Quantum harmonic oscillator algebras as non-relativistic limits of multiparametric $gl(2)$ quantizations

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Abstract

Multiparametric quantum $gl(2)$ algebras are presented according to a classification based on their corresponding Lie bialgebra structures. From them, the non-relativistic limit leading to quantum harmonic oscillator algebras is implemented in the form of generalized Lie bialgebra contractions.
1 Introduction

The $gl(2)$ Lie algebra can be viewed as the natural relativistic analogue of the one-dimensional harmonic oscillator algebra $h_4$. Reciprocally, $h_4$ can be obtained from $gl(2)$ through a generalized Inönü-Wigner contraction that translates into mathematical terms the non-relativistic limit $c \to \infty$. Explicitly, if we consider the commutation relations and second-order Casimir of the $gl(2)$ Lie algebra

$$\begin{align*}
[J_3, J_+] &= 2J_+,&
[J_3, J_-] &= -2J_-,&
[J_+, J_-] &= J_3.&
\end{align*}$$

and we apply the map defined by

$$\begin{align*}
A_+ &= \epsilon J_+, &
A_- &= \epsilon J_-,&
N &= (J_3 + I)/2, &
M &= \epsilon^2 I,
\end{align*}$$

then the limit $\epsilon \to 0$ ($\epsilon = 1/c$) leads to the harmonic oscillator algebra $h_4$

$$\begin{align*}
[N, A_+] &= A_+, &
[N, A_-] &= -A_-,&
[A_-, A_+] &= M, &
[M, \cdot] &= 0.
\end{align*}$$

The Casimir of $h_4$ is also obtained by computing $\lim_{\epsilon \to 0} \frac{1}{2\epsilon^2}(-C + I^2)$:

$$C = 2NM - A_+A_- - A_-A_+.$$

Recently, a systematic and constructive approach to multiparametric quantum $gl(2)$ algebras based on the classification of their associated Lie bialgebra structures has been presented. In that paper, the question concerning the generalization of the Lie bialgebra contraction procedure to multiparametric structures has been also solved. Now, we make use of those results in order to obtain several quantum $h_4$ algebras and their associated deformed Casimir operators. We emphasize that all these quantum $h_4$ algebras are endowed with a Hopf algebra structure, which can be related to integrability properties of associated models. In particular, note that the quantum group symmetry of the spin $1/2$ Heisenberg XXZ and XXX chains with twisted periodic boundary conditions is given by quantum $gl(2)$ algebras whose non-relativistic limit will be analysed.

2 Quantum $gl(2)$ algebras

In this section we present some relevant quantum $gl(2)$ Hopf algebras. Deformed Casimir operators, essential for the construction of integrable systems, and quantum $R$-matrices are also explicitly given.

2.1 Family $I_+$ quantizations
2.1.1 Standard subfamily $U_{a_+,a}(gl(2))$ with $a_+ \neq 0, a \neq 0$

The quantum algebra $U_{a_+,a}(gl(2))$ and its Casimir are given by

$$
\Delta(J_3') = 1 \otimes J_3' + J_3' \otimes 1, \quad \Delta(I) = 1 \otimes I + I \otimes 1, \quad \Delta(J_+) = e^{aJ_3'/2} \otimes J_+ + J_+ \otimes e^{-aJ_3'/2},
\Delta(J_-) = e^{aJ_3'/2} \otimes J_- + J_- \otimes e^{-aJ_3'/2},
$$

$$
[J_3', J_+] = 2J_+, \quad [J_3', J_-] = -2J_- - \frac{a_+}{a} \sinh(aJ_3'/2) - \frac{a_+^2}{a^2} J_+,
[I, \cdot] = 0,
\begin{align}
\lim_{a_+ \to 0} C_{a_+,a} &= \frac{2}{a \tanh a} \left( \cosh(aJ_3') - 1 \right) + 2(J_+ J_- + J_- J_+) + \frac{a_+^2}{a^2} J_+^2 \\
&+ \frac{a_+}{a} \left( \sinh(aJ_3'/2) J_+ + J_+ \sinh(aJ_3'/2) \right),
\end{align}

