Minimax Crossover Designs

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Abstract

In crossover experiments, two broad classes of treatment effects are typically considered: direct effects that capture the instantaneous impact of the treatment, and carryover effects that capture the lagged impact of past treatments. Existing approaches to optimal crossover design usually minimize a criterion that relies on an outcome model, and on assumptions that carryover effects are limited, say, to one or two time periods. The latter assumption is problematic when long-range carryover effects are expected, and are of primary interest. In this paper, we derive minimax optimal designs for estimating both direct and carryover effects simultaneously. In contrast to prior work, our minimax designs do not require specifying a model for the outcomes, relying instead on invariance assumptions. This allows us to address problems with arbitrary carryover structure, such as those encountered in digital experimentation.
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1 Introduction

A crossover trial is an experiment performed over multiple time periods, such that an experimental unit may be assigned to different treatments at different time periods. Crossover trials are typical in pharmaceutical and clinical settings (Matthews, 1994; Jones et al., 1999; Senn and Senn, 2002), psychology and behavioral research (Rouanet and Lepine, 1970; Ellis, 1999), consumer preference studies (MacFie et al., 1989; Kunert, 1998), weather control experiments (Neyman, 1967; Gabriel et al., 1967), and in digital experimentation (Hohnhold et al., 2015; Yan et al., 2019).

In general, the goal of a crossover trial is to evaluate direct effects; that is, the effects of treatments at the time when they are assigned. In practice, however, the effect of a treatment often persists for multiple periods, and such carryover effects can bias the analysis of direct effects if they are not accounted for. Accordingly, standard model-based methods in crossover trials include parameters for carryover effects, but with two significant limitations. First, it is assumed that treatment effects carry over to only one or two time periods. This is a strong assumption in practice. For example, Hohnhold et al. (2015) recently showed that carryover effects persist over long periods of time in digital experiments; see also (Senn and Lambrou, 1998; Senn, 2006) for additional discussion on the limitations of standard models of carryover effects. Second, carryover effects are usually handled as nuisance rather than as quantities of interest.

In this paper, we consider the design of crossover trials in settings where both direct and carryover effects are of interest. We make two contributions. First, we adapt the potential outcomes framework of causal inference (Imbens and Rubin, 2015) to the setting of crossover trials. This allows us to define the causal estimands independently of any outcome model specification, and to use Neymanian randomization theory for minimax analysis. Second, we derive minimax crossover designs under conditions of permutation invariance on the potential outcomes. This builds upon the seminal work of Wu (1981) on minimax designs in the cross-sectional setting.

Our paper is organized as follows. Section 2 introduces the potential outcomes framework, and defines the causal estimands. We develop our core theory in Section 3, and consider extensions in Section 4. Finally, we illustrate our theory through simulated studies in Section 5.
2 Setup and notation

In this section, we introduce our notation and define the problem of crossover trial design. In particular, we discuss the definitions of treatment, outcomes, and estimands of interest.

2.1 Crossover designs

In a crossover experiment, an experimental unit may be assigned to different treatments at different points in time. A crossover design thus randomizes units over temporal sequences of treatments (Jones and Kenward, 2014). Formally, we consider a crossover experiment with \( N \) units, taking place over \( T \) discrete periods, indexed by \( t = 1, \ldots, T \), where \( T \) is fixed. At each time \( t \), a unit \( i \) can either receive treatment or control, denoted by \( Z_{it} = 1 \) or \( Z_{it} = 0 \), respectively. Vector \( Z_{i,1:t} = (Z_{i1}, \ldots, Z_{it}) \) \( \in \{0, 1\}^t \) denotes the treatment history of \( i \) up to time \( t \). The full treatment history for unit \( i \) is denoted by \( Z_i = (Z_{i1}, \ldots, Z_{iT}) \), where the dependence on \( T \) is left implicit; \( Z \in \{0, 1\}^{N \times T} \) denotes the \( N \times T \) matrix of population treatment assignments, whose \( i \)-th row is equal to \( Z_i \). A crossover design is thus a distribution on all \( Z \), the set of population treatment assignment matrices.

In our paper, certain assignment vectors play a special role, which requires additional notation. Let \( 1 = (1, \ldots, 1) \) and \( 0 = (0, \ldots, 0) \) be respectively the always-treated and always-control assignment vectors of length \( T \). We call pulse assignment at time \( t \) and denote by \( e_t \) the assignment vector that treats a unit only at time \( t \); i.e.

\[
e_t = (0, \ldots, \underbrace{1}_\text{index } t, 0, \ldots, 0).
\]

Similarly, the wedge assignment at time \( t \), denoted \( w_t \), treats a unit at every time point after \( t \); i.e.

\[
w_t = (0, \ldots, \underbrace{1}_\text{index } t, \ldots, 1) = \sum_{s=t}^T e_s.
\]

The assignment vectors \( 1, 0, \{e_t\}_{t=1}^T \) and \( \{w_t\}_{t=1}^T \) are the building blocks for the following two classes of designs that will be our main focus.
Definition 1 (Pulse and wedge designs). Let $\mathcal{E} = \{1, 0, e_1, \ldots, e_T\}$. Denote by $\mathcal{Z}(\mathcal{E}) = \{Z \in \{0, 1\}^{N \times T} : Z_i \in \mathcal{E}, \forall i = 1, \ldots, N\}$ the set of all assignment matrices for which every unit’s assignment is in $\mathcal{E}$. Similarly, let $\mathcal{E}^* = \{1, 0, w_1, \ldots, w_T\}$ and denote by $\mathcal{Z}(\mathcal{E}^*) = \{Z \in \{0, 1\}^{N \times T} : Z_i \in \mathcal{E}^*, \forall i = 1, \ldots, N\}$ the set of all assignment matrices for which every unit assignment is in $\mathcal{E}^*$. A pulse design is a probability distribution, $\eta(Z)$, with support on $\mathcal{Z}(\mathcal{E})$. A wedge design is a probability distribution, $\eta^w(Z)$, with support on $\mathcal{Z}(\mathcal{E}^*)$. The sets of all possible pulse designs and wedge designs are denoted by $\mathcal{H}$ and $\mathcal{H}^*$, respectively.

Both pulse and wedge designs are the building blocks of our causal estimands in Section 2.2.3, and arise naturally in applications. For example, wedge designs are widely used in clinical trials (Brown and Lilford, 2006; Prost et al., 2015; Hargreaves et al., 2015). A type of wedge design, called the “cookie-cookie-day design”, was recently proposed in digital experimentation to estimate cumulative carryover effects in online advertising (Hohnhold et al., 2015; Yan et al., 2019). In the following section, we focus on pulse designs because they are conceptually simpler than wedge designs. Importantly, the key results on pulse designs hold unchanged for wedge designs (see Remark 3.3).

2.2 Potential outcomes

We adapt the potential outcomes framework of causal inference (Neyman, 1923; Rubin, 1974) to the crossover setting, building on recent related work (Ji et al., 2017; Bojinov and Shephard, 2019).

2.2.1 Notation and assumptions. Let $Y_{it}(Z)$ denote the scalar potential outcome of unit $i$ at time period $t$ under population assignment $Z$. We also denote by $Y_i(Z) = (Y_{i1}(Z), \ldots, Y_{iT}(Z))$ the vector of outcomes over time under $Z$ for unit $i$, and by $Y(Z) = [Y_1(Z), \ldots, Y_N(Z)]^\top$ the $N \times T$ matrix whose $i$-th row is equal to $Y_i(Z)$. We make the following two assumptions.

Assumption 1 (No interference). The outcome of unit $i$ at time $t$ depends only on its own sequence of assignments. That is,

$$\forall Z, Z', Z_i = Z'_i \Rightarrow Y_{it}(Z) = Y_{it}(Z'), \ \forall t = 1, \ldots, T.$$
**Assumption 2** (Non-anticipating outcome). The outcome of unit \( i \) at time \( t \) depends only on the treatment history up to time \( t \).

Assumption 1 extends to the temporal setting the classical “no interference” assumption (Cox, 1958), which is standard in causal inference and econometrics (Imbens and Rubin, 2015). Assumption 2 restricts the potential outcomes at each time period \( t \) to depend only on the assignments up to and including time period \( t \). In other words, the potential outcomes of a unit at time \( t \) are not allowed to depend on future treatments.

**Remark 2.1.** Assumption 2 is often made implicitly, and is plausible in many settings. Exceptions may be settings where experimental units are made aware of (and can therefore anticipate) the future sequence of treatments. Then, the outcomes for units at time \( t \) may reflect not only the treatment effect, but also the units’ expectations of future treatments.

2.2.2 Outcome models. The potential outcomes notation, \( Y_{it}(Z) \), can flexibly accommodate complex dependencies between assignments and outcomes. A simple model satisfying Assumptions 1 and 2, for example, is as follows:

\[
Y_{it}(Z) = \mu + \alpha_i + \beta_t + Z_{it}\delta + \gamma Z_{i(t-1)} + \epsilon_{it},
\]

where \( \alpha_i \) and \( \beta_t \) are fixed effects associated to unit \( i \) and time period \( t \), respectively; \( \mu + \delta \) is the direct effect associated with assignment \( Z_{it} = 1 \), \( \mu \) is a constant term, and \( \gamma Z_{i(t-1)} \) is the carryover effect associated with the treatment assigned to unit \( i \) at time \( t - 1 \) (so \( \gamma Z_{i(t-1)} \) takes value \( \gamma_1 \) if \( Z_{i(t-1)} = 1 \) and \( \gamma_0 \) if \( Z_{i(t-1)} = 0 \)). Such models are arguably the most widely used in the literature of crossover trials (Hedayat et al., 1978; Matthews, 1994; Kunert and Stufken, 2008; Jones and Kenward, 2014), although a number of variants have been considered. For instance, John et al. (2004) surveys a number of alternatives, including the so called “placebo model”:

\[
Y_{it}(Z) = \mu + \alpha_i + \beta_t + Z_{it}\delta + \gamma Z_{i(t-1)} + \gamma^2 Z_{i(t-2)} + \ldots + \epsilon_{it},
\]
where the treatment \( Z_{it} = 0 \) is considered a placebo, and therefore does not carry over to the next time period. The dotted terms could correspond to carryover effects of any desired order—when carryovers are of order one, Equation (2) is a simple reparametrization of Equation (1).

In scenarios where we expect the effect of the active treatment to be reduced if repeated, the model of Equation (2) can be modified into a “treatment decay model”:

\[
Y_{it}(Z) = \mu + \alpha_i + \beta_t + Z_{it} \delta - Z_{it} Z_{i(t-1)} \rho \delta + \epsilon_{it},
\]

where \( \rho \in [0, 1] \) here represents a decay in treatment efficacy if the treatment is repeated between successive time periods. In this context, models allowing for higher-order carryover effects are not commonly used, although a few have been proposed (John et al., 2004; Bose and Dey, 2009; Lakatos and Raghavarao, 1987). For instance, a simple extension of the treatment decay model to higher-order carryover effects can be written as follows:

\[
Y_{it}(Z) = \mu + \alpha_i + \beta_t + Z_{it} \delta - \sum_{k=1}^{K} \left( \prod_{k'=0}^{k-1} Z_{i(t-k')} \right) \rho^{k} + \epsilon_{it}.
\]

In the standard literature, once a model for the outcomes is assumed, optimal design of crossover experiments is possible, usually by minimizing some criterion based on the information matrix of the model parameters (Bose and Dey, 2009). The downside is, of course, that optimality depends on correct model specification. Such assumptions of correct specification are generally strong, especially regarding the specification of the carryover effects (Senn and Lambrou, 1998; Senn, 2006).

In this paper, we adopt a nonparametric, randomization-based perspective of inference. That is, we consider the potential outcomes as fixed quantities, while the only source of randomness is the random assignment. Thus, even though the outcome models in Equations (1)-(4) are useful for intuition and exposition, our results on minimax crossover designs do not use or depend on any outcome model. In the following section, we present our causal estimands for crossover trials based on potential outcomes, which then sets the stage for our minimax analysis.
2.2.3 Estimands. In the potential outcomes framework, the estimands of interest are defined as contrasts of potential outcomes (Rubin, 1974). As mentioned earlier, this is useful because it separates the estimand from any particular outcome model. In our setting, a simple causal estimand is the average treatment effect at time $t$, namely,

$$\text{ATE}(t) = \frac{1}{N} \sum_{i=1}^{N} Y_{it}(1) - Y_{it}(0),$$

which contrasts the potential outcomes at time $t$ between the always-treated and always-control population assignments. Intuitively, this estimand aggregates two effects: the differential effect of the treatment at time $t$ on the outcome at time $t$, and the differential effect of the treatment history up to time $t$ on the outcome at time $t$. Formally,

$$\text{ATE}(t) = \frac{1}{N} \sum_{i=1}^{N} \{Y_{it}(1) - Y_{it}(e_t) + Y_{it}(e_t) - Y_{it}(0)\}$$

$$= \frac{1}{N} \sum_{i=1}^{N} \{Y_{it}(1) - Y_{it}(e_t)\} + \frac{1}{N} \sum_{i=1}^{N} \{Y_{it}(e_t) - Y_{it}(0)\}.$$  

This decomposition leads to the following definitions.

