Regularity of solutions to non-stationary Navier–Stokes equations

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Abstract. This paper covers a non-stationary system of Navier–Stokes equations for incompressible fluids. A regularized problem is considered that factors in the velocity field being relaxed to a solenoidal field; this problem is used to prove the pressure function exists almost everywhere in the domain for the Hopf class solutions. The proposed regularization proves there exist more regular weak solutions to the initial problem that do not impose smallness restrictions on the input data. The theorem of uniqueness is proven for the 2D case.

1. Introduction
Differential equations of incompressible fluids are usually regularized to prove the correctness of the initial-boundary value problems for Navier–Stokes equations. Regularization methods usually add terms with a low parameter $\varepsilon$ from the unknown functions that compensate the time derivative of pressure \cite{1}. A priori estimates are then used to prove the theorem the regularized problem is uniquely soluble; passage to the limit by $\varepsilon \to 0$ is carried out in regularized equations.

Note that all the regularized models are built purely mathematically to generate a priori estimates of the generalized solutions. Papers \cite{2, 3} present numerical analysis of regularized Navier–Stokes equations, where the low regularization parameter characterizes the time for the velocity vector field to relax to a solenoidal field. There emerges a natural pressure boundary condition
\[(\nabla p + f) \cdot n = 0,\]
which means there are no hydrodynamic oscillations on the wall.

In this paper, a regularized system of differential equations is obtained by applying a small correction to the right side of the continuity equation while also adding a boundary conditions for the pressure function. This approximation guarantees that the pressure function exists almost everywhere in the domain for the Hopf class solutions.

To state the main results, introduce the necessary functional spaces. $H^l(\Omega)$, with $l$ being a natural number, denotes the Sobolev space of square-summable functions together with derivatives to the order $l > 0$. Let $H^l_0(\Omega)$ be closure of $C_0^\infty(\Omega)$ with respect to the norm $H^1(\Omega), H^{-1}(\Omega)$ be a space conjugate to $H^1_0(\Omega)$. Denote the norm in the space $L^2(\Omega)$ as $\| \cdot \|$, and the scalar product of elements in $L^2(\Omega)$ as $(\cdot, \cdot)$. Let $X(\Omega; \text{Ker} \{\text{div}\})$ stand for the space of solenoidal vector functions from $X$. Let function spaces consisting of $l$ times continuously...
differentiable functions on $\tilde{\Omega}$ be denoted as $C^l(\tilde{\Omega})$. Introduce the spaces:

$$V = \{ u \in H^1; (u \cdot n)|_{\partial \Omega} = 0, \text{ div } u = 0 \};$$

$$\hat{V} = \{ u \in H^1; (u \cdot n)|_{\partial \Omega} = 0 \}; \quad \hat{H}_R^2 = \{ h \in H^2; (\nabla h \cdot n)|_{\partial \Omega} = 0 \}.$$

Let $\tilde{H}_R^2 = (\tilde{H}_R^2)'$, $\tilde{H}^{-1} = (H^1)'$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial \Omega \in C^2$, $T > 0$, $Q = (0, T) \times \Omega$, $\Sigma = (0, T) \times \partial \Omega$ be the lateral surface of the cylinder $Q$, $n = 2, 3$.

Consider an initial-boundary value problem for Navier–Stokes equations describing the motion of a viscous homogeneous incompressible fluid

$$\partial_t v + (v \cdot \nabla)v = \Delta v - \nabla p + f, \quad \text{ div } v = 0, \quad (t, x) \in Q, \quad v|_{\partial \Omega} = 0, \quad (t; x) \in \Sigma, \quad v|_{t=0} = v_0(x), \quad x \in \Omega,$$

(1)

where $f$ are external mass forces, $\Delta$ is the Laplace operator, $\nabla$ is the gradient operator.

Assume that the known functions have the following properties:

$$f \in L^2(0, T; H^{-1}(\Omega)), \quad v_0 \in L^2(\Omega), \quad (v_0, \nabla q) = 0 \quad \forall q \in H^1(\Omega).$$

(3)

Weak solution to the problem (1), (2) is the element $v \in L^2(0, T; H_0^1(\Omega))$ satisfying a certain integral identity. In this case, the element $v$ is referred to as a Hopf solution to the problem (1)–(2), see [4]. The pressure function is not mentioned [1, p. 266]. Using the terminology from [1, p. 308], refer to the Hopf solution as "a very weak solution".

