Effective action method for the Langevin equation

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Abstract

In this paper we present a formulation of the nonlinear stochastic differential equation which allows for systematic approximations. The method is not restricted to the asymptotic, i.e., stationary, regime but can be applied to derive effective equations describing the relaxation of the system from arbitrary initial conditions. The basic idea is to reduce the nonlinear Langevin equation to an equivalent equilibrium problem, which can then be studied with the methods of conventional equilibrium statistical field theory. A particular well suited perturbative scheme is that developed in quantum field theory by Cornwall, Jackiw and Tomboulis. We apply this method to the study of N component Ginzburg-Landau equation in zero spatial dimension. In the limit of $N \to \infty$ we can solve the effective equations and obtain closed forms for the time evolutions of the average field and of the two-time connected correlation.
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I. INTRODUCTION

Non linear stochastic differential equations (SDE) in one or more variables occur frequently in the description of a wide variety of physical phenomena such as those involving relaxation towards a steady state [1,2]. A celebrated example of SDE is the Langevin equation, where one relates the rate of change of some physical observables to a drift term, i.e. to a deterministic driving force, plus a stochastic noise. Whereas stochastic linear differential equations are in principle amenable to analytic solutions, those of nonlinear type are much more difficult to treat and in most cases one has to resort either to computer simulations or to approximate schemes. Several approximations exist in the literature for the analysis of nonlinear SDE. A wide used method is the so called statistical linearization [3] which consists of the replacement of the nonlinear SDE by equivalent linear ones whose coefficients are determined by some error minimization algorithm. This method in its original form is appropriate in the asymptotic, i.e., stationary, regime. In many physical systems, however, it is the transient regime in which one is interested. To deal with this problem the idea of statistical linearization has been extended to include a possible time dependence in the parameters [4,5]. The drawback of these methods is that they are not derived in a systematic way so it is not simple to improve them.

An alternative procedure is the so called dynamical Hartree approximation. This scheme amounts to neglect correlations of higher order than the second and is exact whenever the probability distribution associated with the problem is Gaussian. This is not the case of strongly interacting systems, so corrections to the Hartree approximation should be taken into account. Also in this case it is not simple to improve the approximation.

In this paper we present a formulation of the nonlinear SDE which allows for systematic approximations. This is achieved by reducing the nonlinear Langevin equation to an equivalent equilibrium problem, which can be analyzed with the methods of conventional equilibrium statistical field theory. In particular we have applied a method originally developed in quantum field theory by Cornwall, Jackiw and Toumoulis [6,7], alternative to
conventional perturbation theory, because a normal coupling constant expansion can only be used for the study of small corrections to the deterministic result. In this respect the present approach is alternative to the field theoretical treatment based on the introduction of fermionic auxiliary fields.

To illustrate the method we shall study an $N$ component Ginzburg-Landau equation in zero spatial dimension with the purpose of deriving systematically the time dependent Hartree equations and the first corrections. The same model has been discussed previously by Scalapino and coworkers [8], who introduced an approximation scheme for the Langevin equation based on the expansion in the small parameter $1/N$. We shall obtain a solution which represents a systematic treatment, and as a bonus we derive a closed equation for the two-time connected correlation function. This will be discussed in Sects. II, IV and V, where the corrections to the $N \to \infty$ solution are considered.

The formalism is established in Sec. II where we construct the generating functional $\Gamma$ for the average value of the observables and its correlations. In Sect. VI we provide an alternative derivation of the Hartree equations, based on a variational principle similar to the Feynman method in equilibrium statistical mechanics, and discuss the relevant case $N = 1$.

II. THE FORMALISM

In this Section we derive the effective action formalism for the Langevin equation. The path integral method constitutes a convenient representation of the Langevin equation for a field $\phi(t)$. Within this approach, the original stochastic differential equation, where the $\phi$ depends on another field $\xi$, called the noise, is reformulated by constructing an effective action for the field $\phi$ only obtained by eliminating the noise. The advantage of this transformation is that one can employ the well known methods of equilibrium statistical field theory. To keep the notation as simple as possible the derivation will be carried out for a single component real field. The extension to $N$-vector fields will be discussed later.
The time evolution of the field $\phi(t)$ is governed by the Langevin equation:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial}{\partial \phi} S[\phi] + \xi$$

(1)

where $S[\phi]$ is an “energy” function, and $\xi$ a Gaussian random variable with

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = \Gamma \delta(t - t').$$

(2)

