Efficient Kernel Transfer in Knowledge Distillation

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Abstract
Knowledge distillation is an effective way for model compression in deep learning. Given a large model (i.e., teacher model), it aims to improve the performance of a compact model (i.e., student model) by transferring the information from the teacher. An essential challenge in knowledge distillation is to identify the appropriate information to transfer. In early works, only the final output of the teacher model is used as the soft label to help the training of student models. Recently, the information from intermediate layers is also adopted for better distillation. In this work, we aim to optimize the process of knowledge distillation from the perspective of kernel matrix. The output of each layer in a neural network can be considered as a new feature space generated by applying a kernel function on original images. Hence, we propose to transfer the corresponding kernel matrix (i.e., Gram matrix) from teacher models to student models for distillation. However, the size of the whole kernel matrix is quadratic to the number of examples. To improve the efficiency, we decompose the original kernel matrix with Nyström method and then transfer the partial matrix obtained with landmark points, whose size is linear in the number of examples. More importantly, our theoretical analysis shows that the difference between the original kernel matrices of teacher and student can be well bounded by that of their corresponding partial matrices. Finally, a new strategy of generating appropriate landmark points is proposed for better distillation. The empirical study on benchmark data sets demonstrates the effectiveness of the proposed algorithm. Code will be released.

Introduction
With the development of deep learning, neural networks make many computer vision tasks applicable for edge devices. Edge devices often have limited computation and storage resources. Therefore, neural networks that contain a small number of FLOPS and parameters are preferred. Lots of efforts are devoted to improving the performance of neural networks with resource constraints (Courbariaux, Bengio, and David 2015; Hinton, Vinyals, and Dean 2015; Sandler et al. 2018). Among various developed strategies, knowledge distillation (KD) is a simple yet effective way to help train compact networks (Hinton, Vinyals, and Dean 2015).

Knowledge distillation in deep learning aims to improve the performance of a small network (i.e., student) with the information from a large network (i.e., teacher). Given a teacher, various information can be transferred to regularize the training of the student network. (Hinton, Vinyals, and Dean 2015) transfers the label information to smooth the label space of the student network. (Romero et al. 2015) and (Zagoruyko and Komodakis 2017) propose to transfer the information of intermediate layers to help training. (Yim et al. 2017) transfers the flow between layers as hints for student networks. (Chen, Wang, and Zhang 2018) improves the performance of metric learning with the rank information from teacher models. Recently, (Liu et al. 2019) and (Park et al. 2019) consider the similarity of examples and transfer the distances between examples to student networks. All of these methods can achieve success in certain applications but there lacks a consistent problem formulation for knowledge distillation in different layers of the neural network.

In this work, we study knowledge distillation from the perspective of kernel matrix. Given two images $x_i$ and $x_j$, their similarity can be computed as

$$k(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle$$
where $\phi(\cdot)$ is the corresponding kernel function that projects examples from the original space to a space that fits the problem better. In an appropriate space, a simple model (e.g., a linear model) can describe the data well. For example, (Williams and Seeger 2000) projects the original examples to an infinite space where different classes become linearly separable. If we consider a neural network as a kernel function and $\phi(x_i)$ becomes the output from a certain layer of the neural network, each layer in the neural network can generate a kernel matrix that captures the similarity between examples. Since the space spanned by the teacher model serves the target task better, we propose to transfer the kernel matrix from the teacher model to the student model.

The main challenge in transferring kernel matrix comes from the size of the kernel matrix (i.e., $n \times n$, where $n$ is the number of total examples). With a large number of training examples, it becomes intractable to transfer the whole matrix directly, especially for training neural networks, where only a mini-batch of examples are accessible at each iteration. If transferring only the sub-matrix in the mini-batch, the optimization can be slow and other loss functions for knowledge distillation have to be designed to achieve a desired performance as shown in (Park et al. 2019). Therefore, we propose to apply the Nyström method (Williams and Seeger 2000) to obtain a low-rank approximation of the original kernel matrix with landmark points. Then, we can minimize the difference between the compact kernel matrices that are calculated between examples and landmark points, to transfer the information from the teacher effectively. Fig. 1 illustrates the proposed strategy.

Compared with the whole kernel matrix whose size is $O(n^2)$, the transferred one is only $O(n)$ in our method. Besides, since the number of landmark points is small, we can keep them in the network, which makes the optimization with mini-batch effective. Considering that the selection of landmark points is important for approximating the original kernel matrix, we propose to apply class centers as landmark points for better distillation. More importantly, our theoretical analysis shows that the difference between original kernel matrices from teacher and student can be well bounded by that of the corresponding partial matrices. The empirical study on benchmark data sets and popular neural networks confirms that the proposed method with a single loss for distillation can transfer the knowledge from different layers well.

