Many-Help-One Problem for Gaussian Sources with a Tree Structure on Their Correlation

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Abstract—In this paper we consider the separate coding problem for \( L + 1 \) correlated Gaussian memoryless sources. We deal with the case where \( L \) separately encoded data of sources work as side information at the decoder for the reconstruction of the remaining source. The determination problem of the rate distortion region for this system is the so called many-help-one problem and has been known as a highly challenging problem. The author determined the rate distortion region in the case where the \( L \) sources working as partial side information are conditionally independent if the remaining source we wish to reconstruct is given. This condition on the correlation is called the CI condition. In this paper we extend the author’s previous result to the case where \( L + 1 \) sources satisfy a kind of tree structure on their correlation. We call this tree structure of information sources the TS condition, which contains the CI condition as a special case. In this paper we derive an explicit outer bound of the rate distortion region when information sources satisfy the TS condition. We further derive an explicit sufficient condition for this outer bound to be tight. In particular, we determine the sum rate part of the rate distortion region for the case where information sources satisfy the TS condition. For some class of Gaussian sources with the TS condition we derive an explicit recursive formula of this sum rate part.

Index Terms—Multiterminal source coding, many-help-one problem, Gaussian, rate-distortion region, CEO problem.

I. INTRODUCTION

In multi-user source networks separate coding systems of correlated information sources are significant from both theoretical and practical point of view. The first fundamental result on those coding systems was obtained by Slepian and Wolf [1]. They considered a separate source coding system of two correlated information sources. Those two sources are separately encoded and sent to a single destination, where the decoder reconstruct the original sources.

In the above source coding system, we can consider the situation, where the decoder wishes to reproduce one of two sources. We call this source the primary source. In this case the remaining source that we call the auxiliary source works as a partial side information at the decoder for the reconstruction of the primary source. Wyner [2], Ahlswede and Körner [3] determined the admissible rate region for this system, the set that consists of a pair of transmission rates for which the primary source can be decoded with an arbitrary small error probability.

We can naturally extend the system studied by Wyner, Ahlswede and Körner to the one where there are several separately encoded data of auxiliary sources serving as side informations at the decoder. The determination of the admissible rate region for this system is called the many-help-one problem. In this sense Wyner, Ahlswede and Körner solved the so called one-helps-one problem. The many-help-one problem has been known as a highly challenging problem. To date, partial solutions given by Körner and Marton [4], Gelfand and Pinsker [5], Oohama [8], [10], and Tavildar et al. [11] are known.

Gelfand and Pinsker [5] studied an interesting case of the many-help-one problem. They determined the admissible rate region in the case, where the auxiliary sources are conditionally independent if the primary source is given. We hereafter say the above correlation condition on the information sources the CI condition.

In Oohama [8], the author extended the many-help-one problem studied by Gelfand and Pinsker [5] to a continuous case. He considered the many-help-one problem for \( L + 1 \) correlated memoryless Gaussian sources, where \( L \) auxiliary sources work as partial side information at the decoder for the reconstruction of the primary source. The mean square error was adopted as a distortion criterion between the decoded output and the original primary source output. The rate distortion region was defined by the set of all transmission rates for which the average distortion can be upper bounded by a prescribed level. In [8], the author determined the rate distortion region when information sources satisfy the CI condition. This result contains the author’s previous works for Gaussian one-helps-one problem [6] and Gaussian CEO problem [7].

The problem still remains open for Gaussian sources with general correlation. Pandya et al. [9] studied the general case and derived an outer bound of the rate distortion region using some variant of bounding technique the author [6] used to prove the converse coding theorem for Gaussian one-helps-one problem. However, their bounding method was not sufficient to provide a tight result.

In Oohama [10], the author extended the result of [8]. He considered a case of correlation on Gaussian sources, where \( L + 1 \) sources satisfy a kind of tree structure on their correlation. The author called this tree structure of information sources the TS condition. The TS condition contains the CI condition as a special case. In [10], he derived an explicit outer bound of the rate distortion region for Gaussian sources satisfying TS condition. Furthermore, he had shown that for \( L = 2 \), this outer bound coincides with the rate distortion region. The author also presented a sufficient condition for the outer bound to coincide with the rate distortion region.

Subsequently, Tavildar et al. [11] extended the TS condition to a binary Gauss Markov tree structure condition. They...
studied a characterization of the rate distortion region for Gaussian source with the complete binary tree structure and succeeded in it. To derive their result, they made the full use of the complete binary tree structure of the source. They further determined the rate distortion region for Gaussian sources with general tree structure.

In Oohama [10], the analysis for matching condition of the rate distortion region and the derived outer bound was not sufficient, so that the author could not realize that there exists a part of the rate distortion region where the outer bound derived by him coincides with the rate distortion region. In this paper we give a further analysis on matching condition for the outer bound derived by Oohama [10] to coincide with the rate distortion region and derive a condition much stronger than the matching condition in [10]. Through this analysis we obtain an insight on a way of examining the sum rate part of the rate distortion region to show that for Gaussian sources with the TS condition the minimum sum rate part of the outer bound given by Oohama [10] is tight. This result implies that in Oohama [10], the author had already obtained an explicit characterization of the sum rate part of the rate distortion region before the work by Tavildar et al. [11]. On this optimal sum rate we derive its explicit recursive formula for some class of Gaussian sources with the TS condition. Our formula contains the result of Oohama [2] for Gaussian CEO problem as a special case.

The rest of this paper is organized as follows.

In Section II, we present a problem formulation and state the previous works.

In Section III, we give our main result. We first derive an explicit outer bound of the rate distortion region when information sources satisfy the TS condition. This outer bound is essentially the same as the author’s previous outer bound in [10], but it has a form more suitable than the previous one for analysis of a matching condition. Using the derived outer bound, we present an explicit sufficient condition for the outer bound to coincide with the inner bound.

In Section IV, we investigate the sum rate part of the rate distortion region. We show that for the outer bound in this paper and that in [10], their sum rate parts coincide with the sum rate part of the inner bound. Hence, in the case where information sources satisfy the TS condition, we establish an explicit characterization of the sum rate part of the rate distortion region. This optimal sum rate has a form of optimization problem. For some class of the Gaussian source with the TS condition, we solve this optimization problem to establish an explicit recursive formula of the optimal sum rate.

In Section V, we give the proofs of the results. Finally, in Section VI, we conclude the paper.

II. PROBLEM STATEMENT AND PREVIOUS RESULTS

In this section we state the problem formulation and previous results. We first state some notations used throughout this paper. Let $\Phi = \{1, 2, \cdots, |\Phi|\}$ and $A_i, i \in \Phi$ be arbitrary sets. Consider a random variable $A_i, i \in \Phi$ taking values in $A_i$. We write $n$ direct product of $A_i$ as $A_i^n = A_i \times \cdots \times A_i$. Let a random variable $A_i$ be denoted by $A_i = A_{i,1}A_{i,2} \cdots A_{i,n}$.

We write an element of $A_i^n$ as $a_i = a_{i,1}a_{i,2} \cdots a_{i,n}$. Let $S$ be an arbitrary subset of $\Phi$. Let $A_S$ and $A_i$ denote random vectors $(A_i)_{i \in S}$ and $(A_i)_{i \in S}$, respectively. Similarly, let $a_S$ denote a vector $(a_i)_{i \in S}$. When $S = \{k, k+1, \cdots, l\}$, we also use the notation $A_{k,l}$ for $A_S$ and use similar notations for other vectors or random variables. When $k = 1$, we sometimes omit subscript 1. Throughout this paper all logarithms are taken to the natural.

A. Formal Statement of the Problem

Let $X_i, i = 0, 1, 2, \cdots, L$ be correlated zero mean Gaussian random variables taking values in real lines $X_i$. Let $\Lambda = \{1, 2, \cdots, L\}$. The CI condition Oohama [8] treated corresponds to the case where $X_1, X_2, \cdots, X_L$ are independent if $X_0$ is given. In this paper we deal with the case where $X_1, \cdots, X_L$ have some correlation when $X_0$ is given. Let $\{(X_{0,t}, X_{1,t}, \cdots, X_{L,t})\}_{t=1}^{\infty}$ be a stationary memoryless multiple Gaussian source. For each $t = 1, 2, \cdots, (X_{0,t}, X_{1,t}, \cdots, X_{L,t})$ obeys the same distribution as $(X_0, X_1, \cdots, X_L)$.

The multiterminal source coding system treated in this paper is depicted in Fig. 1. For each $i = 0, 1, \cdots, L$, the data sequence $X_i$ is separately encoded to $\phi_i(X_i)$ by encoder function $\phi_i$. The encoded data $\phi_i(X_i), i = 0, 1, \cdots, L$ are sent to the information processing center, where the decoder observes them and outputs the estimation $\hat{X}_0$ of $X_0$ by using the decoder function $\psi$. The encoder functions $\phi_i, i = 0, 1, \cdots, L$ are defined by

$$\phi_i : X_i^n \rightarrow M_i = \{1, 2, \cdots, M_i\}$$

and satisfy rate constraints

$$\frac{1}{n} \log M_i \leq R_i + \delta$$

where $\delta$ is an arbitrary prescribed positive number. The decoder function $\psi$ is defined by

$$\psi : M_0 \times M_1 \times \cdots \times M_L \rightarrow X_0^n.$$ 

Denote by $\mathcal{F}(\delta_L(R_0, R_1, \cdots, R_L))$ the set that consists of all the $(L+2)$ tuple of encoder and decoder functions $(\phi_0, \phi_1, \cdots, \phi_L, \psi)$ satisfying (1)-(3). Let $d(x, \hat{x}) = (x-\hat{x})^2$, $(x, \hat{x}) \in A_0^n$ be a square distortion measure. For $X_0$ and its
estimation $\hat{X}_0 = \psi(\varphi_0(X_0), \varphi_1(X_1), \ldots, \varphi_L(X_L))$, define the average distortion by

$$\Delta(X_0, \hat{X}_0) \triangleq \frac{1}{n} \sum_{i=1}^{n} \text{Ed}(X_{0,i}, \hat{X}_{0,i}).$$

For a given $D > 0$, the rate vector $(R_0, R_1, \ldots, R_L)$ is admissible if for any positive $\delta > 0$ and any $n$ with $n \geq n_0(\delta)$, there exists $(\varphi_0, \varphi_1, \ldots, \varphi_L, \psi) \in F_{\delta}^{(n)}(R_0, R_1, \ldots, R_L)$ such that $\Delta(X_0, \hat{X}_0) \leq D + \delta$. Let $\mathcal{R}_L(D)$ denote the set of all the admissible rate vector. Our aim is to characterize that there exists $D > 0$ such that $\hat{X}_0$ satisfies the following correlations:

$$\hat{X}_0 \in \mathcal{R}_L(D).$$

By the rate-distortion theory for single Gaussian sources, when $R_0 \geq \frac{1}{2} \log \left[ \frac{\sigma_X^2}{2D} \right]$, $R_1 = R_2 = \cdots = R_L = 0$ is admissible. Here $\log^+ a = \max\{\log a, 0\}$. Hence, we have

$$\mathcal{R}_L(D) \cap \left\{ R_0 \geq \frac{1}{2} \log^+ \left[ \frac{\sigma_X^2}{2D} \right] \right\} = \left\{(R_0, R_1, \ldots, R_L) : R_0 \geq \frac{1}{2} \log^+ \left[ \frac{\sigma_X^2}{2D} \right], R_i \geq 0, i \in \Lambda \right\}.$$}

Throughout this paper we assume that $D \leq \sigma_X^2$ and $R_0 < \frac{1}{2} \log \left[ \frac{\sigma_X^2}{2D} \right]$.

### B. Tree Structure of Gaussian Sources

In this subsection we explain the tree structure of Gaussian source which is an important class of correlation. Consider the case where the $L+1$ random variables $X_0, X_1, \ldots, X_L$ satisfy the following conditions:

$$\begin{align*}
Y_0 &= X_0, \\
Y_l &= Y_{l-1} + Z_l, 1 \leq l \leq L, \\
X_l &= Y_l + N_l, 1 \leq l \leq L-1, \\
X_L &= Y_L, N_L = Z_L,
\end{align*}$$

(4)

where $Z_l$, $i \in \Lambda$ are $L$ independent Gaussian random variables with mean 0 and variance $\sigma_Z^2$, and $N_i$, $i=1, 2, \cdots, L-1$ are $L-1$ independent Gaussian random variables with mean 0 and variance $\sigma_N^2$. We assume that $Z_L$ is independent of $X_0$ and that $N^L-1$ is independent of $X_0$ and $Z_L$. We can see that the above $(X_0, X_1, \ldots, X_L)$ has a kind of tree structure(TS). We say that the source $(X_0, X_1, \ldots, X_L)$ satisfies the TS condition when it satisfies (4). The TS condition contains the CI condition as a special case by letting $\sigma_Z^2 = 0$ and $\sigma_N^2 = 0$.

### C. Previous Results

In this subsection we state the previous results on the determination problem of $\mathcal{R}_L(D)$. Let $U_i, i=0, 1, \ldots, L$ be random variables taking values in real lines $U_i$. For $S \subseteq \Lambda$, define

$$\mathcal{G}(D) \triangleq \{(U_0, U_L) : (U_0, U_L) \text{ is a Gaussian random vector that satisfies} \}$$

$$\begin{align*}
U_L &\to X_L \to X_0 \to U_0, \\
U_S &\to X_S \to U_S, \\
\text{for any } S \subseteq \Lambda \text{ and } &\text{E}[X_0 - \hat{\psi}(U_0, U_L)]^2 \leq D
\end{align*}$$

for some linear mapping $\hat{\psi} : U_0 \times U_L \to X_0$.

where $S^c \triangleq \Lambda - S$. Let

$$\pi = \left( \begin{array}{cccc} 1 & \cdots & i & \cdots & L \\ \pi(1) & \cdots & \pi(i) & \cdots & \pi(L) \end{array} \right)$$

be an arbitrary permutation on $\Lambda$ and $\Pi$ be a set of all permutations on $\Lambda$. For $S \subseteq \Lambda$, we set $\pi(S) \triangleq \{\pi(i)\}_{i \in S}$. Define $L$ subsets $S_i, i=1, 2, \ldots, L$ of $\Lambda$ by $S_i \triangleq \{i, i+1, \ldots, L\}$. Set

$$\mathcal{R}_{\pi, L}(D) \triangleq \{(R_0, R_1, \ldots, R_L) : \text{There exists a random vector } (U_0, U_L) \in \mathcal{G}(D) \text{ such that} \}$$

$$\begin{align*}
R_0 &\geq I(X_0; U_0), \\
R_\pi(i) &\geq I(X_{\pi(i)}; U_{\pi(i)}|U_{\pi(S^c)}) \\
&\text{for } i = 1, 2, \ldots, L,
\end{align*}$$

Fig. 2. TS condition in the case of $L = 4$.

