Linearization of analytic order relations

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Abstract

We prove that if $\preceq$ is an analytic partial order then either $\preceq$ can be extended to a $\Delta^1_2$ linear order similar to an antichain in $2^{<\omega}$ ordered lexicographically or a certain Borel partial order $\leq_0$ embeds in $\preceq$. Some corollaries for analytic equivalence relations are given, for instance, if $E$ is a $\Sigma^1_1[z]$ equivalence relation such that $E_0$ does not embed in $E$ then $E$ is determined by intersections with $E$-invariant Borel sets coded in $L[z]$.

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Introduction

It is a simple application of Zorn’s lemma that any partial order can be extended to a linear order on the same domain. More generally any partial quasi-order admits a linearization.  

A much more difficult problem is to provide a descriptive characterization of the linear order in the assumption that one has such for the given p. q.-o.. For instance, not every Borel p. q.-o. is Borel linearizable.

Example 1. Recall that $E_0$ is an equivalence relation on $2^\omega$ defined as follows: $a E_0 b$ if and only if $a(k) = b(k)$ for all but finite $k$. Let $\preceq$ be $E_0$ considered as a p. q.-o.. Then $\preceq$ is not Borel linearizable. Indeed any linearization $h$ satisfies $a E_0 b \iff h(a) = h(b)$, but it is known that Borel maps $h$ with such a property do not exist (see Harrington, Kechris, Louveau [2]).

Example 2. Example 1 can be converted to a partial order. Define the anti-lexicographical p. o. $\leq_0$ on $2^\omega$ as follows: $a \leq_0 b$ if and only if either $a = b$ or there is $m \in \omega$ such that $a(k) = b(k)$ for all $k > m$ and $a(m) < b(m)$. Clearly $a \leq_0 b$ implies $a E_0 b$ and $\leq_0$ linearly orders each $E_0$-equivalence class similarly to the integers $\mathbb{Z}$, except for the class of $\omega \times \{0\}$ (ordered as $\omega$) and the class of $\omega \times \{1\}$ (ordered as $\omega^{-1}$ – the inverted $\omega$).

Finally $\leq_0$ is not Borel linearizable (see Subsection 7.1).

There are Borel-non-linearizable Borel orders of different nature, e. g. the p. q.-o. $a \preceq b$ if and only if $a(k) \leq_0 b(k)$ for all but finite $k$ on $2^\omega$ or the dominance relation on $\omega^\omega$. However by the next theorem the relation $\leq_0$ of Example 2 is actually a minimal Borel-non-linearizable Borel order. (Compare with the “Glimm–Effros” theorem of Harrington, Kechris, Louveau [2] saying that $E_0$ is a minimal non-smooth Borel equivalence relation.)

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1 Notation. Several notions related to orders are sometimes understood differently, so let us take a space to fix an unambiguous meaning.

A binary relation $\preceq$ on a set $X$ is a partial quasi-order, or p. q.-o. in brief, on $X$, iff $x \preceq y \wedge y \preceq z \implies x \preceq z$, and $x \preceq x$ for any $x \in X$. In this case, $\approx$ is the associated equivalence relation, i.e. $x \approx y$ iff $x \preceq y \wedge y \preceq x$.

If $x \approx x \implies x = x$ for any $x$ then $\preceq$ is a partial order, or p. o.. If in addition $x \preceq y \lor y \preceq x$ for all $x, y \in X$ then $\approx$ is a linear order (l. o.).

Let $\preceq$ and $\preceq'$ be p. q.-o.’s on resp. $X$ and $X'$. A map $h : X \rightarrow X'$ will be called half order preserving, or h. o. p., iff $x \preceq y \implies h(x) \preceq' h(y)$.

Finally a linearization is any h. o. p. map $h : \langle X; \preceq \rangle \rightarrow \langle X'; \preceq' \rangle$, where $\preceq'$ is a l. o., satisfying $x \approx y \iff h(x) = h(y)$.

2 If one enlarges $<_0$ so that, in addition, $a <_0 b$ whenever $a, b \in 2^\omega$ are such that $a(k) = 1$ and $b(k) = 0$ for all but finite $k$ then the enlarged relation can be induced by a Borel action of $\mathbb{Z}$ on $2^\omega$, such that $a <_0 b$ iff $a = zb$ for some $z \in \mathbb{Z}$, $z > 0$. 

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Theorem 1 (Kanovei [11]) Suppose that \( \preceq \) is a Borel p. q.-o. on \( N = \omega^\omega \). Then exactly one of the following two conditions is satisfied:

(I\( ^B \)) \( \preceq \) is Borel linearizable -- moreover \( \preceq \) in this case there are an ordinal \( \alpha < \omega_1 \) and a Borel linearization \( h : \langle N; \preceq \rangle \rightarrow \langle 2^\alpha; \leq_{\text{lex}} \rangle \);

(II\( ^B \)) there exists a continuous h. o. p. 1–1 map \( F : \langle 2^\omega; \leq_0 \rangle \rightarrow \langle N; \preceq \rangle \) such that \( a \neq E_0 b \Rightarrow F(a) \neq F(b) \).

Example 3. Let \( W_0 = \{ x \in N : x \text{ codes an ordinal} \}; \) for \( x \in W_0 \) let \( |x| \) be the ordinal coded by \( x \). Define a \( \Sigma_1^1 \) p. q.-o. \( x \prec y \) iff either \( y \notin W_0 \) or \( x, y \in W_0 \) and \( |x| \leq |y| \). (I\( ^A \)) is impossible for \( \preceq \) even via a non-Borel map \( h \) since orders \( \langle 2^\alpha; \leq_{\text{lex}} \rangle, \alpha < \omega_1 \), do not admit strictly increasing \( \omega_1 \)-chains. (II\( ^A \)) is also impossible via analytic maps \( F \) by the restriction theorem. Thus Theorem 1 fails for analytic relations.

Following ideas of Hjorth and Kechris [7], we involve longer orders, \( 2^{<\omega_1} \) and \( 2^\omega_1 \), to match the nature of analytic p. q.-o.'s. A set \( A \subseteq 2^{<\omega_1} \) will be called an antichain when it consists of pairwise \( \subseteq \)-incomparable elements.

Theorem 2 Suppose that \( \preceq \) is a \( \Sigma_1^1 \) p. q.-o. on \( N \). Then at least one of the following two conditions is satisfied:

(I\( ^A \)) There is a linearization \( h : \langle N; \preceq \rangle \rightarrow \langle 2^\omega_1; \leq_{\text{lex}} \rangle \) such that for any \( \gamma < \omega_1 \) the map \( x \mapsto h(x) | \gamma \) is Borel, and has an \( \omega_1 \)-Borel \( ^4 \) code in \( L[z] \) provided \( \preceq \) is \( \Sigma_1^1 \) \( [z] \). In addition in each of the two \( \^ \) following cases there is an antichain \( A \subseteq 2^{<\omega_1} \) and a \( \Delta_1^1 \) in the codes linearization \( h : \langle N; \preceq \rangle \rightarrow \langle A; \leq_{\text{lex}} \rangle \):

(a) for any \( x \) the set \( [x]_\approx = \{ y : y \approx x \} \) is Borel; \( ^6 \)

(b) the universe is a set generic \( ^7 \) extension of a class \( L[z_0] \), \( z_0 \in N \).

(II\( ^A \)) As \( \Pi^1 \) of Theorem 4.

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3 Harrington e. a. [3] proved that any Borel l. o. is Borel order isomorphic to a l. o. \( \langle X; \leq_{\text{lex}} \rangle \), where \( X \subseteq 2^\alpha \) for some \( \alpha < \omega_1 \) and \( \leq_{\text{lex}} \) is the lexicographical order.

4 Means: \( \lambda \)-Borel for an ordinal \( \lambda < \omega_1 \) which is not necessarily countable in \( L[z] \).

5 An obvious parallel with the “Ulm classification” theorem in Hjorth and Kechris [3] allows to conjecture that the additional assertion is also true in the assumption of the existence of “sharps”, or an even weaker assumption in Friedman and Velickovic [4]. However the most interesting problem is to prove the additional assertion in \( \text{ZFC} \).

6 This applies \( e. g. \) when \( \preceq \) is a p. o.. Recall that \( x \approx y \) iff \( x \approx y \land y \approx x \).

7 Via any kind of set forcing. Compare with a theorem on thin \( \Sigma_1^1 \) equivalence relations in Hjorth [3].
Take notice that \([I^A]\) and \([II^A]\) here are compatible for instance in the assumption \(V = L\). There possibly exist reasonable sufficient conditions (like: all \(\Delta^1_2\) sets are Lebesgue measurable) for \([I^A]\) and \([II^A]\) to be incompatible.

### Applications for analytic equivalence relations

Theorem 2 applies for analytic equivalence relations viewed as a particular case of p.q.-o.'s.

**Corollary 3** Let \(E\) be a \(\Sigma^1_1\) equivalence relation on \(\mathbb{N}\). Then at least one of the following two conditions is satisfied:

\((I^E)\) There is a map \(h : \mathbb{N} \rightarrow 2^{\omega_1}\) such that \(x E y \iff h(x) = h(y)\) and for any \(\gamma < \omega_1\) the map \(x \mapsto h(x) | \gamma\) is Borel, and, provided \(\preceq\) is \(\Sigma^1_1[z]\), has an \(\omega_1\)-Borel code in \(L[z]\). In addition in each of the two following cases there is an antichain \(A \subseteq 2^{<\omega_1}\) and a \(\Delta^1_2\) in the codes map \(h : \mathbb{N} \rightarrow A\) such that \(x E y \iff h(x) = h(y)\):

(a) for any \(x\) the set \([x]_E = \{y : y E x\}\) is Borel;

(b) the universe is a set generic extension of a class \(L[z_0], z_0 \in \mathbb{N}\).

\((II^E)\) There exists a continuous \(1 - 1\) function \(F : 2^{\omega} \rightarrow \mathbb{N}\) such that \(a E_0 b \iff F(a) E F(b)\). □

This result, with \([I^E]\) in the additional form, has been obtained by Hjorth and Kechris \[7\] in the subcase \((I^E)(a)\) (as well as in the assumption of existence of sharps), by Friedman and Velickovic \[1\] in a hypothesis connected with weakly compact cardinals, and by Kanovei \[10\] in the subcase \((I^E)(b)\).

(Recall that a map \(F\) as in \((II^E)\) is called an embedding of \(E_0\) in \(E\) – a continuous embedding in this case. A map \(h\) as in \([I^E]\) is called a reduction of \(E\) to the equality on \(2^{\omega_1}\) or \(2^{<\omega_1}\).

**Corollary 4** Assume that \(E\) is a \(\Sigma^1_1[z]\) equivalence relation, \(z \in \mathbb{N}\), and \((II^E)\) of Corollary 3 fails. Then \(x E y\) iff we have \(x \in X \iff y \in X\) for every \(E\)-invariant Borel set \(X \subseteq \mathbb{N}\) with an \(\omega_1\)-Borel code in \(L[z]\). □

Hjorth and Kechris \[7\] proved that any analytic \(E\) which does not satisfy \((II^E)\) of Corollary 3 admits an effective reduction \(h : \mathbb{N} \rightarrow 2^{\omega_1}\) (i.e. we have \(x E y \iff h(x) = h(y)\)), however it is not clear whether the property mentioned in \([I^E]\) of Corollary 3 holds for the reduction given in \[7\] and equally whether the reduction in \[7\] directly leads to Corollary 4.

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8 Hjorth and Kechris told the author in April 1997 that they had known the result.
Organization of the proofs

The following theorem stands behind the results above.

Recall that if $T$ is a tree on $\omega \times \omega \times \lambda$ then

$$[T] = \{\langle x, y, f \rangle \in N^2 \times \lambda^\omega : \forall m T(x | m, y | m, f | m)\}$$

and $p[T] = \{\langle x, y \rangle : \exists f [T](x, y, f)\}$.

