PROLONGATIONS, INVARIANTS, AND FUNDAMENTAL IDENTITIES OF GEOMETRIC STRUCTURES

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ABSTRACT. Working in the framework of nilpotent geometry, we give a unified scheme for the equivalence problem of geometric structures which extends and integrates the earlier works by Cartan, Singer-Sternberg, Tanaka, and Morimoto. By giving a new formulation of the higher order geometric structures and the universal frame bundles, we reconstruct the step prolongation of Singer-Sternberg and Tanaka. We then investigate the structure function $\gamma$ of the complete step prolongation of a proper geometric structure by expanding it into components $\gamma = \kappa + \tau + \sigma$ and establish the fundamental identities for $\kappa, \tau, \sigma$. This then enables us to study the equivalence problem of geometric structures in full generality and to extend applications largely to the geometric structures which have not necessarily Cartan connections.

Among all we give an algorithm to construct a complete system of invariants for any higher order proper geometric structure of constant symbol by making use of generalized Spencer cohomology group associated to the symbol of the geometric structure. We then discuss thoroughly the equivalence problem for geometric structure in both cases of infinite and finite type.

We also give a characterization of the Cartan connections by means of the structure function $\tau$ and make clear where the Cartan connections are placed in the perspective of the step prolongations.

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INTRODUCTION

The equivalence problem of geometric structures is to find criteria and to determine whether two arbitrarily given geometric structures are equivalent or not. The theory for the equivalence problem has been developed over more than a hundred years and has played an important role in differential geometry.
Recently motivated partly by a necessity to improve the theory for applications to concrete problems, in particular, in complex geometry and in subriemannian geometry, and partly for the unity and completeness of the theory itself, we are led to reconsider the equivalence problem of geometric structures.

It was S. Lie who first posed this problem in a general form under his theory of continuous transformation groups, and developed a method to obtain the differential invariants of a geometric structure or a system of differential equations by integration of the completely integrable systems on certain jet spaces defined by the action of a continuous transformation group ([22]).

From 1904 to 1909 in his series of papers ([4], [5], [6]) É. Cartan developed the theory of continuous infinite groups, in the course of which he invented a heuristic method for general equivalence problems, by using the method of bundles of moving frames and the theory of Pfaff systems in involution, nowadays called the Cartan-Kähler theory.

This ingenious idea of Cartan then found rigorous foundations through the modern theories of equivalence problem as developed, in particular, in I. M. Singer and S. Sternberg ([36], N. Tanaka ([38], [39]) and the second-named author ([27], [30]).

Fundamental notions such as principal fiber bundles, G-structure, etc., were first used by Ehresmann and Chern, and the algebraic nature of the systems in involution was maid clear by M. Kuranishi, D. C. Spencer, S. Sternberg, V. Guillemin, J. P. Serre, D. Quillen, H. Goldschmidt, and others ([21], [37], [35], [16], [35], [14]).

Singer-Sternberg treated the equivalence problem of geometric structures as that of G-structures and gave a foundation of an aspect of Cartan’s prolongation procedure, which we call here the step prolongation.

Tanaka introduced a method of nilpotent approximation by making use of differential systems and began to develop nilpotent geometry ([38]). First he extended the step prolongation of Singer-Sternberg to a nilpotent version to prove the finite dimensionality of the automorphism group of a geometric structure of finite type in the sense of nilpotent geometry. Next he developed extensively Cartan’s espace généralisé ([8]) by constructing Cartan connections to geometric structures associated with simple graded Lie algebras ([39]), which plays a fundamental role in parabolic geometry ([8]).

A general theoretical method to solve completely the equivalence problem of geometric structures was given by the second-named author ([27]), by introducing the notion of higher order non-commutative frame bundles. This, in turn, applied to nilpotent geometry, gave a unified method to construct Cartan connections as well as a best possible criterion for constructing Cartan connections ([30]). It was also given a nilpotent version of the notion of involutive geometric structure in terms of generalized Spencer cohomology groups.

In the present paper, working in the framework of nilpotent geometry, we give a unified scheme for the equivalence problem of geometric structures which extends and integrates the earlier works ([36], [38], [39], and [27]), and which is well adapted to applications.

The base spaces of the geometric structures that we consider in this paper are filtered manifolds of constant symbols: A **filtered manifold** is a differentiable manifold \( M \) equipped with a tangential filtration \( \{ F^p \}_{p \in \mathbb{Z}} \) of depth \( \mu \),

\[
TM = F^{-\mu} \supset F^{-\mu+1} \supset \cdots \supset F^{-1} \supset F^0 = 0
\]
satisfying $[E^p, F^q] \subset F^{p+q}$, where $F$ denotes the sheaf of sections of $F$ (Section 1.1.1).

To every point of a filtered manifold $(M, F)$ there is attached a nilpotent graded Lie algebra $\text{gr} F_x$ called the symbol (algebra) of the filtered manifold at the point. If the symbol algebra $\text{gr} F_x$ is isomorphic to a graded Lie algebra $\mathfrak{g}_- = \oplus_{p<0} \mathfrak{g}_p$ for all $x \in M$, we say that the filtered manifold $(M, F)$ has constant symbol of type $\mathfrak{g}_-$. A usual differentiable manifold $M^n$ is regarded as a trivial filtered manifold $(M, F)$ with the trivial filtration $F^{-1} = TM$ of depth 1, of which the symbol algebra is isomorphic to the abelian Lie algebra $\mathbb{R}^n$.

The frame bundle $\mathcal{S}^{(0)}(M, F)$ of order 0 of a filtered manifold $(M, F)$ of constant symbol $\mathfrak{g}_-$ is the set of all graded Lie algebra isomorphisms

$$z : \mathfrak{g}_- \rightarrow \text{gr} F_x \quad (x \in M).$$

It is a principal fiber bundle over $M$ with structure group $G_0(\mathfrak{g}_-)$, the group of all graded Lie algebra automorphisms of $\mathfrak{g}_-$.

A $G$-structure on a filtered manifold $(M, F)$ (in the sense of nilpotent geometry) is a $G_0$-principal subbundle of $\mathcal{S}^{(0)}(M, F)$, where $G_0$ is a Lie subgroup of $G_0(\mathfrak{g}_-)$. We know many important examples of $G$-structures in the usual sense. And there are also many interesting $G$-structures in the nilpotent sense, such as contact structures, CR-structures, and various geometric structures linked to differential systems. Indeed, most of the geometric structures studied in differential geometry are $G$-structures in the extended nilpotent sense, which we define in this paper as geometric structures of order 0.

There are as well geometric structures which should be regarded as order 1 such as projective structures and contact projective structures.

Our first task in this paper is to give a general definition of higher order geometric structures, which is indispensable to find higher order differential invariants of a given geometric structure that may be even of lower order. How to define the geometric structures depends on how to derive the differential invariants.

In order to well formulate the step prolongation in a clearer setting, we present a new category of higher order geometric structures by broadening the category of towers introduced in [30].

A geometric structure $Q^{(k)}$ of order $k \geq -1$ of type (or of symbol) $(\mathfrak{g}_-, G_0, \ldots, G_k)$ is defined to be a series of principal fiber bundles

$$Q^{(k)} \xrightarrow{G_k} Q^{(k-1)} \xrightarrow{G_{k-1}} \ldots \xrightarrow{G_1} Q^{(0)} \xrightarrow{G_0} Q^{(-1)}$$

satisfying that (1) $Q^{(-1)}$ is a filtered manifold $(M, F)$ of type $\mathfrak{g}_-$, and (2) for $0 \leq i \leq k$, $Q^{(i)}$ is a geometric structure of order $i$ of type $(\mathfrak{g}_-, G_0, \ldots, G_i)$ and is a principal sub-bundle of the universal frame bundle $\mathcal{S}^{(0)}Q^{(i-1)}$ of order $i$ of $Q^{(i-1)}$, while the universal frame bundle is defined by the properties of naturalness and universality (Definition 2.1). Therefore, geometric structures of order 0 are $G$-structures in the sense of nilpotent geometry as mentioned above. However, the class of higher-order geometric structures is quite large and includes “virtual” geometric structures that do not appear in “real” geometry but are theoretically necessary.
Setting $\mathcal{S}^{(\ell)} Q^{(k)} = \mathcal{S}^{(\ell)} \mathcal{S}^{(\ell-1)} Q^{(k)}$ and passing to the projective limit, we obtain the 
completed universal frame bundle
\[ \mathcal{S} Q^{(k)} = \lim_{\ell} \mathcal{S}^{(\ell)} Q^{(k)}, \]
which proves to be a principal fibre bundle over $Q^{(k-1)}$ equipped with a canonical 1-form
$\theta = \theta_{\mathcal{S} Q^{(k)}}$ giving an isomorphism
\[ \theta_z : T_z \mathcal{S} Q^{(k)} \to E(g_{-}, g_{0}, \ldots, g_{k}) \quad \text{for all} \ z \in \mathcal{S} Q^{(k)} \]
and then defining an absolute parallelism on $\mathcal{S} Q^{(k)}$, where $E(g_{-}, g_{0}, \ldots, g_{k})$ is the universal \[ \text{graded vector space determined by} \ (g_{-}, g_{0}, \ldots, g_{k}), \] which we denote simply by $E$.

The first fundamental observation is that the equivalence problem of the geometric \[ \text{structures} \ (Q^{(k)}) \] reduces to the equivalence problem of the absolute parallelism $(\mathcal{S} Q^{(k)}, \theta_{\mathcal{S} Q^{(k)}})$ of the completed universal frame bundles (Theorem 2.1).

The invariants of the absolute parallelism are given by its structure function $\gamma$, a \[ \text{Hom}(\wedge^2 E, E) \text{-valued function on} \ \mathcal{S} Q^{(k)} \text{defined by:} \]
\[ d\theta + \frac{1}{2} \gamma (\theta \wedge \theta) = 0. \]

All information on $Q^{(k)}$ is encoded in the structure function of $\mathcal{S} Q^{(k)}$. However, the completed universal frame bundle $\mathcal{S} Q^{(k)}$ is infinite dimensional and of large magnitude. Following the idea of Cartan, we make a reduction of the universal frame bundle by using the structure function to obtain a smaller subbundle representing the invariants more effectively, which is carried out in Section 3.

To do that, we study basic properties of structure function. In particular, we have the following decomposition:
\[ \gamma = \gamma_I + \gamma_{II} + \gamma_{III} = \kappa + \tau + \sigma, \quad \text{with} \ \gamma_I = \kappa, \ \gamma_{II} = \tau, \ \gamma_{III} = \sigma \]
according to the direct sum decomposition
\[ \text{Hom}(\wedge^2 E, E) = \text{Hom}(E_{-} \wedge E_{-}, E) \oplus \text{Hom}(E_{+} \wedge E_{-}, E) \oplus \text{Hom}(E_{+} \wedge E_{+}, E), \]
where $E_{-} = \bigoplus_{p < 0} E_p, E_{+} = \bigoplus_{p \geq 0} E_p$. We also decompose $\gamma$ as
\[ \gamma = \sum \gamma_p, \ \gamma^{(l)} = \sum_{p \leq l} \gamma_p, \quad \text{and} \quad \gamma = \sum \gamma_{[p]}, \ \gamma^{[l]} = \sum_{p \leq l} \gamma_{[p]}, \]
according to the homogeneous degree and modified homogeneous degree of Hom$(\wedge^2 E, E)$, respectively. For the definition, see Section 2.5.2. We then show that the truncated structure function $\gamma^{[l]}$ on $\mathcal{S} Q^{(k)}$ is a function on $\mathcal{S}^{(l)} Q^{(k)}$.

In Section 3 we first introduce the class of proper geometric structures. We say that a geometric structure $Q^{(k)}$ of type $(g_{-}, G_{0}, \ldots, G_{k})$ is proper if $(g_{-}, g_{0}, \ldots, g_{k})$ forms a truncated transitive graded Lie algebra $g[k]$. It is the proper geometric structure that has moderate magnitude and appears actually in real geometry. See the remark after Definition 3.1.

Now let us make the key procedure of $W$-normal reduction. Given a truncated transitive graded Lie algebra $g[k] = \bigoplus_{p \leq k} g_p$ and Lie groups $G_{0}, \ldots, G_{k}$ with Lie algebras $g_{0}, \ldots, g_{k}$, let $g = \bigoplus g_p$ be the prolongation of $g[k]$, $g[l] = \bigoplus_{p \leq l} g_p$, and $G_{\ell}$ be the vector
group \(g_\ell\) for \(\ell > k\). Here, the vector group \(g_\ell\) means the same vector space viewed as a commutative Lie group. Fix complementary subspaces \(W = \{W^1_\ell, W^2_\ell\}_{\ell \geq k}\) such that
\[
\text{Hom}(\wedge^2 g_-, g)_{\ell + 1} = W^2_{\ell + 1} \oplus \partial \text{Hom}(g_-, g)_{\ell + 1}
\]
\[
\text{Hom}(g_-, g)_{\ell} = W^1_{\ell} \oplus \partial g_\ell.
\]

For any proper geometric structure \(Q^{(k)}\) of type \((g_-, G_0, \ldots, G_k)\), set \(\mathcal{S}_W^{(k)} = Q^{(k)}\) and
\[
\mathcal{S}^{(\ell + 1)}_W Q^{(k)} = \{ z \in \mathcal{S}^{(\ell + 1)} \mathcal{S}_W^{(\ell)} Q^{(k)} : \kappa^{[\ell + 1]}(z) \in W^2_{\ell + 1}, \tau^{[\ell + 1]}(z) \in \text{Hom}((\oplus_{i=0}^{\ell - 1} g_i, W^1_{\ell})) \}.
\]
Then \(\mathcal{S}_W^{(\ell)} Q^{(k)}\) is a proper geometric structure of type \(g[\ell]\) and the projective limit \(\mathcal{S}_W Q^{(k)} = \lim_\ell \mathcal{S}_W^{(\ell)} Q^{(k)}\) is endowed with a canonical \(g\)-valued 1-form \(\theta = \theta \mathcal{S}_W\) which gives an absolute parallelism on \(\mathcal{S}_W Q^{(k)}\).

We call \(\mathcal{S}_W Q^{(k)}\) the \(W\)-normal step prolongation of \(Q^{(k)}\) or the \(W\)-normal reduction of \(\mathcal{S}^{(k)}\). We then have

**Theorem I** (Theorem 3.1). The equivalence problem of the proper geometric structures \(Q^{(k)}\) of type \((g_-, G_0, \ldots, G_k)\), reduces to the equivalence problem of the absolute parallelisms \((\mathcal{S}_W Q^{(k)}, \theta)\) of the step prolongations. Moreover if \(g[k]\) is of finite type, that is, the prolongation \(g\) is finite dimensional, then so is \(\mathcal{S}_W Q^{(k)}\) and \(\text{dim} \mathcal{S}_W Q^{(k)} = \text{dim} g\).

This is one of the main theorems of [36] and [38]. The \(W\)-normal step prolongation \(\mathcal{S}_W^{(\ell)} Q^{(0)}\) essentially coincides with what Tanaka constructed. We thus reformulate the construction of step prolongation in our new scheme of the complete universal frame bundles.

By virtue of the conceptional construction of step prolongations, we are naturally led to the fundamental identities: In Section 4 we prove the following theorem, which plays a key role in our prolongation scheme.

**Theorem II** (Theorem 4.1). Let \(Q^{(k)}\) be a proper geometric structure of order \(k\) and let \(\mathcal{S}_W Q^{(k)}\) be the \(W\)-normal step prolongation of \(Q^{(k)}\), and let \(\gamma = \gamma_I + \gamma_{II} + \gamma_{III}\) be its structure function. Then we have
\[
\begin{align*}
1) \quad \partial \gamma_I[k] &= \Psi_I[k](\gamma_I[i], \gamma_{III}[i]; i < k) \\
2) \quad \partial \gamma_{II}[k] &= \Psi_{II}[k](\gamma_I[i], \gamma_{III}[i], \gamma_{III}[i]; i < k) \\
3) \quad \partial \gamma_{III}[k] &= \Psi_{III}[k](\gamma_I[i], \gamma_{III}[i]; i < k)
\end{align*}
\]
where \(\Psi_X[k]\) for \(X \in \{I, II, III\}\) is a polynomial in \(\gamma_Y[i]\) with \(Y \in \{I, II, III\}\) and \(i < k\), and their covariant derivatives.

In the above theorem, we have written the fundamental identities simply by trivially extending the Spencer cochain complex. We provide more precise and explicit formulas, which give the identities for the invariants of a geometric structure in the most general form (Section 4). In the special case of Riemannian structures they are known as the first and second Bianchi identities. We remark also that if there is no \(\gamma_{II}\) and \(\gamma_{III}\), the above identities reduce to those known in the geometry of Cartan connection.

From the fundamental identities it follows many important consequences as described in Section 5 – Section 7 each of which is of independent interest.
In Section 5 we show that, as an immediate consequence of Theorem I and II, and on account of the vanishing of the generalized Spencer cohomology group for higher degree, we obtain generators of a fundamental system of invariants after a finite number of steps in prolongation procedure even in case of a geometric structure of infinite type.

**Theorem III** (Theorem 5.1). Let $\mathfrak{g}[k]$ be a truncated transitive graded Lie algebra and $\mathfrak{g}$ be its prolongation. Let

\[
I^1 = \{i \in \mathbb{Z}_{>0} : H_1^i(\mathfrak{g} - , \mathfrak{g}) \neq 0\}
\]

\[
I^2 = \{i \in \mathbb{Z}_{>0} : H_2^i(\mathfrak{g} - , \mathfrak{g}) \neq 0\},
\]

which are finite sets by Theorem [1.1]

Let $Q^{(k)}$ be a proper geometric structure of type $\mathfrak{g}[k]$. Then the following invariants, called the set of essential invariants of $Q^{(k)}$,

\[
\{D^m\kappa[i] : i \in I^2, m \geq 0\}\] and \[
\{D^m\tau[i] : i - 1 \in I^1, m \geq 0\}
\]

form a fundamental system of invariants of $Q^{(k)}$.

Thus we have given an algorithm to find a fundamental system of invariants for any proper geometric structure, which solves the first half of the equivalence problem (Necessity).

In Section 6 we consider the second half of equivalence problem (Sufficiency) and pose the following question:

[Equivalence Problem (Sufficiency)] Let $P^{(k)}$ and $Q^{(k)}$ be two proper geometric structures of order $k$. Assume that there is an isomorphism between the (fundamental system of, or essential) invariants of $P^{(k)}$ and those of $Q^{(k)}$ (in a suitable sense). Then, is there an isomorphism between $P^{(k)}$ and $Q^{(k)}$?

We say that a proper geometric structure $P^{(k)}$ is involutive if it is quasi-involutive (i.e., $H_1^\ell(\mathfrak{g} - , \mathfrak{g}) = H_2^{\ell+1}(\mathfrak{g} - , \mathfrak{g}) = 0$ for $\ell \geq k$) and the structure function $\gamma^{[k]}$ of $P^{(k)}$ is constant. We then prove

**Theorem IV** (Theorem 6.2). The answer to the equivalence problem (Sufficiency) is affirmative for the involutive geometric structures in the analytic category, and for those of finite type in the $C^\infty$-category.

For the proof we rely on the Cartan-Kähler theorem or its generalization to a nilpotent version. For the geometric structure of finite type the theorem holds in the $C^\infty$-category, which is easily proved by the Frobenius theorem.

In particular, we obtain as a corollary that if the fundamental invariants vanish then the geometric structure is equivalent to a standard one, under the assumption of analyticity if it is of infinite type. For the finite type analytic geometric structures we have a more general theorem worth noting: The equivalence problem (Sufficiency) is affirmative for the analytic geometric structures of finite type ([Theorem 6.4]).

In Section 7 we study relations between the $W$-normal step prolongations and the Cartan connections by using fundamental identities, and make clear where the Cartan connections are placed in the perspective of step prolongations.
For this, we revisit the notion of tower and reformulate it to define the category of principal geometric structures, which is parallel and in a sharp contrast to that of geometric structures introduced in Section 2. This then allows us to characterize the proper principal geometric structure among the proper geometric structures, and then the Cartan connection among the \( W \)-normal step prolongations in terms of the structure function \( \tau \).

We first show that Cartan connections are almost equal to complete proper geometric structures with constant structure function \( \tau \) (Theorem 7.1). More precisely and specifically we prove:

**Theorem V (Theorem 7.2).** Let \( Q^{(0)} \) be a geometric structure of order 0 with a connected structure group \( G_0 \) and \( \mathcal{J}_W Q^{(0)} \) the \( W \)-normal step prolongation. If \( \tau^{[k+1]} \) is flat, then \( \mathcal{J}_W^{(k)} Q^{(0)} \rightarrow M \) is a proper principal geometric structure of order \( k \). If the structure function \( \tau \) of \( \mathcal{J}_W Q^{(0)} \) is flat, then \( \mathcal{J}_W Q^{(0)} \) is a Cartan connection.

Here, we say that \( \tau^{[k+1]} \) is flat if \( \tau^{[\ell][m]} \) is zero for any \( \ell > 0 \) and \( m \leq k + 1 \) and that \( \tau \) is flat if \( \tau^{[\ell][m]} \) is zero for any \( \ell > 0 \) and \( m \).

From this we obtain a more precise result well adapted to applications in complex geometry: the fundamental invariants can be regarded as sections of bundles on the base manifold inductively as in the case when there exists a Cartan connection associated with a given geometric structure (Theorem 7.4).

We also prove:

**Theorem VI (Theorem 7.3).** If the structure group \( G_0 \) of a geometric structure \( Q^{(0)} \) of order 0 satisfies the condition (C), then \( W \)-normal step prolongation \( \mathcal{J}_W Q^{(0)} \) coincides with the \( W \)-normal Cartan connection \( \mathcal{R}_W Q^{(0)} \) constructed in [30].

Thus the \( W \)-normal step prolongations contain, as special cases, the Cartan connections, and therefore have vast applications to the geometric structures which do not necessarily admit Cartan connections.

In Section 8 we explain how our method is applied to subriemannian geometry and complex geometry.

We give a mention to the works [2], [41] which provide alternative proofs of Tanaka’s step prolongation. In our opinion, without the fundamental identities, the prolongation scheme would not be as clear and as effective as shown in the present paper.

It is to intrinsic geometry that our present paper is concerned. In the paper [12] a general theory on extrinsic geometry is developed and it is made clear the similarity existing between intrinsic and extrinsic geometry. The present paper will make more visible and highlight this similarity.

In this paper we have confined ourselves to geometric structures of constant symbols. But if we use the method in [27] to treat prolongations of \( G \)-structures with varying structure group, we can extend the prolongation scheme of the present paper to the geometric structures of non-constant symbol, which will be an object of our forthcoming paper.

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1. Preliminaries

1.1. Geometric preliminaries.

1.1.1. A. filtered manifold is a differentiable manifold \( M \) equipped with a filtration \( F = \{ F^p \}_{p \in \mathbb{Z}} \) of the tangent bundle \( TM \) of \( M \) satisfying:

1. \( F^p \) is a subbundle of \( TM \) and \( F^p \supseteq F^{p+1} \) for all \( p \in \mathbb{Z} \);
2. \( \cup_{p \in \mathbb{Z}} F^p = TM \) and \( F^0 = 0 \);
3. \( [F^p, F^q] \subseteq F^{p+q} \) for all \( p, q \in \mathbb{Z} \), where \( F^\bullet \) is the sheaf of sections of \( F^\bullet \).

The minimal integer \( \mu \) such that \( F^{-\mu} = TM \) is called the depth of \( F \). A filtered manifold will be denoted as \( (M, F) \), or \( (M, F_M) \), or simply as \( M \). The filtration is written as \( \{ F^p \} \), or alternatively, as \( \{ F_M^p \} \), \( \{ F^{pTM} \} \) or \( \{ T^pM \} \).

Let \( (M, F) \) be a filtered manifold. For each \( x \in M \) we set

\[
\text{gr } F_x = \bigoplus_p \text{gr}_p F_x, \quad \text{gr}_p F_x = F_x^p / F_x^{p+1}
\]

where \( F_x^p \) denotes the fiber of \( F^p \) over \( x \). The bracket operation of vector fields induces a bracket

\[
[\cdot, \cdot] : \text{gr}_p F_x \times \text{gr}_q F_x \rightarrow \text{gr}_{p+q} F_x.
\]

Then \( \text{gr } F_x \) becomes a nilpotent graded Lie algebra and is called the symbol (algebra) of \( (M, F) \) at \( x \).

An isomorphism of a filtered manifold \( M \) onto another \( N \) is a diffeomorphism \( \varphi : M \rightarrow N \) which preserves the filtrations, that is, \( \varphi_* F^p_M = F^p_N \). It should be noted that if \( \varphi : (M, F_M) \rightarrow (N, F_N) \) is an isomorphism of filtered manifolds, then \( \varphi \) induces a graded Lie algebra isomorphism \( \text{gr } \varphi : \text{gr } F_{M,x} \rightarrow \text{gr } F_{N,\varphi(x)} \).

We say that a filtered manifold \( (M, F) \) is of (constant symbol of) type \( g_- \) if \( g_- = \bigoplus_{p < 0} g_p \) is a graded Lie algebra and \( \text{gr } F_x \) is isomorphic to \( g_- \) as graded Lie algebras for all \( x \in M \).

In this paper, we will be almost exclusively concerned with filtered manifolds which are of constant symbol. Unless otherwise stated, this will always be assumed.

1.1.2. Let \( g_- = \bigoplus_{p < 0} g_p \) be a graded Lie algebra and \( (M, F) \) be a filtered manifold of type \( g_- \). For each \( x \in M \) we set

\[
\mathcal{S}_x^{(0)}(M) := \{ z : g_- \rightarrow \text{gr } F_x : \text{graded Lie algebra isomorphisms} \}
\]

and \( \mathcal{S}^{(0)}(M) = \cup_{x \in M} \mathcal{S}_x^{(0)}(M) \). We denote by \( G_0(g_-) \) the Lie group consisting of all graded Lie algebra automorphisms of \( g_- \). Then \( \mathcal{S}^{(0)}(M) \) is a principal fiber bundle over \( M \) with structure group \( G_0(g_-) \). The bundle \( \mathcal{S}^{(0)}(M) \) is called the (reduced) frame bundle of \( (M, F) \) (of order 0) in the sense of nilpotent geometry.

It is a quotient space of the (reduced) frame bundle \( \widehat{\mathcal{S}}^{(0)}(M) \) of \( (M, F) \) (of order 0) in the usual sense defined as follows. We regard the graded Lie algebra \( g_- \) as a filtered vector space with the filtration \( \{ F^p g_- \} \) given by \( F^p g_- = \bigoplus_{i \geq p} g_i \). For \( x \in M \), let \( \widehat{\mathcal{S}}_x^{(0)}(M) \) be the set of all linear isomorphisms \( \zeta : g_- \rightarrow T_x M \) such that \( \zeta \) preserves the
filtration and 
\[ [\zeta] : \text{gr } g_- (= g_-) \to \text{gr } F_x \]
is an isomorphism of graded Lie algebras, and let \( \hat{\mathcal{S}}(0)(M) = \cup_{x \in M} \hat{\mathcal{S}}_x(0)(M) \). Then \( \hat{\mathcal{S}}(0)(M) \) is a principal fiber bundle over \( M \) with structure group \( \hat{G}_0(g_-) \) consisting of all filtration preserving automorphisms \( \alpha \) of \( g_- \) such that the induced map \( \alpha \) is contained in \( G_0(g_-) \).

For a vector space \( V \) with a filtration \( \{F^pV\}_p \), Hom\( (V,V) \) and GL\( (V) \) are filtered as follows:

\[
F^p\text{Hom}(V,V) = \{ A \in \text{Hom}(V,V) : A(F^qV) \subset F^{p+q}V \text{ for all } q \}
\]

\[
F^p\text{GL}(V) = \{ a \in \text{GL}(V) : a - \text{id}_V \in F^p\text{Hom}(V,V) \}.
\]

We see that
\[
G_0(g_-) = \hat{G}_0(g_-)/F^1, \quad \hat{\mathcal{S}}(0)(M) = \hat{\mathcal{S}}(0)(M)/F^1,
\]
where \( F^1(= F^1\hat{G}_0(g_-)) \) is the subgroup of \( \hat{G}_0(g_-) \) defined by the induced filtration.

We remark that the definition of frame bundles \( \mathcal{S}(0)(M) \) and \( \hat{\mathcal{S}}(0)(M) \) depends on the choice of a graded Lie algebra \( g_- \). To avoid the ambiguity we shall fix, once and for all, one representative \( g_- \) for each equivalence class of graded Lie algebras.

Then for each isomorphism \( \varphi : M \to N \) of filtered manifolds, we have the associated bundle isomorphisms, called the lift or the prolongation of \( \varphi \):

\[
\hat{\mathcal{S}}(0)\varphi : \hat{\mathcal{S}}(0)M \to \hat{\mathcal{S}}(0)N, \quad \mathcal{S}(0)\varphi : \mathcal{S}(0)M \to \mathcal{S}(0)N
\]

where \( \mathcal{S}(0)\varphi(\zeta) = \varphi_* \circ \zeta, \mathcal{S}(0)\varphi(z) = \text{gr}\varphi_* \circ z \) for \( \zeta \in \hat{\mathcal{S}}(0)M, z \in \mathcal{S}(0)M \).

### 1.1.3. We define a geometric structure of order 0 on a filtered manifold \((M,F)\) of type \( g_- \) to be a reduction of the frame bundle \( \mathcal{S}(0)(M) \) to a subgroup \( G_0 \subset G_0(g_-) \), that is, a \( G_0 \)-principal subbundle \( Q_0(0) \) of \( \mathcal{S}(0)(M) \). Two geometric structures \( P_0 \to (M,F_M) \) and \( Q_0 \to (N,F_N) \) are said to be isomorphic, or equivalent if there is an isomorphism \( \varphi : (M,F_M) \to (N,F_N) \) such that the induced map \( \mathcal{S}(0)\varphi : \mathcal{S}(0)(M) \to \mathcal{S}(0)(N) \) maps \( P_0 \) onto \( Q_0 \).

Thus a geometric structure of order 0 may be called a \( G \)-structure in the sense of nilpotent geometry: if the underlying filtered manifold is of depth 1, i.e., a trivial filtered manifold, it is just what is usually called a \( G \)-structure.

We know various important examples of \( G \)-structures, and almost all geometric structures that we treat in differential geometry are geometric structures of order 0 of the above definition or their generalizations to higher orders. In the proceeding sections we shall study the equivalence problem of geometry structures of order 0 or higher order.

### 1.2. Algebraic preliminaries.

#### 1.2.1. An infinitesimal algebraic model of a transitive geometric structure on a filtered manifold is represented by a transitive filtered Lie algebra defined as follows: A transitive filtered Lie algebra is a Lie algebra \( L \) equipped with a decreasing filtration \( \{F^pL\} \) satisfying the following conditions:

1. \( [F^pL,F^qL] \subset F^{p+q}L \)
2. \( \dim L/F^0L < \infty \)
3. \( \cap_pF^pL = 0 \)
(4) (transitivity) If \( i \geq 0, X \in F^i L \) and if \([X, F^a L] \subset F^{i+a+1} L\) for all \( a < 0\), then \( X \in F^{i+1} L\).

(5) \( L\) is complete with respect to the topology which makes \( \{F^p\} \) as a fundamental system of neighbourhoods of the origin.

Passing to the graded object \( grL = \bigoplus F^p L/F^{p+1} L\), we have a transitive graded Lie algebra, that is, a Lie algebra \( g\) equipped with a grading \( g = \bigoplus g_i\) satisfying the following conditions:

1. [\( [g_i, g_j] \subset g_{i+j}\) ]
2. \( \dim g_- < \infty\), where we put \( g_- = \bigoplus_{p<0} g_p\).
3. (transitivity) If \( X \in g_i \) (\( i \geq 0\)) and \([X, g_-] = 0\), then \( X = 0\).

This (or its completion) gives an infinitesimal algebraic model of a flat geometric structure.

1.2.2. A truncated transitive graded Lie algebra of order \( k \) (\( k \geq -1\)) is a graded vector space \( g[k] = \bigoplus_{i \leq k} g_i\) endowed with a bracket operation \([\cdot , \cdot] : g_i \otimes g_j \rightarrow g_{i+j}\) which is defined only for \( i, j, i+j \leq k\) and satisfies the Jacobi identity whenever it makes sense, and moreover satisfies the conditions (2) and (3) in the definition of a transitive graded Lie algebra.