Fig. 3. CI condition in the case of $L = 4$. 
where conv\{A\} denotes a convex hull of the set A. Then, we have the following.

**Theorem 1 (Oohama [10]):** For Gaussian sources with general correlation
\[
\tilde{R}_L^{(i)}(D) \subseteq R_L(D).
\]

For Gaussian sources with the CI condition the inner bound \(\tilde{R}_L^{(i)}(D)\) is tight, that is
\[
\tilde{R}_L^{(i)}(D) = R_L(D).
\]

The above inner bound \(\tilde{R}_L^{(i)}(D)\) can be regarded as a variant of the inner bound which is well known as the inner bound of Berger [12] and Tung [13]. Theorem 1 contains the solution that Oohama [6] obtained to the one-helps-one problem for Gaussian sources as a special case. When \(R_0 = 0\), the second result of Theorem 1 has some implications for the Gaussian CEO problem studied by Viswanathan and Berger [14] and Oohama [7] and source coding problem for multiterminal communication systems with a remote source investigated by Yamamoto and Itoh [15] and Flynn and Gray [16].

The notion of TS condition for Gaussian sources was first introduced by Oohama [10]. Tavildar et al. [11] extended the TS condition to a binary Gauss Markov tree structure condition. They studied a full characterization of the rate distortion region for Gaussian sources with a binary tree structure. In the next section we shall state the results of Tavildar et al. [11] and compare them with our results.

### III. Results on the Rate Distortion Region

In this section, we state our main results on inner and outer bounds of \(R_L(D)\) in the case where \((X_0, X_1, \ldots, X_L)\) satisfies the TS condition.

#### A. Definition of Functions and their Properties

In this subsection we define several functions which are necessary to describe our results and present their properties. Let \(r_i, i \in \Lambda\) be nonnegative numbers. Define the sequence of nonnegative functions \(\{f_i(r_i^L)\}\) by the following recursion:
\[
\begin{align*}
&f_{L-1}(r_{L-1}^L) = \frac{1-e^{-2r_{L-1}}}{2} + \frac{1-e^{-2r_{L-1}}}{\sigma_{N_{L-1}}^2}, \\
f_i(r_i^L) = & \frac{f_{i+1}(r_{i+1}^L)}{1+\sigma_{Z_i}^2} + \frac{1-e^{-2r_i^L}}{\sigma_{N_i}^2}, \\
&L - 2 \leq i \leq 1, \\
f_0(r_0^L) = & \frac{f_1(r_1^L)}{1+\sigma_{Z_1}^2}.
\end{align*}
\]

Next, we define the sequence of nonnegative functions
\[
\{g_l(D, r_0)\}_{l=0,1} \cup \{g_l(D, r_0, r_{l-1})\}_{l=2}^{L-1}
\]
by the following recursion:
\[
\begin{align*}
g_0(D, r_0) &= \frac{e^{-2r_0^L}}{D} - \frac{1}{\sigma_{X_0}^2}, \\
g_1(D, r_0) &= \frac{g_0(D, r_0)}{1-\sigma_{Z_1}^2 g_0(D, r_0)}, \\
g_{l+1}(D, r_0, r_l^L) &= \left[ g_l(D, r_0, r_{l-1}) + \frac{1-\sigma_{Z_{l+1}}^2}{\sigma_{N_{l+1}}^2} \left( 1-e^{-2r_{l+1}} \right) \right], \\
&1 \leq l \leq L - 2,
\end{align*}
\]
where \([a]^+ = \max\{a, 0\}\). Let \(B_L(D)\) be the set of all nonnegative vectors \(r_0^L\) that satisfy
\[
f_0(r_0^L) \geq g_0(D, r_0) = \frac{e^{-2r_0^L}}{D} - \frac{1}{\sigma_{X_0}^2}.
\]
Let \(\partial B_L(D)\) be the boundary of \(B_L(D)\), that is, the set of all nonnegative vectors \(r_0^L\) that satisfy
\[
f_0(r_0^L) = g_0(D, r_0) = \frac{e^{-2r_0^L}}{D} - \frac{1}{\sigma_{X_0}^2}.
\]
We can easily show that the functions we have defined satisfy the following property.

**Property 1:**
- a) For each \(i \in \Lambda\), \(f_0(r_i^L)\) is a monotone increasing function of \(r_i\). For each \(1 \leq l \leq L\) and for each \(i = l, l+1, \ldots, L\), \(f_i(r_i^L)\) is a monotone increasing function of \(r_i\).
- b) For each \(2 \leq l \leq L - 1\) and for each \(i = 0, 1, \ldots, l - 1\), \(g_l(D, r_0, r_{l-1})\) is a monotone decreasing function of \(r_i\).
- c) If \(r_0^L \in \partial B_L(D)\), then, for \(0 \leq l \leq L - 1\),
\[
g_l(D, r_0, r_{l-1}) \leq f_l(r_l^L).
\]

In the above \(L\) inequalities the equalities simultaneously hold if and only if \(r_0^L \in \partial B_L(D)\).

Define
\[
F(r_i^L) \triangleq \prod_{l=1}^{L-1} \left[ 1 + \sigma_{Z_l}^2 f_l(r_l^L) \right],
\]
\[
G(D, r_0, r_i^{L-2}) \triangleq \prod_{l=1}^{L-1} \left[ 1 + \sigma_{Z_l}^2 g_l(D, r_0, r_{l-1}) \right].
\]
For \(S \subseteq \Lambda\), define
\[
f_0(r_S) \triangleq \left. f_0(r_i^L) \right|_{r_S \rightarrow 0}, \quad F(r_S) \triangleq \left. F(r_i^L) \right|_{r_S \rightarrow 0}.
\]
We can easily show that the functions \(F(r_i^L)\) and \(G(D, r_0, r_i^{L-2})\) satisfy the following property.

**Property 2:**
- a) For each \(i \in S\), \(F(r_S)\) is a monotone increasing function of \(r_i\).
- b) For each \(i = 0, 1, \ldots, L - 2\), \(G(D, r_0, r_i^{L-2})\) is a monotone decreasing function of \(r_i\).
- c) If \(r_0^L \in \partial B_L(D)\), then
\[
G(D, r_0, r^{L-2}) \leq F(r_L^L).
\]

The equality holds if and only if \(r_0^L \in \partial B_L(D)\).
For $D > 0, r_i \geq 0, i \in \Lambda$ and $S \subseteq \Lambda$, define

$$J_S(D, r_0, r^{L-2}, r_S|r_S) \triangleq \frac{1}{2} \log^+ \left[ \frac{G(D, r_0, r^{L-2}, r_S|S)}{F(r_S)} \cdot \left\{ \frac{1+\sigma^2_S}{1+\sigma^2_S} \right\} D \cdot \prod_{i \in S} e^{2r_i} \right],$$

$$K_S(r_S|r_S) \triangleq \frac{1}{2} \log \left[ \frac{F(r_S)}{F(r_S)} \cdot \left\{ \frac{1+\sigma^2_S}{1+\sigma^2_S} \right\} D \cdot \prod_{i \in S} e^{2r_i} \right].$$

We can show that for $S \subseteq \Lambda$, $K_S(r_S|r_S)$ and $J_S(D, r_0, r^{L-2}, r_S|r_S)$ satisfy the following two properties.

**Property 3:**

a) If $r^L_0 \in B_L(D)$, then, for any $S \subseteq \Lambda$,

$$J_S(D, r_0, r^{L-2}, r_S|r_S) \leq K_S(r_S|r_S).$$

The equality holds when $r^L_0 \in \partial B_L(D)$.

b) Suppose that $r^L \in B_L(D)$. If $r^L|_{r_S=0}$ still belongs to $B_L(D)$, then,

$$J_S(D, r_0, r^{L-2}, r_S|r_S)|_{r_S=0} = K_S(r_S|r_S)|_{r_S=0} = 0.$$

**Property 4:** Fix $r^L \in B_L(D)$. For $S \subseteq \Lambda$, set

$$\rho_S = \rho_S(r_S|r_S) \triangleq J_S(D, r_0, r^{L-2}, r_S|r_S).$$

By definition it is obvious that $\rho_S, S \subseteq \Lambda$ are nonnegative. We can show that $\rho \triangleq \{ \rho_S \}_{S \subseteq \Lambda}$ satisfies the following:

a) $\rho_0 = 0$.

b) $\rho_A \leq \rho_B$ for $A \subseteq B \subseteq \Lambda$.

c) $\rho_A + \rho_B \leq \rho_{A \cup B} + \rho_{A \cap B}$.

In general $(\Lambda, \rho)$ is called a co-polymatroid if the nonnegative function $\rho$ on $2^\Lambda$ satisfies the above three properties. Similarly, we set

$$\tilde{\rho}_S = \tilde{\rho}(r_S|r_S) \triangleq K_S(r_S|r_S), \quad \tilde{\rho} = \{ \tilde{\rho}_S \}_{S \subseteq \Lambda}.$$

Then, $(\Lambda, \tilde{\rho})$ also has the same three properties as those of $(\Lambda, \rho)$ and becomes a co-polymatroid.

**B. Results**

In this subsection we present our results on inner and outer bounds of $R_L(D)$. In the previous work [10], we derived an outer bound of $R_L(D)$. We denote this outer bound by $\hat{R}_L^{(out)}(D)$. According to [10], $\hat{R}_L^{(out)}(D)$ is given by

$$\hat{R}_L^{(out)}(D) = \{ (R_0, R^L) : \text{There exists a nonnegative vector} \ (r_0, r^L) \text{ such that}$$

$$R_0 \geq r_0 \geq \frac{1}{2} \log^+ \left[ \frac{\sigma^2_0}{1+\sigma^2_0} \right] D,$$

$$R_i \geq r_i \text{ for any } i \in \Lambda,$$

$$R_0 + \sum_{i \in S} R_i \geq \frac{1}{2} \log^+ \left[ \frac{G(D, r_0, r^{L-2})\sigma^2_0}{F(r_S)\{1+\sigma^2_0\}} D \right] + \sum_{i=1}^L r_i$$

for any $S \subseteq \Lambda \}.$

Set

$$\mathcal{R}_L^{(out)}(D, r^L_0) \triangleq \{ (R_0, R_1, \cdots, R_L) :$$

$$R_0 \geq r_0, \sum_{i \in S} R_i \geq J_S(D, r_0, r^{L-2}, r_S|r_S), \text{ for any } S \subseteq \Lambda \},$$

$$\mathcal{R}_L^{(in)}(r^L_0) \triangleq \{ (R_0, R_1, \cdots, R_L) :$$

$$R_0 \geq r_0, \sum_{i \in S} R_i \geq K_S(r_S|r_S), \text{ for any } S \subseteq \Lambda \},$$

$$\mathcal{R}_L^{(out)}(D) \triangleq \bigcup_{r^L_0 \in B_L(D)} \mathcal{R}_L^{(out)}(D, r^L_0),$$

$$\mathcal{R}_L^{(in)}(D) \triangleq \bigcup_{r_0 \in B_L(D)} \mathcal{R}_L^{(in)}(r^L_0).$$

Our main result is as follows.

**Theorem 2:** For Gaussian sources with the TS condition

$$\mathcal{R}_L^{(in)}(D) \subseteq \mathcal{R}_L^{(out)}(D) \subseteq \mathcal{R}_L(D) \subseteq \hat{R}_L^{(out)}(D) \subseteq \mathcal{R}_L^{(out)}(D).$$

Proof of this theorem will be given in Section V. The inclusion $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(out)}(D)$ and an outline of proof of this inclusion was given in Oohama [10]. Furthermore, by Theorem 1 we have $\mathcal{R}_L^{(in)}(D) \subseteq \mathcal{R}_L(D)$. Hence, it suffices to show $\mathcal{R}_L^{(out)}(D) \subseteq \mathcal{R}_L^{(out)}(D)$ and $\mathcal{R}_L^{(in)}(D) \subseteq \mathcal{R}_L^{(in)}(D)$ to prove Theorem 2. Proofs of those two inclusions will be given in Section V. We can directly prove $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(out)}(D)$ in a manner similar to that of Oohama [10]. For the detail of the direct proof of $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(out)}(D)$, see Appendix B.

An essential difference between $\mathcal{R}_L^{(out)}(D)$ and $\mathcal{R}_L^{(in)}(D)$ is the difference between $J_S(D, r_0, r^{L-2}, r_S|r_S)$ in the definition of $\mathcal{R}_L^{(out)}(D)$ and $K_S(r_S|r_S)$ in the definition of $\mathcal{R}_L^{(in)}(D)$. By Property 4 part a) and the definitions of $\mathcal{R}_L^{(out)}(D, r^L_0)$ and $\mathcal{R}_L^{(in)}(r^L_0)$, if $r^L_0 \in B_L(D)$, then,

$$\mathcal{R}_L^{(out)}(D, r^L_0) = \mathcal{R}_L^{(in)}(r^L_0).$$
This gap suggests a possibility that in some cases those two bounds match. In the following we present a sufficient condition for $R_L^{(\text{out})}(D) \subseteq R_L^{(\text{in})}(D)$. We consider the following condition on $G(D, r_0, r^{L-2})$.

Condition: For each $l = 1, 2, \ldots, L-2$, $e^{2r_l}G(D, r_0, r^{L-2})$ is a monotone increasing function of $r_l$.

We call the above condition the MI condition. The following proposition.

Fig. 4. TS conditions in the case of $L = 2$ and the case of $L = 3$ and $Z_2 = 0$.

For each $l = 1, 2, \ldots, L-2$, $e^{2r_l}G(D, r_0, r^{L-2})$ is a monotone increasing function of $r_l$.

We call the above condition the MI condition. The following proposition.

Proposition 1: If

$$\sum_{k=0}^{L-2} \frac{\sigma^2_{Z_k+1}}{\sigma^2_{N_l}} \left(1 + \sigma^2_{Z_{k+1}} f^*_{k+1}\right) \prod_{j=0}^{k} \left(1 + \sigma^2_{Z_j} f^*_{j}\right)^2 \leq 1 \quad (8)$$

hold for $l = 1, 2, \ldots, L-2$, then, $G(D, r_0, r^{L-2})$ satisfies the MI condition.

Proof of the proposition will be given in Appendix A. It can be seen that the proposition says that for $L \geq 3$, the MI condition holds for relatively small values of $\sigma^2_{Z_l}, l = 2, \ldots, L-1$. In particular, when $L = 3$, the sufficient condition given by (8) is

$$\frac{\sigma^2_{Z_2}}{\sigma^2_{N_1}} \left(1 + \sigma^2_{Z_2} \left(\frac{1}{\sigma^2_{N_1}} + \frac{1}{\sigma^2_{N_2}}\right)\right) \leq 1.$$

Solving the above inequality with respect to $\sigma^2_{Z_2}$, we have

$$\sigma^2_{Z_2} \leq \frac{2}{1 + 4\sigma^2_{N_1} \left(\frac{1}{\sigma^2_{N_1}} + \frac{1}{\sigma^2_{N_2}}\right)} \cdot \sigma^2_{N_1}.$$

The TS condition in the case of $L = 3$ is shown in Fig. 5.

