Lump, lump-stripe, and breather wave solutions to the (2 + 1)-dimensional Sawada-Kotera equation in fluid mechanics

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A B S T R A C T
The present study investigates the lump, one-stripe, lump-stripe, and breather wave solutions to the (2+1)-dimensional Sawada-Kotera equation using the Hirota bilinear method. For lump and lump-stripe solutions, a quadratic polynomial function, and a quadratic polynomial function in conjunction with an exponential term are assumed for the unknown function \( f \) giving the solution to the mentioned equation, respectively. On the other hand, only an exponential function is considered for one-stripe solutions. Besides, the homoclinic test approach is adopted for constructing breather wave solutions. The propagations of the attained lump, lump-stripe, and breather wave solutions are shown through some graphical illustrations. The graphical outputs demonstrate that the lump wave moves along the straight line \( y = -x \) and exponentially decreases away from the origin of the spatial domain. On the other hand, lump-kink solutions illustrate the fission and fusion interaction behaviors upon the selection of the free parameters. Fission and fusion processes show that the stripe soliton splits into a stripe soliton and a lump soliton, and then the lump soliton merges into one stripe soliton. In addition, the achieved breather waves illustrate the periodic behaviors in the \( x y \)-plane. The outcomes of the study can be useful for a better understanding of the physical nature of long waves in shallow water under gravity.

1. Introduction

Solitary wave solutions including rational type analytic solutions to nonlinear partial differential equations (PDEs) are frequently used as reliable tools in describing various quantitative and qualitative phenomena in many research fields, such as optical fiber communication system, fluid mechanics, water waves, and nonlinear sciences [1, 2, 3, 4, 5]. Nowadays, rational type analytic solutions of nonlinear PDEs have been paid much attention to the research community, for they can explain the behavior of localized waves (lumps, solitons, rogues, and breathers) and their interactions [1]. A lump solution (LS) is a special kind of rational function solution localized in all spatial directions, whereas a soliton solution is an analytic solution that is exponentially localized in all directions in space and time [6, 7, 8]. LSs are found when surface tension dominates the shallow water surface [8]. The interaction between two different classes of solutions, such as interactions between lumps and stripe solitons, rogue waves and solitary waves, etc., can explain the natural nonlinear wave phenomena in more detail [9]. Symbolic computation of LSs starts with the study on the (2+1)-dimensional (hereafter (2+1)-D) Kadomtsev-Petviashvili (KP) equation via the Hirota bilinear method (HBM) in ref. [10]. In soliton theory, the HBM is considered as one of the most straightforward direct methods that can extract different types of analytic solutions to integrable nonlinear evolution equation (NLEEs) arising in Mathematics and Physics [11]. In order to justify the efficiency and straightforwardness of the HBM, many scholars constructed the lump, mixed lump-kink, lump-soliton, rogue, and breather wave solutions to some integrable NLEEs incorporating a variety of test functions [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. For example, Ma and Zhou [12] systematically established a general theory for finding the LSs with the assist of the Hirota bilinear form (HBF) to a variety of nonlinear wave equations. Based on the Hirota formulation, Yang and Ma [13] constructed a class of LSs to the (2+1)-D BKP equation, whereas mixed lump-kink solutions were obtained to the same equation by Zhang

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and Ma [14]. Huang and Chen [15] investigated (2+1)-D Sawada Kotera (SK) equation for positive quadratic function solution, whereas Huang and Chen [16] investigated nonlocal symmetry and similarity reductions for the Drinfeld–Sokolov–Satsuma–Hirota system. In ref. [17], the authors examined (3+1)-D Jimbo-Miwa (JM) equation for lump-type, kinkly breather degeneracy, periodic degeneracy, stripe-lump-type solutions based on standardization of lump-type solutions. In ref. [18], the authors studied lump solutions and the interaction solutions between lump and resonance stripe solitons of the (2+1)-D SK equation. In ref. [7], the authors investigated (2+1)-D generalized Bogoyavlensky-Konopelchenko (gBK) equation for lump-type and lump solutions. Ma [6] considered (3+1)-D linear PDE for the lump, lump-soliton, lump-periodic solutions, Cheng et al. [8] studied (3+1)-D KP equation for lump-type solutions, Ma [19] inspected (2+1)-D linear PDE for the lump, lump-kink, and lump-soliton solutions, Manafian and Lakestani [20] considered (2+1)-D bidirectional SK equation for lump-kink, lump-soliton, periodic kink-wave, periodic soliton, periodic wave solutions, Wang et al. [21] determined lump, mixed lump-stripe, and rogue wave-stripe solutions to a (3+1)-D nonlinear wave equation for a liquid with gas bubbles, Yang et al. [22] studied combined soliton equation involving three fourth-order nonlinear terms in (2+1)-D dispersive waves and determined its lump solutions, Ma et al. [23] computed lump solutions to a combined fourth-order equation involving three types of nonlinear terms in (2+1)-D dispersive waves, Du et al. [24] constructed the multi-soliton solutions of the modified Zakharov-Kuznetsov (mZK) equation, and Ma and Chen [25] applied some direct search approaches to find analytic solutions to the nonlinear Schrödinger (NLS) equation. More recently, based on the Hirota trilinear form of the second member in the KP hierarchy, Ma and Zhang [26] investigated lump solutions in nonlinear dispersive waves with higher-order rational dispersion relations.

