ON FORMALITY OF GENERALISED SYMMETRIC SPACES

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Abstract. We prove that all generalised symmetric spaces of compact simple Lie groups are formal in the sense of Sullivan. Nevertheless, many of them, including all the non-symmetric flag manifolds, do not admit Riemannian metrics for which all products of harmonic forms are harmonic.

1. Introduction

In this paper we discuss formality properties of certain compact homogeneous spaces $G/H$, with $G$ a compact connected Lie group and $H$ a closed subgroup. We shall discuss formality in the sense of Sullivan’s rational homotopy theory [16] and geometric formality in the sense of [9].

There are some classes of compact homogeneous spaces which are well-known to be formal in the sense of Sullivan, for example the symmetric spaces and those homogeneous spaces with $\text{rk} G = \text{rk} H$. We shall see that it is an immediate consequence of earlier work of the second author [19] that in fact all generalised symmetric spaces$^{1}$ of compact simple Lie groups are formal in the sense of Sullivan.

The notion of geometric formality was introduced by the first author in [9]. A smooth manifold is said to be geometrically formal if it admits a Riemannian metric for which all wedge products of harmonic forms are harmonic. This clearly implies formality in the sense of Sullivan, and is even more restrictive. As compact symmetric spaces are the classical examples of geometrically formal manifolds, it is natural to explore this notion in the context of generalised symmetric spaces. Although these turn out to be formal in the sense of Sullivan and also satisfy all the restrictions on geometrically formal manifolds found in [9], we shall

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$^{1}$These spaces are sometimes called $k$-symmetric.
prove here that many of them are *not* geometrically formal. Some of our examples have \( \text{rk} \, G = \text{rk} \, H \), whereas others do not.

At the time of writing it remains unclear whether there is a reasonable class of non-symmetric compact homogeneous spaces which are geometrically formal.

In Section 2 we collect some classical results on the cohomology of compact homogeneous spaces, and we summarise the results we shall need from \([18, 19]\) on the classification of generalised symmetric spaces and their cohomology. These results are used in Section 3 to conclude that all generalised symmetric spaces of compact simple Lie groups are formal in the sense of Sullivan. Section 4 makes explicit the additive generators and multiplicative relations between them for the cohomology algebras of the flag manifolds. This is then used in Section 5 to prove that the non-symmetric flag manifolds and several other classes of generalised symmetric spaces are not geometrically formal.

2. The real cohomology of compact homogeneous spaces

Let \( G \) be compact connected Lie group, and \( H \subset G \) a connected closed subgroup. We denote by \( \mathfrak{t} \) and \( \mathfrak{s} \) the maximal abelian subalgebras of the Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \) respectively, and by \( BG \) the classifying space of \( G \).

By the Hopf theorem \([1]\), \( H^\ast(G) \) is an exterior algebra on universal transgressive elements \( z_1, \ldots, z_n \). The Cartan-Chevalley theorem \([1]\) implies that \( H^\ast(BG) \) is the ring of \( W \)-invariant polynomials on \( \mathfrak{t} \) with real coefficients, where \( W \) is the Weyl group of \( \mathfrak{g} \) relative to \( \mathfrak{t} \). For all compact simple Lie groups the generators of the Weyl invariants are well-known \([11]\). Coordinates \( x_1, \ldots, x_n \) on \( \mathfrak{t} \) expressing the Weyl invariant polynomials in classical form will be called canonical coordinates. Let \( y_1, \ldots, y_n \) correspond to \( z_1, \ldots, z_n \) by transgression in the universal \( G \)-bundle over \( BG \). Then \( H^\ast(BG) \) is generated by \( y_1, \ldots, y_n \) \([1]\).

We consider the map \( \rho^\ast(H, G) : \mathbb{R}[\mathfrak{t}]^{W_G} \to \mathbb{R}[\mathfrak{s}]^{W_H} \) assigning to each polynomial in \( \mathbb{R}[\mathfrak{t}]^{W_G} \) its restriction to \( \mathfrak{s} \). The Cartan algebra of the homogeneous space \( G/H \) is the algebra \( C = \mathbb{R}[\mathfrak{s}]^{W_H} \otimes H^\ast(G) \) endowed with the following differential \( d \):

\[
d(1 \otimes z_i) = \rho^\ast(H, G)y_i \otimes 1 \quad (1 \leq i \leq n),
\]

\[
d(b \otimes 1) = 0 \text{ for } b \in \mathbb{R}[\mathfrak{s}]^{W_H}.
\]

The name derives from the following celebrated result:
Theorem 1 (Cartan). The real cohomology algebra of the homogeneous space $G/H$ is isomorphic to the cohomology algebra of its Cartan algebra $(C, d)$.

This theorem in principle computes the cohomology of $G/H$. In practice, however, one still needs information about the map $\rho^*(H, G)$ in order to obtain an explicit result.

Before giving a summary of the calculations for generalised symmetric space that we shall need, we recall some earlier applications of Cartan’s theorem.

2.1. Homogeneous spaces with $\text{rk } G = \text{rk } H$. In his classical paper [1], Borel studied the cohomology rings of homogeneous spaces with $\text{rk } G = \text{rk } H$. For these he showed that $\rho^*(H, G)$ is injective, which implies:

Theorem 2 ([1]). The real cohomology algebra of a compact homogeneous space $G/H$ with $\text{rk } G = \text{rk } H$ is given by

$$H^*(G/H) \cong \mathbb{R}[t]^{W_H}/\langle \mathbb{R}[t]^{W_G} \rangle^+,\]$$

where $\langle \mathbb{R}[t]^{W_G} \rangle^+$ is the ideal in $\mathbb{R}[t]^{W_H}$ generated by the elements of $\mathbb{R}[t]^{W_G}$ of positive degrees.

Note that, in order to obtain a more explicit formula, one also requires information about the transition functions between canonical coordinates of the group $G$ and its subgroup $H$.

