LONGITUDINAL DIELECTRIC PERMITTIVITY OF QUANTUM MAXWELL COLLISIONAL PLASMAS

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The kinetic equation of Wigner – Vlasov – Boltzmann with collision integral in relaxation BGK (Bhatnagar, Gross and Krook) form in coordinate space for quantum non–degenerate (Maxwellian) collisional plasma is used. Exact expression (within the limits of considered model) is found. The analysis of longitudinal dielectric permeability is done. It is shown that in the limit when Planck’s constant tends to zero of expression for dielectric permittivity transforms into the classical case of dielectric permittivity. At small values of wave number it has been received the solution of the dispersion equation. Damping of plasma oscillations has been analyzed. The analytical comparison with the dielectric Mermin’ function received with the use of the kinetic equation in momentum space is done. Graphic comparison of the real and imaginary parts of dielectric permittivity of quantum and classical plasma is done also.

Key words: collisional plasma, BGK equation, electric conductivity, dielectric permittivity, Lindhard’s formula, Landau’s damping.

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1. Introduction

In the present work formulas for electric conductivity and for dielectric permittivity of quantum electronic non–degenerate Maxwellian plasma are deduced.

Dielectric permittivity in the collisionless quantum gaseous plasma was studied by many authors (see, for example, [11]-[29]). G. Manfredi [6] investigated one-dimensional case of the quantum plasma. In this article G. Manfredi has noted the importance of carrying out of analysis of the dielectric permittivity deduced with help of the quantum kinetic equation
with collision integral in coordinate space. The present work is devoted to performance of this problem for Maxwellian plasma.

In the present work for derivation of dielectric permittivity the quantum kinetic Wigner — Vlasov — Boltzmann equation (WVB–equation) with collision integral in the form of $\tau$–models in coordinate space is applied. Such collision integral is named the BGK collision integral.

The WVB–equation is written for Wigner function, which is an analogue of distribution function of electrons for quantum plasma (see [10], [11] and [30]).

The most widespread method of investigation of quantum plasma is the method of Hartree — Fock or a method equivalent to it, namely, the method of Random Phase Approximation [16], [17]. In the work [21] this method has been applied for receiving of the expression for dielectric permittivity of quantum degenerate plasma in $\tau$–approach. However, in the work [23] it is shown, that expression received in [21] is noncorrect, as it does not turn into classical expression under a condition, when quantum amendments can be neglected. Thus in the work [23] empirically corrected expression for dielectric permittivity of quantum plasma, free from the specified lack has been offered. By means of this expression authors investigated quantum amendments to optical properties of metal [24], [25]. Friedel’s oscillations in quantum plasma also have been investigated already for more than half a century (see, for example, [18]-[20]).

In the theory of quantum plasma two essentially different possibilities of construction of the relaxation kinetic equation in $\tau$ - approach exist: in momentum space (in the space of Fourier images of distribution function) and in coordinate space. On the basis of the relaxation kinetic equation in the space of momentum Mermin [22] has carried out consistent derivation of the dielectric permeability for quantum collisional plasma in 1970 for the first time.

In the present work expression for the longitudinal dielectric permittivity for non–degenerate plasma with the use of the relaxation equations
in space of coordinates is deduced. If in the received expression we make Planck constant converge to zero ($\hbar \to 0$), we will receive exactly the classical expression of dielectric permittivity of non–degenerate plasma. Various limiting cases of the dielectric permittivity are investigated. Comparison with Mermin’s result is carried out also.

2. Solution of the kinetic equation

We consider the kinetic Wigner — Vlasov — Boltzmann equation \[27\] with collisional integral in the form of BGK–model

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} = \frac{ie}{\hbar} W[f] + \nu[f_{eq}(r, p, t) - f(r, p, t)].$$  \tag{2.1}

This equation describes evolution of the Wigner function for electrons in quantum plasma.

Here $e$ is the charge of electron, $\hbar$ is the Planck’s constant, $\nu$ is the effective collision frequency of electrons with ions and neutral atoms, $f(r, p, t)$ is the Wigner function for electrons.

The function $f_{eq}(r, p, t)$ is the equilibrium Maxwell distribution function of electrons,

$$f_{eq}(r, p, t) = n(r, t) \left( \frac{m}{2\pi \kappa T} \right)^{3/2} \exp \left( -\frac{p^2}{2mT} \right),$$

or

$$f_{eq}(r, p, t) = \frac{n(r, t)m^3}{\pi^{3/2}p_T^3} \exp \left( -\frac{p^2}{p_T^2} \right),$$

where $n(r, t)$ is the number density (concentration) of electrons, $\kappa$ is the Boltzmann’s constant, $m$ is the electron mass, $p = mv$ is the electron momentum, $p_T = mv_T$ is the thermal momentum of electrons, $v_T = \frac{1}{\sqrt{\beta}}$ is the thermal electron velocity, $\beta = \frac{m}{2\kappa T}$, $W[f]$ is the Wigner – Vlasov functional,

$$W[f] = \frac{1}{(2\pi)^3} \int \left[ U(r - \frac{\hbar b}{2}, t) - U(r + \frac{\hbar b}{2}, t) \right] \times$$

$$\times f(r, p', t) e^{ib(p' - p)} d^3b d^3p'. \tag{2.2}$$
The Wigner function is an analogue of function of distribution for quantum systems. It is widely used in the diversified questions of physics. Wigner’s function was investigated, for example, in the works [29] and [30].

