Quantum reflection and dwell times of metastable states

N. G. Kelkar

Departamento de Física, Universidad de los Andes,
Cra.1E No.18A-10, Santafe de Bogota, Colombia

Abstract

The concept of phase and dwell times used in tunneling is extended to quantum collisions to derive a relation between the phase and dwell time delays in scattering. This relation can be used to remove the near threshold s-wave singularities in the Wigner-Eisenbud delay times and amounts to an extension of the concept of quantum reflection to strong interactions. Dwell time delay emerges as the quantity which gives the correct behaviour of the density of states of a metastable state at all energies. This fact is demonstrated by investigating some recently found metastable states of mesic-nuclei.

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In an attempt to answer the question of how long does a particle take to tunnel through a barrier, physicists have given rise to several definitions of tunneling times. For example, the dwell time is considered a measure of the average time spent by a particle in a given region of space. A phase time is defined in terms of the energy derivatives of the phases of the reflection and transmission amplitudes. A traversal time was defined by Büttiker and Landauer as the interaction time of a particle transmitted through a barrier \([1]\). Though there exist extensive reviews \([2, 3]\) on these and some others such as complex and Larmor times, one does find contradictory remarks regarding the physical interpretation of some of them and the subject in general has remained controversial. It is however not the objective of the present work to attempt to resolve any of the controversies in tunneling and related phenomena like the Hartmann effect \([4]\). Instead, we start with an established relation between the phase time and dwell time which follows from a derivation in a tunneling problem \([2, 5]\) and derive a similar relation in the context of metastable states in quantum collisions. The self-interference term appearing in the phase time of the tunneling problem due to the interference of the reflected and incident flux at a barrier re-appears in our derivation in terms of a transition matrix in elastic scattering. Such a derivation becomes possible only after interpreting the reflection appearing in tunneling in terms of ‘quantum reflection’ by a strong potential in the scattering process. The dwell time delay expression derived, is useful in extracting information on the metastable states close to threshold, in contrast to the usual phase time delay (often addressed simply as ‘time delay’ of Wigner \([6]\) which as we shall see becomes singular in the \(s\)-wave scattering near threshold.

We illustrate the above relation by applying it to the realistic case of the quantum mechanical scattering of \(\eta\) mesons on light nuclei for the location of unstable \(\eta\)-mesic nuclear states. The attractive nature of the \(\eta\)-nucleon interaction arising due to the proximity of the \(\eta\)-nucleon threshold to the \(s\)-wave nucleon resonance \(N^*(1535)\), motivated the searches of \(\eta\)-mesic nuclei. The results on the \(\eta\)-mesic states obtained within the various theoretical models of the \(\eta\)-nucleon interaction are relevant in the light of recent experimental claims \([7]\) of such states and speculations based on \(\eta\)-production data \([8]\).

We start by considering a relation derived in \([5]\) in context with tunneling through a potential barrier in one dimension (1D) (though the final results would have relevance for the case of three dimensions (3D), as also discussed in \([5]\)). For a particle of energy \(E = \hbar^2 k^2 / 2\mu\)
\( \hbar k \) is the momentum, incident on the barrier, it was shown in [5] that,

\[
\tau_\phi(E) = \tau_D(E) - \hbar \frac{\text{Im}(R)}{k} \frac{dk}{dE},
\]

where \( \tau_\phi(E) \) which is given in terms of a weighted sum of the energy derivative of the reflection and transmission phases follows from the standard definition [2, 5] of phase time. The first term on the right hand side is the ‘dwell time’ [5] which is a measure of the time spent by the particle in the barrier region regardless of whether it is ultimately transmitted or reflected. The second term on the right is the self-interference term which arises due to the overlap of the incident and reflected waves in front of the barrier [5]. As we shall see later, it is this term which relates to threshold singularities in the phase time delay for 3D \( s \)-wave collisions. Eq. (1) and its physical interpretation is discussed extensively in [9] and the case of scattering in 3D in [10].

