Empirical study and model of personal income

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Summary. Personal income distributions in Japan are analyzed empirically and a simple stochastic model of the income process is proposed. Based on empirical facts, we propose a minimal two-factor model. Our model of personal income consists of an asset accumulation process and a wage process. We show that these simple processes can successfully reproduce the empirical distribution of income. In particular, the model can reproduce the particular transition of the distribution shape from the middle part to the tail part. This model also allows us to derive the tail exponent of the distribution analytically.

Keywords. Personal income, Power law, Stochastic model

1 Introduction

Many economists and physicists have studied wealth and income. About one hundred years ago, Pareto found a power law distribution of wealth and income [13]. However, afterwards, Gibrat clarified that the power law is applicable to only the high wealth and income range, and the remaining part follows a lognormal distribution [7]. This characteristic of wealth and income was later rediscovered [2][10][16][17]. Today, it is generally believed that high wealth and income follow a power law distribution. However, the remaining range of the distribution has not been settled. Recently an exponential distribution [5] and a Boltzmann distribution [20] has been proposed.

To explain these characteristics of wealth and income, some mathematical models have been proposed. One of them is based on a stochastic multiplicative process (SMP). For example, the SMP with lower bound [9], the SMP with additive noise [15][19], the SMP with wealth exchange [4], and the generalized Lotka-Voltera model [3][14].

This paper is organized as follows. In Sec. 2, we empirically study the personal income distribution in Japan. In Sec. 3, we propose a two-factor stochastic model to explain income distribution. The last section is devoted to a summary and discussion.
2 Empirical study of the personal income distribution

In this article we use three data sets. We call them employment income data, self-assessed income data, and income tax data for top taxpayers. The employment income data is coarsely tabulated data for the distribution of wages in the private sector. This is reported by the National Tax Agency of Japan (NTAJ) [11]. This is composed of two kinds of data. One is for employment income earners who worked for less than a year, and we can acquire the data since 1951. For example, a log-log plot of the rank-size distribution of the data in 1999 is shown by the open circles in the left panel of Fig. 1. The other is for employment income earners who worked throughout the year, and we can acquire the data since 1950. For example, the distribution in 1999 is shown by the open squares in the left panel of Fig. 1. In this figure the crosses are the sum of these two data, and are almost the same as the distribution of employment income earners who worked throughout the year.

The self-assessed income data is also reported by NTAJ. This is also coarsely tabulated data, and we can acquire this since 1887. The income tax law was changed many times, and so the characteristics of this data also changed many times. However, this data consistently contains high income earners. In Japan, in recent years, persons who have some income source, who earned more than 20 million yen, and who are not employees must declare their income. For example, the distribution in 1999 is shown by the open triangles in the right panel of Fig. 1. In this figure the filled circles are the sum of the employment income data and the self-assessed income data. However, we use only the self-assessed income data in the range greater than 20 mil-
lion yen. This is because persons who earned more than 20 million yen must declare their income, even if they are employees and have only one income source. This figure shows that the distribution of middle and low income is almost the same as that of the employment income. This means that the main income source of middle and low income earners is wages.

In Japan, if the amount of one’s income tax exceeds 10 million yen, the individual’s name and the amount of income tax are made public by each tax office. Some data companies collect this and produce income tax data for top taxpayers. We obtained this data from 1987 to 2000. For example, the distribution in 1999 is shown by the open diamonds in the right panel of Fig. 1. To understand the whole image of distribution, we must convert income tax to income. We know from the self-assessed income data that the income of the 40,623th person is 50 million yen. On the other hand we also know from the income tax data for top taxpayers that the income tax of the 40,623th person is 13.984 million yen. Hence, if we assume a linear relation between income and income tax, we can convert income tax to income by multiplying 3.5755 by the income tax. The dots in Fig. 1 represent the distribution of converted income tax. This clearly shows the power law distribution in the high income range, and the particular transition of the distribution shape from the middle part to the tail part.

