Cyber-physical systems (CPS) are a class of systems characterized by high penetration of computational and communication resources, providing increased performance and efficiency. Recent events have highlighted that the security of these systems, which aptly describe power grids, transport networks, and many other industrial plants, is of critical importance. Indeed, some high profile cases have made it evident that the integration of communication within control systems have also introduced the possibility of malicious agents penetrating these systems and performing attacks (Falliere et al., 2011; Lee, 2008).

Over the past decade, a growing body of research has been devoted to the detection, isolation, and mitigation of these CPS. Of the possible ways to classify research on cyber-attack detection, one is to distinguish between passive and active techniques (Weerakkody et al., 2019): while in passive methods detection algorithms rely on the effect of the attack and knowledge of the nominal behavior of the system to perform detection, in active architectures specific signals are constructed to enhance the detection capabilities of a diagnostic tool. Example of the methods that can be classified as active are additive watermarking (Weerakkody and Sinopoli, 2015), switching multiplicative watermarking (Ferrari and Teixeira, 2020), as well as moving target defense strategies (Griffioen et al., 2020). Here, we focus on developing a design procedure for switching multiplicative watermarking.

The main concept behind the strategy of multiplicative watermarking is to modulate the signals that are to be transmitted over communication networks through two dynamical systems which modulate these signals, increasing the effect of a data injection attack, thus improving detection capabilities. Differently to additive watermarking, where a signal is added to the system input to verify its presence in the measured output, multiplicative watermarking does not involve the controlled plant. Rather, modulation of the signal occurs through a watermark generator, before transmission across a communication link. After transmission, the original signal is reconstructed via a watermark remover, before being used for control and diagnostic purposes. Thus, if the two systems are appropriately designed, the performance of the control system remains unchanged. Introduced in Ferrari and Teixeira (2020), switching multiplicative watermarking presupposes that, at certain time instances, the parameters of the watermark generator and remover are changed, synchronously allowing for further detection capabilities against more sophisticated classes of attacks.

In this paper, we aim to design a safe switching function, requiring minimal secret information to be shared between the watermark generator and remover, thus enhancing its overall security. We build a switching function based on modular arithmetic of elliptic curves, which are used in public key exchange cryptography because of the difficulty of solving the discrete logarithm problem on them. Our contributions are:

a. we introduce the basic characteristics of elliptic curves, giving a tutorial overview of their properties, and how they may be used for secure control;
b. we define a switching function for a multiplicative watermarking scheme similar to that proposed in Ferrari and Teixeira (2020), providing evidence as to its security properties;
c. we provide guarantees that the switching function results in a stable watermark generator-remover pair, for finite-impulse response generators.

The remainder of the paper is structured as follows: in Section 2 we formally introduce the problem formulation, giving an overview of the modeling of a cyber-physical system, as well as that of the switching multiplicative watermarking detection strategy; in Section 3 we give an overview of the mathematical properties of elliptic curves, and specifically of modular arithmetic over elliptic curves; in Section 4 we present the algorithm defining the

Abstract: In this paper we present a novel switching function for multiplicative watermarking systems. The switching function is based on the algebraic structure of elliptic curves over finite fields. The resulting function allows for both watermarking generator and remover to define appropriate system parameters, sharing only limited information, namely a private key. Given the definition of the switching function, we prove that the resulting watermarking parameters lead to a stable watermarking scheme.
the measurements. We suppose that all matrices are of
the unmodeled disturbances affecting the process and
output. The vectors where
the control input, and

\[ x(k+1) = A_x x(k) + B_x y(k) \]
\[ y(k) = C_x x(k) + D_x y(k) \]

(2)

w is introduced to highlight the possibility of an
injected signal in the communication, as will be formally defined in
Section 5. We provide a numerical implementation of the algorithm.

2. BACKGROUND AND PROBLEM FORMULATION

In this paper we consider a linear time-invariant (LTI) CPS, represented in Figure 1. The closed loop system is
composed of a plant \( P \) and a controller \( C \). We consider
that the output of the plant, \( y_p \), is transmitted over some
form of communication network to \( C \), and is therefore
exposed to malicious tampering. Specifically, we consider
a man-in-the-middle attack, capable of eavesdropping the
communicated signal, as well as injecting false data into
the channel. To detect the presence of this attack, we
suppose the system is equipped with an anomaly detector
\( D \), as well as a watermark generator and remover pair, \( W \)
and \( Q \), to enhance the detection properties. Specifically,
the measurement output \( y_p \) is modulated via an LTI
system, and the resulting output, \( y_w \), is transmitted over the
communication network. Once received, the signal \( \tilde{y}_w \)
1 is demodulated via the watermark remover system \( Q \), and
the resulting output \( y_q \) is used as an input to the controller.
Note that we implicitly assume that the control signal \( u \)
from the controller to the plant is secure.

In the remainder of this section, we introduce the closed
loop dynamics of the CPS, highlighting the properties of
\( W \) and \( Q \) which make them an appropriate multiplicative
watermarking pair, and we formally define the problem.

