A NOTE ON THE PAIR CORRELATION OF FAREY FRACTIONS

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Abstract. The pair correlations of Farey fractions with denominators $q$ satisfying $(q, m) = 1$, respectively $q \equiv b \pmod{m}$ with $(b, m) = 1$, are shown to exist and are explicitly computed.

1. Introduction

The Farey fractions sequence $\mathfrak{F}_Q := \{ \frac{a}{q} : 0 < a \leq q \leq Q, (a, q) = 1 \}$ arises in several problems in mathematics. The elements of $\mathfrak{F}_Q$ are well known to be uniformly distributed in $[0, 1]$ as $Q \to \infty$ [19], with discrepancy exactly $\frac{1}{Q}$ [11]. The distribution of Farey fractions is of major interest, due in part to the connection with the distribution of zeros of the Riemann zeta function [13, 17] or of Dirichlet $L$-functions [16].

Although the major problems in the area remain widely open, the spacing statistics of Farey fractions are more accessible. The gap distribution of $h$-tuples of consecutive gaps between elements of $\mathfrak{F}_Q$ was computed in [14] for $h = 1$ and in [3] for $h \geq 2$. More recently, the correlations of $\mathfrak{F}_Q$, shown to exist and explicitly computed by Zaharescu and the first author [9], turned out to play a key role in the study of the moments of eigenvalues of large sieve matrices [8].

Motivated by Huxley’s work [16], a number of papers investigated various features (such as discrepancy or gap distribution) of the distribution of Farey fractions with denominators subjected to various constraints [1, 2, 4, 7, 15, 18].

For every finite set $F \subseteq \mathbb{R}$ of cardinality $N(F)$ and every interval $I$, define

$$G_F(I) := \frac{1}{N(F)} \# \left\{ (x, y) \in F^2 : y \neq x, y - x \in \frac{1}{N(F)} I + \mathbb{Z} \right\}.$$ (1.1)

The pair correlation measure of an increasing sequence $(F_n)$ of finite subsets of $\mathbb{R}$ is defined (when it exists) by

$$G(I) := \lim_{n} G_{F_n}(I) \quad (I \text{ interval}).$$

If, in addition,

$$G(\lambda) := \mathcal{G}([0, \lambda]) = \int_0^\lambda g(x) \, dx,$$

then $g$ is called the pair correlation function of $(F_n)$.

The pair correlation function of $\mathfrak{F}_Q$ was shown in [9] to be given by

$$g(\lambda) = \frac{1}{\zeta(2) \lambda^2} \sum_{1 \leq \Delta \leq 2\zeta(2) \lambda} \varphi(\Delta) \log \frac{2\zeta(2) \lambda}{\Delta}, \quad \forall \lambda > 0. \hspace{1cm} (1.2)$$

This formula was useful in [8] to recognize the connection between the pair correlation of $\mathfrak{F}_Q$ and the expression of the main term of the second moment of the large sieve matrix, provided in [20].

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The proof of formula \((1.2)\) given in \([9]\) relies essentially on the Poisson summation formula. The original motivation of this note was to re-prove \((1.2)\) using some different counting arguments that also provide effective estimates. Our direct approach turns out to also work well in the case of two important subsets of \(\mathcal{F}_Q\), obtained by imposing congruence conditions on the denominators:

\[
\mathcal{F}_Q^{(m)} := \left\{ \gamma = \frac{a}{q} \in \mathcal{F}_Q : (q,m) = 1 \right\},
\]

\[
\mathcal{F}_Q^{(m,b)} := \left\{ \frac{a}{q} \in \mathcal{F}_Q : q \equiv b \pmod{m} \right\},
\]

where \(m \in \mathbb{N}\), \(b \in \mathbb{Z}\) and \((b,m) = 1\).

Set

\[
N_{Q,m;\alpha,\beta} := \#(\mathcal{F}_Q^{(m)} \cap (\alpha,\beta)), \quad \text{and} \quad N_{Q,m} := N_{Q,m;0,1}.
\]

The following constant will appear several times in this paper:

\[
C_m := \frac{\varphi(m)}{\zeta(2)m} \prod_{\substack{p|m \text{ prime} \quad \text{prime}}} \left( 1 - \frac{1}{p^2} \right)^{-1} = \frac{\varphi(m)}{m} \prod_{\substack{p|m \text{ prime}}} \left( 1 - \frac{1}{p^2} \right).
\]

As noticed at the beginning of Section 2, for every \(0 \leq \alpha < \beta \leq 1\) we have

\[
N_{Q,m;\alpha,\beta} = (\beta - \alpha)N_{Q,m} + O_\delta(Q^{1+\delta})
\]

\[
= \frac{(\beta - \alpha)C_m}{2} Q^2 + O_\delta(Q^{1+\delta}), \quad \forall \delta > 0,
\]

which gives an effective estimate for the uniform distribution of \(\mathcal{F}_Q^{(m)}\).

