Optimal frame designs for multitasking devices with energy restrictions

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Abstract

Let $d = (d_j)_{j \in I_m} \in \mathbb{N}^m$ be a finite sequence (of dimensions) and $\alpha = (\alpha_i)_{i \in I_n}$ be a sequence of positive numbers (weights), where $I_k = \{1, \ldots, k\}$ for $k \in \mathbb{N}$. We introduce the $(\alpha, d)$-designs i.e., families $\Phi = (F_j)_{j \in I_m}$ such that $F_j = \{f_{ij}\}_{i \in I_n}$ is a frame for $C_d$, $j \in I_m$, and such that the sequence of non-negative numbers $(\|f_{ij}\|^2)_{j \in I_m}$ forms a partition of $\alpha_i$, $i \in I_n$. We show, by means of a finite-step algorithm, that there exist $(\alpha, d)$-designs $\Phi^{op} = (F^{op}_j)$ that are universally optimal; that is, for every convex function $\varphi : [0, \infty) \to [0, \infty)$ then $\Phi^{op}$ minimizes the joint convex potential induced by $\varphi$ among $(\alpha, d)$-designs, namely

$$\sum_{j \in I_m} P_\varphi(F^{op}_j) \leq \sum_{j \in I_m} P_\varphi(F_j)$$

for every $(\alpha, d)$-design $\Phi = (F_j)_{j \in I_m}$, where $P_\varphi(F) = \text{tr}(\varphi(S_F))$; in particular, $\Phi^{op}$ minimizes both the joint frame potential and the joint mean square error among $(\alpha, d)$-designs. This corresponds to the existence of optimal encoding-decoding schemes for multitasking devices with energy restrictions.

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Contents

1 Introduction 2
2 Preliminaries 4
   2.1 Majorization 4
   2.2 Frames and convex potentials 5
3 On the optimal $(\alpha, d)$-design problem 7
   3.1 Modeling the problem 7
   3.2 Main results: existence and spectral structure 9
4 Algorithmic construction of an optimal weight partition 14
   4.1 The water-filling construction revisited 15
   4.2 The Algorithm 19
5 Some consequences of the algorithmic construction 26
   5.1 Proof of Theorem 3.8 26
   5.2 Spectral monotonicity 30

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1 Introduction

A finite sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ of vectors in $\mathbb{C}^d$ is a frame for $\mathbb{C}^d$ if $\mathcal{F}$ is a (possibly redundant) system of generators for $\mathbb{C}^d$. In this case, it is well known that there exist finite sequences $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ in $\mathbb{C}^d$ - the so-called duals of $\mathcal{F}$ - such that

$$f = \sum_{i \in \mathbb{I}_n} \langle f, f_i \rangle g_i = \sum_{i \in \mathbb{I}_n} \langle f, g_i \rangle f_i \quad \text{for} \quad f \in \mathbb{C}^d. \quad (1)$$

Thus, we can encode/decode the vector $f$ in terms of the inner products $\langle (f, f_i) \rangle_{i \in \mathbb{I}_n} \in \mathbb{C}^n$: (see [7, 12, 13] and the references therein). These redundant linear encoding-decoding schemes are of special interest in applied situations, in which there might be noise in the transmission channel: in this context, the linear relations between the frame elements can be used to produce simple linear tests to verify whether the sequence of received coefficients has been corrupted by the noise of the channel. In case the received coefficients are corrupted we can attempt to correct the sequence and obtain a reasonable (in some cases perfect) reconstruction of $f$ (see [6, 19]).

Given a finite sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ in $\mathbb{C}^d$, the frame operator $S_{\mathcal{F}} \in M_d(\mathbb{C})^+$ is given by

$$S_{\mathcal{F}} f = \sum_{i \in \mathbb{I}_n} \langle f, f_i \rangle f_i \quad \text{for} \quad f \in \mathbb{C}^d. \quad (2)$$

If $S_{\mathcal{F}}$ is invertible (i.e. if $\mathcal{F}$ is a frame) the canonical dual of $\mathcal{F}$ is given by $g_i = S_{\mathcal{F}}^{-1} f_i$ for $i \in \mathbb{I}_n$; this dual plays a central role in applications since it has several optimal (minimal) properties within the set of duals of $\mathcal{F}$. Unfortunately, the computation of the canonical dual depends on finding $S_{\mathcal{F}}^{-1}$, which is a challenging task from the numerical point of view. A way out of this problem is to consider those frames $\mathcal{F}$ for which $S_{\mathcal{F}}^{-1}$ is easy to compute (e.g. tight frames). In general, the numerical stability of the computation of $S_{\mathcal{F}}^{-1}$ depends on the spread of the eigenvalues of $S_{\mathcal{F}}$. In [4] Benedetto and Fickus introduced a convex functional called the frame potential of a sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ given by

$$\text{FP}(\mathcal{F}) = \sum_{i, j \in \mathbb{I}_n} \|\langle f_i, f_j \rangle\|^2 \geq 0. \quad (3)$$

In [4] the authors showed that under some normalization conditions, $\text{FP}(\mathcal{F})$ provides an scalar measure of the spread of the eigenvalues of $\mathcal{F}$. More explicitly, the authors showed that the minimizers of $\text{FP}$ among sequences $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ for which $\|f_i\| = 1$, $i \in \mathbb{I}_n$, are exactly the $n/d$-tight frames. It is worth pointing out that these minimizers are also optimal for transmission through noisy channels (in which erasures of the frame coefficients may occur, see [6, 19]).

In some applications of frame theory, we are drawn to consider frames $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ such that $\|f_i\|^2 = \alpha_i$, $i \in \mathbb{I}_n$, for some prescribed sequence $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{>0})^n$. In practice, we can think of frames with prescribed norms as designs for encoding-decoding schemes to be applied by a device with some sort of energy restrictions (e.g. a device with limited access to energy power): in this case, control of the norms of the frame elements amounts to control the energy needed to apply the linear scheme.

It is then natural to wonder whether there are tight frames with norms prescribed by $\alpha$. This question has motivated the study of the frame design problem (see [1] [8] [10] [11] [14] [15] [16] [20] and [17] [18] [22] [21] [24] [25] [20] for the more general frame completion problem with prescribed norms). It is well known that in some cases there are no tight frames in the class of sequences in $\mathbb{C}^d$ with norms prescribed by $\alpha$; in these cases, it is natural to consider minimizers of the frame potential within this class, since the eigenvalues of the frame operator of such minimizers have
minimal spread (thus, inducing more stable linear reconstruction processes). These considerations lead to the study of optimal designs with prescribed structure. In [9], the authors compute the structure of such minimizers and show it resembles that of tight frames.

It is worth pointing out that there are other measures of the spread of the spectra of frame operators (e.g. the mean squared error (MSE)). It turns out that both the MSE and the FP lie within the class of convex potentials introduced in [23]. It is shown in [23] that there are solutions \( F^{\text{op}} \) to the frame design problem which are structural in the sense that they are minimizers of every convex potential (e.g. MSE and FP) among frames with squared norms prescribed by \( \alpha \). A fundamental tool to show the existence of such structural optimal frame designs is the so-called majorization in \( \mathbb{R}^n \), which is a partial order used in matrix analysis (see [5]).

In the present paper we consider an extension of the optimal frame design problem as follows: given a finite sequence (of dimensions) \( d = (d_j)_{j \in \mathbb{I}_m} \in \mathbb{N}^m \) and a sequence of positive numbers (weights) \( \alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in \mathbb{R}^\odot_0 \), we consider the set \( D(\alpha, d) \) of \( (\alpha, d) \)-designs. i.e. sequences \( \Phi = (F_j)_{j \in \mathbb{I}_m} \) such that each \( F_j = \{f_{ij}\}_{i \in \mathbb{I}_n} \) is a frame for \( \mathbb{C}^{d_j} \), for \( j \in \mathbb{I}_m \) and such that

\[
\sum_{j \in \mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n. \tag{4}
\]

Notice that the restrictions on the norms above involve vectors in the (possibly different) spaces \( f_{ij} \in \mathbb{C}^{d_j} \) for \( j \in \mathbb{I}_m \). As in the case of frames with prescribed norms, \( (\alpha, d) \)-designs can be considered as encoding-decoding schemes to be applied by a multitasking device with some sort of energy restriction (e.g. due to isolation, or devices that are far from energy networks); in this context, the frames \( \Phi = (F_j)_{j \in \mathbb{I}_m} \) induce linear schemes in the spaces \( (\mathbb{C}^{d_j})_{j \in \mathbb{I}_m} \) that run in parallel. In this case, we want to control the overall energy needed (in each step of the encoding-decoding scheme) to apply simultaneously the \( m \) linear schemes, through the restrictions in Eq.(4). It is natural to consider those \( (\alpha, d) \)-designs that give rise to the more stable multitasking processes. In order the measure the overall stability of the family \( \Phi = (F_j)_{j \in \mathbb{I}_m} \) we can consider the joint frame potential of \( \Phi \) or the joint MSE of \( \Phi \) given by

\[
\text{FP}(\Phi) = \sum_{j \in \mathbb{I}_m} \text{FP}(F_j), \quad \text{MSE}(\Phi) = \sum_{j \in \mathbb{I}_m} \text{MSE}(F_j) \quad \text{respectively}.
\]

More generally, given a convex function \( \varphi : [0, \infty) \to [0, \infty) \) we introduce the joint convex potential \( P_\varphi(\Phi) \) induced by \( \varphi \) (see Section 3.1 for details; this family of convex potentials (that contains the joint frame potential and joint MSE) provides with natural measures of numerical stability of the family \( \Phi = (F_j)_{j \in \mathbb{I}_m} \). The main problem that we study in this paper is the construction of \( (\alpha, d) \)-designs that are optimal in \( D(\alpha, d) \) with respect to every joint convex potential. The kernel of this problem is the computation of optimal weight partitions, in the following sense: Consider the set of \( (\alpha, m) \)-weight partitions given by

\[
P_{\alpha, m} = \{ A \in M_{n,m}(\mathbb{R}^\geq_0) : A \mathbb{I}_m = \alpha \},
\]

where \( \mathbb{I}_m = (1, \ldots, 1) \in \mathbb{R}^m \). Given \( A \in P_{\alpha, m} \), consider the set of \( A \)-designs, given by

\[
D(A) = \{ \Psi = (F_j)_{j \in \mathbb{I}_m} \in D(\alpha, d) : (\|f_{ij}\|^2)_{i \in \mathbb{I}_n, j \in \mathbb{I}_m} = A \} \subseteq D(\alpha, d),
\]

which can be considered as a slice of \( D(\alpha, d) \). For each slice, a water-filling process works, and it produces the spectral structure (defined in Remark 3.5) of \( (\alpha, d) \)-designs that are minimizers in \( D(A) \) of every joint convex potential (see [23] or Theorem 2.6). These frames can be computed by a finite-step algorithm (see Remark 2.7).

In order to solve our main problem, we compute an optimal weight partition \( A_0 \in P_{\alpha, m} \) in terms of an iterative multi-water-filling process. Within the slice \( D(A_0) \), the previously mentioned minimizers are structural solutions to the optimal \( (\alpha, d) \)-designs, in the sense that their spectral structure
is majorized by those of sequences in the whole set $D(\alpha, d)$. We further obtain the uniqueness of the spectral structure of these universally optimal $(\alpha, d)$-designs (while the optimal $(\alpha, m)$-weight partitions $A_0 \in P_{\alpha, m}$ are not necessarily unique), and some monotonicity properties of the spectra of this optimal $(\alpha, d)$-designs with respect to the initial weights $\alpha = (\alpha_i)_{i \in I_n}$; thus, our results generalize the results in [4][9][24].

We point out that the existence of optimal $(\alpha, d)$-designs as above settles in the affirmative a conjecture in [2] Section 4.2.] regarding the existence of optimal finitely generated shift invariant systems (for a finitely generated shift invariant subspace of $L^2(\mathbb{R}^d)$) with norm restrictions, with respect to convex potentials (see also [3]).

Our approach to the existence of optimal $(\alpha, m)$-weight partitions and $(\alpha, d)$-designs is constructive. Indeed, we introduce a recursive finite-step algorithm that produces an optimal $(\alpha, m)$-weight partition, based on the existence of an associated optimal $(\alpha', m')$-weight partition of smaller order. Along the way we (inductively) show that the output of this algorithm has certain specific features, so that the recursive process is well defined. Moreover, we include several numerical examples of optimal $(\alpha, d)$-designs obtained with the implementation of our algorithm in MATLAB.

The paper is organized as follows. In Section 2 we recall the notion of majorization together with some fundamental results about this pre-order. We also include some notions and results related with finite frame theory and convex potentials. In Section 3 we formalize the notion of $(\alpha, m)$-weight partitions, $(\alpha, d)$-designs and describe in detail our main goals. In Section 3.2 we give a detailed description of our main results, that include the existence of (universal) optimal designs. In order to show this last result, we point out the existence of some special designs; the proof of the existence of such special designs is presented in Section 5.1 In Section 4.1 we establish some properties of the water-filling construction for vectors; in Section 4.2 we describe a recursive algorithm (based on the water-filling technique) that produces a particular $(\alpha, m)$-weight partition. In Section 5.1 we show that this particular $(\alpha, m)$-weight partition give rise to the special designs whose existence was claimed in Section 3.2 In Section 5.2 we obtain some further properties of the optimal $(\alpha, d)$-designs. The paper ends with Section 6 in which we present several numerical examples that exhibit the properties of the optimal $(\alpha, d)$-designs computed with a finite step algorithm.

2 Preliminaries

In this section we introduce the notation, terminology and results from matrix analysis and frame theory that we will use throughout the paper. General references for these results are the texts [5] and [7][12][13].

2.1 Majorization

In what follows we adopt the following

**Notation and terminology.** We let $\mathcal{M}_{k,d}(S)$ be the set of $k \times d$ matrices with coefficients in $S \subset \mathbb{C}$ and write $\mathcal{M}_{d,d}(\mathbb{C}) = \mathcal{M}_d(\mathbb{C})$ for the algebra of $d \times d$ complex matrices. We denote by $\mathcal{H}(d) \subset \mathcal{M}_d(\mathbb{C})$ the real subspace of selfadjoint matrices and by $\mathcal{M}_d(\mathbb{C})^+ \subset \mathcal{H}(d)$ the cone of positive semidefinite matrices. We let $\mathcal{U}(d) \subset \mathcal{M}_d(\mathbb{C})$ denote the group of unitary matrices. For $d \in \mathbb{N}$, let $I_d = \{1, \ldots, d\}$ and let $I_d = (1)_{i \in I_d} \in \mathbb{R}^d$ be the vector with all its entries equal to 1. Given $x = (x_i)_{i \in I_d} \in \mathbb{R}^d$ we denote by $x^\downarrow = (x_i^\downarrow)_{i \in I_d}$ (respectively $x^\uparrow = (x_i^\uparrow)_{i \in I_d}$) the vector obtained by rearranging the entries of $x$ in non-increasing (respectively non-decreasing) order. We denote by $(\mathbb{R}^d)^\downarrow = \{x^\downarrow : \ x \in \mathbb{R}^d\}$, $(\mathbb{R}_0^d)^\downarrow = \{x^\downarrow : \ x \in \mathbb{R}_0^d\}$ and analogously for $(\mathbb{R}^d)^\uparrow$ and $(\mathbb{R}_0^d)^\uparrow$. Given a matrix $A \in \mathcal{H}(d)$ we denote by $\lambda(A) = \lambda^\downarrow(A) = (\lambda_i(A))_{i \in I_d} \in (\mathbb{R}^d)^\downarrow$ the eigenvalues of $A$ counting
Notation and terminology

In what follows we adopt the following

2.2 Frames and convex potentials

Let $\mathcal{M}_{d,n}(\mathbb{C})$ denote the synthesis operator of $\mathcal{F}$ given by $T_{\mathcal{F}} \cdot (\alpha_i)_{i \in \mathbb{Z}_d} = \sum_{i \in \mathbb{Z}_d} \alpha_i f_i$. Then,

1. $T_{\mathcal{F}} \in \mathcal{M}_{d,n}(\mathbb{C})$ denotes the synthesis operator of $\mathcal{F}$ given by $T_{\mathcal{F}} \cdot (\alpha_i)_{i \in \mathbb{Z}_d} = \sum_{i \in \mathbb{Z}_d} \alpha_i f_i$.
2. $T_{\mathcal{F}}^* \in \mathcal{M}_{n,d}(\mathbb{C})$ denotes the analysis operator of $\mathcal{F}$ and it is given by $T_{\mathcal{F}}^* \cdot f = (\langle f, f_i \rangle)_{i \in \mathbb{Z}_d}$. 

Next we recall the notion of majorization between vectors, that will play a central role throughout our work.

**Definition 2.1.** Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^d$. We say that $x$ is submajorized by $y$, and write $x \prec_w y$, if

$$\sum_{i \in I_j} x_i^j \leq \sum_{i \in I_j} y_i^j \quad \text{for every} \quad 1 \leq j \leq \min\{n, d\}.$$ 

If $x \prec_w y$ and $\text{tr } x = \sum_{i \in \mathbb{Z}_d} x_i = \sum_{i \in \mathbb{Z}_d} y_i = \text{tr } y$, then $x$ is majorized by $y$, and write $x < y$. \triangle 

Given $x, y \in \mathbb{R}^d$ we write $x \leq y$ if $x_i \leq y_i$ for every $i \in \mathbb{Z}_d$. It is a standard exercise to show that $x \leq y \implies x^+ \leq y^+$.

**Remark 2.2.** Let $\gamma_1 \geq \ldots \geq \gamma_p \in \mathbb{R}$ and consider $\alpha = (\gamma_1 \mathbb{I}_{r_1}, \ldots, \gamma_p \mathbb{I}_{r_p}) = (\alpha_i)_{i \in \mathbb{Z}_p} \in (\mathbb{R}^+)^d$, where $r = \sum_{i \in \mathbb{Z}_p} r_i$. Set $s_k = \sum_{j \in \mathbb{Z}_k} r_j$, for $k \in \mathbb{Z}_p$. Given $\beta \in (\mathbb{R}^+)^d$ such that $\text{tr}(\alpha) = \text{tr}(\beta)$ then

$$\alpha \prec \beta \iff \sum_{i \in \mathbb{Z}_k} \gamma_i r_i \leq \sum_{j \in \mathbb{Z}_k} \beta_j \quad \text{for} \quad k \in \mathbb{Z}_{p-1}. \quad (5)$$

Indeed, if the right conditions hold and there exists $0 \leq k \leq p - 1$ with $s_k < t < s_{k+1}$ $(s_0 = 0)$ and such that $\sum_{j \in \mathbb{Z}_k} \beta_j > \sum_{j \in \mathbb{Z}_k} \beta_j$, it is easy to see that

$$\sum_{j=s_k+1}^{s_{k+1}} \beta_j < \sum_{j=s_k+1}^{s_{k+1}} \alpha_j = (t-s_k) \gamma_{k+1} \implies \beta_k < \gamma_{k+1} \implies \sum_{j \in \mathbb{Z}_{k+1}} \beta_j < \sum_{i \in \mathbb{Z}_{k+1}} \gamma_i r_i,$$

which contradicts our assumption $(5)$. Therefore $\alpha \prec \beta$. \triangle 

It is well known that majorization is intimately related with tracial inequalities of convex functions. The following result summarizes these relations (see for example [5]):

**Theorem 2.3.** Let $x, y \in \mathbb{R}^d$. If $\varphi : \mathcal{R} \rightarrow \mathbb{R}$ is a convex function defined on an interval $\mathcal{R} \subseteq \mathbb{R}$ such that $x, y \in \mathcal{R}$ then:

1. If $x < y$, then $\text{tr } \varphi(x) = \sum_{i \in \mathbb{Z}_d} \varphi(x_i) \leq \sum_{i \in \mathbb{Z}_d} \varphi(y_i) = \text{tr } \varphi(y).$
2. If only $x \prec_w y$, but $\varphi$ is an increasing convex function, then still $\text{tr } \varphi(x) \leq \text{tr } \varphi(y).$
3. If $x < y$ and $\varphi$ is a strictly convex function such that $\text{tr } \varphi(x) = \text{tr } \varphi(y)$ then, $x^+ = y^+$. \hfill \Box

### 2.2 Frames and convex potentials

In what follows we adopt the following

**Notation and terminology:** let $\mathcal{F} = \{f_i\}_{i \in \mathbb{Z}_n}$ be a finite sequence in $\mathbb{C}^d$. Then,

1. $T_{\mathcal{F}} \in \mathcal{M}_{d,n}(\mathbb{C})$ denotes the synthesis operator of $\mathcal{F}$ given by $T_{\mathcal{F}} \cdot (\alpha_i)_{i \in \mathbb{Z}_n} = \sum_{i \in \mathbb{Z}_n} \alpha_i f_i$.
2. $T_{\mathcal{F}}^* \in \mathcal{M}_{n,d}(\mathbb{C})$ denotes the analysis operator of $\mathcal{F}$ and it is given by $T_{\mathcal{F}}^* \cdot f = (\langle f, f_i \rangle)_{i \in \mathbb{Z}_n}$.
3. \( S_F \in M_d(\mathbb{C})^+ \) denotes the frame operator of \( F \) and it is given by \( S_F = T_F T_F^* \). Hence,

\[
S_F f = \sum_{i \in I_n} \langle f, f_i \rangle f_i = \sum_{i \in I_n} f_i \otimes f_i(f) \quad \text{for} \quad f \in \mathbb{C}^d.
\]

4. We say that \( F \) is a frame for \( \mathbb{C}^d \) if it spans \( \mathbb{C}^d \); equivalently, \( F \) is a frame for \( \mathbb{C}^d \) if \( S_F \) is a positive invertible operator acting on \( \mathbb{C}^d \). In this case we have the canonical reconstruction formula

\[
f = \sum_{i \in I_n} \langle f, S_F^{-1} f_i \rangle f_i = \sum_{i \in I_n} \langle f, f_i \rangle S_F^{-1} f_i \quad \text{for} \quad f \in \mathbb{C}^d
\]

in terms of the so-called canonical dual frame \( \{S_F^{-1} f_i\}_{i \in I_n} \).

