Anisotropic cosmological models with perfect fluid and dark energy

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We consider a self-consistent system of Bianchi type-I (BI) gravitational field and a binary mixture of perfect fluid and dark energy. The perfect fluid is taken to be the one obeying the usual equation of state, i.e., \( p = \zeta \epsilon \), with \( \zeta \in [0, 1] \) whereas, the dark energy density is considered to be either the quintessence or the Chaplygin gas. Exact solutions to the corresponding Einstein equations are obtained.

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I. INTRODUCTION

The description of the different phases of the Universe concerning the time evolution of its acceleration field is among the main objectives of the cosmological models. There is mounting evidence that the Universe at present is dominated by the so-called dark energy. Although the nature of the dark energy (DE) is currently unknown, it is felt that DE is non-baryonic in origin \([1]\). It is also believed that the dark energy has large negative pressure that leads to accelerated expansion of the Universe.

In view of its importance in explaining the observational cosmology many authors have considered cosmological models with dark energy. The simplest example of dark energy is a cosmological constant, introduced by Einstein in 1917 \([2]\). The discovery that the expansion of the Universe is accelerating \([3]\) has promoted the search for new types of matter that can behave like a cosmological constant \([4, 5]\) by combining positive energy density and negative pressure. This type of matter is often called quintessence. Zlatev et al. \([6]\) showed that ”tracker field”, a form of quintessence, may explain the coincidence, adding new motivation for the quintessence scenario.

An alternative model for the dark energy density was used by Kamenshchik et al. \([7]\), where the authors suggested the use of some perfect fluid but obeying ”exotic” equation of state. This type of matter is known as Chaplygin gas. The fate of density perturbations in a Universe dominated by the Chaplygin gas, which exhibit negative pressure was studied by Fabris et al. \([8]\). Model with Chaplygin gas was also studied in the Refs. \([9, 10]\). In a recent paper Kremer \([11]\) has modelled the Universe as a binary mixture whose constitutes are described by a van der Waals fluid and by a dark energy density. In doing so the authors considered mainly a spatially flat, homogeneous and isotropic Universe described by a Friedmann-Robertson-Walker (FRW) metric.

The theoretical arguments and recent experimental data, which support the existence of an anisotropic phase that approaches an isotropic one, lead to consider the models of Universe with
anisotropic back-ground. The simplest of anisotropic models, which nevertheless rather completely describe the anisotropic effects, are Bianchi type-I (BI) homogeneous models whose spatial sections are flat but the expansion or contraction rate is direction-dependent. Moreover, a BI universe falls within the general analysis of the singularity given by Belinskii et al. [12] and evolves into a FRW universe [13] in presence of a matter obeying the equation of state \( p = \zeta \varepsilon \), \( \zeta < 1 \).

Since the modern-day Universe is almost isotropic at large, this feature of the BI universe makes it a prime candidate for studying the possible effects of an anisotropy in the early Universe on present-day observations. In a number of papers, e.g., [14,15], we have studied the role of a nonlinear spinor and/or a scalar fields in the formation of an anisotropic Universe free from initial singularity. It was shown that for a suitable choice of nonlinearity and the sign of \( \Lambda \) term the model in question allows regular solutions and the Universe becomes isotropic in the process of evolution. Recently Khalatnikov et al. [16] studied the Einstein equations for a BI Universe in the presence of dust, stiff matter and cosmological constant. In a recent paper [17] the author studied a self-consistent system of Bianchi type-I (BI) gravitational field and a bimnary mixture of perfect fluid and dark energy given by a cosmological constant. The perfect fluid in that paper was chosen to be the one obeying either the usual equation of state, i.e., \( p = \zeta \varepsilon \), with \( \zeta \in [0, 1] \) or a van der Waals equation of state. In this paper we study the evolution of an initially anisotropic Universe given by a BI spacetime and a binary mixture of a perfect fluid obeying the equation of state \( p = \zeta \varepsilon \) and a dark energy given by either a quintessence or a Chaplygin gas.

II. BASIC EQUATIONS

The gravitational field in our case is given by a Bianchi type I (BI) metric in the form
\[
ds^2 = dt^2 - a^2 dx^2 - b^2 dy^2 - c^2 dz^2,
\]
with the metric functions \( a, b, c \) being the functions of time \( t \) only.