Let’s consider, that distribution electron function depends on one spatial coordinate $x$, time $t$ and momentum $p$, and the electric scalar potential depends on one spatial coordinate $x$ and time $t$. Then we can write down the equations (2.1) and (2.2) in the form

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} = \frac{ie}{\hbar} W[f] + \nu [f_{eq}(x, p, t) - f(x, p, t)], \quad (2.3)$$

$$W[f] = \frac{1}{(2\pi)^3} \int \left[ U(x - \frac{\hbar b_x}{2}, t) - U(x + \frac{\hbar b_x}{2}, t) \right] \times \times f(x, p', t) e^{ib(p' - p)} d^3bd^3p'. \quad (2.4)$$

We will linearize the locally equilibrium function $f_{eq}$ in terms of absolute Maxwell’s distribution

$$f_M(c) = n_0 \left( \frac{\beta}{\pi} \right)^{3/2} e^{-\beta c^2} = \frac{n_0}{\pi^{3/2}v_T^3} \exp \left( - c^2 \right),$$

where $c$ is the module of dimensionless electron velocity $c = \frac{v}{v_T}$.

We take the scalar potential in the form

$$U(x, t) = U_0 e^{i(\omega t - kx)}. \quad (2.5)$$

Let’s search the electron distribution function in the following form:

$$f = f_M(p) \left[ 1 + U(x, t) h(p) \right]. \quad (2.6)$$

Linearization of $f_{eq}$ leads us to expression

$$f_{eq} = f_M(p) \left( 1 + \frac{n_1(x, t)}{n_0} \right),$$

where

$$n_1(x, t) \equiv \delta n(x, t) = n(x, t) - n_0.$$
From the law of conservation of number of particles
\[
\int (f_{eq} - f) d^3v = 0
\]
we find that
\[
n_1(x, t) = U(x, t) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(\mu) d\mu, \quad \mu = c_x = \frac{p_x}{p_T}.
\] (2.7)

Now we can write down the equation (2.3) in the following form
\[
f_M(v) h(v_x) \left[ \nu + i(kv_x - \omega) \right] U(x, t) =
\]
\[
= \frac{ie}{\hbar} W[f] + \nu U(x, t) f_M(v) A,
\] (2.8)

where
\[
A = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} h(\mu) d\mu.
\] (2.9)

The expression (2.4) in linear approximation has the form
\[
W[f] = \frac{1}{(2\pi)^3} \int \left[ U(x - \frac{\hbar b_x}{2}, t) - U(x + \frac{\hbar b_x}{2}, t) \right] \times
\]
\[
\times f_M(p') e^{ib(p' - p)} d^3b d^3p'.
\] (2.10)

Now we receive for potential (2.5)
\[
U(x - \frac{\hbar b_x}{2}, t) - U(x + \frac{\hbar b_x}{2}, t) =
\]
\[
= U_0 e^{i(kx - \omega t)} \left[ \exp(-i \frac{\hbar b_x}{2}) - \exp(i \frac{\hbar b_x}{2}) \right].
\]

Let’s calculate internal integral in (2.2). Considering the last equality, we will integrate in (2.10) by $d^3b$. We have
\[
\frac{1}{(2\pi)^3} \int \left[ \exp(-i \frac{\hbar b_x}{2}) - \exp(i \frac{\hbar b_x}{2}) \right] \times
\]
\[
\times e^{ib_x(p_x' - p_x)} e^{ib_y(p_y' - p_y)} e^{ib_z(p_z' - p_z)} db_x db_y db_z =
\]
\[
= \delta(p_y - p_y') \delta(p_z - p_z') \left[ \delta(p_x - p_x' + \frac{\hbar k}{2}) - \delta(p_x - p_x' - \frac{\hbar k}{2}) \right].
\]
Substituting this equality in (2.10), we receive, that

\[ W[f] = U(x, t) \int \delta(p_y - p'_y) \delta(p_z - p'_z) \times \]
\[ \times \left[ \delta(p_x - p'_x + \frac{\hbar k}{2}) - \delta(p_x - p'_x - \frac{\hbar k}{2}) \right] f_M(p') dp'_x dp'_y dp'_z. \]  
(2.11)

It is necessary to us to integrate by momentums. Let’s notice, that

\[ \int_{-\infty}^{\infty} \delta(p_y - p'_y) e^{-p_y'^2/p_y'^2} dp'_y = e^{-p_y^2/p_y^2} = e^{-c_y^2}, \]
\[ \int_{-\infty}^{\infty} \delta(p_z - p'_z) e^{-p_z'^2/p_z'^2} dp'_z = e^{-p_z^2/p_z^2} = e^{-c_z^2}, \]
\[ \int_{-\infty}^{\infty} \delta(p_x - p'_x ± \frac{\hbar k}{2}) e^{-p_x'^2/p_x'^2} dp'_x = \]
\[ = \exp \left( - \frac{(p_x ± \frac{\hbar k}{2})^2}{p_x'^2} \right) = \exp \left( - \left( \mu ± \frac{\hbar k}{2m v_T} \right)^2 \right) = e^{-(\mu ± q/2)^2}, \]

where \( \mu = c_x, \ q = k/k_T, \ k_T = mv_T/\hbar \) is the thermal wave electron number.

According to (2.11) we receive further

\[ W[f] = U(x, t) \left[ f_M^+ - f_M^- \right], \]  
(2.12)

where

\[ f_M^± = f_M^±(c) = \frac{n_0}{\pi^{3/2} v_T^3} \exp \left[ - (\mu ± q/2)^2 - c_y^2 - c_z^2 \right]. \]

By means of (2.11) we will rewrite the equation (2.8) in the form

\[ h(\mu) \left( 1 - i\omega \tau + ik_1 \mu \right) = \frac{ie}{\hbar v} \frac{f_M^+ - f_M^-}{f_M} + A, \]  
(2.13)

where \( k_1 \) is the dimensionless wave number, \( k_1 = k/l, l = v_T \tau \) is the mean free path of electrons.