C. Binary Tree Structure Condition

As a corollary property of Gaussian source Tavildar et al. [1] introduced a binary Gauss Markov tree structure condition. They studied a full characterization of the rate distortion region for Gaussian sources with this binary tree structure. In this subsection we describe their result and compare it with our results.

We first explain the binary tree structure introduced by them. Let $k$ be a positive integer. We consider the case where $L = 2^k$. Let $\eta_{i,j}, 1 \leq i \leq 2^j, 1 \leq j \leq k$, be zero mean independent Gaussian random variables with variance $\sigma^2_{\eta_{i,j}}$. Those $2^{k+1} - 2$ random variables are independent of $X_0$. Define the sequence of Gaussian random variables $\{Y_{i,j}\}_{1 \leq i \leq 2^j, 0 \leq j \leq k}$ by the following recursion:

$$Y_{i,0} = X_0,$$
$$Y_{i,j} = Y_{i,j-1} + \eta_{i,j},$$
$$X_i = Y_{i,k},$$

for $1 \leq i \leq 2^j, 0 \leq j \leq k$.

where $[a]$ stands for the smallest integer not below $a$. We say that for $L = 2^k$ the Gaussian source $(X_0, X_1, \ldots, X_L)$
proof of Theorem 4. We think that the method of Tavildar we have integer property as the TS condition in the case of \( N = 0 \).

The BTS condition in this case is shown in Fig. 7. In general the set of Gaussian sources satisfying the TS condition and \( Z_1 = 0 \) can be embedded into the set of Gaussian sources satisfying BTS condition.

The communication system treated by Tavildar et al. is shown in Fig. 8. It can be seen from this figure that their \( X_1 \) becomes independent of \( (X_2, X_3, X_4) \) and \( (X_2, X_3, X_4) \) has the same correlation property as the TS condition in the case of \( L = 3 \) and \( Z_1 = 0 \). The BTS condition in this case is shown in Fig. 7. In general the set of Gaussian sources satisfying the TS condition and \( Z_1 = 0 \) can be embedded into the set of Gaussian sources satisfying BTS condition.

The communication system treated by Tavildar et al. is shown in Fig. 8. It can be seen from this figure that their problem set up is slightly different from ours. In their communication system there is no encoder that can directly access to the source \( X_0 \). Tavildar et al. studied a characterization of the rate distortion region \( R_L(D) \cap \{ R_0 = 0 \} \) for Gaussian sources with the binary tree structure and succeeded in it. Their result is the following.

**Theorem 4 (Tavildar et al. [11]):** When \( L = 2^k \) for some integer \( k \) and \( (X_0, X_1, \cdots, X_L) \) satisfies the BTS condition, we have

\[
R_L(D) \cap \{ R_0 = 0 \} = \tilde{R}_L^{(in)}(D) \cap \{ R_0 = 0 \}.
\]

From the above theorem we have the following corollary.

**Corollary 1 (Tavildar et al. [11]):** When \( (X_0, X_1, \cdots, X_L) \) satisfies the TS condition and \( Z_1 = 0 \), we have

\[
R_L(D) \cap \{ R_0 = 0 \} = \tilde{R}_L^{(in)}(D) \cap \{ R_0 = 0 \}.
\]

The BTS condition differs from the TS condition in its symmetrical property, which plays an essential role in the proof of Theorem 4. We think that the method of Tavildar et al. [11] is applicable to the general case where \( Z_1 \) is not constant and \( R_0 > 0 \) and that \( R_L(D) = \tilde{R}_L^{(in)}(D) \) still holds in this general case.

Unfortunately, our approach developed in [10] and this paper cannot establish \( R_L(D) = \tilde{R}_L^{(in)}(D) \) for Gaussian sources satisfying the TS condition without requiring the condition on the variances of \( Z_i, 2 \leq i \leq L - 1 \) and \( N_i, 1 \leq i \leq L \), specified with (8) in Proposition 1. However, we think that our work in [10] had provided an important step toward the full characterization of the rate distortion region established by Tavildar et al. [11].

IV. SUM RATE PART OF THE RATE DISTORTION REGION

In this section we state our result on the rate sum part of \( R_L(D) \). Set

\[
R_{sum,L}^{(l)}(D, R_0) \triangleq \min_{r^L : f_0(r^L) \geq D, R_0} \frac{J_L(D, R_0, r^{L-2}, r^L)},
\]

\[
R_{sum,L}^{(u)}(D, R_0) \triangleq \min_{r^L : f_0(r^L) \geq D, R_0} K_L(r^L).
\]

Let \( \hat{R}_{sum,L}^{(l)}(D, R_0) \) be the minimum sum rate for \( \tilde{R}_L^{(out)}(D) \), that is,

\[
\hat{R}_{sum,L}^{(l)}(D, R_0) \triangleq \min_{(R_0, R_1, \cdots, R_L) \in \tilde{R}_L^{(out)}} \left\{ \sum_{i=1}^{L} R_i \right\}.
\]

Then, it immediately follows from Theorem 4 that we have the following corollary.

**Corollary 2:** For Gaussian sources with the TS condition

\[
R_{sum,L}^{(l)}(D, R_0) \leq \hat{R}_{sum,L}^{(l)}(D, R_0) \leq R_{sum,L}^{(u)}(D, R_0).
\]

On the other hand, we have the following lemma.

**Lemma 2:** For Gaussian sources with the TS condition, we have

\[
R_{sum,L}^{(l)}(D, R_0) \geq R_{sum,L}^{(u)}(D, R_0).
\]

Proof of this lemma will be given in Section V. Combining Corollary 2 and Lemma 2 we have the following.
Theorem 5: For Gaussian sources with the TS condition
\[ R_{\text{sum},L}(D, R_0) = R_{\text{sum},L}^{(a)}(D, R_0) = R_{\text{sum},L}^{(l)}(D, R_0) \]
\[ = \min_{r^L : f_0(r^L) = g_0(D, R_0)} \left[ \sum_{l=1}^{L} r_l + \frac{1}{2} \log \sigma_0^2 D \right] \]
\[ - R_0 + \frac{1}{2} \log \frac{\sigma_0^2}{D}. \]

The optimal sum rate \( R_{\text{sum},L}(D, R_0) \) has a form of optimization problem. In the remaining part of this section we deal with this optimization problem. For \( 1 \leq l \leq L \), set \( \sigma_N^2 = \sigma_1^2, \sigma_0^2 = \epsilon_l \sigma_1^2 \). By the TS condition \( \epsilon_\ell \) should be one. For some class of Gaussian source satisfying TS condition, we solve the optimization problem to derive a parametric form of \( R_{\text{sum},L}(D, R_0) \). The recursion (5) is
\[ f_l(r_L) = \frac{1}{\sigma_l^2} \left( e^{2r_l} - 1 \right), \]
\[ f_{l-1}(r_{l-1}) = \frac{f_l(r_L)}{1 + \epsilon_l \sigma_l^2 f_l(r_L)^2} + \frac{1}{\sigma_{l-1}^2} \left( 1 - e^{-2r_{l-1}} \right) \]
for \( L \geq l \geq 2 \),
\[ f_0(r^L) = \frac{f_l(r_L)}{1 + \epsilon_l \sigma_l^2 f_l(r_L)^2}. \]

The optimization problem presenting \( R_{\text{sum},L}(D, R_0) \) is
\[ R_{\text{sum},L}(D, R_0) = \min_{r^L : f_0(r^L) = g_0(D, R_0)} \left[ \sum_{l=1}^{L} r_l + \frac{1}{2} \log \left( 1 + \epsilon_l \sigma_l^2 f_l(r_L)^2 \right) \right] \]
\[ - R_0 + \frac{1}{2} \log \frac{\sigma_0^2}{D}. \]

Set
\[ \alpha_l \triangleq \frac{\sigma_l^2 f_l(r_L)^2}{1 + \epsilon_l \sigma_l^2 f_l(r_L)^2}, \quad \text{for } L \geq l \geq 1. \]

By the above transformation, we transform the variable \( r^L \) into \( \alpha^L \). From (11), we have
\[ f_l = f_l(r_L) = \frac{1}{\sigma_l^2} \cdot \frac{\alpha_l}{1 - \epsilon_l \alpha_l}, \quad \text{for } L \geq l \geq 1. \]

Note that \( f_l \geq 0 \) for \( L \geq l \geq 1 \). Then, form (12), \( \alpha_l, L \geq l \geq 1 \) must satisfy \( 0 \leq \alpha_l \leq \epsilon_l^{-1} \). For \( L \geq l \geq 2 \), set \( \tau_l \triangleq \sigma_l^2 / \sigma_{l-1}^2 \). From (10) and (12), we have
\[ e^{-2r_{l-1}} = 1 - \frac{\alpha_{l-1}}{1 - \epsilon_l \alpha_{l-1}} + \frac{\alpha_l}{\tau_l}, \quad \text{for } L \geq l \geq 2. \]

Since \( r_{l-1} \geq 0 \) for \( L \geq l \geq 2 \) and (13), \( \alpha_l, L \geq l \geq 2 \) must satisfy
\[ 0 \leq \alpha_l \leq \frac{\tau_l \alpha_{l-1}}{1 - \epsilon_l \alpha_{l-1}}, \]
\[ \tau_l \left( \frac{1}{1 - \epsilon_l \alpha_{l-1}} - 1 \right) < \alpha_l < \epsilon_l^{-1}. \]

Let \( \prod_{l=1}^{L} [0, \epsilon_l^{-1}] \) be a \( L \) direct product of the semi-open intervals \( [0, \epsilon_l^{-1}], 1 \leq l \leq L \). Let \( A_L \) be a set of all \( L \) dimensional vectors \( \alpha^L \in \prod_{l=1}^{L} [0, \epsilon_l^{-1}] \) that satisfy (14).

Using \( \alpha^L \), \( R_{\text{sum},L}(D, R_0) \) is rewritten as
\[ R_{\text{sum},L}(D, R_0) = \min_{\alpha_L \in A_L} \left\{ \left| \sum_{l=1}^{L} \left( \log \left( 1 - \frac{\alpha_l}{1 - \epsilon_l \alpha_l} + \frac{\alpha_{l+1}}{\tau_{l+1}} \right) \right) \right| + \log(1 - \epsilon_l \alpha_l) \right\} + \log(1 - \epsilon_l \alpha_l). \]

Then, we have the following lemma.

Lemma 3: For \( \alpha^L \in A(\alpha), \zeta(\alpha^L) \) is strictly concave with respect to \( \alpha^L \).

Proof of this lemma will be given in Appendix C. It can be seen from this lemma that if we can find \( \theta^L_{\tau} \) satisfying \( \nabla \zeta |_{\alpha^L = \theta^L_{\tau}} = 0 \) and \( \theta^L_{\tau} \in A_L(\sigma_1^2 g_0(D, R_0)) \), this \( \theta^L_{\tau} \) is the unique vector which attains \( R_{\text{sum},L}(D, R_0) \). We shall give such \( \theta^L_{\tau} \) in an explicit form of recursion. Let \( \omega \in [0, 1] \). Define the sequence of functions \( \{ \theta_l(\omega) \}^L_{l=1} \) by the following recursion:
\[ \theta_L(\omega) = \omega, \]
\[ \theta_{L-1}(\omega) = \frac{\theta_L(\omega) - 1}{1 + \epsilon_l \theta_{L-1}(\omega)} + \tau_l \]
\[ \theta_{L-1}(\omega) = \frac{\theta_L(\omega) - 1}{1 + \epsilon_l \theta_{L-1}(\omega)} + \tau_l \]
for \( L - 1 \geq l \geq 2 \).
b) \[ \nabla \zeta_{|\alpha^2_{L} = \theta^2_{L} (\omega)} = 0. \]

c) For each \( L - 1 \geq l \geq 1 \), \( \theta_l (\omega) \) is differentiable with respect to \( \omega \in [0, 1] \) and satisfies the following:

\[
\frac{d \theta_l}{d \omega} \geq \left[ \frac{2 (L-l)}{1 + \epsilon_l \omega \tau_l} \left( \frac{\sigma_l^2}{\sigma_l^2} + 1 \right) \right] - \frac{L}{1 + \epsilon_l \omega \tau_l} \left( \frac{1}{1 + \epsilon_l \omega \tau_l} \right)^2
\times \sigma_l^2 \prod_{j=L+1}^{l-1} \left( 1 + \frac{1}{1 + \epsilon_j \omega \tau_j} \right)^2
\geq (L - l + 1) \sigma_l^2 \prod_{j=L+1}^{l-1} \left( 1 + \frac{1}{1 + \epsilon_j \omega \tau_j} \right)^2 > 0.
\]

This implies that for each \( 1 \leq l \leq L \), the mapping \( \omega \in [0, 1] \to \theta_l (\omega) \) is an injection.

Proof of this lemma is given in Appendix D. From this lemma, we obtain the following theorem.

Theorem 6: Let \( \{ \theta_l (\omega) \}_l \) be a sequence of functions defined by (15). Suppose that the Gaussian source satisfies the TS condition and the condition (16) stated in Lemma 4. Then, we have the following parametric form of \( R_{\text{sum}, L}(D, R_0) \) with the parameter \( \omega \in [0, 1] \):

\[
\sigma_l^2 g_0 (D, R_0) = \sigma_l^2 \left[ \frac{-2 \tau_0}{\sigma_0^2} - \frac{1}{\sigma_0^2} \right] = \theta_1 (\omega),
\]

\[
R_{\text{sum}, L}(D, R_0) = \left( -\frac{1}{2} \right) \sum_{l=1}^{L-1} \left\{ \log \left( 1 - \epsilon_l \theta_l (\omega) + \frac{\theta_{l+1} (\omega)}{\tau_{l+1}} \right) + \log(1 - \epsilon_l \theta_l (\omega)) \right\} + \log(1 - \omega) - R_0 + \frac{1}{2} \log \frac{\sigma_0^2}{D}.
\]

When \( \epsilon_l = 0 \) for \( L - 1 \geq l \geq 1 \) and \( \tau_l = 1 \) for \( L \geq l \geq 2 \), the recursion (15) becomes the following:

\[
\begin{align*}
\theta_L (\omega) &= \omega, \theta_{L-1} (\omega) = 2 \omega, \\
\theta_l (\omega) &= 2 \omega - \frac{\theta_{l+1} (\omega)}{\tau_{l+1}},
\end{align*}
\]

for \( L - 1 \geq l \geq 2 \).

