Parton densities beyond Perturbation Theory

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Abstract
I compute non-perturbative corrections to the kernel governing the evolution of non-singlet parton densities. The model used is QED in the limit of many charged particles. I find an ultraviolet renormalon corresponding to a term of order $Q^2/\Lambda^2$, where $\Lambda$ is the pole of the coupling constant. This term has a non-trivial dependence on the variable $x = -q^2/(2pq)$ and its coefficient scales as $x^3/(1-x)^2$ ($p$ is the momentum of the hadron and $q$ is the momentum transfer). An extrapolation of my results to QCD implies a breakdown of the parton model near the elastic region.

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1 Introduction

One of the main processes for the study of the hadron structure is deep inelastic scattering:

\[ l + H \rightarrow l + X \] (1)

where \( H \) is a hadron, \( l \) is a lepton and \( X \) is any hadronic final state.

In the parton model picture, the hadron is viewed as a gas of fast moving, real, collinear and non-interacting partons (quarks and gluons), which share the energy-momentum of the hadron. Perturbative QCD accounts for hard gluon radiation and produces logarithmic corrections to the naive parton model. The cross section \( \sigma \) can be written as the product of a short-distance ('hard') cross section times a parton density:

\[ \sigma(p,q) = \sigma_{\text{hard}}(xp,q) \cdot q(x,Q^2) \] (2)

where \( Q^2 = -q^2 > 0 \).

In perturbative QCD one can compute the hard cross section and the evolution of the parton densities from a scale \( Q^2 \) to a different scale \( Q'^2 \) \([1, 2]\). The parton densities at a given scale cannot be predicted and have to be determined experimentally.

There are corrections to the parton model which originate from the intrinsic transverse momenta of the partons, their virtualities, as well as from parton-parton interactions. These phenomena are all characterized by the hadronic scale \( \Lambda_{\text{QCD}} \sim 300 \text{ MeV} \). A kinematical analysis shows that taking these effects into account produces power-suppressed corrections to the factorized cross section (2), of the form \([3]\)

\[ \left( \frac{\Lambda_{\text{QCD}}^2}{Q^2} \right)^n, \quad \text{where} \quad n = 1, 2, 3, ..., k, ... \] (3)

It is crucial to have an estimate of these non-perturbative terms for a trustworthy application of perturbative QCD.

The occurrence of power suppressed corrections can also be demonstrated with the operator product expansion (OPE) \([4, 5]\). The cross-section of the process (1) is proportional to the imaginary part of the forward scattering amplitude

\[ \langle H | h_{\mu\nu} | H \rangle \] (4)
where

\[ h_{\mu\nu} = i \int d^4x e^{-iq\cdot x} T J_\mu(x)J^\dagger_\nu(0) \]  

(5)

is a hadronic tensor and \( J_\mu \) is an electromagnetic or a weak current. The tensor \( h_{\mu\nu} \) can be expanded in the limit,

\[ Q^2, p \cdot q \to \infty, \quad x = \text{constant}, \]  

(6)

in a tower of covariants of progressively higher twist \( T \) (\( T = \text{dimension of the operator - spin} \)):

\[ h_{\mu\nu} = \sum_i C_i(Q^2/\mu^2)O_i(\mu^2) \]  

(7)

where \( C_i(Q^2/\mu^2) \) are coefficient functions, \( O_i(\mu^2) \) are local operators and \( \mu \) is a subtraction point.

The lowest twist operators (\( T = 2 \))

\[ \bar{\psi}\gamma_\mu\lambda_\alpha D_\nu D_{\mu_1}D_{\mu_2}...D_{\mu_n}\psi - \text{traces}, \quad F_{\mu\rho}D_{\mu_1}D_{\mu_2}...D_{\mu_n}F^{\rho}_{\nu} - \text{traces} \]  

(8)

correspond to the parton model, while higher twist operators produce corrections of order \( 1/Q^2 \) (\( \lambda_\alpha \) is a matrix in flavor space).

Arguments based on parton kinematics or the OPE can provide only order of magnitude estimates of the terms (3). It is the purpose of this paper to give a semiquantitative estimate of the power suppressed corrections, based on the renormalon calculus. In general, there are power suppressed corrections to the evolution kernel and to the hard cross section and the observable effect is a convolution of both terms. We work however in the leading logarithmic approximation (see eq. (40)), in which the power suppressed corrections to the hard cross section are absent. The power suppressed corrections to the evolution kernel are instead non-vanishing and have a direct physical meaning.

The idea of the renormalon approach is that the complete perturbative series reflects non-perturbative properties of the process [6]. The chain of arguments is the following. The perturbative series is strongly divergent in all the known cases, i.e. it diverges for every non-zero value of the coupling constant. The coefficients have typically a factorial growth,

\[ |c_n| \sim n!. \]  

(9)
A meaning can be assigned to the perturbative series through a resummation of the series itself. This operation requires in general an arbitrary prescription, which introduces an ambiguity. The latter is a size of the intrinsic non-perturbative corrections.