In the first part of Section 2 we prove

**Theorem 1.** The pair correlation function \(g_{(m)}(\lambda)\) of \(\mathcal{F}_Q^{(m)}\) exists and

\[
g_{(m)}(\lambda) = \frac{\varphi(m)}{m} \frac{C_m}{\lambda^2} \sum_{1 \leq \Delta \leq \frac{2\lambda}{C_m}} \varphi(\Delta) \frac{(\Delta,m)}{\varphi((\Delta,m))} \log \frac{2\lambda}{C_m\Delta}.
\]

In particular, the support of the function \(g_{(m)}(\lambda)\) is the interval \([\frac{1}{2}C_m, \infty)\).

Next, we extend the equality \(\lim_{\lambda \to \infty} g_{(1)}(\lambda) = 1\), due to R. R. Hall and presented in \([9]\), by proving that

\[
\lim_{\lambda \to \infty} g_{(m)}(\lambda) = 1, \quad \forall m \in \mathbb{N}.
\]

In Section 3 we investigate the pair correlation of \(\mathcal{F}_Q^{(m,b)}\) under the assumption \((b,m) = 1\).

The cardinality of \(\mathcal{F}_Q^{(m,b)}\) is given by

\[
N_{Q,(m,b)} = \frac{C_m}{2\varphi(m)} Q^2 + O_m(Q \log Q).
\]

Furthermore, we have

\[
\#(\mathcal{F}_Q^{(m,b)} \cap (\alpha,\beta)) = (\beta - \alpha)N_{Q,(m,b)} + O_\delta(Q^{1+\delta}), \quad \forall \delta > 0,
\]

showing effectively that the elements of \(\mathcal{F}_Q^{(m,b)}\) are uniformly distributed.

We prove

**Theorem 2.** The pair correlation function of \(\mathcal{F}_Q^{(m,b)}\) exists, is independent of \(b\), and is given by

\[
\tilde{g}_{(m)}(\lambda) = g_{(m)}(\varphi(m)\lambda).
\]
Our approach also allows us to prove that the pair correlation function of \( F^{(m)}_Q \cap I \), respectively \( F^{(m,b)}_Q \cap I \), coincides with \( g(m) \), respectively \( \tilde{g}(m) \), for every interval \( I \subseteq [0, 1] \). It also gives effective asymptotic formulas in \( Q \) for the quantities \( G_{F^{(m)}_Q}([0, \lambda]) \), \( G_{F^{(m,b)}_Q}([0, \lambda]) \) and \( G_{\delta^{(m,b)}_Q}([0, \lambda]) \).

A related result is contained in [21, case \( n = 1 \) of Theorem 3.2]. However, our result involves the extra coprimality condition \((a,q) = 1\) in the definition of \( F^{(m,b)}_Q \), which is not included in formulas (3.3.1) and (3.3.2) of [21].

\[
N_{Q,m} = 1 + \sum_{(k,m) = 1}^{Q} \varphi(k) = C_m \frac{Q^2}{2} + O(Q \log Q).
\]

When restricting to \([0, \beta]\), the number of new fractions in the \( k \)th step is not \( \varphi(k) \) anymore, but rather

\[
\sum_{n=1}^{[k\beta]} 1 = \frac{\varphi(k)}{k} [k\beta] + O_{\delta}(k^\delta),
\]

where one can use, for example, [10, Lemma A.1]. Thus

\[
N_{Q,m,0,\beta} = \beta N_{Q,m} + O_{\delta}(Q^{1+\delta}),
\]

\[
N_{Q,m,\alpha,\beta} = N_{Q,m,0,\beta} - N_{Q,m,0,\alpha} = (\beta - \alpha) N_{Q,m} + O_{\delta}(Q^{1+\delta}).
\]
Set
\[ H_{Q,m;\beta}(\lambda) := \# \left\{ (\gamma, \gamma') : \gamma, \gamma' \in \mathfrak{S}^{(m)}_Q, 0 < \gamma' - \gamma \leq \frac{\lambda}{Q^2}, \gamma' \leq \beta \right\}, \]
\[ \overline{H}_{Q,m;\alpha}(\lambda) := \# \left\{ (\gamma, \gamma') : \gamma, \gamma' \in \mathfrak{S}^{(m)}_Q, 0 < \gamma' - \gamma \leq \frac{\lambda}{Q^2}, \gamma \leq \alpha \right\}. \]

If the limit exists, set
\[ G_{(m;\alpha,\beta)}(\lambda) := \lim_{Q \to \infty} \frac{1}{N_{Q,m;\alpha,\beta}(Q)} \# \left\{ (\gamma, \gamma') : 0 < \gamma' - \gamma \leq \frac{\lambda}{N_{Q,m;\alpha,\beta}(Q)} \right\}. \]