where $J_3' = J_3 - \frac{a_+}{a} J_+$. This quantum algebra is just a superposition of the standard and non-standard deformations of $sl(2, \mathbb{R})$ since the underlying standard classical $r$-matrix is $r = \frac{1}{2} (a_+ J_3' \wedge J_+ - 2a_+ J_+ \wedge J_-)$. This fact can be clearly appreciated by considering the $4 \times 4$ quantum $R$-matrix associated to $U_{a_+,a}(gl(2))$ [2]:

$$
R = \begin{pmatrix}
1 & h & -qh & h^2 \\
0 & q & 1 - q^2 & qh \\
0 & 0 & q & -h \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad q = e^a, \quad h = \frac{a_+}{2} \left( \frac{e^a - 1}{a} \right). \tag{2.6}
$$

The limit $a_+ \to 0$ yields the standard $R$-matrix of $sl(2, \mathbb{R})$, while taking $a \to 0$ gives rise to the non-standard one. This quantum algebra underlies the construction of non-standard $R$-matrices out of standard ones introduced in [7, 8].

2.1.2 Non-standard subfamily $U_{a_+,b_+}(gl(2))$ with $a_+ \neq 0$

The Hopf algebra $U_{a_+,b_+}(gl(2))$, whose Lie bialgebra is generated by the triangular classical $r$-matrix $r = \frac{1}{2} (a_+ J_3 \wedge J_+ + b_+ J_+ \wedge I)$, is given by

$$
\Delta(J_+) = 1 \otimes J_+ + J_+ \otimes 1, \quad \Delta(I) = 1 \otimes I + I \otimes 1, \quad \Delta(J_3) = 1 \otimes J_3 + J_3 \otimes e^{a_+ J_+} - b_+ I \otimes \left( \frac{e^{a_+ J_+} - 1}{a_+} \right),
\Delta(J_-) = 1 \otimes J_- + J_- \otimes e^{a_+ J_+} - \frac{b_+}{a_+} \left( J_3 - \frac{b_+}{a_+} I \right) \otimes I e^{a_+ J_+},
$$

$$
[J_3, J_+] = 2 \frac{e^{a_+ J_+} - 1}{a_+}, \quad [J_3, J_-] = -2J_- + \frac{a_+}{2} \left( J_3 - \frac{b_+}{a_+} I \right)^2,
[J_+, J_-] = J_3 + b_+ I \frac{e^{a_+ J_+} - 1}{a_+}, \quad [I, \cdot] = 0,
\begin{align}
C_{a_+,b_+} &= \left( J_3 - \frac{b_+}{a_+} I \right) e^{-a_+ J_+} \left( J_3 - \frac{b_+}{a_+} I \right) + 2 \frac{b_+}{a_+} J_3 I.
\end{align}
$$

\[3\]
\[+2 \frac{1 - e^{-a+J_+}}{a_+} J_- + 2J_- \frac{1 - e^{-a+J_+}}{a_+} + 2(e^{-a+J_+} - 1).\]

This quantum algebra has been also obtained in [9, 10, 11] and its universal quantum \( R \)-matrix can be found in [10, 11].

### 2.2 Family II quantizations

#### 2.2.1 Standard subfamily \( U_{a,b}(gl(2)) \) with \( a \neq 0 \)

The corresponding coproduct, commutation rules and Casimir are given by

\[
\begin{align*}
\Delta(I) &= 1 \otimes I + I \otimes 1, \\
\Delta(J_3) &= 1 \otimes J_3 + J_3 \otimes 1, \\
\Delta(J_+) &= e^{(aJ_3 - bI)/2} \otimes J_+ + J_+ \otimes e^{-(aJ_3 - bI)/2}, \\
\Delta(J_-) &= e^{(aJ_3 + bI)/2} \otimes J_- + J_- \otimes e^{-(aJ_3 + bI)/2}, \\
\end{align*}
\]

\[
(J_3, J_+) = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = \frac{\sinh aJ_3}{a}, \quad [I, \cdot] = 0,
\]

\[
\mathcal{C}_a = \cosh a \left( \frac{\sinh(aJ_3/2)}{a/2} \right)^2 + 2 \frac{\sinh a}{a} (J_+J_- + J_-J_+).
\]

