FREE SPACES
OVER COUNTABLE COMPACT METRIC SPACES

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Abstract. We prove that the Lipschitz-free space over a countable compact metric space is isometric to a dual space and has the metric approximation property.

1. Introduction

Let \((M, d)\) be a pointed metric space, that is to say, a metric space equipped with a distinguished origin, denoted 0. The space \(\text{Lip}_0(M)\) of Lipschitz functions from \(M\) to \(\mathbb{R}\) vanishing at 0 is a Banach space equipped with the Lipschitz norm:

\[
\|f\|_{\text{L}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.
\]

Its unit ball is compact with respect to the pointwise topology, thus \(\text{Lip}_0(M)\) is a dual space. In [3], its predual is called the Lipschitz-free space over \(M\), denoted \(F(M)\), and it is the closed linear span of \(\{\delta_x, x \in M\}\) in \(\text{Lip}_0(M)^*\). One can prove that the map \(\delta : M \to F(M)\) is an isometry. For more details on the basic theory of the spaces of Lipschitz functions and their preduals, called Arens-Eells space there, see [14].

Very little is known about the structure of Lipschitz-free spaces. For instance \(F(\mathbb{R})\) is isomorphically isometric to \(L_1\), but A. Naor and G. Schechtman [11] proved that \(F(\mathbb{R}^2)\) is not isomorphic to any subspace of \(L_1\). The study of the Lipschitz-free space over a Banach space is useful to learn more about the structure of this Banach space. For example G. Godefroy and N. Kalton [3] proved, using this theory, that if a separable Banach space \(X\) isometrically embeds in a Banach space \(Y\), then \(Y\) contains a linear subspace which is linearly isometric to \(X\).

We recall that a Banach space \(X\) is said to have the approximation property (AP) if for every \(\varepsilon > 0\) and every compact set \(K \subset X\), there is a bounded finite-rank linear operator \(T : X \to X\) such that \(\|Tx - x\| \leq \varepsilon\) for every \(x \in K\). If moreover there exists \(1 \leq \lambda < +\infty\) not depending on \(\varepsilon\) or \(K\) such that \(\|T\| \leq \lambda\), then \(X\) has the \(\lambda\)-bounded approximation property (\(\lambda\)-BAP) and \(X\) has the bounded approximation property (BAP) if it has the \(\lambda\)-BAP for some \(\lambda\). Finally \(X\) has the metric approximation property (MAP) if \(\lambda = 1\).

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It is already known that $\mathcal{F}(\mathbb{R}^n)$ has the MAP \cite{3} and that if $M$ is a doubling metric space, then $\mathcal{F}(M)$ has the BAP \cite{9}. Moreover, E. Pernecká and P. Hájek \cite{7} proved that $\mathcal{F}(\ell_1)$ and $\mathcal{F}(\mathbb{R}^n)$ have a Schauder basis. However, Godefroy and N. Ozawa \cite{4} constructed a compact metric space $K$ such that $\mathcal{F}(K)$ fails the AP.

In the first part of this article we will prove that the Lipschitz-free space over a countable compact metric space $K$ is isometrically isomorphic to the dual space of $lip_0(K) \subset Lip_0(K)$. Let $\omega_1$ be the first uncountable ordinal. We will prove, by induction on $\alpha < \omega_1$ such that $K^{(\alpha)}$ is finite, that $\mathcal{F}(K)$ has the MAP. This will rely on a theorem of A. Grothendieck \cite{6} asserting that any separable dual having the BAP has the MAP, and a decomposition of the space $K$ due to Kalton \cite{8}. This provides a negative answer to Question 2 in \cite{4}, which was originally asked by G. Aubrun to G. Godefroy during a seminar in Lyon about his paper with N. Ozawa.

\section{Duality}

For any pointed metric space $(M, d)$ we denote by $lip_0(M)$ the subspace of $Lip_0(M)$ defined as follows: $f \in lip_0(M)$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for $x, y \in M$, $d(x, y) < \delta$ implies $|f(x) - f(y)| \leq \varepsilon d(x, y)$.

The main result of this section is the following:

**Theorem 2.1.** If $(K, d)$ is a countable compact metric space, then $\mathcal{F}(K)$ is isometrically isomorphic to a dual space, namely $lip_0(K)^*$.

**Definition 2.2.**

1. Let $X$ be a Banach space. A subspace $S$ of $X^*$ is called separating if $x^*(x) = 0$ for all $x^* \in S$ implies $x = 0$.
2. For $(M, d)$ a pointed metric space, $lip_0(M)$ separates points uniformly if there exists a constant $c \geq 1$ such that for every $x, y \in M$, some $f \in lip_0(M)$ satisfies $\|f\|_L \leq c$ and $|f(x) - f(y)| = d(x, y)$.