We shall derive a generating functional from which the correlations can be obtained. Proceeding in the standard way, see e.g. Ref. [9], we introduce an external source $J$ and define the generating functional

$$Z[J] = N \int \mathcal{D}''\phi \mathcal{D}\xi \mathcal{P}[\phi(0)] \delta(\phi - \phi_\xi)
\times \exp \left[ - \int_0^\tau dt \int_0^\tau dt' J\phi(t) \exp \left[ - \int_0^\tau dt \frac{\xi^2}{2\Gamma} \right] \right]$$

(3)

where $\phi_\xi$ is the solution of stochastic eq. (1) subject to some set of initial value conditions $\phi(0)$ assigned with probability $\mathcal{P}[\phi(0)]$, and $N$ is a normalizing constant. The functional integral on $\phi$ in eq. (3) includes integration over $\phi(0)$ and $\phi(\tau)$. We denote it by the double quote: $\mathcal{D}''\phi$. The $\delta$-function stands for

$$\delta(\phi - \phi_\xi) = \delta \left[ \frac{\partial \phi}{\partial t} + \frac{\partial S}{\partial \phi} - \xi \right] \det \left| \frac{\delta \xi}{\delta \phi} \right|$$

(4)

where $\det \frac{\delta \xi}{\delta \phi}$ is the Jacobian of the transformation $\xi \rightarrow \phi$. With well-known manipulations, see e.g. Ref. [9,10], one has

$$\det \left| \frac{\delta \xi}{\delta \phi} \right| = \exp \left[ \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \frac{\partial^2 S}{\partial \phi^2} \right].$$

(5)

In deriving eq. (5) we have used the forward time propagation Green function $\theta(t - t')$ of the operator $\partial_t$. Moreover we have used the definition

$$\theta(0) = \frac{1}{2}.$$

(6)

This choice corresponds to the “physical” regularization of the noise term $\xi$ as

$$\langle \xi(t) \xi(t') \rangle = \Gamma \eta(t - t').$$

(7)
where $\eta(t)$ is an even function sharply peaked at $t = 0$, whose integral from $-\infty$ to $+\infty$ is equal to 1. The $\delta$-correlated noise is obtained in the limit of vanishing width. In terms of stochastic differential equations this corresponds to the Stratonovich formalism [11].

At this stage one eliminates the noise field by inserting eq. (5) into eq. (3) and performing the $\xi$ integral over the noise obtaining,

$$Z[J] = N \int D\phi \mathcal{P}[\phi(0)] \exp \left[ -\int_0^\tau dt \frac{1}{2\Gamma} \left( \frac{\partial \phi}{\partial t} + \frac{\partial S}{\partial \phi} \right)^2 \right.$$

$$+ \frac{1}{2} \int_0^\tau dt \frac{\partial^2 \phi}{\partial \phi^2} - \int_0^\tau dt J\phi \right]. \quad (8)$$

The argument of the exponential can be simplified by performing the integration of the term $\int_0^\tau dt \dot{\phi} \frac{\partial S}{\partial \phi} = S[\phi(\tau)] - S[\phi(0)]$, so we finally have

$$Z[J] = N \int D\phi(0) \mathcal{P}[\phi(0)] e^{S[\phi(0)]/2\Gamma} \mathcal{D}\phi(\tau) e^{S[\phi(\tau)]/2\Gamma}$$

$$\times \mathcal{D}\phi \exp \left[ -I(\phi) - \int_0^\tau dt J\phi \right] \quad (9)$$

where the action $I(\phi)$ is given by

$$I(\phi) = \frac{1}{2\Gamma} \int_0^\tau dt \left[ \left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial S}{\partial \phi} \right)^2 \right] - \frac{1}{2} \int_0^\tau dt \frac{\partial^2 S}{\partial \phi^2}. \quad (10)$$

and $\mathcal{D}\phi$ denotes integration over all paths starting at $\phi(0)$ for $t = 0$ and ending at $\phi(\tau)$ for $t = \tau$. It is defined as

$$\mathcal{D}\phi = \lim_{N \to \infty} \prod_{i=1}^{N-1} d\phi(t_i) \quad (11)$$

where $\phi(t_i)$ is the field at time $t_i = i\epsilon$, having sliced the interval 0 to $\tau$ in $N$ parts of size $\epsilon = \tau/N$.

