ON BLOWING UP EXTREMAL KÄHLER MANIFOLDS

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Abstract. We show that the blowup of an extremal Kähler manifold at a relatively stable point in the sense of GIT admits an extremal metric in Kähler classes that make the exceptional divisor sufficiently small, extending a result of Arezzo-Pacard-Singer. We also study the K-polystability of these blowups, sharpening a result of Stoppa in this case. As an application we show that the blowup of a Kähler-Einstein manifold at a point admits a constant scalar curvature Kähler metric in classes that make the exceptional divisor small, if it is K-polystable with respect to these classes.

1. Introduction

Let \((M,\omega)\) be a compact Kähler manifold of dimension \(m\), such that \(\omega\) is an extremal metric in the sense of Calabi [6]. This means that the gradient of the scalar curvature of \(\omega\) is a holomorphic vector field, so important special cases are constant scalar curvature (cscK) metrics and Kähler-Einstein metrics. Following Arezzo-Pacard [2, 3] and Arezzo-Pacard-Singer [4] we study the problem of constructing extremal metrics on the blowup of \(M\) in one or more points, in Kähler classes which make the exceptional divisors sufficiently small. To state the result precisely, we make a few definitions. The condition that \(\omega\) is extremal implies that the Hamiltonian vector field \(X_s\) corresponding to the scalar curvature \(s(\omega)\) is a Killing field. Let \(G\) be the group of Hamiltonian isometries of \((M,\omega)\), and write \(\mathfrak{g}\) for its Lie algebra. We fix a moment map \(\mu : M \to \mathfrak{g}^\ast\) for the action of \(G\) on \(M\), such that for any vector field \(X \in \mathfrak{g}\) the function \(\langle \mu, X \rangle\) has zero mean on \(M\). We will also identify \(\mathfrak{g}\) with its dual \(\mathfrak{g}^\ast\) using the inner product

\[\langle X, Y \rangle = \int_M \langle \mu, X \rangle \langle \mu, Y \rangle \omega^n.\]

for \(X, Y \in \mathfrak{g}\), so we will think of elements in \(\mathfrak{g}^\ast\) as vector fields. Our first main result is then as follows.

**Theorem 1.** Choose distinct points \(p_1, \ldots, p_n \in M\) and numbers \(a_1, \ldots, a_n > 0\) such that the vector fields \(X_s\) and \(\sum_i a_i^{m-1} \mu(p_i)\) vanish at the \(p_i\). Then there exists \(\varepsilon_0 > 0\) such that for \(\varepsilon \in (0, \varepsilon_0)\) the blowup \(\text{Bl}_{p_1, \ldots, p_n} M\) admits an extremal metric in the Kähler class

\[\pi^\ast[\omega] - \varepsilon^2 (a_1[E_1] + \ldots + a_n[E_n]),\]

where \(E_i\) are the exceptional divisors and \(\pi\) is the blowdown map to \(M\).

To compare with the earlier results, we now describe the theorem proved by Arezzo-Pacard-Singer in [4]. As above \((M,\omega)\) is an extremal Kähler manifold. We choose \(K\) to be any group of Hamiltonian isometries of \((M,\omega)\) such that its Lie
algebra \( k \) contains the vector field \( X_s \). Now \( G \) is the group of Hamiltonian isometries commuting with \( K \), and \( g \) is its Lie algebra. We define \( g' = g \cap \mathfrak{k} \), and define \( g'' \) to be the orthogonal complement, so

\[
g = g' \oplus g''.
\]

Then the most general result in [4] is the following.

**Theorem 2** (Arezzo-Pacard-Singer). With notation as above, let \( p_1, \ldots, p_n \in M \) be points where each vector field in \( k \) vanishes. Suppose that

(i) **(Balancing condition)** we choose \( a_1, \ldots, a_n > 0 \) such that

\[
\sum_{j=1}^{n} a_j^{m-1} \mu(p_j) \in g'^{*},
\]

(ii) **(Genericity condition)** the projections of \( \mu(p_1), \ldots, \mu(p_n) \) onto \( g''^* \) span \( g''^* \),

(iii) **(General position condition)** there is no nontrivial element of \( g'' \) that vanishes at \( p_1, \ldots, p_n \).

Then there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) there is a \( K \)-invariant extremal \( K \)ähler metric on the blowup \( \text{Bl}_{p_1, \ldots, p_n} M \) whose \( \text{Kähler} \) class is

\[
[\omega] - \varepsilon^2 (a_1[E_1] + \ldots + a_n[E_n]),
\]

where the \( E_i \) are the exceptional divisors.

In addition condition (iii) can be removed if we allow losing control of the \( \text{Kähler} \) class a bit (see [4] for more details).

Note that since the vector field \( X_s \) and also any vector field in \( g' \) is contained in \( \mathfrak{k} \), the assumptions of Theorem 2 imply those of Theorem 1. In particular we do not need conditions (ii) and (iii). Although once the number of points blown up is large enough the conditions (ii) and (iii) are satisfied generically, it is clearly of interest to obtain results that work for fewer points. We also see that condition (i) seems to be weakened, but in fact if we choose \( K \) to be the largest possible fixing the points \( p_1, \ldots, p_n \) then condition (i) is equivalent to the vanishing of \( \sum_{i=1}^{n} a_i^{m-1} \mu(p_i) \) at the points \( p_i \).

The new ingredient in the proof of Theorem 1 is fairly simple so we describe it here briefly, focusing on the case of blowing up just one point. Starting with an extremal metric on \( M \), in [4] the authors try to directly construct an extremal metric on the blowup \( \text{Bl}_p M \) in suitable \( \text{Kähler} \) classes, whereas we try to solve a slightly more general equation instead. More precisely for suitably small \( \varepsilon \) we find a metric \( \omega_{p, \varepsilon} \) on \( \text{Bl}_p M \) in the \( \text{Kähler} \) class \([\omega] - \varepsilon^2 [E]\) together with a vector field \( h_{p, \varepsilon} \in \mathfrak{g} \) such that if the vector field \( h_{p, \varepsilon} \) vanishes at the point \( p \), then \( \omega_{p, \varepsilon} \) is an extremal metric. So the problem becomes to analyse when \( h_{p, \varepsilon} \) vanishes at \( p \), but this is a finite dimensional problem. Varying \( p \) we obtain a map

\[
h_\varepsilon : M \to \mathfrak{g}
\]

for small \( \varepsilon \), and the crucial point is that \( h_\varepsilon \) is a perturbation of the moment map \( \mu \). Then a perturbation argument shows that if \( \mu(p) \) vanishes at \( p \) then there is a point \( q \) in the same orbit of the complexified group \( G^C \) as \( p \), such that \( h_\varepsilon(q) \) vanishes at \( q \). This means that we have an extremal metric on the blowup \( \text{Bl}_p M \), but this is biholomorphic to the blowup \( \text{Bl}_q M \) and this concludes the proof. The actual proof will be slightly different, since for technical reasons we will work on a
suitable submanifold of $M$ instead of all of $M$. The idea of separating the problem in this way into an infinite dimensional problem that is easier to solve than the original, together with a finite dimensional “obstruction” problem is well known (see for example Hong [13, 14] for a similar technique used to construct constant scalar curvature metrics on ruled manifolds).

Recently there has been much work on relating the existence of extremal metrics to algebro-geometric conditions on the underlying complex manifold. This work is centered around the following conjecture.

**Conjecture 3** (Yau-Tian-Donaldson). Let $L$ be an ample line bundle over a compact complex manifold $M$. Then there exists an extremal metric in $c_1(L)$ if and only if the pair $(M, L)$ is relatively K-polystable.

This conjecture goes back to Yau [34] in the case of Kähler-Einstein metrics, and the first results are due to Tian [33]. Donaldson [9] extended the question to the case of constant scalar curvature metrics and proved the conjecture in the case of toric surfaces [11]. Two recent surveys on this topic are Thomas [32] and Phong-Sturm [26]. Note that it is essentially known that if an extremal metric exists then the manifold is relatively K-polystable (see Donaldson [10], Stoppa [28, 30], Mabuchi [21]), although this depends on the precise definition of stability being used.

A natural problem is to verify the conjecture for the type of blowups we are considering, building on existence results like Theorem 1 and 2. Let us suppose that $\omega$ is an extremal metric on $M$, and $\omega \in c_1(L)$ for some ample line bundle $L \to M$. For simplicity let us focus on the case of blowing up just one point. We want to characterize the points $p \in M$ for which the blowup $\text{Bl}_p M$ admits an extremal metric in the class $c_1(\pi^* L - \varepsilon E)$ for small rational $\varepsilon$. Note that some partial results on this question for cscK metrics on multiple blowups were obtained by Arezzo-Pacard [1].

The choice of moment map gives a lifting of the infinitesimal action of $G$ to the line bundle $L$. Replacing $L$ by a large power if necessary we can assume that we obtain a global action of $G$, and moreover we can extend this to an action of the complexified group $G^c$. In this case Theorem 1 can be reformulated as follows.

**Corollary 4.** Suppose $p \in M$ is such that $X_s$ vanishes at $p$, and $p$ is relatively stable for the action of $G^c$ on $M$ with respect to the polarization $L$. Then $\text{Bl}_p M$ admits an extremal metric in the class $c_1(\pi^* L - \varepsilon E)$ for sufficiently small $\varepsilon$.

In order to verify Conjecture 3 we need to relate the stability of $p \in M$ for the action of $G^c$, to relative K-polystability of $\text{Bl}_p M$ with respect to the class $\pi^* L - \varepsilon E$ for small $\varepsilon$. In this direction, Stoppa [29] and Della Vedova [8] showed that if $p$ is relatively strictly unstable for the action of $G^c$, then $\text{Bl}_p M$ is not relatively K-stable with the polarization $\pi^* L - \varepsilon E$ for small $\varepsilon$, and so it does not admit an extremal metric in these classes. The remaining question is what happens when $p$ is strictly semistable. We now focus on the cscK case, for which we have the following.

**Theorem 5.** Let $(M, L)$ be a polarized manifold and fix $p \in M$. Suppose that for all sufficiently small rational $\varepsilon > 0$, the blowup $\text{Bl}_p M$ is K-polystable with respect to the polarization $\pi^* L - \varepsilon E$. Then $p \in M$ is stable for the action of $G^c$ with respect to the linearization $L_\delta$ for all sufficiently small rational $\delta > 0$, where $L_\delta$ is the following, depending on the dimension:
• If \( m > 2 \) then \( L_\delta = L + \delta K_M \), where \( K_M \) is the canonical bundle.
• If \( m = 2 \) and \( K_X \cdot L \geq 0 \) then \( L_\delta = L + \delta K_M \).
• If \( m = 2 \) and \( K_X \cdot L < 0 \) then \( L_\delta = L - \delta K_M \).

In comparison, Stoppa’s result \([29]\) implies that if \( Bl_p M \) is K-polystable for the polarizations \( \pi^* L - \varepsilon E \) for sufficiently small \( \varepsilon \), then \( p \in M \) is semistable with respect to the linearization \( L \). But if \( p \in M \) is stable with respect to the linearization \( L + \varepsilon K_M \) for all sufficiently small \( \varepsilon > 0 \), then by letting \( \varepsilon \to 0 \) it follows easily that \( p \in M \) is semistable with respect to \( L \), so our result is slightly stronger. In fact we expect Theorem \([5]\) to be sharp at least when \( m > 2 \). In order to show this, we would need the following strengthening of Theorem \([1]\) in the cscK case, stated for the blowup in just one point.

**Conjecture 6.** Let \( \dim M = m > 2 \). Suppose that \( M \) admits a cscK metric in \( c_1(L) \), and let \( p \in M \). There exist \( \delta_0, \varepsilon_0 > 0 \) such that if \( \mu(p) + \delta \Delta \mu(p) = 0 \) for some \( \delta \in (0, \delta_0) \) then for all \( \varepsilon \in (0, \varepsilon_0) \) the manifold \( Bl_p M \) admits a cscK metric in the Kähler class \( c_1(\pi^* L - \varepsilon E) \). If \( m = 2 \) then we can ask for an analogous result to hold, just using \( \mu(p) \pm \delta \Delta \mu(p) \) with the sign in accordance with the signs in Theorem \([3]\).

Here \( \Delta \mu \) is the Laplacian of \( \mu \) taken componentwise after identifying \( g^* \) with \( R^l \) for some \( l \). Note that \( \mu + \delta \Delta \mu \) is a moment map for the action of \( G \) on \( M \) with respect to the Kähler form \( \omega - \delta \rho \), where \( \rho \) is the Ricci form of \( \omega \), so by the Kempf-Ness theorem we can find a zero of \( \mu + \delta \Delta \mu \) in the \( G^\ast \)-orbit of \( p \) if and only if \( p \) is stable with respect to the linearization \( L + \delta K_M \) (see Lemma \([28]\) in Section \([6]\) for this). So Theorem \([5]\) and Conjecture \([4]\) together imply that if \( M \) admits a cscK metric in \( c_1(L) \) and \( Bl_p M \) is K-polystable for the polarization \( \pi^* L - \varepsilon E \) for some sufficiently small \( \varepsilon \), then \( Bl_p M \) admits a cscK metric in \( c_1(\pi^* L - \varepsilon E) \). At the end of Section \([5]\) we indicate why we expect this conjecture to hold at least when \( m > 2 \).

There is one case when Conjecture \([6]\) follows directly from Theorem \([1]\) namely when \( (M, \omega) \) is Kähler-Einstein. The only interesting case is when \( M \) is Fano, since otherwise \( M \) does not admit Hamiltonian holomorphic vector fields, so the blowup at any point admits a cscK metric by the theorem in \([2]\). In the Fano case we can scale the metric so that \( \rho = \omega \), in which case \( \Delta \mu = -\mu \). The statement of the conjecture then reduces to that of Theorem \([1]\) so we obtain the following.

