Spinless fermions ladders at half filling

P. Donohue,1,† M. Tsuchiizu,2,† T. Giamarchi,1,§ and Y. Suzumura2,3,∗∗

1Laboratoire de Physique des Solides, CNRS-UMR 8502, Université Paris-Sud, Bât. 510, 91405 Orsay, France
2Department of Physics, Nagoya University, Nagoya 464-8602, Japan
3CREST, Japan Science and Technology Corporation (JST), Japan

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We study a half filled ladder of spinless fermions. We show that contrarily to a single chain, the ladder becomes a Mott insulator for arbitrarily small repulsive interactions. We obtain the full phase diagram and physical quantities such as the charge gap. We show that there is only a single insulating phase for repulsive interactions, regardless of the strength of the interchain hopping and single chain Mott gap. There is thus no confinement-deconfinement transition in this system but a simple crossover. We show that upon doping the system becomes a Luttinger liquid with a universal parameter $K = 1/2$ different from the one of the single chain ($K = 1/4$).

I. INTRODUCTION

One dimensional systems are one of the few known examples of non-fermi liquid behaviors. It is thus of utmost theoretical importance to understand how one can go from a one dimensional situation to a more conventional high (typically three) dimensional one by coupling one dimensional systems, allowing particles to jump from chain to chain. This question is far from being elucidated despite several theoretical attempts. For commensurate one dimensional system another phenomenon appears: such systems are Mott insulators. This leads to a direct competition between interactions and hopping. The insulating behavior of the one dimensional system tends to kill the interchain hopping and thus to confine the electrons on individual chains. Conversely, a large interchain hopping destroys the one-dimensional character and thus weakens the Mott transition considerably, turning the system into a metal. This competition between the Mott transition and interchain hopping has in addition to its theoretical importance, implications for organic compounds that are three dimensional stacks of quarter filled chains.

Unfortunately studying an infinite number of coupled chains is extremely difficult, so to understand such phenomenon it is interesting to investigate simpler systems with a finite number of coupled chains. Such systems are the so-called ladders. They present the advantage to allow a careful study of the effects of hopping by being tractable by powerful analytical and numerical techniques. For commensurate ladders with spin, the relevance of interchain hopping was studied by renormalization group techniques. Depending on the ratio between the single chain Mott gap and the interchain hopping (suitably renormalized by the interactions) a very different flow of the single particle hopping was observed, reminiscent of the confinement-deconfinement transition expected for the infinite number of chains, even if in the ladder there is no real transition but a simple crossover. In addition, interchain hopping was shown to drastically modify the critical properties of the Mott transition compared to the one of a single chain. Despite these studies on commensurate ladders a detailed description of the phase diagram and of the nature of the Mott transition is still lacking.

In the present paper we investigate these issues on a ladder of spinless fermions. Spinless fermions exhibit extremely interesting behavior since a single chain needs a finite repulsive interaction before turning into a Mott insulator, contrarily to the spinful chain for which any repulsive interaction freezes the charge leaving only the spin degrees of freedom. One could thus naively think to be able to go from an insulating phase, dominated by the single chain gap, to a metallic phase even for repulsive interactions. In fact, quite interestingly, for the spinless ladder the Mott transition is pushed in the vicinity of the non-interacting point, invalidating this naive picture. Quite fortunately the fact that the Mott transition is now in the vicinity of the non interacting point allows to study it using standard renormalization group technique, and extract the complete properties of the transition.

The plan of the paper is as follows. In section II we introduce the model for the two leg spinless ladder. In section III we study this model using the renormalization group technique. We show that the Mott transition occurs now for arbitrary repulsive interactions and compute the various physical parameters (charge gap, Luttinger liquid parameters) both analytically and by a numerical integration of the RG equations. We analyse the phase diagram in section IV. We show that in the ladder the confinement-deconfinement is in fact a crossover. We also investigate the properties of the slightly doped ladder and point out the differences that exist compared to a doped single chain. Conclusions can be found in section V. Finally some technical details can be found in the appendix.
II. MODEL

We start from spinless electrons on a two leg ladder, described by the Hamiltonian

\[ H = -t \sum_{i,\alpha} (c_{i+1,\alpha}^\dagger c_{i,\alpha} + \text{h.c.}) + V \sum_{i,\alpha} n_{i,\alpha} n_{i+1,\alpha} \]

\[ -t_\perp \sum_{i} (c_{i,1}^\dagger c_{i,2} + \text{h.c.}) \]  \hspace{1cm} (1)

where \( \alpha = 1, 2 \) is the chain index, \( t \) and \( t_\perp \) are respectively the intra and interchain hopping, and \( V \) the repulsion between nearest neighbors particles.

