EXCLUSION PROCESS WITH SLOW BOUNDARY

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ABSTRACT. We present the hydrodynamic and hydrostatic behavior of the Simple Symmetric Exclusion Process with slow boundary. The slow boundary means that particles can be born or die only at the boundary with rate proportional to $N^{-\theta}$, where $\theta \geq 0$ and $N$ is the scale parameter, while in the bulk the particles exchange rate is equal to 1. More precisely, in the hydrostatic scenario, we obtain three different linear profiles, depending on the value of the parameter $\theta$. We also prove that the time evolution of the spatial density of particles, in the diffusive scaling, is given by the unique weak solution of the heat equation with boundary conditions, which also depend on the range of the parameter $\theta$. If $\theta \in [0, 1)$, we get Dirichlet boundary condition; if $\theta = 1$, we get Robin boundary condition; and, if $\theta \in (1, \infty)$, we get Neumann boundary condition.

1. INTRODUCTION

The problem we address in the present paper is a complete characterization of the hydrostatic and hydrodynamic scenario for the Symmetric Simple Exclusion Process (SSEP) with slow boundary. The SSEP is described by particles that move as independent random walks in $\{1, \ldots, N-1\}$, under the exclusion rule, which says that two particles can not occupy the same site at the same time (the so-called fermions in Physics). Let us describe this process in terms of clocks: at each bond we associate a different Poisson clock with parameter 1, and suppose all of them are independent. When a clock rings, either the occupation at the sites connected by the corresponding bond are exchanged, if one of the sites is occupied and the other is not, or nothing happens if both sites are occupied or both are empty. The parameters of the Poisson clocks are called exchange rates. By slow boundary we mean that particles can enter or leave the system at the site 1 with rate equal to $\alpha/N^\theta$ or $(1-\alpha)/N^\theta$, respectively, while at the site $N-1$ particles can also enter or leave the system with rate equal to $\beta/N^\theta$ or $(1-\beta)/N^\theta$. We consider the parameters $\alpha, \beta$ in $(0, 1)$ and $\theta \geq 0$.

![Figure 1. The model.](image)

If one wants to produce mass transfer between two infinite reservoirs of particles with different densities, one way to do that is by connecting them. This connection will allow particles to change reservoirs and produce the desired mass transfer. There are many works that consider similar models. For example, in [4, 5, 6] the authors consider a model where deaths can only happen in an interval around the left boundary and births in an interval around the right boundary. Their model has some similarity with the case $\theta = 1$ described above. Two other similar models are presented in [8, 14]: they correspond to the case $\theta = 0$, which means that there is no slow boundary. Namely, in [14] they study the SSEP in $\{1, \ldots, N-1\}$ with births and deaths occurring only in the sites 1 and $N-1$ with fixed rates: in the left boundary particles are born with rate $\alpha$ and die with rate $1-\alpha$, while in the right boundary this rates are $\beta$ and $1-\beta$.

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In [3] the SSEP is defined in \(Z\) and the boundary is free, in a sense that particles can be born or die in different sites, depending on the current configuration. In that work there is no slow boundary either.

Finally, we also have the work [2], where there are no reservoirs, but it is worth to be mentioned here, because it presents a "battery effect" that produces a current of particles through the system. Such a system takes place in the discrete torus \(Z/NZ\), with \(N\) points, and the "battery effect" is due to a single modified bond \((N,1)\), where the jump rates to the right and to the left are different.

In the present paper, our goal is to understand the collective behavior of the time evolution of the microscopic system described above. In order to do this, we study the limit of the time evolution of the spatial density of particles, when we rescale the time and the space in a suitable way. This scaling procedure limit leads to the so-called hydrodynamic limit, which is usually characterized by the weak solution of some partial differential equation (PDE) called the hydrodynamic equation.

We observe that the model analysed here was motivated by a process considered in [11, 12], the SSEP with slow bond. In these two papers the authors consider the SSEP in the discrete torus, \(Z/NZ\), with \(N\) points, where an usual bond has an exchange rate equal to 1 and an unique bond (the slow bond) has a rate proportional to \(N^{-\theta}\). The papers [11, 12] show different hydrodynamic behaviors depending on the range of \(\theta\). In all of them the hydrodynamic equation is the heat equation, and boundary conditions vary depending on the value of \(\theta\), comprising three different boundary conditions. The intuitive idea is that if we "open" the discrete torus in the slow bond, then this bond will behave like a "boundary". In the case \(\theta = 1\), the boundary conditions found in [12] show a connection between the extremes 0 and 1. This motivates us to study the model considered in the present paper, where the boundary is disconnected.

Now, let us concentrate on the results of the present paper: as in [11, 12], the model we study here has three different hydrodynamical behaviors, depending on the range of \(\theta\): the first case is \(\theta \in [0,1)\), where we obtain as hydrodynamic equation the heat equation with Dirichlet boundary condition:

\[
\begin{align*}
\partial_t \rho(t,u) &= \partial_u^2 \rho(t,u), \\
\rho(t,0) &= \alpha, \quad \rho(t,1) = \beta.
\end{align*}
\]

We notice that the case \(\theta = 0\) has already been considered in [9]. Here in our paper the rate in the slow boundary is smaller than in the case \(\theta = 0\), but not small enough to modify the hydrodynamic equation, and the system behaves in the same way, with the same Dirichlet boundary condition obtained in [9]. We observe that our techniques can also be used in the case \(\theta = 0\).

However, when we analyse the second case, where the parameter \(\theta\) belongs to the set \((1, \infty)\), the rate in the slow boundary is much smaller than in the previous case, and this makes births and deaths of particles rare. In the macroscopic level, the boundary is isolated, and we get Neumann boundary condition:

\[
\begin{align*}
\partial_t \rho(t,u) &= \partial_u^2 \rho(t,u), \\
\partial_u \rho(t,0) &= \partial_u \rho(t,1) = 0.
\end{align*}
\]

We also have a third case, which is the critical value: \(\theta = 1\). As in [11, 12], here we have an interesting case, where the model has a scale that produces a different macroscopic behavior, given by the heat equation with the following Robin boundary condition:

\[
\begin{align*}
\partial_t \rho(t,u) &= \partial_u^2 \rho(t,u), \\
\partial_u \rho(t,0) &= \rho(t,0) - \alpha, \\
\partial_u \rho(t,1) &= \beta - \rho(t,1).
\end{align*}
\]

As we already observed, the model analysed in [4, 5] has some relations with our model, but while they consider a case where the rate of births and deaths in the vicinity of the boundary corresponds to our case \(\theta = 1\), they obtain a Dirichlet boundary condition, which is different of the Robin boundary condition we have here. This is not a contradiction, because the models are different: in [4, 5] there is only entrance of particles in the right side of the boundary and leaving in the left side, while in ours we have both entrance and leaving in each side, but with different rates.

Also, the boundary condition in (3) shows that the rate of mass transfer through each side of the boundary in our model depend only on what is happening in that side of the boundary. This is a different behavior when compared to the boundary condition obtained in the case \(\theta = 1\) for the slow
be denoted by \( \partial_x \rho (t, 0) = \rho (t, 0) - \rho (t, 1) = \partial_x \rho (t, 1) \). We stress the fact that in the slow bond model the two sides of the boundary are connected, which is not what happens in the case considered here.

There is another important difference between the slow bond and the slow boundary processes. This difference is that in the SSEP with slow bond the Bernoulli product measure is an invariant measure, but, here, for the SSEP with slow boundary, the Bernoulli product is only an invariant measure if \( \alpha = \beta \). In the general case, \( \alpha \neq \beta \), the invariant measure is given in [7].

As one can think, in the proof of the hydrodynamic limit the main difficulty is in the characterization of the limit points, and this difficulty is caused by the boundary terms. We overcome this difficulty using Replacement Lemmas and Energy Estimates. However, as we already said, our model differs from the one considered in [11, 12] by the fact that in our model the invariant measure is not the Bernoulli product measure. Even if the Bernoulli product measure is not an invariant measure in the general case (\( \alpha \neq \beta \)), we use it to prove the Replacement Lemma and Energy Estimates. The use of another measure instead of the invariant measure has some cost, but fortunately it is small and vanishes in the limit. For the Replacement Lemma in the case \( \theta \in [0, 1) \), we need to take the Bernoulli product with a special parameter as in the work [9].

In order to understand the behavior of these invariant measures, we look at the profile associated to them. This is what we mean by hydrostatic limit. Even if [7] gives a characterization for these measures, we choose to follow a different path that does not rely on their explicit expression. As it is expected, the invariant measures are associated to the initial conditions of the hydrodynamic equations, which have solutions that are stationary in time.

The unexpected case is \( \theta = 1 \), where the profile \( \rho \) is linear with \( \rho (0) = (2\alpha + \beta)/3 \) and \( \rho (1) = (\alpha + 2\beta)/3 \). Here, it is possible to observe the influence that each side of the boundary creates. The case \( \theta = 0 \) has been considered in [14], where the authors proved fluctuations for the model. We get, for \( \theta \in [0, 1) \), a profile which is equal to theirs: it is linear with \( \rho (0) = \alpha \) and \( \rho (1) = \beta \). If \( \theta > 1 \) the profile is constant equal to \( (\alpha + \beta)/2 \).

The present work is divided as follows. In Section 2, we introduce notation. In Section 3, we prove the hydrostatic behavior stated in the Theorem 3.2. In Section 4, we make precisely the scaling limit and sketch the proof of the Theorem 4.5. The remaining of this paper is dedicated to the proof of Theorem 4.5. In Section 5, we prove tightness for any range of the parameter \( \theta \). In Section 6, we prove the Replacement Lemma and we establish the Energy Estimates, which are fundamental steps towards the proof. In Section 7, we characterize the limit points as weak solutions of the corresponding partial differential equations. Finally, uniqueness of weak solutions is refereed to Section 8.

2. Notations

For any \( N \geq 1 \), let \( I_N := \{1, \ldots, N-1\} \) be a subset of \( \mathbb{N} \) with \( N-1 \) points. The sites (points) of \( I_N \) will be denoted by \( x, y \) and \( z \), while the macroscopic variables (points of the interval \([0, 1]\)) will be denoted by \( u, v \) and \( w \). The microscopic state will be denoted by \( \{0, 1\}^{I_N} \); elements of \( \{0, 1\}^{I_N} \), which are called configurations, will be denoted by \( \eta \) and \( \xi \). Therefore, \( \eta (x) \in \{0, 1\} \) represents the number of particles in site \( x \) for the configuration \( \eta \). Given a Markov process with initial measure \( \mu \), we denote by \( \mathbb{P}_\mu \) or \( \mathbb{Q}_\mu \) the probability measure induced in the trajectory space by the process with initial distribution \( \mu \) and by \( E_\mu \), the expectation with relation to \( \mathbb{P}_\mu \) or \( \mathbb{Q}_\mu \).

The exclusion process with slow boundary can be informally described in the following way: there exist at most one particle in each site of \( I_N \), and this particle can move to any one of the two sites of its neighborhood, if such sites are empty, with rate 1 (for each empty neighbor site). Also, a particle which is in the left border (site 1) can leave the system with rate \((1-\alpha)/N^\theta\), while the site 1, if empty, can receive a particle with rate \( \alpha/N^\theta \). Analogously, we have the same behavior in the right border (site \( N-1 \)), with rate of leaving \((1-\beta)/N^\theta\) and rate of entering \( \beta/N^\theta \). The parameters \( \alpha, \beta \in (0, 1) \) and \( \theta \geq 0 \) are fixed constants.

The purpose of this work is to study of the hydrodynamical limit of this process. For doing that, we consider the process in \( \{1/N, 2/N, \ldots, (N-1)/N\} \) and take the limit \( N \to \infty \). We will show that the density of particles in \([0, 1]\) (which will be precisely defined in the following) converges (in a sense
that will also be defined) to the (unique) weak solution of an evolution PDE, depending on the value of \( \theta \geq 0 \).

We can define the one-dimensional exclusion process with slow open boundary as a Markov process defined in the state space \( \{0, 1\}^{I_N} \), which has an infinitesimal generator given by the operator \( L_N \), that sends \( f : \{0, 1\}^{I_N} \to \mathbb{R} \) to

\[
(L_N f)(\eta) = (L_{N,0} f)(\eta) + (L_{N,b} f)(\eta),
\]

where

\[
(L_{N,0} f)(\eta) = \sum_{x=1}^{N-2} [f(\eta^{x,x+1}) - f(\eta)],
\]

with

\[
(\eta^{x,x+1})(y) = \begin{cases} 
\eta(x+1), & \text{if } y = x, \\
\eta(x), & \text{if } y = x+1, \\
\eta(y), & \text{otherwise},
\end{cases}
\]

is the infinitesimal generator of the bulk dynamic, and

\[
(L_{N,b} f)(\eta) = \left[ \frac{\alpha}{N^2} (1 - \eta(1)) + \frac{1 - \alpha}{N^2} \eta(1) \right] [f(\eta^1) - f(\eta)] + \left[ \frac{\beta}{N^2} (1 - \eta(N-1)) + \frac{1 - \beta}{N^2} \eta(N-1) \right] [f(\eta^{N-1}) - f(\eta)],
\]

with

\[
(\eta^x)(y) = \begin{cases} 
1 - \eta(x), & \text{if } y = x, \\
\eta(y), & \text{otherwise},
\end{cases}
\]

is the infinitesimal generator of the boundary dynamic.

Denote by \( \{\eta_t^N := \eta_t^N : t \geq 0\} \) the Markov process with state space \( \{0, 1\}^{I_N} \) with infinitesimal generator \( L_N \) speeded up by \( N^2 \). Although \( \eta_t^N \) depends on \( \alpha, \beta \) and \( \theta \), we shall omit these index in order to simplify the notation. We can also write \( \eta_t \) for the process \( \eta_t^N \) when it will not cause ambiguities. Notice that we can view \( \eta_t^N \) as a random function in \( D(\{0, 1\}^{I_N} \mathbb{R}^+) \), it is the space of right continuous trajectories with left limits taking values in \( \{0, 1\}^{I_N} \), where for each time \( t \) we associate the configuration \( \eta_t^N \in \{0, 1\}^{I_N} \).

### 3. Hydrostatic Limit

A straightforward computation shows that, when \( \alpha = \beta \), the Bernoulli product measure with parameter \( \alpha \), defined by the fact that the random variables \( \{\eta(x), x \in I_N\} \) are independent and have distribution Bernoulli(\( \alpha \)), is reversible for the dynamics. However, when \( \alpha \neq \beta \) the Bernoulli product measure is not invariant.

In this section we prove that the invariant measures are associated to a linear profile depending on \( \theta \), in the following sense:

**Definition 3.1.** A sequence of probability measures \( \{\mu_N : N \geq 1\} \) in the space \( \{0, 1\}^{I_N} \) is associated to the density profile \( \gamma : [0, 1] \to [0, 1] \) if, for all \( \delta > 0 \) and continuous function \( H : [0, 1] \to \mathbb{R} \) the following relation holds:

\[
\lim_{N \to \infty} \mu_N \left[ \eta : \left| \frac{1}{N} \sum_{x=1}^{N-1} H(\frac{x}{N}) \eta(x) - \int_{[0,1]} H(u) \gamma(u) \, du \right| > \delta \right] = 0.
\]

**Theorem 3.2.** Let \( \mu_N \) be the probability measure in \( \{0, 1\}^{I_N} \) invariant for the Markov process with infinitesimal generator given by \( N^2 L_N \), this operator was defined in (4). Then the sequence \( \mu_N \) is associated to the profile \( \bar{\gamma} : [0, 1] \to \mathbb{R} \) given by

\[
\bar{\gamma}(u) = \begin{cases} 
(\beta - \alpha)u + \alpha, & \text{if } \theta \in [0, 1), \\
\frac{\beta - \alpha}{3} u + \alpha + \frac{\beta - \alpha}{3}, & \text{if } \theta = 1, \\
\frac{\beta + \alpha}{3}, & \text{if } \theta \in (1, \infty),
\end{cases}
\]

for all \( u \in [0, 1] \).
The profiles in (5) are precisely the stationary solutions of the hydrodynamic equations (1), (3) and (2), these equations will be presented in the next section in a complete way.

The idea of the proof is to compare \( \eta(x) \) with \( \rho^N(x) := E_{\mu_N}[\eta(x)] \) and \( \rho^N \) with \( \bar{\rho} \). In order to do this, we will need characterized \( \rho^N \), it is done in Lemma 3.3. Moreover, we will need to study the covariances \( \varphi^N(x,y) := E_{\mu_N}[\eta(x)\eta(y)] - E_{\mu_N}[\eta(x)]E_{\mu_N}[\eta(y)] \), it is done in Lemma 3.4.

We start with Lemma 3.3, where we compute explicitly the mean \( \rho^N(x) \), which turns out to be a discrete approximation to the stationary solution of the hydrodynamic equation.

