Note on Archimedean property in ordered vector spaces

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Abstract: It is shown that an ordered vector space $X$ is Archimedean if and only if $\inf_{\tau \in \{\tau\}, y \in L} (x_\tau - y) = 0$ for any bounded decreasing net $x_\tau \downarrow$ in $X$, where $L$ is the collection of all lower bounds of $\{x_\tau\}_\tau$. We give also a characterization of the almost Archimedean property of $X$ in terms of existence of a linear extension of an additive mapping $T: Y_+ \rightarrow X_+$ of the positive cone $Y_+$ of an ordered vector space $Y$ into $X_+$.

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1 Introduction

In this note we deal with vector spaces over reals. A subset $K$ of a vector space $X$ is called a cone if it satisfies

$$K \cap (-K) = \{0\}, \quad K + K \subseteq K \quad \text{and} \quad rK \subseteq K$$

for all $r \geq 0$. A cone $K$ is said to be generating if $K - K = X$. Given a cone $X_+$ in $X$, we say that $(X, X_+)$

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is an ordered vector space. The partial ordering $\leq$ on $X$ is defined by

$$x \leq y \text{ if } y - x \in X_+.$$ 

The space $(X, X_+)$ is also denoted by $(X, \leq)$ or simply by $X$ if $\leq$ is well understood.

In what follows, we denote by $X$ an ordered vector space. For every $x, y \in X$, the (possibly empty if $x \not\leq y$) set

$$[x, y] = \{z : x \leq z \leq y\}$$

is called order interval. A subset $A \subseteq X$ is said to be order convex if for all $a, b \in A$, we have $[a, b] \subseteq A$. Given $B \subseteq X$, the smallest order convex subset $[B]$ of $X$ containing $B$ is called the order convex hull of $B$. It is immediate to see that

$$[B] = (B + X_+) \cap (B - X_+) = \bigcup_{a,b \in B} [a, b].$$

Any order convex vector subspace of $X$ is called an order ideal. In the case when $Y \subseteq X$ is an order ideal, the quotient space $X/Y$ is partially ordered by:

$$[0] \leq [f] \text{ if } \exists q \in Y \text{ with } 0 \leq f + q .$$

The order convexity of $Y$ is needed for the property:

$$[f] \leq [g] \leq [f] \Rightarrow [f] = [g].$$
which guarantees that the canonical image $X_+$ is a cone in $X/Y$. Indeed if $[f] \leq [g] \leq [f]$, then

$$q_1 \leq g - f \quad \text{and} \quad -q_2 \leq f - g$$

for some $q_1, q_2 \in Y$. Thus, $q_1 \leq g - f \leq q_2$. Since $Y$ is order convex, then $g - f \in [q_1, q_2] \subseteq Y$ and $[g] = [f]$.

Let $(x_n)$ be a sequence in $X$ and $u \in X_+$. The sequence $(x_n)$ is said to be **uniformly convergent** to a vector $x \in X$, in symbols $x_n \overset{(u)}{\rightarrow} x$, if

$$x_n - x \in [-\varepsilon_n u, \varepsilon_n u]$$

for some sequence $(\varepsilon_n)$ of reals such that $\varepsilon_n \downarrow 0$. A subset $A \subseteq X$ is said to be **uniformly closed** in $X$ if it contains all limits of all $u$-uniformly convergent sequences $(x_n) \subseteq A$ for all $u \in X_+$.