We receive now from the equation (2.13)

\[ e^{-\mu^2} h(\mu) = \frac{ie}{\hbar v} \cdot \frac{e^{-(\mu+q/2)^2} - e^{-(\mu-q/2)^2}}{1 - i\omega \tau + ik_1 \mu} + \frac{A e^{-\mu^2}}{1 - i\omega \tau + ik_1 \mu}. \]  
(2.14)
For finding of the constant $A$ we will substitute (2.14) in (2.9). As a result we receive

$$A = -\frac{ie}{\hbar \nu} \frac{J_1(\omega, k, q)}{1 - T_0(\omega, k)},$$

(2.15)

where

$$T_0 = T_0(\omega, k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{1 - i\omega \tau + ik_1 \mu},$$

$$J_1 = J_1(\omega, k, q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-(\mu-q/2)^2} - e^{-(\mu+q/2)^2}}{1 - i\omega \tau + ik_1 \mu} d\mu.$$  

3. Conductivity and permittivity

We consider the connection between electric field and potential

$$\mathbf{E}(x, t) = -\text{grad} \mathbf{E}(x, t).$$

Therefore

$$E_x(x, t) = -\frac{\partial U(x, t)}{\partial t} = -ik U(x, t).$$

From the definition of the longitudinal electric conductivity $j_x(x, t) = \sigma_l E_x(x, t)$, we find that $j_x(x, t) = -ik \sigma_l U(x, t)$, hence $\frac{\partial j_x}{\partial x} = \sigma_l k^2 U(x, t)$.

From the equation of continuity for current and charge densities

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} = 0$$

according to the last equality we receive that

$$\frac{\partial \rho}{\partial t} = -\frac{\partial j_x}{\partial x} = -\sigma_l k^2 U(x, t).$$

On the other hand, from the definition of a charge of a density

$$\rho = e \int f d\Omega = e \int f_M(v)[1 + U(x, t) h(v_x)] d^3v,$$

we find

$$\frac{\partial \rho}{\partial t} = -i\omega e U(x, t) \int f_M(v) h(v_x) d^3v.$$
Now we receive the relation for longitudinal electroconductivity

\[ \sigma_l = \frac{ie\omega}{k^2} \int f_M h(v_x) \, d^3v = \frac{ie\omega n_0}{k^2} \int_{-\infty}^{\infty} e^{-\mu^2} h(\mu) d\mu = \frac{ie\omega n_0}{k^2} A. \]

Substituting the expression (2.15) instead of \(A\), we receive the expression for conductivity of quantum plasma

\[ \sigma_l = \frac{e^2 n_0 \omega \tau}{k^2 \hbar} \frac{J_1}{1 - T_0} = \sigma_0 \frac{q \omega}{k_1 \nu} \frac{J_1}{1 - T_0}. \] (3.1)

Let’s transform integrals \(T_0\) and \(J_1\). We receive, that

\[ T_0 = \frac{1}{ik\tau v_T} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{\mu - \frac{\omega - i\nu}{kv_T}} = -i\nu kv_T t(z) = -i k_1 t(z), \]

where

\[ t(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{\mu - z}, \quad z = \frac{\omega + i\nu}{kv_T}. \]

Integral \(J_1\) we will present in the form

\[ J_1 = \frac{1}{ik\tau v_T} J(z, q) = -i\nu \frac{q}{kv_T} J(z, q) = -i \frac{k_1}{k_1} J(z, q), \]

where

\[ J(z, q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\mu-q/2)^2} - e^{-(\mu+q/2)^2} \frac{d\mu}{\mu - z}. \]

Hence, electric conductivity of quantum plasma according to (3.1) is equal to

\[ \sigma_l = -i \sigma_0 \frac{q \omega}{k_1^2 \nu} \frac{J(z, q)}{1 + it(z)/k_1}, \quad z = \frac{\omega + i\nu}{kv_T}. \] (3.2)

Dielectric permittivity of plasma we will find, if we use (3.1) or (3.2). According to definition

\[ \varepsilon_l = 1 + \frac{4\pi i}{\omega} \sigma_l \]

we find accordingly

\[ \varepsilon_l = 1 + i \frac{\omega^2 q}{\nu^2 k_1} \frac{J_1}{1 - T_0}. \] (3.3)
or
\[ \varepsilon_l = 1 + \frac{\omega_p^2 q}{\nu^2 k_1^2} \frac{J(z, q)}{1 + it(z)/k_1}. \] (3.4)

Here \( \omega_p \) is the own plasma (Langmuir) frequency
\[ \omega_p^2 = \frac{4\pi e^2 n_0}{m}. \]

Let’s show that in a limit when \( \hbar \to 0 \), the expression \( \varepsilon_l \) for longitudinal permittivity of quantum plasmas passes into corresponding expression for classical plasma
\[ \varepsilon_l^0 = 1 + \frac{2\omega_p^2}{k^2 v_T^2} \frac{\lambda_0(z)}{1 - T_0}, \]
where \( \lambda_0(z) \) is the well known (see [31]) dispersion plasma function entered by Van Kampen
\[ \lambda_0(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{\mu - \frac{\omega + i\nu}{kv_T}}. \]

We notice that
\[ \lim_{\hbar \to 0} \frac{e^{-(\mu-q/2)^2} - e^{-(\mu+q/2)^2}}{\hbar} = \frac{2k e^{-\mu^2}}{mv_T}. \]

By means of this relationship we receive
\[ \lim_{\hbar \to 0} \frac{J}{\hbar} = \frac{2k}{mv_T} \lambda_0(z). \]

Thus, passing to a limit at \( \hbar \to 0 \) in the expression (3.4), we receive in accuracy the relationship for conductivity of classical plasma.

