CONFORMAL VECTOR FIELDS ON TANGENT BUNDLE OF A RIEMANNIAN MANIFOLD*

S. HEDAYATIAN AND B. BIDABAD**

Faculty of Mathematics, Amir-Kabir University of Technology, Hafez Ave. 15914, Tehran, I. R. of Iran
Emails: s_hedayatian@aut.ac.ir, bidabad@aut.ac.ir

Abstract – Let $M$ be an $n$-dimensional Riemannian manifold and $TM$ its tangent bundle. The conformal and fiber preserving vector fields on $TM$ have well-known physical interpretations and have been studied by physicists and geometers. Here we define a Riemannian or pseudo-Riemannian lift metric $\tilde{g}$ on $TM$, which is in some senses more general than other lift metrics previously defined on $TM$, and seems to complete these works. Next we study the lift conformal vector fields on $(TM, \tilde{g})$ and prove among the others that, every complete lift conformal vector field on $TM$ is homothetic, and moreover, every horizontal or vertical lift conformal vector field on $TM$ is a Killing vector.

Keywords – Complete lift metric, Conformal, Homothetic, Killing and Fiber-preserving vector fields.

1. INTRODUCTION

Let $M$ be an $n$-dimensional differential manifold with a Riemannian metric $g$ and $\phi$ be a transformation on $M$. Then $\phi$ is called a conformal (resp. projective) transformation if it preserves the angles (resp. geodesics). Let $V$ be a vector field on $M$ and $\{\varphi_t\}$ be the local one-parameter group of local transformations on $M$ generated by $V$. Then $V$ is called an infinitesimal conformal (resp. projective) transformation on $M$ if each $\varphi_t$ is a local conformal (resp. projective) transformation of $M$. It is well known that $V$ is an infinitesimal conformal transformation or conformal vector field on $M$ if and only if there is a scalar function $\rho$ on $M$ such that $\mathcal{L}_V g = 2 \rho g$, where $\mathcal{L}_V$ denotes Lie derivation with respect to the vector field $V$. $V$ is called homothetic if $\rho$ is constant and is called an isometry or Killing vector field when $\rho$ vanishes.

Let $TM$ be the tangent bundle over $M$, and $\Phi$ be a transformation on $TM$. Then $\Phi$ is called a fiber preserving transformation if it preserves the fibers. Fiber preserving transformations have well known applications in Physics. Let $X$ be a vector field on $TM$ and $\{\Phi_t\}$ the local one parameter group of local transformation on $TM$ generated by $X$. Then $X$ is called an infinitesimal fiber preserving transformation or fiber preserving vector field on $TM$ if each $\Phi_t$ is a local fiber preserving transformation of $TM$.

Let $\tilde{g}$ be a Riemannian or pseudo-Riemannian metric on $TM$. The conformal vector field $X$ on $TM$ is said to be essential if the scalar function $\Omega$ on $TM$ in $\mathcal{L}_V \tilde{g} = 2 \Omega \tilde{g}$ depends only on $(g^b)$.
(with respect to the induced coordinates \((x', y')\) on \(TM\), and is said to be \textit{inessential} if \(\Omega\) depends only on \((x')\). In other words, \(\Omega\) is a function on \(M\).

There are some lift metrics on \(TM\) as follows:

- \textit{complete} lift metric or \(g_2\), \textit{diagonal} lift metric or \(g_1 + g_3\), lift metric \(g_2 + g_3\) and lift metric \(g_1 + g_2\).

In this area the following results are well known:

Let \((M, g)\) be a Riemannian manifold. If we consider \(TM\) with metrics \(g_1 + g_3\) or \(g_2 + g_3\), then every infinitesimal fiber preserving conformal transformation on \(TM\) is homothetic, and induces a homothetic vector field on \(M\) [1].

Let \((M, g)\) be a complete, simply connected Riemannian manifold. If we consider \(TM\) with metric \(g_1\), and \(TM\) admits an essential infinitesimal conformal transformation, then \(M\) is isometric to the standard sphere [2].