Then
\[ H_{Q,m;\beta}(\lambda) = \sum_{1 \leq \Delta \leq \lambda} \# S_{Q,m;\beta}(\Delta, \lambda) = \sum_{1 \leq \Delta \leq \lambda} \# \overline{S}_{Q,m;\beta}(\Delta, \lambda), \]
\[ \overline{H}_{Q,m;\alpha}(\lambda) = \sum_{1 \leq \Delta \leq \lambda} \# \overline{S}_{Q,m;\alpha}(\Delta, \lambda), \]

where we used the variables \( x = a', v = q', u = q, y = a \) to get
\[ S_{Q,m;\beta}(\Delta, \lambda) = \left\{ (u, v, x, y) \in \mathbb{N}^4 : \frac{x_u - y_v}{u} \leq \frac{\lambda}{Q^2}, (u,m) = 1 \right\}, \]
\[ \overline{S}_{Q,m;\beta}(\Delta, \lambda) = \left\{ (u, v, x, y) \in \mathbb{N}^4 : \frac{x_u - y_v}{u} \leq \frac{\lambda}{Q^2}, (u,m) = 1 \right\}, \]
\[ \overline{S}_{Q,m;\alpha}(\Delta, \lambda) = \left\{ (u, v, x, y) \in \mathbb{N}^4 : \frac{x_u - y_v}{u} \leq \frac{\lambda}{Q^2}, (u,m) = 1 \right\}. \]

Observe that \( xu - yv = \Delta \iff \frac{u}{v} = \frac{y}{x} + \frac{\Delta}{uv} \) implies
\[ \# S_{Q,m;\beta}(\Delta, \lambda) \leq \# \overline{S}_{Q,m;\beta}(\Delta, \lambda) \leq \# S_{Q,m;\beta + \frac{\Delta}{Q^2}}(\Delta, \lambda), \]
so that \( \overline{H}_{Q,m;\beta}(\lambda) \) is asymptotically the same as \( H_{Q,m;\beta}(\lambda) \) as \( Q \to \infty \). Thus it suffices to estimate \( \# \overline{S}_{Q,m;\beta}(\Delta, \lambda) \) as follows:
\[ \# \overline{S}_{Q,m;\beta}(\Delta, \lambda) = \sum_{\Delta \leq v \leq Q} \sum_{\Delta \leq u \leq Q} \mu(d) \left[ \sum_{x \leq \beta v, (x,v) = 1} \sum_{y \leq \alpha u, \Delta \mod v, (y,u) = 1} 1 \right], \]
\[ = \sum_{\Delta \leq v \leq Q} \sum_{\Delta \leq u \leq Q} \mu(d) \left[ \sum_{x \leq \beta v, (x,v) = 1} \sum_{y \leq \alpha u, \Delta \mod v, (y,u) = 1} 1 \right]. \]

To estimate the innermost sum on the right hand side in (2.1), we need to check that [10] Proposition A.3] carries over with the additional condition \( (q, m) = 1 \). Set
\[ N_{q,h,m}(I_1, I_2) = \{ (x, y) \in I_1 \times I_2 : (x, q) = 1, (y, m) = 1, xy \equiv h \mod q \}. \]
Proposition 3. Assuming \((q, m) = 1\), for any intervals \(I_1, I_2\) and any integer \(h\) we have
\[
\#{\mathcal N}_{q, h, m}(I_1, I_2) = \frac{\varphi(q)}{q^2} \cdot \frac{\varphi(m)}{m} |I_1||I_2| + O_{\delta, m}\left(q^{1/2+\delta}(h, q)^{1/2}\left(1 + \frac{|I_1|}{q}\right)(1 + \frac{|I_2|}{q})\right).
\]
Proof. For \(x\) such that \((x, q) = 1\), let \(\overline{x}\) denote the unique inverse of \(x\) mod \(q\). We have that
\[
\#{\mathcal N}_{q, h, m}(I_1, I_2) = \sum_{(x,y)\in I_1 \times I_2} 1 = \sum_{(x,y)\in I_1 \times I_2} \sum_{y \equiv h \mod q} e\left(\frac{k(y - \overline{x}h)}{q}\right).
\]
We distinguish the cases \(k = 0\) and \(k > 0\):
\[
M := \frac{1}{q} \sum_{x \in I_1} \sum_{y \in I_2 (y, m) = 1} 1,
E := \frac{1}{q} \sum_{y \in I_2 (y, m) = 1} \sum_{k = 1}^{q-1} e\left(\frac{ky}{q}\right) \sum_{x \in I_1 (x, q) = 1} e\left(-\frac{\overline{x}hk}{q}\right).
\]
For the term \(M\), two successive applications of \([10]\) Lemma A1 give
\[
M = \frac{1}{q} \sum_{x \in I_1 (x, q) = 1} \left(\frac{\varphi(m)}{m} |I_2| + O_{\delta}(m^\delta)\right)
= \frac{1}{q} \cdot \frac{\varphi(m)}{m} |I_2| \sum_{x \in I_1 (x, q) = 1} 1 + O_{\delta}\left(m^\delta |I_1| + 1\right)
= \frac{1}{q} \cdot \frac{\varphi(m)}{m} |I_2| \left(\frac{\varphi(q)}{q} |I_1| + O_{\delta}(q^\delta)\right) + O_{\delta}\left(m^\delta |I_1| + 1\right)
= \frac{\varphi(q)}{q^2} \cdot \frac{\varphi(m)}{m} |I_1||I_2| + O_{\delta, m}\left(\frac{|I_1| + |I_2| + 1}{q^{1-\delta}}\right).
\]
Now, following the notation and approach from \([10]\), which makes essential use of the Weil-Salié type estimates derived in \([12]\ (5)\), we have
\[
E = \frac{1}{q} \sum_{y \in I_2 (y, m) = 1} \sum_{k = 1}^{q-1} e\left(\frac{ky}{q}\right) S_{I_1}(0, -hk; q)
= \frac{1}{q} \sum_{d | m} \mu(d) \sum_{k = 1}^{q-1} S_{I_1}(0, -hk; q) \sum_{y \in I_2 \frac{d | y}{d}} e\left(\frac{ky}{q}\right)
= \frac{y = dt}{q} \sum_{d | m} \mu(d) \sum_{k = 1}^{q-1} S_{I_1}(0, -hk; q) \sum_{t \leq \frac{1}{2} I_2} e\left(\frac{kdt}{q}\right).
\]
We distinguish the cases \(q | kd\) and \(q \nmid kd\). The former cannot occur because \(d | m, (q, m) = 1\) and \(q > k\). For the latter, we use \([10]\) Lemma A2 to estimate \(S_{I_1}(0, -hk; q)\). Here, we do not
necessarily have $I_1 \subseteq [0, q)$, so we get the extra factor in the final formula:

$$S_{I_1}(0, -hk; q) \ll (hk, q)^{1/2}q^{1/2+\delta} \left(1 + \frac{|I_1|}{q}\right)$$

$$\leq (h, q)^{1/2}(k, q)^{1/2}q^{1/2+\delta} \left(1 + \frac{|I_1|}{q}\right).$$

(2.2)

Since $S_{I_1}(0, -hk; q) \ll (hk, q)^{1/2}q^{1/2+\delta}$, the first estimate in (2.2) can be improved to

$$S_{I_1}(0, -hk; q) \ll (hk, q)^{1/2}q^{1/2+\delta} + (hk, q)^{1+\delta} \frac{|I_1|}{q}.$$ 

Combine (2.2) with the geometric sum and the inequality $|\sin \pi x| \geq 2||x||$ to get

$$E = \frac{1}{q} \sum_{d|\ell} \mu(d) \sum_{k=1}^{q-1} S_{I_1}(0, -hk; q) \sum_{\ell \in \frac{1}{q}I_2} e\left(\frac{k\ell}{q}\right)$$

$$\ll \frac{(h, q)^{1/2}q^{1/2+\delta}}{q} \left(1 + \frac{|I_1|}{q}\right) \sum_{d|\ell} \sum_{k=1}^{q-1} \frac{(k, q)^{1/2}}{||kd||}$$

$$\leq \frac{(h, q)^{1/2}q^{1/2+\delta}}{q} \left(1 + \frac{|I_1|}{q}\right) \sum_{d|\ell} \sum_{k=1}^{q-1} \frac{(kd, q)^{1/2}}{||kd||}.$$ 

Since

$$\{kd : 1 \leq k < q, q \nmid kd\} \subseteq \{n = cq + r : 1 \leq r < q, 0 \leq c < d\},$$

we further get

$$E \ll \frac{(h, q)^{1/2+\delta}}{q} \left(1 + \frac{|I_1|}{q}\right) \sum_{d|\ell} \sum_{c=0}^{d-1} \sum_{r=1}^{q-1} \frac{(cq + r, q)^{1/2}}{||c + \frac{r}{q}||}$$

$$= \frac{(h, q)^{1/2}q^{1/2+\delta}}{q} \left(1 + \frac{|I_1|}{q}\right) \sum_{d|\ell} \sum_{r=1}^{q-1} \frac{(r, q)^{1/2}}{||\frac{r}{q}||}$$

$$\leq \frac{(h, q)^{1/2}q^{1/2+\delta}}{q} \left(1 + \frac{|I_1|}{q}\right) \sum_{d|\ell} \sum_{s=0}^{q-1} \sum_{\ell|s \leq \frac{q}{\ell}} \frac{2q^{1/2}}{\ell s}$$

$$\ll_m (h, q)^{1/2}q^{1/2+3\delta} \left(1 + \frac{|I_1|}{q}\right).$$
Thus, under the correspondence $v \leftrightarrow q, x \leftrightarrow x, w \leftrightarrow y, \frac{\lambda}{d} \leftrightarrow h$, relation (2.1) becomes

$$\#\tilde{S}_{Q,m;\beta}(\Delta, \lambda) = \sum_{\substack{d|\Delta \atop (d,m)=1}} \mu(d) \sum_{\substack{\Delta \leq v \leq Q \atop (v,m)=1}} \left( N_{v,\frac{\Delta}{d},m}(\{0, \beta v\}, [0, \frac{Q}{d}]) - N_{v,\frac{\Delta}{d},m}(\{0, \beta v\}, [0, \frac{Q^2}{d\lambda v}]) \right)$$

