AN ISOMORPHISM BETWEEN THE COMPLETION OF AN ALGEBRA AND ITS CARATHEODORY EXTENSION

JUN TANAKA

Abstract. Let $\Omega$ denote an algebra of sets and $\mu$ a $\sigma$-finite measure. We then prove that the completion of $\Omega$ under the pseudometric $d(A,B) = \mu^*(A \triangle B)$ is $\sigma$-algebra isomorphic and isometric to the Caratheodory Extension of $\Omega$.

1. Introduction

This paper shows a new result by combining two papers authored by P.F. Mcloughlin and myself ([4], [5]).

Let $\mu$ be a $\sigma$-finite measure and let $\Omega$ denote an algebra of sets; i.e., $\Omega$ is closed under unions and complements. Let $(X, \Omega, \mu)$ denote a measure space where $\mu(X)$ is finite from the $\sigma$-finite property. Let $\mu^*$ denote the outer measure defined by $\mu^*(A) = \inf\{\sum \mu(A_i) \mid E \subseteq \cup A_i \text{ and } A_i \in \Omega \text{ for all } i \geq 1\}$, for any $A \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ is the power set of $X$. Clearly, $d(A, B) = \mu^*(A \triangle B)$ is a pseudometric, where $\triangle$ is the symmetric difference of sets. In addition, $d$ is a metric on $\mathcal{P}(X)/\sim$, where $A \sim B$ iff $\mu^*(A \triangle B) = 0$. From [1], pg 292, $\mu^*|_{\Omega} = \mu$.

In [5], we defined a $\mu$-Cauchy sequence $\{B_n\}$, $B_n \in \Omega$, if $\lim \mu(B_n \triangle B_m) \to 0$ as $n, m \to \infty$. Let $\tilde{\Sigma} = \{S \in \mathcal{P}(X) \mid \exists \mu$-Cauchy sequence $\{B_n\}$ s.t. $\lim \mu^*(B_n \triangle S) = 0\}$. In the first joint paper [5], we proved that $\tilde{\Sigma}$ is a $\sigma$-algebra where, for any $\mu$-Cauchy sequence $\{B_n\}$ such that $\lim \mu^*(B_n \triangle S) = 0$, the measure $\bar{\mu}(S)$ on $\tilde{\Sigma}$ is defined as $\bar{\mu}(S) = \lim \mu(B_n)$. In addition, we proved that $\bar{\mu}$ is a countably additive measure on $\tilde{\Sigma}$. Thus, $(\bar{\mu}, \tilde{\Sigma})$ is a measure space. We showed that the Carathéodory Extension of $\Omega$ can be expressed as the set of limit points of $\mu$-Cauchy sequences under the pseudometric $d(A, B) = \mu^*(A \triangle B)$. Moreover, when the measure is a sigma finite measure, we obtained an equivalent expression of the Carathéodory Extension, $\{S \in \mathcal{P}(X) \mid \exists \mu$-Cauchy sequence $\{B_n\}$ s.t. $\lim \mu^*(B_n \triangle S) = 0\}$.

Theorem 2 in [5] shows that $E$ is a measurable set iff $E$ is in $\tilde{\Sigma}$. Thus, the measure space $(\bar{\mu}, \tilde{\Sigma})$ agrees with the Carathéodory Extension when $\mu$ is a finite measure. Moreover, it shows that measurable sets are exactly limit points of $\mu$-Cauchy sequences. The $\sigma$-finite case follows from the finite case.

From the second joint paper [4], we denoted by $(\tilde{d}, \tilde{\Omega})$ the completion of $(d, \Omega_{/\triangle})$. Let $S$ be the set of all $\mu$-Cauchy sequences in $(d, \Omega_{/\triangle})$. By the completion procedures, we know $\{B_n^\mu\} \sim \{B_n^\bar{\mu}\}$ iff $\lim d(B_n^\mu, B_n^\bar{\mu}) = 0$ defines an equivalence relation on $S$. Moreover, $\tilde{\Omega} = S_{/\sim}$ and $\tilde{d}(\{B_n^\mu\}, \{B_n^\bar{\mu}\}) = \lim d(\{B_n^\mu\}, \{B_n^\bar{\mu}\})$, where

