CHARACTERS OF SOLVABLE GROUPS, HILBERT–SCHMIDT STABILITY AND DENSE PERIODIC MEASURES

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Abstract. We study the character theory of metabelian and polycyclic groups. It is used to investigate Hilbert–Schmidt stability via the character-theoretic criterion of Hadwin and Shulman. There is a close connection between stability and dynamics of automorphisms of compact abelian groups. Relying on this, we deduce that finitely generated virtually nilpotent groups, free metabelian groups, lamplighter groups as well as upper triangular groups over certain rings of algebraic integers are Hilbert–Schmidt stable.

1. Introduction

Characters play a key role in harmonic analysis on finite and on abelian groups. Thoma discovered a fruitful approach that allows to study characters of general infinite groups [Tho64]. It has since been developed and attracted considerable attention over the last two decades.

Definition. A trace on a discrete group $G$ is a positive-definite, conjugation-invariant function $\varphi : G \to \mathbb{C}$ normalized so that $\varphi(e) = 1$. A character of $G$ is a trace which is not a proper convex combination of traces.

If a group is abelian then its character space coincides with its Pontryagin dual. Other relevant examples of characters are the normalized traces $\frac{1}{n} (\text{tr} \circ \pi)$ where $\pi : G \to \text{U}(n)$ is any irreducible finite dimensional unitary representation.

A main part of this work is concerned with developing the character theory of solvable groups, focusing on the metabelian and polycyclic cases. A recurring theme is monomiality, namely all characters are induced from abelian (or virtually abelian) subgroups. Prior to discussing this theory in any greater detail, we mention the notion of stability where our results find a perhaps unexpected application.

Stability and dense periodic measures. A discrete group $G$ is called Hilbert–Schmidt stable if every “almost homomorphism” of $G$ into a finite-dimensional unitary group is “nearby” an actual homomorphism with respect to the Hilbert–Schmidt metric. The formal definition is outlined in §9. For a survey see [Lon21].

Hilbert–Schmidt stability provides a potential direction in the search for a non-hyperlinear group [Pes08, CLP15]. Quantitative versions of Hilbert–Schmidt stability play a crucial role in the proof of MIP* = RE and the refutation of the Connes embedding problem [Con76, JNV+21, dlS22].

In the realm of amenable groups there is an elegant and powerful criterion for Hilbert–Schmidt stability that relies on characters.

Theorem (Hadwin–Shulman [HS18]). An amenable group $G$ is Hilbert–Schmidt stable if and only if every trace on $G$ is a pointwise limit of normalized traces of finite dimensional unitary representations.
Hadwin and Shulman use their criterion to show that virtually abelian groups as well as the discrete Heisenberg group are Hilbert-Schmidt stable (see also [Gle10] for abelian groups). We are not aware of other Hilbert–Schmidt stability results for amenable groups in the literature, prior to this work.

The problem of determining whether this criterion holds for a given amenable group $G$ turns out to be intimately related to topological dynamics. To illustrate, assume that $G$ admits an abelian normal subgroup $N$. Any trace $\varphi$ on the group $G$ determines via the Bochner theorem a $G$-invariant Borel probability measure $\mu_\varphi$ on the Pontryagin dual $\hat{N}$. If the trace $\varphi$ is indeed a pointwise limit of finite-dimensional traces then the measure $\mu_\varphi$ is a weak-$\ast$ limit of $G$-invariant probability measures of finite support. We are led to introduce the following notion.

**Definition.** A topological dynamical system $(G, X)$ has **dense periodic measures** if every $G$-invariant Borel probability measure on the compact space $X$ is a weak-$\ast$ limit of $G$-invariant probability measures with finite supports.

For the most part we will assume that the space $X$ is a compact abelian group and that the group $G$ is acting on $X$ by continuous automorphisms. The above discussion suggests a relationship between Hilbert–Schmidt stability and the density of periodic measures for the action on certain Pontryagin dual groups.

**Proposition A.** Let $G$ be a Hilbert–Schmidt stable amenable group. Then for any abelian normal subgroup $N$ of $G$ the dynamical system $(G, \hat{N})$ has dense periodic measures.

To argue from the density of periodic measures to stability, we require a better understanding of the character theory of the group in question.

**Metabelian groups.** Our analysis of solvable groups depends in an essential way on **induced characters**, a notion introduced and developed in §3.

**Theorem B.** Let $G$ be a metabelian group. Then any character of $G$ is induced from the abelianization $H^{\text{ab}}$ of some normal subgroup $H \triangleleft G$.

Characters of particular metabelian groups were previously classified in [Gui63, BdlH20]. We deduce the following stability criterion.

**Theorem C.** Let $G$ be a finitely generated metabelian group. Assume that the topological dynamical system $(G, \hat{H}^{\text{ab}})$ has dense periodic measures for any normal subgroup $H \triangleleft G$. Then the group $G$ is Hilbert–Schmidt stable.

We find the question of determining which automorphisms of compact abelian groups have dense periodic measures to be very natural and interesting. Unfortunately we don’t know the answer to this question in complete generality.

Known classes of topological dynamical systems admitting dense periodic measures include Bernoulli shifts [Par61] as well as torus automorphisms which are ergodic with respect to the Haar measure [Mar80]. Loosely speaking, these systems enjoy a much stronger dynamical property called **periodic specification** [Bow71, Rue73, Lin79, Lin82, LS99, KLO16].

\[\text{Theorems 4.1 and 10.1 respectively are slightly more general formulations of Theorems 19 and 20.}\]
Corollary D. The following metabelian groups are Hilbert–Schmidt stable:

1. Finitely generated free metabelian groups.
2. Wreath products $A \wr \mathbb{Z}^d$ where $A$ is any finitely generated abelian group, e.g. the lamplighter groups.
3. The Baumslag–Solitar groups $BS(1,n)$ for all non-zero $n \in \mathbb{Z}$.
4. The semidirect products $\mathbb{Z} \ltimes_\alpha \mathbb{Z}^k$ where the automorphism $\alpha \in \text{GL}_k(\mathbb{Z})$ is ergodic with respect to the Haar measure on the torus.

The property of stability for the class of finitely generated metabelian groups has a dynamical counterpart.

Theorem E. The following two statements are equivalent:

1. All finitely generated metabelian groups are Hilbert–Schmidt stable.
2. Any topological dynamical system $(G,X)$ where $G$ is a finitely generated abelian group acting on a compact abelian group $X$ by automorphisms and satisfying the descending chain condition has dense periodic measures.

We are unable to determine whether these statements hold true in general. We remark that in statement (1) it suffices to consider only split metabelian groups.

For topological dynamical systems as the ones in Theorem E the descending chain condition (see p. [35]) implies that periodic points are dense [Sch12, Theorem 5.7]. Foregoing the descending chain condition, it is not hard to find dynamical systems without dense periodic measures. This leads to the following examples of infinitely generated metabelian groups that are not Hilbert–Schmidt stable in light of Proposition A.

Corollary F. Let $k$ be a non-Archimedean local field with ring of integers $\mathcal{O}$. Let $A$ be an infinite subgroup of $\mathcal{O}^*$. Then the group $G = A \ltimes \hat{\mathcal{O}}$ is not Hilbert–Schmidt stable.

For example, the group $\mathbb{Z} \ltimes \mathbb{Z}(p^{\infty})$ is not Hilbert–Schmidt stable for any pair of distinct primes $p$ and $q$ where $\mathbb{Z}(p^{\infty})$ is the Prüfer $p$-group and the $\mathbb{Z}$-action corresponds to multiplication by powers of $q$. This example demonstrates that Hilbert–Schmidt stability is not a local group property.

Polycyclic groups. Our main contribution to the character theory of solvable groups is Theorem G. This is, to the best of our knowledge, the first result dealing with characters of general polycyclic groups.

We need to introduce a few notions first. The FC-center of a group $G$ is the characteristic subgroup $FC(G)$ given by the union of the finite conjugacy classes. Given a virtually polycyclic group $G$ we let $vFit(G)$ denote the maximal virtually nilpotent normal subgroup of $G$. The characteristic subgroup $vFit(G)$ contains the Fitting subgroup $Fit(G)$ with finite index (see §6 for further details). Lastly, the kernel of a trace $\varphi$ is $\ker \varphi = \varphi^{-1}(1)$. This is always a normal subgroup.

Theorem G. Let $G$ be a virtually polycyclic group. For every character $\varphi \in \text{Ch}(G)$ there is a finite index subgroup $H$ of $G$ and a quotient $\overline{H}$ of $H$ such that the character $\varphi$ is induced from the subquotient $FC(vFit(H))$.

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Finitely generated amenable non-residually finite groups are never Hilbert–Schmidt stable [Dog21, Proposition 2.5], but this line of reasoning does not apply to infinitely generated groups.
This generalizes the classical theorem of Howe \cite{How}- any character of a finitely generated nilpotent group \( G \) is induced from \( \text{FC}(G/\ker \varphi) \).

Our proof of Theorem \( \text{G} \) relies on a statement reminiscent of Mackey’s theorem for induced characters that we find interesting in its own right (Theorem 3.3).

The leniency of being able to pass to a finite index subgroup as well as considering \( \text{vFit} \) rather than \( \text{Fit} \) and the FC-center rather than the center is necessary (even when dealing only with solvable groups).

In keeping with our philosophy and relying on Theorem \( \text{G} \) we get the following dynamical criterion for stability.

**Theorem H.** Let \( G \) be a virtually polycyclic group. Assume that any quotient \( L \) of any finite index subgroup of \( G \) acts on the Pontryagin dual of its subgroup \( \text{Z}(\text{vFit}(L)) \) with dense periodic measures. Then the group \( G \) is Hilbert–Schmidt stable.

The action of any group on the character space of its center is trivial and therefore has dense periodic orbits. We immediately deduce:

**Corollary I.** Any finitely generated virtually nilpotent group is Hilbert–Schmidt stable.

For any unipotent subgroup \( G \leq \text{GL}_k(\mathbb{Z}) \) the semidirect product \( G \rtimes \mathbb{Z}^k \) is a finitely generated nilpotent group. Therefore the dynamical system \( (G, \mathbb{T}^k) \) has dense periodic measures as a consequence of Corollary \( \text{I} \) and of Proposition \( \text{A} \) (the same can be derived from Ratner’s measure classification theorem \cite{Rat}).

Recall that a finitely generated group \( G \) has finite conjugacy classes if and only if \( [G : Z(G)] < \infty \) \cite[§15.1]{Sc}. Together with Theorem \( \text{I} \) this says that any character of a virtually polycyclic group is induced from some virtually abelian normal group. The dual topological dynamical system associated to this situation turns out to be topologically conjugate to an action on a torus, see §7. Taking all this into account allows us to deal with a particular family of polycyclic groups.

**Theorem J.** Let \( \mathcal{O} \) be the ring of algebraic integers of some number field \( k \). Assume that the group of units \( \mathcal{O}^* \) has rank one. Then the group of invertible upper triangular matrices over the ring \( \mathcal{O} \) is Hilbert–Schmidt stable.

Stability for the class of virtually polycyclic groups turns out to be equivalent to a dynamical property of torus automorphisms (note the analogy with Theorem \( \text{E} \)).

**Theorem K.** The following four statements are equivalent:

(1) All groups of the form \( \mathbb{Z}^d \rtimes \mathbb{Z}^k \) for some \( d, k \in \mathbb{N} \) are Hilbert–Schmidt stable.

(2) All virtually polycyclic group \( \mathbb{G} \) are Hilbert–Schmidt stable.

(3) The topological dynamical system \( (G, \mathbb{T}^k) \) has dense periodic measures for any abelian subgroup \( G \leq \text{GL}_k(\mathbb{Z}) \) and all \( k \geq 2 \).

(4) The topological dynamical system \( (G, \mathbb{T}^k) \) has dense periodic measures for any amenable subgroup \( G \leq \text{GL}_k(\mathbb{Z}) \) and all \( k \geq 2 \).

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\(^3\) Works on characters of nilpotent groups include \cite{Kan80, CMS, BKM, TM04, Kan06}.

\(^4\) Dirichlet’s unit theorem says that \( \text{rank}(\mathcal{O}^*) = 1 \) if and only if \( k \) is either a real quadratic field, a complex cubic field or a totally imaginary number field of degree 4.

\(^5\) The class of virtually polycyclic groups coincides with the class of finitely generated amenable subgroups of \( \text{GL}_k(\mathbb{Z}) \).
Surprisingly, to show that all virtually polycyclic groups are Hilbert–Schmidt stable it will be enough to consider the metabelian ones as in statement (1).

We do not know whether Statement (3) of Theorem K is true. For a single toral automorphism this seems likely — ergodic automorphisms are covered by the work of Marcus \([\text{Mar80}]\) and unipotent automorphisms were discussed after Corollary \([\text{I}]\). The case of multiple commuting automorphisms seems to be much deeper. It is closely related to the higher rank measure rigidity conjecture \([\text{Fur67}], \text{KS96} \text{Main Conjecture}], \text{Mar00 Conjecture 3}], \text{Lin21 Conjecture 4}]. Here is one explicit formulation of this conjecture.

**Conjecture (Mar00).** Let \(\alpha \) be an almost minimal \(\mathbb{Z}^d\)-action on the torus \(T^k\), i.e. \(T^k\) has no proper closed infinite \(\mathbb{Z}^d\)-invariant subsets. Then the only non-atomic invariant probability measure on \(T^k\) is the Haar measure.

Any \(\mathbb{Z}^d\)-action on the torus \(T^k\) that satisfies the measure rigidity conjecture clearly has dense periodic measures. Here is an example of a conditional application of the conjecture to get stability.

**Proposition L.** Assume that the measure rigidity conjecture holds true. Let \(k\) be a totally real number field with ring of algebraic integers \(\mathcal{O}\). Then the metabelian polycyclic group \(\mathcal{O}^* \ltimes \mathcal{O}\) is Hilbert–Schmidt stable.

Irreducible genuinely partially hyperbolic higher rank actions \([\text{KN11, §2.2.7}]\) are not covered by the measure rigidity conjecture, i.e. they admit plenty non-atomic invariant probability measures singular with respect to the Haar measure on the torus. See Question \([\text{I}]\) below for more details.

**Remarks.**

1. Not long after this work was announced, Eckhardt and Shulman \([\text{ES23}]\) announced their work which has a certain overlap with ours. For instance Eckhardt and Shulman also establish Hilbert–Schmidt stability of finitely generated nilpotent groups, albeit in a very different way.

2. The product of two Hilbert–Schmidt stable groups stays Hilbert–Schmidt stable provided one of the groups is amenable \([\text{IS21 Corollary D}].\) So any new example of a Hilbert–Schmidt stable amenable group provides non-amenable examples. Works on Hilbert–Schmidt stability of non-amenable groups include \([\text{Atk18], HS18], [\text{ISW20], GS21], Ioa21}.\) See also \([\text{DCOT19}, \text{AD22}]\) for a uniform version of Hilbert–Schmidt stability.

3. Parallel to Hilbert–Schmidt stability is stability in permutations \([\text{AP15}].\) where the target groups are taken to be finite symmetric rather than unitary ones. This property is related to sofic groups \([\text{Gro99], [Wci00], [Pes08], [GR09}].\) The Hadwin–Shulman criterion has a close parallel in that setting — a finitely generated amenable group \(G\) is stable in permutations if and only if any \(G\)-invariant Borel probability measure on its space of closed subgroups is a weak-* limit of \(G\)-invariant measures supported on finite index subgroups \([\text{BLT19}].\) This dynamical criterion was used to show that lamp-lighter groups are stable in permutations \([\text{LL19a}].\) All virtually polycyclic groups are stable in permutations \([\text{BLT19 Corollary 8.2}].\)

4. It is natural to investigate the connection between permutation and Hilbert–Schmidt stability. The preprint \([\text{Bur21}].\) shows that a weakly permutation stable group must be "weakly Hilbert–Schmidt stable". Recall that any
A finitely generated amenable group is weakly permutation stable in the sense of [AP15]. To the best of our understanding, our Hilbert–Schmidt stability results do not follow from the methods of [Bur21]. The recent paper [ES23] includes a first example of an amenable (indeed solvable) group which is Hilbert–Schmidt stable but is not permutation stable [ES23, Theorem 5.23].

An incomplete list of works dealing with the character theory of other groups includes [DM14, Bek07, PT16, TTD18, DM19, BF20, LL20, BBH21, Tho22]. In most known cases all characters are almost monomial, i.e. are induced from virtually abelian subgroups.

Open questions. Here are some open problems suggested by our work.

Question 1. Let $H$ be a finite index subgroup of $G$. Assume that the system $(G, X)$ has dense periodic measures. Is the same true for the system $(H, X)$?

The converse direction to Question 1 is true, see Lemma 8.2 below.

Question 2. Do all actions on the torus generated by a single automorphism admit dense periodic measures?

The answer to Question 2 is positive if the action is totally irreducible. The remaining case is essentially that of reducible transformations of “mixed type” (i.e. hyperbolic and unipotent). We find the following problem concerning genuinely partially hyperbolic higher rank $\mathbb{Z}^d$-action to be particularly intriguing.

Question 3. Let $A, B \in \text{GL}_6(\mathbb{Z})$ be the pair of partially hyperbolic commuting matrices from [KN11, Example 2.2.20]. The corresponding dynamical system $(\mathbb{Z}^2, T^6)$ is not almost minimal. Does it admit dense periodic measures?

Question 4. Consider the finitely generated metabelian group $\mathbb{Z}^2 \rtimes \mathbb{Z}[1/6]$ closely related to Furstenberg’s famous $\times 2, \times 3$ problem. Is it Hilbert–Schmidt stable? Does the topological dynamical system $(\mathbb{Z}^2, \hat{\mathbb{Z}}[1/6])$ admit dense periodic measures?

A vast generalization of Questions 2, 3 and 4 is the following.

Question 5. Are all finitely generated metabelian groups Hilbert–Schmidt stable? Equivalently, do all systems $(\mathbb{Z}^d, X)$ satisfying the descending chain condition have dense periodic measures?

The authors intend to address the following problem in a forthcoming work.

Question 6. Is every solvable minimax group almost monomial?

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2. Characters and trace representations

We define traces and characters and discuss their relation to trace representations via von Neumann algebras. We introduce the notions of relative and dominated traces. Let $G$ be a countable group. We use the notation $g^h = h^{-1}gh$ for any pair of elements $g, h \in G$. This makes conjugation a right action.

**Definition.** A trace on the group $G$ is a function $\varphi : G \to \mathbb{C}$ so that

1. $\varphi$ is positive definite, i.e.
   \[
   \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \varphi(g_i^{-1}g_j) \geq 0
   \]
   for all $n \in \mathbb{N}$ and any choice of elements $g_1, ..., g_n \in G$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$,
2. $\varphi$ is conjugation-invariant, i.e. $\varphi(g^h) = \varphi(g)$ for all elements $g, h \in G$, and
3. $\varphi$ is normalized, i.e. $\varphi(\epsilon) = 1$.

The kernel of a trace $\varphi$ on the group $G$ is the normal subgroup $\ker \varphi = \varphi^{-1}(1)$. We say that the trace $\varphi$ is faithful if the subgroup $\ker \varphi$ is trivial. Any trace $\varphi$ on the group $G$ descends to a faithful trace on the quotient group $G/\ker \varphi$ [BdlH20, Lemma 12.1].

Let $\text{Tr}(G)$ denote the space of traces on the group $G$ endowed with the topology of pointwise convergence. This makes $\text{Tr}(G)$ a compact convex subset of $C^\infty(G)$. Let $\text{Ch}(G)$ denote the set of extreme points of $\text{Tr}(G)$. The members of $\text{Ch}(G)$ are called characters.

It is known that $\text{Tr}(G)$ is a metrizable Choquet simplex [Tho64]. This means that the barycenter map

\[
\text{Prob}(\text{Ch}(G)) \to \text{Tr}(G), \quad \mu \mapsto \varphi = \int_{\text{Ch}(G)} \psi \, d\mu(\psi)
\]

is a continuous affine bijection. We refer to this map as the Fourier transform. See [Phe01] for an overview of Choquet theory.

