I. A STUDY OF THE $E(J_{\text{max}})$ INTERACTION

This is an extension of work done with Apolodor Raduta and Elvira Moya de Guerra when they visited Rutgers on a NATO grant in 2001 [1]. In contrast to work on $J = 0$ pairing, we here study, in a single $j$ shell, an interaction $E(J_{\text{max}})$ for which all two-body matrix elements vanish except for $J = J_{\text{max}} = 2j$. In this work we will focus on the model itself. Comparisons with realistic interactions are given in Ref. [2].

For a two-particle (or two-hole) system in a single $j$ shell, the even-$J$ states have isospin $T = 1$ (triplets) and the odd-$J$ states have isospin $T = 0$ (singlets). Thus, an interaction acting in only $J_{\text{odd}}$ states, including $J = J_{\text{max}}$, cannot occur for two neutrons or two protons in a single $j$ shell—only in the neutron–proton system. This leads to some simplifications.

We will consider a system of two protons and two neutrons (or holes), e.g. $^{96}$Cd in the $g_{9/2}$ shell. The basis states are $|pp(J_p)nn(J_n)|^J_J$, where $I$ is the total angular momentum. To satisfy the Pauli principle, $J_p$ and $J_n$ must be even.

For $I = 0$, the secular Hamiltonian is separable and $J_n = J_p$:

$$H_{J_pJ_n} = V(J_{\text{odd}})f(J_p)f(J_p'),$$

where $f(J_p)$ is twice the unitary 9-$j$ symbol (U9-$j$):

$$f(J_p) = 2((jj)^{J_p}(jj)^{J_p'})|_{J_{\text{odd}}}(jj)^{J_{\text{odd}}}|_{J_{\text{odd}}} = 0 = 2(2J_p + 1)(2J_{\text{odd}} + 1) \begin{array}{ccc} j & j & J_p \\ j & j & J_p \\ J_{\text{odd}} & J_{\text{odd}} & 0 \end{array}$$

If we write the wave function as $\sum X_{J_pJ_n} |pp(J_p)nn(J_p)|^J_J$, then it was shown in Ref. [1] that $X_{J_pJ_n}$ is proportional to $f(J_p)$. The other $I = 0^+$ eigenstates are degenerate and, if $V(J_{\text{odd}})$ is negative, they are at higher energies. In other words, what we have shown in Ref. [1] is that the wave-function components $X_{J_pJ_n}$ of the lowest $I = 0^+$ state are equal within a normalization to the U9-$j$ coefficients:

$$\sqrt{2}((jj)^{J_p}(jj)^{J_p'})|_{J_{\text{odd}}}(jj)^{J_{\text{odd}}}|_{J_{\text{odd}}} = 0.$$

The eigenvalue is given by

$$E(I = 0^+) = V(J_{\text{odd}})\sum f(J_p)X_{J_pJ_n}^2$$

Note that our very simple interactions are charge independent. This means that the lowest (non-degenerate) $I = 0^+$ state has good isospin, presumably $T = 0$. It is amusing that we can assign the isospin quantum number to a wave function with U9-$j$ coefficients.

Similarly, we can show that there is also a very simple expression for the $I = 1^+$ lowest eigenfunction:

$$2((jj)^{J_p}(jj)^{J_n} |jj)^{J_{\text{max}}}(jj)^{J_{\text{max}}-1}|)^J = 1.$$

However, for states of angular momentum 2 or higher, the secular Hamiltonian is no longer separable. The eigenvalue equation is

$$4 \sum_{J_x} ((jj)^{J_p}(jj)^{J_n} |jj)^{J_x}(jj)^{J_x})^J \times \sum_{J_p'J_n'} ((jj)^{J_p'}(jj)^{J_n'} |jj)^{J_x}(jj)^{J_x})^J X_{J_p'J_n'} = \lambda X_{J_pJ_n},$$

where $\lambda$ is the eigenvalue and $X_{J_pJ_n}$ stands for the eigenfunction components. For $I = 2^+$ there are two terms corresponding to $J_x = 7$ and 9; for $I = 3^+$ the values are $J_x = 6$ and 8, etc.