2.2. Symmetric spaces. Borel [1] also calculated the cohomology algebras of the symmetric spaces $SU(n)/SO(n)$ and $SU(2m)/Sp(m)$, which have $\text{rk } H < \text{rk } G$. The cohomology algebras of the remaining three kinds of symmetric spaces, $SO(2l)/(SO(2m+1) \times SO(2l-2m-1))$, $E_6/F_4$ and $E_6/PSp(4)$, were calculated by Takeuchi [17].

2.3. Homogeneous spaces of Cartan type. There is a larger class of compact homogeneous spaces for which Cartan’s theorem can be used directly, which we call homogeneous spaces of Cartan type. They are called normal position homogeneous spaces in [7], and Cartan pairs $(G, H)$ in [1]. We say that the homogeneous space $G/H$ with $\text{rk } G = n$ and $\text{rk } H = k$ is of Cartan type if one can choose generators $P_1, \ldots, P_n$ of $\mathbb{R}[t]^{W_G}$ in such a way that $\rho^*(H, G)P_{k+1}, \ldots, \rho^*(H, G)P_n$ belong to the ideal in $\mathbb{R}[t]^{W_H}$ generated by $\rho^*(H, G)P_1, \ldots, \rho^*(H, G)P_k$. The following theorem is proved in [3, 13]:
**Theorem 3.** Let \( G/H \) be a compact homogeneous space of Cartan type with \( \text{rk} \, G = n \) and \( \text{rk} \, H = k \). Then its cohomology algebra is given by

\[
H^*(G/H) \cong \mathbb{R}[s]^W_H / \langle \rho^*(H, G)(\mathbb{R}[t]^W_G) \rangle \otimes \wedge(z_{k+1}, \ldots, z_n),
\]

where the \( z_i \) are universal transgressive generators of \( H^*(G) \).

We shall refer to the first and second factors in (1) as the polynomial and the exterior algebra parts of the cohomology.

Note that deciding whether a homogeneous space \( G/H \) is of Cartan type, or not, is almost equivalent to calculating the map \( \rho^*(H, G) \), and is therefore quite difficult in general.

**Remark 1.** The Poincaré polynomial for a homogeneous space \( G/H \) of Cartan type is given by

\[
p(G/H, t) = \prod_{i=1}^{k} \frac{1 - t^{2k_i}}{1 - t^{2l_i}} \prod_{i=k+1}^{n} (1 + t^{2k_i-1}),
\]

where \( k_i (1 \leq i \leq n) \) are the exponents of \( G \) and \( l_i (1 \leq i \leq k) \) are the exponents of the subgroup \( H \). Compare [6].

The following lemma provides an important fact about fibrations between homogeneous spaces.

**Lemma 1.** Let \( G/H \) and \( G/L \) be Cartan type homogeneous spaces with the same exterior algebra parts of their cohomologies, and such that \( H \subset L \). Then the restriction to the fiber \( L/H \) of the fibration \( G/H \to G/L \) is surjective in real cohomology.

**Proof.** Since the spaces \( G/L \) and \( G/H \) have the same exterior algebra parts of their cohomologies, obviously \( \text{rk} \, L = \text{rk} \, H \) and from (2) it follows that

\[
p(G/H, t) = p(G/L, t) \cdot p(L/H, t).
\]

Thus, the spectral sequence of the fibration collapses. Now the Leray-Hirsch theorem implies that the restriction to the fiber is surjective in real cohomology.

\[\square\]

2.4. **Generalised symmetric spaces.** There are several ways of generalising the notion of a symmetric space. The spaces we shall consider here have been studied by many authors, see e. g. [3, 11, 20, 21]. They are sometimes called \( k \)-symmetric, where \( k \) is the order, which we prefer to denote by \( m \) below.
Definition 1. A generalised symmetric space of order $m$ is a triple $(G, H, \Theta)$, where $G$ is a connected Lie group, $H \subset G$ is a closed subgroup, and $\Theta$ is an automorphism of finite order $m$ of the group $G$ satisfying
\[ G^\Theta_0 \subset H \subset G^\Theta, \]
where $G^\Theta$ is the fixed point set of $\Theta$ and $G^\Theta_0$ is its identity component.

Obviously, for $m = 2$ these are the usual symmetric spaces. The “space” underlying a generalised symmetric space is the homogeneous space $G/H$. Just as in the case of symmetric spaces, generalised symmetric spaces of order $m$ in the sense of the above definition can be characterised as Riemannian manifolds admitting certain geodesic symmetries of order $m$, see for example [5].

The class of generalised symmetric spaces is a lot larger than that of symmetric spaces; it is easy to see that many generalised symmetric spaces do not have the homotopy type of any symmetric space.

For semi-simple and simply connected Lie groups $G$ all fixed point subgroups are connected, and there is a bijection between generalised symmetric spaces and generalised symmetric algebras [14]. One can then discuss triples $(\mathfrak{g}, \mathfrak{g}^\Theta, \Theta)$, for simple Lie algebras $\mathfrak{g}$ of compact Lie groups with a finite order automorphism $\Theta$. Assuming simplicity, one can appeal to the classification of Lie algebras. In this way, using the results of V. Kac on automorphisms of Lie algebras (cf. [8]), an explicit list of all the generalised symmetric spaces of compact simple simply connected Lie groups is given in [18].

Even when $G$ is not simply connected, by [18] one has a list of possible generalised symmetric spaces given by the classification of the generalised symmetric Lie algebras, or, equivalently, the generalised symmetric spaces of the simply connected groups.

From the classification one concludes that generalised symmetric spaces $G/H$ with $G$ compact, simple and simply connected and with $\text{rk} H < \text{rk} G$ occur only for the groups $SU(n)$, $Spin(2n)$ and $E_6$, compare [19].

For generalised symmetric spaces, the application of Cartan’s theorem is made possible by describing the inclusion of the maximal abelian subalgebra of the subgroup $H$ into the maximal abelian subalgebra of $G$. More precisely, in [19] the second author gave an explicit formula expressing, for an arbitrary automorphism $\Theta$, a basis of $\mathfrak{t}^\Theta$ through a basis of $\mathfrak{t}$. This formula makes it possible to proceed to explicit calculations of the map $\rho^*(H, G)$ for the generalised symmetric spaces. The first result is:
Theorem 4 ([19]). All generalised symmetric spaces of simple compact Lie groups are of Cartan type.

