Graph coloring for determining angklung distribution

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Abstract. Angklung is a traditional instrument originated from West Java, Indonesia. It is played by shaking it and each angklung can only produce one note. A set of angklungs are needed in order to produce a whole song. An angklung player can hold no more than seven angklungs because of the size of the instrument. A player also cannot hold two overlapping angklungs since shaking more than one angklung at the same time is nearly impossible. These restrictions create problems that can be modeled as mathematical problem. It is called the angklung distribution problem, which is the problem of distributing angklungs to each player based on a given song so that no player holds two or more overlapping angklungs with the least possible number of players. In this paper, we model this problem into a graph theoretical problem where we use the concept of graph coloring to solve this problem. We also develop a method to lower the number of players suggested from the graph coloring by viewing the song as an array of symbols.

1. Introduction

Graph theory is a field that has many real world applications, where real world problems are modeled by using graphs and solved by using methods developed in the study of graphs. In particular, graph coloring problem is a problem in graph theory that has been used to model some practical problems such as planning, scheduling, map coloring, and pattern matching. Some interesting connections have previously been found between graph theory and the study of musical instrument, such as in [1] and [2]. Connections between graph theory and various aspects in music theory in general also have been studied in [3], [4], [5], [6], and [7]. In this paper, we study the connection between graph theory and a traditional musical instrument called angklung by applying the concept of graph coloring to the problem of distributing angklungs.

1.1 Graph

Throughout this paper, a graph is defined to be the pair \((V,E)\) of finite vertex set \(V\) and edge set \(E\) (see [8]), where \(E\) is just the subset of the set of all 2-subsets of \(V\). If \(\{a,b\} \in E\), we say that the vertices \(a\) and \(b\) are adjacent. If \(G = (V,E)\) is a graph, a \(k\)-coloring of \(G\) is a function \(f: V \to \{1,2,\ldots,k\}\), where the elements in the set \(\{1,2,\ldots,k\}\) are called colors, such that adjacent vertices are mapped by \(f\) to different colors. Thus, the value \(f(v)\) serves as the color of vertex \(v\). If a \(k\)-coloring
exists for a graph, the graph is said to be \( k \)-colorable. The \textbf{chromatic number} of a graph \( G \), denoted by \( \chi(G) \), is the least \( k \) such that \( G \) is \( k \)-colorable.

A graph \( H = (V',E') \) is called a \textbf{subgraph} of the graph \( G = (V,E) \) if \( V' \subseteq V \) and \( E' \subseteq E \). A \textbf{complete graph} is a graph where the edge set is the whole set of all 2-subsets of the vertex set. A \textbf{clique} is a complete subgraph of a graph, where a largest clique is a clique that has the most vertices. The \textbf{clique number} \( \omega(G) \) is the number of vertices of a largest clique of \( G \).

\[ \text{1.2. Angklung} \]

Angklung is a traditional musical instrument from Indonesia that originated in West Java and Banten provinces, Indonesia (see [9]). In the past, angklung was played for honoring a goddess or for signaling prayer time. It was in 1930's that the angklung people know today existed and turned into an instrument that can play many kinds of modern music by Daeng Sutigna. Indonesian Angklung was officially recognized by UNESCO as a Masterpiece of Oral and Intangible Heritage of Humanity on November 18, 2010. Following this recognition, angklung performances have become more popular especially in Indonesia and effort is continuously being made to spread and introduce the music of angklung to the world.

Angklung is a percussion instrument where the strokes are produced by shaking two bamboo tubes. More precisely, it consists of two half-open bamboo tubes that produce two notes that differ by an octave when shaken, the base tube that is also made of bamboo, and the bamboo frame that connects the three tubes. When angklung is shaken, the frame holds the upper parts of the two half-open tubes and their bottom parts hit the base back and forth, producing sound from the two half-open tubes. An angklung player commonly holds the instrument in one hand at the mid section of the frame and shakes it with the other hand, preferably the dominant one, at the side of the base. One player can hold more than one angklung by putting his hand through the mid section of the frame of the bigger angklung and add the smaller angklung after that. Due to the size of angklung and the way to play it, each player can only hold about five to seven angklungs. Details are shown in Figure 1.

![Figure 1. Angklung and its parts.](image)

One angklung is considered to be representing only one note, which is the note produced by the bigger tube. This means that one player can only play about up to seven notes. Hence, angklung needs to be played in an ensemble if one wants to cover a full song. There are many angklung ensembles performing musical composition specifically arranged for the instrument every year. The uniqueness of angklung performance has made more people interested in attending concerts of
angklung, thus increasing the number of performances and the number of ensembles almost every year.

Considering the amount of performances held and ensembles performing every year, methods of organizing angklung performance have been studied by each ensemble. Especially when preparing for a concert, an angklung ensemble must face many technical problems and solve them before finally having a successful performance. One of these problems is the angklung distribution problem, which is arguably one of the most important problems in preparing a performance. It is the problem of distributing angklungs to as few players as possible such that no player has two overlapping angklungs. Overlapping angklungs exist when there is a player holding these angklungs but he/she cannot shake them at once since these angklungs represent notes that must be played nearly at the same time. Forcing to shake them at once will decrease the quality of the sound production significantly.

