Next-to-leading order corrections to the elastic scattering of an electron off of a static scattering center

Abdullah Khalil and W. A. Horowitz
Department of Physics, University of Cape Town, Private Bag X3, Rondebosch 7701, South Africa
E-mail: abdullah@aims.ac.za and wahorowitz@gmail.com

Abstract. We calculate the elastic scattering cross section for an electron off of a classical point source in weak-coupling perturbative quantum electrodynamics at next-to-leading order accuracy in the \(\overline{\text{MS}}\) renormalization scheme. Since we use the \(\overline{\text{MS}}\) renormalization scheme, our result is valid up to arbitrary large momentum transfers between the source and the scattered electron.

1. Introduction
Quantum Chromodynamics (QCD) is the accepted theory for describing the interaction between quarks, antiquarks and gluons [1]. One of the fundamental predictions of QCD is the existence of a state of matter called the Quark Gluon Plasma (QGP) [2, 3]. The QGP is predicted to be formed at very high energy densities, and temperatures exceeding the energy density inside the atomic nuclei by an order of magnitude \(1 \rightarrow 10 \text{GeV/} \text{fm}^3\) and at a temperatures of order \((\sim 180)\text{MeV}\).

One of the methods to study QCD is on the lattice [4,5]. A significant result of this theory is the temperature dependence of the energy density in QCD. Which shows a rapid change in the value of the energy density at the critical temperature \(T_c\). This rapid change can be interpreted as a change in the number of degrees of freedom in the system. Well below \(T_c\), there are three hadronic degrees of freedom due to the three lightest hadrons: \(\pi^+\), \(\pi^-\) and \(\pi^0\). Well above \(T_c\), there are \(2(N_c^2 - 1) + 2 \times 2 \times N_c \times N_f\) degrees of freedom from the fundamental gluons and quarks of the theory.

The high \(p_{\perp}\) data from RHIC and LHC have been interpreted as evidence that jet quenching is due to final state energy loss, which is qualitatively well described by leading order pQCD methods [6,7]. We wish to check the self-consistency of these pQCD results, and to make the pQCD calculation more quantitative. To accomplish these two goals, we must compute the next-to-leading order contribution to the energy loss of partons in a QGP. As a first step towards this NLO pQCD calculation, we compute in this paper the NLO corrections to the elastic scattering of an electron off of a static source.
2. The leading order of the differential cross section

We consider the Lagrangian describing an electron interacting with a classical current

\[ \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial - m)\psi - e\bar{\psi}\gamma^{\mu}\psi A_{\mu} + eJ_{\mu}A^{\mu}, \]  

with \( J^{\mu} = V^{\mu}\delta^{(4)}(\vec{x} - \vec{V}x^0) \). \hspace{1cm} (1)

where \( V^{\mu} = (1, \vec{0})^{\mu} \) is the four velocity vector. We can easily write down the Feynman rules for this Lagrangian which will be exactly the same rules for the normal QED Lagrangian \(^1\) in addition to that for each external source we write \(-ieV^{\mu}\).

Let the momenta of the incoming electron to be \( p \) and for the outgoing electron to be \( p' \), then the momentum transfer would be \( q = p' - p \). Also when we define the cross section we always have the delta function \( \delta(E_{p'} - E_p) \) which requires \( E_{p'} = E_p = E \). Now we write down the amplitude of the leading term using the Feynman rules

\[ iM_0 = \begin{pmatrix} p \\ p' \end{pmatrix} = \frac{i e^2}{q^2} \pi s(p') \gamma^0 \bar{u}(p). \hspace{1cm} (2) \]

We recall the identity \( \sum s u^s(p) \bar{u}^s(p) = \delta + m \), the trace technology and the properties of the \( \gamma \)-matrices. The leading term of the cross section will be

\[ \left( \frac{d\sigma}{d\Omega} \right)_0 = \frac{e^4}{8\pi^2q^4} (2E^2 - p' \cdot p + m^2). \hspace{1cm} (3) \]

3. Renormalization of the Lagrangian

Calculating the NLO correction to the cross section require including diagrams which contain either fermion or photon loops. These loops require integrations over the loop momentum which usually diverges in 4-dimensions. In order to remove the divergences from our calculations we first should define the divergent parts by using the dimensional regularization to regularize the UV divergences and the mass regularization for the IR divergences. Then we need to renormalize our Lagrangian using the systematic way of renormalization, at the end we apply the renormalization scheme to get ride of the UV divergences.

