Generalized Uncertainty Principle and the Ramsauer-Townsend Effect

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Received September 23, 2011; in final form, January 23, 2012

Abstract—The scattering cross-section of electrons in noble gas atoms exhibits a minimum value at electron energies of approximately 1 eV. This is the Ramsauer-Townsend effect. In this letter, we study the Ramsauer-Townsend effect in the framework of the Generalized Uncertainty Principle.

DOI: 10.1134/S0202289312030097

1. INTRODUCTION

Various approaches to quantum gravity, such as string theory and loop quantum gravity, as well as black hole physics, predict a minimum measurable length of the order of the Planck length, \( \ell_p = \sqrt{\hbar/c^3} \approx 10^{-35} \text{ m} \). In the presence of this minimal observable length, the Heisenberg’s standard Uncertainty Principle attains an important modification leading to the so-called Generalized Uncertainty Principle (GUP). As a result, the corresponding commutation relations between positions and momenta are generalized too [1]. In the recent years, a lot of attention has been attracted to extending the fundamental problems of physics in this framework (see, e.g., [2–25]). Since in the GUP framework one cannot probe distances smaller than the minimum measurable length at finite time, we expect that it modifies the Hamiltonian of systems too. Recently it has been shown that the GUP affects the Lamb shift, Landau levels, reflection and transmission coefficients of a potential step and a potential barrier [26]. In addition, the authors speculated on the possibility of extracting measurable predictions of the GUP in future experiments. In this work we will follow the procedure introduced in [26], but we are going to address the effect of GUP on the Ramsauer-Townsend (RT) effect. The latter can be observed as long as the scattering does not become inelastic by exciting the first excited state of the atom. This condition is best fulfilled by the closed-shell noble gas atoms. Physically, the RT effect may be thought of as a diffraction of the electron around a rare-gas atom, in which the wave function inside the atom is distorted in such a way that it fits on smoothly to an undistorted wave function outside. The effect is analogous to perfect transmission found at particular energies in one-dimensional scattering from a square well. A one-dimensional treatment of scattering from a square well and also a three-dimensional treatment using partial-wave analysis can be found in [27]. We generalize the one-dimensional treatment of scattering from a square well to the GUP framework. We also address the condition for interference in the Fabry-Perot interferometer in the GUP framework.

2. THE GENERALIZED UNCERTAINTY PRINCIPLE

Quantum mechanics with a modification of the usual canonical commutation relations has been investigated intensively in the last few years (see [23] and references therein). Such works, motivated by several independent streamlines of investigations in string theory and quantum gravity, suggest the existence of a finite lower bound to the possible resolution \( \Delta X \) of spacetime points. The following deformed commutation relation has attracted much attention in the recent years [1]:

\[
[X, P] = i\hbar(1 + \beta P^2),
\]

and it was shown that it implies the existence of a minimal resolution length \( \Delta X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} \geq \hbar\sqrt{\beta} \). This means that there is no possibility to measure the coordinate \( X \) with an error smaller than \( \hbar\sqrt{\beta} \). Since in the context of string theory the minimum observable distance is the string length, we conclude that \( \sqrt{\beta} \) is proportional to this length. If we set \( \beta = 0 \), the usual Heisenberg algebra is recovered. The use of the deformed commutation relation (1) brings new difficulties in solving the quantum problems. A part of difficulties is related to the breakdown of the...
where $H_0 = p^2/(2m) + V(x)$ and $H_1 = p^4/(3m)$.

In the quantum domain, this Hamiltonian results in the following generalized Schrödinger equation in the quasi-position representation:

$$
\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + \beta \frac{\hbar^4}{3m} \frac{\partial^4 \psi(x)}{\partial x^4} + V(x)\psi(x) = E\psi(x),
$$

(6)

where the second term in the left-hand side is due to the generalized commutation relation (1). This is a fourth-order differential equation which in principle admits four independent solutions. Therefore, solving this equation in the $x$ space and separating the physical solutions is not an easy task. With these preliminaries, in the next section we solve Eq. (6) for a quantum well to address the RT effect and the Fabry-Perot interferometer resonance condition in the presence of a minimal observable length.