Mimicking an argument from \cite{2} we will use a theorem due to Petunin and Plíchko \cite{13} saying that if $(X, \|\cdot\|)$ is a separable Banach space and $S$ a closed subspace of $X^*$ contained in $NA(X)$ (the subset of $X^*$ consisting of all linear forms which attain their norm) and separating points of $X$, then $X$ is isometrically isomorphic to $S^*$. Theorem 3.3.3 in \cite{14} gives the same result but in a less general case.

We start with two lemmas taken from \cite{2}.

**Lemma 2.3.** For any $(K, d)$ compact pointed metric space, the space $lip_0(K)$ is a subset of $NA(\mathcal{F}(K))$.

**Proof.** We can see $lip_0(K)$ as the subset of $Lip_0(K)$ containing all $f$ such that for every $\varepsilon > 0$, the set $K^2 := \{(x, y) \in K^2, x \neq y, |f(x) - f(y)| \geq \varepsilon d(x, y)\}$ is compact.

Let $f \in lip_0(K)$; we may assume that $f \neq 0$. Then there exists $\varepsilon > 0$ such that

$$
\|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} = \sup_{(x, y) \in K^2} \frac{|f(x) - f(y)|}{d(x, y)} = \max_{(x, y) \in K^2} \frac{|f(x) - f(y)|}{d(x, y)}.
$$

Thus there exist $x \neq y$ such that $\|f\|_L = \frac{|f(x) - f(y)|}{d(x, y)}$, and setting $\gamma = \frac{1}{d(x, y)}(\delta_x - \delta_y)$ we obtain $\gamma \in \mathcal{F}(K)$ and $\|f\|_L = |f(\gamma)|$, with $\|\gamma\|_{\mathcal{F}(K)} = 1$ because $\delta$ is an isometry. Then $f$ is norm attaining and $lip_0(K) \subset NA(\mathcal{F}(K))$.  \qed
Lemma 2.4. For any \((K, d)\) compact pointed metric space, if \(\text{lip}_0(K)\) separates points uniformly, then it is separating.

Proof. Using the Hahn-Banach theorem, one can prove that \(\text{lip}_0(K)\) is separating if and only if it is weak*- dense in \(\text{Lip}_0(K)\).

Now assume \(\text{lip}_0(K)\) separates points uniformly. Then there exists \(c \geq 1\) such that for every \(F \subset K\), \(F\) finite, and every \(f \in \text{lip}_0(K)\), \(\|g\|_L \leq c\|f\|_L\), such that \(f|_F = g|_F\) (see Lemma 3.2.3 in [14]), and it is easy to deduce that \(\overline{\text{lip}_0(K)}^{\text{wo}} = \text{Lip}_0(K)\).

These lemmas allow us to reduce the problem. We need to prove that the little Lipschitz space over a countable compact metric space separates points uniformly.

For this proof we will use a characterization of countable compact metric spaces with the Cantor-Bendixon derivation: for a metric space \((M, d)\) we denote

- \(M'\) the set of accumulation points of \(M\).
- \(M^{(\alpha)} = (M^{(\alpha-1)})'\), for a successor ordinal \(\alpha\).
- \(M^{(\alpha)} = \bigcap_{\beta < \alpha} M^{(\beta)}\), for a limit ordinal \(\alpha\).

A compact metric space \((K, d)\) is countable if and only if there is a countable ordinal \(\alpha\) such that \(K^{(\alpha)}\) is finite.

Proof of Theorem 2.1. Let us prove that

\[ \exists c \geq 1, \forall x \neq y \in K, \exists h \in \text{lip}_0(K), \|h\|_L \leq c, |h(x) - h(y)| = d(x, y). \]

Let \(x \neq y \in K\) and set \(a = d(x, y)\). Since \(K\) is countable and compact, the closed ball \(\overline{B}(x, \frac{a}{2})\) of center \(x\) and radius \(\frac{a}{2}\) is countable and compact and there exists a countable ordinal \(\alpha_0\) such that \(\overline{B}(x, \frac{a}{2})^{(\alpha_0)}\) is finite and nonempty: there exist \(k_1 \in \mathbb{N}, y_1^1, \ldots, y_1^{k_1} \in K\) such that \(\overline{B}(x, \frac{a}{2})^{(\alpha_0)} = \{y_1^1, \ldots, y_1^{k_1}\}\). We denote \(a_1^i = d(y_1^i, x)\), for \(1 \leq i \leq k_1\). Then we can find \(r_1\) and \(v_1^1 < \cdots < v_1^{r_1}\) such that \(\{a_1^1, \ldots, a_1^{k_1}\} = \{v_1^1, \ldots, v_1^{r_1}\}\). Now set

