Critical edge behavior and the Bessel to Airy transition in the singularly perturbed Laguerre unitary ensemble

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Abstract
In this paper, we study the singularly perturbed Laguerre unitary ensemble

$$\frac{1}{Z_n}(\det M)^\alpha e^{-\text{tr} V_t(M)} dM, \quad \alpha > 0,$$

with $V_t(x) = x + t/x$, $x \in (0, +\infty)$ and $t > 0$. Due to the effect of $t/x$ for varying $t$, the eigenvalue correlation kernel has a new limit instead of the usual Bessel kernel at the hard edge 0. This limiting kernel involves $\psi$-functions associated with a special solution to a new third-order nonlinear differential equation, which is then shown equivalent to a particular Painlevé III equation. The transition of this limiting kernel to the Bessel and Airy kernels is also studied when the parameter $t$ changes in a finite interval $(0, d]$. Our approach is based on Deift-Zhou nonlinear steepest descent method for Riemann-Hilbert problems.

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1 Introduction and statement of results

For \( n \in \mathbb{N}, \alpha > 0 \) and \( t > 0 \), we consider the following unitary random matrix ensemble

\[
\frac{1}{Z_n} (\det M)^\alpha e^{-\text{tr} V_t(M)} dM, \quad dM = \prod_{i=1}^{n} dM_{ii} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} d\text{Re} M_{ij} d\text{Im} M_{ij}
\]

on the space of \( n \times n \) positive definite Hermitian matrices \( M = (M_{ij})_{n \times n} \). Here

\[
Z_n = \int (\det M)^\alpha e^{-\text{tr} V_t(M)} dM
\]

is the normalization constant, and

\[
V_t(x) := x + \frac{t}{x}, \quad x \in (0, \infty), \quad t > 0.
\]

When \( t = 0 \), we have \( V_0(x) = x \), and (1.1) is reduced to the well-known Laguerre unitary ensemble (LUE); see, e.g., Forrester [19, Chap. 3]. In this paper, by introducing the extra term \( t/x \) in (1.3), we call (1.1) the perturbed Laguerre unitary ensemble (pLUE). We note that this pLUE has recently been considered by Chen and Its [6], where a relation with the Painlevé III (PIII, for short) function was discovered.

It is well-known, see e.g. [10, 19, 30], that the eigenvalue correlation kernel for the ensemble (1.1) has the following form

\[
K_n(x, y; t) = x^{\alpha} y^{\alpha} e^{-V_t(x) + V_t(y)} \sum_{k=0}^{n-1} p_k(x)p_k(y),
\]

where \( p_k(x) \) denotes the \( k \)-th degree orthonormal polynomial with respect to the weight

\[
w(x) = w(x; t) = x^\alpha e^{-V_t(x)}, \quad x \in (0, \infty), \quad t > 0, \quad \alpha > 0.
\]

Using the famous Christoffel-Darboux formula, (1.4) can be put into the following closed form

\[
K_n(x, y; t) = \gamma_n^{-1} \sqrt{w(x)w(y)} \frac{\pi_n(x)\pi_n(y) - \pi_{n-1}(x)\pi_n(y)}{x - y},
\]

where \( \gamma_n \) is the leading coefficient of \( p_k(x) \), and \( \pi_k(x) \) is the monic polynomial such that \( p_k(x) = \gamma_k \pi_k(x) \).

In the study of random matrices, there is a lot of interest in the limit of the correlation kernel \( K_n(x, y) \) when the matrix size \( n \) tends to infinity. For the LUE case \( (t = 0) \), the limiting mean eigenvalue density is

\[
\psi_V(x) = \lim_{n \to \infty} 4K_n(4nx, 4nx; 0) = \frac{2}{\pi} \sqrt{\frac{1 - x}{x}}, \quad x \in (0, 1);
\]

see e.g. [19, p.106]. Note that the above density is independent of \( \alpha \), and this is a typical example of the Marčenko-Pastur law; see [29]. Moreover, it is well-known that the limiting behavior of \( K_n \) is given by the sine kernel

\[
\mathcal{S}(x, y) := \frac{\sin \pi(x - y)}{x - y}
\]
in the bulk of the spectrum [20-33], by the Airy kernel
\[
A(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}
\tag{1.9}
\]
at the soft edge of the spectrum [18-39], and by the Bessel kernel
\[
J_\alpha(x, y) := \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y})\sqrt{x}J'_\alpha(\sqrt{x})}{2(x - y)}
\tag{1.10}
\]
at the hard edge of the spectrum [18-40]. In general, when \(V_0(x)\) is a polynomial instead of the simplest monomial \(x\) in (1.3), Vanlessen [41, Thm. 2.7] proved that the above limiting kernels hold as well. Of course, one may consider an even more general case by assuming that \(V_0(x)\) is a real analytic function and satisfies
\[
\lim_{x \to +\infty} \frac{V_0(x)}{\ln(x^2 + 1)} = +\infty,
\]
and the above limiting kernels are also expected. Such a phenomenon is called universality in random matrix theory. To prove these universality results, the Deift-Zhou nonlinear steepest descent method is a very powerful tool; for example, see [10, 12, 26]. It is also worth pointing out that, some kernels involving Painlevé functions have appeared in certain critical situations, see e.g. [2, 8, 9, 22, 44].

In the present paper, we will focus on the case when \(V_t(x)\) is not real analytic. To be precise, we will work on one of the simplest non-analytic cases, in which \(V_t(x)\) possess a simple pole at the hard edge 0; cf. (1.3). The exponent \(t/x\) induces an infinitely strong zero of the weight \(w(x; t)\) at the origin; cf. (1.5). Therefore, it is natural to expect that the distribution of eigenvalues near 0 may change dramatically due to the perturbation \(e^{-t/x}\), and the limiting kernel at the hard edge may no longer be the Bessel kernel \(J_\alpha\) in (1.10). Indeed, we will show that the limiting kernel is related to a third-order nonlinear differential equation. The third-order nonlinear differential equation is integrable and its Lax pair and the corresponding Riemann-Hilbert problem (RHP, for short) will be provided in Section 2 below. With the initial values adapted, later in Section 2.2, the third-order equation is shown equivalent to a particular PIII equation.

Note that, even though the perturbed weight (1.5) can be extended to a \(C^\infty(\mathbb{R})\) function when \(t > 0\), it has an essential singularity at the origin with respect to complex variable \(x\). In recent years, matrix models whose weight function has an essential singularity like (1.5) have appeared in several different areas of mathematics and physics; see, e.g., Berry and Shukla [1] in the study of statistics for zeros of the Riemann zeta function, Lukyanov [28] in a calculation of finite temperature expectation values in integrable quantum field theory, and [4, 32, 38] in the study of the Wigner time delay in quantum transport. Wigner delay time stands for the average time that an electron spends when scattered by an open cavity and is of fundamental importance in the theory of mesoscopic quantum dots. As suggested in [32, 38], the distribution of the Wigner delay time is far from being understood and several interesting questions remain open.

In the Laguerre ensemble, the Wigner time delay is given by the sum of random variables \(\tau_j\) such that \(1/\tau_j\) are distributed like the eigenvalues of matrices; see [4, 38]. The partition function, i.e., the quantity defined in (1.2), serves as the moment generating function of the probability density of the Wigner delay time \(\tau\) [8, 32].

Due to the influence of the essential singularity in the weight function, the asymptotic analysis of the above matrix models is very different from analytic cases as well as other singular
cases, such as weights with jump discontinuities (see, e.g., [21, 12]), and weights with weak or algebraic singularities (see, e.g., [22, 23, 43]). The first attempt to study asymptotics of matrix models with an essential singularity was done by Mezzadri and Mo [31] and Brightmore et al. [3] when they are considering asymptotic properties of the partition function (the normalization constant $Z_n$ in our notation) associated with the following weight

$$w(x; z, s) = \exp \left( -\frac{z^2}{2x^2} + \frac{s}{x} - \frac{x^2}{2} \right), \quad z \in \mathbb{R} \setminus \{0\}, \quad 0 \leq s < \infty, \quad x \in \mathbb{R}. \quad (1.11)$$

Here $x = 0$ is an essential singular point. As pointed out in [3, 31], when $\alpha = \pm \frac{1}{2}$ and $s = 0$, the system of polynomials orthogonal with respect to (1.5) and (1.11) can be mapped to each other by a change of variables, however, the respective partition functions are still different. In [3], Brightmore et al. showed that a phase transition emerges as the matrix size $n \to \infty$ and $s, z = O(1/\sqrt{n})$. They also obtained asymptotics of the partition function $Z_n$, which is characterized by a solution of a PIII equation. Although the PIII equation in [3] is not the same as that in (1.20) below, similar phase transitions are observed; c.f. Theorems 1-3. We think this may reflect a new class of universality under the effect of the essential singularity. It is worth mentioning that, as in the current paper, the Deift-Zhou nonlinear steepest descent method is also used as one of the main tools in [3]. The interested reader may compare different model RH problems in Section 2 and [3, Sec. 4.3], which are employed in the construction of local parametrices near the essential singular points.

It is also interesting to study this problem from a polynomial point of view. In fact, Chen and Its [6] use (1.5) as a concrete and important example of the Pollaczek-Szegö type orthogonal polynomials, supported on infinite intervals. The Hankel determinant, which is the normalizing constant $Z_n$ in (1.2), plays a fundamental role in [6], upon which the main results are derived and stated. A relation is also found between Hankel determinant and the Jimbo-Miwa-Ueno isomonodromy $\tau$-function.

For orthogonal polynomials with a certain weight $w(x)$ supported on $[-1, 1]$, if the Szegö condition

$$\int_{-1}^{1} \frac{\ln w(x)}{\sqrt{1-x^2}} dx > -\infty$$

is fulfilled, then the weight is said to be of Szegö class. The classical and the modified Jacobi weights belong to the Szegö class; cf. [25]. While the Pollaczek weight furnishes a well-studied non-Szegö class example, defined as

$$w(x; a, b) = \frac{e^{(2\theta - \pi)h(\theta)}}{\cosh[\pi h(\theta)]}, \quad \theta = \arccos x, \quad x \in (-1, 1),$$

where $h(\theta) = \frac{a \cos \theta + b}{2 \sin \theta} = \frac{ax + b}{2 \sqrt{1-x^2}}$, and $a, b$ are real constants such that $|b| < a$. Asymptotics for the corresponding orthogonal polynomials can be found in [37, pp. 296-312]; see also [46], where a Riemann-Hilbert analysis has been carried out.

It is readily seen that the Pollaczek weight just violates the Szegö condition since

$$w(x; a, b) \sim 2e^{a+b} e^{-\frac{C \pm}{\sqrt{1-x}}} \quad \text{as} \quad x \to 1^-, \quad \text{and} \quad w(x; a, b) \sim 2e^{a-b} e^{-\frac{C \pm}{\sqrt{1+x}}} \quad \text{as} \quad x \to -1^+, \quad \text{where} \quad C_\pm = \pi(a \pm b)/\sqrt{2} > 0.$$
The non-Szegő class weights demonstrate a singular behavior, as compared with the classical polynomials; see Szegő [37, pp. 296-312]. For example, such a singular behavior is shown in the extreme zeros. For the Pollaczek polynomials, the gap between the largest zero and the endpoint 1 is of the order of magnitude $O(1/n)$, much bigger that the distance $O(n^{-4/3})$ between consecutive extreme zeros. This fact is noticeable as compared with the Jacobi polynomials: in the Jacobi case, both quantities are of the same order of magnitude $O(1/n^2)$. Hence, distinguishing the soft edge with the hard edge is not necessary in the Jacobi case. While it is not so in the Pollaczek case. Such a singular behavior is closely connected to the determination of the equilibrium measure, and to the 'soft edge' appearing in later sections; cf. $\alpha_n$ in (6.30).

It is worth mentioning that certain Szegő class weights may also show a singular behavior, very similar to the Pollaczek case; see [45] for an asymptotic analysis in such cases, and see [44] for an application in random matrix theory.

Now we can see from a polynomial point of view that for $t > 0$ fixed, the weight $w(x; t)$ in (1.5) is of non-Szegő class, and the asymptotic behavior of the polynomials at the edge $x = 0$ is expected to be described in terms of the Airy function, as in the Pollaczek case; cf. [16]. While in the limiting case $t = 0$, the weight in (1.5) reduces to the classical Laguerre weight, a typical Szegő class case, and the asymptotics are described in terms of the Bessel functions; see [25, 26]. The really interesting piece here, might be the Bessel to Airy transition, as the parameter $t$ shifts from $t = 0$ to a fixed positive number. To achieve such a transition, it is desirable to carry out a large-$n$ asymptotic analysis for the orthogonal polynomials as $t = t_n \to 0$, or, eventually, uniformly for $t \in (0, d]$, with $d$ fixed and positive. Note that, the Bessel to Airy transition in our paper is totally different from that in Claeys and Kuijlaars [8], where the PII functions appear.

In the same paper [6], along with an investigation of the Hankel determinants and several relevant statistic quantities, an observation is made, that for fixed degree $n$, the corresponding polynomials are related to the PIII equation in a straightforward manner. Indeed, Chen and Its [3] apply the Riemann-Hilbert formulation of the orthogonal polynomials and the theory of Jimbo-Miwa, to represent the polynomials orthogonal with respect to weight (1.5) via the Jimbo-Miwa Lax pair for the PIII equation, with parameters depending on the polynomial degree $n$. Yet it is not easy to extract asymptotic approximations from such a PIII representation. It is noted in [3] that the asymptotics of the polynomials for large degree $n$ will provide valuable insight into the asymptotics of the Painlevé transcendent related. However, to the best of our knowledge, there is no results on the asymptotics of the polynomials so far.

There are other examples where Painlevé equations PI-PVI are involved. For instance, the three-term recurrence coefficients are shown related to PV in [5] for a Pollaczek-Jacobi type weight, and to PIV in [14] for the semi-classic Laguerre weight. Also, in recent papers [7, 11, 24], transition type uniform asymptotics of Hankel determinants have been considered.

The main objective of the present paper is to obtain the limit behavior of the kernel $K_n(x, y)$ at the edge of the spectrum for $t \in (0, d]$. Much attention will be paid to the transition of the edge behavior between the Bessel kernel and the Airy kernel, as the parameter $t = t_n$ varies between $t = 0$ and a fixed $d > 0$. To achieve our goal, we derive the large degree asymptotic behavior of the orthogonal polynomials with respect to the weight (1.5), uniformly for $t = t_n \in (0, d]$, where $d$ is a positive constant. We use the Deift-Zhou nonlinear steepest descent method (also termed the Riemann-Hilbert approach) to serve the purpose.

It is also of interest to further consider, in a separate paper, several asymptotic quantities such as the Hankel determinant, the three-term recurrence coefficients, the leading coefficients and extreme zeros of the corresponding orthogonal polynomials, in these quantities the switch-on and switch-off of a singular behavior might be observed, as the parameter $t$ varies in $(0, d]$. 
1.1 A third-order nonlinear differential equation and its reduction to PIII

To state our results, we need to introduce a scalar function \( r(s) \) for \( s \in [0, +\infty) \). This function solves the following third-order nonlinear differential equation

\[
2s^2r'r''' - s^2r''^2 + 2sr'r'' - 4sr'^3 + \left(2r + 2l - \frac{1}{4}\right)r'^2 + 1 = 0. \tag{1.12}
\]

The above equation is integrable. Its Lax pair is given as follows.

**PROPOSITION 1.** The equation (1.12) is the compatibility condition \( \Psi_{s\zeta} = \Psi_{\zeta s} \) of the following Lax pair

\[
\Psi_{\zeta}(\zeta, s) = \left( A_0(s) + \frac{A_1(s)}{\zeta} + \frac{A_2(s)}{\zeta^2} \right) \Psi(\zeta, s), \tag{1.13}
\]

\[
\Psi_s(\zeta, s) = \frac{B_1(s)}{\zeta} \Psi(\zeta, s), \tag{1.14}
\]

where

\[
A_0(s) = \begin{pmatrix} 0 & 0 \\ \frac{i}{2} & 0 \end{pmatrix} := \frac{i}{2} \sigma_-, \quad A_1(s) = \begin{pmatrix} -\frac{1}{4} + \frac{1}{2}r(s) & -\frac{i}{2} \\ -i\varphi(s) & \frac{1}{4} - \frac{1}{2}r(s) \end{pmatrix}, \quad A_2(s) = -sB_1(s). \tag{1.15}
\]

and

\[
B_1(s) = \begin{pmatrix} q'(s) & -ir'(s) \\ iq'(s) & -q'(s) \end{pmatrix}. \tag{1.16}
\]

Here the functions \( t(s) \) and \( q(s) \) are determined in terms of \( r(s) \) by

\[
t'(s) = \frac{1 - q'(s)^2}{r'(s)}, \tag{1.17}
\]

\[
q(s) = -sr'(s) + \frac{1}{2}r(s) + \frac{1}{2}r^2(s) + l, \tag{1.18}
\]

and \( l \) is a constant.