In several applied situations it is desired to construct a finite sequence \( G = \{g_i\}_{i \in I_n} \subset (\mathbb{C}^d)^n \), in such a way that the spectra of the frame operator of \( G \) is given by some \( \lambda \in (\mathbb{R}_{\geq 0})^d \) and the squared norms of the frame elements are prescribed by a sequence of positive numbers \( \alpha = (\alpha_i)_{i \in I_n} \). This is known as the (classical) frame design problem and it has been studied by several research groups (see for example [11, 13, 14, 15, 16, 20]). The following result characterizes the existence of such frame designs in terms of majorization relations.

**Theorem 2.4 ([11, 22])**. Let \( \lambda \in (\mathbb{R}_{\geq 0})^d \) and consider \( \alpha = (\alpha_i)_{i \in I_n} \subset (\mathbb{R}_{> 0})^d \). Then there exists a sequence \( G = \{g_i\}_{i \in I_n} \subset \mathbb{C}^d \) with \( \|g_i\|^2 = \alpha_i \) for \( i \in I_n \) and such that \( \lambda(S_G) = \lambda \) if and only if \( \alpha \prec \lambda \).

The previous result shows the flexibility of structured frame designs, which is important in applied situations. Also, numerical stability of the encoding-decoding scheme induced by a frame plays a role in applications; hence, a central problem in this area is to describe the structured frame designs that maximize the stability of their encoding-decoding scheme. One of the most important (scalar) measures of stability is the so-called frame potential introduced by Benedetto and Fickus in [4] given by

\[
\text{FP}(F) = \sum_{i,j \in I_n} |\langle f_i, f_j \rangle|^2 = \text{tr}(S_F^2) \quad \text{for} \quad F = \{f_i\}_{i \in I_n} \subset (\mathbb{C}^d)^n.
\]

Benedetto and Fickus have shown that (under certain normalization conditions) minimizers of the frame potential induce the most stable encoding-decoding schemes. More generally, we can measure the stability of the scheme induced by the sequence \( F = \{f_i\}_{i \in I_n} \subset (\mathbb{C}^d)^n \) in terms of convex potentials. In order to introduce these potentials we consider the sets

\[
\text{Conv}(\mathbb{R}_{\geq 0}) = \{ \varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} : \varphi \text{ is a convex function} \}
\]

and \( \text{Conv}_s(\mathbb{R}_{\geq 0}) = \{ \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) : \varphi \text{ is strictly convex} \} \).

**Definition 2.5.** Following [23] we consider the convex potential \( P_\varphi \) associated to \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \), given by

\[
P_\varphi(F) = \text{tr} \varphi(S_F) = \sum_{i \in I_d} \varphi(\lambda_i(S_F)) \quad \text{for} \quad F = \{f_i\}_{i \in I_n} \subset (\mathbb{C}^d)^n,
\]

where the matrix \( \varphi(S_F) \) is defined by means of the usual functional calculus.

Convex potentials allow us to model several well known measures of stability considered in frame theory. For example, in case \( \varphi(x) = x^2 \) for \( x \in \mathbb{R}_{\geq 0} \) then \( P_\varphi \) is the Benedetto-Fickus frame potential; in case \( \varphi(x) = x^{-1} \) for \( x \in \mathbb{R}_{\geq 0} \) then \( P_\varphi \) is known as the mean squared error (MSE).
Going back to the problem of stable designs, it is worth pointing out the existence of structured designs that are optimal with respect to every convex potential. Indeed, given $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in \mathbb{R}_{\geq 0}^n$ and $d \in \mathbb{N}$ with $d \leq n$, let

$$B_{\alpha,d} = \{ \mathcal{F} = \{ f_i \}_{i \in \mathbb{I}_n} \in (\mathbb{C}^d)^n : \| f_i \|^2 = \alpha_i, \ i \in \mathbb{I}_n \}.$$  \hspace{1cm} (6)

We endow $B_{\alpha,d}$ (which is a product space) with the product metric. The structure of (local) minimizers of convex potentials in $B_{\alpha,d}$ has been extensively studied. The first results were obtained for the frame potential in $[4]$ and in a more general context in $[9]$. The case of general convex minimizers of convex potentials in $\mathbb{C}^d$ has been extensively studied. The first results were obtained in $[17, 18, 21, 22, 23, 24, 25, 26]$ (in some cases in the more general setting of frame completion problems with prescribed norms).

**Theorem 2.6** ([9, 23, 24, 25]). Let $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)\downarrow$ and let $d \in \mathbb{N}$ be such that $d \leq n$. Then, there exists $\gamma_{\alpha,d}^{op} = \gamma^{op} = (\gamma_i^{op})_{i \in \mathbb{I}_d} \in (\mathbb{R}_{\geq 0}^d)\downarrow$ such that:

1. There exist $\mathcal{F}_{\alpha,d}^{op} \in B_{\alpha,d}$ such that $\lambda(\mathcal{S}_{\mathcal{F}_{\mathcal{F}_{\alpha,d}}^{op}}) = \gamma^{op}$.

2. If $\# \{ i \in \mathbb{I}_n : \alpha_i > 0 \} \geq d$ then $\gamma_{\alpha,d}^{op} \in (\mathbb{R}_{\geq 0}^d)\downarrow$ (so $\mathcal{F}_{\alpha,d}^{op}$ is a frame for $\mathbb{C}^d$).

3. If $\mathcal{F}_{\alpha,d}^{op} \in B_{\alpha,d}$ is such that $\lambda(\mathcal{S}_{\mathcal{F}_{\alpha,d}^{op}}) = \gamma_{\alpha,d}^{op}$ then for every $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$ we have that

$$P_{\varphi}(\mathcal{F}_{\alpha,d}^{op}) \leq P_{\varphi}(\mathcal{F}) \quad \text{for every} \quad \mathcal{F} \in B_{\alpha,d}.$$  \hspace{1cm} (7)

4. If we assume further that $\varphi \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ and $\mathcal{F} \in B_{\alpha,d}$ is a local minimizer of $P_{\varphi} : B_{\alpha,d} \to \mathbb{R}_{\geq 0}$ ($B_{\alpha,d}$ endowed with the product metric) then $\lambda(\mathcal{S}_{\mathcal{F}}) = \gamma_{\alpha,d}^{op}$. \hspace{1cm} \text{□}

**Remark 2.7.** The vector $\gamma_{\alpha,d}^{op}$ of Theorem 2.6 can be described and computed by means of the so called water-filling construction of the vector $\alpha$ in dimension $d$ (see Definition 4.2). We shall study this construction with detail in subsection 4.1. In particular, we shall give a short proof of almost all items of Theorem 2.6 using the majorization properties of the water-filling construction (see Remark 4.5).

Once the vector $\gamma_{\alpha,d}^{op}$ is computed, we can apply the one-sided Bendel-Mickey algorithm (see [10, 11, 14, 19]) to compute $\mathcal{F}_{\alpha,d}^{op} \in B_{\alpha,d}$ i.e. a finite sequence of vectors in $\mathbb{C}^d$ with prescribed norms and prescribed spectra of its frame operator. \hspace{1cm} \text{△}

## 3 On the optimal $(\alpha, d)$-design problem

We begin this section by introducing notation and terminology that allow us to model the optimal design problem. Then, we give a detailed description of our main results, including the existence of optimal designs with norm restrictions.

### 3.1 Modeling the problem

Recall that given a finite sequence of non-negative real numbers $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in \mathbb{R}_{\geq 0}^n$ and $d \in \mathbb{N}$ with $d \leq n$, we consider

$$B_{\alpha,d} = \{ \mathcal{F} = \{ f_i \}_{i \in \mathbb{I}_n} \in (\mathbb{C}^d)^n : \| f_i \|^2 = \alpha_i, \ i \in \mathbb{I}_n \}.$$  \hspace{1cm} (6)

We now introduce some new notions

**Definition 3.1.** Let $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in \mathbb{R}_{\geq 0}^n$, $m \in \mathbb{N}$ and let $d = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)\downarrow$ be such that $d_1 \leq n$. We consider
1. the set of \((\alpha, m)\)-weight partitions given by
\[ P_{\alpha, m} = \{ A \in \mathcal{M}_{n,m}(\mathbb{R}_{\geq 0}) : A \mathbf{1}_m = \alpha \} . \]

2. the set of \((\alpha, d)\)-designs given by
\[ \mathcal{D}(\alpha, d) = \bigcup_{A \in P_{\alpha, m}} \prod_{j=1}^m B_{c_j(A), d_j} \]
where \( c_j(A) = (a_{ij})_{i \in \mathbb{I}_n} \in \mathbb{R}_{\geq 0}^n \) denotes the \( j \)-th column of \( A = (a_{ij})_{i \in \mathbb{I}_n, j \in \mathbb{I}_m} \), for \( j \in \mathbb{I}_m \).

**Remark 3.2.** Consider the notation and terminology of Definition 3.1. Notice that

1. Given \( A = (a_{ij})_{i \in \mathbb{I}_n, j \in \mathbb{I}_m} \in \mathcal{M}_{n,m}(\mathbb{C}) \) then
\[ A \in P_{\alpha, m} \iff a_{ij} \geq 0 \quad \text{and} \quad \sum_{j \in \mathbb{I}_m} a_{ij} = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n . \]

2. \( \mathcal{D}(\alpha, d) \) is the set of all finite sequences \((\mathcal{F}_j)_{j \in \mathbb{I}_m}\), where \( \mathcal{F}_j = \{ f_{ij} \}_{i \in \mathbb{I}_n, j \in \mathbb{I}_m} \in P_{\alpha, m} \), i.e.
\[ \sum_{j \in \mathbb{I}_m} \| f_{ij} \|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n . \]

We point out that (in order to simplify our description of the model) we consider \((\alpha, d)\)-designs in a broad sense; namely, if \( \Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, d) \) then \( \mathcal{F}_j \) is not necessarily a frame for \( \mathbb{C}^{d_j} \), for \( j \in \mathbb{I}_m \).

\[ \triangle \]

In order to compare the overall stability of the linear encoding-decoding schemes induced by an \((\alpha, d)\)-design we introduce the following

**Definition 3.3.** Let \( \alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in \mathbb{R}_{\geq 0}^n \), \( m \in \mathbb{N} \) and let \( d = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^+ \) be such that \( d_1 \leq n \).
Given \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \) we consider the joint potential induced by \( \varphi \) on \( \Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, d) \) given by
\[ P_\varphi(\Phi) = \sum_{j \in \mathbb{I}_m} P_\varphi(\mathcal{F}_j) \left( = \sum_{j \in \mathbb{I}_m} \text{tr} \left( \varphi(S_{\mathcal{F}_j}) \right) \right) = \sum_{j \in \mathbb{I}_m} \sum_{i \in d_j} \varphi(\lambda_i(S_{\mathcal{F}_j})) . \]

\[ \triangle \]

Consider the notation and terminology of Definitions 3.1 and 3.3. We can now describe the main problems that we consider in this work as follows:

**P1.** Show that there exist \((\alpha, d)\)-designs \( \Phi^{\text{op}} = (\mathcal{F}^{\text{op}}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, d) \) that are optimal in the following structural sense: for every \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \) then \( \Phi^{\text{op}} \) minimizes the joint convex potential \( P_\varphi \) in \( \mathcal{D}(\alpha, d) \), that is
\[ P_\varphi(\Phi^{\text{op}}) = \min \{ P_\varphi(\Phi) : \Phi \in \mathcal{D}(\alpha, d) \} . \quad (8) \]
In this case we say that \( \Phi^{\text{op}} \) is an \textit{optimal} \((\alpha, d)\)-design.

**P2.** Describe an algorithmic procedure that computes optimal \((\alpha, d)\)-designs.

**P3.** Characterize the optimal \((\alpha, d)\)-designs in terms of some structural properties.
P4. Study further properties of optimal \((\alpha, d)\)-designs.

We will solve problems P1.-P3. and study some (monotone) dependence of optimal \((\alpha, d)\)-designs on the initial weights \(\alpha\). In particular, we will show that is \(\Phi^{op} = (F^{op}_j)_{j \in \mathbb{I}_m}\) is an optimal \((\alpha, d)\)-design then, \(F^{op}_j\) is a frame for \(\mathbb{C}d_j\) for each \(j \in \mathbb{I}_m\) (see Section 3.2).

**Remark 3.4.** There is a reformulation of our problem in a more concise model. Let \(\alpha\) and \(d\) be as in Definition 3.1. Set \(|d| = \text{tr } d\) and assume that \(H = \mathbb{C}^{|d|} = \bigoplus_{j \in \mathbb{I}_m} H_j\) for some subspaces with \(\dim H_j = d_j\), for \(j \in \mathbb{I}_m\). Let us denote by \(P_j : H \to H_j \subset H\) the corresponding projections.

Notice that a sequence \(G = \{g_i\}_{i \in \mathbb{I}_m} \in \mathcal{B}_{\alpha, |d|} \subset \mathcal{H}^n \iff \) the sequence \(\Phi = (F_j)_{j \in \mathbb{I}_m}\) given by \(F_j = P_j(G)\) (i.e. \(f_{ij} = P_j(g_i) \in H_j \cong \mathbb{C}d_j\), \(i \in \mathbb{I}_n\)) for \(j \in \mathbb{I}_m\), satisfies that \(\Phi \in \mathcal{D}(\alpha, d)\).

Consider the pinching map \(C_d : \mathcal{M}_{|d|}(\mathbb{C}) \to \mathcal{M}_{|d|}(\mathbb{C})\) given by \(C_d(A) = \sum_{j \in \mathbb{I}_m} P_j A P_j\), for every \(A \in \mathcal{M}_{|d|}(\mathbb{C})\). Then, for each \(\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})\) we can define a \(d\)-pinched potential

\[
P_{\varphi, d}(\mathcal{G}) \overset{\text{def}}{=} \text{tr } \varphi(C_d(S_G)) \quad \text{for every} \quad \mathcal{G} \in \mathcal{H}^n ,
\]

which describes simultaneously the behavior of the projections of \(G\) to each subspace \(H_j\). Actually, with the previous notations,

\[
P_{\varphi, d}(\mathcal{G}) = \sum_{j \in \mathbb{I}_m} \text{tr } \varphi(P_j S_G P_j) = \sum_{j \in \mathbb{I}_m} P_{\varphi}(F_j) = P_{\varphi}(\Phi) .
\]

Therefore the problem of finding optimal \((\alpha, d)\)-designs (and studying their properties) translates to the study of sequences \(G \in \mathcal{B}_{\alpha, |d|}\) which minimize the \(d\)-pinched potentials \(P_{\varphi, d}\).

We point out that for \(\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})\) and \(\mathcal{G} \in \mathcal{H}^n\)

\[
P_{\varphi, d}(\mathcal{G}) \neq P_{\varphi}(\mathcal{G})
\]

in general, where \(P_{\varphi}(\mathcal{G}) = \text{tr } \varphi(S_G)\) (see Definition 2.5). Indeed, previous results related with the structure of minimizers of convex potentials in \(\mathcal{B}_{\alpha, |d|}\) (e.g. 23) do not apply to the \(d\)-pinched potential and we require a new approach to study this problem. \(\triangle\)

### 3.2 Main results: existence and spectral structure

In this section we give a detailed description of our main results; these include the existence of \((\alpha, d)\)-designs with an special spectral structure, which turn out to be optimal designs in the sense of Problem (P1). We further show the uniqueness of the spectral structure of optimal \((\alpha, d)\)-designs.

The following definition introduces a vector associated to every \((\alpha, d)\)-design, that allow us to prove the existence of optimal designs in terms of majorization relations (see Theorem 3.9 below).

**Definition 3.5.** Let \(\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0})^n\), \(m \in \mathbb{N}\) and let \(d = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}_{\geq 0})^m\) with \(d_1 \leq n\). Let \(\Phi = (F_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, d)\) and let \(S_j = S_{F_j} \in \mathcal{M}_{d_j}(\mathbb{C})^+\) denote the frame operators of \(F_j\), for \(j \in \mathbb{I}_m\). We define the vector

\[
\Lambda_{\Phi} = (\lambda(S_1), \ldots , \lambda(S_m)) \in \mathbb{R}_{\geq 0}^{|d|}
\]

where \(|d| = \sum_{j \in \mathbb{I}_m} d_j\) and \(\lambda(S_j) \in (\mathbb{R}_{\geq 0})^{d_j}\) denotes the vector of eigenvalues of \(S_j\), for \(j \in \mathbb{I}_m\). \(\triangle\)

**Remark 3.6.** Consider the notation in Definition 3.5. If \(\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})\) and \(P_{\varphi}\) denotes the joint convex potential induced by \(\varphi\) (see Definition 2.5), then,

\[
P_{\varphi}(\Phi) = \sum_{j \in \mathbb{I}_m} P_{\varphi}(F_j) = \sum_{j \in \mathbb{I}_m} \text{tr } (\varphi(\Lambda(S_j))) = \sum_{\ell \in \mathbb{I}_{|d|}} \varphi((\Lambda_{\Phi})_{\ell}) =: \text{tr } (\varphi(\Lambda_{\Phi})) . \tag{9}
\]
Therefore, by Theorem 2.3 and Eq. (9), the existence of an (optimal) \((\alpha, d)\)-design satisfying Eq. (8) for every \(\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})\) is equivalent to the existence of \(\Phi^0 = (\mathcal{F}^0_j)_{j \in I_m} \in \mathcal{D}(\alpha, d)\) such that
\[
\Lambda_{\Phi^0} \prec \Lambda_{\Phi} \quad \text{for every} \quad \Phi = (\mathcal{F}_j)_{j \in I_m} \in \mathcal{D}(\alpha, d).
\]

\[\triangle\]

**Remark 3.7.** Consider the notation in Definition 3.5. In what follows we show the existence of \((\alpha, d)\)-designs \(\Phi^{op} = \{\mathcal{F}^{op}_j\}_{j \in I_m}\) that are optimal with respect to every joint convex potential. It turns out that these optimal designs have some special features; indeed, if we let
\[
\gamma^{op}_j = (\gamma^{op}_{ij})_{i \in I_{dj}} = \lambda(S_{\mathcal{F}^{op}_j}) \in (\mathbb{R}^d_{\geq 0})^j
\]
denote the eigenvalues of the frame operators of \(\mathcal{F}^{op}_j\), for \(j \in I_m\), then:
\[
\gamma^{op}_{ij} = \gamma^{op}_{ik} \quad \text{for} \quad j \leq k, \ i \in I_k.
\]

We can picture this situation as follows:

![Diagram](image)

Figure 1: A graphic example of the structure of \((\gamma^{op}_j)_{j \in I_m}\) \((m = 4\) and \(d = (6, 5, 4, 2))\).