In this study, we investigate an integrable nonlinear PDE, viz. the (2+1)-D SK equation, which reads [27, 28]:

\[
\left(u_{xxx} + 5uu_x + \frac{5}{3} u^3 + S u_y \right)_x - u_x + 5uu_y - 9 \int u_yd(x + Su_x) + \int u_x dx = 0,
\]

(1)

where \( u = u(x, y, t) \) represents a real differentiable function of the spatial variables \( x, y \) and the temporal variable \( t \), and the subscripts represent the partial derivatives with respect to the independent variables.

The (2+1)-D SK equation is the extended form of the Korteweg-de Vries (KdV) equation that can explain long-wave phenomena. Equation (1) was proposed by Sawada and Kotera [29], and also by Caudrey et al. [30] independently. According to Liu [27] and Li et al. [31], Eq. (1) has been studied well due to its wide applications in many research fields of science and engineering, such as the gravitational force field, the conformal field, the two-dimensional quantum gravity gauge field, and the long waves in shallow water under gravity.

It is worth noting that the Burgers-type and Boussinesq–Burgers type equations also emerge in the study of fluid flow and represent the proliferation of shallow water oceanic waves [4, 5]. It is pertinent to note here that Burgers system, with the proper choice of the parameters involved, can be shown to be similar to the well-known incompressible Navier-Stokes equations excluding the pressure and continuity that can be used in the models for studying hydrodynamical turbulence and wave processes in non-linear media [4]. Gao et al. [4] worked out on hetero-Bäcklund (non-auto-Bäcklund) transformation for an extended (2+1)-D coupled Burgers system in fluid mechanics, whereas Gao et al. [5, 32, 33] worked out on the auto- and non-auto-Bäcklund transformations for a higher-order Boussinesq-Burgers system, a generalized (2+1)-D dispersive long-wave system, and a variable-coefficient nonlinear dispersive-wave system, respectively, for studying oceanic waves including the Earth, Enceladus, and Titan.

However, we concentrate on studying the (2+1)-D SK equation for exploring its unreported wave structures. It is of interest to note that by means of the HBMs [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25], the equation has been investigated for different positive quadratic and periodic wave solutions in several studies in the past. Zhang and Ma [34] studied the (2+1)D SK equation for LSs with its bright and dark characteristics, where the unknown function, as we will see later, was assumed as a positive quadratic function \( f = g^2 + h^2 \) with \( g = g_1 x + g_2 y + g_3 t + g_4, h = a x + a_4 y + a_5 t + a_6 ) \). Huang and Chen [15], in their study, discussed lump and lump-stripe solutions for the (2+1)-D SK equation, where the unknown function was similar to that presented in ref. [34] for obtaining lump and lump-stripe solutions (the unknown function was assumed as \( f = g^2 + h^2 + n x^2 + a_1 x + a_2 y + a_3 t + a_4, h = a x + a_4 y + a_5 t + a_6, n = k_1 x + k_2 y + k_3 t + k_4 \)). In refs. [15, 34], an inelastic interaction between a lump and a stripe was found to be drowned or swallowed by the stripe solution. It is claimed in several studies that LSs for some nonlinear PDEs restore their shapes, velocities, and amplitudes after the interaction with solitons, which means the interaction is considered to be completely elastic [15, 16, 35, 36]. Inelastic interactions can also be found to happen under certain conditions in different fields of nonlinear science and engineering [15, 34, 35, 36, 37, 38, 39, 40]. However, Li et al. [7] considered a quadratic polynomial function for obtaining lump type solutions to the (2+1)-D gBK equation, which is more general than the other considered positive-quadratic functions executed in refs. [15, 34].

The present study undertakes the assumption due to Li et al. [7] for unknown function, where a quadratic polynomial unknown function \( f \) is assumed for LSs, as we will see later. As far as to the authors’ knowledge, Li et al.’s [7] assumption is relatively a new approach for constructing the lump and lump-stripe solutions to the (2+1)-D SK equation. Also, there is no record to study the mentioned equation via the homoclinic test approach (HTA) available in ref. [41, 42]. Accordingly, the main intention of this study is to explore lump and lump-stripe solutions following the assumption due to Li et al. [7] with the fission and fusion characteristics [43, 44, 45, 46] between the lump and stripe solutions, and the breather wave solutions via the HTA [41, 42] to the (2+1)-D SK equation. Thus, along with the approach adopted in the study, the breather wave solutions via the HTA are the novelty of the study that may enrich the collection of nonlinear features to the (2+1)-D SK equation.

