Abstract. We study the maximum cardinality problem of a set of few distances in the Hamming and Johnson spaces. We formulate semidefinite programs for this problem and extend the 2011 works by Barg-Musin and Musin-Nozaki. As our main result, we find new parameters for which the maximum size of two- and three-distance sets is known exactly.

1. Introduction

Let $M$ be a metric space with metric $d$ that defines the distance between any pair of points in $M$. For a finite set $X \subset M$, its distance set is defined as $D(X) = \{d(x, y) : x \neq y \in X\}$, which is the set of distinct distances in $X$. We say that $X$ is an $s$-distance set in $M$ if $|D(X)| = s$. For instance, the eight vertices of a cube in $\mathbb{R}^3$ form a three-distance set.

The problem of determining the maximum size of an $s$-distance set in $M$, denoted by $A(M, s)$, has been studied for a range of metric spaces. For example, the only maximum three-distance set in $\mathbb{R}^3$ is formed by the 12 vertices of a regular icosahedron, and thus $A(\mathbb{R}^3, 3) = 12$; similarly $A(\mathbb{R}^3, 5) = 20$, attained by the vertex set of a dodecahedron (see [NS21], which also contains other examples).

The problem of finding the maximum cardinality of $s$-distance sets was first studied by Einhorn and Schoenberg [ES66a], [ES66b] in the Euclidean space. For $s = 2$, Larman, Rogers, and Seidel proved a necessary integrality condition for the distances of a 2-distance set of a sufficiently large size [LRS77]. Namely, if the distances in a 2-distance set $X \subset \mathbb{R}^n$ are $a$ and $b$ with $a < b$ and $|X| \geq 2n + 4$, then $\frac{a^2}{b^2} = \frac{k-1}{k}$ for some integer $k$ such that $2 \leq k \leq \frac{1 + \sqrt{2n}}{2}$. As shown by Neumaier [Neu81], the minimum size of $X$ for the LRS theorem to hold can be reduced from $2n + 4$ to $2n + 2$. This statement, called below the LRS theorem, serves as a constraint on the distances of a 2-distance set and is helpful in estimating the maximum cardinality of such sets. More than three decades after the LRS paper, Nozaki [Noz11] generalized their result to $s$-distance sets for an arbitrary $s$, again for sets $X$ of sufficiently large size. Recently Yeh and Yu [YY21] relaxed the condition for the minimum size of $X$ sufficient for the result [Noz11] to hold.

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The maximum cardinality problem for $s$-distance sets has also been studied for metric spaces commonly considered in coding theory such as the binary Hamming and Johnson spaces, as well as the sphere in $\mathbb{R}^n$. In this paper we will focus on the binary Hamming space $H_2^n$ and the binary Johnson space $J_2^{n,w}$, formed of all binary vectors of length $n$ and all binary vectors of length $n$ and weight $w$, respectively.

Following the convention of coding theory, we call an $s$-distance set in the Hamming and Johnson spaces an $s$-code. The maximum cardinality of an $s$-code can be bounded above using Delsarte’s linear programming (LP) method [Del73a]; however this approach does not yield closed-form results. Delsarte [Del73a, Del73b] also proved a general upper bound on the size of $s$-codes, called the harmonic bound, and Nozaki and Shinohara recently improved it in [NS10]. As shown in [MN11], an LRS-type theorem holds also for the Hamming and Johnson spaces.

Relying on the harmonic bound, Barg and Musin [BM11] determined the maximum cardinality $A(H_2^n, 2)$ and $A(J_2^{n,w}, 2)$ in a number of cases, and Musin and Nozaki [MN11] extended their results to $s = 3$ and $4$. The central idea of these papers is first to derive general lower bounds $A(H_2^n, s) \geq \sum_{i=0}^{\lfloor \frac{s}{2} \rfloor} (n - 2i)$ and $A(J_2^{n,w}, s) = \left( \binom{n-w+s}{s} \right)$ using simple code constructions and then to rule out the existence of codes of larger size using LP and certain other tools. However, these works have left a number of exceptional parameter sets unresolved.

Our objective in this work is to address these exceptions relying on Schrijver’s semidefinite programming (SDP) method [Sch05]. The SDP bound forms a refinement of the LP method, so it is at least as strong as LP, and in some cases has been shown to provide strict improvements. Already [Sch05] gave some examples, and later Barg and Yu have applied the SDP method for spherical codes to determine the maximum cardinality of spherical two-distance sets [BY13b] and equiangular lines [BY13a] for several dimensions $n$ in $\mathbb{R}^n$. There are also many other examples related to kissing numbers and other problems. All of this motivates us to study SDP bounds on the size of $s$-codes in the Hamming and Johnson spaces.