Let \(\sigma(S_{\mathcal{F}^{op}_j}) = \{\gamma_1, \ldots, \gamma_p\}\) be the distinct eigenvalues of \(S_{\mathcal{F}^{op}_j}\), such that \(\gamma_1 > \ldots > \gamma_p \geq 0\); let \(g_0 = 0 < g_1 < \ldots < g_p = d_1\) be such that
\[
\{i \in I_{d_1} : \gamma_{i1} = \gamma_{\ell}\} = \{i : g_{\ell-1} + 1 \leq i \leq g_{\ell}\} \quad \text{for} \quad \ell \in I_p,
\]
and let
\[
h_i := \#\{j \in I_m : d_j \geq i\} \quad \text{for} \quad i \in I_{d_1}.
\]

Then, using the relations in Eq. (10) we get that
\[
\Lambda_{\Phi^{op}}^\perp = (\gamma^{op}_j 1_{d_j})_{\ell \in I_p} \quad \text{where} \quad r_{\ell} = \sum_{i=g_{\ell-1}+1}^{g_{\ell}} h_i, \ \ell \in I_p.
\]
For example, if we consider the situation described in Figure 1 above, and assume that
\[
\gamma_{11}^{op} = \gamma_{21}^{op} = \gamma_{31}^{op} = \gamma_1 \quad \text{and} \quad \gamma_{41}^{op} = \gamma_{51}^{op} = \gamma_2 \quad \text{and} \quad \gamma_{61}^{op} = \gamma_3 \quad \text{with} \quad \gamma_1 > \gamma_2 > \gamma_3
\]
then we have: \( g_0 = 0, g_1 = 3, g_2 = 5 \) and hence, \( r_1 = 11, r_2 = 5, r_3 = 1 \); therefore, we compute \( \Lambda_{\Phi^{op}} = (\gamma_1 1_{11}, \gamma_2 1_5, \gamma_3 1_1) \in \mathbb{R}_{>0}^{17} \) in this case. \( \triangle \)

In our first main result we state the existence of \((\alpha, d)\)-designs with special features, as described in Remark 3.7 above.

**Theorem 3.8.** Let \( \alpha = (\alpha_i)_{i \in I_n} \in (\mathbb{R}^n_{>0})^\perp \), \( m \in \mathbb{N} \) and let \( d = (d_j)_{j \in I_m} \in (\mathbb{N}^m)^\perp \) with \( d_1 \leq n \). Consider \( h_i \), for \( i \in I_{d_1} \), as in Eq. (11). Then, there exist: \( p \in I_{d_1} \),

1. \( \gamma_1 > \ldots > \gamma_p > 0 \);
2. \( g_1, \ldots, g_p \in \mathbb{N} \) such that \( g_0 = 0 < g_1 < \ldots < g_p = d_1 \) and
3. \( \Phi^{op} = (F_j^{op})_{j \in I_m} \in \mathcal{D}(\alpha, d) \)

such that: if we let \( r_\ell = \sum_{i=g_{\ell-1}+1}^{g_\ell} h_i \), for \( \ell \in I_p \), then

\[
\begin{align*}
(a) & \quad \Lambda_{\Phi^{op}} \uparrow = (\gamma_\ell 1_{r_\ell})_{\ell \in I_p} \in (\mathbb{R}^{|d|}_{>0})^\perp; \\
(b) & \quad r_\ell \gamma_\ell = \sum_{i=g_{\ell-1}+1}^{g_\ell} \alpha_i, \quad \text{for} \quad \ell \in I_{p-1} \quad \text{and} \quad r_p \gamma_p = \sum_{i=g_{p-1}+1}^{n} \alpha_i.
\end{align*}
\]

In particular, \( F_j^{op} \) is a frame for \( \mathcal{C}^{d_j} \), for \( j \in I_m \). \( \square \)

We will develop the proof of Theorem 3.8 in Section 5.1 next, we derive several consequences from this result.

The following is our second main result.

**Theorem 3.9.** Let \( \alpha = (\alpha_i)_{i \in I_n} \in (\mathbb{R}^n_{>0})^\perp \), \( m \in \mathbb{N} \) and let \( d = (d_j)_{j \in I_m} \in (\mathbb{N}^m)^\perp \) with \( d_1 \leq n \). Let \( \Phi^{op} = (F_j^{op})_{j \in I_m} \in \mathcal{D}(\alpha, d) \) be as in Theorem 3.8. If \( \Phi = (F_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \), then

\[
\Lambda_{\Phi^{op}} < \Lambda_{\Phi},
\]

where \( \Lambda_{\Phi^{op}}, \Lambda_{\Phi} \in \mathbb{R}^{d} \) are as in Definition 3.5.

**Proof.** Let \( \Phi = (F_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \) be such that \( F_j = \{ f_{ij} \}_{i \in I_n} \), for \( j \in I_m \). Hence, by construction we have that

\[
\sum_{j \in I_m} \| f_{ij} \|^2 = \alpha_i \quad \text{for} \quad i \in I_n. \quad (12)
\]

On the other hand, we also have that

\[
\| f_{ij} \|^2 = \lambda(S_{f_j}) = (\lambda_{ij})_{i \in I_{d_j}} \quad \text{for} \quad j \in I_m.
\]

Hence, we conclude that

\[
\sum_{i \in I_n} \| f_{ij} \|^2 \leq \sum_{i=1}^{\min\{s, d_j\}} \lambda_{ij} \quad \text{for} \quad s \in I_n \quad \text{and} \quad j \in I_m. \quad (13)
\]

Let \( \Phi^{op} \in \mathcal{D}(\alpha, d) \) be as in Theorem 3.8; hence, we also consider \( p \in \mathbb{N}, \gamma_1 > \ldots > \gamma_p > 0, \) \( g_0 = 0 < g_1 < \ldots < g_p = d_1 \) and \( r_1, \ldots, r_p \in \mathbb{N} \) such that they satisfy properties (a) and (b) from this result.
We introduce the index set
\[ I_d = \{(i, j) : i \in I_{d_j}, j \in I_m\}. \]

Since \( \lambda_{\Phi}^{\uparrow} = (\gamma \lambda_{\Phi})_{i \in I_m} \), then, by Remark 2.2 in order to check the majorization relation \( \lambda_{\Phi}^{\uparrow} < \Lambda_{\Phi} \) it is sufficient to check that

\[ \sum_{i \in I_{d_i}} r_i \leq \sum_{(i, j) \in S(q)} \lambda_{ij} \quad \text{for some } S(q) \subset I_d, \quad \#S(q) = \sum_{i \in I_{d_i}} r_i \quad \text{for } q \in \mathbb{I}_{p-1}. \]  

(14)

For \( q \in \mathbb{I}_{p-1} \), let \( S(q) = \{(i, j) : 1 \leq i \leq \min\{g_q, d_j\}, j \in I_m\} \).

Using the relations
\[ r_i = \sum_{i = g_{\ell-1} + 1}^{g_{\ell}} h_i = \sum_{j \in I_m} \min\{g_{\ell}, d_j\} - g_{\ell-1})^+ \quad \text{for } \ell \in \mathbb{I}_p, \]

we see that
\[ S(q) = \bigcup_{\ell \in I_{d_{\ell}}} \{(i, j) : g_{\ell-1} + 1 \leq i \leq \min\{g_{\ell}, d_j\}, j \in I_m\} \implies \#S(q) = \sum_{\ell \in I_{d_{\ell}}} r_i. \]

Hence, using Eqs. (12) and (13) we have that
\[ \sum_{(i, j) \in S(q)} \lambda_{ij} = \sum_{j \in I_m} \sum_{i = 1}^{\min\{g_q, d_j\}} \lambda_{ij} \geq \sum_{j \in I_m} \sum_{i \in I_{d_{\ell}}} \|f_{ij}\|^2 = \sum_{i \in I_{d_{\ell}}} \alpha_i \]
\[ = \sum_{\ell \in I_{d_{\ell}}} \left( \sum_{i = g_{\ell-1} + 1}^{g_{\ell}} \alpha_i \right) = \sum_{\ell \in I_{d_{\ell}}} r_i \gamma_{\ell} \]

\[ \square \]

Theorem 3.9 together with the argument in Remark 3.6 allow us to obtain our third main result.

**Theorem 3.10.** Let \( \alpha = (\alpha_i)_{i \in I_m} \in (\mathbb{R}_+^n)^1 \), \( m \in \mathbb{N} \) and let \( d = (d_j)_{j \in I_m} \in (\mathbb{N}^m)^1 \) with \( d_1 \leq n \). Let \( \Phi^{\uparrow} = (\Phi_{j}^{\uparrow})_{j \in I_m} \in \mathcal{D}(\alpha, d) \) be as in Theorem 3.8. If \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \) then we have that

\[ P_{\varphi}(\Phi^{\uparrow}) \leq P_{\varphi}(\Phi) \quad \text{for every } \Phi = (\Phi_j)_{j \in I_m} \in \mathcal{D}(\alpha, d). \]  

(15)

Moreover, if \( \Phi = (\Phi_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \) is such that there exists \( \varphi \in \text{Conv}_{\alpha}(\mathbb{R}_{\geq 0}) \) for which equality holds in Eq. (15), then \( \lambda(S_{\Phi_j}) = \gamma_{\uparrow}^{\Phi} \), for \( j \in I_m \).

**Proof.** Let \( \Phi = (\Phi_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \) and let \( \Lambda_{\Phi} = (\lambda(S_{\Phi_j}))_{j \in I_m} \in \mathbb{R}_{\geq 0}^{d_j} \), where \( d = \sum_{j \in I_m} d_j \). If we let \( \Lambda_{\Phi^{\uparrow}} = (\lambda(S_{\Phi_{j}^{\uparrow}}))_{j \in I_m} \in \mathbb{R}_{\geq 0}^{d_j} \) then, by Theorem 3.9 and Remark 3.6 we get that for every \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \):

\[ \Lambda_{\Phi^{\uparrow}} < \Lambda_{\Phi} \implies P_{\varphi}(\Phi^{\uparrow}) = \sum_{j \in I_m} P_{\varphi}(\Phi_{j}^{\uparrow}) = \text{tr}(\varphi(\Lambda_{\Phi^{\uparrow}})) \leq \text{tr}(\varphi(\Lambda_{\Phi})) = \sum_{j \in I_m} P_{\varphi}(\Phi_j) = P_{\varphi}(\Phi). \]

Assume further that \( \varphi \in \text{Conv}_{\alpha}(\mathbb{R}_{\geq 0}) \) and \( \Phi = (\Phi_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \) is such that equality holds in Eq. (15). We introduce the set
\[ \mathcal{M} = \{\Lambda_{\Phi} : \Psi = (G_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \} \subset \mathbb{R}_{\geq 0}^{d_j}. \]
We claim that \( \mathcal{M} \) is a convex set: indeed, let \( t \in [0, 1] \) and \( \Psi = (\mathcal{G}_j)_{j \in I_m}, \Theta = (\mathcal{H}_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \) be such that \( \mathcal{G}_j = (g_{ij})_{i \in I_n} \) and \( \mathcal{H}_j = (h_{ij})_{i \in I_n} \). Set
\[
a_{ij} = \|g_{ij}\|^2 \quad \text{and} \quad b_{ij} = \|h_{ij}\|^2 \quad \text{for} \quad i \in I_n \quad \text{and} \quad j \in I_m.
\]
Further, set \( a_j = (a_{ij})_{i \in I_n}, b_j = (b_{ij})_{i \in I_n} \) and \( c_j = (c_{ij})_{i \in I_n} = t a_j + (1 - t) b_j \in \mathbb{R}^n \), for \( j \in I_m \). Using the convexity of \( P_{\alpha, m} \), we see that \( (c_{ij})_{i \in I_n, j \in I_m} \in P_{\alpha, m} \). On the other hand, if \( j \in I_m \) and \( S \subset \{1, \ldots, n\} \) is such that \( \# S = k \) then, if we let \( a_j^\downarrow \) and \( b_j^\downarrow \) denote the re-arrangements of \( a_j \) and \( b_j \) in non-increasing order, we get that
\[
\sum_{i \in S} c_{ij} = \sum_{i \in S} t a_{ij} + (1 - t) b_{ij} \leq \sum_{i \in I_n} t (a_j^\downarrow)_i + (1 - t) (b_j^\downarrow)_i \leq \sum_{i \in I_n} t \lambda_i(S_{\mathcal{G}_j}) + (1 - t) \lambda_i(S_{\mathcal{H}_j})
\]
\[
= \sum_{i \in I_n} (t \lambda(S_{\mathcal{G}_j}) + (1 - t) \lambda(S_{\mathcal{H}_j}))^\downarrow_i.
\]
This last fact shows that \( c_j \prec t \lambda(S_{\mathcal{G}_j}) + (1 - t) \lambda(S_{\mathcal{H}_j}) \) for \( j \in I_m \). Hence, Theorem 2.4 shows that for each \( j \in I_m \) there exists \( K_j = (k_{ij})_{i \in I_n} \in (\mathbb{C}^d)_n \) such that \( \|k_{ij}\|^2 = c_{ij} \), for \( i \in I_n \), and \( \lambda(S_{K_j}) = t \lambda(S_{\mathcal{G}_j}) + (1 - t) \lambda(S_{\mathcal{H}_j}) \). Therefore, \( \Pi = (K_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \) and \( t \Lambda_\Psi + (1 - t) \Lambda_\Theta = (t \lambda(S_{\mathcal{G}_j}) + (1 - t) \lambda(S_{\mathcal{H}_j}))_{j \in I_m} = \Lambda_\Pi \) and the claim follows.

We finally introduce \( F_\varphi : \mathcal{M} \to \mathbb{R}_{\geq 0}, F_\varphi(\Lambda) = tr(\varphi(\Lambda)) \) for \( \Lambda \in \mathcal{M} \). Since \( \varphi \) is strictly convex we immediately see that \( F \) - which is defined on the convex set \( \mathcal{M} \) - is strictly convex as well. Hence, there exists a unique \( \Lambda^{(\varphi)} \in \mathcal{M} \) such that
\[
F(\Lambda^{(\varphi)}) = \min\{F(\Lambda) : \Lambda \in \mathcal{M}\}.
\]
Notice that by hypothesis, we have that \( F(\Lambda_\Phi) = F(\Lambda_{\Phi^{op}}) = \min\{F(\Lambda) : \Lambda \in \mathcal{M}\} \) so then
\[
(\lambda(S_{\mathcal{F}_j}))_{j \in I_m} = \Lambda_\Phi = \Lambda^{(\varphi)} = \Lambda_{\Phi^{op}} = (\lambda(S_{\mathcal{F}_j^{op}}))_{j \in I_m}.
\]
\(\Box\)

**Remark 3.11.** Consider the notation in Theorem 3.10 as a consequence of this result, we see that the spectral structures of \( (\alpha, d) \)-designs that minimize a convex potential (induced by a strictly convex function) on \( \mathcal{D}(\alpha, d) \) coincide with that of \( \Phi^{op} \), so this spectral structure is unique. It is natural to wonder whether the \( (\alpha, m) \)-weight partitions corresponding to such minimizers also coincide. It turns out that this is not the case; indeed, consider the following example: let \( \alpha = 1_6 \in (\mathbb{R}^6_{\geq 0})^\downarrow, m = 2 \) and let \( d = (4, 2) \in \mathbb{N}^2 \).

On the one hand, we consider the weight partition given by
\[
a_1^1 = \frac{4}{6} 1_6 \in (\mathbb{R}^6_{\geq 0})^\downarrow \quad \text{and} \quad a_2^1 = \frac{2}{6} 1_6 \in (\mathbb{R}^6_{\geq 0})^\downarrow.
\]
In this case, the water-filling of the weights in the corresponding dimensions are
\[
\gamma_1^1 = 1_4 \quad \text{and} \quad \gamma_2^1 = 1_2.
\]
Thus, we can construct \( \Phi^1 = (\mathcal{F}_1^1, \mathcal{F}_1^1) \in \mathcal{D}(\alpha, 2) \) in such a way that \( \mathcal{F}_j^{op} \) is a Parseval frame for \( \mathbb{C}^d, \ j = 1, 2 \).

On the other hand, we can consider the weight partition \( a_1^2 = (1, 1, 1, 0, 0) \) and \( a_2^2 = (0, 0, 0, 0, 1, 1) \); moreover, if we let \( \{e_i^{(k)}\}_{i \in I_k} \) denote the canonical basis of \( \mathbb{C}^k \) for \( k \in \mathbb{N} \) and let
\[
\mathcal{F}_1^2 = \{e_1^{(4)}, \ldots, e_4^{(4)}, 0, 0\} \in (\mathbb{C}^4)^6 \quad \text{and} \quad \mathcal{F}_2^2 = \{0, 0, 0, 0, e_1^{(2)}, e_2^{(2)}\} \in (\mathbb{C}^2)^6
\]
then \( \|f_{ij}\|_2^2 = (a_j^i)^2 \) for \( i \in I_6, j = 1, 2 \). Hence, \( \Phi^2 := (\mathcal{F}_{j_1}^2, \mathcal{F}_{j_2}^2) \in \mathcal{D}(\alpha, 2) \) and is such that \( \lambda(S_{\mathcal{F}_j}) = \lambda(S_{\mathcal{F}_j}) \) for \( j = 1, 2 \). Thus, in this case, we have that

\[
\Lambda_{\Phi_1} = \Lambda_{\Phi_2} = \mathbb{1}_6 < \Lambda_{\Phi} \quad \text{for every} \quad \Phi \in \mathcal{D}(\alpha, 2).
\]

Therefore, for every \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \) we have that

\[
P_{\varphi}(\Phi_1) = P_{\varphi}(\Phi_2) \leq P_{\varphi}(\Phi) \quad \text{for every} \quad \Phi \in \mathcal{D}(\alpha, 2).
\]

Nevertheless, \( a_1^j \neq a_2^j \) for \( j = 1, 2 \). That is, these two different weight partitions generate optimal \((\alpha, 2)\)-designs. Thus, weight partitions inducing optimal \((\alpha, 2)\)-designs are not unique. As a final comment, let us mention that the \((\alpha, 2)\)-designs \( \Phi^1 \) and \( \Phi^2 \) are qualitatively different. \( \triangle \)

4 Algorithmic construction of an optimal weight partition

In this section we develop a finite step algorithm that computes a distinguished weight partition. This algorithm will be the key for proving Theorem 3.8 in the next section.

We begin with the following general remarks on our approach to the proof of Theorem 3.8. Let \( \alpha = (\alpha_i)_{i \in I_m} \in (\mathbb{R}_{>0})^\uparrow \) and \( m \in \mathbb{N} \) with \( m \geq 1 \) be given; consider \( d = (d_j)_{j \in I_m} \in (\mathbb{N}^m)^\downarrow \) such that \( d_1 \leq n \). Let \( \lambda^j = (\lambda^j_{i,j})_{i \in d_j} \in (\mathbb{R}_{>0})^\uparrow \), for \( j \in I_m \). Then, Theorem 2.4 provides with a characterization of when there exist \( \Phi = (\mathcal{F}_j)_{j \in I_m} \in \mathcal{D}(\alpha, d) \) such that \( \lambda(S_{\mathcal{F}_j}) = \lambda^j \), for \( j \in I_m \): indeed, the existence of such \( \Phi \in \mathcal{D}(\alpha, d) \) is equivalent to the existence of a \((\alpha, m)\)-weight partition \( A \in P_{\alpha, m} \) such that

\[
c_j(A) < \lambda^j \quad \text{for} \quad j \in I_m,
\]

where \( c_j(A) = (a_{i,j})_{i \in d_j} \) denotes the \( j \)-th column of the matrix \( A \). Nevertheless, determining (in an effective way) the existence of such \((\alpha, m)\)-weight partition \( A \in P_{\alpha, m} \) is a hard problem, in general. Hence, although Theorem 3.8 contains a (partial) description of the spectral structure of the sequences \( \Phi^\text{op} = (\mathcal{F}^\text{op}_j)_{j \in I_m} \) that we want to construct, we cannot expect to use this spectral information to conclude the existence of \( \Phi^\text{op} \in \mathcal{D}(\alpha, d) \).

Our proof of Theorem 3.8 is based on the construction of a distinguished \((\alpha, m)\)-weight partition \( A^\text{op} \in P_{\alpha, m} \). The construction of \( A^\text{op} \) (in terms of a recursive and finite step algorithm) is done in such a way that we can keep track of the water-filling of \( c_j(A^\text{op}) \) in dimension \( d_j \), for each \( j \in I_m \). (see Remark 2.7 and the next remark).