\[ v_1 = \begin{cases} \frac{a}{2}, & \text{if } \overline{B}(x, \frac{a}{2})^{(\alpha_0)} = \{x\} \\ \min \left(\{v_1^1, \frac{a}{2} - v_1^{r_1}\}\{0\}\right) \cup \{v_1^i - v_1^{i-1}, 2 \leq i \leq r_1\}, & \text{otherwise} \end{cases} \]

and define \(\varphi_1 : [0, +\infty] \to [0, +\infty] \) by

\[ \varphi_1(t) = \begin{cases} 0, & t \in \left[0, v_1^1\right] \, \left[0, \frac{a}{12}\right] := V_1^0 \\ v_1^1, & t \in \left[v_1^1 - \frac{a}{12}, v_1^1 + \frac{a}{12}\right] := V_1^1, 1 \leq i \leq r_1 \\ v_1^i - v_1^{i-1}, & t \in \left[v_1^i, v_1^i + \frac{a}{12}\right], 1 \leq i \leq r_1 \end{cases} \]

and \(\varphi_1\) is continuous on \([0, +\infty]\) and affine on each interval of \([0, +\infty] \setminus \bigcup_{i=0}^{r_1+1} V_i^i\).

One can check that the slope of \(\varphi_1\) is at most 2 on each of these intervals, so \(\|\varphi_1\|_L \leq 2\).

With \(f(\cdot) = d(\cdot, x)\) we set \(C_1 = f^{-1}\left([0, +\infty] \setminus \bigcup_{i=0}^{r_1+1} V_i^i\right)\).

If \(C_1\) is finite or empty define \(h(\cdot) = 2 \varphi_1 \circ d(\cdot, x) - \varphi_1(d(0, x))\). It is clear from the definition of \(\varphi_1\) that \(\|h\|_L \leq 4\), \(h(x) = h(y) = d(x, y)\) and \(h(0) = 0\). Now we set

\[ \delta = \begin{cases} v_1/2, & \text{if } C_1 = \emptyset \\ 1/2 \min\{\{v_1, \text{sep}(C_1)\} \cup \{\text{dist}(z, K \setminus C_1), z \in D_1\}\}, & \text{otherwise} \end{cases} \]
where \( \text{sep}(C_1) = \inf\{d(z, t), \ z \neq t, \ z, t \in C_1 \} \) and \( D_1 = f^{-1}\left([0, +\infty \setminus \bigcup_{i=0}^{r_1+1} V_1^i]\right) \).

Note that \( \delta > 0 \). Indeed \( v_1 > 0 \), \( C_1 \) is finite, thus \( \text{sep}(C_1) > 0 \); for any \( z, t \in D_1 \), \( \text{dist}(z, K \setminus C_1) > 0 \) and \( D_1 \) is finite.

It follows that every \( z \neq t \in K \) such that \( d(z, t) \leq \delta \) are not in \( D_1 \) and there exists \( i \leq r_1 \) such that \( z, t \in f^{-1}\left(V_1^i\right) \), so the equality \( h(z) = h(t) \) holds, i.e. \( h \in \text{lip}_0(K) \).

Assume that \( C_1 \) is infinite. Since \( C_1 \subset \overline{B}\left(x, \frac{\alpha}{2}\right) \) we have that for every ordinal \( \alpha, C_1^{(\alpha)} \subset \overline{B}\left(x, \frac{\alpha}{2}\right)^{(\alpha)} \). But \( C_1 \cap \overline{B}\left(x, \frac{\alpha}{2}\right)^{(\alpha_0)} = \emptyset \) so \( C_1^{(\alpha_0)} = \emptyset \). However \( C_1 \) is compact; thus there exists \( 1 \leq \alpha_1 < \alpha_0 \) so that \( C_1^{(\alpha_1)} \) is finite and nonempty. Then there exist \( k_2 \in \mathbb{N} \) and \( y_1^{(k_2)} \cdots y_2^{(k_2)} \in K \) such that \( C_1^{(\alpha_1)} = \{y_1^{(k_2)}, \cdots, y_2^{(k_2)}\} \).

