The Efficient Frontier in Randomized Social Choice

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Abstract

Since the celebrated Gibbard-Satterthwaite impossibility results and Gibbard’s 1977 extension for randomized rules, it is known that strategyproofness imposes severe restrictions on the design of social choice rules. In this paper, we employ approximate strategyproofness and the notion of score deficit to study the possible and necessary trade-offs between strategyproofness and efficiency in the randomized social choice domain. In particular, we analyze which social choice rules make optimal trade-offs, i.e., we analyze the efficient frontier. Our main result is that the efficient frontier consists of two building blocks: (1) we identify a finite set of “manipulability bounds” \( B \) and the rules that are optimal at each of them; (2) for all other bounds not in \( B \), we show that the optimal rules at those bounds are mixtures of two rules that are optimal at the two nearest manipulability bounds from \( B \). We provide algorithms that exploit this structure to identify the entire efficient frontier for any given scoring function. Finally, we provide applications of our results to illustrate the structure of the efficient frontier for the scoring functions \( v = (1, 0, 0) \) and \( v = (1, 1, 0) \).

1. Introduction

When a group of people has to decide on an action to collectively pursue, there is almost always disagreement among its members about what to do, e.g., who should be the next president, whom to side with in an armed conflict, or where to build a new stadium. Social choice rules have a long history of facilitating decisions in these and many other situations. However, it may be possible for individual participants (agents) to manipulate the rule to their own advantage by being insincere about their own preferences. While the goal of a social choice rule is to determine an outcome with high social welfare (efficiency), incentives for truth-telling are a major concern; a rule that receives false input from lying agents will be unable to select an efficient outcome.

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1.1. The Curse of Strategyproofness

The desideratum of strategyproofness requires that a rule makes it a dominant strategy for all agents to reveal their preferences truthfully. The famous impossibility results by Gibbard (1973) and Satterthwaite (1975) established that if there are at least three alternatives and all strict preferences are possible, then the only “reasonable” deterministic social choice rules that are strategyproof are dictatorships. (Gibbard, 1977) extended these insights to social choice rules that involve randomization and showed that all strategyproof randomized social choice rules are probability mixtures of strategyproof unilaterals and duples. However, dictatorial rules or mixtures of unilaterals and duples are undesirable in most applications. Therefore, much research is committed to circumventing these impossibility results. Important approaches include approximation, domain restrictions, limit results in large markets, computational complexity as a barrier to manipulation, or allowing for “a little bit of” manipulability. By relaxing the strategyproofness requirement, mechanism designers hope to achieve better performance of their rules on other dimensions. In this paper, we also relax strategyproofness and analyze what can be gained in return in terms of efficiency. Specifically, we study randomized social choices rules that make this trade-off optimally.

1.2. Measuring Manipulability and (In)efficiency

To measure the degree of manipulability of a non-strategyproof social choice rule we use ε-approximate strategyproofness, a common relaxation of strategyproofness, which puts a quantitative limit (i.e., ε) on the gain that any agent can obtain by lying about its preferences. The economic intuition behind this requirement is that if the potential gain is small, the agents might not be willing to collect the necessary information and deliberate about misreports, but stick with truthful reporting instead. For the ε-approximate strategyproofness concept to be meaningful in the randomized social choice domain, we follow earlier work (e.g., (Birrell and Pass, 2011; Carroll, 2013)), and assume that agents’ preferences over lotteries are given by normalized vNM utility functions. The manipulability of any randomized social choice rule is the smallest bound ε ∈ [0, 1] for which it is ε-approximately strategyproof. By allowing weaker incentive requirements, i.e., higher ε, the design space of possible social choice rules grows, and the question arises how this new freedom can be harnessed to improve efficiency. To measure the efficiency of social choice rules, we use the notion of score, a well known concept from the study of positional scoring rules (see, e.g., (Xia and Conitzer, 2008)). Under a positional scoring rule, alternatives receive scores based on their positions in the preference orders of the agents, and the rule then picks an outcome that maximizes the aggregate score. Many (but not all) prominent social choice rules can be represented as positional scoring rules: giving positive scores only for first choices yields Plurality, giving equal, positive scores for all but the last choices yields the Veto rule, and linear scoring yields the Borda rule. If a rule is not a positional scoring rule (e.g., Random Dictatorship), there is at least one type profile where it selects an outcome with less than maximal score, i.e., the rule incurs a score deficit. In this paper, we follow the approach of Procaccia (2010) to quantify the (in)efficiency of randomized social choice rules in terms of the worst-case score deficit, i.e., the largest score deficit that the
1.3. Trading off Strategyproofness and Efficiency: the Efficient Frontier

Different rules have different manipulability \( \varepsilon \) and worst-case score deficit \( \sigma \), and as mechanism designers, we want to select rules for which both values are low. Figure 1 visualizes this strategyproofness-efficiency-footprint for three rules \( f \), \( g \), and \( h \). Consider Random Dictatorship, which is strategyproof but has a non-zero worst-case score deficit (like \( f \) in Fig. 1). Next, consider Plurality, which has a worst-case score deficit of zero (because it is a positional scoring rule), but it is not strategyproof (like \( g \) in Fig. 1). Choosing between these two rules requires an obvious trade-off between strategyproofness and efficiency. However, both rules may be unacceptable, i.e., the worst-case score deficit of Random Dictatorship may be too high, and the manipulability of Plurality may also be too high. Thus, we may prefer a rule with intermediate performance on both dimensions (like \( h \) in Fig. 1). One way to create intermediate rules is via “mixing:” the \( \beta \)-hybrid of \( f \) and \( g \) is a rule that first collects the agents’ reports, then randomly decides to use \( f \) or \( g \) with probabilities \( \beta \) and \( 1 - \beta \), respectively. We prove appealing guarantees for the footprints of such hybrids and demonstrate how they can be used to make interesting trade-offs between strategyproofness and efficiency. Ultimately, however, as mechanism designers we seek optimal rules, which minimize the worst-case score deficit across all rules that are \( \varepsilon \)-approximately strategyproof for a given \( \varepsilon \). To understand the possible and necessary trade-offs between strategyproofness and efficiency, we need an understanding of the efficient frontier, i.e., the set of all optimal rules. One might expect that the hybrid of two rules on the efficient frontier is again on the efficient frontier. While this is not true in general, we show that hybrids are an essential building block: there exists a finite segmentation \( \varepsilon_0 = 0 < \ldots < \varepsilon_K = 1 \), such that between any two “hinges” \( \varepsilon_k \) and \( \varepsilon_{k+1} \) the efficient frontier consists precisely of the hybrids of two rules that are optimal at \( \varepsilon_k \) and \( \varepsilon_{k+1} \), respectively. We also give algorithms that exploit this structure to identify the entire efficient frontier.
1.4. Overview of Contributions

Our main contributions are as follows:

1. **Determining Optimal Rules (Prop. 2 & Alg. FindOpt):** we prove a novel equivalence between \( \varepsilon \)-approximate strategyproofness and a finite set of linear constraints. This allows the characterization (and computation) of rules on the efficient frontier as solutions to a linear program.

2. **Footprints of Hybrid Rules (Prop. 3, Thm. 1, 2):** first, we show that the footprint of any \( \beta \)-hybrid of rules \( f, g \) is weakly better than the \( \beta \)-convex combination of the footprints of \( f \) and \( g \). Second, as \( \beta \) runs from 0 to 1, the footprints of \( h_\beta \) follow a piecewise-linear path that connects the footprints of \( f \) and \( g \). Third, we give an equivalent condition for when this path is in fact a single linear segment.

3. **Footprints of the Efficient Frontier (Cor. 1, Thm. 3, Alg. FindHinges):** we show that the mapping \( \varepsilon \mapsto \sigma_\varepsilon \) that associates each manipulability \( \varepsilon \) with the lowest attainable worst-case score deficit \( \sigma_\varepsilon \) is well-defined, non-increasing, and convex. Thus, the marginal efficiency improvement from accepting additional manipulability is lower for more manipulable rules along the efficient frontier. We also show that \( \varepsilon \mapsto \sigma_\varepsilon \) is piecewise-linear, and our algorithm FindHinges, exploits this structure to compute the entire efficient frontier with few executions of FindOpt.

4. **Applications (Sec. 7):** we apply our findings to show that anonymity and neutrality come “for free” in terms of strategyproofness and efficiency, while this is not true for Condorcet consistency, Pareto optimality, or unanimity. In a setting with 3 agents, 3 alternatives, strict preferences, and scoring function \( v = (1, 0, 0) \), we show that Uniform Plurality is \( \frac{1}{3} \)-approximately strategyproof, and Random Dictatorship and Uniform Plurality as well as their hybrids are on the efficient frontier. For \( v = (1, 1, 0) \), we show that Uniform Veto is \( \frac{1}{2} \)-approximately strategyproof, Random Duples and Uniform Veto are both on the efficient frontier, but their hybrids are not.

This paper provides a new understanding of the possible and necessary trade-offs between strategyproofness and efficiency in the randomized social choice domain. The insights about the structure of the efficient frontier unlock the set of optimal social choice rules for further analytical, axiomatic, and algorithmic explorations.

2. Related Work

The social choice domain is littered with impossibility results revolving around strategyproofness. The seminal Gibbard-Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) established that if all strict preferences over at least 3 alternatives are possible, then the only unanimous, strategyproof, deterministic rules are dictatorial. Gibbard (1977) extended this result for randomized social choice rules. Much research has since been committed to circumventing these impossibility results. *Approximate strategyproofness* captures the intuition that an agent may
not bother to manipulate as long as the potential gain from manipulation is small. In domains that rule out payments, the definition of approximate strategyproofness requires a normalization of the agents’ underlying cardinal utilities. Carroll (2013) used this approach to quantify the manipulability of rules in different informational environments. Using a distance metric based on vote-corruption, Birrell and Pass (2011) explored how well different rules can be approximated by approximately strategyproof rules. In this paper, we quantify manipulability by the same notion of approximate strategyproofness. Another way to circumvent the impossibility results is to consider restricted preference domains. Moulin (1980) showed that in the single-peaked domain, all strategyproof, anonymous, and efficient rules are variants of the Median rule with additional virtual agents. Ehlers, Peters and Storcken (2002) extended this result to randomized rules. Azevedo and Budish (2012) introduced the desideratum of strategyproofness in the large, where incentive requirements are formulated as properties of the rule in limit settings. In (Mennle and Seuken, 2014) we proposed partial strategyproofness, which arises from an axiomatic relaxation of strategyproofness for assignment mechanisms. Since the proposition of computational complexity as a barrier against manipulation (Bartholdi, Tovey and Trick, 1989), we have obtained a good understanding of the complexity of the manipulation problem under various rules (Xia, 2011). However, the standard worst-case notion of complexity does not prevent manipulation strategies from being effective on average. A recent stream of work (Xia, 2008; Dobzinski and Procaccia, 2008; Friedgut et al., 2011; Isaksson, Kindler and Mossel, 2012; Mossel and Rácz, 2014) established that the manipulation problem has polynomial average complexity. The efficiency properties of social choice rules are traditionally assessed along an array of axioms, e.g., unanimity, Pareto optimality, Condorcet consistency, etc. A more refined evaluation is possible via efficiency notions based on dominance relations (Aziz, Brandt and Brill, 2013; Aziz, Brandl and Brandt, 2014). However, this remains a “binary” measure in the sense that rules either satisfy the requirement or not. Positional scoring rules are rules that select alternatives to maximize a particular score, which is computed based on the agents’ reported preferences (Xia and Conitzer, 2008). Procaccia (2010) considered the notion of a worst-case score deficit, i.e., the deviation from the maximum score, to give bounds on how well strategyproof rule can approximate positional scoring rules. In the present paper, we employ the same measure for efficiency. One of our main findings is that rules on the efficient frontier correspond to solutions of a linear program. This idea is related to the automated mechanism design approach, where desirable properties are encoded as constraints of an optimization problem, which is then solved, e.g., using linear programming, to obtain “good” mechanisms (Sandholm, 2003). Similarly, Brandt and Geist (2014) employed SAT solvers to identify rules with particular properties and prove impossibility results.

3. The Model

We formulate our results for the randomized social choice domain with indifferences.
3.1. Formal Model

$N$ is a set of $n$ agents and $M$ as set of $m$ alternatives. Agents have (weak) preferences over alternatives, where $a \succeq_i b$ indicates that agent $i$ weakly prefers alternative $a$ over alternative $b$. If $a \succeq_i b$ and $b \succeq_i a$, then $i$ is indifferent between $a$ and $b$, which is denoted by $a \sim_i b$. Conversely, if $a \succeq_i b$ and $b \succeq_i a$, then $i$ strictly prefers $a$ to $b$, denoted by $a \succ_i b$. Agents’ preferences over alternatives are extended to lotteries over alternative via von Neumann-Morgenstern utility functions. These utilities are assumed to be normalized, so that we can define a meaningful notion of $\varepsilon$-approximate strategyproofness. A utility function $u : M \to [0,1]$ represents a preference order $\succeq$ if $u(a) \geq u(b)$ whenever $a \succeq b$. We let $U$ denote the space of all utility functions, and for any preference order $\succeq$ the corresponding type $t_\succeq \subseteq U$ is the set of utility functions that represent $\succeq$. Every agent $i$ is endowed with a utility function $u_i \in t_\succeq_i$ that represents its preference order $\succeq_i$. We write $t_i$ to denote $t_{\succeq_i}$, the type of agent $i$, and $T$ to denote the set of all possible types. Furthermore, for any alternative $j \in M$ let $r_{t_i}(j) = \#\{j' \in M | j' >_i j\} + 1$ be the rank of $j$ at $t_i$, i.e., the position of $j$ in the preference order of $i$. 1 is added to the rank so that the first choice alternative receives rank 1 (and not 0). We use the notation $t = (t_1, t_{-1})$ for type profiles, where $t_{-1}$ denotes the types of all the other agents, except $i$. A randomized social choice rule is a mapping $f : T^n \to \Delta(M)$, where $\Delta(M)$ denotes the space of probability distributions over alternatives. Thus, $f$ picks a lottery over the alternatives, and any such lottery $x \in \Delta(M)$ is called an outcome. The present paper studies randomized social choice rules, so we will usually suppress the prefix and simply speak about social choice rules, or just rules.

Remark 1. Unless stated otherwise, the results are formulated for the full domain with indifference, but most proofs do not make particular use of the richness of the domain. They remain valid on (almost) all domain restrictions, including strict preferences, assignment, and two-sided matching. In Section 8 we comment on the generality of our results.

Our goal is to study the trade-off between strategyproofness and efficiency. For this purpose we need measures for the performance of different rules on both dimensions. In this section we recap the notion of the worst-case score deficit as used by Procaccia (2010). The idea of “scoring” is that society associates a certain value “$v(r)$” with selecting an alternative that is some agent’s $r$th choice. A scoring function $v$ is a mapping that assigns this value to each rank $r$, i.e., $v : \{1, \ldots, m\} \to [0,1]$ with $v(r) \geq v(r+1)$ for all $r \in \{1, \ldots, m-1\}$. For the remainder of this paper, we fix the setting $(N, M, v)$, i.e., number of agents, number of alternatives, and scoring function $v$.

**Definition 1** (Score). For an outcome $x \in \Delta(M)$ and a type profile $t \in T^n$ the score of $x$ at $t$ with respect to $v$ is given by

$$sc_v(x, t) = \frac{1}{n} \sum_{i \in N} \sum_{j \in M} x_j \cdot v(r_i(j)). \quad (1)$$

It is interpreted as the expected value from selecting outcome $x$.\(^1\) Naturally, one might wish

\(^1\)The factor $\frac{1}{n}$ normalizes the score for the sake of exposition, but without loss of generality.
to select outcomes that maximize the score, where the score from (1) is equal to $s_{v}^\text{max}(t) = \max_{x \in \Delta(M)} s_{v}(x, t)$. Positional scoring rules are designed to achieve this.

**Definition 2** (Positional Scoring Rule). A rule $f$ is a positional scoring rule if for all type profiles $t \in T^n$ we have that

$$s_{v}(f(t), t) = s_{v}^\text{max}(t).$$

(2)

For example, choosing $v(k) = \frac{k}{m}, k \in \{1, \ldots, m\}$ yields the Borda rule. Any rule that is not a positional scoring rule produces outcomes that deviate from the maximal score at one or more type profiles, i.e., they incur a deficit.

**Definition 3** (Score Deficit, adapted from (Procaccia, 2010)). For an outcome $x \in \Delta(M)$ and a type profile $t \in T^n$ the score deficit of $x$ at $t$ with respect to $v$ is given by

$$\sigma_{v}(x, t) = s_{v}^\text{max}(t) - s_{v}(x, t).$$

(3)

For a rule $f$, the worst-case score deficit is the largest score deficit across all type profiles, i.e.,

$$\sigma_{v}(f) = \max_{t \in T^n} \sigma_{v}(f(t), t).$$

(4)

Since the scoring function $v$ is fixed in our context, we usually suppress the index and write $\sigma(f)$, and for the sake of brevity, we sometimes simply call it deficit. A lower score deficit means that the “damage to efficiency” is low, relative to what could have been achieved by a positional scoring rule. Thus, the worst-case score deficit $\sigma(f)$ gives a bound for how far away from the optimal choice of outcome $f$ can be. $\sigma(f)$ is always weakly positive, since by construction no rule can achieve a better score than a positional scoring rule. Finally, by definition, a rule $f$ is a positional scoring rule if and only if it has a worst-case score deficit $\sigma(f)$ of zero.

### 3.2. Measuring Manipulability

The second dimension for our assessment is manipulability. The most demanding incentive concept is strategyproofness.

**Definition 4** (Strategyproofness). A rule $f$ is strategyproof if for all agents $i \in N$, type profiles $t = (t_i, t_{-i}) \in T^n$, utilities $u_i \in t_i$, and possible misreports $t_i' \in T$

$$\sum_{j \in M} u_i(j) \cdot (f_j(t_i', t_{-i}) - f_j(t_i, t_{-i})) \leq 0,$$

(5)

i.e., deviating from the true type report yields weakly lower expected utility, independent of the other agents’ reports.

The next definition of $\varepsilon$-approximate strategyproofness relaxes Definition 4 by replacing the bound “0” by a small, positive value $\varepsilon$ on the right-hand side of (5). The assumption of normalized utility is needed at this point: with unbounded utilities, the potential gain from manipulation under any non-strategyproof rule would instantly become arbitrarily large (see (Carroll, 2013)).
Definition 5 ($\varepsilon$-Approximate Strategyproofness). For $\varepsilon \in [0, 1]$, a rule $f$ is $\varepsilon$-approximately strategyproof if for all agents $i \in N$, type profiles $t = (t_i, t_{-i}) \in T^n$, utilities $u_i \in t_i$, and possible misreports $t'_i \in T$

$$\varepsilon \left( u_i, (t_i, t_{-i}), t'_i, f \right) = \sum_{j \in M} u_i(j) \cdot \left( f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}) \right) \leq \varepsilon, \quad (6)$$

where $\varepsilon \left( u_i, (t_i, t_{-i}), t'_i, f \right)$ denotes the expected utility gain that $i$ can obtain by switching to the false report $t'_i$.

