Criticality and Bifurcation in the Gravitational Collapse of a Self-Coupled Scalar Field

Eric W. Hirschmann
Department of Physics
University of California
Santa Barbara, CA 93106-9530

Douglas M. Eardley
Institute for Theoretical Physics
University of California
Santa Barbara, CA 93106-4030
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Abstract

We examine the gravitational collapse of a non-linear sigma model in spherical symmetry. There exists a family of continuously self-similar solutions parameterized by the coupling constant of the theory. These solutions are calculated together with the critical exponents for black hole formation of these collapse models. We also find that the sequence of solutions exhibits a Hopf-type bifurcation as the continuously self-similar solutions become unstable to perturbations away from self-similarity.

*Electronic address: ehirsch@dolphin.physics.ucsb.edu
†Electronic address: doug@itp.ucsb.edu
1. INTRODUCTION

The last few years have seen a renewed interest in gravitational collapse, particularly with regard to what numerical relativity is able to teach us about the general phenomenon. Choptuik’s initial discovery of criticality and other behavior strikingly similar to that seen in statistical mechanical systems has suggested a deep property of the gravitational field equations.

A good deal of recent work has shown the existence of collapse solutions exactly at the threshold of the formation of a black hole for a variety of matter fields. These include both real and complex scalar fields, vacuum gravity, a perfect fluid, and an axion-dilaton model from low energy string theory. In each of these models, some common behavior emerges. For example, the growth of the black hole mass just off threshold is described by a power law relation

\[ M_{\text{BH}}(p) \propto \begin{cases} 0, & p \leq p^* \\ (p - p^*)^{\gamma}, & p > p^* \end{cases} \]

where \( p \) is any parameter which can be said to characterize the strength of the initial conditions, and \( p^* \) is the threshold value, i.e., the value for the critical solution. The critical exponent \( \gamma \) is universal within a particular class of matter fields. For example, \( \gamma \approx 0.37 \) for the real scalar field, \( \gamma \approx 0.36 \) for perfect fluid collapse, and \( \gamma \approx 0.2641066 \) for the axion-dilaton (axiodil) system. The solutions may also exhibit an echoing behavior in that the features of the exactly critical solution are repeated on ever decreasing time and length scales. This self-similar behavior of the solutions has been found in both discrete and continuous versions. In particular, for vacuum gravity, discrete self-similarity and echoing are observed, while in fluid collapse, continuous self-similarity with no echoing emerges. In scalar field collapse, both types have been shown to be present.

The main results of this paper unify the discrete vs continuous self-similarity known in the above models. Specifically, we examine a particular non-linear sigma model which smoothly interpolates between the complex scalar field model and the axion-dilaton model as the value of a certain dimensionless coupling constant \( \kappa \) varies. We find a family of continuously self-similar solutions parametrized by \( \kappa \). Using linear perturbation theory, we study the stability of these solutions, and find that the sequence of solutions undergoes a bifurcation at a particular value, \( \kappa \approx 0.0754 \), where the continuously self-similar solutions go from being stable to being unstable. The free complex scalar field (\( \kappa = 0 \)) is found to be on the unstable side of this bifurcation, while the axion-dilaton field (\( \kappa = 1 \)) is on the stable side. This is in agreement with previous results for both of these matter fields. Further, we find that for negative values \( \kappa < -0.28 \), the self-similar solutions become ever more unstable hinting at the possibility of further bifurcations, more complicated dynamics, and perhaps even chaotic behavior in the collapse of these particular models. Since we work only in perturbation theory, these latter results are highly tentative, but they suggest the existence of more exotic behavior than may have previously been observed. For this reason, full scale numerical work on these models would undoubtedly be a very enlightening undertaking.

Prior to our work, Choptuik and Liebling recently studied an apparently different model, namely Brans-Dicke gravity coupled to a free real scalar field, for various values of the dimensionless Brans-Dicke coupling constant, \( -3/2 < \omega_{BD} < \infty \). They use a spherical
collapse code, and their main result is a change of stability at $\omega_{BD} \approx 0$. After the continuously self-similar solution was found in the collapse of an axion-dilaton field [7], they realized that it was their more general Brans-Dicke model for a particular value of $\omega_{BD}$. In fact, we find that their Brans-Dicke model is equivalent to some range of our nonlinear sigma model ($\infty > \kappa \geq 0$), with $\omega_{BD} = \infty$ corresponding to the free complex scalar field, and $\omega_{BD} = -11/8$ corresponding to the axion-dilaton field. The bifurcation in stability we find here in linear perturbation theory then coincides with the change of stability previously found by Choptuik and Liebling; in particular, we agree with their result that, for axion-dilaton collapse, the continuously self-similar critical solution is stable and appears to be the attractor. The range $\kappa < 0$ is not present in the Brans-Dicke model, however.

After the research presented here was completed, but before this paper was posted, Hamade, Horne, & Stuart [10] reported numerical and perturbation results on axion-dilaton collapse in spherical symmetry. Our results in linear perturbation theory agree with theirs with regard to real modes and critical exponents. They also find by a numerical collapse code that the continuously self-similar critical solution is stable and is the attractor, in agreement with the work of Choptuik and Liebling; this is also consistent with our results below on the complex modes.

The outline of this paper is as follows. In Section II, we give general arguments on what kinds of self-coupling of a scalar field may show new critical phenomena in gravitational collapse; likely candidates are the non-linear sigma models. In Section III, we introduce the particular non-linear sigma model to be studied in this paper, and discuss its relationship to matter fields which have been studied previously. Section IV introduces the equations of motion, derives their form in the presence of a continuous self-similarity, and sketches our numerical approach to solving them. Section V discusses the perturbation of the continuously self-similar solutions and the question of stability of these solutions. Section VI presents our results and conclusions.