Schrijver’s SDP method is based on the action of the isometry group of the Hamming (Johnson) space, block diagonalization of the Terwilliger algebra [Sch05] and the ensuing positive semidefinite (p.s.-d.) conditions. While the LP bound considers relations (distances) between pairs of points in the set $X$, Schrijver extended this to relations among triples, and Gijswijt, Mittelmann, and Schrijver went even further, considering relations among 4-tuples of points in $X$ [GMS12]. Another strengthening of the SDP method, proposed recently by Tseng, Lai, and Yu [TLY22], is based on considering distances on subsets of the code coordinates, also called split distance distributions.

In this paper, we use the SDP method to derive an upper bound on the size of an $s$-code in the Hamming or Johnson space. Given an $s$-code $C$ with distance set $\mathcal{D} = \{d_1, \ldots, d_s\}$, we know that the only values of the distances that occur come from $\mathcal{D}$, and these constraints can be added to Schrijver’s semidefinite program. Computing these SDP bounds, we are able to eliminate the existence of several codes that are exceptions to the results of [BM11, MN11].

The current state of the art for 2- and 3-distance sets in $H_2^n$ is as follows.

**Theorem 1.1.** (1) For $6 \leq n \leq 74$ and $n = 78$ with the exceptions of $n = 70$ and $71$, $A(H_2^n, 2) = 1 + \binom{n}{2}$.

(2) For $8 \leq n \leq 37$ and $n = 44$ with the exception of $n = 23$, $A(H_2^n, 3) = n + \binom{n}{3}$. 
Earlier works [BM11, MN11] did not include the dimensions \( n = 47, 53, 59, 65 \) for \( s = 2 \) and \( n = 34, 35 \) for \( s = 3 \), all of which constitute new results. Note that the dual Golay code \( G_23^2 \) yields \( A(H_3^2, 3) \geq 2048 \) which is greater than \( 23 + \binom{23}{3} \) = 1794, so the dimension \( n = 23 \) in part 2 is a true exception [BM11].

Using SDP bounds, we can similarly improve the results of [BM11, MN11] for the Johnson space. The current state of the art for 2-distance sets in \( J_2^n \) is as follows.

**Theorem 1.2.** For \( 6 \leq n \leq 46 \) and \( 3 \leq w \leq \frac{n}{2} \), with the exception of \( (n, w) = (23, 7) \), \( A(J_2^n, w, 2) \leq \left( \frac{n}{2} \right) - n + 1 \).

Earlier works [BM11] Prop. 2.10, [MN11] Thm. 3.5 did not include the dimension \( n = 46 \), and the pairs \( (n, w) = (44, 17), (45, 16) \). Again the case \( (23, 7) \) is a true exception because the binary Golay code yields a 2-code of 253 which is greater than \( (\frac{23}{2}) - 23 + 1 = 231 \).

Additionally, we extend several other results for \( A(J_2^n, w, s) \), \( s = 2, 3, 4 \) in [BM11, MN11]. Using a stronger version of the LRS theorem, we prove

**Proposition 1.3.** \( A(J_2^n, w, 2) = \left( \frac{n-w+2}{2} \right) \) for \( 3(w-1) \leq n \leq 100 \), \( w = 3, 4, 5 \).

Previously the exact value of \( A(J_2^n, w, 2) \) was known only for \( n \leq 46 \) [BM11].

Finally we point out a small improvement of the results in [MN11] where the authors stopped short of completing their analysis. Specifically, we show that \( A(J_2^n, w, 3) = \left( \frac{n-w+3}{3} \right) \) for \( n = 12, w = 5 \) and \( A(J_2^n, w, 4) = \left( \frac{n-w+4}{4} \right) \) for \( (n, w) = (15, 6), (16, 6) \), explaining the details in Sec.[5]

The structure of this paper is as follows. In Section 2 we review known results for \( s \)-distance sets in the Hamming and Johnson spaces. In Section 3 we present SDP bounds for \( s \)-codes in the Hamming and Johnson spaces. Numerical results are presented and discussed in Section 4 and 5. As a conclusion, we state two research problems in Section 6.

2. **Preliminaries on \( s \)-distance sets and linear programming bounds**

2.1. **Hamming space.** The Hamming space \( H_2^n \) is an \( n \)-dimensional vector space over the finite field \( \mathbb{F}_2 \) equipped with the Hamming metric. The Hamming distance \( d_H(x, y) \) between two binary vectors \( x, y \in H_2^n \) is the number of coordinates where the two vectors differ. The (Hamming) weight of a vector \( x \in H_2^n \) is its distance to the zero vector \( d_H(x, 0) \), which is the number of nonzero coordinates in \( x \). A subset \( C \subset H_2^n \) is called a code, and if \( C \) is an \( s \)-distance set, we call it an \( s \)-code. A vector in a code is called a codeword.