$$= \sum_{\substack{d|\Delta \atop (d,m)=1}} \mu(d) \sum_{\substack{\Delta \leq v \leq Q \atop (v,m)=1}} \left( \frac{\varphi(v)}{v} \cdot \frac{\varphi(m)}{m} \beta v \left( \frac{Q}{d} - \frac{\Delta Q^2}{\lambda v d} \right) + O_{\delta,m,\Delta} \left( v^{1/2+\delta} \left( \frac{Q}{v} + \frac{Q^2}{\lambda v^2} \right) \right) \right)$$

$$= \beta Q \frac{\varphi(m)}{m} \sum_{\substack{d|\Delta \atop (d,m)=1}} \frac{\mu(d)}{d} \sum_{\substack{\Delta \leq v \leq Q \atop (v,m)=1}} \frac{\varphi(v)}{v} \left( 1 - \frac{\Delta Q}{\lambda v} \right) + O_{\delta,m,\Delta}(Q^{3/2+\delta}).$$

Note that there is no dependence of $\lambda$ in the error term because the inequality $\gamma' - \gamma \geq \frac{1}{\sqrt{d'}}$, $\forall \gamma, \gamma' \in \mathbb{F}_Q$, $\gamma < \gamma'$ allows for $\lambda \geq 1$.

A simple calculation shows that the function $K_m(n) = \sum_{\substack{d|n \atop (d,n)=1}} \frac{\mu(d)}{d}$ is multiplicative for every $m$, and that at prime powers we have

$$K_m(p^\ell) = \sum_{\substack{d|\ell \atop (d,m)=1}} \frac{\mu(d)}{d} = \begin{cases} 1, & \text{if } p \nmid m \\ 1 - \frac{1}{p} = \frac{\varphi(p^\ell)}{p^\ell}, & \text{if } p \mid m. \end{cases}$$

Therefore

$$K_m(\Delta) = \prod_{p|\Delta \atop p \nmid m} \left( 1 - \frac{1}{p} \right) = \prod_{p|\Delta \atop p|\ell} \left( 1 - \frac{1}{p} \right) = \frac{\varphi(\Delta)}{\Delta} \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \sum_{\substack{\Delta \leq v \leq Q \atop (v,m)=1}} \frac{\varphi(v)}{v} \left( 1 - \frac{\Delta Q}{\lambda v} \right) + O_{\delta,m,\Delta}(Q^{3/2+\delta}).$$

Using [4 Lemma 2.1] twice, we get

$$\#\tilde{S}_{Q,m;\beta}(\Delta, \lambda) = \beta C_m Q^{3} \frac{\varphi(m)}{m} \cdot \frac{\varphi(\Delta)}{\Delta} \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \left( 1 - \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda} \log \frac{\lambda}{\Delta} \right) + O_{\delta,m,\Delta}(Q^{3/2+\delta}).$$

Thus, for every $K \geq 1$ we have, uniformly in $\lambda \in [1, K]$,

$$H_{Q,m;\beta}(\lambda) = \beta C_m Q^{3} \frac{\varphi(m)}{m} \sum_{1 \leq \Delta \leq \lambda} \frac{\varphi(\Delta)}{\Delta} \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \left( 1 - \frac{\Delta}{\lambda} - \frac{\Delta}{\lambda} \log \frac{\lambda}{\Delta} \right) + O_{\delta,m,K}(Q^{3/2+\delta}).$$

This leads in turn to

$$G_{(m,\alpha,\beta)}(\lambda) = 2 \frac{\varphi(m)}{m} \sum_{1 \leq \Delta \leq \frac{2\lambda}{C_m}} \frac{\varphi(\Delta)}{\Delta} \left( 1 - \frac{\Delta C_m}{2\lambda} - \frac{\Delta C_m}{2\lambda} \log \frac{2\lambda}{\Delta C_m} \right). \quad (2.3)$$
Finally, Theorem 1 follows by differentiating in (2.3). For \( m = 1, \alpha = 0, \beta = 1 \), we retrieve Theorem 1 in [9]. For \( m = 2 \) we get

\[
g_{2}(\lambda) = \frac{1}{3\zeta(2)\lambda^{2}} \sum_{1 \leq \Delta \leq 3\zeta(2)\lambda} \varphi(\Delta)(\Delta, 2) \log \frac{3\zeta(2)\lambda}{\Delta}.
\]

In the remaining part of this section we prove equality (1.3). Consider the Dirichlet series

\[
D_{m}(s) := \sum_{\Delta = 1}^{\infty} \frac{K_{m}(\Delta)}{\Delta^{s-1}} = \prod_{\ell = 0}^{\infty} \left( 1 + \frac{K_{m}(p^\ell)}{p^{\ell(s-1)}} + \frac{K_{m}(p^{2\ell})}{p^{2\ell(s-2)}} + \cdots \right) \quad \text{if } \Re s > 2.
\]