**Definition 2.** For a given time period $t$, the cumulative carryover effect, $\lambda_t$, and the direct treatment effect, $\delta_t$, are defined as:

$$\lambda_t = \frac{1}{N} \sum_{i=1}^{N} \{Y_{it}(1) - Y_{it}(e_t)\}, \quad \delta_t = \frac{1}{N} \sum_{i=1}^{N} \{Y_{it}(e_t) - Y_{it}(0)\}.$$  

The cumulative carryover effect, $\lambda_t$, contrasts the potential outcomes under the always-treated assignment and the pulse assignment at $t$. These two assignments have the same treatment at time $t$, so $\lambda_t$ intuitively captures the cumulative effect of treatment history. The direct effect, $\delta_t$, contrasts the potential outcomes under the pulse assignment and the always-control assignment. These two assignments have the same treatment history up to time $t-1$ but differ in their treatment at time $t$. So, $\delta_t$ intuitively captures the effect of the treatment at time $t$.

Although we don’t assume a model for the potential outcomes, it is helpful to make a connection
causal estimands

\[ \begin{array}{c|c|c}
\text{Outcome model} & \text{causal estimands} & \text{c} \\
\hline
\text{Model of Equation (1): standard model} & \gamma_1 - \gamma_0 & \delta \\
\text{Model of Equation (2): placebo model} & \gamma & \delta \\
\text{Model of Equation (3): placebo + decay} & -\delta \rho & \delta \\
\text{Model of Equation (4): placebo + decay (higher order)} & -\delta \left( \frac{1-p^{K+1}}{1-p} - 1 \right) & \delta \\
\end{array} \]

Table 1: Interpretation of the cumulative carryover effect and the direct effect at time \( t \) under different specifications for the outcome model.

to the standard crossover design literature, and consider how to interpret \( \lambda_t \) and \( \delta_t \) with respect to the outcome models discussed in Section 2.2.2. In Table 1 we present the expected values of the cumulative carryover effect and the direct effect under the four models described in Section 2.2.2. This establishes a connection between \( \lambda_t, \delta_t \) and parameters from specific outcome models. For instance, we see that the direct effect, \( \delta_t \), corresponds exactly to the notion of direct effect in the standard models, except that it is allowed to vary with time. The cumulative carryover effect, \( \lambda_t \), has the clearest interpretation under the placebo models, where it corresponds either exactly to the carryover effect as in Equations (2) and (3), or to a sum of carryover effects in Equation (4).

For the rest of this paper, the causal estimands \( \{\lambda_t, \delta_t\}_{t=1}^T \) will be our main focus. As illustrated in Table 1, these estimands are consistent with the standard literature, the difference being one of emphasis: we place equal importance on the direct effects and the carryover effects. Our approach reflects this by not placing any restrictive assumption on the scope of carryover effects.

Remark 2.2. Our definition of cumulative carryover effects is closely related to the concept of “learning effects” in the literature on digital experimentation. For instance, Hohnhold et al. (2015); Yan et al. (2019) study the behavior of users under repeated exposure to different volumes and styles of advertisement. In particular, they attempt to quantify the phenomenon of “ad blindness”, in which users become habituated and less responsive to online advertisements. A similar goal is pursued in ergonomics studies (Kim and Wogalter, 2009), where the objective is to quantify the extent to which visual attention to a stimulus is decreased after repeated exposure.
3 Minimax crossover designs

Standard optimal design in the crossover literature aims to minimize the variance of an appropriate estimator under parametric assumptions (Kiefer, 1975; Matthews, 1994; Bose and Dey, 2009). In contrast, minimax designs seek to minimize the worst-case error of estimators, and are thus more robust. Here, we derive our minimax optimal design for crossover trials. All results are stated in terms of pulse designs, but straightforwardly extend to wedge designs (see Remark 3.3).

3.1 Estimators and risk

To state our minimax results, we need to define estimators for our estimands, \( \delta_t, \lambda_t \). To estimate \( \lambda_t \) we define the following plug-in estimator,

\[
\hat{\lambda}_t = \frac{1}{N_1} \sum_{i=1}^{N} 1(Z_i = 1) Y_{it}(1) - \frac{1}{N_{e_t}} \sum_{i=1}^{N} 1(Z_i = e_t) Y_{it}(e_t), \quad t = 2, \ldots, N,
\]

where \( N_1 = \sum_{i=1}^{N} 1(Z_i = 1) \) is the number of always-treated units, and \( N_{e_t} = \sum_{i=1}^{N} 1(Z_i = e_t) \) is the number of units assigned to a pulse at time \( t \). Variables \( N_1, N_{e_t} \) are random because \( Z \) is random in the experiment. Similarly, define the plug-in estimator of \( \delta_t \) as follows,

\[
\hat{\delta}_t = \frac{1}{N_{e_t}} \sum_{i=1}^{N} 1(Z_i = e_t) Y_{it}(e_t) - \frac{1}{N_0} \sum_{i=1}^{N} 1(Z_i = 0) Y_{it}(0), \quad t = 2, \ldots, N,
\]

where \( N_0 = \sum_{i=1}^{N} 1(Z_i = 0) \) is the number of always-control units. These plug-in estimators are simple and have well-studied sampling properties under randomization (Imbens and Rubin, 2015). In addition, their symmetry makes them amenable to minimax analysis.

The risk of these estimators is a function of the design and potential outcomes. Let \( \mathbf{Y}(z) = [\mathbf{Y}_1(z) \ldots \mathbf{Y}_N(z)]^\top \) denote the \( N \times T \) matrix of potential outcomes under population assignment \( z \), whose \( i \)-th row is equal to \( \mathbf{Y}_i(z) \). The full schedule of potential outcomes, denoted by \( \mathbf{Y} = [\mathbf{Y}(0), \mathbf{Y}(1), \mathbf{Y}(e_2), \ldots, \mathbf{Y}(e_T)] \), therefore contains all the information needed for causal inference, since the causal estimands, \( \lambda_t \) and \( \delta_t \), are deterministic functions of \( \mathbf{Y} \). For a random assignment
\( Z \in \mathcal{Z}(\mathcal{E}) \) and a schedule of potential outcomes \( \mathbf{Y} \), we consider the squared loss function,

\[
L(Z, Y) = \sum_{t=2}^{T} (\hat{\lambda}_t - \lambda_t)^2 + \sum_{t=2}^{T} (\hat{\delta}_t - \delta_t)^2,
\]

which is a function only of \( Z \) and \( Y \), since \( \hat{\lambda}_t \) and \( \hat{\delta}_t \) are deterministic functions of \( Z \) and \( Y \), while \( \lambda_t \) and \( \delta_t \) are functions of \( Y \) only. The risk of a pulse design, \( \eta \in \mathcal{H} \), is then defined as

\[
r(\eta; \mathbf{Y}) = E_\eta\{L(Z, Y)\},
\]

where the expectation is with respect to the randomization distribution in the design, \( \eta(Z) \). As mentioned in Section 2.2, the schedule of potential outcomes, \( \mathbf{Y} \), is considered fixed, all randomness coming from \( Z \). A minimax pulse design, \( \eta^{\text{opt}} \), minimizes the maximum risk over the support of potential outcome schedules, denoted by \( \mathbf{Y} \), in the class of pulse designs \( \mathcal{H} \):

\[
\eta^{\text{opt}} = \arg\min_{\eta \in \mathcal{H}} \max_{\mathbf{Y} \in \mathbf{Y}} \{ r(\eta, \mathbf{Y}) \}. \tag{12}
\]

The task of obtaining a minimax design can therefore be seen as a game between the statistician who chooses a pulse design \( \eta \in \mathcal{H} \) and nature who chooses the worst-case schedule of potential outcomes from \( \mathbf{Y} \). To make progress, we impose an invariance property on \( \mathbf{Y} \), which we describe in the following section.

### 3.2 Permutation invariance of potential outcomes

In the cross-sectional setting, Wu (1981) considered minimax designs over permutation-invariant sets of model parameters to reflect the experimenter’s ignorance at the design stage. We adapt this idea to our setting by introducing a similar notion of permutation invariance for \( \mathbf{Y} \).

Specifically, for some set \( \mathcal{Y} \subset \mathbb{R}^N \), let \( \mathbb{Y}(\mathcal{Y}) = \{ [y_1 \ldots y_T] : y_t \in \mathcal{Y}, \ t = 1, \ldots, T \} \) be the set of all \( N \times T \) matrices whose columns are in \( \mathcal{Y} \), and let \( \mathbf{Y}(\mathcal{Y}) \) be the set of all potential outcomes schedules whose matrices are all elements of \( \mathbb{Y}(\mathcal{Y}) \). Thus, if \( \mathbf{Y} = [Y(0), Y(1), Y(e_2), \ldots, Y(e_T)] \) \( \in \mathbf{Y}(\mathcal{Y}) \) is one such schedule, then \( Y(z) \in \mathbb{Y}(\mathcal{Y}) \) for all \( z \in \mathcal{E} \).
Definition 3 (Permutation-invariant schedule). Consider some $\mathcal{Y} \subset \mathbb{R}^N$ and let $S_N$ be the symmetric group on $N$ elements. A set $Y$ of potential outcomes schedules is called permutation-invariant if $Y = Y(Y)$ for some $\mathcal{Y} \subset \mathbb{R}^N$, such that $S_N \cdot \mathcal{Y} = \mathcal{Y}$, where $S_N \cdot \mathcal{Y}$ is the set in which every element of $\mathcal{Y}$ has been permuted with every element of $S_N$.

The permutation invariance property in Definition 3 captures a symmetry on the units’ outcomes. For example, it implies that for any $z \in \mathcal{E}$ and any $t = 1, \ldots, T$, if it is possible that $y_t(z) = y$, where $y_t(z) = (Y_{1t}(z), \ldots, Y_{Nt}(z))$, it should also be possible that $y_t(z) = \pi \cdot y$, for any permutation $\pi \in S_N$. This property is not a probabilistic statement about the likelihood of $y$ or $\pi \cdot y$, but a statement about the support of the potential outcomes, which is a weaker assumption; see also (Wu, 1981) for an interpretation of permutation invariance in terms of robustness.

3.3 Minimax optimal design

We can now state our first minimax theorem for pulse designs. The resulting design is the solution to an integer optimization problem, which is generally hard to solve. We will therefore discuss a continuous relaxation later in Proposition 1.