II. CRITICAL BEHAVIOR AND SELF-INTERACTION

With the important exception of [2], all the work so far on critical phenomena in gravitational collapse has assumed spherical symmetry. In spherical symmetry, there is no gravitational collapse without matter, from Birkhoff’s theorem. Therefore one might expect that critical behavior would depend importantly on the model of the matter. Indeed, the critical phenomenology and exponents differ among matter models such as real scalar field, ideal gas, complex scalar field, axiodil . . . . However, studying a real scalar field $\phi$, Choptuik found that inclusion of a nonlinear interaction term $V(\phi)$ in the action,

$$L_{\text{matter}} = \frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi), \quad (2a)$$

$$V(\phi) \equiv \mu^2 \phi^2 / 2 + \lambda \phi^4 / 4 \quad (2b)$$

made no difference in the critical solution itself or in its phenomenology.

We can understand this result as follows. At least in all known cases, the critical solution is either “echoing” (discretely self-similar) or continuously self similar (CSS – admitting a homothetic Killing vector field). In either case, by dimensional analysis, the solution cannot
depend on any dimensionful parameters. Here we use dimensional analysis appropriate to classical general relativity, with a unit of length $\ell$ in some system of units where Newton’s gravitational constant $G \equiv 1$. A scalar field $\phi$ (real or complex) then has dimensions $\ell^0$, while a Lagrangian must have units $\ell^{-2}$. It follows that the parameters $\mu$ and $\lambda$ above have dimensions different from zero; in particular, $\mu$ is just the inverse compton wavelength of the particle. Since these parameters are dimensionful, the critical solution cannot depend on them, consistent with the numerical results.

For this reason we turn attention to a different kind of self-coupling, one which multiplies the kinetic term instead of adding to it. The general form is

$$\frac{1}{2} G_{IJ}(\phi^K) \nabla^\alpha \phi^I \nabla_\alpha \phi^J$$

where there are now some number $N$ of scalar fields $\phi^I$ ($I = 1 \ldots N$), and where $G_{IJ}$ is some function of the fields, fixed once and for all to specify the model. The nonlinear functions $G_{IJ}$ take the form of a Riemannian metric on the internal space of the $\phi^I$, the target space. Such models are called non-linear sigma models (or “harmonic map” models, as discussed by Misner [10]), and much is known about them in high energy physics, not least because they often appear in the low energy limit of superstring theory. By dimensional analysis, the scalar fields $\phi^I$ are of dimension $\ell^0$, as is the target space metric $G_{IJ}$. Therefore any parameters appearing in $G_{IJ}$ may also be taken as dimensionless, and we can expect the critical solution to depend on them.

What is the simplest nonlinear sigma model we can study? If $N = 1$ then the matter action can be reduced to that of a free field by a field redefinition; a 1-dimensional Riemannian space is always flat. So the simplest nontrivial value is $N = 2$, wherein the two real scalar fields can be grouped into a single complex scalar field $\phi$. For the target space metric, the simplest cases are the spaces of constant curvature, namely the 2-sphere, flat 2-space, or the 2-hyperboloid, all with homogeneous metrics. This is the model we shall study.

III. THE MODEL

We work with a model defined by the following action

$$S = \int d^4 x \sqrt{-g} \left( R - \frac{2|\nabla F|^2}{(1 - \kappa |F|^2)^2} \right).$$

(4)

The complex field $F(x^\mu)$ is a scalar coupled to Einstein gravity with $\kappa$ a real dimensionless coupling constant,

$$- \infty < \kappa < \infty.$$  

(5)

$^1$Choptuik has also tried adding a conformal coupling $\xi R \phi^2$ to the matter Lagrangian. In contrast, $\xi$ is dimensionless, so that that critical solution should depend on it. This point deserves more investigation.
The model given by Eq. (4) is a nonlinear sigma model. As mentioned above, the target space of the model is a two-dimensional space of constant curvature. The curvature of this internal space is proportional to $-\kappa$ so that the space is hyperbolic for $\kappa > 0$ and a 2-sphere for $\kappa < 0$. For the particular case $\kappa = 1$, our model becomes the axion-dilaton (axiodil) field $\tilde{\tau}$ coupled to gravity

$$F = \frac{1 + i\tilde{\tau}}{1 - i\tilde{\tau}}.$$  (6)

It turns out that the value $\kappa = 1$ is not affected by quantum corrections as it can be protected by extended supersymmetry. For $\kappa = 0$ the model (4) reduces (after a further trivial rescaling of the field) to the free complex scalar field coupled to gravity. Thus this general model smoothly interpolates between the two particular matter models that we have already considered. In fact, for $0 < \kappa < \infty$ we find that this nonlinear sigma model is equivalent to the model of a massless real scalar field coupled to Brans-Dicke theory. Liebling has recently examined this theory using a version of Choptuik’s adaptive mesh refinement algorithm. He finds behavior qualitatively similar to that found by Choptuik for the real scalar field [11]. The connection between the two theories can be seen in the relationship between the Brans-Dicke coupling constant $\omega_{BD}$ and our constant $\kappa$

$$\omega_{BD} = -\frac{3}{2} + \frac{1}{8\kappa}, \quad 0 \leq \kappa < \infty.$$  (7)

This means that the axion-dilaton model ($\kappa = 1$) corresponds to $\omega_{BD} = -11/8$, while the free complex scalar field ($\kappa = 0$) corresponds to $\omega_{BD} = +\infty$. For $-\infty < \kappa < 0$ the model (4) appears not to be equivalent to any Brans-Dicke model; in particular Eq. (7) does not apply. The model behaves in a smooth way as $\kappa$ passes through zero.

