LATTICE STATISTICS ON PLANE, CYLINDER AND TORUS

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Abstract. In this paper the relations among entropy constants of lattice statistics on plane, cylinder and torus are discussed. We show that for many statistical models, the entropy constant of a plane lattice is the same as the entropy constants of the corresponding cylindrical and toroidal lattices. As a result, we get the entropy constants of quadratic lattices on cylinder and torus. Furthermore, we consider three more complex lattices which can not be handled by a single transfer matrix. Using the concept of transfer multiplicity we extend results of quadratic lattice to these three lattices and obtain lower and upper bounds of the entropy constants.

1. Introduction

The lattice statistics in statistical physics has a long history. A typical problem is to count the ways of putting particles in a plane lattice sites so that no two share the same site or are in adjacent sites. Such problems are called the planar lattice gases models [1, 2, 3, 7, 9, 13]. Mathematicians formulated them by the enumeration of the (0,1) matrices which describe the independent sets in a plane quadratic lattice graph (also called a planar grid graph) [4, 8, 16, 17]. In this paper, we use $G_{m,n}$ (where $m$, $n$ always denote positive integers) to denote a plane lattice graph, that is a finite part of a plane lattice, whose vertices are arranged in $(m + 1)$ rows and $(n + 1)$ columns. Given an independent set $S$ of graph $G_{m,n}$, a portion of $S$ that lies in a fixed column of $G_{m,n}$ can be represented by an $(m + 1)$-vector of 0’s and 1’s, where a 1 indicates the vertex is in $S$ and a 0 indicates the vertex is not in $S$.

If $G_{m,n}$ is a plane quadratic lattice graph, then any $(m + 1)$-vector arising this way has no two consecutive 1’s. Clearly, a vertex subset $S$ of $G_{m,n}$ is an independent set only if all its $n + 1$ corresponding $(m + 1)$-vectors represents independent sets. Let $P_m$ denote the set of all $(m + 1)$-vectors of 0’s and 1’s with no two consecutive 1’s. The cardinality of $P_m$ is well known (and can be easily seen) to be $F_{m+3}$, the Fibonacci number (starting from $F_0 = 0$, $F_1 = 1$). We can construct an $(m + 1)$ by $(n + 1)$ matrix $M$ to represent an

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independent set $S$ of $G_{m,n}$ by the gluing procedure described below. First, take a vector from $P_m$ such that it corresponds to the first column of $S$, and denote it as $v_1$. Then, to the right of $v_1$ we glue a vector $v_2$ selected from $P_m$ that corresponds to the second column of $S$. Then we glue a vector $v_3$ from $P_m$ to the right of $v_2$, and so on so forth. Continue this way until the $(n + 1)$-th column is glued and then we obtain the $(m + 1)$ by $(n + 1)$ matrix $M$ representing the independent set $S$. We can get all the possible matrices representing independent sets of $G_{m,n}$ by this procedure of gluing vectors from $P_m$. Note that two vectors of $P_m$ can be glued together if and only if they have no 1’s in common position, i.e., they are orthogonal (their dot product equals zero.) In the above procedure, we glue columns from left to right. Similarly, we can have another procedure that glues rows from top to bottom. Note that for non-grid lattice graphs $G_{m,n}$, these two procedures may lead to different transfer patterns. But for all the lattices considered in this paper, the transfer patterns in the two procedures are the same.

Fig.1 shows an independent set $S$ in the plane quadratic lattice graph $G_{4,5}$. The portions of $S$ that lie in each of the columns can be represented by the respective 5-vectors $(0,0,1,0,1),(0,1,0,0,0),(0,0,1,0,0),(1,0,0,1,0),(0,0,0,0,1),(0,1,0,1,0)$.