It is known that a unique Hopf solution of (1)–(2) exists in the 2D case [1, 5]. In the 3D case, the very weak solution is unique either on a small time interval [6, 7] or in a space narrower than that where the existence has been identified [6, 7].

**Definition 1.** Let the weak solution to the problem (1)–(2) be defined as a pair of $(v; p)$ such that $v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \hat{H}_0^1(\Omega); \text{ Ker } \{ \text{div} \})$, $p \in L^2(0, T; L^2(\Omega))$ in case $n = 2$ or $p \in L^1(0, T; L^2(\Omega))$ in case $n = 3$, $(p, 1) = 0$, which must satisfy the integral identities

$$\int_0^T \left[ \langle \partial_t v + (v \cdot \nabla)v, u \rangle_{H^{-1}(\Omega) \times \hat{H}_0^1(\Omega)} + \langle \text{rot } v, \text{ rot } u \rangle - \langle p, \text{ div } u \rangle \right] dt =$$

$$= \int_0^T \langle f, u \rangle_{H^{-1}(\Omega) \times \hat{H}_0^1(\Omega)} dt, \quad T \int_0^T (\text{div } v, q) dx = 0$$

for the arbitrary $u \in L^\infty(0, T; \hat{H}_0^1(\Omega))$, $q \in L^2(Q)$ and the condition (2) almost everywhere in $\Omega$.

This research has produced the following theorems.

**Theorem 1.** If $n = 2$, then for any $f$, $v_0$ such that (3), there exists a unique weak solution to the problem (1)–(2), for which the estimate is generated

$$\|v\|_{L^\infty(0, T; L^2(\Omega))} + \int_0^T (\|v\|_{H_0^1(\Omega)}^2 + \|p\|^2) dt \leq c(\|f\|_{L^2(0, T; H^{-1}(\Omega))}^2 + \|v_0\|^2).$$

**Theorem 2.** If $n = 3$, then for any $f$, $v_0$ such that (3), there exists a weak solution to the problem (1)–(2), for which the estimate is generated

$$\|v\|_{L^\infty(0, T; L^2(\Omega))}^2 + \int_0^T (\|v\|_{H_0^1(\Omega)}^2 + \|p\|) dt \leq c(\|f\|_{L^2(0, T; H^{-1}(\Omega))}^2 + \|v_0\|^2).$$
2. Regularized Navier–Stokes Equations

To prove the solvability, approximate the problem (1)–(2) by a regularized problem. Let \( \varepsilon > 0 \) and the conditions of (3) hold true. Consider the following initial-boundary value problem: find a pair of functions \((v_\varepsilon, p_\varepsilon)\) defined on \( Q = \Omega \times (0, T) \) such that

\[
\partial_t v_\varepsilon + (v_\varepsilon \cdot \nabla) v_\varepsilon = \Delta v_\varepsilon + \frac{1}{2} (\text{div} v_\varepsilon) v_\varepsilon - \nabla p_\varepsilon + f,
\]

\[
\text{div} v_\varepsilon = \varepsilon \Delta p_\varepsilon,
\]

\[
\left( \frac{\partial v_\varepsilon}{\partial n} \times n \right) = -\frac{1}{\varepsilon} (v_\varepsilon \times n), \quad (v_\varepsilon \cdot n) = 0, \quad (\nabla p_\varepsilon \cdot n) = 0, \quad (t; x) \in \Sigma,
\]

\[
v_\varepsilon|_{t=0} = v_0(x), \quad x \in \Omega.
\]

**Definition 2.** The weak solution to the problem (4)–(7) is the pair \((v_\varepsilon; p_\varepsilon)\) such that \(v_\varepsilon \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; \tilde{V})\), \(p_\varepsilon \in L^*(0, T; \tilde{H}^2)\) satisfying the identity

\[
\frac{d}{dt} \langle v_\varepsilon, u \rangle_{\tilde{V}' \times \tilde{V}} + \langle \nabla v_\varepsilon, \nabla u \rangle - \langle p_\varepsilon, \text{div} u \rangle + \frac{1}{\varepsilon} \int_{\partial \Omega} (v_\varepsilon \cdot u) \, d\sigma = -\langle (v_\varepsilon \cdot \nabla) v_\varepsilon - (1/2)(\text{div} v_\varepsilon) v_\varepsilon, u \rangle_{\tilde{V}' \times \tilde{V}} + \langle f, u \rangle_{\tilde{V}' \times \tilde{V}} \quad \forall u \in \tilde{V},
\]

the equation (5) almost everywhere in \((0, T)\) and the condition (7) in \(\tilde{D}'(\Omega)\).