In eq.(9) the integration over the end points just fixes the boundary conditions at $t = 0$ and $t = \tau$, so without lost of generality, we can consider a ‘reduced’ generating functional

$$Z[J] = N \int_{\phi_0}^{\phi_1} D\phi \exp \left[ -I(\phi) - \int_0^\tau dt J\phi \right] \quad (12)$$

where now the integral runs over all paths which start at $t = 0$ from $\phi(0) = \phi_0$ and end at $t = \tau$ at $\phi(\tau) = \phi_1$. In the limit $\tau \to \infty$, or equivalently $t/\tau \ll 1$, the path becomes independent of the final value $\phi(\tau)$.
We note however that the presence of the additional constraint is necessary to select paths which are solution of the original equation of motion eq. (11). Equations (11) and (12) lead in fact to second order differential equations of motion, whereas the original stochastic equation is of the first order. The additional constraint at \( t = \tau \) makes the problem well defined since the paths in eq. (12) must satisfy the two constraints \( \phi(0) = \phi_0 \) and \( \phi(\tau) = \phi_1 \). Once the two boundaries conditions are imposed the path is also solution of the first order differential equation (1), as can be easily seen in the limit \( \Gamma \to 0 \), i.e., the deterministic limit.

In general the calculation of path integrals like eqs. (11) and (12) is not at all straightforward. Nevertheless quantities of physical interest can be obtained. In our case we are interested into the noise-averaged value of the field \( \langle \phi(t) \rangle \) and correlations \( \langle \phi(t) \phi(t') \rangle \) as a functions of time. An advantage of the present formalism is that self-consistent, systematic variational principles for these quantities can be obtained using a method introduced by Cornwall, Jackiw and Tomboulis in Quantum Field Theory [6]. The basic idea is to derive an effective action which is stationary at the physical values of \( \langle \phi(t) \rangle \) and \( \langle \phi(t) \phi(t') \rangle \).

The method starts by generalizing eq. (12) to account for the composite operator \( \phi(t)\phi(t') \). We then define the generating functional

\[
Z[J, K] = \mathcal{N} \int_{0,\phi_0}^{\tau,\phi_1} \mathcal{D}\phi \exp\left[ -I(\phi) - \int_0^\tau dt J(t) \phi(t) \right. \\
- \left. \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \phi(t) K(t, t') \phi(t') \right]
\]

(13)

where \( J \) and \( K \) are a local and a bilocal source, respectively.

By taking functional derivatives with respect to the external sources the averaged correlations of \( \phi \) can be obtained. In particular, by considering \( W[J, K] = -\ln Z[J, K] \), we have for \( 0 < s < \tau \)

\[
\begin{align*}
\frac{\delta}{\delta J(s)} W[J, K] & = \langle \phi(s) \rangle \equiv q(s) \\
\frac{\delta}{\delta K(s, s')} W[J, K] & = \frac{1}{2} \langle \phi(s) \phi(s') \rangle \equiv \frac{1}{2} [q(s) q(s') + G(s, s')]
\end{align*}
\]

(14)
where the averages are obtained with the weight of eq. (13). In the limit of vanishing external sources $q$ and $G$ become the noise-averaged field $\langle \phi(s) \rangle$ and connected two-point correlation function $\langle \phi(s) \phi(s') \rangle_c$ of the process described by the Langevin equation (1).

By Legendre transforming $W[J, K]$ we can eliminate $J$ and $K$ in favor of $q$ and $G$:

$$
\Gamma[q, G] = W[J, K] - \int_0^\tau ds \, q(s) \, J(s) - \frac{1}{2} \int_0^\tau ds \, \int_0^\tau ds' \, q(s) \, K(s, s') \, q(s') - \frac{1}{2} \int_0^\tau ds \, \int_0^\tau ds' \, G(s, s') \, K(s, s')
$$

(15)

where $J$ and $K$ are eliminated as a function of $q$ and $G$ by the use of eq. (14). It can be shown that $\Gamma[q, G]$ is the generating function of 2PI (two-particle irreducible) Green functions, i.e. it is given by all diagrams that cannot be separated in two pieces by cutting two lines [6,7].

The external sources can be obtained from $\Gamma[q, G]$ as

$$
\begin{align*}
\frac{\delta}{\delta q(s)} \Gamma[q, G] &= -J(s) - \int_0^\tau ds' \, K(s, s') \, q(s) \\
\frac{\delta}{\delta G(s, s')} \Gamma[q, G] &= -\frac{1}{2} K(s, s').
\end{align*}
$$

(16)

The physical process corresponds to vanishing sources, $J = K = 0$. From eq. (16) it follows that in this limit the value of $q$ and $G$ are determined by the stationary point of $\Gamma[q, G]$. We have thus obtained a variational principle for the the noise-averaged field $\langle \phi(s) \rangle$ and connected two-point correlation function $\langle \phi(s) \phi(s') \rangle_c$ of the process described by the Langevin equation (1).