**Theorem 5** Let $\omega \leq \lambda < \omega_1$. Suppose that $T$ and $S$ are trees on $\omega \times \omega \times \lambda$ such that the sets $\leq_T = p[T] \subseteq \leq_S = \overline{\mathcal{C}p[S]}$ are p. q.-o.'s on $N$. Then at least one of the following two conditions is satisfied:

(I) There are $\alpha < \omega_1$ and a $\omega_1$-Borel coded in $L[T, S]$ h. o. p. map $h : \langle N; \leq_T \rangle \rightarrow \langle 2^\alpha; \leq_{\text{lex}} \rangle$ such that $h(x) = h(y) \Rightarrow x \approx_S y$.

(II) There is a continuous 1-1 h. o. p. map $F : \langle 2^\omega; \leq_0 \rangle \rightarrow \langle N; \leq_T \rangle$ such that $a \neq_E b \Rightarrow F(a) \neq_S F(b)$.

The principal technical scheme of the proofs goes back to the papers of Harrington and Shelah [4], Shelah [13], and Horth [5] containing theorems on bi-$\kappa$-Souslin equivalence and order relations. However our version of the technique is free of any use of model theory including admissible sets.

On the other hand we exploit several technical achievements made in the study of the Borel orders (Harrington e. a. [3], Louveau [12]) by means of the Gandi – Harrington topology.

The two technical schemes, the one we use and the one based on the Gandi – Harrington topology, involve different kinds of “effective” sets in the forcing, but have many common points in the construction of the proofs (like a similar definition of the “regular” and “singular” cases, a similar construction of splitting systems etc.), although differ in many details.

As a matter of fact the Gandi – Harrington topology technique proves Theorem [4] shorter than we do here (see Kanovei [11]), but it has problems with the analytic case as it does not capture the proper type of effectiveness.

After some preliminaries in Section [1] (including an effective version of the classical separation theorem) we introduce the dichotomy in Section [2]. Then the proof of Theorem [5] naturally develops itself in sections [3], [4], [5], and [6] (where we show that [11] of Theorem [5] is Shoenfield–absolute).

Theorem [2] (Section [6]) will require a reflection argument saying that an analytic p. q.-o. has uncountably many indices for “upper” Borel approximations which are p. q.-o.’s, together with a delicate reasoning in the case of a generic universe, in Section [5].
1 Preliminaries

The proof of Theorem 5 is the major part of this paper.

We fix an ordinal \( \lambda, \omega \leq \lambda < \omega_1 \), and trees \( T, S \subseteq (\omega \times \omega \times \lambda)^{<\omega} \).

Assume that both \( T \) and \( S \) are constructible. \(^9\)

Suppose that \( \preceq_T = p[T] \subseteq \preceq_S = \mathcal{Cp}[S] \) are p. q.-o.’s on \( N \). Define \( x \approx_S y \) iff \( x \preceq_S y \land y \preceq_S x \) and \( x \approx_T y \) similarly.

1.1 Coding Borel sets

We let \( \mathcal{L}_{\lambda+1,0} \) be the infinitary language containing

(i) constant symbols \( \dot{x}, \dot{y}, \dot{z}, ... \) for indefinite elements of \( N = \omega^\omega \) and constant symbols \( \dot{f}, \dot{g}, ... \) for indefinite elements of the set \( \lambda^\omega \);

(ii) elementary formulas of the form \( \dot{x}(k) = l \) and \( \dot{f}(k) = \alpha \), where \( k, l \in \omega \) while \( \alpha < \lambda \);

(iii) conjunctions and disjunctions of size \( \leq \lambda \), together with the ordinary propositional connectives, but it is assumed that any formula contains only finitely many constant symbols mentioned in (i).

(Quantifiers are not allowed). Thus formulas in \( \mathcal{L}_{\lambda+1,0} \) code \((\lambda+1)\)-Borel subsets of spaces \( N^n \times (\lambda^\omega)^n \). For a formula, say, \( \varphi(\dot{x}, \dot{f}) \) we put

\[ [\varphi] = \{ (x, f) \in N \times \lambda^\omega : \varphi(x, f) \} . \]

For instance \([T](\dot{x}, \dot{y}, \dot{f})\) is a \( \mathcal{L}_{\lambda+1,0} \)-formula; we shall denote it by \( \dot{x} \preceq_T \dot{f} \dot{y} \).

Similarly, the formula \( \neg [S](\dot{x}, \dot{y}, \dot{f}) \) will be denoted by \( \dot{x} \not\preceq_S \dot{f} \dot{y} \). Formulas \( \dot{x} \not\preceq_S \dot{f} \dot{y}, \dot{x} \approx_S \dot{y} \) etc. are derivatives. Then

\[ x \preceq_T y \iff \exists f \in \lambda^\omega \ x \preceq_T f \ y \quad \text{and} \quad x \preceq_S y \iff \forall f \in \lambda^\omega \ x \preceq_S f \ y . \]

1.2 Consistency and separation

A formula \( \varphi \) is consistent if it has a model, i.e. becomes true after one suitably substitutes its constants by elements of \( N \) and \( \lambda^\omega \).

A theory in \( \mathcal{L}_{\lambda+1,0} \) will be any set of formulas of \( \mathcal{L}_{\lambda+1,0} \) containing a common (finite) list \( S \) of constants of type \( [1] \). (We shall usually consider constructible theories \( \Phi \subseteq \mathcal{L}_{\lambda+1,0} \).) A theory is consistent if it has a model.

A theory \( \Phi \) is \( \lambda \)-consistent if every constructible subtheory \( \Phi' \subseteq \Phi \) of cardinality \( \leq \lambda \) in \( L \) is consistent.

\(^9\) Otherwise all entries of \( L \) from now on have to be uniformly changed to \( L[T, S] \).
A theory $\Phi$ $\lambda$-implies a formula $\psi$ if $\Phi \cup \{\psi\}$ is $\lambda$-inconsistent. Other statements like this are to be understood accordingly.

The following theorem has a semblance of the Craig interpolation theorem, but essentially it belongs to the type of separation theorems.

**Theorem 6** Suppose that $\Phi(\dot{x}, \dot{y}, \dot{f}, \ldots)$ and $\Psi(\dot{x}, \dot{y}', \dot{f}', \ldots)$ are constructible theories in $L_{\lambda+1,0}$ having $\dot{x}$ as the only common constant in the (finite) lists of constants. Assume that $\Phi(\dot{x}, \dot{y}, \dot{f}, \ldots) \cup \Psi(\dot{x}, \dot{y}', \dot{f}', \ldots)$ is $\lambda$-inconsistent. Then there is a $L_{\lambda+1,0}$-formula $\pi(\dot{x})$ $\lambda$-separating $\Phi$ from $\Psi$ in the sense that $\Phi(\dot{x}, \ldots)$ $\lambda$-implies $\pi(\dot{x})$ while $\Psi(\dot{x}, \ldots)$ $\lambda$-implies $\neg \pi(\dot{x})$.

**Proof** First of all we can assume that $\Phi$ and $\Psi$ consist of single formulas, resp. $\varphi(\dot{x}, \dot{y}, \dot{f}, \ldots)$ and $\psi(\dot{x}, \dot{y}', \dot{f}', \ldots)$. Let, for the sake of simplicity, $\varphi$ be $\varphi(\dot{x}, \dot{y})$ and $\psi$ be $\psi(\dot{x}, \dot{f})$. Consider the sets

$$P = [\varphi] = \{\langle x, y \rangle : \varphi(x, y)\}, \quad Q = [\psi] = \{\langle x, f \rangle \in N \times \lambda^\omega : \psi(x, f)\}.$$  

The projections $X = \{x : \exists y P(x, y)\}$, $Y = \{x : \exists f Q(x, f)\}$ are disjoint $\Sigma^1_1$ sets by the inconsistency assumption, hence by the classical separation theorem they can be separated by a Borel set. Moreover as we demonstrated in Kanovei [4] (Theorem 7) in this case the separating set can be defined in the form $B = [\pi]$ for an appropriate $L_{\lambda+1,0}$-formula $\pi(\dot{x})$. \qed

### 1.3 Hulls

By $\mathcal{F}(\dot{x})$ we shall denote the (constructible) collection of all $L_{\lambda+1,0}$-formulas $\varphi(\dot{x}) \in L$. For a theory $\Phi(\dot{x}, \dot{y}, \ldots)$, $\mathcal{F}_x[\Phi(\dot{x}, \dot{y}, \ldots)]$ will be the set of all formulas $\varphi(\dot{x}) \in \mathcal{F}(\dot{x})$ which are $\lambda$-implied by $\Phi(\dot{x}, \dot{y}, \ldots)$.

**Lemma 7** Suppose that $\Pi(\dot{x}, \ldots)$ is a constructible theory in $L_{\lambda+1,0}$ while $R(\dot{x}) \subseteq \mathcal{F}(\dot{x})$ is also constructible and satisfies $\mathcal{F}_x[\Pi(\dot{x}, \ldots)] \subseteq R(\dot{x})$. Then the theory $\Pi'(\dot{x}, \ldots) = \Pi(\dot{x}, \ldots) \cup R(\dot{x})$ satisfies $\mathcal{F}_x[\Pi'(\dot{x}, \ldots)] = \mathcal{F}_x[R(\dot{x})]$.

**Proof** Prove that $\mathcal{F}_x[\Pi'(\dot{x}, \ldots)] \subseteq \mathcal{F}_x[R(\dot{x})]$ (the nontrivial direction). Let $\psi(\dot{x}) \in \mathcal{F}_x[\Pi'(\dot{x}, \ldots)]$. By definition there is a set $\Psi(\dot{x}) \in L$, $\Psi(\dot{x}) \subseteq R(\dot{x})$, of cardinality $\leq \lambda$ in $L$, such that $\Pi(\dot{x}, \ldots) \cup \Psi(\dot{x})$ $\lambda$-implies $\psi(\dot{x})$. We conclude that the formula $(\wedge \Psi(\dot{x})) \implies \psi(\dot{x})$ belongs to $\mathcal{F}_x[\Pi(\dot{x}, \ldots)]$, hence to $R(\dot{x})$, which guarantees $\psi(\dot{x}) \in \mathcal{F}_x[R(\dot{x})]$. \qed
2 Monotone Borel functions and the dichotomy

To introduce the dichotomy we have to extend the language $\mathcal{L}_{\lambda+1,0}$ by Borel functions mapping $\mathbb{N}$ in a set of the form $2^\alpha$, where $\alpha < (\lambda^+)^L$. If such an $\alpha$ is fixed, let a function code be a sequence of the form $\vec{\varphi} = \langle \varphi_\gamma(x) : \gamma < \alpha \rangle$ where each $\varphi_\gamma$ is a $\mathcal{L}_{\lambda+1,0}$-formula. Such a sequence defines a function $h_{\vec{\varphi}} : \mathbb{N} \to 2^\alpha$ so that $h_{\vec{\varphi}}(x)(\gamma) = 1$ iff $\varphi_\gamma(x)$.

Define $H_\alpha$ to be the set of all h. o. p. maps $h_{\vec{\varphi}} : \langle \mathbb{N} ; \leq_T \rangle \to 2^\alpha$; $\leq_{\text{lex}}$, where $\vec{\varphi}$ is a constructible function code. Define $H = \bigcup_{\alpha < (\lambda^+)^L} H_\alpha$. Thus every function in $H$ is an $\omega_1$-Borel (even $\lambda+1$-Borel), coded in $L$, map from $\mathbb{N}$ to some $2^\alpha$, $\alpha < (\lambda^+)^L$, satisfying $x \leq_T y \implies h(x) \leq_{\text{lex}} h(y)$.

But as a matter of fact functions in $H$ will be used only via equalities of the form $h(x) = h(y)$ where $h = h_{\vec{\varphi}} \in H_\alpha$ for some $\alpha < (\lambda^+)^L$, viewed as shorthand for $\land_{\gamma < \alpha} (\varphi_\gamma(x) \iff \varphi_\gamma(y))$.