### Related Work

**Knowledge distillation** Knowledge distillation has a long history in ensemble learning and becomes popular for training small-sized neural networks (Chen, Wang, and Zhang 2018; Hinton, Vinyals, and Dean 2015; Liu et al. 2019; Park et al. 2019; Zagoruyko and Komodakis 2017; Tian, Krishnan, and Isola 2020). Various algorithms have been developed to transfer different information from the teacher model to the student model. (Hinton, Vinyals, and Dean 2015) considers the final output of the teacher model as the soft label and regularizes the similarity between the label distribution output from the student model and that of the soft label from the teacher model. (Zagoruyko and Komodakis 2017) transfers the attention maps from intermediate layers, which provides a way to explore more information from the teacher model. The algorithms proposed in (Liu et al. 2019) and (Park et al. 2019) are close to our work, where the Euclidean distances between examples are transferred for knowledge distillation. However, they can only transfer the distance information of pairs within a mini-batch while we aim to transfer the whole kernel matrix to achieve a better performance. Furthermore, we provide a theoretical analysis to demonstrate the effectiveness of the proposed method. Besides these work for classification, some methods are proposed for other tasks, e.g., detection (Chen et al. 2017) and metric learning (Chen, Wang, and Zhang 2018). We focus on classification in this work while the proposed method can be easily extended to metric learning that aims to optimize the performance of the embedding layer.

**Nyström method** Nyström method is an effective algorithm to obtain a low-rank approximation for a kernel matrix (Williams and Seeger 2000). Given a whole kernel matrix, it tries to reconstruct the original one with the randomly sampled columns. The data points corresponding to the selected columns are denoted as landmark points. The approximation error can be bounded even with the randomly sampled landmark points. Later, researchers show that a delicate sampling strategy can further improve the performance (Drineas and Mahoney 2005; Kumar, Mohri, and Talwalkar 2012; Zhang, Tsang, and Kwok 2008). (Drineas and Mahoney 2005) proposes to sample landmark points with a data-dependent probability distribution rather than the uniform distribution. (Kumar, Mohri, and Talwalkar 2012) and (Zhang, Tsang, and Kwok 2008) demonstrate that using clustering centers as landmark points provides the best approximation among different strategies. Note that Nyström method is developed for unsupervised kernel matrix approximation while we can access the label information in knowledge distillation. In this work, we provide an analysis on the selection criterion of landmark points for kernel matrix transfer and develop a supervised strategy accordingly.

### Efficient Kernel Matrix Transfer

Given two image $x_i$ and $x_j$, the similarity between them can be measured with a kernel function as

$$k(x_i, x_j) = \langle \phi(x_i), \phi(x_j) \rangle$$

where $\phi(x_i)$ is a projection function that projects examples from the original space to a space better for the target task.

In this work, we consider each layer in a neural network as a projection function. We denote the student network as $S$ and the teacher network as $T$. The features output from a certain layer of $S$ and $T$ are referred as $f_S(x_i) = x^S_i$ and $f_T(x_i) = x^T_i$, respectively, and the index for the layer is omitted for brevity. Then, the similarity between two image $x_i$ and $x_j$ in the kernel matrix can be computed by

$$K_S(x_i, x_j) = \langle f_S(x_i), f_S(x_j) \rangle = x^S_i x^S_j^\top$$

$$K_T(x_i, x_j) = \langle f_T(x_i), f_T(x_j) \rangle = x^T_i x^T_j^\top$$

Thus, the empirical similarity between two image $x_i$ and $x_j$ can be approximated as

$$K_{approx}(x_i, x_j) = K_S(x_i, x_j) I_{j} - \sum_{s=1}^{S} w_s(x_i) x^S_s x^S_j^\top$$

where $I_{j}$ is the unit matrix with one in the $j$th column and zero in other columns, and $w_s(x_i)$ is the weight of $x_i$ in the $s$th layer. The first term $K_S(x_i, x_j) I_{j}$ represents the full kernel matrix, and the second term $\sum_{s=1}^{S} w_s(x_i) x^S_s x^S_j^\top$ approximates the difference between original and transferred kernel matrices.
Let $K_S$ and $K_T$ denote the $n \times n$ kernel matrices from the student and teacher networks, respectively, where $n$ is the total number of images. We aim to transfer the kernel matrix from the teacher model to the student model. The corresponding loss for knowledge distillation with kernel matrix transfer can be written as

$$
\ell_{S,T} = \|K_S - K_T\|_F = \|X_S^T X_S - X_T^T X_T\|_F
$$

where $X_S \in \mathbb{R}^{d_S \times n}$ and $X_T \in \mathbb{R}^{d_T \times n}$ denote the representations of the entire data set output from the same certain layer of the student and teacher network.