Let \((M, g)\) be a Riemannian manifold and \(V\) a vector field on \(M\) and let \(X^C\), \(X^V\), \(X^H\) be complete, vertical and horizontal lifts of \(V\) to \(TM\) respectively. If we consider \(TM\) with metric \(g_2\), then \(X^C\) is a conformal vector field on \(TM\) if and only if \(V\) is Killing vector on \(M\), then \(X^C\) and \(X^V\) are Killing vectors on \(TM\) [3].

In this paper we are going to replace the cited lift Riemannian or pseudo-Riemannian metrics on \(TM\) by \(\tilde{g}\), that is a combination of diagonal lift and complete lift metrics, where \(a\), \(b\) and \(c\) are certain positive real numbers. More precisely, we prove the following Theorems.

\textbf{Theorem 1.} Let \(M\) be a connected \(n\)-dimensional Riemannian manifold and let \(TM\) be its tangent bundle with metric \(\tilde{g}\). Then every complete lift conformal vector field on \(TM\) is homothetic, and moreover, every horizontal or vertical lift conformal vector field on \(TM\) is a Killing vector.

\textbf{Theorem 2.} Let \(M\) be a connected \(n\)-dimensional Riemannian manifold and \(TM\) be its tangent bundle with metric \(\tilde{g}\). Then every inessential fiber preserving conformal vector field on \(TM\) is homothetic.

\section{2. PRELIMINARIES}

Let \((M, g)\) be a real \(n\)-dimensional Riemannian manifold and \((U, x)\) a local chart on \(M\), where the induced coordinates of the point \(p \in U\) are denoted by its image on \(IR^n\), \(x(p)\) or briefly \((x')\). Using the induced coordinates \((x')\) on \(M\), we have the local field of frames \(\{\frac{\partial}{\partial x_i}\}\) on \(T_pM\). Let \(\nabla\) be a Riemannian connection on \(M\) with coefficients \(\Gamma_{ij}^k\), where the indices \(a, b, c, i, j, k, m, ..., n\) run over the range \(1, 2, ..., n\). The Riemannian curvature tensor is defined by

\[K(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \forall X, Y, Z \in X(M).\]

Locally we have

\[K_{ijk}^m = \partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{im}^m \Gamma_{jk}^l - \Gamma_{jm}^m \Gamma_{ik}^l,\]

where \(\partial_i = \frac{\partial}{\partial x^i}\) and \(K(\partial_i, \partial_j, \partial_k) = K_{ijk}^m \partial_m\).
3. NON-LINEAR CONNECTION

Let $TM$ be the tangent bundle of $M$ and $\pi$ the natural projection from $TM$ to $M$. Consider $\pi_* : TTM \rightarrow TM$ and let us put

$$\ker \pi_* = \{ z \in TTM \mid \pi_*(z) = 0 \}, \forall \nu \in TM.$$

Then the vertical vector bundle on $M$ is defined by $VTM = \bigcup_{\nu \in TM} \ker \pi_*$. A non-linear connection or a horizontal distribution on $TM$ is a complementary distribution for $VTM$ on $TTM$. The non-linear nomination arise from the fact that $HTM$ is spanned by a basis which is completely determined by non-linear functions. These functions are called coefficients of non-linear connection and will be noted in the sequel by $\Gamma_{ij}^k$. It is clear that $HTM$ is a horizontal vector bundle.

By definition, we have decomposition $TTM = VTM \oplus HTM$ [5].

Using the induced coordinates $(x^i, y^j)$ on $TM$, where $x^i$ and $y^j$ are called respectively position and direction of a point on $TM$, we have the local field of frames $\{\partial_{x^i}, \partial_{y^j}\}$ on $TTM$. Let $\{dx^i, dy^j\}$ be the dual basis of $(\partial_{x^i}, \partial_{y^j})$. It is well known that we can choose a local field of frames $\{X_i, \frac{\partial}{\partial y^j}\}$ adapted to the above decomposition, i.e. $X_i \in X(HTM)$ and $\frac{\partial}{\partial y^j} \in X(VTM)$ are sections of horizontal and vertical sub-bundle on $HTM$ and $VTM$, defined by $X_i = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$, where $N_i^j(x,y)$ are functions on $TM$ and have the following coordinate transformation rule in local coordinates $(x^i, y^j)$ and $(x'^i, y'^j)$ on $TM$.

$$N_{i'}^h = \frac{\partial x'^h}{\partial x^i} \Gamma_{ij}^k N_j^h + \frac{\partial^2 x'^h}{\partial x^i \partial x^j} y^j.(\partial_{x^h}).$$

To see a relation between linear and non-linear connections let $\Gamma_{i}^{k}$ be the coefficients of the Riemannian connection of $(M,g)$. Then it is easy to check that $y^i \Gamma_{ij}^k$ satisfies the above relation and thus can be regarded as coefficients of the non-linear connection on $TM$ in the sequel.