This quantum algebra, together with its universal quantum \( R \)-matrix, has been obtained in [12]; it is just the quantum algebra underlying the XXZ Heisenberg Hamiltonian with twisted boundary conditions [5]. This deformation can be thought of as a Reshetikhin twist of the usual standard deformation since in the associated \( r \)-matrix, \( r = -\frac{1}{2}bJ_3 \wedge I - aJ_+ \wedge J_- \), the second term generates the standard deformation and the exponential of the first one gives us the Reshetikhin twist.

#### 2.2.2 Non-standard subfamily \( U_{b+,b}(gl(2)) \)

The coproduct reads

\[
\begin{align*}
\Delta(I) &= 1 \otimes I + I \otimes 1, \\
\Delta(J_3) &= 1 \otimes J_3 + J_3 \otimes 1 + b_+J_+ \otimes \left( \frac{e^{bl} - 1}{b} \right), \\
\Delta(J_+) &= 1 \otimes J_+ + J_+ \otimes e^{-bl} + b_+J_3 \otimes \left( \frac{e^{-bl} - 1}{2b} \right) \\
&\quad + b_+^2J_+ \otimes \left( \frac{1 - \cosh bl}{2b^2} \right), \\
\end{align*}
\]

and the associated commutation rules and Casimir are non-deformed ones (1.1).

A twisted XXX Heisenberg Hamiltonian invariant under \( U_{b+,b}(gl(2)) \) has been constructed in [2]. The \( r \)-matrix is \( r = -\frac{1}{2}(bJ_3 - b_+J_+) \wedge I \) and the universal \( R \)-matrix turns out to be \( R = \exp \{ r \} \), which in the fundamental representation reads

\[
R = \begin{pmatrix}
1 & -e^{-b}p & p & -e^{-b}p^2 \\
0 & e^{-b} & 0 & e^{-b}p \\
0 & 0 & e^b & -p \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad p = \frac{b_+}{2} \left( \frac{e^b - 1}{b} \right).
\]
3 Constructions to quantum oscillator algebras

In the sequel, we work out the contractions from the above quantum $gl(2)$ algebras to quantum $h_4$ algebras (a systematic approach to the latter structures can be found in [13]). In order to contract a given quantum algebra we have to consider the Inönü-Wigner contraction (e.g. [12]) together with a mapping $a = \varepsilon^n a'$ on each initial deformation parameter $a$ where $n$ is any real number and $a'$ is the contracted deformation parameter [14]. The convergency of both the classical $r$-matrix and the cocommutator $\delta$ under the limit $\varepsilon \to 0$ have to be analysed separately, since starting from a coboundary bialgebra, the contraction can lead to either another coboundary bialgebra (both $r$ and $\delta$ converge) or to a non-coboundary one ($r$ diverges but $\delta$ converges). Hence we have to find out the minimal value of the number $n$ such that $r$ converges, the minimal value of $n$ such that $\delta$ converges, and finally to compare both of them.

