Composite Fermions and quantum Hall systems: Role of the Coulomb pseudopotential

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The mean field composite Fermion (CF) picture successfully predicts angular momenta of multiplets forming the lowest energy band in fractional quantum Hall (FQH) systems. This success cannot be attributed to a cancellation between Coulomb and Chern–Simons interactions beyond the mean field, because these interactions have totally different energy scales. Rather, it results from the behavior of the Coulomb pseudopotential \( V(L) \) (pair energy as a function of pair angular momentum) in the lowest Landau level (LL). The class of short range repulsive pseudopotentials is defined that lead to short range Laughlin like correlations in many body systems and to which the CF model can be applied. These Laughlin correlations are described quantitatively using the formalism of fractional parentage. The discussion is illustrated with an analysis of the energy spectra obtained in numerical diagonalization of up to eleven electrons in the lowest and excited LL’s. The qualitative difference in the behavior of \( V(L) \) is shown to sometimes invalidate the mean field CF picture when applied to higher LL’s. For example, the \( \nu = \frac{1}{7} \) state is not a Laughlin \( \nu = \frac{1}{3} \) state in the first excited LL. The analysis of the involved pseudopotentials also explains the success or failure of the CF picture when applied to other systems of charged Fermions with Coulomb repulsion, such as the Laughlin quasiparticles in the FQH hierarchy or charged excitons in an electron–hole plasma.

71.10.Pm, 73.20.Dx, 73.40.Hm

I. INTRODUCTION

The discovery of the integer (von Klitzing, Dorda, and Pepper 1980) and fractional (Tsui, Störner, and Gossard 1982) quantum Hall (IQH and FQH) effects raised great interest in the properties of a two-dimensional electron gas (2DEG) in high magnetic fields. Both IQH and FQH effects are a manifestation of the occurrence of nondegenerate incompressible ground states in the 2DEG spectrum at certain (integral for IQH and fractional for FQH) Landau level (LL) fillings. However, unlike the single particle cyclotron gap responsible for the IQH effect, the gap separating an FQH incompressible ground state from the excited states is due to the electron–electron interactions (Laughlin 1983a). While the occurrence of incompressible ground states of both kinds results in quantization of the Hall conductance, the origin of incompressible ground states in the IQH and FQH effects, i.e. the physics underlying the two quantum Hall effects, is very different.

A simple picture of the FQH states is offered by the mean field composite Fermion (CF) approach (Jain 1989, Lopez and Fradkin 1991, Halperin, Lee, and Read 1993). The CF’s are obtained in the Chern–Simons (CS) gauge transformation, which can be interpreted as attaching to each electron a magnetic flux tube oriented opposite to the external magnetic field \( B \). In the mean field approximation, the magnetic field of these flux tubes is evenly spread over the occupied area. If the attached flux tubes carry an even number of flux quanta, the CS transformation without the mean field approximation leaves the energy spectrum and particle statistics unchanged. When the mean field approximation is made, the effective magnetic field \( B' \) seen by the CF’s is lower than the original field \( B \) seen by the electrons. The incompressible ground states are predicted to occur at fractional electron fillings that correspond to integer fillings of CF LL’s. A gas of strongly interacting electrons is said to behave as a gas of weakly interacting CF’s, and the FQH effect of electrons is interpreted as the IQH effect of CF’s.

The mean field CF picture correctly predicts filling factors at which the FQH effect has been experimentally observed. Also, in almost all cases, the mean field CF predictions of low lying states of finite systems agree with the results of exact numerical calculations in the lowest LL. However, a very fundamental question: ‘Why does the mean field CF picture work so well?’, is not yet completely understood. The original conjecture that Coulomb and CS gauge interactions beyond the mean field cancel each other in the FQH systems cannot possibly be correct because the CS interactions are measured on an energy scale proportional to \( B \), which can be much larger than the energy scale of the Coulomb interactions, proportional to \( \sqrt{B} \). Because so many experimental and numerical results in the lowest LL can be interpreted in terms of CF’s, it is extremely important to understand why the CF picture works.

It is known that the CF picture sometimes fails when applied to other systems of identical charged Fermions interacting through Coulomb like forces. For example, the occurrence of incompressible states only at some of the odd denominator fractional filling factors implies that the CF model is not always valid for Laughlin quasiparticles.
QP’s) in the FQH hierarchy (Haldane 1983, Laughlin 1984, Halperin 1984) or for the CF’s themselves in the CF hierarchy (Sitko et al. 1996). The CF picture also fails for electrons in the lowest LL, when the layer thickness exceeds certain critical value (Shayegan et al. 1990). On the other hand, the numerical experiments show that it is correct for variety of repulsive interaction potentials (e.g., $V(r) \sim -\ln r$ or $-\alpha \ln r$ for $\alpha \geq 1$ or even $-\alpha r^\beta$ with an arbitrary dielectric constant $\epsilon$). The original justification of the CF model rested on the assumption that spontaneously generated gauge interactions canceled to a substantial extent the repulsive interactions between electrons, independent of the exact form of these interactions. While the CF picture can be used to make certain predictions after it has been established that a certain physical system exhibits incompressible fluid ground states with Laughlin like correlations at appropriate conditions (magnetic field, electron density, layer thickness, disorder, material parameters, etc.), it cannot predict its own validity for such a system. Therefore, another very fundamental problem: ‘When does the mean field CF picture work?’, needs to be answered.

In this paper, we explain the connection between the form of the Coulomb pseudopotential (Haldane 1987) $V(L)$, defined as the dependence of the pair interaction energy $V$ on the pair angular momentum $L$, and the occurrence of the incompressible ground states in the lowest LL of an interacting 2DEG at Laughlin–Jain filling factors $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, etc. We present arguments justifying the validity of the mean field CF picture for the lowest LL and show when and why it can be used. It is known that the electrons in Laughlin $\nu = (2p+1)^{-1}$ states avoid a number ($p$) of pair states with largest repulsion (Haldane 1987). The origin of incompressible FQH states at certain other filling factors, such as $\nu = \frac{2}{5}$, has been also attributed to the ability of avoiding strongly repulsive pair states (Halperin 1983, Haldane 1987, Rezayi and MacDonald 1991, Bellkhir and Jain 1993). In order to formally treat the ability to avoid certain pseudopotential parameters in the incompressible many body states we use the formalism of fractional parentage, well established in the nuclear (Shalit and Talmi 1963) and atomic (Cowan 1981) physics. It is shown that the condition for the validity of the mean field CFpicture can be more easily expressed in terms of the behavior of the pseudopotential $V(L)$ than in terms of the behavior of the interaction potential $V(r)$. The condition on the form of interaction pseudopotential necessary for the occurrence of FQH states is given, which defines the class of short range pseudopotentials to which mean field CF picture can be applied. It is shown that the Coulomb interaction in the lowest LL falls in this class, while in higher LL’s the mean field CF picture can be used only below a certain filling factor. Similarly, the success or failure of the mean field CF picture applied to Laughlin QP’s, depending on the type of QP’s and their filling factor (Sitko, Yi, and Quinn 1997), is shown to reflect the behavior of appropriate QP pseudopotentials. It is argued that a QP hierar-

chical picture taking into account the qualitative features of involved pseudopotentials (Wójc and Quinn 2000) should most naturally explain the occurrence and relative stability of observed odd denominator FQH states. We are not discussing even denominator fractions (Willett et al. 1987) which are explained in terms of pairing of electrons (Haldane and Rezayi 1988, Moore and Read 1991), although a pseudopotential approach to the interaction between bound pairs might be possible. The discussion throughout the paper is illustrated by exact numerical calculations of energy spectra and parentage coefficients in Haldane’s spherical geometry (Haldane 1983), for up to eleven electrons at $\nu \sim \frac{1}{3}$ and up to eight electrons at $\nu \sim \frac{1}{5}$ in the lowest and excited LL’s (matrix dimensions up to $3 \cdot 10^6$), using a modified Lanczos algorithm (Lanczos 1950, Haydock 1980).

The paper is organized as follows. Section II gives a brief overview of the numerical (exact diagonalization) calculations on the Haldane sphere. Section III explains the mean field CF picture of the FQH states. The success of the mean field CF approach is illustrated in the energy spectra of the eight electron system in the lowest LL, for filling factors between $\nu = \frac{1}{4}$ and $\frac{1}{4}$. Section IV introduces the interaction pseudopotential. Section V discusses the three electron system. The idea of fractional parentage from pair states is used to characterize the three particle states. The energy spectra in the lowest and excited LL’s are analyzed and interpreted in terms of pseudopotential and fractional parentage. Section VI generalizes the analysis of the three electron case to an arbitrary electron number, and presents the numerical results for up to eleven electrons. Section VII explains the relation between the form of the interaction pseudopotential and the occurrence of many electron incompressible ground states. The Coulomb interaction in different LL’s is compared to the harmonic repulsive interaction and the Coulomb interaction in the atomic shells. The Hund’s rule appropriate for FQH systems is formulated. The short range pseudopotential is defined, to which the CF model can be applied. The prescription for the low energy many electron multiplets is derived, which agrees with predictions of the mean field CF picture. The consequences of the form of pseudopotential for condensation of QP’s in the hierarchy picture is mentioned. Section VIII contains the conclusions.

II. NUMERICAL STUDIES

A. Introduction

In a magnetic field $B$, the lowest LL of a 2DEG can accommodate $N_\phi = BC/\phi_0$ electrons per area $C$ ($\phi_0 = hc/e$ is the magnetic flux quantum). The measure of electron density is the fraction of occupied states, given by the filling factor $\nu = N/N_\phi$, where $N$ is the number
of electrons in the area C. In the absence of electron–electron interactions, the \( N_\phi \) single particle states are degenerate. Therefore, these interactions entirely determine the low energy spectrum of the system at \( \nu < 1 \) and cannot be treated perturbatively. Instead, numerical diagonalization techniques have commonly been employed, which, however, limit the system to a finite (small) number of electrons. Different approaches to restrict motion of a finite number of electrons to a finite area \( C \) to model an infinite 2DEG at a finite density include imposing a lateral (parabolic, hard wall, etc.) confinement (Laughlin 1983a), using periodic boundary conditions (Haldane and Rezayi 1985b), or confining electrons on a closed surface (Haldane 1983). The last approach has proven particularly useful, since it naturally avoids edge effects. Also, the translational symmetry of a (planar) 2DEG is preserved in the form of the rotational symmetry of a sphere. In particular, the pair of good quantum numbers resulting from the translational symmetry of a plane: the center of mass and relative momentum, correspond to the pair of good quantum numbers on a sphere: total angular momentum \( L \) and its projection \( L_z \) (Wójs and Quinn 1998a). Consequently, the degeneracies associated with center of mass excitations on a plane correspond to those associated with different values of \( L_z \) (\( |L_z| \leq L \)) on a sphere, and the nondegenerate incompressible ground states of a planar 2DEG correspond to nondegenerate (\( L = 0 \)) ground states on a sphere.

B. Haldane sphere

The magnetic field \( B \) perpendicular to the surface of the Haldane sphere of radius \( R \) is an isotropic radial field produced by a magnetic monopole placed at the origin. The monopole strength \( 2S \) is defined in the units of elementary flux \( \phi_0 = h/e \), so that the total flux through the sphere is \( 4\pi BR^2 = 2S\phi_0 \). Dirac’s monopole quantization condition requires that \( 2S \) is an integer (Dirac 1931), and positive \( S \) means magnetic field pointing outward. The convenient units of length and energy, magnetic length \( \lambda \) and the cyclotron frequency \( \omega_c \), are given by

\[
\lambda^2 |S| = R^2, \\
\hbar \omega_c = S \frac{\hbar^2}{\mu R^2}.
\]

The eigenstates of the single particle Hamiltonian are denoted by \( |S, l, m \rangle \) and called monopole harmonics (Wu and Yang 1976). They are labeled by angular momentum \( l \) and its projection \( m \). The degenerate angular momentum shells are equivalent to the LL’s of the planar geometry. The eigenenergies are given by

\[
E_n = \frac{\hbar \omega_c}{2S} \left[ l(l + 1) - S^2 \right] \\
= \hbar \omega_c \left[ n + \frac{1}{2} + \frac{n(n + 1)}{2S} \right],
\]

where the shell (LL) index is defined as \( n = l - S = 0, 1, 2, \ldots \). The degeneracy of each shell (LL) is \( N_\phi = 2l + 1 \).

For the FQH states at filling factors \( \nu < 1 \), only the lowest, spin polarized shell (LL) need be considered. It corresponds to \( n = 0 \) (\( l = S \)), and for simplicity its single particle states will be denoted as \( |m \rangle \). The spin polarized FQH states in excited LL’s will also be studied. Due to the high (cyclotron) energy of the inter-LL excitations in high magnetic fields, the FQH states at filling factors \( 2n < \nu < 2n + 1 \) are composed of completely filled LL’s (spin up and down) up to the \( (n-1) \)-st one, and a partially filled \( n \)th LL with the filling factor \( \nu_n < 1 \) (we discuss only the spin polarized states in partially filled excited LL’s). The Hartree–Fock energy describing interaction between an electron in the \( n \)th LL and the underlying completely filled LL’s is a constant. Therefore, the energy spectrum of \( N \) electrons at \( \nu_n < 1 \) in an isolated \( n \)th LL describes (up to this constant) the low energy spectrum of \( N + 2n(2S + n) \) electrons at \( \nu = 2n + \nu_n \). Since states of only one LL with a given \( n \) appear in the ‘reduced’ problem for \( \nu = 2n + \nu_n \), the following simplified notation will be used: filling factor \( \nu_n \) will be denoted as \( \nu \), and states \( |S, l, m \rangle \) will be denoted as \( |m \rangle \).

