Heavy baryon properties with
NLO accuracy in perturbative QCD

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Abstract

We present an analysis of the static properties of heavy baryons at next-to-leading
order in the perturbative expansion of QCD. We obtain analytical next-to-leading order
three-loop results for the two-point correlators of baryonic currents with one finite mass
quark field for a variety of quantum numbers of the baryonic currents. We consider
both the massless limit and the HQET limit of the correlator as special cases of the
general finite mass formula and find agreement with previous results. We present closed
form expressions for the moments of the spectral density. We determine the residues of
physical baryon states using sum rule techniques.
1 Introduction

Baryons form a rich family of particles which have been experimentally studied with high accuracy [1]. With the advent of new accelerators and detectors many properties of baryons containing a heavy quark have been experimentally measured in recent years [1]. A theoretical analysis of these experimental data gives a great deal of information about the structure of QCD and the numerical values of its parameters. The hypothetical limit $N_c \to \infty$ for the number $N_c$ of quark and gluon colours in the symmetry group $SU(N_c)$ was especially successful for baryons [2]. The analysis of this limit is a very powerful tool for the investigation of general properties of gauge interactions. The information about the spectrum of baryons is contained in the correlator of two baryonic currents and the spectral density associated with it. Within the operator product expansion and to leading order in perturbative QCD the correlator is given by a product of $N_c$ fermionic propagators. The diagrams of this topology have been studied in some detail [3, 4, 5]. They are rather frequently used in phenomenological applications [6, 7, 8]. In general, these diagrams represent the leading order of the perturbative expansion for the relevant correlation functions. In some cases, though (especially for the gluon current correlators [9]), they first appear at next-to-leading order. Complete calculations beyond the leading order have not been done for many interesting cases. In this paper we fill this gap.

We report on the results of calculating the $\alpha_s$ corrections to the correlator of two baryonic currents with one finite mass quark and two massless quarks. We present analytical results and discuss the magnitude of the $\alpha_s$ corrections for the physically interesting cases. The high energy (massless quarks) and near-threshold (Heavy Quark Effective Theory, HQET) [10, 11, 12, 13] limits are obtained from our general results as special cases. For these we find agreement with previous results in the literature. We present analytical results for the moments of the spectral density associated with correlators of baryonic currents.

We briefly discuss the impact of our new results for the baryonic correlators on the phenomenology of baryons. However, the main aim of this paper is to present the results of the perturbative calculations in some detail and to show how they have been arrived at. The new technically demanding feature of our calculation is the presence of a finite-mass quark in the correlator which is needed for baryons containing a heavy charm or bottom quark. For the $b$-quark the accuracy of the HQET approximation is rather good. The exact result will help in controlling the precision of the approximation. For the $c$-quark, however, the accuracy of the near threshold approximation is insufficient for physical applications and the use of the exact formulas is unavoidable.

The massless case has been known since long ago [7, 8] and serves as a test of the massless limit of our results. We mention that the mesonic analogue of our baryonic calculation with one finite mass quark and one massless antiquark was completed some time ago [14] and has subsequently provided a rich source of inspiration for many applications in meson physics. Some of the techniques used in this paper have already been usefully employed in the analysis of perturbative corrections to sum rules involving pentaquark states [15].
2 Generalities

In this section we present our choice of interpolating currents for baryons containing one heavy quark. We also introduce two-point correlation functions as the principal tool in our analysis of the static properties of heavy baryons. Finally, we give an outline of the techniques that were used in our calculations.

2.1 Choice of currents and correlators

A generic lowest dimensional baryonic current has the form

\[ j = \epsilon^{abc} (u_b^T C \Gamma d_c) \Gamma' \Psi_a. \]  

The current (1) refers to a baryon with three valence quarks and no gluonic fields and no derivative couplings. \( \Psi \) is a finite mass quark field with the mass parameter \( m \) and \( u \) and \( d \) are massless quark fields. \( C \) is the charge conjugation matrix, \( \epsilon^{abc} \) is the totally antisymmetric tensor and \( a, b, c \) are colour indices of the \( SU(3) \) colour group. \( \Gamma \) and \( \Gamma' \) stand for Dirac matrices or strings of Dirac matrices where possible Lorentz indices on \( \Gamma \) and \( \Gamma' \) such as in \( \gamma^\mu \) or \( \sigma^{\mu\nu} \) are suppressed. In much the same way we have suppressed a possible Lorentz index on \( \Psi_a \) which is needed later on in the discussion of the spin 3/2 field. For \( \Gamma = 1, \Gamma' = \gamma_5 \) the interpolating current has the quantum numbers of a \( J^P = 1/2^+ \) baryon. Other baryonic currents with any given specified quantum numbers are obtained from the current in Eq. (1) by using the appropriate Dirac matrices or strings of Dirac matrices. We first consider the simplest case and take \( \Gamma = 1, \Gamma' = 1 \) corresponding to an interpolating current with quantum numbers \( 1/2^- \). This allows us to explain our techniques and to demonstrate the idiosyncratic features of the calculation. Later in the text we will introduce more general interpolating currents and discuss calculational differences in comparison to those in the simplest case.

The correlator of two baryonic currents can be expanded into a basis of invariant functions. The form of this expansion depends on the Dirac matrices employed. In the simplest case \( \Gamma = 1, \Gamma' = 1 \) there are only two invariant functions \( \Pi^q(q^2) \) and \( \Pi^m(q^2) \) in the expansion which are defined through

\[ i \int \langle T \{ j(x) \bar{j}(0) \} \rangle e^{iqx} dx = m \Pi^m(q^2) + q \Pi^q(q^2). \]  

In the following we shall refer to the two contributions on the r.h.s. of (2) as the mass and the momentum term, respectively. Note that each of the replacements \( \Gamma \rightarrow \Gamma \gamma_5 \) and \( \Gamma' \rightarrow \Gamma' \gamma_5 \) leads to the change \( \Pi^q(q^2) \rightarrow -\Pi^q(q^2) \).

2.2 Basic techniques

The generic correlation function \( \Pi(q^2) \) has a dispersion representation

\[ \Pi(q^2) = \int_{m^2}^{\infty} \frac{\rho(s)ds}{s - q^2} + \text{subtractions} \]  

through its discontinuity \( \rho(s) \) on the physical cut \( s > m^2 \),

\[ \rho(s) = \frac{1}{2\pi i} \left( \Pi(s + i0) - \Pi(s - i0) \right). \]
The discontinuity can also be written as the imaginary part of the correlation function if the phases are properly chosen,
\[ \rho(s) = \frac{1}{\pi} \text{Im}\Pi(s + i0). \] (5)

The expression for the discontinuity or spectral density \( \rho(s) \) is simpler than the expression for the correlation function \( \Pi(q^2) \) itself. The knowledge of \( \rho(s) \) suffices for physical purposes and allows one to recover the whole function \( \Pi(q^2) \) through a one-dimensional integral with a simple weight function as given in Eq. (3). For this reason we concentrate on calculating the spectral density \( \rho(s) \).

The general strategy is rather straightforward. The main part of the calculation is done by using a symbolic manipulation program. One reduces all integrals to some basic master integrals and then one puts them together again to get the result for a particular correlator with any given quantum numbers. This program has been explicitly realized in our evaluation.

The topology of the NLO diagrams is such that at least one line connecting the initial and final points of the diagram is free. If this line is the massive one, the remaining part of the diagram consists of massless lines and can be integrated analytically. Adding the massive line leads to a one-dimensional integration which can be done analytically.

If the massive line is part of the radiative corrections, the basic quantity is a NLO two-point correlator of the meson type with one heavy and one light quark. The spectral density for this NLO correlator is known to be computable. The fact that we only have one finite mass and a special topology of diagrams in the baryon sector therefore makes the analytical computation feasible. Note that the computation with two different finite masses can still be done but requires a numerical calculation while some limiting cases such as the small mass ratio limit can still be done analytically.

In order to explain our main tools, we consider the correlation function of the baryonic current, which, up to NLO, can be written as
\[ \Pi(q^2) = \int dk \Pi_2(q - k)\Pi_1(k) \] (6)
where \( \Pi_{1,2}(k) \) are one- and two-line correlators. If the one-line correlator is massive, the massless two-line correlator can be explicitly integrated and we are left with an integral in \( D \)-dimensional space-time given by
\[ V(\alpha, \beta) = \int \frac{d^Dk}{(m^2 - k^2)^\alpha(-(q - k)^2)^\beta}. \] (7)
This integral can be expressed through hypergeometric functions and is therefore completely known. If the one-line correlator is massless, the spectral density reads
\[ \rho(s) = \int_{m^2}^s ds' \rho_2(s')(1 - s'/s). \] (8)
It is not difficult to obtain \( \rho(s) \) since \( \rho_2(s) \) is known from mesonic type calculations.

All in all the calculation includes no unknown elements in the sense that all necessary blocks (prototypes or masters) are known to be calculable analytically. The main problem is the reduction of the initial diagrams to prototypes and the assembly of the final results from these building blocks. This has been done using the computer.

In the next section we present the calculation and results for the lowest spin baryons which means that the Dirac matrices in the interpolating currents are just unity or \( \gamma_5 \).
3 Lowest spin baryons

In this section we present the results for the simplest choice of the Dirac structure of interpolating baryonic currents. We take $\Gamma = 1$ and $\Gamma' = \gamma_5$ which corresponds to an interpolating current with quantum numbers $J^P = 1/2^+$. We can in fact omit $\gamma_5$ in the process of the calculation because the effect of $\gamma_5$ can later on be easily accounted for by a simple multiplication in Dirac space. In this case the basic baryonic current has the form

$$j = \epsilon^{abc}(u^T_b C d_c)\Psi_a.$$ (9)

For this scalar case with $\Gamma = \Gamma' = 1$, the results for the invariant functions $\Pi^m(q^2)$ and $\Pi^q(q^2)$ in Eq. (2) have already been presented in Refs. [16, 17]. The invariant function $\Pi^\alpha(q^2)$ with $\alpha \in \{q, m\}$ can be represented compactly via the dispersion relation

$$\Pi^\alpha(q^2) = \int_{m^2}^{\infty} \frac{\rho^\alpha(s)ds}{s - q^2}$$ (10)

where $\rho^\alpha(s)$ is the spectral density. All quantities are understood to be appropriately regularized. Since the spectral density is the main object of interest for phenomenological applications, we limit our subsequent discussion to the spectral density

$$\rho^\alpha(s) = \frac{s^2}{128\pi^4} \left\{ \rho^\alpha_0(s) \left( 1 + \frac{\alpha_s}{\pi} \ln \left( \frac{\mu^2}{m^2} \right) \right) + \frac{\alpha_s}{\pi} \rho^\alpha_1(s) \right\},$$ (11)

where $\mu$ is the renormalization scale parameter, $m$ is the pole mass of the heavy quark (see e.g. Ref. [18]), and $\alpha_s = \alpha_s(\mu)$.

3.1 LO analytical results

The leading order two-loop diagram is shown in Fig. 1(a). Note that this topology coincides with what is referred to as sunrise-type diagrams for which a general evaluation method (with arbitrary masses) has been developed in Refs. [4, 19]. Sunrise-type diagrams can be calculated by a variety of methods. In this paper we apply the configuration space technique in a straightforward manner. The result reads

$$\rho^m_0(s) = 1 + 9z - 9z^2 - z^3 + 6z(1 + z) \ln z$$ (12)

$$\rho^q_0(s) = \frac{1}{4} - 2z + 2z^3 - \frac{1}{4}z^4 - 3z^2 \ln z$$ (13)

with $z = m^2/s$.

3.2 NLO analytical results

The contributing three-loop diagrams are shown in Figs. 1(b1) to (c21). They have been evaluated using the advanced algebraic methods for multi-loop calculations along the lines
Figure 1: Two-loop (a1) and three-loop (b11–c21) topologies with one external momentum. Heavy lines represent the heavy quark and light lines massless quarks.

described in Refs. [4]. The result can be obtained analytically. In the $\overline{\text{MS}}$-subtraction scheme one has [16, 17]

$$
\rho_{i}^m(s) = 9 + \frac{665}{9} z - \frac{665}{9} z^2 - 9 z^3 - \left(\frac{58}{9} + 42 z - 42 z^2 - \frac{58}{9} z^3\right) \ln(1 - z)
$$

$$ + \left(2 + \frac{154}{3} z - \frac{22}{3} z^2 - \frac{58}{9} z^3\right) \ln z + \frac{8}{3} \left(1 + 9 z - 9 z^2 - z^3\right) \left(\text{Li}_2(z) + \frac{1}{2} \ln(1 - z) \ln z\right)
$$

$$ + z \left(24 + 36 z + \frac{4}{3} z^2\right) \left(\text{Li}_2(z) - \zeta(2) + \frac{1}{2} \ln^2 z\right) + 24 z(1 + z) \left(\text{Li}_3(z) - \zeta(3) - \frac{1}{3} \text{Li}_2(z) \ln z\right),
$$

$$
\rho_{i}^q(s) = \frac{71}{48} - \frac{565}{36} z - \frac{7}{8} z^2 + \frac{625}{36} z^3 - \frac{109}{48} z^4 - \frac{1}{36} \left(\frac{49}{36} - \frac{116}{9} z + \frac{116}{9} z^3 - \frac{49}{36} z^4\right) \ln(1 - z)
$$

$$ + \left(\frac{1}{4} - \frac{17}{3} z - 11 z^2 + \frac{113}{9} z^3 - \frac{49}{36} z^4\right) \ln z + \frac{2}{3} \left(1 - 8 z + 8 z^3 - z^4\right) \left(\text{Li}_2(z) + \frac{1}{2} \ln(1 - z) \ln z\right)
$$

$$ - \frac{1}{3} z^2 \left(54 + 8 z - z^2\right) \left(\text{Li}_2(z) - \zeta(2) + \frac{1}{2} \ln^2 z\right) - 12 z^2 \left(\text{Li}_3(z) - \zeta(3) - \frac{1}{3} \text{Li}_2(z) \ln z\right) \right)
$$

(14)

where $z = m^2/s$ and $\text{Li}_n(z)$ are polylogarithms

$$
\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}, \quad \text{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n).
$$

(16)

$\zeta(n)$ is Riemann’s zeta function. Note that in the physical region we have $z < 1$. Therefore, no analytic continuation is required. Note the repeated appearance of particular combinations of polylogarithms and logarithms in the form

$$
\text{Li}_3(z) - \zeta(3) - \frac{1}{3} \text{Li}_2(z) \ln z = \frac{1}{3} \int_{1}^{z} \frac{dz'}{z'} \left(2 \text{Li}_2(z') + \ln(1 - z') \ln z'\right),
$$

$$
\text{Li}_2(z) - \zeta(2) + \frac{1}{2} \ln^2 z = \int_{1}^{z} \frac{dz'}{z'} \left(\ln z' - \ln(1 - z')\right) \quad \text{and}
$$

$$
\text{Li}_2(z) + \frac{1}{2} \ln(1 - z) \ln z = -\frac{1}{2} \int_{0}^{z} \frac{dz'}{1 - z'} \left(\frac{\ln z' - \ln(1 - z')}{{z'}^{2}}\right)
$$

(17)

appear as a consequence of the integration of the two-line spectral functions.
4 General baryon case

The baryonic current defined in Eq. (1) has the most general structure concerning possible choices for the Dirac matrices $\Gamma$ and $\Gamma'$. However, not all of this structure has to be kept in order to calculate the different invariant functions of the correlator. In four-dimensional spacetime where the initial (bare) current is defined, Fierz rearrangement can always be used to take the heavy spinor out of the trace in the correlator. When taken out of the trace together with the heavy spinor, the matrix $\Gamma'$ can be considered to be an overall factor that has no effect on the calculation. Indeed,

$$j = \epsilon^{abc}(u_b^T C \Gamma d_c) \Gamma' \Psi_a = \Gamma' j \Gamma$$

(18)

where $j \Gamma$ is the current for $\Gamma' = 1$,

$$j = \epsilon^{abc}(u_b^T C \Gamma d_c) \Psi_a.$$

(19)

The general case is recovered as

$$\Pi(q^2) = i \int \langle T\{j(x)j(0)\}\rangle e^{iqx} dx = \Gamma' \Pi \Gamma(q^2)$$

(20)

with $\Gamma' = \gamma^0 \Gamma^\dagger \gamma^0$ and

$$\Pi \Gamma(q^2) = i \int \langle T j \Gamma(x)j \Gamma(0)\rangle e^{iqx} dx.$$  

(21)

Because the general result can easily be recovered, we can limit ourselves to the case $\Gamma' = 1$. The calculation of the correlator has to be done with its full dependence on the matrix $\Gamma$. We obtain the general expression for the correlator in the form

$$\Pi \Gamma(q^2) = \sum_{i=1}^6 A_i(q^2) \text{tr}_i(\Gamma, q^2)$$

(22)

where

$$\text{tr}_1(\Gamma, q^2) = \text{Tr}(\Gamma q / \bar{\Gamma} q /) m / q^2 \quad \text{tr}_2(\Gamma, q^2) = \text{Tr}(\Gamma \gamma_\alpha \bar{\Gamma} \gamma_\alpha)m$$
$$\text{tr}_3(\Gamma, q^2) = \text{Tr}(\Gamma q / \bar{\Gamma} q /) q / q^2 \quad \text{tr}_4(\Gamma, q^2) = \text{Tr}(\Gamma \gamma_\alpha \bar{\Gamma} \gamma_\alpha) \gamma^\alpha_q$$
$$\text{tr}_5(\Gamma, q^2) = \text{Tr}(\Gamma \gamma_\alpha \bar{\Gamma} \gamma_\alpha) \gamma^\alpha_q \quad \text{tr}_6(\Gamma, q^2) = \text{Tr}(\gamma_\alpha \bar{\Gamma} \gamma_\alpha q / \bar{\Gamma} q /).$$

(23)

The trace in (23) is to be taken only with respect to the $\gamma$-string in the round brackets. We have again omitted possible Lorentz indices on $\Gamma$ such as appear in the next example.