3.1 Standard family II: $U_{a,b}(gl(2)) \to U_{\xi,\vartheta}(h_4)$ with $\xi \neq 0$

Let us illustrate our procedure starting with the quantum algebra $U_{a,b}(gl(2))$. We consider the maps

$$a = -\varepsilon^{n_a} \xi, \quad b = -\varepsilon^{n_b} \vartheta,$$

where $\vartheta, \xi$ are the contracted deformation parameters, and $n_a, n_b$ are real numbers to be determined by imposing the convergency of $r$. We introduce the maps (3.12) and (3.11) in the classical $r$-matrix associated to $U_{a,b}(gl(2))$:

$$r = -\frac{1}{2} b J_3 \wedge I - a J_+ \wedge J_-$$
$$= \varepsilon^{n_a} \vartheta (2N - M \varepsilon^{-2}) \wedge M \varepsilon^{-2} + \varepsilon^{n_a} \xi A_+ \varepsilon^{-1} \wedge A_- \varepsilon^{-1} \quad \text{(3.12)}$$

Hence the minimal values of the indices $n_a, n_b$ which ensure the convergency of $r$ under the limit $\varepsilon \to 0$ are $n_a = 2, n_b = 2$. Now we have to analyse the convergency of the cocommutator $\delta$ associated to $U_{a,b}(gl(2))$. Thus we consider the maps (3.11) and look for the minimal values of $n_a, n_b$ which allow $\delta$ to converge under the limit $\varepsilon \to 0$. It can be checked that they are again $n_a = 2, n_b = 2$, so that the resulting $h_4$ bialgebra is coboundary (both $n_a, n_b$ coincide for $r$ and $\delta$). Therefore the transformations of the deformation parameters so obtained are $a = -\varepsilon^2 \xi$ and $b = -\varepsilon^2 \vartheta$. Finally, we introduce these maps together with (3.12) in $U_{a,b}(gl(2))$ and we obtain the following quantum oscillator algebra $U_{\xi,\vartheta}(h_4)$:

$$\Delta(N) = 1 \otimes N + N \otimes 1, \quad \Delta(M) = 1 \otimes M + M \otimes 1,$$
$$\Delta(A_+) = e^{(\vartheta + \xi)M/2} \otimes A_+ + A_+ \otimes e^{-(\vartheta + \xi)M/2},$$
$$\Delta(A_-) = e^{-(\vartheta - \xi)M/2} \otimes A_- + A_- \otimes e^{(\vartheta - \xi)M/2}, \quad \text{(3.13)}$$

$$[N, A_+] = A_+, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = \frac{\sinh \xi M}{\xi}, \quad [M, \cdot] = 0. \quad \text{(3.14)}$$
The deformed oscillator Casimir comes from \( \lim_{\varepsilon \to 0} \frac{1}{2} \varepsilon^2 (\mathcal{C}_a + (\frac{\sinh(a/2)}{a/2})^2) \):

\[
C_\xi = 2N \frac{\sinh \xi M}{\xi} - A_+ A_+ - A_- A_+.
\]

(3.15)

If \( \vartheta = 0 \), the quantum oscillator introduced in [15, 16] is recovered.

Hereafter we give the transformations of the deformation parameters for the remaining quantum \( gl(2) \) algebras together with the resulting quantum \( h_4 \) algebras; we stress that in all cases the contractions are found to have a coboundary character.

### 3.2 Non-standard family II: \( U_{b_+ b}(gl(2)) \to U_{\beta_+, \vartheta}(h_4) \)

The transformations of the deformation parameters are \( b_+ = 2\varepsilon^3 \beta_+ \) and \( b = -\varepsilon^2 \vartheta \).

The coproduct of the quantum oscillator algebra \( U_{\beta_+, \vartheta}(h_4) \) reads

\[
\Delta(M) = 1 \otimes M + M \otimes 1, \quad \Delta(A_+) = 1 \otimes A_+ + A_+ \otimes e^{-\vartheta M},
\]

\[
\Delta(A_-) = 1 \otimes A_- + A_- \otimes e^{\vartheta M} + \beta_+ M \otimes \left( \frac{e^{\vartheta M} - 1}{\vartheta} \right),
\]

\[
\Delta(N) = 1 \otimes N + N \otimes 1 + \beta_+ A_+ \otimes \left( \frac{1 - e^{-\vartheta M}}{\vartheta} \right).
\]

(3.16)

Commutation rules and Casimir of \( U_{\beta_+, \vartheta}(h_4) \) are the non-deformed ones (1.3).