Minimizing the loss directly is intractable due to the large size of the kernel matrix, especially for the conventional layer of the student and teacher network. We aim to transfer the kernel matrix from the teacher model to the student model. The corresponding loss for knowledge distillation with kernel matrix transfer can be written as

$$
\ell_{S,T} = \|K_S - K_T\|_F = \|X_S^T X_S - X_T^T X_T\|_F
$$

where $X_S \in \mathbb{R}^{d_S \times n}$ and $X_T \in \mathbb{R}^{d_T \times n}$ denote the representations of the entire data set output from the same certain layer of the student and teacher network.

Nyström Approximation

Nyström method is prevalently applied to approximate the kernel matrix (Drineas and Mahoney 2005; Kumar, Mohri, and Talwalkar 2012; Williams and Seeger 2000; Zhang, Tsang, and Kwok 2008). We briefly review it in this subsection.

Given a $n \times n$ kernel matrix $K$, we can first randomly shuffle columns and rewrite it as

$$
K = \begin{bmatrix} W & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix}
$$

where $W \in \mathbb{R}^{m \times m}$. Then, a good approximation for $K$ can be obtained as $\tilde{K} = CW^+ C^T$, where $C = \begin{bmatrix} W & K_{12} \end{bmatrix} \in \mathbb{R}^{n \times m}$ and $W^+$ denotes the pseudo inverse of $W$ (Williams and Seeger 2000).

Let $K_k$ denote the best top-$k$ approximation of kernel matrix $K$ and $k \leq m$. The rank-$k$ approximation derived by the Nyström method can be computed as $\tilde{K}_k = CW_k^+ C^T$, where $W_k$ denotes the best top-$k$ approximation of $W$ and $W_k^+$ is the corresponding pseudo inverse. The performance of the approximation can be demonstrated by the following theorem.

**Theorem 1.** (Kumar, Mohri, and Talwalkar 2012) Let $\tilde{K}_k$ denote the rank-$k$ Nyström approximation with $m$ columns that are sampled uniformly at random without replacement from $K$. We have $\|K - \tilde{K}_k\|_F \leq \|K - K_k\|_F + \varepsilon: \varepsilon = O(1/m^{1/4})\|K\|_F$

The examples corresponding to the selected columns in $C$ are referred as landmark points. Theorem 1 shows that the approximation is applicable even with the randomly sampled landmark points.

Kernel Transfer

With the low-rank approximation, the kernel matrix from a certain layer of the student and teacher network can be computed as

$$
\tilde{K}_S^k = C_S W_{kS}^+ C_{T}^T; \quad \tilde{K}_T^k = C_T W_{kT}^+ C_{T}^T
$$

Compared with the original kernel matrix, the partial matrix $C$ has significantly less number of terms. Let $D_S \in \mathbb{R}^{d_S \times m}$ and $D_T \in \mathbb{R}^{d_T \times m}$ denote the landmark points for the student and teacher kernel matrices, then we have $C_S = X^T_S D_S, C_T = X^T_T D_T$ and $W_S = D_S^T D_S, W_T = D_T^T D_T$. We theoretically show that transferring the compact matrix is up-bounding the distance between original kernel matrices. The detailed proof can be found in the supplementary.

**Corollary 1.** Assuming that $C_S$ and $C_T$ are bounded by a constant $c$ as $\|C_S\|_F, \|C_T\|_F \leq c$ and the smallest eigenvalues in $W_{kS}$ and $W_{kT}$ are larger than $1$, with the Nyström approximation, we can bound the loss in Eqn. 1 by

$$
\|K_S - K_T\|_F \leq 2\varepsilon + 2c^2 \|C_S W_{kS}^+ - C_T W_{kT}^+\|_F
$$

In Corollary 1, the partial kernel matrix will be regularized with the pseudo inverse of $W$. The computational cost of obtaining pseudo inverse is expensive and it can introduce additional noise when the feature space of the student is unstable in the early stage of training.

For efficiency, we aim to bound the original loss in Eqn. 1 solely with $C_S$ and $C_T$ as in the following Corollary.

**Corollary 2.** With the same assumptions in Corollary 1, we can bound the loss in Eqn. 1 by

$$
\|K_S - K_T\|_F \leq 2\varepsilon + o(\|C_S - C_T\|_F)
$$

Corollary 2 illustrates that minimizing the difference between the partial kernel matrices using landmark points can transfer the original kernel matrix from the teacher model effectively. Note that the partial matrices have the size of $n \times m$, where $m$ is the number of landmark points. When $m \ll n$, landmark points $D_S$ and $D_T$ can be kept in GPU memory as parameters of the loss function for optimization, which is much more efficient than transferring the original kernel matrix. Since the selection of landmark points is important for approximation. We elaborate our strategy in the next subsection.

Landmark Selection

We consider a strategy that obtains a single landmark point for each class, which means $m = L$ for a $L$-class classification problem. We will theoretically demonstrate the selection of appropriate landmark points as follows.