3. THE RAMSAUER-TOWNSEND EFFECT WITH GUP

We choose the following geometry of the quantum well (see Fig. 1):

$$
V(x) = \begin{cases}
-V_0 & 0 < x < a, \\
0 & \text{elsewhere},
\end{cases}
$$

(7)

where $V_0$ is a positive constant and we assume $E > 0$.

The eigenfunctions of a particle in this potential well satisfy the generalized Schrödinger equation (6). We need to find the solutions in three different regions which are indicated in Fig. 1. To proceed further, we rewrite Eq. (6) in these regions separately as

$$
d^2\psi(x) + q^2\psi(x) - k^2d^4\psi(x) = 0
$$

(8)

for $0 < x < a$, and

$$
d^2\psi(x) + k^2\psi(x) - \ell_p^2d^4\psi(x) = 0,
$$

(9)

elsewhere, where by definition $d^n \equiv \partial^n/\partial x^n$, $k = \sqrt{2mE/\hbar^2}$, $q = \sqrt{2m(E + V_0)/\hbar^2}$, and $\ell_p = \hbar\sqrt{2\beta/3}$. As stated before, the above equations are forth-order differential equations which in general admit four independent solutions. However, some solutions would be nonphysical and should be removed by imposing boundary conditions. As is shown in [22], alternatively, we can find the equivalent physical solutions by adding the following constraint: the physical solutions should also satisfy the ordinary Schrödinger equation but with different eigenenergy. In fact, for the cases of a free particle and a particle in a box, this additional condition prevents us from doing equivalent but lengthy calculations [22]. Therefore,
we demand that the eigenfunctions also satisfy the following second-order differential equations:

\[ d^2\psi(x) + q'^2\psi(x) = 0 \tag{10} \]

for \(0 < x < a\), and

\[ d^2\psi(x) + k'^2\psi(x) = 0, \tag{11} \]

elsewhere, where \(k' = \sqrt{2mE'/\hbar^2}\) and \(q' = \sqrt{2m(E' + V_0)/\hbar^2}\). The solutions of Eqs. (10) and (11) in the regions I, II and III are

\[
\psi_I = e^{ik'x} + Ae^{-ik'x}, \\
\psi_{II} = Be^{iq'x} + Ce^{-iq'x}, \\
\psi_{III} = De^{ik'x},
\]

respectively. These solutions should also satisfy Eqs. (8) and (9), which result in

\[ k^2 = k'^2 + \ell_p^2k'^4, \quad q^2 = q'^2 + \ell_p^2q'^4. \tag{13} \]

These solutions are similar to those of ordinary quantum mechanics but with modified wavenumbers. Now the boundary conditions are the continuity of the wave functions and their first derivatives at the boundaries. The resulting equations can be solved analytically to obtain the coefficients \(A, B, C,\) and \(D\). For our purposes, the solution for \(A\) is as follows:

\[
A = \frac{(k^2 - q'^2)\sin(q'a)}{(k^2 + q'^2)\sin(q'a) + 2ikq'\cos(q'a)}. \tag{14} \]

So the reflection coefficient is given by

\[
R_\alpha(k', q') \equiv |A|^2 = \frac{(k^2 - q'^2)^2\sin^2(q'a) + 4k^2q'^2\cos^2(q'a)}{(k^2 + q'^2)^2\sin^2(q'a) + 4k^2q'^2\cos^2(q'a)}. \tag{15} \]

