NONAUTONOMOUS VECTOR FIELDS ON $S^3$:
SIMPLE DYNAMICS AND WILD EMBEDDING OF SEPARATRICES

V. Z. Grines* and L. M. Lerman*

We construct new substantive examples of nonautonomous vector fields on a 3-dimensional sphere having simple dynamics but nontrivial topology. The construction is based on two ideas: the theory of diffeomorphisms with wild separatrix embedding and the construction of a nonautonomous suspension over a diffeomorphism. As a result, we obtain periodic, almost periodic, or even nonrecurrent vector fields that have a finite number of special integral curves possessing exponential dichotomy on $\mathbb{R}$ such that among them there is one saddle integral curve (with a $(3, 2)$ dichotomy type) with a wildly embedded 2-dimensional unstable separatrix and a wildly embedded 3-dimensional stable manifold. All other integral curves tend to these special integral curves as $t \to \pm \infty$. We also construct other vector fields having $k \geq 2$ special saddle integral curves with the tamely embedded 2-dimensional unstable separatrices forming mildly wild frames in the sense of Debrunner–Fox. In the case of periodic vector fields, the corresponding specific integral curves are periodic with the period of the vector field, and are almost periodic in the case of an almost periodic vector field.

Keywords: nonautonomous vector field, integral curve, uniform equivalence, exponential dichotomy, separatrix, wild embedding

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1. Introduction

The aim of this paper is to present new substantive examples of nonautonomous periodic, almost periodic, or even nonrecurrent vector fields on a 3-dimensional sphere $S^3$. The main feature of the constructed vector fields is that they have a saddle integral curve (IC) $\gamma$ whose 2-dimensional unstable and 3-dimensional stable invariant manifolds are wildly embedded in the extended phase manifold $S^3 \times \mathbb{R}$. Thus, this provides new invariants of a uniform equivalence of nonautonomous vector fields (see definitions below). Otherwise, from the dynamical standpoint, the vector field has a quite simple structure of its foliation into ICs in $S^3 \times \mathbb{R}$.

It is to be seen from the construction that the method allows obtaining nonautonomous uniformly dissipative vector fields in $\mathbb{R}^3$ with a similar structure whose ICs enter some cylindrical domain of the form $D^3 \times \mathbb{R}$, $D^3 \subset \mathbb{R}^3$, with its boundary manifold $S^2 \times \mathbb{R}$ being uniformly transverse to the ICs.

*National Research University “Higher School of Economics in Nizhny Novgorod,” Nizhny Novgorod, Russia, e-mails: vgrines@yandex.ru, lermanl@mm.unn.ru (corresponding author).

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We first recall the notion of a nonautonomous vector field on a smooth \((\mathcal{C}^\infty)\) connected closed manifold \(M\). Let \(\mathcal{V}(M)\) be the Banach space of \(\mathcal{C}^r\)-smooth, \(r \geq 1\), vector fields on \(M\) endowed with the \(\mathcal{C}^r\)-norm. By a \(\mathcal{C}^r\)-smooth nonautonomous vector field (NVF) on \(M\), we understand a uniformly continuous bounded map \(v: \mathbb{R} \to \mathcal{V}(M)\). We endow the set of NVFs with the supremum norm of the related maps. As a particular case, we may think on a periodic NVF if the map \(v\) is periodic: there exist a positive \(T \in \mathbb{R}\) such that \(v(t + T) \equiv v(t)\) for all \(t \in \mathbb{R}\). If the map \(v\) is almost periodic \([1]\), \([2]\), we speak of almost periodic NVFs.

A solution for an NVF \(v\) is a \(\mathcal{C}^1\)-differentiable map \(x: I \to M\), where \(I\) is an interval of \(\mathbb{R}\) such that for any \(t \in I\), the tangent vector \(x'(t) = D_{x(t)}(1) \in T_{x(t)}M\) coincides with the vector \(v_t(x(t)) = v(x(t), t)\). Here, we standardly identify the tangent space \(T_t\mathbb{R}\) at the point \(t \in \mathbb{R}\) with \(\mathbb{R}\) itself by shifts in \(\mathbb{R}\).

By the standard existence and uniqueness theorem, there is a unique solution passing through any initial point \((x_0, \tau) \in M \times \mathbb{R}\). On a closed manifold \(M\), any solution for an NVF is extended to the entire \(\mathbb{R}\).

The graph of a solution \(x\), i.e., the set \(\bigcup_{t \in I} (x(t), t) \subset M \times \mathbb{R}\), is an IC of \(x\). Thus, every NVF \(v\) generates a foliation \(\mathcal{L}_v\) of the manifold \(M \times \mathbb{R}\) into ICs of \(v\). An example of such a foliation in the case \(M = I\) is plotted in Fig. 1.

![Fig. 1. Foliation into ICs on a segment, \(M = I\).](image)

Following \([3]\) we say that two NVFs \(v_1\) and \(v_2\) on \(M\) are uniformly equivalent if there is an equimorphism (see the Appendix for the definition) \(\Phi: M \times \mathbb{R} \to M \times \mathbb{R}\) that transforms \(\mathcal{L}_{v_1}\) into \(\mathcal{L}_{v_2}\) preserving the orientation in \(\mathbb{R}\). Here, we consider the manifold \(M \times \mathbb{R}\) with the uniform structure of the direct product of a unique uniform structure on \(M\) given by the topology of \(M\) (which is a compact manifold) and the standard uniform structure on \(\mathbb{R}\) invariant with respect to shifts on an Abelian group (see the Appendix for definitions of the uniform structure and equimorphism).

The NVFs that we construct below fall into the class of gradient-like NVFs defined in \([3]\), \([4]\). They satisfy several restrictions on the structure of their foliation \(\mathcal{L}_v\), including the exponential dichotomy condition for any of their solutions on both semiaxes \(\mathbb{R}_+\) and \(\mathbb{R}_-\) (dichotomy types may be distinct for the two semiaxes \([5]\)). This allows obtaining invariant stable manifolds and unstable manifolds. The entire extended phase manifold \(M \times \mathbb{R}\) is thus partitioned into smooth stable manifolds; another partition is generated by unstable manifolds. One more assumption is the finiteness of both partitions (although they can be completely different). We do not go deep into the details of these restrictions (see \([3]\), \([4]\)) because they are specified explicitly in the examples discussed in this paper.

