Concurrence of Stochastic 1-Qubit Maps

Meik Hellmund
Mathematisches Institut, Universität Leipzig, Johannisgasse 26, D-04103 Leipzig, Germany

Armin Uhlmann
Institut für Theoretische Physik, Universität Leipzig, Vor dem Hospitaltore 1, D-04103 Leipzig, Germany

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Explicit expressions for the concurrence of all positive and trace-preserving (“stochastic”) 1-qubit maps are presented. By a new method we find the relevant convex roof pattern. We conclude that two component optimal decompositions always exist. Our results can be transferred to \(2 \times n\)-quantum systems providing the concurrence for all rank two density operators as well as a lower bound for their entanglement of formation.

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INTRODUCTION

In quantum physics a system in a pure state \(\pi = |\psi\rangle\langle\psi|\) may have subsystems in states which are not pure but mixed. These mixed substates are typically correlated in a non-local and non-classical way. The use of this phenomenon of entanglement as a resource for communication and computation is a main feature of quantum information theory \([1]\). This makes the search for a quantitative understanding and characterization of entanglement a central issue \([2, 3]\). Entanglement measures ought to describe single-use or asymptotic capabilities of quantum systems and channels just as the von Neumann entropy describes single-use or asymptotic capabilities of quantum information theory \([1]\). This makes the search for a quantitative understanding and characterization of entanglement a central issue \([2, 3]\). Entanglement measures ought to describe single-use or asymptotic capabilities of quantum systems and channels just as the von Neumann entropy \(S(\rho) = -\text{Tr} \rho \log \rho\) is an asymptotic measure for information content. They are, similar to entropy, non-linear and unitarily invariant functions on the space of states.

Bennett et al. \([4]\) introduced the entanglement of formation \(E_F(\rho)\) as the asymptotic number of ebits (maximally entangled qubit pairs) needed to prepare the entangled bipartite state \(\rho\) by local operations and classical communication (LOCC) and showed that

\[
E_F(\rho) = \min \sum p_j S(\text{Tr}_B(\pi_j))
\]

where \(\text{Tr}_B\) is the partial trace over one of the two subsystems and the minimum is taken over all possible convex \((\sum p_j = 1, p_j > 0)\) decompositions of the state \(\rho\) into pure states

\[
\rho = \sum p_j \pi_j, \quad \pi_j \text{pure.}
\]

Closed formulas for the entanglement of formation, i.e., analytic solutions to the global optimization problem \([1]\) are only known for certain classes of highly symmetric states \([5, 6]\) and for the case of a pair of qubits \((2 \times 2)\) system. In the latter case, the analytic formula for the entanglement of formation was obtained first for special states \([4, 7]\) and later proved for all states of a qubit pair \([8]\). It expresses \(E_F(\rho)\) in terms of another entanglement measure \(C(\rho)\) which was named concurrence \([9]\). The concurrence appeared to be an interesting quantity in itself \([9]\). Many authors, e.g. \([10, 11, 12]\), have obtained bounds for the concurrence of larger bipartite systems.

In the present paper we obtain analytic expressions for the concurrence for general stochastic 1-qubit maps and therefore for general \(2 \times n\) bipartite systems provided the input state \(\rho\) has rank two. For this we employ the convex roof construction \([13, 14]\) as a way to study global optimization problems of the type \([11]\). Our main results are given by Theorems 2 and 3.

Let \(\Phi\) be a positive and trace-preserving (i.e., stochastic) map from a general quantum system into a 1-qubit-system. This setup includes as special case the partial trace \(\text{Tr}_B\) which maps states of a bipartite \(2 \times n\) system to states of the subsystem. For pure input states \(\pi = |\psi\rangle\langle\psi|\) the concurrence is defined as

\[
C_{\Phi}(\pi) = 2\sqrt{\text{det} \, \Phi(\pi)}
\]

and for a general mixed input state \(\rho\) one defines

\[
C_{\Phi}(\rho) = \min \sum p_j C_{\Phi}(\pi_j),
\]

where the minimum is again taken over all possible convex decompositions into pure states. Let us consider the case where \(\rho\) has rank 2 and is therefore supported by a 2-dimensional input subspace. Then we have to consider \([2]\) only pure states supported in the same 2-dimensional supporting input space. By unitary equivalence we are allowed to identify input and output subspaces. Hence, calculating the concurrence of a rank two density operator \(\rho = \sum_{i,j=1}^{2} \rho_{ij} |v_i\rangle\langle v_j|\) of a \(2 \times n\) system is equivalent to computing the concurrence of a certain 1-qubit stochastic map. This map is completely positive and explicitly given by \(\Phi(\rho) = \sum_{i,j} \rho_{ij} D_{ij}\) with \(D_{ij} = \text{Tr}_B |v_i\rangle\langle v_j|\).
However, our construction of the concurrence works for all stochastic 1-qubit maps, not only for completely positive ones. It is therefore suggestive, but not the topic of the present paper, to ask for applications to the entanglement witness problem [15].

