Complex Langevin simulation in condensed matter physics

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Complex Langevin Method

Euclidean path integral:

\[ Z = \int D\Phi \, e^{-S[\Phi]} \]

\( S[\Phi] \neq S^*[\Phi] \) → “sign problem”

ex) chemical potential, electric field, theta term, real time, matrix model, non-relativistic system, spin system
**Complex Langevin Method**

complex Langevin equation:

\[ \Phi \in \mathbb{R} \rightarrow \tilde{\Phi} \in \mathbb{C} \]

\[ \frac{\partial}{\partial \theta} \tilde{\Phi} = -\frac{\delta}{\delta \tilde{\Phi}} S[\tilde{\Phi}] + \eta \]

fictitious time noise field

expectation value:

\[ \langle \hat{O}[\tilde{\Phi}] \rangle = \lim_{\theta \to \infty} \langle \hat{O}[\tilde{\Phi}(\theta)] \rangle_{\eta} \]
Complex Langevin Method

complex Langevin simulations of

Bose system & Fermi system

in condensed matter physics

✓ for the ab-initio analysis of condensed matter systems
✓ for a test of the complex Langevin method
Bose System

non-relativistic Bose gas:

$$S_0[\Phi_1, \Phi_2] = \int d\tau d^3x \left[ \Phi^*(x) \left( \frac{\partial}{\partial \tau} - \mu - \frac{1}{2m} \Delta \right) \Phi(x) + \frac{1}{4} \lambda |\Phi(x)|^4 \right]$$

anti-Hermitian

superfluid $^4$He

from wikipedia

atomic BEC

JILA group experiment
condensate fraction: \[ R = \lim_{|x-y| \to \infty} \frac{\langle \Phi^*(x) \Phi(y) \rangle}{\langle \Phi^*(x) \Phi(x) \rangle} \]
Bose System

condensate fraction: \[ R = \lim_{|x-y| \to \infty} \frac{\langle \Phi^*(x)\Phi(y) \rangle}{\langle \Phi^*(x)\Phi(x) \rangle} \]
Bose System

condensate fraction: \[ R = \lim_{|x-y| \to \infty} \frac{\langle \Phi^*(x)\Phi(y) \rangle}{\langle \Phi^*(x)\Phi(x) \rangle} \]

no gap (no Silver Blaze)
Rotation

rotating Bose gas:

\[ S_\Omega = S_0 - \Omega L_z \]
\[ L_z = -i \int d\tau d^3x \Phi^*(x) \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \Phi(x) \]

**vortex in \(^4\text{He}**

Maryland group experiment

**vortex in atomic BEC**

MIT group experiment
Rotation

circulation: \[ \hat{\Gamma} = \frac{1}{2\pi} \oint \delta \Theta = N_{\text{vortex}} \]
circulation (superfluid dominant)

![Graph showing circulation vs \( \Omega a \)](image)
Rotation

circulation (superfluid dominant)

![Graph showing circulation and configurations](image-url)
rotation (normal-state dominant)
Rotation

circulation (normal-state dominant)

\[ R \approx 0.3 \]

\[ \Gamma \sim 4 \]

\[ \Omega, \alpha \]

\[ \hat{\Gamma}, \hat{\alpha} \]

[Histogram of configurations]
Fermi System

Hubbard model:

\[
S[\Psi^*_\uparrow, \Psi_\uparrow, \Psi^*_\downarrow, \Psi_\downarrow] = \int d\tau \sum_x \left[ \sum_{i=\uparrow,\downarrow} \Psi^*_i(x) \left( \frac{\partial}{\partial \tau} - \mu_i \right) \Psi_i(x) \right.
\]

\[
- \sum_{i=\uparrow,\downarrow} \sum_j t_i (\Psi^*_i(x) \Psi_i(x + \hat{j}) + \Psi^*_i(x + \hat{j}) \Psi_i(x))
\]

\[
+ U \Psi^*_\uparrow(x) \Psi_\uparrow(x) \Psi^*_\downarrow(x) \Psi_\downarrow(x) \right]
\]

high-Tc superconductor
from wikipedia

optical lattice of atom gas
LMU-MPQ group experiment
Fermi System

Hubbard model:

\[ Z = \int D\Phi \; \det K_\uparrow[\Phi] \det K_\downarrow[\Phi] e^{-S_A[\Phi]} \]

\[ K_i[\Phi] = \frac{\partial}{\partial \tau} - \mu_i + \Phi(x) - \sum_j t_i(T_{+j} + T_{-j}) \]

\[ S_A[\Phi] = \int d\tau \sum_x \frac{1}{2|U|} \Phi^2(x) \]
Fermi System

\[ \det K[t, \mu] \in \mathbb{R} \]

balanced:

\[ \det K[t, \mu] \det K[t, \mu] \geq 0 \quad \text{no problem} \]

imbalanced:

\[ \det K[t^+, \mu^+] \det K[t^-, \mu^-] \geq 0 \quad \text{sign problem} \]

cf) QCD

\[ \det D[m, \mu] \det D[m, \mu] \in \mathbb{C} \quad \text{sign problem} \]
average sign in "sign-quenched" Monte Carlo
Fermi System
eigenvalues (free)
Fermi System

eigenvalues (weak coupling)
Fermi System

eigenvalues (strong coupling)

logarithmic singularity at zero eigenvalues

Mollgaard Splittorff ’13, Greensite ’14, Nishimura Shimasaki ’15
probability distribution of $\det K$
Fermi System

Probability distribution of $\det K$

- Sign problem
- Singularity problem
- No problem

$\det K$
Boson is good.

Fermion is bad.