2.1 Cyber-physical system

We consider the dynamics of the plant \( P \) to be modeled
via the following:

\[ P : \begin{cases}
    x_p(k+1) = A_p x_p(k) + B_p u(k) + w \\
    y_p(k) = C_p x_p(k) + v
  \end{cases} \]

(1)

where \( x_p \in \mathbb{R}^{n_p} \) is the state of the plant, \( u \in \mathbb{R}^{n_u} \)
the control input, and \( y_p \in \mathbb{R}^{n_y} \) is the measurement
output. The vectors \( w \in \mathbb{R}^{n_w} \) and \( v \in \mathbb{R}^{n_v} \) represent
the unmodeled disturbances affecting the process and
the measurements. We suppose that all matrices are of

1 Note here that \( \tilde{y}_w \) is introduced to highlight the possibility of an
injected signal in the communication, as will be formally defined in
the following.

the appropriate dimensions. The plant is regulated via a
dynamic controller of the form:

\[ C : \begin{cases}
    x_c(k+1) = A_c x_c(k) + B_c y_q(k) \\
    u(k) = C_c x_c(k) + D_c y_q(k)
  \end{cases} \]

(2)

where \( x_c \in \mathbb{R}^{n_c} \) is the controller state, and \( y_q \in \mathbb{R}^{n_y} \)
is the output of the watermarking remover system \( Q \). As
mentioned previously, \( Q \) is included in the closed-loop
CPS, together with the watermarking generator \( W \), to
enhance the detection capabilities of the anomaly detector
\( D \), to be introduced. The watermarking pair is defined by
the following dynamical systems:

\[ W : \begin{cases}
    x_w(k+1) = A_w(\theta_w(k)) x_w(k) + B_w(\theta_w(k)) y_p(k) \\
    y_w(k) = C_w(\theta_w(k)) x_w(k) + D_w(\theta_w(k)) y_p(k)
  \end{cases} \]

(3a)

\[ Q : \begin{cases}
    x_q(k+1) = A_q(\theta_q(k)) x_q(k) + B_q(\theta_q(k)) \tilde{y}_w(k) \\
    y_q(k) = C_q(\theta_q(k)) x_q(k) + D_q(\theta_q(k)) \tilde{y}_w(k)
  \end{cases} \]

(3b)

where the vectors \( x_w, x_q \in \mathbb{R}^{n_w} \) and \( y_w, y_q \in \mathbb{R}^{n_y} \) are,
respectively, the state and outputs of the watermarking
systems \( W \) and \( Q \).

Finally, the CPS is equipped with an anomaly detection
module \( D \), to detect the presence of anomalies (such as
malicious false-data injection attacks):

\[ D : \begin{cases}
    x_r(k+1) = A_r x_r(k) + B_r w(k) + K_r y_q(k) \\
    y_r(k) = C_r x_r(k) + L_r y_q(k)
  \end{cases} \]

(4)

where \( x_r \in \mathbb{R}^{n_r} \) is the anomaly detector’s internal state,
and \( y_r \in \mathbb{R}^{n_y} \) its output. Note that \( y_r \) is used as a residual
signal, and the detection test

\[ |y_r(k)| \leq \tilde{y}_r(k) \]

(5)

is performed at each time instant to detect whether an
attack is active on the communication network, where
\( \tilde{y}_r(k) \) is an appropriately defined, time-varying detection
threshold, and the inequality is performed component-by-
component. The definition of \( \tilde{y}_r \) is out of the scope of this
paper, but may be found in, e.g., Zhang et al. (2002).

2.2 Attack model

As mentioned previously, we consider the output-side
signal \( y_w \) to be transmitted over a communication network.
Thus it is possibly subject to attacks from some time
\( k_a > 0 \). We define the received signal, appearing in (3b),
as:

\[ \tilde{y}_w(k) = y_w(k) + \beta_a(k - k_a) \varphi_y(Y_{w,[k-N_a,k]}) \]

(6)

where \( \varphi_y() \) is the maliciously defined attack signal,
depending on the matrix:

\[ Y_{w,[k-N_a,k]} := [y_w(k-N_a), y_w(k-N_a+1), \ldots, y_w(k)] \]

(7)

and \( \beta_a() \) is an activation function, which we here consider
to be \( \beta(k) = 0, \forall k < 0 \), and \( \beta(k) = 1 \) otherwise.

2.3 Multiplicative watermarking: some background

Let us now focus on the design of the time-varying systems
\( W \) and \( Q \), i.e., the watermarking pair. The results reported
in this section rely on those in Ferrari and Teixeira (2020).
**Definition 1.** For two systems $W$ and $Q$, defined in (3), to be an appropriate watermarking pair, the following conditions must be satisfied:

a. $W$ is stable and invertible;
b. $Q$ is stable;
c. if $\theta_w = \theta_q$, $Q$ is the inverse of $W$.