We will transform a denominator from formulas (3.3)
\[ 1 - T_0 = 1 + \frac{i\nu}{kv_T} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{\mu - \frac{\omega + i\nu}{kv_T}} = 1 + \frac{i\nu}{kv_T} t(z). \]

We notice that \( \lambda_0(z) = 1 + z \cdot t(z) \). Further we receive
\[ 1 - T_0 = 1 + \frac{i\nu}{\omega + i\nu} \cdot \frac{\omega + i\nu}{kv_T} t(z) = 1 + \frac{i\nu}{\omega + i\nu} z t(z) = \]
\[
\frac{\omega + i\nu + ivzt(z)}{\omega + i\nu} = \frac{\omega + i\nu\lambda_0(z)}{\omega + i\nu}.
\]

Taking into account this identity dielectric permittivity of the quantum and classical plasma accordingly equals

\[
\varepsilon_l = 1 + \frac{\omega_p^2 z^2}{(\omega + i\nu)(\omega + i\nu\lambda_0(z))} \cdot \frac{t(z - q/2) - t(z + q/2)}{q}.
\] (3.5)

and

\[
\varepsilon_l^o = 1 + \frac{2\omega_p^2}{k^2 v_T^2} \cdot \frac{\omega + i\nu\lambda_0(z)}{\omega + i\nu\lambda_0(z)}.
\] (3.6)

The expression (3.6) can be written down in the form

\[
\varepsilon_l^o = 1 + \frac{2\omega_p^2}{\omega + i\nu} \cdot \frac{z^2\lambda_0(z)}{\omega + i\nu\lambda_0(z)}.
\] (3.6’)

The formula (3.5) can be presented in one of the equivalent forms

\[
\varepsilon_l = 1 + \frac{\omega_p^2}{(\omega + i\nu)(\omega + i\nu\lambda_0(z))} \cdot \frac{z^2[t(z - q/2) - t(z + q/2)]}{q},
\] (3.7)

or

\[
\varepsilon_l = 1 + \frac{\omega_p^2(\omega + i\nu)}{k^2 v_T^2(\omega + i\nu\lambda_0(z))} \cdot \frac{t(z - q/2) - t(z + q/2)}{q}.
\] (3.7’)

We present the formula (3.7) in the form

\[
\varepsilon_l = 1 + \frac{\omega_p^2}{(\omega + i\nu)(\omega + i\nu\lambda_0(z))} \cdot \frac{f(z, q)}{q},
\] (3.8)

Here

\[
f(z, q) = z^2[t(z - q/2) - t(z + q/2)].
\]

We notice that

\[
\frac{t(z - q/2) - t(z + q/2)}{q} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{(\mu - z)^2 - q^2/4}.
\]

We will designate

\[
J_0(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{(\mu - z)^2 - q^2/4}.
\]
By means of this designation we will present the formula (3.8) in the form

$$\varepsilon_l = 1 + \frac{\omega_p^2 z^2}{(\omega + i\nu)(\omega + i\nu\lambda_0(z))} \cdot J_0(z),$$

(3.9)
or

$$\varepsilon_l = 1 + \frac{\omega_p^2}{k^2 v_T^2} \cdot \frac{(\omega + i\nu)J_0(z)}{\omega + i\nu\lambda_0(z)}, \quad z = \frac{\omega + i\nu}{kv_T},$$

(3.9′)

Further we will enter dimensionless parameters

$$x = \frac{\omega}{kTv_T} = \frac{\omega}{\nu k_0}, \quad y = \frac{\nu}{kTv_T} = \frac{1}{k_T} = \frac{1}{k_0},$$

where $k_0 = k_T l$ is the dimensionless wave thermal number of electrons.

We will present the formula (3.9′) in the explicit form using the entered dimensionless parameters

$$\varepsilon_l = 1 - \frac{x_p^2}{q^2} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\sqrt{\pi} \left[ (\tau - (x + iy)/q)^2 - q^2/4 \right]}, \quad x + iy \sqrt{\pi} \int_{-\infty}^{\infty} \frac{\tau e^{-\tau^2} d\tau}{\sqrt{\pi} \left[ \tau - (x + iy)/q \right]}$$

(3.10)

where $x_p$ is the dimensionless plasma frequency, $x_p = \frac{\omega_p}{kTv_T}$.

4. Dielectric permittivity properties

Let’s find a long-wave limit (at $k \to 0$) of dielectric permittivity (3.9).

We will present this formula in the form

$$\varepsilon_l = 1 - \frac{\omega_p^2}{(\omega + i\nu)(\omega + i\nu\lambda_0(z))} \cdot z^2 \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{(\mu - z)^2 - q^2/4},$$

We will transform the last factor as follows

$$\frac{z^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{(\mu - z)^2 - q^2/4} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} d\mu}{1 - \frac{2\mu}{z} - \frac{q^2/4 - \mu^2}{z^2}}.$$
From here it is easy to notice, that at $z \to \infty$ the last integral tends to unit. It means, that
\[
\varepsilon_l(\omega, \nu, k = 0) = 1 - \frac{\omega_p^2}{(\omega + i\nu)(\omega + i\nu\lambda_0(z))}.
\] (4.1)

From the expression (4.1) we can see that at $\nu = 0$ we receive classical result for collisionless plasma
\[
\varepsilon_l(\omega, 0, 0) = 1 - \frac{\omega_p^2}{\omega^2}.
\]

Let’s consider special cases of dielectric permittivity.

