**-Trek III:
The Search for Ramond-Ramond Couplings

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Abstract: We give a detailed discussion of the disk amplitudes with one closed string insertion, which we used to construct the supergravity couplings of noncommutative D-branes to the RR potentials, given in hep-th/0104139. We prove the inclusion of Elliott’s formula, the integer-valued modification of the noncommutative Chern character, to all orders in the gauge field. We also give a detailed comparison between the form of the result in which Elliott’s formula is manifest, and the form expressed in Matrix model variables.
1. Introduction

In the past few years, great insights have been gained by studying the couplings of D-branes to bulk closed string modes. Examples include gauge theory dynamics, black holes and the AdS/CFT correspondence. In particular studying Ramond-Ramond (RR) couplings \[1, 2\] has yielded the “branes-within-branes” phenomenon \[3, 4\], K-theory descriptions of D-branes \[5, 6\], Myers effects\[7, 8\] and better understandings of anomalies.

When we turn on a constant background Neveu-Schwarz $B$-field in the presence of a D-brane, the low energy worldvolume theory of the D-brane becomes noncommutative \[9, 10, 11\] (for a review see \[12\]). In this paper we continue to study couplings of noncommutative D-branes to spacetime gravity fields. In previous papers \[13, 14\], we examined the couplings to the fluctuations of the closed string metric $g_{\mu\nu}$, dilaton and the $B$-field. In this paper we will examine the couplings to RR potentials. We have already presented our results in \[15\]. The purpose of this paper is to provide a detailed derivation of the results. Closely related discussions of the couplings of noncommutative D-branes to background closed string modes have appeared in \[16, 17, 18, 19, 20, 21, 22, 23\].

In the presence of a constant Neveu-Schwarz $B$-field, the worldsheet open string propagator is

$$\langle X^\mu(\tau)X^\nu(\tau') \rangle = -\alpha' G^{\mu\nu} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{\mu\nu} \epsilon(\tau - \tau') \quad (1.1)$$

where the open string metric $G$ and the noncommutative parameter $\theta$ are given in terms of the constant background closed string parameters by

$$G_{\mu\nu} = g_{\mu\nu} - (Bg^{-1}B)_{\mu\nu}, \quad (1.2a)$$

$$\theta^{\mu\nu} = -(2\pi\alpha') \left( \frac{1}{g + B} B \frac{1}{g - B} \right)^{\mu\nu}. \quad (1.2b)$$

In particular the mass shell condition for open string modes is $k_\mu G^{\mu\nu} k_\nu \in \mathbb{Z}$. Thus the limit $\alpha' \to 0$, with $\theta$ and $G$ fixed, yields a noncommutative gauge theory with noncommutative parameter $\theta$. Here we shall be interested in the couplings between the RR modes and the noncommutative gauge modes to leading order in $\alpha'$. They are to be extracted from the on-shell disk amplitudes.

Since the open string metric $G$ and the closed string metric $g$ have different scaling limits with respect to $\alpha'$ with fixed $\theta$, the low energy limit on the brane in terms of the open string metric no longer corresponds to the low energy limit in terms of the closed string metric in the bulk. Thus the low energy modes on the brane generically
excite finite momentum RR fields in the bulk, even to lowest order in $\alpha'$. In the D-brane worldvolume theory this is reflected in the presence of an open Wilson line [24, 25, 26, 27] with an appropriate integration prescription [13]. Open Wilson lines are also necessary in order to have off-shell gauge invariance in the noncommutative gauge theory\(^1\). An open Wilson line may be considered as the object which transforms an object in the open string algebra to an element of the closed string algebra. In explicit amplitude calculations with a finite number of external open string modes, the Wilson lines manifest themselves in the form of $n$-ary $*_n$ operations [30, 31, 32, 13].

Microscopically, the appearance of the Wilson line may be understood from the fact that in noncommutative field theories, the elementary quanta are dipoles whose lengths are proportional to their transverse momenta [33, 34, 35]. These dipoles interact by splitting and joining their ends. When a closed string mode scatters off open string modes on a D-brane, all external open string modes join together to form a macroscopic open Wilson line, which then couples to the closed string mode. The Wilson line can thus be viewed as the spacetime image of the worldsheet boundary.

The leading RR couplings we find are topological in nature and contain Elliott’s formula involving the noncommutative Chern character [36, 37]. In particular for D-branes which are described by topologically nontrivial configurations of the world-volume gauge theory of higher dimensional branes, their RR charges are given precisely by Elliott’s formula. This result fits in well with the expectations of [38, 39], based on the K-theory of noncommutative tori. The results we find can also be written in various other forms. Each form is convenient for understanding certain physical aspects, such as the relation with noncommutative K-theory; the Matrix model; and the Seiberg-Witten map between different descriptions.

As pointed out in [11], there are different descriptions of the D-brane dynamics parameterized by a noncommutative parameter $\theta$ or an open string two-form background $\Phi$, via

$$\frac{1}{g + B} = \frac{1}{G + \Phi} + \frac{\theta}{2\pi \alpha'}.$$  \hspace{1cm} (1.3)

The results we find from the on-shell amplitudes correspond to the description $\Phi = 0$ with $\theta$ given by (1.2b). Different descriptions have different open string metrics $G$ and different $\alpha'$ expansions. Since the mass shell conditions for massive open string states are defined in terms of the open string metric $G$ in the $\Phi = 0$ description (1.2a), the leading order results (in terms of $\alpha'$) for other values of $\Phi$ generally involve the

\(^1\)For a discussion of noncommutative gauge invariant operators from a different perspective, see [28]. See also [29] for a discussion of open Wilson lines in noncommutative scalar field theories.
contributions of massive open string modes. The RR couplings in other descriptions, and the relations between them, are discussed in [15].

We believe the results presented here and in [13, 14, 15] give the couplings from which one can read off the CFT operators corresponding to various supergravity fields in the noncommutative version of the AdS/CFT correspondence [40, 41]. These results should also provide clues for understanding the closed string modes in the open string field theory.

This paper is organized as follows. In section 2 we set our conventions and derive the basic correlation functions that we use to compute the amplitude. One interesting result here is the boundary condition for the spacetime spinor fields on the worldsheet. An explicit derivation of this boundary condition is given in appendix B. In section 3, we compute the one and two external open string amplitudes. The corresponding action is given in 4 and its extension to all orders in the gauge field is proposed. In section 5 we present the result in the Matrix model language, which has a rather simple and instructive form. It agrees with results from the Matrix model [18, 20] in the infinite $B$ limit. Finally, in section 6, we extract, from arbitrary order diagrams, all but the “nonabelian” terms in the action—that is, interactions in field strengths and covariant derivatives—thereby providing another strong check of our results. We conclude in section 7.

We have several appendices containing additional details. In appendix A we discuss the RR vertex operator in the picture $(-1/2, -3/2)$ in detail. The worldsheet boundary conditions for the spacetime spinor fields is derived in appendix B. Our results depend on a somewhat complicated $\Gamma$-matrix trace; this is derived in detail in appendix C. The amplitudes with two external open strings require some integrations over vertex operator positions; the exact formulas are listed in appendix D. Finally, we explicitly show how our amplitude leads to the proposed action, in appendix E. In appendix E.2, we explicitly expand out the action through quadratic order, listing all twelve terms.

2. Setting Up the Computation

We will use $M, N = 0, 1, \ldots, 9$ to denote the spacetime indices; $\mu, \nu = 0, 1, \ldots, p$ to denote the worldvolume directions of a D-brane; and $i, j = p+1, \ldots, 9$ for the directions transverse to the brane. In addition to the relations (1.2), there is a relation between the closed and open string couplings, $g_s$ and $G_s$. We have

\[ G_s^{-1} + \frac{\theta}{2\pi\alpha'} = \frac{1}{g_s + B}, \tag{2.1} \]
\[ G_s = g_s \left( \frac{\det G}{\det g} \right)^{1/4}. \] (2.2)

We assume that \( B \) lies only in Neumann directions (that is, along the \( D \)-brane) and that \( g_{MN} \) vanishes for mixed Neumann/Dirichlet directions. Therefore equation (2.1) applies to all spacetime directions; in particular, \( G_{ij} = g_{ij} \) and \( \theta^{ij} = 0 \).

We will put the RR vertex operator in the \((-1/2, -3/2)\) picture; this will soak up all the superghost zero modes, and then we can put all the open string vertex operators in the 0-picture, thereby allowing us to treat them symmetrically. Also, the \((-1/2, -3/2)\) picture involves the RR potentials rather than the field strengths, thereby making it more natural for the purpose of finding the couplings of D-branes. We shall take the worldsheet to be the upper half-plane and use the doubling trick to extend the upper half-plane to the full complex plane.

We are thus interested in computing the disk amplitude

\[ A_n = \left[ \prod_{a=2}^{n} \int_{-\infty}^{\infty} dy_a \right] \left\langle V_{RR}^{-1/2,-3/2}(q; i) V_O^0(a_1, k_1; 0) \prod_{a=2}^{n} V_O^0(a_a, k_a; y_a) \right\rangle \] (2.3)

where \( V_{RR}^{-1/2,-3/2} \) and \( V_O^0 \) are vertex operators for the massless RR potentials and open string modes (gauge bosons and/or transverse scalar fields) respectively. We have used the \( SL(2,\mathbb{R})\)-invariance of the upper half-plane to fix the RR vertex operators at \( z = i \) and one of the open string vertex operators at the origin. The gauge bosons and the transverse scalar fields have momenta \( k_{a\mu} \) and polarizations \( a_M \), with \( a, b, \ldots \) labeling the open string mode; the massless closed string mode has momentum \( q \). Obviously, \( k_a \) are purely longitudinal while \( q \) can also have transverse components because the D-brane breaks translational invariance. Specifically, momentum conservation requires \( q || = -k \equiv -\sum_a k_a \), where \( q || \) and \( q \perp \) denote the components of \( q \) parallel and perpendicular to the brane.

### 2.1 Vertex Operators

As discussed in detail in [42], there are many ways of writing the \((-1/2, -3/2)\) RR vertex operator. In appendix A, we give additional details regarding the choice given here. We use

\[ V_{RR}^{-1/2,-3/2} = \lambda g e^{-\phi/2 - 3\phi/2} \Theta \mathcal{C} \frac{1 - \Gamma^{11}}{2} \Theta \tilde{\Theta} e^{iq \cdot X}, \] (2.4)

\[ V_O^0 = a_M (i \dot{X}^M + 2\alpha' k \cdot \dot{\Psi}^M) e^{ik \cdot X}, \] (2.5)
where $C = \sum_n C^{(n)}$ is a sum of all the RR gauge potentials; and $n$ is odd (even) for Type IIA(B). We use a Feynman-slash notation,

$$C^{(n)} = \frac{1}{n!} C^{(n)}_{M_1 \ldots M_n} \Gamma^{M_1 \ldots M_n},$$

where, of course, $\Gamma^{M_1 \ldots M_n}$ are antisymmetrized $\Gamma$-matrices of Spin(1,9), with unit weight and the convention $\{\Gamma^M, \Gamma^N\} = 2g_{MN}$. Note the appearance of the closed string metric here. The slash is distributive over addition. $\Theta$ and $\tilde{\Theta}$ are the left- and right-moving spin operators, which can be related to the worldsheet fermions $\psi$ and $\tilde{\psi}$ via bosonization.\(^2\) We will usually suppress spinor indices, $A, B, \ldots$. The appearance of $\Gamma^1 = \Gamma^0 \ldots \Gamma^9$ in (2.4) enforces the chirality of the spacetime spinors as required by the GSO projections; this may seem to differ from the usual chirality conditions because the picture-changing operator changes the chirality of $\tilde{\Theta}$ as compared to the $(-1/2, -1/2)$-picture vertex operator involving the field strengths. The charge conjugation matrix is denoted $C$, and $\lambda$ is a normalization constant to be fixed later.

In equation (2.5), the overdot is understood to be the tangential derivative along the worldsheet boundary for Neumann directions (gauge bosons), and the normal derivative for Dirichlet directions (transverse scalar fields). We have absorbed a factor of $g_{YM}$ into the polarization $a_M$. Also, we will suppress Chan-Paton factors from the formulas for notational simplicity, although we will comment on them when appropriate.

In the following, we will largely suppress the superghosts $\phi, \tilde{\phi}$ and the $bc$ ghosts, noting only that they have the same standard OPEs and correlators as when $B = 0$.

Note that in equations (2.4) and (2.5)

$$g^2_{YM} = (2\pi)^{p-2} G_s \alpha'^{p-3}, \quad g_c = \frac{K_{10}}{2\pi} = \sqrt{\pi} g_s^2 (2\pi \alpha')^2,$$

where the open and closed string couplings $G_s$ and $g_s$ are related by (2.2). The overall normalization constant for the disk amplitudes is given by [43]

$$C_{D_2} = \frac{1}{2 g^2_{YM} \alpha'^2} \sqrt{-\det G},$$

and the brane tension is related to the Yang-Mills coupling constant $g_{YM}$ by

$$T_p = \frac{1}{(2\pi)^p g_s \alpha'^{p-1}}, \quad T_p \sqrt{-\det (g + B)} = \frac{1}{g^2_{YM} (2\pi \alpha')^2} \sqrt{-\det G}.$$

\(^2\)Schematically, if $\psi \sim e^{iH}$ then $\Theta \sim e^{isH}$ where $s = \pm \frac{1}{2}$.
The boundary conditions at $\bar{z} = z$ for the worldsheet fields are

$$\bar{\partial} X^M(\bar{z}) = \left( \frac{1}{g - B} (g + B) D \right)_N^M \partial X^N(z), \quad (2.10)$$

$$\bar{\psi}^M(\bar{z}) = \left( \frac{1}{g - B} (g + B) D \right)_N^M \psi^N(z). \quad (2.11)$$

where $D = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$, i.e. the identity matrix in the Neumann directions and minus the identity in the Dirichlet directions. The boundary fermion $\Psi^M$,

$$\Psi^M = \left( \frac{1}{g - B} \right)_N^M \psi^N, \quad (2.12)$$

is the open string supersymmetric partner of $X^M$ that lives on the worldsheet boundary. It thus appears in the open string vertex operator (2.5).