Theorem 1 (Minimax pulse design). Let $Y$ be a bounded, permutation-invariant set of potential outcome schedules. The minimax optimal pulse design, $\eta^\text{opt}$, in Equation (12) is the completely randomized design that assigns $N_0^\text{opt}$, $N_1^\text{opt}$, and $\{N_e^\text{opt}\}_{t=2}^T$ units to the assignments 1, 0, and $\{e_t\}_{t=2}^T$, respectively, where:

$$\begin{align*}
(N_0^\text{opt}, N_1^\text{opt}, \{N_e^\text{opt}\}_{t=2}^T) &= \arg\min_{N_0^\text{opt}, N_1^\text{opt}, \{N_e^\text{opt}\}_{t=2}^T \in \mathbb{N}^+} \left( \frac{T - 1}{N_1^\text{opt}} + \frac{T - 1}{N_0^\text{opt}} + 2 \sum_{t=2}^T \frac{1}{N_e^\text{opt}} \right). 
\end{align*}$$

(13)

Two aspects of this result deserve special mention. First, the minimax design in Theorem 1 is a completely randomized design. This result agrees with the results of Wu (1981) and Li et al. (1983) who proved that complete randomization is minimax optimal in the non-crossover setting. Second, the minimax design in Theorem 1 is not balanced as it does not assign the same number of units to each of the $T + 1$ treatment arms in $\mathcal{E}$. This is made clearer by the following result.
Proposition 1. The relaxation of the integer optimization problem of Equation (13),

\[
(\tilde{N}_0, \tilde{N}_1, \{\tilde{N}_{et}\}_{t=2}^T) = \arg\min_{N_0', N_1', \{N_{et}'\}_{t=2}^T \in \mathbb{R}^+ \ \ s.t. \ N_0' + N_1' + \sum_{t=2}^T N_{et}' = N} \left( \frac{T - 1}{N_1'} + \frac{T - 1}{N_0'} + 2 \sum_{t=2}^T \frac{1}{N_{et}'} \right),
\]

has the following solutions:

\[
\tilde{N}_1 = \tilde{N}_0 = \frac{N}{2 + \sqrt{2(T - 1)}}; \quad \tilde{N}_{et} = \sqrt{\frac{2}{T - 1}} - \frac{1}{\sqrt{2(T - 1)}}, \quad t = 2, \ldots, T.
\]

The solution of the relaxed problem exhibits a partial asymmetry with three notable features. First, \(\tilde{N}_1 = \tilde{N}_0\) while \(\tilde{N}_{et} = \tilde{N}_{et}'\) for all \(1 \leq t, t' \leq T\), and so there is symmetry between always-control and always-treated units, but not across all units. Second, \(\tilde{N}_{et}/\tilde{N}_1 \to 0\) as the time horizon \(T\) grows, and so the number of units assigned to any pulse assignment is asymptotically negligible compared to the always-treated (or always-control) assignments. Third, \(\tilde{N}_1/\sum_{t=2}^T \tilde{N}_{et} \to 0\), and so the pulse assignments taken together dominate the other treatment arms. To get intuition, let us consider the problem with \(T = 30\) and \(N = 10000\). In this case, the minimax design assigns \(\tilde{N}_1 = \tilde{N}_0 \approx 1040\), and \(\tilde{N}_{et} \approx 273\) for \(t = 2, \ldots, 30\). In contrast, a balanced randomized design would assign 322 units to each treatment arm. Such difference between our design and standard balanced designs mainly stem from the definition of our loss function in Equation (10), which takes into account arbitrary carryover effects.

Remark 3.1. Another important concept in the crossover design literature is that of uniformity. Many optimal designs are either uniform on periods (i.e., at each period they allocate the same number of units to control and treatment), or uniform on subjects (i.e., each unit is assigned to treatment and control for the same number of periods), or both (Matthews, 1988). In contrast, the minimax design we obtain from Theorem 1 is neither uniform on periods nor uniform on subjects. As mentioned before, this is a byproduct from the definition of our loss function.

Remark 3.2. Since the optimal design obtained in Theorem 1 is completely randomized, randomization-based analysis of the experiment is straightforward using standard Neymanian theory (Imbens and Rubin, 2015, Chapter 6). See proof in Appendix for details.
Remark 3.3. As mentioned earlier, the results in this section (as well as in Section 3.4) are stated in terms of pulse designs, but they can be extended to wedge designs since by Assumption 2,

\[ Y_{it}(w_{t'}) = Y_{it}(e_{t'}), \quad \forall t, t': t' \geq t. \]

That is, the potential outcomes of a unit under assignments \( w_{t'} \) and \( e_{t'} \) are identical for all periods prior to and including \( t' \). In particular, the estimands, \( \lambda_t \) and \( \delta_t \), can be written in terms of wedge assignments instead of pulse assignments by substituting \( w_t \) for \( e_t \) in Equation (6), and the rest of the analysis remains unchanged.

3.4 Minimax optimal design with augmented controls

The design of Theorem 1 was obtained using plug-in estimators for \( \delta_t \) and \( \lambda_t \). Here, we discuss the minimax design problem using a better estimator for \( \delta_t \). The key idea relies on Assumption 2, which implies that \( Y_{it}(e_{t'}) = Y_{it}(0) \), for all \( t < t' \). In other words, under Assumption 2, at all times prior to \( t' \) a pulse assignment \( e_{t'} \) is indistinguishable from an always-control assignment in the sense that a unit assigned to \( e_{t'} \) behaves as if it had been assigned to \( 0 \), for all \( t < t' \). We can therefore use outcomes from units assigned to pulses for estimation of unknown control outcomes.

The new estimator that replaces the plug-in estimator, \( \hat{\delta}_t \), is defined as follows:

\[ \hat{\gamma}_t = \frac{1}{N_{et}} \sum_i^N 1(Z_i = e_t) Y_{it}(e_t) - \frac{1}{N_t} \sum_i^N 1(i \in C(t)) Y_{it}(0), \tag{15} \]

where \( N_t = |C(t)| \), and \( C(t) = \{ i : Z_i = 0, \text{ or } Z_i = e_{t'}, \ t' > t \} \). Here, \( C(t) \) is the new set of “augmented control” units at \( t \), i.e., the set comprised either of always-control units, or units assigned to pulse at a time \( t' > t \). The new loss function is now defined as follows:

\[ L(Z, Y) = \sum_{t=2}^T (\hat{\lambda}_t - \lambda_t)^2 + \sum_{t=2}^T (\hat{\gamma}_t - \delta_t)^2, \tag{16} \]

which only differs from Equation (11) in using the new estimator, \( \hat{\gamma}_t \). The updated minimax result is stated in the following theorem.
Theorem 2. Let $\mathcal{Y}$ be a bounded, permutation-invariant set of potential outcome schedules. The minimax optimal pulse design, $\eta^{\text{opt}}$, in Equation (12) using the new direct effect estimator $\hat{\gamma}_t$ in Equation (15), is the completely randomized design that assigns $N_{1}^{\text{opt}}$, $N_{0}^{\text{opt}}$, and $\{N_{et}^{\text{opt}}\}_{t=2}^{T}$ units to the assignments $1$, $0$, and $\{e_t\}_{t=2}^{T}$, respectively, where:

$$
(N_{0}^{\text{opt}}, N_{1}^{\text{opt}}, \{N_{et}^{\text{opt}}\}_{t=2}^{T}) = \arg\min_{N_{0}', N_{1}', \{N_{et}'\}_{t=2}^{T} \in \mathbb{R}} \left( \frac{T - 1}{N_{1}'} + 2 \sum_{t=2}^{T} \frac{1}{N_{et}'} + \sum_{t=2}^{T} \frac{1}{N_{t}'} \right), \quad (17)
$$

and for each $t$, $N_{t}' = |\mathcal{C}(t)| = N_0 + \sum_{t'=2}^{T} \mathbbm{1}(t' > t)N_{e_t}$.

The main difference between this theorem and Theorem 1 is that the optimization problem of Equation (17) is replacing that of Equation (14). The following proposition derives the solutions to a continuous relaxation of the problem.

Proposition 2. The following integer relaxation of the optimization problem in Equation (17),

$$
(N_{0}^{\text{opt}}, N_{1}^{\text{opt}}, \{N_{et}^{\text{opt}}\}_{t=2}^{T}) = \arg\min_{N_{0}', N_{1}', \{N_{et}'\}_{t=2}^{T} \in \mathbb{R}} \left( \frac{T - 1}{N_{1}'} + 2 \sum_{t=2}^{T} \frac{1}{N_{et}'} + \sum_{t=2}^{T} \frac{1}{N_{t}'} \right),
$$

has analytical solutions:

\begin{align*}
N_{et}^{\text{opt}} &= N_{0}^{\text{opt}} \sqrt{2} c_t, \quad t = 2, \ldots, T, \\
N_{1}^{\text{opt}} &= N - N_{0}^{\text{opt}} \left[ 1 + \sqrt{2} \sum_{t=2}^{T} c_t \right], \\
N_{0}^{\text{opt}} &= N \left[ 1 + (\sqrt{T - 1} + \sqrt{2})c_2 + \sqrt{2} \sum_{t=3}^{T} c_t \right]^{-1},
\end{align*}

where $\{c_t\}_{t=2}^{T}$ are defined recursively by $c_T = 1$ and $c_t = \left[ \frac{1}{c_{t+1}} + \frac{1}{(1+\sqrt{2}T_{t'}c_{t'})^2} \right]^{-1/2}$, for all $t = 2, \ldots, T - 1$.

The solution described by Proposition 2 offers a sharp contrast to the solution described by Proposition 1. Specifically, the solution of Proposition 2 does not exhibit the same form of partial
symmetry as in Proposition 1 because the new estimator, \( \hat{\gamma}_t \), uses outcomes from pulse treatment as information about control potential outcomes. This reflects the fundamental asymmetry of the loss function in Equation (16) in how it uses always-treated and always-control units. The effect will be illustrated more clearly in the simulations of Section 5, which will give more insight into this minimax design by comparing it to the minimax design of Section 3.3, and to the standard, balanced completely randomized design.

4 Extensions

Here, we consider extensions of the main theory on minimax designs developed in the previous sections. Section 4.1 considers loss functions that assign different importance weights to carryover and direct effects, while in Section 4.2 we consider minimax designs when the scope of carryover effects is restricted.

4.1 Weighted loss functions

The loss functions considered in Equation (10) and Equation (16) put the same weight on the terms involving the direct effects and those involving the cumulative carryover effects. This implicitly assumes that both types of effects are of equal interest, which may not be true in practice. More flexible loss functions assign different weights to the terms involving direct and carryover effects. In this section, we focus on extending the results of Section 3.4; analogous results for Section 3.3 are in the Appendix.

Specifically, for some \( \rho \in [0, 1] \) consider the weighted loss function,

\[
L(Z, Y) = \rho \sum_{t=2}^{T} (\hat{\lambda}_t - \lambda_t)^2 + (1 - \rho) \sum_{t=2}^{T} (\hat{\gamma}_t - \delta_t)^2,
\]

(18)

which generalizes Equation (16). Thus, parameter \( \rho \) controls the relative importance in estimating \( \lambda_t \) or \( \delta_t \). The original loss function in Equation (10) is a special case (up to a multiplicative constant) with \( \rho = 1/2 \). The following theorem extends Theorem 2 to this new loss function, and derives the minimax optimal design as a function of \( \rho \).
Theorem 3. Under the weighted loss function of Equation (18), the minimax optimal design, \( \eta^{\text{opt}} \), is still completely randomized, but with \( N_1^{\text{opt}} \), \( N_0^{\text{opt}} \) and \( \{N_{e_t}^{\text{opt}}\}_{t=2}^T \) solving:

\[
(N_0^{\text{opt}}, N_1^{\text{opt}}, \{N_{e_t}^{\text{opt}}\}_{t=2}^T) = \arg\min_{N_0', N_1', \{N_{e_t}'\}_{t=2}^T} \left( \rho \frac{T-1}{N_1'} + \sum_{t=2}^T \frac{1}{N_{e_t}'} + (1 - \rho) \sum_{t=2}^T \frac{1}{N_t'} \right),
\]

where for each \( t \), \( N_t' = |C(t)| = N_0 + \sum_{t'=2}^T \mathbb{1}(t' > t)N_{e_{t'}} \).

In words, the minimax optimal design under a weighted loss function is still completely randomized: what changes is the number of units assigned to each arm. The following relaxation of the optimization problem in Equation (19) provides more intuition.

Proposition 3. The continuous relaxation of the optimization problem of Equation (19) has solutions, for \( \rho < 1 \):

\[
N_{e_t}^{\text{opt}} = N_0^{\text{opt}} c_t, \quad t = 2, \ldots, T,
\]

\[
N_1^{\text{opt}} = N - N_0^{\text{opt}} \left[ 1 + \ell \sum_{t=2}^T c_t \right],
\]

\[
N_0^{\text{opt}} = N \left[ 1 + \ell (1 + \sqrt{\rho(T-1)}) c_2 + \ell \sum_{t=3}^T c_t \right]^{-1},
\]

where \( \ell = (1 - \rho)^{-1/2} \), \( c_T = 1 \) and \( c_t = \left[ \frac{1}{\ell + 1} + \frac{1}{(1 + \ell \sum_{t'=1}^t c_{t'})^2} \right]^{-1/2} \), for all \( t = 2, \ldots, T - 1 \). The case when \( \rho = 1 \) is obtained by symmetry with \( \rho = 0 \); see Appendix.