**Lemma 3.3.** Let \( x \in I_N := \{1, \ldots, N-1\} \) and \( \rho^N(x) := E_{\mu_N}[\eta(x)] \). Then

\[
\rho^N(x) = a_N x + b_N,
\]

where

\[
a_N = \frac{\beta - \alpha}{2N^\theta + N - 2}
\]

and

\[
b_N = \alpha + a_N(N^\theta - 1).
\]

From the statement of the last lemma is easy to show that \( \lim_{N \to \infty} \max_{x \in I_N} |\rho^N(x) - \bar{\rho}(x/N)| = 0 \) (it is enough to check for \( x = 1 \) and \( x = N - 1 \)). This will be used in the proof of Theorem 3.2.

**Proof of Lemma 3.3** Fix \( x \in I_N \). Let \( f_x : \{0, 1\}^I \to \mathbb{R} \) be given by \( f_x(\eta) = \eta(x) \). Making simple computations, we have that the infinitesimal generator of the bulk dynamic on the function \( f_x \) is equal to

\[
L_{N,0}f_x(\eta) = \begin{cases} 
\eta(x + 1) - 2\eta(x) + \eta(x - 1), & \text{if } 1 < x < N - 1, \\
\eta(2) - \eta(1), & \text{if } x = 1, \\
\eta(N - 2) - \eta(N - 1), & \text{if } x = N - 1.
\end{cases}
\]

and the infinitesimal generator of the boundary dynamic on the function \( f_x \) is equal to

\[
L_{N,b}f_x(\eta) = \begin{cases} 
0, & \text{if } 1 < x < N - 1, \\
\frac{\alpha - \eta(1)}{N^\theta}, & \text{if } x = 1, \\
\frac{\beta - \eta(N - 1)}{N^\theta}, & \text{if } x = N - 1.
\end{cases}
\]

Since \( \mu_N \) is an invariant measure, \( E_{\mu_N}[L_{N,0}f_x] = 0 \). Substituting the formulas for \( L_{N,0}f_x \) and \( L_{N,b}f_x \) given above we get the following recurrence equation for \( \rho^N(x) \):

\[
\begin{align*}
0 &= \rho^N(2) - \rho^N(1) + (\alpha - \rho^N(1))N^{-\theta}, \\
0 &= \rho^N(x + 1) - 2\rho^N(x) - \rho^N(x - 1), & \text{if } 1 < x < N - 1, \\
0 &= \rho^N(N - 2) - \rho^N(N - 1) + (\beta - \rho^N(N - 1))N^{-\theta}.
\end{align*}
\]

A direct verification shows that (7) and (8) solve the above equation, thus concluding the proof.

We now turn to the estimate of the covariances of the occupation variables \( \eta(x) \), a key to the proof of Theorem 3.2.

**Lemma 3.4.** Let \( 1 \leq x < y \leq N - 1 \). Denote by \( \varphi^N(x,y) \) the covariance of the random variables \( \eta(x) \) and \( \eta(y) \) in the stationary state \( \mu_N \):

\[
\varphi^N(x,y) := E_{\mu_N}[\eta(x)\eta(y)] - E_{\mu_N}[\eta(x)]E_{\mu_N}[\eta(y)].
\]

Then

\[
\lim_{N \to \infty} \max_{0 < x < y < N} |\varphi^N(x,y)| = 0. \tag{9}
\]
Proof. Let \(1 \leq x < y \leq N-1\) and \(f_{xy}: \{0,1\}^I \rightarrow \mathbb{R}\) be defined by
\[
f_{xy}(\eta) := [\eta(x) - \rho_N(x)][\eta(y) - \rho_N(y)].
\]

As in the last proof we will compute the infinitesimal generator of the bulk dynamic on the function \(f_{xy}\):
\[
L_{N,0} f_{xy}(\eta) = [\eta(x - 1) - \eta(x)][\eta(y) - \rho_N(y)]1\{x > 1\} + [\eta(x + 1) - \eta(x)][\eta(y) - \rho_N(y)] + [\eta(x) - \rho_N(x)][\eta(y - 1) - \eta(y)] + [\eta(x) - \rho_N(x)][\eta(y + 1) - \eta(y)]1\{y < N-1\}.
\]

And, the infinitesimal generator of the boundary dynamic on the function \(f_{xy}\):
\[
L_{N,b} f_{xy}(\eta) = N^{-\theta}[\alpha(1 - \eta(1)) + (1 - \alpha)\eta(1)] \cdot [1 - 2\eta(1)] \cdot [\eta(y) - \rho_N(y)]1\{x = 1\} + N^{-\theta}[\beta(1 - \eta(N-1)) + (1 - \beta)\eta(N-1)] \cdot [1 - 2\eta(N-1)].
\]

As the measure \(\mu_N\) is invariant, we have \(\mathbb{E}_{\mu_N}[L_{N} f_{xy}] = 0\). Using the formulas above we obtain the following system of equations involving the covariances \(\varphi_N(x, y)\) (below \(||(x', y') - (x, y)||_1 := |x' - x| + |y' - y|\)):
\[
\begin{align*}
4\varphi_N(x, y) &= \sum_{||(x', y')-(x, y)||_1=1} \varphi_N(x', y') , \quad \text{if} \quad 1 < x < y < N-1 \quad \text{and} \quad x + 1 \neq y, \\
(3 + N^{-\theta})\varphi_N(1, y) &= \varphi_N(2, y) + \varphi_N(1, y + 1) + \varphi_N(1, y - 1), \quad \text{if} \quad 2 < y < N-1, \\
(3 + N^{-\theta})\varphi_N(x, N-1) &= \varphi_N(x - 1, N-1) + \varphi_N(x + 1, N-1) + \varphi_N(x, N-2), \quad \text{if} \quad 1 < x < N-2, \\
(2 + 2N^{-\theta})\varphi_N(1, N-1) &= \varphi_N(2, N-1) + \varphi_N(1, N-2), \\
\alpha_N^2 &= \varphi_N(x - 1, x + 1) + \varphi_N(x, x + 2) - 2\varphi_N(x, x + 1), \quad \text{if} \quad 1 < x < N-1, \\
\alpha_N^3 &= \varphi_N(1, 3) - (1 + N^{-\theta})\varphi_N(1, 2), \\
\alpha_N^4 &= \varphi_N(N-3, N-1) - (1 + N^{-\theta})\varphi_N(N-2, N-1).
\end{align*}
\]

This system of equations has a probabilistic interpretation: if we set \(V := \{(x, y) \in \mathbb{Z}^2 : 0 \leq x < y \leq N\}, \partial V := \{(x, y) \in V : x = 0 \text{ or } y = N\}\) and declare \(\varphi_N\) to be zero in \(\partial V\), then the system of equations can be written as
\[
A_N \varphi_N(x, y) = \alpha_N^2 \varphi_N(y = x + 1),
\]
where \(a_N = \rho^N(1) - \rho^N(0)\) and \(A_N\) is the infinitesimal generator of a continuous time random walk on \(V\) which is absorbed at \(\partial V\) and has the following jump rates: if \((x, y) \in V \setminus \partial V\), the random walk jumps from \((x, y)\) to the nearest neighbor \((x', y')\) at rate 1 if \((x', y') \notin \partial V\) and at rate \(N^{-\theta}\) otherwise.

In the classical case \(\theta = 0\), the system (12) is a discrete Poisson equation on \(V\), its unique solution can be easily checked to be
\[
\varphi_N(x, y) = -\frac{\alpha_N^2}{N-1} x(N - y).
\]

In that case (9) is readily verified to be true, see (14).

Let \((X_t)_{t \geq 0}\) be a random walk with infinitesimal generator \(A_N\) starting from \((x, y)\), and denote \(D := \{(x, y) \in V : y = x + 1\}\). Then, for all \(t \geq 0\)
\[
\varphi_N(X_t) - \varphi_N(X_0) - \alpha_N^2 \int_0^t 1\{X_s \in D\} \, ds
\]
is a martingale with respect to the canonical filtration. Taking expectations it follows
\[
\varphi_N(x, y) = \mathbb{E}_{(x, y)}[\varphi_N(X_t)] - \alpha_N^2 \mathbb{E}_{(x, y)} \left[ \int_0^t 1\{X_s \in D\} \, ds \right].
\]
Almost surely, the random walk will be absorbed at \( \partial V \), where \( \varphi^N \) vanishes. Therefore, taking the limit \( t \to \infty \),

\[
\varphi^N(x, y) = -a_N^2 \cdot E_{(x,y)} \left[ \int_0^{\infty} 1\{X_s \in D\} \, ds \right].
\]  

(14)

Now, we need to estimate the occupation time of the diagonal \( D \). We know from equation (13) this occupation time when \( \theta = 0 \). Then, we define a new Markov process \( (Z_s)_{s \geq 0} \) with state space \( V \times N \). Let \( \{B_n\}_{n \in \mathbb{N}} \) be a sequence of independent and identically distributed Bernoulli(\( N - \theta \)) random variables, and \( \{X^n\} \) be a sequence of independent continuous time random walks in \( V \) that are absorbed in \( \partial V \) with rate 1 and are independent of the Bernoulli sequence \( \{B_n\}_{n \in \mathbb{N}} \). We set \( \tau_n = \inf\{s : X^n_s \in \partial V\} \), \( \xi_n = \sum_{k=1}^n \tau_k \) and \( Y = \inf\{k : B_k = 1\} \). We define the process \( Z_s \) beginning in the point \( (x, y, 1) \in V \times N \) as

\[
Z_s = \begin{cases} 
(X^1_s, 1), & \text{if } s < \tau_1, \\
(X^n_{N-\xi_{N-1}}, n), & \text{if } \xi_{N-1} \leq s < \xi_n, \quad 2 \leq n \leq Y, \\
(X^Y_s, Y), & \text{if } s \geq \xi_Y,
\end{cases}
\]

where in the second line above, the process \( X^n \) begins in the point \( (X^N_{\tau_{N-1}}, 1) \in V \).

There are some important things to note about the random walk \( Z_s \). First observe that the time this process spends in the diagonal \( D \times \{n\} \) is the same as the time spent in the diagonal by the random walk \( X^n \), and these times have expectation bounded by \( CN \) independently of the initial point (see equations (13) and (14)). Another important point is that if we begin the process \( X^n \) in a point \( (x, y) \) where \( \partial V \) can be achieved in only one jump (\( x = 1 \) or \( y = N-1 \)), we can bound the expectation of the diagonal occupation time by a constant \( C \). This is the case when \( n \geq 2 \). Besides, it is clear that the distribution of \( Y \) is Geometric(\( N - \theta \)).

For \( i = 1, 2, \ldots \) let

\[
T^{(i)} := \int_0^{\infty} 1\{Z_s \in D \times \{i\}\} \, ds.
\]

By the construction of the process \( (Z_s)_{s \geq 0} \), the distribution of \( T^{(i)} \) is that of the killing time of a simple random walk on \( V \) absorbed in \( \partial V \), whose starting point depends on the past of \( (Z_s)_{s \geq 0} \). Using (13) and (14), we can bound \( E_{(x,y)}[T^{(i)}] \) by a positive constant \( C \), if \( i \geq 2 \) and by \( CN \) if \( i = 1 \), that does not depend on \( x, y \).
By projecting the process $Z_s$ in $V$ we conclude that $T^{(1)} + \cdots + T^{(Y)}$ has the same distribution as $\int_0^\infty 1\{X_s \in D\} \, ds$. We bound
\[
\mathbb{E}_{(x,y)}[T^{(1)} + \cdots + T^{(Y)}] = \sum_{n \geq 1} \mathbb{E}_{(x,y)}[(T^{(1)} + \cdots + T^{(n)}) 1\{Y = n\}] \\
\leq \sum_{n \geq 1} C N \mathbb{P}(Y = n) + \sum_{n \geq 1} C(N-1) \mathbb{P}(Y = n) \\
\leq C(N + N^0).
\]

Now we use equation (14) to get $|\varphi^N(x,y)| \leq C(N + N^0)a_N^2$. It can be checked using (7) that (9) holds, thus concluding the proof.

\[\square\]

**Proof of Theorem 3.2** Adding and subtracting $\frac{1}{N} \sum_{x=1}^{N-1} H(\frac{x}{N}) \rho^N(x)$ and using the Markov’s inequality,
\[
\mu_N \left[ \eta : \frac{1}{N} \sum_{x=1}^{N-1} H(\frac{x}{N}) \eta(x) - \int_{[0,1]} H(u) \rho(u) \, du \right] > \delta \right] \\
\leq \delta^{-1} \mathbb{E}_{\mu_N} \left[ \left| \frac{1}{N} \sum_{x=1}^{N-1} H(\frac{x}{N}) (\eta(x) - \rho^N(x)) \right| \right] \\
+ \delta^{-1} \left| \frac{1}{N} \sum_{x=1}^{N-1} H(\frac{x}{N}) \rho^N(x) - \int_{0}^{1} H(u) \rho(u) \, du \right|.
\]

By Lemma 3.3 we have $\lim_{N \to \infty} \max_{x \in I_N} |\rho^N(x) - \rho(x/N)| = 0$. It follows that the second term above converges to zero as $N \to \infty$ (a way to see this is to approximate the integral by a Riemann sum). For the first term, we use Cauchy-Schwarz and the fact that $|\eta(x) - \rho^N(x)| \leq 1$ to compute
\[
\mathbb{E}_{\mu_N} \left[ \left| \frac{1}{N} \sum_{x=1}^{N-1} H(\frac{x}{N}) (\eta(x) - \rho^N(x)) \right| \right] \leq \mathbb{E}_{\mu_N} \left[ \left( \frac{1}{N} \sum_{x=1}^{N-1} H(\frac{x}{N}) (\eta(x) - \rho^N(x)) \right)^2 \right]^{1/2} \\
\leq \|H\|_{\infty} \left( \frac{1}{N} + \frac{2 \max_{1 \leq x \leq y \leq N-1} |\varphi^N(x,y)|}{1} \right)^{1/2},
\]
which converges to zero by Lemma 3.4.

\[\square\]

4. HYDRODYNAMIC LIMIT

We denote by $\langle \cdot, \cdot \rangle$ the $L^2[0,1]$ inner product. When we consider $L^2[0,1]$ with respect to a measure $\mu$, we denote by $\langle \cdot, \cdot \rangle_\mu$ its inner product. For $I$ an interval in $\mathbb{R}$ and integers $n$ and $m$, we denote by $C^{n,m}([0,T] \times I)$ the set of functions defined on $[0,T] \times I$ that are $n$ times differentiable on the first variable and $m$ on the second one. An index on a function will always denote a variable, not a derivative. For example, $H_s(u)$ means $H(s,u)$. The derivative of $H \in C^{1,2}([0,T] \times I)$ will be denoted by $\partial_s H$ (first variable) and $\partial_u H$ (second variable). We shall write $\Delta H$ for $\partial^2_s H$. We also have to consider the subset $C_{0,0}^{1,2}([0,1] \times [0,1])$ of functions $H \in C^{1,2}([0,1] \times [0,1])$ such that $H_t(0) = 0 = H_t(1)$, for all $t \in [0,T]$.

