Abstract. Bethe ansatz solution of the two-axis two-spin Hamiltonian is derived based on the Jordan–Schwinger boson realization of the SU(2) algebra. It is shown that the solution of the Bethe ansatz equations can be obtained as zeros of the related extended Heine–Stieltjes polynomials. Symmetry properties of excited levels of the system and zeros of the related extended Heine–Stieltjes polynomials are discussed. As an example of an application of the theory, the two equal spin case is studied in detail, which demonstrates that the levels in each band are symmetric with respect to the zero energy plane perpendicular to the level diagram and that excited states are always well entangled.

Keywords: quantum integrability (Bethe ansatz)
1. Introduction

Quantum squeezing of both Bose and Fermi many-body systems [1–12] is effective and useful in quantum metrology and its applications in quantum informatics [13, 14]. Two previous models for dynamical generation of spin-squeezed states are the one-axis twisting and two-axis countertwisting Hamiltonians, respectively [1], of which the latter model gives rise to maximal squeezing with a squeezing angle independent of system size or evolution time. Very recently, two-axis one-spin (2A1S) countertwisting Hamiltonian [1] has been generalized to the two-axis two-spin (2A2S) case [15]. As shown in [15], the 2A2S Hamiltonian produces the spin EPR states, the analog of the two-mode squeezed state for spins, which are able to violate the Bell-CHSH inequality when the quantum numbers of the two spins are finite.

It was shown in our previous works [16, 17] that the 2A1S Hamiltonian is exactly solvable and its solution can be obtained by using the Bethe ansatz method. As noted in [17], the 2A1S Hamiltonian is equivalent to a special case of the Lipkin–Meshkov–Glick (LMG) model [18, 19] after an Euler rotation. Similar to other many-spin systems [20–23], the LMG model can be solved analytically by using the algebraic Bethe ansatz [24, 25]. The same problem can also be solved by using the Dyson boson realization of the SU(2) algebra [18, 19, 26, 27], of which the solution may be obtained from the Riccati differential equations [18, 19]. Recently, the Bethe ansatz method has also been applied to generate exact solution of mean-field plus orbit-dependent non-separable pairing model with two non-degenerate j-orbits [28] and that of the dimer Bose–Hubbard model with multi-body interactions [29]. In these two models [28, 29], there is an additional $S_2$ symmetry with respect to the two species of bosons or quasi-spins, which is helpful in constructing operators involved in the corresponding Bethe ansatz states and the related operator algebra calculations.

The purpose of this work is to construct exact and complete solution of the 2A2S Hamiltonian. In section 2, the 2A2S Hamiltonian is written in terms of boson operators after the Jordan–Schwinger boson realization of the two related SU(2) algebras, which can then be expressed in terms of generators of two copies of SU(1, 1) algebra. Since
there are only two sets of SU(1, 1) generators involved in the Hamiltonian, the technique used in [28, 29] is helpful in constructing the Bethe ansatz states and the related operator algebra calculations. The derivation of the related extended Heine-Stieltjes polynomials is presented, whose zeros are related to the exact solution of the 2A2S Hamiltonian. Symmetry properties of excited levels of the system and zeros of the related extended Heine–Stieltjes polynomials are also discussed. Some numerical examples of the two equal spin case are presented in section 3, which demonstrate the main features of the solution. A brief summary is provided in section 4.

2. The exact solution of the two-axis two-spin Hamiltonian

The 2A2S Hamiltonian is given by [15]

\[ H_{2A2S} = \chi (S_1^+ S_2^- + S_1^- S_2^+), \tag{1} \]

where \( \chi \) is a constant and \( \{S_i^\pm, S_0^i\} \) (\( i = 1, 2 \)) are generators of two copies of the SU(2) algebra satisfying the following commutation relations:

\[ [S_i^0, S_j^\pm] = \pm \delta_{ij} S_i^\pm, \quad [S_i^\pm, S_j^\pm] = \delta_{ij} 2 S_i^0. \tag{2} \]

Though the Hamiltonian (1) is not commutative with \( S_i^\nu \) (\( \nu = +, -, 0 \); \( i = 1, 2 \)), it is commutative with the two SU(2) Casimir invariants \( C_2^{(i)}(\text{SU}(2)) = (S_i^+ S_i^- + S_i^- S_i^+) / 2 + (S_i^0)^2 \) for \( i = 1, 2 \). Thus, the two spins are good quantum numbers of the system, while the total spin and its projection are not.

The generators of the two copies of the SU(2) can be represented using the Jordan–Schwinger boson realization

\[ S_1^+ = a_1^\dagger a_2, \quad S_1^- = a_2^\dagger a_1, \quad S_1^0 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \]
\[ S_2^+ = b_1^\dagger b_2, \quad S_2^- = b_2^\dagger b_1, \quad S_2^0 = \frac{1}{2} (b_1^\dagger b_1 - b_2^\dagger b_2), \tag{3} \]

where \( a_i \) and \( b_i \) (\( a_i^\dagger \) and \( b_i^\dagger \)) are the boson annihilation (creation) operators satisfying

\[ [a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [b_i, b_j] = [a_i, b_j] = 0. \tag{4} \]

By substituting (3) into (1), the Hamiltonian (1) can be expressed in terms of the generators of two copies of SU(1, 1) algebra,

\[ H_{2A2S} = \chi \left( \Lambda_1^+ \Lambda_2^- + \Lambda_1^- \Lambda_2^+ \right) \tag{5} \]

with

\[ \Lambda_1^+ = a_1^\dagger b_1^\dagger, \quad \Lambda_1^- = a_1 b_1, \quad \Lambda_1^0 = \frac{1}{2} (a_1^\dagger a_1 + b_1^\dagger b_1 + 1), \]
\[ \Lambda_2^+ = a_2^\dagger b_2^\dagger, \quad \Lambda_2^- = a_2 b_2, \quad \Lambda_2^0 = \frac{1}{2} (a_2^\dagger a_2 + b_2^\dagger b_2 + 1). \tag{6} \]
Table 1. The allowed $\lambda_1$ and $\lambda_2$ values of the SU$_1(1, 1) \otimes$ SU$_2(1, 1)$ lowest weight states satisfying (7) and the corresponding boson and the two-spin intrinsic quantum numbers, where $\mu_1$ and $\mu_2$ can be taken as arbitrary positive integers or zero.