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6Some authors refer to traces as “characters” and to extreme traces as “indecomposable characters”.
If the discrete group $G$ is abelian then any character of $G$ is multiplicative (see Lemma 7.1). Therefore the character space $\text{Ch}(G)$ coincides with the compact Pontryagin dual group $\hat{G}$. The fact that the barycenter map in Equation (2.1) is a bijection in this case is known as Bochner’s theorem [Fol16]. As the space $\text{Prob}(\text{Ch}(G))$ is compact the barycenter map is a homeomorphism.

**Trace representations.** Von Neumann algebras provide a way to generalize the intimate relationship between characters and representations from finite to infinite groups.

Let $M$ be a von Neumann algebra [Dix11]. A trace on the von Neumann algebra $M$ is a positive linear functional $\tau : M \to \mathbb{C}$ satisfying $\tau(1_M) = 1$ as well as $\tau(xy) = \tau(yx)$ for all elements $x, y \in M$. The trace $\tau$ is called faithful if the condition $\tau(x^*x) = 0$ implies $x = 0$ for all elements $x \in M$. The trace $\tau$ is called normal if $\tau$ is continuous in the ultraweak topology.

A von Neumann algebra $M$ is called a factor if its center $Z(M)$ consists of scalar multiples of $1_M$. A finite factor is a factor von Neumann algebra admitting a trace. A classical result by Murray and von Neumann states that the trace on a finite factor is unique, normal and faithful [MvN37]. For example, the finite dimensional von Neumann algebra $M_d(\mathbb{C})$ of all $d$-by-$d$ complex matrices is a finite factor. Its unique trace is the standard normalized trace $\tau_d : A \mapsto \frac{1}{d} \sum_{i=1}^{d} A_{ii}$.

**Definition.** A trace representation of the group $G$ is a triple $(M, \pi, \tau)$ where

1. $M$ is a von Neumann algebra,
2. $\pi : G \to \mathcal{U}(M)$ is a representation of the group $G$ into the group $\mathcal{U}(M)$ of unitaries of the algebra $M$ such that the image $\pi(G)$ generates $M$ as a von Neumann algebra, and
3. $\tau$ is a normal faithful trace on the von Neumann algebra $M$.

Saying that the image $\pi(G)$ generates the von Neumann algebra $M$ is equivalent to requiring that $\pi(G)^\prime = M$ by the von Neumann double commutant theorem [Arv12, Theorem 1.2.1].

An isomorphism between two trace representations $(M_1, \pi_1, \tau_1)$ and $(M_2, \pi_2, \tau_2)$ of the group $G$ is an isomorphism of von Neumann algebras $f : M_1 \to M_2$ satisfying $f \circ \pi_1 = \pi_2$ and $\tau_1 = \tau_2 \circ f$.

Any trace representation $(M, \pi, \tau)$ gives rise to the trace $\tau \circ \pi \in \text{Tr}(G)$. The GNS construction can be used to obtain the following fundamental correspondence. For the proof see e.g. [Pet13, Theorem 5.1.10].

**Theorem 2.1** (Thoma’s correspondence). The set of traces $\text{Tr}(G)$ on the group $G$ stands in a bijective correspondence with the set of isomorphism classes of trace representations of $G$. Moreover

- the trace representation $(M, \pi, \tau)$ corresponds to the trace $\varphi = \tau \circ \pi$, and
- $\varphi$ is a character if and only if the von Neumann algebra $M$ is a factor.

**Dominated traces.** There is a natural partial ordering on the set of traces on the group $G$. To define this ordering it will be useful to consider the compact convex set $\text{Tr}_{\leq 1}(G)$ consisting of all positive definite conjugation-invariant functions $\varphi : G \to \mathbb{C}$ satisfying $\varphi(e) \leq 1$. 

Given a pair of traces\(^7\) \(\varphi_1, \varphi_2 \in \text{Tr}_{\leq 1}(G)\) we say that \(\varphi_1\) is \textit{dominated} by \(\varphi_2\) if the function \(\varphi_2 - \varphi_1\) is positive definite. This condition is equivalent to saying that \(\varphi_2 - \varphi_1 \in \text{Tr}_{\leq 1}(G)\). We denote this relation by \(\varphi_1 \leq \varphi_2\). The two traces \(\varphi_1, \varphi_2 \in \text{Tr}(G)\) are \textit{disjoint} if the only trace \(\psi \in \text{Tr}_{\leq 1}(G)\) satisfying both \(\psi \leq \varphi_1\) and \(\psi \leq \varphi_2\) is the zero trace \(\psi = 0\).

Let \((M, \pi, \tau)\) be a trace representation of the group \(G\) with corresponding trace \(\varphi = \tau \circ \pi\). For every positive central element \(T \in Z(M)\) consider the function \(\varphi_T\) on the group \(G\) given by
\[
\varphi_T = \tau \circ \text{Ad}_{T^{1/2}} \circ \pi
\]
where \(\text{Ad} : M \to \text{End}(M)\) is the \textit{adjoint representation} defined by
\[
\text{Ad}_x(y) = xy^* \quad \forall x, y \in M.
\]
The mapping \(T \mapsto \varphi_T\) sets up a bijective correspondence between the set of central elements \(T \in Z(M)\) satisfying \(0_M \leq T \leq 1_M\) and the set of traces in \(\text{Tr}_{\leq 1}(G)\) dominated by the given trace \(\varphi\) [BrHa20, Lemma 11.C.3].

\textbf{Relative traces.} Let \(N\) be a normal subgroup of the group \(G\). The group \(G\) admits a left action on the set of traces \(\text{Tr}_{\leq 1}(N)\) via conjugation \(\varphi \mapsto \varphi^g\) where
\[
\varphi^g(n) = \varphi(gn) \quad \forall g \in G, n \in N.
\]
The conjugation action is by affine homeomorphisms. It preserves the partial order relation of dominated traces.

Consider the compact convex subset of \(\text{Tr}(N)\) consisting of traces on \(N\) fixed by the conjugation action of \(G\), namely
\[
\text{Tr}_G(N) = \{\varphi \in \text{Tr}(N) : \varphi^g = \varphi \quad \forall g \in G\}.
\]
The members of \(\text{Tr}_G(N)\) are called \textit{relative traces}. For every trace \(\varphi \in \text{Tr}(G)\) the restriction \(\varphi|_N\) is a relative trace in \(\text{Tr}_G(N)\). The Fourier transform of the restriction \(\varphi|_N \in \text{Tr}(N)\) determines a \(G\)-invariant Borel probability measure \(\mu_\varphi \in \text{Prob}(\text{Ch}(N))\) such that \(\varphi|_N = \int \psi d\mu_\varphi(\psi)\). The probability measure \(\mu_\varphi\) is ergodic if and only if the trace \(\varphi\) is a character.

\textbf{Lemma 2.2.} Let \((M, \pi, \tau)\) be a trace representation of the group \(G\) corresponding to the trace \(\varphi = \tau \circ \pi\). Let \(N\) be a normal subgroup of \(G\). Consider the relative trace \(\psi = \varphi|_N \in \text{Tr}_G(N)\). Then
\begin{enumerate}
\item The von Neumann subalgebra \(Q = \pi(N)'' \leq M\) is \(G\)-invariant.
\item \((Q, \pi|_N, \tau_Q)\) is the trace representation corresponding to the trace \(\psi\).
\item The mapping
\[
T \mapsto \psi_T = \tau \circ \text{Ad}_{T^{1/2}} \circ \pi|_N
\]
is a \(G\)-equivariant order-preserving bijection from the set of central elements \(T \in Z(Q)\) satisfying \(0 \leq T \leq 1_Q\) to the set of traces in \(\text{Tr}_{\leq 1}(N)\) dominated by the relative trace \(\psi\).
\item The central element \(T\) is a projection if and only if the two traces \(\psi_T\) and \(\psi_{1-N} = \psi - \psi_T\) are disjoint.
\end{enumerate}

Throughout Lemma 2.2 we regard the von Neumann algebra \(M\) with the \(G\)-action given by the adjoint representation defined in Equation (2.3). In other words \(x^* = \text{Ad}_{x(g)}x\) for all \(x \in M\) and all elements \(g \in G\).

\footnote{We abuse our standard terminology and refer to the elements of \(\text{Tr}_{\leq 1}(G)\) as traces, even through these functions need not be normalized.}
Proof of Lemma 2.2. Statements (1) and (2) are immediate. The fact that the mapping \( T \mapsto \psi_T \) is a bijection between the two sets in question has been discussed after Equation (2.3). This mapping is \( G \)-equivariant as
\[
(2.6) \quad \psi_T(n) = (\tau \circ \text{Ad}_{T^{-1}g} \circ \pi)(n) = \tau(T\pi(n)) = \tau(T\pi(g)^{-1}n) = \tau(T\pi(g)^{-1}ng) = (\tau \circ \text{Ad}_{T^{-1}g} \circ \pi)(n^g) = \psi_T(n^g) = \psi_T^g(n)
\]
holds true for all elements \( g \in G \) and \( n \in N \). This gives statement (3).

For statement (4) observe that the bijection \( T \mapsto \psi_T \) is additive in the sense that \( \psi_{T_1+T_2} = \psi_{T_1} + \psi_{T_2} \) holds true for any pair of central elements \( T_1, T_2 \in \mathbb{Z}(Q) \). Moreover \( \psi_{1_Q} = \psi = \psi_{0_Q} = 0 \). With this in mind it is clear that the bijection \( T \mapsto \psi_T \) is order-preserving. Therefore the two traces \( \psi_{T_1} \) and \( \psi_{T_2} \) are disjoint if and only if the two elements \( T_1 \) and \( T_2 \) are disjoint in the sense that \( T_1T_2 = 0 \). It follows that the two traces \( \psi_T \) and \( \psi_{I-T} \) are disjoint if and only if \( T(I-T) = 0 \). This last condition is equivalent to \( T \) being a projection. □

3. Induction of Characters

We discuss induction of traces and characters. Let \( G \) be a discrete group. For the purpose of the following discussion fix a subgroup \( H \) of \( G \).

Definition. A trace \( \varphi \) on the subgroup \( H \) is almost \( G \)-invariant if the subgroup
\[
(3.1) \quad G_{\varphi} = \{ g \in N_G(H) : \varphi^g = \varphi \}
\]
has finite index in \( G \).

Almost \( G \)-invariant traces can be induced from the subgroup \( H \) to the group \( G \). Let us explain how this is done. The trivial extension of any function \( \varphi : H \rightarrow \mathbb{C} \) is the function \( \tilde{\varphi} : G \rightarrow \mathbb{C} \) defined by
\[
(3.2) \quad \tilde{\varphi}(g) = \begin{cases} 
\varphi(g) & g \in H \\
0 & g \notin H.
\end{cases}
\]
If \( \varphi \) is a positive definite function on the subgroup \( H \) then \( \tilde{\varphi} \) is a positive definite function on the group \( G \), see e.g. [BluH20 Proposition 1.F.10]. In particular if the subgroup \( H \) is normal and \( \varphi \in \text{Tr}_G(H) \) then \( \tilde{\varphi} \in \text{Tr}(G) \). The general induction procedure is as follows. A very similar definition of induced characters is studied in [Kan06].

Definition. Let \( \varphi \in \text{Tr}(H) \) be an almost \( G \)-invariant trace on the subgroup \( H \). The induced trace \( \text{Ind}^G_H \varphi \) in \( \text{Tr}(G) \) is given by
\[
(3.3) \quad \text{Ind}^G_H \varphi = \frac{1}{[G : G_{\varphi}]} \sum_{g \in G/G_{\varphi}} \tilde{\varphi}^g.
\]

Note that \( \varphi \mapsto \varphi^g \) defines a left action. The expression on the right-hand side of Equation (3.3) is independent of the particular choice of the coset representatives for \( G/G_{\varphi} \). The formula in Equation (3.3) continues to hold true if the subgroup \( G_{\varphi} \) is replaced by any other finite index subgroup \( L \) of \( G \) contained in \( G_{\varphi} \).

The function \( \text{Ind}^G_H \varphi \) is normalized and positive definite since it is a convex combination of finitely many normalized positive definite functions of the form \( \tilde{\varphi}^g \). This
function is conjugation-invariant as

\[(3.4) \quad (\text{Ind}_H^G \varphi)^{g'} = \frac{1}{[G : G \varphi]} \sum_{g' \in G/G \varphi} (\tilde{\varphi})_{g'} = \frac{1}{[G : G \varphi]} \sum_{g' \in G/G \varphi} \tilde{\varphi}^{g'} = \text{Ind}_H^G \varphi \]

for any element \(g' \in G\). We conclude that \(\text{Ind}_H^G \varphi\) is indeed a trace on \(G\).

This notion of induced traces generalizes the following situations:

- If \(H\) is a normal subgroup of \(G\) and \(\varphi \in \text{Tr}_G(H)\) is a relative trace, then \(\text{Ind}_H^G \varphi = \tilde{\varphi}\).
- If \(H\) is a finite index subgroup of \(G\) then any trace \(\varphi \in \text{Tr}(H)\) is almost \(G\)-invariant and

\[\text{Ind}_H^G \varphi = \frac{1}{[G : H]} \sum_{g \in G/H} \tilde{\varphi}^g.\]

- Generalizing the two previous examples, if \([G : N_G(H)] < \infty\) (i.e. \(H\) is an almost normal subgroup of \(G\)) then any trace \(\varphi \in \text{Tr}_{N_G(H)}(H)\) is almost \(G\)-invariant and

\[\text{Ind}_H^G \varphi = \frac{1}{[G : N_G(H)]} \sum_{g \in G/N_G(H)} \tilde{\varphi}^g.\]

The induced trace has several very useful properties.

**Lemma 3.1** (Induction in stages). Let \(H\) and \(L\) be a pair of subgroups of \(G\) satisfying \(H \leq L \leq G\) and \([G : L] < \infty\). Let \(\varphi \in \text{Tr}(H)\) be an almost \(G\)-invariant trace. Then

\[(3.5) \quad \text{Ind}_L^G \text{Ind}_H^L \varphi = \text{Ind}_H^G \varphi. \]

**Proof.** To begin with note that \(\varphi\) is an almost \(L\)-invariant trace and \(\text{Ind}_H^L \varphi\) is an almost \(G\)-invariant trace so that the left-hand side of Equation (3.5) is well defined. Write

\[
\text{Ind}_L^G \left(\text{Ind}_H^L \varphi\right) = \frac{1}{[G : L]} \sum_{g \in G/L} \left(\text{Ind}_H^L \varphi\right)^g = \frac{1}{[G : L]} \frac{1}{[L : L \varphi]} \sum_{g \in G/L} \sum_{\ell \in L/L \varphi} \tilde{\varphi}^{g \ell} = \frac{1}{[G : L \varphi]} \sum_{h \in G/L \varphi} \tilde{\varphi}^h = \text{Ind}_H^G \varphi. \]

\[\square\]

We remark that Lemma 3.1 remains true with a similar proof, if the assumption that the subgroup \(L\) has finite index in \(G\) is replaced by the more general assumption that the trivial extension of \(\varphi\) to \(L\) is almost \(G\)-invariant. We will not have occasion to use this more general statement.

**Lemma 3.2** (Continuity of induction). Let \(\varphi_n \in \text{Tr}(H)\) be a sequence of traces on the subgroup \(H\) converging to the trace \(\varphi \in \text{Tr}(H)\) in the pointwise topology. Assume that the subgroup

\[(3.6) \quad L = \bigcap_{n \in \mathbb{N}} G_{\varphi_n}\]
has finite index in $G$. Then the sequence of traces $\text{Ind}_H^G \varphi_n$ converges to $\text{Ind}_H^G \varphi$ in the pointwise topology.

**Proof.** The assumption implies that the traces $\varphi_n$ as well as the trace $\varphi$ are all almost $G$-invariant. It is clear that the sequence of positive definite functions $\tilde{\varphi}_g^n$ converges to the positive definite function $\tilde{\varphi}_g$ for any fixed element $g \in G$. Therefore

$$\text{Ind}_H^G \varphi_n = \frac{1}{[G : L]} \sum_{g \in G/L} \tilde{\varphi}_g^n \xrightarrow{n \to \infty} \frac{1}{[G : L]} \sum_{g \in G/L} \tilde{\varphi}_g = \text{Ind}_H^G \varphi$$

in the pointwise topology. \qed

**Mackey theory.** We introduce the following Mackey-type criterion which helps to identify when a given character is induced from a finite index subgroup.

**Theorem 3.3.** Let $N$ be a normal subgroup of the discrete group $G$. Let $\varphi$ be a character of $G$. Assume that

$$\varphi|_N = \frac{1}{n}(\psi_1 + \cdots + \psi_n)$$

for some $n \in \mathbb{N}$ and some family of pairwise disjoint traces $\psi_i$ of the group $N$ permuted transitively by the $G$-action. Then each trace $\psi_i$ can be extended to a character $\varphi_i$ of the isotropy group $G_i = \text{stab}_G(\psi_i)$ so that

$$\varphi = \text{Ind}_G^G \varphi_i.$$

**Proof.** Let $\varphi \in \text{Ch} (G)$ be a character with a corresponding trace representation $(M, \pi, \tau)$. Here $M$ is a finite factor von Neumann algebra, $\pi$ is a representation of the group $G$ into the group $U(M)$ of unitaries of $M$ and $\tau$ is a normal faithful trace on $M$ satisfying $\varphi = \tau \circ \pi$.

Let $N$ be a normal subgroup of the group $G$. Assume that the restriction $\varphi|_N$ of the character $\varphi$ can be written as

$$\varphi|_N = \frac{1}{n}(\psi_1 + \cdots + \psi_n)$$

for some family of pairwise disjoint traces $\psi_i$ of the group $N$ that are permuted transitively by the natural $G$-action. Denote $Q = \pi(N)'$ so that $Q$ is a von Neumann subalgebra of $M$ with $1_M = 1_Q \in Q$. Moreover denote $G_1 = \text{stab}_G(\psi_1)$. Let $t_1 \in G$ be a choice of coset representatives for $G_1$ in $G$ so that $\psi_{t_1}^1 = \psi_1$ for all $i$. We may assume without loss of generality that $t_1 = e_G$.

There exist central orthogonal projections $p_1, \ldots, p_n \in Z(Q)$ satisfying

$$\sum_{i=1}^n p_i = 1_Q = 1_M \quad \text{and} \quad p_i p_j = 0 \quad \forall i \neq j$$

as well as

$$\frac{1}{n} \psi_i = (\varphi|_N)_{p_i} = \tau \circ \text{Ad}_{p_i} \circ \pi$$

as traces on the subgroup $N$ for all $i \in \{1, \ldots, n\}$, see Lemma 2.2. Since the correspondence $p_i \leftrightarrow \psi_i$ is $G$-equivariant we have that $G_1 = \text{stab}_G(p_1)$ and that the group $G$ permutes the central projections $p_1, \ldots, p_n$ transitively via the adjoint action. In particular $\tau(p_i) = \tau(p_{t_1}^i) = \tau(p_1)$ so that $\tau(p_i) = \frac{1}{n}$ for all $i \in \{1, \ldots, n\}$. 

We construct a trace representation \((M_1, \pi_1, \tau_1)\) of the finite index subgroup \(G_1\) as follows. Take
\[
(3.13) \quad M_1 = p_1 M p_1, \quad \pi_1 = \Ad_{p_1} \circ \pi \quad \text{and} \quad \tau_1 = \frac{1}{\tau(p_1)} \tau \circ \Ad_{p_1}.
\]
Note that
\[
(3.14) \quad \Ad_{p_1}(\pi(G)) = p_1 \pi(G)p_1 = p_1 \pi(G_1)p_1 = \pi_1(G_1).
\]
The von Neumann subalgebra \(M_1\) is a factor and \(M_1\) is generated by \(\pi_1(G_1)\) \cite{Dix11} Chapter 1 §2. We conclude that \((M_1, \pi_1, \tau_1)\) is a genuine trace representation of the subgroup \(G_1\). Consider the character \(\varphi_1 = \tau_1 \circ \pi_1\) of the subgroup \(G_1\). It extends the trace \(\psi_1\) in the sense that
\[
(3.15) \quad \varphi_1(n) = \frac{\tau(p_1 \pi(n)p_1)}{\tau(p_1)} = \frac{(\varphi_1|N)(n)}{\tau(p_1)} = \psi_1(n)
\]
holds true for all elements \(n \in N\).