Similarly, the spin operators $\Theta$ and $\bar{\Theta}$ are related via the boundary conditions. That is, there exists a matrix $M$ so that at $z = \bar{z}$,

$$\bar{\Theta}(\bar{z}) = M \Theta(z). \quad (2.13)$$

From (2.11) and the consistency of the OPEs one can show that for a D$p$-brane with $B$-field background,

$$M = \frac{\sqrt{-\det g}}{\sqrt{-\det (g + B)}} \mathcal{A}(B) \Gamma^0 \cdots \Gamma^p \left\{ \begin{array}{l} 1, \quad \text{type IIA} \\ \Gamma^{11}, \quad \text{type IIB} \end{array} \right. \quad (2.14)$$

up to a sign convention. A similar formula appeared in [44] from analyzing the boundary states in Type I theory, and we have followed that paper in employing the “antisymmetrized exponential,” $\mathcal{A}(B)$. The equation (2.14) is also familiar as its eigenspinors give the unbroken supersymmetries for D-branes in a background field in supergravity (see e.g. [45]). Note that $\mathcal{A}(B)$, as defined in [44], is the exponential of $B$, but with the $\Gamma$ matrices totally antisymmetrized. (This is equivalent to $e^B$, where wedge products are understood in the definition of the exponential, and where, as in (2.6), the Feynman slash of a sum is the sum of Feynman slashes.) In appendix B we give a worldsheet derivation of (2.14) along with a more explicit expression for $\mathcal{A}(B)$.

2.2 Basic Worldsheet Bosonic Correlators

As usual, the amplitude (2.3) factorizes into bosonic and fermionic amplitudes. The
bosonic amplitudes are evaluated using the Green functions

\[
\langle X^\mu(z, \bar{z}) \, X^\nu(w, \bar{w}) \rangle = -\alpha' \left[ g^{\mu\nu}(\log|z-w| - \log|z-\bar{w}|) + G^{\mu\nu} \log|z-\bar{w}| \right]^2 + \frac{1}{2\pi\alpha'} \theta^{\mu\nu} \log \frac{z-\bar{w}}{z-w} \tag{2.15}
\]

\[
\langle X^i(z, \bar{z}) \, X^j(w, \bar{w}) \rangle = -\alpha' g^{ij}(\log|z-w| - \log|z-\bar{w}|). \tag{2.16}
\]

Using equations (2.15) and (2.16), we can work out some basic correlation functions. For example, with real \( y_a \),

\[
A_n = \left\langle e^{iq \cdot X} \prod_{a=1}^{n} e^{ik_a \cdot X(y_a)} \right\rangle = i 2^{\alpha'k^2} (2\pi)^d \delta(\sum_a k_a + q) \prod_{a<b} |\sin(\pi\tau_{ab})|^{2\alpha'k_a \cdot k_b} \exp \left[ \frac{i}{2} (k_a \times k_b) (2\tau_{ab} - \epsilon(\tau_{ab})) \right] \tag{2.17}
\]

For later convenience we have expressed the above correlator in terms of \( \tau_a \) which are defined by \( y_a = -\cot(\pi\tau_a) \). Note that \( 0 \leq \tau_a \leq 1 \) follows from \( -\infty < y_a < \infty \). When not specified otherwise, the dot product is with respect to the open string metric and the cross product denotes contraction using \( \theta^{\mu\nu} \), i.e. \( a \times k = a_\mu \theta^{\mu\nu} k_\nu \).

Another useful correlation function is

\[
B_n = a_M \left\langle e^{iq \cdot X} \, i\dot{X}^M(y_a) \prod_{b=1}^{n} e^{ik_b \cdot X(y_b)} \right\rangle = a_M V^M(y_a) A_n \tag{2.18}
\]

with

\[
a_M V^M(y_a) = 2\alpha' \left[ -(a \cdot k) \frac{y_a}{1 + y_a^2} + \frac{i}{2\pi\alpha'} (q \times a) \frac{1}{1 + y_a^2} + ia_a g^{ij} q_{1j} \frac{1}{1 + y_a^2} + \sum_{b=1, b\neq a}^{n} (a \cdot k_b) \frac{1}{y_a - y_b} \right] \tag{2.19}
\]

where \( k = \sum_b k_b \). Since we are using pointsplitting regularization on the worldsheet [11], there are no \( \delta \)-functions in (2.19).

One more bosonic correlation function that we will need is

\[
C_n = a_{1M} a_{2N} \left\langle e^{iq \cdot X} \, i\dot{X}^M(y_a) \, i\dot{X}^N(y_b) \prod_{c=1}^{n} e^{ik_c \cdot X(y_c)} \right\rangle = a_{1M} V^M(y_a) a_{2N} V^N(y_b) + \frac{2\alpha'}{(y_a - y_b)^2} a_1 \cdot a_2 \right] A_n. \tag{2.20}
\]

Note that \( a_1 \cdot a_2 = G^{\mu\nu} a_{1\mu} a_{2\nu} + G^{ij} a_{1i} a_{2j} = G^{\mu\nu} a_{1\mu} a_{2\nu} + g^{ij} a_{1i} a_{2j} \).
2.3 Basic Worldsheet Fermionic Correlators

The fermionic correlators are somewhat more complicated. It is well-known that

\[ \langle \psi^M(z) \psi^N(w) \rangle = g^{MN} \frac{1}{z - w}, \quad \langle \Psi^M(y_1) \Psi^N(y_2) \rangle = G^{MN} \frac{1}{y_1 - y_2}, \]  

(2.21)

\[ \langle \Theta_A(z_1) \Theta_B(z_2) \rangle = \frac{C_{AB}^{-1}}{z_{12}^{5/4}}, \]  

(2.22)

and

\[ \langle \psi^\mu(z_1) \Theta_A(z_2) \Theta_B(z_3) \rangle = \frac{(\Gamma^\mu C^{-1})_{AB}}{\sqrt{2} z_{12}^{1/2} z_{13}^{1/2} z_{23}^{5/4}}. \]  

(2.23)

A general correlation function involving an arbitrary number of \( \psi \)'s and two \( \Theta \)s is obtained using the following Wick-like rule. Since the \( \psi \)s are like \( \Gamma \)-matrices (up to normalization), one has

\[ \langle \psi^{\mu_1}(y_1) \ldots \psi^{\mu_n}(y_n) \Theta_A(z) \Theta_B(\bar{z}) \rangle = \frac{1}{2^{n/2}} \left( \frac{z - \bar{z}}{y_1 - z} \right)^{n/2 - 5/4} \left\{ \left( \Gamma^{\mu_n \ldots \mu_1} C^{-1} \right)_{AB} \right. \]

\[ + \psi^{\mu_1}(y_1) \psi^{\mu_2}(y_2) \left( \Gamma^{\mu_n \ldots \mu_3} C^{-1} \right)_{AB} \pm \text{perms} \]

\[ + \psi^{\mu_1}(y_1) \psi^{\mu_2}(y_2) \psi^{\mu_3}(y_3) \psi^{\mu_4}(y_4) \left( \Gamma^{\mu_n \ldots \mu_5} C^{-1} \right)_{AB} \pm \text{perms} \]

\[ + \ldots \left\} \right. \]  

(2.24)

The Wick-like contraction, for \( y_a \) real, is

\[ \psi^{\mu}(y_1) \psi^{\nu}(y_2) = 2 g^{\mu \nu} \text{Re} \left[ \frac{(y_1 - z)(y_2 - \bar{z})}{(y_1 - y_2)(z - \bar{z})} \right], \]  

(2.25)

and in (2.24), the sum is over all possible contractions. The functional dependence in (2.24) and (2.25) follows essentially from the conformal weights, and symmetry. The counterparts of (2.23)–(2.25) for \( \Psi \) can be obtained using (2.12). Incidentally, note that the fractional power in (2.24) will be removed by the ghost amplitude.

From equations (2.13) and (2.14), the fermionic part of (2.3) has the form

\[ \Xi_m = \left\langle \Theta_A(i) \left( C \frac{1}{2} - \Gamma^{11} \right)_{AB} \tilde{\Theta}_B(-i) \Psi^{M_1}(y_1) \ldots \Psi^{M_{2m}}(y_{2m}) \right\rangle \]  

(2.26)

with various values of \( m \). Using (2.14), the above equation can be written

\[ \Xi_m = T_{AB} \left\langle \Theta_A(i) \Theta_B(-i) \Psi^{M_1}(y_1) \ldots \Psi^{M_{2m}}(y_{2m}) \right\rangle \]  

(2.27)
with
\[ T_{AB} = \frac{\sqrt{-\det g}}{\sqrt{-\det(g + B)}} \left( C \frac{1}{2} - \Gamma^{11} \mathcal{E}(B) \Gamma^0 \cdots \Gamma^p \begin{pmatrix} \mathbb{1}, & \text{type IIA} \\ \Gamma^{11}, & \text{type IIB} \end{pmatrix} \right)_{AB}. \] (2.28)

In appendix A we explain how the BRST invariance of the vertex operator (2.4), implies that \( C \) is in a special gauge in which \( C \) and \( F \) are both self-dual: \( \Gamma^{11} C = -C \) and \( \Gamma^{11} F = \Gamma^{11} dC = -F \). Thus equation (2.28) yields an identical form for both the IIA and IIB theories
\[ T_{AB} = \frac{\sqrt{-\det g}}{\sqrt{-\det(g + B)}} (C \mathcal{E}(B) \Gamma^0 \cdots \Gamma^p)_{AB}. \] (2.29)

Using equation (2.12) to turn the correlation function (2.24) involving left-moving fermions into a correlation function for boundary fermions, it can be seen that (2.26) and eventually the amplitude (2.3) can be written as a sum
\[ A_n = \sum_{m=0}^{n} \omega^{M_1 \cdots M_{2m}} \Lambda^{M_1 \cdots M_{2m}}, \] (2.30)
where \( w \cdots \) contain all the dependence on the polarizations and momenta of the open string modes, and \( \Lambda \cdots \) are the Gamma matrix traces
\[ \Lambda^{M_1 \cdots M_{2m}} = \left( \frac{1}{g - B} g \right)^{M_1} N_1 \cdots \left( \frac{1}{g - B} g \right)^{M_{2m}} N_{2m} \text{Tr} \left[ (\Gamma^{N_{2m} \cdots N_1} C^{-1})^T C \mathcal{E}(B) \Gamma^0 \cdots \Gamma^p \right]. \]
\[ = -\left( \frac{1}{g - B} g \right)^{M_1} N_1 \cdots \left( \frac{1}{g - B} g \right)^{M_{2m}} N_{2m} \text{Tr} \left( \Gamma^{N_1 \cdots N_{2m}} C \mathcal{E}(B) \Gamma^0 \cdots \Gamma^p \right). \] (2.31)
Note that the Tr above is taken only with respect to the spacetime spinor indices and that we have used the property of the charge conjugation matrix \( C \),
\[ (\Gamma^{N_{2m} \cdots N_1} C)^{-1} = -\Gamma^{N_1 \cdots N_{2m}} C^{-1}. \] (2.32)

By evaluating the trace in (2.31) explicitly, in appendix C we show that (up to a sign that depends only on \( p \))
\[ \frac{\omega_{\mu_1 \cdots \mu_q} \chi_{\nu_1 \cdots \nu_r}}{q! r!} \Lambda^{\mu_1 \cdots \mu_q \nu_1 \cdots \nu_r} = 32 \left\{ * \left[ (e^{-i \theta/(2 \pi \alpha')} \omega) (t_{\pi} C) e^B \right] \right\}_0 \text{form} \] (2.33)
where wedge products are implied and the subscript means that we keep only the scalar part of the right-hand side. The Hodge dual, \( * \), is the Hodge dual in the worldvolume of the D-brane, with respect to the closed string metric. The notation \( t_T \) denotes contraction with respect to the antisymmetric tensor \( T \) of rank \( m \); i.e.
\[ (t_T C^{(n)})_{M_1 \cdots M_{n-m}} = \frac{1}{m!} T^{N_m \cdots N_1} C_{N_1 \cdots N_m M_1 \cdots M_{n-m}}. \] (2.34)
In (2.33), \( \chi \) only contracts with the transverse indices of \( C \) and
\[
e^{-i_\theta \omega} = \sum_n \frac{(-1)^n}{n!} n \text{ copies} \ i_\theta \cdots i_\theta \omega. \tag{2.35}
\]
Note that in equation (2.33), \( i_\theta \) does not contract with objects outside the parentheses.

3. The Amplitudes

In this section we will compute the amplitudes with one and two open strings explicitly. The contribution to (2.3) from ghosts is always
\[
\mathcal{A}_{\text{ghost}} = \langle c(i) \tilde{c}(-i)c(0) \rangle \langle e^{-\phi/2(i)} e^{-3\tilde{\phi}/2(-i)} \rangle = (2i)^{1/4}. \tag{3.1}
\]
where we have used
\[
\langle e^{-\phi/2(z)} e^{-3\tilde{\phi}/2(\bar{z})} \rangle = \frac{1}{(z - \bar{z})^{3/4}}, \quad \langle c(z_1) \tilde{c}(\bar{z}_2)c(z_3) \rangle = (z_1 - \bar{z}_2)(\bar{z}_2 - z_3)(z_1 - z_3) \tag{3.2}
\]

3.1 One Open String

We wish to evaluate
\[
\mathcal{A}_1 = (2i)^{1/4} \lambda g_c a_M \left( \Theta(i) \mathcal{C} \frac{1}{2} \Gamma^{11} \tilde{\Theta}(-i) e^{iq \cdot X(i)} \left( i \hat{X}^M(0) + 2\alpha' k \cdot \Psi(0) \Psi^M(0) \right) e^{ik \cdot X(0)} \right), \tag{3.3}
\]
where the factor \((2i)^{1/4}\) comes from (3.1). Using the rules of the previous section, it is straightforward to evaluate this. One finds,
\[
\mathcal{A}_1 = \frac{1}{2\pi} \lambda g_c C_{D_2} \sqrt{-\text{det} g} (2\pi)^{p+1} \delta^{(p+1)}(k + q) \left[ i(2\pi\alpha') k_\mu a_M \Lambda^{\mu M} + \mathcal{M} \right],
\]
\[
= \frac{\lambda}{2} \kappa_{10} \mu_p \sqrt{-\text{det} g} (2\pi)^{p+1} \delta^{(p+1)}(k + q) \left[ i(2\pi\alpha') k_\mu a_M \Lambda^{\mu M} + \mathcal{M} \right], \tag{3.4}
\]
with
\[
\mathcal{M} = iq \times a + i(2\pi\alpha') a \cdot q_\perp. \tag{3.5}
\]
\( \Lambda^{\mu M} \) and \( \Lambda \) were defined in equation (2.31) and in the second line we have used equations (2.7)–(2.9) to obtain
\[
g_c C_{D_2} \sqrt{-\text{det} g} = \pi \kappa_{10} \mu_p \sqrt{-\text{det} g} \tag{3.6}
\]
where \( \mu_p = T_p \frac{1}{(2\pi)^p g_s \alpha' \frac{p+1}{2}} \) is the RR charge density of a Dp-brane.