Despite the complexity of the above equation, there are some surprising results. The eigenfunction components of the lowest $2^+$ state are numerically extraordinarily close to the single U9-$j$ symbols $\sqrt{2}((jj)^{J_p}(jj)^{J_n} |jj)^{J_x}(jj)^{J_x})^J = 2$. Furthermore, the next $2^+$ state has also components exceedingly close to $2((jj)^{J_p}(jj)^{J_n} |jj)^{J_x}(jj)^{J_x})^J = 2$. This is by no means obvious because, as mentioned above, the interaction involves a sum of two separable terms corresponding to $J_x = 7$ and 9.

We can explain this result by performing a calculation of the overlap of the two U9-$j$‘s of the last paragraph. We restrict the sum to even $J_p$ and even $J_n$. We first note schematically

$$\sum_{\text{even } J_pJ_n} (1 + (-1)^{J_n})(1 + (-1)^{J_n}) = \sum_{\text{even } J_pJ_n} (-1)^{J_p} = \sum_{\text{even } J_pJ_n} (-1)^{J_p + J_n}$$

Using orthogonality relations for 9-$j$ symbols, we can see that the first term vanishes. In the last term, one of the U9-$j$‘s has two rows that are the same, which means that the only non-vanishing terms in the sum have $(J_p + J_n)$ even. Thus, the last term is the same as the first—zero. The two middle terms are the same, so we get the
overlap of the two $U9$-$j$’s to be

$$\sum_{even_{J_p,J_n}} \frac{1}{2} \sum_{even_{J_p,J_n}} (-1)^{J_p} \langle (jj)^9| (jj)^9 \rangle I^2 = \sum_{even_{J_p,J_n}} (-1)^{J_p} \langle (jj)^9| (jj)^9 \rangle I^2 =$$

$$= \frac{1}{2} \langle (jj)^9| (jj)^9 \rangle I^2.$$  \hfill (6a)

$$\times \langle (jj)^9| (jj)^9 \rangle I^2 = \frac{1}{2} \langle (jj)^9| (jj)^9 \rangle I^2.$$  \hfill (6b)

$$= - \frac{1}{2} \langle (jj)^9| (jj)^9 \rangle I^2.$$  \hfill (6c)

We obtain the above by using again orthogonality relations for $9j$-symbols.

Using similar arguments, one can show that the normalization for the $||pp(9)n(9)||=2$ state is such that its normalization factor is

$$N(9)^{-2} = \frac{1}{2} - \frac{1}{2} \langle (jj)^9| (jj)^9 \rangle I^2 =$$

$$= \frac{1}{2} - \frac{1}{2} \times 0.00001209813 = 0.499993950935.$$  \hfill (7)

For the $||pp(9)n(9)||=2$ state, we obtain

$$N(7)^{-2} = \frac{1}{4} - \frac{1}{4} \langle (jj)^7| (jj)^7 \rangle I^2 =$$

$$= \frac{1}{4} + \frac{1}{4} \times 0.000075253477 = 0.250376267385.$$  \hfill (8)

To get this latter result, we use the following relationship

$$\sum_{J_p,J_n} (-1)^{J_p+J_n} | \langle (jj)^9| (jj)^9 \rangle I^2 |^2 = 0$$

From Eqs. (7) and (8), we find that the normalizations are 1.414222 and 1.998497, only slightly different from $\sqrt{2}$ and 2 respectively. Therefore, we obtain that the term in Eq. (6c) is exceedingly small for the $9g_2/2$ shell, namely 0.00009113 and, if we include the exact normalization factors, we get 0.00025756.

We can see in Table I that the results for matrix diagonalization for both $I=2^+$ states yield wave function components which are very close to the normalized $U9$-$j$ coefficients. In fact, they are so close that one could wonder if they are exactly the same. But they are not. As seen in Eq. (6c), the two $U9$-$j$ sets corresponding to $J_e = 9$ and $J_e = 7$ are very nearly orthogonal, but not quite.

It turns out that all the other lowest even-$I$ states have eigenfunctions close although not exactly equal to $\sqrt{2}^2 \langle (jj)^9| (jj)^9 \rangle I^2$. In Table II we compare, as an example, the wave function of the $I=8^+$ state. In the second column, we give the single $U9$-$j$ symbols (normalized) and in the third column we give results of diagonalizing the E(9) interaction. Since the coefficient $[J_p, J_n]$ is the same as $[J_n, J_p]$, we list only one of them. The overlap of the two wave functions is 0.9998.