Remark 4.1. Let \( \alpha = (\alpha_i)_{i \in I_m} \in (\mathbb{R}_{>0})^\uparrow \) and \( m \in \mathbb{N} \) with \( m \geq 1 \) be given; consider \( d = (d_j)_{j \in I_m} \in (\mathbb{N}^m)^\downarrow \) such that \( d_1 \leq n \). Let \( A = (a_{i,j})_{i \in I_m, j \in I_m} \in P_{\alpha, m} \) be a fixed \((\alpha, m)\)-weight partition. We can consider the set of \( A \)-designs given by

\[
\mathcal{D}(A) = \bigcap_{j=1}^m B_{c_j(A), d_j}
\]

where \( c_j(A) = (a_{i,j})_{i \in d_j} \in \mathbb{R}_{>0}^\uparrow \) denotes the \( j \)-th column of \( A \), for \( j \in I_m \). Notice that \( \mathcal{D}(A) \subset \mathcal{D}(\alpha, d) \) can be considered as a slice of \( \mathcal{D}(\alpha, d) \). By Theorem 2.6, for each \( j \in I_m \) there exists \( \mathcal{F}_j^0 \in B_{c_j(A), d_j} \) that can be computed by a finite-step algorithm (see Remark 2.7), such that for every \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \),

\[
P_{\varphi}(\mathcal{F}_j^0) \leq P_{\varphi}(\mathcal{F}_j) \quad \text{for every} \quad \mathcal{F}_j \in B_{c_j(A), d_j}.
\]

Hence, \( \Phi^0 := (\mathcal{F}_j^0)_{j \in I_m} \in \mathcal{D}(A) \) is such that for every \( \varphi \in \text{Conv}(\mathbb{R}_{\geq 0}) \),

\[
P_{\varphi}(\Phi^0) = \sum_{j \in I_m} P_{\varphi}(\mathcal{F}_j^0) \leq \sum_{j \in I_m} P_{\varphi}(\mathcal{F}_j) = P_{\varphi}(\Phi) \quad \text{for every} \quad \Phi = (\mathcal{F}_j)_{j \in I_m} \in \mathcal{D}(A).
\]
That is, once we fix $A \in P_{\alpha,m}$ then there is an structural solution $\Phi^0$ for the optimization of convex potentials in (the slice) $D(A)$. Moreover, for each $j \in \mathbb{I}_m$ the vector $\lambda(S_{j\alpha})$ coincides with the water-filling of $c_j(A) \in \mathbb{R}^n$ in dimension $d_j$ (see Remark 2.7).

The previous comments show that the problem of computing optimal structural $(\alpha, d)$-designs $\Phi^{op}$ (as in Eq. (8)) reduces to the problem of finding optimal $(\alpha, m)$-weight partitions $A^{op} \in P_{\alpha,m}$ - in terms of a finite-step algorithm - in the sense that the structural optimal solution $\Phi^{op} \in D(A^{op})$ (as described above) is an structural solution for the optimization of convex potentials in the set $D(\alpha, d)$ of all $(\alpha, d)$-designs. Notice that the structural solution $\Phi^{op} = \{ \gamma^{op}_j \}_{j \in \mathbb{I}_m}$ is characterized in terms of the spectra $\lambda(S_{j\alpha}^{op})$ which are obtained by water-filling of the vectors $c_j(A^{op})$ in dimension $d_j$, for $j \in \mathbb{I}_m$. Thus, in order to warrant the optimality properties of $A^{op}$, we will construct $A^{op}$ in such a way that we keep track of the water-fillings of its columns. Therefore, in the next section we develop some properties of the water-filling construction.

**Theorem 4.3 (23, 24)**. Let $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$, let $d \in \mathbb{N}$ be such that $n \geq d$, and let $\gamma \in (\mathbb{R}^d_{\geq 0})^\downarrow$ denote the water-filling of $\alpha$ in dimension $d$. Then,

1. $\alpha \prec \gamma$;
2. If $\beta \in \mathbb{R}^d_{\geq 0}$ is such that $\alpha \prec \beta$ then $\gamma \prec \beta$. 

**4.1 The water-filling construction revisited**

In this section, we develop some properties of the water-filling construction that we will need in the sequel. Indeed, consider the optimal $(\alpha, d)$-design problem in case $m = 1$. Hence, we let $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ and let $d = (d_1) \in \mathbb{N}$ be such that $d_1 \leq n$. In this case

$$D(\alpha, d) = B_{\alpha, d_1} \subset (\mathbb{S}^{d_1})^n,$$

and the existence of optimal $(\alpha, d_1)$-designs is a consequence of Theorem 2.6. In order to give an explicit description of the vector $\gamma^{op}_{\alpha, d}$ in Theorem 2.6 we introduce the following construction, that will also play a central role in our present work.

**Definition 4.2** (The water-filling construction). Let $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ and let $d \in \mathbb{N}$ be such that $d \leq n$. We define the water-filling of $\alpha$ in dimension $d$ as the vector

$$\gamma = (\max\{\alpha_i, c\})_{i \in \mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow$$

where $c \geq \alpha_d$ is uniquely determined by $\operatorname{tr} \gamma = \operatorname{tr} \alpha$.

In this case we say that $c$ is the water-level of $\gamma$; notice that $c$ is determined by the equation

$$\sum_{i \in \mathbb{I}_d} \max\{\alpha_i, c\} = \sum_{i \in \mathbb{I}_n} \alpha_i,$$

or equivalently by the equation

$$\sum_{i \in \mathbb{I}_d} (c - \alpha_i)^+ = \sum_{i = d+1}^{n} \alpha_i. \quad (18)$$

In Figure 2 there is a graphic description of the water-filling construction using the vector $a = (10, 8.5, 7, 5, 3.8, 3.8, 2.4, 2, 1.7, 0.8)$ and dimension $d = 6$. Notice that Eq. (18) means that the striped regions have the same area.

The spectra of optimal $(\alpha, d_1)$-designs is computed in terms of the water-filling construction (see Remark 2.5). Hence, it is not surprising that the water-filling construction plays a key role in our construction of optimal $(\alpha, d)$-designs for $m > 1$ (and general $d = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$). Thus, in this section we explore some properties of this construction that we will use in the next section. One of the main motivation for considering the water-filling comes from its relation with majorization.

**Theorem 4.3 (23, 24)**. Let $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$, let $d \in \mathbb{N}$ be such that $n \geq d$, and let $\gamma \in (\mathbb{R}^d_{\geq 0})^\downarrow$ denote the water-filling of $\alpha$ in dimension $d$. Then,

1. $\alpha \prec \gamma$;
2. If $\beta \in \mathbb{R}^d_{\geq 0}$ is such that $\alpha \prec \beta$ then $\gamma \prec \beta$. 

\[ \square \]
Remark 4.4. Theorem 4.3 can be deduced from Definition 4.2 and Remark 2.2. Indeed, notice that the water-filling of $\alpha$ in dimension $d$ is, by construction, a particular case of a vector with a structure as described in Remark 2.2 with $d_1 = d_2 = \ldots = d_{p-1} = 1$ and $d_p = d - p + 1$ (here, $p - 1$ would be $p - 1 = \max\{i: \alpha_i \geq c\}$). Thus, it is easy to see that any vector $\beta \in \mathbb{R}^d_{\geq 0}$ such that $\alpha \prec \beta$ satisfies the corresponding inequalities of Eq. (5). Therefore, item 2 follows by Remark 2.2. On the other hand, item 1 follows from the definition of majorization. △

Remark 4.5. In order to show how the water-filling/majorization interacts with the optimal frame design problems, we give a short proof of items 1. and 3. in Theorem 2.6 in terms of Theorems 2.3, 2.4 and 4.3 (notice that the proof of item 2. in Theorem 2.6 is a direct consequence of the water-filling construction). Indeed, consider the notation of Theorem 2.6: in this case, if we let $\gamma^{\text{op}}$ denote the water-filling of $\alpha$ in dimension $d$ then the first item in Theorem 4.3 together with Theorem 2.4 show that there exists $F^{\text{op}} \in B_{\alpha,d}$ such that $\lambda(SF^{\text{op}}) = \gamma^{\text{op}}$. Moreover, if $F \in B_{\alpha,d}$ then Theorem 2.4 shows that $\alpha \prec \lambda(SF)$ so $\lambda(SF^{\text{op}}) \prec \lambda(SF)$ by Theorem 4.3 now we see that Eq. (7) follows from Theorem 2.3 (and Definition 2.5). The spectral structure of local minimizers of strictly convex potentials is a more delicate issue (see [25]). Nevertheless, we can show the uniqueness of the spectral structure of global minimizers of strictly convex potentials as follows: assume that $F \in B_{\alpha,d}$ is such that $P_\varphi(F^{\text{op}}) = P_\varphi(F)$. Then the equality $\lambda(SF) = \gamma^{\text{op}}$ is a consequence of the majorization relation $\lambda(SF^{\text{op}}) \prec \lambda(SF)$ and Theorem 2.3. △

Figure 2: Water-filling with $\alpha = (10, \ 8.5, \ 7, \ 5, \ 3.8, \ 3.8, \ 2.4, \ 2, \ 1.7, \ 0.8)$ and $d = 6$

In what follows we state and prove several properties of the water-filling construction that we will need in the next subsection.

Proposition 4.6. Let $\alpha = (\alpha_i)_{i \in I_n} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ and let $d \in \mathbb{N}$ be such that $d \leq n$. Then

1. Let $t \geq 0$ and $\gamma(t) \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ denote the water-filling of $t \cdot \alpha$ in dimension $d$. Then, we have that $\gamma(t) = t \cdot \gamma(1)$.

2. Let $\beta = (\beta_i)_{i \in I_n} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ be such that $\alpha_i \geq \beta_i$ for $i \in I_n$. If $\gamma = (\gamma_i)_{i \in I_d}$ and $\delta = (\delta_i)_{i \in I_d}$ denote the water-fillings in dimension $d$ of $\alpha$ and $\beta$ respectively, then $\gamma_i \geq \delta_i$ for $i \in I_d$.
3. Assume that \( d' \in \mathbb{N} \) is such that \( d' \leq d \). If there exists \( c \in \mathbb{R}_{>0} \) such that the water-filling of \( \alpha \) in dimension \( d \) is \( c \cdot \mathbb{L}_d \) then there exists \( c' \geq c \) such that the water-filling of \( \alpha \) in dimension \( d' \) is \( c' \cdot \mathbb{L}_{d'} \).

**Proof.**
1. The case \( n = d \) is trivial so we assume that \( d < n \). Let \( \gamma = \gamma(1) \) be the water-filling of \( \alpha \) in dimension \( d \). Hence there exists a unique \( c \geq \alpha_d \) such that \( \gamma = (\max\{\alpha_i, c\})_{i \in \mathbb{L}_d} \) where

\[
\sum_{i=d+1}^{n} \alpha_i = \sum_{i \in \mathbb{L}_d} (c - \alpha_i) = \sum_{i=r}^{d} c - \alpha_i,
\]

for \( r = \min\{j \in \mathbb{L}_d : \ c \geq \alpha_j\} \in \mathbb{L}_d \). Hence, for \( t > 0 \) we see that

\[
\sum_{i=d+1}^{n} t \alpha_i = \sum_{i=r}^{d} t c - t \alpha_i = \sum_{i \in \mathbb{L}_d} (t c - t \alpha_i),
\]

since \( r = \min\{j \in \mathbb{L}_d : \ t c \geq t \alpha_j\} \in \mathbb{L}_d \). Therefore, \( \gamma(t) = (\max\{t \alpha_i,\ t c\})_{i \in \mathbb{L}_d} = t \gamma \). Notice that in case \( t = 0 \) the result is trivial.

2. The case \( n = d \) is trivial so we assume that \( d < n \). By construction, there exists \( c \geq \alpha_d \) and \( e \geq \beta_d \) such that \( \gamma = (\max\{\alpha_i, e\})_{i \in \mathbb{L}_d} \) and \( \delta = (\max\{\beta_i, e\})_{i \in \mathbb{L}_d} \) where

\[
\sum_{i \in \mathbb{L}_d} (c - \alpha_i) = \sum_{i=d+1}^{n} \alpha_i \quad \text{and} \quad \sum_{i \in \mathbb{L}_d} (e - \beta_i) = \sum_{i=d+1}^{n} \beta_i.
\]

Assume that \( e > c \).

If we assume that \( e > \beta_d \) then

\[
\sum_{i=d+1}^{n} \beta_i = \sum_{i \in \mathbb{L}_d} (e - \beta_i) > \sum_{i \in \mathbb{L}_d} (c - \beta_i) \geq \sum_{i \in \mathbb{L}_d} (c - \alpha_i) = \sum_{i=d+1}^{n} \alpha_i \geq \sum_{i=d+1}^{n} \beta_i,
\]

which is a contradiction.

If we assume that \( e = \beta_d \) then

\[
0 = \sum_{i=d+1}^{n} \beta_i = \sum_{i \in \mathbb{L}_d} (e - \beta_i) \geq \sum_{i \in \mathbb{L}_d} (e - \alpha_i) \geq \sum_{i \in \mathbb{L}_d} (c - \alpha_i) = \sum_{i=d+1}^{n} \alpha_i \geq 0.
\]

Hence \( c = \alpha_d \) and \( e = \beta_d \leq \alpha_d = c \), which contradicts our previous assumption.

Hence, we conclude that \( e \leq c \) and therefore \( \delta_i = \max\{\beta_i, e\} \leq \max\{\alpha_i, c\} = \gamma_i \), for \( i \in \mathbb{L}_d \).

3. Notice that \( d' \leq d \leq n \). On the other hand,

\[
\sum_{i \in \mathbb{L}_d} \max\{\alpha_i, c\} = \sum_{i=d+1}^{n} \alpha_i \implies \sum_{i \in \mathbb{L}_{d'}} \max\{\alpha_i, c\} \leq \sum_{i=d'+1}^{n} \alpha_i.
\]

Therefore, if \( \delta = (\max\{\alpha_i, c'\})_{i \in \mathbb{L}_{d'}} \) is the water-filling of \( \alpha \) in dimension \( d' \) we see that \( c' \geq c \) and hence \( \delta = c' \mathbb{L}_{d'} \), since \( c' \geq c \geq \alpha_i \) for \( i \in \mathbb{L}_d \). \( \square \)

**Definition 4.7.** Let \( a' = (a'_i)_{i \in \mathbb{n}_n} \in (\mathbb{R}_{\geq 0}^n)^i \) and let \( \gamma' = (\gamma'_i)_{i \in \mathbb{L}_d} \) be its water-filling in dimension \( d \leq n \), with water-level \( c' \). We define the functions \( \alpha_i(t) : [0, \gamma'_i] \to [0, \gamma'_i] \) for \( i \in \mathbb{n}_n \) as follows:

\[
a_i(t) = \frac{\min\{t, c'\}}{c'} \min\{a'_i, \max\{t, c'\}\}.
\]

Notice that \( a(t) := (a_i(t))_{i \in \mathbb{n}_n} \in (\mathbb{R}_{\geq 0}^n)^{\mathbb{L}_d} \), for \( t \in [0, \gamma'_i] \). \( \triangle \)
In the next section, we will make use of the functions \( a_i(t) \) introduced in Definition 4.7 above to build an algorithm that constructs \((\alpha, m)\)-weight partitions (see Algorithm 4.10). Thus, we study some of the elementary properties of these functions.

**Lemma 4.8.** Consider the notation of Definition 4.7. If we let \( \gamma(t) \in (\mathbb{R}_{\geq 0}^d)^{\downarrow} \) denote the water-filling of \( a(t) \in (\mathbb{R}_{\geq 0}^n)^{\downarrow} \) in dimension \( d \), then

\[
\gamma(t) = (\min\{\gamma_i', t\})_{i \in \mathbb{I}_d} \quad \text{for} \quad t \in [0, \gamma'_1].
\]

**Proof.** Since \( \gamma' \) is the water-filling of \( a' \) in dimension \( d \), then

\[
\gamma' = (\max\{a'_i, c'\})_{i \in \mathbb{I}_d} \in (\mathbb{R}_{\geq 0}^d)^{\downarrow} \quad \text{for} \quad c' \geq a'_d \quad \text{such that} \quad \sum_{i=d+1}^{n} (c' - a'_i) = \sum_{i=d+1}^{n} a'_i.
\]

On the other hand, if we let \( \gamma(t) \in (\mathbb{R}_{\geq 0}^d)^{\downarrow} \) denote the water-filling of \( a(t) \) in dimension \( d \) then

\[
\gamma(t) = (\max\{a_i(t), c(t)\})_{i \in \mathbb{I}_d} \quad \text{for} \quad c(t) \geq a_d(t) \quad \text{such that} \quad \sum_{i=d+1}^{n} (c(t) - a_i(t)) = \sum_{i=d+1}^{n} a_i(t).
\]

Then, considering Definition 4.7

1. If \( c' \leq t \leq \gamma'_1 \), then \( a_i(t) = \min\{a'_i, t\} \), for \( i \in \mathbb{I}_n \). Hence \( c(t) = c' \) and
   \[
   \gamma(t) = (\max\{a_i(t), c'\})_{i \in \mathbb{I}_d} = (\max\{\min\{a'_i, t\}, c'\})_{i \in \mathbb{I}_d} = (\min\{\gamma'_i, t\})_{i \in \mathbb{I}_d}.
   \]

2. If \( 0 \leq t < c' \), then \( a_i(t) = \frac{t}{c'} \min\{a'_i, c'\} \), for \( i \in \mathbb{I}_n \). If \( \delta \in \mathbb{R}^d \) denotes the water-filling of \( b = (\min\{a'_i, c'\})_{i \in \mathbb{I}_n} \) then it is clear that \( \delta = \frac{t}{c'} I_d \). Since \( \gamma(t) \) coincides with the water-filling of \( \frac{t}{c'} b \) then, by Proposition 4.6, we see that
   \[
   \gamma(t) = \frac{t}{c'} (c' \mathbb{1}_d) = t \mathbb{1}_d = (\min\{\gamma'_i, t\})_{i \in \mathbb{I}_d}.
   \]

\( \square \)

**Lemma 4.9.** Let \( a' = (a'_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^{\downarrow} \) and let \( \gamma' = (\gamma'_i)_{i \in \mathbb{I}_d} \) be its water-filling in dimension \( d \), with water-level \( c' \). Let \( a_i(t) : [0, \gamma'_1] \rightarrow [0, \gamma'_1] \), for \( i \in \mathbb{I}_n \), be as in Definition 4.7. Assume that \( a'_1 \geq c' \) and set \( a^{(2)} = (a^{(2)}_i)_{i=2}^{n} \in (\mathbb{R}_{\geq 0}^{n-1})^{\downarrow} \). Then

1. \( \gamma^{(2)} = (\gamma^{(2)}_i)_{i=2}^{d-1} \in (\mathbb{R}_{\geq 0}^{d-1})^{\downarrow} \) is the water-filling of \( a^{(2)} \) in dimension \( d - 1 \).

2. If we let \( a^{(2)}_i(t) : [0, \gamma'_2] \rightarrow [0, \gamma'_2] \) for \( 2 \leq i \leq n \), be constructed as in Definition 4.7 with respect to \( a^{(2)} \) and \( d' = d - 1 \) then
   \[
a_i(t) = a^{(2)}_i(t) \quad \text{for} \quad t \in [0, \gamma'_2] \quad \text{and} \quad 2 \leq i \leq n.
   \]

3. If \( \gamma(t) = (\gamma_i(t))_{i \in \mathbb{I}_d} \) denotes the water-filling of \( a(t) = (a_i(t))_{i \in \mathbb{I}_n} \) in dimension \( d \) then
   \[
t = \gamma_1(t) = a_1(t) \quad \text{for} \quad t \in [0, \gamma'_1].
   \]

**Proof.** Notice that the first claim is straightforward. In order to prove the second claim, notice that the water-level of \( \gamma^{(2)} \) is \( c' \). We consider the following two cases:

Case 1: \( a'_2 \geq c' \), so that \( \gamma'_2 = a'_2 \geq c' \). In this case, for \( 2 \leq i \leq n \):

\[
18
\]
• If \( c' \leq t \leq \gamma_2' \): \( a_i^{(2)}(t) = \min\{a_i', t\} = a_i(t) \);

• If \( 0 \leq t \leq c' \): \( a_i^{(2)}(t) = \frac{t}{\gamma_2'} \min\{a_i', c'\} = a_i(t) \).