Delsarte showed that the size of a code is bounded above by the value of a linear program defined in the following theorem [Del73a].

**Theorem 2.1.** [Del73a] Let \( C \subset H_2^n \) be an \( s \)-code with distances \( D = \{d_1, \ldots, d_s\} \). Then \( |C| \leq \text{LP}_H(n, D) \), where

\[
\text{LP}_H(n, D) := \max \left\{ 1 + \sum_{j=1}^s f_j : f_j \geq 0 \text{ for } j = 1, \ldots, s, \right. \\
\left. \quad \sum_{j=1}^s f_j \phi_k^n(d_j) \geq -\binom{n}{k}, \text{ for } k = 0, \ldots, n \right\},
\]

and

\[
\phi_k^n(x) = \sum_{i=0}^{\left\lfloor \frac{n}{k} \right\rfloor} \binom{i}{k} \binom{n-i}{n-k} x^i (1-x)^{n-i}.
\]
and
\[
\phi^n_k(x) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j}
\]
are the binary Krawtchouk polynomials \[MS77\ Ch.5]; \[Del73a\].

The linear programming bound \(LP_H(n, D)\) has been one of the main tools in estimating the value of \(A(M, s)\) for \(M = H^o_2\), \(J^2_{n,w}\) as well as for spherical codes. Direct application of this bound is difficult because of the need to exhaust all possible subsets \(D\), and here the LRS-type integrality conditions have proved instrumental.

Delsarte \[Del73a, Del73b\] also proved another upper bound on \(A(M, s)\), called the harmonic bound (it is derived relying on the dimension of the space of spherical harmonics).

We quote this bound in its improved form due to Nozaki and Shinohara \[NS10\]. Note that \((\phi^n_k, k = 0, 1, \ldots n)\) form an orthogonal basis of the set of functions \(f: \{0, 1, \ldots, n\} \rightarrow \mathbb{R}\) with respect to the weight \(n^i\), so any such \(f\) has a unique Krawtchouk expansion.

**Theorem 2.2.** \[NS10\] Let \(C \subset H^o_2\) be an \(s\)-code with distances \(D = \{d_1, \ldots, d_s\}\). Consider the polynomial \(f(t) = \prod_{i} \frac{d_i-t}{d_i}\). Suppose that the Krawtchouk expansion of \(f(t)\) is \(f(t) = \sum_{k=1}^{s} f_k \phi^n_k(t)\). Then
\[
|C| \leq \sum_{i: f_i > 0} \binom{n}{i}.
\]

Barg and Musin derived a concrete form of the above theorem under the condition that the sum of the code’s distances does not exceed \(1/2 sn\).

**Theorem 2.3.** \[BM11, Theorem 11\] Let \(C \subset H^o_2\) be an \(s\)-code with distances \(D = \{d_1, \ldots, d_s\}\). Suppose that \(\sum_{i=1}^{s} d_i \leq \frac{1}{2} sn\). Then
\[
|C| \leq \sum_{i=0}^{s-2} \binom{n}{i} + \binom{n}{s}.
\]

This inequality is proved by showing that the assumption \(\sum_{i=1}^{s} d_i \leq \frac{1}{2} sn\) forces the coefficient \(f_{s-1}\) of the Krawtchouk expansion to be nonpositive, and thus the term \(\binom{n}{s-1}\) is missing from the sum of Theorem 2.2.

Musin and Nozaki \[MN11\] extended the LRS integrality conditions to the case of arbitrary \(s\).

**Theorem 2.4.** (Generalized LRS theorem for the Hamming space, \[MN11\]) Let \(C \subset H^o_2\) be an \(s\)-code with distances \(D = \{d_1, \ldots, d_s\}\). Let \(N(H^o_2, s) = \sum_{i=0}^{s-1} \binom{n}{i}\) and
\[
K_i = \prod_{j \neq i} \frac{d_j}{d_j - d_i} \quad \text{for } i = 1, \ldots, s.
\]
If \(|C| \geq 2N(H^o_2, s)\), then \(K_i\) is an integer and
\[
|K_i| \leq \left[ \frac{1}{2} + \sqrt{\frac{N(H^o_2, s)^2}{2N(H^o_2, s) - 2} + \frac{1}{4}} \right] \quad \text{for } i = 1, \ldots, s.
\]

In particular, note that the minimum code size for which these conditions are applicable is \(2n+2\) for \(s = 2\) and \(n^2 + 3n + 2\) for \(s = 3\). To show tightness of this bound in a number of cases, the authors also gave an easy construction of \(s\)-codes in \(H^o_2\).
Proposition 2.5. [MN11, Equation (3.1)] For $2s \leq n$, we have

$$A(H_n^{2s}, s) \geq \sum_{i=0}^{\left\lfloor \frac{n}{2s} \right\rfloor} \binom{n}{s - 2i}.$$  

Proof. For $s \leq 2n$, the set $C$ of all binary vectors of Hamming weight $k \leq s$ such that $k = s \pmod{2}$ is an $s$-code with distance set $\mathcal{D}(C) = \{2, 4, \ldots, 2s\}$. □

Observe that for $s = 2$ the upper bound in Theorem 2.3 matches the lower bound of this proposition. We use this fact to establish some of the numerical results in Section 4.