To conclude the proof it remains to show that \(\varphi = \Ind_G^G \varphi_1\). With this goal in mind consider the von Neumann algebra \(M_n(M_1)\) of all \(n\)-by-\(n\) matrices with entries in \(M_1\). This von Neumann algebra admits the normal faithful trace
\[
(3.16) \quad \hat{\tau} : x \mapsto \frac{1}{n} \sum_{i=1}^n \tau_1(x_{ii}) \quad \forall x = (x_{ij}) \in M_n(M_1).
\]
On the other hand, the map \(f : M \to M_n(M_1)\) given by
\[
(3.17) \quad f(x)_{ij} = p_1 \pi(t_i^{-1}) xt_j p_1 \quad \forall x \in M, \forall i, j \in \{1, \ldots, n\}
\]
is a normal unital *-homomorphism. Since the von Neumann algebra \(M\) is a factor it follows that \(f\) must be an embedding. Therefore \(\hat{\tau} \circ f\) is a faithful normal trace on the von Neumann algebra \(M\). The uniqueness of traces on finite factors implies that \(\tau = \hat{\tau} \circ f\). Write
\[
(3.18) \quad \varphi(g) = \tau \circ \pi(g) = \hat{\tau} \circ f \circ \pi(g) = \frac{1}{n} \sum_{i=1}^n \tau_1(p_1 \pi(t_i^{-1} g t_i)p_1) = \frac{1}{n} \sum_{i=1}^n \left\{ \begin{array}{ll} \tau_1 \circ \pi_1(t_i^{-1} g t_i) & g \in G_1^1 \\
0 & g \notin G_1^1 \end{array} \right\} = \Ind_{G_1}^G \varphi_1(g)
\]
for every element \(g \in G\). The first equality on the second line of Equation \[(3.18)\] is due to the observation that the condition \(g_0 \notin G_1\) implies that \(p_{i_0} = p_i\) for some central projection \(p_i\) satisfying \(p_1 p_i = 0\). In that case \(p_1 \pi(g_0)p_1 = 0\).

**Vanishing of characters.** The following two lemmas are instrumental towards our study of the character theory of solvable groups.

**Lemma 3.4 (Bekka).** Let \(N\) and \(H\) be a pair of normal subgroups of the group \(G\) satisfying \(N \leq H\). Let \(\varphi \in \text{Tr}(G)\) be a trace vanishing on \(H \setminus N\). Let \(g \in G\) be any element. If there is a sequence of elements \(x_n \in G\) such that the commutators \([g, x_n]\) belong to pairwise distinct cosets of \(N\) and such that \([g, x_n] \in H\) for all \(n \in \mathbb{N}\) then \(\varphi(g) = 0\).

**Proof.** The case where the subgroup \(N\) is trivial is \cite[Lemma 16]{Bek07}. See \cite[Lemma 4.13]{LL20} for the general case. \(\square\)
Lemma 3.5 (Bekka–de la Harpe). Let $N$ be an abelian normal subgroup of the 
group $G$ with Pontryagin dual $\hat{N}$. Let $\varphi \in \text{Tr}(G)$ be a trace whose restriction 
to $N$ corresponds to the Borel probability measure $\mu_\varphi \in \text{Prob}(\hat{N})$ via the Fourier 
transform. Then $\varphi(g) = 0$ for every element $g \in G$ that acts $\mu_\varphi$-essentially freely 
on $\hat{N}$.

Proof. This statement is part of the “second case” of the proof of [BdlH20, Theorem 12.D.1]. It deals specifically with the Baumslag–Solitar group $G = \mathbb{Z} \times \mathbb{Z} [1/n]$ and 
its normal subgroup $N = \mathbb{Z} [1/n]$. However the same argument applies verbatim in 
the general setting. \hfill $\square$

4. Characters of metabelian groups

We study the character theory of metabelian groups and show that all of their 
characters are induced from abelian subquotients.

Theorem 4.1. Let $G$ be a metabelian group admitting an abelian normal subgroup 
$N$ such that $G/N$ is abelian. Then any character of the group $G$ is induced from a 
trace on the abelianization $H^{ab}$ of some subgroup $H$ satisfying $N \leq H \leq G$.

Proof. Let $\varphi$ be any character of the metabelian group $G$. We may assume without 
loss of generality that the character $\varphi$ is faithful, for otherwise we may regard $\varphi$ as 
a faithful character of the metabelian quotient group $G/\ker \varphi$.

Consider the ergodic $G$-invariant Borel probability measure $\mu_\varphi$ on the Pontryagin 
dual $\hat{N}$ associated to the Fourier transform of the restriction $\varphi|_N$. Denote $Q = G/N$ 
and let $\text{Sub}(Q)$ be the Chabauty space of all subgroups of $Q$. Consider the Borel 
stabilizer map

$$(4.1) \quad \text{stab} : \hat{N} \rightarrow \text{Sub}(Q), \quad \chi \mapsto \text{stab}_Q(\chi) \quad \forall \chi \in \hat{N}.$$ 

This form of the stabilizer map with range $\text{Sub}(Q)$ is well defined as the normal 
subgroup $N$ is abelian. Moreover the map $\text{stab}$ is $G$-invariant since the conjugation 
action of the group $G$ on its quotient $Q$ is trivial. The $G$-ergodicity of the measure 
$\mu_\varphi$ implies that there is some normal subgroup $M$ with $N \leq M \leq G$ such that 
$M = \text{stab}_Q(\chi)$ holds true for $\mu_\varphi$-almost every character $\chi \in \hat{N}$. We obtain that 
$\varphi = \hat{\varphi}|_M = \text{Ind}_M^G(\hat{\varphi}|_M)$ according to Lemma [3.5].

Observe that any pair of elements $m \in M$ and $n \in N$ satisfy $[m, n] \in N$ and 

$$(4.2) \quad \varphi([m, n]) = \int_{\hat{N}} \chi([m, n]) \, d\mu_\varphi(\chi) = \int_{\hat{N}} \chi^m(n)\chi^{-1}(n) \, d\mu_\varphi(\chi) = 1.$$ 

Since the character $\varphi$ is faithful by our assumption it follows that $[M, N] = \{e\}$. In 
other words $M \leq C_G(N)$ and the subgroup $M$ is two-step nilpotent.

Let $\text{Sub}(M)$ denote the Chabauty space of all subgroups of the group $M$. Consider the 
Borel map

$$(4.3) \quad \mathcal{Z} : \text{Ch}(M) \rightarrow \text{Sub}(M), \quad \theta \mapsto \mathbb{Z}(M/\ker \theta).$$ 

Note that $N \leq \mathcal{Z}(\theta)$ holds true for all characters $\theta \in \text{Ch}(M)$. In particular the 
map $\mathcal{Z}$ is in fact $G$-invariant. It follows that the map $\mathcal{Z}$ is essentially constant with 
respect to the ergodic $G$-invariant Borel probability measure on the space $\text{Ch}(M)$ 
associated via the Fourier transform to the restriction $\varphi|_M$. Therefore there is some 
subgroup $N \leq H \leq M$ satisfying $H = \mathcal{Z}(\theta) = \mathbb{Z}(M/\ker \theta)$ almost surely.
Howe’s lemma implies that any character $\theta$ of the two-step nilpotent group $M$ is induced from its central subgroup $H$, see e.g. [CMS4 Proposition 2.6]. We conclude that $\varphi = \tilde{\varphi}|_H = \text{Ind}_G^H(\varphi|_H)$ where $\varphi|_H$ is a $G$-invariant trace on the subgroup $H$. The restriction $\varphi|_H$ factorizes through the abelian subquotient $H^{ab}$ as required. □

The above is a restatement of Theorem B of the introduction.

5. Characters of Noetherian groups

Recall that a group is called Noetherian if it satisfies the ascending chain condition on subgroups. This condition is equivalent to saying that every subgroup is finitely generated. A solvable group is Noetherian if and only if it is polycyclic.

In the current section we study the character theory of Noetherian groups in general and of virtually nilpotent groups in particular.

Totally faithful characters. We introduce a property of characters which is a strengthening of being faithful. It will play a central role in our analysis.

Definition. A character $\varphi$ of a group $G$ is totally faithful if $\varphi$ is not induced from a non-faithful character of any finite index subgroup.

The usefulness of totally faithful characters has to do with the fact that any character of a Noetherian group is induced from a totally faithful character of some subquotient.

Proposition 5.1 (Noetherian induction principle). Let $G$ be a Noetherian group and $\varphi \in \text{Ch}(G)$ a character. Then there is a finite index subgroup $H$ of $G$ and a character $\psi \in \text{Ch}(H)$ satisfying $\varphi = \text{Ind}_G^H \psi$ such that $\psi$ is totally faithful when regarded as a character of the subquotient $H/\ker \psi$.

Proof. We construct inductively a descending sequence of finite index subgroups $G_i \leq G$ equipped with characters $\varphi_i \in \text{Ch}(G_i)$ satisfying $\varphi = \text{Ind}_G^G \varphi_i$ for all $i \in \mathbb{N}$. The fact that the group $G$ is Noetherian will be used to ensure that this process terminates after finitely many steps and that we arrive at some totally faithful character. The base of the induction is given by taking the group $G_0 = G$ and the character $\varphi_0 = \varphi \in \text{Ch}(G)$.

Let us consider the induction step. Assume that $G_i \leq G$ is some finite index subgroup and $\varphi_i \in \text{Ch}(G_i)$ is a character satisfying $\varphi = \text{Ind}_G^G \varphi_i$ for some $i \in \mathbb{N}$. There are two possible cases:

- If $\varphi_i$ is totally faithful regarded as a character of the subquotient $G_i/\ker \varphi_i$ then we may conclude the proof taking $H = G_i$ and $\psi = \varphi_i$.
- Otherwise there exist a further finite index subgroup $G_{i+1} \leq G_i$ containing $\ker \varphi_i$ and a character $\varphi_{i+1} \in \text{Ch}(G_{i+1})$ satisfying $\ker \varphi_i \leq \ker \varphi_{i+1}$ and $\varphi_i = \text{Ind}_{G_{i+1}}^{G_i} \varphi_{i+1}$. Therefore $\varphi = \text{Ind}_{G_{i+1}}^{G_i} \varphi_{i+1}$ by induction in stages (Lemma 3.1).

Proceed inductively to define the finite index subgroups $G_i$ and the characters $\varphi_i \in \text{Ch}(G_i)$ for all $i \in \mathbb{N}$ as in the above paragraph. As the group $G$ is Noetherian, the resulting ascending sequence of kernel subgroups $\ker \varphi_i$ must eventually stabilize. This completes the proof. □

Lemma 5.2. Let $G$ be a Noetherian group and $\varphi \in \text{Ch}(G)$ be a totally faithful character. Let $N$ be a normal subgroup of $G$ such that $\mu_{\varphi} \in \text{Prob}(\text{Ch}(N))$ is the
Fourier transform of the restriction $\varphi|_N$. Then $\mu_\varphi$-almost every character of $N$ is faithful.

Proof. Let $\text{Sub}(N)$ denote the Chabauty space of all subgroups of $N$. As the group $N$ is Noetherian the space $\text{Sub}(N)$ is countable. The group $G$ acts on the space $\text{Sub}(N)$ by conjugation. There is a natural $G$-equivariant kernel map

$$\kappa : \text{Ch}(N) \to \text{Sub}(N), \chi \mapsto \ker \chi. \tag{5.1}$$

The pushforward measure $\kappa_*\mu_\varphi$ is a $G$-invariant ergodic Borel probability measure on the countable space $\text{Sub}(N)$. As such there are finitely many normal subgroups $K_1, \ldots, K_m$ of $N$ such that $G$-acts transitively on this family and $\mu_\varphi$ is the uniform probability measure on this orbit. In particular $\kappa_*\mu_\varphi(\{K_i\}) = \frac{1}{m}$ for all $i$.

Write $\varphi|_N = \frac{1}{m}(\psi_1 + \cdots + \psi_m)$ where each trace $\psi_i \in \text{Tr}(N)$ is given by

$$\psi_i = m \cdot \int_{\kappa^{-1}(K_i)} \psi \, d\mu_\varphi(\psi) \tag{5.2}$$

for $i \in \{1, \ldots, m\}$. The traces $\psi_1, \ldots, \psi_m$ are pairwise disjoint and the group $G$ permutes them transitively. It follows from Theorem 3.3 that $\varphi = \text{Ind}_{N_{G}(K_1)}^{G} \varphi_1$ for some character $\varphi_1 \in \text{Ch}(N_G(K_1))$ extending the trace $\psi_1$.

The assumption that the the character $\varphi$ is totally faithful implies that the character $\varphi_1$ must be faithful. This means that the subgroup $K_1$ is trivial. In other words $m = 1$ so that $\mu_\varphi$-almost every character $\psi \in \text{Ch}(N)$ is faithful. \hfill $\square$

Rather than assuming that the group $G$ itself is Noetherian in Lemma 5.2 it would have sufficed to assume that its normal subgroup $N$ is Noetherian.

Total faithfulness is useful to obtain a vanishing criterion for characters of Noetherian groups.

Lemma 5.3. Let $G$ be a Noetherian group and $\varphi \in \text{Ch}(G)$ be a totally faithful character. Let $N$ be an abelian normal subgroup of $G$. Then $\varphi(g) = 0$ for any element $g \in G \setminus C_G(N)$.

Proof. Let $g \in G$ be any element satisfying $g \notin C_G(N)$. We claim that $g$ acts $\mu_\varphi$-essentially freely on the Pontryagin dual $\widehat{N}$ where $\mu_\varphi \in \text{Prob}(\widehat{N})$ is the measure corresponding to the restriction $\varphi|_N$ via the Fourier transform. To establish this claim assume towards contradiction that

$$\mu_\varphi(\{\chi \in \widehat{N} : \chi^g = \chi\}) > 0. \tag{5.3}$$

Since the character $\varphi$ is totally faithful we know that $\mu_\varphi$-almost every multiplicative character $\chi \in \widehat{N}$ is faithful by Lemma 5.2. In particular there exists some faithful multiplicative character $\chi \in \widehat{N}$ with $\chi^g = \chi$. So

$$\chi^g = \chi \implies \chi(n^g) = \chi(n) \quad \forall n \in N \implies n^g = n \quad \forall n \in N. \tag{5.4}$$

This means that $g \in C_G(N)$ contrary to the assumption. The claim follows. We conclude that $\varphi(g) = 0$ by Lemma 5.3 on vanishing of characters. \hfill $\square$

Virtually nilpotent groups. Let us review the character theory of nilpotent groups alongside some new observations. Nilpotent groups are a basic special case of polycyclic ones. More importantly, the Fitting subgroup of a general polycyclic group is nilpotent and it plays a crucial role in our approach.
Theorem 5.4 (Howe [How77]). Let $G$ be a finitely generated nilpotent group. Then any faithful character $\varphi \in \text{Ch}(G)$ is induced from $\text{FC}(G)$.

The statement of Theorem 5.4 holds true more generally for any nilpotent group $G$ such that the quotient $G/\text{FC}(G)$ has finite rank, in the sense that any finitely generated subgroup of $G$ is contained in some $r$-generated subgroup for some fixed rank $r \in \mathbb{N}$ [CM84, Theorem 4.5].

The rest of this section is dedicated to the proof of Theorem 5.8 which generalizes Theorem 5.4 to the virtually nilpotent case. We begin with a few useful lemmas.

Lemma 5.5 (Kaniuth). Let $G$ be a finitely generated group admitting a nilpotent normal subgroup $H$ of finite index. Let $\varphi$ be a faithful character of $G$ such that the restriction $\varphi|_H$ is a character of $H$. Then the character $\varphi$ is induced from $\text{FC}(G)$.

We emphasize that saying that the restriction $\varphi|_H$ is a character means that is not a proper convex combination of traces of $H$.

Proof of Lemma 5.5. This is [Kan80, Lemma 2].

Lemma 5.6. Let $G$ be a group admitting a finitely generated virtually nilpotent normal subgroup $N$. Let $\varphi \in \text{Ch}(G)$ be a totally faithful character. Then the restriction $\varphi|_N$ is induced from $\text{FC}(N)$.

Proof. Let $\mu_{\varphi} \in \text{Prob}(\text{Ch}(N))$ denote the measure corresponding to the restriction $\varphi|_N$ via the Fourier transform. The measure $\mu_{\varphi}$ is $G$-invariant and ergodic.

Fix a finite index nilpotent subgroup $H$ of $N$ which is normal in $G$. For every character $\zeta \in \text{Ch}(N)$ the restriction $\zeta|_H$ is a convex combination of finitely many characters of $H$. Choose a $\mu_{\varphi}$-measurable mapping taking a character $\zeta \in \text{Ch}(N)$ to an arbitrary character $\psi_\zeta \in \text{Ch}(H)$ appearing in this restriction.

Consider the subgroups

\begin{equation}
S(\zeta) = \text{stab}_N(\psi_\zeta)
\end{equation}

defined for $\mu_{\varphi}$-almost every character $\zeta$ of $N$. The fact that the group $N$ has countably many subgroups combined with the ergodicity of the measure $\mu_{\varphi}$ implies that the subgroups $S(\zeta)$ all belong to a single conjugacy class. This allows us to assume that $S(\zeta) = S$ for some fixed subgroup $S$ satisfying $H \leq S \leq N$ up to modifying the $\mu_{\varphi}$-measurable mapping $\psi_\zeta$ where necessary.

Consider the collection of characters $M(\zeta)$ given for $\mu_{\varphi}$-almost every character $\zeta \in \text{Ch}(N)$ by

\begin{equation}
M(\zeta) = \{\chi \in \text{Ch}(S) : \chi|_H = \psi_\zeta, \text{Ind}_S^N \chi = \zeta\}.
\end{equation}

The set $M(\zeta)$ is $\mu_{\varphi}$-almost surely non-empty by Theorem 5.3 and finite by [BV22].

Let $\chi_\zeta \in M(\zeta)$ be an arbitrary character chosen in a $\mu_{\varphi}$-measurable manner.

As the character $\varphi$ is totally faithful it follows that the character $\psi_\zeta \in \text{Ch}(H)$ is faithful $\mu_{\varphi}$-almost surely, see Lemma 5.2. In particular the kernel of the character $\chi_\zeta$ is finite $\mu_{\varphi}$-almost surely so that

\begin{equation}
\text{FC}(S/\ker \chi_\zeta) = \text{FC}(S).
\end{equation}

Every character $\chi_\zeta \in \text{Ch}(S)$ restricts $\mu_{\varphi}$-almost surely to a character of $H$ and is therefore induced from the subquotient $\text{FC}(S/\ker \chi_\zeta)$ by Lemma 5.5. In other words the character $\chi_\zeta$ is $\mu_{\varphi}$-almost surely induced from the subgroup $\text{FC}(S)$.

Observe that $\text{FC}(S) \leq \text{FC}(N)$ as $[N:S] < \infty$. Therefore $\mu_{\varphi}$-almost every character $\zeta$ is induced from the subgroup $\text{FC}(N)$ according to the definition of the
family \( M(\zeta) \) given in Equation (5.6) and using induction in stages (Lemma 3.1). It follows that the restriction \( \varphi|_N = \int \zeta \, d\mu_\varphi(\zeta) \) vanishes outside the normal subgroup \( \text{FC}(N) \). \qed

The special case of Lemma 5.6 where the group \( G \) is itself virtually nilpotent gives the following.

**Corollary 5.7.** Let \( G \) be a finitely generated virtually nilpotent group. Any totally faithful character \( \varphi \in \text{Ch}(G) \) is induced from \( \text{FC}(G) \).

We are ready to obtain the following result.

**Theorem 5.8.** Let \( G \) be a finitely generated virtually nilpotent group. Then any character \( \varphi \) of the group \( G \) is induced from \( \text{FC}(H/\ker \varphi) \) for some finite index subgroup \( H \) of \( G \).

