Nonrelativistic Fermions in Magnetic Fields: a Quantum Field Theory Approach

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The statistical mechanics of nonrelativistic fermions in a constant magnetic field is considered from the quantum field theory point of view. The fermionic determinant is computed using a general procedure that contains all possible regularizations. The nonrelativistic grand-potential can be expressed in terms polylogarithm functions, whereas the partition function in 2+1 dimensions and vanishing chemical potential can be compactly written in terms of the Dedekind eta function.

I. INTRODUCTION

The motion of fermions in magnetic fields is an old problem of quantum mechanics that plays a role in a wide variety of physical problems, ranging from neutron stars to the quantum Hall effect. From the point of view of quantum field theory, several studies have been performed, related to the chiral anomaly, effective actions and applications to anyons systems.

In this paper we would like to study another aspect of this problem, namely, following the approach proposed in [8], we shall discuss the statistical mechanics of nonrelativistic fermions embedded in a constant magnetic field.

Our approach requires the explicit computation of a fermionic determinant, and this is carried out in terms of polylogarithm and Dedekind functions. Additionally our calculation, in a particular case, allows us to study the strong and weak magnetic field limits quite easily, due the modular invariance of the Dedekind function. The paper is organized as follows: in section 2 we review the approach proposed in [8], in section 3 we formulate the problem in the context of the path integral method, in section 4 we present an explicit computation of the nonrelativistic grand potential, and in section 5 we study the partition function in the case of 2+1 dimensions. Our conclusions are given in section 6.

II. EFFECTIVE ACTION AT LOW ENERGIES

In reference [8], a method was proposed to compute effective actions at low energies, based on a path integral derivation of the Foldy-Wouthuysen transformation. This method can be used for nonrelativistic fermions as well. Let us consider the lagrangian

\[ \mathcal{L} = \bar{\psi} [iD - m] \psi, \]

where \( D = \partial + igA \).

Following the procedures developed in [8], we redefine the origin of the energy by the rescaling

\[ \psi(x) = e^{-imt} \phi(x), \]

to write the lagrangian as

\[ \mathcal{L} = \bar{\phi} (iD - m(1 - \gamma_0)) \phi. \]

Now we decompose the spinor \( \phi \) into ‘large’ (\( \varphi \)) and ‘small’ (\( \chi \)) components, in whose terms

\[ \mathcal{L} = \varphi^\dagger iD_0 \varphi + \chi^\dagger [iD_0 + 2m] \chi + \varphi^\dagger i \bar{\sigma} \cdot D \chi \]

\[ + \chi^\dagger i \bar{\sigma} \cdot D \varphi, \]

where the Dirac representation for the \( \gamma \)-matrices has been used.

The next step is to diagonalize the lagrangian in \( \varphi \) by means of the change of variables

\[ \varphi' = \varphi, \]

\[ \varphi'^\dagger = \varphi^\dagger, \]

\[ \chi' = \chi + [iD_0 + 2m]^{-1} i \bar{\sigma} \cdot D \varphi, \]

\[ \chi'^\dagger = \chi^\dagger + \varphi^\dagger i \bar{\sigma} \cdot D [iD_0 + 2m]^{-1}. \]

This change of variables has a Jacobian equal to unity and the effective lagrangian, under this transformations, becomes (omitting the primes)

\[ \mathcal{L} = \varphi^\dagger [iD_0 + \bar{\sigma} \cdot D (iD_0 + 2m)^{-1} \bar{\sigma} \cdot D] \varphi, \]

\[ + \chi^\dagger [iD_0 + 2m] \chi. \]

This lagrangian describes the non-local dynamics of the fermions in terms of two-components spinors. One should note that \( \varphi \) and \( \chi \) decouple and, after expanding \( (iD_0 + 2m)^{-1} \) in powers of \( 1/m \), the partition function becomes

\[ Z = \int D\varphi^\dagger D\varphi e^{iS}, \]

where

\[ S = \int \mathcal{L} d^4x. \]

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where
\[ S_\phi = \int \! d^4 x \chi \chi^\dagger [iD_0 + 2m] \chi, \]
\[ S_\varphi = \int \! d^4 x \varphi \varphi^\dagger [iD_0 + \frac{1}{2m} D^2 + \frac{g}{2m} \vec{\sigma} \cdot \vec{B}] \varphi + O(1/m^2). \]
(8)

\( S_\varphi \) is the nonrelativistic action for fermions interacting with a magnetic field. \( S_\chi \) is the action associated to the lower contribution of the spinor and can be neglected in the nonrelativistic limit.