The rest of the paper is structured as follows. Section 2 deals with the Hirota bilinear representation of the (2+1)-D SK equation. Lump, one-stripe soliton, and lump-stripe solutions are illustrated in section 3 and its subsections. Section 4 and its subsections deal with the different wave profiles of the solutions obtained by using the HTA. Finally, conclusions based on the obtained results are presented in section 5.

2. Hirota form of the (2+1)-D SK equation

In this study, the following transformation is used to obtain the equivalent HBF of the (2+1)-D SK equation represented by (1):

\[
u = 6(lnf)_{t_x} = \frac{6(f_{xx}f-f_{x}^2)}{f^2},
\]

(2)

where \( f \) is an unknown function associated with the independent variables \( x, y, \) and \( t \). Equation (1), by dint of the transformation specified by Eq. (2), can be brought to the following form:

\[
\left(D_{t}^6 - D_{t}D_{x} + 5D_{t}D_{y} - 5D_{t}^2 \right) f.f = 0.
\]

(3)
which yields
\[
2 \left[ (f f_{xxxxx} - 6 f_x f_{xxx} + 15 f_{xx} f_{xx} - 10 f_{xxx}^2 ) - (f f_{xx} - f_x f_x) + 5 \left( 3 f_{xx} f_{xx} - 3 f_x f_{xxx} + f f_{xxxx} - f_{xxx} f_x - 5 \left( f f_{xy} - f_{xy}^2 \right) \right) = 0.
\]

In Eq. (3), \( D \) represents the Hirota bilinear operator (HBO). Its subscripts and superscripts indicate the Hirota bilinear derivative (HBD) with respect to the corresponding independent variable and order of the derivative, respectively. The HBOs \( D_x \), \( D_y \), and \( D_t \) are defined as [34]
\[
D_x^m D_y^n D_t^q (f(x,y,t), g(x',y',t'))
= \left( \frac{\partial}{\partial x} \right)^m \left( \frac{\partial}{\partial y} \right)^n \left( \frac{\partial}{\partial t} \right)^q f(x,y,t)g(x',y',t')|_{x=x', y=y', t=t'}.
\]

To attain different types of nonlinear characteristics of the (2+1)-D SK equation, the suitable choice of the unknown function \( f \) can be used.

3. Lump, single stripe, and lump-stripe solutions

3.1. Lump solutions

Following Li et al. [7], the unknown function \( f \) is assumed in the following form:
\[
f = X^T A X + \alpha,
\]
where \( X = (x, y, t)^T \) and \( A = (a_{ij})_{3\times3} \) is a symmetric matrix over the real field \( R \). Here \( m_{ij} \) (\( i,j = 1,\ldots,4 \)) and \( \alpha \) are real constants to be determined. By means of the unknown function \( f \) and Eq. (3), a polynomial equation in \( x, y, \) and \( t \) is obtained. Setting the coefficients of the polynomial equation to zero, a set of nonlinear algebraic equations in \( m_{ij} (i,j = 1,\ldots,4) \) and \( \alpha \) is obtained. The solution of the nonlinear algebraic equations can be obtained as
\[
\begin{align*}
m_{14} &= \frac{-m_{22} m_{33} a_{12} a_{13} - 5 m_{22} m_{23}^2 - 2 m_{12} m_{23} a_{24} + 10 m_{12} m_{32} a_{33}}{2 m_{23}} , \\
m_{24} &= \frac{-10 m_{12}^2 a_{12} + 5 m_{23} m_{13} - 3 m_{13} m_{23} a_{24}}{m_{23} (m_{13} - 5 m_{23})}, \\
m_{34} &= \frac{5 m_{23}^2 (m_{23} - 5 m_{13})}{m_{23}}, \\
m_{44} &= \frac{-2 m_{23}^2 (m_{33} - 5 m_{13})}{m_{23}}, \\
\alpha &= -m_{11} \frac{-m_{22} m_{33} a_{12} a_{13} - 2 m_{12} m_{23} a_{24} + 5 m_{12} m_{32} a_{33}}{2 m_{23}}.
\end{align*}
\]

where \( P = (m_{23}^2 - 25 m_{33}^2), Q = (m_{22} - 10 m_{33}), R = (m_{12} m_{23} - m_{13} m_{23}) \), and \( S = (m_{13} a_{24} + 30 m_{33}^3) \), wherein \( m_{11}, m_{12}, m_{13}, m_{23}, m_{24}, \) and \( m_{33} \) are free parameters. It is clear from Eq. (5) that \( m_{23} \neq 0 \) and \( m_{24}^2 - 25 m_{33}^2 \neq 0 \). These are the nonzero conditions for the function to be analytic.

