the general polynomial potential, Eqs. (3.3), (3.4), (3.7), (3.8), and (3.9):

\[
A = \frac{2\chi}{\chi - 1}, \quad B_n = \frac{2(n - \chi)}{1 - \chi}, \quad \eta = -G, \quad \sigma_n = -\frac{(-1)^n}{(2n)!} \frac{\lambda^{n-1}}{m^{2(n-2)}}, \quad n = 2, 3, 4 \cdots, \quad \text{and} \quad n \neq \chi. \quad (4.21)
\]

The duality transformations (3.5) and (3.6) then become

\[
\phi \rightarrow \varphi^{\frac{1}{1-\chi}}, \quad (4.22)
\]

\[
x^\mu \rightarrow \frac{1}{1-\chi} y^\mu. \quad (4.23)
\]

Substituting the duality relation into Eq. (3.2) and summing up the series give the dual potential of \( V(\phi) \)

\[
U(\varphi) = -G\varphi^{2\chi - \frac{1}{1-\chi}} - \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\lambda^{n-1}}{m^{2(n-2)}} \varphi^{2n(\frac{1}{1-\chi})} \quad (4.24)
\]

\[
= -G\varphi^{2\chi - \frac{1}{1-\chi}} - \frac{m^4}{\lambda} \varphi^{\frac{2}{1-\chi}} \left[ \cos \left( \frac{\sqrt{\lambda} \varphi^{\frac{1}{1-\chi}}}{m} \right) - 1 \right] + \frac{1}{2} m^2 \varphi^2. \quad (4.25)
\]

The field equation is then

\[
\Box \varphi = \frac{2\chi}{\chi - 1} G\varphi^{\frac{1}{1-\chi}} - \frac{2\chi}{\chi - 1} \frac{m^4}{\lambda} \varphi^{\frac{2}{1-\chi}} \left[ \cos \left( \frac{\sqrt{\lambda} \varphi^{\frac{1}{1-\chi}}}{m} \right) - 1 \right] + \frac{1}{1-\chi} \frac{m^3}{\sqrt{\lambda}} \varphi^{\frac{3}{1-\chi}} \sin \left( \frac{\sqrt{\lambda} \varphi^{\frac{1}{1-\chi}}}{m} \right) = 0. \quad (4.26)
\]
Substituting the solution (4.10) into Eq. (3.10) gives

\[ G = \frac{2m^4}{\lambda}. \]  

Then we arrive at

\[ U(\varphi) = -\frac{2m^4}{\lambda} \varphi^{\frac{2\chi}{\chi-1}} - \sum_{n=2}^{\infty} \frac{(-1)^n \lambda^{n-1}}{(2n)! \, m^{2(n-2)}} \varphi^{\frac{2(n-\chi)}{\chi-1}} = -\frac{m^4}{\lambda} \varphi^{\frac{2\chi}{\chi-1}} \left( 1 + \cos \left( \frac{\sqrt{\lambda}}{m \, \varphi^{\frac{1}{\chi-1}}} \right) - \frac{1}{2} m^2 \varphi^2 - \left( \frac{\lambda}{m^4} \right)^{\chi-1} \frac{m^{2\chi}}{(2\chi)!} \right). \]  

The field equation of the dual field then reads

\[ \Box \varphi - \frac{2\chi}{\chi-1} \frac{m^4}{\lambda} \varphi^{\frac{2\chi}{\chi-1}} \left[ \cos \left( \frac{\sqrt{\lambda}}{m \, \varphi^{\frac{1}{\chi-1}}} \right) + 1 \right] + \frac{1}{1 - \chi} \frac{m^2}{\sqrt{\lambda}} \varphi^{\frac{\chi}{\chi-1}} \sin \left( \frac{\sqrt{\lambda}}{m \, \varphi^{\frac{1}{\chi-1}}} \right) = 0. \]

Substituting the duality relations (4.22) and (4.23) into the solution (4.10) gives the solution of the dual potential:

\[ \varphi = \left[ \frac{4m}{\sqrt{\lambda}} \arccot \left( i \cot \left( \frac{1}{1 - \chi} \frac{\beta_\mu y^\mu}{2\sqrt{-\beta^2}} m \right) \right) \right]^{1-\chi}. \]  

5 Fields with general potentials

For nonpolynomial potentials, like that in the case of the sine-Gordon fields, we can first expand the potential as a series, then construct the dual field with the help of the duality relation of the general polynomial potential given in section 3, and sum up the series.