Let $\dot{x} \equiv_H \dot{y}$ be the theory $\{ h(\dot{x}) = h(\dot{y}) : h \in H \}$. (Thus $\equiv_H$ defines an equivalence relation extending $\equiv_T$.)

We have two cases.\footnote{There is a point of dissatisfaction in the distribution on the two cases we use. It would be more natural to define Case 1 as that $\dot{x} \leq_H \dot{y}$ $\lambda$-implies $\dot{x} \leq_S \dot{y}$, where $\dot{x} \leq_H \dot{y}$ is the theory $\{ h(\dot{x}) \leq_{\text{lex}} h(\dot{y}) : h \in H \}$, which would improve [1] of Theorem 3 to the existence of a h. o. p. map satisfying $h(x) \leq h(y) \implies x \leq_S y$. However then the arguments for Case 2, especially the key lemmas in the next section, do not go through.}

**Case 1**: the theory $\dot{x} \equiv_H \dot{y} \lambda$-implies $\dot{x} \equiv_{S,j} \dot{y}$.

Then clearly there is a single function $h \in H$ such that $h(\dot{x}) = h(\dot{y})$ already implies $\dot{x} \equiv_{S,j} \dot{y}$. Then $h$ satisfies [1] of Theorem 3.

**Case 2**: the theory $(\dot{x} \equiv_H \dot{y}) \cup \{ \dot{x} \not\equiv_{S,j} \dot{y} \}$ is $\lambda$-consistent.

Assuming this we shall work towards [2] of Theorem 3. We begin with a study of an important class of “conditionally downward closed” formulas.

Let $\mathcal{H}(\dot{x})$ be the (constructible as above) set of all formulas $\eta(\dot{x}) \in \mathcal{F}(\dot{x})$ satisfying the following: $\dot{x} \equiv_H \dot{x}' \lambda$-implies $\eta(\dot{x}) \wedge \dot{x}' \equiv_{T,j} \dot{x} \implies \eta(\dot{x}')$.

**Lemma 8** Suppose that $\eta(\dot{x}) \in \mathcal{H}(\dot{x})$. Then there is a function $h \in H_{\alpha+1}$ for some $\alpha < (\lambda^+)^L$ such that $\eta(x) \iff h(x) = 0$. In particular the theory $\eta(\dot{x}) \wedge \neg \eta(\dot{y}) \wedge \dot{x} \equiv_H \dot{y}$ is then $\lambda$-inconsistent.

**Proof** By definition there exists a function $h_0 \in H_\alpha$ for some $\alpha < (\lambda^+)^L$ satisfying $h_0(x) = h_0(y) \land \eta(x) \land x' \equiv_T x \implies \eta(x')$. Define $h(x) = h_0(x)^\lambda 0$ whenever $\eta(x)$ and $h(x) = h_0(x)^\lambda 1$ otherwise.\qed

For a theory $\Phi(\dot{x}, \dot{y}, ...)$, let $\mathcal{H}_\dot{x}[\Phi(\dot{x}, \dot{y}, ...)] = \mathcal{F}_\dot{x} [\Phi(\dot{x}, \dot{y}, ...)] \cap \mathcal{H}(\dot{x})$.  

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3 The basic forcing

Let $\Xi(\dot{x})$ denote the (constructible) set of all formulas $\xi(\dot{x}) \in \mathcal{F}(\dot{x})$ which are $\lambda$-implied by the theory $\dot{x} \equiv_H \dot{y} \cup \{\dot{x} \not\equiv_{S,j} \dot{y}\}$.

Let $\Pi$ be the set of all $\lambda$-consistent theories $\Pi(\dot{x}) \in L$, $\Pi \subseteq \mathcal{F}(\dot{x})$, including $\Xi(\dot{x})$. Then $\Pi \subseteq L$, so we can view it as a forcing notion over $L$.

**Lemma 9** Let $G \subseteq \Pi$ be $\Pi$-generic over $L$. Then there is a unique real $x = x_G \in L[G]$ such that $\pi(x)$ holds in $L[G]$ for any formula $\pi(\dot{x}) \in \cup G$.

**Proof** Note that, for any $n$, the set $D_n$ of all theories $\Pi(\dot{x}) \in \Pi$ which include $\dot{x}(n) = m$ for some $m$ is dense in $\Pi$ and belongs to $L$, hence $D_n \cap G \neq \emptyset$. The rest of the proof is standard. \hfill \Box

3.1 Key lemmas

**Lemma 10** Let $\Pi(\dot{x})$ be a theory in $\Pi$. Then the theory $\Phi_H(\dot{x}, \dot{y}, \dot{f}) =_{df} \Pi(\dot{x}) \cup \Pi(\dot{y}) \cup \dot{x} \equiv_H \dot{y} \cup \{\dot{x} \not\equiv_{S,j} \dot{y}\}$ is $\lambda$-consistent and satisfies the equalities $\mathcal{F}_x[\Phi_H(\dot{x}, \dot{y}, \dot{f})] = \mathcal{F}_x[\Pi(\dot{x})]$ and $\mathcal{F}_y[\Phi_H(\dot{x}, \dot{y}, \dot{f})] = \mathcal{F}_y[\Pi(\dot{y})]$.

**Proof** Let us first prove the consistency. Otherwise there is a formula $\pi(\dot{x}) \in \mathcal{F}_x[\Pi(\dot{x})]$ and a function $h \in H$ such that the formula $\pi(\dot{x}) \land \pi(\dot{y}) \land h(\dot{x}) = h(\dot{y}) \land \dot{x} \not\equiv_{S,j} \dot{y}$ is $\lambda$-inconsistent.

The plan is to find functions $h', h'' \in H$ such that the theories $\pi(\dot{x}) \land h'(\dot{x}) = h'(\dot{y}) \land \dot{y} \not\equiv_{S,j} \dot{x}$ and $\pi(\dot{x}) \land h''(\dot{x}) = h''(\dot{y}) \land \dot{x} \not\equiv_{S,j} \dot{y}$ are $\lambda$-inconsistent: then the formula $\neg \pi(\dot{x})$ belongs to $\Xi$, which is a contradiction because $\Pi$ includes $\Xi$.

Consider the first theory (the other one is similar). By Lemma 8 it suffices to get a formula $\psi(\dot{x}) \in \mathcal{K}(\dot{x})$ such that $X = [\pi] \subseteq U = [\psi]$ and, for all $x \in X$ and $u \in U$, $h(x) = h(u)$ implies $u \preceq_S x$.

Let $Z = \{z : \forall x \in X (h(z) = h(x) \implies z \preceq_S x)\}$. Then $X \subseteq Z$ by the inconsistence assumption above.

Define a sequence of sets $X = X_0 \subseteq U_0 \subseteq X_1 \subseteq U_1 \subseteq \ldots \subseteq Z$ and formulas $\varphi_n(\dot{x}) \in \mathcal{F}$ so that $U_n = \{u : \exists x \in X_n (h(x) = h(u) \land u \preceq_T x)\}$, $X_n = [\varphi_n(\dot{x})]$, and the sequence of formulas $\varphi_n$ is constructible.

Now, $\psi(\dot{x}) = \bigvee_n \varphi_n(\dot{x})$ is the formula required. (Note that $[\psi] = \bigcup_n X_n = \bigcup_n U_n$.) It remains to carry out the construction of $X_n, U_n, \varphi_n$.

Suppose that $X_n = [\varphi_n(\dot{x})] \subseteq Z$ has been defined. Define $U_n$ by the equality above. Then $X_n \subseteq U_n$, and $U_n \subseteq Z$. (Assume that $u \in U_n$, so
Corollary 12 Suppose that \( \Pi(x) \) and \( R(y) \) belong to \( \mathcal{P} \) and \( \mathcal{H}_x[\Pi(x)] = \mathcal{H}_x[R(x)] \). Then \( \Psi_{\mathcal{H}}(\hat{x}, \hat{y}, \hat{f}) =_{df} \Pi(\hat{x}) \cup R(\hat{y}) \cup \hat{x} \equiv_H \hat{y} \cup \{\hat{x} \not\lesseq_T \hat{y}\} \) is a \( \lambda \)-consistent theory satisfying the equalities \( \mathcal{F}_\hat{x}[\Psi_{\mathcal{H}}(\hat{x}, \hat{y}, \hat{f})] = \mathcal{F}_\hat{x}[\Pi(\hat{x})] \) and \( \mathcal{F}_\hat{y}[\Psi_{\mathcal{H}}(\hat{x}, \hat{y}, \hat{f})] = \mathcal{F}_\hat{y}[R(\hat{y})] \).

Proof As in the previous lemma, it suffices to prove the consistency. Suppose otherwise. Then there exist formulas \( \pi(\hat{x}) \in \mathcal{F}_\hat{x}[\Pi(\hat{x})] \) and \( \rho(\hat{y}) \in \mathcal{F}_\hat{y}[R(\hat{y})] \), and a function \( h \in H \) such that the formula \( \pi(\hat{x}) \land \rho(\hat{y}) \land h(\hat{x}) = h(\hat{y}) \land \hat{x} \not\lesseq_T \hat{y} \) is inconsistent. In other words we have \( x \not\lesseq_T y \) whenever \( x \in X = [\pi] \) and \( y \in Y = [\rho] \) satisfy \( h(x) = h(y) \).

Define \( Z = \{z : \forall y \in Y \ (h(y) = h(z) \Rightarrow z \not\lesseq_T y\} \), so that \( X \subseteq Z \) but \( Y \cap Z = \emptyset \). The same iterated procedure as in the proof of Lemma [11] (with \( U_n = \{u : \exists x \in X_n (h(x) = h(u) \land x \not\lesseq_T u)\} \) yields a formula \( \psi(\hat{x}) \in \mathcal{H}(\hat{x}) \) such that the set \( U = \mathbb{C}[\psi] \) satisfies \( X \subseteq U \subseteq Z \). But this contradicts the assumption \( \mathcal{H}_x[\Pi(\hat{x})] = \mathcal{H}_x[R(\hat{x})] \).

Corollary 12 Suppose that \( \Pi(\hat{x}) \), \( R(\hat{y}) \) belong to \( \mathcal{P} \) and \( \mathcal{H}_x[\Pi(\hat{x})] = \mathcal{H}_x[R(\hat{x})] \). Then \( U(\hat{x}, \hat{y}, \hat{f}) =_{df} \Pi(\hat{x}) \cup R(\hat{y}) \cup \hat{x} \equiv_H \hat{y} \cup \{\hat{x} \not\lesseq_S \hat{y}\} \) is a \( \lambda \)-consistent theory satisfying the equalities \( \mathcal{F}_\hat{x}[U(\hat{x}, \hat{y}, \hat{f})] = \mathcal{F}_\hat{x}[\Pi(\hat{x})] \) and \( \mathcal{F}_\hat{y}[U(\hat{x}, \hat{y}, \hat{f})] = \mathcal{F}_\hat{y}[R(\hat{y})] \).

Proof It suffices, as above, to prove the consistency. Suppose otherwise. Then the theory \( \Phi_\Pi(\hat{x}, \hat{z}, \hat{f}) \cup \Psi_{\mathcal{H}}(\hat{y}, \hat{z}, \hat{g}) \) is \( \lambda \)-inconsistent as well. (Otherwise we have reals \( x, y, z \) satisfying \( \Pi(x), \Pi(z), x \equiv_H z, R(y), \)
and \( y \equiv_{T} z \), hence \( x \equiv_{T} z \), and, in addition, \( x \not\equiv_{S} z \) and \( y \not\equiv_{T} z \), hence \( x \not\equiv_{S} y \).) Theorem \( 3 \) yields a formula \( \pi(z) \in \mathcal{F}(z) \) \( \lambda \)-implied by \( \Phi_{\Pi}(\dot{x}, \dot{z}, \dot{f}) \) but \( \lambda \)-inconsistent with \( \Psi_{R\Pi}(\dot{y}, \dot{z}, \dot{g}) \), which is a contradiction as \( \mathcal{F}_{\dot{z}}[\Phi_{\Pi}(\dot{x}, \dot{z}, \dot{f})] = \mathcal{F}_{\dot{z}}[\Pi(\dot{z})] = \mathcal{F}_{\dot{z}}[\Psi_{R\Pi}(\dot{y}, \dot{z}, \dot{g})] \) by lemmas \([10, 11] \). \( \square \)

**Corollary 13** Suppose that \( \Pi(\dot{x}) \), \( R(\dot{x}) \) belong to \( P \) and \( \mathcal{H}_{\dot{x}}[\Pi(\dot{x})] = \mathcal{H}_{\dot{x}}[R(\dot{x})] \). Then there are theories \( \Pi'(\dot{x}), R'(\dot{x}) \in P \) such that \( \Pi \subseteq \Pi' \), \( R \subseteq R' \), still \( \mathcal{H}_{\dot{x}}[\Pi'(\dot{x})] = \mathcal{H}_{\dot{x}}[R'(\dot{x})] \), and \( \Pi'(\dot{x}) \cup R'(\dot{x}) \) is \( \lambda \)-inconsistent.