We note that \( \tau_\phi \) (in 1D or 3D), is not the same as the ‘time delay’ in [6, 11]. The scattering phase shift derivative in Wigner’s time delay, namely, \( \tilde{\tau}_\phi(E) = 2\hbar \frac{d\delta}{dE} \), is related through the Beth-Uhlenbeck formula to the density of states [12],

\[
\sum_l g_l(E) - g_0^l(E) = \sum_l \frac{2l + 1}{\pi} \frac{d\delta_l(E)}{dE},
\]

where \( g_l(E) \) and \( g_0^l(E) \) are the densities of states with and without interaction respectively. In this sense, \( \tau_\phi(E) \) and \( \tau_D(E) \) are the times spent ‘with interaction’ and not a difference which represents time delay. There also exists a general connection between the dwell times and the density of states in the barrier region for a 3D [13] and a 1D system [14]. The \( \tau_D(E) \) is also a measure of the time spent in the barrier region and hence is ‘the density of states with interaction’. Though both the dwell and phase times are densities of states with interaction, the interaction in the two cases differs. In the case of dwell time it is related to the time spent inside the barrier or the time spent by colliding particles within a certain distance say ‘\( r \)’ of each other. In the case of the phase time, it is the difference between the time of arrival of the wave packet at the barrier and the time when it leaves. In this case, the incident wave has mixed with the reflected one and this affects the time spent in the barrier. In fact, numerical simulation shows that the traversal time of a wave packet through a barrier agrees with the phase time rather well [15].

Subtracting the density of states without interaction for a given region [11] (or in other
words the density of states of free particles, say $\tau^0(E)$) from both sides of (1), we get,

$$\tilde{\tau}_\phi(E) = \tilde{\tau}_D(E) - \hbar \frac{Im(R)}{k} \frac{dk}{dE}$$

(3)

where, $\tilde{\tau}_\phi(E) = \tau_\phi(E) - \tau^0(E)$ and $\tilde{\tau}_D(E) = \tau_D(E) - \tau^0(E)$ are now the phase and the dwell time delay respectively. In 3D, $\tilde{\tau}_\phi(E) = 2\hbar \frac{d\delta}{dE}$ is what is generally referred to as Wigner’s time delay \(^6\). It is the ‘delay times’ which are strongly peaked at energy values corresponding to a metastable state \(^{11}\), and not simply the times (see \(^{16}\) and the references therein). It would now be right to ask if relation (3) could act as a general relation for any type of interaction, an attractive strong one for example without the existence of a Coulomb barrier from which the particle would get reflected. To understand this, we shall briefly dwell upon the concept of ‘quantum reflection’ below.

Recent research involving ultra cold atoms and molecules has drawn great attention to the phenomenon of quantum reflection \(^{17}\). The term ‘quantum reflection’ describes the classically forbidden reflection of a particle in a classically allowed region (without classical turning points). This could happen above potential barriers or at the edge of attractive potential tails and is often relevant at very low energies. The phenomenon of reflection in 1D can be viewed as a back scattering in 3D s-wave collisions. With no angle dependence of the scattering amplitude in the case of s-waves, the s-wave 3D motion can be viewed as a 1D motion in the radial coordinate. The importance of the reflection term in (3) at low energies becomes obvious as one examines the relation between the reflection amplitude, $R$, and the scattering matrix, $S$, namely, $R = |R| e^{i\phi} = -S = -e^{2i\delta}$, where $\delta$ is the scattering phase shift (and can in general be complex \(^{18}\)). In \(^{18}\), while discussing the effective range theory for quantum reflection amplitudes, it was further pointed out that at low energies ($k \to 0$), $|R|$ approaches unity and the phase $\phi \sim \pi - 2ak$, where $a$ is the scattering length and is related to the transition matrix, $t$, at zero energy. A similar connection between the $S$-matrix and the reflection amplitude and its phase has been used in \(^{19}\). With $R = -S$ and $S$ related to the complex transition matrix in scattering as, $S = 1 - i\mu k (t_R + it_I)/\pi$, where $t_R$ and $t_I$ are the real and imaginary parts of the $t$-matrix respectively and $\mu$ is the reduced mass of the system, we obtain the ‘self-interference’ term of (3) in terms of $t$ as,

$$-\hbar \frac{Im(R)}{k} \frac{dk}{dE} = -\hbar \mu \frac{t_R}{\pi} \frac{dk}{dE}.$$  

(4)

Replacing the low energy behaviour of the reflection amplitude mentioned above ($R \sim e^{i(\pi-2ak)}$), $dk/dE = \mu/\hbar^2 k$ and the definition of the complex scattering length $a = \ldots$
\( a_R + ia_I, \) namely, \( t(E = 0) = -2\pi a/\mu, \) we see that \(-\hbar \frac{\text{Im}(R)}{k} \frac{dk}{dE} \sim \frac{2aR}{\hbar k} \) as well as \(-\hbar \mu \frac{t_R dE}{dE} \sim \frac{2aR}{\hbar k}. \) This is indeed also the threshold singularity present in the definition of the time delay in s-wave collisions near threshold. The real scattering phase shift for s-waves, \( \delta \to k a_R \) close to threshold and Wigner’s time delay, \( \tilde{\tau}_\phi(E) = 2\hbar \frac{d\delta}{dE} k^{-\eta} \sim \frac{2aR}{\hbar k}. \) Putting together Eqs (3) and (4), i.e., identifying the reflection amplitude in (3) as a quantum reflection amplitude, we can evaluate the “dwell time delay”, \( \tilde{\tau}_D(E), \) once the scattering matrix in elastic collisions is known. Thus, in scattering processes,

\[
\tilde{\tau}_D(E) = \tilde{\tau}_\phi(E) + \hbar \mu \frac{t_R}{\pi} \frac{dk}{dE}.
\]