Let us put $X_i = \frac{\partial}{\partial x^i} - y^i \Gamma_{ij}^m \frac{\partial}{\partial y^j}$ and $X_\pi = \frac{\partial}{\partial y^i}$. Then $\{X_i, X_\pi\}$ is the adapted local field of frames of $TM$ and let $\{dx^i, dy^j\}$ be the dual basis of $\{X_i, X_\pi\}$, where $\delta y^h = dy^h + y^h \Gamma_{ij}^h dx^i$ and the indices $i, j, h, \ldots$ and $i, j, \bar{h}, \ldots$ run over the range 1, 2, ...n.

4. THE RIEMANNIAN OR PSEUDO-RIEMANNIAN METRIC $\bar{g}$ ON TANGENT BUNDLE

Let $(M,g)$ be a Riemannian manifold. The Riemannian metric $g$ has components $g_{ij}$, which are functions of variables $x^i$ on $M$, and by means of the above dual basis it is well known that [3]; $g_1 := g_{ij}dx^i dx^j$, $g_2 := 2g_{ij}dx^i \delta y^j$ and $g_3 := g_{ij}dy^i \delta y^j$ are all bilinear differential forms defined globally on $TM$.

The tensor field:

$$\bar{g} = a g_1 + b g_2 + c g_3,$$

on $TM$ where $a$, $b$ and $c$ are certain positive real numbers, has components

$$\begin{pmatrix}
a_{ij} & b_{ij} \\
b_{ij} & c_{ij}
\end{pmatrix},$$
with respect to the dual basis of the adapted frame of $TM$. From linear algebra we have $\det \tilde{g} = (ac - b^2)^p \det g^2$. Therefore $\tilde{g}$ is nonsingular if $ac - b^2 \neq 0$ and positive definite if $ac - b^2 > 0$ and define, respectively, pseudo-Riemannian or Riemannian lift metrics on $T(M)$.

5. LIE DERIVATIVE

Let $M$ be an $n$-dimensional Riemannian manifold, $V$ a vector field on $M$, and $\{\phi_t\}$ any local group of local transformations of $M$ generated by $V$. Take any tensor field $S$ on $M$, and denote by $\phi_t^*(S)$ the pull-back of $S$ by $\phi_t$. Then Lie derivation of $S$ with respect to $V$ is a tensor field $\mathcal{L}_V S$ on $M$ defined by

$$\mathcal{L}_V S = \frac{\partial}{\partial t} \phi_t^*(S) \mid_{t=0} = \lim_{t \to 0} \frac{\phi_t^*(S) - (S)}{t},$$

on the domain of $\phi_t$. The mapping $\mathcal{L}_V$ which maps $S$ to $\mathcal{L}_V(S)$ is called the Lie derivative with respect to $V$.

Suppose that $S$ is a tensor field of type $(n, m)$. Then the components $(\mathcal{L}_V S)_{i_1 \ldots i_n}^{h_1 \ldots h_m}$ of $\mathcal{L}_V S$ may be expressed as [6]

$$(\mathcal{L}_V S)_{i_1 \ldots i_n}^{h_1 \ldots h_m} = V^a \partial_a S_{i_1 \ldots i_n}^{h_1 \ldots h_m} + \sum_{k=1}^m \partial_{h_k} V^a S_{i_1 \ldots i_n h_k}^{h_1 \ldots h_m} - \sum_{k=1}^n \partial_{i_k} V^a S_{i_1 \ldots i_k h_k \ldots h_m}^{h_1 \ldots h_m},$$

where $S_{i_1 \ldots i_n}^{h_1 \ldots h_m}$ and $V^a$ denote the components of $S$ and $V$.

