Advances in wave turbulence: rapidly rotating flows

C Cambon\textsuperscript{1}, R Rubinstein\textsuperscript{2} and F S Godeferd\textsuperscript{1}

\textsuperscript{1}LMFA, Ecole Centrale de Lyon, France
\textsuperscript{2}NASA Langley Research Center, Hampton, VA 23681, USA
E-mail: claude.cambon@ec-lyon.fr

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Abstract. At asymptotically high rotation rates, rotating turbulence can be described as a field of interacting dispersive waves by the general theory of weak wave turbulence. However, rotating turbulence has some complicating features, including the anisotropy of the wave dispersion relation and the vanishing of the wave frequency on a non-vanishing set of ‘slow’ modes. These features prevent straightforward application of existing theories and lead to some interesting properties, including the transfer of energy towards the slow modes. This transfer competes with, and might even replace, the transfer to small scales envisioned in standard turbulence theories.

In this paper, anisotropic spectra for rotating turbulence are proposed based on weak turbulence theory; some evidence for their existence is given based on numerical calculations of the wave turbulence equations. Previous arguments based on the properties of resonant wave interactions suggest that the slow modes decouple from the others. Here, an extended wave turbulence theory with non-resonant interactions is proposed in which all modes are coupled; these interactions are possible only because of the anisotropy of the dispersion relation. Finally, the vanishing of the wave frequency on the slow modes implies that these modes cannot be described by weak turbulence theory. A more comprehensive approach to rotating turbulence is proposed to overcome this limitation.
1. Introduction

Rotation of the reference frame is an important factor in certain mechanisms of flow instability and in fields as diverse as engineering (e.g. turbomachinery and reciprocating engines with swirl and tumble), geophysics and astrophysics. Effects of mean curvature or of advection by a large eddy can also be tackled using methods developed for rotating flows. In these applications, effects of rotation are often combined with those of mean shear, mean strain and solid boundaries; thus, the problem of ‘pure’ homogeneous rotating turbulent flow without physical boundaries must be considered as a canonical flow very far from applications. Despite these idealizations, homogeneous rotating turbulence is a very challenging test-case for checking subgrid scale and statistical models [1, 2]. Even if mean velocity and temperature gradient effects are entirely excluded, the role of Coriolis forces in a rotating frame of reference is subtle and difficult to model.

The modification of turbulence dynamics by rotation is due to the presence of inertial waves in rotating flows [3]. Although these wave motions arise from linear modifications of the equations of motion, in contrast with shear flows, there is no direct production of energy by linear effects (in the language of linear stability theory, inertial waves are linearly neutral), and the alteration of the distribution of energy is controlled instead by nonlinear mechanisms such as resonant wave interactions.
Since inertial waves are dispersive, linear mechanisms alone cause the temporal decorrelation of any ensemble of waves with random wavevectors. When the rotation is sufficiently rapid, linear decorrelation can dominate the nonlinear decorrelation of the wave field; in this limit of weakly nonlinear interactions of dispersive waves, the infinite hierarchy of moment equations familiar from the turbulence closure problem admits a perturbatively derived ‘intrinsic closure’ which is often called the weak turbulence (WT) or weak wave-turbulence theory [4]–[6].

This theory provides an appropriate framework in which to study rapidly rotating turbulence. In the WT theory, individual modes are of the form

$$u_i(x, t) = a_i(t) \exp[i(k \cdot x - \sigma_k t)]. \quad (1)$$

The dispersion relation $\sigma_k = \sigma(k)$ is known from the linear part of the equations of motion, and the nonlinear theory gives equations of motion for the slowly varying wave amplitudes $a_i(t)$. Because of their formal resemblance to the Boltzmann equation, these equations are often referred to as ‘wave kinetic equations’.

WT studies have generally emphasized isotropic dispersion laws such as $\sigma_k = |k|^\alpha$ for which Kolmogorov spectra can be constructed, with the key properties of a constant and isotropic energy flux across scales associated with wavenumber $k = |k|$ [7]. In contrast, in geophysical flows, the dispersion laws are anisotropic because a rotation axis or the direction of gravity introduces a preferred unit vector $\mathbf{n}$. Defining

$$k_\parallel = k \cdot \mathbf{n},$$
$$k_\perp = \sqrt{k^2 - k_\parallel^2}, \quad (2)$$

we have $\sigma = \pm \beta k_\perp / k^2$ for Rossby waves, $\sigma = \pm 2\Omega k_\parallel / k$ for inertial waves and $\sigma = \pm N k_\perp / k$ for gravity waves. An important common feature of all of these problems is that the wave frequency vanishes on a nonempty set of modes [8], which therefore form a ‘slow manifold’ for each problem.

The combination of anisotropy and degeneracy causes novel effects that are entirely absent in problems of isotropic WT. The most important is a tendency of nonlinear interactions to transfer energy towards the slow manifold [8]; the question remains open whether such ‘angular’ transfer might not replace transfer to small scales entirely. A consequence of this transfer is the preferential growth of correlation lengths parallel to the rotation axis, which might be conceived as a tendency toward ‘two-dimensionalization’ of the velocity field statistics. In the special case of rotation, transfer towards the slow manifold is sometimes incorrectly explained by invoking the Proudman theorem. However, contrary to this often-cited interpretation, the Proudman theorem only confirms that the ‘slow manifold’ is indeed the two-dimensional manifold $k_\parallel = 0$ at high rotation rates; it cannot predict the transition from three-dimensional (3D) to two-dimensional (2D) turbulence, which requires non-linear transfer of energy from all possible modes towards the 2D modes.

It is natural to ask whether these remarkable features of rotating turbulence are the result of resonant nonlinear interactions alone, i.e. whether they can be explained by an anisotropic WT theory. In [9] (BGSC hereinafter), such a theory was derived from earlier EDQNM (eddy-damped quasi-normal Markovian) based closures for rotating turbulence; this theoretical development is explained in [10] (CMG hereinafter), for example. In particular, the EDQNM$_3$ closure, which combines a fully anisotropic description of linear rotation effects with the effects of nonlinear
eddy damping, was considered in the limit of vanishing nonlinearity. The result, which was called the AQNM (asymptotic quasi-normal Markovian) closure proves to coincide with the anisotropic WT theory of rotating turbulence [5].

The resulting equations were numerically integrated in the case of decaying rotating turbulence. Starting from isotropic initial data with a narrow-band energy spectrum, the spectral energy function eventually becomes self-similar and angle-dependent. The angle dependence confirms the transfer of energy from ‘rapid’ to ‘slow’ modes noted previously. The energy-containing cone reaches a finite, nonzero angle, so that the angular distribution of spectral energy remains far from the 2D dependence on \( \delta(k_\parallel) \) expected from a naive application of the Proudman theorem to rotating turbulence. The main quantitative result is the emergence of a \( k^{-3} \) law for the classical spherically averaged energy spectrum which is discussed further in section 3.2. This spectral scaling is in agreement with the long-time results found in large-eddy simulation (LES) of decaying rotating turbulence by Yang and Domaradzki [2].

The present paper continues the WT analysis of rotating turbulence initiated in BGSC as follows.

(i) In isotropic WT theory, Kolmogorov spectra with power-law scaling can be derived by dimensional analysis, and shown to be exact solutions of the wave kinetic equations. Analogous results for anisotropic WT have not been found except for very special, but generally nonphysical, dispersion relations. Following previous work by Galtier [5], a suitably restricted class of wave interactions is introduced for which approximate anisotropic scaling solutions of the WT wave kinetic equations can be constructed. Tentative evidence for the existence of these solutions is found from AQNM calculations.

(ii) An interesting feature of anisotropic WT theory is the appearance of certain volume integrals which automatically vanish in isotropic WT theory. These volume integrals were ignored in the formulation of AQNM but are reconsidered here. In rotating turbulence, resonant interactions cannot transfer energy from fast \( (k_\parallel \neq 0) \) modes to slow, 2D \( (k_\parallel = 0) \) modes [8]. This property of resonant interactions has been formulated as a ‘decoupling’ of the 2D part of the motion from inertial wave dynamics. This issue is discussed in depth by Smith and Waleffe [11]. We propose instead that this exact decoupling does not occur, and suggest that volume integral contributions can play a role in coupling inertial waves to the 2D motions.

(iii) The vanishing of the wave frequency on the slow manifold suggests that WT may not be uniformly valid. Numerically, the AQNM solution seems to become singular when \( k_\parallel = 0 \), although this singularity appears to be integrable and the contribution of the plane \( k_\parallel = 0 \) to the total energy is therefore small [12]. Nevertheless, the singularity suggests that eddy damping may be required to regularize the solution near this plane. Accordingly, we outline a possible asymptotic theory of the nearly slow modes in which nonlinear eddy damping is restored. This work is based on the previous EDQNM3 model. We suggest the possibility of a matched asymptotic expansions description of rotating turbulence, in which WT provides the ‘outer’ solution, and a simplified EDQNM3 model provides the ‘inner’ solution. The inner solution thereby has the role of a ‘2D boundary layer’ in the spectral dynamics of rotating turbulence.

(iv) EDQNM-based models are not theoretically self-contained, because they rely on an exogenous specification of a relaxation time. The only wholly satisfactory theoretical approach is to compute the nonlinear damping from a comprehensive turbulence theory.
like the direct interaction approximation (DIA) or Lagrangian renormalized approximation (LRA) \[13\]. These theories, while considerably more difficult to analyse and compute than WT or EDQNM, make no assumptions about the relative strength of nonlinear and linear decorrelation. Thus, they apply equally to weak and strong rotation and, most importantly, to all modes at any rotation rate. These theories are therefore also consistent with nonlinear ‘renormalization’ of the rotation rate on the one hand, and with rotational modification of nonlinear damping on the other. We conclude by formulating a complete strategy for moving from weak to strong turbulence, based on the following hierarchy of models/theories:

Rapid distortion theory (linear theory) \(\subset\) Wave turbulence (weakly nonlinear theory) \(\subset\) EDQNM\(_3\) (strongly nonlinear theory with empirical eddy damping) \(\subset\) LRA \[13\] (completely self-contained theoretical description of rotating turbulence).

