Active-Site Motion and Pattern Formation in Self-Organised Interface Depinning

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We study a dynamically generated pattern in height gradients, centered around the active growth site, in the steady state of a self-organised interface depinning model. The pattern has a power-law tail and depends on interface slope. An approximate integral equation relates the profile to local interface readjustments and long-ranged jumps of the active site. The pattern results in a two-point correlation function saturating to a finite value which depends on system size. Pattern formation is generic to systems in which the dynamics leads to correlated motion of the active site.

PACS numbers: 47.54.+r, 68.10.Gw, 05.40.+j, 47.55.Mh

A common feature of many self-organised systems is that their dynamics is extremal \[1\]: that is, activity is triggered at that spot where a link is weakest, or a force the strongest, and not statistically uniformly over the full system. As a result, the active site performs an erratic motion with nontrivial correlations in space and time \[2\]. In this Letter, we demonstrate an intriguing and generic feature of such systems: the existence of a well-defined spatial pattern around this dynamical center of activity.

We study this phenomenon in a model of a self-organizing interface in a random medium. In steady state, we find that there is a nonzero time-averaged spatial pattern of height differences, provided that it is referred always to the location of the active site at that instant. Figure 1 is a schematic representation of the interface profile, time-averaged in a frame which keeps the active site at the origin. Interfaces in which \( r \to -r \) symmetry is present (Fig. 1a), or absent (Fig. 1b), are shown, symmetry-breaking being induced by tilting the interface \( \Psi \). In the former case, the profile has a cusp at the origin, while in the latter case the dominant feature is a larger slope near the origin — a local amplification of the broken symmetry. The activity-centered pattern (ACP) is defined in terms of height gradients,

\[
\Psi(r) \equiv \frac{1}{2}\langle [\nabla h(r + R(t))] - m \rangle,
\]

where \( R(t) \) is the position of the active site at time \( t \), \( h(r') \) denotes the height at the site \( r' \), \( \langle \ldots \rangle \) is a time average and \( m \) is the overall slope of the interface. \( \Psi(r) \) has a long-ranged (power law) tail with exponents which differ in tilted and untilted interfaces. We also find that the ACP leads to an unusual finite size effect, namely the saturation of a two-point correlation function with increasing separation. Finally, we present an approximate integral equation which relates the ACP to the local dynamics of interface readjustment and the long-ranged dynamics of active-site motion. This provides an understanding of the form of the induced pattern as a function of spatial correlations in the activity.

In our model the interface is taken to be a directed path on a square lattice (Fig. 2), with tilted cylindrical boundary conditions \( \Psi \) which ensure that the mean slope is preserved. To every bond \( k \) on the lattice is pre-assigned a fixed random number \( f_k \) drawn from the interval \([0,1]\). Motion of the interface is initiated at that bond, on the forward perimeter of the interface, which carries the smallest random number. In order that the length of the interface not change (an effect of surface tension), its directed walk character is preserved by local re-adjustments; if the chosen minimal bond has a positive (negative) slope, the sequence of links with negative (positive) slope just below (on the left) also advance \( \Psi \), as illustrated in Fig. 2. This model resembles that introduced by Sneppen \( \Psi \) but differs from it in that the length of the interface is a strict constant of the motion and the \( f_k \)'s are associated with bonds rather than sites.

The interface dynamics is equivalent to the dynamics of a system of hard-core particles on a ring, with a positive-slope link of the interface represented by a particle \( (n_1 = 1) \) and a negative-slope link by a hole \( (n_j = 0) \); see Fig. 2. The difference in height of the interface between sites \( j_1 \) and \( j_2 \) is given by \( h_{j_2} - h_{j_1} = \sum_{j=j_1}^{j_2} (2n_j - 1) \). Each site \( j \) of the ring also carries the random number \( f_j \) assigned to the bond in front of the corresponding spot on the interface. In each time step, activity is initiated at the site with the minimum \( f_j \). If this site contains a particle (hole), it exchanges with the first hole (particle) on the left (right). All sites hopped over, including the two which exchange the particle and hole, are refreshed by assigning a new set of \( f_j \)'s. The overall particle density \( \rho \) is strictly conserved, and determines the mean slope \( m = 2\rho - 1 \) of the interface. There is a nonzero current, which translates into a forward-advancing interface.

