A note on actions of the symplectic group $\text{Sp}(2g, \mathbb{Z})$ on homology spheres

Bruno P. Zimmermann

Università degli Studi di Trieste
Dipartimento di Matematica e Informatica
34127 Trieste, Italy
zimmer@units.it

Abstract. The symplectic group $\text{Sp}(2g, \mathbb{Z})$ is a subgroup of the linear group $\text{SL}(2g, \mathbb{Z})$ and admits a faithful action on the sphere $S^{2g-1}$, induced from its linear action on Euclidean space $\mathbb{R}^{2g}$. Generalizing corresponding results for linear groups, we show that, if $m < 2g - 1$ and $g > 2$, any continuous action of $\text{Sp}(2g, \mathbb{Z})$ on a homology $m$-sphere, and in particular on $S^m$, is trivial.

1. Introduction

The linear group $\text{SL}(n, \mathbb{Z})$ admits a faithful, linear action on Euclidean space $\mathbb{R}^n$, on the sphere $S^{n-1}$ and on the torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$. On the other hand, by [We] any smooth action of $\text{SL}(n, \mathbb{Z})$ on the torus $T^m$ is trivial if $m < n$ and $n > 2$, and by results of Parwani [Pa] and the author [Z1], any smooth action of the linear group $\text{SL}(n, \mathbb{Z})$ on a mod 2 homology $m$-sphere is trivial if $m < n - 1$ and $n > 2$. We note that the results in [Pa] are stated for arbitrary continuous actions but it is noticed in [BV, Remarks 4.16 and 4.17] that some of the arguments involving Smith fixed point theory are not correct in this more general setting; in [BV], the result is obtained for continuous actions as a consequence of a corresponding result for actions of the automorphism group of a free group of rank $n$ on homology spheres.

Our main result is the following analogue for actions of the symplectic group $\text{Sp}(2g, \mathbb{Z})$ (which admits a faithful, linear action on $S^{2g-1}$).

**Theorem.** For $m < 2g - 1$ and $g \geq 3$, any continuous action of the symplectic group $\text{Sp}(2g, \mathbb{Z})$ on a homology $m$-sphere, and in particular on $S^m$, is trivial (for $g = 2$, it factors through the action of a finite group).

In fact, we will prove the Theorem for mod 3 homology $m$-spheres, i.e. for manifolds with the mod 3 homology of the $m$-sphere (homology with coefficients in the integers mod 3).
It should be noted that the proofs of most of these results depend strongly on the existence of torsion, that is of certain types of finite subgroups in the groups considered (3-torsion in the case of the Theorem), and that the real challenge is to prove such results for arbitrary, possibly torsion-free subgroups of finite index. For the case of $S^1$, Witte [Wi] has shown that every continuous action of a subgroup of finite index in $\text{SL}(n,\mathbb{Z})$ on the circle $S^1$ factors through a finite group action, for $n > 2$. In the context of the Zimmer program for actions of irreducible lattices in semisimple Lie groups of $\mathbb{R}$-rank at least two, it is conjectured by Farb and Shalen ([FS]) that any smooth action of a finite-index subgroup of $\text{SL}(n,\mathbb{Z})$ on a compact $m$-manifold factors through the action of a finite group if $m < n - 1$ and $n > 2$.

Abelianization of the fundamental group $\pi_1(F_g)$ of a closed, orientable surface $F_g$ of genus $g$ induces a surjection

$$\text{MC}(F_g) \rightarrow \text{Sp}(2g,\mathbb{Z})$$

of the mapping class group $\text{MC}(F_g) \cong \text{Out}_+(\pi_1(F_g))$ of isotopy classes of orientation-preserving homeomorphisms of $F_g$ onto the symplectic group $\text{Sp}(2g,\mathbb{Z})$, and the following question naturally arises: What is the minimal dimension of a non-trivial (or infinite image, or faithful) action of the mapping class group $\text{MC}(F_g)$ on a sphere or a homology sphere? See [BV] for the case of the outer automorphism group $\text{Out}_+(F_n)$ of a free group $F_n$ of rank $n$. The case of mapping class groups appears more difficult, due to the irregular distribution of torsion depending on the genus; we present the following partial result for genus three. Note that, via the surjection $\text{MC}(F_g) \rightarrow \text{Sp}(2g,\mathbb{Z})$, the mapping class group $\text{MC}(F_g)$ admits a non-trivial, linear action on $S^{2g-1} \subset \mathbb{R}^{2g}$.

**Proposition.** Any smooth action of the mapping class group $\text{MC}(F_3)$ on a homology sphere of dimension less than five is trivial.

But even in this case the minimal dimension of a faithful action, or of an action with infinite image, remains open; note that $\text{MC}(F_3)$ acts faithfully on the Teichmüller space in genus three, homeomorphic to $\mathbb{R}^{12}$, and on its boundary homeomorphic to $S^{11}$.