In these different approaches, the relationship between what is exact and what comes from assumptions needs the following clarification. Linear theory only provides the anisotropic dispersion law and the definition of eigenmodes, inertial modes here, in the unbounded flow; it amounts to constant amplitudes \(a_i\) in equation (1). The quasi-normal (QN) relation, amounting to zero fourth-order cumulants of amplitudes, is pivotal in EDQNM, as well as in sophisticated endogenous statistical theories (LRA here), which all use expansions around a Gaussian field. The QN assumption in EDQNM is classically complemented in strong turbulence with additional heuristic procedures: non-zero eddy damping (ED) and Markovianization (M). Only in the limit of weak nonlinearity can QN be almost completely justified by mathematical developments as an ‘intrinsic closure’ \[4\], so that ED may vanish and M is consistent with the separation of slow-amplitude statistics and rapid-phase statistics in EDQNM\(_3\). Hence, WT can appear as a by-product of EDQNM\(_3\), and gives an exact limiting case of EDQNM\(_3\), except in the vicinity of the slow manifold. Finally, given the difficulty to exploit fully anisotropic LRA using the relevant eigenmodes, this kind of theory could at least provide guidelines for improving the eddy damping and possibly renormalizing the rotation rate, when recourse to strong turbulence is needed.

The paper is organized as follows. Background equations for the EDQNM\(_3\) model and its asymptotic AQNM limit are reviewed in section 2. Comparison of analytical laws with AQNM numerical results is done in section 3, and the dynamics in the vicinity of the slow manifold is discussed in section 4. A better treatment of the \(k_\parallel \rightarrow 0\) limit is also addressed in section 4, and a general discussion, with perspectives, is presented in section 5. A short synthesis is provided in section 6.

2. EDQNM\(_3\) and AQNM models for rotating turbulence

2.1. The helical modes decomposition

The analysis of rotating turbulence is simplest using a coordinate system moving with the rotating frame. In this non-Galilean frame, rotation introduces centrifugal and Coriolis forces. Since the centrifugal force can be incorporated in the pressure term, only the Coriolis force appears explicitly in the Navier–Stokes equations

\[
(\partial_t + u \cdot \nabla)u + 2\Omega n \times u + \nabla p - \nu \nabla^2 u = 0, \tag{3}
\]

\[
\nabla \cdot u = 0, \tag{4}
\]
where $\mathbf{u}$ is the fluctuating velocity and $p$ the pressure divided by a mean reference density. The unit vector $\mathbf{n}$ is aligned with the angular velocity of the rotating frame so that $\Omega = \Omega \mathbf{n}$. Without loss of generality, coordinates are chosen so that $n_i = \delta_{i3}$.

The linearized inviscid equation is

$$\frac{\partial \mathbf{u}}{\partial t} + 2\Omega \times \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0.$$  

Since the Coriolis force is not divergence-free, the pressure term makes a nontrivial contribution to maintaining the incompressibility constraint. Without pressure, the horizontal motion is modified by a circular periodic motion due to the rotation vector $\Omega$, but propagating waves do not occur. Hence, the fluctuating pressure is responsible both for anisotropic dispersivity and for horizontal-vertical coupling.

The linear or rapid distortion theory (RDT) problem is simplified by introducing 3D Fourier space in which an analytical solution can be found for the Fourier transform $\hat{\mathbf{u}}(k, t)$:

$$\frac{\partial \hat{u}_i}{\partial t} + 2\Omega P_{in} \epsilon_{n3} \hat{u}_j = 0$$  

with $P_{in} = \delta_{in} - k_i k_n / k^2$ being the projection operator ensuring the incompressibility constraint $\hat{\mathbf{u}} \cdot \mathbf{k} = 0$. Given this constraint, it is easier to project the equation onto the local frame ($\mathbf{e}^{(1)}, \mathbf{e}^{(2)}$) normal to $k$ using the so-called Craya–Herring frame defined in CMG. The linear solution expresses the rotation of the initial Fourier component $\hat{\mathbf{u}}(k, 0)$ about the axis $k$ through an angle $\sigma_k t$. A diagonal form of the RDT Green’s function is found in terms of the two complex eigenvectors $\mathbf{N} = \mathbf{e}^{(2)} - i\mathbf{e}^{(1)}$ and $\mathbf{N}^* = \mathbf{N}(-k) = \mathbf{e}^{(2)} + i\mathbf{e}^{(1)}$:

$$G^{RDT}_{ij}(k, t, t') = \sum_{s=\pm 1} N_i(s k) N_j(-s k) e^{i\sigma_k(s-t)}$$  

with

$$\sigma_k = 2\Omega \frac{k_\parallel}{k}$$  

being the dispersion frequency for inertial waves in an unbounded domain. $G^{RDT}_{ij}$ defines the general solution of (5) as

$$\hat{u}_i(k, t) = G^{RDT}_{ij}(k, t, t') \hat{u}_j(k, t').$$  

Equation (6) displays the eigenmodes $\mathbf{N}(k)$ and $\mathbf{N}(-k) = \mathbf{N}^*(k)$, which have been used by different authors for over two decades (Cambon, Waleffe, Morinishi, Turner, in the same context and surely many others), and are called here ‘helical modes’ following Waleffe [8]. This diagonalization of the linear part of the problem is particularly useful in the context of pure rotation, but $\mathbf{N}$ and $\mathbf{N}^*$ also generate the eigenmodes of the curl operator, and thereby form a useful basis in which to expand the full nonlinear Navier–Stokes equations.

Random ‘slow’ wave amplitudes $a_s$, $s = \pm 1$, are defined by

$$\hat{u}_i(k, t) = \sum_{s=\pm 1} a_s(k, t) e^{i\sigma_k t} N_i(s k).$$
This representation improves on equation (1) by satisfying the divergence-free condition automatically; hence, the velocity amplitude is described by two independent amplitudes, instead of by three amplitudes subject to one linear condition. These wave amplitude variables, which are constant in the linear theory, can be used instead of the velocities $\hat{u}_i$ themselves in any dynamical or statistical approach. The nonlinear equation for the wave amplitudes derived from (3) and (4) is

$$\dot{a}_s = \sum_{s',s''=\pm 1} \int_{k+p+q=0} \exp\left[2i\Omega\left(\frac{k_{||}}{k} + s\frac{p_{||}}{p} + s'\frac{q_{||}}{q}\right)t\right] M_{ss's''}(k,p)a^*_s(p,t)a^*_s(q,t)\,d^3p$$

in which the diagonalized linear operator acting on $a_s(k,t)$ is absorbed into the nonlinear term through an integrating factor. In terms of the new variables, the quadratic nonlinearity retains its convolution structure which is reflected in the triadic relationship $q = -k - p$, but a new influence matrix $M_{ss's''}(k,p)$ (given in CMG) arises. We will note from equation (9) the importance of resonant triads, $\sigma_k \pm \sigma_p \pm \sigma_q = 0$, which correspond to zero value of the phase term on the right-hand side of (9), or

$$F_{ss's''}(k,p) = 2\Omega\left(\frac{k_{||}}{k} + s\frac{p_{||}}{p} + s'\frac{q_{||}}{q}\right) = 0 \quad \text{with } s, s', s'' = \pm 1 \text{ and } k + p + q = 0.$$ (10)

2.2. Equations for second-order correlations

The second-order spectral tensor, which is defined by

$$\langle \hat{u}_i^+(k',t)\hat{u}_j(k,t) \rangle = \Phi_{ij}(k,t)\delta(k-k')$$

can be expressed in terms of three scalars $e, \zeta, h$ (energy, polarization anisotropy, helicity, see CMG)\(^3\)

$$\Phi_{ij} = eP_{ij} + \mathfrak{M}[\zeta N_i N_j] + i\epsilon_{ijn}k_n/k.$$ (11)

This decomposition can be expressed in terms of products of helical modes alone

$$\Phi_{ij} = \frac{1}{2} \sum_{s=\pm 1} [(e + sh)N_i(-sk) + \zeta(sk)N_i(sk)]N_j(sk)$$

in agreement with the identity $N_i^* N_j = P_{ij} + i\epsilon_{ijn}k_n/k$. Accordingly, the second-order correlations of slow amplitudes, $A_{ss'}$, defined by

$$\langle a_s^*(k',t)a_{s'}(k,t) \rangle = A_{ss'}(k,t)\delta(k-k')$$

are related to the scalars $(e, \zeta, h)$ as follows:

$$e = \frac{1}{2}(A_{++} + A_{--}), \quad \zeta = 2A_{+-}e^{-2i\sigma t}, \quad h = \frac{1}{2}(A_{++} - A_{--}).$$

\(^3\) To avoid confusion, the variable previously called $z$ or $Z$ by Cambon and co-workers is denoted $\zeta$ hereinafter, and $h$ denotes the helicity spectrum divided by $k$.

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Only the ‘slow’ part of the polarization anisotropy \( Z = \zeta e^{-2i\omega t} = 2A_{+-} \) will be considered hereinafter. Equations for \((e, Z, h)\) are written as

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) e = T^{(e)}, \tag{13}
\]

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) Z = T^{(z)}, \tag{14}
\]

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) h = T^{(h)}, \tag{15}
\]

where the right-hand sides are transfer-like terms which involve triple velocity correlations. The EDQNM closure for the transfer terms will not be discussed in detail here for the sake of brevity; only three main ingredients are briefly recalled as follows: the equations for triple velocity correlations involve fourth-order correlations which are closed as products of second-order correlations according to the QN (quasi-normal) assumption. This assumption, which yields zero fourth-order cumulants, is then improved by replacing the fourth-order cumulants by damping of the triple correlations: the ED (eddy damping) introduced for this purpose modifies the ‘bare’ viscous Green’s function. This step can be considered to be an empirical evaluation of the nonlinear part of Kraichnan’s response tensor. Finally, the time memory of the triple correlations is reduced by Markovianization (M).

In the classical EDQNM for isotropic turbulence [14], the eddy damping is pivotal, the viscous + damping kernel in the time integral \( ED(t, t') \) is rapidly decreasing with time separation \( t - t' \), so that the contribution from double correlations can be ‘frozen’ at \( t' = t \). For turbulence in the presence of mean velocity gradients or body forces, the Markovianization procedure is less obvious, because of the presence of a nontrivial Green’s function in the time integral, and different versions can be proposed [15]. Since Markovianization amounts to selecting which terms \( T(t, t') \) have to be considered as ‘rapid’ (not altered) and which terms have to be considered as ‘slow’ (hence ‘frozen’ as \( T(t, t) \)), it is clear that there exists an optimal way to ‘Markovianize’ in WT, by only ‘freezing’ the contributions from slow amplitudes, or \( A_{zx}(t') \to A_{zx}(t) \) in (12), all other time-dependent contributions being unaltered. Roughly, these time contributions other than the slow amplitudes are connected to rapidly oscillating terms which induce a rapid damping when integrated in Fourier space, because of dispersivity and associated phase mixing.