Case 2: \( a_2' < c' = \gamma_2' \) and hence \( \gamma_2' = c' \). In this case, if \( 0 \leq t \leq \gamma_2' \) and \( 2 \leq i \leq n \):

\[
a_i^{(2)}(t) = \frac{t}{\gamma_2'} a_i = \frac{t}{c'} \min\{a_i', c'\} = a_i(t),
\]

since \( c' = \gamma_2' \geq a_2' \geq a_i' \), for \( 2 \leq i \leq n \).

The proof of the third follows by the fact that \( a_i' \geq c' \) implies \( a_i' = \gamma_i' \). Therefore, using Definition 4.7 and Lemma 4.8, we have \( a_1(t') = t = \min\{\gamma_1', t\} = \gamma_1(t) \) for \( t \in [0, \gamma_1'] \).

4.2 The Algorithm

Let \( \alpha = (\alpha_i)_{i \in I_n} \in (\mathbb{R}_{\geq 0}^n)^{+} \) and \( m \in \mathbb{N} \) with \( m \geq 1 \) be given and consider \( d = (d_j)_{j \in I_m} \in (\mathbb{N}^m)^{+} \).

In this section we describe a finite-step algorithm whose input are \( \alpha \) and \( d \) and whose output are the sequences \( a_j^{op} = (a_{ij}^{op})_{i \in I_n} \in (\mathbb{R}_{\geq 0}^n)^{+} \) for \( j \in I_m \), such that \((a_{ij}^{op})_{i \in I_n, j \in I_m} \in P_{\alpha, m}\).

We further construct \( \gamma_j^{op} = (\gamma_{ij}^{op})_{i \in I_n} \in (\mathbb{R}_{\geq 0}^n)^{+} \) that is the water-filling of \( a_j^{op} \) in dimension \( d_j \), for \( j \in I_m \).

The procedure is recursive in \( m \) i.e. assuming that we have applied the algorithm to \( \alpha \) and the dimensions \( d_1 \geq \ldots \geq d_{m-1} \) with a certain output, we use it to construct the output for \( \alpha \) and \( d_1 \geq \ldots \geq d_{m-1} \geq d_m \geq 1 \). Along the way, we (inductively) assume some specific features of the output; we will show that the recursive process is well defined. These features will allow us to prove Theorem 3.8 at the end of this section.

Algorithm 4.10.

INPUT:

- \( \alpha = (\alpha_i)_{i \in I_n} \in (\mathbb{R}_{\geq 0}^n)^{+} \);
- \( m \in \mathbb{N} \) with \( m \geq 1 \) and \( d_1 \geq \ldots \geq d_m \geq 1 \), with \( n \geq d_1 \).

ALGORITHM:

- In case \( m = 1 \) we set \( a_i^{op} = \alpha_i \) for \( i \in I_n \), so \( \gamma_i^{op} \in (\mathbb{R}_{\geq 0}^{d_1})^{+} \).
- In case \( m > 1 \): assume that we have constructed \( a_j' = (a_{ij}')_{i \in I_n} \) for \( j \in I_{m-1} \) according to the algorithm, using the input \( \alpha = (\alpha_i)_{i \in I_n} \) and the dimensions \( d_1 \geq \ldots \geq d_{m-1} \). Thus, \((a_{ij}')_{i \in I_n, j \in I_{m-1}} \) is an \( (\alpha, m-1)\)-weight partition. Let \( \gamma_j' = (\gamma_{ij}')_{i \in I_n} \in (\mathbb{R}_{\geq 0}^{d_j})^{+} \) be the vector obtained by water-filling of \( a_j' \) in dimension \( d_j \), for \( j \in I_{m-1} \). We denote by \( c_j' \) the water-level of \( \gamma_j' \).

We assume (Inductive Hypothesis) that \( a_j' \in (\mathbb{R}_{\geq 0}^{n_j} \in (\mathbb{R}_{\geq 0}^{d_j})^{+} \), and that for \( 1 \leq r \leq s \leq m - 1 \)

\[
\gamma_{ir}' = \gamma_{is}' \quad \text{for} \quad i \in I_n.
\]

1. For \( 0 \leq t \leq \gamma_{11}' \) we introduce the partitions \( A^{(1)}(t) = (a_{ij}^{(1)}(t))_{i \in I_n, j \in I_m} \in P_{\alpha, m} \) as follows:

   For \( j \in I_{m-1} \), \( a_{ij}^{(1)}(t) \) is as in Definition 4.7 applied to \( a_j' \), i.e.:

   \[
a_{ij}^{(1)}(t) = \frac{\min\{t, c_j'\}}{c_j'} \min\{a_{ij}', \max\{t, c_j'\}\} \quad \text{for} \quad i \in I_n.
\]
Notice that \( \mathbf{a}_j^{(1)}(t) := (a^{(1)}_{ij}(t))_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0})^{\mathbb{I}_n} \), for \( 0 \leq t \leq \gamma_{11}^{\prime} \) and \( j \in \mathbb{I}_{m-1} \).

We set:

\[
a^{(1)}_{im}(t) = \alpha_i - \sum_{j \in \mathbb{I}_{m-1}} a^{(1)}_{ij}(t) \quad \text{for} \quad 0 \leq t \leq \gamma_{11}^{\prime} \quad \text{and} \quad i \in \mathbb{I}_n.
\]

Claim 1 (see Remark 4.11): \( \mathbf{a}^{(1)}_m(t) = (a^{(1)}_{im}(t))_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0})^{\mathbb{I}_n} \).

2. For \( 0 \leq t \leq \gamma_{11}^{\prime} \) and \( j \in \mathbb{I}_m \) we also set:

\[
\gamma_j^{(1)}(t) = \gamma_{ij}^{(1)}(t)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0})^{\mathbb{I}_n}
\]

that is obtained by water-filling of \( \mathbf{a}_j^{(1)}(t) \) in dimension \( d_j \).

3. Claim 2 (see Remark 4.11): with the previous definitions:

\[
\gamma_j^{(1)}(t) = t \quad \text{for} \quad 0 \leq t \leq \gamma_{11}^{\prime} \quad \text{and} \quad j \in \mathbb{I}_{m-1}.
\]  

Claim 3 (see Remark 4.11): the functions \( a^{(1)}_{im}(t) \) are non-increasing and \( \gamma^{(1)}_{1m}(t) \) is strictly decreasing in \([0, \gamma_{11}^{\prime}]\). Moreover, \( \gamma^{(1)}_{1m}(0) \geq \gamma_{11}^{\prime} \) and \( \gamma^{(1)}_{1m}(\gamma_{11}^{\prime}) = 0 \).

Therefore, there exists a unique value \( t_1 \in [0, \gamma_{11}^{\prime}] \) such that

\[
\gamma^{(1)}_{1m}(t_1) = t_1 \quad \text{for} \quad j \in \mathbb{I}_{m-1}.
\]

For this \( t_1 \) we consider two cases:

Case 1: assume that \( \gamma^{(1)}_{m}(t_1) \neq \gamma^{(1)}_{1m}(t_1) \cdot I_{d_m} \) so that \( d_m > 1 \). In this case:

- (a) We set \( a_{ij}^{(1)} := a_{ij}^{(1)}(t_1) \) for \( j \in \mathbb{I}_m \);
- (b) We re-initialize the algorithm by setting \( d_j := d_j - 1 \), for \( j \in \mathbb{I}_m \) and considering \( (a_{ij})_{i=2}^{n} \in (\mathbb{R}_{\geq 0})^{\mathbb{I}_n} \) for \( j \in \mathbb{I}_{m-1} \), which forms an \((\alpha_i)_{i=2}^{n}, m-1\)-weight partition (see Remark 4.13).

(c) Hence, we apply the construction of step 1 to \( (a_{ij})_{i=2}^{n}, j \in \mathbb{I}_{m-1} \), together with the new dimensions and compute:

\[
\mathbf{a}_j^{(2)}(t) = (a_{ij}^{(2)}(t))_{i=2}^{n} \in (\mathbb{R}_{\geq 0})^{\mathbb{I}_{m-1}} \quad \text{and} \quad \gamma_j^{(2)}(t) = (\gamma_{ij}^{(2)}(t))_{i=2}^{n} \in (\mathbb{R}_{\geq 0})^{\mathbb{I}_{m-1}}.
\]

Thus, there exists a unique \( t_2 \in [0, \gamma_{21}^{\prime}] \) such that

\[
\gamma_{2m}(t_2) = t_2 = \gamma_{2j}(t_2) \quad \text{for} \quad j \in \mathbb{I}_{m-1}.
\]

In particular, we define (at least)

\[
a_{2j}^{(1)} = a_{ij}^{(2)}(t_2) \quad \text{for} \quad j \in \mathbb{I}_m.
\]

Case 2: assume that \( \gamma^{(1)}_{m}(t_1) = \gamma^{(1)}_{1m}(t_1) \cdot I_{d_m} \).

In this case we set \( a_{ij}^{(2)} := a_{ij}^{(1)}(t_1) \), for \( i \in \mathbb{I}_n \) and \( j \in \mathbb{I}_m \). Thus we compute the optimal weights \( \mathbf{a}^{(2)}_j = (a_{ij}^{(2)})_{i \in \mathbb{I}_n} \), for \( j \in \mathbb{I}_m \). The algorithm stops. This case shall be subsequently referred to as “the Algorithm stops in the first iteration” assuming that the process starts computing the weight \( a_{1m}^{(2)} \).

OUTPUT:

Notice that the algorithm stops at some point, having defined a partition \( A^{op} = (a^{op}_{ij})_{i \in \mathbb{I}_n, j \in \mathbb{I}_m} \). In this case we set \( \mathbf{a}_j^{op} = (a_{ij}^{op})_{i \in \mathbb{I}_n} \) and \( \gamma_j^{op} = (\gamma_{ij}^{op})_{i \in \mathbb{I}_{d_j}} \) that is obtained by water-filling of \( \mathbf{a}_j^{op} \in (\mathbb{R}_{\geq 0})^{\mathbb{I}_n} \) in dimension \( d_j \), for \( j \in \mathbb{I}_m \).
In the next remark we prove Claims 1,2 and 3 stated in Algorithm 4.10. After that, we consider some results in order to show that the inductive hypothesis, assumed in Algorithm 4.10 holds for m groups (see Theorem 4.18) so that the recursive process is well defined.

**Remark 4.11.** Consider the notation and definitions of Algorithm 4.10.

Proof of Claim 1. For \( j \in \mathbb{I}_{m-1} \) we consider \( b_{ij}(t) = a_{ij}^{(1)}(t) - a_{ij}^{(1)}(t) \), for \( 0 \leq t \leq \gamma_{11}' \) and \( i \in \mathbb{I}_n \). Thus, for \( j \in \mathbb{I}_{m-1} \) we have that:

(a1) If \( c_j' \leq t \leq \gamma_{11}' \) then \( b_{ij}(t) = (a_{ij}^{(1)} - t)^+ \) for \( i \in \mathbb{I}_n \).

(a2) If \( 0 \leq t < c_j' \) then \( b_{ij}(t) = (a_{ij}^{(1)} - c_j')^+ + \frac{c_j' - t}{c_j'} \min \{a_{ij}^{(1)}, c_j'\} \) for \( i \in \mathbb{I}_n \).

Hence, we see that \( b_{ij}(t) = (b_{ij}(t))_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\mathbb{I}_n \), for \( 0 \leq t \leq \gamma_{11}' \) and \( j \in \mathbb{I}_{m-1} \). Also, for \( j \in \mathbb{I}_{m-1} \) and \( i \in \mathbb{I}_n \) we have that the function \( b_{ij}(t) \) is non-increasing for \( t \in [0, \gamma_{11}'] \).

By definition, we have that for \( 0 \leq t \leq \gamma_{11}' \) and \( i \in \mathbb{I}_n \)

\[
a_{im}^{(1)}(t) = \alpha_i - \sum_{j \in \mathbb{I}_{m-1}} a_{ij}^{(1)}(t) = \sum_{j \in \mathbb{I}_{m-1}} (a_{ij}^{(1)} - a_{ij}^{(1)}(t)) = \sum_{j \in \mathbb{I}_{m-1}} b_{ij}(t),
\]

which shows that \( a_{im}^{(1)}(t) = (a_{im}^{(1)}(t))_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\mathbb{I}_n \).

Proof of Claim 2. Notice that the functions \( a_{ij}(t) \) introduced in Definition 4.7 allows to describe a sub-routine that computes the vectors \( a_{ij}^{(1)}(t) = (a_{ij}^{(1)}(t))_{i \in \mathbb{I}_n} \) for \( j \in \mathbb{I}_{m-1} \) - as described in Algorithm 4.10 - in terms of the vectors \( a_j' = (a_{ij}')_{i \in \mathbb{I}_n} \) for \( j \in \mathbb{I}_{m-1} \). Hence, the claim follows from Lemma 4.8.

Proof of Claim 3. An analysis similar to that considered in the proof of Claim 1 above shows that: \( a_{im}^{(1)}(t) \) are non-increasing while \( a_{im}^{(1)}(t) \) and \( \text{tr}(a_{im}^{(1)}(t)) \) are strictly decreasing functions in \([0, \gamma_{11}']\). Notice that by construction, \( a_{ij}^{(1)}(0) = 0_n \in \mathbb{R}^n \) is the zero vector, for \( j \in \mathbb{I}_{m-1} \) and hence \( a_{im}^{(1)}(0) = \alpha_i \) for \( i \in \mathbb{I}_n \). Therefore, since \( a_j' \leq \alpha \) and \( d_1 \geq d_m \) it is straightforward to check that

\[
\gamma_{im}^{(1)}(0) \geq \gamma_{11}' \quad \text{for} \quad i \in \mathbb{I}_{dm} \implies \gamma_{im}^{(1)}(0) \geq \gamma_{11}' .
\]

On the other hand, using (the inductive hypothesis) we see that

\[
\gamma_{ij} = \gamma_{11}' \quad \text{for} \quad j \in \mathbb{I}_{m-1} \implies a_{ij}^{(1)}(\gamma_{11}') = a_j' \quad \text{for} \quad j \in \mathbb{I}_{m-1} .
\]

This last fact shows that \( a_{ij}^{(1)}(\gamma_{11}') = 0_n \in \mathbb{R}^n \), so that \( a_{im}^{(1)}(\gamma_{11}') = 0 \).

\[ \Delta \]

**Lemma 4.12.** With the notation, notions and constructions of Algorithm 4.10, assume that there exists \( t \in \mathbb{I}_{m-1} \) such that \( c_t' \geq a_{it}' \). Then, the algorithm stops in the first iteration.

**Proof.** Notice that the assumptions above imply that \( \gamma_{it}' = c_t' \mathbb{I}_{dt} \). Using the inductive hypothesis we conclude that

\[
\gamma_{ij} = c_t' \quad \text{for} \quad i \in \mathbb{I}_{dm} \quad \text{and} \quad j \in \mathbb{I}_{m-1}
\]

where we are using that \( d_m \leq d_j \) for \( j \in \mathbb{I}_{m-1} \). Therefore, the water-filling of \( b_{ij}(t) = (b_{ij}(t))_{i \in \mathbb{I}_n} \) in dimension \( d_m \) is a multiple of \( \mathbb{I}_{dm} \) for \( t \geq 0 \); indeed, let \( j \in \mathbb{I}_{m-1} \) and notice that by inductive hypothesis

\[
\gamma_{ij}' = c_t' \geq c_j' .
\]

Hence, we consider the following cases (see the proof of Claim 1 in Remark 4.11):
1. In case $\gamma'_{ij} = c'_i > c'_j$: we have that $a'_{ij} = \gamma'_{ij} = c'_i$ for $i \in \mathbb{I}_d$. We now consider the following subcases:

(a1) If $c'_j \leq t \leq \gamma'_{ij}$: then $b_{ij}(t) = (a'_{ij} - t)^+$ for $i \in \mathbb{I}_n$. Hence, in particular, $b_{ij}(t) = (c'_i - t)^+$ for $i \in \mathbb{I}_m$. This last fact shows that the water-filling of $b_{ij}(t) \in (\mathbb{R}^n_{\geq 0})^d$ in dimension $d_m$ is a multiple of $\mathbb{I}_{d_m}$.

\[ b_{ij}(t) = (c'_i - t)^+ + \frac{c'_i - t}{c'_j} \min\{a'_{ij}, c'_j\} \quad \text{for} \quad i \in \mathbb{I}_n. \]

In particular, $b_{ij}(t) = (c'_i - c'_j)^+ + c'_j - t = (c'_i - t)^+$ for $i \in \mathbb{I}_m$. Again, this shows that the water-filling of $b_{ij}(t) \in (\mathbb{R}^n_{\geq 0})^d$ in dimension $d_m$ is a multiple of $\mathbb{I}_{d_m}$.

2. Case 2: $\gamma'_{ij} = c'_i = c'_j$: then, $\gamma'_{ij} = c'_j$ for $i \in \mathbb{I}_d$. Hence, if $0 \leq t < c'_j$ then

\[ b_{ij}(t) = \frac{c'_j - t}{c'_j} a'_{ij} \quad \text{for} \quad i \in \mathbb{I}_n. \]

Then, by item 1 in Proposition 4.6 we see that the water-filling of $b_{ij}(t)$ in dimension $d_j$ is a multiple of $\mathbb{I}_{d_j}$. Finally, by item 3 in Proposition 4.6 we see that the water-filling of $b_{ij}(t)$ in dimension $d_m$ is a multiple of $\mathbb{I}_{d_m}$.

In this case it is straightforward to check that the water-filling of $a^{(1)}_{m}(t) = \sum_{j \in \mathbb{I}_{m-1}} b_{ij}(t)$ in dimension $d_m$ is a multiple of $\mathbb{I}_{d_m}$ for $t \in [0, \gamma'_{11}]$. Therefore, (according to case 2 in Algorithm 4.10) the algorithm stops in the first step.

**Remark 4.13.** Consider the notation, notions and constructions of Algorithm 4.10. Assume that Algorithm 4.10 does not stop in the first iteration (notice that in this case $d_m \geq 2$). We show (inductively on $m$) that $(a^{(\alpha)}_{i})_{i=1}^{n}$ for $j \in \mathbb{I}_{m}$ coincides with the output of the algorithm for the weights $(\alpha_{i})_{i=1}^{n}$ and the dimensions $d_1 - 1 \geq \ldots \geq d_{m-1}$.

We first point out that the case $m = 1$ is straightforward. Indeed, in this case the output of the algorithm for $\alpha$ and $d_1$ is $\alpha$. Analogously, the output of the algorithm for $(\alpha_{i})_{i=1}^{n}$ and $d_1 - 1$ is $(\alpha_{i})_{i=1}^{n}$.

We now assume that $m \geq 2$. Recall that the algorithm is based on $a'_{j} = (a'_{ij})_{i \in \mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$, which is the output of the algorithm for $\alpha$ and the $m - 1$ dimensions $d_1 \geq \ldots \geq d_{m-1}$. If the algorithm for computing $a'_{j} = (a'_{ij})_{i \in \mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$ stops in the first iteration then we have that

\[ \gamma'_{m-1} = (\gamma'_{(m-1)j})_{i \in \mathbb{I}_{d_{m-1}}} = c'_{m-1} \mathbb{I}_{d_{m-1}}. \]

But in this case $c'_{m-1} \geq a'_{(m-1)j}$, so Lemma 4.12 shows that the algorithm based on $a'_{j} = (a'_{ij})_{i \in \mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$ - that computes $a^{(\alpha)}_{j}$ for $j \in \mathbb{I}_{m}$ - stops in the first iteration; this last fact contradicts our initial assumption. Therefore, we can apply the inductive hypothesis and conclude that the output of the algorithm with initial data $(\alpha_{i})_{i=1}^{n}$ and dimensions $d_1 - 1 \geq \ldots \geq d_{m-1} - 1$ is $(a^{(\alpha)}_{ij})_{i=1}^{n}$ for $j \in \mathbb{I}_{m-1}$.

