Detecting Multipartite Classical States and their Resemblances

Lin Chen, Eric Chitambar, Kavan Modi, and Giovanni Vacanti

Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, 117542, Singapore

Department of Physics and Department of Electrical & Computer Engineering, University of Toronto, Toronto, Ontario, M5S 3G4, Canada

(Dated: May 8, 2019)

We study various types of multipartite states lying near the quantum-classical boundary. The class of so-called classical states are precisely those in which each party can perform a projective measurement to identify a locally held state without disturbing the global state, a task known as non-disruptive local state identification (NDLID). We introduce a new class of states called generalized-classical states which allow for NDLID when the most general quantum measurements are permitted. A simple analytic method as well as a physical criterion are presented for detecting whether a multipartite state is classical. To decide whether a state is generalized-classical, we provide a semi-definite programming algorithm which can be adapted for use in other unrelated contexts such as signal processing.

PACS numbers: 03.67.-a, 03.65.Ud, 03.67.Mn

Introduction.—There are many ways in which composite quantum systems can exhibit non-classical properties. The correlations between entangled states have generated some of the most puzzling paradoxes in quantum theory; however even unentangled, or separable, states possess correlations that cannot be simulated by classical systems and thus defy our intuition. Recently, much interest has been raised concerning the properties of these non-classical correlations with applications to a variety of fields [1–8]. Of particular note is the DQC1 quantum computation model which runs exponentially faster than its best-known classical counterpart by using a highly mixed state possessing quantum correlations but no entanglement [3, 4]. This supports a hypothesis that non-mixed state possessing quantum correlations but no entanglement can be simulated by classical systems. Of particular note is the DQC1 quantum computation model which runs exponentially faster than its best-known classical counterpart by using a highly mixed state possessing quantum correlations but no entanglement [3, 4]. This supports a hypothesis that non-mixed state possessing quantum correlations but no entanglement can

In light of this, several measures have been designed to isolate and quantify precisely the non-classical nature of a quantum state such as quantum discord [9], quantum deficit [1], measurement induced disturbance [10], and similar quantities [2, 11, 12]. One common feature of all these measures is that they vanish for fully classical states [9]. As a result, the possibility for a given state to undergo some sort of non-disruptive local state identification (NDLID) by the possibility for a given state to undergo some sort of non-disruptive local state identification (NDLID) by

NLDI and a Hierarchy of Separable States.—As a motivating example, consider the fully classical state \( \rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \). Each party can perform a projective measurement in the computational basis and learn his/her local state to be either \( |0\rangle \) or \( |1\rangle \). When these results are not recorded or kept secret, the post-measurement state is still \( \rho \), and the parties have thus identified their state without perturbing the overall state. The ability for each party to perform such an information-gathering process without failure is not particular to this example but, in fact, completely characterizes the set of fully classical states [9]. As a result, the possibility for a given state to undergo some sort of NDLID can be regarded as a signature of “classicalness.”

In general, we will say a state \( \rho \) allows for NDLID by party \( k \) if there exists a decomposition \( \rho = \sum_i p_i \rho_i^{(k)} \otimes |\phi^{(k)}_i\rangle\langle \phi^{(k)}_i| \) and local measurement \( \{M^{(k)}_i\}_{i=1}^n \) with \( \sum_{i=1}^n M^{(k)}_i M^{(k)}_i = 1^{(k)} \) such that

\[
M^{(k)}_i |\phi^{(k)}_j\rangle\langle \phi^{(k)}_j| M^{(k)}_i = \lambda \delta_{ij} |\phi^{(k)}_j\rangle\langle \phi^{(k)}_j| \quad (1)
\]

for some \( 0 < \lambda \leq 1 \). Upon outcome \( i \), party \( k \) can then

\[
\sum_{i=1}^n M^{(k)}_i |\phi^{(k)}_j\rangle\langle \phi^{(k)}_j| M^{(k)}_i = \lambda \langle \phi^{(k)}_j | \phi^{(k)}_j \rangle
\]