The result of this ‘Markovianization limited to slow-amplitude terms’ was called EDQNM3, after EDQNM1 and EDQNM2 versions presented in [15]. Fortunately, EDQNM2 and EDQNM3 only differ in their treatment of \( Z \), so that EDQNM3 equations (given in the appendix) are easily derived from EDQNM2 equations in the case of pure rotation without initial helicity (CMG, appendix). As a bonus, the empirical eddy damping is much less important than it is in ‘strong’ turbulence, since rapid damping results from wave dispersivity. In the limit of very high rotation rate, the role of the viscous + damping term is only to regularize the resonance operators. Accordingly, the asymptotic limit in which the nonlinear eddy damping of EDQNM2 and EDQNM3 vanishes, called AQNM by BGSC, is independent of both eddy damping and viscosity, and gives essentially the same equations as Eulerian WT.

From now on, EDQNM3 equations will be considered with zero helicity (see the appendix) for the sake of simplicity, since helicity remains zero if initially zero (this results from a general
invariance property of NS equations in a rotating frame, and not from any peculiarities of EDQNM3). In EDQNM3 equations, \( T^{(e,z)} \) are given by volume integrals. The integrands are completely expressed in terms of \((e, Z)\) through quadratic terms involving triads.

Removing all terms which involve rapidly oscillating factors \( e^{\pm 2i\sigma t} \) with \( \sigma = \sigma_k, \sigma_p, \sigma_q, \sigma_p \pm \sigma_q \) in equations (A.5) and (A.6), only two contributions remain:

\[
T^{(e)} = \frac{1}{4} \sum_{s'} \int C_{kpq}^2 \left[ \frac{A_1(k, s'p, s'q)}{\mu_{kpq} + i(\sigma_k + s'\sigma_p + s'\sigma_q)} e(q, t)(e(k, t) - e(p, t)) \right] d^3 p, \tag{16}
\]

\[
T^{(z)} = \frac{1}{4} Z(k, t) \sum_{s'} \int C_{kpq}^2 \left[ \frac{A_1(k, -s'p, -s''q)}{\mu_{kpq} + i(-\sigma_k + s'\sigma_p + s''\sigma_q)} e(q, t) \right] d^3 p. \tag{17}
\]

The integrals in equations (16) and (17) are assumed to be over wavevector triples satisfying \( k + p + q = 0 \).

The AQNM equations are then derived using Plemelj’s formula \[16\]^4

\[
\lim_{\mu \to 0} \frac{1}{\mu + i x} \frac{1}{x} = \pi \delta(x) - i \mathcal{P} \left( \frac{1}{x} \right). \tag{18}
\]

Accordingly

\[
\lim_{\mu \to 0} \int A(k, p) \frac{d^3 p}{\mu + i F_s' s''} = \int_{S_{s'} s''} \frac{A(k, p)}{\alpha_{s'} s''} dS - i \int_{\mathbb{R}^3} \frac{A(k, p)}{F_s' s''} d^3 p, \tag{19}
\]

where \( \alpha_{s'} s'' \) denotes the gradient of the function \( F_{s'} s'' \) (see (10)) at the surface \( S_{s'} s'' \) defined by \( F_{s'} s'' = 0 \). It is therefore angle-dependent and involves the group velocity \( C_s \) as

\[
\alpha_{s'} s'' = \frac{1}{\pi} |s' C_s(p) - s'' C_s(q)| \tag{20}
\]

with

\[
C_s(k) = -\frac{2\Omega}{k} e^2(k) \frac{k_\perp}{k}. \tag{21}
\]

The final AQNM equations are equations (13) and (14) with

\[
T^{(e)} = \sum_{s'} \int_{S_{s'} s''} \frac{g_{s' s''}}{\alpha_{s'} s''} (e'(e'' - e)) dS, \tag{22}
\]

\[
T^{(z)} = -Z \sum_{s'} \left[ \int_{S_{s'} s''} \frac{g_{s' s''}}{\alpha_{s'} s''} e' dS + i \int_{\mathbb{R}^3} \frac{g_{s' s''}}{F_{s' s''}} e' d^3 p \right]. \tag{23}
\]

^4 This formula, mentioned in [17] expresses the limit of vanishing \( \mu \) in the (weak) sense of distributions, turning any integral of which the integrand is weighted by the left-hand side, into both a real Dirac contribution and an (imaginary) principal value integral. All details are reviewed in [18] for those who are not acquainted with this formalism.

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where \( e' = e(q, t), e'' = e(p, t) \) and \( e = e(k, t) \). The geometric factor in both equations is
\[
g_{s',s''}(k, p, q) = \frac{1}{16k^2p^2q^2}[(p + q)^2 - k^2][k^2 - (p - q)^2][s'p - k][s'p - s''q][k + s'p + s''q]^2. \tag{24}
\]

Strong anisotropy results from the angle-dependent \( \alpha_{s',s''} \) term and from the topology of the resonance surfaces themselves; both effects reflect the appearance of a Dirac function \( \delta(F_{s',s''}) \) in energy and helicity transfers (see also Galtier [5]). Because of the link of \( \alpha_{s',s''} \) to the group velocity, the transfer scales as \( 1/\Omega \), which replaces the eddy damping term \( 1/\mu_{kpq} \) in classical EDQNM versions with no rotation.

3. Analytical derivation of angular dependence and comparison with AQNM numerical results

3.1. Analytical bihomogeneous energy spectrum

Closure theories such as AQNM and EDQNM\textsuperscript{3} provide important insights into the nonlinear dynamics of energy transfer in rotating turbulence, but at the expense of potentially elaborate and complex computations. Purely analytical arguments have been used [7] to find scaling laws, inertial range constants and flux directions from isotropic weak turbulence theory. This section considers some recent attempts to extend those analytical arguments to the anisotropic problem of rotating turbulence.

The dispersion relation of inertial waves, equation (7), is homogeneous of order 0 in \( k \). Whereas an isotropic dispersion relation with this scaling would not admit three-wave resonances [7], the anisotropy of the dispersion relation of inertial waves makes three-wave resonances possible. The (entirely formal) isotropic weak turbulence spectrum for three-wave resonances in a problem in which the dispersion law is of homogeneity order zero is found by counting powers of \( k \) in the constant flux condition
\[
\epsilon = \int_{k \leq k_0} d^3k \, T^{(e)}(k) \tag{25}
\]
using equation (22) for \( T^{(e)} \). Since
\[
g \sim k^2, \quad \alpha \sim \Omega/k, \quad dS \sim k^2, \quad d^3k \sim k^3 \tag{26}
\]
the result is
\[
\epsilon \sim k^8\Omega^{-1}e(k)^2 \tag{27}
\]
so that \( \epsilon \sim \sqrt{\epsilon \Omega k^{-4}} \), and the classical spherically averaged energy spectrum \( E(k) \sim \sqrt{\epsilon \Omega k^{-2}} \), as in the phenomenological model of Zhou [19] and the closure theory of Canuto [20]. Although this spectral law appears to be supported by some experimental data [21], DNS [22] and some simplified closure computations (see equation (2.18) in [10]) any argument based on equation (27) can be questioned, because although it completely ignores anisotropy, equation (27) requires anisotropy to postulate an energy balance based on three-wave resonances rather than the energy...
balance based on four-wave resonances

\[ \epsilon = k^{13} \Omega^{-3} e(k)^3, \]  

(28)

which must be used when three-wave resonances are impossible [7, 23]. Thus, a more fundamental analytical theory of rotating turbulence cannot avoid treating anisotropy explicitly.

Zakharov et al [7] observe that corresponding to a hypothetical bihomogeneous dispersion relation of the form

\[ \sigma(k) \sim k^a k^b, \]  

(29)

the WT wave kinetic equation admits a steady-state bihomogeneous solution of the form

\[ e(k) \sim k^c k^d. \]  

(30)

The scaling exponents c and d, inertial range constants and flux directions can all be established by analytical arguments similar to those used in the isotropic case. It should be noted that this analysis assumes a tensorially isotropic form for the correlation function so that, in the notation of equation (11), \( h = \zeta = 0 \); this paper will discuss the possibility that in rotating turbulence, the correlation function must depend on \( \zeta \) because the wave frequency vanishes when \( k \parallel = 0 \).

The anisotropic theory does not apply directly, because the dispersion relation of inertial waves, equation (7), is not of the required bihomogeneous form (29). Nevertheless, following Zakharov et al [7], this theory can be applied to waves with nearly horizontal wavevectors for which \( k \parallel \approx 0 \) and \( k \approx k \perp \). In this limit,

\[ \sigma(k) = 2 \Omega \frac{k_\parallel}{k_\perp} \left\{ 1 - \frac{1}{2} \left( \frac{k_\parallel}{k_\perp} \right)^2 \pm \cdots \right\} \approx 2 \Omega \frac{k_\parallel}{k_\perp}, \]  

(31)

leading to an approximate bihomogeneous dispersion relation of the form (29) with \( a = +1 \) and \( b = -1 \).

A detailed derivation of the corresponding anisotropic energy spectrum has recently been completed by Galtier [5]. These arguments and their implications for scaling laws in rotating turbulence will be considered next. An analysis of a similar problem in stratified turbulence, the weak turbulence of internal waves with nearly vertical wavevectors, is given by Caillol and Zeitlin [17].

We again refer to the constant flux condition (25), but with equation (26) replaced by

\[ g \sim k^2_\perp, \quad \alpha \sim \Omega/k_\perp, \quad dS \sim k^2_\perp, \quad d^3k \sim k_\parallel k^2_\perp. \]  

(32)

Therefore [7], evaluating \( T^{(e)} \) for the scaling solution (30),

\[ T^{(e)}(k) = k_\parallel^{2c+1} k_\perp^{2d+5} I(c, d), \]  

(33)

where \( I(c, d) \) is a function of the scaling exponents alone, and the energy flux scales as

\[ \epsilon \sim k_\parallel^{2c+1} k_\perp^{2d+7} I(c, d). \]  

(34)
In a constant flux state, necessarily \( c = -1/2 \) and \( d = -7/2 \) [5], therefore in view of equation (30),

\[
e(k_{\parallel}, k_{\perp}) \sim k_{\parallel}^{-1/2} k_{\perp}^{-7/2}.
\]

(35)

It should be stressed that this approximate WT spectrum, similar to the isotropic spectrum with \( e(k) \sim k^{-4} \), contains the prefactor \( \sqrt{\epsilon/\Omega_1} \).