Let us present a physical argument of why the investigation of the power suppressed corrections (3) requires the knowledge of the complete perturbative series. Consider a quantity that has a simple expansion in powers of \( \alpha_s(Q^2) \) (i.e. an infrared safe quantity). Since

\[
\alpha_s(Q^2) \sim \frac{1}{\log Q^2} \quad \text{for} \quad Q^2 \rightarrow \infty,
\]

any power of \( \alpha_s(Q^2) \) decays more slowly than any power of \( 1/Q^2 \):

\[
\left( \frac{1}{Q^2} \right)^m \ll \frac{1}{\log^n Q^2} \quad \text{for any positive } n \text{ and } m
\]

Therefore, it is consistent to look at the power suppressed corrections only after the inclusion of all the perturbative terms. The renormalon calculus cannot be applied to QCD because we are not able to compute the coefficients of any observable for any \( n \). The use of models cannot be avoided and various kinds of \( 1/N \) expansions have been used. We consider a \( 1/N \) expansion of QED, where \( N \) is the number of charged particles \( \bar{3} \). This model is indeed solvable and involves tree diagrams containing strings of self-energy corrections to the photon lines (see fig.1). Since QED contains massless vector quanta like QCD, collinear singularities are the same in both theories. Therefore, perturbative physics is the same as in QCD as far as charge flow effects are neglected.

Actually, QCD can be modelled by changing the sign of the beta function at the end of our QED-like computation \( \bar{3} \). In such a way the infrared properties of QCD are characterized by the perturbative growth of the coupling constant at small momentum transfer. Loosely speaking, we are dealing with tree diagrams in which the bare coupling is replaced by the QCD one-loop running coupling constant.

The paper is organized as follows. Section (2) contains a short review of the Borel transform technique and of the renormalon calculus. In section (3) we compute the input diagrams for the kernel \( P_{qq} \). The latter is derived in section (4). Section (5) contains a discussion of the results and section (6)
traces the conclusions. There are also two appendices. In appendix A the
technique to compute the integrals of the Borel transform is described, while
in appendix B we prove some properties of the planar gauge.

2 The Borel transform

The perturbative expansion of an observable $O$ in quantum field theory is
given by a series of the form

$$O(\alpha) = c + \sum_{n=0}^{\infty} c_n \alpha^{n+1}. \quad (12)$$

This series is strongly divergent in all known cases because the coefficients
have a factorial growth with $n$: $|c_n| \sim n!$.

As a 0-dimensional model of quantum field theory, we can consider the integral:

$$I_k(\alpha) = \int_{-\infty}^{+\infty} dx \ x^k e^{-x^2-\alpha x^4} \quad (13)$$

There is an instability for $\alpha < 0$ and the expansion in powers of $\alpha$ is divergent.

By explicit computation one finds $c_n = (-1)^{n+1} \Gamma(2n+k/2+5/2)/\Gamma(n+2)$.

For large $n$ there is a factorial-like growth of the coefficients with alternating
sign: $c_n \sim (-1)^{n+1} 2^{2n+k/2+3/2} e^{(k-1)/2}(n+1)!$.

The simplest interpretation of the expansion (12) is that of an asymptotic
series. It is sensitive to truncate the series at its smallest term, which is
a measure of the error. A more detailed analysis aims to reconstruct the
original function from the knowledge of the $c_n$'s. This is achieved by means
of the Borel transform technique. The Borel transform is defined by:

$$\tilde{O}(t) = c \delta(t) + \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n \quad (14)$$

We assume that the series on the right-hand side defines an analytic function
of $t$ in a neighborhood of $t = 0$, which can be analytically continued in all the
Borel plane. The observable $O$ can be reconstructed from the Borel trasform
by means of the formula:

$$O^B(\alpha) = \int_0^{\infty} dt \ e^{-t/\alpha} \tilde{O}(t) \quad (15)$$
If, for example, $c_n = (-1)^n n!$, the series $\sum_{n=0}^{\infty} c_n x^n$ has zero radius of convergence, while the Borel transform defines an analytic function in all the $t$-plane, given by $\tilde{O}(t) = 1/(1 + t)$. Since $\tilde{O}(t)$ has no singularities for $t \geq 0$, the observable can be reconstructed unambiguously by means of the inversion formula. In such a case we say that the perturbative series is Borel summable. On the contrary, if $c_n = n!$, the Borel transform is given by $\tilde{O}(t) = 1/(1 - t)$ and there is a singularity at $t = 1$. The integral in eq. (15) must be supplemented with a prescription to deal with the pole. The arbitrariness of the prescription reflects the intrinsic ambiguity in the reconstruction of $O(\alpha)$.