In general, with the phase shift, \( \delta \propto k^{2l+1} \) (as given in standard scattering theory books [20]), or rather \( \delta \propto E^{l+1/2} \) (where \( E = E_{CM} - E_{th} \) and \( E_{CM} \) and \( E_{th} \) are the centre of mass and threshold energies respectively), the energy derivative \( d\delta/dE \propto E^{l-1/2}. \) Hence this derivative which is the phase time delay, blows up for \( l = 0, \) when \( E_{CM} \to E_{th}. \) This makes locating resonances near threshold difficult using \( \tilde{\tau}_\phi(E). \)

With the \( S \) matrix given by \( S = 1 + 2itT, \) with \( T = -(\mu k/2\pi)t \) and using the delay time in elastic collisions, defined as \( \tilde{\tau}_\phi(E) = \Re \{ -i\hbar (S^{-1}dS/dE) \}, \) we get [16, 21, 22],

\[
\tilde{\tau}_\phi(E) = \frac{2\hbar}{A} \left[ -\frac{\mu k}{2\pi} \frac{dt_R}{dE} - \frac{\mu^2 k^2}{2\pi^2} \left( t_I \frac{dt_R}{dE} - t_R \frac{dt_I}{dE} \right) - \frac{\mu}{2\pi} t_R \frac{dk}{dE} \right],
\]

with \( A = 1 + (2\mu kt_I/\pi) + (\mu^2 k^2(t_R^2 + t_I^2)/\pi^2). \) For elastic scattering in the absence of inelasticities, the factor \( A = 1 \) and the last term can be identified with the self-interference delay given in [4]. In the presence of inelasticities, \( R = \eta e^{2i\delta_R}, \) where \( \delta_R \) is the real part of the scattering phase shift and \( \eta = e^{-2\delta_I}, \) with \( \delta_I \) the imaginary part of the phase shift. In the context of the discussion in [5] above their Eq. (9), this would be the case of a complex potential with losses, where \( |R|^2 + |T|^2 \) is not unitary. One could write \( |R|^2 + |T|^2 = 1 - |A|^2 = |\xi|^2, \) where \( |A|^2 \) is an absorption factor and for a lossless barrier, \( |\xi|^2 \) is unity. However, due to the fact that time is real, the additional term which arises, does not contribute to the delay times or the density of states and the general relation between the phase and dwell times in (1) and hence the delay times in (3) remains the same.

We shall now use Eq. (3) to evaluate the dwell time delay distribution for the \( \eta^{-3}\text{He} \) and \( \eta^{-4}\text{He} \) systems. The model of the \( t \)-matrix for \( \eta^{-3}\text{He} \) and \( \eta^{-4}\text{He} \) elastic scattering has already been developed in [21, 22] using few body equations within the finite rank approximation (FRA) and hence we shall not repeat it here. Considering the uncertainty in the knowledge
of the eta-nucleon ($\eta N$) interaction, in [21], these distributions were evaluated for different models of the elementary $\eta N$ interaction. Since the nucleus in $\eta$-nucleus scattering remains in its ground state in the intermediate state in the FRA of few body equations, we shall focus on the near threshold region. Besides, the $\eta$-mesic states of interest are also expected to be close to threshold, according to some recent experimental claims [7]. One of the objectives of these calculations is also to demonstrate the extension of the concept of “quantum reflection in nuclear physics”. To the best of our knowledge, quantum reflection phenomena have been observed [17] in the case of long ranged attractive potentials. With the absence of a Coulomb ‘barrier’ for the neutral $\eta$ mesons, one can understand the singular term related to the reflection amplitude only in terms of a ‘quantum reflection’.

