QCD and a Holographic Model of Hadrons

Joshua Erlich,1 Emanuel Katz,2 Dam T. Son,3 and Mikhail A. Stephanov4

1Department of Physics, College of William and Mary, Williamsburg, Virginia 23187-8795, USA
2Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309, USA
3Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA
4Department of Physics, University of Illinois, Chicago, Illinois 60607-7059, USA

Abstract

We propose a five-dimensional framework for modeling low-energy properties of QCD. In the simplest three parameter model we compute masses, decay rates and couplings of the lightest mesons. The model fits experimental data to within 10%. The framework is a holographic version of the QCD sum rules, motivated by the anti-de Sitter/conformal field theory (AdS/CFT) correspondence. The model naturally incorporates properties of QCD dictated by chiral symmetry, which we demonstrate by deriving the Gell-Mann–Oakes–Renner relationship for the pion mass.

Introduction.—QCD has eluded an analytic solution, despite extensive efforts applied to this problem in the past 30 years. Recently, the gravity/gauge, or anti-de Sitter/conformal field theory (AdS/CFT) correspondence, has revived the hope that QCD can be reformulated as a solvable string theory. So far, theories which can be solved using AdS/CFT techniques differ substantially from QCD, most notably by the strong coupling in the ultraviolet (UV) regime and the lack of asymptotic freedom. Nevertheless, certain important properties of QCD, such as confinement and chiral symmetry breaking, are present in many of these theories, and the gravity/gauge duality provides a new approach to studying the resulting dynamics. An important development in the prototypical example of \(N = 4\) super Yang-Mills (SYM) theory has been the introduction of fundamental quarks using probe D7 branes. The mesons that appear in these theories behave in many ways similarly to the mesons in QCD.

Inspired by the gravity/gauge duality we propose the following complementary approach. Rather than deform the SYM theory to obtain QCD, we start from QCD and attempt to construct its five-dimensional (5D) holographic dual. In this Letter, we present an exploratory study of a simple holographic model of QCD. The field content of the 5D theory is chosen to reproduce holigraphically the dynamics of chiral symmetry breaking in QCD, the boundary theory. The model has four free parameters, one of which is fixed by the number of colors; the remaining three parameters can be fitted using three well-measured observables, e.g., the p meson mass, the pion mass, and the pion decay constant. The model then predicts other low-energy hadronic observables with surprisingly good accuracy.

Such an approach is similar in spirit to the construction of the QCD moose theory in Ref. \(\text{[1]}\), where the holographic description arises in the continuum limit of infinitely many hidden local symmetries (see also Ref. \(\text{[2]}\)). As in Ref. \(\text{[3]}\), vector meson dominance and QCD sum rules are natural consequences of our model. Hence, the success of the model is not coincidental, but a result of linking several proven approaches through the AdS/CFT correspondence. We expect the success of our model to diminish above roughly the scale given by the mass of the lightest isospin-carrying spin-2 resonance, namely the \(a_2\) \((1318\text{ MeV})\). In particular, we are completely neglecting stringy physics which becomes important at higher energies, and we have not included in our description any modes with spin larger than 1. At this stage, we also neglect running of the QCD coupling, which is likely a poor approximation for a larger range of energies. While our model is too simple to provide a complete dual description of QCD, its success seems to suggest that there is a quantitatively useful reformulation of QCD as a string theory in a higher-dimensional curved space.

Field content.—Table \(\text{[1]}\) illustrates the field content of our model. The choice of the 5D fields is dictated by a principle of the AdS/CFT correspondence: each operator \(\mathcal{O}(x)\) in the 4D field theory corresponds to a field \(\phi(x,z)\) in the 5D bulk theory. The 5D theory dual to QCD should, therefore, contain an infinite number of fields corresponding to the infinite number of operators in QCD. There is, however, a small number of operators that are important in the chiral dynamics: the left- and right-handed currents corresponding to the SU\((N_f)_L\times SU(N_f)_R\) chiral flavor symmetry, and the chiral order parameter (see Table \(\text{[1]}\)). We shall include in our model only the 5D fields which correspond to these operators and neglect all other fields.