**Theorem 3.** For any \(f, v_0\) satisfying (3), there exists a weak solution to the problem (4)–(7), for which the estimate is generated

\[
\|v_\varepsilon(t)\|^2 + \int_0^t \left( \|\nabla v_\varepsilon\|^2 + \varepsilon \|\nabla p_\varepsilon\|^2 \right) \, ds \leq c(\|f\|_{L^2(0, T; H^{-1}(\Omega))}^2 + \|v_0\|^2).
\]

Here the constant \(c\) does not depend on \(\varepsilon\). If \(n = 2\), the solution is unique.

For the fixed parameter \(\varepsilon > 0\) the solvability of the problem (4)–(7) is provable by a standard Galerkin procedure.

Theorem 3 and Lemma 1.1 from [1] mean that

\[
\partial_t v_\varepsilon \in L^s(0, T; \tilde{V}'),
\]

where the parameter \(s = 2\) at \(n = 2\), \(s = 1\) at \(n = 3\) and the vector function \(v_\varepsilon\) almost everywhere equals a certain continuous function from \([0, T]\) in \(\tilde{V}'\). Thus, the condition (7) is meaningful.

Obtain an additional a priori estimate for \(p_\varepsilon\). First, show that \(p_\varepsilon\) satisfies the homogeneous initial condition. Theorem 3 states the condition (7) must be interpreted as follows:

\[
\langle v_\varepsilon(t), u \rangle_{\tilde{V}' \times \tilde{V}} \to \langle v_0, u \rangle_{\tilde{V}' \times \tilde{V}} \quad \text{at } t \to 0
\]

for an arbitrary \(u \in \tilde{V}\).

Let

\[
\tilde{H}^2 = \{ h \in \tilde{H}^2_1; (h, 1) = 0 \}, \quad \tilde{H}^{-2} = (\tilde{H}^2)',
\]

The continuous embeddings hold \(\tilde{H}^2 \subset L^2(\Omega) = (L^2(\Omega))' \subset \tilde{H}^{-2}\).

The theory of solvability of boundary elliptic problems means there exists an operator

\[
R: \{ r \in L^2(\Omega), \ (r, 1) = 0 \} \mapsto \tilde{H}^2
\]
such that \( b = R(r) \), if
\[
\Delta b = r, \quad (b, 1) = 0; \quad (\nabla b \cdot n)|_{\partial \Omega} = 0.
\] (8)

Then
\[
\| R(r) \|_{H^2}^2 \leq C\| r \|^2.
\] (9)

Let \( g \in L^2(\Omega) \). For \( \pi \in \tilde{H}^2 \) satisfying the conditions
\[
\Delta \pi = g, \quad (\pi, 1) = 0, \quad (\nabla \pi \cdot n)|_{\partial \Omega} = 0,
\] (10)

the estimate holds true
\[
\| \pi \|^2 \leq C\| g \|_{H_{-2}}^2.
\] (11)

Indeed. Let \( r \in L^2(\Omega) \), and \( b \) be a solution to (8). Use the Green formula to find the following from (10)
\[
(\pi, r) = (g, b).
\]

Given (9), obtain
\[
\| \pi \| = \sup_{\| r \| = 1} (\pi, r) = \sup_{\| r \| = 1} \langle g, b \rangle_{\tilde{H}_{-2}^2} \leq \sup_{\| r \| = 1} \{ \| g \|_{\tilde{H}_{-2}} \| b \|_{\tilde{H}_{2}} \} \leq \sup_{\| r \| = 1} \{ c \| g \|_{\tilde{H}_{-2}} \| r \| \} \leq c \| g \|_{\tilde{H}_{-2}}.
\]

Consider the variation problem. Let \( g \in \tilde{H}_{-2} \). Find \( \pi \in L^2(\Omega) \) such that
\[
(\pi, \Delta b) = \langle g, b \rangle_{\tilde{H}_{-2}^2} \quad \forall b \in \tilde{H}^2.
\] (12)