Given a transitive graded Lie algebra \( g = \bigoplus p g_p\) with the Lie bracket \([\cdot , \cdot]\), the graded vector space \( \bigoplus_{p \leq k} g_p\) with the bracket operation \([\cdot , \cdot]^{(k)}\) defined by

\[
[X, Y]^{(k)} = \begin{cases} 
[X, Y] & \text{if } X \in g_i, Y \in g_j, \text{ and } i+j \leq k \\
0 & \text{otherwise,}
\end{cases}
\]

is a truncated transitive graded Lie algebra of order \( k\), denoted by \( \text{Trun}^{(k)}(g)\). We call \( \text{Trun}^{(k)}(g)\) a truncation of \( g\).

Note that \( g_-\) and \( g_- \oplus g_0\) are truncated transitive graded Lie algebras, where \( g_-\) is a graded Lie algebra negatively concentrated and of finite dimension, and \( g_0\) is a Lie subalgebra of the Lie algebra \( \text{Der}_0(g_-)\) of all derivations of degree 0 of \( g_-\). Fundamental is the following:

**Proposition 1.1** ([36, 38]). For a truncated transitive graded Lie algebra \( g[k] = \bigoplus_{p \leq k} g_p\) there exists, uniquely up to isomorphisms, a maximal transitive graded Lie algebra \( \bar{g} = \bigoplus \bar{g}_p\) such that \( g[k] \) is a truncation of \( \bar{g}\).

In fact, we can construct the transitive graded Lie algebra \( \bar{g} = \bigoplus \bar{g}_p\) by setting \( \bar{g}_p = g_p\) for \( p \leq k\) and then inductively so as to satisfy

\[
\bar{g}_{p+1} = \{ \alpha \in \text{Hom}(g_-, \bigoplus_{i \leq p} g_i)_{p+1} : \alpha([u, v]) = [\alpha(u), v] + [u, \alpha(v)], \ u, v \in g_-\}
\]

as well as the Jacobi identity, whenever it is defined.

The transitive graded Lie algebra \( \bar{g}\) is called the prolongation of \( g[k]\) and denoted by \( \text{Prol}(g[k])\).

1.2.3. Let \( g\) be a transitive graded Lie algebra and \( g_- = \bigoplus_{p<0} g_p\) be its negative part. The adjoint representation of \( g_-\) on \( g\) gives rise to the cohomology group \( H(g_-, g) = \bigoplus H^q(g_-, g)\), called the generalized Spencer cohomology group. It is the cohomology group associated to the following chain complex:

\[
0 \rightarrow g \xrightarrow{\partial} \cdots \xrightarrow{\partial} \text{Hom}(\wedge^q g_-, g) \xrightarrow{\partial} \text{Hom}(\wedge^{q+1} g_-, g) \xrightarrow{\partial} \cdots.
\]
The coboundary operator is defined as follows: For \( c \in \text{Hom}(\wedge^q g_-, g) \), define \( \partial c \in \text{Hom}(\wedge^{q+1} g_-, g) \) by
\[
\partial c(v_1, \ldots, v_{q+1}) = \sum_i (-1)^{i+1} [v_i, c(v_1, \ldots, \hat{v}_i, \ldots, v_{q+1})]
+ \sum_{i<j} (-1)^{i+j} c([v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{q+1}).
\]

Let \( \text{Hom}(\wedge^q g_-, g)_r \) denote the set of all maps of degree \( r \). Then, since the coboundary operator preserves the degree, we have the subcomplex:
\[
0 \to g_r \xrightarrow{\partial} \cdots \xrightarrow{\partial} \text{Hom}(\wedge^q g_-, g)_r \xrightarrow{\partial} \text{Hom}(\wedge^{q+1} g_-, g)_r \xrightarrow{\partial} \cdots.
\]

The associated cohomology group is denoted by \( H^q(g_-, g) \). Then we have
\[
H(g_-, g) = \bigoplus H^q(g_-, g).
\]

It is \( H^q_+(g_-, g) := \bigoplus_{r>0} H^q_r(g_-, g) \) where \( q = 1, 2 \) that plays an important role in our intrinsic geometry. Note that \( g \) is the prolongation of its truncation \( g[k] \) if and only if \( H^1_r(g_-, g) = 0 \) for \( r > k \).

We have the following finitude of the cohomology group.

**Theorem 1.1 (Theorem of finitude, [28]).** For every transitive graded Lie algebra \( g \) there exists a positive integer \( r_0 \) such that
\[
H^q_r(g_-, g) = 0 \quad \text{for all } q \geq 1 \text{ and } r > r_0.
\]

A finitude theorem concerning the prolongation of exterior differential systems proved by Kuranish opened a way to modern theory of infinite Lie pseudo-groups. The finitude theorem in this form was proved in the case of depth 1 by Singer-Sternberg. The general case is due to the second-named author.

## 2. GEOMETRIC STRUCTURES OF HIGHER ORDER AND UNIVERSAL FRAME BUNDLES

In this section we introduce a category of higher order geometric structures which represents the most general geometric structures of constant symbols and which will turn out to be well adapted to the step prolongations to be studied in the next section.

### 2.1. Geometric structures of order \( k \).

**Definition 2.1.** A geometric structure \( Q^{(k)} \) of order \( k \) and its universal frame bundle \( \mathcal{S}^{(k+1)}Q^{(k)} \) of order \( k+1 \) are defined inductively for \( k \geq -1 \) by the following properties.

1. A geometric structure \( Q^{(k)} \) of order \( k \geq -1 \) on a filtered manifold \( (M, F) \) (of type \( (g_-, G_0, \cdots, G_k) \)) is a step-wise principal fiber bundle over \( (M, F) \) of type
\[
Q^{(k)} \xrightarrow{G_k} Q^{(k-1)} \to \cdots \to Q^{(0)} \xrightarrow{G_0} Q^{(-1)} = M,
\]
(that is, each \( Q^{(i)} \xrightarrow{G_i} Q^{(i-1)} \) is a principal fiber bundle with structure group \( G_i \) for \( 0 \leq i \leq k \) satisfying:
(a) A geometric structure \( Q^{(-1)} \) of order \( -1 \) on \( (M, F) \) is the filtered manifold \( (M, F) \) itself.
(b) If \( k \geq 0 \), the truncated sequence \( Q^{(i-1)} = (Q^{(i-1)} \to \ldots Q^{(0)} \to Q^{(-1)} = M) \) is a geometric structure of order \( i - 1 \) for \( i = k \) (and then consequently for \( 0 \leq i \leq k \) by induction).

(c) If \( k \geq 0 \), \( Q^{(i)} \to Q^{(i-1)} \) is a principal subbundle of the universal frame bundle \( \hat{Q}^{(i)} \to \hat{Q}^{(i-1)} \) of \( Q^{(i-1)} \) of order \( i \) for \( i = k \) and then consequently for \( 0 \leq i \leq k \) by induction.

(2) To every geometric structure \( \hat{Q}^{(k)} \) of order \( k \geq -1 \) there is associated a principal fiber bundle \( \hat{\mathcal{Q}}^{(k+1)} Q^{(k)} \to \hat{Q}^{(k)} \) which is called the universal linear frame bundle of \( \hat{Q}^{(k)} \) of order \( k + 1 \). Its structure group \( \hat{G}_{k+1} \) is endowed with a filtration \( \{ F^\ell \hat{G}_{k+1} \} \). The quotient

\[
\mathcal{Q}^{(k+1)} Q^{(k)} := \mathcal{Q}^{(k+1)} Q^{(k)}/F^{k+2} \hat{G}_{k+1}
\]

is a principal fiber bundle over \( Q^k \) with structure group

\[
G_{k+1} := \hat{G}_{k+1}/F^{k+2} G_{k+1},
\]

which is regarded as a geometric structure of order \( k + 1 \) and called the universal frame bundle of \( \hat{Q}^{(k)} \) of order \( k + 1 \).

(3) The universal linear frame bundle \( \hat{\mathcal{Q}}^{(k+1)} Q^{(k)} \) for a geometric structure \( Q^{(k)} \) of order \( k \geq -1 \) is defined as follows:

(a) The bundle \( \hat{\mathcal{Q}}^{(k+1)} Q^{(k)} \). We set

\[
E^{(-1)} := \mathfrak{g}_- \quad \text{and} \quad E^{(k)} := \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k \quad \text{if} \quad k \geq 0,
\]

where \( \mathfrak{g}_i \) is the Lie algebra of the structure group \( G_i \) for \( 0 \leq i \leq k \). We regard \( E^{(k)} \) not only as a graded vector space but also as a filtered vector space in the standard way. We note also that the tangent space \( T_z Q^{(i)} \) \( (-1 \leq i \leq k) \) has a canonical filtration \( \{ F^\ell T_z Q^{(i)} \} \) \( \ell \in \mathbb{Z} \) defined inductively by the following conditions for \( 0 \leq i \leq k \): \( F^{i+1} T_z Q^{(i)} = 0 \) and

\[
0 \to F^\ell T_z Q^{(i)} \to T_z Q^{(i)} \to T_{z^{-1}} Q^{(i-1)}/F^\ell T_{z^{-1}} Q^{(i-1)} \to 0 \quad \text{exacts for} \quad \ell \leq i,
\]

where \( z^{-1} \in Q^{(i-1)} \) is the projection of \( z^k \in Q^{(k)} \).

It being prepared, we define the fiber \( \hat{\mathcal{Q}}_{z^k}^{(k+1)} Q^{(k)} \) over \( z^k \in Q^{(k)} \) to be the set of all filtration preserving isomorphisms

\[
\zeta^{k+1} : E^{(k)} \to T_{z^k} Q^{(k)}
\]

which satisfy the following conditions:

If \( k = -1 \), \( [\zeta^0] = \text{gr} \zeta^0 : \mathfrak{g}_- \to \text{gr} T_{z^{-1}} Q^{(-1)} \) is a graded Lie algebra isomorphism.

If \( k \geq 0 \), the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \to & \mathfrak{g}_k \\
\downarrow & & \downarrow \zeta^{k+1} \\
0 & \to & E^{(k)} \quad E^{(k-1)} \to 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \to & (\mathfrak{g}_k)_{z^k} \\
\downarrow & & \downarrow \zeta^k \\
0 & \to & T_{z^k} Q^{(k)} \quad T_{z^{k-1}} Q^{(k-1)} \to 0
\end{array}
\]
where \( \tilde{\cdot} \) denote the map which sends \( A \in \mathfrak{g}_k \) to \( \frac{d}{dt}|_{t=0}(z^k \exp tA) \in (\tilde{\mathfrak{g}}_k)_{z^k} \), and \( \zeta^k \) is the truncation of \( \zeta^{k+1} \), and

\[
z^k = [z^k] = \zeta^k / F^{k+1}GL(E^{(k-1)})
\]

and \( z^{k-1} \) is the image of \( z^k \) by the projection map \( Q^{(k)} \to Q^{(k-1)} \).

(b) The group \( \tilde{G}_{k+1} \) consists of all filtration preserving linear isomorphisms \( \alpha^{k+1} : E^{(k)} \to E^{(k)} \) which satisfy the following conditions:

If \( k = -1 \), \( [\alpha^0] = \text{gr} \alpha^0 : \mathfrak{g}_- \to \mathfrak{g}_- \) is a graded Lie algebra isomorphism.

If \( k \geq 0 \), the following diagram is commutative:

\[
\begin{array}{cccc}
0 & \rightarrow & \mathfrak{g}_k & \rightarrow & E^{(k)} & \rightarrow & E^{(k-1)} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \alpha^{k+1} & \downarrow & \alpha^k & & & \\
0 & \rightarrow & \mathfrak{g}_k & \rightarrow & E^{(k)} & \rightarrow & E^{(k-1)} & \rightarrow & 0 \\
\end{array}
\]

and \( [\alpha^k] = \alpha^k / F^kGL(E^{(k-1)}) = 1 \). The group \( \tilde{G}_{k+1} \) is endowed with a natural filtration induced from that of \( GL(E^{(k)}) \).

(c) The action of \( \tilde{G}_{k+1} \) on \( \tilde{\mathcal{F}}^{(k+1)}Q^{(k)} \). For \( \alpha^{k+1} \in \tilde{G}_{k+1} \) and \( \zeta^{k+1} \in \tilde{\mathcal{F}}^{(k+1)}Q^{(k)} \), \( \zeta^{k+1} \cdot \alpha^{k+1} \) is given by the following commutative diagram:

\[
\begin{array}{ccc}
E^{(k)} & \xrightarrow{\alpha^{k+1}} & T_{\zeta^{k+1}}Q^{(k)} \\
\downarrow & & \downarrow \text{id} \\
E^{(k)} & \xrightarrow{\zeta^{k+1}} & T_{\zeta^{k+1}}Q^{(k)}
\end{array}
\]

where \( z^k = [z^k] \) is the projection of \( \zeta^{k+1} \) to \( Q^{(k)} \).

This completes the definition. Indeed, the three properties (1) (2) (3) above being bound each to each, the inductive definition is well carried out in the order of:

\[
Q^{(-1)}, \tilde{\mathcal{F}}^{(0)}Q^{(-1)}, \tilde{\mathcal{F}}^{(0)}Q^{(-1)}, Q^{(0)}, \tilde{\mathcal{F}}^{(1)}Q^{(0)}, \tilde{\mathcal{F}}^{(1)}Q^{(0)}, Q^{(1)}, \ldots
\]

We remark that for a geometric structure of order \(-1\), \( Q^{(-1)} = (M, F) \), the universal frame bundle \( \mathcal{F}^{(0)}Q^{(-1)} \) is clearly identified with the (reduced) frame bundle \( \mathcal{F}^{(0)}(M, F) \) of \( (M, F) \) introduced in Section 1, so that the geometric structures of order \( 0 \) are the \( G \)-structures in the sense of nilpotent geometry. We remark also that \( \tilde{\mathcal{F}}^{(k+1)}Q^{(k)} = \mathcal{F}^{(k+1)}Q^{(k)} \) if the filtration of the filtered manifold \( Q^{(-1)} \) is trivial.

In Definition 22.1, the filtered vector space \( E^{(k)} \) and the filtered group \( G_{k+1} \) depend on the graded Lie algebra \( \mathfrak{g}_- \) and the sequence of structure groups \( (G_0, \ldots, G_k) \) of \( Q^{(k)} \), and may be written more precisely as

\[
E^{(k)}(\mathfrak{g}_-, G_0, \ldots, G_k) \quad \text{and} \quad G_{k+1}(\mathfrak{g}_-, G_0, \ldots, G_k).
\]

Let us see more concretely what \( G_{k+1}(\mathfrak{g}_-, G_0, \ldots, G_k) \) is. Recall that \( G_0(\mathfrak{g}_-) \) is the group of automorphisms of \( \mathfrak{g}_- \) as graded Lie algebra, and \( \mathfrak{g}_0(\mathfrak{g}_-) \) is the Lie algebra of all derivation of \( \mathfrak{g}_- \) preserving the degree. If we take a Lie subgroup \( G_0 \subset G_0(\mathfrak{g}_-) \), we see that the Lie algebra \( \mathfrak{g}_1(\mathfrak{g}_-, \mathfrak{g}_0) \) of \( G_1(\mathfrak{g}_-, \mathfrak{g}_0) \) is abelian and can be identified as:

\[
\mathfrak{g}_1(\mathfrak{g}_-, \mathfrak{g}_0) = \bigoplus_{p<0} \text{Hom}(\mathfrak{g}_p, \mathfrak{g}_{p+1})
\]
and $G_1(g_-, G_0) = \exp g_1(g_-, g_0)$, where we view an element of $g_1(g_-, g_0)$ and $G_1(g_-, G_0)$ as a matrix in $\text{Hom}(g_- \oplus g_0, g_- \oplus g_0)$ modulo the filtration $F^2 \text{Hom}(g_- \oplus g_0, g_- \oplus g_0)$ and $\exp$ denote the usual exponential map of matrix.

Taking successively subgroups

$$G_i \subset G_i(g_-, G_0, \ldots, G_{i-1})$$

for $i = 0, 1, \ldots, k$

we get a “symbol” $(g_-, G_0, \ldots, G_k)$ of a geometric structure of order $k$. For $k \geq 1$, the Lie algebra $g_{k+1}(g_-, G_0, \ldots, g_k)$ of $G_{k+1}(g_-, G_0, \ldots, G_k)$ can be identifies as:

$g_{k+1}(g_-, G_0, \ldots, g_k) = (\oplus_{p=0}^k \text{Hom}(g_p, g_{p+k+1})) \oplus (\oplus_{i=0}^{k-1} \text{Hom}(g_i, g_k))$

and $G_{k+1}(g_-, G_0, \ldots, G_k)$ is .... Then $g_{k+1}(g_-, G_0, \ldots, g_k)$ and $G_{k+1}(g_-, G_0, \ldots, G_k)$ can be represented in matrix as illustrated below (Figure 1) for $g_- = g_{-2} \oplus g_{-1}$ and $k = 2$.

| $g_3(g_-, g_0, g_1, g_2)$ | $G_3(g_-, g_0, G_1, G_2)$ |
|---------------------------|-----------------------------|
| $g_2$ | $g_2$ | Id |
| $g_1$ | $g_1$ | Id |
| $g_0$ | $g_0$ | Id |
| $g_1$ | $\alpha_1^{-2}$ | $\beta_1^2$ | $\beta_2^2$ |
| $g_2$ | $\alpha_2^{-1}$ | $\beta_0^2$ | $\beta_2^2$ |

where $\alpha = \alpha_1^{-2} + \alpha_2^{-1} \in \oplus_{p=0}^k \text{Hom}(g_p, g_{p+k+1})$ and $\beta = \beta_0^2 + \beta_2^2 \in \oplus_{i=0}^{k-1} \text{Hom}(g_i, g_k)$.

**Figure 1.** $g_3(g_-, g_0, g_1, g_2)$ and $G_3(g_-, g_0, G_1, G_2)$

**Notations.** For simplicity of notation, we will write a geometric structure $Q^{(k)} = (Q^{(k)} \rightarrow Q^{(k-1)} \rightarrow \ldots \rightarrow Q^{(1)})$ (respectively, $E^{(k)}(g_-, g_0, \ldots, g_k)$ and $G_{k+1}(g_-, G_0, \ldots, G_k)$) simply as $Q^{(k)} \xrightarrow{G_k} Q^{(k-1)}$ (respectively, $E^{(k)}(G_k)$ and $G_{k+1}(G_k)$) or $Q^{(k)}$ (respectively, $E^{(k)}$ and $G_{k+1}$) when no confusion can arise. The Lie algebra $g_{k+1}(g_-, G_0, \ldots, g_k)$ of $G_{k+1}(g_-, G_0, \ldots, G_k)$ will be also written as $g_{k+1}(g_k)$ or $g_{k+1}$.

Similarly, we will write $\mathcal{F}^{(k+1)}Q^{(k)}$ and $\mathcal{J}^{(k+1)}Q^{(k)}$ simply as $\mathcal{F}^{(k+1)}Q^{(k)}$ and $\mathcal{J}^{(k+1)}Q^{(k)}$ when no confusion can arise.

### 2.2. Completed universal frame bundles.

For a geometric structure $Q^{(k)} \xrightarrow{G_k} Q^{(k-1)}$ we define $\mathcal{J}^{(\ell+1)}Q^{(k)}$ and $G_{\ell+1}(G_k)$ for $\ell \geq k$ inductively as

$$\mathcal{J}^{(\ell+1)}Q^{(k)} = \mathcal{J}^{(\ell+1)}\mathcal{J}^{(\ell)}Q^{(k)} \text{ and } G_{\ell+1}(G_k) = G_{\ell+1}(G_{\ell}(G_k)).$$

Then $\mathcal{J}^{(\ell+1)}Q^{(k)} \xrightarrow{G_{\ell+1}(G_k)} \mathcal{J}^{(\ell)}Q^{(k)} \rightarrow \ldots \rightarrow \mathcal{J}^{(k+1)}Q^{(k)} \rightarrow Q^{(k)} \rightarrow \ldots \rightarrow Q^{(1)}$ is a geometric structure of order $\ell$ for $\ell \geq k$. In this subsection we shall show that $\mathcal{J}^{(\ell+1)}Q^{(k)} \rightarrow \ldots \rightarrow Q^{(k)} \rightarrow Q^{(k-1)}$ is not only a step-wise principal fiber bundle but also a principal fiber bundle over $Q^{(k-1)}$ for $\ell \geq k$. 
Define $G$ by induction on the exact sequence

$$G_{k+1}(G_k) \to G_k \to 1$$

Next, let us show that the group $G_k$ acts on $\mathcal{S}(G_k)$, denoted $\mathcal{S}(G_k)$. This completes the inductive definition of $G_k$.

First, let us define the structure group $G^{(k+1)}(G_k)$ of $\mathcal{S}(G_k)$, where $G^{(k+1)}(G_k)$ is inductively given by $G^{(k+1)}(G_k)/\mathcal{F}_k$.

Note that for $\ell \geq k + 1$, the Lie algebra $g_\ell$ of $G\ell(G_k)$ is inductively given by

$$g_\ell = \text{Hom}(g_{\ell-1}, g_\ell \oplus \bigoplus_{i=0}^{\ell-1} g_i).$$

where $g_i$ denotes $g_i$ for $0 \leq i \leq k$ and $g_i(G_k)$ for $k + 1 \leq i \leq \ell - 1$. Set

$$E^{(\ell)}(g_k) := g_\ell \oplus g_0 \oplus \cdots \oplus g_k \oplus g_{k+1}(g_k) \oplus \cdots \oplus g_\ell(G_k).$$

Define $\tilde{G}^{(\ell+1)}(G_k)$ by the group of all filtration preserving isomorphisms $\alpha^{\ell+1} : E^{(\ell)}(g_k) \to E^{(\ell)}(g_k)$ which makes the following diagram commutative:

$$\begin{array}{cccccc}
0 & \to & g_\ell(g_k) & \to & E^{(\ell)}(g_k) & \to & E^{(\ell-1)}(g_k) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & g_\ell(g_k) & \to & E^{(\ell)}(g_k) & \to & E^{(\ell-1)}(g_k) & \to & 0
\end{array}$$

with $a^{\ell} = [\alpha^{\ell}] \in G^{(\ell)}(G_k)$. Here, $\alpha^{\ell}$ is the induced map from $\alpha^{\ell+1}$ and $[\alpha^{\ell}]$ denotes $\alpha^{\ell}/\mathcal{F}^{\ell+1}GL(E^{(\ell-1)})$. Note also that by the inductive assumption we have the following exact sequence:

$$1 \to G^{(k)}(G_k) \to G^{(\ell)}(G_k) \to G^{(\ell-1)}(G_k) \to 1$$

so that $a^{\ell} \in G^{(\ell)}(G_k)$ acts on $g_\ell(g_k)$, denote it by $\text{Ad}(a^{\ell})$. The group $G^{(\ell+1)}(G_k)$ is defined by the quotient

$$G^{(\ell+1)}(G_k) = \tilde{G}^{(\ell+1)}(G_k)/\mathcal{F}^{\ell+2}GL(E^{(\ell)}).$$

Then we get an exact sequence

$$1 \to G^{(\ell)}(G_k) \to G^{(\ell+1)}(G_k) \to G^{(\ell)}(G_k) \to 1.$$ 

This completes the inductive definition of $G^{(\ell)}(G_k)$.

Next, let us show that the group $G^{(\ell)}(G_k)$ acts on $\mathcal{S}(\ell)(G_k)$, where $G^{(\ell)}(G_k)$ is defined, we will show that we can define it for $\ell + 1$. For $\alpha^{\ell+1} \in G^{(\ell+1)}(G_k)$ and $\zeta^{\ell+1} \in \mathcal{S}(\ell+1)(G_k)$ we define the right action $\zeta^{\ell+1} \cdot \alpha^{\ell+1}$ by the following commutative diagram:

$$\begin{array}{cccc}
E^{(\ell)} & \to & E^{(\ell)} \times [\zeta^{\ell+1} \cdot a^{\ell+1}] Q^{(\ell)} \\
\downarrow & & \downarrow \quad \alpha^{\ell+1} \quad R_{a^{\ell+1}} \\
E^{(\ell)} & \to & E^{(\ell+1)} \times \mathcal{S}(\ell+1) Q^{(\ell)}
\end{array}$$

where $z^{\ell} = [\zeta^{\ell}]$ and $a^{\ell} = [\alpha^{\ell}]$ are projections of $\zeta^{\ell+1}$ and $\alpha^{\ell+1}$ to $\mathcal{S}(\ell)(G_k)$ and $G^{(\ell)}(G_k)$ respectively, and $z^{\ell} \cdot a^{\ell}$ is the associated action of $G^{(\ell)}(G_k)$ on $\mathcal{S}(\ell)(G_k)$. 


Since $G^{(\ell)}(G_k)$ acts on $\mathcal{F}^{(\ell)}Q^{(k)}$, we have
\[(R_{a\ell})_\ast \tilde{A}_{\alpha\ell} = \text{Ad}(a\ell)^{-1}A_{\alpha\ell}a\ell\]
for $A \in \mathfrak{g}^{(\ell)}(g_k)$. Therefore, we see that $\zeta^{\ell+1} : G^{(\ell+1)}Q^{(k)}$. It then follows that $\mathcal{F}^{(\ell+1)}(G_k)$ acts on $\mathcal{F}^{(\ell+1)}$. Passing to the quotients, we see that $G^{(\ell+1)}(G_k)$ acts on $\mathcal{F}^{(\ell+1)}Q^{(k)}$.

Hence we have proven the following:

**Proposition 2.1.** For a geometric structure $Q^{(k)} \xrightarrow{G_k} Q^{(k-1)}$, 
$\mathcal{F}^{(\ell+1)}Q^{(k)} \rightarrow Q^{(k-1)}$

is a principal fiber bundle over $Q^{(k-1)}$ with structure group $G^{(\ell+1)}(G_k)$.

**Definition 2.2.** Passing to the projective limit, we set 
$$
\mathcal{F}Q^{(k)} := \lim_{\leftarrow \ell} \mathcal{F}^{(\ell)}Q^{(k)} \text{ and } G(G_k) := \lim_{\leftarrow \ell} G^{(\ell)}(G_k) \text{ and } E(g_k) := \lim_{\leftarrow \ell} E^{(\ell)}(g_k).
$$

Then $\mathcal{F}Q^{(k)} \rightarrow Q^{(k-1)}$ is a principal bundle with structure group $G(G_k)$. We call $\mathcal{F}Q^{(k)}$ the completed universal frame bundle of $Q^{(k)}$ and $\mathcal{F}^{(\ell)}Q^{(k)}$ the universal frame bundle of $Q^{(k)}$ of order $\ell$ (Figure 2).

We remark that the completed frame bundle $\mathcal{F}Q^{(k)}$ is of infinite dimension. However, since it is the projective limit of a sequence of finite dimensional manifolds, we can deal with it almost equally as in the finite dimensional case. In this paper we will freely use standard terminologies concerning finite dimensional manifolds also for these infinite dimensional objects, such as Lie groups, fiber bundles, tangent spaces, smooth or analytic differential forms, absolute parallelism etc. The reader may refer to Section 2.1 (p.275–p.279) of [30] for a brief summary of this convention.

2.3. **Canonical Pfaff class and canonical Pfaff form.**

2.3.1. Here we give the definition of an isomorphism of geometric structures.

**Definition 2.3.** An isomorphism of geometric structures is defined inductively by the following conditions:

1. An isomorphism $\Phi : Q^{(-1)}(= (M, F)) \rightarrow Q^{(-1)}(= (M', F'))$ of geometric structures of order $-1$ is an isomorphism of filtered manifolds.

2. If $Q^{(k)}$ and $Q^{(k)}$ are geometric structures of order $k \geq -1$ of same type $(g_-, G_0, \cdots G_k)$ and if $\Phi : Q^{(k)} \rightarrow Q^{(k)}$ is an isomorphism of the geometric structures, then there is induced a bundle isomorphism $\mathcal{F}^{(k+1)}\Phi : \mathcal{F}^{(k+1)}Q^{(k)} \rightarrow \mathcal{F}^{(k+1)}Q^{(k)}$ which lifts $\Phi$.

3. Let $Q^{(k+1)}$ and $Q^{(k+1)}$ be geometric structures of order $k+1 \geq 0$ of type $(g_-, G_0, \cdots G_{k+1})$. A bijection $\Phi^{(k+1)} : Q^{(k+1)} \rightarrow Q^{(k+1)}$ is an isomorphism of the geometric structures if and only if there exists an isomorphism $\Phi^{(k)} : Q^{(k)} \rightarrow Q^{(k)}$ such that the lift $\mathcal{F}^{(k+1)}\Phi^{(k)}$ maps $Q^{(k+1)}$ to $Q^{(k+1)}$ and the restriction of $\mathcal{F}^{(k+1)}\Phi^{(k)}$ to $Q^{(k+1)}$ coincides with $\Phi^{(k+1)}$.

To complete the definition, we just indicate how to define the canonical lift $\mathcal{F}^{(k+1)}\Phi^{(k)} : \mathcal{F}^{(k+1)}Q^{(k)} \rightarrow \mathcal{F}^{(k+1)}Q^{(k)}$ for an isomorphism $\Phi^{(k)} : Q^{(k)} \rightarrow Q^{(k)}$: For $\zeta^{(k+1)} \in$
we get a well-defined map \( \hat{S}(k+1) \Phi(k) : \hat{S}(k+1)Q(k) \to \hat{S}(k+1)Q'(k) \). Then, passing to the quotient, we have the canonical lift \( \hat{S}(k+1) \Phi(k) : \hat{S}(k+1)Q(k) \to \hat{S}(k+1)Q'(k) \).

Now we proceed to the definition of the canonical Pfaff class.

The canonical Pfaff class \([\theta(k-1)]\) of a geometric structure \( Q^{(k)} \to Q^{(k-1)} \) is the equivalence class of the Pfaff form under the action of the group \( E^{k+1}GL(E^{(k-1)}) \). Recall that
for \( z^k \in Q^{(k)} \), there is a filtration preserving isomorphism \( \zeta^k : E^{(k-1)} \rightarrow T_{z^k}Q^{(k-1)} \) with \( z^k = \zeta^k/F^{k+1}GL(E^{(k-1)}) \), where \( z^{k-1} \) is the image of \( z^k \) by the projection map \( \pi : Q^{(k)} \rightarrow Q^{(k)} \). The composition

\[
\theta^{(k-1)} : T_{z^k}Q^{(k)} \xrightarrow{\pi^*} T_{z^{k-1}}Q^{(k-1)} \xrightarrow{\zeta^{k-1}} E^{(k-1)},
\]

depends on a choice of \( \zeta^k \). The canonical Pfaff class \([\theta^{(k-1)}]\) at \( z^k \) is the equivalent class of \( \theta^{(k-1)} \) by the action of \( F^{k+1}GL(E^{(k-1)}) \).

**Proposition 2.2.** Let \( Q^{(k)} \) and \( \overline{Q}^{(k)} \) be geometric structures of type (\( g_-; G_0, \ldots, G_k \)). If \( \Phi : Q^{(k)} \rightarrow \overline{Q}^{(k)} \) is an isomorphism, then \( \Phi \) preserves the canonical Pfaff classes \([\theta^{(k-1)}]\) and \([\overline{\theta}^{(k-1)}]\) of \( Q^{(k)} \) and \( \overline{Q}^{(k)} \), respectively, that is \( \Phi^*[\theta^{(k-1)}] = [\theta^{(k-1)}] \). Conversely, if \( G_0, \ldots, G_k \) are connected, then a diffeomorphism \( \Phi : Q^{(k)} \rightarrow \overline{Q}^{(k)} \) such that \( \Phi^*[\theta^{(k-1)}] = [\theta^{(k-1)}] \) is an isomorphism of the geometric structures.

**Proof.** The first statement is clear, i.e., if \( \Phi : Q^{(k)} \rightarrow \overline{Q}^{(k)} \) is an isomorphism, then \( \Phi \) preserves the canonical classes. For the second statement, we will use the induction on \( \ell \) (\( 0 \leq \ell \leq k \)) and show that if \( \Phi^{(\ell)} : Q^{(\ell)} \rightarrow \overline{Q}^{(\ell)} \) satisfies \( \Phi^{(\ell)}^*[\theta^{(\ell-1)}] = [\theta^{(\ell-1)}] \), then \( \Phi^{(\ell)} \) is an isomorphism.