For \( 1 \leq i \leq k_2 \), we denote \( a_2^i = d(y_2^i, x) \). We can find \( r_2 \) and \( v_2 \) such that
\[
\{a_2^1, \cdots, a_2^{k_2}\} = \{v_2^1, \cdots, v_2^{r_2}\}.
\]

Now set
\[
v_2 = \min\left(\{v_1, v_1^1\} \cup \{v_2^i - v_2^{i-1}, \ 2 \leq i \leq r_2\}\right)
\]
and define \( \varphi_2 : [0, +\infty[ \to [0, +\infty] \) continuous by
\[
\varphi_2(t) = \begin{cases} \varphi_1(t), & t \in \bigcup_{i=0}^{r_1+1} V_1^i \\ \varphi_1(v_2^i), & t \in [v_2^i - \frac{v_2}{2}, v_2^i + \frac{v_2}{2}] \end{cases}
\]
and \( \varphi_2 \) is affine on each interval of \([0, +\infty] \setminus (\bigcup_{i=0}^{r_1+1} V_1^i) \cup (\bigcup_{i=0}^{r_2} V_2^i) \).

The Lipschitz constant of \( \varphi_2 \) equals the maximum between \( \|\varphi_1\|_L \) and new slopes of \( \varphi_2 \). It is easy to check that \( \|\varphi_2\|_L \leq 2 \times \left(1 + \frac{1}{3}\right) = \frac{8}{3} \).

Set \( C_2 = f^{-1}\left(\left[\frac{v_1}{2}, \frac{v_2}{2} - \frac{v_1}{4}\right]\setminus (\bigcup_{i=0}^{r_2} V_2^i) \cup (\bigcup_{i=1}^{r_2} V_2^i)\right) \).

If \( C_2 \) is finite or empty, then setting \( h(\cdot) = 2 (\varphi_2 \circ d(\cdot, x) - \varphi_2(d(0, x))) \), we obtain \( \|h\|_L \leq \frac{16}{3} \), \( |h(x) - h(y)| = d(x, y), h(0) = 0 \), and with
\[
0 < \delta = \begin{cases} v_2/2, & \text{if } C_2 = \emptyset \\ \frac{1}{2} \min\{\{v_2, \text{sep}(C_2)\} \cup \{\text{dist}(z, K \setminus C_2), z \in D_2\}\}, & \text{otherwise}
\end{cases}
\]
where \( D_2 = f^{-1}\left(\left[\frac{v_2}{4}, \frac{v_2}{2} - \frac{v_1}{4}\right]\setminus (\bigcup_{i=1}^{r_2} V_1^i) \cup (\bigcup_{i=1}^{r_2} V_1^i)\right) \).

When \( z, t \in K \) are such that \( d(z, t) \leq \delta \), then \( h(z) = h(t) \), i.e. \( h \in \text{lip}_0(K) \).

If \( C_2 \) is infinite we proceed inductively in a similar way until we get \( C_n \) finite, which eventually happens because we have a decreasing sequence of ordinals.

The function \( h \) we obtain verifies \( h(0) = 0, \ |h(y) - h(x)| = d(x, y) \) and
\[
\|h\|_L \leq 2 \prod_{j=1}^{n} \left(1 + \frac{1}{2j - 1}\right) \leq 2 \prod_{j=1}^{+\infty} \left(1 + \frac{1}{2j - 1}\right) := c
\]
where \( c \) does not depend on \( x \) and \( y \). Moreover, setting
\[
0 < \delta = \begin{cases} v_n/2, & \text{if } C_n = \emptyset \\ \frac{1}{2} \min\{v_n, \text{sep}(C_n)\} \cup \{\text{dist}(z, K \setminus C_n), z \in D_n\}, & \text{otherwise}
\end{cases}
\]
if \( z, t \in K \) are such that \( d(z, t) \leq \delta \), then \( h(z) = h(t) \), i.e. \( h \in \text{lip}_0(K) \). This concludes the proof. \( \square \)
3. Metric approximation property

**Theorem 3.1.** Let \((K,d)\) be a countable compact metric space. Then \(\mathcal{F}(K)\) has the metric approximation property.

Before starting the proof let us recall a construction due to Kalton \[8\]. Let \((K,d)\) be an arbitrary pointed metric space and set

\[K_n = \{x \in K, \ d(0,x) \leq 2^n\}\] and \(O_n = \{x \in K, \ d(0,x) < 2^n\}\), \(n \in \mathbb{Z}\),

\[F_N = K_{N+1} \setminus O_{-N-1}, \ N \in \mathbb{N}.\]

Then, for every \(n \in \mathbb{Z}\), we can define a linear operator \(T_n : \mathcal{F}(K) \to \mathcal{F}(K)\) by

\[T_n \delta(x) = \begin{cases} 
0 & , x \in K_{n-1} \\
(\log d(0,x) - (n-1)) \delta(x) & , x \in K_{n} \setminus K_{n-1} \\
(\log d(0,x) - n) \delta(x) & , x \in K_{n+1} \setminus K_{n} \\
0 & , x \notin K_{n+1}.
\end{cases}\]