An observation shows that the equation (1.12) can be reduced to a certain PIII equation. A proof will be given in Section 2.2.

**PROPOSITION 2.** By a change of unknown function

\[
v(s) = sr'(s), \tag{1.19}
\]

the third-order equation (1.12) for \( r(s) \) is reduced to a particular PIII equation for \( v(s) \), namely,

\[
v'' = \frac{v'^2}{v} - \frac{v'}{s} + \frac{v^2}{s^2} + \frac{\alpha}{s} - \frac{1}{v}, \tag{1.20}
\]

cf., e.g., [6, (3.12)] for the PIII equation.

We note that the coefficients \( l \) and \( \alpha \), respectively in (1.12) and (1.20), are determined by initial values of \( r(s) \); see Section 2.2 below. In the present paper, the constant \( l \) in (1.12) and (1.18) is equal to 0.

We need a special solution of (1.12) which has no poles on \((0, +\infty)\). Indeed, we have
PROPOSITION 3. There exists a solution \( r(s) \) of (1.12), analytic for \( s \in (0, +\infty) \), with the following boundary behaviors

\[
r(0) = \frac{1}{8}(1 - 4\alpha^2), \quad \text{and} \quad r(s) = \frac{3}{2}s^{\frac{3}{2}} - \alpha s^{\frac{3}{4}} + O(1) \quad \text{as} \quad s \to +\infty. \tag{1.21}
\]

The existence of such an \( r(s) \) follows directly from the vanishing lemma of the corresponding RH problem in Section 2.3 below. The initial value \( r(0) \) is determined in (5.20), and the asymptotic behavior at infinity is given in (6.29).

Given \( r(s), q(s) \) and \( t(s) \), the solutions of

\[
\frac{\partial}{\partial \zeta} \begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix} = \begin{pmatrix} A_0(s) + \frac{A_1(s)}{\zeta} + \frac{A_2(s)}{\zeta^2} \end{pmatrix} \begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix} \tag{1.22}
\]

are analytic in the complex \( \zeta \)-plane with an essential singularity and a possible branch point at \( \zeta = 0 \). We define \( \begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix} \) as the unique solution of the above equation with asymptotics

\[
\begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix} = \begin{bmatrix} I + O(\zeta^{-1}) \zeta^{-\frac{1}{2}\alpha_3} I + i\sigma_1 \sqrt{2} e^{\frac{1}{2}i(\alpha - 1)\sigma_3} (1) \\ 1 \end{bmatrix} \quad \text{as} \quad \zeta \to \infty, \tag{1.23}
\]

and

\[
\begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix} = e^{\frac{1}{2}i(\alpha - 1)\zeta^{\frac{1}{2}}} e^{\frac{i}{2}Q(s)} \begin{bmatrix} (1) \\ 0 \end{bmatrix} + O(\zeta) \quad \text{as} \quad \zeta \to 0. \tag{1.24}
\]

Both approximations are uniform in the sector \(-\pi + \delta \leq \arg \zeta \leq -\frac{\pi}{2} - \delta \) for small \( \delta > 0 \). In the above two formulas, the branches of powers of \( \zeta \) are chosen such that \( \arg \zeta \in (-\pi, \pi) \), \( \sigma_1 \) and \( \sigma_3 \) are the Pauli matrices; cf. (2.4) below, and \( Q(s) \) is independent of \( \zeta \), such that \( \det Q(s) = 1 \). The uniqueness of the pair of functions is justified since they are recessive solutions at the origin in the above sector, so long as \( s > 0 \); cf. (1.24). The functions \( \psi_1(\zeta, s) \) and \( \psi_2(\zeta, s) \) can be extended to analytic functions in \( \arg \zeta \in \mathbb{R} \), and will appear in our main results; see also (4.4) and (2.1)-(2.3) for an alternative definition. Note that they are well-defined for \( s \in (0, +\infty) \) due to Proposition 3.

1.2 Main results

Now we are ready to present out main results.

Limiting kernel at the hard edge

The first main result is the \( \Psi \)-description of the limit of the re-scaled kernel \( 4n K_n(4nx, 4ny; t) \), where \( K_n(x, y; t) \) is the polynomial kernel appeared in (1.6), associated with the weight (1.5). We focus on the large-\( n \) behavior of the kernel near the edge \( x = 0 \):

**THEOREM 1.** Let \( K_n(x, y; t) \) be the kernel given in (1.6), then it has the \( \Psi \)-kernel asymptotic approximation

\[
\frac{1}{4n} K_n \left( \frac{u}{4n}, \frac{v}{4n}; t \right) = K_\Psi(u, v, 2nt) + O \left( \frac{1}{n^2} \right) \tag{1.25}
\]

as \( n \to \infty \), uniformly for \( u, v \) in compact subsets of \((0, \infty)\) and uniformly for \( t \) in \([0, d]\), where \( d \) is a positive constant. And the \( \Psi \)-kernel is given by

\[
K_\Psi(u, v, s) = \frac{\psi_1(-v, s)\psi_2(-u, s) - \psi_1(-u, s)\psi_2(-v, s)}{2\pi i(u - v)} \tag{1.26}
\]
where the scalar function $\psi_k(\zeta, s)$, $k = 1, 2$, are defined in (1.22)-(1.24).

Accordingly, the following result holds:

**COROLLARY 1.** Let $K_n(x, y)$ be the kernel given in (1.6). If the parameter $t \to 0$ and $n \to \infty$ in the way such that

$$\lim_{n \to \infty} 2nt = \tau, \quad \tau \in (0, \infty),$$

we have the double scaling limit for $K_n(x, y)$ given in terms of the $\Psi$-kernel defined in (1.26)

$$\lim_{n \to \infty} \frac{1}{4n} K_n \left( \frac{u}{4n}, \frac{v}{4n}; t \right) = K_\Psi(u, v, \tau)$$

(1.27)

uniformly for $u$, $v$ and $\tau$ in compact subsets of $(0, \infty)$.

**REMARK 1.** The reader may find it a little confusing to see the quantity $2nt$ on the right-hand-side of (1.25). The reason why we put (1.25) in its current form is to describe the phase transition when the parameter $t$ varies in the interval $(0, d]$ (or, equivalently the parameter $s$ in the interval $(0, +\infty)$ if we let $s = 2nt$). As one can see in the results here, when $t \sim \frac{1}{n}$ (i.e. $s \sim 1$), we simply get the $\Psi$-kernel in (1.26); when $t = o\left(\frac{1}{n}\right)$ (i.e. $s \to 0+$), the $\Psi$-kernel is reduced to the Bessel kernel in (1.10); when $nt \to \infty$ (i.e. $s \to +\infty$) as $n \to \infty$, the $\Psi$-kernel is then reduced to the Airy kernel in (1.9).

**Transition to the Bessel kernel**

The case dealt with in Theorem 1 is for the parameter $s = 2nt$ in compact subsets of $(0, \infty)$. It is of interest to consider the possible transition of the $\Psi$-kernel in (1.25) as $s \to 0^+$ and $s \to +\infty$. Indeed, by a nonlinear steepest descent analysis of the model RH problem for small $s$, we have

**THEOREM 2.** We obtain the Bessel type limit for small parameter.

(a) The $\Psi$-kernel is approximated by the Bessel kernel as $s \to 0^+$

$$K_\Psi(u, v, s) = \mathbb{J}_\alpha(u, v) + O(s),$$

(1.28)

where the error term is uniform for $u$ and $v$ in compact subsets of $(0, \infty)$. The Bessel kernel $\mathbb{J}_\alpha$ is defined in (1.10).

(b) If the parameter $t \to 0^+$ and $n \to \infty$ such that

$$\lim_{n \to \infty} 2nt = 0,$$

we have the Bessel kernel limit for $K_n$:

$$\lim_{n \to \infty} \frac{1}{4n} K_n \left( \frac{u}{4n}, \frac{v}{4n}; t \right) = \mathbb{J}_\alpha(u, v),$$

(1.29)

uniformly for $u$ and $v$ in compact subsets of $(0, \infty)$. 

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Transition to the Airy kernel

It is even more interesting to study the asymptotic properties of the model RH problem for \( \Psi(\zeta, s) \) when \( s \to \infty \), and to describe the transition of the \( \Psi \)-kernel to the Airy kernel. Indeed, when the \( \Psi \) function is analyzed in the scale \( \Psi(s^{2/3} \lambda, s) \) for large \( s \), the behavior of it can be described at infinity in elementary functions, and at \( \lambda = -1 \) via the Airy function. From this fact we eventually obtain an Airy kernel limit as \( s \to \infty \). The results are summarized in the following theorem.

**THEOREM 3.** We obtain the Airy type limit for large parameter.

(a) The \( \Psi \)-kernel is approximated by the Airy kernel as \( s \to +\infty \)

\[
\frac{s^{4/9}}{c} K_\Psi \left( s^{2/3} \left( 1 - \frac{u}{cs^{2/9}} \right), s^{2/3} \left( 1 - \frac{v}{cs^{2/9}} \right), s \right) = \mathbb{A}(u, v) + O \left( s^{-2/9} \right),
\]

where the error term is uniform for \( u \) and \( v \) in compact subsets of \(( -\infty, \infty )\), \( c = \left( \frac{3}{2} \right)^2 \) and the Airy kernel \( \mathbb{A} \) is defined in (1.9).

(b) If the parameter \( t \in (0, d] \) and \( n \to \infty \) such that

\[
\lim_{n \to \infty} 2nt = \infty,
\]

we have the Airy kernel limit for \( K_n \):

\[
\lim_{n \to \infty} \frac{\alpha_n}{cs^{2/9}} K_n \left( \alpha_n \left( 1 - \frac{u}{cs^{2/9}} \right), \alpha_n \left( 1 - \frac{v}{cs^{2/9}} \right); t \right) = \mathbb{A}(u, v),
\]

where \( s = 2nt, \alpha_n = s^{2/3}/(4n) = 2^{-4/3}n^{-1/3}t^{2/3}, c = \left( \frac{3}{2} \right)^2 \) and the formula holds uniformly for \( u \) and \( v \) in compact subsets of \(( -\infty, \infty )\).

**REMARK 2.** We note that the constant \( \alpha_n \) appearing in (1.31) indicates the position of the soft edge. Indeed, the equilibrium measure with potential \( \frac{1}{n} \left( x + \frac{t}{2x} \right), x \in (0, \infty) \) can be computed, and the support of the equilibrium measure turn out to be \(( \alpha_n, 4n )\). From the perspective of Riemann-Hilbert approach to the universality of random matrices, the asymptotic behavior of the kernel \( K_n \) at the soft edge is expected to be described in terms of the Airy kernel.

The rest of the paper is arranged as follows. In Section 2 we formulate the model RH problem for \( \Psi(\zeta, s) \), prove its solvability for \( x \in (0, \infty) \). We also derived a Lax pair for \( \Psi(\zeta, s) \), and show that the compatibility of the Lax pair leads to a third-order nonlinear ordinary differential equation. In Section 3 we carry out, in full details, the Riemann-Hilbert analysis of the polynomials orthogonal with respect to the weight functions (1.5). Section 4 will be devoted to the proof of Theorem 1 based on the asymptotic results of the model problem \( \Psi(\zeta, s) \) and of the RH problem associated with the weight (1.5). In Section 5 we investigate the \( \Psi \)-kernel to Bessel kernel transition and prove Theorem 2. In the last section, Section 6 we consider the \( \Psi \)-kernel to Airy kernel transition and prove Theorem 3. Thus we complete the Bessel to Airy transition as the parameter \( t \) in (1.5) varies from left to right in a finite interval \((0, d]\).

2 A model Riemann-Hilbert problem

We formulate a model RH problem, which will play a crucial role later in the steepest descent analysis. The model problem for \( \Psi(\zeta) = \Psi(\zeta, s) \) is the following:
Figure 1: Contours and regions for the model RH problem for \( \Psi \) in the \( \zeta \)-plane, where both sectors \( \Omega_2 \) and \( \Omega_3 \) have an opening angle \( \pi/3 \).

(a) \( \Psi(\zeta) \) is analytic in \( \mathbb{C} \setminus \bigcup_{j=1}^{3} \Sigma_j \), where \( \Sigma_j \) are illustrated in Figure 1.

(b) \( \Psi(\zeta) \) satisfies the jump condition

\[
\Psi^+(\zeta) = \Psi^-(\zeta)
\]

\[
\begin{cases}
\left( \begin{array}{cc}
1 & 0 \\
e^{\pi i \alpha} & 1
\end{array} \right), & \zeta \in \Sigma_1, \\
\left( \begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array} \right), & \zeta \in \Sigma_2, \\
\left( \begin{array}{cc}
1 & 0 \\
e^{-\pi i \alpha} & 1
\end{array} \right), & \zeta \in \Sigma_3;
\end{cases}
\]

(2.1)

(c) The asymptotic behavior of \( \Psi(\zeta) \) at infinity is

\[
\Psi(\zeta, s) = \left[ I + \frac{C_1(s)}{\zeta} + O\left( \frac{1}{\zeta^2} \right) \right] \zeta^{-\frac{1}{2} \sigma_3} I + i \sigma_1 e^{\sqrt{\zeta} \sigma_3}, \quad \arg \zeta \in (-\pi, \pi), \quad \zeta \to \infty,
\]

where \( C_1(s) \) is a matrix independent of \( \zeta \);

(d) The asymptotic behavior of \( \Psi(\zeta) \) at \( \zeta = 0 \) is

\[
\Psi(\zeta, s) = Q(s) \left\{ I + O(\zeta) \right\} e^{\frac{\zeta}{2} \sigma_3} \zeta^\sigma_3
\]

\[
\begin{cases}
I, & \zeta \in \Omega_1 \cup \Omega_4, \\
\left( \begin{array}{cc}
1 & 0 \\
e^{\pi i \alpha} & 1
\end{array} \right), & \zeta \in \Omega_2, \\
\left( \begin{array}{cc}
1 & 0 \\
e^{-\pi i \alpha} & 1
\end{array} \right), & \zeta \in \Omega_3
\end{cases}
\]

(2.2)
for \( \arg \zeta \in (-\pi, \pi) \), as \( \zeta \to 0 \), where \( \Omega_1 - \Omega_4 \) are depicted in Figure 1. \( Q(s) \) is a matrix independent of \( \zeta \), such that \( \det Q(s) = 1 \), and \( \sigma_j \) are the Pauli matrices, namely,

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

(2.4)

2.1 Proof of Proposition 1 and the Lax pair for \( \Psi \)

In this subsection, we will prove Proposition 1 and show that the above RH problem gives us the Lax pair given in (1.13) and (1.14). The idea of the proof can be found in Fokas et al. [15, Chap. 5].

Proof of Proposition 1. Since \( \det \Psi(\zeta,s) \equiv 1 \) by Liouville’s theorem, we have \( \text{tr} C_1(s) = 0 \) in (2.2). Therefore, we may denote

\[
C_1(s) = \begin{pmatrix} q(s) & -ir(s) \\ it(s) & -q(s) \end{pmatrix}.
\]

(2.5)

All jumps in (2.1) are independent of \( \zeta \) and \( s \). Hence we see that both \( \Psi_\zeta \Psi^{-1} \) and \( \Psi_s \Psi^{-1} \) are analytic in the whole complex \( \zeta \)-plane, except two possible isolated singularities at \( \zeta = 0 \) and \( \zeta = \infty \).