From time to time, the interface aligns along directed spanning paths in a percolation problem, as in the Sneppen model \( \Psi \). In our case, the corresponding percolation problem is simpler, viz. directed bond percolation (DP) \( \Psi \) on the dual lattice, or equivalently, diode-resistor percolation (DRP) \( \Psi \) on the original lattice. For a given configuration \( \{f_k\} \) and a trial threshold value \( f_o \), occupy all bonds \( k' \) with \( f_{k'} > f_o \) by diodes (one-way connections pointing against the direction of advance of the interface, see Fig. 2), and place resistors (two-way connections) on the remaining bonds. The diode concentration is \( p = 1 - f_o \). Infinite connected paths
(stoppers) first form along the easy direction of DP at $p = p_{DP} \simeq 0.6445$ on the square lattice. An untilted interface ($m = 0$) aligns along stoppers in the course of its motion. For $m \neq 0$, the corresponding threshold $p_m(m) > p_{DP}$ is that value of $p$ for which the edge of the DP cone has slope $m$. As it evolves, a tilted interface of slope $m$ aligns along stoppers of that slope.

The pattern defined in (1) is associated with a spatial profile of the density in the particle-hole model, as the height gradient maps on to the density. The defining equation (1) now reads $\Psi(r) = \langle n(r + R(t)) \rangle - \rho$. We studied $\Psi(r)$ by Monte Carlo simulation. In the tilted case, $\Psi(r)$ is an odd function (Fig. 3a) decaying asymptotically as a power law $|r|^{-\theta}$ with $\theta = 0.90 \pm 0.03$. In the tilted case (Fig. 3b), as there is no $r \to -r$ symmetry, $\Psi(r)$ does not have a definite parity. It is useful to separately analyse $\Psi_{\pm}(r) \equiv (\Psi(r) \pm \Psi(-r))/2$. The odd part decays as $\Psi_{-}(r) \sim |r|^{-\theta_{-}}$ with $\theta_{-} = 1.04 \pm 0.05$. The even part $\Psi_{r}(r) \approx -b(L) + a |r|^{-\theta_{+}}$ where $\theta_{+} = 0.46 \pm 0.05$ and $b(L) \to 0$ as the lattice size $L \to \infty$.

The density profiles of Fig. 3 correspond to the height patterns shown in Fig. 1. In the untilted case, on average, the active site is located at the peak (Fig. 1a) where $f_R$'s which have not been sampled earlier are most likely to occur. In the tilted case there is also a bootstrap effect at work. If $\rho \geq 0.5$, the active site is more likely to contain a refreshed. Thus regions to the left are more often refreshed, making the active site likely to move in this direction and find itself amidst a particle cluster. This leads to densities higher than $\rho$ on both sides of the active site (Fig. 3b) for $\rho$ sufficiently different from 0.5.

More quantitatively, we may write an integral equation for $\Psi(r)$ for untilted and tilted interfaces. If the pattern is centered at $R(t)$ at time $t$, the dynamics causes two changes at the next instant: (i) A short-ranged readjustment of the interface changes the profile near $R(t)$. As a result, the local density to the right of the active site becomes less than the density to its left. The average density change at site $R(t) + r$ denotes the density-increment function $\Phi(r)$. (ii) The active site jumps a distance $l \equiv R(t+1) - R(t)$; there is a substantial probability $P(l)$ for large jumps. Since the ACP is centered at the active site, the result of (i) followed by (ii) is that the average profile reproduces itself, except that it is centered at the shifted site $R(t) + l$. Both effects are incorporated into the integral equation

$$\Psi(r) = \int_{-\infty}^{+\infty} \left[ \Psi(r - l) + \Phi(r - l) \right] P(l) dl.$$

(2)

$P(l)$ decays as a power for large $l$: $P(l) \sim |l|^{-\pi}$ (Fig. 4). In the tilted case $P(l) = P(-l)$ holds because of $r \to -r$ symmetry. We find $\pi = 2.25 \pm 0.05$ which compares well with earlier determined values of $\pi$ for the Sneppen model. In the tilted case ($\rho \neq 1/2$), $P(l)$ is not a symmetric function (Fig. 4) and it is convenient to separately analyse the even and odd parts $P_{\pm}(l) \equiv (P(l) \pm P(-l))/2$. We find that the even part, $P_{\pm}(l)$, decays asymptotically as $P_{\pm}(l) \sim |l|^{-\pi_{\pm}}$ with $\pi_{\pm} = 2.00 \pm 0.02$. The odd part $P_{-}(l)$ changes sign (as implied by the crossing of the curves in Fig. 4) and asymptotically follows $P_{-}(l) \sim |l|^{-\pi_{-}}$ with $\pi_{-} = 2.49 \pm 0.06$. We verified that the values of $\pi_{+}$ and $\pi_{-}$ are the same for various $\rho \neq 1/2$.