Returning to the full model, the field equations in covariant form as derived from the action in Eq. (4) are,

$$R_{ab} = \frac{1}{(1 - \kappa|F|^2)^2} \left( \nabla_a F \nabla_b F^* + \nabla_a F^* \nabla_b F \right),$$  (8a)

$$\nabla^a \nabla_a F = \frac{-2\kappa F^*}{1 - \kappa|F|^2} \nabla_a F \nabla^a F.$$  (8b)

In this form, these equations are invariant under a global $U(1)$ group of transformations for a constant $\Lambda$

$$F' = e^{i\Lambda} F, \quad -\infty < \Lambda < \infty$$  (9)

and which leave the metric unchanged.

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2 Notation: We use $\tilde{\tau}$ here for the axiodil field, instead of $\tau$ as we did in [7], to avoid confusion with logarithmic time coordinate $\tau$ below.

3 As $\omega_{BD} \to -3/2^+$, we have $\kappa \to +\infty$; however this may be a funny limit.
For $\kappa > 0$, this model also has an extra global symmetry not present in general relativity, namely an $SL(2, \mathbb{R})$ symmetry that acts on $F$, but leaves the spacetime metric invariant; this is a classical version of the conjectured $SL(2, \mathbb{Z})$ symmetry of heterotic string theory called $S$-duality \[15\]. For the axiodil, $\kappa = 1$, this symmetry acts on $\tilde{\tau}$ as

$$\tilde{\tau} \rightarrow \frac{a\tilde{\tau} + b}{c\tilde{\tau} + d}, \quad (10)$$

where $(a, b, c, d) \in \mathbb{R}$ with $ad - bc = 1$, while leaving $g_{\mu\nu}$ invariant. The corresponding transformation of $F$ for general $\kappa > 0$ is

$$F \rightarrow \frac{1}{\sqrt{\kappa}} \alpha \sqrt{\kappa} F + \beta \sqrt{\kappa} F + \alpha^*,$$

where $(\alpha, \beta) \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 = 1$.

In the case where $\kappa = 0$, the extra global symmetry consists of translations in the two flat directions of the target space. Finally, for $\kappa < 0$, the group of motions on the 2-sphere, $SO(3)$, constitutes the extra global symmetry.

### IV. THE CONTINUOUSLY SELF-SIMILAR SOLUTIONS

We briefly review the process of setting up the equations such that they are compatible with a continuous self-similarity. To begin, we work in spherical symmetry so the metric can be taken as

$$ds^2 = (1 + u) \left[-b^2 dt^2 + dr^2\right] + r^2 d\Omega^2 \quad (12)$$

where $b(t, r)$ and $u(t, r)$ are the metric functions. This is essentially Choptuik’s metric in radial gauge with some minor redefinitions. The timelike coordinate $t$ is chosen so that the collapse on the axis of spherical symmetry happens at $t = 0$ and the metric is regular for $t < 0$.

We are interested in finding collapsing solutions of our model. In particular we ask whether, as in the complex scalar, axiodil, and fluid collapse cases, there exist continuously self-similar (CSS) solutions to these equations for arbitrary $\kappa$. That a spacetime admits a continuous self-similarity is described covariantly by the existence of a homothetic Killing vector field $\xi$ satisfying

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 2g_{ab}, \quad (13)$$

where $\mathcal{L}$ denotes the Lie derivative. A coordinate system better adapted to our assumption of self-similarity involves the coordinates $z = -r/t$ and $\tau = \ln | -t |$. In these coordinates, the metric takes the form

$$ds^2 = e^{2\tau} \left((1 + u) \left[-(b^2 - z^2) d\tau^2 + 2zd\tau dz + dz^2\right] + z^2 d\Omega^2\right), \quad (14)$$

and the homothetic Killing vector is then expressed simply in these coordinates as
\[ \xi^\alpha \partial_\alpha = \partial_\tau. \]  

(15)

In these coordinates, Eqs. (8) can be written as:

\[ zu' - \dot{u} = \frac{z(u + 1)}{\rho^2} \left[ F'(zF' - \dot{F})^* + F'^*(zF' - \dot{F}) \right] \]  

(16a)

\[ u' = \frac{z(u + 1)}{\rho^2} \left[ |F'|^2 + \frac{1}{b^2} |zF' - \dot{F}|^2 \right] - \frac{u(u + 1)}{z} \]  

(16b)

\[ b' = \frac{ub}{z} \]  

(16c)

\[ 0 = F'' \Delta - \dot{F} + 2z\dot{F}' + F' \left[ z(u - 2) + \frac{b^2}{z}(u + 2) - \frac{\dot{b}}{b} \right] \]  

(16d)

\[ + \dot{F} \left( \frac{\dot{b}}{b} + 1 - u \right) + \frac{2\kappa}{\rho} F^*(\Delta F'^2 + 2zF'\dot{F} - \dot{F}^2) \]  

(16e)

where the overdot here means \( \partial/\partial \tau \) and the prime denotes \( \partial/\partial z \) and we define the functions

\[ \Delta = b^2 - z^2 \quad \rho = 1 - \kappa |F|^2. \]  

(17)

The boundary conditions we use are that the solution is regular on the time axis \( z = 0 \) and on the so called similarity horizon \( \Delta = b^2 - z^2 = 0 \). Regularity on the time axis \( z = 0 \) at the center of spherical symmetry allows us to write the boundary conditions for the metric functions \( b(\tau, z) \) and \( u(\tau, z) \) as

\[ b(\tau, 0) = 1 \quad u(\tau, 0) = 0 \]  

(18)

The hypersurface defined by \( \Delta = 0 \) is where the homothetic Killing vector becomes null. As this hypersurface is in the Cauchy development of the initial data, we expect everything to be perfectly regular there even though this is a singular point of Eqs. (16).