![Figure 1](image_url)

From a plane quadratic lattice graph $G_{m,n}$ ($n > 1$), identifying its $m$ edges on the left with its $m$ edges on the right, correspondingly, we get a cylindrical quadratic lattice graph $H_{n,m}$ (Note that here the graph can be seen as drawn on a vertical cylinder.) Let $C_n$ denote the set of $n$-vectors of 0’s and 1’s with the property that no two consecutive 1’s occur in cyclic order. Then, similar to the discussion on $G_{m,n}$, we can see that any independent set $S$ of $H_{n,m}$ can be represented by an $(m + 1)$ by $n$ matrix $M$ whose rows are from the set $C_n$, and that all the representing matrices $M$ can be obtained by the row gluing procedure (from top to bottom) similar to the column gluing procedure (from left to right) for $G_{m,n}$. It is not difficult to see that the cardinality of the set $C_n$ is $F_{n-1} + F_{n+1}$. 
Similarly, by identifying the top and bottom boundary cycles of a cylindrical quadratic lattice graph $H_{n,m}$, we get a toroidal quadratic lattice graph $S_{n,m}$ that can be seen as drawn on a torus.

In general, for any plane lattice graph $G_{m,n}$, the cylindrical lattice graph and the toroidal lattice graph obtained by the identifications as above will also be denoted $H_{n,m}$ and $S_{n,m}$.

For a given plane quadratic lattice graph $G_{m,n}$, the transfer matrix $T_m$ is a $F_{m+3} \times F_{m+3}$ matrix of 0’s and 1’s, defined as follows. The rows and columns of $T_m$ are indexed by vectors of $P_m$, and the entry of $T_m$ in position $(\alpha, \beta)$ is 1 if the vectors $\alpha, \beta$ are orthogonal, and is 0 otherwise. Note that the matrix depends only on $m$.

For example when $m = 3$, the possible column vectors of $P_3$ are $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(1,0,1,0)$, $(0,0,0,1)$, $(0,1,0,1)$. If we index the rows and columns in this order, then the transfer matrix of $G_{m,n}$ is

\[
T_3 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}.
\]

Similarly, for cylindrical quadratic lattice graph $H_{n,m}$, its transfer matrix $T_n$ is an $(F_{n-1} + F_{n+1}) \times (F_{n-1} + F_{n+1})$ symmetric matrix of 0’s and 1’s, defined as follows. The rows and columns of $T_n$ are indexed by vectors of $C_n$, and the entry of $T_n$ in position $(\alpha, \beta)$ is 1 if the vectors $\alpha, \beta$ are orthogonal, and is 0 otherwise. For $n = 4$ the possible row vectors of $C_4$ are $(0,0,0,0)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,0,1,0)$, $(0,1,0,1)$. If we index the row and column in this order, then the transfer matrix of $H_{n,m}$ is
For a plane quadratic lattice graph $G_{m,n}$, by adding two crossed diagonals in each square inner face we get a crossed quadratic lattice graph, which will be denoted as $\tilde{G}_{m,n}$. The transfer matrix of $\tilde{G}_{m,n}$ can be defined in a similar way. We say that $G_{m,n}$ and $\tilde{G}_{m,n}$ are lattice graphs with transfer multiplicity one, since computing the number of independent sets for each of them only needs to employ one transfer matrix. This may not hold for other lattices. For example, in [19], for an aztec diamond we need to introduce two transfer matrices, each of which is the transpose of the other. So the transfer multiplicity of an aztec diamond is two. There are also lattices with transfer multiplicity three; one example is the 8.8.4 lattice that will be discussed later in section 4.

The entropy constant of a plane lattice is defined by $\eta = \lim_{m,n \to \infty} f(m,n)^{1/k(m,n)}$ where $f(m,n)$ denotes the number of independent sets of $G_{m,n}$ and $k(m,n)$ denotes the number of vertices of $G_{m,n}$. The entropy constants of cylindrical and toroidal lattices can be defined similarly. As in [4], we have $f(m,n) = \sum_{u,v \in P_m} T_{u,v}^n = 1 \cdot T^n 1$ for $G_{m,n}$ and $H_{n,m}$. Clearly, $\text{Trace}(T^n) = \sum_{u \in P_m} T_{u,u}^n$.