Equation (2) can be solved by Fourier transform. Defining $\tilde{\Psi}(q) \equiv \int_{-\infty}^{+\infty} e^{2\pi i qr} \Psi(r) dr$ etc we find

$$\Psi(q) = \frac{\Phi(q) \tilde{P}(q)}{1 - \tilde{P}(q)}.$$

This equation indicates that the ACP arises as a nonlocal response to the local density-increment function $\Phi(r)$. Notice that the denominator vanishes as $q \to 0$. We have solved (3) numerically, but it is more instructive to examine the small-$q$ (large-$r$) behaviour.

In the untilted case, the $r \to -r$ symmetry implies that $P(l)$ is even, while $\Phi(r)$ and the profile $\Psi(r)$ are both odd. $\Phi(r)$ is a short-ranged function with finite first moment $\phi_1$. Hence, to first order, $\Phi(q) \approx i\phi_1 q$. The power-law tails in $P(l)$ imply that $\tilde{P}(q) \approx 1 - A|q|^{\pi_{+} - 1}$. Thus we find $\Psi(r) \sim \text{sgn}(r) |r|^{-(3-\pi_{+})}$ i.e. the profile is an odd function, with a power law tail.

In the tilted case, the absence of $r \to -r$ symmetry implies that none of $P(l)$, $\Phi(r)$ and $\Psi(r)$ has a definite parity. Since $\Phi(r)$ is short-ranged, $\Phi(q) \approx i\phi_1 q + \phi_2 q^2$ as $q \to 0$. There is no $\phi_0$ term, as the elementary step of hopping a particle or hole conserves particle number, implying $\int \Phi(r) dr = 0$. The $q \to 0$ behaviour of $\tilde{P}(q)$ is determined by the asymptotic power law decays of the even and odd parts $P_{\pm}(l)$ as $|l| \to \infty$. Thus we have $\tilde{P}_{\pm}(q) \approx 1 - A|q|^{\pi_{\pm} - 1}$. We might have expected $\tilde{P}_{\pm}(q) \approx Bq + C \text{sgn}(q) |q|^{\pi_{\pm} - 1}$, but in fact the mean velocity $\int l P(l) dl$ of the active site vanishes implying $B = 0$. Thus the integral equation predicts that to leading order, both $\Psi_{+}(r)$ and $\Psi_{-}(r)$ decay as powers $\sim |r|^{-\theta_{\pm}}$, with $\theta_{+} + 2\pi_{+} - \pi_{-} = 3$ and $\theta_{-} + \pi_{+} = 3$. The prediction $\pi_{+} = \pi_{-} = 3$ compares quite well with the numerical values 3.15 for the untilted case and 3.04 for $\Psi_{\pm}(r)$ in the tilted case. For $\Psi_{+}(r)$, however, the numerically determined value of $\pi_{+} + 2\pi_{+} - \pi_{-} \approx 1.97$ deviates more from the predicted value 3. The reason behind the discrepancy is that (2) implicitly ignores correlations e.g. in successive jump lengths of the active site.

To check this point, we numerically studied a model with no correlations in successive jump lengths. At every instant the jump length of the active site was taken from a power-law distribution $P(l)$ chosen suitably so as to implement the zero velocity constraint. Short-ranged adjustments were taken to follow the same particle-hole hopping rules as the extremal model. This model is similar in spirit to the Lévy flight interface model considered in 1. The results in this case were fully consistent with the predictions $\theta_{+} + 2\pi_{+} - \pi_{-} = \theta_{-} + \pi_{+} = 3$. Further, we also studied cases in which $P(l)$ is not a power law. For short ranged $P(l)$, the ACP has the form of a kink
function, with a particle (hole) - rich region on the left (right) of the active site. In the limit of strictly infinite ranged $P(l)$, tantamount to normal stochastic evolution \footnote{3}, the ACP ceases to exist. In all cases we compared our numerical results with those predicted by (2) and found good agreement.

The ACP has several interesting consequences. For instance, we expect there to be a larger length of interface in a region of fixed size $x$ around the active site, than in a region opposite it. Accordingly, we monitored mean squared fluctuations of the height around the instantaneous average, in regions around and opposite the active site, and found a pronounced difference (factor $\simeq 2$, for tilted and untilted cases with $x = 256, L = 4096$). This effect is smaller at stops, indicating that the ACP itself is suppressed there. The excess length of interface associated with the ACP may provide a useful way to identify the active region in experiment.