**Definition 4.1.** The space $L^2(0,T;\mathcal{H}^1(0,1))$ is the set of measurable functions $f : [0,T] \to \mathcal{H}^1(0,1)$ such that
\[
\int_0^T \|f_t\|_{\mathcal{H}^1(0,1)}^2 \, dt < +\infty.
\]
Definition 4.2. Let \( \gamma : [0, 1] \to \mathbb{R} \) be a continuous function. We say that a bounded function \( \rho : [0, T] \times [0, 1] \to \mathbb{R} \) is a weak solution of the heat equation with Dirichlet boundary condition

\[
\begin{align*}
    \partial_t \rho &= \Delta \rho, \\
    \partial_a \rho_t(0) &= -\alpha + \rho_t(0), \quad \forall t \in [0, T], \\
    \partial_a \rho_t(1) &= \beta - \rho_t(1), \quad \forall t \in [0, T], \\
    \rho(0, \cdot) &= \gamma(\cdot),
\end{align*}
\]

if \( \rho \in L^2(0, T; \mathcal{H}^1(0, 1)) \) and for \( t \in [0, T] \) and \( H \in C^{1,2}_0([0, T] \times [0, 1]), \rho_t(\cdot) \) satisfies

\[
\begin{align*}
    \langle \rho_t, H_t \rangle - \langle \gamma, H_0 \rangle &= \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle ds \\
    &+ \int_0^t \left\{ \rho_s(0)\partial_a H_s(0) - \rho_s(1)\partial_a H_s(1) \right\} ds \\
    &+ \int_0^t \left\{ \beta \partial_a H_s(1) - \alpha \partial_a H_s(0) \right\} ds.
\end{align*}
\]

Definition 4.3. Let \( \gamma : [0, 1] \to \mathbb{R} \) be a continuous function. We say that a bounded function \( \rho : [0, T] \times [0, 1] \to \mathbb{R} \) is a weak solution of the heat equation with Robin boundary condition

\[
\begin{align*}
    \partial_t \rho &= \Delta \rho, \\
    \partial_a \rho_t(0) &= -\alpha + \rho_t(0), \quad \forall t \in [0, T], \\
    \partial_a \rho_t(1) &= \beta - \rho_t(1), \quad \forall t \in [0, T], \\
    \rho(0, \cdot) &= \gamma(\cdot),
\end{align*}
\]

if \( \rho \in L^2(0, T; \mathcal{H}^1(0, 1)) \) and for \( t \in [0, T] \) and \( H \in C^{1,2}_0([0, T] \times [0, 1]), \rho_t(\cdot) \) satisfies

\[
\begin{align*}
    \langle \rho_t, H_t \rangle - \langle \gamma, H_0 \rangle &= \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle ds \\
    &+ \int_0^t \left\{ \rho_s(0)\partial_a H_s(0) - \rho_s(1)\partial_a H_s(1) \right\} ds \\
    &+ \int_0^t \left\{ \beta \partial_a H_s(1) - \alpha \partial_a H_s(0) \right\} ds.
\end{align*}
\]

Definition 4.4. Let \( \gamma : [0, 1] \to \mathbb{R} \) be a continuous function. We say that a bounded function \( \rho : [0, T] \times [0, 1] \to \mathbb{R} \) is a weak solution of the heat equation with Neumann boundary condition

\[
\begin{align*}
    \partial_t \rho &= \Delta \rho, \\
    \partial_a \rho_t(0) &= 0, \quad \forall t \in [0, T], \\
    \partial_a \rho_t(1) &= 0, \quad \forall t \in [0, T], \\
    \rho(0, \cdot) &= \gamma(\cdot),
\end{align*}
\]

if \( \rho \in L^2(0, T; \mathcal{H}^1(0, 1)) \) and for \( t \in [0, T] \) and \( H \in C^{1,2}_0([0, T] \times [0, 1]), \rho_t(\cdot) \) satisfies

\[
\begin{align*}
    \langle \rho_t, H_t \rangle - \langle \gamma, H_0 \rangle &= \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle ds \\
    &+ \int_0^t \left\{ \rho_s(1)\partial_a H_s(1) - \rho_s(0)\partial_a H_s(0) \right\} ds.
\end{align*}
\]

Now we are able to enunciate our second statement:

**Theorem 4.5.** Let \( \gamma : [0, 1] \to [0, 1] \) be a fixed initial profile and \( \{\mu_N : N \geq 1\} \) a sequence of probability measures \( \{0, 1\}^I \) associated to \( \gamma \), in the sense of Definition 3.1. For each \( t \in [0, T] \), \( \delta > 0 \) and for all functions \( H \in C([0,1]), \) we have that

\[
\lim_{N \to +\infty} \mathbb{P}_{\mu_N} \left[ H^N : \frac{1}{N} \sum_{x \in I^N} H \left( \frac{x}{N} \right) \eta_t(x) - \int_{[0,1]} H(u) \rho(t, u) du > \delta \right] = 0,
\]

holds, where \( \rho(t, \cdot) \) is the unique weak solution of the equation (15) if \( \theta \in [0,1) \), or (17) if \( \theta = 1 \), or (19) if \( \theta \in (1, \infty) \).
In order to prove the theorem, we begin by defining the empirical measure associated to a configuration $\eta$: let

$$\pi^N(\eta, du) = \frac{1}{N} \sum_{x=1}^{N-1} \eta(x) \delta_x(du),$$

where $\delta_x$ denotes the Dirac mass at $u$. Thus, given a Markov process in $D_{\{0,1\}^I} [0, T]$, we can consider the empirical process $\pi^N(du) = \pi^N(\eta, du)$ in $D_M[0, T]$, where $M$ is the set of positive measures on $[0, 1]$ with total mass bounded by 1 endowed with the weak convergence topology. Observe that $\pi^N$ is also a Markov process.

For an integrable function $H : [0, 1] \to \mathbb{R}$, $(\pi^N, H)$ denotes the integral of $H$ with respect to the measure $\pi^N$:

$$\langle \pi^N, H \rangle = \frac{1}{N} \sum_{x \in I_N} H(\frac{x}{N}) \eta_t(x).$$

When $\pi_t$ has a density $\rho(\pi(t, du) = \rho(t, u)du)$, we will write $\langle \rho_t, H \rangle$ instead of $\langle \pi_t, H \rangle$.

Fix $T > 0$ and a value of $\theta \geq 0$. Given a probability measure $\mu_N$ on $\{0,1\}^{I_N}$, consider the Markov process $\pi^N$ in $D_M[0, T]$ associated to the process $\eta^N$ in $D_{\{0,1\}^{I_N}}[0, T]$, which has $\mu_N$ as initial distribution. Denote by $Q_N$ the probability measure on $D_M[0, T]$ induced by $\pi^N$.

Fix an initial continuous profile $\gamma : [0, 1] \to [0, 1]$ and consider the sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0,1\}^{I_N}$ associated to $\gamma$. Let $Q_\gamma$ be the probability measure on $D_M[0, T]$ which gives mass 1 to the path $\pi(t, du) = \rho(t, u)du$, where $\rho(t, \cdot)$ is the weak solution of the corresponding PDE (remember that the PDE depends on the value of $\theta$). Now, we are in position to state the next proposition, we would stress that this proposition implies the Theorem 4.5.

**Proposition 4.6.** The sequence of probabilities $\{Q_N\}_{N \in \mathbb{N}}$ converges weakly to $Q_\gamma$, when $N \to +\infty$.

The proof of this result is divided in three main steps. In the next section we prove that the sequence $\{Q_N\}_{N \in \mathbb{N}}$ is tight. In section 7 the limit points are characterized as probabilities concentrated on Lebesgue absolutely continuous measures with densities $\rho_t(u)$, which are weak solutions of the corresponding PDE. The last step is to verify the uniqueness of the weak solution of the three possible PDEs. This is done in Section 8.

5. Tightness

In this section we prove the following proposition:

**Proposition 5.1.** The sequence of probabilities $\{Q_N\}_{N \in \mathbb{N}}$ is tight in the Skorohod topology of $D_M[0, T]$.

The proof is divided in 2 cases: $\theta \geq 1$ and $\theta \in [0, 1)$. We begin by the first one.

**Tightness for $\theta \geq 1$.** In order to prove the assertion, it is enough to prove that, for all $\varepsilon > 0$,

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \epsilon_{\mu_N} \left[ \sup_{|t-s| \leq \delta} \left| \langle \pi^N, H \rangle - \langle \pi, H \rangle \right| > \varepsilon \right] = 0,$$  \hspace{1cm} (21)

holds for all functions $H$ in a dense subset of $C[0, 1]$. We will consider $H \in C^2[0, 1]$.

By Dynkin’s formula we know that

$$M^N_t := \langle \pi^N, H \rangle - \langle \pi, H \rangle - \int_0^t N^2 L_N(\pi_s^N, H) \, ds$$  \hspace{1cm} (22)

is a martingale with respect to the natural filtration $F_t := \sigma(\eta_s : s \leq t)$. By the previous expression, (21) holds if we prove that

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \epsilon_{\mu_N} \left[ \sup_{|t-s| \leq \delta} \left| M^N_t - M^N_s \right| \right] = 0,$$  \hspace{1cm} (23)

and

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \epsilon_{\mu_N} \left[ \sup_{0 \leq t-s \leq \delta} \left| \int_s^t N^2 L_N(\pi^N_r, H) \, dr \right| \right] = 0,$$  \hspace{1cm} (24)
In order to verify (23), we will use the quadratic variation of $M^N_t(H)$, which we denote by $\langle M^N_t(H) \rangle$. We have

$$E_{\mu_N} \left[ \sup_{|t-s| \leq \delta} |M^N_t(H) - M^N_s(H)| \right] \leq 2 E_{\mu_N} \left[ \sup_{0 \leq t \leq T} |M^N_t(H)| \right],$$

$$\leq 2 E_{\mu_N} \left[ \sup_{0 \leq t \leq T} |M^N_t(H)|^2 \right]^{\frac{1}{2}},$$

$$\leq 4 E_{\mu_N} [M^N_t(H)]^2,$$

$$= 4 E_{\mu_N} [(M^N_t(H))]^\frac{1}{2},$$

where the third inequality is Doob’s inequality and the last equality comes from the fact that

$$\{M^N_t(H)^2 - \langle M^N_t(H) \rangle \}_{0 \leq t \leq T}$$

is a martingale that is equal to zero when $t = 0$.

We now prove that the quadratic variation $\langle M^N_t(H) \rangle$ converges to zero uniformly in $t \in [0, T]$, when $N \to \infty$. In order to do that, we remember that

$$\langle M^N_t(H) \rangle = \int_0^t N^2 [L_N(\pi^N_s, H)^2 - 2(\pi^N_s, H)L_N(\pi^N_s, H)] ds,$$

and therefore we just have to handle the last integral. It is easy to compute the following expression

$$N^2 [L_{N,0}(\pi^N_s, H)^2 - 2(\pi^N_s, H)L_{N,0}(\pi^N_s, H)]$$

$$= \frac{1}{N^2} \sum_{x=1}^{N-2} (\pi_s(x) - \pi_s(x+1))^2 \left( \frac{H(\frac{x+1}{N}) - H(\frac{x}{N})}{N-1} \right)^2.$$

By the mean value theorem, the term in parentheses is bounded by $\|H'\|^2_\infty$. The last expression is then bounded by $\frac{1}{N} \|H'\|^2_\infty$, and we conclude that

$$\lim_{N \to \infty} \int_0^t N^2 [L_{N,0}(\pi^N_s, H)^2 - 2(\pi^N_s, H)L_{N,0}(\pi^N_s, H)] ds = 0,$$

uniformly in $t \in [0, T]$.

The next step consists in verifying that

$$\lim_{N \to \infty} \int_0^t N^2 [L_{N,b}(\pi^N_s, H)^2 - 2(\pi^N_s, H)L_{N,b}(\pi^N_s, H)] ds = 0,$$  \(25\)

uniformly in $t \in [0, T]$. Then, we have

$$N^2 [L_{N,b}(\pi^N_s, H)^2 - 2(\pi^N_s, H)L_{N,b}(\pi^N_s, H)]$$

$$= \frac{1}{N^2} [(1-\alpha)\eta_s(1) + \alpha(1 - \eta_s(1))]H(\frac{1}{N})^2$$

$$\quad + \frac{1}{N^2} [(1 - \beta)\eta_s(N-1) + \beta(1 - \eta_s(N-1))]H(\frac{N-1}{N})^2.$$

The absolute value of the last expression is bounded by $2\|H^2\|_\infty N^{-6}$, and therefore we have (25). This concludes the verification of (23).

**Remark.** The quantity $E_{\mu_N} \left[ \sup_{0 \leq t \leq T} |M^N_t(H)|^2 \right]^{\frac{1}{2}}$ converges to zero when $N \to +\infty$. The Markov’s inequality implies that, for all $\delta > 0$

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} |M^N_t(H)| > \delta \right] = 0.$$  \(26\)

The last expression and (23) imply tightness for the sequence $\{M^N_t(H); t \in [0, T]\}_{N \in \mathbb{N}}$. We will use this fact later.
In order to verify (24), we claim that we can find a constant $C := C(H) > 0$ such that
\[ |N^2L_N\langle \pi_s^N, H \rangle| \leq C. \] (27)
This implies
\[ \left| \int_s^t N^2L_N\langle \pi_r^N, H \rangle \, dr \right| \leq C|t - s|, \]
and (24) easily follows.
To prove (27), we begin by handling
\[ N^2L_N,0(\pi_s^N, H) \]
\[ = \frac{1}{N} \sum_{x=2}^{N-2} \eta_s(x)N^2 \{ H \left( \frac{x+1}{N} \right) + H \left( \frac{x-1}{N} \right) - 2H \left( \frac{x}{N} \right) \} + \eta_s(N-1)N \left( H \left( \frac{N-2}{N} \right) - H \left( \frac{N-1}{N} \right) \right) \]
\[ = \frac{1}{N} \sum_{x=2}^{N-2} \eta_s(x) \Delta_N H \left( \frac{x}{N} \right) - \eta_s(N-1) \nabla_N^{-1} H \left( \frac{N-1}{N} \right) + \eta_s(1) \nabla_N^+ H \left( \frac{1}{N} \right), \]
where
\[ \Delta_N H(u) = N^2 \left( H \left( u + \frac{1}{N} \right) + H \left( u - \frac{1}{N} \right) - 2H(u) \right), \]
\[ \nabla_N^+ H \left( \frac{1}{N} \right) = N \left( H \left( \frac{2}{N} \right) - H \left( \frac{1}{N} \right) \right) \]
and
\[ \nabla_N^{-1} H \left( \frac{N-1}{N} \right) = -N \left( H \left( \frac{N-2}{N} \right) - H \left( \frac{N-1}{N} \right) \right). \]
Now, the mean value theorem implies that the last expression is bounded by $2\|H''\|_\infty + 2\|H'\|_\infty$.
To finish the proof of (27), we need only to verify that $N^2L_N,0(\pi_s^N, H)$ is bounded by $2\|H\|_\infty$: this is a consequence of
\[ N^2L_N,0(\pi_s^N, H) = N^{1-\theta}(\alpha - \eta_s(1))H \left( \frac{1}{N} \right) + N^{1-\theta}(\beta - \eta_s(N-1))H \left( \frac{N-1}{N} \right). \] (29)

\[ \square \]

5. Tightness for $\theta \in [0, 1)$. The proof for this case is similar to the one in the case $\theta \geq 1$. If we try to apply the same computations, everything remains valid, with the only exception of the bound in (29) which is no longer true in general. This is solved by observing that using an $L^1$ approximation, we may suppose that our functions $H$ are in the set $C_0\theta[0, 1]$. With this assumption, equation (29) is bounded by $2N^{-\theta}\|H'\|_\infty$.

\[ \square \]

6. Replacement Lemma and Energy Estimates

In this section we establish two technical results needed in the proof of the hydrodynamic limit. These are the Replacement Lemma and the Energy Estimates, one can find them in Lemmas 6.5, 6.7 and Proposition 6.9 The standard method for proving statements such as Lemma 6.5 proceeds in three steps: first, one uses the entropy inequality to change the measure to a more convenient one; then, the relative entropy between the old and new measures is estimated. Finally, the Feynmann-Kac’s inequality reduces the problem to estimate the new expectation by the Dirichlet form of density functions.

Usually one changes the measure to a reversible measure to make easy the calculations of the Dirichlet form. Here, when $\alpha \neq \beta$, we do not use the reversible measure. The solution, following [9] is to change to a Bernoulli product measure associated to a function $\gamma : [0, 1] \to [0, 1]$, which is constant equal to $\alpha$ near 0 and constant near to $\beta$ near 1. Near 0, this measure “looks like” a Bernoulli product measure with constant parameter $\alpha$ near 0, so it is “almost reversible” for the left boundary dynamics. The estimates for the Dirichlet form of the left boundary dynamics explore that heuristic by conditioning with respect to the occupation at the leftmost site, and an analogous procedure is carried out for the rightmost dynamics. This is used in the Replacement Lemma for $\theta \in [0, 1)$. The case $\theta \geq 1$ is handled by changing to a Bernoulli product measure with constant parameter.
Without loss of generality in the following we will suppose that $\alpha \leq \beta$. Here we need to use that $0 < \alpha$ and $\beta < 1$. Let $\gamma : [0,1] \to [0,1]$ be a smooth function such that $\alpha \leq \gamma(u) \leq \beta$, for all $u \in [0,1]$, and assume that there exists a neighborhood of $0$ where the function $\gamma$ is constant equal to $\alpha$, and there exists a neighborhood of $1$, where the function $\gamma$ is constant equal to $\beta$. Let $\nu^N_u$ be the Bernoulli product measure on $\{0,1\}^I$ with marginals given by $\nu^N_u \{ \eta(x) = 1 \} = \gamma(\eta(x))$.

Denote by $H_N(\mu_N | \nu^N_u)$ the relative entropy of a probability measure $\mu_N$ on $\{0,1\}^I$ with respect to the probability $\nu^N_u$. For a precise definition and properties of the relative entropy, we refer the reader to [13]. In Proposition A.1, we show that there exists a finite constant $K$ such that

$$H_N(\mu_N | \nu^N_u) \leq K_0N, \tag{30}$$

for any probability measure $\mu_N$ on $\{0,1\}^I$.

Denote by $(\cdot, \cdot)_{\nu^N_u}$ the scalar product of $L^2(\nu^N_u)$ and consider the Dirichlet form of $f$, which is defined by

$$\langle -L_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_u},$$

where $f$ is a probability density with respect to $\nu^N_u$ (i.e. $f \geq 0$ and $\int fd\nu^N_u = 1$).