**Corollary 7.** Suppose that \( (M, \omega) \) is Fano and Kähler-Einstein, and let \( p \in M \). For sufficiently small rational \( \varepsilon > 0 \) the following are equivalent:

(i) The blowup \( Bl_p M \) admits a cscK metric in the class \( \pi^* [\omega] - \varepsilon [E] \).
(ii) \( Bl_p M \) is K-polystable with respect to the polarization \( \pi^* K_M^{-1} - \varepsilon E \), for test-configurations that are invariant under a maximal torus.
(iii) The point \( p \in M \) is GIT polystable for the action of the automorphism group of \( M \), with respect to the polarization \( K_M^{-1} \).

In the second statement we need to use test-configurations invariant under a maximal torus for the implication (i)⇒(ii) because in \([30]\) we were not yet able to prove full K-polystability assuming the existence of a cscK metric.

The outline of the paper is as follows. We first give the proof of the finite dimensional perturbation result in Section \([2]\) together with a related result in geometric invariant theory. In Section \([3]\) we discuss some background material on extremal metrics. Then in Section \([4]\) we set up the equation that we want to solve and give
the proof of Theorem [1] assuming that we can solve this equation. In Section [3] we solve the equation using a gluing method similar to [4], completing the proof of Theorem [1]. Our approach is slightly different than that of Arezzo-Pacard-Singer, but the technical ingredients are more or less the same. We could also have adapted the proof in [4] more directly to our slightly more general setting. Finally in Section [5] we discuss the algebro-geometric side of the problem, and we prove Theorem [6] and Corollary [7].

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2. Deforming relatively stable points in GIT

Let $U$ be a Kähler manifold (open, or compact without boundary) with Kähler form $\omega$, and suppose that a compact group $H$ acts on $U$, preserving $\omega$. Let $H^\C$ be the complexification of $H$, and $\mathfrak{h}$ be the Lie algebra of $H$. The action of $H$ extends to a partial action of $H^\C$, i.e. if $x \in U$ and $\xi \in \mathfrak{h}$ is sufficiently small then $e^{i\xi}x \in U$. For any $x \in U$ write $\mathfrak{h}_x$ for the stabilizer of $x$. Also, let

$$\mu : U \to \mathfrak{h}$$

be a moment map for the $H$ action on $U$, where we have identified $\mathfrak{h}$ with its dual using an invariant inner product.

Proposition 8. Suppose that $x \in U$ satisfies $\mu(x) \in \mathfrak{h}_x$. Let $\mu_\varepsilon : U \to \mathfrak{h}$ be a family of maps such that $\mu_\varepsilon \to \mu$ in $C^0$ as $\varepsilon \to 0$, and for each $y \in U$ and $\varepsilon > 0$ the element $\mu_\varepsilon(y)$ commutes with the stabilizer $\mathfrak{h}_y$. Then for sufficiently small $\varepsilon$ there exists $\xi \in \mathfrak{h}$ such that $y = e^{i\xi}x$ satisfies $\mu_\varepsilon(y) \in \mathfrak{h}_y$.

Proof. This is essentially an application of the implicit function theorem. Let us first treat the case when $\mu(x) = 0$. For every $y \in U$ write $\mathfrak{h}^\perp_y$ for the orthogonal complement of $\mathfrak{h}_y$, and let $\Pi_y$ be the orthogonal projection onto $\mathfrak{h}^\perp_y$. For $\xi$ in a small ball $B \subset \mathfrak{h}^\perp_x$ we define the projection $P_\xi : \mathfrak{h} \to \mathfrak{h}^\perp_x$ by

$$P_\xi(\eta) = \Pi_\eta \Pi_y \xi(\eta).$$

Then $P_\xi(\mu(e^{i\xi}x)) = 0$ means that $\mu(e^{i\xi}x) \in \mathfrak{h}^\perp_x$. For small $\varepsilon$ define the map $F_\varepsilon$ by

$$F_\varepsilon : B \subset \mathfrak{h}^\perp_x \to \mathfrak{h}^\perp_x \quad \xi \mapsto P_\xi(\mu_\varepsilon(e^{i\xi}x)).$$

Then $F_0(0) = 0$ and $DF_0$ at $0$ is just the derivative of $\mu$ at $x$ (we use here that $\mu(x) = 0$). Since this is an isomorphisms $\mathfrak{h}_x^\perp \to \mathfrak{h}_x^\perp$, we know that $F_0(B)$ contains a small ball $B_\delta$ around the origin. It then follows from degree considerations that for sufficiently small $\varepsilon$, the image $F_\varepsilon(B)$ contains a small ball $B_{\delta'}$ around the origin, in particular $F_\varepsilon(\xi) = 0$ for some $\xi$. But this means that $y = e^{i\xi}x$ satisfies $\mu(y) \in \mathfrak{h}_y$.

If $\mu(x) = \xi \neq 0$, then we reduce to the previous case as follows. Let $T \subset H$ be the closure of the subgroup of $H$ generated by $\xi$, so $T$ is a torus. Write $H_T$ for the centraliser of $T$ in $H$, and $\mathfrak{h}_T$ for its Lie algebra. Inside this Lie algebra let $\mathfrak{h}_{T^\perp}$ be the orthogonal complement of the Lie algebra of $T$, and let $H_{T^\perp}$ be the corresponding subgroup of $H$. We then look at the action of $H_{T^\perp}$ on $U$, for which the moment map $\mu_T$ is simply the projection of $\mu$ onto $\mathfrak{h}_{T^\perp}$. But then $\mu_T(x) = 0$,
so we can apply the previous argument to find \( \xi \in \mathfrak{h}_{T^*} \) such that \( y = e^{i \xi} x \) satisfies \( \mu_{t, \xi}(y) \in \mathfrak{h}_y \). Here \( \mu_{t, \xi} \) is the projection of \( \mu_t \) onto \( \mathfrak{h}_{T^*} \). Since \( \mu_t(y) \) commutes with elements in the stabiliser \( \mathfrak{h}_y \), in particular it commutes with \( t \). Hence \( \mu_{t, \xi}(y) \) differs from the projection \( \mu_{t, \xi}(y) \) by an element in \( t \), so \( \mu_{t, \xi}(y) \in \mathfrak{h}_y \). \( \square \)

It is helpful to put this result into the context of relative stability. In this setting \( U \) is a compact Kähler manifold and the symplectic form \( \omega \) represents the first Chern class of a \( \mathbb{Q} \)-line bundle \( L \) over \( U \). Moreover the choice of moment map \( \mu \) corresponds to a choice of lifting of the action to some power of \( L \), called a choice of linearization.

**Definition 9.** A point \( x \in U \) is relatively stable if there exists a point \( y \) in the \( H^* \)-orbit of \( x \) for which \( \mu(y) \in \mathfrak{h}_y \).

The relationship of this definition using moment maps to geometric invariant theory is a version of the Kempf-Ness theorem [16] and is worked out in [31] (see also Kirwan [17]). Using this terminology, Proposition 8 says that if \( x \) is relatively stable for a certain choice of line bundle and linearization then it is still relatively stable for small perturbations. It is more general however, because we do not need to know that the \( \mu_t \) are also moment maps.

In the rest of this section we study what more we can say about the stability of points in the sense of GIT as we deform the polarization. We will only consider a very simple kind of deformation, namely we have two line bundles \( L \) and \( K \) on \( M \), such that \( L \) is ample. We suppose that a complex reductive group \( G \) acts on \( M \) and we choose linearizations of the action on \( L \) and \( K \). Fix a point \( p \in M \), and let \( \lambda \) be a one-parameter subgroup in \( G \). Let
\[
q = \lim_{t \to 0} \lambda(t)p.
\]
Then \( \lambda \) fixes the point \( q \) and we write \( w_L(p, \lambda) \) for the weight of the action of \( \lambda \) on the fiber \( L_q \), and \( w_K(p, \lambda) \) for the weight on \( K_q \). The following is well-known, see Mumford-Fogarty-Kirwan [24].

**Proposition 10** (Hilbert-Mumford criterion). The point \( p \in M \) is semistable with respect to the polarization \( L \) if and only if \( w_L(p, \lambda) \geq 0 \) for all one-parameter subgroups \( \lambda \). If \( w_L(p, \lambda) > 0 \) for all \( \lambda \) which does not fix \( p \), then \( p \) is polystable.

The result we will need is the following version of this for the deformed polarization \( L + \varepsilon K \).

**Proposition 11.** There exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) the point \( p \) is polystable with respect to the polarization \( L + \varepsilon K \) if and only if \( p \) is semistable with respect to \( L \) and for every one-parameter subgroup \( \lambda \) for which \( w_L(p, \lambda) = 0 \) and \( \lambda \) does not fix \( p \), we have \( w_K(p, \lambda) > 0 \). In particular whether \( p \) is polystable with respect to \( L + \varepsilon K \) or not is independent of the choice of \( \varepsilon \in (0, \varepsilon_0) \).

**Proof.** First let us assume that \( K \) is ample. Choose a maximal torus \( T \subset G \) and let \( t \) be its Lie algebra. We will use Proposition 2.14 in [24]. This says that there are a finite number of rational linear functionals \( l_i, m_j \in t^* \), such that for every point \( x \in M \) and one-parameter subgroup \( \lambda \subset T \) we have
\[
w_L(x, \lambda) = \max \{ l_i(\lambda) \mid i \in I(x) \},
w_K(x, \lambda) = \max \{ m_j(\lambda) \mid j \in J(x) \}.
\]
Here we identified $\lambda$ with its generator in $t$ and $I(x)$, $J(x)$ are finite index sets depending on $x$.

If we have a one-parameter subgroup $\lambda \subset G$ which is not in $T$, then we can always find a conjugate $\gamma \lambda \gamma^{-1} \subset T$, and we have

$$w_L(x, \lambda) = w_L(\gamma x, \gamma \lambda \gamma^{-1}) = \max \{ l_i(\gamma \lambda \gamma^{-1}) | i \in I(\gamma x) \},$$

$$w_K(x, \lambda) = w_K(\gamma x, \gamma \lambda \gamma^{-1}) = \max \{ m_j(\gamma \lambda \gamma^{-1}) | j \in J(\gamma x) \}.$$  

We want to show that for sufficiently small $\varepsilon$, the weight $w_L(p, \lambda) + \varepsilon w_K(p, \lambda)$ is positive for all $\lambda$ which does not fix $p$. It is enough to check this for $\lambda \subset T$, because allowing conjugate one-parameter subgroups $\gamma \lambda \gamma^{-1}$ amounts to replacing $p$ by $\gamma p$, but in the weight computation all that matters is the index sets $I(\gamma p)$ and $J(\gamma p)$. Since there are only finitely many of these, if we find an $\varepsilon$ that works for each case separately, then we can take the minimum of these.

Restricting attention to one-parameter subgroups $\lambda \subset T$, we can extend the definition of $w_L(x, \lambda)$ continuously to $\lambda \in \mathfrak{t}_{\mathbb{R}}$. By taking a smaller torus we can also assume that no element in $T$ fixes $p$. The main point is that then the set of $\lambda$ for which $w_L(p, \lambda) = 0$ is a convex cone $C \subset \mathfrak{t}_{\mathbb{R}}$, whose extremal rays are rational. By our assumption $w_K(p, \lambda) > 0$ for $\lambda \in C \cap \mathfrak{t}_{\mathbb{Q}}$, but then this is true for all $\lambda \in C$ because of the rationality of the $\gamma_{m_j}$. Let us write $\partial B \subset t$ for the unit sphere with respect to some fixed norm. We then have $w_K(p, \lambda) > 0$ on $C \cap \partial B$, but the latter is compact, so there exists an open neighbourhood $U \subset \partial B$ of $C \cap \partial B$ for which

$$w_K(p, \lambda) > 0 \quad \text{for } \lambda \in U.$$  

At the same time $|w_K(p, \lambda)| < C$ for some constant $C$ and all $\lambda \in \partial B$. In addition there exists $\delta > 0$ such that $w_L(p, \lambda) > \delta$ for $\lambda \not\in U$. Finally it follows that if $\varepsilon < \delta/C$, then

$$w_L(p, \lambda) + \varepsilon w_K(p, \lambda) > 0$$  

for all $\lambda$.

For the converse direction, we note that $|w_K(p, \lambda)| < C$ for some constant $C$ and all $\lambda \in \partial B$, with $C$ independent of $p$. If $p$ is not semistable with respect to $L$, then there is a one-parameter subgroup $\lambda$ for which $w_L(p, \lambda) = -\delta < 0$. But then for all $\varepsilon < \delta/(C|\lambda|)$ we have

$$w_L(p, \lambda) + \varepsilon w_K(p, \lambda) < -\delta + C \varepsilon |\lambda| < 0,$$

so $p$ is not stable with respect to $L + \varepsilon K$ for any sufficiently small $\varepsilon$. The fact that we need $w_K(p, \lambda) > 0$ for all $\lambda$ such that $w_L(p, \lambda) = 0$ is immediate.