Let us discuss now how renormalons are related to power suppressed corrections in quantum field theory. Consider a Borel transform of the form:

$$\tilde{O}(t) = \frac{K}{t - n} + \text{regular function.}$$

The ambiguity in the observable is given by:

$$\delta O = K e^{-n/\alpha(Q^2)}$$

where we have taken the pole residue as a measure of the ambiguity. Expressing the coupling constant in terms of the relevant scales (dimensional transmutation),

$$\alpha(Q^2) = \frac{1}{-b_0 \log Q^2 / \Lambda^2},$$

we have:

$$\delta O = K \left( \frac{\Lambda^2}{Q^2} \right)^{-b_0 n},$$

where $b_0$ is the first coefficient of the $\beta$-function, defined including the factor $1/(4\pi)$. In QED, $b_0 = 1/(3\pi)$. The position of the renormalon pole determines the power of the non-perturbative correction, while the pole residue sets the coefficient of the correction. In general, the Borel transform has a collection of singularities along the real axis and the leading ambiguity is related to the pole on the positive axis closest to the origin.

Up to now, the existence of renormalons has not been proved in any realistic theory (such as QED, QCD, $\lambda\phi^4$, etc.). We consider QED at order $1/N$. In this model the existence of renormalons is related to the Landau pole, as
shown by the following intuitive argument. The amplitudes are given by loop integrals containing the one-loop running coupling constant \( \alpha \)

\[
\int \frac{d^2 k}{2\pi^2} \frac{1}{k^2} \frac{1}{(k + p_1)^2} \frac{1}{(k + p_2)^2} \ldots
\]

(20)

where \( p_1, p_2 \ldots \) denote external momenta. The coupling constant diverges at the Landau pole, producing a divergence of the integral. The latter has to be regularized, and this operation introduces ambiguities in the amplitude. There is also a more explicit way of understanding the relation between Landau pole and renormalons. Expanding the running coupling constant in powers of \( \alpha \) at a fixed scale \( \mu \),

\[
\alpha(k^2) = \frac{\alpha(\mu^2)}{1 - b_0 \alpha(\mu^2) \log(k^2/\mu^2)} = \sum_{n=0}^{\infty} (b_0 \log \frac{k^2}{\mu^2})^n \alpha(\mu^2)^{n+1},
\]

(21)

gives rise to a coefficient of \( \alpha(\mu^2)^{n+1} \) of the form

\[
\int_0^1 dx \ (\log \frac{1}{x})^n = n!
\]

(22)

There is a fixed sign factorial divergence resulting in renormalons in the Borel plane.

3 The computation

In this section we evaluate the input diagrams for the derivation of the evolution kernel \( P_{qq} \). The model is QED with \( N \) flavors, and we work at order \( 1/N \). Quarks are assumed massless and the incoming quark is taken on-shell, i.e. \( p^2 = 0 \). Soft singularities (infrared and collinear) are regulated by the Borel variable \( t \) itself. For \( t > 0 \) the gluon propagator is indeed less singular than in the usual case (see appendix A for a formal proof of the regulator). It is convenient to use the planar gauge with the gauge vector directed along the final quark direction. In this gauge collinear singularities decouple from the final quark leg and the only relevant diagram are the ladder ones (see...
The free photon propagator is given by:

\[ i \Delta_{\mu\nu}(k) = \frac{i S_{\mu\nu}(k)}{k^2 + i \epsilon} \]  

where

\[ S_{\mu\nu}(k) = -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{n \cdot k}, \]  

is the polarization sum, and \( n = xp + q \) is a light-like vector, \( n^2 = 0 \).

The lowest order diagram is given in fig.2, and is of order \( (1/N)^0 \). The rate is given by:

\[ w_0 = \frac{\pi}{2E(p)Q^2} \delta(1-x) \text{Tr}[x \hat{p} \gamma_\nu(x \hat{p} + \hat{q}) \gamma_\mu] \]  

where \( E(p) \) is the energy of the incoming quark.

The next order diagrams involve a single photon exchange, and are therefore of order \( \alpha = a/N \). Inserting bubbles into the photon line involves the coupling \( a = \alpha N \). Diagrams with multiple gluon emissions are of order \( \alpha^k = a^k/N^k \) with \( k \geq 2 \). We work at order \( 1/N \) and to all orders in \( a \).

This implies that we have to consider the diagram of fig.3, with the photon propagators containing strings of an arbitrary number of bubbles.

The proper self-energy at one-loop is given by:

\[ i \Pi_{\mu\nu}(k) = i(k_\mu k_\nu - g_{\mu\nu} k^2) \Pi(k^2) \]  

where

\[ \Pi(k^2) = -ab_0 \log \left( \frac{-k^2 - i\epsilon}{\mu^2} \right) \]  

in the momentum subtraction scheme.