The existence of the homothetic Killing vector simplifies these equations somewhat. For the general collapse problem without self-similarity, the metric coefficients \( u \) and \( b \) will be functions of \( z \) and \( \tau \), but our assumed symmetry restricts these coefficients to be functions of \( z \) alone. We could also let the field \( F \) be invariant under the action of the vector field \( \xi \), but that would then fail to incorporate the \( U(1) \) symmetry which the field equations also possess. To allow for more interesting dynamics to occur, we let a \( U(1) \) transformation accompany the scale transformations (translations in \( \tau \)). Infinitesimally, this amounts to

\[ \mathcal{L}_\xi F = \xi^\alpha \partial_\alpha F = i\omega F. \]  

(19)

This allows us to give the form of \( F \) under our assumption of self-similarity as

\[ F(\tau, z) = e^{i\omega \tau} f(z). \]  

(20)

\[ ^4 \text{For completeness, we have included the field equations in the } (t, r) \text{ coordinates in the appendix. However, they are not crucial to our current discussion.} \]
The continuously self-similar (CSS) fields are now

\[ F(\tau, z) = e^{i \omega \tau} f_0(z) \]  
\[ b(\tau, z) = b_0(z) \]  
\[ u(\tau, z) = u_0(z), \]  

where \( \omega \) is an eigenvalue, determined by solving the field equations. The subscript \( _0 \) that we have appended denotes unperturbed values in anticipation of our eventually perturbing the exactly self-similar solution.

Our equations are now just Eqs. (16) with the \( \tau \) derivatives of \( u(z) \) and \( b(z) \) vanishing, \( F \) and \( F' \) being replaced by \( f_0 \) and \( f'_0 \), and \( \dot{F} \) and \( \ddot{F} \) being replaced by \( i \omega f_0 \) and \( -\omega^2 f_0 \) respectively. Note that with \( \dot{u}_0 = 0 \), we can eliminate \( u'_0 \) and we are left with an algebraic relation for \( u_0(z) \). The equations of motion now reduce to

\[ b'_0 = b_0 u_0 \]  
\[ \Delta_0 f''_0 = f'_0 \left( -2i \omega z - z(u_0 - 2) - \frac{b_0^2}{z} (u_0 + 2) - \frac{4i \kappa \omega z}{\rho_0} |f_0|^2 \right) \]  
\[ - f_0 \left( \omega^2 + i \omega (1 - u_0) \right) - \frac{2\kappa}{\rho_0} f'_0 \left( \Delta_0 f'^2_0 + \omega^2 f'^2_0 \right) \]

where we have defined

\[ \Delta_0 = b_0^2 - z^2 \]  
\[ \rho_0 = 1 - \kappa |f_0|^2 \]  
\[ u_0 = \frac{z^2}{\rho_0^2} \left( \frac{1}{b_0} |i \omega f_0 - z f'_0|^2 + |f'_0|^2 \right) + \]  
\[ \frac{z}{\rho_0^2} \left( f'_0 (i \omega f_0 - z f'_0)^* + f'^*_0 \right) \]

and where the prime now denotes \( d/dz \).

The boundary conditions at \( z = 0 \) for the CSS problem now reduce to

\[ b_0(0) = 1, \quad f_0 = \text{free real constant}, \quad f'_0(0) = 0, \]

where we have used our \( U(1) \) phase symmetry to fix \( f_0 \) as real. We define the value of \( z \) where \( \Delta_0 \) vanishes as \( z_2 \). As mentioned earlier, we demand regularity at \( \Delta_0(z_2) = 0 \) and this leads to the additional boundary conditions

\[ b_0(z_2) = z_2 = \text{free real const} \quad f_0(z_2) = \text{free complex const} \]

with the constant \( f'_0(z_2) \) being determined by Eq. (22) at the similarity horizon.

\[ ^{5}\text{It is worth pointing out that our notation here is more closely aligned with what we use in our paper} \[ ^{10}\text{on the axiodil, } \kappa = 1, \text{ than the notation in our papers} \[ ^{6,14}\text{on the complex scalar field, } \kappa = 0.} \]
Now with the equations and boundary conditions, we can numerically integrate these equations. Once we reduce our second order ODE to two first order ODEs and include the real eigenvalue $\omega$ we have five real equations and five real unknowns. We use our standard technique of solving this two-point boundary value problem by shooting with an adaptive ODE solver from both boundary points to a point $z_1$ in the middle. The free boundary values are then found using a Newton’s solver for the nonlinear matching conditions. [8]

We then follow the CSS solution as $\kappa$ varies, and we find that a CSS solution exists for

$$-0.60 \lesssim \kappa < +\infty;$$

for $\kappa = 0, 1$ the CSS solution is the same one found in previous work. Our computations actually only extend to $\kappa \leq 15$, but the behavior is smooth and the CSS solutions seem likely to extend all the way to $\kappa = \infty$. On the other hand, our calculations of CSS solutions appear to terminate somehow at $\kappa \approx -0.60$. We are unsure what exactly goes wrong there, but we tend to believe that our numerical routine fails and it is not the case that the CSS solutions cease to exist for smaller $\kappa$. It is, however, worth noting that Maison found that his sequence of CSS gas collapses terminated at a maximal value $k_{\text{max}} \approx 0.88$ where $k$ appears in the equation of state for an Eulerian fluid $p = k\rho$. The reason in his case was a change in the nature of the eigenvalues associated with the singular sonic point. At $k_{\text{max}}$, two of the eigenvalues degenerate. But we have no evidence that a similar thing occurs here.

