On the quantum information entropies and squeezing associated with the eigenstates of the isotonic oscillator

A Ghasemi¹, M R Hooshmandasl¹ and M K Tavassoly²

¹ Department of Mathematical Sciences, Yazd University, Yazd, Iran
² Atomic and Molecular Group, Faculty of Physics, Yazd University, Yazd, Iran

E-mail: mktavassoly@yazduni.ac.ir

Received 9 March 2011
Accepted for publication 27 July 2011
Published 15 August 2011
Online at stacks.iop.org/PhysScr/84/035007

Abstract

In this paper we calculate the position and momentum space information entropies for the quantum states associated with a particular physical system, i.e. the isotonic oscillator Hamiltonian. We present our results for its ground states, as well as for its excited states. We observe that the lower bound of the sum of the position and momentum entropies expressed by the Beckner, Bialynicki-Birula and Mycielski (BBM) inequality is satisfied. Moreover, there exist eigenstates that exhibit squeezing in the position information entropy. In fact, entropy squeezing, which occurs in position, will be compensated for by an increase in momentum entropy, such that the BBM inequality is guaranteed. To complete our study we investigate the amplitude squeezing in \( x \) and \( p \)-quadratures corresponding to the eigenstates of the isotonic oscillator and show that amplitude squeezing, again in \( x \), will be revealed as expected, while the Heisenberg uncertainty relationship is also satisfied. Finally, our numerical calculations of the entropy densities will be presented graphically.

PACS numbers: 03.65.−w, 42.50.Lc, 42.50.Dv

(Some figures in this article are in colour only in the electronic version.)

1. Introduction

The probability densities of the position and momentum of a single-particle system in one dimension are expressed as

\[
\rho(x) = |\psi(x)|^2, \quad \xi(p) = |\phi(p)|^2,
\]

where \( \psi(x) \) is the solution of the time-independent Schrödinger equation and \( \phi(p) \) is its Fourier transform in momentum space. Using the above expressions, Shannon information entropy of the position and momentum space entropies are respectively defined as follows [1]:

\[
S(\rho) = -\int \rho(x) \ln \rho(x) dx,
\]

\[
S(\xi) = -\int \xi(p) \ln \xi(p) dp.
\]

These definitions have been used frequently in the literature [2–5]. These entropies can be helpful in particular cases, for instance one may point to the reconstruction of the charge and momentum densities of atomic and molecular systems [6, 7] by means of a maximum entropy procedure. It is believed that the above entropies for one-dimensional systems lead to a stronger version of the Heisenberg uncertainty relationship written as (see p 28 of [8])

\[
S(\rho) + S(\xi) \geq \ln(e\pi),
\]

which is a consequence of a well known inequality in Fourier analysis, first conjectured by Everett in [9], Hirschman in [10] and then proved by Bialynicki-Birula, Mycielski [11] and Beckner (BBM) in [12]. The inequality (4) indicates that the sum of entropies is bounded from below by the value 2.1447... for one dimensional systems. This lower bound is obtained from the Gaussian wavefunction of the ground state of the harmonic oscillator. It ought to be mentioned that the relationship (4) can be expressed for any two conjugate operators. For instance, recently we have examined
the entropy uncertainty relationship for the phase-number operators of nonlinear coherent states corresponding to solvable quantum systems with discrete spectra [13].