The definition of the systems $W$ and $Q$ in (3) are parametrized by the vectors $\theta_w, \theta_q \in \mathbb{R}^{n_w}$ defining the dynamics of these systems. These parameters are piecewise constant, and are updated only at specific switching times, to be defined later, with the updates given by:

$$ W: \begin{cases} \theta_w^k(k) = \sigma_w(\mathcal{I}_w(k)) & \text{if } \tau_w(y_w(k)) = 1 \\ x_w^k(k) = \rho_w(x_w^k(k), y_w(k), \theta_w^k(k)) \end{cases} \quad (8a) $$

$$ Q: \begin{cases} \theta_q^k(k) = \sigma_q(\mathcal{I}_q(k)) & \text{if } \tau_q(y_q(k)) = 1 \\ x_q^k(k) = \rho_q(x_q^k(k), y_q(k), \theta_q^k(k), \theta_q^+(k)) \end{cases} \quad (8b) $$

where $\mathcal{I}_w(k), \mathcal{I}_q(k)$ are the sets of information available to $W$ and $Q$ at time $k$, as defined in Definitions 3 and 4; $\rho_i : \mathbb{R}^{n_{w_i}} \times \mathbb{R}^{n_{w_i}} \times \mathbb{R}^{n_{w_i}} \times \mathbb{R}^{n_{w_i}} \rightarrow \mathbb{R}^{n_{w_i}}, i \in \{w, q\}$ is a jump map of $W$ and $Q$, $\sigma_i : \mathbb{R}^{n_{w_i}} \rightarrow \mathbb{R}^{n_{w_i}}, i \in \{w, q\}$ is a switching map, and the superscripts $+$ and $-$ denote the values of the vectors before and after a switch. Finally, $\tau_i, i \in \{w, q\}$ are trigger functions inducing the switch.

**Definition 2.** The functions $\tau_w, \tau_q \in \mathbb{R}^{n_w} \rightarrow \{0, 1\}$ are said to be trigger functions of $W$ and $Q$ if the triggering sets $\mathcal{C}_w \doteq \{ y_w : \tau_w(y_w) = 1 \}$ and $\mathcal{C}_q \doteq \{ y_q : \tau_q(y_q) = 1 \}$ are convex and open. The sequences $K_w \doteq \{ k : \tau_w(y_w(k)) = 1 \}$ and $K_q \doteq \{ k : \tau_q(y_q(k)) = 1 \}$ are called the switching time sequences of $W$ and $Q$.

Given that $W$ and $Q$ are not colocated, it is important to properly define what information is available to $W$ and $Q$. We therefore formally introduce the information sets, $\mathcal{I}_w$ and $\mathcal{I}_q$, defined in terms of input and output data over a window of size $N_I \geq 1, N_I \in \mathbb{Z}_+$, as well as $\theta_k, i \in \{w, q\}$.

**Definition 3.** (Information at $W$). The information available to the watermark generator $W$ is defined as:

$$ \mathcal{I}_w(k) = \{ Y_{p_k-N_I, k, x_w(k-N_I, \theta_w(k))} \}. \quad (9) $$

**Definition 4.** (Information at $Q$). The information available to the watermark remover $Q$ is defined as:

$$ \mathcal{I}_q(k) = \{ Y_{w_k-N_I, k, x_q(k-N_I, \theta_q(k))} \}. \quad (10) $$

in addition, for $\kappa_w \in K_w, \mathcal{I}_q^+(\kappa_w) = \mathcal{I}_q(\kappa_w) \cup \{ y_q^+(\kappa_w) \}$. \quad (11)

**Remark 1.** Given the definition of the information sets, $\mathcal{I}_q \subset \mathcal{I}_w$ holds, in nominal conditions, for all $k$.

**2.4 Synchronized switching**

To guarantee that the multiplicative watermarking scheme does not influence the closed-loop performance of the system, the following must hold:

a. $K_w = K_q$ (synchronized switching times);
b. the output of $\sigma_w(\kappa) = \sigma_q(\kappa)$ and $\rho_{w_\kappa}(\kappa) = \rho_{q_\kappa}(\kappa)$, for all $\kappa \in K_w$ (synchronized switches and jumps);
c. $y_q^+(\kappa) = y_q(\kappa)$ (synchronized output).

Here we have slightly abused notation, writing $\sigma_w(\kappa) = \sigma_q(\kappa)$, rather than $\sigma_w(\mathcal{I}_w(k)) = \sigma_q(\mathcal{I}_q(k))$. We consider the same induced synchronization scheme presented in Ferrari and Teixeira (2020), by which $W$ initializes the switch triggering and $Q$ updates its parameters, without any additional communicated data.

**Definition 5.** (Synchronized watermarking). The watermark generator $W$ and remover $Q$ are said to be synchronized if at switching time $k$ they are:

a. trigger synchronized, i.e. $\tau_w(y_w(k)) = \tau_q(y_q(k)) = 1$;
b. switch synchronized, i.e. $\theta_w^k(\kappa) = \theta_q^k(\kappa)$;
c. jump synchronized, i.e. $x_w^+(\kappa) = x_q^+(\kappa)$;
d. output synchronized, i.e. $y_q^+(\kappa) = y_q(\kappa)$.

