On Rigid, Hard and Soft Problems and Results in Arithmetic Geometry

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Abstract: Rigid, hard and soft problems and results in arithmetic geometry are presented. ”Soft” and ”hard” in our paper are limited to the framework of solutions of quadratic forms over rings of integers of local and global fields, the Hardy-Littlewood-Kloosterman method. Next we consider the notion of rigidity. In the framework we give review of some novel results in the area.

Key–Words: Diophantine equation, rigidity, Dirichlet character, ergodic method, hermition, Hardy-Littlewood-Kloosterman method, convolution of measures, group action, uniform rigidity, superrigidity

1 Introduction

We review some novel results and methods on rigidity. These include (but not exhaust) methods and results by H. Furstenberg, G.A. Margulis, G. D. Mostow, J. Bourgain, A. Furman, A. Lindenstrauss, S. Mozes, J. James, T. Koberda, K. Lindsey, C. Silva, P. Speh, A. Ioana, K. Kedlaya, J. Tuitman [1, 2, 3, 4], and others. M. Gromov[5] in his talk at the International Congress of Mathematicians in Berkeley have presented problems and results of soft and hard symplectic geometry. In this connection we will present some soft and hard problems and results in arithmetic geometry. ”Soft” and ”hard” in our talk are limited to the framework of solutions of algebraic equations over rings of integers of local and global fields and the elements of Hardy-Littlewood-Kloosterman methods.

It is well known that the projective space is rigid [6]. The set of integer solutions of a Diophantine equation is a hard or a rigid object. Lattices in spaces are rigid objects.

2 Sums of squares

Let \( \Lambda \) be a lattice in \( n \)-dimensional real euclidean space that is defined by congruences. Davenport, Mordell, Cassels and others used the lattices and Minkowski’s convex body theorem for proving results about existence of nontrivial solutions of some Diophantine equations. We will give examples below. Recall the case of positive quadratic forms. Let \( \tau \) be a complex number, \( \text{Im} \, \tau > 0, q = \exp \pi i \tau, \theta_3(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2} \) the Jacobi function. Denote by \( \mathbb{Z}^n \) the \( d \)-dimensional integer lattice. Let \( r_n(m) \) be the number of ways of writing \( m \) as a sum \( f(x_1 \cdots x_n) = f \) of \( n \) squares. Put \( \Theta_{\mathbb{Z}^n} = \theta_3(\tau)^n \).

2.1 Sums of two squares

Let \( p \equiv 1 \pmod{4} \). In the case there is the integer \( l \) such that \( l^2 + 1 \equiv 0 \pmod{p} \). The lattice \( \Lambda \) of pairs \((a, b)\) of integer numbers is defined by congruences \( a \equiv lb \pmod{p} \) and has determinant \( d(\Lambda) \leq p \). From this and Minkowski’s convex body theorem follow that every prime \( p \equiv 1 \pmod{4} \) is the sum of two squares.

Let \( \chi \) be the nontrivial Dirichlet character mod 4, integer \( m > 0 \). There is the well known

**Proposition 1** The number of integer solutions of the equation \( x_1^2 + x_2^2 = m \) is equal \( 4 \sum_{d|m} \chi(d) \).

In the framework of the function \( \Theta_{\mathbb{Z}^2} \) we have \( \Theta_{\mathbb{Z}^2} = \sum_{m=0}^{\infty} r_2(m) q^m \).

2.2 Sums of three squares

In the case and in the case \( n = 4 \) it is possible to use quaternions (hermitians) but for simplicity we will formulate the well known result by \( \Theta_{\mathbb{Z}^3} \) and \( r_3(m) \).
Proposition 2 \( \Theta_{Z^4} = \sum_{m=0}^{\infty} r_3(m)q^m \).

2.3 Sums of four squares

The quadratic form \( x_1^2 + x_2^2 + x_3^2 + x_4^2 \) represents all positive numbers (Lagrange). The number of solutions of the equation \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = m \), where \( m \) is a positive integer is given by Jacobi.

Proposition 3 The number of integer solutions of the equation \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = m \), where \( m \) is a positive integer is given by Jacobi.

\( \Theta_{Z^4} = \sum_{m=0}^{\infty} r_3(m)q^m \).

2.4 Sums of squares greater than four

Recall elements of Hardy-Littelwood-Kloosterman method in the case. This is valid also in the previous case \( n = 4 \). Consider a function of complex variable \( u, |u| < 1 \)

\[ \vartheta(f, u) = \sum_{x_1, \ldots, x_n = -\infty}^{\infty} u^{f(x_1, \ldots, x_n)} \]

Then the number \( r_n(m) \) of ways of writing \( m \) as a sum of \( n \) squares by Cauchy’s integral formula is given as

\[ r_n(m) = \frac{1}{2\pi i} \oint_{\Gamma} \vartheta(f, u)u^{-m-1}du \]

where \( \Gamma \) is the circle \( |u| = \exp(-\frac{1}{m}) \). We omit here the very important step of the dividing \( \Gamma \) into Farey-arcs.