Applying dimensional regularization requires replacing the 4 momentum integral by the integral over the momentum in \( d \)-dimensions. Which also requires rescaling the electron charge \( e \) by the factor \( \mu^{4-d} \), where \( \mu \) is any mass scale to make \( e \) dimensionless again \(^8\). At the end of the calculations, the physics (the physical parameters) should not depend on this scale.

Now let us define the Lagrangian from equation \((1)\) in terms of the bare parameters as follows

\[ \mathcal{L}_0 = -\frac{1}{4}F_0^{\mu\nu}F_0^{\mu\nu} + \bar{\psi}_0(i\partial - m_0)\psi_0 - e_0\bar{\psi}_0\gamma^{\mu}\psi_0A_0^{\mu} + e_0J_{0\mu}A_0^{\mu}. \hspace{1cm} (4) \]

Next we renormalize the fields \( \psi_0 \) and \( A_0^{\mu} \) and the parameters \( e_0, m_0 \) and \( J_{0\mu} \) such that

\[ \psi_0 = Z_\psi^{1/2}\psi, \hspace{0.5cm} A_0^{\mu} = Z_A^{1/2}A^{\mu}, \hspace{0.5cm} Z_\psi m_0 = Z_m m, \hspace{0.5cm} e_0 Z_e Z_A^{1/2} = \mu^{4-d} Z_e \hspace{0.5cm} \frac{Z_\psi}{Z_e} J_{0\mu} = Z J. \hspace{1cm} (5) \]

Next we expand the renormalization parameters around 1 in terms of the counter terms \( (\delta_\psi, \delta_A, \delta_e, \delta_m, \delta_J) \). The Lagrangian becomes

\[ \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial - m)\psi - e\bar{\psi}\gamma^{\mu}\psi A_{\mu} + eJ_{\mu}A^{\mu} \]

\[ -\frac{1}{4}\delta_AF_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\delta_\psi - m\delta_m)\psi - e\delta_e\bar{\psi}\gamma^{\mu}\psi A_{\mu} + e\delta_JJ_{\mu}A^{\mu}. \hspace{1cm} (6) \]
After renormalizing our theory we would be able to write the Feynman rules for the renormalized Lagrangian which will be exactly the same rules of the renormalized QED \cite{1}.

4. NLO Corrections
In this section we compute the amplitudes due to the NLO diagrams which are the vertex correction, vacuum polarization, bremsstrahlung correction, self energy correction and the box diagram. We choose $\overline{MS}$ renormalization scheme to get ride of the UV divergences, the reason of choosing $\overline{MS}$ is that we need to extend these calculations for QCD calculations where we will deal with light quarks (i.e nearly zero mass). So we will calculate the cross section with massive electron and using $\overline{MS}$ allows us to set $m_e = 0$ safely giving a finite form of the cross section \cite{9}.

4.1. Vacuum Polarization
The amplitude for the vacuum polarization is given by

$$iM_p = \frac{p}{q} \frac{p'}{q'} \frac{k}{k+q} = -\frac{8i\alpha^2}{q^2} \xi(q^2) \bar{u}^s(p')\gamma^0 u^s(p),$$

(7)

where

$$\xi(q^2) \approx \frac{5}{18} + \frac{1}{6} \log \left( \frac{\mu^2}{-q^2} \right), \quad -q^2 >> m^2. \quad (8)$$

Now the cross section of the interference between the leading and the vacuum polarization diagrams will be

$$\left( \frac{d\sigma}{d\Omega} \right)_{PL} \approx \left( \frac{d\sigma}{d\Omega} \right)_0 \frac{\alpha}{\pi} \left( -\frac{2}{3} \log \left( \frac{\mu^2}{-q^2} \right) - \frac{10}{9} \right). \quad (9)$$

4.2. Vertex Correction
The amplitude of the vertex correction is given by

$$iM_V = \frac{p-k}{p'-k} \frac{p'}{q} = \frac{4i\pi\alpha}{q^2} \bar{u}^s(p') \left( \gamma^0 \cdot F_1(q^2) + \frac{i\sigma^0q^\nu}{2m} F_2(q^2) \right) u^s(p),$$