Another consequence of the ACP is that being a one-point correlation function, albeit an unusual one, it has a strong effect on the customary, space-time averaged two-point correlation function $C(\Delta r) \equiv \{ \langle n(r') \rangle / n(r + \Delta r) \} - \{ \langle n(r) \rangle / n(r + \Delta r) \}$. Here $r'$ is a fixed site on the lattice, $\langle \ldots \rangle$ stands for a time average and $\{ \ldots \}$ stands for an average over all sites $r'$. Numerical results for $C(\Delta r)$ (Fig. 5) show that it saturates at a value $C_{\text{sat}}$ which decreases with increasing size $L$ \footnote{4}. This unusual behaviour can be understood in terms of the ACP. Consider the correlation function $\Gamma(r, \Delta r) = \langle \delta n(r + R(t)) \delta n(r + \Delta r + R(t)) \rangle$, where $R(t)$ is the distance from the active site, $r$ is the distance from the active site and $\delta n(r + R(t)) = n(r + R(t)) - \rho - \Psi(r)$ is the fluctuation around the average ACP. A reasonable expectation is that the $\delta n$'s are independent for large separations $\Delta r$, i.e. $\Gamma(r, \Delta r) \to 0$ as $\Delta r \to \infty$. On averaging over $r$ (which has the same effect as the average over space-fixed sites $r'$), we see that $\{ \langle n(r) \rangle / n(r + \Delta r) \} - \{ \langle \rho + \Psi(r) \rangle / (\rho + \Psi(r + \Delta r)) \}$ approaches zero as $\Delta r \to \infty$. This predicts the saturation value $C_{\text{sat}}$ to be $\{ \langle \rho + \Psi(r) \rangle / (\rho + \Psi(r + \Delta r)) \} - \rho^2$. To test this, we subtracted this from $C(\Delta r)$ and found that the saturation effect is in fact suppressed strongly (Fig. 5), supporting our interpretation \footnote{5}. Customarily, the large-separation saturation of $C(\Delta r)$ is associated with nonzero values of $\langle n(r') \rangle - \rho$; the unusual aspect here is that there is saturation even though $\langle n(r') \rangle = \rho$.

Though we have focussed on the ACP associated with a conserved quantity like the density, a conservation law is not essential for the occurrence of a nontrivial ACP. For instance, on monitoring the distribution of $f$'s, with $r$ referred to the active site, we find profiles with power-law tails in the interface depinning model with and without tilt \footnote{6}. We also find similar patterns in the Bak-Sneppen model of evolution \footnote{7} and a modified asymmetric version of it \footnote{8}. This indicates that pattern formation is a generic feature of systems evolving through extremal dynamics.

In summary, we have explored a new aspect of self-organised interface depinning, namely the existence of a spatial pattern centered around the active site. We have shown that the pattern in height gradients has power law tails which are sensitive to $r \to -r$ symmetry, broken by tilting the interface, and that it explains the saturation of the two-point correlation function with increasing separation. The integral equation (2) provides an approximate description of the formation of the pattern. We have also shown that power-law patterns exist in other quantities, for instance the distribution of random numbers around the minimum, in both the interface and evolution models. Our studies suggest that systems which evolve through extremal dynamics are likely to exhibit activity-centered patterns.

We thank Gautam Menon and Deepak Dhar for useful discussions, and the referee for his comments.

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Figure Captions

FIG. 1. Schematic view of averaged interface profile around the active site A in (a) untilted (b) tilted interfaces. Tilt breaks $r \rightarrow -r$ symmetry, an effect that is amplified near A.

FIG. 2. The bond model. The tilted interface $II'$ advances along the extremal perimeter bond $A$ and locally readjusts to align along the dashed line. At the next instant, the activity moves from $A$ to $A'$. The corresponding configuration and local moves for the particle-hole model are also shown. $SS'$ is a stopper, whose perimeter is fully occupied by diodes.

FIG. 3. Density profiles in the (a) untilted ($\rho = 0.5$) and (b) tilted ($\rho = 0.75$) cases. Inset: The odd part of the profile in tilted (open circles) and untilted (filled circles) cases. We used $L = 16384$ and averaged over $10^6$ configurations.

FIG. 4. Monte Carlo results for the probability distribution of the jump of the active site for three different densities, $\rho = 0.5$ (plus sign), $\rho = 0.75$ (circles) and $\rho = 0.84375$ (triangles). If $\rho \neq 0.5$, there is a left-right asymmetry and the asymptotic slope differs from that for $\rho = 0.5$. We used $L = 65536$ and averaged over $3 \times 10^6$ configurations.

FIG. 5. Density-density correlations for $L = 4096$ (open circles) and for $L = 16384$ (squares). The saturation is reduced strongly (filled circles) on subtracting the contribution of the ACP. We averaged over $10^6$ configurations.