We can plot the coupling of Eq. (6c) for various shells, as we can see in Table III. We see that the coupling $U9$-$j$ decreases at least exponentially as we go up in shell.

| $[J_p, J_n]$ | $U9$-$j$ | E(9) |
|-------------|---------|-----|
| [0, 2]      | 0.5534  | 0.1349 |
| [2, 2]      | -0.4707 | -0.5569 |
| [2, 4]      | 0.3035  | 0.538 |
| [4, 4]      | -0.1388 | 0.6300 |
| [6, 4]      | 0.0531  | 0.1320 |
| [6, 6]      | -0.0137 | 0.1350 |
| [8, 8]      | -0.0003 | 0.0011 |
| [0, 8]      | 0.0630  | 0.0644 |
| [2, 6]      | 0.4299  | 0.4271 |
| [2, 8]      | -0.0522 | -0.0513 |
| [4, 4]      | 0.7444  | 0.7456 |
| [4, 6]      | -0.1803 | -0.1729 |
| [4, 8]      | 0.0256  | 0.0280 |
| [6, 6]      | 0.0521  | 0.0657 |
| [6, 8]      | -0.0076 | -0.0012 |
| [8, 8]      | 0.0011  | 0.0047 |

Experts say that this type of 9-$j$ lies in the non-classical region.

### A. Degeneracies

With the $E(J_{\text{max}})$ interaction for $9g_2/2$ [that is, E(9)], we get several degenerate states with an absolute energy zero. In some detail, for $I=0^+$ there are five states, three with isospin $T = 0$ and two with $T = 2$. There is one non-degenerate state at an energy $2V(9) (V(9)$ is negative). The other four $I=0^+$ states have zero energy. For $I=1^+$ all states have isospin $T = 1$. There is a single
non-degenerate state at \( V(9) \), the other three have zero energy. For \( I = 2^+ \) there are twelve states—six have \( T = 0 \), four have \( T = 1 \), and two have \( T = 2 \). There are two non-degenerate \( T = 0 \) states with approximate energies \( 2V(9) \) and \( V(9) \) respectively, and one non-degenerate \( T = 1 \) state with energy \( V(9) \). The other nine states have zero energy. To understand this, take a wave function

\[
|\Psi^\alpha\rangle = \sum_{J_pJ_n} C^\alpha(J_p,J_n)[|pp(J_p)nn(J_n)|^I]
\]

and the corresponding energies \( E^\alpha = \langle \Psi^\alpha | H | \Psi^\alpha \rangle \). Consider the sum \( \sum \alpha E^\alpha \). We have

\[
\sum_{J_pJ_n} C^\alpha(J_p,J_n)C^\alpha(J'_p,J'_n) = \delta_{J_p,J'_p} \delta_{J_n,J'_n} \tag{10}
\]

Thus

\[
\sum \alpha E^\alpha = \sum_{J_pJ_n} \langle [pp(J_p)nn(J_n)]^I | H | [pp(J_p)nn(J_n)]^I \rangle = 4V(9) \sum_{J_pJ_n} \sum_{J_e} | \langle (jj)^{J_p} (jj)^{J_n} | (jj)^{J_p} (jj)^{J_n} \rangle |^2 \tag{11}
\]

This expression does not depend on the detailed wave functions. Referring to Eqs. \((7)\) and \((8)\) and neglecting the very small correction terms, we see that \( N^{-2} \) is equal to 1/2 for \( J_x = 9 \) and to 1/4 for all other \( J_x \). Basically then Eq. \((11)\) becomes \( 4V(9) \sum N(J_x)^{-2} \). Hence we obtain \( \sum \alpha E^\alpha = 2V(9) \) for \( I = 0 \), \( V(9) \) for \( I = 1 \), and \( 4V(9) \) for \( I = 2 \). But we can alternately show, using the explicit wave functions, that for \( I = 0 \) the energy of the lowest state is \( 2V(9) \). Hence, all the other states must have zero energy. A similar story for \( I = 1 \). The \( I = 2 \) state is a bit more complicated because of the coupling between two states, however small it is. Still one can work it through and see that the \( 4V(9) \) energy is exhausted by the two \( T = 0 \) and the one \( T = 1 \) non-degenerate states.