The local expression of the Lie derivative $\mathcal{L}_V(S)$ in terms of covariant derivatives on a Riemannian manifold for a tensor field of type $(1, 2)$ is given by:

$$\mathcal{L}_V S_{ij}^a = v^c \nabla_c S_{ij}^a - S_{ij}^b \nabla_c v^b + S_{ij}^a \nabla_c v^a + S_{ij}^b \nabla_c v^b,$$  \hspace{1cm} (1)

where, $S_{ij}^a$ and $v^b$ are components of $S$ and $V$, and $\nabla_c v^a$ are components of covariant derivatives of $S$ and $V$, respectively [1, 3, 6].

**Lemma 1.** [1], [7] The Lie bracket of adapted frame of $TM$ satisfies the following relations

$$[X_i, X_j] = y^r K_{jir}^m X_m,$$

$$[X_i, X_j] = \Gamma_i^m j X_m,$$

$$[X_i, X_j] = 0,$$

where $K_{jir}^m$ denotes the components of a Riemannian curvature tensor of $M$.

**Lemma 2.** [1] Let $X$ be a vector field on $TM$ with components $(X^h, X^\pi)$ with respect to the adapted frame $\{X_h, X^\pi\}$. Then $X$ is fiber-preserving vector field on $TM$ if and only if $X^h$ are functions on $M$.

Therefore, every fiber-preserving vector field $X$ on $TM$ induces a vector field $V = X^h \frac{\partial}{\partial x^h \mid_{x^h}}$ on $M$.

**Definition 1.** [1], [3] Let $V$ be a vector field on $M$ with components $V^h$. We have the following vector fields on $TM$ which are called respectively, complete, horizontal and vertical lifts of $V$:
\[ X^C := V^h X_h + y^m (\Gamma^a_m V^a + \partial_a V^h) X^a, \]
\[ X^H := V^h X_h, \]
\[ X^V := V^h X^\pi. \]

From Lemma 2 we know that \( X^C, X^H \) and \( X^V \) are fiber-preserving vector fields on \( TM \).

**Lemma 3.** [1] Let \( X \) be a fiber-preserving vector field on \( TM \). Then the Lie derivative of the adapted frame and its dual basis are given by:

I) \( \mathcal{L}_X X_a = (\partial_b X^a) X_a + \left( y^b X^e K_{hec}^a - X^b \Gamma^a_{b h} - X_h (X^\pi) \right) X_\pi \), II) \( \mathcal{L}_X X_\pi = \{ X^b \Gamma^a_{b h} - X_h (X^\pi) \} X_a \),

III) \( \mathcal{L}_X dx^h = (\partial_m X^h) dx^m \),

IV) \( \mathcal{L}_X \delta y^m = -\{ y^b X^e K_{mec}^h - X^b \Gamma^h_{m b} - X_m (X^\pi) \} dx^m - \{ X^b \Gamma^h_{m b} - X_m (X^\pi) \} \delta y^m. \)

**Lemma 4.** [8] Let \( X \) be a fiber-preserving vector field on \( TM \), which induces a vector field \( V \) on \( M \). Then Lie derivatives \( \mathcal{L}_x g_{ij} \), \( \mathcal{L}_x g_{22} \) and \( \mathcal{L}_x g_{33} \) are given by:

I) \( \mathcal{L}_x g_{ij} = (\mathcal{L}_x g_{ij}) dx^i dx^j \),

II) \( \mathcal{L}_x g_{22} = 2 \{ \partial_m \left( y^b X^e K_{meh}^i - X^b \Gamma^i_{m b} - X_m (X^\pi) \right) dx^i dx^j + \}

\[ \{ \mathcal{L}_x g_{ij} - \partial_m \nabla_i X^m + g_{jm} X_j (X^\pi) \} dx^j dy^i \],

III) \( \mathcal{L}_x g_{33} = -2 \{ \partial_m \left( y^b X^e K_{meh}^i - X^b \Gamma^i_{m b} - X_m (X^\pi) \right) dx^i dy^j + \}

\[ \{ \mathcal{L}_x g_{ij} - \partial_m \nabla_i X^m + 2 g_{mj} X_j (X^\pi) \} dy^i dy^j, \]

where \( \mathcal{L}_x g_{ij} \) and \( \nabla_i X^m \) denote the components of \( \mathcal{L}_x g \) and the covariant derivative of \( V \) respectively.