The next step is to evaluate $\Gamma[q, G]$. Following Ref. [3, 4] $\Gamma[q, G]$ can be written as

$$
\Gamma[q, G] = I(q) + \frac{1}{2} \text{Tr} \ln G^{-1} + \frac{1}{2} \text{Tr} \mathcal{D}^{-1}(q) G + \Gamma_2(q, G) + \text{const.}
$$

(17)

where $I(q)$ is given by eq. (14) with $\phi \to q$,

$$
\mathcal{D}^{-1}(q) \equiv \left. \frac{\delta^2 I(\phi)}{\delta \phi(s) \delta \phi(s')} \right|_{\phi=q} = D^{-1} + \left. \frac{\delta^2 I_{\text{int}}(\phi)}{\delta \phi(s) \delta \phi(s')} \right|_{\phi=q}
$$

(18)

with $D^{-1}$ propagator of the “free” theory. The functional $\Gamma_2$ is given by the sum of all 2PI vacuum diagrams of a theory with interactions determined by $I_{\text{int}}$ and propagators $G$. The interaction term is defined by the shifted action

$$
\text{8}
$$
\[ I(q + \phi) - I(q) - \phi \frac{\delta I(\phi)}{\delta \phi} \bigg|_{\phi=q} = \frac{1}{2} \phi D^{-1}(q) \phi + I_{\text{int}}(\phi, q). \] (19)

This procedure corresponds to a dressed loop expansion with vertices which depend on \( \phi \), and can thus exhibit non-perturbative effects even for a small number of dressed loops. The crucial point is that it in no sense corresponds to a perturbation theory in physical amplitudes. The physics is obtained by going to the stationary point of the expansion with respect to both variables \( \phi \) and \( G \). This leads to nonlinear dynamical equations for \( \phi \) and \( G \). If one could sum up the whole series, the exact value of \( \phi \) and \( G \) will emerge from the stationary point. If the series is truncated one gets approximate values of \( \phi \) and \( G \). The bonus is that one can get equations describing nonperturbative behaviors. The drawback is that in general there is no systematic knowledge about errors occurred.

In the next section we apply the above formalism to a \( O(N) \) problem where the leading contribution to \( \Gamma_2 \) can be extracted in the \( N \to \infty \) limit.

III. THE MODEL

To illustrate the formalism introduced in the previous section we consider an \( N \)-component Ginzburg-Landau time-dependent field \( \phi_i \) with quadratic local interaction in zero spatial dimension. When discussing fluctuations effects to any given order in a perturbation expansion one is not usually able to justify the neglect of yet higher orders. However for theories with large \( N \) internal symmetry group there exists another perturbative scheme, the \( 1/N \) expansion. The model is specified by the evolution equation

\[ \frac{\partial \phi_i}{\partial t} = - \frac{\partial}{\partial \phi_i} S[\phi] + \xi_i \] (20)

\[ S(\phi) = \frac{a}{2} \phi^2 + \frac{\lambda}{4!N}(\phi^2)^2 \] (21)

where Gaussian random noise is defined by:

\[ \langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t) \xi_j(t') \rangle = \Gamma \delta_{ij} \delta(t-t') \] (22)
and we assume \( a < 0 \) and \( \lambda > 0 \). Generalizing eq. (10) to \( N \)-component fields one finds the following action

\[
I(\phi) = \frac{1}{\Gamma} \int_0^\tau dt \left[ \frac{\dot{\phi}_1^2}{2} + \frac{m^2}{2} \phi_1^2 + \frac{\lambda_0}{4!N} (\phi_1^2)^2 + \frac{g_0}{6!N^2} (\phi_1^2)^3 - \frac{aN}{2} \right] \tag{23}
\]

where the parameters \( m^2 \), \( \lambda_0 \) and \( g_0 \) are related to the original constants by:

\[
m^2 = a^2 - \frac{\lambda}{6} - \frac{\lambda}{3N}, \tag{24a}
\]

\[
\lambda_0 = 4 a \lambda, \tag{24b}
\]

\[
g_0 = 10 \lambda^2. \tag{24c}
\]

The last term in eq. (23) does not depend on \( \phi \) and can be absorbed into the definition of the normalizing constant \( N \) in eq. (12).

In the limit \( N \to \infty \) we can calculate explicitly the leading order term of the functional (17) following the same steps of Ref. [12]. The "action" (23) corresponds to a classical \( \phi^6 \)-theory in one spatial dimension.