C. Many body problem

The object of numerical studies is to diagonalize the electron–electron interaction Hamiltonian

\[
\hat{H} = \sum_{m_1m_2m_3m_4} c_{m_1}^\dagger c_{m_2} c_{m_3} c_{m_4} \langle m_1, m_2 | V | m_3, m_4 \rangle
\]

within the Hilbert space \( \mathcal{H}_{MB} = N_\phi! [N! (N_\phi - N)!]^{-1} \) degenerate antisymmetric \( N \) electron states of a given \( (N_\phi\text{-fold degenerate}) \) LL. In the above, \( c_{m}^\dagger \) (\( c_{m} \)) creates (annihilates) an electron in the state \( |m \rangle \). The two body Coulomb matrix elements have a particularly simple form in the lowest LL (Fano et al. 1986), but they can also be evaluated analytically for a general case of inter- or intra-LL scattering. The \( N \) electron Hilbert space \( \mathcal{H}_{MB} \) is spanned by single particle configurations \( |m_1, m_2, \ldots, m_N \rangle \), classified by the total angular momentum projection \( M = m_1 + m_2 + \ldots + m_N \). Taking advantage of the Wigner–Eckart theorem, each \( (M) \) subspace \( \mathcal{H}_{MB}(M) \) can be further block diagonalized into \( (M, L) \) subspaces \( \mathcal{H}_{MB}(M, L) \) corresponding to different values of the total angular momentum \( L \). The Wigner–Eckart theorem tells us that because the interaction Hamiltonian is a scalar, its matrix element between angular momentum eigenstates \( |L, M, \alpha \rangle \) can be written as

\[
\langle L', M', \alpha' | \hat{H} | L, M, \alpha \rangle = \delta_{LL'} \delta_{MM'} V_{\alpha \alpha'}(L),
\]

i.e., in terms of a reduced matrix element

\[
V_{\alpha \alpha'}(L) = \langle L, \alpha' | \hat{H} | L, \alpha \rangle
\]
TABLE I. The dimension $N_\phi$ of the single particle Landau level, dimension $N_{MB}$ of the total many body Hilbert space $H_{MB}$, dimension $N_{MB}(0)$ of the largest $(M)$ subspace $H_{MB}(0)$, dimension $N_{MB}^{MAX}(M, L)$ of the largest $(M, L)$ subspace $H_{MB}^{MAX}(M, L)$, and dimension $N_{MB}(0, 0)$ of the $(M, L)$ subspace containing the Laughlin $L = 0$ ground state, $H_{MB}(0, 0)$, of $N = 6$ to 11 electrons at the filling factor $\nu = \frac{1}{3}$.

| $N$ | $N_\phi$ | $N_{MB}$ | $N_{MB}(0)$ | $N_{MB}^{MAX}(M, L)$ | $N_{MB}(0, 0)$ |
|-----|----------|----------|-------------|----------------------|----------------|
| 6   | 16       | 8,008    | 338         | 24                   | 6              |
| 7   | 19       | 50,388   | 1,656       | 86                   | 10             |
| 8   | 22       | 319,770  | 8,512       | 352                  | 31             |
| 9   | 25       | 2,042,975| 45,207      | 1,533                | 84             |
| 10  | 28       | 13,123,110| 246,448    | 7,069                | 319            |
| 11  | 31       | 84,672,315| 1,371,535  | 33,787               | 1,160          |

which is independent of $M$. Here, index $\alpha$ distinguishes different states in the same space $H_{MB}(M, L)$. The typical dimensions are given in table I, where we list the dimension of the total Hilbert space $H_{MB}$, of the largest $(M)$ subspace $H_{MB}(0)$, of the largest $(M, L)$ subspace $H_{MB}^{MAX}(M, L)$, and of the $(M, L)$ subspace containing the Laughlin $L = 0$ ground state, $H_{MB}(0, 0)$, for between six and eleven electrons at the filling factor $\nu = \frac{1}{3}$. Even when both $M$ and $L$ are resolved, exact diagonalization becomes difficult when $N$ exceeds 10 and $N_\phi$ exceeds 28.

The calculations give the eigenenergies $E$ as a function of total angular momentum $L$. The numerical results for the lowest LL always show one or more $L$ multiplets forming a low energy sector (or low energy band). The spectra for $N$ in the range 6–20 (depending on the filling factor) are available in literature and have been extensively analyzed. As an example, in figures 1 and 2 we show the energy spectra obtained for eight electrons in the lowest LL, at values of $2S$ between 21 and 37; the spectra for $2S < 21$ can be found in earlier numerical studies (He, Xie, and Zhang 1992). The Laughlin filling factors $\nu = \frac{1}{2}$ and $\frac{1}{3}$ occur at $2S = 21$ and 35, and the Jain filling factors $\nu = \frac{3}{4}$ and $\frac{5}{6}$ occur at $2S = 26$ and 30, respectively. At $2S = 28$, an even denominator filling of $\nu = \frac{5}{4}$ occurs. The low energy bands are marked with open circles. For some values of $2S$ these bands contain subbands marked with dashed lines. The physical meaning of bands marked in figures 1 and 2 will be explained in section III B.

III. COMPOSITE FERMION APPROACH

A. Introduction

In the Chern–Simons (CS) transformation, an equal and even number ($2p$) of elementary fluxes $\phi_0$ (a fictitious flux tube of strength $2p\phi_0$) oriented opposite to the original magnetic field $B$ is attached to each electron. The composite Fermions (CF’s) obtained in this way carry electric charge and magnetic flux. The CS transformation is a gauge transformation and thus the CF energy spectrum is identical to the original electron spectrum. Since attached fluxes are localized on electrons and the magnetic field acting on each electron is unchanged, the classical Hamiltonian of the system is also unchanged. However, the quantum Hamiltonian includes additional terms describing an additional charge–flux (CS) interaction, which arises from the Aharonov–Bohm phase attained when one electron’s path encircles the flux tube attached to another electron. One difficulty in treatment of the CS interaction results from the fact that it contains both two and three body terms; another is the absence of a small parameter with which to construct a perturbation expansion.

FIG. 1. The energy spectra of eight electrons in the lowest Landau level at the monopole strength $2S$ between 21 and 28. (a) $2S = 21$ corresponds to the filling factor $\nu = \frac{1}{2}$, the lowest energy state at $L = 0$ is the Laughlin ground state; (b) $2S = 22$, $\nu = \frac{3}{4}$, Jain ground state at $L = 0$. (h) $2S = 28$, $\nu = \frac{5}{4}$. The low energy states selected by the Chern–Simons transformation with $p = 1$ and $p = 2$ are marked with open circles and dashed lines, respectively.
It has been shown that the mean field Hamiltonian of noninteracting CF’s gives a good qualitative description of the low lying states of interacting electrons in the lowest LL. The Jain sequence of incompressible ground states is predicted at filling factors $\nu$ for which $\nu^*$ is an integer (Jain 1989), and the $\nu^* = 1$ states correspond to Laughlin $\nu = (2p + 1)^{-1}$ states (Laughlin 1983a). If $\nu^*$ is not an integer, the low lying states contain a number of QP’s ($N_{QP} \leq N$) in the neighboring incompressible state with integer $\nu^*$.

On a sphere, an effective CF monopole strength is

$$2S^* = 2S - 2p(N - 1), \quad (9)$$

and $l^* = |S^*|$ plays the role of the angular momentum of the lowest CF shell (Chen and Quinn 1996). If $n$ lowest CF LL’s at $2S^*$ are filled completely by N CF’s, the corresponding N electron state at $2S$ is incompressible. The states at other values of $2S$ are compressible and contain $N_{QP}$ QP’s in the neighboring incompressible state of an equal number of $N$ electrons at $2S_{INC}$,

$$N_{QP} = n(|2S_{INC}| - |2S^*|). \quad (10)$$

Here $2S_{INC}$ is the effective monopole strength calculated for the incompressible state, i.e. $2S_{INC} = 2S_{INC} - 2p_{INC}(N - 1)$, and $n$ is an integral number of completely filled CF LL’s. Positive $N_{QP}$ corresponds to quasielectrons (QE’s) in the $(n + 1)$st (lowest unoccupied) CF shell, each with angular momentum $l_{QE} = l^* + n$. Negative $N_{QP}$ corresponds to quasiholes (QH’s) in the $n$th (highest occupied) CF shell, each with angular momentum $l_{QH} = l^* + n - 1$. Different values of $2S$ that lead to the same value of $l^* = |S^*|$ are equivalent and their low energy bands contain the same $L$ multiplets.

It is noteworthy that the CS transformation applied to the state at $2S$ can have a different flux strength ($2p$) than that ($2p_{INC}$) applied to the incompressible state $2S_{INC}$. Consequently, alternative pictures of the $(N,2S)$ state, containing different numbers and/or types of QP’s, can be obtained (Yi et al. 1996). Writing $p_{INC}$ and $p$ explicitly, equation (11) can be written as

$$N_{QP} = n(|2S_{INC} - 2p_{INC}(N - 1)| - |2S - 2p(N - 1)|). \quad (11)$$

The original spectrum of interacting electrons is similar to that of noninteracting mean field CF’s in a sense that (i) the lowest band of angular momentum multiplets contains states of the minimum number of QP’s consistent with the values of $N$ and $2S$, and (ii) neighboring excited bands contain additional QE–QH pairs.

Let us illustrate the success of the mean field CF approach in predicting the lowest band of multiplets on the example of an eight electron system. The sequence of incompressible states is given in table II. Eight mean field CF’s fill completely one CF LL ($n = 1$) at $|2S^*| = 7$ and two CF LL’s ($n = 2$) at $|2S^*| = 2$. Following equation (11), the sequences of incompressible states for CF fillings $n = 1$ and 2 are generated by varying $p = 0, \pm 1,

B. Mean field approximation

In the mean field approach, the magnetic field due to attached flux tubes is evenly spread over the occupied area $C$. The mean field CF’s obtained in this way move in an effective magnetic field $B^* = B - 2p\phi_0 N/C$. An effective filling factor $\nu^*$ seen by one CF is defined as

$$(\nu^*)^{-1} = \nu^{-1} - 2p, \quad (7)$$

so that

$$B^* \nu^* = B\nu = \frac{N}{C}\phi_0. \quad (8)$$

Negative $\nu^*$ means negative $B^*$ (oriented opposite to $B$). It has been shown that the mean field Hamiltonian of

FIG. 2. The energy spectra of eight electrons in the lowest Landau level at the monopole strength $2S$ between 30 and 37. (a) $2S = 30$ corresponds to the filling factor $\nu = \frac{4}{7}$, the lowest energy state at $L = 0$ is the Jain ground state; (f) $2S = 35$, $\nu = \frac{5}{7}$, Laughlin ground state at $L = 0$. The low energy states selected by the Chern–Simons transformation with $p = 2$ and $p = 3$ are marked with open circles and dashed lines, respectively.
Table II. The incompressible states of eight electrons; filling factor $\nu \geq \frac{1}{3}$.

| $n$ | $S^* = 7$ | $n = 2, S^* = 2$ |
|-----|-----------|-----------------|
| $p$ | $|\nu|$ | $|\nu|$ |
| -2  | 21        | 1/3             | 26   | 2/7             |
| -1  | 7         | 1               | 12   | 2/3             |
| 0   | 7         | 1               | 2    | 2               |
| 1   | 21        | 1/3             | 16   | 2/5             |
| 2   | 35        | 1/5             | 30   | 2/9             |

$\pm 2, \ldots$. States listed in table II ($\nu = 2, 1, \frac{2}{3}, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}$, and $\frac{1}{5}$) are all the incompressible eight electron states at filling factors greater than or equal to $\nu = \frac{1}{5}$ (filling of more than two CF LL’s requires $N > 8$). The states outside the incompressible sequence of $2S_{\text{INC}} = 2, 7, 12, 16, 21, 26, 30, 35, \ldots$ are compressible and contain an appropriate number of QP’s, given by equation (1).

The spectra of an eight electron system in the lowest LL for values of $2S$ between 21 and 37, i.e. for the filling factors $\nu$ from $\frac{1}{3}$ down to below $\frac{1}{5}$, are shown in figures 1 and 2. In figure 1, the open circles and dashed lines mark bands of multiplets predicted in the mean field CF picture as the lowest energy states of CF’s for $p = 1$ and $p = 2$, respectively. In figure 2, all shown states belong to the lowest band corresponding to $p = 1$, and the open circles and dashed lines mark bands obtained for $p = 2$ and $p = 3$, respectively. The range of $2S$ shown in figure 1 alone covers all values of $l^2$ from $N - 1$ to 0 (for $p = 1$) and thus exhausts all possible configurations of QP’s for the eight electron system. Let us analyze the spectra in figure 1 in greater detail.

At $2S = 21$ the CS transformation with $p = 1$ gives $2S^* = 7$. The lowest CF LL is completely filled ($\nu^* = n = 1$) and the Laughlin incompressible $\nu_{\text{INC}} = \frac{1}{3}$ state with $L = 0$ is formed. The CS transformation with $p = 2$ gives $2S^* = -7$ and the equivalent interpretation of the ground state. At $2S = 22$ the CS transformation with $p = 1$ gives $2S^* = 8$. The lowest CF LL has degeneracy of $2S^* + 1 = 9$ so it holds $N = 8$ CF’s and one QH with $l_{\text{QH}} = 4$ (QH in the $\nu_{\text{INC}} = \frac{1}{3}$ state). Therefore, the low energy band contains a single multiplet with $L = 4$. The CS transformation with $p = 2$ gives $2S^* = -6$ which corresponds to a completely filled lowest CF LL and one QH with $l_{\text{QH}} = 4$ in the first excited CF LL. Depending on the applied CS transformation, the $L = 4$ ground state can be viewed as a state of either a single QE or a single QH in the appropriate CF LL (Yi et al. 1996). The low energy multiplets obtained using the CS transformation with $p = 1$ at $2S = 23, 24, \ldots, 28$ contain 2, 3, 3, 5, 7 QH’s in the lowest CF LL (i.e. in the $\nu_{\text{INC}} = \frac{1}{3}$ state), each with angular momentum $l_{\text{QH}} = \frac{5}{2}, 5, \ldots, 7$, respectively. For example, at $2S = 24$ the band of states of three QH’s each with $l_{\text{QH}} = 5$ contains the following multiplets: $L = 0, 2, 3, 4^2, 5, 6^2, 7, 8, 9, 10, 12$. At $2S \geq 23$ the CS transformation with $p = 2$ selects a subset of multiplets out of those obtained with $p = 1$, and the low energy subband corresponding to $p = 2$ develops in the $p = 1$ band. For example, the low energy $p = 2$ subband predicted for $2S = 23$ ($2S^* = -5$) contains two QE’s each with $l_{\text{QE}} = \frac{2}{7}$, and thus $L = 0, 2, 4, 6$. At $2S = 26$ the CF monopole strength for $p = 2$ is $2S^* = -2$ and two lowest CF LL’s are completely filled ($\nu^* = n = 2$). The ground state is the incompressible Jain $\nu_{\text{INC}} = \frac{1}{2}$ state with $L = 0$. At $2S = 25$ the CF monopole strength for $p = 2$ is $2S^* = -3$ and at $2S = 27$ it is $2S^* = -1$. In both cases, the low energy band contains two QP’s each with $l_{\text{QP}} = \frac{2}{7}$ in the $\nu_{\text{INC}} = \frac{1}{2}$ state (two QE’s at $2S = 25$ and two QH’s at $2S = 27$). For $2S = 24$ one obtains $2S^* = -4$ and the lowest energy band contains three QE’s each with $l_{\text{QE}} = 3$. Finally, for $2S = 28$ one obtains $2S^* = 0$ and one QH with $l_{\text{QH}} = 2$ in the second excited CF LL. The effective magnetic field acting on the CF’s vanishes, and this state is assigned an even denominator filling factor $\nu = \frac{2}{7}$.