(10)

where $F_1(q^2)$ and $F_2(q^2)$ in the limit $-q^2 >> m^2$ will be

$$F_1(q^2) \approx \frac{\alpha}{2\pi} \left( -\frac{1}{2} \log \left( \frac{-q^2}{\mu^2} \right) - \log \left( \frac{-q^2}{m^2} \right) \log \left( \frac{-q^2}{m^2} \right) + \frac{1}{2} \right),$$

$$F_2(q^2) \approx \frac{\alpha}{2\pi} \left( \frac{m^2}{-q^2} \log \left( \frac{-q^2}{m^2} \right) \right), \quad (11)$$

where $m_\gamma$ is the photon mass to regularize the expected IR divergences. Finally the cross section of the interference between the leading and the vertex diagrams will be

$$\left( \frac{d\sigma}{d\Omega} \right)_{VL} \approx \left( \frac{d\sigma}{d\Omega} \right)_0 \frac{\alpha}{\pi} \left( \frac{1}{2} \log \left( \frac{\mu^2}{-q^2} \right) - \log \left( \frac{-q^2}{m^2} \right) \log \left( \frac{-q^2}{m^2} \right) + \frac{1}{2} \right). \quad (12)$$

We note that equation (12) contains IR divergence as we send $m_\gamma = 0$ which appear as a single pole in addition to the double pole when we send both $m_e$ and $m_\gamma$ to zero.
4.3. Soft Bremsstrahlung Correction

Usually the detectors can not differentiate between the photon emitted from the vertex and the Bremsstrahlung radiation \[10\] which means we should add the bremsstrahlung correction. The amplitude for the soft Bremsstrahlung is given by

\[
i M_B = \frac{ie^3}{q^2} \bar{u}(p') \gamma_0 \left( \frac{p - k + m}{(p - k)^2 - m^2} \gamma^\alpha \gamma^\mu (k) + \gamma^\alpha \gamma^\mu (k) \left( \frac{p' + k + m}{(p' + k)^2 - m^2} \gamma_0 \right) \right) u(p).
\] (13)

For the soft photon approximation \(|k| << |p' - p|\). The amplitude of the soft bremsstrahlung will be given by

\[
i M_B' = ie M_0 \left( \frac{p' \cdot \gamma_r}{p' \cdot k} - \frac{p \cdot \gamma_r}{p \cdot k} \right),
\] (14)

then the cross section due to the emission of a soft photon becomes \[8,10\]

\[
\left( \frac{d\sigma}{d\Omega} \right)_B = \left( \frac{d\sigma}{d\Omega} \right)_0 \frac{\alpha}{\pi} \log \left( -\frac{q^2}{m_\gamma^2} \right) \log \left( \frac{-q^2}{m^2} \right) - \log \left( \frac{-q^2}{m_\gamma^2} \right).
\] (15)

4.4. Electron Self Energy

The amplitude of the electron self energy is given by

\[
-i \Sigma_2(p) = \begin{array}{ccc}
  p & \rightarrow & p \\
  p & \rightarrow & k \\
  k & \rightarrow & k \\
\end{array}.
\] (16)

Applying the usual procedure of our renormalization scheme. The amplitude for the electron-self energy becomes

\[
\Sigma_2(\not{p}) = \frac{\alpha}{4\pi} \left( (\not{p} - 2m) + \int_0^1 dx \left( 4m - 2x\not{p} \right) \log \left( \frac{m^2}{(1-x)m^2 - x(1-x)p^2 + x m_\gamma^2} \right) \right).
\] (17)

The Fourier transform of the two point correlation function of the electron self energy is given by \[1\]

\[
\int d^4x \langle \Omega | T(\psi(x)\bar{\psi}(0)) | \Omega \rangle e^{ip \cdot x} = \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m} \left( \frac{\Sigma(\not{p})}{\not{p} - m} \right) + \frac{i}{\not{p} - m} \left( \frac{\Sigma(\not{p})}{\not{p} - m} \right)^2 + \ldots
\] = \frac{i}{\not{p} - m - \Sigma(\not{p})}.
\] (18)

This means that the pole is shifted by \(\Sigma(\not{p})\), so the renormalized mass is not the physical mass and the residue of this pole is no longer one \[9\]. So our goal now is to find the correction to the residue and the relation between the renormalized mass \(m\) and the physical mass \(m_e\) and
the correction to the residue. The physical mass can be given by the position of the pole where we have

\[ m_e = m \left( 1 + \frac{\alpha}{4\pi} \left( 4 + 3 \log \left( \frac{\mu^2}{m^2} \right) \right) + \mathcal{O}(\alpha^2) \right). \]  