Expand the potential \( V(\phi) \) as

\[ V(\phi) = \lambda \phi^a + \sum_n \kappa_n \phi^b_n. \]  

Using the duality relations (3.3), (3.4), (3.7), (3.8), and (3.9), we obtain the series of the dual potential,

\[ U(\varphi) = -G \varphi^{\frac{2a}{2-a}} + \sum_n \kappa_n \varphi^{\frac{2a}{2-a}} \varphi^{\frac{2b_n}{a-2}} = -G \left( \varphi^{\frac{2}{2-a}} \right)^{-a} + \left( \varphi^{\frac{2}{2-a}} \right)^{-a} \sum_n \kappa_n \left( \varphi^{\frac{2}{2-a}} \right)^{b_n}. \]  

By Eq. (5.1) we have \( \sum_n \kappa_n \phi^b_n = V(\phi) - \lambda \phi^a \), so the series in Eq. (5.2) can be summed up as

\[ U(\varphi) = -G \left( \varphi^{\frac{2}{2-a}} \right)^{-a} + \left( \varphi^{\frac{2}{2-a}} \right)^{-a} \left[ V \left( \varphi^{\frac{2}{2-a}} \right) - \lambda \left( \varphi^{\frac{2}{2-a}} \right)^a \right] \]

\[ = -G \varphi^{\frac{2}{2-a}} + \varphi^{\frac{2}{2-a}} V \left( \varphi^{\frac{2}{2-a}} \right) - \lambda. \]  

This is a general result on dual fields.
6 Constructing the dual fields from the solution

In this section, we discuss an approach of constructing the dual field from the solution. In the implicit solution of the field equation (1.2),

\[ \beta_\mu x^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right]}} d\phi = 0, \quad (6.1) \]

by writing the duality transformation in the following general form,

\[ \phi \to f(\varphi), \quad (6.2) \]
\[ x^\mu \to g^\mu(y), \quad (6.3) \]

the solution (6.1) becomes

\[ \beta_\mu g^\mu(y) + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 f^2(\varphi) + V(f(\varphi)) - G \right]}} f'(\varphi) d\varphi = 0. \quad (6.4) \]

In order that Eq. (6.4) is still a solution of a field equation, a simple choice is

\[ g^\mu(y) = \chi y^\mu \quad (6.5) \]

with \( \chi \) a constant. Then Eq. (6.4) with the duality transformation (6.5) becomes

\[ \beta_\mu y^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 \chi^2 f^2(\varphi) + \chi^2 V(f(\varphi)) - G \chi^2 \right]}} d\varphi = 0. \quad (6.6) \]

Requiring that Eq. (6.6) is still a solution of the field equation and comparing Eqs. (2.15) and (6.6), we can choose (1) one of the three terms as the mass term which should be in proportion to \( \varphi^2 \), (2) one of the other terms as the constant term, and (3) the remaining term as the potential term \( U(\varphi) \). Different choices give different dual fields.

7 The duality of polynomial fields: examples

In this section, we give some examples of the method of solving the field equation by the duality transformation.

7.1 The self-duality: free fields and \( \phi^4 \)-fields

The self-duality means that the dual field of a field is the field itself. The free field and the \( \phi^4 \)-field are self-dual. This can be seen directly from the duality relation (2.3): if \( a = 0 \), then \( A = 0 \); if \( a = 4 \), then \( A = 4 \).

Take the \( \phi^4 \)-field as an example. The field equation of the \( \phi^4 \)-field is

\[ \Box \phi + m^2 \phi + 4 \lambda \phi^3 = 0, \quad (7.1) \]
which has a soliton solution [38],
\[
\phi = \frac{im}{2\sqrt{\lambda}} \tanh \left( \alpha t + \beta x_1 + \gamma x_2 - \frac{x_3}{2} \sqrt{4\alpha^2 - 4\beta^2 - 4\gamma^2 - 2m^2 + \delta} \right). \tag{7.2}
\]

For the \( \phi^4 \)-field potential, by Eqs. (2.4) and (2.5), the duality transformations are
\[
\begin{align*}
\phi &\to \varphi^{-1}, \tag{7.3} \\
x^\mu &\to -y^\mu. \tag{7.4}
\end{align*}
\]
The field equation (7.1) under the duality transformation becomes
\[
-\varphi^{-2} \Box \varphi + 2\varphi^{-3} \partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^{-1} + 4\lambda \varphi^{-3} = 0. \tag{7.5}
\]

The coupling constant \( \lambda \), by the duality relation (2.6) and Eq. (2.9), should be replaced by
\[
\lambda \to -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \eta \varphi^4. \tag{7.6}
\]
Substituting the replacement (7.6) into the field equation (7.5) gives
\[
\Box \varphi + m^2 \varphi + 4\eta \varphi^3 = 0, \tag{7.7}
\]
which is the field equation with \( U(\varphi) = \eta \varphi^4 \).