**Proof** The theory \( \Phi(\dot{x}, \dot{y}) =_{df} \Pi(\dot{x}) \cup R(\dot{x}) \cup (\dot{x} \equiv_{H} \dot{y}) \cup \{ \dot{x} \neq \dot{y} \} \) is \( \lambda \)-consistent by the previous corollary. It easily follows that there exist numbers \( m \) and \( k_{x} \neq k_{y} \) such that \( \Phi'(\dot{x}, \dot{y}) =_{df} \Phi(\dot{x}, \dot{y}) \cup \{ \dot{x}(m) = k_{x} \} \cup \{ \dot{y}(m) = k_{y} \} \) is still \( \lambda \)-consistent. Now set \( \Pi'(\dot{x}) = \mathcal{H}_{\dot{x}}[\Phi'(\dot{x}, \dot{y})] \) and \( R'(\dot{y}) = \mathcal{H}_{\dot{y}}[\Phi'(\dot{x}, \dot{y})] \). \( \square \)

### 3.2 Two-dimensional modifications

There are two related forcing notions which produce pairs of reals.

Let \( P(2) \) be the (constructible) set of all \( \lambda \)-consistent theories \( \Delta(\dot{x}, \dot{y}) \in L \) such that \( \Xi(\dot{x}) \cup \Xi(\dot{y}) \subseteq \Delta(\dot{x}, \dot{y}) \).

**First modification.** Recall that \( \preceq_{T} = p[T] \). The idea is to define a forcing which leads to pairs of reals satisfying \( x \preceq_{T} y \).

We let \( T_{P}(2)(\dot{x}, \dot{y}, \dot{f}) \) be the set of all \( \lambda \)-consistent theories \( \mathcal{T}(\dot{x}, \dot{y}, \dot{f}) \) of the form \( \Delta(\dot{x}, \dot{y}) \cup \mathcal{F} \cup \dot{x} \equiv_{H} \dot{y} \cup \{ \dot{x} \preceq_{T,j} \dot{y} \} \), where \( \Delta \in P(2) \) and \( \mathcal{F} \) is a finite collection of formulas \( \hat{f}(k) = \alpha \) (where \( k \in \omega \) and \( \alpha < \lambda \)).

For instance the theory \( \Xi(\dot{x}) \cup \Xi(\dot{y}) \cup \dot{x} \equiv_{H} \dot{y} \cup \{ \dot{x} \preceq_{T,j} \dot{y} \} \) (which is \( \lambda \)-consistent by Lemma \([11] \) belongs to \( T_{P}(2) \).

**Lemma 14** Let \( G \subseteq T_{P}(2) \) be \( T_{P}(2) \)-generic over \( L \). There is a unique triple \( \langle x, y, f \rangle \in L[G] \cap (\mathbb{N} \times \mathbb{N} \times \lambda^{\omega}) \) such that \( \tau(\dot{x}, \dot{y}, \dot{f}) \) holds for any formula \( \tau(\dot{x}, \dot{y}, \dot{f}) \in \bigcup G \). In particular we have \( x \preceq_{T} y \).

**Proof** Analogous to Lemma \([3] \). \( \square \)

**Second modification.** Recall that \( \preceq_{S} = \mathbb{C}p[S] \). Now the intension is to force pairs of reals \( \langle x, y \rangle \) satisfying \( x \not\preceq_{S} y \). We let \( S_{P}(2)(\dot{x}, \dot{y}, \dot{f}) \) be the set of all \( \lambda \)-consistent theories \( \Sigma(\dot{x}, \dot{y}, \dot{f}) \) of the form \( \Delta(\dot{x}, \dot{y}) \cup \mathcal{F} \cup \dot{x} \equiv_{H} \dot{y} \cup \{ \dot{x} \not\equiv_{S,j} \dot{y} \} \), where \( \mathcal{F} \) and \( \Delta \) are as in the definition of \( T_{P}(2) \).

To see that \( S_{P}(2) \neq \emptyset \) note that \( \Xi(\dot{x}) \cup \Xi(\dot{y}) \cup \dot{x} \equiv_{H} \dot{y} \cup \{ \dot{x} \not\equiv_{S,j} \dot{y} \} \) is a \( \lambda \)-consistent theory by Lemma \([11] \).
Lemma 15 Let $G \subseteq {}^b\mathbb{P}^{(2)}$ be $\mathbb{P}^{(2)}$-generic over $L$. There is a unique triple $(x, y, f) \in L[G] \cap (N \times N \times \lambda^\omega)$ such that $\sigma(x, y, f)$ holds for any elementary formula $\sigma(\dot{x}, \dot{y}, \dot{f}) \in \bigcup G$. In particular we have $x \not\leq_S y$.

Proof Analogous to Lemma 9.

4 The product forcing

The forcing notion $\mathbb{P}^{(2)}$ executes too a tight control over generic reals. Fortunately generic $\leq_S$-incomparable pairs can be obtained by another forcing, which connects the components in a much looser way, so that it is rather a kind of product forcing, with the factors equal to $\mathbb{P}$.

We let $\mathbb{P} \times_T \mathbb{P}$ be the set of all theories $\Upsilon(\dot{x}, \dot{z})$ of the form $\Pi(\dot{x}) \cup R(\dot{z})$, where $\Pi$ and $R$ belong to $\mathbb{P}$ and satisfy $\mathcal{H}_{\dot{z}}[\Pi(\dot{x})] = \mathcal{H}_{\dot{z}}[R(\dot{x})]$. The set $\mathbb{P} \times_T \mathbb{P}$ is constructible and non-empty.

Theorem 16 Let $G \subseteq \mathbb{P} \times_T \mathbb{P}$ be $\mathbb{P} \times_T \mathbb{P}$-generic over $V$. There is a unique pair $(x, z) \in V[G] \cap N^2$ such that $v(x, z)$ holds for any formula $v(\dot{x}, \dot{z}) \in \bigcup G$. Moreover we have $x \not\leq_S z$.

Pairs $(x, z)$ as in the theorem will be denoted by $(x_G, z_G)$ and called $\mathbb{P} \times_T \mathbb{P}$-generic over $V$. ($V$ is the universe of all sets as usual.)

Proof Let us concentrate on the proof that $x_G$ and $z_G$ are $\leq_S$-incomparable; the rest is analogous to the above.

Suppose on the contrary that $x_G \leq_S z_G$ is $\mathbb{P} \times_T \mathbb{P}$-forced over $V$ by a “condition” $\Upsilon_0(\dot{x}, \dot{z}) = \Pi_0(\dot{x}) \cup R_0(\dot{z}) \in \mathbb{P} \times_T \mathbb{P}$, where $\Pi_0$ and $R_0$ belong to $\mathbb{P}^{(2)}$ and satisfy $\mathcal{H}_{\dot{z}}[\Pi_0(\dot{x})] = \mathcal{H}_{\dot{z}}[R_0(\dot{x})]$.

We shall define a generic “rectangle” of reals $x, z, x', z'$, such that the following will be forced: $x \leq_S z$ and $x' \leq_S z'$ — by Lemma 14 and $x \not\leq_S z'$ — by Lemma 13, which is a contradiction. The forcing $\mathbb{P}$ used to get a required “rectangle” consists of forcing conditions of the following general form:

$$p = (\Upsilon(\dot{x}, \dot{z}), \Upsilon'(\dot{x}', \dot{z}', \dot{f}), \Upsilon''(\dot{x}', \dot{z}', \dot{f}))$$

such that the theories $\Upsilon(\dot{x}, \dot{z}) = \Pi(\dot{x}) \cup R(\dot{z})$ and $\Upsilon'(\dot{x}', \dot{z}') = \Pi'(\dot{x}') \cup R'(\dot{z}')$ belong to $\mathbb{P} \times_T \mathbb{P}$, $\Upsilon$ belongs to $T\mathbb{P}^{(2)}$, $\Sigma$ belongs to $S\mathbb{P}^{(2)}$, and

$$\Pi(\dot{x}) = F_{\dot{z}}[\Sigma(\dot{x}, \dot{z}', \dot{f})], \quad R(\dot{z}) = F_{\dot{z}}[\Upsilon(\dot{z}, \dot{x}', \dot{f})]$$

$$\Pi'(\dot{x}') = F_{\dot{z}}'[\Upsilon'(\dot{z}', \dot{x}', \dot{f})], \quad R'(\dot{z}') = F_{\dot{z}}'[\Sigma(\dot{x}, \dot{z}', \dot{f})]$$

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Order $\mathfrak{P}$ componentwise: $p_1$ is stronger than $p_2$ iff $\Upsilon_2(x, z) \subseteq \Upsilon_1(x', z')$, $T_2(z, z', f) \subseteq T_1(z, z', f)$, $\Upsilon_2 \subseteq \Upsilon_1$, and $\Sigma_2 \subseteq \Sigma_1$.

To get a condition in $\mathfrak{P}$ we start with the given theory $\Upsilon_0(x, z) = \Pi_0(x) \cup R_0(z) \in \mathfrak{P} \times T \mathfrak{P}$. By definition $\mathcal{H}_x[\Pi_0(x)] = \mathcal{H}_x[R_0(z)]$.

It can be supposed that $F_x[\Pi_0(x)] = \Pi_0(x)$ (otherwise replace $\Pi_0(x)$ by $F_x[\Pi_0(x)]$) and $F_x[R_0(z)] = R_0(z)$.

The theory $\Sigma_0(x, z', f) =_{df} \Pi_0(x) \cup R_0(z') \cup x \equiv \exists z \forall x' z' \neq S_f z'$ belongs to $\mathfrak{P}(x, z')$ and satisfies the equalities $F_x[\Sigma_0(x, z', f)] = \Pi_0(x)$ and $F_x'[\Sigma_0(x, z', f)] = R_0(z')$ by Corollary $[\mathfrak{P}]$. Similarly, by Lemma $[\mathfrak{P}]$ the theory $T_0(z, z', f) =_{df} R_0(z) \cup \Pi_0(x') \cup z \equiv H x' \cup z \leq T_f x'$ belongs to $\mathfrak{P}(z, z')$ and satisfies $F_x[T_0(z, z', f)] = R_0(z)$ and $F_x'[T_0(z, z', f)] = \Pi_0(x')$.

Now $p_0 = \langle \Upsilon_0(x, z), T_0(z, z', f), \Upsilon_0(x, z', z'), \Sigma_0(x, z', f) \rangle$ belongs to $\mathfrak{P}$.

**Assertion** Suppose that $p = \langle T, T, T, \Sigma \rangle \in \mathfrak{P}$, $\Upsilon_1(x, z) \subseteq \mathfrak{P} \times T \mathfrak{P}$, and $\Upsilon_1(x, z) \subseteq \Upsilon_1(x, z)$. Then there exists $p_1 = \langle \Upsilon_1, T_1, \Upsilon_1', \Sigma_1 \rangle \in \mathfrak{P}$ stronger than $p$ (i.e. $T_1 \subseteq T_1$, $\Upsilon_1' \subseteq \Upsilon_1'$, and $\Sigma_1 \subseteq \Sigma_1$).