From (2.2) and (2.3), it is easily verified that \( \Psi_s \Psi^{-1} \) has a removable singularity at \( \infty \) and at most a simple pole at the origin. This gives us \( \Psi_s = \left( B_0(s) + \frac{B_1(s)}{\zeta} \right) \Psi \). Moreover, a careful calculation from (2.2) yields

\[
\Psi_s(\zeta)\Psi^{-1}(\zeta) \sim \frac{C_1'(s)}{\zeta} \quad \text{as} \quad \zeta \to \infty,
\]

which implies that \( B_0(s) = 0 \) and \( B_1(s) = C_1'(s) \). Therefore, (1.14) follows. In addition, from (2.3) we have

\[
\Psi_s(\zeta)\Psi^{-1}(\zeta) \sim \frac{1}{\zeta} Q(s)\sigma_3 Q(s)^{-1} \quad \text{as} \quad \zeta \to 0.
\]

(2.6)

Hence we obtain another representation for \( B_1 \), namely, \( B_1(s) = Q(s)\sigma_3 Q(s)^{-1} \), so that \( \det C_1(s) = \det B_1(s) = -1 \), which in turn gives

\[
q'(s)^2 + r'(s)t'(s) = 1.
\]

(2.7)

Similarly, from (2.2) and (2.3) we see that \( \Psi_\zeta \Psi^{-1} \) has a removable singularity at infinity and a possible double pole at \( \zeta = 0 \). Writing \( \Psi_\zeta(\zeta,s) = \left( A_0(s) + \frac{A_1(s)}{\zeta} + \frac{A_2(s)}{\zeta^2} \right) \Psi(\zeta,s) \), from (2.2) we have

\[
\Psi_\zeta(\zeta)\Psi^{-1}(\zeta) \sim i\sigma_- + \frac{1}{\zeta} \left( -\frac{1}{4} \sigma_3 + \frac{i}{2} [C_1, \sigma_-] - \frac{i}{2} \sigma_+ \right) \quad \text{as} \quad \zeta \to \infty,
\]

with \([A,B] = AB - BA\), and, from (2.3)

\[
\Psi_\zeta(\zeta)\Psi^{-1}(\zeta) \sim -\frac{s}{\zeta^2} Q(s)\sigma_3 Q(s)^{-1} \quad \text{as} \quad \zeta \to 0.
\]

The above two formulas give us (1.13), with coefficient matrices given in (1.15). Here use has been made of the relation \( A_2(s) = -sQ(s)\sigma_3 Q(s)^{-1} \).
Next, we will derive the differential equation for $r(s)$ in (1.12) from the compatibility condition of (1.13) and (1.14). The compatibility condition, namely $\Psi s = \Psi s$, is now equivalent to
\[
\frac{\partial}{\partial s} \left( A_0(s) + \frac{A_1(s)}{\zeta} + \frac{A_2(s)}{\zeta^2} \right) - \frac{\partial}{\partial \zeta} \left( B_1(s) \right) + \left[ A_0(s) + \frac{A_1(s)}{\zeta} + \frac{A_2(s)}{\zeta^2}, B_1(s) \right] = 0,
\]
that is,
\[
A_0'(s) + \frac{A_1'(s) + [A_0(s), B_1(s)]}{\zeta} + \frac{A_2'(s) + B_1(s) + A_1(s), B_1(s)}{\zeta^2} + \frac{[A_2(s), B_1(s)]}{\zeta^3} = 0.
\]
Using (1.16) and (1.15), we see that only the $\zeta^{-2}$ term remains, and the last equation is thus reduced to
\[
- s C_1''(s) + [A_1(s), B_1(s)] = 0.
\]
The above equation, together with (2.7), gives us the following equations for unknown scalar functions $q(s), r(s)$ and $t(s)$
\[
\begin{align*}
qs''(s) &= q(s)r'(s) + \frac{1}{2}t'(s) \\
qs'r''(s) &= -q'(s) - \frac{1}{2}r'(s) + r(s)r'(s) \\
qs''(s) &= -2q(s)q'(s) + \frac{1}{2}t'(s) - r(s)t'(s) \\
q'(s)^2 + r'(s)t'(s) &= 1.
\end{align*}
\]
(2.8)

Note that, although there are 4 equations for 3 unknown functions, one of the equations is redundant. For example, the third equation in (2.8) can be deduced from a combination of the first two equations, with the equation obtained by taking derivative of the fourth equation, and hence the third equation may be removed. Now integrating the second equation in (2.8) yields
\[
q(s) = -sr'(s) + \frac{1}{2}r^2(s) + \frac{1}{2}r^2(s) + l,
\]
where $l$ is a constant. In the present case, $l = 0$, as can be seen from the initial conditions in (5.20). Eliminating $t(s)$ from the first and the last equation in (2.8) gives
\[
-qs''(s)r'(s) + \frac{1}{2} = -q(s)r'^2(s) + \frac{1}{2}q'^2(s).
\]
Further eliminating $q(s)$ from the last two equations yields the following third-order nonlinear differential equation for $r(s)$
\[
2s^2r''' - s^2r'' + 2sr' - 4sr^3 + \left(2r^2 + 2l - \frac{1}{4}\right)r^2 + 1 = 0,
\]
where the constant $l = 0$.

### 2.2 Proof of Proposition 2: Reduction to PIII

As is shown in Chen and Its [6], various quantities, such as the three-term recurrence coefficients of the associated orthogonal polynomials, are expressed in terms of a specific solution to a PIII equation. On the other hand, in the present paper, the previous derivation indicates that the third-order equation would play the same role. Hence, a reduction of (1.12) to (1.20) is expected, as established in Proposition 2.

Now we prove Proposition 2 by showing (1.12) and (1.20) are equivalent.
Reduction \(1.20 \implies 1.12\)

Substituting \(v = sr'\) into \(1.20\), we obtain an alternative third-order equation
\[
s^2 r'' - s^2 r'' + sr'' - sr^3 - \alpha r' + 1 = 0.
\] (2.10)

Multiplying \(2r''/r^3\), and grouping the terms, we can put (2.10) into the form of a differential
\[
\frac{d}{ds} \left\{ \frac{s^2 r''}{r^2} - 2sr' + \frac{2\alpha}{r} - \frac{1}{r^2} + 2r \right\} = 0.
\] (2.11)

Integrating (2.11) yields
\[
s^2 r'' - 2sr'^3 + 2\alpha r' - 1 + 2rr'^2 = \left( \frac{1}{4} - 2l \right) r'^2,
\] (2.12)

with an integral constant
\[
l = \frac{1}{8} - r(0) - \frac{\alpha}{r'(0)} + \frac{1}{2r'(0)^2}.
\] (2.13)

Multiplying (2.10) by 2 and adding it to (2.12) give us (1.12). Here, we note that in the present case, with the special initial values \(r(0) = \frac{1}{8} (1 - 4\alpha^2)\) and \(r'(0) = \frac{1}{\alpha}\), the constant \(l\) in (2.13) vanishes.

Reduction \(1.12 \implies 1.20\)

Reversely, we proceed to show that (1.12) implies (2.10). To this aim, we denote by \(\Lambda\) the left hand side of (2.10). In view of the linear dependence of (2.9), (2.10) and (2.12) mentioned above, paying attention to the equivalence of (2.10) and (2.11), we see that
\[
\left( \frac{2}{r^2} \Lambda \right)' = -\frac{2r''}{r^3} \Lambda.
\] (2.14)

Solving the equation, we have
\[
\Lambda = \alpha_1 r', \quad \alpha_1 = \alpha - \frac{1}{r'(0)},
\] (2.15)

where the constant \(\alpha_1\) is determined by comparing both sides at \(s = 0\). For the chosen initial value \(r'(0) = \frac{1}{\alpha}\), the constant vanishes, thus we get \(\Lambda = 0\), which is the third-order equation (2.10). Finally, substituting \(v = sr'\), or, equivalently, \(r' = v/s\), into (2.10), we obtain the PIII equation (1.20).

2.3 Solvability of the model Riemann-Hilbert problem

We proceed to justify the solvability of the RH problem for \(\Psi(\zeta,s)\), by proving a vanishing lemma.

**Lemma 1.** Assume that the homogeneous RH problem for \(\Psi^{(1)}(\zeta,s)\) adapting the same jump conditions (2.1) and the same boundary condition (2.3) as \(\Psi(\zeta,s)\), with the behavior (2.2) at infinity being altered to
\[
\Psi^{(1)}(\zeta,s) = O \left( \frac{1}{\zeta} \right) \zeta^{-\frac{1}{4} \sigma_3} \frac{I + i\sigma_1}{\sqrt{2}} e^{\sqrt{2} \sigma_3}, \quad \text{arg}\zeta \in (-\pi, \pi), \quad \zeta \to \infty.
\] (2.16)
If the parameter $s \in (0, +\infty)$, then $\Psi(\zeta, s)$ is trivial, that is, $\Psi \equiv 0$.

**Proof.** First, we remove the exponential factor at infinity and eliminate the jumps on $\Sigma_1$ and $\Sigma_3$ by defining

$$
\Psi^{(2)}(\zeta) = \begin{cases} 
\Psi^{(1)}(\zeta)e^{-\sqrt{\zeta}\sigma_3}, & \zeta \in \Omega_1 \cup \Omega_4, \\
\Psi^{(1)}(\zeta)e^{-\sqrt{\zeta}\sigma_3}\left( \begin{array}{cc} 1 & \frac{1}{e^{\pi i\alpha}e^{-2\sqrt{\zeta}}} \\ e^{\pi i\alpha}e^{-2\sqrt{\zeta}} & 1 \end{array} \right), & \zeta \in \Omega_2, \\
\Psi^{(1)}(\zeta)e^{-\sqrt{\zeta}\sigma_3}\left( \begin{array}{cc} 1 & \frac{1}{-e^{-\pi i\alpha}e^{-2\sqrt{\zeta}}} \\ -e^{-\pi i\alpha}e^{-2\sqrt{\zeta}} & 1 \end{array} \right), & \zeta \in \Omega_3;
\end{cases}
$$

(2.17)

cf. Figure 1 for the regions $\Omega_1 - \Omega_4$, where $\arg \zeta \in (-\pi, \pi)$.

It is easily verified that $\Psi^{(2)}(\zeta)$ solves the following RH problem:

(a) $\Psi^{(2)}(\zeta)$ is analytic in $\zeta \in \mathbb{C} \setminus \Sigma_2$ (see Figure 1);

(b) $\Psi^{(2)}(\zeta)$ satisfies the jump condition

$$
\left( \Psi^{(2)} \right)_+(\zeta) = \left( \Psi^{(2)} \right)_-(\zeta) \left( \begin{array}{cc} e^{-(2\sqrt{\zeta} - \pi i\alpha)} & 1 \\
0 & e^{(2\sqrt{\zeta} - \pi i\alpha)} \end{array} \right), \quad \zeta \in \Sigma_2,
$$

(2.18)

where $\arg \zeta \in (-\pi, \pi)$, and $\sqrt{\zeta} = i|\zeta|$ for $\zeta \in \Sigma_2$;

(c) The asymptotic behavior of $\Psi^{(2)}(\zeta)$ at infinity is

$$
\Psi^{(2)}(\zeta) = O\left( \zeta^{-\frac{3}{4}} \right), \quad \arg \zeta \in (-\pi, \pi), \quad \zeta \to \infty;
$$

(2.19)

(d) The behavior of $\Psi^{(2)}(\zeta)$ at the origin is

$$
\Psi^{(2)}(\zeta) = O(1) e^{\frac{2}{5} \sigma_3} \frac{1}{\zeta^{\frac{3}{4}} \sigma_3}, \quad \arg \zeta \in (-\pi, \pi), \quad \zeta \to 0.
$$

(2.20)

**Remark 3.** It is worth noting that a consistency check of (2.18) and (2.20) yields $s > 0$. Indeed, it is seen that the $O(1)$ factor in (2.20) stands for an invertible matrix, bounded and with determinant 1, and thus having an $O(1)$ inverse. Substituting (2.20) into (2.18) leads eventually to

$$
\begin{pmatrix}
|\zeta|^{-2s/|\zeta|} & e^{-2\sqrt{\zeta}} \\
0 & e^{2\sqrt{\zeta}}
\end{pmatrix} = O(1), \quad s \in (-\infty, 0)
$$

is thus declined. Instead, we consider the solvability of $\Psi(\zeta, s)$ and the analyticity in $s$ only for $s > 0$.

We carry out yet another transformation to move the oscillating entries in the jump matrices to off-diagonal, as follows:

$$
\Psi^{(3)}(\zeta) = \begin{cases} 
\Psi^{(2)}(\zeta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \text{for } \Im \zeta > 0, \\
\Psi^{(2)}(\zeta), & \text{for } \Im \zeta < 0.
\end{cases}
$$

(2.21)

Then $\Psi^{(3)}(\zeta)$ solves a RH problem with jumps

$$
\left( \Psi^{(3)} \right)_+(\zeta) = \left( \Psi^{(3)} \right)_-(\zeta) J^{(3)}(\zeta), \quad \zeta \in \mathbb{R},
$$

(2.22)
where
\[
J^{(3)}(\zeta) = \begin{cases} 
\left( \begin{array}{cc} \frac{1}{e^{(2\sqrt{\zeta}-\pi\alpha)}} & -e^{-(2\sqrt{\zeta}-\pi\alpha)} \\
0 & 0 
\end{array} \right), & \zeta \in (-\infty, 0), \\
\left( \begin{array}{cc} 0 & -1 \\
1 & 0 
\end{array} \right), & \zeta \in (0, \infty);
\end{cases}
\] (2.23)

Furthermore, the behavior of \(\Psi^{(3)}\) at infinity is still of the form [2.19], while the condition at \(\zeta = 0\) now takes
\[
\Psi^{(3)}(\zeta) = O(1)e^{\frac{\zeta^2}{2}}\zeta^{\frac{\alpha}{2}}, \quad \text{Im} \zeta > 0,
\] and \(\Psi^{(3)}(\zeta) = O(1)e^{\frac{\zeta^2}{2}}\zeta^{\frac{\alpha}{2}}, \quad \text{Im} \zeta < 0,\) (2.24)

recalling that \(-i\sigma_2 = \left( \begin{array}{cc} 0 & -1 \\
1 & 0 \end{array} \right).\)

It is readily seen that
\[
(J^{(3)}(\zeta))^* + J^{(3)}(\zeta) = 2 \left( \begin{array}{cc} 1 & 0 \\
0 & 0 
\end{array} \right), \quad \zeta \in (-\infty, 0),
\] (2.25)

where \(X^*\) denotes the Hermitian conjugate of the matrix \(X.\)

Next, we define an auxiliary matrix function
\[
H(\zeta) = \Psi^{(3)}(\zeta) \left( \Psi^{(3)}(\bar{\zeta}) \right)^*, \quad \zeta \not\in \mathbb{R}.
\] (2.26)

Then \(H(\zeta)\) is analytic in \(\mathbb{C}\setminus\mathbb{R}.\) Substituting (2.19) and (2.24) to (2.26) gives
\[
H(\zeta) = O \left( \zeta^{-\frac{3}{2}} \right) \text{ as } \zeta \to \infty,
\]

and
\[
H(\zeta) = O(1) \text{ as } \zeta \to 0.
\]

Thus, by Cauchy’s integral formula, we have
\[
\int_{\mathbb{R}} H_+(\zeta)d\zeta = 0.
\] (2.27)

Now in view of (2.25), and adding to (2.27) its Hermitian conjugate, we have
\[
2 \int_{-\infty}^{0} \left( \Psi^{(3)} \right)_-(\zeta) \left( \begin{array}{cc} 1 & 0 \\
0 & 0 \end{array} \right) \left( \Psi^{(3)} \right)^*_-(\zeta)d\zeta = 0.
\]

A straightforward consequence is that the first column of \(\left( \Psi^{(3)} \right)_-(\zeta)\) vanishes for \(\zeta \in (-\infty, 0).\) Furthermore, it follows from (2.22) that the second column of \(\left( \Psi^{(3)} \right)_+(\zeta)\) vanishes, also for \(\zeta \in (-\infty, 0).\)

The jump \(J^{(3)}(\zeta)\) in (2.23) admits an analytic continuation in a neighborhood of \((-\infty, 0).\) Accordingly,
\[
\tilde{\Psi}^{(3)}(\zeta) := \begin{cases} 
\Psi^{(3)}(\zeta), & \arg \zeta \in (0, \pi), \\
\Psi^{(3)}(\zeta e^{-2\pi i}) \left( \begin{array}{cc} 1 & -e^{-(2\sqrt{\zeta}-\pi\alpha)} \\
e^{(2\sqrt{\zeta}-\pi\alpha)} & 0 \end{array} \right), & \arg \zeta \in (\pi, 2\pi)
\end{cases}
\]
defines an analytic function in the cut-plane $\arg \zeta \in (0, 2\pi)$, such that $\hat{\Psi}^{(3)}(\zeta) = \Psi^{(3)}(\zeta)$ for $\Im \zeta > 0$, and $\tilde{\Psi}^{(3)}(\zeta) = (\Psi^{(3)})^+(\zeta)$ for $\zeta \in (-\infty, 0)$. Hence we have

$$
\left( \Psi^{(3)} \right)_{12}(\zeta) = \left( \Psi^{(3)} \right)_{22}(\zeta) = 0, \quad \Im \zeta > 0.
$$

Similarly, we can also obtain

$$
\left( \Psi^{(3)} \right)_{11}(\zeta) = \left( \Psi^{(3)} \right)_{21}(\zeta) = 0, \quad \Im \zeta < 0.
$$

The reader is referred to [42] and [43] for a similar argument.