Let $\mu$ be a measure on $\{0,1\}^I$ and $g : \{0,1\}^I \to \mathbb{R}$ be a bounded function, define

$$D_{N,0}(g, \mu) := \frac{1}{2} \sum_{x=1}^{N-2} \int \left( g(\eta^{x+1}) - g(\eta) \right)^2 d\mu(\eta),$$

$$D_{N,\beta}(g, \mu) := \frac{1}{2} \int \left[ \frac{\partial}{\partial N}(1 - \eta(N-1)) + \frac{1 - \beta}{\partial N} \eta(N-1) \right] \left( g(\eta^{N-1}) - g(\eta) \right)^2 d\mu(\eta),$$

and

$$D_{N,\beta}(g, \mu) := \frac{1}{2} \int \left[ \frac{\partial}{\partial N}(1 - \eta(N-1)) + \frac{1 - \beta}{\partial N} \eta(N-1) \right] \left( g(\eta^{N-1}) - g(\eta) \right)^2 d\mu(\eta).$$

**Lemma 6.1.** Let $f$ be a density with respect to the measure $\nu^N_u$, defined above. Then, there exists a constant $C_0 > 0$ such that

$$\langle L_{N,0} \sqrt{f}, \sqrt{f} \rangle_{\nu^N_u} \leq -\frac{1}{2} D_{N,0}(\sqrt{f}, \nu^N_u) + \frac{C_0}{N}.$$

**Proof:** One can write $\langle L_{N,0} \sqrt{f}, \sqrt{f} \rangle_{\nu^N_u}$ as

$$-D_{N,0}(\sqrt{f}, \nu^N_u) - \frac{1}{2} \sum_{x=1}^{N-2} \int_{\{0,1\}^I} \left( f(\eta^{x+1}) \right)^2 \left[ \frac{\nu^N_u(\eta^{x+1})}{\nu^N_u(\eta)} - 1 \right] \nu^N_u(\eta).$$

Then, we just need to handle the term on the right hand side of the above equation. In order to do this, we split this term in two equal parts and we change variables, $\eta^{x+1} \mapsto \eta$, in the second part, getting

$$-\frac{1}{4} \sum_{x=1}^{N-2} \int_{\{0,1\}^I} \left( f(\eta^{x+1}) \right)^2 \left[ \frac{\nu^N_u(\eta^{x+1})}{\nu^N_u(\eta)} - 1 \right] \nu^N_u(\eta)$$

and

$$-\frac{1}{4} \sum_{x=1}^{N-2} \int_{\{0,1\}^I} \left( f(\eta) \right)^2 \left[ \frac{\nu^N_u(\eta)}{\nu^N_u(\eta^{x+1})} - 1 \right] \nu^N_u(\eta^{x+1}).$$

If we rewrite the second term above, the last expression becomes

$$\frac{1}{4} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I} \left( f(\eta^{x+1}) \right)^2 \left( \frac{1}{\nu^N_u(\eta^{x+1})} \right) \nu^N_u(\eta) - \frac{1}{4} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I} \left( f(\eta) \right)^2 \left( \frac{1}{\nu^N_u(\eta)} \right) \nu^N_u(\eta^{x+1}).$$
Now, using that \( a^2 - b^2 = (a-b)(a+b) \leq \frac{4}{3}(a-b)^2 + \frac{1}{2A}(a+b)^2 \), for all \( A > 0 \), the integral above is bounded from above by

\[
\frac{A}{8} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I_N} \left( \sqrt{f(\eta^{x,x+1})} - \sqrt{f(\eta)} \right)^2 \nu_{\gamma(c)}^N(\eta) + \frac{1}{8A} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I_N} \left( \sqrt{f(\eta^{x,x+1})} + \sqrt{f(\eta)} \right)^2 \left[ 1 - \frac{\nu_{\gamma(c)}^N(\eta^{x,x+1})}{\nu_{\gamma(c)}^N(\eta)} \right]^2 \nu_{\gamma(c)}^N(\eta).
\]

Choosing \( A = 2 \), the first term above is equal to \( \frac{1}{2} D_{N,0}(\sqrt{f}, \nu_{\gamma(c)}^N) \). Now, a simple computation shows that

\[
\left| 1 - \frac{\nu_{\gamma(c)}^N(\eta^{x,x+1})}{\nu_{\gamma(c)}^N(\eta)} \right| \leq \frac{\|\gamma'\|_{\infty}}{\alpha(1 - \beta) N}. \tag{31}
\]

Thus \( \langle L_{N,0}\sqrt{f}, \sqrt{f} \rangle_{\nu_{\gamma(c)}^N} \) is bounded from above by

\[
- \frac{1}{2} D_{N,0}(\sqrt{f}, \nu_{\gamma(c)}^N) - \frac{C}{16N^2} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I_N} \left( \sqrt{f(\eta^{x,x+1})} + \sqrt{f(\eta)} \right)^2 \nu_{\gamma(c)}^N(\eta). \tag{32}
\]

Using the inequality \((a+b)^2 \leq 2a^2 + 2b^2\) and the fact that \( f \) is a density with respect to \( \nu_{\gamma(c)}^N \), the second term in \( \text{(32)} \) is less than

\[
\frac{C}{8N} + \frac{C}{8N^2} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I_N} \left( \sqrt{f(\eta)} \right)^2 \nu_{\gamma(c)}^N(\eta).
\]

To finish the proof, we need to bound the second term in the expression above by \( k/N \), for some \( k > 0 \). For that, we perform the change of variables \( \eta \mapsto \eta^{x,x+1} \), and rewriting that term as

\[
\frac{C}{8N^2} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I_N} \left( \sqrt{f(\eta)} \right)^2 \frac{\nu_{\gamma(c)}^N(\eta^{x,x+1})}{\nu_{\gamma(c)}^N(\eta)} \nu_{\gamma(c)}^N(\eta).
\]

Since \( \nu_{\gamma(c)}^N(\eta^{x,x+1})/\nu_{\gamma(c)}^N(\eta) \leq C(\alpha, \beta) \) and \( f \) is a density with respect to \( \nu_{\gamma(c)}^N \), we get

\[
\frac{C}{8N^2} \sum_{x=1}^{N-2} \sum_{\eta \in \{0,1\}^I_N} \left( \sqrt{f(\eta)} \right)^2 \frac{\nu_{\gamma(c)}^N(\eta^{x,x+1})}{\nu_{\gamma(c)}^N(\eta)} \nu_{\gamma(c)}^N(\eta) \leq \frac{C}{8N} C(\alpha, \beta),
\]

it finishes the proof.

**Lemma 6.2.** Let \( f \) be a density with respect to the measure \( \nu_{\gamma(c)}^N \), defined above. Then, there exists a constant \( C_i > 0 \) such that

\[
\langle L_{N,b}^I \sqrt{f}, \sqrt{f} \rangle_{\nu_{\gamma(c)}^N} = -D_{N,b}^I(\sqrt{f}, \nu_{\gamma(c)}^N) \leq -\frac{1}{C_i N^g} \text{Var}(\sqrt{f_1}),
\]

where \( f_1(\eta(1)) := E_{\nu_{\gamma(c)}^N}[f|\eta(1)]. \)

**Proof.** The first equality in the statement of this lemma follows from the choice of function \( \gamma \), which is constant equal to \( \alpha \) near the point \( 0 \) and the fact that the Bernoulli product with parameter constant equal \( \alpha \) is invariant for \( L_{N,b}^I \). In order to prove the inequality, we denote the variable \( \eta = (\eta(1), \bar{\eta}) \), where \( \bar{\eta} \in \{0,1\}^{2 \cdots N-1} \), the measure \( \nu_{\gamma(c)}^N(\eta) = \nu_{\alpha}^1(\eta(1)) \nu_{\gamma(c)}^N(\bar{\eta}) \), and we rewrite the function \( f_1 \) as

\[
f_1(\eta(1)) = E_{\nu_{\gamma(c)}^N}[f|\eta(1)] = \sum_{\bar{\eta}} f(\eta(1), \bar{\eta}) \nu_{\gamma(c)}^N(\bar{\eta}).
\]
Now we compute the variance for $\sqrt{f_1}$:

$$
\text{Var}(\sqrt{f_1}) = \sum_{\eta(1)} \left( \sqrt{f_1(\eta(1))} - E_{\nu_\alpha}[\sqrt{f_1}] \right)^2 \nu_\alpha^1(\eta(1))
$$

$$
= (1 - \alpha)^2 \alpha \left( \sqrt{f_1(1) - f_1(0)} \right)^2 + (1 - \alpha)^2 \alpha \left( \sqrt{f_1(0) - f_1(1)} \right)^2
$$

$$
\leq c(\alpha) N^\gamma \sum_{\eta(1)} \left[ \frac{1 - \eta(1)}{\alpha N^\gamma(1)} + \frac{\eta(1)}{\alpha N^\gamma(1)} \right] \left( \sqrt{f_1(1) - f_1(1)} - \sqrt{f_1(1)(1)} \right)^2 \nu_\alpha(\eta(1)),
$$

where $c(\alpha) = \alpha \vee (1 - \alpha)$. Keep in mind the definition of $f_1$ and observe that

$$
\left( \sqrt{f_1(1) - f_1(1)} \right)^2 = \left( \left( \sum_{\eta} f(1 - \eta(1), \eta) \nu_{\gamma(1)}^N(\eta) \right)^{1/2} - \left( \sum_{\eta} f(\eta(1), \eta) \nu_{\gamma(1)}^N(\eta) \right)^{1/2} \right)^2
$$

$$
\leq \sum_{\eta} \left( \left( f(1 - \eta(1), \eta) \right)^{1/2} - \left( f(\eta(1), \eta) \right)^{1/2} \right)^2 \nu_{\gamma(1)}^N(\eta).
$$

The last inequality follows from Lemma A.2. Thus, using this inequality in the expression (33), we bound $\text{Var}(\sqrt{f_1})$ by

$$
c(\alpha) N^\gamma \sum_{\eta} \left[ \frac{1 - \eta(1)}{\alpha N^\gamma(1)} + \frac{\eta(1)}{\alpha N^\gamma(1)} \right] \left( \sqrt{f(\eta(1))} - \sqrt{f(\eta)} \right)^2 \nu_{\gamma(1)}^N(\eta).
$$

Then, the statement of this lemma follows from the last expression combined with the expression for $D_{N,b}^2(\sqrt{f}, \nu_{\gamma(1)}^N)$.

\[\square\]

**Lemma 6.3.** Let $f$ be a density with respect to the measure $\nu_{\gamma(1)}^N$, defined above. Then, there exists a constant $C_\gamma > 0$ such that

$$
\langle L_{N,b}^N \sqrt{T}, \sqrt{T} \nu_{\gamma(1)}^N \rangle = - D_{N,b}^2(\sqrt{T}, \nu_{\gamma(1)}^N) \leq \frac{1}{C_\gamma N^\gamma} \text{Var}(\sqrt{f_{N-1}}),
$$

where $f_{N-1}(\eta(N-1)) := E_{\nu_{\gamma(1)}^N}[f(\eta(N-1))].$

**Proof:** It is analogous to the proof of Lemma 6.2. \[\square\]

### 6.1. Replacement Lemma for $\theta \in [0, 1]$. Remember that we denote the variable $\eta$ as $(\eta(1), \bar{\eta})$, the measure $\nu_{\gamma(1)}(\eta) = \nu_{\gamma(1)}^1(\eta(1)) \nu_{\gamma(1)}^N(\bar{\eta})$, and

$$
f_1(\eta(1)) = E_{\nu_{\gamma(1)}^N}[f(\eta(1))] = \sum_{\eta} f(\eta(1), \bar{\eta}) \nu_{\gamma(1)}^N(\bar{\eta}).
$$

We write $f_{N-1}(\eta(N-1))$ in the same way.

**Lemma 6.4.** There exist constants $C(\alpha) > 0$ and $C(\beta) > 0$ such that, for all density functions $f$ with respect to $\nu_{\gamma(1)}^N$ and for all $B > 0$, we have

$$
\int \{\eta(1) - \alpha\} f(\eta) \, d\nu_{\gamma(1)}^N(\eta) \leq BC(\alpha) + \frac{1}{B} \text{Var}(\sqrt{f_1})
$$

and

$$
\int \{\eta(N-1) - \beta\} f(\eta) \, d\nu_{\gamma(1)}^N(\eta) \leq BC(\beta) + \frac{1}{B} \text{Var}(\sqrt{f_{N-1}}).
$$
Proof. We will prove only the inequality (34), because the other one is similar. From definition of the function $f_1$, we have
\[\int \{ \eta(1) - \alpha \} f(\eta) d\nu^N_{\gamma(1)}(\eta) = \sum_{\eta(1)} \{ \eta(1) - \alpha \} \sum_{\eta} f(\eta(1), \tilde{\eta}) \nu^N_{\gamma(1)}(\tilde{\eta}) \nu^1_\alpha(\eta(1)) = \sum_{\eta(1)} \{ \eta(1) - \alpha \} f_1(\eta(1)) \nu^1_\alpha(\eta(1)).\]
Since the function $\{ \eta(1) - \alpha \}$ has zero mean, the last expression becomes
\[\sum_{\eta(1)} \{ \eta(1) - \alpha \} \{ f_1(\eta(1)) - E_{\nu^1_\alpha}[\sqrt{f_1}]^2 \} \nu^1_\alpha(\eta(1)).\]
Using the inequality $ab \leq Ba^2 + B^{-1}b^2$, the sum above is bounded from above by
\[\frac{B}{\alpha} \sum_{\eta(1)} \{ \eta(1) - \alpha \}^2 \{ \sqrt{f_1(\eta(1))} + E_{\nu^1_\alpha}[\sqrt{f_1}] \}^2 \nu^1_\alpha(\eta(1)) \leq \frac{1}{B} \text{Var}(\sqrt{f_1}).\]
Now, as $f$ is a density with respect to $\nu^N_{\gamma(1)}$, we have
\[\sum_{\eta(1)} \{ \eta(1) - \alpha \}^2 \{ \sqrt{f_1(\eta(1))} + E_{\nu^1_\alpha}[\sqrt{f_1}] \}^2 \nu^1_\alpha(\eta(1)) \leq 2\alpha(1 - \alpha),\]
which finishes the proof.

Lemma 6.5 (Replacement Lemma). Fix $\theta \in [0, 1]$. For all functions $G \in C^{1,2}([0, T] \times [0, 1])$ and all $t \in [0, T]$, we have
\[\lim_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t G_s(0) \{ \eta_\theta(1) - \alpha \} \, ds \right| \right] = 0\]
and
\[\lim_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t G_s(1) \{ \eta_\theta(N - 1) - \beta \} \, ds \right| \right] = 0.\]

Proof. We will prove the first limit, because the second one is analogous. From Jensen’s inequality and the definition of entropy, for any $A \in \mathbb{R}$ (which will be chosen large), the expectation considered in the statement of the lemma is bounded from above by
\[\frac{H_N(\mu_N|\nu^N_{\gamma(1)})}{AN} + \frac{1}{AN} \log \mathbb{E}_{\nu^N_{\gamma(1)}} \left[ \exp \left\{ AN \left| \int_0^t G_s(0) \{ \eta_\theta(1) - \alpha \} \, ds \right| \right\} \right]. \tag{35}\]
By Proposition A.1
\[H_N(\mu_N|\nu^N_{\gamma(1)}) \leq K_0 N,\]
so we need to analyse the second term above. Since $e^{x^2} \leq e^x + e^{-x}$ and
\[\lim_{N \to \infty} \frac{1}{N} \log(\alpha N + b_N) = \max \left\{ \lim_{N \to \infty} \frac{1}{N} \log a_N, \lim_{N \to \infty} \frac{1}{N} \log b_N \right\}, \tag{36}\]
for all sequences $\{a_N\}$ and $\{b_N\}$ of positive numbers, we can remove the absolute value inside the exponential. By Feynman-Kac’s formula (see Lemma A 1.7.2 of [13]), the second term in (35) is less than or equal to
\[\int_0^t \sup_f \mathbb{E}_{\nu^N_{\gamma(1)}} \left[ \int \{ \eta_\theta(1) - \alpha \} f(\eta) d\nu^N_{\gamma(1)}(\eta) + N \langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\gamma(1)}} \right] \, ds.\]
Using the Lemmas 6.1, 6.2 and 6.3 we have that $\langle L_N \sqrt{f}, \sqrt{f} \rangle_{\nu^N_{\gamma(1)}}$ is bounded from above by
\[-\frac{1}{2} D_{\alpha,0}(\sqrt{f}, \nu^N_{\gamma(1)}) + \frac{C_0}{N} - D_{\alpha,b}(\sqrt{f}, \nu_\alpha) - \frac{1}{C_1 N^\theta} \text{Var}(\sqrt{f_1}) \leq -\frac{1}{C_1 N^\theta} \text{Var}(\sqrt{f_1}) + \frac{C_0}{N},\]
for some constants $C_0, C_1$. Now, from the Lemma 6.4 we have that the expression inside the braces in the supremum above can be bounded by
\[\|G\|_\infty B C(\alpha) + \left( \frac{\|G\|_\infty}{B} - \frac{N^{1-\theta}}{ACl} \right) \text{Var}(\sqrt{f_1}) + \frac{C_0}{A},\]
in the limit $N \to \infty$. Therefore, returning to (35), we have
\[\lim_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \left| \int_0^t G_s(0) \{ \eta_\theta(1) - \alpha \} \, ds \right| \right] = 0.\]
Taking $B = \frac{A\|G\|_{\infty}}{N^1-\theta}$, we have that the last expression is bounded by
\[
\frac{AC(\alpha)C_l}{N^{1-\theta}} + \frac{C_0}{A}.
\]
Since $\theta \in [0,1)$ and $A$ is any positive real number, our statement follows.