2. **Proof of the Theorem**

Our main reference for the symplectic group $\text{Sp}(2g,\mathbb{Z})$ is the book of Newman [N].

Fixing a basis $a_1, b_1, \ldots, a_g, b_g$ of the free abelian group $\mathbb{Z}^{2g}$, one considers the antisymmetric or symplectic bilinear form on $\mathbb{Z}^{2g}$ defined by

$$a_i \times b_i = 1, \quad a_i \times b_j = 0 \text{ for } i \neq j, \quad a_i \times a_j = 0,$$

that is with matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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with respect to the chosen basis (so $J$ is a $(2g \times 2g)$ matrix consisting of $g$ blocks).

The symplectic group $\text{Sp}(2g, \mathbb{Z})$ is the group of automorphisms of $\mathbb{Z}^{2g}$ which preserve this bilinear form, or equivalently the subgroup of integer $2g \times 2g$ matrices $A$ of the linear group $\text{SL}(2g, \mathbb{Z})$ such that $AJAT = J$ (where $A^T$ denotes the transposed matrix); note that $\text{Sp}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})$.

For a fixed $i$, we consider the symplectic automorphism $\phi_i$ of order three of $\mathbb{Z}^{2g}$ which fixes all $a_j, b_j$ with $j \neq i$, and with $\phi_i(a_i) = -a_i - b_i$, $\phi_i(b_i) = a_i$, that is with matrix

$$
\begin{pmatrix}
-1 & 1 \\
-1 & 0
\end{pmatrix}
$$

of order three on the subgroup $\mathbb{Z}^2 = \langle a_i, b_i \rangle$ of $\mathbb{Z}^{2g}$. We denote by $\mathbb{Z}_3^g$ the elementary abelian subgroup of $\text{Sp}(2g, \mathbb{Z})$ generated by the symplectic automorphisms $\phi_1, \ldots, \phi_g$.

Let $g \geq 3$. We consider an action of $\text{Sp}(2g, \mathbb{Z})$ on a mod 3 homology sphere $M$ of dimension less than $2g - 1$, and have to show that the action is trivial. We denote by $U$ the normal subgroup of $\text{Sp}(2g, \mathbb{Z})$ consisting of all elements which act trivially on $M$. Crucial for the proof of the Theorem is the following well-known result from Smith fixed point theory ([S]; see also [BV] for a useful discussion of the concept of a generalized manifold suited for the proofs).

**Proposition 1.** For a prime $p$, the minimal dimension of a faithful, continuous action of the elementary abelian group $\mathbb{Z}_p^k$ on a mod $p$ homology sphere is $2k - 1$ if $p$ is odd, and $k - 1$ if $p = 2$.

By Proposition 1, some element $u$ of order three in $\mathbb{Z}_3^g$ acts trivially on $M$, that is lies in the kernel $U$ of the action of $\text{Sp}(2g, \mathbb{Z})$ on $M$; note that $u$ is non-central in $\text{Sp}(2g, \mathbb{Z})$. By the congruence subgroup property for the symplectic groups ([Me], and [BMS] for a more general version), for $g > 1$ any non-central, normal subgroup $U$ of the symplectic group $\text{Sp}(2g, \mathbb{Z})$ has finite index and contains a congruence subgroup $C(k)$, for some positive integer $k$; here $C(k)$ denotes the kernel of the canonical map $\text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}/k\mathbb{Z})$ which is surjective by [N, Theorem VII.21], so we have an exact sequence

$$1 \to C(k) \to \text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}/k\mathbb{Z}) \to 1.$$

Hence the action of $\text{Sp}(2g, \mathbb{Z})$ on $M$ factors through the action of a finite group $G$, with $G \cong \text{Sp}(2g, \mathbb{Z})/U$, and there is a surjection from the finite group $\text{Sp}(2g, \mathbb{Z})/C(k) \cong \text{Sp}(2g, \mathbb{Z}/k\mathbb{Z})$ to $\text{Sp}(2g, \mathbb{Z})/U \cong G$ which we denote by

$$\Phi : \text{Sp}(2g, \mathbb{Z}/k\mathbb{Z}) \to G.$$

Remark. In order to see that the action of $\text{Sp}(2g, \mathbb{Z})$ on $M$ factors through the action of a finite group $G$, one may apply also the Margulis finiteness theorem which states
that an irreducible lattice in a semisimple Lie group of real rank at least two is almost simple, that is any normal subgroup of the lattice is either of finite index, or contained in the center of the semisimple Lie group (see [Ma] or [Z, Theorem 8.1.2]). Note that the Margulis finiteness theorem applies to the irreducible lattice $\text{Sp}(2g, \mathbb{Z})$ in the semisimple Lie group $\text{Sp}(2g, \mathbb{R})$, for $g \geq 2$.