Of course, the entire amplitude (3.4) consists of contact terms that will give us the field theory action. This will not be true of the amplitude with two open strings, for which there are also poles corresponding to intermediate states. In particular, the two open string amplitude will have \( \alpha' \) corrections, while equation (3.4) is exact in \( \alpha' \).

### 3.2 Two Open Strings

Now we wish to evaluate

\[
\mathcal{A}_2 = (2i)^{1/4} \lambda g_c \int_{-\infty}^{\infty} dy \Theta(i) \mathcal{C} \left( \frac{\Gamma^{11}}{2} \gamma \right) e^{i\eta \cdot X(i)}
\]

\[
\times \left( i\dot{X}^M(0) + 2\alpha' k_1 \cdot \Psi(0) \Psi^M(0) \right) e^{ik_1 \cdot X(0)} \left( i\dot{X}^N(y) + 2\alpha' k_2 \cdot \Psi(y) \Psi^N(y) \right) e^{ik_2 \cdot X(y)} a_{1M} a_{2N}
\]

\[
= \lambda \pi \kappa_{10} \mu_p \sqrt{-\det g} \int_{-\infty}^{\infty} dy \left[ I_0(y) + I_2(y) + I_4(y) \right]
\]

(3.7)

where \( I_{0,2,4} \) correspond to the correlators involving respectively zero, two and four \( \Psi \)s, and we have extracted from them an appropriate prefactor.

Using the formulas of the previous section, it is straightforward to find

\[
I_0 = -\frac{i}{2} C_2(y) \Lambda
\]

\[
= -\frac{i}{2} A_2(y) \Lambda \left[ 2\alpha'(a_1 \cdot a_2) \frac{1}{y^2} + \frac{1}{\pi^2} \mathcal{M}_1 \mathcal{M}_2 \frac{1}{1 + y^2} - (2\alpha')^2 (a_1 \cdot k_2)(a_2 \cdot k_1) \frac{1}{y^2(1 + y^2)} \right.
\]

\[
\left. - \frac{2\alpha'}{\pi} [(a_1 \cdot k_2) \mathcal{M}_2 - (a_2 \cdot k_1) \mathcal{M}_1] \frac{1}{y(1 + y^2)} \right]
\]

(3.8)

where \( C_2, A_2 \) and \( \Lambda \) were given in equations (2.20), (2.17) and (2.31) respectively and \( \mathcal{M}_a \) was defined in (3.5) with \( a \) labeling the gauge boson. Similarly,

\[
I_2 = \frac{\alpha'}{\pi} A_2(y) \mathcal{M}_2 + Q_2 \mathcal{M}_1 - 2\alpha' \frac{A_2(y)}{y(1 + y^2)} [(a_1 \cdot k_2) Q_2 - (a_2 \cdot k_1) Q_1]
\]

(3.9)

where we have defined a short-hand notation

\[
Q_a = k_{a\mu} a_{a\nu} \Lambda^{\mu\nu}.
\]

The expression for \( I_4 \) is more complicated. According to (2.24) it is convenient to split it into three parts \( I_4 = I_{40} + I_{42} + I_{44} \) where the second index denotes the number
of $\Gamma$-matrices appearing in (2.24). We find that

$$I_{40} = -2i\alpha'^2 \frac{A_2(y)}{y^2(1 + y^2)} \Lambda [(a_1 \cdot k_2)(a_2 \cdot k_1) - (k_1 \cdot k_2)(a_1 \cdot a_2)],$$

(3.11)

$$I_{42} = -2\alpha'^2 \frac{A_2(y)}{y(1 + y^2)} [(a_1 \cdot k_2)k_{\mu 2}a_{1N}\Lambda^{\mu N} - (a_1 \cdot a_2)k_{\mu 2}k_{\nu 2}\Lambda^{\mu \nu} - (a_2 \cdot k_1)k_{\mu 1}a_{1N}\Lambda^{\mu N} - (k_1 \cdot k_2)a_{1M}a_{2N}\Lambda^{MN}],$$

(3.12)

$$I_{44} = 2i\alpha'^2 \frac{A_2(y)}{1 + y^2} k_{\mu 1}a_{1M}k_{\nu 2}a_{2N}\Lambda^{\mu \nu M N}.$$  

(3.13)

Since both $I_0$ and $I_{40}$ are proportional to $\Lambda$ we may combine them to obtain,

$$I_0 + I_{40} = -A_2(y) \Lambda \left[ \frac{i}{2\pi^2} M_1 M_2 \frac{1}{1 + y^2} - \frac{i\alpha'}{\pi} [(a_1 \cdot k_2)M_2 - (a_2 \cdot k_1)M_1] \frac{1}{y(1 + y^2)} \right. 

+ \left. i\alpha'(a_1 \cdot a_2) \left( (1 + \alpha't) \frac{1}{y^2} - \alpha't \frac{1}{1 + y^2} \right) \right],$$

(3.14)

with $t = -2k_1 \cdot k_2$.

We now proceed to evaluate the $y$-integrals in (3.7), which can be found in Gradshteyn and Ryzhik [46] equations (3.631.9) and (3.633.1). We find

$$\int_{-\infty}^{\infty} \frac{dy}{1 + y^2} = \pi i \frac{\sin k_1 \times k_2}{k_1 \times k_2} + O(\alpha't),$$

(3.15a)

$$\int_{-\infty}^{\infty} \frac{dy}{y(1 + y^2)} = \frac{2}{\alpha't} \sin k_1 \times k_2 \frac{2}{2} + O(\alpha't),$$

(3.15b)

$$(1 + \alpha't) \int_{-\infty}^{\infty} \frac{dy}{y^2} = \pi i \alpha't \frac{\sin k_1 \times k_2}{k_1 \times k_2} - \frac{2i}{\pi} \frac{k_1 \times k_2}{\alpha't} \sin \frac{k_1 \times k_2}{2} + O(\alpha't),$$

(3.15c)

where $A_2(y)$ is given by (2.17) with $n = 2$ and $y_1 = 0$. On the right-hand side of equation (3.15) we have, for notational simplicity, suppressed the factor $(2\pi)^{p+1} \delta(k_1 + k_2 + q_\parallel)$.

We see from (3.15) that there are $\alpha'$ corrections to the amplitudes. We shall be interested only in the lowest order contact terms in (3.7). The terms containing poles in $\alpha't$ may be understood from processes involving exchanging intermediate Yang-Mills modes using vertices (3.4) and those of noncommutative Yang-Mills theory. Substituting (3.15) into (3.9)–(3.14) and collecting only the lowest order contact terms we

---

3We give the complete expressions in appendix D.
find

\begin{align*}
A_2 &= \frac{\lambda}{2} \kappa_{10} \mu_p \sqrt{-\det g} \left\{ \sin \frac{k_1 \times k_2}{2} \left[ \mathcal{M}_1 \mathcal{M}_2 \Lambda - (2\pi\alpha')^2 k_{1\mu} a_{1M} k_{2\nu} a_{2N} \Lambda^{\mu\nu} \right. \\
&\quad+ i(2\pi\alpha') (k_{1\mu} a_{1N} + k_{2\mu} a_{2N} \mathcal{M}_1) \Lambda^{\mu N} \left. \right] - 4\pi\alpha' \sin \frac{k_1 \times k_2}{2} a_{1M} a_{2N} \Lambda^{MN} \right\} \quad (3.16)
\end{align*}

where we again have suppressed the factor \((2\pi)^{p+1}\delta(k_1 + k_2 + q_{\parallel})\).

### 3.3 The Total Amplitude

The last term of (3.16) is proportional to \(\sin \frac{k_1 \times k_2}{2}\) and it precisely combines with the first term in (3.4) to give

\begin{align*}
\frac{\lambda}{2} \kappa_{10} \mu_p \sqrt{-\det g} (\pi\alpha') f_{MN} \Lambda^{MN}, \quad (3.17)
\end{align*}

with

\begin{align*}
f_{\mu\nu} &= i(k_{\mu} a_{\nu} - k_{\nu} a_{\mu}) - i [a_{\mu}, a_{\nu}]_*, \\
f_{\mu\lambda} &= -f_{\mu\lambda} = D_{\mu} a_{\lambda} = i k_{\mu} a_{\lambda} - i [a_{\mu}, a_{\lambda}]_*, \\
f_{ij} &= -i [a_i, a_j]_* \
\end{align*}

where in momentum space

\begin{align*}
[f(k_1), g(k_2)]_* &= -2i \sin \frac{k_1 \times k_2}{2} f(k_1) g(k_2) \quad (3.19)
\end{align*}

We recognize the factor \(\frac{\sin \frac{k_1 \times k_2}{2}}{\frac{k_1 \times k_2}{2}}\) in (3.16) as the \(*_2\)-operation [30, 31], which in momentum space is

\begin{align*}
f(k_1) *_2 g(k_2) &= f(k_1) \frac{\sin \frac{k_1 \times k_2}{2}}{\frac{k_1 \times k_2}{2}} g(k_2) \quad (3.20)
\end{align*}

We will now combine the two amplitudes (3.4) and (3.16). Using (3.18) and (3.20) we find that \(A_1 + A_2\) corresponds to the action

\begin{align*}
S &= \frac{\lambda}{2} \kappa_{10} \mu_p \int \sqrt{-\det g} \left\{ (1 + \mathcal{M} + \frac{1}{2} \mathcal{M} *_2 \mathcal{M}) \Lambda \\
&\quad+ \pi\alpha' (f_{MN} + \mathcal{M} *_2 f_{MN}) \Lambda^{MN} + \frac{1}{2} (\pi\alpha')^2 f_{MN} *_2 f_{PQ} \Lambda^{MPNQP} \right\}. \quad (3.21)
\end{align*}

Equation (3.21) contains some terms that are cubic and quartic in open string modes which therefore do not follow directly from the amplitudes we computed. But, of
course, we expect these terms by gauge invariance. Also, we have inserted a \( \Lambda \) by hand, corresponding to a tadpole diagram with just the RR field.

We recognize the factors of \( M \) appearing in (3.21) precisely as expected from the expansion of a straight open Wilson line (with the substitution \( a_i \rightarrow A_i \), \( 2\pi \alpha' a_i \rightarrow X^i \))

\[
W(x, C_q) = \text{Ps exp} \left[ i \int_0^1 d\tau (q_\mu \theta^{\mu\nu} A_\nu(x + \xi(\tau)) + q_{\perp i} X^i(x + \xi(\tau))) \right] \tag{3.22}
\]

with the path \( \xi : C_q \hookrightarrow \mathbb{R}^{p+1} \) given by \( \xi^\mu(\tau) = \theta^{\mu\nu} q_\nu \tau \). We also see the \( n \)-ary operations (in this case \( *_2 \)) appearing just as we expect from smearing the Yang-Mills operators along a Wilson line [13].

4. The Full WZ Term

We now attempt to extract the full Wess-Zumino (WZ) coupling from the first few terms given in (3.21). For simplicity, we first focus on the “zero-momentum” couplings (i.e. setting \( q \), the momentum of the RR potential, to zero, in which case \( M = 0 \)),

\[
S = \frac{\lambda}{2} \kappa_{10} \mu_p \int \sqrt{-g} \text{Tr} \left[ \Lambda + 2\pi \alpha' \frac{f_{MN}}{2!} \Lambda^{MN} + \frac{1}{2} (2\pi \alpha')^2 \frac{f_{MN} f_{PQ}}{2!} \Lambda^{MNPQ} \right]. \tag{4.1}
\]

Although in the last section we have not included the Chan-Paton factors explicitly, they can be added in straightforwardly. For this reason we have included a Tr over the \( U(N) \) indices in (4.1) for the case of \( n \) D-branes. Using equation (2.33) for the \( \Lambda \)-s, we find that (4.1) can be precisely reproduced by expanding the formula

\[
S = \mu_p \text{STr} \int \left( e^{-i \theta/(2\pi \alpha')} e^{2\pi \alpha' f P} \right) e^{-2\pi \alpha' \iota_{(p,0)} C} e^B, \tag{4.2}
\]

to terms quadratic in open string modes. In reaching (4.2) we have set the value of the normalization constant \( \lambda = 1/16 \), and for notational convenience we have made the substitution \( a_i \rightarrow \phi_i \) and absorbed the factor of \( \kappa_{10} \) into \( C \). In (4.2) when evaluating products of open-string fields, the \( * \)-product is implied and \( \text{STr} \) is the symmetrized trace over both the \( U(N) \) matrices and \( * \)-product. As usual wedge products are implied in expanding the exponentials and in product of forms, and the integration extracts only the \( (p+1) \)-form in the integrand. \( \iota_T \) denotes contraction with respect to an antisymmetric tensor as in the definition (2.34). The notation \( P \) denotes the pullback; e.g.

\[
P \omega^{(2)}_{\mu
u} = \omega^{(2)}_{\mu
u} + 2(D_\nu X^i)\omega^{(2)}_{\mu i} + D_\mu X^i D_\nu X^j \omega^{(2)}_{ij}, \tag{4.3}
\]
where \( D_\mu X^i = 2\pi\alpha' D_\mu \phi^i \) should be understood as the coordinate space version of (3.18b). The parenthesis in (4.2) enforce that \( \theta \) can contract with the longitudinal indices coming from the pullback, but not with \( C \) or \( B \). For example, with \( \mathcal{C}^{(2)} = \left[ \frac{1}{2} C_{\mu\nu} dx^\mu dx^\nu + C_{\mu i} dx^\mu dx^i + \frac{1}{2} C_{ij} dx^i dx^j \right] \),

\[
\left( e^{-i\theta} \mathcal{P} \right) \mathcal{C}^{(2)} = \left[ \frac{1}{2} C_{\mu\nu} dx^\mu dx^\nu + C_{\mu i} dx^\mu dx^i + \frac{1}{2} C_{ij} dx^i dx^j \right] + D_\mu X^i \left[ C_{i\nu} dx^\mu dx^\nu + C_{ij} dx^i dx^j \right] + D_\mu X^i D_\nu X^j C_{ij} dx^\mu dx^\nu - \frac{1}{2} \theta^{\sigma\tau} D_\sigma X^i D_\tau X^j C_{ij}.
\]

(4.4)

An explicit expansion of (4.2) and its comparison with (4.1) is given in appendix E. Since there are a rather large number of terms involved, we see this as very strong evidence that (4.2) gives the correct couplings involving an arbitrary number of open string modes. Note that at quadratic order in open string modes, \( \text{STr} \) in (4.2) reduces to the normal trace and the \( * \)-product reduces to the ordinary product as in (4.1); thus at this order we have only checked the tensor structure and not the product and ordering structure. In section 6 we give further evidence for equation (4.2) by looking at amplitudes with an arbitrary number of open string modes.