From eq. (23) it follows that the leading contributions to eq. (17) for \( N \to \infty \) are

\[
\frac{1}{2} \text{Tr} \mathcal{D}^{-1} G = \frac{N}{2} \int_0^\tau dt \int_0^\tau dt' \left[ -\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N} q(t)^2 + \frac{g_0}{5!N^2} q^4(t) \right] G(t, t') \delta(t-t') \tag{25}
\]

and

\[
\mathcal{D}^{-1}(t, t') = \left[ -\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N} q(t)^2 + \frac{g_0}{5!N^2} q^4(t) \right] \delta(t-t'). \tag{26}
\]

The leading order 2PI diagrams \( N \to \infty \), shown in Fig. 1, lead to

\[
\Gamma_2(q, G) = \frac{N \lambda_0}{4!} \int_0^\tau dt G^2(t, t) + \frac{N g_0}{6!} \int_0^\tau dt G^3(t, t) + \frac{3 g_0}{6!} \int_0^\tau dt q^2(t) G^2(t, t) \tag{27}
\]

where \( q(t) = \langle \phi_1(t) \rangle \) assuming that the symmetry is broken along the direction '1'.

Stationarity of the functional \( \Gamma[q, G] \) with respect to \( q(t) \) and \( G(t, t') \) yields the dynamical equations for the order parameter and its fluctuations which read, respectively
\[
\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N} q^2(t) + \frac{g_0}{5!N^2} q^4(t) \right. \\
\left. + \frac{\lambda_0}{3!} G(t, t) + \frac{g_0}{5!} G^2(t, t) + \frac{2g_0}{5!N} q^2(t) G(t, t) \right] q(t) = 0
\]  
(28)

and

\[
\left[-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!N} q^2(t) + \frac{g_0}{5!N^2} q^4(t) \right. \\
\left. + \frac{\lambda_0}{3!} G(t, t) + \frac{g_0}{5!} G^2(t, t) + \frac{2g_0}{5!N} q^2(t) G(t, t) \right] G(t, t') = \Gamma \delta(t - t').
\]  
(29)

These coupled dynamical equations are exact to leading order in \(N\).

The effective dynamical equations (28) and (29) can be also derived by a variational approach to the path integral (8), or Hartree approximation, where one seeks for the best quadratic approximation for the action.

**IV. THE SOLUTION**

To solve the coupled effective dynamical equations (28) and (29) we rewrite them as

\[
\frac{\partial^2 q(t)}{\partial t^2} = F[q(t), G(t, t)] q(t)
\]  
(30)

\[
\frac{\partial^2 G(t, t')}{\partial t^2} = F[q(t), G(t, t)] G(t, t') - \Gamma \delta(t - t')
\]  
(31)

where

\[
F[q(t), G(t, t)] = m^2 + \frac{\lambda_0}{3!N} q^2(t) + \frac{g_0}{5!N^2} q^4(t) \\
+ \frac{\lambda_0}{3!} G(t, t) + \frac{g_0}{5!} G^2(t, t) + \frac{2g_0}{5!N} q^2(t) G(t, t)
\]  
(32)

We are eventually interested into solutions for times \(t\) such that \(t/\tau \to 0\). Under this assumption the effective dynamical equations (30) and (31) can be reduced to simpler first order non linear differential equations by using the following representation for \(q(t)\) and \(G(t, t')\):

\[
q(t) = q(0) f_1(t)
\]  
(33)
and

\[ G(t, t') = f_1(t') f_2(t) \theta(t' - t) + f_1(t) f_2(t') \theta(t - t') \]  \hspace{1cm} (34)

with

\begin{align*}
  f_1(t) &= e^{- \int_0^t d\tau R(\tau)} \\
  f_2(t) &= \Gamma e^{- \int_0^t e^{2 \int_0^\tau d\tau' R(\tau')}}
\end{align*}  \hspace{1cm} (35) \quad (36)

where the function \( R(t) \) is solution of the first order non linear differential equation

\[ \frac{\partial R(t)}{\partial t} = R^2(t) - F[q(t), G(t, t)] \]  \hspace{1cm} (37)

By inspection of eqs. (32) and (37) it follows that \( R(t) \) should have the functional form

\[ R(t) = \alpha C(t) + \beta q^2(t) + \gamma \]  \hspace{1cm} (38)

where \( C(t) = \lim_{t \to t'} G(t, t') \). The parameters \( \alpha, \beta \) and \( \gamma \) are determined by substituting \( R(t) \) from eq. (38) into eq. (37), and eliminating \( dC(t)/dt \) and \( dq(t)/dt \) with the help of

\begin{align*}
  \frac{\partial q(t)}{\partial t} &= -R(t)q(t) \\
  \frac{\partial C(t)}{\partial t} &= -2R(t)C(t) + \Gamma
\end{align*}  \hspace{1cm} (39) \quad (40)

obtained from eqs. (33) - (36). We obtain the following solution for \( R(t) \)

\[ R(t) = \frac{\lambda}{6} C(t) + \frac{\lambda}{6N} q^2(t) + a. \]  \hspace{1cm} (41)

In principle there exists another set of parameters, but it leads to an asymptotically unsteady state.