### III. NONRELATIVISTIC FERMIONS IN MAGNETIC FIELDS

Using (8) and (9) one can explicitly study the nonrelativistic quantum field theory of fermions at finite temperature.

The partition function associated to this problem is
\[ Z_\varphi = \int \! D\varphi D\varphi^\dagger e^{iS_\varphi}, \]
\[ = \det \left[ iD_0 + \frac{1}{2m} D^2 + \frac{g}{2m} \vec{\sigma} \cdot \vec{B} + \mu \right], \]
\[ = \prod_n \lambda_n, \]
(9)
(10)

where \( g \) stands for the electric charge of the fermions, \( \mu \) is the chemical potential, and the infinite product runs over all the eigenvalues \( \lambda_n \) of the operator that appears in (8),
\[ [iD_0 + \frac{1}{2m} D^2 + \frac{g}{2m} \vec{\sigma} \cdot \vec{B} + \mu] \phi_n = \lambda_n \phi_n, \]
(11)

subject to the usual antiperiodic boundary conditions in the imaginary time direction.

For a constant magnetic field one can choose the gauge
\[ \mathbf{A} = (-B_0 y, 0, 0), \]
\[ A_0 = 0, \]
in which case the fermionic determinant can be computed from
\[ [i\partial_t + \frac{1}{2m} D^2 + \frac{g}{2m} B_0 + \mu] \phi_n^\pm = \lambda_n^\pm \phi_n^\pm, \]
(12)

and thus the partition function (9) becomes
\[ Z_\varphi = \det [i\partial_t - H^+ + \mu] \det [i\partial_t - H^- + \mu], \]
\[ = \prod_n \lambda_n^+ \prod_n \lambda_n^-, \]
(13)
(14)

where the \( H^\pm \) can be read off from (12).

Each determinant in (13) is evaluated by explicitly solving the eigenvalue equation (12) by means of the Ansatz \( \phi_n^\pm (x, t) = f_n^\pm (x) T(t) \), which yields
\[ \left[ -\frac{1}{2m} D^2 + \frac{g}{2m} B_0 - \mu \right] f_n^\pm = (\Omega - \lambda_n^\pm) f_n^\pm, \]
(15)

since the operator \( \frac{1}{2m} D^2 \pm \frac{g}{2m} B_0 \) is time-independent. Equation (15) is just Schrödinger's equation for the Landau problem, whose eigenvalues are known,
\[ E_n^\pm = \Omega - \lambda_n^\pm = (n + \frac{1}{2} \pm \frac{1}{2}) \omega + \frac{p^2}{2m} - \mu, \]
(16)

with \( \omega = gB_0/m \). The equation for \( T(t) \) has a solution only if
\[ \Omega = \Omega_m = \frac{\pi}{T} (2m + 1), \]
(17)

where \( T \) is the period and \( m \) an integer, in virtue of the antiperiodic boundary condition on \( T(t) \).

Thus, the eigenvalues in (14) are given by
\[ \lambda_n^\pm = \frac{\pi}{T} (2m + 1) - E_n^\pm. \]
(18)

The statistical mechanics of the fermionic system under consideration is described by the grand potential – basically the logarithm of the partition function after going to Euclidean space, i.e. replacing \( T \) by \( i \beta \), where \( \beta = 1/T \) is the inverse temperature. The logarithm of the partition function is
\[ \log Z_\varphi = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \int \! dp_z [\log(\lambda^+_n) + \log(\lambda^-_{m,n})], \]
\[ \equiv \sum_{n=0}^{\infty} \int \! dp_z [L^+_n + L^-_n], \]
(19)

where \( L^\pm_n \) are infinite sums defined as
\[ L^\pm_n = \sum_{m=-\infty}^{\infty} \log(\lambda^\pm_{m,n}). \]
(20)
(21)

Although the series in (21) are divergent, they can be computed by using a definite regularization prescription. Let us start considering the divergent series
\[ L(a, b) = \sum_{m=-\infty}^{\infty} \log \left( a m + b \right), \]
(22)

where \( a \) and \( b \) are constants.