6.2. **Replacement Lemma for $\theta \in [1,\infty)$.**

**Lemma 6.6.** Let $\nu^N_\beta$ the Bernoulli product measure in $\{0,1\}^N$ with constant parameter $\beta \in (0,1)$. Suppose that $\theta \geq 1$. Then
\[
\langle N^2 L_{N,0} \sqrt{f}, \sqrt{f} \rangle_{\nu^N_\beta} = -\frac{N^2}{2} \sum_{x=1}^{N-2} \int (\sqrt{f}(\eta^{x\to x+1}) - \sqrt{f}(\eta))^2 d\nu^N_\beta(\eta),
\]
and there exists a positive constant $C_0 = C(\alpha, \beta)$ such that
\[
\langle NL_{N,b} \sqrt{f}, \sqrt{f} \rangle_{\nu^N_\beta} \leq C_0,
\]
for all $f$ density with respect to $\nu^N_\beta$.

**Proof:** The first statement follows from the reversibility of $\nu^N_\beta$ for the dynamics of the infinitesimal generator $L_{N,0}$.

In order to prove the second statement, we will consider separately the dynamics at the right and left boundaries, writing $L_{N,b} = L_{N,b}^L + L_{N,b}^R$, where
\[
(L_{N,b}^L \sqrt{f})(\eta) = \left[ \frac{\alpha}{N^\theta}(1 - \eta(1)) + \frac{1 - \alpha}{N^\theta} \eta(1) \right] \left[ \sqrt{f}(\eta^L) - \sqrt{f}(\eta) \right]
\]
\[
(L_{N,b}^R \sqrt{f})(\eta) = \left[ \frac{\beta}{N^\theta}(1 - \eta(N-1)) + \frac{1 - \beta}{N^\theta} \eta(N-1) \right] \left[ \sqrt{f}(\eta^{N-1}) - \sqrt{f}(\eta) \right].
\]

The measure $\nu^N_\beta$ is reversible for the right boundary dynamics, so $\langle NL_{N,b}^R \sqrt{f}, \sqrt{f} \rangle_{\nu^N_\beta} \leq 0$. It is therefore sufficient to estimate $\langle NL_{N,b}^L \sqrt{f}, \sqrt{f} \rangle_{\nu^N_\beta}$. We have
\[
\langle NL_{N,b}^L \sqrt{f}, \sqrt{f} \rangle_{\nu^N_\beta} = \sum_{\eta} N^{1-\theta} \nu^N_\beta(\eta) \sqrt{f}(\eta) [\alpha(1 - \eta(1)) + (1 - \alpha)\eta(1)] \cdot [\sqrt{f}(\eta^L) - \sqrt{f}(\eta)]
\]
\[
\leq \sum_{\eta} \nu^N_\beta(\eta) \sqrt{f}(\eta) \sqrt{f}(\eta^L) \leq \left( \sum_{\eta} \nu^N_\beta(\eta) f(\eta) \right)^{1/2} \cdot \left( \sum_{\eta} \nu^N_\beta(\eta) f(\eta^L) \right)^{1/2} \leq C_0.
\]

For the following we will need to define the empirical density at the boundaries $\{1,N-1\}$ as
\[
\eta^N(1) = \frac{1}{[\varepsilon N]} \sum_{y=1}^{[\varepsilon N]} \eta(y) \quad \text{and} \quad \eta^N(N-1) = \frac{1}{[\varepsilon N]} \sum_{y=N-1-[\varepsilon N]}^{N-1} \eta(y).
\]

The next lemma allows us to replace the random variables $\eta_s(1)$ and $\eta_s(N-1)$ by functions of the empirical measure, namely $\eta^N(1)$ and $\eta^N(N-1)$.

**Lemma 6.7 (Replacement Lemma).** Suppose $\theta \geq 1$. Let $F_N : [0,1] \to \mathbb{R}$, $N \in \mathbb{N}$ be a sequence of functions such that $\|F_N\|_{\infty} \leq L$ for all $N \in \mathbb{N}$, where $L > 0$ is a constant. Then, for $x \in \{1,N-1\}$, we have
\[
\limsup_{\varepsilon \downarrow 0} \limsup_{N \to \infty} \mathbb{E}_{\mu^N} \left[ \left| \int_0^t F_N(s) [\eta^N_s(x) - \eta_s(x)] \, ds \right| \right] = 0.
\]
Proof. We will work in the case \( x = 1 \). The other one is similar.

Fix \( \gamma > 0 \). Using the entropy inequality we can bound the expectation in the statement by

\[
\frac{1}{\gamma N} H(\mu_N | \nu_{\beta}^N) + \frac{1}{\gamma N} \log \mathbb{E}_{\nu_{\beta}^N} \left[ \exp \left\{ \gamma N \int_0^t F_N(s)[\eta_s^N(x) - \eta_s(x)] ds \right\} \right].
\]  

We start the proof considering the second term. Using that \( \epsilon^x \leq e^x + e^{-x} \) and (36), we can remove the absolute value inside the exponential in (37). Since \( F_N \) is arbitrary, we only need to estimate

\[
\frac{1}{\gamma N} \log \mathbb{E}_{\nu_{\beta}^N} \left[ \exp \left\{ \gamma N \int_0^t F_N(s)[\eta_s^N(x) - \eta_s(x)] ds \right\} \right].
\]  

Using the Feynman-Kac’s formula, we can bound (38) by

\[
\int_0^t \sup_f \text{density} \left\{ \int F_N(s)[\eta_s^N(1) - \eta(1)] f(\eta) d\nu_{\beta}^N(\eta) - \frac{N}{2\gamma} \sum_{x=1}^{N-2} \left[ (\sqrt{\mathcal{J}(\epsilon^x, x+1)} - \sqrt{\mathcal{J}(\eta^x, x+1)})^2 d\nu_{\beta}^N(\eta) + \frac{N}{\gamma} \{ \langle L_N, \sqrt{\mathcal{J}} \rangle \sqrt{\mathcal{J}} \nu_{\beta}^N \} ds \right. 
\]
\]

Now, we analyse the first term in the supremum in (39):

\[
\int F_N(s)[\eta_s^N(1) - \eta(1)] f(\eta)^2 d\nu_{\beta}^N(\eta) = \frac{1}{\epsilon N} \int F_N(s) \sum_{x=1}^{\epsilon N} [\eta(x) - \eta(1)] f(\eta)^2 d\nu_{\beta}^N(\eta) + o(1).
\]

The error term appears because in the definition of \( \eta^N \) we are dividing by \( |\epsilon N| \) and here we are considering \( \epsilon N \) instead of \( \lfloor \epsilon N \rfloor \). But, it is not important, then we can omit the error term for simplicity and we rewrite the last expression as

\[
\frac{F_N(s)}{2\epsilon N} \sum_{x=1}^{\epsilon N} \sum_{y=1}^{x-1} [\eta(y+1) - \eta(y)] f(\eta) d\nu_{\beta}^N(\eta)
\]

\[
+ \frac{F_N(s)}{2\epsilon N} \sum_{x=1}^{\epsilon N} \sum_{y=1}^{x-1} [\eta(y) - \eta(y+1)] f(\eta^{y+1}) d\nu_{\beta}^N(\eta)
\]

\[
= \frac{F_N(s)}{2\epsilon N} \sum_{x=1}^{\epsilon N} \sum_{y=1}^{x-1} [\eta(y+1) - \eta(y)] [f(\eta) - f(\eta^{y+1})] d\nu_{\beta}^N(\eta).
\]

Let \( A > 0 \). Writing \( f(\eta) - f(\eta^{y+1}) = [\sqrt{\mathcal{J}(\eta)} - \sqrt{\mathcal{J}(\eta^{y+1})}] [\sqrt{\mathcal{J}(\eta)} + \sqrt{\mathcal{J}(\eta^{y+1})}] \), using the inequality \( ab \leq a^2 + b^2 / A \), for all \( A > 0 \), and the fact that there is at most one particle per site, we can bound the last expression by

\[
\frac{1}{2\epsilon N} \sum_{x=1}^{\epsilon N} \sum_{y=1}^{x-1} A \left( \sqrt{\mathcal{J}(\eta)} - \sqrt{\mathcal{J}(\eta^{y+1})} \right)^2 d\nu_{\beta}^N(\eta)
\]

\[
+ \frac{F_N(s)}{2\epsilon N} \sum_{x=1}^{\epsilon N} \sum_{y=1}^{x-1} \frac{1}{A} \left( \sqrt{\mathcal{J}(\eta)} + \sqrt{\mathcal{J}(\eta^{y+1})} \right)^2 d\nu_{\beta}^N(\eta).
\]

Letting the sum in \( y \) run from 1 to \( N - 2 \), the first term is bounded by

\[
\frac{A}{2} \sum_{y=1}^{N-1} \left( \sqrt{\mathcal{J}(\eta)} - \sqrt{\mathcal{J}(\eta^{y+1})} \right)^2 d\nu_{\beta}^N(\eta).
\]

Choosing \( A = N / \gamma \), the last term will cancel with the second term in the supremum (39).
Now, let us study the second term in (40). Using that \( f \) is a density for \( \nu_N^r \) and that \( \|F_N\|_\infty \leq L \), the second term in (40) is bounded by
\[
2L^2 \frac{|\varepsilon N|}{\varepsilon N} \sum_{x=1}^{N} \sum_{y=1}^{N} \frac{1}{A} \leq \frac{2\varepsilon NL^2}{A}.
\]

By Lemma 6.6 we can find a constant \( C = C(\alpha, \beta) \) such that \( \langle NL_N, \sqrt{f}, \sqrt{\mathcal{C}} \rangle \leq C \), for all \( f \) density for \( \nu_N^r \). The supremum in (39) is bounded by \( 2\varepsilon \gamma L^2 + C/\gamma \). The limit, as \( \varepsilon \to 0 \), (38) is bounded by \( C/\gamma \).

To finish the proof, we observe that by the Proposition A.1, we have \( H(\mu_N|\nu_N^r) \leq K_0 N \), for some constant \( K_0 \). This shows that the expression (37) is bounded by \( (K_0 + C)/\gamma \). As \( \gamma \) is arbitrary, the proof is finished. \( \square \)

6.3. Energy Estimates. Here we will prove the following proposition:

**Proposition 6.8.** Let \( \mathcal{Q}_* \) be a limit point of \( \{Q_N\} \). The support of \( \mathcal{Q}_* \) is contained in the set of trajectories \( \pi \in \mathcal{D}_M[0, T] \) satisfying:

1. \( \pi_t \) is a Lebesgue absolutely continuous measure with density \( \rho_t \);
2. The function \( \rho : [0, T] \to \mathcal{H}^1(0, 1) \) is in \( L^2(0, T; \mathcal{H}^1(0, 1)) \).

**Proof.** The proof of (1) is standard, see [13]. In order to prove (2), we define a linear functional \( \ell_\rho \) in \( C^{0,1}_c([0, T] \times (0, 1)) \) given by
\[
\ell_\rho(H) = \int_0^T \int_0^1 \partial_u H(s, u) \rho(s, u) \, du \, ds = \int_0^T \int_0^1 \partial_u H(s, u) \, d\pi_s(u) \, ds.
\]

We will prove that this functional can be extended to \( L^2([0, T] \times (0, 1)) \) in such a way that it is \( \mathcal{Q}_* \) almost surely continuous. If we do that, we can use Riesz Representation Theorem to find \( \xi \in L^2([0, T] \times (0, 1)) \) satisfying
\[
\int_0^T \int_0^1 \partial_u H(s, u) \rho(s, u) \, du \, ds = -\int_0^T \int_0^1 H(s, u) \xi(s, u) \, du \, ds,
\]
for all \( H \in C^{0,1}_c([0, T] \times (0, 1)) \), which implies \( \xi \in L^2(0, T; \mathcal{H}^1(0, 1)) \).

Our main tool for achieving this objective is the next proposition, which is also the energy estimate for the process. More specifically, as in [11] [12], one can use the energy estimate to prove that the functional (41) is \( \mathcal{Q}_* \) almost surely continuous. \( \square \)

**Proposition 6.9.** For all \( \theta \geq 0 \). If \( \ell_\rho \) is defined as in (41), then there exist positive constants \( C \) and \( c \) such that
\[
\mathbb{E}_* \left[ \sup_{H} \left\{ \ell_\rho(H) - c \|H\|_{L^2([0, T] \times (0, 1))}^2 \right\} \right] \leq C < \infty,
\]
where the supremum above is taken in the set \( L^2([0, T] \times (0, 1)) \).

We will denote by \( \|H\|_2 \) the norm of a function \( H \in L^2([0, T] \times (0, 1)) \).

**Proof.** By density and the Monotone Convergence Theorem, it is enough to prove that
\[
\mathbb{E}_* \left[ \max_{k \leq m} \left\{ \ell_\rho(H^k) - c \|H^k\|_2^2 \right\} \right] \leq K_0,
\]
for each fixed \( m \), where \( \{H^m\}_{m \in \mathbb{N}} \) is a dense subset of \( C^{0,2}_c([0, T] \times (0, 1)) \) and the bound \( K_0 \) does not depend on \( m \).

Now, the function
\[
\Phi(\pi) = \max_{k \leq m} \left\{ \int_0^T \int_0^1 \partial_u H^k(s, u) \, d\pi_s(u) \, ds - c \|H^k\|_2 \right\}
\]
is continuous and bounded in the Skorohod topology of $D_{\mathbb{R}}[0,T]$, which gives us
\[
E_\ast \left[ \max_{k \leq m} \{ \ell_\rho (H^k) - c \| H^k \|_2^2 \} \right] = \\
= \lim_{N \to +\infty} E_{\mu_N} \left[ \max_{k \leq m} \left\{ \frac{1}{N} \int_0^T \sum_{x=1}^{N-1} \partial_u H_s^k \left( \frac{x}{N} \right) \eta_s(x) \, ds - c \| H^k \|_2^2 \right\} \right] \\
= \int E_\eta \left[ \max_{k \leq m} \left\{ \frac{1}{N} \int_0^T \sum_{x=1}^{N-1} \partial_u H_s^k \left( \frac{x}{N} \right) \eta_s(x) \, ds - c \| H^k \|_2^2 \right\} \right] \, d\mu_N(\eta).
\]

The relative entropy inequality gives the following upper bound for the last expression:
\[
\frac{H(\mu_N | \nu_{\gamma(\cdot)}^N)}{N} + \frac{1}{N} \log \exp \left[ \max_{k \leq m} \left( \int_0^T \sum_{x=1}^{N-1} \partial_u H_s^k \left( \frac{x}{N} \right) \eta_s(x) \, ds - cN \| H^k \|_2^2 \right) \right] \, d\nu_{\gamma(\cdot)}^N(\eta),
\]
where $\nu_{\gamma(\cdot)}^N$ is the Bernoulli product measure with parameter $\gamma(\frac{x}{N})$ at the site $x$, considered in the beginning of this section.

By Lemma A.1, $H(\mu_N | \nu_{\gamma(\cdot)}^N) \leq K_0 N$, where $K_0$ is a constant that depends only on $\alpha$ and $\beta$. This is the upper bound of the first term in the sum above. For the second term, we use Jensen’s inequality and $\exp \{ \max_{n \leq m} a_n \} \leq \sum_{n=1}^m e^{a_n}$ to get the bound
\[
\frac{1}{N} \log E_{\nu_{\gamma(\cdot)}^N} \left[ \sum_{k=1}^m \exp \left( \int_0^T \sum_{x=1}^{N-1} \partial_u H_s^k \left( \frac{x}{N} \right) \eta_s(x) \, ds - cN \| H^k \|_2^2 \right) \right].
\]

Using (36), it is enough to bound
\[
\limsup_{N \to +\infty} \frac{1}{N} \log E_{\nu_{\gamma(\cdot)}^N} \left[ \exp \left( \int_0^T \sum_{x=1}^{N-1} \partial_u H_s \left( \frac{x}{N} \right) \eta_s(x) \, ds - cN \| H \|_2^2 \right) \right],
\]
for a fixed function $H \in C_{c,0}^0([0,T] \times (0,1))$.