In the formulas (3.7) or (3.9) we will put $\nu = 0$. Then for the dielectric permittivity in collisionless plasma we receive
\[
\varepsilon_l(\omega, k, \nu = 0) = 1 - \frac{\omega_p^2 k^2 v_T^2 \sqrt{\pi}}{q^2 \pi} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{(\tau - \frac{\omega}{kv_T})^2 - \left(\frac{\hbar k}{2mv_T}\right)^2}.
\] (4.2)

We present the formula (4.2) in dimensionless parameters
\[
\varepsilon_l(x, q) = 1 - \frac{x_p^2}{q^2 \sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{(\tau - x/q)^2 - q^2/4},
\] (4.3)
or
\[
\varepsilon_l(x, q) = 1 + \frac{x_p^2}{q^3 \sqrt{\pi}} \left[ \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau - x/q + q/2} - \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau - x/q - q/2} \right].
\] (4.4)

Now we will consider a low-frequency limit ($\omega \to 0$) in collisional form ($\nu \neq 0$). In this limit we receive
\[
\varepsilon_l(\omega = 0, k, \nu) = 1 + \frac{\omega_p^2}{k^2 v_T^2 \lambda_0(z)} J_0(z), \quad z = \frac{i\nu}{kv_T}.
\] (4.5)

We can rewrite down the formula (4.5) in the form
\[
\varepsilon_l(0, y, q) = 1 + \frac{x_p^2}{q^2 \lambda_0(iy/q)} J_0(iy/q).
\] (4.6)
We transform the formula (4.6) into the following explicit form
\[
\varepsilon_l(0, y, q) = 1 - \frac{x_p^2}{q^3} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{(\tau - iy/q)^2 - q^2/4} - \frac{iy}{q\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} d\tau}{\tau - iy/q}.
\]  
(4.7)

5. Plasma oscillations in long–wave limit

Let’s consider the linearization of the dielectric permittivity by quantum parameter \( q = \frac{\hbar k}{mv_T} = \frac{k}{k_T} \) for the case when \(|z| \gg 1\), and \( q \ll 1 \), where
\[
z = \frac{\omega + i\nu}{kv_T}.
\]

We will expand the function
\[
f(z, q) = z^2[t(z - q/2) - t(z + q/2)].
\]
at large \( z \) and small \( q \)
\[
\frac{f(z, q)}{q} = -1 - \frac{3}{2z^2} - \frac{q^2}{4z^2} - \frac{15}{4z^4}.
\]

We consider the integral in the complex plane
\[
J(z, q) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-(\mu-q/2)^2} - e^{-(\mu+q/2)^2}}{\mu - z} d\mu.
\]  
(5.1)

We will present the subintegral function in the form
\[
\frac{e^{-(\mu-q/2)^2} - e^{-(\mu+q/2)^2}}{\mu - z} = \frac{e^{-\mu^2-q^2/4}}{\mu - z} \left[ e^{q\mu} - e^{-q\mu} \right] = 2q\mu \left[ 1 + \frac{1}{6} (q\mu)^2 \right] \frac{e^{-\mu^2-q^2/4}}{\mu - z}.
\]

Hence, we can present the integral (5.1) in the form
\[
\frac{J}{2q} = \frac{e^{-q^2/4}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} \mu d\mu}{\mu - z} + \frac{q^2 e^{-q^2/4}}{6\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\mu^3 e^{-\mu^2} d\mu}{\mu - z}.
\]  
(5.2)
We will notice, that
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} \mu^3}{\mu - z} d\mu = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mu^2} (\mu^2 - z^2 + z^2)}{\mu - z} d\mu = \\
= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2} \left[ \mu (\mu + z) + z^2 \frac{\mu}{\mu - z} \right] d\mu = \frac{1}{2} + z^2 \lambda_0(z).
\]

Thus, the expression (5.2) can be written down in the following manner
\[
\frac{J}{2q} = e^{-q^2/4} \left\{ \lambda_0(z) + \frac{q^2}{6} \left[ \frac{1}{2} + z^2 \lambda_0(z) \right] \right\}.
\]

(5.3)

Now we will return to the dielectric permittivity (3.9) and with the help of (5.3) we will present it in the form
\[
\varepsilon_l = 1 + \frac{2\omega_p^2 \alpha}{(\omega + i\nu)(\omega + i\nu \lambda_0(z))} \cdot e^{-q^2/4} \left\{ \lambda_0(z) + \frac{q^2}{6} \left[ \frac{1}{2} + z^2 \lambda_0(z) \right] \right\}.
\]

(5.4)

In the expansion (5.4) by degrees \(k^2\) we will keep members with degrees not above than \(k^4\), taking into account the equalities proportional to \(k\)
\[
q = \frac{\hbar}{mv_T} \cdot k \quad \text{and} \quad \frac{1}{z} = \frac{v_T}{\omega + i\nu} \cdot k.
\]

We will search for the solution of the dispersion equation
\[
\varepsilon_l(\omega, k) = 0
\]
for small \(k\), i.e. we seek the solution \(\omega = \omega(k)\) of the equation (5.5) for the case when
\[
|z| = \left| \frac{\omega + i\nu}{kv_T} \right| \gg 1, \quad \nu \ll \omega.
\]

(5.6)