**Remark 2.** Under synchronized watermarking $K_w = K_q$ holds, and therefore $\mathcal{I}_q^+(\kappa_w) \subset \mathcal{I}_w(\kappa_w), \forall \kappa_w \in K_w$. \quad (12)

**2.5 Problem formulation**

While in Ferrari and Teixeira (2020) detailed techniques are presented to define $\rho_i(\cdot), i \in \{w, q\}$, as well as a minimally visible \footnote{Interested readers are referred to Ferrari and Teixeira (2020) for a definition of switch visibility.} definition of $y_q^+(\kappa_w), \kappa_w \in K_w$, the definition of the switching maps $\sigma_i, i \in \{w, q\}$ are left unspecified. Thus, the objective of this paper is as follows:

**Problem 1.** Given a switching multiplicative watermarking scheme defined by (3)-(8), define switching functions $\sigma_w(\mathcal{I}_w(k))$ and $\sigma_q(\mathcal{I}_q(k))$ such that:

a. the entire sequence must not be known a priori;
b. $\theta_w^k(\kappa) = \theta_q^k(\kappa)$, for all $k \in K$;
c. $\theta_q^k(\kappa)$ is such that $W$ and $Q$ satisfy the conditions in Definition 1. \quad (13)

In the following section, we give some background on the mathematics of elliptic curves, and how they have been used in cryptography to generate private keys for encryption in the IT-security literature. After this, in Section 4, we present the algorithm used to generate the parameters of $W$ and $Q$, highlighting the required information.

**3. ELLIPTIC CURVES ON FINITE FIELDS: SOME BACKGROUND**

Let us now present some background on the arithmetic of elliptic curves. We here rely on overviews presented in Wohlwend (2010); López and Dahab (2000). Elliptic curves are abelian varieties, which have had large success in the field of cryptography. Indeed, as outlined in Wohlwend (2010); López and Dahab (2000), elliptic curve cryptography (ECC), is a form of asymmetric or public key cryptography, which guarantees higher levels of security than the Diffie-Hellman-Merkle key exchange, or RSA. We exploit the mathematical properties of elliptic curves to define a switching function common to $W$ and $Q$ which, although still including a shared secret, does not require the entire switching sequence to be defined a priori.

**3.1 Some fundamentals in group theory**

To properly introduce both the group defined by the elliptic curve and its operations, let us introduce some definitions.
Definition 6. A group is defined as a set \( G \) together with a binary operation \( \circ \) closed in \( G \), i.e., for any \( a, b \in G \), \( a \circ b \in G \), such that the following axioms (the so-called group axioms) hold:

a. the operation \( \circ \) is associative, i.e., \( (a \circ b) \circ c = a \circ (b \circ c) \);

b. there exists an identity element in \( G \), \( e \), such that \( a \circ e = e \circ a = a \);

c. there exists an inverse element of \( a \in G \), denoted \( a^{-1} \), under the group operation, such that \( a \circ a^{-1} = a^{-1} \circ a = e \).

Definition 7. A group \( G \) is abelian if its binary operator \( \circ \) is commutative, i.e., \( a \circ b = b \circ a, \forall a, b \in G \).

Definition 8. A group \( G \) is said to be cyclic if, for an element \( h \in G \), every element \( g \in G \) satisfies \( g = xh \) or \( g = h^x \), \( x \in \mathbb{Z}_+ \), depending on whether the group operation is additive or multiplicative. The element \( h \) is said to be the generator of the group.

Definition 9. A set \( H \subseteq G \) is said to be a cyclic subgroup if it is cyclic, given some generator \( h \).

Definition 10. Suppose \( a \in G \), and \( e \) is the identity element. The order of \( a \) is the smallest integer \( n \) such that:

\[
a \circ a \circ \cdots \circ a = e. \tag{11}
\]

The set \( \{ a, a^2, \ldots, a^n \} \) (or \( \{ a, 2a, \ldots, na \} \)) forms a cyclic subgroup of \( G \) of order \( n \), with \( a \) as its generator.

Definition 11. A field is an algebraic structure composed of a set \( \mathbb{F} \) together with the binary addition and multiplication operations, + and \( \times \), satisfying the following:

a. \( \mathbb{F} \) is an abelian group under addition +;

b. \( \mathbb{F}\setminus\{0\} \) is an abelian group under multiplication \( \times \);

c. \( \times \) is distributive over addition: \( a \times (b+c) = a \times b + a \times c \).

A field is said to be finite if \( |\mathbb{F}| < \infty \).

Examples of fields are the set of rational, real, complex numbers \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \). A field that is fundamental in cryptography and used in this paper, is the set of integers modulo \( l, \mathbb{Z}/l\mathbb{Z} \), with \( l \) prime. We use the notation \( \mathbb{F}_l = \mathbb{Z}/l\mathbb{Z} \) for compactness; this must not be confused with the set of \( l \)-adic integers.

3.2 Elliptic curves

Given the preliminary definitions in Section 3.1, we can now introduce elliptic curves. Although we are interested in elliptic curves over finite fields, as is further highlighted in the following, we start by presenting the general concept of an elliptic curve over a (possibly infinite) field \( \mathbb{F} \); an elliptic curve \( E(\mathbb{F}) \) is the set of points in \( \mathbb{F} \), satisfying

\[
y^2 = x^3 + ax + b. \tag{13}
\]

There are a number of special elliptic curves, of which the Weierstrass normal form:

\[
y^2 = x^3 + ax + b, \tag{13}
\]

with \( a, b \in \mathbb{F} \), is commonly used in cryptography. In Figure 2 we present an example of an elliptic curve defined for \( \mathbb{F} = \mathbb{R} \), with \( a = 1, b = -1 \).