### 3.3 Standard family I+ : \( U_{a_+, a}(gl(2)) \to U_{\beta_+, \xi}(h_4) \to U_{\xi}(h_4) \) with \( \xi \neq 0 \)

In this case, the maps \( a_+ = 2\varepsilon^3 \beta_+ \) and \( a = -\varepsilon^2 \xi \) lead to \( U_{\beta_+, \xi}(h_4) \):

\[
\Delta(M) = 1 \otimes M + M \otimes 1, \quad \Delta(A_\pm) = e^{\xi M/2} \otimes A_\pm + A_\pm \otimes e^{-\xi M/2},
\]

\[
\Delta(N) = 1 \otimes N + N \otimes 1 + \beta_+ \left( \frac{1 - e^{\xi M/2}}{\xi} \right) \otimes A_+ + \beta_+ A_+ \otimes \left( \frac{1 - e^{-\xi M/2}}{\xi} \right),
\]

\[
[N, A_+] = A_+, \quad [A_+, A_-] = \frac{\sinh \xi M}{\xi}, \quad [M, \cdot] = 0,
\]

(3.17)

\[
[N, A_-] = -A_- + \beta_+ \left( \frac{\sinh \xi M}{\xi} - \frac{\sinh(\xi M/2)}{\xi/2} \right),
\]

\[
C_{\beta_+, \xi} = 2N \frac{\sinh \xi M}{\xi} - A_+ A_- - A_- A_+ + 2A_+ \frac{\beta_+ \xi}{\xi} \left( \frac{\sinh \xi M}{\xi} - \frac{\sinh(\xi M/2)}{\xi/2} \right),
\]

where the Casimir is provided by \( \lim_{\varepsilon \to 0} \frac{1}{2} \varepsilon^2 (-C_{a_+, a} + (\frac{\sinh(a/2)}{a/2})^2) \). However the parameter \( \beta_+ \) is irrelevant and it can be removed from (3.17) by applying the change of basis defined by

\[
N' = N + \beta_+ A_+ + \frac{\beta_+}{\xi} A_+, \quad A_+ = A_+ + \frac{\beta_+}{\xi} \frac{\sinh(\xi M/2)}{\xi/2}, \quad M' = M.
\]

(3.18)

Thus we recover \( U_{\xi}(h_4) \), already obtained in sec. 3.1 as \( U_{\vartheta, \xi}(h_4) \to U_{\vartheta = 0, \xi}(h_4) \).
3.4 Non-standard family I$_+$: $U_{a_+,b_+}(gl(2)) \to U_{\alpha_+}(h_4)$ with $\alpha_+ \neq 0$

The transformations of the deformation parameters turn out to be $a_+ = \varepsilon \alpha_+$ and $b_+ = -\varepsilon \alpha_+$. Hence, we obtain the “Jordanian $q$-oscillator” $U_{\alpha_+}(h_4)$:

$$\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1, \quad \Delta(M) = 1 \otimes M + M \otimes 1,$$
$$\Delta(A_-) = 1 \otimes A_- + A_- \otimes e^{\alpha_+ A_+} + \alpha_+ N \otimes Me^{\alpha_+ A_+},$$
$$\Delta(N) = 1 \otimes N + N \otimes e^{\alpha_+ A_+}, \quad [M, \cdot] = 0,$$
$$[N, A_+] = \frac{e^{\alpha_+ A_+} - 1}{\alpha_+}, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = Me^{\alpha_+ A_+}. \quad (3.19)$$

The quantum Casimir is computed as $\lim_{\varepsilon \to 0} \varepsilon^2 (\frac{1}{2}C_{a_+,b_+} + I^2)$, and reads,

$$C_{\alpha_+} = 2NM + \frac{e^{-\alpha_+ A_+} - 1}{\alpha_+}A_- + A_- \frac{e^{-\alpha_+ A_+} - 1}{\alpha_+}. \quad (3.20)$$

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