It is straightforward to see that when \( \rho = 1/2 \) we recover the results of Proposition 2. Additional intuition can be obtained by examining boundary values of \( \rho \). Consider, for example, the case when \( \rho = 1 \), such that the loss function only involves cumulative carryover effects. Then, \( N_0^{\text{opt}} = 0 \). This is reasonable: if we are only interested in cumulative carryover effects, then always-control units are not needed. Similarly, when the loss function only involves direct effects (\( \rho = 0 \)), the minimax optimal design assigns no units to the always-treated arm (\( N_1 = 0 \)).
4.2 Recycling units

A fundamental premise of our approach so far is that we do not restrict the temporal scope of carryover effects. As we discussed in Section 1, this is an important difference with standard models in crossover designs, which typically assume that carryover effects may only be of order one or two time periods.

In some cases, however, it may be reasonable to assume that carryover effects wear off after a certain number of time periods has passed. We state this assumption formally, and then derive the resulting minimax design.

**Assumption 3** \((k\text{-order carryovers})\). For all \(i = 1, \ldots, N\) and all \(t = 1, \ldots, T\),

\[
Y_{it}(e_{t'}) = Y_{it}(0), \quad \forall 0 < t' \leq t - k.
\]

Assumption 3 implies that after \(k\) periods following a pulse treatment, \(e_{t'}\), the effects of the pulse assignment are indistinguishable from those of an always-control treatment. Thus, a unit assigned to a pulse \(e_{t'}\) behaves as if it had been assigned to 0, for all time periods after (and including) \(t' + k\). Taken together, Assumption 2 and Assumption 3 suggest a new estimator of the direct effect that generalizes the “recycling estimator” of Section 3.4.

In particular, for some time period \(t\), and \(k \in \{1, \ldots, T\}\) with \(t - k > 0\), let \(C_k^{(t)}(Z) = \{i : Z_i = 0\text{ or }Z_i = e_{t'}, \text{ with } t' \leq t - k\text{ or }t' > t\}\) be the new set of “augmented controls”, comprised of units assigned either to always-control, to pulses before \(t - k\), or to pulses after \(t\). We replace the estimator of the direct effect, \(\hat{\delta}_t\), by the following estimator:

\[
\hat{\beta}_t = \frac{1}{N_{e_t}} \sum_i 1(Z_i = e_{t})Y_{it}(e_{t}) - \frac{1}{N_{t,k}} \sum_i 1(i \in C_k^{(t)})Y_{it}(0),
\]

where \(N_{t,k} = |C_k^{(t)}|\). This new estimator is “recycling” units that are assigned to pulses in order to estimate control outcomes (term \(C_k^{(t)}\) in Equation (20)). The loss and risk functions are as in Section 3, but with \(\hat{\beta}_t\) instead of \(\hat{\delta}_t\) (or \(\hat{\gamma}_t\)). We can now state the minimax result under the new estimator of the direct effect.

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Theorem 4. Let $\mathcal{Y}$ be a bounded, permutation-invariant set of potential outcome schedules. The minimax optimal pulse design, $\eta^{\text{opt}}$, in Equation (12) using the new direct effect estimator $\hat{\beta}_t$ in Equation (20) is the completely randomized design that assigns $N_0^{\text{opt}}$, $N_1^{\text{opt}}$ and $(N_{e_t}^{\text{opt}})_{t=2}^T$ units to the assignments $1$, $0$, and $(e_t)_{t=2}^T$, respectively, where:

$$\begin{align*}
(N_0^{\text{opt}}, N_1^{\text{opt}}, (N_{e_t}^{\text{opt}})_{t=2}^T) &= \arg\min_{N_0', N_1', (N_{e_t}'_{t=2}) \in \mathbb{N}} \left( \frac{T - 1}{N_1'} + \sum_{t=2}^T \frac{1}{N_{e_t}'} + \sum_{t=2}^T \frac{1}{N_{t,k}'} \right)
\end{align*}$$

As before the optimal design is completely randomized but the number of units assigned to each treatment arm differs from previous designs. The integer optimization problem it involves (as well as its relaxation) is difficult to solve analytically. We plan to address this problem in future work.

Remark 4.1. The results of this section are specific to pulse designs in contrast to the results of Sections 3.3, 3.4 and 4.1, which also apply to wedge designs. Indeed, the fundamental idea of “recycling” units is that if we wait long enough after a pulse, the units assigned to the pulse behave like control units. This does not apply when wedge designs are used, since units remain treated after the pulse.

5 Simulations

The goal of this section is to compare three designs: our minimax design of Section 3.3; our augmented minimax design of Section 3.4; and the balanced completely randomized design (BCRD) that assigns the same number of units to all $T + 1$ treatment arms. We consider the following dimensions for our comparison: the number of units allocated to each treatment (Section 5.1), the maximum risk (Section 5.2), and other metrics from the risk distribution under various models for the potential outcomes (Section 5.3).

5.1 Treatment allocation

We start by illustrating visually the optimal allocations obtained analytically in Proposition 1 and Proposition 2. Figure 1 shows the optimal allocation for the BCRD, the minimax design and the
Figure 1: Unit allocations for BCRD, minimax design, and minimax design with augmented control. Blue horizontal line corresponds to BCRD, as a benchmark. In x-axis, $N_1, N_0$ correspond to always treated and always control, the numbers $2, \ldots, T$ correspond to the pulse designs $N_{e_2}, \ldots, N_{e_T}$.

augmented minimax design for $N = 10000$ and for the time horizons $T = 5, 10, 15$.

As expected, the BCRD produces a fully symmetric allocation, shown as horizontal blue line. The standard minimax design of Section 3.3 produces an allocation that is only partially symmetric between two groups: $N_1 = N_0$ and $N_{e_t} = N_{e_t}'$ for all $2 \leq t, t' \leq T$; see Proposition 1 and the subsequent discussion for details on such symmetry. We also confirm visually that the minimax optimal design allocates more units than the BCRD to the always-treated arm ($z = 1$) and always-control arm ($z = 0$), but less in the pulses arms ($z = e_t$). On the other hand, the minimax design with augmented controls does not exhibit a symmetry. It allocates very few units to the always-control arms: this is expected since in this setting, some pulse units can be used as controls at each time $t$. We also see that the number of units assigned to the pulse assignments $\{e_t\}_{t=2}^T$ increases for larger values of $t$. This again is consistent with our intuition: as $t$ increases, the number of time periods for which a pulse can be used as control increases.

5.2 Maximum risk

The risk under the augmented minimax design should be lower than the risk under the minimax design, which in turn should be lower than the risk under BCRD. In this section, we illustrate the magnitude of that reduction. Figure 2 plots the maximum risk for the minimax and augmented minimax designs relative to the maximum risk of BCRD, for $N = 1000$ and $T = 10, 20, 30, 40, 50$. 

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Figure 2: Relative ratio of maximum risk using BCRD used as the baseline. Left: only augmented minimax design has augmented units. Right: both designs have augmented units.

We consider two settings, as shown in Figure 2. In the left panel, the estimator $\hat{\gamma}$ is used for the risk under the augmented minimax but not for the risk under the other designs; in the right panel, the estimator $\hat{\gamma}$ is used with all three designs. In both settings, the maximum risk is always lower for the augmented minimax design as predicted by theory. Compared to BCRD, the augmented minimax design reduces the risk by up to 20% compared to when $\hat{\lambda}$ is used for all designs. Given the trend further reduction would be expected for longer time horizons.

5.3 Risk

Here, we study the performance of our minimax design as measured in terms of the expected risk under one of the following outcome models from Table 1:

- **Standard**: standard/placebo model corresponding to Model of Equation (1) and (2). Recall that when the carryover effect of the placebo model is of order one, the two models are equivalent but with different parametrization.

- **Decay**: placebo + treatment decay model with one-order carryover effect, corresponding to Model of Equation (3).

We set the parameters values to $\mu = 0$, $\alpha_i = \log(i)$, $\beta_t = \log(t)$, $\delta = 1.0$, $\gamma = -1$, $\rho = 0.5$, $\epsilon_{it} \sim \mathcal{N}(0, 1)$. We consider settings with $N = 50, 250, 500$ units, $T = 10, 15, 20, 25, 30$ maximum
time periods, and perform 100 runs for every experimental setting. Figure 3 displays the distribution of the risk values defined in Equation (11) for the minimax and BCRD designs.

From the results of Figure 3, we see that the minimax design achieves, in general, smaller risk values than BCRD, especially as the number of units, $N$, increases. The difference can be as much as fourfold in favor of the minimax design. For fixed $N$, an increase in $T$ generally leads to an increase in risk values. This is expected since there are effectively fewer data to estimate the individual estimands, $\lambda_t, \delta_t$ (the effect is more evident in the $N = 50$ panel). This also explains why the variance for both designs decreases in general as $N$ increases.

In summary, the main takeaway from the experiments of this section is that our minimax design not only performs better than the BCRD in a worst-case sense, which is guaranteed by theory – it also seems to perform well in the average sense compared to the standard, completely randomized design.
6 Concluding remarks

In this paper, we constructed minimax optimal crossover designs with carryover effects of arbitrary order. Our construction uses the potential outcomes framework of causal inference, and is non-parametric. We have shown in simulation studies that our minimax designs reduce not only the maximum risk, but also the expected risk with respect to three popular models.

There are several open questions for future work. First, from a technical perspective, it would be interesting to obtain analytical solutions for the recycling designs in Theorem 4. These should outperform the augmented minimax design because they wouldn’t throw away useful data. Second, in future work we would like to connect our estimands in this paper ($\lambda_t, \delta_t$ in Definition 2) to estimands of long-term effects in user behavior experiments, which are popular in digital experimentation (Hohnhold et al., 2015; Yan et al., 2019; Kohavi et al., 2009). Finally, we are looking at extensions of our approach, allowing covariate information to be incorporated in the design.

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Yan, J., B. Tiwana, S. Ghosh, H. Liu, and S. Chatterjee (2019, Jan). Measuring Long-term Impact of Ads on LinkedIn Feed. arXiv e-prints, arXiv:1902.03098.
A. Proof of results in Section 3.3

A.1 Intermediate results and lemmas

The proof of Theorem 1 has a number of intermediate steps, which we will state as lemmas.

First, we define:

\[ L_\lambda(Z, Y) = \sum_{i=2}^T (\lambda_i - \lambda_1)^2; \quad L_\delta(Z, Y) = \sum_{i=2}^T (\delta_i - \delta_1)^2 \]

so that we have \( L(Z, Y) = L_\lambda(Z, Y) + L_\delta(Z, Y) \). Next, we need to define the permutation actions on vectors and matrices.

**Definition 4.** Let \( S_N \) be the symmetric group on \( N \) elements, and \( \pi \in S_N \). If \( v \) is a vector of length \( N \), then the action of \( \pi \) on \( v \), denoted \( \pi \cdot v \), is the vector obtained by permuting the indices of \( v \) according to \( \pi \). Similarly, for any population assignment \( Z = [Z_1, \ldots, Z_N] \in \mathcal{Z} \subset \{0, 1\}^{N \times T} \),

\[ \pi \cdot Z \equiv [Z_{\pi^{-1}(1)}, \ldots, Z_{\pi^{-1}(N)}], \]

where \( \pi^{-1}(i) \) is the element \( j \) that is mapped to \( i \) through \( \pi \).

**Definition 5.** For a potential outcomes schedule, \( Y = [Y(0), Y(1), Y(e_2), \ldots, Y(e_T)] \), we define

\[ \pi \cdot Y \equiv [\pi \cdot Y(0), \pi \cdot Y(1), \pi \cdot Y(e_2), \ldots, \pi \cdot Y(e_T)] \]

where, as above, we have \( \pi \cdot Y(z) \equiv (Y_{\pi^{-1}(1)}(z), \ldots, Y_{\pi^{-1}(N)}(z)) \), for every assignment \( z \in \mathcal{E} \).

We can now state and prove a sequence of lemmas. Throughout, we denote \( I = \{1, \ldots, N\} \). It is immediate to see that if \( \pi \in S_N \), then \( \pi \cdot I \equiv \{\pi(i), i \in I\} = I \) and thus \( \pi^{-1} \cdot I = I \). Our first lemma is to show that the loss function, \( L \), is permutation-invariant in its arguments.

**Lemma 1.** For any \( \pi \in S_N \), and any assignment matrix \( Z \) and schedule \( Y \),

\[ L(\pi \cdot Z, \pi \cdot Y) = L(Z, Y) \]

**Proof.** We show that \( L_\delta(\pi \cdot Z, \pi \cdot Y) = L_\delta(Z, Y) \). The proof for \( L_\lambda \) is identical, and so the proof for \( L \) follows immediately.