The Einstein field equations for the BI space-time we write in the form
\[
\begin{align*}
\frac{\ddot{b}}{b} + \frac{\dot{b}}{c} + \frac{\dot{c}}{bc} &= \kappa T_1^1, \\
\frac{\ddot{c}}{c} + \frac{\dot{c}}{a} + \frac{\dot{a}}{ca} &= \kappa T_2^2, \\
\frac{\ddot{a}}{a} + \frac{\dot{a}}{b} + \frac{\dot{b}}{ab} &= \kappa T_3^3, \\
\frac{\dot{a} \dot{b}}{ab} + \frac{\dot{b} \dot{c}}{bc} + \frac{\dot{c} \dot{a}}{ca} &= \kappa T_0^0.
\end{align*}
\]

Here \( \kappa \) is the Einstein gravitational constant and over-dot means differentiation with respect to \( t \).

The energy-momentum tensor of the source is given by
\[
T_{\mu}^\nu = (\varepsilon + p)u_\mu u_\nu - p\delta_\mu^\nu,
\]
where \( u^\mu \) is the flow vector satisfying
\[
g_{\mu\nu}u^\mu u^\nu = 1.
\]

Here \( \varepsilon \) is the total energy density of a perfect fluid and/or dark energy density, while \( p \) is the corresponding pressure. \( p \) and \( \varepsilon \) are related by an equation of state which will be studied below in
detail. In a co-moving system of coordinates from (2.3) one finds

\[ T^0_0 = \epsilon, \quad T^1_1 = T^2_2 = T^3_3 = -p. \]  \hspace{1cm} (2.5)

In view of (2.5) from (2.2) one immediately obtains \[14\]

\[ a(t) = D_1 \tau^{1/3} \exp[X_1 \int \frac{dt}{\tau(t)}], \]  \hspace{1cm} (2.6a)

\[ b(t) = D_2 \tau^{1/3} \exp[X_2 \int \frac{dt}{\tau(t)}], \]  \hspace{1cm} (2.6b)

\[ c(t) = D_3 \tau^{1/3} \exp[X_3 \int \frac{dt}{\tau(t)}]. \]  \hspace{1cm} (2.6c)

Here \( D_i \) and \( X_i \) are some arbitrary constants obeying

\[ D_1 D_2 D_3 = 1, \quad X_1 + X_2 + X_3 = 0, \]

and \( \tau \) is a function of \( t \) defined to be

\[ \tau = abc. \]  \hspace{1cm} (2.7)

From (2.2) for \( \tau \) one find

\[ \frac{\dot{\tau}}{\tau} = \frac{3\kappa}{2}(\epsilon - p). \]  \hspace{1cm} (2.8)

On the other hand the conservation law for the energy-momentum tensor gives

\[ \dot{\epsilon} = -\frac{\dot{\tau}}{\tau}(\epsilon + p). \]  \hspace{1cm} (2.9)

After a little manipulations from (2.8) and (2.9) we find

\[ \dot{\tau} = \pm \sqrt{C_1 + 3\kappa \epsilon \tau^2}, \]  \hspace{1cm} (2.10)

with \( C_1 \) being an integration constant. On the other hand rewriting (2.9) in the form

\[ \frac{\dot{\epsilon}}{(\epsilon + p)} = -\frac{\dot{\tau}}{\tau}, \]  \hspace{1cm} (2.11)

and taking into account that the pressure and the energy density obey a equation of state of type \( p = f(\epsilon) \), we conclude that \( \epsilon \) and \( p \), hence the right hand side of the Eq. (2.8) is a function of \( \tau \) only, i.e.,

\[ \tau = \frac{3\kappa}{2}(\epsilon - p) \tau \equiv \mathcal{F}(\tau). \]  \hspace{1cm} (2.12)

From the mechanical point of view Eq. (2.12) can be interpreted as an equation of motion of a single particle with unit mass under the force \( \mathcal{F}(\tau) \). Then the following first integral exists \[18\]:

\[ \tau = \sqrt{2\mathcal{E} - \mathcal{U}(\tau)}. \]  \hspace{1cm} (2.13)