We will use the expansions
\[
\lambda_0(z) = -\frac{1}{2z^2} - \frac{3}{4z^4} - \frac{15}{8z^6} - \cdots, \quad z \to \infty,
\]
\[
\frac{1}{2} + z^2 \lambda_0(z) = -\frac{3}{4z^2} - \frac{15}{8z^4}.
\]

By means of these expansions we receive
\[
2z^2 e^{-q^2/4} \left\{ \lambda_0(z) + \frac{q^2}{6} \left[ \frac{1}{2} + z^2 \lambda_0(z) \right] \right\} =
\]
\[
= -\left\{1 + \frac{3}{2z^2} + \frac{15}{4z^4} + \frac{q^2}{4z^2}\right\}.
\]

Let’s rewrite the equation (5.5) in the explicit form

\[
(\omega + i\nu)(\omega + i\nu\lambda_0(z)) -
\]

\[-\omega_p^2\left(1 + \frac{3}{2z^2} + \frac{q^2}{4z^2} + \frac{15}{4z^4} - \frac{q^4}{16}\right) = 0,
\]

or, taking into account (5.6), we have

\[
\omega^2 - \omega_p^2\left(1 + \frac{3}{2z^2} + \frac{q^2}{4z^2} + \frac{15}{4z^4}\right) = 0,
\]

(5.7)

Let’s consider the real part of the equation (5.7) at conditions (5.6) near to plasma resonance, i.e. we will search for the solution (5.7) in the form

\[
\omega_k^2 = \omega_p^2(1 + \varepsilon), \quad |\varepsilon| \ll 1.
\]

(5.8)

Substituting (5.8) in the equation (5.7), we come to the following equation

\[
\varepsilon = \frac{3}{2z^2} + \frac{q^2}{4z^2} + \frac{15}{4z^4}.
\]

Let’s write down this equation in the explicit form

\[
\varepsilon = \frac{3k^2v_T^2(1 - \varepsilon)}{2\omega_p^2} + \frac{\hbar^2k^4v_T^2(1 - \varepsilon)}{4m^2v_T^2\omega_p^2} + \frac{15k^4v_T^4(1 - 2\varepsilon)}{4\omega_p^4}.
\]

From here we receive

\[
\varepsilon = \frac{3k^2}{k_D^2} + \frac{6k^4}{k_D^4} + \frac{\hbar^2k^4}{2m^2v_T^2k_D^4},
\]

where \(k_D\) is the Debye wave number inverse proportional to the Debye radius \(r_D\),

\[
k_D = \frac{\sqrt{2}\omega_p}{v_T}, \quad k_D = \frac{1}{r_D}.
\]

According to (5.8) we have found the dependence

\[
\omega^2(k) = \omega_p^2\left(1 + \frac{3k^2}{k_D^2} + \frac{6k^4}{k_D^4} + \frac{\hbar^2k^4}{2m^2v_T^2k_D^4}\right).
\]

(5.9)
Let’s rewrite (5.9) in two equivalent forms

\[ \omega^2(k) = \omega_p^2 \left( 1 + \frac{3\kappa T}{m\omega_p^2} k^2 + \frac{6\kappa^2 T^2}{m^2\omega_p^4} k^4 + \frac{\hbar^2 k^4}{4m^2\omega_p^2} \right), \]

or

\[ \omega^2(k) = \omega_p^2 + \frac{3\kappa T}{m} k^2 + \frac{6\kappa^2 T^2}{m^2\omega_p^2} k^4 + \frac{\hbar^2 k^4}{4m^2}. \] (5.10)

\[ \omega^2(k) = \omega_p^2 + \frac{3\kappa T}{m} k^2 + \frac{6\kappa^2 T^2}{m^2\omega_p^2} k^4 \left( 1 + \frac{Q^2}{24} \right), \quad Q = \frac{\hbar\omega_p}{\kappa T}. \] (5.10a)

Here \( Q \) is the quantum parameter, showing how much essential quantum amendments to quantity of frequency of plasma oscillations are.

With use of thermal electron velocity \( v_T \) the formula (5.10) can be written down in the form

\[ \omega^2(k) = \omega_p^2 + \frac{3v_T^2}{2\omega_p^2} k^2 + \frac{3v_T^4}{2\omega_p^4} k^4 + \frac{\hbar^2 k^4}{4m^2}. \]

The last member in the formula (5.10) (the quantum amendment) in accuracy coincides with a corresponding member from the formula (5.8) from Manfredi’s work [6], deduced for degenerate plasma.

From the expression (5.10) one can see that when \( \hbar = 0 \) it passes into classical expression:

\[ \omega_k^2 = \omega_p^2 \left( 1 + \frac{3k^2}{k_D^2} + \frac{6k^4}{k_D^4} \right). \] (5.11)

Now we will find damping of plasma oscillations in quantum plasma. Let’s present the dispersion equation (5.5) by means of (5.4) in the form

\[ (\omega + i\nu)(\omega + i\nu\lambda_0(z)) + \omega_p^2 e^{-q^2/4} \left[ 2z^2 \lambda_0(z) \left[ 1 + \frac{(qz)^2}{6} \right] \right] = 0. \] (5.12)