The analytical form of the position and momentum space entropies in (2) and (3) have been found for a few particular solvable quantum mechanical systems. For instance, the momentum and position entropies associated with the ground state of the harmonic oscillator were calculated exactly, where it is shown that the BBM inequality is saturated: $S(\rho) = 1.07236 = S(\xi)$ (it is a fact that the vacuum state and the canonical coherent state also minimize the Heisenberg uncertainty relationship). However, for excited states of the harmonic oscillator the related entropies may only be approximated [2]. As a second example, one may refer to the position and momentum space entropies associated with Pöshl–Teller potentials. These quantities for the ground state of this system are evaluated exactly, but even for the first excited states the results have again been given numerically [14]. The Morse potential is another physical system whose quantum information entropies were previously considered and discussed. All of the related results have been given numerically [15]. In all of the above mentioned systems, the lower bound, which is predicted by the BBM inequality, is guaranteed. On the other side, in addition to the above physical systems, Shannon information entropies for some classical orthogonal polynomials such as Hermite, Laguerre and Gegenbauer polynomials have also been introduced recently [3, 16]. Of late, the evolution of entropy squeezing in quadratures of a single-mode field in the Jaynes–Cummings model in the presence of a nonlinear effect has been studied and discussed in [17]. Entropy squeezing has also been investigated using Shannon information entropy for the solutions of the Hamiltonian describing the interaction between a single two-level atom and two electromagnetic fields in the framework of a modified Jaynes–Cummings model [18].

This paper deals with the same studies, i.e. investigates the position and momentum entropies corresponding to a special quantum system known as the isotonic oscillator [19], through which we find that the BBM inequality is satisfied. However, surprisingly, unlike the previously discussed cases in the literature [2, 14, 15], entropy squeezing occurs in position for some special eigenstates of the isotonic oscillator, i.e. it becomes less than that of the harmonic oscillator. Clearly, a large enough increase in the momentum entropy, corresponding to the eigenstates whose entropy squeezing in $x$ has occurred, compensates for the reduction of entropy in position, such that the BBM inequality (4) is not violated. Note that, Pöshl–Teller and Morse potential entropy squeezing is not reported in any of the previously discussed physical systems in the literature, such as the harmonic oscillator. Meanwhile, our considered system is a real physical oscillator, which possesses entropy squeezing in position space.

It was established in [4] that for all quantum states having a squeezing property in one of the quadratures (double squeezing is a very rare phenomenon [20, 21]), the corresponding entropy is also squeezed. Hence, apart from the above mentioned results on entropy squeezing emphasized in this paper, in view of the connection between the BBM inequality and the Heisenberg uncertainty relationship, we have calculated the variances of the two quadratures of the field associated with the isotonic oscillator. In light of our further numerical calculations, which are presented later in the paper, we may say that the ground state of the isotonic oscillator (as a real physical potential) can reveal a quadrature squeezing feature; i.e. its spatial variance falls below that of the vacuum. Therefore, our results confirm that whenever squeezing in one of the quadratures is detected, the corresponding information entropy is also squeezed. Nevertheless, as expected there are states that possess entropic squeezing in position, while quadrature squeezing is not seen. This is also in agreement with the statement reported in [4].

In view of the increasing interest in the non-classical state in quantum optics and related fields of research, we wish to establish the non-classical features of some of the eigenstates of the isotonic oscillator that have not been pointed out up to now. In addition, we have found a set of (entropic or/and quadrature) squeezed states corresponding to a well known physical potential.

The paper is organized as follows. In the next section we will address the entropies of the eigenstates of the isotonic oscillator Hamiltonian. In section 3 we present the numerical results and make some comments on the isotonic oscillator system. Finally, in section 4 we outline the conclusion.

2. Entropies of the eigenfunctions of the isotonic oscillator

There exist a few potentials corresponding to physical systems whose exact solutions are known. Among these solvable systems, the isotonic oscillator is an interesting model with Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2 + A, \quad A \geq 0,$$

acting in Hilbert space $L_2(0, \infty)$. The Hamiltonian in (5) admits exact solutions as follows [19, 22],

$$\psi_m^n(x) = (-1)^m \frac{2(\gamma_m)^{x - \frac{1}{2}} e^{-x^2}}{m! \Gamma(\gamma)} F_1(-m; \gamma; x^2)$$

$$= \sum_{k=0}^{m} (-1)^m \frac{2(\gamma_m)^{x - \frac{1}{2}}}{m! \Gamma(\gamma)} (-m)_k \frac{e^{-x^2}}{(\gamma_k)^{\frac{1}{2}}} x^{2k+\gamma-\frac{3}{2}},$$