Now we proceed to examine the other entries of $\Psi^{(3)}(\zeta)$ by appealing to Carlson’s theorem (cf. [36, p.236]). To this aim, for $k = 1, 2$, we define scalar functions

$$
g_k(\zeta) = \begin{cases} 
(\Psi^{(3)}(\zeta))_{k1}, & \text{for } 0 < \arg \zeta < \pi, \\
(\Psi^{(3)}(\zeta))_{k2}, & \text{for } -\pi < \arg \zeta < 0.
\end{cases}
$$

From (2.23) and (2.28)-(2.30), we see that each $g_k(\zeta)$ is analytic in $\mathbb{C}\setminus(-\infty, 0]$, and satisfies the jump conditions

$$
(g_k)_+(\zeta) = (g_k)_-(\zeta)e^{2\sqrt{-\pi i} \zeta}, \quad \zeta \in (-\infty, 0).
$$

The sector of analyticity of $g_k(\zeta)$ can be extended as follows:

$$
\hat{g}_k(\zeta) = \begin{cases} 
g_k(e^{-2\pi i} \zeta)e^{-\pi i} e^{2\sqrt{-\pi i} \zeta}, & \text{for } \pi \leq \arg \zeta < 2\pi, \\
g_k(e^{2\pi i} \zeta)e^{\pi i} e^{2\sqrt{-\pi i} \zeta}, & \text{for } -2\pi < \arg \zeta \leq -\pi.
\end{cases}
$$

Thus $\hat{g}_k(\zeta)$ is now analytic in a sector $-2\pi < \arg \zeta < 2\pi$. It is worth noting that the function $\hat{g}_k(\zeta)$, so defined, is actually analytic in a larger sector $-3\pi < \arg \zeta < 3\pi$, and the exponential term $|e^{\sqrt{-\pi i} \zeta}| \leq 1$ for $\pi \leq \arg \zeta \leq 2\pi$ and $-2\pi < \arg \zeta \leq -\pi$.

If we put

$$
h_k(\zeta) = \hat{g}_k((\zeta + 1)^4) \quad \text{for } \arg \zeta \in [-\pi/2, \pi/2],
$$

then the above discussion implies that $h_k(\zeta)$ is analytic in $\Re \zeta > 0$, continuous and bounded in $\Re \zeta \geq 0$, and satisfies the decay condition on the imaginary axis

$$
|h_k(\zeta)| = O \left( e^{-|\zeta|^2} \right), \quad \text{for } \Re \zeta = 0 \text{ as } |\zeta| \to \infty.
$$

Hence Carlson’s theorem applies, and we have $h_k(\zeta) \equiv 0$ for $\Re \zeta > 0$. Tracing back, we see that all entries of $\Psi^{(3)}(\zeta)$ vanish for $\zeta \notin \mathbb{R}$. Therefore, $\Psi^{(3)}(\zeta)$ vanishes identically, which implies that $\Psi^{(1)}(\zeta)$ vanishes identically. This completes the proof of the vanishing lemma.

The solvability of the RH problem for $\Psi_0$ follows from the vanishing lemma. As briefly indicated in [15, p.104], the RH problem is equivalent to a Cauchy-type singular integral equations, the corresponding singular integral operator is a Fredholm operator of index zero. The vanishing lemma states that the null space is trivial, which implies that the singular integral equation (and thus $\Psi_0$) is solvable as a result of the Fredholm alternative theorem. More details can be found in [22, Proposition 2.4]; see also [10, 12, 15, 17] for standard methods connecting RH problems with integral equations.

Now we have the following solvability result:

**Lemma 2.** For $s \in (0, \infty)$, there exists a unique solution to the RH problem (2.1)-(2.3) for $\Psi(\zeta, s)$.
3 Nonlinear steepest descent analysis

This whole section will be devoted to the asymptotic analysis of the orthogonal polynomials with respect to the weight \( w(x; t) \) given in (1.5). We begin with a RH formulation \( Y(z) \) of the orthogonal polynomials. Such a remarkable connection between the orthogonal polynomials and RH problems is observed by Fokas, Its and Kitaev [16]. Then, we apply the nonlinear steepest descent analysis developed by Deift and Zhou et al. [12, 13] to the RH problem for \( Y \); see also Bleher and Its [2]. The idea is to obtain, via a series of invertible transformations \( Y \rightarrow T \rightarrow S \rightarrow R \), eventually the RH problem for \( R \), with jumps close to the identity matrix, where

- \( Y \rightarrow T \) is to re-scale the variable, to accomplish a normalization of \( Y(z) \) at infinity, and to remove the exponential factor \( e^{-t/x} \) in the weight function \( w \). As a result, \( T(z) \) solves an oscillatory RH problem, normalized at infinity.

- \( T \rightarrow S \) is based on a factorization of the oscillatory jump, and a deformation of the contours. \( S(z) \) solves a RH problem without oscillation, yet the contours are self-intersected.

- \( S \rightarrow R \), the final transformation, leads to a RH problem for \( R(z) \) with all jumps close to \( I \), and \( R(z) \) can then be expanded on the whole complex plane into a Neumann series. We use only the leading term in the present paper, though. To apply the transformation, a parametrix at the outside region, and local parametrices at the origin and at the soft edge \( z = 1 \) have to be constructed.

Tracing back, the uniform asymptotics of the orthogonal polynomials in the complex plane is obtained for large polynomial degree \( n \). Technique difficulties lie in the construction of the local parametrix in a neighborhood of the origin \( z = 0 \). The parametrix possesses irregular singularity both at infinity and at the origin.

3.1 Riemann-Hilbert problem for orthogonal polynomials

Initially, the RH problem for orthogonal polynomials is as follows (cf. [16]).

\( Y(z) \) is analytic in \( \mathbb{C}\setminus[0, \infty) \);

\( Y(z) \) satisfies the jump condition

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (0, \infty),
\]

(3.1)

where \( w(x) = w(x; t) = x^\alpha e^{-x-t/x} \) is the weight function defined in (1.5);

\( Y(z) \) at infinity is

\[
Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad \text{as } z \to \infty;
\]

(3.2)

\( Y(z) \) at the end points \( z = 0 \) are

\[
Y(z) = \begin{pmatrix} O(1) & O(1) \\ O(1) & O(1) \end{pmatrix}, \quad \text{as } z \to 0.
\]

(3.3)
By virtue of the Plemelj formula and Liouville's theorem, it is known that the above RH problem for $Y$ has a unique solution

$$\begin{align*}
Y(z) &= \begin{pmatrix}
\pi_n(z) \\
-2\pi i \gamma_{n-1} \pi_{n-1}(z)
\end{pmatrix} + \begin{pmatrix}
\frac{1}{2\pi i} \int_0^\infty \frac{\pi_n(s) w(s)}{s-z} ds \\
-\gamma_{n-1}^2 \int_0^\infty \frac{\pi_{n-1}(s) w(s)}{s-z} ds
\end{pmatrix},
\end{align*}$$ (3.4)

where $\pi_n(z)$ is the monic polynomial, and $p_n(z) = \gamma_n \pi_n(z)$ is the orthonormal polynomial with respect to the weight $w(x) = w(x; t)$; cf., e.g., [10] and [16].

### 3.2 The first transformation $Y \rightarrow T$

The first transformation is to normalize the above RH problem for $Y$ at infinity. Beforehand, we write down the equilibrium measure with the external field $V(x) = 4x, x > 0$, that is,

$$\psi(x) = \frac{2}{\pi} \sqrt{\frac{1-x}{x}}, \quad 0 < x < 1;$$ (3.5)

cf. [11], see also [35]. For later use, we defined several other auxiliary functions

$$g(z) = \int_0^1 \ln(z-x) \psi(x) dx,$$ (3.6)

where the branch is chosen such that $\arg(z-x) \in (-\pi, \pi)$, and

$$\phi(z) = 2 \int_0^z \sqrt{s-1} ds, \quad z \in \mathbb{C}\setminus[0, \infty),$$ (3.7)

where $\arg z \in (0, 2\pi)$, such that the Maclaurin expansion $\phi(z) = 4i\sqrt{z} \left\{1 - \frac{x}{6} + \cdots \right\}$ holds for $|z| < 1$.

The first transformation $Y \rightarrow T$ is defined as

$$T(z) = (4n)^{-(n+\frac{3}{2})} e^{-n\frac{1}{2}} \sigma_3 \left\{ \psi + \cdots \right\} \pi_n(z) e^{-n(g(z)-\frac{1}{2}l) \sigma_3} e^{-\frac{4n}{\pi n^2} \sigma_3 (4n)^{\frac{3}{2}} \sigma_3},$$ (3.8)

for $z \in \mathbb{C}\setminus(0, \infty)$, where $l = -2(1 + \ln 4)$ is the Euler-Lagrange constant. The purpose of the transformation is threefold: to re-scale the variable, to accomplish a normalization of $Y(z)$ at infinity, and to remove the exponential factor in the weight function $w$. Here $t = t_n$ indicates the dependence of the parameter $t$ on $n$, the polynomial degree. Then $T$ solves the RH problem:

(T1) $T(z)$ is analytic in $\mathbb{C}\setminus[0, \infty)$;

(T2) The jump condition is

$$T_+(x) = T_-(x) \begin{pmatrix}
e^{n(g_+(x)-g_+(x))} x^\alpha e^{-n(-4x+g_+(x)+g_-(x)-l)} \\
x^\alpha e^{n(-4x+g_+(x)+g_-(x)-l)}
\end{pmatrix}, \quad x \in (0, \infty);$$ (3.9)

(T3) The asymptotic behavior of $T(z)$ at infinity is

$$T(z) = I + O(1/z) \text{ as } z \rightarrow \infty;$$ (3.10)
The asymptotic behavior of $T(z)$ at the end points $z = 0$ is

$$T(z) = O(1)e^{-\frac{4n}{8nz_3}}. \quad (3.11)$$

From (3.6) and (3.7) it is readily seen that

$$g_+(x) - g_-(x) = 2\pi i - 2\phi_+(x), \quad x \in (0, 1).$$

Also, one of the phase conditions reads

$$-4x + g_+(x) + g_-(x) - l = 0, \quad x \in (0, 1),$$

with the Euler-Lagrange constant involved. Hence, the jumps in (3.9) can be represented in $\phi$, as follows:

$$T_+(x) = T_-(x) \begin{cases} 
1 & x^\alpha e^{-2n\phi(x)} \\
0 & 1 \\
\phi(x) & x^\alpha e^{2n\phi_-(x)} \\
0 & e^{2n\phi_+(x)} 
\end{cases}, \quad x \in (1, \infty);$$

$$T_+(x) = T_-(x) \begin{cases} 
0 & x^\alpha e^{-2n\phi(x)} \\
1 & 1 \\
\phi(x) & e^{2n\phi_-(x)} \\
0 & z^\alpha e^{2n\phi_+(x)} 
\end{cases}, \quad x \in (0, 1). \quad (3.12)$$

### 3.3 The second transformation $T \rightarrow S$

Let us take a closer look at the function $\phi(z)$, defined in (3.7) for $z \in \mathbb{C}\backslash[0, \infty)$. We see that $\phi(x) > 0$ for $x > 1$, Re $\phi(z) < 0$ in the lens-shaped domains; cf. Figure 2 and that $\phi_+(x) = \pm 2i \int_0^x \sqrt{1-z} ds$, purely imaginary, for $x \in (0, 1)$. Hence the RH problem for $T$ is oscillatory, in the sense that the jump matrix in (3.12) has oscillating diagonal entries on the interval $(0, 1)$. To remove the oscillation, we introduce the second transformation $T \rightarrow S$, based on a factorization of the oscillatory jump matrix and a deformation of contours. We define

$$S(z) = \begin{cases} 
T(z), & \text{for } z \text{ outside the lens shaped region}; \\
T(z) \begin{pmatrix} 
1 & 0 \\
-z^\alpha e^{2n\phi(z)} & 1 
\end{pmatrix}, & \text{for } z \text{ in the upper lens region}; \\
T(z) \begin{pmatrix} 
1 & 0 \\
z^\alpha e^{2n\phi(z)} & 1 
\end{pmatrix}, & \text{for } z \text{ in the lower lens region}, 
\end{cases} \quad (3.13)$$

where arg $z \in (-\pi, \pi)$. Then $S$ solves the RH problem:

(S1) $S(z)$ is analytic in $\mathbb{C}\backslash\Sigma_S$, where $\Sigma_S = \bigcup_{k=1}^3 \gamma_k \cup (1, \infty)$, illustrated in Figure 2.

(S2) The jump conditions are

$$S_+(z) = S_-(z) \begin{cases} 
\begin{pmatrix} 
1 & 0 \\
-z^\alpha e^{2n\phi(z)} & 1 
\end{pmatrix}, & z \in \gamma_1 \cup \gamma_3; \\
\begin{pmatrix} 
0 & x^\alpha \\
-x^\alpha & 1 
\end{pmatrix}, & z = x \in \gamma_2, \\
\begin{pmatrix} 
1 & z^\alpha e^{-2n\phi(z)} \\
0 & 1 
\end{pmatrix}, & z \in (1, +\infty); 
\end{cases} \quad (3.14)$$
Figure 2: Contours for the RH problem for $S(z)$ in the $z$-plane.

(S3) The asymptotic behavior at infinity is

$$S(z) = I + O(1/z), \quad \text{as} \quad z \to \infty; \quad (3.15)$$

(S4) The asymptotic behavior at the origin is sector-wise. As $z \to 0$,

$$S(z) = O(1) e^{-\frac{\pi}{8n^2} \sigma_3} \begin{cases} I, & \text{outside the lens-shaped regions,} \\ \left( \begin{array}{cc} 1 & 0 \\ -z^{-\alpha} e^{2n\phi(z)} & 1 \end{array} \right), & \text{in the upper lens region,} \\ \left( \begin{array}{cc} 1 & 0 \\ z^{-\alpha} e^{2n\phi(z)} & 1 \end{array} \right), & \text{in the lower lens region.} \end{cases} \quad (3.16)$$

3.4 Global parametrix

From (3.14), we see that the jump matrix for $S$ is of the form $J_{S,j} = I$, plus an exponentially small term for fixed $z \in \gamma_1 \cup \gamma_3 \cup (1, \infty)$. Neglecting the exponential small terms, we arrive at an approximating RH problem for $N(z)$, as follows:

(N1) $N(z)$ is analytic in $\mathbb{C} \setminus [0, 1]$;

(N2)

$$N_+(x) = N_-(x) \left( \begin{array}{cc} 0 & x^\alpha \\ -x^{-\alpha} & 0 \end{array} \right) \text{ for } x \in (0, 1); \quad (3.17)$$

(N3)

$$N(z) = I + O(1/z), \quad \text{as} \quad z \to \infty. \quad (3.18)$$

A solution to the above RH problem can be constructed explicitly,

$$N(z) = D_\infty^\sigma M^{-1} a(z)^{-\sigma_3} M D(z)^{-\sigma_3}, \quad (3.19)$$

where $M = (I + i\sigma_1)/\sqrt{2}$, $a(z) = \left( \frac{z - 1}{z} \right)^{1/4}$ with $\arg z \in (-\pi, \pi)$ and $\arg(z - 1) \in (-\pi, \pi)$, and the Szegö function

$$D(z) = \left( \frac{z}{\varphi(2z - 1)} \right)^{\alpha/2}, \quad \varphi(z) = z + \sqrt{z^2 - 1},$$

the branch is chosen such that $\varphi(z) \sim 2z$ as $z \to \infty$, and $D_\infty = 2^{-\alpha}$.