Equation (4) can be put in the following form:
\[
f = m_{22} x^2 + 2 m_{23} x y + 2 m_{24} x t + m_{33} y^2 + 2 m_{34} y t + m_{44} t^2 + 2 m_{12} x + 2 m_{13} y + 2 m_{14} t + m_{11} + \alpha.
\]

Substituting the computed values of the coefficients given by Eq. (5) into Eq. (6), the unknown function \( f \) is obtained for LS in the following form:
\[
f = \frac{-10 m_{12}^2 m_{23} + 5 m_{33}^2 (m_{23}^2 - 2 m_{24} x y + 2 m_{24} x t + m_{33} y^2 + 2 m_{34} y t + m_{44} t^2 + 2 m_{12} x + 2 m_{13} y + 2 m_{14} t + m_{11} + \alpha)}{m_{23}},
\]

where \( m_{11}, m_{12}, m_{13}, m_{23}, m_{24}, \) and \( m_{33} \) are free parameters. Equation (7) with the help of the transform specified by Eq. (2) yields the following solution:
\[
u(x, y, t) = 6 \left( \frac{f_x x}{f} - \frac{f_x x}{f^2} \right) = 12 \left[ \frac{-10 m_{12}^2 m_{33}^2 m_{23} - 5 m_{24} x y + 2 m_{24} x t + m_{33} y^2 + 2 m_{34} y t + m_{44} t^2 + 2 m_{12} x + 2 m_{13} y + 2 m_{14} t + m_{11}}{f^2} \right].
\]

To visualize the propagation characteristics of the lump wave graphically, we set the values of the free parameters as \( m_{11} = 1, m_{12} = 3, m_{13} = 1, m_{23} = \frac{1}{7}, m_{24} = 1, \) and \( m_{33} = 1 \) in Eq. (8). This leads to
\[
u(x, y, t) = \frac{15}{2B} \left( \frac{6}{4} x + y + 2t + 6 \right)^2,
\]

where \( B = \frac{1}{2} x^2 + xy + 2xt + y^2 - 8yt + 40x^2 + 6x + 2y + 32t + 923 \).

It is to be noted here that one must maintain the nonzero conditions in choosing the values of the free parameters for \( \alpha \) to be analytic. It is computed that \( u \rightarrow 0 \) as \( \sqrt{x^2 + y^2} \rightarrow \infty \) in all directions in the \( xy \)-plane. The maximum and minimum values of \( u \) are also calculated. For maxima or
obtained where

\[ Eq.(9). \]

It is also perceived from Fig. 1(c) that the lump wave moves along the straight line \( y = -x \) and exponentially decreases away from the origin of the spatial domain. The 2D line graphs displayed in Figs. 1(d)-(f) show the projected wave profiles on the coordinate line with the variations in the independent variables.

### 3.2. Single stripe soliton solutions

In order to search for one-stripe solutions to the (2+1)-D SK equation, the unknown function \( f \) is assumed in the following way:

\[ f = 1 + \beta e^{ax + yd_1 + zd_2 + d_3 + d_4}. \]

(10)

where \( d_i \) (i = 1, 2, 3, 4) and \( \beta \neq 0 \) are real constants to be determined. A similar process mentioned above has been maintained for getting the values of the parameters and it yielded

\[ d_3 = \frac{d_1^6 + 5d_1^4d_2 + 5d_2^2}{d_1^2}. \]

(11)

where \( d_1, d_2, \) and \( d_4 \) are free parameters with the nonzero condition \( d_1 \neq 0 \). As an outcome, the following class of one-stripe soliton solutions is obtained from Eq. (2) by substituting Eq. (10) with the obtained result in Eq. (11):
Fig. 2. Single-stripe wave profile of the solution given by Eq. (13) through different plots; (a) 3D plot (at \( t = 0 \)); (b) the density plot of (a); (c) the contour plot of (a); (d) the projection of the wave profile on the \( y \)-axis at \( t = 0 \), where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles for \( x = -10 \), \( x = 0 \), and \( x = 10 \), respectively; (e) the projection of the wave profile on the \( x \)-axis at \( t = 0 \), where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles for \( y = -10 \), \( y = 0 \), and \( y = 10 \), respectively; (f) the projection of the wave profile on the \( x \) - \( y \) plane.