First, let us show that it holds for \( \ell = 0 \). In fact, since the fibers of \( \overline{Q}^{(0)} \rightarrow \overline{Q}^{(-1)} \) are leaves of \( \overline{\theta}^{(-1)} = 0 \) and the fibers are connected, there exists \( \Phi^{(-1)} : Q^{(-1)} \rightarrow \overline{Q}^{(-1)} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
Q^{(0)} & \xrightarrow{\Phi^{(0)}} & \overline{Q}^{(0)} \\
\downarrow & & \downarrow \\
Q^{(-1)} & \xrightarrow{\Phi^{(-1)}} & \overline{Q}^{(-1)}. \\
\end{array}
\]

Let \( z^{(0)} \in \overline{Q}^{(0)} \), which may be viewed as an isomorphism \( z^{(0)} : g_- \rightarrow T_x\overline{Q}^{(-1)} \) of graded Lie algebras. We see that the inverse map \( (z^{(0)})^{-1} \) can be identified with the canonical class \([\overline{\theta}^{(-1)}]_{z^{(0)}} \). Then the assumption that \( \Phi^{(0)}^*[\overline{\theta}^{(-1)}] = [\theta^{(-1)}] \) implies that \( \Phi^{(-1)} \) is a filtration preserving map. Moreover, we see that the lift \( \mathcal{S}^{(0)}\Phi^{(-1)} : \mathcal{S}^{(0)}Q^{(-1)} \rightarrow \mathcal{S}^{(0)}\overline{Q}^{(-1)} \) coincides with \( \Phi^{(0)} \) on \( Q^{(0)} \).

Now assuming the statement valid for \( \ell - 1 \), we prove it for \( \ell \). By the assumption \( \Phi^{(\ell)}^*[\overline{\theta}^{(\ell-1)}] = [\theta^{(\ell-1)}] \), we see again that there exists \( \Phi^{(\ell-1)} \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
Q^{(\ell)} & \xrightarrow{\Phi^{(\ell)}} & \overline{Q}^{(\ell)} \\
\downarrow & & \downarrow \\
Q^{(\ell-1)} & \xrightarrow{\Phi^{(\ell-1)}} & \overline{Q}^{(\ell-1)}. \\
\end{array}
\]

Moreover, we see that \( \Phi^{(\ell-1)}^*[\overline{\theta}^{(\ell-2)}] = [\theta^{(\ell-2)}] \). Therefore, \( \Phi^{(\ell-1)} \) is an isomorphism by induction assumption.

Now take \( z^{(\ell)} \in Q^{(\ell)} \) and \( \overline{z}^{(\ell)} \in \overline{Q}^{(\ell)} \) with \( \overline{z}^{(\ell)} = \Phi^{(\ell)}(z^{(\ell)}) \), and let \( z^{(\ell-1)} \) and \( \overline{z}^{(\ell-1)} \) be their projections on \( Q^{(\ell-1)} \) and \( \overline{Q}^{(\ell-1)} \), respectively. Consider the following commutative
2.3.2. Now we will show that there is an $E(\mathfrak{g}_k)$-valued 1-form $\theta$ canonically defined on the completed frame bundle $\mathcal{F}Q(k)$ of $Q(k) \xrightarrow{\mathcal{G}_k} Q^{(k-1)}$, called the *canonical Pfaff form* on $\mathcal{F}Q(k)$.

For each $\ell$ define the canonical Pfaff class $[\theta^{(\ell-1)}]$ of $\mathcal{F}Q^{(k)} \rightarrow \mathcal{F}Q^{(\ell-1)}Q^{(k)}$ in a similar way as we define $[\theta^{(k-1)}]$. For a tangent vector $X \in T_zQ$, we can assign a sequence $\{v(i) \in E(i)\}$ such that

$$\langle \pi_{E(i)} \theta^{(j)}, \pi_{Q(j+1)}X \rangle = v(i)$$

for any representative $\theta^{(j)}$ of the canonical Pfaff class $[\theta^{(j)}]$ of $Q^{(j+1)}$ for any $j$ large enough, where $\pi_{E(i)}, \pi_{Q(j)}$ denote the projections on $E(i)$ and $Q(j)$, respectively, and $\pi_{E(i)} v^{(j)} = v^{(i)}$ for all $j \geq i$. We then set

$$\langle \theta, X \rangle := \lim_{i} v^{(i)}$$

and thus define a Pfaff form $\theta$ on $\mathcal{F}Q(k)$ taking values in $E$.

In fact, $\theta$ can be also defined in the following way. The universal frame bundle $\mathcal{F}Q(k)$ of order $\ell$ has a tangential filtration regular of type $E^{(\ell)}(\mathfrak{g}_k)$ and $\mathcal{F}^{(\ell+1)}Q(k)$ is a principal subbundle of the frame bundle of $\mathcal{F}Q(k)$, so that $\mathcal{F}^{(\ell+1)}Q(k)$ has an $E^{(\ell)}(\mathfrak{g}_k)$-valued one form, the restriction of the canonical one-form of the frame bundle on $\mathcal{F}Q(k)$, denoted by $\theta^{(\ell)}(\mathfrak{g}_k)$ and called the *canonical form* of $\mathcal{F}^{(\ell+1)}Q(k)$.

For a given $\ell$, there is a projection map $\mathcal{F}^{(\ell+m)}Q(k) \rightarrow \mathcal{F}^{(\ell+1)}Q(k)$ for sufficiently large $m$, so that we have a projection map $\mathcal{F}Q(k) \rightarrow \mathcal{F}^{(\ell+m)}Q(k) \rightarrow \mathcal{F}^{(\ell+1)}Q(k)$. Let $\theta^{(\ell)}(\mathfrak{g}_k)$ be the pull-back of the canonical form on $\mathcal{F}^{(\ell+1)}Q(k)$ under the projection map $\mathcal{F}Q(k) \rightarrow \mathcal{F}^{(\ell+1)}Q(k)$.

Passing to the projective limit, we set

$$\theta(\mathfrak{g}_k) := \lim_{\ell \to \ell} \theta^{(\ell)}(\mathfrak{g}_k).$$

Then $\theta(\mathfrak{g}_k)$ is an $E(\mathfrak{g}_k)$-valued one-form on $\mathcal{F}Q(k)$. 

\[
\begin{array}{c}
\begin{array}{c}
T_{z(\ell-1)}Q^{(\ell-1)} \xrightarrow{\Phi^{(\ell-1)}} T_{\mathcal{F}(\ell-1)}Q^{(\ell-1)} \\
\eta(\ell) \downarrow \downarrow \zeta(\ell) \\
E^{(\ell-1)} \quad E^{(\ell-1)},
\end{array}
\end{array}
\]

where we choose an $\eta(\ell)$ so that $[\eta^{(\ell)}] = z^{(\ell)}$ and define $\zeta(\ell)$ by the commutative diagram. Then the assumption that $\Phi^{(\ell)} \circ \Phi^{(\ell-1)} = \theta^{(\ell-1)}$ implies that $[\zeta^{(\ell)}] = \varpi^{(\ell)}$. This means that $\varpi^{(\ell)} = \mathcal{F}(\ell) \Phi^{(\ell-1)}(z^{(\ell)})$. Hence the restriction of $\mathcal{F}(\ell) \Phi^{(\ell-1)}$ to $Q^{(\ell)}$ is $\Phi^{(\ell)}$, which completes the induction. \qed

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]
Semi-canonical embeddings.

2.4. Semi-canonical embeddings.

**Proposition 2.3.** We use the notations $\mathcal{S}$, $\theta$, $E$, $\mathcal{G}$, for $\mathcal{S}Q^{(k)}$, $\theta(\mathfrak{g}_k)$, $E(\mathfrak{g}_k)$, $G(G_k)$, respectively. Then the canonical Pfaff form $\theta$ on $\mathcal{S}$ satisfies the following properties.

1. $\theta : T_z \mathcal{S} \to E$ is a filtration preserving isomorphism for every $z \in \mathcal{S}$.
2. $\text{gr} \theta_z : \text{gr} T_z \mathcal{S} \to \mathfrak{g}_-$ is a graded Lie algebra isomorphism for every $z \in M$ and $\text{gr} \theta_z : \text{gr} V_z \mathcal{S} \to \text{gr} F^0 E$ is the canonical isomorphism $\mathcal{S}$ for every $z \in \mathcal{S}$, where $V \mathcal{S}$ denotes the vertical tangent space of $\mathcal{S} \to M$.
3. $R^a \theta = a^{-1} \theta$ for $a \in \mathcal{G}$.

**Proof.** By the same argument as in the proof of Theorem 2.3.1 of [30], together with the properties:

(i) $\hat{\theta}(\mathfrak{g}) = A/F^{\ell+1}$ for $A \in \hat{\mathfrak{g}}^{\ell+1}(\mathfrak{g}_k)$;
(ii) $R^a \hat{\theta}(\ell) = a^{-1} \hat{\theta}(\ell)$ for $a \in \hat{G}^{\ell+1}(G_k)$,
we get the desired results.

**Theorem 2.1.** Let $P^{(k)}$ and $Q^{(k)}$ be geometric structures of order $k$ of type $(\mathfrak{g}_-, G_0, G_1, \ldots, G_k)$. Set $P^{(\ell)} = \mathcal{S}^{(\ell)} P^{(k)}$ and $Q^{(\ell)} = \mathcal{S}^{(\ell)} Q^{(k)}$ for $\ell \geq k + 1$.

1. If there is an isomorphism $\varphi^{(k)} : P^{(k)} \to Q^{(k)}$ of geometric structures, then it induces an isomorphism $\mathcal{S}^{(\ell)} \varphi^{(k)} : \mathcal{S}^{(\ell)} P^{(k)} \to \mathcal{S}^{(\ell)} Q^{(k)}$ of geometric structures, i.e., a diffeomorphism satisfying $(\mathcal{S}^{(\ell)} \varphi^{(k)})^* [\theta^{(\ell-1)}]_Q = [\theta^{(\ell-1)}]_P$ for all $\ell = k + 1, k + 2, \ldots, \infty$.

2. Conversely, assuming that $G_0, G_1, \ldots, G_k$ are connected, if there is a diffeomorphism $\Phi^{(i)} : P^{(i)} \to Q^{(i)}$ for some $i \geq 0$ such that $(\Phi^{(i)})^* [\theta^{(i-1)}]_Q = [\theta^{(i-1)}]_P$, then $\Phi^{(i)}$ induces an isomorphism $\varphi^{(i-1)} : P^{(i-1)} \to Q^{(i-1)}$ such that the canonical lift $\mathcal{S}^{(i)} \varphi^{(i-1)} : \mathcal{S}^{(i)} P^{(i-1)} \to \mathcal{S}^{(i)} Q^{(i-1)}$ sends $P^{(i)}$ onto $Q^{(i)}$ and its restriction $\mathcal{S}^{(i)} \varphi^{(i-1)}|_{P^{(i)}}$ to $P^{(i)}$ is $\Phi^{(i)}$.

In particular, the equivalence problem of geometric structure $Q^{(k)}$ reduces to the equivalence problem of the absolute parallelism $(\mathcal{S} Q^{(k)}, \theta)$ on the complete universal frame bundle $\mathcal{S} Q^{(k)}$.

**Proof.** Use Proposition 2.2.

2.4. Semi-canonical embeddings.
Proposition 2.4 (W-canonical embedding). Let \( Q^{(k+1)} \xrightarrow{G_{k+1}} Q^{(k)} \) be a geometric structure of order \( k + 1 \). A choice of a direct sum decomposition
\[
g_{k+1}(g_k) = g_{k+1} \oplus W
\]
determines (filtration preserving) injective homomorphisms
\[
G(G_{k+1}) \xrightarrow{\iota_{W}} G(G_k), \quad E(G_{k+1}) \xrightarrow{\iota_{W}} E(G_k),
\]
and a principal fiber bundle embedding
\[
\mathcal{J}Q^{(k+1)} \xrightarrow{\iota_{W}} \mathcal{J}Q^{(k)}
\]
satisfying \( \iota_{W}^* \theta_{\mathcal{J}Q^{(k)}} = \iota_{W} \circ \theta_{\mathcal{J}Q^{(k+1)}} \).

Proof. Fix a complementary subspace \( W \) such that \( g_{k+1}(g_k) = g_{k+1} \oplus W \). Note that \( g_{k+1}(g_k) = \text{Hom}(g_k, E^{(k)}(g_k))_{k+1} \oplus (\oplus_{i=0}^{k-1} \text{Hom}(g_i, g_k)) \).

We claim that for each \( \ell \geq k \) there is an embedding \( \mathcal{J}(\ell+1)Q^{(k+1)} \xrightarrow{\iota_{\ell+1}} \mathcal{J}(\ell+1)Q^{(k)} \) such that the following diagram
\[
\begin{array}{ccc}
\mathcal{J}(\ell+1)Q^{(k+1)} & \xrightarrow{\iota_{\ell+1}} & \mathcal{J}(\ell+1)Q^{(k)} \\
\downarrow & & \downarrow \\
\mathcal{J}(\ell)Q^{(k+1)} & \xrightarrow{\iota_{\ell}} & \mathcal{J}(\ell)Q^{(k)}
\end{array}
\]
is commutative, as well as splittings:
\[
g_{\ell+1}(g_k) = g_{\ell+1}(g_k) \oplus W_{\ell+1} \text{ and } E^{(\ell+1)}(g_k) = E^{(\ell+1)}(g_{k+1}) \oplus V_{\ell+1},
\]
where \( V_{\ell+1} = \oplus_{i=k+1}^{\ell+1} W_i \), such that for \( \xi^{\ell+1} \in \mathcal{J}(\ell+1)Q^{(k+1)} \), \( \iota^{(\ell)}([\xi^{\ell+1}])[V_{\ell+1}] \) is vertical with respect to \( \mathcal{J}(\ell+1)Q^{(k)} \rightarrow Q^{(k)} \).

For \( \ell = k \), we have already had
\[
Q^{(k+1)} \rightarrow \mathcal{J}(k+1)(Q^{(k)})
\]
\[
\begin{array}{c}
\downarrow \theta_{g_{k+1}} \\
\downarrow \text{identity}
\end{array}
\]
\[
Q^{(k)} \rightarrow Q^{(k)}
\]
and \( g_{k+1}(g_k) = g_{k+1} \oplus W \), so that we can take \( W_{k+1} = W \).

Suppose that we have embeddings as in the above claim for \( \ell = m \geq k \). We will show that there are such embeddings for \( \ell = m + 1 \).
Since $E$ is satisfying that $g$

Therefore, there is a subspace $W$ as $E, \theta$

By the induction assumption there is an embedding $\mathcal{I}^{(m+1)}Q^{(k+1)} \xrightarrow{\iota^{(m+1)}} \mathcal{I}^{(m+1)}Q^{(k)}$
and decompositions $g_{m+1}(G_k) = g_{m+1}(G_k) \oplus W_{m+1}$ and $E^{(m+1)}(G_k) = E^{(m+1)}(G_k) \oplus V_{m+1}$, where $V_{m+1} = \oplus_{i=k+1}^{m+1} W_i$. Note that

\begin{align*}
g_{m+2}(G_{k+1}) &= \text{Hom}(g_-, E^{(m+1)}(G_{k+1}))_{m+2} \oplus \oplus_{i=0}^{m} \text{Hom}(g_i, g_{m+1}(G_{k+1})) \\
g_{m+2}(G_k) &= \text{Hom}(g_-, E^{(m+1)}(G_k))_{m+2} \oplus \oplus_{i=0}^{m} \text{Hom}(g_i, g_{m+1}(G_k))
\end{align*}

Therefore, there is a subspace $W_{m+2}$ such that $g_{m+2}(G_k) = g_{m+2}(G_{k+1}) \oplus W_{m+2}$.

Since $E^{(m+2)}(G_k) = E^{(m+1)}(G_k) \oplus g_{m+2}(G_k)$ and $E^{(m+2)}(G_k) = E^{(m+1)}(G_k) \oplus g_{m+2}(G_k)$, we have $E^{(m+2)}(G_k) = E^{(m+2)}(G_{k+1}) \oplus V_{m+2}$, where $V_{m+2} = V_{m+1} \oplus W_{m+2} = \oplus_{i=k+1}^{m+2} W_i$.

For $\zeta^{m+2} \in \mathcal{I}^{(m+2)}Q^{(k+1)}$ define $\tilde{\zeta}(\zeta^{m+2})$ by

\[\tilde{\zeta}((\zeta^{m+2})(X) = \begin{cases} \iota^{(m+1)}_\zeta X & \text{for } X \in E^{(m+1)}(G_{k+1}) \\ \iota^{(m+1)}_\zeta X & \text{for } X \in V_{m+1}. \end{cases}\]

2.5. Structure equations and structure functions.

Let $Q^{(k)} \xrightarrow{\iota_k} Q^{(k-1)}$ is a geometric structure of order $k$ and $\mathcal{J}Q^{(k)}$ be its completed universal frame bundle. We write $E(G_k), \theta(G_k), G(G_k), g(G_k) = \oplus G_i(G_k)$ simply as $E, \theta, \mathcal{G}, \mathcal{G} = \oplus \mathcal{G}_i$.

2.5.1. By Proposition 2.3 (1) there is a unique function $\gamma : \mathcal{J}Q^{(k)} \to \text{Hom}(\wedge^2 E, E)$ satisfying that

\[d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0.\]

We call $\gamma$ the structure function of $\mathcal{J}Q^{(k)}$.

**Proposition 2.5.** For $z \in \mathcal{J}Q^{(k)}$, $a \in \mathcal{G}$ and $X, Y \in E$

\[\gamma(za)(X, Y) = a^{-1}\gamma(z)(aX, aY).
\]

In other words, $R_a^*\gamma = \rho(a)^{-1}\gamma$ for $a \in \mathcal{G}$. Here, the action of $\mathcal{G}$ on $\text{Hom}(\wedge^2 E, E)$ is given by $(\rho(a)\varphi)(X, Y) = a\varphi(a^{-1}X, a^{-1}Y)$ for $a \in \mathcal{G}$ and $X, Y \in E$. 

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Proof. It follows from Proposition 2.3 (3). Indeed, \(0 = d\theta(z) + \frac{1}{2}\gamma(z) \circ (d(\theta(z)), \theta(z)) = d(a^{-1}\theta(z)) + \frac{1}{2}\gamma(z)(a^{-1}\theta(z), a^{-1}\theta(z))\) because \(\theta(z) = a^{-1}\theta\). Thus
\[
d\theta(z) + \frac{1}{2}a\gamma(z)(a^{-1}\theta, a^{-1}\theta) = 0.
\]
Hence \(a\gamma(z)(a^{-1}, a^{-1}) = \gamma(z)(, )\). \qed

Proposition 2.6 (Bianchi identity).
\[
\gamma(\theta, \theta) = d\gamma(\theta, \theta)
\]
Proof. By differentiating \(d\theta + \frac{1}{2}\gamma(\theta, \theta) = 0\) and replacing \(d\theta\) by \(-\frac{1}{2}\gamma(\theta, \theta)\), we get
\[
\gamma(\theta, \theta) = d\gamma(\theta, \theta).
\]
\qed

2.5.2. Let us consider various pieces of the structure function \(\gamma\) of \(\mathcal{S}Q^{(k)}\), by decomposing the vector space \(\text{Hom}(\wedge^2 E, E)\) in which \(\gamma\) takes values.

Let \(\pi_{ab}\) denote the projection \(\text{Hom}(\wedge^2 E, E) \rightarrow \text{Hom}(\overline{\mathfrak{g}}_a \wedge \overline{\mathfrak{g}}_b, \overline{\mathfrak{g}}_c)\) as well as the image \(\pi_{ab} \text{Hom}(\wedge^2 E, E)\) (we shall assume the latter for a projection and its image). We have the natural notion of degree: \(\alpha \in \text{Hom}(\wedge^2 E, E)\) is of homogeneous degree \(r\) if \(\alpha(\overline{\mathfrak{g}}_a \wedge \overline{\mathfrak{g}}_b) \subset \overline{\mathfrak{g}}_{a+b+r}\) for all \(a, b \in \mathbb{Z}\). But we shall also use a modified degree which is well adapted to our setting: \(\alpha \in \text{Hom}(\wedge^2 E, E)\) is of modified degree \(s\) if \(\alpha(\overline{\mathfrak{g}}_a \wedge \overline{\mathfrak{g}}_b) \subset \overline{\mathfrak{g}}_{a+b+s}\) for all \(a, b \in \mathbb{Z}\), where we set \(\bar{a} = \min\{a, -1\}\). Then
\[
\pi(r) = \sum_{a,b} \pi_{ab}^{a+b+r}, \quad \pi(s) = \sum_{a,b} \pi_{ab}^{a+b+s}
\]
are projections to the elements of homogeneous degree \(r\), of modified degree \(s\), respectively. Then the image of \(\pi(r)\) is \(\text{Hom}(\wedge^2 E, E)_r\).

We define the corresponding filtrations \(\{F^i\}\) and \(\{\bar{F}^j\}\) by
\[
F^i = \sum_{r \geq i} \pi(r), \quad \bar{F}^j = \sum_{s \geq j} \pi(s)
\]
and we identify
\[
\text{Hom}(\wedge^2 E, E) = \text{Hom}(\wedge^2 E, E)/F^{\ell+1} \quad \text{with} \quad \pi(\ell) := \sum_{r \leq \ell} \pi(r)
\]
\[
\text{Hom}(\wedge^2 E, E) = \text{Hom}(\wedge^2 E, E)/\bar{F}^{m+1} \quad \text{with} \quad \pi(m) := \sum_{s \leq m} \pi(s).
\]

Another projection is made by the direct sum decomposition:
\[
\text{Hom}(\wedge^2 E, E) = \text{Hom}(\wedge^2 \mathfrak{g}, E) \oplus \text{Hom}(\mathfrak{g} \wedge E, E) \oplus \text{Hom}(\wedge^2 E, E)
\]
where \(E = \bigoplus_{a \geq 0} \overline{\mathfrak{g}}_a\). We denote the projections to each component by \(\pi_I, \pi_{II}, \pi_{III}\), respectively. Noting that all projections commute, we write
\[
\pi_{II} \circ \pi[m] = \pi_{II}^{[m]}, \quad \pi_I \circ \pi(r) \circ \pi[s] = \pi_{II}^{(r,s)}, \text{ etc.}
\]
Applying various projections, we define
\[
\gamma^c_{ab} = \pi_{ab}^c \circ \gamma, \quad \gamma^m_I = \pi_{i}^m \circ \gamma, \quad \gamma_{II(r)}[s] = \pi_{II(r)}[s] \circ \gamma, \text{ etc.}
\]
It being said, we have
Proposition 2.7.

(1) \( \gamma_{I(d)} = 0 \) for \( d < 0 \) and \( \gamma_{I(0)} \) coincides with the Lie bracket of \( \mathfrak{g}_- \).
(2) \( \gamma_{II(d)} = 0 \) for \( d < 0 \) and \( \gamma_{II(0)} \) coincides with the action of \( \mathfrak{g}_- \) on \( E \).
(3) \( \gamma_{IIIab} = 0 \) if \( a, b \geq 0 \) and \( c < \max\{a, b\} - 1 \).

Proof. By Proposition 2.4 there is an embedding
\[
(\mathcal{S}Q^{(k)}, \theta(\mathfrak{g}_k)) \xrightarrow{\iota} (\mathcal{S}Q^{(0)}, \theta(\mathfrak{g}_0)),
\]
and (filtration preserving) injective morphisms
\[
E(\mathfrak{g}_k) \xhookrightarrow{\iota} E(\mathfrak{g}_0), \quad G(G_k) \xrightarrow{\iota} G(G_0)
\]
such that \( \iota^* \theta(\mathfrak{g}_0) = \iota \circ \theta(\mathfrak{g}_k) \). Since (1) and (2) hold for \( \mathcal{S}Q^{(0)} \), so do for \( \mathcal{S}Q^{(k)} \).

For the statement (3), note that
\[
\overline{\mathfrak{g}}_a \subset \text{Hom}(\mathfrak{g}_-, \mathfrak{g}_- \oplus (\otimes_{i=0}^{a-1} \overline{\mathfrak{g}}_i))_a \oplus \text{Hom}(\otimes_{i=0}^{a-2} \overline{\mathfrak{g}}_i, \overline{\mathfrak{g}}_{a-1}).
\]
From \( R_s^{} \theta = a^{-1} \theta \) for \( a \in \overline{G} \), it follows that \( L_A^{} \theta = -\rho(A) \theta \) for \( A \in \mathfrak{g} \). Since \( L_A^{} \theta = A_\ell^{} d\theta + d(A_\nu^{} \theta) \), we have \( -A_\ell^{} d\theta = d(\theta(A)) + \rho(A) \theta \). By Proposition 2.3 (2), for \( A \in \overline{\mathfrak{g}}_a \), \( \theta(A) \mod F^{a+1} \) and thus \( -A_\ell^{} d\theta = \rho(A) \theta \mod F^{a+1} \). \( \square 

2.5.3. On account of Proposition 2.7 we define a subspace \( \overline{\pi} = \text{Hom}(\wedge^2 E, E) \) by the condition:
\[
\pi_{I(1)} = \pi_{II(1)} = 0 \quad \text{and} \quad \pi_{IIIab} = 0 \quad \text{for} \quad a, b, c \quad \text{such that} \quad a, b \geq 0 \quad \text{and} \quad c < \max\{a, b\} - 1.
\]
Recall that our group \( \overline{G} \) acts on \( E \) and it naturally induces an action on \( \text{Hom}(\wedge^2 E, E) \), denoted by \( \rho \). It is easy to see that \( \overline{G} \) leaves invariant the filtration \( \{\overline{F}^j\} \) and the subspace \( \text{Hom}(\wedge^2 E, E) \). Hence \( \overline{G} \) acts on \( \text{Hom}(\wedge^2 E, E)/\overline{F}^{k+1} =: \text{Hom}(\wedge^2 E, E)[k] \).

Lemma 2.1. The action of \( F^{k+1} \overline{G} \) on \( \text{Hom}(\wedge^2 E, E)[k] \) is trivial for \( k \geq k + 1 \).

Proof. We may assume that the group \( \overline{G} \) is connected. Then it suffices to prove: For \( A \in F^{k+1} \overline{\mathfrak{g}}_a \),
\[
\pi_{|m|}(\rho(A) \gamma) = 0 \quad \text{for} \quad m \leq \ell.
\]
Let us verify this by showing that its projections by \( \pi_I, \pi_{II}, \pi_{III} \) vanishes.

(1) For \( X \in \overline{\mathfrak{g}}_p \) and \( Y \in \overline{\mathfrak{g}}_q \), where \( p, q < 0 \), we have
\[
\pi_{|m|}(\rho(A) \gamma)(X, Y) = \pi_{p+q+m}(\rho(A) \gamma)(X, Y) = \pi_{p+q+m}(A \gamma(X, Y) - \gamma([A, X], Y) - \gamma(X, [A, Y])) = 0 \quad \text{because} \quad \pi_{p+q+m} F^{p+q+\ell+1} = 0 \quad \text{for} \quad m \leq \ell.
\]

(2) For \( X \in \overline{\mathfrak{g}}_p \) and \( Y \in \overline{\mathfrak{g}}_q \), where \( p < 0 \) and \( a \geq 0 \), we have
\[
\pi_{|m|}(\rho(A) \gamma)(X, Y) = \pi_{p-1+m}(\rho(A) \gamma)(X, Y) - \gamma(A \cdot X, Y) - \gamma(X, A \cdot Y)) = 0 \quad \text{because} \quad \pi_{p-1+m} (F^{\ell+p} + F^{\ell+p} + F^{\ell+p}) = 0.
\]

(3) For \( X \in \overline{\mathfrak{g}}_a \) and \( Y \in \overline{\mathfrak{g}}_b \), where \( a, b \geq 0 \), we have
\[
\pi_{|m|}(\rho(A) \gamma)(X, Y) = \pi_{-2+m}(\rho(A) \gamma)(X, Y) = 0 \quad \text{because} \quad \pi_{-2+m} F^{\ell+1} = 0.
\]
Here, we use the fact that \( F^{\ell+1} \overline{\mathfrak{g}}_a \cdot E_+ \subset F^\ell \). \( \square \)
Then we have immediately

**Proposition 2.8.** The structure function $\gamma$ of $\mathcal{Q}^{(k)}$ induces a $G$-equivariant map

$$\gamma^{[\ell]} : \mathcal{I}^{(\ell)}Q^{(k)} \to \tilde{\text{Hom}}(\wedge^2 E, E)^{[\ell]}$$

for $\ell \geq k$.

**Proposition 2.9.** If $\varphi^{(k)} : P^{(k)} \to Q^{(k)}$ preserves the canonical Pfaff classes $[\theta^{(k-1)}]_P$ and $[\theta^{(k-1)}]_Q$, then the canonical lift $\mathcal{I}^{(\ell)}\varphi^{(k)} : \mathcal{I}^{(\ell)}P^{(k)} \to \mathcal{I}^{(\ell)}Q^{(k)}$ preserves the structure functions $\gamma^{[\ell]}_P$ and $\gamma^{[\ell]}_Q$ for any $\ell \geq k + 1$.

**Proof.** An isomorphism $\varphi^{(k)} : P^{(k)} \to Q^{(k)}$ lifts to an isomorphism $\mathcal{I}^{(\ell)}\varphi^{(k)} : \mathcal{I}^{(\ell)}P^{(k)} \to \mathcal{I}^{(\ell)}Q^{(k)}$ for any $\ell \geq k + 1$, and thus an isomorphism $\mathcal{I}^{(\ell)}\varphi^{(k)} : \mathcal{I}P^{(k)} \to \mathcal{I}Q^{(k)}$. Therefore, $\mathcal{I}\varphi^{(k)}$ preserves the structure functions, $\gamma_P$ and $\gamma_Q$, and thus $\mathcal{I}^{(\ell)}\varphi^{(k)}$ preserves the structures function $\gamma^{[\ell]}_P$ and $\gamma^{[\ell]}_Q$. \qed

Since $\gamma^{[k+1]} = \sum_{s \leq k+1} \gamma^{[s]}$, the map $\gamma^{[k+1]} : \mathcal{I}^{(k+1)}Q^{(k)} \to \tilde{\text{Hom}}(\wedge^2 E, E)^{[k+1]}$ induces a function

$$\gamma^{[k+1]} : \mathcal{I}^{(k+1)}Q^{(k)} \to \tilde{\text{Hom}}(\wedge^2 E, E)^{[k+1]}$$

which will play an important role in the following section.

### 3. W-normal Step prolongations

#### 3.1. Proper geometric structures.

**Definition 3.1.** A geometric structure $Q^{(k)}$ of type $(g_-, G_0, \ldots, G_k)$ is said to be proper if $E^{(k)} := g_- \oplus g_0 \oplus \cdots \oplus g_k$ forms a truncated transitive graded Lie algebra (TTGLA), which we denote by $g[k]$.

Let us explain more precisely what the above definition means. Recall that $(g_-, g_0, \ldots, g_k)$ satisfies

- $g_0 \subset g_0(g_-)$ and
- $g_i \subset g_1(g_{i-1}) = (\oplus_{p<0} \text{Hom}(g_p, g_{p+i})) \oplus (\oplus_{a=0}^{i-2} \text{Hom}(g_{a}, g_{i-1}))$ for $1 \leq i \leq k$.

Saying the $E^{(k)} = g_- \oplus g_0 \oplus \cdots \oplus g_k$ becomes a truncated transitive graded Lie algebra requires the following conditions to be satisfied: for $0 \leq i \leq k$,

1. $g_i \subset \oplus_{p<0} \text{Hom}(g_p, g_{p+i})$;
2. $g_i \subset \text{Prol}(g[i-1])$, that is, $E^{(i-1)} = g_- \oplus g_0 \oplus \cdots \oplus g_{i-1}$ is a truncated transitive graded Lie algebra $g[i-1]$ and $g_i$ is a subspace of the $i$-th component of $\text{Prol}(g[i-1])$;
3. $E^{(i)} = g_- \oplus g_0 \oplus \cdots \oplus g_i$ is closed under the truncated Lie bracket defined in $\text{Trun}^{(i)}(\text{Prol}g[i-1])$ and then forms a truncated transitive graded Lie algebra $g[i]$.

Note that a geometric structure $Q^{(0)}$ of order 0 is always proper. If $Q^{(k)}$ is a $W$-normal prolongation of $Q^{(0)}$ (that we are going to study), then $Q^{(k)}$ is proper.

By (3), $g_i$ $(0 \leq i \leq k)$ acts trivially on $E^{(i-1)} := g_0 \oplus \cdots \oplus g_{i-1}$ and by (2), $g_i$ is contained in the prolongation of $g_- \oplus g_0 \oplus \cdots \oplus g_{i-1}$.