2.2. Johnson space. The results in the previous section in fact apply more generally, and can be extended to other two-point homogeneous spaces of coding theory. Here we focus on the Johnson space $J_2^{n,w}$, which is the set of all $n$-dimensional binary vectors of Hamming weight $w$, $2w \leq n$ with the Johnson distance given by $d_J(x, y) = \frac{1}{2}d_H(x, y)$ for $x, y \in J_2^{n,w}$.

A constant weight code $C \subset J_2^{n,w}$ such that $|\mathcal{D}(C)| = s$ is called a constant weight $s$-code. Since the diameter of $J_2^{n,w}$ is $w$, we necessarily have $s \leq w$ for all such codes.

Delsarte’s LP bound for constant weight $s$-codes has the following form.

Theorem 2.6. [Del73a] Let $C \subset J_2^{n,w}$ be a constant weight $s$-code with distances $\mathcal{D} = \{d_1, \ldots, d_s\}$. Then

$$|C| \leq \text{LP}_{J} (\mathcal{D}),$$

where

$$\text{LP}_{J} (\mathcal{D}) := \max \left\{ \sum_{i=0}^{s} f_i : f_0 = 1; f_j \geq 0, j = 0, \ldots, s \right\},$$

and

$$f_j \psi_k(d_j) \geq 0, k = 0, 1, \ldots, w,$$

are certain Hahn polynomials.

The Hahn polynomials form an orthogonal basis in the function space $f : \{0, 1, \ldots, w\} \to \mathbb{R}$.

The (improved) harmonic bound for the Johnson space has the following form.

Theorem 2.7. [NS10] Let $C \subset J_2^{n,w}$ be a constant weight $s$-code with distances $\mathcal{D} = \{d_1, \ldots, d_s\}$. Consider the polynomial $f(t) = \prod_i \frac{d_i - t}{d_i}$. Suppose that the Hahn expansion of $f(t)$ is $f(t) = \sum_{k=1}^{s} f_k \psi_k^n(t)$. Then

$$|C| \leq \sum_{i: f_i > 0} \binom{n}{i - 1} \frac{n - 2i + 1}{i}.$$

A specialization of this theorem for $s$-codes of sufficiently large length has the following form.

Theorem 2.8. [BM11] Let $C \subset J_2^{n,w}$ be an $s$-code with $n \geq (w - s + 1)(s + 1)$. Then

$$|C| \leq \binom{n - w + s}{s}.$$
The generalized LRS theorem for the Johnson space has the following form.

**Theorem 2.9.** (Generalized LRS theorem for the Johnson space, [MN11]) Let $C \subset J_{2}^{n,w}$ be an $s$-code with distances $\{d_1, \ldots, d_s\}$. Let $N(J_{2}^{n,w}, s) = \binom{n}{s-1}$ and

$$K_i = \prod_{j \neq i} \frac{d_j}{d_j - d_i} \text{ for } i = 1, \ldots, s.$$ 

If $|C| \geq 2N(J_{2}^{n,w}, s)$, then $K_i$ is an integer and

$$|K_i| \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{N(J_{2}^{n,w}, s)^2}{2N(J_{2}^{n,w}, s) - 2}} + \frac{1}{4} \right\rfloor \text{ for } i = 1, \ldots, s.$$ 

Barg and Musin proved the following upper bound for $s$-codes.

**Theorem 2.10.** [BM11, Theorem 8] Let $C \subset J_{2}^{n,w}$ be an $s$-code with distances $D = \{d_1, \ldots, d_s\}$. Suppose that

$$\sum_{i=1}^{s} (w - d_i) \geq \frac{s(w^2 - (s - 1)(2w - \frac{n}{2}))}{n - 2(s - 1)}.$$ 

Then

$$|C| \leq \binom{n}{s} - \binom{s}{s-1} \frac{n - 2s + 3}{n - s + 2}.$$ 

Similarly to Theorem 2.9, there is a general construction of $s$-codes in the Johnson space that implies a lower bound on $A(J_{2}^{n,w}, s)$.