Recall that \( \delta_t(Y) = N^{-1} \sum_i Y_{it}(e_t) - N^{-1} \sum_i Y_{it}(0) \). We have:

\[
\delta_t(\pi \cdot Y) = N^{-1} \sum_{i \in I} (\pi \cdot Y(e_t))_{it} - N^{-1} \sum_{i \in I} (\pi \cdot Y(0))_{it} \\
= N^{-1} \sum_{i \in I} Y_{\pi^{-1}(i), t}(e_t) - N^{-1} \sum_{i \in I} Y_{\pi^{-1}(i), t}(0) \\
= N^{-1} \sum_{j \in \pi^{-1} \cdot I} Y_{jt}(e_t) - N^{-1} \sum_{j \in \pi^{-1} \cdot I} Y_{jt}(0) \\
= N^{-1} \sum_{j \in I} Y_{jt}(e_t) - N^{-1} \sum_{j \in I} Y_{jt}(0) \\
= \delta_t(Y). \quad (22)
\]
We now turn to the estimator \( \hat{\delta}_t \),

\[
\hat{\delta}_t(Z, Y) = N^{-1} \sum_{i \in I} 1(Z_i = e_t)Y_{it}(e_t) - N^{-1} \sum_{i \in I} 1(Z_i = 0)Y_{it}(0_t)
\]

\[= \hat{\delta}_t(e) - \hat{\delta}_t(0) \equiv \hat{\delta}_t(Z, Y).
\]

With a similar argument as before:

\[
\hat{\delta}_t(e)(\pi \cdot Z, \pi \cdot Y) = N^{-1} \sum_{i \in I} 1((\pi \cdot Z)_i = e_t)Y_{\pi^{-1}(i), t}(e_t) = \hat{\delta}_t(\pi \cdot Z)\]

The exact same reasoning leads to \( \hat{\delta}_t(0)(\pi \cdot Z, \pi \cdot Y) = \hat{\delta}_t(0)(Z, Y) \), and so:

\[
\hat{\delta}_t(\pi \cdot Z, \pi \cdot Y) = \hat{\delta}_t(Z, Y).
\]

Putting together Equation (22) and Equation (23) we have:

\[
L_\delta(\pi \cdot Z, \pi \cdot Y) = \sum_{t=1}^{T} (\hat{\delta}_t(\pi \cdot Z, \pi \cdot Y) - \delta_t(\pi \cdot Y))^2 = \sum_{t=1}^{T} (\hat{\delta}_t(Z, Y) - \delta_t(Y))^2 = L_\delta(Z, Y).
\]

\[\square\]

**Lemma 2.** For a pulse design \( \eta \in H \) and \( \pi \in S_N \), let \( \eta_\pi \) be the design such that \( \eta_\pi(Z) = \eta(\pi \cdot Z) \), and let \( \hat{\eta} = (N!)^{-1} \sum_{\pi \in S_N} \eta_\pi \). Then, if \( Y \) is permutation-invariant, we have:

\[
\max_{Y \in \mathcal{Y}(\mathcal{Y})} \{ r(\hat{\eta}; Y) \} \leq \max_{Y \in \mathcal{Y}(\mathcal{Y})} \{ r(\eta; Y) \}.
\]

**Proof.** We have:

\[
r(\hat{\eta}; Y) = \sum_{Z \in \mathcal{Z}(\mathcal{L})} \hat{\eta}(Z) L(Z, Y)
\]

\[= \sum_{Z \in \mathcal{Z}(\mathcal{L})} \left[(N!)^{-1} \sum_{\pi \in S_N} \eta_\pi(Z) \right] L(Z, Y)
\]

\[= (N!)^{-1} \sum_{\pi \in S_N} \sum_{Z \in \mathcal{Z}(\mathcal{L})} \eta_\pi(Z) L(Z, Y).
\]
It follows that
\[
\max_{\mathbf{Y} \in \mathcal{Y}} r(\tilde{\eta}; \mathbf{Y}) \leq (N!)^{-1} \sum_{\pi \in S_N} \max_{\mathbf{Y} \in \mathcal{Y}} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta_\pi(\mathbf{Z}) L(\mathbf{Z}, \mathbf{Y}) \right\}
\]
\[
= (N!)^{-1} \sum_{\pi \in S_N} \max_{\mathbf{Y} \in \mathcal{Y}} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\pi \mathbf{Z}) L(\pi \mathbf{Z}, \mathbf{Y}) \right\}
\]
\[
= (N!)^{-1} \sum_{\pi \in S_N} \max_{\mathbf{Y} \in \mathcal{Y}} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\pi \mathbf{Z}) L(\pi \mathbf{Z}, \mathbf{Y}) \right\} \quad \text{[by Lemma 1]}
\]
\[
= (N!)^{-1} \sum_{\pi \in S_N} \max_{\mathbf{Y} \in \mathcal{Y}} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\pi \mathbf{Z}) L(\mathbf{Z}', \mathbf{Y}) \right\}
\]
\[
= (N!)^{-1} \sum_{\pi \in S_N} \max_{\mathbf{Y} \in \mathcal{Y}} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\mathbf{Z}) L(\mathbf{Z}, \mathbf{Y}) \right\},
\]
where the last equality follows from the fact that the support \( \mathcal{Z}(\mathcal{E}) \) is permutation invariant. Now we use the fact that \( \mathcal{Y}(\mathcal{Y}) \) is permutation invariant:
\[
\max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\mathbf{Z}) L(\mathbf{Z}, \mathbf{Y}) \right\} = \max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\mathbf{Z'}) L(\mathbf{Z'}, \mathbf{Y}) \right\}
\]
\[
= \max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} \left\{ \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\mathbf{Z}) L(\mathbf{Z}, \mathbf{Y}) \right\}
\]
\[
= \max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} r(\tilde{\eta}; \mathbf{Y}).
\]

Putting everything together, we obtain:
\[
\max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} r(\tilde{\eta}; \mathbf{Y}) \leq (N!)^{-1} \sum_{\pi \in S_N} \max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} \left\{ r(\eta; \mathbf{Y}) \right\}
\]
\[
= \max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} \left\{ r(\eta; \mathbf{Y}) \right\} \cdot (N!)^{-1} \sum_{\pi \in S_N} 1
\]
\[
= \max_{\mathbf{Y} \in \mathcal{Y}(\mathcal{Y})} r(\eta; \mathbf{Y}).
\]

\[
\square
\]

Next, we prove a representation lemma.

**Lemma 3.** Let \( \delta_{\mathbf{Z}} \) be the design that assigns mass 1 at the assignment \( \mathbf{Z} \). Let \( \eta \in \mathcal{H} \). Then:
\[
\tilde{\eta} = \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\mathbf{Z}) \delta_{\mathbf{Z}},
\]
where \( \tilde{\delta}_{\mathbf{Z}} = (N!)^{-1} \sum_{\pi \in S_N} \xi_{\pi, \mathbf{Z}} \) and \( \xi_{\pi, \mathbf{Z}}(\mathbf{Z}') = \delta_{\pi} \mathbf{Z}' \).

**Proof.** From \( \pi \cdot \mathcal{Z}(\mathcal{E}) = \mathcal{Z}(\mathcal{E}) \) it follows, for all \( \pi \in S_N \), that
\[
\eta_\pi = \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\pi \cdot \mathbf{Z}) \delta_{\mathbf{Z}} = \sum_{\mathbf{Z}' \in \pi \cdot \mathcal{Z}(\mathcal{E})} \eta(\mathbf{Z'}) \delta_{\pi^{-1} \cdot \mathbf{Z}'} = \sum_{\mathbf{Z} \in \mathcal{Z}(\mathcal{E})} \eta(\mathbf{Z}) \delta_{\pi^{-1} \cdot \mathbf{Z}}.
\]

By definition, \( \delta_{\pi^{-1} \cdot \mathbf{Z}} = \xi_{\pi, \mathbf{Z}} \) since both function out mass 1 at the treatment \( \pi^{-1} \cdot \mathbf{Z} \). Then, from its definition
in Lemma 2:
\[
\hat{\eta} = (N!)^{-1} \sum_{\pi \in S_N} \eta_\pi = \sum_{Z \in \mathcal{Z}(E)} \eta(Z)(N!)^{-1} \sum_{\pi \in S_N} \hat{\xi}_{\pi,Z} = \sum_{Z \in \mathcal{Z}(E)} \eta(Z)\hat{\delta}_Z.
\]

\[\square\]

**Lemma 4.** Let \(\eta \in \mathcal{H}\) and \(\mathcal{Y}(\mathcal{Y})\) be a permutation-invariant schedule of potential outcomes, where \(\mathcal{Y}\) is bounded. Then,
\[
\max_{Y \in \mathcal{Y}(\mathcal{Y})} \{r(\hat{\eta}; \hat{Y})\} = V^* \sum_{Z \in \mathcal{Z}(E)} \left( \frac{T - 1}{N_1(Z)} + \frac{T - 1}{N_0(Z)} + 2 \sum_{t=2}^T \frac{1}{N_{c_t}(Z)} \right).
\]

where
\[
V^* = \max_{\epsilon \in \mathcal{Y}} \frac{1}{N - 1} \sum_{i=1}^N (x_i - \bar{x})^2 = O(1).
\]

**Proof.** We have:
\[
r(\hat{\eta}; \hat{Y}) = \sum_{Z' \in \mathcal{Z}(E)} \hat{\eta}(Z')L(Z', \hat{Y}) = \sum_{Z' \in \mathcal{Z}(E)} \sum_{Z \in \mathcal{Z}(E)} \eta(Z)\hat{\delta}_Z(Z')L(Z', \hat{Y}) \quad \text{[from Lemma 3]}
\]
\[
= \sum_{Z \in \mathcal{Z}(E)} \left\{ \sum_{Z' \in \mathcal{Z}(E)} \eta(Z)\hat{\delta}_Z(Z')L(Z', \hat{Y}) \right\}
\]
\[
= \sum_{Z \in \mathcal{Z}(E)} \eta(Z)r(\hat{\delta}_Z; \hat{Y}),
\]

where \(\hat{\delta}_Z\) is the CRD that assigns \(N_0(Z)\) units to \(0\), \(N_1(Z)\) units to \(1\), and \(N_{c_t}(Z)\) units to pulse \(e_t\), for \(t = 2, \ldots, T\). The usual randomization sampling results hold (Imbens and Rubin, 2015):
\[
r(\hat{\delta}_Z; \hat{Y}) = \sum_{t=2}^T \left\{ \text{bias}_{\hat{\delta}_Z}(\hat{\lambda}_t, \lambda_t) \right\}^2 + \left\{ \text{bias}_{\hat{\delta}_Z}(\hat{\delta}_t, \delta_t) \right\}^2 + \sum_{t=2}^T \left[ V_\delta(\hat{\lambda}_t) + V_\delta(\hat{\delta}_t) \right] = \sum_{t=2}^T \left[ V_0(\hat{\lambda}_t) + V_0(\hat{\delta}_t) \right],
\]

where we defined:
\[
V_\delta(\hat{\lambda}_t) = \frac{V_1(t)}{N_1(Z)} + \frac{V_{c_t}(t)}{N_{c_t}(Z)} - \frac{V_0(t)}{N_0(Z)}; \quad V_\delta(\hat{\delta}_t) = \frac{V_0(t)}{N_0(Z)} + \frac{V_{c_t}(t)}{N_{c_t}(Z)} - \frac{V_0(t)}{N_{c_t}(Z)},
\]

and
\[
V_h(t) = (N - 1)^{-1} \sum_{i=1}^N \{Y_i(t) - \bar{Y}_t(t)\}^2, \quad h = 0, e_t, \quad t = 2, \ldots, T
\]
\[
V_{0,c_t}(t) = (N - 1)^{-1} \sum_{i=1}^N \{[Y_i(t) - Y_i(0)] - [\bar{Y}_t(t) - \bar{Y}_t(0)]\}^2,
\]
\[
V_{1,c_t}(t) = (N - 1)^{-1} \sum_{i=1}^N \{[Y_i(t) - Y_i(0)] - [\bar{Y}_t(t) - \bar{Y}_t(0)]\}^2,
\]

where \(\bar{Y}_t\) indicates averaging over all units. Occasionally, we will write \(V_h(t)(\hat{Y}), V_{0,c_t}(\hat{Y}), V_{1,c_t}(\hat{Y})\) to empha-
size that these are functions of the potential outcomes schedule, \( \mathbf{Y} \). Note also that the bias terms are zero because the estimators, \( \hat{\lambda}_i, \hat{\delta}_i \) are unbiased. Putting everything together, we obtain:

\[
    r(\hat{\eta}; \mathbf{Y}) = \sum_{t=2}^{T} \left[ V_1^{(t)} \sum_{\mathbf{z} \in \mathcal{Z}(\xi)} \frac{\eta(\mathbf{Z})}{N_1(\mathbf{Z})} + V_0^{(t)} \sum_{\mathbf{z} \in \mathcal{Z}(\xi)} \frac{\eta(\mathbf{Z})}{N_0(\mathbf{Z})} + 2V_{e_t}^{(t)} \sum_{\mathbf{z} \in \mathcal{Z}(\xi)} \frac{\eta(\mathbf{Z})}{N_{e_t}(\mathbf{Z})} - \frac{V_{0,e_t}^{(t)}}{N} - \frac{V_{1,e_t}^{(t)}}{N} \right].
\]