Because of Theorem 2, calculations of the cohomology are of interest only in the cases where \( \text{rk} \, H < \text{rk} \, G \). By the classification, in almost all cases there is then only one possibility for \( \text{rk} \, H \), the exception being \( G = \text{Spin}(8) \). For these spaces one has:

**Theorem 5 ([19]).** The real cohomology algebra of a generalised symmetric space \( G/H \) with \( \text{rk} \, H < \text{rk} \, G \) is as follows:

1. If \( G = SU(l + 1) \), then
   - for \( l = 2n, \, n \geq 1 \)
     \[ H^*(G/H) \cong (\mathbb{R}[s]^{W_H} / \langle \rho^*(H,G)\sigma_j(x_1^2, \ldots, x_n^2) \rangle) \otimes \wedge(z_3, \ldots, z_{2n+1}) \]
   - for \( l = 2n - 1, \, n \geq 3 \)
     \[ H^*(G/H) \cong (\mathbb{R}[s]^{W_H} / \langle \rho^*(H,G)\sigma_j(x_1^2, \ldots, x_n^2) \rangle) \otimes \wedge(z_3, \ldots, z_{2n-1}) \]
   where \( \sigma_j \) are the elementary symmetric functions, the \( z_i \) are universal transgressive generators of \( H^*(G) \) corresponding to \( \sigma_j \) by transgression in the universal bundle for \( G \).

2. If \( G = \text{Spin}(2n + 2), \, n \geq 2, \) and \( \text{rk} \, H = n \), then
   \[ H^*(G/H) \cong (\mathbb{R}[s]^{W_H} / \langle \rho^*(H,G)\sigma_j(x_1^2, \ldots, x_n^2) \rangle) \otimes \wedge(z_{n+1}) \]
   where \( z_{n+1} \) is a universal transgressive generator of \( H^*(G) \) corresponding to \( x_1 \ldots x_n \).

3. If \( G = \text{Spin}(8) \) and \( \text{rk} \, H = 2 \), then
   \[ H^*(G/H) \cong (\mathbb{R}[s]^{W_H} / \langle \rho^*(H,G)\sigma_1(x_1^2, x_2^2, x_3^2) \sigma_2(x_1^2, x_2^2, x_3^2) \rangle) \otimes \wedge(z_2, z_4) \]
   where \( z_2, z_4 \) are universal transgressive generators of \( H^*(G) \) corresponding to \( \sigma_2 \) and \( x_1x_2x_3x_4 \) respectively.

4. If \( G = E_6 \), then
   \[ H^*(G/H) \cong (\mathbb{R}[s]^{W_H} / \langle \rho^*(H,E_6)I_2, \rho^*(H,E_6)I_6, \rho^*(H,E_6)I_8, \rho^*(H,E_6)I_{12} \rangle) \otimes \wedge(z_5, z_9) \]
   where \( I_2, I_6, I_8, I_{12} \) are the generators of the Weyl invariants given in [17], and \( z_5, z_9 \) are universal transgressive generators of \( H^*(E_6) \) corresponding to \( I_5, I_9 \).

The above theorem, together with Lemma 1, implies the following.

**Lemma 2.** For any two generalised symmetric spaces \( G/H \) and \( G/L \) of a simple compact Lie group \( G \) such that \( \text{rk} \, H = \text{rk} \, L \) and \( H \subset L \), the fibration \( G/H \to G/L \) with fiber \( L/H \) has the property that restriction to the fiber is a surjection in cohomology.
3. Formality in the sense of Sullivan

We will show in this section that for generalised symmetric spaces formality in the sense of Sullivan is an immediate consequence of their cohomology structure.

Recall that a differentiable manifold is said to be formal in the sense of Sullivan if its de Rham algebra of differential forms and its cohomology algebra endowed with the zero differential are weakly equivalent, meaning that they can be connected by a sequence of quasi-isomorphisms, compare \[16\] or \[7\].

In the mid-1970s it became clear \[12\] that the Cartan algebra of a homogeneous space contains more information on its topology than that given by Cartan’s theorem. More precisely, it turned out that the algebra of differential forms on a homogeneous space is weakly equivalent to its Cartan algebra. Thus, for homogeneous spaces formality is equivalent to formality of its Cartan algebra \[13\]. However, formality of the Cartan algebra can be described in terms of its cohomology:

**Theorem 6.** For the Cartan algebra \((C, d)\) of a compact homogeneous space \(G/H\) with \(\text{rk} G = n\) and \(\text{rk} H = k\) the following conditions are equivalent:

1. \(H^*(C) = H^*(BH)/\langle \rho^*(H, G)H^*(BG) \rangle \otimes (z_{k+1}, \ldots, z_n)\),
2. \((C, d)\) is formal.

A more general statement on the formality of Cartan algebras can be found in \[6, 13\] in the context of formal algebras.

**Remark 2.** Theorem \[6\] implies that a compact homogeneous space is formal if and only if it is of Cartan type. It follows immediately that all homogeneous spaces \(G/H\) with \(\text{rk} G = \text{rk} H\) are formal.

**Remark 3.** From the cohomology calculations in \[1\] and \[17\] described in subsection 2.2 above it follows that all symmetric spaces are of Cartan type. Together with Theorem \[6\] this shows that compact symmetric spaces are formal. This is usually proved by showing that they are geometrically formal, as in \[8\], but the cohomological proof seems to be closer in spirit to Sullivan’s theory of minimal models.

Combining Theorem \[6\] with Theorem \[4\] we obtain:

**Theorem 7.** All generalised symmetric spaces of simple compact Lie groups are formal in the sense of Sullivan.

**Remark 4.** Theorem \[7\] extends partial results due to Dumańska-Malyszko, Stepień and Tralle \[8\]. We were recently informed by A. Tralle that a different proof of Theorem \[7\] has been given by Z. Stepień \[15\].
4. The cohomology structure of flag manifolds

Our proof that the non-symmetric flag manifolds are not geometrically formal requires detailed, explicit information about the generators and relations of their cohomology rings. This section provides these details, starting from the theorems of Section 2. We originally obtained the formulas for the relations using computer calculations with Groebner basis algorithms. Having found the formulas, they are then easy to prove by elementary arguments not invoking Groebner bases.