**Theorem 2.11.** [MN11] For $s \leq n - w$ we have

$$A(J_{2}^{n,w}, s) \geq \binom{n - w + s}{s}.$$ 

**Proof.** For $s \leq n - w$, consider the set of binary vectors in $J_{2}^{n,w}$ with ones in the first $w-s$ coordinates. Clearly, it forms an $s$-code of size $\binom{n - w + s}{s}$ with distances $\{1, \ldots, s\}$. □

### 3. Semidefinite programming bounds for $s$-codes in the Hamming and Johnson spaces

Schrijver’s SDP method [Sch05] forms a far-reaching extension of Delsarte’s LP bounds. In this section we specialize general results of [Sch05] for $s$-codes in $H_2^n$ or $J_{2}^{n,w}$.

#### 3.1. Semidefinite program for $s$-codes in the Hamming space

Consider an $s$-code $C$ in $H_2^n$ with distances $\{d_1, \ldots, d_s\}$. Its triple distance distribution is defined as

$$x_{t,ij} = \frac{1}{|C| \binom{n}{i-t,j-t,t}} \binom{|\{(u,v,z) \in C^3 : d_H(u,v) = i, d_H(u,z) = j, d_H(v,z) = i + j - 2t\}|}{1}.$$ 

The variables $x_{t,ij}$ characterize the distance relations among triples of codewords. Together with the fact the $C$ has only $s$ distances, we obtain the following form of the SDP bound.

**Theorem 3.1.** Let $C \subset H_2^n$ be an $s$-code with distance set $\emptyset = \{d_1, \ldots, d_s\}$. Suppose that $x_{t,ij} \in [0,1]$ for $i,j,t \in \{0, \ldots, n\}$ are variables satisfying the following conditions:
Remark 3.2. Here only positive definiteness constitutes a nontrivial condition whereas all the other restrictions are straightforward consequences of the definition. Additionally, the obvious constraints \( i - t, j - t, i + j - t \in \{0, \ldots, n\} \) enable a speedup of the implementation.

3.2. Semidefinite program for scodes in the Johnson space. Consider a constant weight \( s \)-code \( C \) in \( J_2^{n,w} \) with distances \( \{d_1, \ldots, d_s\} \). Let \( \chi^A \) be the indicator vector of a set \( A \subset \{1, \ldots, n\} \). Define a semidistance between two vectors \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathbb{F}_2^n \) as an \( n \)-dimensional vector
\[
\bar{d}(u, v) = \chi^{\text{supp}(u) \setminus \text{supp}(v)}
\]
where \( \text{supp}(u) = \{i : u_i = 1\} \) is the support of \( u \). Then the triple distance distribution of \( C \) is defined as
\[
x_{i,j}^{t,r} = \frac{1}{|C|} \sum_{i-t,j-t \in C^3} \chi_{i,j}^{t,r}
\]
where
\[
\chi_{i,j}^{t,r} = |\{(u, v, z) \in C^3 : d_j(u, v) = i, d_j(u, z) = j, \bar{d}(u, v) \cdot \bar{d}(u, z) = t, \bar{d}(v, u) \cdot \bar{d}(z, u) = r\}|
\]
Next we formulate Schrijver’s SDP bound for the Johnson space, taking account of the fact that \( C \) is an \( s \)-code and of the ensuing additional constraints on the distance coefficients \( x_{i,j}^{t,r} \).

Theorem 3.3. Let \( C \subset J_2^{n,w} \) be an \( s \)-code with distance set \( \mathcal{D} = \{d_1, \ldots, d_s\} \). Suppose that \( x_{i,j}^{t,s} \in [0, 1] \) for \( i, j, t, r \in \{0, \ldots, \min\{w, n-w\}\} \) are variables satisfying the following conditions:
\[
\begin{align*}
\text{i.} & \quad x_{0,0}^{0,0} = 1; \\
\text{ii.} & \quad 0 \leq x_{i,j}^{t,r} \leq x_{0,0}^{0,0} \quad \text{and} \quad x_{i,0}^{0,0} + x_{j,0}^{0,0} \leq 1 + x_{i,j}^{t,r}; \\
\text{iii.} & \quad x_{i,j}^{t,r} = x_{i',j'}^{t',r'} \quad \text{if} \ t - r = t' - r' \quad \text{and} \ (i, j, i + j - t - r) \text{ is a permutation of} \ (i', j', i' + j' - t' - r'); \\
\text{iv.} & \quad x_{i,j}^{t,r} = 0 \quad \text{if one of} \ i, j, \text{and} \ i + j - t - r \text{ is not in} \ \mathcal{D}.
\end{align*}
\]
For \( k = 0, \ldots, \lfloor \frac{w}{2} \rfloor \), \( l = 0, \ldots, \lfloor \frac{n-w}{2} \rfloor \), define