| $\lambda_1$ | $\lambda_2$ | $n_{a_1}$ | $n_{a_2}$ | $n_{b_1}$ | $n_{b_2}$ | $S_{1}^{\text{in}}$ | $M_{1}^{\text{in}}$ | $S_{2}^{\text{in}}$ | $M_{2}^{\text{in}}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|----------------|----------------|----------------|----------------|
| $\frac{\mu_1+1}{2}$ | $\frac{\mu_2+1}{2}$ | $\mu_1$ | $\mu_2$ | 0 | 0 | $\frac{1}{2}(\mu_1 + \mu_2)$ | $\frac{1}{2}(\mu_1 - \mu_2)$ | 0 | 0 |
| $\mu_1$ | $\mu_2$ | 0 | 0 | $\frac{1}{2}(\mu_1 + \mu_2)$ | $\frac{1}{2}(\mu_1 - \mu_2)$ | $\frac{1}{2}(\mu_1 + \mu_2)$ | $\frac{1}{2}(\mu_1 - \mu_2)$ | 0 | 0 |
| $\mu_2$ | $\mu_1$ | 0 | 0 | $\frac{1}{2}(\mu_1 + \mu_2)$ | $\frac{1}{2}(\mu_1 - \mu_2)$ | $\frac{1}{2}(\mu_1 + \mu_2)$ | $\frac{1}{2}(\mu_1 - \mu_2)$ | 0 | 0 |

which satisfy the commutation relations

$$[\Lambda_{\rho}^{0}, \Lambda_{\rho'}^{\pm}] = \pm \delta_{\rho\rho'} \Lambda_{\rho}^{\pm}, \quad [\Lambda_{\rho}^{\pm}, \Lambda_{\rho'}^{-}] = -\delta_{\rho\rho'} 2\Lambda_{\rho}^{0}. \quad (7)$$

As shown in the following, the SU$(1, 1)$ type Bethe ansatz states for the Hamiltonian (5) can be written as

$$|\eta, k; \lambda_1, \lambda_2\rangle = \prod_{i=1}^{k} \Lambda_{\pm}^{+(x_i^{(0)})}|\lambda_1, \lambda_2\rangle, \quad (8)$$

where $\eta$ is an additional label needed, $|\lambda_1, \lambda_2\rangle$ is one of the lowest weight states of SU$_1(1, 1) \otimes$ SU$_2(1, 1)$ satisfying

$$\begin{pmatrix} \Lambda_{i}^{0} \\ \Lambda_{i}^{\pm} \end{pmatrix} |\lambda_1, \lambda_2\rangle = \begin{pmatrix} \lambda_i \\ 0 \end{pmatrix} |\lambda_1, \lambda_2\rangle \quad \text{for } i = 1, 2, \quad (9)$$

and

$$\Lambda_{\pm}^{+}(x) = \Lambda_{\pm}^{+} + x \Lambda_{\pm}^{\pm} \quad (10)$$

with variable $x$ to be determined. The allowed $\lambda_1$ and $\lambda_2$ values are provided in table 1.

The single- and double-commutators of the Hamiltonian (5) with the operator (10) can be expressed as

$$[H_{2A2S/\chi}, \Lambda_{\pm}^{+}(x)] = 2x\Lambda_{1}^{+}\Lambda_{2}^{0} + 2\Lambda_{2}^{+}\Lambda_{1}^{0}, \quad (11)$$

$$[[H_{2A2S/\chi}, \Lambda_{\pm}^{+}(x)], \Lambda_{\pm}^{+}(y)] = 2(xy + 1)\Lambda_{1}^{+}\Lambda_{2}^{+}, \quad (12)$$

while other higher order commutators of the Hamiltonian (5) with the operator (10) vanish. Once these commutators are obtained, the corresponding polynomials $G_1(x)$ and $G_2(x, y)$ in $\Lambda_{1}^{+}$ and $\Lambda_{2}^{+}$ on the lowest weight states of SU$_1(1, 1) \otimes$ SU$_2(1, 1)$, defined as [29]

$$G_1(x)|\lambda_1, \lambda_2\rangle = [H_{2A2S/\chi}, \Lambda_{\pm}^{+}(x)]|\lambda_1, \lambda_2\rangle,$$

$$G_2(x, y)|\lambda_1, \lambda_2\rangle = [[H_{2A2S/\chi}, \Lambda_{\pm}^{+}(x)], \Lambda_{\pm}^{+}(y)]|\lambda_1, \lambda_2\rangle, \quad (13)$$

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can be expressed in the form

\[ G_1(x) = \alpha(x)\Lambda^+(g) + \beta(x)\Lambda^+(x), \]
\[ G_2(x, y) = a(x, y)\Lambda^+(g)\Lambda^+(x) + b(x, y)\Lambda^+(g)\Lambda^+(y) + c(x, y)\Lambda^+(x)\Lambda^+(y). \]

(14)

(15)

Here \( g \) is a free parameter whose allowed values will be determined later and

\[ \alpha(x) = \frac{2(\lambda_2x^2 - \lambda_1)}{x - g}, \quad \beta(x) = \frac{2(\lambda_1 - \lambda_3xg)}{x - g}, \]
\[ a(x, y) = \frac{2y(1 + xy)}{(g - y)(y - x)}, \quad b(x, y) = a(y, x) = \frac{2x(1 + xy)}{(g - x)(x - y)}, \]
\[ c(x, y) = c(y, x) = -\frac{2g(1 + xy)}{(g - x)(g - y)}. \]

(16)

(17)

It can be observed that the single- and double-commutators of the Hamiltonian (5) with the operator (10) are quite similar to the corresponding ones appearing in the Richardson–Gaudin type models [28, 29]. Therefore, the SU(1, 1) type Bethe ansatz (8) works for this case.