The jump matrices of $SN^{-1}$ are not uniformly close to the unit matrix near the end-points 0 and 1, thus local parametrices have to be constructed in neighborhoods of the end-points.
3.5 Local parametrix \( P^{(1)}(z) \) at \( z = 1 \)

The local parametrix at the right end-point \( z = 1 \) is the same as that of the Hermite polynomials or the Laguerre polynomials at the soft edge. More precisely, the parametrix is to be constructed in \( U(1,r) = \{ z \mid |z-1| < r \} \), \( r \) being a fixed positive number, such that

(a) \( P^{(1)}(z) \) is analytic in \( U(1,r)\setminus \Sigma_S \), see Figure 2 for the contours \( \Sigma_S \);
(b) In \( U(1,r) \), \( P^{(1)}(z) \) satisfies the same jump conditions as \( S(z) \) does; cf. (3.14);
(c) \( P^{(1)}(z) \) fulfills the following matching condition on \( \partial U(1,r) \):

\[
P^{(1)}(z)N^{-1}(z) = I + O(1/n). \tag{3.20}
\]

The parametrix can be constructed, out of the Airy function and its derivative, as in Section 6.1 below, and in [41 (3.74)]; see also [10, 13, 35].

3.6 Local parametrix \( P^{(0)}(z) \) at the origin

In this subsection, we focus on the construction of the parametrix at \( z = 0 \). The parametrix, to be constructed in the neighborhood \( U(0,r) = \{ z \mid |z| < r \} \) for sufficiently small \( r \), solves a RH problem as follows:

(a) \( P^{(0)}(z) \) is analytic in \( U(0,r)\setminus \Sigma_S \);
(b) In \( U(0,r) \), \( P^{(0)}(z) \) satisfies the same jump conditions as \( S(z) \) does; cf. (3.14);
(c) \( P^{(0)}(z) \) fulfills the following matching condition on \( \partial U(0,r) = \{ z \mid |z| = r \} \):

\[
P^{(0)}(z)N^{-1}(z) = I + O(n^{-1/3}) \quad \text{as } n \to \infty; \tag{3.21}
\]

(d) The behavior at the center \( z = 0 \) is the same as that of \( S(z) \), as described in (3.16).

Now we apply a transformation to convert all the jumps of the RH problem for \( P^{(0)}(z) \) to constant jumps by defining

\[
P^{(0)}(z) = \hat{P}^{(0)}(z)(-z)^{-\frac{\alpha}{2}}e^{\pi i \phi(z)}\sigma_3, \quad z \in U(0,r)\setminus \Sigma_S, \tag{3.22}
\]

where \( \text{arg}(-z) \in (-\pi, \pi) \). It is readily seen that \( \hat{P}^{(0)} \) solves the RH problem

(a) \( \hat{P}^{(0)}(z) \) is analytic in \( U(0,r)\setminus \Sigma_S \);
(b) In \( U(0,r) \), \( \hat{P}^{(0)}(z) \) satisfies the jump conditions

\[
\hat{P}^{(0)}_+(z) = \hat{P}^{(0)}_-(z) \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\ e^{-\pi i \alpha} & 1 \\
\end{array} \right), & z \in \gamma_3 \cap U(0,r), \\
\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \\
\end{array} \right), & z \in (0,r), \\
\left( \begin{array}{cc} 1 & 0 \\ e^{\pi i \alpha} & 1 \\
\end{array} \right), & z \in \gamma_1 \cap U(0,r); 
\end{cases} \tag{3.23}
\]
(c) The behavior at the center \( z = 0 \) is, as \( z \to 0 \),

\[
\hat{P}^{(0)}(z) = O(1)e^{-\frac{16}{\pi n^2} \sigma_3}(-z)^{\frac{1}{2} \sigma_3}e^{-n\phi(z) \sigma_3} \begin{cases} 
I, & \text{outside the lens}, \\
\begin{pmatrix} 1 & 0 \\
-e^{-i\pi \alpha} & 1 
\end{pmatrix}, & \text{upper lens}, \\
\begin{pmatrix} 1 & 0 \\
e^{i\pi \alpha} & 1 
\end{pmatrix}, & \text{lower lens}.
\end{cases}
\tag{3.24}
\]

We are now in a position to bring together \( \hat{P}^{(0)} \) and the model problem \((2.1)-(2.3)\), treated in the previous section. To this aim, we note first that

\[ \zeta = n^2 \phi^2(z) \tag{3.25} \]

is a conformal mapping in the \( z \)-neighborhood \( U(0, r) \), for \( r \) sufficiently small, such that \( \zeta \approx -16n^2z \) for small \( z \); cf. (3.7). Specifying the parts of the contours \( \gamma_1 - \gamma_3 \) within \( U(0, r) \) so that they map respectively to the rays \( \Sigma_1 - \Sigma_3 \); see Figure 1 and Figure 2 for the contours. Attention should be paid to the orientation of the contours, which has been reversed after the conformal mapping.

Based on these, we seek a solution \( \hat{P}^{(0)}(z) \) of the following form

\[ \hat{P}^{(0)}(z) = E(z)\Psi(n^2 \phi^2, 2nt_n)e^{-\frac{3}{2}i\sigma_3}, \tag{3.26} \]

where \( E(z) \) is an analytic matrix in \( U(0, r) \), taken to meet the matching condition \((3.21)\), and the factor \( e^{-\frac{3}{2}i\sigma_3} \) is appended in accordance with the now reversed orientation of \( \Sigma_1 - \Sigma_3 \); cf. (3.7) and the fact that the conformal mapping \( \zeta = n^2 \phi^2 \approx -16n^2z \).

From (3.26) and (2.1)-(2.3), it is readily verified that the jumps (3.23) and the behavior at the origin (3.24) is well fulfilled. What is more, the analytic factor can be determined by the matching condition (3.21) and the asymptotic behavior of \( \Psi(\zeta) \) for \( \zeta \to \infty \). Indeed, we can take

\[ E(z) = N(z)e\frac{1}{2}i\sigma_3 (-z)^{\frac{1}{2} \sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \left\{ n^2 \phi^2(z) \right\}^{\frac{1}{2} \sigma_3}. \tag{3.27} \]

where \( \arg(-z) \in (-\pi, \pi) \), and \( \arg\left\{ n^2 \phi^2(z) \right\} \in (-\pi, \pi) \). The matching condition (3.21) now follows from (3.19), (3.26)-(3.27) and (2.2); see also Remark 4 below.

At last, we show that, so-defined \( E(z) \) is an analytic function in \( U(0, r) \). Since \( E(z) \) is analytic in \( U(0, r) \setminus \gamma_2 \), it suffices to show that

\[ E_+(x) = E_-(x) \quad \text{for } x \in \gamma_2, \]

and that \( E(z) \) possesses at most a weak singularity at \( z = 0 \), both can be verified straightforward from (3.27) and (3.19). Here use may also be made of the facts that \( \arg(-z) = \mp \pi \) respectively on the positive and negative side of \( \gamma_2 \), and also that \( \arg(n^2 \phi^2(z)) = \mp \pi \), from above or below \( \gamma_2 \).

### 3.7 The final transformation \( S \to R \)

Now we bring in the final transformation by defining

\[ R(z) = \begin{cases} 
S(z)N^{-1}(z), & z \in \mathbb{C} \setminus \{ U(0, r) \cup U(1, r) \cup \Sigma_S \}; \\
S(z)(P^{(0)})^{-1}(z), & z \in U(0, r) \setminus \Sigma_S; \\
S(z)(P^{(1)})^{-1}(z), & z \in U(1, r) \setminus \Sigma_S.
\end{cases} \tag{3.28} \]

Then, \( R(z) \) solves the following RH problem:
Figure 3: Contours for the RH problem for $R(z)$ in the $z$-plane.

(R1) $R(z)$ is analytic in $\mathbb{C}\setminus \Sigma_R$ (see Figure 3 for the contours);

(R2) $R(z)$ satisfies the jump conditions

$$R_+(z) = R_-(z)J_R(z), \quad z \in \Sigma_R, \quad (3.29)$$

where

$$J_R(z) = \begin{cases} 
P^{(0)}(z)N^{-1}(z), & z \in \partial U(0,r), \\
P^{(1)}(z)N^{-1}(z), & z \in \partial U(1,r), \\
N(z)J_S(z)N^{-1}(z), & \Sigma_R \setminus \partial(U(0,r) \cup U(1,r));
\end{cases}$$

(R3) $R(z)$ demonstrates the following behavior at infinity:

$$R(z) = I + O\left(1/z\right), \quad \text{as } z \to \infty. \quad (3.30)$$

It follows from the matching condition (3.21) of the local parametrices and the definition of $\phi$ that

$$J_R(z) = \begin{cases} 
I + O\left(n^{-1/3}\right), & z \in \partial U(0,r) \cup U(1,r), \\
I + O(e^{-cn}), & z \in \Sigma_R \setminus \partial(U(0,r) \cup U(1,r)),
\end{cases} \quad (3.31)$$

where $c$ is a positive constant, and the error term is uniform for $z$ on the corresponding contours. Hence we have

$$\|J_R(z) - I\|_{L^2 \cap L^\infty(\Sigma_R)} = O(n^{-1/3}). \quad (3.32)$$

Then, applying the now standard procedure of norm estimation of Cauchy operator and using the technique of deformation of contours (cf. [10, 13]), it follows from (3.32) that

$$R(z) = I + O(n^{-1/3}), \quad (3.33)$$

uniformly for $z$ in the whole complex plane.

This completes the nonlinear steepest descent analysis.

**REMARK 4.** Indeed, for $s = 2nt$ in compact subsets of $(0, \infty)$, the error in (3.21) and in (3.31) can be made uniformly $O(1/n)$. For $t$ in larger range, $t \in (0,d]$ for fixed $d$, detailed calculation shows that the error term takes the weaker form $O(n^{-1/3})$, with the uniformity preserved, as will be confirmed by Theorem 2 and Theorem 3, see also (5.19) and (6.23) below.
4 Proof of Theorem \([1]\): Limiting kernel at the edge

In terms of the matrix-valued function \(Y(z)\) defined in \([3.4]\), the kernel \(K_n(x, y)\) in \([1.6]\) can be written as

\[
K_n(x, y) = \frac{\sqrt{w(x)w(y)}}{2\pi i(x - y)} \{Y_+^{-1}(y)Y_+(x)\}_{21},
\]

where \(x\) and \(y\) belong to the support of the equilibrium measure, which in this case is \([0, 4n]\). So it is natural to introduce a re-scaling of the variable to fix the support to \([0, 1]\), and to consider the re-scaled kernel \(4nK_n(4nx, 4ny)\), such that

\[
\tilde{K}_n(x, y) := 4nK_n(4nx, 4ny) = \frac{\sqrt{w(4nx)w(4ny)}}{2\pi i(x - y)} \{Y_+^{-1}(4ny)Y_+(4nx)\}_{21}, \quad x, y \in (0, 1). \quad (4.1)
\]

We note that such a re-scaling amounts to considering a so-called varying weight, in this case, the weight is an re-scaled version of \([1.5]\), namely,

\[
\tilde{w}(x; t_n) = 4n \frac{w(4nx; t_n)}{(4n)^{\alpha + 1}x^\alpha e^{-4nx - t_n/(4nx)}} \quad \text{for} \quad x \in (0, \infty).
\]

However, in the asymptotic analysis conducted in the previous section, we have chosen to analyze the original \(w(x; t)\) in \([1.5]\).

Represented in \([4.1]\), the asymptotics of the kernel, or, the large-\(n\) limit of it, can be derived from the Riemann-Hilbert analysis. We note that the large-\(n\) limit for \(\tilde{K}_n(x, y)\) is expect to be the sine kernel for \(x, y\) in the interior of the equilibrium measure, namely, \(x, y \in (0, 1)\), and at the soft edge \(x = 1\), the kernel is expected to be approximated by the Airy kernel. The reader is referred to \([11]\) for a detailed analysis. In the present paper, we focus on the edge behavior at the end-point \(z = 0\), where there is an essential singularity of the weight function \(w(x; t)\).

Attention will be paid to the dependence on \(t = t_n\) of the statistic quantities.

Tracing back the transformations \(R \to S \to T \to Y\), and combining \([3.8]\), \([3.13]\) and \([3.28]\) with \([3.22]\) and \([3.26]\), we have

\[
Y_+(4nx) = c_n^{\sigma_3}R(x)E(x)\Psi_-(f_n(x), s)\left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array}\right) \left[ \sqrt{w(4nx)} \right]^{-\frac{1}{2} \sigma_3}, \quad (4.2)
\]

for \(0 < x < r\), where

\[
f_n(x) = n^2 \varphi^2(x), \quad s = 2nt_n, \quad c_n = (-1)^n (4n)^{\alpha + \frac{2}{2}} e^{\frac{1}{2} nl},
\]

and use has been made of the fact that \(e^{n\pi i \sigma_3} = (-1)^n I\),

\[
g_+(x) + \phi_+(x) - \frac{1}{2} t = 2x + \pi i \quad \text{for} \quad x \in (0, 1),
\]

as can be seen from Section \([3.2]\) and that the boundary value on the positive side \(P_+^{(0)}\) corresponds to the value \(\Psi_-\) on the negative side, as can be seen from the correspondence \(\zeta \approx -16n^2z\), and the orientation of \(\gamma_2\) and \(\Sigma_2\); cf. Figures \([1]\) and \([2]\). Substituting \([4.2]\) into \([4.1]\), we have

\[
\tilde{K}_n(x, y) = \frac{-\psi_2(f_n(y)) \psi_1(f_n(y)) E_-(y)R_-(y)R(x)E(x) \left(\psi_1(f_n(x)), \psi_2(f_n(x))\right)^T}{2\pi i(x - y)}, \quad (4.3)
\]

where \(X^T\) stands for the transpose matrix of \(X\), and

\[
\left(\begin{array}{c} \psi_1(\zeta) \\ \psi_2(\zeta) \end{array}\right) = \left(\begin{array}{c} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{array}\right) = (\Psi_-(\zeta, s) \left(\begin{array}{c} e^{\frac{\pi i \alpha}{2}} \\ e^{-\frac{\pi i \alpha}{2}} \end{array}\right)) \quad \text{for} \quad \zeta \in (-\infty, 0). \quad (4.4)
\]
Now let \( x = \frac{u}{16n^2} \) and \( y = \frac{v}{16n^2} \), where both \( u \) and \( v \) are positive and of the size \( O(1) \). In view of (3.7), we obtain
\[
f_n(z) = n^2 \phi^2(z) = n^2 \left[-16z + O(z^2)\right]
\]
for small \( z \). In particular, we have
\[
f_n(x) = -u \left[1 + O\left(\frac{1}{n^2}\right)\right], \quad \text{and} \quad f_n(y) = -v \left[1 + O\left(\frac{1}{n^2}\right)\right].
\]

The analyticity of \( E(z) \) in \( U(1, r) \) implies that both \( E(z) \) and \( E^{-1}(z) \) are bounded in a neighborhood of the origin, and
\[
E^{-1}(y)E(x) = I + E^{-1}(y)(E(x) - E(y)) = I + O(x - y) = I + (u - v)O(n^{-2})
\]
for bounded \( u, v \). Similarly, since \( R(z) \) is a matrix function analytic in \( U(0, r) \), we have
\[
R^{-1}(y)R(x) = I + (u - v)O(n^{-2}).
\]
Here again, the error term is uniform for \( u \) and \( v \) lying in compact subsets of \((0, \infty)\). Also we have
\[
\psi_k(f_n(x), s) = \psi_k(-u, s) + O\left(n^{-2}\right)
\]
for \( k = 1, 2 \), and the error bound is uniform for both \( u \) and \( s \) in compact subsets of \((0, \infty)\).