We have to show that the finite quotient $G$ of $\text{Sp}(2g, \mathbb{Z})$ is trivial; assume, by contradiction, that it is not. Then, since $\text{Sp}(2g, \mathbb{Z})$ is perfect for $g \geq 3$, also $G$ is perfect and hence non-solvable (more generally, by [Po], $\text{MC}(\mathcal{F}_g)$ is perfect for $g \geq 3$ whereas the abelianization of $\text{MC}(\mathcal{F}_2)$ is cyclic of order 10).

Considering the prime decomposition $k = p_1^{r_1} \ldots p_s^{r_s}$ of $k$, one has

$$\text{Sp}(2g, \mathbb{Z}/k\mathbb{Z}) \cong \text{Sp}(2g, \mathbb{Z}/p_1^{r_1}\mathbb{Z}) \times \ldots \times \text{Sp}(2g, \mathbb{Z}/p_s^{r_s}\mathbb{Z})$$

(see [N, Theorem VII.26]). The restriction of $\Phi : \text{Sp}(2g, \mathbb{Z}/k\mathbb{Z}) \to G$ to some factor $\text{Sp}(2g, \mathbb{Z}/p_i^{r_i}\mathbb{Z}) = \text{Sp}(2g, \mathbb{Z}/p^{r}\mathbb{Z})$ has to be non-trivial and induces a surjection

$$\Phi_0 : \text{Sp}(2g, \mathbb{Z}/p^{r}\mathbb{Z}) \to G_0$$

onto a perfect, non-solvable subgroup $G_0$ of $G$; we denote by $U_0$ the kernel of $\Phi_0$ (the elements of $\text{Sp}(2g, \mathbb{Z}/p^{r}\mathbb{Z})$ acting trivially on $M$).

Let $K$ denote the kernel of the canonical surjection $\text{Sp}(2g, \mathbb{Z}/p^r\mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$, so $K$ consists of all matrices in $\text{Sp}(2g, \mathbb{Z}/p^r\mathbb{Z})$ which are congruent to the identity matrix $I = I_{2g}$ when entries are taken mod $p$. By performing the binomial expansion of $(I + pA)^{p^{r-1}}$ one checks that $K$ is a $p$-group, in particular $K$ is solvable.

Let $K_0$ denote the kernel of the surjection from $\text{Sp}(2g, \mathbb{Z}/p^r\mathbb{Z})$ to the central quotient $\text{PSp}(2g, \mathbb{Z}/p\mathbb{Z})$ of $\text{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$; also $K_0$ is solvable and, since $g \geq 3$, $\text{PSp}(2g, \mathbb{Z}/p\mathbb{Z})$ is a non-abelian simple group ($\text{PSp}(4, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to the symmetric group $S_6$).

Note that the element $u$ of order three in $\text{Sp}(2g, \mathbb{Z})$ considered above injects into the successive quotients $\text{Sp}(2g, \mathbb{Z}/k\mathbb{Z})$, $\text{Sp}(2g, \mathbb{Z}/p^r\mathbb{Z})$, $\text{Sp}(2g, \mathbb{Z}/p\mathbb{Z})$ and $\text{PSp}(2g, \mathbb{Z}/p\mathbb{Z})$, hence the normal subgroup $U_0$ of $\text{Sp}(2g, \mathbb{Z}/p^r\mathbb{Z})$ surjects onto the finite simple group $\text{PSp}(2g, \mathbb{Z}/p\mathbb{Z})$.

We consider the two exact sequences

$$1 \to K_0 \to \text{Sp}(2g, \mathbb{Z}/p^r\mathbb{Z}) \to \text{PSp}(2g, \mathbb{Z}/p\mathbb{Z}) \to 1,$$

$$1 \to U_0 \cap K_0 \to U_0 \to \text{PSp}(2g, \mathbb{Z}/p\mathbb{Z}) \to 1.$$
This completes the proof of the Theorem.

2. Proof of the Proposition

The mapping class group MC(F_3) has a finite subgroup isomorphic to the linear fractional group PSL(2, Z/7Z) of order 168, induced from the Hurwitz action of the smallest Hurwitz group PSL(2, Z/7Z) on Klein’s quartic of genus three (i.e., of maximal possible order 84(g – 1)). By [Z3, Proposition 1] and [MZ, Proposition 2.1], the simple group PSL(2, Z/7Z) does not admit a non-trivial, smooth action on a homology sphere of dimension less than five (see also [Z4]). On the other hand, by [Z2, Lemma 1] any non-trivial homomorphism from MC(F_3) to an arbitrary group has to inject PSL(2, Z/7Z), hence any action of MC(F_3) on a homology sphere of dimension less than five is trivial.

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