Higher energy bands, containing multiplets with additional QE-QH pairs, are more difficult to identify in figures 1 and 2 than the lowest ones. However, for $2S = 21$ one can easily notice the low lying band of states at $L = 2, 3, 4, 5, 6, 7, 8$, and 8, which correspond to the states of one QE-QH pair ($l_{\text{QE}} = \frac{7}{2}$ and $l_{\text{QH}} = \frac{4}{7}$) in the mean field CF picture. Similarly, the band of QE–QH pair states for $2S = 26$ occurs at $L = 2, 3, 4, 5$ ($l_{\text{QE}} = 3$ and $l_{\text{QH}} = 2$). For $2S = 25$ the lowest band contains two QH’s each with $l_{\text{QH}} = \frac{5}{2}$ in the $\nu^* = 2$ CF state ($L = 0, 2, 4$). The first excited band has two subbands at the same CF energy. One contains states corresponding to three QH’s each with $l_{\text{QH}} = \frac{5}{2}$ and one QE with $l_{\text{QE}} = \frac{2}{7}$. The allowed multiplets of such CF system are $L = 2^2, 2^3, 3^2, 3^3, 4^3, 5^3, 6^2, 7$, and 8. The other contains states of one QH in the lowest CF LL ($l_{\text{QH}} = \frac{5}{2}$) and one QH in the first excited CF LL ($l_{\text{QH}} = \frac{5}{2}$). The allowed multiplets in this subband are $L = 1, 2, 3, 4$. One can identify in figure 2(a) (a few multiplets with highest angular momenta ($L = 8, 7, 6^2, \ldots$) of this band.

The bands of states containing an increasing number of QE–QH pairs are much better visible in the density of states (DOS), $dN(E)/dE$, plotted in figure 3. Frames (a) and (b) show the data for $2S = 21$ (Laughlin $\nu = \frac{1}{3}$ ground state) and $2S = 22$ (one QH in the ground state), respectively. The continuous DOS is obtained by broadening of discrete energy levels with Gaussians,

$$
\frac{dN(E)}{dE} = \frac{\sqrt{\pi}}{\delta} \sum_{L+1}(2L+1) \exp\left[-\frac{|E-E_{\text{Inc}}|^2}{\delta^2}\right]
$$

where summation goes over all $L$ multiplets (distinguished by different $\alpha$), and the normalization prefactor guarantees that $\int dN(E)/dE \ dE = N$, the total number of states. The thin lines were obtained for $\delta = 0.001 \epsilon^2/\lambda$ and the thick lines correspond to $\delta = 0.02 \epsilon^2/\lambda$. The thick lines, free of noise characteristic of the discrete spectrum, reveal a series of equidistant peaks and/or steps in the DOS. The peaks corresponding to the ground states are hardly visible and their positions have been marked.
with open circles. A number of higher peaks (at lower energies) or plateaus plateaus (at higher energies) are the remnants of the CF bands with increasing numbers of QP’s. The quasiperiodic character of the DOS spectrum is even more pronounced in the derivatives of the DOS, shown in the insets (calculated only for \( \delta = 0.02 e^2/\lambda \)). The plateaus in \( dN/dE \) and the minima in \( d^2N/dE^2 \) correspond to the bands of states with an increasing number of quasielectron–quasihole pairs.

C. Energy scales and fluctuations beyond mean field approximation

Despite the success of the mean field CF approach in describing the low energy spectra of interacting electrons in many numerical (exact) calculations carried out for finite systems, the reason for its success still remains a puzzle. The original conjecture that the CF transformation converts a system of strongly interacting electrons into one of weakly interacting CF’s cannot possibly be correct because the CS interactions among fluctuations are measured on an energy scale proportional to \( \hbar \omega_c \propto B \), which can be much larger than the energy scale of the Coulomb interactions, proportional to \( e^2/\lambda \propto \sqrt{B} \). This is demonstrated in figure 4, where the original energy spectrum of free electrons is compared to that of noninteracting mean field CF’s (note that the degeneracy of multiplets is not shown). Clearly, inclusion of the electron–electron Coulomb interaction with characteristic energy as small as marked in figure 4 with a shaded rectangle cannot reproduce the separation of levels present in the mean field CF spectrum. Because so many results can be successfully interpreted in terms of composite Fermions, the understanding of the actual reason for the success of the mean field CF model, as well as defining its limitations and range of applicability, is extremely important.

IV. PSEUDOPOTENTIAL OF COULOMB INTERACTION

The two body interaction Hamiltonian of the many body system can be expressed as

\[
\hat{H} = \sum_{i<j} \sum_{L} V(L) \hat{P}_{ij}(L). \tag{13}
\]

Here, \( V(L) \) is the two particle interaction pseudopotential (Haldane 1987) defined as the interaction energy of a pair in the eigenstate \( |L\rangle \) of angular momentum \( L \),

\[
\hat{H} |L\rangle = V(L) |L\rangle, \tag{14}
\]

and \( \hat{P}_{ij}(L) \) is the projection operator onto the subspace with the pair \( ij \) in the state \( |L\rangle \). Pair angular momentum \( L \) measures the average squared electron-electron distance \( \vec{d}^2 \). It can be shown that within the \( n \)th LL of the Haldane sphere.
calculated for the lowest and the first two excited LL’s with

\[ R \]

associated with the relative coordinate \( r \) on a plane it is equal to the angular momentum \( r \), and on a plane it is equal to the angular momentum as-

dermines many body eigenstates and eigenenergies.

Using the relative angular momentum \( \hat{S} \) for meaningful comparison of the pseudopotentials in the eigenvalue parameters, \( V \) enters the Hamiltonian limited, and the electron–electron interaction potential one (lowest) LL, the number of pair states is strongly

dane sphere with a given \( 2S, l, \) and \( R \). All pseudopotentials \( V(R) \) for all values of \( R \). The \( V_{HC} \) is an ‘ideal’ short range pseudopotential (the class of short range pseudopotentials leading to the similar, Laughlin like short range correlations will be formally defined in section VII E).

The conditions \( \frac{d^2 V}{dR^2} \geq 0 \) can be rewritten as \( dV/dR < 0 \) and \( d^2 V/dR^2 \gg 0 \), where the derivatives are to be understood as finite differences. Clearly, in the low lying many body eigenstates of \( V_{HC} \), electrons must avoid as much as possible pair states with largest repulsion, i.e. pair states with the smallest separation or smallest values of \( R \). The many body states which avoid certain values of \( R \) can be constructed explicitly using parentage or grandparentage coefficients. In the following sections we shall investigate in detail the connection between the low lying states of the FQH systems and the avoiding of pair states with largest repulsion.

\[ V_{HC}(R) \gg V_{HC}(R + 2), \]

\[ V_{HC}(R - 2) - V_{HC}(R) \gg V_{HC}(R) - V_{HC}(R + 2) \]

for all values of \( R \).

FIG. 5. The pseudopotentials \( V \) of the Coulomb interaction in the lowest (a), first excited (b), and second excited (c) Landau levels, as a function of the relative angular momen-
tum \( L \). Open circles: plane; full triangles, diamonds and circles: Haldane sphere with \( l = \frac{1}{2}, \) 10, and \( \frac{25}{2} \), respectively.

Notice that \( 0 < d^2 < (2R)^2 \) and \( d^2 \equiv 2R^2 \) for \( 2S = 0 \).

Due to the confinement of single particle states to one (lowest) LL, the number of pair states is strongly limited, and the electron–electron interaction potential enters the Hamiltonian \( H \) only through a small set of pseudopotential parameters. This reveals the magnetic field quantization of electron–electron interaction, i.e., electron–electron separation (Laughlin 1983b). On a Haldane sphere with a given \( 2S \), a finite number of these parameters, \( V(2l - R) \), where \( R \leq 2l \) is an odd integer, determines many body eigenstates and eigenenergies.

Using the relative angular momentum \( R \) instead of the eigenvalue \( L \) of total angular momentum \( \hat{L} = \hat{l}_1 + \hat{l}_2 \) to label pair states and pseudopotential coefficients allows for meaningful comparison of the pseudopotentials in the planar system and in spherical systems with different \( l \) (or \( 2S \)). On a sphere, \( R \) is defined as

\[ R = 2l - L, \]

and on a plane it is equal to the angular momentum associated with the relative coordinate \( r = r_1 - r_2 \). In both cases, larger \( R \) means larger separation (see equation 12 for the sphere). Figure 5 shows pseudopotentials \( V(R) \) calculated for the lowest and the first two excited LL’s (\( n = 0, 1, \) and \( 2 \)) for the plane and for the Haldane sphere with \( l = \frac{1}{2}, \) 10, and \( \frac{25}{2} \). All pseudopotentials \( V(R) \) in figure 5 decrease with increasing \( R \).

The important part of the pseudopotential spectrum is where its slope is the highest. It follows from equation 12 that each pair state with a given \( L \) corresponds to a certain average separation \( d \) and, roughly, \( d \propto R \). Large slope \( dV/dR \) means large energy gradient, i.e. large effective force, that would describe two point charges at a distance \( d \). This effective force is solely due to the Coulomb force, but takes into account different spread of electron wavefunctions in pair states for different \( 2S, l, \) and \( R \). As will be shown later, the crucial difference between the lowest LL (a) and excited LL’s (b,c) is that in the former case \( V(R) \) decreases more quickly at the smallest values of \( R \).

Let us define a model hard core pseudopotential \( V_{HC} \) for which

\[ V_{HC}(R) \gg V_{HC}(R + 2), \]

\[ V_{HC}(R - 2) - V_{HC}(R) \gg V_{HC}(R) - V_{HC}(R + 2) \]

for all values of \( R \). The \( V_{HC} \) is an ‘ideal’ short range pseudopotential (the class of short range pseudopotentials leading to the similar, Laughlin like short range correlations will be formally defined in section VII E).

The conditions \( \frac{dV}{dR} \ll 0 \) and \( d^2 V/dR^2 \gg 0 \), where the derivatives are to be understood as finite differences. Clearly, in the low lying many body eigenstates of \( V_{HC} \), electrons must avoid as much as possible pair states with largest repulsion, i.e. pair states with the smallest separation or smallest values of \( R \). The many body states which avoid certain values of \( R \) can be constructed explicitly using parentage or grandparentage coefficients. In the following sections we shall investigate in detail the connection between the low lying states of the FQH systems and the avoiding of pair states with largest repulsion.

V. THREE ELECTRON SYSTEM

A. Coefficients of fractional parentage

We begin the discussion of the three electron case by listing in table 11 all possible \( L \) multiplets appearing in the spectrum for a given single particle angular momentum \( l \). An eigenfunction of three electrons each of angular momentum \( l \) whose total angular momentum is \( L \) will be denoted by \( \langle \ell^3; L \alpha \rangle \), with an index \( \alpha \) distinguishing different multiplets with the same \( L \). This state can be written as

\[ \langle \ell^3; L \alpha \rangle = \sum_{L_{12}} F_{\ell^3 L \alpha}(L_{12}) \langle \ell^2; L_{12}; l, L \rangle, \]

a combination of product states \( \langle \ell^2; L_{12}; l, L \rangle \) in which \( l_1 = l_2 = l \) are added to obtain pair angular momentum \( L_{12} \), and then \( l_3 = l \) is added to \( L_{12} \) to obtain total angular momentum \( L \) (Shalit and Talmi 1963, Cowan 1981). Note that state \( \langle \ell^3; L \alpha \rangle \) is antisymmetric under

8
interchange of any pair of particles 1, 2, and 3, while states $|l^2, l_{12}; l, L\rangle$ are antisymmetric only under interchange of particles 1 and 2. The factor $F_{L\alpha}(L_{12})$, or $F_{L\alpha}(R)$ where $R = 2l - L_{12}$, is called the coefficient of fractional parentage (CFP) associated with pair angular momentum $L_{12}$.

The two particle interaction matrix element can be conveniently expressed through the CFP’s and the pseudopotential coefficients (Sitko et al. 1996),

$$
\langle \beta, L\alpha | V | \beta, L\beta \rangle = 3 \sum_{R} F_{L\alpha}(R) F_{L\beta}(R) V(R). \tag{19}
$$

If state $|\beta, L\alpha \rangle$ is an eigenstate of the interacting system, its energy is

$$
E_{L\alpha} = 3 \sum_{R} F_{L\alpha}(R) V(R), \tag{20}
$$

where $F_{L\alpha} = |F_{L\alpha}|^2$. The CFP’s for three particles with given $l$ can be derived analytically or found in nuclear (Shalit and Talmi 1963) or atomic (Cowan 1981) physics books. Note however that the squared CFP’s, which appear in equation (20) and measure the probability that a pair of electrons $ij$ are in the pair state of angular momentum $R$ can be expressed as

$$
F_{L\alpha}(R) = \langle L\alpha | \hat{P}_{ij}(R) | L\alpha \rangle. \tag{21}
$$

It follows from equation (21) that they can be calculated quite easily for any state $|L\alpha \rangle$ as the expectation value of the ‘selective interaction’ Hamiltonian $\hat{H}_{R}$, whose only nonvanishing pseudopotential parameter is $V(R) = 1$, i.e.

$$
F_{L\alpha}(R) = \frac{1}{3} \langle L\alpha | \hat{H}_{R} | L\alpha \rangle. \tag{22}
$$

or with a number of pseudopotential parameters associated with the strongest repulsion, $V(1), V(3), \ldots$.

For three electrons (Fermions), the angular momenta of states in which $R \geq 3$, 5, ... for all pairs can be predicted from the following argument (Wójc and Quinn 1998b). If we choose $R = 1$ for the pair of electrons 1 and 2 (i.e. $L_{12} = 2l - 1$), and add to $L_{12}$ the same single particle angular momentum $l$ of the third electron, then the total angular momentum $L$ must satisfy the vector addition rule, $|L_{12} - l| \leq L \leq L_{12} + l$. The antisymmetrization of the total wavefunction will eliminate some of the values of $L$ from this range, but it is guaranteed that no states with $L$ smaller than the minimum value, $L < l - 1$, can have nonvanishing parentage from $R = 1$. In Table II we have underlined the three electron states with $L < l - 1$, which must therefore have $R \geq 3$ for all pairs. The next higher value of $R$ to avoid is 3, and, using the same argument as above, we obtain that all states with $L < l - 3$ must have $R \geq 5$ (double underlined in Table II). Further, states with $L < l - 5$ must all have $R \geq 7$ (triple underlined in Table II), and so on.

In Table IV we list the values of $2L$ for which the CFP with $R = 1$ or with $R \leq 3$ or with $R \leq 5$ must vanish, i.e. $R \geq 3$, 5, or 7, respectively. The $L = 0$ states for $2S = 6, 10$, and 14 are the Laughlin ground states with $\nu = \frac{1}{3}, \frac{1}{5},$ and $\frac{1}{7}$, respectively.

Note that the multiplets listed at $2l$ with $R \geq R_{\text{MIN}}$ are always the same as those at $2l - 2p(N - 1)$ with $R \geq R_{\text{MIN}} - 2p$. But for the lowest LL ($l = S$), $2S - 2p(N - 1)$ is just $2S^*$, the effective monopole strength of CFP’s! This very important result remains true for any number of electrons, and will be discussed in more detail in section VII B.

At $2S = 8$, two $L = 3$ multiplets occur (see Table III) and the interparticle interaction must be diagonalized in
For the lowest LL, the Coulomb interaction plotted in figure 6(a) behaves like the hard core repulsion $V_{HC}$ defined in equation (3). The energy spectrum in figure 6(a) splits into bands of states with no parentage from pair states with $\mathcal{R} < 7$ (diamonds), $\mathcal{R} < 5$ (open circles), $\mathcal{R} < 3$ (full circles), and remaining states with parentage from all pair states including $\mathcal{R} = 1$ (squares). The CFP’s which are expected to vanish identically for any pseudopotential (see the last column in table IV) or which would vanish for the eigenstates of the interaction $V_{HC}$ defined in equation (3) indeed vanish or are very small ($\mathcal{F} < 0.01$) for the eigenstates of the Coulomb interaction. This means that the Coulomb interaction within the lowest LL acts like $V_{HC}$ and the two interactions have essentially identical eigenstates.