For couplings to RR potentials with nonzero momentum \( q \), as already discussed around equation (3.22), we include a straight open Wilson line (3.22) with its length given by \( \theta^{\mu\nu} q_\nu \) and smear the operators in (4.2) over the Wilson line [13, 16]. The full action is therefore,

\[
S = \mu_p \int d^{10}q \left\{ \int d^{p+1}x L_* \left[ W(x, C_q) \left( e^{-i\theta/(2\pi\alpha')} e^{2\pi\alpha' f \mathcal{P}} \right) e^{-2\pi\alpha' i[\phi, \phi]} e^{i q \cdot x} \right] C(q) e^B \right\},
\]

(4.5)

where \( L_* \) denotes the prescription of smearing of all operators in the integrand over the Wilson line with the path ordering in terms of the \( * \)-product. For this purpose, \( f_{\mu\nu}, D_\mu \phi \) and \( [\phi, \phi] \) are considered individual operators. For example,

\[
\int d^{p+1}x L_* \left[ W(x, C_q) f_{\mu\nu}(x) f_{\lambda\rho}(x) \right] * e^{i q \cdot x^\mu} = \int d^{p+1}x \int_0^1 d\tau_1 d\tau_2 P_* \left[ W(x, C_q) f_{\mu\nu}(x + \xi(\tau_1)) f_{\lambda\rho}(x + \xi(\tau_2)) \right] * e^{i q \cdot x^\mu} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^{p+1}x \int_0^1 d\tau_1 \cdots \int_0^1 d\tau_{n+2} P_* \left[ f_{\mu\nu}(x + \xi(\tau_1)) * f_{\lambda\rho}(x + \xi(\tau_2)) * \cdots \right. \left. \mathcal{M}(x + \xi(\tau_3)) * \cdots * \mathcal{M}(x + \xi(\tau_{n+2})) \right] * e^{i q \cdot x}.
\]

(4.6)

where \( \mathcal{M} = i q \times a + i (2\pi\alpha') q_\perp \cdot \phi \) comes from the expansion of the Wilson line (3.22) and \( P_* \) denotes path ordering. On performing the \( \tau \) integrations, equation (4.6) can
be written in terms of a power series in $\mathcal{M}$ using $n$-ary operations $*_n$ [30, 31]. These factors of $\mathcal{M}$ were seen in equation (3.21). In appendix E we verify also that the quadratic expansion of (4.5) gives precisely (3.21). The definition and properties of the $*_n$ $n$-ary operations, and their the relations to the expansion of open Wilson lines, was given in [13]. Note that it is manifest from the above equations that the $L_\ldots$ prescription completely symmetrizes the integrand. Again, in section 6 we shall give further evidence for equation (4.5) by looking at higher order terms.

We conclude this section with some remarks:

1. As we take $\theta \to 0$, the Wilson line collapses to a point and we recover the standard commutative result, including the factor $e^{-2\pi \alpha' i\phi}$. found in [7, 8].

2. As we take the zero momentum limit, $q \to 0$, the Wilson line collapses to a point and the $*_n$ operations become symmetrized $\ast$-products. More explicitly,

$$\text{Tr} \int d^{p+1}x \, _n[f_1(x) \cdots f_n(x)] = \text{STr} \int d^{p+1}x \, (f_1(x) \ast \cdots \ast f_n(x)) \quad (4.7)$$

where on the right hand of the equation STr denotes the symmetrized trace prescription denoting the normalized sum over all possible permutations.

3. When we take $q = 0$ and set the transverse scalar fields to zero, (4.5) becomes

$$S_{WZ} = \mu_p \, \text{Tr}_\theta \left( e^{-i\theta e^f} \right) \, e^{BC} \quad (4.8)$$

where for convenience we have set $2\pi \alpha' = 1$ and defined $\text{Tr}_\theta = \text{Tr} \int$. The combination of $Ce^B$ is a consequence of T-duality.[47, 48] Equation (4.8) implies that the RR charges of the lower dimensional D-branes generated by the topologically nontrivial configurations of the gauge theory are given by

$$\text{Tr}_\theta \left( e^{-i\theta e^f} \right) \quad (4.9)$$

which is called the Elliott formula in noncommutative geometry. This agrees very well with the noncommutative geometry result that Elliott’s formula is integer valued, while the Chern characters are not. That (4.9) gives the right charges for D-branes was anticipated in refs. [38, 39] based on the K-theory of the noncommutative torus. For more discussion of equations (4.8) and (4.9), see section 3 of [15].

4. The result (4.5) was motivated from the on-shell amplitudes and corresponds to the $\Phi = 0, \theta = -(2\pi \alpha')^{1/g+B}B^{1/g-B}$ description of the D-brane couplings. In [15] we argued that it actually applies to every $\theta$-descriptions (see also the next section).
5. Comparison to the Matrix Model

In this section we shall rewrite the couplings (4.2) and (4.5) in various other forms. In particular we shall make connections to the results of [19, 17, 20] which describe the couplings in the $\theta = \frac{1}{B}, \Phi = -B$ description. For convenience in this section, we will set $2\pi \alpha' = 1$ and assume that $\theta$ and $B$ have maximal rank. Then, for $p$ odd (IIB), we shall consider a Euclidean world-volume with all longitudinal directions noncommutative, while for $p$ even (IIA) all longitudinal directions but time are noncommutative.

In [19, 17, 20], motivated from the connection between noncommutative gauge theory and the Matrix model (see e.g. [49]), the zero momentum RR couplings were argued to be

$$S_{\text{WZ}} = \frac{\mu_p}{\text{Pf}(\theta)} \text{STr} \int \star \left( e^{-i[i(x,x)]} C \right),$$

where now $\star$ is the Hodge dual in the noncommutative directions of the brane, still with respect to the closed string metric. That is, for odd $p$ (IIB), we take the Hodge dual in the brane, whereas for even $p$, we take the Hodge dual along the spatial directions of the brane. Thus, for odd $p$, equation (5.1) is equivalent to

$$S_{\text{WZ}} = \frac{\mu_p}{\text{Pf}(\theta)} \text{STr} \int d^{(p+1)}x \sqrt{g} \left( e^{-i[i(x,x)]} C \right).$$

For even $p$, equation (5.1) can be similarly rewritten as the integral of a time-like one-form. Equation (5.1) is essentially the RR coupling of D-instantons expanded around a background (5.2) which is noncommutative. As noted in [49], the Matrix model description corresponds to the $\theta = \frac{1}{B}, \Phi = -B$ description of the D-brane. Note that

$$[X^\mu, X^\nu] = i (\theta - \theta f \theta)^{\mu\nu}, \quad [X^\mu, X^i] = i \theta^{\mu\nu} D^\nu \phi^i. \quad (5.3)$$

In the following we shall show that equation (4.2) can be written in terms of Matrix model-type variables (5.2) as

$$S_{\text{WZ}} = \frac{\mu_p}{\text{Pf}(\theta)} \text{STr} \int \star \left( e^{-i[i(x,x)]} e^{B - \frac{1}{2} C} \right),$$

It is remarkable that (5.1) and (5.4) are so tantalizingly close to each other in this form\(^4\). In particular if we take $\theta = \frac{1}{B}$ in (5.4) we precisely recover (5.1). The reason may be understood from two aspects:

\(^4\)This apparent similarity is actually somewhat deceptive. Explicit expansion when $B$ is far from infinite, shows that they actually differ significantly since the $\mu, \nu$ indices in $[X^\mu, X^\nu]$ and $[X^\mu, X^i]$ also contract with $e^{B - \frac{1}{2} C}$. Thus the on-shell amplitudes which follow from (5.1) and (5.4) are very different for generic values of $B$, agreeing only in the limit of infinite $B$. 

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1. When we take the large $B$ limit, the worldsheet boundary conditions reduce to that of D-instantons and $\theta = -\frac{1}{g+B}B - \frac{1}{g-B} \rightarrow \frac{1}{B}$. Thus we precisely recover (5.1) from (5.4) in this limit.

2. In [15] based on the topological nature of the terms in (5.4), we argued that (5.4) (while derived in the $\Phi = 0$ description) gives the leading term (in an $\alpha'$ expansion) for every $\theta$-description. In particular it applies to the Matrix description with $\theta = \frac{1}{B}, \Phi = -B$. We emphasize that while different descriptions are related by field redefinitions, they give rise to the same on-shell amplitudes only after we sum over all $\alpha'$ corrections.

To derive (5.4) from (4.2) we introduce a shorthand notation for the pullback. With a mild notational abuse,
\[
P = e^{D_{i\phi}} = e^{D_{i\chi}}. \tag{5.5}
\]
where $D_{i\phi} = D_\mu \phi^i dx^\mu$ is considered as a one-form in the worldvolume and a contracted vector in the transverse dimensions. That is, we can think of $D_{i\phi}$ as an operator which acts on forms to the right, by contracting the vector index and antisymmetrizing the form index, thereby preserving the dimension of the form on which it acts. This reproduces (4.3).

Note the following identities for the manipulations of forms and contractions\footnote{We emphasize that the Hodge dual, $\star$, is only in the (Euclidean) noncommutative directions of the brane. In particular, the $n$ in equation (5.6) is the dimension of the form along the noncommutative directions. Equations (5.7) and (5.8) are consequences of (5.6) and $\int (\star \omega)(\star \chi) = \int \omega \chi$ where the integral, here and in the text, is over the entire worldvolume of the brane. Finally, we should note that, in a basis in which $\theta$ is skew-diagonal, our convention for the Pfaffian of a $(2\Delta)$-dimensional antisymmetric matrix $M$ is $\text{Pf} M = \frac{(-1)^{\Delta}}{2^{2\Delta} \Delta!} \epsilon_{\mu_1 \cdots \mu_{2\Delta}} M^{\mu_1 \mu_2} \cdots M^{\mu_{2\Delta-1} \mu_{2\Delta}}$. That is, to compute $\text{Pf} \theta$, one can go to a basis in which $\theta$ is skew diagonal, and multiply over the skew eigenvalues of $\theta$ in the lower-left corners of the $2 \times 2$ blocks.}
\[
e^{-i\theta \omega^{(n)}} = (-1)^{n+\frac{\Delta(\Delta+1)}{2}} \text{Pf}(\theta) \star \left( \left( \star e^{-\theta^{-1}} \right) \left( \star \omega^{(n)} \right) \right), \tag{5.6}
\]
\[
\int (e^{-i\theta \chi}) \omega = \int (e^{-i\theta \omega}) \chi \tag{5.7}
\]
\[
\int \star (e^{-i\theta \omega}) = (-1)^{\frac{\Delta(\Delta+1)}{2}} \text{Pf}(\theta) \int e^{-\theta^{-1}} \omega. \tag{5.8}
\]

Using
\[
[D_{i\phi}, t_\theta] = -t_\theta D_{i\phi}, \quad [D_{i\phi}, t_\theta D_{j\phi}] = 2t_\theta (D_{i\phi})^2, \quad [t_\theta D_{i\phi}, \theta^{-1}] = -D_{i\phi}. \tag{5.9}
\]
in the Campbell-Baker-Hausdorff formula, where these equations are meant to act on forms, we also find that
\[ e^{D\phi}e^{-i\theta} = e^{-i\theta}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}}, \]  
where \((\theta D\phi)^{\mu i} = \theta^{\mu\nu}D_{\nu}\phi^{i}\). Using these formulas, we obtain
\[
\int \left( e^{-i\theta}e^{fP} \right) e^{-i\mu e^{D\phi}Ce^{B}}
\]
[via (5.5), (5.7)]
\[
= \int e^{f}e^{D\phi}e^{-i\theta}e^{-i\mu e^{D\phi}Ce^{B}}
\]
[via (5.10)]
\[
= \int e^{f}e^{-i\theta}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}
\]
[via (5.8)]
\[
= (-1)^{\frac{p(p+1)}{2}} \int Pf(f) \star \left[ e^{f_{j-1}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}} \right]
\]
[via (5.8)]
\[
= \int \frac{Pf(\theta - \theta f\theta)}{Pf(\theta)} e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}
\]
[via (5.8)]
\[
= \frac{1}{Pf(\theta)} \int \star \left( e^{i\theta - \theta f\theta} e^{i\theta D\phi} e^{-i\mu e^{D\phi}Ce^{B}} \right)
\]
[via (5.3)]
\[
= \frac{1}{Pf(\theta)} \int \star \left[ e^{-i\mu e^{D\phi}Ce^{B}} \right]
\]
where we have used the identities 
\[ \frac{1}{Pf(-f^{-1})} = Pf(f) \text{ and } Pf(\theta - f^{-1}) Pf(\theta) Pf(f) = Pf(\theta - \theta f\theta). \]
The above manipulations use the commutative product structure between various open string fields, which applies to (4.2) under the symmetrized trace \(STr\), and generally follows from full symmetry inside \(L_{s}\).

Now we give another form of (4.2) and its finite momentum version (4.5), which is also very useful. Using (5.7) and (5.11) we find that
\[
S_{WZ} = \mu_{p} \int \left( e^{-i\theta}e^{f} \right) \left( e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{B}} \right)
\]
\[
= \mu_{p} \int \sqrt{\det(1 - \theta f)} e^{f_{j-1}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}}
\]
Note that now \(e^{-i\theta}\) acts only on \(e^{f}\). In the second line above we have used an identity
\[ e^{-i\theta}e^{f} = \sqrt{\det(1 - \theta f)} e^{f_{j-1}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}}. \]

The identity is derived using (5.6):
\[ e^{-i\theta}e^{f} = e^{-i\theta} \star [(\star e^{f})(1)] = (-1)^{\frac{p(p+1)}{2}} Pf(f)e^{-i\theta}e^{f_{j-1}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}} \]
\[ = (-1)^{\frac{p(p+1)}{2}} Pf(f) Pf(f^{-1} - \theta) \star [(\star e^{f_{j-1}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}})(1)] = \sqrt{\det(1 - \theta f)} e^{f_{j-1}e^{\frac{1}{2}e^{i\theta}D\phi e^{-\frac{1}{2}i\theta}e^{-i\mu e^{D\phi}Ce^{B}}}}. \]
In the last step we take $f$ to be sufficiently small relative to $\theta^{-1}$ so that the sign is unambiguous.