Take \( \zeta_{m}(s) := \prod_{\ell = 0}^{\infty} \left( 1 - \frac{1}{p^{\ell}} \right)^{-1} \), \( \Re s > 1 \). We have

\[
\frac{\zeta_{m}(s-1)}{\zeta_{m}(s)} = \prod_{\ell = 0}^{\infty} \frac{1 - \frac{1}{p^{\ell}}}{1 - \frac{1}{p^{s-1}}} = \prod_{\ell = 0}^{\infty} \frac{p^{s} - 1}{p^{s} - p}
\]

and

\[
\sum_{\ell = 0}^{\infty} \frac{K_{m}(p^\ell)}{p^{\ell(s-1)}} = \begin{cases} 1 + \frac{1}{p^{s-1}} \cdot \frac{1}{1 - \frac{1}{p^{s-1}}} = 1 + \frac{p-1}{p^{s-1} - p} = \frac{p^{s-1} - 1}{p^{s-1} - 1} & \text{if } p \nmid m, \\ (1 - \frac{1}{p^{s-1}})^{-1} & \text{if } p \mid m,
\end{cases}
\]

leading to

\[
D_{m}(s) = \frac{\zeta_{m}(s-1)}{\zeta_{m}(s)} \prod_{\ell = 0}^{\infty} \left( 1 - \frac{1}{p^{s-1}} \right)^{-1} = \frac{\zeta(s-1)}{\zeta(s)} \prod_{\ell = 0}^{\infty} \frac{p^{s} - p}{p^{s} - 1} \prod_{\ell = 0}^{\infty} \frac{p^{s-1} - 1}{p^{s-1} - 1}
\]

\[
= \frac{\zeta(s-1)}{\zeta(s)} c_{m}(s), \quad \text{where } c_{m}(s) := \prod_{\ell = 0}^{\infty} \frac{1}{1 - \frac{1}{p^{\ell}}}
\]

Next we follow closely the final part of [9]. By Perron’s formula [9, (4.14)] we infer

\[
\sum_{1 \leq \Delta \leq x} \Delta K_{m}(\Delta) \log \frac{x}{\Delta} = \frac{1}{2\pi i} \int_{\sigma_{0} - i\infty}^{\sigma_{0} + i\infty} D_{m}(s) \frac{x^{s}}{s^{2}} ds
\]

\[
= \frac{1}{2\pi i} \int_{\sigma_{0} - i\infty}^{\sigma_{0} + i\infty} \frac{\zeta(s-1)}{\zeta(s)} c_{m}(s) \frac{x^{s}}{s^{2}} ds \quad (\sigma_{0} > 2).
\]

Moving the contour at \( \Re s = 1 \) and employing the notation from [9] we get

\[
\frac{\zeta(it)}{\zeta(1 + it)} \prod_{\ell = 0}^{\infty} \left( 1 - \frac{1}{p^{\ell}} \right) = \chi(it) \frac{\zeta(1 - it)}{\zeta(1 + it)} \prod_{\ell = 0}^{\infty} \left( 1 - \frac{1}{p^{\ell}} \right),
\]

\[
\zeta(1 - it) = \overline{\zeta(1 + it)},
\]

\[
\text{Res}_{s=2} \frac{\zeta(s-1)}{\zeta(s)} c_{m}(s) \frac{x^{s}}{s^{2}} = \lim_{s \to 2} \frac{1}{s-2} \zeta(s-1) \zeta(s) c_{m}(s) \frac{x^{s}}{4}
\]

\[
= \frac{c_{m}(2)}{\zeta(2)} \cdot \frac{x^{2}}{4} \quad (s = 2 \text{ is a simple pole}).
\]
Estimating the error as in [9] we find
\[
\sum_{1 \leq \Delta \leq x} \Delta K_m(\Delta) \log \frac{x}{\Delta} = \text{Res}_{s=2} \frac{\zeta(s-1)}{\zeta(s)} c_m(s) \frac{x^s}{s^2} + O_m(x) \\
= \frac{c_m(2)}{\zeta(2)} \cdot \frac{x^2}{4} + O_m(x).
\]
Setting \(\mu := \frac{2\lambda}{C_m}\) we get \(\lambda = \frac{C_m \mu}{2}\) and (as \(\lambda \to \infty\))
\[
g_{(m)}(\lambda) = g_{(m)}(\frac{1}{2} C_m \mu) = \frac{\varphi(m)}{m} C_m \frac{4}{C_m \mu^2} \sum_{1 \leq \Delta \leq \mu} \Delta K_m(\Delta) \log \frac{\mu}{\Delta} \\
= \frac{4 \varphi(m)}{m C_m \mu^2} \left( \prod_{p|m} \left( 1 - \frac{1}{p^2} \right)^{-1} \frac{\mu^2}{4 \zeta(2)} + O_m(\mu) \right) \\
= \frac{1}{\zeta(2)} \prod_{p|m} \left( 1 - \frac{1}{p^2} \right)^{-1} \prod_{p|m} \left( 1 - \frac{1}{p^2} \right)^{-1} + O_m(\mu^{-1}) \\
= 1 + O_m(\lambda^{-1}).
\]