Similar to what is shown in [28], using the commutation relations (11) and (12), and the expressions shown in (14) and (15), we can directly check that

\[ (H_{2A2S}/\chi)|\eta, k; \lambda_1, \lambda_2\rangle = \sum_{i=1}^{k} \alpha(x_i^{(\eta)})\Lambda^+(g)\prod_{\rho(\neq i)}^{k} \Lambda^+(x_\rho^{(\eta)})|\lambda_1, \lambda_2\rangle \]
\[ + \sum_{i=1}^{k} \beta(x_i^{(\eta)})\prod_{\rho}^{k} \Lambda^+(x_\rho^{(\eta)})|\lambda_1, \lambda_2\rangle \]
\[ + \sum_{i}^{k} \sum_{i'\neq i}^{k} a(x_i^{(\eta)}, x_{i'}^{(\eta)})\Lambda^+(g)\prod_{\rho(\neq i)}^{k} \Lambda^+(x_\rho^{(\eta)})|\lambda_1, \lambda_2\rangle \]
\[ + \sum_{i}^{k} \sum_{i'+1}^{k} c(x_i^{(\eta)}, x_{i'}^{(\eta)})\prod_{\rho}^{k} \Lambda^+(x_\rho^{(\eta)})|\lambda_1, \lambda_2\rangle. \]

(18)

It is clear that the second and the fourth terms in (18) are proportional to the Bethe ansatz state (8), which is assumed to be the eigenstate of the system. Therefore, the eigen-energy of the 2A2S Hamiltonian is given by

\[ E_{k,\mu_1,\mu_2}^{(\eta)} = \chi \sum_{i=1}^{k} \left( \beta(x_i^{(\eta)}) + \sum_{i'+1}^{k} c(x_i^{(\eta)}, x_{i'}^{(\eta)}) \right), \]

(19)

as long as the first and the third terms proportional to \( \Lambda^+(g)\prod_{\rho(\neq i)}^{k} \Lambda^+(x_\rho^{(\eta)})|\lambda_1, \lambda_2\rangle \) in (18) for given \( i \) are cancelled out, which leads to the corresponding equations in determining...
the \( k \) variables \( \{x_{i_{1}}^{(1)}, x_{2}^{(1)}, \ldots, x_{k}^{(1)}\} \):

\[
\alpha(x_{i_{1}}^{(1)}) + \sum_{i' \neq i}^{k} a(x_{i'}^{(1)}, x_{i_{1}}^{(1)}) = 0 \quad \text{for} \quad i = 1, 2, \ldots k. \quad (20)
\]

Using \( \alpha(x_{i_{1}}^{(1)}), \beta(x_{i}^{(1)}), a(x_{i'}^{(1)}, x_{i}^{(1)}) \) and \( c(x_{i}^{(1)}, x_{i'}^{(1)}) \) in (16) and (17), the eigen-energy can be simplified to

\[
E_{k,\mu_{1},\mu_{2}}^{(1)} = 2\chi \left( \sum_{i=1}^{k} \frac{\lambda_{1}g x_{i}^{(1)} - \lambda_{1}}{g - x_{i}^{(1)}} - g \sum_{i=1}^{k} \sum_{i' = i+1}^{k} \frac{1 + x_{i}^{(1)} x_{i'}^{(1)}}{(g - x_{i}^{(1)})(g - x_{i'}^{(1)})} \right), \quad (21)
\]

where the \( k \) variables \( \{x_{i}^{(1)}\} \) should satisfy

\[
\frac{\lambda_{1}}{x_{i}^{(1)}} - \lambda_{2} x_{i}^{(1)} + \sum_{i' \neq i}^{k} \frac{1 + x_{i}^{(1)} x_{i'}^{(1)}}{x_{i}^{(1)} - x_{i'}^{(1)}} = 0 \quad \text{for} \quad i = 1, 2, \ldots k, \quad (22)
\]

under the condition that \( g \neq x_{i}^{(1)} \forall \eta, i \). It is obvious that \( g = x_{i}^{(1)} \) for any \( \eta \) and \( i \) is the singular point of (21) and should be avoided. It can be verified that root components \( \{x_{i}^{(1)}\} \) of (22), which are always real and unequal one another, lie in the two open intervals \((-\infty, 0) \cup (0, +\infty)\).

By using (22), (21) can be expressed as

\[
E_{k,\mu_{1},\mu_{2}}^{(1)} = 2\chi \left( \sum_{i=1}^{k} \frac{\lambda_{1}}{x_{i}^{(1)}} + \frac{1}{2} \sum_{i=1}^{k} \sum_{i' \neq i}^{k} \frac{1 + x_{i}^{(1)} x_{i'}^{(1)}}{x_{i}^{(1)} - x_{i'}^{(1)}}, \sum_{i=1}^{k} \sum_{i' = i+1}^{k} \frac{1 + x_{i}^{(1)} x_{i'}^{(1)}}{(g - x_{i}^{(1)})(g - x_{i'}^{(1)})} \right). \quad (23)
\]

It is obvious that the first part \( 1/(x_{i}^{(1)} - x_{i'}^{(1)}) \) within the double sum over \( i \) and \( i' \) in (23) is antisymmetric, while the last part is symmetric with respect to the permutation \( i \leftrightarrow i' \), which ensures that the second term in (23) is zero. Hence, the eigen-energies (21) are independent of \( g \) as long as \( g \neq x_{i}^{(1)} \forall \eta, i \), and can be further simplified to the form

\[
E_{k,\mu_{1},\mu_{2}}^{(1)} = 2\chi \sum_{i=1}^{k} \frac{\lambda_{1}}{x_{i}^{(1)}} = 2\chi \sum_{i=1}^{k} \frac{\lambda_{1}}{x_{i}^{(1)}}. \quad (24)
\]

It is now clear that \( \eta \) labels the \( \eta \)-th solution of (22). In addition, though the eigenstates provided in (8) are not normalized, they are always orthogonal with

\[
\langle \eta' ; k' ; \lambda'_{1}, \lambda'_{2} | \eta ; k ; \lambda_{1}, \lambda_{2} \rangle = (N(\eta, k ; \lambda_{1}, \lambda_{2}))^{-\delta_{\eta \eta'}} \delta_{k k'} \delta_{\lambda_{1} \lambda'_{1}} \delta_{\lambda_{2} \lambda'_{2}}, \quad (25)
\]

where \( N(\eta, k ; \lambda_{1}, \lambda_{2}) \) is the corresponding normalization constant.