By this observation we have the following.
Proposition 3.1. If $Q^{(k)}$ is proper, then the structure function $\gamma$ of the universal frame bundle $Q^{(k)}$ satisfies that $\gamma_{01}^{c}$ vanishes for $0 \leq a, b \leq k$ and $c < \max\{a, b\}$.

3.2. $W$-normal reductions.

Let $Q^{(k)}$ be a proper geometric structure of type $\mathfrak{g}[k]$, $Q^{(k)}$ the universal frame bundle, and $\gamma$ its structure function. Let $E$, $\mathcal{G}$, $\mathfrak{g}$, $\theta$, $\gamma$, $\gamma^{[k+1]}$, $\gamma_{[k+1]}$ be given as in Section 2.5. We will often write alternatively $\kappa, \tau, \sigma$ for $\gamma I, \gamma II, \gamma III$. Then $\kappa_{[k+1]}$, $\tau_{[k+1]}$, $\sigma_{[k+1]}$ are functions on $Q^{(k+1)}Q^{(k)}$ defined as follows.

\[
\kappa_{[k+1]} = \gamma_{[k+1]}^{w \otimes g_{-}}, \quad \tau_{[k+1]} = \sum_{\ell+a=b-1, a \geq 0, \ell \geq 0} \gamma_{[a]}^{w \otimes g_{-}}, \quad \sigma_{[k+1]} = \sum_{\ell+a+b=k-1, a, b \geq 0} \gamma_{[a]}^{w \otimes g_{-}}.
\]

Recalling the construction of the universal frame bundles, we see easily that Lie algebra $\overline{q}_{k+1}$ of the structure group of the principal bundle $Q^{(k+1)}Q^{(k)} \to Q^{(k)}$ is

\[
q_{k+1} + r_{k+1},
\]

where $q_{k+1} = \text{Hom}(g_{-} \oplus (\oplus_{i=0}^{k} \mathfrak{g}_{i}))_{k+1}$ and $r_{k+1} = \text{Hom}\left((\oplus_{i=0}^{k-1} g_{i}, g_{k})\right)$ (Section 2.1).

For $\varphi \in q_{k+1}$ and $\psi \in r_{k+1}$ denote by $a_{\varphi}$ and $a_{\psi}$ the elements in $\mathcal{G}_{k+1}$ corresponding to $\varphi$ and $\psi$. Then for $X \in \mathfrak{g}_{p}$ ($p < 0$) and $A \in \mathfrak{g}_{a}$ ($a \geq 0$),

\[
a_{\varphi}X \equiv X + \varphi X \mod F^{p+k+2},
\]

\[
a_{\psi}A \equiv A + \psi A \mod F^{k+1}.
\]

Proposition 3.2. The structure function $\gamma_{[k+1]} : Q^{(k+1)}Q^{(k)} \to \text{Hom}(\wedge^{2}E, E)_{k+1}$ satisfies that, for $\varphi \in q_{k+1}$ and $\psi \in r_{k+1}$, we have

1. \(\kappa_{[k+1]}(za_{\varphi}) = \kappa_{[k+1]}(z) + \partial \varphi\)
2. \(\tau_{[k+1]}(za_{\psi})(A, \cdot) = \tau_{[k+1]}(z)(A, \cdot)\)
3. \(\tau_{[k+1]}(za_{\psi})(A, \cdot) = \tau_{[k+1]}(z)(A, \cdot) + \partial \psi(A),\) where $A \in (\oplus_{i=0}^{k} g_{i})$.

Proof. (1) Let $X \in \mathfrak{g}_{p}$ and $Y \in \mathfrak{g}_{q}$, where $p < 0$ and $q < 0$. From $\gamma(za_{\varphi})(X, Y) = a_{\varphi}^{-1}(z) \gamma(a_{\varphi}X, a_{\varphi}Y)$ (Proposition 2.5) it follows that

\[
\gamma(za_{\varphi})(X, Y) = \gamma(z)(X, Y) + \gamma(z)(X, \varphi Y) + \gamma(z)(\varphi X, Y) + \gamma(z)(\varphi X, \varphi Y) + \ldots
\]

\[
- \varphi(\gamma(z)(X, Y) + \gamma(z)(X, \varphi Y) + \gamma(z)(\varphi X, Y) + \gamma(z)(\varphi X, \varphi Y) + \ldots) + \ldots
\]

By comparing $g_{p+q+k+1}$-component, we get

\[
\kappa_{[k+1]}(za_{\varphi})(X, Y) = \kappa_{[k+1]}(z)(X, Y) + [X, \varphi Y] + [\varphi X, Y] - \varphi[X, Y].
\]

Here, we use that $\gamma_{0}^{w \otimes g_{-}}$ and $\gamma_{0}^{w \otimes g_{-}}$ are given by the Lie bracket $[,]$. It follows that $\kappa_{[k+1]}(za_{\varphi}) = \kappa_{[k+1]}(z) + \partial \varphi$.

(2) Let $Y \in \mathfrak{g}_{q}$ for some $q < 0$ and $A \in \mathfrak{g}_{a}$ for some $0 \leq a \leq k - 1$. As in (1), from $\gamma(za_{\varphi})(A, Y) = a_{\varphi}^{-1}(z) \gamma(za_{\varphi}A, a_{\varphi}Y)$ it follows that

\[
\gamma(za_{\varphi})(A, Y) = \gamma(z)(A, Y) + \gamma(z)(A, \varphi Y) + \gamma(z)(\varphi A, Y) + \gamma(z)(\varphi A, \varphi Y) + \ldots
\]

\[
- \varphi(\gamma(z)(A, Y) + \gamma(z)(A, \varphi Y) + \gamma(z)(\varphi A, Y) + \gamma(z)(\varphi A, \varphi Y) + \ldots) + \ldots
\]

By comparing the $g_{q+k+1}$-component, we get

\[
\tau_{[k+1]}(za_{\varphi})(A, Y) = \tau_{[k+1]}(z)(A, Y),
\]

\[
\tau_{[k+1]}(za_{\psi})(A, Y) = \tau_{[k+1]}(z)(A, Y) + [\psi(A), Y].
\]
Here, we use Proposition 3.1 in the first identity. Indeed, $\gamma(z)(A, \varphi Y)$ belongs to $F^{\max(a,q+k+1)}$ if $q+k+1 \geq 0$, and to $F^{a+q+k+1}$ if $q+k+1 < 0$, and thus their $g_{q+k}$-component is zero. \hfill \Box

Now let us make the key procedure of $W$-normal reduction consisting of $\gamma_{II}$-reduction ($\tau$-reduction) and $\gamma_{I}$-reduction ($\kappa$-reduction).

**Proposition 3.3.** Let $g[k] = \oplus_{i \leq k} g_i$ be a truncated transitive graded Lie algebra and $g = \oplus_{i} g_i$ be its prolongation. Fix subspaces $W^1_k$ and $W^2_{k+1}$ such that

\[
\begin{align*}
\text{Hom}(g_-, g)_k &= W^1_k \oplus \partial g_k \\
\text{Hom}(\wedge^2 g_-, g)_{k+1} &= W^2_{k+1} \oplus \partial \text{Hom}(g_-, g)_{k+1}.
\end{align*}
\]

Let $Q^{(k)} \xrightarrow{G_k} Q^{(k-1)}$ be a proper geometric structure of type $g[k]$ on a filtered manifold $M$. Let $\mathcal{S}^{(k+1)}Q^{(k)}$ be the universal frame bundle of order $k + 1$. Let $\kappa_{[k+1]}$ and $\tau_{[k+1]}$ be the structure functions of $\mathcal{S}^{(k+1)}Q^{(k)}$.

1. ($\gamma_{II}$-reduction) Define a subbundle $\mathcal{S}_W^{(k+1)}Q^{(k)}$ of $\mathcal{S}^{(k+1)}Q^{(k)}$ by

\[
\mathcal{S}_W^{(k+1)}Q^{(k)} = \left\{ z \in \mathcal{S}^{(k+1)}Q^{(k)} : \tau_{[k+1]}(z)(A, \cdot) \in W^1_k \text{ for any } A \in \bigoplus_{i=0}^{k-1} g_i \right\}.
\]

Then $\mathcal{S}_W^{(k+1)}Q^{(k)} \to Q^{(k)}$ is a principal $\Omega_{k+1}$-subbundle of $\mathcal{S}^{(k+1)}Q^{(k)} \to Q^{(k)}$, where $\Omega_{k+1}$ is the maximal subgroup of $G_{k+1}$ whose Lie algebra is $a_{k+1} \subset g_{k+1}$.

2. ($\gamma_{I}$-reduction) Define a subbundle $\mathcal{S}_W^{(k+1)}Q^{(k)}$ of $\mathcal{S}^{(k+1)}Q^{(k)}$ by

\[
\mathcal{S}_W^{(k+1)}Q^{(k)} = \left\{ z \in \mathcal{S}^{(k+1)}Q^{(k)} : \kappa_{[k+1]}(z) \in W^2_{k+1} \right\}.
\]

Then $\mathcal{S}_W^{(k+1)}Q^{(k)} \to Q^{(k)}$ is a principal $G_{k+1}$-subbundle of $\mathcal{S}_W^{(k+1)}Q^{(k)} \to Q^{(k)}$, where $G_{k+1}$ is the maximal subgroup of $G_{k+1}$ whose Lie algebra is $g_{k+1} \subset g_{k+1}$.

Consequently, $\mathcal{S}_W^{(k+1)}Q^{(k)} \to Q^{(k)}$ is a proper geometric structure of order $k + 1$ of type $g[k+1] = g[k] \oplus g_{k+1}$.

**Proof.** (1) For $A \in E_+^{(k-1)} = \oplus_{i=0}^{k-1} g_i$, consider

\[
\tau_{[k+1]}(A, \cdot) : \mathcal{S}^{(k+1)}Q^{(k)} \to \text{Hom}(g_-, g)_k.
\]

Take any $z \in \mathcal{S}^{(k+1)}Q^{(k)}$ and write

\[
\tau_{[k+1]}(z)(A, \cdot) = \partial \psi(A) + \alpha(A)
\]

where $\psi \in \text{Hom}(E_+^{(k-1)} \times g_k)$ and $\alpha \in \text{Hom}(E_+^{(k-1)} \times W^1_k)$. Then by Proposition 3.2 (2)(b), we have $\tau_{[k+1]}(za_{\varphi})(A, \cdot) = \alpha(A) \in W^1_k$. Therefore, $\mathcal{S}_W^{(k+1)}Q^{(k)}$ is nonempty.

By Proposition 3.2 (2)(a), if $z$ is an element of $\mathcal{S}_W^{(k+1)}Q^{(k)}$, then so is $za_{\varphi}$ for any $\varphi \in a_{k+1}$. On the other hand, if both $z$ and $za_{\varphi}$, where $\psi \in a_{k+1}$, are contained in $\mathcal{S}^{(k+1)}Q^{(k)}$, then, by Proposition 3.2 (2)(b) we have

\[
-\partial \psi(A) = \tau_{[k+1]}(z)(A, \cdot) - \tau_{[k+1]}(za_{\varphi})(A, \cdot) \in W^1_k.
\]

Since $W^1_k \cap \partial g_k = 0$, $\partial \psi(A)$ is zero, and hence $\psi(A)$ is zero. Therefore, $\mathcal{S}_W^{(k+1)}Q^{(k)}$ intersects $\{ za_{\varphi} : \psi \in a_{k+1} \}$ only at one point. Therefore, $\mathcal{S}_W^{(k+1)}Q^{(k)} \to Q^{(k)}$ is a principal subbundle of $\mathcal{S}^{(k+1)}Q^{(k)} \to Q^{(k)}$ whose structure group has Lie algebra $a_{k+1}$.\hfill 27
(2) Consider
\[ \kappa_{[k+1]} : \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \rightarrow \text{Hom}(\wedge^2 g_-, g)_{k+1}. \]
Take any \( z \in \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \) and write
\[ \kappa_{[k+1]}(z) = \partial \varphi_{k+1} + \alpha \]
where \( \varphi_{k+1} \in \text{Hom}(g_-, g)_{k+1} \) and \( \alpha \in W^2_{k+1} \). Then by Proposition 3.2 (1), we have \( \kappa_{[k+1]}(z a - \varphi_{k+1}) = \alpha \in W^2_{k+1} \). Therefore, \( \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \) is nonempty.

Furthermore, if \( z \in g_{k+1} \) are contained in \( \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \), then by Proposition 3.2 (1), we have \( \partial \varphi = \kappa_{[k+1]}(z a - \varphi_{k+1}) - \kappa_{[k+1]}(z) \in W^2_{k+1} \). Therefore, Since \( \partial g_{k+1} \cap W^2_{k+1} = 0 \), \( \partial \varphi \) is zero. Thus \( \varphi \) is contained in the kernel of \( \partial : g_{k+1} \rightarrow \text{Hom}(\wedge^2 g_-, g)_{k+1} \), which is \( g_{k+1} \). Consequently, \( \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \) is a principal subbundle of \( \mathcal{S}_{W^1}^{(k+1)} (Q^{(k)}) \) whose structure group has Lie algebra \( g_{k+1} \). □

**Proposition 3.4.** We use the same notations as in Proposition 3.3

1. If there is an isomorphism \( \varphi^{(k)} : P^{(k)} \rightarrow Q^{(k)} \) of proper geometric structures \( P^{(k)} \) and \( Q^{(k)} \), then there is a unique isomorphism \( \mathcal{S}_{W^1}^{(k+1)} \varphi_{k+1} : \mathcal{S}_{W^1}^{(k+1)} P^{(k)} \rightarrow \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \) prolonging \( \varphi^{(k)} \).

2. Conversely, if \( G_0, G_1, \ldots, G_k \) are connected and there is a diffeomorphism \( \Phi^{(k+1)} : \mathcal{S}_{W^1}^{(k+1)} P^{(k)} \rightarrow \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \) such that \( (\Phi^{(k+1)})^* [\theta^{(k)}]_P = [\theta^{(k)}]_Q \), then there exists an isomorphism \( \varphi^{(k)} : P^{(k)} \rightarrow Q^{(k)} \) such that the restriction of the lift \( \mathcal{S}_{W^1}^{(k+1)} \varphi_{k+1} \) to \( \mathcal{S}_{W^1}^{(k+1)} P^{(k)} \) is \( \Phi^{(k+1)} \).

**Proof.** (1) If \( \varphi^{(k)} : P^{(k)} \rightarrow Q^{(k)} \) is an isomorphism of proper geometric structures \( P^{(k)} \) and \( Q^{(k)} \), then there is an isomorphism \( \mathcal{S}_{W^1}^{(k+1)} \varphi_{k+1} : \mathcal{S}_{W^1}^{(k+1)} P^{(k)} \rightarrow \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \) prolonging \( \varphi^{(k)} \). Hence \( \mathcal{S}_{W^1}^{(k+1)} \varphi_{k+1} \) preserves the canonical Pfaff classes \( [\theta^{(k)}]_P \) and \( [\theta^{(k)}]_Q \) of \( \mathcal{S}_{W^1}^{(k+1)} P^{(k)} \) and \( \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \), and thus preserves their structure functions \( \gamma_P^{(k+1)} \) and \( \gamma_Q^{(k+1)} \) (Proposition 2.9). Now that \( \mathcal{S}_{W^1}^{(k+1)} P^{(k)} \) (\( \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \), respectively) is defined by the condition that \( \kappa_P^{[k+1]}(z) \in W^2_{k+1} \) and \( \tau_P^{[k+1]}(z)(A, \cdot) \in W^1_{k+1} \) \((\kappa_Q^{[k+1]}(z) \in W^2_{k+1} \) and \( \tau_Q^{[k+1]}(z)(A, \cdot) \in W^1_{k+1} \), respectively), \( \mathcal{S}_{W^1}^{(k+1)} \varphi_{k+1} \) maps \( \mathcal{S}_{W^1}^{(k+1)} P^{(k)} \) onto \( \mathcal{S}_{W^1}^{(k+1)} Q^{(k)} \) isomorphically.

(2) follows from Proposition 2.2 □

**Theorem 3.1.** Let \( g[k] \) be a truncated transitive graded Lie algebra and \( g \) be its prolongation. Fix a set of subspaces \( W = \{ W^1, W^2, \ldots \} \) such that
\[
\text{Hom}(g_-, g)_{\ell} = W_{\ell}^2 \oplus \partial g_{\ell}
\]
\[
\text{Hom}(\wedge^2 g_-, g)_{\ell+1} = W_{\ell+1}^2 \oplus \partial \text{Hom}(g_-, g)_{\ell+1}.
\]
Then to every proper geometric structure \( Q^{(k)} \) of type \( g[k] \) there is canonically associated a series of proper geometric structures \( \{ \mathcal{S}_{W^{(\ell)}}^{(k)} Q^{(k)} \}_{\ell=k+1, \ldots, \infty} \) of type \( g[\ell] \) which is uniquely determined by the condition that \( \mathcal{S}_{W^{(\ell)}}^{(k)} Q^{(k)} \) is a maximal proper geometric structure prolonging \( Q^{(k)} \) whose structure function \( \gamma_{[\ell]}^{(k)} \) satisfies
\[
\kappa_{[\ell]}^{(k)}(z) \in W_{\ell}^2 \text{ and } \tau_{[\ell]}^{(k)}(z) \in \text{Hom}(E_{+}^{(\ell-1)}, W_{\ell-1}^1).
\]

Furthermore, we have the following.
(1) If there is an isomorphism \( \varphi^{(k)} : P^{(k)} \to Q^{(k)} \) of proper geometric structures \( P^{(k)} \) and \( Q^{(k)} \), then there is a unique isomorphism \( \mathcal{S}_W^{(\ell)} \varphi^{(k)} : \mathcal{S}_W^{(\ell)} P^{(k)} \to \mathcal{S}_W^{(\ell)} Q^{(k)} \) prolonging \( \varphi^{(k)} \) for \( \ell = k + 1, \ldots, \infty \).

(2) Conversely, assume that \( G_0, G_1, \ldots, G_k \) are connected. For \( \ell = k + 1, \ldots, \infty \), if there is a diffeomorphism \( \Phi^{(\ell)} : \mathcal{S}_W^{(\ell)} P^{(k)} \to \mathcal{S}_W^{(\ell)} Q^{(k)} \) such that \( (\Phi^{(\ell)})^* [\theta^{(\ell-1)}]_Q = [g^{(\ell-1)}]_P \), then there exists an isomorphism \( \varphi^{(k)} : P^{(k)} \to Q^{(k)} \) such that the restriction of the lift \( \mathcal{S}_W^{(\ell)} \varphi^{(k)} \) to \( \mathcal{S}_W^{(\ell)} P^{(k)} \) is \( \Phi^{(\ell)} \).

Therefore, the equivalence problem of the proper geometric structure \( Q^{(k)} \) of type \( g[k] \) reduces to that of the absolute parallelism of \( (\mathcal{S}_W Q^{(k)}, \theta) \).

We call \( \mathcal{S}_W^{(\ell)} Q^{(k)} \) the \( W \)-normal step prolongation of \( Q^{(k)} \) of order \( \ell \) for \( \ell \geq k + 1 \). We denote the projective limit \( \lim_{\ell \to \infty} \mathcal{S}_W^{(\ell)} Q^{(k)} \) by \( \mathcal{S}_W Q^{(k)} \) and call it the \( W \)-normal complete step prolongation of \( Q^{(k)} \).

Proof. Apply Proposition 3.3 inductively to get \( \mathcal{S}_W^{(\ell)} Q^{(k)} \) for \( \ell \geq k + 1 \) such that

\[
\kappa_{[\ell]}(z) \in W^2_{\ell} \quad \text{and} \quad \tau_{[\ell]}(z) \in \text{Hom}(E_{+}^{(\ell-1)}, W_{\ell-1}^1).
\]

The statements (1) and (2) follows from Proposition 3.4

3.3. Proper geometric structure of finite type.

Definition 3.2. A proper geometric structure \( Q^{(k)} \) of type \( g[k] = g_- \oplus g_0 \oplus \cdots \oplus g_k \) \((k \geq 0)\) is said to be of finite type if the prolongation \( \text{Prol} (g[k]) \) is finite dimensional, that is, there exists an integer \( k_0 \geq 0 \) such that \( g_i = 0 \) for \( i > k_0 \), where we set \( \text{Prol} (g[k]) = \oplus g_i \).

Corollary 3.1. The equivalence problem of the proper geometric structure \( Q^{(k)} \) of finite type reduces to that of the absolute parallelism \( (\mathcal{S}_W Q^{(k)}, \theta) \) of finite dimension.

Corollary 3.2. The automorphism group \( \text{Aut}(Q^{(k)}) \) of a proper geometric structure \( Q^{(k)} \) of finite type is a finite dimensional Lie group. The dimension of \( \text{Aut}(Q^{(k)}) \) is less than or equal to the dimension of \( \text{Prol} (g[k]) \).

It was Cartan who invented the ingenious idea to study the equivalence problem of geometric structures through prolongation and reduction of bundles of frames adapted to geometric structures. It seems that he had a very general and deep ideas, which were, however, of heuristic nature, and carried applications in various concrete problems. Afterwards, the theoretical aspects were developed, to certain extent, rigorously in modern mathematics. A modern formulation of the step prolongation is due to Singer and Sternberg (35) and the generalization to the nilpotent geometry is due to Tanaka (38). Their main results were Corollary 3.1 and Corollary 3.2.

4. Fundamental identities

From now on, we will write alternatively \( \kappa, \tau, \sigma \) for \( \gamma_I, \gamma_{II}, \gamma_{III} \). We denote by the same notation their restrictions to \( \mathcal{S}_W^{(k+1)} Q^{(k)} \). Recall that we introduce the following notation to keep the information on the degree of structure functions.
Definition 4.1. Set \( \tau_{(\ell)[k+1]} \) to be the component of \( \tau \) of degree \( \ell \) and modified degree \( k + 1 \), so that

\[
\tau_{(\ell)[k+1]} := \gamma_{\ell} \big|_{g_{k-\ell} \wedge g_{-}}
\]

for \( 0 \leq \ell \leq k \). We say that \( \tau_{[k+1]} \) is flat if \( \tau_{(\ell)[k+1]} \) is zero for all \( \ell > 0 \), so that \( \tau_{[k+1]} = \tau_{(0)[k+1]} \).

Similarly, we set \( \sigma_{(\ell)[k+1]} \) to be the component of \( \sigma \) of degree \( \ell \) and modified degree \( k + 1 \), so that

\[
\sigma_{(\ell)[k+1]} := \sum_{i+j=k-1-\ell} \gamma_{\ell} \big|_{g_i \wedge g_j}
\]

for any \( \ell \in \mathbb{Z} \). We say that \( \sigma_{[k+1]} \) is flat if \( \sigma_{(\ell)[k+1]} \) is zero for any \( \ell \neq 0 \) and \( \sigma_{(0)[k+1]} \) is given by the Lie bracket [ , ].

We define

\[
\kappa_{[k+1]} := \sum_{0 \leq i \leq k+1} \kappa_{[i]}
\]

\[
\tau_{[k+1]} := \sum_{1 \leq i \leq k+1} \tau_{[i]}
\]

\[
\sigma_{[k+1]} := \sum_{2 \leq i \leq k+1} \sigma_{[i]}
\]

We say that \( \tau_{[k+1]} \) (\( \sigma_{[k+1]} \), respectively) is flat if \( \tau_{[i]} \) (\( \sigma_{[i]} \), respectively) is flat for all \( i \leq k + 1 \).

In this section we will prove a recursive formula for \( \kappa_{[k]}, \tau_{[k]}, \) and \( \sigma_{[k]} \).

Theorem 4.1 (Fundamental identities).

1. For \( X \in g_x, Y \in g_y, \) and \( Z \in g_z, \) where \( x, y, z < 0, \) and for a nonnegative integer \( k, \) we have

\[
(\partial \kappa_{[k]})(X, Y, Z) = \mathcal{G}_{X,Y,Z} \left\{ \sum_{d_1 + d_2 = k, d_1, d_2 > 0} \left\{ \kappa_{[d_1]}(\kappa_{[d_2]}(X, Y, -), Z) + \tau_{(d_1)[k+x+y+1]}(\kappa_{[d_2]}(X, Y, +), Z) \right\} - D_X \kappa_{[k+x]}(Y, Z) \right\}.
\]

2. For \( X \in g_x, Y \in g_y, \) and \( A \in g_a, \) where \( x, y < 0 \) and \( a \geq 0, \) and for \( k = d+a+1, \) where \( d \) is a nonnegative integer, we have

\[
(\partial \tau_{(d)[k]}(A, \cdot, \cdot))(X, Y) \]

\[
= (D_A \kappa_{[k-1]} + \rho(A)\kappa_{[d]})(X, Y) - \mathcal{A}_{X,Y,\tau_{(d)[k+x]}([A, X, +], Y) + [\kappa_{[d]}(X, Y, +), A]} - \sigma_0(\kappa_{[d]}(X, Y, +), A) \]

\[
- \sum_{\substack{-a \leq \delta_1 < d \text{ and } \delta_2 \neq 0 \\text{ or } \delta_2 \leq d+a-k-1 \\text{ and } \delta_1 = 0}} \left\{ \tau_{(d_1)[k-1]}(\kappa_{[d_2]}(X, Y, -), A) + \sigma_{(\delta_1)[k+x+y+1]}(\kappa_{[d_2]}(X, Y, +), A) \right\}
\]

\[
+ \mathcal{A}_{X,Y} \sum_{d_1+d_2=d, d_1, d_2 > 0} \left\{ \kappa_{[d_1]}(\tau_{(d_2)[k-d_1]}(A, X, -), Y) + \tau_{(d_1)[k+x]}(\tau_{(d_2)[k-d_1]}(A, X, +), Y) \right\}
\]

\[
+ \mathcal{A}_{X,Y} D_Y \tau_{(d+y)[k+y]}(A, X).
\]
(3) For $A_a \in \mathfrak{g}_a$, $B \in \mathfrak{g}_b$, $X \in \mathfrak{g}_x$, where $a, b \geq 0$ and $x < 0$, and for $k = d + a + b + 2$, where $d$ is an integer, we have

$$
[\sigma_{(d|k)}(A,B),X] = \mathcal{A}_{A,B} \sum_{d_1 + d_2 = d} \left\{ \sigma_{(d_1)|k+x-d_2}(A,\tau_{(d_2)}[k_1-(d_1+a)](B,X)_+) + \tau_{(d_1)}[k_1-(d_2+b)](A,\tau_{(d_2)}[k_1-(d_1+a)](B,X)_-) \right\}
$$

$$
+ \sum_{\delta_1 + \delta_2 = d} \tau_{(\delta_1)}[k_1-1](X,\sigma_{(\delta_2)}[k_1-\delta_1](A,B))
$$

$$
+ D_X \sigma_{(d+x)}[k+x](A,B) + \mathcal{A}_{A,B} DA \tau_{(d+a)}[k_1-1](B,X).
$$

(4) For $A \in \mathfrak{g}_a$, $B \in \mathfrak{g}_b$, $C \in \mathfrak{g}_c$, where $a, b, c \geq 0$, and for $k = d + a + b + c + 3$, where $d$ is an integer, we have

$$
\mathcal{S}_{A,B,C} \left\{ \sum_{d_1 + d_2 = d} \sigma_{(d_1)}[k_1-1] \left[ \sigma_{(d_2)}[k_1-c-d_1](A,B),C \right] - D_C \sigma_{(d+c)}[k_1-1](A,B) \right\} = 0.
$$

Here, $\mathcal{S}_{X,Y,Z} F(X,Y,Z)$ denotes the cyclic sum $F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y)$ and $\mathcal{A}_{X,Y} G(X,Y)$ denotes the alternating sum $G(X,Y) - G(Y,X)$. Also, we define $\kappa(,)_+ := \max\{\kappa(,),0\}$ and $\kappa(,)_- := \min\{\kappa(,),0\}$, so that $\kappa(,)=\kappa(,)_+ + \kappa(,)_-$, and define $\tau(,)_\pm$ and $\sigma(,)_\pm$ similarly.

Proof. Recall the Bianchi identity:

$$
\gamma \circ \gamma - D\gamma = 0.
$$

(1) Let $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$ and $Z \in \mathfrak{g}_z$ with $x, y, z < 0$. The $g_{k+x+y+z}$-component of $(\gamma \circ \gamma - D\gamma)(X,Y,Z)$, where $k$ is any positive integer, is

$$
\mathcal{S}_{X,Y,Z} \left\{ \sum_{i+j=k} \gamma_i(\gamma_j(X,Y),Z) - \sum_{-i+j=k} D_X \gamma_j(Y,Z) \right\}
$$

$$
= \mathcal{S}_{X,Y,Z} \left\{ \gamma_0(\gamma_k(X,Y),Z) + \gamma_k(\gamma_0(X,Y),Z) \right\}
$$

$$
+ \mathcal{S}_{X,Y,Z} \left\{ \sum_{0<i,j<k} \gamma_i(\gamma_j(X,Y),Z) - \sum_{-i+j=k} D_X \gamma_j(Y,Z) \right\}.
$$

The 1st line is

$$
\mathcal{S}_{X,Y,Z} \{\gamma_0(\gamma_k(X,Y),Z) + \gamma_k(\gamma_0(X,Y),Z)\} = \mathcal{S}_{X,Y,Z} \{[\gamma_k(X,Y),Z] + \gamma_k([X,Y],Z)\}
$$

$$
= -(\partial \kappa_{[k]})(X,Y,Z)
$$
and the 2nd line is
\[
\mathcal{S}_{X,Y,Z} \left\{ \sum_{d_1+d_2=k \atop d_1,d_2>0} \left\{ \kappa_{d_1} \left( \kappa_{d_2}(X,Y)\right) - D_X \kappa_{k+x}(Y,Z) \right\} - \mathcal{S}_{X,Y,Z} \left\{ \sum_{d_1+d_2=k \atop d_1,d_2>0} \left\{ \kappa_{d_1} \left( \kappa_{d_2}(X,Y)\right) - D_X \kappa_{k+x}(Y,Z) \right\} \right\},
\]
from which we get the desired identity.

(2) Let \( X \in g_x,Y \in g_y \) and \( A \in g_a \) with \( x, y < 0 \) and \( a \geq 0 \). The \( g_{d+x+y+a} \)-component of \( (\gamma - \gamma - D\gamma)(X,Y,A) \), where \( d \) is any integer, is
\[
\sum_{\delta_1+\delta_2=d} \gamma_{\delta_1} \left( \gamma_{\delta_2}(X,Y), A \right) + \mathcal{S}_{X,Y} \sum_{d_1+d_2=d} \gamma_{d_1} \left( \gamma_{d_2}(A,X), Y \right) + \{ D_A \gamma_{d+a}(X,Y) + \mathcal{S}_{X,Y} D_Y \gamma_{d+y}(A,X) \}
\]

Each line can be written as follows.

- The 1st line \( = \gamma_0(\gamma_{d}(X,Y), A) + \gamma_d([X,Y], A) + \mathcal{S}_{X,Y} [\gamma_{d}(A,X), Y] + \gamma_d([A,X], Y) \)
- \( = \partial_{\mathcal{S}}(\delta^1_d) \{ (A,\gamma)(X,Y) - (\rho(A) \kappa_{\delta_d})(X,Y) + \sigma_0(\kappa_{\delta_d}(X,Y)) + A \}
+ \mathcal{S}_{X,Y} \tau_{\mathcal{S}_{\delta_d}}([A,X], Y), \)

where \( \delta^1_d = d + a + 1 = k \) and \( \delta^2_d = d + a + 1 = k + x \). Here, we use the formula
\[
(\rho(A) \kappa_{[k-a-1]})(X,Y) = [A, \kappa_{[k-a-1]}(X,Y)] - \kappa_{[k-a-1]}([A,X],Y) - \kappa_{[k-a-1]}(X,[A,Y]).
\]

- The 2nd line \( = \sum_{\delta_1+\delta_2=d \atop -a \leq \delta_1 < 0 \text{ and } \delta_2 \neq 0} \gamma_{\delta_1} \left( \gamma_{\delta_2}(X,Y), A \right) + \mathcal{S}_{X,Y} \sum_{d_1+d_2=d \atop 0 < d_1 < d} \gamma_{d_1} \left( \gamma_{d_2}(A,X), Y \right)
+ \mathcal{S}_{X,Y} \sum_{d_1+d_2=d \atop 0 < d_1 < 0} \left\{ \tau_{\mathcal{S}_{\delta_d}}(\kappa_{\delta_d}(X,Y), A) + \sigma_{\mathcal{S}_{\delta_d}}(\kappa_{\delta_d}(X,Y), A) \right\}
\]

where \( \delta^1_d = \delta_1 + a < d + a = k - 1 \), \( \delta^2_d = \delta_1 + \delta_2 + x + y + z + 2 < k + x + y + 1 \),
\( \delta^2_d = d_2 + a + 1 < k \).

