THE $W^*$–CURVATURE TENSOR ON RELATIVISTIC SPACE-TIMES

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Abstract. This paper aims to study the $W^*$–curvature tensor on relativistic space-times. The energy-momentum tensor $T$ of a space-time is semi-symmetric given that the $W^*$–curvature tensor is semi-symmetric whereas energy-momentum tensor $T$ of a space-time having a divergence free $W^*$–curvature tensor is of Codazzi type. A space-time having a traceless $W^*$–curvature tensor is Einstein. A $W^*$–curvature flat space-time is Einstein. Perfect fluid space-times which admits $W^*$–curvature tensor are considered.

1. Introduction

In [12–16], the authors introduced some curvature tensors similar to the projective curvature tensor [9]. They investigated their geometrical properties and physical significance. These tensors have been recently studied in different ambient spaces [1, 4, 5, 11, 17, 18, 20]. However, we noticed that little attention is paid to the $W^*$–curvature tensor. This tensor is a $(0, 4)$ tensor defined as

$$W^*_3(U, V, Z, T) = R(U, V, Z, T) - \frac{1}{n-1} \left[ g(V, Z) \text{Ric}(U, T) - g(V, T) \text{Ric}(U, Z) \right],$$

where $R(U, V, Z, T) = g(R((U, V) Z, T)$, $R(U, V) Z = \nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U, V]} Z$ is the Riemann curvature tensor, $\nabla$ is the Levi-Civita connection, and $\text{Ric}(U, V)$ is Ricci tensor. For the simplicity, we will denote $W^*_3$ by $W^*$. In the local coordinates, it is

$$W^*_{ijkl} = R_{ijkl} - \frac{1}{n-1} \left[ g_{jk} R_{il} - g_{jl} R_{ik} \right].$$

The $W^*$–curvature tensor does not have neither symmetry nor cyclic properties.

A semi-Riemannian manifold $M$ is semi-symmetric [19] if

$$R(\zeta, \xi) \cdot R = 0,$$

where $R(\zeta, \xi)$ acts as a derivation on $R$. $M$ is Ricci semi-symmetric [8] if

$$R(\zeta, \xi) \cdot \text{Ric} = 0,$$

where $R(\zeta, \xi)$ acts as a derivation on $\text{Ric}$. A semi-symmetric manifold is known to be Ricci semi-symmetric as well. The converse does not generally hold. On the same line of the above definitions we say that $M$ has a semi-symmetric $W^*$–curvature tensor if

$$R(\zeta, \xi) \cdot W^* = 0,$$

where $R(\zeta, \xi)$ acts as a derivation on $W^*$.

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This study was designed to fill this observed gap. The relativistic significance of the $W^\star$-curvature tensor is investigated. First, it is shown that space-times with semi-symmetric $W^\star_{jkl} = g^{il}W^\star_{ijkl}$ tensor have Ricci semi-symmetric tensor and consequently the energy-momentum tensor is semi-symmetric. The divergence of the $W^\star$-curvature tensor is considered and it is proved that the energy-momentum tensor $T$ of a space-time $M$ is of Codazzi type if $M$ has a divergence free $W^\star$-curvature tensor. If $M$ admits a parallel $W^\star$-curvature tensor, then $T$ is a parallel. Finally, a $W^\star$-flat perfect fluid space-time performs as a cosmological constant. A dust fluid $W^\star$-flat space-time satisfies Einstein's field equation is a vacuum space.

2. $W^\star$-SEMI-SYMMETRIC SPACE-TIMES

A 4-dimensional relativistic space-time $M$ is said to have a semi-symmetric $W^\star$-curvature tensor if

$$R(\xi, \eta) \cdot W^\star = 0,$$

where $R(\xi, \eta)$ acts as a derivation on the tensor $W^\star$. In local coordinates, one gets

$$
(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) W^\star_{ijkl} = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{ijkl} - \frac{1}{3} (g_{jk} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{il} - g_{jl} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{ik}).
$$

(2.1)

Contracting both sides with $g^{il}$ yields

$$
(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) W^\star_{jk} = \frac{4}{3} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{jk},
$$

(2.2)

where $W^\star_{jk} = g^{il}W^\star_{ijkl}$. Thus we have the following theorem.

**Theorem 1.** $M$ is Ricci semi-symmetric if and only if $W^\star_{jk} = g^{il}W^\star_{ijkl}$ is semi-symmetric.

The following result is a direct consequence of this theorem.