3 Elements of history of rigidity

The history of rigidity is reflected in papers by A. Selberg, E. Calabi, E. Vesentini, A. Weil, H. Furstenberg, G. Mostow, G. A. Margulis and their colleagues \([7, 8, 9, 10, 11, 12]\). There are interesting surveys by D. Fisher \([13]\) and R. Spatzier \([14]\).

Let \( G \) be a finitely generated group, \( D \) a topological group, and \( h : G \to D \) a homomorphism. Follow to \([13]\)

Definition 4 Given a homomorphism \( h : G \to D \), it is said that \( h \) is locally rigid if any other homomorphism \( h' \) which is close to \( h \) is conjugate to \( h \) by a small element of \( D \).

Recall follow to \([8, 9]\) in framework of \([14]\) the Local Rigidity Theorem.

Theorem 5 Cocompact discrete subgroups \( H \) in semisimple Lie groups without compact nor \( SL(2, \mathbb{R}) \) nor \( SL(2, \mathbb{C}) \) local factors is deformation rigid.

The notion of uniform rigidity was introduced as a topological version of rigidity by S. Glasner and D. Maon \([15]\).

4 Uniformly rigid and measurable weak mixing

Authors of the paper \([4]\) investigate properties of uniformly rigid transformations and analyze the compatibility of uniform rigidity and measurable weak mixing along with some of their asymptotic convergence properties.

This interesting survey includes some recent results on genericity of rigid and multiply recurrent infinite measure preserving and nonsingular transformations by O. Ageev and C. Silva \([16]\) and on measurable sensitivity by J. James, T. Koberda, K. Lendszy, C. Silva, P. Speh \([17]\). All spaces of the paper \([4]\) are considered simultaneously as topological spaces and as measure spaces. Presented results concern either the measurable dynamics on the spaces or the interplay between the measurable and topological dynamics. After some introductory section, second section of the paper \([4]\) considers functional analytic properties of uniform rigidity that is similar to the properties of rigidity. Authors formulate and prove

Theorem 6 Theorem 1. Every totally ergodic finite measure-preserving transformation on a Lebesgue space has a representation that is not uniformly rigid, except in the case where the space consists of a single atom.
The proof of the theorem connects with results of authors of the paper [4] that uniform rigidity and weak mixing are mutually exclusive notions on a Cantor set, and follows from the Jewett-Krieger Theorem by [18].

Third section concerns with uniform rigidity and measurable weak mixing. Author motivation for this topic is that a (nontrivial) measure-preserving weakly mixing transformation that is uniformly rigid would yield an example of a measurable sensitive transformation that is not strongly measurably sensitive. For a subset Y of a metric space X and a measurable transformation of X authors of the paper [4] define the notion of uniformly rigid transformation on Y and prove Theorem 3.4 that is reminiscent of Egorovs Theorem by P. Halmos [19]. In forth section authors present asymptotic convergence behavior. Let X be a compact metric space and let T be a finite measure-preserving ergodic transformation. Authors prove:

**Proposition 7** If T is uniformly rigid, then the uniform rigidity sequence has zero density.

The aim of section five is to study group action and generalized uniform rigidity. Let G be a countable group endowed with the discrete topology acting faithfully on a finite measure space by measure-preserving transformations. Following authors of the paper [4] the action of G is uniformly rigid if there exists a sequence \( \{g_i\} \) of group elements that leaves every compact \( K \subset G \), denoted \( g_i \to \infty \), such that \( d(x, g_i \cdot x) \to 0 \) uniformly. The main result of the section is Theorem 5.3:

**Theorem 8** Let X admit a weakly mixing group action and a uniformly rigid action by nontrivial subgroups of a fixed group of automorphisms G. Then there exists a G-action on X that is simultaneously weakly mixing and uniformly rigid.

Authors formulate several interesting questions that arise under investigations of weak mixing and uniform rigidity.

Some results and methods that are connected with topics of this and next section are considered in the paper [20].

5 Actions of groups and semigroups

Furstenberg and Berent investigate the action of abelian semigroups on the torus \( T^d \) for \( d = 1 \) and \( d > 1 \) respectively. The authors of the paper [2] extend to the noncommutative case some results of Furstenberg and Berent. Author’s results answer problems raising by H. Furstenberg [21] and by Y. Guivarc’h [private communication to authors of the paper [2]].