- The 3rd line \( = D_A \kappa_{k-1}(X,Y) + \mathcal{S}_{X,Y} D_Y \tau_{(d+y)(d+y)}(A,X), \)

where \( (d + y)^d = d + y + a + 1 = k + y. \)
Taking the sum of all terms in three lines, we get the identity (2).

(3) Let \( A \in g_a, B \in g_b \) and \( X \in g_x \), where \( a, b \geq 0 \) and \( x < 0 \). For \( d \in \mathbb{Z} \), set \( k = d + a + b + 2 \). The \( g_{d+a+b+2} \)-component of \( (\gamma \circ \gamma - D\gamma)(A, B, X) \) is

\[
\sum_{\delta_1 + \delta_2 = d} \gamma_{\delta_1} (\gamma_{\delta_2}(A, B), X) + \mathcal{A}_{A, B} \sum_{d_1 + d_2 = d} \gamma_{d_1} (\gamma_{d_2}(B, X), A) \\
+ \{D_X \gamma_{d+x}(A, B) + \mathcal{A}_{A, B} D_B \gamma_{d+b}(X, A)\}
\]

= \sum_{\delta_1 + \delta_2 = d} \gamma_{\delta_1} (\gamma_{\delta_2}(A, B), X)

+ \sum_{\delta_1 + \delta_2 = d} \gamma_{\delta_1} (\gamma_{\delta_2}(A, B), X) + \mathcal{A}_{A, B} \sum_{d_1 + d_2 = d} \gamma_{d_1} (\gamma_{d_2}(B, X), A)

+ \{D_X \gamma_{d+x}(A, B) + \mathcal{A}_{A, B} D_B \gamma_{d+b}(X, A)\}.

Each line can be written as follows.

- The 1st line = \([\gamma_d(A, B), X]\)
- The 2nd line = \([\sigma(d)[k](A, B), X]\), where \( d + b + 1 = k - 1 - a < k \)
- The 3rd line = \(D_X \sigma_{d+x}[k+x](A, B) + \mathcal{A}_{A, B} D_A \tau_{d+a}[k-1](B, X)\).

Here, we use Proposition 2.7 to get the bounds \( 0 < \delta_1 \leq d + \min(a, b) \) and \( 0 \leq d_2 \leq d + a \).

For example,

- if \(-\delta_2 = \delta_1 - d > \min(a, b)\), then \(\sigma_{(d)[k-\delta_1]}(A, B) = 0\) by Proposition 2.7 (3).
- if \(-d_1 = d_2 - d > a\), then \(\sigma_{(d_1)[k+x]}(\cdot, A) = 0\) by Proposition 2.7 (3) and \(\tau_{(d_1)[k-1-(d_2+b)]}(\cdot, A) = 0\) by Proposition 2.7 (2). In fact, the latter vanishes if \(-d_1 > 0\).

This completes the proof of Theorem A.1 (3).

(4) Use

\[
\mathcal{G}_{A, B, C} \left\{ \sum_{d_1 + d_2 = d} \sigma_{(d_1)[d_1]}(\sigma_{(d_2)[d_2]}(A, B), C) - D_C \sigma_{(\delta)[\delta]}(A, B) \right\} = 0.
\]

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Now for $d \in \mathbb{Z}$, set $k = d + a + b + c + 3$. Put $u = a + b + d_2$ and $w = a + b + c + d$. Then $d_1 = w - u - c \geq -c$ and $d_2 = u - a - b \geq - \min\{a, b\}$. Furthermore, we have

\[
\begin{align*}
d_1^2 &= w + 2 - d + a + b + c + 2 \\
      &= k - 1 \\
d_2^2 &= u + 2 - d_2 + a + b + 2 \\
      &= k - d_1 - c - 1 \\
      &= k - 1 - (c + d_1) \\
      &\leq k - 1 \\
\delta &= w - a - b \\
      &= d + c \\
\delta^2 &= w + 2 \\
      &= k - 1.
\end{align*}
\]

This completes the proof of Theorem 4.1. □

Let us see more concretely the fundamental identities for lower orders.

**Order 0.** We see that $\kappa_0$ is the bracket of $g_-$ and the identity (1) turns out to be

\[\partial \kappa_0 = 0\]

which says that $\kappa_0$ satisfies the Jacobi identity.

Next recall that $\tau[k]$ is the $\text{Hom}(g_+, \text{Hom}(g_-, g)_{k-1})$-component of $\gamma$ where $g_+ = \bigoplus_{i \geq 0} g_i$, so that the identity (2) is trivial. The structure function $\sigma[k]$ is the $\text{Hom}(\wedge g_+, g_{k-1})$-component of $\gamma$, and therefore $\sigma[0] = 0$ and the identity (3) is trivial.

**Order 1.** We have

\[\partial \kappa_1 = 0\]

because the term $D \kappa$ vanishes in this case.

The identity (2) becomes

\[\partial \tau_1(A_0, \cdot) = D_{A_0} \kappa_0 + \rho(A_0) \kappa_0 \text{ for } A_0 \in g_0.\]

Therefore, $\partial \tau_1(A_0, \cdot) = 0$. We see also that the right hand side of the above identity vanishes. We see also that $\sigma_1 = 0$.

**Order 2.** In this case we have

\[
\partial \kappa_2 = \kappa_1 \circ \kappa_1 - \sum_{p < 0} D_{\pi_p} \kappa_{2+p} \quad \text{(*)1}
\]

where

\[
\kappa_1 \circ \kappa_1(X,Y,Z) := \mathcal{S}_{X,Y,Z} \kappa_1(\kappa_1(X,Y), Z)
\]

\[
(D_{\pi_p} \kappa_{2+p})(X,Y,Z) := \mathcal{S}_{X,Y,Z} D_{\pi_p} \kappa_{2+p}(Y,Z).
\]

For $\tau_2$ we have

\[
\partial \tau_2(A_0, \cdot) = D_{A_0} \kappa_1 + \rho(A_0) \kappa_1 \text{ for } A_0 \in g_0. \quad \text{(*)2}
\]
We have also
\[ \partial \tau_2(A_1, \cdot) = D_{A_1} \kappa_1 + \rho(A_1) \kappa_0 \] for \( A_1 \in g_1 \).
But \( \tau_2(A_1, X) = [A_1, X] \) and thus the both sides vanish.

We remark that the identities \( *1 \) and \( *2 \) appeared already in [10] in his style. Singer-Sternberg gave its interpretation in [36] especially when the structure function is constant.

Theorem 4.1 has many important implications. We give here as corollaries some of them which follow immediately from it and will be used in later sections.

**Corollary 4.1.**

1. \( \partial \kappa_{[1]} = 0 \).
2. For \( k \geq 2 \), if \( \kappa_{[k-1]} \) is flat, then \( \partial \kappa_{[k]} = 0 \) and \( \partial \tau_{[k]}(A, \cdot) = 0 \) for \( A \in g_a \) where \( 0 \leq a \leq k \).

**Proof.** It follows from Theorem 4.1 (1) and (2). \( \square \)

**Corollary 4.2.** If the structure function \( \tau_{[k-1]} \) is flat for some integer \( k \), then so is \( \sigma_{[k]} \).

**Proof.** Note first that \( \sigma_{[1]} \) is flat and \( \sigma_{[2]} = \sigma_{(0)[2]} \). Applying Theorem 4.1 (3) for \( d = 0 \) and \( a = b = 0 \), we have
\[ [\sigma_{(0)[2]}(A, B), X] = [A, [B, X]] - [B, [A, X]] = [[A, B], X]. \]
Thus \( \sigma_{(0)[2]}(A, B) = [A, B] \), which shows that \( \sigma_{[2]} \) is flat.

Now assume that \( k \geq 3 \) and \( \tau_{[k+1]} \) is flat. Then by induction, we deduce that \( \sigma_{[k-1]} \) is flat.

Let \( A \in g_a, B \in g_b \) and \( X \in g_x \) with \( a, b \geq 0 \) and \( x < 0 \) and put \( d = k - (a + b + 2) \), to which we apply Theorem 4.1 (3). Then we see that \( [\sigma_{(d)[k]}(A, B), X] \) is written as a sum of the following forms of terms:
\[ \sigma_{(d_1)[d_1]} \circ \tau_{(d_2)[d_2]} \circ \tau_{(d_3)[d_3]} \circ \tau_{(d_4)[d_4]} \circ \tau_{(d_5)[d_5]} \circ \sigma_{(d_6)} \]
where \( d_1 + d_2 = \delta_1 + \delta_2 = d \) and \( \delta_1 > 0 \) and \( d_1^1, d_2^1, d_3^1, d_4^1, d_5^1, \delta_1^1 < k \), as well as vanishing terms:
\[ D_X \sigma_{[m]}, \ D_A \tau_{[i]}, \ D_B \tau_{[j]}, \]
where \( i, j, m < k \).

Among them non-vanishing terms can occur only from \( \sigma_{(0)} \circ \tau_{(0)} \) and \( \tau_{(0)} \circ \sigma_{(0)} \), from which it follows that
\[ \sigma_{(d)[k]}(A, B) = 0 \] if \( d \neq 0 \)
and
\[ [\sigma_{(0)[k]}(A, B), X] = [A, [B, X]] - [B, [A, X]], \]
that is, \( \sigma_{(0)[k]}(A, B) = [A, B] \). Therefore, \( \sigma_{[k]} \) is flat. \( \square \)

**Corollary 4.3.** If \( \tau_{[k-1]} \) is flat, then
\[ \partial \tau_{[k]}(A, \cdot))(X, Y) = (\rho(A) \kappa_{[k-1]} + D_A \kappa_{[k-1]}) (X, Y) \]
for \( A \in g_a, X \in g_x, v \in g_y \) with \( 0 \leq a \leq k - 2 \) and \( x, y < 0 \).
Proof. By Theorem 4.1 (2) and Corollary 4.2 we get the desired result.

**Corollary 4.4.** Let $k \geq 2$. If $\tau^{[k-1]}$ is constant, then

1. $\sigma^{[k]}$ is constant;
2. $\sigma^{[d]} = 0$ for $d < 0$;
3. $\sigma^{[k]}(A,B) = [A,B]$ for $A \in g_a, B \in g_b$, where $a \geq 0, b \geq 0$ and $a + b \leq k - 2$.

Proof. The statement (1) follows immediately from the third fundamental identity (Theorem 4.1 (3)). The assertion (2) can be verified by induction as follows. Let $d$ be a negative integer and assume that

(i) $\sigma^{[d]} = 0$ for $d' < d$
(ii) $\sigma^{[d]} = 0$ for $k' < k$.

Now let $a, b$ be non-negative integer such that $k = d + a + b + 2$. Let $A \in g_a$ and $B \in g_b$ and $X \in g_x$ ($x < 0$). Then by Theorem 4.1 (3), we have

$$[\sigma^{[d]}(A,B), X] = \sigma^{[d]}[A,B] + \sigma^{[d]}(A,B)[X]$$

By the inductive assumption (i),

$$\sigma^{[d]}(A,B) = \sigma^{[d]}[A,B] = \sigma^{[d]}(A,B) = 0$$

Therefore, the 2nd and the 3rd terms of the right hand side of the above identity vanish.

For the 1st term, we have

$$(\tau_{d_1} + \sigma_{d_1})(\tau_{d_2}(A,X), B) = \sigma_{d_1}(\tau_{d_2}(A,X), B) = \sigma_{d_1}[d_1 + d+a+b+x](\tau_{d_2}(A,X), B) = \sigma_{(d)[k-2+x]}([A,X], B)$$

Hence the 1st term vanishes.

The last term is

$$D_A\tau_{d+a}(B,X) = D_A\tau_{(d+a)[d+a+b+1]}(B,X) = D_A\tau_{(d+a)[k-1]}(B,X).$$

But by the assumption $\tau^{[k-1]}$ is constant, therefore, the last term also vanishes. Hence $\sigma_{(d)}^{[k]} = 0$, which proves (2).

By a similar argument we have

$$[\sigma_0^{[k]}(A,B), X] = \sigma_0^{[k]}([A,X], B) + \sigma_0^{(k)}(A, [B,X]),$$
from which follows the assertion (3).

Remark 4.1. In general \( \sigma_d^{[k]} \) does not necessarily vanish even for \( d < 0 \). We have \( \gamma_{11} = \) and \( \gamma_{22} = 0 \) but \( \gamma_{21} \) does not always vanish.

5. Invariants

On the basis of Section 3 and Section 4, we show how to obtain the invariants of an arbitrarily given proper geometric structure.

5.1. Fundamental system of invariants.

Let \( Q^{(k)} \) be a proper geometric structure of type \((g_-, G_0, \ldots, G_k)\). Choose complementary subspaces \( W = \{ W_1, W_{2} \}_{\ell \geq k} \) as in Section 3 and let \( \mathcal{S}_w Q^{(k)} \) be the \( W \)-normal complete prolongation of \( Q^{(k)} \) and \( \gamma \) be its structure function, which is a \( \text{Hom}(\wedge^2 E, E) \)-valued function on \( \mathcal{S}_w Q^{(k)} \). Define \( \text{Hom}(\wedge^m E, \text{Hom}(\wedge^2 E, E)) \)-valued function \( D^m \gamma \) by

\[
D^0 \gamma = \gamma
\]

\[
d(D^{m-1} \gamma) = (D^m \gamma) \circ \theta \quad \text{for } m > 0.
\]

Then \( \{D^m \gamma\}_{m \geq 0} \) forms a set of invariants of \( Q^{(k)} \), that is, if two proper geometric structures \( Q^{(k)} \) and \( \overline{Q}^{(k)} \) are isomorphic, then \( D^m \gamma(z) = D^m \gamma(\overline{z}) \) for any \( z \in \mathcal{S}_w Q^{(k)} \) and for the corresponding point \( \overline{z} \in \mathcal{S}_w \overline{Q}^{(k)} \).

Let us call \( \{D^m \gamma\}_{m \geq 0} \) the complete system of invariants of \( Q^{(k)} \) determined by the \( W \)-normal prolongation since these invariants \( \{D^m \gamma\}_{m \geq 0} \) completely determine the formal equivalence class of geometric structure \( Q^{(k)} \) (Theorem 6.1).

Now Theorem 6.1 gives a recursive formula how to determine \( \gamma \). To see it more closely, we give the following.

Definition 5.1. Let \( Q^{(\ell)} \) be a proper geometric structure of type \( g[\ell] \) and let \( g \) be the prolongation of \( g[\ell] \). We say that \( Q^{(\ell)} \) is quasi-involutive if

\[
H^r_{\ell}(g_-, g) = 0 \quad \text{and} \quad H^2_{r+1}(g_-, g) = 0 \quad \text{for } r \geq \ell.
\]

By Theorem 6.1 there is an integer \( \nu \) \((\geq k)\) such that \( \mathcal{S}_w^{(\ell)} Q^{(k)} \) is quasi-involutive for \( \ell \geq \nu \).

By the fundamental identities (Theorem 4.1), if \( \ell > \nu \), then \( \kappa_{[\ell]}, \tau_{[\ell]}, \sigma_{[\ell]} \) and all their covariant derivatives are uniquely determined by \( \{\kappa_{[j]}, \tau_{[j]}, \sigma_{[j]} ; j \leq \nu \} \) and their covariant derivatives through polynomial functions determined only by the graded Lie algebra \( g \). Indeed, if \( H^2_{\ell}(g_-, g) = 0 \), then the restriction of \( \partial \) to \( W^2_{\ell} \) is injective. Since \( \kappa_{[\ell]} \) has its value in \( W^2_2 \), \( \partial \kappa_{[\ell]} \) determines \( \kappa_{[\ell]} \). Similarly, if \( H^1_{\ell-1}(g_-, g) = 0 \), then \( \partial \tau_{[\ell]}(A, \cdot) \) determines \( \tau_{[\ell]}(A, \cdot) \).

In this regard, we say that \( \{D^m \gamma^{[\nu]}\}_{m \geq 0} \) forms a fundamental system of invariants of \( Q^{(k)} \).

More specifically, we define the set of essential invariants of \( Q^{(k)} \) as the set

\[
\{D^m \tau_i : i \in I^1, m \geq 0\} \cup \{D^m \kappa_i : i \in I^2, m \geq 0\}
\]
where

\[ I^1 := \{ i \in \mathbb{Z}_{\geq 0} : H^1_{i-1}(g-, g) \neq 0 \} \]
\[ I^2 := \{ i \in \mathbb{Z}_{\geq 0} : H^2_i(g-, g) \neq 0 \}. \]

Summarizing the above discussion we have the following.

**Theorem 5.1.** Let \( Q^{(k)} \) be a proper geometric structure and \( \mathcal{S}_W Q^{(k)} \) its \( W \)-normal prolongation. If \( \mathcal{S}_W^{(\ell)} Q^{(k)} \) is quasi-involutive, then \( \{ D^m \gamma^{[\ell]} \}_{m \geq 0} \) forms a fundamental system of invariants of \( Q^{(k)} \). Furthermore, the set of essential invariants of \( Q^{(k)} \) forms a fundamental system of invariants of \( Q^{(k)} \).

We remark that, though \( \kappa_\ell, \tau_{[\ell]} \) are functions on \( \mathcal{S}_W^{(\ell)} Q^{(0)} \), their covariant derivatives \( D^i \kappa_\ell, D^i \tau_{[\ell]} \) are functions on \( \mathcal{S}_W^{(\ell+\mu)} Q^{(k)} \) and the essential invariants \( \{ D^i \kappa_\ell, D^i \tau_{[\ell]} \} \) should be regarded as functions on \( \mathcal{S}_W Q^{(k)} \).

Let us see what the above theorem says, in particular, for a geometric structure \( Q^{(0)} \) of order 0.

**Corollary 5.1.** Let \( Q^{(0)} \) be a geometric structure of order 0. If \( \mathcal{S}_W^{(\ell)} Q^{(0)} \) is quasi-involutive, then \( \kappa_1, \ldots, \kappa_\ell \) form a fundamental system of invariants.

**Proof.** In this case it holds that \( H^1_r(g-, g) = 0 \) for \( r > 0 \). Therefore the essential invariants are composed of

\[ \kappa_0, \kappa_1, \ldots, \kappa_\ell, \tau_{[0]}, \tau_{[1]} \]

and their covariant derivatives. Now \( \tau_{[0]} = 0 \) and \( \kappa_0, \tau_{[1]} \) comes from the bracket of the graded Lie algebra \( g \). Therefore non-trivial invariants can appear only from \( \kappa_1, \ldots, \kappa_\ell \).

\[ \square \]

This is a well-known phenomenon for the case of Cartan connection. It is remarkable that the same folds for the step prolongation \( \mathcal{S}_W Q^{(0)} \).

5.2. **Involutive geometric structures.**

Again from Theorem 1.1 we have the following important consequence:

**Proposition 5.1.** Let \( Q^{(k)} \) be a proper quasi-involutive geometric structure and \( Q \) be \( W \)-normal complete step prolongation of \( Q^{(k)} \). If the structure function \( \gamma^{[k]} \) is constant, then the structure function \( \gamma \) is constant.

**Definition 5.2.** A proper geometric structure \( Q^{(k)} \) is called involutive if

1. \( Q^{(k)} \) is quasi-involutive.
2. The structure function \( \gamma^{[k]} \) is constant.

If \( Q^{(k)} \) is involutive, we have seen that the structure function \( \gamma \) of \( Q \) is constant, but not only this, we see by Corollary 4.4 that \( \sigma_{(d)} = 0 \) for \( d < 0 \) and that \( \tau_{(0)} \) and \( \sigma_{(0)} \) coincide with the bracket of the graded Lie algebra \( g \). Then we see that \( (g, \gamma) \), where \( \gamma \in \text{Hom}(\wedge^2 g, g) \) is the structure function of \( Q \), becomes a transitive filtered Lie algebra \( L \) such that \( \text{gr} L \cong g \). To summarize, we have the following proposition.
Proposition 5.2. If a proper geometric structure $Q^{(k)}$ of type $\mathfrak{g}[k]$ is involutive, then the structure function $\gamma$ of the $W$-normal complete step prolongation $Q$ of $Q^{(k)}$ defines on the prolongation $\mathfrak{g}$ of $\mathfrak{g}[k]$ a structure of transitive filtered Lie algebra $L = (\mathfrak{g}, \gamma)$ such that $\text{gr} L \simeq \mathfrak{g}$.

For a proper geometric structure $Q$, let $\text{LocAut}(Q)$ be the Lie pseudo-group of all local isomorphisms $\varphi$ of $Q$, $\varphi$ being defined on an open set of the base space $M$ of $Q$, and let $\text{InfAut}(Q)$ be the Lie algebra sheaf on $M$ of all infinitesimal automorphisms $X$ of $Q$, $X$ being a local vector field on $M$ such that $\text{Exp} X \in \text{LocAut}(Q)$.

If $Q$ is analytic and $\gamma$ is constant, then $\text{LocAut}(Q)$ and $\text{InfAut}(Q)$ are transitive on $M$ provided that $M$ is connected (Theorem 5.2), so that each stalk $\mathcal{L}_x$ of $\text{InfAut}(Q)$ at $x \in M$ is isomorphic to each other. We see that the completion $L_x$ of $\mathcal{L}_x$ with respect to the natural filtration $\{\mathcal{L}_x^p\}$ turns out to be a transitive filtered Lie algebra, which we call the formal algebra of $\text{InfAut}(Q)$.

5.3. Transitive models of geometric structures.

In the previous subsection we have seen that an involutive geometric structure (or a geometric structure of constant structure function) yields a transitive filtered Lie algebra. In this subsection we consider the converse.

Let $(L, \{L^p\})$ be a transitive filtered Lie algebra and $\mathfrak{g} = \oplus \mathfrak{g}_p$ its associated graded Lie algebra $\text{gr} L$. Choose complementary subspaces $\{\mathfrak{g}_p\}_{p \in \mathbb{Z}}$ such that

$$L^p = \mathfrak{g}_p \oplus L^{p+1},$$

and then define $\gamma \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ by

$$\gamma(X, Y) := [X, Y] \text{ for } X, Y \in \mathfrak{g} = \oplus \mathfrak{g}_p.$$  

Then identifying $\mathfrak{g}$ with $\mathfrak{g}$, we define $\gamma \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ to be the bilinear map corresponding to $\gamma$.

Theorem 5.2 (The 3rd fundamental theorem of Lie). Let $L$ be a transitive filtered Lie algebra and $\mathfrak{g} = \text{gr} L$ and $\gamma \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ as above. Then there exists an involutive geometric structure $Q_L$ with constant structure function equal to $\gamma$ such that the (formal) Lie algebra of $\text{InfAut}(Q_L)$ is $L$.

This is a local version of Lie’s third fundamental theorem that for a Lie algebra $\mathfrak{g}$ there exists a Lie group $G$ corresponding to $\mathfrak{g}$, which is well-known if $\mathfrak{g}$ is finite dimensional. In the infinite dimensional case, however, we have to state it locally as in the above theorem because we have no satisfactory formulation of infinite dimensional Lie groups. The involutive geometric structure plays a role of local Lie group.

In [4] and then in [10], Cartan proved the following theorem in a classical form:

Theorem (C). Let $(V, \mathfrak{g}_0, c)$ be a triple consisting of a finite dimensional vector space $V$, a Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{gl}(V)$ and $c \in \text{Hom}(\wedge^2 V, V)$ which satisfy

(i) $c \circ c \in \partial \text{Hom}(\wedge^2 V, \mathfrak{g}_0)$

(ii) $\rho(A)c \in \partial \text{Hom}(V, \mathfrak{g}_0)$ for $A \in \mathfrak{g}_0$

(iii) $\mathfrak{g}_0$ is involutive, that is, $H^1_r(V, \mathfrak{g}_0) = H^2_{r+1}(V, \mathfrak{g}_0) = 0$ for $r > 0$,

where the coboundary operator $\partial$ and the cohomology group $H^p(V, \mathfrak{g}_0)$ are those defined by the Spencer complex associated to the prolongation of $(V, \mathfrak{g}_0)$. Then there exists an
analytic manifold \( \tilde{M} \) equipped with a \( V \)-valued 1-form \( \theta \) and \( g \)-valued 1-form \( \pi \) on \( \tilde{M} \) satisfying

1. \( \, d\theta + \frac{1}{2} \mathfrak{c}(\theta, \theta) + \pi \wedge \theta = 0 \)
2. \( (\theta + \pi)_z : T_z \tilde{M} \to V \oplus g \) is an isomorphism for all \( z \in \tilde{M} \).

The above formulation of Theorem (C) and its rigorous proof based on Cartan-Kähler Theorem, may be found in Part 1 of [26]. It is also proved there Theorem 5.2 in the case where the depth of the filtered Lie algebra \( (L, \{ L^p \}) \) is one as follows.

Let \( k \) be an integer such that

\[
H^1_r((\text{gr } L)_-, \text{gr } L) = H^2_{r+1}((\text{gr } L)_-, \text{gr } L) = 0 \text{ for } r > k.
\]

Let \( g \) be \( \text{gr } L \) and \( E^{(k-1)} \) be \( g / F^k \). Denote by \( \gamma^{(k-1)} \in \text{Hom}(\wedge^2 E^{(k-1)}, E^{(k-1)}) \) the truncated bracket of \( L \) with respect to an identification of \( L/F^k \) with \( E^{(k-1)} \). Then \( (E^{(k-1)}, g_k, \gamma^{(k-1)}) \) satisfies the assumptions (i) – (iii) of Theorem (C). By Theorem (C), there exist a manifold \( U^{(k)} \) equipped with an \( E^{(k)} \)-valued 1-form \( \theta^{(k-1)} \) and a \( g_k \)-valued 1-form \( \omega_k \) satisfying the conditions (1) and (2) in the theorem. The filtration of \( E^{(k-1)} \) then induces a filtration of \( U^{(k)} \) and it is shown that \( U^{(k)} \) can be realized as an open set of an involutive geometric structure of order \( k \).

**Proof of Theorem 5.2.** For a transitive filtered Lie algebra \( (L, \{ L^p \}) \) of arbitrary depth, we define a filtration \( \{ L^p \} \) of \( L \) of depth one by setting

- \( L^p = L \) for \( p < 0 \) and \( L^0 = L^0 \);
- \( L^p > 0, L^{p+1} = \{ X \in L^p : [X, L] \subset L^p \} \).

Then we see that

(a) \( [L^p, L^q] \subset L^{p+q} \);
(b) for any \( L^p \) there exists \( q \) such that \( L^p \supset L^q \);
(c) for any \( L^p \) there exists \( p \) such that \( L^p \supset L^p \).

Thus \( (L, \{ L^p \}) \) is a transitive filtered Lie algebra of depth one. Applying the result in the above to \( (L, L^p) \), we get \( (U^{(k)}, \theta^{(k-1)}, \omega_k) \).

Let \( \text{LocAut}(U^{(k)}, \theta^{(k-1)}) \) be the pseudo-group of local diffeomorphisms of \( U^{(k)} \) which preserve \( \theta^{(k-1)} \). Note it is transitive on \( U^{(k)} \) by Theorem 6.2. On account of the structure equation for \( \theta \) and the filtration on \( U^{(k)} \), we may assume that there is a fibration \( U^{(k)} \to M \) with \( \text{dim } M = \text{dim } L/L^0 \) and \( \text{LocAut}(Q^{(k)}, \theta^{(k-1)}) \) to \( M \) to define a pseudo-group \( \mathcal{P} \) on \( M \) isomorphic to \( \text{LocAut}(U^{(k)}, \theta^{(k-1)}) \). Moreover we see that there is a filtration \( \{ F^p \} \) on \( M \) invariant by \( \mathcal{P} \) defined from \( L^p \) (\( p \leq 0 \)).

Now consider \( \mathcal{S}^{(0)}(M, F) \) and lift \( \mathcal{P} \) to \( \mathcal{S}^{(0)}(M, F) \) to get a pseudo-group \( \mathcal{P}_{\mathcal{S}^{(0)}} \) on \( \mathcal{S}^{(0)}(M, F) \). Then take one orbit \( Q^{(0)} \) and let \( Q^{(0)} \) be the restriction of \( Q^{(0)} \) to \( Q^{(0)} \). Next consider \( \mathcal{S}^{(1)}(Q^{(0)}) \) and the lift \( P_{\mathcal{S}^{(1)}} \) of \( Q^{(0)} \) and take an orbit \( Q^{(1)} \). Repeating this process, we obtain an involutive geometric structure \( Q^{(\ell)} \) after a finite number of steps. Then by step prolongation of \( Q^{(\ell)} \) we get an involutive geometric structure \( Q^{(\infty)} = \mathcal{S}_W Q^{(\ell)} \).

In each of the above construction we can choose an orbit \( Q^{(i)} \) so that the structure function of \( Q^{(\infty)} \) coincides with the prescribed \( \gamma \). This completes the proof of Theorem 5.2. \( \square \)
The involutive geometric structure $Q_L = Q^{(\infty)}$ thus constructed may be called the transitive model corresponding to $(L, \gamma)$. Then by Theorem 6.2 which we will prove in Section 6 we have:

**Theorem 5.3.** The transitive model $Q_L$ is uniquely determined up to local analytic isomorphism.

### 6. Equivalence problems

In the previous section, we have studied how to obtain the invariants of a geometric structure. In this section we study the converse, that is, we consider whether the invariants that we have found in Section 5 is sufficient to determine the equivalence.

#### 6.1. Formal equivalence.

Let $X$ be a differentiable or analytic manifold and $(x^1, \ldots, x^n)$ be a local coordinate system. Given a differentiable function $f$ in a neighborhood of a point $p \in X$, the series of derivatives $\left\{ \frac{\partial^{i_1+\cdots+i_n} f}{\partial(x^1)^{i_1}\cdots\partial(x^n)^{i_n}}(p) \right\}$ determines a formal structure at $p$, which we denote by $f[[X, p]]$ or $f[[p]]$.

Let $Y$ be another manifold with a coordinate system $(y^1, \ldots, y^n)$ and $h[[Y, q]]$ be a formal structure at $q$. We say $f[[X, p]]$ and $h[[Y, q]]$ are formally equivalent and write $f[[X, p]] \sim h[[Y, q]]$ if there exists a formal (power series) invertible map $\phi: X \to Y$ given by a formal power series $x^i = \phi^i(y^1, \ldots, y^n)$ ($i = 1, \ldots, n$) such that $\phi^* h[[Y, q]] = f[[X, p]]$.

Let $Q^{(k)}$ be a geometric structure of type $g[k]$. Denote by $Q^{(\ell)}$ ($\ell > k$) and $Q$ the $W$-normal step prolongation $S_W Q^{(k)}$ and $S_W Q^{(k)}$. Then the formal structure $f[[z]]$ at $z \in Q$ may be given by the series of covariant derivatives evaluated at $z$: $\{(D^i f)(z), i = 0, 1, 2, \ldots \}$.

Furthermore, all the relations among $f^{ij \ldots k}$ come from $d^2 f = 0$, $d^2 f_i = 0$, etc.

From $d^2 f = 0$ it follows $f_{ij} - f_{ji} = \sum_k \gamma^k_{ij} f_k$.

Similarly all the relations among $f^{ij \ldots k}$ are determined by $\gamma^k_{ij}$ and their derivatives.

**Theorem 6.1.** Let $Q^{(k)}$ and $\overline{Q}^{(k)}$ be proper geometric structures of the same type, and let $\mathcal{S}_W Q^{(k)}$ and $\mathcal{S}_W \overline{Q}^{(k)}$ be their $W$-normal complete prolongations. Let $(z, \overline{z}) \in \mathcal{S}_W Q^{(k)} \times \mathcal{S}_W \overline{Q}^{(k)}$. Then there exists a formal (power series) invertible map $\phi: X \to Y$ given by a formal power series $x^i = \phi^i(y^1, \ldots, y^n)$ ($i = 1, \ldots, n$) such that $\phi^* \overline{Q}^{(k)}(\overline{z}) = Q^{(k)}(z)$ if and only if $f[[X, p]] \sim h[[Y, q]]$ for some formal (power series) invertible map $\phi: X \to Y$.
$\mathcal{W}Q^{(k)}$. If $(D^i\gamma)(z) = D^i\Psi(z)$ for all $i \geq 0$, then there exists a formal isomorphism

$$
\Phi^{(k)} : [[Q^{(k)}, z^{(k)}]] \to [[\overline{Q}^{(k)}, \overline{z}^{(k)}]]
$$

where $(z^{(k)}, \overline{z}^{(k)}) \in Q^{(k)} \times \overline{Q}^{(k)}$ is the projection of $(z, \overline{z})$.