Now suppose that $K$ is not ample. We can choose a large constant $c$ such that $cL + K$ is ample. Then note that for small $\varepsilon$

$$L + \varepsilon K = (1 - \varepsilon c) \left( L + \frac{\varepsilon}{1 - \varepsilon c} (cL + K) \right),$$

so if $p$ is polystable with respect to $L + \varepsilon K$ then is is also polystable with respect to $L + \frac{\varepsilon}{1 - \varepsilon c} (cL + K)$, where $cL + K$ is ample. If $\varepsilon$ is sufficiently small, then we can apply what we just proved. So $p$ is polystable with respect to $L + \varepsilon K$ if and only if $p$ is semistable with respect to $L$ and for every one-parameter subgroup $\lambda$ for which $w_L(p, \lambda) = 0$ and $\lambda$ does not fix $p$, we have $w_K(p, \lambda) = w_{cL + K}(p, \lambda) > 0$. \hfill \Box

### 3. Background on extremal metrics

In this section we collect some material which we will need later on.
3.1. The extremal metric equation. As we said before, the basic strategy to obtain an extremal metric on a blowup $\text{Bl}_p M$ is to first use the extremal metric $\omega$ on $M$ and a simple gluing argument to obtain an approximately extremal metric $\omega_\varepsilon$ on $\text{Bl}_p M$ and then to try perturbing this to an extremal metric. When we set up the problem more precisely later we will have a maximal torus of automorphisms $T$ acting on $\text{Bl}_p M$, preserving the approximate solution $\omega_\varepsilon$ and we will seek an extremal metric of the form

$$\omega_\varepsilon + i \partial \bar{\partial} \varphi,$$

where $\varphi$ is $T$-invariant. Let us write $\mathfrak{t} \subset C^\infty(\text{Bl}_p M)$ for the space of Hamiltonian functions generating elements of $T$, which includes the constants. Note that $\dim \mathfrak{t} = \dim \mathfrak{t} + 1$ where $\mathfrak{t}$ is the Lie algebra of $T$, and $\mathfrak{t}$ consists of the smooth $T$-invariant functions in the kernel of the Lichnerowicz operator on $\text{Bl}_p M$, defined in Section 3.2. We need the following which can also be found in [4].

**Lemma 12.** Suppose that $\varphi \in C^\infty(\text{Bl}_p M)^T$ and $f \in \mathfrak{t}$ such that

$$s(\omega_\varepsilon + i \partial \bar{\partial} \varphi) - \frac{1}{2} \nabla f \cdot \nabla \varphi = f,$$

where the gradient and inner product are computed with respect to the metric $\omega_\varepsilon$. Then $\omega_\varepsilon + i \partial \bar{\partial} \varphi$ is an extremal metric.

**Proof.** Let $X$ be the holomorphic vector field on $\text{Bl}_p M$ with Hamiltonian function $f$, ie.

$$df = \iota_X \omega_\varepsilon.$$

At the same time we can compute

$$\iota_X (i \partial \bar{\partial} \varphi) = \frac{1}{2} d(JX(\varphi)).$$

Since $JX = \nabla f$, by combining the previous two formulae we get

$$\iota_X (\omega_\varepsilon + i \partial \bar{\partial} \varphi) = d \left(f + \frac{1}{2} \nabla f \cdot \nabla \varphi\right) = ds(\omega_\varepsilon + i \partial \bar{\partial} \varphi).$$

This means that $\omega_\varepsilon + i \partial \bar{\partial} \varphi$ is an extremal metric. □

In order to solve Equation (1) as a perturbation problem, we will write it in the form

$$s(\omega_\varepsilon + i \partial \bar{\partial} \varphi) - \frac{1}{2} \nabla (\hat{s} + \hat{f}) \cdot \nabla \varphi = \hat{s} + \hat{f},$$

where $\hat{s}, \hat{f} \in \mathfrak{t}$, and $\hat{s}$ is chosen so that the holomorphic vector field $\nabla \hat{s}$ is the natural holomorphic lift of the vector field $\nabla s(\omega)$ on $M$. In addition we can normalise $\hat{s}$ so that it agrees with $s(\omega)$ outside a small ball around $p$, where the metrics $\omega$ and $\omega_\varepsilon$ coincide. The advantage of this is that we now seek $\varphi$ and $\hat{f}$ which are small, or in other words, setting $\varphi = 0$ and $\hat{f} = 0$ we get an approximate solution to the equation.

For any metric $\tilde{\omega}$ let us define the operators $L_{\tilde{\omega}}$ and $Q_{\tilde{\omega}}$ by

$$s(\tilde{\omega} + i \partial \bar{\partial} \varphi) = s(\tilde{\omega}) + L_{\tilde{\omega}}(\varphi) + Q_{\tilde{\omega}}(\varphi),$$

where $L$ is the linearized operator. A simple computation shows that

$$L_{\tilde{\omega}}(\varphi) = \Delta_{\tilde{\omega}}^2 \varphi + \text{Ric}(\tilde{\omega})^{ij} \varphi_{ij},$$

where $\Delta_{\tilde{\omega}}$ is the Laplacian with respect to the metric $\tilde{\omega}$.
and analysing this operator will be crucial later on. We are using the complex Laplacian here which is half of the usual Riemannian one.

At the same time note that the linear operator appearing in the linearization of Equation (2) is

\[(\phi, \tilde{f}) \mapsto L_{\omega_{\epsilon}}(\phi) - \frac{1}{2} \nabla \tilde{s} \cdot \nabla \phi - \tilde{f},\]

which is closely related to the Lichnerowicz operator.

3.2. The Lichnerowicz operator. For any Kähler metric \(\tilde{\omega}\) on a manifold \(X\) we have the operator

\[D_{\tilde{\omega}} : C^\infty(X) \rightarrow \Omega^{0,1}(T^{1,0}X),\]

given by \(D(\phi) = \bar{\partial}^* \nabla^1 \phi\) where \(\bar{\partial}\) is the natural \(\bar{\partial}\)-operator on the holomorphic tangent bundle. The Lichnerowicz operator is then the fourth order operator

\[D_{\tilde{\omega}}^* D_{\tilde{\omega}} : C^\infty(X) \rightarrow C^\infty(X),\]

whose significance is that the kernel consists of precisely those functions whose gradients are holomorphic vector fields. The relation to the operator in Equation (4) is that a computation (see eg. LeBrun-Simanca [19]) shows that

\[(5)\]

\[D_{\tilde{\omega}}^* D_{\tilde{\omega}}(\phi) = L_{\tilde{\omega}}(\phi) - \frac{1}{2} \nabla s(\tilde{\omega}) \cdot \nabla \phi,\]

but note that in general \(s(\tilde{\omega})\) is not equal to \(\tilde{s}\).

3.3. Burns-Simanca metric. The approximate metric \(\omega_{\epsilon}\) on \(\text{Bl}_p M\) is constructed by gluing the extremal metric \(\omega\) on \(M\) to a rescaling of a suitable model metric on \(\text{Bl}_0 C^m\), i.e. on the blowup of \(C^m\) at the origin. This model metric is a scalar flat metric found by Burns (see LeBrun [18]) for \(m = 2\) and by Simanca [27] for \(m \geq 3\).

Away from the exceptional divisor it can be written in the form

\[\eta = \frac{i}{2} \partial \bar{\partial} \left( \frac{1}{2} |z|^2 + \psi(z) \right),\]

where \(z = (z_1, \ldots, z_m)\) are standard coordinates on \(C^m\). For \(m = 2\) we have \(\psi(z) = \log |z|\) while for \(m > 2\) we have

\[\psi(z) = -|z|^{4-2m} + O(|z|^{3-2m})\]

for large \(|z|\). The quantity \(O(|z|^{3-2m})\) is a function in the space \(C^{k,\alpha}_{3-2m}(Bl_0 C^m)\) in the notation of section 5 for any \(k\) and \(\alpha \in (0,1)\). See Lemma 24 for a sharper asymptotic expansion.

4. THE MAIN ARGUMENT

Suppose as before that \(\omega\) is an extremal Kähler metric on \(M\). Let \(X_\alpha\) be the Hamiltonian vector field corresponding to the scalar curvature \(s(\omega)\). Write \(G\) for the Hamiltonian isometry group of \((M, \omega)\), so the Lie algebra \(\mathfrak{g}\) of \(G\) consists of holomorphic Killing fields with zeros.

Choose a point \(p \in M\) where the vector field \(X_\alpha\) vanishes, and let \(T \subset G\) be a maximal torus of the subgroup fixing \(p\). Let \(H \subset G\) consist of the elements commuting with \(T\) and let us write \(\mathfrak{h} \subset C^\infty(M)\) for the space of Hamiltonian functions of vector fields in the Lie algebra of \(H\). Note that \(\mathfrak{h}\) contains the constants as well. Let us also write \(\mathfrak{t} \subset \mathfrak{h}\) for the Hamiltonian functions corresponding to the subgroup \(T \subset H\).
Given a small parameter $\varepsilon > 0$, we will construct an approximate solution to our problem on $B_p M$ in the Kähler class $[\omega] - \varepsilon^2 d_m[E]$, for some constant $d_m$ depending on the dimension, so that $d_m^{n-1}$ is the volume of the exceptional divisor of $Bl_0 \mathbb{C}^m$ with the Burns-Simanca metric $\eta$ from Section 3.3. Of course we could make $d_m = 1$ by rescaling $\eta$. For simplicity assume that the exponential map is defined on the unit ball in the tangent space $T_p M$ (if not, we can scale up the metric $\omega$). Choose local normal coordinates $z$ near $p$ such that the group $T$ acts by unitary transformations on the unit ball $B_1$ around $p$ (this is possible by linearizing the action, see Bochner-Martin [5] Theorem 8). In these coordinates we can write

$$\omega = i\partial \bar{\partial}(|z|^2/2 + \varphi(z)),$$

where $\varphi = O(|z|^4)$. At the same time recall the Burns-Simanca metric

$$\eta = i\partial \bar{\partial}(|z|^2/2 + \psi(z)).$$

We glue $\varepsilon^2 \eta$ to $\omega$ using a cutoff function in the annulus $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$ in $M$, where the dependence of $r_\varepsilon$ on $\varepsilon$ will be chosen later. To do this, let $\gamma : \mathbb{R} \to [0, 1]$ be smooth such that $\gamma(x) = 0$ for $x < 1$ and $\gamma(x) = 1$ for $x > 2$ and then define

$$\gamma_1(r) = \gamma(r/r_\varepsilon),$$

and write $\gamma_2 = 1 - \gamma_1$. Then we can define a Kähler metric $\omega_\varepsilon$ on $B_p M$ which on the annulus $B_1 \setminus B_{r_\varepsilon}$ is given by

$$\omega_\varepsilon = i\partial \bar{\partial} \left( \frac{|z|^2}{2} + \gamma_1(|z|)\varphi(z) + \gamma_2(|z|)\varepsilon^2 \psi(\varepsilon^{-1}z) \right).$$

Moreover outside $B_{2r_\varepsilon}$ the metric $\omega_\varepsilon = \omega$ while inside the ball $B_{r_\varepsilon}$ we have $\omega_\varepsilon = \varepsilon^2 \eta$. Note that the action of $T$ lifts to $B_p M$ giving biholomorphisms, and that $\omega_\varepsilon$ is $T$-invariant.

It will be important to lift functions in $\mathcal{F}$ to $B_p M$. Only elements in $\mathcal{F}$ have a natural lifting, since they correspond to holomorphic vector fields vanishing at $p$, so we give the following definition.

**Definition 13.** We define a linear map

$$1 : \mathcal{H} \to C^\infty(B_p M)$$

as follows. First let us decompose $\mathcal{H}$ into a direct sum $\mathcal{H} = \mathcal{F} \oplus \mathcal{H}'$, where we can assume that each function in $\mathcal{H}'$ vanishes at $p$. Any $f \in \mathcal{F}$ corresponds to a holomorphic vector field $X_f$ on $M$ vanishing at $p$. For such $f$ we define $1(f)$ to be the Hamiltonian function of the holomorphic lift of the vector field $X_f$ to $B_p M$, with respect to the symplectic form $\omega_\varepsilon$, normalized so that $f = 1(f)$ outside $B_1$. For $f \in \mathcal{H}'$ we simply let $1(f) = \gamma_1 f$ near $p$ using the cutoff function $\gamma_1$ from before, and we think of this $1(f)$ as a function on $B_p M$. Finally define the lift of general elements in $\mathcal{H}$ by linearity.

We can now state the main technical result we need, whose proof will be given in Section 5.

**Proposition 14.** Suppose that the point $p \in M$ is chosen so that the vector field $X_\varepsilon$ vanishes at $p$. Then there are constants $\varepsilon_0, c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we can find $u \in C^\infty(B_p M)^T$ and $f \in \mathcal{F}$ satisfying the equation

$$s(\omega_\varepsilon + i\partial \bar{\partial}u) - \frac{1}{2} \nabla 1(f) \cdot \nabla u = 1(f).$$


In addition the element \( f \in \mathfrak{h} \) has an expansion
\[
(7) \quad f = s + \varepsilon^{2m-2}(\lambda + c_m \mu(p)) + f_\varepsilon,
\]
where \( c_m \) is a constant depending only on the dimension, \( \lambda = \text{Vol}(M)^{-1}c_m \) is another constant, and \( |f_\varepsilon| \leq \varepsilon^\kappa \) for some \( \kappa > 2m - 2 \).

Note that in this proposition, \( I(f) \) corresponds to a Hamiltonian vector field \( X_{I(f)} \) on \( Bl_p M \), and if this vector field is holomorphic, then the metric \( \omega + i\partial\overline{\partial} u \) above is extremal by Lemma \( \text{[12]} \). Moreover \( X_{I(f)} \) is holomorphic if and only if \( f \in \mathfrak{h} \), i.e. if the vector field \( X_f \) on \( M \) vanishes at \( p \). Given this proposition we can now prove Theorem \( \text{[1]} \).

**Proof of Theorem \( \text{[1]} \).** We will give the proof for the blowup of one point to simplify the notation, since blowing up several points does not give rise to essential new difficulties. Let us use the notation from before, so that \( G \) is the Hamiltonian isometry group of \( (M, \omega) \), \( p \in M \) and \( T \) is a maximal torus in the stabilizer of \( p \).

The subgroup \( H \subset G \) consists of the elements commuting with each element of \( T \), and let \( \mathfrak{h}, \mathfrak{t} \) be the Lie algebras of \( H, T \). Note that in this case \( \mathfrak{h}_p = \mathfrak{t} \), where \( \mathfrak{h}_p \) is the stabilizer of \( p \) in \( \mathfrak{h} \).