Let \( \nu \) be a probability measure on \( SL_d(\mathbb{Z}) \) satisfying the moment condition \( \mathbb{E}_\nu(\|g\|^p) < \infty \) for some \( p \). The authors of the paper [2] show that if the group generated by the support of \( \nu \) is large enough, in particular if this group is Zariski dense in \( SL_d \), for any irrational \( x \in T^d \) the probability measures \( \nu^{n \ast} \ast \delta_x \) tend to the uniform measure on \( T^d \). If in addition \( x \) is Diophantine generic, authors show this convergence is exponentially fast.

This interesting survey includes resent results on rigidity theory by M. Einsiedler, E. Lindenstrauss [22] and by G.A. Margulis [23], convolution of measures, on \( \nu \)-stiff action, on Fourier coefficients of measures and on notions of coarse dimension.

Let the action of semigroup \( \Gamma \) on \( T^d \) satisfy the following three conditions: (\( \Gamma \)−0) \( \Gamma \subset SL_d(\mathbb{R}) \), (\( \Gamma \)−1) \( \Gamma \) acts strongly irreducibly on \( \mathbb{R}^d \), (\( \Gamma \)−2) \( \Gamma \) contains a proximal element: there is some \( g \in \Gamma \) with a dominant eigenvalue which is a simple root of its characteristic polynomial.

In Section 1 authors formulate main result of the paper.

**Theorem 9** Let \( \Gamma < SL_d(\mathbb{R}) \) satisfy (\( \Gamma \)−0) and (\( \Gamma \)−2) above, and let \( \nu \) be a probability measure supported on a set of generators of \( \Gamma \) satisfying \( \sum_{g \in T} \nu(g) \|g\| < \infty \) for some \( \epsilon > 0 \). Then for any \( 0 < \lambda < \lambda_1(\nu) \) there is a constant \( C = C(\nu, \lambda) \) so that if for a point \( x \in T^d \) the measure \( \mu_n = \nu^{n \ast} \ast \delta_x \) satisfies that for some \( a \in \mathbb{Z}^d \setminus \{0\} \) \( \| \dot{\mu}_n(a) \| > t > 0 \), with \( n > C \cdot \log(\frac{2d}{\epsilon d}) \), then \( x \) admits a rational approximation \( p/q \) for \( p \in \mathbb{Z}^d \) and \( q \in \mathbb{Z}_+ \) satisfying \( \| x - \frac{p}{q} \| < \exp^{-\lambda n} \) and \( q \| < \left( \frac{2d}{\epsilon} \right)^C \).

Authors of [2] denote the theorem as Theorem A.

Section 2 is devoted to the deduction of two corollaries from Theorem A. Let in the corollaries \( \Gamma \) and \( \nu \) be as in theorem A.

**Corollary 10** Let \( x \in T^d \setminus (\mathbb{Q}/\mathbb{Z})^d \). Then the measures \( \mu_n = \nu^{n \ast} \ast \delta_x \) converge to the Haar measure \( m \) on \( T^d \) in weak-* topology.

This is authors [2] Corollary B. Next corollary is the authors [2] Corollary C:

**Corollary 11** Let \( x \in T^d \) and \( \mu_n = \nu^{n \ast} \ast \delta_x \). Then there are \( c_1, c_2 \) depending only on \( \nu \) so that the following holds: (1) Assume \( x \) is Diophantine generic in the sense that for some \( M \) and \( Q \) \( \| x - \frac{p}{q} \| > q^{-M} \) for all integers \( q \geq Q \) and \( p \in \mathbb{Z}^d \). Then for \( n > c_1 \log Q \max_{b \in \mathbb{Z}^d, \|b\| < B} | \dot{\mu}_n(b) | < B \exp^{-c_2 n/M} \). (2) Assume \( x \notin \mathbb{Q}/\mathbb{Z}^d \). Then there is a sequence \( n_i \to \infty \) along which \( \max_{b \in \mathbb{Z}^d, \|b\| < \exp^{c_2 n_i}} | \dot{\mu}_n(b) | < \exp^{-c_2 n_i} \).
Section 3 gives the deduction of authors’ solution of Furstenberg problem from the authors’ Proposition 3.1:

**Proposition 12** Let $\Gamma$ and $\nu$ be as in theorem A, $0 < \lambda < \lambda_1(\nu)$. Then for some constant $C$ depending on $\nu$, $\lambda$ the following holds: for any probability measure $\mu_0$ on $\mathbb{Z}^d$, if $\mu_n = \nu^\lambda \ast \mu_0$ has a nontrivial Fourier coefficient $a \in \mathbb{Z}^d \setminus \{0\}$, $|\hat{\mu}_n(a)| > t$, with $n > C \cdot \log(\frac{\lambda n}{t})$, then $\mu_0(W_{Q, \exp(-\lambda n)}) > \left(\frac{1}{2}\right)^C$ where $Q = (\frac{2n}{\lambda})^C$.