The 5D masses \(m_5\) of the fields \(A_L^\mu\), \(A_R^\mu\), and \(X\) are determined via the relation \(\Delta(p)(\Delta + p - 4) = m_5^2\), where \(\Delta\) is the dimension of the corresponding \(p\)-form operator—see Table \(\text{[1]}\). We assumed here that these operators keep their canonical dimensions, which is true

| TABLE I: Operators/fields of the model |
|--------------------------------------|
| 4D: \(\mathcal{O}(x)\) | 5D: \(\phi(x,z)\) | \(p\) | \(\Delta\) | \(m_5^2\) |
| \(q_L\gamma^\mu t^aq_L\) | \(A_L^\mu\) | 1 | 3 | 0 |
| \(q_R\gamma^\mu t^aq_R\) | \(A_R^\mu\) | 1 | 3 | 0 |
| \(\tau_L\bar{q}_L\) | \((2/z)X^{a\dot{a}}\) | 0 | 3 | -3 |

(Dated: January 2005)
The matrix $\Sigma$ is determined by the IR boundary condition on $X$. Instead of specifying this condition we shall choose $\Sigma$ as an input parameter of the model. The meaning of $\Sigma$ in QCD can be found by calculating the variation of the vacuum energy with respect to $M$ \cite{12}: $\Sigma^{\alpha\beta} = \langle \bar{q}^\alpha q^\beta \rangle$. We shall assume, as usual, $\Sigma = \sigma I$ and take $M = m_q I$.

At this stage the model has four free parameters: $m_q$, $\sigma$, $z_m$, and $g_5$. The gauge coupling $g_5$ will be fixed by the QCD operator product expansion (OPE) for the product of currents, leaving three adjustable parameters.

We will focus on the $N_f = 2$ lightest flavors and neglect effects of $O(m_q^2)$. Therefore, in Table I $\alpha, \beta = 1, 2$; $a, b = 1, 2, 3$ and $t^a = \sigma^a/2$, where $\sigma^a$ are the Pauli matrices.

**Matching the 5D gauge coupling.**—We will use the holographic duality to relate of the 5D coupling $g_5$ in (4) to the number of colors $N_c$ in QCD. The precise sense of the holographic correspondence is the equivalence between the generating functional of the connected correlators in the 4D theory $W_{4D}[\phi_0(x)]$ and the effective action of the 5D theory $S_{5D,\text{eff}}[\phi(x, \epsilon)]$, with UV boundary values of the 5D bulk fields set to the value of the sources in 4D theory:

$$W_{4D}[\phi_0(x)] = S_{5D,\text{eff}}[\phi(x, \epsilon)] \quad \text{at} \quad \phi(x, \epsilon) = \phi_0(x).$$

QCD Green’s functions can therefore be obtained by differentiating the 5D effective action with respect to the sources. In the case that stringy effects can be neglected, $S_{5D,\text{eff}}$ is simply given by Eq. (4). The action is evaluated on solutions to the 5D equations of motion subject to the condition that the value of each bulk field at the boundary $z = \epsilon \to 0$ be given by the source $\phi$ of the corresponding 4D operator $O$ (see Table I).

We may now fix the 5D gauge coupling by comparing the result for the vector current two-point function obtained from the above prescription with that of QCD. Introducing the vector field as \( V_{\mu} = (A_L + A_R)/2 \), one finds, in the \( V_\perp = 0 \) gauge, the equation of motion for the transverse part of the gauge field:

$$\left[ \partial_z \left( \frac{1}{z} \partial_z V_\mu^\alpha(q, z) \right) + \frac{q^2}{z} V_\mu^\alpha(q, z) \right] = 0. \quad (5)$$

Here \( V_\mu^\alpha(q, z) \) is the 4D Fourier transform of \( V_\mu^\alpha(x, z) \). The equations of motion are linearized, as is appropriate for determination of two-point functions. Evaluating the action on the solution leaves only the boundary term

$$S = -\frac{1}{2g_5^2} \int d^4x \left( \frac{1}{z} V_\mu^\alpha \partial_z V^{\mu\alpha} \right)_{z=\epsilon}. \quad (6)$$

If \( V_\mu^\alpha(q) \) is the Fourier transform of the source of the vector current \( J_\mu = \bar{q} \gamma_\mu t^a q \) at the boundary then letting \( V_\mu^\alpha(q, z) = V(q, z) V_\mu^\alpha(q) \), we require that \( V(q, \epsilon) = 1 \). Differentiating twice with respect to the source \( V_0 \), we
arrive at the vector current two-point function,

\[ \int e^{iqx} \langle J_\mu^a(x) J_\nu^b(0) \rangle = \delta^{ab} (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi_V(Q^2), \] (7a)

\[ \Pi_V(-q^2) = -\frac{1}{g_s^2 Q^2} \frac{\partial_z V(q, z)}{z} \bigg|_{z=\epsilon}, \] (7b)

where \( Q^2 = -q^2 \). For large Euclidean \( Q^2 \) we only need to know \( V(q, z) \) near the boundary,

\[ V(Q, z) = 1 + \frac{Q^2 z^2}{4} \ln(Q^2 z^2) + \ldots \] (8)

which up to contact terms gives

\[ \Pi_V(Q^2) = -\frac{1}{2g_s^2} \ln Q^2. \] (9)