We will work on the \( H^c \)-orbit of \( p \), so let us write \( U = H^c \cdot p \). Then \( U \subset M \) is an \( H \)-invariant complex submanifold. If \( \mu(p) \in \mathfrak{h}_p \), then the stabilizer of \( p \) in \( H^c \) is \( T^c \). This can be seen using the structure of the stabilizer group of relatively stable points (analogous to Calabi’s structure theorem for the automorphism groups of extremal metrics \([11]\)). Since every element in \( H^c \) commutes with \( T \), it follows that for every \( q \in U \) the stabilizer of \( q \) in \( H \) is \( T \). We can therefore apply Proposition \( \text{[13]} \) to each point \( q \in U \) with \( T \) as the maximal torus. We can first replace \( U \) by a relatively compact complex submanifold \( U' \subset U \) which is still \( H \)-invariant and contains \( p \), to ensure that we can choose \( \varepsilon, \epsilon \) in the proposition uniformly over \( U' \). Note that the solution of the equation in Proposition \( \text{[13]} \) is obtained using the contraction mapping theorem, so although the solution is not unique, the various choices can be made so that it depends smoothly on the data.

For a suitably small \( \varepsilon \) we therefore have a smooth map
\[
\mu_\varepsilon : U' \to \mathfrak{h}
\]
\[
\mu_\varepsilon(q) = \mu(q) + t^{-1} \varepsilon^{2-2m} f_\varepsilon,
\]
where \( f_\varepsilon \in \mathfrak{h} \) is given by the Proposition \( \text{[13]} \) applied at the point \( q \) (so \( f_\varepsilon \) depends on \( q \)). Since \( |f_\varepsilon| \leq \varepsilon^\kappa \) for some \( \kappa > 2m - 2 \), it follows that
\[
\lim_{\varepsilon \to 0} \mu_\varepsilon = \mu.
\]

Applying Proposition \( \text{[8]} \) we see that if the vector field \( \mu(p) \) vanishes at the point \( p \) then for sufficiently small \( \varepsilon > 0 \) we can find a point \( q \) in the \( H^c \)-orbit of \( p \) such that \( \mu_\varepsilon(q) \) vanishes at \( q \). Note that \( X_s \) is in the center of \( \mathfrak{g} \), so if the vector field \( X_s \) vanishes at \( p \) then it also vanishes at \( q \). This means that when applying Proposition \( \text{[13]} \) at the point \( q \), the element \( f \in \mathfrak{h} \) is actually in \( \mathfrak{t} \), i.e. \( X_f \) vanishes at \( q \). By Lemma \( \text{[12]} \) we therefore obtain an extremal metric on the blowup \( Bl_q M \) in the Kähler class
\[
[\omega] - \varepsilon^2 d_m[E].
\]
Since \( q \) is in the \( H^c \)-orbit of \( p \), the manifolds \( Bl_q M \) and \( Bl_p M \) are biholomorphic so we have constructed an extremal metric on the blowup \( Bl_p M \). \( \square \)
5. THE GLUING ARGUMENT

In this section we prove Proposition 14. As before we will only blow up one point, but the proof of the general case is identical apart from more complicated notation. We will mainly focus on the case \(m \geq 3\) since the case \(m = 2\) needs special care but we will make brief comments on how to adapt the arguments when \(m = 2\). We first need some analytic preliminaries.

5.1. The Lichnerowicz operator on weighted spaces. The key to solving our equation using a perturbation method is to construct an inverse to the linear operator (4) and to control its inverse acting between suitable Banach spaces. It turns out that weighted Hölder spaces are suitable spaces to work in and in order to understand the mapping properties of the operator (4) between these spaces on the blowup \(\text{Bl}_p M\) we first need to understand the behaviour of the Lichnerowicz operator on weighted spaces on the manifolds \(M \setminus \{p\}\) and \(\text{Bl}_p C^m\). This is the fundamental tool in Arezzo-Pacard [2, 3] and Arezzo-Pacard-Singer [4] and we follow their treatment here. See also Lockhart-McOwen [20], Mazzeo [22], Melrose [23] or Pacard-Riviére [25] for more details on weighted spaces.

First we look at \(M_p = M \setminus \{p\}\) with the metric \(\omega\). For functions \(f : M_p \to \mathbb{R}\) we define the weighted norm

\[
\|f\|_{C^{k,\alpha}_\delta(M_p)} = \|f\|_{C^{k,\alpha}_\omega(M \setminus B_{1/2})} + \sup_{r < 1/2} r^{-\delta} \|f\|_{C^{k,\alpha}_{r^{-2}\omega}(B_{2r} \setminus B_r)}.
\]

Here the subscripts \(\omega\) and \(r^{-2}\omega\) indicate the metrics used for computing the corresponding norm. The weighted space \(C^{k,\alpha}_\delta(M_p)\) consists of functions on \(M \setminus \{p\}\) which are locally in \(C^{k,\alpha}\) and whose \(\| \cdot \|_{C^{k,\alpha}_\delta}\) norm is finite.

The main result we need is the following.

Proposition 15. If \(\delta < 0\) and \(\alpha \in (0, 1)\) then the operator

\[
C^{4,\alpha}_{\delta}(M_p)^T \times \overline{h} \to C^{0,\alpha}_{\delta-4}(M_p)^T
\]

\((\varphi, f) \mapsto D^* \varphi D\omega \varphi - f\)

has a bounded right-inverse. Here \(T\) is a torus of isometries of \((M, \omega)\) and \(\overline{h}\) is the space of \(T\)-invariant Hamiltonian functions of holomorphic killing fields.

Proof. This follows from the duality theory in weighted spaces. The image of

\[
D^* \mathcal{D} : C^{4,\alpha}_{\delta}(M_p) \to C^{0,\alpha}_{\delta-4}(M_p)
\]

is the orthogonal complement of the kernel of

\[
D^* \mathcal{D} : C^{4,\alpha}_{4-2m-\delta}(M_p) \to C^{0,\alpha}_{-2m-\delta}(M_p).
\]

If \(\delta < 0\) then \(4 - 2m - \delta > 4 - 2m\), so we need to see that if \(h \in \text{Ker} D^* \mathcal{D}\) is such that \(h \in C^{\gamma,\alpha}_{\delta}(M_p)^T\) for some \(\gamma > 4 - 2m\), then \(h\) is smooth. This follows from the regularity theory in weighted spaces since there are no indicial roots in \((4 - 2m, 0)\). \(\square\)

Let us turn now to the manifold \(\text{Bl}_0 C^m\) with the Burns-Simanca metric \(\eta\). The relevant weighted Hölder norm is now given by

\[
\|f\|_{C^{k,\alpha}_{\delta}(\text{Bl}_0 C^m)} = \|f\|_{C^{k,\alpha}_{\eta}(B_2)} + \sup_{r > 1} r^{-\delta} \|f\|_{C^{k,\alpha}_{r^{-2}\eta}(B_{2r} \setminus B_r)}.
\]
Here we abused notation slightly by writing $B_r \subset Bl_0 \mathbb{C}^m$ for the set where $|z| < r$ (ie. the pullback of the $r$-ball in $\mathbb{C}^m$ under the blowdown map).

The key result here is the following.

**Proposition 16.** For $\delta > 4 - 2m$ and $\alpha \in (0,1)$ the operator
\[
C^{4,\alpha}_{\delta}(Bl_0 \mathbb{C}^m)^T \to C^{0,\alpha}_{\delta-4}(Bl_0 \mathbb{C}^m)^T
\]
\[\varphi \mapsto \mathcal{D}^\ast_{\eta} \mathcal{D}_{\eta}(\varphi)\]
has a bounded inverse. If $m = 2$ then we should instead choose $\delta \in (3-2m,4-2m)$. In that case if we let $\chi$ be a compactly supported function on $Bl_0 \mathbb{C}^m$ with non-zero integral, then the operator
\[
C^{4,\alpha}_{\delta}(Bl_0 \mathbb{C}^m)^T \times \mathbb{R} \to C^{0,\alpha}_{\delta-4}(Bl_0 \mathbb{C}^m)^T
\]
\[(\varphi, t) \mapsto \mathcal{D}^\ast_{\eta} \mathcal{D}_{\eta}(\varphi) + t\chi
\]
has bounded inverse.

**Proof.** This is also a consequence of duality theory in weighted spaces. Once again the image of
\[
\mathcal{D}^\ast \mathcal{D} : C^{4,\alpha}_{\delta}(Bl_0 \mathbb{C}^m) \to C^{0,\alpha}_{\delta-4}(Bl_0 \mathbb{C}^m)
\]
is the orthogonal complement of the kernel of
\[
\mathcal{D}^\ast \mathcal{D} : C^{4,\alpha}_{4-2m-\delta}(Bl_0 \mathbb{C}^m) \to C^{0,\alpha}_{2m-\delta}(Bl_0 \mathbb{C}^m).
\]
If $\delta > 4 - 2m$, then $4 - 2m - \delta < 0$. If $h \in \text{Ker} \mathcal{D}^\ast \mathcal{D}$ and $h \in C^{4,\alpha}_{\gamma}(Bl_0 \mathbb{C}^m)$ for some $\gamma < 0$ then we must have $h = 0$ (for the proof see [2]). This implies that our operator is surjective.

When $m = 2$ then the same argument shows that the image of $\mathcal{D}^\ast_{\eta} \mathcal{D}_{\eta}$ when $\delta \in (3 - 2m, 4 - 2m)$ has codimension 1, and more precisely the image is the subspace of functions with integral zero. It follows that the operator (3) is surjective.

5.2. Weighted spaces on $Bl_p M$. We will need to do analysis on the blown-up manifold $Bl_p M$ endowed with the approximately extremal metric $\omega_\varepsilon$ we constructed in Section 4. For this we define the following weighted spaces, which are simply glued versions of the above weighted spaces on $M \setminus \{p\}$ and $Bl_0 \mathbb{C}^m$.

We define the weighted Hölder norms $C^{k,\alpha}_\delta$ by
\[
\|f\|_{C^{k,\alpha}_\delta} = \|f\|_{C^{k,\alpha}_\omega(M\setminus B_1)} + \sup_{\varepsilon \leq r \leq 1/2} r^{-\delta} \|f\|_{C^{k,\alpha}_{k-2\varepsilon}}(B_{2r} \setminus B_r) + \varepsilon^{-\delta} \|f\|_{C^{k,\alpha}_\omega(B_1)}.
\]

The subscripts indicate the metrics used to compute the relevant norm. This is a glued version of the two spaces defined in the previous section in the following sense. If $f \in C^{k,\alpha}_{\omega}(Bl_p M)$ and we think of $Bl_p M$ as a gluing of $M \setminus \{p\}$ and $Bl_0 \mathbb{C}^m$ then $\gamma_1 f$ and $\gamma_2 f$ can naturally be thought of as functions on $M \setminus \{p\}$ and $Bl_0 \mathbb{C}^m$ respectively. Then the norm $\|f\|_{C^{k,\alpha}_{\omega}(Bl_p M)}$ is comparable to
\[
\|\gamma_1 f\|_{C^{k,\alpha}_\omega(M\setminus\{p\})} + \varepsilon^{-\delta} \|\gamma_2 f\|_{C^{k,\alpha}_\omega(Bl_0 \mathbb{C}^m,\eta)}.
\]

Another way to think about the norm is that if $\|f\|_{C^{k,\alpha}_\delta} \leq c$ then $f$ is in $C^{k,\alpha}_{\delta}(Bl_p M)$ and also for $i \leq k$ we have
\[
|\nabla^i f| \leq c \quad \text{for } r \geq 1
\]
\[
|\nabla^i f| \leq cr^{\delta-i} \quad \text{for } \varepsilon \leq r \leq 1
\]
\[
|\nabla^i f| \leq c\varepsilon^{\delta-i} \quad \text{for } r \leq \varepsilon.
\]
The norms here are computed with respect to the metric $\omega_\varepsilon$, and note that on $B_\varepsilon$ we have $\omega_\varepsilon = \varepsilon^2 \eta$.

Sometimes we will restrict this norm to subsets such as $C^k_\delta(M \setminus B_{r_\varepsilon})$ and $C^k_\delta(B_{2r_\varepsilon})$. A crucial property of these weighted norms is that

\begin{equation}
\| \gamma_i \|_{C^{k_\alpha}_\delta} \leq c
\end{equation}

for some constant $c$ independent of $\varepsilon$.

In addition we need the following lemma about lifting elements of $\mathfrak{h} \subset C^\infty(M)$ to $C^\infty(B_l p M)$ according to Definition 13.

Lemma 17. For any $f \in \mathfrak{h}$ its lifting satisfies

$$\| F(f) \|_{C^0_{0,\alpha}} \leq c |f|$$

and also $| X(f) \omega_\varepsilon | \leq c |f|$ for some constant $c$ independent of $\varepsilon$. Here $| \cdot |$ is any fixed norm on $\mathfrak{h}$.

Proof. Recall that we defined the lifting in Definition 13 using a decomposition $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{h}'$, where the functions in $\mathfrak{h}'$ vanish at $p$. Suppose first that $f \in \mathfrak{h}'$. Since $f$ vanishes at $p$, we have

$$\| f \|_{C^1_{0,\alpha}(M_p)} \leq c |f|,$$

where $c$ is independent of $f$. It follows from the multiplication properties of weighted spaces and (9) that

$$\| F(f) \|_{C^0_{0,\alpha}} \leq c |f|,$$

from which the required inequalities follow.