To analyze the long distance properties of this model it is convenient to use the boson representation of fermions operators, valid in one dimension. Two basis are possible: (i) one can start in the original chain basis and bosonize each chain; (ii) one can use the bonding and antibonding basis. Each basis has advantages and drawbacks and we will need both to tackle the Mott transition in the ladder, so we give both boson representations below.

A Chain basis

We refer the reader to the literature for the boson mapping and recall here only the main steps to fix the notations. Taking a linearized energy dispersion at the Fermi level, we use the following expressions for right and left moving fermions:

\[ \Psi_{R,L} = \frac{\eta_{R,L}}{\sqrt{2\pi a}} e^{-i(\pm \theta - \phi) e^{i k_F x}} \]

\[ \phi \text{ and } \Pi \text{ are canonically conjugate operators and } \Pi = \frac{1}{2} \partial_\theta \phi. \nabla \phi \text{ and } \nabla \theta \text{ give respectively the long wavelength fluctuations of the density and current. We note } \eta_{R,L} \text{ Klein factors which one must introduce to reproduce the anticommutation properties of several fermion species.} \]

\[ a \text{ is a short distance cutoff of the order of the lattice spacing. With these operators the single chain Hamiltonian takes the form:} \]

\[ H = H_0 + H_{\text{int}} + H_u \]

\[ H_0 = \frac{v_F}{2\pi} \int dx \left[ (\pi \Pi)^2 + (\partial_x \phi)^2 \right] \]

\[ H_{\text{int}} = g \int dx (\partial_x \phi_s)^2 \]

\[ H_u = -4g_u \int \frac{dx}{(2\pi a)^2} \cos(4\phi) \]

where \( v_F \) and \( g \) are respectively the bare Fermi velocity and the interaction. They are given by

\[ v_F = 2ta \sin(k_F a) \]

\[ g = \frac{(1 - \cos(2k_F a))aV}{\pi^2} \]  \hspace{1cm} (4)

\[ \phi_s = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) \]  \hspace{1cm} (9)

\[ \phi_a = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2) \]  \hspace{1cm} (10)

The umklapp part, \( H_u \), only appears in this form at half-filling (i.e. when \( 4k_F a = 2\pi \)) and is responsible for the Mott transition of a single chain. The interaction \( g \) can be absorbed in the quadratic part to give the Luttinger Hamiltonian

\[ H = \frac{1}{2\pi} \int dx \left[ uK(\pi \Pi)^2 + \frac{u}{K}(\partial_x \phi)^2 \right] \]  \hspace{1cm} (6)

The parameters of the Hamiltonian are the renormalized Fermi velocity \( u \), the Luttinger \( K \) parameter and the non-universal umklapp coupling constant. For small \( V \) these parameters can be perturbatively computed:

\[ uK = v_F \]

\[ \frac{u}{K} = v_F + aV \frac{2}{\pi}(1 - \cos(2k_F a)) \]  \hspace{1cm} (7)

\[ g_u = aV \]

However, the description (3) is much more general and is valid even at large coupling provided the proper renormalized coupling constant are used. In fact in the following we will not assume such relations for the coupling constants and take \( g_u \) as a free parameter (the parameters may be tuned at will with for example a second nearest-neighbor interaction).

Using the bosonized expression (3) for the single chain, we can write the two uncoupled chains in (1) as

\[ H_{\text{int}} = g \int dx ((\partial_x \phi_s)^2 + (\partial_x \phi_a)^2) \]  \hspace{1cm} (8)

\[ H_u = -4g_u \int \frac{dx}{(2\pi a)^2} \cos(\sqrt{2}\phi_s) \cos(\sqrt{2}\phi_a) \]

where we define

\[ \phi_s = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) \]  \hspace{1cm} (9)

\[ \phi_a = \frac{1}{\sqrt{2}}(\phi_1 - \phi_2) \]  \hspace{1cm} (10)

The interchain hopping in (1) reads in this basis

\[ -t_\perp \frac{2}{\pi a} \cos(\sqrt{2}\phi_a) \cos(\sqrt{2}\phi_a) \]

\[ \text{This basis has the advantage to treat very simply the umklapp term, but has the drawback not to reproduce easily the band picture of free fermions.} \]