The curve \( E(\mathbb{F}) \) forms an abelian group together with an addition operator +, with the so-called point at infinity \( O \) as its identity. Specifically, addition in \( E(\mathbb{F}) \) satisfies:

\[
P + O = O + P = P; \tag{15}
\]

- for \( P,Q \in E(\mathbb{F}) \), \( P \neq \pm Q \), the point \( R = P + Q \) is the point satisfying \( P + Q - R = O \);

- if \( P = (x,y) \in E(\mathbb{F}) \), then the negative of \( P \), \( \neg P = (x,\neg y) \), is such that \( P - \neg P = O \).

Coordinates \((x_R,y_R) = P + Q \) can be computed via:

\[
\begin{align*}
x_R &= \lambda^2 - x_P - x_Q, \quad \lambda = \frac{y_Q - y_P}{x_Q - x_P}, \\
y_R &= \lambda(x_P - x_Q) - y_P.
\end{align*}
\tag{14}
\]

with \( P = (x_P,y_P) \) and \( Q = (x_Q,y_Q) \). This can be interpreted geometrically by taking the line through \( P \) and \( Q \) and finding where it intersects the elliptic curve: this is \( -P \). Thus, taking the inverse of the \( y \) coordinate, \( R = P + Q \) is found.

Given operator +, we are also interested in the so called doubling operation, i.e., computing \( 2P = P + P \). Clearly, in this case the geometric interpretation we provided for addition over the elliptic curve does not hold. Instead, \( 2P \) can be interpreted geometrically by taking the tangent to \( P \), which crosses the elliptic curve in one point, \( -2P \). Then point \( 2P \) is found by inverting the \( y \) coordinate. Geometric representations for addition and doubling are given in Figure 2. In coordinate terms, \((x_{2P},y_{2P}) = 2P \) can be found through:

\[
\begin{align*}
x_{2P} &= \lambda^2 - 2x_P, \\
y_{2P} &= \lambda(x_P - x_{2P}) - y_P, \quad \lambda = \frac{3x_P^2 + a}{2y_P}.
\end{align*}
\tag{15}
\]

We can also define scalar multiplication over \( E(\mathbb{F}) \), by repeatedly adding a point \( P \in E(\mathbb{F}) \) to itself:

\[
sP = \underbrace{P + \cdots + P}_{n}.
\tag{16}
\]

with \( s \in \mathbb{Z}_+ \). For elliptic curves over finite fields, to be used in the definition of the switching function in the following, this operation can be efficiently computed in \( O(\log(s)) \) using the Double and Add algorithm (Wohlwend, 2016).

Scalar multiplication can also be used to define a cyclic subgroup with generator \( P \) and order \( n \).

Definition 12. The order of a point \( P \in E(\mathbb{F}) \) is defined as the minimum \( n \in \mathbb{Z}_+ \) such that:

\[
P + \cdots + P = O. \tag{17}
\]
Definition 13. The cofactor of a point \( P \in E(\mathbb{F}_p) \) of order \( n \) is defined as:

\[
h = \frac{|E(\mathbb{F}_p)|}{n}.
\]  

Lagrange’s theorem states that the order of a subgroup must be a divisor of the order of the group; thus, \( h \in \mathbb{Z}_+ \) for all \( P \in E(\mathbb{F}_p) \) (Birkhoff and Mac Lane, 2017).

Until now we have described elliptic curves on a generic field \( \mathbb{F} \). However, for the purpose of our contribution, we are interested in elliptic curves defined on finite fields, specifically \( \mathbb{F}_p \), the field of integers modulo \( p \). These curves are different to that shown in Figure 2, and an example is found in Figure 3. This field is used also in Elliptic Curve Cryptography (ECC), a successful public key cryptographic scheme, summarized briefly in Section 3.3.

3.3 Elliptic curve cryptography

One of the main applications that elliptic curves have found is that of elliptic curve cryptography (ECC). Similar to Diffie-Hellman-Merkle private key cryptography, it is based on the difficulty of solving the discrete logarithm problem, i.e., even if \( P \) and \( S = lP \) are known, there are no efficient solutions to find \( l \).

The Diffie-Hellman key exchange on an elliptic curve has the following structure: suppose Alice and Bob want to share a key, without any shared secrets. Additionally, suppose they agree on the parameters of a shared (public) elliptic curve \( E(\mathbb{F}_p) = \{ \mathbb{F}_p, P, n, h, a, b \} \), where \( \mathbb{F}_p \) is the field on which the elliptic curve is defined, \( P \) is a generator point, \( n \) and \( h \) are respectively the order and cofactor of \( P \), and \( a, b \) are the parameters of the elliptic curve (13). Alice defines a private key \( k_a \), and computes \( Q_a = k_aP \), her public key. Similarly, Bob defines a private key \( k_b \), and generates public key \( Q_b = k_bP \). Thus, Alice and Bob exchange their public keys, and compute the shared key, \( Q_{ab} = k_aQ_b = k_bQ_a = (k_ak_b)P \). This shared information is reached exploiting private information that has not been transmitted over a (possibly insecure) communication channel. Any malicious agent capable of eavesdropping the communication between Alice and Bob will have access to \( P, Q_a \), and \( Q_b \), but cannot reconstruct \( k_a \) or \( k_b \), as solving the so-called discrete logarithm on elliptic curves does not have an efficient algorithmic solution on (non-quantum) computers. In the following we show how this is increases security when using elliptic curves to define the switching function \( \sigma(\cdot) \), i.e., in \( \{ w, q \} \).