After the first iteration of the algorithm with initial data $\alpha$ and $d_1 \geq \ldots \geq d_m$, the algorithm defines $(a^{(\alpha)}_{ij})_{j \in \mathbb{I}_m}$ and re-initializes (case 1 (b) in Algorithm 4.10) using $(a^{(\alpha)}_{ij})_{i=1}^{n}$ for $j \in \mathbb{I}_{m-1}$ and the dimensions $d_1 - 1 \geq \ldots d_{m-1} - 1 \geq 1$, and iterates until computing $(a^{(\alpha)}_{ij})_{i=1}^{n}$ for $j \in \mathbb{I}_{m}$ with this data. But, by the comments above, we see that $(a^{(\alpha)}_{ij})_{i=1}^{n}$ for $j \in \mathbb{I}_{m}$ is actually being computed by applying the algorithm to the output of the algorithm with initial data $(\alpha_{i})_{i=1}^{n}$ and $(m - 1)$ blocks with dimensions $d_1 - 1 \geq \ldots \geq d_{m-1} - 1$; the claim now follows from this last fact.

\[ \triangle \]
Lemma 4.14. Consider the notation, notions and constructions of Algorithm 4.10. Assume that the algorithm does not stop in the first iteration. Therefore, there exists $t_1$ and $t_2$ such that

$$t_1 = \gamma_{ij}^{(1)}(t_1) \quad \text{and} \quad t_2 = \gamma_{ij}^{(2)}(t_2) \quad \text{for} \quad j \in \mathbb{I}_m$$

where $\gamma_{ij}^{(2)}(t_2) = (\gamma_{ij}^{(2)}(t_2))_{i=2}^{d_1} \in (\mathbb{R}_{\geq 0}^{d_1-1})^d$ denotes the water-filling of $a_{ij}^{(2)}(t_2)$ obtained in the second iteration of the algorithm. In this case:

1. $a_{ij}^{(1)}(t) = a_{ij}^{(2)}(t)$ for $2 \leq i \leq n, \ j \in \mathbb{I}_m$ and $t \in [0, \gamma_{ij}^{(1)}]$: hence,

2. $a_{ij}^{op} = a_{ij}^{(1)}(t_1)$ for $i = 1, 2$ and $j \in \mathbb{I}_m$.

3. $a_{ij}^{op} = \gamma_{ij}^{(1)}(t_1) = \frac{\alpha_j}{m}$ for $j \in \mathbb{I}_m$.

4. $t_1 \geq t_2$.

Proof. We first notice that in our case $d_m > 1$ and $c'_\ell < a'_{i\ell}$ for $\ell \in \mathbb{I}_{m-1}$; otherwise, Lemma 4.12 shows that the algorithm stops in the first step. Notice that the functions $a_i(t)$ are actually computed using the construction in Definition 4.7 based on $a_j^{(2)} = (a_j^{(2)})_{i=2}^{d_1}$, for $j \in \mathbb{I}_{m-1}$. The previous comments also show that Lemma 4.9 applies and therefore, if $a_j^{(2)}(t) = (a_j^{(2)}(t))_{i=2}^{d_1}$ then

$$a_{ij}^{(1)}(t) = a_{ij}^{(2)}(t) \quad \text{for} \quad 2 \leq i \leq n, \ j \in \mathbb{I}_m \quad \text{and} \quad t \in [0, \gamma_{ij}^{(1)}]$$

Since the algorithm does not stop in the first step we see that

$$t_1 = \gamma_{ij}^{(1)}(t_1) = a_{1m}^{(1)}(t_1) \quad \text{for} \quad j \in \mathbb{I}_m$$

The previous comments together with the third item in Lemma 4.9 show that

$$a_{ij}^{(1)}(t_1) = \gamma_{ij}^{(1)}(t_1) = t_1, \ j \in \mathbb{I}_m \implies \alpha_1 = \sum_{j \in \mathbb{I}_m} a_{ij}^{(1)}(t_1) = m t_1 \quad \text{and} \quad a_{ij}^{op} = a_{ij}^{(1)}(t_1) = \frac{\alpha_1}{m}$$

We now assume that $t_1 < t_2$ and reach a contradiction. We consider the following two cases:

Case 1: the algorithm stops in the second step. In this case,

$$t_2 = \gamma_{2m}^{(2)}(t_2) \implies \gamma_{2m}^{(2)}(t_2) = t_2 \mathbf{1}_{d_{m-1}} \quad \text{and} \quad \sum_{i=2}^{\mathbf{1}} a_{ij}^{(2)}(t_2) = (d_m - 1) t_2$$

Recall that the functions $a_{im}(t)$ are non-increasing in $[0, \gamma_{11}^{(1)}]$: hence, using that $t_1 < t_2$ then

$$a_{im}^{(1)}(t_1) \geq a_{im}^{(1)}(t_2) = a_{im}^{(2)}(t_2) \quad \text{for} \quad 2 \leq i \leq n$$

On the other hand, $a_{1m}^{(1)}(t_1) = t_1 < t_2$. Then,

$$\frac{1}{d_m} \sum_{i \in \mathbb{I}_n} a_{im}^{(1)}(t_1) \geq \frac{1}{d_m} \left( t_1 + \sum_{i=2}^{n} a_{im}^{(2)}(t_2) \right) = \frac{1}{d_m} (t_1 + (d_m - 1) t_2) > t_1$$

These facts show that $\gamma_{im}^{(1)}(t_1)$ should be a multiple of $\mathbf{1}_{d_m}$ (since $\gamma_{im}^{(1)}(t_1)$ is the water-filling of $a_{im}^{(1)}(t_1)$ in dimension $d_m$) which contradicts the assumption that the algorithm does not stop in the first step.

Case 2: the algorithm does not stop in the second step. In this case, if $t_1 < t_2$ then

$$t_2 = \gamma_{2m}^{(2)}(t_2) = a_{2m}^{(2)}(t_2) = a_{2m}^{(1)}(t_2) \leq a_{1m}^{(1)}(t_2) \leq a_{1m}^{(1)}(t_1) = t_1$$

since $a_{1m}^{(1)}(t) \in (\mathbb{R}_{\geq 0}^d)^d$ and $a_{1m}^{(1)}(t)$ is a non-increasing function; therefore $t_2 \leq t_1$ which is a contradiction. 

□
Proposition 4.15. Consider the notation, notions and constructions of Algorithm 4.10 for $m \geq 2$. Assume that the algorithm computing $a_{ij}^{op} \in (\mathbb{R}_+^n)^j$ for $j \in \mathbb{I}_m$, stops in the $k$-th iteration (so $k \leq d_m$). Let $t_1, \ldots, t_k$ be constructed in each iteration of Algorithm 4.10. Then:

1. $a_{ij}^{(t)} = a_{ij}^{(t)}(t)$ for $t \in [0, \gamma^{(1)}_{ij}], i \leq \ell \leq n$, $i \in \mathbb{I}_n$ and for $j \in \mathbb{I}_m$;

2. Let $\gamma^{(i)}_{ij}(t_i) = (\gamma^{(i)}_{ij}(t_i))_i$ be the water-filling of $(a_{ij}^{(1)}(t_i))_i$ in dimension $d_j - i + 1$, for $i \in \mathbb{I}_{k-1}$. Then, for $j \in \mathbb{I}_m$ we have that
   \[
   a_{ij}^{op} = a_{ij}^{(1)}(t_i) = \gamma^{(i)}_{ij}(t_i) = \frac{\alpha_i}{m}, \quad i \in \mathbb{I}_{k-1} \quad \text{and} \quad a_{ij}^{op} = a_{ij}^{(1)}(t_k), \quad k \leq i \leq n. \tag{20}
   \]

3. $t_1 \geq t_2 \ldots \geq t_k \geq 0$.

Moreover, we have that $a_{ij}^{op} \in (\mathbb{R}_+^n)^j$, for $j \in \mathbb{I}_m$.

Proof. In case the algorithm stops in the first step (i.e. $k = 1$) then $a_{ij}^{op} = a_{ij}^{(1)}(t_1) \in (\mathbb{R}_+^n)^j$, (according to Claim 1 in Algorithm 4.10 for $j = m$, while this property clearly holds by construction for $j \in \mathbb{I}_{m-1}$).

In case $k > 1$ then, with the notation and terminology from Lemma 4.14 we have that $t_2 \leq t_1$ and $a_{ij}^{(1)}(t) = a_{ij}^{(2)}(t)$ for $2 \leq i \leq n$, $j \in \mathbb{I}_m$. In particular, $a_{ij}^{op} = a_{ij}^{(1)}(t_i)$ for $i = 1, 2$ and $j \in \mathbb{I}_m$. If $j \in \mathbb{I}_{m-1}$ then $a_{ij}^{(1)}(t)$ is a non-decreasing function, and hence
   \[
   a_{ij}^{op} = a_{ij}^{(1)}(t_2) \leq a_{ij}^{(1)}(t_2) \leq a_{ij}^{(1)}(t_1) = \gamma^{(1)}_{ij}(t_1) = a_{ij}^{op},
   \]
where we have also used that $a_{ij}^{(1)}(t_2) \in (\mathbb{R}_+^n)^j$. On the other hand,
   \[
   a_{1m}^{op} = \gamma^{(1)}_{1m}(t_1) = t_1 \geq t_2 = a_{2m}^{(2)}(t_2) \geq a_{2m}^{(2)}(t_2) = a_{2m}^{op},
   \]
Using Remark 4.13 we can repeat the previous argument together with Lemma 4.14 ($k - 1$) times (applied to subsequent truncations of the initial weights and dimensions) and conclude that $t_1 \geq t_2 \geq \ldots \geq t_{k-1}$; hence, for $j \in \mathbb{I}_m$ we have that
   \[
   a_{ij}^{(t)} = a_{ij}^{(t)}(t) \quad \text{for} \quad i \leq \ell \leq n, \quad i \in \mathbb{I}_k \quad \text{and} \quad a_{ij}^{op} \geq a_{ij}^{(1)}(t_i) = \gamma^{(i)}_{ij}(t_i) = \frac{\alpha_i}{m} \quad \text{for} \quad 2 \leq i \leq k - 1. \tag{22}
   \]
Since the algorithm stops in the $k$-th step we see that
   \[
   a_{ij}^{op} = a_{ij}^{(k)}(t_k) = a_{ij}^{(1)}(t_k) \quad \text{for} \quad k \leq i \leq n \quad \text{and} \quad j \in \mathbb{I}_m. \tag{23}
   \]
Eq. (20) together with the fact that $a_{ij}^{op} \in (\mathbb{R}_+^n)^j$ is a consequence of Eqs. (22) and (23). \hfill \Box

Remark 4.16. Consider the notation, notions and constructions of Algorithm 4.10 for $m \geq 2$. Assume that the algorithm stops in the $k$-th iteration, for $k \geq 2$. In this case, in the $i$-th iteration (for $i \in \mathbb{I}_k$) the algorithm defines the functions $a_{ij}^{(i)}(t)$ for $i \leq \ell \leq n$ and $j \in \mathbb{I}_m$. Proposition 4.15 now shows that we only need to define the functions $a_{ij}^{(1)}(t)$ for $i \in \mathbb{n}$ and $j \in \mathbb{I}_m$. This simplifies considerably the complexity of the algorithm. \hfill \triangle

Proposition 4.17. Consider the notation, notions and constructions of Algorithm 4.10 for $m \geq 2$. Assume that the algorithm computing $a_{ij}^{op} \in (\mathbb{R}_+^n)^j$, for $j \in \mathbb{I}_m$, stops in the $k$-th iteration. If $t_1, \ldots, t_k$ are constructed in each iteration of the Algorithm 4.10 then:
1. For \( j \in \mathbb{I}_m \), \( a_{ij}^\text{op} = \gamma_{ij} (t_i) = \frac{\alpha_i}{m} \) for \( i \in \mathbb{I}_{k-1} \).

2. For \( j \in \mathbb{I}_{m-1} \) then \( \gamma_{ij}^\text{op} = \min \{ \gamma_{ij}, t_k \} \) for \( k \leq i \leq d_j \); \( \gamma_{im} = t_k \) for \( k \leq i \leq d_m \).

Moreover, for \( 1 \leq r \leq s \leq m \) we have that

\[
\gamma_{ir}^\text{op} = \gamma_{is}^\text{op} \quad \text{for} \quad i \in \mathbb{I}_d.
\] (24)

**Proof.** In case the algorithm stops in the first iteration (i.e. \( k = 1 \)), then let \( 0 < t_1 \leq \gamma_{11}' \) be such that

\[
t_1 = \gamma_{11}'(t_1) \quad \text{for} \quad j \in \mathbb{I}_m \quad \text{and} \quad \gamma_{1m}'(t_1) = t_1 \mathbb{I}_{d_m}.
\] (25)

Recall that \( a_{ij}^{(1)}(t) = (a_{ij}^{(1)}(t))_{i \in \mathbb{I}_n} \) is constructed as in Definition 4.7 based on \( a'_j = (a'_j)_{i \in \mathbb{I}_n} \) and that \( \gamma_{ij}' = (\gamma_{ij}')_{i \in \mathbb{I}_d} \) is the water-filling of \( a'_j \) in dimension \( d_j \), for \( j \in \mathbb{I}_{m-1} \). Hence, by Lemma 4.8 applied to \( a_{ij}^{(1)}(t_1) \) for \( j \in \mathbb{I}_{m-1} \), we conclude that

\[
\gamma_{ij}^\text{op} = \gamma_{ij}'(t_1) = \min \{ \gamma_{ij}', t_1 \} \quad \text{for} \quad j \in \mathbb{I}_{m-1}.
\] (26)

If we assume that \( t_1 > \gamma_{d_m}' \), then, notice that \( \gamma_{ij}' = \gamma_{1j}' \geq t_1 > \gamma_{d_m}' = \gamma_{d_m,j} \) for \( j \in \mathbb{I}_{m-1} \). Thus, if \( c_j' \) denotes the water-level of \( \gamma_{ij}' \) then

\[
a_{ij}^{op} = \gamma_{ij}'(t_1) \geq t_1 > \max \{ a_{d_m,j}, c_j' \} \quad \text{for} \quad j \in \mathbb{I}_{m-1} \quad \Rightarrow \quad a_{d_m}^{(1)}(t_1) = 0.
\]

But in this case, \( \gamma_{d,m,m} = 0 \) contradicting Eq. (25). Therefore, \( t_1 \leq \gamma_{d_m}' \) and using Eq. (26)

\[
\gamma_{ij}^\text{op} = t_1 = \gamma_{im} \quad \text{for} \quad j \in \mathbb{I}_{m-1} \quad \text{and} \quad i \in \mathbb{I}_{d}.
\]

In case \( k = 1 \) the result follows from these remarks.

In case \( k > 1 \) then, by Proposition 4.14, we get that

\[
a_{ij}^{op} = \frac{\alpha_i}{m} = \gamma_{ij}^{(k)}(t_i) = a_{ij}^{(1)}(t_i) \quad \text{for} \quad i \in \mathbb{I}_{k-1} \quad \text{and} \quad a_{ij}^{op} = a_{ij}^{(1)}(t_k) \quad \text{for} \quad k \leq i \leq n,
\]

for \( t_1 \geq \ldots \geq t_k > 0 \).

Claim: for \( j \in \mathbb{I}_m \) we have that

\[
\gamma_{ij}^\text{op} = (\frac{\alpha_1}{m}, \ldots, \frac{\alpha_{k-1}}{m}, \gamma_{k,j}, \ldots, \gamma_{d,j}) \in (\mathbb{R}^{d_j}_{\geq 0})^+,
\]

where \( \gamma_{ij} = (\gamma_{ij})_{i=k} \) is the water-filling of \( a_{ij}^{(1)}(t_k)_{i=k} \).

Consider first \( j \in \mathbb{I}_{m-1} \). Let \( c_j(t_i) \) denote the water-level of the water-filling \( \gamma_{ij}(t_i) \in \mathbb{R}^{d_j-i+1}_{\geq 0} \) of \( a_{ij}^{(1)}(t_i) = (a_{ij}^{(1)}(t))_{i \in \mathbb{I}_n} \) in dimension \( d_j - i + 1 \), for \( i \in \mathbb{I}_k \) (notice that \( \gamma_{ij} = \gamma_{ij}(t_k) \)). Since \( a_{ij}^{(1)}(t_{k-1}) = \gamma_{(k-1)}^{(1)}(t_{k-1}) \geq c_j(t_{k-1}) \) then, by Lemma 4.9, we get that \( (\gamma_{ij}(t_{k-1})_{i=k})_{i=k} \) is the water-filling of \( (a_{ij}^{(1)}(t_{k-1}))_{i=k} \). On the other hand, using that \( t_{k-1} \geq t_k \geq 0 \) and that \( a_{ij}^{(1)}(t) \) is a non-decreasing function of \( t, k \leq \ell \leq n \) and \( j \in \mathbb{I}_{m-1} \), then Proposition 4.6 (item 2) shows that

\[
\frac{\alpha_{k-1}}{m} = a_{ij}^{(1)}(t_{k-1}) = (\gamma_{ij}(t_{k-1}))_{i=k} \geq \gamma_{ij}(t_{k-1}) \geq \gamma_{ij}(t_k) \geq c_j(t_k).
\]

This last fact shows that the water-filling of \( (a_{ij}^{(1)}(t_{k-1}), a_{k,j}^{(1)}(t_k), \ldots, a_{nj}^{(1)}(t_k)) = (a_{ij}^{op})_{i=k-1} \) in dimension \( d_j - k + 2 \) is \( (\frac{\alpha_{k-1}}{m}, \gamma_{k,j}^{(k)}, \ldots, \gamma_{d,j}) \), which proves the claim above.
We now consider the case $j = m$. In this case, since the algorithm stops in the $k$-th iteration we have that
\[ a_{op}^m = \left( \frac{\alpha_1}{m}, \ldots, \frac{\alpha_{(k-1)}}{m}, a_{km}^{(1)}(t_k), \ldots, a_{nm}^{(1)}(t_k) \right) \quad \text{and} \quad \gamma_m^{(k)} = t_k \mathbb{1}_{d_m-k+1}. \]

The claim now follows from the facts that $\frac{\alpha_{(k-1)}}{m} = t_{(k-1)} \geq t_k$ and that $\gamma_m^{(k)}$ is the water-filling of $(a_{im}^{(1)}(t_k))_{n=i}^k$ in dimension $d_m - k + 1$.

**Theorem 4.18.** The recursive process in Algorithm 4.10 is well defined

**Proof.** Propositions 4.15 and 4.17 show that the inductive hypothesis (assumed during the construction of the algorithm for $m$ blocks in terms of the output of the algorithm for $m-1$ blocks) in Algorithm 4.10 is also verified for the output of the algorithm for $m$ blocks. Hence, the recursive process is well defined and always constructs an output.

## 5 Some consequences of the algorithmic construction

In this section we develop a complete proof of Theorem 3.8. Our approach is based on the previously stated algorithm, and in the analysis of its multi-waterfilling processes. As a byproduct, we show a monotonicity property of the spectra of optimal $(\alpha, d)$-designs with respect to the initial weights.

### 5.1 Proof of Theorem 3.8

**Definition 5.1.** Let $\alpha = (\alpha_i)_{i \in I_n} \in (\mathbb{R}_{\geq 0})^I$, $m \in \mathbb{N}$ and let $d = (d_j)_{j \in I_m} \in (\mathbb{N}^m)^I$ with $d_1 \leq n$. Let $a_{op}^j = (a_{ij}^{op})_{j \in I_m}$, for $j \in I_m$, be the output of Algorithm 4.10 for the input $\alpha$ and $d$. Let $\gamma_{op}^j = (\gamma_{ij}^op)_{j \in I_m} \in (\mathbb{R}^{d_j})^I$ be the water-filling of $a_{op}^j$ in dimension $d_j$, for $j \in I_m$. We then consider the vector $\Gamma_{op}(\alpha, d) = \Gamma_{op}$ given by
\[
\Gamma_{op} = (\gamma_{ij}^op)_{j \in I_m} \in \mathbb{R}^{\sum_{j \in I_m} d_j} \quad \text{where} \quad |d| = \sum_{j \in I_m} d_j.
\]

The following result, that gives a detailed description of the vector $\Gamma_{op}$ introduced in Definition 5.1, plays a key role in the proof of Theorem 3.8.

**Proposition 5.2.** Consider the notation in Definition 5.1. Then, there exist $p \in I_{d_1}$,

1. $\gamma_1 \geq \ldots \geq \gamma_p > 0$;
2. $r_1, \ldots, r_p \in \mathbb{N}$ such that $\sum_{\ell \in I_p} r_\ell = |d|$ and
3. $g_1, \ldots, g_p \in \mathbb{N}$ such that $g_0 = 0 < g_1 \leq \ldots \leq g_p = d_1$,

such that:

(a) $(\Gamma_{op})^j = (\gamma_{ij}^op)_{j \in I_p} \in (\mathbb{R}_{\geq 0})^j$;

(b) $r_\ell = \sum_{j \in I_m} (\min\{g_\ell, d_j\} - g_{\ell-1})^+$ for $\ell \in I_p$;

(c) $r_\ell \gamma_\ell = \sum_{\ell = g_{\ell-1}+1} \alpha_i$ for $\ell \in I_{p-1}$ and

(d) $r_p \gamma_p = \sum_{i = g_{p-1}+1} \alpha_i$. 