It is obvious that the Hamiltonian (1) is invariant under the permutations of the two spin operators with \( S_{1}^{\nu} \leftrightarrow S_{2}^{\nu} \) for \( \nu = 0, +, - \), which corresponds to the permutation of \( a \)-bosons with \( b \)-bosons. The SU(1, 1) operators \( \Lambda^{\nu}(x_{i}) \) used in (10) are invariant under the permutation of \( a \)-bosons with \( b \)-bosons. Therefore, the SU(1, 1) lowest weight states \( |\lambda_{1}, \lambda_{2}\rangle \) should be invariant under the permutation of \( a \)-bosons with \( b \)-bosons. As clearly

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shown in table 1, the first two and the last two two-spin intrinsic states \[ |S_{\text{in}1}^1 S_{\text{in}2}^1 \rangle \]
and \[ |S_{\text{in}2}^2 S_{\text{in}1}^2 \rangle \]
are indeed the same SU(1, 1) lowest weight state \[ |\lambda_1, \lambda_2 \rangle \]
when both \( \mu_1 \) and \( \mu_2 \neq 0 \). The two-spin intrinsic state is unique with \[ |S_{\text{in}1}^1 S_{\text{in}2}^1 \rangle = |0 0 \rangle \]
only when \( \mu_1 = \mu_2 = 0 \). It is also obvious that the two-spin intrinsic states with the same \( M_{\text{in}} = M_{\text{in}1} + M_{\text{in}2} \) value are the same SU(1, 1) lowest weight state \[ |\lambda_1, \lambda_2 \rangle \].

Once the Bethe ansatz equation (22) is solved, the eigenstates (8), up to the two-spin permutation and a normalization constant, can be expressed in terms of uncoupled two-spin states as

\[
|\eta, k; \lambda_1, \lambda_2 \rangle = \sum_{\rho=0}^{k} S_{\rho}^{(k, \eta)} \sqrt{(k + \mu_1 - \rho)!(\mu_2 + \rho)!(k - \rho)!} \rho! 
\]

\[
\times \left( \begin{array}{c} \frac{1}{2}(k + \mu_1 + \mu_2) & k \\ \frac{1}{2}(k + \mu_1 - \mu_2) - \rho & k - \rho \\end{array} \right) \left( \begin{array}{c} \frac{1}{2}(k + \mu_1) & \frac{1}{2}(k + \mu_2) \\ \frac{1}{2}(k + \mu_1) - \rho & \frac{1}{2}(k - \mu_2) - \rho \\end{array} \right),
\]

where

\[
S_{0}^{(k, \eta)} = 1, \quad S_{\rho}^{(k, \eta)} = \sum_{1 \leq \mu_1 \neq \cdots \neq \mu_\rho \leq k} x_{\mu_1}^{(\eta)} \cdots x_{\mu_\rho}^{(\eta)}
\]  

are symmetric functions of \( \{x_{1}^{(\eta)}, \ldots, x_{k}^{(\eta)}\} \).

Moreover, it can be verified directly that (22) and the corresponding eigen-energy (21) are invariant under the simultaneous interchanges \( \lambda_1 \leftrightarrow \lambda_2 \) and \( x_{i}^{(\eta)} \leftrightarrow 1/x_{i}^{(\eta)} \), which corresponds to the permutation of the two copies of the SU(1, 1) generators, \( \Lambda_{\mu}^{\eta} \leftrightarrow \Lambda_{-\mu}^{\eta} \) for \( \mu = +, -, 0 \). It is obvious that the 2A2S Hamiltonian (5) is also invariant under the permutation \( \Lambda_{\mu}^{\eta} \leftrightarrow \Lambda_{-\mu}^{\eta} \). Therefore, when \( \lambda_1 \neq \lambda_2 \), the eigenvalue of the 2A2S Hamiltonian with \( \{x_{1}^{(\eta)}\} \) built on the SU(1, 1) lowest weight state \( |\lambda_1, \lambda_2 \rangle \) and that with \( \{(x_{1}^{(\eta)})^{-1}\} \) built on the \( |\lambda_2, \lambda_1 \rangle \) are the same. Thus, when \( \lambda_1 = \lambda_2 \), if \( x_{i}^{(\eta)} \) is a solution, \( \{(x_{i}^{(\eta)})^{-1}\} \) gives the same solution. Namely, for fixed \( i \), if \( x_{i}^{(\eta)} \) is one of the root component, \( x_{i}^{(\eta)} = (x_{i}^{(\eta)})^{-1} \) is also a root component of the same root. We can also verify that the roots of (22) have the mirror symmetry. Namely, if \( \{x_{i}^{(\eta)}\} \) is a solution, \( \{-x_{i}^{(\eta)}\} \) is also a solution. Thus, if the eigen-energy is nonzero, the sign of the eigen-energy with \( \{x_{i}^{(\eta)}\} \) is opposite to that with \( \{-x_{i}^{(\eta)}\} \).