Solving (18), we obtain \( \theta_l (\omega) = (L - l + 1) \omega \). The parametric form of \( R_{\text{sum}, L}(D, R_0) \) becomes

\[
\sigma_l^2 g_0 (D, R_0) = \theta_1 (\omega) = L \omega,
\]

\[
R_{\text{sum}, L}(D, R_0) = \left( -\frac{L}{2} \right) \log(1 - \omega) - R_0 + \frac{1}{2} \log \frac{\sigma_0^2}{D}.
\]

From (19), we have

\[
R_{\text{sum}, L}(D, R_0) = \left( -\frac{L}{2} \right) \log \left( 1 - \frac{\sigma_l^2}{L} g_0 (D, R_0) \right) - R_0 + \frac{1}{2} \log \frac{\sigma_0^2}{D}. \tag{20}
\]

In particular, by letting \( R_0 = 0 \) and \( L \to \infty \) in (20), we have

\[
\lim_{L \to \infty} R_{\text{sum}, L}(D, 0) = \frac{1}{2} \sigma_0^2 g_0 (D, 0) + \frac{1}{2} \log \frac{\sigma_0^2}{D} = \frac{\sigma_1^2}{2 \sigma_0^2} \left[ \frac{\sigma_0^2}{D} - 1 \right] + \frac{1}{2} \log \frac{\sigma_0^2}{D}.
\]

The above formula coincides with the rate distortion function for the quadratic Gaussian CEO problem obtained by Oohama [7]. Hence, our solution to \( R_{\text{sum}, L}(D, R_0) \) includes the previous result on the Gaussian CEO problem as a special case.

V. PROOFS OF THE RESULTS

In this section, we prove Theorem 2 and Lemma 1 stated in Section III and prove Lemma 2 stated in Section IV.

A. Derivation of the Outer Bound

In this subsection, we prove \( R_{L}^{(\text{out})} (D) \subseteq R_{L}^{(\text{out})} (D) \) stated in Theorem 2.

Proof of \( R_{L}^{(\text{out})} (D) \subseteq R_{L}^{(\text{out})} (D) \): Set

\[
\Delta = \frac{1}{2} \log \left[ \frac{G(D, r_0, r^{L-2}) \sigma_0^2}{D} \right] + \sum_{i=1}^{L} r_i - R_0
\]

We first observe that

\[
\Delta = \frac{1}{2} \log \left[ \frac{G(D, r_0, r^{L-2}) \sigma_0^2}{D} \right] + \sum_{i=1}^{L} 2 r_i - 2 r_0 + \sum_{i=1}^{L} 2 r_i - 2 r_0
\]

\[
\geq \frac{1}{2} \log \left[ \frac{G(D, r_0, r^{L-2}) \sigma_0^2}{D} \right] + \sum_{i=1}^{L} 2 r_i - 2 r_0
\]

\[
= J_S (D, r_0, r^{L-2}, r_S | r_S, R_0).
\]

Then, we have the following.

\[
\vec{R}_{L}^{(\text{out})} (D) \subseteq \begin{cases} (R_0, R^L) : & \text{There exists a nonnegative vector} \\
 & (r_0, r^L) \text{ such that} \\
 & R_0 \geq r_0 \geq \frac{1}{2} \log \left[ \frac{\sigma_0^2}{D} \right], \\
 & \sum_{i \in S} R_i \geq \hat{J}_S (D, r_0, r^{L-2}, r_S | r_S, R_0) \\
 & \text{for any } S \subseteq \Lambda. \}
\end{cases}
\]

\[
\begin{cases} (R_0, R^L) : & \text{There exists a nonnegative vector} \\
 & r^L \text{ such that} \\
 & R_0 \geq \frac{1}{2} \log \left[ \frac{\sigma_0^2}{D} \right], \\
 & \sum_{i \in S} R_i \geq \hat{J}_S (D, R_0, r^{L-2}, r_S | r_S, R_0) \\
 & \text{for any } S \subseteq \Lambda. \}
\end{cases}
\]
\[ \leq \left\{ (R_0, R^L) : \text{There exists a nonnegative vector} \right. \\
\left. (r_0, r^L) \text{ such that} \right. \\
R_0 \geq r_0 \geq \frac{1}{2} \log \left[ \frac{\sigma^2_0}{1 + \sigma^2_0 \rho_0(r^L)} \right], \\
\sum_{i \in S} R_i \geq \tilde{J}_S(D, r_0, r^L - 2, \{r \}_{r \leq \infty}, r_0) \\
\text{for any } S \subseteq \Lambda. \right\}. \\
\leq \left\{ (R_0, R^L) : \text{There exists a nonnegative vector} \right. \\
(r_0, r^L) \text{ such that} \right. \\
R_0 \geq r_0 \geq \frac{1}{2} \log \left[ \frac{\sigma^2_0}{1 + \sigma^2_0 \rho_0(r^L)} \right], \\
\sum_{i \in S} R_i \geq \tilde{J}_S(D, r_0, r^L - 2, \{r \}_{r \leq \infty}, r_0) \\
\text{for any } S \subseteq \Lambda. \right\}. \\
(23)

Step (a) follows from the definition of \( \tilde{J}_S(D, r_0, r^L - 2, \{r \}_{r \leq \infty}, r_0) \) and the nonnegative property of \( R^L \). Step (b) follows from that \( \tilde{J}_S(D, r_0, r^L - 2, \{r \}_{r \leq \infty}, r_0) \) is a monotone decreasing function of \( r_0 \). Step (c) follows from (21). Thus \( \mathcal{R}^\text{(out)}_L(D) \subseteq \mathcal{R}^\text{(out)}_L(D) \) is proved.

B. Derivation of the Inner Bound

In this subsection we prove \( \mathcal{R}^\text{(in)}_L(D) \subseteq \mathcal{R}^\text{(in)}_L(D) \) stated in Theorem [2]. We first derive a preliminary result on a form of \( \mathcal{R}^\text{(in)}_L(D) \). Fix \( R_0 \geq r_0 \) and set

\[
\mathcal{R}^\text{(in)}_L(r^L_0 | R_0) \triangleq \left\{ (R_1, \ldots, R_L) : \right. \\
(R_0, R_1, \ldots, R_L) \in \mathcal{R}^\text{(in)}_L(r^L_0) \right\}.
\]

Let \( (\Lambda, \tilde{\rho}) \) be a co-polymatroid defined in Property [4]. Expression of \( \mathcal{R}^\text{(in)}_L(r^L_0 | R_0) \) using \( (\Lambda, \tilde{\rho}) \) is

\[
\mathcal{R}^\text{(in)}_L(r^L_0 | R_0) = \left\{ (R_1, \ldots, R_L) : \sum_{i \in S} R_i \geq \tilde{\rho}_S (r_0 | r^L) \right. \\
\text{for any } S \subseteq \Lambda. \right\}.
\]

The set \( \mathcal{R}^\text{(in)}_L(r^L_0 | R_0) \) forms a kind of polytope, which is called a co-polymatroidal polytope in the terminology of matroid theory. It is well known as a property of this kind of polytope that the polytope \( \mathcal{R}^\text{(in)}_L(r^L_0 | R_0) \) consists of \( L! \) end-points whose components are given by

\[
R_{\pi(i)} = \tilde{\rho}(\pi|\pi(1), \ldots, \pi(L)) [r(\pi(1), \ldots, \pi(L)) | r(\pi(1), \ldots, \pi(L))], \\
\text{for } i = 1, 2, \ldots, L - 1, \\
R_{\pi(L)} = \tilde{\rho}(\pi|\pi(L)) [r(\pi(1), \ldots, \pi(L)) | r(\pi(1), \ldots, \pi(L))],
\]

where \( \pi = (1, \pi(1), \ldots, \pi(i), \ldots, \pi(L)) \in \Pi \)

is an arbitrary permutation on \( \Lambda \). For each \( \pi \in \Pi \) and \( r^L_0 \in \mathcal{B}_L(D) \), let \( \mathcal{R}^\text{(in)}_{\pi, L}(r^L_0) \) be the set of nonnegative vectors \( (R_0, R_1, \ldots, R_L) \) satisfying

\[
\begin{align*}
R_0 & \geq r_0 \\
R_{\pi(i)} & \geq \tilde{\rho}(\pi(i), \pi(i+1), \ldots, \pi(L)) [r(\pi(i), \pi(i+1), \ldots, \pi(L)) | r(\pi(i), \pi(i))], \\
& \text{for } i = 1, 2, \ldots, L - 1, \\
R_{\pi(L)} & \geq \tilde{\rho}(\pi(L)) [r(\pi(1), \ldots, \pi(L)) | r(\pi(1), \ldots, \pi(L - 1))].
\end{align*}
\]

Then, we have

\[
\mathcal{R}^\text{(in)}_L(r^L_0) = \text{conv} \left\{ \bigcup_{\pi \in \Pi} \mathcal{R}^\text{(in)}_{\pi, L}(r^L_0) \right\}.
\]

Proof of \( \mathcal{R}^\text{(in)}_L(D) \subseteq \mathcal{R}^\text{(in)}_L(D) \): Fix \( \pi \in \Pi \) and \( r^L_0 \in \mathcal{B}_L(D) \) arbitrary. By (23), it suffices to show that for \( r^L_0 \in \mathcal{B}_L(D) \), \( \mathcal{R}^\text{(in)}_L(r^L_0) \subseteq \mathcal{R}^\text{(in)}_L(D) \) to prove \( \mathcal{R}^\text{(in)}_L(D) \subseteq \mathcal{R}^\text{(in)}_L(D) \). Let \( V_i, i \in \{0\} \cup \Lambda \) be independent Gaussian random variables \( \mu \) and \( \sigma^2 \). Suppose that \( V^L_0 \) is independent of \( X^L_0 \). Define the Gaussian random variables \( U_i, i \in \{0\} \cup \Lambda \) by

\[
U_i \triangleq X_i + V_i, \quad i \in \{0\} \cup \Lambda.
\]

From the above definition it is obvious that

\[
\begin{align*}
U^L_0 & \Rightarrow X^L_0 \Rightarrow X_0 \Rightarrow U_0, \\
U^L_S & \Rightarrow X^L_{S \cup \{0\}} \Rightarrow X^L_S \Rightarrow U^L_S, \\
\text{for any } S \subseteq \Lambda. 
\end{align*}
\]

For given \( r_i \geq 0, i \in S \) \( D > 0 \), set \( \frac{1}{\sigma^2_i} = \frac{e^{2r_i - 1}}{\sigma^2_i} \) \( \text{when } r_i > 0 \). When \( r_i = 0 \), we choose \( U_i \) so that \( U_i \) takes the constant value zero. Define the sequence of random variables \( \{\Omega_i\}_{i=0} \) by

\[
\begin{align*}
\Omega_{L-1} & = \frac{1 - e^{-2r_i - 1}}{\sigma^2_{L-1}} \cdot U_{L-1} + \frac{1 - e^{-2r_i}}{\sigma^2_U} \cdot U_L \\
\Omega_i & = \frac{1}{1 + \frac{e^{-2r_i}}{\sigma^2_{L-1}}} \cdot \Omega_{L-1} + \frac{1 - e^{-2r_i}}{\sigma^2_{L-1}} \cdot U_i \\
\Omega_0 & = \frac{1}{1 + \frac{e^{-2r_i}}{\sigma^2_U}} \cdot \Omega_{L-1}.
\end{align*}
\]

Note that \( \Omega_0 = \Omega_0(U^L_0) \) is a linear function of \( U^L_0 \). Then, by an elementary computation, we have

\[
X_0 = \frac{1}{\sigma^2_U} + \frac{1}{\sigma^2_{L-1}} + f_0(r^L_0), \\
\frac{1}{\sigma^2_0}, + \frac{1}{\sigma^2_{L-1}} + f_0(r^L_0)
\]

where \( \tilde{N}_0 \) is a zero mean Gaussian random variable with variance

\[
\left( \frac{1}{\sigma^2_{L-1}} + \frac{1}{\sigma^2_U} + f_0(r^L_0) \right)^{-1}.
\]

\( \tilde{N}_0 \) is independent of \( (U_0, U^L_0) \). Since \( r^L_0 \in \mathcal{B}_L(D) \), we have

\[
\frac{e^{-2r_0}}{D} \leq f_0(r^L_0) \leq \frac{1}{\sigma^2_{L-1}}.
\]

We put

\[
\frac{1}{\sigma^2_{L-1}} = \frac{e^{-2r_0}}{D}.
\]
Then, from (27) and (28), we have
\[
\left[\frac{1}{\sigma_0} + \frac{1}{\sigma_0 r_0} + f_0(r_L) \right]^{-1} = \left[\frac{1}{\sigma_0} + \frac{1-e^{-2\rho_0}}{D} + f_0(r_L) \right]^{-1} \leq D.
\] (29)

Based on (26), (28), and (29), define the linear function \( \tilde{\psi} \) of \((U_0, U_L)\) by
\[
\tilde{\psi}(U_0, U_L) = \left[\frac{1}{\sigma_0} + \frac{1-e^{-2\rho_0}}{D} + f_0(r_L) \right]^{-1} X_0 - \frac{1}{\sigma_0} Y_0 + \Omega_0(U_L).
\]
Then, we obtain
\[
E \left[ X_0 - \tilde{\psi}(U_0, U_L) \right]^2 = \text{Var} \left[ \tilde{N}_0 \right] = \left[\frac{1}{\sigma_0} + \frac{1-e^{-2\rho_0}}{D} + f_0(r_L) \right]^{-1} \leq D.
\] (30)

From (24) and (30), we have \((U_0, U_L) \in G(D)\). By simple computations, we can show that
\[
\begin{align*}
\rho_0 &= I(X_0; U_0|U_L), \\
r_i &= I(X_i; U_i|X_0Y^{L-1}),
\end{align*}
\] for any \(i \in \Lambda\),
\[
\frac{1}{2} \log \left[ F_S(r_S) \cdot \left(1 + \sigma_0^2 f_0(r_S) \right) \right] = I(X_0Y^{L-1}; U_S),
\] for any \(i \in \Lambda\).