**Corollary 1.** $M$ is Ricci semi-symmetric if the $W^\star$-curvature is semi-symmetric.

A space-time manifold is conformally semi-symmetric if the conformal curvature tensor $C$ is semi-symmetric.

**Theorem 2.** Assume that $M$ is a space-time admitting a semi-symmetric $W^\star_{jk} = g^{il}W^\star_{ijkl}$. Then, $M$ is conformally semi-symmetric if and only if it is semi-symmetric i.e. $\nabla_{[\mu} \nabla_{\nu]} R_{ijkl} = 0 \Leftrightarrow \nabla_{[\mu} \nabla_{\nu]} C_{ijkl} = 0$.

The Einstein’s field equation is

$$
R_{ij} - \frac{1}{2} g_{ij} R + g_{ij} \Lambda = k T_{ij},
$$

(2.3)

where $\Lambda$, $R$, $k$ are the cosmological constant, the scalar curvature, and the gravitational constant. Then

$$
(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{ij} = k (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T_{ij},
$$

(2.4)

i.e., $M$ is Ricci semi-symmetric if and only if the energy-momentum tensor is semi-symmetric.

**Theorem 3.** The energy-momentum tensor of a space-time $M$ is semi-symmetric if and only if $W^\star_{jk} = g^{il}W^\star_{ijkl}$ is semi-symmetric.
Remark 1. A space-time $M$ with semi-symmetric energy-momentum tensor has been studied by De and Velimirovic in [2].

It is clear that $\nabla_\mu W_{ijkl}^\ast = 0$ implies $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) W_{ijkl}^\ast = 0$. Thus the following result rises.

Corollary 2. Let $M$ be a space-time having a covariantly constant $W^\ast$–curvature tensor. Then $M$ is conformally semi-symmetric and the energy-momentum tensor is semi-symmetric.

A space-time is called Ricci recurrent if the Ricci curvature tensor satisfies

$$\nabla_\mu R_{ij} = b_\mu R_{ij},$$

where $b$ is called the associated recurrence 1–form. Assume that the Ricci tensor is recurrent, then

$$\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_{ij} = \nabla_\mu (\nabla_\nu R_{ij}) - \nabla_\nu (\nabla_\mu R_{ij})$$

$$= \nabla_\mu (b_\nu R_{ij}) - \nabla_\nu (b_\mu R_{ij})$$

$$= (\nabla_\mu b_\nu) R_{ij} + b_\nu \nabla_\mu R_{ij} - (\nabla_\nu b_\mu) R_{ij} - b_\mu \nabla_\nu R_{ij}$$

$$= [\nabla_\mu b_\nu - \nabla_\nu b_\mu] R_{ij}. \quad (2.6)$$

Corollary 3. The following conditions on a space-time $M$ are equivalent

1. The Ricci tensor is recurrent with closed recurrence one form,
2. $T$ is semi-symmetric, and
3. $W_{jk}^\ast = g^{kl} W_{ijkl}^\ast$ is semi-symmetric.

3. Space-times admitting divergence free $W^\ast$–curvature tensor

The tensor $W_{jkl}^\ast$ of type $(1, 3)$ is given by

$$W_{jkl}^\ast = g^{hi} W_{ijkl}^\ast$$

$$= R_{jkl} - \frac{1}{3} [g_{jk} R_{il} - g_{jl} R_{ik}].$$

Consequently, one defines its divergence as

$$\nabla_h W_{jkl}^\ast = \nabla_h R_{jkl} - \frac{1}{3} [g_{jk} \nabla_h R_{il} - g_{jl} \nabla_h R_{ik}]$$

$$= \nabla_h R_{jkl} - \frac{1}{3} [g_{jk} \nabla_l R - g_{jl} \nabla_k R]. \quad (3.1)$$

It is well known that the contraction of the second Bianchi identity gives

$$\nabla_h R_{jkl} = \nabla_l R_{jk} - \nabla_k R_{jl}.$$

Thus, Equation $(3.1)$ becomes

$$\nabla_h W_{jkl}^\ast = \nabla_l R_{jk} - \nabla_k R_{jl} - \frac{1}{3} [g_{jk} \nabla_l R - g_{jl} \nabla_k R]. \quad (3.2)$$

If the $W^\ast$–curvature tensor is divergence free, then Equation $(3.2)$ turns into

$$0 = \nabla_l R_{jk} - \nabla_k R_{jl} - \frac{1}{3} [g_{jk} \nabla_l R - g_{jl} \nabla_k R].$$

Multiplying by $g^{jk}$ we have

$$\nabla_l R = 0. \quad (3.3)$$
Thus, the tensor $R_{ij}$ is a Codazzi tensor and $R$ is constant. Conversely, assume that the Ricci tensor is a Codazzi tensor. Then
\[
\nabla_h W_{ijkl}^{\star h} = -\frac{1}{3} [g_{jk} \nabla_{i} R - g_{jl} \nabla_{k} R]
\]
However, the last equation implies that $\nabla_{l} R = 0$. Consequently, the $W^{\star}$—curvature tensor has zero divergence.