Theorem A follows from Proposition 3.1.

Section 4 is devoted to random matrix products. It includes estimates of the metric on $\mathbb{P}^{d-1}$ and random walks. In Section 5 two notions of coarse dimension are discussed. Section 6 describes the structure of the set of $t$–large Fourier coefficients. The last section "Granulated measures" gives the prove of Proposition 3.1.

The results of the paper will be of use to specialists interested in Diophantine approximation, measure theory and algebraic dynamics.

### 6 Rigid cohomology

Let $p$ be a prime, $n$ a positive integer, and $F_q$ the finite field with $\mathbb{F}_q$ elements. Let $Q_n$ denote the unique unramified extension of degree $n$ of the field of $p$-adic numbers. Let $U$ be an open dense subscheme of the projective space $\mathbb{P}^1_{Q_n}$ with nonempty complement $Z$. Let $V$ be the rigid analytic subspace of $\mathbb{P}^1_{Q_n}$ which is the complement of the union of the open disks of radius 1 around the points of $Z$. A Frobenius structure on $\mathcal{E}$ with respect to $\sigma$ is an isomorphism $\mathcal{F} : \sigma^* \mathcal{E} \simeq \mathcal{E}$ of vector bundles with connection defined on some strict neighborhood of $V$.

A meromorphic connection on $\mathbb{P}^1$ over a $p$-adic field admits a Frobenius structure defined over a suitable rigid analytic subspace. Authors of the paper[1] give an effective convergence bound for this Frobenius structure by studying the effect of changing the Frobenius lift. They also give an example indicating that their bound is optimal.

The techniques used are computational. This is a good place to see the interplay between matrix representation of a Frobenius structure and a Gauss-Manin connection.

The theory of rigid $p$-adic cohomology is developed by Berthelot[24] and others. Rigid cohomology in some sense extends crystalline cohomology. Review of some novel results and applications of crystalline cohomology is given in paper[25].

### 7 Superrigidity

The notion of property (T) for locally compact groups was defined by D. Kazhdan[26] and the notion of relative property (T) for inclusion of countable groups $\Gamma_0 \leq \Gamma$ was defined by G. Margulis[27].

The concept of superrigidity was introduced by G. D. Mostow[28] and by G. A. Margulis[29] in the context of studying the structure of lattices in rank one and higher rank Lie groups respectively. The first result on orbit equivalent (OE) super-rigid actions was obtained by A. Furman[30], who combined the cocycle superrigidity by R. Zimmer[31] with ideas from geometric group theory to show that the actions $SL_n(Z) \to T^n(n \geq 3)$ are OE superrigid. The deformable actions of rigid groups are OE superrigid by S. Popa[32].

The paper[3] presents a new class of orbit equivalent superrigid actions. The main result of the paper[3] is the Theorem A on orbit equivalence (OE) superrigidity. As a consequence of Theorem A the author can construct uncountable many non-OE profinite actions for the arithmetic groups $SL_n(Z)(n \geq 3)$, as well as for their finite subgroups, and for the groups that are semi direct products of groups $SL_m(Z)$ and $\mathbb{Z}^m(m \geq 2)$. The author deduces Theorem A as a consequence of the Theorem B on cocycle superrigidity.

Let $\Gamma \to X$ be a free ergodic measure-preserving profinite action (i.e., an inverse limit of actions $\Gamma \to X_n$ with $X_n$ finite) of a countable property (T) group $\Gamma$ (more generally, of a group $\Gamma$ which admits an infinite normal subgroup $\Gamma_0$) such that the inclusion $\Gamma_0 \subset \Gamma$ has relative property (T) and $\Gamma/\Gamma_0$ is finitely generated) on a standard probability space $X$. The author prove that if $\omega : \Gamma \times X \to \Lambda$ is a measurable cocycle with values in a countable group $\Lambda$, then $\omega$ is a cohomologous to a cocycle $\omega'$ which factors through the map $\Gamma \times X \to \Gamma / \Gamma_0 \times X_n$, for some $n$. As a corollary, he shows that any free ergodic measure-preserving action $\Lambda \to Y$ comes from a (virtual) conjugacy of actions.

### 8 Conclusion

Rigid, hard and soft problems and results in arithmetic geometry have presented. Diverse notions of rigidity and respective novel results are reviewed.
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