To summarize, with the scalar fields set to zero we have three equivalent expressions that yield the charge coupling to the RR fields, namely,

$$S_{WZ} = \mu_p \text{STR} \int \left( e^{-i\theta} e^f \right) C e^B$$

$$= \mu_p \text{STR} \int \sqrt{\det(1 - \theta f)} e^{f - \frac{1}{2} \theta} C e^B$$

$$= \mu_p \text{STR} \int \ast \left[ e^{-i\mu [X,X]} e^{-\frac{1}{2} \theta} C e^B \right]$$

(5.17)

Again we emphasize that the RR charge of a gauge configuration should be measured using $Ce^B$.

Finally note that the open Wilson line (3.22) can be written in Matrix Theory variables (5.2) as [50]

$$W(x, C_q) * e^{iqx} = \exp (iq \cdot X) = \exp \left( i q_\mu X^\mu + i q_i X^i \right)$$

(5.18)

Using this, the complete finite momentum action can now be written as

$$S_{WZ} = \mu_p \frac{1}{\text{Pf}(\theta)} L_s \int d^{10} q \ast \left( e^{-i\mu [X,X]} e^{-\frac{1}{2} \theta} C(q) e^B e^{iq \cdot X} \right).$$

(5.19)

6. An Analysis of Higher Powers

In the previous section, we have verified the first few terms of (4.2) to the amplitudes derived in section 3. Here we will verify some of the higher order terms. Specifically, for the terms that we have found in (4.1), we can verify the presence of the Wilson line and the $L_s$ prescription to all orders in the gauge field. We can also verify the presence of Elliott’s formula in the action to all orders in the gauge field, albeit without the interaction term in the field strength. That is, we can verify

$$\left( e^{-i\theta/(2\pi\alpha')} e^{2\pi\alpha' da} \right) C e^B.$$ 

This verifies the zero-momentum action (4.2) to all orders in the gauge field, in the absence of transverse scalars, assuming the completion of $da$ to $f$. This is a reasonable assumption by gauge invariance. We can also incorporate, to all orders, the pullback in equation (4.2), although again our computation is not sensitive to the presence of covariant, rather than partial, derivatives in the definition of the pullback.

Note that none of the aforementioned verifications involve interactions of the gauge field amongst themselves. Those are more complicated to verify. In particular, the Myers term is an interaction term, so we are, unfortunately, not going to verify it to higher order (for the commutative theory, some higher order Myers terms were checked in [51]).
6.1 The Wilson Line

In this subsection we verify that the terms in the quadratic action, (4.1), are attached to an open Wilson line with the \( L^* \) prescription. This will demonstrate equation (4.5) to quadratic order. Our strategy is to isolate, from the amplitude with \( n \) open strings, a particular subset of terms which will be identified with the Wilson line completion of (4.1). The discussion of this subsection will closely parallel section 4 of [14].

For an amplitude with \( n \) open strings (2.3), we can split off one or two open strings and obtain correlation functions like those that gave rise to (3.4) or (3.7). For the remaining factor, we consider the subset of terms that involve \( \dot{X} \); specifically, those that give rise to factors of the form (2.19). In particular, we focus on the middle two terms,

\[
\frac{1}{2\pi\alpha'} \frac{1}{1+y_a^2} \mathcal{M}_a; \quad \mathcal{M}_a = i(q \times a) + i(2\pi\alpha')a \cdot q_\perp.
\]

(6.1)

Naïvely, this, along with the overall exponential phase factor

\[
\exp \left[ \sum_{a<b} \frac{i}{2} (k_a \times k_b) (2\tau_{ab} - \epsilon(\tau_{ab})) \right],
\]

(6.2)

from (2.17) in the \( \alpha'k_a \cdot k_b \to 0 \) limit, is precisely what is required to reproduce the Wilson line. Specifically, the factors of \( \frac{1}{1+y_a^2} = \pi \frac{dy_a}{\cot(\pi y_a)} \) give the measure for the change of variables \( y_a = -\cot(\pi \tau_a) \); then the integration of (6.2) gives the \(*_n\) \( n \)-ary operation that we expect from the expansion of the Wilson line [13]. Thus, (6.1) essentially shows that we obtain the Wilson line; the numerical factor for the exponentiation follows from the combinatorics in converting an amplitude to an action.

Strictly speaking, however, in order for this to work properly, we have to make sure that \( y_1 \) and \( y_2 \) also appear properly. In particular, because poles can appear in the amplitudes—or equivalently, because the amplitudes are generally defined by analytic continuation of the \( y_a \) integrals—it is generally troublesome to take the \( \alpha' \to 0 \) limit before integrating. However, as in [14], one can check that the integrand is regular if we take all \( \alpha'k_a \cdot k_b \to 0 \) except \( \alpha'k_1 \cdot k_2 \). This leaves, in addition to the contribution from (6.2), an overall trigonometric factor involving \( \tau_2 \) (we fix \( \tau_1 = \frac{1}{2} \)), which is generically singular as \( \alpha'k_1 \cdot k_2 \to 0 \), but can be made nonsingular, except for a possible explicit factor of \( \frac{1}{\alpha'k_1 \cdot k_2} \), via an integration by parts. Then we can also take \( \alpha'k_1 \cdot k_2 \to 0 \), and throwing out the explicit poles, recover the \(*_n\) kernel (6.2).

\(^6\)See also [17] for a nice discussion of the Wilson line structure from the amplitudes in the \( \alpha' \to 0 \) limit. Our discussions here and in [14] are slightly more general.
For example, for the generalization of equation (3.7) to \(n\) open strings, the terms of interest, after taking the \(\alpha' k_a \cdot k_b \to 0\) limit, are of the form

\[
\int_0^1 d\tau_2 \cdots \int_0^1 d\tau_n \prod_{a<b} \exp \left[ \frac{i}{2} (k_a \times k_b) \left(2\tau_{ab} - \epsilon(\tau_{ab})\right) \right] |\cos(\pi\tau_2)|^{2\alpha' k_1 \cdot k_2} T M_3 \cdots M_n
\]

(6.3)

where \(T\) denotes the integrand of (3.7) with \(A_2\) removed, i.e. the sum of (3.8)–(3.13) with the factor \(A_2(y_2)\) removed. There are three types of terms in \(T\) classified according to their dependence on \(y_2\) or \(\tau_2\), which we shall now analyze one by one:

**Terms depending only on \(\frac{1}{1+y_2}\):** These terms give rise to the contact terms in our discussion in section 3. Since there is no singularity in \(y\), we can take the limit \(\alpha' k_1 \cdot k_2 \to 0\) in the integrand and the integrations (6.3) precisely give \(*_n\) operations, which corresponds to the affixation of an open Wilson line with \(L_s\) ordering.

**Terms singular at \(y_2 = 0\) and without an explicit \(\alpha' k_1 \cdot k_2\):** These terms gave rise to pole terms in section 3 and were discarded. It is easy to argue that they also contain poles in (6.3). The idea is that we can perform an integration by parts in the integrand so that the singularity in \(y_2\) is replaced by a pole in \(\alpha' k_1 \cdot k_2\). For example to take care of a simple pole in \(y_2\)—or a factor of \(\tan \pi\tau_2\)—we use the identity

\[
|\cos(\pi\tau_2)|^{2\alpha' k_1 \cdot k_2} \tan(\pi\tau_2) = -\frac{1}{2\pi \alpha' k_1 \cdot k_2} \frac{\partial}{\partial \tau_2} |\cos(\pi\tau_2)|^{2\alpha' k_1 \cdot k_2}. \quad (6.4)
\]

Since the remainder of the integrand is nonsingular, we can integrate by parts, and find that the resulting integrand is \(\frac{1}{2\pi \alpha' k_1 \cdot k_2}\) times a nonsingular integrand.

**Terms singular at \(y_2 = 0\) and with an explicit \(\alpha' k_1 \cdot k_2\):** In section 3, these terms gave rise to contact terms, thereby yielding the \([a_M, a_N]_*\) completion to the noncommutative field strength, (covariant) pullback, and Myers type terms. Using the same trick (6.4), the singular part in the integrand can again be written as a pole in \(\alpha' k_1 \cdot k_2\) times a nonsingular integrand, for which we can take the limit \(\alpha' k_1 \cdot k_2 \to 0\) before doing the integration. Now the pole precisely cancels with the multiplicative factor \(\alpha' k_1 \cdot k_2\) and gives rise to a contact term. A careful integration by parts yields precisely \([a_M, a_N]_*\) attached to a Wilson line. Note that if we had naively taken \(\alpha' k_a \cdot k_b \to 0\) inside the integrals, we would have missed the contributions of these terms.
As the procedure just outlined is rather subtle, we will demonstrate this explicitly. The requisite integration, after employing the trick (6.4), is

\[
I = \int_0^1 d\tau_n \cdots \int_0^1 d\tau_2 \left( \frac{\partial}{\partial \tau_2} |\cos \pi \tau_2|^{2\alpha' k_1 \cdot k_2} \right) \exp \left[ \frac{i}{2} \sum_{a<b} (k_a \times k_b) (2\tau_{ab} - \epsilon(\tau_{ab})) \right] ;
\]

(6.5)

recall that \( \tau_1 = \frac{1}{2} \). We now integrate by parts. Note that there are many surface terms, since pointsplitting regularization on the worldsheet implies that we exclude the points \( \tau_2 = \tau_a \) (for \( a \neq 2 \)) from the integration. So we have

\[
I = \int_0^1 d\tau_n \cdots \int_0^1 d\tau_3 \sum_{\tau_2 \in S} s_{\tau_2} |\cos \pi \tau_2|^{2\alpha' k_1 \cdot k_2} \exp \left[ \frac{i}{2} \sum_{a<b} (k_a \times k_b) (2\tau_{ab} - \epsilon(\tau_{ab})) \right] - \int_0^1 d\tau_n \cdots \int_0^1 d\tau_2 |\cos \pi \tau_2|^{2\alpha' k_1 \cdot k_2} \frac{\partial}{\partial \tau_2} \exp \left[ \frac{i}{2} \sum_{a<b} (k_a \times k_b) (2\tau_{ab} - \epsilon(\tau_{ab})) \right] .
\]

(6.6)

with the first term the surface contributions. Here \( S \) is the set \( \{0, \frac{1}{2}^+, \tau_3^+, \cdots, \tau_n^+, 1\} \) (recall that \( \tau_1 = \frac{1}{2} \)), where, of course, the superscript denotes whether we approach from above or below, and the signs \( s_{\tau_2} \) are \( s_{\tau_2=\frac{1}{2}^+} = 1, s_{\tau_2=0} = -1 \) and \( s_{\tau_2=1} = 1 \). Since the \( \tau_2 \) integration in the second term above is now nonsingular, we can take \( \alpha' k_1 \cdot k_2 \to 0 \) in the integrand thereby obtaining a total derivative. So we now have,

\[
I = \int_0^1 d\tau_n \cdots \int_0^1 d\tau_3 \sum_{\tau_2 \in S} s_{\tau_2} \left\{ |\cos \pi \tau_2|^{2\alpha' k_1 \cdot k_2} - 1 \right\} \times \exp \left[ \frac{i}{2} \sum_{a<b} (k_a \times k_b) (2\tau_{ab} - \epsilon(\tau_{ab})) \right] .
\]

(6.7)

Of course, away from the zeros of the cosine—that is, away from \( \tau_2 = \frac{1}{2}^+ \)—the quantity in curly brackets vanishes as \( 2\alpha' k_1 \cdot k_2 \to 0 \). Thus we are left with the contribution from \( \tau_2 = (\frac{1}{2})^+ \) in (6.7),

\[
I = 2i \sin \frac{k_1 \times k_2}{2} \int_0^1 d\tau_n \cdots \int_0^1 d\tau_3 \exp \left[ \frac{i}{2} \sum_a [(k_1 + k_2) \times k_a] (2\tau_{1a} - \epsilon(\tau_{1a})) + \frac{i}{2} \sum_{2<a<b} (k_a \times k_b) (2\tau_{ab} - \epsilon(\tau_{ab})) \right] .
\]

(6.8)
We recognize that the integration in (6.8) gives \(*_{n-1}\); thus we obtain
\[*_{n-1} [[a_{1M}(k_1), a_{2N}(k_2)], \mathcal{M}_3(k_3), \cdots, \mathcal{M}_n(k_n)]. This completes the demonstration; the reader can check that the numerical factors work properly. Again note that taking \(\alpha' k_1 \cdot k_2 \to 0\) prematurely would have resulted in the omission of this important term.

6.2 Elliott’s Formula and The Pullback

In this subsection we check the couplings (4.5) to all orders up to the nonlinear terms in the field strengths, i.e. up to terms of the form \([a_M, a_N]_*\).

The simplest correlation functions are those for which we ignore both the \(\dot{X}\) part of the vertex operator (2.5), and the Wick contractions (2.25). Then the correlation function simply gives (suppressing the \(\tau_a\) integrations and setting \(\mu_p \kappa_{10} = 1\))

\[
\frac{\lambda}{2i} (2\pi i \alpha')^n A_n k_{1\mu_1} e_{1M_1} \cdots k_{n\mu_n} e_{nM_n} A^{\mu_1 M_1 \cdots \mu_n M_n}. \tag{6.9}
\]

As in the previous subsection, in the \(\alpha' k_a \cdot k_b \to 0\) limit, the factor of \(A_n\) gives rise precisely to the \(*_n\) \(n\)-ary operation between the open string modes in the expression; there is no subtlety here because the only \(y_a\)-dependence is that in equation (6.2).

If we take only longitudinal polarizations in equation (6.9), (and take the low energy limit) then equation (2.33) gives us

\[
(2\pi \alpha')^n \star \left[ (e^{-i\theta/(2\pi \alpha')} \star_n [da_1, \cdots, da_n]) C e^B \right]_{0\text{-form}} \tag{6.10}
\]

where the wedge product of the field strengths is combined with the \(*_n\) \(n\)-ary operation. Summing over \(n\), with a \(1/n!\) indistinguishability factor, gives Elliott’s formula,

\[
\left( e^{-i\theta/(2\pi \alpha')} e^{2\pi \alpha' f} \right) C e^B. \tag{6.11}
\]

as given in (4.5). Here we have made the replacement \(da \to f\); the extra \(a \star a\) terms should come from a Wick contraction that we ignored, as it did for equation (3.12)—see, in particular, the comments just preceding equation (3.17).