The second derivative of \( L(a, b) \) is
\[ \frac{d^2 L(a, b)}{db^2} = - \sum_{m=-\infty}^{\infty} \frac{1}{(am + b)^2}, \]
\[ = -\pi^2 \csc^2(b \pi /a) \frac{1}{a^2}, \]
(23)

so that upon integration one finds
\[ L(a, b) = \log \left( e^{c_1 + b c_2} \sin \left( \frac{\pi b}{a} \right) \right), \]  

where \( c_1 \) and \( c_2 \) are two arbitrary integration constants.

In our case, \( a \) and \( b \) can be read from (19) and (21). Therefore

\[ L_n^\pm = \log \left[ e^{c_1^\pm + c_2^\pm |z|^2} \cosh \left( \frac{E_n^\pm}{2} \right) \right] \]

\[ \sim c_1^\pm - i c_2^\pm \frac{\pi}{\beta} + ( - c_2^\pm + \frac{\beta}{2} ) E_n^\pm + \log(1 + e^{-\beta E_n^\pm}). \]  

Clearly, the constants \( c_1^\pm \) and \( c_2^\pm \) parametrize the arbitrariness of the regularization. Since the Euclidean space effective action must be real, we choose \( c_{1,2}^\pm \) in such a way that (26) has no imaginary part.

At this point we can compare our result (26) with the nonrelativistic limit of the general result given in (3) for the effective action. The contribution

\[ c_1^\pm - i c_2^\pm \frac{\pi}{\beta} + ( - c_2^\pm + \frac{\beta}{2} ) E_n^\pm \]

indeed corresponds – after adding the analogous positron contribution to the first term, Tr[\( \mathcal{E} \)], in equation (18) of reference (10) – and the dependence on the arbitrary constants \( c_{1,2}^\pm \) reflects the regularization that needs to be performed in order to define the trace. For instance the choice \( c_1^\pm = 0 = c_2^\pm \) is consistent with \( \zeta \)-function regularization (10).

The second contribution, \( \log(1 + e^{-\beta E_n^\pm}) \), which is finite and independent of the constants \( c_{1,2}^\pm \), coincides (up to a factor \( \beta \)) with the grand potential in nonrelativistic statistical mechanics.

IV. GRAND-POTENTIAL

In this section we compute the nonrelativistic grand potential (\( \Omega \)) directly from its thermodynamical definition, rather than deriving it form the partition function (Z).

The nonrelativistic grand potential is given by

\[ \Omega(\beta, \mu) = -\frac{\tilde{g}}{\beta} \sum_{\ell = +, -} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \log(1 + e^{-\beta E_n^\ell}), \]

\[ = \Omega^+(\beta, \mu) + \Omega^- (\beta, \mu), \]  

where \( \tilde{g} \) is the degeneracy factor \( \tilde{g} = (g B_0/4 \pi^2) V \).

Clearly, we need to consider the following object

\[ S(A, b) \equiv \sum_{n=0}^{\infty} \log \left( 1 + A e^{-bn} \right). \]  

In terms of \( S \) we have

\[ \Omega^\pm(T, \mu) = -\frac{\tilde{g}}{\beta} \int_{-\infty}^{\infty} dp_z S(A^\pm, \beta \omega) \]  

with

\[ A^+ = e^{-\beta(\omega-\mu)} e^{-\beta p_z^2/2m}, \quad A^- = e^{\beta \mu} e^{-\beta p_z^2/2m}, \]  

The function \( S(A, b) \) has the following Taylor series in the variable \( A \):

\[ S(A, b) = \log(1 + A) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n ( e^{bn} - 1 )} A^n, \]

\[ = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n (1 - e^{-bn})} A^n. \]  

Since \( A^n \) has the form \( a^n e^{-n \gamma^2} \), the \( p_z \) integral can be performed explicitly to yield

\[ \int_{-\infty}^{\infty} dp_z S(A^\pm, b) = \sqrt{\frac{2 \pi m}{\beta}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^{3/2} (1 - e^{-bn})} (a^\pm)^n. \]  