In [4], Calkin and Wilf proved the existence of the entropy constant of plane quadratic lattice and established its upper and lower bounds. Two natural problems are to consider the entropy constants for lattices on cylinder or torus. Note that the method of Calkin and Wilf’s is valid for the lattices with the same symmetric transfer matrices in both horizontal and vertical directions. In this paper, we will investigate the relationship among the three cases (plane, cylinder and torus) for several lattices. We will show that for each type of four lattices, the entropy constant is the same no matter the lattice is on plane, cylinder or torus. Upper and lower bounds of the entropy constant will be established for the lattices discussed in this paper.
2. LATTICES WITH TRANSFER MULTIPLICITY ONE

The entropy constant of the plane quadratic lattice was already discussed in [4]. Now we consider the crossed quadratic lattice, which is obtained from the plane quadratic lattice by adding two crossed diagonals to each square inner face. Figure 2 shows the crossed quadratic lattice graph $\mathcal{G}_{4,5}$ that is a part of the crossed quadratic lattice, where an independent set $S$ is indicated by small circles. The portions of $S$ that lie in each of the columns can be represented by 5-vectors $(0,1,0,0,0)$, $(0,0,0,1,0)$, $(1,0,1,0,0)$, $(0,0,0,0,1)$, $(1,0,0,0,0)$, $(0,0,0,0,1)$. 

![Figure 2.](image)

Note that the plane crossed lattice graphs are no longer planar graphs. However, it is easy to see that the transfer matrix of the plane crossed lattice graph $\mathcal{G}_{m,n}$ is an $F_{m+3} \times F_{m+3}$ matrix of 0’s and 1’s. When $m = 3$, the possible column vectors of $P_m$ are $(0,0,0,1)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,0,1)$. If we index the rows and columns in this order, then the transfer matrix of $\mathcal{G}_{m,n}$ is

$$
T_3 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}.
$$

Clearly, the plane crossed lattice is a non-planar lattice with transfer multiplicity one.

For a given $\mathcal{G}_{m,n}$, by identifying its left edges with the right edges correspondingly, we get the cylindrical crossed lattice graph $\mathcal{H}_{n,m}$. The transfer matrix of $\mathcal{H}_{n,m}$ is an
$(F_{n-1} + F_{n+1}) \times (F_{n-1} + F_{n+1})$ symmetric matrix of 0’s and 1’s. When $n = 4$ the possible row vectors of its $C_n$ are $(0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 0, 1, 0), (0, 1, 0, 1)$. If we index the rows and columns in this order, then the transfer matrix of $\tilde{H}_{n,m}$ is

$$B_4 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}.$$ 

In the following lemma, we will establish a relation between the transfer matrix of $G_{m,n}$ (cylindrical lattice $H_{n,m}$) and the number of independent sets of the corresponding cylinder lattice graph $H_{n,m}$ (toroidal lattice graph $S_{n,m}$). Then we will prove our main theorem and provide numerical upper and lower bounds of the entropy constants for the quadratic and crossed lattices on cylinder and torus.

**Lemma 2.1.** For each plane lattice graph $G_{m,n}$ with transfer matrix $T_{m}$, transfer multiplicity one and a positive integer $n$, the trace of $T_{n}^{m}$ is equal to the number of independent sets of the corresponding cylinder lattice graph $H_{n,m}$. For each cylinder lattice graph $H_{n,m}$ with transfer matrix $T_{n}$, transfer multiplicity one and a positive integer $m$, the trace of $T_{m}^{n}$ is equal to the number of independent sets of the corresponding torus lattice graph $S_{n,m}$.

**Proof.** Recall that $H_{n,m}$ can be obtained by identifying the left column and the right column of $G_{m,n}$. Thus there is a bijection between the independent sets of $H_{n,m}$ and the independent sets of $G_{m,n}$ whose left and right column vector are the same. And the latter is the trace of $T_{n}^{m}$.

Similarly, $S_{n,m}$ can be obtained by identifying the top row and the bottom row of $H_{n,m}$. Thus there is a bijection between the independent sets of $S_{n,m}$ and the independent sets of $H_{n,m}$ whose corresponding top and bottom row vectors are the same. And the latter is the trace of $T_{m}^{n}$. □

**Theorem 2.2.** For a lattice with transfer multiplicity one, if in both directions the transfer matrices are the same real symmetric matrix, then its entropy constants on plane, cylinder and torus are the same.
Proof. Let $T$ be the transfer matrix of columns of $G_{m,n}$ with the characteristic polynomial $f(x)$, and let

$$1 = (1, 1, \cdots, 1)_{1 \times g(m)}$$

where $g(m)$ equals the number of independent sets on the first column of $G_{m,n}$. Particularly, for the quadratic lattice and the crossed lattice, $g(m) = F_{m+3}$.