2.1 Income sources

Understanding income sources is important for the modeling of the income process. As we saw above, the main income source of middle and low income earners is wages. We can also see the income sources of high income earners from the report of NTAJ. The top panel of Fig. 2 shows a number of high income earners who earned income greater than 50 million yen in each year from 2000 to 2003. In this figure income sources are divided into the 14 categories of business income, farm income, interest income, dividends, rental income, wages & salaries, comprehensive capital gains, sporadic income, miscellaneous income, forestry income, retirement income, short-term separate capital gains, long-term separate capital gains, and capital gains of stocks. The bottom panel of this figure shows the amount of income for each income source. These figures show that the main income sources of high income earners are wages and capital gains.

2.2 Change of distribution

The rank-size distribution of all acquired data is shown in the top panel of Fig. 3. The gap found in this figure reflects the change of the income tax law. We fit distributions in the high income range by the power law distribution, for which a probability density function is given by

\[ p(x) = Ax^{-(\alpha-1)}, \]
where $A$ is a normalization factor. Here $\alpha$ is called the Pareto index. The small $\alpha$ corresponds to the unequal distribution. The change of $\alpha$ is shown by the open circles in the bottom panel of Fig. 3. The mean value of the Pareto index is $\bar{\alpha} = 2$, and $\alpha$ fluctuates around it.

It is recognized that the period of modern economic growth in Japan is from the 1910s to the 1960s. It has been reported that the gross behavior of the Gini coefficient in this period looks like an inverted U-shape [18]. This behavior of the Gini coefficient is known as Kuznets’s inverted U-shaped relation between income inequality and economic growth [8]. This postulates that in the early stages of modern economic growth both a country’s economic growth and its income inequality rises, and the Gini coefficient becomes large. For developed countries income inequality shows a tendency to narrow, and the Gini coefficient becomes small. Figure 3 shows that the gross behavior of the Pareto index from the 1910s to the 1960s is almost the inverse of that of the Gini coefficient, i.e., U-shaped. This means that our analysis of the Pareto index also supports the validity of Kuznets’s inverted U-shaped relation.

We assume that the change of the Pareto index in the 1970s is responsible for the slowdown in the Japanese economic growth and the real estate boom. In Fig. 3 we can also see that $\alpha$ decreases toward the year 1990 and
increases after 1990, i.e., V-shaped relation. In Japan, the year 1990 was the peak of the asset-inflation economic bubble. Hence the Pareto index decreases toward the peak of the bubble economy, and it increases after the burst of the economic bubble. The correlation between the Pareto index and risk assets is also clarified in Ref. [16].

We fit distributions in the low and middle income range by log-normal distribution, for which the probability density function is defined by

\[ p(x) = \frac{1}{x \sqrt{2\pi \sigma^2}} \exp \left[ -\frac{\log^2 (x/x_0)}{2\sigma^2} \right], \]

where \( x_0 \) is mean value and \( \sigma^2 \) is variance. Sometimes \( \beta \equiv 1/\sqrt{2\sigma^2} \) is called the Gibrat index. Since the large variance means the global distribution of the income, the small \( \beta \) corresponds to unequal distribution. The change of \( \beta \) is shown by the crosses in the bottom panel of Fig. 3. This figure shows that \( \alpha \) and \( \beta \) correlate with each other around the years 1960 and 1980. However, they have no correlation in the beginning of the 1970s and after 1985. Especially after 1985, \( \beta \) stays almost the same value. This means that the variance of the low and middle income distribution does not change. We assume that capital gains cause different behaviors of \( \alpha \) and \( \beta \), and \( \alpha \) is more sensitive to capital gains than \( \beta \).
The top panel of Fig. 3 shows that the distribution moves to the right. This motivates us to normalize distributions by quantities that characterize the economic growth. Though many candidates exist, we simply normalize distributions by the average income. The left panel of Fig. 4 is a log-log plot of the cumulative distributions of normalized income from 1987 to 2000, and the right panel is a semi-log plot of them. These figures show that distributions almost become the same, except in the high income range. Though distributions in the high income range almost become the same, distributions of some years apparently deviate from the stational distribution. In addition the power law distribution is not applicable to such a case. This behavior happens in an asset-inflation economic bubble [6].

3 Modeling of personal income distribution

The empirical facts found in the previous section are as follows.

1. The distribution of high income earners follows the power law distribution, and the exponent, Pareto index, fluctuates around $\alpha = 2$.
2. The main income sources of high income earners are wages and capital gains.
3. Excluding high income earners, the main income source is wages.
4. The distribution normalized by the average income is regarded as the stational distribution.