In some cases the convex roof for the concurrence appears to be a flat convex roof. In these cases optimal decompositions for the concurrence also provide optimal decompositions for the entanglement of formation and therefore $E_F(\rho)$ can be expressed as a function of the concurrence $C(\rho)$, exactly as in the case of a pair of qubits [8]. If the roof of the concurrence is not flat, our results for the concurrence provide a lower bound for the entanglement of formation.

We illustrate our procedure by explicit formulas for the cases of bistochastic and of axial symmetric 1-qubit maps. In both cases the result is of a surprising transparency.

THE CONVEX ROOF CONSTRUCTION

In the following, all linear combinations are understood as convex combinations, i.e., the \{p_j\} always satisfy $\sum p_j = 1$ and $p_j > 0$. Solutions to the optimization problem eq. (4) can be characterized as so-called convex roofs. Let $\Omega$ denote the convex set of density operators $\rho$ and let $g(\pi)$ be a continuous real-valued function on the set of pure states.

Theorem 1 (see [13, 14]). There exists exactly one function $G(\rho)$ on $\Omega$ which can be characterized uniquely by each one of the following two properties:

1. $G(\rho)$ is the solution of the optimization problem

   $$ G(\rho) = \min_{\rho = \sum p_j \pi_j} \sum p_j g(\pi_j). \quad (5) $$

2. $G(\rho)$ is convex [16] and a roof, i.e., for every $\rho \in \Omega$ exists an extremal decomposition $\rho = \sum p_j \pi_j$ such that

   $$ G(\rho) = \sum p_j g(\pi_j). \quad (6) $$

Furthermore, given $\rho$, the function $G$ is linear on the convex hull of all those pure states $\pi_j$ which appear in the decomposition (9) of $\rho$. Therefore, $G$ provides a foliation of $\Omega$ into leaves such that a) each leaf is the convex hull of some pure states and b) $G$ is linear on each leaf. If $G$ is not only linear but even constant on each leaf, it is a flat roof.

STOCHASTIC 1-QUBIT MAPS

The space $\mathcal{M}_2$ of hermitian $2 \times 2$ matrices $\rho = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix}$ is isomorphic to Minkowski space $\mathbb{R}^{1,3}$ via

$$ x = (x_0, \vec{x}) \iff \rho = \frac{1}{2} (x_0 I + \vec{x} \cdot \vec{\sigma}) = \frac{1}{2} \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}. \quad (7) $$

We have $\det \rho = \frac{1}{4} (x_0^2 - x_1^2 - x_3^2 - x_2^2) = \frac{1}{2} \cdot x \cdot x$ where the dot between 4-vectors denotes the Minkowski space inner product and $\text{Tr} \rho = x_0$. Therefore the cone of positive matrices is just the forward light cone and the state space $\Omega$ of a qubit, the Bloch ball, is the intersection of this cone with the hypersurface $V$ defined by $x_0 = 1$. In this picture mixed states correspond to time-like vectors and pure states to light-like vectors, both normalized to $x_0 = 1$.

A trace-preserving positive linear map $\Phi : \mathcal{M}_2 \to \mathcal{M}_2$ can be parameterized as [17]

$$ \Phi(\rho) = \Phi \left( \frac{1}{2} (x_0 I + \vec{x} \cdot \vec{\sigma}) \right) = \frac{1}{2} (x_0 I + (x_0 \vec{t} + \Lambda \vec{x}) \cdot \vec{\sigma}) \quad (8) $$

where $\Lambda$ is a $3 \times 3$ matrix and $\vec{t}$ a 3-vector.

We consider the quadratic form $q$ on $\mathcal{M}_2$ defined by

$$ q^\Phi_w(x) = 4(\det \Phi(\rho) - w \cdot \det \rho) = \Phi(x) \cdot \Phi(x) - w \cdot x \cdot x = \sum_{i,j=0}^4 q_{ij} x_i x_j \quad (9) $$

where $w$ is some real parameter. For pure states, i.e., on the boundary of the Bloch ball where $x \cdot x = 0$, the form $q(x)$ equals the square of the concurrence $C = 2 \sqrt{\det \Phi(\rho)}$.