The NVF on \(S^3\) that we construct has four special ICs, each with exponential dichotomy on all of \(\mathbb{R}\). One such IC, \(\gamma_\alpha\), is exponentially unstable on \(\mathbb{R}\); there is also a saddle-type IC, \(\gamma_\sigma\), with an exponential
type-(3, 2) dichotomy on \( \mathbb{R} \), which means that such an IC has a 3-dimensional stable manifold in \( S^3 \times \mathbb{R} \) and a 2-dimensional unstable manifold. Finally, our NVF has two ICs that are exponentially stable on \( \mathbb{R} \), their stable manifolds being 4-dimensional (of dichotomy type \((1, 4)\)).

We clarify the term “wild embedding.” For any section \( t = t_0 \) in \( S^3 \times \mathbb{R} \), its intersection with a wildly embedded 2-dimensional unstable manifold is a 1-dimensional ray in \( S^3_{t_0} \) wildly embedded in the topological sense [6]–[8] (see below). The topological limit of this ray is a point, which is the trace of an exponentially stable IC \( \gamma_{\omega_2} \). All the ICs in this half of the unstable manifold tend to \( \gamma_{\sigma} \) as \( t \to -\infty \) and to \( \gamma_{\omega_2} \) as \( t \to \infty \).

Fig. 2. Phase portrait of a diffeomorphism on \( S^3 \) with wildly embedded separatrices.

Also, the trace of the 3-dimensional stable manifold of \( \gamma_{\sigma} \) on \( M_{t_0} \) is an embedded \( \mathbb{R}^2 \) and its closure in \( M_{t_0} \) is an embedded sphere that is wild at one point, the trace of the IC \( \gamma_\alpha \).

The construction of such NVFs relies on two ideas. One belongs to Pixton [7] and was developed further by Bonatti, Grines, Pochinka, and others [8]–[10]. In [7], a simple 3-dimensional Morse–Smale diffeomorphism on \( S^3 \) was constructed whose nonwandering set consists of only four hyperbolic fixed points: one source \( \alpha \), one saddle \( \sigma \) of type \((2, 1)\), and two sinks \( \omega_1 \) and \( \omega_2 \). Moreover, the closure of the stable manifold \( w^s_\sigma \) of \( \sigma \) is homeomorphic to the sphere smoothly embedded at each point except the point \( \alpha \), where it is wildly embedded (see Fig. 2). The point \( \sigma \) divides the unstable manifold \( w^u_\sigma \) into two separatrices \( l^u_1, l^u_2 \). The closure of one of these separatrices \((l^u_2 \text{ in Fig. 2})\) is a simple arc that is wildly embedded at the point \( \omega_2 \), but the closure of the other arc \((l^u_1)\) is tamely embedded at each point. The separatrix \( l^u_2 \) tends to the sink \( \omega_2 \) such that the fundamental domain near this sink, after identifying points on its boundary, contains the image of the separatrix that makes up a nontrivial knot in the associated quotient space (which is homeomorphic to \( S^2 \times S^1 \)). This is explained in more detail in Sec. 2. The nontriviality of this knot implies the wild embedding of the separatrix of the diffeomorphism; moreover, if two different diffeomorphisms have nonhomeomorphic knots in \( S^2 \times S^1 \), then they are not topologically conjugated. This fact leads to the existence of an infinite number of topologically nonconjugated Morse–Smale diffeomorphisms of the considered type [8].

The second idea is borrowed from [11]. It uses the construction of a so-called nonautonomous suspension over a diffeomorphism introduced in [12] and developed further in [11]. To recall this construction, we suppose that \( f: M \to M \) is a diffeomorphism of a smooth \((C^\infty)\) closed manifold \( M \). To avoid a discussion on the class of smoothness for the suspension, we assume \( f \) to be \( C^\infty \)-smooth. Its (usual) suspension [13] is a smooth closed manifold \( M_f \) of dimension \( \dim M + 1 \) defined as follows. In the cylinder \( M \times I \), \( I = [0, 1] \), we identify points \((x, 1) \) and \((f(x), 0) \). It is more convenient to consider the manifold \( M \times \mathbb{R} \) with an action \( F \) of the group \( \mathbb{Z} \) defined by the following rule: for \( m \in \mathbb{Z} \), the corresponding \( F^m \) acts as \( F^m(x, s) = (f^m(x), s - m) \). This action is free and discrete (any orbit of the action has no accumulation points). Thus, the quotient \( M_f = (M \times \mathbb{R})/F \) is a smooth manifold, being a smooth bundle over the
circle $S^1$, $p: M_f \to S^1$, with a fiber $M$. A vector field is generated on $M_f$ by the constant vector field $V = (0, 1)$ on $M \times \mathbb{R}$ (its orbits are straight lines $(x, t)$, $t \in \mathbb{R}$). After taking the quotient, we obtain a smooth vector field $v_f$ on $M_f$ with a global cross section, which can be chosen as any $M_\theta = p^{-1}(\theta)$, $\theta \in S^1$. The Poincaré map defined on this cross section is conjugated to $f$. This construction allows obtaining vector fields with the dynamics similar to that of iterations of the diffeomorphism $f$ (see [13]).

To proceed, we consider the covering manifold $\widetilde{M}_f$ for $M_f$ generated by the standard covering $\mathbb{R} \to S^1$, $t \to \exp[2\pi it]$, which gives the commutative diagram

$$
\begin{array}{ccc}
\widetilde{M}_f & \xrightarrow{\exp} & M_f \\
\downarrow p & & \downarrow p \\
\mathbb{R} & \xrightarrow{\exp} & S^1
\end{array}
$$

Topologically, $\widetilde{M}_f$ is homeomorphic to $M \times \mathbb{R}$ because $\mathbb{R}$ is contractible. The manifold $M_f$ is compact and hence has a unique uniform structure compatible with its topology [14]. The uniform structure in $M_f$ is defined by lifting the uniform structure in $M_f$ by the map $\exp$. This is easier to understand if we endow $M_f$ with a smooth Riemannian metric and lift this metric to $\widetilde{M}_f$ by the covering map $\exp$. Because $\exp$ is a local diffeomorphism, we obtain a Riemannian metric on $\widetilde{M}_f$ such that $\exp$ is a local isometry. The foliation in $M_f$ into orbits of the vector field $v_f$ is lifted as some foliation $\mathcal{L}_{v_f}$ into infinite curves in $\widetilde{M}_f$. This foliation is homeomorphic to the foliation of $M \times \mathbb{R}$ into straight lines $(x, t)$, $t \in \mathbb{R}$, because $\widetilde{M}_f$ is homeomorphic to $M \times \mathbb{R}$, but generically the foliation in $\widetilde{M}_f$ is not equimorphic to the foliation into straight lines. Moreover, even the manifold $\widetilde{M}_f$ itself with its uniform structure lifted from $M_f$ is not always equimorphic to $M \times \mathbb{R}$. For instance, this is the case for $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with $f$ being the Anosov diffeomorphism (see the details in [11]). The next proposition, which is Corollary 4.1 in [11], is useful in what follows.