Now suppose that $f \in \mathfrak{t}$, and write $X_f$ for the holomorphic vector field on $M$ corresponding to $f$. On the ball $B_{r_\varepsilon} \subset M$, the action of $X_f$ is given by unitary transformations, and the size of the lifting to $B_{r_\varepsilon} \subset B_l p M$ is determined by the size of $X_f$ on $\partial B_{r_\varepsilon}$. Outside $B_{r_\varepsilon}$ the vector field is unchanged and the metrics $\omega$ and $\omega_\varepsilon$ are uniformly equivalent. From these observations we can check that $| X(f) \omega_\varepsilon | \leq c |f|$ for some constant $c$. This in turn bounds $\nabla F(f)$, from which the bound on $\| F(f) \|_{C^0_{0,\alpha}}$ follows.

\[ \Box \]

5.3. The linearized operator. We now want to start studying the linearized operator (4). The constants that appear below will be independent of $\varepsilon$ unless the dependence is made explicit.

Recall that for any metric $\tilde{\omega}$ we defined

$$L_{\tilde{\omega}}(\varphi) = \Delta^2_{\tilde{\omega}} \varphi + \text{Ric}(\tilde{\omega}) \varphi_{ij}.$$

We want to first study how this varies as we change the metric. For this we have

**Proposition 18.** Suppose $\delta < 0$. There exist constants $c_0, C > 0$ such that if

$$\| \varphi \|_{C^2_{\delta,\alpha}} < c_0$$

then

$$\| L_{\omega_\varepsilon}(f) - L_{\omega_\varepsilon}(f) \|_{C^{0,\alpha}_{\delta-4}} \leq C \| \varphi \|_{C^2_{0,\alpha}} \| f \|_{C^4_{\delta,\alpha}},$$

where $\omega_\varepsilon = \omega_\varepsilon + i \partial \bar{\partial} \varphi$.

Proof. In the proof $c$ will denote a constant that may change from line to line, but is always independent of $\varepsilon$. Let us write $g, g_\varphi$ and for the Riemannian metrics
Lemma 19. Suppose that \( \delta < 0 \). There exist \( c_0, C > 0 \) such that if 
\[
\| \phi \|_{C^{4,\alpha}_\delta}, \| \psi \|_{C^{4,\alpha}_\delta} \leq c_0,
\]
then
\[
\| Q_{\omega_\delta}(\phi) - Q_{\omega_\delta}(\psi) \|_{C^{0,\alpha}_{\delta-4}} \leq C \left\{ \| \phi \|_{C^{4,\alpha}_\delta} + \| \psi \|_{C^{4,\alpha}_\delta} \right\} \| \phi - \psi \|_{C^{4,\alpha}_\delta}.
\]
Proof. By the mean value theorem there exists some $\chi$, which is a convex combination of $\varphi$ and $\psi$, such that

$$Q_{\omega_\epsilon}(\varphi) - Q_{\omega_\epsilon}(\psi) = DQ_{\omega_\epsilon,\chi}(\varphi - \psi).$$

From Equation (10) we see that $DQ_{\omega_\epsilon,\chi} = L_{\omega_\epsilon} - L_{\omega_\epsilon}$, so if $c_0$ is sufficiently small, then from the previous proposition we get

$$\|DQ_{\omega_\epsilon,\chi}(\varphi - \psi)\|_{C^0,\alpha_\delta} \leq C\|\chi\|_{C^4,\alpha}\|\varphi - \psi\|_{C^4,\alpha}.$$

But $\|\chi\|_{C^4,\alpha} \leq \|\varphi\|_{C^4,\alpha} + \|\psi\|_{C^4,\alpha}$ so the required inequality holds. □

Next we want to study the invertibility of the linearized operator (4) of our problem on $Bl_p M$. Let us write $X = \nabla I(s)$, where $I(s)$ is the lift to $Bl_p M$ of the scalar curvature $s(\omega)$.

**Proposition 20.** For sufficiently small $\epsilon$ and $\delta \in (4 - 2m, 0)$ the operator

$$G : (C^4_{\delta}, T) \times \overline{\mathbb{H}} \to (C^0_{\delta}, T)$$

$$(\varphi, f) \mapsto L_\omega(\varphi) - \frac{1}{2}X(\varphi) - 1(f)$$

has a right inverse $P$, with bounded operator norm $\|P\| < C$ for some constant $C$ independent of $\epsilon$.

When $m = 2$ and we choose $\delta = 4 - 2m - \theta$ for $\theta > 0$ small, then we obtain a right inverse $P$ with $\|P\| < C\epsilon^{-\theta}$.

**Proof.** This follows a standard argument for gluing solutions of linear problems, by first constructing an approximate inverse. See for example Chapter 7 in Donaldson-Kronheimer [12].

We will use the cutoff functions $\gamma_1, \gamma_2$ from before, where $\gamma_1 + \gamma_2 = 1$, the function $\gamma_1$ is supported on $M \setminus B_{r_\epsilon}$, $\nabla \gamma_1$ is supported on $B_{2r_\epsilon} \setminus B_{r_\epsilon}$ and

$$\|\gamma_1\|_{C^4,\alpha} \leq c.$$

We will also need a cutoff function $\beta_1$ which is equal to 1 on the support of $\gamma_1$, such that $\nabla \beta_1$ is supported on a set slightly smaller than $B_{r_\epsilon} \setminus B_{r_\epsilon}$ and $\beta_1 = 0$ on $B_{\epsilon}$. We will later choose $a < 1$ such that $r_\epsilon = \epsilon^a$, and for now let $\Omega$ be such that $a < \Omega < 1$.

Then we can define

$$\beta_1(z) = \beta \left( \frac{\log |z|}{\log \epsilon} \right),$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is a fixed cutoff function such that $\beta(r) = 1$ for $r < a$ and $\beta(r) = 0$ for $r > \Omega$. The key point is that with this definition

$$\|\nabla \beta_1\|_{C^3,\alpha} \leq \frac{c}{\log \epsilon},$$

and also the support of $\nabla \beta_1$ is in $B_{r_\epsilon} \setminus B_{r_\epsilon}$.

Similarly we define $\beta_2$ so that $\beta_2 = 1$ on the support of $\gamma_2$, but we want the support of $\nabla \beta_2$ to be slightly smaller than $B_{1} \setminus B_{2r_\epsilon}$. Namely we want $\nabla \beta_2$ to be supported on $B_{a} \setminus B_{2r_\epsilon}$, where $0 < a < a$. Again we can define

$$\beta_2(z) = \tilde{\beta} \left( \frac{|z|/2}{\log r_\epsilon} \right),$$
where \( \tilde{\beta} : \mathbb{R} \to \mathbb{R} \) is a cutoff function such that \( \beta(r) = 0 \) for \( r < a/\epsilon \) and \( \beta(r) = 1 \) for \( r > 1 \). Once again, we obtain

\[
\|\nabla \tilde{\beta}\|_{C^{2,0}_\delta} \leq \frac{c}{|\log \epsilon|} \leq \frac{c'}{|\log \epsilon|},
\]

for sufficiently small \( \epsilon \).

Let \( \varphi \in (C^{0,0}_{\delta - a})^T \). The function \( \gamma_1 \varphi \) can be thought of as being defined on \( M_p \). Since \( \|\gamma_1\|_{C^{0,0}_\delta} \leq c \) and the metrics \( \omega_\varphi \) and \( \omega \) are uniformly equivalent, we have

\[
\|\gamma_1 \varphi\|_{C^{0,0}_{\delta - a}(M_p)} \leq c\|\varphi\|_{C^{0,0}_\delta}.
\]

It follows from Proposition 15 that there exists some \( f \in \mathbb{R} \) and \( P_1(\gamma_1 \varphi) \) with

\[
\|P_1(\gamma_1 \varphi)\|_{C^{2,0}_{\delta - a}(M_p)} + |f| \leq c\|\varphi\|_{C^{0,0}_\delta}
\]

for which

\[
L_{\omega} P_1(\gamma_1 \varphi) - \frac{1}{2} \nabla s(\omega) : \nabla P_1(\gamma_1 \varphi) - f = \gamma_1 \varphi.
\]

Similarly the function \( \gamma_2 \varphi \) can be thought of as a function on \( B_{l_0} \mathbb{C}^m \), and from the definition of our norm we have

\[
\|\gamma_2 \varphi\|_{C^{0,0}_{\delta - 4}(B_{l_0} \mathbb{C}^m)} \leq c \epsilon^\delta \|\varphi\|_{C^{0,0}_\delta}.
\]

From Proposition 10 we have some \( P_2(\gamma_2 \varphi) \) with

\[
\|P_2(\gamma_2 \varphi)\|_{C^{1,0}_{\delta - 4}(B_{l_0} \mathbb{C}^m, \eta)} \leq c \epsilon^\delta \|\gamma_2 \varphi\|_{C^{0,0}_{\delta - 4}(B_{l_0} \mathbb{C}^m, \eta)} \leq c \epsilon \|\varphi\|_{C^{0,0}_{\delta - 4}},
\]

for which

\[
L_{\omega} P_2(\gamma_2 \varphi) = \epsilon^4 \gamma_2 \varphi,
\]

so we also have

\[
L_{\omega, \gamma_1} P_2(\gamma_2 \varphi) = \gamma_2 \varphi.
\]

We can think of \( \beta_2 P_2(\gamma_2 \varphi) \) as a function on \( B_{\delta^2} \mathbb{C}^m \).

We then define

\[
P(\varphi) = \beta_1 P_1(\gamma_1 \varphi) + \beta_2 P_2(\gamma_2 \varphi),
\]

where we are thinking of the annulus \( B_1 \setminus B_\epsilon \) as a subset of \( M_p, B_{l_0} \mathbb{C}^m \) and \( B_{\delta^2} \mathbb{C}^m \) at the same time. The bounds (11) and (13) imply that

\[
\|P(\varphi)\|_{C^{1,0}_{\delta - 4}} \leq c \|\varphi\|_{C^{0,0}_{\delta - 4}}.
\]

We want to show that the operator \( \varphi \mapsto (P(\varphi), f) \) gives an approximate inverse to the operator \( G \).

**Claim.** For sufficiently small \( \epsilon \) we have

\[
\left\| L_{\omega_\varphi}(P \varphi) - \frac{1}{2} X(P \varphi) - I(f) - \varphi \right\|_{C^{0,0}_{\delta - 4}} \leq \frac{1}{2} \|\varphi\|_{C^{0,0}_{\delta - 4}}.
\]

To prove this note that we can write the expression we want to estimate as

\[
L_{\omega_\varphi}(\beta_1 P_1(\gamma_1 \varphi)) - \frac{1}{2} X(\beta_1 P_1(\gamma_1 \varphi)) - \gamma_1 I(f) - \gamma_1 \varphi
+ L_{\omega_\varphi}(\beta_2 P_2(\gamma_2 \varphi)) - \frac{1}{2} X(\beta_2 P_2(\gamma_2 \varphi)) - \gamma_2 I(f) - \gamma_2 \varphi,
\]

where the terms on the top row are all supported in \( M \setminus B_{\delta^2} \mathbb{C}^m \) and the terms on the bottom row are supported in \( B_{\delta^2} \mathbb{C}^m \).
Therefore it remains to show that for sufficiently small $\varepsilon$ we have
\[
\rho(z) = \gamma_2(|z|)(-\varphi(z) + \varepsilon^2\psi(\varepsilon^{-1}z)).
\]
It follows that on the complement of $B_\varepsilon$ we have
\[
\|\rho\|_{C^{1,\alpha}_2(\mathbb{R}^n\setminus B_\varepsilon)} \leq c(r^2 + \varepsilon^{2m-2}) = o(1),
\]
where by $o(1)$ we mean a constant going to zero as $\varepsilon \to 0$. By the argument in Proposition 18 this implies that on the complement of $B_\varepsilon$ we have
\[
\|L_{\omega_x} - L_{\omega_0}\| = o(1).
\]
At the same time $s(\omega) - I(s)$ is supported on $B_{2r_x}$ and is bounded in $C^{1,\alpha}_0$ by Lemma 17. It follows from this that
\[
\|(X - \nabla s(\omega))\varphi\|_{C^{1,\alpha}_2} \leq cr_\varepsilon^2 \|\varphi\|_{C^{1,\alpha}_2}.
\]
Similarly, inside $B_{2r_x}$ we have
\[
\left\|L_\omega + \frac{1}{2} X - L_{\varepsilon^2 j}\right\| = o(1).
\]
Therefore it remains to show that for sufficiently small $\varepsilon$ we have
\[
\|L_\omega(\beta_1 P_1(\gamma_1 \varphi)) - \frac{1}{2} \nabla s(\omega) \cdot \nabla (\beta_1 P_1(\gamma_1 \varphi)) - \gamma_1 l(f) - \gamma_1 \varphi\|_{C^{1,\alpha}_2} \leq \frac{1}{4} \|\varphi\|_{C^{1,\alpha}_2},
\]
and
\[
\|L_{\varepsilon^2 j}(\beta_2 P_2(\gamma_2 \varphi)) - \gamma_2 l(f) - \gamma_2 \varphi\|_{C^{1,\alpha}_2} \leq \frac{1}{4} \|\varphi\|_{C^{1,\alpha}_2}.
\]
For the first inequality note, using Equation (12) that
\[
L_\omega(\beta_1 P_1(\gamma_1 \varphi)) - \frac{1}{2} \nabla s(\omega) \cdot \nabla (\beta_1 P_1(\gamma_1 \varphi)) - \gamma_1 l(f) - \gamma_1 \varphi = \beta_1 \gamma_1 \varphi + \beta_1 f - \gamma_1 l(f) - \gamma_1 \varphi + D_3^3(\nabla \beta_1 \ast P_1(\gamma_1 \varphi))
\]
\[
= \beta_1 f - \gamma_1 l(f) + D_3^3(\nabla \beta_1 \ast P_1(\gamma_1 \varphi)),
\]
where $D_3^3$ denotes a 3rd order differential operator with the coefficient of $\nabla^3$ bounded in $C^{1,\alpha}_3(M_\varepsilon)$ and $\ast$ is a bilinear algebraic operator. Since $\beta_1 f - \gamma_1 l(f)$ is supported in $B_{2r_x}$ and is bounded in $C^{1,\alpha}_0$ by $c|f|$, we get
\[
\|\beta_1 f - \gamma_1 l(f)\|_{C^{1,\alpha}_2} \leq cr_\varepsilon^3 \|\varphi\|_{C^{1,\alpha}_2}.
\]
Finally we have
\[
\|D_3^3(\nabla \beta_1 \ast P_1(\gamma_1 \varphi))\|_{C^{1,\alpha}_2} \leq c\|\nabla \beta_1\|_{C^{3,\alpha}_1} \|P_1(\gamma_1 \varphi)\|_{C^{1,\alpha}_2} = o(1) \|\varphi\|_{C^{1,\alpha}_2},
\]
so for small enough $\varepsilon$ the Inequality (10) holds.