Remark 3. The order and cofactor of a generator point \( P \) plays a role in how difficult\(^3\) it is to compute the discrete logarithm problem. For cryptographic problems, it is common to select generator points \( P \) such that their cofactor \( h \leq 4 \) (Wohlwend, 2016).

4. A SWITCHING FUNCTION BASED ON ELLIPTIC CURVES

Having presented the fundamentals of elliptic curves on finite fields, let us now present the algorithm that defines the switching functions \( \sigma \). We drop subscripts to improve readability, which are however to be considered for both watermark generator and remover. We then highlight the information to be shared between \( W \) and \( Q \) for \( \sigma_w = \sigma_q \).

4.1 Switching function

Recall from (8) that \( \sigma(\cdot) \) plays a role in changing the parameters of the watermarking pair, once the triggering function \( \tau(y_p(\kappa)) = 1 \), for \( \kappa \in \mathcal{K} \). Here, we take \( y_p(\kappa-1) \) as the input to the switching function \( \sigma(\cdot) \). In the following, we presume that \( n_y = 1 \), and therefore that \( y_p \in \mathbb{R} \). The procedure can be easily extended for \( n_y > 1 \).

Remark 4. It is necessary to take \( y_p(\kappa-1) \) rather than \( y_p(\kappa) \) because \( y_p(\kappa-1) = y_p(\kappa-1) \in \mathbb{Z}_w \cap \mathbb{Z}_q \), while \( y_p(\kappa) \in \mathbb{Z}_q \), although \( y_p(\kappa) \in \mathbb{Z}_w \). Indeed, the watermark remover \( Q \) requires knowledge of the new parameter vector \( \theta^+(\kappa) \) to recover the plant’s measured output.

Remark 5. The switching function depends on \( y_p \) to introduce some randomness to the algorithm, thus making it more complex to identify for an attacker. Indeed, the use of physical quantities to generate true random numbers (rather than random number generators) is common in the selection of private keys in public key cryptography.

The switching function \( \sigma(\cdot) \) is the result of the following:

a. the projection of \( y_p(\kappa-1) \) onto \( P \in E(\mathbb{F}_s) \), the elliptic curve defined on the field of integers modulo \( s \), with \( s \) prime;

b. the computation of \( S = lP, S \in E(\mathbb{F}_s) \), for some \( l \in \mathbb{Z}_+ \);

c. the mapping of \( S \) onto \( \Theta \), the constrained parameter set guaranteeing that the resulting watermarking pair \( \{ W, Q \} \) satisfies conditions in Definition 1.

In the first step the measurement output is “projected” to a point on the elliptic curve \( E(\mathbb{F}_s) \), then used as the generator of a cyclic subgroup. We define function \( \alpha(\cdot) : \mathbb{R}^n \rightarrow E(\mathbb{F}_s) \) as this function. In turn \( \alpha(\cdot) \) can be seen as \( \alpha(\cdot) = \alpha_2(\alpha_1(\cdot)) \), where \( \alpha_1(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}_s \times \mathbb{R}_s \) scales the output, where \( \mathbb{R}_s \) is the set of real numbers modulo \( s \), defining the coordinates of a point \( \hat{P} = (x_\hat{P}, y_\hat{P}) \in \mathbb{R}_s \times \mathbb{R}_s \), whilst \( \alpha_2 : \mathbb{R}_s \times \mathbb{R}_s \rightarrow E(\mathbb{F}_s) \) maps the point \( \hat{P} \) to \( P \in E(\mathbb{F}_s) \). Specifically:

\[
\alpha_1 = (\alpha_{1,x}, \alpha_{1,y}),
\]

where \( \alpha_{1,x} \) and \( \alpha_{1,y} \) may have similar structure, but should have different parameters, such that \( x_\hat{P} \neq y_\hat{P} \), in general. These functions may be defined in several ways, e.g., that given in Section 5.

Once \( \hat{P} \) is computed, \( \alpha_2(\cdot) : \mathbb{F}_s \times \mathbb{F}_s \rightarrow E(\mathbb{F}_s) \) can be defined as follows:

\[
P = \alpha_2(\hat{P}) = \arg \min_{\Lambda \in E(\mathbb{F}_s)} ||\Lambda - \hat{P}||^2_2,
\]

thus defining a generator point \( P \in E(\mathbb{F}_s) \).

Remark 6. It is important to highlight the importance of scaling the measurement output \( y_p \). In nominal operations,
i.e., whilst in steady state, the output of the plant varies only slightly in a (likely small) neighborhood of some nominal output, \( r \), which may be given by a reference to the controller. Thus, without scaling by \( \alpha(\cdot) \), the result of the projection of \( y_p \) onto \( E(\mathbb{F}_q) \) would always be onto a subset \( \mathcal{E} \subset E(\mathbb{F}_q) \), with \( |\mathcal{E}| \ll |E(\mathbb{F}_q)| \), and possibly \( |\mathcal{E}| = 1 \). This is undesirable, as it would imply that \( \theta^+ = \theta \). Thus, by scaling \( y_p \), even small changes in the measured output lead to a difference in \( \mathcal{E} \), and therefore to increased variability in \( \theta^+ \).