**Proof.** Let $Q$ denote $\mathcal{W}Q^{(k)}$ and $[[Q, z]]$ the formal power series ring of $Q$. Define

$$
\varepsilon : f \in [[Q, z]] \mapsto (D^i f)(z)_{i \geq 0} \in \otimes E^*,
$$

where $E$ is the vector space in which $\theta$ takes values. All the relations that the derivatives $(D^i f)(z)$ should satisfy are uniquely determined from $\{(D^i \gamma)(z)\}$ and so is the image of $\varepsilon$, which we denote by $(\otimes E^*)_{\gamma}$.

With a similar notation for $\mathcal{W}Q^{(k)}$, we have an isomorphism

$$
\varepsilon : [[Q, z]] \to (\otimes E^*)_{\gamma}.
$$

But, by the assumption that $(D^i \gamma)(z) = (D^i \Psi)(z)$ for $i \geq 0$, we see that $(\otimes E^*)_{\gamma} = (\otimes E^*)_{\overline{\gamma}}$.

Now define an isomorphism $\Psi$ by the following commutative diagram:

$$
\begin{array}{ccc}
[[Q, z]] & \xrightarrow{\Psi} & [[\overline{Q}, \overline{z}]] \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
(\otimes E^*)_{\gamma} & \xrightarrow{id} & (\otimes E^*)_{\overline{\gamma}}.
\end{array}
$$

Then the ring isomorphism satisfies

$$
\Psi \overline{\Psi} = \theta.
$$

Take a coordinate system $x = (x^1, x^2, \ldots)$ of $Q$ and let $\overline{\Psi}$ be the corresponding coordinate system of $\overline{Q}$. Then

$$
dx = (Dx) \circ \theta \quad \text{and} \quad d\overline{\Psi} = (\overline{D\Psi}) \circ \overline{\theta}
$$

and

$$
dx = \Phi^* dx \quad \text{and} \quad Dz = \Phi^* \overline{Dz}.
$$

Furthermore, $Dx$ and $\overline{Dz}$ are invertible. Hence we conclude $\Psi^* \overline{\Psi} = \theta$. Thus we have a formal isomorphism

$$
\Psi : [[\mathcal{W}Q^{(k)}, \theta, z]] \to [[\mathcal{W}Q^{(k)}, \overline{\theta}, \overline{z}]].
$$

On the other hand, we have the following categorical isomorphisms:

$$
(Q^{(k)}, \text{geom.str.}) \simeq (\mathcal{W}Q^{(k)}, \text{geom.str.}) \simeq (\mathcal{W}Q^{(k)}, \theta).
$$

Therefore, the formal isomorphism $\Psi$ induces a formal isomorphism

$$
\Psi : [[Q^{(k)}, \text{geom.str.}, z^{(k)}]] \to [[\overline{Q}^{(k)}, \text{geom.str.}, \overline{z}^{(k)}]].
$$

\[\square\]

**Corollary 6.1.** Let $Q^{(k)}$ and $\overline{Q}^{(k)}$ be proper geometric structures of the same type, and let $\mathcal{W}Q^{(k)}$ and $\mathcal{W}\overline{Q}^{(k)}$ be their $W$-normal complete prolongations. If $\chi(z) = \overline{\chi}(\overline{z})$ for any essential invariant $\chi$ of $Q^{(k)}$ and the corresponding one $\overline{\chi}$ of $\overline{Q}^{(k)}$ for a pair $(z, \overline{z}) \in \mathcal{W}Q^{(k)} \times \mathcal{W}\overline{Q}^{(k)}$, then there exists a formal isomorphism

$$
\Phi^{(k)} : [[Q^{(k)}, z^{(k)}]] \to [[\overline{Q}^{(k)}, \overline{z}^{(k)}]].
$$
where \((z^{(k)}, \overline{z}^{(k)}) \in Q^{(k)} \times Q^{(k)}\) is the projection of \((z, \overline{z})\).

6.2. Involutive geometric structures.

The quasi-involutivity is not enough to solve the equivalence problem for geometric structure of infinite type. It is the involutive geometric structure that we can solve the equivalence problem perfectly.

**Theorem 6.2.** Let \(Q^{(k)}\) and \(Q^{(k)}\) be proper geometric structures of type \(g[k]\) with structure functions \(\gamma[k]\) and \(\overline{\gamma}[k]\). Assume that \(Q^{(k)}\) and \(Q^{(k)}\) are involutive and \(\gamma[k] = \overline{\gamma}[k]\).

1. If, moreover, \(g(k)\) is of finite type, then in the \(C^\infty\)-category, there exists a \(C^\infty\)-local isomorphism \((Q^{(k)}, z_Q^{(k)}) \rightarrow (Q^{(k)}, z_Q^{(k)})\) for any \((z_Q^{(k)}, z_Q^{(k)}) \in Q^{(k)} \times Q^{(k)}\).

2. In the analytic category, there exists an analytic local isomorphism \((Q^{(k)}, z_Q^{(k)}) \rightarrow (Q^{(k)}, z_Q^{(k)})\) for any \((z_Q^{(k)}, z_Q^{(k)}) \in Q^{(k)} \times Q^{(k)}\).

**Proof.** The assertion (1) is easily obtained by applying Frobenius Theorem. In fact, it follows immediately from the second fundamental theorem of Lie: Two absolute parallelism \((Q, \theta)\) and \((\overline{Q}, \overline{\theta})\) with constant structure functions \(\gamma\) and \(\overline{\gamma}\) are locally isomorphic if and only if \(\gamma = \overline{\gamma}\).

The assertion (2) is more involved because we have to deal with general system of partial differential equations.

In the case where the filtration of the base manifold is trivial, it was first proved by Cartan ([4], [5], [6]) by using nowadays called Cartan-Kähler theorem, and rigorously settled by Singer-Sternberg ([36]).

In the case of general filtered manifold of depth > 1, we have two proofs: One is to reduce the proof to the case of trivial filtration by geometric consideration. This kind of proof was given in Theorem 3.6.2 of [30].

Another proof uses, instead of the classical Cartan-Kähler theorem, its nontrivial generalization to a setting of nilpotent analysis ([29], Theorem 3.3 of [32]). This proof has the advantage that it makes distinct and clear geometric, algebraic, and analytic essences which are mixed in the classical Cartan-Kähler theorem.

However, because of the nature of this theory, we have to impose an additional assumption that the base filtered manifold satisfy the Hörmander condition, that is, \(g_-\) is generated by \(g_{-1}\).

Now let us give the second proof.

Let \(Q\) and \(Q\) be geometric structures on filtered manifolds \((M, F)\) and \((\overline{M}, \overline{F})\) respectively. Assume that they are of the same type \(g = \oplus g_i\) and both involutive at order \(k\) having the same structure constants \(\gamma[k] = \overline{\gamma}[k]\).

Let \(M \times \overline{M} \rightarrow M\) be a fibred manifold, whose local smooth section \(\sigma : x \in U \mapsto (x, \varphi(x)) \in U \times \overline{U}\) is identified with a smooth map \(U \rightarrow \overline{U}\), where \(U, \overline{U}\) are open sets of \(M, \overline{M}\), respectively.

Let \(\mathcal{J}_\ell(M \times \overline{M})\) denote the set of all weighted \(\ell\)-jets \(j_\ell^x \varphi\) of local smooth maps \(\varphi : M \rightarrow \overline{M}\) with respect to the filtered manifolds \((M, F)\) and \((\overline{M}, \overline{F})\). Let \(\Gamma_\ell(M \times \overline{M})\) denote the set of local sections of \(M \times \overline{M} \rightarrow M\) consisting of local diffeomorphisms

\[\varphi : U(\subset M) \rightarrow \overline{U}(\subset \overline{M}).\]
Recall that if \( \varphi \in \Gamma^0(M \times \overline{M}) \) preserves the filtrations, that is, \( \varphi_* F^p \subset \overline{F}^p \), then it induces the lift

\[
\mathcal{J}(\ell)\varphi : \mathcal{J}(\ell)(M)|_U \to \mathcal{J}(\ell)(\overline{M})|_\mathcal{U}.
\]

Consider the equation

\[
(\mathcal{J}(\ell)\varphi)(Q(\ell)) \subset \overline{Q}(\ell),
\]

where we regard \( Q(\ell) \) as a submanifold of \( \mathcal{J}(\ell)(M) \) and \( \overline{Q}(\ell) \) as a submanifold of \( \mathcal{J}(\ell)(\overline{M}) \) by fixing the complementary subspaces necessary to the embeddings commonly to \( Q \) and \( \overline{Q} \).

Since the relation (\( \dagger \)) depends only on \( \ell \)-jet \( j^{(\ell)}\varphi \) of \( \varphi \), it defines a submanifold

\[
\mathcal{R}(\ell) \subset \mathcal{J}(\ell)(M \times \overline{M}),
\]

which may be a system of differential equations for \( \varphi \) to define an isomorphism \( \mathcal{J}(\ell)\varphi : Q(\ell) \to \overline{Q}(\ell) \). If \( \mathcal{J}(\ell)\varphi : (Q(\ell), z^\ell) \to (\overline{Q}(\ell), \overline{z}^\ell) \) is a local or formal isomorphism, then \( j^x_\ell \varphi \) belongs to \( \mathcal{R}_x(\ell) \), where \( x \in M \) is the projection of \( z^\ell \).

If we denote by \( p^{(\ell)}\mathcal{R}(k) (\ell \geq k) \) the prolongation of \( \mathcal{R}(k) \) to \( \mathcal{J}(\ell)(M \times \overline{M}) \), we see that \( p^{(\ell)}\mathcal{R}(k) \supset \mathcal{R}(\ell) \). Moreover, since \( Q(\ell) \) and \( \overline{Q}(\ell) \) are prolongations of \( Q(k) \) and \( \overline{Q}(k) \), respectively, we see that \( p^{(\ell)}\mathcal{R}(k) = \mathcal{R}(\ell) \) for \( \ell \geq k \).

Now take \( z \in Q \) and set \( z^\ell \) and \( x \) its projections to \( Q(\ell) \) and \( M \). Define a map

\[
\mathcal{R}_x(\ell) \xrightarrow{\iota} \overline{Q}(\ell)
\]

by \( \iota(j^x_\ell \varphi) = (\mathcal{J}(\ell)\varphi)(z^\ell) \). Then clearly \( \iota \) is injective. Furthermore, \( \iota \) is surjective for \( \ell \geq k \). Indeed, since \( Q(k) \) and \( \overline{Q}(k) \) are involutive with the same structure constant, for any \( \ell \geq k \) and \( z^\ell \in \overline{Q}(\ell) \), there exist \( z \in \overline{Q}(\ell) \) which projects to \( z^\ell \) and formal isomorphisms \( \mathcal{J}\varphi : (Q, z) \to (\overline{Q}, z) \) and \( \mathcal{J}(\ell)\varphi : (Q(\ell), z^\ell) \to (\overline{Q}(\ell), \overline{z}^\ell) \) (Theorem 6.1) which make the following diagram commutative:

\[
\begin{array}{ccc}
(Q, z) & \xrightarrow{\mathcal{J}\varphi} & (\overline{Q}, \overline{z}) \\
| & & | \\
(Q(\ell), z^\ell) & \xrightarrow{\mathcal{J}(\ell)\varphi} & (\overline{Q}(\ell), \overline{z}^\ell)
\end{array}
\]

with \( (\mathcal{J}(\ell)\varphi)(z^\ell) = \overline{z}^\ell \).

We say, according to Malgrange, that a \( u^k \in \mathcal{R}(k) \) is strongly prolongable if for \( \ell \geq k \), \( u^\ell \in \mathcal{R}(\ell) \) projects to \( u^k \), then there is \( u^{\ell+1} \in \mathcal{R}(\ell+1) \) which projects to \( u^\ell \). Since \( \iota(\mathcal{R}(\ell)) = \overline{Q}(\ell) \) and \( \overline{Q}(\ell+1) \to \overline{Q}(\ell) \) is surjective for \( \ell \geq k \), for our \( \mathcal{R} \), any \( u^\ell \in \mathcal{R}(\ell) (\ell \geq k) \) is strongly prolongable.

Now by a generalized Cartan-Kähler Theorem (29) if the given data are all analytic, then for any \( \ell \geq k \) and \( u^\ell \in \mathcal{R}_x(\ell) \), there exists a formal solution \( \varphi \in \mathcal{R}_x \) which projects to \( u^\ell \) and satisfies the formal Geverey estimate, that is, \( \varphi \) is a formal map \( (M, x) \to (\overline{M}, \overline{x}) \) satisfying the following estimate: There exist \( C, \rho > 0 \) such that

\[
|\langle X_I \varphi \rangle(0)| \leq Cw(I)!\rho^{w(I)} \text{ for all } I
\]
where \( \{X_1, \ldots, X_n\} \) is an admissible local basis of the filtered vector bundle \((TM, F)\) and \(X_I = X_{i_1} \cdots X_{i_r} \) for \( I = (i_1, \ldots, i_r) \), \( r \geq 0 \), \( 1 \leq i_1, \ldots, i_r \leq n \) and \( w(I) \) is the associated weight of \( I \).

Then under the assumption that \( g_- \) is generated by \( g_{-1} \), we conclude that the formal Gevrey map \( \varphi : (M, x) \to (M, \mathbf{r}) \) is analytic by virtue of Theorem 2 in \([31]\) which may be stated equivalently in the following form:

**Theorem 6.3.** Let \( \{X_1, \ldots, X_r\} \) be independent analytic vector fields defined in a neighborhood of the origin 0 in \( \mathbb{R}^n \) or \( \mathbb{C}^n \) satisfying the Hörmander condition, that is,
\[
X_1, \ldots, X_r, [X_i, X_j], [[X_i, X_j], X_k], \ldots
\]
generate the tangent space. If \( F \) is a formal function at 0 satisfying:
\[
\exists C, \rho > 0 \text{ such that } |(X_{i_1} \cdots X_{i_\ell} F)(0)| \leq C \ell! \rho^\ell \text{ for all } \ell > 0, 1 \leq i_1, \ldots, i_\ell \leq r,
\]
then \( F \) is analytic.

This completes the proof of Theorem 6.2. \( \square \)

**Remark 6.1.** The equivalence problem is very delicate in the \( C^\infty \)-category for the geometric structure of infinite type. A general theory for \( C^\infty \)-integrability was initiated by Kumpera-Spencer and Goldschmidt (cf. \([20], [15]\)). As shown by J. Conn (\([11]\), Theorem 6.2 (2) does not hold in general in the \( C^\infty \)-category for a geometric structure of infinite type.

### 6.3. Analytic geometric structures of finite type.

We have the following notable theorem which applies to all geometric structures of finite type in the analytic category.

**Theorem 6.4.** Let \( Q^{(k)} \) and \( \overline{Q}^{(k)} \) be analytic proper geometric structures of the same type and of finite type. If \( (D^i)_{\gamma}(z) = (\overline{D})_{\gamma}(\overline{z}) \) for all \( i \geq 0 \) for a pair \( (z, \overline{z}) \in \mathcal{A}_W Q^{(k)} \times \mathcal{A}_W \overline{Q}^{(k)} \), then there exists an analytic local isomorphism
\[
\varphi^{(k)} : (Q^{(k)}, z^{(k)}) \to (\overline{Q}^{(k)}, \overline{z}^{(k)}).
\]

Theorem 6.4 follows from Theorem 6.1 and the following fact.

**Proposition 6.1.** Let \( X_1, \ldots, X_n \) be \( n \)-independent analytic vector fields defined on a neighborhood of 0 in \( \mathbb{R}^n \) or \( \mathbb{C}^n \). A formal function \( f \) at 0 is convergent in a neighborhood of 0 if and only if there exists positive constants \( C \) and \( \rho \) such that
\[
|(X_{i_1} X_{i_2} \cdots X_{i_\ell} f)(0)| \leq C \ell! \rho^\ell \text{ for } \ell = 0, 1, 2, \ldots, (i_1, i_2, \ldots, i_\ell) \in \{1, 2, \ldots, n\}^\ell.
\]

This proposition is a special case of Theorem 6.3. We, however, give an elementary proof in Appendix at the end of this paper.

**Proof of Theorem 6.4.** We keep the same notation as in the proof of Theorem 6.1. If \( (D^i)_{\gamma}(z) = (\overline{D})_{\gamma}(\overline{z}) \) for \( (z, \overline{z}) \in \mathcal{A}_W Q^{(k)} \times \mathcal{A}_W \overline{Q}^{(k)} \), then, by the proof of Theorem 6.1, there is a ring isomorphism
\[
\Psi : [Q, z] \to [\overline{Q}, \overline{z}]
\]
such that \( \Psi \overline{\theta} = \theta \). Now we assume that \( Q \) and \( \overline{Q} \) are finite dimensional analytic manifolds.
Suppose that $\overline{f} \in [(Q, z)]$ is convergent at $z$. Then by Proposition 6.1 \{$(D^i\overline{f})(z)$\} satisfies the estimate in this proposition. Then so does \{$(D^i f)(z)$\}, where $f = \Psi \overline{f}$. This implies that $\Psi$ is a local analytic isomorphism
\[
\Psi : (\mathcal{W} Q^{(k)}, z) \rightarrow (\mathcal{W} \overline{Q}^{(k)}, \overline{z})
\]
satisfying $\Psi^* \overline{f} = \theta$, from which immediately follows the theorem.

**Corollary 6.2.** Let $Q^{(k)}$ and $\overline{Q}^{(k)}$ be analytic proper geometric structures of the same type and of finite type. If $\chi(z) = \overline{\chi(z)}$ for any essential invariant $\chi$ of $Q^{(k)}$ and the corresponding one $\overline{\chi}$ of $\overline{Q}^{(k)}$ for a pair $(z, \overline{z}) \in \mathcal{W} Q^{(k)} \times \mathcal{W} \overline{Q}^{(k)}$, then there exists an analytic local isomorphism
\[
\varphi^{(k)} : (Q^{(k)}, z^{(k)}) \rightarrow (\overline{Q}^{(k)}, \overline{z}^{(k)}).
\]

**7. Cartan Connections from the Viewpoint of Step Prolongation**

In this section we investigate conditions for our $W$-normal complete prolongation $\mathcal{W} Q^{(0)}$ with the canonical Pfaff form $\theta$ to become a Cartan connection of type $G/G^0$.

**7.1. L’espace généralisé.**

In 1922 Cartan introduced the notion of espace généralisé as a curved version of Klein geometry ([8]). It has been playing an important role in geometry, especially in the equivalence problem of geometric structures. The following formulation is a standard one.

**Definition 7.1.** A Cartan connection of type $L/L^0$ is a principal $L^0$-bundle $P$ on a manifold $M$ with a $l$-valued 1-form $\theta$ on $P$ satisfying the following properties.

1. $\theta : T_z P \rightarrow l$ is an isomorphism for all $z \in P$
2. $R^*_a \theta = \text{Ad}(a)^{-1} \theta$ for $a \in L^0$
3. $\theta(A) = A$ for $A \in l^0$.

In the above definition it is not the Lie group $L$ but the Lie algebra $l$ that actually appear, so that we may refer to it a Cartan connection of type $(l, L^0)$. We may slightly weaken the condition on $l$ not assuming that $l$ has a Lie algebra structure.

**Definition 7.2.** Let $L^0$ be a filtered Lie algebra and $E$ an $L^0$-module which contains the Lie algebra $l^0$ of $L^0$ as an $L^0$-adjoint submodule. We call $(E, L^0)$ a transitive filtered pre-Lie algebra if the module $E$ is endowed with a filtration \{$F^pE\}_{p \in \mathbb{Z}}$ such that

1. $F^pE \supset F^{p+1}E$, $F^0E = l^0$
2. $\rho(F^0 l^0) F^i E \subset F^{i+p} E$ for $i \geq 0$ and $p \in \mathbb{Z}$
3. $\text{gr} E$ is a transitive graded Lie algebra.

**Definition 7.3.** Let $(E, L^0)$ be a transitive filtered pre-Lie algebra. A pre-Cartan connection of type $(E, L^0)$ is a principal $L^0$-bundle $P$ on a manifold $M$ with an $E$-valued 1-form $\theta$ on $P$ satisfying the following properties.

1. $\theta : T_z P \rightarrow E$ is an isomorphism for all $z \in P$
2. $R^*_a \theta = \rho(a)^{-1} \theta$ for $a \in L^0$
3. $\theta(A) = A$ for $A \in l^0$. 

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Note that if there is a pre-Cartan connection \( \tilde{\gamma} \) of type \( (E, L^0) \) which has a constant structure function \( \gamma \), then \( \gamma \) defines a Lie algebra structure on \( E \) extending that of \( \mathfrak{l}^0 \), so that any pre-Cartan connection \( P \) of type \( (E, L^0) \) is a Cartan connection.

### 7.2. Principal geometric structures and principal prolongations.

In this subsection we define a principal geometric structure and its universal frame bundle, and introduce the category of the principal geometric structures which is a little more restrictive than that of geometric structures introduced in Section 2.1, and which is well adapted to study the Cartan connections.

To do this we have only to replace “step wise principal fiber bundle” in Definition 2.1 by “principal fiber bundle”.

**Definition 7.4.** A principal geometric structure \( \tilde{\mathcal{Q}}^{(k)} \) of order \( k \) and its universal principal frame bundle \( \mathcal{E}(k+1)|\mathcal{Q}^{(k)} \) of order \( k+1 \) are defined inductively for \( k \geq -1 \) by the following properties.

1. A principal geometric structure \( \tilde{\mathcal{Q}}^{(-1)} \) of order \( -1 \) on \((M,F)\) is the filtered manifold \((M,F)\) itself.

2. If \( k \geq 0 \), \( \tilde{\mathcal{Q}}^{(i-1)} := \tilde{\mathcal{Q}}^{(i)}/F^{i+1} \) is a principal geometric structure of order \( i-1 \) (of type \( (\mathfrak{g}_-^{(i)}, \mathcal{G}^{(i)}) \)) on \((M,F)\) for \( 0 \leq i \leq k \).

3. If \( k \geq 0 \), \( \tilde{\mathcal{Q}}^{(i)} \to M \) is a principal subbundle of the universal principal frame bundle \( \mathcal{E}(i)|\tilde{\mathcal{Q}}^{(i-1)} \to M \) of \( \tilde{\mathcal{Q}}^{(i-1)} \) of order \( i \) for \( 0 \leq i \leq k \).

4. To every principal geometric structure \( \tilde{\mathcal{Q}}^{(k)} \to M \) of order \( k \geq 0 \) there is associated a principal fiber bundle \( \mathcal{E}(k+1)|\tilde{\mathcal{Q}}^{(k)} \to M \) which is called the universal linear frame bundle of \( \tilde{\mathcal{Q}}^{(k)} \). Its structure group \( \mathcal{G}^{(k+1)} \) is endowed with a filtration \( \{F^k\} \). The quotient

\[
\mathcal{E}(k+1)|\tilde{\mathcal{Q}}^{(k)} := \mathcal{E}(k+1)|\tilde{\mathcal{Q}}^{(k)}/F^{k+2}
\]

is a principal fiber bundle on \( M \) with structure group

\[
\mathcal{G}^{(k+1)} := \mathcal{G}^{(k+1)}/F^{k+2},
\]

which is called the universal frame bundle of \( \tilde{\mathcal{Q}}^{(k)} \) of order \( k+1 \).

5. The universal linear frame bundle \( \mathcal{E}(k+1)|\tilde{\mathcal{Q}}^{(k)} \) for a principal geometric structure \( \tilde{\mathcal{Q}}^{(k)} \) of order \( k \geq -1 \) on \( M \) is defined as follows.

   a) The bundle \( \mathcal{E}(k+1)|\tilde{\mathcal{Q}}^{(k)} \). We set

\[
\tilde{E}^{(k+1)} := \mathfrak{g}_- \quad \text{and} \quad \tilde{E}^{(k)} := \mathfrak{g}_- \oplus \mathfrak{g}_+^{(k)} \quad \text{if} \quad k \geq 0,
\]
where \( \hat{\mathfrak{g}}^{(k)} \) is the Lie algebra of \( \hat{G}^{(k)} \). We regard \( \hat{E}^{(k)} \) as a filtered vector space. We note also that the tangent space \( T_{z^k} \hat{Q}^{(k)} \) has a canonical filtration \( \{ F^\ell T_{z^k} \hat{Q}^{(k)} \}_{\ell \in \mathbb{Z}} \).

The fiber \( \hat{\mathfrak{F}}^{(k+1)}_{z^k} \) over \( z^k \) for \( z^k \in \hat{Q}^{(k)} \), is the set of all filtration preserving isomorphisms

\[
\zeta^{k+1} : \hat{E}^{(k)} \rightarrow T_{z^k} \hat{Q}^{(k)}
\]

which satisfy the following conditions:

\[
\zeta^{k+1}(A) = \hat{A}_{z^k} \quad \text{for} \quad A \in \hat{\mathfrak{g}}^{(k)} \quad \text{and} \quad [\zeta^k] = z^k,
\]

where \( \zeta^k \) is the truncation of \( \zeta^{k+1} \), defined by the following commutative diagram

\[
\begin{array}{ccc}
\hat{E}^{(k)} & \xrightarrow{\zeta^{k+1}} & \hat{E}^{(k-1)} \\
\downarrow \zeta^k & & \downarrow \zeta^k \\
T_{z^k} \hat{Q}^{(k)} & \xrightarrow{} & T_{z^{k-1}} \hat{Q}^{(k-1)},
\end{array}
\]

and \( z^{k-1} \) is the image of \( z^k \) by the projection map \( \hat{Q}^{(k)} \rightarrow \hat{Q}^{(k-1)} \).

(b) The structure group \( \hat{G}^{(k+1)} \) of \( \hat{\mathfrak{F}}^{(k+1)} \hat{Q}^{(k)} \) consists of all filtration preserving linear isomorphisms \( \alpha^{k+1} \) which make the following diagram commutative:

\[
\begin{array}{ccc}
\hat{E}^{(k)} & \xrightarrow{\alpha^{k+1}} & \hat{E}^{(k-1)} \\
\downarrow \alpha^k & & \downarrow \alpha^k \\
\hat{E}^{(k)} & \xrightarrow{} & \hat{E}^{(k-1)}
\end{array}
\]

with \( [\alpha^k] = a^k \in \hat{G}^{(k)} \), and which satisfy

\[
\alpha^{k+1}(A) = \text{Ad}(a^k)A \quad \text{for} \quad A \in \hat{\mathfrak{g}}^{(k)}.
\]

(c) The action of \( \hat{G}^{(k+1)} \) on \( \hat{\mathfrak{F}}^{(k+1)} \hat{Q}^{(k)} \). For \( \alpha^{k+1} \in \hat{G}^{(k+1)} \) and \( \zeta^{k+1} \in \hat{\mathfrak{F}}^{(k+1)} \hat{Q}^{(k)} \) we define the right action \( \zeta^{k+1} \cdot \alpha^{k+1} \) by the following commutative diagram:

\[
\begin{array}{ccc}
\hat{E}^{(k)} & \xrightarrow{\zeta^{k+1} \cdot \alpha^{k+1}} & T_{z^k} \hat{Q}^{(k)} \\
\downarrow \alpha^k & & \downarrow \alpha^k \\
\hat{E}^{(k)} & \xrightarrow{} & T_{z^k} \hat{Q}^{(k)}
\end{array}
\]

where \( z^k = [\zeta^k] \) is the projection of \( \zeta^{k+1} \) to \( \hat{Q}^{(k)} \) and \( a^k = [\alpha^k] \) similarly.

**Notations.** In Definition 7.4 the filtered vector space \( \hat{E}^{(k)} \) and the filtered group \( \hat{G}^{(k+1)} \) depend on the structure group \( \hat{G}^{(k)} \) of \( \hat{Q}^{(k)} \). When we need to emphasize it, we write \( \hat{E}^{(k)}(\hat{\mathfrak{g}}^{(k)}) \) and \( \hat{G}^{(k+1)}(\hat{G}^{(k)}) \) for \( \hat{E}^{(k)} \) and \( \hat{G}^{(k+1)} \).

Let \( \hat{Q}^{(k)} \rightarrow M \) be a principal geometric structure of order \( k \). By setting for \( \ell > k \)

\[
\hat{\mathfrak{F}}^{(\ell)}(\hat{Q}^{(k)}) = \hat{\mathfrak{F}}^{(\ell)}(\hat{\mathfrak{F}}^{(\ell-1)} \hat{Q}^{(k)}), \quad \hat{G}^{(\ell)}(\hat{G}^{(k)}) = \hat{G}^{(\ell)}(\hat{G}^{(\ell-1)}(\hat{G}^{(k)}))
\]

and by passing to the limit

\[
\hat{\mathfrak{F}} \hat{Q}^{(k)} = \lim_{\ell \rightarrow \ell} \hat{\mathfrak{F}}^{(\ell)} \hat{Q}^{(k)}, \quad \hat{G}(\hat{G}^{(k)}) = \lim_{\ell \rightarrow \ell} \hat{G}^{(\ell)}(\hat{G}^{(k)}),
\]
we obtain a principal fiber bundle \( \check{\mathcal{Q}}^{(k)} \to (M, F) \) called the complete universal principal frame bundle of \( \check{\mathcal{Q}}^{(k)} \), which is endowed with an absolute parallelism defined by the canonical form \( \check{\theta} \).

The canonical form \( \check{\theta} \) is an \( \check{\theta}(\check{g}^{(k)}) = g_- \oplus \text{gr} \check{\xi}^{(k)} \)-valued 1-form on \( \check{\mathcal{Q}}^{(k)} \). Write \( \check{E}(\check{g}^{(k)}) \) simply as \( \check{E} \). We then have the structure function

\[
\check{\gamma} : \check{\mathcal{Q}}^{(k)} \to \text{Hom}(\wedge^2 \check{E}, \check{E})
\]
defined by \( d\check{\theta} + \frac{1}{2} \check{\gamma}(\check{\theta}, \check{\theta}) = 0 \).

The canonical form \( \check{\theta} \) and the structure function \( \check{\gamma} \) of \( \check{\mathcal{Q}}^{(k)} \) enjoy the same property as those of \( \mathcal{Q}^{(k)} \). More rigidly, we have

\[
\check{R}^a_{\check{\theta}} = \rho(a)^{-1} \check{\theta} \quad \text{for } a \in \check{G}(\check{\xi}^{(k)})
\]

and

\[
\check{\gamma}(A, X) = \rho(A)X \quad \text{for } A \in \check{g}(\check{\xi}^{(k)}) \text{ and } X \in \check{E}(\check{g}^{(k)})
\]

where \( \rho \) denotes the action of \( \check{G}(\check{\xi}^{(k)}) \) or \( \check{g}(\check{\xi}^{(k)}) \) on \( \check{E}(\check{g}^{(k)}) \). We have thus seen the similarity and the difference between \( \{Q^{(k)}, \mathcal{Q}^{(k)}\} \) and \( \{\check{Q}^{(k)}, \check{\mathcal{Q}}^{(k)}\} \).

**Remark 7.1.** The notions of principal geometric structure and universal principal frame bundle were introduced in [27] and [30] under different terminologies as non-commutative frame bundle, Cartan bundle, or tower, and there played a fundamental role to study the equivalence problem of geometric structures and, in particular, to construct Cartan connections.

It is in order to study the step prolongation that we have introduced in this paper in Section 2.1 the larger category of geometric structures by extending the category of principal geometric structures.

This distinction and similarity of two categories will then make clear the relation between step prolongation and Cartan connection as shown in the next sub section.

Parallel to Definition 3.1 we give the following:

**Definition 7.5.** A principal geometric structure \( \check{Q}^{(k)} \to M \) of type \( (g_-, \check{G}^{(k)}) \) is called proper if \( g_- \oplus \text{gr} \check{\xi}^{(k)} \) has a compatible structure of a transitive truncated graded Lie algebra, where \( \text{gr} \check{\xi}^{(k)} \) is the graded Lie algebra associated with the Lie algebra \( \check{g}^{(k)} \) of \( \check{G}^{(k)} \).

### 7.3. The structure function \( \tau \) and Cartan connections.

In this subsection we first show that a principal geometric structure can be viewed as a geometric structure whose structure function \( \tau \) is constant.