For \( I = 0 \) we have two \( T = 0 \) and two \( T = 2 \) states, all degenerate. One can remove the degeneracies of \( T = 0 \) and \( T = 2 \) by adding to the Hamiltonian an interaction \( b \langle i | t(j) \rangle \). This will not affect the wave functions of the non-degenerate states but will shift the \( T = 2 \) states away from the formerly degenerate \( T = 0 \) states.

The number of neutron-proton pairs with even \( J \) for a system of two neutrons and two protons is given by

\[
\sum_{J_n} |D^J(J_n)J_p|^2 (\delta_{T,0} + 4 \delta_{T,2}) \tag{12}
\]

where \( D^J(J_pJ_n) \) is the probability amplitude that in a state of total angular momentum \( I \), the protons couple to \( J_p \) and the neutrons to \( J_n \).

**II. HOW TO HANDLE DEGENERACIES—PAIRING AND \( Q \cdot Q \)**

We show how degeneracies, accidental or otherwise, can obscure some interesting physics. But we further show how one can get around this problem.

In a 2006 publication, Escuderos and Zamick \[4\] found some interesting behaviour in the \( g_{9/2} \) shell. Unlike the lower shells, e.g. \( f_{7/2} \), seniority is not a good quantum number in the \( g_{9/2} \) shell. Despite this, it was found that in a matrix diagonalization with four identical particles in the \( g_{9/2} \) shell with total angular momentum \( I = 4 \) or 6, one unique state emerged no matter what interaction was used. This problem was also addressed by others. Before the mixing, one has two states with seniority \( v = 4 \) and one with \( v = 2 \). The surprise was that, after the diagonalization, one gets a unique state that is always the same independently of the interaction used. This unique state has seniority \( v = 4 \). The components of the wave function are given in the fourth column of Table \[V\] (labelled \( "T = 2, v = 4\) unique\)). The problem to be dealt with was not only why this state did not mix with the \( v = 2 \) state, but also why it does not mix with the other \( v = 4 \) state. But this will not concern us here. Rather we will use this as an example of how degeneracies can obscure interesting physics.

We first consider how the unique \( T = 2 \) wave function looks like for a system of three protons and one neutron. This is shown in Table \[IV\]. No matter what interaction is used, this appears as a unique state.

**Table IV: A unique \( J = 4, v = 4 \) cfp for \( j = 9/2 \).**

| \( J_0 \) | \( (j^3J_0) J = 4, v = 4 \) |
|---|---|
| 3/2 | 0.1222 |
| 5/2 | 0.0548 |
| 7/2 | 0.6170 |
| 9/2 \((v = 1)\) | 0.0000 |
| 9/2 \((v = 3)\) | 0.0000 |
| 11/2 | −0.4043 |
| 13/2 | −0.6148 |
| 15/2 | −0.1597 |
| 17/2 | 0.1853 |

It was already commented on in a later paper by Zamick \[5\] that cfp’s for identical particles are usually calculated using a pairing interaction. With such an interaction, the two \( v = 4 \) states are degenerate, i.e. they have the same energy. This means that any linear combination of the two states can emerge in a matrix diagonalization. Thus, the emergence of a unique state gets completely lost. The problem was also addressed in Refs. \[6\] \[8\].

In this work we consider a less obvious example: a matrix diagonalization of two proton holes and two neutron holes in the \( g_{9/2} \) shell, i.e. we consider \( ^{96}\text{Cd} \) rather than \(^{96}\text{Pd} \), the latter consisting of four proton holes (whether we consider holes or particles does not matter). We use a quadrupole–quadrupole interaction \( Q \cdot Q \) for the matrix diagonalization. The two-body matrix elements in units of MeV from \( J = 0 \) to \( J = 9 \) are: 

\(-1.0000, -0.8788, -0.6516, -0.3465, -0.0152, 0.2879, 0.4849, 0.4849, 0.1818, \) and \(-0.5454\).
We show some relevant results in Table IV. For $I = 4$, we get 14 eigenfunctions, but we list only two of them in the first two columns. The reason we single these out is that they are degenerate—both are at an excitation energy of 3.5284 MeV.