If we take the open string modes in (6.9) to be purely transverse, then (2.33) gives

\[
(2\pi \alpha')^n \star \left[ (e^{-i\theta/(2\pi \alpha')} \star_n [d\phi_1^{i_1}, \cdots, d\phi_n^{i_n}]) C_{i_1 \cdots i_n} e^B \right]_{0\text{-form}} \tag{6.12}
\]

where the longitudinal indices are suppressed and treated as forms. On replacing the partial derivatives with covariant derivatives, this precisely reproduces the \(n\)th term of the pullback of \(C e^B\), as given in equation (4.5).
More generally, we can take $q$ of the open string modes to be longitudinal and $r$ to be transverse. Then we find

$$(2\pi\alpha')^{q+r} \ast \left[ (e^{-i\theta/(2\pi\alpha')} \ast_{q+r} [da_1, \ldots, da_q, d\phi^i_1, \ldots, d\phi^i_r]) C_{i_1 \ldots i_r} e^B \right]_{0\text{-form}}.$$  (6.13)

Summing over $q$ and $r$, and including the indistinguishability factors, exponentiates these precisely to give an action

$$\int \left[ e^{-i\theta/(2\pi\alpha')} e^{da_\mathcal{P}\phi} \right] C e^B,$$  (6.14)

where the subscript on the pullback is to emphasize that here we do not have the covariant derivative in the definition of the pullback. Also, we have suppressed the $n$-ary operation. This differs from (4.5) only by pure open string interaction terms $a \ast a$, $a \ast \phi$ and $\phi \ast \phi$, and the explicit inclusion of the Wilson line. Of course, we checked the Wilson line for the first couple of open-string powers in the previous subsection.

7. Conclusion

We believe we have presented convincing evidence that equation (4.5) gives the WZ terms for noncommutative D-branes. In particular, it precisely matches amplitude calculations to quadratic order in the gauge field, as shown in section 4. Furthermore, in section 6, we have found the Wilson line to all orders in the gauge field. We have confirmed the presence, in the action, of both Elliott’s formula and the pullback, up to interaction terms in the field strength and covariant derivatives. We have also shown that our result for the WZ term that we have given here, agrees with that derived from the Matrix model.

While we have not discussed it explicitly in the paper, it is a simple matter to check that the on-shell amplitudes we computed in section 3 are invariant under a gauge transformation of the RR potential. However to check that the off-shell extension we proposed in section 4 is gauge invariant—or equivalently, that the expression coupled to $C$ in equation (4.5) is closed—is rather complicated. In [18], it was argued that the action, written in the Matrix model form (5.19), is gauge invariant. Also, some simple limits of equation (4.5) are obviously gauge invariant. For example, in the commutative limit it is known to be gauge invariant. Also, Elliott’s formula is closed, thereby implying gauge invariance at zero momentum, in the absence of transverse scalar fields.

We should note that although the WZ term we have given here involves the $B$-field, in principle this is merely the background value of the $B$-field; we have no guarantee that equation (4.5) properly incorporates fluctuations of the $B$-field. This is because
we have only computed disk amplitudes with the insertion of one closed string mode. It is interesting to note that, because the commutative limit of equation (4.5) precisely reproduces the commutative WZ term, in this limit fluctuations of $B$ are incorporated in the action. However, there is a general argument [52] that the Wilson line prescription for couplings of two or more closed string modes to an open string operator is more complicated than that in (4.5). It would be interesting to analyze the problem of $B$-field fluctuations more thoroughly.

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A. On the RR Vertex Operator

We wish to write down a $(-1/2, -3/2)$-picture vertex operator for the RR fields. In [42], it was shown that, under picture-changing, the known $(-1/2, -1/2)$-picture operator is reproduced by the vertex operator

$$V_{RR}^{-1/2,-3/2} = \frac{2g_c}{\alpha'} e^{-\phi/2-3\tilde{\phi}/2} \Theta C \frac{\Gamma^{11}}{2} \tilde{\Theta} e^{i\eta \cdot X}, \quad (A.1)$$

but that this $V_{RR}^{-1/2,-3/2}$ is BRST closed only when $d\hat{\star} C^{(n)} = 0$; i.e. only for vanishing field strengths. Here and throughout this paper, $\hat{\star}$ is the 10-dimensional Hodge dual with respect to the closed string metric. If we add in another RR potential, and write

$$V_{RR}^{-1/2,-3/2} = \frac{2g_c}{\alpha'} e^{-\phi/2-3\tilde{\phi}/2} \Theta C \frac{\Gamma^{11}}{2} \left[ C^{(n+2)} \tilde{\Theta} e^{i\eta \cdot X} + C^{(n+2)} \tilde{\Theta} e^{i\eta \cdot X} \right], \quad (A.2)$$

then BRST invariance requires [42]

$$d\hat{\star} C^{(n)} = 0, \quad dC^{(n)} - \hat{\star} d\hat{\star} C^{(n+2)} = 0, \quad dC^{(n+2)} = 0. \quad (A.3)$$

Via a gauge transformation, $\delta C^{(n+2)} = d\Lambda^{(n+1)}$, we can always satisfy the middle equation (since one can always solve the “Poisson” equation); thus we now have an RR vertex operator for arbitrary $F^{(n+1)} = dC^{(n)}$ field strength. So, by adding in all the RR potentials as in (2.4), we have a BRST invariant vertex operator for arbitrary field...
strengths. The analogues of the first and last equations of (A.3) are automatically satisfied since there are no eleven-forms in a ten-dimensional theory.\textsuperscript{7}

We note that the operator $\frac{1}{2}(I - \Gamma^{11})$ imposes a chirality condition on the spinors which then projects out the anti-self dual part of $C = \sum C^{(n)}$, keeping only $\Gamma^{11}C = -C$. This is a self duality condition since on an $n$-form $\omega^{(n)}$,

$$\Gamma^{11}\omega^{(n)} = (-1)^{\frac{n(n+1)}{2}}\hat{\star}\omega^{(n)}.$$  \hfill (A.4)

It is convenient to include this sign as part of the definition of self vs. anti-self duality. Note also that the closed string metric arises because of the closed string $\Gamma$-matrices.

One might wonder about the relationship—or even the compatibility—of self duality of $C$ and the statement from supergravity—or from the $(-1/2, -1/2)$-picture—that the on-shell field strengths are self dual: $\Gamma^{11}F = -F$.\textsuperscript{8} It turns out that self duality of the field strength is equivalent (up to a gauge transformation) to self duality of the potential in the gauge implied by the generalization of (A.3),

$$dC^{(n)} - \hat{\star}\hat{d}\hat{\star}C^{(n+2)} = 0.$$  \hfill (A.5)

In particular, we want to show that self duality of $C$ implies self duality of $F$. So suppose that $C$ is self dual. Then,

$$F^{(n)} = dC^{(n-1)} = -(-1)^{\frac{n(n+1)}{2}}\hat{\star}dC^{(11-n)} = (-1)^{\frac{n(n-1)}{2}}\hat{\star}dC^{(9-n)} = (-1)^{\frac{n(n-1)}{2}}\hat{\star}F^{(10-n)},$$  \hfill (A.6)

where in the second step we used self duality of $C$ and in the third step, we used (A.5). Thus, we see that $F$ is self dual: $F^{(10-n)} = -(-1)^{\frac{n(n+1)}{2}}\hat{\star}F^{(n)} \Rightarrow \Gamma^{11}F = -F$ by equation (A.4).

Conversely, suppose the field strength is self dual. Then,

$$dC^{(n)} = F^{(n+1)} = (-1)^{\frac{n(n+1)}{2}}\hat{\star}F^{(9-n)} = (-1)^{\frac{n(n+1)}{2}}\hat{\star}dC^{(8-n)} = (-1)^{\frac{n(n-1)}{2}}\hat{\star}dC^{(10-n)},$$  \hfill (A.7)

and so $C^{(n)} = (-1)^{\frac{n(n-1)}{2}}\hat{\star}dC^{(10-n)}$, up to an exact form (i.e. a gauge transformation). This is equivalent to $\Gamma^{11}C = -C$ and completes the proof.

Note that while the vertex operator implies a special choice of gauge for $C^{(n)}$, the final on-shell amplitudes are gauge invariant and do not depend on the choice of the gauge.

\textsuperscript{7}For IIA, we note that $F^{(10)}$ always vanishes in perturbation theory.

\textsuperscript{8}Note that with this sign, equation (A.4) indeed gives $F^{(5)} = \hat{\star}F^{(5)}$. Note that since $(\Gamma^{11})^2 = I$, the (anti-)self duality of even-dimensional forms is well-defined here.
B. A Derivation of the Fermionic Boundary Condition

We will show that for a Dp-brane with B-field background, the worldsheet boundary conditions for the spin operators are \( \tilde{\Theta}(\bar{z}) = M\Theta(z) \) with \( M \) given by

\[
M = \frac{\sqrt{-\det g}}{\sqrt{-\det (g + B)}} \mathcal{A}(B) \Gamma^0 \ldots \Gamma^p \begin{cases} 1, & \text{type IIA} \\ \Gamma^{11}, & \text{type IIB} \end{cases},
\]

(2.14)

\( \mathcal{A}(B) \), as defined in [44], is the exponential of \( B \), but with the \( \Gamma \) matrices totally antisymmetrized. (This is equivalent to \( e^B \), where wedge products are understood in the definition of the exponential, and where, as in (2.6), the Feynman slash of a sum is the sum of Feynman slashes.) Our derivation of (2.14) here generalizes those in [53, 54] to non-zero \( B \) (see also [55]).

The consistency of the OPEs

\[
\Theta_A(z_1)\Theta_B(z_2) \sim \frac{C^{-1}_{AB}}{z_1^{5/4}}, \quad \tilde{\Theta}_A(\bar{z}_1)\tilde{\Theta}_B(\bar{z}_2) \sim \frac{C^{-1}_{AB}}{\bar{z}_1^{5/4}},
\]

(B.1)

\[
\psi^M(z_1)\Theta_A(z_2) \sim \frac{\Gamma^M_{AB}\Theta_B(z_2)}{2z_1^{1/2}}, \quad \tilde{\psi}^M(\bar{z}_1)\tilde{\Theta}_A(\bar{z}_2) \sim \frac{\Gamma^M_{AB}\tilde{\Theta}_B(\bar{z}_2)}{2\bar{z}_1^{1/2}}
\]

(B.2)

with the boundary conditions at \( z = \bar{z} \)

\[
\tilde{\Theta}(\bar{z}) = M\Theta(z), \quad \tilde{\psi}^\mu(\bar{z}) = \left( \frac{g + B}{g - B} \right)^{\mu}_\nu \psi^\nu(\bar{z}), \quad \tilde{\psi}^i(\bar{z}) = -\psi^i(\bar{z})
\]

(B.3)

requires that \( M \) satisfies

\[
C^{-1} = MC^{-1}M^T,
\]

(B.4)

\[
\Gamma^\mu M = \left( \frac{g + B}{g - B} \right)^{\mu}_\nu \nu M\Gamma^\nu, \quad \Gamma^i M = -M\Gamma^i,
\]

(B.5)

where \( C_{AB} \) is the charge conjugation matrix, for which

\[
C\Gamma^\mu C^{-1} = -(\Gamma^\mu)^T.
\]

(B.6)

Equation (B.5) states that \( M \) is the action, in a spinor basis, of the transformation which acts as a Lorentz rotation \( \left( \frac{g + B}{g - B} \right)^{\mu}_\nu \) in the longitudinal directions and an inversion of all coordinates in the transverse directions. Since \( B \) only affects the longitudinal directions, we decompose \( M = M_0 S \), where \( M_0 \) is a solution of (B.4) and (B.5) with
\( B = 0 \) and \( S \) acts on spinors as a Lorentz transformation \((\sigma^\mu + B^\mu)\) in the longitudinal directions. The expression for \( M_0 \) is well-known, and is given by

\[
M_0 = \Gamma^0 \cdots \Gamma^p \begin{cases} 1, & \text{IIA} \\ \Gamma^{11}, & \text{IIB} \end{cases}, \tag{B.7}
\]

while \( S \) has the form

\[
S_{AB} = \exp \left( i \frac{\omega_{\mu\nu} \Sigma^{\mu\nu}}{2} \right) \tag{B.8}
\]

where \( \Sigma^{\mu\nu} = -\frac{i}{4} [\Gamma^\mu, \Gamma^\nu] \) are Lorentz generators in the spinor basis. Since \( S \) always contains an even number of longitudinal \( \Gamma \)-matrices, it satisfies

\[
C^{-1} = SC^{-1}S^T, \quad \Gamma^i S = S \Gamma^i. \tag{B.9}
\]

and commutes with \( M_0 \). Thus we have shown that \( M = M_0 S \) satisfies (B.4) and (B.5).

To find the explicit form of \( S \) in terms of \( B \), it is convenient to choose a basis so that \( g_{\mu\nu} = \eta_{\mu\nu} \) and \( B \) is skew diagonal. Without loss of generality let us look at a Euclidean two-dimensional block \( a, b = 2, 3 \) with \( B = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \). Then

\[
\left( \frac{1}{g-B} \right) (g + B) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta \cos \theta \end{pmatrix} \tag{B.10}
\]

with \( \lambda = \tan \frac{\theta}{2} \). Now we can immediately write down the transformation in the spinor basis,

\[
S = \exp \left[ \frac{i}{2} \theta \Gamma^2 \Gamma^3 \right] = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \Gamma^2 \Gamma^3 \tag{B.11}
\]

Thus when we include all directions

\[
S = \frac{1}{\sqrt{\det (\eta + B)}} \prod_a \left( 1 + \frac{1}{2} B_{ab} \Gamma^a \Gamma^b \right) \tag{B.12}
\]

\[
= \frac{\sqrt{-\det g}}{\sqrt{-\det (g + B)}} \left\{ 1 + \frac{1}{2} B_{\mu\nu} \Gamma^{\mu\nu} + \frac{1}{2!2^2} B_{\mu\nu} B_{\rho\sigma} \Gamma^{\mu\nu\rho\sigma} + \frac{1}{3!2^3} B_{\mu\nu} B_{\rho\sigma} B_{\tau\lambda} \Gamma^{\mu\nu\rho\sigma\tau\lambda} \\
+ \frac{1}{4!2^4} B_{\mu\nu} B_{\rho\sigma} B_{\tau\lambda} B_{\alpha\beta} \Gamma^{\mu\nu\rho\sigma\tau\lambda\alpha\beta} + \frac{1}{5!2^5} B_{\mu\nu} B_{\rho\sigma} B_{\tau\lambda} B_{\alpha\beta} B_{\gamma\delta} \Gamma^{\mu\nu\rho\sigma\tau\lambda\alpha\beta\gamma\delta} \right\}
\]
where in the first line the product runs over all $2\times 2$ blocks, and the second line is obtained from expanding out the product in the prior line and writing it in a covariant form. The last line follows from the explicit expansion of $\mathcal{E}(B)$. (Of course many of the terms in the second line vanish trivially for $p < 9$.)