\[
\begin{align*}
    u(x, y, t) &= \frac{6\beta d_1 e^{x t_1} y + \beta d_2 e + \beta d_3 e^{-x t_1} + \beta d_4 e^{-y t_1}}{1 + \beta e^{x t_1} y} \cdot \frac{e^{x t_1} y + \beta d_2 e + \beta d_3 e^{-x t_1} + \beta d_4 e^{-y t_1}}{1 + \beta e^{x t_1} y} \\
    &\quad - \frac{6\beta^2 d_t^2 e^{x t_1} y + \beta d_2 e + \beta d_3 e^{-x t_1} + \beta d_4 e^{-y t_1}}{1 + \beta e^{x t_1} y} \cdot \frac{e^{x t_1} y + \beta d_2 e + \beta d_3 e^{-x t_1} + \beta d_4 e^{-y t_1}}{1 + \beta e^{x t_1} y}.
\end{align*}
\]

(12)

Setting the values of the free parameters, namely \( d_1 = 1 \), \( d_2 = \frac{1}{2} \), \( d_3 = 1 \), and \( \beta = \frac{1}{2} \) in Eq. (12), it can be brought to the following form:

\[
\begin{align*}
    u(x, y, t) &= \frac{9e^{x t_1} y + \beta d_2 e + \beta d_3 e^{-x t_1} + \beta d_4 e^{-y t_1}}{1 + \beta e^{x t_1} y} \cdot \frac{e^{x t_1} y + \beta d_2 e + \beta d_3 e^{-x t_1} + \beta d_4 e^{-y t_1}}{1 + \beta e^{x t_1} y} \\
    &\quad - \frac{27e^{2(x y + x t_1 + y t_1)}}{2(1 + \frac{1}{4} e^{x t_1} y)}
\end{align*}
\]

(13)

From Eq. (12), one can easily see that the function is localized along all directions. For a clear understanding, Fig. 2 is presented via the solution given by Eq. (13), from where one can see that the profile of the solution specified by Eq. (13) is a bright soliton, which is presented in the \( (x, y) \)-plane. The 3D plot shows that the solution is localized in all spatial directions. It is clear from Fig. 2 that the single soliton moves in the negative \( x \) and \( y \) directions with uniform velocity and amplitude with the change of \( x \), \( y \), and \( t \), respectively (see Fig. 2(d)-(f)).

3.3. Lump-stripe solutions

In order to explore lump-stripe solutions, a new unknown function \( f \) is assumed as a linear combination of two different unknown functions, viz. the right-hand side of Eq. (4) and that of Eq. (10). Thus, the new \( f \) can be put in the following form:

\[
    f = X^T A X + \beta e^{x t_1} y + \beta e^{-x t_1} y + a.
\]

(14)

where the description of the involved parameters are aforementioned. It is worth noting that the same process mentioned above was undertaken to obtain the values of the parameters. The obtained values are given by
Among the fifteen parameters, $m_{11}$, $m_{12}$, $m_{13}$, $m_{22}$, $d_1$, $d_2$, and $d_4$ are found to be free with the nonzero conditions $m_{22} \neq 0$, $d_1 \neq 0$, and $d_1^3 + 4d_2 \neq 0$. Substituting the parametric values represented by (15) in Eq. (14), the unknown function $f$ is obtained. Thus, the lump-stripe solution to the (2+1)-D SK equation is obtained via the specified transformation given by Eq. (2). As we have seen in the previous subsections. Since the explored analytic solution is very large in size, we think the presentation of it will only be space consuming. Thus, the computed analytic solution is not given here, but the particular solution for suitable values of the parameters, namely $m_{11} = 2$, $m_{12} = 1$, $m_{13} = 1$, $m_{21} = 1$, $m_{22} = 10$, $d_4 = 0$, and $\beta = 1$ is conferred below:

\[
\begin{align*}
\alpha &= -m_{11} - 4d_1^2m_{11} - 4d_1^2m_{13} + 4d_2^2m_{11}^2 - 8d_1^2m_{11}m_{13} - 12d_1d_2m_{11} - 4d_1m_{11} - d_2m_{12}^2, \\
\beta &= d_1^3 + 5d_1^2d_2 - 5d_2^3, \\
m_{14} &= 5(d_1^2m_{12} + d_1m_{13} + 2d_2^2)d_2m_{12} - 2d_1m_{11} + d_1^2m_{12}^2), \\
m_{23} &= \frac{5}{d_1^2} \left[m_{12}(d_1^2 + 4d_1d_2 + d_2^2) - 2d_2^2 - 2d_1^2 \right], \\
m_{24} &= \frac{5}{d_1^2} \left[m_{12}(d_1^2 + 4d_1d_2 + d_2^2) - 2d_1^2 \right], \\
m_{25} &= \frac{5}{d_1^2} \left[m_{12}(d_1^2 + 4d_1d_2 + d_2^2) - 2d_1^2 \right], \\
m_{44} &= \frac{5}{d_1^2} \left[m_{12}(d_1^2 + 4d_1d_2 + d_2^2) - 2d_1^2 \right].
\end{align*}
\]