Using (24) and (31), the \(L+1\) inequalities of (23) are rewritten as
\[
R_0 \geq I(X_0; U_0|U_L),
\]
\[
R_{\pi(i)} \geq I(X_0Y^{L-1}; U_{\pi(S_i)}|U_{\pi(S_i')}),
\]
\[
+ I(X_{\pi(i)}; U_{\pi(i)}|X_0Y^{L-1}) - I(X_0Y^{L-1}; U_{\pi(S_i+1)}|U_{\pi(S_i+1)})
\]
\[
= I(X_0Y^{L-1}; U_{\pi(i)}; U_{\pi(S_i)}),
\]
\[
I(X_{\pi(i)}; U_{\pi(i)}|X_0Y^{L-1}U_{\pi(S_i)}) = I(X_0Y^{L-1}X_{\pi(i)}; U_{\pi(i)}|U_{\pi(S_i)}),
\]
\[
I(X_{\pi(i)}; U_{\pi(i)}|U_{\pi(S_i')}) = I(X_0Y^{L-1}X_{\pi(i)}; U_{\pi(i)}|U_{\pi(S_i')}),
\]
\[
R_{\pi_L} \geq 0, \quad \text{for } i = 1, 2, \ldots, L.
\]

Thus, we conclude that \((R_0, R_{\pi(1)}, \ldots, R_{\pi(L)} \in R^{(\text{in})}_{\pi(L)}(D)\).

\[\blacksquare\]

\textbf{C. Proofs of Lemmas 2 and 3}

In this subsection we prove Lemmas 2 and 3. We first present a preliminary observation on \(R^{(\text{out})}_L(D)\). Fix \(R_0 \geq r_0\) arbitrary and set
\[
R^{(\text{out})}_L(D, r_L^0|R_0) = \{(R_1, \ldots, R_L) : (R_0, R_1, \ldots, R_L) \in R^{(\text{out})}_L(D, r_L^0)\}.
\]
Let \((\Lambda, \rho) = \{\rho_S(r_S|r_{S'})\} \subseteq \Lambda\) be a co-polymatroid defined in Property 3. Expression of \(R^{(\text{out})}_L(D, r_L^0|R_0)\) using \((\Lambda, \rho)\) is
\[
R^{(\text{out})}_L(D, r_L^0|R_0) = \{(R_1, \ldots, R_L) : \sum_{i \in S} R_i \geq \rho_S(r_S|r_{S'}) \text{ for all } S \subseteq \Lambda \}\).
\]
The set \(R^{(\text{out})}_L(D, r_L^0|R_0)\) forms a co-polymatroidal polytope. The polytope \(R^{(\text{out})}_L(D, r_L^0|R_0)\) consists of \(L!\) end-points whose components are given by
\[
\begin{align*}
R_{\pi(i)} &= \rho(\pi(i), \ldots, \pi(1), r_L, \pi(L)) \left(r_{\pi(1)}, \ldots, r_{\pi(L)} \right) \\
&\quad - \rho(\pi(i+1), \ldots, r_L, \pi(L)) \left(r_{\pi(1)}, \ldots, r_{\pi(L)} \right) \\
&\quad \text{for } i = 1, 2, \ldots, L - 1, \\
R_{\pi_L} &= \rho(\pi(L)) \left(r_{\pi(1)}, \ldots, r_{\pi(L-1)} \right),
\end{align*}
\] (32)
where
\[
\pi = \left(1, \ldots, i, \pi(1), \ldots, \pi(L) \right) \in \Pi.
\]
For each \(\pi \in \Pi\) and \(l = 1, 2, \ldots, L\), set
\[
B_{\pi,i}(D) = \{r_L^0 : r_L^0 \in B_{\pi,i}(D) \} \quad \text{and} \quad \partial B_{\pi,i}(D) = \{r_L^0 : r_L^0 \in \partial B_{\pi,i}(D) \}.
\]

In particular, when \(\pi\) is the identity map, we omit \(\pi\) to write \(B_i(D)\) and \(\partial B_i(D)\). By Property 3 when \(r_L^0 \in B_{\pi,i}(D)\), the end-point given by (32) becomes
\[
\begin{align*}
R_{\pi(i)} &= \rho(\pi(i), \ldots, \pi(1), r_L, \pi(L)) \left(r_{\pi(1)}, \ldots, r_{\pi(L)} \right) \\
&\quad - \rho(\pi(i+1), \ldots, r_L, \pi(L)) \left(r_{\pi(1)}, \ldots, r_{\pi(L)} \right) \\
&\quad \text{for } i = 1, 2, \ldots, L - 1, \\
R_{\pi_L} &= \rho(\pi(L)) \left(r_{\pi(1)}, \ldots, r_{\pi(L-1)} \right),
\end{align*}
\] (33)
Next, we present a lemma on a property of \(G(D, r_0, r_L^{L-1})\).

\textbf{Lemma 5.} For \(r_L^0 \in B_L(D)\), \(G(D, r_0, r_L^{L-2})\) is computed as
\[
G(D, r_0, r_L^{L-2})|_{r_L^0=0} = \prod_{k=1}^l \left[1 + \sigma_0^2 g_k(D, r_0, r_{k-1}^L) \right].
\]
\textbf{Proof:} By Property 2(c), \(l + 1 \leq k \leq L\),
\[
0 \leq g_k(D, r_0, r_{k-1}^L) \leq f(r_L^0) = 0.
\]

Hence, the result of Lemma 5 follows.

\textbf{Proof of Lemma 7} Fix \(\pi \in \Pi\) and \(r_L^0 \in B_L(D)\) arbitrary. Let \((R_0, R_L)\) be a nonnegative rate vector such that \(R_0 \geq r_0\) and \(L\) components of \(R_L\) satisfy (32). To prove Lemma 1 it suffices to show that this nonnegative vector belongs to \(R^{(\text{in})}_L(D)\). For \(l = 1, 2, \ldots, L\), we prove the claim that under the MI condition, if \(r_L^0 \in B_{\pi,i}(D)\), then the rate vector \((R_0, R_L)\) satisfies \(R_0 \geq r_0\) and (33) belongs to \(R^{(\text{in})}_L(D)\). We prove this claim by induction with respect to \(l\). When \(l = 1\), from (33), we have
\[
\begin{align*}
R_{\pi(1)} &= \rho(\pi(1), r_L), \\
R_{\pi(i)} &= 0, \quad \text{for } i = 2, \ldots, L.
\end{align*}
\] (34)
The function $\rho(|\pi(1)|)(r_{\pi(1)})$ is computed as
\[
\rho(|\pi(1)|)(r_{\pi(1)}) = \frac{1}{2} \log^+ \left[ \frac{\sigma_{\min}^2 \sigma^2 e^{-2r_{\pi(1)}^* r_{\pi(1)}}}{\sigma_{\max}^2 \sigma^2 e^{-2r_{\pi(1)}^* r_{\pi(1)}}} \right].
\] (35)

By the above form of $\rho(|\pi(1)|)(r_{\pi(1)})$ and
\[
\frac{\sigma_{\min}^2 \sigma^2 e^{-2r_{\pi(1)}^* r_{\pi(1)}}}{\sigma_{\max}^2 \sigma^2 e^{-2r_{\pi(1)}^* r_{\pi(1)}}} \geq 1,
\]
$\rho(|\pi(1)|)(r_{\pi(1)})$ is positive. Since $r_{0}^L \in B_{\pi,l}(D)$, we can decrease $r_{\pi(1)}$ keeping $r_{0}^L \in B_{\pi,l}(D)$ so that it arrives at $r_{\pi(1)}^* = 0$ or a positive $r_{\pi(1)}^*$ satisfying
\[
(r_{0}^*, r_{\pi(1)}^*, r_{\pi(1)}^* - e^{r_{\pi(1)}^* - r_{0}^L}) = (r_{0}^*, r_{\pi(1)}^*, 0, \ldots, 0) \in \partial B_{\pi,l}(D).
\] (36)

Let $(R_0, R_{\pi(1)}^*, \ldots, R_{\pi(L)}^*)$ be a rate vector corresponding to $(r_0^*, r_{\pi(1)}^*, r_{\pi(1)}^*)$. If $r_{\pi(1)}^* < 0$, then by Property 3 part b), $\rho(|\pi(1)|)(r_{\pi(1)}^*)$ must be zero. This contradicts the fact that $\rho(|\pi(1)|)(r_{\pi(1)}^*)$ is positive. Therefore, $r_{\pi(1)}^*$ must be positive. Then, from (36), we have
\[
(R_0, R_{\pi(1)}^*, \ldots, R_{\pi(L)}^*) \in \mathcal{R}_{\pi,(L)}^{(in)}(D).
\]

On the other hand, by Lemma 5 we have
\[
\frac{G(D, r_0^L, r_{\pi(1)}^* - e^{r_{\pi(1)}^* - r_{0}^L})}{2} \leq \frac{G(D, r_0^L, r_{\pi(1)}^* - e^{r_{\pi(1)}^* - r_{0}^L})}{2} = \sum_{k=1}^{\infty} \left[ 1 + \sigma_{\min}^2 g_k(D, r_0^L, r_{\pi(1)}^* - e^{r_{\pi(1)}^* - r_{0}^L}) \right].
\] (37)

From (35) and (37), we can see that $G(D, r_0^L, r_{\pi(1)}^* - e^{r_{\pi(1)}^* - r_{0}^L})$ does not depend on $r_{\pi(1)}$. This implies that $\rho(|\pi(1)|)(r_{\pi(1)}^*)$ is a monotone increasing function of $r_{\pi(1)}$. Then, we have $R_{\pi(1)}^* \geq R_{\pi(1)}$. Hence, we have
\[
(R_0, R_{\pi(1)}, \ldots, R_{\pi(L)}) \in \mathcal{R}_{\pi,(L)}^{(in)}(D).
\]

Thus, the claim holds for $l = 1$. Assume that the claim holds for $l - 1$. Since $\rho_l(r_{\pi(l)}^*)$ is a monotone increasing function of $r_{\pi(l)}$ on $B_{\pi,l}(D)$, we can decrease $r_{\pi(l)}$ keeping $r_{\pi(l)}^* \in B_{\pi,l}(D)$ so that it arrives at $r_{\pi(l)}^* = 0$ or a positive $r_{\pi(l)}^*$ satisfying
\[
(r_0^*, r_{\pi(l)}^*, r_{\pi(l)}^* - e^{r_{\pi(l)}^* - r_{0}^L}) = (r_0^*, r_{\pi(l)}^*, 0, \ldots, 0) \in \partial B_{\pi,l}(D).
\] (38)

Let $(R_0, R_{\pi(1)}^*, \ldots, R_{\pi(L)}^*)$ be a rate vector corresponding to $(r_0^*, r_{\pi(l)}^*, r_{\pi(l)}^*)$. By Property 4 part b) and the MI condition, the $l$ functions
\[
\rho(|\pi(l)|)(r_{\pi(l)|\pi(l-1)}^*(r_{\pi(l)|\pi(l-1)}, \ldots, r_{\pi(1)|\pi(1)}^*(r_{\pi(1)|\pi(1)}, \ldots, r_{\pi(i-1)|\pi(i-1)}^*)) - \rho(|\pi(i-1)|)(r_{\pi(i-1)|\pi(i-1)}^*(r_{\pi(i-1)|\pi(i-1)}, \ldots, r_{\pi(1)|\pi(1)}^*(r_{\pi(1)|\pi(1)}, \ldots, r_{\pi(i-1)|\pi(i-1)}^*))
\]
for $i = 1, 2, \ldots, l - 1$, appearing in the right members of (33) are monotone increasing functions of $r_{\pi(i)}$. Then, from (33), we have
\[
R_{\pi(i)}^* \geq R_{\pi(i)}^* \text{ for } i = 1, 2, \ldots, l,
\]
\[
R_{\pi(l)}^* = R_{\pi(l)}^* = 0 \text{ for } i = l + 1, \ldots, L.
\] (39)

When $r_{\pi(l)}^* = 0$, we have $(r_0^*, r_{\pi(l)}^*, r_{\pi(l)}^*) \in B_{\pi,l}(D)$. Then, by induction hypothesis, we have
\[
(R_0, R_{\pi(1)}^*, \ldots, R_{\pi(L)}^*) \in \mathcal{R}_{\pi,(L)}^{(in)}(D).
\]

When $r_{\pi(l)}^* > 0$, from (38), we have
\[
(R_0, R_{\pi(1)}^*, \ldots, R_{\pi(L)}^*) \in \mathcal{R}_{\pi,(L)}^{(in)}(D).
\]

Hence, by (39), we have
\[
(R_0, R_{\pi(1)}^*, \ldots, R_{\pi(L)}^*) \in \mathcal{R}_{\pi,(L)}^{(in)}(D).
\]

Thus, the claim holds for $l$. This completes the proof of Lemma 2.

Proof of Lemma 2: For $R_0 > 0$ and for $1 \leq l \leq L$, set
\[
\mathcal{B}_l(D|R_0) \triangleq \{ r^L : (R_0, r^L) \in \mathcal{B}_l(D) \},
\]
\[
\partial \mathcal{B}_l(D|R_0) \triangleq \{ r^L : (R_0, r^L) \in \partial \mathcal{B}_l(D) \}.
\]

We first observe that
\[
R_{\sum,L}(D, R_0) = \min_{1 \leq l \leq L} \left[ \min_{r^L \in \mathcal{B}_l(D|R_0)} J_A(D, R_0, r^L - e^{r_{\pi(l)}^* - r_{0}^L}) \right],
\]
\[
R_{\sum,L}^{(i)}(D, R_0) = \min_{1 \leq l \leq L} \left[ \min_{r^L \in \partial \mathcal{B}_l(D|R_0)} K_A(r^L) \right].
\]

We compute $J_A(D, R_0, r^L - e^{r_{\pi(l)}^* - r_{0}^L}) \forall r^L \in \mathcal{B}_l(D|R_0)$.