As far as we know, there is only one eigenvalue $\omega$ possible for the CSS solution for a given $\kappa$; however we have not looked very carefully for others.

We also mention that although we describe the spacetime only up to the similarity horizon, the spacetime can be continued in these coordinates to $z = +\infty$. This corresponds to the spacelike hypersurface $t = 0$. We expect everything to be regular on this hypersurface except at the axis of spherical symmetry since it too is in the Cauchy development of the initial data. Thus the apparent singularity in our equations at $z = +\infty$ is merely a coordinate singularity. By changing coordinates, we can continue the spacetime through $t = 0$. We do not construct this extension here, but as it was possible to make this continuation for the complex scalar field and the axiodil cases, we fully expect that such a construction should be possible [6,7].

V. PERTURBATIONS AND STABILITY

As interesting as the CSS solutions are, they do not tell us everything we would like to know about the gravitational collapse. After all, these are the exactly critical solutions $p = p^*$ and comprise a set of measure zero in the space of initial conditions of the collapse. To reach them, the initial conditions must be tuned with exquisite care. In addition, such things as the critical exponents of the black hole scaling relation are found only with information gained by collapse slightly away from the critical solution.

For these reasons, we look to perturbation theory for additional understanding of the CSS solutions. It too is not the last word, but it can shed some light on questions of stability and in particular allow us to calculate the critical exponents of the black hole growth.

As described in [14], the very construction of a Choptuon involves stabilization – a balancing between subcritical dissipation and supercritical black hole formation with the critical
exponent $\gamma$ measuring the strength of this black hole/dissipation instability. More specifically, for initial data close to, but not exactly on the critical solution, the critical solution serves as an intermediate attractor with near-critical solutions approaching it but eventually running away from it to form a black hole or dissipate the field to infinity. However, in addition to this particular instability, we would like to know if there are additional instabilities which would rather drive the near-critical solutions completely away from the Choptuik to another, perhaps very different, attractor. Thus by appealing to perturbation theory, we are looking for both the black hole instability (i.e. the critical exponent) and possibly other instabilities indicating the existence of other, stronger attractors.

So, with the continuously self-similar solutions in hand, we now carry out a linear perturbation analysis of the CSS solutions, still in spherical symmetry. We define the perturbed fields as

\begin{align}
    b(\tau, z) &\approx b_0(z) + \epsilon b_1(\tau, z) \\
    u(\tau, z) &\approx u_0(z) + \epsilon u_1(\tau, z) \\
    F(\tau, z) &\approx e^{i\omega \tau} (f_0(z) + \epsilon f_1(\tau, z))
\end{align}

where again, $b_0$ denotes the 0th order critical solution, $b_1$ denotes the 1st order perturbation, $\omega$ is the (unique) eigenvalue of the unperturbed equations (which depends on the coupling constant $\kappa$), and where $\epsilon > 0$ is an infinitesimal constant, a measure of how far away the solution is from the critical solution in the space of initial conditions. Using Choptuik’s terminology, we consider the supercritical regime for infinitesimal $\epsilon \propto p - p^*$. (28)

We now perturb the Einstein equations through 1st order in $\epsilon$, to obtain a set of linear partial differential equations for the perturbed fields $b_1$, $u_1$, $f_1$, in the independent variables $\tau, z$. Following the standard approach, we Fourier transform the 1st order fields with respect to the ignorable coordinate $\tau = \log(-t)$,

\begin{align}
    \hat{u}_1(\sigma, z) &= \int e^{i\sigma \tau} u_1(\tau, z) d\tau, \\
    \hat{b}_1(\sigma, z) &= \int e^{i\sigma \tau} b_1(\tau, z) d\tau, \\
    \hat{f}_1(\sigma, z) &= \int e^{i\sigma \tau} f_1(\tau, z) d\tau;
\end{align}

throughout, $\hat{\cdot}$ will denote such a Fourier transform. The transform coordinate $\sigma$ is in general complex. The 1st order field equations now become ordinary differential equations (ODE’s) in $z$, and under appropriate boundary conditions, become an eigenvalue problem for $\sigma$. Solutions of the eigenvalue problem are then normal modes of the critical solution. Generally speaking, there will be many different normal modes $\hat{f}_1$, each belonging to a different eigenvalue $\sigma$. Eigenvalues in the lower half plane $\text{Im}\sigma < 0$ belong to unstable (growing) normal modes. Eigenvalues in the upper half $\sigma$ plane correspond to quasi-normal (dying) modes of the critical solution. The eigenvalue $\sigma$ is related to the critical exponent by $\gamma = -1/\text{Im}\sigma$. (4, 5, 14)

We now want to integrate our equations numerically so we need to determine the boundary conditions. It is important to bear in mind that in addition to solving the equation for
\( \hat{f}_1(\sigma, z) \), we must also solve the analogous equation for \( \hat{f}_1(-\sigma^*, z)^* \). Thus, we will have two second order ODE’s which must be reduced to four first order ODE’s, we will have a total of six complex equations to integrate. For the perturbation problem, the boundary conditions at \( z = 0 \) are found to be

\[
\hat{b}_1(0) = 0, \quad \hat{u}_1(0) = 0, \quad \hat{f}_1'(\sigma, 0) = 0, \quad \hat{f}_1''(\sigma^*, 0) = 0,
\]

(30)

\[
\hat{f}_1(\sigma, 0) = \text{free complex constant}, \quad \hat{f}_1'(\sigma^*, 0) = \text{free complex constant}.
\]

(31)