(15)

In order to explain the inherent mechanism of the solution specified by Eq. (16), the interaction between one lump soliton and one stripe soliton is described graphically. One lump and one stripe soliton interactions yield a lump-off wave solution. Before and after a specific time, lump-off solutions have the property that the lump waves are cut by the one stripe soliton waves. The negative and positive values of the free parameters $d_1$ and $d_2$ of the attained solutions presented in Eq. (16) can signify the reasons for the fission and fusion interactions between one lump and one stripe soliton. In this case, four assumptions can be made, such as (i) $d_1 > 0$ and $d_2 > 0$, (ii) $d_1 < 0$ and $d_2 > 0$, (iii) $d_1 < 0$ and $d_2 < 0$, and (iv) $d_1 > 0$ and $d_2 < 0$. Figs. 3 and 4 present such types of interaction solutions in the $(x, y)$-plane with $d_1, d_2 > 0$ and $d_1, d_2 < 0$, respectively. Figs. 3(a)-(c) display the 3D plots in the $(x, y)$-plane when $t = -5, 0, 5$, whereas Figs. 3(d)-(f) represent the surface views of Figs. 3(a)-(c), respectively. It can be seen from Fig. 3(a) that there is only one-stripe soliton, while the lump soliton is hidden at $t = -5$. With the evolution of time ($t = 0.5$), the lump wave is viable and it becomes enlarged gradually and finally, the lump soliton is separated from one stripe soliton (see Figs. 3(b)-(c)). Such phenomenon is known as fission interaction between one lump and one stripe solitons [40, 43, 44, 45, 46], which is more clearly observable in Figs. 3(d)-(f). When $d_1, d_2 < 0$, the opposite characteristics as that obtained for $d_1, d_2 > 0$ can be perceived from Fig. 4. It is seen from Fig. 4(a) that the lump wave is fully observable and completely separated from one stripe soliton when $t = -5$. In such a situation, the amplitude of the lump wave is maximum. It can also be seen from the figure (Figs. 4(b)-(c)) that the lump wave is gradually integrated into the soliton and disappears as time goes on ($t = 0.5$). Such phenomenon is known as fusion interaction between a lump soliton and one stripe soliton [40, 43, 44, 45, 46], which is visible in Figs. 4(d)-(f). From these figures, it is also perceived that during the fusion interaction process, the amplitude of the lump decreases. However, the amplitude of the one-stripe soliton remains unchanged. Thus, it is justifiable to say that the energy of the lump wave is transmitted into the one stripe soliton. For $d_1 < 0, d_2 > 0$ and $d_1 > 0, d_2 < 0$, the multi-stripe phenomena have also been observed via the solution presented in Eq. (16), respectively. However, the figures are not included for the sake of brevity. From the figures (Figs. 3 and 4), it is seen that the fusion-fission interaction is completely inelastic. After the interaction of lump with stripe, the amplitude of the lump varies as the time increases, but the dynamic characteristics (amplitude and velocity) of the stripe soliton remain unchanged. It is mentioned here that the speed of the lump wave was about ten times higher than that of the stripe wave.

4. Homoclinic test approach

For obtaining breather and multiple wave profiles, the HTA is adopted. The unknown function in this regard has been chosen in the following form [41, 42]: 
where

\[
\begin{align*}
&y_3: \quad \text{free process} \\
&\alpha \cos b c i_0 = c_1 = -9 c_2 = -10 b c_1 = 2 b c_1 + c_3 = 0 \\
&\beta = \{1, 2, 3\}, \quad a \neq 0, \quad \text{and} \quad \beta \neq 0 \quad \text{are real constants to be evaluated. The coefficients constitute the following four sets of values, where the same process mentioned above was taken into consideration in obtaining the following set of values.}
\end{align*}
\]

\[
\begin{align*}
\text{Set 1:} & \quad \& \beta = \left\{ \begin{array}{l}
\text{for } \alpha = 0 \Rightarrow \text{set } (a) \text{ and } (b) \text{ of } \text{the surface views of } (a)-(c), \text{ respectively.}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&f = a + \cos(b_0 x + b_1 y + b_2 t + b_3) e^{-((x+y) + 2t + y) + \beta e^{-2b_0 (x+y) + 2t + y)},
\end{align*}
\]