**Proposition 1.** If $\widetilde{M}_f$ is equimorphic to $M \times \mathbb{R}$, then there is an $n \in \mathbb{Z}$ such that $f^n$ is homotopic to $\text{id}_M$.

Given a diffeomorphism $f: M \to M$ of a smooth ($C^\infty$) closed manifold $M$, an important question is whether an NVF $v$ exists on $M$ such that its foliation $\mathcal{L}_v$ into ICs in $M \times \mathbb{R}$ (with its uniform structure of the direct product) is equimorphic to the foliation $\mathcal{L}_{v_f}$ into infinite curves generated by $v_f$ in $\widetilde{M}_f$. Evidently, the first condition for this to be true is that the uniform spaces $M \times \mathbb{R}$ and $\widetilde{M}_f$ be equimorphic. This gives a meaning to a definition introduced in [11].

**Definition 1.** A diffeomorphism $f: M \to M$ is said to be reproduced by an NVF $v$ on $M$ (or, equivalently, $v$ is said to reproduce the structure of $f$) if the foliations $\mathcal{L}_v$ in $M \times \mathbb{R}$ and $\mathcal{L}_{v_f}$ in $\widetilde{M}_f$ are equimorphic. In particular, the uniform spaces $M \times \mathbb{R}$ and $\widetilde{M}_f$ are then equimorphic.

**Remark 1.** It follows from Proposition 2.5(a) in [11] that the diffeomorphisms $f$ and $f^n$ are reproduced simultaneously for any $n \in \mathbb{Z}$.

**Remark 2.** As is known, for a given diffeomorphism $f: M \to M$, its suspension $M_f$ is not necessarily diffeomorphic to the direct product $M_{f^n}$, but it is for some of its iterations $f^n$. The manifold $M_{f^n}$ is in fact a $k$-fold covering of $M_f$. As a simplest example of this situation, we can take $M = S^1$ with the coordinate $\varphi$ (mod $2\pi$) and the diffeomorphism $f(\varphi) = 2\pi - \varphi$. Then $S^1_2$ is the Klein bottle, but $f^2(\varphi) = \varphi$, and hence $S^1_2 = S^1 \times S^1 = \mathbb{T}^2$ (a 2-dimensional torus). Thus, according to Remark 1 the manifold $S^1_2$ is equimorphic to $M \times \mathbb{R}$.
We let $\pi_M: M \times \mathbb{R} \to M$ denote the standard projection onto the first factor. For any NVF $v$ on $M$, the map $\Phi_v: M_0 \to \pi_M(M_t)$ from the section $M_0 = M \times \{0\}$ to the manifold $M_t = M \times \{t\}$, generated by solutions of $v$ with initial points on $M_0$, $\pi_M(M_t) = M$, is diffeotopic to the identity map $\text{id}_M$ for all $t \in \mathbb{R}$.

In particular, if $v$ is a periodic vector field on $M$, then its Poincaré map over the period is diffeotopic to the identity map $\text{id}_M$.

Our first result is Theorem 1 below, which gives a sufficient condition for a diffeomorphism $f$ to be reproduced by the flow generated by a nonautonomous periodic vector field $v$. We first formulate an obvious lemma.

**Lemma 1.** If $f$ is diffeotopic to $\text{id}_M$, then there is a diffeotopy $F_t: M \to M$, $t \in [0, 1]$, joining $\text{id}_M$ and $f$ such that the diffeomorphisms $F_t$ depend smoothly on $t$ and for some sufficiently small $\varepsilon > 0$, we have $F_t \equiv \text{id}_M$ for $t \in [0, \varepsilon]$ and $F_t \equiv f$ for $t \in [1 - \varepsilon, 1]$.

**Theorem 1.** Let for some $n \in \mathbb{N}$ a diffeomorphism $f^n: M \to M$ be diffeotopic to the identity map $\text{id}_M$. Then

1. $M_f$ is fiber-wise diffeomorphic to $M \times S^1$, and
2. there is a periodic vector field $v$ on $M$ such that $v$ reproduces the structure of $f$.

**Proof.** To simplify the exposition, we assume that $f$ itself is diffeotopic to $\text{id}_M$. At the first step, we construct a 1-periodic vector field $v$ on $M$ such that the vector field on $M \times S^1$ given by $(v_1, 1)$ is diffeomorphic to the vector field of the suspension of $f$ on $M_f$. For this, we need to explicitly endow the manifold $M_f$ with a direct-product structure. This means that we need to define two foliations of $M_f$. One of them is given by fibers of a bundle over $S^1$ that are diffeomorphic to $M$. The second foliation into closed curves is defined as follows. We assume $F_t: M \to M$, $t \in [0, 1]$, to be a diffeotopy joining $\text{id}_M$ and $f$, that is, $F_t$ are diffeomorphisms such that $F_0 = \text{id}_M$ and $F_1 = f$. We assume, by Lemma 1, that $F_t = \text{id}_M$ for $t \in [0, \varepsilon]$ and $F_t = f$ for $t \in [1 - \varepsilon, 1]$. If $p: M_f \to S^1$ is the bundle map, then for any point $x \in p^{-1}(0)$ we define the curve passing through a point $(x, 0) \in M \times \mathbb{R}$ as $(F_t^{-1}(x), t)$ for $t \in [0, 1]$, and then we apply the quotient map using the identification $(x, t) = (f(x), t - 1)$.