Let’s calculate decrement of decrease \( \gamma_k \). Let’s consider for this purpose frequency \( \omega \) as complex value and assume also, that

\[ \omega = \omega_k + i\gamma_k, \quad \omega_k = \omega(k), \quad |\gamma_k| \ll \omega_k. \] (5.13)
Let’s substitute the expression (5.13) in the equation (5.12) and we will mark out in the received equation an imaginary part, using the inequalities (5.6). As a result we receive the equation

$$2\omega_k \gamma_k + \omega_k \nu = -2\sqrt{\pi} \omega_p^2 \frac{\omega_k^3}{v_T^3} \exp\left(-\omega_k^2 \frac{\nu}{k^2 v_T^2} - \alpha^2\right) \left(1 + \frac{2}{3}(\alpha z)^2\right),$$

from which we receive the decrement of damping

$$\gamma_k = -\frac{\nu}{2} - \sqrt{\frac{\pi}{8}} \omega_p \frac{k_0^3}{k^3} \exp\left(-\frac{3}{2} - \frac{k_0^2}{2k^2}\right) \left(1 - \frac{q^2}{4}\right) \left(1 + \frac{q^2 z^2}{6}\right). \quad (5.14)$$

It is obvious, that at \(\hbar = 0\) the formula (5.14) passes into the well known classical result

$$\gamma_k = -\frac{\nu}{2} - \sqrt{\frac{\pi}{8}} \omega_p \frac{k_D^3}{k^3} \exp\left(-\frac{3}{2} - \frac{k_D^2}{2k^2}\right). \quad (5.15)$$

Believing in (5.15) \(\nu = 0\), we come to the well known Landau damping formula

$$\gamma_k = -\sqrt{\frac{\pi}{8}} \omega_p \frac{k_D^3}{k^3} \exp\left(-\frac{3}{2} - \frac{k_D^2}{2k^2}\right). \quad (5.16)$$

6. Comparison with Mermin’s result

Mermin (see Mermin N.D. [22]) has been received the general expression of dielectric function

$$\varepsilon^M(\omega, k) = 1 + \frac{(\omega + i\nu) \left[\varepsilon^0(\omega + i\nu, k) - 1\right]}{\omega + i\nu \left[\varepsilon^0(\omega + i\nu, k) - 1\right]} \varepsilon^0(0, k) - 1. \quad (6.1)$$

The formula (3.1) is received on the basis of the kinetic equation for one-partial matrix of density \(\rho\)

$$\frac{\partial \rho}{\partial t} + i[\mathcal{E} + V, \rho] = \frac{\rho^l.e. - \rho}{\tau}$$

with relaxation time \(\tau\) and local equilibrium density matrix \(\rho^l.e.\), and \(\mathcal{E}\) is the kinetic energy of electrons, \(V\) is the self-consistent potential, \([\cdot, \cdot]\) is the
commutator, \( \rho_{l.e.} \) is the local equilibrium distribution function of electrons (l.e. \( \equiv \) local equilibrium),
\[
\rho = \frac{1}{1 + \exp(\varepsilon - \mu - \delta\mu)},
\]
\( \mu \) is the dimensionless (normalized) chemical potential.

Dielectric function is calculated in the momentum representation
\[
f(p, q, t) = \langle p + \frac{q}{2} | \rho | p - \frac{q}{2} \rangle,
\]
where \( | p \rangle \) is the eigen momentum state \( p \). For convenience we will transform the density matrix to Wigner's distribution in phase space
\[
f(p, R, t) = \frac{1}{(2\pi)^{3}} \int e^{i\cdot qR} f(p, q, t) d^{3}q,
\]
where \( R \) is the spatial coordinate.

In the formula (6.1) the designation \( \varepsilon^{\circ}(\omega, k) \) is entered, it is so-called Lindhard' dielectric function, i.e. the dielectric function received for collisionless plasmas, expression \( \varepsilon^{\circ}(\omega+i\nu, k) \) means, that argument of Lindhard' dielectric function \( \omega \) is replaced formally by \( \omega+i\nu \). According to (3.7)
\[
\varepsilon^{\circ}_{l}(\omega + i\nu, k) - 1 = \frac{2\omega_{p}^{2}}{k^{2}v_{T}^{2}2q} \left[ \frac{\omega + i\nu}{k^{}v_{T}} - \frac{q}{2} \right] - \frac{t(\omega + i\nu)}{k^{}v_{T}} + \frac{q}{2} \right] , \quad (6.2)
\]
\[
\varepsilon^{\circ}_{l}(0, k) - 1 = \frac{2\omega_{p}^{2}}{k^{2}v_{T}^{2}2q} \left[ t(-q/2) - t(q/2) \right]. \quad (6.3)
\]

For comparison we will write out the received dielectric function for collisional plasma (3.7) in the explicit form
\[
\varepsilon_{l}(k, \omega) = 1 + \frac{2\omega_{p}^{2}(\omega + i\nu)}{k^{2}v_{T}^{2}2q} \cdot \frac{t(\frac{\omega + i\nu}{k^{}v_{T}} - \frac{q}{2}) - t(\frac{\omega + i\nu}{k^{}v_{T}} + \frac{q}{2})}{\omega + i\nu\lambda_{0}(\frac{\omega + i\nu}{k^{}v_{T}})}, \quad (6.4)
\]
From formulas (6.2) and (6.3) we can see that
\[
\frac{\varepsilon^{\circ}(\omega + i\nu, k) - 1}{\varepsilon^{\circ}(0, k) - 1} = \frac{t(z - q/2) - t(z + q/2)}{t(-q/2) - t(q/2)}, \quad z = \frac{\omega + i\nu}{k^{}v_{T}}. \quad (6.5)
\]
By means of (6.2)–(6.5) Mermin’s formula (6.1) in our designations will be written in the form

$$\varepsilon^M = 1 + \frac{2\omega^2(\omega + i\nu)}{k^2v_T^2 2q} \cdot \frac{t(z - q/2) - t(z + q/2)}{\omega + i\nu \frac{t(z - q/2) - t(z + q/2)}{t(-q/2) - t(q/2)}}. \tag{6.6}$$

From (6.4) and (6.6) we can see that the received formula for dielectric function differs from the corresponding Mermin’ function by that in Mermin’s formula the relation

$$\frac{t(z - q/2) - t(z + q/2)}{t(-q/2) - t(q/2)}, \quad z = \frac{\omega + i\nu}{k v_T},$$

is necessary to replace with dispersion plasma function of Van Kampen $\lambda_0(z)$.