Since $V(1) - V(3) > V(3) - V(5) > \ldots$ in figure 6(a), the gap between the highest energy band ($\mathcal{R} \geq 1$) and the lower ones is the largest, the gap below the $\mathcal{R} = 3$ band is the next largest, etc. The lowest band ($\mathcal{R} \geq 7$) consists of only one state at $L = 0$. This is the Laughlin $\nu = \frac{1}{7}$ ground state. The excitation gap above the $\nu = \frac{1}{7}$ state is governed by $V(5) - V(7)$ and, as might be expected, it is almost unobservable. Note also that the first excited band in figure 6(a) containing states with $\mathcal{R} \geq 5$ consists of triplets at $L = 2, 3, 4, 6$, in contrast with the mean field CF prediction ($L = 1, 2$, and 3).

The inset in figure 6(a) shows the spectrum for $l = 3$. The $L = 0$ ground state has $\mathcal{F}(1) = 0$ (see the first column in table IV); this is the Laughlin $\nu = \frac{1}{7}$ state. The structure of energy spectrum for $l = 3$ is very similar to that within the two lowest bands for $l = 7$. This is because the Coulomb interaction for $n = 0$ acts like hard core repulsion and decreasing angular momentum by $p(N - 1)$ is equivalent to introduction of a hard core which forbids pair states with $\mathcal{R} < 2p + 1$ (see figure 7 and the discussion in the following section).

The Coulomb pseudopotentials for $n = 0$ in figure 6(a) and for $n = 1$ in figure 6(b) behave similarly for $\mathcal{R} \geq 3$. In consequence, the two lowest bands of states in figure 6(a) and (b) look similar. The CFP’s which are expected to be small, are found to be smaller than 0.01 for both $n = 0$ and 1. However, for the smallest $\mathcal{R}$, the condition $V(1) - V(3) \gg V(3) - V(5)$ is no longer satisfied for $n = 1$. Close to $\mathcal{R} = 1$, the Coulomb pseudopotential for $n = 1$ decreases too slowly with increasing $\mathcal{R}$, and its eigenstates, having some parentage from the $\mathcal{R} = 1$ pair state, are significantly different from those of the hard core repulsion. For example, the states at $L = 10$ and 12 marked with full dots in figure 6(b) both have significant parentage from $\mathcal{R} = 1$, $\mathcal{F}(1) \approx 0.11$, while the two other states with $L = 10$ and 12, marked with squares, both have $\mathcal{F}(1) \approx 0.23$, only twice as large. For the same reason, there is almost no gap above the $\mathcal{R} \geq 3$ band for $n = 1$, in contrast to the $n = 0$ spectrum.

Different behavior of $V(\mathcal{R})$ for $n = 1$ at small values of $\mathcal{R}$ has much more pronounced effect on the $l = 3$ spectrum shown in the inset. The $L = 0$ state must have
\[ E_{La} = \frac{N(N-1)}{2} \sum_{R} G_{La}(R) V(R), \] (26)

where the coefficient

\[ G_{La}(R) = \sum_{L'\alpha'} |G_{La,L'\alpha'}(R)|^2 \] (27)

gives the probability that a pair of electrons \(ij\) are in the pair state of a given \(R\). The derivation of the CFGP’s for arbitrary \(N\) and \(l\) is rather tedious. Note however that the coefficients \(G(R)\) can be expressed as (compare equation 21)

\[ |G_{La}(R)|^2 = \langle La | \hat{P}_{ij}(R) | La \rangle \] (28)

and calculated as the expectation value of the ‘selective interaction’ Hamiltonian \(\hat{H}_R\), whose only nonvanishing pseudopotential parameter is \(V(R) = 1\) (compare equation 22),

\[ G_{La}(R) = \frac{2}{N(N-1)} \langle La | \hat{H}_R | La \rangle . \] (29)

From the orthonormality of functions \(|l^N, La\rangle\) it is also apparent that

\[ \sum_{R} \sum_{L'\alpha'} G_{La,L'\alpha'}(R) G_{L,\beta,L'\alpha'}(R) = \delta_{\alpha\beta}. \] (30)

B. Dynamical symmetry of hard core repulsion

The angular momentum addition argument fails for more than three electrons, and there are no \(L\) multiplets for \(N > 3\) whose CFGP for \(R = 1, 3, \ldots\) would vanish regardless of the form of interaction pseudopotential. However, the many electron Hilbert space \(\mathbb{H}\) still contains subspaces \(\mathbb{H}_p\) holding many body states with grandparentage only from pair states with \(R \geq 2p + 1\), for which \(G(1) = G(3) = \ldots = G(2p - 1) = 0\),

\[ \mathbb{H} \equiv \mathbb{H}_0 \supset \mathbb{H}_1 \supset \mathbb{H}_2 \supset \ldots \] (31)

The total Hilbert space splits thus into subspaces \(\mathbb{H}_p = \mathbb{H}_p \setminus \mathbb{H}_{p+1}\), containing many body states that do not have grandparentage from pair states with \(R < 2p + 1\), but have some grandparentage from \(R = 2p + 1\),

\[ \mathbb{H} = \mathbb{H}_0 \oplus \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \ldots \] (32)

For \(N\) electrons on a Haldane sphere each with angular momentum \(l\), there is more than one subspace (subspace \(\mathbb{H}_1\) is not empty) for \(2l \geq 3(N-1)\), i.e. for filling factors \(\nu \leq \frac{1}{3}\). In general, \(\mathbb{H}_p\) is not empty (some states with \(R \geq 2p + 1\) can be constructed) for \(\nu \leq (2p + 1)^{-1}\).

The subspaces \(\mathbb{H}_p\) are the eigenspaces of the hard core repulsive potential \(V_{HC}\) defined in equation 17.
the spectrum is the appearance of an additional band \( N \) is increased by \( S \) and \( L \) Landau level at different monopole strength: 2 (hard core like) Coulomb pseudopotential. When four electrons in the lowest \( LL \) interacting through the \( \sim \) the same as the set of multiplets in the \( n \) which make the \( \sim \) between the 0th band and the 1st band, the next largest \( q \) band \( (q, L) \) \( \geq 0 \), 1, and 2, respectively; \( \geq 1 \), i.e. \( G(1) \approx 0 \) and \( G(3) \geq 0 \); open circles: \( \geq 3 \), i.e. \( G(1) \approx 0 \) and \( G(3) > 0 \); and open squares: \( \geq 1 \), i.e. \( G(1) > 0 \).

whose low energy states have to avoid grandparentage from pair states with large repulsion (small \( R \)). Consequently, as for three electrons, the energy levels in the many electron spectrum with hard core interaction form bands corresponding to subspaces \( \tilde{H}_q \). For given \( N \) and \( l \), i.e. for a given filling factor \( \nu \) such that \( (2p + 3)^{-1} < \nu \leq (2p + 1)^{-1} \), there are \( (p + 1) \) bands, and the \( q \)th band \( (q = 0, 1, \ldots, p) \) corresponds to \( \tilde{H}_q \). The \( p \)th band is the lowest energy band with the maximum number of \( CF \)'s vanishing, and the 0th band is the highest energy band containing states with some grandparentage from the \( R = 1 \) pair state. The energy gap between the \( q \)th band and the \( (q + 1) \)-st band is of the order of \( V(2q + 1) - V(2q + 3) \). Hence, the largest gap is that between the 0th band and the 1st band, the next largest is that between the 1st band and 2nd band, etc.

Importantly, the set of angular momentum multiplets which make the \( q \)th band \( \tilde{H}_q \) subspace) of the spectrum of \( N \) electrons each with angular momentum \( l \) is always the same as the set of multiplets in the \( (q + 1) \)-st band \( \tilde{H}_{q+1} \) subspace) of \( N \) electrons each with angular momentum \( l + (N - 1) \). This is demonstrated in figure 7 for four electrons in the lowest \( LL \) interacting through the (hard core like) Coulomb pseudopotential. When \( l = S \) is increased by \( N - 1 \), the only significant difference in the spectrum is the appearance of an additional band at high energy. The structure of the low energy part of the spectrum is completely unchanged. All bands and multiplets in the spectrum for \( 2S \) correspond directly to appropriate bands and multiplets in the spectrum for the monopole strength \( 2S + 2(N - 1) \). For example, all three allowed multiplets at \( 2S = 5 \) \((L = 0, 2, \) and \( 4) \) form the low energy band at \( 2S = 11, 17, \) and 23, where they span the \( H_1, H_2 \) and \( H_3 \) subspaces, respectively. Similarly, the first excited band at \( 2S = 11 \) (open squares in frame b) is repeated in the spectra for \( 2S = 17 \) and 23, where it corresponds to \( H_1 \) and \( H_2 \) subspace, respectively.

Let us repeat that the fact that identical multiplets occur in subspaces \( H_q \) for single electron angular momentum \( l \), and in subspace \( H_{q+1} \) for \( l \) replaced by \( l + (N - 1) \), does not depend on the form of interaction, and follows solely from the rules of addition of angular momenta of identical Fermions. However, if the interaction pseudopotential has the hard core properties as in equation [3] then the many body interaction Hamiltonian has a new, dynamical symmetry, as a result of which: (i) the subspaces \( H_q \) are the eigensubspaces and the subspace (band) index \( q \) is a good quantum number; (ii) the energy bands corresponding to \( H_q \) with higher \( q \) lie below those of lower \( q \); (iii) the spacing between neighboring bands is governed by a difference between appropriate pseudopotential coefficients; and (iv) the wavefunctions and the relative position of energy levels within each \( (q, l) \) band do not depend on the details of interaction (it will be shown later that they repeat the spectrum of \( G(2q + 1) \); see figure 15). Replacing the model hard core pseudopotential by a ‘softer’ one (the measure of the ‘hard core’ character \( \beta \) will be specified in section VII F) leads to: (i) coupling between subspaces \( H_q \); (ii) mixing, overlap, or even order reversal of bands; (iii) deviation of wavefunctions and the structure of energy levels within bands from those of the hard core repulsion (and thus their dependence on details of the interaction pseudopotential).

The reoccurrence of \( L \) multiplets forming the low energy band when \( l \) is replaced by \( l \pm (N - 1) \) has the following crucial implication. The lowest energy, \( p \)th band contains \( L \) multiplets which are all the allowed multiplets of \( N \) electrons each with angular momentum \( l - p(N - 1) \). This is because if \( (2p + 3)^{-1} < \nu_{N,l} \leq (2p + 1)^{-1} \), then \( \frac{1}{3} < \nu_{N,l-p(N-1)} \leq 1 \) and there is only one, \( p \)th band in the spectrum. As for three electrons, for the lowest \( LL \) with \( l = S \) this means that the lowest energy band at the monopole strength \( 2S \) contains a subset of low energy multiplets which are all the allowed multiplets at a smaller monopole strength \( 2S - 2p(N - 1) \). But \( 2S - 2p(N - 1) \) is just \( 2S^* \), the effective monopole strength of \( CF \)'s! The mean field CS transformation which attaches \( 2p \) fluxes (vortices) to each electron selects the same \( L \) multiplets from the entire spectrum as does the introduction of a hard core, which forbids a pair of electrons to be in a state with \( R < 2p + 1 \).

The success of the mean field CF picture in prediction of the low lying band of states in the many electron spectrum relies on the fact that the Coulomb interaction
within the lowest LL acts like the hard core repulsion. For filling factors \( \nu \) such that \( (2p + 3)^{-1} < \nu \leq (2p + 1)^{-1} \), the states predicted by the mean field CF picture as the states of an appropriate number of QH’s in the Laughlin \( \nu = (2p + 1)^{-1} \) ground state are the states which for the hard core interaction have the maximum number \( (p) \) of vanishing CFGP’s associated with the highest pseudopotential parameters. These are the states with \( R \geq 2p + 1 \) spanning subspace \( \hat{H}_p \). In particular, there is always only one state with \( R \geq 2p + 1 \) \( (\hat{H}_p \) is one dimensional) at the filling factor \( \nu = (2p + 1)^{-1} \). This state has \( L = 0 \) and it is the Laughlin incompressible ground state, separated from other states by the gap \( \Delta \) which is of the order of \( \Delta = V(2p - 1) - V(2p + 1) \).

As long as the eigenstates of the Coulomb interaction are approximately those of the hard core repulsive interaction, the incompressible ground states are associated with appearance of states with significantly lower CFGP’s than all other states in the spectrum. The Laughlin \( \nu = (2p + 1)^{-1} \) ground states are the only states with \( G(1) \approx G(3) \approx \ldots \approx G(2p - 1) \approx 0 \) in their Hilbert spaces (the CFGP’s do not vanish identically due to the weak mixing between \( \hat{H}_p \) subspaces). All other states have some (significant) grandparentage from pair states with \( R < 2p + 1 \). The Jain states at filling factors \( \nu \) in the range \( (2p + 3)^{-1} < \nu < (2p + 1)^{-1} \) are those of all states with \( G(1) \approx G(3) \approx \ldots \approx G(2p - 1) \approx 0 \), for which \( G(2p + 1) \), the first nonvanishing CFGP, is significantly smaller than for other states (Wojs and Quinn 1999a).

What is the condition for the interaction pseudopotential to behave like the hard core repulsion and have the energy spectrum characteristic of the FQH effect? In the following sections we answer this question and explain why the hard core type (FQH) ground states occur for the Coulomb interaction within the lowest LL. We also show that due to a different form of the Coulomb pseudopotential in higher (spin polarized) LL’s, the FQH ground states for \( n > 0 \) occur only at lower densities, when, at low energy, only the hard core like part of the pseudopotential (at high \( R \)) contributes to the Hamiltonian given by equation (13).

C. Coulomb interaction in lowest and excited Landau levels

Figure 8 shows the Coulomb energy as a function of \( L \) for the system of four electrons each with \( \nu = \frac{1}{4} \). Three frames correspond to the lowest LL \( (n = 0) \) and two excited LL’s \( (n = 1 \) and \( 2) \), and the insets show the spectra for \( \nu = \frac{1}{4} \). Figure 8 is similar to figure 6 and demonstrates that the conclusions drawn for the simple three electron system remain valid for an arbitrary \( N \).

As for three electrons, the Coulomb interaction within the lowest LL \( (n = 0) \) behaves like the hard core interaction and the energy spectrum splits into bands of states with \( R \geq 5 \) (open circles), \( R \geq 3 \) (full circles), and \( R \geq 1 \) (squares). The \( R \geq 5 \) band contains only the Laughlin \( \nu = \frac{1}{4} \) ground state. For \( N > 3 \), none of CFP’s vanish identically for an arbitrary interaction, but the CFP’s which would vanish for the eigenstates of the hard core interaction defined in equation (17) indeed vanish or are very small \( (G < 0.01) \) for the eigenstates of the Coulomb interaction. The inset in figure 8(a) shows the spectrum for \( \nu = \frac{1}{2} \), with the Laughlin \( \nu = \frac{1}{4} \) ground state. The energy spectrum for \( \nu = \frac{1}{2} \) repeats main features of the two lowest energy bands for \( \nu = \frac{1}{4} \).