As an example, let us exhibit the structure of the expressions for the interesting and important case of the “vector” current

$$j^\mu = \epsilon^{abc}(u_b^T \gamma^\mu d_c) \Psi_a.$$ 

(24)

In the above notation this means that $\Gamma' = 1$. The expansion of the correlator reads

$$\Pi^{\mu\nu}(q^2) = i \int \langle T j^\mu(x)j^\nu(0)\rangle e^{iqx} dx.$$ 

(25)
The correlator can be expanded along a set of ten covariants. The expansion reads

\[
\Pi^{\mu\nu}(q^2) = m \left( A_1^m q^\mu q^\nu + A_2^m q^2 g^{\mu\nu} + A_3^m g q^{\mu} q^\nu + A_4^m q^{\mu} \gamma_5 q^\nu + A_5^m q^2 \gamma^\mu \gamma^\nu \right) / q^2 \\
+ g \left( A_1^m q^\mu q^\nu + A_2^m q^2 g^{\mu\nu} + A_3^m g q^{\mu} q^\nu + A_4^m q^{\mu} \gamma_5 q^\nu + A_5^m q^2 \gamma^\mu \gamma^\nu \right) / q^2. 
\]

(26)

All invariant amplitudes \( A_i^m \) have been calculated. The above expansion of the vector correlator is only one of several possible expansions. By making use of the symmetry properties of the result we shall later on use a different expansion in terms of nine covariants whose mixing property is simpler.

### 4.1 LO analytical results

Differing from our earlier calculation [16, 17], where the mass and momentum parts were calculated separately, we have now learned to calculate the mass and momentum parts in one go. The reason is that no explicit traces have to be taken when extracting the mass or momentum part. Instead, the traces are kept to the very end. It is then not difficult to interpret one part of the expression as the mass part and the other part as the momentum part depending on the occurrence or absence of an explicit factor of \( m \). The result for the leading order diagram (a1) in \( D = 4 - 2\epsilon \) space-time dimensions reads

\[
\rho_{a1}(s) = \frac{(D - 2)G(1, 1)N_c}{16(4\pi)^D(D - 1)^2} s^{D-2} \hat{\rho}_{a1}(m^2/s) 
\]

where

\[
\hat{\rho}_{a1}(z) = \sum_{i=1}^6 \hat{\rho}^i_{a1}(z) \text{tr}_i(\Gamma, s).
\]

(27)

Here \( G(1, 1) = G/\epsilon \) and \( G = \Gamma(1 + \epsilon)\Gamma(1 - \epsilon)^2/\Gamma(2 - 2\epsilon) \). Note that we use hatted spectral functions whenever we present them as a function of \( z = m^2/s \). The corrections are

\[
\hat{\rho}^1_{a1}(z) = D \left\{ \hat{\rho}_V(1, \epsilon - 2; z) + (1 - z)^2 \hat{\rho}_V(1, \epsilon; z) - 2 \left( \frac{D - 2}{D} - z \right) \hat{\rho}_V(1, \epsilon - 1; z) \right\},
\]

\[
\hat{\rho}^2_{a1}(z) = - \left\{ \hat{\rho}_V(1, \epsilon - 2; z) + (1 - z)^2 \hat{\rho}_V(1, \epsilon; z) + 2 \left( \frac{3D - 4}{D - 2} + z \right) \hat{\rho}_V(1, \epsilon - 1; z) \right\},
\]

\[
\hat{\rho}^3_{a1}(z) = \frac{D + 2}{2} \left\{ \hat{\rho}_V(1, \epsilon - 3; z) + \left( \frac{D - 2}{D + 2} + z \right) (1 - z)^2 \hat{\rho}_V(1, \epsilon; z) + \left( \frac{6 - D}{D + 2} + 3z \right) \hat{\rho}_V(1, \epsilon - 2; z) + \left( \frac{2Dz}{D + 2} - \frac{2D}{D + 2} + 3z^2 \right) \hat{\rho}_V(1, \epsilon - 1; z) \right\},
\]

\[
\hat{\rho}^4_{a1}(z) = - \frac{1}{2} \left\{ \hat{\rho}_V(1, \epsilon - 3; z) - (1 - z)^3 \hat{\rho}_V(1, \epsilon; z) + (1 + 3z) \hat{\rho}_V(1, \epsilon - 2; z) - (1 - z)(1 + 3z) \hat{\rho}_V(1, \epsilon - 1; z) \right\},
\]

\[
\hat{\rho}^5_{a1}(z) = - \frac{1}{2} \left\{ \hat{\rho}_V(1, \epsilon - 3; z) - (1 - z)^3 \hat{\rho}_V(1, \epsilon; z) + (1 + 3z) \hat{\rho}_V(1, \epsilon - 2; z) - (1 - z)(1 + 3z) \hat{\rho}_V(1, \epsilon - 1; z) \right\},
\]

\[9\]
\[ \hat{\rho}_{a1}^6(z) = -\frac{1}{2} \left\{ \hat{\rho}_V(1, \varepsilon - 3; z) + (1 - z)^2(1 + z)\hat{\rho}_V(1, \varepsilon; z) + \right. \\
+ \left( \frac{7D - 10}{D - 2} + 3z \right) \hat{\rho}_V(1, \varepsilon - 2; z) + \\
+ \left( \frac{7D - 10}{D - 2} + 2 \frac{3D - 4}{D - 2} z + 3z^2 \right) \hat{\rho}_V(1, \varepsilon - 1; z) \right\}. \] (28)

For later reference we also need starred elements \( \hat{\rho}_{a1}^{1\ast}(z) \), \( \hat{\rho}_{a1}^{2\ast}(z) \), \( \hat{\rho}_{a1}^{3\ast}(z) \), \( \hat{\rho}_{a1}^{4\ast}(z) \), \( \hat{\rho}_{a1}^{5\ast}(z) \), and \( \hat{\rho}_{a1}^{6\ast}(z) \). These can be obtained from the corresponding unstarred elements by replacing \( \varepsilon \) in the argument of the spectral functions \( \hat{\rho}_V \) by \( 2\varepsilon \). The basic spectral functions \( \hat{\rho}_V \) are given by

\[ \hat{\rho}_V(n_1, n_2; z) = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int_1^1 (1-x)^{D/2-n_2-1}x^{n_2-1}(x-z)^{D/2-n_1-n_2}dx \] (29)

where \( \Gamma(z) \) is Euler’s gamma function. Note that the singularity given by the factor \( G(1, 1) \) in Eq. (27) cancels against the singularity of the second gamma functions in the denominator if the appropriate arguments \( n_2 = \varepsilon - n \) (or \( n_2 = 2\varepsilon - n \) in case of the starred elements) with \( n \geq 0 \) are used. In the limit \( D = 4 \) we find

\[ \rho_{a1}(s) = \frac{s^2}{512\pi^4} \sum_{i=1}^6 \hat{\rho}_{a1}^{i0}(m^2/s) \text{tr}(\Gamma, s) \] (30)

where

\[ \hat{\rho}_{a1}^{10}(z) = \frac{1}{2} + \frac{5}{3}z - 3z^2 + z^3 - \frac{1}{6}z^4 + 2z \ln z \]
\[ \hat{\rho}_{a1}^{20}(z) = \frac{1}{8} + \frac{11}{6}z - \frac{3}{2}z^2 - \frac{1}{2}z^3 + \frac{1}{24}z^4 + \left( z + \frac{3}{2}z^2 \right) \ln z \]
\[ \hat{\rho}_{a1}^{30}(z) = \frac{1}{10} - \frac{1}{2}z + z^2 - z^3 + \frac{1}{2}z^4 - \frac{1}{10}z^5 \]
\[ \hat{\rho}_{a1}^{40}(z) = \frac{1}{40} - \frac{1}{6}z - \frac{1}{2}z^2 + \frac{1}{2}z^3 - \frac{1}{8}z^4 + \frac{1}{60}z^5 - \frac{1}{2}z^2 \ln z \]
\[ \hat{\rho}_{a1}^{50}(z) = \frac{1}{40} - \frac{1}{6}z - \frac{1}{2}z^2 + \frac{1}{2}z^3 - \frac{1}{8}z^4 + \frac{1}{60}z^5 - \frac{1}{2}z^2 \ln z \]
\[ \hat{\rho}_{a1}^{60}(z) = \frac{1}{40} - \frac{1}{6}z - \frac{1}{2}z^2 + \frac{1}{2}z^3 - \frac{1}{8}z^4 + \frac{1}{60}z^5 - \frac{1}{2}z^2 \ln z \] (31)

Note that \( \hat{\rho}_{a1}^{40}(z) = \hat{\rho}_{a1}^{50}(z) = \hat{\rho}_{a1}^{60}(z) \). Eqs. (30) and (31) give the full answer for the leading order contribution to the baryonic correlators for all possible configurations of Dirac gamma matrices. In this sense this completes the leading order calculation of correlators for baryons with any quantum numbers as long as there are no derivative couplings in the interpolating currents.

### 4.2 NLO contributions

The NLO contributions result from four different diagrams. In the calculation of these diagrams we have used different techniques depending on their topologies and on the location of the massive line (cf. Fig. [I]).
• The NLO light contributions result from two diagrams, the self energy correction of one of the massless lines (b21) and the diagram with gluon exchange between two massless lines (c11) which we term “light fish”. The technique of calculating these diagrams consists in first analytically calculating the massless part and then adding the massive fermion line. Advantage is taken of the fact that massless two-loop diagrams are explicitly calculable. In this case one cannot avail of a convolution of spectral functions.

• The massive line self energy contribution (b11) was calculated again by using the explicit evaluation of the massless part and the dispersion relation for the mass operator of the massive quark at leading order. Here we define the heavy quark mass through its pole mass which is most convenient for the calculation using cuts.

• The most demanding semi-massive fish diagram (c21) was calculated by making full use of the decomposition into prototypes and the convolution of spectral functions. For this diagram the use of symbolic manipulation programs is indispensable as the number of terms in intermediate expressions involving different structures are quite large.

These three main parts constitute the whole calculation.

4.3 Light contributions (b21 and c11)

The NLO light contributions result from two diagrams, the self energy correction of one of the massless lines (b21) and the diagram with gluon exchange between the two massless lines (c11), called “light fish”. More details on the calculation of these two contributions are found in Appendix A. It turns out that the dominant singular part is proportional to the leading order contribution. The result for the spectral density reads

\[ \rho_{\text{light}}(s) = \frac{\alpha_s N_c C_F}{4\pi} \left\{ \left( \frac{B_0}{\varepsilon^2} + \frac{B_1}{\varepsilon} + B_2 \right) \rho_{a1}^*(s) + \left( \frac{B'_1}{\varepsilon} + B'_2 \right) \rho_{a1}^{*'}(s) \right\} \]  

(32)

where

\[ \rho_{a1}^*(s) = \frac{(D - 2)G^2 s^{3D/2 - 4}}{16(4\pi)^D(D - 1)^2} \sum_{i=1}^{6} \hat{\rho}_{a1}^{si}(m^2/s) \text{tr}_i(\Gamma, s), \]

\[ \rho_{a1}^{*'}(s) = \frac{G^2 s^{3D/2 - 4}}{2(4\pi)^D D} \left( \hat{\rho}_{a1}^{m*}(m^2/s) \text{tr}_2(\Gamma, s) + \hat{\rho}_{a1}^{q*}(m^2/s) \text{tr}_6(\Gamma, s) \right) \]  

(33)

and where \( \hat{\rho}_{a1}^{si}(z) \) are the starred elements introduced after Eq. (28). The spectral functions

\[ \hat{\rho}_{a1}^{m*}(z) = -\hat{\rho}_V(1, 2\varepsilon - 1; z), \]

\[ \hat{\rho}_{a1}^{q*}(z) = -\frac{1}{2} \left\{ (1 + z)\hat{\rho}_V(1, 2\varepsilon - 1; z) + \hat{\rho}_V(1, 2\varepsilon - 2; z) \right\} \]  

(34)

are known from the scalar calculation, and

\[ \frac{B_0}{\varepsilon^2} + \frac{B_1}{\varepsilon} + B_2 = \left( \frac{1}{\varepsilon^2} - \frac{1}{6\varepsilon} + \frac{17}{12} \right) - \frac{C_B}{C_F} \left[ \frac{c_t^2}{4} \left( \frac{1}{\varepsilon^2} + \frac{1}{3\varepsilon} - \frac{1}{6} \right) + 3 \left( \frac{1}{\varepsilon} + \frac{19}{6} - 4\zeta(3) \right) \right] \],

\[ \frac{B'_1}{\varepsilon} + B'_2 = \frac{2}{3} \left( \frac{1}{\varepsilon} + 2 \right) - \frac{C_B}{C_F} \left[ \frac{c_t^2}{6} \left( \frac{1}{\varepsilon} + \frac{5}{2} \right) + 1 \right] \]  

(35)
where \( C_F = (N_c^2 - 1)/2N_c = 4/3 \), \( C_B = (N_c + 1)/2N_c = 2/3 \) for \( N_c = 3 \) colours. The occurrence of \( C_F \) in the denominator of the second parts in Eq. (35) results from the fact that \( C_F \) is factored out in Eq. (32). The parameter \( c_F \) is defined by

\[
\gamma_0 \bar{\Gamma} \gamma^\alpha = c_F \bar{\Gamma}, \quad c_F = s_F(D - 2r_F) \quad (36)
\]

and can be expressed by the signature \( s_F = \pm 1 \) of the matrix according to whether one has an even or odd number of \( \gamma \)-matrices (including \( \gamma_5 \)) and the number \( r_F \) of Dirac matrices other than \( \gamma_5 \). The fact that the most singular term is proportional to the leading order contribution allows one to extract a common renormalization factor. The result reads

\[
\rho_i(s) = \frac{N_c!s^2}{18(4\pi)^4} \left\{ 1 + \frac{\alpha_s C_F}{4\pi} C^{r(0)} g_1(m^2/s) + \frac{\alpha_s C_F}{4\pi} \left( C^{r(1)} g_1^{(0)}(m^2/s) + C^{r(0)} g_2^{(1)}(m^2/s) + C^{r(0)} g_2^{(0)}(m^2/s) \right) \right\}. \quad (37)
\]

where

\[
C^{r(0)} = 2B_0 G, \quad C^{r(1)} = 2 \left( B_0 \ln \left( \frac{\mu^2}{s} \right) + B_0 + B_1 \right) G, \quad C^{r(0)} = 2B_1 G. \quad (38)
\]

The definition of the functions \( \hat{g}_i^\alpha(z) \) can be found in Appendix A.

### 4.4 The massive line self energy contribution (b11)

We divide the leading order contribution in Eq. (27) into a mass part

\[
\rho_{a1}^m(s) = \frac{(D - 2)G(1,1)N_c!}{16(4\pi)^D(D - 1)^2} s^{D - 2} \rho_{a1}^m(m^2/s), \quad \text{where}
\]

\[
\hat{\rho}_{a1}^m(m^2/s) = \sum_{i=1}^{2} \hat{\rho}_{a1}^i(m^2/s) \text{tr}_i(\Gamma, s), \quad (39)
\]

and a momentum part

\[
\rho_{a1}^\mu(s) = \frac{(D - 2)G(1,1)N_c!}{16(4\pi)^D(D - 1)^2} s^{D - 2} \rho_{a1}^\mu(m^2/s), \quad \text{where}
\]

\[
\hat{\rho}_{a1}^\mu(m^2/s) = \sum_{i=3}^{6} \hat{\rho}_{a1}^i(m^2/s) \text{tr}_i(\Gamma, s). \quad (40)
\]

The result for the self energy correction of the massive line is obtained by a convolution of these leading order contributions. One obtains \( \rho_{b11}(s) = \rho_{b11}^m(s) + \rho_{b11}^\mu(s) \) where

\[
\rho_{b11}^m(s) = \frac{\alpha_s G(1,1)N_c!C_F(D - 2)s^{D - 2}}{16(4\pi)^D(D - 1)^2} \int_x^1 \left( -\frac{\hat{\rho}_{a1}^b(x)}{x(1 - x)} + 2\hat{L}_b(x) \frac{d}{dx} \right) \hat{\rho}_{a1}^m(z/x) dx,
\]

\[
\rho_{b11}^\mu(s) = \frac{\alpha_s G(1,1)N_c!C_F(D - 2)s^{D - 2}}{16(4\pi)^D(D - 1)^2} \int_x^1 \left( -\frac{\hat{\rho}_{a1}^b(x)}{x(1 - x)} + 2\hat{L}_b(x) \frac{d}{dx} \right) \hat{\rho}_{a1}^\mu(z/x) dx \quad (41)
\]
and where again \( z = m^2 / s \). We have introduced the functions

\[
\hat{\rho}_a(z) = (1 + z) \hat{\rho}_V(1, 1; z), \quad \hat{\rho}_b(z) = \left( \frac{D + 2}{2} - \frac{D - 2}{2} z \right) \hat{\rho}_V(1, 1; z), \tag{42}
\]

\[
\left( \hat{\rho}_{a+b}(z) = \hat{\rho}_a(z) + \hat{\rho}_b(z) \right),
\]

and

\[
\hat{L}_b(z) = \int_0^z \frac{\hat{\rho}_b(z') dz'}{(1 - z')^2}. \tag{43}
\]

The details of the calculation can be found in Appendix B.