**Proof.** Let \( G \) be a finitely generated virtually nilpotent group. Consider any character \( \varphi \) of \( G \). The Noetherian induction principle allows us to find a finite index subgroup \( H \) of \( G \) and a character \( \psi \) of \( H \) satisfying \( \varphi = \text{Ind}^G_H \psi \) so that \( \psi \) is totally faithful regarded as a character of the subquotient \( H/\ker \psi \). The totally faithful character \( \psi \) is induced from \( \text{FC}(H/\ker \psi) \) according to Corollary 5.7. The statement follows by induction in stages (Lemma 3.1). \qed

### 6. Characters of virtually polycyclic groups

The goal of the current section is to prove Theorem 5 from the introduction dealing with characters of virtually polycyclic groups.

To set the stage for the study of the character theory of polycyclic groups we first make a brief algebraic digression.

**The Fitting subgroup.** Recall that the *Fitting subgroup* \( \text{Fit}(G) \) of the group \( G \) is the characteristic subgroup generated by all normal nilpotent subgroups of \( G \). If the group \( G \) is Noetherian then \( \text{Fit}(G) \) is nilpotent [LR04, 1.2.9]. In particular \( \text{Fit}(G) \) is nilpotent provided that \( G \) is virtually polycyclic.

**Lemma 6.1.** Let \( G \) be a virtually polycyclic group. Let \( g \in G \) be an element whose projection to the quotient \( G/\text{Fit}(G) \) has infinite order. Then the subgroup \( L = \langle g \rangle \cdot \text{Fit}(G) \) is not nilpotent.

**Proof.** The quotient group \( G/\text{Fit}(G) \) is virtually abelian by a theorem of Mal’cev [Mal56]. Let \( H \) be a normal finite index subgroup of \( G \) containing \( \text{Fit}(G) \) such that the quotient group \( H/\text{Fit}(G) \) is abelian. The nilpotent subgroup \( \text{Fit}(H) \) is normal in \( G \) so that \( \text{Fit}(H) \leq \text{Fit}(G) \). On the other hand \( L \cap H \triangleleft H \) and \( \text{Fit}(H) \leq L \cap H \) by the assumption on the element \( g \). We conclude that the subgroup \( L \cap H \) is not nilpotent. Therefore the group \( L \) itself is not nilpotent. \qed

We are interested in studying the center of the Fitting subgroup \( \text{Fit}(G) \). Note that if \( G \) is solvable then \( Z(\text{Fit}(G)) = C_G(\text{Fit}(G)) \) [LR04, 1.2.10]. The following result gives more refined information.

**Lemma 6.2.** Let \( G \) be a virtually polycyclic group. Let \( g \in C_G(Z(\text{Fit}(G))) \) be an element whose projection to the quotient \( G/\text{Fit}(G) \) has infinite order. Then there is an element \( x \in \text{Fit}(G) \) such that the commutators \( [g, x^n] \) are pairwise distinct modulo the subgroup \( \text{FC}(\text{Fit}(G)) \) for all \( n \in \mathbb{N} \).
Proof. Consider the upper central series

$$
\{e\} = Z_0 \leq Z_1 = Z(Fit(G)) \leq Z_2 \leq \cdots \leq Z_k = Fit(G)
$$

for the nilpotent subgroup $Fit(G)$ of nilpotence degree $k \in \mathbb{N}$. An automorphism $\alpha$ of the group $Fit(G)$ is said to stabilize this central series if $\alpha(xZ_i) = xZ_i$ for all $i \in \{0, \ldots, k-1\}$ and all elements $x \in Z_{i+1}$. Any subgroup of $Aut(Fit(G))$ which stabilizes a central series for the group $Fit(G)$ must be nilpotent [LR04, 1.2.7].

We know that the subgroup $L = (g)Fit(G)$ is not nilpotent by Lemma [6.1]. In particular the central quotient $\overline{L} = L/Z(L)$ is not nilpotent as well. Moreover

$$
C_L(Fit(G)) = C_G(Fit(G)) \cap L = Z(Fit(G)) \cap L = Z(Fit(G)).
$$

This means that the central quotient $\overline{L}$ embeds into $Aut(Fit(G))$ via its action by inner automorphisms. Therefore the element $g$ does not centralize some factor of the upper central series in Equation (6.1) above. We wish to make this observation a bit more precise, as follows.

Write $Z_i/Z_{i-1} = T_i \oplus A_i$ where $T_i$ is the torsion subgroup of the abelian factor $Z_i/Z_{i-1}$ and $A_i$ is some direct complement for each index $i \in \{1, \ldots, k\}$. There is some $m \in \mathbb{N}$ such that the power $g^m$ centralizes the torsion subgroup $T_i$ and that the condition $[g, A_i] \subset T_i$ implies $[g^m, A_i] \subset Z_{i-1}$ for all $i$. Arguing as in the previous paragraph with respect to the power $g^m$, we deduce that the element $g$ satisfies $\{g, A_i\} \not\subset T_i$ for some fixed index $i \in \{2, \ldots, k\}$. The possibility of $i = 1$ here is excluded by the assumptions.

Let $z \mapsto \overline{z}$ denote the quotient map from the subgroup $Z_i$ to the abelian factor $Z_i/Z_{i-1}$. Take an element $x \in Z_i$ such that $\overline{x} \in A_i$ and that the commutator $y = [g, x]$ has $\overline{y} \not\in T_i$.

To conclude the proof it remains to show that the commutators $y_n = [g, x^n]$ are pairwise distinct modulo the subgroup $FC(Fit(G))$ for all $n \in \mathbb{N}$. Note that the center $Z(Fit(G))$ has finite index in the subgroup $FC(Fit(G))$ [Kan80, p. 98]. In particular the projection of $FC(Fit(G)) \cap Z_i$ to the factor group $Z_i/Z_{i-1}$ lies inside the torsion subgroup $T_i$. Therefore it will suffice to verify that the $\overline{y}_n$’s are pairwise distinct modulo the torsion subgroup $T_i$ regarded as elements of the subquotient $Z_i/Z_{i-1}$. Relying on standard commutator identities we write

$$
y_n = [g, x^n] = [g, x^{n-1}] \cdot [g, x]^{n-1} = y_{n-1} \cdot [g, x^{n-1}]
$$

for all $n \in \mathbb{N}$. As $[g, x^{n-1}] \in Z_{i-1}$ we have $\overline{y}_n = \overline{y}_{n-1} \cdot \overline{y}$ by Equation (6.3). This gives $\overline{y}_n = \overline{y}^n$ by induction. The proof is complete. \qed

**Crystallographic groups.** For an arbitrary group $G$ let $F(G)$ denote the characteristic subgroup\footnote{This subgroup is called the *polyfinite radical of $G$* and denoted $W(G)$ in [Cor15].} generated by all finite normal subgroups. If the group $G$ is Noetherian then its subgroup $Fit(G)$ is finite. Let $vFit(G)$ denote the characteristic subgroup of $G$ corresponding to $F(G/Fit(G))$. See [D194].

From now on assume that the group $G$ is virtually polycyclic. This means that $[vFit(G) : Fit(G)] < \infty$ so that $vFit(G)$ is virtually nilpotent. In fact $vFit(G)$ is the maximal virtually nilpotent normal subgroup of $G$.

Recall that the quotient $G/Fit(G)$ is virtually abelian [Mal50]. Therefore the quotient $\Gamma(G) = G/vFit(G)$ is a virtually abelian group without non-trivial finite normal subgroups. It follows that $\Gamma(G)$ is a *crystallographic group* [D194, Theorem 1.1]. In other words, the quotient $\Gamma(G)$ is a uniform lattice in the group of isometries
of the Euclidean space $\mathbb{E}^d$ in some dimension $d \in \mathbb{N}$. We shall use the following elementary fact regarding crystallographic groups.

**Lemma 6.3.** Let $\Gamma$ be a crystallographic group and $g \in \Gamma$ be a non-trivial torsion element. Then there is a translation $h \in \Gamma$ so that the commutators $[g, h^n]$ are pairwise distinct translations in $\Gamma$ for all $n \in \mathbb{N}$.

**Proof.** The fixed point set $\text{Fix}(g) = \{x \in \mathbb{E}^d : gx = x\}$ of the torsion element $g$ is a proper affine subspace of the Euclidean space $\mathbb{E}^d$. The subgroup of translations in $\Gamma$ acts co-compactly on $\mathbb{E}^d$. Therefore there is some non-trivial translation $h \in \Gamma$ which does not preserve the set $\text{Fix}(g)$. In particular the elements $g$ and $h$ do not commute. The element $t = [g, h]$ is a non-trivial translation (since translations form a normal subgroup of $\Gamma$). Using standard commutator identities and relying on the fact that the subgroup of translations is abelian we get

\[
[g, h^n] = [g, h^{n-1}] [g, h]^{h^{-1}} = [g, h^{n-1}] t h^{-1} = [g, h^{n-1}] t
\]

for all $n \in \mathbb{N}$. Arguing by induction gives $[g, h^n] = t^n$ for all $n \in \mathbb{N}$. Therefore the elements $[g, h^n]$ are pairwise distinct. \qed

**Characters of virtually polycyclic groups.** We begin by establishing Theorem 6.4 under the additional assumption that the character in question is totally faithful.

**Proposition 6.4.** Let $G$ be a virtually polycyclic group. Then any totally faithful character of $G$ is induced from $\text{FC}(\text{vFit}(G))$.

**Proof.** Let $\varphi \in \text{Ch}(G)$ be a totally faithful character. Fix an arbitrary element $g \in G \setminus \text{FC}(\text{vFit}(G))$. We will show that $\varphi(g) = 0$.

To begin with, by Lemma 5.2 that the restriction $\varphi|_{\text{vFit}(G)}$ is induced from the subgroup $\text{FC}(\text{vFit}(G))$. Therefore we may assume that $g \notin \text{vFit}(G)$. Likewise Lemma 5.3 allows us to assume that $g \in C_G(Z(\text{Fit}(G)))$.

Assume that the image of the element $g$ in the crystallographic group quotient $\Gamma = G/\text{vFit}(G)$ has infinite order. According to Lemma 6.2 there is an element $x \in \text{Fit}(G)$ so that the commutators $[g, x^n] \in \text{Fit}(G)$ are pairwise distinct modulo the subgroup $\text{FC}(\text{Fit}(G))$. However another application of Lemma 5.4 with respect to the restriction $\varphi|_{\text{Fit}(G)}$ shows that this restriction is induced from $\text{FC}(\text{Fit}(G))$. We conclude that $\varphi(g) = 0$ from the vanishing result for characters stated in Lemma 5.4.

Finally assume that the image of the element $g$ in the crystallographic group quotient $\Gamma = G/\text{vFit}(G)$ has finite order. There exists an element $h \in G \setminus \text{vFit}(G)$ projecting to a translation in the crystallographic group quotient $\Gamma$ so that the commutators $[g, h^n]$ project to pairwise distinct translations in $\Gamma$, see Lemma 6.3. We have seen in the previous paragraph that the character $\varphi$ vanishes on all elements of $G$ that project onto a translation element in the crystallographic quotient $\Gamma$. We conclude that $\varphi(g) = 0$ by relying again on Lemma 5.4 (with the normal subgroup of all translations in $\Gamma$ playing the role of the subgroup $H$). \qed

We conclude the proof of our main result in the character theory of polycyclic groups.

**Proof of Theorem 6.4.** Let $\varphi \in \text{Ch}(G)$ be any character of the virtually polycyclic group $G$. According to the Noetherian induction principle (Proposition 5.4) there is some finite index subgroup $H \leq G$ and some character $\psi \in \text{Ch}(H)$ satisfying
Let \( \varphi = \text{Ind}_H^G \psi \) such that \( \psi \) is totally faithful when regarded as a character of the subquotient \( H / \ker \psi \). The character \( \psi \) is induced from FC (\( \text{vFit} (H / \ker \psi) \)) according to Proposition 6.4. The result follows by induction in stages (Lemma 3.1).

7. Characters of virtually central groups

The point of departure for the current section is the following well-known fact.

**Lemma 7.1** (Schur’s lemma for characters). The restriction of any character of a countable group \( G \) to its center \( Z(G) \) is a multiplicative character.

**Proof.** Let \( G \) be a countable group. Consider any character \( \varphi \in \text{Ch} (G) \) with a corresponding trace representation \((M, \pi, \tau)\). The unitary operators \( \pi(Z(G)) \) are contained in the center \( Z(M) \) of the von Neumann algebra \( M \). Since \( \varphi \) is a character the von Neumann algebra \( M \) is a factor. Therefore \( \pi(z) \) is a unimodular complex scalar denoted \( \chi (z) \) for each element \( z \in Z(G) \). It follows that \( \varphi|_Z = \chi \) is a multiplicative character of the center \( Z(G) \).

**Restriction is a covering map.** Let \( G \) be a virtually central group, i.e. a group satisfying \( [G : Z(G)] < \infty \). Fix a central subgroup \( Z \leq Z(G) \) with \( [G : Z] < \infty \). We obtain the restriction map

\[
(7.1) \quad r : \text{Ch} (G) \rightarrow \text{Ch} (Z), \quad r : \varphi \mapsto \varphi|_Z.
\]

The restriction map \( r \) is continuous, surjective \([\text{Tho64}, \text{Lemma 16}]\) and has finite fibers \([\text{LL20}, \text{Proposition 4.8.(1)}]\). A deeper property of the map \( r \) is as follows.

**Proposition 7.2.** The restriction map \( r : \text{Ch} (G) \rightarrow \text{Ch} (Z) \) is open.

**Proof.** Denote \( Q = G / Z \) so that the quotient group \( Q \) is finite. Consider the short exact sequence

\[
(7.2) \quad 1 \rightarrow Z \rightarrow G \twoheadrightarrow Q \rightarrow 1.
\]

Let \( s : Q \rightarrow G \) be an arbitrary section to the projection map \( p \) satisfying \( s(e_Q) = e_G \). Consider the cohomology class \([c] \in H^2(Q, Z)\) naturally associated to the central extension given in Equation (7.2). The class \([c]\) is represented by the 2-cocycle \( c : Q \times Q \rightarrow Z \) determined by

\[
(7.3) \quad s(q_1)s(q_2) = c(q_1, q_2)s(q_1q_2) \quad \forall q_1, q_2 \in Q.
\]

Every multiplicative character \( \chi \in \text{Ch} (Z) \) determines the cohomology class \([c_\chi] \in H^2(Q, S^1)\) represented by the 2-cocycle \( c_\chi = \chi \circ c \). We observe that the second cohomology group \( H^2(Q, S^1) \) is finite. Indeed \( H^2(Q, S^1) \) can be regarded as a subgroup the cohomology group \( H^2(Q, \mathbb{C}^*) \) and this latter group is finite by \([\text{Karp85}, \text{Chapter 2, Theorem 3.22.(ii)}]\). It follows that the group of coboundaries \( B^2(Q, S^1) \) is an open subgroup of the group of cocycles \( Z^2(Q, S^1) \) when both are regarded as abelian topological groups.

Note that the correspondence \( \eta \mapsto \beta_\eta \) taking a point \( \eta \in (S^1)^Q \) to the coboundary \( \beta_\eta \in B^2(Q, S^1) \) determined by

\[
(7.4) \quad \beta_\eta(q_1, q_2) = \eta(q_1)\eta(q_2)\eta(q_1q_2)^{-1}
\]

is an homomorphism of compact abelian groups. This establishes the following topological group isomorphism

\[
(7.5) \quad B^2(Q, S^1) \cong (S^1)^Q / \text{Hom}(Q, S^1).
\]
Fix a multiplicative character $\chi \in \text{Ch}(Z)$ of the central subgroup $Z$ and let $\varphi \in r^{-1}(\chi)$ be any character of the group $G$ that extends $\chi$. Consider a character $\chi_1 \in \text{Ch}(Z)$ which is close to $\chi$, i.e. so that $\chi^{-1}\chi_1$ lies in some small neighborhood of the identity in $\text{Ch}(Z)$. Our goal is to find a character $\varphi_1 \in r^{-1}(\chi_1)$ which is close to $\varphi$.

Let $(M, \pi, \tau)$ be the trace representation corresponding to the character $\varphi$. We will define a new representation $\pi_1$ of the group $G$ into the group of unitaries of the von Neumann algebra $M$ so that the desired character $\varphi_1$ will be given by $\tau \circ \pi_1$.

Consider the set-theoretical map $\pi : Q \to M$ given by $\pi = \pi \circ s$. The map $\pi$ satisfies
\[(7.6) \quad \pi(q_1)\pi(q_2) = c_\chi(q_1, q_2)\pi(q_1q_2) \quad \forall q_1, q_2 \in Q.
\]

Next, consider the map $\bar{\pi}_1 : Q \to M$ given by
\[(7.7) \quad \bar{\pi}_1(q) = \eta(q)\pi(q)
\]
for some choice of a map $\eta \in (S^1)^Q$. Define
\[(7.8) \quad \pi_1 : G \to M, \quad \pi_1(s(q)z) = \pi(q)\chi_1(z) \quad \forall q \in Q, z \in Z.
\]

The map $\pi_1$ defined in Equation (7.3) is a group representation if and only if the condition
\[(7.9) \quad \bar{\pi}_1(q_1)\pi_1(q_2) = c_{\chi_1}(q_1, q_2)\pi_1(q_1q_2) \quad \forall q_1, q_2 \in Q
\]
holds where $c_{\chi_1} = \chi_1 \circ c$. Substituting Equation (7.7) into Equation (7.9) gives
\[(7.10) \quad \eta(q_1)\eta(q_2)\pi(q_1)\pi(q_2) = \eta(q_1q_2)c_{\chi_1}(q_1, q_2)\pi(q_1q_2).
\]
Rearranging and using the definition of the cocycle $c_\chi$ as in Equation (7.6) we obtain that the map $\pi_1$ is a group representation if and only if
\[(7.11) \quad [c_{\chi_1}^{-1}] = [\beta_0] = 0 \in H^2(Q, S^1).
\]

As the 2-cocycle $[c_\chi]$ depends continuously on the choice of the character $\chi \in \text{Ch}(Z)$ and as the co-boundaries $B^2(Q, S^1)$ form an open subgroup of the group of co-chains $Z^2(Q, S^1)$ we deduce that $\pi_1$ can be made into a group representation for all characters $\chi'$ sufficiently close to $\chi$ and for some suitable choice of $\eta \in (S^1)^Q$.

In order to find a character $\varphi_1 \in r^{-1}(\chi_1)$ which is close to the given character $\varphi$, the point $\eta$ has to be close to the identity in the compact abelian group $(S^1)^Q$. This can always be achieved taking into account Equation (7.3) and as a quotient map of topological groups is open. We conclude that the restriction map $r$ is open. \(\square\)

**Characters of FC-groups.** Recall that a finitely generated group is virtually central if and only if it is an FC-group.

**Theorem 7.3.** Let $G$ be a finitely generated FC-group and $Z \leq G$ be a central subgroup satisfying $[G : Z] < \infty$. Then the restriction map $r : \text{Ch}(G) \to \text{Ch}(Z)$ is a finite-sheeted cover.

**Proof of Theorem 7.3.** The restriction map $r$ is continuous, surjective and has finite fibers. The restriction map $r$ is open by Proposition 7.2. Hence $r$ is a local homeomorphism. Generally speaking, any proper local homeomorphism of connected, locally path-connected metric spaces is a covering map [Lee10 p. 303]. \(\square\)

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9 A group $G$ is called an FC-group if every conjugacy class of $G$ is finite. See [Sclo12 §15.1] for more information on FC-groups.
We turn to studying the dynamics of the dual action on the space of characters of FC-groups. It turns out that the action of a certain finite index subgroup of automorphisms on each connected component of this space is conjugate to toral automorphisms.

In order to state a more precise conclusion we need some terminology. An action \( \alpha_1 \) on the torus \( \mathbb{T}^k \cong \mathbb{R}^k / \mathbb{Z}^k \) by toral automorphisms is called a finite algebraic factor of some other action \( \alpha_2 \) if there is a homomorphism \( f : \mathbb{T}^k \to \mathbb{T}^k \) with finite fibers such that \( \alpha_1 \circ f = f \circ \alpha_2 \). Two such actions \( \alpha_1 \) and \( \alpha_2 \) are called weakly algebraically isomorphic if each one is a finite algebraic factor of the other. This is an equivalence relation. It is not hard to see that if \( \alpha_1 \) is a finite algebraic factor of \( \alpha_2 \) then the two actions \( \alpha_1 \) and \( \alpha_2 \) are in fact weakly algebraically isomorphic.