The duality transformation of the solution is straightforward. Substituting the solution (7.2) into Eq. (2.8) gives
\[
G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2 + \lambda \phi^4 = -\frac{m^4}{16\lambda}. \tag{7.8}
\]
Substituting the transformations (7.3) and (7.4) into the solution (7.2) and using Eq. (2.7) give the solution of \( U(\varphi) = \eta \varphi^4 \):
\[
\varphi = -\frac{2\sqrt{\lambda}}{im} \coth \left( \alpha \tau + \beta y_1 + \gamma y_2 - \frac{y_3}{2} \sqrt{4\alpha^2 - 4\beta^2 - 4\gamma^2 - 2m^2 - \delta} \right). \tag{7.9}
\]
Putting \( \frac{m^4}{16\lambda} = \eta \), we obtain the solution of \( U(\varphi) = \eta \varphi^4 \):
\[
\varphi = \frac{im}{2\sqrt{\eta}} \coth \left( \alpha \tau + \beta y_1 + \gamma y_2 - \frac{y_3}{2} \sqrt{4\alpha^2 - 4\beta^2 - 4\gamma^2 - 2m^2 - \delta} \right)
= \frac{im}{2\sqrt{\eta}} \tanh \left( \alpha \tau + \beta y_1 + \gamma y_2 - \frac{y_3}{2} \sqrt{4\alpha^2 - 4\beta^2 - 4\gamma^2 - 2m^2 + \delta'} \right) \tag{7.10}
\]
with the constant \( \delta' = -\delta + i\frac{\pi}{2} \).

7.2 The \( \phi^1 \)-field and the \( \phi^{-2} \)-field

The dual field of the \( \phi^1 \)-field, by the duality relation (2.3), is the \( \phi^{-2} \)-field, i.e., the fields
\[
\begin{align*}
V(\phi) &= \lambda \phi, \tag{7.11} \\
U(\varphi) &= \eta \varphi^{-2}. \tag{7.12}
\end{align*}
\]
are dual to each other. The duality transformations, by Eqs. (2.4), (2.5), and (2.7), are

\[ \phi \rightarrow \varphi^2, \]
(7.13)
\[ x^\mu \rightarrow 2y^\mu, \]
(7.14)
\[ \eta \rightarrow -G. \]
(7.15)

For the field \( \phi \) with the potential (7.11),

\[ G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi. \]
(7.16)

**7.2.1 The 1 + 3-dimensional solution**

The field equation of \( \phi \), Eq. (7.11), has a solution

\[ \phi = \exp \left( \alpha t + \beta x_1 + \gamma x_2 + i \sqrt{-m^2 - \alpha^2 + \beta^2 + \gamma^2} x_3 \right) - \frac{\lambda}{m^2}. \]
(7.17)

Substituting the solution (7.17) into Eq. (7.16) gives

\[ G = -\frac{\lambda^2}{2m^2}. \]
(7.18)

Substituting the duality transformations (7.13) and (7.14) into the solution (7.17) and using the duality relation (7.15), we arrive at the solution of \( U(\varphi) = \frac{\lambda^2}{2m^2} e^{-2 \varphi^2}; \)

\[ \varphi = \left[ \exp \left( 2\alpha t + 2\beta y_1 + 2\gamma y_2 + 2i \sqrt{-m^2 - \alpha^2 + \beta^2 + \gamma^2} y_3 \right) - \frac{\lambda}{m^2} \right]^{1/2}. \]
(7.19)

This is just the solution of the dual potential \( U(\varphi) = \eta \varphi^{-2} \) with \( \frac{\lambda^2}{2m^2} = \eta; \)

\[ \varphi = \left[ \exp \left( 2\alpha t + 2\beta y_1 + 2\gamma y_2 + 2i \sqrt{-m^2 - \alpha^2 + \beta^2 + \gamma^2} y_3 \right) - \frac{\sqrt{2\eta m}}{\lambda} \right]^{1/2}. \]
(7.20)