Here \( \mathcal{E} \) can be viewed as energy and \( \mathcal{U}(\tau) \) is the potential of the force \( \mathcal{F} \). Comparing the Eqs. (2.10) and (2.13) one finds \( \mathcal{E} = C_1/2 \) and

\[ \mathcal{U}(\tau) = \frac{3}{2} \kappa \epsilon \tau^2. \]  \hspace{1cm} (2.14)
Finally, rearranging (2.10), we write the solution to the Eq. (2.8) in quadrature:

$$\int \frac{d\tau}{\sqrt{C_1 + 3K\epsilon \tau^2}} = t + t_0,$$

(2.15)

where the integration constant $t_0$ can be taken to be zero, since it only gives a shift in time.

In what follows we study the Eqs. (2.8) and (2.9) for perfect fluid and/or dark energy for different equations of state obeyed by the source fields.

### III. UNIVERSE AS A BINARY MIXTURE OF PERFECT FLUID AND DARK ENERGY

In this section we thoroughly study the evolution of the BI Universe filled with perfect fluid and dark energy in details. Taking into account that the energy density ($\epsilon$) and pressure ($p$) in this case comprise those of perfect fluid and dark energy, i.e.,

$$\epsilon = \epsilon_{pf} + \epsilon_{DE}, \quad p = p_{pf} + p_{DE}$$

the energy momentum tensor can be decomposed as

$$T^\nu_\mu = (\epsilon_{DE} + \epsilon_{pf} + p_{DE} + p_{pf})u^\mu u_\nu - (p_{DE} + p_{pf})\delta^\nu_\mu.$$  

(3.1)

In the above equation $\epsilon_{DE}$ is the dark energy density, $p_{DE}$ its pressure. We also use the notations $\epsilon_{pf}$ and $p_{pf}$ to denote the energy density and the pressure of the perfect fluid, respectively. Here we consider the case when the perfect fluid in question obeys the following equation of state

$$p_{pf} = \zeta \epsilon_{pf}.$$  

(3.2)

Here $\zeta$ is a constant and lies in the interval $\zeta \in [0, 1]$. Depending on its numerical value, $\zeta$ describes the following types of Universes [13]

- $\zeta = 0$, (dust Universe),
- $\zeta = 1/3$, (radiation Universe),
- $\zeta \in (1/3, 1)$, (hard Universes),
- $\zeta = 1$, (Zeldovich Universe or stiff matter).

(3.3a-3.3d)

In a comoving frame the conservation law of the energy momentum tensor leads to the balance equation for the energy density

$$\dot{\epsilon}_{DE} + \dot{\epsilon}_{pf} = -\frac{\dot{\tau}}{\tau} (\epsilon_{DE} + \epsilon_{pf} + p_{DE} + p_{pf}).$$  

(3.4)

The dark energy is supposed to interact with itself only and it is minimally coupled to the gravitational field. As a result the evolution equation for the energy density decouples from that of the perfect fluid, and from Eq. (3.4) we obtain two balance equations

$$\dot{\epsilon}_{DE} + \frac{\dot{\tau}}{\tau}(\epsilon_{DE} + p_{DE}) = 0,$$  

(3.5a)

$$\dot{\epsilon}_{pf} + \frac{\dot{\tau}}{\tau}(\epsilon_{pf} + p_{pf}) = 0.$$  

(3.5b)
In view of the Eq. (3.2) from (3.5b) one easily finds
\[
\varepsilon_{\text{pf}} = \varepsilon_0 / \tau^{(1+\zeta)}, \quad p_{\text{pf}} = \varepsilon_0 \zeta / \tau^{(1+\zeta)},
\]
where \(\varepsilon_0\) is the integration constants. In absence of the dark energy one immediately finds
\[
\tau = C t^{2/(1+\zeta)},
\]
with \(C\) being some integration constant. As one sees from (2.6), in absence of a \(\Lambda\) term, for \(\zeta < 1\) the initially anisotropic Universe eventually evolves into an isotropic FRW one, whereas, for \(\zeta = 1\), i.e., in case of stiff matter the isotropization does not take place.

In what follows we consider the case when the Universe is filled with the dark energy as well.