with the Dirichlet boundary condition $\psi_0^0(0) = 0$, where $\gamma = 1 + \frac{1}{4} \sqrt{1 + 4A}$ and the eigenvalues are exactly obtained as $\epsilon_m = 2(m + \gamma)$. Also notice that $F_1(-m; \gamma; x^2)$ denotes the Kummer confluent hypergeometric function, $\Gamma(\gamma)$ denotes the Gamma function and $m+1 \equiv \gamma + 1 + 2 + \cdots + (m + 1)$ is the Pochhammer symbol with $(\gamma_0) = 1$. The condition $A \geq 0$ stated in (5) is equivalent to $\gamma \geq 3/2$ in (6). It is obvious that adding the term $Ax^2$ to the harmonic oscillator Hamiltonian does not change the equidistant aspect of the spectrum of the linear harmonic oscillator. A word is necessary about the completeness of the eigenstates in (6) [22]. It was proved by Hall et al that these eigenfunctions satisfy the orthonormality condition in $L_2(0, \infty)$, that is $\int_0^\infty \psi_m^n(x) \psi_{m'}^{n'}(x) dx = \delta_{m,n} \delta_{m',n'}$ where $m, n = 0, 1, 2, 3, \ldots$ So the set of $\{|\psi_m^n(x)|_{m,n=0}^\infty\}$-functions is a complete orthonormal basis for the Hilbert space $L_2(0, \infty)$. 

2
Our numerical results which establish the BBM inequality for the isotonic oscillator when some fixed values of \( m \) and \( \gamma \) are considered.

| \( \gamma \) | \( m \) | \( S'_m(\rho) \) | \( S'_m(\xi) \) | \( S'_m(\rho) + S'_m(\xi) \) | \( 1 + \ln \pi \) |
|---|---|---|---|---|---|
| \( \frac{1}{2} \) | 0 | 0.6496 | 1.5807 | 2.2303 | 2.1447 |
| 1 | 0.9166 | 1.9052 | 2.8218 | 2.1447 |
| 2 | 1.0749 | 2.0839 | 3.1588 | 2.1447 |
| 3 | 1.1889 | 2.0797 | 3.3968 | 2.1447 |
| \( \frac{3}{2} \) | 0 | 0.6852 | 1.4941 | 2.1793 | 2.1447 |
| 1 | 0.9546 | 1.8167 | 2.7623 | 2.1447 |
| 2 | 1.0985 | 2.0018 | 3.1003 | 2.1447 |
| 3 | 1.2087 | 2.1329 | 3.3416 | 2.1447 |
| \( \frac{5}{2} \) | 0 | 0.6984 | 1.4663 | 2.1647 | 2.1447 |
| 1 | 0.9591 | 1.7797 | 2.7388 | 2.1447 |
| 2 | 1.1108 | 1.9628 | 3.0736 | 2.1447 |
| 3 | 1.2198 | 2.2042 | 3.4240 | 2.1447 |

The Fourier transform of \( \phi_m(x) \) in (6) is denoted by \( \phi_m(p) \) and it is given by

\[
\phi_m(p) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{m} C_k(\gamma, m) \times \left[ \Gamma\left(\frac{1}{4} + k + \frac{\gamma}{2}\right) F_1\left(\frac{1}{4} + k + \frac{\gamma}{2}, \frac{1}{2}, -\frac{p^2}{2}\right) + i \sqrt{2} p \Gamma\left(\frac{3}{4} + k + \frac{\gamma}{2}\right) F_1\left(\frac{3}{4} + k + \frac{\gamma}{2}, \frac{3}{2}, -\frac{p^2}{2}\right) \right],
\]

where we set

\[
C_k(\gamma, m) = \frac{(-1)^m 2^{-\frac{1}{2}k + \frac{\gamma}{2}} (-m)_k}{k! (\gamma)_k} \sqrt{\frac{(\gamma)_m}{\pi m! \Gamma(\gamma)}}.
\]