**7.2.2 The 1 + 1-dimensional solution**

The field equation of \( \phi \), Eq. (7.11), has a 1 + 1-dimensional solution,

\[ \phi = e^{\alpha t} \sinh \left( \sqrt{\alpha^2 + m^2} x \right) - \frac{\lambda}{m^2}. \]
(7.21)

Substituting the solution (7.21) into Eq. (7.16) gives

\[ G = -\frac{\lambda^2}{2m^2} - \frac{\alpha^2 + m^2}{2} e^{2\alpha t}. \]
(7.22)

Substituting the duality transformations (7.13) and (7.14) into the solution (7.21) and using the duality relation (7.15), we arrive at the solution of \( U(\varphi) = \left( \frac{\lambda^2}{2m^2} + \frac{\alpha^2 + m^2}{2} e^{2\alpha t} \right) \varphi^{-2}; \)

\[ \varphi = \left[ e^{2\alpha t} \sinh \left( 2\sqrt{\alpha^2 + m^2} y \right) - \frac{\lambda}{m^2} \right]^{1/2}. \]
(7.23)
7.3 The $\phi^3$-field and the $\phi^6$-field

The dual field of the $\phi^3$-field, by the duality relation (2.3), is the $\phi^6$-field, i.e., the fields

$$V(\phi) = \lambda \phi^3,$$
$$U(\phi) = \eta \phi^6$$

are dual to each other. The duality transformations by Eqs. (2.4), (2.5), and (2.7) are

$$\phi \rightarrow \varphi^{-2},$$
$$x^\mu \rightarrow -2y^\mu.$$  

and

$$\eta \rightarrow -G.$$  

The field equation of $\phi$, Eq. (7.24), has a solution

$$\phi = -\frac{m^2}{6\lambda} \left[ 3 \tanh^2 \left( \alpha x_1 + \beta x_2 + \gamma x_3 + \frac{\sqrt{4\alpha^2 + 4\beta^2 + 4\gamma^2 + m^2}}{2} t \right) - 1 \right],$$

For the field $\phi$ with the potential (7.24),

$$G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi^3 = \frac{m^6}{54\lambda^2}.$$  

Substituting the duality transformations (7.26) and (7.27) into the solution (7.29) and using the duality relation (7.28), we obtain the solution of $U(\varphi) = -\frac{m^6}{54\lambda^2} \phi^6$:

$$\varphi = \left\{ -\frac{m^2}{6\lambda} \left[ 3 \tanh^2 \left( 2\alpha y_1 + 2\beta y_2 + 2\gamma y_3 + \sqrt{4\alpha^2 + 4\beta^2 + 4\gamma^2 + m^2} t \right) - 1 \right] \right\}^{-1/2}.$$  

We then arrive at the solution of $U(\varphi) = \eta \phi^6$ with $-\frac{m^6}{54\lambda^2} = \eta$:

$$\varphi = \left\{ \frac{i\sqrt{6\eta}}{2m} \left[ 3 \tanh^2 \left( 2\alpha y_1 + 2\beta y_2 + 2\gamma y_3 + \sqrt{4\alpha^2 + 4\beta^2 + 4\gamma^2 + m^2} t \right) - 1 \right] \right\}^{-1/2}.$$  

7.4 $V(\phi) = \lambda \phi^3 + \kappa \phi$ and its duality

The massless scalar field with the polynomial potential

$$V(\phi) = \lambda \phi^3 + \kappa \phi$$

has two dual potentials. Taking $a = 3$ in the duality relations (3.3) and (3.4) gives $A = 6$ and $B_n = 4$, i.e., the dual potential is

$$U(\varphi) = \eta \phi^6 + \sigma \varphi^4.$$  

Taking $a = 1$ in the duality relations (3.3) and (3.4) gives $A = -2$ and $B_n = 4$, i.e., the dual potential is

$$U(\varphi) = \eta \varphi^{-2} + \sigma \varphi^4.$$  

– 14 –
Obviously, the fields with the potentials (7.34) and (7.35) are also dual.

The massless field equation of the potential (7.33) has a solution
\[
\phi = i \sqrt{\frac{\kappa}{\lambda}} \left[ \frac{2}{\sqrt{3}} - \sqrt{3} \tanh^2 \left( \frac{\alpha t - x}{2} \right) \sqrt{4 \alpha^2 - 2i (3\kappa \lambda)^{1/2}} \right].
\] (7.36)

For the field \(\phi\) with the potential (7.33),
\[
G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi^3 = \frac{2i\kappa^{3/2}}{3\sqrt{3}\lambda}.
\] (7.37)

From the solution of the potential (7.33), we can obtain the solution of its dual potentials, Eqs. (7.34) and (7.35), by means of the duality relation.