### 4.5 The semi-massive fish (c21)

The two diagrams obtained by connecting the massive line with one of the massless lines (see Fig. III(c21)) are called semi-massive fish diagrams. Summing up the results for these two diagrams we obtain

\[
\rho_{c21}(s) = \frac{g_s^2 D/2 - 3}{16(4\pi)^{3D/2}(D - 2)(D - 1)^2} \int_0^s ds_1 s_1^{D-4}(s - s_1) \hat{\rho}_V(1, 1; s_1/s) \times
\]

\[
\times \left[ 4(D - 2) \left( D - 2 + \frac{D s_1}{s} \right) (\hat{\rho}_m(z_1) \text{tr}_1(\Gamma, s_1) + \hat{\rho}_m'(z_1) \text{tr}_1'(\Gamma, s_1)) +
\right.
\]

\[
+ 4(D - 2) \left( 1 - \frac{s_1}{s} \right) (\hat{\rho}_m(z_1) \text{tr}_2(\Gamma, s_1) + \hat{\rho}_m'(z_1) \text{tr}_2'(\Gamma, s_1)) +
\]

\[
+ \left( D - 2 + 2(D - 2) \frac{s_1}{s} + (D + 2) \frac{s_1^2}{s^2} \right) (\hat{\rho}_q(z_1) \text{tr}_3(\Gamma, s_1) + \hat{\rho}_q'(z_1) \text{tr}_3'(\Gamma, s_1)) +
\]

\[
+ \left( 1 - \frac{s_1}{s} \right) \left( 1 + \frac{s_1}{s} \right) \left( \hat{\rho}_q(z_1) \frac{1}{2} (\text{tr}_4(\Gamma, s_1) + \text{tr}_5(\Gamma, s_1)) + \hat{\rho}_q'(z_1) \text{tr}_4'(\Gamma, s_1) \right) +
\]

\[
- \left( 1 - \frac{s_1}{s} \right)^2 \left( \hat{\rho}_q(z_1) \frac{1}{2} (\text{tr}_4(\Gamma, s_1) + \text{tr}_5(\Gamma, s_1)) + \hat{\rho}_q'(z_1) \text{tr}_5'(\Gamma, s_1) \right) +
\]

\[
+ \left( 1 - \frac{s_1}{s} \right) \left( 1 + \frac{s_1}{s} \right) (\hat{\rho}_q(z_1) \text{tr}_6(\Gamma, s_1) + \hat{\rho}_q'(z_1) \text{tr}_6'(\Gamma, s_1)) +
\]

\[
+ 4(D - 1) \frac{s_1}{s} \left( \hat{\rho}_q''(z_1) \frac{1}{2} (\text{tr}_4(\Gamma, s_1) + \text{tr}_5(\Gamma, s_1)) + \hat{\rho}_q'''(z_1) \text{tr}_4'(\Gamma, s_1) \right) \right] \tag{44}
\]

where \( z_1 = m^2 / s_1 \) and

\[
\hat{\rho}_m(z_1) = 2(D - 1) B_1, \quad \hat{\rho}_m'(z_1) = B_2,
\]

\[
\hat{\rho}_q(z_1) = 2(D - 2) \left( (D - 2 + D z_1) B_4 - 2(D - 1) z_1 B_6 \right) - D(2B_5 + (D - 2) B_7),
\]

\[
\hat{\rho}_q'(z_1) = 2(D - 2) \left( (1 - z_1) B_4 + (D - 1)(B_8 + z_1 B_6) \right) + D(B_5 + (D - 2) B_7),
\]

\[
\hat{\rho}_q''(z_1) = D B_5 + (D - 2) B_7, \quad \hat{\rho}_q'''(z_1) = -B_5 \tag{45}
\]

with

\[
B_1 = -\hat{\rho}_V(1, 0, 0, 1, 1; z_1) + 2(1 - z_1)\hat{\rho}_V(1, 0, 1, 1, 1; z_1) +
\]
Appendix C. The additional traces that do not appear in Eq. (23) are given by

\[ B_2 = -\hat{\rho}_V(1, 0, 0, 1, 1; z_1) + \hat{\rho}_V(1, 0, 1, 0, 1; z_1) - \hat{\rho}_V(1, 0, 1, 1, 1; z_1), \]
\[ B_4 = -2(1 + z_1)\hat{\rho}_V(1, 0, 1, 1, 1; z_1) + +2(1 + z_1)\hat{\rho}_V(1, 1, 1, 0, 1; z_1) - 2\hat{\rho}_V(1, 1, 1, 1, 1; z_1), \]
\[ B_5 = 2\hat{\rho}_V(1, -1, 0, 1, 1; z_1) + 2\hat{\rho}_V(1, 0, -1, 1, 1; z_1) + +2(1 - z_1)\hat{\rho}_V(1, 0, 0, 1, 1; z_1) - 2(1 - z_1)\hat{\rho}_V(1, 0, 1, 0, 1; z_1) + -2\hat{\rho}_V(1, 1, -1, 1, 1; z_1) + (1 - z_1)^2\hat{\rho}_V(1, 1, 1, 1, 1; z_1), \]
\[ B_6 = 2\hat{\rho}_V(1, 1, 0, 1, 1; z_1) - \hat{\rho}_V(1, 1, 1, 1, 1; z_1), \]
\[ B_7 = -2\hat{\rho}_V(1, 0, 0, 1, 1) + 2\hat{\rho}_V(1, 0, 1, 1, 1) - 2\hat{\rho}_V(1, 0, 1, 1, 0; z_1) + -2\hat{\rho}_V(1, 1, 1, 1, 1; z_1) - 2(1 - z_1)\hat{\rho}_V(1, 1, 1, 1, 1; z_1). \]

The spectral functions \( \hat{\rho}_V(n_1, n_2, n_3, n_4, n_5; z_1) \) are so-called prototypes which are defined in Appendix C. The additional traces that do not appear in Eq. (23) are given by

\[ \text{tr}'_i(\Gamma, q^2) = \frac{m}{4q^2} \left\{ \text{Tr}(\Gamma \bar{q}[\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} - \sigma^{\mu\nu} \text{Tr}([\sigma_{\mu\nu}, \Gamma]q\bar{q}) \right\}, \]
\[ \text{tr}'_2(\Gamma, q^2) = \frac{m}{4} \left\{ \text{Tr}(\Gamma \bar{q}[\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} - \sigma^{\mu\nu} \text{Tr}([\sigma_{\mu\nu}, \Gamma]q\bar{q}) \right\}, \]
\[ \text{tr}'_3(\Gamma, q^2) = \frac{1}{4q^2} \left\{ \text{Tr}(\Gamma \bar{q}[\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} - \sigma^{\mu\nu} \text{Tr}([\sigma_{\mu\nu}, \Gamma]q\bar{q}) \right\}, \]
\[ \text{tr}'_4(\Gamma, q^2) = \frac{1}{4} \left\{ \text{Tr}(\Gamma \bar{q}[\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} - \gamma^\alpha \text{Tr}(\Gamma q\bar{q} [\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} + \sigma^{\mu\nu} \text{Tr}(\Gamma q\bar{q} [\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} \right\}, \]
\[ \text{tr}'_5(\Gamma, q^2) = \frac{1}{4} \left\{ \gamma^\alpha \text{Tr}(\Gamma q\bar{q} [\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} - \text{Tr}(\Gamma \bar{q}[\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} + \sigma^{\mu\nu} \gamma^\alpha \text{Tr}(\Gamma \bar{q}[\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} \right\}, \]
\[ \text{tr}'_6(\Gamma, q^2) = \frac{1}{4} \left\{ \text{Tr}(\Gamma q\bar{q} [\sigma_{\mu\nu}, \Gamma]q)\sigma^{\mu\nu} - \sigma^{\mu\nu} \text{Tr}([\sigma_{\mu\nu}, \Gamma]q\bar{q}) q \right\}. \]

The traces \( \text{tr}'_i(\Gamma, q^2) \) correspond to traces which are absent in the leading order term. They enter the calculation in the course of renormalizing the results.

### 4.6 Renormalization

The baryonic currents need to be renormalized. In general, there will be mixing under renormalization and therefore one has to construct the whole matrix of renormalization constants. Within our technique of parameterizing the results for arbitrary \( \Gamma \) matrices this is a straightforward procedure. The genuine first-order vertex correction for the current \( j \) is given by

\[ j^B = \varepsilon^{abc}(u_a^T C(\gamma_\mu \gamma_\nu \Gamma + \Gamma \gamma_\nu \gamma_\mu) u_b) \gamma^\mu \gamma^\nu \Gamma' \Psi_c \]
which (using $\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$) can be written as

$$j^B = \varepsilon^{abc} \left\{ u_a^T C (g_{\mu\nu} \Gamma + \Gamma g_{\mu\nu}) d_b \right\} \gamma^\mu \gamma^\nu \Gamma' \Psi_c - i (u_a^T C (\sigma_{\mu\nu} \Gamma - \Gamma \sigma_{\mu\nu}) d_b) \gamma^\mu \gamma^\nu \Gamma' \Psi_c \right\} = \varepsilon^{abc} \left\{ 2D (u_a^T C \Gamma d_b) \Gamma' \Psi_c - (u_a^T C (\sigma_{\mu\nu} \Gamma - \Gamma \sigma_{\mu\nu}) d_b) \sigma^{\mu\nu} \Gamma' \Psi_c \right\}. \quad (49)$$

In calculating the correlator we therefore expect objects of the form

$$\frac{1}{4D^2} \langle j^B_1 j^B_2 \rangle = \text{Tr}(\Gamma \circ \Gamma \circ) \circ - \frac{1}{2D} \text{Tr}(\Gamma \circ (\sigma_{\mu\nu} \Gamma - \Gamma \sigma_{\mu\nu}) \circ) \sigma^{\mu\nu} + \frac{1}{2D} \sigma^{\mu\nu} \text{Tr}\left((\sigma_{\mu\nu} \Gamma - \Gamma \sigma_{\mu\nu}) \circ \tilde{\Gamma} \circ\right). \quad (50)$$

The open circles stand for further Dirac structures in the calculation. For example, if the first term on the r.h.s. of (50) is given by $\text{Tr}(\Gamma \circ \Gamma \circ) \circ = q^2 \text{tr}_3 = \text{Tr}(\Gamma \Psi \Psi \Psi \Psi)$, the whole right hand side of Eq. (50) reads

$$\text{Tr}(\Gamma \Psi \Psi \Psi \Psi) \Psi - \frac{1}{2D} \text{Tr}\left(\Gamma \Psi (\sigma_{\mu\nu} \Gamma - \Gamma \sigma_{\mu\nu}) \Psi \right) \Psi \sigma^{\mu\nu} + \frac{1}{2D} \sigma^{\mu\nu} \text{Tr}\left((\sigma_{\mu\nu} \Gamma - \Gamma \sigma_{\mu\nu}) \Psi \right) \Psi = q^2 \left(\text{tr}_3 - \frac{2}{D} \text{tr}_3\right). \quad (51)$$

where we have used the trace definitions (23) and (17). The left hand side of Eq. (50) represents the singular contribution of the diagram. On the other hand, the singular parts of the spectral functions of the basic structures $\text{tr}_i$ are $2\alpha_s / 3\pi \varepsilon$ times the LO term, whereas the spectral functions of the primed structures $\text{tr}_i'$ are $-\alpha_s / 6\pi \varepsilon$ times the LO result of the corresponding basic structure. Note, finally, that we need to consider only the LO singularities within the $\overline{\text{MS}}$-scheme. Therefore, we need not specify $\Gamma$ at this point. If we write the total result for the semi-massive fish as

$$\sum_{i=1}^{6} \left[ \rho^i_b \text{tr}_i + \rho^i'_{b'} \text{tr}_i' \right] = \sum_{i=1}^{6} \left[ \left( \rho^i_{00} + \rho^i_{01} \varepsilon + \frac{\alpha_s}{\pi} \left( \rho^i_{10} \frac{1}{\varepsilon} + \rho^i_{11} \right) \right) \text{tr}_i + \frac{\alpha_s}{\pi} \left( \rho^i_{01} \frac{1}{\varepsilon} + \rho^i_{11} \right) \text{tr}_i' \right] \quad (52)$$

with $\rho^i_{10} = \frac{2}{3} \rho^i_{00}$ and $\rho^i'_{10} = -\frac{1}{6} \rho^i'_{00}$, we can extract the renormalization factors and obtain

$$\sum_{i=1}^{6} \left[ \left( 1 + \frac{2\alpha_s}{3\pi \varepsilon} \right) \left( \rho^i_{00} + \rho^i_{01} \varepsilon + \frac{\alpha_s}{\pi} \left( \rho^i_{11} - \frac{2}{3} \rho^i_{01} \right) \right) \text{tr}_i + \left( -\frac{\alpha_s}{6\pi \varepsilon} \rho^i_{00} + \frac{\alpha_s}{\pi} \rho^i_{11} \right) \text{tr}_i' \right] \quad (53)$$

Up to $O(\alpha_s)$ we have

$$\rho^i_\Gamma = \rho^i_{00} + \rho^i_{01} \varepsilon + \frac{\alpha_s}{\pi} \left( \rho^i_{11} - \frac{2}{3} \rho^i_{01} \right). \quad (54)$$

If we substitute this expression for $\rho^i_{00}$ in the coefficient of $\text{tr}_i'$, the next-to-leading order contribution can be skipped while the term proportional to $\varepsilon$ leads to a subtraction of the finite term,

$$\sum_{i=1}^{6} \left[ \left( 1 + \frac{2\alpha_s}{3\pi \varepsilon} \right) \rho^i_\Gamma \text{tr}_i + \left( -\frac{\alpha_s}{6\pi \varepsilon} \rho^i_\Gamma + \frac{\alpha_s}{\pi} \left( \rho^i_{11} + \frac{1}{6} \rho^i_{01} \right) \right) \text{tr}_i' \right]. \quad (55)$$

In total we have

$$\rho^i_b \text{tr}_i + \rho^i'_{b'} \text{tr}_i' = \left( 1 + \frac{2\alpha_s}{3\pi \varepsilon} \right) \rho^i_\Gamma \text{tr}_i + \left( -\frac{\alpha_s}{6\pi \varepsilon} \rho^i_\Gamma + \rho^i_{11} \right) \text{tr}_i' \quad (56)$$

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where
\[
\rho_i'^\nu = \frac{\alpha_s}{\pi} \left( \rho_i'^{\nu} + \frac{1}{6} \rho_i'^0 \right). \tag{57}
\]
The coefficient of the primed structure has no LO contribution. Nevertheless, the renormalization works in the same formal manner as for the basic structure. As these calculations show, the renormalization of the mixing operators for the basic and primed structure is accomplished by a renormalization matrix,
\[
\begin{pmatrix}
\rho_b^i \\
\rho_b'^i
\end{pmatrix} = \begin{pmatrix}
1 + \frac{2\alpha_s}{3\pi\varepsilon} & 0 \\
-\frac{\alpha_s}{6\pi\varepsilon} & 1
\end{pmatrix}
\begin{pmatrix}
\rho_r^i \\
\rho_r'^i
\end{pmatrix}. \tag{58}
\]
One can easily invert this renormalization matrix to compute the renormalized quantities. One also needs the bare quantities which are given by
\[
\rho_i^r = \left( 1 - \frac{2\alpha_s}{3\pi\varepsilon} \right) \rho_b^i, \quad \rho_i'^r = \rho_b'^i + \frac{\alpha_s}{6\pi\varepsilon} \rho_b^i. \tag{59}
\]
The results given in Appendices A to C are already renormalized ones. In order to obtain the total result we have to combine the spectral functions giving
\[
\hat{\rho}_0^i(z) = \hat{\rho}_\text{leading}(z), \quad \hat{\rho}_1^i(z) = \hat{\rho}_\text{light}(z) + \hat{\rho}_\text{massi}(z) + \hat{\rho}_\text{fish}(z), \quad \hat{\rho}_1'^i(z) = \hat{\rho}_\text{fish}(z). \tag{60}
\]
Using these spectral functions and the traces defined in Eqs. (23) and (47), the spectral density is given by
\[
\rho(s) = \frac{s^2}{512\pi^4} \sum_{i=1}^6 \left[ \hat{\rho}_0^i(m^2/s) \left\{ 1 + \left[ \left( n_m^i + \frac{3}{3} + \frac{r^2}{r^1} \right) \frac{\alpha_s}{\pi} \text{tr}_i - \frac{\alpha_s}{6\pi} \text{tr}_i' \right] \ln \left( \frac{m^2}{\mu^2} \right) \right\} + \frac{\alpha_s}{\pi} \left( \hat{\rho}_1^i(m^2/s) \text{tr}_i + \hat{\rho}_1'^i(m^2/s) \text{tr}_i' \right) \right]. \tag{61}
\]
where \( n_m^i = 0, 1 \) depending on whether there is a factor of \( m \) in \( \text{tr}_i \) or not. Explicitly one has \( n_m^{1,2} = 1 \) and \( n_m^{3-6} = 0 \).

5 Some properties of the spectral densities at NLO

In this section we consider two limiting cases of the NLO spectral densities. First we consider the large energy or equivalently the mass zero limit. Second we analyze the near threshold limit relevant for a comparison with HQET results. Both limits are interesting and physically relevant. The limiting cases for the lowest spin baryons have been discussed before in Refs. [16, 17]. We therefore concentrate on the case of the “vector” current [24] in the following.