Within the first excited LL \( (n = 1) \), only the lowest band with \( R \geq 5 \) can be distinguished. The two higher bands \( (R \geq 3 \) and \( R \geq 1) \) overlap. Also, some of the coefficients \( G(1) \) which would be zero for the hard core repulsion, are quite large \((0.1) \) for \( n = 1 \). In the inset, the two \( L = 0 \) states have \( G(1) = 0.08 \) and 0.26, and the Laughlin like \( \nu = \frac{1}{4} \) state with a smaller \( G(1) \) (full circle) is the one with higher energy. Even though the ground state has \( L = 0 \), it is not the state with the Laughlin like correlations, with electrons avoiding pair states with the largest repulsion (i.e. smallest average separation, see equation (13)). The gap above this ground state is not associated with the energy \( V(1) - V(3) \), and hence the \( \nu = 2 + \frac{1}{3} \) state is unlikely to be an incompressible ground state in the thermodynamic limit.

For \( n = 2 \), neither the Laughlin like \( \nu = \frac{1}{4} \) state in the main frame \( (R \geq 5, \) open circle), nor the Laughlin like \( \nu = \frac{1}{3} \) state in the inset \( (R \geq 3, \) full circle) is the ground
state. This suggests that neither the \( \nu = 4 + \frac{1}{5} \) state nor the \( \nu = 4 + \frac{1}{5} \) state is an incompressible ground state in the thermodynamic limit.

We have calculated the energy spectra analogous to those in figure 8 for different numbers of electrons and conclude that the Laughlin like \( L = 0 \) state with \( \nu = \frac{1}{3} \), which is the only state with \( \mathcal{R} \geq 3 \) in its spectrum, is the ground state only within the lowest LL (\( n = 0 \)). Similarly, the Laughlin like \( \nu = \frac{1}{5} \) state with \( \mathcal{R} \geq 5 \) is the ground state only for \( n \leq 1 \).

The angular momentum \( \vec{L} \) of the ground state of \( N \) electrons at the monopole strength \( 2S \) corresponding to the \( \nu = \frac{1}{3} \) filling within the LL of \( n > 0 \) or to the \( \nu = \frac{1}{5} \) filling within the LL of \( n > 1 \) depends on \( N \). Even though \( L = 0 \) (ground state is nondegenerate) for some values of \( N \), the low lying spectra do not resemble those in the lowest LL, and the excitation is not associated with energy \( V(1) - V(3) \). In order to verify if the \( L = 0 \) ground states with \( \nu = 2 + \frac{1}{3}, 2 + \frac{1}{5}, 4 + \frac{1}{3}, \text{ and } 4 + \frac{1}{5} \) are incompressible ground states in the thermodynamic limit, we have calculated the energy gaps above these states for different values of \( N \). The energy spectra of up to eleven electrons at the filling factor \( \nu = \frac{1}{3} \) in the lowest and first excited LL’s are presented in figure 8. The energy scales for \( n = 0 \) and 1 are different, and the bar in the bottom right corner of each \( n = 1 \) graph on the right shows the energy gap of the corresponding system in the lowest LL on the left. Figure 8 shows the dependence of the gap \( \Delta_{L=0} \) from the lowest \( L = 0 \) state to the lowest state of \( L > 0 \), as a function of \( N^{-1} \). For filling factors \( \nu = 2n + \frac{1}{3} \) (full circles), \( N \) varies between four and eleven, and for \( \nu = 2n + \frac{1}{5} \) (open circles) \( N \) goes up to eight. Negative \( \Delta_{L=0} \) means that the ground state is degenerate (has \( L > 0 \)). In such case, \( |\Delta_{L=0}| \) gives the excitation energy from this degenerate ground state to the lowest state at \( L = 0 \).

For \( n = 0 \), the ground states at both \( \nu = \frac{1}{3} \) and \( \frac{1}{5} \) are Laughlin incompressible states. The gap \( \Delta \) persists for \( N \to \infty \), and the estimates obtained from the best linear fits (dashed lines) are \( \Delta_{\nu=1/3} = 0.0632 e^2/\lambda \) and \( \Delta_{\nu=1/5} = 0.0105 e^2/\lambda \). For \( n = 1 \), the \( L = 0 \) state at \( \nu = 2 + \frac{1}{3} \) is the Laughlin like state and the gap above it seems to converge to a finite value; the linear fit gives \( \Delta_{\nu=2+1/3} = 0.0116 e^2/\lambda \), very close to \( \Delta_{\nu=1/5} \). On the other hand, the dependence of the gap \( \Delta \) above the (non Laughlin like) states at \( \nu = 2 + \frac{1}{3}, 4 + \frac{1}{3}, \text{ and } 4 + \frac{1}{5} \) on the electron number \( N \) is quite different than those for Laughlin states. No conclusive statement about the incompressibility (or even the sign of \( \Delta \), i.e. the nondegeneracy) of these states in the thermodynamic limit can be made based our finite-size calculations for up to eleven electrons. Since at least at \( \nu = 2 + \frac{1}{3} \) the FQH plateau has been observed experimentally (Willett et al. 1987), we have to restrict ourselves to repeating the statement (MacDonald and Girvin 1986) that the nature of the low lying states at \( \nu = 2 + \frac{1}{3}, 4 + \frac{1}{3}, \text{ and } 4 + \frac{1}{5} \) is different than of the Laughlin \( \nu = \frac{1}{3} \) and \( \frac{1}{5} \) states. In general, low

FIG. 9. The energy spectra of eight (top) to eleven (bottom) electrons in the lowest (left) and first excited (right) Landau level at the filling factor \( \nu = \frac{1}{3} \).

lying states in the lowest and nth LL’s have Laughlin like correlations only below the filling factor \( \nu = (2n + 1)^{-1} \). At fillings \( \nu \geq (2n+1)^{-1} \) in the nth LL, the correlations are different, possible incompressibility has a different origin, the excitation gap is not simply related to the difference between appropriate pseudopotential parameters, and the excitations do not contain Laughlin QP’s.

A clear signature of the non Laughlin like character of the \( n = \frac{1}{3} \) state in excited LL’s is the lack of QP type excitations at neighboring filling factors. In figure 11 we compare the energy spectra of ten electrons at equal fillings (near \( \nu = \frac{1}{3} \) ) of the lowest and first excited LL’s. In the lowest LL, lowest energy states (marked with lines) contain two QE’s (a), one QE (c), one QH (e) and two QH’s (g) in the Laughlin \( \nu = \frac{1}{3} \) state, while in the first excited LL no similar low lying states occur (note also that the energy axes in \( n = 1 \) frames are stretched by a factor of two compared to the \( n = 0 \) ones). Note also that the energies connected with lines in figure 11(a) and (g) define the pseudopotentials of a pair of appropriate interacting QP’s in the Laughlin \( \nu = \frac{1}{3} \) state.
FIG. 10. The energy gap $\Delta_{L=0}$ from the lowest $L = 0$ state to the lowest state at $L > 0$ as a function of the inverse electron number $N^{-1}$, for the lowest (a), first excited (b), and second excited (c) Landau level. Full circles: $\nu = 2n + \frac{1}{3}$; open circles: $\nu = 2n + \frac{1}{5}$. The dashed lines give linear fits for the Laughlin like incompressible ground states at $\nu = \frac{2}{5}$, $\nu = \frac{1}{3}$, and $\nu = 2 + \frac{1}{3}$, and $4 + \frac{1}{3}$ are unlikely to be incompressible for $N \to \infty$.

For a complete report of our numerical results for the lowest LL, let us add a few numbers to the tables published earlier (Fano et al. 1986). In table V we list the Laughlin ground state energy per particle (calculated including interaction with a charge compensating background, $-N^2e^2/(2R)$, the angular momentum and excitation energy of the magnetoroton minimum, and the "proper" QE and QH energies (calculated including additional fractional charge $\pm e/m$ in the background; Haldane and Rezayi 1985a, Fano et al. 1986), for $N = 10$ and 11 electrons at filling factor $\nu = \frac{1}{5}$ and for $N = 7$ and 8 electrons at $\nu = \frac{1}{3}$. The limiting values for $N \to \infty$ have been calculated using data for these and smaller values of $N$. For example, the QE and QH energies agree very well with extrapolation of the Monte Carlo results in disk geometry: $\varepsilon_{\text{QE}} = 0.073$ and $\varepsilon_{\text{QH}} = 0.0268$ (Morf and Halperin 1986).

TABLE V. The ground energy per particle $E/N$ of the Laughlin ground state, the angular momentum $L$ and excitation energy $\Delta$ of the magnetoroton minimum, and the proper quasielectron and quasi-hole energies, $\varepsilon_{\text{QE}}$ and $\varepsilon_{\text{QH}}$, for $N$ electrons at a filling factor $\nu$.

| $\nu$ | $N$ | $E/N$ | $L$ | $\Delta$ | $\varepsilon_{\text{QE}}$ | $\varepsilon_{\text{QH}}$ |
|-------|-----|-------|-----|----------|----------------|----------------|
| $1/3$ | 10  | $-0.432841$ | 5   | 0.074715 | 0.085675       | 0.030501       |
|       | 11  | $-0.430623$ | 5   | 0.075706 | 0.084658       | 0.030092       |
|       | $\infty$ | $-0.415948$ | $\infty$ | 0.063177 | 0.073724       | 0.025813       |
| $1/5$ | 7   | $-0.353494$ | 4   | 0.016245 | 0.020188       | 0.009068       |
|       | 8   | $-0.350066$ | 5   | 0.015572 | 0.019278       | 0.008510       |
|       | $\infty$ | $-0.332850$ | $\infty$ | 0.010516 | 0.014912       | 0.006288       |

FIG. 11. The energy spectra of ten electrons in the lowest (left) and first excited (right) Landau level at filling factors $\nu = \frac{1}{5}$, $\nu = \frac{1}{3}$, $\nu = 2 + \frac{1}{3}$, $\nu = 4 + \frac{1}{3}$, and $\nu = 4 + \frac{1}{5}$.
D. Grandparentage coefficients of low lying states

Typical dependences of $G_{\Lambda \alpha}$ on $\mathcal{R}$ for low lying states are plotted in figure 13 for a six electron system at $l = \frac{11}{2}$ ($\nu = \frac{2}{3}$) and $l = \frac{7}{2}$ ($\nu = \frac{1}{3}$), in the lowest and first two excited LL’s. In each frame, thicker line and larger symbols (dots) mark data corresponding to the state at $L = 0$. The CFGP profile $G(\mathcal{R})$ can be regarded as a pair correlation function, except that the probability $G$ is given as a function of a pair quantum number $\mathcal{R}$ rather than of a pair distance. In figure 13(a), the $L = 0$ state is the Jain $\nu = \frac{2}{3}$ ground state and the states with $L = 2, 3, 4$ contain a single QE–QH pair. Similarly, in figure 13(b), the $L = 0$ state is the Laughlin $\nu = \frac{1}{3}$ ground state and the states of a single QE–QH pair have $L = 2, 3, 4, 5, 6$. Typically for the low energy states in the lowest LL (or for any other short range interaction pseudopotential) at $\nu \geq \frac{1}{3}$, $G(1)$ is small, $G(3)$ is large, and for higher $\mathcal{R}$, $G$ decreases when $\mathcal{R}$ increases up to the maximum allowed value. The Jain incompressible ground states always have $G(1)$ smaller than all other states (by at least 0.035 for $N = 6$ and $\nu = \frac{2}{3}$). For Laughlin states, $G(1)$ is always negligible (less than 0.0008 for $N = 6$ and $\nu = \frac{1}{3}$). The strong maximum of $G(\mathcal{R})$ at $\mathcal{R} = 3$ means that a large number of pairs are in the ‘$\nu = \frac{1}{3}$’ pair state, on a plane given by the Laughlin correlation factor $(z_1 - z_2)^3$.

In higher LL’s, the $G_{\Lambda \alpha}(\mathcal{R})$ profiles in figure 13(c–f) differ from those in the lowest LL, but they are rather similar for different fillings ($\nu = \frac{2}{3}$ and $\frac{1}{3}$). Clearly, at any filling or $n$, the low lying states must maximally avoid parentage from pair states of highest repulsion. However, because the pseudopotential $V(\mathcal{R})$ in higher LL’s does not increase sufficiently quickly with decreasing $\mathcal{R}$ in its entire range, it appears energetically favorable to minimize total parentage from a number of pair states with lowest $\mathcal{R}$, rather from a single highest energy state with $\mathcal{R} = 1$. It appears that requirement of having small total parentage from a number of pair states of smallest $\mathcal{R}$ (smallest separation) rather than from a single pair state at $\mathcal{R} = 1$ for a density at which only one pair state can be completely avoided implies having large parentage from the $\mathcal{R} = 1$ state. As a result, the maximum of $G(\mathcal{R})$ shifts from $\mathcal{R} = 3$ (for $n = 0$) to $\mathcal{R} = 5$ (for $n = 1$) or $\mathcal{R} = 7$ (for $n = 2$). Similarly, the minimum at $\mathcal{R} = 1$ (for $n = 0$) shifts to $\mathcal{R} = 3$ (for $n = 1$) or $\mathcal{R} = 5$ (for $n = 2$). The occurrence of a large number of pairs in certain pair states of small $\mathcal{R}$ (at certain small average distance) and avoiding others defines a different type of short range correlation in the $\nu = \frac{2}{3}$ or $\nu = \frac{1}{3}$ states in higher LL’s. The natural interpretation of the maximum at $\mathcal{R} = 1$ for $n > 0$ instead of the strong minimum as for $n = 0$ seems to be the formation of electron pairs (Haldane and Rezayi 1988, Moore and Read 1991). Since the electron–electron interaction is repulsive, the formation of such pairs is a many body phenomenon and the stability of a pair requires the presence of a surrounding electron gas at an appropriate density. For a given pseudopotential, the pairs could be formed if putting two electrons in a pair state with strong repulsion greatly reduces their interaction with other electron pairs. As a result, the gain in total interaction energy in equation 20 due to reducing the contribution from pair states of intermediate $\mathcal{R}$ can
At the smallest value of \( \mathcal{R} \), the largest term in equation 26. This term is the one that contributes completely, and the dominant contribution to the energy is avoided to have short range. At a given filling factor \( \nu \), a number of pair states with largest repulsion are avoided completely, and the dominant contribution to the energy is the largest term in equation 26. This term is the one at the smallest value of \( \mathcal{R} \), for which \( \mathcal{G}(\mathcal{R}) \) does not vanish. There is a strong correlation between energy and the lowest order nonvanishing CFGP, \( \mathcal{G}(2p+1) \) for \((2p+1)^{-1} < \nu \leq (2p+1)^{-1}\). The low energy states always have significantly smaller \( \mathcal{G}(2p+1) \) than all other states with \( \mathcal{R} \geq 2p+1 \). As an example, in figure 14 we plot energies and coefficients \( \mathcal{G}(1) \) and \( \mathcal{G}(3) \) for the eigenstates of six electrons in the lowest LL at \( 2S = 19 \). The band of multiplets marked with open circles have \( \mathcal{G}(1) < 0.0043 \), and all other states have \( \mathcal{G}(1) > 0.037 \). The energy gap between the two bands in frame (a) is the result of the CFGP gap in frame (b). The states with \( \mathcal{G}(1) \approx 0 \), are approximate zero energy eigenstates of the hard core pseudopotential with \( V(1) > 0 \) and all other parameters vanishing. In the mean field CF picture, these states contain four QH's in the Laughlin \( \nu = \frac{5}{2} \) state, each with angular momentum \( l_{\text{QH}} = \frac{9}{2} \). The angular momentum dependence of energy within this band in frame (a) is very similar to that of \( \mathcal{G}(3) \) in frame (c). In particular, the \( L = 0 \) ground state, which is the \( \nu = \frac{5}{2} \) Jain state in the mean field CF picture, has the lowest \( \mathcal{G}(3) \) of all states in this band.