7.4.1 \(U (\varphi) = \eta \varphi^6 + \sigma \varphi^4\)

The duality transformations given by Eqs. (3.5), (3.6), (3.8), and (3.9) are
\[
\phi \rightarrow \varphi^{-2},
\]
\[
x^\mu \rightarrow -2y^\mu,
\]
\[
\eta \rightarrow -G,
\]
\[
\sigma \rightarrow \kappa.
\] (7.38)

Substituting the duality relation into the solution (7.36) gives the solution of \(U (\varphi) = \eta \varphi^6 + \sigma \varphi^4\) when \(-\frac{2i\kappa^{3/2}}{3\sqrt{3}\lambda} = \eta\) and \(\kappa = \sigma\):
\[
\varphi = \left( \frac{\sigma}{3\eta} \right)^{1/2} \left[ 1 - \frac{3}{2} \tanh^2 \left( -2\alpha \tau + 2y \sqrt{\alpha^2 + \frac{\sigma^2}{3\eta}} \right) \right]^{1/2}.
\] (7.40)

7.4.2 \(U (\varphi) = \eta \varphi^{-2} + \sigma \varphi^4\)

Similarly, the solution of the potential (7.35) can be achieved by the duality transformations (3.5), (3.6), (3.8), and (3.9):
\[
\phi \rightarrow \varphi^2,
\]
\[
x^\mu \rightarrow 2y^\mu,
\]
\[
\eta \rightarrow -G,
\]
\[
\sigma \rightarrow \lambda.
\] (7.41)

Substituting into the solution (7.36) gives the solution of \(U (\varphi) = -\frac{2i\kappa^{3/2}}{3\sqrt{3}\lambda} \varphi^{-2} + \lambda \varphi^4\),
\[
\varphi = \left[ 2i \sqrt{\frac{\kappa}{3\lambda}} - i \sqrt{\frac{3\kappa}{\lambda}} \tanh^2 \left( 2\alpha \tau - y \sqrt{4\alpha^2 - 2i (3\kappa \lambda)^{1/2}} \right) \right]^{1/2}.
\] (7.42)
This is the solution of the dual potential $U(\varphi) = \eta \varphi^{-2} + \sigma \varphi^4$ when $-\frac{2n^2}{3\sqrt{3n}} = \eta$ and $\sigma = \lambda$

$$\varphi = \left(\frac{\eta}{2\sigma}\right)^{1/6} \left[3 \tanh^2 \left(2\alpha \tau - y \sqrt{4\alpha^2 + 6 \left(\frac{\sigma^2 \eta}{2}\right)^{1/3}}\right) - 2\right]^{1/2}.$$  \hspace{1cm} (7.43)

### 7.5 $V(\phi) = \lambda \phi^n + \kappa_1 \phi^{2n-2} + \kappa_2$ and its duality

The massive scalar field with the polynomial potential

$$V(\phi) = \lambda \phi^n + \kappa_1 \phi^{2n-2} + \kappa_2, \quad n \geq 3,$$  \hspace{1cm} (7.44)

has the following dual potentials. By the duality relations (3.3) and (3.4), $a = n$ corresponds to $A = \frac{2n}{n-2}$, $B_1 = -2$ and $B_2 = \frac{2n}{n-2}$, i.e., the dual potential is

$$U(\varphi) = \eta \varphi^{\frac{2n}{n-2}} + \sigma_1 \varphi^{-2} + \sigma_2 \varphi^{\frac{2n}{n-2}};$$  \hspace{1cm} (7.45)

$a = 2n - 2$ corresponds to $A = \frac{2n-2}{n-2}$, $B_1 = 1$, and $B_2 = \frac{2n-2}{n-2}$, i.e., the dual potential is

$$U(\varphi) = \eta \varphi^{\frac{2n-2}{n-2}} + \sigma_1 \varphi + \sigma_2 \varphi^{\frac{2n-2}{n-2}}.$$  \hspace{1cm} (7.46)

The fields with the potentials (7.45) and (7.46) are also dual.

The field equation with the potential (7.43) has a solution [38]

$$\phi = \left[\sqrt{\lambda^2 - 2\kappa_1 m^2} \cos \left(2m (\tau \cosh \alpha + y \sinh \alpha) + \beta\right) - \frac{\lambda}{m^2}\right]^{\frac{1}{n-2}}.$$  \hspace{1cm} (7.47)

For the field $\phi$ with the potential (7.44),

$$G = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \lambda \phi^3 = \kappa_2.$$  \hspace{1cm} (7.48)

From the solution of the potential (7.47), we can obtain the solution of its dual potentials (7.45) and (7.46) by means of the duality relation.