The curve in $M \times \mathbb{R}$ with the initial point $(x, 0)$ has the extreme point $(F_1^{-1}(x), 1) = (f^{-1}(x), 1)$ at $t = 1$. After the identification, this point becomes $(f \circ f^{-1}(x), 0) = (x, 0)$. Thus, all curves constructed in the manifold $M_f$ are closed and we obtain a homeomorphism $h$ between $M_f$ and $M \times S^1$. The map $h$ is defined as follows. We take any point $a \in M_f$ and let $l_a$ denote a closed second-foliation curve passing through $a$. We define a map $p_1: M_f \to M_0$, $M_0 = p^{-1}(0)$, by the rule $p_1(a) = l_a \cap M_0$. We then have a map $h: a \to (p_1(a), p(a))$, which is a homeomorphism in general because the dependence of $F_t$ on $t$ can be only continuous. But we need a diffeomorphism between bundles $M_f$ and $M \times S^1$ in order that the orbits of the suspended flow be transformed to smooth curves in $M \times S^1$. Lemma 1 guarantees that if $f: M \to M$ is diffeotopic to $\text{id}_M$, then a diffeotopy $F_t$ exists joining $\text{id}_M$ and $f$ such that the curves constructed above give a smooth foliation, that is, the curves are smooth and their dependence on the points is smooth, and hence the map $p_1$ is smooth. This proves the first part of the theorem.

We now construct a periodic vector field $v$ on $M$ such that its foliation $\mathcal{L}_v$ is uniformly diffeomorphic to the foliation $\mathcal{L}_{v_f}$ into infinite curves in $M_f$. The diffeomorphism $h: M_f \to M \times S^1$ defined above allows us to identify $M_f$ with $M \times S^1$. Thus, the suspended vector field is given by $(V(x, t), n(x, t))$ with $V, n$ being 1-periodic in $t$ and $n > 0$. Its flow therefore has a cross section, (e.g., $t = 0$). The Poincaré

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1. Two diffeomorphisms $f$ and $g$ of a smooth manifold $M$ are diffeotopic if they can be joined by a continuous arc $F_t$ ($F_0 = f$, $F_1 = g$) such that each $F_t$ is a diffeomorphism of $M$.
2. The term “fiber-wise” means the existence of a diffeomorphism $\Psi: M_f \to M \times S^1$ acting as $(x, s) \to (\psi(x, s), s)$. 

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map \( g: M_0 \to M_0 \) on this cross section is evidently conjugated to \( f \). Hence, instead of the diffeomorphism \( f \) on \( M \), we can consider the diffeomorphism \( g \). Because \( f \) and \( g \) are conjugated, their nonautonomous suspensions are equimorphic, as are their corresponding foliations. The compactness of \( M \) and \( S^1 \) implies that \( n \) is strictly positive. We can define a periodic vector field on \( M \) as \( v(x,t) = (x,t)/n(x,t) \). Integral curves in \( M \times \mathbb{R} \) of this periodic vector field coincide with orbits the vector field \((V(x,t), n(x,t))\) because they are obtained by a change of time, uniformly bounded from above and below. The second part of the theorem is thus proved. ■

2. Diffeomorphisms with wildly embedded separatrices

We list some definitions and results contained in book [10]. We give them here for the convenience of the reader.

2.1. Wild embedding.

**Definition 2.** A topological embedding \( \lambda: X \to Y \) of an \( m \)-dimensional manifold \( X \) into an \( n \)-dimensional manifold \( Y \) \((m \leq n)\) is said to be locally flat at a point \( \lambda(x) \in Y \) if there is a chart \((U, \psi)\), \( \lambda(x) \in U \), \( \psi: U \to \mathbb{R}^n \), in the manifold \( Y \) such that \( \psi(\lambda(X) \cap U) = D^m \subset \mathbb{R}^m \), where \( \mathbb{R}^m \subset \mathbb{R}^n \) is the set of points for which the last \( n - m \) coordinates are equal to zero, or \( \psi(\lambda(X) \cap U) = \mathbb{R}_m^+ \), where \( \mathbb{R}_m^+ \subset \mathbb{R}^m \) is the set of points with a nonnegative last coordinate.

An embedding \( \lambda \) is said to be tame and the manifold \( X \) is said to be tamely embedded if \( \lambda \) is locally flat at every point \( \lambda(x) \in Y \). Otherwise, the embedding \( \lambda \) is said to be wild and the manifold \( X \) is said to be wildly embedded. If the embedding \( \lambda \) is not locally flat at a point \( \lambda(x) \), this point is said to be a point of wildness.

It is worth noting that the definition of a tamely embedded manifold coincides with the definition of a topological submanifold.

Every topological embedding into the space \( \mathbb{R}^2 \) (and \( S^2 \)) is tame. In \( \mathbb{R}^3 \) (and \( S^3 \)), there are wild arcs and wild 2-spheres. As an example of a wild arc, we recall the construction by Artin and Fox [6]. The corresponding arc is smooth everywhere except at its boundary point.

We consider a linear contraction \( \phi: \mathbb{R}^3 \to \mathbb{R}^3 \) defined in spherical coordinates \((\rho, \varphi, \theta)\) as \( \phi(\rho, \varphi, \theta) = (\rho/2, \varphi, \theta) \), and let \( L \subset \mathbb{R}^3 \) denote a spherical layer defined by the inequalities \( 1/2 \leq \rho \leq 1 \). Its boundary spheres are \( V_{1/2} = \{ (\rho, \varphi, \theta) \mid \rho = 1/2 \} \) and \( V_1 = \{ (\rho, \varphi, \theta) \mid \rho = 1 \} \).

![Fig. 3. Constructions of wild curves in \( \mathbb{R}^3 \).](image)
Let $a, b, c \subset L$ be pairwise disjoint simple arcs with their respective boundary points $\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma_1, \gamma_2$ (see Fig. 3a) such that

1. $\alpha_1, \alpha_2, \gamma_1 \subset V_1; \beta_1, \beta_2, \gamma_2 \subset V_1/2;$
2. $\phi(\alpha_1) = \gamma_2, \phi(\alpha_2) = \beta_1, \phi(\gamma_1) = \beta_2.$

We choose arcs $a, b, c$ such that the arc $\ell_O \subset \mathbb{R}^3$ defined as $\ell_O = \bigcup_{k \in \mathbb{Z}} \phi^k(a \cup b \cup c) \cup O$ (see Fig. 3b) is smooth at every point except $O$. Artin and Fox proved that $\ell_O$ is wildly embedded into $\mathbb{R}^3$ and $O$ is a point of wildness. This fact also follows from the following criterion, which was proved in [5].

**Proposition 2.** Let $\ell$ be a compact arc in $\mathbb{R}^3$ that is smooth everywhere except at its boundary point $O$. Then $\ell$ is locally flat at $O$ if and only if for every $\varepsilon$-ball $B_r(O)$ centered at $O$ there is a subset $U \subset B_r(O)$ diffeomorphic to the closed 3-ball such that $O$ is an interior point of $U$, and the intersection $\partial U \cap \ell$ is the only point.