We use Feynman-Kac’s formula to bound the last expression by $\limsup_{N \to +\infty} \int_0^T \frac{1}{N} \Gamma_s \, ds$, with
\[
\frac{1}{N} \Gamma_s = \sup_{f \text{ density}} \left\{ \frac{1}{N} \int \sum_{x=1}^{N-1} \partial_u H_s \left( \frac{x}{N} \right) \eta(x) f(\eta) \, d\nu_{\gamma(\cdot)}(\eta) - c \int_0^1 H_s(u)^2 \, du + N \langle L_N \sqrt{f} , \sqrt{f} \rangle_{\nu_{\gamma(\cdot)}^N} \right\}.
\]

Using the Lemmas 6.1, 6.2 and 6.3, the expression above is bounded from above by
\[
\sup_{f \text{ density}} \left[ \frac{1}{N} \int \sum_{x=1}^{N-1} \partial_u H_s \left( \frac{x}{N} \right) \eta(x) f(\eta) \, d\nu_{\gamma(\cdot)}(\eta) - c \int_0^1 H_s(u)^2 \, du - \frac{N}{2} D_{N,0}(\sqrt{f} , \nu_{\gamma(\cdot)}^N) + C_0 \right].
\]

Now, we compute the first term above. If we denote
\[
A = \frac{1}{N} \int \sum_{x=1}^{N-1} \partial_u H_s \left( \frac{x}{N} \right) \eta(x) f(\eta) \, d\nu_{\gamma(\cdot)}(\eta),
\]
using that $H \in C_{c,0}^0([0,T] \times (0,1))$, we can write
\[
A = \frac{1}{N} \int \sum_{x=1}^{N-1} N \left[ H_s \left( \frac{x}{N} \right) - H_s \left( \frac{x-1}{N} \right) \right] \eta(x) f(\eta) \, d\nu_{\gamma(\cdot)}(\eta) + o(1).
\]

After some simple computations, we get the following expression:
\[
A = \sum_{x=1}^{N-2} \int H_s \left( \frac{x+1}{N} \right) (\eta(x) - \eta(x+1)) f(\eta) \, d\nu_{\gamma(\cdot)}^N + o(1),
\]

\[20\]
where $o(1)$ depends only on $\|H''\|_\infty$ by the Mean Value Theorem and not on the function $f$. Now we write $A = \frac{1}{2}A + \frac{1}{2}A$ and make the change of variables $\eta \mapsto \eta^{x,x+1}$ (which is not invariant for $\nu_\gamma^{N}$) in order to get

$$A = \frac{1}{2} \sum_{x=1}^{N-2} \int H_s \left( \frac{x}{N} \right) \left( \eta(x) - \eta(x+1) \right) (f(\eta) - f(\eta^{x,x+1})) \, d\nu_\gamma^{N}(\eta) + o(1)$$

$$- \frac{1}{2} \sum_{x=1}^{N-2} \int H_s \left( \frac{x}{N} \right) (\eta(x) - \eta(x+1)) f(\eta^{x,x+1}) \left( \frac{\nu_\gamma^{N}(\eta^{x,x+1})}{\nu_\gamma^{N}(\eta)} - 1 \right) \, d\nu_\gamma^{N}(\eta)$$

$$=: A_1 + A_2 + o(1).$$

Since $2ab \leq B^2 + \frac{a^2}{B}$, for any real number $B > 0$, and $|\eta(x) - \eta(x+1)| \leq 1$, we have

$$A_1 \leq \frac{B}{4} \sum_{x=1}^{N-2} \int \left( H_s \left( \frac{x}{N} \right) \right)^2 \left( \sqrt{f(\eta)} + \sqrt{f(\eta^{x,x+1})} \right)^2 \, d\nu_\gamma^{N}(\eta) + \frac{1}{4B} \sum_{x=1}^{N-2} \int \left( \sqrt{f(\eta)} - \sqrt{f(\eta^{x,x+1})} \right)^2 \, d\nu_\gamma^{N}(\eta).$$

Choosing $B := 1/2N$ and using $\int \sqrt{f(\eta)} + \sqrt{f(\eta^{x,x+1})} \, d\nu_\gamma^{N}(\eta) \leq 4$, we have

$$A_1 \leq \frac{1}{2N} \sum_{x=1}^{N-2} \left( H_s \left( \frac{x}{N} \right) \right)^2 + \frac{N}{2} D_{N,0}(\sqrt{f}, \nu_\gamma^{N}).$$

To handle the term $A_2$ in (45), we use again the inequality $ab \leq \frac{1}{2N}a^2 + \frac{N}{2}b^2$:

$$A_2 \leq \frac{1}{4N} \sum_{x=1}^{N-2} \left( H_s \left( \frac{x}{N} \right) \right)^2 \int f(\eta^{x,x+1}) \, d\nu_\gamma^{N}(\eta) + \frac{1}{4} \sum_{x=1}^{N-2} \int f(\eta^{x,x+1}) N \left( \frac{\nu_\gamma^{N}(\eta^{x,x+1})}{\nu_\gamma^{N}(\eta)} - 1 \right)^2 \, d\nu_\gamma^{N}(\eta).$$

Since $f$ is a density, and the bound (31), we have

$$A_2 \leq \frac{1}{4N} \sum_{x=1}^{N-2} \left( H_s \left( \frac{x}{N} \right) \right)^2 + C.$$ 

Combining (45), (46) and (47) gives

$$A \leq \frac{3}{4N} \sum_{x=1}^{N-2} \left( H_s \left( \frac{x}{N} \right) \right)^2 + \frac{N}{2} D_{N,0}(\sqrt{f}, \nu_\gamma^{N}) + C + o(1).$$

Substituting in (44) and choosing $c = 3/4$, we have

$$\frac{1}{N} \Gamma_s \leq \sup_{f \text{ density}} \left\{ \frac{3}{4N} \sum_{x=1}^{N-2} \left( H_s \left( \frac{x}{N} \right) \right)^2 - \frac{3}{4} \int_0^1 (H_s(u))^2 \, du + C + o(1) \right\}.$$ 

Since $\frac{1}{N} \sum_{x=1}^{N-2} \left( H_s \left( \frac{x}{N} \right) \right)^2 \to \int_0^1 (H_s(u))^2 \, du$, when $N \to +\infty$, we obtain

$$\limsup_{N \to \infty} \int_0^T \frac{1}{N} \Gamma_s \, ds \leq CT.$$ 

It follows that

$$E_{\mathbb{P}} \left[ \sup_{H} \left\{ \ell_\rho(H) - \frac{3}{4} \left\| H \right\|_{L^2([0,T] \times (0,1))}^2 \right\} \right] \leq K_0 + CT.$$ 

7. Characterization of the Limit Points

This section deals with the characterization of limit points in the three ranges of $\theta \geq 0$. We will focus in the case $\theta = 1$, because this is the critical case. The other ones have similar proofs. We will also present the proof of the cases $\theta \in (0,1]$ and $\theta \in (1,\infty)$ pointing out the main differences between these cases and the proof of the case $\theta = 1$. 

$$\text{□}$$
7.1. Characterization of the limit points for $\theta = 1$. Now we look at the limit points of the sequence $\{Q_N\}_{N \in \mathbb{N}}$. We would stress that by Proposition 6.3, if $Q^*$ is a limit point of $\{Q_N\}_{N \in \mathbb{N}}$, then the support of $Q_*$ is contained in the set of trajectories $\pi \in D_M[0, T]$ such that $\pi_t$ is a Lebesgue absolutely continuous measure with density $\rho_t$ in $\mathcal{H}^1(0,1)$, for almost surely $t \in [0,T]$.

**Proposition 7.1.** If $Q^*$ is a limit point of $\{Q_N\}_{N \in \mathbb{N}}$, then it is true that

$$Q^* \left[ \pi : \langle \rho_t, \gamma \rangle - \langle \gamma, H_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle \, ds - \int_0^t \rho_s(0)\partial_u H_s(0) - \rho_s(1)\partial_u H_s(1) \right] ds$$

$$- \int_0^t \{ H_s(0)(\alpha - \rho_s(0)) + H_s(1)(\beta - \rho_s(1)) \} \, ds = 0, \quad \forall t \in [0,T], \quad \forall H \in C^{1,2}([0,T] \times [0,1]) \right] = 1. \quad \text{(49)}$$

By density, all we need to verify is that, for $\delta > 0$ and $H \in C^{1,3}([0,T] \times [0,1])$ fixed,

$$Q^* \left[ \pi : \sup_{0 \leq t \leq T} \left| \langle \rho_t, H_t \rangle - \langle \gamma, H_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle \, ds - \int_0^t \rho_s(0)\partial_u H_s(0) - \rho_s(1)\partial_u H_s(1) \right] ds$$

$$+ \int_0^t \rho_s(1)[\partial_u H_s(1) + H_s(1)] \, ds - \int_0^t (\alpha H_s(0) + \beta H_s(1) \, ds > \delta \right] = 0. \quad \text{(50)}$$

We would like to work with the probabilities $Q_N$, using Portmanteau’s Theorem. Unfortunately the set inside the above probability is not an open set in the Skorohod space. In order to avoid this problem, we will substitute $\rho_0(0)$ and $\rho_1(1)$ by the averages $\frac{1}{\varepsilon} \pi_s[0, \varepsilon]$ and $\frac{1}{\varepsilon} \pi_s[1 - \varepsilon, 1]$. We start bounding from above the probability in (50) by the sum of the following terms

$$Q^* \left[ \pi : \sup_{0 \leq t \leq T} \left| \langle \pi_t, H_t \rangle - \langle \pi_0, H_0 \rangle - \int_0^t \langle \pi_s, (\partial_s + \Delta)H_s \rangle \, ds - \int_0^t \frac{1}{\varepsilon} \pi_s[0, \varepsilon][\partial_u H_s(0) - H_s(0)] \right] ds$$

$$+ \int_0^t \frac{1}{\varepsilon} \pi_s[1 - \varepsilon, 1][\partial_u H_s(1) + H_s(1)] \, ds - \int_0^t (\alpha H_s(0) + \beta H_s(1) \, ds > \delta / 4 \right], \quad \text{(51)}$$

$$Q^* \left[ \pi : \sup_{0 \leq t \leq T} \left| \langle \pi_t, H_t \rangle - \langle \gamma, H_0 \rangle \right| > \delta / 4 \right], \quad \text{(52)}$$

$$Q^* \left[ \pi : \sup_{0 \leq t \leq T} \left| \int_0^t [\partial_u H_s(0) - H_s(0)] \left[ \frac{1}{\varepsilon} \pi_s[0, \varepsilon] - \rho_s(0) \right] \, ds \right| > \delta / 4 \right] \quad \text{(53)}$$

and

$$Q^* \left[ \pi : \sup_{0 \leq t \leq T} \left| \int_0^t [\partial_u H_s(1) + H_s(1)] \left[ \frac{1}{\varepsilon} \pi_s[1 - \varepsilon, 1] - \rho_s(1) \right] \, ds \right| > \delta / 4 \right]. \quad \text{(54)}$$

The probability (52) is equal to zero, because $Q^*$ is a limit point of $\{Q_N\}_{N \in \mathbb{N}}$, each measure $Q_N$ has as initial measure $\mu_N$ and $\{\mu_N\}_{N \in \mathbb{N}}$ is a sequence associated to the initial profile $\gamma : [0,1] \to \mathbb{R}$.

The next step is to prove that (53) and (54) converge to zero when $\varepsilon \to 0$. This is a consequence of the following lemma, which is proved in the end of this subsection.

**Lemma 7.2.** Let $\kappa > 0$. Then

$$\lim_{\varepsilon \to 0} Q^* \left[ \sup_{0 \leq t \leq T} \int_0^t \left| \frac{1}{\varepsilon} \pi_s[0, \varepsilon] - \rho_s(0) \right| \, ds > \kappa \right] = 0,$$

$$\lim_{\varepsilon \to 0} Q^* \left[ \sup_{0 \leq t \leq T} \int_0^t \left| \frac{1}{\varepsilon} \pi_s[1 - \varepsilon, 1] - \rho_s(1) \right| \, ds > \kappa \right] = 0.$$

Now, we look at (51). It is not yet possible to apply Portmanteau’s Theorem directly. The next lemma address this question.
Lemma 7.3. The probability in (51) is bounded by

\[ \liminf_{N \to \infty} \mathbb{Q}_N \left[ \sup_{0 \leq t \leq T} \left| \langle \pi_t, H_s \rangle - \langle \pi_0, H_0 \rangle - \int_0^t \langle \pi_s, (\partial_s + \Delta)H_s \rangle \, ds \right| \right. \]
\[ \left. - \int_0^t \frac{1}{\varepsilon} \pi_s[0, \varepsilon][\partial_u H_s(0) - H_s(0)] \, ds + \int_0^t \frac{1}{\varepsilon} \pi_s[1 - \varepsilon, 1][\partial_u H_s(1) + H_s(1)] \, ds \right. \]
\[ \left. - \int_0^t \alpha H_s(0) + \beta H_s(1) \, ds \right| > \delta/16 \]. (55)

The proof of this lemma is also postponed for the end of this subsection.

Summing and subtracting \( \int_0^t N^2 L_N \langle \pi_s^N, H_s \rangle \) from the expression above, we bound it by the sum of

\[ \limsup_{N \to \infty} \mathbb{Q}_N \left[ \sup_{0 \leq t \leq T} |M_t^N(H)| > \delta/32 \right] \] (56)

and

\[ \limsup_{N \to \infty} \mathbb{Q}_N \left[ \sup_{0 \leq t \leq T} \left| \int_0^t N^2 L_N \langle \pi_s^N, H_s \rangle - \int_0^t \langle \pi_s^N, \Delta H_s \rangle \, ds \right. \]
\[ \left. - \int_0^t \frac{1}{\varepsilon} \pi_s[0, \varepsilon][\partial_u H_s(0) - H_s(0)] \, ds + \int_0^t \frac{1}{\varepsilon} \pi_s[1 - \varepsilon, 1][\partial_u H_s(1) + H_s(1)] \, ds \right. \]
\[ \left. - \int_0^t \alpha H_s(0) + \beta H_s(1) \, ds \right| > \delta/32 \], (57)

where \( M_t^N(H) \) was defined in (22).

The remark in Section 5 (see (26)) help us to conclude that (56) is zero. Now we only have to estimate (57), which may be rewritten as

\[ \limsup_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t N^2 L_N \langle \pi_s^N, H_s \rangle - \int_0^t \langle \pi_s^N, \Delta H_s \rangle \, ds \right. \]
\[ \left. - \int_0^t \eta_s^N(1)[\partial_u H_s(0) - H_s(0)] \, ds + \int_0^t \eta_s^N(N - 1)[\partial_u H_s(1) + H_s(1)] \, ds \right. \]
\[ \left. - \int_0^t \alpha H_s(0) + \beta H_s(1) \, ds \right| > \delta/32 \], (58)

where \( \pi_s^N \) is the empirical measure and \( \eta_s^N(1) (\eta_s^N(N - 1)) \) is the mean of the box in the right (left) of site \( x = 1 (x = N - 1) \). Recalling (28) and (29), the expression \( N^2 L_N \langle \pi_s, H_s \rangle \) can be rewritten as

\[ N^2 L_N \langle \pi_s, H_s \rangle = \frac{1}{N} \sum_{x=2}^{N-2} \eta_s(x) \Delta_N H_s \left( \frac{x}{N} \right) + \eta_s(1) \left[ \nabla_N H_s \left( \frac{1}{N} \right) - H_s \left( \frac{1}{N} \right) \right] \]
\[ - \eta_s(N - 1) \left[ \nabla_N H_s \left( \frac{N-1}{N} \right) + H_s \left( \frac{N-1}{N} \right) \right] + \left( \alpha H_s \left( \frac{1}{N} \right) + \beta H_s \left( \frac{N-1}{N} \right) \right). \]

Finally it is possible to bound the expression inside the probability in (58) by the sum of the following terms

\[ \sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \frac{1}{N} \sum_{x=2}^{N-2} \eta_s(x) \Delta_N H_s \left( \frac{x}{N} \right) - \langle \pi_s^N, \Delta H_s \rangle \right\} \, ds \right|, \] (59)

\[ \sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \eta_s^N(1)[\partial_u H_s(0) - H_s(0)] - \eta_s(1) \left[ \nabla_N H_s \left( \frac{1}{N} \right) - H_s \left( \frac{1}{N} \right) \right] \right\} \, ds \right|, \] (60)

\[ \sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \eta_s^N(N - 1)[\partial_u H_s(1) + H_s(1)] - \eta_s(N - 1) \left[ \nabla_N H_s \left( \frac{N-1}{N} \right) + H_s \left( \frac{N-1}{N} \right) \right] \right\} \, ds \right| \] (61)
and

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \alpha(H_s(\frac{1}{t^2}) - H_s(0)) + \beta(H_s(\frac{N}{t}) - H_s(1)) \right\} ds \right|. \tag{62}
\]

Now, as \( H \in C^{1,3}(\{0, T\} \times [0, 1]), \Delta_N H_s(u) \to \Delta_N H_s(u) \) uniformly in \( s \) and in \( u \). If we look at the definition of \( \pi^N_s \) and keep in mind that \( \eta_s(x) \) only takes values \( \{0, 1\}, \) it is easy to see that \( (61) \) goes to zero, uniformly in \( \eta_s \). By the same reason \((H \in C^{1,3}(\{0, T\} \times [0, 1]))\), it follows that \( (62) \) also goes to zero. The other two terms are similar, then we will treat only \( (60) \). Let us begin by

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \left[ \partial_u H_s(0) - \nabla^+_N H_s(\frac{1}{N}) \right] ds \right| + \sup_{0 \leq t \leq T} \int_0^t \left[ \nabla^+_N H_s(\frac{1}{N}) - H_s(0) \right] \eta_s(1) ds
\]

As \( H \) is continuously differentiable in the first variable we have that \( \nabla^+_N H_s(\frac{1}{N}) \to \partial_u H_s(0) \) uniformly in \( s \). This implies that the first term above converges to zero as \( N \to \infty \). The second term converges for similar reasons. We can handle \( (61) \) in the same way. Then, instead of \( (58) \), we may only look at the following expression as \( N \to \infty \), for \( x \in \{1, N-1\} \) and \( \delta > 0 \)

\[
\mathbb{P}_{\mu_N} \left[ \eta_s : \sup_{0 \leq t \leq T} \left| \int_0^t \left[ \nabla^2_N H_s^2(\frac{1}{N}) - H_s(\frac{k}{N}) \right] \eta_s(x) ds \right| > \delta \right],
\]

where \( a = +, if x = 1 \) and \( a = 1, if x = N-1 \). Applying the Proposition \( 5.7 \) (Replacement Lemma) we conclude that, if \( x \in \{1, N-1\} \) the above expression converges to 0 if \( N \to \infty \) and \( \varepsilon \to 0 \). This concludes the proof of \( (59) \).