On the other hand, we can compute \( \Pi_V \) from QCD by evaluating Feynman diagrams \[13\]. The leading-order diagram is the quark bubble,

\[ \Pi_V(Q^2) = -\frac{N_c}{24\pi^2} \ln Q^2. \] (10)

This leads to the identification

\[ g_s^2 = \frac{12\pi^2}{N_c}, \] (11)

which completes the definition of the action \[2\].

Hadrons.—The hadrons of QCD correspond to the normalizable modes of the 5D fields. These normalizable modes satisfy the linearized equation of motion and decay sufficiently rapidly near the boundary \( z \to 0 \) so as to have a finite action. The IR boundary condition gives rise to a discrete tower of normalizable modes. The eigenvalue of a normalizable mode is the squared mass of the corresponding meson, and the derivative of the mode near the UV boundary yields the decay constant.

To illustrate the above, consider the tower of the \( \rho \) mesons. A \( \rho \) wavefunction, \( \psi_\rho(z) \), is a solution to Eq. (15) for an arbitrary component of \( V_\mu \) with \( q^2 = m_\rho^2 \), subject to \( \psi_\rho(\epsilon) = 0, \partial_z \psi_\rho(z_m) = 0 \) and normalized as \( \int (dz/z) \psi_\rho(z)^2 = 1 \). Consider the Green's function corresponding to Eq. (16) for an arbitrary component of \( V_\mu \):

\[ G(q; z, z') = \sum_\rho \frac{\psi_\rho(z) \psi_\rho(z')}{q^2 - m_\rho^2 + i\epsilon}. \] (12)

(The \( i\epsilon \) prescription, among other things, guarantees the positivity of the spectral function, contrary to the claim of Ref. \[16\].) One can show that \( V(q, z') \) of Eq. (17) is given by \(-1/z\partial_z G(q; z, z')\) at \( z = \epsilon \). Now from \[16\] we find:

\[ \Pi_V(-q^2) = -\frac{1}{g_s^2} \sum_\rho \frac{[\psi_\rho'(\epsilon)/\epsilon]^2}{(q^2 - m_\rho^2 + i\epsilon) m_\rho^2}. \] (13)

This allows us to extract the decay constants \( F_\rho \):

\[ F_\rho^2 = \frac{1}{g_s^2} [\psi_\rho'(\epsilon)/\epsilon]^2 = \frac{1}{g_s^2} [\psi_\rho''(0)]^2, \] (14)

where \( F_\rho \) is defined by \( \langle 0| J_\rho^a | \rho \rangle = F_\rho \delta^{ab} \epsilon_\mu \) for a \( \rho \) meson with polarization \( \epsilon_\mu \). Eqs. \[15\] and \[16\] are the holographic version of the QCD sum rules.

In the axial sector (\( a_1 \) and \( \pi \) mesons), the action to quadratic order is

\[ S = \int d^5x \left[ -\frac{1}{4g_s^2} F_\mu^a F^a_\mu + \frac{v(z)^2}{2z^3} (\partial \xi^a - A^a_z)^2 \right]. \] (15)

where we have defined \( v(z) = m_q + z^3, A = (A_L - A_R)/2 \), and \( X = X_0 \exp(2\pi\alpha\mu) \). In the \( A_2 = 0 \) gauge, the resulting equations of motion in 4D momentum space are \((A_\mu = A_{\mu\perp} + \partial_\mu \varphi)\)

\[ \left[ \partial_z \left( \frac{1}{z} \partial_z A_\mu^a \right) + \frac{q^2}{z} A_\mu^a - \frac{g_s^2 v^2}{z^3} A_\mu^a \right] = 0; \] (16)

\[ \partial_z \left( \frac{1}{z} \partial_z \varphi^a \right) + \frac{g_s^2 v^2}{z^3} (\xi^a - \varphi^a) = 0; \] (17)

\[ -q^2 \partial_z \varphi^a + \frac{g_s^2 v^2}{z^3} \partial_z \xi^a = 0. \] (18)

The \( a_1 \), being a spin-1 particle, is the solution to Eq. (16) with \( \psi_{a_1}(0) = \partial_z \psi_{a_1}(z_m) = 0 \). The \( a_1 \) decay constant, \( F_{a_1} \), is given by an expression similar to Eq. (14), but with \( \rho \) replaced by \( a_1 \).