5.1 High energy expansion

In the high energy (or, equivalently, small mass) limit \( z \to 0 \) the spectral density reads
\[
\rho_{\mu\nu}(s) = \frac{s^2}{512\pi^4} \left[ 4 \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{59}{12} + \ln \left( \frac{\mu^2}{s} \right) \right) \right\} m q^\mu q^\nu - 3 \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{13}{3} + \frac{2}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} m g^\mu g^\nu +
\]
\begin{align}
+ \frac{4}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{343}{180} - \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} q^\mu q^\nu + \frac{1}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{821}{360} - \frac{4}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} \gamma^\mu \gamma^\nu + \\
+ \frac{1}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{313}{180} - \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} \gamma^\mu q^\nu - \frac{4}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{82}{45} - \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} \gamma^\mu \gamma^\nu + \\
- \frac{\alpha_s}{\pi} \left( \frac{25}{36} + \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \left( 2m \tilde{g} \gamma^\mu \gamma^\nu - \gamma^\mu q^\nu + \gamma^\mu \gamma^\nu \right) - \frac{\alpha_s}{\pi} \left( \frac{9}{4} + \frac{2}{3} \ln \left( \frac{\mu^2}{s} \right) \right) m \gamma^\mu \gamma^\nu \right\} \tag{62}
\end{align}

where \( m = m_{\overline{\text{MS}}} (\mu) \) is the \( \overline{\text{MS}} \) mass. When one compares \((62)\) with the results of an \textit{ab initio} (multiplicatively renormalized) massless calculation one does not obtain full agreement. We discuss an alternative route. Instead of expanding \( \rho^{\mu\nu} \) along the set of ten covariants given in Eq. \((26)\) we exploit the symmetry of the problem and expand along an alternative set of only nine covariants

\[
\{ \rho^{\mu\nu} - \rho_{\text{LO}}^{\mu\nu}, \gamma^\mu q^\nu /s, \gamma^\mu q^\nu /s, \gamma^\mu q^\nu, \gamma^\mu \gamma^\nu, \gamma^\mu \gamma^\nu, \gamma^\mu \gamma^\nu, \gamma^\mu \gamma^\nu \}.
\]

In this case we obtain

\begin{align}
\rho^{\mu\nu}(s) &= \frac{s^2}{512 \pi^4} \left[ 4 \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{59}{12} + \ln \left( \frac{\mu^2}{s} \right) \right) \right\} m \frac{q^\mu q^\nu}{s} - 3 \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{13}{3} + \frac{2}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} m \gamma^\mu \gamma^\nu + \\
+ \frac{4}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{343}{180} - \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} \rho^{\mu\nu} + \frac{1}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{313}{180} - \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} \gamma^\mu \gamma^\nu + \\
+ \frac{1}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{313}{180} - \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} \gamma^\mu q^\nu - \frac{4}{5} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{82}{45} - \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \right\} \gamma^\mu \gamma^\nu + \\
- \frac{\alpha_s}{\pi} \left( \frac{25}{36} + \frac{1}{3} \ln \left( \frac{\mu^2}{s} \right) \right) \left( m \frac{\gamma^\mu \gamma^\nu {\bar q} /s - \gamma^\mu q^\nu} - \frac{\alpha_s}{\pi} \left( \frac{14}{9} + \frac{2}{3} \ln \left( \frac{\mu^2}{s} \right) \right) m \gamma^\mu \gamma^\nu \right] \tag{63}
\end{align}

where the first six contributions which contain both LO and NLO parts are reproduced by the massless calculation. It cannot be expected that a multiplicatively renormalized massless result can reproduce the remaining three contributions. In addition, the massless calculation cannot reproduce terms such as \( z \ln(z) \) as they appear for instance in the full NLO results in Eqs. \((14)\) and \((15)\). These terms can be parametrized with condensates of local operators. \footnote{For a discussion of this point cf. Ref. \[16\].}

### 5.2 Near-threshold expansion

As concerns the \( \Lambda \)-type baryons we have already compared the near-threshold limit with the HQET result in Ref. \[16\]. Here we repeat the exercise for the \( \Sigma \)-type \( J^P = 1/2^+ \) baryon with \( \Gamma = \gamma^\mu, \Gamma' = \gamma_\mu \gamma_5 \), i.e. \( s_\Gamma = -1 \), and \( r_\Gamma = 1 \). In the near-threshold limit \( E \to 0 \) (\( s = (m + E)^2 \)) Eq. \((61)\) gives

\[
\rho^{\mu\nu}(m, E) = \frac{E^5}{40 \pi^4 m} \left[ 2 \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{42}{5} + \frac{4 \pi^2}{9} + 2 \ln \left( \frac{\mu}{2E} \right) \right) \right\} \frac{E^{\mu\nu}}{s} + \\
- \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{121}{15} + \frac{4 \pi^2}{9} + 2 \ln \left( \frac{\mu}{2E} \right) + \frac{1}{3} \ln \left( \frac{m^2}{\mu^2} \right) \right) \right\} \left( m + \bar{q} \right) g^{\mu\nu} + \\
\right.\left. \right]
\]
\[- \frac{\alpha_s}{3\pi} (q^\mu \gamma^\nu + \gamma^\mu q^\nu) - \frac{\alpha_s}{6\pi} \left( 1 - \ln \left( \frac{m^2}{\mu^2} \right) \right) m T \frac{\gamma^\mu \gamma^\nu}{s} + \]
\[- \frac{\alpha_s}{6\pi} \left( 1 - \ln \left( \frac{m^2}{\mu^2} \right) \right) m \gamma^\mu \gamma^\nu + \frac{\alpha_s}{3\pi} \left( 1 - \ln \left( \frac{m^2}{\mu^2} \right) \right) \gamma^\mu \gamma^\nu \]  
(64)

where \( m \) is the pole mass. The HQET contribution is obtained by projecting on both sides with \((1 + \bar{\nu})/2 = (m + \bar{q})/2m \). Omitting the projector \((1 + \bar{\nu})/2 \) itself we obtain
\[
\rho^{\mu\nu}_{\text{HQET}}(m, E) = \frac{E^5}{20\pi^4 m} \left\{ 2 \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{42}{5} + \frac{4\pi^2}{9} + 2 \ln \left( \frac{\mu}{2E} \right) \right) \right\} \frac{q^\mu q^\nu}{s} + \right. 
\left. - \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{121}{15} + \frac{4\pi^2}{9} + 2 \ln \left( \frac{\mu}{2E} \right) + \frac{1}{3} \ln \left( \frac{m^2}{\mu^2} \right) \right) \right\} \gamma^\mu \gamma^\nu + \frac{\alpha_s}{3\pi} \left( 1 - \ln \left( \frac{m^2}{\mu^2} \right) \right) \left( \gamma^\mu q^\nu - \frac{q^\mu q^\nu}{m} \right) \right\} + \]
\[
\left. \frac{\alpha_s}{6\pi} \left( 1 - \ln \left( \frac{m^2}{\mu^2} \right) \right) \left( \frac{q^\mu q^\nu}{s} - \gamma^\mu \gamma^\nu + \frac{q^\mu q^\nu}{m} \right) \right\}. \quad (65)
\]

We now proceed to compare (65) with the corresponding result derived from the HQET current [22]
\[
\tilde{j}_{\Sigma 1} = (q^T C \gamma^\mu q) \gamma^\nu \gamma_5 \bar{Q}, \quad (66)
\]
where \( \gamma^\mu_5 = \gamma^\nu (g^{\mu\nu} - q^{\mu} q^{\nu}/q^2) \). One needs to extract the transverse piece from Eq. (65) using the transverse projection \( g_{\mu\nu} - q_{\mu} q_{\nu}/q^2 \).
\[
m\rho_{\Sigma 1}(m, E) = -\frac{3E^5}{20\pi^4} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{136}{15} + \frac{4\pi^2}{9} + 2 \ln \left( \frac{\mu}{2E} \right) - \frac{2}{3} \ln \left( \frac{m^2}{\mu^2} \right) \right) \right\}. \quad (67)
\]

This can be compared with the HQET result [23]
\[
\tilde{\rho}_{\Sigma 1}(E, \mu) = -\frac{3E^5}{20\pi^4} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{116}{15} + \frac{4\pi^2}{9} + 2 \ln \left( \frac{\mu}{2E} \right) \right) \right\} \quad (68)
\]
where the explicit mass factor \( m \) appearing on the left hand side of Eq. (67) has been absorbed in the definition of \( \tilde{\rho}_{\Sigma 1}(E, \mu) \) in Eq. (68). After addition of the matching coefficient [22]
\[
C_{\Sigma 1}(m/\mu, \alpha_s) = 1 + \frac{\alpha_s}{3\pi} \left( 2 - \ln \left( \frac{m^2}{\mu^2} \right) \right) \quad (69)
\]
one obtains \( m\rho_{\Sigma 1}(m, E) = C_{\Sigma 1}(m/\mu, \alpha_s)^2 \tilde{\rho}_{\Sigma 1}(\mu, E) \). In this case the matching procedure allows one to recover the near-threshold limit of the full correlator starting from the simpler effective theory near threshold [24]. Note that the higher order \( E/m \) corrections to Eq. (64) can be easily obtained from the explicit result given in Eq. (65). Indeed, the next-to-leading order correction in the low energy threshold expansion reads
\[
\Delta \rho_{\Sigma 1}(m, E) = -\frac{11E^6}{120m^2} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{1744}{165} + \frac{4\pi^2}{9} - \frac{5}{3} \ln \left( \frac{m^2}{\mu^2} \right) + \frac{74}{33} \ln \left( \frac{m}{2E} \right) \right) \right\} m + \]
\[
-\frac{17E^6}{120m^2} \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{512}{51} + \frac{4\pi^2}{9} - \frac{28}{51} \ln \left( \frac{m^2}{\mu^2} \right) + \frac{110}{51} \ln \left( \frac{m}{2E} \right) \right) \right\} \tilde{\rho}, \quad (70)
\]

\footnote{Note that we have to take into account the contraction with \( \Gamma' \cdots \Gamma' \) as well.}
To obtain this result starting from HQET is a more difficult task requiring the analysis of contributions from higher dimension operators.

5.3 Interpolation

We now discuss some quantitative features of the correction given in Eq. (61). Of interest is whether the two limiting expressions (the massless and threshold limit) can be used to characterize the full function for all energies. To this end we compare components of the baryonic spectral density up to NLO. In Fig. 2 we show the ratio $\rho_{i}^{1}(s)/\rho_{0}^{i}(s)$ for $i = 3$. Fig. 2 shows that one would obtain a rather good approximation for the full NLO order correction for all values of $s$ if one were to use an interpolation between the two limiting cases.

![Figure 2: The ratio $\rho_{i}^{3}(s)/\rho_{0}^{i}(s)$ up to NLO as a function of the squared energy $s$.](image)

5.4 Moments

An instructive set of observables are the negative moments of the spectral density (see e.g. [25]):

$$M_{-n} = \int_{m^2}^{\infty} s^{-n} \rho(s) ds = \frac{m^{6-2n}}{512\pi^4(n-3)} \sum_{i=1}^{6} \left( M_{n-4}^{i} \text{tr}_{i} + M_{n-4}^{i'} \text{tr}_{i}' \right),$$

$$M_{n}^{(i)} = (n+1) \int_{0}^{1} z^{n} \hat{\rho}^{(i)}(z) dz \quad (71)$$

where $\rho(s)$ is taken from Eq. (61) and $\hat{\rho}^{i}(m^2/s)$ and $\hat{\rho}^{i'}(m^2/s)$ are the coefficients of $\text{tr}_{i}$ and $\text{tr}_{i}'$. One has

$$M_{n}^{i} = M_{n}^{(i)} \left\{ 1 + \frac{\alpha_{s}}{\pi} \left( n_{m}^{i} + \frac{3 - 4r_{T} + r_{T}^{2}}{3} \right) \ln \left( \frac{\mu^{2}}{m^{2}} \right) + \delta_{n}^{i} \right\} \quad (70)$$

\[3\text{We have normalized } M_{n}^{i} \text{ such that } M_{n}^{i} = 1 \text{ for } \hat{\rho}^{i}(z) = 1.\]
where

\begin{align*}
M_{n}^{(0)} &= \frac{12}{(n+2)^2(n+3)(n+4)(n+5)}, \\
M_{n}^{(2)} &= \frac{6}{(n+2)^2(n+3)^2(n+4)(n+5)}, \\
M_{n}^{(3)} &= \frac{12}{(n+2)(n+3)(n+4)(n+5)(n+6)}, \\
M_{n}^{(4)} &= \frac{6}{(n+2)(n+3)^2(n+4)(n+5)(n+6)} = M_{n}^{(5)} = M_{n}^{(6)}
\end{align*}

(73)

and \( \delta_{n}^{i} = \delta_{n}^{(a)} + \delta_{n}^{(b)} + \delta_{n}^{(c)} \) with

\begin{align*}
\delta_{n}^{(a)} &= \frac{2}{3} \left( \frac{2 - 4r_{\Gamma} + r_{\Gamma}^2}{2(n+1)(n+2)} \right) \left( \psi(n+1) + \gamma_{E} - \frac{3n+1}{4(n+1)(n+2)} \right) + \frac{2}{3} r_{\Gamma} = \delta_{n}^{2(a)}, \\
\delta_{n}^{(b)} &= \frac{2}{3} \left( \frac{2 - 4r_{\Gamma} + r_{\Gamma}^2}{2(n+1)(n+2)} \right) \left( \psi(n+1) + \gamma_{E} - \frac{n(3n+5)}{4(n+1)(n+2)} \right) + \frac{2}{3} r_{\Gamma} = \delta_{n}^{4(a)} = \delta_{n}^{5(a)} = \delta_{n}^{6(a)}, \\
\delta_{n}^{(c)} &= \frac{4}{3} \left( \psi(n+1) + \gamma_{E} + \frac{n^2 - 2}{2(n+1)(n+2)} \right) = \delta_{n}^{2(b)}, \\
\delta_{n}^{(d)} &= \frac{4}{3} \left( \psi(n+1) + \gamma_{E} + \frac{2n^2 + 2n + 1}{4(n+1)(n+2)} \right) = \delta_{n}^{4(b)} = \delta_{n}^{5(b)} = \delta_{n}^{6(b)}, \\
\delta_{n}^{(e)} &= \frac{2}{3(n+4)} \left( \psi(n+1) + \gamma_{E} + \frac{n^2 - 2n - 2}{4(n+1)} \right) = \delta_{n}^{\psi}, \\
\delta_{n}^{(f)} &= \frac{2}{3(n+5)} \left( \psi(n+1) + \gamma_{E} + \frac{n^2 - n - 5}{4(n+2)} \right) = \delta_{n}^{\psi} = \delta_{n}^{\psi}.
\end{align*}

(74)

Further \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the digamma function, \( \psi'(z) \) its first derivative (first polygamma function) and \( \gamma_{E} = 0.577 \ldots \) is the Euler–Mascheroni constant. In order to combine the three
As mentioned before, all moments defined in (71) are normalized to one in the sense that
\[ M^i_n = 1 \] for \( \hat{\rho}^i(z) = 1 \). Note that the difference \( \delta_n - \delta_N \) is scheme-independent. This feature was used in the high precision analysis of heavy quark properties \[26\] within NRQCD (see e.g. Ref. \[27\]). One can now easily find the actual magnitude of the correction. Indeed, for any desired precision and range of \( n \), a set of perturbatively commensurate moments can be found. We therefore present differences of consecutive \( \delta^i_n \) in the second part of the columns of Table 1.

Note that because of the \( s \)-integration in Eq. (71), moments represent massive vacuum bubbles, i.e. diagrams without external momenta with massive lines. These diagrams have been comprehensively analyzed in Refs. \[28, 29\]. The analytical results for the first few moments at the three-loop level can be checked independently with existing computer programs (see e.g. Ref. \[30\]).
6 Applications to physics: Sum rules

A phenomenological application is given by QCD sum rules (see e.g. Ref. [31, 32]) or finite energy sum rules (FESR, see e.g. Ref. [33]). In general, the main quantity of interest is the residue

\[ \langle |j_B(0)| \Lambda(p, \sigma) \rangle = \lambda_B u(p, \sigma) \]  

(78)

where \( \Lambda(p, \sigma) \) is a baryon state and \( u(p, \sigma) \) is the spinor that satisfies the free equation of motion \( (\not{p} - m_B)u(p, \sigma) = 0 \). The contribution of the ground state baryon \( B \) to the spectral density of the correlator reads

\[ \rho_B(0)(s) = \lambda_B^2 \delta(s - m_B^2) \]  

(79)

while the excited states contribute to the spectral density \( \rho_B(s) \) that were calculated before. We define a threshold value \( s_0 \) where the excited states and the continuum start contributing. If we take moments, i.e. integrals over the phenomenological spectral density \( \rho_B(0)(s) + \theta(s - s_0) \rho_B(s) \) with different powers of \( s \), we obtain the sum rule condition

\[ \int_0^\infty \rho_B(s) s^n ds = \int_0^\infty \left( \rho_B(0)(s) + \theta(s - s_0) \rho_B(s) \right) s^n ds = \lambda_B^2 m_B^{2n} + \int_{s_0}^\infty \rho_B(s) s^n ds \]  

(80)

or, equivalently, the sum rule

\[ \lambda_B^2 m_B^{2n} = \int_0^{s_0} \rho_B(s) s^n ds =: M_n(s_0). \]  

(81)

The two unknown parameters \( s_0 \) and \( \lambda_B \) (\( m_B \) is assumed to be known) can be determined in turn by the sum rule analysis. For this purpose we calculate the ratio of nearby moments,

\[ \frac{M_n(s_0)}{M_{n-1}(s_0)} = \frac{\int_0^{s_0} \rho_B(s) s^n ds}{\int_0^{s_0} \rho_B(s) s^{n-1} ds} = m_B^n \]  

(82)

and adjust \( s_0 \). Once \( s_0 \) is determined we can calculate \( \lambda_B^2 \) by using the zeroth moment,

\[ \lambda_B^2 = M_0(s_0) = \int_0^{s_0} \rho_B(s) ds. \]  

(83)

The second mass scale occurring in the problem is the mass \( m \) of the heavy constituent quark occurring in \( \rho_B(s) \). It is fixed by assuming a value of 500 MeV for the difference \( \Lambda_B = m_B - m \) between baryon and quark mass. Finally, we have to decide which of the sum rules we use. Here we have decided to use the \( n = 0 \) sum rule. For \( \rho(s) \) we use Eq. (61) in the scalar setting (\( \Lambda \)-type baryon: \( \Gamma = 1, \Gamma' = \gamma_5 \)) or vector settings (\( \Sigma \)-type baryon: \( \Gamma = \gamma^\mu, \Gamma' = \gamma^\mu \gamma_5, \Sigma^* \)-type baryon: \( \Gamma = \gamma^\mu, \Gamma' = 1 \) with \( \Psi \rightarrow \Psi^\mu \)). \( \alpha_s \) is the running QCD coupling with a scale set by the heavy quark mass.