The full photon propagator is therefore given by:

\[ i \Delta_{\mu\nu}(k) = \frac{i S_{\mu\nu}(k)}{k^2 + i \epsilon} + \frac{i S_{\mu\rho}(k)}{k^2 + i \epsilon} \frac{i \Pi^{\rho\sigma}(k) i S_{\sigma\nu}(k)}{k^2 + i \epsilon} + ... \]

\[ = \frac{i S_{\mu\nu}(k)}{[k^2 + i \epsilon] [1 + \Pi(k^2)]} \]

The inclusion of the bubbles modifies the free propagator by the factor \( 1/[1 + \Pi((p - k)^2)] \).
The diagrams involved at order $1/N$ are given in fig.4. For diagram (a) we have:

\[
M_a = e \bar{u}(k + q)\gamma_\mu \frac{k}{k^2 + i\epsilon} \gamma_\mu u(p) \frac{S^{\rho\nu}(p - k)}{1 + \Pi((p - k)^2)} \epsilon_\nu (p - k) \tag{29}
\]

The contribution of this diagram to the transition probability per unit time is given by:

\[
w_a = \frac{1}{4\pi NE(p)} \int d^4k \ Tr[k\gamma_\rho \gamma_\sigma \gamma_\mu (\hat{k} + \hat{q})\gamma_\nu] \times \times f(a) \delta^+((k + q)^2) \delta^+((p - k)^2) S^{\rho\sigma}(p - k) \tag{30}
\]

where $\delta^+(x^2) = \delta(x^2) \theta(x_0)$ and we defined

\[
f(a) = \frac{a}{1 + \Pi((p - k)^2)^2}. \tag{31}
\]

The dependence on $a$ is contained in this function, which can be expanded as

\[
f(a) = \sum_{n,k=0}^{\infty} \frac{(2n + k + 1)!}{(2n + 1)! k!} (-b_0^2 \pi^2)^n \left( b_0 \log \left( \frac{(p - k)^2}{\mu^2} \right) \right)^k a^{2n+k+1}. \tag{32}
\]

According to the definition in eq. (14), the Borel transform is given by:

\[
\tilde{f}(t) = \left( \cos \pi u - \frac{\sin \pi u}{\pi} \log \frac{(p - k)^2}{\mu^2} \right) \left( \frac{\mu^2}{(p - k)^2} \right)^u \tag{33}
\]

where $u = -b_0 t$ (notice a difference of sign with respect to the usual definition).

The Borel transform $\tilde{w}_a$ is computed substituting $\tilde{f}(t)$ in eq. (33) in the place of $f(a)$. Because of the $\delta^+((p - k)^2)$, $\tilde{w}_a$ vanishes for $u < 0$. By analytic continuation (we assume that the Borel transform is an analytic function), the Borel transform vanishes in all the Borel plane:

\[
\tilde{w}_a = 0 \tag{34}
\]

This result is somewhat paradoxical because it implies the vanishing of the contribution of the basic ladder diagram, which is known to produce the one-loop scaling violations in QCD. There exists however a physical explanation
to eq. (34). The dynamics in our model is controlled by the distribution of invariant masses \((p - k)^2\) flowing into the fermion bubbles. In diagram (a), we have that \((p - k)^2 = 0\), implying that any non-trivial contribution to the Borel transform is impossible.

The amplitude of diagram (b) is given by:

\[
M_b = e^2 \frac{\hat{k}}{k^2 + i\epsilon} \gamma_\mu \left( \frac{S_{\rho\sigma}(p - k)}{\pi(p - k - l)\gamma_\sigma v(l)} \right)
\]  

(35)

The partial rate is given by:

\[
w_b = \frac{1}{12\pi^2 N E(p)} \int d^4k \frac{S_{\rho\sigma}(p - k)}{(p - k)^2} \times \frac{1}{(k^2)^2} \text{Tr}[(\hat{k} + \hat{q})\gamma_\mu \hat{k}\gamma_\rho \hat{p}\gamma_\sigma \hat{k}\gamma_\nu] \delta^+((k + q)^2) \theta^+((p - k)^2)
\]

(36)

where \(\theta^+(x^2) = \theta(x^2)\theta(x_0)\) and we defined

\[
g(a) = \frac{a^2}{1 + \Pi((p - k)^2)^2}
\]

(37)

The phase-space integral over the final quark-antiquark pair has already been done.

The dependence of \(w_b\) on the coupling constant \(a\) is in

\[
g(a) = \sum_{n,k=0}^{\infty} \frac{(2n + k + 1)!}{(2n + 1)!k!} (-b_0^2\pi^2)^n \left( b_0 \log \frac{(p - k)^2}{\mu^2} \right)^k a^{2n+k+2}
\]

(38)

whose Borel transform is given by [3]:

\[
\tilde{g}(t) = -\frac{\sin \pi u}{\pi b_0} \left( \frac{\mu^2}{(p - k)^2} \right)^u
\]

(39)

To compute \(\tilde{w}_b\), we substitute the expression of \(\tilde{g}(t)\) in eq. (36) in the place of \(g(a)\).