Once a generator point \( P \in E(\mathbb{F}_q) \) is computed, a scalar multiplication is performed to define \( S = lP \). Note that, given the definition of time-varying generator \( P \), it may be that for some values of \( P, l \) is the order of \( P \), and thus that \( S = O \), which is undesirable. Thus, if \( S = O \), some heuristic can be selected, such as \( S = P \) or \( S = -P = (l-1)P \), to avoid the result \( S = O \).

**Remark 7.** Note that here we assume \( l \) to be defined a priori and to be time invariant. This is done because, if \( l \) were to be time-varying, it would either have to be a function of \( y_p(k-1) \), or to satisfy another switching function \( \eta(\cdot) \). Similarly to \( \eta(\cdot) \), the attacker would not be capable of reconstructing \( S \), and therefore necessary that:

\[
\{\alpha(\cdot), \eta(\cdot), l, E(\mathbb{F}_q)\} \subset \mathcal{I}_a.
\]  

What is particularly interesting, and a direct consequence of using elliptic curves, is that even if all functions were known to the attacker, so long as

\[
l \notin \mathcal{I}_a,
\]

the attacker would not be capable of reconstructing \( \theta^+ \). Indeed, even if an eavesdropping attacker were to be able to estimate \( \theta \) and \( \theta^+ \), after some delay following the switch, and through knowledge of \( \alpha(\cdot) \) and \( \eta(\cdot) \), to reconstruct \( P \) and \( S \), finding \( l \) solving \( S = lP \) is the solution to the discrete logarithm over elliptic curves, for which there are no known algorithms capable of finding a solution in polynomial time (Wohlwend, 2016).

### 4.3.3 Watermark pair stability: an example with finite impulse response filters

Let us now focus on the definition of the parameter set \( \Theta \) such that if \( \theta^+ \in \Theta \), the resulting watermarking pair is guaranteed to satisfy the conditions in Definition 1. We restrict the set of parameters by giving the watermark generator and remover some structure, namely, for the purpose of this paper, we suppose that \( W \) is composed of \( n_y \) parallel FIR filters of order \( n_k \); thus, \( n_q = n_q \cdot n_k \). For the sake of maintaining notation streamlined, and without loss of generality, for the remainder of this subsection we suppose that \( n_y = 1 \). The output of \( W \) in (3a) is thus:

\[
y_w(k) = \sum_{l=0}^{n_k} b_h y_p(i(k - i)),
\]

with \( b_h \in \mathbb{R}, \forall h \in \{0, 1, \ldots, n_k\} \). This formulation guarantees that \( W \) is stable, with \( n_k \) poles at the origin. Finding \( \Theta \subseteq \mathbb{R}^{n_k} \) is equivalent to finding the set of parameters \( b_h \) for which \( W \) is invertible, with stable inverse.

**Theorem 1.** Suppose that \( W \) is an LTI system with dynamics defined by the FIR filter in (25). Thus, if the parameters are such that

```plaintext
Algorithm 1 Switching function \( \sigma_w(y_p(\kappa_w - 1)) \)
1: Input: \( y_p(\kappa_w - 1), E(\mathbb{F}_q), l, \alpha(\cdot), \eta(\cdot) \)
2: Output: \( \theta^+_w(\kappa_w) \)

3: Compute the generator of the elliptic curve by computing \( P = \alpha(y_p(\kappa_w - 1)) \)
4: Given \( P \), compute \( S = lP \)
5: Define \( \theta^+_w(\kappa_w) = \eta(S) \)
6: return: \( \theta^+_w(\kappa_w) \)
```
hold, the resulting watermarking pair \{W, Q\} is guaranteed to satisfy Definition 1, with \(\theta_w = \theta_q = \text{col}_i[b_i]\).

**Proof.** We start by noting that if \(W\) is defined by (25), then it is by definition stable. Indeed, FIR filters have all poles at the origin. Furthermore, if (26a) is satisfied, \(W\) admits an inverse. We therefore must prove that the watermark remover \(Q = W^{-1}\) is stable. The dynamics of \(y_w\) defined in (25) can be written in state-space form (3a) with the following matrices:

\[
A_w = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \\
B_w = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
C_w = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \\
D_w = b_0.
\]

Recalling the definition of \(Q\) as the inverse of \(W\), given the definition of the inverse of a system (Zhou et al., 1996, Lemma 3.15), we write the matrices defining the system dynamics (3b) as:

\[
A_q = \frac{1}{b_0} \begin{bmatrix} -b_1 & -b_2 & \cdots & -b_n \\ b_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_0 \end{bmatrix}, \\
B_q = b_0^{-1}B_w, \\
C_q = b_0^{-1}C_w, \\
D_q = b_0^{-1}.
\]

Thus, by Geršgorin’s circle theorem (Horn and Johnson, 1985), it is sufficient that conditions (26) hold for \(Q\) to be stable. Hence, \(\{W, Q\}\) satisfy Definition 1.