If we set for \(N \in \mathbb{N}\), \(S_N = \sum_{n=-N}^{N} T_n\), then Lemma 4.2 in \[8\] gives:

**Lemma 3.2.** For every \(N \in \mathbb{N}\), we have \(\|S_N\| \leq 72\), \(S_N (\mathcal{F}(K)) \subset \mathcal{F}(F_N)\) and for every \(\gamma \in \mathcal{F}(K)\), \(\lim_{N \to +\infty} S_N \gamma = \gamma.\)

In order to prove Theorem 3.1 we need the following classical lemma. We will give its proof for the sake of completeness.

**Lemma 3.3.** If for \(\alpha\) countable ordinal there exist \(F_1, \cdots, F_n\) clopen subsets of \(K^{(\alpha)}\), mutually disjoint, such that \(K^{(\alpha)} = F_1 \cup \cdots \cup F_n\), then there exist \(G_1, \cdots, G_n\) clopen subsets of \(K\), mutually disjoint, such that \(K = G_1 \cup \cdots \cup G_n\) and \(G_i^{(\alpha)} = F_i\).

**Proof.** We proceed by induction on \(\alpha < \omega_1\) such that \(K^{(\alpha)} = F_1 \cup \cdots \cup F_n\), for all \(1 \leq i \neq j \leq n\), \(F_i\) is clopen in \(K^{(\alpha)}\) and \(F_i \cap F_j = \emptyset\).

The result is clear for \(\alpha = 0\).

Assume that the result is true for \(\alpha < \omega_1\) and suppose that \(\{F_i\}_{1 \leq i \leq n}\) is a clopen partition of \(K^{(\alpha+1)}\). Each \(F_i\) is closed in \(K^{(\alpha)}\) which is compact; then we can find \(O_i\) open subset of \(K^{(\alpha)}\) such that \(F_i \subset O_i\), \(O_i' = F_i\), and \(O_i \cap O_j = \emptyset\), for \(i \neq j\). Set \(O = K^{(\alpha)} \setminus \bigcup_{i=1}^{n} O_i\), \(U_1 = O_1 \setminus O\) and \(U_i = O_i\), for \(2 \leq i \leq n\). Then \(K^{(\alpha)} = \bigcup_{i=1}^{n} U_i\), \(U_i' = F_i\), and every \(U_i\) is clopen in \(K^{(\alpha)}\). Indeed we defined \(O_i\), \(2 \leq i \leq n\), as open subsets of \(K^{(\alpha)}\) so every \(U_i\) is open in \(K^{(\alpha)}\). Moreover points in \(O\) are isolated points of \(K^{(\alpha)}\), thus \(O\) and then \(U_1\) are open in \(K^{(\alpha)}\). Finally \(K^{(\alpha)} = \bigcup_{i=1}^{n} U_i\); then every \(U_i\) is closed.

We can apply the induction hypothesis to find \(G_1, \cdots, G_n\) clopen subsets of \(K\), mutually disjoint, such that \(K = G_1 \cup \cdots \cup G_n\) and \(G_i^{(\alpha)} = U_i\); that is, \(G_i^{(\alpha+1)} = F_i\), \(1 \leq i \leq n\).

Finally we assume \(\alpha\) is a limit ordinal and \(K^{(\alpha)} = F_1 \cup \cdots \cup F_n\), disjoint union of clopen sets in \(K^{(\alpha)}\). There exist \(O_1, \cdots, O_n\) open subsets of \(K\) such that \(F_i \subset O_i\), \(O_i^{(\alpha)} = F_i\), and \(O_i \cap O_j = \emptyset\) for \(i \neq j\).

Set \(F = K^{(\alpha)} \setminus \bigcup_{i=1}^{n} O_i\); then \(\bigcap_{\beta < \alpha} F \cap K^{(\beta)} = F \cap K^{(\alpha)} = \emptyset\). But \(F\) is compact, so there exists \(\beta < \alpha\) such that \(F \cap K^{(\beta)} = \emptyset\), that is to say, \(K^{(\beta)} \subset \bigcup_{i=1}^{n} O_i\).
Finally $K^{(β)}$ is the disjoint union of $O_i \cap K^{(β)}$, $1 \leq i \leq n$, clopen sets in $K^{(β)}$, so we can use the induction hypothesis to write $K = G_1 \cup \cdots \cup G_n$, $G_i$ mutually disjoint and clopen in $K$ and $G_i^{(β)} = O_i \cap K^{(β)} = O_i^{(β)}$. Moreover we have $β < α$; thus $G_i^{(α)} = \bigcap_{γ < α} G_i^{(γ)} = \bigcap_{γ < α} O_i^{(γ)} = F_i$. □

Proof of Theorem 3.1. We proceed by induction on $α < ω_1$ such that $K^{(α)}$ is finite.