In the theory of Zakharov et al [7], the local flux vector has two components given by

\[
P_{\parallel} = \frac{\partial I}{\partial \alpha} \left( -\frac{1}{2}, -\frac{7}{2} \right) \frac{1}{k_{\perp}^2} \text{sgn}(k_{\parallel}),
\]

(36)

\[
P_{\perp} = \frac{\partial I}{\partial \beta} \left( -\frac{1}{2}, -\frac{7}{2} \right) \frac{1}{k_{\perp} k_{\parallel}} \text{sgn}(k_{\parallel}).
\]

(37)

The flux through a region bounded by the planes \( k_{\parallel} = \pm a \) is given by

\[
\epsilon_{\parallel} = \int_{k_{\parallel}=\pm a} \pm P_{\parallel} k_{\perp} \, dk_{\perp},
\]

where a wave source is concentrated on the plane \( k_{\parallel} = 0 \), and the flux through a cylinder \( k_{\perp} = b \) is given by

\[
\epsilon_{\perp} = \int_{k_{\perp}=b} P_{\perp} \, dk_{\perp} \, dk_{\parallel},
\]

where a wave source is concentrated on the line \( k_{\perp} = 0 \).

Since

\[
\frac{P_{\parallel}}{P_{\perp}} = \frac{\partial I/\partial \alpha}{\partial I/\partial \beta} \frac{k_{\parallel}}{k_{\perp}},
\]

(40)

the flux is isotropic if

\[
\frac{\partial I}{\partial \alpha} \left( -\frac{1}{2}, -\frac{7}{2} \right) = \frac{\partial I}{\partial \beta} \left( -\frac{1}{2}, -\frac{7}{2} \right)
\]

(41)

but, otherwise, it is anisotropic. This argument gives theoretical support to at least the possibility of an angular energy flux in anisotropic rotating turbulence and shows, at least in principle, how an anisotropic flux could be computed. Unfortunately, the computations required to evaluate the flux components are rather complicated. Galtier [5] observes that the fluxes are very likely to be positive, but the question of whether the flux might have a component which moves energy towards the \( k_{\parallel} = 0 \) plane remains open.

An important property of the energy spectrum is its locality, namely the convergence of the integral representing the energy flux when the proposed scaling form applies to all modes. Indeed, the formal calculations leading from equation (32) to the scaling solution (35) implicitly assumes the convergence of all integrals. The form of the closure for \( T^{(e)} \) in equation (22) suggests a possible divergence of \( T^{(e)} \), and hence of the energy flux, when \( q \approx 0 \) and consequently \( e' \to \infty \).
In this limit, \( p \to k \). Obviously then, \( g \approx k_\perp^2 \) and \( \alpha \approx k_\perp^{-1} \) as before, and these terms have no role in the convergence or divergence of \( T^{(e)} \). Expanding \( e - e'' \) in a Taylor series,

\[
e(k) - e(p) \approx \frac{\partial^2 e}{\partial k^2} q_\perp^2.
\] (42)

Next, we must consider how \( q_\parallel \) and \( q_\perp \) are related on the resonance surface. Writing the approximate resonance condition as

\[
k_\parallel \approx \pm \frac{k_\parallel - q_\parallel}{|k_\perp - q_\perp|} \pm q_\perp
\] (43)

and solving for \( q_\parallel \), we obtain

\[
q_\parallel = \pm q_\perp \frac{k_\parallel |k_\perp - q_\perp|}{k_\parallel q_\perp - |k_\perp - q_\perp|}.
\] (44)

The different choices of sign give two different solutions

\[
q_\parallel = \pm k_\parallel q_\perp \frac{k_\parallel \cdot q_\perp}{k_\perp^2} + O(q_\perp^2)
\] (45)

and

\[
q_\parallel = \pm k_\parallel \frac{2q_\perp}{k_\perp} + O(q_\perp^2).
\] (46)

The assumption made to derive the anisotropic spectrum, \( q_\parallel \ll q_\perp \), requires the solution (45). Since \( q_\parallel = O(q_\perp^2) \), we may use \( dq_\perp \sim q_\perp dq_\perp \) to integrate over the resonance surface near the point where \( q_\perp = 0 \). Near this point,

\[
T^{(e)} \sim \int q_\perp^2 q_\perp^{-1/2} q_\perp^{-7/2} dq_\perp dq_\perp.
\] (47)

Substituting \( q_\parallel \sim q_\perp^2 \) into equation (47), the divergent expression

\[
T^{(e)} \sim \int q_\perp^{2-1-7/2+1} dq_\perp = \int q_\perp^{-3/2} dq_\perp
\] (48)

is obtained. Note that the reason for the divergence is the combination of the factor \( q_\perp^{-1/2} \) in the bihomogeneous spectrum with the local parametrization \( q_\parallel \sim q_\perp^2 \).

The conclusion is that the flux integral is divergent, and that the divergence is proportional to \( k_0^{-1/2} \) where \( k_0 \) is a cutoff in the \( k_\perp \) plane. This divergence implies that the spectrum actually scales as

\[
e(k) \sim k_0^{-1/2} k_\perp^{-1/2} k_\perp^{-3}
\] (49)

although this formal solution will still induce a weak divergence requiring a logarithmic correction, which we will ignore. Zakharov et al [7] note that anisotropic spectra are usually non-local; the conclusion of Caillol and Zeitlin [17] that the anisotropic spectrum of inertial waves

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is non-local also agrees with this observation. Numerical evidence from AQNM calculations for the spectrum, equation (49), and the significance of non-locality will be considered in the next section.

Although the numerical results, to be discussed in the next subsection, support the tentative conclusion that a $k^{-1/2}k^{-3}$ range exists in the AQNM approximation for nearly horizontal modes with $k_\parallel \ll k_\perp$, certain theoretical questions about equation (49) remain. In [7], it is observed that computing a spectrum using the approximate dispersion relation (31) requires that the modes for which the approximation is valid must interact among themselves more strongly than they interact with modes for which the approximation is not valid. Since this restriction is a locality condition, the nonlocality of the proposed bihomogeneous spectrum might undermine the very assumption on which the derivation was based. We conclude that although the numerical AQNM results encourage looking for approximate scaling laws in anisotropic rotating turbulence, equation (49) for rotating turbulence can only be advanced tentatively at present.

3.2. Angular-dependent energy spectrum from numerical AQNM

In our numerical implementation of the AQNM model, the transfer (22), which sets the rate of change of the energy spectrum, is an integral over resonant surfaces, which are approximated by piecewise-linear surface elements. These surface elements are defined from the intersection of a spherical grid in $k$ space with the resonant surfaces. We have used a typical discretization of 300 points in each direction (radius $k$, azimuthal and latitude angles) which is enough for achieving convergence of the method. When needed, linear interpolation of the energy density spectrum is used for computing the integrand.

We will now exploit the axisymmetric energy density distribution $e(k, \cos \theta)$ obtained from numerically solving AQNM, at the largest time for which the inertial range is stably constructed. Time $t_f$ is omitted in the following for the sake of brevity. Analytical 'bihomogeneous' laws, equations (49) and (35), with and without cutoff will be referred to as (R) and (G), respectively.

Both spherical and cylindrical coordinates are relevant, so that we will also work with

$$\tilde{e}(k_\parallel, k_\perp) = e(k, \cos \theta).$$

More classically, integrated energy spectra are considered, as the spherically averaged one:

$$E(k) = 4\pi \int_0^1 k^2 e(k, x) \, dx \quad (50)$$

or

$$E(k) = 4\pi \int_0^1 k^2 \tilde{e}(kx, k\sqrt{1-x^2}) \, dx. \quad (51)$$

Figure 1 shows the spectrum $E(k)$ obtained by the numerical resolution of AQNM. Initial conditions are chosen to be a narrow-band spectrum, so as not to influence the future development of the inertial subrange, which comes out to scale as $k^{-3}$ when the spectrum begins to evolve in a self-similar way, i.e. when it is fully developed. The spectrum at this moment is shown in figure 1.

Figure 2 shows the spectrum obtained by the AQNM numerical resolution, closest to the horizontal plane. Owing to the accurate discretization, the corresponding angle is only $\pi/150$ rads away from the $\pi/2$ angle for the horizontal manifold. Figure 2 shows the spectrum $e$ along this
Figure 1. Energy spectrum $E(k)$ as a function of $k$ at time $t = 1.05$ (shortly after turbulence has reached a fully self-similar decaying regime). The straight line indicates a $k^{-3}$ power law.

Figure 2. Energy density spectrum $k^2 \tilde{e}(k, \cos(\theta = 149\pi/300))$ as a function of $k$. This spectrum is the closest to the equatorial plane available from the numerical resolution of the AQNM model. The straight lines indicate $k^{-3/2}$ and $k^{-2}$ power laws.
radius, multiplied by $k^2$ to dimensionalize it as the integrated spectrum. We observe a $k^{-2}$ power law, which was expected by Galtier [5].

In figures 3 and 4, we consider the dependence of $e$ respectively with $k_\perp$ and $k_\parallel$, at the same instant as figure 1. Each figure shows a series of spectra obtained by setting $k_\parallel$ or $k_\perp$ to different constant values, so as to examine the dependence of the spectrum independently of the other component. This corresponds to cutting slices of the spectral domain $(k, \cos \theta)$ along the horizontal or vertical axes, and obtaining the corresponding spectra. Figure 3 shows that the $k_\perp^{-3}$ dependency of $\tilde{e}$ is obtained only for spectra at small given values $k_\parallel$, i.e. in the region close to the quasi 2D boundary layer, where energy accumulates. When considering larger values of $k_\parallel$, the energy decays very fast, and no such power law is observed. The $k_\perp^{-3}$ law seems to be recovered only for the smallest values of $k_\parallel$, for which, unfortunately, the numerical cut-off allows only the extraction of the spectrum over a narrow wavenumber band. It is however clear enough to be identified without doubt.

The $k_\parallel^{-1/2}$ dependency of $\tilde{e}$ can be easily observed in figure 4, where spectra at minimum $k_\perp$ follow quite nicely this power law, then exhibit the classical viscous cut-off at larger $k_\parallel$. When $k_\perp$ increases, the spectra depart from the scaling of equation (46).