B Two band basis

Another basis is the bonding-antibonding band basis. We first diagonalize the kinetic energy in (1) with

\[ c_{i,0} = \frac{1}{\sqrt{2}}(c_{i,1} + c_{i,2}) \]  \hspace{1cm} (12)

\[ c_{i,\pi} = \frac{1}{\sqrt{2}}(c_{i,1} - c_{i,2}) \]  \hspace{1cm} (13)
and the corresponding boson fields. In this basis the interchain hopping is diagonal:

\[ -t_\perp \sum_i (c_i^{\dagger \perp} c_{i+1}^{\perp} - c_i^{\perp} c_i^{\dagger \perp}) \]  

however the interaction term is less simple to formulate. Rather than to use the bonding and antibonding boson fields it is again convenient to introduce the symmetric and antisymmetric combination that we denote now \( \phi_p = (\phi_0 + \phi_\pi)/\sqrt{2} \) and \( \phi_s = (\phi_0 - \phi_\pi)/\sqrt{2} \) to distinguish them from the ones in the chain basis. This leads to the simple bosonized expression:

\[ -\frac{t_\perp \sqrt{2}}{\pi} \int dx \partial_x \phi_s \]  

The change of basis for the interaction term can be most easily performed using the transformation formulas for the total charge current.

\[ -\frac{1}{\pi} \partial_x \phi_s = -\frac{1}{\pi} \partial_x \phi_p \]

\[ -1/\pi \partial_x \phi_s = \frac{\sqrt{2}}{\pi a} \cos \sqrt{2} \phi_s \cos \sqrt{2} \theta_s \]

Since the interaction terms are quadratic in currents we also need an operator product expansion to extract the most relevant operators.

\[ \cos n\phi^2 = \frac{1}{2} (1 - \frac{n^2 a^2}{2} (\partial_x \phi)^2 + \cos 2n\phi) \]

The expression for the anti-symmetric current is however misleading since it does not include Klein factors required to identify bosonic exponents to anticommuting fermions. To obtain the exponential terms it is necessary to read the bosonized expression on the transformed four fermion operators. With these boson operators the interaction term takes the following form:

\[ H_{\text{int}} = \frac{2V a}{\pi^2} \int dx (-\partial_x \phi_s^2) \]

\[ -\frac{V_\perp}{\pi^2} \int dx (\pi^2 \Pi^2 + (\partial \phi_\sigma)^2) \]

\[ + \frac{V a}{\pi^2} \int dx (\frac{\cos \sqrt{8} \phi_s}{a} - \cos \sqrt{8} \phi_\sigma) \]

\[ - \cos \sqrt{8} \phi_\sigma \cos \sqrt{8} \theta_\sigma \]

and the umklapp term is

\[ H_u = -\frac{g_\sigma}{2\pi^2 a^2} \int dx (\cos (\sqrt{8} \phi_p) (\cos (\sqrt{8} \phi_\sigma) + \cos (\sqrt{8} \theta_\sigma)) \]

We now define the coupling constant \( g_\sigma \) associated with the operator \( \cos (\sqrt{8} \theta_\sigma) \).

\[ \delta H = 2g_\sigma \int \frac{dx}{(2\pi a)^2} \cos (\sqrt{8} \theta_\sigma) \]

### III. MOTT TRANSITION

Let us now analyse the Mott transition. The single chain needs a finite strength interaction in order to become an insulator. Indeed, as can be seen from (3), for a single chain the umklapp term has a dimension \( 2 - 4K \). It thus opens a gap in the charge sector and leads to a Mott insulating phase for \( K < 1/2 \) and a metallic (Luttinger liquid) phase for \( K > 1/2 \). For the t-V model this corresponds to \( V = 2t \). In the ladder the presence of interchain hopping dramatically modifies this. As we will show, the ladder is an insulator for infinitesimal repulsive interactions. This can be easily seen by looking in the chain basis at the operators generated by \( t_\perp \). Obviously \( t_\perp \) will generate terms such as \( \cos (\sqrt{8} \phi_\sigma) \) and \( \cos (\sqrt{8} \theta_\sigma) \). Note that these terms only contain the antisymmetric field. However when combined with the umklapp (3) the \( \cos (\sqrt{8} \phi_\sigma) \) will generate a \( \cos (\sqrt{8} \theta_\sigma) \). This term has the dimension \( 2 - 2K_s \), and in contrast with the single chain umklapp, is relevant for \( K_s < 1 \) leading to a gap in the symmetric sector and hence to a Mott insulating phase. The perpendicular hopping thus reinforces the insulating character of the system.