For $\varepsilon = 0$, Definition 5 is equivalent to full strategyproofness. Since the utility of the agents in our model is normalized, the gain will never exceed 1. Thus, for $\varepsilon = 1$, constraint (6) is trivially satisfied by any rule. Obviously, if $f$ is $\varepsilon$-approximately strategyproof, then it is also $\varepsilon'$-approximately strategyproof for any $\varepsilon' \geq \varepsilon$.

Definition 6. (Manipulability, adapted from (Carroll, 2013; Birrell and Pass, 2011)) For a rule $f$ its manipulability $\varepsilon(f)$ is the largest gain that any agent could obtain by misreporting, i.e.,

$$\varepsilon(f) = \max_{i \in N, t \in T^n, u_i \in t_i, t'_i \in T} \varepsilon \left( u_i, (t_i, t_{-i}), t'_i, f \right). \quad (7)$$

4. Efficient Frontier

In this paper we study the efficient frontier, i.e., the set of rules with the lowest deficit $\sigma(f)$, subject to $\varepsilon$-approximate strategyproofness for some $\varepsilon \in [0, 1]$. We now formalize this efficient frontier and give our first main results.

4.1. Footprint of Rules

To formalize the relation between the deficit $\sigma(f)$ and the manipulability $\varepsilon(f)$ of $f$, we introduce the notion of a footprint.

Definition 7 (Footprint). The tuple $(\varepsilon(f), \sigma(f))$ is called the footprint of $f$.

The footprint allows a convenient graphical representation of the strategyproofness and efficiency properties. Since strategyproofness is equivalent to 0-approximate strategyproofness, the footprint of any strategyproof rule must have value 0 in the first component. On the other hand, the second component of the footprint of any positional scoring rule is 0. Figure 2 (i) gives examples of footprints. The “ideal” rule would have footprint $(0, 0)$ in the bottom left. However, positional scoring rules are not strategyproof in general (see Lemma 1), and this footprint is usually unattainable.
4.2. Optimal Rules

When trading off strategyproofness and efficiency, the mechanism designer may accept some more manipulability in exchange for better efficiency, e.g., when accepting a manipulability of $\varepsilon$, she would naturally prefer a rule that has the lowest deficit among all $\varepsilon$-approximately strategyproof rules. We denote this lower bound by $\sigma_\varepsilon = \inf \{\sigma(f')|f' \ \varepsilon$-approximately strategyproof}. Optimal rules attain this bound.

**Definition 8 (Optimal Rule).** A rule $f$ is optimal at $\varepsilon \in [0,1]$ if $f$ is $\varepsilon$-approximately strategyproof, and $\sigma(f) = \sigma_\varepsilon$. We let $\text{Opt}(\varepsilon)$ denote the set of all rules that are optimal at $\varepsilon$, and any optimal rule $f \in \text{Opt}(\varepsilon)$ is a representative of $\text{Opt}(\varepsilon)$.

**Definition 9 (Efficient Frontier).** The efficient frontier consists of all rules that are optimal at some $\varepsilon \in [0,1]$, denoted by

$$\text{Opt}([0,1]) = \bigcup_{\varepsilon \in [0,1]} \text{Opt}(\varepsilon).$$

(8)

It is not obvious that the efficient frontier is non-empty, since the lower bound $\sigma_\varepsilon$ in Definition 8 may not actually be attained by any $\varepsilon$-approximately strategyproof rules. Proposition 1 establishes that $\text{Opt}(\varepsilon)$ is indeed never empty.

**Proposition 1.** The mapping $\varepsilon \mapsto \sigma_\varepsilon$ that associates each manipulability $\varepsilon$ with the lowest attainable deficit is well-defined and non-increasing in $\varepsilon$.

**Proof idea.** (formal proof in Appendix C.1.1) To show that $\text{Opt}(\varepsilon)$ is non-empty, we construct a sequence of $\varepsilon$-approximately strategyproof rules for which the deficits approach $\sigma_\varepsilon$. A compactness argument yields convergence (in the space of $\varepsilon$-approximately strategyproof rules), and the limit rule attains the minimal deficit.

Figure 2 (ii) show examples of what the graph of $\varepsilon \mapsto \sigma_\varepsilon$ can and cannot look like.
4.3. Computing Optimal Rules

We now present our first main results, the representation of $\text{OPT}(\varepsilon)$ as the set of solutions of a particular linear program. The automated mechanism design approach (Sandholm, 2003) proposed encoding desirable properties as linear constraints, so “good” mechanisms can be found by solving a linear optimization problem. The challenge with this approach in our context is that $\varepsilon$-approximate strategyproofness imposes restrictions for all utility functions $u_i \in t_i$. Since $t_i$ contains an infinite number of utilities, there is no straightforward way to incorporate this requirement. The following Proposition yields an equivalent set of conditions for this purpose.

**Proposition 2.** For $\varepsilon \in [0, 1]$ and a rule $f$ the following are equivalent:

1. $f$ is $\varepsilon$-approximately strategyproof.

2. For all type profiles $t \in T^n$, agents $i \in N$, misreports $t'_i \in T$, and ranks $k \in \{1, \ldots, m\}$
   \[ \sum_{j \in M : r_j(i) \leq k} f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}) \leq \varepsilon. \]

**Proof idea.** (formal proof in Appendix C.1.2) The key is the representation of any utilities as a subset of the convex hull of particular binary utilities. (9) is precisely the $\varepsilon$-constraints for these utilities. A limit argument is needed for (1) $\Rightarrow$ (2).

The solutions to the following linear program correspond to optimal rules, where the variables (in typewriter font) have the obvious interpretation.

**Linear Program 1** ($\text{FindOpt}(\varepsilon)$).

\[
\begin{align*}
\text{minimize} & \quad s_{\text{max}} \\
\text{subject to} & \quad \sum_{j \in M} f_j(t) = 1 \quad \forall t \in T^n \quad \text{(Objective)} \\
& \quad s_{\text{max}}(t) - \sum_{i \in N, j \in M} v(r_i(j)) \cdot f_j(t) \leq s_{\text{max}} \quad \forall t \in T^n \quad \text{(Probability)} \\
& \quad \sum_{j \in M : r_j(i) \leq k} f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}) \leq \varepsilon \quad \forall i \in N, t_i, t'_i \in T, t_{-i} \in T^{n-1}, k \in \{1, \ldots, m\} \quad \text{(Score deficit at t)} \\
& \quad f_j(t) \in [0, 1] \quad \forall t \in T^n, j \in M \quad \text{(\varepsilon-approximate SP)} \\
& \quad s_{\text{max}} \in [0, 1] \quad \text{(Outcome variables)} \\
& \quad \left(\varepsilon\right) \text{-approximate SP (Worst-case score deficit)}
\end{align*}
\]

Note that $\text{FindOpt}$ does not “just determine the outcome” for some type profile. Instead, it considers the entire space of rules. This is necessary, because $\varepsilon$-approximate strategyproofness imposes restrictions on how the outcomes differ across type reports.

5. Hybrid Rules

Ultimately, we are interested in the structure of the entire efficient frontier, i.e., the sets $\text{OPT}(\varepsilon)$ for each $\varepsilon \in [0, 1]$ and their relationship. So far, we have learned that for any given $\varepsilon$ we can determine $\text{OPT}(\varepsilon)$ using a linear program. We will now introduce hybrids, which, together with the linear program $\text{FindOpt}$, form the two essential building blocks for the efficient frontier.
5.1. Construction of Hybrid Rules

The intuition of hybrid rules is that to select a rule with good incentives and another rule with good efficiency and mix them to obtain a new rule with intermediate properties.

**Definition 10 (Hybrid).** For $\beta \in [0,1]$ and rules $f, g$, the $\beta$-hybrid is given by

$$h_\beta = \beta g + (1 - \beta) f,$$

i.e., for any type profile $t \in T^n$ the outcome $h_\beta(t)$ is the $\beta$-convex combination of the outcomes $f(t)$ and $g(t)$.

Hybrid rules are straightforward to run: after the reports are collected, the toss of a $\beta$-coin determines whether to apply $g$ (probability $\beta$) or $f$ (probability $1 - \beta$).

5.2. Guarantees for Hybrid Rules

Our second main result is a structural analysis of the trade-offs between strategyproofness and efficiency from the use of hybrids. The following proposition gives guarantees for the footprint of any hybrid.

**Proposition 3.** For any $\beta \in [0,1]$ and any rules $f, g$ we have

1. $\varepsilon(h_\beta) \leq \beta \varepsilon(g) + (1 - \beta) \varepsilon(f)$, and
2. $\sigma(h_\beta) \leq \beta \sigma(g) + (1 - \beta) \sigma(f)$.

**Proof idea.** (formal proof in Appendix C.2.1) We write out the definitions of $\varepsilon(h_\beta)$ and $\sigma(h_\beta)$, each of which involves taking a max. The inequalities (1) and (2) are then obtained with the help of the triangle inequality.

Proposition 3 yields that the footprint of the hybrid $h_\beta$ is weakly better than the $\beta$-convex combination of the footprints of $f$ and $g$. Figure 3 (i) illustrates this: the footprint of the $\beta$-hybrid must lie somewhere in the shaded area. A surprising consequence of Proposition 3 is an insight about the value of randomization: if a mechanism designer knows two different rules with the same footprint, using a hybrid of the two (i.e., randomizing between them) will yield a rule with a weakly better footprint. Recall what we learned from Proposition 1 about the efficient frontier: the mapping $\varepsilon \mapsto \sigma_\varepsilon$ that associates each manipulability $\varepsilon$ with the lowest attainable deficit $\sigma_\varepsilon$ is well-defined and non-increasing. By Proposition 3 that it is also convex.

**Corollary 1.** The mapping $\varepsilon \mapsto \sigma_\varepsilon$ in convex.

In Figure 2 (ii) we can therefore also rule out (c) as possible shapes of the graph, since it is not convex. A formal proof of Corollary 1 is given in Appendix C.2.2.
5.3. Path Properties

After recognizing the guarantees for the footprint of a single hybrid, we now explore the evolution of the footprints as we “move from $f$ to $g$” using hybrids. The following definitions are needed to formalize our results.

**Definition 11.** A path is a function $P : [0, 1] \rightarrow [0, 1]^2$ that maps any $\alpha \in [0, 1]$ to some footprint. We let $P_\varepsilon$ and $P_\sigma$ denote the $\varepsilon$- and $\sigma$-component of $P$, respectively. The set of points $P([0, 1]) = \{P(\alpha)|\alpha \in [0, 1]\} \subset [0, 1]^2$ is called the curve of $P$.

We will analyze the strategyproofness and efficiency properties of sets of rules (i.e., hybrid rules and the efficient frontier) by describing the properties of suitably defined paths. To make our statements precise, we introduce the following properties.

**Definition 12.** $P$ is connected if it is a continuous mapping from $[0, 1]$ to $[0, 1]^2$.

**Definition 13.** $P$ is piecewise linear if there exists a finite segmentation $\alpha_0 = 0 < \ldots < \alpha_K = 1$, such that for all $\gamma \in [0, 1]$ and $k \in \{0, \ldots, K-1\}$

$$P(\gamma \alpha_{k+1} + (1 - \gamma) \alpha_k) = \gamma P(\alpha_{k+1}) + (1 - \gamma) P(\alpha_k),$$

i.e., on each interval $[\alpha_k, \alpha_{k+1}]$, the image of the $\gamma$-convex combination of the end-points is equal to the $\gamma$-convex combination of the images of the two end-points.

The values $\{\alpha_0, \ldots, \alpha_K\}$ in Definition 13 are the hinges of $P$.

**Definition 14.** $P$ is linear if it is piecewise linear with only two hinges, i.e., $K = 2$.

**Definition 15.** $P$ is oriented if for all $\alpha, \alpha', \gamma \in [0, 1]$

$$P(\gamma \alpha' + (1 - \gamma) \alpha) \leq \max(P(\alpha), P(\alpha')).$$
$P$ is strongly oriented if it is oriented and
\[ P(\gamma \alpha' + (1 - \gamma) \alpha) \geq \min(P(\alpha) , P(\alpha')) , \tag{13} \]
where the comparison “\(\leq\)” is by components.

**Definition 16.** $P$ has convex components if the mappings $\alpha \mapsto P_\epsilon(\alpha)$ and $\alpha \mapsto P_\sigma(\alpha)$ are convex, i.e., for all $\alpha, \alpha', \gamma \in [0, 1]$
\[ P(\gamma \alpha' + (1 - \gamma) \alpha) \leq \gamma P(\alpha') + (1 - \gamma) P(\alpha) , \tag{14} \]
where the comparison “\(\leq\)” is by components.

Figure 3 (ii) illustrate the different properties. Note that the plots show curves, i.e., images of $P$, which do not reflect all aspects of the properties of a path, e.g., a single line segment could be the curve of a discontinuous path.

**Remark 2.** The following implications hold between the path properties, while the reverse implications are not true in general:

1. Linear $\Rightarrow$ piecewise linear $\Rightarrow$ connected.
2. Linear $\Rightarrow$ convex components $\Rightarrow$ oriented.
3. Linear $\Rightarrow$ strongly oriented $\Rightarrow$ oriented.

Formal proofs are given in Appendix C.2.3.

### 5.4. Path of Hybrid Rules

In addition to the guarantees from Proposition 3, we can now provide a more detailed understanding of the footprints of hybrids. For rules $f, g$ we consider the path $P_{Hyb}$ defined for each $\beta \in [0, 1]$ by $P_{Hyb}(\beta) = (\varepsilon(h_\beta), \sigma(h_\beta))$, i.e., $\beta$ is mapped to the footprint of the $\beta$-hybrid of $f$ and $g$.

**Theorem 1.** For rules $f, g$ the path $P_{Hyb}$ is connected, piecewise linear, oriented, and has convex components. However, $P_{Hyb}$ may fail to be strongly oriented or linear.

**Proof idea.** (formal proof in Appendix C.2.4) At each type profile $t \in T^n$ both rules $f$ and $g$ incur score deficits $\sigma(f(t), t)$ and $\sigma(g(t), t)$, and the score deficit of the hybrid is exactly equal to the $\beta$-convex combination $\sigma(h_\beta(t), t) = \beta \sigma(g(t), t) + (1 - \beta) \sigma(f(t), t)$. The worst-case score deficit for $f$ may be attained at some type profile $t_1$, for $g$ at $t_3$, and for $h_\beta$ with $\beta \in (0, 1)$ at yet another profile $t_2$. The bold line in Figure 4 (i) marks the worst-case score deficit for each $\beta$. Since there are only finitely many type profiles, $\beta \mapsto \sigma(h_\beta)$ is piecewise-linear and convex. The intuition for manipulability (i.e., $\beta \mapsto \varepsilon(h_\beta)$) is similar, using Proposition 2 to ensure finiteness of the constraints. Finally, we lift all properties from components to paths (Lemma 3).
Figure 4: (i) worst-case score deficit of hybrids, (ii) plots of \( P_{\text{Hyb}} \)

Figure 4 (ii) shows what curves of \( P_{\text{Hyb}} \) can look like in the light of Theorem 1, where the dash-dot lines illustrate violations of the properties. The fact that \( P_{\text{Hyb}} \) is not linear in general raises the question when - if ever - \( P_{\text{Hyb}} \) has this simple, linear structure, i.e., when the only hinges of \( P_{\text{Hyb}} \) are 0 and 1. Our next result characterizes these situations.

**Theorem 2.** For rules \( f, g \) the following are equivalent:

1. \( P_{\text{Hyb}} \) is linear.
2. There exists a type profile \( t \in T^n \) such that

\[
\sigma(f(t), t) = \sigma_{\text{max}}(f) \quad \text{and} \quad \sigma(g(t), t) = \sigma_{\text{max}}(g),
\]

and there exist \( k \in \{1, \ldots, m\} \), \( i \in N \), \( (t_i, t_{-i}) \in T^n, t'_i \in T \) such that

\[
\sum_{j \in M \setminus r_i(j) \leq k} f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}) = \varepsilon(f) \quad \text{and} \quad \sum_{j \in M \setminus r_i(j) \leq k} g_j(t'_i, t_{-i}) - g_j(t_i, t_{-i}) = \varepsilon(g).
\]

In words, \( f \) and \( g \) attain their worst-case score deficit at the same type profile \( t \) and the worst manipulability at the same combination \((k, (t_i, t_{-i}), t'_i)\).

**Proof idea.** (formal proof in Appendix C.2.5) For the direction \( \Rightarrow \) we show that if a constraint binds for both \( f \) and \( g \), it binds for all their hybrids. Consequently, the mappings \( \beta \mapsto \varepsilon(h_\beta) \) are \( \beta \mapsto \sigma(h_\beta) \) are linear, and so is the path \( P_{\text{Hyb}} \). A convexity argument yields \( \Leftarrow \). \( \Box \)

In conclusion, we have shown that for any two rules \( f, g \), the \( \beta \)-hybrid has a weakly better footprint than the \( \beta \)-combination of the footprints of \( f \) and \( g \). Furthermore, we have proven that the path describing the footprints of \( h_\beta \) as \( \beta \) runs from 0 to 1 has a piecewise linear structure, is oriented, and has convex components, and we have given an equivalent condition for when this path is in fact linearity. In the next section, we present a similar analysis for the efficient frontier.
6. Structure of the Efficient Frontier

Our third main result is a structural analysis of the efficient frontier. It would be particularly simple if the optimal rules at some $\varepsilon \in [0, 1]$ were just the $\varepsilon$-hybrids of optimal rules at 0 and 1, i.e., $\text{OPT}(\varepsilon) = \varepsilon \text{OPT}(1) + (1 - \varepsilon)\text{OPT}(0)$. While this is not true in general, we show that the efficient frontier does in fact have such a structure over each interval $[\varepsilon_k, \varepsilon_{k+1}]$ for a finite segmentation $\varepsilon_0 = 0 < \ldots < \varepsilon_K = 1$. Consequently, we have identified the building blocks of the efficient frontier:

1. the sets of optimal rules $\text{OPT}(\varepsilon_k)$ for finitely many values $\varepsilon_k, k \in \{0, \ldots, K\}$, corresponding to the solutions of the linear programs $\text{FINDOPT}(\varepsilon_k)$,

2. and hybrid rules, which provide the missing link at $\varepsilon \neq \varepsilon_k, k \in \{0, \ldots, K\}$.

6.1. Footprint of the Efficient Frontier

Similar to the analysis of hybrid rules, we formulate our results in terms of properties of a particular path: $P^\text{EF} : [0, 1] \rightarrow [0, 1]^2$ is defined by $P^\text{EF}(\varepsilon) = (\varepsilon, \sigma_\varepsilon)$.