The same is true when we strengthen any of the other three components.

**Proof** By definition $\Upsilon_1(x, z) = \Pi_1(x) \cup R_1(z)$ where $\Pi_1$ and $R_1$ belong to $\mathfrak{P}(2)$ and $\mathcal{H}_x[\Pi_1(x)] = \mathcal{H}_x[R_1(z)]$.

Let $T_1(z, z', f) = T(z, z', f) \cup R_1(z)$. Lemma $[\mathfrak{P}]$ yields $\mathcal{H}_x[T_1(z, z', f)] = \mathcal{H}_x[R_1(z)]$. Let $\Pi_1(z', f) = F_x[T_1(z, z', f)]$. Now $\mathcal{H}_x[\Pi_1(z)] = \mathcal{H}_x[R_1(z)]$ by Lemma $[\mathfrak{P}]$ because even $T_1(z, z', f)$ includes $z \equiv H z'$ by the definition of $\mathfrak{P}(2)$. Hence $\mathcal{H}_x[\Pi_1(z)] = \mathcal{H}_x[\Pi_1(x)]$ by the above.

We define $R_1(z')$ using the other arks of the rectangle. Let $\Sigma_1(x, z', f) = \Sigma(x, z', f) \cup \Pi_1(x)$ and $R_1'(z') = F_x'[\Sigma_1(x, z', f)]$. Then $\mathcal{H}_x[R_1'(z')] = \mathcal{H}_x[R_1(x)]$ by Lemma $[\mathfrak{P}]$ and $[\mathfrak{P}]$ as above. Thus in particular we have $\mathcal{H}_x[R_1'(z')] = \mathcal{H}_x[R_1'(z')]$, so that $\Upsilon_1(x, z', z') = \Pi_1(z') \cup R_1'(z')$ belongs to $\mathfrak{P} \times T \mathfrak{P}$, closing the diagram. Clearly $\Upsilon_1 \subseteq \Upsilon_1$. It easily follows from the construction that $p_1 = \langle \Upsilon_1, T_1, \Upsilon_1', \Sigma_1 \rangle \in \mathfrak{P}$ is as required.

We continue the proof of Theorem $[\mathfrak{P}]$. Consider a $\mathfrak{P}$-generic extension $V[\mathfrak{S}]$ such that the generic set $\mathfrak{S} \subseteq \mathfrak{P}$ contains the condition $p_0$ defined above. It easily follows from the assertion just proved that $\mathfrak{S}$ results in a “rectangle” of reals $x, z, x', z' \in V[\mathfrak{S}]$ such that

1) The pair $\langle z, x' \rangle$ is $\mathfrak{P}(2)$-generic over $V$, therefore we have $z \leq_T x'$ in $V[\mathfrak{S}]$ by Lemma $[\mathfrak{P}]$.

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2) The pair \( \langle x, z' \rangle \) is \( S^{P(2)} \)-generic over \( V \), therefore we have \( x \not\leq_S y \) in \( V[\Theta] \) by Lemma 15.

3) The pairs \( \langle x, z \rangle \) and \( \langle x', z' \rangle \) are \( P \times T \)-generic over \( V \), and moreover, the corresponding generic subsets of \( P \times T \) contain the “condition” \( T_0(x, \dot{z}) \) fixed above; hence we have \( x \leq_S z \) and \( x' \leq_S z' \) in \( V[\Theta] \) by the choice of \( \Theta_0 \).

This is a contradiction because \( \leq_T \subseteq \leq_S \) in \( V[\Theta] \) by absoluteness. \( \square \)

5 The construction of an embedding

We are going to define a continuous 1 − 1 map \( F : 2^\omega \rightarrow N \) satisfying [III] of Theorem 1 of Theorem 1. Our strategy will be to prove the existence of such a map in a \( \kappa \)-collapsing generic extension \( V^+ \) of \( V \), the universe of all sets, where \( \kappa \) is the cardinal \( 2^{2^\omega} \) in \( V \). This suffices to conclude that [III] of Theorem 1 holds in \( V \) by Lemma 18 of the next section.

5.1 Generic splitting family of theories

Let a crucial pair be any (ordered) pair \( \langle u, v \rangle \) such that \( u, v \in 2^m \) for some \( m \) and \( u = 1^k \wedge 0^w , v = 0^k \wedge 1^w \) for some \( k < m \) and \( w \in 2^{m-k-1} \).

By the choice of \( V^+ \) the sets \( P, T^{P(2)}, P \times T \) have only \( V^+ \)-countably many subsets in \( V \). Let \( \{D(n) : n \in \omega\}, \{D^2(n) : n \in \omega\}, \{D_2(n) : n \in \omega\} \) be enumerations, in \( V^+ \), of the collections of all dense (by dense we mean open dense) subsets of resp. \( P, P \times T \), \( T^{P(2)} \). It will be assumed that each dense set has infinitely many indices in the relevant enumeration.

We shall define, in \( V^+ \), a family of theories \( \Pi_u(x) \in T \) (where \( u \in 2^{<\omega} \) ) and \( T_{uv}(x, \dot{y}, \dot{f}) \in T^{P(2)} \) (where \( \langle u, v \rangle \) is a crucial pair in some \( 2^n \) ) satisfying the following conditions, for all \( u \in 2^{<\omega} \) and \( i = 0, 1 \):

(i) \( \Pi_u \in D(n) \) whenever \( u \in 2^n \); \( \Pi_u(x) \subseteq \Pi_{u \land i}(x) \);

(ii) if \( \langle u, v \rangle \) is a crucial pair in \( 2^n \) then \( T_{uv} \in D_2(n) ; T_{uv}(x, \dot{y}, \dot{f}) \subseteq T_{u \land i, v \land i}(x, \dot{y}, \dot{f}) \);

(iii) if \( u, v \in 2^n \) and \( u(n-1) \neq v(n-1) \) then \( \Pi_u(x) \cup \Pi_v(\dot{z}) \in P \times T \), moreover \( \in D^2(n) \), and \( \Pi_u(x) \cup \Pi_v(\dot{z}) \) is \( \lambda \)-inconsistent;

(iv) \( F_x[T_{uv}(x, \dot{y}, \dot{f})] = \Pi_u(x) \) and \( F_{\dot{y}}[T_{uv}(x, \dot{y}, \dot{f})] = \Pi_v(\dot{y}) \) — then in particular \( F_x[\Pi_u(x)] = \Pi_u(x) \) for all \( u \).
Remark 17 Since theories in $^T\mathbb{P}(2)$ contain $\dot{x} \equiv_H \dot{y}$, it follows from $[\text{iv}]$ by Lemma 8 that $\mathcal{H}_\dot{x}[\Pi_u(\dot{x})] = \mathcal{H}_\dot{x}[\Pi_v(\dot{x})]$ for all crucial pairs $\langle u, v \rangle$. Therefore $\mathcal{H}_\dot{x}[\Pi_u(\dot{x})] = \mathcal{H}_\dot{x}[\Pi_v(\dot{x})]$ for all $u, v \in 2^n$ and $n \in \omega$ as any two tuples $u, v \in 2^n$ are connected by a (unique) chain of crucial pairs. □

Let us first of all demonstrate that the existence of such a system yields a continuous map $F$ in $V^+$ which witnesses $[\Pi]$ of Theorem 3.

Lemma 3 and $[\Pi]$ imply that for any $a \in 2^\omega$ there is a unique real, denoted by $F(a)$, satisfying every formula in $\bigcup_{n \in \omega} \Pi_{a \downharpoonright n}(\dot{x})$, and the map $F$ is continuous. Moreover $F$ is $1 - 1$ by $[\text{iii}]$.

Suppose that $a, b \in 2^\omega$ and $a \equiv_0 b$, so that $a(n) \neq b(n)$ for infinitely many $n$. It follows then from $[\text{iii}]$ and Theorem 6 that $F(a) \neq F(b)$.

Let us check that $F$ satisfies $[\Pi]$ of Theorem 3. Suppose that $a, b \in 2^\omega$ are $\leq_n$-neighbours, i.e. $a = 1^k \wedge 0^\omega c$ and $b = 0^k 1^c$ for some $k \in \omega$ and $c \in 2^\omega$. Then $\langle a \upharpoonright n, b \upharpoonright n \rangle$ is a crucial pair for all $n > k$. Therefore, by $[\Pi]$ and Lemma 14, there is a unique triple $\langle x, y, f \rangle \in \mathbb{N}^2 \times \mathbb{N}^\omega$ which satisfies every formula in $\bigcup_{n \in \omega} \mathcal{T}_{a \downharpoonright n, b \downharpoonright n}(\dot{x}, \dot{y}, \dot{f})$, and now $x = F(a), y = F(b)$ by $[\text{iv}]$. This implies $F(a) \equiv_T F(b)$ by $[\Pi]$.

5.2 The construction of theories

We argue in $V^+$.

To define $\Pi_\lambda$ (where $\lambda$ is the empty sequence, the only member of $2^0$) consider first the theory $\Xi(\dot{x})$, see Subsection 3. As clearly $\Xi(\dot{x}) \in \mathbb{P}$, there is a theory $\Pi(\dot{x}) \in D(0)$ including $\Xi(\dot{x})$. Let $\Pi_\lambda(\dot{x}) = \Pi(\dot{x})$.

Suppose that the construction has been completed up to a level $n$, and expand it to the next level.

To start with we set $\Pi_{\lambda \upharpoonright i}(\dot{x}) = \Pi_\lambda(\dot{x})$ for all $s \in 2^n$ and $i = 0, 1$, and $\mathcal{T}_{s \upharpoonright i, t \upharpoonright i}(\dot{x}, \dot{y}, \dot{f}) = \mathcal{T}_{s}(\dot{x}, \dot{y}, \dot{f})$ whenever $i = 0, 1$ and $\langle s, t \rangle$ is a crucial pair in $2^n$. For the “initial” pair $\langle \emptyset \upharpoonright 0, 0^n \upharpoonright 1 \rangle$, let $\mathcal{T}_{1^n \wedge 0, 0^n \wedge 1}$ be the theory $\Pi_{0^n}(\dot{x}) \cup \Pi_{0^n}(\dot{y}) \cup \{\dot{x} \in_H \dot{y}\} \cup \{\dot{x} \equiv_T \dot{y}\} \cup \{\dot{x} \equiv_\omega \dot{y}\}$. Then, by Lemma 11, $\mathcal{T}_{1^n \wedge 0, 0^n \wedge 1 \upharpoonright i} \equiv_T \mathcal{T}_{1^n \wedge 0, 0^n \wedge 1 \upharpoonright i}(\dot{x}, \dot{y}, \dot{f}) = \Pi_{0^n \wedge 0}(\dot{x})$, $\mathcal{F}_0[\mathcal{T}_{1^n \wedge 0, 0^n \wedge 1 \upharpoonright i}(\dot{x}, \dot{y}, \dot{f})] = \Pi_{0^n \wedge 1}(\dot{y})$.

This ends the definition of “initial values” at the $n + 1$-th level. The plan is to gradually enforce the theories in order to fulfill the requirements.