The term \( \lambda^{-1/2} \) is a normalization factor that may be determined for fixed \( m, \gamma \), i.e. \( \int_{-\infty}^{\infty} |\phi_m(p)|^2 dp = 1 \). Inserting the expressions obtained in (6) and (7) in the following integrals,

\[
S'_m(\rho) = -\int_{-\infty}^{\infty} |\psi_m^\prime(x)|^2 \ln |\psi_m^\prime(x)|^2 dx,
\]

\[
S'_m(\xi) = -\int_{-\infty}^{\infty} |\phi_m^\prime(p)|^2 \ln |\phi_m^\prime(p)|^2 dp, \quad m = 0, 1, 2, \ldots,
\]

yields the position and momentum information entropies \([2, 5]\). Note that while in the integration procedure in (8) the limits are from 0 to \( \infty \) (which is consistent with the corresponding Hilbert space \( L_2(0, \infty) \)), the limits of integration in (9) are from \(-\infty\) to \(+\infty\). The latter is due to the fact that the particle confined in the well-potential can travel back and forth (positive and negative). Seemingly, solving the above integrals in the closed form to obtain the information entropies for arbitrary values of \( \gamma \) and \( m \) is very hard if not impossible, so we confine ourselves to some particular values of \( \gamma \) and \( m \) in the rest of the paper.

### Table 2. Our numerical results for calculating the Heisenberg uncertainty relationship \((\Delta x)^2(\Delta p)^2 \geq 0.25\) for the isotonic oscillator with some fixed values of \( m \) and \( \gamma \).

| \( \gamma \) | \( m \) | \( (\Delta x)^2 \) | \( (\Delta p)^2 \) | \( (\Delta x)(\Delta p) \) | \( \min((\Delta x)^2(\Delta p)^2) \) |
|---|---|---|---|---|---|
| \( \frac{1}{2} \) | 0 | 0.2268 | 1.4640 | 0.3320 | 0.2500 |
| 1 | 0.6352 | 3.4456 | 2.1887 | 0.2500 |
| 2 | 1.0238 | 5.4317 | 5.5608 | 0.2500 |
| 3 | 1.4074 | 7.4199 | 10.4424 | 0.2500 |
| \( \frac{3}{2} \) | 0 | 0.2365 | 1.6666 | 0.3759 | 0.2500 |
| 1 | 0.6746 | 3.6655 | 2.1362 | 0.2500 |
| 2 | 1.0869 | 5.6633 | 5.6152 | 0.2500 |
| 3 | 1.4875 | 7.6661 | 10.6597 | 0.2500 |
| \( \frac{5}{2} \) | 0 | 0.2405 | 1.1000 | 0.2646 | 0.2500 |
| 1 | 0.6939 | 3.1000 | 2.1511 | 0.2500 |
| 2 | 1.1225 | 5.1000 | 5.7235 | 0.2500 |
| 3 | 1.5367 | 7.0999 | 10.9108 | 0.2500 |

### 3. Numerical results

We will now study the entropies of position and momentum for the eigenstates of the isotonic oscillator Hamiltonian. In general, the analytical solutions of the entropies defined in \((8)\) and \((9)\) are not possible for arbitrary values of \( \gamma \) and \( m \). Instead, one can work distinctly with, for instance, \( \gamma = 3/2, 5/2, 7/2, \ldots \), also fixed values of \( m \) should be taken into account. The numerical results for the entropies in \((8)\) and \((9)\) and their sum, for three different values of \( \gamma \) were expressed in table 1, where four different values of \( m \) are considered for each \( \gamma \). From table 1 it is clear that for any choice of \( \gamma \), the entropies and their sum increase with increasing \( m \). Interestingly, for any fixed \( \gamma \), there exist some values of \( m \) (ground and first excited states) for which the information entropies in position become less than 1.07236, i.e. the ground state’s value of the harmonic oscillator. In other words the entropy squeezing occurs in \( x \). Altogether, it is observed that the BBM inequality for arbitrary values of \( m \) and \( \gamma \) is satisfied in all cases for the states displayed in table 1. Indeed, an increase in the momentum entropy, relative to that of the ground state of the harmonic oscillator, is observed for any \( \gamma \) (with \( m = 0, 1 \)), which is enough to compensate for the reduction in position entropy. Consequently, our results confirm the stated result in \([23]\), i.e. ‘there are no physical states which violate the BBM inequality’.