The first case to consider is that of the classical flag manifolds $SU(n + 1)/T^n$. From Theorem 2 we have:

$$H^*(SU(n + 1)/T^n) \cong \mathbb{R}[x_0, \ldots, x_n]/\langle S^+(x_0, \ldots, x_n) \rangle,$$

where the $S^+(x_0, \ldots, x_n)$ are the symmetric functions of positive degrees.

Here is an explicit form:

**Proposition 1.** The classes represented by $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, $0 \leq \alpha_i \leq i$, $1 \leq i \leq n$ form a basis for the cohomology of $SU(n + 1)/T^n$ as a vector space.

Multiplicative relations between the $x_1, \ldots, x_n$ are given by:

$$\sum_{i_1 + \ldots + i_p = n-p+2} x_{n-p+1}^{i_p}x_{n-p+2}^{i_{p-1}}\ldots x_{n-1}^{i_2}x_1^{i_1} = 0, \quad 1 \leq p \leq n.$$

**Proof.** Define $s_k(x_0, \ldots, x_n)$ to be the sum of all the monomials in the $x_i$ which are homogeneous of degree $k$. This is clearly a symmetric polynomial. The relations (II) that we have to prove amount to

$$s_k(x_{k-1}, \ldots, x_n) = 0, \quad 2 \leq k \leq n + 1.$$

We shall prove more, namely that

$$s_m(x_{k-1}, \ldots, x_n) = 0, \quad 2 \leq k \leq n + 1,$$

holds for all $m \geq k$.

First we prove that $s_m(x_1, \ldots, x_n) = 0$ for all $m \geq 2$. We know

$$0 = s_m(x_0, \ldots, x_n) = x_0s_{m-1}(x_0, \ldots, x_n) + s_m(x_1, \ldots, x_n),$$

because all the symmetric functions in $x_0, \ldots, x_n$ vanish. Thus the vanishing of $s_{m-1}(x_0, \ldots, x_n)$ implies the vanishing of $s_m(x_1, \ldots, x_n)$.

We now prove (II) by induction on $k$. The case $k = 2$ is what we just proved. Suppose we have proved the statement up to $k$. Then consider

$$s_{m+1}(x_{k-1}, \ldots, x_n) = x_{k-1}s_m(x_{k-1}, \ldots, x_n) + s_{m+1}(x_k, \ldots, x_n).$$

As soon as $m \geq k$ both the left hand side and the first summand on the right hand side vanish by the induction hypothesis. Therefore the
Having proved the multiplicative relations, it remains to prove the statement about the vector space basis of the cohomology. This can be proved by induction on the degree. A vector space basis for $H^2$ is given by $x_1, \ldots, x_n$. Suppose we have proved the statement up to degree $2k$. Now $H^{2k+2}$ is linearly generated by all homogeneous monomials of degree $k + 1$ in the $x_1, \ldots, x_n$. However, by induction there are linear relations expressing $x_1^2$ as a linear combination of monomials containing $x_1$ at most linearly, expressing $x_2^3$ as a linear combination of monomials containing $x_2$ at most in the second power, and so on up to $x_{k-1}^k$. We also have a new relation in this degree, namely $s_{k+1}(x_k, \ldots, x_n) = 0$. This allows us to replace $x_{k+1}^k$ by a linear combination of monomials containing only smaller powers of $x_k$.

We now have eliminated all monomials not listed in (3). The remaining ones must be linearly independent because their number in each degree is seen to equal the respective Betti number by inspection of the Poincaré polynomial.

Next we consider the flag manifolds $\text{Spin}(2n+1)/T^n = \text{SO}(2n+1)/T^n$ and $\text{Sp}(n)/T^n$. Theorem 2 implies $H^\ast(\text{Spin}(2n+1)/T^n) = H^\ast(\text{Sp}(n)/T^n) \cong \mathbb{R}[x_1, \ldots, x_n]/\langle S^+(x_1^2, \ldots, x_n^2) \rangle$.

We can use the same argument as in the proof of the previous proposition to obtain:

**Proposition 2.** The classes represented by

$$x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \quad 0 \leq \alpha_i \leq 2i - 1, \quad 1 \leq i \leq n$$

form a vector space basis for the cohomology of $\text{Spin}(2n+1)/T^n$ and of $\text{Sp}(n)/T^n$. Multiplicative relations between the $x_1, \ldots, x_n$ are given by:

$$\sum_{i_1 + \cdots + i_p = n-p+1} x_{n-p+1}^{2i_p}x_{n-p+2}^{2i_{p-1}}\cdots x_{n-1}^{2i_1}x_n^{2i_1} = 0, \quad 1 \leq p \leq n.$$

Finally, we consider $\text{Spin}(2n)/T^n = \text{SO}(2n)/T^n$. In this case Theorem 2 gives:

$$H^\ast(\text{Spin}(2n)/T^n) \cong \mathbb{R}[x_1, \ldots, x_n]/\langle S^+(x_1^2, \ldots, x_n^2, x_1 \cdots x_n) \rangle.$$

More explicitly:

**Proposition 3.** A vector space basis for the cohomology of $\text{Spin}(2n)/T^n$ is given by

$$x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n},$$
with the coefficients $\alpha_i$ satisfying: $0 \leq \alpha_i \leq 2i - 1$ for $1 \leq i \leq n - 1$, $0 \leq \alpha_n \leq 2n - 2$ and $\alpha_i = 2i - 1$ implies $\alpha_{i+1}\ldots\alpha_n = 0$.

Multiplicative relations between the $x_1, \ldots, x_n$ are given by the formulas

\begin{align*}
(10) \quad & \sum_{i_1 + \ldots + i_p = n-p+1} x_{n-p+1}^{2i_p} x_{n-p+2}^{2i_{p-1}} \ldots x_{n-1}^{2i_2} x_n^{2i_1} = 0, \quad 1 \leq p \leq n, \\
(11) \quad & \sum_{i_1 + \ldots + i_p = n} x_{n-p+1}^{2i_p-1} x_{n-p+2}^{2i_{p-1}-1} \ldots x_{n-1}^{2i_2-1} x_n^{2i_1-1} = 0, \quad 1 \leq p \leq n.
\end{align*}

Proof. To prove the relations (10) we can proceed as in the proof of Propositions 1 and 2.