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$\alpha \in \{ B_n^\gamma \}$ is the class of $\{ B_n^\gamma \}$. Let $E_\alpha = \{ B_n^\gamma \}$ and $E_A = \{ A \}$ when $A \in \Omega$. Let $\overline{\mu}(E_\alpha) = d(E_\alpha, E_\emptyset) = \lim d(B_n^\gamma, \emptyset) = \lim \mu(B_n^\gamma)$. Note that in [4] $d(A, B) := \mu(A \triangle B)$; the completion of $\Omega$ will remain the same due to the property $\mu^*|_\Omega = \mu$. Note that $\overline{\mu}(E_A) = \mu(A)$ when $A \in \Omega$. In [4], we defined set-theoretic notations for unions, intersections, and complements on $\Omega$ as follows: $\bigcup : \Omega \times \Omega \to \Omega$ where $\bigcup(E_\alpha \times E_\gamma) = E_\alpha \cup E_\gamma = \{ B_n^\alpha \cup B_n^\gamma \}$; similarly for intersections on $\Omega$. $\triangle : \Omega \to \Omega$ where $((B_n^\alpha)^C) = \{ (B_n^\alpha)^C \}$ and we showed the set theoretic notations are well defined on $\Omega$ in [4].

We showed the set theoretic notations are well defined on $\overline{\Omega}$ in [4]. Note that $E_\alpha \cap E_\gamma = E_\emptyset$ if $\overline{\mu}(E_\alpha \cap E_\gamma) = 0$ iff $\lim \mu(B_n^\alpha \cap B_n^\gamma) = 0$. We say $E_\alpha$ and $E_\gamma$ are disjoint iff $E_\alpha \cap E_\gamma = E_\emptyset$. Thus, if $E_\alpha$ and $E_\alpha'$ are disjoint, then $\overline{\mu}(E_\alpha \cup E_\alpha') = \overline{\mu}(E_\alpha) + \overline{\mu}(E_\alpha')$.

As for the infinite union on $\overline{\Omega}$; if $E_{\alpha_i} \in \overline{\Omega}$ for $i \geq 1$, there exists a unique $E := \bigcup_{i=1}^\infty E_{\alpha_i}$ in $\overline{\Omega}$ such that $\bigcup_{i=1}^n E_{\alpha_i} \subseteq E$ for all $n$, and $\lim \overline{\mu}(E \cap \bigcup_{i=1}^n E_{\alpha_i}) = 0$.

In addition, we showed that for any $\mu$-Cauchy sequence $\{ B_n \}$, there exists a $f(n) > n$ such that $\lim \mu^*(B_n \triangle \liminf B_{f(n)}) = 0$.

In this paper, I define a $\sigma$-algebra isomorphism between two $\sigma$-algebras, and define a map $F : \overline{\Omega} \to \mathbf{P}(X)$ given by $F(\{ B_n \}) = \liminf B_{f(n)}$ where $f(n)$ is defined as above. We will show that $F$ is an isometry and a $\sigma$-algebra isomorphism between the completion $\overline{\Omega}$ and the Carathéodory Extension of $\Omega$ under the equivalence relation $\sim$ defined as $A \sim B$ iff $\mu^*(A \triangle B) = 0$.

2. Main Result

**Definition 1.** For $A, B$ in $\mathbf{P}(X)$, $A = B$ a.e. iff $\mu^*(A \triangle B) = 0$.

**Definition 2.** Define a map $F : \overline{\Omega} \to \mathbf{P}(X)$ given by $F(\{ B_n \}) = \liminf B_{f(n)}$ where $\lim \mu^*(B_n \triangle \liminf B_{f(n)}) = 0$. Note that such $f(n)$ always exists by Lemma 20 in [4].

**Remark 1.** $F$ is a map into $\overline{S}$ by the definition of $\overline{S}$.

**Lemma 1.** $F$ is well defined.

**Proof.** Suppose that $\{ A_n \} = \{ B_n \}$.

There exist $f(n)$ and $g(n)$ such that $\lim \mu^*(A_n \triangle \liminf A_{f(n)}) = 0$ and $\lim \mu^*(B_n \triangle \liminf B_{g(n)}) = 0$ by Lemma 20 in [4].

$$\mu^*(\liminf A_{f(n)} \triangle \liminf B_{g(n)}) \leq \mu^*(\liminf A_{f(n)} \triangle A_{f(n)}) + \mu^*(A_{f(n)} \triangle B_{g(n)}) + \mu^*(B_{g(n)} \triangle \liminf B_{g(n)})$$

by the triangle inequality.

By taking the limit on both sides, $\mu^*(\liminf A_{f(n)} \triangle \liminf B_{g(n)}) = 0$.

Thus, $\lim A_{f(n)} = \liminf B_{g(n)}$ a.e. Therefore, $F$ is well-defined.