### 7.5.1 $U(\varphi) = \kappa_1 \varphi^{-2}$

The duality transformations given by Eqs. (3.5), (3.6), (3.8), and (3.9) are

$$\phi \rightarrow \varphi^{\frac{2}{n-2}},$$
$$x^\mu \rightarrow \frac{2}{2-n} y^\mu,$$
$$\eta \rightarrow -G,$$
$$\sigma_1 \rightarrow \kappa_1,$$
$$\sigma_2 \rightarrow \kappa_2.$$  \hspace{1cm} (7.49)

Substituting into the solution (7.47) gives the solution of $U(\varphi) = -\kappa_2 \varphi^{\frac{2n}{n-2}} + \kappa_1 \varphi^{-2} + \kappa_2 \varphi^{\frac{2n-2}{n-2}} = \kappa_1 \varphi^{-2}$:

$$\varphi = \left[\sqrt{\lambda^2 - 2\kappa_1 m^2} \cos \left(2m (\tau \cosh \alpha + y \sinh \alpha) + \beta\right) - \frac{\lambda}{m^2}\right]^{\frac{1}{2}}.$$  \hspace{1cm} (7.50)
### 7.5.2 $U(\phi) = \lambda \phi$

Similarly, the solution of the potential (7.46) can be achieved by the duality transformation given by Eqs. (3.5), (3.6), (3.8), and (3.9):

$$
\begin{align*}
\phi &\to \varphi^{\frac{1}{2n}} , \\
x^\mu &\to \frac{1}{2-n} y^\mu , \\
\eta &\to -G , \\
\sigma_1 &\to \lambda , \\
\sigma_2 &\to \kappa_2 .
\end{align*}
$$

(7.51)

Substituting into the solution (7.47) gives the solution of $U(\phi) = -\kappa_2 \phi^{\frac{2n}{2n-2}} + \lambda \phi + \kappa_2 \phi^{\frac{2n}{2n-2}} = \lambda \phi$:

$$
\varphi = \sqrt{\lambda^2 - 2\kappa_1 m^2} \cos (m (\tau \cosh \alpha + y \sinh \alpha) + \beta) - \frac{\lambda}{m^2} .
$$

(7.52)

### 8 Conclusion

In this paper, we reveal a duality between fields. The dual fields are related by the duality relation.

It is shown that fields who are dual to each other form a duality family. For polynomial potentials, the duality family consists of finite number of dual fields, for nonpolynomial potentials, e.g., the sine-Gordon field, the duality family consists of infinite number of dual fields. Some fields, e.g., the free field and the $\phi^4$-field, are self-duality.

The existence of the duality family inspires us to classify fields based on the duality. A duality family is a duality class. In a duality class, the different of fields is only a duality transformation.

The duality of fields provides a high-efficiency approach for solving field equations. Once one field equation is solved, all of its dual fields are solved by the duality relation. That is, in a duality family, we only need to solve one of its member.

As examples, we solve the $\phi^{-2}$-field from the solution of its dual field, $\phi^1$-field, solve the field equation of the dual field of the sine-Gordon field from the sine-Gordon field, etc.

In further work, we will consider the quantum theory of dual fields. For example, the relation of the Feynman rule of dual fields. Especially, in quantum field theory we will consider the duality in the heat kernel method [39] and in the scattering spectrum method [40–42]. In these methods we can calculate the one-loop effective action and the vacuum energy [43, 44]. We may observe the relation of the one-loop effective action and the vacuum energy of dual fields. Moreover, we will consider the duality of spinor fields and vector fields.

### A A solution of the scalar field equation

In this appendix we show the field equation

$$
\Box \phi + m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} = 0 ,
$$

(A.1)
has an implicit solution
\[
\beta_\mu x^\mu + \int \frac{\sqrt{-\beta^2}}{\sqrt{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right]}} d\phi \equiv F_1(x^\mu, \phi) = 0. \quad \text{(A.2)}
\]

From the solution (A.2) we obtain
\[
\partial_\mu \phi = - \frac{\partial F_1(x^\mu, \phi)}{\partial x^\mu} = \beta_\mu \frac{\sqrt{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right]}}{\sqrt{-\beta^2}}. \quad \text{(A.3)}
\]