In the third wave function column, we have the unique state, one that emerges, as we said above, with any interaction, however complicated, e.g. CCGI [3]. But now we have to modify the phrase “any interaction”. We do not see this unique state when we use the $Q \cdot Q$ interaction—none of the 14 states looks like the one in column 3. Learning from our experience with the pairing interaction, we suspect that the problem lies with the two degenerate states at 3.5284 MeV. We assumed that the two states were mixtures of one $T = 0$ and one $T = 2$ state.

We can remove the degeneracy without altering the wave functions of the non-degenerate states by adding a $t(1) \cdot t(2)$ interaction to the Hamiltonian. This will shift energies of states of different isospin. What we actually did was equivalent to this. We added $-1.000$ MeV to the two-body $T = 0$ matrix elements. These had odd spin $J = 1, 3, 5, 7, 9$. What emerged is shown in columns 3 and 4. The degeneracy is removed. We have a $T = 2$ state in the third column shifted up by 3 MeV and in the fourth column a $T = 0$ state unshifted. The wave function components are different from what they are in the first two columns. The $T = 2$ state is the unique state we were talking about—one that emerges with any interaction, e.g. CCGI or delta. It is the double analog of a state of four identical proton holes ($^{96}$Pd). The $T = 0$ state in the fourth column has an interesting structure with vanishing components for $|0, 4\rangle$. However, when interactions other than $Q \cdot Q$ are used, it is no longer an eigenfunction.

In the last column, we list the other $T = 2, v = 4$ state. One sees this on the list when one uses a seniority-conserving interaction such as a delta interaction. However, for a general interaction, it does not appear. This is because it gets mixed with the $T = 2, v = 2$ state. Only the state in the third column remains unscathed when we turn on some arbitrary interaction—and only that state does not end up being degenerate with some other state.

There is also a unique $J = 6^+, v = 4$ state. With the pairing interaction, this is degenerate with another $J = 6^+, v = 4$ state and so the uniqueness gets obscured. However, with the $Q \cdot Q$ interaction, unlike the case for $J = 4^+$, the unique $J = 6^+, v = 4$ state is not degenerate with another state. Hence even with $Q \cdot Q$ this unique state appears in the calculation.

There are other examples of confusions. The electric dipole moment of the neutron would vanish if parity conservation holds. But at a more important level, it vanishes if time reversal invariance holds.

III. THE FIRST $J = 1^+, T = 0$ STATES IN A SINGLE-$j$-SHELL CONFIGURATION IN EVEN–EVEN NUCLEI.

The first even–even nucleus for which there are $J = 1^+$, $T = 0$ states in a single-$j$-shell configuration is $^{48}$Cr. If we limit ourselves to single $j$, there are no $M1$ transitions from these states to any $J = 0^+, T = 0$ states.

A. Absence of $J = 1^+, T = 2$ states in $j^4$ configurations

In early shell model calculations by McCullen et al. [10] and Ginocchio and French [11], it was noted that, in the $f_{7/2}$ shell, certain combinations of spin and isospin did not exist. For example, there were no $J = 0^+, T = 1$ states in $^{44}$Ti and no $J = 1^+$ states with $T = T_{\text{min}} + 1$, where $T_{\text{min}} = |N - Z|/2$. There were also no $J = 1^+$ states with $T = T_{\text{max}}$. However, those states are analogous to states of a system of identical particles, i.e. calcium isotopes. Explanations for some of the missing states can be shown by simple techniques as will be discussed later.

In the mid eighties, papers were published which counted the states in a more systematic way. They include the works of I. Talmi on recursion relations for the single $j$ shell. To make things more concrete, consider the first two columns. The unique $J = 0^+, T = 2$ state unshifted. The wave function is $|0, 4\rangle$. However, when interactions other than $Q \cdot Q$ are used, it is no longer an eigenfunction.

In the third column, we list the other $T = 2, v = 4$ state. One sees this on the list when one uses a seniority-conserving interaction such as a delta interaction. However, for a general interaction, it does not appear. This is because it gets mixed with the $T = 2, v = 2$ state. Only the state in the third column remains unscathed when we turn on some arbitrary interaction—and only that state does not end up being degenerate with some other state.