We next consider a standard sphere

$$S^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$ 

We call the point $N(0, 0, 0, 1) \in S^3$ (respectively, $S(0, 0, 0, -1) \in S^3$) the north (respectively, south) pole.

For each point $x \in S^3 \setminus \{N\}$, there is a unique straight line in $\mathbb{R}^4$ containing $N$ and $x$. This line intersects the plane $(x_1, x_2, x_3, 0)$ at exactly one point $\vartheta(x)$:

$$\vartheta(x_1, x_2, x_3, x_4) = \frac{x_1}{1 - x_4}, \frac{x_2}{1 - x_4}, \frac{x_3}{1 - x_4}.$$

The stereographic projection of the point $x$ is defined as $\vartheta(x)$. The stereographic projection is a diffeomorphism of $S^3 \setminus \{N\}$ to $\mathbb{R}^3$ (see Fig. 4, where the stereographic projection from $S^2 \setminus \{N\}$ to $\mathbb{R}^2$ is shown).

![Fig. 4. Stereographic projection.](image)

Let $\ell = \vartheta^{-1}(\ell_O) \cup S$ (see Fig. 5a). Then the arc $\ell_N (\ell_S)$ in Fig. 5 is a subarc of the arc $\ell$ running from the point $\vartheta^{-1}(\alpha_1)$ to the point $N$ (from $\vartheta^{-1}(\alpha_1)$ to $S$). The arc $\ell_N (\ell_S)$ is wildly embedded into $S^3$.

Now we can “blow up” the arcs on Fig. 5a,b to obtain closed 3-balls whose boundaries are 2-spheres wildly embedded into $S^3$ and whose wild points are the poles.

### 2.2. Pixton-type diffeomorphisms on $S^3$.

Let $V$ be a smooth closed orientable 3-manifold whose fundamental group admits a nontrivial homomorphism $\eta_V: \pi_1(V) \to \mathbb{Z}$. We let $(V, \eta_V)$ denote the manifold $V$ equipped with the homomorphism $\eta_V$.

**Definition 3.** Manifolds $(V, \eta_V)$ and $(V', \eta_{V'})$ are said to be equivalent if there is a homeomorphism $\varphi: V \to V'$ such that $\eta_{V'} \varphi_* = \eta_V$.

**Definition 4.** Two smooth submanifolds $a \subset V$ and $a' \subset V'$ are said to be equivalent if there is a homeomorphism $\varphi: V \to V'$ such that $\eta_{V'} \varphi_* = \eta_V$ and $\varphi(a) = a'$.
Definition 5. A smooth submanifold \( a \subset V \) is said to be \( \eta \)-essential if \( \eta(a \ast (\pi_1(a))) \neq 0 \), where \( i_a: a \to V \) is the inclusion map.

We illustrate these definitions for the manifold \( S^2 \times S^1 \). We represent the manifold \( S^2 \times S^1 \) as the orbit space of the homothety \( a^s(x) = 0.5x \) \((x = (x_1, x_2, x_3))\), \((\mathbb{R}^3 \setminus \{O\})/a^s\). It is easy to verify that the natural projection \( p: \mathbb{R}^3 \setminus O \to S^2 \times S^1 \) is a covering map, which induces the epimorphism \( \eta_{S^2 \times S^1}: \pi_1(S^2 \times S^1) \to \mathbb{Z} \).

We set \( \hat{\gamma}_0 = p(Ox_1^+) \), \( \hat{\lambda}_0 = p(Ox_2x_3 \setminus O) \), where \( Ox_1^+ \) is the positive semiaxis and \( Ox_2x_3 \) is the coordinate plane \( x_1 = 0 \). In Fig. 6, we show a spherical layer bounded by spheres of radii 1 and 0.5. Identifying points that lie on the boundary of the spherical layer and belong to the same ray passing through \( O \), we obtain the manifold \( S^2 \times S^1 \). Moreover, if we identify the extreme points of the segment with the same numbers (1), we obtain the knot \( \hat{\gamma}_0 \), and if we identify extreme points lying on the same ray that belongs to circles with the same numbers (2) and the bounding 2-annulus, we obtain the torus \( \hat{\lambda}_0 \) (\( \hat{\gamma}_0 \) and \( \hat{\lambda}_0 \) are embedded into \( S^2 \times S^1 \)).

It is easy to verify that \( \hat{\gamma}_0 \) (respectively, \( \hat{\lambda}_0 \)) is an \( \eta_{S^2 \times S^1} \)-essential knot (respectively, torus) in the manifold \((S^2 \times S^1, \eta_{S^2 \times S^1})\).

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3We take the homotopy class \([c] \in \pi_1(S^2 \times S^1)\) of a loop \( c: R/\mathbb{Z} \to S^2 \times S^1 \). Then \( c: [0, 1] \to S^2 \times S^1 \) lifts to a curve \( \tilde{c}: [0, 1] \to \mathbb{R}^3 \setminus \{O\} \) joining \( x \) with the point \((a^s)^n(x)\) for some \( n \in \mathbb{Z} \), where \( n \) is independent of the lift. We then set \( \eta_{S^2 \times S^1}(\tilde{c}) = n \).
**Definition 6.** A knot (torus) \( \dot{\gamma}(\lambda) \) in the manifold \((S^2 \times S^1, \eta_{S^2 \times S^1})\) is said to be *trivial* if it is equivalent to the knot (torus) \( \dot{\gamma}_0(\lambda_0) \).

**Proposition 3.** Every \( \eta_{S^2 \times S^1}^s \)-essential torus \( \lambda \subset (S^2 \times S^1, \eta_{S^2 \times S^1}^s) \) bounds a solid torus in \( S^2 \times S^1 \).

**Proposition 4.** A knot \( \dot{\gamma} \) (torus \( \lambda \)) in the manifold \((S^2 \times S^1, \eta_{S^2 \times S^1}^s)\) is trivial if and only if there is a tubular neighborhood \( N(\dot{\gamma}) \) of it in the manifold \( S^2 \times S^1 \) such that the manifold \((S^2 \times S^1) \setminus N(\dot{\gamma})(S^2 \times S^1) \setminus N(\lambda)\) is homeomorphic to the solid torus (a pair of the solid tori).