**6. MAIN RESULTS**

**Proposition 1.** Let \( X \) be a complete (resp. horizontal or vertical) lift conformal vector field on \( TM \). Then the scalar function \( \Omega(x, y) \) in \( \mathcal{L}_x \tilde{g} = 2 \Omega \tilde{g} \) is a function of position alone (resp. \( \Omega = 0 \)).

**Proof:** Let \( TM \) be the tangent bundle over \( M \) with Riemannian metric \( \tilde{g} \) and \( X \) be a complete (resp. horizontal or vertical) lift conformal vector field on \( TM \). By definition, there is a scalar function \( \Omega \) on \( TM \) such that

\[ \mathcal{L}_x \tilde{g} = 2 \Omega \tilde{g}. \]

Since the complete horizontal and vertical lift vector fields are fiber preserving, by applying \( \mathcal{L}_x \) to the definition of \( \tilde{g} \), using Lemma 4 and the fact that \( dx^i dx^j, \ dx^i dy^j \) and \( dy^i dy^j \) are linearly independent, we have following three relations

\[ a(\mathcal{L}_x g_{ij} - 2 \Omega g_{ij}) = b g_{im} \left( y^b X^e K_{meh}^i - X^b \Gamma^i_{m b} - X_m (X^\pi) \right) \]

\[ + g_{jm} \left( y^b X^e K_{meh}^i - X^b \Gamma^i_{m b} - X_m (X^\pi) \right) \] (2)

\[ b(\mathcal{L}_x g_{ij} - 2 \Omega g_{ij}) = b g_{im} \left( \nabla_j X^m - X_j (X^\pi) \right) \]

\[ + c g_{jm} \left( y^b X^e K_{meh}^i - X^b \Gamma^i_{m b} - X_m (X^\pi) \right). \] (3)
HEDAYATIAN S.          BIDABAD B.

Using relation 1, we have 
\[ 2\Omega g_{ij} = g_{mj}X_i(X^m) + g_{mi}X_j(X^m). \]  
(4)

Applying \( X_\tau \) to the relation 4 and using the fact that \( g_{ij} \) is a function of position alone, we have 
\[ 2g_{ij}X_\tau(\Omega) = g_{mj}X_\tau X_i(X^m) + g_{mi}X_\tau X_j(X^m). \]  
(5)

By means of definition 1 for complete lift vector fields, and by replacing the value of \( X^m \) in relation 5, we have
\[ 2g_{ij}X_\tau(\Omega) = g_{mj}X_\tau X_i(\gamma^j(\Gamma^m_i V^a + \partial_i V^m)) + g_{mi}X_\tau X_j(\gamma^j(\Gamma^m_i V^a + \partial_i V^m)). \]

Since the coefficients of the Riemannian connection on \( M \), and components of vector field \( V \) are functions of position alone, the right hand side of the above relation becomes zero, from which we have \( X_\tau(\Omega) = 0 \). This means that the scalar function \( \Omega(x,y) \) on \( TM \) depends only on the variables \( (x^k) \).

Similarly, for vertical lift vector fields, by using the fact that the components of \( V \) are functions of position alone and from relation 4, we have \( \Omega = 0 \). Finally, for horizontal lift vector field by means of relation 4, we have \( \Omega = 0 \).

**Proposition 2.** Let \( M \) be a connected manifold and \( X \) be a complete lift conformal vector field on \( TM \). Then the scalar function \( \Omega(x,y) \) in \( L_\gamma \bar{g} = 2\Omega \bar{g} \) is constant.

**Proof:** Let \( X \) be a complete lift conformal vector field on \( TM \) with components \( (X^h,X^\bar{h}) \), with respect to the adapted frame \( \{X_h,X_{\bar{h}}\} \).

Let us put 
\[ A^m_a = \Gamma^m_{a h}X^h + \partial_a X^m. \]

The coordinate transformation rule implies that \( A^m_a \) are the components of \((1,1)\) tensor field \( A \). Then its covariant derivative is 
\[ \nabla_i A^m_a = \partial_i A^m_a + \Gamma^m_{ik}A^k_a - \Gamma^m_{ia}A^m_k, \]

where \( \nabla_i A^m_a \) is the component of the covariant derivative of tensor field \( A \).