By Hamilton-Cayley Theorem we have

$$f(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_{g(m)} T^{g(m)} = 0.$$ 

Put $f_0 = 1$ and $f_{n+1} = T f_n$, then

$$f_n = T^n 1.$$ 

Let

$$u_{m,n} = 1 \cdot f_n = 1 \cdot T^n 1,$$ 

where $u_{m,n}$ is the sum of all entries of the matrix $T^n$, namely the number of independent sets of $G_{m,n}$.

Thus

$$1 \cdot f(T) 1 = a_0 u_{m,0} + a_1 u_{m,1} + a_2 u_{m,2} + \cdots + a_{g(m)} u_{m,g(m)} = 0.$$ 

In general $u_{m,n}$ satisfies the following recurrence relation:

$$a_0 u_{m,n} + a_1 u_{m,n+1} + a_2 u_{m,n+2} + \cdots + a_{g(m)} u_{m,n+g(m)} = 0.$$ 

By a well known theorem on difference equations (see, for example, [11]), the characteristic polynomial of this linear recurrence relation can be written as

$$f(x) = (x - \lambda_1)^{e_1} (x - \lambda_2)^{e_2} \cdots (x - \lambda_s)^{e_s}$$ 

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s$. 

Note that $\lambda_s$ depends on $m$. Then
for all $n$ where $p_i(n)$ is a polynomial with degree at most $e_i - 1$ in $n$. The coefficients $p_i(n)$ of the polynomial $u_{m,n}$ are determined by the initial values

$$u_{m,0}, u_{m,1}, \ldots, u_{m,g(m)-1}.$$

Note that the first row and the first column of $T$ are both vectors of all 1’s. Thus the matrix $T$ is non-negative, irreducible (its corresponding digraph is strongly connected) and prime. Hence the spectral radius of $T$ is a simple positive real eigenvalue $\lambda_s$ with magnitude greater than any other eigenvalues' (see [15]) and $p_s(n)$ is a positive constant.

It is not difficult to see that

$$\lim_{n \to \infty} (u_{m,n})^{1/n} = \lambda_s.$$ 

Since in both directions the real and symmetric transfer matrices are the same, by the same method in [4], one can show that the following double limit exists:

$$\eta_1 = \lim_{m,n \to \infty} (u_{m,n})^{1/m} = \lim_{m \to \infty} \lambda_s^m.$$ 

Now by Lemma 2.1 we can see that the trace of the $n$-th power of the transfer matrix of $G_{m,n}$ is equal to $v_{m,n}$, the number of independent sets of $H_{n,m}$.

Since the trace of $T^n$ is the sum of its eigenvalue of $T^n$ and each one is an $n$-th power of an eigenvalue of $T$, we see that if

$$v_{m,1} = \text{Trace}(T) = \sum_{i=1}^{s} e_i \lambda_i,$$

then

$$v_{m,n} = \text{Trace}(T^n) = \sum_{i=1}^{s} e_i \lambda_i^n,$$

where $e_i$ is the multiplicity of eigenvalue $\lambda_i$.

Since $\lambda_s$ is the simple largest eigenvalue, then

$$\lim_{n \to \infty} (v_{m,n})^{1/n} = \lambda_s.$$
So, the entropy constant of $H_{n,m}$

$$\eta_2 = \lim_{m,n \to \infty} \left( \frac{v_{m,n}}{m} \right)^{1/(n-1)} = \lim_{m \to \infty} \frac{\lambda_s^{1/m}}{m} = \eta_1,$$

That is, the entropy constant of $G_{m,n}$ and $H_{n,m}$ are the same. Similarly, using the second conclusion of Lemma 2.1 we can prove that entropy constants of $H_{n,m}$ and $S_{n,m}$ are the same. This completes the proof. \[\square\]

In [4] Calkin and Wilf already obtained a good estimate for the entropy constant of the plane quadratic lattice. Now, by Theorem 2.2 we immediately see that the entropy constants of the quadratic lattice on plane, cylinder and torus are all between $1.503047782\ldots$ and $1.5035148\ldots$.