3. The pair correlation of \(\delta_{Q}^{(m,b)}\)

We will employ the following estimate ([5] Lemma 3.3):

**Lemma 4.** Assuming \((b, m) = 1\) and \(V \in C^1[0, Q]\), we have
\[
\sum_{\substack{q=1 \\ q \equiv b \ (\text{mod} \ m)}} (q) V(q) = \frac{C_m}{\varphi(m)} \int_0^Q V + O\left(||V||_\infty + T_0^N V \log Q\right).
\]

In particular this gives [1.4].

Given \(\lambda > 0\), we are interested in estimating the following three quantities as \(Q \to \infty\):

\[
S_{Q; m, b, \Delta}(\lambda) := \# \left\{ (\gamma, \gamma') : \gamma, \gamma' \in \delta_Q^{(m,b)}, \gamma' - \gamma \leq \frac{\lambda}{Q^2} \right\},
\]

\[
H_{Q; m, b}(\lambda) := \# \left\{ (\gamma, \gamma') : \gamma, \gamma' \in \delta_Q^{(m,b)}, 0 < \gamma' - \gamma \leq \frac{\lambda}{Q^2} \right\} = \sum_{1 \leq \Delta \leq \lambda} S_{Q; m, b, \Delta}(\lambda),
\]

\[
G_{Q; m, b}(\lambda) := \frac{1}{N_{Q,(m,b)}} \# \left\{ (\gamma, \gamma') : \gamma, \gamma' \in \delta_Q^{(m,b)}, 0 < \gamma' - \gamma \leq \frac{\lambda}{N_{Q,(m,b)}} \right\}.
\]

As in the previous section we can write
\[
S_{Q; m, b, \Delta}(\lambda) = \sum_{d | \Delta} \mu(d) \sum_{\Delta \leq x < \lambda} \sum_{v \equiv b \ (\text{mod} \ m)} \frac{\Delta \lambda^2}{x v \leq \lambda} T_{Q; m, b, \Delta}(v, \lambda), \quad (3.1)
\]

We write \(u = dw\) and observe that the assumption \((b, m) = 1\) implies \((d, m) = (v, m) = 1\).

Hence we get
\[
S_{Q; m, b, \Delta}(\lambda) = \sum_{\substack{d | \Delta \\ (d, m) = 1}} \mu(d) \sum_{\Delta \leq x < \lambda} \sum_{v \equiv b \ (\text{mod} \ m)} T_{Q; m, b, \Delta}(v, \lambda), \quad (3.2)
\]
where
\[ T_{Q,m,b,\Delta}(v, \lambda) := \sum_{\Delta Q \leq w \leq \Delta x \equiv b \pmod{m} \atop x \leq v, (x,v) = 1} 1. \] (3.3)

We write
\[ T_{Q,m,b,\Delta}(v, \lambda) = \sum_{x \in [0,v], (x,v) = 1} \frac{1}{v} \sum_{k \pmod{v}} e\left(\frac{k(y - \frac{\Delta}{v} \bar{x})}{v}\right), \] (3.4)
where \( \bar{d} \) is the multiplicative inverse of \( d \pmod{m} \) and \( \bar{x} \) the multiplicative inverse of \( x \pmod{v} \),
\[ M_{Q,m,b,\Delta}(v, \lambda) := \frac{1}{v} \sum_{x \in [0,v], (x,v) = 1} \sum_{\Delta Q \leq w \leq \Delta x \equiv b \pmod{m} \atop y \equiv \bar{d} \pmod{m}} 1, \] (3.5)
\[ E_{Q,m,b,\Delta}(v, \lambda) := \frac{1}{v} \sum_{k=1}^{v-1} S_{[1,v]}(0, -\frac{\Delta}{v} k; v) \sum_{y \in \left[\Delta Q^2 \frac{Q}{\Delta}, \frac{\Delta Q^2}{\Delta}\right] \atop y \equiv \bar{d} \pmod{m}} e\left(\frac{ky}{v}\right). \] (3.6)

We employ the bound\[1\]
\[ S_{[1,v]}(0, -\frac{\Delta}{d} k; v) \ll \left(\frac{\Delta}{d} k, v\right)^{1/2} v^{1/2+\delta} \leq \frac{\Delta}{d} (k,v)^{1/2} v^{1/2+\delta}, \] (3.7)
\((v,m) = 1\), and the geometric series with ratio \( e\left(\frac{km}{v}\right) \) to gather
\[ E_{Q,m,b,\Delta}(v, \lambda) \ll_{\delta, \Delta} v^{-1/2+\delta} \sum_{k=1}^{v-1} (k,v)^{1/2} \left| \sum_{y \in \left[\Delta Q^2 \frac{Q}{\Delta}, \frac{\Delta Q^2}{\Delta}\right] \atop y \equiv \bar{d} \pmod{m}} e\left(\frac{ky}{v}\right) \right| \ll v^{-1/2+\delta} \sum_{k=1}^{v-1} (k,v)^{1/2}. \] (3.8)