\[
B_{k,l} = \left( \sum_t \alpha(n,i,j,k,l) \beta_{i,j,k,l}^{t,w} \gamma_{i,j}^{n-w,}\delta_{i,j}^{r,t} x_{i,j}^{(t,w)} \right)_{(i,j)=(k,l)}
\]

where

\[
\beta_{i,j,k}^a = \sum_{u=0}^a (-1)^{a-u} \binom{a}{u} \binom{a-2k}{u-k} \binom{a-k-u}{i-k} \binom{a-k-u}{j-u}.
\]

for \( a = w, n-w \), and

\[
\alpha(n,i,j,k,l) = \binom{n-2k}{i-k} \binom{n-2k}{j-k} \binom{n-2l}{i-l} \binom{n-2l}{j-l}^{-\frac{1}{2}}.
\]

Then

\[
|C| \leq \max \left\{ \sum_{i=0}^n \binom{w}{i} \binom{n-w}{i} x_{i,0,0} : B_{k,l} \text{ is p.s.-d. for all } k,l \right\}.
\]

**Proof.** Only condition (iv) is new compared to Schrijver’s conditions for constant weight codes in \([\text{Sch05}]\). To justify it, check that for \( (u,v,z) \in C^3 \) such that \( d_J(u,v) = i, d_J(u,z) = j, d(v,u) = t, d(z,u) = r \), the distance \( d_J(v,z) = i+j-t-r \). Therefore, \( x_{i,j}^t \) is nonzero only if the numbers \( i, j, \) and \( i+j-t-r \) are in \( \mathcal{D} \). \( \square \)

**Remark 3.4.** As for the Hamming case, the obvious conditions \( i-t, j-t, i+j-t \in \{0, \ldots, w\}, i-r, j-r, i+j-r \in \{0, \ldots, n-w\} \) aide in speeding up the SDP implementation.

4. Numerical Results

4.1. Implications of the SDP bound for the Hamming space. The best known results concerning bounds for \( A(\mathcal{H}_s^2, s) \), \( s = 2, 3 \) prior to this work were due to \([\text{BM11}]\) \([\text{MN11}]\). In particular, for \( s = 2 \), Theorem \( 2.3 \) from \([\text{BM11}]\) gives an upper bound \( A(\mathcal{H}_2^2, 2) \leq 1 + \binom{n}{2} \) under the condition that \( d_1 + d_2 \leq n \). If the pairs of distances that violate this condition can be eliminated using the LP bound, then we have that \( A(\mathcal{H}_2^2, 2) = 1 + \binom{n}{2} \) by the construction in Theorem \( 2.3 \). Exceptions arise when the LP bound is insufficient, and this is where we bring in SDP to handle them. Similarly, for \( s = 3 \) we aim to eliminate exceptions to the equality \( A(\mathcal{H}_2^3, s) = n + \binom{n}{3} \) relying on the SDP bound.

Our findings are listed in Tables 2 and 3 below, where the numerical results for the LP and SDP bounds are rounded up to the nearest integer. Table 2 concerns the two-distance case, where we list the dimensions that we were able to solve using SDP. The pairs of distances that were previously unresolved are listed in column \( \mathcal{D} \). The LP bound for these distances is given in the column LP, and it exceeds \( 1 + \binom{n}{2} \), while the SDP bound, shown next to it, helps to finish the argument.

The implementation of SDP for large values of \( n \) involves matrices of large dimensions, and is prone to numerical issues. Specifically, the constraints involve large numbers, and the MATLAB SDP solver cannot handle this problem very well. As a remedy, we rescale the problematic constraints by an appropriate constant factor, which yields a stable numerical solution. This modification results in an equivalent optimization problem. The scaling factor is listed in the column “Constant” in the tables.
Example 1. Applying the described approach for the Hamming space with $n = 47$ and $\mathcal{D} = \{16, 32\}$, we obtain the following results.

| $n$ | $1 + \binom{n}{2}$ | $\mathcal{D}$ | LP | SDP | Constant |
|-----|-------------------|----------------|-----|-----|----------|
| 47  | 1082              | $\{16, 32\}$  | 1689| 620.2962| $10^{-2}$|
| 47  | 1082              | $\{16, 32\}$  | 1689| 620.3703| $10^{-3}$|
| 47  | 1082              | $\{16, 32\}$  | 1689| 620.4560| $10^{-2}$|
| 47  | 1082              | $\{16, 32\}$  | 1689| 620.4290| $10^{-5}$|

Table 1. Bounds on 2-codes in the Hamming space

The results of our calculations for 2- and 3-codes in the Hamming space are listed in the next two tables.