For all this see [KN11, §2.2.1].

**Proposition 7.4.** Let \( G \) be a finitely generated FC-group. Then there exists a finite index subgroup of \( \text{Aut}(G) \) whose dual action preserves each connected component of the space \( \text{Ch}(G) \) and its action on each connected component is topologically conjugate to a linear action on a torus. Moreover these linear actions are weakly algebraically isomorphic to the dual action on the torus \( \text{Ch}(\mathbb{Z}(G)) \).

We precede the proof of Proposition 7.3 with a general result on equivariant covers of tori.

**Lemma 7.5.** Let \( T_1 \) and \( T_2 \) be a pair of tori. Let \( \Gamma \) be a group acting on \( T_1 \) as well as on \( T_2 \) by homeomorphisms such that there is a finite-sheeted \( \Gamma \)-equivariant covering map \( r : T_1 \to T_2 \). Assume that the \( \Gamma \)-action on the torus \( T_2 \) is by toral automorphisms. Then \( \Gamma \) admits a finite-index subgroup \( \Gamma_0 \) such that the \( \Gamma_0 \)-action on the torus \( T_1 \) is topologically conjugate to toral automorphisms.

**Proof.** Assume that the two tori \( T_1 \) and \( T_2 \) are \( k \)-dimensional for some \( k \in \mathbb{N} \). Let \( p_1 : \mathbb{R}^k \to T_1 \) be the universal covering map of the torus \( T_1 \). The composition \( p_2 = r \circ p_1 \) is the universal covering map of the torus \( T_2 \). Consider the finite-index subgroup \( \Lambda = r_* \pi_1(T_1) \) of the fundamental group \( \pi_1(T_2) \cong \mathbb{Z}^k \). Up to topological conjugacy, we may identify the pair of tori \( T_1 \) and \( T_2 \) with the two quotient spaces \( \mathbb{R}^k / \Lambda \) and \( \mathbb{R}^k / \mathbb{Z}^k \) respectively. In particular the point \( 0 \in \mathbb{R}^k \) satisfies \( p_1(0) = [0] \in \mathbb{R}^k / \Lambda \) and \( p_2(0) = [0] \in \mathbb{R}^k / \mathbb{Z}^k \).

The action of the group \( \Gamma \) on the torus \( T_2 \) by toral automorphisms lifts to a linear action on the universal cover \( \mathbb{R}^k \) that fixes the point \( 0 \in \mathbb{R}^k \). Such an action is given by some homomorphism \( f : \Gamma \to \text{GL}_k(\mathbb{Z}) \). The two covering maps \( p_1 \) and \( p_2 \) are \( \Gamma \)-equivariant with respect to this action.

Let \( \Gamma_0 \) be the stabilizer of the point \( [0] \in \mathbb{R}^k / \Lambda \cong T_1 \) for the action of the group \( \Gamma \) on the torus \( T_1 \). Since the \( \Gamma \)-action on torus \( T_2 \) fixes the point \( [0] \in \mathbb{R}^k / \mathbb{Z}^k \cong T_2 \) and as the covering \( r \) is finite-sheeted we have that \( [\Gamma : \Gamma_0] < \infty \). The restricted \( \Gamma_0 \)-action on the universal cover \( \mathbb{R}^k \) preserves the \( \Lambda \)-orbit of the point \( 0 \in \mathbb{R}^k \). This orbit can be naturally identified with \( \Lambda \) itself regarded as a subgroup of the universal cover \( \mathbb{R}^k \). Finally, the restriction \( f|_{\Gamma_0} : \Gamma_0 \to \text{GL}_k(\mathbb{Z}) \) descends to an action of the subgroup \( \Gamma_0 \) on the torus \( T_1 \) by toral automorphisms.

**Proof of Proposition 7.4.** Let \( d \in \mathbb{N} \) denote the rank of center \( \mathbb{Z}(G) \) of the group \( G \). The dual space \( \text{Ch}(\mathbb{Z}(G)) \) can be identified with a disjoint union of finitely many \( d \)-dimensional tori. The group \( G \) satisfies \( [G : \mathbb{Z}(G)] < \infty \). Therefore the space \( \text{Ch}(G) \) is a finite-sheeted cover of \( \text{Ch}(\mathbb{Z}(G)) \) by Theorem 7.3. As such \( \text{Ch}(G) \) is...
also a disjoint union of $d$-dimensional tori. The group $\text{Aut}(G)$ acts on the space $\text{Ch}(G)$ by homeomorphisms. Let $\Gamma \leq \text{Aut}(G)$ be a finite index subgroup that preserves each connected component of the space $\text{Ch}(G)$. We conclude the proof by relying on Lemma 7.5 individually with respect to each connected component of this cover. The fact that the action on each connected component is weakly algebraically isomorphic to the dual action on the torus $\text{Ch}(\mathbb{Z}(G))$ follows from the discussion on [KN11, p. 54]. □

Characters with finite index kernel. We consider the behaviour of characters of FC-groups whose kernel has finite index.

**Lemma 7.6.** Let $A$ be a discrete abelian group and $\Gamma \leq \text{Aut}(A)$ be any subgroup such that $\Gamma \ltimes A$ is finitely generated. Let $\varphi \in \text{Ch}(A)$ be a $\Gamma$-invariant character. Then there is a sequence of $\Gamma$-invariant characters $\varphi_n \in \text{Ch}(A)$ with $[A : \ker \varphi_n] < \infty$ satisfying $\varphi = \lim_n \varphi_n$ in the pointwise topology.

**Proof.** Let $A_{\Gamma}$ denote the $\Gamma$-invariant subgroup of $A$ generated by all elements of the form $\gamma_1 a - \gamma_2 a$ where $a \in A$ and $\gamma_1, \gamma_2 \in \Gamma$. The group of co-invariants $A/A_{\Gamma}$ is finitely generated as the group $\Gamma \ltimes A$ is assumed to be finitely generated. Note that $\varphi|_{A_{\Gamma}} = 1$. The Pontryagin dual of the quotient $A/A_{\Gamma}$ can be identified with the closed subgroup

$$
\hat{A}_{\Gamma} = \{ \chi \in \hat{A} : \gamma \chi = \chi \forall \gamma \in \Gamma \}
$$

of the Pontryagin dual $\hat{A}$ consisting of the $\Gamma$-invariant characters. The action of the group $\Gamma$ on the quotient $A/A_{\Gamma}$ and on its dual $\hat{A}_{\Gamma}$ is trivial.

The character $\varphi$ can be regarded as a character of the quotient group $A/A_{\Gamma}$. As the quotient $A/A_{\Gamma}$ is finitely generated the character $\varphi$ is a pointwise limit of characters of $A/A_{\Gamma}$ with finite index kernels converging to $\varphi$. □

**Lemma 7.7.** Let $G$ be a finitely generated FC-group. Let $\Gamma \leq \text{Aut}(G)$ be any subgroup of automorphisms. Then there is a finite index subgroup $\Gamma_0 \leq \Gamma$ such that any finitely supported $\Gamma_0$-invariant probability measure on the space $\text{Ch}(G)$ is a weak-$\ast$ limit of $\Gamma_0$-invariant probability measures supported on finitely many characters whose kernels all have finite index in $G$.

**Proof.** Consider the restriction map $r : \text{Ch}(G) \to \text{Ch}(\mathbb{Z}(G))$. A given character $\varphi \in \text{Ch}(G)$ satisfies $[G : \ker \varphi] < \infty$ if and only if $[\mathbb{Z}(G) : \ker r(\varphi)] < \infty$. The latter condition holds true if and only if the restriction $r(\varphi)$ is a torsion element of the dual group $\hat{\mathbb{Z}(G)}$.

Let $\Gamma_0$ denote the intersection of the subgroup $\Gamma$ with the finite index subgroup of $\text{Aut}(G)$ provided by Proposition 7.4. The group $\Gamma_0$ preserves every connected component of the space $\text{Ch}(G)$ and acts on it linearly (up to topological conjugacy). Additionally it follows from Proposition 7.4 that the preimage under the restriction map $r$ of the torsion subgroup of $\text{Ch}(\mathbb{Z}(G))$ can be identified with the torsion subgroup of every connected component of $\text{Ch}(G)$ regarded as a compact abelian group. The desired conclusion follows from Lemma 7.6. □
8. Dense periodic measures

Let $G$ be a discrete group acting on a compact metrizable space $X$ by homeomorphisms. The pair $(G, X)$ is called a topological dynamical system. We restrict our attention to the situation where the acting group $G$ is amenable.

Definition. The system $(G, X)$ has dense periodic measures if every $G$-invariant Borel probability measure on the space $X$ is a weak-$\ast$ limit of $G$-invariant probability measures with finite supports.

For the purpose of showing that the system $(G, X)$ has dense periodic measures it suffices to consider ergodic $G$-invariant probability measures on $X$. In that case, it is always possible to take the limiting $G$-invariant probability measures to be the uniform probability measures supported on certain finite $G$-orbits [LL19a, §2]¹⁰.

Remark 8.1. Throughout this work it will most often be the case that the space $X$ is a compact abelian group and that the group $G$ is acting on $X$ by group automorphisms.

Hereditary properties. We discuss the behaviour of the density of periodic measures for topological dynamical systems with respect to some basic operations.

Lemma 8.2. Let $H \leq G$ a finite index normal subgroup. If $(H, X)$ has dense periodic measures then so does $(G, X)$.

Proof. Let $\mu$ be an ergodic $G$-invariant probability measure on $X$. Consider the ergodic decomposition of the probability space $(X, \mu)$ with respect to the $H$-action, see e.g. [ETW13, §4.2]. The space $Z$ of the $H$-ergodic components admits an ergodic $G/H$-action. Therefore the space $Z$ is isomorphic to the coset space $G/K$ with the uniform probability measure for some finite index subgroup $K$ satisfying $H \leq K \leq G$. It follows that there is an $H$-ergodic probability measure $\nu$ on $X$ so that

$\mu = \frac{1}{[G : K]} \sum_{g \in G/K} g_* \nu = \frac{1}{[G : H]} \sum_{g \in G/H} g_* \nu.$

The system $(H, X)$ has dense periodic measures by assumption. Therefore there is a sequence $\nu_n$ of finitely supported $H$-invariant probability measures on $X$ converging to $\nu$ in the weak-$\ast$ topology. The sequence of finitely supported $G$-invariant probability measures

$\mu_n = \frac{1}{[G : K]} \sum_{g \in G/K} g_* \nu_n$

converges to the measure $\mu$ in the weak-$\ast$ topology. □

The converse direction of Lemma 8.2 seems more challenging, see Question 1 in the introduction.

Proposition 8.3. Let $(G, X)$ and $(G, Y)$ be a pair of topological dynamical systems and $p : X \to Y$ a continuous, surjective and $G$-equivariant map. Assume that there is a $G$-equivariant Borel map $Y \to \text{Prob}(X)$ taking a point $y \in Y$ to a probability measure $\nu_y \in \text{Prob}(X)$ with $\text{supp}(\nu_y) \subset p^{-1}(y)$. If $(G, X)$ has dense periodic measures then so does $(G, Y)$.

¹⁰The paper [LL19a] deals with the special case of the topological dynamical system $(G, \text{Sub}(G))$, however the proof of the above mentioned fact is true more generally.
Proof. Let $\mu$ be any $G$-invariant probability measure on the space $Y$. We wish to show that $\mu$ is a limit of finitely supported $G$-invariant probability measures. To do so, consider the $G$-invariant probability measure $\nu$ on the space $X$ given by

$$\nu = \int_Y \nu_y \, d\mu.$$  

Note that $p_*\nu = \mu$. As the system $(G, X)$ has dense periodic measures there is a sequence of $G$-invariant finitely supported probability measures $\nu_n$ on $X$ converging to $\nu$ in the weak-* topology. The $G$-invariant pushforward measures $p_*\nu_n$ have finite supports and converge to the measure $\mu$ on the space $Y$ in the weak-* topology. \hfill $\square$

**Corollary 8.4.** Let $(G, X)$ and $(G, Y)$ be a pair of topological dynamical systems and $p : X \to Y$ a continuous, surjective and $G$-equivariant map. Assume either that

1. there is a $G$-equivariant Borel section $s : Y \to X$ of the map $p$, or
2. the system $(G, X)$ is a compact group extension of the system $(G, Y)$, or
3. the fibers of the map $p$ are all finite.

If the system $(G, X)$ has dense periodic measures then so does the system $(G, Y)$.

**Proof.** Assume that the system $(G, X)$ has dense periodic measures. We will deduce the density of periodic measures for the system $(G, Y)$ from Proposition 8.3 by constructing a suitable map $Y \to \text{Prob}(X)$.

In case (1) let $s : Y \to X$ be a $G$-equivariant Borel section of the map $p$. It is clear that the map $Y \to \text{Prob}(X)$ defined by $y \mapsto \nu_y = \delta_{s(y)}$ is Borel and $G$-equivariant. It satisfies $\text{supp}(\nu_y) = \{s(y)\} \subseteq p^{-1}(y)$ as required.

In case (2) there is a compact group $K$ admitting a $G$-equivariant continuous action on the space $X$ such that $Y = X/K$ [Gla03 p. 15]. There is a Borel measurable section $s : Y \to X$ by the Jankov–von Neumann theorem, see e.g. [Gla03 Theorem 2.10]. Let $\mu$ denote the Haar probability measure on the compact group $K$. Define a Borel map $Y \to \text{Prob}(X)$ taking each point $y \in Y$ to the pushforward $\nu_y$ of the Haar measure $\mu$ via the action map $K \to Ks(y)$. It remains to verify that the map $y \mapsto \nu_y$ is $G$-equivariant. Let $y_1, y_2 \in Y$ be a pair of points with $gy_1 = y_2$ for some element $g \in G$. It follows that $s(y_2) = k_0gs(y_1) = gk_0s(y_1)$ for some element $k_0 \in K$. To conclude observe that

$$\nu_{y_2} = \mu \ast \delta_{s(y_2)} = \mu \ast \delta_{gk_0s(y_1)} = g\left(\mu \ast k_0\delta_{s(y_1)}\right) = g\nu_{y_1}.$$  

In case (3) assume that the surjective map $p : X \to Y$ has finite fibers. We need to show that the $G$-equivariant map $Y \to \text{Prob}(X)$ assigning to each point $y \in Y$ the uniform measure $\nu_y$ on the finite set $p^{-1}(y)$ is Borel. We may write the compact space $Y$ as a disjoint union $Y = \bigsqcup_{n \in \mathbb{N}} Y_n$ of $G$-invariant Borel sets so that $|p^{-1}(y)| = n$ for all points $y \in Y_n$. Apply the Kuratowski–Ryll-Nardzewski selection theorem [Kec12 Theorem 12.13] to find Borel maps $f_{n,i} : Y_n \to X$ for all $n \in \mathbb{N}$ and all $i \in \{1, \ldots, n\}$ satisfying

$$p^{-1}(y) = \{f_{n,1}(y), f_{n,2}(y), \ldots, f_{n,n}(y)\} \quad \forall n \in \mathbb{N}, \forall y \in Y_n.$$  

For any continuous function $F \in C(X)$ and any point $y \in Y$ the integral $\nu_y(F)$ can be evaluated as

$$\nu_y(F) = \frac{1}{n} \sum_{i=1}^{n} F(f_{n,i}(y)) \quad \forall n \in \mathbb{N}, \forall y \in Y_n.$$  

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It follows that the map $y \mapsto \nu_y$ is indeed Borel measurable with respect to the weak-$\ast$ topology on the space $\text{Prob}(X)$. This concludes the proof. □

**The specification property.** Any $\mathbb{Z}^d$-action which enjoys the so called *periodic specification property* has dense periodic measures. For an in-depth discussion of various flavors of this property we refer the reader to [KLO16].

Bowen introduced the specification property and established it for Axiom A diffeomorphisms, a class of dynamical systems including hyperbolic torus automorphisms [Bow71]. See also Sigmund’s treatment in [Sig70, Sig74]. Lind and Schmidt obtained necessary and sufficient conditions for a very general family of commuting automorphisms of compact abelian groups to have specification [LS99].

**Ergodic torus automorphisms.** Non-hyperbolic ergodic torus automorphisms do not enjoy specification in the strict sense [Lin79]. Nevertheless there is a useful periodic analogue of *weak specification* for such systems. It was used by Marcus [Mar80] to obtain the following.

**Theorem 8.5 (Marcus).** Let $A$ be an automorphism of the torus $\mathbb{T}^k$ ergodic with respect to the Haar measure. Then the topological dynamical system $(\mathbb{Z}, \mathbb{T}^k)$ has dense periodic measures.

**Bernoulli shifts.** Topological dynamical systems arising from Bernoulli shifts have dense periodic measures. Most of these facts are well-known and appear in the literature.

**Proposition 8.6.** Let $G$ be any residually finite amenable group. Then for any compact set $K$ the Bernoulli system $(G, K^G)$ has dense periodic measures with respect to the shift action.

The case where the acting group is $\mathbb{Z}$ was established in [Par61, Theorem 3.3]. The proof we present below relies on the pointwise ergodic theorem for amenable groups [Lin01] in the manner of [Sig70, Lemma 1] and [Mar80, Main Theorem]. Moreover it uses ideas from [LL19a, Theorem 3.10].

**Proof of Proposition 8.6** Fix a descending chain $N_i$ of finite index normal subgroups of the group $G$ with trivial intersection. The main result of [Wei01] allows us to find a Følner sequence $F_i$ of coset transversals to the subgroups $N_i$. For each index $i \in \mathbb{N}$ and every element $g \in G$ let $f_i(g) \in F_i$ denote the unique group element satisfying $f_i(g)N_i = gN_i$. Given any point $x \in K^G$ we define a sequence of new points $x_i \in K^G$ via

$$x_i(g) = x(f_i(g)) \quad \forall g \in G.$$  

(8.7)

Observe that the point $x_i$ is $N_i$-periodic for every $i \in \mathbb{N}$, namely $gx_i = x_i$ holds true for all elements $g \in N_i$.

Let $\mu$ be an arbitrary $G$-invariant probability measure on the space $K^G$. We may assume without loss of generality that the measure $\mu$ is ergodic. The sequence of finitely supported probability measures $\mu_{x,i} = \frac{1}{|F_i|} \sum_{g \in F_i} \delta_{gx}$ weak-$\ast$ converges to the probability measure $\mu$ for $\mu$-almost every point $x$ by the pointwise ergodic theorem, see [LL19b, Theorem 3.9] for details. On the other hand, as the sequence $F_i$ is Følner, the sequence of $G$-invariant finitely supported probability measures

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$\dagger$ A torus automorphism is ergodic with respect to the Haar measure if and only if all of its eigenvalues are not roots of unity.
\[ \nu_{x,i} = \frac{1}{|F_i|} \sum_{g \in F_i} \delta_{g x_i} \] satisfies \( d(\mu_{x,i}, \nu_{x,i}) \to 0 \) with respect to any compatible metric \( d \) on the space \( \text{Prob}(K^G) \) and any point \( x \in K^G \). The desired conclusion follows. \( \square \)

**Proposition 8.7.** Let \( S_p = \lim \leftarrow \mathbb{R}/p^n \mathbb{Z} \) be \( p \)-adic solenoid. Consider the dynamical system \((\mathbb{Z}, S_p)\) where the group \( \mathbb{Z} \) is acting via multiplication by \( p \). The system \((\mathbb{Z}, S_p)\) has dense periodic measures.