According to the Heine–Stieltjes correspondence [28, 30–32], the second-order Fuchsian equation of the extended Heine–Stieltjes polynomials whose zeros are the roots of
(22) can be established. By using the identity
\[
\sum_{i\neq i'} \frac{x_i^{(n)} - x_{i'}^{(n)}}{x_i^{(n)} - x_{i'}^{(n)}} = x_i^{(n)} \sum_{i \neq i'} \frac{x_{i'}^{(n)} - x_i^{(n)}}{x_{i'}^{(n)} - x_i^{(n)}} - k + 1,
\]
(28)

it can be verified that the related extended Heine–Stieltjes polynomials \( y_k^{(n)}(x) \) of degree \( k \) should satisfy
\[
(x + x^3) \frac{d^2 y_k^{(n)}(x)}{dx^2} + (2\lambda_1 - 2(\lambda_2 + k - 1)x^2) \frac{dy_k^{(n)}(x)}{dx} + V^{(n)}(x)y_k^{(n)}(x) = 0,
\]
(29)

where the Van Vleck polynomial \( V^{(n)}(x) \) is simply a binomial to be determined by (29). Write \( y_k^{(n)}(x) = \sum_{n=0}^{k} f_n^{(n)} x^n \) and \( V^{(n)}(x) = v_0^{(n)} + v_1^{(n)} x \), where \( \eta \) labels the \( \eta \)th polynomial. It can be verified directly that the expansion coefficients \( f_n^{(n)} (n = 0, \ldots, k) \) should satisfy the following three-term recurrence relations:
\[
(n + 1)(2\lambda_1 + n)f_{n+1}^{(n)} + v_0^{(n)} f_n^{(n)} + \left( (n - 1)(n - 2\lambda_2 - 2k) + v_1^{(n)} \right) f_{n-1}^{(n)} = 0
\]
(30)

with
\[
v_1^{(n)} = k(2\lambda_2 + k - 1),
\]
(31)

which is independent of \( \eta \). Instead of solving the three-term recurrence relations (30) for the expansion coefficients \( f_n^{(n)} (n = 0, \ldots, k) \) and \( v_0^{(n)} \), we can construct the corresponding \((k+1) \times (k+1)\) bidiagonal matrix \( A \) with entries
\[
A_{n,n'} = [(n - 1)(n - 2\lambda_2 - 2k) + k(2\lambda_2 + k - 1)] \delta_{n,n-1} + (n + 1)(2\lambda_1 + n) \delta_{n',n+1}.
\]
(32)

Let \( \mathbf{F}^{(n)} = (f_0^{(n)}, f_1^{(n)}, \ldots, f_{k}^{(n)})^T \), where \( T \) stands for the transpose operation. The \( k + 1 \)-dimensional vector \( \mathbf{F}^{(n)} \) is the \( \eta \)th eigenvector of the bidiagonal matrix \( A \) with
\[
A \mathbf{F}^{(n)} = (-v_0^{(n)}) \mathbf{F}^{(n)},
\]
(33)

where \( -v_0^{(n)} \) is the corresponding eigenvalue, which clearly shows that there are \( k + 1 \) sets of solutions of (22) for the eigen-energies (24) and the corresponding eigenstates (26) with \( \eta = 1, 2, \ldots, k + 1 \). It is also obvious that not only the construction of the matrix \( A \), but also its diagonalization is easier with smaller size of the matrix and thus more efficient than the direct diagonalization of the 2A2S Hamiltonian (1) in the original uncoupled two-spin basis, especially when \( k \) is large. Once the \( \eta \)th set of the expansion coefficients \( f_n^{(n)} (n = 0, \ldots, k) \) are known from the eigen-equation (33), the \( k \) zeros \( \{x_i^{(n)}\} (i = 0, \ldots, k) \) of the polynomial \( y_k^{(n)}(x)/f_k^{(n)} = \sum_{n=0}^{k} f_n^{(n)} x^n \), where, up to an overall factor, \( \tilde{f}_n^{(n)} = f_n^{(n)}/f_k^{(n)} (n = 1, \ldots, k) \), can easily be calculated due to the fact that \( y_k^{(n)}(x) \) is one-variable polynomial and \( f_k^{(n)} \) is always nonzero. In addition, \( y_k^{(n)}(x)/f_k^{(n)} \) can

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also be expressed in terms of the zeros \( \{ x_j^{(q)} \} \) \( (j = 1, \ldots, k) \) as

\[
y_k^{(q)}(x)/f_k^{(q)} = \prod_{j=1}^{k} (x - x_j^{(q)}) = \sum_{q=0}^{k} (-1)^q S_q^{(k,n)} x^{-q},
\]

where \( S_q^{(k,n)} \) is the symmetric function defined in (27). Comparing (34) with \( y_k^{(q)}(x)/f_k^{(q)} = \sum_{n=0}^{k} f_n^{(q)} x^n \), we get

\[
S_q^{(k,n)} = (-1)^q f_{k-q}^{(q)},
\]

which can be used to avoid unnecessary computation of \( S_q^{(k,n)} \) from \( \{ x_1^{(q)}, \ldots, x_k^{(q)} \} \) needed in the eigenstates (26). Furthermore, using the three-term recurrence relations (30), we have

\[
v_0^{(q)} = -2\lambda_1 f_1^{(q)}/f_0^{(q)} = 2\lambda_1 S_{k-1}^{(k,n)} / S_k^{(k,n)} = E_{k,\mu_1,\mu_2}/\chi,
\]

where the relation (35) is used for the second equality, which shows that the constant term \( v_0^{(q)} \) in the Van Vleck polynomial equals exactly to the corresponding eigen-energy \( E_{k,\mu_1,\mu_2}/\chi \) of the 2A2S Hamiltonian.

Finally, we prove that the solutions of the Bethe ansatz equation (22) are complete. When the two spins are unequal with \( S_1 > S_2 \), there are three sets of uncoupled two-spin states used in the eigenstates (26): case 1 with \( S_1 = \frac{1}{2}(k+\mu_1+\mu_2) \) and \( S_2 = \frac{1}{2}k \) corresponding to the upper one in (26); case 2 with \( S_1 = \frac{1}{2}(k+\mu_1) \) and \( S_2 = \frac{1}{2}(k+\mu_2) \) corresponding to the lower one in (26); and case 3 with \( S_1 = \frac{1}{2}(k+\mu_2) \) and \( S_2 = \frac{1}{2}(k+\mu_1) \) corresponding to the two-spin permutation \( S_1 \leftrightarrow S_2 \) of the lower one in (26).