Furthermore, we denote by $Q$ the linear map $Q : x_i \mapsto \sum q_{ij} x_j$ corresponding to the quadratic form $q$ via polar-
Theorem 2. Let the quadratic form \( q \) and therefore the matrix \( Q \) be positive semidefinite and degenerate, i.e., \( Q \geq 0 \) and \( \dim \ker Q > 0 \). If \( \ker Q \) contains a non-zero vector \( n \) which is space-like or light-like, \( n \cdot n \leq 0 \), then \( q^{1/2} \) is a convex roof. Furthermore, this roof is flat if such an \( n \) exists with \( n_0 = 0 \).

Theorem 3. For every positive trace-preserving map \( \Phi \) exists a unique value \( w_0 \) for the parameter \( w \) such that the conditions of Theorem 2 are fulfilled. Therefore the concurrence of an arbitrary stochastic 1-qubit map \( \Phi \) is given by \( C_\Phi(\rho) = \sqrt{q^w_\Phi(\rho)} \).

Let us sketch the proof of Theorems 2 and 3. The square root \( \sqrt{q} \) of a positive semidefinite form \( q \) on a linear space provides a seminorm on this space and is therefore convex. According to Theorem 1 we need to show that it is also a roof, i.e., there is a foliation of the space into leaves such that \( q^{1/2} \) is linear on each leaf. Let \( n = (n_0, \vec{n}) \) be a non-zero vector in \( \ker Q \). Then for all vectors \( m \) we have

\[
q(m + n) = (m + n)Q(m + n) = mQm + q(m). \tag{11}
\]

Let us start with the case where \( n_0 = 0 \). Then \( \vec{n} \) gives a direction in \( V \) along which \( q \) is constant. Therefore, \( \sqrt{q} \) is a flat convex roof.

\[
\begin{align*}
M_2 = &\text{FIG. 1: The embedding of the Bloch ball into } M_2 \text{ and its foliation by a flat convex roof.}
\end{align*}
\]

\[
\begin{align*}
\text{Let us now consider the case where } \ker Q \text{ does not contain a vector } n \text{ with } n_0 = 0. \text{ Then we have } \dim \ker Q = 1 \text{ and this line intersects } V \text{ in one point which we call } n. \text{ Every other point } m \text{ in } V \text{ can be connected to the point } n \text{ by a line lying in } V. \text{ Then } q^{1/2} \text{ is linear along the half-line } \mathbb{R}^+ \ni s \mapsto s m + (1 - s)n \text{ since}
\end{align*}
\]

\[
q(s m + (1 - s)n) = (s m + (1 - s)n)Q(s m + (1 - s)n) = s^2 q(m) \tag{12}
\]

This concludes the proof of Theorem 2.

For the proof of Theorem 3 we note that the space \( \mathcal{P} \) of stochastic maps is itself a convex space. It can be parameterized as follows [18]: Let \( \xi \) be a unit 3-vector and \( \alpha, \beta, \omega_1, \omega_2, \omega_3 \) be parameters taking values between zero and one: \( 0 \leq \alpha \leq 1; \ 0 \leq \beta \leq 1; \ 0 \leq \omega_1 \leq \omega_2 \leq \omega_3 = 1 \).

With the abbreviation \( \nu = \sqrt{\sum_{i=1}^{3} \xi_i^2 \omega_i^2} \) we can represent stochastic maps [3] up to orthogonal transformations by \( \bar{t} = (t_1, t_2, t_3) \), \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \) where

\[
\begin{align*}
t_i &= \beta \xi_i (1 - \alpha \omega_i^2) \tag{13}
\end{align*}
\]

\[
\begin{align*}
\lambda_i &= \alpha \nu \omega_i \tag{14}
\end{align*}
\]

Furthermore, the boundary \( \partial \mathcal{P} \) is given by \( \beta = 1 \). In this case, the unit vector \( \xi \) represents the touching point (or one of the touching points in more degenerate cases) between the unit sphere and its image. Let \( \Phi \in \partial \mathcal{P} \), so \( \beta = 1 \). Then it is easy to check that \( w_0 = \alpha \nu^2 \) makes \( Q \) positive semidefinite since it permits a Cholesky decomposition \( Q = R R^T \) into a triangular matrix with a zero on the diagonal:

\[
\begin{align*}
R = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-\omega_1 \xi_1 \mu_1 & \nu \mu_1 & 0 & 0 \\
-\omega_2 \xi_2 \mu_2 & 0 & \nu \mu_2 & 0 \\
-\omega_3 \xi_3 \mu_3 & 0 & 0 & \nu \mu_3
\end{pmatrix} \tag{15}
\end{align*}
\]

where \( \mu_i = \sqrt{\alpha(1 - \alpha \omega_i^2)} \). Furthermore, \( n = (1, \frac{1}{\nu} \xi_i \omega_i) \) is a lightlike vector in \( \ker Q \).