From the above formula, we can see that for $r^L \in \mathcal{B}_l(D|R_0)$, $G(D, R_0, r^L - e^{r_{\pi(l)}^* - r_{0}^L}) \forall r^L \in \mathcal{B}_l(D|R_0)$ is a function of $r_{\pi(l)}$. We denote this function by $G(D, R_0, r_{\pi(l)} - e^{r_{\pi(l)}^* - r_{0}^L})$, that is,
\[
G(D, R_0, r_{\pi(l)} - e^{r_{\pi(l)}^* - r_{0}^L}) \triangleq \prod_{k=1}^{l} \left[ 1 + \sigma_{\min}^2 g_k(D, R_0, r_{\pi(l)} - e^{r_{\pi(l)}^* - r_{0}^L}) \right].
\]

Then, for $r^L \in \mathcal{B}_l(D|R_0)$,
\[
J_A(D, R_0, r^L - e^{r_{\pi(l)}^* - r_{0}^L}) \triangleq \frac{1}{2} \log^+ \left[ \frac{G(D, R_0, r^L - e^{r_{\pi(l)}^* - r_{0}^L})}{2} \right] \left[ \frac{\sigma_{\min}^2 \sigma^2 e^{-2r_{\pi(l)}^* r_{\pi(l)}}}{\sigma_{\max}^2 \sigma^2 e^{-2r_{\pi(l)}^* r_{\pi(l)}}} \right] \left[ \frac{1 + \sigma_{\min}^2 g_k(D, R_0, r_{\pi(l)} - e^{r_{\pi(l)}^* - r_{0}^L})}{2} \right].
\]

We denote the right member of (40) by $J_A(D, R_0, r_{\pi(l)} - e^{r_{\pi(l)}^* - r_{0}^L})$. Using this function, $R_{\sum,L}^{(i)}(D, R_0)$ can be written as
\[
R_{\sum,L}^{(i)}(D, R_0) = \min_{1 \leq i \leq L} \left[ \min_{r^L \in \mathcal{B}_l(D|R_0)} J_A(D, R_0, r_{\pi(l)} - e^{r_{\pi(l)}^* - r_{0}^L}) \right].
\]
Note here that \( J_{\Lambda}(D, R_0, r^{l-1}, r^l) \) is a monotone increasing function of \( r_l \). To prove \( R_{\text{sum}, l}(D, R_0) \geq R_{\text{sum}, l}(D, R_0) \), it suffices to show that for \( 1 \leq l \leq L \),

\[
\min_{r^l_1 \in \partial B_l(D|R_0)} J_{\Lambda}(D, R_0, r^{l-1}, r^l) \geq \frac{1}{2} \log^+ \left( \frac{1+\sigma_{2R_l}(D|R_0) \sigma_{2R_l} e^{-2\sigma_{2R_0} r^l}}{D} \right) \cdot \frac{\sigma_{2R_0} e^{-2\sigma_{2R_0} r^l}}{1-\sigma_{2R_l}(D|R_0)} D \cdot \frac{\sigma_{2R_0} e^{-2\sigma_{2R_0} r^l}}{D} .
\]

We prove this claim by induction with respect to \( l \). When \( l = 1 \), the function \( J_{\Lambda}(D, R_0, r_1) \) is computed as

\[
J_{\Lambda}(D, R_0, r_1) = \frac{1}{2} \log^+ \left( \frac{1+\sigma_{2R}(D,R_0) \sigma_{2R} e^{-2\sigma_{2R_0} r^1}}{D} \right) .
\]

Since \( \frac{\sigma_{2R} e^{-2\sigma_{2R_0} r^1}}{D} > 1 \), \( J_{\Lambda}(D, R_0, r_1) \) is positive. Since \( J_{\Lambda}(D, R_0, r_1) \) is a monotone increasing function of \( r_1 \), the minimum of this function is attained by \( r^*_1 = 0 \) or a positive \( r^*_1 \) satisfying \( r^*_1 \in \partial B_1(D|R_0) \). If \( r^*_1 = 0 \), then, by Property 3 part b), \( J_{\Lambda}(D, R_0, r_1) \) must be zero. This contradicts that \( J_{\Lambda}(D, R_0, r_1) \) is positive. Therefore, \( r^*_1 \) must be positive. Then, by \( r^*_1 \in \partial B_1(D|R_0) \), we have

\[
J_{\Lambda}(D, R_0, r_1) \geq J_{\Lambda}(D, R_0, r^*_1) = K_\Lambda(r^*_1) = \frac{1}{2} \log^+ \left( \frac{1+\sigma_{2R_0} e^{-2\sigma_{2R_0} r^*_1}}{D} \right) \cdot \frac{\sigma_{2R_0} e^{-2\sigma_{2R_0} r^*_1}}{1-\sigma_{2R_l}(D|R_0)} D .
\]

Thus, the claim holds for \( l = 1 \). We assume that the claim holds for \( l - 1 \). Since \( J_{\Lambda}(D, R_0, r^{l-1}, r^l) \) is a monotone increasing function of \( r_l \), the minimum of this function is attained by \( r^*_l = 0 \) or a positive \( r^*_l \) satisfying \( (r^{l-1}, r^l) \in \partial B_l(D,R_0) \). When \( r^*_l = 0 \), we have \( r^{l-1} \in \partial B_{l-1}(D|R_0) \) and

\[
J_{\Lambda}(D, R_0, r^{l-1}, r^l) \geq J_{\Lambda}(D, R_0, r^{l-1}, r^l) \quad (41)
\]

Computing \( J_{\Lambda}(D, R_0, r^{l-1}, r^l) \), we obtain

\[
J_{\Lambda}(D, R_0, r^{l-1}, r^l) = \frac{1}{2} \log^+ \left( \frac{G(D, R_0, r^{l-2}) \cdot \sigma_{2R_0} e^{-2\sigma_{2R_0} r^l}}{D} \right) \cdot \frac{\sigma_{2R_0} e^{-2\sigma_{2R_0} r^l}}{1-\sigma_{2R_l}(D|R_0)} D .
\]

Combining (41) and (42), we have

\[
J_{\Lambda}(D, R_0, r^{l-1}, r^l) \geq J_{\Lambda}(D, R_0, r^{l-2}, r^{l-1}) .
\]

On the other hand, by induction hypothesis, we have

\[
\min_{r^{l-1} \in \partial B_{l-1}(D|R_0)} K_\Lambda(r^{l-1}) .
\]

Combining (43) and (44), we have

\[
J_{\Lambda}(D, R_0, r^{l-1}, r^l) \geq \min_{r^{l-1} \in \partial B_{l-1}(D|R_0)} K_\Lambda(r^{l-1}) .
\]

When \( r^*_l > 0 \), we have

\[
J_{\Lambda}(D, R_0, r^{l-1}, r^l) \geq J_{\Lambda}(D, R_0, r^{l-1}, r^l) = K_\Lambda(r^{l-1}) .
\]

where the second equality follows from \((r^{l-1}, r^l) \in \partial B_l(D|R_0)\). Thus, the claim holds for \( l \), completing the proof.

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VI. CONCLUSIONS

We have considered the Gaussian many-help-one problem and given a partial solution to this problem by deriving explicit outer bound of the rate distortion region for the case where information sources satisfy the TS condition. Furthermore, we established a sufficient condition under which this outer bound is tight. We have determined the sum rate part of the rate distortion region for the case where information sources satisfy the TS condition.

For the case that information sources do not satisfy the TS condition we cannot derive an outer bound having a similar form of \( R(\text{out})_l(D) \) since the proof of the converse coding theorem depends heavily on this property of information sources. Hence the complete solution is still lacking for Gaussian information sources with general correlation.

APPENDIX

A. PROOF OF PROPOSITION 7

In this appendix we prove Proposition 7. To prove this proposition we give some preparations. For \( 0 \leq l \leq L - 2 \), we set

\[
\eta_l = \eta_l(D, r_0, r^l) \triangleq \begin{cases} g_0(D, r_0), & \text{for } l = 0, \\
 (g_l(D, r_0, r^{l-1}) - \frac{1}{\sigma_{2R_l}} (1 - e^{-2r^l}) , & \text{for } 1 \leq l \leq L - 2 .
\end{cases}
\]

For \( 1 \leq l \leq L - 2 \), and \( a < \frac{1}{\sigma_{2R_l}} \), define

\[
\tau_l(a) = \frac{|a|^2}{1-\sigma_{2R_l}^2 |a|^2} - \frac{1}{\sigma_{2R_l}} (1 - e^{-2r^l}) .
\]

Then, \( \{\eta_l\}_{l=0}^{L-2} \) satisfies the following:

\[
\eta_l(D, r_0, r^l) = \tau_l(D, r_0, r^{l-1})
\]

for \( 1 \leq l \leq L - 2 \). (45)

Fix \( a < \frac{1}{\sigma_{2R_l}} \) and set

\[
p_k(a) = \sup \left( p : \log \frac{1-\sigma_{2R_l}^2 |a|^2}{1-\sigma_{2R_l}^2 |b|^2} \geq p(b - a) \right.
\]

for any \( b < \frac{1}{\sigma_{2R_l}} \).

By a simple computation we have

\[
p_k(a) = \begin{cases} \frac{1}{1-\sigma_{2R_l}^2 |a|^2} , & \text{for } 0 \leq a < \frac{1}{\sigma_{2R_l}} , \\
 0 , & \text{for } a < 0 , \\
 \frac{\sigma_{2R_l}^2 |a|^2}{1-\sigma_{2R_l}^2 |a|^2} , & \text{for } a < \frac{1}{\sigma_{2R_l}} .
\end{cases}
\]

Fix \( a < \frac{1}{\sigma_{2R_l}} \) and set

\[
g_j(a) = \sup \left( q : r_j(b) - r_j(a) \geq q(b - a) \right.
\]

for any \( b < \frac{1}{\sigma_{2R_l}} \).
By a simple computation we have
\[q_j(a) = \begin{cases} \frac{1}{(1 - \sigma_{z_j}^2)^2}, & \text{for } 0 \leq a < \frac{1}{\sigma_{z_j}}, \\ \frac{1}{(1 - \sigma_{z_j}^2)^2}, & \text{for } a < 0 \end{cases},\]
\[\leq \frac{1}{(1 - \sigma_{z_j}^2)^2}, \text{ for } a < \frac{1}{\sigma_{z_j}}.\] \hfill (47)

**Proof of Proposition 1** Let \(\mathcal{L}\) be a set of integers \(l\) such that \(\eta_l(D, r_0, r^k)\) is positive for some \(r^k_L \in \mathcal{B}_L(D)\). From (45), there exists a unique integer \(1 \leq L^* \leq L - 2\) such that \(\mathcal{L} = \{0, 1, \ldots, L^*\}\). Using \(\eta_l(D, r_0, r^{L^*-2})\) and \(L^*\), \(\log G(D, r_0, r^{L^*-2})\) can be rewritten as
\[
\log G(D, r_0, r^{L^*-2}) = \sum_{k=1}^{L^*-1} \log \left\{ 1 + \sigma_{z_k}^2 g(D, r_0, r^{k}) \right\} = \sum_{k=1}^{L^*-1} \log \left[ \frac{1}{1 - \sigma_{z_k}^2 \eta_k(D, r_0, r^{k})} \right] = \sum_{k=0}^{L^*-2} \log \left[ \frac{1}{1 - \sigma_{z_{k+1}}^2 \eta_k(D, r_0, r^{k+1})} \right] = \sum_{k=0}^{L^*-2} \log \left[ \frac{1}{1 - \sigma_{z_{k+1}}^2 \eta_k(D, r_0, r^{k+1})} \right]. \hfill (48)
\]

Fix nonnegative vector \(r^L\). For each \(s_l \geq r_l\), \(1 \leq l \leq L - 2\), let \(G(s_l)\) be a function obtained by replacing \(r_l\) in \(G(D, r_0, r^{L^*-2})\) with \(s_l\), that is
\[G(s_l) \triangleq G(D, r_0, r^{L^*-1}, s_l, r^{L^*-2}_{l+1}).\]

It is obvious that when \(s_l = r_l\),
\[G(r_l) = G(D, r_0, r^{L^*-1}, r_l, r^{L^*-1}_{l+1}) = G(D, r_0, r^{L^*-2}).\]

By Property 2 part b), we have \(G(s_l) \leq G(r_l)\) for \(1 \leq l \leq L - 2\). For each \(s_k \geq r_k\), \(l \leq k \leq L - 2\), let \(\eta_k(s_l)\) be a function obtained by replacing \(r_l\) in \(\eta_k(D, r_0, r^{k})\) with \(s_l\), that is
\[\eta_k(s_l) \triangleq \eta_k(D, r_0, r^{L^*-1}, s_l, r^{L^*-2}_{l+1}).\]

It is obvious that when \(s_l = r_l\),
\[\eta_k(r_l) = \eta_k(D, r_0, r^{L^*-1}, r_l, r^{L^*-1}_{l+1}) = \eta_k(D, r_0, r^{k}).\]

By Property 1 part b), we have \(\eta_k(s_l) \leq \eta_k(r_l)\) for \(1 \leq k \leq L - 2\). For each \(l = 1, \cdots, L^*\), we evaluate an upper bound of \(\log G(s_l) - \log G(r_l)\). Using (49), we have
\[
\log G(s_l) = \sum_{k=0}^{L^*-2} \log \left[ \frac{1}{1 - \sigma_{z_{k+1}}^2 \eta_k(s_l)} \right] = \sum_{k=0}^{L^*-2} \log \left[ \frac{1}{1 - \sigma_{z_{k+1}}^2 \eta_k(r_l)} \right] \hfill (49)
\]

By definition of \(p_k(\cdot)\), we have
\[
\log \left[ \frac{1}{1 - \sigma_{z_{k+1}}^2 \eta_k(r_l)} \right] \geq p_k(\eta_k(r_l)) [\eta_k(s_l) - \eta_k(r_l)], \hfill (50)
\]

where the last inequality follows from \(\eta_k(s_l) \leq \eta_k(r_l)\) and (46). From (49) and (50), we have
\[
\log G(s_l) - \log G(r_l) \geq \sum_{k=0}^{L^*-2} \sigma_{z_{k+1}}^2 \eta_k(r_l) [\eta_k(s_l) - \eta_k(r_l)]. \hfill (51)
\]

By definition of \(q_j(\cdot)\) and (48), for \(l + 1 \leq j \leq k\), we have
\[
\eta_j(s_l) - \eta_j(r_l) \geq \frac{1}{1 - \sigma_{z_{j+1}}^2 \eta_j(r_l)} [\eta_j(s_l) - \eta_j(r_l)] \geq \frac{1}{1 - \sigma_{z_{j+1}}^2 \eta_j(r_l)} [\eta_j(s_l) - \eta_j(r_l)]. \hfill (52)
\]

where the last inequality follows from \(\eta_j(s_l) \leq \eta_j(r_l)\) and (47). Using (52) iteratively for \(l + 1 \leq j \leq k\), we obtain
\[
\eta_k(s_l) - \eta_k(r_l) \geq (\eta_k(s_l) - \eta_k(r_l)) \prod_{j=l+1}^{k} \frac{1}{1 - \sigma_{z_{j+1}}^2 \eta_j(r_l)}. \hfill (53)
\]

Observe that
\[
\eta_k(s_l) - \eta_k(r_l) = \frac{1}{\sigma_{z_k}} \left[ e^{-2s_l} - e^{-2r_l} \right] \geq -\frac{2e^{-2s_l}}{\sigma_{z_k}} (s_l - r_l). \hfill (54)
\]

From (53) and (54), we have
\[
\eta_k(s_l) - \eta_k(r_l) \geq -\frac{2e^{-2s_l}}{\sigma_{z_k}} (s_l - r_l) \prod_{j=l+1}^{k} \frac{1}{1 - \sigma_{z_{j+1}}^2 \eta_j(r_l)}. \hfill (55)
\]

From (51) and (55), we have
\[
\frac{1}{2} \log G(s_l) - \log G(r_l) \geq -(s_l - r_l) \sum_{k=0}^{L^*-1} \sigma_{z_{k+1}}^2 \eta_k(s_l) \prod_{j=l+1}^{k} \left( \frac{1}{1 - \sigma_{z_{j+1}}^2 \eta_j(r_l)} \right)^2. \hfill (56)
\]

By Property 1 part b) and the definition of \(\eta_j\), we have
\[
\eta_j \leq f_j - \frac{1}{\sigma_{z_j}} (1 - e^{-2r_j}) = \frac{f_{j+1}}{1 + \sigma_{z_{j+1}}^2 f_{j+1}}, \hfill (57)
\]

from which we have
\[
\frac{1}{1 + \sigma_{z_{j+1}}^2 f_{j+1}} \leq 1 + \sigma_{z_{j+1}}^2 f_{j+1} \leq 1 + \sigma_{z_{j+1}}^2 f_j^* \hfill (57)
\]

From (56) and (57), we have
\[
\frac{1}{2} \log G(s_l) - \log G(r_l) \geq -(s_l - r_l) \sum_{k=0}^{L^*-1} \sigma_{z_{k+1}}^2 \eta_k(s_l) \prod_{j=l+1}^{k} \left( 1 + \sigma_{z_j}^2 f_j^* \right)^2 \hfill (58)
\]

If
\[
\sum_{k=0}^{L^*-1} \sigma_{z_{k+1}}^2 \left( 1 + \sigma_{z_{k+1}}^2 f_{k+1}^* \right) \prod_{j=l+1}^{k} \left( 1 + \sigma_{z_j}^2 f_j^* \right)^2 \leq 1 \hfill (59)
\]
Theorem 2. We first present a lemma necessary for the proof for the MI condition. Let the encoded

Lemma 6:

Lemma 8:

From the above lemma we immediately obtain the follow-

Lemma 7:

I(X_0; W_0) \leq \frac{n}{2} \log \left[ 1 + \sigma^{2}_{x_0} f_0(r_0) \right].