Let $D_S = [d_{S1}^1, \ldots, d_{S1}^L]$ and $D_T = [d_{T1}^1, \ldots, d_{T1}^L]$ denote the landmark points for a certain layer of the student and teacher network, respectively. First, considering the similarity between an arbitrary pair of examples, we can bound the difference between that from the teacher and student as follows.

**Lemma 1.** Given an arbitrary pair $(i, j)$, let $d_{S i}^j$ and $d_{T i}^j$ denote the corresponding landmark points for the $i$-th example in the space of student and teacher network, respectively. Assume the norm of $x_S^i, x_T^i, x_S^j, x_T^j$ are bounded by $c$. Then we have

$$
\|K_S(x_i, x_j) - K_T(x_i, x_j)\| \\
\leq c\|x_S^i - d_{S i}^j\| + c\|x_T^i - d_{T i}^j\| + \|d_{S i}^j x_S^i - d_{T i}^j x_T^i\|
$$
Lemma 1 provides the bound on a single pair. The bound for the kernel matrix can be accumulated over all pairs.

**Theorem 2.** With the assumptions in Lemma 1, we have
\[
\|K_S - K_T\|_F \leq n \sum_i \|x_i^T - d_i^T\| + n \sum_i \|x_i^T - d_i^T\|_A + \sum_{i,j} |d_i^T x_j - d_j^T x_i|_B \tag{2}
\]

According to Theorem 2, we can find that the transfer loss comes from two aspects. Term A in Eqn. 2 contains the distance from each example to its corresponding landmark point. Since the corresponding landmark point for \(x_i^S\) can be obtained by \(\arg \min_{1 \leq l \leq L} \{\|x_i^T - d_i^T\|\}\), we can rewrite the problem of minimizing the original term as
\[
\min \sum_i \min_l \{\|x_i^T - d_i^T\|\}
\]

Apparently, this objective is a standard clustering problem. It inspires us to use cluster centers as the landmark points for both the student and teacher networks. Unlike the conventional Nyström method, which is often in an unsupervised learning setting, we can access the label information in knowledge distillation. When we set the number of clusters to be the number of classes, the landmark point becomes the center in each class and can be computed by averaging examples within the class as
\[
d_S^l = \frac{1}{n_l} \sum_{i : y_i = l} x_i^S; \quad d_T^l = \frac{1}{n_l} \sum_{i : y_i = l} x_i^T \tag{3}
\]

where \(y_i\) is the class label of the \(i\)-th example and \(n_l\) denotes the number of examples in the \(l\)-th class.

Term B from Eqn. 2, in fact, indicates the difference between the student and teacher kernel matrices defined by the landmark points as in Corollary 2. With landmark points \(D_S\) and \(D_T\) obtained from optimizing the term A, we can formulate the Knowledge Distillation problem by transferring Approximated kernel matrix (KDA) as
\[
\min_{X_S} \ell_{KDA}(X_S^T D_S - X_T^T D_T) \tag{4}
\]

where \(\ell_{KDA}(X_S^T D_S - X_T^T D_T) = \sum_{i=1}^{n} \ell(d_S^l x_i^S - d_T^l x_i^T)\) and we adopt \(\ell(\cdot)\) as the smoothed \(L_1\) loss for the stable optimization as \(\ell(z) = \begin{cases} |z| - 0.5 & z > 1 \\ 0.5z^2 & o.w. \end{cases}\).

With the KDA loss, we propose a novel knowledge distillation algorithm that works in an alternative manner. In each iteration, we first compute the landmark points with features of examples accumulated from the last epoch by Eqn. 3. Then, the KDA loss defined by the fixed landmark points in Eqn. 4 will be optimized along with a standard cross-entropy loss for the student. The proposed algorithm is summarized in Alg. 1. Since at least one epoch will be spent on collecting features for computing landmark points, we will minimize the KDA loss after \(H\) epochs of training, where \(H \geq 1\).

**Algorithm 1 Knowledge Distillation by Approximated Kernel Transfer (KDA)**

**Input:** Data set \(\{x_i\}\), a student model \(S\), a teacher model \(T\), total epochs \(T\), warm-up epochs \(H\)

**Initialize** \(\{d_S^0\} = \emptyset, \{d_T^0\} = \emptyset\)

for \(t = 1\) to \(H\) do

Optimize \(S\) without KDA loss

Compute \(\{d_S^t\}, \{d_T^t\}\) as in Eqn. 3

end for

for \(t = H + 1\) to \(T\) do

Optimize \(S\) with KDA loss defined on \(\{d_S^{t-1}\}, \{d_T^{t-1}\}\)