**Proof.** Let \( X_p = \{0, \ldots, p-1\} \mathbb{Z} \) be a Bernoulli system with the shift action. There is a \( \mathbb{Z} \)-equivariant continuous surjection \( X_p \to S_p \) with finite fibers. We conclude that the system \((\mathbb{Z}, S_p)\) has dense periodic measures by combining Proposition 8.6 with Corollary 8.4. \( \square \)

### 9. Characters and Hilbert–Schmidt stability

We formulate the precise definition of Hilbert–Schmidt stability for discrete groups. Let \( U(n) \) denote the unitary group of degree \( n \) for each \( n \in \mathbb{N} \). The normalized Hilbert–Schmidt norm \( \|A\|_{\text{HS}} \) on the group \( U(n) \) is

\[
\|A\|_{\text{HS}} = \sqrt{\frac{1}{n} \text{tr}(A^* A)} \quad \forall A \in U(n).
\]

The corresponding normalized Hilbert–Schmidt metric on the group \( U(n) \) is given by

\[
d_{\text{HS}}(A, B) = \|A - B\|_{\text{HS}} \quad \forall A, B \in U(n).
\]

This metric is bi-invariant in the sense that

\[
d_{\text{HS}}(A, B) = d_{\text{HS}}(CAD, CBD) \quad \forall A, B, C, D \in U(n).
\]

**Definition.** Let \( G \) be a discrete group. An asymptotic homomorphism of the group \( G \) is a sequence of set-theoretic maps \( f_n : G \to U(n) \) for all \( n \in \mathbb{N} \) satisfying

\[
d_{\text{HS}}(f_n(g) f_n(h), f_n(gh)) \xrightarrow{n \to \infty} 0 \quad \forall g, h \in G.
\]

The group \( G \) is Hilbert–Schmidt stable if for any asymptotic homomorphism \( f_n : G \to U(n) \) there is a sequence of group homomorphisms \( \varphi_n : G \to U(n) \) such that

\[
d_{\text{HS}}(f_n(g), \varphi_n(g)) \xrightarrow{n \to \infty} 0 \quad \forall g \in G.
\]

The Hadwin–Schulman criterion [HS18] relates Hilbert–Schmidt stability to characters. We proceed with a detailed analysis of this criterion.

**Finite dimensional traces.** Let \( G \) be a discrete group. A trace representation \((M, \pi, \tau)\) of the group \( G \) is called finite dimensional if the von Neumann algebra \( M \) is finite dimensional. A trace on the group \( G \) is called finite dimensional if the trace representation corresponding to it via Theorem 2.1 is finite dimensional. We denote by \( \text{Tr}_{\text{id}}(G) \) the set of all finite dimensional traces on the group \( G \).

**Lemma 9.1.** \( \text{Tr}_{\text{id}}(G) \) is a face of the convex set \( \text{Tr}(G) \).

A convex subset \( F \) of a convex set \( C \) is called a face if \( y, z \in F \) whenever \( x = ty + (1-t)z \) for some points \( x \in F \), \( y, z \in C \) and some \( 0 < t < 1 \), see [Phe01, Definition 16.5].

\[ ^{12} \text{By abuse of notation, we omit the the index } n \text{ from the notations of the Hilbert–Schmidt norm and metric.} \]
Proof of Lemma 9.1. Consider a pair of traces $\varphi_1, \varphi_2 \in \text{Tr}(G)$. Let $\varphi$ be a non-trivial convex combination of the traces $\varphi_1$ and $\varphi_2$ so that $\varphi = t\varphi_1 + (1-t)\varphi_2$ for some $t \in (0, 1)$.

We may write $\varphi_1 = s_1\varphi'_1 + (1-s_1)\psi$ and $\varphi_2 = s_2\varphi'_2 + (1-s_2)\psi$ for some real numbers $s_1, s_2 \in [0, 1]$ such that the three traces $\varphi'_1, \varphi'_2$ and $\psi$ are pairwise disjoint. Let $(M_1, \pi_1, \tau_1), (M_2, \pi_2, \tau_2)$ and $(M_3, \pi_3, \tau_3)$ be the trace representations corresponding to $\varphi'_1, \varphi'_2$ and $\psi$ respectively. A direct computation shows that the trace representation corresponding to the trace $\varphi$ is given by $(M, \pi, \tau)$ where

$$ (9.6) \quad \pi = \pi_1 \oplus \pi_2 \oplus \pi_3 \quad \text{and} \quad \tau = ts_1\tau_1 \oplus (1-t)s_2\tau_2 \oplus (t-1)s_1 + (1-t)(1-s_2)\tau_3. $$

The von Neumann algebra $M$ is given by $M = \pi(G)'' \subseteq M_1 \oplus M_2 \oplus M_3$. The fact that the traces $\varphi'_1, \varphi'_2$ and $\psi$ are pairwise disjoint implies that $M = M_1 \oplus M_2 \oplus M_3$.

A similar argument shows that the von Neumann algebras corresponding to the two traces $\varphi_1$ and $\varphi_2$ are $M_1 \oplus M_3$ and $M_2 \oplus M_3$ respectively. We conclude that the trace $\varphi$ is finite dimensional if and only if both traces $\varphi_1$ and $\varphi_2$ are finite dimensional. This means that $\text{Tr}_{fd}(G)$ is convex and is a face of the simplex $\text{Tr}(G)$, as required.

The Hadwin–Shulman criterion is given in terms of the functions $\frac{1}{\dim \pi} \text{tr} \circ \pi$ where $\pi : G \to U(n)$ is some finite dimensional unitary representation. Any such function is a finite dimensional trace on $G$. Indeed its corresponding trace representation is given by $(\pi(G)'', \pi, \frac{1}{\dim \pi} \text{tr})$. However, not every finite dimensional trace is a normalized trace of some finite dimensional unitary representation. An example for the group $G = \mathbb{Z}/2\mathbb{Z}$ is provided by the trace $\varphi = t\chi_1 + (1-t)\chi_{-1}$ for any irrational $t \in [0, 1]$. This nuance is clarified by the following lemma.

**Lemma 9.2.** Let $\varphi : G \to \mathbb{C}$ be a function. The following two conditions are equivalent.

1. $\varphi = \frac{1}{\dim \pi} \text{tr} \circ \pi$ for some finite dimensional unitary representation $\pi$.
2. $\varphi$ is a finite rational convex combination of finite dimensional characters.

**Proof.** Assume that the function $\varphi$ is given by $\frac{1}{\dim \pi} \text{tr} \circ \pi$ for some finite dimensional unitary representation $\pi$ of the group $G$. Let $\pi = \bigoplus_{i=1}^d \pi_i$ be the decomposition of the representation $\pi$ into isotypic components, that is, each subrepresentation $\pi_i$ is the sum of all the irreducible subrepresentations of $\pi$ sharing the same isomorphism type. By Schur’s Lemma each von Neumann algebra $\pi_i(G)''$ is a factor. Therefore each function $\varphi_i = \frac{1}{\dim \pi} \text{tr} \circ \pi_i$ is a finite dimensional character. We have

$$ (9.7) \quad \varphi = \frac{1}{\dim \pi} \text{tr} \circ \pi = \frac{1}{\dim \pi} \sum_{i=1}^d \text{tr} \circ \pi_i = \sum_{i=1}^d \frac{\dim \pi_i}{\dim \pi} \varphi_i $$

so that the function $\varphi$ is a finite rational convex combination of finite dimensional characters.

Consider the converse direction. Assume that the function $\varphi$ is a finite rational convex combination of finite dimensional characters. Therefore $\varphi = \frac{1}{d} \sum_{i=1}^d \varphi_i$ for some $d \in \mathbb{N}$ and some finite dimensional characters $\varphi_1, \ldots, \varphi_d \in \text{Ch}(G)$, possibly with repetitions. Let $(M_i, \pi_i, \tau_i)$ be a trace representation corresponding to each character $\varphi_i$. Since $\varphi_i$ is a finite dimensional character the von Neumann algebra $M_i$ is a finite dimensional factor. In other words $M_i \cong M_{\dim \pi_i}(\mathbb{C})$. Therefore we
must have $\tau_i = \frac{1}{\dim \pi_i} \tr$ and $\varphi_i = \tau_i \circ \pi_i$. Let $m \in \mathbb{N}$ be the least common multiple of the dimensions $\dim \pi_1, \ldots, \dim \pi_d$. Consider the finite dimensional unitary representation $\pi = \bigoplus_{i=1}^d \frac{m}{\dim \pi_i} \pi_i$. Its dimension is $dm$. We conclude that

\begin{equation}
\varphi = \frac{1}{d} \sum_{i=1}^d \varphi_i = \frac{1}{\dim \pi} \sum_{i=1}^d \frac{m}{\dim \pi_i} \tr \circ \pi_i = \frac{1}{\dim \pi} \tr \circ \pi
\end{equation}

as required. □

The above analysis can be used to reformulate the Hadwin–Shulman criterion in the following manner.

**Corollary 9.3** (Hadwin–Shulman [HS18]). Let $G$ be an amenable group. The following conditions are equivalent.

1. The group $G$ is Hilbert–Schmidt stable.
2. Any character of the group $G$ is a pointwise limit of normalized traces of finite dimensional unitary representations.
3. Any trace on the group $G$ is a pointwise limit of normalized traces of finite dimensional unitary representations.
4. The face $\operatorname{Tr}_{fd}(G)$ is dense in the compact convex set $\operatorname{Tr}(G)$.

Indeed the equivalence of (2), (3) and (4) holds true in general and does not depend on amenability.

**Proof of Corollary 9.3**. The equivalence of conditions (1), (2) and (3) is established in [HS18, Theorem 4] combined with [HS18, Lemma 1]. The implication (3) $\Rightarrow$ (4) is clear as normalized traces of finite dimensional unitary representations are a special form of finite dimensional traces. It remains to establish the implication (4) $\Rightarrow$ (3). Recall that the subset $\operatorname{Tr}_{fd}(G)$ of the finite dimensional traces is a face of the convex set $\operatorname{Tr}(G)$ of all traces, see Lemma 9.1. In particular the extreme points of $\operatorname{Tr}_{fd}(G)$ are precisely the finite dimensional characters. The Krein–Milman theorem implies that the subset of the face $\operatorname{Tr}_{fd}(G)$ consisting of all finite rational convex combinations of finite dimensional characters is dense in $\operatorname{Tr}_{fd}(G)$. But this subset coincides with the subset of all normalized traces of finite dimensional unitary representations, see Lemma 9.2. This completes the proof. □

**Dense periodic measures on character spaces.**

**Lemma 9.4.** Let $\varphi$ be a trace on the discrete group $G$ with Fourier transform $\mu_\varphi \in \operatorname{Prob}(\operatorname{Ch}(G))$. If the trace $\varphi$ is finite dimensional then $\operatorname{supp}(\mu_\varphi)$ is finite. The converse direction holds true assuming that the group $G$ is virtually abelian.

**Proof.** Let $\varphi \in \operatorname{Tr}(G)$ be a finite dimensional trace with a corresponding trace representation $(M, \pi, \tau)$. So $M$ is a finite dimensional von Neumann algebra. Then

\begin{equation}
M = \bigoplus_{i=1}^m M_{d_i}(\mathbb{C}), \quad \pi = \bigoplus_{i=1}^m \pi_i \quad \text{and} \quad \tau = \bigoplus_{i=1}^m \alpha_i \tau_i
\end{equation}

for some number $m \in \mathbb{N}$, some dimensions $d_i \in \mathbb{N}$, some finite dimensional unitary representations $\pi_i$ and some real numbers $\alpha_1, \ldots, \alpha_m \in [0,1]$ satisfying $\sum_{i=1}^m \alpha_i = 1$. Here $\tau_i$ is the unique normalized trace $\frac{1}{\alpha_i} \tr$ on the matrix algebra $M_{d_i}(\mathbb{C})$. In particular each function $\varphi_i = \tau_i \circ \pi_i$ is a character and $\varphi = \sum_{i=1}^m \alpha_i \varphi_i$. We conclude that the probability measure $\mu_\varphi$ is supported on the finite set $\{\varphi_1, \ldots, \varphi_m\} \subset \operatorname{Ch}(G)$. 

For the converse direction, assume that the group $G$ is virtually abelian and that $\text{supp}(\mu_\varphi) = \{\varphi_1, \ldots, \varphi_m\} \subseteq \text{Ch}(G)$ for some $m \in \mathbb{N}$. This means that $\varphi = \sum_{i=1}^{m} \alpha_i \varphi_i$ for some real numbers $\alpha_i \in [0, 1]$ satisfying $\sum_{i=1}^{m} \alpha_i = 1$.

As the group $G$ is virtually abelian it has type I and all of its irreducible representations are finite dimensional. Let $(M_i, \pi_i, \tau_i)$ be a trace representation corresponding to each character $\varphi_i$. Each algebra $M_i$ is a type I factor, i.e. $M_i$ is isomorphic to the algebra of all bounded operators on some Hilbert space [Ball12] Chapter 6]. With respect to this isomorphism each representation $\pi_i$ is irreducible. Therefore each $\pi_i$ is finite dimensional so that each algebra $M_i$ is isomorphic to the matrix algebra $M_{d_i}(\mathbb{C})$ for some dimension $d_i \in \mathbb{N}$ and $\tau_i = \frac{1}{d_i} \text{tr}$ is the unique normalized trace on $M_i$. The trace representation corresponding to the character $\varphi$ is given by $(M, \pi, \tau)$ where $\pi = \bigoplus_{i=1}^{m} \pi_i$, $M = \pi'(G) \leq \bigoplus_{i=1}^{m} M_i$ and $\tau = \bigoplus_{i=1}^{m} \alpha_i \tau_i$. In particular the trace $\varphi$ is finite dimensional.

We are now able to deduce the necessary condition for Hilbert–Schmidt stability from the introduction.

**Proof of Proposition**. Let $N$ be an abelian normal subgroup of the group $G$. Consider the dual action of $G$ on $\hat{N}$ by continuous automorphisms. We will assume that the group $G$ is Hilbert–Schmidt stable and infer that the topological dynamical system $(G, \hat{N})$ has dense periodic measures.

Let $\mu$ be any $G$-invariant Borel probability measure on the space $\text{Ch}(N) \cong \hat{N}$. It gives rise to the relative trace $\psi \in \text{Tr}_G(N)$ by taking the Fourier transform of the measure $\mu$. Consider the trivial extension $\varphi = \hat{\psi} \in \text{Tr}(G)$ as defined in Equation 6.22. Since the group $G$ is Hilbert–Schmidt stable the trace $\varphi$ is a pointwise limit of a sequence $\varphi_n$ of finite dimensional traces, see Corollary 9.3. Let $\mu_n$ denote the $G$-invariant Borel probability measure on the space $\text{Ch}(N)$ corresponding to the restrictions $(\varphi_n)|N$. The measures $\mu_n$ have finite supports by Lemma 9.4.

The barycenter map $\text{Prob}(\text{Ch}(N)) \rightarrow \text{Tr}(N)$ is a homeomorphism in this case since the space of characters $\text{Ch}(N) \cong \hat{N}$ is compact. In particular the probability measures $\mu_n$ converge to the measure $\mu$ in the weak-* topology. We conclude that the dynamical system $(G, \hat{N})$ has dense periodic measures.

**Approximation of induced finite dimensional traces.** The following approximation technique will play an important role in our applications towards Hilbert–Schmidt stability. Recall that a subgroup is *profinitely closed* if it is an intersection of finite index subgroups.

**Proposition 9.5.** Let $H$ be a subgroup of $G$ and $\psi \in \text{Tr}(H)$ be an almost $G$-invariant trace. Assume that $[H : \ker \psi] < \infty$ and that $\ker \psi$ is profinitely closed in $G$. Then the trace $\text{Ind}^G_H \psi$ is a limit of finite dimensional traces on the group $G$.

**Proof.** To begin with assume that the subgroup $H$ is normal in $G$ and that the trace $\psi$ is precisely $G$-invariant, i.e. $\psi \in \text{Tr}_G(H)$. These additional assumptions imply that $\ker \psi$ is a normal subgroup of $G$. Up to replacing the group $G$ by its quotient $G/\ker \psi$, we will assume without further loss of generality that the trace $\psi$ is faithful, the subgroup $H$ is finite and the group $G$ is residually finite.

Fix a descending sequence $K_n$ of finite index normal subgroups of the group $G$ with trivial intersection. Up to discarding finitely many subgroups from this sequence one has $K_n \cap H = \{e\}$ for all $n \in \mathbb{N}$. 

Consider the subgroups $G_n = K_n H$. The subgroup $G_n$ is a direct product of the two subgroups $K_n$ and $H$ for all $n \in \mathbb{N}$. Moreover $[G : G_n] < \infty$ for all $n \in \mathbb{N}$. Consider the family of functions $\psi_n : G_n \to \mathbb{C}$ given by

$$\psi_n(kh) = \psi(h) \quad \forall k \in K_n, \forall h \in H.$$  

(9.10)

Each function $\psi_n$ is the composition of the natural projection from $G_n$ to $H$ with the trace $\psi$ on the subgroup $H$. In particular each $\psi_n$ is a finite dimensional trace factoring though the finite group $H$. Denote $\varphi_n = \text{Ind}^G_{G_n} \psi_n$. Each $\varphi_n$ is a finite dimensional trace on the group $G$ by Lemma 9.6 below. Observe that

$$\varphi_n(g) = \text{Ind}^G_{G_n} \psi_n(g) = \widetilde{\psi}_n(g) = \psi_n(g) = \psi(g) \quad \forall n \in \mathbb{N}$$  

(9.11)

for all elements $g \in H$ and that

$$\lim_{n \to \infty} \varphi_n(g) = \lim_{n \to \infty} \text{Ind}^G_{G_n} \psi_n(g) = \lim_{n \to \infty} \widetilde{\psi}_n(g) = 0$$  

(9.12)

for all elements $g \in G \setminus H$. Equations (9.11) and (9.12) put together imply the desired conclusion (given the additional assumptions).

In the general case consider the finite index subgroup $G_\psi$ defined in Equation (3.1). The subgroup $G_\psi$ normalizes the subgroup $H$ and the trace $\psi$ is $G_\psi$-invariant. Relying on the previous paragraphs we find a sequence $\varphi_n$ of finite dimensional traces on the group $G_\psi$ converging to $\text{Ind}^G_H \psi \in \text{Tr}(G_\psi)$. Induction in stages (Lemma 3.1) combined with the continuity of induction (Lemma 3.2) gives

$$\text{Ind}^G_H \psi = \text{Ind}^G_{G_\psi} \text{Ind}^G_{G_\psi} \psi = \text{Ind}^G_{G_\psi} \left( \lim_{n \to \infty} \varphi_n \right) = \lim_{n \to \infty} \text{Ind}^G_{G_\psi} \varphi_n.$$  

(9.13)

The functions on the right-hand side of Equation (9.13) are finite dimensional traces on the group $G$ by Lemma 9.6 below.

Proposition 9.5 is to be compared with [HST18, Theorem 7] as well as with [BLT19, Proposition 8.1].

**Lemma 9.6.** Let $H$ be a finite index subgroup of the group $G$. If $\varphi \in \text{Tr}(H)$ is a finite dimensional trace on the group $H$ then $\text{Ind}^G_H \varphi \in \text{Tr}(G)$ is a finite dimensional trace on the group $G$.

**Proof.** Let $\varphi$ be a finite dimensional trace on the group $H$. This means that $\varphi = \tau \circ \pi$ for some trace representation $(M, \pi, \tau)$ where the von Neumann algebra $M$ is finite dimensional. The induced representation $\text{Ind}^G_H \pi$ of the group $G$ can be viewed as a representation into the group of unitaries of the von Neumann algebra $\mathcal{M}_d(M)$ of $d$-by-$d$ matrices with entries in $M$ where $d = [G : H]$. By comparing the definitions of induced traces and induced representations we see that

$$\text{Ind}^G_H \varphi = \frac{1}{d} (\text{tr} \otimes \tau) \circ \text{Ind}^G_H \pi.$$  

(9.14)

It follows that the trace representation corresponding to $\text{Ind}^G_H \varphi$ is finite dimensional. Therefore the trace $\text{Ind}^G_H \varphi$ is finite dimensional.

## 10. Metabelian Groups and Hilbert–Schmidt Stability

Let $G$ be a finitely generated metabelian group. Assume that $N$ is an abelian normal subgroup of $G$ so that the quotient $G/N$ is abelian. We provide sufficient conditions for the group $G$ to be Hilbert–Schmidt stable.
Theorem 10.1. If the topological dynamical system \((G, \hat{H}^{ab})\) has dense periodic measures for every subgroup \(H\) with \(N \leq H \leq G\), then the metabelian group \(G\) is Hilbert–Schmidt stable.