In the framework of a \(1/N\) expansion, the leading log approximation (LLA) involves the resummation of all the terms of the form:

\[
\left( \frac{1}{N} \log \frac{Q^2}{\mu^2} \right)^k, \quad k = 0, 1, 2, 3..., n, ...
\]

(40)
In the evaluation of the kernel, only those terms in $\tilde{w}_b$ which diverge logarithmically at $u = 0$ have to be considered. This means that we need only to keep the terms in the trace proportional to $k^2$ or to $k \cdot p$. It is easy to see that $k \cdot p$ is of the same order as $k^2$. The collinear singularity is indeed produced by the virtual states of the photon with a small invariant mass, i.e. $(p - k)^2 \sim 0$, which implies $2p \cdot k \sim k^2$ (see appendix A for a rigorous proof). The terms proportional to $(k^2)^2$, $k^2 k \cdot p$, etc. do not give rise to any logarithmic singularity and therefore do not contribute to the evolution kernel (they contribute to the $O(1/N)$ hard cross section). Therefore, once a power of $k^2$ or $k \cdot p$ has been extracted, the trace can be evaluated in the collinear limit, $k^2 = 0$. This condition together with $(k + q)^2 = 0$, implies

$$k = xp.$$  \hspace{1cm} (41)

The rate is given by:

$$\tilde{w}_b = \frac{\sin \pi u \mu^2 u}{6\pi^2 b_0 N E(p)} \left[ \frac{1}{x} I_1 - I_2 + \frac{Q^2}{x} I_3 \right] \text{Tr}[x \hat{p}\gamma_\mu(x \hat{p} + \hat{q})\gamma_\mu] \hspace{1cm} (42)$$

where $Q^2 = -q^2 > 0$, and the integrals $I_1$, $I_2$ and $I_3$ are given by:

$$I_1 = \int d^4k \frac{\delta^+(k^2)}{k^2} \frac{\theta^+((p-k)^2)}{(p-k)^2)^{1+u}}$$

$$I_2 = \int d^4k \frac{\delta^+(k^2)}{(k^2)^2} \frac{\theta^+((p-k)^2)}{(p-k)^2)^{1+u}} 2p \cdot k$$

$$I_3 = \int d^4k \frac{\delta^+(k^2)}{k^2} \frac{\theta^+((p-k)^2)}{(p-k)^2)^{1+u}} n \cdot (p-k) \hspace{1cm} (43)$$

The final result is (the computation of the integrals is described in detail in appendix A):

$$\tilde{w}_b = \frac{\sin \pi u \mu^2 u}{12\pi^2 b_0 N E(p)} \text{Tr}[x \hat{p}\gamma_\mu(x \hat{p} + \hat{q})\gamma_\mu] \frac{x^{u+1}}{(Q^2)^{u+1}(1-x)^{u-1}} A(x, u) \hspace{1cm} (44)$$

where

$$A(x, u) = \frac{1}{xu^2} + \frac{1}{xu} - \frac{1}{u(u-1)} \left( 1 + u - \frac{1}{x} \right) F(1, 1-u, 2-u; 1-x) +$$

$$+ 2 \frac{\log x}{(1-x)^2} \frac{1}{u} F(1, -u, 1-u,-x) +$$

$$+ \frac{4}{(1-x)^2} \text{LerchPhi}(-x, 2, -u) + \frac{2}{(1-x)^2} h(x, u). \hspace{1cm} (45)$$
The special functions are defined in appendix A.

Let us end this section with some comment about the decay rate (44). The single pole at \( u = 0 \) is the collinear singularity and replaces, loosely speaking, the pole in \( \epsilon \) of usual dimensional regularization. Notice that \( A(x, u) \) has a double pole at \( u = 0 \), which is converted into a single pole by the factor \( \sin \pi u \) coming from the transform of \( g(a) \) (cf. eq. (39)). The softening of the singularity occurs because infrared singularities come from the integration over \( x \) around the elastic region. The complete box diagram involves instead the Borel transform of

\[
\frac{a}{1 + \Pi((p - k)^2)},
\]

given by

\[
\left( \frac{\mu^2}{(p - k)^2} \right)^u \text{ for } (p - k)^2 < 0
\]

(47)

and

\[
\left( \frac{\mu^2}{(p - k)^2} \right)^u e^{i\pi u} \text{ for } (p - k)^2 > 0
\]

(48)

In this case the factor \( \sin \pi u \) is absent and there is a double pole at \( u = 0 \) in the final result, coming from the product of the infrared and the collinear singularity.

4 The kernel

Collecting formulas (25), (34) and (44), we have for the total rate:

\[
\tilde{w} = \tilde{w}_b + \tilde{w}_a + \tilde{w}_b = \\
\frac{\pi}{2E(p) Q^2} \text{Tr}[x\hat{p}\gamma_\nu(x\hat{p} + \hat{q})\gamma_\mu] q(x, Q^2/\mu^2; t)
\]

(49)

where \( q(x, Q^2/\mu^2; t) \) is the parton density of a quark in the Borel plane,

\[
q(x, Q^2/\mu^2; t) = \delta(1 - x)\delta(t) - \frac{\sin \pi u}{6\pi^3 b_0 N} \frac{\mu^{2u}}{(Q^2)^u (1 - x)^{u-1}} A(x, u).
\]

(50)

The evolution kernel is computed by taking the derivative with respect to \( \log Q^2 \) of the parton density:

\[
P_{qq}(x, Q^2/\mu^2; t) = Q^2 \frac{\partial}{\partial Q^2} q(x, Q^2/\mu^2; t) =
\]
\[
\frac{1}{6\pi^3 b_0 N} \left( \frac{\mu^2}{Q^2} \right)^u \frac{x^{u+1}}{(1-x)^{u-1}} u \sin \pi u \ A(x, u) \tag{51}
\]

Notice that the evolution kernel in the coupling constant space and in the Borel space are computed in the same way, because taking the derivative with respect to \( \log Q^2 \) commutes with the Borel transform.