- If $K$ is finite, then $F(K)$ is finite dimensional, so has trivially the MAP, and the property is true for $α = 0$.
- Let $α$ be a countable ordinal and assume that for every $β < α$, if $(K, d)$ is a compact metric space so that $K^{(β)}$ is finite, then $F(K)$ has the MAP.

Now let $(K, d)$ be a compact metric space such that $K^{(α)}$ is finite.

First $F(K)$ is linearly isometric to $lip_0(K)^*$, and a theorem of Grothendieck [6] (see also Theorem 1.e.15 in [10]) asserts that a separable Banach space which is isometric to a dual space and which has the AP has the MAP, so it is enough to prove that $F(K)$ has the BAP.

Secondly, if $K$ is such that $K^{(α)} = \{a_1, \cdots, a_n\}$, singletons $\{a_i\}$ are clopen in $K^{(α)}$ and Lemma 3.3 gives $G_1, \cdots, G_n$ mutually disjoint clopen subsets of $K$ such that $∀i ≤ n, G_i^{(α)} = \{a_i\}$ and $K = G_1 \cup \cdots \cup G_n$. Moreover $F(K)$ is isomorphic to $(\bigoplus_{i=1}^n F(K_i))_{ℓ_1}$, where $K_i = G_i \cup \{0\}$, $1 ≤ i ≤ n$.

Indeed if $a = \min dist(G_i, G_j), where$

$$dist(G_i, G_j) = \inf \{d(x, y) : x \in G_i, y \in G_j\},$$

by compactness we have $a ≥ 0$. Then the operator

$$Φ : Lip₀(K) → (\bigoplus_{i=1}^n Lip₀(K_i))_{∞}$$

$$f → (f|_{K_i})_{i=1}^n$$

is onto, linear, weak*-continuous, and for $f \in Lip₀(K)$, we have

$$a \over 2 diam(K)||f||_L ≤ ||Φ(f)||_∞ ≤ ||f||_L.$$ 

Hence $F(K)$ is isomorphic to $(\bigoplus_{i=1}^n F(K_i))_{ℓ_1}$.

The BAP is stable with respect to finite $ℓ_1$-sums and isomorphisms; then it is enough to prove that for any $i ∈ \{1, \cdots, n\}, F(K_i)$ has the BAP. In other words we need to prove that when $K^{(α)}$ is a singleton, then $F(K)$ has the BAP.

Suppose as we may that $K^{(α)} = \{0\}$. Using the construction due to Kalton [8] we have a sequence of linear operators $S_N : F(K) → F(FN)$, $||S_N|| ≤ 72$ and for every $γ ∈ F(K)$, \lim_{N→+∞} S_Nγ = γ.

Moreover, for every $N ∈ \mathbb{N}$, there exists $β < α$ such that $F_N^{(β)}$ is finite and then $F(F_N)$ has the MAP: since $F(F_N)$ is separable, for every $N ∈ \mathbb{N}$, there exists a sequence of finite-rank linear operators $R_p^N : F(F_N) → F(F_N)$ so that for every $γ ∈ F(F_N)$, \lim_{p→+∞} R_p^Nγ = γ and $||R_p^N|| ≤ 1$ for every $p ∈ \mathbb{N}$ [12]; see also Theorem 1.e.13 in [10].

Setting $Q_{N,p} = R_p^N ∘ S_N$ we deduce that the range of $Q_{N,p}$ is finite dimensional as the range of $R_p^N$, $||Q_{N,p}|| ≤ ||R_p^N|| ||S_N|| ≤ 72$ and for every
Corollary 4.1. Let $K$ be a compact metric space which is not perfect (i.e. $K \neq K'$). Then for every countable ordinal $\alpha \geq 1$, the space $F(K/K(\alpha))$ has the MAP.

Proof. Remark that for every compact metric space $(K,d)$ and every countable ordinal $\alpha \geq 1$, the quotient space $K/K(\alpha)$ is compact and countable because $(K/K(\alpha))^{(\alpha)}$ is empty or $\{0\}$. Then this result is a consequence of Theorem 3.1.

Remark 4.2. (1) If $K$ is perfect, then $F(K/K(\alpha)) = \{0\}$.