**Theorem 4.** The $W^{\star}$—curvature tensor has zero divergence if and only if the Ricci tensor is a Codazzi tensor. In both cases, the scalar curvature is constant.

The divergence of the Weyl curvature $C$ tensor is given by
\[
\nabla_h C_{ijkl} = \frac{n-3}{n-2} [\nabla_k R_{ij} - \nabla_j R_{ik}] + \frac{1}{2(n-1)} [g_{ij} \nabla_k R - g_{lk} \nabla_j R].
\]

**Remark 2.** Since divergence free of $W^{\star}$—curvature tensor implies that $R_{ij}$ is a Codazzi tensor, the conformal curvature tensor has zero divergence.

Equation (2.3) yields
\[
\nabla_l R_{ij} - \frac{1}{2} g_{ij} \nabla_l R = k \nabla_l T_{ij},
\]
The above theorem now implies the following result.

**Corollary 4.** The energy-momentum tensor is a Codazzi tensor if and only if the $W^{\star}$—curvature tensor has zero divergence. In both cases, the scalar curvature is constant.

Einstein’s field equation infers
\[
(3.4) \quad k (\nabla_l T_{ij} - \nabla_i T_{lj}) = \nabla_l \left( R_{ij} - \frac{1}{2} g_{ij} R \right) - \nabla_i \left( R_{lj} - \frac{1}{2} g_{lj} R \right)
\]
\[
(3.5) \quad = \nabla_l R_{ij} - \nabla_i R_{lj} - \frac{1}{2} (g_{ij} \nabla_l R - g_{lj} \nabla_i R)
\]
\[
(3.6) \quad = \nabla_h W_{ijkl}^{\star h} - \frac{1}{6} (g_{ij} \nabla_l R - g_{lj} \nabla_i R).
\]
Now, it is noted that the above theorem may be proved using this identity.

4. $W^{\star}$—symmetric space-times

A space-time $M$ is called $W^{\star}$—symmetric if
\[
\nabla_m W_{ijkl}^{\star m} = 0.
\]
Applying the covariant derivative on the both sides of equation (1.1), one gets
\[
\nabla_m W_{ijkl} = \nabla_m R_{ijkl} - \frac{1}{n-1} [g_{jk} \nabla_m R_{il} - g_{jl} \nabla_m R_{ik}].
\]
If $M$ is a $W^{\star}$—symmetric space-time, then
\[
\nabla_m R_{ijkl} = \frac{1}{3} [g_{jk} \nabla_m R_{il} - g_{jl} \nabla_m R_{ik}].
\]
Multiplying the both sides by $g^{il}$, we get
\[
\nabla_m R_{jk} = \frac{1}{3} [g_{jk} \nabla_m R - \nabla_m R_{jk}],
\]
and hence
\( (4.2) \quad \nabla_m R_{jk} = \frac{1}{4} g_{jk} \nabla_m R. \)

Now, the following theorem rises.

**Theorem 5.** Assume that \( M \) is a \( W^* \)-symmetric space-time, then \( M \) is a Ricci symmetric if the scalar curvature is constant.

The second Bianchi identity for \( W^* \)-curvature tensor is
\[
\nabla_m W_{ijkl}^* + \nabla_k W_{ijlm}^* + \nabla_l W_{ijmk}^* = -\frac{1}{3} [g_{jk} (\nabla_m R_{il} - \nabla_l R_{im}) + g_{jl} (\nabla_k R_{im} - \nabla_m R_{ik})] - \frac{1}{3} g_{jm} (\nabla_l R_{ik} - \nabla_k R_{il}).
\]
(4.3)

If the Ricci tensor satisfies \( \nabla_m R_{il} = \nabla_l R_{im} \), then
\[ (4.4) \quad \nabla_m W_{ijkl}^* + \nabla_k W_{ijlm}^* + \nabla_l W_{ijmk}^* = 0. \]

Conversely, if the above equation holds, then Equation (4.3) implies
\[ (4.5) \quad g_{jk} (\nabla_m R_{il} - \nabla_l R_{im}) + g_{jl} (\nabla_k R_{im} - \nabla_m R_{ik}) + g_{jm} (\nabla_l R_{ik} - \nabla_k R_{il}) = 0. \]

Multiplying the both sides with \( g^{ik} \), then we have
\[ (4.6) \quad \nabla_m R_{jl} = \nabla_l R_{jm}, \]
which means that the Ricci tensor is of Codazzi type.