\[
\begin{align*}
&\& b_2 = \left\{ \begin{array}{l}
\text{for } \alpha = 0 \Rightarrow \text{set } (a) \text{ and } (b) \text{ of } \text{the surface views of } (a)-(c), \text{ respectively.}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\text{Set 2:} & \quad \& b_0 = b_0 - 3 b_0 c_0^2, \\
&\& b_1 = \left\{ \begin{array}{l}
\text{for } \alpha = 0 \Rightarrow \text{set } (a) \text{ and } (b) \text{ of } \text{the surface views of } (a)-(c), \text{ respectively.}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
&\& b_2 = \left\{ \begin{array}{l}
\text{for } \alpha = 0 \Rightarrow \text{set } (a) \text{ and } (b) \text{ of } \text{the surface views of } (a)-(c), \text{ respectively.}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\text{Set 3:} & \quad \& b_1 = \left\{ \begin{array}{l}
\text{for } \alpha = 0 \Rightarrow \text{set } (a) \text{ and } (b) \text{ of } \text{the surface views of } (a)-(c), \text{ respectively.}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\text{with free parameters } a, \beta, b_0, b_1, c_0, c_1, \text{ and } c_3. \text{ It is of interest to note here that the set of solutions is independent of nonzero conditions.}
\end{align*}
\]

\[
\begin{align*}
\text{Set 3:} & \quad \& b_1 = \left\{ \begin{array}{l}
\text{for } \alpha = 0 \Rightarrow \text{set } (a) \text{ and } (b) \text{ of } \text{the surface views of } (a)-(c), \text{ respectively.}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\text{where } a, \beta, b_0, b_1, c_0, c_1, \text{ and } c_3 \text{ are free parameters and } a, b_0 \neq 0.
\end{align*}
\]
Fig. 5. The periodic breather wave profile for the solution given by Eq. (22) through different plots; (a) 3D plot (at \(t = 0\)); (b) the density plot of (a); (c) the contour plot of (a); (d) the projection of the wave profile on the \(y\)-axis at \(t = 0\), where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles for \(x = -1, x = 0,\) and \(x = 1,\) respectively; (e) the projection of the wave profile on the \(x\)-axis at \(t = 0\), where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles for \(y = -1, y = 0,\) and \(y = 1,\) respectively; (f) the projection of the wave profile on the \(x\)-axis for \(y = 0,\) where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles at \(t = -1, t = 0,\) and \(t = 1,\) respectively.

Set 4:

\[
\begin{align*}
&b_0 = 0, \\
&b_2 = \frac{5b_1^2(18b_0^6+5b_1^2) - 4b_1^2}{4b_1^2}, \\
&c_1 = -\frac{12b_1^6+2b_1^2 - 4b_1^2}{12b_1^6}, \\
&c_2 = \frac{129b_1^6+18b_1^2 - 144b_1^2 + 5b_1^2 - 40b_1^2 + 80b_1^2}{144b_1^6}, \\
&c_3 = \frac{186800}{144b_1^6},
\end{align*}
\]

where \(a, \beta, b_1, b_3, c_0,\) and \(c_3\) are free parameters and \(\beta, c_0 \neq 0.\)

4.1. Breather wave profile

Substituting Eq. (18) in Eq. (17), the unknown function for the breather wave profile was obtained for Set 1, and hence from Eq. (2), the solution \(u\) for the breather wave profile was obtained. For the sake of brevity, the solution is avoided to present here. A specific \(u\) for the suitable choice of the values of the free parameters, namely \(a = \frac{1}{2}, b_0 = 3, b_1 = 2, b_3 = 1, c_0 = 1, c_1 = \frac{1}{2},\) and \(c_3 = 1\) can be represented in the following form:

\[
\begin{align*}
u_1(x,y,t) = 6 \left( \frac{-\frac{1}{2} \cos(P) - 6 \sin(P) + \frac{129299}{186800} e^Q}{1 + \frac{1}{2} \cos(P) e^Q + \frac{129299}{186800} e^2Q} \right) e^Q - 6 \left( \frac{\frac{1}{2} \sin(P) - \cos(P) - \frac{129299}{4800} e^Q}{1 + \frac{1}{2} \cos(P) e^Q + \frac{129299}{186800} e^2Q} \right)^2 e^{2Q},
\end{align*}
\]

where \(P = -3x - 2y + \frac{12389}{133} t - 1\) and \(Q = -2x - \frac{1}{2} y + \frac{129}{133} t - 1.\)

The wave profile for the solution given by Eq. (22) is presented graphically through several types of plots in Fig. 5 for a better understanding of the inherent physical characteristics of such a solution. It is seen from the figure (Fig. 5) that the solution specified by Eq. (22) represents a periodic breather wave. The obtained solution is actually the interaction of periodic and solitary wave solutions.

Similarly, for Set 2 (Eq. (19)), one can obtain the following solution, where the values of the free parameters are chosen as \(a = 2, \beta = \frac{3}{2}, b_0 = \frac{1}{2}, b_1 = 1, c_0 = 1, c_1 = \frac{1}{2},\) and \(c_3 = 1:\)
Similarly, the projection of the wave profile on the x-axis at $t = 1$ is seen in Fig. 6 that the explored solution via the HTA represents a periodic breather wave profile, which is similar to the one presented in refs. [44, 45].