To go beyond this simple argument let us now investigate the full RG flow using a two scale analysis of the relevant operators in the Hamiltonian.

#### A Mott insulator for \( 1/2 < K < 1 \)

Let us first focus for values of \( K \) for which the single chain would be metallic. In that case single chain umklapp operator (3) is irrelevant. The gap in this regime thus results from the competition of the initial decrease of the umklapp constant in the flow and its subsequent growth after \( t_\perp \) has blocked the transverse fluctuations.

To investigate the gap let us write the full RG equations, which include the umklapp term. Near the non-interacting point, if \( t_\perp \) is small the flow is given by

\[ \frac{dg_u}{dl} = (2 - 2K_a - 2K_s) g_u \]

\[ \frac{dg_s}{dl} = (2 - 2K_s) g_s + \frac{1}{\pi} g_u g_a \]

\[ \frac{dg_a}{dl} = (2 - 2K_a) g_a + \pi(K^{-1} - K)(t_\perp a)^2 + \frac{1}{\pi} g_u g_s \]

\[ \frac{dg_f}{dl} = (2 - 2K_s) g_f + \pi(K^{-1} - K)(t_\perp a)^2 \]

\[ \frac{dK_a}{dl} = \frac{1}{2\pi^2} (g_f^2 - g_s^2 - 2g_u^2) \]

\[ \frac{dK_s}{dl} = -\frac{1}{2\pi^2} (g_f^2 + 2g_u^2) \]

\[ \frac{dt_\perp}{dl} = (2 - \frac{1}{2}(K_a + 1/K_s)) t_\perp \]  

where we have set \( g/u \to g \) (\( u = 1 \)) to keep the notations simple. For \( g_u = 0 \) these equations reduce to the
ones obtained in Ref. 13. We have introduced couplings which are generated during the flow, though they are not present in the bare hamiltonian:

$$\delta H = 2g_0 \int \frac{dx}{(2\pi a)^2} \cos \sqrt{8} \phi_a - 2g_f \int \frac{dx}{(2\pi a)^2} \cos \sqrt{8} \theta_a + 2g_s \int \frac{dx}{(2\pi a)^2} \cos \sqrt{8} \phi_s \tag{22}$$

These new couplings are $g_s$, which is a density-density interaction between the two chains and $g_f$ which is the umklapp part of this interaction. The coupling $g_f$ transfers a pair of fermion from one chain to the other and corresponds to a Josephson coupling between the two chains.

For a finite value of the renormalised $t_\perp$ that depends on the initial fermi velocity and interactions, one can neglect the interaction terms that couple the two bands in a non-resonant way. Hence we can use the above flow equations up to a lengthscale $l_1$ where $t_\perp$ reaches this finite value, $t_\perp \sim O(1)$. Above the scale $l_1$, it is more convenient to switch to the two band basis in order to study the flow. By definition of $l_1$ we discard any term that contains $\cos \sqrt{8} \phi_a$, since such terms transfer momentum among the bands. Above $l_1$ the flow becomes

$$\frac{dK_\sigma}{dl} = \frac{1}{2\pi^2}(g_\sigma^2 + \frac{1}{2}g_u^2) \tag{23}$$

$$\frac{dg_\sigma}{dl} = (2 - 2K_\rho - 2K_\sigma^2)g_\sigma$$

$$\frac{dg_u}{dl} = (2 - 2K_\rho)g_u \tag{24}$$

In the $\sigma$-sector the remaining $\cos \sqrt{8} \theta_\sigma$ term opens a gap. We note $l_\rho$, the scale when the coupling term $\cos \sqrt{8} \theta_\sigma$ has flowed to a value of order one, and the gap amplitude is evaluated as $\Delta_\sigma = e^{-l_\sigma}$. Above the scale $l_\sigma$ the umklapp operator becomes relevant since it is reduced to a simple $\cos \sqrt{8} \phi_\rho$.

These equations describe completely the Mott transition in the ladder system. An analytical solution can be given both in the limit of very small interchain hopping and for large interchain hopping but in the limit of very small interactions. A numerical integration of the equations allows to obtain the gap for arbitrary initial parameters.

$$l \quad t_\perp \to 0$$

Since the umklapp is irrelevant for a single chain, one can replace the flow, when $t_\perp$ is the smallest scale in the problem, by

$$\frac{dg_u}{dl} = (2 - 4K^*)g_u \quad \frac{dt_\perp}{dl} = (2 - \frac{1}{2}(K^* + 1/K^*))t_\perp \tag{25}$$

where $K^*$ is the renormalized Luttinger parameter for a single chain. For the $t - V$ model $K^*$ can be obtained exactly.13 When $t_1 \sim O(1)$ we switch to the second set of flow equations (23). We do not know the precise renormalization of the umklapp constant between $l_1$ and $l_\sigma$. But since $l_\sigma - l_1$ does not depend on $t_\perp$, this gives a simple multiplicative constant. Above $t_\sigma$ we use (24).