To prove the relations (11) we will proceed by backward induction on $p$. For $p = n$ the left hand side is $x_1 \ldots x_n$, which obviously vanishes. Now suppose we have proved the statement down to $p-1 \leq n$. Consider the relation

$$\sum_{i_1 + \ldots + i_{p+1} = n-p} x_{n-p}^{2i_{p+1}} x_{n-p+1}^{2i_{p}} \ldots x_{n-1}^{2i_2} x_n^{2i_1} = 0,$$

which was proved already. Multiplying it by $x_{n-p+1} \ldots x_n$ and splitting the resulting sum into two sums corresponding to the cases $i_{p+1} \neq 0$ and $i_{p+1} = 0$ we get

$$x_{n-p} \sum_{i_1 + \ldots + i_{p+1} = n-p} x_{n-p}^{2i_{p+1}-1} x_{n-p+1}^{2i_{p}+1} \ldots x_{n}^{2i_2+1} x_n^{2i_1+1}$$

$$+ \sum_{i_1 + \ldots + i_p = n-p} x_{n-p+1}^{2i_{p+1}} x_{n-p+2}^{2i_{p-1}+1} \ldots x_n^{2i_2+1} x_n^{2i_1+1} = 0.$$

The first sum vanishes by the induction hypothesis. Therefore the second sum vanishes, which is what we need to prove in the inductive step.

To prove the statement about the vector space basis of the cohomology we can use the same argument as in the proof of Proposition 1. □

Remark 5. We originally obtained the formulas discussed above using Groebner basis algorithms. It can be proved that the polynomials defining the relations (4), (8) and (10), (11) give Groebner bases for the ideals $(S^+(x_0, \ldots, x_n))$, $(S^+(x_1^2, \ldots, x_n^2))$ and $(S^+(x_2^2, \ldots, x_n^2), x_1 \ldots x_n)$ respectively. This also implies that the polynomials given by (3), (7) and (9) form vector space bases for the corresponding cohomology algebras.
5. Failure of geometric formality

In this section we prove that various generalised symmetric spaces \( G/H \) are not geometrically formal in the sense of [9], i.e. that they do not admit Riemannian metrics for which all products of harmonic forms are harmonic. Note that we do not assume that the metrics are \( G \)-invariant.

The simplest result, which however illustrates a main part of the argument for the flag manifolds as well, is the following theorem.

**Theorem 8.** The 6-symmetric space \( G_2/T^2 \) is not geometrically formal.

**Proof.** The real cohomology of \( X = G_2/T^2 \) was calculated by Borel [1]; we use the presentation in [2]. There are two linearly independent generators \( x \) and \( y \in H^2(X, \mathbb{R}) \), which satisfy the relations

\[
x^2 + 3xy + 3y^2 = 0 \tag{12}
\]

and

\[
x^6 = y^6 = 0 \tag{13}
\]

On the other hand, \( xy^5 \) generates the top-dimensional cohomology \( H^{12}(X, \mathbb{R}) \).

Suppose that \( X \) admits a formal Riemannian metric. By an obvious abuse of notation, we denote by \( x \) and \( y \) the harmonic representatives of the above cohomology classes. Then the above relations for \( x \) and \( y \) hold at the level of differential forms. In particular \( x \wedge y^5 \) is a volume form on \( X \).

On the other hand, it follows from (13) that both kernel distributions

\[
N_x = \{ v \in TM \mid i_v x = 0 \}
\]

and

\[
N_y = \{ w \in TM \mid i_w y = 0 \}
\]

have rank at least 2. Therefore, we can locally choose linearly independent vector fields \( v \in N_x \) and \( w \in N_y \). It follows from (12) that \( i_w x \wedge i_v y = 0 \). But this implies \( i_v i_w (x \wedge y^5) = 0 \), contradicting the fact that \( x \wedge y^5 \) is a volume form. \( \square \)

We now consider the flag manifolds.

**Theorem 9.** For all \( n \geq 2 \) the classical flag manifolds \( SU(n+1)/T^n \) are not geometrically formal.

By the results of [18], \( SU(n+1)/T^n \) is a generalised symmetric space of order \( n + 1 \). For \( n = 1 \) it is the symmetric space \( S^2 \), which is of course geometrically formal.
The proof of Theorem 9 uses the same idea as that of Theorem 8, together with induction over \( n \). To carry this out, we need the explicit relations from Proposition 4. First, we will prove the following lemma.

**Lemma 3.** Let \( M \) be a differentiable manifold of dimension \( n^2 + n \), with \( n \geq 2 \). Suppose there are \( n \) closed two-forms \( x_1, \ldots, x_n \) on \( M \) satisfying relations (4). Then \( x_1 \wedge x_2 \wedge \ldots \wedge x_n \) vanishes identically. In particular, it is not a volume form on \( M \).

**Proof.** Note that the first relation in (4), for \( p = 1 \), gives \( x_{n+1}^n = 0 \). Using this, the second relation, for \( p = 2 \), implies \( x_{n+1}^{n+1} = 0 \).

Assume that, under the assumptions of the lemma, \( x_1 \wedge x_2 \wedge \ldots \wedge x_n \) is a volume form on some open subset \( U \subset M \). Then \( x_{n+1}^n = 0 \), but \( x_n^n \) is not zero on \( U \). Thus \( x_n \) is of constant rank 2\( n \) in \( U \). By the same argument, \( x_{n-1} \) is of constant rank equal to 2\( n-2 \) or 2\( n \) in \( U \). The kernel distributions

\[
N_{x_n} = \{ w \in TM \mid i_w x_n = 0 \} \\
N_{x_{n-1}} = \{ v \in TM \mid i_v x_{n-1} = 0 \}
\]

are of ranks \( \text{rk}(N_{x_n}) = n^2 - n \) and \( \text{rk}(N_{x_{n-1}}) = n^2 - n + 2 \) or \( n^2 - n \). As \( x_n \) and \( x_{n-1} \) are closed, the Frobenius theorem implies that the kernel distributions are integrable.