Let \( \mathcal{P} \) denote the class of the Morse–Smale diffeomorphisms whose nonwandering set consists of the source \( \alpha_f \), the saddle \( \sigma_f \), and the sinks \( \omega_1, \omega_2 \). The phase portrait of a diffeomorphism of the class \( \mathcal{P} \) is shown in Fig. 7. Pixton has constructed an example from the class \( \mathcal{P} \), and we therefore call the class \( \mathcal{P} \) the *Pixton class*. We omit the index \( f \) in the notation for fixed points in what follows.

![Fig. 7. Phase portrait of a diffeomorphism of the class \( \mathcal{P} \).](image)

A surprising fact is the existence of a countable set of nonconjugated diffeomorphisms in the class \( \mathcal{P} \).

To understand this, we describe the topological knot invariant suggested in [8]. This invariant, moreover, explains the existence in the class \( \mathcal{P} \) of diffeomorphisms for which a saddle fixed point has wildly embedded 1-dimensional and 2-dimensional separatrices.

We let \( \ell_1, \ell_2 \) denote the unstable 1-dimensional separatrices of the point \( \sigma \). It follows from [13] that the closure \( \text{cl}(\ell_i), i = 1, 2 \) is homeomorphic to a simple compact arc that consists of the separatrix itself and two of its extreme points: \( \sigma \) and the sink (see Proposition 2.3 in [10]). Moreover, the closures of the separatrices \( \ell_1 \) and \( \ell_2 \) contain different sinks (see Corollary 2.2 in [10]). For definiteness, let \( \omega_i \) belong to \( \text{cl}(\ell_i) \) (see Fig. 7). For \( i = 1, 2 \), let \( V_i = W^s(\omega_i) \backslash \{\omega_i\} \). Let \( \dot{V}_i = V_i/\sim \) be the corresponding orbit space and \( p_i : V_i \rightarrow \dot{V}_i \) the natural projection, which is the covering map inducing an epimorphism \( \eta_i : \pi_1(\dot{V}_i) \rightarrow \mathbb{Z} \). For the sink \( \omega_i \), the restriction \( f|_{\dot{V}_i} \) is topologically conjugated to the diffeomorphism \( a : \mathbb{R}^3 \setminus \{O\} \rightarrow \mathbb{R}^3 \setminus \{O\} \), and hence the manifold \( (\dot{V}_i, \eta_i) \) is equivalent to the manifold \((S^2 \times S^1, \eta_{S^2 \times S^1}^s)\) and the set \( \dot{\ell}_i = p_i(\ell_i) \) is an \( \eta_i \)-essential knot in the manifold \( \dot{V}_i \) such that \( \eta_i(i_{\dot{\ell}_i}(\pi_1(\dot{\ell}_i))) = \mathbb{Z} \) (see Theorem 2.3 in [10]).

It was proved in [7] (Theorem 1) that at least one of the knots \( \dot{\ell}_1, \dot{\ell}_2 \) is trivial (see also [10], Proposition 4.3). For definiteness, we assume the knot \( \dot{\ell}_1 \) to be trivial in what follows.

The next result was proved in [8] (Theorem 3) (see also [10], Theorem 4.3).

**Proposition 5.** Diffeomorphisms \( f, f' \in \mathcal{P} \) are topologically conjugated if and only if the knots \( \dot{\ell}_2(f) \) and \( \dot{\ell}_2(f') \) are equivalent.

Therefore, the equivalence class of a knot \( \dot{\ell}_2(f) \) is a complete topological invariant for diffeomorphisms from the Pixton class. Moreover, the following realization theorem holds (see Theorem 2 in [8] and Theorem 4.4 in [10]).
Proposition 6. For every knot $\hat{\ell} \subset (S^2 \times S^1, \eta_{S^2 \times S^1}^x)$ such that

$$\eta_{S^2 \times S^1}^x(i_{\hat{\ell}}(\pi_1(\hat{\ell}))) = \mathbb{Z},$$

there is a diffeomorphism $f: S^3 \to S^3$ from the class $P$ such that the knots $\hat{\ell}$ and $\hat{\ell}^2(f)$ are equivalent.

Masur constructed an example of an essential and nontrivial knot embedded into $S^2 \times S^1$ [16]. According to Proposition 6, there exists a Pixton-class diffeomorphism $f$ such that exactly one unstable 1-dimensional separatrix and a stable 2-dimensional separatrix of the saddle point $\sigma$ are wildly embedded.

In Fig. 8, we show Masur’s knot $\hat{l}_{\sigma}^u$, which appears in the quotient space $\hat{W}^s(\omega_2)$, and an essential torus $\hat{l}_s^u$ embedded into $\hat{W}^u(\alpha)$, which is tubular neighborhood of the Masur knot.

![Phase portrait of a diffeomorphism of the class $P$ and the projection of saddle separatrices in the quotient spaces.](image)

**Fig. 8.** Phase portrait of a diffeomorphism of the class $P$ and the projection of saddle separatrices in the quotient spaces.

2.3. Diffeomorphisms with wildly embedded frames.

**Definition 7.** For $k \in \mathbb{N}$, a $k$-frame $F_k$ in $\mathbb{R}^n$ at a point $p$ is the union of $k$ simple curves $A_1, \ldots, A_k$, $F_k = \bigcup_{i=1}^{k} A_i$, with a single common point $p$ such that $p$ is a boundary point of each $A_k$, $k \geq 1$, and $A_i \cap A_j = p, i \neq j$.

**Definition 8.** A $k$-frame $F_k = \bigcup_{i=1}^{k} A_i$ is said to be

- **standard** if each arc $A_i$ lies in the plain $Ox_1x_2$ and is defined as $\varphi = 2\pi(i - 1)/k$, where $\rho, \varphi$ are polar coordinates in the plain $Ox_1x_2$;
- **tame** if there is a homeomorphism $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(F_k)$ is standard. Otherwise, $F_k$ is said to be **wild**;
- **mildly wild** if the frame $F_k \setminus (A_i \setminus p)$ is tame for every $i \in \{1, \ldots, k\}$. 

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We can easily construct a wild $k$-frame if we assume the arc $A_1$ to be the wild arc $\tilde{\ell}$ in Artin–Fox's example [6]. But the fact that each arc $A_i$ is tame does not mean that the frame $F_k$ is tame. Figure 9b shows an example of a wild 2-frame. The boundary points $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ of the respective arcs $\alpha, \beta$ are glued by $\phi(\alpha_1) = \alpha_2$, $\phi(\beta_1) = \beta_2$ and $A_1 = \bigcup_{k \in \mathbb{Z}} \phi^k(a) \cup O$, $A_2 = \bigcup_{k \in \mathbb{Z}} \phi^k(b) \cup O$, $F_2 = A_1 \cup A_2$. It follows from Proposition 2 that both $A_1$, $A_2$ are tame. Debrunner and Fox [17] presented the construction of a mildly wild $k$-frame for every $k > 1$.