(24)

Now is the key part of the argument. First, \( V_h^{(t)} \) is a function only of \( y_t(h) = (Y_{1,t}(h), \ldots, Y_{N_t}(h)) \), say, \( V_h^{(t)} = V(y_t(h)) \). Furthermore, \( y_t(h) \) takes values from \( \mathbf{Y} \). Therefore,

\[
    \text{argmax} V(y_t(h)) = \text{constant} \equiv \mathbf{Y}^{\text{opt}}, \text{ for all } h, t;
\]

here, we allow argmax to return a set. In particular, \( \mathbf{Y}^{\text{opt}} \) is a set because there might be many vectors that maximize \( V \); the set can be a singleton, but it is not empty. Now, for every element \( \mathbf{y} \in \mathbf{Y}^{\text{opt}} \), it holds

\[
    \max_{\mathbf{Y} \in \mathbf{Y}(\mathbf{Y})} V_h^{(t)} = V(h),
\]

(25)

since \( \mathbf{Y}(\mathbf{Y}) \) contains all potential outcomes schedules, such that the columns of every matrix in every schedule are from \( \mathbf{Y} \). Take any vector \( \mathbf{y} \in \mathbf{Y}^{\text{opt}} \). Define \( \mathbf{Y}^{\text{opt}} \) as the \( N \times T \) matrix where each column is equal to \( \mathbf{y} \), and define \( \mathbf{Y}^{\text{opt}} = [\mathbf{Y}^{\text{opt}}, \ldots, \mathbf{Y}^{\text{opt}}] \) the potential outcomes schedule containing \( T + 1 \) copies of \( \mathbf{Y}^{\text{opt}} \). Then, for all \( h = 0, 1, e_t, \) and all \( t = 2, \ldots, T \),

\[
    \max_{\mathbf{Y} \in \mathbf{Y}(\mathbf{Y})} V_h^{(t)} = V_h^{(t)}(\mathbf{Y}^{\text{opt}}) = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \bar{y})^2 \equiv V^* = O(1),
\]

(26)

since the potential outcomes are bounded. It then follows that:

\[
    \mathbf{Y}^{\text{opt}} \in \text{argmax} \mathbf{Y} \in \mathbf{Y}(\mathbf{Y}) \left\{ \sum_{t=2}^{T} \left[ V_1^{(t)} \sum_{\mathbf{z} \in \mathcal{Z}(\xi)} \frac{\eta(\mathbf{Z})}{N_1(\mathbf{Z})} + V_0^{(t)} \sum_{\mathbf{z} \in \mathcal{Z}(\xi)} \frac{\eta(\mathbf{Z})}{N_0(\mathbf{Z})} + 2V_{e_t}^{(t)} \sum_{\mathbf{z} \in \mathcal{Z}(\xi)} \frac{\eta(\mathbf{Z})}{N_{e_t}(\mathbf{Z})} \right] \right\},
\]

since we can maximize \( V_1^{(t)}, V_0^{(t)}, V_{e_t}^{(t)} \) separately for every \( t \) by construction of the \( \mathbf{Y}^{\text{opt}} \) and the argument in Equation (25).

Now, we need to turn our attention to the negative terms in Equation (24). First, for all \( t \),

\[
    V_{0,e_t}^{(t)}(\mathbf{Y}^{\text{opt}}) = V_{1,e_t}^{(t)}(\mathbf{Y}^{\text{opt}}) = 0,
\]

by definition of \( \mathbf{Y}^{\text{opt}} \). But we also have:

\[
    \max_{\mathbf{Y} \in \mathbf{Y}(\mathbf{Y})} \{-V_{0,e_t}^{(t)}(\mathbf{Y})\} = \max_{\mathbf{Y} \in \mathbf{Y}(\mathbf{Y})} \{-V_{1,e_t}^{(t)}(\mathbf{Y})\} = 0,
\]

because the \( V \)-functions are non-negative. It follows that

\[
    \mathbf{Y}^{\text{opt}} \in \text{argmax} r(\hat{\eta}; \mathbf{Y}).
\]

(27)
We conclude that
\[
\max_{\mathbf{Z} \in \mathbb{Y}(Y)} r(\tilde{\eta}_e; \mathbf{Y}) = \sum_{t=2}^T \left[ V^* \sum_{Z \in \mathbb{Z}(\mathcal{E})} \eta(Z) \left\{ \frac{T-1}{N_1(Z)} + \frac{T-1}{N_0(Z)} + 2 \sum_{t=2}^T \frac{1}{N_{e_t}(Z)} \right\} \right]
\]
\[
= V^* \sum_{Z \in \mathbb{Z}(\mathcal{E})} \eta(Z) \left\{ \frac{T-1}{N_1(Z)} + \frac{T-1}{N_0(Z)} + 2 \sum_{t=2}^T \frac{1}{N_{e_t}(Z)} \right\},
\]
where we combined Equation (26) and Equation (27).

\section*{A.2 Proof of Theorem 1}

\begin{proof}
If \( \eta \) is minimax optimal, then by Lemma 2, \( \tilde{\eta} \) is also minimax; our strategy, therefore, will be to find the design of the form \( \eta \) that achieves the minimax risk. For a design \( \eta \), we have, by Lemma 4,
\[
\max_{\mathbf{Y} \in \mathbb{Y}(Y)} r(\tilde{\eta}_e; \mathbf{Y}) = V^* \sum_{Z \in \mathbb{Z}(\mathcal{E})} \eta(Z) \left\{ \frac{T-1}{N_1(Z)} + \frac{T-1}{N_0(Z)} + 2 \sum_{t=2}^T \frac{1}{N_{e_t}(Z)} \right\}
\]
\[
= \sum_{Z \in \mathbb{Z}(\mathcal{E})} \eta(Z) \left\{ \frac{T-1}{N_1(Z)} + \frac{T-1}{N_0(Z)} + 2 \sum_{t=2}^T \frac{1}{N_{e_t}(Z)} \right\},
\]
since \( V^* \) does not depend on \( Z \). It follows that \( \min_{\eta} \max_{\mathbf{Y} \in \mathbb{Y}(Y)} r(\tilde{\eta}_e; \mathbf{Y}) \) is attained by the design \( \eta \) that satisfies:
\[
\eta(Z) > 0 \Rightarrow (N_1(Z), N_0(Z), \{N_{e_t}(Z)\}) = \arg\min_{N_1', N_0', \{N_{e_t}'\}} \left\{ \frac{T-1}{N_1'} + \frac{T-1}{N_0'} + 2 \sum_{t=2}^T \frac{1}{N_{e_t}'} \right\}.
\]
(28)

Let \( \eta^{opt} \) be a design satisfying Equation (28), and consider \( \tilde{\eta}^{opt} \). For any \( Z \) such that \( \eta^{opt}(Z) > 0 \), we have:
\[
\tilde{\eta}^{opt}_{\pi}(Z) = \eta^{opt}(\pi \cdot Z)
\]
\[
= (N!)^{-1} \sum_{\pi' \in S_N} \eta^{opt}(\pi' \cdot Z)
\]
\[
= (N!)^{-1} \sum_{\pi' \in S_N} \eta^{opt}(\pi' \cdot \pi \cdot Z)
\]
\[
= (N!)^{-1} \sum_{\pi' \in S_N} \eta^{opt}(\pi'' \cdot Z)
\]
\[
= \tilde{\eta}^{opt}(Z),
\]
for any permutation \( \pi \in S_N \); that is, \( \tilde{\eta}^{opt}(\pi \cdot Z) = \tilde{\eta}^{opt}(Z) = c \), for any permutation \( \pi \in S_N \). But the permutations of \( Z \) are the assignments such that:
\[
(N_1(Z), N_0(Z), \{N_{e_t}(Z)\}) = \arg\min_{N_1', N_0', \{N_{e_t}'\}} \left\{ \frac{T-1}{N_1'} + \frac{T-1}{N_0'} + 2 \sum_{t=2}^T \frac{1}{N_{e_t}'} \right\}.
\]
(29)

It is easy to verify that \( \tilde{\eta}^{opt} \) assigns mass zero to the assignments \( Z \) that do not satisfy Equation (29), and so in conclusion, \( \tilde{\eta}^{opt} \) is the completely randomized design with \( N_1^{opt}, N_0^{opt} \) and \( \{N_{e_t}^{opt}\} \) satisfying Equation (29).
\end{proof}
A.3 Proof of Proposition 1

Proof. The Lagrangian function corresponding to the objective function is:

\[
\mathcal{L}(N'_0, N'_1, \{N'_{et}\}_{t=2}^T, \lambda) = \left( \frac{T-1}{N'_0} + \frac{T-1}{N'_1} + 2 \sum_{t=2}^{T} \frac{1}{N'_{et}} \right) + \lambda \left( N'_0 + N'_1 + \sum_{t=2}^{T} N_{et} - N \right)
\]

Setting the partial derivatives of the Lagrangian to zero yields the following system of equations:

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial N'_0} &= -\frac{T-1}{N'_0^2} + \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial N'_1} &= -\frac{T-1}{N'_1^2} + \lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial N'_{et}} &= -2 \frac{1}{(N'_{et})^2} + \lambda = 0, \quad t = 2, \ldots, T
\end{align*}
\]

whose solution we write as a function of \( \lambda \):

\[
N'_0 = \sqrt{\frac{T-1}{\lambda}}, \quad N'_1 = \sqrt{\frac{T-1}{\lambda}}, \quad N'_{et} = \sqrt{\frac{2}{\lambda}}, \quad t = 2, \ldots, T.
\]

But the total number of units assigned must add up to \( N \), which allows us to solve for \( \lambda \):

\[
N'_0 + N'_1 + \sum_{t=2}^{T} N_{et} = 2 \sqrt{\frac{T-1}{\lambda}} + (T-1) \sqrt{\frac{2}{\lambda}} = N \quad \implies \quad \sqrt{\frac{1}{\lambda}} = \frac{N}{2\sqrt{\frac{T-1}{\lambda}} + \sqrt{2(T-1)}}
\]

leading to the following solution:

\[
\begin{align*}
\tilde{N}_1 &= \frac{N}{2 + \sqrt{2(T-1)}}, \\
\tilde{N}_0 &= \frac{N}{2 + \sqrt{2(T-1)}}, \\
\tilde{N}_{et} &= \sqrt{\frac{2}{T-1} + \frac{N}{2 + \sqrt{2(T-1)}}}, \quad t = 2, \ldots, T.
\end{align*}
\]

This completes the proof. \( \square \)

B Proof of results in Section 3.4

B.1 Proof of Theorem 2

The proof of Theorem 2 follows exactly the same lines as that of Theorem 4 which we prove below in Appendix D.