We proceed by induction on \( n \). For \( n = 2 \) formula (4) gives the following relation between \( x_1 \) and \( x_2 \):

\[
x_1^2 + x_1 \wedge x_2 + x_2^2 = 0.
\]

From the above discussion, the ranks of \( N_{x_1} \) and \( N_{x_2} \) are \( \geq 2 \). Thus locally there are linearly independent vectors \( v \in N_{x_1} \) and \( w \in N_{x_2} \). From (4) it follows that \( i_w x_1 \wedge i_v x_2 = 0 \), which implies \( i_w i_v (x_1 \wedge x_2^2) = 0 \). This implies that \( x_1 \wedge x_2^2 \) can not be a volume form anywhere, and therefore vanishes identically.

Assume that the statement holds for \( n-1 \geq 2 \). Let us consider a manifold \( M \) of dimension \( n^2 + n \) and let \( x_1, \ldots, x_n \) be forms on \( M \) satisfying (4), such that \( x_1 \wedge x_2^2 \wedge \ldots \wedge x_n \) is a volume form on some open subset \( U \subset M \). Since \( N_{x_n} \) is integrable, it defines a foliation. Let \( N \) be a leaf of this foliation. Then \( N \) is of dimension \( n^2 - n = (n-1)^2 + (n-1) \), and the forms \( x_1, \ldots, x_{n-1} \) restricted to \( N \) satisfy relations (4) and \( x_1 \wedge x_2^2 \wedge \ldots \wedge x_{n-1}^{n-1} \) is a volume form on \( N \). This contradicts the induction hypothesis.

**Proof of Theorem 9.** Assume that the flag manifold \( SU(n+1)/T^n \) is geometrically formal, that is, there is a metric for which all products of harmonic forms are harmonic. If for the classes \( x_1, \ldots, x_n \) we choose
their harmonic representatives (denoted by the same letters), geometric formality implies that the relations \( (4) \) hold at the level of differential forms. The dimension of \( SU(n+1)/T^n \) is \( n^2 + n \), and from formula \( (3) \) we see that \( x_1 \wedge x_2^3 \wedge \ldots \wedge x_n^n \) is a volume form on \( SU(n+1)/T^n \). This gives a contradiction with the above lemma.

**Theorem 10.** For all \( n \geq 2 \) the flag manifolds \( \text{Spin}(2n+1)/T^n \) and \( \text{Sp}(n)/T^n \) are not geometrically formal.

By [18], these spaces are generalised symmetric of order \( 2n \). For \( n = 1 \) we again obtain the 2-sphere.

As before, we first prove a lemma about closed forms satisfying certain relations.

**Lemma 4.** Let \( M \) be a smooth manifold of dimension \( 2n^2 \), with \( n \geq 2 \). Suppose there are \( n \) closed two-forms \( x_1, \ldots, x_n \) on \( M \) satisfying the relations \( (8) \). Then \( x_1 \wedge x_2^3 \wedge \ldots \wedge x_n^{2n-1} \) vanishes identically. In particular, it is not a volume form on \( M \).

**Proof.** We proceed as in the proof of Lemma 3. The first relation in \( (8) \), for \( p = 1 \), gives \( x_n^n = 0 \). Using this, the second relation, for \( p = 2 \), implies \( x_{n-1}^{2n-1} = 0 \).

If we assume that \( x_1 \wedge x_2^3 \wedge \ldots \wedge x_n^{2n-1} \) is a volume form on some open subset \( U \subset M \), then in \( U \) we conclude that \( x_n \) is of constant rank \( 2(2n-1) \) and \( x_{n-1} \) is of constant rank equal to \( 2(2n-3) \), \( 2(2n-2) \) or \( 2(2n-1) \). So, the kernel distributions

\[
N_{x_n} = \{ w \in TM \mid i_w x_n = 0 \}
\]

\[
N_{x_{n-1}} = \{ v \in TM \mid i_v x_{n-1} = 0 \}
\]

are of ranks \( \text{rk}(N_{x_n}) = 2n^2 - 4n + 2 \) and \( \text{rk}(N_{x_{n-1}}) = 2n^2 - 4n + 6 \), \( 2n^2 - 4n + 4 \) or \( 2n^2 - 4n + 2 \), and are integrable.

We proceed by induction on \( n \). For \( n = 2 \) formula \( (8) \) gives the following relation between \( x_1 \) and \( x_2 \):

\[
x_1^2 + x_2^2 = 0.
\]

From the above discussion, the rank of \( N_{x_1} \) is \( \geq 2 \), thus locally there is a non-zero vector field \( v \in N_{x_1} \). From [13] it follows that \( x_2 \wedge i_v x_2 = 0 \), which implies \( i_v (x_1 \wedge x_2^3) = 0 \). This implies that \( x_1 \wedge x_2^3 \) can not be a volume form.

Assume that the statement holds for \( n - 1 \geq 2 \). Let us consider a manifold \( M \) of dimension \( 2n^2 \) and let \( x_1, \ldots, x_n \) be forms on \( M \) satisfying \( (8) \), such that \( x_1 \wedge x_2^3 \wedge \ldots \wedge x_n^{2n-1} \) is a volume form on \( M \). Since \( N_{x_n} \) is integrable, it defines a foliation. Let \( N \) be a leaf of this
foliation. Then $N$ is of dimension $2n^2 - 4n + 2 = 2(n - 1)^2$, and the forms $x_1, \ldots, x_{n-1}$ restricted to $N$ satisfy relations (8) and $x_1 \wedge x_2^3 \wedge \ldots \wedge x_n^{2n-3}$ is a volume form on $N$. This contradicts the induction hypothesis.