There is also a unique $J = 6^+, v = 4$ state. With the pairing interaction, this is degenerate with another $J = 6^+, v = 4$ state and so the uniqueness gets obscured. However, with the $Q \cdot Q$ interaction, unlike the case for $J = 4^+$, the unique $J = 6^+, v = 4$ state is not degenerate with another state. Hence even with $Q \cdot Q$ this unique state appears in the calculation.

There are other examples of confusions. The electric dipole moment of the neutron would vanish if parity conservation holds. But at a more important level, it vanishes if time reversal invariance holds.

B. The first occurrence of $J = 1^+, T = 0$ states in the single $j$ shell—$^{48}$Cr

There are no $J = 1^+, T = 0$ states for four nucleons in the single $j$ shell. To make things more concrete, consider $^{44}$Ti. The two $f_{7/2}$ protons can have angular momenta 0, 2, 4, and 6, all occurring once; likewise the two neutrons. The $|J_m, J_n\rangle$ configurations that can add up to a total $J = 1$ are $|2, 2\rangle$, $|4, 4\rangle$ and $|6, 6\rangle$. Thus, there are three $J = 1^+$ states. The possible isospins are 0, 1, and 2. Let us next consider $^{48}$Sc. The three neutrons can have angular momenta 3/2, 5/2, 7/2, 9/2, 11/2, and 15/2, all occurring once. The states that add up to one are $|7/2, 5/2\rangle$, $|7/2, 7/2\rangle$, and $|7/2, 9/2\rangle$. Again we have three states. However, since $^{48}$Sc has $|T_{\text{min}}| = 1$, the isospins can only be one or two. Hence, there are no $J = 1^+, T = 0$ states in $^{44}$Ti which are of the $(f_{7/2})^4$ configuration.

We next consider $^{48}$Cr. The possible states of four protons, including seniority labels are: $v = 0, J = 0; v = 2, J = 2, 4, 6; v = 4, J = 2^+, 4^+, 5, 8$. The possible $J = 1^+$ states are $|2, 2\rangle$, $|4, 4\rangle$, $|6, 6\rangle$, $|2^+, 2^+\rangle$, $|4^+, 4^+\rangle$, $|5, 5\rangle$, $|8, 8\rangle$, $|2, 2^+\rangle$, $|2^+, 2\rangle$, $|4, 4^\rangle$, $|4^+, 4\rangle$, $|4, 5\rangle$, $|5, 4\rangle$, $|4^+, 5\rangle$, $|5, 4^\rangle$, $|5, 6\rangle$, $|6, 5\rangle$. There are 17 such states with a priori possible isospins $T = 0, 1, 2, 3, 4$. We next consider $^{48}$V, which consists of three protons and five neutrons. The
latter can also be regarded as three neutron holes, so the possible states are the same for neutrons and protons. The three proton states are $3/2, 5/2, 7/2, 9/2, 11/2, and 15/2$, all occurring only once. The possible $J = 1^+$ states are $[3/2, 5/2], [5/2, 7/2], [7/2, 9/2], [9/2, 11/2], [3/2, 7/2], [5/2, 9/2], [7/2, 11/2], [9/2, 13/2]$. There are 14 such states and they all must have isospins greater than zero. Hence, the number of $J = 1^+, T = 0$ states of the $(f_{7/2})^2$ configuration is $(17 - 14) = 3$.

The wave functions of these states are included in a larger compilation by A. Escuderos, L. Zamick, and B.F. Bayman [14]. It is there noted that because both protons and neutrons are at mid shell, the quantity $s = (-1)^v$ is a good quantum number, where $v = (v_p + v_n)/2$. Referring to Ref. [10] for $J = 0^+$, $T = 0$, there are four states with $s = +1$ and two with $s = -1$. All $J = 0^+$, $T = 1$ states have $s = -1$, while all $T = 2$ and $T = 4$ states have $s = +1$. There are two $J = 1^+$, $T = 0$ states with $s = -1$ at energies of 7.775 and 9.258 MeV. There is one $s = +1$ state at 9.037 MeV with a rather simple wave function: $1/\sqrt{2}(4^*5) + (5, 4^*)$. The lowest $J = 0^+, T = 0$ state has $s = +1$.