Using \( \{km : 1 \leq k \leq v-1\} \subseteq \{\ell : 1 \leq \ell \leq mv, v \nmid \ell\} \) and \((k,v)^{1/2} \leq (km,v)^{1/2}\) this yields
\[ E_{Q,m,b,\Delta}(v, \lambda) \ll_{\delta, \Delta} v^{-1/2+\delta} \sum_{\ell=1}^{mv} \frac{(\ell,v)^{1/2}}{\|\ell\|} = mv^{-1/2+\delta} \sum_{\ell=1}^{v-1} \frac{(\ell,v)^{1/2}}{\|\ell\|}, \]
\[ \leq 2mv^{-1/2+\delta} \sum_{1 \leq \ell < \frac{v}{2}} \frac{(\ell,v)^{1/2}}{\ell} \ll_{\delta, \Delta, m} v^{1/2+3\delta}. \]

---

\[1\] Here \( S_{[1,v]}(0, \ell; v) \) coincides with the Ramanujan sum \( c_\ell(v) \ll_{\delta} (\ell,v)^{1+\delta} \). On replacing \( v \) by \( \beta v \) with \( \beta \in (0,1) \) the inequality \( (3.7) \) follows from \( (2.2) \).
From (3.4), (3.5) and (3.8) we now infer

\[ S_{Q;m,b,\Delta}(\lambda) = \frac{Q}{m} \sum_{d | \Delta} \frac{\mu(d)}{d} \left( \sum_{\frac{\Delta Q}{v} \leq Q \atop v \equiv b \pmod{m}} \frac{\varphi(v)}{v} - \frac{\Delta Q}{\lambda} \sum_{\frac{\Delta Q}{v} \leq Q \atop v \equiv b \pmod{m}} \frac{\varphi(v)}{v^2} \right) + O_{\delta,m,\Delta}(Q^{3/2+\delta}). \]

Next, an application of Lemma 4 and the equality

\[ \sum_{d | \Delta} \frac{\mu(d)}{d} = \frac{\varphi(\Delta)}{\Delta} \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \]

lead to

\[ S_{Q;m,b,\Delta}(\lambda) = \frac{C_m Q^2}{m \varphi(m)} \cdot \frac{\varphi(\Delta)}{\Delta} \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \left( 1 - \frac{1}{\Delta} - \frac{1}{\lambda} \log \frac{\lambda}{\Delta} \right) + O_{\delta,m,\Delta}(Q^{3/2+\delta}), \quad (3.9) \]

and so we get, for every \( K \geq 1 \) and uniformly in \( \lambda \in [1, K] \),

\[ H_{Q;m,b}(\lambda) = \frac{C_m Q^2}{m \varphi(m)} \sum_{1 \leq \Delta < \lambda} \varphi(\Delta) \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \left( 1 - \frac{1}{\Delta} - \frac{1}{\lambda} \log \frac{\lambda}{\Delta} \right) + O_{\delta,m,K}(Q^{3/2+\delta}). \quad (3.10) \]

Estimates (3.10), (1.4), and the definitions of \( G_{Q;m,b} \) and \( H_{Q;m,b} \) now yield

\[ G_{Q;m,b}(\lambda) = \frac{1}{N_{Q;m,b}} H_{Q;m,b} \left( \frac{Q^2}{N_{Q;m,b}} \right) = G_{(m)}(\lambda) + O_{\delta,m}(Q^{-1/2+\delta}), \quad (3.11) \]

where

\[ G_{(m)}(\lambda) = \frac{2}{m} \sum_{1 \leq \Delta \leq K_m \lambda} \varphi(\Delta) \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \left( 1 - \frac{1}{\Delta} - \frac{1}{K_m \lambda} \log \frac{K_m \lambda}{\Delta} \right); \]

with \( K_m := \frac{2 \varphi(m)}{C_m} \).

This shows that the pair correlation function of \( \hat{S}_{Q,m,b}^{(m,b)} \) exists and is given by

\[ \hat{g}_{(m)}(\lambda) = G_{(m)}'(\lambda) = \frac{C_m}{m \varphi(m)} \cdot \frac{1}{\lambda^2} \sum_{1 \leq \Delta \leq K_m \lambda} \varphi(\Delta) \cdot \frac{(\Delta, m)}{\varphi((\Delta, m))} \log \frac{K_m \lambda}{\Delta}. \quad (3.12) \]

Comparing (3.12) with the formula for \( g_{(m)} \) given in Theorem 1 we find that

\[ \tilde{g}_{(m)}(\lambda) = g_{(m)}(\varphi(m) \lambda). \]

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