The proof of the second inequality is similar, just note that $\gamma_2 l(f)$ is supported in $B_{2r_x}$ so
\[
\|\gamma_2 l(f)\|_{C^{1,\alpha}_2} \leq cr_\varepsilon^3 \|\varphi\|_{C^{1,\alpha}_2}.
\]
This proves the claim, so if we write $\hat{P}(\varphi) = (P \varphi, f)$, this means that the operator norm $\|G \circ \hat{P} - I\| \leq \frac{1}{2}$, which implies that we have a uniformly bounded inverse $(G \circ \hat{P})^{-1}$. This in turn shows that $G$ has a right inverse $\hat{P} \circ (G \circ \hat{P})^{-1}$ whose norm is bounded independently of $\varepsilon$, which is what we wanted.
When $m = 2$ then we first work with the operator
\[
G_0 : (C^{4,\alpha}_\delta)^T \times \overline{h}_0 \times \mathbb{R} \to (C^{0,\alpha}_{\delta-4})^T
\]
\[
(\varphi, f, t) \mapsto D_\omega^* D_\omega(\varphi) - I(f) + t \chi,
\]
where $\chi$ is the function from Proposition 19. We are thinking of $\chi$ as a function on $Bl_p M$ using the identification of $B_\varepsilon \subset Bl_p M$ with $B_1 \subset Bl_0 C^m$ and in addition $\overline{h}_0$ denotes the functions $f \in \overline{h}$ for which the lifting $I(f)$ has zero mean. The same argument as above can be used to show that if $\varepsilon$ is small then $G_0$ is invertible, with the inverse bounded independently of $\varepsilon$. Since $D^* D(\varphi)$ and $I(f)$ have zero mean, this implies that the operator
\[
G_1 : (C^{4,\alpha}_\delta)^T \times \overline{h}_0 \to (C^{0,\alpha}_{\delta-4})^T
\]
\[
(\varphi, f) \mapsto D_\omega^* D_\omega(\varphi) - I(f)
\]
also has bounded inverse, where $(C^{0,\alpha}_{\delta-4})^T$ consist of functions with zero mean. Now note that if $\delta < 4 - 2m$ then
\[
\|\varphi\|_{L^1} \leq \varepsilon^{\delta-(4-2m)} \|\varphi\|_{C^{0,\alpha}_{\delta-4}}.
\]
It follows, by applying $G_1^{-1}$ to $\varphi - \overline{\varphi}$ and then absorbing the mean value $\overline{\varphi}$ into $f$, that the operator
\[
G_2 : (C^{4,\alpha}_\delta)^T \times \overline{h} \to (C^{0,\alpha}_{\delta-4})^T
\]
\[
(\varphi, f) \mapsto D_\omega^* D_\omega(\varphi) - I(f)
\]
also has bounded inverse, but we only get $\|G_2^{-1}\| < C\varepsilon^{\delta-(4-2m)} = C\varepsilon^{-\theta}$. Finally we can compare $G_2$ with the operator $G$ in the statement of the proposition using (15), and we find that $G$ is also invertible with the same bound, if $\theta$ is small enough so that $r^2 \ll \varepsilon^\theta$.

5.4. The nonlinear equation. We are now ready to solve the equation of Proposition 14 (see also Equation (2)), i.e. we want to find a $T$-invariant function $u \in C^\infty(B_l M)$ and $f \in \overline{h}$ satisfying
\[
s(\omega + i\partial\overline{\partial} u) = -\frac{1}{2} \nabla I(f + s) \cdot \nabla u = I(f + s).
\]
Recall that here $I(f)$ and $I(s)$ are our lifts of the functions $f, s(\omega) \in \overline{h}$ to the blowup $Bl_p M$, defined in Definition 12. As before we will write $X = \nabla I(s)$. Let us write the equation as
\[
L_{\omega_\varepsilon}(u) - \frac{1}{2} X(u) - I(f) = I(s) - s(\omega_\varepsilon) - Q_{\omega_\varepsilon}(u) + \frac{1}{2} \nabla I(f) \cdot \nabla u.
\]
Following [1] we first modify $\omega$ on $M \setminus \{p\}$ so that it matches up with the Burns-Simanca metric to higher order. For this let $\Gamma$ be a $T$-invariant solution of the linear equation
\[
D_\omega^* D_\omega \Gamma = h \quad \text{on } M \setminus \{p\}
\]
for some $h \in \overline{h}$, such that $\Gamma$ has an expansion
\[
\Gamma(z) = -|z|^{4-2m} + \tilde{\Gamma},
\]
where $\tilde{\Gamma} = O(|z|^{5-2m})$ for $m > 2$ and $\Gamma$ has leading term $\log |z|$ when $m = 2$. It follows from this expansion that $\Gamma$ is a distributional solution of
\[
D_\omega^* D_\omega \Gamma = h - c_m \delta_p.
\]
where $c_m > 0$ is a constant depending on the dimension and $\delta_\rho$ is the delta function at $p$. We then find that for all $g \in \mathfrak{g}$ we have
\[
\int_M gh \omega^m = c_m g(p),
\]
so $h = c_m \mu(p) + \lambda$, where $\lambda = \Vol(M)^{-1} c_m$ is a constant.

Define the metric $\tilde{\omega} = \omega + i \partial \overline{\partial} \mu$ on $M \setminus \{p\}$, so
\[
\tilde{\omega} = i \partial \overline{\partial} \left( \frac{|z|^2}{2} + \varepsilon^{2m-2} \Gamma(z) + \varphi(z) \right),
\]
where recall that $\omega = i \partial \overline{\partial}(|z|^2/2 + \varphi(z))$ near $p$. We can then write
\[
\tilde{\omega} = i \partial \overline{\partial} \left( \frac{|z|^2}{2} - \varepsilon^{2m-2} |z|^{4-2m} + \varepsilon^{2m-2} \tilde{\Gamma}(z) + \varphi(z) \right)
\]
when $m > 2$. At the same recall that the Burns-Simanca metric $\eta$ has an expansion
\[
\eta = i \partial \overline{\partial} \left( \frac{|z|^2}{2} - |z|^{4-2m} + \tilde{\psi}(z) \right),
\]
where $\tilde{\psi} = O(|z|^{-3-2m})$ for large $z$. We define the metric $\tilde{\omega}_\varepsilon$ by gluing $\tilde{\omega}$ to $\varepsilon^2 \eta$ as before, to get on the annulus $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$
\[
\tilde{\omega}_\varepsilon = i \partial \overline{\partial} \left( \frac{|z|^2}{2} - \varepsilon^{2m-2} |z|^{4-2m} + \gamma_1(|z|) \left[ \varepsilon^{2m-2} \tilde{\Gamma}(z) + \varphi(z) \right] + \gamma_2(|z|) \varepsilon^2 \tilde{\psi}(\varepsilon^{-1} z) \right).
\]
Moreover outside $B_{2r_\varepsilon}$ we have $\tilde{\omega}_\varepsilon = \tilde{\omega}$ while inside $B_{r_\varepsilon}$ we have $\tilde{\omega}_\varepsilon = \varepsilon^2 \eta$. Note that in terms of our previous approximate metric $\omega_\varepsilon$
\[
\omega_\varepsilon = \omega_\varepsilon + i \partial \overline{\partial} \left[ \varepsilon^{2m-2} \gamma_1(|z|) \Gamma(z) \right].
\]
If $m = 2$ then we can glue $\tilde{\omega}$ to $\varepsilon^2 \eta$ in the same way.

We want to find a solution to Equation (17) as a perturbation of $\tilde{\omega}_\varepsilon$ so we write
\[
u = \varepsilon^{2m-2} \gamma_1 \Gamma + v
\]
\[f = \varepsilon^{2m-2} h + g.
\]
Substituting this into Equation (17) and rearranging we get
\[
L_{\omega_\varepsilon}(v) - \frac{1}{2} X(v) - I(g) = I(s) - s(\omega_\varepsilon) - Q_{\omega_\varepsilon}(u) + \frac{1}{2} \nabla(I) \cdot \nabla u
\]
\[\quad - L_{\omega_\varepsilon}(\varepsilon^{2m-2} \gamma_1 \Gamma) + \frac{1}{2} X(\varepsilon^{2m-2} \gamma_1 \Gamma) + \varepsilon^{2m-2} I(h).
\]
We can write this as a fixed point problem
\[
(v, g) = \mathcal{N}(v, g)
\]
where we use the inverse $P$ constructed in Proposition (20) and
\[
\mathcal{N}(v, g) = P \left\{ I(s) - s(\omega_\varepsilon) - Q_{\omega_\varepsilon}(\varepsilon^{2m-2} \gamma_1 \Gamma + v) + \frac{1}{2} \nabla(\varepsilon^{2m-2} I(h) + I(g)) \cdot \nabla (\varepsilon^{2m-2} \gamma_1 \Gamma + v) - L_{\omega_\varepsilon}(\varepsilon^{2m-2} \gamma_1 \Gamma) + \frac{1}{2} X(\varepsilon^{2m-2} \gamma_1 \Gamma) + \varepsilon^{2m-2} I(h) \right\}
\]
Lemma 21. There exist constants $c_0,\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ the operator $\mathcal{N}$ is a contraction on the set

$$\{(v,g) : \|v\|_{C^{4,\alpha}_2}, |g| < c_0\}$$

with constant $1/2$.

Proof. Suppose $m > 2$. Since $P$ is bounded independently of $\varepsilon$, we need to control

$$Q_{\omega_\varepsilon}(u_1) - \frac{1}{2} \nabla f_1 \cdot \nabla u_1 - Q_{\omega_\varepsilon}(u_2) + \frac{1}{2} \nabla f_2 \cdot \nabla u_2,$$

where

$$u_i = \varepsilon^{2m-2} \gamma_1 \Gamma + v_i$$

$$f_i = \varepsilon^{2m-2} h + g_i.$$

First note that

$$\|\varepsilon^{2m-2}\Gamma\|_{C^{\alpha,\alpha}_2} \leq c(\varepsilon \varepsilon^{-1})^{2m-2} = o(1)$$

as $\varepsilon \to 0$. Hence for any $\lambda > 0$ we can choose sufficiently small $c_0$ and $\varepsilon$ for which Lemma 19 implies that

$$\|Q_{\omega_\varepsilon}(u_1) - Q_{\omega_\varepsilon}(u_2)\|_{C^{4,\alpha}_2} \leq \lambda \|u_1 - u_2\|_{C^{4,\alpha}_2} = \lambda \|v_1 - v_2\|_{C^{4,\alpha}_2}.$$

On the other hand we have

$$\|\nabla f_1 \cdot \nabla u_1 - \nabla f_2 \cdot \nabla u_2\|_{C^{\alpha,\alpha}_2} \leq$$

$$\leq \|\nabla f_1 \cdot (\nabla u_1 - \nabla u_2)\|_{C^{\alpha,\alpha}_2} + \|\nabla (f_1) - \nabla (f_2)\|_{C^{\alpha,\alpha}_2} \nabla u_2\|_{C^{\alpha,\alpha}_2}$$

$$\leq c(f_1 \cdot \|u_1 - u_2\|_{C^{4,\alpha}_2} + \|u_2\|_{C^{4,\alpha}_2} \nabla (f_1 - f_2),$$

where we used Lemma 17. From this it is clear that by choosing $c_0$ and $\varepsilon$ sufficiently small, $\mathcal{N}$ is a contraction with constant $1/2$.

If $m = 2$ then $P$ is not bounded independently of $\varepsilon$, but if we choose $\delta < 4 - 2m$ very close to $4 - 2m$ then the bound only blows up slowly as $\varepsilon \to 0$ and the same argument works.

Next we need to bound $\mathcal{N}(0,0)$, which is the same as estimating $\|F\|_{C^{4,\alpha}_2}$, where $F$ is the function

$$F = \frac{1}{2} \nabla \varepsilon^{2m-2\gamma_1} \Gamma + 1 \nabla \varepsilon^{2m-2\gamma_1} \Gamma$$

(22)

$$- L_{\omega_\varepsilon}(e^{2m-2\gamma_1} \Gamma) + \frac{1}{2} X(e^{2m-2\gamma_1} \Gamma) + \varepsilon^{2m-2\gamma_1} \Gamma.$$

Lemma 22. Choose $\delta$ very close to $4 - 2m$ with $\delta > 4 - 2m$ for $m > 2$ and $\delta < 4 - 2m$ for $m = 2$. Let $r_\varepsilon = \varepsilon^{2m-1}$. Then we have the estimate

$$\|F\|_{C^{4,\alpha}_2} \leq c\varepsilon^{4-\delta},$$

where $F$ is defined by Equation (22).
Proof. To prove this we look at three different pieces of $B_\rho M$ separately, namely $M \setminus B_{2r_\varepsilon}, B_{2r_\varepsilon} \setminus B_r$, and $B_r$. First of all in $B_r$ we have $F = I(s) + \varepsilon^{2m-2} l(h)$, but note that by Lemma 17
\[ ||I(s)||_{C^0,\alpha}, \ ||I(h)||_{C^0,\alpha} \leq c, \]
which implies that $||F||_{C^0,\alpha}(B_r) \leq c r_\varepsilon^{4-\delta}$.