| $n$ | $1 + \binom{n}{2}$ | $\mathcal{D}$ | LP | SDP | Constant | weakened SDP |
|-----|-------------------|----------------|-----|-----|----------|--------------|
| 47  | 1082              | $\{16, 32\}$  | 1689| 621 | $10^{-2}$| No           |
| 53  | 1379              | $\{18, 36\}$  | 1856| 833 | $10^{-3}$| Yes          |
| 59  | 1712              | $\{20, 40\}$  | 2160| 1210| $10^{-4}$| Yes          |
| 65  | 2081              | $\{22, 44\}$  | 2635| 2036| $10^{-5}$| Yes          |

Table 2. Bounds on 2-codes in the Hamming space

| $n$ | $n + \binom{n}{3}$ | $\mathcal{D}$ | LP | SDP | Constant | weakened SDP |
|-----|-------------------|----------------|-----|-----|----------|--------------|
| 34  | 6018              | $\{8, 12, 20\}$| 6723| 2388| 1        | No           |
| 35  | 6580              | $\{8, 12, 20\}$| 8523| 2628| 1        | No           |

Table 3. Bounds on 3-codes in the Hamming space

Combining these calculations with the earlier results in [BM11, Proposition 13(a)], [MN11, Theorem 3.2(1)], we obtain Theorem 1.1.

As a remark, we also tried to construct a stronger LP problem by adding Delsarte’s constraints for the split Hamming weight distribution of an s-code [Sim95]. However, this did not yield any further improvements of the known results.

4.2. SDP bounds for the Johnson space. For the Johnson space, we similarly examine the exceptions to $A(J_2^n, 2) \leq \binom{n}{2} - n + 1$ left in earlier works, resolving several previously unsolved cases. Our results are shown in Table 4 (organized similarly to Tables 2 and 3). Combined with the previously known results, they yield Theorem 1.2 stated above.

| $(n, w)$ | $\binom{n}{2} - n + 1$ | $\mathcal{D}$ | LP | SDP | Constant | weakened SDP |
|----------|-------------------------|----------------|-----|-----|----------|--------------|
| (44, 17) | 903                     | $\{7, 14\}$   | 949 | 466 | $10^{-3}$| No           |
| (45, 16) | 946                     | $\{7, 14\}$   | 1034| 390 | $10^{-3}$| No           |
| (46, 16) | 990                     | $\{7, 14\}$   | 1430| 503 | $10^{-3}$| No           |

Table 4. Bounds on 2-codes in the Johnson space
5. Further results for $s$ codes in the Johnson space

In this section, we extend some earlier results in [BM11][MN11] for 2-codes in the Johnson space without relying on the SDP bound. Rather, we use the LP approach for larger dimensions than in earlier works, extending the region of exact answers for all $47 \leq n \leq 100$ from $n \leq 46$ known previously. Together with [BM11] Proposition 10 we have the following results.

**Theorem 5.1.** $A(J^{n, w}, 2) = \binom{n-w+2}{2}$ if $n$ and $w$ satisfy one of the following conditions:
1. $6 \leq n \leq 8$ and $w = 3$;
2. $9 \leq n \leq 11$ and $3 \leq w \leq 4$;
3. $12 \leq n \leq 14$ or $25 \leq n \leq 34$ and $3 \leq w \leq 5$;
4. $15 \leq n \leq 24$ or $35 \leq n \leq 100$ and $3 \leq w \leq 6$.

Our results are obtained as follows. Recall that the generalized LRS theorem (Theorem 2.3) gives necessary integrality conditions for the distances of an $s$ code of sufficiently large size. These conditions underlie the application of LP bounds in [MN11]. For the distances that satisfy these conditions, the authors of [MN11] used the LP and harmonic bounds, arriving at their results. At the same time, for the other cases, i.e., when the conditions are not satisfied, they used a trivial upper bound $2N(J^{n, w}, s) - 1$ on the size of the code. At the same time, it is still possible to obtain a tighter upper bound. Namely, we check all the possible distance sets when the $2N(J^{n, w}, s) - 1$ is larger than $\binom{n-w+s}{s}$ in Theorem 2.11 using the LP method, obtaining the improved results mentioned in Theorem 5.1.

Similarly, for 3-codes we have the following.

**Theorem 5.2.** $A(J^{n, w}, 3) = \binom{n-w+3}{3}$ if $n$ and $w$ satisfy one of the following conditions:
1. $n = 11$ and $w = 4$;
2. $12 \leq n \leq 13$ and $4 \leq w \leq 5$;
3. $16 \leq n \leq 17$ and $4 \leq w \leq 6$;
4. $20 \leq n \leq 24$ and $4 \leq w \leq 7$;
5. $25 \leq n \leq 58$ and $4 \leq w \leq 8$.

Compared to [MN11], this adds a new case $(n, w) = (12, 5)$ left out earlier as an exception.