One-dimensional (1D) spectra are also common, e.g. from measurements and observations. They correspond to integrals of the second-order spectral tensor over planes. Particularly relevant in our axisymmetric case are those integrated on horizontal planes $k_\parallel = \text{constant}$:

$$E_{33}(k_\parallel) = \int_0^\infty \int_0^{2\pi} \Phi_{33} k_\perp \, dk_\perp \, d\phi,$$
Figure 4. Energy density spectrum $\tilde{e}(k_{\parallel}, k_{\perp})$ as a function of $k_{\parallel}$ at constant $k_{\perp}$. The spectrum decreases with $k_{\perp}$ which takes the following values (nondimensionalized by $k_{\text{min}} = 0.1$): 10, 20, 30, 70, 140. The straight line is a $k^{-1/2}$ power law.

which is simplified as

$$E_{33}(k_{\parallel}) = E_v(k_{\parallel}) = 2\pi \int_0^\infty \tilde{e}(k_{\parallel}, k_{\perp}) \frac{k_{\perp}^2}{k_{\parallel}^2 + k_{\perp}^2} k_{\perp} \, dk_{\perp}$$  \hspace{1cm} (52)

or

$$E_{33}(k_{\parallel}) = E_v(k_{\parallel}) = 2\pi \int_{k_{\parallel}}^\infty e \left( \frac{k_{\parallel}}{k} \right) \left( 1 - \frac{k_{\parallel}^2}{k^2} \right) k \, dk$$  \hspace{1cm} (53)

using axisymmetry, $Z = 0$ in (11), and $kd \, k = k_{\parallel} \, dk_{\perp}$. Similarly

$$E_{11}(k_{\parallel}) + E_{22}(k_{\parallel}) = E_h(k_{\parallel}) = 2\pi \int_0^\infty \tilde{e}(k_{\parallel}, k_{\perp}) \left( 1 + \frac{k_{\parallel}^2}{k_{\parallel}^2 + k_{\perp}^2} \right) k_{\perp} \, dk_{\perp}$$  \hspace{1cm} (54)

or

$$E_h(k_{\parallel}) = E_{11}(k_{\parallel}) + E_{22}(k_{\parallel}) = 2\pi \int_{k_{\parallel}}^\infty e \left( \frac{k_{\parallel}}{k} \right) \left( 1 + \frac{k_{\parallel}^2}{k^2} \right) k \, dk.$$  \hspace{1cm} (55)

1D spectra integrated over vertical planes are more complicated to calculate and especially to interpret, except for the limiting case $k_{\perp} = 0$ in which typical integral length scales can be
derived. A relevant alternative is to evaluate spectra integrated over cylinders with vertical axis, as proposed by Waite and Bartello [24], since they are well adapted to axisymmetry (in counterpart classical length scales cannot be derived from them). One defines

\[ E_v(k_\perp) = E_{33}(k_\perp) = 2\pi k_\perp \int_0^\infty \tilde{e}(k_\perp, k_\perp) \frac{k_\perp^2}{k_\parallel^2 + k_\perp^2} \, dk_\parallel \]  

(56)

or

\[ E_v(k_\perp) = E_{33}(k_\perp) = 2\pi k_\perp \int_{k_\perp}^\infty e \left( k, \sqrt{\frac{k^2 - k_\perp^2}{k}} \right) \frac{k_\perp^2}{k^2} \, dk \]  

(57)

and

\[ E_h(k_\perp) = E_{11}(k_\perp) + E_{22}(k_\perp) = 2\pi k_\perp \int_0^\infty \tilde{e}(k_\perp, k_\perp) \left( 1 + \frac{k_\parallel^2}{k_\perp^2 + k_\parallel^2} \right) \, dk_\parallel \]  

(58)

or

\[ E_h(k_\perp) = E_{11}(k_\perp) + E_{22}(k_\perp) = 2\pi k_\perp \int_{k_\perp}^\infty e \left( k, \sqrt{\frac{k^2 - k_\perp^2}{k}} \right) \left( 1 + \frac{k_\parallel^2 - k_\perp^2}{k^2} \right) \, dk. \]  

(59)

We can discuss now AQNM results and consequences of the analytical distributions (G) and (R) for these different spectra. For \( E(k) \), the \( k^{-3} \) law is the most salient AQNM result. In turn, the (G) distribution yields \( E \sim k^{2-1/2-7/2} = k^{-2} \) and the (R) distribution yields \( E \sim k_0^{-1/2} k^{2-1/2-3} = k_0^{-1/2} k^{-3/2} \). Looking at 1D spectra, but only for the sum \( E_v + E_h = E' \) for the sake of simplicity, one may look for scalings of \( E'(k_\parallel) \) from AQNM, but \( E' \) diverges due to the divergence \( k_\parallel = 0 \) of \( \int_0^\infty k^{-5/2} \, dk \) using (G), and \( E'(k_\parallel) \sim k_\parallel^{-1/2} k_0^{-3/2} \) using (R). In the same way, looking at \( E'(k_\perp) \), and \( E'(k_\perp) \) diverges because of the divergence \( k_\parallel = \infty \) of \( \int_0^\infty k^{1/2} \, dk \), in both (G) and (R), which can explain bad results for \( E(k_\perp) \).

4. Revisiting the vicinity of the slow mode

4.1. The slow manifold: general problem

In the strict limit \( k_\parallel = k \cos \theta = 0 \), the relevant resonance surfaces collapse to the 2D plane, so that only planar\(^5\) triads are involved in the AQNM transfer for \( e \) and \( h \), as in pure 2D turbulence. This result is consistent with the ‘decoupling’ of the pure 2D mode predicted by Babin et al [25]. Nevertheless, the corresponding AQNM energy transfer always scales as \( 1/\Omega \) as previously noted, whereas the rotation term would disappear in the pure 2D case. Even the geometric coefficient in AQNM at \( \cos \theta = 0 \), which results from \( g_{s's''}/\alpha_{s's''} \), does not coincide with its counterpart in e.g. 2D EDQNM [26]. Note that the case \( k_\parallel = 0 \) was not calculated in numerical AQNM. Numerical computation, however, accounted for the closest angular directions, with 150

\[^5\] The other surface characterized by \( p_x \neq 0, p = q \) has no contribution since all geometric coefficients identically vanish, as the influence matrix itself in (9) does. Problems of possible double resonances are not considered here.
polar angles \( \theta \) for \( k \), in \([0, \pi/2]\). Currently, attempts to solve AQNM numerically at \( k_\parallel = 0 \) seem to blow up [12]. Perhaps this observation is consistent with the \( k_-^{1/2} \) dependence noted in the bihomogeneous spectra of the previous section.

On the other hand, AQNM \( T^{(Z)} \) is linear in term of \( Z \), and it is the only term which does not reduce to a surface integral. Nevertheless, its contribution in (23) exactly vanishes at \( k_\parallel = 0 \), due to reflectional symmetry, as will be discussed subsequently. In any case, \( Z \) is unimportant in AQNM since it remains zero if it is initially zero (as in canonical 3D isotropic initial data).

Looking at the complete zero-helicity EDQNM\(_3\) equations in the appendix, it is clear that many other quadratic terms involving \( Z \) were discarded in AQNM when removing rapidly oscillating terms, namely, those terms which are multiplied by \( e^{\pm 2i\sigma t} \), with \( \sigma = \sigma_k, \sigma_p, \sigma_q, \sigma_p \pm \sigma_q \). These assumption should be questioned as soon as the frequencies \( \sigma_k, \sigma_p, \sigma_q \) are not large enough even at very large rotation frequency, i.e. if the cosines \( \cos \theta = k_\parallel / k, \cos \theta_p = p_\parallel / p, \cos \theta_q = q_\parallel / q \) are of order \( O(1/\Omega) \). This problem is not accounted for in classical WT theory, in which vanishing of the wave frequency is not considered. Although WT predicts concentration of energy towards the slow manifold, it cannot describe the vicinity of the slow manifold [11], but the problem is twofold.

On the one hand, it is trivial to show that removal of terms associated with explicit time dependence, as \( e^{\pm 2i\sigma t} \), is incorrect. Applying equation (18) without a priori removing these terms will give an ‘extended WT’ theory, in which \( Z \) will be an unavoidable component of the correct anisotropic description. These \( Z \)-type terms, which represent cross-correlations between wave amplitudes of opposite polarities, will involve new relevant imaginary contributions in equation (18). Both ingredients (imaginary parts, \( Z \)-correlation) are systematically ignored in the kinetic equation formalism of classical WT theory [5].

On the other hand, and this is more subtle, it can be proved [12] that equation (18) no longer applies when \( e \) and \( Z \) vary so rapidly across the slow manifold \( x = \cos \theta = O(Ro) \) that they no longer can be interpreted as ‘slow’ variables in the ‘inner’ calculation. The full EDQNM\(_3\) equations would have to be solved in the inner calculation, without taking the limit \( \mu_{kpq} \rightarrow 0 \), as it is done when applying equation (18).

In other words, it is necessary to consider a kind of ‘matched asymptotic expansion’ in which WT has the role of an ‘outer solution’ which matches to an ‘inner solution’ in which eddy damping is relevant. The inner solution has the role of a ‘quasi-2D boundary layer’ for the slow manifold. As will be discussed in the next section, this analysis is made possible by the existence of a general EDQNM\(_3\) model which underlies the AQNM model. It will be shown how \( Z \) can be fed by strong nonlinearities. The first step will be to derive at \( k_\parallel = 0 \) those terms which were unjustifiably discarded in AQNM, using an ‘extended WT theory’.

4.2. The extended WT theory

Let us now consider when removing the EDQNM\(_3\) terms (see the appendix) with time-dependent phase factors is not justified, namely at sufficiently small \( k_\parallel \). As a first step, we will systematically apply equation (18), even if it is known that it can also be questioned, and then remove the time-dependent phase terms. In other words, it is possible to go one step beyond AQNM by applying the limit of vanishing \( \mu \) before removing the rapidly oscillating terms—all those terms which contain \( e^{-\mu_{kpq} t} \) contributions in the transfer (see the appendix)—thus reversing the procedure in BGSC. The result can be seen as an ‘extended WT theory’ containing volume integral contributions.