6.1 \( \Lambda \)-type baryons

The \( \Lambda \)-type baryon ground state masses are given by [1]

\[ m_{\Lambda^+} = 2286.46 \pm 0.14 \text{ MeV}, \quad m_{\Lambda^0_b} = 5624 \pm 9 \text{ MeV}. \]  

(84)
We first do the sum rule analysis for the LO contribution. Using the sum rule

\[
\int_{m^2}^{s_0} \rho(s) ds = \int_{m^2}^{s_0} \rho(s) \frac{ds}{s} = m_\Lambda^2
\]

(85)

with \( z_0 = m^2 / s_0 \), we obtain the sum rule value for \( z_0 \). Reinserting this value into \( M_0(s_0) \), we obtain the value for \( \lambda_{\Lambda^+}^2 \) and \( \lambda_{\Lambda^0}^2 \). The results are collected in Tab. 2. In going from LO to NLO, the value of \( z_0 \) does not change significantly. However, the values for \( \lambda_{\Lambda^+}^2 \) and \( \lambda_{\Lambda^0}^2 \) change by a factor of 1.7 (\( \Lambda^+ \)) and 1.6 (\( \Lambda^0 \)) which disqualifies the sum rule method as an appropriate method to determine the baryon’s parameters. Theoretically, we can improve the convergence behaviour by tuning the mass logarithm \( \ln(m^2 / \mu^2) \) to values of roughly 4.5 (\( m \)-part) or 6.0 (\( q \)-part). However, the necessary values of \( \mu \) (less than 200 MeV) are far from being realistic.

### 6.2 \( \Sigma \)-type baryons

For the \( \Sigma \)-type baryons, the three ground state masses read [1]

\[
m_{\Sigma^{++}} = 2454.02 \pm 0.18 \text{ MeV}, \quad m_{\Sigma^{+}} = 2452.9 \pm 0.4 \text{ MeV}, \quad m_{\Sigma^0} = 2453.76 \pm 0.18 \text{ MeV}. \quad (86)
\]

We take the mass average \( m_{\Sigma_c} = 2453.56 \text{ MeV} \) as input parameter. For the recently found \( \Sigma_b \) state we take the average of the mass of the positive and negative charged state and obtain \( m_{\Sigma_b} = 5811.5 \text{ MeV} \) [36]. Tab. 3 gives an overview over the results. The NLO/LO ratio for the residue is 1.6 for \( \Sigma_c \) and 1.46 for \( \Sigma_b \).

### 6.3 \( \Sigma^* \)-type baryons

In order to extract the vector baryon components of highest spin 3/2 we use the Rarita–Schwinger formalism (see e.g. [37]). Instead of the spinor \( u(p, \sigma) \) in Eq. (78) we have to use the Rarita–Schwinger spinor \( u_\mu(p, \sigma) \) (\( \sigma \) takes the four values \( \pm 3/2 \) and \( \pm 1/2 \)). The equation of motion for the highest spin state leads to the two additional constraints

\[
\gamma^\mu u_\mu(p, \sigma) = 0 \quad \text{and} \quad p^\mu u_\mu(p, \sigma) = 0. \quad (87)
\]

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Table 2: Sum rule analysis for the } \Lambda \text{-type baryons with } & & & \\
& z_0 \text{ (LO)} & z_0 \text{ (NLO)} & \lambda \text{ (LO)} & \lambda \text{ (NLO)} \\
\hline
\Lambda^+_c & 0.565550 & 0.564193 & 1.77978 \times 10^{-3} \text{ GeV}^3 & 3.01893 \times 10^{-3} \text{ GeV}^3 \\
& 0.564191 & 0.562752 & 1.60523 \times 10^{-3} \text{ GeV}^3 & 2.72316 \times 10^{-3} \text{ GeV}^3 \\
\hline
\Lambda^0_c & 0.805403 & 0.804903 & 1.81465 \times 10^{-3} \text{ GeV}^3 & 2.86564 \times 10^{-3} \text{ GeV}^3 \\
& 0.805088 & 0.804575 & 1.74339 \times 10^{-3} \text{ GeV}^3 & 2.75323 \times 10^{-3} \text{ GeV}^3 \\
\hline
\end{array}
\]
Using the ansatz
\[
\Pi^{\mu\nu} = A q^\mu q^\nu + B g^{\mu\nu} + \frac{1}{i} \sigma^{\alpha\beta} D^{\mu\nu}_{\alpha\beta}
\]
(88)
where
\[
D^{\mu\nu}_{\alpha\beta} = R(g^\mu_{\alpha}g^\nu_{\beta} - g^\nu_{\alpha}g^\mu_{\beta}) + X_1 q^\mu (g^\nu_{\alpha}q_{\beta} - g^\nu_{\beta}q_{\alpha}) + X_2 q^\nu (g^\mu_{\alpha}q_{\beta} - g^\mu_{\beta}q_{\alpha}),
\]
(89)
we can contract from the left with \(\gamma^\mu\) in order to satisfy the first constraint \(\gamma^\mu \Pi^{\mu\nu} = 0\). We obtain \(X_1 = -X_2\), \(A + 4X_2 = 0\) and
\[
R = \frac{1}{6} \left( \frac{A}{2} q^2 - B \right).
\]
Further imposing the condition \(q_\mu \Pi^{\mu\nu} = 38\), we can determine all coefficients up to one for which we choose \(A\). This gives rise to a basis of projectors \((\mathcal{P}^{3/2})^{\mu\nu}\), \((\mathcal{P}^{1/2})^{\mu\nu}\), and \((\mathcal{P}^{1/2})^{\mu\nu}\) and nilpotent operators \((\mathcal{P}^{1/2})^{\mu\nu}\) of the form [37, 39]
\[
(\mathcal{P}^{3/2})^{\mu\nu} = q^{\mu\nu} - \frac{2}{3} \frac{q^\mu q^\nu}{q^2} - \frac{1}{3} \gamma^\mu \gamma^\nu + \frac{1}{3q^2} (\gamma^\mu q^\nu - \gamma^\nu q^\mu) q^\nu,
\]
\[
(\mathcal{P}^{1/2})^{\mu\nu} = \frac{q^\mu q^\nu}{q^2},
\]
\[
(\mathcal{P}^{1/2})^{\mu\nu} = \sqrt{\frac{3}{q^2}} \frac{1}{3q^2} (-q^\mu + \gamma^\mu q^\nu) q^\nu,
\]
\[
(\mathcal{P}^{1/2})^{\mu\nu} = \sqrt{\frac{3}{q^2}} \frac{1}{3q^2} q^\nu (-q^\mu + \gamma^\nu q^\nu).
\]
(91)

In order to obtain the highest spin component we contract with \((\mathcal{P}^{3/2})^{\mu\nu}\). If we use the above results for a sum rule analysis of the \(\Sigma^*\)-type baryons with ground state masses [1]
\[
m_{\Sigma^{*++}} = 2518.4 \pm 0.6 \text{ MeV}, \quad m_{\Sigma^{*+}} = 2517.5 \pm 2.3 \text{ MeV}, \quad m_{\Sigma^{*-}} = 2518.0 \pm 0.5 \text{ MeV}
\]
(92)
with an average value of \(m_{\Sigma^c} = 2518.0 \text{ MeV}\), we obtain the values in Table 4. For the recently found \(\Sigma^*_c\)-baryon we take the average of the mass of the positive and negative charged state and obtain \(m_{\Sigma^*_c} = 5832.7 \text{ MeV} ([36])\). The ratio NLO/LO is 1.55 for \(\Sigma^*_c\) and 1.45 for \(\Sigma^*_b\).
Table 4: Sum rule analysis for the $\Sigma^*$-type baryons with $E_{\Sigma^*} = m_{\Sigma^*} - m = 500$ MeV

7 Conclusions

To conclude, we have computed the NLO perturbative corrections to the correlators of finite mass baryons containing one heavy quark and two massless quarks for a variety of quantum numbers of the baryonic currents. Technically this is a genuine three loop calculation with the two mass scales $s$ and $m^2$. We have considered both the massless limit and the threshold HQET limit of the correlator as special cases of the general finite mass formula. The two respective limiting expressions agree with previous massles and HQET results in the literature. From threshold to high energies the exact spectral density interpolates nicely between the leading order (leading in $1/m_Q$) HQET result close to threshold and the asymptotic mass zero result. This raises the hope that one can find a similar interpolation formula at the four loop NNLO level using the massless and the HQET four-loop results. These can be calculated using existing computational algorithms [40, 41].

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A The light contributions

The light contribution consists of contributions from the two diagrams (b21, light self energy) and (c11, light fish). These contributions are calculated in turn.

A.1 The light self energy (b21)

The result has the same structure as the leading order contribution,

$$
\rho_{b21}(s) = \frac{g_s^2G^2}{16(4\pi)^{3D/2}s^{3D/2-4}} \sum_{i=1}^{6} \rho^{i}_{b21}(m_i^2/s) \text{tr}_i(\Gamma, s)
$$

(A1)
where $\text{tr}_i(\Gamma, s)$ are given by Eqs. (23) and

$$
\begin{align*}
\hat{\rho}_{b21}^1(z) &= \frac{D - 2}{2(D - 1)^2} \left( \frac{1}{\varepsilon^2} - \frac{1}{6\varepsilon} + \frac{17}{12} \right) \rho_{a1}^1(z), \\
\hat{\rho}_{b21}^2(z) &= \frac{D - 2}{2(D - 1)^2} \left( \frac{1}{\varepsilon^2} - \frac{1}{6\varepsilon} + \frac{17}{12} \right) \rho_{a1}^2(z) - \frac{2}{3} \left( \frac{1}{\varepsilon} + \frac{5}{2} \right) \rho_{a1}^m(z), \\
\hat{\rho}_{b21}^3(z) &= \frac{D - 2}{2(D - 1)^2} \left( \frac{1}{\varepsilon^2} - \frac{1}{6\varepsilon} + \frac{17}{12} \right) \rho_{a1}^3(z), \\
\hat{\rho}_{b21}^4(z) &= \frac{D - 2}{2(D - 1)^2} \left( \frac{1}{\varepsilon^2} - \frac{1}{6\varepsilon} + \frac{17}{12} \right) \rho_{a1}^4(z), \\
\hat{\rho}_{b21}^5(z) &= \frac{D - 2}{2(D - 1)^2} \left( \frac{1}{\varepsilon^2} - \frac{1}{6\varepsilon} + \frac{17}{12} \right) \rho_{a1}^5(z), \\
\hat{\rho}_{b21}^6(z) &= \frac{D - 2}{2(D - 1)^2} \left( \frac{1}{\varepsilon^2} - \frac{1}{6\varepsilon} + \frac{17}{12} \right) \rho_{a1}^6(z) - \frac{2}{3} \left( \frac{1}{\varepsilon} + \frac{5}{2} \right) \rho_{a1}^m(z).
\end{align*}
$$

(A2)

A.2 The light fish (c11)

For the light fish the traces are more complicated. However, using the parameter $c_T$ introduced in Eq. (36), the result again has again the same structure as the leading order contribution,

$$
\rho_{c11}(s) = \frac{g_s^2 G^2}{16(4\pi)^{3D/2}} s^{3D/2-4} \sum_{i=1}^{6} \hat{\rho}_{c11}^i (m^2/s) \text{tr}_i(\Gamma, s)
$$

(A3)

where

$$
\begin{align*}
\hat{\rho}_{c11}^1(z) &= \left( \frac{6}{\varepsilon} + 19 - 24\zeta(3) + \frac{c_T^2}{12} \left( \frac{6}{\varepsilon^2} + \frac{2}{\varepsilon} - 1 \right) \right) \rho_{a1}^1(z), \\
\hat{\rho}_{c11}^2(z) &= \left( \frac{6}{\varepsilon} + 19 - 24\zeta(3) + \frac{c_T^2}{12} \left( \frac{6}{\varepsilon^2} + \frac{2}{\varepsilon} - 1 \right) \right) \rho_{a1}^2(z) - \left( 2 + c_T^2 \left( \frac{1}{3\varepsilon} + 1 \right) \right) \rho_{a1}^m(z), \\
\hat{\rho}_{c11}^3(z) &= \left( \frac{6}{\varepsilon} + 19 - 24\zeta(3) + \frac{c_T^2}{12} \left( \frac{6}{\varepsilon^2} + \frac{2}{\varepsilon} - 1 \right) \right) \rho_{a1}^3(z), \\
\hat{\rho}_{c11}^4(z) &= \left( \frac{6}{\varepsilon} + 19 - 24\zeta(3) + \frac{c_T^2}{12} \left( \frac{6}{\varepsilon^2} + \frac{2}{\varepsilon} - 1 \right) \right) \rho_{a1}^4(z), \\
\hat{\rho}_{c11}^5(z) &= \left( \frac{6}{\varepsilon} + 19 - 24\zeta(3) + \frac{c_T^2}{12} \left( \frac{6}{\varepsilon^2} + \frac{2}{\varepsilon} - 1 \right) \right) \rho_{a1}^5(z), \\
\hat{\rho}_{c11}^6(z) &= \left( \frac{6}{\varepsilon} + 19 - 24\zeta(3) + \frac{c_T^2}{12} \left( \frac{6}{\varepsilon^2} + \frac{2}{\varepsilon} - 1 \right) \right) \rho_{a1}^6(z) - \left( 2 + c_T^2 \left( \frac{1}{3\varepsilon} + 1 \right) \right) \rho_{a1}^m(z).
\end{align*}
$$

(A4)

Note that the parameter $c_T$ only appears in squared form, so that the signature of the current (i.e. whether or not there is a $\gamma_5$) is irrelevant for the result.

A.3 Merger with the leading order diagram

When one combines the results for the two light diagrams and adds the leading order contribution, the divergences can be factored out. When one combines the contributions one has to
take into account colour and combinatorial factors. The relevant factors are $N_c!$ for the leading order diagram \((a_1)\), $2N_c!C_F$ for the light self energy diagram \((b_21)\), and $-N_c!C_B$ for the light fish \((c_11)\). The results read

$$
\rho_{\text{leading}}(s) = N_c! \rho_{a_1}(s) = N_c! \frac{(D - 2)Gs^{2 - 2\varepsilon}}{16(4\pi)^D(D - 1)^2} \frac{1}{\varepsilon} \hat{\rho}_{a_1}(m^2/s),
$$

$$
\rho_{\text{light}}(s) = \frac{\alpha_s N_c! C_F}{4\pi} \left\{ \left( \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} + B_2 \right) \rho_{a_1}(s) + \left( \frac{2}{\varepsilon} + B'_2 \right) \rho_{a_1}^*(s) \right\} = \frac{\alpha_s N_c! C_F}{4\pi} \left\{ \frac{(D - 2)G^2s^{2 - 3\varepsilon}}{16(4\pi)^D(D - 1)^2} \left( \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} + B_2 \right) \rho_{a_1}(m^2/s) + \frac{G^2s^{2 - 3\varepsilon}}{2(4\pi)^D} \left( \frac{2}{\varepsilon} + B'_2 \right) \rho_{a_1}^*(m^2/s) \right\}
$$

(A5)

where the coefficients $B_i$, $B'_i$ are given in Eq. (35), and

$$
\hat{\rho}_{a_1}(m^2/s) := \sum_{i=1}^{6} \hat{\rho}_{i_{a_1}}^r(m^2/s) \text{tr}_i(\Gamma, s),
$$

$$
\hat{\rho}_{a_1}^*(m^2/s) := \sum_{i=1}^{6} \hat{\rho}_{i_{a_1}}^r(m^2/s) \text{tr}_i(\Gamma, s),
$$

$$
\hat{\rho}_{a_1}^{r*}(m^2/s) := \hat{\rho}_{a_1}^{m*}(m^2/s) \text{tr}_2(\Gamma, s) + \hat{\rho}_{a_1}^{g*}(m^2/s) \text{tr}_0(\Gamma, s).
$$

(A6)

The functions $\hat{\rho}_{a_1}(z)$, $\hat{\rho}_{a_1}^r(z)$, and $\hat{\rho}_{a_1}^{r*}(z)$ that occur in the spectral densities for the leading and light contributions vanish for $\varepsilon \to 0$. This can be seen by retracing the construction down to the elements $\hat{\rho}_{a_1}^r(z)$ and $\hat{\rho}_{a_1}^{r*}(z)$ \((i = 1, 2, \ldots, 6)\). These elements are given by a linear combination of spectral functions $\hat{\rho}_V(1, n\varepsilon - p; z)$ as defined in Eq. (29). Using the abbreviations

$$
C_n^\varepsilon := \frac{1}{\Gamma(n\varepsilon - 1)\Gamma(3 - (n + 1)\varepsilon)}, \quad C_1^n := \frac{\Gamma(n\varepsilon - 1)\Gamma(3 - (n + 1)\varepsilon)}{\Gamma(n\varepsilon - p)\Gamma(p + 2 - (n + 1)\varepsilon)}
$$

(A7)

for the overall and relative factor, the elements can now be written as

$$
\hat{\rho}_{a_1}^r(z) = C_1^n \hat{g}_{a_1}^r(z), \quad \hat{\rho}_{a_1}^{r*}(z) = C_2^n \hat{g}_{a_1}^{r*}(z),
$$

$$
\hat{\rho}_{a_1}^r(z) = C_1^n \hat{g}_{a_1}^r(z), \quad \hat{\rho}_{a_1}^{r*}(z) = C_2^n \hat{g}_{a_1}^{r*}(z), \quad i = 1, 2, \ldots, 6
$$