**Theorem 3.** The path $P^\text{EF}$ is connected, piecewise linear, strongly oriented, has convex components, and $P^\text{EF}(1) = (1, 0)$. However, it may fail to be linear.

**Proof idea.** (formal proof in Appendix C.3.1) We exploit that $\text{OPT}(\varepsilon)$ corresponds to the solutions of a linear program with feasible set $F_\varepsilon = \{x \mid Dx \leq d, Ax \leq \varepsilon\}$, where neither $D, d$, nor $A$ depend on $\varepsilon$. First, we show that if a set of constraints is binding for $F_\varepsilon$, then it is binding for $F_{\varepsilon'}$ for all $\varepsilon' \in [\varepsilon^-, \varepsilon^+]$. With finiteness of constraints, this yields a finite segmentation of $[0, 1]$. The $\gamma$-representations of $F_\varepsilon$ and $S_\varepsilon$ can then be used to show that on each segment $[\varepsilon_k, \varepsilon_{k+1})$, the solutions $S_\varepsilon = \min F_\varepsilon$ and $s^\text{max}$ are $\gamma$-convex combinations of $S_{\varepsilon_k}$ and $S_{\varepsilon_{k+1}}$ with $\gamma = \frac{\varepsilon - \varepsilon_k}{\varepsilon_{k+1} - \varepsilon_k}$.

Figure 5 shows what the curve of $P^\text{EF}$ can look like ((a) & (b)). One might hope that the hybrid of two optimal rules is again optimal, which is, however, not necessarily the case. What we learn from piecewise linearity in Theorem 3 is that we can segment $[0, 1]$ into compact intervals, such that $P^\text{EF}$ is in fact linear across each interval. The optimal rules on each segment are exactly the hybrids of the optimal rules at the two end-points of that segment, i.e., the hinges $\varepsilon_k, \varepsilon_{k+1}$.  

**Corollary 2.** For any two consecutive hinges $\varepsilon_k, \varepsilon_{k+1}$ of $P^\text{EF}$ and $\gamma \in [0, 1]$

\[
\text{OPT}(\gamma \varepsilon_{k+1} + (1 - \gamma)\varepsilon_k) = \gamma \text{OPT}(\varepsilon_{k+1}) + (1 - \gamma)\text{OPT}(\varepsilon_k). \tag{16}
\]

---

2 There always exists some positional scoring rule with minimal manipulability $\varepsilon_{\text{par}}$. Therefore, the minimal deficit $\sigma_\varepsilon$ is 0 for any $\varepsilon \geq \varepsilon_{\text{par}}$, i.e., the $\sigma$-component of $P^\text{EF}$ is zero on $[\varepsilon_{\text{par}}, 1]$. Furthermore, $P^\text{EF}(0) = (0, x)$ with $x > 0$ as positional scoring rules are (usually) not strategyproof.
A second useful insight for the mechanism designer is that the marginal benefit of sacrificing one additional unit of strategyproofness is highest in the beginning and weakly decreases as the rules become more manipulable.

**Corollary 3.** For any \( \varepsilon, \varepsilon' \in [0, 1] \) with \( \varepsilon \leq \varepsilon' \) we have \( \sigma_\varepsilon - \sigma_{\varepsilon + \delta} \geq \sigma_{\varepsilon'} - \sigma_{\varepsilon' + \delta} \).

### 6.2. Determining the Efficient Frontier

The structural understanding from Theorem 3 allows the algorithmic identification of the complete efficient frontier.

#### 6.2.1. Finding Optimal Representatives

First, we consider the following algorithmic problem: suppose, we know the hinges \( \varepsilon_0 = 0 < \ldots < \varepsilon_K = 1 \) of \( P_{\text{EF}} \), and for each hinge \( \varepsilon_k \) we also know at least one representative \( f_k \in \text{OPT}(\varepsilon_k) \). For any given \( \varepsilon \in [0, 1] \), can we determine an optimal rule \( f_\varepsilon \in \text{OPT}(\varepsilon) \) without running the linear program \( \text{FindOPT}(\varepsilon) \)? By virtue of Corollary 2, the solution is simple.

**Corollary 4.** For any \( \varepsilon \in [0, 1] \) the rule \( f_\varepsilon \) with

\[
f_\varepsilon = \begin{cases} 
    f_k, & \text{if } \varepsilon = \varepsilon_k \text{ for } k \in \{0, \ldots, K\}, \\
    f_K, & \text{if } \varepsilon > \varepsilon_K, \\
    \gamma f_{k+1} + (1 - \gamma) f_k, & \text{if } \varepsilon \in (\varepsilon_k, \varepsilon_{k+1}) \text{ for } k \in \{0, \ldots, K - 1\}, \\
    \gamma = \frac{\varepsilon - \varepsilon_k}{\varepsilon_{k+1} - \varepsilon_k}, 
\end{cases}
\]

is optimal at \( \varepsilon \), i.e., \( f_\varepsilon \in \text{OPT}(\varepsilon) \).

#### 6.2.2. Finding Hinges Larger than \( \delta > 0 \)

Our goal is to identify all hinges of \( P_{\text{EF}} \) (and corresponding optimal rules). A naïve approach would be to run the linear program \( \text{FindOPT} \) from Section 4.3 for many different values of
and plot the curve. However, this approach has two drawbacks: first, we cannot be sure that we computed \( \text{OPT}(\varepsilon) \) for the actual hinges of \( P^{\text{EF}} \). Thus, the hybrids obtained from (17) may not be on the efficient frontier. Second, to obtain even an approximation of the efficient frontier, we would have to run \( \text{FindOPT} \) many times, but each run is costly. The algorithm \( \text{FindHinges} \) (Algorithm 1 in Appendix B) exploits the structure of \( P^{\text{EF}} \) to guess possible hinges. It uses \( \text{FindOPT} \) repeatedly to compute \( \text{OPT}(\varepsilon) \) and \( \sigma_\varepsilon \) for different values of \( \varepsilon \). In each step, it assumes that \( P^{\text{EF}} \) has the “coarsest” possible structure that is consistent with the points \((\varepsilon, \sigma_\varepsilon)\) of \( P^{\text{EF}} \) identified so far. It then interpolates the possible location of an additional hinge and attempts to verify this. The key to success is that \( \text{FindHinges} \) computes at most three points between any two consecutive true hinges of \( P^{\text{EF}} \).

**Proposition 4.** \( \text{FindHinges} \) is correct and complete for the problem of finding all hinges in \((\delta, 1]\). If there are \( K \) hinges in \((\delta, 1]\), \( \text{FindHinges} \) requires at most \( 4K + 1 \) executions of \( \text{FindOPT} \).

A formal proof of Proposition 4 is given in Appendix C.3.2. For technical reasons, \( \text{FindHinges} \) only identifies hinges that are larger than some value \( \delta > 0 \), but we address this issue next.

**6.2.3. Finding all Hinges of \( P^{\text{EF}} \)**

\( \text{FindHinges} \) relies on the assumption that there are no hinges below \( \delta \). However, our goal is to identify the entire efficient frontier. If we ran \( \text{FindHinges} \) with \( 0 < \delta < \varepsilon_1 \) (\( \varepsilon_1 \) is the smallest non-zero hinge), it would identify all hinges. \( \text{FindDelta} \) (Algorithm 2 in Appendix B) computes such a \( \delta \) while keeping the number of additional executions of \( \text{FindOPT} \) low. The idea is to guess a value for \( \delta \) and try to verify that it is small enough. If not, divide the guess by two and try again.

**Proposition 5.** Let \( \varepsilon_1 \) be the smallest non-zero hinge of \( P^{\text{EF}} \), then \( \text{FindDelta} \) is correct and complete for the problem of finding a bound \( 0 < \delta < \varepsilon_1 \). \( \text{FindDelta} \) requires at most \( 3 + \log_2 (1/\varepsilon_1) \) executions of \( \text{FindOPT} \).

A formal proof is given in Appendix C.3.3.

**7. Applications**

We now present three applications of our results: in Section 7.1, we show that anonymity and neutrality come at no cost in terms of strategyproofness and efficiency. We also show that this is not the case for Condorcet consistency, Pareto optimality, or even unanimity. In Section 7.2, we describe the efficient frontier in the concrete setting with \( n = m = 3 \), strict preferences, and with respect to the scoring function \( v = (1, 0, 0) \). In this setting, we show that \( \text{Plurality with uniform tie-breaking} \) is the least manipulable of all positional scoring rules, \( \text{Random Dictatorship} \) has the lowest deficit of all strategyproof rules, and all hybrids of these two rules lie on the efficient frontier. In Section 7.3, we describe the efficient frontier in the
same setting, but with respect to another scoring function \( v = (1, 1, 0) \). We show that \textit{Veto with uniform tie-breaking} is the least manipulable of all positional scoring rules and \textit{Random Duple} has the lowest deficit of all strategyproof rules. However, none of the hybrids of Uniform Veto and Random Duple lie on the efficient frontier.

### 7.1. Incorporating Additional Requirements

Imposing any additional requirements will necessarily reduce the design space for admissible rules. This will have a weakly negative impact on the efficient frontier, i.e., optimal rules could become infeasible and the footprint of the best feasible rules could become worse. Using our insights about hybrid rules, we make the interesting observation that anonymity and neutrality come for free. Let \( \Pi = \{ \pi : N \to N \text{ bijection} \} \) and \( \Phi = \{ \varphi : M \to M \text{ bijection} \} \) be the sets of permutations of \( N \) and \( M \), respectively. For \( \pi \in \Pi \) let \( f^\pi \) denote the rule that re-orders the agents’ reports using \( \pi \) and applies \( f \) to this new type profile, i.e., \( f^\pi(t_1, \ldots, t_n) = f(t_{\pi(1)}, \ldots, t_{\pi(n)}) \). Similarly, for \( \varphi \in \Phi \) let \( f^\varphi \) be the rule that renames the outcomes using \( \varphi \), applies \( f \) to the new type profiles, and translates the outcome back via \( \varphi^{-1} \), i.e., \( f^\varphi_j(t_1, \ldots, t_n) = f_{\varphi(j)}(\varphi(t_1), \ldots, \varphi(t_n)) \), where \( \varphi(t_i) \) is the type for which \( \varphi(j) \succeq \varphi(t_i) \varphi(j') \) whenever \( j \succeq i, j' \).

**Definition 17** (Anonymity). A rule \( f \) is \textit{anonymous} if \( f = f^\pi \) for any \( \pi \in \Pi \).

**Definition 18** (Neutrality). A rule \( f \) is \textit{neutral} if \( f = f^\varphi \) for any \( \varphi \in \Phi \).

The following result shows that the footprint of the efficient frontier does not become worse when we add anonymity or neutrality as requirements.

**Proposition 6.** In the full domain (with or without indifferences), for any rule \( f \) there exists an anonymous, neutral rule \( f^* \) with weakly better footprint, i.e., \( \varepsilon(f^*) \leq \varepsilon(f) \) and \( \sigma(f^*) \leq \sigma(f) \).

**Proof idea.** (formal proof in Appendix C.4.1) We consider the rule \( f^* \) defined by \( f^*(t) = \frac{1}{n!m!} \sum_{\pi \in \Pi, \varphi \in \Phi} f^\pi,\varphi(t) \). It is anonymous and neutral by construction, and since it is a hybrid of many rules, we can apply Proposition 3 to obtain the inequalities.

Note that for anonymity and neutrality to be meaningfully defined, the type space must be sufficiently rich. Proposition 6 can be generalized to domains that are \textit{agent symmetric} and \textit{alternative symmetric} (see Appendix D). Anonymity and neutrality can be included in the linear program \textsc{FindOpt} via the following additional constraints.

\[
\begin{align*}
  f_j(t_1, \ldots, t_n) &= f_j(t_{\pi(1)}, \ldots, t_{\pi(n)}) \quad \forall t \in T^n, \pi \in \Pi, j \in M \quad \text{(Anonymity)} \\
  f_j(t_1, \ldots, t_n) &= f_{\varphi(j)}(\varphi(t_1), \ldots, \varphi(t_n)) \quad \forall t \in T^n, \varphi \in \Phi, j \in M \quad \text{(Neutrality)}
\end{align*}
\]

It is also possible to include \textit{Condorcet consistency}, \textit{Pareto optimality}, and \textit{unanimity} in the linear program (details in Appendix E). However, in contrast to anonymity and neutrality, they do not come “for free.”

---

\(^3\)Since the proof is constructive, running \( f^* \) is straightforward: after collecting the reports, randomly rename the agents and alternatives, run \( f \) on the change type profile, and convert the outcome back.
Proposition 7. There exists a setting \((N, M, v)\) such that any rule that is strategyproof and Condorcet consistent (or Pareto optimal, or unanimous) has a strictly higher deficit than any strategyproof, optimal rule.

Example 1 in Appendix E yields the proof.

7.2. Plurality and the Efficient Frontier

This section we study the efficient frontier for a concrete setting. We give formal proofs for our results, but they can also be obtained by running the Algorithms FindDelta & FindHinges for the respective setting. The setting we consider has \(n = 3\) agents with only strict preferences, \(m = 3\) alternatives, and the scoring function \(v = (1, 0, 0)\). The motivation for this choice is, first, to allow formal proofs and retain algorithmic tractability, and second, that in this case the efficient frontier consists of the hybrids of interesting, well-known rules.

Definition 19. Plurality with uniform tie-breaking (UnifPlurality):

1. Collect all type reports.
2. Determine all alternatives that are ranked as first choice by a maximal number of agents (denoted \(M^{\text{max}} \subseteq M\)).
3. Pick an alternative from \(M^{\text{max}}\) uniformly at random.

Definition 20. Random Dictatorship (RandDict):

1. Collect all type reports.
2. Select an agent \(i \in N\) uniformly at random.
3. Pick \(i\)'s first-ranked alternative.

The following results provide insights about the efficient frontier in the chosen setting: first, of all positional scoring rules, UnifPlurality is the least manipulable (Prop. 8). Second, of all strategyproof rules, RandDict has lowest deficit (Prop. 9). Third, all hybrids of RandDict and UnifPlurality are on the efficient frontier (Prop. 10).

Proposition 8. For \(n = m = 3\), \(v = (1, 0, 0)\), and strict preferences, UnifPlurality is \(\frac{1}{3}\)-approximately strategyproof, and any positional scoring rule (including UnifPlurality) violates \(\varepsilon\)-approximate strategyproofness for any \(\varepsilon < \frac{1}{3}\).

Proposition 9. For \(n = m = 3\), \(v = (1, 0, 0)\), and strict preferences, RandDict \(\in\) Opt(0), i.e., it has the lowest deficit among all strategyproof rules.

Proposition 10. For \(n = m = 3\), \(v = (1, 0, 0)\), and strict preferences, all hybrids of RandDict and UnifPlurality are on the efficient frontier.

In particular, hybrids of RandDict and UnifPlurality can be used to generate optimal rules with intermediate strategyproofness and efficiency, which means that their hybrids make optimal trade-offs. Figure 6 (i) shows the curve of \(P^\text{EF}\). Formal proofs of the Propositions are given in Appendix C.4.2, C.4.3, and C.4.4.
7.3. Veto and the Efficient Frontier

As in the previous section, we describe the efficient frontier for a setting with $n = m = 3$ and strict preferences. This time, we consider the scoring function $v = (1, 1, 0)$.

**Definition 21.** Veto with uniform tie-breaking (UnifVeto):

1. Collect all type reports.
2. Determine all alternatives that are ranked as last choice by a minimal number of agents (denoted $M^{\text{min}} \subseteq M$).
3. Pick an alternative from $M^{\text{min}}$ uniformly at random.

**Definition 22.** Random Duple (RandDuple):

1. Collect all type reports.
2. Select two different alternatives $a, b \in M$ uniformly at random.
3. If more agents prefer $a$ to $b$, pick $a$, else pick $b$.

Our next results show the following: first, of all positional scoring rules, UnifVeto is the least manipulable (Proposition 11). Second, of all strategyproof rules, RandDuple is the most efficient (Proposition 12). Third, none of the hybrids of RandDuple and UnifVeto lie on the efficient frontier, except at the extreme points $\beta = 0$ and $\beta = 1$ (Proposition 13). The last result is in contrast the previous findings for $v = (1, 0, 0)$ and highlights the fact that hybrids of optimal rules are not necessarily optimal.
Proposition 11. For \( n = m = 3, v = (1,1,0) \), and strict preferences, \( \text{UnifVeto} \) is \( \frac{1}{2} \)-approximately strategyproof, and any positional scoring rule (including \( \text{UnifVeto} \)) violates \( \varepsilon \)-approximate strategyproofness for any \( \varepsilon < \frac{1}{2} \).

Proposition 12. For \( n = m = 3, v = (1,1,0) \), and strict preferences, \( \text{RandDuple} \in \text{Opt}(0) \), i.e., it has the lowest deficit among all strategyproof rules.

Proposition 13. For \( n = m = 3, v = (1,1,0) \), and strict preferences, the hybrids of \( \text{RandDuple} \) and \( \text{UnifVeto} \) are not on the efficient frontier for \( \beta \neq 0,1 \).

An application of the Algorithms \text{FindDelta} & \text{FindHinges} \) reveals that the hinges of \( P_{\text{EF}} \) are \( \{0, \frac{1}{21}, \frac{1}{12}, \frac{1}{2}, 1\} \). Figure 6 (ii) shows the curve of \( P_{\text{EF}} \) for the scoring function \( v = (1,1,0) \). The circles mark the hinges. Formal proofs of the Propositions are given in Appendix C.4.5, C.4.6, and C.4.7.

8. Generality of Results

The results from Sections 4, 5, and 6 do not make any use of the particular aspects of the preferences, and they continue to hold in many domain restrictions.

8.1. Restricting the Type Profiles

Taking the full domain with indifferences as a starting point, we can consider domain restrictions where some type profiles are not admissible, e.g., agent 2 may only rank \( a \) over \( b \) if agent 2 is indifferent between \( a \) and \( c \). Our results will remain valid under any domain restrictions that rule out arbitrary subsets of the type profiles. One domain that can be obtained in this fashion is the full domain of strict preferences, at which we arrive by excluding all type profiles where any agent is indifferent between any two alternatives. Similarly, we can reduce the domain to the assignment (i.e., one-sided matching) or the two-sided matching domain, where agents are indifferent between any outcomes that leave their individual assigned object or matching partner unchanged.