B.2 Proof of Proposition 2

Proof. Consider the objective function:

\[
\rho(N_0, N_{e_2}, \ldots, N_{e_T}) = \frac{T-1}{N - N_0 - \sum_{t=2}^{T} N_{et}} + 2 \sum_{t=2}^{T} \frac{1}{N_{et}} + \sum_{t=2}^{T} \frac{1}{N'_t}
\]

33
Therefore, we can write the same equation for $t$

$$\frac{dp}{dN_{e_t}} = \frac{T - 1}{[N - N_0 - \sum_{t'=2}^T N_{e_{t'}}]^2} - \frac{2}{(N_{e_t})^2} \sum_{t'=2}^T \mathbb{1}(t' < t) \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > t') N_{e_{t''}}]^2}$$

$$\frac{dp}{dN_{e_{t-1}}} = \frac{T - 1}{[N - N_0 - \sum_{t'=2}^T N_{e_{t'}}]^2} - \frac{2}{(N_{e_{t-1}})^2} \sum_{t'=2}^T \mathbb{1}(t' < t - 1) \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > t') N_{e_{t''}}]^2}$$

Make $\frac{dp}{dN_{e_t}} = 0$ and $\frac{dp}{dN_{e_{t-1}}} = 0$, and the terms cancel between them by subtraction as follows:

$$\frac{2}{(N_{e_t})^2} = \frac{2}{(N_{e_{t-1}})^2} - \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > t - 1) N_{e_{t''}}]^2}$$

Therefore, we can write the same equation for $t = 3, \ldots, T$

$$\frac{2}{(N_{e_T})^2} = \frac{2}{(N_{e_{T-1}})^2} - \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > T - 1) N_{e_{t''}}]^2}$$

$$\frac{2}{(N_{e_{T-1}})^2} = \frac{2}{(N_{e_{T-2}})^2} - \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > T - 2) N_{e_{t''}}]^2}$$

$$\vdots$$

$$\frac{2}{(N_{e_2})^2} = \frac{2}{(N_{e_1})^2} - \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > 2) N_{e_{t''}}]^2}$$

Add all these equations on the left and right sides, we have

$$\frac{2}{(N_{e_T})^2} = \frac{2}{(N_{e_1})^2} - \sum_{t'=2}^{T-1} \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > t') N_{e_{t''}}]^2}$$

(30)

Also, we have

$$\frac{dp}{dN_0} = \frac{T - 1}{[N - N_0 - \sum_{t'=2}^T N_{e_{t'}}]^2} - \sum_{t'=2}^T \frac{1}{[N_0 + \sum_{t''=2}^T \mathbb{1}(t'' > t') N_{e_{t''}}]^2} = 0$$

(31)

$$\frac{dp}{dN_{e_1}} = \frac{T - 1}{[N - N_0 - \sum_{t'=2}^T N_{e_{t'}}]^2} - \frac{2}{(N_{e_2})^2} = 0$$

(32)

Plug these two equations to Equation 30, we have

$$\frac{2}{(N_{e_T})^2} = \frac{1}{(N_0)^2} \Rightarrow N_{e_T} = N_0 \sqrt{2} e_T$$

where $c_T = 1$. Plug this to the $\frac{2}{(N_{e_T})^2} = \frac{2}{(N_{e_{T-1}})^2} - \frac{1}{[N_0 + N_{e_{T-1}}]^2}$, we have

$$N_{e_{T-1}} = N_0 \sqrt{2} e_{T-1},$$

where $c_{T-1} = \left[\frac{1}{c_T^2} + \frac{1}{(1+\frac{1}{\sqrt{2} e_T})^2}\right]^{-1/2}$. So, do the same operation for other equations, we have

$$N_{e_t} = N_0 \sqrt{2} c_t, \ t = 2, \ldots, T$$
where
\[
c_t = \left[ \frac{1}{c_t^2} + \frac{1}{(1 + \sqrt{2} \sum_{t' > t} c_{t'})^2} \right]^{-1/2}, \quad t = 2, \ldots, T - 1
\]
and
\[c_T = 1\]
Plug these \(\{N_t\}\) to Equation 32, we have
\[
\frac{T - 1}{[N - N_0 - \sum_{t'=2}^{T} N_{t'}]^2} - \frac{2}{(N_{e2})^2} = 0 \implies 2c_2^2 N_0^2 = \frac{T - 1}{[N - N_0 - \sqrt{2} \sum_{t'=2}^{T} c_{t'} N_0]^2}
\]
Solve this equation, we get
\[
N_0 = \left[ 1 + (\sqrt{T - 1} + \sqrt{2}) c_2 + \sqrt{2} \sum_{t=2}^{T} c_t \right]^{-1}
\]
Finally, we have
\[
N_1 = N - N_0 \left[ 1 + \sqrt{2} \sum_{t=2}^{T} c_t \right]
\]

C Proof of results in Section 4.1

C.1 Proof of Theorem 3

Lemma 1 through Lemma 3 hold trivially with the weighted loss function. Adapting Lemma 4 requires more care.

Lemma 5 (Analog to Lemma 4). Let \(\eta \in H\) and \(Y(\eta)\) where \(Y\) is bounded and permutation-invariant, then:
\[
\max_{\eta \in \mathcal{Y}(\eta)} V^* \sum_{Z \in \mathcal{Z}(\mathcal{E})} \left[ \rho \frac{T - 1}{N_1(Z)} + (1 - \rho) \sum_{t=2}^{T} \frac{1}{N_t(Z)} + \sum_{t=2}^{T} \frac{1}{N_{e1}(Z)} \right]
\]
where \(V^*\) is as in Lemma 4.

Proof of Lemma 5. The proof requires to make some of same modifications made in the proof of Lemma 7. Here, we give a high-level view of some of the changes in the proof. We have:
\[
r(\hat{\delta}_Z; Y) = \sum_{t=2}^{T} \left( \rho \text{Bias}(\hat{\lambda}_t) + (1 - \rho)\text{Bias}(\hat{\delta}_t) \right) + \sum_{t=2}^{T} \left( \rho \text{Var}_{\delta Z}(\hat{\lambda}_t) + (1 - \rho)\text{Var}_{\delta Z}(\hat{\delta}_t) \right)
\]
\[
= \sum_{t=2}^{T} \left( \rho \text{Var}_{\delta Z}(\hat{\lambda}_t) + (1 - \rho)\text{Var}_{\delta Z}(\hat{\delta}_t) \right)
\]
since the bias is zero under complete randomization. We therefore have (as in the proof of Lemma 4):
\[
r(\hat{\eta}; Y) = \sum_{t=2}^{T} \left[ V_{1(t)}(Z) \rho \sum_{Z \in \mathcal{Z}(\mathcal{E})} \frac{\eta(Z)}{N_1(Z)} + V_0(t) \sum_{Z \in \mathcal{Z}(\mathcal{E})} (1 - \rho) \frac{\eta(Z)}{N_{e1}(Z)} + V_{t(t)} \sum_{Z \in \mathcal{Z}(\mathcal{E})} (\rho + 1 - \rho) \frac{\eta(Z)}{N_{e1}(Z)} - \frac{V_{0,e1}}{N} - \frac{V_{1,e1}}{N} \right]
\]
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and it follows that:

\[
\max_{V \in \mathcal{V}(\mathcal{Z})} \frac{\sum_{t \in \mathcal{Z}^{(t)}} \left[ \phi T - 1 \frac{T - 1}{N_t(Z)} + (1 - \rho) \sum_{t' = 2}^T 1 \frac{1}{N_{t'}(Z)} + \sum_{t = 2}^T \frac{1}{N_t(Z)} \right] }{N_t - N_0 - \sum_{t' = 2}^T N_{t'}}
\]

which concludes the proof.

Proof of Theorem 3. The proof follows exactly the lines of the proof of Theorem 1 with the modified lemmas.

C.2 Proof of Proposition 3

The proof is similar to that of Proposition 2, so we focus on the parts that change:

Proof of Proposition 3. Let

\[
\phi(N_0, N_{e_2}, \ldots, N_{e_T}) = \rho \frac{T - 1}{N - N_0 - \sum_{t' = 2}^T N_{e_{t'}}} + \sum_{t' = 2}^T \frac{1}{N_{e_{t'}}} + (1 - \rho) \frac{1}{N_0 + \sum_{t'' = 2}^T \mathbf{1} \{ t'' > t \} N_{e_{t''}}}
\]

where we use \( \phi \) instead of \( \rho \) for the objective function, since \( \rho \) is already used to denote the weight in this section. Now notice that:

\[
\frac{d\phi}{dN_{e_t}} = \rho \frac{T - 1}{(N - N_0 - \sum_{t' = 2}^T N_{e_{t'}})^2} - \frac{1}{N_{e_t}^2} - \sum_{t' = 2}^T \frac{1}{N_{e_{t'}}} \mathbf{1} \{ t' < t \} \frac{(1 - \rho)}{(N_0 + \sum_{t'' = 2}^T \mathbf{1} \{ t'' > t \} N_{e_{t''}})^2}
\]

and

\[
\frac{d\phi}{dN_{e_{t-1}}} = \rho \frac{T - 1}{(N - N_0 - \sum_{t' = 2}^T N_{e_{t'}})^2} - \frac{1}{N_{e_{t-1}}^2} - \sum_{t' = 2}^T \frac{1}{N_{e_{t'}}} \mathbf{1} \{ t' < t - 1 \} \frac{(1 - \rho)}{(N_0 + \sum_{t'' = 2}^T \mathbf{1} \{ t'' > t - 1 \} N_{e_{t''}})^2}
\]

and so:

\[
\frac{d\phi}{dN_{e_t}} = \frac{d\phi}{dN_{e_{t-1}}} = 0 \iff \frac{d\phi}{dN_{e_t}} - \frac{d\phi}{dN_{e_{t-1}}} = 0
\]

\[
\iff \frac{1}{N_{e_{t-1}}^2} - \frac{1}{N_{e_t}^2} - \frac{1}{N_{e_{t-1}}} - \frac{1}{N_{e_t}} \mathbf{1} \{ t' < t - 1 \} \frac{(1 - \rho)}{(N_0 + \sum_{t'' = 2}^T \mathbf{1} \{ t'' > t - 1 \} N_{e_{t''}})^2} = 0
\]

\[
\iff \frac{1}{N_{e_t}^2} = \frac{1}{N_{e_{t-1}}^2} - \frac{1}{N_{e_{t-1}}} - \frac{1}{N_{e_t}} \mathbf{1} \{ t' < t - 1 \} \frac{(1 - \rho)}{(N_0 + \sum_{t'' = 2}^T \mathbf{1} \{ t'' > t - 1 \} N_{e_{t''}})^2}.
\]

By telescoping the sums, we have

\[
\frac{1}{N_{e_t}^2} = \frac{1}{N_{e_{t-1}}^2} - \frac{1}{N_{e_{t-1}}} - \sum_{t' = 2}^{T-1} \frac{1}{(N_0 + \sum_{t'' = 2}^T \mathbf{1} \{ t'' > t \} N_{e_{t''}})^2}
\]

In addition, we have:

\[
\frac{dp}{dN_0} = \rho \frac{T - 1}{[N - N_0 - \sum_{t' = 2}^T N_{e_{t'}}]^2} - (1 - \rho) \frac{T - 1}{[N_0 + \sum_{t'' = 2}^T \mathbf{1} \{ t'' > t \} N_{e_{t''}}]^2}
\]

\[
\frac{dp}{dN_{e_2}} = \rho \frac{T - 1}{[N - N_0 - \sum_{t' = 2}^T N_{e_{t'}}]^2} - \frac{1}{N_{e_2}^2}
\]
Combining with the previous equation, as in the proof of Proposition 2 yields:

\[
\frac{1}{N_{eT}^2} = (1 - \rho) \frac{1}{N_0^2}
\]

and so \(N_{eT} = N_0 \ell c_T\) where \(\ell = (1 - \rho)^{-1/2}\) and \(c_T = 1\). Plugging this into the recursive definition of \(N_{eT}\), starting from \(N_{eT}\) we get

\[
\frac{1}{N_{eT-1}^2} = \frac{1}{N_{eT}^2} + \frac{(1 - \rho)}{(N_0 + N_{eT})^2}
\]

which implies:

\[
N_{eT-1} = N_0 (1 - \rho)^{-1/2} \left[ \frac{1}{(1 - \rho) \ell^2 c_T^2} + \frac{1}{(1 + \ell c_T)^2} \right]^{-1/2}
\]

\[
= N_0 \ell \left[ \frac{1}{c_T^2} + \frac{1}{(1 + \ell c_T)^2} \right]^{-1/2}
\]

where \(c_{T-1} = \left[ \frac{1}{c_T^2} + \frac{1}{(1 + \ell c_T)^2} \right]^{-1/2}\). Now reasoning by recurrence on \(t\), assume that \(N_{e_{t'}} = N_0 \ell c_{t'}\) with \(c_{t'} = \left[ \frac{1}{c_{t+1}^2} + \frac{1}{(1 + \ell \sum_{t'=t+1}^t c_{t'})^2} \right]^{-1/2}\) for all \(t' > t\). We then have:

\[
\frac{1}{N_{e_{t-1}}^2} = \frac{1}{N_{e_t}^2} + (1 - \rho) \frac{1}{(N_0 + \sum_{t'=2}^T \mathbb{1}\{t' > t - 1\} N_{e_{t'}}) (N_0 + \sum_{t'=2}^T \mathbb{1}\{t' > t - 1\} N_{e_{t'}})^2}
\]

\[
= \frac{1}{N_0^2 (1 - \rho)} \left[ \frac{1}{(1 - \rho) \ell^2 c_t^2} + \frac{1}{(1 + \sum_{t'=2}^T \mathbb{1}\{t' > t - 1\} \ell c_{t'})^2} \right]
\]

\[
= \frac{1}{N_0^2 (1 - \rho)} \left[ \frac{1}{c_t^2} + \frac{1}{(1 + \sum_{t'=2}^T \mathbb{1}\{t' > t - 1\} \ell c_{t'})^2} \right]
\]

and therefore:

\[
N_{e_{t-1}}^2 = N_0 (1 - \rho)^{-1/2} \left[ \frac{1}{c_t^2} + \frac{1}{(1 + \sum_{t'=2}^T \mathbb{1}\{t' > t - 1\} \ell c_{t'})^2} \right]^{-1/2}
\]

\[
= N_0 \ell c_{t-1}.
\]