**Lemma 1.** For an arbitrary \( g \in \tilde{H}_{-2} \) the problem (12) has a unique solution \( \pi \in L^2(\Omega) \) that satisfies the estimate
\[
\| \pi \|^2 \leq C\| g \|_{H_{-2}}^2.
\]

Proof. The uniqueness follows at once if we set \( g = 0 \), and \( \Delta b = \pi \) in (12). Prove the existence. Let \( g \in \tilde{H}_{-2} \). The embedding \( L^2(\Omega) \subset \tilde{H}_{-2} \) is dense. Therefore, there exists a sequence \( \{ g_n \} \subset L^2(\Omega) \) such that \( g_n \rightarrow g \) in \( \tilde{H}_{-2} \). Let \( \pi_n \in \tilde{H}^2 \) satisfy the conditions:
\[
\Delta \pi_n = g_n, \quad (\pi_n, 1) = 0, \quad (\nabla \pi_n \cdot n)|_{\partial \Omega} = 0.
\]

The estimate (11) holds true, meaning there exist \( \tilde{\pi} \) and \( \pi_n \rightarrow \tilde{\pi} \) is weak in \( L^2(\Omega) \). Pass to the limit by \( n \rightarrow \infty \) in the equality
\[
(\pi_n, r) = (g_n, b) \quad \forall r \in L^2(\Omega),
\]
where \( b \) is a solution to (8) and given the uniqueness of the problem (12), find that \( \tilde{\pi} = \pi \). The lemma is proven.

According to Theorem 3, \( v_e(t) \in \tilde{V}' \) for all \( t \in [0, T] \). It means that \( \text{div} \ v_e(t) \in \tilde{H}_{-2}^2 \subset \tilde{H}^2 \) for any \( t \in (0, T) \).

Let \( r \in L^2(\Omega) \), \( b \) be the solution to (8). Multiply the equation (5) to \( b \in \tilde{H}^2 \) and use the Green formula to obtain the equality
\[
(\varepsilon \tilde{p}_e(t), r) = (\text{div} \ v_e(t), b)_{\tilde{H}_{-2}^2} \quad \forall r \in L^2(\Omega),
\] (13)
where \( \tilde{p}_e = p_e - (\text{mes} \Omega)^{-1} v_e, 1 \). Given Lemma 1, (13) means that \( \tilde{p}_e(t) \in L^2(\Omega) \). From (13) find
\[
-(\varepsilon \tilde{p}_e(t), r) = (v_e(t), \nabla b)_{\tilde{V}' \times \tilde{V}} \rightarrow (v_0, \nabla b) = 0 \quad \forall r \in L^2(\Omega),
\]
where \( b \) is the solution to (8).

Thus,
\[
\tilde{p}_e|_{t=0} = 0 \quad \text{a. e. in } \Omega.
\]  

(14)

Let
\[
s = (\nabla \cdot \nabla) \tilde{v}_e - (1/2)(\nabla \cdot \tilde{v}_e) \tilde{v}_e - \nabla \cdot \nabla \tilde{v}_e - f.
\]

(15)

According to Theorem 1, the first condition in (3) \( s \in L^4(0, T; \tilde{V}') \), where the parameter \( s = 2 \) if \( n = 2 \), while \( s = 1 \) if \( n = 1 \).

The further goal is to generate the a priori estimate \( \nabla \cdot \tilde{v} \) in the space \( L^4(0, T; \tilde{H}^2_R) \). Use (15) to find from (4):
\[
\langle \partial_t \tilde{v}_e + \operatorname{rot} \operatorname{rot} \tilde{v}_e + s + \nabla p_e, w \rangle_{\tilde{V}' \times \tilde{V}} = 0 \quad \forall w \in L^\infty(0, T; \tilde{V}).
\]

(16)

Theorem 3 states that \( \partial_t \tilde{v}_e \in L^4(0, T; \tilde{V}') \), thus \( \partial_t (\nabla \cdot \tilde{v}_e) \in L^4(0, T; \tilde{H}^2_R) \).