At the similarity horizon, \( z = z_2 \), the boundary conditions are as follows. Both \( \hat{b}_1(z_2) \) and \( \hat{u}_1(z_2) \) are free complex constants. Either \( \hat{f}_1(\sigma, z_2) \) or \( \hat{f}_1'(\sigma, z_2) \) is a free complex constant with the other describable in terms of the other boundary condition at \( z_2 \). We chose to let \( \hat{f}_1'(\sigma, z_2) \) to be free and \( \hat{f}_1(\sigma, z_2) \) fixed as this facilitated examining the lower half complex \( \sigma \) plane. The same is true for the values \( \hat{f}_1''(\sigma^*, z_2)^* \) and \( \hat{f}_1(-\sigma^*, z_2)^* \). Counting the eigenvalue \( \sigma \), we now have seven pieces of complex boundary data to go with the six complex equations we need to integrate. Since the perturbation equations are linear, we expect the solutions to scale, so the extra piece of data is merely a reflection of the linearity of the equations. Solutions will come in families which will be parameterized by a single complex parameter. Thus we have an eigenvalue problem which is well posed and which should yield a discrete spectrum of eigenvalues \( \sigma \).

To solve the 1st order problem we used a Runge-Kutta integrator with adaptive stepsize as part of a standard two point shooting method [8], shooting from \( z = 0 \) and from both boundaries and matching in the middle \( z = z_1 \). For convenience we solved the 0th order system, Eqs. (22), and the 1st order system, Eqs. (A2) simultaneously with the same steps in \( z \). As discussed elsewhere, the similarity horizon \( z_2 \) is a demanding place to enforce a boundary condition, and a second order Taylor expansion of the regular solution was used for this purpose.

To solve the 1st order system, we collected all the boundary values but \( \sigma \) into a complex 6-vector \( X \equiv (\hat{f}_1(\sigma, 0), \hat{f}_1(-\sigma^*, 0)^*, \hat{b}_1(z_2), \hat{u}_1(z_2), \hat{f}_1'(\sigma, z_2), \hat{f}_1''(\sigma^*, z_2)^*) \). Because the equations are linear, the matching conditions at \( z = z_1 \) are likewise linear in the boundary values. A solution is found when the values at \( z_1 \) of \( (\hat{b}_1, \hat{u}_1, \hat{f}_1(\sigma), \hat{f}_1(-\sigma^*)^*, \hat{f}_1'(\sigma), \hat{f}_1''(-\sigma^*)^*) \) upon integrating from \( z = 0 \) match with those found by integrating from \( z = z_2 \), for some boundary values \( X \). We can express this matching condition

\[
A(\sigma)X = 0
\]

(32)

where \( A(\sigma) \) is a \( 6 \times 6 \) complex matrix which is a nonlinear function of \( \sigma \), constructed numerically by integrations of the 1st order equations, Eqs. (A2), for six linearly independent choices of boundary values \( X \). The condition on \( \sigma \) for a solution is then

\[
\det A(\sigma) = 0.
\]

(33)

Once a value for \( \sigma \) was found that satisfies this condition, the corresponding boundary values \( X \) were found as a zero eigenvector of the matrix \( A \); these come in one (complex) parameter families, as observed above. Solution of Eqs. (A2) with boundary values \( X \) yields the normal mode itself. Now, \( A(\sigma) \) has been carefully constructed so that it is a complex
analytic solution of \( \sigma \). This follows from the fact that all equations leading to \( A \) contain \( \sigma \) but not \( \sigma^* \), together with some standard theorems about ODE’s. Moreover, \( A(\sigma) \) has no singularities in the closed lower half \( \sigma \) plane. These properties allow us to use a number of ideas from scattering theory to study \( \det A(\sigma) \). In particular, there is a theorem for counting the number \( N_C \) of zeros of \( \det A \) within any closed contour \( C \) in the closed lower half \( \sigma \) plane:

\[
\Delta_C \text{Arg} \det A = 2\pi N_C
\]  

(34)

where \( \text{Arg} \det A \) is the phase of \( \det A \), and \( \Delta_C \text{Arg} \det A \) is the total phase wrap (in radians) around the closed contour \( C \), a result similar to Levinson’s theorem for counting resonances in quantum scattering theory.

Furthermore, a conjugacy relation holds,

\[
A^*(-\sigma^*) = A(\sigma),
\]

(35)

which means that \( A \) need only be evaluated for \( \text{Re} \sigma \geq 0 \) in the lower half plane.

The nonlinear equation \( \det A(\sigma) = 0 \) was solved by the secant variant of Newton’s method [8]. The equation being complex-analytic, the 1-complex-dimensional realization of the method was used, and it performed well.

Since our field equations possess gauge invariance due to general coordinate invariance and global \( U(1) \) phase invariance, some unphysical pure gauge modes will appear at 1st order, to the extent that the gauge conditions implicit in our boundary conditions Eqs. (A0,31) fail to be unique.

A pure gauge mode arises from an infinitesimal phase rotation \( \phi \to e^{i\epsilon \phi} \) in the 0th order critical solution:

\[
\begin{align*}
\hat{b}_1(z) &= 0, \\
\hat{u}_1(z) &= 0, \\
\hat{f}_1(z) &= if_0(z).
\end{align*}
\]

(36a)

(36b)

(36c)

This gives a time independent solution of Eqs. (A2) that satisfies the boundary conditions; hence it corresponds to an unphysical mode at \( \sigma = 0 \).