The first order non linear differential equations (39) - (41) gives the full description of
the model (20) - (21) in the limit \( N \to \infty \) for all times \( t \) such that \( t/\tau \to 0 \).

If \( q(t) \) is not identically equal to zero it is not straightforward to solve analytically the
set of equations (33) - (41). Nevertheless these can be easily solved numerically for any set
of initial conditions. We note that this is not the case for eqs. (28) and (29). These, indeed, suffer of strong numerical instability and one has to resort clever algorithm to handle them.

On the contrary if \( q(t) = 0 \) for all times, then we can find a closed analytical solution. We note that from the structure of the equations one can see that the \( O(N) \) symmetry dictates the equilibrium value of \( q(t) \): \( \lim_{t \to \infty} q(t) = 0 \). Thus if we assume that \( q(0) = 0 \) then the solution is \( q(t) = 0 \). In this case the equation for \( G(t, t') \) can also be solved, and if we take as initial condition \( C(t) = 0 \), the solution reads

\[
\begin{align*}
\mathcal{f}_1(t) &= e^{-at/2} \sqrt{\frac{1 - (\alpha/\beta)}{e^{\lambda \Delta t/3} - (\alpha/\beta) e^{-\lambda \Delta t/3}}} \\
\mathcal{f}_2(t) &= \frac{3 \Gamma}{\lambda \Delta} e^{at/2} \sqrt{\frac{1 - (\alpha/\beta)}{e^{\lambda \Delta t/3} - (\alpha/\beta) e^{-\lambda \Delta t/3}}} \sinh(\lambda \Delta t/3)
\end{align*}
\]

where

\[
\alpha = -3 \frac{a}{\lambda} + \sqrt{\gamma^2 + 3 \frac{\Gamma}{\lambda}} \quad \beta = -3 \frac{a}{\lambda} - \sqrt{\gamma^2 + 3 \frac{\Gamma}{\lambda}}
\]

Substitution of eqs. (42) – (44) into eq. (34) leads to the solution for \( G(t, t') \). In the limit \( t' \to t \) we recover the result of Ref. [8] for \( C(t) \).

V. BEYOND THE HARTREE APPROXIMATION

The next leading terms of order \( 1/N \) can be included systematically by evaluating diagrams not included in eq. (27). In the case \( q(t) = 0 \) the work is simplified since one does not have to consider separately transverse and parallel components of the correlation function \( \langle \phi_i(t) \phi_j(t') \rangle_c \).

The diagrams contributing to the first corrections to \( \Gamma_2 \) are shown in Fig. 1 (a) and (b) and Fig. 2 and yields

\[
\Gamma_2(G) = \frac{2}{4!} \lambda_0 \int dt \ G^2(t, t) + \frac{6}{6!} g_0 \int dt \ G^3(t, t) \\
- \frac{\lambda_0^2}{12^2} \int dt \ \int dt' \ G^4(t, t') - \frac{36}{(6!)^2} g_0^2 \int dt \ \int dt' \ G^4(t, t') \ C(t, t) G(t', t') \\
- \frac{\lambda_0 g_0^2}{720} \int dt \ \int dt' \ G^4(t, t') \ C(t, t) \quad (45)
\]
To improve systematically this result, and the Hartree approximation, one has to consider an infinite series of diagrams. A complete summation of the series, see Figs. 3 and 4, can be performed, however, only in the case of a system at equilibrium where time translational invariance holds. The final result for $\Gamma_2$ is valid to all orders in $\lambda$ and to first order in $1/N$ and reads [13,14]:

$$\Gamma_2(G)/\tau = \frac{N + 2}{4!} \lambda_0 \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \tilde{G}^{(\omega_1)} \tilde{G}^{(\omega_2)} + \frac{N + 6}{6!} g_0 \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int \frac{d\omega_3}{2\pi} \tilde{G}^{(\omega_1)} \tilde{G}^{(\omega_2)} \tilde{G}^{(\omega_3)}$$

$$+ \frac{1}{2} \int \frac{d\omega}{2\pi} \ln \left[ 1 + \left( \frac{\lambda_0}{3!} + \frac{g_0}{60} G(0,0) \right) \tilde{\Pi}(\omega) \right]$$

$$- \frac{1}{2} \int \frac{d\omega}{2\pi} \left( \frac{\lambda_0}{3!} + \frac{g_0}{60} G(0,0) \right) \tilde{\Pi}(\omega) \right) \tilde{G}^{(\omega_1)} \tilde{G}^{(\omega - \omega_1)} + \tilde{\Pi}(\omega_1) \right]$$

where $\tilde{\Pi}(\omega)$ is the so called vacuum polarization propagator:

$$\tilde{\Pi}(\omega) = \int \frac{d\omega}{2\pi} \tilde{G}(\eta) \tilde{G}(\eta + \omega). \quad (47)$$