In this Letter, we take an alternative approach to the sharpening of the quantum-classical boundary region; instead of grouping states in this region according to some numerical distance away from the set of classical states, we identify a state as “nearly” classical if it possesses a well-defined trace of some purely classical property. Specifically, we address the following two questions: (i) in what physical ways can general quantum states resemble classical states, and (ii) how can one detect whether a given state is classical or at least resemblant to one in the sense of question (i)? One answer to the first question, which we investigate below, involves a state’s ability to undergo non-disruptive local state identification (NDLID). In the remainder of this letter, we will first give a precise description of NDLID and characterize the states which exhibit this property. NDLID capable states are found to occupy a measure zero volume of state space and belong to the class of so-called minimal length separable states. After that, we will proceed to answer question (ii) by providing computational and experimental methods for deciding whether or not a given multipartite state is classical or even just similar to one in its ability for NDLID. Our detection algorithm can be efficiently implemented which differs drastically from the best known methods of detecting separability.
conclude that his/her system is in state $|\phi_j^{(k)}\rangle$ among
the ensemble $\{|\phi_j^{(k)}\rangle\}$, while the rest of the system is in
state $|\rho_{\text{E}}\rangle$. Furthermore, it can easily be seen that under
the action of this measurement, the global state remains
invariant: $\sum_i^n p_i (I^{(E)} \otimes M_i^{(k)}) \rho (I^{(E)} \otimes M_i^{(k)}) = \rho$.

From Eq. (1), it immediately follows that the task
of NDLID is equivalent to unambiguous state discrimi-
nation among the states $|\phi_j^{(k)}\rangle$ with a post-selection rate of $\lambda$. A
well-known necessary and sufficient condition for accomplishing this feat is that the $|\phi_j^{(k)}\rangle$ are
linearly independent \cite{13}. In this case, the measure-
ment operators take the form $M_i^{(k)} = |\phi_{i_k}^{(k)}\rangle\langle\phi_{i_k}^{(k)}|$ where
$\langle\phi_j^{(k)}|\phi_{i_k}^{(k)}\rangle = \delta_{ij}\lambda$ for some $0 < \lambda \leq 1$. Furthermore, we
have $\lambda = 1$ if and only if the $|\phi_j^{(k)}\rangle$ are orthogonal and
the NDLID can be performed by a complete projective
measurement. These facts motivate the following classi-
fications of multipartite separable states.

**Definition 1** Let $\{|\phi_i\rangle\} = \{|\phi_1^{(1)} \phi_2^{(2)} \cdots \phi_N^{(N)}\rangle\}$ denote a
product state basis.

a. A multipartite state $\rho$ is called separable if it is di-
agonal in some product state basis; i.e.

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|,$$

b. The state $\rho$ is called generalized-classical for the $k^{\text{th}}$
party if it is diagonal in some product state basis in
which the states $\{|\phi_i^{(k)}\rangle\}$ are linearly independent.

c. The state $\rho$ is called classical for the $k^{\text{th}}$ party if it is
diagonal in some product state basis in which
the states $\{|\phi_i^{(k)}\rangle\}$ are orthogonal.

d. The state $\rho$ is called fully generalized-classical or
fully classical if it is diagonal in some product state basis in
which statements b or c are true respec-
tively for all parties.

From the discussion preceding Definition 1, generalized-
classical states are nearly classical in the following sense:

A state is classical (resp. generalized-
classical) with respect to party $k$ iff party $k$
can perform NDLID by a projective (resp. generalized)
measurement.

There exists an even broader class of separable states
still hovering close to the quantum-classical border. An
$N$-partite state $\rho$ of rank $r$ will be called a mini-
mal length separable state if it has a decomposition
$\rho = \sum_{i=1}^r \lambda_i |\phi_1^{(1)} \cdots \phi_i^{(N)}\rangle\langle\phi_1^{(1)} \cdots \phi_i^{(N)}|$ \cite{13}. It is quite
easy to see from the following lemma that any fully
generalized-classical state is also a minimal length sep-
arable state.

**Lemma 2** For some multi-index $(i_1, \ldots, i_N)$, if up to rep-
etition of states the $|\phi_{i_1}^{(1)}\rangle$ are linearly independent for all
parties $j$, then the product states $|\phi_{i_1}^{(1)} \cdots \phi_i^{(N)}\rangle$ are also
linearly independent.