Step 1. Take care of item $[\Pi]$. Consider an arbitrary $u_0 = s_0 \upharpoonright i_0 \in 2^{n+1}$. As $D(n)$ is dense there is a theory $\Pi'(\dot{x}) \in D(n)$ including $\Pi_{u_0}(\dot{x})$. We can assume that $\Pi'(\dot{x}) = \mathcal{F}_x[\Pi'(\dot{x})]$ for otherwise change $\Pi'(\dot{x})$ by $\mathcal{F}_x[\Pi'(\dot{x})]$. The intension is to take $\Pi'(\dot{x})$ as the “new” $\Pi_{u_0}$. But this change has to be expanded through the net of crucial pairs, in order to preserve $[\text{iv}]$. (Fortunately the tree of all crucial pairs in $2^{n+1}$ is a chain.)
Thus put \( \Pi'_u(\hat{x}) = \Pi'(\hat{x}) \). Suppose that \( \Pi'_u \) has been defined, includes \( \Pi_u \), the older version, and satisfies \( \mathcal{F}_x[\Pi'_u(\hat{x})] = \Pi'_u(\hat{x}) \), for some \( u \in 2^{n+1} \) which is connected by a crucial pair with a not yet encountered \( v \in 2^{n+1} \). Define \( \mathcal{T}'_{uv}(\hat{x}, \hat{y}, \hat{f}) \) to be \( \Pi'_u(\hat{x}) \cup \mathcal{T}_{uv}(\hat{x}, \hat{y}, \hat{f}) \) and \( \Pi'_v(\hat{y}) \) to be \( \mathcal{F}_y[\mathcal{T}'_{uv}(\hat{x}, \hat{y}, \hat{f})] \). Note that \( \Pi'_v(\hat{y}) \) includes \( \Pi_v(\hat{y}) \) because [iv] is assumed for the old theories \( \Pi_u, \Pi_v, \mathcal{T}_{uv} \). Note that [iv] holds for the new theories \( \Pi'_u, \Pi'_v, \mathcal{T}'_{uv} \): indeed \( \mathcal{F}_x[\mathcal{T}'_{uv}(\hat{x}, \hat{y}, \hat{f})] = \Pi'_u(\hat{x}) \) follows from Lemma 7.

The construction describes how the change from \( \Pi_{u_0} \) to \( \Pi'_u \) spreads through the chain of crucial pairs in \( 2^{n+1} \), resulting in a system of new theories, \( \Pi'_u \) and \( \mathcal{T}'_{uv} \), which satisfy [ii] for the particular \( u_0 \in 2^{n+1} \).

We iterate this construction consecutively for all \( u_0 \in 2^{n+1} \), getting finally a system of theories satisfying [i] (fully) (and [iv]) which we shall denote by \( \Pi_u \) and \( \mathcal{T}_{uv} \) from now on.

Step 2. Take care of item [iii]. Let us fix a pair of \( u_0 \) and \( v_0 \) in \( 2^{n+1} \), such that \( u_0(n) = 0 \) and \( v_0(n) = 1 \). By the density of \( D^2(n) \), there is a theory \( \Pi_{u_0}(\hat{x}) \cup \Pi'_{v_0}(\hat{y}) \in D^2(n) \) which includes \( \Pi_{u_0}(\hat{x}) \cup \Pi_{v_0}(\hat{y}) \). We may assume that \( \Pi'_{u_0}(\hat{x}) = \mathcal{F}_x[\Pi_{u_0}(\hat{x})] \) and \( \Pi'_{v_0}(\hat{y}) = \mathcal{F}_y[\Pi_{v_0}(\hat{y})] \). We can also assume, by Corollary 13, that \( \Pi_{u_0}(\hat{x}) \cup \Pi'_{v_0}(\hat{y}) \) is \( \lambda \)-inconsistent.

Spread the change from \( \Pi_{u_0} \) to \( \Pi'_{u_0} \) and from \( \Pi_{v_0} \) to \( \Pi'_{v_0} \) through the chain of crucial pairs in \( 2^{n+1} \) until the two waves of spreading meet each other at the pair \( \langle 1^n \land 0^n \land 1 \rangle \). This leads to a system of theories \( \Pi'_u \) and \( \mathcal{T}'_{uv} \) which satisfy [iii] for the particular pair \( \langle u_0, v_0 \rangle \) and still satisfy [iv] with the exception of the “meeting” crucial pair \( \langle 1^n \land 0^n \land 1 \rangle \) (for which basically \( \mathcal{T}'_{1^n \land 0^n \land 1} \) is not yet defined for this step).

Take notice that the construction of Step 1 has left \( \mathcal{T}'_{1^n \land 0^n \land 1} \) in the form \( \Pi_{1^n \land 0^n \land 1}(\hat{x}) \cup \Pi_{0^n \land 1}(\hat{y}) \cup (\hat{x} \equiv_{1^n} \hat{y}) \cup \{ \hat{x} \not\in_{T_f} \hat{y} \} \) (where \( \Pi_{1^n \land 0^n \land 1} \) and \( \Pi_{0^n \land 1} \) are the “versions” at the end of Step 1). We now have new \( \lambda \)-consistent theories, \( \Pi_{1^n \land 0^n \land 1} \) and \( \Pi_{0^n \land 1} \), including resp. \( \Pi'_{1^n \land 0^n \land 1} \) and \( \Pi'_{0^n \land 1} \) and satisfying \( \mathcal{H}_x[\Pi'_{1^n \land 0^n \land 1}(\hat{x})] \) and \( \mathcal{H}_x[\Pi'_{0^n \land 1}(\hat{x})] \). (See Remark 17; recall that \( \mathcal{H}_x[\Pi'_{1^n \land 0^n \land 1}(\hat{x})] \) and \( \mathcal{H}_x[\Pi'_{0^n \land 1}(\hat{x})] \) for the initial pair simply because \( \Pi_{u_0}(\hat{x}) \cup \Pi'_{v_0}(\hat{y}) \in \mathcal{P}_x \land \mathcal{P}_y \).) Now the theory \( \Pi'_{1^n \land 0^n \land 1}(\hat{x}) \cup \Pi'_{0^n \land 1}(\hat{y}) \cup (\hat{x} \equiv_{1^n} \hat{y}) \cup \{ \hat{x} \not\in_{T_f} \hat{y} \} \) taken as \( \mathcal{T}'_{1^n \land 0^n \land 1} \) belongs to \( \mathcal{T'}_{P_2} \) and satisfies [iv] for the pair \( \langle 1^n \land 0^n \land 1 \rangle \) by Lemma 14. This ends the consideration of the pair \( \langle u_0, v_0 \rangle \).

Applying this construction consecutively for all pairs of \( u_0 \in P_0 \) and \( v_0 \in P_1 \) (including the pair \( \langle 1^n \land 0^n \land 1 \rangle \)) we finally get a system of theories satisfying [i], [iii], and [iv], which will be denoted still by \( \Pi_u \) and \( \mathcal{T}_{uv} \).

Step 3. We finally take care of [ii]. Consider a particular crucial pair \( \langle u_0, v_0 \rangle \) in \( 2^{n+1} \). By the density of \( D_2(n) \), there is a theory \( \mathcal{T}'_{u_0,v_0}(\hat{x}, \hat{y}, \hat{f}) \) in \( D_2(n) \) including \( \mathcal{T}_{u_0,v_0}(\hat{x}, \hat{y}, \hat{f}) \).
Define $\Pi'_{u_0}(\dot{x}) = T_{\dot{x}}[T'_{u_0,v_0}(\dot{x},\dot{y},\dot{f})]$ and $\Pi'_{v_0}(\dot{y}) = T_{\dot{y}}[T'_{u_0,v_0}(\dot{x},\dot{y},\dot{f})]$ and spread this change through the chain of crucial pairs in $2^{n+1}$. (Note that $\mathcal{H}_{\dot{x}}[\Pi'_{u_0}(\dot{x})] = \mathcal{H}_{\dot{x}}[\Pi'_{v_0}(\dot{x})]$ because theories in $\mathcal{T}_{P(2)}$ include $\dot{x} \equiv_{\mathcal{H}} \dot{y}$. This implies $\mathcal{H}_{\dot{x}}[\Pi'_{u_0}(\dot{x})] = \mathcal{H}_{\dot{x}}[\Pi'_{v_0}(\dot{x})]$ for all $u, v \in 2^{n+1}$ after the spreading.)

Executing this construction for all crucial pairs in $2^{n+1}$ we finally end the construction, in $\mathcal{V}^+$, of a system of theories satisfying $\mathfrak{I}$ through $\mathfrak{IV}$.

$\square$ (Theorem 3)

6 Why embedding $\leq_0$ is absolute

The aim of this section is to prove that $\mathfrak{I}$ of Theorem 6 and $\mathfrak{II}$ of Theorem 6 are absolute statements. By the way this fills the gap left in the proof of Theorem 6 (see the beginning of Section 6).

Lemma 18 If $p \in \mathcal{N}$ and $\mathcal{E}$ is a $\Sigma^1_1(p)$ p. q.-o. then $\mathfrak{II}$ of Theorem 6 is equivalent to a $\Sigma^1_2(p)$ statement uniformly in $p$. Similarly if $\mathcal{E} \subseteq S$ are resp. $\Sigma^1_1(p)$ p. q.-o. and $\Pi^1_1(p)$ p. q.-o. then $\mathfrak{I}$ of Theorem 6 is equivalent to a $\Sigma^1_2(p)$ statement uniformly in $p$.

Proof $\square$ We consider only $\mathfrak{I}$ of Theorem 6, the other statement is pretty similar. The aim does not seem easy: at the first look the statement is $\Sigma^1_3$. To reduce it to $\Sigma^1_2$ we use Borel approximations of $\mathcal{E}$ and $\mathcal{E}$. Recall that $\mathcal{W} = \{z \in \mathcal{N} : z$ codes an ordinal$\}$; for $z \in \mathcal{W}$ let $|z|$ be the ordinal coded by $z$, and $\mathcal{W}_\nu = \{z \in \mathcal{W} : |z| = \nu\}$.

Being a $\Sigma^1_1$ subset of $\mathcal{N}^2$, the relation $\mathcal{E}$ classically has the form $\mathcal{E} = \bigcup_{\nu < \omega_1} \mathcal{E}_{T,\nu}$ where $(\mathcal{E}_{T,\nu} : \nu < \omega_1)$ is an increasing sequence of Borel subsets of $\mathcal{E}_T$. Moreover there is a $\Pi^1_1$ formula $\pi(z,x,y)$ (containing $p$ as a parameter) such that $x \mathcal{E}_{T,\nu} y \iff \pi(z,x,y)$ whenever $z \in \mathcal{W}_\nu$. (There also exists a $\Sigma^1_1$ formula with the same property.)

Similarly $\mathcal{E}_S = \bigcap_{\nu < \omega_1} \mathcal{E}_{S,\nu}$, where $(\mathcal{E}_{S,\nu} : \nu < \omega_1)$ is an increasing sequence of Borel supersets of $\mathcal{E}_S$, and there is a $\Sigma^1_1$ formula $\sigma(z,x,y)$ such that $x \mathcal{E}_{S,\nu} y \iff \sigma(z,x,y)$ whenever $z \in \mathcal{W}_\nu$.

The following statement is clearly $\Sigma^1_2(p)$ (use formulas $\pi$ and $\sigma$):

(II') There is a continuous $1-1$ map $F' : 2^\omega \to \mathcal{N}$ and a countable ordinal $\nu$ such that

(a) $a \leq_0 b$ implies $F'(a) \mathcal{E}_{T,\nu} F'(b)$;

(b) $a \mathcal{E}_0 b$ implies $\neg F'(a) \equiv S,\nu F'(b)$.

\footnote{11 The proof involves an idea communicated to the author by G. Hjorth with a reference to Hjorth and Kechris [6], Section 3, where the idea is realized in terms of category.}
Thus it remains to prove that $\text{(II)}$ of Theorem 3 is equivalent to $\text{(II')}$ of Theorem 5. To prove this direction we consider a $\kappa$-collapse generic extension $V^+$ of $V$, the universe of all sets, where $\kappa$ is $2^{\aleph_0}$ in $V$. As $\text{(III)}$ is $\Sigma_3^1(p)$ while $\text{(II')}\text{ is } \Sigma_2^1(p)$, it suffices to prove that $\text{(II)}$ implies $\text{(II')}\text{ in } V^+$.