(A8)

where \(C_1^n = 1\)

$$
\hat{g}_{a_1}^1(z) = D \left\{ C_n^2 \hat{g}_{n2}(z) + C_n^0 (1 - z)^2 \hat{g}_{n0}(z) - 2 \left( \frac{D - 2}{D} - z \right) \hat{g}_{n1}(z) \right\},
$$

$$
\hat{g}_{a_1}^2(z) = - \left\{ C_n^2 \hat{g}_{n2}(z) + C_n^0 (1 - z)^2 \hat{g}_{n0}(z) + 2 \left( \frac{3D - 4}{D - 2} + z \right) \hat{g}_{n1}(z) \right\},
$$

$$
\hat{g}_{a_1}^3(z) = \frac{D + 2}{2} \left\{ C_n^2 \hat{g}_{n3}(z) + C_n^0 \left( \frac{D - 2}{D + 2} + z \right) (1 - z)^2 \hat{g}_{n0}(z) + 
\quad + C_n^2 \left( \frac{6 - D}{D + 2} + 3z \right) \hat{g}_{n2}(z) + \left( \frac{2 - D}{D + 2} - \frac{2Dz}{D + 2} + 3z^2 \right) \hat{g}_{n1}(z) \right\},
$$

\(27\)
\[ \dot{g}_n(z) = -\frac{1}{2} \left\{ C_n^3 \hat{g}_{n-3}(z) - C_n^0(1 - z)^3 \hat{g}_{n0}(z) + C_n^2(1 + 3z) \hat{g}_{n2}(z) - (1 - z)(1 + 3z) \hat{g}_{n1}(z) \right\}, \]

\[ \dot{g}_n(z) = -\frac{1}{2} \left\{ C_n^3 \hat{g}_{n3}(z) - C_n^0(1 - z)^3 \hat{g}_{n0}(z) + C_n^2(1 + 3z) \hat{g}_{n2}(z) - (1 - z)(1 + 3z) \hat{g}_{n1}(z) \right\}, \]

\[ \dot{g}_n(z) = -\frac{1}{2} \left\{ C_n^3 \hat{g}_{n3}(z) + C_n^0(1 - z)^3(1 + z) \hat{g}_{n0}(z) + +C_n^2 \left( \frac{7D - 10}{D - 2} + 3z \right) \hat{g}_{n2}(z) + \left( \frac{7D - 10}{D - 2} + \frac{3D - 4}{D - 2} z + 3z^2 \right) \hat{g}_{n1}(z) \right\}. \]

\[ \dot{g}_n(z) = \dot{g}_n^0(z) = \dot{g}_n^4(z) = \dot{g}_n^5(z) = 0, \]

\[ \dot{g}_n^2(z) = -\dot{g}_{n1}(z), \]

\[ \dot{g}_n^6(z) = -\frac{1}{2} \left\{ (1 + z) \dot{g}_{n1}(z) + C_n^2 \hat{g}_{n2}(z) \right\} . \tag{A9} \]

We can cast this into a more compact form by writing

\[ \rho^{(1)}_{a1}(z) = C_1 \epsilon \dot{g}^{(1)}_1(z), \quad \rho^{(1)}_{a1}(z) = C_2 \epsilon \dot{g}^{(1)}_2(z) \tag{A10} \]

where

\[ \dot{g}^{(0)}_n(m^2/s) = \sum_{i=1}^{6} \dot{g}^{(0)i}_n(m^2/s) \text{tr}(\Gamma, s). \tag{A11} \]

These expressions can then be inserted into the sum of leading and light contributions. One obtains

\[ \rho(s) := \rho_{\text{leading}}(s) + \rho_{\text{light}}(s) = \frac{G N_c^!}{16(4\pi)^D} s^{2-2\epsilon} \frac{(D - 2)}{(D - 1)^2} C_1 \times \]

\[ \times \left\{ \dot{g}_1(m^2/s) + \frac{\alpha_s C_F}{4\pi} s^{-\epsilon} \frac{C_G}{C_1} (B_0 + B_1 \epsilon + B_2 \epsilon^2) \dot{g}_2(m^2/s) + +\frac{\alpha_s C_F}{4\pi} s^{-\epsilon} \frac{8\epsilon(1 - 2)^2 \epsilon^2}{D(D - 2) C_1} (B_1' + B_2' \epsilon) \dot{g}_2(m^2/s) \right\} = \tag{A12} \]

\[ = \frac{G N_c^!}{16(4\pi)^D} s^{2-2\epsilon} \frac{(D - 2)}{(D - 1)^2} C_1 \left\{ \dot{g}_1(m^2/s) + \frac{\alpha_s C_F}{4\pi} C^r \dot{g}_2(m^2/s) + \frac{\alpha_s C_F}{4\pi} C^{r*} \dot{g}_2(m^2/s) \right\}. \]

The coefficients \( C^r \) and \( C^{r*} \) can be expanded in \( \epsilon \), viz.

\[ C^r := s^{-\epsilon} \frac{C_G}{C_1} (B_0 + B_1 \epsilon + B_2 \epsilon^2) = C^{r(0)} + C^{r(1)} \epsilon + O(\epsilon^2), \]

\[ C^{r*} = s^{-\epsilon} \frac{8\epsilon(1 - 2)^2 \epsilon^2}{D(D - 2) C_1} (B_1' + B_2' \epsilon) = C^{r*(0)} + O(\epsilon). \tag{A13} \]

The main singularity of the NLO contribution is proportional to \( \dot{g}_2(z) \). But \( \dot{g}_2(z) \) is similar to \( \dot{g}_1(z) \). Indeed, one obtains

\[ \dot{g}_{21}(z) := \dot{g}_2(z) - \dot{g}_1(z) = \dot{g}_{21}^{(1)}(z) \epsilon + O(\epsilon^2). \tag{A14} \]
Therefore, the leading singular NLO part is added and subtracted where \( \hat{g}_2(z) \) is replaced by \( \hat{g}_1(z) \). The leading singular part of this new contribution is then combined with the LO part. Expanding
\[
g_2(z) = g_2^{(0)}(z) + O(\varepsilon), \quad g'_2(z) = g_2'^{(0)}(z) + O(\varepsilon)
\]
and finally using \( D = 4 \), one obtains
\[
\rho_i(s) = \frac{G_N c!}{16(4\pi)^D s^{2-2\varepsilon}} \frac{(D-2)}{(D-1)^2} C_1 \left\{ \left( 1 + \frac{\alpha_s C_F}{4\pi \varepsilon} C^{r(0)} \right) \hat{g}_1(m^2/s) + \right.
\]
\[
+ \frac{\alpha_s C_F}{4\pi} \left( C^{r(1)} \hat{g}_1(m^2/s) + C^{r(0)} \hat{g}_2^{(1)}(m^2/s) + C^{r(0)} \hat{g}_2^{(0)}(m^2/s) \right) \right\} =
\]
\[
= \frac{2N_c s^2}{9(4\pi)^4} \left\{ \left( 1 + \frac{\alpha_s C_F}{4\pi \varepsilon} C^{r(0)} \right) \hat{g}_1(m^2/s) + \right.
\]
\[
+ \frac{\alpha_s C_F}{4\pi} \left( C^{r(1)} \hat{g}_1(m^2/s) + C^{r(0)} \hat{g}_2^{(1)}(m^2/s) + C^{r(0)} \hat{g}_2^{(0)}(m^2/s) \right) \right\}. \quad (A16)
\]
The LO contributions \( \hat{\rho}_{a1}^i(z) \) are given by Eqs. \( (A11) \). For the remaining light contributions one has
\[
\hat{\rho}_i^{\text{light}}(z) = \frac{1}{3} \left( 2 - 4 r_\Gamma + r_\Gamma^2 \right) \hat{\rho}_a^{i}(z) + \frac{2}{3} r_\Gamma \hat{\rho}_a^{01}(z), \quad (A17)
\]
where
\[
\hat{\rho}_a^1(z) = \frac{19}{24} + \frac{41}{18} z - \frac{29}{6} z^2 + \frac{13}{6} z^3 - \frac{29}{72} z^4 - \left( 1 + \frac{10}{3} z - 6 z^2 + 2 z^3 - \frac{1}{3} z^4 \right) \ln(1 - z) +
\]
\[
+ \left( \frac{1}{2} + \frac{7}{3} z - 6 z^2 + 2 z^3 - \frac{1}{3} z^4 \right) \ln z + 4 z \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right),
\]
\[
\hat{\rho}_a^2(z) = \frac{25}{96} + \frac{41}{9} z - \frac{47}{12} z^2 - z^3 + \frac{29}{288} z^4 - \left( \frac{1}{4} + \frac{1}{11} z - 3 z^2 - z^3 + \frac{1}{12} z^4 \right) \ln(1 - z) +
\]
\[
+ \left( \frac{1}{8} + \frac{19}{6} z - \frac{3}{2} z^2 - z^3 + \frac{1}{12} z^4 \right) \ln z + (2 z + 3 z^2) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right),
\]
\[
\hat{\rho}_a^3(z) = \frac{107}{600} - \frac{59}{120} z + \frac{47}{30} z^2 - \frac{67}{30} z^3 + \frac{149}{120} z^4 - \frac{157}{600} z^5 +
\]
\[
+ \left( \frac{1}{5} - z + 2 z^2 - 2 z^3 + z^4 - \frac{1}{5} z^5 \right) \ln(1 - z) +
\]
\[
+ \left( \frac{1}{10} - \frac{1}{2} z + 2 z^2 - 2 z^3 + z^4 - \frac{1}{5} z^5 \right) \ln z,
\]
\[
\hat{\rho}_a^4(z) = \hat{\rho}_a^5(z) = \hat{\rho}_a^6(z) = \frac{137}{2400} - \frac{107}{240} z - \frac{47}{180} z^2 + \frac{9}{10} z^3 - \frac{47}{160} z^4 + \frac{157}{3600} z^5 +
\]
\[
+ \left( \frac{1}{20} - \frac{1}{2} z - \frac{1}{3} z^2 - \frac{1}{4} z^3 + \frac{1}{30} z^5 \right) \ln(1 - z) +
\]
\[
+ \left( \frac{1}{40} - \frac{1}{4} z - \frac{1}{3} z^2 + z^3 - \frac{1}{4} z^4 + \frac{1}{30} z^5 \right) \ln z - z^2 \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right). \quad (A18)
\]
The first order self energy correction of the massive line is given by the inclusion of the master bubble

$$\Pi_B(k^2) = \int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 - m^2)(k - l)^2} = \frac{(m^2)^{D/2-2}}{(4\pi)^{D/2}} V(1, 1; k^2/m^2).$$  \hspace{1cm} (B1)

The corresponding spectral density reads

$$\rho_B(s) = \frac{(m^2)^{D/2-2}}{(4\pi)^{D/2}} \rho_V(1, 1; s/m^2) = \frac{s^{D/2-2}}{(4\pi)^{D/2}} \rho_V(1, 1; m^2/s).$$  \hspace{1cm} (B2)

With the help of the dispersive representation it is easy to see that the mass and momentum part of the self energy correction can be written as

$$\Sigma_m(k^2) = D g_s^2 \int \frac{\rho_B(s) ds}{s - k^2},$$
$$\Sigma_p(k^2) = \frac{2 - D}{2} g_s^2 \int \left(1 + \frac{m^2}{s}\right) \frac{\rho_B(s) ds}{s - k^2}.$$  \hspace{1cm} (B3)

In order to enact the renormalization and to absorb the singular parts of these NLO contributions in the renormalization factors for the mass and the wave function, we use an expansion for the propagator–type factor,

$$\frac{i(1 + a)}{\not{k} - m(1 + b)} \approx \frac{i}{\not{k} - m} + \frac{i}{\not{k} - m} \left(-i \not{k} a - i m(b - a)\right) \frac{i}{\not{k} - m}.$$  \hspace{1cm} (B4)

One obtains

$$a(k^2) = \Sigma_p(k^2) = \frac{2 - D}{2} g_s^2 \int \frac{\rho_B(s) ds}{s - k^2} \left(1 + \frac{m^2}{s}\right) =: \int \frac{\rho_a(s) ds}{s - k^2},$$
$$b(k^2) = \Sigma_p(k^2) + \Sigma_m(k^2) = g_s^2 \int \frac{\rho_B(s) ds}{s - k^2} \left(\frac{D + 2}{2} - \frac{D - 2 m^2}{2 s}\right) =: \frac{\rho_b(s) ds}{s - k^2}.$$  \hspace{1cm} (B5)

Using momentum subtraction at $k^2 = m^2$, the singular parts can be split off,

$$a(k^2) = \int \frac{\rho_a(s) ds}{s - k^2} = \int \frac{\rho_a(s) ds}{s - m^2} + \int \left(\frac{1}{s - k^2} - \frac{1}{s - m^2}\right) \rho_a(s) ds =$$
$$= \int \frac{\rho_a(s) ds}{s - m^2} + (k^2 - m^2) \int \frac{\rho_a(s) ds}{(s - m^2)(s - k^2)} =: a(m^2) + a_f(k^2),$$
$$b(k^2) = \int \frac{\rho_b(s) ds}{s - m^2} + (k^2 - m^2) \int \frac{\rho_b(s) ds}{(s - m^2)(s - k^2)} =: b(m^2) + b_f(k^2)$$  \hspace{1cm} (B6)

where

$$a(m^2) = \int \frac{\rho_a(s) ds}{s - m^2} = -\frac{g_s^2}{(4\pi)^{D/2}} \left(\frac{\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \frac{G}{\epsilon} \left(1 + \zeta(2)\epsilon^2 + O(\epsilon^3)\right),$$
$$b(m^2) = \int \frac{\rho_b(s) ds}{s - m^2} = -\frac{g_s^2}{(4\pi)^{D/2}} \left(\frac{\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \frac{G}{\epsilon} \left(3 \epsilon + 3\zeta(2)\epsilon^2 + O(\epsilon^3)\right).$$  \hspace{1cm} (B7)
Having absorbed the divergent parts into the renormalization of mass and wave function, the
finite parts can be expanded again,

$$\frac{i(1 + a_f)}{k - m(1 + b_f)} \approx \frac{i k}{k^2 - m^2} \left(1 + a_f + \frac{2m^2 b_f}{k^2 - m^2}\right) + \frac{im}{k^2 - m^2} \left(1 + a_f + b_f + \frac{2m^2 b_f}{k^2 - m^2}\right) = \frac{i k}{k^2 - m^2} \left(1 + P(k^2)\right) + \frac{im}{k^2 - m^2} \left(1 + M(k^2)\right). \quad (B8)$$

For the leading order diagram one obtains

$$V_{a1}(q^2) = -\frac{2G(1, 1)}{(4\pi)^{D/2}} \int \frac{d^D k}{(2\pi)^D} \frac{i}{k - m} \left(-(q - k)^2\right)^{D/2 - 1}$$

(B9)

where $G(1, 1) = G/\varepsilon$ is the massless master bubble. The corrections to the propagator results
in the use of an effective propagator

$$D_{\text{eff}}(k^2) = \frac{i k}{k^2 - m^2} D^k_{\text{eff}}(k^2) + \frac{im}{k^2 - m^2} D^m_{\text{eff}}(k^2) \quad (B10)$$

with $(\rho_{a+b}(s) = \rho_a(s) + \rho_b(s))$

$$D^k_{\text{eff}}(k^2) = \frac{i}{k^2 - m^2} - i \int \frac{\rho_a(s) ds}{(s - m^2)(k^2 - s)} + 2im^2 \int \frac{\rho_b(s) ds}{(s - m^2)^2} \left(\frac{1}{k^2 - m^2} - \frac{1}{k^2 - s}\right),$$

$$D^m_{\text{eff}}(k^2) = \frac{i}{k^2 - m^2} - i \int \frac{\rho_{a+b}(s) ds}{(s - m^2)(k^2 - s)} + 2im^2 \int \frac{\rho_b(s) ds}{(s - m^2)^2} \left(\frac{1}{k^2 - m^2} - \frac{1}{k^2 - s}\right) \quad (B11)$$

One finally obtains

$$V_{a1}(q^2) + V_{b11}(q^2) = -\frac{2G(1, 1)}{(4\pi)^{D/2}} \int \frac{d^D k}{(2\pi)^D} \frac{D_{\text{eff}}(k^2)}{(-(q - k)^2)^{1-D/2}}. \quad (B12)$$

It is therefore obvious how to calculate the contribution $V_{b11}(q^2)$ from the self energy correction
of the massive line and its spectral density $\rho_{b11}(s) = \hat{\rho}_{b11}(s) + m\rho_{b11}^m(s)$. After integrating by
parts the final result can be seen to be a convolution of the leading order contribution with
specified weight functions

$$\rho_{b11}^q(s) = \frac{g_s^2 G(1, 1)(D - 2)s^{D-2}}{16(4\pi)^3D/2(D - 1)} \int_z^1 \left(-\frac{\hat{\rho}_a(x)}{x(1-x)} + 2\hat{L}_b(x) \frac{d}{dx}\right) \hat{\rho}_{a1}(z/x) dx,$$

$$\rho_{b11}^m(s) = \frac{g_s^2 G(1, 1)(D - 2)s^{D-2}}{16(4\pi)^3D/2(D - 1)} \int_z^1 \left(-\frac{\hat{\rho}_{a+b}(x)}{x(1-x)} + 2\hat{L}_b(x) \frac{d}{dx}\right) \hat{\rho}_{a1}(z/x) dx \quad (B13)$$

where

$$\rho_a(s) = \frac{g_s^2}{(4\pi)^{D/2}} \hat{\rho}_a(m^2/s), \quad \rho_{a+b}(s) = \frac{g_s^2}{(4\pi)^{D/2}} \hat{\rho}_{a+b}(m^2/s) \quad (B14)$$

and

$$L_b(s) = \int_s^\infty \frac{m^2 \rho_b(s_1)}{(s_1 - m^2)^2} ds_1 = \frac{g_s^2}{(4\pi)^{D/2}} \hat{L}_b(m^2/s). \quad (B15)$$