For case 1, \( S_1 = \frac{1}{2}(k+\mu_1+\mu_2) > S_2 = \frac{1}{2}k \) with \( \mu_1 = 2S_1 - 2S_2 - \mu_2 \geq 0 \), where \( \mu_2 \) can be taken as 0, 1, 2, …, 2\( S_1 - 2S_2 \). The number of solutions provided by (22) for a fixed \( \mu_2 \) is \( k + 1 = 2S_2 + 1 \). Thus, the total number of solutions for \( S_1 = \frac{1}{2}(k+\mu_1+\mu_2) \) and \( S_2 = \frac{1}{2}k \) case shown by (26) with the upper uncoupled two-spin states is \( (2S_2 + 1)(2S_1 - 2S_2 + 1) \), which includes \( \mu_2 = 0 \) case.

For case 2, \( S_1 = \frac{1}{2}(k+\mu_1) > S_2 = \frac{1}{2}(k+\mu_2) \) with \( \mu_1 = \frac{1}{2}(2S_1 - 2S_2) \) and \( \mu_2 > 0 \) because \( \mu_2 = 0 \) case is already considered in case 1, the number of solutions provided by (22) is \( k + 1 = 2S_2 - \mu_2 + 1 \) for a fixed \( \mu_2 \), where \( \mu_2 \) can be taken as 1, 2, …, 2\( S_2 \). Hence, the total number of solutions for this case shown by (26) with the lower uncoupled two-spin states is \( \sum_{\mu_2=1}^{2S_2} (2S_2 - \mu_2 + 1) = (2S_2 + 1)S_2 \), which excludes the \( \mu_2 = 0 \) case.

For case 3, \( S_1 = \frac{1}{2}(k+\mu_2) > S_2 = \frac{1}{2}(k+\mu_1) \) corresponding to the two-spin permutation \( S_1 \leftrightarrow S_2 \) of the lower one in (26), where \( \mu_1 = \mu_2 - 2S_1 + 2S_2 \geq 0 \) and \( 2S_1 \geq \mu_2 \geq 2S_1 - 2S_2 \), the number of solutions provided by (22) is \( k + 1 = 2S_1 - \mu_2 + 1 \) for a fixed \( \mu_2 \). Since \( \mu_2 = 2S_1 - 2S_2 \) with \( \mu_1 = 0 \) case is already considered in case 2, the total number of solutions for this case is \( \sum_{\mu_2=2S_1-2S_2+1}^{2S_1} (2S_1 - \mu_2 + 1) = (2S_2 + 1)S_2 \), which excludes the \( \mu_2 = 2S_1 - 2S_2 \) case.

It is now obvious that the total number of the solutions provided by the three cases equals exactly to \( (2S_1 + 1)(2S_2 + 1) \), which is the dimension of the uncoupled two-spin states for \( S_1 > S_2 \). This conclusion also applies to \( S_1 < S_2 \) case by the permutation \( S_1 \leftrightarrow S_2 \).

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For the two equal spin case with $S_1 = S_2 = k/2$, which is exemplified in the next section, the upper and lower uncoupled two-spin states shown in (26) are the same. According to (26), the two-spin symmetric or anti-symmetric eigenstates of the 2A2S Hamiltonian in this case can be written uniformly as

$$|\eta, k - \mu; (\mu + 1)/2, (\mu + 1)/2\rangle_{S,A}$$

$$= \sum_{\rho=0}^{k-\mu} S_{\rho}^{(k-\mu, \eta)} \sqrt{(k-\rho)! (\mu+\rho)! (k-\mu-\rho)! \rho!} \left| \begin{array}{c} \frac{k}{2} \\ \frac{k}{2} - \rho \\ \frac{k}{2} - \mu - \rho \end{array} \right\rangle_{S,A}$$

for $\mu = 0, 1, \ldots, k$, where

$$\left| \begin{array}{c} \frac{k}{2} \\ \frac{k}{2} - \rho \\ \frac{k}{2} - \mu - \rho \end{array} \right\rangle_{S} = \sqrt{\frac{1}{2(1+\delta_{\rho0})}} \left( \left| \begin{array}{c} \frac{k}{2} \\ \frac{k}{2} - \rho \\ \frac{k}{2} - \mu - \rho \end{array} \right\rangle + \left| \begin{array}{c} \frac{k}{2} \\ \frac{k}{2} - \rho \\ \frac{k}{2} - \mu - \rho \end{array} \right\rangle \right)$$

$$\left| \begin{array}{c} \frac{k}{2} \\ \frac{k}{2} - \rho \\ \frac{k}{2} - \mu - \rho \end{array} \right\rangle_{A} = \frac{1}{\sqrt{2}} \left( \left| \begin{array}{c} \frac{k}{2} \\ \frac{k}{2} - \rho \\ \frac{k}{2} - \mu - \rho \end{array} \right\rangle - \left| \begin{array}{c} \frac{k}{2} \\ \frac{k}{2} - \rho \\ \frac{k}{2} - \mu - \rho \end{array} \right\rangle \right)$$

are the symmetric and anti-symmetric two-spin states, respectively. The number of solutions of (22) with the replacement: $k \rightarrow k - \mu$ with a fixed $\mu$ for (37) is $k - \mu + 1$. Due to the two-fold degeneracy of the $\mu \neq 0$ states, the total number of solutions provided by (37) is $2\sum_{\mu=1}^{k}(k - \mu + 1) + k + 1 = (k + 1)^2$.