Our theory has all the consequences of chiral symmetry built in. Let us derive the Gell-Mann–Oakes–Renner (GOR) relation,

\[ m_\pi^2 f_\pi^2 = (m_u + m_d)(\bar{q}q) = 2m_q\sigma. \] (19)

Since \( \langle 0| A_\mu | \pi \rangle = if \pi \delta_\mu \), the axial current correlator in the \( m_\pi = 0 \) limit has a singularity at \( q^2 = 0 \): \( \Pi_A(-q^2) \to -f_\pi^2/q^2 \). Using the holographic recipe [cf. Eq. (17)],

\[ f_\pi^2 = -\frac{1}{g_s^2} \frac{\partial_z A_0(0, z)}{z} \bigg|_{z=\epsilon}, \] (20)

where \( A_0(0, z) \) is the solution to Eq. (16) with \( q^2 = 0 \), satisfying \( A'(0, z_m) = 0, A(0, \epsilon) = 1 \). The pion is the solution to Eqs. (17) and (18), subject to \( \varphi'(z_m) = \varphi(\epsilon) = \pi(\epsilon) = 0 \). We may construct such a solution perturbatively in \( m_\pi \) by letting \( \varphi(z) = A(0, z) - 1 \). Then, from Eq. (18), to leading order in \( m_\pi^2 \),

\[ \pi(z) = m_\pi^2 \int_0^z du \frac{u^4}{v(u)^2} \frac{1}{g_s^2} \partial_u A(0, u). \] (21)

The function \( u^3/v(u)^2 \) has a significant support only for \( u \sim z_\epsilon \equiv \sqrt{m_\pi/\sigma} \). The function \( \partial_u A(0, u)/(g_s^2 u) \) for such small values of \( u \) can be replaced by its value at \( u = \epsilon \), which is related to \( f_\pi \) via (20). Performing the integral
one finds that \( \pi = -m_p^2 f_\pi^2/(2m_q\sigma) \) for \( z \gg z_c \). Equations (16) and (17) are solved by \( \varphi = A(0, z) - 1 \) and \( \pi = \text{const} \) for \( z \gg z_c \) only if \( \pi = -1 \), hence \( m_p^2 f_\pi^2 = 2m_q\sigma + O(m_q^2) \).

Meson interactions and \( g_{\rho\pi\pi} \).—The meson interactions can be read from the nonlinear terms in the 5D action. For example, we find that the \( \pi-\rho \) coupling is given by

\[
g_{\rho\pi\pi} = g_5 \int dz \psi_\rho(z) \left( \frac{\phi'(z)^2}{g_5^2 z} + \frac{v(z)^2(\pi - \phi')^2}{z^3} \right). \tag{22} \]

The normalization of \( \pi \) is fixed by the pion kinetic term: integrating the function in parentheses in Eq. (22) gives 1. One must be aware that this 3-meson amplitude could be sensitive to the \( F^3 \) terms not yet included in our model.

**Predictions.**—From Eq. (4) and the Dirichlet boundary conditions, the \( \rho \) wavefunctions are Bessel functions with masses determined by zeroes of \( J_0(q z_m) \). Hence, \( m_\rho = 2.405/z_m = 776 \text{ MeV} \) fixes \( z_m = 1/(332 \text{ MeV}) \). \( m_q \) and \( \sigma \) can then be fit to the experimental values of \( m_\pi \) and \( f_\pi \), yielding \( m_q = 2.29 \text{ MeV} \) and \( \sigma = (327 \text{MeV})^3 \). These parameters correspond to Model A in Table II.

The rms error, \( \epsilon_{\text{rms}} = \left( \sum \delta O/O \right)^{1/2} \) (where \( \delta O/O \) is the fractional error of an observable \( O \) and \( n = 4 \) equals the number of observables minus the number of parameters) for Model A is 15%.

A global fit to all seven observables (Model B) yields the parameters, \( z_m = 1/(346 \text{ MeV}) \), \( m_q = 2.3 \text{ MeV} \) and \( \sigma = (308 \text{ MeV})^3 \). The last column of Table II lists the calculated observables in this model. The rms error of Model B is a remarkably small 9%.