Nevertheless, due to the relationship that exists between the BMM inequality and the Heisenberg uncertainty relation we have been motivated to evaluate the variances of the quadratures of the field defined as

\[
(\Delta x)^2 = \langle z^2 \rangle - \langle z \rangle^2, \quad z = x \text{ or } p.
\]

Notice that \([x, p] = i\), the Heisenberg uncertainty reads \((\Delta x)^2(\Delta p)^2 \leq 0.25\). Therefore, a state is squeezed if \((\Delta x)^2 < 0.5 \) or \((\Delta p)^2 < 0.5 \). In the squeezing phenomenon, the reduction in the uncertainties of one of the quadratures below the value of the vacuum occurs with the cost of an increase in the conjugate quadrature. We have used \((6)\) and \((7)\), respectively, for the calculation of squeezing in \( x \) and \( p \). Our results in table 2 on noise fluctuations show that, for the ground states (where the entropy squeezing in position has occurred), squeezing in \( x \)-quadratures of the field is visible. Further calculations show that for
Figure 1. The position space entropy densities of the isotonic oscillator for \( m = 0, 1, 2, 3 \) and \( \gamma = 3/2 \).

\( \psi_{m=0}(x) \) one has \((\Delta x)^2 = 0.2427, 0.2441, 0.2450, \) and \((\Delta p)^2 = 1.0714, 1.0555, 1.0455, \) respectively, for \( \gamma = 9/2, 11/2, 13/2 \). Hence, the multiplication of \((\Delta x)^2(\Delta p)^2\) yields \( \approx 0.25 \). Continuing these computations, we observe that the squeezing effect in \( x \) (\( p \)) tends to the finite value 0.25 (1) with increasing \( \gamma \). Meanwhile, it ought to be mentioned that these states are rather different from the well known squeezed states in quantum optics, and have been obtained from the action of the squeezing operator on the vacuum of the field. Hence, we may point out that all of the ground states (with arbitrary \( \gamma \)) of the isotonic oscillator, as a physical potential, are indeed squeezed states. Thus, since all of the ground states of the isotonic oscillator possess this property, independent of the values of \( \gamma \), i.e. they can reasonably be considered as a class of (non-classical) squeezed states that are directly related to a well known physical system. In addition, some special sets of eigenstates of an isotonic oscillator with \( m = 0 \) may be called the ‘ideal squeezed coherent states’, since they saturate the Heisenberg inequality (where we have used the definition of ideal squeezed coherent states as introduced by Scully and Zubairy in [25]). At this stage, it is worth mentioning that it has recently been proved that the quadrature squeezing exhibition is always accompanied by a corresponding entropy reduction below the vacuum entropy level \( \approx 1.07236 \) [4]. This feature was not be observed in previously considered physical systems, such as the harmonic oscillator, Pöschl-Teller and Morse potentials.

To this end, instead of the evaluation of quantum information entropies, we have plotted entropic densities defined by

\[
\Psi_{m}^{x}(x) \equiv -|\psi_{m}^{x}(x)|^{2}\ln|\psi_{m}^{x}(x)|^{2}
\]

and

\[
\Phi_{m}^{p}(p) \equiv -|\phi_{m}^{p}(p)|^{2}\ln|\phi_{m}^{p}(p)|^{2},
\]