Compute \(\{d_S^t\}, \{d_T^t\}\) as in Eqn. 3

end for

**return** \(S\)

**Connection to Conventional KD**

In the conventional KD method (Hinton, Vinyals, and Dean 2015), only the output from the last layer in the teacher model are adopted for the student. By setting an appropriate parameter, (Hinton, Vinyals, and Dean 2015) illustrates that the loss function for KD can be written as
\[
\ell(X_S) = \sum_i \|x_i^S - x_i^T\|^2
\]

where \(x_i^S, x_i^T \in \mathbb{R}^C\) denote the logits before the SoftMax operator. With the identity matrix \(I\), the equivalent formulation is
\[
\ell(X_S) = \|X_S^T I - X_T^T I\|^2_F
\]

Compared to the KDA loss in Eqn. 4, the conventional KD can be considered as applying one-hot label vectors as landmark points to transfer the kernel matrix of the teacher network. However, it lacks the constraints on the similarity between each example and its corresponding landmark point as illustrated in Theorem 2, which may degenerate the performance of knowledge distillation.

**Experiments**

We conduct experiments on two benchmark data sets to illustrate the effectiveness of the proposed KDA algorithm. We employ ResNet-34 (He et al. 2016) as the teacher network. ResNet-18, ResNet-18-0.5 and ShuffleNetV2 (Ma et al. 2018) are adopted as student networks, where ResNet-18-0.5 denotes ResNet-18 with a half number of channels. We apply the standard stochastic gradient descent (SGD) with momentum to train the networks. Specifically, we set the size of mini-batch to 256, momentum to 0.9 and weight decay to 5e-4 in all experiments. The student models are trained with 120 epochs. The initial learning rate is 0.1 and cosine decay is adopted with \(H = 5\) epochs for warm-up.

Three baseline knowledge distillation methods are included in the comparison

- KD (Hinton, Vinyals, and Dean 2015): a conventional knowledge distillation method that constrains the KL-divergence between the output label distributions of the student and teacher networks.
• AT (Zagoruyko and Komodakis 2017): a method that transfers the information from intermediate layers to accelerate the training of the student network.
• RKD (Park et al. 2019): a recent work that regularizes the similarity matrices between student and teacher networks. Unlike the method proposed in this work, they focus on transferring the similarity matrices within a mini-batch. Note that a similar method is proposed in (Liu et al. 2019). Every algorithm will minimize the combined loss from both the distillation and the standard cross entropy loss for classification. For RKD, we transfer the features before the last fully-connected (FC) layer for comparison. Note that AT transfers the attention map of the teacher, so we adopt the feature before the last pooling layer for distillation. Besides, we let “Baseline” denote the method that trains the student without information from the teacher. Our method is referred as “KDA”. We search the best parameters for all methods in the comparison and keep the same parameters for different experiments.

Ablation Study
We perform the ablation study on CIFAR-100 (Krizhevsky 2009). This data set contains 100 classes, where each class has 500 images for training and 100 for test. Each image is a color image with size of $32 \times 32$.

In this subsection, we set ResNet-34 as the teacher and ResNet-18 as the student. During the training, each image is first padded to be $40 \times 40$, and then we randomly crop a $32 \times 32$ image from it. Besides, random horizontal flipping is also adopted for data augmentation.

Effect of Landmark Points
First, we evaluate the strategy for generating landmark points. As illustrated in Corollary 2, the randomly selected landmark points can achieve a good performance. So we compare the KDA with class centers to that with random landmark points in Fig. 2. In this experiment, we adopt the features before the last FC layer for transfer.

From Fig. 2, we can observe that with landmark points, two variants of KDA perform significantly better than the baseline. It demonstrates that kernel matrix is informative for training student models and transferring kernel matrix from teacher can help improve the performance of student. Furthermore, KDA with randomly sampled landmark points can surpass baseline by a large margin. It is consistent with Corollary 2 that even with the random landmark points, Nyström method can guarantee a good approximation of the kernel matrix. Finally, KDA with class centers as the landmark points shows the best performance among different methods, which confirms the criterion suggested in Theorem 2. We will use class centers as landmark points in the remaining experiments.

Effect of Kernel Transfer
Then, we compare the difference between kernel matrices from a teacher and its student models. The performance of transferring a kernel matrix is measured by $\|K_S - K_T\|_F/\|K_T\|_F$, which calculates the fraction of information that has not been transferred. We investigate features from two layers in the comparison: the one before the last FC layer and that after the FC layer. The kernel transfer performance of different layers are illustrated in Fig. 3 (a) and (b), respectively.

![Figure 3: Comparison of kernel transfer performance measured by $\|K_S - K_T\|_F/\|K_T\|_F$.](image)

Fig. 3 (a) compares the kernel transfer performance of the baseline and that of those methods which can transfer information from the layer before the last FC layer. First, it is obvious that both RKD and KDA are better than the baseline (i.e., less information loss during the transferring). It indicates that minimizing the difference between kernel matrices can effectively transfer appropriate information from the teacher. Second, RKD transfers the similarity matrix defined by examples in a mini-batch only and shows a larger transfer loss than KDA. Considering the massive number of pairs, optimizing with all of these pairs in RKD is intractable. Note that the number of pairs can be up to $O(n^2)$ while the number of pairs in a mini-batch is only $O(r^2)$, where $r$ is the size of a mini-batch. To visit all pairs only once, it requires at least $O(n/r)$ epochs.