**Conditions (26) in Theorem 1** define \(\Theta\) for the class of FIR filters. We can therefore now define an example for \(\eta(\cdot)\). Take \(\eta(\cdot) : E(F_\gamma) \to \mathbb{R}^{n_h}\) to be any vector of \(n_h\) nonlinear functions, and define:

\[
b^- = \eta(S).
\]

Define each element of \(b^- \in \mathbb{R}^{n_h}\) as \(b^-_i\). To ensure that \(b = \eta_2(b^-)\) satisfies (26) in Theorem 1:

i. condition (26a) is satisfied if

\[
\eta_2,0(S) \neq 0, \forall S \in E(F_\gamma);
\]

ii. condition (26b) holds if

\[
\eta_2,1(b^-_i) = \frac{b^-_i}{|b^-_i| + \eta},
\]

with \(\eta \in \mathbb{R}_+\) arbitrary;

iii. condition (26c) is satisfied for

\[
\eta_2,2(b^-) = \xi(b^-)b^-_i,
\]

where \(\xi(b^-)\) is an auxiliary variable:

\[
\xi(b^-) = \frac{1 - |b^-_i| - \varepsilon}{\sum_{k=1}^{n_h} \frac{|b^-_k|}{|b^-_i|}},
\]

with \(\varepsilon \in (0, 1 - |b^-_i|)\) arbitrary.

5. **NUMERICAL RESULTS**

Having presented the switching function, and given an overview of its properties, let us now focus on considerations for the practical implementation of \(\sigma(\cdot)\). We start by discussing the consequences of the definition of both the elliptic curve and of the functions \(\alpha(\cdot)\) and \(\eta(\cdot)\). Following this, we highlight the effect that different operating points have on the sensitivity of \(\sigma(\cdot)\) to changes in its inputs.

The elliptic curve we choose for this example is that defined on \(F_{17}\), shown in Figure 3. This is an elliptic curve, of order \(|E(F_{17})| = 19\), on which the solution to the discrete logarithm problem does not require a lot of computation; indeed, in cryptographic settings, FIPS 186-4 recommends using elliptic curves defined on fields where the prime integer used for the modulo operation is at least 192 bits long. However, its structure makes it suitable for illustrating some fundamental characteristics of \(\sigma(\cdot)\); indeed, because its order is a prime number, the order of each of its points \(P \in E(F_{17})\) is also 19, as the coefficient \(h\) of \(P\), defined in Definition 13, is always an integer (Birkhoff and Mac Lane, 2017). Furthermore, by selecting \(l\) such that \(l \mod 19 \neq 0\), it is possible to guarantee that \(S = lP \neq O\), the point at infinity, for all \(P \in E(F_{17})\).

Moreover, the order of \(E(F_{17})\) also determines the number of parameters \(\theta_w = \theta_q \in \Theta\) that define \(\{W, Q\}\). It is worth noting that the points \(P \in E(F_{17})\) do not partition the space uniformly. To show this, in Figure 3 we include the Voronoi diagram generated by taking the points in \(E(F_{17})\) as seeds: there are points (and thus parameters) that have a higher likelihood of being selected by a given \(\gamma \in \mathbb{R}^{n_h}\).

We can now focus our attention on the definition of \(\alpha(\cdot)\) and \(\eta(\cdot)\), supposing \(n_h = 1\). As discussed previously, both of these can be seen as the combination of a scaling and a projection function. While the latter have been discussed in Section 4.1, a possible definition of the former is:

\[
\alpha_{1,i}(\gamma) = a_{i,0}\text{atan}(a_{i,1}\gamma) + \sum_{j=2}^{n_h} a_{i,j}\gamma_j^j
\]

\[
\eta_i(H) = \sum_{j=0}^{n_h} b_j||H||^2
\]

with \(H \in E(F_{17})\), and where \(a_{i,j}, i \in \{x, y\}, j \in \{0, 1, \ldots, n_h\}\) and \(b_j, j \in \{0, 1, \ldots, n_h\}\) are time-invariant parameters, shared between \(W\) and \(Q\); moreover, recall from Section 4.1 that \(\alpha_1(\cdot) \equiv (\alpha_{1,x}(\cdot), \alpha_{1,y}(\cdot))\).

Finally, we are interested in examining the sensitivity of the output of \(\sigma(\cdot)\) with respect to (small) variations of the plant measurement outputs \(y_p\). Results are presented in Figure 4. We consider the measurement of the plant \(y_p = r + \epsilon\), where \(r \in \mathbb{R}^{n_h}\) is an output reference for \(P\), and \(\epsilon\) is the tracking error. Specifically, we suppose \(r \in \{0, 1, 10, 100\}\), and for each we consider 500 realizations of \(v\), each taken from a uniform distribution with limits \([-0.05, 0.05]\). In Figure 4 we show how frequently different points \(P \in E(F_{17})\) are reached for changes in \(y_p\), relative to the total number of values of \(v\) taken. We see that, irrespective of the operating point, it is possible to reach all cells, which implies that for all operating conditions all 19 values that \(\theta^v\) can take are reachable. However, there are large differences between the behaviors relative
there is sufficient sensitivity of the switching function $\sigma$ to those points that are to be tracked by the plant, to ensure the "driver" of this sensitivity, is to be appropriately tuned for implementation of input. This gives us important insight for the practical applicability of this scheme, testing it on real world hardware, considering computational and numerical problems.

In future work, we are interested in testing the real-world applicability of this scheme, testing it on real world hardware, considering computational and numerical problems.

6. CONCLUSION

In this work, we have presented a method to define the switching function for switching multiplicative watermarking based on elliptic curves, guaranteeing that the switching signal may remain secure. We show how, even for attackers with large amounts of information, the switching signal may remain secure.

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