Upon differentiating with respect to $\tilde{G}(\omega)$ the functional $\Gamma[G]$ with the $1/N$ corrections included one obtains [12]:

$$\tilde{G}^{-1}(\omega) = \omega^2 + m^2 + \frac{\lambda_0}{3!} G(0,0) + \frac{g_0}{5!} G^2(0,0)$$

$$+ \frac{1}{N} \int \frac{d\omega_1}{2\pi} \frac{1}{1 + \left( \frac{\lambda_0}{3!} + \frac{g_0}{60} G(0,0) \right) \tilde{\Pi}(\omega_1)} \tilde{G}(\omega - \omega_1) + \tilde{\Pi}(\omega_1) \right]$$

Equation (48) represents the spectrum of the equilibrium fluctuations correct to order $1/N$. The study of these corrections will be the subject of a future publication.

VI. VARIATIONAL APPROACH

A well-known approximation often employed in statistical mechanics is obtained by applying the Peierls-Feynman-Bogolubov inequality:

$$- \ln Z \leq - \ln Z_0 + \langle \mathcal{L} - \mathcal{L}_0 \rangle_0 = W^{(1)}$$

(49)
where $\mathcal{L}$ is action density given by

$$I(\phi) = \int_0^\tau dt \mathcal{L}$$

(50)

and $\mathcal{L}_0$ is an arbitrary action density. The average $\langle \cdots \rangle_0$ in (49) is done with respect to the probability distribution corresponding to $\mathcal{L}_0$ and $Z_0$ is the partition function associated with $\mathcal{L}_0$. The Hartree method consists of choosing the arbitrary action $\mathcal{L}_0$ to be the most general quadratic form:

$$\int_0^\tau dt \mathcal{L}_0 = \frac{1}{2\Gamma} \int_0^\tau dt \int_0^\tau dt' \sum_{ij} \phi_i(t) u_{ij}(t, t') \phi_j(t') - \int_0^\tau dt \sum_i v_i(t) \phi_i(t)$$

(51)

where $u_{ij}$ is a positive definite kernel. Recalling that:

$$W^{(0)}(0) = -\ln Z_0 = -\frac{1}{2} \ln ||u^{-1}(t, t')|| - \frac{\Gamma}{2} \int_0^\tau dt \int_0^\tau dt' v(t) u^{-1}(t, t') v(t')$$

(52)

Taking the derivatives of $W^{(0)}$ with respect to $v(t)$ we find:

$$\frac{\delta W^{(0)}}{\delta v(t)} = -q(t) = -\Gamma \int_0^\tau dt' u^{-1}(t, t') v(t')$$

(53)

$$\frac{\delta^2 W^{(0)}}{\delta v(t) \delta v(t')} = -G(t, t') = -\Gamma u^{-1}(t, t')$$

(54)

which can be inverted to give:

$$\int_0^\tau dt' u(t, t') q(t') = \Gamma v(t)$$

(55)

$$\int_0^\tau dt' u(t, t') G(t', t'') = \Gamma \delta(t - t'')$$

(56)

The arbitrary functions $u_{ij}$ and $v_i$ are determined by looking for the minimum of $W^{(1)}$ to have the best estimate of $\ln Z$.

For the case under study by applying the above method to (21) one finds

$$u(t, t') = [-\frac{\partial^2}{\partial t'^2} + m^2 + \frac{\lambda_0}{3!N} q^2(t) + \frac{g_0}{5!N^2} q^4(t)$$

$$+ \frac{\lambda_0}{3!} G(t, t) + \frac{g_0}{5!} G^2(t, t) + \frac{2g_0}{5!N} q^2(t) G(t, t)] \delta(t - t').$$

(57)

and
Thus inserting (57) and (58) in eqs. (53) and (54) we find the same result as eqs. (28) and (29).

Before concluding the discussion of the Hartree method we shall analyze what happens in the case $N = 1$. Its variational nature in fact justifies its application even when the problem under scrutiny does not contain a natural small parameter around which to perform some sort of expansion.