On the set $M \setminus B_{2r_\varepsilon}$ note that $\omega_\varepsilon = \omega$, so $I(s) = s(\omega_\varepsilon)$ and $I(h) = h$. In addition
\[ D_\omega^* D_\omega \Gamma = L_\omega \Gamma - \frac{1}{2} X(\Gamma) = h \]
using Equations (18) and (5). This means that on the set $M \setminus B_{2r_\varepsilon}$
\[ F = -Q_\omega(\varepsilon^{2m-2}\Gamma) + \frac{1}{2} \varepsilon^{4m-4} \nabla h \cdot \nabla \Gamma. \] (23)
It is useful to note that $||\gamma_1 \Gamma||_{C^1,\alpha}$ is bounded by $cr_\varepsilon^{4-2m-w}$ for $w > 4 - 2m$ and by $c$ for $w < 4 - 2m$. For the second term in (23) we have
\[ \varepsilon^{4m-4} \nabla h \cdot \nabla \Gamma||_{C^0,\alpha}(M \setminus B_{2r_\varepsilon}) \leq \varepsilon^{4m-4} ||\nabla \Gamma||_{C^0,\alpha} \leq \varepsilon^{4m-4} ||\Gamma||_{C^1,\alpha} \leq \varepsilon^{4m-4} \ll r_\varepsilon^{4-\delta}, \]
as long as $\delta$ is close to $4 - 2m$. For the term involving $Q_\omega$ we use Proposition 23 below. Indeed
\[ ||Q_\omega(\varepsilon^{2m-2}\Gamma)||_{C^0,\alpha}(M \setminus B_{2r_\varepsilon}) \leq \varepsilon^{4m-4} r_\varepsilon^{4-2m-\delta} \ll c r_\varepsilon^{4-\delta}, \]
as long as $m \geq 2$.

Finally on the annulus $A_\varepsilon = B_{2r_\varepsilon} \setminus B_r$ we first note that by Equation (20) we have
\[ s(\omega_\varepsilon) + L_\omega(\varepsilon^{2m-2}\gamma_1 \Gamma) + Q_\omega(\varepsilon^{2m-2}\gamma_1 \Gamma) = s(\tilde{\omega}_\varepsilon). \]

The other terms in the expression for $F$ can be dealt with as before, so we only need to show that on the annulus $A_\varepsilon$
\[ ||s(\tilde{\omega}_\varepsilon)||_{C^0,\alpha} \leq cr_\varepsilon^{4-\delta}. \]

This is where we use that $\tilde{\omega}$ matches up with the Burns-Simanca metric to leading order at $p$. We use the formula given in Equation (14), which we write as
\[ \tilde{\omega}_\varepsilon = i \partial \bar{\partial} \left( \frac{|z|^2}{2} + g(z) \right). \]

On $A_\varepsilon$ we then have
\[ s(\tilde{\omega}) = L_\eta(g), \]
where $\eta = i \partial \bar{\partial} \left( |z|^2/2 + t g(z) \right)$ for some $t \in [0, 1]$. We have
\[ s(\tilde{\omega}) = \Delta^2 g + (L_\eta - L_0) g, \]
where $L_0 = \Delta^2_0$ is the linearized operator at the flat metric. At the same time
\[ ||g||_{C^2,\alpha}(A_\varepsilon) \leq \epsilon \varepsilon^{2m-2} \ll r_\varepsilon^{2-\delta}, \]
so Proposition 23 implies that
\[ ||L_\eta(g) - L_0(g)||_{C^0,\alpha}(A_\varepsilon) \leq cr_\varepsilon^{3-2} \left( \epsilon^{2m-2} r_\varepsilon^{4-2m-\delta} \right)^2 \leq c r_\varepsilon^{4-\delta}. \]
Finally for $\Delta^2 g$ note that $\Delta^2 g(z) = -\varepsilon^{2m-2} |z|^{4-2m} + \tilde{g}(z)$, so writing
\[ g(z) = -\varepsilon^{2m-2} |z|^{4-2m} + \tilde{g}(z) \]
we have $\Delta^2 g = \Delta^2 \tilde{g}$. At the same time
\[ \|\tilde{g}\|_{C^{4,\alpha}(A_{\varepsilon})} \leq C \varepsilon^{2m-1} r_\varepsilon^{3-2m-\delta} = cr_\varepsilon^{4-\delta}, \]
so
\[ \|\Delta^2 \tilde{g}\|_{C^{4,\alpha}(A_{\varepsilon})} \leq cr_\varepsilon^{4-\delta}, \]
which gives the result we wanted. \(\square\)

We used the following result, whose proof is identical to that of Proposition 18.

**Proposition 23.** There exist constants $c_0, C > 0$ such that if $U \subset Bl_p M$ and $\|u\|_{C^{4,\alpha}_p(U)} \leq c_0$ then for any $v$ we have
\[ \|L_{\omega_u}(v) - L_{\omega_v}(v)\|_{C^{0,\alpha}_p(U)} \leq C \|u\|_{C^{4,\alpha}_p(U)} \|v\|_{C^{4,\alpha}_p(U)}, \]
where $\omega_u = \omega + i\partial \bar{\partial} u$. It follows that
\[ \|Q_{\omega_u}(v)\|_{C^{4,\alpha}_p(U)} \leq C \|u\|_{C^{4,\alpha}_p(U)} \|u\|_{C^{4,\alpha}_p(U)}. \]

We can now complete the proof of Proposition 14. Let us choose $\delta$ close to $4 - 2m$ with $\delta > 4 - 2m$ for $m > 2$ and $\delta < 4 - 2m$ for $m = 2$, and let $r_\varepsilon = \varepsilon^{\theta}$ as above. First by Lemma 22 and our bound on the inverse $P$ from Proposition 20 we have
\[ \|N(0, 0)\|_{C^{4,\alpha}_p} \leq c_1 r_\varepsilon^{4-\delta} \varepsilon^{-\theta} \]
for some constant $c_1$ independent of $\varepsilon$, as long as $\varepsilon$ is sufficiently small. Here $\theta = 0$ if $m > 2$ and $\theta = 4 - 2m - \delta$ if $m = 2$. Define
\[ S = \{(v, g) : \|v\|_{C^{4,\alpha}_p}, |g| \leq 2c_1 r_\varepsilon^{4-\delta} \varepsilon^{-\theta}\}. \]
For $(v, g) \in S$ we have
\[ \|v\|_{C^{4,\alpha}_p} \leq C r_\varepsilon^{4-\delta} \varepsilon^{-2m}, \]
so if $\varepsilon$ and $\theta$ are small enough then Lemma 21 implies that $N$ is a contraction with constant $1/2$ on $S$. In particular $N$ then maps $S$ to itself, since if $(v, g) \in S$ then
\[ \|N(v, g)\| \leq \|N(v, g) - N(0, 0)\| + \|N(0, 0)\| \leq \frac{1}{2} \|v, g\| + c_1 r_\varepsilon^{4-\delta} \varepsilon^{-\theta} \leq 2c_1 r_\varepsilon^{4-\delta} \varepsilon^{-\theta}. \]
It follows that for small enough $\varepsilon$ there is a fixed point of $N$ in the set $S$. This gives a solution $(v, g)$ to our equation, with $|g| \leq 2c_1 r_\varepsilon^{4-\delta} \varepsilon^{-\theta}$. Finally if $\delta$ is sufficiently close to $4 - 2m$, we find that $r_\varepsilon^{4-\delta} < \varepsilon^{\kappa}$ for some $\kappa > 2m - 2$. Hence from the expansion 21 and the fact that $h = c_m h(p)$, we obtain the required expansion in Equation 17.

5.5. **A remark on Conjecture 6** A natural problem is to compute more terms in the expansion 21 of the element $f$ above. Examining the argument we see that the key point was to first perturb the extremal metric $\omega$ away from $p$, so that it matches up with the Burns-Simanca metric to higher order. To see what the next term should be we need the following.
Lemma 24. If \( m \geq 3 \) then the Kähler potential for a suitable scaling of the Burns-Simanca metric
\[
\eta = i\bar{\partial}\bar{\partial}(\frac{|z|^2}{2} + \psi(z))
\]
satisfies
\[
\psi(z) = -|z|^{4-2m} + a|z|^{2-2m} + O(|z|^{6-4m}),
\]
where \( a > 0 \).

Proof. This can be seen by finding the first few terms in the power series expansion of the solution of the ODE for scalar flat \( U(n) \) invariant metrics on \( Bl_{0}\mathcal{O}^{m} \), written down in [2], Section 7 (see also [27]). Following [2] let us write \( \eta = i\bar{\partial}\bar{\partial}(|z|^2) \) and let \( s = |z|^2 \). Let us also introduce the variable \( t = s^{-1} \) and define the function \( \xi(t) = \partial_{s}A(s) \). From the equations given in [2] one can check that \( \xi \) satisfies the equation
\[
\xi^{m-1}(t)\xi'(t) - (m-1)t^{m-2}\xi(t) + (m-2)t^{m-1} = 0.
\]
Moreover we want \( \xi(0) = 1/2 \). It is then straightforward to check that the first few terms in the expansion of \( \xi \) around \( t = 0 \) are
\[
\xi(t) = \frac{1}{2} + \frac{2m-2}{m}t^{m-1} - \frac{m-2}{m}t^{m-1} + O(t^{m+1}).
\]
From this we can recover \( A(s) \), and finally by scaling the variable \( z \) and the metric, we obtain the first two terms in (24), with \( a > 0 \).

To show that the next term is \( O(|z|^{6-4m}) \) we can either compute more terms in the expansion of \( \xi \), or instead we can follow the argument in [2], Lemma 7.2. The scalar curvature of \( \eta = i\bar{\partial}\bar{\partial}(\frac{|z|^2}{2} + \psi(z)) \) is given by
\[
s(\eta) = \Delta^2\psi + Q(\psi),
\]
where \( \Delta \) is the Laplacian for the flat metric (we use the Kähler Laplacian and half the Riemannian scalar curvature, so the coefficient of \( \Delta^2 \) differs from that in [2]). It is shown in [2] that if \( \psi \in C^{\delta}_{\alpha}(Bl_{0}\mathcal{O}^{m}) \) for some \( \delta < 2 \) then
\[
Q(\psi) \in C^{0,\alpha}_{\delta} \mathcal{O}^{m}.
\]
If \( s(\eta) = 0 \) then also \( \Delta^2\psi \in C^{0,\alpha}_{\delta-2} \mathcal{O}^{m} \), so from the regularity theory for the Laplacian acting between weighted spaces we get that
\[
\psi \in C^{\delta}_{\alpha-\delta,\alpha} \mathcal{O}^{m} \oplus \text{span}\{1, |z|^{4-2m}, |z|^{2-2m}\}.
\]
The reason why we only get these powers of \( |z| \) is that these (together with \( |z|^2 \)) are the only \( U(n) \) invariant elements in the kernel of \( \Delta^2 \). Or in other words if we work with \( U(n) \) invariant spaces then the indicial roots are 2, 0, 4 \(- 2m \) and 2 \(- 2m \). Subtracting a constant we can therefore suppose that \( \psi \in C^{\delta}_{4-2m} \mathcal{O}^{m} \) so we can apply the above with \( \delta = 4 - 2m \). Then \( 2\delta - 2 = 6 - 4m \) so the result follows from (25). \( \square \)

In order to match with the metric \( \varepsilon^{2}\eta \) we therefore need to perturb \( \omega \) to \( \omega + i\bar{\partial}\bar{\partial}\Gamma \), where
\[
\Gamma = -\varepsilon^{2m-2}|z|^{4-2m} + \varepsilon^{2m}a|z|^{2-2m} + \text{lower order terms}
\]
and \( \mathcal{D}_{\omega}^{*}\mathcal{D}_{\omega}\Gamma = h \) on \( M \setminus \{p\} \) for some \( h \in H \). By changing the lower order terms if necessary (we can use a term of the order of \( \varepsilon^{2m}|z|^{4-2m} \) to cancel the contribution of \( \text{Ric}^{\omega}\Gamma_{ij} \)) we can assume that \( \Gamma \) is a distributional solution of
\[
\mathcal{D}_{\omega}^{*}\mathcal{D}_{\omega}\Gamma = h - \varepsilon^{2m-2}c_{m}\delta_{p} - \varepsilon^{2m}ac_{m}\Delta\delta_{p},
\]
where $c_m, c_m > 0$ are constants depending on the dimension. Taking the $L^2$ product of both sides with all $g \in C$ as before, we find that
\[ h = \varepsilon^{2m-2}(\lambda + c_m\mu(p)) + \varepsilon^{2m}a c_m\Delta\mu(p), \]
where $\lambda = \text{Vol}(M)^{-1}c_m$. Under the assumptions of Conjecture 6 if $\varepsilon$ is sufficiently small we can assume that $h = \varepsilon^{2m-2}\lambda$. It seems reasonable to expect that one can deform this metric $\omega + i\partial\bar{\partial}\Gamma$ to a cscK metric, but we have not been successful with this so far. Note also that when $m = 2$, then the potential for the Burns-Simanca metric is given by $\psi = \log |z|$ with no lower order terms, so it is not clear where the expression $\mu(p) \pm \varepsilon\Delta\mu(p)$, which we see in the algebro-geometric calculations in the next section, comes from in this case.

6. K-stable blowups

In this section we give the proof of Theorem 5 which is an extension of a result in Stoppa [29]. First we need to review the notion of K-stability introduced by Donaldson [9].