Since the transfer matrix of crossed quadratic lattice is symmetric, the same approach in [4] and the proof of theorem 2.1 can be processed here. Thus we can use the same method in [4] to get the upper and lower bound of the entropy constant of crossed quadratic lattice. The lower bound of $\lim_{m,n \to \infty} f(m,n)^{1/(mn)}$ is $(\frac{\lambda_{p+2q}}{\lambda_{2q}})^{1/p}$ where $\lambda$’s are the largest eigenvalues of corresponding $T$’s and the upper bound of $\lim_{m,n \to \infty} f(m,n)^{1/(mn)}$ is $(\xi_{2k})^{1/2k}$ where $\xi$’s are the largest eigenvalues of corresponding $B$’s. Let $p = 4$, $q = 4$ and $k = 6$, we have $1.342542258\ldots \leq \eta \leq 1.342652572\ldots$.

### 3. LATTICES WITH TRANSFER MULTIPLICITY TWO

In this section we consider the lattices with transfer multiplicity two, i.e., the lattices for which we need two transfer matrices to compute the number of independent sets of the lattice graphs. One of such lattices is inspired by the famous aztec diamonds. The study of enumeration of perfect matchings, spanning trees and independent sets of an aztec diamond can be found in [5, 6, 12] and the references cited therein.

Let $L_i$ be the path with $i$ vertices $1, 2, \cdots, i$. The tensor product of two paths $L_m \otimes L_n$ is the graph on $m \times n$ vertices $\{(x,y) : 1 \leq x \leq m, 1 \leq y \leq n\}$, with $(x,y)$ adjacent to $(x',y')$ if and only if $|x - x'| = |y - y'| = 1$. This graph consists of two connected components, the one with the vertices $\{(x,y)|x+y\text{ is odd}\}$, denoted $O(L_{2m+1} \otimes L_{2m+1})$, is called the aztec diamond of order $m$. More generally $O(L_{2m+1} \otimes L_{2n+1})$ is called the generalized aztec diamond of order $m \times n$ introduced by the author in [19]. An independent set of a generalized aztec diamond can be represented by an ordered list of column vectors. Fig.3 shows an example. For $O(L_9 \otimes L_9)$, the given $S$ (the big dots) can be represented
by the ordered list of 9 column vectors: 

(0, 1, 0, 0), (1, 0, 0, 1, 1), (0, 1, 0, 0), (0, 0, 0, 1, 0), 
(1, 1, 0, 0), (0, 0, 0, 0, 0), (0, 0, 0, 1), (0, 1, 0, 0), (0, 0, 0, 1), (1, 1, 1, 0, 0).

Figure 3.

By identifying the top row and the bottom row of a generalized aztec diamond, we get a cylindrical generalized aztec diamond. And, the toroidal generalized aztec diamonds can be obtained by identifying the left cycle and the right cycle of a cylindrical generalized aztec diamond.

Now we consider the transfer matrix of $O(L_{2m+1} \otimes L_{2n+1})$. It is clear that the generation of each independent set of $O(L_{2m+1} \otimes L_{2n+1})$ involves $2n$ assembling steps after the first column $v_1$ is established (For simplicity, the assembling of the $(i + 1)$-th column $v_{i+1}$ to the $ith$ column $v_i$ is called step $i$.) Step one is to assemble $v_2$ to the right side of $v_1$. The transfer matrix representing step one, denoted $T_{m_1}$, can be constructed as follows. Let $R_m$ be the set of all possible vectors $v_1$. Obviously, $R_m$ consists of all $m$-dimensional vectors of 0’s and 1’s, so it has $2^m$ vectors. Similarly, the set of all possible $v_2$ is the set $R_{m+1}$ of all $(m + 1)$-dimensional vectors of 0’s and 1’s, and $R_{m+1}$ has $2^{m+1}$ vectors.

Then the transfer matrix $T_{m_1} = (T_{v_1,v_2})$ is a $2^m \times 2^{m+1}$ matrix whose rows are indexed by vectors of $R_m$ and columns are indexed by vectors of $R_{m+1}$, where $T_{v_1,v_2} = 1$ if $v_1$ and $v_2$ represent possible consecutive pair of columns in an independent set of $O(L_{2m+1} \otimes L_{2n+1})$, and $T_{v_1,v_2} = 0$ otherwise. Similarly, the transfer matrix for step two is a $2^{m+1} \times 2^m$ matrix $T_{m_2}$ which is the transpose of $T_{m_1}$. It is easily seen that $T_{m_1}$ is the transfer matrix for every step $i$ where $i$ is odd and $T_{m_2}$ is the transfer matrix for every step $i$ where $i$ is even. Thus, if we take the transfer matrix of generalized aztec diamond to be $T_m = T_{m_1}T_{m_2}$ then it is a $2^m \times 2^m$ symmetric matrix. Furthermore, the transfer multiplicity of $T_m$ can be considered as one and the results of Theorem 2.2 hold.