Now we have to prove the technical lemmas used above.

**Proof of Lemma 7.2.** Let us show the first equation. The second verification is analogous. We start with an estimate.

\[
\left| \frac{1}{\varepsilon} \pi_s[0, \varepsilon] - \rho_s(0) \right| = \frac{1}{\varepsilon} \int_0^\varepsilon \left| \rho_s(u) - \rho_s(0) \right| du
\]

Using the inequality \( ab \leq Ca^2 + b^2/C \), \( \forall a, b \in \mathbb{R} \) and \( C > 0 \), we get

\[
\left| \frac{1}{\varepsilon} \pi_s[0, \varepsilon] - \rho_s(0) \right| \leq \frac{C}{\varepsilon} \| \partial_u \rho_s \|_{L^2[0,1]}^2 + \frac{1}{\varepsilon C} \int_0^\varepsilon (\varepsilon - v)^2 dv = \frac{C}{\varepsilon} \| \partial_u \rho_s \|_{L^2[0,1]}^2 + \frac{\varepsilon^2}{3C}.
\]

Taking \( C = \varepsilon^{3/2} \), we have

\[
\left| \frac{1}{\varepsilon} \pi_s[0, \varepsilon] - \rho_s(0) \right| \leq \sqrt{\varepsilon} \left( \| \partial_u \rho_s \|_{L^2[0,1]}^2 + \frac{1}{3} \right).
\]

Then

\[
\sup_{0 \leq t \leq T} \int_0^t \left| \frac{1}{\varepsilon} \pi_s[0, \varepsilon] - \rho_s(0) \right| ds \leq \sqrt{\varepsilon} \| \partial_u \rho_s \|_{L^2([0,T] \times [0,1])}^2 + \frac{T \sqrt{\varepsilon}}{3}.
\]

In the second equality we assume \( \rho_s \) absolutely continuous with derivative \( \partial_u \rho_s \) (see [1], Theorem 8.2).
From Corollary A.5 in [12], we obtain that
\[ \|\partial_u \rho\|_{L^2([0,T] \times [0,1])}^2 = \frac{1}{3} \sup_H \left\{ \int_0^t \int_0^1 \partial_u H_s(u) \rho_s(u) \, du \, ds - \frac{3}{4} \int_0^T \int_0^1 (H_s(u))^2 \, du \, ds \right\} \]
where the supreme above is taken over the set of functions \( H \in C_c^{1,2}([0, T] \times (0, 1)) \).

Finally, we calculate
\[ Q^* \left[ \sup_{0 \leq t \leq T} \left\{ \int_0^t \frac{1}{\varepsilon} \pi_s[0, \varepsilon] - \rho_s(0) \, ds \geq \kappa \right\} \right] \leq Q^* \left[ \sqrt{\varepsilon} \|\partial_u \rho\|_{L^2([0,T] \times [0,1])}^2 + \frac{T \sqrt{\varepsilon}}{3} > \kappa \right] \]
\[ = Q^* \left[ \|\partial_u \rho\|_{L^2([0,T] \times [0,1])}^2 > \frac{\kappa}{\sqrt{\varepsilon}} - \frac{T}{3} \right] . \]

Take \( \varepsilon \) such that \( \frac{\kappa}{\sqrt{\varepsilon}} - \frac{T}{3} > 0 \). The above probability is equal to
\[ Q^* \left[ \sup_H \left\{ \int_0^t \int_0^1 \partial_u H_s(u) \rho_s(u) \, du \, ds - \frac{3}{4} \int_0^T \int_0^1 (H_s(u))^2 \, du \, ds \right\} \right] > 8 \left( \frac{\kappa}{\sqrt{\varepsilon}} - \frac{T}{3} \right) . \]

By the Markov’s inequality and (42) we bound the last expression by \( \frac{K_0}{8} \left( \frac{\kappa}{\sqrt{\varepsilon}} - \frac{T}{3} \right)^{-1} \), which converges to 0 as \( \varepsilon \to 0 \) and finishes the proof of the lemma.

\[ \square \]

**Proof of Lemma 7.3** Let \( \gamma > 0 \). Define functions \( f_\gamma : [0, 1] \to \mathbb{R} \) and \( g_\gamma : [0, 1] \to \mathbb{R} \) such that \( f_\gamma \) is equal to \( 1/\varepsilon \) in \([0, \varepsilon], 0 \) in \([\varepsilon + \gamma, 1]\) and it is linear in \([\varepsilon, \varepsilon + \gamma]\) while \( g_\gamma \) is equal to \( 1/\varepsilon \) in \([1 - \varepsilon, 1], 0 \) in \([0, 1 - \varepsilon - \gamma]\) and it is linear in \([1 - \varepsilon - \gamma, 1 - \varepsilon]\).

To simplify notation, define, for \( t \in [0, T] \) and \( \pi, \pi' \in \mathcal{D}_M[0, T] \),
\[ \varphi(t, \pi, \pi') := \langle \pi_t, H_t \rangle - \langle \pi_0, H_0 \rangle - \int_0^t \langle \pi_s, (\partial_s + \Delta) H_s \rangle \, ds - \int_0^t \frac{1}{\varepsilon} \pi_s[0, \varepsilon] H_s(0) \, ds \]
\[ + \int_0^t \frac{1}{\varepsilon} \pi_s[1 - \varepsilon, 1] \partial_s H_s(1) \, ds - \int_0^t \alpha(H_s(0) + H_s(1)) \, ds \] (63)
and
\[ \varphi_\gamma(t, \pi, \pi') := \langle \pi_t, H_t \rangle - \langle \pi_0, H_0 \rangle - \int_0^t \langle \pi_s, (\partial_s + \Delta) H_s \rangle \, ds - \int_0^t \partial_s H_s(0) \, ds \]
\[ + \int_0^t \partial_s H_s(1) \, ds - \int_0^t f_\gamma(u) \, ds \] (64)

Using this notation the statement of the lemma becomes
\[ Q^*[\pi, \sup_{0 \leq t \leq T} |\varphi(t, \pi, \pi')| > \delta/4] \leq \liminf_{N \to \infty} Q_N[\pi, \sup_{0 \leq t \leq T} |\varphi(t, \pi, \pi')| > \delta/16] . \]

The strategy of proof is to change \( \varphi \) by \( \varphi_\gamma \) and show that \( \pi \mapsto \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi)| \) is lower semi-continuous. Notice that
\[ Q^*[\pi, \sup_{0 \leq t \leq T} |\varphi(t, \pi, \pi')| > \delta/4] \leq Q^*[\pi, \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi, \pi')| > \delta/8] + Q^*[\pi, \sup_{0 \leq t \leq T} |\varphi(t, \pi, \pi') - \varphi_\gamma(t, \pi, \pi')| > \delta/8] . \] (65)

Making simple computations one can estimate the difference \( |\varphi(t, \pi, \pi') - \varphi_\gamma(t, \pi, \pi')| \) by
\[ \frac{Ct}{\varepsilon} \left[ \sup_{0 \leq t \leq T} \pi_t(\varepsilon, \varepsilon + \gamma) + \sup_{0 \leq t \leq T} \pi_t(1 - \varepsilon - \gamma, 1 - \varepsilon) \right] , \]
where \( C \) is a bound for \( H \) and \( \partial_s H \) in \([0, T] \times [0, 1] \). Since \( Q^*\)-almost surely \( \pi \) has density \( \rho \) bounded by 1, for \( \gamma \) small enough the second term in the right hand side of the inequality (65) becomes negligible.

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The next step is to show that the function \( \pi \mapsto \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi)| \) is lower semi-continuous. In order to do that let us consider \((\pi^m)_{m \in \mathbb{N}}\) a sequence in \(D_M[0, T]\) converging to \(\pi\). Let us show that
\[
\liminf_{m \to \infty} \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi^m)| \geq \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi)|.
\]

Indeed, let \(\kappa > 0\), there exists \(t_0 \in [0, T]\) such that
\[
|\varphi_\gamma(t_0, \pi)| - \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi)| < \kappa. 
\]
Since \(\varphi_\gamma\) is a right-continuous function in the first coordinate, then for \(t'_0 \geq t_0\) and sufficiently close to \(t_0\)
\[
|\varphi_\gamma(t_0, \pi)| - |\varphi_\gamma(t'_0, \pi)| < \kappa. 
\]
(66)

Since \((\pi^m)_{m \in \mathbb{N}}\) is a sequence in \(D_M[0, T]\) converging to \(\pi\), then \(\pi^m \to \pi_t\), for almost surely \(t \in [0, T]\). Thus, it is possible to find \(t'_0\) such that \(\pi^m \to \pi'_t, \varphi_\gamma(t'_0, \pi^m) \to \varphi_\gamma(t'_0, \pi)\) and (66) holds.

Taking \(\kappa \to 0\) we finish the proof of lower semi-continuity of \(\sup_{0 \leq t \leq T} |\varphi_\gamma(t, \cdot)|\). Finally, we can use the Portmanteau’s Theorem to write
\[
\mathbb{Q}^*[\pi : \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi)| > \delta/8] \leq \liminf_{N \to \infty} \mathbb{Q}_N[\pi : \sup_{0 \leq t \leq T} |\varphi_\gamma(t, \pi)| > \delta/8] 
\]
\[
\leq \liminf_{N \to \infty} \mathbb{Q}_N[\pi : \sup_{0 \leq t \leq T} |\varphi(t, \pi)| + \sup_{0 \leq t \leq T} |\varphi(t, \pi) - \varphi_\gamma(t, \pi)| > \delta/8].
\]
Recall that \(\mathbb{Q}_N\)-almost surely \(\pi_t(a, b) \leq (b - a)\), for all \((a, b) \in [0, 1]\). Thus, taking \(\gamma\) small enough, we get
\[
\sup_{0 \leq t \leq T} |\varphi(t, \pi) - \varphi_\gamma(t, \pi)| < \delta/16,
\]
and the lemma follows.

\(\square\)

7.2. Characterization of Limit Points for \(\theta \in [0, 1]\). As in the last section, we will look at the limit points of the sequence \(\{\mathbb{Q}_N\}_{N \in \mathbb{N}}\).

**Proposition 7.4.** If \(\mathbb{Q}^*\) is a limit point of \(\{\mathbb{Q}_N\}_{N \in \mathbb{N}}\), we have
\[
\mathbb{Q}^* \left[ \pi : \left\langle \rho_t, H_t \right\rangle - \langle \gamma, H_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle ds + \int_0^t \left\{ \beta \partial_u H_s(1) - \alpha \partial_u H_s(0) \right\} ds = 0, \right. \forall t \in [0, T], \forall H \in C^{1, 2}_0([0, T] \times [0, 1]) \right]\ = 1. 
\]
(67)

As in the case \(\theta = 1\), using density, it is enough to verify that, for \(\delta > 0\) and \(H \in C^{1, 3}_0([0, T] \times [0, 1])\)
fixed,
\[
\mathbb{Q}^* \left[ \pi : \sup_{0 \leq t \leq T} \left\langle \rho_t, H_t \right\rangle - \langle \gamma, H_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle ds + \int_0^t \left\{ \beta \partial_u H_s(1) - \alpha \partial_u H_s(0) \right\} ds \geq \delta \right] = 0. 
\]
(68)

Since the set considered above is an open set, we can use the Portmanteau’s Theorem directly and bound the last probability by
\[
\liminf_{N \to \infty} \mathbb{Q}_N \left[ \pi : \sup_{0 \leq t \leq T} \left\langle \rho_t, H_t \right\rangle - \langle \gamma, H_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle ds + \int_0^t \left\{ \beta \partial_u H_s(1) - \alpha \partial_u H_s(0) \right\} ds \geq \delta \right].
\]
(69)
Following the same steps of the last section, we need only to bound

\[
\liminf_{N \to \infty} Q_N \left[ \pi : \sup_{0 \leq t \leq T} \left| \int_0^t N^2 L_N \langle \pi_s^N, H_s \rangle \, ds - \int_0^t \langle \pi_s^N, \Delta H_s \rangle \, ds + \int_0^t \{ \beta \partial_u H_s(1) - \alpha \partial_u H_s(0) \} \, ds \right| > \delta / 2 \right],
\]

(70)

where \( \pi_s^N \) is the empirical measure. Using the expressions (28) and (29) the expression of \( N^2 L_N \langle \pi_s, H_s \rangle \) can be rewritten as

\[
N^2 L_N \langle \pi_s, H_s \rangle = \frac{1}{N} \sum_{x=2}^{N-2} \eta_s(x) \Delta_N H_s \left( \frac{x}{N} \right) + \eta_s(1) \nabla_N H_s \left( \frac{1}{N} \right) - \eta_s(N-1) \nabla_N H_s^- \left( \frac{N-1}{N} \right)
\]

\[
+ N^{1-\theta} \left( \alpha - \eta_s(1) \right) H_s \left( \frac{1}{N} \right) + N^{1-\theta} \left( \beta - \eta_s(N-1) \right) H_s \left( \frac{N-1}{N} \right).
\]

(71)

Finally, it is possible to bound the expression inside the probability in (70) by the sum of the following terms

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \frac{1}{N} \sum_{x=2}^{N-2} \eta_s(x) \Delta_N H_s \left( \frac{x}{N} \right) - \langle \pi_s^N, \Delta H_s \rangle \right\} \, ds \right|,
\]

(72)

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \eta_s(1) \nabla_N H_s \left( \frac{1}{N} \right) - \alpha \partial_u H_s(0) \right\} \, ds \right|,
\]

(73)

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \beta \partial_u H_s(1) - \eta_s(N-1) \nabla_N H_s^- \left( \frac{N-1}{N} \right) \right\} \, ds \right|,
\]

(74)

and

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ N^{1-\theta} \left( \alpha - \eta_s(1) \right) H_s \left( \frac{1}{N} \right) + N^{1-\theta} \left( \beta - \eta_s(N-1) \right) H_s \left( \frac{N-1}{N} \right) \right\} \, ds \right|,
\]

(75)

Since \( H \in C^1(0, T) \times [0, 1] \), and keeping in mind that \( \eta_s(x) \) only takes values in \( \{0, 1\} \), it is easy to see that (75) goes to zero, uniformly in \( \eta \). By the same reason as in the case \( \theta = 1 \), it follows that (59) also goes to zero. The other two terms are similar. Let us begin bounding (73) by

\[
\sup_{0 \leq t \leq T} \left| \int_0^t \left[ \eta_s(1) \left[ \nabla_N H_s \left( \frac{1}{N} \right) - \partial_u H_s(0) \right] \right] \, ds \right|
\]

\[
+ \sup_{0 \leq t \leq T} \left| \int_0^t \partial_u H_s(0) \left[ \eta_s(1) - \alpha \right] \, ds \right|.
\]

As \( H \) is continuously differentiable in the first variable, we have \( \nabla_N H_s \left( \frac{1}{N} \right) \to \partial_u H_s(0) \) uniformly in \( s \). This implies that the first term above converge to zero as \( N \to \infty \). We can treat (61) in the same way. Then, instead of (70), we may only look at the following expression as \( N \to \infty \)

\[
\mathbb{P}_{\mu_N} \left[ \eta : \sup_{0 \leq t \leq T} \left| \int_0^t \partial_u H_s(0) \left[ \eta_s(1) - \alpha \right] \, ds \right| > \delta \right]
\]

and

\[
\mathbb{P}_{\mu_N} \left[ \eta : \sup_{0 \leq t \leq T} \left| \int_0^t \partial_u H_s(1) \left[ \eta_s(N-1) - \beta \right] \, ds \right| > \tilde{\delta} \right],
\]

for all \( \delta > 0 \).