3. Some numerical examples of the solution

To demonstrate the method and results presented above, in this section, we consider the two equal spin case with $S_1 = S_2 = k/2$ as studied in [15]. The two-spin symmetric or anti-symmetric eigenstates of the 2A2S Hamiltonian are given by (37). It is obvious that the $\mu = 0$ eigenstates of the 2A2S Hamiltonian are always symmetric, while both symmetric and anti-symmetric states are possible for $\mu \neq 0$. Moreover, the corresponding eigen-energies $E_{k-\mu,\mu}/\chi$ ($\eta = 1, 2, \ldots, k - \mu + 1$) of both two-spin symmetric and anti-symmetric cases are symmetric with respect to the the sign change. Namely, if the $k - \mu + 1$ level energies are arranged as $E_{k-\mu,\mu}/\chi < E_{k-\mu,\mu}/\chi < \cdots < E_{k-\mu,\mu}/\chi$, then

$$E_{k-\mu,\mu}/\chi = -E_{k-\mu,\mu}/\chi$$

$$E_{k-\mu,\mu}/\chi$$

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Figure 1. The level pattern of the 2A2S Hamiltonian for $S_1 = S_2 = k/2$ with $k = 4$, where the number on the right of each level is the degeneracy due to the two-spin permutation symmetry, and the dashed line indicates the $E = 0$ plane perpendicular to the level diagram as a mirror, with which the levels in each band labelled by $\mu$ are mirror symmetric.

For

$$q = \begin{cases} 
1, 2, \ldots, (k - \mu + 1)/2 & \text{when } k - \mu + 1 \text{ is even}, \\
1, 2, \ldots, (k - \mu + 2)/2 & \text{when } k - \mu + 1 \text{ is odd},
\end{cases} \quad (41)$$

which shows that the middle level energy $E^{(k-\mu+2)/2}$ is zero when $k - \mu + 1$ is odd. As the consequence, for given $k$, the degeneracy of the zero eigen-energy level is $k + 1$.

The Heine–Stieltjes polynomials $y^{(q)}_k(x)$ and the corresponding coefficient $v^{(q)}_0$ in the Van Vleck polynomials (19) up to $k = 4$ are shown in table 2, while the $k = 16$ and $\mu = 0$ case is provided in table 3. It should be noted that the degeneracy of the corresponding level energy of the 2A2S Hamiltonian shown in the last column of tables 2 and 3 is 1 for $\mu = 0$ case and 2 for $\mu \neq 0$ case due to the two-spin permutation symmetry. For any case, it can be verified that any zero of $y^{(q)}_{k-\mu}(x)$ is always real and lies in one of the intervals $(-\infty, 0)$ and $(0, \infty)$, which ensures that eigenvalues (24) are always real.

Figure 1 provides the level pattern of the 2A2S Hamiltonian for $k = 4$ with the level band labelled by $\mu$, where the number on the right of each level is the degeneracy due to the two-spin permutation symmetry, which clearly shows that the mirror symmetry of the $k - \mu + 1$ levels in each band with respect to the $E = 0$ plane perpendicular to the level diagram and that the total number of levels for given $k$ equals exactly to $(k + 1)^2$, which is the total dimension of the two-spin basis. In addition, when quantum numbers of the two spins of the system are small, the 2A2S Hamiltonian can easily be diagonalized within the two-spin basis, with which one can check that the eigen-energies shown in tables 2 and 3 are exactly the same as those obtained from the direct diagonalization, which validates the Bethe ansatz solution presented in section 2.

As an application of the solution, the entanglement measure of all possible 2A2S pure states (37) with $k = 40$ is calculated, which, for given $\mu$ and $\eta$, is quantified by the

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von Neumann entropy

\[
\text{Ent}(\mu, \eta) = -\text{Tr}(\rho_1^{(\mu, \eta)} \log_N \rho_1^{(\mu, \eta)}) = -\text{Tr}(\rho_2^{(\mu, \eta)} \log_N \rho^{(\mu, \eta)})2,
\]

(42)

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Table 3. The same as table 2, but for \( k = 16 \) and \( \mu = 0 \).

| \( \eta \) | \( y^{(\eta)}_{16}(x) \) | \( \nu^{(\eta)}_0 = E^{(\eta)}_{16,0,0}/\chi \) |
|---|---|---|
| 1 | \((x + 0.0108)(x + 0.0553)(x + 0.1298)(x + 0.2288)(x + 0.3497)(x + 0.4933)(x + 0.6648)(x + 0.8749)\) | -132.862 |
| 2 | \((x - 1)(x + 0.0130)(x + 0.0656)(x + 0.1515)(x + 0.2635)(x + 0.3988)(x + 0.5600)(x + 0.7550)\) | -110.346 |
| 3 | \((x - 1.3550)(x - 0.7380)(x + 0.0160)(x + 0.0791)(x + 0.1787)(x + 0.3055)(x + 0.4574)(x + 0.6396)\) | -89.3684 |
| 4 | \((x - 1.7302)(x - 1)(x - 0.5780)(x + 0.0203)(x + 0.09718)(x + 0.2129)(x + 0.3563)(x + 0.5276)\) | -69.9638 |
| 5 | \((x - 2.1755)(x - 1.2622)(x - 0.7922)(x - 0.4597)(x + 0.0268)(x + 0.1217)(x + 0.2558)(x + 0.4180)\) | -52.189 |
| 6 | \((x + 0.6121)(x + 0.8531)(x + 1.1721)(x + 1.6338)(x + 2.3924)(x + 3.9092)(x + 8.2178)(x + 37.2658)\) | -36.145 |
| 7 | \((x - 2.7484)(x - 1.5548)(x - 1)(x - 0.6432)(x - 0.3639)(x + 0.0375)(x + 0.1552)(x + 0.3095)\) | -22.032 |
| 8 | \((x - 3.5500)(x - 1.9054)(x - 1.2235)(x - 0.8173)(x - 0.5248)(x - 0.2817)(x + 0.0564)(x + 0.2007)\) | -10.114 |
| 9 | \((x - 4.8049)(x - 2.3557)(x - 1.4807)(x - 1)(x - 0.6753)(x - 0.4245)(x - 0.2081)(x + 0.0900)\) | 0 |
| 10 | \((x - 7.0260)(x - 2.9726)(x - 1.7931)(x - 1.2039)(x - 0.8307)(x - 0.5577)(x - 0.3364)(x - 0.1423)\) | 10.114 |