When $m = 3$, if we index the rows and columns in increasing order in binary numbers, then the transfer matrix of $G_{m,n}$ is an $8 \times 8$ matrix given by the product of $T_{3_1}$ and $T_{3_2}$. Here
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\[
T_{3_1} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
T_{3_2} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

Thus,

\[
T_3 = \begin{pmatrix}
16 & 4 & 4 & 2 & 4 & 1 & 2 & 1 \\
4 & 4 & 2 & 2 & 1 & 1 & 1 & 1 \\
4 & 2 & 4 & 2 & 2 & 1 & 2 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}.
\]
Let $R_n$ denote all $n$-dimensional vectors of 0’s and 1’s. The cylindrical generalized aztec diamond, obtained by identifying the left and right columns of the generalized aztec diamond $O(L_{2m+1} \otimes L_{2n+1})$, can also be seen as obtained by beginning with some vector in $R_n$, gluing to the top a new one in $R_n$ such that the vertices represented by the 1’s in these two vectors are not adjacent until $2m + 1$ vectors are glued. It is clear that for each $i$, step $i$ and step $i + 2$ can be represented by the same transfer matrix. Let $B_{n_1}$ ($B_{n_2}$, resp) denote the transfer matrix of the cylindrical generalized aztec diamond, which represents every step $i$ for $i$ odd (even, resp). Clearly, $B_{n_1}$ is a $2^n \times 2^n$ matrix of 0’s and 1’s whose rows and columns are indexed by vectors of $R_n$, the entry of $B_{n_1}$ in position $(\alpha, \beta)$ is 1 if the vectors represent possible consecutive pair of rows in an independent set of $O(L_{2m+1} \otimes L_{2n+1})$ on cylinder, and is 0 otherwise. It is no difficult to see that $B_{n_2}$ is the transpose of $B_{n_1}$. Thus we get the transfer matrix of cylindrical aztec diamond as $B_n = B_{n_1} B_{n_2}$, which is a $2^n \times 2^n$ symmetric matrix. Furthermore, the transfer multiplicity of $B_n$ is one.

Take $n = 3$ as an example. If we index the rows and columns in increasing order in binary numbers, then the transfer matrix of $H_{n,m}$ is an $8 \times 8$ matrix given by the product of $B_{3_1}$ and $B_{3_2}$, where

$$
B_{3_1} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
Thus

\[
B_{32} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Thus

\[
B_3 = \begin{pmatrix}
8 & 2 & 2 & 1 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 2 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Since the generalized aztec diamond lattice has the same symmetric transfer matrices in both horizontal and vertical directions, the same approach in [4] can be taken here. So, the lower bound of \( \lim_{m,n \to \infty} f(m, n)^{1/mn} \) is \( \left( \lambda^2 + 2 \right)^{1/p} \) where \( \lambda \)'s are the largest eigenvalues of corresponding \( T \)'s, and the upper bound of \( \lim_{m,n \to \infty} f(m, n)^{1/mn} \) is \( \left( \xi^2 k \right)^{1/2k} \) where \( \xi \)'s are the largest eigenvalues of corresponding \( B \)'s. Taking \( p = 2, q = 4 \) and \( k = 5 \), we get

\[
2.259132578... \leq \lim_{m,n \to \infty} f(m, n)^{1/mn} \leq 2.259154406....
\]

Note that the entropy constant of the generalized aztec diamond lattice is

\[
\eta = \lim_{m,n \to \infty} f(m, n)^{1/(2mn+m+n)} = \lim_{m,n \to \infty} f(m, n)^{1/2mn} \text{ where } 2mn + m + n \text{ is the number of vertices of vertices of generalized aztec diamond. Then we see that the entropy constant of the generalized aztec diamond lattice is } \left( \lim_{m,n \to \infty} f(m, n)^{1/mn} \right)^{1/2}, \text{ which is between } 1.503041110... \text{ and } 1.503048371....
\]