Let $L \to M$ be an ample line bundle. A test-configuration for the pair $(M, L)$ is a flat $C^\infty$-equivariant family $\pi : \mathcal{M} \to \mathbb{C}$ together with a $C^\infty$-equivariant relatively ample line bundle $\mathcal{L} \to \mathcal{M}$, such that the fiber $(\pi^{-1}(1), \mathcal{L}|_{x^{-1}(1)})$ is isomorphic to $(M, L^r)$ for some $r > 0$. Let us denote by $\alpha$ the induced $C^\infty$-action on the central fiber $(M_0, L_0)$. This gives rise to a $C^\infty$-action on the space of sections $\mathcal{H}^0(M_0, L_0^k)$ for each $k$. Let us write $d_k = \dim \mathcal{H}^0(M_0, L_0^k)$ and $w_k$ for the total weight of the action on $\mathcal{H}^0(M_0, L_0^k)$. Define the numbers $a_0, a_1, b_0, b_1$ to be the coefficients in the expansions
\[ d_k = a_0k^m + a_1k^{m-1} + \ldots \]
\[ w_k = b_0k^{m+1} + b_1k^m + \ldots, \]
valid for large $k$. We define the Futaki invariant of the action $C^\infty$-action $\alpha$ on $(M_0, L_0)$ to be
\[ \text{Fut}(\alpha, M_0, L_0) = \frac{a_1}{a_0}b_0 - b_1. \]
Since the $C^\infty$-action is induced by the test-configuration $(\mathcal{M}, \mathcal{L})$, we also write
\[ \text{Fut}(\mathcal{M}, \mathcal{L}) = \text{Fut}(\alpha, M_0, L_0). \]

Definition 25. The polarized manifold $(M, L)$ is K-polystable if for all test-configurations $\text{Fut}(\mathcal{M}, \mathcal{L}) \geq 0$ with equality only if the central fiber of the test-configuration is isomorphic to $M$.

We want to study the K-stability of the blowup $Bl_p M$ with the polarization $\pi^*L - \varepsilon E$ for sufficiently small $\varepsilon$. For this the key calculation is to compute the Futaki invariants of $C^\infty$-actions on $Bl_p M$.

Let us fix a Kähler metric $\omega \in c_1(L)$, and suppose that the vector field $v = J\nabla h$ generates a holomorphic $S^1$-action for some $h \in C^\infty(M)$. Replacing $L$ by a large power if necessary, a choice of $h$ gives rise to a lifting of the $S^1$-action to $L$ which can be extended to a holomorphic $C^\infty$-action. We define the Futaki invariant of $v$ with the same formula as above. If the vector field $v$ vanishes at $p$ then it has a holomorphic lift $\tilde{v}$ to $Bl_p M$. Consider the $\mathbb{Q}$-line bundle $\mathcal{L}_\varepsilon = \pi^*L - \varepsilon E$ on $Bl_p M$ for small rational $\varepsilon$, where $E$ is the exceptional divisor. We have a $C^\infty$-action on the space of sections $\mathcal{H}^0(Bl_p M, \mathcal{L}_\varepsilon^k)$, and so we can define constants $\tilde{a}_0, \tilde{a}_1, \tilde{b}_0, \tilde{b}_1$ corresponding to this action as above, which depend on $\varepsilon$ (if we take $k$ for which
$k \varepsilon$ is an integer, then $L^k_p$ is a line bundle). The following lemma is an extension of the calculation in Stoppa [29].

**Lemma 26.** For the action on the blowup we have

\begin{align*}
\tilde{a}_0 &= a_0 - \frac{\varepsilon^m}{m!} \\
\tilde{a}_1 &= a_1 - \frac{\varepsilon^{m-1}}{2(m-2)!} \\
\tilde{b}_0 &= b_0 + \frac{\varepsilon^m}{m!} h(p) + \frac{\varepsilon^{m+1}}{(m+1)!} \Delta h(p) \\
\tilde{b}_1 &= b_1 + \frac{\varepsilon^{m-1}}{2(m-2)!} h(p) + \frac{(m-2)\varepsilon^m}{2m!} \Delta h(p),
\end{align*}

where $\Delta h$ is the Laplacian of $h$ with respect to the metric $\omega$.

**Proof.** Let us write $I_p$ for the ideal sheaf of $p \in M$. For large $k$ we have an isomorphism

$$H^0(Bl_p M, L^k_p) = H^0(M, I_p^k \otimes L^k_p).$$

To study this space, we use the exact sequence

$$0 \rightarrow I_p^k \otimes L^k_p \rightarrow L^k_p \rightarrow \mathcal{O}_{k^p} \otimes L^k_p \rightarrow 0.$$ 

As before, let us write $d_k$ and $w_k$ for the dimension of $H^0(M, L^k_p)$ and the weight of the action on this space. Similarly write $\tilde{d}_k$ and $\tilde{w}_k$ for the dimension of, and weight of the action on, $H^0(Bl_p M, L^k_p)$. From the exact sequence we have

$$d_k = d_k - \dim \mathcal{O}_{k^p}$$

$$\tilde{d}_k = d_k - \dim \mathcal{O}_{k^p}$$

$$\tilde{w}_k = w_k - w(\mathcal{O}_{k^p} \otimes L^k_p).$$

Here the weight $w(\mathcal{O}_{k^p} \otimes L^k_p)$ is given by

$$w(\mathcal{O}_{k^p} \otimes L^k_p) = w(\mathcal{O}_{k^p}) - kh(p) \dim(\mathcal{O}_{k^p}),$$

since the weight of the action on the fiber $L_p$ is $-h(p)$. The sign here depends on our convention that the real part of the $C^*$-action corresponding to $h$ is generated by $\nabla h$.

We can think of $\mathcal{O}_{l^p}$ for an integer $l > 0$ as being the space of $(l-1)$-jets of functions at $p$, ie.

$$\mathcal{O}_{l^p} = \mathbb{C} \oplus T^*_{l^p} \oplus \ldots \oplus S^{l-1}T^*_{l^p},$$

where $S^i$ is the $i^{th}$ symmetric product. The dimension of $\mathcal{O}_{l^p}$ is therefore given by

$$\dim(\mathcal{O}_{l^p}) = \binom{m+l-1}{m} = \frac{1}{m!} \left( \frac{m(m-1)}{2} l^{m-1} + O(l^{m-2}) \right).$$

Similarly if we write $w$ for the weight of the action on $T^*_{l^p}$, then we can compute that

$$w(\mathcal{O}_{l^p}) = \left[ \frac{m+l-1}{m+1} \right] w = \frac{w}{(m+1)!} \left( l^{m+1} + \frac{(m-2)(m+1)}{2} l^{m} + O(l^{m-1}) \right).$$
Substituting $k \varepsilon$ for $l$ and using the formulas \[eqref{26}\] we get
\[
d_k = d_k - \frac{\varepsilon^m}{m!} k^m - \frac{\varepsilon^{m-1}}{2(m-2)!} k^{m-1} + O(k^{m-2})
\]
\[
\bar{w}_k = w_k - \left( \frac{w^m_{m+1}}{(m+1)!} - \frac{h(p) \varepsilon^m}{m!} \right) k^{m+1}
- \left( \frac{(m-2) w^m_{m}}{2m!} - \frac{h(p) \varepsilon^{m-1}}{2(m-2)!} \right) k^m + O(k^{m-1})
\]
The only thing that remains is to see that the weight of the action on $T_p^*$ is given by $w = -\Delta h(p)$. This follows from the fact that by our convention the induced action on the tangent space $T_p$ is given by the Hessian of $h$ at $p$.

A simple calculation then gives

**Corollary 27.** If the Futaki invariant $\text{Fut}(v, M, L) = 0$ on $M$ then on the blowup we have
\[
\text{Fut}(\tilde{v}, Bl_p M, L_c) = \frac{a_1}{a_0} \tilde{b}_0 - \tilde{b}_1
\]
\[eqref{27}\]
\[
= -\frac{\varepsilon^{m-1}}{2(m-2)!} h(p) - \frac{\varepsilon^m}{m!} \left( \frac{m-2}{2} \Delta h(p) - \frac{a_1}{a_0} h(p) \right) + O(\varepsilon^{m+1}),
\]
if $m \geq 3$,
\[
\text{Fut}(\tilde{v}, Bl_p M, L_c) = -\frac{\varepsilon}{2} h(p) + \frac{\varepsilon^2 a_1}{2a_0} h(p) + \frac{\varepsilon^3}{2a_0} \left( \frac{a_1}{3} \Delta h(p) - \frac{h(p)}{2} \right) + O(\varepsilon^4),
\]
if $m = 2$ and $a_1 \neq 0$ and finally
\[
\text{Fut}(\tilde{v}, Bl_p M, L_c) = -\frac{\varepsilon}{2} h(p) + \frac{\varepsilon^3}{4a_0} h(p) - \frac{\varepsilon^4}{12a_0} \Delta h(p) + O(\varepsilon^5),
\]
if $m = 2$ and $a_1 = 0$.

In each case if $h(p) = \Delta h(p) = 0$, then $\text{Fut}(\tilde{v}, Bl_p M, L_c) = 0$ for all $\varepsilon$.

Combining this with Proposition \[eqref{11}\] from Section \[eqref{2}\] we can prove Theorem \[eqref{5}\].

**Proof of Theorem \[eqref{3}\].** Let us assume that $m > 2$, since the argument in the $m = 2$ case is essentially identical. We argue by contradiction. We can also assume that $p$ is semistable with respect to the polarization $L$ since if it were strictly unstable, then Stoppa’s result \[eqref{29}\] implies that the pair $(Bl_p M, L_c)$ is K-unstable for all sufficiently small $\varepsilon$.

So suppose that $p$ is semistable with respect to $L$, but it is not polystable with respect to $L + \varepsilon K_M$ for sufficiently small $\varepsilon$. Then by Proposition \[eqref{11}\] there exists a one-parameter subgroup $\lambda$ such that $\lim_{t \to 0} \lambda(t) \cdot p = q$ and $w_L(q, \lambda) = 0$, but $w_{K_M}(q, \lambda) \leq 0$. By blowing up the trivial family $M \times C$ in the closure of the $C^\ast$-orbit of $(p, 1)$ under the action $t(p, 1) = (\lambda(t)p, t)$ we obtain a test-configuration for $(Bl_p M, L_c)$ with central fiber $(Bl_q M, L_c)$, and the action on $Bl_q M$ is simply the lifting of the action $\lambda$. Suppose that this $\lambda$ is generated by a holomorphic vector field with Hamiltonian function $h$. Then $w_{L}(q, \lambda) = -h(q) = 0$ and $w_{K_M}(q, \lambda) = -\Delta h(q) \leq 0$. If $-\Delta h(q) < 0$ then from the formula \[eqref{27}\] we see that for small enough $\varepsilon$ the Futaki invariant of this test-configuration is negative. Whereas if $\Delta h(q) = 0$ then we get that the Futaki invariant is zero. In both cases this means that $(Bl_p M, L_c)$ is not K-polystable.
To deal with the different situations when \( m = 2 \), note that by the Hirzebruch-Riemann-Roch formula \( a_1 = -\frac{1}{2}K_M \cdot L \).

To relate GIT stability with respect to \( L + \delta K_M \) (or \( L - \delta K_M \)) to moment maps, we need the following.

**Lemma 28.** Let \( X \) be a holomorphic Killing field on \((M, \omega)\). If \( \iota_X \omega = dh \), then \( \iota_X \rho = -d\Delta h \), where \( \rho \) is the Ricci form. It follows that if \( p \) is polystable with respect to the polarization \( L + \delta K_M \) for the action of the Hamiltonian isometry group \( H \), then there is a point \( q \) in the complex orbit \( H^c \cdot p \) such that \( \mu(q) + \delta \Delta \mu(q) = 0 \).

**Proof.** We can compute

\[
2\iota_X \rho = \iota_X (dJ d\log \det(\omega)) = -d\iota_X (J d\log \det(\omega)) = d(\mathcal{L}_{JX} \log \det(\omega)) = d\Lambda \mathcal{L}_{JX} \omega,
\]

where \( \mathcal{L} \) is the Lie derivative and \( \Lambda \) is taking the trace with respect to \( \omega \).

It follows that if \( \mu \) is a moment map for the action of \( H \) with respect to the symplectic form \( \omega \) then \( \mu + \delta \Delta \mu \) is a moment map with respect to \( \omega - \delta \rho \). Since \( \omega - \delta \rho \in c_1(L + \delta K_M) \), the second statement in the lemma follows from the Kempf-Ness theorem.

Finally we give the proof of Corollary 7.

**Proof of Corollary 7**. We will prove the following three implications.

(i)\(\Rightarrow\)(ii) This follows from the main theorem in [30].

(ii)\(\Rightarrow\)(iii) This follows from Theorem 5, even though we are using a more restrictive version of K-polystability. The reason is that in finite dimensions when using the Hilbert-Mumford criterion for testing stability of a point, it is enough to look at one-parameter subgroups which commute with a torus fixing the point. This follows for example from the theory of optimal destabilizing one-parameter subgroups (see Kempf [15]), or also from the Kempf-Ness theorem and the observation that \( \mu(p) \) is always in the center of the stabilizer of \( p \) since \( \mu \) is equivariant.

(iii)\(\Rightarrow\)(i) This follows from Theorem 1. Namely if \( p \in M \) is GIT polystable then by replacing \( p \) by a different point in its \( H^c \)-orbit, we can assume that \( \mu(p) = 0 \), so \( Bl_p M \) admits an extremal metric in the class \( \pi^* [\omega] - \varepsilon E \) for small \( \varepsilon \). Since \((M, \omega)\) is Kähler-Einstein, the Futaki invariant of any vector field on \( M \) is zero with respect to the class \([\omega]\). Since \( \mu(p) = 0 \) and \( \rho = \omega \), Lemma 28 implies that \( \Delta \mu(p) = 0 \), so from Corollary 27 the Futaki invariant of every vector field on \( Bl_p M \) vanishes in the classes \( \pi^* [\omega] - \varepsilon E \).

This means that the extremal metric we obtain actually has constant scalar curvature.

\[\square\]

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