8.2. Different Measures of Efficiency

Our results also remain true when we use different measures of efficiency besides the worst-case score deficit. First, we can consider the average (or any weighted average) of the score deficits, as long as the weights do not depend on the type profiles. This would be interesting if we were to search for optimal rules with some prior distribution over agents’ types in mind. Second, we can consider a scaled worst-case score deficit, where the score deficit at each type profile is scaled according to the maximum achievable score at that profile. The definition of \( \sigma \) in equation (4) would be replaced by the following

\[
\sigma(f) = \max_{t \in T^n} \left( \frac{\sigma(f(t), t)}{s_{\text{max}}} \right).
\]  

(18)
Thus, instead of having low absolute deviations from the best score, optimal rules would have low relative deviations. Again, using these different objectives would not change the character of our results, though the efficient frontiers may certainly consist of different rules and their paths may look different. Third, our results also hold in situations when the efficiency requirement towards the rules is not expressible via a scoring function. Instead, suppose that at each type profile there is a set of “good” alternatives \( A(t) \subseteq M \), e.g., the Pareto optimal alternatives or the Condorcet winners at \( t \). In this case, a rule could be considered more efficient if the alternative selected at \( t \) is outside \( A(t) \) with low probability, i.e., we can define a quantitative measure for inefficiency by setting

\[
\sigma(f, t) = 1 - \sum_{j \in A(t)} f_j(t),
\]

and

\[
\sigma(f) = \max_{t \in T^n} \sigma(f, t).
\]

Thus, \( \sigma(f) \) is a tight upper bound for the probability that \( f \) selects a undesirable alternative at any given type profile. This measure can replace the worst-case score deficit and all our results in Sections 4, 5, and 6 remain valid (all proofs are analogous). In addition, we could consider the ex-ante probability of selecting an undesirable alternative with respect to some known prior \( P \) over type profiles, i.e.,

\[
\sigma(f, P) = \sum_{t \in T^n} \sigma(f, t) \cdot P[t].
\]

9. Conclusion

In this paper, we have provided a structural analysis of the efficient frontier in the randomized social choice domain. To measure the performance of rules with respect to strategyproofness and efficiency, we have adopted the notions of \( \varepsilon \)-approximate strategyproofness and worst-cases score deficit, such that the footprints \( (\varepsilon(f), \sigma(f)) \) of social choice rules allow for a meaningful comparison along both dimensions. First, we have shown that for any manipulability \( \varepsilon \in [0, 1] \), the set of optimal rules \( \text{Opt}(\varepsilon) \) can be characterized by the solutions of the linear program \( \text{FindOpt}(\varepsilon) \). The main challenge in formulating this LP was to capture the \( \varepsilon \)-approximate strategyproofness requirement (over an infinite utility space) via a finite number of linear constraints. Second, we have introduced hybrid rules, which are convex combinations of two component rules \( f \) and \( g \). We have shown that the footprint of any \( \beta \)-hybrid is weakly better than the \( \beta \)-convex combination of the footprints of \( f \) and \( g \). This has afforded us with a simple procedure to define rules with intermediate strategyproofness and efficiency properties. Moreover, as the mixing factor \( \beta \) runs from 0 to 1, the footprints of hybrids follow a piecewise linear, oriented path, and we have given an equivalent condition for all situations in which this path is simply a linear segment. Third, we have shown that the footprints of the efficient frontier of social choice rules follow a piecewise linear, strongly oriented path. In combination

\footnote{This would correspond to the original approximation ratio as studied in (Procaccia, 2010).}
with our previous findings, this yields the two building blocks of the efficient frontier: (1) the sets of optimal rules $\text{OPT}(\varepsilon_k)$ for finitely many hinges $\varepsilon_k, k \in \{0, \ldots, K\}$, and (2) hybrid rules, which provide the link at $\varepsilon \notin \{\varepsilon_0, \ldots, \varepsilon_K\}$. The algorithms \textsc{FindDelta} and \textsc{FindHinges} exploit this structure to compute the entire efficient frontier, which enables mechanism designers to make informed decisions how to trade off strategyproofness and efficiency. Finally, we have applied our findings: first, we have shown that anonymity and neutrality come “for free” in terms of strategyproofness and efficiency, while this is not true for Condorcet consistency, Pareto optimality, or unanimity. Next, for a concrete setting with 3 agents, 3 alternatives, strict preferences, and the scoring function $v = (1, 0, 0)$, we have shown that Uniform Plurality is $\frac{1}{3}$-approximately strategyproof, and Random Dictatorship and Uniform Plurality as well as all their hybrids are on the efficient frontier. For the same setting, but with $v = (1, 1, 0)$, Uniform Veto is $\frac{1}{2}$-approximately strategyproof, and that Random Duples and Uniform Veto are both on the efficient frontier. However, in contrast to the previous example, the hybrids of Random Duples and Uniform Veto are not on the efficient frontier, except for $\beta = 0$ or $\beta = 1$. Our findings provide a novel understanding of the efficient frontier in randomized social choice and unlock the set of optimal rules for further analytical, axiomatic, and algorithmic exploration. We also expect our findings to provide interesting directions for future research: first, the computational complexity of automated mechanism design is an ongoing concern, and the linear program \textsc{FindOPT} is no exception. One promising idea to significantly reduce the computational cost is to exploit additional axioms, such as anonymity or neutrality, in the construction of the LP. Second, our insights may be extended and refined, e.g., for matching or assignment, or for generalized scoring rules.

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APPENDIX

A. Manipulability of Positional Scoring Rules

Lemma 1. For \( n = m = 3 \), no positional scoring rule is strategyproof.

Proof. Without loss of generality, we can assume that the positional scoring rule \( f \) is anonymous and neutral (Proposition 6). Then at the type profile

\[
\begin{align*}
    t_1 & : a > b > c, \\
    t_2 & : b > c > a, \\
    t_3 & : c > a > b,
\end{align*}
\]

all alternatives have equal probability of \( \frac{1}{3} \). If \( v = (1, p, 0) \) with \( p < 1 \), then agent 3 can enforce implementation of its second choice \( a \) by swapping \( a \) with \( c \). If \( u_3(a) > \frac{1}{2}u_3(c) \), then this is a beneficial manipulation for 3. If \( p = 1 \), then consider the type profile

\[
\begin{align*}
    t_1 & : a > b > c, \\
    t_2 & : b > c > a, \\
    t_3 & : c > b > a,
\end{align*}
\]

where \( b \) is implemented by \( f \). Agent 3 can swap \( b \) and \( a \) in its ranking and change the outcome to the uniform distribution over alternatives. If \( u_3(c) > 2u_3(b) \), this is a beneficial manipulation for agent 3, which concludes the proof.

B. The Algorithms FindHinges and FindDelta

\textsc{FindHinges} and \textsc{FindDelta} are given by Algorithms 1 and 2, respectively. Note that the computation of \( P_{\text{EF}} \) for every new value of \( e \) (in \textsc{FindHinges}) or \( \delta \) (in \textsc{FindDelta}) corresponds to an execution of the LP \textsc{FindOpt}. Furthermore, we would like to point out the following technical detail about \textsc{FindHinges}: it identifies all hinges in \((\delta, 1]\) correctly. In addition, it finds 0, which always the hinge associated with all strategyproof rules. In addition, if there exist any hinges \( \varepsilon_k < \delta \), then \textsc{FindHinges} will find a hinge at \( \delta \), which is to be understood as a placeholder for all hinges in the interval \((0, \delta]\).
ALGORITHM 1: FINDHINGS

Input: bound \( \delta > 0 \)

Variables: set of hinges \( \text{hinges} \), stacks of unverified, verified, and outer segments \( \text{segments.u} \), \( \text{segments.v} \), \( \text{segments.o} \)

begin
  \( \text{hinges} \leftarrow \{0, 1\} \)
  \( \text{segments.u} \leftarrow \{(P^{EF}(\delta), P^{EF}(1))\} \)
  \( \text{segments.v} \leftarrow \emptyset \)
  \( \text{segments.o} \leftarrow \{(P^{EF}(0), P^{EF}(\delta)), (P^{EF}(1), P^{EF}(2))\} \)
  while \( \text{segments.u} \neq \emptyset \) do
    \( (P^-, P^+) \leftarrow \text{pop}(\text{segments.u}) \)
    \( (P^-, P^-), (P^+, P^{++}) \in \text{segments.v} \cup \text{segments.u} \cup \text{segments.o} \)
    \( e \leftarrow \text{affine.hull}([P^-, P^-] \cap \text{aff}([P^-, P^-]) \epsilon \text{EF}(e)) \)
    \( P \leftarrow P^{EF}(e) \)
    if \( P \in \text{affine.hull}([P^-, P^-] \cap \text{aff}([P^+, P^{++}])) \) then
      \( \text{hinges} \leftarrow \text{hinges} \cup \{P\} \)
      \( \text{segments.v} \leftarrow \text{segments.v} \cup \{(P^-, P^-), (P^+, P^{++}), (P^-, P), (P, P^+)\} \)
    end
    else if \( P \in \text{affine.hull}(P^-, P^+) \) then
      \( \text{segments.v} \leftarrow \text{segments.v} \cup \{(P^-, P^+)\} \)
    end
    else
      \( \text{segments.u} \leftarrow \text{segments.u} \cup \{(P^-, P), (P, P^+)\} \)
    end
  end
  return \( \text{hinges} \)
end

C. Omitted Proofs

C.1. Proofs from Section 4

C.1.1. Proof of Proposition 1

The mapping \( \varepsilon \mapsto \sigma_\varepsilon \) that associates each manipulability \( \varepsilon \) with the lowest attainable deficit is well-defined and non-increasing in \( \varepsilon \).

Proof. Since the score deficit of any rule is upper bounded by 1, and there exist strategyproof rules (e.g., constant rules), \( \sigma_\varepsilon \leq 1 \) for all \( \varepsilon \in [0, 1] \). Next, suppose that for some \( \varepsilon \in [0, 1] \) there exists no rule \( f_\varepsilon \) such that

\[
\sigma(f_\varepsilon) = \sigma^{\inf} = \inf \{ \sigma(f') | f' \ \varepsilon\text{-approximately strategyproof} \} \tag{28}
\]

Then there must exist a sequence of rules \( (f^k)_{k \geq 1} \) such that \( \sigma(f^k) \to \sigma^{\sup} \) as \( k \to \infty \). Since all \( f_k \) are rules, they are uniformly bounded functions from the finite set \( T^n \) of type profiles to the compact set \( \Delta(M) \). Therefore, \( f^k \to \hat{f} \) for some rule \( \hat{f} \). But since all incentive constraints and the worst-case score deficit arise from weak inequalities, we have that \( \varepsilon(f^k) \to \varepsilon(\hat{f}) \) and
For $\varepsilon \in [0, 1]$ and a rule $f$ the following are equivalent:

1. $f$ is $\varepsilon$-approximately strategyproof.

2. For all type profiles $t \in T^n$, agents $i \in N$, misreports $t'_i \in T$, and ranks $k \in \{1, \ldots, m\}$

   \[
   \sum_{j \in M \cap (j) \leq k} f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}) \leq \varepsilon.
   \]  

(29)

Proof. Fix a type profile $t \in T^n$, an agent $i \in N$, and a misreport $t'_i \in T$. The admissible set of utility functions for agent $i$ is $t_i$, i.e., all the utilities $u_i : M \rightarrow [0, 1]$ for which $u_i(j) \geq u_i(j')$ whenever $j \succeq_i j'$. Let $B^{(0,1)}(t_i)$ denote the set of binary utilities associated with $t_i$, i.e.,

\[
B^{(0,1)}(t_i) = \{u : M \rightarrow \{0, 1\} | u(j) \geq u(j') \text{ whenever } j \succeq_i j' \}.
\]  

(30)

We first show the direction “$\Leftarrow$”, i.e., the condition in Proposition 2 implies $\varepsilon$-approximate strategyproofness. Let $u \in B^{(0,1)}(t_i)$, then the incentive constraint (6) for this particular utility function has the form

\[
\varepsilon (u, (t_i, t_{-i}), t'_i, f) = \sum_{j \in M} u(j) \cdot (f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}))
\]  

(31)

\[
= \sum_{j \in M : u(j) = 1} f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i})
\]  

(32)

\[
= \sum_{j \in M : r (j) \leq k} f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i})
\]  

(33)

C.1.2. Proof of Proposition 2

For $\varepsilon \in [0, 1]$ and a rule $f$ the following are equivalent:

1. $f$ is $\varepsilon$-approximately strategyproof.

2. For all type profiles $t \in T^n$, agents $i \in N$, misreports $t'_i \in T$, and ranks $k \in \{1, \ldots, m\}$

ALGORITHM 2: FINDDELTA

Variables: footprints $P^0, P, P^+$, bound $\delta$

begin
\[
\delta \leftarrow 1/2
\]
\[
P^0 \leftarrow P^{\text{EF}}(0), P^+ \leftarrow P^{\text{EF}}(1), P \leftarrow P^{\text{EF}}(\delta)
\]
while $P \notin \text{affine hull}(P^0, P^+)$ do
\[
P^+ \leftarrow P, \delta \leftarrow \delta/2, P \leftarrow P^{\text{EF}}(\delta)
\]
end
return $\delta$
end

\[\sigma(f^k) \rightarrow \sigma(\tilde{f}) \text{ as } k \rightarrow \infty.\]
Thus, $\tilde{f}$ is $\varepsilon$-approximately strategyproof and $\sigma(\tilde{f}) = \sigma^{\text{inf}}$. This means that using min (instead of inf) in Definition 8 is possible, i.e., $\text{OPT}(\varepsilon) \neq \emptyset$. Next, suppose that there are two rule $f_\varepsilon, f'_\varepsilon \in \text{OPT}(\varepsilon)$ with $\sigma(f_\varepsilon) < \sigma(f'_\varepsilon)$. Then $f'_\varepsilon$ is by definition not an optimal rule. Finally, we have already observed that an $\varepsilon$-approximately strategyproof rule is also $\varepsilon'$-approximately strategyproof for any $\varepsilon' \geq \varepsilon$. Thus, the choice set of rules only becomes larger as $\varepsilon$ grows, and therefore, the worst-case score deficit can not increase. $\square$
for some \( k \in \{1, \ldots, m\} \). By the condition (9) from Proposition 2, this term is always upper bounded by \( \varepsilon \). By Lemma 2, \( t_i \subseteq \text{Conv}(B^{(0,1)}(t_i)) \), which means that any \( u_i \in t_i \) can be represented as a convex combination of utility functions in \( B^{(0,1)}(t_i) \), i.e.,

\[
    u_i = \sum_{l=1}^{L} \alpha_l u'_l
\]

for \( u' \in B^{(0,1)}(t_i) \) and \( \alpha_l \geq 0 \) for all \( l \in \{1, \ldots, L\} \) and \( \sum_{l=1}^{L} \alpha_l = 1 \). By linearity of the incentive constraint (6) we get that

\[
\varepsilon \left(u_i, (t_i, t_{-i}), t'_i, f\right) = \sum_{l=1}^{L} \alpha_l \varepsilon \left(u'_l, (t_i, t_{-i}), t'_i, f\right)
\leq \sum_{l=1}^{L} \alpha_l \varepsilon = \varepsilon.
\]

This proves the direction “\( \Rightarrow \)”. Next, we prove the direction “\( \Leftarrow \)” of Proposition 2. Towards contradiction, assume that the constraint (9) is violated for some \( k \in \{1, \ldots, m\} \), i.e.,

\[
    \sum_{j \in M: r_i(j) \leq k} f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}) = \varepsilon + \delta
\]

with \( \delta > 0 \). Let \( u \in B^{(0,1)}(t_i) \) be the binary utility function with

\[
    u(j) = \begin{cases} 
        1, & \text{if } r_i(j) \leq k, \\
        0, & \text{else.}
    \end{cases}
\]

Then

\[
\varepsilon \left(u, (t_i, t_{-i}), t'_i, f\right) = \varepsilon + \delta.
\]

Choose any utility function \( u' \in t_i \) and let \( \beta = \frac{\delta/2}{\varepsilon + \delta + 1} \). The utility function constructed by \( \tilde{u} = (1 - \beta)u + \beta u' \) is in \( t_i \) and we have

\[
\varepsilon \left(\tilde{u}, (t_i, t_{-i}), t'_i, f\right) = \left(1 - \beta\right)\varepsilon \left( u, (t_i, t_{-i}), t'_i, f\right) + \beta \varepsilon \left( u', (t_i, t_{-i}), t'_i, f\right)
\geq \left(1 - \beta\right)(\varepsilon + \delta) - \beta
= -\beta(\varepsilon + \delta + 1) + (\varepsilon + \delta) = \varepsilon + \delta/2,
\]

since the change in utility from manipulation is lower bounded by \(-1\). Thus, the \( \varepsilon \)-approximate strategyproofness constraint is violated (for the utility function \( \tilde{u} \), a contradiction.

**Lemma 2.** For any type \( t \in T \) define the set of binary utilities associated with \( t \) by

\[
    B^{(0,1)}(t) = \{ u : M \to \{0, 1\} \mid u(j) \geq u(j') \text{ whenever } j \succeq_{t} j' \}.
\]

Then \( t \subseteq \text{Conv}(B^{(0,1)}(t)) \), where Conv denotes the convex hull.

28
Proof. First, suppose that the preference ordering corresponding to $t$ is strict, i.e., $u(j) \neq u(j')$ for all $j \neq j'$, and without loss of generality,

$$j_1 >_t j_2 >_t \ldots j_{m-1} >_t j_m.$$  \hspace{1cm} (44)

In this case, $B^{[0,1]}(t)$ consists of all the functions $u^r : M \to \{0, 1\}$ with

$$u^k(j_r) = \begin{cases} 1, & \text{if } r \leq k, \\ 0, & \text{else}. \end{cases}$$  \hspace{1cm} (45)

With $k \in 0, \ldots, m$. Let $\Delta u(0) = 1 - u(j_1)$, $\Delta u(k) = u(j_k) - u(j_{k+1})$ for all $k \in \{1, \ldots, m-1\}$, and $\Delta u(m) = u(j_m)$. Then represent $u$ by

$$u(j_r) = \sum_{k=r}^m \Delta u(k).$$  \hspace{1cm} (46)

Now we construct the utility function

$$\tilde{u} = \sum_{k=0}^m \Delta u(k) \cdot u^k.$$  \hspace{1cm} (47)

First note that $\sum_{k=0}^m \Delta u(k) = 1$ and $\Delta u(k) \geq 0$ for all $k$, so that $\tilde{u}$ is a convex combination of elements of $B^{[0,1]}(t)$. Furthermore, for any $j_r \in M$, we have that

$$\tilde{u}(j_r) = \sum_{k=0}^m \Delta u(k) \cdot u^k(j_r)$$  \hspace{1cm} (48)

$$= \sum_{k=r}^m \Delta u(k) = u(j_r),$$  \hspace{1cm} (49)

i.e., $u = \tilde{u}$. This establishes the Lemma for types corresponding to strict preference orders. For arbitrary types (i.e., with indifferences) the proof can be easily extended by combining all alternatives about which an agent of type $t$ is indifferent into a single virtual alternative. Then we apply the proof for the strict case. The utility functions in $B^{[0,1]}(t)$ will be exactly those that put equal value on the groups of alternatives between which an agent of type $t$ is indifferent. This concludes the proof of the Lemma. \hfill $\blacksquare$

This concludes the proof of Proposition 2. \hfill $\blacksquare$ \hfill $\blacksquare$

C.2. Proofs from Section 5

C.2.1. Proof of Proposition 3

For any $\beta \in [0, 1]$ and any rules $f, g$ we have
1. $\varepsilon(h_\beta) \leq \beta \varepsilon(g) + (1 - \beta) \varepsilon(f)$, and
2. $\sigma(h_\beta) \leq \beta \sigma(g) + (1 - \beta) \sigma(f)$.