On the contrary, the loss from KDA is only about 23% of that in RKD. KDA optimizes the partial kernel matrix with landmark points and the total number of pairs is linear in that of original examples. Due to a small number of landmark points, the partial matrix $C$ is much more compact than the original one. For example, there are 50,000 examples in CIFAR-100. When applying 100 landmark points for distillation, the partial matrix contains 0.2% terms of the original one. Besides, since we keep class centers in the memory as the parameters of the loss function, the whole kernel matrix can be approximated in a single epoch. Therefore, SGD can optimize the KDA loss sufficiently.

Then, we compare the performance of transfer after the last FC layer as shown in Fig. 3 (b). For KDA, we compute the kernel matrix with features before the SoftMax operator.
From the comparison, we can observe that both of KD and KDA have much less transfer loss than the baseline. As illustrated in the discussion of “Connection to Conventional KD”, conventional KD is equivalent to transferring the partial kernel matrix with one-hot landmark points. Therefore, it can reduce the difference between teacher and student effectively. However, the landmark points adopted by KD fail to satisfy the property illustrated in Theorem 2. By equipping class centers as landmark points, KDA can further reduce the transfer loss from 0.23 in KD to 0.2, which confirms the effectiveness of transferring kernel matrix with appropriate landmark points.

Finally, we demonstrate that the difference between partial kernel matrices is closely correlated with that between whole kernel matrices as suggested in Corollary 2. Features from the layer before the last FC layer are adopted for evaluation. Fig. 4 illustrates how the ground-truth transfer loss \( \| C_S - C_T \|_F / \| K_T \|_F \) (i.e., our estimation) and \( \| K_S - K_T \|_F / \| K_T \|_F \) (i.e., ground-truth).

![Figure 4: Relationship between \( \| C_S - C_T \|_F / \| K_T \|_F \) (i.e., our estimation) and \( \| K_S - K_T \|_F / \| K_T \|_F \) (i.e., ground-truth).](image)

![Figure 5: Smallest eigenvalues of \( W_S \) (i.e., student) and \( W_T \) (i.e., teacher).](image)

**Effect of Different Layers**

Now, we illustrate the performance of transferring the kernel matrix from different layers. ResNet consists of 5 convolutional layer groups and we compare the performance of the last 3 groups (i.e., “conv3_x”, “conv4_x” and “conv5_x”) and the one after the last FC layer. The definition of groups can be found in Table 1 of (He et al. 2016). For each group, we adopt the last layer for transfer. Before transfer, we add a pooling layer to reduce the dimension of the feature map. Note that after pooling, the last layer of “conv5_x” becomes the layer before the last FC layer.

Table 1: Comparison of accuracy (%) on CIFAR-100 when transferring the kernel matrix from different layers.

| S | conv3_x | conv4_x | Before FC | After FC | T |
|---|---------|---------|-----------|---------|---|
| 77.2 | 77.7 | 78.1 | 79.6 | 79.4 | 80.3 |

Table 1 shows the performance of transferring information from different layers. First, transferring information from teacher always improves the performance of student, which demonstrates the effectiveness of knowledge distillation. Besides, the information from the later layers is more helpful for training student. It is because later layers contain more semantic information that is closely related to the target task. We will focus on the layers before and after the FC layer in the rest experiments.

**CIFAR-100**

In this subsection, we compare the proposed KDA method to other methods on CIFAR-100. The results of different meth-
ods can be found in Table 2. Methods in the comparison are repeated 3 times and the average results with standard deviation are reported. First, all methods with knowledge distillation outperform the student network without a teacher, which shows that knowledge distillation can improve the performance of student models significantly.

By transferring the kernel matrix before the last FC layer, KDA surpasses RKD by 1.3% when ResNet-34 is the teacher and ResNet-18 is the student. The observation is consistent with the comparison in the ablation study, which confirms that \( \| K_S - K_T \|_F / \| K_T \|_F \) is an appropriate metric to evaluate the transfer loss of similarity matrix transfer. Moreover, KDA shows a significant improvement on different student networks, which implies that the proposed method can be applicable for different teacher-student configurations.

Furthermore, when transferring the kernel matrix after the last FC layer, both of KD and KDA can demonstrate good performance with the student model. It is due to the fact that these methods transfer the kernel matrix with landmark points, which is efficient for optimization. Besides, KDA can further improve the performance compared to KD. The superior performance of KDA demonstrates the effectiveness of the proposed strategy for generating landmark points.