Because of the smallness of the Planck length, we can obtain \(k' \approx k(1 - \frac{1}{2}\ell_p^2k^2)\) and \(q' \approx q(1 - \frac{1}{2}\ell_p^2q^2)\) from Eq. (13) and write the reflection coefficient in terms of the physical wavenumbers:

\[
R_\alpha(k, q) = (k^2 - q'^2)^2 \left[1 - 2\ell_p^2(k^2 + q^2)\right] \times \\
sin^2 \left[ q \left( 1 - \frac{1}{2}\ell_p^2q^2 \right) a \right] \\
\times \left\{ (k^2 + q'^2)^2 - 2\ell_p^2(k^4 + q^4) \right\} \\
\times \sin^2 \left[ q \left( 1 - \frac{1}{2}\ell_p^2q^2 \right) a \right] \\
+ 4k^2q'^2 \left[ 1 - \ell_p^2(k^2 + q^2) \right] \\
\times \cos^2 \left[ q \left( 1 - \frac{1}{2}\ell_p^2q^2 \right) a \right] \right)^{-1}. \tag{16} \]

Fig. 2 shows the variation of the reflection coefficient versus the energy. It also compares the ordinary quantum-mechanical result with the corresponding result in the presence of a minimal observable length.

At this point, it is worth mentioning that the rectangular potential well is an idealization, and it would be desirable to evaluate the admitted deviations of a real potential from this ideal one. Indeed, in reality, the sharp edges are changed to smoothed-out edges. A proper candidate for this case is the Woods–Saxon potential which has the following functional form [28]:

\[
V(x) = -V_0 \left[ \frac{\theta(-x + L/2)}{1 + e^{-\alpha x}} + \frac{\theta(x - L/2)}{1 + e^{\alpha(x - L/2)}} \right], \tag{17} \]

where \(\alpha\) and \(L\) are real and positive, and \(\theta(x)\) is the Heaviside step function. For \(\alpha L \gg 1\) this potential closely resembles a rectangular well with smooth edges and size \(L\) (Fig. 3). This potential is exactly solvable in relativistic and non-relativistic cases, and the solutions can be written in terms of the hypergeometric functions [28–30]. Since these solutions smoothly converge to the plane-wave solutions at \(\alpha L \gg 1\), we expect that the GUP-corrected reflection coefficient of this system also continuously tends to Eq. (16) in this limit. However, in this case the wave
function cannot satisfy both the ordinary and GUP-corrected Schrödinger equations simultaneously. It is due to the fact that the potential is not constant inside the well. This problem needs a further investigation, and we are going to study it in a separate program.

In the particular case where \( \sin[q(1 - \frac{1}{2} \ell^2 q^2) a] = 0 \), there is no reflection, that is, \( R = 0 \) and therefore we have a maximum transmission. This is the Ramsauer-Townsend effect. In this case,

\[
q \left( 1 - \frac{1}{2} \ell^2 q^2 \right) = \frac{n\pi}{a}.
\]  

(18)

In ordinary quantum mechanics this effect occurs at those wavenumbers that satisfy the condition \( q_{\text{ord}} = n\pi/a \). This feature shows that there is a shift \( (\Delta q = q_{\text{GUP}} - q_{\text{ord}}) \) in the wavenumber of the transmission resonance, and this shift itself is wavenumber-dependent. Up to the first-order in the GUP parameter, we find

\[
\Delta q \approx \frac{1}{2} \ell^2 \left( \frac{n\pi}{a} \right)^3.
\]  

(19)

We also note that, in the ordinary quantum-mechanical description, the resonance condition is \( \lambda_{\text{ord}} = 2\pi/q = 2a/n \), which is the same condition as in the Fabry-Perot interferometer. In the presence of a minimal observable length, this condition modifies as follows:

\[
\lambda' = \frac{2\pi}{q'} \approx \frac{2\pi}{q} \left( 1 + \frac{1}{2} \ell^2 q^2 \right) = \lambda_{\text{ord}} \left( 1 + \frac{1}{2} \ell^2 q^2 \right).
\]  

(20)

Therefore, in the presence of a minimal length, the condition for interference in Fabry-Perot interferometer changes too. Amazingly, this change is itself wavelength-dependent.