Another pure gauge mode results by adding an infinitesimal constant to time \( t \to t + \epsilon \) at constant \( r \) in the 0th order critical solution. This is possible because our coordinate conditions, Eqs. (A3) normalize \( t \) to proper time along the negative time axis \( (t < 0, z = 0) \), but the zero of time is not specified. Then the solution is perturbed by

\[
\begin{align*}
b_1(\tau, z) &= \frac{\partial b_0}{\partial t}\big|_r = -(z/t)b'(\zeta) = e^{-\tau}zb'(\zeta), \\
u_1(\tau, z) &= \frac{\partial u_0}{\partial t}\big|_r = -(z/t)u'(\zeta) = e^{-\tau}zu'(\zeta), \\
f_1(\tau, z) &= e^{-i\omega \tau} \frac{\partial (e^{i\omega f_0})}{\partial t}\big|_r = e^{-\tau}(-i\omega f_0(z) + zf_0'(z)).
\end{align*}
\]

(37a)

(37b)

(37c)

This pure gauge mode has time dependence \( e^{-i\sigma \tau} = e^{-\tau} \) and so has negative imaginary \( \sigma = -i \).

There are also two more gauge modes which appear as a pair on the real axis. These come from the addition of an infinitesimal complex constant, \( c \), to our zeroth order solution: \( F \to F + \epsilon c \). The perturbed fields are then
\begin{equation}
\begin{aligned}
b_1(\tau, z) &= 0, \\
u_1(\tau, z) &= 0, \\
f_1(\tau, z) &= ce^{-i\omega \tau}.
\end{aligned}
\tag{38a,b,c,d}
\end{equation}

This mode has a time dependence of \(e^{-i\sigma \tau} = e^{-i\omega \tau}\) and so has \(\sigma = \omega\). Of course, since we have \(A^*(-\sigma^*) = A(\sigma)\), the value \(\sigma = -\omega\) will also solve the equation \(\det A = 0\) and be the fourth gauge mode.\(^6\) Thus there are these four gauge modes in the \(\sigma\) plane and no others. These modes should appear as numerical solutions, but are unphysical.

\section*{VI. RESULTS AND CONCLUSIONS}

On integrating and solving for the eigenvalues, \(\sigma(\kappa)\), we found some novel behavior. We confirmed the existence of the gauge modes thereby checking the consistency of our method. We also found the critical exponent \(\gamma(\kappa)\) over the range of \(\kappa\) values for which we found a solution. Figure 1 is a graph of this exponent as a function of the coupling constant. As can be seen, the critical exponent for the CSS solution depends strongly on the value of the coupling constant.

In addition, we evaluated \(\det A(\sigma)\) around a large rectangular contour in the lower half plane and used Eq. (34) to count the zeros lying within. This allowed us to determine if there were additional modes in the lower half \(\sigma\) plane. Our results were as follows. We did find many more modes in the complex \(\sigma\) plane. These additional modes are initially in the upper half plane for large positive \(\kappa\) and approach the real axis as \(\kappa\) decreases. Once one of these modes crosses the axis into the lower half plane we infer that the leading normal mode of the CSS solution has a change of stability. This first occurs at \(\kappa \approx 0.0754\). We thus have the following

\begin{equation}
\begin{aligned}
0.0754 &\lesssim \kappa < +\infty, & \text{CSS stable} \\
-0.60 &\lesssim \kappa \lesssim 0.0754, & \text{CSS unstable}
\end{aligned}
\end{equation}

(39)

This confirms the original discovery by Choptuik and Liebling of a change of stability at \(\omega_{BD} \approx 0\); from Eq. (7) the value would be \(\omega_{BD} \approx 0.158\).\(^{11,17}\) Note that these results are in good agreement with earlier work. The CSS solution for the complex scalar field \((\kappa = 0)\) was shown to be unstable by a similar analysis\(^{14}\) while the CSS solution for the axion-dilaton field \((\kappa = 1)\) was recently shown in\(^{16}\) to be the attractor in gravitational collapse and hence agrees with what we have found here, namely that the solution found in\(^7\) is stable. An important question is if the CSS solution becomes unstable at \(\kappa \approx 0.0754\), what is the attractor for the collapse. Our conjecture, borne out by collapse calculations of Choptuik and Liebling, is that the attractor between \(0 < \kappa \lesssim 0.0754\) (i.e. \(\omega_{BD} \gtrsim 0\)) is the

\(^6\)When we did a similar analysis for the complex scalar field, we were insensitive to these modes since we worked with the derivatives of \(\phi(\tau, z)\) so that these modes vanished identically.
more dynamically interesting discretely self-similar (DSS) or echoing solution analogous to the echoing solution originally seen by Choptuik in the collapse of a real scalar field.

Since everything in our model is smooth at $\kappa = 0$, as we decrease $\kappa$ below zero, we expect the relevant attractor for the collapse to continue to be the echoing solution. However, the above mentioned unstable mode turns out not to be the only mode to move into the lower half plane. We have evidence for more modes going unstable by $\kappa \approx -0.28$. We have been able to construct these perturbation modes and they appear to be legitimate solutions of the perturbation equations and not numerical artifacts. The presence of these additional modes suggests that the model becomes ever more unstable, that is more nonlinear, as $\kappa$ decreases. So what happens in gravitational collapse as $\kappa$ decreases below $\approx -0.28$? The CSS solution will certainly not be the attractor and the existence of additional unstable modes may trigger further bifurcations in the echoing solution. Since our calculations are limited to perturbation theory, we can not say this with certainty, but our expectation is that the echoing solution will become unstable and bifurcate into an even more dynamically complicated solution. One way to determine what happens here with greater assurance would be to take a numerical solution for the DSS solution and perform a perturbation analysis. That this would be feasible is suggested by Gundlach’s results in which he calculates the echoing solution as an eigenvalue problem resulting from the assumption of discrete self-similarity in the collapse of a real scalar field. However, a more direct approach would be to perform a full scale numerical collapse calculation in order to understand what is going on in this regime.