A closer inspection of figure 14 reveals a general tendency for the energy to increase with increasing \( \nu \), which does not show up in the \( \mathcal{G}(2p+1) \) spectrum. The \( \mathcal{G}(2p+1) \) spectrum predicts very well relative positions of energy levels with neighboring \( \nu \)'s, but, on the average, energy increases more quickly than \( \mathcal{G}(2p+1) \) when \( \nu \) is increased. This is clearly visible in figure 14(a), which shows energy of six electrons at \( l = \frac{5}{4} \) (left) and \( l = \frac{6}{4} \) (right) as a function of the grandparentage coefficient \( \mathcal{G}(1) \), calculated for the lowest (top), first excited (middle), and second excited (bottom) Landau level. Full circles, open circles, diamonds, full squares, and open squares mark data for angular momenta \( L = 0, 3, 8, 12, \) and 18, respectively (only selected values of \( L \) are shown).

![Figure 14](image1.png)

**FIG. 14.** The energy \( E( \nu) \) (a), and grandparentage coefficients \( \mathcal{G}(1) \) (b), and \( \mathcal{G}(3) \) (c), as a function of angular momentum \( L \) for the system of six electrons in the lowest Landau level at \( 2S = 19 \). Open circles: states with \( \mathcal{R} \geq 3 \), i.e. \( \mathcal{G}(1) \approx 0 \) and \( \mathcal{G}(3) > 0 \); full circles: states with \( \mathcal{R} \geq 1 \), i.e. \( \mathcal{G}(1) > 0 \).

![Figure 15](image2.png)

**FIG. 15.** The Coulomb energy of six electrons each with \( l = \frac{5}{4} \) (left) and \( l = \frac{6}{4} \) (right) as a function of the grandparentage coefficient \( \mathcal{G}(1) \), calculated for the lowest (top), first excited (middle), and second excited (bottom) Landau level. Full circles, open circles, diamonds, full squares, and open squares mark data for angular momenta \( L = 0, 3, 8, 12, \) and 18, respectively (only selected values of \( L \) are shown).
tributions of lower order terms in equation \(26\). It will be apparent from equation \(33\) that the second highest term, \(G(2p+3) V(2p+3)\), increases with \(L\) like, roughly, \(G(2p+3) \sim L(L+1)\).

The similarity of the energy and \(G(2p+1)\) spectra makes it clear that a model pseudopotential with only one nonvanishing pseudopotential parameter, \(V(1) > 0\), reproduces the main features of the spectrum for \(\nu \geq \frac{1}{4}\). Similarly, the spectrum of a model pseudopotential with a hard core, \(V(1) = \infty\), one finite parameter, \(V(3) > 0\), and all higher parameters vanishing, resembles the low energy band of the Coulomb spectrum for \(\frac{1}{4} \geq \nu \geq \frac{1}{4}\). In general, for the filling factor \(\nu\) in the range \((2p+3)^{-1} < \nu \leq (2p+1)^{-1}\), the finite energy eigenstates of the hard core pseudopotential defined as

\[
\begin{align*}
V_{HC}^{(p)}(R < 2p+1) &= \infty, \\
V_{HC}^{(p)}(R = 2p+1) &= 1, \\
V_{HC}^{(p)}(R > 2p+1) &= 0
\end{align*}
\]

are very close to those of the Coulomb pseudopotential.

The dependence of finite eigenenergies of \(V_{HC}^{(p)}\) on angular momentum \(L\) reproduces main features of the lowest band of the Coulomb spectrum.

Due to different behavior of the pseudopotential, the above conclusion does not generally hold for higher LL's. The correlation between energy and \(G(1)\) for the same filling factors \(\nu = \frac{1}{3}\) and \(\frac{1}{3}\) within the first excited LL \((n = 1)\), plotted in figure 15(c,d), is much worse than that for \(n = 0\) in figure 15(a,b). In particular, the lowest energy \(L = 0\) state is no longer the state with the smallest \(G(1)\) at either filling. Also, the Coulomb eigenstates in figure 15(c,d) are not similar to those of a hard core repulsion. For example, there is no Laughlin like state at \(\nu = \frac{1}{3}\) with \(R \geq 3\) (instead, \(G(1) > 0.06\) for all states) and no Jain like state at \(\nu = \frac{2}{3}\) with \(G(1) \approx 0.12\) (instead, \(G(1) > 0.19\) for all states).

As shown in figure 15(e,f), the correlation between energy and \(G(1)\) reappears in the second excited LL (Haldane and Rezayi 1985a). However, it is reversed and the low energy states have high values of \(G(1)\). At \(\nu = \frac{2}{3}\), the Jain like state with \(G(1) \approx 0.12\), maximally avoiding pair states with the smallest average separation and largest repulsion, is the highest energy state in its \(L = 0\) subspace. Similarly, the highest \(L = 0\) state at \(\nu = \frac{1}{3}\) is the Laughlin state with \(G(1) \approx 0.02\).

The approximation of the Coulomb pseudopotential by the hard core pseudopotential, which gives almost exact many body eigenstates in the lowest LL and predicts the sequence of the Laughlin incompressible ground states, becomes valid in higher LL's at lower density (filling factor). For \(n = 1\) and at fillings \(\nu \leq \frac{1}{2}\), the second lowest band \((R \geq 3)\) couples to the next higher one \((R \geq 1)\). Interband coupling means here that the actual eigenstates are linear combinations of hard core eigenstates from both bands and the eigenstates originating from the \(R \geq 3\) band of the hard core spectrum have some grandparentage from the \(R = 1\) pair state. However, as seen in figure 8 for only one (ground) state, the band originating from the \(\mathcal{R} \geq 5\) band is (to a good approximation) uncoupled, i.e. its eigenstates indeed all have \(\mathcal{R} \geq 5\) and are very close to the corresponding hard core states. This occurs because the decoupling of the lowest band from the rest of the spectrum depends on the behavior of the pseudopotential \(V\) at \(\mathcal{R} \geq 3\), where \(V\) of \(n = 1\) is similar to that of \(n = 0\) (see figure 9). Figure 16 shows the energy spectra of six electrons each with angular momentum \(l = \frac{25}{5}\) (top) and \(l = \frac{25}{6}\) (bottom), in the lowest (left) and first excited (right) Landau level. Open circles: states maximally avoiding pairs with largest repulsion. Dashed lines: states with one quasielectron-quasihole pair.
VII. RELATION BETWEEN PSEUDOPOTENTIAL AND OCCURRENCE OF INCOMPRESSIBLE GROUND STATES

A. Total angular momentum vs. average pair angular momentum

A very useful operator identity

\[ \sum_{i<j} \hat{L}_{ij}^2 = \hat{L}^2 + N(N - 2) \hat{l}^2 \]  \hspace{1cm} (34)

is straightforward to prove (Wójc and Quinn 1999a). Here \( \hat{L} = \sum_i \hat{l}_i \) and \( \hat{L}_{ij} = \hat{l}_i + \hat{l}_j \). Taking the expectation value of equation (34) in the state \( |l^N, L\alpha\rangle \) gives

\[ \langle \sum_{i<j} \hat{L}_{ij}^2 \rangle = L(L + 1) + N(N - 2) l(l + 1), \]  \hspace{1cm} (35)

which is independent of which multiplet \( \alpha \) of a given angular momentum \( L \) is being considered. From equation (24) we also have

\[ \langle \sum_{i<j} \hat{L}_{ij}^2 \rangle = \frac{N(N - 1)}{2} \sum_{L\alpha} \mathcal{G}_{L\alpha}(L_{12}) L_{12}(L_{12} + 1). \]  \hspace{1cm} (36)

Combining the above two equations, a nontrivial condition on the allowed values of CFGP’s is obtained. Adding the normalization condition following from equation (30), we have the following pair of constraints on the allowed CFGP’s profiles \( \mathcal{G}_{L\alpha}(\mathcal{R}) \) in a multiplet of a given \( L \)

\[ \sum_{L\alpha} \mathcal{G}_{L\alpha}(L_{12}) L_{12}(L_{12} + 1) = \frac{L(L + 1) + N(N - 2) l(l + 1)}{N(N - 1)/2}, \]  \hspace{1cm} (37)

\[ \sum_{L\alpha} \mathcal{G}_{L\alpha}(L_{12}) = 1. \]  \hspace{1cm} (38)

The minimization of the total interaction energy in a Hilbert space of a given \( N, l, M, \) and \( L \) occurs through the optimization of the CFGP profile \( \mathcal{G}(\mathcal{R}) \) (i.e., the pair correlation function), and must conform to the above constraints.

B. Harmonic repulsive interaction

It follows from equations (26), (33), and (36) that if the pseudopotential were given by

\[ V_{H}(L_{12}) = c_1 + c_2 L_{12}(L_{12} + 1), \]  \hspace{1cm} (39)

all different multiplets with the same value of total angular momentum \( L \) would be degenerate at the energy

\[ E_{L\alpha} = c_1 N(N - 1)/2 + c_2 [L(L + 1) + N(N - 2) l(l + 1)]. \]  \hspace{1cm} (40)

What is the physical meaning of the pseudopotential \( V_{H} \) which is linear in \( L_{12}^2 \)? From equation (15), \( V_{H} \) is the harmonic interaction,

\[ V_{H}(|r_i - r_j|) = c_1 - c_2 \frac{|r_i - r_j|^2}{R^2}, \]  \hspace{1cm} (41)

and, using equation (34), the many body harmonic interaction Hamiltonian can be written as

\[ H_{H} = c_1 N(N - 1)/2 + c_2 N(N - 2) l(l + 1) + B \hat{L}^2, \]  \hspace{1cm} (42)

i.e., for the harmonic repulsive interaction within an isolated LL, each \( L \) subspace is degenerate and the energy increases linearly with increasing \( L(L + 1) \).

The difference between the harmonic and actual pseudopotentials, the ‘anharmonic’ contribution \( V_{AH} = V - V_{H} \), lifts this degeneracy and the actual values of \( E_{L\alpha} \) depend on how the values of \( \mathcal{G}_{L\alpha}(L_{12}) \) are distributed, not just on the average value of \( L_{12}^2 \) for that value of \( L \). However, if the anharmonic correction \( V_{AH} \) is small, the ground state will have the lowest available value of angular momentum, \( L = L^{\text{MIN}} \). If \( V_{AH} \) is not small, different multiplets with the same \( L \) repel one another, and the splittings caused by \( V_{AH} \) can become large when \( N_L \), the number of times the multiplet \( L \) occurs, is large. As a result, the lowest multiplet with certain angular momentum \( L \) can have lower energy than multiplets of a smaller neighboring \( L' \), for which \( N_{L'} \ll N_L \). In this case, a state with \( L \) larger than \( L^{\text{MIN}} \) can become the ground
state. For example, for the system of eight electrons at $2S = 22$, the lowest energy multiplet at $L = 4$ has lower energy than the multiplets at $L = 0, 1, 2, \text{ and } 3$ (see figure 18(b) and table VII). Even if $V_{AH}$ is not small, if only $V(L_{12})$ increases with increasing $L_{12}$, then states with low angular momentum $L$ (and thus low average pair angular momentum $L_{12}$) will tend to have low energy, and states with high $L$ will tend to have high energy.

How close is the actual Coulomb pseudopotential to the harmonic one? The plots of $V$ given as a function of squared pair angular momentum $L(L+1)$, are shown in figure 18. The pseudopotentials for $n = 0$ increase more quickly than linearly with increasing $L(L+1)$ in the entire range of $L$. For $n = 1$, they do so at low values of $L$, and the dependence is almost linear close to $R = 1$. And for $n = 2$, $V$ becomes a sublinear function of $L(L+1)$ at high energy. The dashed lines give the pseudopotentials of a harmonic interaction which correspond to the best fit to the six electron spectra.

Examples of energy spectra of the six electron system in the lowest ($n = 0$) and two excited ($n = 1$ and 2) LL’s approximated by the harmonic interaction are shown in figure 18 for $l = 5$. The general trend for the energy to increase with $L$ as well as the effects due to level repulsion caused by the anharmonicity of the pseudopotentials are visible. In all frames, the highest energy state is the one with the highest $L$, and the lowest energy states have low $L$. The spectrum for $n = 1$ is less distorted from its harmonic fit than the spectra for $n = 0$ and 2. This reflects the fact that the corresponding pseudopotential, marked with squares in figure 18(b), is closer to a harmonic one than the other two, also marked with squares in figure 18(a) and (c). For $n = 1$ and 2, the ground state has the lowest available angular momentum $L = L_{\text{MIN}} = 1$. For $n = 0$, the anharmonicity of the pseudopotential is sufficiently large for the state with $L = 3 > L_{\text{MIN}}$, to become the ground state due to the level repulsion ($N_3 = 4$ is larger than $N_1 = 2$ or $N_2 = 1$). Open circles in figure 18 mark the two states at $L = 1$ and 3, which have the lowest $G(1)$ of all states in the spectrum. For $n = 0$ these states are predicted by the mean field CF picture as the states of two QE’s in the $\nu = \frac{2}{5}$ state.

### C. Comparison with atomic system: Hund’s rule

The problem of electrons in a high magnetic field, occupying single particle states of the $n$th LL (monopole harmonics with $2S > 0$, shell index $n < S$ and angular momentum $l = S + n$), can be compared to that of electrons in an atomic shell, occupying atomic states (spherical harmonics with $S = 0$ and $l = n$). In both cases the problem is that of $N$ electrons each with angular momentum $l$ in a degenerate shell of states with different values of $m$. However, the pseudopotential $V(R)$ behaves very differently in the two systems. The comparison be-
between the extreme \( n = 0 \) and \( S = 0 \) cases is presented in figure 21. The pseudopotentials for the lowest LL shell \( V_{n=0} \) and for the atomic shell \( V_{S=0} \), calculated for the same \( l = S + n \), look quite similar when \( V_{S=0} \) is plotted as a function of pair angular momentum \( R \), and \( V_{n=0} \) is plotted as a function of relative pair angular momentum \( R = 2l - L \). Therefore, while \( V_{n=0} \) decreases quickly with increasing \( R \) and attains the highest value at \( R = 1 \), the \( V_{S=0} \) does just the opposite.