Proof. The manipulability of a rule is determined by the maximum manipulability across all agents $i \in N$, type profiles $(t_i, t_{-i}) \in T^n$, and misreports $t'_i \in T$. Using the triangle inequality for this max-operator, we get that

$$
\varepsilon(h_\beta) = \max \varepsilon(u, (t_i, t_{-i}), t'_i, h_\beta)
\leq \beta \max \sum_{j \in M} u_i(j) \cdot ((h_\beta)_j(t'_i, t_{-i}) - (h_\beta)_j(t_i, t_{-i}))
\leq \beta \max \sum_{j \in M} u_i(j) \cdot (g_j(t'_i, t_{-i}) - g_j(t_i, t_{-i}))
+ (1 - \beta) \max \sum_{j \in M} u_i(j) \cdot (f_j(t'_i, t_{-i}) - f_j(t_i, t_{-i}))
= \beta \varepsilon(g) + (1 - \beta) \varepsilon(f).
$$

Using linearity of the score deficit function $\sigma_v : \Delta(M) \times T^n \to [0, 1]$ in the first component, the proof for (2) is analogous. \hfill \square

C.2.2. Proof of Corollary 1

The mapping $\varepsilon \mapsto \sigma_\varepsilon$ in convex.

Proof. Assume towards contradiction that there exist $\varepsilon_0, \varepsilon_1, \gamma \in [0, 1]$ such that

$$
\sigma(f_{\gamma \varepsilon_1 + (1 - \gamma) \varepsilon_0}) > \gamma \sigma(f_{\varepsilon_1}) + (1 - \gamma) \sigma(f_{\varepsilon_0})).
$$

Then by Proposition 3 the $\gamma$-hybrid $h_\gamma$ of $f_{\varepsilon_0}$ and $f_{\varepsilon_1}$ has

$$
\sigma(h_\gamma) \leq \gamma \sigma(f_{\varepsilon_1}) + (1 - \gamma) \sigma(f_{\varepsilon_0})) < \sigma(f_{\gamma \varepsilon_1 + (1 - \gamma) \varepsilon_0}),
$$

a contradiction to the optimality of $f_{\gamma \varepsilon_1 + (1 - \gamma) \varepsilon_0}$ at $\varepsilon' = \gamma \varepsilon_1 + (1 - \gamma) \varepsilon_0$. \hfill \square

C.2.3. Proof of Remark 2

The following implications hold between the path properties, while the reverse implications are not true:

1. Linear $\Rightarrow$ piecewise linear $\Rightarrow$ connected.
2. Linear $\Rightarrow$ convex components $\Rightarrow$ orientated.
3. Linear $\Rightarrow$ strongly orientated $\Rightarrow$ orientated.
Proof. The implications in (1) are obvious, and also that the reverse does not hold. The implication in (2) follows from the fact that \( \gamma y + (1 - \gamma)x \leq \max(x, y) \). To see that the reverse implication does not hold, consider the following example: let \( P_\alpha(x) = x + 1 \) and \( P_\sigma(x) = 1 - x^2 \). Then \( P_\sigma \) is not a convex function, but the path \( P \) is oriented. Finally, (3) is obvious. \( \square \)

C.2.4. Proof of Theorem 1

For rules \( f, g \) the path \( P_{\text{Hyb}} \) is connected, piecewise linear, oriented, and has convex components. However, \( P_{\text{Hyb}} \) may fail to be strongly oriented or linear.

Proof. In the proof we make explicit use of the fact that \( \varepsilon \)-approximate strategyproofness can be expressed via a finite set of linear constraints (Proposition 2). For a fixed type profile \( t \in T \) we have that the score deficit of \( h_\beta \) is given by

\[
\sigma(h_\beta(t), t) = \beta \sigma(g(t), t) + (1 - \beta)\sigma(f(t), t).
\]  

(58)

The largest of the values in (58) determine \( \sigma(h_\beta) \). Figure 7 (i) shows the score deficits at three different type profiles and for all \( \beta \in [0, 1] \). The red shade indicates to the maximum over all three score deficits, i.e., the worst-case score deficit, which is relevant for \( \sigma(h_\beta) \). We say that a type profile \( t \in T \) is \( \sigma \)-binding at \( \beta \) if

\[
\sigma(h_\beta(t), t) = \sigma_{\max}(h_\beta).
\]  

(59)

With this notion in mind, we observe that for each type profile \( t \in T \) the set of \( \beta \in [0, 1] \) is a compact interval (indicated by \( \text{Int} \) in Figure 7 (i). At the end points of this interval, some other type profiles become binding over other intervals. It follows that the path \( \beta \mapsto (\beta, \sigma(h_\beta)) \) is connected, piecewise linear, convex, and oriented. We proceed analogously for manipulability: denote the left side of the constraint (9) in Proposition 2 as

\[
\varepsilon(k, (t_i, t_{-i}), t'_i, h_\beta) = \sum_{j \in M: r_i(j) \leq k} (h_\beta)_j(t'_i, t_{-i}) - (h_\beta)_j(t_i, t_{-i}).
\]  

(60)

Then for any fixed agent \( i \in N \), type profile \( (t_i, t_{-i}) \in T \), misreport \( t'_i \in T \), and \( k \in \{1, \ldots, m\} \), we have

\[
\varepsilon(k, (t_i, t_{-i}), t'_i, h_\beta) = \beta \varepsilon(k, (t_i, t_{-i}), t'_i, g) + (1 - \beta)\varepsilon(k, (t_i, t_{-i}), t'_i, f).
\]  

(61)

The largest of the values in (61) determine \( \varepsilon(h_\beta) \). It follows that the path \( \beta \mapsto (\varepsilon(h_\beta), \beta) \) is also connected, piecewise linear, oriented, and \( P_{\text{Hyb}} \) has convex components. This is illustrated in the Figure 7 (ii). By Lemma 3 the path \( P_{\text{Hyb}} \) with \( \beta \mapsto (\varepsilon(h_\beta), \sigma(h_\beta)) \) is also connected, piecewise linear, and oriented. To see that \( P_{\text{Hyb}} \) can fail to be linear and strongly oriented,
consider the following example: let $v = (1, 0, 0)$ be the scoring function in a setting with $N = \{1\}$ and $M = \{a, b, c\}$. Let $f$ be the rule with

$$f_j((t_1)) = \begin{cases} \frac{1}{3}, & \text{if } r_1(j) = 1, \\ \frac{2}{3}, & \text{if } r_1(j) = 2, \\ 0, & \text{if } r_1(j) = 3. \end{cases}$$

(62)

$f$ has a score deficit of $\frac{2}{3}$ at all type profiles, and agent 1 can gain at most $\frac{1}{3}$ by ranking its first choice in second place (if $u_1 = (1,0,0)$). Thus, $\varepsilon(f) = \frac{1}{3}$ and $\sigma(f) = \frac{2}{3}$. Next let $g$ be the rule with

$$g_j((t_1)) = \begin{cases} \frac{1}{2}, & \text{if } r_1(j) = 1, \\ 0, & \text{if } r_1(j) = 2, \\ \frac{1}{2}, & \text{if } r_1(j) = 3. \end{cases}$$

(63)

g has a score deficit of $\frac{1}{2}$ at all type profiles, and agent 1 can gain at most $\frac{1}{2}$ by ranking its second choice in last place (if $u_1 = (1,1,0)$). Thus, $\varepsilon(g) = \frac{1}{2}$ and $\sigma(g) = \frac{1}{2}$. However, the $\frac{1}{2}$-hybrid of $f$ and $g$ is strategyproof, since

$$\frac{1}{2} \left( \frac{1}{3}, \frac{2}{3}, 0 \right) + \frac{1}{2} \left( \frac{1}{2}, 0, \frac{1}{2} \right) = \left( \frac{5}{12}, \frac{4}{12}, \frac{3}{12} \right),$$

(64)
i.e., $\varepsilon(h_{\frac{1}{2}}) = 0$, and $\sigma(h_{\frac{1}{2}}) = \frac{7}{12}$. Figure 8 illustrates the footprints of $f$, $g$, and $h_{\frac{1}{2}}$. If $P^{\text{Hyb}}$ was linear, $p(h_{\frac{1}{2}})$ would lie on the dark green line connecting $p(f)$ and $p(g)$, and if it was strongly oriented, $p(h_{\frac{1}{2}})$ would lie inside the light green square. Since neither is the case, we conclude that $P^{\text{Hyb}}$ is not strongly oriented and not linear.
Lemma 3. Let $P_1, P_2 : [0, 1] \to [0, 1]^2$ be paths such that $P_1(\alpha) = (p_1(\alpha), \alpha)$ and $P_2(\alpha) = (\alpha, p_2(\alpha))$. If $P_1$ and $P_2$ are connected/piecewise linear/linear/oriented/strongly oriented, then so is the path $P$ with $P(\alpha) = (p_1(\alpha), p_2(\alpha))$.

Proof. Suppose that $P_1$ and $P_2$ both have the respective property.

Connectedness If the $P_i$ are connected, the mappings $\alpha \mapsto p_i(\alpha)$ are continuous. The combination of continuous mappings is continuous, thus $P$ is connected.

Linearity If $P_i, i = 1, 2$ are linear, the mapping $\alpha \mapsto p_i(\alpha)$ are linear, i.e.,

$$p_i(\alpha) = m_i^\alpha + c_i, i = 1, 2.$$  

Thus, $P(\alpha) = (m_1^\alpha, m_2^\alpha) \alpha + (c_1, c_2)$, which implies linearity of $P$.

Piecewise linearity Let $I^i = \{I^i_k | k \geq 1\}, i = 1, 2$ be the decomposition of $[0, 1]$ into compact intervals $I^i_k$ on which the paths $P^i$ are linear. Since $P^i$ is piecewise linear, $I^i$ is finite and has $K^i$ elements, the intersection of any two elements contains at most a single point, and the union of the elements of $I^i$ is equal to $[0, 1]$. Let

$$A^i = \{\alpha^i \mid \alpha^i = I^i_k \cap I^i_{k'}, \text{ for } k, k' \in \{1, \ldots, K^i\}\}$$  

be the set of unique points where two intervals intersect, and relabel the elements of $A = A^1 \cup A^2 \cup \{0, 1\}$ such that $A = \{\alpha_0, \ldots, \alpha_K\}$, where $K \leq K_1 + K^2 + 1$ and $\alpha_0 = 0$, $\alpha_K = 1$, and $\alpha_k < \alpha_{k+1}$ for all $k \in \{0, \ldots, K-1\}$. It follows that $P^1$ and $P^2$ are both linear on each interval $[\alpha_k, \alpha_{k+1}]$. Thus, $P$ is also linear on the intervals $[\alpha_k, \alpha_{k+1}]$ by the arguments that showed linearity of $P$ before. But then $P$ is piecewise linear.

Orientation and Strong orientation Orientation and strong orientation are defined by component. Since the first component of $P^1$ is is oriented/strongly oriented by assumption, and
the same holds for the second component of \( P^2 \), the path \( P \) must also be oriented/strongly oriented.

This concludes the proof of the Lemma.

This concludes the proof of Theorem 1.

\[ \square \]

C.2.5. Proof of Theorem 2

For rules \( f, g \) the following are equivalent:

1. \( P^{\text{Hyb}} \) is linear.

2. There exists a type profile \( t \in T^n \) such that

\[ \sigma(f(t), t) = \sigma^{\text{max}}(f) \quad \text{and} \quad \sigma(g(t), t) = \sigma^{\text{max}}(g), \]  \hspace{1cm} (67)

and there exist \( k \in \{1, \ldots, m\}, i \in N, (t_i, t_{i-}) \in T^n, t'_i \in T \) such that

\[ \sum_{j \in M_r(i), (j) \leq k} f_j(t'_i, t_{i-}) - f_j(t_i, t_{i-}) = \epsilon(f) \quad \text{and} \quad \sum_{j \in M_r(i), (j) \leq k} g_j(t'_i, t_{i-}) - g_j(t_i, t_{i-}) = \epsilon(g). \]

In words, \( f \) and \( g \) attain their worst-case score deficit at the same type profile \( t \) and the worst manipulability at the same combination \((k, (t_i, t_{i-}), t'_i)\).

\[ \square \]

\[ \square \]

Proof. Recall Figure 4 in the proof of Theorem 1: if some type profile \( t \in T^n \) is \( \sigma \)-binding at \( \beta = 0 \) and \( \beta = 1 \), then it is \( \sigma \)-binding on the entire interval \([0, 1]\). Thus, \( \sigma^{\text{max}}(h_\beta) = \sigma(h_\beta(t), t) \).

Similarly, if the same constellation \((k, (t_i, t_{i-}), t'_i)\) is \( \varepsilon \)-binding at \( \beta = 0 \) and \( \beta = 1 \), we have that \( \epsilon(h_\beta) = \epsilon(k, (t_i, t_{i-}), t'_i, h_\beta) \) for all \( \beta \in [0, 1] \). Consequently, \( P^{\text{Hyb}} \) is linear, because the only hinges are \( \beta_0 = 0 \) and \( \beta_1 = 1 \). To see that the reverse implication also holds, assume towards contradiction that some type profile \( t \in T^n \) is \( \sigma \)-binding on some interval \([0, \beta_1]\) but not at \( \beta = 1 \), while another type profile \( t^* \in T^n \) is \( \sigma \)-binding at \( \beta = 1 \). Then

\[ \sigma(h_{\beta_1}) = \beta_1 \sigma(g(t), t) + (1 - \beta_1) \sigma(f(t), t), \]  \hspace{1cm} (68)

since \( t \) is \( \sigma \)-binding at \( \beta_1 \) by assumption. Linearity of \( P^{\text{Hyb}} \) implies that

\[ \sigma(h_{\beta_1}) = \beta_1 \sigma(g) + (1 - \beta_1) \sigma(f), \]  \hspace{1cm} (69)

and therefore

\[ \sigma(g(t^*), t^*) = \sigma(g) \]  \hspace{1cm} (70)

\[ = \frac{1}{\beta_1} (\sigma(h_{\beta_1}) - (1 - \beta_1) \sigma(f)) \]  \hspace{1cm} (71)

\[ = \frac{1}{\beta_1} (\beta_1 \sigma(g(t), t) + (1 - \beta_1) \sigma(f(t), t) - (1 - \beta_1) \sigma(f)) \]  \hspace{1cm} (72)

\[ = \sigma(g(t), t). \]  \hspace{1cm} (73)
This is a contradiction to the assumption the \( t \) was not \( \sigma \)-binding at \( \beta = 1 \). If we assume instead that two different constellations \( (k, (t_i, t_{i-1}), t'_i) \) and \( (k^*, (t^*_i, t^*_{i-1}), (t^*_{i})') \) are \( \varepsilon \)-binding on \([0, \beta_1] \) and at \( \beta = 1 \), respectively, we can make the analogous argument to get a contradiction, which concludes the proof.

C.3. Proofs from Section 6

C.3.1. Proof of Theorem 3

The path \( P^{EF} \) is connected, piecewise linear, strongly oriented, has convex components, and \( D^{EF}(1) = (1,0) \). However, it may fail to be linear.

Proof. From Section 4.3 we know that for each \( \varepsilon \in [0,1] \) we can write \( \text{Opt}(\varepsilon) \) as the set of solutions to a linear program, i.e.,

\[
\text{Opt}(\varepsilon) = \arg\min_{x} \langle w, x \rangle \quad \text{s.t.} \quad Dx \leq d, \quad Ax \leq \varepsilon, \quad (74)
\]

where \( D \) and \( A \) are matrices, \( w \) and \( d \) are vectors, \( \varepsilon \) is a vector with all entries equal to \( \varepsilon \), and \( x \) is a vector of variables of dimension \( L \). Observe that \( \varepsilon \) enters the constraints only as the upper bound in a number of linear inequalities. The proof utilizes this characterization of \( \text{Opt}(\varepsilon) \). Before we proceed with the proof of Theorem 3, we require a number of definitions. Denote by \( F_{\varepsilon} \) the set of feasible points at \( \varepsilon \), i.e.,

\[
F_{\varepsilon} = \{ x | Dx \leq d, Ax \leq \varepsilon \}, \quad (77)
\]

and denote by \( S_{\varepsilon} \) the set of solutions at \( \varepsilon \), i.e.,

\[
S_{\varepsilon} = \arg\min_{x \in F_{\varepsilon}} \langle w, x \rangle. \quad (78)
\]

A constraint is a row \( C_l \) of either the matrix \( A \) or the matrix \( D \) with the corresponding bound \( c_l \) equal to respective entry of \( d \) or \( \varepsilon \). A feasible point \( x \in F_{\varepsilon} \) is an extreme point of \( F_{\varepsilon} \) if there exist \( L \) independent constraints \( C_1, \ldots, C_L \) such that

\[
C_l x = c_l \quad (79)
\]

for all \( l \in \{1, \ldots, L\} \), i.e., the constraints are satisfied with equality at \( x \). \( x \) is then said to be an extreme point of \( F_{\varepsilon} \) with respect to \( (C_1, \ldots, C_L) \). We say that the set of constraints \( (C_1, \ldots, C_L) \) is restrictive at \( \varepsilon \) if they are independent and there exists an extreme point in \( F_{\varepsilon} \) with respect to these constraints. Let

\[
\mathcal{R}(\varepsilon) = \{ (C_1, \ldots, C_L) | (C_1, \ldots, C_L) \text{ is restrictive at } \varepsilon \} \quad (80)
\]
be the set of all sets of constraints that are restrictive at $\varepsilon$. A set of restrictive constraints $C \in \mathcal{R}(\varepsilon)$ is binding at $\varepsilon$ if the extreme point $x \in F_\varepsilon$ where the constraints of $C$ are satisfied with equality is a solution, i.e., $x \in S_\varepsilon$. Let

$$
\mathcal{B}(\varepsilon) = \{C \in \mathcal{R}(\varepsilon) | C \text{ is binding at } \varepsilon\}
$$

be the set of all sets of constraints that are binding at $\varepsilon$. We denote by $E(F_\varepsilon)$ and $E(S_\varepsilon)$ the extreme points of $F_\varepsilon$ and $S_\varepsilon$, respectively. Observe that since $F_\varepsilon$ and $S_\varepsilon$ are polyhedrons bounded by finitely many hyperplanes, the extreme points $E(F_\varepsilon)$ and $E(S_\varepsilon)$ form minimal $\mathcal{V}$-representations of $F_\varepsilon$ and $S_\varepsilon$ (see, e.g., p.51ff in (Grünbaum, 2003)). Thus $\text{Conv}(E(F_\varepsilon)) = F_\varepsilon$ and $\text{Conv}(E(S_\varepsilon)) = S_\varepsilon$. Furthermore, since $S_\varepsilon \subseteq F_\varepsilon$, any extreme point of $S_\varepsilon$ is also an extreme point of $F_\varepsilon$, i.e., $E(S_\varepsilon) \subseteq E(F_\varepsilon)$. Finally, each extreme point is uniquely determined by the set of constraints with respect to which it extreme, i.e., if there exists an extreme points with respect a to set of constraints $C \in \mathcal{R}(\varepsilon)$, then this point is unique.