4.2. Multi-profile soliton solutions

Through Eqs. (17) and (20) along with the transformation specified by Eq. (2), the following solution is attained for choosing the suitable values of the free parameters as $a = \frac{3}{2}$, $\beta = \frac{1}{2}$, $b_0 = 1$, $b_1 = 0$, $c_1 = \frac{1}{2}$, $c_2 = 0$:

$$u_3(x,y,t) = -\frac{3\cos(P)e^Q}{1 + \frac{3}{2}\cos(P)e^Q + \frac{1}{2}e^{2Q}} - \frac{3\sin(P)^2 e^{2Q}}{2 \left(1 + \frac{1}{2}\cos(P)e^Q + \frac{1}{2}e^{2Q}\right)^2}$$

(24)

where $P = -x + \frac{19}{12}y + \frac{2611}{252}t$ and $Q = -\frac{1}{2}y + \frac{123}{12}t$.

The solution given by Eq. (24) is presented graphically in Fig. 7 through the 3D and surface plots. Figs. 7(a)-(c) represent the 3D view of the multiple lump wave profile in the $(x,y)$, $(x,t)$, and $(y,t)$-planes, respectively, whereas Figs. 7(d)-(f) illustrate the surface views of Figs. 7(a)-(c), respectively. The 3D and surface views in this regard clearly show the existence of a series of periodic lump waves. A breather wave profile is one kind of periodic lump wave solution, which is made from the superposition of single lump wave solutions. Furthermore, it can be seen from Fig. 7 that the kink breather wave is not only the space-periodic breather but also the time-periodic.

Similarly, for Set 4, one can obtain the following soliton wave solution, where the values of the free parameters are chosen as $a = 2$, $\beta = \frac{1}{2}$, $b_1 = 3$, $b_3 = 1$, $c_0 = 1$, $c_1 = \frac{1}{2}$, and $c_2 = 1$:

$$u_4(x,y,t) = \frac{6\left(2\cos(S)e^T + 6e^{2T}\right)}{1 + 2\cos(S)e^T + \frac{5}{2}e^{2T}} - \frac{6\left(-2\cos(S)e^T - 3e^{2T}\right)^2}{\left(1 + 2\cos(S)e^T + \frac{5}{2}e^{2T}\right)^2}$$

(25)
with $S = 3y + 15t + 1$ and $T = -x - 46t - 1$. The solution specified by Eq. (25) is presented graphically in Fig. 8 through several types of plots for a better understanding of its intrinsic features. It is observed from Fig. 8 that it has multiple peaks and valleys. These behaviors are consistent with periodic lump wave solutions or breather wave solutions.

5. Conclusions

In this study, the (2+1)-D SK equation has been investigated through the HBM. As outcomes, the lump, solitary, lump-stripe, and breather wave solutions are obtained. The stripe soliton is found to be localized in all directions in the space. Also, the lump wave is found to be drowned or swallowed by the stripe soliton. Furthermore, in the fission interaction process (between one lump soliton and one stripe soliton), the stripe soliton splits into one stripe soliton and one lump soliton, and in the fusion interaction process, one lump soliton and one stripe soliton merge into one stripe soliton, which signifies that the interaction solution is completely inelastic. During the fission-fusion interaction process, the lump only changes its amplitude and phase. On the other hand, in this study, the breather wave (periodic lump wave) profile is explored by the HTA. It is found from the graphical outcomes of the breather waves that they are localized in space and time. This is the novel result of the study and the attained breather wave solutions have not yet been reported in other literature [13, 15, 27, 31, 32]. The outcomes emanated from the study might be helpful to researchers in understanding the propagation behaviors of the shallow water waves of small amplitude and long wavelength, stratified internal waves in an ocean, and ion-acoustic waves in a plasma.

 Declarations

Author contribution statement

Md. Emran Ali, G. C. Paul: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.
F. Bilkis, H. Naher: Analyzed and interpreted the data; Wrote the paper.
D. Kumar: Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

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Fig. 8. The wave profile of the solution given by Eq. (25) through different plots; (a) 3D plot (at \( t = 0 \)); (b) the density plot of (a); (c) the contour plot of (a); (d) the projection of the wave profile on the \( y \)-axis at \( t = 0 \), where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles for \( y = -1 \), \( y = 0 \), and \( y = 1 \), respectively; (e) the projection of the wave profile on the \( x \)-axis at \( t = 0 \), where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles for \( x = -1 \), \( x = 0 \), and \( x = 1 \), respectively; (f) the projection of the wave profiles on the \( x \)-axis at \( y = 0 \), where the dark-orange, light-sea-green, and medium-violet-red curves denote the wave profiles at \( t = -1 \), \( t = 0 \), and \( t = 1 \), respectively.

Data availability statement

Data will be made available on request.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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