The factor \( u \sin \pi u \) cancels the single as well as the double pole at \( u = 0 \) in \( A(x, u) \). The evolution kernel is therefore finite at \( u = 0 \), as it should be for an infrared safe quantity. Notice that the parton density itself has a simple pole at \( u = 0 \), coming from the collinear singularity. The function \( u \sin \pi u \) has simple zeros at the integers \( u = 1, 2, 3, ..., n, ... \). The singularities of the Borel transform on the real axis originate therefore only from the term proportional to the LerchPhi function:

\[
P_{qq}(x, t) = \frac{2}{3\pi^3 b_0 N} \left( \frac{\mu^2}{Q^2} \right)^u \frac{x^{u+1}}{(1-x)^u} u \sin \pi u \ \text{LerchPhi}(-x, 2, -u) + \text{(regular function)} \tag{52}
\]

There are ultraviolet renormalons corresponding to simple poles at the integers \( u = 1, 2, 3, ..., n, ... \). The leading renormalon is at \( u = 1 \) and contributes to the evolution kernel by a term of the form:

\[
P_{qq}(x, t) = \frac{2}{3\pi^3 b_0 N} \frac{\mu^2}{Q^2} \frac{x^3}{(1-x)^2} \frac{1}{u-1} + \text{(higher order renormalons)} \tag{53}
\]

The leading non-perturbative correction coming from the ultraviolet renormalon at \( u = 1 \) is therefore given by:

\[
\delta P_{qq}(x, a) = -\frac{2}{3\pi^3 b_0^2 N} \frac{x^3}{(1-x)^2} \frac{\mu^2}{Q^2} e^{1/b_0 a} \tag{54}
\]

where we have taken the pole residue as a measure of the ambiguity.

We can have a qualitative picture of the ambiguity in QCD by changing the sign of the \( \beta \)-function, i.e. considering the case \( b_0 < 0 \):

\[
\delta P_{qq}(x, a) = -\frac{2}{3\pi^2 b_0^2 N} \frac{x^3}{(1-x)^2} \frac{\mu^2}{Q^2} e^{-1/|b_0|a(\mu^2)} + \text{(h.o. corrections)} = -\frac{2}{3\pi^2 b_0^2 N} \frac{x^3}{(1-x)^2} \frac{\Lambda_{QCD}^2}{Q^2} + \text{(h.o. corrections)} \tag{55}
\]

where by higher order corrections we mean terms of the form \( (\Lambda_{QCD}^2/Q^2)^n \) with \( n > 1 \).
5 Discussion

Eq. (55) is our main result. It says that the evolution kernel has corrections of order $1/Q^2$. The most interesting result is that the coefficient of the power correction is a function of the Bjorken variable $x$, and scales as

$$\frac{x^3}{(1-x)^2}.$$  \hspace{1cm} (56)

The behaviour (56) does depend on the dynamics of the model and cannot be derived with kinematical methods. Non-perturbative corrections grow with $x$ and diverge as $1/(1-x)^2$ for $x \to 1$, i.e. in the elastic region. This is a stronger divergence than that of the perturbative kernel (computed up to two loops [10]), given by $1/(1-x)$. There exists therefore a critical value

$$x_c \sim 1 - \frac{\Lambda_{QCD}^2}{Q^2}(57)$$

at which non-perturbative effects dominate over perturbative dynamics and the parton model becomes irrelevant. There is a consequent breakdown of the factorization in the cross section (2). We believe therefore that our analysis has not only an interest of principle but may have also phenomenological applications, identifying the domain of perturbative QCD. The values of $x_c$ are quite close to one for any reasonable value of $Q^2$. For example, $x_c \sim 0.97$ at $Q^2 = 3 \text{ GeV}^2$, while $x_c \sim 0.999$ at $Q^2 = 100 \text{ GeV}^2$.

On the contrary, non-perturbative corrections vanish as $x^3$ for $x \to 0$. They are therefore completely negligible with respect to the perturbative terms, which do not vanish in the small $x$ region. This conclusion however has to be taken with care because we believe that QED at order $1/N$ is not completely appropriate to describe small $x$ physics.