By this lemma and Definition 1, if $\rho$ is fully
generalized-classical, it has a decomposition $\rho = \sum_{i=1}^\delta |\phi_1^{(1)} \cdots \phi_i^{(N)}\rangle\langle\phi_1^{(1)} \cdots \phi_i^{(N)}|$ with $\delta \geq r$ and each
$|\phi_{i_1}^{(1)} \cdots \phi_i^{(N)}\rangle$ linearly independent. This last property
implies that $r = \delta$ and so we see that each fully
generalized-classical state is a minimal length state. Fur-
thermore, in the bipartite case, if a state is generalized-
classical with respect to just one of the parties, it will
be of minimal length. The following chain of inclusions
summarizes the main parsings described in this letter:

minimal length $\supset$ fully generalized-classical
$\supset$ fully classical $\supset$ product.

Here, product states refer to states of the form $\rho = \rho_1 \otimes
\cdots \otimes \rho_N$.

There are two reasons to consider minimal length
states as also lying near the quantum-classical border.
First, it is known that only non-minimal length states
constitute the opposite end of the spectrum at the
separable/non-separable boundary \cite{14}. While this alone
does not imply a closeness between minimal length and
classical states, such an interpretation becomes further
justified when considering the volumes of each set in state
space. Separable states possess a nonzero volume \cite{15}
while minimal length states are of measure zero \cite{16}. This
final point has an even greater relevance to our discussion
since it implies that fully generalized-classical states are
also of measure zero. In other words, nearly all multi-
partite quantum states lack the property of non-disruptive
local state identification. Also note that this provides an
alternative proof for the result in Ref. \cite{17} which shows
a generic state to have a nonzero discord (i.e. is non-
classical).

**Decision Algorithms for Classical and Generalized-
Classical States.**— In the last portion of this letter we
address the question of deciding whether a given mul-
tipartite state is classical or generalized-classical. Our
results, discovered independently, generalize the recent
works on this topic \cite{7,18,21} in which necessary and suffi-
cient conditions have been provided for deciding the non-
classical bipartite states. The techniques we use are sim-
ilar to those in Ref. \cite{20} in that both our algorithms in-
volve checking commutation relations. Interestingly, we
find that deciding whether a state is generalized-classical
reduces to a problem similar in nature to those well-
studied in the field of signal processing \cite{22,23}. Hence,
our use of semi-definite programming (SDP) in detect-
ing generalized-classical states may be of interest to re-
searchers in that subject, as well as the linear algebra
community at large. From a computational complex-
Lemma 3 The state $\rho$ is fully generalized-classical (resp. classical) if it is generalized-classical (resp. classical) for all parties.

Proof. We will prove this for the bipartite case, but the idea immediately generalizes to arbitrary number of parties. Suppose $\rho = \sum_i \rho_i \otimes |a_i\rangle\langle a_i|$ where the $|a_i\rangle$ are linearly independent (resp. orthonormal). Then we see that each $\rho_i$ is a linear combination of the $|a_i\rangle\langle a_i|$ so that $|a_i\rangle\otimes|b_j\rangle$ is a product basis in which $\rho$ is diagonal.

By Lemma 3 it will be sufficient to only consider bipartite systems in the following discussion. So introduce Alice and Bob and let $d_A$ and $d_B$ denote the dimensions of their subsystems respectively. Assume that some state $\rho$ is classical or generalized-classical with respect to Bob. By definition, there exists some basis $|b_i\rangle$ such that

$$\rho = \sum_i \rho_i |b_i\rangle\langle b_i|,$$

while for classical states, the $|b_i\rangle$ are orthogonal. Note that in both cases, the contraction $\langle \phi^{(A)}_i | \rho | \phi^{(A)}_j \rangle$ will be diagonal in the basis $|b_i\rangle$ for any two states $|\phi^{(A)}_i\rangle, |\phi^{(A)}_j\rangle \in \mathcal{H}_A$. This fact leads to the following theorem.

**Theorem 4** Let $\{|\phi^{(A)}_i\rangle\}$ be any orthonormal basis for $\mathcal{H}_A$. Then $\rho$ is generalized-classical (resp. classical) if and only if

$$\rho^{(B)}_{ij} := \langle \phi^{(A)}_i | \rho | \phi^{(A)}_j \rangle$$

is diagonal in the same (resp. orthonormal) basis $\{|b_i\rangle\}$ for all $i,j$.