Finally, compared to benchmark methods, KDA can distill the information from different layers with the uniform formulation in Eqn. 4. The proposed method provides a systematic perspective to understand a family of knowledge distillation methods that aim to transfer kernel matrices. If transferring the kernel matrix before and after the FC layer simultaneously, the performance of KDA can be slightly improved too as illustrated by “Combo” in Table 2.

**Tiny-ImageNet**

Then, we compare different methods on Tiny-ImageNet data set\(^1\). There are 200 classes in this data set and each class provides 500 images for training and 50 for validation. We report the performance on the validation set. Since the size of images in Tiny-ImageNet is \( 64 \times 64 \) that is larger than CIFAR-100, we replace the random crop augmentation with a more aggressive version as in (He et al. 2016), and keep other settings the same.

Table 3 summarizes the comparison. We can observe the similar results as on CIFAR-100. First, all methods with the information from a teacher model can surpass the student model without a teacher by a large margin. Second, compared with the baseline methods that try to transfer the similarity matrix, KDA outperforms them no matter in which layer the transfer happens. Finally, KDA with information combined from two layers can further improve the performance, which implies that the information from different layers can be complementary. Note that CIFAR-100 and Tiny-ImageNet have very different data formats, which demonstrates the applicability of the proposed algorithm in various real-world applications.

**Conclusion**

In this work, we investigate the knowledge distillation problem from the perspective of kernel matrix. Since kernel matrix is closely related to the performance on the target task, we propose to transfer the kernel matrix from the teacher model to the student model. Considering the number of terms in the kernel matrix is quadratic in the number of training examples, we adopt the Nyström method and propose a strategy to obtain the landmark points for efficient optimization. The proposed method not only improves the efficiency of transferring kernel matrix, but also has the theoretical guarantee for the efficacy. Experiments on the benchmark data sets verify the effectiveness of the proposed algorithm. Besides kernel matrix, there are many existing methods that transfer different information from the teacher to the student. Combining the proposed KDA loss with other knowledge for distillation can be our future work.

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\(^1\)https://tiny-imagenet.herokuapp.com

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### Table 2: Comparison of accuracy (%) on CIFAR-100.

| Teacher     | Student     | S     | Before Last FC | After FC | Combo | T  |
|-------------|-------------|-------|---------------|---------|-------|----|
| ResNet34    | ResNet18    | 77.2  | 78.1±0.3      | 79.6±0.1| 79.7±0.1| 80.3|
| ResNet34    | ResNet18-0.5| 73.5  | 75.0±0.1      | 75.6±0.3| 75.9±0.2| 80.3|
| ResNet34    | ShuffleNet  | 71.7  | 73.0±0.1      | 74.0±0.3| 74.2±0.3| 80.3|

### Table 3: Comparison of accuracy (%) on Tiny-ImageNet.

| Teacher     | Student     | S     | Before Last FC | After FC | Combo | T  |
|-------------|-------------|-------|---------------|---------|-------|----|
| ResNet34    | ResNet18    | 63.4  | 64.4±0.1      | 65.2±0.3| 65.5±0.1| 66.6|
| ResNet34    | ResNet18-0.5| 60.3  | 61.0±0.1      | 61.7±0.3| 62.2±0.2| 66.6|
| ResNet34    | ShuffleNet  | 60.6  | 61.3±0.1      | 62.0±0.2| 62.4±0.1| 66.6|
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Theoretical Analysis

Proof of Corollary 1

Proof.

\[ \| K_S - K_T \|_F = \| K_S - \tilde{K}_S - (K_T - \tilde{K}_T) \|_F + \| \tilde{K}_S - \tilde{K}_T \|_F \]

\[ \leq 2\varepsilon + \| C_S W_{kS}^{+} C_S^\top - C_T W_{kT}^{+} C_T^\top \|_F \]

\[ \leq 2\varepsilon + 2\varepsilon^2 \| C_S W_{kS}^{+} - C_T W_{kT}^{+} \|_F \]

Then, we want to show that \( \| W_{kS}^{+} - W_{kT}^{+} \|_F \leq \| C_S - C_T \|_F \). Note that \( \| W_{kS} - W_{kT} \|_F \leq \| C_S - C_T \|_F \) due to the fact that \( W_k \) is a partial matrix from \( C \). We can prove \( \| W_{kS}^{+} - W_{kT}^{+} \|_F \leq \| W_{kS} - W_{kT} \|_F \) instead.