4. Lattices with transfer multiplicity three

The 8.8.4 lattice graphs \( G_{m,n} \), as shown in Fig 4, are finite subgraphs of the 8.8.4 tiling of Euclidean plane. Some study of the properties of 8.8.4 lattice graphs can be found in [14, 18].
The cylindrical 8.8.4 lattice graphs are obtained by identifying the top row and the bottom row of the 8.8.4 lattice graphs $G_{m,n}$. The toroidal 8.8.4 lattice graphs can be obtained by identifying the left cycle and the right cycle of the cylindrical 8.8.4 lattice graphs.

Consider the transfer matrix of the 8.8.4 lattice graph $G_{m,n}$. Define the assembling of the $(i + 1)$th column $v_{i+1}$ to the $i$th column $v_i$ as step $i$ as we did before. We can see that the generation of each independent set of $G_{m,n}$ involves $3n - 3$ steps after the first column $v_1$ determined. Step one is to assemble $v_2$ to the right side of $v_1$. The transfer matrix representing step one, denoted $T_{m,1}$, can be constructed as follows. Let $R_m$ denote the set of all possible vectors which can appear as $v_1$. Clearly $R_m$ consists of $(2m - 2)$-vectors of 0’s and 1’s in which no consecutive 1’s occupy the positions of the $2k - 1$-th and $2k$-th entries, for $1 \leq k \leq m - 1$. Since there are three possibilities at each pair of consecutive $2k - 1$-th and $2k$-th 1 positions, the set $R_m$ has $3^{m-1}$ vectors. The set of all possible vectors $v_2$, denoted as $K_m$, consists of all $m$-vectors of 0’s and 1’s. So $K_m$ has $2^m$ vectors.

Thus the transfer matrix $T_{m1} = (T_{v1,v2})$ is a $3^{m-1} \times 2^m$ matrix whose rows are indexed by vectors of $R_m$ and columns are indexed by vectors of $K_m$, where $T_{v1,v2} = 1$ if $v_1$ and $v_2$ represent possible consecutive pair of columns in an independent set of $G_{m,n}$ and $T_{v1,v2} = 0$ otherwise. Similarly, the transfer matrix for step three is a $2^m \times 3^{m-1}$ matrix $T_{m3}$ which is the transpose of $T_{m1}$, and the transfer matrix for step two is a $2^m \times 2^m$ matrix $T_{m2}$ whose rows and columns are indexed by vectors in $K_m$, $T_{m2}$’s entry in position $(\alpha, \beta)$ is 1 if the vectors represented by $\alpha, \beta$ are orthogonal, and is 0 otherwise. Note that $T_{m1}$ is the transfer matrix for every step $i$ when $i = 3k + 1 (0 \leq k \leq n - 2)$, $T_{m2}$ is the transfer matrix for every step $i$ when $i = 3k + 2 (0 \leq k \leq n - 2)$ and $T_{m3}$ is the transfer matrix for every step $i$ when $i = 3k (0 \leq k \leq n - 1)$. Thus, the transfer multiplicity of $G_{m,n}$ is 3. Thus, if we take the transfer matrix of $G_{m,n}$ to be $T_3 = T_{31}T_{32}T_{33}$, then it is a $2^m \times 2^m$ symmetric matrix with transfer multiplicity one.
When \( m = 2 \), if we index the rows and columns in increasing order in binary numbers, then the transfer matrix of \( G_{m,n} \) is a \( 3 \times 3 \) matrix given by the product of \( T_{21} \), \( T_{22} \) and \( T_{23} \).

\[
T_{21} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},
\]

\[
T_{22} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]

\[
T_{23} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]

Thus \( T_2 = \begin{pmatrix} 9 & 6 & 6 \\ 6 & 3 & 4 \\ 6 & 4 & 3 \end{pmatrix} \).