Proof. Necessity follows from the above observation. For sufficiency, suppose that $\rho^{(B)}_{ij} = \sum_m b_{ijm} |b_m\rangle\langle b_m|$ where $\{|b_m\rangle\}$ is any linearly independent (resp. orthonormal) set spanning $\mathcal{H}_B$. From the general expansion $\rho = \sum_{ijmn} c_{ijmn} |\phi^{(A)}_i\rangle \langle \phi^{(A)}_j | \otimes |b_m\rangle\langle b_n|$, we see that $c_{ijmn} = \delta_{mn} b_{ijm}$ and so

$$\rho = \sum_{ijm} b_{ijm} |\phi^{(A)}_i\rangle \langle \phi^{(A)}_j | \otimes |b_m\rangle\langle b_m| = \sum_m \rho_m \otimes |b_m\rangle\langle b_m|$$

where $\rho_m = \sum_{ij} b_{ijm} |\phi^{(A)}_i\rangle \langle \phi^{(A)}_j | = \sum_{ij} b_{ijm} |b_j\rangle\langle b_j|$. The last equation implies that $\rho_m$ is semidefinite positive. Hence the state $\rho$ is generalized-classical (resp. classical) as defined in Eq. 2.

Theorem 4 implies that to decide whether $\rho$ is generalized-classical for Bob, we need to check whether $\frac{1}{2} d_A (d_A - 1)$ matrices $\{|\phi^{(A)}_i\rangle | \phi^{(A)}_j\rangle \rangle_{1 \leq i \neq j \leq d_A}$ of size $d_B \times d_B$ are simultaneously congruent to diagonal matrices. In a more general form, this problem asks for some set $\{|A_i\rangle\}_{i=0..m}$ of $n \times n$ matrices whether there exists an invertible matrix $P$ such that $PA_i P^\dagger = \Lambda_i$ is diagonal for all $i$. This is a natural question to ask in linear algebra studies and we have already alluded to practical situations in which it arises outside of quantum information. We thank Yaoyun Shi for his assistance with the following. To our knowledge, SDP is a previously unrecognized approach to solving the described problem.

**Lemma 5** [Shi] Deciding if nonsingular $P$ exists such that $PA_i P^\dagger = \Lambda_i$ can be achieved by a semi-definite program (SDP).

To construct the algorithm, we first assume without loss of generality that the $A_i$ are hermitian. For we can always write $A_i = A_i^\dagger + i A_i''$ where $A_i^\dagger$ and $A_i''$ are hermitian. Then $PA_i P^\dagger$ is diagonal if and only if $PA_i P^\dagger$ is diagonal. Conversely, if both $PA_i P^\dagger$ and $PA_i'' P^\dagger$ are diagonal, then $A_i$ being hermitian, the $PA_i P^\dagger$ are hermitian and if $PA_i P^\dagger = \Lambda_i$, the $PA_i P^\dagger$ are simultaneously diagonalized and therefore $[PA_i P^\dagger, PA_j P^\dagger] = 0$ for all $i,j$. Conversely, if this latter condition holds, then there exists a unitary $U$ such that $U P A_i P^\dagger U^\dagger = PA_i P^\dagger = \Lambda_i$ for all $i$. So the question is whether $[PA_i P^\dagger, PA_j P^\dagger] = 0$ for all $i,j$. Or in other words, $A_i W A_j = A_j W A_i$ where $W$ is a positive-definite matrix. Note that if $W$ is positive-definite, then we can scale appropriately so that $W \geq I$. Thus, we have the SDP feasibility problem:

$$\begin{align*}
\text{Find} & \quad W \\
\text{subject to} & \quad A_i W A_j = A_j W A_i \quad \text{for all } i,j \\
& \quad W - I \geq 0.
\end{align*}$$

Known algorithms based on the ellipsoid and interior-point methods can efficiently solve this problem.

To decide whether $\rho$ is classical for Bob, the situation is easier. We first begin by choosing any basis $\{|\phi^{(A)}_i\rangle\}$ for Alice and checking whether $\rho^{(B)}_{ij}$ is diagonalizable for all $i,j$. In total, there will be $\frac{d_A^2 - d_A}{2}$ matrices to check. If these are not diagonalizable, then by Theorem 3, $\rho$ is not classical. If so, $\rho$ is classical if and only if the commutation $[\rho^{(B)}_{ij}, \rho^{(B)}_{kl}]$ vanishes for all $i,j,k,l$, which amounts to at most $\frac{1}{4} d_A^2 (d_A^2 - 1)$ commutation relations to check. In the case that all operators commute, a common
eigenbasis $\{|a_j\rangle\}$ can be easily computed; the sufficiency of Theorem 3 proves $\rho$ to be classical.