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First, the exact ‘slow’ limit, or $k_\parallel = 0$ is addressed. To apply equation (18) to the EDQNM3 equations in this limit, it is necessary to calculate both $F_{s's''}$ and its gradient $\alpha_{s's''}$ at $k_\parallel = 0$. For instance,

$$F_{s's''} = s'\sigma_p + s''\sigma_q = 2\Omega (s' \cos \theta_p + s'' \cos \theta_q) = 2\Omega \sin \lambda C_{kpq}(s'p - s''q), \quad (60)$$

$$\alpha_{s's''} = \frac{2\Omega}{\pi pq} |s'p - s''q| \quad (61)$$

(see the appendix). Only the $F_{s's''}$ denominator in the principal value volume integral contains the angle $\lambda$—which rotates the triad about $k$ to the horizontal plane—in addition to $k$, $p$, $q$. Accordingly, the equations for $k_\parallel = 0$ reduce to

$$T^{(c)} = T^{(c)}_{\text{planar}} + \frac{1}{2\Omega} \sum_{s',s''} \int d^3p d^3q B_3(k, -s'p, -s''q) \cos \lambda e(q, t) [e(k, t) - e(p, t)] d^3p. \quad (62)$$

$$T^{(c)} = T^{(c)}_{\text{planar}} - \frac{1}{2\Omega} \sum_{s',s''} \int d^3p d^3q B_3(k, -s'p, -s''q) \cos \lambda e(q, t) [e(k, t) - e(p, t)] d^3p. \quad (63)$$

In both equations, the surface contributions are the ‘endogenous’ contributions from planar triads ($p\parallel = q\parallel = 0$) alone, and are denoted by $T^{(c)}_{\text{planar}}$ and $T^{(c)}_{\text{planar}}$ in the equations above. In addition, unavoidable volume contributions exhibit $e/Z$ coupling from the entire wavespace, except for nearly planar triads, since the principal value was introduced to remove this zone, given the singularity of $F_{s's''} = 0$. The coefficient $B_3$ results from dividing the geometric coefficient $A_3(k, s'p, s''q)$ by $s'p \pm s''q$. The angle $\lambda$ is the angle by which the plane of the triad is rotated around $k$ as in CMG, so that the elementary volume could be expressed as $d^3p = dp_1dp_2dp_3 = \frac{\pi}{3} dp dq d\lambda$ using $p$, $q$, $\lambda$ as integration variables.

Note that the volume contribution $Z(k, t)$ vanishes in $T^{(c)}$ (second term in (23) at $k_\parallel = 0$ due to reflectional symmetry). Similarly, the $e''(e - e')$ contribution vanishes in $T^{(c)}$. All other possible volume contributions remain weighted by (really!) rapidly oscillating terms, so that they were neglected. Removing these terms have to be justified by a refined analysis, but it seems to be a reasonable simplification.

Since the surface contributions are characterized by pure planar triads $\sigma_k = \sigma_p = \sigma_q = 0$, no rapidly oscillating term occurs in the limit $k_\parallel = 0$, even at very high rotation rate, so that all contributions inherited from EDQNM3 are present. Thus,

$$T^{(c)}_{\text{planar}} = \frac{1}{2\Omega} \sum_{s',s''} \int \frac{pqC_{kpq}^2}{2\Omega} [B_1(k, s'p, s''q)e''(e - e') - B_2(k, s'p, s''q)\cos(\lambda e(q, t)] d^3p \quad (64)$$

$$- B_3(k, s'p, s''q)\cos(\lambda e(q, t)] + B_5(k, s'p, s''q)e'Z(s''q) + B_4(k, s'p, s''q)Z(s''q)(Z(k) - Z(s'p))] d^3p.$$

$B_1$–$B_5$ geometric factors are derived from their full EDQNM3 (or EDQNM2) counterparts $A_1$–$A_5$ in dividing them by the factor $|s'p \pm s''q|$ linked to $\alpha_{s's''}$ (see the appendix). The first term

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is simply the AQNM term in (22), with the particular value of $\alpha_s$, $\alpha'_s$, $\alpha''_s$ for planar triads; all the others are new.

Similarly,

$$T_{\text{planar}}^{(2)} = \frac{1}{2\Omega} \sum_{s,s'} \int p q C_{kpq}^2 \left[ B_3(k, -s' p, -s'' q)e''(e - e') + B_4(k, -s' p, -s'' q)e Z(s'' q) \right.
\left. - B_1(k, -s' p, -s'' q)e Z(k) - B_5(k, -s' p, -s'' q)e'' Z(s'' p) \right.
\left. + B_2(k, s' p, s'' q)Z(s'' q)(Z(k) - Z(s' p)) \right] d^2 p \tag{65}$$

in which the linear $Z$-term with geometric coefficient $B_1$ corresponds to the first AQNM term in (23) for planar triads. All other terms are new. As in full 2D EDQNM, the elementary area $d^2 p$ can be replaced by $d p d q / \sin(p, q)$ (see [10], appendix).

Finally, surface contributions to the transfer for energy $e$ and slow polarization $Z$ in the 2D mode are 2D-‘endogenous’ but reflect a strong $e/Z$ coupling (all EDQNM$_3$ contributions are involved), with $Z$ becoming important even if it is initially zero.

It is important to notice that these ‘planar’ contributions differ essentially from the pure 2D limit. The strict 2D limit is formulated as

$$e(k, 0) = \frac{E(k)}{2\pi k} \delta(k x), \quad Z(k, 0) = C_{\text{pol}}(k) \frac{E(k)}{2\pi k} \delta(k x).$$

Substituting the above equations in the full EDQNM$_3$ (see the appendix, or equivalently in EDQNM$_2$) recovers the classical 2D EDQNM transfer [26] for $T^{(e)} - T^{(c)}$ with $C_{\text{pol}} = -1$ (two-component, 2D limit) as shown in CMG appendix. The term $\delta(k x)$ selects only planar triads and also removes all $\Omega$ dependence, in agreement with the rotation-independent 2D limit. A slightly more general 2D model was proposed by Cambon and Godeferd [27], by keeping the polarization coefficient $C_{\text{pol}}$ as a free parameter. Again, a purely 2D EDQNM equation was recovered for $T^{(e)} - T^{(c)}$ in terms of $e - Z$ only. In addition, analysis of the transfer term $T^{(e)} + T^{(c)}$ showed that $e + Z$ had the role of a passive scalar in 2D turbulence, a conclusion which is ultimately not too far from that of Babin et al [25] found using a different formalism.

Instead of developing these oversimplified ‘full 2D, 2C or 3C’ models further, we will only insist on the importance of $Z$, which is real at $k_\parallel = 0$, so that $e(k, 0) - Z(k, 0)$ corresponds to the energy density of the vertically averaged horizontal motion, and $e(k, 0) + Z(k, 0)$ to the energy density of the vertically averaged vertical motion. The role of $Z$ will be rediscussed at the end of this section with respect to evidence from physical and numerical experiments.

### 4.3. Mathematical and physical relevance of coupling terms with polarization anisotropy $Z$

It is now crucial to stress that equation (18) cannot be justified in the ‘inner’ calculation of the ‘slow’ manifold. On the one hand, equations (64) and (65) must be replaced by the full EDQNM$_3$ equations given in the appendix for the inner calculation, with non-planar triads explicitly involved, and in particular both $\mu$ (in contrast with WT) and $\Omega$ (in contrast with pure 2D or oversimplified ‘decoupling’ arguments) being nonzero. Of course, our equations above remain a cartoon model for a correct treatment of the inner zone. Full EDQNM$_3$ equations will conserve all these $e/Z$ couplings, but in an even more complex way.

On the other hand, the $A_3$ related EDQNM$_3$ terms in both $T^{(e)}$ and $T^{(c)}$ yield principal value volume integral ‘exogenous’ contributions, again reinforcing the $e/Z$ coupling. It is reasonable to consider that these volume contributions in (62) and (63) remain valid for a correct ‘outer’
treatment of the vicinity of the slow manifold, at least at \( k_\parallel = 0 \). Since the principal values exclude the case of pure planar triads, they can be evaluated from the outside of the zone, therefore with the smooth \( e \) -distribution inherited from AQNM. In addition, these outer contributions display only the two, relatively simple, integrals
\[
\int_{\mathbb{R}^3} C_{kpq} B_3 (k, s'' q, s' p) \cos \lambda e(p, \cos \theta_p, t) \, d^3 p
\]
and
\[
\int_{\mathbb{R}^3} C_{kpq} B_3 (k, -s'' q, -s' p) \cos \lambda e(p, \cos \theta_q, t) e(q, \cos \theta_q, t) \, d^3 p
\]
with \( \theta_p \) and \( \theta_q \) given in the appendix and \( d^3 p \) possibly replaced by \( (pq/k) \, dp \, dq \, d\lambda \).

The arguments which justify a possible ‘decoupling’ of the ‘2D mode’ ignore the \( Z \) contribution and focus exclusively on resonant triads instead. The volume contributions couple the inner domain of the slow manifold to the outer one. In this connection, it may be useful to review the ‘nonlinear Proudman Theorem’ proposed by Babin et al [25]. This theorem is formulated in terms of the \( \text{vertically averaged field} \)
\[
\bar{u}(x_\perp, t) = \lim_{L \to \infty} \frac{1}{L} \int_{-L}^{L} dz \bar{u}(x, t)
\]
and states the commutativity of evolution of strongly rotating turbulence with vertical averaging. That is, if a given field \( u(x) \) is vertically averaged to produce \( \bar{u}(x_\perp) \), and \( \bar{u}(x_\perp) \) is used as an initial condition which is evolved to time \( T \) under the equations of 2D turbulence with \( u_z = \bar{u}(x_\parallel) \) evolved as a passive scalar, the result approximately equals the result of evolving the original field \( u(x) \) to time \( T \) under the equations of strongly rotating turbulence and vertically averaging the result.

This result is an elegant restatement of the approximate decoupling of Fourier with \( k_\parallel = 0 \) from all other Fourier modes under resonant interactions alone although it is based upon a sophisticated mathematical analysis. In distinguishing vertical and horizontal ‘vertically averaged’ motion, it is also consistent with the oversimplified 2D model by Cambon and Godeferd [27] in terms of \( e - Z \) and \( e + Z \), reviewed above.