Note that since $S$ generates a Lorentz transformation, it satisfies $SS^T = 1$, which implies an interesting and useful identity for $\mathcal{E}$,

$$
\mathcal{E}(B), \mathcal{E}(-B) = \frac{\det(g + B)}{\det g}.
$$

(B.13)

We finally note that equations (B.4) and (B.5) determine $M$ up to an arbitrary sign.

C. Trace Formulas

We wish to derive equation (2.33), and, in the process, derive some other useful formulas. The main such useful formula is

$$
\Lambda^{\mu_1 \ldots \mu_q \ldots \mu_r} = 32 \sum_{n,m} \frac{q!(-1)^{r(r-1)}}{2^{m}m!\left(q-2m\right)!\left(n-r\right)!\left(p+1-n-q+r+2m\right)!\left(\frac{p+1-n}{2} - \frac{q-r}{2} + m\right)!} \frac{\theta^{\mu_1 \mu_2}}{2\pi \alpha'} \cdots \frac{\theta^{\mu_{2m-1} \mu_{2m}}}{2\pi \alpha'} \epsilon^{\mu_{2m+1} \ldots \mu_q} \sigma_1 \ldots \sigma_{n-r} \tau_1 \ldots \tau_{p+1-n-q+r+2m} \times C^{(n)}_{1 \cdots i_r \sigma_1 \ldots \sigma_{n-r}} \left[B^{\frac{p+1-n}{2} - \frac{q-r}{2} + m}\right]_{\tau_1 \ldots \tau_{p+1-n-q+r+2m}},
$$

(C.1)

for $q + r$ even (otherwise $\Lambda^{\mu_1 \ldots \mu_q \ldots \mu_r}$ vanishes, of course). We denote by $\epsilon$ the antisymmetric volume element for the D$p$-brane. Note that the sum is only over $n$ with the same parity as $p + 1$; that is over odd (even) $n$ for type IIA (IIB).

Once we have established equation (C.1)—as we shortly will—then obtaining (2.33) is merely an exercise in combinatorics. Namely,

$$
\frac{i^{i_1 \cdots i_r} \chi_{i_1 \cdots i_r}}{q!} \Lambda^{\mu_1 \ldots \mu_q \ldots \mu_r} = 32 \sum_{n,m} \frac{1}{2^{m}m!\left(q-2m\right)!\left(n-r\right)!\left(p+1-n-q+r+2m\right)!\left(\frac{p+1-n}{2} - \frac{q-r}{2} + m\right)!} \frac{\theta^{\mu_1 \mu_2}}{2\pi \alpha'} \cdots \frac{\theta^{\mu_{2m-1} \mu_{2m}}}{2\pi \alpha'} \epsilon^{\mu_{2m+1} \ldots \mu_q} \sigma_1 \ldots \sigma_{n-r} \tau_1 \ldots \tau_{p+1-n-q+r+2m} \left(-1\right)^{r(r-1)} \chi^{i_1 \cdots i_r} \frac{C^{(n)}_{1 \cdots i_r \sigma_1 \ldots \sigma_{n-r}}}{(n-r)!} \times \left[B^{\frac{p+1-n}{2} - \frac{q-r}{2} + m}\right]_{\tau_1 \ldots \tau_{p+1-n-q+r+2m}}
$$

(C.2)
It is not hard to see that this is an expanded version of equation (2.33)
\[
\frac{\omega_{\mu_1 \ldots \mu_q}}{q!} \chi_{i_1 \ldots i_r} \Lambda^{\mu_1 \ldots \mu_q i_1 \ldots i_r} = 32 \left\{ \ast \left[ (e^{-i\theta/(2\pi\alpha')} \omega) (\tau_\chi C) e^B \right] \right\}_{0\text{-form}}^{(2.33)}
\]
Specifically, we can recognize
\[
\frac{1}{(q-2m)!} \frac{\omega_{\mu_1 \mu_2} \ldots \omega_{\mu_{2m-1} \mu_{2m}}}{2^{m} \pi \alpha'} \omega_{\mu_1 \ldots \mu_q} dx^{\mu_2m+1} \ldots \wedge dx^{\mu_q}
\]
as the \(m\)th term in the expansion of \(e^{-i\theta/(2\pi\alpha')} \omega\), where the minus sign in the exponential comes from interchanging the order of the indices as per the definition (2.34). Similarly, it is clear that
\[
\left( -1 \right)^{1/2} r! \chi_{i_1 \ldots i_r} (C^{(n)})_{i_1 \ldots i_r} dx^{\sigma_1} \wedge \ldots \wedge dx^{\sigma_{n-r}} = \tau_\chi C^{(n)}.
\]
Finally, the Hodge dualization and the factor of \(e^B\) is also obvious. The fact that the number of powers of \(B\) in the expansion of \(e^B\) is related to the number of powers of \(\theta\) in \(e^{\theta/(2\pi\alpha')}\), is due to the restriction to the zero-form, since \(\frac{\omega_{\mu_1 \ldots \mu_q}}{q!} \chi_{i_1 \ldots i_r} \Lambda^{\mu_1 \ldots \mu_q i_1 \ldots i_r}\) is a scalar.

Deriving equation (C.1) is quite lengthy and tedious. We start by considering the object
\[
\tilde{\Lambda}_{\tilde{n}}^{\mu_1 \ldots \mu_q i_1 \ldots i_r} = \text{Tr} \left( \Gamma^{\mu_1 \ldots \mu_q i_1 \ldots i_r} C^{(n)} E(B) \Gamma^0 \ldots \Gamma^p \right),
\]
and we observe that since the only transverse \(\Gamma\) matrices, aside from the \(\Gamma^i\)s, are attached to \(C^{(n)}\), we must have
\[
\tilde{\Lambda}_{\tilde{n}}^{\mu_1 \ldots \mu_q i_1 \ldots i_r} = \left( -1 \right)^{r/(p+1)} \frac{\chi_{i_1 \ldots i_r}}{(n-r)!} \text{Tr} \left( \Gamma^{\mu_1 \ldots \mu_q} \frac{C^{(n)}_{i_1 \ldots i_r}}{\sigma_1 \ldots \sigma_{n-r}} \Gamma^{\sigma_1 \ldots \sigma_{n-r}} E(B) \Gamma^0 \ldots \Gamma^p \right).
\]
Thus, it is sufficient to consider
\[
\tilde{\Lambda}_{\tilde{n}}^{\mu_1 \ldots \mu_q} = \frac{\epsilon_{\tau_0 \ldots \tau_p}}{(p+1)!} \text{Tr} \left( \Gamma^{\mu_1 \ldots \mu_q} \frac{C^{(\tilde{n})}_{\sigma_1 \ldots \sigma_0}}{\sigma_1 \ldots \sigma_{\tilde{n}}} \sum_m \frac{B^{m}_{\rho_1 \ldots \rho_2m}}{m!(2m)!} \Gamma^{\rho_1 \ldots \rho_{2m}} \Gamma_{\tau_0 \ldots \tau_p} \right).
\]
for arbitrary \(\tilde{n}, q\), with \(\tilde{n} + q\) having the opposite parity of \(p\), and where \(B^{m}\) means \(\underbrace{B \wedge \ldots \wedge B}_m\). By replacing \(\tilde{n}\) with \(n - r\) and adding \(r\) transverse indices to \(C^{(\tilde{n})} \rightarrow C^{(n)}\), where \(r\) and \(q\) have the same parity, we will recover \(\left( -1 \right)^{r/2} \tilde{\Lambda}_{\tilde{n}}^{\mu_1 \ldots \mu_q i_1 \ldots i_r}\), where here \(n\) and \(p + 1\) must also have the same parity. In the following we will drop the tilde on \(\tilde{n}\).
We next observe that
\[
\epsilon_{\tau_0 \cdots \tau_p} \Gamma_{\tau_0 \cdots \tau_p} \Gamma_{\mu_1 \cdots \mu_q} = (-1)^{\frac{q(q-1)}{2}} \frac{(p+1)!}{(p+1-q)!} \epsilon_{\tau_0 \cdots \tau_{p-q}} \mu_{1 \cdots \mu_q} \Gamma_{\tau_0 \cdots \tau_{p-q}}, \tag{C.6}
\]
where indices are raised and lowered with the closed string metric $g$. Therefore,\(^9\)
\[
\tilde{\Lambda}_{n}^{\mu_1 \cdots \mu_q} = (-1)^{\frac{q(q-1)}{2}} \sum_m \frac{\epsilon_{\tau_0 \cdots \tau_{p-q}} \mu_{1 \cdots \mu_q} C_{\sigma_1 \cdots \sigma_n}^{(n)} B_{\rho_1 \cdots \rho_{2m}}^{m} \Gamma_{\tau_0 \cdots \tau_{p-q} \rho_1 \cdots \rho_{2m}}^{\tau_0 \cdots \tau_{p-q} \rho_1 \cdots \rho_{2m-1}}}{n!m!(2m)!(p+1-q)!} \text{Tr} (\Gamma_{\sigma_1 \cdots \sigma_n}^{\sigma_1 \cdots \sigma_n} B_{\rho_1 \cdots \rho_{2m}}^{m} \Gamma_{\tau_0 \cdots \tau_{p-q}}^{\rho_1 \cdots \rho_{2m} \tau_0 \cdots \tau_{p-q}}), \tag{C.7}
\]
The usual trace rules tells us that, say, the first $s \sigma$s should be identified with the last $s \tau$s and that the remaining $\sigma$s should be identified with the first $n-s \rho$s. Identifying the remaining $\rho$s with the remaining $\tau$s gives $s = \frac{p+1+n-q}{2} - m$. Including the combinatorics and signs gives
\[
\tilde{\Lambda}_{n}^{\mu_1 \cdots \mu_q} = 32 \sum_m (-1)^{\frac{n(n-1)}{2} + \frac{p(p+1)}{2} + pq + \frac{q(q+1)}{2} - \frac{m(m+1)}{2} - m(\frac{p+1+n-q}{2} - m-1)} \epsilon_{\tau_0 \cdots \tau_{p-q} \mu_1 \cdots \mu_q} \Gamma_{\tau_0 \cdots \tau_{p-q} \mu_1 \cdots \mu_q}^{\Gamma_{\tau_0 \cdots \tau_{p-q} \rho_1 \cdots \rho_{2m}}^{\Gamma_{\tau_0 \cdots \tau_{p-q} \rho_1 \cdots \rho_{2m-1}}}}{n!m!(2m)!(p+1-q)!} \text{Tr} (\Gamma_{\tau_0 \cdots \tau_{p-q} \rho_1 \cdots \rho_{2m}}^{\tau_0 \cdots \tau_{p-q} \rho_1 \cdots \rho_{2m-1}}) \tag{C.8}
\]
Equation (C.8), however, is not the most convenient form to work with; it would be more convenient for $B$ to never contract with $C$. Since every possible longitudinal index is attached to $\epsilon$, it follows that $\rho_1 \cdots \rho_{n+q-2} + m$ are equal to an appropriate subset of the $\mu$s. Thus, we can replace the $\rho$s with $\mu$s, and then replace the $\mu$s on $\epsilon$ and $C^{(n)}$ with $\rho$s. In that way, we have shifted some of the $\mu$s from $\epsilon$ to $B$, and $C^{(n)}$ is fully contracted with $\epsilon$. Including permutations and combinatorics gives
\[
\tilde{\Lambda}_{n}^{\mu_1 \cdots \mu_q} = 32 \sum_m \frac{q!(-1)^{\frac{m(m+1)}{2} + \frac{p+q(p+q+1)}{2}}}{n!(q-m)!(p+1-n-m)!(\frac{p+1+q-n}{2} - m)!} \epsilon_{\sigma_1 \cdots \sigma_n} \Gamma_{\sigma_1 \cdots \sigma_n}^{\Gamma_{\sigma_1 \cdots \sigma_n}^{\sigma_1 \cdots \sigma_n} B_{\rho_1 \cdots \rho_{2m}}^{m} \Gamma_{\tau_0 \cdots \tau_{p-q-n} - m}}{\frac{p+1+q-n}{2} - m} \tag{C.9}
\]
where we have also redefined $m \to \frac{p+1+q-n}{2} - m$.