(2) Otherwise $F(K)/F(K(\alpha))$ is linearly isometric to $F(K/K(\alpha))$. (We write $F(K)/F(K(\alpha)) \equiv F(K/K(\alpha))$.)

Indeed we can assume that $0 \in K(\alpha)$. Then
\[
\{ f \in Lip_0(K) ; \forall x,y \in K(\alpha), f(x) = f(y) \}
= \{ f \in Lip_0(K) ; \forall x \in K(\alpha), f(x) = 0 \}.
\]

And since $F(K(\alpha)) = \text{vect} \{ \delta_x, x \in K(\alpha) \}$, we have
\[
\{ f \in Lip_0(K) ; \forall x \in K(\alpha), f(x) = 0 \} = F(K(\alpha))^{\perp},
\]
which is isometric to $(F(K)/F(K(\alpha)))^*$. To sum up,
\[
\{ f \in Lip_0(K) ; \forall x,y \in K(\alpha), f(x) = f(y) \} \equiv \left( F(K)/F(K(\alpha)) \right)^*.
\]

From Propositions 1.4.3 and 1.4.4 in [14], there exists an isometry $\Phi$ from $\{ f \in Lip_0(K) ; \forall x,y \in K(\alpha), f(x) = f(y) \}$ onto $Lip_0(K/K(\alpha))$. Moreover $Lip_0(K/K(\alpha))$ is linearly isometric to $F(K/K(\alpha))^*$, so the space $(F(K)/F(K(\alpha))^*$ is isomorphically isometric to $F(K/K(\alpha))^*$. One can easily check that $\Phi$ is weak*-continuous, and finally $F(K)/F(K(\alpha))$ is linearly isometric to $F(K/K(\alpha))$.

To finish this paper we will use Corollary 4.1 and the previous remark to prove the following: in order to obtain that every countable compact metric space has the BAP it is not possible to use the three-space property due to Godefroy and Saphar [5], asserting:

If $M$ is a closed subspace of a Banach space $X$ so that $M^\perp$ is complemented in $X^*$ and $X/M$ has the BAP, then $X$ has the BAP if and only if $M$ has the BAP.
Indeed we can construct a compact metric space $K$ so that $K^{(2)} = \{0\}$; in particular $\mathcal{F}(K)$, $\mathcal{F}(K')$ and $\mathcal{F}(K) / \mathcal{F}(K')$ have the MAP, but $\mathcal{F}(K')$ is not complemented in $\text{Lip}_0(K)$.

To construct this space we need a proposition similar to Proposition 7 in [4]:

**Proposition 4.3.** For any $\lambda > 0$, there exist a finite metric space $H_\lambda$ and a subset $G_\lambda$ of $H_\lambda$ such that if $P : \text{Lip}_0(H_\lambda) \to \mathcal{F}(G_\lambda)^\perp$ is a bounded linear projection, then $\|P\| \geq \lambda$.

**Proof.** Assume that for some $\lambda_0 > 0$ and for all pairs $(G, H)$ of finite metric spaces with $G \subset H$ we can construct $P : \text{Lip}_0(H) \to \mathcal{F}(G)^\perp$ linear projection with norm bounded by $\lambda_0$.

Let $K$ be the compact metric space such that $\mathcal{F}(K)$ fails AP appearing in Corollary 5 of [4]. There exists $(G_n)_{n \in \mathbb{N}}$ an increasing sequence of finite subsets of $K$ such that $\bigcup_{n \in \mathbb{N}} G_n = K$.

Then for every $n \in \mathbb{N}$ and every $k \geq n$, there exists $P_n^k : \text{Lip}_0(G_k) \to \mathcal{F}(G_n)^\perp$ a linear projection of norm less than $\lambda_0$, where $\mathcal{F}(G_n)^\perp \subset \text{Lip}_0(G_k)$.

Fix $n \in \mathbb{N}$. For $k \in \mathbb{N}$, let $E_k : \text{Lip}_0(G_k) \to \text{Lip}_0(K)$ be the nonlinear extension operator which preserves the Lipschitz constant given by the inf-convolution formula:

$$\forall f \in \text{Lip}_0(K), \forall x \in K, E_k f(x) = \inf_{y \in G_k} \{ f(y) + \| f \| \text{Lip}_0 d(x, y) \}.$$  

For $f \in \text{Lip}_0(K)$, we set

$$\widetilde{P}_n^k(f) = \left\{ \begin{array}{ll} E_k P_n^k(f|_{G_k}) & , k \geq n, \\ 0 & , k < n. \end{array} \right.$$  

Then $\| \widetilde{P}_n^k(f) \| \text{Lip}_0 \leq \lambda_0 \| f \| \text{Lip}_0$, for every $f \in \text{Lip}_0(K)$.