The notation \(H^{ab}\) stands for abelianization, i.e. \(H^{ab} = H/[H, H]\).

Proof of Theorem 10.1. Assume that the dynamical systems \((G, \hat{H}^{ab})\) as in the statement of the theorem all have dense periodic measures. We will rely on the Hadwin–Shulman criterion by showing that any character of \(G\) is a pointwise limit of finite dimensional traces on the group \(G\), see Corollary 9.3.

Let \(\phi\) be a character of the group \(G\). The classification of characters of metabelian groups (see Theorem 4.1) says that there exists a subgroup \(H\) with \(N \leq H \leq G\) and a \(G\)-invariant trace \(\psi\) on the abelianization \(H^{ab}\) such that \(\phi = \text{Ind}_{H}^{G} \psi\). Let \(\mu\) be the \(G\)-invariant Borel probability measure on the Pontryagin dual \(\hat{H}^{ab}\) corresponding to the trace \(\psi\) via the Fourier transform. The dynamical system \((G, \hat{H}^{ab})\) has dense periodic measures by assumption. Therefore there is a sequence \(\mu_{n}\) of finitely supported \(G\)-invariant probability measures on \(\hat{H}^{ab}\) converging to the measure \(\mu\) in the weak-\(\ast\) topology. Let \(\psi_{n} \in \text{Tr}(H^{ab})\) be the Fourier transform of the measure \(\mu_{n}\) for each \(n\). The traces \(\psi_{n}\) are \(G\)-invariant and satisfy \(\lim_{n} \psi_{n} = \psi\) by the continuity of the Fourier transform over abelian groups. Furthermore each trace \(\psi_{n}\) is finite dimensional by Lemma 9.4. Continuity of induction (Lemma 3.2) gives

\[
\lim_{n} \text{Ind}_{H}^{G} \psi_{n} = \text{Ind}_{H}^{G} \lim_{n} \psi_{n} = \text{Ind}_{H}^{G} \psi = \phi.
\]

Denote \(\Gamma = G/H\) and regard \(\Gamma\) as a subgroup of \(\text{Aut}(H^{ab})\). The semidirect product \(\Gamma \times H^{ab}\) is finitely generated. In this situation Lemma 7.6 allows us to assume without loss of generality that \([H^{ab} : \ker \psi_{n}] < \infty\) for all \(n\). Each \(\ker \psi_{n}\) is a normal subgroup of the finitely generated metabelian group \(G\). As such each \(\ker \psi_{n}\) is profinitely closed in the group \(G\) by a classical theorem of Hall [Hal59]. We conclude that each induced trace \(\text{Ind}_{H}^{G} \psi_{n}\) is a limit of finite dimensional traces by making use of Proposition 9.6.

\[\square\]

Theorem D of the introduction is a special case of the above Theorem 10.1.

Hilbert–Schmidt stable metabelian groups. We are ready to present the proofs of the theorems from the introduction dealing with metabelian groups.

Proposition 10.2. Let \(G = \mathbb{Z} \times N\) be a finitely generated metabelian group. The topological dynamical system \((\mathbb{Z}, \hat{N})\) has dense periodic measures if and only if the group \(G\) is Hilbert–Schmidt stable.

Proof. Assume that \((\mathbb{Z}, \hat{N})\) has dense periodic measures. Let \(H\) be any fixed normal subgroup of \(G\) satisfying \(N \leq H \leq G\). It will suffice to show that the dynamical system \((G, \hat{H}^{ab})\) has dense periodic measures according to Theorem 10.1. There are two separate cases to consider. If \(H = N\) then \(\hat{H}^{ab} = \hat{N}\) and the system in question has dense periodic measures by assumption. Otherwise \(N \leq H\), or equivalently \([G : H] < \infty\). In that case the \(G\)-action on the space \(\hat{H}^{ab}\) factors through the finite quotient group \(G/H\) and as such it certainly has dense periodic measures. The converse direction is an immediate consequence of Proposition 10.1. \(\square\)
Proof of Corollary [7]. We show that the metabelian groups listed in the statement are Hilbert–Schmidt stable.

(1) Let $M_k$ be the free metabelian group of rank $k$. In other words $M_k \cong F_k/F_k''$ where $F_k$ is the free group of rank $k$. Let $H$ be any subgroup of the free metabelian group satisfying $H' \leq M_k$. To prove that the group $M_k$ is Hilbert–Schmidt stable it suffices by Theorem [10.1] to show that the dynamical system $(M_k, \widehat{H}^{ab})$ has dense periodic measures. Consider the subgroup $R$ of the free group $F_k$ satisfying $F_k' \leq R \leq F_k$ and corresponding to the subgroup $H$ via the correspondence theorem. Since $F_k' \leq R$ we have $F_k'' \leq R''$. In particular the subgroup $R'' \leq F_k$ corresponds to the subgroup $H'/M_k$ via the correspondence theorem. The third isomorphism theorem says that $H/H' \cong R/R''$. In other words $H^{ab}$ and $R^{ab}$ are isomorphic as $M_k$-modules. Using the method of free differential calculus one can show that the $M_k$-module $R^{ab}$ is isomorphic to a submodule of the $M_k$-module $\bigoplus_{i=1}^k \mathbb{Z}[A]$ where $A \cong M_k/H \cong F_k/R$, see [Loh97] Chapter 11, Theorem 1 for details. The Bernoulli dynamical system $(M_k, \prod_{A} \mathbb{T}^k)$ is a compact extension of the dynamical system $(M_k, \widehat{H}^{ab})$. Any Bernoulli dynamical system has dense periodic measures by Proposition [8.6]. The same is true for the dual dynamical system $(M_k, \widehat{H}^{ab})$ by Item (2) of Proposition [8.3].

(2) Consider the wreath product $G = A \wr \mathbb{Z}^d$ where $A$ is any finitely generated abelian group. Denote $N = \bigoplus_{\mathbb{Z}^d} A$, so that the normal subgroup $N$ as well as the quotient group $G/N$ are abelian. Let $H$ be any subgroup with $N \leq H \leq G$. To prove that the group $G$ is Hilbert–Schmidt stable it suffices by Theorem [10.1] to show that the dynamical system $(G, \widehat{H}^{ab})$ has dense periodic measures. We claim that $[H,N] = [H,H]$. Indeed given any pair of elements $qn, rm \in G$ with $q,r \in \mathbb{Z}^d$ and $n,m \in N$ standard commutator identities give

$$[qn, rm] = [q, m] - [r, n].$$

The claim follows. As the quotient $H/N$ is a free abelian group we obtain the direct sum $H^{ab} \cong (H/N) \oplus N/[H,N]$. The dual dynamical system $(G, \widehat{H}^{ab})$ is isomorphic to the product system $(G, \widehat{H}/\widehat{N} \times X)$ where $\widehat{H}/\widehat{N}$ is a finite dimensional torus and

$$X = \widehat{N}^H = \{x \in \widehat{N} : hx = x \ \forall h \in H\}.$$ 

The $G$-action is trivial in the first factor and corresponds to the dual action in the second factor. Note that the topological dynamical system $(G/H, X)$ is a Bernoulli system over a certain compact abelian group. At this point we may conclude exactly as in case (1).

(3) The Baumslag–Solitar group $BS(1, n)$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}[1/n]$. The Pontryagin dual of the abelian group $\mathbb{Z}[1/n]$ is the $n$-adic solenoid $S_n$ and the dual action corresponds to multiplication by $n$. The dynamical system $(Z, S_n)$ has dense periodic measures by Proposition [8.7]. We conclude relying on Proposition [10.2].

(4) Consider the group $G = \mathbb{Z} \rtimes_n \mathbb{Z}^d$ where $n \in GL_d(\mathbb{Z})$ is an ergodic torus automorphism. The topological dynamical system $(Z, \mathbb{T}^d)$ has dense periodic measures by Theorem [8.5]. We conclude relying on Proposition [10.2].

More generally, consider any metabelian group of the form $G = \mathbb{Z} \rtimes_n \mathbb{Z} \lceil x, x^{-1} \rceil / \langle p \rangle$ where $p \in \mathbb{Z} \langle x, x^{-1} \rangle$ is not of the form $p(x, x^{-1}) = x^n c(x^m)$ for some cyclotomic polynomial $c$ and some $n, m \in \mathbb{Z}$. Similar methods can be used to show that the group $G$ is Hilbert–Schmidt stable. Indeed the corresponding topological dynamical
system is expansive and has completely positive entropy \[\text{Sch12}\]. Therefore it has periodic specification \[\text{LS99, Theorem 5.2}\]. As such it has dense periodic measures.

**Proof of Corollary** \[A\] Let \( k \) be a non-Archimedean local field with ring of integers \( \mathcal{O} \). Let \( A \) be any infinite subgroup of \( \mathcal{O}^* \). We will show that the group \( G = A \ltimes \hat{\mathcal{O}} \) is not Hilbert–Schmidt stable by showing that the topological dynamical system \( (A, \hat{\mathcal{O}}) \) does not have dense periodic measures, see Proposition \[A\]. Indeed the singleton \( \{0\} \) is the only finite \( A \)-orbit in the dual action on the Pontryagin dual group \( \hat{\mathcal{O}} \). In particular the Haar measure on \( \hat{\mathcal{O}} \) is certainly not a weak-\(^*\) limit of \( G \)-invariant probability measures of finite support. \( \square \)

Let \( G \) be a discrete group and \( X \) a compact abelian group admitting a \( G \)-action by continuous automorphisms. This action is said to satisfy the *descending chain condition* (d.c.c) if every descending chain of closed \( G \)-invariant subgroups of \( X \) stabilizes. The topological dynamical system \((G, X)\) satisfies the descending chain condition if and only if the Pontryagin dual abelian group \( \hat{X} \) is a Noetherian module over the group ring \( \mathbb{Z}[G] \). See \[\text{Sch12}\] for more on this property.

**Proof of Theorem** \[A\] \( (1) \Rightarrow (2) \). Assume that all finitely generated metabelian groups are Hilbert–Schmidt stable. Let \((G, X)\) be a topological dynamical system where \( G \) is a finitely generated abelian group and \( X \) a compact abelian group admitting a \( G \)-action by automorphisms satisfying the d.c.c. Choose any finite index torsion-free subgroup \( G_0 \leq G \). The metabelian group \( G_0 \ltimes \hat{X} \) is finitely generated and hence Hilbert–Schmidt stable by the assumption. Therefore the topological dynamical system \((G_0, X)\) has dense periodic measures by Proposition \[A\]. The density of periodic measures for the system \((G, X)\) follows from Lemma \[8.2\].

\( (2) \Rightarrow (1) \). Assume that any \( \mathbb{Z}^d \)-action by automorphisms on any compact abelian group satisfying the d.c.c has dense periodic measures. Let \( G \) be any finitely generated metabelian group. The assumption implies that the topological dynamical system \((G, \hat{H}^{ab})\) has dense periodic measures for any normal subgroup \( H \triangleleft G \). It follows from Theorem \[10.1\] that the group \( G \) is Hilbert–Schmidt stable. \( \square \)

## 11. Polycyclic groups and Hilbert–Schmidt stability

This last section deals with various questions concerning the Hilbert–Schmidt stability of virtually polycyclic groups.

We start by developing a sufficient condition for such groups to be stable in terms of the density of periodic measures for certain associated dynamical systems, see Theorem \[11.1\]. We proceed by constructing a family a non-metabelian polycyclic stable groups via a certain algebraic idea of rank-one modules, see Theorem \[11.4\]. Finally we explore the connection between the higher-rank measure rigidity conjecture and stability relying on some algebraic number theory.

For the reader’s convenience we restate Theorem \[H\] of the introduction.

**Theorem 11.1.** Let \( G \) be a virtually polycyclic group. Assume that for any finite index subgroup \( H \leq G \) and any quotient \( H \to L \) the dual dynamical system
\[(11.1) \quad (L, \text{Ch}(\mathbb{Z}^{\text{vFit}(L)}))\]
has dense periodic measures. Then the group \( G \) is Hilbert–Schmidt stable.
Proof. Let \( \varphi \) be any character of the group \( G \). In light of the Hadwin–Shulman criterion formulated in Corollary \([9,3]\) it will suffice to show that the character \( \varphi \) is a pointwise limit of finite dimensional traces on the group \( G \).

According to our study of the character theory of virtually polycyclic groups (Theorem \([??]\)) there exist a finite index subgroup \( H \leq G \), a quotient \( L \) of the subgroup \( H \) and an \( L \)-invariant trace \( \psi \) on the subquotient \( F = \text{FC}(v\text{Fit}(L)) \) such that \( \varphi = \text{Ind}^G_F\psi \).

We claim that the dynamical system \((L, \text{Ch}(F))\) has dense periodic measures. Let \( L_0 \) be any finite index normal subgroup of the group \( L \) whose action on every connected component of the character space \( \text{Ch}(F) \) is weakly algebraically isomorphic to the \( L_0 \)-action on \( \text{Ch}(Z(v\text{Fit}(L))) \), see Proposition \([7,4]\). The two subgroups \( Z(v\text{Fit}(L_0)) \) and \( Z(v\text{Fit}(L)) \) are commensurable (i.e. their intersection is of finite index in both). Therefore the \( L_0 \)-actions on \( \text{Ch}(Z(v\text{Fit}(L))) \) and on \( \text{Ch}(Z(v\text{Fit}(L_0))) \) are weakly algebraically isomorphic. The latter \( L_0 \)-action has dense periodic measures by the assumption. So the dynamical system \((L_0, \text{Ch}(F))\) has dense periodic measures by the transitivity of weak algebraic isomorphisms and by statement (2) of Corollary \([8,4]\). The claim follows from Lemma \([8,2]\).

Let \( \mu \) be the \( L \)-invariant probability measure on the character space \( \text{Ch}(F) \) determined by the Fourier transform of the trace \( \psi \). The above claim says that there is a sequence of \( L \)-invariant probability measures \( \mu_n \in \text{Prob}(\text{Ch}(F)) \) with finite supports that converges to the measure \( \mu \) in the weak-* topology. We may assume without loss of generality that the corresponding traces \( \psi_n = \hat{\mu}_n \in \text{Tr}(F) \) satisfy \( [F : \ker \psi_n] < \infty \) in light of Lemma \([7,7]\). Denote \( \varphi_n = \text{Ind}_F^G \psi_n \) for all \( n \in \mathbb{N} \).

Recall that any subgroup of the virtually polycyclic group \( H \) is profinitely closed \([\text{Mal56}]\) (see also \([\text{LR04}, 1.3.10]\)). Therefore we may apply Proposition \([7,5]\) and deduce that each trace \( \varphi_n \) is a pointwise limit of finite dimensional traces on the group \( G \). As the sequence of traces \( \psi_n \) converges pointwise to the trace \( \psi \), the sequence of traces \( \varphi_n \) converges pointwise to the trace \( \varphi \) by the continuity of induction (Lemma \([8,2]\)). The desired conclusion follows.

Proof of Corollary \([1] \) Let \( G \) be a finitely generated virtually nilpotent group. The group \( G \) is in particular virtually polycyclic so that the stability criterion of Theorem \([11,11]\) applies. Any subquotient \( L \) of the group \( G \) is virtually nilpotent. In particular \( L = v\text{Fit}(L) \). The topological dynamical system in question \((L, \widehat{\text{Z}(L)})\) certainly has dense periodic measures as the \( L \)-action on the character space of its center \( \mathbb{Z}(L) \) is trivial.

Proof of Theorem \([\Box]\) The two implications \((2) \Rightarrow (1) \) and \((4) \Rightarrow (3) \) are immediate.

\((1) \Rightarrow (3) \). Let \( G \) be an abelian subgroup of the group of automorphisms \( \text{GL}_d(\mathbb{Z}) \) for some \( d \in \mathbb{N} \). The corresponding semidirect product \( \tilde{G} = G \rtimes \mathbb{T}^d \) is Hilbert–Schmidt stable by assumption. Proposition \([\Box]\) implies that the dynamical system \((\tilde{G}, \mathbb{T}^d)\) has dense periodic measures. Equivalently the dynamical system \((G, \mathbb{T}^d)\) has dense periodic measures, as required.

\((2) \Rightarrow (1) \). Let \( G \) an amenable subgroup of the group of automorphisms \( \text{GL}_d(\mathbb{Z}) \) for some \( d \in \mathbb{N} \). The linear group \( G \) is virtually solvable by the Tits alternative \([\text{Tit72}]\). As such the group \( G \) is virtually polycyclic by \([\text{Mal56}]\), see also \([\text{LR04}, \S3.2]\). Therefore the semidirect product \( \tilde{G} = G \rtimes \mathbb{Z}^d \) is Hilbert–Schmidt stable by assumption. We conclude exactly as in the previous paragraph.
(3) ⇒ (2). Let $G$ be a virtually polycyclic group. We will rely on Theorem 11.1 to deduce that the group $G$ is Hilbert–Schmidt stable.

With this goal in mind, consider the topological dynamical system $(L, \hat{Z})$ where $L$ is a quotient of some finite index subgroup of the group $G$ and $\hat{Z}$ is the subquotient $Z(\text{vFit}(L))$. The $L$-action on the space $\hat{Z}$ factors through some virtually abelian quotient. Indeed the action of the subquotient $\text{vFit}(L)$ on the space $\hat{Z}$ is trivial and the quotient $L/\text{vFit}(L)$ is virtually abelian. The dynamical system $(L, \hat{Z})$ has dense periodic measures by assumption. This completes the proof. □

**Rank one type and upper triangular matrices.** We study a class of polycyclic non-metabelian groups for which we are able to show Hilbert–Schmidt stability relying on the methods of this work. In particular we establish Theorem J of the introduction.

Fix a finite dimensional rational vector space $V$. Let $A$ be an abelian subgroup of $\text{GL}(V)$ generated by semisimple transformations. It follows that every element of the group $A$ is semisimple [Bor12, §4].

Consider the $\mathbb{Q}$-algebra $A = \mathbb{Q}A$ spanned by the group $A$ inside the $\mathbb{Q}$-algebra $\text{End}_\mathbb{Q}(V)$. As $A$ is diagonalizable it has no non-zero nilpotent elements. Moreover $A$ is commutative and Artinian. These conditions imply that the $\mathbb{Q}$-algebra $A$ is semisimple [FD12, Corollary 2.5].

**Definition.** A simple $A$-module $M$ has rank one if the restricted action of the group $A$ on $M$ factors through some homomorphism $f : A \to \mathbb{Z}$. An $A$-module $M$ has rank one type if every simple $A$-submodule of $M$ has rank one. The group $A$ itself has rank one type if the $A$-module $V$ has rank one type.

**Proposition 11.2.** Let $M$ be an $A$-module of rank one type. Then any $A$-submodule $N \leq M$ as well as any quotient $A$-module $M \twoheadrightarrow L \cong M/N$ has rank one type.

**Proof.** The $A$-module $M$ has a unique decomposition $M = \bigoplus_{i=1}^l M_i$ into its isotypic components $M_1, \ldots, M_l$. Any $A$-submodule $N$ of $M$ must be of the form $N = \bigoplus_{i=1}^l N_i$ for some $A$-submodules $0 \leq N_i \leq M_i$. It follows from this description that the $A$-submodule $N$ as well as the quotient $A$-module

$$(11.2) \quad L \cong M/N \cong \bigoplus_{i=1}^l M_i/N_i$$

have rank one type. □

**Lemma 11.3.** Let $A$ be a commutative subgroup of $\text{GL}_d(\mathbb{Z})$ generated by semisimple transformations. If the group $A$ has rank one type then the dynamical system $(A, T^d)$ has dense periodic measures.

**Proof.** Consider the rational vector space $V = \mathbb{Q}^d$ regarded as a module over the semisimple $\mathbb{Q}$-algebra $A = \mathbb{Q}A$. There is a uniquely determined decomposition $V = \bigoplus_{i=1}^l V_i$ into isotypic components for some $l \in \mathbb{N}$. Each $A$-submodule $V_i$ is in particular a rational subspace of the rational vector space $V$.