To see this more precisely, let \( \check{Q}^{(k)} \to M \) be a principal geometric structure. For \( 0 \leq i \leq k \), let \( \check{g}^{(i)} \) be the Lie algebra of \( \check{G}^{(i)} = \check{G}^{(k)}/F^{i+1} \) and set \( g^{(i)} = \oplus_{a=0}^{i} g_a = \text{gr} \check{g}^{(i)} \), \( \check{E}^{(i)} = g_- \oplus \check{g}^{(i)} \) and \( E^{(i)} = g_- \oplus g^{(i)} \).

Choose complementary subspace \( \{W^i\}_{1 \leq i \leq k} \) such that

\[
\text{Hom}(g_-, E^{(i-1)})_i = g_+ \oplus W^i.
\]

Then the choice of \( \{W^i\} \) determines in a natural manner identifications

\[
\phi_W^{(i)} : g^{(i)} \xrightarrow{\cong} \check{g}^{(i)} \quad \text{and} \quad \phi_W^{(i)} : E^{(i)} \xrightarrow{\cong} \check{E}^{(i)}
\]
and further constants $\circ \tau_{[i+1]} \in \text{Hom}(\mathfrak{g}^{(i)}, W^1_i)$.

Indeed, the construction can be done by induction as follows. Suppose that we have the isomorphism $\phi^{(i-1)}_W : \mathfrak{g}^{(i-1)} \to \check{\mathfrak{g}}^{(i-1)}$ and consider the following diagram:

\[
\begin{array}{ccccccc}
0 & \to & \mathfrak{g}_i & \to & \check{\mathfrak{g}}^{(i)} & \to & \check{\mathfrak{g}}^{(i-1)} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Hom}(\mathfrak{g}_-, \check{E}^{(i-1)})_i & \to & \check{\mathfrak{g}}^{(i)} & \to & \check{\mathfrak{g}}^{(i-1)} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & W^1_i & & \pi_W & & W^1_i & & 0
\end{array}
\]

where the first column is induced from the isomorphism $\phi^{(i-1)}_W : E^{(i-1)} \to \check{E}^{(i-1)}$ and the split exact sequence:

$0 \to \mathfrak{g}_i \to \text{Hom}(\mathfrak{g}_-, E^{(i-1)})_i \to W^1_i \to 0$

and the second column is the inclusion map $\iota_{\check{\mathfrak{g}}} : \check{\mathfrak{g}}^{(i)} \hookrightarrow \check{\mathfrak{g}}^{(i)} = \check{\mathfrak{g}}^{(i)} \oplus W^1_i \oplus \mathfrak{g}_i$.

Recalling the definition of the principal prolongation $\check{\mathfrak{g}}^{(i)}$ of $\check{\mathfrak{g}}^{(i-1)}$ and taking into account the isomorphism $\phi^{(i-1)}_W : E^{(i-1)} \to \check{E}^{(i-1)}$, we see that the second row has a natural splitting (as vector spaces). Since the first column and the second row split so does the first row, which then yields an isomorphism $\phi^{(i)}_W : \mathfrak{g}^{(i)} \to \check{\mathfrak{g}}^{(i)}$.

A diagram chasing gives a well-defined map

$w_{[i+1]} : \check{\mathfrak{g}}^{(i-1)} \to W^1_i$

such that

$A + w_{[i+1]}(A) \in \check{\mathfrak{g}}^{(i)}$ for $A \in \check{\mathfrak{g}}^{(i-1)}$

according to the identifications:

$\check{\mathfrak{g}}^{(i)} \subset \check{\mathfrak{g}}^{(i)} = \check{\mathfrak{g}}^{(i-1)} \oplus W^1_i \oplus \mathfrak{g}_i$.

We then define $\circ \tau_{[i+1]} : \mathfrak{g}^{(i)} \to \text{Hom}(\mathfrak{g}_-, E^{(i-1)})_i$ by

$\circ \tau_{[i+1]}|_{\mathfrak{g}^{(i-1)}} = w_{[i+1]} \circ \phi^{(i-1)}_W$, \quad $\circ \tau_{[i+1]}|_{\mathfrak{g}_i} = id_{\mathfrak{g}_i}$.

The notation being as above, we have

**Proposition 7.1.** Let $\check{Q}^{(k)} \xrightarrow{G^{(k)}} M$ be a principal geometric structure of type $(\mathfrak{g}_-, \check{\mathfrak{g}}^{(k)})$. Then a choice of $\{W^1_i\}_{1 \leq i \leq k}$ determines a natural identification of $\check{Q}^{(k)}$ with a geometric structure $Q^{(k)}$ with the structure function $\tau^{[k]} = \circ [k]$ as follows: There exists a geometric structure

$Q^{(i)} \xrightarrow{G_i} Q^{(i-1)} \to \cdots \to Q^{(0)} \to M$
of type \((g_-, G_0, \ldots, G_i)\) and bundle isomorphisms \(\iota_W^{(i)} : Q^{(i)} \to Q^{(i)}\) and \(\mathcal{H}t_W^{(i)} : \mathcal{H}^{(i+1)}Q^{(i)} \to \mathcal{H}^{(i+1)}Q^{(i)}\) for \(0 \leq i \leq k\) such that

1. the structure function \(\tau_{[i+1]}\) of \(\mathcal{H}^{(i+1)}Q^{(i)}\) equals to \(\tau_{[i+1]}\), and
2. the following diagram is commutative

\[
\begin{array}{c}
\mathcal{H}^{(i+1)}Q^{(i)} \\
\downarrow \mathcal{H}^{(i+1)}Q^{(i)} \\
\downarrow \mathcal{H}^{(i+1)}Q^{(i)} \\
Q^{(i)} \end{array}
\]

Here, we ignore the first row if \(i = k\).

**Proof.** The isomorphism \(\phi_W^{(i)} : E^{(i)} \to \tilde{E}^{(i)}\) induces a map \(\mathcal{H}t_W^{(i)} : \mathcal{H}^{(i+1)}Q^{(i)} \to \mathcal{H}^{(i+1)}Q^{(i)}\) as follows. Let \(\tilde{z}^{i+1} = [\tilde{\zeta}^{i+1}] \in \mathcal{H}^{(i+1)}Q^{(i)}\). Define \(\zeta^{i+1} = [\zeta^{i+1}] \in \mathcal{H}^{(i+1)}Q^{(i)}\) by

\[
\begin{array}{rcl}
\tilde{E}^{(i)} & \xrightarrow{\tilde{\tau}_{(i)}} & \tilde{Q}^{(i)} \\
\phi_W^{(i)} & \xrightarrow{\iota_W^{(i)}} & \iota_W^{(i)} \\
E^{(i)} & \xrightarrow{\tau_{(i)}} & Q^{(i)}
\end{array}
\]

and take the reduction \(\mathcal{H}^{(i+1)}Q^{(i)}\) of \(\mathcal{H}^{(i+1)}Q^{(i)}\) as the inverse image of \(\mathcal{H}^{(i+1)}Q^{(i)}\). Then \(\mathcal{H}^{(i+1)}Q^{(i)}\) coincides exactly with the image of \(\mathcal{H}^{(i+1)}Q^{(i)}\). \(\square\)

**Theorem 7.1.** Let \(Q\) be a proper geometric structure of type \((g_-, G_0, \ldots, G_k, \ldots)\). If the structure function \(\tau\) of \(Q\) is constant and the structure groups \(G_i\) are connected for \(i \geq 0\), then \(Q\) is a principal proper geometric structure and \((Q, \theta)\) is a pre-Cartan connection.

**Proof.** Let \(Q\) be a proper geometric structure of type \((g_-, G_0, \ldots, G_k, \ldots)\) with constant structure function \(\tau\). Let \(g = \bigoplus_p \mathfrak{g}_p\) be the transitive graded Lie algebra associated to \(Q\), which we also denote by \(E\). We also write \(g^0 = \bigoplus_{p \geq 0} \mathfrak{g}_p\), the graded subalgebra of \(g\).

From the fundamental identity it follows that the structure function \(\sigma\) is also constant (Corollary 13). The Bianchi identity implies that the structure function \(\sigma \in \text{Hom}(\bigwedge^2 g^0, g^0)\) satisfies the Jacobi identity. Hence \((g^0, \sigma)\) defines a filtered Lie algebra \((\tilde{g}, \{F^p \tilde{g}\})\) such that \([F^p \tilde{g}, F^q \tilde{g}] \subseteq F^{p+q} \tilde{g}\) and \(\text{gr} g^0\) is isomorphic to \(g^0\). Again by the Bianchi identity, we see that the structure function \(\tau + \sigma \in \text{Hom}(g^0 \otimes E, E)\) determines an action of \(\tilde{g}^0\) on \(E\), which acts on \(g^0 \subseteq E\) as the adjoint action, so we also write \(E = \tilde{E} = g_- \oplus \tilde{g}^0\). Then we have \(F^p \tilde{g}^0 \cdot F^p \tilde{E}^q \subset F^{p+1} \tilde{E}\), and thus we have an injective Lie algebra homomorphism

\[
\iota : \tilde{g}^{(k)} \to F^0 \mathfrak{gl}(\tilde{E}^{(k-1)})/F^{k+1}
\]

where \(\tilde{g}^{(k)} = \tilde{g}/F^{k+1}\) and \(\tilde{E}^{(k-1)} = \tilde{E}/F^{k}\). We then identify \(\tilde{g}^{(k)}\) with its image \(\iota(\tilde{g}^{(k)})\).

Now we take the connected Lie subgroup \(\tilde{G}^{(k)}\) of \(F^0 \text{GL}(\tilde{E}^{(k-1)})/F^{k+1}\) with Lie algebra \(\tilde{g}^{(k)}\). Let \(\tilde{G} = \lim_{\longrightarrow} G^{(k)}\), which then satisfies: \(F^k G/F^{k+1} \cong G_k\). Then via the canonical form \(\theta\) we can easily see that \(\tilde{Q}^{(k)} = Q^{(k)} = Q/F^{k+1}\) admits \(\tilde{G}^{(k)}\)-action, which together
with the canonical Pfaff class $[\theta^{(k-1)}]$ makes $Q^{(k)}$ a principal proper geometric structure of order $k$. The rest of assertions of the theorem then follows immediately. \hfill $\square$

**Corollary 7.1.** An involutive proper geometric structure $Q^{(k)}$ of order $k$ is a principal geometric structure, provided that the structure group $G_i$ of $Q^{(i)} \to Q^{(i-1)}$ is connected for $0 \leq i \leq k$.

**Proof.** The step prolongation $\mathcal{S}_W Q^{(k)}$ has constant structure function. Hence $\tau$ is constant. \hfill $\square$

**Remark 7.2.** The definition of involutive geometric structure given in [30] and in the present paper thus infinitesimally coincide. In [30] the geometric structures are assumed a priori to be principal.

Now let us consider the case where the structure function $\tau$ is flat and demonstrate a slightly stronger statement than that of Theorem 7.1.

Let $g_\omega = \bigoplus_{p<0} g_p$ be a graded Lie algebra. Let $G_0$ be a connected subgroup of the connected component $\text{Aut}(g_\omega)^0$ of the automorphism group of the graded Lie algebra and $g_0$ be its Lie algebra. Let $g = \bigoplus_{p\in\mathbb{Z}} g_p$ be the prolongation of $g_\omega \oplus g_0$. Let $G^{(k)}, G_k$ be the Lie groups with Lie algebra $g^{(k)} = \bigoplus_{i=0}^k g_i$ and $g_k$, respectively, constructed as follows. Write $G^{(\infty)}$ also as $G^0$.

We construct $G^{(k)}$ inductively. Suppose that we have constructed $G^{(k-1)}$ in such a way that

$$G^{(k-1)} = G_0 \times N(g_1 \oplus \cdots \oplus g_{k-1}) \subset F^0GL(E^{(k-2)})/F_k.$$  

Here, $G_0$ is regarded as a closed subgroup of $F^0GL(E^{(k-2)})/F_k$ and $N(g_1 \oplus \cdots \oplus g_{k-1})$ is a closed normal connected subgroup of $F^0GL(E^{(k-2)})/F_k$ with Lie algebra $g_1 \oplus \cdots \oplus g_{k-1}$, defined by

$$N(g_1 \oplus \cdots \oplus g_{k-1}) := \text{Exp}(g_1 \oplus \cdots \oplus g_{k-1})/F_k$$

where $\text{Exp} : gl(E^{(k-2)}) \to GL(E^{(k-2)})$ is the exponential map.

Let $G^{(k)}_g = G^{(k)}(G^{(k-1)})$ be the principal prolongation of $G^{(k-1)}$. Then $G_0$ is regarded as a closed subgroup of $G^{(k)}_g$ and

$$G^{(k)}_g = G_0 \times N(g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k)$$

where $q_k$ is $\text{Hom}(g_\omega, E^{(k-1)})_k$ and $N(g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k)$, define by

$$N(g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k) := \text{Exp}(g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k)/F_{k+1},$$

is a closed normal connected subgroup with Lie algebra $g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k$. Here, $\text{Exp} : gl(E^{(k-1)}) \to GL(E^{(k-1)})$ is the exponential map. Since $g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k$ is contained in the lower triangular matrices, the exponential map $\text{Exp} : gl(E^{(k-1)}) \to GL(E^{(k-1)})$ maps $g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k$ onto a closed subgroup (homeomorphic to an Euclidean space), and the quotient map

$$\text{Exp}(g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k) \to \text{Exp}(g_1 \oplus \cdots \oplus g_{k-1} \oplus q_k)/F_{k+1}$$

is bijective.

For the subalgebra $g_1 \oplus \cdots \oplus g_{k-1} \oplus g_k \subset g_1 \oplus \cdots \oplus g_{k-1} \oplus g_k$, the image $N(g_1 \oplus \cdots \oplus g_{k-1} \oplus g_k)$ of the map

$$g_1 \oplus \cdots \oplus g_{k-1} \oplus g_k \to \text{Exp}(g_1 \oplus \cdots \oplus g_{k-1} \oplus g_k) \to \text{Exp}(g_1 \oplus \cdots \oplus g_{k-1} \oplus g_k)/F_{k+1}$$
is a closed normal subgroup of $G^{(k)}$. Set $G^{(k)} = G_0 \times N(g_1 \oplus \cdots \oplus g_{k-1} \oplus g_k)$. Then there is a filtration $\{F^i G^{(k)}\}$ on $G^{(k)}$ so that $G^{(k)}/F^{k+1} = G^{(k-1)}$, and there is an embedding $g^{(k)} \to g^{(k)}(g^{(k-1)})$, where $g^{(k-1)}$ is the Lie algebra of $G^{(k-1)}$.

Fix subspaces $\{W^1_i\}_{i \geq 0}$ and $\{W^2_{i+1}\}_{i \geq 0}$ such that

\[
\text{Hom}(g_-, g)_i = W^1_i \oplus \partial g_i, \\
\text{Hom}(\wedge^2 g_-, g)_{i+1} = W^2_{i+1} \oplus \partial \text{Hom}(g_-, g)_{i+1}.
\]

**Theorem 7.2.** Let $Q^{(0)} \xrightarrow{G_0} (M, F)$ be a proper geometric structure of order 0 with $G_0$ being connected. Let $Q^{(k)}$ be the $W$-normal step prolongation of $Q^{(0)}$ of order $k$ and let $Q$ be the $W$-normal complete step prolongation of $Q^{(0)}$. If $\tau^{[k+1]}$ is flat on $Q^{(k+1)}$, then $Q^{(k)} \to M$ is a proper principal geometric structure of order $k$.

If $\tau$ is flat, i.e., $\tau[i]$ is flat for all $i$, then $Q \to M$ is a Cartan connection of type $G/G^0$.

**Proof.** Let $Q^{(0)} \xrightarrow{G_0} (M, F)$ be a geometric structure of order 0 of type $(g_-, G_0)$ with $G_0$ connected and let $Q^{(k)}$ be the $W$-normal step prolongation of $Q^{(0)}$ of order $k$ for $k \geq 1$.

We will prove by induction on $\ell$ that the following holds:

\[(*) \quad \text{If } \tau^{[\ell+1]} \text{ is flat on } Q^{(\ell+1)}, \text{ then } Q^{(\ell)} \to M \text{ is a principal subbundle of } \mathcal{R}^{(\ell)} Q^{(\ell-1)} \text{ with structure group } G^{(\ell)}.\]

For the case when $\ell = 0$, there is noting to prove: $\tau^{[1]}$ is flat and $Q^{(0)} \to M$ is a principal subbundle of $\mathcal{R}^{(0)}(M) = \mathcal{S}^{(0)}(M)$. Furthermore, all three bundles $\mathcal{S}^{(1)} Q^{(0)}$, $\mathcal{S}^{(1)} Q^{(0)}$, $\mathcal{R}^{(1)} Q^{(0)}$ are the same.

Assuming that the statement (*) holds for $\ell < k$, we will prove that the statement (*) holds for $\ell = k$. If $\tau^{[k+1]}$ is flat, then $\tau[k]$ is flat. By the inductive assumption, $Q^{(k-1)} \to M$ is a proper principal geometric structure of order $k - 1$ with structure group $G^{(k-1)}$. The isomorphism from the Lie algebra $g^{(k-1)}$ with the graded Lie algebra $\oplus_{i=0}^{k-1} g_i$ induces an embedding of $\mathcal{R}^{(k)} Q^{(k-1)}$ into $\mathcal{S}^{(k)} Q^{(k-1)}$ whose image is $\mathcal{R}^{(k)} Q^{(k-1)}$ as in the proof of Proposition 7.1. We identify $\mathcal{S}^{(k)} Q^{(k-1)}$ with $\mathcal{R}^{(k)} Q^{(k-1)}$ which is a principal bundle over $M$ with structure group $G^{(k)}(G^{(k-1)})$.

Note that the group $G^{(k)}(G^{(k-1)})$ contains $G^{(k)}$ as a subgroup. Therefore, $G^{(k)}$ acts on $\mathcal{R}^{(k)} Q^{(k-1)}$, and the structure function $\kappa[k]$ on $\mathcal{R}^{(k)} Q^{(k-1)}$ satisfies

\[R^*_a \kappa[k] = \rho(a)^{-1} \kappa[k] \quad \text{for } a \in G^{(k)}.\]

Thus

\[(1) \quad \kappa[k](z^{(k)} a) = \rho(a)^{-1} \kappa[k](z^{(k)})\]

for any $z^{(k)} \in \mathcal{R}^{(k)} Q^{(k-1)}$ and $a \in G^{(k)}$.

Now $Q^{(k)} = \mathcal{S}^{(k)} Q^{(k-1)}$ is a subbundle of $\mathcal{S}^{(k)} Q^{(k-1)} \to Q^{(k-1)}$ defined by

\[Q^{(k)} = \left\{ z \in \mathcal{S}^{(k)} Q^{(k-1)} : \kappa[k](z) \in W^2_k \right\}.
\]

We will show that the action of $G^{(k)}$ on $\mathcal{R}^{(k)} Q^{(k-1)} = \mathcal{S}^{(k)} Q^{(k-1)}$ leaves invariant $Q^{(k)}$.  

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Let $Q$ be the $W$-normal complete step prolongation of $Q^{(0)} \to M$ and let $\theta$ denote the canonical Pfaff form on $Q$. Take a section $\sigma$ of $Q \to Q^{(k)}$ and set $\overline{\theta} := \sigma^* \theta$ and $\theta(k) := \sigma^* \theta(k)$, where $\theta(k)$ is the $E(k)$-component of $\theta$, that is, $\theta(k) = \pi(k) \theta$, where $\pi(k) : E \to E^{(k)} = E/F^{k+1}E$.

Let $V_{z(k)}Q^{(k)}$ denote the vertical tangent space at $z^{(k)}$. Then

$$
\overline{\theta}_{z(k)} : V_{z(k)}Q^{(k)} \to F^0E^{(k)} = \mathfrak{g}^{(k)}
$$

is an isomorphism of filtered vector space and

$$
\text{gr} \overline{\theta}^{(k)} : \text{gr} V \to \text{gr} F^0E^{(k)}
$$

is the canonical isomorphism.

For $A \in \mathfrak{g}^{(k)}$, let $A^*$ denote the vertical vector field on $Q^{(k)}$ determined by $\overline{\theta}^{(k)}(A^*) = A$ and let $a(t) = \exp(tA) \in G^{(k)}$.

Let $z^{(k)}$ be an element of $Q^{(k)}$ and let $z^{(k)}(t)$ be the integral curve of $A^*$ with $z^{(k)}(0) = z^{(k)}$. By Corollary 4.3 the flatness of $\tau^{[k+1]}$ implies

$$
A^* \kappa^{[k]} = -\rho(A) \kappa^{[k]}.
$$

Integrating this, we have

$$
\kappa^{[k]}(z^{(k)}(t)) = \rho(a(t))^{-1} \kappa^{[k]}(z^{(k)}).
$$

By (11) and (2), we have

$$
\kappa^{[k]}(z^{(k)}(t)) = \kappa^{[k]}(z^{(k)}a(t)).
$$

Note that $\{z(t)^{(k)}\}$ is a curve in $Q^{(k)}$ and $\{z^{(k)}a(t)\}$ is a curve in $\mathcal{S}^{(k)}Q^{(k-1)} = \mathcal{S}^{(k)}W^1Q^{(k-1)}$, and that both $Q^{(k)}$ and $\mathcal{S}^{(k)}Q^{(k-1)}$ are principal bundles on $Q^{(k-1)}$. Furthermore, both $z^{(k)}(t)$ and $z^{(k)}a(t)$ project to the same point in $Q^{(k-1)}$.

Recall that $Q^{(k)} = \mathcal{S}^{(k)}W^1Q^{(k-1)}$ is defined by imposing a condition on the value of $\kappa^{[k]}$:

$$
Q^{(k)} = \left\{ z \in \mathcal{S}^{(k)}W^1Q^{(k-1)} : \kappa^{[k]}(z) \in W^2_k \right\}.
$$

Since $z^{(k)}(t)$ is contained in $Q^{(k)}$, so is $z^{(k)}a(t)$. Hence the action of $G^{(k)}$ on $\mathcal{S}^{(k)}Q^{(k-1)}$ leaves invariant $Q^{(k)}$. It follows that $Q^{(k)} \to M$ is a principal subbundle of $\mathcal{S}^{(k)}Q^{(k-1)}$ with structure group $G^{(k)}$.

If $\tau$ is flat, then $Q^{(k)} \to M$ is a proper principal geometric structure for any $k$ and thus its projective limit $Q \to M$ with the canonical Pfaff form $\theta$ is a Cartan connection. $\square$

### 7.4. Condition (C).

Now assume that $G_0$ satisfies the condition (C), that is, there exists a $G^0$-invariant subspace $W^2 = \oplus_{i \geq 1} W^2_i$ such that

$$
\text{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g})_i = \partial \text{Hom}(\mathfrak{g}_-, \mathfrak{g}), \oplus W^2_i \quad \text{for } i \geq 1.
$$

Then by Theorem 3.10.1 of [30] to each geometric structure $P^{(0)} \to M$ of order 0 of type $(\mathfrak{g}_-, G_0)$ there is associated a series of principal bundles $P^{(k)} G^{(k)} \to M$ defined by

$$
P^{(k)} = \left\{ z^k \in \mathcal{S}^{(k)}P^{(k-1)} : \kappa^{[k]}(z^k) \in W^2_k \right\}
$$

and the projective limit $P := \lim_{\to k} P^{(k)}$ is equipped with an absolute parallelism $\theta_P$ which defines a Cartan connection of type $(\mathfrak{g}_-, G^0)$. The principal $G^{(k)}$-bundle $P^{(k)} \to M$
Theorem 7.3. Assume that associated a series of principal bundles \(Q(k)\) and \(Q(k-1)\) and the projective limit \(\mathcal{W}Q(0)\), the \(W\)-normal complete prolongation of \(Q(0)\) constructed in Section 3.2.

Theorem 7.3. Assume that \(G_0\) satisfies the condition (C). Let \(Q(0)\) be a geometric structure of type \((g_-, G_0)\). Then there is an isomorphism

\[\iota_W : \mathcal{R}_W Q(0) \rightarrow \mathcal{W}Q(0).\]

Proof. Let \(Q(k)\) denote the \(W\)-normal step prolongation of order \(k\) of \(Q(0)\) and \(P(k)\) denote the \(W\)-normal principal prolongation of order \(k\) of \(P(0) = Q(0)\). We will show that there is an isomorphism \(\iota_W : P(k) \rightarrow Q(k)\) for any \(k \geq 1\).

As in the proof of Theorem 7.2 we define inductively,

\[
\begin{array}{ccc}
\mathcal{R}(k+1) P(k) & \xrightarrow{\iota_W} & \mathcal{W} Q(k+1) \\
\downarrow & & \downarrow \\
P(k) & \xrightarrow{\iota_W} & Q(k).
\end{array}
\]

Furthermore, \(P(k+1)\) and \(Q(k+1)\) are defined by the same condition:

\[
P(k+1) = \{ z^{k+1} \in \mathcal{R}(k+1) P(k) : \kappa_{[k+1]}(z^{k+1}) \in W_{k+1} \}
\]

\[
Q(k+1) = \{ z^{k+1} \in \mathcal{W} Q(k) : \kappa_{[k+1]}(z^{k+1}) \in W_{k+1} \}.
\]

Thus \(\mathcal{R}(k)\) sends \(P(k)\) onto \(Q(k+1)\). This completes the induction and gives the isomorphism

\[\iota_W : \mathcal{R}_W P(0) \rightarrow \mathcal{W}Q(0).\]

We remark that in Theorem 7.3 \(\tau^{[k+1]}\) on \(\mathcal{W}Q(0)\) is flat for any \(k \geq 0\) by Proposition 7.1.

For the sake of consistence of our description, let us prove again directly not relying on Theorem 3.10.1 of [30] that the \(G_{k+1}\)-principal bundle \(Q(k+1) \rightarrow Q(k)\) constructed as above turns out to be a principal \(G^{(k+1)}\)-bundle \(Q(k+1) \rightarrow M\).

Suppose that \(Q(k) \rightarrow M\) is a principal \(G(k)\)-bundle. Construct \(\mathcal{R}^{(k+1)} Q(k)\) and \(\mathcal{W} Q(k)\), and identify \(\mathcal{R}^{(k+1)} Q(k)\) with \(\mathcal{W} Q(k)\). Then \(Q(k+1)\) is defined by

\[
Q(k+1) = \{ z^{k+1} \in \mathcal{R}^{(k+1)} Q(k) = \mathcal{R}^{(k+1)} Q(k) : \kappa_{[k+1]}(z^{k+1}) \in W_{k+1} \}.
\]

Let \(\mathcal{R} Q(k)\) be the complete universal principal frame bundle of \(Q(k)\). Denote by \(G_{\mathcal{R}}\) and \(G_{\mathcal{R}^{(k+1)}}\) the structure group of \(\mathcal{R} Q(k)\) and \(\mathcal{R}^{(k+1)} Q(k)\) respectively. Then the structure
function $\gamma$ of $\mathcal{A}Q^{(k)}$ satisfies
\[ \gamma(\lambda a) = \rho(a)^{-1}\gamma(\lambda) \quad \text{for } \lambda \in \mathcal{A}Q^{(k)} \text{ and } a \in G_{\mathcal{R}}, \]
which descends to the structure function $\gamma^{[k+1]}$ of $\mathcal{A}^{(k+1)}Q^{(k)}$ satisfying
\[ \gamma^{[k+1]}(z^{k+1}a^{k+1}) = \rho(a^{k+1})^{-1}\gamma^{[k+1]}(z^{k+1}) \quad \text{for } z^{k+1} \in \mathcal{A}^{(k+1)}Q^{(k)} \text{ and } a^{k+1} \in G_{\mathcal{R}}^{(k+1)}. \]
Since $\tau^{[k+1]}$ is flat, so in $\sigma^{[k+2]}$ by Corollary 4.12, and thus we can write
\[ \gamma^{[k+1]} = \kappa^{[k+1]} + \gamma_0^{[k+1]} \]
where $\kappa^{[k+1]} = \sum_{1 \leq i \leq k+1} \kappa_i^{[k+1]}$ and $\gamma_0^{[k+1]}$ is the bracket of $g_- \oplus g^{(k+1)}$. Now that $G^0$ preserves the bracket of $g$, $G^{(k+1)}$ preserves $\gamma_0^{[k+1]}$. It then follows that
\[ \rho(a^{k+1})^{-1}(\kappa^{[k+1]} + \gamma_0^{[k+1]}) = \rho(a^{k+1})^{-1}\kappa^{[k+1]} + \gamma_0^{[k+1]}. \]
By the assumption that $W^2$ is $G^0$-invariant, $W^2, [k+1] = \oplus_{1 \leq i \leq k+1} W_i^2$ is $G^{(k+1)}$-invariant. Hence we deduce that if $z^{k+1} \in Q^{(k+1)}$ and $a^{k+1} \in G^{(k+1)}$, then
\[ \kappa^{[k+1]}(z^{k+1}a^{k+1}) = \rho(a^{k+1})^{-1}\kappa^{[k+1]}(z^{k+1}) \in W^2, [k+1] \]
and thus $z^{k+1}a^{k+1}$ belongs to $Q^{(k+1)}$. This proves that $Q^{(k+1)}$ is a principal $G^{(k+1)}$-bundle.

By Theorem 7.3 if the condition (C) is satisfied, then the $W$-normal complete step prolongation $\mathcal{A}_W Q^{(0)}$ produces a Cartan connection. We remark that $\mathcal{A}_W Q^{(0)}$ does not depend on the choice of $\{W_i\}$ because $\tau$ is flat in this case.

7.5. Inductive vanishing of the structure function $\kappa$.

**Proposition 7.2.** Let $Q^{(0)} \xrightarrow{G_0} (M, F)$ be a geometric structure of order 0 of type $(g_-, G_0)$ with $G_0$ being connected. Let $Q^{(k)}$ be the $W$-normal step prolongation of order $k$ of $Q^{(0)} \xrightarrow{G_1} (M, F)$. If $\kappa^{[k]}$ is flat, then

1. $\tau^{[k+1]}$ is flat and
2. $Q^{(k)}$ is a principal bundle on $M$ with structure group $G^{(k)}$ and
3. $\kappa^{[k+1]}$ induces a section $s_{\kappa^{[k+1]}}$ of the associated vector bundle
\[ \mathcal{H}^{2}_{k+1} := Q^{(0)} \times_{G_0} H^2(g_-, g)_{k+1} \]
on $M$, the vanishing of which implies the vanishing of $\kappa^{[k+1]}$.

**Proof.** Note that $\kappa^{[1]}$ is a function on $Q^{(1)}$ with values in $\Hom(\wedge^2 g_-, g)_1$. From Corollary 4.1, we get $\partial \kappa^{[1]} = 0$. By Proposition 3.2 $\kappa^{[1]}(z(1 + A)) = \kappa^{[1]}(z) + \partial A$ for $A \in g_1$ and thus $\kappa^{[1]}(za) \equiv \kappa^{[1]}(z) \mod \partial g_1$ for $a \in G_1$. Therefore, $\kappa^{[1]}$ induces a function
\[ \overline{\kappa}_1 : Q^{(0)} \rightarrow H^2(g_-, g)_1. \]
Since $\overline{\kappa}_1(za) = a^{-1}\overline{\kappa}_1(z)$ for $a \in G_0$, the function $\overline{\kappa}_1$ induces a section $s_{\overline{\kappa}_1}$ of the vector bundle $\mathcal{H}^2_1 := Q_0 \times_{G_0} H^2(g_-, g)_1$ on $M$. If the section $s_{\overline{\kappa}_1}$ vanishes, then the function $\overline{\kappa}_1$ vanishes and thus $\kappa_1 = \partial A$ for some $\chi : Q_0 \rightarrow \Hom(g_-, g)_1$. Since $\kappa^{[1]}$ has values in $W^2_i$ and $W^2_i \cap \partial \Hom(g_-, g)_1 = 0$, we have $\kappa^{[1]} = 0$. 