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The corresponding spectral functions are

\[
\rho_{b11}^1(z) = \frac{49}{36} + \frac{116}{27} z - \frac{74}{9} z^2 + \frac{28}{9} z^3 - \frac{59}{108} z^4 + \left( \frac{1}{3} + \frac{44}{9} z - 4z^2 + \frac{4}{3} z^3 - \frac{2}{9} z^4 \right) \ln z +
- \left( \frac{2}{3} + \frac{20}{9} z - 4z^2 + \frac{4}{3} z^3 - \frac{2}{9} z^4 \right) \ln(1 - z) + 2z \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right),
\]

\[
\rho_{b11}^2(z) = \frac{55}{144} + \frac{329}{54} z - \frac{46}{9} z^2 - \frac{3}{2} z^3 + \frac{59}{432} z^4 + \left( \frac{1}{12} + \frac{34}{9} z + \frac{3}{2} z^2 - \frac{2}{3} z^3 + \frac{1}{18} z^4 \right) \ln z +
- \left( \frac{1}{6} + \frac{22}{9} z - 2z^2 - \frac{2}{3} z^3 + \frac{1}{18} z^4 \right) \ln(1 - z) +
+ \frac{2}{3} (2 + 3z) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right),
\]

\[
\rho_{b11}^3(z) = -\frac{77}{1800} - \frac{433}{360} z + \frac{289}{90} z^2 - \frac{299}{90} z^3 + \frac{613}{360} z^4 - \frac{623}{1800} z^5 +
- \left( \frac{1}{30} + \frac{5}{6} z - \frac{4}{3} z^2 - \frac{2}{3} z^3 + \frac{2}{15} z^4 \right) \ln z +
- \left( \frac{2}{15} - \frac{2}{3} z + \frac{4}{3} z^2 - \frac{2}{3} z^3 - \frac{2}{25} z^5 \right) \ln(1 - z),
\]

\[
\rho_{b11}^4(z) = \rho_{b11}^5(z) = \rho_{b11}^6(z) = -\frac{107}{7200} - \frac{709}{720} z^2 - \frac{173}{1080} z^3 + \frac{91}{60} z^4 - \frac{199}{480} z^5 + \frac{623}{10800} z^5 +
- \left( \frac{1}{120} + \frac{5}{12} z - \frac{47}{36} z^2 - \frac{2}{3} z^3 + \frac{2}{3} z^4 - \frac{2}{3} z^5 \right) \ln z +
- \left( \frac{1}{30} - \frac{z}{3} - \frac{2}{9} z^2 + \frac{2}{3} z^3 - \frac{2}{3} z^4 + \frac{2}{45} z^5 \right) \ln(1 - z) +
- \frac{2}{3} z^2 \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right). \tag{B16}
\]

These results have already been renormalized. In addition to Eq. \([B13]\) we have to take into account a further finite contribution coming from the singular parts of \(a\) and \(b\). Since

\[
a_s = -\frac{g_s^2}{(4\pi)^{D/2}} \left( \frac{\mu^2}{m^2} \right)^\varepsilon \frac{G}{\varepsilon} \left( 1 + O(\varepsilon^2) \right), \quad a_s + b_s = \frac{g_s^2}{(4\pi)^{D/2}} \left( \frac{\mu^2}{m^2} \right)^\varepsilon \frac{G}{\varepsilon} \left( 2 - 2\varepsilon + O(\varepsilon^2) \right) \tag{B17}
\]

we can absorb the singularity into the renormalization factor. This is the case for the \(\overline{\text{MS}}\)-mass where the finite constants have to be added to the result. If we absorb the finite constants as well we end up with the pole mass. This is preferable because then we do not have to take care of the numerator singularity containing \(b\). In any case, we obtain expressions for \(\ln(\mu^2/m^2)\) which have the same coefficients as the poles.

\[\text{C} \quad \text{The semi-massive fish contribution (c21)}\]

In order to determine the semi-massive fish contribution one has to calculate a number of scalar two-loop integrals. These so-called \(\text{prototypes}\) are spectral functions \(\rho_V(n_1, n_2, n_3, n_4, n_5; s/m^2)\).
These spectral functions are determined by the discontinuities of the correlator functions $V(n_1, n_2, n_3, n_4, n_5; q^2/m^2)$ given by

$$\frac{1}{(4\pi)^D}(m^2)^{D-n_1-n_2-n_3-n_4-n_5} V(n_1, n_2, n_3, n_4, n_5; q^2/m^2) =$$

$$:= \int \frac{d^D k}{(2\pi)^D} \frac{d^D l}{(2\pi)^D} \frac{1}{(k^2 + m^2)^{n_1}((q-k)^2)^{n_3}((q-l)^2)^{n_4}((k-l)^2)^{n_5}}$$

(C1)

(for convenience written in the Euclidean domain). For later use it is convenient to use the representation

$$\hat{\rho}_V(n_1, n_2, n_3, n_4, n_5; z) := z^{D-n_1-n_2-n_3-n_4-n_5} \rho_V(n_1, n_2, n_3, n_4, n_5; 1/z).$$

(C2)

A subset of the prototypes turn out to be reducible to scalar one-loop integrals using the spectral representation $\hat{\rho}_V(n_1, n_2; z) := z^{D/2-n_1-n_2} \rho_V(n_1, n_2; 1/z)$, where the spectral function $\rho_V(n_1, n_2; s/m^2)$ is given by the discontinuity of the correlator function $V(n_1, n_2; q^2/m^2)$. One obtains

$$\frac{1}{(4\pi)^{D/2}}(m^2)^{D/2-n_1-n_2} V(n_1, n_2; q^2/m^2) := \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 + m^2)^{n_1}((q-k)^2)^{n_2}}.$$  

(C3)

An alternative approach is to relate them to the massless one-loop integrals $G(n_1, n_2)$ with

$$\frac{1}{(4\pi)^{D/2}}(q^2)^{D/2-n_1-n_2} G(n_1, n_2) := \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2)^{n_1}((q-k)^2)^{n_2}}.$$  

(C4)

### C.1 The proper fish and the spectacle prototype

We start with the most difficult prototype $\hat{\rho}_V(1, 1, 1, 1; z)$, the *proper fish* or tetrahedron (see Fig. 3(a) and (b) for two different representations of this topology). The corresponding correlator function can be obtained as a limiting case of an expression taken from the literature [14, 42]. The discontinuity of this expression turns out to be finite. One obtains

$$\hat{\rho}_V(1, 1, 1, 1; z) = 4 \left( \text{Li}_2(z) + \frac{1}{2} \ln(1 - z) \ln z \right) + O(\varepsilon).$$

(C5)
If the last entry “1” is replaced by “0”, the correlator consists of two master bubbles. This diagram is termed the spectacle diagram (cf. Fig 3(c) and (d)) where the name derives from the pictorial representation Fig 3(d). We calculate the spectacle diagram by adding a further scalar line,

\[
\left(\frac{m^2}{(4\pi)^{D/2}}\right)^{1-\varepsilon} V(1, 1, 1, 0, 1; q^2/m^2) = \int \frac{d^D p}{(2\pi)^D (q - p)^2} \int V(1, 1; p^2/m^2) V(1, 1; p^2/m^2). \tag{C6}
\]

Note, however, that we have to subtract the infrared divergence before we can make use of the dispersive representation. We insert

\[
V(1, 1; p^2/m^2) = \int \frac{\rho_V(1, 1; s/m^2)}{s + p^2} ds = V(1, 1; -1) - (p^2 + m^2) \int \frac{\rho_V(1, 1; s/m^2) ds}{(s - m^2)(s + p^2)}, \tag{C7}
\]

where we have chosen the momentum subtraction at the point \( p^2 = -m^2 \) in the Minkowskian domain (cf. Appendix B). One obtains

\[
\left(\frac{m^2}{(4\pi)^{D/2}}\right)^{1-\varepsilon} V(1, 1, 1, 0, 1; q^2/m^2) = \int \tilde{\lambda}_a(q^2, s_1) \rho_V(1, 1; s_1/m^2) ds_1 + \int \tilde{\lambda}_b(q^2, s_1, s_2) \rho_V(1, 1; s_1/m^2) \rho_V(1, 1; s_2/m^2) ds_1 ds_2.
\]

The convolution functions are given by

\[
\tilde{\lambda}_a(q^2, s_1) = -2 \int \frac{d^D p}{(2\pi)^D (s_1 - m^2)(s_1 + p^2)(q - p)^2} = 2V(1, 1; -1) \tilde{\lambda}_0(q^2, s_1),
\]

\[
\tilde{\lambda}_b(q^2, s_1, s_2) = \int \frac{d^D p}{(2\pi)^D (s_1 - m^2)(s_2 - m^2)(s_2 - s_1)(s_1 + p^2)(s_2 + p^2)(q - p)^2}
\]

\[
= \frac{(s_1 - m^2)(s_2 - m^2)}{(s_1 - m^2)(s_2 - s_1)} \int \frac{d^D p}{(2\pi)^D} \left( \frac{(p^2 + m^2)^2}{(s_1 + p^2)(q - p)^2} - \frac{(p^2 + m^2)^2}{(s_2 + p^2)(q - p)^2} \right)
\]

\[
= \frac{(s_1 - m^2)^2 \tilde{\lambda}_0(q^2, s_1) - (s_2 - m^2)^2 \tilde{\lambda}_0(q^2, s_2)}{(s_1 - m^2)(s_2 - m^2)(s_2 - s_1)} \tag{C9}
\]

where

\[
\tilde{\lambda}_0(q^2, s) = \int \frac{d^D p}{(2\pi)^D (s + p^2)(q - p)^2} = \frac{1}{(4\pi)^{D/2}} V(1, 1; p^2/s). \tag{C10}
\]

Note that for the reduction to the fundamental convolution function \( \tilde{\lambda}_0(q^2, s) \) one can use the fact that each non-negative integer power of \( p^2 \) occuring in the integrand of this function can be effectively replaced by \(-s\). Calculating the spectral function we obtain

\[
\left(\frac{m^2}{(4\pi)^{D/2}}\right)^{1-\varepsilon} \rho_V(1, 1, 1, 0, 1; s/m^2) = \int \lambda_a(s, s_1) \rho_V(1, 1; s_1/m^2) ds_1 + \int \lambda_b(s, s_1, s_2) \rho_V(1, 1; s_1/m^2) \rho_V(1, 1; s_2/m^2) ds_1 ds_2
\]

where

\[
\lambda_a(s, s_1) = 2V(1, 1; -1) \lambda_0(s, s_1), \quad \lambda_0(s, s_1) = \frac{1}{(4\pi)^{D/2}} \rho_V(1, 1; s/s_1),
\]

\[
\lambda_b(s, s_1, s_2) = \frac{(s_1 - m^2)^2 \lambda_0(s, s_1) - (s_2 - m^2)^2 \lambda_0(s, s_2)}{(s_1 - m^2)(s_2 - m^2)(s_2 - s_1)}. \tag{C12}
\]
In order to combine both parts into an integral including a unique convolution function and a single integrand (which later on will be identified with the prototype $\rho_V(1,1,1,0; s/m^2)$) we use the fact that the second integral is symmetric in $s_1$ and $s_2$. We further make use of the expression
\[
\rho_V(1,1;1/z) = \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} \varepsilon^\varepsilon (1-z)^{1-2\varepsilon}
\] (C13) at $D = 4$ (i.e. $\varepsilon = 0$) for the one-loop spectral function to perform one of the integrations as a principal value integral. One obtains
\[
\lambda_b(s,s_1) := \int \lambda_b(s,s_1,s_2) \rho_V(1,1; s_2/m^2) ds_2 = -2\lambda_0(s,s_1) \frac{s_1 - m^2}{s_1} \ln \left( \frac{s_1 - m^2}{m^2} \right),
\] (C14) such that
\[
\frac{(m^2)^{1-\varepsilon}}{(4\pi)^{D/2}} \rho_V(1,1,1,0,1; s/m^2) = \int 2\lambda_0(s,s_1) \left( V(1,1;-1) - \left( 1 - \frac{m^2}{s_1} \right) \ln \left( \frac{s_1}{m^2} - 1 \right) \right) \rho_V(1,1; s_1/m^2) ds_1.
\] (C15) We identify
\[
\rho_V(1,1,1,0;1/z) = 2 \left( V(1,1;-1) - (1-z) \ln \left( \frac{1}{z} - 1 \right) \right) \rho_V(1,1;1/z)
\] (C16) and finally obtain
\[
\hat{\rho}_V(1,1,1,0;z) = 2V(1,1;-1)z^{-\varepsilon} \hat{\rho}_V(1,1;z) + 2(1-z)^2 (\ln(1-z) - \ln z).
\] (C17) Next we calculate the prototype $\hat{\rho}_V(1,1,1,1,1; z)$. For such prototypes with negative entries we need the vector integral $V'(1,1; p^2/m^2)$ defined by
\[
\frac{1}{(4\pi)^{D/2}} (m^2)^{D/2-2} p^\mu V'(1,1;p^2/m^2) = \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu}{(k^2 + m^2)(p-k)^2}.
\] (C18) One obtains
\[
V'(1,1;p^2/m^2) = \frac{1}{2} \left( 1 - \frac{m^2}{p^2} \right) V(1,1;p^2/m^2) - \frac{m^2}{2p^2} V(1,0;-1).
\] (C19) Again, we use momentum subtraction at the point $p^2 = -m^2$,
\[
V'(1,1;p^2/m^2) = V'(1,1;-1) - (p^2 + m^2) \int \frac{\rho'_V(1,1; s/m^2) ds}{(s + p^2)(s - m^2)}
\] (C20) where the spectral function is given by
\[
\rho'_V(1,1; s/m^2) = \frac{1}{2} \left( 1 + \frac{m^2}{s} \right) \rho_V(1,1; s/m^2), \quad \hat{\rho}'_V(1,1; z) = \frac{1}{2} (1+z) \hat{\rho}_V(1,1;z).
\] (C21)
In terms of the above vector integral we calculate
\[ V(1, 1, 1, 1, -1; p^2/m^2) = -2V(1, 1, 1, 1, 0; p^2/m^2) - \frac{2p^2}{m^2} V'(1, 1; p^2/m^2)^2, \tag{C22} \]
and obtain
\[
\hat{\rho}_V(1, 1, 1, -1; z) = -2z\rho_V(1, 1, 1, 0; z) + 4V'(1, 1; -1)z^{-\varepsilon}\hat{\rho}_V(1, 1; z) + (1 + z)(1 - z)^2 \left( 1 + (1 + z)\ln \left( \frac{1}{z} - 1 \right) \right) = -4V(1, 1; -1)z^{1-\varepsilon}\hat{\rho}_V(1, 1; z) + 4V'(1, 1; -1)z^{-\varepsilon}\hat{\rho}_V(1, 1; z) + (1 + z)(1 - z)^2 - (1 - z)^4 (\ln(1 - z) - \ln z). \tag{C23} \]

### C.2 Prototypes of the class \( \hat{\rho}_V(1, 1, 0, 1, 1; z) \)

Prototypes with one vanishing massless propagator reduce to a nested integral. For the general case that we need to consider here we obtain
\[
\frac{1}{(4\pi)^D} (m^2)^{D-2-n_4-n_5} V(1, 1, 0, n_4, n_5) = \int \frac{d^D k}{(2\pi)^D (k^2 + m^2)((p - k)^2)^{n_4}} \int \frac{d^D l}{(2\pi)^D (l^2 + m^2)((k - l)^2)^{n_5}} V(1, n_5; k^2/m^2) = \int \frac{d^D k}{(2\pi)^D (k^2 + m^2)((p - k)^2)^{n_4}} V(1, n_5; k^2/m^2) \tag{C24} \]
Again making use of momentum subtraction
\[ V(1, n_5; k^2/m^2) = V(1, n_5; -1) - (k^2 + m^2) \int \frac{\rho_V(1, n_5; s/m^2)ds}{(s - m^2)(s + k^2)} \tag{C25} \]
one ends up with
\[ V(1, 1, 0, n_4, n_5; p^2/m^2) = V(1, n_5; -1) V(1, n_4; p^2/m^2) - \int \frac{\rho_V(1, n_5; s/m^2)ds}{s - m^2} V(1, n_4; p^2/s) ds, \]
\[ \rho_V(1, 1, 0, n_4, n_5; s/m^2) = V(1, n_5; -1) \rho_V(1, n_4; s/m^2) - \int \frac{\rho_V(1, n_5; s_1/m^2)ds_1}{s - m^2} V(1, n_4; s/s_1) ds_1 \tag{C26} \]
and
\[ \hat{\rho}_V(1, 1, 0, n_4, n_5; z) = z^{1-n_5-\varepsilon} \left( V(1, n_5; -1) \hat{\rho}_V(1, n_4; z) - \int \frac{\hat{\rho}_V(1, n_5; z_1)dz_1}{z_1^{n_4-n_5+1}(1 - z_1)} \hat{\rho}_V(1, n_4; z_1/z)dz_1 \right). \tag{C27} \]
Because of \( \hat{\rho}_V(1, 0; z) = 0 \) we obtain
\[ \hat{\rho}_V(1, 1, 0, 1, 1; z) = V(1, 1; -1) z^{-\varepsilon} \hat{\rho}_V(1, 1; z) + 1 - z + \ln z, \]
\[ \hat{\rho}_V(1, 1, 0, 1, 0; z) = V(1, 0; -1) z^{1-\varepsilon} \hat{\rho}_V(1, 1; z) + (1 - z)z, \]
\[ \hat{\rho}_V(1, 1, 0, 0, 1; z) = 0 \tag{C28} \]
In order to calculate the final result, we consider the remaining spectral functions. For the special cases that occur in our calculations we obtain

\[ \hat{\rho}_V(1, 1, -1, 1, 1; z) = (1 + z)\rho_V(1, 1, 0, 1, 1; z) = 2z^{-\varepsilon}V'(1, 1; -1)\hat{\rho}_V(1, 1; z) - 2 \int_z^1 \frac{\hat{\rho}_V(1, 1; z)}{z(1 - z)} \hat{\rho}_V(1, 1; z/2) dz. \]  
(C29)