Expanding now

\[ (1 - e^{-bn})^{-1} = \sum_{k=0}^{\infty} (e^{-bn})^k, \]

we find

\[ \int_{-\infty}^{\infty} dp_z S(A^\pm, b) = -\sqrt{\frac{2 \pi m}{\beta}} \sum_{k=0}^{\infty} \frac{1}{k^{3/2}} \text{Li}_{3/2}(-a^\pm e^{-kb}), \]

where we have introduced the polylogarithm function, defined as the analytic continuation to the whole complex \( z \)-plane of the series

\[ \text{Li}_s(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \]  

defined for \( |z| < 1 \) and \( s \in \mathbb{C} \). Therefore,

\[ \Omega^+(T, \mu) = V \frac{g B_0}{4 \pi^2} \sqrt{\frac{2 \pi m}{\beta^3 \gamma^2}} \sum_{k=0}^{\infty} \text{Li}_{3/2}(-e^{\beta \mu} e^{-(k+1) \beta \omega}), \]

and

\[ \Omega^-(T, \mu) = V \frac{g B_0}{4 \pi^2} \sqrt{\frac{2 \pi m}{\beta^3 \gamma^2}} \text{Li}_{3/2}(-e^{\beta \mu}) + \Omega^+(T, \mu). \]  

It is instructive to check that our result for \( \Omega = \Omega^+ + \Omega^- \) reduces to the correct result in the limit of zero temperature (\( \beta \rightarrow \infty \)). Depending on the sign of \( \mu - k \omega \), the argument of the polylog function will tend either to zero (in which case \( \text{Li}_{3/2}(0) = 0 \) and there is no contribution) or infinity. To get the corresponding contribution in the latter case we need the asymptotic behavior of \( \text{Li}_{3/2}(z) \) as \( z \rightarrow \infty \). This can be obtained from Jonquière’s relation (14),
\[ \text{Li}_s(z) + e^{i\pi} \text{Li}_s(1/z) = \frac{(2\pi)^s}{\Gamma(s)} e^{i\pi/2} \zeta(1-s) \log z. \] (39)

In particular, for \( x > 0 \) we have
\[ \text{Li}_{3/2}(-e^{\beta x}) \rightarrow \left( \frac{2\pi}{\Gamma(3/2)} \right)^{3/2} e^{3\pi/4} \zeta(-\frac{1}{2} - \frac{1}{2}) + \frac{\beta x}{2\pi i} \] (40)
\[ \rightarrow -\frac{2}{3\pi(3/2)} e^{3\pi/2}, \] (41)

because of the asymptotic behavior \( \zeta(-\frac{1}{2}, q) \rightarrow -\frac{2\pi}{3} q^{3/2} \) of the Hurwitz zeta function. With \( \Gamma(3/2) = \sqrt{\pi}/2 \), we obtain the correct nonrelativistic result
\[ \frac{1}{V} \Omega(T = 0, \mu) = \left[ -\frac{2gB_0}{3\pi^2} \sqrt{2m} \right] \sum_{k=0}^{[\mu/\omega]} (\mu - k\omega)^{3/2} - \frac{1}{2} \mu^3/2]. \] (42)

Here \([x]\) is the floor of the real number \( x \).

A similar calculation can be performed in 2+1-dimensions, but then the result one obtains is nothing more than the starting expression \( (28) \), with the corresponding 2+1-degeneracy factor \( \tilde{g}_2 = (gB_0/2\pi) L^2 \), \( L^2 \) being the area of the 2-dimensional quantization box:
\[ \Omega_{2+1}(T, \mu) = -\frac{\tilde{g}_2}{\beta} \left[ \sum_{k=0}^{\infty} \log \left( 1 + e^{\beta \mu} e^{-(k+1)\beta \omega} \right) + \log \left( 1 + e^{\beta \omega} \right) \right]. \] (43)

V. PARTITION FUNCTION IN 2+1-DIMENSIONS

In section we compute the partition function in 2+1-dimensions. In order to do that, one can proceed as follows: the product \((43)\) is first written as
\[ Z_\varphi = (Z_0)^{\tilde{g}_2}, \] (44)
in view of the degeneracy \( \tilde{g}_2 \) of each Landau eigenvalue. Using \( \zeta \)-function regularization the reduced partition function \( Z_0 \) can be put into the form
\[ Z_0 = \prod_{n=0}^{\infty} \cosh \left( \frac{\theta n + \delta^+}{\beta} \right) \cosh \left( \frac{\delta^+ - \beta \omega}{\beta} \right). \] (45)