**Claim 1.** For $\varepsilon_0, \varepsilon_1 \in [0, 1]$ with $\varepsilon_0 < \varepsilon_1$, if $x_0 \in F_{\varepsilon_0}$ and $x_1 \in F_{\varepsilon_1}$, then for any $\gamma \in [0, 1]$ and $\varepsilon = \gamma\varepsilon_1 + (1 - \gamma)\varepsilon_0$ we have that

$$
x = \gamma x_1 + (1 - \gamma) x_0 \in F_\varepsilon.
$$

**Proof.** By assumption, $D_k x_0 \leq d_k$ and $D_k x_1 \leq d_k$ for all $k$. Thus,

$$
D_k (\gamma x_1 + (1 - \gamma) x_0) = \gamma D_k x_1 + (1 - \gamma) D_k x_0 \leq d_k.
$$

Furthermore, $A_k x_0 \leq \varepsilon_0$ and $A_k x_1 \leq \varepsilon_1$ for all $k$, which implies

$$
A_k (\gamma x_1 + (1 - \gamma) x_0) = \gamma A_k x_1 + (1 - \gamma) A_k x_0 \leq \gamma \varepsilon_1 + (1 - \gamma) \varepsilon_0 = \varepsilon.
$$

\hfill \Box

**Claim 2.** For $\varepsilon_0, \varepsilon_1 \in [0, 1]$ with $\varepsilon_0 < \varepsilon_1$, if $C \in \mathcal{R}(\varepsilon_0)$ and $C \in \mathcal{R}(\varepsilon_1)$, then for any $\gamma \in [0, 1]$ and $\varepsilon = \gamma\varepsilon_1 + (1 - \gamma)\varepsilon_0$ we have that $C \in \mathcal{R}(\varepsilon)$, and the $\gamma$-convex combination of the extreme points at $\varepsilon_0$ and $\varepsilon_1$ with respect to $C$ is the unique extreme point at $\varepsilon$ with respect to $C$.

**Proof.** Since $C \in \mathcal{R}(\varepsilon_0)$ and $C \in \mathcal{R}(\varepsilon_1)$, there exist unique extreme points $x_0 \in E(F_{\varepsilon_0})$ and $x_1 \in E(F_{\varepsilon_1})$ with respect to $C$. By Claim 1 the point $x = \gamma x_1 + (1 - \gamma) x_0$ is feasible at $\varepsilon$. For any $l \in \{1, \ldots, L\}$ if $C_l = D_k$ for some $k$ we have that

$$
C_l x = \gamma D_k x_1 + (1 - \gamma) D_k x_0 = \gamma d_k + (1 - \gamma) d_k = d_k = c_l,
$$

i.e., $x$ satisfies the constraint $C_l$ with equality. If $C_l = A_k$, then the constraint is also satisfied with equality, since

$$
C_l x = \gamma A_k x_1 + (1 - \gamma) A_k x_0 = \gamma \varepsilon_1 + (1 - \gamma) \varepsilon_0 = \varepsilon.
$$

Consequently, $x$ is the unique extreme point at $\varepsilon$ with respect to $C$. This in turn implies that $C$ is restrictive at $\varepsilon$. \hfill \Box
Claim 3. There exists a finite decomposition

$$0 = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_K = 1$$

(87)

of the interval $[0, 1]$, such that on each interval $[\varepsilon_k, \varepsilon_{k+1}]$ we have that $R(\varepsilon) = R(\varepsilon')$ for all $\varepsilon, \varepsilon' \in [\varepsilon_k, \varepsilon_{k+1}]$.

Proof. By Claim 2, if some set of $L$ independent constraints $C$ is restrictive at some $\varepsilon \in [0, 1]$, then the set of $\varepsilon' \in [0, 1]$ where $C$ is also restrictive must be compact interval $[\varepsilon_C^-, \varepsilon_C^+] \subseteq [0, 1]$ with $\varepsilon \in [\varepsilon_C^-, \varepsilon_C^+]$. Since there is a finite number of constraints, there is also a finite number of constraint sets $C$. Consider the set

$$\{\varepsilon_0, \ldots, \varepsilon_K\} = \bigcup_{C \text{ set of } L \text{ indep. constraints}} \{\varepsilon_C^-, \varepsilon_C^+\}.$$  

(88)

Observe that by construction, a set of $L$ independent constraints $C$ becomes restrictive or un-restrictive only at one of the finitely many $\varepsilon_k$. This proves the claim. \hfill \Box

Claim 4. On each interval $[\varepsilon_k, \varepsilon_{k+1}]$ from Claim 3 and for any $\varepsilon \in [\varepsilon_k, \varepsilon_{k+1}]$, if $C \in R(\varepsilon)$, then $C \in R(\varepsilon_k) \cap R(\varepsilon_{k+1})$.

Proof. Assume towards contradiction that $C \in R(\varepsilon)$, but $C \notin R(\varepsilon_k)$. Then there exists an $\varepsilon' \in (\varepsilon_k, \varepsilon) \subseteq (\varepsilon_k, \varepsilon_{k+1})$, where $C$ become restrictive for the first time. Then $\varepsilon' \in \{\varepsilon_0, \varepsilon_k\}$, and therefore $[\varepsilon_k, \varepsilon_{k+1}]$ cannot be one of the intervals in the decomposition. Instead, it would be split by $\varepsilon'$, a contradiction. \hfill \Box

Claim 5. On each interval $[\varepsilon_k, \varepsilon_{k+1}]$ from Claim 3 and for any $\gamma \in [0, 1]$ with $\varepsilon = \gamma \varepsilon_{k+1} + (1 - \gamma)\varepsilon_k$ we have that

$$F_\varepsilon = \gamma F_{\varepsilon_{k+1}} + (1 - \gamma) F_{\varepsilon_k},$$

(89)

i.e., the set of feasible points at $\varepsilon$ is equal to the $\gamma$-convex combination of feasible points at $\varepsilon_k$ and $\varepsilon_{k+1}$.

Proof. By Claim 1 we have

$$F_\varepsilon \supseteq \gamma F_{\varepsilon_{k+1}} + (1 - \gamma) F_{\varepsilon_k}.$$  

(90)

By Claim 4 the extreme points of $F_\varepsilon$ are the $\gamma$-convex combinations of extreme points of $F_{\varepsilon_k}$ and $F_{\varepsilon_{k+1}}$. Since $F_\varepsilon = \text{Conv}(E(F_\varepsilon))$, this implies

$$F_\varepsilon \subseteq \gamma F_{\varepsilon_{k+1}} + (1 - \gamma) F_{\varepsilon_k},$$  

(91)

which concludes the proof of the claim. \hfill \Box

Claim 6. On each interval $[\varepsilon_k, \varepsilon_{k+1}]$ from Claim 3, if $C \in B(\varepsilon)$ for some $\varepsilon \in (\varepsilon_k, \varepsilon_{k+1})$, then $C \in B(\varepsilon_k) \cap B(\varepsilon_{k+1})$. Furthermore, the extreme point of $S_\varepsilon$ with respect to $C$ is the $\gamma$-convex combination of the extreme points of $S_{\varepsilon_k}$ and $S_{\varepsilon_{k+1}}$ with respect to $C$ with $\gamma = \frac{\varepsilon - \varepsilon_k}{\varepsilon_{k+1} - \varepsilon_k}$.

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Proof. Since \( C \in \mathcal{B}(\varepsilon) \), there exists a unique extreme point \( x \in \mathcal{E}(S_{\varepsilon}) \) such that \( x \) is extreme at \( \varepsilon \) with respect to \( C \). By Claim 5, we can represent \( x = \gamma x_1 + (1 - \gamma)x_0 \) with \( x_0 \in F_{\varepsilon_k}, x_1 \in F_{\varepsilon_{k+1}} \). By Claim 2, \( x_0 \) and \( x_1 \) are extreme points of \( F_{\varepsilon_k} \) and \( F_{\varepsilon_{k+1}} \), respectively. Suppose towards contradiction that \( x_0 \notin S_{\varepsilon_k} \). Then there exists \( x'_0 \in S_{\varepsilon_k} \) such that
\[
\langle w, x'_0 \rangle < \langle w, x_0 \rangle.
\]
By Claim 1, \( x' = \gamma x_1 + (1 - \gamma)x'_0 \) is in \( F_\varepsilon \) and
\[
\begin{align*}
\langle w, x' \rangle &= \gamma \langle w, x_1 \rangle + (1 - \gamma)\langle w, x'_0 \rangle \\
&< \gamma \langle w, x_1 \rangle + (1 - \gamma)\langle w, x_0 \rangle \\
&= \langle w, x \rangle,
\end{align*}
\]
i.e., \( x' \) is feasible at \( \varepsilon \) and has lower objective than \( x \). This contradicts the assumption that \( x \in S_{\varepsilon} \). A similar argument yields \( x_1 \in S_{\varepsilon_{k+1}} \), which concludes the proof of the claim.

Claim 7. On each interval \([\varepsilon_k, \varepsilon_{k+1}]\) from Claim 3 and any \( \gamma \in [0, 1] \) with \( \varepsilon = \gamma \varepsilon_{k+1} + (1 - \gamma)\varepsilon_k \) we have that
\[
S_{\varepsilon} = \gamma S_{\varepsilon_{k+1}} + (1 - \gamma)S_{\varepsilon_k},
\]
i.e., the set of solutions at \( \varepsilon \) is equal to the \( \gamma \)-convex combination of solutions at \( \varepsilon_k \) and \( \varepsilon_{k+1} \).

Proof. By Claim 1 we have
\[
S_{\varepsilon} \supseteq \gamma S_{\varepsilon_{k+1}} + (1 - \gamma)S_{\varepsilon_k}.
\]
By Claim 6 the extreme points of \( S_{\varepsilon} \) are the \( \gamma \)-convex combinations of extreme points of \( S_{\varepsilon_k} \) and \( S_{\varepsilon_{k+1}} \). Since \( S_{\varepsilon} = \text{Conv}(\mathcal{E}(S_{\varepsilon})) \), this implies
\[
S_{\varepsilon} \subseteq \gamma S_{\varepsilon_{k+1}} + (1 - \gamma)S_{\varepsilon_k},
\]
which concludes the proof of the claim.

Claim 7 is the main step in the proof of Theorem 3: every solution \( x \in S_{\varepsilon} \) corresponds to some optimal rule \( f \in \text{OPT}(\varepsilon) \). Furthermore, by the nature of the representation of rules in the linear program, the convex combination of two solutions corresponds to the hybrid rule of the two rules. Linearity of the objective function yields that the path \( P_{\text{EF}} \) is linear on each of intervals \([\varepsilon_k, \varepsilon_{k+1}]\), which establishes piecewise linearity, which in turn implies connectedness. The first component of \( P_{\text{EF}} \), i.e., \( \varepsilon \mapsto \varepsilon \) is obviously a convex function. The convexity of the second component, \( \varepsilon \mapsto \sigma_\varepsilon \) is a consequence of Theorem 1: if for any \( \varepsilon, \varepsilon', \gamma \in [0, 1] \) we had that
\[
\gamma \sigma(f_{\varepsilon'}) + (1 - \gamma)\sigma(f_{\varepsilon}) < \sigma(f_{\gamma \varepsilon' + (1 - \gamma)\varepsilon}),
\]
then we could simply use the \( \gamma \)-hybrid of \( f_{\varepsilon} \) and \( f_{\varepsilon'} \) and achieve a strictly lower deficit. Therefore, \( f_{\gamma \varepsilon' + (1 - \gamma)\varepsilon} \notin \text{OPT}(\gamma \varepsilon' + (1 - \gamma)\varepsilon) \), a contradiction. An analogous argument yields strong orientedness. In Section 7.3 we provide an example where the path \( P_{\text{EF}} \) is not linear, which concludes the proof.

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C.3.2. Proof of Proposition 4

**FindHinges** is correct and complete for the problem of finding all hinges in $[\delta, 1]$. If there are $K$ hinges in $[\delta, 1]$, **FindHinges** requires at most $4K + 1$ executions of **FindOpt**.

**Proof.** A segment consists of two points $(P^\text{EF}(\varepsilon), P^\text{EF}(\varepsilon'))$. In addition, there are two outer segments $(P^\text{EF}(0), P^\text{EF}(\delta))$ and $(P^\text{EF}(1), P^\text{EF}(2))$, which are needed to help the algorithm get started. The algorithm maintains a decomposition of the interval $[\delta, 1]$, which initially consists of a single unverified segment. In every execution of the while-loop, **FindHinges** selects an unverified segment $(P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^+))$. Then the segments to the left and right $(P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^-))$ and $(P^\text{EF}(\varepsilon^+), P^\text{EF}(\varepsilon^+))$ to “guess” the position of a new hinge. This guess $e$ is the $\varepsilon$-value of the point of the intersection of affine hulls of the two segments, i.e.,

$$e = \left(\text{aff}\left(\{P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^-)\}\right) \cap \text{aff}\left(\{P^\text{EF}(\varepsilon^+), P^\text{EF}(\varepsilon^+)\}\right)\right)_\varepsilon,$$

where $\text{aff}$ denotes the affine hull. This value is unique and lies inside the open interval $(\varepsilon^-, \varepsilon^+)$ (by Lemma 4). Now $P = P^\text{EF}(e)$ is computed and one of three cases can occur:

1. $P$ may be equal to the unique point

$$\text{aff}\left(\{P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^-)\}\right) \cap \text{aff}\left(\{P^\text{EF}(\varepsilon^+), P^\text{EF}(\varepsilon^+)\}\right).$$

In this case, $e$ is a hinge of $P^\text{EF}$ and the segments

$$\left(P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^-)\right), \left(P^\text{EF}(\varepsilon^-), P\right), \left(P, P^\text{EF}(\varepsilon^+)\right), \text{ and } \left(P^\text{EF}(\varepsilon^+), P^\text{EF}(\varepsilon^+)\right).$$

By Lemma 6, they are all part of $P^\text{EF}$ and there are no other hinges in the interval $(\varepsilon^-, \varepsilon^+)$. **FindHinges** marks the four segments as verified and includes $e$ in the collection of hinges.

2. $P$ lies in the affine hull of the segment $(P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^+))$. Then by Lemma 5,

$$\text{Conv}\left(\{P^\text{EF}(\varepsilon^-), P\}\right) \cup \text{Conv}\left(\{P, P^\text{EF}(\varepsilon^+)\}\right) \subseteq P^\text{EF},$$

and there are no hinges inside the interval $(\varepsilon^-, \varepsilon^+)$. **FindHinges** marks the segment $(P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^+))$ as verified.

3. In any other case, **FindHinges** splits the segment $(P^\text{EF}(\varepsilon^-), P^\text{EF}(\varepsilon^+))$ and creates two unverified segments

$$\left(P^\text{EF}(\varepsilon^-), P\right) \text{ and } \left(P, P^\text{EF}(\varepsilon^+)\right).$$

We first show correctness of **FindHinges**, then completeness:

**Correctness** **FindHinges** stops running when there are no more unverified segments. Assume towards contradiction that there exists a hinge $\varepsilon \in (\delta, 1)$ that has not been identified. Then this hinge lies in some segment $[\varepsilon^-, \varepsilon^+]$ that was verified. If the verification happened in some case (1), Lemma 6 ensures that there is other but inside $P$ inside
the interval \((\varepsilon^-, \varepsilon^+)\), and \(P\) is added to the collection of hinges. If the verification happened in some case (2), Lemma 5 ensures that \(\varepsilon \notin (\varepsilon^-, \varepsilon^+)\), so that (without loss of generality) \(\varepsilon = \varepsilon^-\). The segment \((P^{\text{EF}}(\varepsilon^-), P^{\text{EF}}(\varepsilon^-))\) was not a verified segment at this time, otherwise this would be a case (1). Thus, at some future step some segment \((P^{\text{EF}}(\tilde{\varepsilon}), P^{\text{EF}}(\tilde{\varepsilon}))\) with a right end-point in \(P^{\text{EF}}(\varepsilon^-)\) was verified. But at this step \(P^{\text{EF}}(\varepsilon^-)\) was on the intersection of the affine hulls of the segments \((P^{\text{EF}}(\tilde{\varepsilon}), P^{\text{EF}}(\tilde{\varepsilon}))\) and \((P^{\text{EF}}(\varepsilon^-), P^{\text{EF}}(\varepsilon^+))\). This creates a case (1), and thus \(\varepsilon\) was identified as a hinge in this step.

**Completeness** It remains to be shown that \(\text{FindHinges}\) stops at some point. By Lemma 7, for ever two adjacent hinges \(\varepsilon_i, \varepsilon_{i+1}\) of \(P^{\text{EF}}\), \(\text{FindHinges}\) computes at most three points on \(P^{\text{EF}}(\varepsilon'), P^{\text{EF}}(\varepsilon''), P^{\text{EF}}(\varepsilon''')\) with \(\varepsilon', \varepsilon'', \varepsilon''' \in (\varepsilon_i, \varepsilon_{i+1})\). Since there is a finite number of hinges, \(\text{FindHinges}\) loops at most \(3 + 1 = 4\) times per interval, which establishes completeness and the run-time bound.