Thus, substituting (4.5), (4.6) and (4.7) into (4.3), we have
\[
\frac{1}{4n}K_n\left(\frac{u}{4n}, \frac{v}{4n}\right) = \frac{\psi_1(f_n(y), s)\psi_2(f_n(x), s) - \psi_1(f_n(x), s)\psi_2(f_n(y), s)}{2\pi i(u - v)} + O\left(\frac{1}{n^2}\right)
\]
for large \( n \).

We introduce an auxiliary function with two variables
\[
H(\xi, \eta) = \frac{\psi_1(\eta, s)\psi_2(\xi, s) - \psi_1(\xi, s)\psi_2(\eta, s)}{2\pi i(\eta - \xi)}, \quad \xi, \eta \in (-\infty, 0).
\]
For \( s \in (0, \infty) \) fixed, \( \psi_k(\eta, s), \ k = 1, 2 \) can be extended to an analytic function. It is thus easily seen that \( H(\xi, \eta) \) is \( C^\infty \) in \((-\infty, 0) \times (-\infty, 0)\), noting that there is no singularity on the ray \( \xi = \eta \). Therefore we have
\[
H(\xi, \eta) = H(\xi_0, \eta_0) + O(|\xi - \xi_0| + |\eta - \eta_0|).
\]
Substituting \( \xi = f_n(x) = f_n\left(\frac{u}{16n^2}\right), \eta = f_n(y) = f_n\left(\frac{v}{16n^2}\right) \), \( \xi_0 = -u \) and \( \eta_0 = -v \) into the above equation, we obtain
\[
\frac{1}{4n}K_n\left(\frac{u}{4n}, \frac{v}{4n}\right) = \frac{\psi_1(-v, s)\psi_2(-u, s) - \psi_1(-u, s)\psi_2(-v, s)}{2\pi i(u - v)} + O\left(\frac{1}{n^2}\right)
\]
for large \( n \), where the error term \( O\left(n^{-2}\right) \) is uniform for \( u \) and \( v \) in compact subsets of \((0, \infty)\) and uniformly for \( s \in (0, \infty) \). In deriving the last formula, use has also been made of the fact that
\[
\frac{\xi - \eta}{v - u} = 1 + O\left(\frac{1}{n^2}\right),
\]
uniformly for \( u, v \) belong to compact subsets of \((0, \infty)\) and for large \( n \).

This is exactly (1.25). It thus completes the proof of Theorem 1.
5 Proof of Theorem 2: Transition to the Bessel kernel as $s \to 0^+$

In this section, we study the asymptotics of the model RH problem for $\Psi(\zeta, s)$ for small positive parameter $s$. Then we apply the results to reduce the $\Psi$-kernel in (1.25) to a Bessel kernel, and to obtain initial conditions for the equations of $r$, $t$ and $q$, derived in (2.8) and (2.9). A similar discussion can be found in [43].

5.1 Nonlinear steepest descend analysis of the model RH problem as $s \to 0^+$

If $s = 0$, in the model RH problem for $\Psi(\zeta, s)$; cf. (2.1)-(2.3), the essential singularity at the origin vanishes. So, to consider the approximation of $\Psi(\zeta, s)$ for small $s$, it is natural to ignore temporarily the exponential term $e^{s\zeta}$ in (2.3), and to consider a limiting RH problem $\Psi_0(\zeta)$. 

(a) $\Psi_0(\zeta)$ is analytic in $\mathbb{C} \cup \bigcup_{j=1}^{3} \Sigma_j$ (see Figure 5);

(b) $\Psi_0(\zeta)$ satisfies the jump condition

$$
(\Psi_0)_+ (\zeta) = (\Psi_0)_- (\zeta) \begin{cases}
\begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, & \zeta \in \Sigma_1, \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \zeta \in \Sigma_3;
\end{cases}
$$

(5.1)

(c) The asymptotic behavior of $\Psi_0(\zeta)$ at infinity is

$$
\Psi_0(\zeta) = \left(I + O \left(\frac{1}{\zeta}\right)\right) \zeta^{-\frac{1}{2} \sigma_3} I + i \sigma_1 \sqrt{2} e^{\sqrt{2} \zeta}, \quad \arg \zeta \in (-\pi, \pi), \quad \zeta \to \infty.
$$

(5.2)

A function $\Phi(\zeta)$, satisfying the jump (5.1) and the behavior (5.2) at infinity, can be constructed in terms of the modified Bessel functions as (cf. [25])

$$
\Phi(\zeta) = M_1 \pi \frac{1}{2} \sigma_3 \begin{cases}
\begin{pmatrix} I_\alpha(\sqrt{\zeta}) & i \pi K_\alpha(\sqrt{\zeta}) \\ \pi i \sqrt{\zeta} I'_\alpha(\sqrt{\zeta}) & -\sqrt{\zeta} K'_\alpha(\sqrt{\zeta}) \end{pmatrix}, & \zeta \in \Omega_1 \cup \Omega_4, \\
\begin{pmatrix} I_\alpha(\sqrt{\zeta}) & i \pi K_\alpha(\sqrt{\zeta}) \\ \pi i \sqrt{\zeta} I'_\alpha(2\sqrt{\zeta}) & -\sqrt{\zeta} K'_\alpha(\sqrt{\zeta}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, & \zeta \in \Omega_2, \\
\begin{pmatrix} I_\alpha(\sqrt{\zeta}) & i \pi K_\alpha(\sqrt{\zeta}) \\ \pi i \sqrt{\zeta} I'_\alpha(\sqrt{\zeta}) & -\sqrt{\zeta} K'_\alpha(\sqrt{\zeta}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \zeta \in \Omega_3,
\end{cases}
$$

(5.3)

where $\arg \zeta \in (-\pi, \pi)$, and $M_1 = \begin{pmatrix} 1 & 0 \\ \frac{i \pi}{4 \alpha^2 + 3} & 1 \end{pmatrix}$, the regions are illustrated in Figure 5.

Indeed, referring to [34] (10.27.6) and (10.27.9), it is easily seen that (5.1) is satisfied by $\Phi(\zeta)$. By expanding the modified Bessel functions for large $\zeta$ (see [34] §10.27 and §10.40), we have a more precise version of (5.2)

$$
\Phi(\zeta) = \left[I + \frac{4 \alpha^2 - 1}{128 \zeta} \begin{pmatrix} 1 & 0 \\ \frac{i \pi}{4 \alpha^2 - 9} & 1 \end{pmatrix} \begin{pmatrix} 4 \alpha^2 - 9 & 16i \\ 9 - 4 \alpha^2 \end{pmatrix} + O \left(\frac{1}{\zeta^2}\right)\right] \zeta^{-\frac{1}{2} \sigma_3} M e^{\sqrt{2} \zeta},
$$

(5.4)
where $M = (I + i\sigma_1)/\sqrt{2}$.

However, at the origin, the asymptotic behavior of $\Phi(\zeta)$ is significantly different from that of $\Psi(\zeta, s)$, as long as $s \neq 0$. When $\alpha \notin \mathbb{Z}$, the behavior of the Bessel model RH problem at $\zeta = 0$ takes the form

$$
\Phi(\zeta) = \hat{\Phi}(\zeta)\zeta^{\sigma_3/2} \begin{pmatrix} 1 & 1 \overline{\sin(\alpha \pi)} \\ 0 & 1 \end{pmatrix} J, \quad \text{arg}\,\zeta \in (-\pi, \pi), \quad \zeta \to 0, \quad (5.5)
$$

where

$$
J = \begin{cases} 
I & \zeta \in \Omega_1 \cup \Omega_4, \\
\begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix} & \zeta \in \Omega_2, \\
\begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix} & \zeta \in \Omega_3,
\end{cases}
$$

and

$$
\hat{\Phi}(\zeta) = \begin{pmatrix} \zeta^{-\alpha/2}I_{\alpha}(\sqrt{\zeta}) & \pi i \zeta^{(1-\alpha)/2}I'_{\alpha}(\sqrt{\zeta}) \\
\pi i \zeta^{(1-\alpha)/2}I'_{\alpha}(\sqrt{\zeta}) & \frac{\pi}{\sin(\alpha \pi)} e^{\alpha/2} I_{\alpha}(\sqrt{\zeta}) \end{pmatrix}
$$

is an entire matrix function, as can be seen from the convergent series expansion of the modified Bessel functions

$$
I_{\alpha}(z) = \left(\frac{z}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k! \Gamma(\alpha + k + 1)}, \quad \text{and} \quad K_{\alpha}(z) = \frac{\pi}{2} \sin(\alpha \pi) \frac{I_{\alpha}(z) - I_{-\alpha}(z)}{\sin(\alpha \pi)};
$$

cf. [34] (10.25.2) and (10.27.4).

It is clear that $\Psi(\zeta, s)$ is not approximated by $\Phi(\zeta)$ near the origin for $s > 0$. So we need to construct a local parametrix $F(\zeta)$, defined in $U(0, \epsilon)$ for a small $\epsilon$, and matches $\Phi(\zeta)$ on $|\zeta| = \epsilon$.

More precisely, $F(\zeta)$ is supposed to solve the following RH problem:

(a) $F(\zeta)$ is analytic in $U(0, \epsilon) \setminus \bigcup_{j=1}^{3} \Sigma_j$ (see Figure 1 for $\Sigma_j$, $j = 1 - 3$);

(b) $F(\zeta)$ satisfies the same jump condition as $\Psi(\zeta, s)$, that is,

$$
F_+(\zeta) = F_-(\zeta)\begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \quad \zeta \in \Sigma_1 \cap U(0, \epsilon),
F_+(\zeta) = F_-(\zeta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \zeta \in \Sigma_2 \cap U(0, \epsilon),
F_+(\zeta) = F_-(\zeta)\begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \quad \zeta \in \Sigma_3 \cap U(0, \epsilon); \quad (5.6)
$$

(c) On the circular boundary $|\zeta| = \epsilon$, a matching condition is fulfilled, such that

$$
F(\zeta) = \left[I + O(s) + O\left(s^{\alpha+1}\right)\right] \Phi(\zeta) \quad \text{as} \quad s \to 0^+; \quad (5.7)
$$

(d) The asymptotic behavior of $F(\zeta)$ at the origin is the same as that of $\Psi(\zeta, s)$ in [2.3], namely,

$$
F(\zeta) = O(1)\zeta^{\sigma_3} e^{\frac{\alpha}{2}\sigma_3} J, \quad (5.8)
$$

where $J$ is the constant factor introduced in [5.5], defined sector-wise.
We seek a solution $F(\zeta)$ of the form

$$F(\zeta) = \Phi(\zeta) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \frac{\alpha e^{2s/\tau}}{\tau - \zeta},$$

(5.9)

where $\Phi(\zeta)$ is the same entire function as in (5.5), and the scalar function $f(\zeta)$, to be determined, is analytic in $U(0, \epsilon) \setminus \Sigma_2$, such that

$$|f_+(\zeta) - f_-(\zeta)| = |\zeta|^{\alpha} e^{2s/\zeta} \cos \alpha \pi$$

for $\zeta \in (-\epsilon, 0)$. (5.10)

Keeping in mind the matching condition (5.7), we chose

$$f(\zeta) = -\frac{1}{4\pi \sin(\alpha \pi)} \int_{\Gamma} \frac{\tau^{\alpha} e^{2s/\tau}}{\tau - \zeta} d\tau,$$

(5.11)

where $\arg \tau \in [-\pi, \pi]$ in the integrand, the integration path $\Gamma$ consisting of the line segments along the upper and lower edge of $[-1, 0]$, and the circle $|\tau| = 1$ joining them, as illustrated in Figure 4. We proceed to show that $f(\zeta)$ is what we are looking for.

First, we see that the jump condition (5.10) is satisfied. Indeed, we may separate the path and rewrite

$$f(\zeta) = f_A(\zeta) - \frac{1}{4\pi \sin(\alpha \pi)} \left[ -\int_{-\epsilon}^{0} \frac{\tau^{\alpha} e^{2s/\tau}}{\tau - \zeta} d\tau - \int_{-\epsilon}^{0} \frac{\tau^{\alpha} e^{2s/\tau}}{\tau - \zeta} d\tau \right],$$

or, in a compact form

$$f(\zeta) = f_A(\zeta) + \frac{1}{2\pi i} \int_{-\epsilon}^{0} \frac{|\tau|^{\alpha} e^{2s/\tau}}{\tau - \zeta} d\tau,$$

28

Figure 4: The integration path $\Gamma$ in the complex $\tau$-plane: The solid bold path consisting of the line segments along the upper and lower edge of $[-1, 0]$, and the circle $|\tau| = 1$ joining them. $\Gamma_\delta$: The closed loop resulted from replacing the line segments $[-\delta, 0]$ with the dashed bold circular part $|\tau| = \delta$. The dotted circle is $|\zeta| = \epsilon$, on which $\Phi(\zeta)$ and $F(\zeta)$ match.
where \( f_A(\zeta) \) denotes the integral on \( \Gamma \backslash [-\epsilon, 0] \). Clearly, the Cauchy integral \( f_A(\zeta) \) is analytic in \( U(0, \epsilon) \). The jump condition (5.10) follows accordingly from the Plemelj-Sokhotski formula.

Next, we show that
\[
f(\zeta) = \frac{\zeta^\alpha}{2i \sin(\alpha \pi)} \left[ 1 + O\left( \frac{s}{\epsilon} \right) + O\left( \frac{\delta}{\epsilon} \alpha+1 \right) \right], \quad |\zeta| = \epsilon,
\]
(5.12)
as \( s/\delta = O(1) \) and \( \alpha > -1 \), and the error term is uniform on the circle. To this aim, we deform the integration path \( \Gamma \) in (5.11) to a closed loop \( \Gamma_\delta \) by replacing the line segments \([-\delta, 0]\) with the circular part \(|\tau| = \delta\), where \( \delta \) may depend on \( s \), such that \( 0 < \delta \ll \epsilon < 1 \); cf. the dashed circle in Figure 4. As a result one has
\[
f(\zeta) = f_\delta(\zeta) + O\left( \epsilon^{-1}\delta^{\alpha+1} \right), \quad |\zeta| = \epsilon,
\]
(5.13)
where
\[
f_\delta(\zeta) = -\frac{1}{4\pi \sin(\alpha \pi)} \int_{\Gamma_\delta} \frac{\tau^\alpha e^{2s/\tau} d\tau}{\tau - \zeta},
\]
and the integrals on the segment \( \tau \in [-\delta, 0] \) and \(|\tau| = \delta\) contribute to the error term in (5.13). Indeed,
\[
\left| \int_{-\delta}^{0} \frac{\tau^\alpha e^{2s/\tau} d\tau}{\tau - \zeta} \right| \leq \frac{1}{\epsilon - \delta} \int_{-\delta}^{0} |\tau|^\alpha e^{-2s/|\tau|} d\tau \leq \frac{\delta^{\alpha+1}}{\epsilon - \delta} e^{2s/\delta} = O\left( \epsilon^{-1}\delta^{\alpha+1} \right),
\]
so long as \( \delta \ll \epsilon \) and \( s/\delta = O(1) \). We obtain the same estimate for the circular part \(|\tau| = \delta\). The formula (5.13) is thus justified.

Using Cauchy’s integral formula, for \( \zeta \in \partial U(0, \epsilon) \), we have
\[
f_\delta(\zeta) = \frac{1}{2i \sin(\alpha \pi)} \zeta^\alpha e^{2s/\zeta} = \frac{\zeta^\alpha}{2i \sin(\alpha \pi)} \left( 1 + O\left( \frac{s}{\epsilon} \right) \right),
\]
(5.14)
where the branch is chosen such that \( \arg \zeta \in (-\pi, \pi) \). A combination of (5.13) and (5.14) gives (5.12).

Noting that \( e^{\zeta^{\alpha+1}} = I + O(s/\epsilon) \) for \(|\zeta| = \epsilon\), substituting (5.12) into (5.9) and in view of (5.5), we see that the matching condition (5.7) is well fulfilled. Here we have set \( \delta = s \) and \( \epsilon \) a small positive constant not depending on \( s \). Hence \( F(\zeta) \) defined in (5.9) does solve the RH problem.