For $1 \leq l \leq L - 1$, we have

\begin{align*}
\frac{n}{2} \log \left[ 1 + \sigma^{2}_{x_i} g_i(r_0, r_l^{-1}, \xi) \right] & \leq I(Y_l; W_l | Y_{l-1}) \leq \frac{n}{2} \log \left[ 1 + \sigma^{2}_{x_i} f_i(r_l) \right].
\end{align*}

From the above lemma we immediately obtain the follow-

Lemma 8:

I(X_0; W_S) \leq \frac{n}{2} \log \left[ 1 + \sigma^{2}_{x_0} f_0(r_0) \right],

I(Y^{L-1}; W_S | X_0) \leq \frac{n}{2} \log F(r_S), S \subseteq \Lambda,

I(Y^{L-1}; W_S | X_0) \geq \frac{n}{2} \log G(\xi, r_0, r^{-2}).

We prove $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(out)}(D)$ by Lemmas 6 and 8 and standard arguments for the proof of the converse coding theorem.

Proof of $\mathcal{R}_L(D) \subseteq \mathcal{R}_L^{(out)}(D)$: We first observe that by virtue of the TS condition,

\[ W_S \rightarrow X_S \rightarrow (X_0, Y^{L-1}) \rightarrow X_{S^c} \rightarrow W_{S^c} \quad (60) \]

hold for any subset $S$ of $\Lambda$. Assume $(R_0, R_1, \ldots, R_L) \in \mathcal{R}_L(D)$. Then, for any $\delta > 0$, there exists an integer $n_0(\delta)$ such that for $n \geq n_0(\delta)$ and for $i \in \Lambda$, we obtain the following chain of inequalities:

\[ n(R_0 + \delta) \geq \log M_0 \geq H(W_0) \geq H(W_0 | W^L) = I(X_0; W_0 | W^L) = n r_0. \quad (61) \]

Furthermore, for any subset $S \subseteq \Lambda$, we obtain

\[ n r_0 + \sum_{i \in S} (R_i + \delta) \geq I(X_0; W_0 | W^L) + \sum_{i \in S} H(W_i) \] 

\[ = H(W_0 | W^L) + \sum_{i \in S} H(W_i) \] 

\[ \geq H(W_0 | W_2 W_{S^c}) + H(W_2 | W_{S^c}) = H(W_0 W_{S^c} | W_{S^c}) \] 

\[ = I(X_0 Y^{L-1}; W_0 W_{S^c}) + H(W_0 W_{S^c} | W_{S^c}) \] 

\[ = I(X_0 Y^{L-1}; W_0 W_{S^c}) + \sum_{i \in S} H(W_i | X_0 Y^{L-1}) \] 

\[ = I(X_0 Y^{L-1}; W_0 W_{S^c}) + \sum_{i \in S} I(X_i; W_i | Y^{L-1}). \quad (62) \]

Step (a) follows from (60). On the other hand, by Lemma 6, we have for $n \geq n_0(\delta)$,

\[ I(X_0; W_0 W^L) = \frac{n}{2} \log \left( \frac{\sigma_{x_0}^2}{\xi} \right) \] 

\[ \geq I(X_0; \tilde{X}_0) \geq \frac{n}{2} \log \left( \frac{\sigma_{x_0}^2}{\delta + \xi} \right), \]

which together with (61), (62), and Lemma 8 yields the following lower bounds of $I(X_0; W_0 | W^L)$ and $I(X_0 Y^{L-1}; W_0 W_{S^c})$

\[ I(X_0; W_0 | W^L) = I(X_0; W_0 W^L) - I(X_0; W^L) \] 

\[ \geq \frac{n}{2} \log \left( \frac{\sigma_{x_0}^2}{\{1 + \sigma_{x_0}^2 f_0(r_0)\} \xi} \right) \] 

\[ \geq \frac{n}{2} \log \left( \frac{\sigma_{x_0}^2}{\{1 + \sigma_{x_0}^2 f_0(r_0)\} (D + 1)} \right), \quad (63) \]

\[ I(X_0 Y^{L-1}; W_0 W_{S^c}) = I(X_0 Y^{L-1}; W_0 W_{S^c}) - I(X_0 Y^{L-1}; W_{S^c}) \] 

\[ = I(X_0 Y^{L-1}; W_0 W^L) + I(Y^{L-1}; W^L | X_0) \] 

\[ - I(X_0; W_{S^c}) - I(Y^{L-1}; W_{S^c} | X_0) \] 

\[ \geq \frac{n}{2} \log \left( \frac{\sigma_{x_0}^2 G(\xi, r_0, r_{S^c}^{L-2})}{F(r_{S^c}) \{1 + \sigma_{x_0}^2 f_0(r_{S^c})\} (D + 1)} \right). \quad (64) \]

From (62) and (64), we have

\[ \sum_{i \in S} (R_i + \delta) \geq \frac{1}{2} \log \left( \frac{\sigma_{x_0}^2 G(\xi, r_0, r_{S^c}^{L-2})}{F(r_{S^c}) (D + 1) \{1 + \sigma_{x_0}^2 f_0(r_{S^c})\}} \right) \] 

\[ + \sum_{i \in S} r_i - r_0. \quad (65) \]
Note here that $\sum_{i \in S} (R_i + \delta)$ are nonnegative. Hence, from (61), (65) and (65), we obtain
\[
R_0 + \delta \geq r_0 \geq \frac{1}{2} \log \left( \frac{\sigma_{N_0}^2 \sigma_{N_1}^2}{\varphi_{S_0}^2 + \sigma_{N_1}^2} \left( D + \delta \right) \right)
\]  
(66)
and for $S \subseteq \Lambda$
\[
\sum_{i \in S} (R_i + \delta) \geq J_S (D + \delta, r_0, r_L - 2, r_S | r_S^c) .
\]
The inequality (66) implies that $r_0^* \in B_1(D + \delta)$. Thus, by letting $\delta \to 0$, we obtain $(R_0, R_1, \cdots, R_L) \in R_{\text{out}}(D)$. ■

Finally, we prove Lemma 7. For $n$ dimensional random vector $U$ with density, let $h(U)$ be a differential entropy of $U$. The following two lemmas are some variants of the entropy power inequality.

**Lemma 9:** Let $U_i$, $i = 1, 2, 3$ be $n$ dimensional random vectors with densities and let $T$ be a random variable taking values in a finite set. We assume that $U_3$ is independent of $U_1$, $U_2$, and $T$. Then, we have
\[
\frac{1}{2 \pi e} e^{\frac{1}{2} h(U_2 + U_3 | U_1, T)} \geq \frac{1}{2 \pi e} e^{\frac{1}{2} h(U_1 | U_2)} \frac{1}{2 \pi e} e^{\frac{1}{2} h(U_3)} .
\]

**Proof of Lemma 7:** Define the sequence of $n$ dimensional random vectors $\{S_i\}_{i=1}^{L-1}$ by
\[
S_l = \frac{1}{\sigma_{N_l}} X_l + \frac{1}{\sigma_{Z_l+1}} Y_{l+1}, 1 \leq l \leq L - 1.
\]  
(67)
By an elementary computation, we obtain
\[
\begin{aligned}
X_0 &= \frac{1}{\sigma_{N_0}^2} Y_0 + \hat{N}_0, \\
Y_l &= \frac{1}{\sigma_{N_l}^2} Y_{l-1} + \frac{1}{\sigma_{Z_l+1}^2} S_l + \hat{N}_l, 1 \leq l \leq L - 1.
\end{aligned}
\]  
(68)
where $\hat{N}_l$, $0 \leq l \leq L - 1$ is an $n$ dimensional random vector whose components are $n$ independent copies of a Gaussian random variable with mean zero and variance $\sigma_{N_l}^2$. $\hat{N}_0$ is independent of $Y_1$. For each $1 \leq l \leq L - 1$, $\hat{N}_l$ is independent of $Y_{l-1}$ and $S_l$. The variance $\sigma_{N_l}^2$, $0 \leq l \leq L - 1$ have the following form:
\[
\begin{aligned}
\frac{1}{\sigma_{N_l}^2} &= \frac{1}{\sigma_{N_0}^2} + \frac{1}{\sigma_{Z_l+1}^2}, \\
\frac{1}{\sigma_{Z_l+1}^2} &= \frac{1}{\sigma_{N_l}^2} + \frac{1}{\sigma_{N_l}^2} + \frac{1}{\sigma_{Z_l+1}^2}, 1 \leq l \leq L - 1.
\end{aligned}
\]  
(69)
Set
\[
\begin{aligned}
\lambda_0 &\triangleq \frac{1}{2 \pi e} e^{\frac{1}{2} h(X_0 | W_L)}, \\
\mu_0 &\triangleq \frac{1}{2 \pi e} e^{\frac{1}{2} h(Y_1 | W_L)}, \\
\lambda_l &\triangleq \frac{1}{2 \pi e} e^{\frac{1}{2} h(Y_l | Y_{l-1}, W_L)}, 1 \leq l \leq L, \\
\mu_l &\triangleq \frac{1}{2 \pi e} e^{\frac{1}{2} h(S_l | Y_{l-1}, W_L)}, 1 \leq l \leq L - 1.
\end{aligned}
\]
We can easily verify that
\[
\mu_0 = \mu_0 \lambda_0 \frac{1}{\sigma_{N_0}^2}, \mu_l = \mu_l \lambda_l \frac{1}{\sigma_{N_l}^2}, 1 \leq l \leq L - 1.
\]  
(70)
Applying Lemma 9 to (68), we obtain
\[
\begin{aligned}
\lambda_0 &\geq \frac{\sigma_{N_0}^2}{\sigma_{Z_l+1}^2} \mu_0 + \sigma_{N_l}^2, \\
\mu_l &\geq \sigma_{N_l}^2, 1 \leq l \leq L - 1.
\end{aligned}
\]  
(71)
From (70) and (71), we obtain
\[
\lambda_0 \geq \frac{1}{\sigma_{Z_l+1}^2} + \frac{1}{\sigma_{N_l}^2} \left( 1 - \frac{\lambda_l^*}{\sigma_{Z_l+1}^2} \right) , \\
\lambda_l \geq \frac{1}{\sigma_{N_l}^2} + \frac{1}{\sigma_{Z_l+1}^2} - \mu_l, 1 \leq l \leq L - 1.
\]  
(72)
On the other hand, we note that for each $1 \leq l \leq L - 1$, the five random variables $W_l, X_l, Y_l, Y_{l+1}$, and $W_{l+1}$ form a Markov chain $W_l \to X_l \to Y_l \to Y_{l+1} \to W_{l+1}$ in this order. Then, applying Lemma 10 to (67), we obtain
\[
\mu_l \geq \sigma_{Z_l+1} e^{2r_l} \frac{1}{\sigma_{Z_l+1}^2} + \frac{1}{\sigma_{N_l}^2} \lambda_{l+1}^* , 1 \leq l \leq L - 1.
\]  
(73)
Combining (72) and (73), we obtain for $1 \leq l \leq L - 1$,
\[
\lambda_l \leq \frac{1}{\sigma_{Z_l+1}^2} + \frac{1}{\sigma_{N_l}^2} \left( 1 - e^{-2r_l} \right) + \frac{1}{\sigma_{Z_l+1}^2} \left( 1 - \frac{\mu_l^*}{\sigma_{Z_l+1}^2} \right). 
\]  
(74)
Set $\nu_l^0 = \lambda_l^* - \frac{1}{\sigma_{X_l}^2}, \nu_l = \lambda_l^* - \frac{1}{\sigma_{N_l}^2}, 1 \leq l \leq L - 1$. Then, we have
\[
\begin{aligned}
I(X_0, W_L) &= \frac{n}{L} \log(1 + \sigma_{N_0}^2, \nu_0), \\
I(Y_l, W_l | Y_{l-1}) &= \frac{n}{L} \log(1 + \sigma_{Z_l}^2, \nu_l), 1 \leq l \leq L - 1, \\
I(Y_L, W_L | Y_{L-1}) &= \frac{n}{L} \log(1 + \sigma_{Z_L}^2, \nu_L) = n \nu_L.
\end{aligned}
\]
Note that $\nu_l, 0 \leq l \leq L - 1$ are nonnegative. From (72) and (74), $\{\nu_l\}_{l=0}^L$ satisfies the following recursion:
\[
\begin{aligned}
\nu_{L-1} &= \frac{n}{L} \log(1 + \sigma_{N_{L-1}}^2, \nu_l) \\
&\geq \frac{n}{L} \log(1 + \sigma_{Z_{L-1}}^2, \nu_{l+1}), 1 \leq l \leq L - 1, \\
\nu_{L-1} &= \frac{n}{L} \log(1 + \sigma_{Z_{L-1}}^2, \nu_{L-2}), L - 2 \geq l \geq 1, \\
\nu_0^0 &\leq \frac{n}{L} \log(1 + \sigma_{Z_0}^2, \nu_0) = e^{-2r_0} - \frac{1}{\sigma_{X_0}^2}.
\end{aligned}
\]  
(75)
From (73), we obtain the upper bounds of $I(X_0; W_L)$ and $I(Y_l; W_l | Y_{l-1}), 1 \leq l \leq L - 1$ in Lemma 7. On the other hand, from (77), (78), and the nonnegative property of $\nu_l, 0 \leq l \leq L - 1$, we have
\[
\nu_0^0 = \left[ e^{-2r_0} - \frac{1}{\sigma_{X_0}^2} \right]^+, 1 \leq l \leq L - 1.
\]  
(79)
From (79) and (80), we obtain the lower bound of $I(Y_l; W_l | Y_{l-1}), 1 \leq l \leq L - 1$ in Lemma 7. ■
C. Proof of Lemma 4

Let $\alpha^2, \beta^2 \in A_L(\alpha_1)$. Then, we have the following chain of inequalities:

$$t \zeta(\alpha^2) + (1-t)\zeta(\beta^2)$$

$$= \sum_{l=1}^{L-1} \left\{ t \log \left( 1 - \frac{\alpha_l}{1 - \epsilon \alpha_l + \frac{\alpha_{l+1}}{\tau_{l+1}}} \right) + (1-t) \log \left( 1 - \frac{\beta_l}{1 - \epsilon \beta_l + \frac{\beta_{l+1}}{\tau_{l+1}}} \right) \right\}$$

$$+ 1-t \log \left( 1 - \frac{\beta_1}{1 - \epsilon \beta_1 + \frac{\beta_{l+1}}{\tau_{l+1}}} \right)$$

Step (a) follows from the strict concavity of the logarithm function. Step (b) follows from the strict concavity of $\log \left( 1 - \frac{\alpha_l}{1 - \epsilon \alpha_l + \frac{\alpha_{l+1}}{\tau_{l+1}}} \right)$ for $a > 0$.