Hence, it is reasonable to regard income as the sum of wages and capital gains. However, to model capital gains, we must model the asset accumulation process. In the following we explain an outline of our model. Details of our model are found in Ref. [12].
3.1 Wage process

We denote the wages of the \(i\)-th person at time \(t\) as \(w_i(t)\), where \(i = 1 \sim N\). We assume that the wage process is given by

\[
w_i(t + 1) = \begin{cases} \frac{uw_i(t) + s\epsilon_i(t)}{\overline{w}(t)} & \text{if } uw_i(t) + s\epsilon_i(t) > \overline{w}(t), \\ \overline{w}(t) & \text{otherwise}, \end{cases}
\]

where \(u\) is the trend growth of wage, and reflects an automatic growth in nominal wage. In this article we use \(u = 1.0422\). This is an average inflation rate for the period from 1961 to 1999. In Eq. (1), \(\epsilon_i(t)\) follows a normal distribution with mean 0 and variance 1, i.e., \(N(0, 1)\). In Eq. (1), \(s\) determines the level of income for the middle class. We choose \(s = 0.32\) to fit the middle part of the empirical distribution. In Eq. (1), \(\overline{w}(t)\) is the reflective lower bound, which is interpreted as a subsistence level of income. We assume that \(\overline{w}(t)\) grows deterministically,

\[
\overline{w}(t) = v^t\overline{w}(0).
\]

Here we use \(v = 1.0673\). This is a time average growth rate of the nominal income per capita.

3.2 Asset accumulation process

We denote the asset of the \(i\)-th person at time \(t\) as \(a_i(t)\). We assume that the asset accumulation process is given by a multiplicative process,

\[
a_i(t + 1) = \gamma_i(t)a_i(t) + w_i(t) - c_i(t),
\]

where the log return, \(\log \gamma_i(t)\), follows a normal distribution with mean \(y\) and variance \(x^2\), i.e., \(N(y, x^2)\). We use \(y = 0.0595\). This is a time-average growth rate of the Nikkei average index from 1961 to 1999. We use \(x = 0.3122\). This is a variance calculated from the distribution of the income growth rate for high income earners. In Eq. (2), we assume that a consumption function, \(c_i(t)\), is given by

\[
c_i(t) = \overline{w}(t) + b \{a_i(t) + w_i(t) - \overline{w}(t)\}.
\]

In this article we chose \(b = 0.059\) from the empirical range estimated from Japanese micro data.

3.3 Income distribution derived from the model

We denote the income of the \(i\)-th person at time \(t\) as \(I_i(t)\), and define it as

\[
I_i(t) = w_i(t) + E[\gamma_i(t) - 1]a_i(t).
\]

The results of the simulation for \(N = 10^6\) are shown in Fig. 5. The left panel of Fig. 5 is a log-log plot of the cumulative distribution for income normalized
Fig. 5. A log-log plot of the cumulative distributions of normalized income in 1999 and simulation results (left), and the Lorenz curve in 1999 and simulation results (right).

by an average. The right panel of Fig. 5 is the simulation results for the Lorenz curve. These figures show that the accountability of our model is high.

In our model, the exponent in the power law part of the distribution is derived from the asset accumulation process. From Eq. (1), we can analytically derive

$$\alpha = 1 - \frac{2\log(1 - z/g)}{x^2} \approx 1 + \frac{2z}{gx^2},$$

(3)

where $z$ is a steady state value of $[w(t) - c(t)]/\langle a(t) \rangle$. Here $\langle a(t) \rangle$ is the average assets. In Eq. (3), $g$ is a steady state value of the growth rate of $\langle a(t) \rangle$. Equation (3) shows that $\alpha$ fluctuates around $\alpha = 2$, if $2z \sim gx^2$.

4 Summary

In this article we empirically studied income distribution, and constructed a model based on empirical facts. The simulation results of our model can explain the real distribution. In addition, our model can explain the reason why the Pareto index fluctuate around $\alpha = 2$. However there are many unknown facts. For example, we have no theory that can explain the income distribution under the bubble economy, that can determine the functional form other than the high income range, and that can explain the shape of the income growth distribution, etc.

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