Integration of the RG equations yields:

$$g_u(l_1) = g_u(0)e^{(2-4K^*)l_1}$$

$$g_u(l_\sigma) = g_u(l_1)e^\beta(l_\sigma - l_1)$$

$$g_u(l_\rho) = g_u(l_\sigma)e^{(2-2K^*)(l_\rho - l_\sigma)} \tag{26}$$

We have used $\beta$ as a constant that takes into account the variation of $K_\sigma$ between $l_1$ and $l_\sigma$ as it flows to zero. This does not change the $t_\perp$ dependence. Collecting the intermediate results, we can give an asymptotic dependence of the total charge gap on $t_\perp$:

$$\Delta_\rho \propto t_\perp^{1 - \frac{K^*}{2(2 - (K^* + 1/K^*))}} \tag{27}$$

While deriving this result we have neglected terms generated to second order by $t_\perp$, such as $g_u, g_f$ and $g_s$. In the appendix we show that including those terms do not affect the dependence of $\Delta_\rho$ on $t_\perp$.

$$2 \quad V \to 0, \text{ large } t_\perp$$

Another interesting limit is when $t_\perp$ is comparable to $t$. Note that we always remain in the limit where $t_\perp < t$ in order to keep four points at the Fermi level, otherwise the problem would be the trivial one of a single filled band of fermions. In that case the initial flow does not exist and we start directly with (23). The umklapp operator has an initial dimension of $t - 4K$ and thus initially decreases in the flow. However combined with the operator $g_\sigma \cos \sqrt{8} \theta_\sigma$, it generates from second order in the interaction expansion a term of the form $g_\rho \cos \sqrt{8} \phi_\rho$. This new operator is relevant, its scaling dimension being $2 - 2K_\rho$. The flow equation of this operator reads:

$$\frac{dg_\rho}{dl} = (2 - 2K_\rho)g_\rho + \frac{1}{2\pi}g_u g_\sigma \tag{28}$$

Using the flow equations at large $t_\perp$, we have $g_u(l) \approx V e^{(2-V)l}$ and $g_\rho(l) = V + O(1/V^{1/2})$, these expressions are valid at the beginning of the flow and we have taken into account only the leading dependence on $V$. This leads to an approximate expression for $g_\rho$ using the fact that for small interactions $2 - 2K_\rho \propto V$:

$$g_\rho(l) = \frac{V^2}{2} (e^{2Vl} - e^{-2l}) \tag{29}$$

The coupling $g_\rho$ starts at zero and is driven by $g_u$ which is decreasing rapidly. We may now determine the scale $l^*$ where $g_\rho$ becomes larger than $g_u$. Using (25) one gets:

$$\frac{V^2}{2} e^{Vl^*} = Ve^{-(2-V)l^*} \tag{30}$$
FIG. 1: Phase diagram as a function of interaction and interchain hopping ($t_{\perp}$). The ladder is an insulator for any repulsive interaction ($K^* < 1$), while the single chain is insulating for $K^* < 1/2$. The dashes represent a crossover between a region where $t_{\perp}$ is relevant in the RG sense and a region where it is irrelevant ($\Delta_{1ch}$ is the gap on a single chain). For $1/2 < K^* < 1$ the charge gap, $\Delta_\rho$, vanishes as a power law of $t_{\perp}$ for small $t_{\perp}$ and as $\ln \Delta_\rho \propto \ln V/V$ for small interaction (see text).

which yields $l^* \propto \ln \frac{1}{l}$. We note that at this scale $g_\sigma$ has hardly changed at all, which validates the expression for $g_\rho$. Indeed $g_\sigma$ reaches a value of order one at $l_\sigma \propto \frac{1}{l}$ which for asymptotically small interactions is much larger than $l^*$. The charge gap is dominated by $g_\rho$ and we may safely drop the original umklapp operator which gives only subdominant contributions. The gap in the total charge sector is given by the scale where $g_\rho$ reaches a value of order one. This gives a gap

$$\ln \Delta_\rho \propto \ln \frac{V}{V}$$  \hspace{1cm} (31)

We note that the gap in the charge sector decreases faster than the gap in $\sigma-$sector, where $\ln \Delta_\sigma \propto V^{-1}$.