### Table V: Selected $I = 4^+$ states in $^{96}$Cd with a $(g_{9/2})^4$ configuration. On the second row we give the energies in MeV.

| $(J_p, J_n)$ | Mix $T = 0, 2$ | Mix $T = 0, 2$ | $T = 2, v = 4$ unique | $T = 0$ untangled | other $T = 2, v = 4$ |
|-------------|---------------|---------------|---------------------|-------------------|---------------------|
| [0, 4]      | 0.0000        | 0.0000        | 0.0000              | 0.0000            | 0.0000              |
| [2, 2]      | −0.3250       | −0.4170       | −0.4270             | 0.3123            | −0.0255             |
| [2, 4]      | −0.2364       | −0.2472       | −0.2542             | 0.2289            | −0.1986             |
| [2, 6]      | 0.2168        | 0.3043        | 0.3107              | −0.2076           | −0.1976             |
| [4, 4]      | 0.0207        | 0.2390        | 0.2395              | −0.0135           | −0.3313             |
| [4, 6]      | −0.1826       | −0.1364       | −0.1418             | 0.1784            | 0.2245              |
| [4, 8]      | 0.0934        | 0.1540        | 0.1567              | −0.0888           | 0.3874              |
| [6, 6]      | −0.1312       | 0.1678        | 0.1638              | 0.1362            | 0.5645              |
| [6, 8]      | −0.1343       | 0.0357        | 0.0316              | 0.1353            | 0.0247              |
| [8, 8]      | −0.7421       | 0.5881        | 0.5625              | 0.7594            | −0.1087             |

C. **M1 selection rules**

There is a modern twist to what we are here doing. There has been an extensive review of M1 excitations, including spin-flip modes, scissors modes etc., by K. Heyde, P. Von Neumann-Cosel, and A. Richter [13]. The mode we are here considering has, to the best of our knowledge, not yet been studied experimentally. There have been studies of M1 $T = 0 \rightarrow T = 0$ transitions, e.g. the electro-excitation of $J = 1^+, T = 0$ excited states of $^{12}$C, but these involve more than one shell. Isospin impurities are very important for these transitions because the isovector $M1$ coupling constants are much larger than the isoscalar ones.

One simple selection rule for M1 transitions in this limited model space is that $M1 \ (T = 0 \rightarrow T = 0)$ equals zero. To see this, we note that in the single-$j$-shell space we can replace the $M1$ operator by $g_J J$. The $M1$ matrix element for a $T = 0 \rightarrow T = 0$ transition is thus proportional to $(g_{1p} + g_{1n})$, i.e. the isoscalar sum. But if such a term is non-zero, it would imply that the total angular momentum operator $J$ (obtained by setting the two $g'$s above each equal to 1/2) could induce an M1 transition, which, of course, it cannot.

Another “midshell” selection rule is that the quantum number $s$ has to be the same for the initial $J = 1^+$, $T = 0$ state and for any final state, e.g $J = 1^+, T = 1$ or $J = 2^+, T = 1$.

Although not necessary, it is nevertheless instructive to show in more detail why the $T = 0 \rightarrow T = 0$ matrix element vanishes. Consider a transition from $s = -1$ to $s = -1$. In the wave functions, there will be no amplitude of the configuration $(J_p, J_n) = (2, 2)$, but there will be of $(2, 2^n)$ and $(2^n, 2)$. The transition matrix element will have the form $\langle (2, 2)^2 + (2^*, 2^*) | M1 | (2, 2)^2 - (2^n, 2^n) \rangle$. This is equal to $\langle (2, 2)^2 | [M1](2, 2)^2 - (2^n, 2^n) \rangle$. Since in the single $j$ shell one can replace $M1$ by $g_J J$, the matrix element $\langle 2| [M1]|2 \rangle$ is equal to $\langle 2^* | M1 | 2^* \rangle$. We thus see that the complete matrix element vanishes.