**Theorem 1.** $F$ is an isometry between $\overline{\Omega}$ and $\overline{S}_\sim$.

**Proof.** First, we show $F$ is onto $\overline{S}$. Let $X \in \overline{S}$. Then there exists a $\mu$-Cauchy sequence $\{ B_n \}$ such that $\lim \mu^*(B_n \triangle X) = 0$.

Then there exist $f(n)$ such that $\lim \mu^*(B_n \triangle \liminf B_{f(n)}) = 0$. Thus $F(\{ B_n \}) = \liminf B_{f(n)} = X$ a.e. Therefore, $F$ is onto.

Second, we will show $F$ preserves the metric. Let $\{ A_n \}, \{ B_n \} \in \overline{\Omega}$. Then we have $f(n)$ and $g(n)$ as before.
\[ \mu(A_{f(n)} \triangle B_{g(n)}) = \mu^*(A_{f(n)} \triangle B_{g(n)}) \]
\[ \leq \mu^*(A_{f(n)} \triangle \lim A_{f(n)}) + \mu^*(\lim A_{f(n)} \triangle \lim B_{g(n)}) + \mu^*(\lim B_{g(n)} \triangle B_{g(n)}) \]

By taking the limit on both sides,
\[ \lim \mu(A_n \triangle B_n) = \lim \mu(A_{f(n)} \triangle B_{g(n)}) \leq \mu^*(\lim A_{f(n)} \triangle \lim B_{g(n)}). \]

In addition,
\[ \mu^*(\lim A_{f(n)} \triangle \lim B_{g(n)}) \]
\[ \leq \mu^*(A_{f(n)} \triangle \lim A_{f(n)}) + \mu^*(A_{f(n)} \triangle B_{g(n)}) + \mu^*(\lim B_{g(n)} \triangle B_{g(n)}). \]

By taking the limit on both sides,
\[ \mu^*(\lim A_{f(n)} \triangle \lim B_{g(n)}) \leq \lim \mu(A_{f(n)} \triangle B_{g(n)}). \]

Therefore, \[ d(\{A_n\}, \{B_n\}) = \lim \mu(A_n \triangle B_n) = \mu^*(\lim A_{f(n)} \triangle \lim B_{g(n)}) \]
\[ = \mu^*(F(\{A_n\}) \triangle F(\{B_n\})) = d(F(\{A_n\}), F(\{B_n\})). \]

Lastly, we will show that \( F \) is one to one. Let \( F(\{A_n\}), F(\{B_n\}) \in \hat{S} \) such that \( F(\{A_n\}) = F(\{B_n\}) \) a.e..

Then \( \lim A_{f(n)} = \lim B_{g(n)} \) a.e. implies \( \mu^*(\lim A_{f(n)} \triangle \lim B_{g(n)}) = 0 \). Then, as in the proof of \( F \) being onto, \( \lim A_n \triangle B_n = \mu^*(\lim A_{f(n)} \triangle \lim B_{g(n)}) \). Thus \( \{A_n\} = \{B_n\} \). Thus, \( F \) is one to one. Therefore, \( F \) is an isometry between \( \overline{\Omega} \) and \( \overline{S} \).

**Definition 3.** Suppose \( X \) and \( Y \) are \( \sigma \)-algebras, and \( F: X \rightarrow Y \) is a one to one, onto well defined map. Then \( F \) is called a \( \sigma \)-algebra isomorphism if
\[ F(\bigcup_{i=1}^{\infty} E_i) = F(\cdot \cup \cdot) = \bigcup_{i=1}^{\infty} F(E_i), \]
\[ F(\bigcap_{i=1}^{n} E_i) = F(\cdot \cap \cdot) = \bigcap_{i=1}^{n} F(E_i), \]
\[ F(\cdot C) = F(\cdot)^C. \]

**Lemma 2.** Let \( E_i = \{B_{n_i}^k \} \in \overline{\Omega} \) for \( i \geq 1 \) and by following Lemma 8 in [5], construct \( Y_L = \bigcup_{i=1}^{\infty} B_{K_L}^i \) for each \( L \) such that
\[ \mu^*(\bigcup_{i=1}^{\infty} S_i \triangle \bigcup_{i=1}^{N_L} B_{K_L}^i) \leq \mu^*(\bigcup_{i=1}^{N_L} S_i \triangle \bigcup_{i=1}^{N_L} B_{K_L}^i) + \frac{1}{L}. \]

Then \( \bigcup_{i=1}^{\infty} E_i \).