The physical relevance of \( Z \)—first visualized from EDQNM2 results by Cambon and Jacquin [28]—also appears clearly from the definition of integral length scales, especially when multiplied by the related Reynolds stress components, and emphasized first in the experiment by Jacquin et al [29]. Since \( Z = \zeta \) is real at \( k_\parallel = 0 \), with \( e - Z \) and \( e + Z \) related to purely horizontal and vertical motions respectively, one has

\[
\mathcal{E}_v = \langle u_3^2 \rangle L_{33}^{(3)} = \pi \int (e + Z)_{k_\parallel=0} \, d^2 k
\]
and

\[
\mathcal{E}_h = 2 \langle u_1^2 \rangle L_{11}^{(3)} = 22 \langle u_2^2 \rangle L_{22}^{(3)} \pi \int (e - Z)_{k_\parallel=0} \, d^2 k.
\]

Rotation was shown to ‘decouple’ the two quantities above, consistent with results from experimental and numerical experiments (see [1, 2] and references in [10]), so that \( \mathcal{E}_h \) was shown not to decay (as without rotation), with only the vertical integral length scale \( L_{11}^{(3)} = L_{22}^{(3)} \) affected by a strong increase in the presence of rotation. This also suggests to reintroduce \( Z \) or \( \zeta = Z(k, x)e^{-2i\omega t} \), with \( x = \cos \theta \), in all definitions of 1D spectra in section 3.2, once a plausible \( Z(k, x) \) distribution is available.
5. General discussion

5.1. Review of main results

- AQNM provides the full angular distribution of spectral energy density \( e(k, x) = \tilde{e}(k_{\parallel}, k_{\perp}) \). At large elapsed time, the spectrum becomes self-similar, with a well defined \( k^{-3} \) scaling of the spherically averaged spectrum \( E(k) \) defined in equation (50). The numerical results, valid except when \( x = \cos \theta_k = 0 \) exactly, appear consistent with the nonlocal analytical law \( e \sim k_0^{-1/2} k_{\parallel}^{-1/2} k_{\perp}^{-3} \), at least for \( x \) not too large. The AQNM slope obtained from \( 4\pi k^2 e(k, x) \) at the smallest fixed \( x \) is \( k^{-2} \) (as would be predicted by Galtier [5] without cut-off) rather than \( k^{-3/2} \) derived from the (R) analytical law. 1D spectra from AQNM \( e(k, x) \) will be presented in a forthcoming paper.

- The \( k^{-2} \) power law for the spherically averaged energy spectrum is marginally relevant in our opinion, as more and more results confirm the validity of a \( k^{-3} \) energy spectrum. However, this spectrum is not the result of 2D dynamics, but reflects instead the typical anisotropy of the angular distribution of \( e(k, x) \).

- We believe that we have a solid basis for radically questioning the ‘decoupling’ of the slow mode. The volume integrals of equations (62) and (63) can couple the ‘slow modes’ \( e(k, x = 0) \) and \( Z(k, 0) \) to the entire wavevector domain. Simpler arguments as in [8, 25] ignore not only these volume integrals, but also the break down of the resonance conditions in the inner zone of the ‘slow’ manifold. The classical 2D dynamics, where such arguments imply, would be recovered in our EDQNM-type equations by assuming distribution \( e = e \delta(kx), Z = Z \delta(kx) \); however, we have never obtained such distributions even at high rotation rate and long elapsed times.

- The slow polarization term \( Z \), which Cambon and Jacquin [28] first called attention, is essential; it cannot be ignored even in extended wave-turbulence theory (as in our equations (62)–(65), not to mention full statistical theories like EDQNM3 and LRA). Its physical relevance is ascertained from physical and numerical experiments, especially for explaining the ‘decoupling’ (caveat, not the same as before) of the vertically averaged horizontal motion, with spectral energy \( e(k, 0) - Z(k, 0) \), from its vertical counterpart \( e(k, 0) + Z(k, 0) \). Looking at quantities available from numerical and even from physical experiments, typical integral length scales \( L_{h,v} \) (vertical separation, horizontal or vertical fluctuating motion) reflect the strong difference between \( e - Z \) (horizontal velocity) and \( e + Z \) (vertical velocity) dynamics of the slow mode. However \( Z \) is only significant in the domain of small \( x \).

Concerning modelling, the related \( e, \zeta, h \) set in general (\( Z \) being the slow contribution to \( \zeta \) in the particular case of pure rotation) is relevant for any anisotropic homogeneous turbulence. For instance, the advanced structure-based modelling by Kassinos et al [30] could be underlied by the \( e, Z \), angle-dependent, spectral description, for both definition and dynamics of additional new tensors, as the ‘Dimensionality’ tensor (involving \( e(., x) \)), the ‘Circulicity’ (involving \( e \) and \(-\Re Z\), in the same way \( e \) and \( \Im Z \) are involved in the Reynolds stress tensor itself) and the ‘Stropholysis’ one (involving \( \Im Z \)).

- Wave-turbulence theory cannot be disconnected from statistical modelling/theory using closure assumptions, in the presence of a slow manifold, as \( k_{\parallel}/k = \mathcal{O}(Ro) \) in our case. The EDQNM3 eddy damping is eventually removed in AQNM and even in our ‘extended’ wave-turbulence approach of section 4, but it would regain importance for correctly solving the inner zone of
the slow manifold, using full EDQNM$_3$ equations (see appendix) with nonzero $\mu_{kpq}$, not to mention more sophisticated closure theories (as LRA) to give an optimal evaluation of $\mu$ in the slow manifold.

5.2. Remaining open problem in rotating turbulence

Since resonance conditions and the related approximation which eventually discards $\mu$, do not apply within the slow manifold, our equations (64) and (65) are just a cartoon model for displaying $e/Z$ coupling. Therefore, it is necessary to solve the full EDQNM$_3$ equations (A.5) and (A.6) for future ‘inner’ calculation of the slow manifold. The ‘outer’ calculation, however, is well understood now. A crucial problem will be to evaluate the ‘spectral thickness’ $\Delta k_\parallel$, or alternatively $\Delta x$, of the slow manifold. The spectral thickness of the ‘resonant’ manifold, normal to the direction of the resonance surface, is for instance given by the coefficient $\alpha_{s,s'}$ in (20), when resonance condition apply. This information is lacking for the slow manifold.

Recent experimental approaches to rotating turbulence [21] and to turbulence near the core of a strong organized vortex [31] bring new results and raise new questions, in terms of the physics of intermittency and the departure from Kolmogorov phenomenology and scaling. A significant departure of the exponents $\zeta_n$ of the $n$-order structure function from $\zeta_n = n/3$ (K-41) to $\zeta_n = n/2$ was observed [21, 31], reflecting a significant change in the physics. This has nothing to do with intermittency, as classically claimed to justify anomalous exponents. At least for the second and third orders, our $e$ and $T(e)$ (not to mention $Z$) spectral terms could be used to evaluate the structure functions, probably for different orientations of increments and two-point separation, in our anisotropic (axisymmetry without mirror symmetry) case with rotation. This was never seriously undertaken, although specialists of (formal) intermittency are becoming aware of anisotropy. In addition, there may be no connection between the new $\zeta_2 = 1$, $\zeta_3 = 3/2$ [21] with the claim to observe a $k^{-2}$ power law for the energy spectrum: the scaling of energy spectra can depend strongly on how an anisotropic energy distribution like $e(k, x)$ is integrated. One can expect different slopes for the spherically averaged spectrum and for a 1D spectrum (section 3.2). A question more relevant than ‘what slope is shown?’ is ‘what spectrum is really approached by the measurements?’.

5.3. Towards a general strategy combining WT, EDQNM$_3$ and LRA

One can conclude that it is worthwhile to extend wave turbulence theory by means of statistical theories like DIA and LRA to those conditions under which wave turbulence theory no longer applies. EDQNM$_3$-type models provide a useful formalism which overcomes almost all the limitations of wave turbulence theory, but they require an heuristic specification of the so-called ‘eddy damping’. Keeping this parameter adjusted according to classical procedures in isotropic EDQNM (not recalled here for the sake of brevity) yields excellent agreement with DNS/LES [10, 32], but such approximations are probably too coarse in the domain outside DNS tractability and/or LES credibility: quasi-infinite Reynolds number, high anisotropy with angular dependence. The additional information about the nonlinear part of the ‘response tensor’ can only be provided by more sophisticated, completely self-contained or ‘endogenous’ theories, like LRA. This recourse to renormalized (nonlinearly altered response tensor) statistical theory can be illustrated in two cases.
On the one hand, when the dispersion law displays a slow mode, classical or ‘extended’ wave turbulence needs a full statistical theory to treat the ‘inner’ slow mode, especially if resonances concentrate energy towards this slow mode. Stably stratified turbulence is another example, provided that the mode which is non-oscillating and slow for almost any wavevector (rediscussed below) is a priori removed, as in [17].

On the other hand, WT is only marginally relevant in cases in which a slow non-oscillating mode (which has to be considered as ‘slow’ for any wavevector) coexists with wavy modes, the latter being affected by rapid oscillations in general, but for particular orientations. In the examples of stably stratified turbulence, the decomposition in terms of linear eigenmodes is very close to the poloidal/toroidal/VSHF one for the fluctuating velocity field, in physical space, or Craya–Herring in 3D Fourier space. The first Craya–Herring mode $e^1$ corresponds to the toroidal mode, but for $k_h = 0$ where it matches the vertically sheared horizontal flow (VSHF); similarly the $e^2$ mode corresponds to the poloidal mode, but for $k_h = 0$ where it matches also the VSHF. Accordingly, the true mode of ‘rapid’ gravity waves is the $e^2$ (poloidal-type mode) combined with a buoyancy mode, whereas the VSHF or $k_h = 0$ is also the slow mode of gravity waves. However, the essential difference with rotating turbulence is the presence of the $e^1$ (toroidal + VSHF) mode (except if it is a priori neglected, but this is highly unphysical). Starting from e.g. isotropic initial data, the kinetic energy is equally distributed among the ‘poloidal’ and ‘toroidal’ modes, and the dynamics which consists of concentrating energy towards the VSHF ($k_h = 0$) is completely dominated by pure ‘toroidal + VSHF’ nonlinear interactions (as shown by our anisotropic EDQNM, in way to be confirmed by DNS [32]). Such a concentration towards the VSHF is qualitatively consistent with horizontal layering and can be observed even at moderate times, in terms of Brünt–Väisälä period; this renders a marginal possible concentration towards the slow manifold of gravity waves, through resonant triads of pure (poloidal-type motion) gravity waves, which could establish only at very long times.

A final example is weakly compressible isentropic turbulence, in which acoustic waves are not dispersive but contain a very small amount of spectral energy for wavenumbers in which the solenoidal energy spectrum is relevant. The solenoidal motion corresponds to the non-oscillating (everywhere) slow mode in this situation. Interesting predictions for spectra of dilatational energy and pressure variance were proposed by Fauchet et al [33] in the presence of a given, established, solenoidal energy spectrum with $k^{-5/3}$ inertial range. They should be confirmed and refined according to the general strategy outlined here.