Applying the Proposition 6.5 (Replacement Lemma) we conclude that the above expressions converge to 0 as \( N \to \infty \) and this finishes the proof of (68).
7.3. Characterization of Limit Points for $\theta \in (1, \infty)$. As before we will look at the limit points of the sequence $\{Q_N\}_{N \in \mathbb{N}}$.

**Proposition 7.5.** If $Q^*$ is a limit point of $\{Q_N\}_{N \in \mathbb{N}}$, it is true that

$$Q^* \left[ \pi : \langle \rho_t, H_t \rangle - \langle \gamma, H_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta) H_s \rangle ds + \int_0^t \{ \rho_s(1) \partial_s H_s(1) - \rho_s(0) \partial_s H_s(0) \} ds = 0, \right. $$

$$ \left. \forall t \in [0, T], \ \forall H \in C_{0}^{1,2}([0, T] \times [0, 1]) \right] = 1. \quad (76)$$

Reasoning in the same way as above, it is enough to verify that, for $\delta > 0$ and $H \in C^{1,3}([0, T] \times [0, 1])$ fixed, we have

$$Q^* \left[ \pi : \sup_{0 \leq t \leq T} \left| \langle \rho_t, H_t \rangle - \langle \gamma, H_0 \rangle - \int_0^t \langle \rho_s, (\partial_s + \Delta) H_s \rangle ds + \int_0^t \{ \rho_s(1) \partial_s H_s(1) - \rho_s(0) \partial_s H_s(0) \} ds \right| > \delta \right] = 0. \quad (77)$$

Here, we have boundary terms in $\rho$, then to be able to use the Portmanteau’s Theorem we need to do the same as in Subsection 7.1, where we changed the boundary terms $\rho_s(0)$ $(\rho_s(1))$ by $\eta^N_s(1)$ $(\eta^N_s(N-1))$. After that, we can apply the Portmanteau’s Theorem and use, as usual, (52) to handle the two first terms. Then, to handle the other terms we sum and subtract $\int_0^t N^2 L_N(\pi, H_s)$ in order to get

$$\liminf_{N \to \infty} Q_N \left[ \pi : \sup_{0 \leq t \leq T} \left| \int_0^t N^2 L_N(\pi^N_s, H_s) ds - \int_0^t \langle \pi^N_s, \Delta H_s \rangle ds \right. \right.$$

$$\left. + \int_0^t \{ \eta^N_s(N-1) \partial_s H_s(1) - \eta^N_s(N-1) \partial_s H_s(0) \} ds \right| > \delta \right]. \quad (78)$$

Recalling the expression for $N^2 L_N(\pi, H_s)$ given in (71), the expression inside the probability in (70) can be bounded by the sum of the following terms

$$\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \frac{1}{N} \sum_{x=2}^{N-2} \eta_s(x) \Delta_N H_s \left( \frac{x}{N} \right) - \langle \pi^N_s, \Delta H_s \rangle \right\} ds \right|. \quad (79)$$

$$\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \eta_s(1) \nabla_N^\perp H_s \left( \frac{1}{N} \right) - \eta^N_s(N-1) \partial_s H_s(0) \right\} ds \right|, \quad (80)$$

$$\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ \eta^N_s(N-1) \partial_s H_s(1) - \eta(N-1) \nabla_N H_s \left( \frac{N-1}{N} \right) \right\} ds \right| \quad (81)$$

and

$$\sup_{0 \leq t \leq T} \left| \int_0^t \left\{ N^{1-\theta} \left( \alpha - \eta_s(1) \right) H_s \left( \frac{1}{N} \right) + N^{1-\theta} \left( \beta - \eta(N-1) \right) H_s \left( \frac{N-1}{N} \right) \right\} ds \right|. \quad (82)$$

Since $H \in C^{1,3}([0, T] \times [0, 1])$ and $\theta \in (1, \infty)$, we have that (79) and (82) go to zero, uniformly in $\eta$, when $N \to \infty$.

We observe the difference between (75) and (82) implies that we need to work with different test functions in the different cases. In the first one as $\theta \in [0, 1)$, we need that $H \in C_{0}^{1,3}([0, T] \times [0, 1])$ to eliminated the term (75), but here, in the case $\theta \in (1, \infty)$ the term (82) converges to zero, without any extra condition.

The other two terms are similar. In both, as above, we will use that $\nabla_N^\perp H_s \left( \frac{1}{N} \right) \to \partial_s H_s(0)$ uniformly in $s$ and the Proposition 6.7 (Replacement Lemma) to conclude that the above expressions converge to 0 as $N \to \infty$. This finishes the proof of (77).
8. Uniqueness of Weak Solutions

The uniqueness of weak solutions of (15) is standard and, for instance, we refer to [13] for a proof. For the equation (19), a proof of uniqueness can be found in [11]. It remains to prove uniqueness of weak solutions of the parabolic differential equation (17), and we will do that in the next subsection.

8.1. Uniqueness of weak solutions of (17). Now we head to our last statement: the weak solution of (17) is unique. To prove this, it is enough to consider \( \alpha = \beta = 0 \), and that our solution \( \rho \) satisfies

\[
\langle \rho_t, H_t \rangle = \int_0^t \langle \rho_s, (\partial_s + \Delta)H_s \rangle ds \\
+ \int_0^t \{ \rho_s(0)\partial_u H_s(0) - \rho_s(1)\partial_u H_s(1) \} ds - \int_0^t \left\{ H_s(0)\rho_s(0) + H_s(1)\rho_s(1) \right\} ds,
\]

for all \( H \in C^{1,2}([0, T] \times [0, 1]) \).

We begin by considering the set \( \mathcal{H} \) of functions \( H \in C^1[0, 1] \) such that

1. \( \partial_u H \) is an absolutely continuous function;
2. \( \Delta H \in L^2[0, 1] \);
3. \( \partial_u H(0) = H(0) \);
4. \( \partial_u H(1) = -H(1) \).

Observe that the operator \( \Delta : \mathcal{H} \to L^2[0, 1] \) is injective in its domain. Given \( g \in L^2[0, 1] \), we define

\[
(-\Delta)^{-1}g(u) := \int_0^1 G(r, u)g(r)dr,
\]

where

\[
G(r, u) := (u + 1) \left( \frac{2 - r}{3} \right) - (u - r)1_{[0,u]}(r).
\]

It is easy to see that the following properties hold for \( g, h \in L^2[0, 1] \):

1. \( (-\Delta)^{-1}g \in \mathcal{H} \);
2. \( (-\Delta)(-\Delta)^{-1}g = g \);
3. \( \langle (-\Delta)^{-1}g, h \rangle = \langle g, (-\Delta)^{-1}h \rangle \) and \( \langle (-\Delta)^{-1}g, g \rangle \geq 0 \).

If \( \rho \) is a classical solution, the following identity is clear:

**Proposition 8.1.** Let \( \rho \) be a weak solution of (17). For any \( t \in [0, T] \) we have

\[
\langle \rho_t, (-\Delta)^{-1} \rho_t \rangle = -2 \int_0^t \langle \rho_s, \rho_s \rangle ds.
\]

The proof of this proposition is in the end of the section. It implies the uniqueness of weak solutions in the following way: if we combine Proposition 8.1 and the third property of \( (-\Delta)^{-1} \), we get

\[
\langle \rho_t, (-\Delta)^{-1} \rho_t \rangle = 0.
\]

As \( -\Delta \) is a bijection, there exists \( f_t \in \mathcal{H} \) such that \( \rho_t = -\Delta f_t \). We have then

\[
0 = \langle -\Delta f_t, f_t \rangle = f_t(1)^2 + f_t(0)^2 + \langle \partial_u f_t, \partial_u f_t \rangle, \text{ and } \partial_u f_t = 0, \text{ which implies } \rho_t = 0.
\]

Now we aim towards the proof of Proposition 8.1.

**Lemma 8.2.** Let \( H \in \mathcal{H} \) and \( \rho \) a weak solution of (17). We have

\[
\langle \rho_t, H \rangle = \int_0^t \langle \rho_s, \Delta H \rangle ds.
\]

**Proof.** Take \( \{ g_n \} \in C[0, 1] \) a sequence that converges in \( L^2 \) to \( \Delta H \in L^2[0, 1] \). Define the functions

\[
G_n(u) = H(0) + u\partial_u H(0) + \int_0^u \int_0^v g_n(r) dr dv,
\]

where \( G_n \) converges to \( H \) in \( L^2 \). Then

\[
\int_0^t \langle \rho_s, \Delta H \rangle ds = \int_0^t \langle \rho_s, \Delta G_n \rangle ds.
\]

Since \( \rho_s \) is a weak solution, we have

\[
\langle \rho_s, \Delta G_n \rangle = \langle \Delta \rho_s, G_n \rangle - \langle \rho_s, \Delta G_n \rangle.
\]

Now, for the convergence of \( g_n \), we have

\[
\langle \Delta \rho_s, G_n \rangle - \langle \rho_s, \Delta G_n \rangle \to 0.
\]

Hence, we get

\[
\int_0^t \langle \rho_s, \Delta G_n \rangle ds = \int_0^t \langle \rho_s, \Delta G_n \rangle ds = \int_0^t \langle \rho_s, \Delta H \rangle ds.
\]

This proves the lemma.
and observe that

\[
\langle \rho_t, G_n \rangle = \int_0^t \langle \rho_s, g_n \rangle \, ds
\]
\[+ \int_0^t \rho_s(0) \partial_u H(0) - \rho_s(1) \left( \partial_u H(0) + \int_0^1 g_n(r) \, dr \right) \, ds
\]
\[\quad - \int_0^t H(0) \rho_s(0) + \rho_s(1) \left( H(0) + \partial_u H(0) + \int_0^1 \int_0^v g_n(r) \, dr \, dv \right) \, ds.\]  \tag{85}

Now, as \(G_n \rightarrow H\) in \(L^2\), we have

\[
\partial_u H(0) + \int_0^1 g_n(r) \, dr \rightarrow \partial_u H(1)
\]

and

\[
H(0) + \partial_u H(0) + \int_0^1 \int_0^v g_n(r) \, dr \, dv \rightarrow H(1).
\]

Once \(H \in \mathcal{H}\) expression (85) reduces to

\[
\langle \rho_t, H \rangle = \int_0^t \langle \rho_s, \Delta H \rangle \, ds
\]

as \(n \to \infty\). \(\square\)

**Proof of Proposition 8.7** We begin by taking a partition \(0 = t_0 < t_1 < \cdots < t_n = t\) of \([0, t]\). A telescopic sum allows us to write

\[
\langle \rho_t, (\Delta)^{-1} \rho_t \rangle = \langle \rho_t, (\Delta)^{-1} \rho_t \rangle - \langle \rho_0, (\Delta)^{-1} \rho_0 \rangle
\]
\[= \sum_{k=0}^{N-1} \langle \rho_{t_{k+1}}, (\Delta)^{-1} (\rho_{t_{k+1}} - \rho_{t_k}) \rangle + \sum_{k=0}^{N-1} \langle \rho_{t_{k+1}} - \rho_{t_k}, (\Delta)^{-1} \rho_{t_k} \rangle.
\]

We will prove that the second sum above converges to \(-\int_0^t \langle \rho_s, \rho_s \rangle \, ds\) when the mesh of the partition \(\{t_i\}\) goes to zero. Once the operator \((\Delta)^{-1}\) is symmetric, the first sum will converge to the same limit, and we get the result.

By Lemma 8.2 we can write

\[
\sum_{k=0}^{N-1} \langle \rho_{t_{k+1}} - \rho_{t_k}, (\Delta)^{-1} \rho_{t_k} \rangle = -\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \langle \rho_s, \rho_s \rangle \, ds + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \langle \rho_s - \rho_{t_k}, \rho_s \rangle \, ds.
\]  \tag{86}

Now we only need to show that the last sum above goes to zero as the mesh of the partitions \(\{t_i\}\) goes to zero. We now approximate \(\rho_s\) by a function in \(C_c^\infty(0, 1)\): given \(\delta > 0\), define

\[
\rho_s^\delta := \rho_s \ast \Phi_\delta,
\]

where \(\Phi_\delta\) is a mollifier and \(\Phi_\delta\) is a non-negative function in \(C_c^\infty(\mathbb{R})\) that is equal to 1 in \([2\delta, 1 - 2\delta]\) and is zero outside of \([\delta, 1 - \delta]\). We know that \(\rho_s^\delta\) converges to \(\rho_s\) in \(L^2(\mathbb{R})\). We write the second sum in (86) as

\[
\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \langle \rho_s - \rho_{t_k}, \rho_s^\delta \rangle \, ds + \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \langle \rho_s - \rho_{t_k}, \rho_s - \rho_s^\delta \rangle \, ds.
\]  \tag{87}

Now we bound the second sum above by

\[
\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \|\rho_s - \rho_{t_k}\|_{L^2[0, 1]} \|\rho_s - \rho_s^\delta\|_{L^2[0, 1]} \, ds.
\]

Once \(\rho\) is a bounded function, we can use the Dominated Convergence Theorem to conclude that the expression above goes to zero as \(\delta \to 0\). Now we fix \(\delta > 0\) and estimate the first sum in (87). By the
definition of weak solution, we get the bound
\[
\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \langle \rho_r, \Delta \rho^k \rangle \, dr \, ds \leq \sum_{k=0}^{N-1} \int_{t_k}^{s} \| \rho_r \|_{L^2([0,1])}^2 \| \Delta \rho^k \|_{L^2([0,1])}^2 \, dr \, ds.
\]
Once again we use that \( \rho \) is bounded, and we conclude that \( C(\rho, \delta) \sum_{k=0}^{N-1} (t_{k+1} - t_k)^2 \) is an upper bound for the first sum in (87), and this bound converges to zero if we take the limit on the mesh of the partition.

APPENDIX A.

Proposition A.1. Denote by \( H_N(\mu_N | \nu_\alpha) \) the entropy of a probability measure \( \mu_N \) with respect to a stationary state \( \nu_\alpha \). Then, there exists a finite constant \( K_0 := K_0(\alpha, \beta) \) such that
\[
H_N(\mu_N | \nu_\alpha) \leq K_0 N,
\]
for all probability measures \( \mu_N \).

Proof. Recall that \( \nu_\alpha \) is Bernoulli product of parameter \( \gamma(\frac{x}{N}) \) at site \( x \), and that \( \alpha \leq \gamma(u) \leq \beta \), for all \( u \in [0,1] \). By the explicit formula given in Theorem A 1.8.3 of [13], we have
\[
H_N(\mu_N | \nu_\alpha) = \sum_{\eta \in \{0,1\}^N} \mu_N(\eta) \log \frac{\mu_N(\eta)}{\nu_\alpha(\eta)} \leq \sum_{\eta \in \{0,1\}^N} \mu_N(\eta) \log \frac{1}{\prod_{x \in I_N} \gamma(\frac{x}{N}) \wedge (1 - \gamma(\frac{x}{N}))} \leq \sum_{\eta \in \{0,1\}^N} \mu_N(\eta) \log \frac{1}{[\alpha \wedge (1 - \beta)]^{N-1}} \leq N (-\log[\alpha \wedge (1 - \beta)]).
\]

Lemma A.2. Let \( \{a_i\}_i, \{b_i\}_i \) and \( \{x_i\}_i \) be sequences of non-negative real numbers. Then,
\[
\left( \sum_{i=1}^{n} a_i x_i \right)^{1/2} - \left( \sum_{i=1}^{n} b_i x_i \right)^{1/2} \leq \sum_{i=1}^{n} (\sqrt{a_i} - \sqrt{b_i})^2 x_i,
\]
for all \( n \geq 2 \).

Proof. Consider the set \( E = \{1, \ldots, n\} \), the measure \( \mu \) on \( E \), given by \( \mu(i) = x_i \), for all \( i \in E \) and define the functions \( f_j : E \to \mathbb{R}, j = 1,2 \), as \( f_1(i) = \sqrt{a_i} \) and \( f_2(i) = \sqrt{b_i} \), for all \( i \in E \). Then the inequality (88) can be rewritten as
\[
\| f_1 \|_{L^2(\mu)} - \| f_2 \|_{L^2(\mu)} \leq \| f_1 - f_2 \|_{L^2(\mu)}.
\]

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