**Proof.** Note that \( E_i = \{B_{n_i}^L \} = \{B_{K_L}^L \} \).
\( (\bigcup_{i=1}^{n} E_i) \bigcap \{Y_L\} = \bigcup_{i=1}^{n} B_{K_L}^i \bigcap Y_L = \bigcup_{i=1}^{n} B_{K_L}^i = \bigcup_{i=1}^{n} E_i \) for any \( n \).
Let \( N_L > n \).
\[ \mu(\bigcup_{i=1}^{N_L} B_{K_L}^i \bigcap (\bigcup_{i=1}^{n} B_{K_L}^i)^C) = \mu(\bigcup_{i=1}^{N_L} B_{K_L}^i \bigcap \bigcup_{i=1}^{n} B_{K_L}^i) + \mu(\bigcup_{i=1}^{N_L} B_{K_L} \bigcup \bigcup_{i=1}^{n} B_{K_L}^i) \]
\[ \leq \mu(\bigcup_{i=1}^{N_L} S_i \triangle \bigcup_{i=1}^{N_L} B_{K_L}^i) + \mu(\bigcup_{i=1}^{N_L} S_i \bigcup \bigcup_{i=1}^{n} B_{K_L}^i) + \frac{1}{L}. \]

This implies that \( \lim \mu(\overline{\{Y_L\} \bigcap (\bigcup_{i=1}^{n} E_{\alpha_i})^C}) = 0 \). Therefore by the uniqueness of \( \bigcup_{i=1}^{\infty} E_i, \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} E_i. \)

**Theorem 2.** \( F \) is a \( \sigma \)-algebra isomorphism between \( \overline{\Omega} \) and \( \overline{S} \).
Proof. We already showed that $F$ is a one to one, onto map in Theorem 1. Since, in general, \( \lim A_n \cup B_n = \lim A_n \cup \lim B_n \), \( F(\cdot \cup \cdot) = F(\cdot) \cup F(\cdot) \) follows immediately.

Let \( \{B_n\} \in \Omega \).

Then,

\[
F(\{B_n\}^c) = F(\{(B_n)^c\}) = \lim(B_{f(n)})^c = (\lim B_{f(n)})^c \text{ a.e.}
\]

Note: by the construction of \( f(n) \), \( \lim B_{f(n)}^c \) a.e. Thus, \( F(\cdot)^c = F(\cdot)^c \) in \( \tilde{S} \).

Similarly, \( F(\cdot \cap \cdot) = F((\cdot \cup \cdot)^c) = [F(\cdot) \cup F(\cdot)]^c = F(\cdot) \cap F(\cdot) \).

Let \( E_{\alpha_i} \in \Omega \) for \( i \geq 1 \) and \( E_{\alpha_i} = \{(B_{\alpha_i})^c\} \).

Then for each \( i \), there exists a \( S_i = \lim B_{f(n)}^\alpha \in \tilde{S} \) such that \( \lim \mu^*(B_{f(n)}^\alpha \triangle S_i) = 0 \).

Now suppose we have \( \{Y_L\} \) in the same manner as Lemma 2. By design, \( \{Y_L\} \) converges to \( \cup_{i=1}^\infty S_i \). Then \( \{Y_L\} = \bigcup_{i=1}^\infty E_{\alpha_i} \) by Lemma 2. Now we have

\[
F(\bigcup_{i=1}^\infty E_{\alpha_i}) = F(\bigcup_{i=1}^\infty (B_{\alpha_i}^c)) = F(\bigcup_{i=1}^\infty Y_L) = \lim Y_{f(L)}.
\]

Since \( \lim \mu^*(Y_L \triangle \lim Y_{f(L)}) = 0 \) and \( \lim \mu^*(Y_L \triangle \cup_{i=1}^\infty S_i) = 0 \), we have \( \lim Y_{f(L)} = \cup_{i=1}^\infty S_i \text{ a.e.} \). In addition, \( \cup_{i=1}^\infty S_i = \cup_{i=1}^\infty F(\{B_{\alpha_i}^c\}) \).

Thus,

\[
F(\bigcup_{i=1}^\infty E_{\alpha_i}) = \cup_{i=1}^\infty F(E_{\alpha_i}).
\]

Therefore, the claim follows.

\[\square\]

3. Conclusion

Theorem 1 and 2 show that the completion of \( \Omega \) is isometric and \( \sigma \)-algebra isomorphic to \( \tilde{S} \). Thus the completion of \( \Omega \) is isometric and \( \sigma \)-algebra isomorphic to the Caratheodory Extension under the equivalence relation \( \sim \) by the conclusion in [5].

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University of California, Riverside, USA

E-mail address: juntanaka@math.ucr.edu, yonigeninnin@gmail.com, junextension@hotmail.com