Consider the transfer matrix of \( H_{n,m} \) which is obtained from \( G_{m,n} \) by identifying its left column and right column. The transfer matrix \( B_{n1} \), which represents every \((3k+1)\)-th \((0 \leq k \leq n-2)\) step, can be defined as a \( 3^n-1 \times 2^{n-1} \) matrix of 0’s and 1’s as follows. The rows of \( B_{n1} \) are indexed by vectors of \( R_n \) and columns are indexed by vectors of \( K_{n-1} \), and the entry of \( B_{n1} \) in position \((\alpha, \beta)\) is 1 if \( \alpha, \beta \) represent possible consecutive pair of rows in an independent set of \( H_{n,m} \), and is 0 otherwise. Let \( B_{n3} \) denote the transfer matrix that represents every \( 3k \)-th \((0 \leq k \leq n-1)\) step. It is no difficult to see that \( B_{n3} \) is the transpose of \( B_{n1} \). The transfer matrix \( B_{n2} \) that represents every \((3k+2)\)-th \((1 \leq k \leq n-2)\) step is a \( 2^{n-1} \times 2^{n-1} \) matrix whose rows and columns are indexed by vectors of \( K_{n-1} \). The entry of \( B_{n2} \) in position \((\alpha, \beta)\) is 1 if \( \alpha, \beta \) are orthogonal, and is 0 otherwise. Thus if we take the transfer matrix of \( H_{n,m} \) to be \( B_n = B_{n1}B_{n2}B_{n3} \), then it is a \( 2^{n-1} \times 2^{n-1} \) symmetric matrix with transfer multiplicity one.

When \( n = 3 \), if we index the rows and columns in increasing order in binary numbers, then the transfer matrix of \( H_{n,m} \) is a \( 8 \times 8 \) matrix given by product of \( B_{31}, B_{32} \) and \( B_{33} \), where
\[ B_{3_1} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix} , \\
B_{3_2} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 
\end{pmatrix} . \\
Thus \\
B_{3_3} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} . \\
B_3 = \begin{pmatrix}
9 & 6 & 6 & 6 & 4 & 6 & 6 & 6 & 4 \\
6 & 3 & 4 & 4 & 2 & 4 & 3 & 3 & 2 \\
6 & 4 & 3 & 3 & 2 & 3 & 4 & 4 & 2 \\
6 & 4 & 3 & 3 & 2 & 3 & 4 & 4 & 2 \\
4 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 \\
6 & 4 & 3 & 3 & 2 & 3 & 4 & 4 & 2 \\
6 & 3 & 4 & 4 & 2 & 4 & 3 & 3 & 2 \\
6 & 3 & 4 & 4 & 2 & 4 & 3 & 3 & 2 \\
4 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 1 
\end{pmatrix} . \\
Note that \( G_{m,n} \) has the same symmetric transfer matrix in both horizontal and vertical directions. So the same approach in [4] can be taken here. Then we can easily get the following results. The lower bound of \( \lim_{m,n \to \infty} f(m,n)^{1/mn} \) is \( (\frac{\lambda_q+2q}{\lambda_{2q}})^{1/p} \) where \( \lambda \)'s are the largest eigenvalues of corresponding \( T \)'s. And the upper bound of \( \lim_{m,n \to \infty} f(m,n)^{1/mn} \) is \( (\xi_{2k})^{1/2k} \) where \( \xi \)'s are the largest eigenvalues of corresponding \( B \)'s. Letting \( p = 1, q = 4 \) and \( k = 4 \), we have
4.631583395... \leq \lim_{m,n \to \infty} f(m,n)^{1/mn} \leq 5.765456528....

Let \( f(m, n) \) denote the number of independent sets of the 8.8.4 lattice graph \( G_{m,n} \). Since the number of vertices of \( G_{m,n} \) is \( 4mn - 2m - 2n \), by the similar reason as for the aztec diamonds, we can see that the entropy constant of the 8.8.4 lattice is

\[
( \lim_{m,n \to \infty} f(m,n)^{1/\frac{1}{mn}} )^{1/4},
\]

which is between 1.467007628... and 1.549560101....

**Remark 1.** In this paper we show that for many statistical models, the entropy constant of a plane lattice is the same as the entropy constants of the corresponding cylindrical and toroidal lattices. But this phenomenon may disappear for some other models. As shown in [18], for the dimer problem the entropy constants of cylindrical and toroidal lattices are different.

**Remark 2.** For many statistical models, using a single transfer matrix is not enough. So, the approach and the concept of transfer multiplicity introduced in this paper can be used to deal with more complex statistic models.

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