The pseudopotentials in both frames in figure 20 describe the same, Coulomb electron-electron interaction \( V(r) = e^2/r \), and the origin of this difference lies in the very different Hilbert spaces. The density profiles \( \varrho_m(\cos \theta) \) for the single particle states in both cases are shown in figure 21. The \( \theta \) is the standard spherical coordinate; \( z = R \cos \theta \). The average value of \( z \) is (Wojs and Quinn 1998a)

\[
\langle S, l, m | z | S, l, m \rangle = \frac{mS}{l(l+1)} R,
\]

and \( \varrho_m(z) = \varrho_m(-z) \) for the monopole harmonics, and \( \varrho_m(z) = \varrho_m(z) \) for the spherical harmonics.

The two electron state \( |L, M\rangle \) with maximum pair angular momentum \( L = 2l - 1 \) and \( M = L \) is the single particle configuration \(|m_1 = l, m_2 = -l - 1\rangle \). For the monopole harmonics, it has high Coulomb energy, as it corresponds to two electrons tightly packed around the north pole of the sphere. On the other hand, the two electron state with minimum pair angular momentum, \( L = 0 \), can be written as \(|L = 0, M = 0\rangle = \sum_m \zeta_m |m_1 = m, m_2 = -m\rangle \), i.e. in each contributing single particle configuration \(|m_1, m_2\rangle \) the two electrons have opposite \( m \)'s. Opposite \( m \)'s mean opposite \( \langle z \rangle \)'s and large spatial separation, and therefore the pair state with \( L = 0 \) must have low interaction energy.

For the spherical harmonics, a similar analysis gives opposite answers. The state \(|m_1 = l, m_2 = -l - 1\rangle \) with maximum allowed \( L \) corresponds to two electrons spread over a large part of the sphere and avoiding one another (high density for \( m = l \) occurs at \( z \) corresponding to low density for \( m = l - 1 \), and vice versa). Therefore this state must have low Coulomb energy. In the state with minimum \( L = 0 \), built of single particle configurations \(|m_1 = m, m_2 = -m\rangle \), opposite \( m \)'s mean equal density profiles \( \varrho(\cos \theta) \), and thus small average separation and high interaction energy.

In the case of an atomic system, the reasoning based on equation 44 and the pseudopotential profile leads to the Hund’s rule. The multiplets with larger total angular momentum \( L \) have, on the average, larger pair angular momenta \( L_{ij} \) and thus lower energy. There is only one multiplet with the maximum allowed total angular momentum \( L = L_{\text{MAX}} = N - N(N - 1)/2 \); it is a single particle ‘compact droplet’ (maximum density) configuration, for \( M = L \) equal to \(|m_1, m_2, \ldots, m_N\rangle = |l, l - 1, \ldots, l - N + 1\rangle \). It has the highest value of the average pair angular momentum and hence it is very likely to be the ground state. A transition to a ground state at a neighboring lower \( L \) would require strong anharmonicity of the pseudopotential. Since relatively low multiplicities \( N_L \) occur at \( L \)'s close to \( L_{\text{MAX}} \) (\( N_{L_{\text{MAX}}-2} = 0, N_{L_{\text{MAX}}-3} \leq 1 \), etc.), \( V_{\text{AH}} \) does not affect the ordering of the levels at high \( L \). Despite this strong indication that the state with the largest \( L \) has the lowest energy in atomic systems, Hund’s rule is considered an empirical rule, that can be rigorously justified only by detailed numerical calculations. It is also noteworthy that the atomic Hund’s rule is usually of interest only for

FIG. 20. The pseudopotentials \( V(r) = e^2/r \) for a pair of electrons each with angular momentum \( l \): (a) lowest Landau level, monopole harmonics, \( n = 0 \) and \( l = S \), \( V \) plotted as a function of relative pair angular momentum \( R \); (b) atomic shell, spherical harmonics, \( S = 0 \) and \( l = n \), calculated for a radial wavefunction which localizes electrons at radius \( R \), \( V \) plotted as a function of pair angular momentum \( L \).

FIG. 21. The single particle density profiles: (a) lowest Landau level, monopole harmonics, \( n = 0 \), \( l = S \); (b) atomic shell, spherical harmonics, \( S = 0 \), \( l = n \), calculated for a radial wavefunction which localizes the electron at radius \( R \).
rather low values of \( l \) (up to the atomic \( g \) or \( h \) shell).

By analogy, the opposite rule can be formulated for monopole harmonics (FQH system on a Haldane sphere), saying that the state with the maximum \( L \) has the highest energy. Since for monopole harmonics the low energy states have low values of angular momentum \( L \) (with large multiplicities \( N_L \)), the direct analog of the atomic Hund’s rule (selecting the ground state) requires that correction \( V_{\text{AH}} \) is negligible. Under this assumption it states that the state with lowest available energy with multiplicity \( N_L \) is the state with lowest available \( L \) has the lowest energy. Both rules have been verified numerically.

For the Coulomb interaction acting in the space of monopole harmonics in the lowest \( LL \), the assumption that \( V_{\text{AH}} \) is negligible does not hold and the multiplicity \( N_L \) at low \( L \) play a crucial role in determining low energy \( L \) multiplets. In such a general case, knowing which multiplet is the ground state or which multiplets form the low energy sector without performing detailed numerical calculations is a considerably more difficult task. The prescription that the low energy states are found at those of low values of \( L \) which correspond to large \( N_L \) can be thought of as a more appropriate analog to the atomic Hund’s rule. As is the case with the atomic Hund’s rule, it is an empirical rule that must be verified numerically.

Importantly, the \( L \) multiplets for which \( N_L \) is relatively large tend to reoccur at the same values of angular momentum \( L \) when \( 2S \) is replaced by \( 2S^* = 2S - 2p(N-1) \).

In Table VI, we present, as an example, \( N_L \) as a function of \( L \) and \( 2S \) for a system of eight electrons. The values of \( 2S \) go from zero to twenty two; the values of \( L \) are shown up to eight. The \( L \) spaces which are predicted by the CF picture to form the lowest energy band are underlined. Clearly, they coincide with relatively high multiplicities \( N_L \) at the lower values of \( L \). Notice, for example, that the high \( N_L \) values at \( 2S = 19, 20, \) and 21 appear at the same angular momenta \( L \) as the allowed multiplets at \( 2S^* = 5, 6, \) and 7, respectively.

### D. Connection between Hund’s rule and avoiding pair states of large repulsion

What is the connection between the two predictions of low energy states discussed earlier, (i) the Hund’s rule argument selecting multiplets at low \( L \) with high \( N_L \) and (ii) the argument selecting multiplets that avoid large fractional grandparentage from pair states with largest repulsion? Let us first notice that whether a many body state without grandparentage from certain pair states belongs to the Hilbert space of given \( N, l \), and \( L \) depends critically on \( N_L \). It follows from equations [24] and [27] that a multiplet with \( G(\mathcal{R}) = 0 \) (e.g., for \( \mathcal{R} = 1 \)) can be constructed if the degeneracy \( N_L \) exceeds \( N_{\mathcal{R}} \), the number of terms \( (L', \alpha') \) in equation [24] with \( L_{12} \) corresponding to \( \mathcal{R} \). For example, for \( L = 0, \) the addition of angular momentum vectors, \( \mathbf{L} = L_{12} + L' \), selects only one value of \( L' \) equal to \( L_{12} \). In this case, it is guaranteed that \( N_{\mathcal{R}} \) does not exceed \( N'_{L_{12}} \), the number of all \( L' = L_{12} \) multiplets of \( N - 2 \) electrons each with angular momentum \( L \). The actual value of \( N_{\mathcal{R}} \) can be smaller than \( N'_{L_{12}} \) because of the Pauli exclusion principle, which eliminates some of the combinations of \( L' \) and \( L_{12} \). However, \( N_{L=0} > N'_{L_{12}} \) guarantees that a multiplet \( \{|N, 0\alpha\} \), a linear combination of terms in equation [24], can be constructed, for which the coefficients \( G_{0\alpha,L_{12}\alpha'}(\mathcal{R}) \) vanish simultaneously for all \( \alpha' \) and therefore so does the coefficient \( G_{0\alpha}(\mathcal{R}) \).

In general, it is difficult to determine \( N_{\mathcal{R}} \) by adding dimensions of all relevant \( L' \) spaces of \( N - 2 \) electrons because of the Pauli principle which imposes additional constraints on CFP’s in equation [24]. However, one can calculate the matrix \( \langle \alpha \mid L' \alpha' \rangle \) of coefficients \( G_{L \alpha,L' \alpha'}(\mathcal{R}) \) for all multiplets of given \( L \) (for any choice of basis states \( \alpha \), not necessarily the interaction eigenstates), and determine \( N_{\mathcal{R}} \) directly. It is clear that \( N_L \) must exceed certain minimum value for the occurrence of \( L \) multiplets which avoid grandparentage from certain (strongly repulsive) pair states. It is also clear that the minimum \( N_L \) that is required to exceed \( N'_{L_{12}} \) increases with increasing \( L \) since a larger number of angular momenta \( L' \) satisfy the addition rule, \( |L' - L_{12}| \leq L \leq L' + L_{12} \), for larger \( L \). If the multiplets with \( \mathcal{R} \geq 3, 5, \ldots \) can be constructed (belong to the Hilbert space of given \( N, l, \) and \( L \), they will be the lowest energy eigenstates of the hard core interaction defined in equation [17].

Hence, the above discussion explains the occurrence of such eigenstates at those of low

| \( 2S \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| 0   |   |   |   |   |   |   |   |   |   |
| 1   | 1 |   |   |   |   |   |   |   |   |
| 2   | 1 |   |   |   |   |   |   |   |   |
| 3   | 1 |   |   |   |   |   |   |   |   |
| 4   |   | 1 |   |   |   |   |   |   |   |
| 5   |   |   | 1 |   |   |   |   |   |   |
| 6   |   |   |   | 1 |   |   |   |   |   |
| 7   |   |   |   |   | 1 |   |   |   |   |
| 8   |   |   |   |   |   | 1 |   |   |   |
| 9   |   |   |   |   |   |   | 1 |   |   |
| 10  |   |   |   |   |   |   |   | 1 |   |
| 11  |   |   |   |   |   |   |   |   | 1 |
| 12  | 1 | 1 |   |   |   |   |   |   |   |
| 13  | 1 | 1 | 1 |   |   |   |   |   |   |
| 14  | 1 | 1 |   | 1 |   |   |   |   |   |
| 15  | 1 |   | 1 | 1 |   |   |   |   |   |
| 16  |   | 1 | 1 | 1 |   |   |   |   |   |
| 17  |   |   | 1 | 1 | 1 |   |   |   |   |
| 18  |   |   |   | 1 | 1 | 1 |   |   |   |
| 19  |   |   |   |   | 1 | 1 | 1 |   |   |
| 20  |   |   |   |   |   | 1 | 1 | 1 |   |
| 21  |   |   |   |   |   |   | 1 | 1 | 1 |
| 22  |   |   |   |   |   |   |   | 1 | 1 |

TABLE VI. The number \( N_L \) of independent multiplets at angular momentum \( L \) for eight electrons as a function of \( 2S \) for \( 0 \leq 2S \leq 22 \). Only values of \( L \) up to 8 are included in the table.
values of $L$ which have high multiplicity $N_L$.

Another problem that still needs clarification is whether the multiplets with $R \geq 3, 5, \ldots$ are the eigenstates of the actual (not strictly hard core) interaction pseudopotential $V(R)$ (e.g., the Coulomb interaction in a given $L$), and if they have low energy. In other words, what is the relevant measure of the ‘short range’ character of electron–electron interaction in the lowest $L$? Or, what is the condition for $V(R)$ to act like hard core repulsion and have the energy spectrum characteristic of the FQH effect, with low energy states that have $R \geq 3, 5, \ldots$? Clearly, whether the ground state and other low lying multiplets tend to avoid grandparentage from pair states with $R = 1, 3, \ldots$ depends not only on whether $V(R)$ is a decreasing function of $R$, but on how quickly it decreases with $R$ as well. This is because the sequence of CFGP’s of a given eigenstate $|L\alpha\rangle$ are mutually connected through the normalization condition given by equations 17, and the nontrivial condition 18. For example, it turns out that the $\nu = \frac{1}{2}$ state with $G(1) \approx 0$ always has the largest $G(3)$ of all states. Therefore, $V(R)$ must decrease sufficiently quickly with increasing $R$ for the state with $R \geq 3$ to be the ground state at the $\nu = \frac{1}{2}$ filling.

E. Definition of short range pseudopotential

The condition for the occurrence of the Laughlin incompressible $\nu = (2p + 1)^{-1}$ ground states with $G(R < 2p + 1) \approx 0$ (and generally, for the occurrence of low energy states with $G(R < 2p + 1) \approx 0$ and low $G(2p + 1)$ for $\nu < (2p + 1)^{-1}$), is that pseudopotential $V(L)$ increases more quickly than linearly with increasing $L(L + 1)$. In the two top frames of figure 22 we show the energy spectra of a system of six electrons each with angular momentum $l = \frac{1}{2}$, calculated for a model pseudopotential

$$V_\beta(L) = [L(L + 1)]^\beta$$

with $\beta > 1$ and $\beta < 1$. In the two bottom frames we plot the corresponding spectra of the CFGP corresponding to the highest pseudopotential parameter, $G(1)$. The $G(1)$ spectra look quite similar for $\beta = 1.1$ and 0.9. In particular, in both cases there is one state in the spectrum (marked with a large open circle) whose $G(1)$ almost vanishes. At first sight, the energy spectra also look similar. Both of them reveal overall tendency to increase energy with increasing $L$, and in both of them the larger width of $L$ subspectra coincides with larger $N_L$. However, a closer inspection shows that the two spectra look like one another’s vertical reflections. For $\beta > 1$, the states with low $G(1)$ tend to have low energy. For example, within the $L = 0$ subspace, the state with $G(1) \approx 0$ (large open circle, this is the Laughlin like $\nu = \frac{1}{2}$ state) has the lowest energy, and the state with the maximum $G(1) \approx 0.3$ (large open square) has the highest energy. On the contrary, for $\beta < 1$, the states with low $G(1)$ tend to have high energy. For example, for $L = 0$, the state with minimum $G(1)$ has the highest energy and vice versa. Clearly, the behavior of energy as a function of $G(1)$ is opposite for $\beta > 1$ and $\beta < 1$. This can be demonstrated even more clearly by comparing the expectation values of the $V_\beta$ interaction in the same states (instead of comparing the eigenspectra). In this case the ordering of energies within each $L$ subspace is exactly reversed.