Furthermore, by Eq. (A.3), letting
\[
\partial_\mu \phi + \beta_\mu \sqrt{\frac{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right]}{-\beta^2}} \equiv F_2 (x^\mu, \phi, \partial_\mu \phi) = 0, \quad \text{(A.4)}
\]
we arrive at
\[
\Box \phi = \partial^\mu \partial_\mu \phi = \frac{\partial}{\partial x_\mu} \partial_\mu \phi
= - \frac{\partial F_2(x^\mu, \phi, \partial_\mu \phi)}{\partial x_\mu} = \beta_\mu \frac{\sqrt{2 \left[ \frac{1}{2} m^2 \phi^2 + V(\phi) - G \right]}}{\sqrt{-\beta^2}} \partial_\phi
= - \frac{\partial F_2 (x^\mu, \phi, \partial_\mu \phi)}{\partial \phi} \partial_\mu \phi
= - \left( m^2 \phi + \frac{\partial V(\phi)}{\partial \phi} \right). \quad \text{(A.5)}
\]

This is the field equation (A.1).

Acknowledgments

We are very indebted to Dr G. Zeitrauman for his encouragement. This work is supported in part by NSF of China under Grant No. 11575125 and No. 11675119.

References

[1] M. H. Zarei, Strong-weak coupling duality between two perturbed quantum many-body systems: Calderbank-Shor-Steane codes and Ising-like systems, Physical Review B 96 (2017), no. 16 165146.

[2] J.-Y. Chen and M. Zimet, Strong-weak Chern-Simons-matter dualities from a lattice construction, Journal of High Energy Physics 2018 (2018), no. 8 15.

[3] W. Bae, Y. Cho, and D. Pak, Electric-magnetic duality in the QED effective action, Physical Review D 64 (2001), no. 1 017303.

[4] N. Seiberg, Electric-magnetic duality in supersymmetric non-Abelian gauge theories, Nuclear Physics B 435 (1995), no. 1-2 129–146.
[5] M. Hatsuda, K. Kamimura, and S. Sekiya, Electric-magnetic duality invariant Lagrangians, Nuclear Physics B 561 (1999), no. 1-2 341–353.

[6] N. Boulanger, S. Cnockaert, and M. Henneaux, A note on spin-s duality, Journal of High Energy Physics 2003 (2003), no. 06 060.

[7] Y. Igarashi, K. Itoh, and K. Kamimura, Electric-magnetic duality rotations and invariance of actions, Nuclear Physics B 536 (1998), no. 1-2 454–468.

[8] M. J. Duff and R. R. Khuri, Four-dimensional string/string duality, Nuclear Physics B 411 (1994), no. 2 473–486.

[9] A. Font, L. E. Ibanez, D. Lüst, and F. Quevedo, Strong-weak coupling duality and non-perturbative effects in string theory, Physics Letters B 249 (1990), no. 1 35–43.

[10] J. Maldacena, The Large N Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys. 2 (1997), no. hep-th/9711200 231–252.

[11] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. 2 (1998), no. IASSNS-HEP-98-21 505–532.

[12] E. Witten, Anti-de Sitter space and holography, Advances in Theoretical and Mathematical Physics 2 (1998) 253–291.

[13] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Large N field theories, string theory and gravity, Physics Reports 323 (2000), no. 3 183–386.

[14] E. D’Hoker and D. Z. Freedman, Supersymmetric gauge theories and the AdS/CFT correspondence, in Strings, Branes and Extra Dimensions: TASI 2001, pp. 3–159. World Scientific, 2004.

[15] I. Bredberg, C. Keeler, V. Lysov, and A. Strominger, From Navier-Stokes To Einstein, Journal of High Energy Physics 2012 (2012), no. 7 146.

[16] V. E. Hubeny, The fluid/gravity correspondence: a new perspective on the membrane paradigm, Classical and quantum gravity 28 (2011), no. 11 114007.

[17] G. Compère, P. McFadden, K. Skenderis, and M. Taylor, The holographic fluid dual to vacuum Einstein gravity, Journal of High Energy Physics 2011 (2011), no. 7 50.

[18] X. Hao, B. Wu, and L. Zhao, Flat space compressible fluid as holographic dual of black hole with curved horizon, Journal of High Energy Physics 2015 (2015), no. 2 30.

[19] I. Quiros, Dual geometries and spacetime singularities, Physical Review D 61 (2000), no. 12 124026.

[20] S. Bhattacharyya, S. Minwalla, V. E. Hubeny, and M. Rangamani, Nonlinear fluid dynamics from gravity, Journal of High Energy Physics 2008 (2008), no. 02 045.