From definition 1, \( X^m = A^m_{a b^a} \). By means of relation 3, we have 
\[ b[L_i g_{ij} - 2\Omega g_{ij} - g_{km}(\nabla_j X^m - A^m_j)] = c g_{jm}[y^a X^c K_{ica}^m - \Gamma^m_{k}A^k_ay^a - \Gamma^m_{ia}A^m_k - \Omega(x,y)] \]

Note that the components of \( A \) are functions of position alone, from which the right hand side of this relation becomes
\[ c g_{jm}[y^a X^c K_{ica}^m - \Gamma^m_{k}A^k_ay^a - \Gamma^m_{ia}A^m_k - \Omega(x,y)] = c y^a (X^c K_{ica} - g_{ij} \nabla_i A^m_a). \]
Thus we have

\[ b[L_{i}g_{ij} - 2\Omega g_{ij} - g_{mi}(\nabla_{j}X^{m} - A_{j}^{m}i)] = cy^{n}(X^{c}K_{icaj} - g_{mj}\nabla_{i}A_{a}^{m}). \]

By means of Proposition 1 the left hand side of the above relation is a function of position alone. Applying \( X_{\tau} = \frac{\partial}{\partial y^{\tau}} \) to this relation gives

\[ X^{c}K_{icaj} - g_{mj}\nabla_{i}A_{a}^{m} = 0, \]

Or

\[ X^{a}K_{icaj} = \nabla_{i}A_{a}. \]

From which

\[ \nabla_{i}A_{ja} + \nabla_{i}A_{aj} = 0. \tag{6} \]

Now by replacing \( X^{m} \) in relation 4

\[ 2\Omega g_{ij} = g_{mj}X_{\tau}\{(y^{b}(\Gamma_{h}^{m}aX^{a} + \partial_{h}X^{m})\} + g_{mi}X_{\tau}\{(y^{b}(\Gamma_{h}^{m}aX^{a} + \partial_{h}X^{m})\} \]

\[ = g_{mj}(\Gamma_{i}^{a}aX^{a} + \partial_{i}X^{m}) + g_{mi}(\Gamma_{j}^{a}aX^{a} + \partial_{j}X^{m}) \]

\[ = g_{mj}A_{ji}^{m} + g_{mi}A_{j}^{m}. \]

Applying covariant derivation \( \nabla_{k} \) to this relation gives

\[ 2g_{\Omega}\nabla_{k}\Omega = \nabla_{k}A_{ji} + \nabla_{k}A_{ij}. \]

From relation 6, we get \( \nabla_{k}\Omega = \frac{\partial}{\partial y^{\tau}}\Omega = 0 \).

Since \( M \) is connected, the scalar function \( \Omega \) is constant.

**Theorem 1.** Let \( M \) be a connected \( n \)-dimensional Riemannian manifold and \( TM \) be its tangent bundle with metric \( \tilde{g} \). Then every complete lift conformal vector field on \( TM \) is homothetic, moreover, every horizontal or vertical lift conformal vector field on \( TM \) is a Killing vector.

**Proof:** Let \( M \) be an \( n \)-dimensional Riemannian manifold, \( TM \) its tangent bundle with the metric \( \tilde{g} \) and \( X \) a complete (resp. horizontal or vertical) lift conformal vector field on \( TM \). Then by means of Proposition 1 the scalar function \( \Omega(x,y) \) in \( L_{\tilde{g}} \bar{g} = 2\Omega \tilde{g} \) is a function of position alone (resp. \( \Omega = 0 \)), and by means of Proposition 2 it is constant. Thus, every complete lift conformal vector field on \( TM \) is homothetic and every horizontal or vertical lift conformal vector field on \( TM \) is a Killing vector.

**Theorem 2.** Let \( M \) be a connected \( n \)-dimensional Riemannian manifold and \( TM \) be its tangent bundle with metric \( \tilde{g} \). Then every inessential fiber preserving conformal vector field on \( TM \) is homothetic.