26
\textbf{Proof.} We argue by induction on $m$.

In case $m = 1$ then $a_1^{\text{op}} = \alpha = (\alpha_i)_{i \in I_n}$ and $\gamma_1^{\text{op}}$ is the water-filling of $\alpha$ in dimension $d_1$. Therefore we consider the cases:

Case 1: $\alpha_1 > \frac{tr(\alpha)}{d_1}$. In this case there exists $s \in \mathbb{I}_{d_1 - 1}$ and $c > 0$ such that

$$\gamma_1^{\text{op}} = (\alpha_1, \ldots, \alpha_s, c \mathbb{1}_{d_1 - s}) \in (\mathbb{R}^{d_1}_{\geq 0})^k \quad \text{with} \quad \sum_{i \in \mathbb{I}_{d_1}} (c - \alpha_i)^+ = \sum_{i = d_1 + 1}^{n} \alpha_i. \quad \text{(27)}$$

Then, in particular, we conclude that

$$(d_1 - s) c = \sum_{i = s + 1}^{n} \alpha_i.$$ 

If we set $p = s + 1$, $g_i = i$, for $0 \leq i \leq s$, and $g_{s+1} = d_1$ then:

$$(\min\{g_\ell, d_1\} - g_{\ell - 1})^+ = 1 \quad \text{for} \quad \ell \in I_s,$$

while $(\min\{g_{s+1}, d_1\} - g_s)^+ = (d_1 - s)$. If we set $\gamma_i = \alpha_i$ and $r_i = 1$, for $i \in \mathbb{I}_s$, $\gamma_{s+1} = c$ and $r_{s+1} = (d_1 - s)$, then properties (a)-(d) now follow from Eq. (27) and the previous remarks.

Case 2: $\alpha_1 \leq \frac{tr(\alpha)}{d_1}$. In this case there exists $c > 0$ such that $\gamma_1^{\text{op}} = c \mathbb{1}_{d_1}$ and $d_1 = tr(\alpha)$. In this case we set $p = 1$, $g_0 = 0$, $g_1 = d_1$, $r_1 = d_1$, so that $(\min\{g_1, d_1\} - g_0)^+ = r_1 = d_1$, $\gamma_1 = c$; then, properties (a)-(d) hold in this case.

Assume that the result holds for $m - 1$. Let $a'_j = (a'_{ij})_{i \in I_n}$ for $j \in \mathbb{I}_{m-1}$, denote the output of the algorithm for $\alpha$ and $d' = (d_1, \ldots, d_{m-1}) \in (\mathbb{N}^{m-1})^k$. Let $\gamma'_j = (\gamma'_{ij})_{i \in I_{d_j}}$ denote the water-filling of $a'_j$ in dimension $d_j$ for $j \in \mathbb{I}_{m-1}$. If we let $\Gamma' = (\gamma'_{ij})_{j \in \mathbb{I}_{m-1}} \in \mathbb{R}^{[d']}$, where $|d'| = \sum_{j \in \mathbb{I}_{m-1}} d_j$, then the inductive hypothesis implies that there exists $p' \in \mathbb{N}$, $\delta_1 \geq \ldots \geq \delta_{p'} > 0$, $s_1, \ldots, s_{p'} \in \mathbb{N}$ and $h_1, \ldots, h_{p'} \in \mathbb{N}$ with $0 = h_0 < h_1 < \ldots < h_{p'} = d_1$, such that

$$(\Gamma')^i = (h_\ell \mathbb{1}_{s_\ell})_{\ell \in \mathbb{I}_{p'}} \in \mathbb{R}^{[d']}, \quad s_\ell = \sum_{j \in \mathbb{I}_{m-1}} \min\{h_\ell, d_j\} - h_{\ell - 1})^+ \quad \text{for} \quad \ell \in \mathbb{I}_{p'},$$

$$s_\ell \delta_\ell = \sum_{i = h_{\ell - 1} + 1}^{h_\ell} \alpha_i \quad \text{for} \quad \ell \in \mathbb{I}_{p' - 1} \quad \text{and} \quad s_{p'} \delta_{p'} = \sum_{i = h_{p'} + 1}^{n} \alpha_i.$$ 

Assume now that the algorithm that computes $a_j^{\text{op}}$ for $j \in \mathbb{I}_m$ stops in $k$-th iteration. We consider the following cases:

Case $k = 1$: by Proposition \ref{prop:water-filling}, there exists $0 \leq t_1 \leq \gamma_{11}'$ such that

$$\gamma_j^{\text{op}} = (\min\{\gamma_{ij}', t_1\})_{i \in I_{d_j}} \quad \text{for} \quad j \in \mathbb{I}_{m - 1} \quad \text{and} \quad \gamma_m^{\text{op}} = t_1 \mathbb{1}_{d_m}.$$ 

Notice that in this case we have that $t_1 > 0$. Let

$$\mathcal{M} = \{\ell \in \mathbb{I}_{p'} : \delta_\ell \geq t_1\}.$$ 

Notice that $\mathcal{M} \neq \emptyset$, since $\delta_1 = \gamma_{11}' \geq t_1$, by construction. Using that $\delta_1 \geq \ldots \geq \delta_{p'} > 0$ we see that there exists $q \in \mathbb{I}_{p'}$ such that $\mathcal{M} = \{1, \ldots, q\}$. Set $p := p' - q + 1$ and

1. $\gamma_1 := t_1$ and $\gamma_\ell := \delta_{\ell + q - 1}$ for $2 \leq \ell \leq p$; hence, $\gamma_p > 0$.
2. $r_1 = \sum_{\ell \in I_q} s_\ell + d_m$ and $r_\ell = s_{\ell + q - 1}$ for $2 \leq \ell \leq p$;
3. \( g_0 = 0, g_1 = h_q \) and \( g_\ell = h_{\ell+q-1} \) for \( 2 \leq \ell \leq p \).

Using the inductive hypothesis and the previous definitions it is straightforward to check that property (a) holds i.e., \( \Gamma^\dagger = (\gamma_\ell \mathbb{I}_{r_\ell})_{\ell \in \mathbb{I}_p} \).

We claim that in this case \( g_1 = h_q \geq d_m \); indeed, if \( h_q < d_m \) then we get that \( h_\ell \leq h_q < d_m \leq d_j \) for \( \ell \in \mathbb{I}_q \) and \( j \in \mathbb{I}_{m-1} \); hence, if \( \ell \in \mathbb{I}_q \) then

\[
s_\ell = \sum_{j \in \mathbb{I}_{m-1}} (\min\{h_\ell, d_j\} - h_{\ell-1})^+ = (m-1)(h_\ell - h_{\ell-1}).
\]

Therefore,

\[
r_1 \geq \sum_{\ell \in \mathbb{I}_q} s_\ell + d_m = (m-1) \sum_{\ell \in \mathbb{I}_q} (h_\ell - h_{\ell-1}) + d_m = (m-1) h_q + d_m < m \cdot d_m \tag{28}
\]

On the other hand, since \( \gamma_\ell^{\mathbb{op}} \equiv t_1 \mathbb{I}_{d_m} \) then Proposition 4.17 shows that \( \gamma_\ell^{\mathbb{op}} = t_1 \) for \( i \in \mathbb{I}_{d_m} \) and \( j \in \mathbb{I}_m \). Thus,

\[
\#\{(i, j) : \gamma_\ell^{\mathbb{op}} = t_1, i \in \mathbb{I}_{d_j}, j \in \mathbb{I}_m\} \geq m \cdot d_m.
\]

Since \( h_q < d_m \) then \( q < p' \) and then \( p \geq 2 \): in this case, by construction of \((\gamma_i)_{i \in \mathbb{I}_p}\), we have that \( \gamma_\ell < \gamma_1 \) for \( 2 \leq \ell \leq p \). The previous facts together with property (a) show that \( r_1 \geq m \cdot d_m \), which contradicts Eq. (28) and the claim follows.

We now check property (b): notice that if \( j \in \mathbb{I}_m \) then

\[
\sum_{\ell \in \mathbb{I}_q} (\min\{h_\ell, d_j\} - h_{\ell-1})^+ = \min\{h_q, d_j\}.
\]

Hence, using the inductive hypothesis and the previous identity we have that

\[
r_1 = \sum_{\ell \in \mathbb{I}_q} s_\ell + d_m = \sum_{\ell \in \mathbb{I}_q} \sum_{j \in \mathbb{I}_{m-1}} (\min\{h_\ell, d_j\} - h_{\ell-1})^+ + d_m = \sum_{j \in \mathbb{I}_m} \min\{h_q, d_j\} = \sum_{j \in \mathbb{I}_m} (g_1, d_j - g_0)^+,
\]

where we have used that \( g_1 = h_q \) and \( \min\{g_1, d_m\} = d_m \). In case \( p > 1 \) we further have that for \( 2 \leq \ell \leq p \) then

\[
r_\ell = s_{\ell+q-1} = \sum_{j \in \mathbb{I}_{m-1}} (\min\{h_{\ell+q-1}, d_j\} - h_{\ell+q-2})^+ = \sum_{j \in \mathbb{I}_m} (\min\{g_\ell, d_j\} - g_{\ell-1})^+,
\]

where we have used the inductive hypothesis and the fact that \( \min\{g_\ell, d_m\} - g_{\ell-1})^+ = 0 \), since \( g_\ell \geq g_{\ell-1} \geq g_1 \geq d_m \). Therefore, property (b) holds in this case.

We now check properties (c) and (d). We consider two cases:

Case \( p = 1 \). In this case we only need to check (d), which is straightforward.

Case \( p > 1 \). In this case,

\[
r_\ell \gamma_p = s_{\ell+q-1} \delta_\ell \gamma_p = \sum_{i=h_{\ell-1}+1}^{h_{\ell+q-1}} \alpha_i = \sum_{i=g_{\ell-1}+1}^{g_{\ell}} \alpha_i
\]

and property (d) holds. Similarly, if \( 2 \leq \ell \leq p - 1 \) then

\[
r_\ell \gamma_\ell = s_{\ell+q-1} \delta_\ell \gamma_\ell = \sum_{i=h_{\ell+q-1}+1}^{h_{\ell+q-1}} \alpha_i = \sum_{i=g_{\ell}+1}^{g_{\ell}} \alpha_i.
\]

28
Finally, using that $\Gamma = (\gamma^\ell \mathbb{I}_{r^\ell})_{\ell \in \mathbb{I}^p} = (\gamma^\ell_j)_{j \in \mathbb{I}^m}$ and the previous facts we get that

$$r_1 \gamma_1 = \text{tr}(\Gamma) - \sum_{\ell=2}^p r_\ell \gamma_\ell = \sum_{i \in \mathbb{I}^n} \alpha_i - \left( \sum_{i=g_{\ell-1}+1}^{g_\ell} \alpha_i + \sum_{i=g_{\ell-1}+1}^n \alpha_i \right)$$

$$= \sum_{i \in \mathbb{I}^n} \alpha_i - \sum_{i=g_{\ell-1}+1}^n \alpha_i = \sum_{i=g_{\ell-1}+1}^n \alpha_i.$$  

Thus, properties (a)-(d) hold in this case.

Case 1 < k: so that 1 < k ≤ d_m. Recall that by Proposition 4.17 we have that

$$\gamma^\ell_{ij} = \frac{\alpha_i}{m} \quad \text{for} \quad i \in \mathbb{I}_{k-1}, \ j \in \mathbb{I}_m \quad \text{and} \quad \gamma^\ell_{ij} = \min\{\gamma'_{ij}, t_k\} \quad \text{for} \quad k \leq i \leq d_j, \ j \in \mathbb{I}_m.$$

By iterating the argument in Remark 4.13 we see that $\gamma^\ell_{ij} = 0$ for $j \in \mathbb{I}_m$ is the output of the algorithm with initial data given by $(\alpha_i)_{i=1}^n$ and dimensions $d' = (d_j-k+1)_{j \in \mathbb{I}_m} \in (\mathbb{N}_m)^d$; further, the algorithm with the previous initial data stops in the first iteration (by definition of $k$). Hence, by the first part of the proof (i.e. applying the inductive hypothesis to $(\alpha_i)_{i=1}^n$ and $(d_j-k+1)_{j \in \mathbb{I}_m}$) there exists $u \geq 1, \eta_1 \geq \ldots \geq \eta_u > 0, v_1, \ldots, v_u \in \mathbb{N}$ such that $\sum_{\ell \in \mathbb{I}^n} v_\ell = |d| - m \cdot (k - 1) := |d'|$, and $w_1, \ldots, w_u \in \mathbb{N}$ with $k - 1 = w_0 < w_1 < \ldots < w_u = d_1$, such that:

$$((\gamma^\ell_{ij})_{j=1}^k)_{j \in \mathbb{I}^m}^{\ell=1} \neq (\eta_\ell \mathbb{I}_{v_\ell})_{\ell \in \mathbb{I}^u} \mathbb{I}^{d'|}, \ v_\ell = \sum_{j \in \mathbb{I}_m} \min\{\gamma'_{ij}, d_j+d_k-1\} - w_{k-1})^{\ell} \quad \text{for} \quad \ell \in \mathbb{I}_u$$

We then set $p := k - 1 + u$

$$\gamma_\ell := \begin{cases} \frac{\alpha_i}{m} & \text{if} \ \ell \in \mathbb{I}_{k-1}, \\ \eta_{k-\ell+1} & \text{if} \ k \leq \ell \leq p; \end{cases} \quad r_\ell := \begin{cases} m & \text{if} \ \ell \in \mathbb{I}_{k-1}, \\ v_\ell-\ell+1 & \text{if} \ k \leq \ell \leq p; \end{cases}$$

and $g_\ell := \begin{cases} \ell & \text{if} \ 0 \leq \ell \leq k - 1, \\ w_\ell-\ell+1 & \text{if} \ k \leq \ell \leq p. \end{cases}$

With this explicit definitions and arguments similar to those already considered, it is straightforward - although rather tedious - to check that the parameters satisfy properties (a)-(d).

**Remark 5.3.** Consider the notation in Definition 5.1 and let $p \in \mathbb{I}_{d_j}$. Notice that as a consequence of Proposition 5.2, we can further obtain a representation of $\Gamma^\ell = (\gamma^\ell_{ij})_{i \in \mathbb{I}_p}$, for which $\gamma^\ell_{ij} > \ldots > \gamma^\ell_{ij} > 0$; indeed, we just have to group together the indexes $i, j \in \mathbb{I}_p$ for which $\gamma_i = \gamma_j$ in the representation of $\Gamma^\ell$ given in Proposition 5.2.

We are now ready to prove our first main result from Section 3.2.

**Proof of Theorem 3.2.** We consider $\alpha = (\alpha_i)_{i \in \mathbb{I}^n} \in (\mathbb{R}^{\geq 0})^n$, $m \in \mathbb{N}$ and $d = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}_m)^d$ with $d_1 \leq n$. Let $a^\ell_{ij} = (a^\ell_{ij})_{i \in \mathbb{I}_j}$, for $j \in \mathbb{I}_m$, be the output of Algorithm 4.10 for the input $\alpha$ and $d$. Let $\gamma^\ell_{ij} = (\gamma^\ell_{ij})_{i \in \mathbb{I}_j} \in (\mathbb{R}^{d_j})^n$ be the water-filling of $a^\ell_{ij}$ in dimension $d_j$, for $j \in \mathbb{I}_m$. By construction, $a^\ell_{ij} < \gamma^\ell_{ij}$ so, by Theorem 2.4, there exists $\mathcal{F}^\ell_{ij} \subset (\mathbb{C}^{d_j})^n$ such that $|f^\ell_{ij}|^2 = a^\ell_{ij}$ for $i \in \mathbb{I}_p$ and $\lambda(S_{\mathcal{F}^\ell_{ij}}) = \gamma^\ell_{ij}$, for $j \in \mathbb{I}_m$. Therefore, $\Phi^\ell := (\mathcal{F}^\ell_{ij})_{j \in \mathbb{I}_m} \in D(\alpha, d)$ and $A_{\Phi^\ell} = \Phi^\ell \in (\mathbb{R}^{d_j})^n$, by construction. This last fact, together with the description of $\Gamma^\ell$ in Proposition 5.2 and Remark 5.3, completes the proof. In particular, since all entries of $A_{\Phi^\ell}$ are positive, then every frame operator $S_{\mathcal{F}^\ell_{ij}}$ is invertible; thus, $\mathcal{F}$ is a frame for $d_{\mathcal{F}^\ell_{ij}}$, for $j \in \mathbb{I}_m$. □
Remark 5.4 (Finite-step algorithm for constructing optimal \((\alpha, d)\)-designs). Consider the notation in Theorem 3.10. We apply Algorithm 4.10 and obtain \(a_j^{\text{op}}\) for \(j \in \mathbb{I}_m\). Then, we compute the optimal spectra \(\gamma_j^{\text{op}}\) for \(j \in \mathbb{I}_m\) by water-filling, in terms of a simple finite step-algorithm. Finally, we can apply well known algorithms (see [10, 11, 14, 16]) to compute \(F_j^{\text{op}} = \{f_{ij}^{\text{op}}\}_{i \in \mathbb{I}_n}\) such that \(\lambda(S_{F_j^{\text{op}}}) = \gamma_j^{\text{op}}\) and such that \((\|f_{ij}^{\text{op}}\|^2)_{i \in \mathbb{I}_n} = a_j^{\text{op}}\) for \(j \in \mathbb{I}_m\). In this case, \(F_j^{\text{op}} = (F_j^{\text{op}})_{j \in \mathbb{I}_m}\) is an optimal \((\alpha, d)\)-design, as in Theorem 3.10. Thus, the conjunction of these routines allows us to effectively compute \(F_j^{\text{op}}\) in a finite number of steps (see Section 6 for numerical examples obtained by this method).

\[\square\]

5.2 Spectral monotonicity

In the following result we show that there is a monotonic dependence of the (unique) spectra of optimal \((\alpha, d)\)-designs with respect to the initial weights.

**Theorem 5.5.** Let \(\alpha_i \geq \beta_i > 0\) for \(i \in \mathbb{I}_n\) and let \(d = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^n)\). Let \(a_j^{\text{op}}\) (respectively \(b_j^{\text{op}}\)) for \(j \in \mathbb{I}_m\) be the output of the algorithm with the input \(\alpha = (\alpha_i)_{i \in \mathbb{I}_n}\) (respectively with the input \(\beta = (\beta_i)_{i \in \mathbb{I}_n}\) and \(d\). Let \(\gamma_j^{\text{op}} = (\gamma_{ij}^{\text{op}})\) (respectively \(\delta_j^{\text{op}} = (\delta_{ij}^{\text{op}})\)) be the water-filling of \(a_j^{\text{op}}\) (respectively \(b_j^{\text{op}}\)) in dimension \(d_j\), for \(j \in \mathbb{I}_m\). Then

\[
\gamma_{ij}^{\text{op}} \geq \delta_{ij}^{\text{op}} \quad \text{for} \quad i \in \mathbb{I}_{d_j} \quad \text{and} \quad j \in \mathbb{I}_m.
\]

**Proof.** We argue by induction on \(d\). If \(m = 1\) then \(\gamma^{\text{op}}\) and \(\delta^{\text{op}}\) coincide with the water-fillings of \(\alpha\) and \(\beta\) in dimension \(d_1\), respectively. Hence, the result follows from Proposition 4.6.

Assume that the result holds for \(m - 1 \geq 1\) and let \(\alpha, \beta\) and \(d = (d_j)_{j \in \mathbb{I}_m}\) be as above. Let \(a_j' = (a_j')_{i \in \mathbb{I}_n}\) and \(b_j' = (b_j')_{i \in \mathbb{I}_n}\) be the outputs of the algorithm with the inputs \(\alpha\) and \(\beta\) respectively, and \(d_1 \geq \ldots \geq d_{m-1}\). If we let \(\gamma_j' = (\gamma_{ij}')_{i \in \mathbb{I}_{d_j}}\) and \(\delta_j' = (\delta_{ij}')_{i \in \mathbb{I}_{d_j}}\) denote the water-filling of \(a_j'\) and \(b_j'\) in dimension \(d_j\) respectively, then the inductive hypothesis implies that

\[
\gamma_{ij}' \geq \delta_{ij}' \quad \text{for} \quad i \in \mathbb{I}_{d_j} \quad \text{and} \quad j \in \mathbb{I}_{m-1}.
\]

We now check the result for \(\gamma_{ij}^{\text{op}}\) and \(\delta_{ij}^{\text{op}}\), for \(j \in \mathbb{I}_m\).