[21] S. Bhattacharyya, R. Loganayagam, S. Minwalla, S. Nampuri, S. P. Trivedi, and S. R. Wadia, Forced fluid dynamics from gravity, Journal of High Energy Physics 2009 (2009), no. 02 018.

[22] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla, and A. Sharma, Conformal nonlinear fluid dynamics from gravity in arbitrary dimensions, Journal of High Energy Physics 2008 (2008), no. 12 116.

[23] T. Ashok, Forced fluid dynamics from gravity in arbitrary dimensions, Journal of High Energy Physics 2014 (2014), no. 3 138.
[24] S. Bhattacharyya, S. Minwalla, and S. R. Wadia, *The incompressible non-relativistic Navier-Stokes equation from gravity*, *Journal of High Energy Physics* **2009** (2009), no. 08 059.

[25] G. Compere, P. McFadden, K. Skenderis, and M. Taylor, *The relativistic fluid dual to vacuum Einstein gravity*, *Journal of High Energy Physics* **2012** (2012), no. 3 76.

[26] N. Pinzani-Fokeeva and M. Taylor, *Towards a general fluid/gravity correspondence*, *Physical Review D* **91** (2015), no. 4 044001.

[27] X. Wu, Y. Ling, Y. Tian, and C. Zhang, *Fluid/gravity correspondence for general non-rotating black holes*, *Classical and Quantum Gravity* **30** (2013), no. 14 145012.

[28] N. Dadhich and Z. Y. Turakulov, *The most general axially symmetric electrovac spacetime admitting separable equations of motion*, *Classical and Quantum Gravity* **19** (2002), no. 11 2765.

[29] M. Nouri-Zonoz, N. Dadhich, and D. Lynden-Bell, *A spacetime dual to the NUT spacetime*, *Classical and Quantum Gravity* **16** (1999), no. 3 1021.

[30] N. Dadhich, *Electromagnetic duality in general relativity*, *General Relativity and Gravitation* **32** (2000), no. 6 1009–1023.

[31] N. Dadhich, L. Patel, and R. Tikekar, *A duality relation for fluid spacetime*, *Classical and Quantum Gravity* **15** (1998), no. 4 L27.

[32] N. Dadhich and L. Patel, *Gravoelectric dual of the Kerr solution*, *Journal of Mathematical Physics* **41** (2000), no. 2 882–890.

[33] E. Alvarez, L. Alvarez-Gaume, and Y. Lozano, *An introduction to T-duality in string theory*, *Nuclear Physics B-Proceedings Supplements* **41** (1995), no. 1-3 1–20.

[34] A. Giveon, M. Porrati, and E. Rabinovici, *Target space duality in string theory*, *Physics Reports* **244** (1994), no. 2-3 77–202.

[35] R. Rajaraman, *Solitons and instantons: an introduction to solitons and instantons in quantum field theory*. North-Holland, 1987.

[36] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Method for solving the sine-Gordon equation*, *Physical Review Letters* **30** (1973), no. 25 1262.

[37] M. J. Ablowitz, M. Ablowitz, P. Clarkson, and P. A. Clarkson, *Solitons, nonlinear evolution equations and inverse scattering*, vol. 149. Cambridge university press, 1991.

[38] A. Polyanin and V. Zaitsev, *Handbook of Nonlinear Partial Differential Equations, Second Edition*. A Chapman et Hall book. CRC Press, 2016.

[39] D. V. Vassilevich, *Heat kernel expansion: user’s manual*, *Physics reports* **388** (2003), no. 5 279–360.

[40] N. Graham, M. Quandt, and H. Weigel, *Spectral methods in quantum field theory*, vol. 777. Springer, 2009.

[41] H. Pang, W.-S. Dai, and M. Xie, *Relation between heat kernel method and scattering spectral method*, *The European Physical Journal C* **72** (2012), no. 5 1–13.

[42] W.-D. Li and W.-S. Dai, *Heat-kernel approach for scattering*, *The European Physical Journal C* **75** (2015), no. 6.

[43] W.-S. Dai and M. Xie, *The number of eigenstates: counting function and heat kernel*, *Journal of High Energy Physics* **2009** (2009), no. 02 033.
[44] W.-S. Dai and M. Xie, *An approach for the calculation of one-loop effective actions, vacuum energies, and spectral counting functions*, *Journal of High Energy Physics* **2010** (2010), no. 6 1–29.