An ordered vector space $X$ is said to be **Archimedean** (we say also that $X_+$ has the **Archimedean property**) if

$$[(\forall n \geq 1) \ n y \leq x \in X_+] \Rightarrow [y \leq 0].$$

(1)

It is easy to see that in (1) a vector $x \in X_+$ can be replaced by any $x \in X$. Any subspace of an Archimedean ordered vector space is Archimedean. It is well known that $X$ is Archimedean iff $\inf_{n \geq 1} \frac{1}{n} y = 0$ for every $y \in X_+$. It is worth to remark that if $X$ admits a linear topology $\tau$ for which its cone $X_+$ is closed, then $X$ is Archimedean.
Indeed, assume $ny \leq x$ for all $n \geq 1$ and some $x, y \in X$. Then
\[ \frac{1}{n}x - y \in X_+ \quad \text{and} \quad \frac{1}{n}x - y \to -y \]
imply $-y \in X_+$ or $y \leq 0$.

$X$ is called **almost Archimedean** if
\[ ((\forall n \in \mathbb{Z}) \ ny \leq x \in X_+) \Rightarrow [y = 0], \]
for every $x \in X_+$. Clearly, $X$ is almost Archimedean iff
\[ \bigcap_{n \geq 1} \left[ -\frac{1}{n}x, \frac{1}{n}x \right] = \{0\}, \]
for every $x \in X_+$. It follows immediately, that any subcone of an almost Archimedean cone is almost Archimedean. A standard example of an ordered vector space which is not almost Archimedean is $(\mathbb{R}^2, \leq_{lex})$, the Euclidean plain with the lexicographic ordering. If $Y \subseteq X$ is an order ideal such that $X/Y$ is almost Archimedean, then $Y$ is uniformly closed. Indeed, let $y_n \xrightarrow{u} x$ for some $(y_n) \subseteq Y$, $u \in X_+$, and $x \in X$. We may assume that
\[ y_n - x \in \left[ -\frac{1}{n}u, \frac{1}{n}u \right] \quad (\forall n \geq 1). \]
Hence
\[ [-x] = [y_n - x] \in \left[ -\frac{1}{n}[u], \frac{1}{n}[u] \right] \quad (\forall n \geq 1). \]
Since $X/Y$ is almost Archimedean, then $[-x] = [0]$, and thus $x \in Y$. If $Y = \{0\}$, the converse is obviously true. Thus $X$ is almost Archimedean iff $\{0\}$ is uniformly
closed. In general, the question whether or not $X/Y$ is almost Archimedean assuming an order ideal $Y$ to be uniformly closed is rather nontrivial (it has a positive answer, for example in vector lattice setting (cf. [3, Thm.60.2])). Any Archimedean ordered vector space is clearly almost Archimedean. The converse is not true even when $\dim(X) = 2$.

**Example 1.** Let $\Gamma$ a set containing at least two elements and let $Y$ be the space of all bounded real functions on $\Gamma$, partially ordered by:

$$f \leq g \text{ if either } f = g \text{ or } \inf_{t \in \Gamma} [g(t) - f(t)] > 0.$$ 

The space $(Y, \leq)$ is almost Archimedean but not Archimedean. Indeed, it can be seen easily that $\inf_{n \geq 1} \frac{1}{n} f$ does not exist for any $0 \neq f \in Y_+$.

An ordered vector space $X$ is said to be a **vector lattice** (or a **Riesz space**) if every nonempty finite subset of $X$ has a least upper bound. $X_+$ is said to be **minihedral** if $X$ is a vector lattice. Any almost Archimedean vector lattice is Archimedean, indeed

$$\left[ (\forall n \geq 1)(ny \leq x \in X_+) \right] \Rightarrow$$

$$\left[ (\forall n \geq 1) [-x \leq n \sup(y, 0) \leq \sup(x, 0) = x] \right] \Rightarrow$$

$$\left[ \sup(y, 0) = 0 \right] \Rightarrow \left[ y \leq 0 \right].$$

For further details on ordered vector spaces we refer to the book [1].
2 A characterization of Archimedean ordered vector spaces

The following characterization of the Archimedean property is well known in the vector lattice case (see, for example, [3, Thm.22.5]). In the general setting of ordered vector spaces, it has been published recently [2, Prop.2] as an auxiliary fact with an incorrect proof of the implication \((b) \Rightarrow (a)\). Below, we fill the gap in the proof.