**Proof:** Let \( X \) be an inessential fiber preserving conformal vector field on \( TM \) with components \( (X^{h},X^{\tau}) \), with respect to the adapted frame \( \{X_{h},X_{\tau}\} \). Using the same argument in proof of Proposition 1, it is obvious that we have relations 2, 3 and 4. From relation 4, we have

\[ \Omega g_{ji} = g_{mi}X_{\tau}(X^{\tau}). \]
Since $\Omega(x, y)$ in $\mathcal{L}_g \tilde{g} = 2\Omega \tilde{g}$ is supposed to be a function of position alone, by applying $X_i$ to the above relation we have

$$X_i(X_i(X^m)) = 0.$$ 

Applying $X_i$ to relation 4 again and using above relation gives

$$X_i(X_i(X^m)) = 0.$$ 

Thus we can write

$$X^m = \alpha^m_a y^a + \beta^m,$$  

(7)

where $\alpha^m_a$ and $\beta^m$ are certain functions of position alone. Replacing relation 7 in relation 3, we have

$$b(\mathcal{L}_g g_{ij} - 2\Omega g_{ij}) = bg_{im}(\nabla_j X^m - \alpha^m_j) + cg_{jm}(y^b X^c K_{icb}^m - y^d \alpha^b_a X^c_{iab}^m - \beta^b_a \Gamma^c_{b i}^m -$$

$$y^a \frac{\partial}{\partial x^a} \alpha^m_a - \frac{\partial}{\partial x^a} \beta^m + y^e \Gamma^c_{a e}^b \alpha^m_a)$$

$$= bg_{im}(\nabla_j X^m - \alpha^m_j) + cg_{jm}(y^b X^c K_{icb}^m - y^d \nabla_i \alpha^m_a) - cg_{jm} \nabla_i \beta^m.$$

Therefore

$$b(\mathcal{L}_g g_{ij} - 2\Omega g_{ij} - g_{im}(\nabla_j X^m - \alpha^m_j)) + cg_{jm} \nabla_i \beta^m = cg_{jm} y^a (X^c K_{ica}^m - \nabla_i \alpha^m_a).$$

The left hand side of this relation is a function of position alone. From which by applying $X_\tau$ we have

$$X^c K_{ica}^m = \nabla_i \alpha^m_a.$$  

(8)

Replacing relation 7 in relation 4 we find

$$2\Omega g_{ij} = \alpha_{ij} + \alpha_{ij}.$$ 

The covariant derivative of this relation and using relation 8 gives

$$\nabla_\tau \Omega = \frac{\partial}{\partial x^\tau} \Omega = 0.$$ 

Since $M$ is connected, then the scalar function $\Omega$ on $M$ is constant. This completes the proof of Theorem 2.

REFERENCES

1. Yamauchi, K. (1995). On infinitesimal conformal transformations of the tangent bundles over Riemannian manifolds. Ann Rep. Asahikawa. Med. Coll., 16, 1-6, and (1996). Ann. Rep. Asahikawa. Med. Coll. 17, 1-7, and (1997). Ann. Rep. Asahikawa. Med. Coll., 18, 27-32.
2. Hasegawa, I. & Yamauchi, K. (2003). Infinitesimal conformal transformations on tangent bundles with the lift metric $1 + 2$. Scientiae Mathematicae Japanicae 57, (1), 129-137, e7, 437-445.
3. Yano, K. & Ishihara, S. (1973). Tangent and Cotangent Bundles. Department of Mathematics Tokyo Institute of Technology, Marcel Dekker, Tokyo, Japan.
4. Yano, K. & Kobayashi, H. (1996). Prolongations of tensor fields and connection to tangent bundle I, General theory. Jour. Math. Soc. Japan, 1894-210.
5. Bejancu, A. (1990). Finsler geometry and applications. *Ellis Horwood Limited publication*.

6. Nakahara, M. (1990). *Geometry Topology and Physics*. Physics institute, Faculty of Liberal Arts Shizuoka, Japan., Bristol and New York, Adam Hilger.

7. Miron, R. (1981). Introduction to the theory of Finsler spaces. *Proc. Nat. Sem. On Finsler spaces, Brasov* (131-183). & (1987). Some connections on tangent bundle And their applications to the general relativity. *Tensor N. S.* 46, 8-22.

8. Yawata, M. (1991). Infinitesimal isometries of frame bundles with natural Riemannian metric. *Tohoku Math. J.* (2), 43(1), 103-115.