**REMARK 5.** In deriving (5.12), we only need \( \alpha > -1 \). When \( \alpha \) is a nonnegative integer, a logarithmic behavior may occur for the Bessel model problem \( \Phi(\zeta) \) at the origin. Indeed, instead of (5.5), now the behavior at \( \zeta = 0 \) is
\[
\Phi(\zeta) = \hat{\Phi}(\zeta) \zeta^{\alpha+1} \left( 1 + \frac{i}{\pi} (-1)^{\alpha+1} \ln \left( \sqrt{\zeta}/2 \right) \right), \quad \arg \zeta \in (-\pi, \pi), \quad \zeta \to 0,
\]
where \( \hat{\Phi}(\zeta) \) is the same as in (5.5), and \( \hat{\Phi}(\zeta) \) is an entire matrix function, as can be explicitly determined by the ascending series of the modified Bessel functions, cf. [34] (10.25.2), (10.31.1). Still, we can express the parametrix \( F(\zeta) \) in the form of (5.9), with
\[
f(\zeta) = \frac{(-1)^{\alpha+1}}{2\pi^2} \int_{\Gamma} \frac{\tau^\alpha e^{2s/\tau} \ln(\sqrt{\tau}/2) d\tau}{\tau - \zeta}, \quad \zeta \in U(0, \epsilon) \backslash [-\epsilon, 0],
\]
29
where $\Gamma$ is the same integration path employed in (5.11). Following the steps (5.9)- (5.14), it is readily verified that such a function $F(\zeta)$ solves the RHP (5.6)-(5.8) for $\alpha = 0, 1, 2, \ldots$. Slight modifications are needed here, for example, (5.12) now reads

$$f(\zeta) = \left(\frac{-1}{\pi i}\right)^\alpha \zeta \ln \left(\frac{\sqrt{\zeta^2}}{2}\right) \left(1 + O\left(\frac{\delta}{\epsilon}\right) + O\left(\left(\frac{\delta}{\epsilon}\right)^{\alpha+1} \ln \delta\right)\right)$$

as $|\zeta| = \epsilon$, and, accordingly, the matching condition (5.7) now takes the form

$$F(\zeta) = [I + O(s) + O(s^{\alpha+1} \ln s)] \Phi(\zeta) \text{ for } |\zeta| = \epsilon, \text{ as } s \to 0^+.$$

**Remark 6.** We have options in choosing $\epsilon$ and $\delta$. The previous estimation works as long as $s/\epsilon \ll 1$, $\delta/\epsilon \ll 1$, and $s/\delta = O(1)$, as $s \to 0^+$. Here we choose $\delta = s$, and $\epsilon$ small fixed. The choice makes sense since the corresponding Bessel functions, upon them the Bessel kernel is built, have no zeros in $U(0, \epsilon)$ for $\epsilon$ small enough. Then the approximation of $Ψ(\zeta, s)$ in $U(0, \epsilon)$ is in a sense of no significance, and one may focus on the Bessel-type approximation outside of the neighborhood $U(0, \epsilon)$.

Finally, we consider

$$R_0(\zeta) = \begin{cases} Ψ(\zeta, s)Φ^{-1}(\zeta), & |\zeta| > \epsilon, \\ Ψ(\zeta, s)F^{-1}(\zeta), & |\zeta| < \epsilon. \end{cases}$$

(5.15)

The matrix function $R_0(\zeta)$ is analytic in $|\zeta| \neq \epsilon$, approaching $I$ at infinity, and the jump on the circle is

$$J_{R_0}(\zeta) = I + O(s) + O(s^{\alpha+1}), \quad |\zeta| = \epsilon;$$

(5.16)

see Figure 5 for the circular contour. So, by an argument similar to Section 3.7 we have

$$R_0(\zeta) = \begin{cases} I + O(s^\mu), & s \to 0^+, \text{ uniform for bounded } \zeta, \\ I + O(s^\mu \zeta^{-1}), & \zeta \to \infty, s \to 0^+, \end{cases}$$

(5.17)

where $\mu = 1$ for $\alpha > 0$, and $\mu = \alpha + 1$ for $-1 < \alpha < 0$.

This completes the nonlinear steepest descend analysis of $Ψ(\zeta, s)$ as $s \to 0^+$.

### 5.2 Proof of Theorem 2

Now we have obtained approximations to $Ψ(\zeta, s)$ as $s \to 0^+$. In the present subsection, we will reduce $Ψ(\zeta, s)$ to the solution $Φ(\zeta)$ of the Bessel model problem for small $s$, and, as a by-product, obtain the initial conditions for the nonlinear equations (2.8) and (2.9), derived from the compatibility conditions of the Lax pair of $Ψ(\zeta, s)$; cf. (1.13) and (1.14).

It follows from (5.3), (5.15) and (5.17) that

$$Ψ(\zeta, s) = [I + O(s^\mu/\zeta)]Φ(\zeta), \quad s \to 0^+,$$

(5.18)

for $|\zeta| > \epsilon$, where $\mu = 1$ for $\alpha \geq 0$ and $\mu = \alpha + 1$ for $-1 < \alpha < 0$, $\epsilon$ is a small positive constant, and $Φ(\zeta)$ is the solution to the model Bessel problem, explicitly given in (5.3). Then a combination of (5.4) with (5.18) gives

$$Ψ(\zeta, s)e^{-\sqrt{\alpha}M^{-1}\zeta^{3/2}} = I + \frac{4\alpha^2 - 1}{12\epsilon} \left(\frac{4\alpha^2 - 9}{12(4\alpha^2 - 9)(4\alpha^2 - 13)} 16i \right) 9 - 4\alpha^2) + O(s^\mu/\zeta) + O(\zeta^{-3/2})$$

(5.19)
as $\zeta \to \infty$, uniformly for $s \in (0, \varepsilon]$ with $\varepsilon$ small positive, where $M = (I + i\sigma_1)/\sqrt{2}$, $\mu = 1$ for $\alpha \geq 0$ and $\mu = \alpha + 1$ for $\alpha \in (-1, 0)$.

Let $s \to 0^+$, in view of (2.2) and (2.5), we obtain the initial condition for the unknown functions $q(s)$, $r(s)$ and $t(s)$, as follows:

**COROLLARY 2.** The initial values for the nonlinear equations for $q(s)$, $r(s)$ and $t(s)$ in (2.8) and (2.9) can be determined, namely,

$$
\begin{align*}
q(0) &= \frac{1}{128}(4\alpha^2 - 1)(4\alpha^2 - 9) \\
r(0) &= \frac{1}{8}(1 - 4\alpha^2) \\
t(0) &= \frac{1}{1536}(4\alpha^2 - 1)(4\alpha^2 - 9)(4\alpha^2 - 13).
\end{align*}
$$

To complete the proof of Theorem 2, we substitute (5.3) and (5.15) into (4.4), and obtain

$$
\begin{pmatrix}
\psi_1(\zeta, s) \\
\psi_2(\zeta, s)
\end{pmatrix}
= R_0(\zeta)M_1\pi^{\alpha/2}e^{\pi i(\alpha - 1)}
\begin{pmatrix}
I_\alpha(\sqrt{|\zeta|}e^{-\pi i/2}) \\
\pi \sqrt{|\zeta|}I_\alpha'(\sqrt{|\zeta|}e^{-\pi i/2})
\end{pmatrix},
$$

where we have used the formula $e^{\frac{1}{2}\pi i\alpha}I_\alpha(z) = J_\alpha(z e^{\frac{1}{2}\pi i})$ for $\arg z \in (-\pi, \pi/2]$; cf. [34] (10.27.6), and an approximation for $R_0(\zeta)$ is provided in (5.18).

We note that for an arbitrary matrix $\tilde{M}$ with $\det \tilde{M} = 1$, it holds

$$
\tilde{M}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$
Here, as before, $\tilde{M}^T$ denotes the transpose of $\tilde{M}$. Substituting the above representation for $\psi_k$ into (4.9) yields
\[ \frac{1}{4n} K_n \left( \frac{u}{4n}, \frac{v}{4n} \right) = \frac{\psi_1(-v,s)\psi_2(-u,s) - \psi_1(-u,s)\psi_2(-v,s)}{2\pi i(u-v)} + O \left( \frac{1}{n^2} \right) \]
\[ = \mathbb{J}_n(u,v) + O(s^6) + O \left( \frac{1}{n^2} \right), \]
with $s = 2nt = 2nt_n \to 0^+$, $t$ is the parameter appeared in the weight (1.5), where $\mathbb{J}_n$ is the Bessel kernel defined in (1.10). The formula in (5.22) holds uniformly for $u$ and $v$ in compact subsets of $(0, \infty)$ and uniformly for $s \in (0, \varepsilon]$.

Thus we complete the proof of Theorem 2.

6 Proof of Theorem 3: Transition to the Airy kernel as $s \to +\infty$

In this section, an asymptotic analysis of the model RH problem for $\Psi(\zeta, s)$ is carried out as the parameter $s \to \infty$. The results are then applied to the reduction of the $\Psi$-kernel in (4.9) to the Airy kernel as $s \to \infty$. Attention will also be paid to the large-$s$ asymptotics of the equations in (2.9) and (2.8). A similar discussion can be found in [23].

6.1 Nonlinear steepest descent analysis of the model RH problem as $s \to +\infty$

Taking a normalization of $\Psi(\zeta, s)$ (cf. (2.1)-(2.3)) at both the infinity and the origin, that is
\[ U(\lambda, s) = \left( \begin{array}{c} 1 \\ \frac{3}{2} i s \frac{1}{3} \\ 0 \end{array} \right) s^{-1/3} \Psi(s^{2/3}, s)e^{-s^{1/3} \theta(\lambda) \sigma_3}, \quad \theta(\lambda) = (\lambda + 1)^{3/2}/\lambda, \]
where arg$(\lambda + 1) \in (-\pi, \pi)$ and arg$\lambda \in (-\pi, \pi)$, we see that $U(\lambda, s)$ ($U(\lambda)$, for short) solves the following RH problem:

(a) $U(\lambda)$ is analytic in $\mathbb{C} \setminus \cup_{j=1}^3 \Sigma_j$ (see Figure 6);

(b) $U(\lambda)$ satisfies the jump conditions
\[ U_+(\lambda) = U_-(\lambda) \]
\[ \left\{ \begin{array}{ll}
\left( \begin{array}{cc} e^{i\pi} e^{-2s^{1/3} \theta(\lambda)} & 0 \\
0 & 1 \end{array} \right), & \lambda \in \Sigma_1, \\
\left( \begin{array}{cc} 0 & 1 \\
-1 & 0 \end{array} \right), & \lambda \in (-\infty, -1), \\
\left( \begin{array}{cc} 0 & e^{2s^{1/3} \theta(\lambda)} \\
e^{-2s^{1/3} \theta(\lambda)} & 0 \end{array} \right), & \lambda \in (-1, 0), \\
\left( \begin{array}{cc} e^{-i\pi} e^{-2s^{1/3} \theta(\lambda)} & 0 \\
1 & 1 \end{array} \right), & \lambda \in \Sigma_3;
\end{array} \right. \]

(c) The asymptotic behavior of $U(\lambda)$ at infinity is
\[ U(\lambda) = \left( I + O \left( \frac{1}{\lambda} \right) \right) \lambda^{-1/4} \sigma_3 I + i\sigma_1 \frac{1}{\sqrt{2}}, \]

\[ \text{(6.3)} \]
(d) The behavior of $U(\lambda)$ at the origin is, as $\lambda \to 0$,

\[
U(\lambda) = O(1)\lambda^{\frac{2}{3}\sigma_3}
\begin{cases}
I, & \lambda \in \Omega_1 \cup \Omega_4, \\
\left( -e^{\alpha i\pi}e^{-2s^{1/3}\theta(\lambda)} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), & \lambda \in \Omega_5, \\
\left( e^{-\alpha i\pi}e^{-2s^{1/3}\theta(\lambda)} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), & \lambda \in \Omega_6;
\end{cases}
\]  

(6.4)

see Figure 6 for the regions involved.

The second transformation $U \to X$ is to move the jumps on $\Sigma_1$ and $\Sigma_3$ to contours passing through $\lambda = -1$, defined as

\[
X(\lambda) = X(\lambda, s) =
\begin{cases}
U(\lambda)\left( e^{\alpha i\pi}e^{-2s^{1/3}\theta(\lambda)} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), & \lambda \in \Omega_5 \\
U(\lambda)\left( e^{-\alpha i\pi}e^{-2s^{1/3}\theta(\lambda)} \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), & \lambda \in \Omega_6, \\
U(\lambda), & \text{otherwise.}
\end{cases}
\]  

(6.5)

Then $X$ solves the RH problem:

(a) $X(\lambda)$ is analytic in $\mathbb{C}\setminus\Sigma_X$ ($\Sigma_X = (-\infty, -1) \cup (-1, 0) \cup \Sigma_1^- \cup \Sigma_3^-$; see Figure 6).
(b) \( X(\lambda) \) satisfies the jump conditions

\[
X_+(\lambda) = X_-(\lambda) \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\
e^{\alpha i\pi} e^{-2s^{1/3} \theta(\lambda)} & 1 \end{array} \right), & \lambda \in \Sigma^-_1, \\
\left( \begin{array}{cc} 0 & 1 \\
0 & e^{-\alpha i\pi} \end{array} \right), & \lambda \in (-\infty, -1), \\
\left( \begin{array}{cc} e^{\alpha i\pi} e^{-2s^{1/3} \theta(\lambda)} & 0 \\
0 & e^{-\alpha i\pi} \end{array} \right), & \lambda \in (-1, 0), \\
\left( \begin{array}{cc} 1 & 0 \\
e^{-\alpha i\pi} e^{-2s^{1/3} \theta(\lambda)} & 1 \end{array} \right), & \lambda \in \Sigma^-_3; 
\end{cases} \tag{6.6}
\]

(c) The asymptotic behavior of \( X(\lambda) \) at infinity is kept

\[
X(\lambda) = \left( I + O\left(\frac{1}{\lambda}\right) \right) \lambda^{-\frac{1}{2} \sigma_3} \frac{I + i \sigma_1}{\sqrt{2}}, \quad \arg \lambda \in (-\pi, \pi), \ \lambda \to \infty; \tag{6.7}
\]

(d) The behavior of \( X(\lambda) \) at the origin is

\[
X(\lambda) = O(1) \lambda^\frac{\sigma}{2} \sigma_3, \quad \arg \lambda \in (-\pi, \pi), \ \lambda \to 0. \tag{6.8}
\]

Ignoring the exponentially small entries in the jumps (6.6), we get an approximating RH problem:

(a) \( \tilde{N}(\lambda) \) is analytic in \( \mathbb{C} \setminus (-\infty, 0) \) (see Figure 6);

(b) \( \tilde{N}(\lambda) \) satisfies the jump conditions

\[
\tilde{N}_+(\lambda) = \tilde{N}_-(\lambda) \begin{cases} 
e^{\pi i\alpha \sigma_3}, & \zeta \in (-1, 0), \\
\left( \begin{array}{cc} 0 & 1 \\
-1 & 0 \end{array} \right), & \zeta \in (-\infty, -1); 
\end{cases} \tag{6.9}
\]

(c) The asymptotic behavior of \( \tilde{N}(\lambda) \) at infinity is

\[
\tilde{N}(\lambda) = \left( I + O\left(\frac{1}{\lambda}\right) \right) \lambda^{-\frac{1}{2} \sigma_3} \frac{I + i \sigma_1}{\sqrt{2}}; \tag{6.10}
\]

(d) The asymptotic behavior of \( \tilde{N}(\lambda) \) at \( \lambda = 0 \) is

\[
\tilde{N}(\lambda) = O(1) \lambda^\frac{\sigma}{2} \sigma_3. \tag{6.11}
\]

The RH problem for \( \tilde{N}(\lambda) \) is an analogue of the global parametrix \( N(z) \) treated in Section 3.4, and, just like \( N \), a solution to the approximating RH problem can be constructed explicitly as (cf. [23, (2.16)])

\[
\tilde{N}(\lambda) = \left( \begin{array}{cc} 1 & 0 \\
i \alpha & 1 \end{array} \right) (\lambda + 1)^{-\frac{\sigma_3}{2}} M \left( \frac{\sqrt{\lambda + 1} + 1}{\sqrt{\lambda + 1} - 1} \right)^{-\frac{\sigma}{2} \sigma_3}, \tag{6.12}
\]
where $M = \frac{1}{\sqrt{2}}(I + i \sigma_1)$, the branches are chosen as $\arg(\lambda + 1) \in (-\pi, \pi)$, $\arg \lambda \in (-\pi, \pi)$, and such that the last factor in (6.12) behaves
\[
\left( \frac{\sqrt{\lambda + 1} + 1}{\sqrt{\lambda + 1} - 1} \right)^{-\frac{2}{\sqrt{\lambda}}} = \left( 1 + \frac{\alpha^2}{2\sqrt{\lambda}} \right) I - \frac{\alpha}{\sqrt{\lambda}} \sigma_3 + O(\lambda^{-3/2}) \quad \text{for large } \lambda.
\]

Comparing $X(\lambda)$ and $\tilde{N}(\lambda)$, the jumps of them have only an exponentially small difference, yet $\tilde{N}$ has an extra singularity at $\lambda = -1$. Hence, to approximate $X(\lambda)$, a local parametrix has to be constructed in a neighborhood, say, $U(-1, r)$, of $\lambda = -1$, where $r$ is a sufficiently small positive constant. The parametrix shares the same jumps (6.6) with $X$ in the neighborhood, bounded at $\lambda = -1$, and matches with $\tilde{N}(\lambda)$ on $|\lambda + 1| = r$.