The qualitative features of (56) can be understood with the following qualitative considerations. As $x \to 1$, the invariant mass of the hadronic (partonic) final state and the typical virtuality of the gluon propagator $k^2$ reduce progressively. The effective coupling constant $\alpha_s(k^2)$ is therefore evaluated at a scale progressively closer to the pole: $k^2 \to \Lambda^2$. The result is that non-perturbative effects increase without bound. On the contrary, the invariant mass of the final state and the typical virtuality of the gluon propagator diverge as $x \to 0$. The coupling constant is evaluated at a scale progressively larger with respect to the pole, and non-perturbative effects tend to zero.
6 Conclusions

We analyzed non-perturbative effects in deep inelastic scattering of a lepton off a hadron. The idea is that the complete perturbative series hides non-perturbative properties of the process. Generally speaking, these corrections turn out to be small, thereby validating the consistency of the perturbative approach. I am also able to explain semiquantitatively such phenomena as the precox scaling, i.e. the scaling at quite small momentum transfer.

There is however a strong divergence of the non-perturbative terms near the elastic region. This implies a breakdown of the parton model, as it is conventionally understood.

We believe that our analysis can be extended to other non infrared-safe processes, such as $\gamma - \gamma$ or hadron-hadron collisions.

Acknowledgement

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A Borel transform integrals

In this appendix we describe the computation of the integrals $I_1$, $I_2$ and $I_3$, given in eqs. (43). We also prove that the Borel variable $u$ is a regulator of the soft divergences (i.e. infrared and collinear). For concreteness, let us consider $I_1$; the computation of $I_2$ and $I_3$ is analogous.

Making use of the identity

$$\frac{\theta^+((p - k)^2)}{((p - k)^2)^{1+u}} = \int_0^\infty \frac{d\mu^2}{(\mu^2)^{1+u}} \delta^+((p - k)^2 - \mu^2),$$

the integral can be written as

$$I_1 = \int_0^\infty \frac{d\mu^2}{(\mu^2)^{1+u}} J(\mu^2)$$

where

$$J(\mu^2) = \int d^4k \frac{\delta^+((k + q)^2) \delta^+((p - k)^2 - \mu^2)}{k^2}$$
$J(\mu^2)$ is a two body phase space integral with a collinear singularity regulated by the 'gluon mass' $\mu \neq 0$. The original integral $I_1$ involves an integration over $\mu^2$ with a weight function given by $(\mu^2)^{-u-1}$. By taking $u$ negative and large enough we can suppress the contribution of $J(\mu^2)$ at small $\mu^2$, where the singularity occurs, in such a way to render the integral finite. Since (as well known) the singularities are at most powers of logs, it is sufficient to take $u < 0$.

$J(\mu^2)$ is computed by changing the loop momentum to $k' = p - k$ and going in the reference frame in which $\vec{p} + \vec{q} = 0$ and $\vec{q}$ is directed along the $z$ axis. In this coordinate system $p_0 = |\vec{p}| = |\vec{q}| = q_s$ and $q_s \cdot k = -\vec{p} \cdot k = q_s k \cos \theta$. Integrating the massless delta function, the integral becomes:

$$J(\mu^2) = \int \frac{d^3k \delta^+((q_0 + q_s)^2 - 2(q_0 + q_s)k - \mu^2)}{q^2 - 2q_0k - 2q_s k \cos \theta}$$

(61)

Going to an adimensional momentum defined by $y = 2k/(q_0 + q_s)$ and integrating the massive $\delta$-functions, we have the result:

$$J(\mu^2) = \frac{\pi}{4q_s(q_0 + q_s)} \log \frac{(1-v)r}{2-(1+v)r}$$

(62)

where $v = q_0/q_s < 1$ and $r = \mu^2/(q_0 + q_s)^2$. The finite terms for $\mu^2 \to 0$ generate simple poles at $u = 0$ in $I_1$, while the logarithmic term generates a double pole.

We perform now the last integration over $\mu^2$ and express $q_0$ and $q_s$ in terms of $x$ and $Q^2$ according to the formulas

$$Q^2 = q_s^2 - q_0^2, \quad x = \frac{1}{2} \left( 1 - \frac{q_0}{q_s} \right)$$

(63)

whose inverse are

$$q_0 = \left( \frac{1}{2} - x \right) \left( \frac{Q^2}{x(1-x)} \right)^{1/2}, \quad q_s = \frac{1}{2} \left( \frac{Q^2}{x(1-x)} \right)^{1/2}$$

(64)

The result is:

$$I_1 = \frac{\pi}{2} \frac{x^{u+2}}{(Q^2)^{u+1}(1-x)^{u-1}} \left[ \frac{1}{x(1-x)} \frac{u^2}{u-1} F(1,1-u,2-u;1-x) \right]$$

(65)

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where \( F = {}_2F_1(a, b, c; z) \) is the hypergeometric function,

\[
F(a, b, c; z) = 1 + \frac{a b}{c \cdot 1} z + \frac{a(a + 1) b(b + 1)}{c(c + 1) \cdot 1 \cdot 2} z^2 + \ldots
\]  

(66)

The evaluation of \( I_2 \) is analogous to that one of \( I_1 \). To simplify the angular integration, it is convenient to write \( 2k \cdot p = k^2 - \mu^2 \). The integral splits as

\[
I_2 = I_1 + I_2'
\]  