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Now assume that \( \kappa^{[k]} \) is flat, i.e., \( \kappa^{[i]} = 0 \) for all \( 1 \leq i \leq k \). Then by the induction hypothesis, \( \tau^{[k]} \) is flat. By Corollary 4.1 we have

\[
\begin{align*}
(i) & \quad \partial \kappa^{[k+1]} = 0; \\
(ii) & \quad (\partial (\tau^{[k+1]} (A, \cdot))) = 0 \text{ for any } A \in g_\alpha, \text{ where } 0 \leq a \leq k - 1.
\end{align*}
\]

By (ii) \( \tau^{[k+1]} (A, \cdot) \) is contained in \( \text{Ker} \partial \). Since \( g \) is the prolongation of \((g_-, g_0)\), we have \( H^1_+(g_-, g) = 0 \). Thus \( \tau^{[k+1]} (A, \cdot) \) is contained in \( \partial g_k \). From \( \tau^{[k+1]} (A, \cdot) \in W^1_k \) and \( W^1_k \cap \partial g_k = 0 \), it follows that \( \tau^{[k+1]} (A, \cdot) = 0 \) for any \( A \in g_\alpha \), where \( 0 \leq a \leq k - 1 \). By Theorem 7.2 \( Q^{(k)} \to M \) is a proper principal geometric structure with structure group \( G^{(k)} \). Furthermore, we have

\[
R^*_a (\kappa_0 + \kappa_1 + \cdots + \kappa_{[k+1]}) = \rho(a)^{-1} (\kappa_0 + \kappa_1 + \cdots + \kappa_{[k+1]}) \text{ for } a \in G^{(k)}.
\]

Hence we have

\[
\begin{align*}
R^*_a \kappa^{[k+1]} & = \rho(a)^{-1} \kappa^{[k+1]} \text{ for } a_0 \in G_0 \\
\tilde{A}_k \kappa^{[k+1]} & = -\rho(A) \kappa^{[k+1]} \text{ for } A \in g_\alpha.
\end{align*}
\]

By the assumption that \( \kappa^{[i]} = 0 \) for all \( 1 \leq i \leq k \), for \( a \in G^{(k)} \) we have

\[
R^*_a \kappa^{[k+1]} = \rho(a)^{-1} \kappa^{[k+1]}
\]

where \( a_0 \) is the \( G_0 \)-component of \( a \).

By Proposition 3.2 \( \kappa^{[k+1]} (z (1 + A)) = \kappa^{[k+1]} (z) + \partial A \) for \( A \in g_{k+1} \), and thus \( \kappa^{[k+1]} (z a) \equiv \kappa^{[k+1]} (z) \) \( \mod \partial g_{k+1} \) for \( a \in G_{k+1} \). Together with (i), it follows that \( \kappa^{[k+1]} : Q^{(k+1)} \to \text{Hom}(\Lambda^2 g_-, g)_{k+1} \) induces a function

\[
\tau^{[k+1]} : Q^{(k)} \to H^2_+ (g_-, g)_{k+1},
\]

which again induces a section \( s_{\kappa^{[k+1]}} \) of the vector bundle \( H^2_{k+1} := Q^{(k)} \times_{G_0} H^2_+ (g_-, g)_{k+1} \) on \( M \). By the same arguments as in the case when \( k = 0 \), if \( s_{\kappa^{[k+1]}} \) vanishes, then \( \kappa^{[k+1]} \) vanishes. This completes the proof of Theorem 7.2 \( \square \)

Theorem 7.4. Let \( Q^{(0)} \overset{G_0}{\to} (M, F) \) be a geometric structure of order 0 with \( G_0 \) being connected. Assume that there is no nontrivial section of \( \mathcal{H}_k = Q^{(0)} \times_{G_0} H^2_+ (g_-, g)_{k+1} \) for all \( k \geq 1 \). Then \( (\mathcal{F}_W Q^{(0)}, \theta) \) is a Cartan connection of type \( G/G^0 \) which is flat, i.e., whose curvature vanishes.

Proof. It follows from Proposition 7.2 and Theorem 7.2 \( \square \)

In the smooth category, there always exists a nontrivial smooth section of the smooth vector bundle \( \mathcal{H}_k = Q^{(0)} \times_{G_0} H^2_+ (g_-, g)_{k+1} \) unless \( H^2_+ (g_-, g)_{k+1} \) is zero. Thus Theorem 7.4 cannot be applied, and we need an explicit formula for structure functions to show their flatness or to compare them. Relations among structure functions as in Theorem 4.1 will indeed play a more important role in this category. However, in the complex analytic category, certain conditions imposed on the geometry of the base manifold enforce the vanishing of holomorphic sections of the holomorphic vector bundle \( \mathcal{H}_k \), which will be explained with a concrete example in subsection 8.2.

8. Applications to Subriemannian geometry and complex geometry

8.1. Subriemannian geometry.
8.1.1. A subriemannian manifold \((M, D, s)\) is a smooth manifold \(M\) equipped with a subbundle \(D \subset TM\) and a fibre metric \(s\) of \(D\), that is, a smooth section \(s\) of the symmetric tensor product \(S^2D^*\) of the dual vector bundle \(D^*\). Since around 1980 subriemannian geometry has been studied from many different domains, nilpotent geometry and analysis, riemannian and symplectic geometry, control theory, etc. ([1], [25], [23]).

Subriemannian geometry, as a generalization of riemannian geometry, inherits from it to some extent main notions and properties. For instance, we may speak of a length minimizing curve \(\gamma\) joining two points of a subriemannian manifold \((M, D, s)\) among the integral curves of \(D\), and we may also speak of the Hamiltonian flow of the energy function associated with a subriemannian metric.

A big surprise was brought by the discovery that there exists a minimizing curve of a subriemannian manifold called abnormal geodesic which appears depending only on the distribution and does not satisfy the usual geodesic equation.

Another difference between two geometries comes from the fact that the first order approximation at a point of a riemannian manifold is nothing but an Euclidean vector space, while the first order approximation of a subriemannian manifold is a pair \((g_{-1}, \sigma)\) of a nilpotent graded Lie algebra \(g_{-1} = \bigoplus_{p < 0} g_p\) and an inner product \(\sigma\) on \(g_{-1}\), which has a great deal of variety. The distributions \(D\) are highly locally non-trivial. This gives to subriemannian geometry much more variety than to riemannian geometry.

Moreover, it had not been clear how to define the curvatures of a subriemannian manifold. In the next paragraph, as an application of our general method, we show how to define the curvature of a subriemannian manifold of constant symbol along with a general algorithm to compute it.

8.1.2. We define a subriemannian filtered manifold \((M, F, s)\) to be a filtered manifold \((M, F)\) equipped with a fibre metric \(s\) on \(F^{-1}\).

When we consider a subriemannian manifold \((M, D, s)\), the distribution \(D\) is usually assumed bracket generating. There is a bijective correspondence between the subriemannian manifolds \((M, D, s)\) with regularly bracket generating and the subriemannian filtered manifolds \((M, F, s)\) with \(F\) being generated by \(F^{-1}\).

Hereafter we consider subriemannian manifolds \((M, F, s)\) whose metric \(s\) is non-degenerate, but not necessarily positive definite.

For each \(x \in M\) there is associated a pair \((\text{gr}F_x, s_x)\) called the subriemannian symbol of \((M, F, s)\) at \(x\), where \(s_x\) is viewed as an inner product on \(\text{gr}_{-1}F_x\). We say that \((M, F, s)\) has constant symbol of type \((\mathfrak{g}_-, \sigma)\), where \(\mathfrak{g}_- = \bigoplus_{p < 0} \mathfrak{g}_p\) is a nilpotent graded Lie algebra and \(\sigma\) an inner product on \(\mathfrak{g}_{-1}\), if there exists for every \(x\) a graded Lie algebra isomorphism \(z : \mathfrak{g}_- \to \text{gr}F_x\) such that \(z|_{\mathfrak{g}_{-1}}^* s_x = \sigma\).

Now consider a subriemannian filtered manifold \((M, F, s)\) of constant symbol \((\mathfrak{g}_-, \sigma)\). Let \(\mathcal{S}^{(0)}(M, F)\) be the universal frame bundle of \((M, F)\) of order 0 and let \(Q^{(0)}(M, F, s)\) be a principal subbundle of \(\mathcal{S}^{(0)}(M, F)\) defined by

\[
Q^{(0)}(M, F, s)_x := \{ z : \mathfrak{g}_- \to \text{gr}F_x \mid \text{graded Lie algebra isomorphism satisfying } z|_{\mathfrak{g}_{-1}}^* s_x = \sigma \}.
\]

The structure group of \(Q^{(0)}(M, F, s)\) is \(G_0(\mathfrak{g}_-, \sigma) := \{ a : \mathfrak{g}_- \to \mathfrak{g}_- \mid \text{graded Lie algebra automorphism satisfying } a|_{\mathfrak{g}_{-1}}^* \sigma = \sigma \}\).
and we denote its Lie algebra by \( g_0(g_-, \sigma) \).

Consider the prolongation of the truncated transitive graded Lie algebra \( g_- \oplus g_0(g_-, \sigma) \) and denote it by \( g(g_-, \sigma) \).

\[
g(g_-, \sigma) = \oplus g_p(g_-, \sigma).
\]

**Proposition 8.1** ([33]). If \( g_- \) is generated by \( g_{-1} \) and \( \sigma \) is positive definite, then \( g_p(g_-, \sigma) \) vanishes for \( p > 0 \).

For the proof we use Yatsui’s result on completely reducible graded Lie algebra ([40]).

If \( \sigma \) is neither positive definite nor negative definite, the above proposition does not hold. We know examples of indefinite \((g_-, \sigma)\) for which \( g_1(g_-, \sigma) \) does not vanish. However, we have:

**Proposition 8.2.** If \( g_- \) is generated by \( g_{-1} \), then \((g_-, \sigma)\) is of finite type, that is, the prolongation \( g_p(g_-, \sigma) \) vanishes for large enough \( p \).

This follows from Tanaka’s criterion that \( g_- \oplus g_0 \) \((g_- \text{ being generated by } g_{-1})\) is of finite type if and only if \( f_{-1} \oplus f_0 \) is of finite type, where \( f_{-1} = g_- \) and \( f_0 = \{ A \in g_0 : [A, X_p] = 0 \text{ for } X_p \in g_p \text{ and } p < -1\} \) ([33]).

To study the equivalence problem of subriemannian structures, the following is fundamental.

**Theorem 8.1** ([33]). Let \( g_- = \bigoplus_{p<0} g_p \) be a graded Lie algebra generated by \( g_{-1} \) and \( \sigma \) a positive definite inner product on \( g_{-1} \). Then for every subriemannian filtered manifold \((M, F, s)\) having constant symbol of type \((g_-, \sigma)\), there exists canonically a Cartan connection \((P, G_0, \theta)\) of type \((g_- \oplus g_0(g_-, \sigma), G_0(g_-, \sigma))\).

The construction of the canonical Cartan connection is made according to the general construction ([30]) by applying the criterion, the condition (C) to this case. Hence for positive definite case, the curvature of the Cartan connection associated with a subriemannian structure gives a generalization of riemannian curvature.

While in the case of indefinite metrics, the condition (C) being not assured, we have no hope to have Cartan connection, in general, associated with subriemannian \( G_0 \)-structure \( Q^{(0)} \), but it is our step prolongation \( \mathcal{S}_W Q^{(0)} \) and fundamental identities that enable us to define the subriemannian curvature and give a theoretical basis for studying the equivalence problem of the indefinite subriemannian structures. Indefinite subriemannian structures are not yet studied much but we know interesting examples in subriemannian contact geometry and in subriemannian geometry associated with Clifford modules ([13]).

As we see even in the simplest example of subriemannian contact structure, there are a great variety of subriemannian symbols, and it is rather restrictive to treat only subriemannian structure of constant symbols, and we are naturally led to consider subriemannian structures of nonconstant symbol.

This is one of motivations for us to extend the present scheme to geometric structures of nonconstant symbol.

8.2. Complex geometry.

8.2.1. One of the import geometric structures of order 0 is \( G_0^\sharp \)-structure of type \( g_- \), which was studied by Tanaka in [39]. Let \( g = \bigoplus_{\mu_0} g_\mu \) be a simple graded Lie algebra over \( \mathbb{R} \) or \( \mathbb{C} \). Let \( g_- \) be the negative part of \( g \) and let \( G_0^\sharp \) be the closed subgroup \( G_0 \cdot N^0 \),
where $G_0$ is a Lie subgroup of the automorphism group of $\mathfrak{g}$ and $N^0$ is the subgroup of $GL(\mathfrak{g}_-)$ consisting of all $a \in GL(\mathfrak{g}_-)$ such that

$$aX \equiv X \mod \sum_{j=p+1}^{-1} \mathfrak{g}_j \text{ for all } X \in \mathfrak{g}_p, \text{ where } p < 0.$$ 

A $G_0^\sharp$-structure of type $\mathfrak{g}_-$ on a manifold $M$ is a reduction $P^\sharp$ of the linear frame bundle of $M$, whose structure group is $G_0^\sharp$.

To the simple graded Lie algebra $\mathfrak{g}$, there is associated a homogeneous space $G/G^0$, where $G$ is a Lie group with Lie algebra $\mathfrak{g}$ and $G^0$ is the subgroup of $G$ corresponding to the nonnegative part $\mathfrak{g}^0 = \oplus_{i \geq 0} \mathfrak{g}_i$. Under the assumption that $\mathfrak{g}$ is the prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$, the equivalence problem for $G_0^\sharp$-structure $P^\sharp$ of type $\mathfrak{g}_-$ can be solved by associating a normal Cartan connection $(P, \theta)$ of type $G/G^0$ to $P^\sharp$ (Theorem 2.7 of [39]). Furthermore, the harmonic part $H(K)$ of the curvature $K$ of $(P, \theta)$ gives a fundamental system of invariants, i.e., the vanishing of $H(K)$ implies the vanishing of $K$, and vice versa (Theorem 2.9 of [39]).

A $G_0^\sharp$-structure of type $\mathfrak{g}_-$ can be defined via a fiber subbundle of the projective tangent bundle $\mathbb{P}(TM)$. Let $\mathcal{S} \subset \mathbb{P}(\mathfrak{g}_-)$ denote the closed $G_0^\sharp$-orbit in $\mathbb{P}(\mathfrak{g}_-)$. A fiber subbundle $\mathcal{S}$ of the projective tangent bundle $\mathbb{P}(TM)$ defines a $G_0^\sharp$-structure if the embedding $\mathcal{S}_x \subset \mathbb{P}(T_x M)$ is projectively equivalent to the embedding $\mathcal{S} \subset \mathbb{P}(\mathfrak{g}_-)$.

The theory of Tanaka applied to give a characterization of the homogeneous space $G/G^0$ in the complex analytic category as follows.

**Theorem 8.2.** [17] Let $X$ be a homogeneous space $G/G^0$ associated to a long simple root. Let $M$ be a Fano manifold of Picard number one. Assume that the variety $\mathcal{C} \subset \mathbb{P}(TM)$ of minimal rational tangents of $M$ defines a $G_0^\sharp$-structure on $M$, that is, the embedding $\mathcal{C}_x \subset \mathbb{P}(T_x M)$ is projectively equivalent to the embedding $\mathcal{S} \subset \mathbb{P}(\mathfrak{g}_-)$ for a general point $x \in M$. Then $M$ is biholomorphic to $G/G^0$.

8.2.2. The theory of Tanaka on simple graded Lie algebra $\mathfrak{g}$ is generalized to the case when $(\mathfrak{g}_-, \mathfrak{g}_0)$ satisfies the condition (C) in [30]. Another development on the construction of Cartan connection in the complex analytic category is given as follows.

**Theorem 8.3** ([19]). Let $X = G/G^0$ be a symplectic Grassmannian $Gr_\omega(k, V)$ and let $\mathcal{S} \subset \mathbb{P}(\mathfrak{g}_-)$ be the variety of minimal rational tangents of $X$ at a general point $a$. Let $M$ be a Fano manifold of Picard number one and $\mathcal{C} \subset \mathbb{P}(TM)$ be the variety of minimal rational tangents associated to a choice of minimal rational component. If the embedding $\mathcal{C}_x \subset \mathbb{P}(T_x M)$ is projectively equivalent to $\mathcal{S} \subset \mathbb{P}(\mathfrak{g}_-)$ for a general point $x \in M$, then $M$ is biholomorphic to $X$.

In fact, Theorem 8.3 also hold for an odd symplectic Grassmannian $X$ ([19]). Here, by an odd symplectic Grassmannian, we mean the space of all isotropic $k$-dimensional subspace of vector space $V$ of dimension $2n+1$ with a skew-symmetric form $\omega$ of maximal rank. Then the odd symplectic Grassmannian $Gr_\omega(k, V)$ with dim $V = 2n+1$ is no longer a homogeneous space, while the symplectic Grassmannian $Gr_\omega(k, V)$ with dim $V = 2n$ is a homogeneous space $G/G^0$.

An odd symplectic Grassmannian is one of the examples of almost homogeneous manifolds, a compact complex manifold on which its automorphism group has an open orbit.
There are only a few results on equivalence problem associated with an almost homogeneous manifold. The next simplest example of an almost homogeneous manifold is a smooth horospherical variety of Picard number one, classified by Pasquier [34]. An odd symplectic Grassmannian is one of this kind of examples. We can now apply the theory developed in this paper to study the rigidity of horospherical varieties. According to Theorem 7.4 a manifold with a geometric structure $Q^{(0)}$ modeled on $X^0 = G/G^0 \subset X$ is locally equivalent to the model space if there is no nontrivial holomorphic section of $\mathcal{H}_k = Q^{(0)} \times_{G_0} H^2(g_-, g)_k$ for all $k \geq 1$, as in the case of $G^2_0$-structure of type $g_-$ studied by Tanaka. Instead of trying to confirm the condition (C), by showing that any holomorphic section of $\mathcal{H}_k = Q^{(0)} \times_{G_0} H^2(g_-, g)_k$ vanishes for all $k \geq 1$, we get a characterization of smooth horospherical varieties of Picard number one as in Theorem 8.2 and Theorem 8.3 ([18]).

**Theorem 8.4 ([18]).** Let $X$ be a smooth horospherical variety of Picard number one and let $S \subset \mathbb{P}(g_-)$ be the variety of minimal rational tangents of $X$ at a general point $o$. Let $M$ be a Fano manifold of Picard number one and $C \subset \mathbb{P}(TM)$ be the variety of minimal rational tangents associated to a choice of minimal rational component. If the embedding $C_x \subset \mathbb{P}(T_x M)$ is projectively equivalent to $S \subset \mathbb{P}(g_-)$ for a general point $x \in M$, then $M$ is biholomorphic to $X$.

**Appendix**

In this Appendix we give an elementary proof of the following proposition:

**Proposition 6.1** Let $X_1, \ldots, X_n$ be $n$-independent analytic vector fields defined on a neighborhood of $0$ in $\mathbb{R}^n$ or $\mathbb{C}^n$. A formal function $f$ at $0$ is convergent in a neighborhood of $0$ if and only if there exists positive constants $C$ and $\rho$ such that

$$(*) \quad |(X_{i_1}X_{i_2} \cdots X_{i_\ell} f)(0)| \leq C\ell! \rho^\ell$$

for $\ell = 0, 1, 2, \ldots, (i_1, i_2, \ldots, i_\ell) \in \{1, 2, \ldots, n\}^\ell$.

**Proof of Proposition 6.1.** Note first that it is easy to see the proposition holds if $\{X_1, \ldots, X_n\}$ is a coordinate frame $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$, where $(x^1, x^2, \ldots, x^n)$ is a coordinate system of $\mathbb{R}^n$ or $\mathbb{C}^n$. We will therefore show that if $(*)$ holds for $u$ with respect to $X_1, \ldots, X_n$, then a similar estimate for $u$ holds with respect to $\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\}$.

Let $V$ be an $n$-dimensional vector space with a basis $\{e_1, \ldots, e_n\}$. Write

$$X = e_1 \otimes X_1 + \cdots + e_n X_n = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \quad D = e_1 \otimes D_1 + \cdots + e_n \otimes D_n = \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix}$$

where $D_i = \frac{\partial}{\partial x^i}$ for $i = 1, \ldots, n$. Then we can write

$$D = AX$$

with $GL(V)$-valued holomorphic function $A = A(x^1, \ldots, x^n)$. By successive differentiation we have

$$D^nu = AX^nu$$
$$D^2u = (DA)Xu + A^2X^2u$$
$$D^3u = (D^2A)Xu + ((DA)A + D(A^2))X^2u + A^3X^3u$$

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and in general
\[ D^k u = \Phi^k_1 X u + \Phi^k_2 X^2 u + \cdots + \Phi^k_k X^k u \]
with
\[ \Phi^k_i = \Phi^{k-1}_{i-1} A + D \Phi^{k-1}_i \]
\[ \Phi^k_i = 0 \text{ for } i \leq 0 \text{ or } i > k. \]
In the above formula \( D^k = \otimes^k D \) and \( X^k = \otimes^k X \) should be regarded as sections of \( \otimes^k V \otimes D \), where \( D \) denotes the sheaf of differential operators on \( \mathbb{C}^n \).

Consider a directed graph whose vertex set is \( \{(i, j) : i \geq j, 1 \leq i, j \leq k\} \) and whose edge set consists of all arrows \((i, j) \rightarrow (i', j')\) satisfying either (\( \searrow \)) \( i+1 = i' \) and \( j+1 = j' \) or (\( \downarrow \)) \( i+1 = i' \) and \( j = j' \).

\[ (1,1) \xrightarrow{\searrow} (2,1) \xrightarrow{\downarrow} (2,2) \]
\[
\vdots
\]
\[ (k-1,1) \rightarrow (k,1) \rightarrow \cdots \rightarrow (k,i) \rightarrow \cdots \rightarrow (k,k) \]

Thus any path \( \Gamma \) from \((1,1)\) to \((k,i)\) has \((i-1)\) edges of the first type \( \searrow \) and \((k-i)\) edges of the second type \( \downarrow \). By the recursion formula we see that \( \Phi^k_i \) is the sum of all elements \( \varphi^k_{i;\Gamma} \) for every path \( \Gamma \) from \((1,1)\) to \((k,i)\), which is created from \( A \) by multiplying \( A \) on the right when the path \( \Gamma \) passes an edge of the first type \( \searrow \) and by applying differentiation \( D \) when it passes an edge of the second type \( \downarrow \).

We endow \( \otimes^k V \) with a norm defined by
\[ \| \sum a_{i_1 \ldots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \| := \sup |a_{i_1 \ldots i_k}|. \]
Then (*) may be written as \( \| X^k u(0) \| \leq C k! \rho^k \), and we want to show
\[ \| D^k u(0) \| \leq \overline{C} k! \overline{\rho}^k \]
for some positive constants \( \overline{C} \) and \( \overline{\rho} \). For that we will show that
\[ |\Phi^k(0)| \leq C_1 k(k-1) \cdots (i+1) \rho_1^k \] for all \( k \) where \( C_1 \) and \( \rho_1 \) are positive constants independent of \( k \).
To prove this we use the following norm $|\cdot|_r$ with $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{> 0}$ for formal power series $F = \sum_{k=0}^{n} F_k$, $F_k = \sum_{|\alpha| = k} f_\alpha x^\alpha$, $f_\alpha \in \mathbb{C}$:

$$
|F_k|_r := \sup_{|\alpha|=k} \left( \frac{\alpha!}{|\alpha|!} |f_\alpha| r^\alpha \right) \\
|F|_r := \sum |F_k|_r.
$$

Lemma A.

1. If $F$ is convergent, then there is $r$ such that $|F|_r < \infty$.
2. $|F_k G_\ell|_r \leq |F_k|_r |G_\ell|_r$, where $F_k$ and $G_\ell$ are homogeneous polynomials of degree $k$ and $\ell$, respectively.
3. $|FG|_r \leq |F|_r |G|_r$.
4. $|DF_k|_r \leq \frac{k}{r_{\text{min}}} |F_k|_r$ for homogeneous polynomial of degree $k$, where $r_{\text{min}} := \min\{r_1, \ldots, r_n\}$.

Proof. Let us prove (2). Let $F_k = \sum_{|\alpha|=k} f_\alpha x^\alpha$, $G_\ell = \sum_{|\beta|=\ell} g_\beta x^\beta$, and $H_m = \sum_{|\gamma|=m} h_\gamma x^\gamma$, where $m = k + \ell$ and $H_m = F_k G_\ell$. Recall that $|H_m|_r = \sup_{|\gamma|=m} \left( \frac{\gamma!}{|\gamma|!} |h_\gamma| r^\gamma \right)$. But for any $\gamma$ with $|\gamma| = m$ we have

$$
\frac{\gamma!}{|\gamma|!} |h_\gamma| r^\gamma = |\gamma|! \left( \sum_{|\alpha|=k, |\beta|=\ell} f_\alpha g_\beta \right) r^\gamma \\
\leq \frac{\gamma!}{|\gamma|!} \left( \sum_{|\alpha|=k, |\beta|=\ell} \frac{|\alpha||\beta|!}{|\alpha|!|\beta|!} \left( \frac{\alpha!}{|\alpha|!} |f_\alpha| r^\alpha \right) \left( \frac{\beta!}{|\beta|!} |g_\beta| r^\beta \right) \right) \\
\leq \frac{k!\ell!}{m!} \left( \sum_{|\alpha|=k, |\beta|=\ell} \frac{\gamma!}{|\alpha|!|\beta|!} \right) |F_k|_r |G_\ell|_r \\
= |F_k|_r |G_\ell|_r
$$

because of the identity

$$
\sum_{|\alpha|=k, |\beta|=\ell} \frac{\gamma!}{|\alpha|!|\beta|!} = \frac{m!}{k!\ell!}
$$

which is derived from the binary expansion.

The other assertions (1), (3), (4) are easy to verify. \qed

Now let us return to the proof of Proposition 6.3. Take $r = (r_1, \ldots, r_n)$ such that $|A|_r < \infty$. We see that $|\Phi A|_r \leq K|\Phi|_r |A|_r$.

Denote by $\text{Trun}^{(\ell)} F := \sum_{i \leq \ell} F_i$ for $F = \sum F_\ell$ and set $k\Psi_\ell^k = \text{Trun}^{(k-\ell+1)} \Phi_\ell^k$.

Then $\Phi_\ell^k(0) = k \Psi_\ell^k(0)$. By Lemma A (4) we have

$$
|D^{(k-\ell+1)} \Phi_\ell^k|_r \leq \frac{k - \ell + 1}{r_{\text{min}}} |k\Psi_\ell^k|_r.
$$
Thus we see that
\[ |\varphi_{k+1}^k(0)| \leq L_k(k - 1) \ldots (i + 1) \rho_{mk}. \]
Since the number of path \( \Gamma \) from \((1, 1)\) to \((k, i)\) is less than \(2^k\), we have finally
\[ |\Phi_k^i(0)| \leq C_1 k(k - 1) \ldots (i + 1) \rho_{1k}, \]
which completes the proof of Proposition 6.1. \( \square \)

References

[1] A. Agrachev and Y. Sachkov, Control theory from the geometric viewpoint, Encyclopaedia of Mathematical Sciences, 87. Control Theory and Optimization, II. Springer-Verlag, Berlin, 2004
[2] D. Alekseevsky and L. David, Prolongation of Tanaka structures: an alternative approach, Annali di Mathematica 196 (2017) 1137–1164
[3] A. Čap and J. Slovák, Parabolic Geometries. I. Background and general theory Mathematical Surveys and Monographs, 154. American Mathematical Society, Providence, RI, 2009.
[4] É. Cartan, Sur la structure des groupes infinites de transformations, Ann. Sci. École Norm. Sup. (3) 21 153–206 (1904)
[5] É. Cartan, Sur la structure des groupes infinites de transformation (suite), Ann. Sci. École Norm. Sup. (3) 22 219–308 (1905)
[6] É. Cartan, Les sous-groupes des groupes continus de transformations, Ann. Sci. École Norm. Sup. (3) 25 57–194 (1908)
[7] É. Cartan, Les groupes de transformations continus, infinis, simples, Ann. Sci. École Norm. Sup. (3) 26 93–161 (1909)
[8] É. Cartan, Sur les espace généralisés et la théorie de la relativité C. R. Acad. Sci. t. 174 pp. 734–737 (1922)
[9] É. Cartan, Les problèmes d’équivalence, Séminaire de mathématiques, t. 4 (1936-1937) exp n. 4 pp 1–40 ou Oeuvres, Partie II volume 2 pp. 1311–1334
[10] É. Cartan, La structure des groupes infinis, Séminaire de mathématiques, t. 4 (1936-1937) exp n. 6 pp 1–43 ou Oeuvres, Partie II volume 2 pp. 1335–1384
[11] J. F. Conn, A new class of counterexamples to the integrability problem Proc. Nat. Acad. Sci. U.S.A. 74 (1977), no. 7, 2655–2658.
[12] B. Doubrov, Y. Machida and T. Morimoto, Extrinsic Geometry and Linear Differential Equations SIGMA 17 (2021), 061, 60 pages
[13] K. Furutani, M. Godoy Molina, I. Markina, T. Morimoto, and A. Vasil’ev, Lie algebras attached to Clifford modules and simple graded Lie algebras, Journal of Lie Theory, Vol. 28 (2018) 843–864
[14] H. Goldschmidt, Existence theorems for analytic linear partial differential equations, Ann. of Math., 86 (1967), 246–270
[15] H. Goldschmidt, On the non-linear cohomology of Lie equations V. J. Diff. Geom. n. 16 595–674 (1981)
[16] V. W. Guillemin and S. Sternberg, An algebraic model of transitive differential geometry Bull. Amer. Math. Soc. 70 (1964), 16–47.
[17] J. Hong and J.-M. Hwang, Characterization of the rational homogeneous space associated to a long simple root by its variety of minimal rational tangents, Algebraic Geometry in East Asia-Hanoi 2005, Advanced Studies in Pure Mathematics 50 (2008) 217-236.
[18] J. Hong and S. Kim, Characterizations of smooth projective horospherical varieties of Picard number one, arXiv:2203.10313
[19] J.-M. Hwang and Q. Li, Characterizing symplectic Grassmannians by varieties of minimal rational tangents, J. Differential Geom. 119(2), (2021) 309–381.
[20] A. Kumpera and D. Spencer, Lie equations Volume I: General theory, Princeton University Press and University of Tokyo Press, Princeton, New Jersey 1972
[21] M. Kuranishi, On E. Cartan’s prolongation theorem of exterior differential systems Amer. J. Math. 79 (1957), 1–47.
[22] S. Lie, Über Differentialinvarianten, Math. Ann. 24 (1884), 537–578, English translation: Ackerman M., Hermann R., Sophus Lie’s 1884 differential invariant paper, Math Sci Press, Brookline, Mass., 1975.

[23] W. Liu and H. Sussman, Shortest paths for sub-riemannian metrics on rank two-distributions, Mem. Amer. Math. Soc. 118 (1995), no. 564

[24] B. Malgrange, Équations de Lie II, J. of differential geom., 7 (1972), 117–142

[25] R. Montgomery, A tour of subriemannian geometries, their geodesics and applications, Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, 2002

[26] T. Morimoto, On transitive infinite Lie pseudo-groups (in Japanese) 1969, Kyoto University.

[27] T. Morimoto, Sur le problème d’équivalence des structures géométriques, Japan. J. Math. 9 no. 2 (1983) 293–372

[28] T. Morimoto, Transitive Lie algebras admitting differential systems, Hokkaido Math. J., bf 17 (1988), 45–81

[29] T. Morimoto, Théorèm de Cartan-Kähler dans une class de fonctions formelles Gevrey, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990), no. 7, 433–436.

[30] T. Morimoto, Geometric structures on filtered manifolds, Hokkaido Mathematical Journal 22 (1993) 263–347

[31] T. Morimoto, Théorème d’existence de solutions analytiques pour des systèmes d’équations aux dérivées partielles non-linéaires avec singularités, C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), no. 11, 1491–1496.

[32] T. Morimoto, Lie algebras, geometric structures and differential equations on filtered manifolds, Advanced Studies in Pure Mathematics 37 (2002) 205–252

[33] T. Morimoto, Cartan connection associated with a subriemannian structure, Differential geometry and its applications, 26 75–78 (2008)

[34] B. Pasquier, On some smooth projective two-orbit varieties with Picard number 1, Mathematische Annalen 344 (2009) 963-987.

[35] D. G. Quillen, Formal theory of linear overdetermined systems of partial differential equations Thesis (Ph.D.) Harvard University. 1964.

[36] I. M. Singer and S. Sternberg, The infinite groups of Lie and Cartan. I. The transitive groups, J. Analyse Math. 15 (1965) 1-114

[37] D. C. Spencer, Deformation of structures on manifolds defined by transitive, continuous pseudogroups. II. Deformations of structure Ann. of Math. (2) 76 (1962), 399–445.

[38] N. Tanaka, On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. 10 (1970) 1–82

[39] N. Tanaka, On the equivalence problems associated with simple graded Lie algebras, Hokkaido Math. J. 8 (1979) 23–84

[40] T. Yatsui, On pseudo-product graded Lie algebras, Hokkaido Math. J. 17 (1988) 333-343

[41] I. Zelenko, On Tanaka’s Prolongation Procedure for Filtered Structures of Constant Type, SIGMA 5 (2009), 094, 21 pages

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