**Proof of Theorem 10.** Assume that $M = \text{Spin}(2n+1)/T^n$ or $\text{Sp}(n)/T^n$ is geometrically formal. If for the classes $x_1, \ldots, x_n$ we choose their harmonic representatives, geometric formality implies that the relations (8) hold at the level of differential forms. The dimension of $M$ is $2n^2$, and from formula (9) we see that $x_1 \wedge x_2^3 \wedge \ldots \wedge x_n^{2n-1}$ is a volume form on $M$. This contradicts the above lemma.

**Theorem 11.** For all $n \geq 4$ the flag manifolds $\text{Spin}(2n)/T^n$ are not geometrically formal.

By (18), $\text{Spin}(2n)/T^n$ is generalised symmetric of order $2n - 2$. For $n = 2$ we obtain the symmetric space $S^2 \times S^2$. For $n = 3$ we obtain $SU(4)/T^3$, which by Theorem 3 is not geometrically formal.

The proof of Theorem 11 is more complicated than that of the previous ones, because the cohomology algebra is more complicated. We shall first prove the following:

**Proposition 4.** Let $M$ be a smooth manifold of dimension $2n^2 - 2n$, with $n \geq 4$. Suppose there are $n$ closed two-forms $x_1, \ldots, x_n$ on $M$ satisfying relations (15) and (14). Then $x_2 \wedge x_3^3 \wedge \ldots \wedge x_n^{2n-2}$ vanishes identically. In particular, it is not a volume form on $M$.

Note that for $p = 1$ the relation (14) becomes $x_n^{2n-1} = 0$. Also from (14) the following relation can easily be obtained by backward induction on $k$:

$$x_k^{2k-1} x_{k+1}^{2k-1} x_{k+2}^{2k-1} \ldots x_{n-1}^{2n-5} x_n^{2n-3} = 0, \quad 2 \leq k \leq n - 1. \tag{16}$$

Put $M_{n+1} = M$ and recursively define $M_k$ to be a leaf of the kernel foliation of $x_k$ restricted to $M_{k+1}$, for all $2 \leq k \leq n$.

**Lemma 5.** Let $M$ be a smooth manifold of dimension $2n^2 - 2n$, with $n \geq 4$. Suppose there are $n$ closed 2-forms satisfying relations (15) such that $x_n^2 \wedge \ldots \wedge x_n^{2n-2}$ is a volume form on $M$. Then $x_k^{2k-1}$ vanishes identically on $M_{k+1}$ and $x_2^2 \wedge \ldots \wedge x_{k-1}^{2(k-1)-2}$ is a volume form on $M_k$.

**Proof.** We will proceed by backward induction on $k$. To prove this for $k = n$ note that $x_n^{2n-1} = 0$ on $M$ and the assumption that $x_2^2 \wedge \ldots \wedge x_{k-1}^{2n-2}$ is a volume form on $M$ implies that $x_n$ has constant rank equal to $4n - 4$, so, being a leaf of its kernel foliation, $M_n$ has dimension $2(n - 1)^2 - 2(n - 1)$ and $x_2^2 \wedge \ldots \wedge x_{k-1}^{2(n-1)-2}$ is a volume form on $M_n$. 

Assume that the lemma has been proved for all \( k + 1 \geq 4 \). Since \( x_2^2 \wedge \ldots \wedge x_{k-1}^{2k-2} \) is a volume form on \( M_{k+1} \) we conclude that \( \dim M_{k+2} - \dim M_{k+1} = 4k \), for all \( k + 1 \geq 4 \). As \( M_k \subset M_{k+1} \), denote by \( D_k \) a distribution complementary to \( TM_k \) in \( TM_{k+1} \). Relation (10) implies that the form \( x_k^{2k-1} \wedge x_{k+1}^{2k-1} \wedge \ldots \wedge x_n^{2n-3} \) vanishes identically on \( M \). If we evaluate this form on \( 2(2k - 1) \) vectors from \( TM_{k+1} \), \( 2(2k - 1) \) vectors from \( D_{k+1} \) on which \( x_k^{2k-1} \) does not vanish, \( 2(2k + 1) \) vectors form \( D_{k+2} \) on which \( x_k^{2k+1} \) does not vanish, and so on, and finally \( 2(2n - 3) \) vectors from \( D_n \) on which \( x_n^{2n-3} \) does not vanish, we conclude that \( x_k^{2k-1} \) vanishes identically on \( M_{k+1} \). Since \( \dim D_{k+1} = 4k \), for \( k + 1 \geq 4 \), the choice of such a vectors is always possible.

Since \( x_2^2 \wedge \ldots \wedge x_k^{2k-2} \) is a volume form on \( M_{k+1} \) it follows that \( x_k \) restricted to \( M_{k+1} \) has constant rank equal to \( 2k - 2 \). Thus, \( \dim M_k = 2(k-1)^2 - 2(k-1) \), so \( x_2^2 \wedge \ldots \wedge x_{k-1}^{2(k-1)-2} \) is a volume form on \( M_k \).

**Proof of Proposition 4.** Assume that under the conditions given in the proposition, \( x_2^2 \wedge x_3^4 \ldots \wedge x_n^{2n-2} \) is a volume form on \( M \). Then the above lemma implies that we have the following situation: a manifold \( M \) of dimension \( 2n^2 - 2n \) and \( n \) closed 2-forms satisfying relations (10) such that \( x_k \) restricted to \( M_{k+1} \) has constant rank equal to \( 2k - 2 \) and \( x_2^2 \wedge \ldots \wedge x_{k-1}^{2(k-1)-2} \) is a volume form on \( M_k \). Note that the forms \( x_2, \ldots, x_k \) satisfy the relations (10) on \( M_{k+1} \).

We prove by induction on \( n \) that this situation gives a contradiction. For \( n = 4 \) we have that \( x_2^2 \wedge x_3^4 \) is a volume on \( M_4 \) and (11) implies that on \( M_4 \) we have following relations:

\[
x_2^4 + x_2^2 \wedge x_3^2 + x_3^4 = 0, \quad x_3^6 = 0.
\]

As in the proofs of Lemmas 3 and 4 this gives a contradiction.