Fig. 9. Construction of a wild 2-frame in $\mathbb{R}^3$.

Let $f$ be the Morse–Smale diffeomorphism. We suppose that its nonwandering set $NW(f)$ has a sink $\omega$ such that there is a 1-dimensional unstable separatrix $\ell_\sigma$ of some saddle $\sigma \in NW(f)$ such that the closure of $\ell_\sigma$ contains exactly two extreme points $\omega$ and $\sigma$. Let $L_\omega = \{\ell_1, \ldots, \ell_k\}$ be the set of all such unstable 1-dimensional separatrices attached to $\omega$. Because $W_\omega^s$ is homeomorphic to $\mathbb{R}^3$ and the set $L_\omega \cup \omega$ is the union of simple arcs with a unique common point $\omega$, we call $L_\omega \cup \omega$ the frame of 1-dimensional unstable separatrices, similar to a frame of arcs in $\mathbb{R}^3$ [17].

**Definition 9.** A frame of separatrices $L_\omega \cup \omega$ is called tame if there is a homeomorphism $\psi_\omega: W_\omega^s \to \mathbb{R}^3$ such that $\psi_\omega(L_\omega \cup \omega)$ is the standard frame of arcs in $\mathbb{R}^3$. Otherwise, the frame of separatrices is called wild.

In [18], a method similar to that described in Sec. 2.2 was used to construct a Morse–Smale diffeomorphism with a mildly wild frame of 1-dimensional separatrices (see Fig. 10).

Fig. 10. Phase portrait of a Morse–Smale diffeomorphism on $S^3$ with a mildly wild frame of separatrices.
3. Periodic vector field on $S^3$ with a wildly embedded separatrix set

We now present a periodic vector field on $S^1$ with a wild embedding of a 2-dimensional unstable separatrix manifold and a 3-dimensional stable separatrix manifold for a saddle IC with exponential dichotomy on $\mathbb{R}$ of type $(3, 2)$. We also present another periodic vector field on $S^3$ that has a mildly wild frame of 2-dimensional separatrix manifolds.

We start with some Pixton-class diffeomorphism $f$ on $S^3$ that has one hyperbolic source $\alpha$, one saddle $\sigma$ of type $(2, 1)$ (a 2-dimensional stable manifold and a 1-dimensional unstable one), and two hyperbolic sinks $\omega_1$ and $\omega_2$. The closure of the stable 2-dimensional manifold of $\sigma$ contains the point $\alpha$, that is, all orbits of $f$ with initial points on $W^s(\sigma)$, except $\sigma$ itself, have the only $\alpha$-limit point $\alpha$ and the $\omega$-limit point $\sigma$. The closure of $W^s(\sigma)$ is a topologically embedded sphere $\Sigma$ in $S^3$, being the boundary of two open 3-balls $D_1$, $D_2$ in $S^3$. The fixed point $\omega_1$ (sink) lies inside the ball $D_1$, and the other sink $\omega_2$ lies inside $D_2$. We suppose the 1-dimensional separatrix of $\sigma$ that enters $D_2$ to be wildly embedded. This implies that the stable manifold $W^s(\sigma)$ is also wildly embedded (see Fig. 2).

We now consider the suspension over $f$. As follows from the results in [19], any two orientation-preserving diffeomorphisms of $S^3$ can be joined by a smooth arc. This implies that $f$ is diffeotopic to id$_{S^3}$, and the manifold $M_f$ is homeomorphic to the direct product $S^3 \times S^1$; moreover, this direct product structure can be chosen by means of some diffeomorphism (see above). We fix this product structure and henceforth consider the suspension as the standard $S^3 \times S^1$. Then the suspension flow in $S^3 \times S^1$ has one totally unstable periodic orbit, one saddle periodic orbit of type $(3, 2)$ and two totally stable periodic orbits, all of them being hyperbolic periodic orbits. The projection of any of these periodic orbits onto the base is a 1 : 1 correspondence.

We recall that the suspension flow is a Morse-Smale flow. All of its periodic orbits are hyperbolic and any other orbit tends to some of these periodic orbits as $t \to \pm \infty$. This implies, by construction, that all four periodic ICs of the NVF on $S^3$ have an exponential dichotomy on $\mathbb{R}$. The exponential dichotomy types are different: two stable periodic orbits give rise to two completely stable periodic ICs, their exponential dichotomy type being $(4, 1)$ (four is the dimension of their stable manifolds), the saddle periodic orbit on $S^3 \times S^1$ gives rise to a saddle periodic IC with the dichotomy on $\mathbb{R}$ of type $(3, 2)$, and the completely unstable periodic orbit gives rise to an IC with the dichotomy of type $(1, 4)$. All other ICs tend to these four ICs and hence have an exponential dichotomy on $\mathbb{R}_-$ and $\mathbb{R}_+$ separately, depending on which of four ICs they approach to.

We also recall that the diffeomorphism $f$ has a smooth curve, the unstable separatrix $W^u(\sigma)$ for the saddle point $\sigma$. For the suspended flow in $S^3 \times S^1$, we obtain a 2-dimensional smooth unstable submanifold $W^u(\gamma_\sigma)$ of the saddle periodic orbit $\gamma_\sigma$. The manifold $W^u(\gamma_\sigma)$ is the direct product $W^u(\sigma) \times S^1$, as follows from the suspension construction. If one of the two unstable separatrices of $\sigma$ is wildly embedded into $S^3$ (see above), then one of the connected components of the intersection $S^3 \times S^1 \cap W^u(\gamma_\sigma)$, $\tau \in S^1$ (to be denoted by $\Sigma_\tau$) is a wildly embedded curve in $S^3 \times S^1$.

We assume for definiteness that the stable sink $\omega_2$ is the $\omega$-limit set for all orbits of the wildly embedded separatrix of $\sigma$. We say that the related component of $W^u(\sigma) \times S^1 \setminus \gamma_\sigma$ is wildly embedded in $S^3 \times S^1$. The characterization of this wild embedding is as follows. We choose any smooth 3-disk $D$ that is transverse to the periodic orbit $\gamma_{\omega_2}$ at some point of the orbit. Then the wildly embedded component intersects this disk along a smooth ray with the extreme point $D \cap \gamma_{\omega_2}$, which is a point of wildness.