**Lemma 4.** \(e \in (\varepsilon^-, \varepsilon^+)\)

**Proof.** By convexity of \(\varepsilon \mapsto \sigma_{\varepsilon}\), we get that \(e \in [\varepsilon^-, \varepsilon^+]\). Now suppose that \(e = \varepsilon^-\). Then \(\varepsilon^- \in \text{aff}([P^{\text{EF}}(\varepsilon^+), P^{\text{EF}}(\varepsilon^+)])\). Since \(\varepsilon^- < \varepsilon^+\), the segments \((P^{\text{EF}}(\varepsilon^-), P^{\text{EF}}(\varepsilon^+))\) and \((P^{\text{EF}}(\varepsilon^+), P^{\text{EF}}(\varepsilon^+))\) would have been verified in a previous step. But this is a contradiction to the assumption that \((P^{\text{EF}}(\varepsilon^-), P^{\text{EF}}(\varepsilon^+))\) was an unverified segment. \(\square\)

**Lemma 5.** For \(0 \leq \varepsilon_- < \varepsilon_0 < \varepsilon_1 \leq 1\), if

\[
P^{\text{EF}}(\varepsilon_0) \in \text{Conv} \left( \{ P^{\text{EF}}(\varepsilon_-), P^{\text{EF}}(\varepsilon_1) \} \right),
\]

then

\[
\text{Conv} \left( \{ P^{\text{EF}}(\varepsilon_-), P^{\text{EF}}(\varepsilon_1) \} \right) \subseteq P^{\text{EF}}.
\]

**Proof.** Assume towards contradiction that

\[
\text{Conv} \left( \{ P^{\text{EF}}(\varepsilon_-), P^{\text{EF}}(\varepsilon_1) \} \right) \not\subseteq P^{\text{EF}}.
\]

Then by convexity of \(\varepsilon \mapsto \sigma_{\varepsilon}\) there exists \(\gamma \in (0, 1)\) with

\[
\sigma(f_{\varepsilon_1 + (1-\gamma)\varepsilon_-}) < \gamma \sigma(f_{\varepsilon_1}) + (1-\gamma)\sigma(f_{\varepsilon_-}).
\]

If \(\varepsilon' = \gamma \varepsilon_1 + (1-\gamma)\varepsilon_- > \varepsilon_0\), then for \(\gamma' = \frac{\varepsilon_0 - \varepsilon_-}{\varepsilon_1 - \varepsilon_-}\) we get

\[
\gamma' \sigma(f_{\varepsilon'}) + (1-\gamma')\sigma(f_{\varepsilon_-}) < \sigma(f_{\varepsilon_0}) = \sigma(f_{\varepsilon_1 + (1-\gamma)\varepsilon_-}),
\]

a contradiction to convexity of \(\varepsilon \mapsto \sigma_{\varepsilon}\). The argument for \(\varepsilon' < \varepsilon_0\) is analogous. \(\square\)

**Lemma 6.** For \(0 \leq \varepsilon_- < \varepsilon_1 < \varepsilon_0 < \varepsilon_2 < 1\), if

\[
\{ P^{\text{EF}}(\varepsilon) \} = \text{aff} \left( \{ P^{\text{EF}}(\varepsilon_-), P^{\text{EF}}(\varepsilon_1) \} \right) \cap \text{aff} \left( \{ P^{\text{EF}}(\varepsilon_0), P^{\text{EF}}(\varepsilon_2) \} \right),
\]

then

\[
\text{Conv} \left( \{ P^{\text{EF}}(\varepsilon_0), P^{\text{EF}}(\varepsilon_2) \} \right) \subseteq P^{\text{EF}}.
\]
Proof. The lemma follows by applying Lemma 5 twice.

Lemma 7. For any two adjacent hinges \( \varepsilon_k, \varepsilon_{k+1} \) of \( P^E \), \( \text{FindHinges} \) computes at most three points on \( P^E(\varepsilon'), P^E(\varepsilon''), P^E(\varepsilon'''') \) with \( \varepsilon', \varepsilon'', \varepsilon''' \in (\varepsilon_k, \varepsilon_{k+1}) \)

Proof. Suppose that \( P^E(\varepsilon'), P^E(\varepsilon'') \), and \( P^E(\varepsilon'''') \) are computed in this order. If \( \varepsilon''' < \min(\varepsilon', \varepsilon'') \), then \( \varepsilon' \) must be a hinge by convexity of \( \varepsilon \mapsto \sigma_\varepsilon \), which is a contradiction. The same holds if \( \varepsilon''' > \max(\varepsilon', \varepsilon'') \). If \( \varepsilon''' \in (\min(\varepsilon', \varepsilon''), \max(\varepsilon', \varepsilon'')) \), the segment

\[
(P^E(\min(\varepsilon', \varepsilon'')), P^E(\max(\varepsilon', \varepsilon'')))
\]

is verified (case (2)). Another guess \( \varepsilon''' \) that lies withing from \( [\varepsilon_k, \varepsilon_{k+1}] \) involve the segment \( (P^E(\varepsilon'), P^E(\varepsilon'')) \). Thus, by convexity of \( \varepsilon \mapsto \sigma_\varepsilon \), \( \varepsilon''' \) is a hinge of \( P^E \).

This concludes the proof of Proposition 4

C.3.3. Proof of Proposition 5

Let \( \varepsilon_1 \) be the smallest non-zero hinge of \( P^E \), then \( \text{FindDelta} \) is correct and complete for the problem of finding a bound \( 0 < \delta < \varepsilon_1 \). \( \text{FindDelta} \) requires at most \( 3 + \log_2 (1/\varepsilon_1) \) executions of \( \text{FindOpt} \).

Proof. By Theorem 3, there exists a smallest hinge \( \varepsilon_1 > 0 \). Since \( \text{FindDelta} \) tries out \( \delta = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \), it will eventually compute \( P^E(\delta) \) and \( P^E(\delta/2) \) for some \( \delta \leq \varepsilon_1 \). At this time, the point \( P^E(\delta/2) \) will be on the straight line connecting \( P^E(0) \) and \( P^E(\delta) \) (otherwise, there would be hinge below \( \delta \). Thus, the algorithm always stops with a correct value for \( \delta \). At the start, \( P^E(0) \) and \( P^E(1/2) \) are computed using the LP. \( P^E(1) \) comes for free as it will always be the point \((1,0) \). The algorithm terminates when \( 2\delta \leq \varepsilon_1 \). If \( \frac{1}{2^n} \leq \varepsilon_1 < \frac{1}{2^{n+1}} \), it will take at most \( k \) iterations of the loop until this happens. Since \( k \geq \log_2 \left( \frac{1}{\varepsilon_1} \right) > k - 1 \), the algorithm needs at most \( \log_2 \left( \frac{1}{\varepsilon_1} \right) + 3 \) computations of \( P^E \) for new \( \delta \).

C.4. Proofs from Section 7

C.4.1. Proof of Proposition 6

In the full domain (with or without indifferences), for any rule \( f \) there exists an anonymous, neutral rule \( f^* \) with weakly better footprint, i.e., \( \varepsilon(f^*) \leq \varepsilon(f) \) and \( \sigma(f^*) \leq \sigma(f) \).

Proof. \( \Pi = \{ \pi : N \to N \text{ bijection} \} \) and \( \Phi = \{ \varphi : M \to M \text{ bijection} \} \) are the possible permutations that re-order the agents and rename the alternatives, respectively. Then \( \#\Pi = n! \) and \( \#\Phi = m! \). For a given rule \( f \) and \( \pi \in \Pi, \varphi \in \Phi \) let \( f^\pi,\varphi \) be the rule defined by

\[
f^\pi,\varphi_j(t_1, \ldots, t_n) = f_{\varphi(j)}(\varphi(t_{\pi(1)}), \ldots, \varphi(t_{\pi(n)}))
\]

(113)
for every alternatives $j \in M$. Since manipulability and deficit are independent of the particular order names of agents and alternatives, we have $\varepsilon(f^{\pi,\varphi}) = \varepsilon(f)$ and $\varphi(f^{\pi,\varphi}) = \sigma(f)$. Define $f^*$ by

$$f^*(t) = \frac{1}{n!m!} \sum_{\pi \in \Pi, \varphi \in \Phi} f^{\pi,\varphi}(t).$$  (114)

This rule is anonymous and neutral by construction. It is also a hybrid of multiple rules $f^{\pi,\varphi}$, all of which have the same manipulability and deficit. Thus, by Proposition 3, the inequalities follow.

C.4.2. Proof of Proposition 8

For $n = m = 3$, $v = (1, 0, 0)$, and strict preferences, UnifPlurality is $\frac{1}{3}$-approximately strategyproof, and any positional scoring rule (including UnifPlurality) violates $\varepsilon$-approximate strategyproofness for any $\varepsilon < \frac{1}{3}$.

Proof. Without loss of generality we can consider anonymous and neutral rules only (by Proposition 6). Consider an arbitrary positional scoring rule. At any type profile where an alternative is ranked first by two agents or more, this alternative is implemented with certainty. Thus, if two agents have the same first choice, the third agent has no opportunity to manipulate, because it cannot change the outcome. The two agents with the same first choice are already receiving their favorite outcome, which makes manipulation useless for them as well. Now consider the type profile

- $t_1 : a > b > c,$
- $t_2 : b > c > a,$
- $t_3 : c > a > b.$

Renaming the alternatives is equivalent to renaming the agents. Thus, an anonymous and neutral rule has to treat all alternatives equally and must therefore select each alternative with probability $\frac{1}{3}$. Now suppose that agent 1 has utility function close to the binary utility $u(a) = u(b) = 1, u(c) = 0$. Then by swapping $a$ and $b$, agent 1 can enforce the implementation $b$ with certainty. Its gain from this manipulation is

$$1 \cdot u(b) - \frac{1}{3} (u(a) + u(b) + u(c)) = \frac{1}{3}.$$  (118)

Thus, any positional scoring rule will have manipulability $\varepsilon(f) \geq \frac{1}{3}$. Now consider the positional scoring rule UnifPlurality. At the above type profile agents cannot change the outcome, unless they change their first choice. By anonymity and neutrality, it suffices to show that agent 1 cannot do any better than $\frac{1}{3}$ by manipulating. However, the only other possible misreport that has any effect on the outcome is to bring $c$ forward and enforce it as the outcome, which would yield no benefit for agent 1.
C.4.3. Proof of Proposition 9

For \( n = m = 3 \), \( v = (1, 0, 0) \), and strict preferences, \( \text{RandDict} \in \text{Opt}(0) \), i.e., it has the lowest deficit among all strategyproof rules.

Proof. Since \( \text{RandDict} \) is a lottery of unilateral, strategyproof rules, it is obviously strategyproof. At any type profile where all agents have the same first choice, \( \text{RandDict} \) will select this alternative and achieve zero score deficit. At any type profile where all agents have different first choices, all alternatives have the same score and therefore, any outcome has zero score deficit. Finally, consider the case where two agents agree on a first choice, \( a \) say, but the third agent has a different first choice, \( b \) say. In this case, selecting \( a \) would yield a maximal score of \( \frac{2}{3} \). However, Random Dictatorship will only select alternative \( a \) with probability \( \frac{2}{3} \) and \( b \) otherwise. This leads to a score of \( \frac{4}{9} \), and therefore \( \sigma_{\text{RandDict}} \geq \frac{2}{9} \).

Now let \( f \) be a strategyproof, anonymous, neutral rule. Suppose, the symmetric decomposition of \( f \) contains a strategyproof duple \( d_{a,b} \). By the symmetric decomposition, we can assume that \( d_{a,b} \) is anonymous. By the characterization of strategyproofness via swap monotonicity, upper invariance, and lower invariance, it follows that the outcome of \( d_{a,b} \) can only depend on the relative rankings of \( a \) and \( b \), so that \( d_{a,b} \) has the form

\[
d_{a,b} = \begin{cases} 
(p_3, 1 - p_3, 0), & \text{if } a >_i b \text{ for all agents } i, \\
(p_2, 1 - p_2, 0), & \text{if } a >_i b \text{ for two agents } i, \\
(p_1, 1 - p_1, 0), & \text{if } a >_i b \text{ for one agent } i, \\
(p_0, 1 - p_0, 0), & \text{if } a >_i b \text{ for one agent } i,
\end{cases}
\]

(119)

where the vector \((p_1, 1 - p_1, 0) = (f_a, f_b, f_c)\) denotes the outcome and \( p_3 \geq p_2 \geq \frac{1}{2} \) and \( p_0 \leq p_1 \leq \frac{1}{2} \). Again by symmetry of the symmetric decomposition, it must also contain the anonymous duple \( d_{a,b}^\varphi \) for any permutation of the alternatives \( \varphi : M \rightarrow M \). Consider the type profile

\[
t_1 : \quad a > b > c, \\
t_2 : \quad a > c > b, \\
t_3 : \quad b > c > a.
\]

(120) (121) (122)

The following table shows what outcomes the different duples will select.

| Duple  | \( a \) | \( b \) | \( c \) |
|--------|--------|--------|--------|
| \( d_{a,b} \) | \( p_2 \) | \( 1 - p_2 \) | 0      |
| \( d_{a,c} \) | \( p_2 \) | 0      | \( 1 - p_2 \) |
| \( d_{b,c} \) | 0      | \( p_2 \) | \( 1 - p_2 \) |
| \( d_{b,a} \) | \( 1 - p_1 \) | \( p_1 \) | 0      |
| \( d_{c,a} \) | \( 1 - p_1 \) | 0      | \( p_1 \) |
| \( d_{c,b} \) | 0      | \( 1 - p_1 \) | \( p_1 \) |

A uniform lottery over these duples assigns probability \( 2 \cdot \frac{1}{6} = \frac{1}{3} \) to alternative \( b \), and consequently, this rule has score deficit \( \frac{4}{9} - \frac{2}{3} = \frac{2}{9} \) at this particular type profile. This is the same score deficit the \( \text{RandDict} \) has at this profile, which means that including any strategyproof
duplicates in the symmetric decomposition will not improve the deficit of \( f \). Suppose now that the symmetric decomposition of \( f \) contains a strategyproof unilateral \( u_i \). By the symmetric decomposition, we can assume that \( u_i \) is neutral. By the characterization of strategyproofness via swap monotonicity, upper invariance, and lower invariance, it follows that \( u_i \) must pick an outcome \( (p_1, p_2, 1 - p_1 - p_2) \) where \( p_1 \geq p_2 \geq 1 - p_1 - p_2 \). Again by symmetry of the symmetric decomposition, it must also contain the neutral unilateral \( u_i^\pi \) for any permutation of the agents \( \pi : N \to N \). Consider again the type profile

\[
\begin{align*}
t_1 : & \quad a > b > c, \\
t_2 : & \quad a > c > b, \\
t_3 : & \quad b > c > a.
\end{align*}
\]

The following table shows what outcomes the different unilaterals will select.

| Unilateral | a   | b   | c       |
|------------|-----|-----|---------|
| \( u_1 \)  | \( p_1 \) | \( p_2 \) | \( 1 - p_1 - p_2 \) |
| \( u_2 \)  | \( p_1 \) | \( 1 - p_1 - p_2 \) | \( p_2 \) |
| \( u_3 \)  | \( 1 - p_1 - p_2 \) | \( p_1 \) | \( p_2 \) |

A uniform lottery over these unilaterals assigns probability \( 1 \cdot \frac{1}{3} = \frac{1}{3} \) to alternative \( b \), and consequently, this rule has score deficit \( \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \) at this particular type profile. This is the same score deficit the \( \text{RandDict} \) has at this profile, which means that including any strategyproof unilaterals in the symmetric decomposition will not improve the deficit of \( f \). This means that there exists no strategyproof, anonymous, neutral rule that has a lower score deficit at \( (t_1, t_2, t_3) \) than \( \frac{2}{9} \), and therefore, \( \text{RandDict} \) has minimal deficit, i.e., \( \text{RandDict} \in \text{Opt}(0) \).

**C.4.4. Proof of Proposition 10**

For \( n = m = 3, \) \( v = (1, 0, 0) \), and strict preferences, all hybrids of \( \text{RandDict} \) and \( \text{UnifPlurality} \) are on the efficient frontier.

**Proof.** We have already seen that \( \text{RandDict} \in \text{Opt}(0) \) with \( \varepsilon(\text{RandDict}) = 0 \) and \( \sigma(\text{RandDict}) = \frac{2}{9} \). Furthermore, \( \text{UnifPlurality} \in \text{Opt}(1/3) \) and no other positional scoring rule has lower manipulability. We will show that for \( \varepsilon = \frac{1}{6} \), all optimal rules have deficit \( \frac{1}{5} \). By applying Theorem 3 we find that the only hinges of the path \( P_{\text{EF}} \) are 0, \( \frac{1}{3} \), and 1. Consequently, all hybrid rules of optimal rules are also optimal, and in particular any hybrid of \( \text{RandDict} \) and \( \text{UnifPlurality} \). Suppose that \( f \) is \( \frac{1}{5} \)-approximately strategyproof, anonymous, and neutral. It follows from anonymity and neutrality that at the type profile

\[
\begin{align*}
t_1 : & \quad a > b > c, \\
t_2 : & \quad b > c > a, \\
t_3 : & \quad c > a > b.
\end{align*}
\]
the outcome must be $\left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$ for $a$, $b$, $c$, respectively. If agent 1 changes its report to

$$t'_1 : \quad b > a > c,$$

(129)

the outcome becomes $\left( \frac{1}{3} - \delta_a, \frac{1}{3} + \delta_a + \delta_c, \frac{1}{3} - \delta_c \right)$ and the score deficit at $(t'_1, t_2, t_3)$ is

$$\frac{2}{3} \left( 1 - \frac{1}{3} - \delta_a - \delta_c \right) \geq \frac{2}{3} \left( 1 - \delta_c \right) = \frac{2}{9} - \frac{2}{3} \delta_c.$$

(130)

Suppose not that agent 1 had a utility close to the binary utility $(1,1,0)$. The gain from misreporting for agent 1 would then be (arbitrarily close to)

$$\left( \frac{1}{3} - \delta_a \right) u(a) + \left( \frac{1}{3} + \delta_a + \delta_c \right) u(b) - \frac{1}{3} (u(a) + u(b)) = \delta_c u(b) - \delta_a (u(a) - u(b)) \approx \delta_c,$$

(131)

(132)

Thus, if $f$ is $\frac{1}{6}$-approximately strategyproof, then $\delta_c \leq \frac{1}{6}$, and consequently, the score deficit at $(t'_1, t_2, t_3)$ is at least

$$\frac{2}{9} - \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}.$$

(133)

Therefore, the deficit of $f$ cannot be smaller than $\frac{1}{9}$, which concludes the proof.

C.4.5. Proof of Proposition 11

For $n = m = 3$, $v = (1,1,0)$, and strict preferences, UnifVeto is $\frac{1}{2}$-approximately strategyproof, and any positional scoring rule (including UnifVeto) violates $\varepsilon$-approximate strategyproofness for any $\varepsilon < \frac{1}{2}$.

Proof. First, we show that UnifVeto is $\frac{1}{2}$-approximately strategyproof. Consider the type profile

$$t_1 : \quad a > b > c,$$

(134)

$$t_2, t_3 : \quad \ldots > c.$$

(135)

UnifVeto selects $a$ and $b$ with probability $\frac{1}{2}$ each. Agent 1 can rank $b$ last and obtain $a$ with certainty. Its gain from this manipulation would be

$$u_1(a) - \frac{1}{2} (u_1(a) + u_1(b)) = \frac{1}{2} (u_1(a) - u_1(b)),$$

(136)

which is at most $\frac{1}{2}$ for $u_1(b)$ close to 0. Suppose now that all agents have different last choices. In this case, UnifVeto selects any of the alternatives with probability $\frac{1}{3}$. By ranking another alternative last, an agent could ensure the implementation of its third choice with certainty, which is not a beneficial manipulation. Finally, suppose that two agents have the same last choice, while a third agent has a different last choice. We have the following cases:
• Case I:

\[ t_1 : \quad a > b > c, \]  
\[ t_2, t_3 : \quad \ldots > a. \]

In this case UnifVeto implements \( b \) with certainty. Agent 1 can only enforce \( c \) by ranking \( b \) last, or rank \( a \) last and obtain \( b \) and \( c \) with probability \( \frac{1}{2} \). Neither of these moves will make agent 1 better off.