Let

\[ W_{kS} = \sum_{j=1}^{k} \alpha_j \mu_j \nu_j^\top; \quad W_{kT} = \sum_{j=1}^{k} \beta_j \nu_j \nu_j^\top \]

where \( \alpha_1 \geq \cdots \geq \alpha_k > 1 \) and \( \beta_1 \geq \cdots \geq \beta_k > 1 \). Let \( M \in \mathbb{R}^{k \times k} \), where \( M_{ij} = (\mu_i \nu_j^\top, \nu_j \nu_j^\top) \). Then we have

\[ \| W_{kS} - W_{kT} \|_F^2 = \sum_j \alpha_j^2 + \sum_j \beta_j^2 - 2 \sum_{s,t} \alpha_s \beta_t M_{s,t} \]

\[ \| W_{kS}^{+} - W_{kT}^{+} \|_F^2 = \sum_j \frac{1}{\alpha_j^2} + \sum_j \frac{1}{\beta_j^2} - 2 \sum_{s,t} \frac{1}{\alpha_s \beta_t} M_{s,t} \]

So

\[ \| W_{kS}^{+} - W_{kT}^{+} \|_F^2 - \| W_{kS} - W_{kT} \|_F^2 \]

\[ = \sum_j \frac{1}{\alpha_j^2} + \sum_j \frac{1}{\beta_j^2} - \sum_j \alpha_j^2 + \sum_j \beta_j^2 \]

\[ + 2 \sum_{s,t} (\alpha_s \beta_t - \frac{1}{\alpha_s \beta_t}) M_{s,t} \]
Finally, we have

\[ \forall i, \quad \sum_{j}^{k} M_{i,j} = \mu_i (\sum_{j} \nu_j \nu_j^T) \mu_i^T \leq 1; \quad \forall j, \quad \sum_{i}^{k} M_{i,j} \leq 1 \]

Since \( M \) is a doubly stochastic matrix, we can show that

\[ \sum_{s,t} (\alpha_s \beta_t - \frac{1}{\alpha_s \beta_t}) M_{s,t} \leq \sum_{s} (\alpha_s - \frac{1}{\alpha_s}) \]

It can be proved by contradiction. If the optimal solution \( M^* \) has a larger result than the R.H.S., we can denote the first column index of the non-zero off-diagonal element as \( j > i \) (i.e., \( M_{i,j} \)), and the corresponding row index as \( k > i \) (i.e., \( M_{k,i} \)). Let \( A_{i,j} = \alpha_i \beta_j - \frac{1}{\alpha_i \beta_j} \) and we have

\[
A_{i,j} + A_{k,i} - A_{k,j} \\
= \alpha_i \beta_j - \frac{1}{\alpha_i \beta_j} + \alpha_k \beta_i - \frac{1}{\alpha_k \beta_i} \\
- (\alpha_i \beta_i - \frac{1}{\alpha_i \beta_i}) - (\alpha_k \beta_j - \frac{1}{\alpha_k \beta_j}) \\
= -(\alpha_i - \alpha_k)(\beta_i - \beta_j) + \frac{1}{\alpha_i} - \frac{1}{\alpha_k}(\frac{1}{\beta_i} - \frac{1}{\beta_j}) \\
= -(\alpha_i - \alpha_k)(\beta_i - \beta_j) + \frac{(\alpha_i - \alpha_k)(\beta_i - \beta_j)}{\alpha_i \beta_i \beta_j} < 0
\]

It shows that the assignment with the diagonal element can achieve a larger result than the optimal assignment, which contradicts the assumption.

With the optimal results from the assignment of \( M \), we obtain that

\[
\| W_{kS}^+ - W_{kT}^+ \|_F - \| W_{kS} - W_{kT} \|_F^2 \leq \sum_{j}^{k} \left( \frac{1}{\alpha_j} - \frac{1}{\beta_j} \right)^2 - (\alpha_j - \beta_j)^2
\]

For each term, it is easy to show that

\[ \forall j, \quad \left( \frac{1}{\alpha_j} - \frac{1}{\beta_j} \right)^2 - (\alpha_j - \beta_j)^2 \leq 0 \]

Finally, we have

\[ \| W_{kS}^+ - W_{kT}^+ \|_F \leq \| W_{kS} - W_{kT} \|_F \leq \| C_S - C_T \|_F \]

\[ \square \]

**Proof of Lemma 1**

**Proof.** For an arbitrary pair, we have

\[
\| K_S(x_i, x_j) - K_T(x_i, x_j) \| = \| x_i^T x^T_S x^T_S - x^T_i x^T_T x^T_T \| \\
= \| (x^T_S - d^T_S) x^T_S x^T_S + (x^T_T - d^T_T) x^T_T x^T_T \| \\
\leq \| (x^T_S - d^T_S) x^T_S x^T_S \| + \| (x^T_T - d^T_T) x^T_T x^T_T \| \\
+ \| d^T_S x^T_S - d^T_T x^T_T \| \\
\leq e \| x^T_S - d^T_S \| + e \| x^T_T - d^T_T \| + \| d^T_S x^T_S - d^T_T x^T_T \|
\]

\[ \square \]