3 Numerical solution of the equations

These asymptotic behaviors and the phase diagram are shown in Figure 1. To go beyond the asymptotics either at small $t_{\perp}$ or at the transition $V \rightarrow 0$, one needs to numerically integrate the flow (22,23,24). One example of such a flow is shown on Figure 2. One clearly sees the initial decrease of the umklapp in the initial phases of the flow, followed by its subsequent increase when $g_f \sim O(1)$. The charge gap is given in Figure 3.

FIG. 2: Typical RG flow, for $1/2 < K < 1$, showing the two regime: below $l_1$ (scale where $t_{\perp}$ becomes unity) and above where some couplings are cut by momentum conservation enabling the umklapp term to flow to strong coupling. The quantities $l_\sigma$ and $l_\rho$ show the scales where $g_\sigma/\pi u$ and $g_\rho/\pi u$ become unity at $l = l_\sigma$ and $l = l_\rho$ respectively. The dashed curve represents the flow for $t_{\perp} = 0$ and the arrow denotes $K^*(\sim 0.59)$.

FIG. 3: Charge gap as a function of $t_{\perp}$ for finite interactions characterized by the Luttinger parameter $K$. The inset shows the power law behavior of the charge gap, for small interchain hopping.
whether $\Delta$ is the same in the whole insulating phase, regardless of out of phase configurations (b). Such a ground state chains, (a) and (b). The interchain hopping selects the

FIG. 4: Degenerate ground states for two uncoupled chain. For small $t_\perp$ the model is conveniently studied in the chain basis, using the flow (21). Because the field $\phi_{1,2}$ orders, the single particle hopping between the chains is irrelevant (for small $t_\perp$). The only remaining relevant coupling is thus the generated interchain density-density interaction $g_\alpha$. This coupling opens a gap in the antisymmetric sector and orders $\phi_\alpha$. The physics is quite clear: in the absence of interchain coupling each chain has a charge density wave ground state that is double degenerate. In the presence of $t_\perp$, particles still cannot hop from chain to chain because of the Mott gap in each chain, but the virtual hops tend to lock the two CDW relative to each other. This is shown on Figure 4. The Mott gap is thus essentially here the single chain gap. The gap as a function of $K$ obtained by numerical integration of the flow is shown in Figure 5.

IV. PHASE DIAGRAM

The results of section III allow to draw the phase diagram of the commensurate ladder, as shown on Figure 4. The very existence of an insulating phase in the ladder for $1/2 < K < 1$ prompts for several questions.

A Confinement vs crossover

Apparentely we have to face two very different behaviors depending on the strength of $t_\perp$. For $K < 1/2$ and small $t_\perp$, $t_\perp$ renormalizes to zero as was shown in section III B, whereas for large $t_\perp$ (or for $1/2 < K < 1$ where the single chain gap is absent) $t_\perp$ renormalizes to large values. One thus seems to have a confinement-deconfinement transition, similar to the one expected for an infinite number of chains, induced by the competition between $t_\perp$ and the single chain Mott gap $\Delta_{1ch}$. The change of behavior occurs when

$$t_\perp^{\text{eff}} = \Delta_{1ch}$$

B Gap for $K < 1/2$

The picture is quite different in this case since without interchain hopping a gap would be present on each chain. For small $t_\perp$ the model is conveniently studied in the chain basis, using the flow (21). Because the field $\phi_{1,2}$ orders, the single particle hopping between the chains is irrelevant (for small $t_\perp$). The only remaining relevant coupling is thus the generated interchain density-density interaction $g_\alpha$. This coupling opens a gap in the antisymmetric sector and orders $\phi_\alpha$. The physics is quite clear: in the absence of interchain coupling each chain has a charge density wave ground state that is double degenerate. In the presence of $t_\perp$, particles still cannot hop from chain to chain because of the Mott gap in each chain, but the virtual hops tend to lock the two CDW relative to each other. This is shown on Figure 4. The Mott gap is thus essentially here the single chain gap. The gap as a function of $K$ obtained by numerical integration of the flow is shown in Figure 5.