This strategy consists of following a strict hierarchy of models/theories which are embedded in each other, as in the introduction section,

Rapid distortion $\subset$ Wave turbulence (AQNM) $\subset$ EDQNM$_3$ $\subset$ LRA.

6. Conclusions

AQNM provides tentative support for the existence of bihomogeneous weak turbulence spectra in rotating turbulence. The apparent nonlocality of these spectra requires further careful investigation of their validity.

An important open question in the dynamics of rotating turbulence is the relation between 2D modes and inertial waves. Arguments based on resonant interactions alone suggest that they decouple entirely; the present work suggests instead that volume integral terms related to ‘polarization anisotropy’ can couple them.

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A complete theory of rotating turbulence cannot be based on weak turbulence arguments alone, because the wave frequency vanishes on the 2D modes. Although complete closure theories such as LRA offer a solution, it is difficult to extract information from such theories, even by numerical integration. A description modelled on matched asymptotic expansions is proposed, in which weak turbulence theory has the role of an outer solution, and an inner solution is constructed using an approximate strongly nonlinear model such as EDQNM₃.

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Appendix. EDQNM₃ equations with zero helicity

In EDQNM₃ equations (13)–(15), $T^{(e,z,h)}$ are given by volume integrals close to the ones in the CMG appendix. Helicity is ignored here as in CMG, for the sake of brevity. The integrands are completely expressed in terms of $(e, Z)$ through quadratic terms involving triads. The most laborious calculation is for deriving five geometric factors, denoted $A_1(k,p,q), \ldots, A_5(k,p,q)$. Fortunately, these factors were calculated once for all, and play the same role in EDQNM₂ and EDQNM₃.

The way to simply move from EDQNM₂ to EDQNM₃, with no helicity, is found as follows.

The only explicit (in addition to the time dependence of the $e - Z$ variables themselves) time-dependent term in the EDQNM₂ integrand of $T^{(e,z)}$ is

$$\exp[-z_{kpq}(t - t')] = \exp[(-\mu_{kpq} - i\Omega_{kpq})(t - t')], \quad \Omega_{kpq} = s\sigma_k + s'\sigma_p + s''\sigma_q, \quad (A.1)$$

and its integral gives

$$\int_{-\infty}^t e^{-z_{kpq}(t-t')} \, dt' = \frac{1}{z_{kpq}}. \quad (A.2)$$

The polarization anisotropy is now denoted by $\zeta$ to avoid confusion with its slow counterpart, $Z$, which is only relevant here, with the relationship

$$\zeta(sk, t') = Z(sk, t') e^{-2i\sigma_k t'}. \quad (A.3)$$

Only $Z$ has to be considered as ‘slow’, so that it has to be frozen to $t' = t$ in the temporal integral over $t'$ resulting from EDQN. Accordingly, the related phase term in $\zeta$ will give an additional (versus EDQNM₂) contribution to the temporal integrand, with the following modifications:

(i) There is no modification (from EDQNM₂ to EDQNM₃) for the terms which do not include $\zeta$ in $T^{(e)}$.

(ii) Terms containing $\zeta$ in $T^e$ are altered in replacing $1/z_{kpq}$ by

$$\int_{-\infty}^t e^{-z_{kpq}(t-t')-i\Omega_c t'} \, dt' = \frac{e^{-i\Omega_c t}}{z_{kpq} - i\Omega_c} \quad (A.4)$$

with $\Omega_c = 2s''\sigma_q$ for $Z''$-type term, $\Omega_c = 2s\sigma_k$ for $Z$-type term, $\Omega_c = 2s''\sigma_q + 2s\sigma_k$ for $ZZ''$-type terms and $\Omega_c = 2s''\sigma_q + 2s'\sigma_p$ for $Z'Z''$-type terms.
Consequently, the EDQNM$_3$ version without helicity of $T^{(e)}$ becomes

\[
T^{(e)} = \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 \left[ \frac{A_1(sk, s'p, s'q)}{\mu + i(\sigma_k + s'\sigma_p + s''\sigma_q)} e''(e - e') \right] d^3p \\
+ \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 \left[ \frac{A_2(sk, s'p, s'q)}{\mu + i(\sigma_k + s'\sigma_p + s''\sigma_q)} e^{2i\lambda'(-\sigma_q t)} eZ(s''q) \right] d^3p \\
+ \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 \left[ \frac{A_3(sk, s'p, s'q)}{\mu + i(\sigma_k + s'\sigma_p + s''\sigma_q)} e^{2i\lambda(\sigma_q t)} e'Z(sk) \right] d^3p \\
- \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 \left[ \frac{A_4(sk, s'p, s'q)}{\mu + i(\sigma_k - s'\sigma_p - s''\sigma_q)} e^{2i\lambda(-\sigma_q t)} e'Z(s''q) Z(sk) \right] d^3p \\
+ \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 \left[ \frac{A_5(sk, s'p, s'q)}{\mu + i(\sigma_k + s'\sigma_p + s''\sigma_q)} e^{2i\lambda(-\sigma_q t)} eZ(s''q) Z(s'p) \right] d^3p
\]

with the geometric factors $A_1$ to $A_5$ given in the CMG appendix, and recalled below. Equations are very symmetric. With respect to EDQNM$_2$, the presence of a $Z$, or $Z'$, $Z''$ factor, results in changing the corresponding sign in the term $\pm\sigma_k \pm \sigma_p \pm \sigma_q$, and to add a specific phase factor, oscillating in time, as $e^{-2i\omega t}$. $T^{(e)}$ being real, it is possible to only retain $s = 1$ and to replace complex contributions by twice their real part. Of course, $e = e(k, t)$, $e' = e(p, t)$, $e'' = e(q, t)$.

The EDQNM$_3$ version of $T^{(z)}$, given just below, is derived from its EDQNM$_2$ counterpart in a similar way, except that the whole term is multiplied, in addition, by the oscillating term $e^{2i\lambda_1t}$,

\[
T^{(z)} = \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 e^{2i(\sigma_1t - \lambda)} \left[ \frac{A_3(k, -s'p, -s''q)}{\mu + i(\sigma_k + s'\sigma_p + s''\sigma_q)} e''(e' - e) \right] d^3p \\
+ \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 e^{2i(\sigma_1t - \lambda)} \left[ \frac{A_4(k, -s'p, -s''q)}{\mu + i(\sigma_k + s'\sigma_p + s''\sigma_q)} e^{2i\lambda(\sigma_q t)} eZ(s''q) \right] d^3p \\
+ \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 e^{2i(\sigma_1t - \lambda)} \left[ \frac{A_1(k, -s'p, -s''q)}{\mu + i(-\sigma_k + s'\sigma_p + s''\sigma_q)} e^{2i\lambda(-\sigma_q t)} e''Z(k) \right] d^3p \\
- \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 e^{2i(\sigma_1t - \lambda)} \left[ \frac{A_2(k, -s'p, -s''q)}{\mu + i(-\sigma_k + s'\sigma_p - s''\sigma_q)} e^{2i\lambda(\sigma_q t)} eZ(s''q) Z(k) \right] d^3p \\
+ \frac{1}{2^3} \sum_{s's''s'} \int C_{kpq}^2 e^{2i(\sigma_1t - \lambda)} \left[ \frac{A_5(k, -s'p, -s''q)}{\mu + i(-\sigma_k - s'\sigma_p + s''\sigma_q)} e^{2i\lambda(-\sigma_q t)} e''Z(s''q) Z(s'p) \right] d^3p
\]

Accordingly all explicit time-dependent oscillating terms cancel for the $T^{(z)}$ term which depends on $Z(k)$ (the third one).
Let us recall the definition of geometric coefficients:

\[ C_{kpq} = \frac{\sin(p, q)}{k} = \frac{\sin(k, q)}{p} = \frac{\sin(k, p)}{q} \]  

(A.7)

and

\[ A_1(k, p, q) = -(p - q)(k - q)(k + p + q)^2, \]  

(A.8)

\[ A_2(k, p, q) = -(p - q)(k + q)(k + p + q)(k - p - q), \]  

(A.9)

\[ A_3(k, p, q) = (p - q)(k + q)(k + p + q)(-k + p + q), \]  

(A.10)

\[ A_4(k, p, q) = (p - q)(k - q)(k + p + q)(k - p + q), \]  

(A.11)

\[ A_5(k, p, q) = -(p - q)(p + q)(k + p + q)(k - p - q) \]  

(A.12)

(note that the additional factor \(2p/k\) was a mistake in the CMG appendix).

The other geometric coefficients which depend not only on the triad geometry (via moduli \(k, p, q\)), but also on the orientation of its plane are only \(\lambda, \lambda', \lambda''\) terms. Following previous authors \([8, 28, 34]\) they are displayed by substituting to the local frames related to the helical (or complex Craya–Herring) decomposition \((N(sk), N(s'p), N(s''q))\), alternative ones having their polar axis normal to the plane of the triad rather than to the plane of rotation, so that

\[ N(sk) = e^{s\lambda}(\beta + is\gamma), \quad N(s'p) = e^{s'\lambda'}(\beta' + is'\gamma), \quad N(s''q) = e^{s''\lambda''}(\beta'' + is''\gamma) \]  

(A.13)

in which \(\gamma\) is the unit vector normal to the plane of the triad, whereas \(\beta, \beta'\) and \(\beta''\) are unit vectors all located in the plane of the triad, and normal to \(k, p, q\), respectively. Accordingly, the scalar products in terms of \(k, p, q, W, W'\) and \(W''\) only depend on \(k, p, q\) (moduli). These scalar products generate all the \(A_1\)–\(A_5\) terms.

The last equations, derived from the previous one, which are used in section 4, are

\[ \cos \theta_p = \frac{p}{\parallel p} = -z \cos \theta_k + \sqrt{1 - z^2} \sin \lambda, \]  

(A.14)

\[ \cos \theta_q = \frac{q}{\parallel q} = -y \cos \theta_h - \sqrt{1 - y^2} \sin \lambda \]  

(A.15)

with \(y = \cos(k, q), z = \cos(k, p), \sin(k, q) = C_{kpq}p, \sin(k, p) = C_{kpq}q\). Accordingly,

\[ p \cos \theta_p = -q \cos \theta_q = pqC_{kpq} \sin \lambda \]  

(A.16)

at \(k_{\parallel} = 0\).

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