\(^9\)Here we clearly have $0 \leq m \leq \frac{p+1}{2}$; however, in practice some of these terms may vanish. At any rate, the combinatorics will take care of the region of summation, so we will never write it explicitly.
Of course, what we really want is not $\tilde{\Lambda}$, but

$$\Lambda^{\mu_1 \cdots \mu_q} = - \left( \frac{1}{g-B^2} \right)^{\mu_1} \nu_1 \cdots \left( \frac{1}{g-B^2} \right)^{\mu_q} \nu_q \sum \tilde{\Lambda}_n^{\nu_1 \cdots \nu_q}$$

$$= \left( \frac{1}{g-B^2} \right)^{\mu_1} \nu_1 \cdots \left( \frac{1}{g-B^2} \right)^{\mu_q} \nu_q \sum_{n,m,n!m!(q-m)!(p+1-n-q+m)!(\frac{p+1-n-q}{2}+m)!} 32 q! (-1)^{\frac{m(m+1)}{2} + q + \frac{p+1}{2} + 1} \epsilon^{\sigma_1 \cdots \sigma_n} T_0 \cdots T_{p-n-q+m} \nu^{\nu_1 \cdots \nu_q} C^{(n)}_{\sigma_1 \cdots \sigma_n} \left( B^{\frac{p+1-n-q}{2}+m} \right)^{\nu_1 \cdots \nu_q} T_0 \cdots T_{p-n-q+m}, \quad (C.10)$$

where we have replaced $m \to q-m$ and rearranged the $\nu$ indices. To evaluate this, it is convenient to distinguish between an even and odd number of $\mu$s attached to $B^-$; i.e. $m$ even or odd. The point is that

$$\left( B^{\frac{p+1-n-q}{2}+2m} \right)^{\nu_1 \cdots \nu_{2m}} T_0 \cdots T_{p-n-q+2m} \cdot \quad (C.11a)$$

and

$$\left( B^{\frac{p+3-n-q}{2}+2m} \right)^{\nu_1 \cdots \nu_{2m+1}} T_0 \cdots T_{p+1-n-q+2m} \cdot \quad (C.11b)$$

So substituting the identities (C.11) into (C.10), gives\(^{10}\)

$$\Lambda^{\mu_1 \cdots \mu_q} = 32 \left( \frac{1}{g-B^2} \right)^{\mu_1} \nu_1 \cdots \left( \frac{1}{g-B^2} \right)^{\mu_q} \nu_q \sum_{n,m} \left\{ q! (-1)^{m+s+pq} \epsilon^{\sigma_1 \cdots \sigma_n} T_0 \cdots T_{p-n-q+2m} T_{p-n-q+2m+1} \nu^{\nu_1 \cdots \nu_q} C^{(n)}_{\sigma_1 \cdots \sigma_n} \
\times B^{\nu_1 \nu_2} \cdots B^{\nu_{2m+2} \nu_{2m+2}} T_{p-n-q+2m+1} \right\}.$$

\(^{10}\)We drop an overall sign of $(-1)^{\frac{p+1}{2}}$; this is of no physical consequence.
Now we set $m = t + s$, and also cyclically permute the $\tau$-indices between the last set of $B$s and the $g$s, to rewrite equation (C.12) as

$$\Lambda^{\mu_1 \cdots \mu_q} = 32 \left( \frac{1}{g-B} \right)^{\mu_1[\nu_1} \cdots \left( \frac{1}{g-B} \right)^{[\mu_q]\nu_q]} C_{\sigma_1 \cdots \sigma_n}^{(n)} \times \sum_{n,s,t} q! \left( -1 \right)^{t+pq} n! (q - 2t - 2s - 1)! 2^{\frac{p+1-n-q}{2} + 2t} \left( \frac{p+1-n-q}{2} + t \right)! (2s)! t!$$

$$\times \epsilon^{\sigma_1 \cdots \sigma_n \tau_0 \cdots \tau_{t-p-n}} B_{\nu_1\nu_2} \cdots B_{\nu_{2t+1}\nu_{2t+2}} B_{\nu_{2t+3}\tau_0} \cdots B_{\nu_{2t+3+s-1}\tau_{2t+1}}$$

$$\times g^{\nu_2 \nu_3 + 2t + 1 \tau_{2t+2}} \cdots g^{\nu_q \tau_{q-2t-1}} B_{\tau_{q-2t} \tau_{q-2t}} \cdots B_{\tau_{t-p-n} \tau_{t-p-n}}$$

$$= 32 \sum_{n,m} q! \frac{\theta^{[\mu_1\mu_2]} \cdots \theta^{[\mu_{2t+1}\mu_{2t+2}]} \left( \frac{1}{g-B} \right)^{\mu_{2t+1}[\nu_{2t+1}]} \cdots \left( \frac{1}{g-B} \right)^{[\mu_q]\nu_q]}}{n! 2^{[p+1-n-q]/2 + 2m} (q - 2m)! m! 2\pi \alpha'} \times \epsilon^{[\mu_{2m+1} \cdots \mu_q]} [\sigma_1 \cdots \sigma_n \tau_0 \cdots \tau_{t-p-n-q+2m} C_{\sigma_1 \cdots \sigma_n}^{(n)} B_{\tau_0 \tau_1} \cdots B_{\tau_{t-p-n-q+2m-1} \tau_{t-p-n-q+2m}} (C.14)$$

where we have noted that $\frac{1}{g-B} B_0 = -\frac{\theta}{2\pi \alpha'}$, and recognized the binomial theorem in the second step.

Replacing $n$ with $n - r$ and adding in the transverse indices plus the extra sign, $(-1)^{\frac{q}{2} r(r-1)}$, as discussed surrounding equation (C.5), turns equation (C.14) into equation (C.1). This is the desired result.
D. Integrals For the Two Open String Amplitude

Here we list the exact formulas for the integrals (3.15). Recall that $y = -\cot(\pi \tau)$.

\[
\int_{-\infty}^{\infty} dy \frac{A_2(y)}{1 + y^2} = \pi i 2^{\alpha' k_1 \cdot k_2} \int_0^1 d\tau |\cos \pi \tau|^{2\alpha' k_1 \cdot k_2} \exp \left\{ \frac{i}{2} (k_1 \times k_2) \left[ 1 - 2\tau - \epsilon \left( \frac{1}{2} - \tau \right) \right] \right\}
\]

\[
= \pi i 2^{\alpha' k_1 \cdot k_2 + 1} \int_0^{\frac{1}{2}} d\tau \cos^{2\alpha' k_1 \cdot k_2} \pi \tau \cos ((k_1 \times k_2)\tau)
\]

\[
= \pi i \frac{\Gamma(2\alpha' k_1 \cdot k_2 + 1)\Gamma(\alpha' k_1 \cdot k_2 + 1 - \frac{k_1 \times k_2}{2\pi})}{\Gamma(\alpha' k_1 \cdot k_2 + 1 + \frac{k_1 \times k_2}{2\pi})}, \quad \text{(D.1a)}
\]

\[
\int_{-\infty}^{\infty} dy \frac{A_2(y)}{y(1 + y^2)} = -\frac{(k_1 \times k_2) \Gamma(2\alpha' k_1 \cdot k_2)}{\Gamma(\alpha' k_1 \cdot k_2 + 1 + \frac{k_1 \times k_2}{2\pi}) \Gamma(\alpha' k_1 \cdot k_2 + 1 - \frac{k_1 \times k_2}{2\pi})}, \quad \text{(D.1b)}
\]

\[
(1 + \alpha') \int_{-\infty}^{\infty} dy \frac{A_2(y)}{y^2} = -\pi i \frac{4\Gamma(2\alpha' k_1 \cdot k_2)}{\Gamma(\alpha' k_1 \cdot k_2 + 1 + \frac{k_1 \times k_2}{2\pi}) \Gamma(\alpha' k_1 \cdot k_2 - \frac{k_1 \times k_2}{2\pi})}. \quad \text{(D.1c)}
\]

On the right-hand sides of equations (D.1), we have suppressed the factor of $(2\pi)^{\nu + 1} \delta(k_1 + k_2 + q_\parallel)$. We obtain equations (3.15) by taking $\alpha' k_1 \cdot k_2 \to 0$ and using the identity

\[
\frac{\Gamma(2\alpha' k_1 \cdot k_2 + 1)}{\Gamma(\alpha' k_1 \cdot k_2 + 1 + \frac{k_1 \times k_2}{2\pi}) \Gamma(\alpha' k_1 \cdot k_2 + 1 - \frac{k_1 \times k_2}{2\pi})} = \frac{\sin \frac{k_1 \times k_2}{2}}{\frac{k_1 \times k_2}{2}} + O(\alpha'). \quad \text{(D.2)}
\]

We recognize the right-hand side as the $\ast_2$-operation.\[30, 31]\]

E. Comparison of the Amplitude to the Action

In this appendix, we compare the action (4.2) to the amplitude (3.21). In section E.1, we rewrite the amplitude (3.21) in such a way as to make the first few terms in an expansion of (4.2) manifest. For the reader's edification, we explicitly expand the action (4.2), in section E.2, to quadratic order in the open string modes.

E.1 The Explicit Form of the Amplitude

The amplitude we computed in section 3 corresponds to the action (4.1), which we write explicitly as

\[
S = \frac{\lambda}{2} k_{10} \mu \rho \int \sqrt{-\det g} \text{Tr} \left[ \Lambda + 2\pi \alpha' f_{\mu\nu} \frac{1}{2!} \Lambda^{\mu\nu} + 2\pi \alpha' D_\mu \phi_1 \Lambda^{\mu i} + 2\pi \alpha' i [\phi_1, \phi_j]_{\ast} \Lambda^{ij} \\
+ \frac{1}{2} (2\pi \alpha')^2 (f \wedge f)_{\mu\nu\rho\sigma} \Lambda^{\mu\nu\rho\sigma} + (2\pi \alpha')^2 f_{\mu\nu} D_\rho \phi_1 \Lambda^{\mu\nu\rho i} + \frac{1}{2} (2\pi \alpha')^2 D_\mu \phi_1 D_\nu \phi_j \Lambda^{\mu\nu\rho i} \right].
\]

\[(E.1)\]
where it is understood that we should only keep terms up to quadratic in open string modes. Recalling equation (2.33),
\[
\omega_{\mu_1 \ldots \mu_q} \chi_{i_1 \ldots i_r} \Lambda_{\mu_1 \ldots \mu_q i_1 \ldots i_r} = 32 \star \left[ (e^{-i\theta/(2\pi\alpha')} \omega) (i\chi C) e^B \right] \bigg|_{0\text{-form}} \tag{2.33}
\]
we can immediately rewrite equation (E.1) as,
\[
S = \kappa_{10p} \text{Tr} \left\{ \left( \int e^{-i\theta/(2\pi\alpha')} \left( 1 + 2\pi\alpha' f + \frac{1}{2} (2\pi\alpha' f)^2 \right) \right) C e^B \right.
\]
\[
+ \int i_{2\pi\alpha' [\phi_i, \phi_j]} C e^B \bigg) \right.
\]
\[
+ \frac{\lambda}{2} \int \sqrt{-\det g} \left[ 2\pi\alpha' D_\mu \phi_i \Lambda^{\mu i} + (2\pi\alpha')^2 \frac{f_{\mu\nu}}{2!} D_\rho \phi_i \Lambda^{\mu\nu\rho i} + \frac{1}{2} (2\pi\alpha')^2 D_\rho \phi_i D_\nu \phi_j \Lambda^{\mu \nu i j} \right] \bigg\}.
\tag{E.2}
\]
All but the last line of equation (E.2) is clearly the quadratic expansion of
\[
S = \kappa_{10p} \text{Tr} \left( e^{-i\theta/(2\pi\alpha')} e^{-2\pi\alpha' f} \right) e^{-i_{2\pi\alpha' [\phi, \phi]} C e^B}. \tag{E.3}
\]
The second line of equation (E.2) will contain the pullback terms; however, this is somewhat more difficult to see because it involves objects which have both longitudinal and transverse indices, while the identity (2.33) separates those in indices. Nevertheless, we can use the identity (2.33) if we include the indices explicitly (cf. equation (C.1)). For the first term of the second line of equation (E.2), we find
\[
\frac{\lambda}{2} \int \sqrt{-\det g} \left[ 2\pi\alpha' D_\mu \phi_i \Lambda^{\mu i} \right] = \int \frac{D_\mu X^i (C e^B)_{i\nu_1 \ldots \nu_p}}{p!} dx^\mu dx^{\nu_1} \cdots dx^{\nu_p} \right]. \tag{E.4}
\]
We indeed recognize equation (E.4) as the first term in the pullback \( \mathcal{P} C e^B \); see equation (4.3). The second term similarly gives
\[
\frac{\lambda}{2} \int \sqrt{-\det g} \left[ (2\pi\alpha')^2 \frac{f_{\mu\nu}}{2!} D_\rho \phi_i \Lambda^{\mu\nu\rho i} \right]
\]
\[
= \int \left[ \frac{2\pi\alpha' f_{\mu\nu} (D_\rho X^i) (C e^B)_{i\sigma_1 \ldots \sigma_{p-2}}}{2!(p-2)!} dx^\mu dx^\nu dx^\rho dx^{\sigma_1} \cdots dx^{\sigma_{p-2}} \right.
\]
\[
\left. + \frac{3!}{2} \frac{\theta_{\mu\nu}}{2\pi\alpha'} \frac{f_{\mu\nu} (D_\rho X^i) (C e^B)_{i\sigma_1 \ldots \sigma_{p}}}{2! p!} dx^\rho dx^{\sigma_1} \cdots dx^{\sigma_p} \right]. \tag{E.5}
\]
Note not only the linear contribution to the pullback, with and without \( f \), but also the contribution from \( e^{-i\theta/(2\pi\alpha')} \) on the pullback (and \( F \)). Furthermore, equations (E.4) and (E.5) contain the only terms linear in the pullback and no more than quadratic in open string modes.

The final term is similar; it gives

\[
\lambda \int \sqrt{-\det g} \frac{1}{2}(2\pi\alpha')^2 D_\mu \phi_i D_\nu \phi_j \Lambda^{ij}
= \int \left[ \frac{D_\mu X^i D_\nu X^j (C e^B)}{2!(p - 1)!} \right] dx^\mu dx^\nu dx^{\sigma_1} \ldots dx^{\sigma_{p-1}}
+ \frac{1}{2\pi\alpha'} \frac{\theta^{\mu\nu} D_\mu X^i D_\nu X^j (C e^B)}{2!(p + 1)!}
\left. \right|_{i j \sigma_1 \ldots \sigma_{p+1}}
\]

and we see the quadratic approximation to the pullback, along with the relevant part of \( e^{-i\theta/(2\pi\alpha')} \), as in equation (4.2). Thus, indeed, equation (4.1) is the quadratic approximation to equation (4.2), as claimed.

### E.2 The Expansion of the Proposed Action

We can explicitly write the action (4.2) term by term. We find,

\[
\int f e^{2\pi\alpha' f(x)p} e^{-2\pi\alpha' u_i \phi \phi_i} C e^B = \int d^{p+1}x \sqrt{-\det g} e^{\mu_1 \ldots \mu_{p+1}}
\]

The final term is similar; it gives

\[
\lambda \int \sqrt{-\det g} \frac{1}{2}(2\pi\alpha')^2 D_\mu \phi_i D_\nu \phi_j \Lambda^{ij}
= \int \left[ \frac{D_\mu X^i D_\nu X^j (C e^B)}{2!(p - 1)!} \right] dx^\mu dx^\nu dx^{\sigma_1} \ldots dx^{\sigma_{p-1}}
+ \frac{1}{2\pi\alpha'} \frac{\theta^{\mu\nu} D_\mu X^i D_\nu X^j (C e^B)}{2!(p + 1)!}
\left. \right|_{i j \sigma_1 \ldots \sigma_{p+1}}
\]

and we see the quadratic approximation to the pullback, along with the relevant part of \( e^{-i\theta/(2\pi\alpha')} \), as in equation (4.2). Thus, indeed, equation (4.1) is the quadratic approximation to equation (4.2), as claimed.
where the omitted terms involve at least three open strings. Note that the explicit metric dependence cancels the metric dependence of the $\epsilon$-tensor.

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