It is readily verified that such a parametrix can be represented as follows
\[
P_1(\lambda) = E_1(\lambda) \Phi_A \left( s^\frac{\pi}{2} f_1(\lambda) \right) e^{-\frac{1}{2} \theta(\lambda) \sigma_3} e^{\pm \frac{i}{2} \pi i \sigma_3}, \quad \text{for } \pm \Im \lambda > 0, \lambda \in U(-1, r), \tag{6.13}
\]
where
\[
f_1(\lambda) = \left( -\frac{3}{2} \theta(\lambda) \right)^{\frac{3}{2}}, \quad \text{such that } f_1(\lambda) \sim \left( \frac{3}{2} \right)^{\frac{3}{2}} (\lambda + 1) \quad \text{for } \lambda \sim -1
\]
serves as a conformal mapping in $U(-1, r) = \{ \lambda \mid |\lambda + 1| < r \}$ for sufficiently small $r$, and $\Phi_A$ is a solution to the Airy model RH problem in Section 3.5, expressed explicitly as
\[
\Phi_A(\zeta) = M_A \begin{cases} 
\left( \begin{array}{cc} \text{Ai}(\zeta) & \text{Ai}(\omega^2 \zeta) \\ \text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta) \end{array} \right) e^{-\frac{\pi i}{2} \sigma_3}, \quad & \zeta \in \Omega_1 \\
\left( \begin{array}{cc} \text{Ai}(\zeta) & \text{Ai}(\omega^2 \zeta) \\ \text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega^2 \zeta) \end{array} \right) e^{-\frac{\pi i}{2} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right), \quad & \zeta \in \Omega_2 \\
\left( \begin{array}{cc} \text{Ai}(\zeta) & -\omega \text{Ai}(\omega \zeta) \\ \text{Ai}'(\zeta) & -\text{Ai}'(\omega \zeta) \end{array} \right) e^{-\frac{\pi i}{2} \sigma_3} \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right), \quad & \zeta \in \Omega_3 \\
\left( \begin{array}{cc} \text{Ai}(\zeta) & -\omega^2 \text{Ai}(\omega^2 \zeta) \\ \text{Ai}'(\zeta) & -\text{Ai}'(\omega^2 \zeta) \end{array} \right) e^{-\frac{\pi i}{2} \sigma_3}, \quad & \zeta \in \Omega_4;
\end{cases}
\]
\[\text{cf. } \left[ 13, (7.9) \right], \text{ where } \omega = e^{\frac{2\pi i}{3}}, \text{ and the constant matrix } M_A = \sqrt{2\pi} e^{\frac{1}{2} \pi i} \left( \begin{array}{cc} 1 & 0 \\ 0 & -i \end{array} \right). \]

Here, as an illustration, we may use Figure 1 to describe the regions $\Omega_1 - \Omega_4$.

What remains is the determination of the factor $E_1(\lambda)$, so that $E_1(\lambda)$ is analytic in $U(-1, r)$ and makes $P_1(\lambda) \approx \tilde{N}(\lambda)$ on $|\lambda + 1| = r$. We choose
\[
E_1(\lambda) = \tilde{N}(\lambda) e^{\frac{\pi i}{2} \alpha i \sigma_3} \frac{I - i \sigma_1}{\sqrt{2}} \left( s^\frac{\pi}{2} f_1(\lambda) \right)^{\frac{1}{2} \sigma_3} \quad \text{for } \pm \Im \lambda > 0, \lambda \in U(-1, r). \tag{6.16}
\]

Indeed, straightforward verification gives $(E_1)_+ (\lambda) = (E_1)_- (\lambda)$ for $\lambda \in (-1 - r, -1 + r)$, and $E_1(\lambda)$ is bounded at $\lambda = -1$, which implies that $E_1(\lambda)$ is analytic in $U(-1, r)$ for $r$ sufficiently small. Moreover, it is readily clarify that, with $E_1(\lambda)$ and $P_1(\lambda)$ so defined, we have the matching condition
\[
P_1(\lambda) \tilde{N}(\lambda)^{-1} = I + O\left( s^{-1/3} \right), \quad |\lambda + 1| = r. \tag{6.17}
\]

To complete the Riemann-Hilbert analysis, we introduce the final transformation $X \to R_1$ as
\[
R_1(\lambda) = \begin{cases} 
X(\lambda, s) \tilde{N}^{-1}(\lambda), \quad & |\lambda + 1| > r, \\
X(\lambda, s) P_1^{-1}(\lambda), \quad & |\lambda + 1| < r.
\end{cases} \tag{6.18}
\]
The matrix function \( R_1(\lambda) \) is analytic in \( \mathbb{C} \setminus \Sigma_{R_1} \), where \( \Sigma_{R_1} \) consists of the parts \( \Sigma_k^+ \setminus U(-1, r) \) for \( k = 1,3 \), and the circular part \( \partial U(-1, r) \), along with the line segment \([-1 + r, 0)\); see Figure 6. \( R_1(\lambda) \) is perfectly normalized at infinity and at \( \lambda = -1 \), is also of \( O(1) \) at \( \lambda = 0 \); cf. (6.8) and (6.11). When \( s \to \infty \), the jump for \( R_1(\lambda) \) on \( \Sigma_{R_1} \) has the following behavior:

\[
J_{R_1}(\lambda) = \begin{cases} 
I + O(s^{-1/3}), & |\lambda + 1| = r, \\
I + O\left(e^{-cs^{1/3}}\right), & \Sigma_{R_1} \setminus \partial U(-1, r),
\end{cases}
\]

where \( c \) is a positive constant. For instance, on \((-1 + r, 0)\), a combination of (6.6), (6.12) and (6.18) yields

\[
J_{R_1}(\lambda) = I + |\lambda|^\alpha e^{4/3\theta(\lambda)}O(1) = I + |\lambda|^\alpha \exp\left(-\frac{\lambda + 1)^{3/2}}{|\lambda|}\right)O(1),
\]

which is of the form given in (6.19), uniformly on \((-1 + r, 0)\). Similarly, jumps on other contours can be estimated. So, by an argument as in Section 3.7 we have

\[
R_1(\lambda) = I + O(s^{-1/3}), \quad s \to \infty,
\]

where the error term is uniform in \( \lambda \), being kept away from \( \Sigma_{R_1} \). Furthermore, for large \( \lambda \), we have

\[
R_1(\lambda) = I + O\left(s^{-\frac{5}{2}}\right), \quad s \to \infty, \quad \lambda \to \infty.
\]

This complete the nonlinear steepest descend analysis of \( \Psi(\zeta, s) \) as \( s \to \infty \).

### 6.2 Proof of Theorem 3

Now we apply the above asymptotic results for \( \Psi(\zeta, s) \) as \( s \to \infty \), obtained by conducting the nonlinear steepest descent analysis, to achieve the transition of \( \Psi \)-kernel to the Airy kernel, and to extract asymptotics of the nonlinear equations in (2.8) and (2.9), derived from the compatibility conditions of the Lax pair (1.13)-(1.14) of \( \Psi(\zeta, s) \).

Tracing back the transformations \( \Psi(\zeta, s) \to U(\lambda, s) \to X(\lambda, s) \to R_1(\lambda) \); cf. (6.1), (6.5) and (6.18), it follows from the approximation (6.20) that

\[
\Psi(s^{2/3}, \lambda, s) = s^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{3}{2}is^{1/3} & 1 \end{pmatrix} \left(I + O\left(s^{-1/3}/\lambda\right)\right) \tilde{N}(\lambda)e^{s^{1/3}\theta(\lambda)\sigma_3}
\]

for large \( \lambda \) and large \( s \). Here use has been made of the fact that \( e^{-2s^{1/3}\theta(\lambda)} \) is exponentially small for \( \lambda \in \Omega_5 \cup \Omega_6 \) as \( \lambda \to \infty \).

Substituting the behavior (6.10) at infinity for \( \tilde{N}(\lambda) \) into (6.22) yields

\[
\Psi(s^{2/3}, \lambda, s)e^{-s^{1/3}/\sqrt{\lambda}\sigma_3} = (s^{2/3}\lambda)^{-\frac{1}{3}\sigma_3} \left[I + O\left(s^{3/2}/\sqrt{\lambda}\right)\right] \frac{I + i\sigma_1}{\sqrt{2}}
\]

for \( \lambda \to \infty \) and \( s \to \infty \).

Similarly, for \( |\lambda + 1| < r \), a combination of (6.1), (6.5) and (6.18)) gives

\[
\Psi(s^{2/3}, \lambda, s) = s^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{3}{2}is^{1/3} & 1 \end{pmatrix} R_1(\lambda)P_1(\lambda)e^{s^{1/3}\theta(\lambda)\sigma_3}
\]

for large \( \lambda \) and \( s \to \infty \).
for \( \arg(\lambda + 1) \in (\frac{2}{3}\pi, \pi) \cup (-\pi, -\frac{2}{3}\pi) \), i.e., for \( \lambda \in \Omega_2 \cup \Omega_3 \); cf. Figure 6 such that \( |\lambda + 1| < r \), where \( R_1(\zeta) = I + O(s^{-1/3}) \); see (6.20), and \( P_1(\zeta) \) is constructed in terms of the Airy function; see (6.13). Similar formulas are also true for \( \lambda \) in other sectors. Thus \( \Psi(s^{2/3}\lambda, s) \) is represented by the solution \( \Phi_A \) to the model Airy RH problem as

\[
\Psi(s^{2/3}\lambda, s) = s^{-\frac{1}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -\frac{3}{2}is^{1/3} & 1 \end{pmatrix} R_1(\lambda)E_1(\lambda)\Phi_A\left(s^{\frac{2}{3}}f_1(\lambda)\right) e^{\pm \frac{i}{2}\alpha_1s^{\sigma_3}},
\]

(6.25) respectively for \( \pm \text{Im}\lambda > 0 \), where \( |\lambda + 1| < r \).

Similar to the derivation leading to (5.22), we obtain (1.30). In deriving (1.30), we need to approximate, case by case, the function \( \Psi(s^{2/3}\lambda, s) \) for \( \zeta \in \Omega_3 \), and for \( \zeta \in \Omega_6 \); cf. Figure 6. Also, we put in use the technique of extracting (4.9) from (4.8). It is noted that let \( \lambda = -1 + \frac{u}{\sqrt{s^3}} \), then the phase variable \( s^{2/9}f_1(\lambda) \) can be expanded into a Maclaurin series in \( u \) for bounded \( u \) and large \( s \), and that \( s^{2/9}f_1(\lambda) = u\left[1 + O\left(u/s^{2/9}\right)\right] \); cf. (6.14). Repeated use has been made of the formula (5.21) as well.

Now we turn to a brief discussion of the asymptotic properties of the nonlinear equation (2.9). For \( \text{Re}\lambda > 0 \), (6.5), (6.18) and (3.33) imply that

\[
U(\lambda) = \left(I + O\left(\frac{s^{-1/3}}{\lambda}\right)\right) \tilde{N}(\lambda) \quad \text{as} \lambda \to \infty.
\]

(6.26)

On the one hand, substituting (2.2) and (2.5) into (6.1), we approximate \( U(\lambda) \) for large \( \lambda \)

\[
U(\lambda)M^{-1}\lambda^{\frac{1}{2}\sigma_3} = I + \frac{1}{\lambda} \left( -\frac{3}{2}r + \frac{9}{8}rs^{\frac{3}{4}} + qs^{\frac{3}{4}} \right) \left( \frac{3}{2}s^{\frac{3}{4}} - rs^{-\frac{1}{2}} \right) + O\left(\frac{1}{\lambda^2}\right).
\]

(6.27)

On the other hand, we expand \( \tilde{N}(\lambda) \) in (6.12) for large \( \lambda \) as

\[
\tilde{N}(\lambda)M^{-1}\lambda^{\frac{1}{4}\sigma_3} = I + \frac{1}{\lambda} \left( \frac{\alpha^2}{2} - \frac{1}{4} \right) - \frac{i\alpha}{4} + O\left(\frac{1}{\lambda^2}\right).
\]

(6.28)

Putting all these together, we obtain

\[
r(s) = \frac{3}{2}s^{\frac{2}{4}} - \alpha s^{\frac{1}{4}} + O(1) \quad \text{as} \ s \to +\infty.
\]

(6.29)

Now we turn to the last part of the proof, that is, the \( \Psi \)-kernel to Airy kernel transition, as \( s \to \infty \). To begin, we calculate a quantity \( \alpha_n \), in the variable \( z \) used in Section 3, \( z = \alpha_n \) corresponding to \( \lambda = -1 \) via the (not re-scaled) conformal mapping \( \zeta = n^2\left\{\phi\left(\frac{z}{4n}\right)\right\}^2 \approx -4nz \) in (3.25), and the re-scaling \( \zeta = s^{2/3}\lambda \) in (6.1). We see that \( \alpha_n = s^{2/3}/4n \) makes \( n^2\left\{\phi\left(\frac{\alpha_n}{4n}\right)\right\}^2 \sim -s^{2/3} \). We note that \( \alpha_n \to 0 \) as \( n \to \infty \), since \( s = 2nt \) and \( t \) is bounded from above.

It is worth pointing out that \( \alpha_n \) plays a role in a direct calculation of the equilibrium measure of the weight (1.1), and serves as a so-called Mhaskar-Rahmanov-Saff (MRS) number, or sometimes termed a soft-edge of the spectrum. So, the edge behavior of statistic quantities, such as the large-\( n \) limit of the kernel \( K_n(x, y) \), requires further investigation; cf. [44] for the determination of the soft-edges in a similar case.

Fortunately, the steepest descent analysis conducted earlier in the present section has constructed an Airy type asymptotic approximation of \( \Psi(s^{2/3}\lambda, s) \); see (6.24) and (6.13).
approximation is given in a normal-sized neighborhood of \( \lambda = -1 \), and, equivalently, a shrinking neighborhood of \( z = \alpha_n \) of size \( O(\alpha_n) \).

Substituting (6.15) and (6.25) into the \( \Psi \)-kernel in (4.9), similar to the derivation leading to (5.22), we have

\[
\lim_{n \to \infty} \frac{\alpha_n}{c_{S^2/3}^2} K_n \left( \alpha_n - \frac{\alpha_n}{c_{S^2/3}^2} u, \alpha_n - \frac{\alpha_n}{c_{S^2/3}^2} v; t \right) = \frac{\Ai(u)\Ai'(v) - \Ai(v)\Ai'(u)}{u - v}
\]

(6.30)
as \( s = 2nt \to +\infty \), where \( \alpha_n = s^{2/3}/(4n) \), \( c = (3/2)^{2/3} \), and the limit is uniformly taken for bounded \( u, v \in \mathbb{R} \). This is exactly (1.31).

Thus we complete the proof of Theorem 3.

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