FIG. 5: Charge gap for finite $t_\perp$ as a function of the Luttinger parameter $K$, the inset shows curves for different values of $t_\perp$. For $K < 1/2$ the Mott gap is essentially the single chain gap, whereas it is strongly $t_\perp$ dependent for $1/2 < K < 1$.

where $t_\perp^{\text{eff}}$ is the renormalized interchain hopping. We thus have to determine here whether we have two different insulating phases. This can be done by looking at the fields that order in each case. In the “confined” phase we showed in section III B that both $\phi_\sigma$ and $\phi_\rho$ acquire mean values. In the “deconfined” phase, as was shown in section III A, $\theta_\sigma$ and $\phi_\rho$ orders. The physical observable corresponding to an out of phase charge density wave takes the following form with the two different set of boson operators:

$$J_{2k_\perp} = \psi_1^{\dagger}\psi_1 - \psi_2^{\dagger}\psi_2$$

$$= \frac{2}{\pi a} \sin \sqrt{2}\phi_\sigma \sin \sqrt{2}\phi_\alpha$$

$$= \frac{2}{\pi a} \sin \sqrt{2}\phi_\sigma \sin \sqrt{2}\theta_\sigma$$

It is easy to see from (33) that the conditions $\langle \cos \sqrt{2}\theta_\sigma \rangle = -1$, $\langle \cos \sqrt{2}\phi_\alpha \rangle = -1$ on the one hand and $\langle \cos \sqrt{2}\phi_\alpha \rangle = -1$, $\langle \cos \sqrt{2}\phi_\sigma \rangle = -1$ on the other both give a non zero value to $\langle J_{2k_\perp} \rangle$. Thus the two “phases” have the same ordered fields (written in a different basis) and there is no transition but a simple crossover. The out of phase charge density waves ground state that arises naturally from the $K < 1/2$ picture and is shown in Figure 4 stays valid even in the other limit.
B Doped ladder

Let us now determine the phase diagram of the ladder system away from half filling. Introducing a chemical potential would change the Hamiltonian into

\[ H = H_{1/2\text{filling}} - \sqrt{2}\mu/\pi \int dx \nabla \rho \]

As we have seen that the operator opening the charge gap is always

\[ H_{\text{Mott ladder}} = \int dx \cos(\sqrt{8}\phi_p) \]

instead of

\[ H_{\text{Mott 1 chain}} = \int dx \cos(4\phi_p) \]

for the single chain. Using the known generic solution of such Mott systems, we can see that in the ladder the doping will destroy the Mott phase when \( \mu = \Delta \). The properties at the transition are the generic properties of the Mott transition in one dimension when varying the doping (Mott-\( \delta \)), namely: (i) a dynamical exponent \( z = 2 \); (ii) a divergent compressibility on the metallic side. The luttinger liquid exponent is universal and is \( K^* = 1/2 \). This values is quite different from the one of a single chain. The differences are recalled in Table I.

The different values of \( K^* \) should be observable in numerical calculations such as exact diagonalization and dmrg. It might be a more clear signature of the difference between the ladder and the single chain that the gap itself specially for \( K < 1/2 \).

These different values of \( K^* \) correspond to two different kinds of elementary excitations. For the gapped single spinless chain the elementary excitations are domain walls separating charge density waves with a different phase, the equivalent of the excitations called spinons in the spin-1/2 Heisenberg model. In a path-integral picture these corresponds to configurations where the field \( \phi \) is step-like with for instance \( \phi(-\infty,t) = 0 \) and \( \phi(+\infty,t) = \pi/2 \). However such a configuration on one chain with a nearly constant configuration on the second chain has an infinite cost when there is a cos \( \sqrt{8}\phi_p \) term in the model, no matter how small the coupling constant. Therefore the low-energy excitations allowed for two coupled chains are kinks (two domain walls confined in the same region). This behavior is reminiscent of the double-sine-Gordon model that appears for a spontaneously dimerized frustrated spin-1/2 chain (next-nearest-neighbor exchange) with an infinitesimal bond alternation perturbation.

A summary of the phase diagram is shown on Figure 6.