In Fig. 1 we have plotted the potentials when the Universe is filled with perfect fluid, perfect fluid plus quintessence and perfect fluid plus Chaplygin gas, respectively. The perfect fluid is given by a radiation. As one sees, these types of potentials allows only infinite motion, i.e., the Universe expands infinitely. The Fig. 2 shows the evolution of the BI Universe. The introduction of dark energy results in accelerated expansion of the Universe. The view of acceleration has been illustrated in Fig. 3.

The Fig. 4 shows the evolution of a BI Universe filled with a perfect fluid and a Chaplygin gas. Here "d", "r", "h" and "s" stand for dust, radiation, hard Universe and stiff matter, respectively. As one sees, even in case of a stiff matter the Universe expands rapid enough to evolve into an isotropic one.

A. Case with a quintessence

Let us consider the case when the dark energy is given by a quintessence. As it was mentioned earlier, a new type of matter, often known as quintessence, can behave like a cosmological constant and was constructed by combining positive energy density and negative pressure and obeys the equation of state
\[
p_q = w_q \varepsilon_q,
\]
FIG. 3: Acceleration of a BI Universe filled with a perfect fluid, perfect fluid plus quintessence and perfect fluid plus Chaplygin gas, respectively.

FIG. 4: Evolution of the BI Universe filled with a perfect fluid and Chaplygin gas.

where the constant $w_q$ varies between $-1$ and zero, i.e., $w_q \in [-1, 0]$. In account of (3.8) from (3.5a) one finds

$$
\varepsilon_q = \varepsilon_{0q}/\tau^{(1+w_q)}, \quad p_q = w_q\varepsilon_{0q}/\tau^{(1+w_q)},
$$

with $\varepsilon_{0q}$ being some integration constant.

Now the evolution equation for $\tau$ (2.8) can be written as

$$
\dot{\tau} = \frac{3\kappa}{2} \left( \frac{(1-\zeta)\varepsilon_0}{\tau^\zeta} + \frac{(1-w_q)\varepsilon_{0q}}{\tau^{w_q}} \right).
$$

(3.10)

As it was mentioned earlier the Eq. (3.10) admits exact solution that can be written in quadrature as

$$
\int \frac{d\tau}{\sqrt{C_1 + 3\kappa(\varepsilon_0 \tau^{(1-\zeta)} + \varepsilon_{0q} \tau^{(1-w_q)})}} = t + t_0.
$$

(3.11)

Here $t_0$ is a constant of integration that can be taken to be trivial.

B. Case with Chaplygin gas

Let us now consider the case when the dark energy is represented by a Chaplygin gas. We have already mentioned that the Chaplygin gas was suggested as an alternative model of dark energy with some exotic equation of state, namely

$$
p_c = -A/\varepsilon_c,
$$

(3.12)

with $A$ being a positive constant. In view of the Eq. (3.12) from (3.5a) one now obtains

$$
\varepsilon_c = \sqrt{\varepsilon_{0c}/\tau^2 + A}, \quad p_c = -A/\sqrt{\varepsilon_{0c}/\tau^2 + A},
$$

(3.13)

with $\varepsilon_{0c}$ being some integration constant.
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Proceeding analogously as in previous case for \( \tau \) we now have

\[
\dot{\tau} = \frac{3\kappa}{2} \left( \frac{(1-\zeta)e_0}{\tau^5} + \sqrt{e_{0c} + A\tau^2 + A/\sqrt{e_{0c} + A\tau^2}} \right). 
\] (3.14)

The corresponding solution in quadrature now has the forms:

\[
\int \frac{d\tau}{\sqrt{C_1 + 3\kappa(e_0\tau^{(1-\zeta)} + \sqrt{e_{0c}\tau^2 + A\tau^4})}} = t, 
\] (3.15)

where the second integration constant has been taken to be zero.

IV. CONCLUSION

A self-consistent system of BI gravitational field filled with a perfect fluid and a dark energy has been considered. The exact solutions to the corresponding field equations are obtained. The inclusion of the dark energy into the system gives rise to an accelerated expansion of the model. As a result the initial anisotropy of the model quickly dies away. Note that the introduction of the dark energy does not eliminate the initial singularity.

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