(67)

where

\[
I_2' = -\int \frac{d\mu^2}{(\mu^2)^u} K(\mu^2)
\]  

(68)

and

\[
K(\mu^2) = \int d^4 k \frac{\delta^+((k + q)^2) \delta^+((p - k)^2 - \mu^2)}{(k^2)^2}
\]  

(69)

\( K(\mu^2) \) has a power singularity of the form \( 1/\mu^2 \) for \( \mu^2 \to 0 \) which, integrated together with the factor \((\mu^2)^{-u}\), generates a pole at \( u = 0 \):

\[
I_2' = \frac{\pi}{2} \frac{x^{u+1}}{(Q^2)^{u+1} (1 - x)^{u-1}} \left[ \frac{1}{x u} - \frac{1}{u - 1} F(1, 1 - u, 2 - u; 1 - x) \right]
\]  

(70)

The total result is:

\[
I_2 = \frac{\pi}{2} \frac{x^{u+1}}{(Q^2)^{u+1} (1 - x)^{u-1}} \left[ \frac{1}{(1 - x) u^2} + \frac{1}{x u} - \frac{u + 1}{u(u - 1)} F(1, 1 - u, 2 - u; 1 - x) \right]
\]  

(71)

\( I_2 \) has the same degree of singularity as \( I_1 \) at \( u = 0 \), given by simple and double poles. This implies that \( k^2 \) and \( 2k \cdot p \) have to be considered of the same order in evaluating the collinear singularities.

For \( I_3 \) we have the formula:

\[
I_3 = \frac{\pi}{2} \frac{x^{u+2}}{(Q^2)^{u+2} (1 - x)^{u-1}} \left[ -2 \frac{\log x}{(1 - x)^2 u} F(1, -u, 1 - u; -x) \right.
\]
\[
- \frac{4}{(1 - x)^2} \text{LerchPhi}(-x, 2, -u) - \frac{2}{(1 - x)^2} h(x, u) \right]
\]  

(72)
where \( \text{LerchPhi}(z, s, a) \) is the Lerch transcendent of \( \Phi \), defined by:

\[
\text{LerchPhi}(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(a + n)^s},
\]

(73)

with the terms for which \( a + n = 0 \) omitted, and \( h(x, u) \) is a function defined by the integral:

\[
h(x, u) = \int_0^1 dy \frac{\log[1 - (1 - x)y]}{y^{1+u}[1 + xy]}
\]

(74)

This function has simple poles at the integers:

\[
h(x, u) = (1 - x) \frac{1}{u - 1} + \frac{(1 - x)(1 - 3x)}{2} \frac{1}{u - 2} + \ldots
\]

(75)

\[\text{B Planar gauge}\]

In this appendix we prove that collinear singularities decouple from the final quark leg in the planar gauge (24), so that the only diagrams contributing to \( P_{qq} \) are box diagrams [11]. In the collinear limit,

\[
k = xp,
\]

(76)

the polarization sum becomes

\[
S_{\mu\nu}(p - k, n) = -g_{\mu\nu} + \frac{p_{\mu}(xp + q)_{\nu} + p_{\nu}(xp + q)_{\mu}}{p \cdot q} + \text{corrections of order } k^2.
\]

(77)

It is a projector orthogonal to the scattering hyperplane, i.e. the hyperplane spanned by \( p \) and \( xp + q \):\n
\[
p_\mu S^{\mu\nu}(p - k, n) = 0, \quad (xp + q)_\mu S^{\mu\nu}(p - k, n) = 0.
\]

(78)

The trace coming from the fermion line is a tensor \( T^{\mu\nu} \) lying in the scattering hyperplane in the limit (76): \( T^{\mu\nu} = T^{\mu\nu}(p, xp + q) \). Therefore, the contraction with \( S_{\mu\nu} \) vanishes:

\[
T^{\mu\nu} S_{\mu\nu} = 0 \quad \text{at } k^2 = 0.
\]

(79)
The diagrams with a photon radiated by the final quark contain a single quasi-real propagator, and the integrals are of the form:

$$\int \frac{d\mathbf{k}^2}{k^2} S_{\mu\nu} T_{\mu\nu}$$

(80)

Because of eq. (79), these diagrams do not contain collinear logs.

In the box-diagrams instead, there are two quasi-real quark propagators, and the integrals are of the form

$$\int \frac{d\mathbf{k}^2}{(k^2)^2} T'_{\mu\nu} S_{\mu\nu}.$$  

(81)

The terms in $T'_{\mu\nu} S_{\mu\nu}$ proportional to $k^2$ give rise to the collinear log. C.V.D.

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FIGURE CAPTIONS

Fig.1: bubble summation in the photon propagator;
Fig.2: lowest order diagram for the scattering of a quark in an external e.m. field;
Fig.3: diagrams of order $\alpha$ for the scattering of a quark in an external e.m. field;
Fig.4: Diagrams of order $1/N$ for the scattering of a quark in an external e.m. field.