In this paper, we have combined a few of the previously studied models of gravitational collapse into a single model of a self-coupled complex scalar field. The model is parameterized by a single coupling constant $\kappa$. In Table 1, we give a summary of some of the key values of $\kappa$. As the value of the coupling constant decreases, the continuously self-similar solution which we find undergoes a change in stability. For the regime where the CSS solution is unstable, we believe that the attractor for gravitational collapse is an echoing and discretely self-similar solution. This change in stability which occurs near $\kappa = 0.0754$ looks like a “Hopf Bifurcation”, as it is known in the dynamics literature. As $\kappa$ continues to decrease, we find evidence for additional instabilities in the model, suggesting that there exists at least another bifurcation of the collapsing solution. From the lore on other dynamical systems, this further conjectural bifurcation might lead to a doubly periodic attractor, or might lead to full blown dynamical chaos in gravitational collapse. Additional work will be able to determine whether that is indeed the case.
Table 1. Range of the model, its relation to the Brans-Dicke model, critical exponents, and stability.

| Nonlinear BD/scalar sigma, $\kappa$ | $\omega_{BD}$ | $\gamma$ | Stability of CSS |
|-------------------------------------|--------------|----------|-----------------|
| $+\infty$                           | $-3/2$       | $\lesssim 0.14(?)$ | Stable?         |
| 10.0                                | $-1.4875$    | 0.1469   | Stable          |
| 1                                   | $-11/8$      | 0.2641   | Stable          |
| $\lesssim 0.0754$                   | $\gtrsim 0.158$ | $\gtrsim 0.373$ | Becomes Unstable |
| 0                                   | $+\infty$    | 0.3871   | Unstable        |
| $\lesssim -0.28$                    | n/a          | $\gtrsim 0.435$ | Becomes Mucho Unstable? |
| $\lesssim -0.60$                    | n/a          |           | (Don’t know if CSS exists) |

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APPENDIX A:

In this appendix, we simply list some of the equations that seemed too cumbersome to burden the main portion of the paper with. We have the general collapse equations for our model (4) in the usual $(t, r)$ coordinates.

$$\dot{u} = \frac{r(u + 1)}{\rho^2} \left[ \dot{F}^* F' + \dot{F} F'' \right]$$  \hspace{1cm} (A1a)

$$u' = \frac{r(u + 1)}{\rho^2} \left[ |F'|^2 + \frac{1}{b^2} |\dot{F}|^2 \right] - \frac{u(u + 1)}{r}$$  \hspace{1cm} (A1b)

$$b' = \frac{ub}{z}$$  \hspace{1cm} (A1c)

$$0 = r^2 \left( \frac{1}{b} \ddot{F} - \frac{b}{b^2} \dot{F} \right) - (r^2 b F'' + 2r b F' + r^2 b' F')$$  \hspace{1cm} (A1d)

$$- \frac{2\kappa r^2}{\rho} F^*(b F'^2 - \frac{1}{b} \dot{F}^2)$$  \hspace{1cm} (A1e)

where $\dot{}$ means $\partial/\partial t$ and $'$ means $\partial/\partial r$, and where

$$\rho = 1 - \kappa |F|^2.$$

The Eqs. (16) when perturbed as given in Eqs. (27) become our original set as well as the following Fourier-transformed first order equations.
\[ z\dot{u}_1 + i\sigma \dot{u}_1 = \]
\[ \frac{z(u_0 + 1)}{\rho_0^2} \left[ f_0' z f_1' - i(\omega + \sigma^*) f_1 \right] + f_0'' z f_1' - i(\omega - \sigma) f_1 \]
\[ \dot{u}_1 = \frac{z(u_0 + 1)}{\rho_0^2} \left[ f_0' z f_1' + f_0'' z f_1' \right]
\[ + \frac{1}{b_0} \left[ (z f_0' - i\omega f_0)(z f_1' - i(\omega + \sigma^*) f_1) + (z f_0' - i\omega f_0)(z f_1' - i(\omega - \sigma) f_1) \right] \]
\[ \dot{b}_1 = \frac{1}{z} (u_0 \dot{b}_1 + u_1 b_0) \]
\[ 0 = \dot{f}_1'' \Delta_0 + \dot{f}_1 \left\{ 2iz \left[ (\omega - \sigma) + \frac{2k\omega}{\rho_0} |f_0|^2 \right] + \frac{1}{z} \left[ z^2(u_0 - 2) + b_0^2(u_0 + 2) + \frac{4k\Delta_0}{\rho_0} f_0^* f_0' \right] \right\} \]
\[ + \dot{f}_1 \left\{ \left( \omega - \sigma \right) \left[ (\omega - \sigma + i(1 - u_0)) + \frac{4k}{\rho_0} f_0^*(iz f_0' + \omega f_0) \right] \right\} \]
\[ + \frac{2k^2}{\rho_0^2} \left( b_0^2 f_0'^2 + (iz f_0' + \omega f_0)^2 \right) \]
\[ + \frac{2k}{\rho_0} \left\{ 2b_0 f_0'' + \frac{4k \rho_0}{\rho_0} f_0' f_0'' + \frac{\sigma}{b_0} (\omega f_0 + iz f_0') + \frac{2b_0}{z} (u_0 + 2) f_0' \right\} \]
\[ + \dot{u}_1 \left\{ -i\omega f_0 + z f_0' + \frac{b_0^2}{z} f_0' \right\} \]
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FIGURES

FIG. 1. This is a graph of the critical exponent, $\gamma$, of the continuously self-similar solution as a function of $\kappa$. 