If $U$ is a nontrivial ultrafilter on $\mathbb{N}$, for every $f \in \text{Lip}_0(K)$ we can define $P_n f$ as the pointwise limit of $\widetilde{P}_n^k(f)$ with respect to $k \in U$. Then $P_n$ is a linear projection onto $\mathcal{F}(G_n)^\perp \subset \text{Lip}_0(K)$ because $P_n^k$ is a projection onto $\mathcal{F}(G_n)^\perp \subset \text{Lip}_0(G_k)$. Moreover $\| P_n f \| \text{Lip}_0 \leq \lambda_0 \| f \| \text{Lip}_0$ and $P_n f$ pointwise converges to 0 for any $f \in \text{Lip}_0(K)$.

Set $Q_n = \text{Id}_{\text{Lip}_0(K)} - P_n : \text{Lip}_0(K) \to \text{Lip}_0(K)$. Then $Q_n$ is a continuous linear projection of finite rank and $\text{Ker} \ Q_n = \mathcal{F}(G_n)^\perp$ is weak$^*$-closed. Therefore $Q_n$ is weak$^*$-continuous. Moreover $\| Q_n \| \leq 1 + \lambda_0$ and for every $f \in \text{Lip}_0(K)$, $Q_n f$ converges pointwise to $f$.

Using Theorem 2 in [1] we deduce that $\mathcal{F}(K)$ has the $(1+\lambda_0)$-BAP, contradicting our assumption on $K$.  

Thanks to that proposition we will construct a compact metric space $K$ such that $K^{(2)} = \{0\}$ and $\mathcal{F}(K')$ is not complemented in $\text{Lip}_0(K)$.

For every $n \in \mathbb{N}$ there exist $A_n \subset B_n$ finite such that for every continuous linear projection $P_n : \text{Lip}_0(B_n) \to \mathcal{F}(A_n)^\perp$, we have $\| P_n \| \geq n$.

Set $\alpha_n = \min \{ d(x, y) \mid x \neq y \in B_n \} > 0$. If we see $B_n$ as a subspace of $\ell^m_{\infty}$, with $m_n$ the cardinality of $B_n$, we can find for every $a \in A_n$, $L_n^a$ a sequence converging to $a$ such that $L_n^a \subset B(a, 2\alpha_n)$.

Define $K_n = \bigcup_{a \in A_n} L_n^a \cup B_n$. We obtain $A_n \subset B_n \subset K_n$ and $K'_n = A_n$. We can assume that the diameter of $K_n$ is less than $8^{-n}$.
Finally we define $K := \left( \bigcup_{n \in \mathbb{N}} \{n\} \times K_n \right) \cup \{0\}$ equipped with the distance:

$$d(0, (n, x)) = 2^{-n},$$
$$d((n, x), (m, y)) = \begin{cases} d_{K_n}(x, y), & n = m \\ |2^{-n} - 2^{-m}|, & n \neq m. \end{cases}$$

Then $K^{(2)} = \{0\}$.

Now assume that there exists $P : \text{Lip}_0(K) \to \mathcal{F}(K') ^{\perp}$ a continuous linear projection. Let $E_n : \text{Lip}_0(B_n) \to \text{Lip}_0(K_n)$, $F_n : \text{Lip}_0(B_n) \to \text{Lip}_0(K)$ and $R_n : \text{Lip}_0(K) \to \text{Lip}_0(B_n)$ be defined as follows:

$$\forall f \in \text{Lip}_0(B_n), \quad (E_n f)(x) = \begin{cases} f(x), & x \in B_n \\ f(a), & x \in L^a_n \end{cases}$$
$$\forall f \in \text{Lip}_0(B_n), \quad (F_n f)(m, x) = \begin{cases} (E_n f)(x), & n = m \\ 0, & n \neq m \end{cases}$$
$$\forall f \in \text{Lip}_0(K), \quad R_n f = f|_{\{n\} \times B_n}.$$

We set $P_n := R_n \circ P \circ F_n : \text{Lip}_0(B_n) \to \mathcal{F}(A_n) ^{\perp}$ and we have that $P_n$ is a continuous linear projection. From Proposition 4.3 we deduce that $\|P_n\| \geq n$. Moreover our choice of $\alpha_n$ implies that $\|F_n\| \leq 3$; then finally $\|P\| \geq n/3$ holds for every $n \in \mathbb{N}$. Therefore $P$ is unbounded and $\mathcal{F}(K') ^{\perp}$ is not complemented in $\text{Lip}_0(K)$.

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