taking into account the position and momentum space representation, respectively. The entropy densities \( \Psi_{m}^{x}(x) \) and \( \Phi_{m}^{p}(p) \) provide a measure of information about the localization of the particle in their respective spaces. The position space entropy densities \( \Psi_{m}^{x}(x) \) are depicted graphically versus position \( x \) in figures 1 and 3 for \( \gamma = 3/2 \) and 7/2, respectively. Also, the momentum space entropy densities \( \Phi_{m}^{p}(p) \) are plotted versus \( p \) in figures 2 and 4 for \( \gamma = 3/2 \) and 7/2, respectively (note that graphs related to the case \( \gamma = 5/2 \), for which the numerical results have been displayed in table 1, are omitted due to economies of space). Each group of figures contains different values of \( m \), i.e. 1, 2, 3. The general (overall) shape of the figures depends strictly on the values of \( \gamma \). From figures 1(a) and 3(a), (which relate to \( m = 0 \)), it is clearly seen that in all cases the densities of position entropies begin from 0, then increase with increasing \( x \), include some oscillations, and finally tend to zero again. Qualitatively, a similar situation holds for the set of figures 1(b) and 3(b) for \( m = 1, 1(c) \) and 3(c) for \( m = 2, \) and 1(d) and 3(d) for \( m = 3 \). There are some differences: (i) the number of oscillations grows up with increasing \( \gamma \), (ii) oscillations occur between 0 and its maximum value \( \approx 0.35 \) for \( m \geq 1 \), with small oscillations occurring on the peak of large oscillations.
Figure 2. The momentum space entropy densities of the isotonic oscillator for $m = 0, 1, 2, 3$ and $\gamma = 3/2$.

Figure 3. The position space entropy densities of the isotonic oscillator for $m = 0, 1, 2, 3$ and $\gamma = 7/2$. 
In figures 2(a) and 4(a), which have been plotted for different values of \( \gamma \), but all for \( m = 0 \), it is observed that there is a maximum at \( p = 0 \) which then tends to 0 with increasing momentum. Figures 2(b) and 4(b) for \( m = 1 \), 2(c) and 4(c) for \( m = 2 \), 2(d) and 4(d) for \( m = 3 \) show that the densities of the momentum entropies have a local extremum at \( p = 0 \), in addition to an extra extremum distributed symmetrically around the vertical axis \( p = 0 \). Also, the number of oscillations increases with increasing \( m \). However, again the general shapes of the figures do not change for each \( m \) with different chosen values of \( \gamma \). Also, it is seen that the maximum heights of the figures of entropic densities, either in \( x \) or \( p \) space are at about \( \approx 0.35 \pm 0.02 \).

### 4. Conclusion

In summary, we investigated squeezing in the position and momentum entropies, as well as in the quadratures of the field for the eigenstates of the isotonic oscillator. Note that ‘coherent states’ corresponding to the isotonic oscillator may be found in recent literature [27], in which the authors used the eigenfunctions of the isotonic oscillator as the basis for their construction. The coherent states and their even and odd states associated with the isotonic oscillator have also been constructed and their non-classical signs were discussed in [28, 29]. However, their approaches to the introduction of non-classical states are essentially different from ours. Meanwhile, as is well known, the main motivation for the construction, generalization and generation of various classes of coherent states lies in searching for non-classical signs in them. We have explored the non-classical signs, especially entropic squeezing, in the eigenfunctions of the isotonic oscillator. The investigation of the information entropy has been carried out for several solvable non-degenerate quantum systems [2, 14, 15], but no squeezing effect has been reported. Summing up, the observation of entropic squeezing, which occurs for the ground and first excited states (of the isotonic oscillator), and the quadrature squeezing which occurs for its ground states, characterizes our contribution from previous work. Interestingly, the strength of the squeezing for \( \psi_{m=0}^{\gamma}(x) \) tends to a finite value 0.25 with increasing \( \gamma \). As a result, all ground states of the isotonic oscillator are squeezed. Obviously, these non-classical signs may also be revealed and observed in the literature for various constructed quantum states. For instance, we may refer to ‘nonlinear coherent states’ which are mostly (quadrature) squeezed [24], but the relation of the evolved deformation function \( f(n) \) (which has a central role in this approach) to a ‘physical potential’ is a problem that has not been transparent up to now, except for the special (linear) case of \( f(n) = 1 \), which is simply connected to the harmonic oscillator (where it does not possess any of the non-classicality features). Even, when one works with coherent states for the solvable quantum systems, it is worth noticing that in these cases one does not deal with the eigenstates of the physical potential, instead, the states are constructed by some specific definitions and requirements, and then the non-classical features are examined [26]. However, we want to emphasize that the non-classical states are directly related to a special, known
physical system. So, the results in this paper are new and interesting to quantum opticians. As a final point, the constant $A$ in the Hamiltonian (5) characterizes the relative strength of the $1/r^2$, i.e. centripetal potential. In a rotating diatomic, for example, this constant would be determined approximately by the rotational energy level. It may be interesting to consider the connection between squeezing in the information entropy and the rotational constant $A$, to motivate future experimental studies of these results.