**Theorem 6.** The Ricci tensor satisfies \( \nabla_m R_{il} = \nabla_l R_{im} \) if and only if the \( W^* \)-curvature tensor satisfies Equation (4.4).

For a purely electro-magnetic distribution, Equation (2.3) reduces to
\[ (4.7) \quad R_{ij} = k T_{ij}. \]

Its contraction with \( g^{ij} \) gives
\[ (4.8) \quad R = -k T. \]

In this case, it is \( T = R = 0 \). Thus Equation (4.2) yields \( \nabla_m T_{jk} = 0. \)

**Theorem 7.** The energy-momentum tensor of a \( W^* \)-symmetric space-time obeying Einstein’s field equation for a purely electro-magnetic distribution is locally symmetric.

5. \( W^* \)-FLAT SPACE-TIMES

Now, we consider \( W^* \)-flat space-times. Multiplying both sides of Equation (1.1) by \( g^{il} \) yields
\[
W_{jk}^* = g^{il} W_{ijkl}^* = \frac{4}{3} \left( R_{jk} - \frac{R}{4} g_{jk} \right).
\]
(5.1)
Thus, a \( W_{jk}^* \)-curvature flat space-time is Einstein, i.e.,
\[
R_{jk} = \frac{R}{4} g_{jk}.
\]

Now, Equation (1.1) becomes
\[
W_{ijkl}^* = R_{ijkl} - \frac{R}{12} [g_{ik} g_{jl} - g_{jl} g_{ik}].
\]
Theorem 8. A space-time manifold $M$ is Einstein if and only if $\mathcal{W}^*_jk = 0$. Moreover, a $\mathcal{W}^*$--flat space-time has a constant curvature.

A vector field $\xi$ is said to be a conformal vector field if

$$L_\xi g = 2\phi g,$$

where $L_\xi$ denotes the Lie derivative along the flow lines of $\xi$ and $\phi$ is a scalar. $\xi$ is called Killing if $\phi = 0$. Let $T_{ij}$ be the energy-momentum tensor defined on $M$. $\xi$ is said to be a matter inheritance collineation if

$$L_\xi T = 2\phi T.$$

The tensor $T_{ij}$ is said to have a symmetry inheritance property along the flow lines of $\xi$. $\xi$ is called a matter collineation if $\phi = 0$. A Killing vector field $\xi$ is a matter collineation. However, a matter collineation is not generally Killing.

Theorem 9. Assume that $M$ is a $\mathcal{W}^*$--flat space-time. Then, $\xi$ is conformal if and only if $L_\xi T = 2\phi T$.

Proof. Using Equations (5.1) and (2.3), we have

$$(5.2) \quad \left(\Lambda - \frac{R}{4}\right) g_{ij} = k T_{ij}.$$  

Then

$$(5.3) \quad \left(\Lambda - \frac{R}{4}\right) L_\xi g = k L_\xi T.$$  

Assume that $\xi$ is conformal. The above two equations lead to

$$2\phi \left(\Lambda - \frac{R}{4}\right) g = k L_\xi T$$  

$$2\phi T = L_\xi T.$$  

Conversely, suppose that the energy-momentum tensor has a symmetry inheritance property along $\xi$. It is easy to show that $\xi$ is a conformal vector field. □

Corollary 5. Assume that $M$ is a $\mathcal{W}^*$--flat space-time. Then, $M$ admits a matter collineation $\xi$ if and only if $\xi$ is Killing.

Equations (5.1) and (2.3) imply

$$(5.4) \quad \left(\Lambda - \frac{R}{4}\right) g_{ij} = k T_{ij}.$$  

Taking the covariant derivative of (5.4) we get

$$(5.5) \quad \nabla_i T_{ij} = \frac{1}{k} \nabla_i \left(\Lambda - \frac{R}{4}\right) g_{ij}.$$  

Since a $\mathcal{W}^*$--curvature flat space-time has $\nabla_i R = 0$, $\nabla_i T_{ij} = 0$.