Exact solution of the two-axis two-spin Hamiltonian
Table 3. continued

| 15 | $(x-62.5073)(x-12.6385)(x-5.5949)(x-3.2735)(x-2.1861)(x-1.5636)(x-1.15671)(x-0.8645)$  
|    | $(x-0.6396)(x-0.4574)(x-0.3055)(x-0.1787)(x-0.0791)(x-0.0160)(x+0.7380)(x+1.3550) 89.3684$  
| 16 | $(x-76.8867)(x-15.2406)(x-6.5992)(x-3.7948)(x-2.5073)(x-1.7858)(x-1.3245)(x-1)$  
|    | $(x-0.3988)(x-0.7550)(x-0.5600)(x-0.2635)(x-0.1515)(x-0.0656)(x-0.0130)(x+1) 110.346$  
| 17 | $(x-92.3648)(x-18.0792)(x-7.7057)(x-4.3702)(x-2.8598)(x-2.0272)(x-1.5042)(x-1.1430)$  
|    | $(x-0.8749)(x-0.6648)(x-0.4933)(x-0.3497)(x-0.2288)(x-0.1298)(x-0.0553)(x-0.0108) 132.862$  

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where $\rho_{1}^{(\mu,\eta)}$ or $\rho_{2}^{(\mu,\eta)}$ is the reduced density matrix of the two-spin symmetric (S) or anti-symmetric (A) eigenstate (37) obtained by taking the partial trace over the subsystem of spin 2 or 1, and the logarithm to the base $N = 2(k - \mu + 1)$ for both the two-spin symmetric (S) and anti-symmetric (A) eigenstates with $\mu \neq 0$ or $N = k + 1$ for the two-spin symmetric (S) eigenstates with $\mu = 0$, which is the total number of modes involved, is used to ensure that the maximum measure is normalized to 1.

Figure 2 shows entanglement measure of all excited states of the system with $k = 40$. It is observed that the excited states are all well entangled with measure $\text{Ent}(\mu, \eta) \geq 0.6476$ in the $k = 40$ case. The entanglement measure for both the two eigenstates in the $(k - 1)$-th band and the single eigenstate in the $k$th band reaches to the maximum value, i.e. $\text{Ent}(k - 1, \eta) = 1$ for $\eta = 1, 2$ and $\text{Ent}(k, 1) = 1$, respectively, and the excited states in other bands with $\mu \leq k - 2$ are also well entangled with $0.9671 \geq \text{Ent}(\mu, \eta) \geq 0.6476$. Moreover, the entanglement measure of the band-head states gradually increases with the increasing of its excitation energy or $\mu$ from $\text{Ent}(0, 1) = 0.6476$ to $\text{Ent}(40, 1) = 1$ in the $k = 40$ case. Additionally, except the last two bands with $\mu = k - 1$ or $k$, the entanglement measure of the excited states in other bands varies almost randomly with the value $\text{Ent}(\mu, \eta) \geq \text{Ent}(\mu, 1)$ for $1 \leq \eta \leq k - \mu + 1$ as shown in figure 2. Therefore, the eigenstates of the 2A2S system are always well entangled.
In this work, a systematic procedure for calculating exact solution of the 2A2S Hamiltonian is presented by using the Bethe ansatz method. It is shown that the 2A2S Hamiltonian can be expressed in terms of the generators of two copies of SU(1, 1) algebra after the Jordan–Schwinger boson realization of the two related SU(2) algebras. Thus, eigenstates of the 2A2S Hamiltonian can be expressed as SU(1, 1) type Bethe ansatz states, by which eigen-energies and the corresponding eigenstates are derived. To avoid solving a set of non-linear Bethe ansatz equations involved, the related extended Heine–Stieltjes polynomials are constructed, whose zeros are just components of a root of the Bethe ansatz equations. Symmetry properties of excited levels of the 2A2S system and those of zeros of the related extended Heine–Stieltjes polynomials are also discussed. As an example, the two equal spin case is analysed in detail. It is shown that the $(2S + 1)^2$-dimensional energy matrix in the uncoupled two-spin basis, where $S = S_1 = S_2$ is the quantum number of the two spins, can be decomposed into $2S - \mu + 1$-dimensional bidiagonal sub-matrices for $\mu = 0, 1, \ldots, 2S$ with two-fold degeneracy for $\mu \neq 0$, which clearly demonstrates the advantages of the Bethe ansatz method over the direct diagonalization in the original uncoupled two-spin basis. Furthermore, in the two equal spin case, it is shown that the levels in each band labelled by $\mu$ are symmetric with respect to the zero energy plane perpendicular to the level diagram and that the excited states are always well entangled.

In comparison to the direct diagonalization of the 2A2S Hamiltonian in the uncoupled two-spin basis with $S_1 = S_2 = k/2$, the size of the energy matrix increases with $k$ quadratically, while the size of the energy sub-matrices constructed by the Bethe ansatz method shown in this paper increases with $k$ linearly, which makes the diagonalization more efficient and doable for large spin cases. For example, when the two spins are large with $k \sim 10^{10}$, the diagonalization of the bidiagonal matrices with $10^{10}$ in size can be carried out on a current day computer, while the direct diagonalization of the energy matrix in the original two-spin basis with $10^{20}$ in size becomes a formidable task. Therefore, the results shown in this paper should be useful for large spin cases in Bose–Einstein condensates [33].

Further analysis of the model using the Bethe ansatz solution, such as computing the overlaps of the eigenstates with the two-spin squeezed state as suggested in [15] and other physical quantities of the system, especially in the thermodynamic limit by using a similar procedure in [34], is beyond the scope of this paper and will be part of our future work to be presented elsewhere.

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