In Fig. 1, the delay times in the elastic scattering reaction, $\eta^3\text{He} \rightarrow \eta^3\text{He}$ are shown as a function of the energy, $E = E_{\eta^3\text{He}} - m_\eta - m_{3\text{He}}$, where $E_{\eta^3\text{He}}$ is the total energy available in the centre of mass of the $\eta^3\text{He}$ system. The solid lines correspond to the Wigner-Eisenbud time delay evaluated using (6). The time delay at negative energies (see [21, 22] for the physical interpretation) corresponds to $k \rightarrow ik$ (hence $E = -k^2/2\mu$) and is useful in locating the bound and quasibound states with negative binding energy. In [21] the correctness of the method for bound states has been demonstrated with the example of the lowest stable nucleus, namely, the deuteron. The one-to-one correspondence between the poles of the $S$-matrix and the peaks in the delay times has been shown for the case of quasi-virtual and quasi-bound states of the $\eta$-meson and the deuteron in [22]. This correspondence holds at all energies for partial waves with $l \neq 0$ and for $l = 0$ only at energies quite away from threshold. At negative energies, in the interference term in (5), $t_R$ gets replaced by $t_I$ and $a_R$ by $a_I$ in the $k \rightarrow 0$ limit. The dashed curves in Fig. 1 are the self-interference delays due to quantum reflection as given in (4). The shaded regions correspond to the dwell time delay which is obtained after subtracting the self-interference term from the delay time $\tilde{\tau}_\phi(E)$ as in (3). The three different plots correspond to three distinct models of the $\eta N$ interaction which lead to the three different $\eta N$ scattering lengths, namely, $a_{\eta N} = (0.88, 0.41) \text{ fm}$ [23], $a_{\eta N} = (0.28, 0.19) \text{ fm}$ [24] and $a_{\eta N} = (0.51, 0.26) \text{ fm}$ [25]. A peak just above threshold (a resonance of $\eta^3\text{He}$ around 0.5 MeV excitation energy) appears for the largest of the scattering lengths. The dwell time delay in this case shows a clear Lorentzian distribution. For the smaller scattering lengths, the peaks become broader and spread over to negative energies. For $a_{\eta N} = (0.28, 0.19) \text{ fm}$, a broad peak appears around −5 MeV, in agreement
with the experimental findings \[7\] of $\eta$-mesic helium.

FIG. 1: Phase time delay (solid lines), dwell time delay (shaded region) and the self-interference delay (dashed lines) in the reaction, $\eta^3\text{He} \rightarrow \eta^3\text{He}$, versus $E = E_{\eta^3\text{He}} - m_\eta - m_{3\text{He}}$, for models of the $\eta N$ interaction with three different $\eta N$ and $\eta^3\text{He}$ scattering lengths, $a_{\eta N}$ and $a_{\eta^3\text{He}}$, in fm.

FIG. 2: Same as Fig. 1 but for the case of $\eta^4\text{He}$ elastic scattering.

In Fig. 2 we show the dwell and phase time delay in $\eta^4\text{He}$ elastic scattering for two $\eta N$ models \[23, 24\]. Once again, for the smaller scattering length, there is a bump at a negative energy of $-2$ MeV. The model with the larger scattering length however displays a clear ‘time advancement’ in the distribution. Such a negative time delay can be interpreted
in terms of a repulsive $\eta^4\text{He}$ interaction $^{[11, 26]}$. This means that the dwell time without interaction is bigger than the one with interaction, thus also implying that the interaction must be repulsive (which is also apparent from the sign of $\Re e [a_{\eta^4\text{He}}] = -3.94 \text{ fm}$).

In summary, we began with an expression connecting the phase and dwell times obtained in connection with quantum tunneling and wrote down the corresponding expression for the phase and dwell time ‘delays’ $^{[3]}$. The relation emerges after noting that the ‘phase time delay’ of Wigner is simply the difference of the density of states with and without interaction as given by the Beth-Uhlenbeck formula. The ‘dwell time delay’ is also a difference of the density of states, however, it is free of the threshold singularity present in the definition of the $s$-wave phase time delay. The self-interference term with a reflection amplitude in tunneling can be identified as a term given in terms of the $t$-matrix in scattering only by introducing the concept of ‘quantum reflection’. Since such a self-interference delay term in scattering depends on the $t$-matrix, it is not necessary to have a barrier, like a Coulomb barrier for example through which the $\alpha$ particle tunnels in the case of $\alpha$ decay. That the dwell time delay gives the correct energy behaviour of metastable states is demonstrated by applying it to the $s$-wave scattering of the neutral $\eta$ mesons and helium nuclei. The practical importance of the relation lies in the separation of the $s$-wave threshold singularity which can be confused with a zero energy resonance. Lorentzian peaks in the dwell time delay distributions, which have some support from experimental data on $\eta$-mesic nuclear states emerge after the subtraction of the self-interference delay from the phase time delay. This method can in general be applied to locate $s$-wave unstable states near threshold.