V. CONCLUSION

We have investigated in this paper commensurate ladders of spinless fermions. We have shown that contrarily to the single chain, that needs a finite repulsion to become a Mott insulator, a two leg ladder turns insulating for arbitrary small repulsion between the fermions. The interchain hopping thus paradoxically reinforces the insulating behavior of the system. We have computed the charge gap as a function of the Luttinger exponent \( K \) of a single chain. For values of \( K \) for which the single chain system would be metallic (\( 1/2 < K < 1 \)) the charge gap vanishes as a power law of the interchain hopping \( t_\perp \) for small \( t_\perp \), and faster than the standard BKT behavior, usually characterizing metal insulator transitions in one dimension, when the interactions tend to zero. The interchain hopping also affects the universal properties of the transition that occur upon doping the system. Indeed for very small doping the Luttinger liquid parameter takes the universal value \( K^* = 1/2 \) instead of \( K^* = 1/4 \) for a single chain (when a Mott gap is present). This difference in Luttinger liquid parameters should be accessible in numerical simulations such as exact-diagonalization or dmrg.

| Parameter          | 1 chain | Ladder |
|--------------------|---------|--------|
| \( K^*_{\text{Mott-U}} \) | \( 1/2 \) | \( 1 \) |
| \( K^*_{\text{Mott-}\delta} \) | \( 1/4 \) | \( 1/2 \) |

FIG. 6: Phase diagram as a function of the chemical potential \( \mu \). The dashed line shows the metal-insulator (commensurate-incommensurate) transition for a single chain (beginning at finite interaction \( V = 2t \)), the continuous line corresponds to the spinless fermionic ladder. \( K^* \) is the universal value of the luttinger liquid parameter at the transition.
Given the fact that when a Mott gap is well established in a single chain the interchain hopping scales to zero, whereas it scales to large values otherwise, it was important to know whether there was a confinement-deconfinement transition. We have shown that there is in the spinless ladder only a crossover and that the two seemingly different insulating phases (large Mott gap on a single chain and small \( t_\perp \) and large \( t_\perp \) and small (or zero) single chain Mott gap) are in fact of the same nature.

This leaves open how this cross-over should evolve for an increasing number of coupled chains to give back the confinement-deconfinement transition expected for an infinite number of chains.

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APPENDIX A: TERMS GENERATED TO SECOND ORDER BY \( T_\perp \)

The RG equations \(^b\) show that during the first part of the flow up to \( l_1 \), \( t_\perp \) generates to second order the term \( g_\rho \cos \sqrt{8} \phi_\rho \). Contracting \( g_\rho \) with \( g_\rho \) gives a contribution to the renormalization of the term \( g_\rho \cos \sqrt{8} \phi_\rho \) (noticing that \( \phi_\rho = \phi_s \), we switch to the notation \( g_\rho \) instead of \( g_\rho \)). This term is implicitly present in the hamiltonian. However it is generated and relevant for any repulsive interaction. One must thus make sure that this term does not grow so fast as to open the gap before the original umklapp term. If it were the case this would change the expression for the charge gap compared to the one given in the text.

To determine the growth of this coupling, we have to determine \( g_\rho (l) \). Naively the scaling dimension of \( g_\rho \) is \( 2 - 2K \), however it is generated by \( t_\perp^2 \), term of dimension \( 4 - (K + 1/K) \). Comparing the two rates shows that the source term dominates. Integration then shows that \( g_\rho (l) \propto t_\perp^2 (l) \). We may now repeat the same analysis for \( g_\rho \), indeed its scaling dimension is \( 2 - 2K \) but it is actually driven by \( g_\rho (l) g_\rho (l) \), term of dimension \( 6 - 5K - 1/K \), dominant in the regime considered here \((K > 1/2)\). Consequently one has \( g_\rho (l) \propto g_\rho (l) g_\rho (l) \). Thus at the scale \( l_1 \), where \( t_\perp \) is of order one, one has \( g_\rho (l_1) \approx g_\rho (l_1) \).

To complete the argument we note that the couplings \( g_\rho (l) \cos \sqrt{8} \phi_\rho \) and \( g_\rho (l) \cos \sqrt{8} \phi_\rho \cos \sqrt{8} \phi_\rho \) are above \( l_1 \) basically the same operator. Indeed in the \( \sigma \)-sector \( g_\sigma \) has flowed to strong coupling above \( l_\sigma \), \( \langle \cos \sqrt{8} \phi_\sigma \rangle \neq 0 \). Recalling that the flow between \( l_1 \) and \( l_\rho \) does not change the asymptotic dependence on \( t_\perp \) and that \( g_\rho (l_1) \) and \( g_\rho (l_1) \) are proportional we recover by following the flow of the coupling \( g_\rho \) the behavior explained in the text.

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\(^\dagger\) Electronic address: honohue@lps.u-psud.fr
\(^\ddagger\) Electronic address: tsuchiiz@edu2.phys.nagoya-u.ac.jp
\(^\ddagger\) Electronic address: giam@lps.u-psud.fr
\(^\dagger\) Electronic address: 43428a@mucc.cc.nagoya-u.ac.jp

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