It is necessary to notice, that at small values of parameter $q$ the Mermin formula is close to true and passes into it when $\hbar \to 0$ (or $q \to 0$).

Indeed, it is required to show, that the following limiting transition is satisfied

$$\lim_{q \to 0} \frac{t(z - q/2) - t(z + q/2)}{t(-q/2) - t(q/2)} = \lambda_0(z).$$

Let’s take advantage of the equalities proved above

$$\lim_{q \to 0} \frac{t(z - q/2) - t(z + q/2)}{2q} = \frac{1}{2} \lim_{q \to 0} J_1(z, q) = \lambda_0(z),$$

$$t(-q/2) - t(q/2) = 2q e^{-q^2/4} \int_0^{q/2} e^{u^2} du.$$

By means of these equalities we receive that

$$\lim_{q \to 0} \frac{t(z - q/2) - t(z + q/2)}{t(-q/2) - t(q/2)} = \lim_{q \to 0} \frac{t(z - q/2) - t(z + q/2)}{2q(1 - q^2/6)} = \lambda_0(z).$$

Thus, at $q \to 0$ (or, that is all the same, $\hbar \to 0$ or $k \to 0$) Mermin’s formula and formula deduced in this work pass into the same dielectric function for classical Maxwellian plasma.

7. Conclusion
In the present work the exact expression for the longitudinal dielectric permittivity of non–degenerate Maxwellian plasma with the account of quantum effects is received. The kinetic Wigner – Vlasov – Boltzmann equation with the collision integral in the relaxation form of $\tau$–model in coordinate space is used.

It is shown, that in the limit, when Planck’s constant tends to zero, the received expression passes into the classical formula of the longitudinal dielectric permittivity of Maxwellian plasma. Various special cases of the dielectric permittivity are investigated. The Lindhard’s dielectric function for collisionless plasma is presented. It is shown that in the long-wave limit dielectric permittivity passes into the known Drude formula. Static limits ($\omega \to 0$) for dielectric permittivity for collisionless, and for collision
plasmas as well are found.

Plasma oscillations are studied in the long-wave limit and near to the plasma resonance. The expression for the plasma oscillation frequency in long-wave limit containing quantum parameter \( Q = \hbar \omega_p/(\kappa T) \) is received. Damping of the plasma oscillations is investigated. The formula for decrement of damping at \( \hbar \rightarrow 0 \) coincide with classical, and when collision frequency is equal to zero, the last formula passes into the known Landau damping formula.

Comparison with classical Mermin’s result for dielectric permittivity is presented. We will notice, that Mermin’s formula is received with use of the relaxation kinetic equation in space of momentum. For collisionless plasma the formula deduced in this work, and Mermin’s formula pass into
the same Lindhard’s formula.

On Figs. 1 and 2 graphic dependence on the quantity $q$ in the case $x_p = 1, y = 0.1$ for real (fig. 1) and imaginary (fig. 2) parts of longitudinal dielectric permittivity are presented. Curves of 1, 2, 3 correspond to values of parameter $x = 1, 0.7, 1.3$.

On Figs. 3 and 4 graphic dependence on the quantity $x$ in the case $x_p = 1, y = 0.1$ for real (fig. 3) and imaginary (fig. 4) parts of longitudinal dielectric permittivity are presented. Curves of 1, 2, 3 correspond to values of parameter $q = 0.5, 0.6, 0.7$.

On Figs. 5 and 6 graphic dependence on the quantity $x$ in the case $x_p = 10, y = 0.01, q = 1$ for real (fig. 5) and imaginary (fig. 6) parts of longitudinal dielectric permittivity are presented. Curves of 1 correspond to classical plasma, and curves 2 correspond to quantum plasma.

On fig. 7 and 8 graphic dependences for real (fig. 7) and imaginary (fig. 8) parts of longitudinal dielectric permittivity on the quantity $x$ for the case $x_p = 1, y = 0.01, q = 1$ are presented. Curves 1 correspond to quantum plasma, and curves 2 correspond to classical plasma.

On fig. 9 and 10 graphic dependences for real (fig. 9) and imaginary (fig. 10) parts of longitudinal dielectric permittivity are presented. These parts depend on quantity $x$ for the case $x_p = 1, y = 0.01, q = 0.5$. Curves 1 correspond to quantum plasma, and curves 2 correspond to classical plasma.
On fig. 11 and 12 graphic dependences for real (fig. 11) and imaginary (fig. 12) parts of longitudinal dielectric permittivity are presented. These parts depend on quantity $y$ ($10^{-5} < y < 10^{-1}$) for the case $x_p = 1$, $x = 1$, $q = 0.5$. Curves 1 correspond to quantum plasma, and curves 2 correspond to classical plasma.

On fig. 13 and 14 graphic dependences for real (fig. 13) and imaginary (fig. 14) parts of longitudinal dielectric permittivity are presented. These parts depend on quantity $q$ ($0 < y < 2.5$) for the case $x_p = 1$, $x = 1$, $y = 0.1$. Curves 1 correspond to quantum plasma, and curves 2 correspond to classical plasma.
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