In the general case \( \beta < 1 \) we have

\[
\begin{align*}
Q_{w_{0\beta}}^{\Phi} = \beta^2 Q_{w_{0\beta - 2}}^{\Phi} + (1 - \beta^2) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \tag{16}
\end{align*}
\]

Therefore, \( Q_{w_{0\beta}}^{\Phi} \) as a sum of two positive semidefinite terms is either positive semidefinite or positive definite. In the first case we are done with \( w_{0\beta} = \alpha \beta^2 \nu^2 \). In the other case we must adjust \( w_{0\beta} \). It is clear that \( Q \) is not positive for \( w \rightarrow \pm \infty \). Therefore due to continuity, we can make \( Q \) positive semidefinite and degenerate by increasing or decreasing \( w \). Let \( w_1 < w_2 \) be the points of degeneration and \( n_1, n_2 \) corresponding vectors in \( \ker Q_{w_j} \).

Then (eq. (10)) \( n_1 Q_0 n_1 = w_1 n_1^2 \) and \( n_2 Q_0 n_2 = w_2 n_2^2 \).
Furthermore, no nonzero vector can be both in Ker $Q_{w_1}$ and Ker $Q_{w_2}$. So, $n_1 Q_{w_1} n_1 > w_2 n_1^2$ and $n_2 Q_{w_2} n_2 > w_1 n_2^2$, providing $(w_2 - w_1) n_1^2 < 0$ and $(w_2 - w_1) n_2^2 > 0$. Therefore, increasing $w$ will make Ker $Q$ time-like and decreasing $w$ will make it space-like. This proofs the claim of Theorem existance of a suitable $w_0$. Uniqueness can be shown easily. It also follows indirectly from the uniqueness of the convex roof extension, Theorem. More details can be found in [19].

**EXPLICIT EXAMPLES**

Let us demonstrate our construction on some examples.

**Bistochastic maps or unital channels**

Unital 1-qubit channels are quite trivial. We have $\vec{t} = 0$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and therefore $w = \max(\lambda_1^2, \lambda_2^2, \lambda_3^2)$ fulfills the conditions of Theorem and provides the roof

$$C(\rho) = q^{1/2}(\rho) = \sqrt{(1-w)x^2_0 + \sum_{i=1}^{3} (w - \lambda_i^2) x_i^2} \quad (17)$$

which is flat in one direction since one of the terms in the sum vanishes.

Nevertheless, this case includes channels of all Kraus lengths between 1 and 4.

**Axial symmetric channels**

Every channel commuting with rotations about the $x_3$-axis is of the form

$$\Phi(\rho) = \begin{pmatrix} \alpha x_{00} + (1 - \gamma) x_{11} & \beta x_{01} \\ \beta x_{10} & \gamma x_{11} + (1 - \alpha) x_{00} \end{pmatrix}. \quad (18)$$

This corresponds to $\Lambda = \text{diag}(\beta, \beta, \alpha + \gamma - 1)$ and $\vec{t} = (0, 0, \alpha - \gamma)$. This family includes many standard channels, e.g.,

- the amplitude-damping channel (length 2, non-unital) for $\gamma = 1$, $\beta^2 = \alpha$;
- the phase-damping channel (length 2, unital) for $\alpha = \gamma = 1$ and
- the depolarizing channel (length 4, unital) for $\alpha = \gamma$, $\beta = 2\alpha - 1$.

Here we find that $q^{1/2}_w$ is a convex roof for $w = \max(\beta_1^2, \beta_2^2)$ with

$$\beta_1^2 = 1 + 2\alpha \gamma - \alpha - \gamma - 2\sqrt{\alpha(1 - \alpha) \gamma(1 - \gamma)}. \quad (19)$$

In the case $\beta^2 \geq \beta_1^2$ we have Ker $Q = \text{Span}\{e_x, e_y\}$ and the resulting roof is flat. In the other case we have a one-dimensional Ker $Q$ generated by $n = (1, 0, 0, 0)$ with $z_0 = \frac{\sqrt{\gamma(1-\gamma) + \alpha(1-\alpha)}}{\sqrt{\gamma(1-\gamma) - \sqrt{\alpha(1-\alpha)}}}$ and a non-flat roof.

**CONCLUSION**

We calculated the concurrence $C_\Phi$ of all trace-preserving positive 1-qubit maps and therefore for general $2 \times n$ bipartite systems with rank-2 input states.

The concurrence is real linear on each member of a unique bundle of straight lines crossing the Bloch ball. The bundle consists either of parallel lines or the lines meet at a pure state, or they meet at a point outside the Bloch ball. Furthermore, $C_\Phi$ turns out to be the restriction of a Hilbert semi-norm to the state space.

More details and applications, including the entanglement of formation in $2 \times n$ systems and the Holevo capacity, will be given in [19].

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