Let us consider the case of a Langevin equation for a scalar field with cubic nonlinearity. The equations of motion for the average value of the field and for the fluctuations are [15]:

\[
\begin{align*}
-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{3!} q^3(t) + \frac{g_0}{5!} q^5(t) \\
+ \frac{\lambda_0}{2} G(t,t) + \frac{g_0}{8} G^2(t,t) + \frac{g_0}{12} q^2(t) G(t,t) \right) q(t) = 0 \\
-\frac{\partial^2}{\partial t^2} + m^2 + \frac{\lambda_0}{2} q^2(t) + \frac{g_0}{4!} q^4(t) \\
+ \frac{\lambda_0}{2} G(t,t) + \frac{g_0}{8} G^2(t,t) + \frac{g_0}{8} q^2(t) G(t,t) \right) G(t,t') = \Gamma \delta(t-t').
\end{align*}
\]

(59)

(60)

With the substitution:

\[ q(t) = e^{-\int_0^t dt R_q(\tau)} \]

(61)

and $G(t,t')$ given by eq. (34) we obtain two equations for $R_q(t)$ and $R(t)$ analogous to eq. (37). Using the trial solution

\[ R_q(t) = \alpha_q C(t) + \beta_q q^2(t) + \gamma_q \]

(62)

and $R(t)$ given by eq.(38) we find, up to terms quadratic in the coupling constant $\lambda$, $\alpha_q = \alpha = \lambda/2$, $\beta_q = \lambda/6$, $\beta = \lambda/2$ and $\gamma_q = \gamma = a$. The value of the coefficients coincides with the value obtained in the so called Langer-Bar on-Miller approximation [16] to the Langevin equation, a result which was also rediscovered few years ago [3], on the basis of a somehow ad hoc variational principle. We believe that the present derivation, being based on a path integral formulation of the stochastic equations makes the underlying physical assumptions more clear.
VII. CONCLUSIONS

Most problems arising in the study of physical problems are most naturally represented in terms of systems of nonlinear stochastic differential equations. Many approximation schemes have been developed to treat the nonlinear aspect of equations. Usually they are based on some reasonable assumptions. The advantage of these approaches is that they may lead to relative simple equations. The drawback is that is is not simple to improve the quality of the approximation. In this paper we presented an alternative approach to the study of nonlinear Langevin equation which allows for systematic development of approximation scheme. The basic idea is to reduce the nonlinear Langevin equation to an equivalent equilibrium problem to which the methods of conventional field theory can be applied. A particular well suited perturbative scheme is that developed in quantum field theory by Cornwall, Jackiw and Tomboulis [6]. The major advantage is that it leads to a variational principle for the physical quantities of interest.

The method is applied to an $N$ component Ginzburg-Landau equation. In the limit of $N \to \infty$ we are able to derive closed forms for the order parameter $q(t)$ and for the two-time connected correlation function $G(t, t')$. We also discuss the first $1/N$ corrections. The study of these is, however, more involved and has not been included in this paper. This is part of future work. It will be also of interest to extend the present approach to higher dimensions and to explore numerically the predictions of the present approach to finite values of $N$.

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REFERENCES

[1] Z. Schuss, *Theory and Applications of Stochastic Differential Equations*, (John Wiley, New York, 1980).

[2] J. D. Mason ed., *Stochastic Differential Equations and Applications*, (Academic Press, New York, 1977).

[3] see e.g. B. J. West, J. Fluid. Mech. 117, 187 (1982) and references therein.

[4] B. J. West, G. Rovner and K. Lindenberg, J. Stat. Phys. 30, 633 (1983).

[5] M. C. Valsakumar, K. P. N. Murthy and A. Ananthakrishna, J. Stat. Phys. 30, 617 (1983).

[6] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. 10, 2428 (1978).

[7] R. W. Haymaker, la Rivista del Nuovo Cimento 14, n. 8 (1991).

[8] J. K. Bhattacharjee, P. Meakin and D. J. Scalapino, Phys. Rev. A 30, 1026 (1984).

[9] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, International Series of Monographs on Physics vol. 77, (Oxford Science Publications, 1990).

[10] E. Gozzi, Phys. Rev. D 28, 1922 (1983).

[11] H. Risken, *The Fokker-Planck Equation*, (Springer Verlag, Berlin, 1989).

[12] D. Dominici and U. Marini Bettolo Marconi, Phys. Lett. B 319, 171 (1993).

[13] A. J. Bray, Phys Rev. Lett. 32, 1413 (1974); A. J. Bray, J. Phys. A 7, 2144 (1974).

[14] P. K. Townsend, Nuclear Phys. 3, 199 (1977).

[15] U. Marini Bettolo Marconi and B. L. Gyorffy, Physica A 159, 221 (1989).

[16] J. S. Langer, M. Bar-on and Harold D. Miller, Phys. Rev. A, 1417 (1975).
FIGURES

FIG. 1. Leading order 2PI diagrams

FIG. 2. 2PI diagrams contribution to the first $1/N$ corrections

FIG. 3. Three vertices 2PI diagrams contributing to the first $1/N$ corrections

FIG. 4. Four vertices 2PI diagrams contributing to the first $1/N$ corrections