Acknowledgment

One of the authors (MKT) would like to thank Professor R Roknizadeh from the Quantum Optics Group of The University of Isfahan for useful discussions.

References

[1] Shannon C E 1948 Bell Syst. Tech. J. 27 623 (1948)
[2] Majerník V and Opatrný T 1996 J. Phys. A: Math. Gen. 29 2187
[3] Dehesa J S, Martínez-Finkelshtein A and Sánchez-Ruiz J 2001 J. Comput. Appl. Math. 133 23
[4] De Nicola S, Fedele R, Man’ko M A and Man’ko V I 2006 Eur. Phys. J. B 52 191
[5] Man’ko M A and Man’ko V I 2011 arXiv:1102.2497
[6] Galindo A and Pascual P 1978 Quantum Mechanics (Berlin: Springer)
[7] Angulo J C, Antolin J, Zarzo A and Cuchi J C 1999 Eur. Phys. J. D 7 479
[8] Dodonov V V and Man’ko V I 1987 Invariants and the evolution of nonstationary quantum systems Proc. P. N. Lebedev Physical Institute vol 183 (Moscow: Nauka) (transl. by Nova Science, New York, 1989)
[9] Everett H 1973 The Many World Interpretation of Quantum Mechanics (Princeton, NJ: Princeton University Press)
[10] Hirschman I I 1957 Am. J. Math. 79 152
[11] Bialynicki-Birula I and Mycielski J 1975 Commun. Math. Phys. 44 129
[12] Beckner W 1975 Ann. Math. 102 159
[13] Honarasa G R, Tavassoly M K and Hatami M 2009 Opt. Commun. 282 2192
[14] Honarasa G R, Tavassoly M K and Hatami M 2009 Phys. Lett. A 373 3931
[15] Atre R, Kumar A and Kumar N 2004 Phys. Rev. A 69 052107
[16] Ekrem A and Cenk O 2008 Int. J. Mod. Phys. B 22 231
[17] Jáñez R J, van Assche W and Dehesa J S 1994 Phys. Rev. A 50 3065
[18] Abdalla M S, Obada A S F and Abdel-Khalek S 2008 Chaos Solitons Fractals 36 405
[19] Abdel-Aty M, Obada A-S F and Abdallah M S 2003 Int. J. Quantum Inform. 3 359
[20] McDermott R J and Solomon A 1994 J. Phys. A: Math. Gen. 27 L-15
[21] Naderi M H, Roknizadeh R and Soltanolkotabi M 2004 Prog. Theor. Phys. 112 811
[22] Hall R L, Saad N and von Keviczky A 2002 J. Math. Phys. 43 94
[23] Hall, R L and Saad N 2000 J. Phys. A: Math. Gen. 33 569
[24] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press) p 65
[25] Gazeau J P and Klauder J R 1999 J. Phys. A: Math. Gen. 32 123
[26] Antoine J P, Gazeau J R, Kluder J R, Monceau P and Penson K A 2001 J. Math. Phys. 42 2349
[27] Thirulogasanthar K and Saad N 2004 J. Phys. A: Math. Gen. 37 4567
[28] Xu Z W 1996 Acta Phys. Sin. 45 1807 (in Chinese)
[29] Wang J-S, Liu T-K and Zhan M-S 2000 J. Opt. B: Quantum Semiclass. Opt. 2 758