**Example 2.** In this example, we apply our method for the parameters $n = 11, w = 4$. First, we use Theorems 2.8 and 2.9 to form a list of cases to be tested, and obtain two possible distance sets $\mathcal{D}_1 = \{1, 2, 3\}$ and $\mathcal{D}_2 = \{1, 3, 4\}$. The LP bound for these two cases yields 120 and 22, respectively. Since $\binom{n-w+3}{3} = 120$, we conclude that $A(J^{n, w}, 3) = \binom{n-w+3}{3}$ for $n = 11$ and $w = 4$.

At the same time, for $n = 12$ and $w = 5$, we have $2N(J^{n, w}_2) > \binom{n-w+3}{3} = 120$, so the LRS-type theorem does not apply, and we need to test all the possible distance sets, listed in Table 5.

| $\mathcal{D}$ | 1, 2, 3 | 1, 2, 4 | 1, 2, 5 | 1, 3, 4 | 1, 3, 5 | 1, 4, 5 | 2, 3, 4 | 2, 3, 5 | 2, 4, 5 | 3, 4, 5 |
|---------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| LP            | 120     | 36      | 57      | 36      | 29      | 28      | 66      | 36      | 32      | 16      |

**Table 5.** LP on distance sets for $n = 12, w = 5$

Using the same approach for $s = 4$, we managed to add two new sets of parameters $(n, w) = (15, 6), (16, 6)$ to the results of [MN11]. A summary of the currently known results appears in the following statement.
Theorem 5.3. For each of the following combinations of the parameters

1. $15 \leq n \leq 19$ and $5 \leq w \leq 6$;
2. $20 \leq n \leq 24$ and $5 \leq w \leq 7$;
3. $25 \leq n \leq 29$ and $5 \leq w \leq 8$;
4. $30 \leq n \leq 34$ or $41 \leq n \leq 47$ and $5 \leq w \leq 9$;
5. $35 \leq n \leq 40$ or $48 \leq n \leq 59$ and $5 \leq w \leq 10$;
6. $60 \leq n \leq 70$ and $5 \leq w \leq 11$

$$A(J^{n,w},4) = \left(\frac{n - w + 4}{4}\right).$$

6. Conclusion

While the primary uses of SDP in coding theory, starting with [Sch05], have been for proving upper bounds on codes, in this work we applied it to the problem of establishing the maximum size of $s$-codes in the Hamming and Johnson spaces.

We conclude with two research problems. Theorems 1.1 and 5.1 seem to suggest closed-form solutions for $A(H^n_2, 2)$ and $A(J^{n,w}_2, 2)$. Accordingly, the following question seems worth further study.

**Problem 6.1.** Is it true that $A(H^n_2, 2) = 1 + \left(\binom{n}{2}\right)$ for sufficiently large $n$?

We believe that the exceptions found by Barg-Musin for 2-codes in the Hamming space are not real exceptions, i.e., they are not supported by actual code constructions. However, because of numerical issues, we managed to eliminate only the cases $n = 47, 53, 59,$ and 65.

**Problem 6.2.** Is it true that $A(J^{n,w}_2, 2) = \left(\frac{n - w + 2}{2}\right)$ for sufficiently large $n$ with $n \geq 3(w - 1)$ and $w \in \{3, 4, 5\}$?

This statement holds true for $n \leq 100$. To deal with this problem, one may try to form an algebraic bound from the semidefinite constraints for $s$-distance sets in the Hamming and Johnson spaces and then analyze its asymptotic behavior. This approach was proved successful in recent works of Kao-Yu [KY22] and Glazyrin-Yu [GY18] devoted to equiangular lines. We also note recent advances in determining the asymptotic size of 2-distance sets on the real sphere [JTY20].

A weaker version of Problem 6.2 can be stated as follows.

**Problem 6.3.** For $w \in \{3, 4, 5\}$, is it true that $A(J^{n,w}_2, 2) = \left(\frac{n - w + 2}{2}\right)$ for infinitely many $n$?

In solving LP problems we relied on Mathematica, while for SDP problems we used MATLAB with CVX toolbox [GB08, GB14] and the MOSEK solver. All of our source codes can be found on github:

[https://github.com/PinChiehTseng/s-distance-set](https://github.com/PinChiehTseng/s-distance-set)

ACKNOWLEDGEMENT

We would like to thank Hiroshi Nozaki for helpful discussions.

AB was supported by NSF grant CCF2104489 and NSF-BSF grant CCF2110113. PCT and CYL were supported by the Ministry of Science and Technology (MOST) in Taiwan under Grant MOST110-2628-E-A49-007, MOST111-2119-M-A49-004, and MOST111-2119-M-001-002. WHY was supported by MOST under Grant109-2628-M-008-002-MY4.
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