There is a $d$-dimensional torus $T$ admitting a product decomposition $T = \prod_{i=1}^l T_i$ into $A$-invariant factors as well as an $A$-equivariant finite-sheeted covering map $T \to T^d$ such that each sub-torus $T_i$ covers the image of the rational subspace $V_i$ inside the torus $T$. 
The action of the group $A$ on each torus $T_i$ factors through a cyclic group $\mathbb{Z}$ generated by a single semisimple transformation $A_i \in \text{Aut}(T_i)$. Up to passing to a finite index subgroup of the group $A$ and reindexing, we may assume that each transformation $A_i$ is ergodic on its respective torus $T_i$ with respect to the Haar measure, expect possibly for a single index $i_0$ for which $A_{i_0}$ is the identity transformation.

We conclude that the dynamical system $(A, T)$ has dense periodic measures by relying on Theorem 8.5. The density of periodic measures for the dynamical system $(A, \mathbb{T}^d)$ follows from Corollary 8.4 concerning finite factors. □

Let $N$ be a finitely generated torsion-free nilpotent group. Recall that there is a well-defined rational Lie algebra $\mathcal{L} = \text{Lie}(N)$ associated to the group $N$ as well as an $\text{Aut}(N)$-equivariant bijection $\log : N \rightarrow N$ with inverse $\exp : N \rightarrow N$, see [Seg03] §6.

**Theorem 11.4.** Let $A$ be an abelian subgroup of $\text{Aut}(N)$. If $A$ is generated by semisimple transformations and has rank one type when regarded as a subgroup of $\text{GL}(\mathcal{L})$ then the polycyclic group $G = A \rtimes N$ is Hilbert–Schmidt stable.

**Proof.** Denote $\mathcal{A} = \mathbb{Q}A$ so that $\mathcal{A}$ is a semisimple $\mathbb{Q}$-algebra. The assumption of the theorem says that the rational Lie algebra $\mathcal{N}$ is a rank one type $\mathcal{A}$-module.

Fix an arbitrary finite index subgroup $H \leq G$ and a normal subgroup $K \triangleleft H$. Denote $\overline{H} = H/K$ and $\overline{Z} = \mathbb{Z}(\text{Fit}(\overline{H}))$. We will conclude that the group $G$ is Hilbert–Schmidt stable by relying on Theorem 11.1 and showing that the topological dynamical system $(\overline{H}, \text{Ch}(\overline{Z}))$ has dense periodic measures.

The subquotient $\overline{Z}$ is a finitely generated abelian group. Therefore $\overline{Z}$ admits some torsion-free characteristic subgroup $\overline{C}$ satisfying $[\overline{Z} : \overline{C}] < \infty$. Consider the subquotient

\[(11.3) \quad \overline{M} = \overline{C} \cap (NK \cap H)/K.\]

Let $\mathcal{C} = \text{Lie}(\overline{C})$ and $\mathcal{M} = \text{Lie}(\overline{M})$ be the rational Lie algebras of the finitely generated torsion-free abelian subquotients $\overline{C}$ and $\overline{M}$ respectively. Clearly $\mathcal{M} \leq \mathcal{C}$. Let $A_0 = N_{A}(H) \cap N_{A}(K)$ so that $A_0$ is a finite index subgroup of the abelian group $A$. In particular $A_0$ spans the $\mathbb{Q}$-algebra $\mathcal{A}$, i.e. $\mathcal{A} = \mathbb{Q}A_0$. This implies that both $\mathcal{M}$ and $\mathcal{C}$ are $\mathcal{A}$-modules. The fact that the $\mathbb{Q}$-algebra $\mathcal{A}$ is semisimple implies that the $\mathcal{A}$-submodule $\mathcal{M}$ admits a complement in $\mathcal{C}$, i.e. there exists some $\mathcal{A}$-submodule $\mathcal{N} \leq \mathcal{C}$ satisfying $\mathcal{C} = \mathcal{M} \oplus \mathcal{N}$.

We claim that the action of the subgroup $A_0$ on the module $\mathcal{N}$ obtained by restricting the action of the algebra $\mathcal{A}$ is trivial. To see this consider any element $x \in \mathcal{C}$ with $\exp x = g \in \overline{C}$. Write $g = anK$ for some pair of elements $a \in A, n \in \mathcal{N}$. Any element $\alpha \in A_0$ satisfies

\[(11.4) \quad g^{-1}\alpha(g) = n^{-1}a^{-1}\alpha(an)K = n^{-1}a^{-1}a\alpha(n)K = n^{-1}\alpha(n)K.\]

It follows that

\[(11.5) \quad \exp(x)^{-1}\alpha(\exp(x)) \in \overline{M}.\]

The $A_0$-equivariance of the exponential map gives $\alpha(x) - x \in \mathcal{M}$ for all elements $x \in \mathcal{C}$ and $\alpha \in A_0$. Therefore $\alpha(y) - y \in \mathcal{M} \cap \mathcal{N} = \{0\}$ for all elements $y \in \mathcal{N}$. The above claim follows.

The $\mathcal{A}$-module $\mathcal{M}$ is a subquotient of the $\mathcal{A}$-module $\mathcal{L}$ and as such has rank one type by Proposition 11.2. The $\mathcal{A}$-module $\mathcal{N}$ certainly has rank one type for
the restricted action of the subgroup $A_0$ on it is trivial. Therefore the $A$-module $C = M \oplus N$ has rank one type as well.

The $H$-actions on the two dual spaces $C_h(C)$ and $C_h(Z)$ are weakly algebraically isomorphic (in the sense discussed in §7). Since the $A$-module $C$ has rank one type the $H$-action on the space $C_h(Z)$ has dense periodic measures according to the two Lemmas 8.2 and 11.3. □

We are ready to prove that upper triangular groups over certain rings of algebraic integers are Hilbert–Schmidt stable.

**Proof of Theorem J.** Let $O$ be the ring of algebraic integers in some number field. Assume that the group of units $O^*$ satisfies $\text{rank}(O^*) = 1$. Let $A$ and $N$ respectively denote the diagonal and upper triangular subgroups of the matrix group $GL_d(O)$ for some fixed $d \in \mathbb{N}$. The group $N$ is finitely generated, nilpotent and torsion free (see [LR04, 1.2.20]). The assumption on the group of units $O^*$ implies that the action of the diagonal group $A$ on the rational Lie algebra $N = \text{Lie}(N)$ has rank one type. The fact that the polycyclic group $G$ is Hilbert–Schmidt stable follows from Theorem 11.4. □

**The higher-rank measure rigidity conjecture.** We discuss the relationship between the higher-rank measure rigidity conjecture and stability.

**Proposition 11.5.** Let $O$ be the ring of algebraic integers in the number field $k$. Assume that $k$ is totally real and that $\text{rank}(O^*) \geq 2$. Then the dual action of group of units $O^*$ on the dual torus $\hat{O}$ is almost minimal, i.e. there are no invariant proper closed infinite subsets.

**Proof.** Consider the dual action of the group of units $O^*$ on the dual torus $\hat{O}$. According to [Ber83] this action is almost minimal provided that the following three conditions are satisfied:

1. There is a unit $u \in O^*$ such that multiplication by the element $u^n$ is rationally irreducible, i.e. admits no non-trivial invariant rational subspace in $k$ for all $n \in \mathbb{N}$ (see [KN11, Proposition 2.2.6] for several equivalent formulations of this statement).
2. There is a unit $u \in O^*$ such that multiplication by the element $u$ is represented by a hyperbolic matrix.
3. There is a pair of units $u, v \in O^*$ such that $u^n \neq v^m$ for all non-zero $n, m \in \mathbb{N}$.

In light of [Seg05, §11, Lemma 2] the dual action of a given unit $u \in O^*$ is totally irreducible if and only if $k = \mathbb{Q}(u)$. Note that there are finitely many intermediate number fields $\mathbb{Q} \subset l \subset k$. On the other hand, since $k$ is totally real and is in particular not a CM-field, the group of units $O^*$ satisfies $\text{rank}(O^* \cap l) < \text{rank}(O^*)$ for each proper number field $l \subsetneq k$, see [Par75]. This shows that it is possible to choose some unit $u \in O^*$ as required in Statement (1).

Statement (2) says that there is some unit $u \in O^*$ whose dual action on the torus $\hat{O}$ is represented by a hyperbolic matrix, i.e. all eigenvalues of the corresponding matrix are not unimodular. These eigenvalues are the algebraic conjugates of $u$, i.e. the roots of the minimal polynomial of $u$. Each eigenvalue has multiplicity $[k : \mathbb{Q}(u)]$. See [Lan12, VI,§5]. Since the number field $k$ is totally real any infinite order unit has the required property.
Statement (3) follows from the fact that rank($O^*$) $\geq$ 2. This concludes the proof.

We are ready to complete the proof of the stability result stated in the introduction conditional on the measure rigidity conjecture.

Proof of Proposition L. Let $O$ be the ring of algebraic integers in some totally real number field $k$. We wish to show that the metabelian polycyclic group $G = O^* \rtimes O$ is Hilbert–Schmidt stable. If rank($O^*$) = 1 then the group $G$ is Hilbert–Schmidt stable by Theorem 11.4. Therefore we will assume that rank($O^*$) $\geq$ 2 from now on.

The conditions of Proposition 11.5 are satisfied so that the dual action of group of units $O^*$ on the dual torus $\hat{O}$ is almost minimal. By conditionally relying on the higher rank measure rigidity conjecture [Mar00] (see the discussion in §1) we may assume that any non-atomic $O^*$-invariant Borel probability measure on the dual group $\hat{O}$ is the Haar measure.

Let $H$ be some normal subgroup of the group $G$ satisfying $O \leq H \leq G$ and consider its abelianization $H^{ab}$. The fact that the action of $O^*$ is irreducible implies that the derived subgroup $[H, H]$ is either equal to the normal subgroup $O \leq G$ or is trivial. In the first case the $G$-action on the abelianization $H^{ab}$ is trivial. In the second case we obtain that $H^{ab} = \hat{O}$. This completes the proof by relying on the higher rank measure rigidity conjecture combined with the criterion given in Theorem 10.1.

References

[AD22] Danil Akhtiamov and Alon Dogon. On uniform Hilbert–Schmidt stability of groups. Proceedings of the American Mathematical Society, 2022.

[AP15] Goulnara Arzhantseva and Liviu Păunescu. Almost commuting permutations are near commuting permutations. Journal of Functional Analysis, 269(3):745–757, 2015.

[Arv12] William Arveson. An Invitation to C*-Algebras. Graduate Texts in Mathematics. Springer New York, 2012.

[Atk18] Scott Atkinson. Some results on tracial stability and graph products. arXiv preprint arXiv:1808.04664, 2018.

[BBH21] Uri Bader, Rémi Boutonnet, and Cyril Houdayer. Charmenability of higher rank arithmetic groups. arXiv preprint arXiv:2112.01337, 2021.

[BdlH20] Bachir Bekka and P. de la Harpe. Unitary representations of groups, duals, and characters. Mathematical Surveys and Monographs. American Mathematical Society, 2020.

[Bek07] Bachir Bekka. Operator-algebraic superrigidity for SL_n(\mathbb{Z}), n \geq 3. Inventiones mathematicae, 169(2):401–425, 2007.

[Ber83] Daniel Berend. Multi-invariant sets on tori. Transactions of the American Mathematical Society, 280(2):509–532, 1983.

[BF20] Bachir Bekka and Camille Francini. Characters of algebraic groups over number fields. arXiv preprint arXiv:2002.07497, 2020.

[BKM97] Lawrence Baggett, Eberhard Kaniuth, and William Moran. Primitive ideal spaces, characters, and Kirillov theory for discrete nilpotent groups. Journal of Functional Analysis, 150(1):175–203, 1997.

[BLT19] Oren Becker, Alexander Lubotzky, and Andreas Thom. Stability and invariant random subgroups. Duke Mathematical Journal, 168(12):2207–2234, 2019.

[Bor12] Armand Borel. Linear algebraic groups, volume 126. Springer Science & Business Media, 2012.

[Bow71] Rufus Bowen. Periodic points and measures for axiom A diffeomorphisms. Transactions of the American Mathematical Society, 154:377–397, 1971.

[Bur21] Peter Burton. Hyperlinear approximations to amenable groups come from sofic approximations. arXiv preprint arXiv:2110.03076, 2021.
[BV22] Uri Bader and Itamar Vigdorovich. Charmenability and stiffness of arithmetic groups. arXiv preprint arXiv:2208.07347, 2022.

[CLP15] Valerio Capraro, Martino Lupini, and Vladimir Pestov. Introduction to sofic and hyperlinear groups and Connes’ embedding conjecture, volume 1. Springer, 2015.

[CM84] Alan Carey and William Moran. Characters of nilpotent groups. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 96, pages 123–137. Cambridge University Press, 1984.

[Con76] Alain Connes. Classification of injective factors cases $II_1$, $II_\infty$, $III_\lambda$, $\lambda \neq 1$. Annals of Mathematics, pages 73–115, 1976.

[Cor15] Yves Cornulier. Commability and focal locally compact groups. Indiana University Mathematics Journal, pages 115–150, 2015.

[DCOT19] Marcus De Chiffre, Narutaka Ozawa, and Andreas Thom. Operator algebraic approach to inverse and stability theorems for amenable groups. Mathematika, 65(1):98–118, 2019.

[DI94] Karel Dekimpe and Paul Igodt. The structure and topological meaning of almost-torsion free groups. Communications in Algebra, 22(7):2547–2558, 1994.

[Dix11] Jacques Dixmier. von Neumann algebras, volume 27. Elsevier, 2011.

[dLS22] Mikael de la Salle. Spectral gap and stability for groups and non-local games. arXiv preprint arXiv:2204.07084, 2022.

[DM14] Artem Dudko and Konstantin Medynets. Finite factor representations of Higman–Thompson groups. Groups, Geometry, and Dynamics, 8(2):375–389, 2014.

[DM19] Artem Dudko and Kostya Medynets. On invariant random subgroups of block-diagonal limits of symmetric groups. Proceedings of the American Mathematical Society, 147(6):2481–2494, 2019.

[Dog21] Alon Dogon. Stability and approximation of group and operator algebras. M.Sc. thesis. Published by the Hebrew University, 2021.

[ES23] Caleb Eckhardt and Tatiana Shulman. On amenable Hilbert–Schmidt stable groups. Journal of Functional Analysis, 285(3):109954, 2023.

[EW13] Manfred Einsiedler and Thomas Ward. Ergodic theory. Springer, 4(4):4–5, 2013.

[FDA12] Benson Farb and Keith Dennis. Noncommutative algebra, volume 144. Springer Science & Business Media, 2012.

[Fol67] Gerd Folland. A course in abstract harmonic analysis, volume 29. CRC press, 2016.

[Fur67] Harry Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. Mathematical systems theory, 1(1):1–49, 1967.

[Gla03] Eli Glasner. Ergodic theory via joinings. Number 101. American Mathematical Soc., 2003.

[Gle10] Lev Glebsky. Almost commuting matrices with respect to normalized Hilbert-Schmidt norm. arXiv preprint arXiv:1002.3082, 2010.

[GR09] Lev Glebsky and Luis Manuel Rivera. Almost solutions of equations in permutations. Taiwanese Journal of Mathematics, 13(2A):493–500, 2009.

[Gro99] Mikhail Gromov. Endomorphisms of symbolic algebraic varieties. Journal of the European Mathematical Society, 1(2):109–197, 1999.

[GS21] Maria Gerasimova and Konstantin Sluchenko. Virtually free groups are $p$-Schatten stable. arXiv preprint arXiv:2107.10032, 2021.

[Gui63] Alain Guichardet. Caractères des algèbres de Banach involutives. In Annales de l’institut Fourier, volume 13, pages 1–81, 1963.

[Hal59] Philip Hall. On the finiteness of certain soluble groups. Proceedings of the London Mathematical Society, 3(4):595–622, 1959.

[How77] Roger Howe. On representations of discrete, finitely generated, torsion-free, nilpotent groups. Pacific Journal of Mathematics, 73(2):281–305, 1977.

[HS18] Don Hadwin and Tatiana Shulman. Stability of group relations under small Hilbert–Schmidt perturbations. Journal of Functional Analysis, 275(4):761–792, 2018.

[Ioa21] Adrian Ioana. Almost commuting matrices and stability for product groups. arXiv preprint arXiv:2108.09589, 2021.

[IS21] Adrian Ioana and Pieter Spaas. $II_1$ factors with exotic central sequence algebras. Journal of the Institute of Mathematics of Jussieu, 20(5):1671–1696, 2021.
Adrian Ioana, Pieter Spaas, and Matthew Wiersma. Cohomological obstructions to lifting properties for full C*-algebras of property (T) groups. *Geometric and Functional Analysis*, 30(5):1402–1438, 2020.

Zhengfeng Ji, Anand Natarajan, Thomas Vidick, John Wright, and Henry Yuen. MIP*=RE. *Communications of the ACM*, 64(11):131–138, 2021.

David Lawrence Johnson. *Presentations of groups*. Number 15. Cambridge university press, 1997.

Eberhard Kaniuth. Ideals in group algebras of finitely generated FC-nilpotent discrete groups. *Mathematische Annalen*, 248(2):97–108, 1980.

Eberhard Kaniuth. Induced characters, Mackey analysis and primitive ideal spaces of nilpotent discrete groups. *Journal of Functional Analysis*, 240(2):349–372, 2006.

Gregory Karpilovsky. Projective representations of finite groups. *New York-Basel*, 1985.

Alexander Kechris. *Classical descriptive set theory*, volume 156. Springer Science & Business Media, 2012.

Dominik Kwietniak, Martha Lacka, and Piotr Oprocha. A panorama of specification-like properties and their consequences. *Contemporary Mathematics*, 669:155–186, 2016.

Anatole Katok and Viorel Niţică. *Rigidity in higher rank abelian group actions: Volume 1, Introduction and Cocycle Problem*, volume 185. Cambridge University Press, 2011.

Anatole Katok and Ralf Spatzier. Invariant measures for higher-rank hyperbolic abelian actions. *Ergodic Theory and Dynamical Systems*, 16(4):751–778, 1996.

Serge Lang. *Algebra*, volume 211. Springer Science & Business Media, 2012.

John Lee. *Introduction to topological manifolds*, volume 202. Springer Science & Business Media, 2010.

Douglas Lind. Ergodic group automorphisms and specification. In *Ergodic Theory*, pages 93–104. Springer, 1979.

Douglas Lind. Dynamical properties of quasihyperbolic toral automorphisms. *Ergodic Theory and Dynamical Systems*, 2(1):49–68, 1982.

Elon Lindenstrauss. Pointwise theorems for amenable groups. *Inventiones mathematicae*, 146(2):49–68, 2001.

Elon Lindenstrauss. Recent progress on rigity properties of higher rank diagonalizable actions and applications. *arXiv preprint arXiv:2101.11114*, 2021.

Arie Levit and Alexander Lubotzky. Infinitely presented permutation stable groups and invariant random subgroups of metabelian groups. *Ergodic Theory and Dynamical Systems*, pages 1–36, 2019.

Arie Levit and Alexander Lubotzky. Uncountably many permutation stable groups. *arXiv preprint arXiv:1910.11722*, 2019.

Omer Lavi and Arie Levit. Characters of the group ELd(R) for a commutative Noetherian ring R. *arXiv preprint arXiv:2007.15547*, 2020.

John Lennox and Derek Robinson. *The theory of infinite soluble groups*. Clarendon press, 2004.

Douglas Lind and Klaus Schmidt. Homoclinic points of algebraic Zd-actions. *Journal of the American Mathematical Society*, 12(4):953–980, 1999.

Anatoly Malcev. On some classes of infinite soluble groups. *Mat. Sbornik*, 28(70):567–588, 1956.

Brian Marcus. A note on periodic points for ergodic toral automorphisms. *Monatshefte für Mathematik*, 89(2):121–129, 1980.

Gregory Margulis. Problems and conjectures in rigidity theory. *Mathematics: frontiers and perspectives*, pages 161–174, 2000.

Francis Murray and John von Neumann. On rings of operators. II. *Transactions of the American Mathematical Society*, 41(2):208–248, 1937.

Kalyanaparam Rangachari Parthasarathy. On the category of ergodic measures. *Illinois Journal of Mathematics*, 5(4):648–656, 1961.

Charles Parry. Units of algebraic numberfields. *Journal of Number Theory*, 7(4):385–388, 1975.