[1] M. Büttiker and R. Landauer, Phys. Rev. Lett. 49, 1739 (1982); ibid, Phys. Scr. 32, 429 (1985).
[2] E. H. Hauge and J. A. Støvneng, Rev. Mod. Phys. 61, 917 (1989); E. H. Hauge, J. P. Falck and T. A. Fjeldly, Phys. Rev. B 36, 4203 (1987).
[3] J. G. Muga, R. Sala-Mayato and I. L. Egusquiza, Time in Quantum Mechanics, New York: Springer (2002); R. Landauer and Th. Martin, Rev. Mod. Phys. 66, 217 (1994); V. S. Olkhovsky and E. Recami, Phys. Rep. 214, 339 (1992); H. Winful, Phys. Rep. 436, 1 (2006).
[4] J. G. Muga, I. L. Egusquiza, J. A. Damborenea and F. Delgado, Phys. Rev. A 66, 042115
(2002); T. E. Hartman, J. Appl. Phys. 33, 3427 (1962).

[5] H. G. Winful, Phys. Rev. Lett. 91, 260401 (2003).

[6] E. P Wigner, Phys. Rev. 98, 145 (1955); E. P. Wigner and L. Eisenbud Phys. Rev. 72, 29 (1947).

[7] M. Pfeiffer et al., Phys. Rev. Lett. 92, 252001 (2004).

[8] B. Mayer et al., Phys. Rev. C 53, 2068 (1996).

[9] C. R. Leavens and G. C. Aers, Phys. Rev. B 39, 1202 (1989); ibid, Solid State Commun. 63, 1107 (1987); M. Böttiker, Phys. Rev. B 27, 6178 (1983);

[10] Ph. A. Martin, Acta Phys. Austrica Suppl. 23, 157 (1981).

[11] F. T. Smith, Phys. Rev. 118, 349 (1960).

[12] E. Beth and G. E. Uhlenbeck, Physics 4, 915 (1937); K. Huang, Statistical Mechanics (Wiley, New York, 1963).

[13] G. Iannaccone, Phys. Rev. B 51, R4727 (1995).

[14] V. Gasparian and M. Pollak, Phys. Rev. B 47, 2038 (1993).

[15] S. Collins, D. Lowe and J. R. Barker, J. Phys. C: Solid State Phys. 20, 6213 (1987).

[16] N. G. Kelkar, M. Nowakowski, K. P. Khemchandani and S. R. Jain, Nucl. Phys. A730, 121 (2004); N. G. Kelkar, M. Nowakowski and K. P. Khemchandani, Mod. Phys. Lett. A 19, 2001 (2004); ibid, Nucl. Phys. A 724, 357 (2003); ibid, J. Phys. G 29, 1001 (2003).

[17] F. Shimizu, Phys. Rev. Lett. 86, 987 (2001); T. Pasquini et al., Phys. Rev. Lett. 93, 223201 (2004); ibid 97, 093201 (2006); H. Oberst, Phys. Rev. A 71, 052901 (2005).

[18] F. Arnecke, H. Friedrich and J. Madroñero, Phys. Rev. A 74, 062702 (2006).

[19] H. Friedrich and A. Jurisch, Phys. Rev. Lett. 92, 103202 (2004); E. Kogan, P. A. Mello and He Liqun, Phys. Rev. E 61, R17 (2000); U. Kuhl, M. Martínez-Mares, R. A. Méndez-Sánchez and H. -J. Stöckmann, Phys. Rev. Lett. 94, 144101 (2005); H. Friedrich and J. Trost, Phys. Rep. 397, 359 (2004).

[20] C. J. Joachain, Quantum Collision Theory (North-Holland, 1975).

[21] N. G. Kelkar, K. P. Khemchandani and B. K. Jain, J. Phys. G 32, L19 (2006).

[22] N. G. Kelkar, K. P. Khemchandani and B. K. Jain, J. Phys. G 32, 1157 (2006).

[23] A. Fix and H. Arenhövel, Eur. Phys. J A 9, 119 (2000); ibid, Nucl. Phys. A 697, 277 (2002).

[24] R. S. Bhalerao and L. C. Liu, Phys. Rev. Lett. 54 (1985) 865.

[25] A. M. Green and S. Wycech, Phys. Rev. C 71, 014001 (2005).
[26] N. G. Kelkar, J. Phys. G: Nucl. Part. Phys. 29, L1 (2003).