**Theorem 1.** For an ordered vector space \(X\), the following are equivalent:

(a) \(X\) is Archimedean.

(b) For any decreasing net \(x_\tau \downarrow \geq d\) in \(X\),

\[
\inf_{\tau \in \{\tau\}, y \in L} (x_\tau - y) = 0,
\]

where \(L = \{y \in X : (\forall \tau \in \{\tau\})[y \leq x_\tau]\}\) is the collection of all lower bounds of \(\{x_\tau\}_\tau\).

**Proof:** \((a) \Rightarrow (b)\): Let \(X\) be Archimedean, \(x_\tau \downarrow \geq d\). Assume \(z \in X\) satisfies \(z \leq x_\tau - y\) for all indexes \(\tau\) and for all \(y \in L\). Since \(0 \leq x_\tau - y\) for all \(\tau\), to complete the proof of the implication, it is enough to show that \(z \leq 0\).

As \(y + z \leq x_\tau\) for all \(\tau\) and all \(y \in L\), we obtain that \(y + z \in L\) for every \(y \in L\). It follows by the induction,

\[y + nz \in L \quad (\forall y \in L)(\forall n \in \mathbb{N}).\]

In particular, \(d + nz \leq x_{\tau_0}\) (and hence, \(nz \leq x_{\tau_0} - d\)) for
some \( \tau_0 \in \{ \tau \} \) and all \( n \in \mathbb{N} \). By the condition
\[
nz \leq x_{\tau_0} - d \quad (\forall n \in \mathbb{N}) ,
\]
the Archimedean property of \( X_+ \) implies \( z \leq 0 \), what is required.

(b) \( \Rightarrow \) (a): Let \( x \in X_+ \), \( ny \leq x \) for all \( n \in \mathbb{N} \). We have to show that \( y \leq 0 \). Denote
\[
L = \left\{ w \in X : (\forall n \geq 1) \ w \leq \frac{1}{n} x \right\} .
\]
Clearly, \( y \in L \). Given \( u \in L \), then \( 0 \leq \frac{1}{n} x - u \) for all \( n \geq 1 \). By the hypothesis applied to the decreasing sequence \( \frac{1}{n} x \downarrow y \), the following infima exist, and
\[
\inf_{n \geq 1, u \in L} \left( \frac{2}{n} x - u \right) = \inf_{n \geq 1, u \in L} \left( \frac{1}{n} x - u \right) = 0 .
\]
Hence,
\[
\inf_{n \geq 1, u \in L} \left[ \left( \frac{2}{n} x - u \right) - y \right] = -y + \inf_{n \geq 1, u \in L} \left( \frac{2}{n} x - u \right) = -y .
\]
Since
\[
0 \leq \left( \frac{1}{n} x - u \right) + \left( \frac{1}{n} x - y \right) = \left( \frac{2}{n} x - u \right) - y
\]
for all \( n \geq 1 \), then
\[
0 \leq \inf_{n \geq 1, u \in L} \left[ \left( \frac{2}{n} x - u \right) - y \right] = -y .
\]
Then, \( y \leq 0 \) and hence \( X \) is Archimedean. ■
3 A characterization of almost Archimedean ordered vector spaces

Here we present a characterization of the almost Archimedean property of $X_+$ in terms of existence of an extension of an additive mapping $T : Y_+ \to X_+$ of the positive cone $Y_+$ of an ordered vector space $Y$ into $X_+$ to a linear operator from $Y$ into $X$. The existence of such an extension is well known in the Archimedean setting (see, for example, [1, Lem.1.26], [4, Lem.83.1]).

**Theorem 2.** For an ordered vector space $X$, the following statements are equivalent:

(i) $X$ is almost Archimedean.

(ii) For any ordered vector space $Y$ and any additive mapping $T : Y_+ \to X_+$, there is an extension of $T$ to a linear operator from $Y$ to $X$.