We can enumerate in $V^+$ by natural numbers all dense subsets of $2^{<\omega}$ and $2^{<\omega} \times 2^{<\omega}$ (the Cohen forcing and its square) which belong to $V$. This allows to define in $V^+$ infinite sequences $\langle u_n : n \in \omega \rangle$ and $\langle v_n : n \in \omega \rangle$ such that $u_n, v_n \in 2^{l(n)}$ for some $l(n)$ for all $n$, and for any $n$:

\[(*) \text{ if } u, v \in 2^l \text{ where } l = n + \sum_{m=0}^{n-1} l(n) \text{ then the pairs } \langle u^\land u_n, v^\land v_n \rangle \text{ and } \langle u^\land v_n, v^\land u_n \rangle \text{ belong to the } n\text{-th dense subset of } 2^{<\omega} \times 2^{<\omega}.\]

Define in $V^+$, for each $a \in 2^\omega$, $G(a) = w_0^\land w_1^\land w_2^\land \ldots$, where $w_n = u_n^\land 0$ whenever $a(n) = 0$ and $w_n = v_n^\land 1$ whenever $a(n) = 1$. Then $G$ is continuous and $1-1$, therefore the map $F'(a) = F(G(a))$ is continuous and $1-1$ as well. (Here $F$ is a map which witnesses $\text{(II)}$ of Theorem 3 in $V^+$. ) Prove that $F'$ witnesses $\text{(II')}$. Suppose that $a, b \in 2^\omega$ and $a \lessdot b$. Then by definition both $a' = G(a)$ and $b' = G(b)$ are Cohen generic over $V$ and $a' \lessdot b'$, $F(a') \leq_T F(b')$ (by the choice of $F$), even in $V[a', b']$, which implies $F(a') \leq_{T, \nu} F(b')$ for an appropriate $\nu < \omega_1$. Since the difference between $a'$ and $b'$ is finite the latter statement is a property of $a'$, hence it is Cohen forced over $V$. It follows that there is a countable in $V[a']$ (hence in $V^+$) ordinal $\nu$ such that we have, in $V^+$, $F'(a) \leq_{T, \nu} F'(b)$ whenever $a, b \in 2^\omega$ satisfy $a \lessdot b$.

Suppose that $a, b \in 2^\omega$ and $a \not\lessdot b$. Then by definition $\langle G(a), G(b) \rangle$ is Cohen generic over $V$, in particular $a' = G(a)$ and $b' = G(b)$ satisfy $a' \not\lessdot b'$, therefore $F(a') \not\lessdot_S F(b')$, which implies $\neg F(a') \not\lessdot_{S, \nu} F(b')$ for an ordinal $\nu < \omega_1$. As above there is a single ordinal $\nu_S < \omega_1$ such that we have, in $V^+$, $\neg F'(a) \not\lessdot_{S, \nu_S} F'(b)$ whenever $a \not\lessdot b$.

It remains to take $\nu = \max \{ \nu_T, \nu_S \}$.

\[\square\]

7 Borel and analytic order relations

This section proves theorems 4 and 5 with the exception of the additional statement in $\text{(II')}\text{ of Theorem 3.}$

7.1 Borel orders: Theorem 4

Let $\not\lessdot$ be a Borel p. q.-o. on $\mathbb{N}$. In view of Theorem 3 it suffices to prove that $\text{(II')}\text{ and } \text{(II')}\text{ of Theorem 4 are incompatible. Suppose otherwise.}
The superposition of the maps $F$ and $h$ is then a Borel h. o. p. map $\phi : \langle 2^\omega ; \leq_0 \rangle \rightarrow \langle 2^\omega ; \leq_{\text{lex}} \rangle$ satisfying the following: $\phi(a) = \phi(b)$ implies that $a \in_E b$, i. e. $a$ and $b$ are $\leq_0$-comparable.

Therefore, as any $E_0$-class is $\leq_0$-ordered as $\mathbb{Z}$, $\omega$, or $\omega^*$, the $\phi$-image $X_a = \phi^{-1}[a]_{E_0}$ of the $E_0$-class of any $a \in 2^\omega$ is $\leq_{\text{lex}}$-ordered as a subset of $\mathbb{Z}$. If $X_a = \{x_a\}$ is a singleton then put $\psi(a) = x_a$.

Assume that $X_a$ contains at least two points. In this case we can effectively pick an element in $X_a$! Indeed there is a maximal sequence $u \in 2^{<\alpha}$ such that $u \leq x$ for each $x \in X_a$. Then the set $X^{\text{left}}_a = \{x \in X : u^\omega 0 \subset x\}$ contains the $\leq_{\text{lex}}$-largest element, which we denote by $\psi(a)$.

To conclude $\psi$ is a Borel reduction of $E_0$ to the equality on $2^\alpha$, i. e. $a \in_E b$ iff $\psi(a) = \psi(b)$, which is impossible.

\( \square \) (Theorem 7)

7.2 The general case of analytic relations

Consider an analytic p. q.-o. $\preceq$ on $\mathbb{N}$. We shall w. l. o. g. assume that $\preceq$ is $\Sigma^1_1$, so that $\preceq = \preceq_T = p[T]$, where $T$ is a recursive tree in $(\omega \times \omega \times \omega)^{<\omega}$. We also suppose that

$$(\dagger) \quad \preceq_T \text{ does not satisfy } \left[ \Pi^1_3 \right] \text{ of Theorem 3}$$

The aim is to prove that then $\preceq_T$ satisfies $\left[ \Pi^1_3 \right]$ (still leaving apart the additional part) of Theorem 3.

Recall that $H_\alpha$ is defined in Section 3. Let $H^* = \bigcup_{\alpha \leq \omega_1} H_\alpha$; this includes the set $H$ also defined in Section 3. By definition each $h \in H^*$ is a Borel function of certain type, coded in $L$. Let us fix a constructible in the codes enumeration $H^* = \{h_\alpha : \alpha < \omega_1\}$. Define the concatenation

$$h(x) = h_0(x) \wedge h_1(x) \wedge \ldots \wedge h_\alpha(x) \wedge \ldots \quad (\alpha < \omega_1).$$

Prove that $h : \langle \mathbb{N} ; \preceq_T \rangle \rightarrow \langle 2^{<\omega_1} ; \leq_{\text{lex}} \rangle$ is a linearization. First of all $h$ is a h. o. p. map from $\langle \mathbb{N} ; \preceq_T \rangle$ to $\langle 2^{<\omega_1} ; \leq_{\text{lex}} \rangle$ because each $h_\alpha$ is h. o. p. by definition. Thus it remains to prove that $h(x) = h(y) \implies x \approx_T y$.

This involves a reflection lemma for analytic p. q.-o.’s.

Being a $\Sigma^1_1$ set, $\preceq_T$ has the form $\preceq_T = \bigcap_{\nu < \omega_1} \preceq^\nu_T$, where each $\preceq^\nu_T$ is an $\omega_1$-Borel set coded in $L$ and $\preceq^\mu_T \subseteq \preceq^\nu_T$ provided $\nu < \mu$. In addition we have the following boundedness principle: if $\preceq_T \subseteq X$, where $X \subseteq \mathbb{N}^2$ is a $\Pi^1_1$ set, then there is $\nu < \omega_1$ such that $\preceq^\nu_T \subseteq X$.

Take notice that the sets $\preceq^\nu_T$ are not necessarily p. q.-o.’s. However
Lemma 19 Assume that $B \subseteq \mathbb{N}^2$ is a Borel set and $\preceq_T \subseteq B$. Then there is $\nu < \omega_1$ such that $\preceq_T^\nu \subseteq B$ and $\preceq_T^\nu$ is a p. q.-o.. \[12\]

Proof Let us first prove a weaker statement: there is $\mu < \omega_1$ such that $x \preceq_T^\mu y \implies \forall x' \forall y' (x' \preceq_T x \wedge y \preceq_T y' \implies x' \preceq_T^\mu y')$.

Note that by the boundedness there exists an ordinal $\mu_0 < \omega_1$ such that $\preceq_T^\mu \subseteq B$. Suppose that an ordinal $\mu_n \geq \mu_0$ has been defined. Put $C(x,y)$ iff $\forall x' \forall y' (x' \preceq_T x \wedge y \preceq_T y' \implies x' \preceq_T^\mu y')$, so that $C$ is a $\Pi_1^1$ set and $\preceq_T \subseteq C \subseteq \preceq_T^\mu$. Using the boundedness principle again we get an ordinal $\mu_n+1 \geq \mu_n$ satisfying $\preceq_T \subseteq \preceq_T^{\mu_n+1} \subseteq C$, so that by definition

$$x \preceq_T^{\mu_n+1} y \implies \forall x' \forall y' (x' \preceq_T x \wedge y \preceq_T y' \implies x' \preceq_T^{\mu_n} y').$$

It remains to define $\mu = \sup_n \mu_n$.

Starting the proof of the lemma, we choose $\nu_0$ so that $\preceq_T^{\nu_0} \subseteq B$ and $x \preceq_T^{\nu_0} y \implies \forall x' \forall y' (x' \preceq_T x \wedge y \preceq_T y' \implies x' \preceq_T^{\nu_0} y')$. \((*)\)

Suppose that an ordinal $\nu_n \geq \nu_0$ satisfying \((*)\) has been defined. Put $C(x,y)$ iff $x \preceq_T^{\nu_n} y \wedge \forall z (y \preceq_T^{\nu_n} z \implies x \preceq_T^{\nu_n} z)$, so that $C$ is a $\Pi_1^1$ set and $\preceq_T \subseteq C \subseteq \preceq_T^{\nu_n}$. As above there is an ordinal $\nu_{n+1} \geq \nu_n$ satisfying $\preceq_T \subseteq \preceq_T^{\nu_{n+1}} \subseteq C$, so that by definition

$$x \preceq_T^{\nu_{n+1}} y \implies \forall z (y \preceq_T^{\nu_n} z \implies x \preceq_T^{\nu_n} z).$$

Now take $\nu = \sup_n \nu_n$. \(\square\)

Now assume $x \not\preceq_T y$ and prove that $h(x) \neq h(y)$. By the lemma there is an ordinal $\nu < \omega_1$ such that $x \not\preceq_T^\nu y$ and $\preceq_T^\nu$ is a p. q.-o.. It is a classical fact that the complement of $\preceq_T^\nu$ (which is the $\nu$-th constituent of the co-analytic set $C(\preceq_T)$) can be presented in the form of the set of all pairs $\langle x, y \rangle$ such that $C(x,y)$ is well-ordered and has the order type $\leq \nu$, where $C : \mathbb{N}^2 \rightarrow \mathbb{Q}$ is a continuous function coded in $L[T] = L$. Therefore there exists a tree $S = S_\nu \in L$, $S \subseteq (\omega \times \omega \times \nu)^{<\omega}$, such that $\preceq_T^\nu = C_p[S]$.

Apply Theorem 3 for the relations $\preceq_T = p[T] \subseteq \preceq_T^\nu = C_p[S]$. We observe that (11) of Theorem 3 fails by the assumption (†). Therefore (1) of Theorem 3 holds, so that there exists an ordinal $\alpha < \omega_1$ such that the map $h_\alpha$ satisfies $h_\alpha(x) = h_\alpha(y) \implies x \preceq_T^\nu y$.

It follows that $h_\alpha(x) \neq h_\alpha(y)$ by the choice of $\nu$, hence $h(x) \neq h(y)$, as required. Thus $h$ witnesses (10) (the general case) of Theorem 3.

\[12\] Compare with the Burgess reflection theorem for $\Sigma_1^1$ equivalence relations.
8 Special cases: Borel classes and generic models

This section is devoted to the “additional” part in [1\(^n\)] of Theorem 2. We continue to argue in the assumptions and notation of Subsection 7.2. In particular we still assume [1\(^n\)] of Subsection 7.2, that is the given \( \Sigma_1^1 \) p. q.-o. \( \simeq_T = p[T] \) does not satisfy [1\(^n\)] of Theorem 2.

We need here to specify definability properties related to the function \( h \). An ordinary argument allows to define the sequence \( \langle h_\alpha : \alpha < \omega_1 \rangle \) to be \( \Delta^L_{i+1} \) hence \( \Delta^\text{HC}_1 \). Then the map \( \langle x, \gamma \rangle \mapsto h(x) \mid \gamma \) is \( \Delta^\text{HC}_1 \), too. Moreover there is a \( \Sigma_1 \) formula \( \Phi(\cdot, \cdot, \cdot) \) such that

<1> If \( M \) is a transitive model of \( \text{ZFC}^- \) (minus Power Sets), \( x \in N \cap M \), and \( \lambda \in M \), \( \lambda < \omega_1 \), then \( u = h(x) \mid \lambda \in M \) and \( u \) is the only member of \( M \) such that \( \Phi(x, \lambda, u) \) holds in \( M \).