Theorem 10. The energy-momentum tensor of a $\mathcal{W}^*$--flat space-time is covariantly constant.

Let $M$ be a space-time and $\mathcal{W}^*_{klm} = g^{ij} \mathcal{W}^*_{jkil}$ be a (1, 3) curvature tensor. According to [3], there exists a unique traceless tensor $\mathcal{B}^i_{klm}$ and three unique (0, 2) tensors $\mathcal{C}_{kl}$, $\mathcal{D}_{kl}$, $\mathcal{E}_{kl}$ such that

$$\mathcal{W}^*_{klm} = \mathcal{B}^i_{klm} + \delta^i_k \mathcal{C}_{lm} + \delta^i_l \mathcal{D}_{km} + \delta^i_m \mathcal{E}_{kl}.$$
All of these tensors are given by

\[ C_{ml} = \frac{1}{33} \left[ 10W_{tml}^{st} - 2 (W_{mtl}^{st} + W_{lmt}^{st}) \right] = 0, \]

\[ D_{km} = \frac{1}{33} \left[ -2 (W_{tkm}^{st} + W_{mtk}^{st}) + 10W_{ktm}^{st} \right] = \frac{1}{9} \left[ R_{km} - \frac{g_{km}}{4} R \right], \]

and

\[ E_{kl} = \frac{1}{33} \left[ 10W_{tkl}^{st} - 2 (W_{tlk}^{st} + W_{ltk}^{st}) \right] = -\frac{1}{9} \left[ R_{kl} - \frac{g_{kl}}{4} R \right]. \]

Assume that the \( W^s \)–curvature tensor is traceless. Then

\[ C_{kl} = D_{kl} = E_{kl} = 0, \]

and consequently

\[ R_{ml} = \frac{g_{ml}}{4} R. \]

**Theorem 11.** Assume that \( M \) is a space-time admitting \( W^s \)–curvature tensor. Then, \( M \) is an Einstein space-time.

For a perfect fluid space-time with the energy density \( \mu \) and isotropic pressure \( p \), it is

\[ T_{ij} = (\mu + p) u_i u_j + p g_{ij}, \]

where \( u_i \) is the velocity of the fluid flow with \( g_{ij} u_i u_j = u_i u^{i} = -1 \) \([6, 7, 10]\). In \([2, \text{Theorem 2.2}]\), a characterization of such space-times is given. This result leads us to.

**Theorem 12.** Assume that the perfect fluid space-time \( M \) is \( W^s \)–semi-symmetric. Then, \( M \) is regarded as inflation and this fluid acts as a cosmological constant. Moreover, the perfect fluid represents the quintessence barrier.

Using Equations (5.2), we have

\[ \left( \Lambda - kp - \frac{R}{4} \right) g_{ij} = k (\mu + p) u_i u_j. \]

Multiplying the both sides by \( g^{ij} \) we get

\[ R = 4\Lambda + k (\mu - 3p). \]

For \( W^s \)–curvature flat space-times, the scalar curvature is constant and consequently

\[ \mu - 3p = \text{constant}. \]

Again, a contraction of Equation (5.7) with \( u^i \) leads to

\[ R = 4 (k\mu + \Lambda). \]

The comparison between (5.8) and (5.10) gives

\[ \mu + p = 0, \]
i.e., the perfect fluid performs as a cosmological constant. Then Equation (5.6) infers
\[ T_{ij} = p g_{ij}. \]

For a $\mathcal{W}^*$-flat space-time, the scalar curvature is constant. Thus $\mu = \text{constant}$ and consequently $p = \text{constant}$. Therefore, the covariant derivative of equation (5.12) implies $\nabla_l T_{ij} = 0$.

**Theorem 13.** Let $M$ be a perfect fluid $\mathcal{W}^*$-flat space-time obeying Equation (2.3), then the $\mu$ and $p$ are constants and $\mu + p = 0$ i.e. the perfect fluid performs as a cosmological constant. Moreover, $\nabla_l T_{ij} = 0$.

The following results are two direct consequences of being $\mathcal{W}^*$-curvature flat.

**Corollary 6.** A $\mathcal{W}^*$-flat space-time $M$ obeying Equation (4.7) is a Euclidean space.

**Corollary 7.** Let $M$ be a dust fluid $\mathcal{W}^*$-flat space-time satisfying Equation (2.3) (i.e. $T_{ij} = \mu u_i u_j$). Then $M$ is a vacuum space-time (i.e. $T_{ij} = 0$).

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