Up to this point, we have addressed the RT effect and the Fabry-Perot interferometer resonance condition in the GUP framework. To complete our treatment of this interesting quantum-mechanical problem, let us consider the negative energy case \(-V_0 < E < 0\) which results in the quantized energy spectrum. In this case, Eqs. (10) and (11) cast into the following equations:

\[
\frac{d^2 \psi(x)}{dx^2} + q^2 \psi(x) = 0
\]  

(21)

for \( 0 < x < a \), and

\[
\frac{d^2 \psi(x)}{dx^2} - \kappa^2 \psi(x) = 0,
\]  

(22)

elsewhere, where by definition \( \kappa = \sqrt{2m|E'}/\hbar \) and \( q = \sqrt{2m(V_0 - |E'|)/\hbar^2} \). So the solutions are

\[
\psi_1 = Ae^{\kappa x},
\]

\[
\psi_2 = Be^{-iq x} + Ce^{-iq x},
\]

\[
\psi_3 = De^{-\kappa x}.
\]  

(23)

If we choose the center of the well as the center of the coordinate system, it is straightforward to check that the energy eigenvalues are given by roots of the equations

\[
\tan(q'a/2) = \kappa'/q',
\]

\[
\cot(q'a/2) = -\kappa'/q'
\]  

(24)

for even and odd eigenstates, respectively. These eigenvalues can also be written in terms of the physical quantities \( k \) and \( q \) as

\[
\tan \left[ q \left( 1 - \frac{1}{2} \ell^2 q^2 \right) a/2 \right] = \sqrt{\frac{2mV_0}{\hbar^2} - q^2 \left[ 1 - \frac{1}{2} \ell^2 q^2 \left( \frac{2mV_0}{\hbar^2} - q^2 \right) \right]/q \left( 1 - \frac{1}{2} \ell^2 q^2 \right)},
\]

(25)

\[
\cot \left[ q \left( 1 - \frac{1}{2} \ell^2 q^2 \right) a/2 \right] = -\sqrt{\frac{2mV_0}{\hbar^2} - q^2 \left[ 1 - \frac{1}{2} \ell^2 q^2 \left( \frac{2mV_0}{\hbar^2} - q^2 \right) \right]/q \left( 1 - \frac{1}{2} \ell^2 q^2 \right)}.
\]  

(26)

So, in the negative energy case, the energy eigenvalues are roots of Eq. (26). We note that there is no trace of the RT effect in the case \(-V_0 < E < 0\).

4. CONCLUSIONS

The scattering cross-section of electrons in noble gas atoms exhibits a minimum value at electron energies of approximately 1 eV, an effect of which is called the Ramsauer-Townsend effect. We have studied the RT effect in the presence of the minimal observable length in the framework of the generalized uncertainty principle. We have shown that in this case there is a shift

\[
\Delta q = q_{\text{GUP}} - q_{\text{ord}} \approx \frac{1}{2} \ell^2 \left( \frac{n\pi}{a} \right)^3
\]  

in the wavenumber of the transmission resonance, and this shift itself is wavenumber-dependent. This shift also affects the resonance in the Fabry-Perot interferometer in such a way that this change is also wavelength-dependent. If in the future experiments one finds a similar shift in the Fabry-Perot interferometer resonance wavelength, it would be an explicit trace of an underlying quantum gravity scenario. We also note that the RT effect is in principle a 3-dimensional effect that needs a 3-dimensional analysis. However, the corresponding calculations are lengthy, and the essential ingredients and outcomes are the same as presented in this one-dimensional
analysis. Finally, we note that this problem can be treated with more real potentials such as the Woods-Saxon potential to have a more realistic situation. This potential is exactly solvable in the relativistic and non-relativistic cases, and the solutions can be written in terms of hypergeometric functions. Since these solutions smoothly converge to the plane-wave solutions in the appropriate limit, we expect that the GUP-corrected reflection coefficient of this system also continuously tends to our result in this limit. We are going to treat this more real situation in a separate research program.

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