8.1 Order relations with Borel classes

Consider the case [1\(^n\)(a) in the “additional” part of Theorem 2. Suppose that every \( \approx_T \)-class \( [x]_{\approx_T} = \{ y : y \approx_T x \} \) is Borel. The aim is to find an antichain \( A \subseteq 2^{<\omega_1} \) and a \( \Delta^\text{HC}_1 \) linearization \( h' : \langle \langle N ; \approx_T \rangle \mapsto \langle A ; \leq \text{lex} \rangle \).

**First attempt.** Let \( x \in N \). As \( [x]_{\approx_T} \) is Borel, there exists, by Lemma 19, an ordinal \( \lambda < \omega_1 \) such that \( x \approx_T y \iff x \approx_T^\lambda y \) and \( \approx_T^\lambda \) is a p. q.-o.. Therefore (see the end of Subsection 7.2) there is an ordinal \( \alpha < \omega_1 \) satisfying \( h_\alpha(x) = h_\alpha(y) \implies x \approx_T^\alpha y \), hence \( h_\alpha(x) = h_\alpha(y) \iff x \approx_T y \) (for the chosen \( x \) and any \( y \)). It follows that \( x \approx_T y \iff h(x) \mid \lambda = h(y) \mid \lambda \), for an ordinal \( \lambda < \omega_1 \). Let \( \lambda_0(x) \) be the least such an ordinal and \( h_0(x) = h(x) \mid \lambda_0(x) \). Now \( A = \{ h_0(x) : x \in N \} \) is an antichain in \( 2^{<\omega_1} \) while \( h_0 : \langle \langle N ; \approx_T \rangle \mapsto \langle A ; \leq \text{lex} \rangle \) is a linearization.

However the definition of \( \lambda_0(x) \) seems not to provide that \( h_0 \) is \( \Delta^\text{HC}_1 \).

**Second attempt.** Let us modify the construction to meet the requirement that \( h_0 \) is \( \Delta^\text{HC}_1 \). The idea is to replace the quantifier \( \forall y \) in the definition of \( \lambda_0(x) \) by: for each \( y \) in every set generic extension of a suitable model containing \( x \), which essentially means: for comeager-many reals \( y \).

Let \( x \in N \). Define \( \lambda_1(x) \) to be the least ordinal \( \lambda \in \omega_1 \) such that

<2> there exists a transitive model \( M \) of \( \text{ZFC}^- \) containing \( \lambda \), and a real \( x' \in M \) satisfying \( x \approx_T x' \) and, for any set generic extension \( M' \) of \( M \) and each real \( z \in M' \), we have \( h(x') \mid \lambda = h(z) \mid \lambda \implies x' \approx_T z \).

(Clearly \( \lambda_1(x) \leq \lambda_0(x) \).)

Define \( h_1(x) = h(x) \mid \lambda_1(x) \). Thus \( h_1 : N \mapsto 2^{<\omega_1} \) is a \( \Delta^\text{HC}_1 \) function.
Lemma 20 We have: \( x \approx_T y \iff h_1(x) = h_1(y) \).

Proof Suppose that \( h_1(x) = h_1(y) = u \in 2^{<\omega_1} \) (the nontrivial direction), in particular \( \lambda_1(x) = \lambda_1(y) = \lambda \). Let \( x' \in M_x \) and \( y' \in M_y \) witness that \( \lambda_1(x) = \lambda_1(y) = \lambda \), in the sense of \([8]\). Prove that \( x' \approx_T y' \).

Let \( f \in \lambda^\omega \) be a \( \lambda \)-collapse function generic over both \( M_x \) and \( M_y \).

Let \( \vartheta_x, \vartheta_y \) be the least ordinals not in resp. \( M_x, M_y \). Suppose that \( \vartheta_x \leq \vartheta_y \). Then both \( M_x[\vartheta_x] \) and \( L[\vartheta_y] \) model \( \text{ZFC}^- \), hence by some version of Shoenfield there is a real \( z \in M_x[\vartheta_x] \) satisfying \( h(z) \restriction \lambda = u \), therefore \( x' \approx_T z \) by the assumed \([8]\). However \( z \) can be also considered in \( L[\vartheta_y] \) which yields \( y' \approx_T z \), hence \( x' \approx_T y' \) as required.

Thus we have defined a \( \Delta^\text{HC}_1 \) map \( h_1 : N \rightarrow 2^{<\omega_1} \) such that, for any \( x, h_1(x) = h(x) \restriction \lambda_1(x) \) for some \( \lambda_1(x) < \omega_1 \), and \( x \approx_T y \iff h_1(x) = h_1(y) \).

However the range \( A_1 = \{ h_1(x) : x \in N \} \) may be not an antichain in \( 2^{<\omega_1} \).

To fix this last problem, we define a new \( \Delta^\text{HC}_1 \) map \( h' : N \rightarrow 5^{<\omega_1} \). First we change all values \( h_1(x)(\gamma) = 1 \) to \( h'(x)(\gamma) = 4 \) and add that \( h'(x)(\lambda_1(x)) = 2 \). If \( \nu < \lambda_1(x) \) is an ordinal of the form \( \nu = \lambda_1(y) \) for some \( y \) then we change \( h'(x)(\nu) \) once again: if \( x \ll_T y \) then put \( h'(x)(\nu) = 1 \) while if \( y \ll_T x \) then put \( h'(x)(\nu) = 3 \). A simple verification shows that \( h' \) satisfies \([1^a]\) of Theorem 2 (except for the fact that \( h' \) takes values in \( 5^{<\omega_1} \) rather than \( 2^{<\omega_1} \) but this can be easily fixed).

8.2 Order relations in a generic universe

This subsection is devoted to subitem \([1^a](b)\) in Theorem 2. In fact we shall assume the following: there exists a real \( z_0 \) such that each real \( x \) in the universe \( V \) belongs to a set \( [^a] \) generic extension of \( L[z_0] \).

It can be assumed that in fact \( z_0 = 0 \), so we simply drop \( z_0 \).

Lemma 21 Let \( x \in N \). There is an ordinal \( \lambda < \omega_1 \) such that

(i) \( L_\lambda[\mathcal{h}(x) \restriction \lambda] \) models \( \text{ZFC}^- \) (minus Power Sets);
(ii) \( x \) belongs to a set generic extension of \( L_\lambda[\mathcal{h}(x) \restriction \lambda] \);
(iii) if \( y, z \) are reals in a set generic extension \( M \) of \( L_\lambda[\mathcal{h}(x) \restriction \lambda] \), and \( \mathcal{h}(y) \restriction \lambda = \mathcal{h}(z) \restriction \lambda \), then \( y \approx_T z \).

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13 It is not clear to what extent class forcing universes can accomodate the reasoning below, in particular the proofs of lemmas 21 and 22.

14 The extensions can be different for different reals \( x \). Moreover the extensions can be Boolean valued extensions of \( L[z] \) rather than factual classes in the universe.
Recall that the map $h$ which are resp. $\lambda$ being set forcing notions in $L$ case $h$ to a set generic extension of $L \lambda$ in $M$ requirements of the the lemma. We put $h$ argument shows that, as $u$ model, and then, by the choice of $\mu$.

Suppose that $h = h(x) | \lambda$ is a class in $L_\lambda[x]$ by $\text{[42]}$. Now it is a known fact (see Lemma 4.4 in Solovay $\text{[14]}$ or Lemma 5 in Kanovei $\text{[10]}$ as particular cases) that $x$ belongs to a set generic extension of $L_\lambda[u]$. 

Prove $\text{[iii]}$. Suppose that reals $y, z$ belong to a set generic extension of $L_\lambda[u]$ and $h(y) | \lambda = h(y) | \lambda$; prove $y \approx_T z$. First of all, a standard forcing argument shows that, as $u = h(x) | \lambda$ is definable in $L_\lambda[x]$ while $x$ belongs to a set generic extension of $L_\lambda$ by the above, we can w.l.o.g. assume that $y, z$ belong to a set generic extension of $L_\lambda$ itself.

Let, for a transitive model $M$ of $\text{ZFC}^-$, $h^M$ denote the map $h$ defined in $M$ (via e.g. the formula $\Phi$ of $\text{[42]}$). For instance $h^V$ is simply $h$. In any case $h^M$ maps reals in $M$ into $2^{\omega^1}$. In particular if each ordinal $\alpha \in M$ is countable in $M$ and $\lambda = \text{Ord} \cap M$ then $h^M(x) \in 2^\lambda$ for any real $x \in M$.

Note that $\text{[42]}$ of Theorem 2 fails in any generic extension $L[G]$ of $L$ because it is essentially a $\Sigma^1_2$ sentence (by Lemma $\text{[18]}$ false in $V$ by $\text{[42]}$). Therefore, by the already proved, in Subsection 7.2, part of Theorem 2, every set forcing notion in $L$ forces that, in the extension $L[G]$, for any two reals $y, z$, if $h^L[G](y) = h^L[G](z)$ then $y \approx_T z$.

A simple forcing argument transfers this result to $L_{\kappa^+}$ as the initial model, and then, by the choice of $\lambda$, to $L_\lambda$, so that for any two reals $y, z$ in a set generic extension $L_\lambda[G]$ of $L_\lambda$, if $h^{L_\lambda}[G](y) = h^{L_\lambda}[G](z)$ then $y \approx_T z$. It remains to note that $h^{L_\lambda[G]}(y) = h(y) | \omega^1[G]$ for any real $y \in L_\lambda[G]$. $\square$

Let, for any $x \in N$, $\lambda_x$ be the least ordinal $\lambda < \omega_1$ satisfying the requirements of the the lemma. We put $h_1(x) = h(x) | \lambda_x$. Apparently $h_1$ is a $\Delta^1_{\text{HC}}$ map.

Lemma 22 We have: $h_1(x) = h_1(y)$ iff $x \approx_T y$.

Proof Suppose that $h_1(x) = h_1(y) = u \in 2^\lambda$ (so that $\lambda = \lambda_x = \lambda_y$) and prove $x \approx_T y$ (the nontrivial direction). Let $x'$ and $y'$ witness that $h_1(x) = h_1(y) = u$, so that they belong to resp. $L_\lambda[u, G_x]$ and $L_\lambda[u, G_y]$, which are resp. $P_x$-generic and $P_y$-generic extensions of $L_\lambda[u]$; $P_x$ and $P_y$ being set forcing notions in $L_\lambda[u]$. In addition, $x \approx_T x'$ and $y \approx_T y'$.

In particular, as $x \approx_T x'$, we have $h(x) = h(x')$, so that $u = h(x') | \lambda$. Recall that the map $h(\cdot) | \lambda$ results in some effective way from the $\Delta^1_{\text{HC}}$
sequence $\langle h_\alpha : \alpha < \lambda \rangle$. Therefore the fact that $h(x') \upharpoonright \lambda = u$ is forced in $L_\lambda[u]$. Thus we can assume that $P_x$ forces in $L_\lambda[u]$ that $h(x') \upharpoonright \lambda = u$.

Consider a set $G \subseteq P_x$ which is $P_x$-generic over both $L_\lambda[u,G_x]$ and $L_\lambda[u,G_y]$. Let $z \in L_\lambda[u,G]$ be produced by $G$ as $x'$ from $G_x$, so that $h(z) \upharpoonright \lambda = u$. Thus two reals, $x'$ and $z$, is the model $L_\lambda[u,G_x,G]$, satisfy $h(z) \upharpoonright \lambda = h(x') \upharpoonright \lambda$. It follows that $x' \approx_T z$ by the choice of $\lambda$.

We similarly prove $y' \approx_T z$, as required. \hfill \Box

It remains to get $h'$ from $h_1$ as in Subsection 8.1. \hfill \Box (Theorem 3)

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