• Case II:

\[ t_1 : \quad a > b > c, \]
\[ t_2, t_3 : \quad \ldots > b. \]

In this case UnifVeto implements \( a \) with certainty, which is already agent 1’s first choice.

• Case III:

\[ t_1 : \quad a > b > c, \]
\[ t_2 : \quad \ldots > b, \]
\[ t_3 : \quad \ldots > c. \]

In this case UnifVeto implements \( a \) with certainty, which is already agent 1’s first choice.

• Case IV:

\[ t_1 : \quad a > b > c, \]
\[ t_2 : \quad \ldots > a, \]
\[ t_3 : \quad \ldots > c. \]

In this case UnifVeto implements \( b \) with certainty. By \( b \) in last place, agent 1 could obtain a probability of \( \frac{1}{3} \) for each alternative instead. Its gain from this manipulation is

\[
\frac{1}{3} (u_1(a) + u_1(b) + u_1(c)) - u_1(b) \leq \frac{1}{3} - \frac{2}{3} u_1(b) \leq \frac{1}{3}.
\]

This shows that UnifVeto is \( \frac{1}{2} \)-approximately strategyproof. To see that any anonymous, neutral positional scoring rule \( f \) has \( \varepsilon(f) \geq \frac{1}{2} \), we consider the type profile

\[ t_1 : \quad a > b > c, \]
\[ t_2 : \quad a > b > c, \]
\[ t_3 : \quad b > a > c. \]
At this profile, \( f \) has to select \( a \) with some probability \( p_a \) and \( b \) with probability \( 1 - p_a \). If \( p_a \geq \frac{1}{2} \), agent 3 can rank \( a \) last and enforce \( b \). If \( u_3(b) = 1, u_3(c) = 0, \) and \( u_3(a) \) is close to 0, its gain will be
\[
1 - (1 - p_a) = p_a \geq \frac{1}{2}. \tag{151}
\]
If \( p_a < \frac{1}{2} \), agent 1 can enforce \( a \) by ranking \( b \) last and obtain a gain of
\[
1 - p_a \geq \frac{1}{2} \tag{152}
\]
with a similar utility function.

**C.4.6. Proof of Proposition 12**

For \( n = m = 3, v = (1, 1, 0) \), and strict preferences, \( \text{RANDDUPLE} \in \text{OPT}(0) \), i.e., it has the lowest deficit among all strategyproof rules.

*Proof.* \( \text{RANDDUPLE} \) is a lottery over strategyproof duples and therefore obviously strategyproof. At any type profile where all agents agree on the last choice, \( \text{RANDDUPLE} \) selects one of the other alternatives, each of which gives maximal score. At any type profile where all agents have different last choices, any outcome has zero score deficit. Finally, consider a type profile with
\[
t_1 : \ldots > c, \tag{153}
\]
\[
t_2 : \ldots > c, \tag{154}
\]
\[
t_3 : \ldots > b. \tag{155}
\]
The score of \( a \) is 1 and the score of \( b \) is \( \frac{2}{3} \), so that the maximum score is 1. The worst case for \( \text{RANDDUPLE} \) is that agents 1 and 2 rank \( b \) first, in which case \( a \) will be selected with probability \( \frac{2}{3} \) and \( b \) with probability \( \frac{1}{3} \). Thus, the deficit of \( \text{RANDDUPLE} \) is
\[
1 - \frac{2}{3} \cdot \frac{2}{3} - \frac{1}{3} \cdot 1 = \frac{2}{9}. \tag{156}
\]
It remains to be prove whether any strategyproof, anonymous, neutral rule \( f \) can achieve a lower deficit. Consider the profile
\[
t_1 : \ a > b > c, \tag{157}
\]
\[
t_2 : \ b > a > c, \tag{158}
\]
\[
t_3 : \ c > a > b, \tag{159}
\]
and let \( u_i \) be a strategyproof, neutral, and unilateral component in the symmetric decomposition of \( f \). \( u_i \) must pick an allocation \((p_1, p_2, 1 - p_1 - p_2)\) with \( p_1 \geq p_2 \geq 1 - p_1 - p_2 \), where \( p_k \) denotes the probability of agent \( i \)’s \( k \)th choice. The symmetric decomposition implies that \( u_1, u_2, \) and \( u_3 \) are equally likely to be chosen. Analogous to the proof of Proposition 9, we get
Thus, \( b \) is selected with probably \( \frac{1}{3} \), which means that the score deficit of \( f \) at \( (t_1, t_2, t_3) \) is not improved by including any unilaterals in the symmetric decomposition. Similarly, if \( d_{a,b} \) is a strategyproof, anonymous duple in the symmetric decomposition of \( f \), it has the form

\[
d_{a,b} = \begin{cases} 
(p_3, 1 - p_3, 0), & \text{if } a >_i b \text{ for all agents } i, \\
(p_2, 1 - p_2, 0), & \text{if } a >_i b \text{ for two agents } i, \\
(p_1, 1 - p_1, 0), & \text{if } a >_i b \text{ for one agent } i, \\
(p_0, 1 - p_0, 0), & \text{if } a >_i b \text{ for one agent } i,
\end{cases}
\]  

(160)

where \( p_3 \geq p_2 \geq \frac{1}{2} \) and \( p_0 \leq p_1 \leq \frac{1}{2} \). Again by symmetry of the symmetric decomposition, it must also contain the anonymous duple \( d_{a,b}^\varphi \), for any permutation of the alternatives \( \varphi : M \to M \). The following table shows what outcomes the different duples will select.

| Duple | \( a \) | \( b \) | \( c \) |
|-------|-------|-------|-------|
| \( d_{a,b} \) | \( p_2 \) | \( 1 - p_2 \) | \( 0 \) |
| \( d_{a,c} \) | \( p_2 \) | \( 0 \) | \( 1 - p_2 \) |
| \( d_{b,c} \) | \( 0 \) | \( p_2 \) | \( 1 - p_2 \) |
| \( d_{b,a} \) | \( 1 - p_1 \) | \( p_1 \) | \( 0 \) |
| \( d_{c,a} \) | \( 1 - p_1 \) | \( 0 \) | \( p_1 \) |
| \( d_{c,b} \) | \( 0 \) | \( 1 - p_1 \) | \( p_1 \) |

Thus, \( b \) is selected with probability \( \frac{1}{3} \), and therefore, including any other duples in the symmetric decomposition of \( f \) will not improve the score deficit at \( (t_1, t_2, t_3) \). Thus, RANDDUPLE is an optimal strategyproof rule.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Duple} & \text{} & \text{\( a \)} & \text{\( b \)} & \text{\( c \)} \\
\hline
\quad & \quad & \quad & \quad & \quad \\
\end{array}
\]

C.4.7. Proof of Proposition 13

For \( n = m = 3, v = (1, 1, 0) \), and strict preferences, the hybrids of RANDDUPLE and UNIFVETO are not on the efficient frontier for \( \beta \neq 0, 1 \).

Proof. Observe that the worst-case score deficit of RANDDUPLE is attained at the type profile \( (t_1, t_2, t_3) \) with

\[
t_1 : \ a > b > c, \\
t_2 : \ a > b > c, \\
t_3 : \ a > b > c.
\]

Since UNIFVETO is a positional scoring rule, its score deficit is zero at all type profiles. Consequently, the worst-case score deficit of any hybrid \( h_\beta \) of RANDDUPLE and UNIFVETO is
Table 1: Executions of the linear program FindOpt when using FindDelta and FindHinges to determine the efficient frontier for \( v = (1, 1, 0) \).

| Algorithm   | \( \varepsilon \) | \( \sigma_\varepsilon \) | Comment                                    |
|-------------|-------------------|----------------------------|--------------------------------------------|
| FindDelta   | 0                 | 2/9                        | found hinges at \( \varepsilon = 0 \) and \( \varepsilon = 1 \) |
| FindDelta   | 1/2               | 0                          |                                            |
| FindDelta   | 1/4               | 1/12                       |                                            |
| FindDelta   | 1/8               | 1/8                        |                                            |
| FindDelta   | 1/16              | 65/432                     |                                            |
| FindDelta   | 1/32              | 13/72                      |                                            |
| FindDelta   | 1/64              | 29/144                     | found \( \delta = 1/64 < \varepsilon \) min |
| FindHinges  | 1/6               | 1/9                        |                                            |
| FindHinges  | 1/18              | 25/162                     |                                            |
| FindHinges  | 3/47              | 28/187                     |                                            |
| FindHinges  | 1/12              | 5/36                       | found hinges at \( \varepsilon = 1/12 \) and \( \varepsilon = 1/2 \) |
| FindHinges  | 1/21              | 10/63                      | found hinge at \( \varepsilon = 1/21 \)    |

determined by the score-deficit at \((t_1, t_2, t_3)\). Furthermore, the manipulability of UnifVeto is highest at the same type profile if agent 1 swaps \( b \) and \( c \) to enforce \( a \) and has a utility close to \((1, 0, 0)\). This misreport leaves the outcome of RandDuple unchanged. Therefore, the manipulability of any hybrid will also be determined by this type and misreport. By Theorem 2, the path \( P^{Hyb} \) must be linear in this case. Thus, if \( P^{EF} \) is not linear, the hybrids will not be on the efficient frontier for \( \beta \neq 0, 1 \). Non-linearity follows from the second statement in the Proposition. To find the hinges we used the algorithm FindDelta to determine a lower bound for the smallest non-zero hinge and then applied FindHinges with this value of \( \delta \). Table 1 gives the points on \( P^{EF} \) in the order in which they were computed.

**D. Agent Symmetric and Alternative Symmetric Domains**

Our result about the non-restrictive nature of anonymity and neutrality in Section 7.1 remains true for certain domain restrictions. Let \( T \) denote the set of admissible type profiles in some restricted domain. To obtain Proposition 6, \( T \) had to be rich enough to actually reflect neutrality and anonymity.

1. \( T \) is *agent symmetric* if for all permutations \( \pi \in \Pi \) of the agents and admissible type profiles \((t_1, \ldots, t_n) \in T\) the type profile \((t_{\pi(1)}, \ldots, t_{\pi(n)})\) is also in \( T \).

2. \( T \) is *alternative symmetric* if for all permutations \( \varphi \in \Phi \) of the alternatives and admissible type profiles \((t_1, \ldots, t_n) \in T\) the type profile \((\varphi(t_1), \ldots, \varphi(t_n))\) is also in \( T \).

Proposition 6 remains true in any domain restriction where the set of admissible type profiles \( T \) is agent and alternative symmetric.
E. Insights about Condorcet Consistency, Pareto Optimality, and Unanimity

Define the following:

1. At a type profile $t \in T^n$, an alternative $j \in M$ is a **unanimity winner** if for all agents $i \in N$ and all other alternatives $j' \in M$ we have $j \succeq_i j'$. Let $M_{\text{Unan}}(t)$ be the set of unanimity winners at $t$ and $M_{\text{Unan}}^{\text{loose}}(t) = M \setminus M_{\text{Unan}}(t)$ the set of non-winners.

2. At a type profile $t \in T^n$, an alternative $j \in M$ *Pareto dominates* another alternative $j' \in M$ if for all agents $i \in N$ we have $j \succeq_i j'$ and for some agent $i'$ we have $j >_{i'} j'$, and $j$ is Pareto optimal if there exists no other alternative $j'$ that Pareto dominates $j$. Let $M_{\text{Pareto}}^{\text{win}}(t)$ denote the set of Pareto optimal alternatives, and let $M_{\text{Pareto}}^{\text{loose}}(t) = M \setminus M_{\text{Pareto}}^{\text{win}}(t)$ be the set of Pareto dominated alternatives.

3. For a type profile $t \in T^n$ and any two alternatives $a, b \in M$ let $n_{a \succ b}(t) = \# \{i \in N \mid a \succ_i b\}$. An alternative $j \in M$ is a **Condorcet winner** at $t$ if for all $j' \in M$ we have $n_{a \succ j}(t) \geq n_{b \succ a}(t)$, $M_{\text{Cond}}^{\text{win}}(t)$ denotes the set of Condorcet winners, and $M_{\text{Cond}}^{\text{loose}}(t) = M \setminus M_{\text{Cond}}^{\text{win}}(t)$ denotes the set of Condorcet losers.

The following linear program shows how Condorcet consistency, Pareto optimality, and unanimity can be incorporated as linear constraints.

**Linear Program 2** (FINDOPT extended).

\[
\begin{align*}
\text{minimize} & \quad s^{\text{max}} \\
\text{subject to} & \quad \ldots \\
& \quad f_j(t) = 0 \quad \forall t \in T^n \text{ such that } M_{\text{Unan}}^{\text{win}}(t) \neq \emptyset \text{ and } j \in M_{\text{Unan}}^{\text{loose}}(t) \quad \text{(Unanimity)} \\
& \quad f_j(t) = 0 \quad \forall t \in T^n, j \in M_{\text{Pareto}}^{\text{win}}(t) \quad \text{(Pareto)} \\
& \quad f_j(t) = 0 \quad \forall t \in T^n \text{ such that } M_{\text{Cond}}^{\text{win}}(t) \neq \emptyset \text{ and } j \in M_{\text{Cond}}^{\text{loose}}(t) \quad \text{(Condorcet)}
\end{align*}
\]

**Example 1.** **Unanimity** requires that if all agents agree on a first choice, then this alternative should be implemented. Suppose that $f$ is strategyproof and unanimous. No duple is unanimous for $m \geq 3$, and unilaterals are unanimous only if they always select the first choice of the agent who’s report is relevant. Thus, $f$ must be a hybrid of dictatorships, and Proposition 6 tells us that $f$ will be no better than RAND_DICT. However, with $v = (1, 1, 0)$, RAND_DICT has a deficit $\sigma(\text{RAND_DICT}) = \frac{4}{3} > \sigma(\text{RAND_DUPLE})$ at type profile

\[
\begin{align*}
t_1 & : \quad a \succ b \succ c, \quad (164) \\
t_2 & : \quad a \succ b \succ c, \quad (165) \\
t_3 & : \quad c \succ b \succ a. \quad (166)
\end{align*}
\]

This means that requiring unanimity strictly decreases the efficiency of the rule. Since Condorcet consistency and Pareto optimality both imply, unanimity, this same problem also occurs for these two constraints.
F. Symmetric Decomposition of Strategyproof, Anonymous, Neutral Rules

In this Section, we present a refinement of Gibbard’s strong characterization of strategyproof rules (Gibbard, 1977). This symmetric decomposition characterizes rules that are strategyproof, anonymous, and neutral. We use this result to establish the efficient frontiers in Sections 7.2 and 7.3, but the symmetric decomposition and may be of independent interest. Consider the full domain of strict preferences. Gibbard (1977) showed that any strategyproof rule is a hybrid of multiple strategyproof simple rules, namely unilaterals and duples.

Definition 23 (Adapted from (Gibbard, 1977)). A rule $f$ is unilateral if the outcome only depends on the report of a single agent, i.e., there exists $i \in N$ such that for all type profiles $t, t' \in T^n$ we have that $t_i = t'_i$ implies $f(t) = f(t')$.

Definition 24 (Adapted from (Gibbard, 1977)). A rule $f$ is duple if only two alternatives are possible, i.e., there exist $a, b \in M$ such that for all type profiles $t \in T^n$ we have $f_j(t) = 0$ for all $j \neq a, b$.

The strong characterization is the following.

Theorem 4 (Adapted from (Gibbard, 1977)). A rule $f$ is strategyproof if and only if it can be written as a hybrid of rules $f^1, \ldots, f^K$, and each component $f^k$ is strategyproof and either unilateral or duple.

Obviously, duples cannot satisfy neutrality (unless $m = 2$) and unilateral rules cannot satisfy anonymity (unless the rule is constant or $n = 1$). Nonetheless, if $f$ is anonymous and neutral, we can provide a more refined understanding of the structure of the lottery.

Theorem 5 (Symmetric Decomposition). For any strategyproof, anonymous, neutral rules $f$ there exist

1. strategyproof, neutral, unilateral rules $u^k, k \in \{1, \ldots, K^u\}$,

2. strategyproof, anonymous, duple rules $d^k, k \in \{1, \ldots, K^d\}$,

3. $\beta^u_k \geq 0, k \in \{1, \ldots, K^u\}$ with $\beta^d_k \geq 0, k \in \{1, \ldots, K^d\}$ and $\sum_{k=1}^{K^u} \beta^u_k + \sum_{k=1}^{K^d} \beta^d_k = 1$,

such that

$$f = \sum_{\pi \in \Pi} \sum_{k=1}^{K^u} \left(\frac{\beta^u_k}{n!}\right) u^{k, \pi} + \sum_{\varphi \in \Phi} \sum_{k=1}^{K^d} \left(\frac{\beta^d_k}{m!}\right) d^{k, \varphi}.$$  \hspace{1cm} (167)

The symmetric decomposition (167) is a consequence of Gibbard’s strong characterization and the fact that for any anonymous, neutral rule we have $f = f^{\pi, \varphi}$.

\footnote{Gibbard further refined his result by using the properties localized and non-pervasive instead of strategyproofness. This characterization is similar in spirit to Theorem 1 from (Mennle and Seuken, 2014). However, our conditions (swap monotonicity, upper invariance, lower invariance) are easier to formalize and verify. We will therefore use our characterization in the proofs in Sections 7.2 and 7.3.}
Proof. By anonymity and neutrality of \( f \) we get that
\[
f = f^\pi,\varphi
\]
for all \( \pi \in \Pi \) and \( \varphi \in \Phi \), which implies
\[
f = \sum_{\pi \in \Pi} \sum_{\varphi \in \Phi} \frac{1}{n!m!} f^\pi,\varphi.
\]
Furthermore, for fixed \( \pi \) and \( \varphi \) we have that
\[
f_{\pi,\varphi} = \sum_{k=1}^{K} \beta_k f_{k,\pi,\varphi}.
\]
Combining these insights yields
\[
f = \sum_{k=1}^{K} \sum_{\pi \in \Pi} \sum_{\varphi \in \Phi} \left( \frac{\beta_k}{n!m!} \right) f_{k,\pi,\varphi}.
\]
If \( f^k \) is a strategyproof duple, then the hybrid
\[
d^k = \sum_{\pi \in \Pi} \left( \frac{\beta_k}{n!} \right) f_{k,\pi}
\]
is strategyproof, anonymous, and a duple. Similarly, if \( f^k \) is strategyproof and unilateral, then the hybrid
\[
u^k = \sum_{\varphi \in \Phi} \left( \frac{\beta_k}{m!} \right) f_{k,\varphi}
\]
is strategyproof, neutral, and unilateral.

The decomposition is symmetric in the sense that for any component \( u^k \) (or \( d^k \)) that occurs with coefficient \( \beta_k^u \) (or \( \beta_k^d \)), the corresponding component \( u^k,\pi \) (or \( d^k,\varphi \)) occurs with the same coefficient. (167) decomposes \( f \) into two parts: a neutral part on the left, that gets “anonymized” by randomization, and an anonymous part on the right that gets “neutralized.”