**Discussion and outlook.**—The holographic model of QCD studied here is quite crude and depends on only three free parameters, but it agrees surprisingly well with the seven experimentally measured observables which we have studied. There are several ways in which we may attempt to extend and improve the model. (i) The glueball spectrum can be calculated from the gravitational and dilaton modes in the theory, which were not included in this study. (ii) It is straightforward to describe power corrections in the current correlators [18].

Here we matched the gauge coupling \( g_5 \) in our model to the leading term—the unit operator—in the OPE of the product of currents. Higher dimension operators also appear in the OPE, suppressed by powers of the Euclidean momentum \( Q \). These corrections can be calculated in QCD [13] but in the holographic model, these corrections arise from trilinear and higher terms in the 5D action, such as \( \int d^5x \sqrt{g} X^2 F^2 \). Matching the QCD OPE coefficients to the coefficients of the 5D action provides a method of building and constraining the effective 5D action.

(iii) Including the strange quark into the model with an approximate SU(3) \( \times \) SU(3) chiral symmetry is a natural extension of the model. (iv) The chiral anomaly can be incorporated via a 5D Chern-Simons term. (v) We can include corrections to the dimensions of the chiral order parameters by varying the mass of the corresponding fields \( X \) in the 5D theory, and we can include running of the gauge coupling via logarithmic corrections to the AdS geometry. It is interesting to note in this context, that those results which follow from partial conservation of the axial current, e.g., the GOR relation, continue to hold as we vary the 5D mass of \( X \) in the model [17].

This work is supported in part by DOE Grants DE-FG0201-ER41195 and DE-AC02-76SF00515. J.E. thanks the College of William and Mary for support. D.T.S. and M.A.S. thank A.P. Sloan Foundation for support.

| Observable | Measured (MeV) | Model A (MeV) | Model B (MeV) |
|------------|---------------|---------------|---------------|
| \( m_\pi \) | 139.6±0.0004 \[8\] | 139.5* | 141 |
| \( m_\rho \) | 775.8±0.5 \[8\] | 775.8* | 832 |
| \( m_{\Delta_1} \) | 1230±40 \[8\] | 1363 | 1220 |
| \( f_\pi \) | 92.4±0.35 \[8\] | 92.4* | 84.0 |
| \( F_{1/2}^1 \) | 345±8 \[15\] | 329 | 353 |
| \( F_{1/2}^2 \) | 433±13 \[6, 16\] | 486 | 440 |
| \( g_{\rho\pi\pi} \) | 6.03±0.07 \[8\] | 4.48 | 5.29 |

[1] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).
[2] A. Karch and E. Katz, JHEP 06 (2002) 043.
[3] M. Kruczenski et al., JHEP 07 (2003) 049.
[4] S. Hong, S. Yoon, and M. J. Strassler, JHEP 04 (2004) 046; hep-th/0409118, hep-th/0410080.
[5] C. Csaki et al., JHEP 01 (1999) 017; L. Girardello et al., Nucl. Phys. B 569, 451 (2000); J. Babington et al., Phys. Rev. D 69, 066007 (2004); M. Kruczenski et al., JHEP 05 (2004) 041; H. Boschﬁlho and N. R. F. Braga, JHEP 05 (2003) 009; G. F. de Teramond and S. J. Brodsky, Phys. Rev. Lett. 94, 201601 (2005).
[6] D. T. Son and M. A. Stephanov, Phys. Rev. D 69, 065020 (2004).
[7] M. Piai, A. Pierce, and J. Wacker, hep-ph/0405242.
[8] S. Eidelman et al. (PDG), Phys. Lett. B 592, 1 (2004).
[9] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B 428, 105 (1998).
[10] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998).
[11] I. R. Klebanov and M. J. Strassler, JHEP 08 (2000) 052.
[12] I. R. Klebanov and E. Witten, Nucl. Phys. B 556, 89 (1999).
[13] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B 147, 385 (1979).
[14] N. Mahajan, Phys. Lett. B 623, 119 (2005).
[15] J. F. Donoghue, E. Golowich, and B. R. Holstein, Dynamics of the Standard Model (Cambridge University Press, Cambridge 1992).
[16] N. Isgur, C. Morningstar, and C. Reader, Phys. Rev. D 39, 1357 (1989).
[17] J. Erlich, E. Katz, D. T. Son, and M. A. Stephanov (to be published).