(iii) For any additive mapping $T : \mathbb{R}_+ \to X_+$, there is an extension of $T$ to a linear mapping from $\mathbb{R}$ to $X$.

**Proof:** $(i) \Rightarrow (ii)$: Since on an algebraic complement $Y_0$ of $Y_+ - Y_+$ in $Y$, an extension of $T$, say $S$, can be chosen as any linear operator, we only need to obtain an extension of an additive mapping $Y_+ \xrightarrow{T} X_+$ to a linear operator $(Y_+ - Y_+) \xrightarrow{S} X$.

For each $y \in (Y_+ - Y_+)$ pick $y_1, y_2 \in Y_+$ with $y = y_1 - y_2$ and put

$$Sy = T(y_1) - T(y_2).$$
It is routine to see that $S$ is well defined and additive. The additivity of $(Y_+ - Y_+) \xrightarrow{S} X$ implies that $S$ is $\mathbb{Q}$-homogeneous. To complete the proof, it is enough to show that $S$ is $\mathbb{R}_+$-homogeneous on $Y_+$, where it coincides with $T$. Thus we need to show that $T : Y_+ \to X_+$ is $\mathbb{R}_+$-homogeneous. We shall use the following elementary remark:

\begin{align*}
[x, y \in [q, p]] &\Rightarrow [x - q, y - q \in [0, p - q]] \\
[x - y = (x - q) - (y - q) \in [-(p - q), p - q]]. & \quad (2)
\end{align*}

Let $\mathbb{Q}_+ \ni r_n \uparrow a \in \mathbb{R}$, $\mathbb{Q}_+ \ni r'_n \downarrow a$, $u \in Y_+$. Then

$$r_n T(u) = T(r_n u) \leq T(au) \leq T(r'_n u) = r'_n T(u).$$

But also:

$$r_n T(u) \leq aT(u) \leq r'_n T(u).$$

That is

$$T(au), aT(u) \in [r_n T(u), r'_n T(u)]. \quad (3)$$

Applying (2) in (3), we obtain

$$T(au) - aT(u) \in [-(r'_n - r_n)Tu, (r'_n - r_n)Tu] \quad (\forall n \in \mathbb{N}).$$

Since $X$ is almost Archimedean and $r'_n - r_n \downarrow 0$, we get

$$T(au) - aT(u) = 0,$$

what is required.

$(ii) \Rightarrow (iii)$: It is trivial.

$(iii) \Rightarrow (i)$: Let $x \in X_+$, $y \in X$ be such that

$$-\frac{1}{n} x \leq y \leq \frac{1}{n} x \quad (\forall n \geq 1). \quad (4)$$
Take a function $f : \mathbb{R} \to \mathbb{R}$ which is $\mathbb{Q}$-linear but not $\mathbb{R}$-linear and define an additive mapping $T : \mathbb{R}_+ \to X$ by

$$T(a) = ax + f(a)y \quad (a \in \mathbb{R}_+).$$

Then $T$ maps $\mathbb{R}_+$ into $X_+$. Indeed, $T(0) = 0$ and if $0 < a$ then $0 \leq a - \frac{|f(a)|}{n_0}$ for some large enough $n_0$. It follows from (4) that $-\frac{|f(a)|}{n}x \leq \pm f(a)y \leq \frac{|f(a)|}{n}x$ for all $n \geq 1$. In particular, $-\frac{|f(a)|}{n_0}x \leq f(a)y$, and therefore

$$0 \leq \left(a - \frac{|f(a)|}{n_0}\right)x \leq ax + f(a)y = T(a)$$

Take a linear extension $S$ of $T$ to all of $\mathbb{R}$. Then $S$ (and hence $T$) must be $\mathbb{R}_+$-homogeneous on $\mathbb{R}_+$ which is only possible if $y = 0$. ■

References

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