Let \( w = \nabla q, q \in L^\infty(0, T; \tilde{H}^2_R) \). Given (5), from (16) find
\[
\langle \varepsilon \partial_t \Delta p_e + \Delta p_e + \nabla s, q \rangle_{\tilde{H}^{-2}_R \times \tilde{H}^2_R} = 0 \quad \forall q \in L^\infty(0, T; \tilde{H}^2_R). 
\]

(17)

for any \( q \in L^\infty(0, T; \tilde{H}^2_R) \). Therefore, given (6), the equality holds true
\[
\langle \nabla s, 1 \rangle_{\tilde{H}^{-2}_R \times \tilde{H}^2_R} = 0 \quad \forall q \in L^\infty(0, T; \tilde{H}^2_R).
\]

(18)

Given (18), write the equality
\[
\langle \nabla s, q \rangle_{\tilde{H}^{-2}_R(\Omega) \times \tilde{H}^2_R(\Omega)} =
\]

\[
= \langle \nabla s, q + 1_\delta \rangle_{\tilde{H}^{-2}_R(\Omega) \times \tilde{H}^2_R(\Omega)} + \langle \nabla s, 1 - 1_\delta \rangle_{\tilde{H}^{-2}_R(\Omega) \times \tilde{H}^2_R(\Omega)} \quad \forall q \in L^\infty(0, T; \tilde{H}^2_R).
\]

(19)

where \( \Omega_{2\delta} \subset \Omega \) is a domain, the boundary of which is set at \( 2\delta \) from the boundary of \( \Omega \), with the arbitrary function \( q \in H^{-1/2}_R(\Omega), 1_\delta \in H^{-1/2}_R(\Omega) \) taking the value 1 in \( \Omega_{2\delta} \), and \( 1_\delta|_{\partial \Omega} = -q \). Apply the Green formula to the first term on the right side of (19) and carry out passage to the limit by \( \delta \to 0 \) in (19), find the following inequality:
\[
\left| \langle \nabla s, q \rangle_{\tilde{H}^{-2}_R(\Omega) \times \tilde{H}^2_R(\Omega)} \right| \leq c\|s\|_{\tilde{V}}\|q\|_{\tilde{V}},
\]

which means
\[
\|\nabla s\|_{\tilde{H}^{-2}_R} = \sup_{\|q\|_{\tilde{H}^2_R} = 1} \langle \nabla s, q \rangle_{\tilde{H}^{-2}_R(\Omega) \times \tilde{H}^2_R(\Omega)} \leq c\|s\|_{\tilde{V}}, \quad \forall q \in L^\infty(0, T; \tilde{H}^2_R).
\]

(20)

Consider the variation problem. Now define the function \( \tilde{p}_1 \in L^4(0, T; L^2(\Omega)) \) such that
\[
\langle \tilde{p}_1, \Delta q_1 \rangle = \langle \nabla s, q_1 \rangle_{\tilde{H}^{-2}_R(\Omega) \times \tilde{H}^2_R(\Omega)} \quad \forall q_1 \in L^\infty(0, T; \tilde{H}^2_R).
\]

According to Lemma 1, the estimate is valid:
\[
\|\tilde{p}_1\| \leq C\|\nabla s\|_{\tilde{H}^{-2}_R} \quad \forall q_1 \in L^\infty(0, T; \tilde{H}^2_R).
\]

Thus, given the continuous embedding of the spaces \( \tilde{H}^{-2}_R \subset \tilde{H}^{-2} \) and estimate (20), find
\[
\|\tilde{p}_1\| \leq C\|\nabla s\|_{\tilde{H}^{-2}_R} \leq C\|s\|_{\tilde{V}}, \quad \forall q_1 \in L^\infty(0, T; \tilde{H}^2_R).
\]