The exponent $\beta$ is the relevant measure of the ‘short range’ character of a pseudopotential $V_{\beta}$. The condition given by equation 17 that has been used to define an ideal short range (hard core) pseudopotential throughout this paper can be rewritten as $\beta \gg 1$. The pseudopotentials with $\beta > 1$ define a class of ‘short range’ repulsive interactions characterized by similar behavior of energy spectra and wavefunctions. For $\beta \to \infty$, the wavefunctions and structure of energy spectra converge to those of the model interaction in equation 17 at the filling factor $\nu = (2p + 1)^{-1}$ the ground state is given exactly by the Laughlin wavefunction (or by its spherical form given in (Haldane 1983)). The pseudopotentials $V_{\beta}$ with $\beta < 1$ belong to a separate class of interactions, characterized by their own (common) behavior of energy spectra and wavefunctions (Wojs and Quinn 1998a), different from those of the short range class with $\beta > 1$. In particular, Laughlin incompressible $\nu = (2p + 1)^{-1}$ ground states with $R \geq 2p + 1$ occur only for $\beta > 1$. The harmonic interaction with $\beta = 1$ separates those two classes and does not belong to either one.
F. Pseudopotentials of other 2D systems

The Coulomb pseudopotential for the lowest LL is not strictly of the form $V_0(L)$. However, as shown in figure 18(a), it increases more quickly than linearly with an increase of $L(L+1)$ in entire range of $L$. In consequence, the low energy states are those with $G(1)\approx G(3)\approx \ldots \approx G(2p-1)\approx 0$ and the lowest value of $G(2p+1)$, and the $L=0$ ground states at $2S=(2p+1)(N-1)$ are Laughlin incompressible $\nu=(2p+1)^{-1}$ states. In general, the low lying states of an interacting many body system at filling factor $\nu\sim (2p+1)^{-1}$ tend have Laughlin correlations (the states with lowest energy have vanishing grandparentage from pair states with $\mathcal{R}<2p-1$ and smallest grandparentage from $\mathcal{R}=2p-1$), if the pseudopotential $V(\mathcal{R})$ decreases as a function of $\mathcal{R}$ in the entire range, and decreases more quickly than the harmonic pseudopotential $V_{\mathcal{H}}$ in the vicinity of $\mathcal{R}=2p+1$. On a sphere, $V_{\mathcal{H}}$ increases linearly as a function of the squared pair angular momentum $L(L+1)$; on a plane it decreases linearly as a function of the angular momentum of the relative motion. The condition for Laughlin correlations can be conveniently expressed in terms of the following anharmonicity parameter

$$\xi(\mathcal{R}) = V(\mathcal{R}) - V_{\mathcal{H}}(\mathcal{R}),$$

where $V_{\mathcal{H}}(\mathcal{R})$ is the harmonic extrapolation of $V(\mathcal{R}+4)$ and $V(\mathcal{R}+2)$ at $\mathcal{R}$. The condition states that Laughlin correlations (avoiding pairs with $\mathcal{R}\leq 2p-1$) occur at $\nu\sim (2p+1)^{-1}$ if $\xi(2p-1)>0$. In figure 23 we plot $\xi(\mathcal{R})$ for a number of different 2D electron systems in a high magnetic field. By analogy to the electron gas in the lowest LL, one could expect Laughlin like correlations in these systems, and we try to interpret them in terms of mean field CF’s.

As shown in figure 23(a) and in figure 18(b,c), the electron pseudopotential in excited LL’s is of the short range type only for $\mathcal{R}\geq 2n+1$. In consequence, the ground states at Laughlin–Jain filling factors $\nu\geq (2n+1)^{-1}$ do not have Laughlin correlations (in contrast to the states at the same filling of the lowest LL). Only at lower filling factors, $\nu< (2n+1)^{-1}$, where the part of the pseudopotential which does not decrease quickly enough with increasing $\mathcal{R}$ is completely avoided and does not affect the lowest energy eigenstates, do these eigenstates have low grandparentage from pair states with large repulsion. In particular, the Laughlin like incompressible ground states occur only at $\nu\sim (2n+1)^{-1}$. This explains the compressibility of the $\nu=2+\frac{1}{3}$, $4+\frac{1}{3}$, and $4+\frac{1}{6}$ ground states (or, at least, different correlations and thus different origin of their incompressibility), and the incompressibility of the $\nu=2+\frac{1}{6}$ ground state, observed in figure 18.

Another example of a system interacting through the short range pseudopotential is the system of charged excitons ($X^{-}$, two electrons bound to a valence hole) or biexcitons ($X_{2}^{+}$, three electrons bound to two valence holes) in the lowest LL, which has been shown to have Laughlin like incompressible ground states (Wójs, Hawrylak, and Quinn 1998c, Wójs, Hawrylak, and Quinn 1999b). This is confirmed in figure 23(b), where we also plot $\xi(\mathcal{R})$ calculated for the interaction of an $X^{-}$ or an $X_{2}^{+}$ with an electron $e^{-}$. Note that for a pair of distinguishable particles, $\mathcal{R}$ can take on any integer value, and that the pseudopotentials involving $X^{-}$ or $X_{2}^{+}$ have hard core ($V=\infty$) at a number of smallest values of $\mathcal{R}$. Clearly, the Laughlin like $e^{-}X^{-}$ or $e^{-}X_{2}^{+}$ correlations described by a Jastrow prefactor in the wavefunction will only occur at odd values of $\mathcal{R}$ (Wójs et al. 1999c).

If electrons are confined in parallel 2D layers separated by a distance $d$, the inter-layer repulsion $V_{d}(r) = e^{2}/\sqrt{r^{2}+d^{2}}$ can result in the inter-layer Laughlin correlations, unless $d$ is larger than the characteristic separation between electrons in each layer ($\sim \sqrt{2R/\nu\lambda}$). The plots of $\xi(\mathcal{R})$ for the pseudopotentials $V_{d}(\mathcal{R})$ in the lowest LL are shown in figure 23(c). When $d$ is large, $V_{d}(r) \approx (1-\frac{1}{2}(r/d)^{2}+\ldots)/d$ becomes essentially harmonic at small $r$, $V_{d}(\mathcal{R})$ becomes harmonic at small $\mathcal{R}$ and the inter-layer correlations disappear. Since $V_{d}(r)$ is a good approximation to an effective 2D potential in a quasi-2D layer of finite width ($\sim 5d$), figure 23(c) shows also the destruction of the FQH effect in a single wide quantum well (Shayegan et al. 1990).

The CF hierarchy uses the mean field approach for the QP’s and therefore should fail when applied to partially filled QP shells unless the QP pseudopotential has short
range. In states with completely filled QE shells (where \( \nu_{\text{QE}} \) is an integer), the gap for creating a new type of QE–QH pair makes the nondegenerate \( L = 0 \) ground state an incompressible fluid state regardless of the form of the QE pseudopotential. For example, the Jain incompressible \( \nu = \frac{1}{3} \) state is obtained when QE’s of the \( \nu = \frac{1}{3} \) parent state fill one shell (\( \nu_{\text{QE}} = 1 \)). For partially filled QP shells, the CF hierarchy correctly predicts daughter incompressible ground states only at certain fractional QP filling factors but not at others. A quick look at the QP pseudopotentials in figure 11(a,g) for ten electrons (as well as in figures 1(c) and 2(d,h) for eight electrons) allows the prediction of filling factors at which the QP’s indeed form a Laughlin ground state. In figure 23(d) we plot \( \xi(\mathcal{R}) \) for QP’s of Laughlin \( \nu = \frac{1}{3} \) and \( \frac{1}{2} \) states, obtained in diagonalization of eleven and eight electron systems, respectively. It can be readily seen that Laughlin QH’s should form a stable Laughlin \( \nu_{\text{QH}} = \frac{1}{3} \) state of their own. It follows from equation 1 that the \( \nu_{\text{QH}} = \frac{1}{3} \) daughter state of the \( \nu = \frac{1}{3} \) parent state corresponds to Jain \( \nu = \frac{1}{3} \) state of electrons. Indeed, this state is an incompressible eight electron ground state in figure 1(f). On the other hand, the \( \nu_{\text{QH}} = \frac{1}{3} \) QH state and the corresponding \( \nu = \frac{1}{3} \) electron state will be compressible. Indeed, the eight electron ground state in figure 1(d) does not even have \( L = 0 \). The \( \nu_{\text{QH}} = \frac{1}{3} \) might be incompressible but with a much smaller gap than that of \( \nu_{\text{QH}} = \frac{1}{3} \), what would lead to weak incompressibility of the \( \nu = \frac{1}{3} \) electron state. Indeed, the gap above the \( L = 0 \) ground state of six electrons at \( 2S = 17 \) is very small. For partially filled QE shells, the \( \nu_{\text{QE}} = \frac{1}{4} \) (\( \nu = \frac{1}{2} \)) and \( \nu_{\text{QE}} = \frac{1}{5} \) (\( \nu = \frac{3}{5} \)) states are expected to be compressible, and the \( \nu_{\text{QE}} = \frac{1}{2} \) (\( \nu = \frac{3}{4} \)) state could be weakly incompressible. These predictions are in perfect agreement with numerical results for finite systems (Wójs and Quinn 2000), and we presume that taking into account the behavior of pseudopotentials of interaction between QE’s and between QH’s in different stable Laughlin states on all levels of hierarchy explains naturally all observed odd denominator FQH fillings and allows the prediction of their relative stability without using trial wavefunctions involving multiple LL’s and projections onto the lowest LL. The inconsistencies of the original QP hierarchy picture (Halvane 1983, Laughlin 1984, Halperin 1984): the appearance of some observed fractions on high hierarchy levels and the actual compressibility of some fractions predicted on lower levels, are removed by noticing that Laughlin QP’s of a given type form incompressible Laughlin states of their own only at certain filling factors.

G. Prescription for low energy multiplets

The discussion presented in the preceding sections can be summarized in the form of a general prescription for the angular momentum multiplets forming the low energy sector in FQH systems.

(i) The pseudopotential \( V(\mathcal{R}) \) describing the Coulomb repulsion in an isolated (lowest or excited) LL decreases when relative pair angular momentum \( \mathcal{R} \) increases, i.e. when the pair angular momentum \( L_{12} \) decreases.

(ii) Multiplets with lower total angular momentum \( L \) have lower expectation value of the pair angular momentum \( L_{12} \), and thus lower energy.

(iii) The energy levels at the same \( L \) repel one another due to the anharmonicity of \( V(\mathcal{R}) \). As a result, low values of total angular momentum \( L \) for which many independent multiplets occur are more likely to have some states at lower energy than neighboring \( L \) values with few multiplets.

(iv) Relatively higher multiplicities \( N_L \) tend to recur at the same values of \( L \) for single particle angular momenta \( l^* = l - p(N - 1) \). These values coincide with predictions of the mean field CF picture.

(v) The many body Hilbert spaces corresponding to low angular momenta \( L \) with large multiplicities \( N_L \) (as predicted by the mean field CF picture) contain some states with small grandparentage from pair states of largest repulsion.

(vi) If \( V(\mathcal{R}) \) decreases more quickly with decreasing \( \mathcal{R} \) than the harmonic pseudopotential, the low lying many body states avoid grandparentage from pair states of largest repulsion, and thus occur at total angular momenta predicted by the mean field CF picture.

(vii) The gap above the low energy states that avoid grandparentage from pair states of largest repulsion is governed by the appropriate difference of pseudopotential parameters. This gap does not collapse in the thermodynamic limit.

(viii) At filling factors at which the low energy band separated from the rest of the spectrum by a gap contains only a nondegenerate (singlet) \( L = 0 \) ground state, the system is incompressible and exhibits the FQH effect.

VIII. SUMMARY

We have shown that the success of the mean field composite Fermion (CF) picture in correctly and simply selecting the band of lowest energy multiplets of fractional quantum Hall (FQH) systems is not due to a cancellation between Coulomb and Chern–Simons interactions among fluctuations, which are described by totally different energy scales. The reason for the success is related to the nature of the Coulomb pseudopotential \( V(\mathcal{R}) \) in the lowest Landau level (LL).

We have identified an exact dynamical symmetry of the hard core repulsive (HCR) pseudopotential. Due to this symmetry, the many body energy spectrum splits into bands of eigenstates which avoid an increasing number of pseudopotential parameters of largest repulsion (the wavefunctions of these eigenstates contain Jastrow prefactors \( \prod_{j<j'}(z_j-z_j')^m \) with increasing exponents \( m \)). The bands are separated by gaps which are associated with
the difference of appropriate pseudopotential parameters and do not collapse in the thermodynamic limit. The incompressibility of Laughlin \( \nu = (2p+1)^{-1} \) states in a system with HCR interactions results from the fact that the nondegenerate \( (L = 0) \) ground state is the only state in its (lowest energy) band at these filling factors. The mean field CF picture can be applied to such systems.

We have defined the class of 'short range' (SR) pseudopotentials \( V(\mathcal{R}) \), for which the Laughlin correlations (avoiding strongly repulsive pair states) minimize the total interaction energy. The occurrence of distinct bands and Laughlin–Jain incompressible ground states in the energy spectrum of systems with SR interactions is a consequence of weakly broken dynamical symmetry of the HCR. The pseudopotential \( V(\mathcal{R}) \) has the SR character in a given range of relative pair angular momentum \( \mathcal{R} \) if \( V(\mathcal{R}) \) decreases in this range more quickly as a function of \( \mathcal{R} \) than the harmonic pseudopotential. The Coulomb repulsion in the lowest LL belongs to the SR class in entire range of \( \mathcal{R} \), and hence Laughlin correlations occur at all Laughlin filling factors \( \nu = (2p+1)^{-1} \).

We have found that the pseudopotentials in excited LL’s decrease more slowly with increasing \( \mathcal{R} \) and do not have SR character at the smallest values of \( \mathcal{R} \). As a result, the Laughlin correlations occur in excited LL’s only below certain filling factor. For example, we have shown that the \( \nu = 2 + \frac{1}{4} \) state does not have Laughlin correlations in the first excited LL, while \( \nu = 4 + \frac{1}{4} \) and \( 4 + \frac{1}{5} \) states do not have such correlations in the second excited LL. On the other hand, the \( \nu = 2 + \frac{1}{4} \) state has Laughlin correlations and an excitation gap comparable to the \( \nu = \frac{3}{2} \) state. Because the mean field CF model describes systems with Laughlin correlations, it is only valid at lower fillings of excited LL’s.

The CF hierarchy uses the mean field approach for the quasiparticles (QP’s) and therefore should fail unless the QP pseudopotential has SR. We have found that QP’s have Laughlin correlations at some of the Laughlin filling factors but not at others. This explains incompressibility of hierarchy ground states at \( \nu = \frac{1}{4} \) and compressibility at such hierarchy fractions as \( \nu = \frac{3}{4} \) or \( \frac{7}{4} \). Also, since the Laughlin quasielectron (QE) and quasihole (QH) energies are governed by different electron pseudopotential parameters, the QE energy is higher than the QH energy.

We have also studied the validity of the atomic Hund’s rule for systems with different pseudopotentials and shown that a modified Hund’s rule remains valid for FQH systems on a Haldane sphere. According to this rule, the FQH states with small total angular momentum \( L \) tend to have lower energy than states with large \( L \). This rule is strict for the harmonic interaction for which energy is completely independent of correlations. Strong anharmonicity of the pseudopotential can invalidate this rule and favor either Laughlin correlated states at low \( L \) with large number of multiplets if the pseudopotential has SR, or other type of correlations (e.g., possible pairing) if the pseudopotential is subharmonic.

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