The final result reads

\[ \hat{\rho}_V(1, 1, -1, 1, 1; z) = (1 + z)V(1, 1; -1)z^{-\varepsilon}\hat{\rho}_V(1, 1; z) + 2V'(1, 1; -1)z^{-\varepsilon}\hat{\rho}_V(1, 1; z) + \frac{5}{4} + z + \frac{1}{4}z^2 - \left(\frac{1}{2} + z\right)\ln z. \]  
(C30)

### C.3 Prototypes of the class \( \hat{\rho}_V(0, 1, 1, 1, 1; z) \)

If one of the massive propagators vanishes, the result is given by the product of a massive and a massless one-loop correlator. The spectral function reads

\[ \hat{\rho}_V(0, n_2, n_3, n_4, n_5; z) = G(n_3, n_5)\hat{\rho}_V(n_2, n_3 + n_4 + n_5 - D/2; z). \]  
(C31)

For the special cases that occur in our calculations we obtain

\[ \hat{\rho}_V(0, 1, 1, 1, 1; z) = G(1, 1)\hat{\rho}_V(1, \varepsilon + 1; z), \]
\[ \hat{\rho}_V(0, 1, 1, 0, 1; z) = G(1, 1)\hat{\rho}_V(1, \varepsilon; z), \]
\[ \hat{\rho}_V(0, 1, 1, -1, 1; z) = G(1, 1)\hat{\rho}_V(1, \varepsilon - 1; z), \]  
(C32)

as well as \( \hat{\rho}_V(0, 0, n_2, n_3, n_4; z) = 0 \). The last prototype \( \hat{\rho}_V(-1, 1, 0, 1; z) \) is more difficult. With a little bit of work one finds

\[ \hat{\rho}_V(-1, 1, 1, 0, 1; z) = -\frac{1}{2}G(1, 1)((1 - z)\hat{\rho}_V(1, \varepsilon; z) + \hat{\rho}_V(1, \varepsilon - 1; z)). \]  
(C33)

In order to calculate the final result, we consider the remaining spectral functions

\[ \hat{\rho}_V(1, \varepsilon - 1; z) = \frac{1}{\Gamma(\varepsilon - 1)\Gamma(3 - 2\varepsilon)} \int_z^1 (1 - x)^{2 - 2\varepsilon} x^{\varepsilon - 2}(x - z)^{2 - 2\varepsilon} dx, \]
\[ \hat{\rho}_V(1, \varepsilon; z) = \frac{1}{\Gamma(\varepsilon)\Gamma(2 - 2\varepsilon)} \int_z^1 (1 - x)^{1 - 2\varepsilon} x^{\varepsilon - 1}(x - z)^{1 - 2\varepsilon} dx, \]
\[ \hat{\rho}_V(1, \varepsilon + 1; z) = \frac{1}{\Gamma(1 + \varepsilon)\Gamma(1 - 2\varepsilon)} \int_z^1 (1 - x)^{-2\varepsilon} x^{\varepsilon}(x - z)^{-2\varepsilon} dx. \]  
(C34)

The first two integrals can be evaluated for \( \varepsilon = 0 \), while for the last member of this family the singularity in \( G(1, 1) \) is not cancelled. However, we can subtract and add \( \hat{\rho}_V(1, 1; z) \) to separate the singular and finite parts. Using

\[ \hat{\rho}_V(1, 1; z) = \frac{1}{\Gamma(1 - \varepsilon)} \int_z^1 (1 - x)^{-\varepsilon}(x - z)^{-\varepsilon} dx \]  
(C35)
and $\Gamma(1 + \varepsilon)\Gamma(1 - 2\varepsilon) = \Gamma(1 - \varepsilon) + O(\varepsilon^2)$, we obtain
\[
\frac{1}{\varepsilon} \left( \hat{\rho}_V(1, \varepsilon + 1; z) - z^{-\varepsilon} \hat{\rho}_V(1, 1; z) \right) = \\
= \frac{1}{\varepsilon} \int_z^1 (1 - x)^{-\varepsilon}(x - z)^{-\varepsilon} \left( (1 - x)^{-\varepsilon}x^\varepsilon(x - z)^{-\varepsilon} - z^{-\varepsilon} \right) dx + O(\varepsilon) = \\
= \int_z^1 (\ln z - \ln(1 - x) + \ln x - \ln(x - z)) dx + O(\varepsilon) = \\
= 1 - z + (1 - 2z) \ln z - 2(1 - z) \ln(1 - z) + O(\varepsilon).
\]

(C36)

C.4 Table containing all needed prototypes

All necessary prototypes are listed in this subsection, starting from the most complicated one, the proper fish prototype, to those that are zero. Using $G(1, 1) = G/\varepsilon$ and
\[
V(1, 1; -1) = \frac{\Gamma(\varepsilon)}{1 - 2\varepsilon} = \frac{G}{\varepsilon} + O(\varepsilon), \\
V(1, 0; -1) = \Gamma(\varepsilon - 1) = -\frac{G}{\varepsilon} + 1 + O(\varepsilon), \\
V'(1, 1; -1) = V(1, 1; -1) + \frac{1}{2} V(1, 0; -1) = \frac{G}{2\varepsilon} + \frac{1}{2} + O(\varepsilon),
\]

one has (in addition to inherent symmetries)
\[
\hat{\rho}_V(1, 1, 1, 1, 1; z) = 4 \left( \text{Li}_2(z) + \frac{1}{2} \ln(1 - z) \ln z \right), \\
\hat{\rho}_V(1, 1, 1, 1, 0; z) = 2 \frac{G}{\varepsilon} z^{-\varepsilon} \hat{\rho}_V(1, 1; z) - 2(1 - z)^2 (\ln(1 - z) - \ln z), \\
\hat{\rho}_V(1, 1, 1, 1, -1; z) = (1 - 3z) \frac{G}{\varepsilon} z^{-\varepsilon} \hat{\rho}_V(1, 1; z) + (1 - z^2)z - (1 - z)^4 (\ln(1 - z) - \ln z), \\
\hat{\rho}_V(1, 1, 1, 0, 0; z) = -\frac{G}{\varepsilon} z^{-\varepsilon} \rho_V(1, 1; z) + (1 - z)z, \\
\hat{\rho}_V(1, 1, -1, 1, 1; z) = -\frac{1}{2} (1 + z) \frac{G}{\varepsilon} z^{-\varepsilon} \hat{\rho}_V(1, 1; z) - \frac{3}{4} z + \frac{1}{2} \frac{z^2}{\varepsilon} + \left( \frac{1}{2} + z \right) \ln z, \\
\hat{\rho}_V(1, 1, 0, 1, 1; z) = \frac{G}{\varepsilon} z^{-\varepsilon} \hat{\rho}_V(1, 1; z) + 1 - z + \ln z, \\
\hat{\rho}_V(0, 1, 1, 1, 1; z) = \frac{G}{\varepsilon} z^{-\varepsilon} \hat{\rho}_V(1, 1; z) + 1 - z + (1 - 2z) \ln z - 2(1 - z) \ln(1 - z), \\
\hat{\rho}_V(0, 1, 1, 0, 1; z) = \frac{1 - z^2}{2} + z \ln z, \\
\hat{\rho}_V(0, 1, 1, -1, 1; z) = -\frac{1}{2} \left( \frac{1}{3} + 3z - 3z^2 - \frac{1}{3} z^3 + 2z(1 + z) \ln z \right), \\
\hat{\rho}_V(-1, 1, 1, 0, 1; z) = -\frac{1}{6} + z - \frac{1}{2} z^2 - \frac{1}{3} z^3 + z^2 \ln z
\]

(C38)

while
\[
\hat{\rho}_V(1, 1, 0, 0, 1; z) = \hat{\rho}_V(0, 1, 1, 1, 0; z) = \hat{\rho}_V(0, 1, 0, 1, 1; z) = \hat{\rho}_V(0, 0, 1, 1, 1; z) = 0,
\]

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\[ \hat{\rho}_V(1,1,0,0,0;z) = \hat{\rho}_V(0,1,1,0,0;z) = \hat{\rho}_V(0,0,1,1,0;z) = \hat{\rho}_V(0,0,0,1,1;z) = 0, \]
\[ \hat{\rho}_V(0,1,0,0,1;z) = \hat{\rho}_V(0,1,0,1,0;z) = 0. \quad (C39) \]

The symmetries are given by
\[ \hat{\rho}_V(n_2, n_1, n_3, n_4, n_5;z) = \hat{\rho}_V(n_1, n_2, n_3, n_4, n_5;z). \quad (C40) \]

### C.5 Spectral functions for the semi-massive fish

The spectral functions that we have obtained in the course of our calculation are
\[ \rho_{c21}^1(z) = \frac{49}{36} + \frac{67}{54}z - \frac{569}{108}z^2 + \frac{169}{54}z^3 - \frac{25}{54}z^4 + \left( \frac{1}{3} + \frac{14}{9}z - \frac{43}{9}z^2 + \frac{32}{9}z^3 - \frac{25}{54}z^4 \right) \ln z + \]
\[ - \left( \frac{31}{18} + \frac{26}{27}z - 6z^2 + \frac{34}{9}z^3 - \frac{25}{54}z^4 \right) \ln(1-z) + \]
\[ + \left( \frac{4}{3} + \frac{40}{9}z - 8z^2 + \frac{8}{3}z^3 - \frac{4}{9}z^4 \right) \left( \text{Li}_2(z) + \frac{1}{2} \ln z \ln(1-z) \right) + \]
\[ + \frac{4}{3} \left( 2z - z^3 + \frac{z^4}{6} \right) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right) + 8z \left( \text{Li}_3(z) - \text{Li}_3(1) - \frac{1}{3} \ln z \text{Li}_2(z) \right), \]
\[ \rho_{c21}^2(z) = -\frac{25}{144} - \frac{13}{54}z + \frac{49}{36}z^2 - \frac{17}{18}z^3 - \frac{z^4}{432} - \left( \frac{1}{12} + \frac{5}{9}z - 2z^2 + \frac{8}{9}z^3 + \frac{z^4}{9} \right) \ln z + \]
\[ + \left( \frac{1}{9} + \frac{8}{9}z - 2z^2 + \frac{8}{9}z^3 + \frac{z^4}{9} \right) \ln(1-z) - \frac{2}{3}z(1-z^2) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right), \]
\[ \rho_{c21}^3(z) = \frac{55}{144} + \frac{665}{216}z - \frac{895}{432}z^2 - \frac{325}{216}z^3 + \frac{25}{216}z^4 + \]
\[ + \left( \frac{1}{12} + \frac{35}{18}z - \frac{77}{36}z^2 - \frac{11}{6}z^3 + \frac{25}{216}z^4 \right) \ln z + \]
\[ - \left( \frac{37}{72} + \frac{115}{27}z - 3z^2 - \frac{17}{9}z^3 + \frac{25}{216}z^4 \right) \ln(1-z) + \]
\[ + \left( \frac{1}{3} + \frac{44}{9}z - 4z^2 - \frac{4}{3}z^3 + \frac{z^4}{9} \right) \left( \text{Li}_2(z) + \frac{1}{2} \ln z \ln(1-z) \right) + \]
\[ + \left( \frac{4}{3} + 5z^2 + \frac{2}{3}z^3 - \frac{z^4}{18} \right) \left( \text{Li}_2(z) - \text{Li}_2(-1) + \frac{1}{2} \ln^2 z \right) + \]
\[ + (4z + 6z^2) \left( \text{Li}_3(z) - \text{Li}_3(1) - \frac{1}{3} \ln z \text{Li}_2(z) \right), \]
\[ \rho_{c21}^4(z) = -\frac{31}{576} - \frac{41}{54}z + \frac{5}{16}z^2 + \frac{z^3}{2} + \frac{z^4}{1728} - \left( \frac{1}{48} + \frac{11}{18}z - \frac{z^2}{8} - \frac{7}{9}z^3 + \frac{z^4}{36} \right) \ln z + \]
\[ + \left( \frac{1}{36} + \frac{7}{9}z - \frac{7}{9}z^3 - \frac{z^4}{36} \right) \ln(1-z) - \frac{z}{3} \left( 1 + 3z + z^2 \right) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right), \]
\[ \rho_{c21}^5(z) = \frac{257}{900} - \frac{349}{900}z + \frac{1273}{900}z^2 - \frac{1321}{300}z^3 + \frac{1507}{450}z^4 - \frac{58}{225}z^5 + \]

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\[
\begin{align*}
\rho_{c_21}(z) &= \frac{287}{3600} - \frac{124}{225} z - \frac{16871}{10800} z^2 + \frac{3097}{1350} z^3 - \frac{1639}{5400} z^4 + \frac{29}{675} z^5 + \\
&\quad + \left( \frac{1}{60} - \frac{2}{15} z - \frac{217}{180} z^2 - \frac{88}{45} z^3 + \frac{17}{120} z^4 + \frac{2}{2700} z^5 \right) \ln z + \\
&\quad - \left( \frac{97}{1800} - \frac{31}{36} z - \frac{35}{27} z^2 + \frac{19}{9} z^3 - \frac{17}{72} z^4 + \frac{17}{2700} z^5 \right) \ln(1 - z) + \\
&\quad + \left( \frac{1}{15} - \frac{2}{3} z - \frac{4}{9} z^2 + \frac{4}{3} z^3 - \frac{4}{3} z^4 + \frac{2}{45} z^5 \right) \left( \text{Li}_2(z) + \frac{1}{2} \ln z \ln(1 - z) \right) + \\
&\quad + \left( \frac{5}{3} z^2 - \frac{4}{3} z^3 + \frac{4}{6} z^4 - \frac{4}{45} z^5 \right) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right) + \\
&\quad - 2z^2 \left( \text{Li}_3(z) - \text{Li}_3(1) - \frac{1}{3} \ln z \text{Li}_2(z) \right),
\end{align*}
\]

\[
\begin{align*}
\hat{\rho}_{c_21}(z) &= \rho_{c_21}(z) = \frac{287}{3600} - \frac{124}{225} z - \frac{16871}{10800} z^2 + \frac{3097}{1350} z^3 - \frac{1639}{5400} z^4 + \frac{29}{675} z^5 + \\
&\quad + \left( \frac{1}{60} - \frac{2}{15} z - \frac{217}{180} z^2 - \frac{88}{45} z^3 + \frac{17}{120} z^4 + \frac{2}{2700} z^5 \right) \ln z + \\
&\quad - \left( \frac{97}{1800} - \frac{31}{36} z - \frac{35}{27} z^2 + \frac{19}{9} z^3 - \frac{17}{72} z^4 + \frac{17}{2700} z^5 \right) \ln(1 - z) + \\
&\quad + \left( \frac{1}{15} - \frac{2}{3} z - \frac{4}{9} z^2 + \frac{4}{3} z^3 - \frac{4}{3} z^4 + \frac{2}{45} z^5 \right) \left( \text{Li}_2(z) + \frac{1}{2} \ln z \ln(1 - z) \right) + \\
&\quad + \left( \frac{5}{3} z^2 - \frac{4}{3} z^3 + \frac{4}{6} z^4 - \frac{4}{45} z^5 \right) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right) + \\
&\quad - 2z^2 \left( \text{Li}_3(z) - \text{Li}_3(1) - \frac{1}{3} \ln z \text{Li}_2(z) \right),
\end{align*}
\]

\[
\begin{align*}
\hat{\rho}_{c_21}(z) &= \hat{\rho}_{c_21}(z) = \rho_{c_21}(z) = -\frac{137}{14400} + \frac{131}{1440} z - \frac{z^2}{4320} - \frac{149}{720} z^3 + \frac{121}{960} z^4 - \frac{7}{21600} z^5 + \\
&\quad - \left( \frac{1}{240} - \frac{7}{72} z^2 + \frac{z^3}{3} - \frac{19}{144} z^4 - \frac{z^5}{90} \right) \ln z + \\
&\quad + \left( \frac{1}{240} - \frac{z}{18} - \frac{5}{36} z^2 + \frac{z^3}{3} - \frac{19}{144} z^4 - \frac{z^5}{90} \right) \ln(1 - z) + \\
&\quad + \frac{z^2}{12} (2 - z^2) \left( \text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z \right),
\end{align*}
\]

\[
\begin{align*}
\hat{\rho}_{c_21}(z) &= \frac{287}{3600} - \frac{221}{900} z - \frac{3821}{10800} z^2 + \frac{3569}{2700} z^3 - \frac{1141}{1350} z^4 + \frac{29}{675} z^5 + \\
&\quad + \left( \frac{1}{60} - \frac{2}{15} z + \frac{53}{180} z^2 + \frac{98}{45} z^3 - \frac{367}{360} z^4 + \frac{17}{2700} z^5 \right) \ln z + \\
&\quad + \left( \frac{1}{60} - \frac{2}{15} z + \frac{53}{180} z^2 + \frac{98}{45} z^3 - \frac{367}{360} z^4 + \frac{17}{2700} z^5 \right) \ln z +
\end{align*}
\]

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\[-\left(\frac{49}{600} - \frac{13}{12} z - \frac{8}{27} z^2 + \frac{7}{3} z^3 - \frac{25}{24} z^4 + \frac{17}{2700} z^5\right) \ln(1 - z) +
\]
\[+ \left(\frac{1}{15} - \frac{2}{3} z - \frac{4}{9} z^2 + \frac{4}{3} z^3 - \frac{z^4}{3} + \frac{2}{45} z^5\right) \left(\text{Li}_2(z) + \frac{1}{2} \ln z \ln(1 - z)\right) +
\]
\[+ \left(-\frac{5}{3} z^2 + \frac{z^4}{2} - \frac{z^5}{45}\right) \left(\text{Li}_2(z) - \text{Li}_2(1) + \frac{1}{2} \ln^2 z\right) +
\]
\[2z^2 \left(\text{Li}_3(z) - \text{Li}_3(1) - \frac{1}{2} \ln z \text{Li}_2(z)\right)\]. (C41)

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