### IV. SELECTED SYSTEMATICS OF ODD–ODD NUCLEAR SPECTRA

We consider $T = 1$ states of four nucleons with three partices (holes) of one kind and one of the other kind, e.g. $^{44}$Sc (three neutrons and one proton) or $^{96}$Ag (three proton holes and one neutron hole). We formulated a $(2j - 1)$ rule which will here be presented in a somewhat different way than in Ref. [2]. The rule is that, for these systems, yrast states with angular momenta $I = (2j - 1)$ lie lower in energy than neighbouring states with angular momenta $(2j - 1) - 1$ or $(2j - 1) + 1$. We give some examples in Table [VI] with energies in MeV.
Table VI: Examples of the \((2j-1)\) mentioned in the text. The energies are given in MeV.

|       | \(I\) | Exp. | Theory |
|-------|------|------|--------|
| \(^{44}\text{Sc}\) | 5    | 1.513 | 1.276  |
|       | 6    | 0.271 | 0.381  |
|       | 7    | 0.968 | 1.272  |
| \(^{52}\text{Mn}\) | 5    | 1.254 | 1.404  |
|       | 6    | 0.000 | 0.000  |
|       | 7    | 0.870 | 1.819  |
| \(^{96}\text{Ag}\) | 7    | ???   | 0.861  |
|       | 8    | 0.000 | 0.000  |
|       | 9    | 0.470 | 0.492  |
| \(h_{11/2} Q \cdot Q\) | 9    | 1.30  |
|       | 10   | 0.21  |
|       | 11   | 0.85  |

A possible explanation of this rule for nuclei at the end of a closed shell is that for such nuclei the value of the rotational quantum number \(K\) is equal to \((2j-1)\). In more detail, the neutron hole has \(k_1 = j\) and the three proton holes have \(k_2 = j-1\), so that \(K = k_1 + k_2 = (2j-1)\).

In Ref. [2], it was noted that it is hard to get two-body matrix elements from experiment. One can get \(T = 1\) matrix elements from the spectrum of \(^{98}\text{Cd}\), but the spectrum of the two-hole nucleus \(^{96}\text{In}\) is not known, so we cannot get the \(T = 0\) two-body matrix elements in a simple direct way. Sorlin and Porquet [10] discussed using a Pandya transformation to get the particle-particle spectrum from the particle-hole spectrum of \(^{90}\text{Nb}\). This is a priori a reasonable thing to try. They used as input the yrast spectrum of \(^{90}\text{Nb}\), except for \(J = 1^+\), where they used the second excited \(1^+\) state. We reproduced the results in [2]. We find that for \(^{96}\text{Cd}\) this method gives a significantly lower excitation energy for \(J = 16^+\) in \(^{96}\text{Cd}\) than does a realistic CCGI interaction [9]: 3.898 MeV vs 5.245 MeV. This may be due to the increasing collectivity as one moves away from the \(N = 50\ Z = 50\) closed shell.

[1] E. Moya de Guerra, A.A. Raduta, L. Zamick, and P. Sarriuguren, Nucl. Phys. A 727, 3 (2003).
[2] L. Zamick and A. Escuderos, Phys. Rev. C 87, 044302 (2013).
[3] L. Zamick, Phys. Rev. C 75, 024307 (2007).
[4] A. Escuderos and L. Zamick, Phys. Rev. C 73, 044302 (2006).
[5] L. Zamick, Phys. Rev. C 75, 064305 (2007).
[6] L. Zamick and P. Van Isacker, Phys. Rev. C 78, 044327 (2008).
[7] Chong Qi, Phys. Rev. C 83, 044307 (2011).
[8] P. Van Isacker and S. Heinze, Phys. Rev. Lett. 100, 052501 (2008).
[9] L. Coraggio, A. Covello, A. Gargano, and N. Itaco, Phys. Rev. C 85, 034335 (2012).
[10] J.D. McCullen, B.F. Bayman, and L. Zamick, Phys. Rev. 134, B515 (1964).
[11] J.N. Ginocchio and J.B. French, Phys. Lett. 7, 137 (1963).
[12] I. Talmi, Phys. Rev. C 72, 037302 (2005).
[13] Y.M. Zhao and A. Arima, Phys. Rev. C 71, 047304 (2005); Phys. Rev. C 72, 064333 (2005).
[14] A. Escuderos, L. Zamick, and B.F. Bayman, arXiv:nucl-th/0506050 (2005).
[15] K. Heyde, P. von Neumann-Cosel, and A. Richter, Rev. Mod. Phys. 82, 2365 (2010).
[16] O. Sorlin and M.-G. Porquet, Prog. Part. Nucl. Phys. 61, 602 (2008).