D. Proof of Lemma 4

Proof of Lemma 4 part a): For the proof we use the following inequality:

$$\frac{1+a}{1+\epsilon(1+a)} - \frac{a}{1+\epsilon a} \leq \frac{1}{1+\epsilon}.$$

(81)

The recursion of (15) is equivalent to

$$\frac{\tau L \theta_{l-1}(\omega)}{1 - \epsilon \tilde{\theta}_{l-1}(\omega)} = 2 \theta_l - \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} + \theta_l$$

(82)

for $L - 1 \geq l \geq 2$. Applying (81) to the second term in the right members of (82) and considering the assumption $\tau_l \geq \frac{1}{1+\epsilon}$ for $L - 1 \geq l \geq 2$, we have

$$\frac{\tau \theta_{l-1}(\omega)}{1 - \epsilon \tilde{\theta}_{l-1}(\omega)} \geq 2 \theta_l - \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} + \theta_l$$

or equivalent to

$$\frac{\theta_{l-1}(\omega)}{1 + \epsilon \tilde{\theta}_{l-1}(\omega)} - \theta_l \geq \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} - \frac{\epsilon \theta_{l+1}(\omega)}{\tau_{l+1}}$$

(83)

for $L - 1 \geq l \geq 2$. We first prove (17) for $l = L$. The equation

$$\frac{\theta_{L-1}(\omega)}{1 + \epsilon \tilde{\theta}_{L-1}(\omega)} = \frac{\theta_{L}(\omega) - \theta_{L-1}(\omega)}{1 + \epsilon \tilde{\theta}_{L-1}(\omega)}$$

(84)

is equivalent to

$$\tau L \left( \frac{\theta_{L-1}(\omega)}{1 - \epsilon \tilde{\theta}_{L-1}(\omega)} - 1 \right) = \theta_{L}(\omega) - \theta_{L-1}(\omega).$$

(85)

It is obvious that

$$0 \leq \theta_{L}(\omega) = \omega - 1 = \epsilon L^{-1}.$$  

(86)

From (85) and (86), we have

$$\tau L \left( \frac{\theta_{L-1}(\omega)}{1 - \epsilon \tilde{\theta}_{L-1}(\omega)} - 1 \right) < \theta_{L}(\omega).$$

From (85) and $\tau L \geq 1$, we have

$$\theta_{L}(\omega) \leq \frac{\frac{\tau L \theta_{L}(\omega)}{1 - \epsilon \tilde{\theta}_{L-1}(\omega)}}{1 - \epsilon \tilde{\theta}_{L-1}(\omega)} \leq \frac{\tau L \theta_{L-1}(\omega)}{1 - \epsilon \tilde{\theta}_{L-1}(\omega)}.$$  

Thus, (17) holds for $l = L$. We assume that (17) holds for some $l + 1$ with $L \geq l + 1$, that is,

$$0 \leq \theta_{l+1}(\omega) \leq \frac{\tau_{l+1} \theta_{l+1}(\omega)}{1 + \epsilon \tilde{\theta}_{l+1}(\omega)} - 1.$$  

(87)

From (87), we obtain

$$\epsilon L^{-1} > \theta_{l+1}(\omega) \geq \frac{\theta_{l+1}(\omega)}{1 + \epsilon \tilde{\theta}_{l+1}(\omega)} > 0,$$

$$\theta_{l+1}(\omega) \leq 1 + \epsilon L^{-1}.$$  

(88)

Using (82), we have

$$\frac{\tau \theta_{l-1}(\omega)}{1 - \epsilon \tilde{\theta}_{l-1}(\omega)} - \theta_l \geq \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} - \frac{\epsilon \theta_{l+1}(\omega)}{\tau_{l+1}}.$$  

(83)

Step (a) follows from the first inequality of (88). Using (83), we have

$$\frac{\tau \theta_{l-1}(\omega)}{1 - \epsilon \tilde{\theta}_{l-1}(\omega)} - \theta_l \geq \theta_{l+1}(\omega) - \frac{\theta_{l+1}(\omega)}{1 + \epsilon \tilde{\theta}_{l+1}(\omega)} \geq 0.$$  

Step (a) follows from the first inequality of (88). Thus, (17) holds for $l$, completing the proof.
Proof of Lemma 4 part b): We first observe that
\[
\zeta(\alpha L^L) = \sum_{l=1}^{L-1} \left\{ \log \left[ 1 - \frac{\alpha_l}{1 - \epsilon_l \alpha_l + \alpha_{l+1} \frac{\alpha_l + 1}{\tau_{l+1}}} \right] + \log(1 - \epsilon_l \alpha_l) \right\} + \log(1 - \alpha_L) = \sum_{l=2}^L \left\{ 1 - \epsilon_l \alpha_{l-1} - \alpha_{l-1} + (1 - \epsilon_l \alpha_{l-1}) \frac{\alpha_l}{\tau_l} \right\} + \log(1 - \alpha_L) = \sum_{l=1}^{L-1} \left\{ 1 + \frac{\alpha_{l+1}}{\tau_{l+1}} - \left\{ 1 + \epsilon_l \left( 1 + \frac{\alpha_{l+1}}{\tau_{l+1}} \right) \right\} \alpha_l \right\} + \log(1 - \alpha_L).
\]
Computing \( \frac{\partial}{\partial \alpha_l} \zeta(\alpha L^L) \) we obtain
\[
\frac{\partial}{\partial \alpha_l} \zeta(\alpha L^L) = \frac{1}{\alpha_L - \tau_l} \left( \frac{\alpha_{l-1}}{1 - \epsilon_{l-1} \alpha_{l-1}} - 1 \right) - \frac{1}{1 - \alpha_l}, \quad \text{for } L - 1 \geq l \geq 2.
\]
From (89), when \( \nabla \zeta(\alpha L^L) = 0 \), \( \alpha L^L \) must satisfy
\[
-2 \alpha_L + 1 + \tau_l \left( \frac{\alpha_{l-1}}{1 - \epsilon_{l-1} \alpha_{l-1}} - 1 \right) = 0, \quad \frac{1 + \epsilon_l}{1 + \alpha_l + 1} - 2 \alpha_l + \tau_l \left( \frac{\alpha_{l-1}}{1 - \epsilon_{l-1} \alpha_{l-1}} - 1 \right) = 0, \quad \text{for } L - 1 \geq l \geq 2.
\]
From (90), we obtain
\[
\alpha_{L-1} = \frac{2 \alpha_{L-2} - 1 + \frac{2 \alpha_{L-2}}{\tau_{L-1}} + 1}{1 + \epsilon_{L-1} \left( \frac{2 \alpha_{L-2}}{\tau_{L-1}} + 1 \right)}, \quad \alpha_{L-1} = \frac{1}{\tau_l} \left[ 2 \alpha_l - \frac{1 + 2 \alpha_l}{1 + \epsilon_l \left( \frac{1 + 2 \alpha_l}{\tau_l} + 1 \right)} + \tau_l \right] \quad \text{for } L - 1 \geq l \geq 2.
\]
The relation (91) implies that \( \nabla \zeta|_{\alpha L^L=\theta L^L(\omega)} = 0 \).

Proof of Lemma 4 part c): For the proof we use the following recursion:
\[
\tau_l \theta_{l-1}(\omega) = 2 \theta_l(\omega) - \frac{1 + \theta_{l-1}(\omega)}{1 + \epsilon_l (1 + \theta_{l+1}(\omega))} + 1.
\]
Taking the derivative of both sides of (92) with respect to \( \omega \), we obtain
\[
\frac{1}{1 - \epsilon_{l-1} \theta_{l-1}(\omega)} \frac{d \theta_{l-1}}{d \omega} = 2 \frac{d \theta_l}{d \omega} - \frac{1 + \theta_{l+1}(\omega)}{1 + \epsilon_l (\theta_{l+1}(\omega) + 1)} \frac{d \theta_{l+1}}{d \omega} \cdot \tau_{l+1}^{-1}.
\]
Since \( \theta L^L(\omega) \in A_L(\theta 1(\omega)) \), we have
\[
\tau_l \left( \frac{\theta_{l-1}(\omega)}{1 - \epsilon_{l-1} \theta_{l-1}(\omega)} - 1 \right) < \theta_l(\omega).
\]
The above inequality is equivalent to
\[
1 + \epsilon_{l-1} \left( \frac{\theta_l(\omega)}{\tau_l} + 1 \right) > \frac{1}{1 - \epsilon_{l-1} \theta_l(\omega)}.
\]
From (23) and (24) we have
\[
\left\{ 1 + \epsilon_{l-1} \left( \frac{\theta_l(\omega)}{\tau_l} + 1 \right) \right\}^2 \frac{d \theta_{l-1}}{d \omega} \cdot \tau_l \geq 2 \left( \frac{1}{\sigma_l^2} \frac{d \theta_l}{d \omega} - \frac{1}{\sigma_l} \right) \left\{ 1 + \epsilon_l \left( \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} + 1 \right) \right\}^2 \times \left( \frac{1}{\sigma_{l+1}} \frac{d \theta_{l+1}}{d \omega} \right).
\]
The above inequality is equivalent to
\[
\left\{ 1 + \epsilon_{l-1} \left( \frac{\theta_l(\omega)}{\tau_l} + 1 \right) \right\}^2 \left( \frac{1}{\sigma_l^2} \frac{d \theta_l}{d \omega} - \frac{1}{\sigma_l} \right) \left\{ 1 + \epsilon_l \left( \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} + 1 \right) \right\}^2 \times \left( \frac{1}{\sigma_{l+1}} \frac{d \theta_{l+1}}{d \omega} \right) \geq \frac{1}{\sigma_l^2} \frac{d \theta_l}{d \omega} - \frac{1}{\sigma_l} \left\{ 1 + \epsilon_l \left( \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} + 1 \right) \right\}^2.
\]
For \( 1 \leq l \leq L \), set
\[
\Phi_l(\omega) \triangleq \left( \frac{1}{\sigma_l^2} \frac{d \theta_l}{d \omega} \right) \prod_{j=2}^{L+1} \left( \frac{1}{1 + \epsilon_{j-1} \left( \frac{\theta_j(\omega)}{\tau_j} + 1 \right)} \right)^2.
\]
Then, by (25), we have
\[
\Phi_{l-1}(\omega) \geq 2 \Phi_l(\omega) - \Phi_{l+1}(\omega) \quad \text{for } 2 \leq l \leq L - 1.
\]
From (77) we have
\[
\Phi_{l-1}(\omega) - \Phi_l(\omega) \geq \Phi_{l-1}(\omega) - \Phi_L(\omega) = \left( \frac{1}{\sigma_{L-1}^2} \frac{d \theta_{L-1}}{d \omega} - \frac{1}{\sigma_L} \right) \left\{ 1 + \epsilon_{l-1} \left( \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} \right) \right\} \times \frac{1}{\sigma_{l+1}^2} \frac{d \theta_{l+1}}{d \omega} \times \prod_{j=2}^{L-1} \left( \frac{1}{1 + \epsilon_{j-1} \left( \frac{\theta_j(\omega)}{\tau_j} + 1 \right)} \right)^2.
\]
Set
\[
A(\omega) \triangleq \left[ \frac{1 + \epsilon_{l-1} \left( \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} \right)}{1 + \epsilon_{l-1} \left( \frac{\theta_{l+1}(\omega)}{\tau_{l+1}} \right)} \right]^2 \times \frac{1}{\sigma_{l+1}^2} \frac{d \theta_{l+1}}{d \omega} \times \prod_{j=2}^{L-1} \left( \frac{1}{1 + \epsilon_{j-1} \left( \frac{\theta_j(\omega)}{\tau_j} + 1 \right)} \right)^2.
\]
Then, by (98), we have

\[ \Phi_l(\omega) \geq \Phi_L(\omega) + (L - l)A(\omega) \]

\[ = \left[ \frac{2(L-l)}{\left(1 + \epsilon_{L-1}\left(\frac{\omega}{L} - 1\right)\right)^2} - \frac{L-(l+1)}{\left(1 + \epsilon_{L-1}\left(\frac{\omega}{L} - 1\right)\right)^2} \right] \times \frac{1}{\sigma_L^2} \prod_{j=2}^{L-1} \left( \frac{1}{1 + \epsilon_{L-1}\left(\frac{1}{\tau_j} + 1\right)} \right) \times \frac{1}{\sigma_{L-l}^2} \prod_{j=l+1}^{L-1} \left( \frac{1}{1 + \epsilon_{L-1}\left(\frac{1}{\tau_j} + 1\right)} \right)^2, \]

from which we obtain

\[ \frac{d\theta_l}{d\omega} \geq \left[ \frac{2(L-l)}{\left(1 + \epsilon_{L-1}\left(\frac{2\omega-1}{L} - 1\right)\right)^2} - \frac{L-(l+1)}{\left(1 + \epsilon_{L-1}\left(\frac{2\omega-1}{L} - 1\right)\right)^2} \right] \times \frac{1}{\sigma_L^2} \prod_{j=2}^{L-1} \left( \frac{1}{1 + \epsilon_{L-1}\left(\frac{1}{\tau_j} + 1\right)} \right) \times \frac{1}{\sigma_{L-l}^2} \prod_{j=l+1}^{L-1} \left( \frac{1}{1 + \epsilon_{L-1}\left(\frac{1}{\tau_j} + 1\right)} \right)^2, \]

completing the proof.

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