This proves that \(N_{e_t} = N_0 \ell c_t\) for all \(t = 2, \ldots, T\). Now since all units are assigned to exactly one arm, we
have:

\[ N_0 + N_1 + \sum_{t=2}^T N_{e_t} = N \quad \Leftrightarrow \quad N_1 = N - N_0 - \sum_{t=2}^T N_{e_t} \]

\[ \Leftrightarrow \quad N_1 = N - N_0 \left[ 1 + \ell \sum_{t=2}^T c_t \right] . \]

The last step is to obtain an expression for \( N_0 \) as a function of \( \{c_t\}_{t=2}^T \) and \( \ell \). Notice that:

\[ \frac{d\rho}{dN_{e_2}} = 0 \quad \Leftrightarrow \quad \rho \frac{T - 1}{(N - N_0 - \sum_{t'=2}^T N_{e_{t'}})^2} - \frac{1}{N_{e_2}^2} = 0 \]

\[ \Leftrightarrow \quad N_{e_2} = [(T - 1)\rho]^{-1/2} \left[ N - N_0 (1 + \ell \sum_{t'=2}^T c_{t'}) \right] \]

\[ \Leftrightarrow \quad [(T - 1)\rho]^{1/2} N_{e_2} = N - N_0 (1 + \ell \sum_{t'=2}^T c_{t'}) \]

\[ \Leftrightarrow \quad [(T - 1)\rho]^{1/2} N_0 \ell c_2 + N_0 (1 + \ell \sum_{t'=2}^T c_{t'}) = N \]

\[ \Leftrightarrow \quad N_0 \left[ 1 + [(T - 1)\rho]^{1/2} \ell c_2 + \ell \sum_{t'=2}^T c_{t'} \right] = N \]

\[ \Leftrightarrow \quad N_0 = N \left[ 1 + [(T - 1)\rho]^{1/2} \ell c_2 + \ell \sum_{t'=2}^T c_{t'} \right]^{-1} \]

\[ \Leftrightarrow \quad N_0 = N \left[ 1 + ((T - 1)\rho)^{1/2} + 1 \right] \ell c_2 + \ell \sum_{t'=3}^T c_{t'} \right]^{-1} \]

which completes the proof.

\[ \square \]

**D Proof of results in Section 4.2**

The proof of Theorem 4 follows the same lines as that of Theorem 1, with a few modifications. We first state and prove an analog to Lemma 1. Lemmas 2 and 3 carry through unchanged (they only depend on the invariance of the loss). We then prove a slightly modified version of Lemma 4. The proof of Theorem 4 follows directly from these modified lemmas. In this section, we use the loss function:

\[ L(Z, Y) = \sum_{t=2}^T (\hat{\lambda}_t - \lambda_t)^2 + \sum_{t=2}^T (\hat{\beta}_t - \delta_t)^2 \]

where:

\[ \hat{\beta}_t = \frac{1}{N_{e_t}} \sum_i^N 1(Z_i = e_t) Y_{it}(e_t) - \frac{1}{N_{tk}} \sum_i^N 1(i \in C^{(t)}_k) Y_{it}(Z_i) \]

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Lemma 7. Under Assumption 3, \( \hat{\beta}_t \) can be rewritten:

\[
\hat{\beta}_t = \frac{1}{N_{ct}} \sum_{i} N_i \mathbf{1}(Z_i = e_t) Y_{it}(e_t) - \frac{1}{N_k} \sum_{i} N(i \in C_k^{(t)}) Y_{it}(0).
\]

We now state a slightly modified version of Lemma 1.

**Lemma 6 (Analog to Lemma 1).** Let \( \pi \in S_N \), \( Z \) and \( Y \). Then:

\[
L(\pi \cdot Z, \pi \cdot Y) = L(Z, Y)
\]

**Proof.** The only element that is changed from Lemma 1 is the use of \( \hat{\beta}_t \) instead of \( \hat{\beta}_t \). More specifically, under Assumption 3, if we let:

\[
\hat{\beta}_t^{(e_t)}(Z, Y) = \frac{1}{N_{ct}} \sum_{i} N_i \mathbf{1}(Z_i = e_t) Y_{it}(e_t); \quad \hat{\beta}_t^{(k)}(Z, Y) = \frac{1}{N_{tk}} \sum_{i} N(i \in C_k^{(t)}) Y_{it}(0)
\]

and \( \hat{\beta}_t = \hat{\beta}_t^{(e_t)}(Z, Y) - \hat{\beta}_t^{(k)}(Z, Y) \), all we need to show is that \( \hat{\beta}_t^{(k)}(\pi \cdot Z, \pi \cdot Y) = \hat{\beta}_t^{(k)}(Z, Y) \) for all \( \pi \) and all \( t = 2, \ldots, T \), since this is the only new term.

First, notice that:

\[
i \in C_k^{(t)}(\pi \cdot Z) \iff (\pi \cdot Z)_i = 0 \text{ or } (\pi \cdot Z)_i = e_{t'} \forall t' \in T^{(t)}
\]

\[
\iff Z_{\pi^{-1}(i)} = 0 \text{ or } Z_{\pi^{-1}(i)} = e_{t'} \forall t' \in T_k^{(t)}
\]

\[
\iff \pi^{-1}(i) \in C_k^{(t)}(Z).
\]

This implies that \( 1(i \in C_k^{(t)}(\pi \cdot Z)) = 1(\pi^{-1}(i) \in C_k^{(t)}) \). Moreover, since \( \pi \) is a permutation, this also implies that:

\[
N_{tk}(Z) = |C_k^{(t)}(Z)| = |C_k^{(t)}(\pi \cdot Z)| = N_{tk}(\pi \cdot Z)
\]

and therefore:

\[
\hat{\beta}_t^{(k)}(\pi \cdot Z, \pi \cdot Y) = \frac{1}{N_{tk}(\pi \cdot Z)} \sum_{i \in I} 1(i \in C_k^{(t)}(\pi \cdot Z))(\pi \cdot Y(0))^{(t)}
\]

\[
= \frac{1}{N_{tk}(Z)} \sum_{i \in I} 1(\pi^{-1}(i) \in C_k^{(t)}(Z)) Y_{\pi^{-1}(i)}^{(t)}(0)
\]

\[
= \frac{1}{N_{tk}(Z)} \sum_{j \in \pi^{-1}I} 1(j \in C_k^{(t)}(Z)) Y_{j}^{(t)}(0)
\]

\[
= \hat{\beta}_t^{(k)}(Z, Y).
\]

\[\square\]

**Lemma 7.** [Analogous to Lemma 4] Let \( \eta \in H \) and \( \tilde{Y}(Y) \) where \( Y \) is bounded and permutation-invariant, then:

\[
\max_{Y \in \mathcal{Y}(Y)} \{ r(\eta; Y) \} = V^* \sum_{Z \in Z(E)} \left( \frac{T - 1}{N_1(Z)} + \frac{1}{N_{tk}(Z)} + 2 \frac{T}{N_2(Z)} \right)
\]

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where

\[
V^* = \max_{x \in Y} \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2
\]

Proof. We only discuss the modifications that need to be made to the proof of Lemma 4.

First, for \( t \), define \( \mathbf{0}^* = \{ \mathbf{0} \text{ or } \mathbf{e}_{t'} : t' \in T_k^{(t)} \} \). Since the treatment is completely randomized, then \( \mathbf{0}^* \) is also completely randomized \((N_{t_k} \text{ out of } N)\). This means that \( \text{Bias}_{\mathbf{0}^*} (\hat{\beta}_t, \delta_t) = 0 \), and:

\[
V_{\mathbf{0}^* Z} = V_{\mathbf{0}^* Z}(\mathbf{0}^*) + V_{\mathbf{0}^* Z}(\mathbf{e}_{t^*}) - V_{\mathbf{0}^* Z}(\mathbf{0}, \mathbf{e}_{t^*}^*).
\]

The rest of the proof follows the lines of the proof of Lemma 4.

Proof of Theorem 4. The proof follows exactly the lines of the proof of Theorem 1, using Lemmas 6 and 7 instead of Lemmas 1 and 4.

E Randomization-inference

In the randomization-based framework we adopt, the potential outcomes are considered fixed, and the only source of randomness is the random assignment \( Z \); in particular, inference is performed with respect to the design \( \eta \) used to randomize the assignment. The minimax optimal designs we obtain lend themselves to straightforward randomization-based inference since they are completely randomized experiments, and the estimator used are differences-in-means. In particular, if \( \eta^{opt} \) is a minimax optimal design as in Theorem 1, then \( \text{Bias}_{\eta^{opt}} (\hat{\lambda}_t, \lambda_t) = 0 \), and \( \text{Bias}_{\eta^{opt}} (\hat{\delta}_t, \delta_t) = 0 \), for \( t = 2, \ldots, T \). The usual variance formulas hold (Imbens and Rubin, 2015):

\[
\text{Var}_{\eta^{opt}} (\hat{\lambda}_t) = \frac{V_{\eta^{opt}}^{(t)}}{N_{\eta^{opt}}^{(t)}} + \frac{V_{\eta^{opt}}^{(t)}_{e_t}}{N_{\eta^{opt}}^{(t)}} - \frac{V_{\eta^{opt}}^{(t)}_{0,e_t}}{N_{\eta^{opt}}^{(t)}}, \quad \text{Var}_{\eta^{opt}} (\hat{\delta}_t) = \frac{V_{\eta^{opt}}^{(t)}}{N_{\eta^{opt}}^{(t)}} + \frac{V_{\eta^{opt}}^{(t)}_{e_t}}{N_{\eta^{opt}}^{(t)}} - \frac{V_{\eta^{opt}}^{(t)}_{0,e_t}}{N_{\eta^{opt}}^{(t)}} ; \quad \text{for } t = 2, \ldots, T
\]

where

\[
V_{\eta^{opt}}^{(t)} = \frac{1}{N-1} \sum_{i=1}^{N} \{ Y_{it}(h) - \bar{Y}_t(h) \}^2, \quad h = 1, 0, e_t
\]

\[
V_{\eta^{opt}}^{(t)}_{e_t} = \frac{1}{N-1} \sum_{i=1}^{N} \{ Y_{it}(h) - \bar{Y}_t(h) \} \{ Y_{it}(e_t) - \bar{Y}_t(e_t) \}, \quad h = 1, 0
\]

If \( \eta^{opt} \) is a minimax optimal design as in Theorem 2, similar results hold but with

\[
\text{Var}_{\eta^{opt}} (\hat{\gamma}_t) = \frac{V_{\eta^{opt}}^{(t)}}{N_{\eta^{opt}}^{(t)}} + \frac{V_{\eta^{opt}}^{(t)}_{e_t}}{N_{\eta^{opt}}^{(t)}} - \frac{V_{\eta^{opt}}^{(t)}_{0,e_t}}{N_{\eta^{opt}}^{(t)}}, \quad \text{for } t = 2, \ldots, T
\]

The variances of the estimators under the designs obtained in our other theorems can be obtained similarly. The standard conservative estimators of these variance elements can be obtained (Imbens and Rubin, 2015) and since \( \hat{\lambda}_t, \hat{\delta}_t \) and \( \hat{\gamma}_t \) are asymptotically normal under mild conditions, conservative confidence intervals can be constructed.