The inverse of the residue is given by

\[ R^{-1} \approx 1 - \frac{\alpha}{4\pi} \left( 2 \log \left( \frac{m^2}{m_e^2} \right) - \log \left( \frac{\mu^2}{m^2} \right) - 4 \right) + \mathcal{O}(\alpha^2). \]  

We should multiply the amplitude by \( R_1/2 \) for each external leg, which means that we should multiply the differential cross section by \( R_2 \) \[9\]. The contribution of the self energy diagram at one loop is zero, since the contribution of the diagram due to the self energy of the incoming electron is exactly the same as the contribution due to the self energy of the outgoing electron with a relative sign difference.

4.5. Box Correction

The amplitude of the box diagram is given by

\[ iM_{BO} = p - k - k = \begin{pmatrix} p \\ k \\ p' - k \end{pmatrix}. \]  

It is obvious that the box diagram does not contain any ultraviolet divergences, so we do not need to perform dimensional regularization. We use the trick made by R. Dalitz \[11\] to simplify the integrals in this diagram. The differential cross section due to the interference between the leading and the box amplitudes will be

\[ \left( \frac{d\sigma}{d\Omega} \right)_{BOL} = -\frac{\pi \alpha^3 E}{|q| q^2} \left( \frac{|q|}{p} - 1 \right). \]  

4.6. The Differential Cross Section to the NLO Correction

We add the differential cross sections due to the leading order and all NLO corrections. But we should be careful because using \( \overline{MS} \) renormalization scheme forces us to multiply the total cross section by the value of \( R^2 \) which is in \( \mathcal{O}(\alpha) \) so the only affected term is the leading term as follows

\[ \left( \frac{d\sigma}{d\Omega} \right) = \left( \frac{d\sigma}{d\Omega} \right)_0 \left[ 1 + \frac{\alpha}{\pi} \left( \frac{3}{2} \log \left( \frac{m^2}{-q^2} \right) - \frac{2}{3} \log \left( \frac{\mu^2}{-q^2} \right) - \frac{47}{18} \right) + \frac{\pi \alpha^3 E}{|q| q^2} \left( \frac{|q|}{p} - 1 \right) + \mathcal{O}(\alpha^4) \right]. \]  

We note that equation (23) is not finite as we send \( m = 0 \), which arises from the emission of photon collinear with the incoming and outgoing electrons through the bremsstrahlung process. Which means we should also including the initial and final state of the hard bremsstrahlung corrections according to the KLN theorem \[12\].

We have mentioned that the physical parameters shouldn’t depend on \( \mu \), but we see that \( \mu \) appears in the final formula for the differential cross section, which means the renormalized parameters depend on \( \mu \) and we find

\[ \beta(\alpha) = \mu \frac{d\alpha}{d\mu} = \frac{2\alpha^2}{3\pi} \quad \text{and} \quad \gamma_m(\alpha) = \mu \frac{dm}{d\mu} = \frac{-3\alpha}{2\pi}. \]  

\[\text{(23)}\]
then it is straightforward to check that
\[
\frac{d\sigma_e}{d\mu} = 0, \quad \frac{d}{d\mu} \left( \frac{d\sigma}{d\Omega} \right) = 0.
\] (25)

5. Conclusion
In this project we calculated the elastic differential cross section, including the next-to-leading order corrections of an electron scattered by a classical static point charge. These corrections come from the photon self energy, vertex, soft Bremsstrahlung and the box diagrams. We found that all ultraviolet divergences due to loop integrals are absorbed by the counter terms in the \( \tilde{\text{MS}} \) renormalization scheme. We also saw that all infrared divergences due to the zero mass of the photon cancel, where the infrared divergence from the vertex correction cancels exactly with the one from the soft and collinear Bremsstrahlung. But we still need to add the hard bremsstrahlung correction due to the final and initial states \([12]\) to obtain finite formula of the cross section when we take the electron mass \( m_e \) to zero; equivalently, we expect a result that is valid up to arbitrary large momentum exchange.

The next step is to extend these calculations to derive a formula for the energy lost by high momentum particles propagating through weakly coupled plasmas in thermal field theory, including the next-to-leading-order corrections due to the emission of very high energy particles. This is very interesting for the study of the QGP and for understanding the nature of the degrees of freedom at energy densities \((1 - 3)T_c\) from lattice QCD.

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