(21)
Let \( p_1 \in L^2(0, T; L^2(\Omega)) \) such that
\[
p_1 = -\varepsilon \partial_t p_\varepsilon - p_\varepsilon.
\]
Then according to (17), obtain the equality
\[
(\Delta p_1, q)_{H^{-2}_R \times H^2_R} = \langle \text{div} \, s, q \rangle_{H^{-2}_R \times H^2_R} \quad \forall q \in L^\infty(0, T; \tilde{H}^2_R).
\]
(17), (22) mean that
\[
(\varepsilon \partial_t \Delta p_\varepsilon + \Delta p_\varepsilon, q)_{H^{-2}_R \times H^2_R} = - (\Delta p_1, q)_{H^{-2}_R \times H^2_R} \quad \text{in} \quad D'(0, T)
\]
for any \( q \in L^\infty(0, T; \tilde{H}^2_R) \). Let \( q = R(\tilde{p}_\varepsilon) + (\text{mes} \, \Omega)^{-1}(q, 1) \), where the operator \( R \) is defined in (8), \( \tilde{p}_\varepsilon = p_\varepsilon - (\text{mes} \, \Omega)^{-1}(p_\varepsilon, 1) \); apply the second Green formula to obtain
\[
\varepsilon (\partial_t \tilde{p}_\varepsilon, \tilde{p}_\varepsilon) + (\tilde{p}_\varepsilon, \tilde{p}_\varepsilon) = - (\tilde{p}_1, \tilde{p}_\varepsilon) \quad \text{in} \quad D'(0, T).
\]
Rewrite the first term of (23) as: \( \varepsilon (\partial_t \tilde{p}_\varepsilon, \tilde{p}_\varepsilon) = \varepsilon \| \tilde{p}_\varepsilon \| \frac{d}{dt} \| \tilde{p}_\varepsilon \| \). From (23), find
\[
\varepsilon \| \tilde{p}_\varepsilon(t) \| \frac{d}{dt} \| \tilde{p}_\varepsilon(t) \| + \| \tilde{p}_\varepsilon(t) \|^2 = (\tilde{p}_1, \tilde{p}_\varepsilon) \leq \| \tilde{p}_1(t) \| \| \tilde{p}_\varepsilon(t) \| \quad \text{in} \quad D'(0, T),
\]
so that either \( \| \tilde{p}_\varepsilon(t) \| = 0 \), or \( \varepsilon \| \tilde{p}_\varepsilon(t) \| + \| \tilde{p}_\varepsilon(t) \|^2 \leq \| \tilde{p}_1(t) \| \). However, \( \| \tilde{p}_\varepsilon(t) \| \) contains the continuous function \( t \), which means that for any \( t \), given (14), we have
\[
\varepsilon \| \tilde{p}_\varepsilon(t) \| + \int_0^t \| \tilde{p}_\varepsilon(s) \| \, ds \leq \int_0^t \| \tilde{p}_1(s) \| \, ds, \quad t \in (0, T).
\]
In case \( n = 2 \)
\[
\varepsilon \| \tilde{p}_\varepsilon(t) \|^2 + \int_0^t \| \tilde{p}_\varepsilon(s) \|^2 \, ds \leq \int_0^t \| \tilde{p}_1(s) \|^2 \, ds, \quad t \in (0, T).
\]
From (21) given (15), find
\[
\| \tilde{p}_1 \|_{L^2(0, T; L^2(\Omega))} \leq \| s \|_{L^2(0, T; \tilde{W}^1)} \leq C \int_0^T \left( \| \nabla v_\varepsilon \|_{L^4(\Omega)} + \| f \|_{H^{-1}(\Omega)} + \| \nabla v_\varepsilon \| \right)^4 \, dt.
\]
Constraints on the first term on the right side of (26) depends on the space dimensionality. To estimate it, use the Sobolev inequalities [8]:
\[
\| u \|_{L^4(\Omega)} \leq \beta(\Omega) \| u \| \left( \| u \| + \| \nabla u \| \right) \quad \forall u \in H^1(\Omega) \quad \text{for} \quad n = 2,
\]
\[
\| u \|_{L^4(\Omega)} \leq \beta(\Omega) \| u \|^{1/4} \left( \| u \| + \| \nabla u \|^{3/4} \right) \quad \forall u \in H^1(\Omega) \quad \text{for} \quad n = 3,
\]
Apply (27) at \( n = 2 \), or (28) at \( n = 3 \) to the right side of (26), obtain the following estimates from (24), (25):
\[
\varepsilon \max_{0 \leq t \leq T} \| \tilde{p}_\varepsilon \|^2 + \int_0^T \| \tilde{p}_\varepsilon \|^2 \, ds \leq c(\| f \|_{L^2(0, T; H^{-1}(\Omega))}^2 + \| v_0 \|^2) \quad \text{at} \quad n = 2,
\]
\[
\varepsilon \max_{0 \leq t \leq T} \| \tilde{p}_\varepsilon \| + \int_0^T \| \tilde{p}_\varepsilon \| \, ds \leq c(\| f \|_{L^2(0, T; H^{-1}(\Omega))}^2 + \| v_0 \|^2) \quad \text{at} \quad n = 3,
\]
where the constant \( c \) does not depend on \( \varepsilon \).

Theorems 1 and 2 are proven on the basis of the obtained a priori estimates followed by passage to the limit by \( \varepsilon \to 0 \) in regularized equations.
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