Notice that Definition 4.7 and Lemma 4.8 show that

\[
\gamma_{ij}^{(1)}(t) = \min\{\gamma_{ij}', t\} \quad \text{and} \quad \delta_{ij}^{(1)}(t) = \min\{\delta_{ij}', t\} \quad \text{for} \quad i \in \mathbb{I}_{d_j}, j \in \mathbb{I}_{m-1}.
\]

On the other hand, notice that

\[
\sum_{j \in \mathbb{I}_m} \text{tr}(\delta_{ij}^{(1)}(t)) = \sum_{i \in \mathbb{I}_n} \alpha_i = \sum_{j \in \mathbb{I}_{m-1}} \text{tr}(\delta_j') = \text{tr}(\delta_{m}^{(1)}(t)) = \sum_{j \in \mathbb{I}_{m-1}} \left(\text{tr}(\delta_j') - \text{tr}(\delta_{ij}^{(1)}(t))\right).
\]

Similarly

\[
\text{tr}(\gamma_{m}^{(1)}(t)) = \sum_{j \in \mathbb{I}_{m-1}} \left(\text{tr}(\gamma_j') - \text{tr}(\gamma_{ij}^{(1)}(t))\right).
\]

If we let \(i \in \mathbb{I}_{d_j}\) for \(j \in \mathbb{I}_{m-1}\) then

\[
\gamma_{ij}' - \gamma_{ij}^{(1)}(t) = \gamma_{ij}' - \min\{\gamma_{ij}', t\} \geq \delta_{ij}' - \min\{\delta_{ij}', t\} = \delta_{ij}' - \delta_{ij}^{(1)}(t),
\]

and hence

\[
\text{tr}(\gamma_j') - \text{tr}(\gamma_{ij}^{(1)}(t)) \geq \text{tr}(\delta_j') - \text{tr}(\delta_{ij}^{(1)}(t)) \iff \text{tr}(\gamma_{m}^{(1)}(t)) \geq \text{tr}(\delta_{m}^{(1)}(t)).
\]
Case 1. Assume now that when computing \((b^\text{op})_{j \in I_m}\), the algorithm stops in the first iteration. Hence, there exists \(s_1\) such that
\[ s_1 = \delta_j^{(1)}(s_1) \quad \text{for} \quad j \in I_m \quad \text{and} \quad \delta_m^{(1)}(s_1) = s_1 \mathbb{I}_{d_m}. \]
In this case, \(\delta_{j}^{\text{op}} = \delta_{j}^{(1)}(s_1)\) for \(j \in I_m\). Similarly, there exists \(t_1\) such that \(t_1 = \gamma_{ij}^{(1)}(t_1) = \gamma_{ij}^{\text{op}}\) for \(j \in I_m\). If \(\gamma_{1m}(s_1) < s_1\) then
\[ s_1 d_m = \text{tr}(\delta_{m}^{(1)}(s_1)) \leq \text{tr}(\gamma_{1m}(s_1)) < s_1 d_m, \]
where we have used (32). The previous contradiction shows that \(\gamma_{1m}(s_1) \geq s_1\) and hence, that \(s_1 \leq t_1\). If the algorithm that computes \((a^\text{op})_{j \in I_m}\) stops in the first iteration then, by Proposition 4.17 and Eq. (30) we get that
\[ \gamma_{ij}^{\text{op}} = \min\{\gamma_{ij}, t_1\} \geq \min\{\delta_{ij}, s_1\} = \delta_{ij}^{\text{op}} \quad \text{for} \quad i \in I_{d_j} \quad \text{and} \quad j \in I_{m-1}. \]
On the other hand, \(\gamma_{im}^{\text{op}} = t_1 \geq s_1 = \delta_{im}^{\text{op}}\) for \(i \in I_{d_m}\) and these facts prove Eq. (29).

If the algorithm that computes \((a^\text{op})_{j \in I_m}\) does not stop in the first iteration, we can argue as before and conclude that \(t_1 = \gamma_{ij}^{(2)} = s_1 = \delta_{ij}^{(1)}\) for \(j \in I_m\). We then consider the auxiliary vectors \(\mu_{j}^{(2)}(t) = (\delta_{ij}(t))_{i=2}^{d_j}\) for \(j \in I_m\). Using that \(\gamma_{ij}^{(2)}(t) = (\gamma_{ij}^{(1)}(t))_{i=2}^{d_j}\) for \(j \in I_m\) we see that these vectors satisfy:
- \(\text{tr}(\gamma_{im}^{(2)}(t)) \geq \text{tr}(\mu_{im}^{(2)})(t)\);
- \(\mu_{j}^{(2)}(s_1) = s_1 \mathbb{I}_{d_m-1} \).

Using these two properties as before we conclude that if \(t_2\) is such that \(t_2 = \gamma_{ij}^{(2)}(t_2)\) for \(j \in I_m\) then \(s_1 \leq t_2\). If the algorithm that computes \((a^\text{op})_{j \in I_m}\) stops in the second iteration then we obtain that
\[ \gamma_{ij}^{\text{op}} = \gamma_{ij}^{(2)}(t_2) = \min\{\gamma_{ij}, t_2\} \geq \min\{\delta_{ij}, s_1\} = \delta_{ij}^{\text{op}} \quad \text{for} \quad 2 \leq i \leq d_j \quad \text{and} \quad j \in I_{m-1}. \]

We also have that \(\gamma_{im}^{\text{op}} = t_2 \geq s_1 = \delta_{im}^{\text{op}}\) for \(2 \leq i \leq d_m\) and these facts prove Eq. (29). Otherwise, we only get \(t_2 = \gamma_{ij}^{(2)} = \delta_{ij}^{(2)} = s_1\) and repeat the previous argument.

Case 2. Assume that when computing \((b^\text{op})_{j \in I_m}\), the algorithm does not stop in the first iteration. Hence, there exists \(s_1\) and \(t_1\) such that
\[ s_1 = \delta_j^{(1)}(s_1) = \delta_j^{\text{op}} \quad \text{and} \quad t_1 = \gamma_{ij}^{(1)}(t_1) = \gamma_{ij}^{\text{op}} \quad \text{for} \quad j \in I_m. \]
As before, if \(\gamma_{1m}(s_1) < s_1\) then
\[ s_1 d_m \leq \text{tr}(\delta_{1m}^{(1)}(s_1)) \leq \text{tr}(\gamma_{1m}(s_1)) < s_1 d_m. \]
Thus, \(\gamma_{1m}(s_1) \geq s_1\) and hence, that \(s_1 \leq t_1\). If the algorithm that computes \((a^\text{op})_{j \in I_m}\) stops in the first iteration then, by Proposition 4.17 and Eq. (30) we get that \(\gamma_{ij}^{\text{op}} = \min\{\gamma_{ij}, t_1\}\) for \(i \in I_{d_j}\) and \(j \in I_{m-1}\), while \(\gamma_{im}^{\text{op}} = t_1 \mathbb{I}_{d_m}\); therefore, \(t_1 = \gamma_{ij}^{\text{op}} \geq \delta_{ij}^{\text{op}} = s_1\). We then consider the auxiliary vectors \(\nu_{j}^{(2)}(t) = (\gamma_{ij}^{(1)}(t))_{i=2}^{d_j}\) for \(j \in I_m\). Using that \(\delta_{j}^{(2)}(t) = (\delta_{ij}^{(1)}(t))_{i=2}^{d_j}\) for \(j \in I_m\) we see that these vectors satisfy:
- \(\text{tr}(\nu_{im}^{(2)}(t)) \geq \text{tr}(\nu_{im}^{(2)})(t)\);
- \(\nu_{j}^{(2)}(t_1) = t_1 \mathbb{I}_{d_m-1} \).
Using these two properties as before we conclude that if \( s_2 \) is such that \( s_2 = \delta_{tj}^{(2)}(s_2) \) for \( j \in \mathbb{I}_m \) then \( s_2 \leq t_1 \). If the algorithm that computes \( (b_j^{\text{op}})_{j \in \mathbb{I}_m} \) stops in the second iteration then we obtain that

\[
\gamma_{ij}^{\text{op}} = \gamma_{ij}^{(2)}(t_1) = \min\{\gamma_{ij}' , t_1\} \geq \min\{\delta_{ij}' , s_2\} = \delta_{ij}^{\text{op}} \quad \text{for} \quad 2 \leq i \leq d_j \quad \text{and} \quad j \in \mathbb{I}_{m-1}.
\]

We also have that \( \gamma_{ij}^{\text{op}} = t_1 = s_2 = \delta_{im}^{\text{op}} \) for \( 2 \leq i \leq d_m \) and these facts prove Eq. \((29)\). Otherwise, we only get \( t_1 = \gamma_{ij}^{(2)} \geq \delta_{ij}^{\text{op}} = s_2 \) for \( j \in \mathbb{I}_m \) and repeat the previous argument. Moreover, by Proposition 4.15 we see that \( t_1 \geq s_1 \geq s_2 \ldots \geq s_k \), where \( s_i \) is the parameter obtained in the \( i \)-th iteration of the algorithm that computes \( (b_j^{\text{op}})_{j \in \mathbb{I}_m} \) with initial input \( \beta \) and \( d \). The result follows from these considerations.

Finally, if the algorithm that computes \((a_j^{\text{op}})_{j \in \mathbb{I}_m}\) does not stop in the first iteration (recall we are assuming that the algorithm that computes \((b_j^{\text{op}})_{j \in \mathbb{I}_m}\) does not stop in the first iteration as well) then, arguing as before, we conclude that \( \gamma_{ij}^{\text{op}} \geq \delta_{ij}^{\text{op}} \) for \( j \in \mathbb{I}_m \). Using Remark 4.13 we can now apply all the previous arguments to the outputs of the reduced problem corresponding to the weights \( \alpha_2 \geq \ldots \geq \alpha_n \) and the dimensions \( d_1 - 1 \geq \ldots \geq d_m - 1 \) and conclude that: in case any of the algorithms that compute \( a_j^{\text{op}} \) for \( j \in \mathbb{I}_m \) or \( b_j^{\text{op}} \) for \( j \in \mathbb{I}_m \) stops in the second iteration then

\[
\gamma_{ij}^{\text{op}} \geq \delta_{ij}^{\text{op}} \quad \text{for} \quad 2 \leq i \leq n \quad \text{and} \quad j \in \mathbb{I}_m,
\]

from which the result follows. In case none of the algorithms that compute \( a_j^{\text{op}} \) for \( j \in \mathbb{I}_m \) or \( b_j^{\text{op}} \) for \( j \in \mathbb{I}_m \) stops in the second iteration then we get that \( \gamma_{2j}^{\text{op}} \geq \delta_{2j}^{\text{op}} \) for \( j \in \mathbb{I}_m \); in this case we repeat the previous argument. We eventually end up showing all instances of Eq. \((29)\). \(\square\)

### 6 Numerical examples

**Example 6.1.** Let \( \alpha = (10, 10, 10, 1, 1) = (10 \mathbb{1}_3, \mathbb{1}_2) \in (\mathbb{R}^5_{\geq 0})^4 \) and \( d = (4, 2) \in (\mathbb{N}^2)^4 \); hence, \( n = 5, m = 2 \). Consider \( A^{\text{op}} = (a_j^{\text{op}})_{j \in \mathbb{I}_m} \) obtained by applying Algorithm 4.10 moreover, let \( \gamma_{\text{op}} \in (R^4)^4 \) and \( \gamma_{\text{op}} \in (R^2)^4 \) denote the water-fillings of \( c_1(A^{\text{op}}) \) and \( c_2(A^{\text{op}}) \) in dimensions \( d_1 = 4 \) and \( d_2 = 2 \) respectively. Hence, in this case

\[
\gamma_{1}^{\text{op}} = (6 \mathbb{1}_3, 2) \in (\mathbb{R}^4_{\geq 0})^4 \quad \text{and} \quad \gamma_{2}^{\text{op}} = (6 \mathbb{1}_2) \in (\mathbb{R}^2_{\geq 0})^4 \quad \implies \quad \Lambda_{\Phi^{\text{op}}} = (6 \mathbb{1}_5, 2) \in (\mathbb{R}^6_{\geq 0})^4,
\]

where \( \Phi^{\text{op}} \in D(\alpha, d) \) is as Theorem 3.8. Notice that in this example, the constants \( \gamma_1 = 6 \) and \( \gamma_2 = 2 \) for the representation of \( \Lambda_{\Phi^{\text{op}}} \) as in Theorem 3.8 are not directly related with the initial data. Hence, in general, there is no simple closed formula for these constants (neither for their multiplicities) in terms of the initial data. \(\triangle\)

The following examples were obtained via an implementation of Algorithm 4.10 using MATLAB, following the scheme described in Remark 5.4.

**Example 6.2.** Consider the family of weights given by \( \alpha = \{9, 8, 7, 5, 4, 2, 5, 2, 2, 1, 5, 0, 6, 0, 5\} \) and suppose that the dimensions to be considered are \( d_1 = 7, d_2 = 5, d_3 = 3 \). Then, the optimal partition of \( \alpha \) is

\[
A^{\text{op}} = \begin{bmatrix}
3 & 3 & 3 \\
2.7583 & 2.7583 & 2.4833 \\
2.7583 & 2.7583 & 1.4833 \\
2.7583 & 1.8135 & 0.4282 \\
2.5267 & 1.1307 & 0.3425 \\
1.5792 & 0.7067 & 0.2141 \\
1.2634 & 0.5654 & 0.1713 \\
1.2634 & 0.5654 & 0.1713 \\
0.9475 & 0.4240 & 0.1285 \\
0.3790 & 0.1606 & 0.0514 \\
0.3158 & 0.1413 & 0.0428 \\
\end{bmatrix}
\]
In this case, the optimal spectra related to this partition are:

\[
\begin{align*}
\gamma_1^\text{op} &= (3, 2.7583, 2.7583, 2.7583, 2.7583, 2.7583) \\
\gamma_2^\text{op} &= (3, 2.7583, 2.7583, 2.7583, 2.7583) \\
\gamma_3^\text{op} &= (3, 2.7583, 2.7583)
\end{align*}
\]

Once we have the optimal partitions and optimal spectra, we can construct examples of frames using these data, applying known algorithms like one-sided Bendel-Mickey algorithm (see \[10, 11, 14, 16\]):

\[
\mathcal{F}_1 = \begin{bmatrix}
0.0705 & 0.1956 & -0.0616 & -0.6865 & -0.3994 & -0.0845 & -0.3280 & -0.1553 & 0.2649 & -0.3180 \\
0.2804 & -0.3211 & -0.2142 & 0.2434 & -0.4716 & 1.2808 & 0.2514 & -0.4197 & 0.3309 & -0.5206 \\
0.0380 & -0.1106 & -0.7528 & 0.8134 & 0.8134 & 0.8134 & 0.2257 & 0.3605 & 0.0342 & 0.9482 & 0.1089 & 0.2125 \\
-0.0084 & -0.1760 & -0.3643 & -0.2125 & -0.2125 & -0.2125 & -0.2125 & -0.2125 & -0.2125 & -0.2125 & -0.2125 & -0.2125 \\
-0.4655 & -0.4260 & -0.5815 & 0.0796 & 0.0796 & 0.0796 & 0.0796 & 0.0796 & 0.0796 & 0.0796 & 0.0796 & 0.0796 \\
0.3120 & 0.2501 & 0.3128 & -0.1034 & -0.1034 & -0.1034 & -0.1034 & -0.1034 & -0.1034 & -0.1034 & -0.1034 & -0.1034 \\
0.0391 & -0.0112 & 0.0316 & 0.0949 & 0.0949 & 0.0949 & 0.0949 & 0.0949 & 0.0949 & 0.0949 & 0.0949 & 0.0949 \\
\end{bmatrix}
\]

\[
\mathcal{F}_2 = \begin{bmatrix}
0.1841 & 0.2017 & 0.3189 & 0.0682 & -0.2093 & -0.2340 & -0.2340 & -0.2340 & -0.2340 & -0.2340 & -0.2340 & -0.2340 \\
0.0249 & 0.0273 & 0.0432 & 0.6049 & 0.6049 & 0.6049 & 0.6049 & 0.6049 & 0.6049 & 0.6049 & 0.6049 & 0.6049 \\
-0.1947 & -0.2192 & -0.3372 & -0.3744 & -0.1430 & -0.1599 & -0.2022 & -0.6973 & 1.2517 & 0.5169 & -0.1238 \\
-0.2625 & -0.2876 & -0.4547 & -0.2253 & 0.1440 & 0.1619 & 0.2037 & -0.9404 & -0.4907 & -0.3351 & 1.0506 \\
-0.0015 & -0.0016 & -0.0025 & -0.0619 & -0.0723 & -0.0808 & -0.1022 & -0.0053 & -0.6598 & 1.5029 & 0.2041 \\
\end{bmatrix}
\]

\[
\mathcal{F}_3 = \begin{bmatrix}
-0.0342 & -0.0375 & -0.0593 & -0.3714 & -0.3952 & -0.4419 & -0.5590 & -0.6249 & -1.1632 & -0.2605 & -0.3888 \\
-0.1953 & -0.2139 & -0.3383 & -0.1805 & 0.0475 & 0.0536 & 0.0678 & 0.0758 & 0.1410 & -1.4873 & 0.5519 \\
0.0592 & 0.0649 & 0.1026 & 0.0281 & -0.1129 & -0.1263 & -0.1597 & -0.1796 & -0.3324 & 0.4511 & 1.5950 \\
\end{bmatrix}
\]

**Example 6.3.** Using the same set of dimensions, take now \(\alpha = \{8.5, 7.6, 4.3, 8.2, 1.6, 1.4, 1, 0.5, 0.4\}\). Notice that we are considering weights that are term by term smaller than previous \(\alpha\). In this case, the Algorithm provides the following optimal spectra:

\[
\begin{align*}
\gamma_1^\text{op} &= (2.8333, 2.3333, 2.3, 2.3, 2.3, 2.3, 2.3) \\
\gamma_2^\text{op} &= (2.8333, 2.3333, 2.3, 2.3, 2.3) \\
\gamma_3^\text{op} &= (2.8333, 2.3333, 2.3)
\end{align*}
\]

illustrating the monotonicity proved in previous section.

**Example 6.4.** When \(\alpha = \{20, 19.5, 10, 5, 4.5, 3, 2.4, 2\}\) and \(d = \{5, 4, 4, 3, 2\}\), Algorithm \[4.10\] provides the following optimal partition:

\[
A^\text{op} = \begin{bmatrix}
4 & 4 & 4 & 4 & 4 \\
3.9 & 3.9 & 3.9 & 3.9 & 3.9 \\
3.3625 & 2.8875 & 2.5 & 1.25 & 0 \\
1.9896 & 1.1354 & 1.25 & 0.625 & 0 \\
1.7907 & 1.0218 & 1.125 & 0.5625 & 0 \\
1.1938 & 0.6812 & 0.75 & 0.375 & 0 \\
0.955 & 0.545 & 0.6 & 0.3 & 0 \\
0.7959 & 0.4541 & 0.5 & 0.25 & 0 \\
\end{bmatrix}
\]

Notice that first two weights \(\alpha_1\) and \(\alpha_2\) are considerably bigger than the rest, this forces the concentration in the first two weights in the last column of the partition \(A^\text{op}\), i.e. the smaller subspace requires only two vectors. Related to this behavior of the optimal partitions, one can see that the optimal spectra is

\[
\begin{align*}
\gamma_1^\text{op} &= (4, 3.9, 3.3625, 3.3625, 3.3625, 3.3625) \\
\gamma_2^\text{op} &= (4, 3.9, 3.3625, 3.3625) \\
\gamma_3^\text{op} &= (4, 3.9, 3.3625, 3.3625) \\
\gamma_4^\text{op} &= (4, 3.9, 3.3625) \\
\gamma_5^\text{op} &= (4, 3.9)
\end{align*}
\]

where the smaller spectrum does not have the water-filling constant 3.3625.
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