Physical detection of classical states—Theorem 4 can be experimentally implemented by a set of projective operations and quantum state tomography. A direct reconstruction of the elements in Eq. 3 is not possible since they are not Hermitian and therefore do not correspond to anything physical. However, these terms can be computed indirectly if Alice makes a set of linearly independent projective operations (observables) that span her Hilbert-Schmidt space: $L = \{\langle \phi_i^{(A)} | \phi_j^{(A)} \rangle, \langle \psi_i^{(A)} | \psi_j^{(A)} \rangle, |\chi_i^{(A)}\rangle \langle \chi_j^{(A)} |\}$ where $|\chi_i^{(A)}\rangle = \frac{1}{\sqrt{2}} (|\phi_i^{(A)} \rangle - i|\phi_j^{(A)} \rangle)$ and $|\psi_i^{(A)} \rangle = \frac{1}{\sqrt{2}} (|\phi_i^{(A)} \rangle + |\phi_j^{(A)} \rangle)$ for $i > j$. With that we have the elements of Eq. 3:

$\langle \phi_i^{(A)} | \rho | \phi_j^{(A)} \rangle = \langle \psi_i^{(A)} | \rho | \psi_j^{(A)} \rangle + i \langle \chi_i^{(A)} | \rho | \chi_j^{(A)} \rangle - \frac{1}{2} \langle \phi_i^{(A)} | \rho | \phi_j^{(A)} \rangle$.

According to Theorem 4 a state $\rho$ is classical if and only if it has the same orthonormal basis for $\langle \phi_i^{(A)} | \rho | \phi_j^{(A)} \rangle$ for all $i, j$. It is clear that if $\text{Tr} A[P_{\rho}]$ is diagonal for all $P \in L$ then $\rho$ is classical. Conversely, $\langle \phi_i^{(A)} | \rho | \phi_j^{(A)} \rangle$ diagonal in some orthonormal basis for all $i, j$ implies that $\text{Tr} A[P_{\rho}]$ is diagonal in same basis for all $P \in L$. As the elements of $L$ span Alice’s space, any POVM she can perform will have operator elements with each being a linear combination of these projectors. Furthermore, if we consider “Alice’s” system as the joint system of $N − 1$ parties, then any local POVM performed by the $N − 1$ parties will have product operators $E_j = \bigotimes_{i=1}^{N-1} E_{ij}$ also being a linear combination of projectors from $L$, and conversely any element of $L$ can be expressed as a linear combination of product operators constituting complete local measurements on the $N − 1$ subsystems. Thus we obtain the following:

**Theorem 6** An $N$-partite state $\rho$ is classical with respect to party $k$ if and only if for any local POVM performed by the other parties,

$$[\rho_{\tilde{i}}, \rho_{\tilde{j}}] = 0 \quad \text{for all } \tilde{i}, \tilde{j}$$

(6)

where $\rho_{\tilde{i}} = \text{Tr}_{\bar{N}}[E_{\tilde{i}} \rho]$.

Quantum mechanics and commuters are intimately related since days of the theory’s foundation. Here, we see that the non-classical nature of a state can be detected precisely by the non-commutativity of reduced states after some local POVM is locally implemented on all but one of the subsystems.

Conclusion.—We have introduced a class of states called generalized-classical which permit the purely classical task of non-disruptive local state identification when general quantum measurements are used. In this sense, generalized-classical states can be said to hover near the quantum-classical boundary. We have provided methods, both analytic and physical, which decide if a state is classical or generalized-classical. For the latter, our algorithm amounts to a seemingly novel way for deciding whether a set of matrices can be simultaneously diagonalized by a general (non-necessarily orthogonal) congruence transformation. Our results hold in the multipartite setting where states can be classical or generalized-classical with respect to one or many of the involved parties. We believe these results are helpful in better understanding the intersection between classical and quantum regimes.

Acknowledgment. LC thank Dr. Ying Li and Prof. Wei Song for helpful discussions. KM thanks B. Dakic, C. Rodriguez-Rosario, and V. Vedral for discussions. EC is partially supported by the U.S. NSF under Awards 0347078 and 0622033. The Center for Quantum Technologies is funded by the Singapore Ministry of Education and the National Research Foundation as part of the Research Centres of Excellence programme.

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