Let us assume that the statement holds for all \( n - 1 \geq 4 \). Consider a manifold \( M \) of dimension \( 2n^2 - 2n \) and assume we have 2-forms \( x_1, \ldots, x_n \) satisfying above conditions. Then, obviously, we have on \( M_n \) the same situation with \( n - 1 \) two-forms \( x_1, \ldots, x_{n-1} \) giving the contradiction.

**Proof of Theorem 4.** Assume that \( M = \text{Spin}(2n)/T^n, n \geq 4 \) is geometrically formal. If for the classes \( x_1, \ldots, x_n \) we choose their harmonic representatives (denoted by the same letters), geometric formality implies that the relations (10) and (11) hold at the level of differential forms. The dimension of \( M \) is \( 2n^2 - 2n \), and from formula (4) we see that \( x_2^2 \wedge x_3^4 \wedge \ldots \wedge x_n^{2n-2} \) is a volume form on \( M \). This contradicts Proposition 4. \( \square \)
So far we have only considered homogeneous spaces $G/H$ where $G$ and $H$ have equal rank. There is a generalisation of these arguments to some generalised symmetric spaces $G/H$ with $\text{rk } G > \text{rk } H$. The simplest case is the following:

**Theorem 12.** The 12-symmetric space $X = \text{Spin}(8)/T^2$ is not geometrically formal.

*Proof.* By Theorem 6 the cohomology algebra of $X$ is the tensor product of a polynomial algebra $P$, which is the cohomology algebra of $G_2/T^2$, and an exterior algebra $E$, which is the cohomology algebra of $S^7 \times S^7$.

The inclusions $T^2 \subset G_2 \subset \text{Spin}(8)$ induce a fibration $X \to Z = \text{Spin}(8)/G_2$ with fiber $Y = G_2/T^2$. As the base and the total space are generalised symmetric spaces, Lemma 2 implies that all cohomology classes on $Y$ are restrictions of classes on $X$.

We shall use the basis for $P$ used in the proof of Theorem 8. Assume that $X$ is geometrically formal and identify all the elements of the cohomology algebra with their harmonic representatives. Then the harmonic 2-forms $x$ and $y$ on $X$ satisfy $x^6 = 0 = y^6$, but $x^5 \neq 0 \neq y^5$, and therefore have kernels of rank $\dim(X) - 10 = 16$. As the codimension of the fiber $Y$ in $X$ is 14, it follows that the restrictions of $x$ and $y$ to $Y$ have kernels of rank at least 2 everywhere.

Thus at every point of a fiber we can find linearly independent local vector fields $v$ and $w$ contained in the kernels of $x$ and $y$ respectively. As the restrictions of $x$ and $y$ to $Y$ satisfy relation (12), we conclude $i_v i_w (x \wedge y^5) = 0$ as in the proof of Theorem 8. This shows that the restriction of $x \wedge y^5$ to $Y$ vanishes identically. This contradicts the fact that restriction to $Y$ is surjective in cohomology. \hfill \Box

Using the theorems about the flag manifolds in a similar way, we also obtain:

**Theorem 13.** The following generalised symmetric spaces are not geometrically formal:

1. $\text{SU}(2n + 1)/T^n$, for $n \geq 2$,
2. $\text{SU}(2n)/T^n$, for $n \geq 3$,
3. $\text{Spin}(2n + 2)/T^n$, for $n \geq 2$.

By [18], these are indeed generalised symmetric spaces of order $4n + 2$, $4n - 2$ and $2n + 2$ respectively. We could consider $n = 2$ in the second case, this would give $\text{SU}(4)/T^2$ which is the same as $\text{Spin}(6)/T^2$ contained in the third case.
Proof. To prove the first statement, let us consider the fibration \( SU(2n+1)/T^n \to SU(2n+1)/SO(2n+1) \) with fiber \( SO(2n+1)/T^n = \text{Spin}(2n+1)/T^n \). The base is a symmetric space, so Lemma 4 shows that the restriction to the fiber is surjective in cohomology. Theorem 5 implies that
\[
H^*(SU(2n+1)/T^n) \cong \mathbb{R}[x_1, \ldots, x_n]/\langle S^+(x_1^2, \ldots, x_n^2) \rangle \otimes (z_3, \ldots, z_{2n+1}) .
\]
Assume that \( SU(2n+1)/T^n \) is geometrically formal. For the cohomology classes \( x_1, \ldots, x_n \) we take their harmonic representatives with respect to a formal metric. Then the relations (8) hold for the harmonic forms, as forms. If we restrict these forms to the fiber \( \text{Spin}(2n+1)/T^n \), Lemma 4 implies that the form \( x_1 \wedge \ldots \wedge x_{2n-1}^2 \) vanishes. This contradicts the fact that the restriction is surjective in cohomology.

For the second case, we consider the fibration \( SU(2n)/T^n \to SU(2n)/Sp(n) \) with fiber \( Sp(n)/T^n \), where, as above, restriction to the fiber is surjective in cohomology. Again Theorem 5 implies
\[
H^*(SU(2n)/T^n) \cong \mathbb{R}[x_1, \ldots, x_n]/\langle S^+(x_1^2, \ldots, x_n^2) \rangle \otimes (z_3, \ldots, z_{2n-1})
\]
and, as in the first case, the assumption of geometric formality for \( SU(2n)/T^n \) contradicts the fact that the restriction to the fiber is surjective in cohomology.

For the third case, we have the fibration \( \text{Spin}(2n+2)/T^n \to \text{Spin}(2n+2)/\text{Spin}(2n+1) \) with fiber \( \text{Spin}(2n+1)/T^n \), and we can proceed as above. \( \square \)

Remark 6. Note that if \( X \) is the total space of a fibration with fiber \( Y \), there is no reason for the restrictions of harmonic forms on \( X \) to be harmonic on \( Y \).

Remark 7. It follows from the classification of generalised symmetric spaces \( G/H \) in \([13]\) that the only such spaces where \( H \) is a torus of rank \( \geq 2 \) and \( G \) is either \( G_2 \) or a simply connected classical simple group are the ones we considered in Theorems 8, 9, 10, 11, 12 and 13. The generalised symmetric spaces of the form \( SO(n)/T^k \) with \( k \geq 2 \) can be treated similarly, using the results of \([8, 9]\).

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