Using the explicit form of the eigenvalues for the Landau problem in 2+1 dimensions \( (p_z = 0) \) from \((17), (18)\) can be written as
\[ Z_0 = \prod_{n=0}^{\infty} \cosh(\theta n + \delta^+) \cosh(\theta n + \delta^-). \] (46)

where
\[ \theta = \frac{\beta \omega}{2}, \]
\[ \delta^+ = \frac{\beta}{2}(\omega - \mu), \]
\[ \delta^- = -\frac{\beta \mu}{2}. \] (47)

Defining \( q = e^{-2\theta} \), \( z = e^{-2\delta^+} \) and \( \tilde{z} = e^{-2\delta^-} \), then \((46)\) becomes
\[ Z_0 = \prod_{n=0}^{\infty} e^{2\theta n} e^{\delta^+ + \delta^-} (1 + zq^n) (1 + \tilde{z}q^n). \] (48)

This is most general expression for the (reduced) partition function in 2+1-dimensions.

A particular situation arises when the chemical potential is zero. In such a case \((48)\) becomes
\[ Z_0 = \left[ \frac{\eta(2i\theta/\pi)}{\eta(i\theta/\pi)} \right]^2, \] (49)

where
\[ \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \] (50)

is the Dedekind eta function \([12]\), with \( q = e^{2i\pi \tau} \).

It is not difficult to see that formula \((49)\) reproduces result \((13)\) in the case of vanishing chemical potential. Indeed, from the definition of the eta function and writing \( 1 - q^{2n} = (1 - q^n)(1 + q^n) \) we find
\[ Z_0 = q^{1/12} \prod_{n=1}^{\infty} (1 + q^n)^2, \] (51)

which, with \( q = e^{-\beta \omega} \) leads to
\[ \log Z_\varphi = -\frac{1}{12} \beta \omega + 2 \sum_{n=1}^{\infty} \log (1 + e^{-n\beta \omega}). \] (52)

This coincides with the result for \(-\beta \Omega_{2+1} \) from \((43)\) at \( \mu = 0 \), up to terms due to the regularization used. The high temperature limit (or equivalently the weak magnetic field limit) corresponds to \( \tilde{g}_2 \rightarrow 0 \), and
\[ \eta(q) \sim q^{1/24}(1 - q)^{-1/2} e^{-\frac{q}{(1 - q)^2}}, \] (53)

and, therefore, the partition function in the high-temperature limit becomes
\[ Z_0|_{\beta \omega \ll 1} \sim \frac{\frac{1}{12} \beta \omega + \frac{2\pi^2}{3\sin^2 \beta \omega}}{\cosh \frac{2\pi \beta \omega}{\beta \omega}}. \] (54)

Moreover, as the Dedekind function satisfies the modular property
\[ \eta(\tau) = \left( \frac{i}{\tau} \right)^{1/2} \eta \left( -\frac{1}{\tau} \right), \]  

one can compute the partition function in the strong magnetic field limit (or low temperature limit) directly from (54). Indeed, by using (54) and (55) we find

\[ Z_0 \bigg|_{\beta \omega \gg 1} \sim e^{-\frac{5 \pi^2}{12 \beta \omega} - \frac{2 \pi^2}{3 \sinh \frac{2 \pi \omega}{\beta}} \cosh \left( \frac{\pi^2}{\beta \omega} \right)}. \]  

VI. CONCLUSIONS

In this paper we have studied the motion of nonrelativistic fermions in a constant magnetic field, following a quantum field theory approach. The main point is the calculation of the fermionic determinant by using a general method that contains all the possible regularizations of the infinite product, in the form of particular choices of the coefficients \( c_{1,2} \).

In addition, we obtained explicit expressions for the nonrelativistic grand-potential, in terms of a series of polylogarithms, and for the partition function in 2+1 dimensions and zero chemical potential, in terms of the Dedekind eta function. In the latter case, the modular properties of the Dedekind function allow us to relate the strong and weak magnetic field limits. This relation could be useful in the dynamics of the quantum Hall effect.

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