**Lemma 2.** If flows $f^t, f'^t \in \mathcal{P}$ are topologically equivalent and $f^t$ has a wildly embedded connected component of the set $W^u(\gamma_S) \setminus \gamma_S$, then the same holds for the flow $f'^t$. 

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We have thus proved the following assertion.

**Theorem 2.** There exists a smooth 1-periodic vector field \( v \) on \( S^3 \) that is gradient-like and has only four 1-periodic ICs with exponential dichotomies on \( \mathbb{R} \): a completely unstable IC (of type \( (1, 4) \)), one saddle IC of type \( (3, 2) \), and two completely stable ones of type \( (4, 1) \). The saddle periodic IC has wildly embedded 2-dimensional and 3-dimensional separatrix sets.

In the same way, starting with a diffeomorphism of \( S^3 \) that has a mildly wild frame of separatrices described above, we obtain a 1-periodic gradient-like vector field on \( S^3 \) such that it has one completely unstable IC, one completely stable IC, and \( n \geq 2 \) saddle periodic ICs with an exponential dichotomy of type \( (3, 2) \) on \( \mathbb{R} \), whose \( n \geq 2 \) 2-dimensional unstable separatrices form a mildly wild frame along with their \( n \) 3-dimensional stable separatrices, which also form a mildly wild frame similar to that plotted in Fig. 10.

The 1-periodic vector field constructed above has a very simple structure of its foliation into ICs. We choose some sufficiently thin uniform neighborhoods \( U_j, j = 1 - 4, \) of these special ICs. They can be chosen such that the passage time from one boundary component of the set \( M \times \mathbb{R} \subset \bigcup_j U_j \) to another component is bounded from above and below by positive constants independent of the choice of the ICs. This allows us to prove that the NVF is structurally stable with respect to sufficiently small uniformly bounded perturbations. We formulate the corresponding assertion concerning the NVFs we have constructed.

**Theorem 3.** Any sufficiently small uniform perturbation of such a vector field \( v \) gives an NVF \( v' \) that is uniformly equivalent to the initial one; in other words, there exists an equimorphism \( h : S^3 \times \mathbb{R} \to S^3 \times \mathbb{R} \) that transforms the foliation \( \mathcal{L}_v \) into \( \mathcal{L}_{v'} \).

Although the very simple structure of \( \mathcal{L}_v \) makes the proof almost evident, it still requires some technique and will be presented elsewhere.

### 4. Perturbations

In this section, we perturb the periodic vector fields constructed in the preceding section such that their uniform structure remains the same, but, depending on the perturbation chosen, the perturbed NVF is almost periodic or even nonrecurrent in time.

By Theorem 3, \( v \) is structurally stable with respect to small uniform perturbations of the form \( v + \varepsilon v_1 \) defined by a bounded uniformly continuous map \( v_1 : \mathbb{R} \to V^r(S^3) \) into the Banach space \( V^r(S^3) \), \( r \geq 1 \). In particular, such a perturbation can be chosen to be almost periodic. In this case, because any of the four periodic ICs has an exponential dichotomy on \( \mathbb{R} \) (of different types), the perturbed almost periodic vector field has an almost periodic IC of the same dichotomy type in a sufficiently small uniform neighborhood of each periodic IC. Moreover, the perturbed vector field is gradient-like and any of its IC tends to some of the four almost periodic ICs and has exponential dichotomy on \( \mathbb{R}_\pm \). Choosing the perturbation to be small but nonrecurrent in time, we obtain a vector field with the same structure of the foliation but with nonrecurrent ICs.

Almost periodic and nonrecurrent NVFs with mildly wild frames of separatrices can be constructed similarly.

### Appendix: Elements of uniform topology

For the convenience of the reader, we recall some notions from uniform topology. The main definitions of the theory of uniform spaces, with all the necessary details, can be found in [14].
A set $X$ is called a uniform space if a collection $\mathcal{U}$ of subsets is defined on $X \times X$ such that the following conditions are satisfied (and $\mathcal{U}$ is then called the uniformity).

1. Each element of $\mathcal{U}$ contains the diagonal $\Delta = \cup_{x \in X} \{(x, x)\}$.

2. If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$, where $U^{-1}$ is the set of all pairs $(y, x)$ for which $(x, y) \in U$.

3. For any $U \in \mathcal{U}$, some $V \in \mathcal{U}$ exists such that $V \circ V \subset U$, where $V \circ V$ denotes composition: $(x, z) \in V \circ V$ if there is $y \in X$ such that $(x, y) \in V$ and $(y, z) \in V$.

4. If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.

5. If $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

If $X$ is a metric space with a metric $d$, then item 1 corresponds to the property $d(x, x) = 0$, and item 2 corresponds to the symmetry of $d$: $d(x, y) = d(y, x)$. Property 3 is of the triangle inequality type: for each point of $x \in X$ and any ball of radius $r$ with centered at $x$, a ball of the radius $r/2$ centered at the same point must exist. Conditions 3 and 5 are similar to the axioms of neighborhoods near a point for the topology defined by uniformity.

The uniformity $\mathcal{U}$ on a given set $X$ can be defined in many ways, which gives different uniform spaces. This was used above where different uniform structures were defined on the set $M \times \mathbb{R}$. If $(X, \mathcal{U}), (Y, \mathcal{V})$ are two uniform spaces, then the notion of a uniformly continuous map $h: X \to Y$ is defined. Namely, a map $h: X \to Y$ is uniformly continuous with respect to $\mathcal{U}, \mathcal{V}$, if for any $V \in \mathcal{V}$ the set $\{(x, y) \mid h(x), h(y)) \in V\}$ belongs to $\mathcal{U}$. If $h: X \to Y$ is one-to-one and both $h$ and $h^{-1}$ are uniformly continuous, then $h$ is called an equimorphism. In this case, the uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are said to be uniformly equivalent or equimorphic.

A uniformity $\mathcal{U}$ on a set $X$ making it a uniform space $(X, \mathcal{U})$ endows $X$ with a certain topology, making it a topological space. This space can have various topological properties. Conversely, each regular* topology $\mathcal{T}$ on $X$ is a uniform topology that corresponds to some uniformity, but such a uniformity is not unique in general. But if the topological space is compact and regular, then there is a unique uniformity generating the topology $\mathcal{T}$.

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* A topological space is regular if and only if for any its point $x$ and any its neighborhood $U$, there is a closed neighborhood $V$ of $x$ such that $V \subset U$. 

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