Overlapping angklungs is one of the main reasons why more people are required to perform a full song. Hence, knowing the minimum number of angklung players required to perform the song is a must. Angklung distribution is usually done by looking for instances of overlapping angklungs, then distribute the angklungs to players one by one manually by considering the angklungs that are overlapping. This time-consuming effort still does not ensure that the number of players required is the least. With the help of graph theoretic model which is motivated by [10], as well as an improved version of the model which uses array of notes model, we demonstrate a solution of modeling the angklung distribution problem with a popular song that serves as an example.

2. Main Results

Overlapping happens when a player holds two angklungs but the notes they represent in the composition are too close to each other. It means that the second note follows the first note almost immediately without any rest, which is a very short but noticeable interval of silence. For example consider the figure below. If these notes are too close to each other or even appear at the same time in the composition, the player has to shake the first angklung that represents the first note and after that immediately shake the second angklung that represents the second note. This is practically almost impossible. The idea of shaking two angklungs at the same time is usually unacceptable as it reduces the quality of the sound produced compared to focusing on shaking just one angklung. Thus, the solution is to find the right angklung distribution that totally avoids the overlapping between angklungs. It is always good to also find a distribution that tells the ensemble the minimum number of players required to play an angklung composition as often only few people are available to perform. To better understand the importance of angklung distribution, consider the figure below.

![Figure 2. A portion of the song “Happy Birthday”.](image)

The notation in Figure 2 is the numbered music notation of a portion of the song “Happy Birthday”. It consists of five bars and each bar consists of three beats. If we consider the notes “6” and “5” in the second bar, both are always considered overlapping in angklung performance. It means that a player cannot hold angklungs that represent the notes “6” and “5”. If a player were to hold these two angklungs, the player would have to shake angklung “6” and then immediately proceed to shake angklung “5”, which in reality is not possible without producing a rest.

Observe that the note “6” in the second bar may or may not be considered overlapping with the note “7” in the third bar. Therefore, we need to determine when it is exactly that two notes are considered overlapping. It will be the beat range of a composition, which is mainly determined by the
tempo of the song or sometimes by the ensemble’s need. For example, if the beat range is three beats or more, then the previous notes “6” and “7” are overlapping because they appear in beat locations separated by less than the beat range. After determining the beat range, we can start finding the distribution. There are exactly five notes appearing in Figure 2. If there are five players, then one player can simply play one note/angklung each. However, as stated previously, it is often necessary to know the minimum number of players required such that the song or the composition can be performed.

To make things precise, we first introduce some notions in angklung distribution problem and present the obtained results before we present an example. An angklung composition, or simply composition is an ordered pair $C = (S,r)$, where $S = [s_{ij}]$ is an m-by-n matrix, which may or may not have zero entries, and $r$ is a positive integer. The matrix $S$ is called the score of $C$. The number $r$ is called the beat range, or simply range of $C$. The entries of $S$ are called the notes (or angklungs).

The column of $S$ are called beats. Let $v_i$ be the $i$-th column of $S$. We define the $i$-th clique submatrix of $S$ to be the submatrix $[v_i v_{i+1} \ldots v_{i+b}]$. Hence, there are $n-b$ clique submatrices of $S$.

Two different notes are said to be overlapping if they belong to the same clique submatrix. Let $I_5 = \{(c,d) \mid c \in \{1,2,\ldots,m\}, d \in \{1,2,\ldots,n\}, s_{cd} \neq 0\}$. Thus, $I_5$ is just the collection of nonzero notes of $S$. The composition $C$ is said to be playable by $l$ players, or simply $l$-playable, if there exists a surjective mapping $f: I_5 \to \{1,2,\ldots,l\}$ such that given two different pairs $(i_1,i_2),(j_1,j_2) \in I_5$, if $s_{i_1i_2}$ and $s_{j_1j_2}$ are different and overlapping, then $f(i_1,i_2) \neq f(j_1,j_2)$. This mapping is called the angklung distribution, or simply distribution of $C$ with $l$ players. Another way of saying that $C$ is playable by $l$ players is that we need $l$ players to play $C$. The elements of $\{1,2,\ldots,l\}$ are called players in this context.

Let us construct a graph generated by $S$, denoted $G_S = (V,E)$, in the following way. Let the vertex set $V$ be the collection of nonzero notes of $S$. Two vertices in $V$ are adjacent if and only if the notes they represent are overlapping.

**Theorem 1.** Every composition $C = (S,r)$ is playable by $\chi(G_S)$ players.

*Proof.* We have to find a distribution of $C$ with $\chi(G_S)$ players. Let $c$ be the $\chi(G_S)$-coloring of $G_S$. Define a mapping $f$ from $I_5$ onto $\{1,2,\ldots,\chi(G_S)\}$ with $f(i,j) = c(s_{ij})$. Clearly, $f$ is well-defined and surjective. If $(i_1,i_2),(j_1,j_2) \in I_5$ and $s_{i_1i_2}$ and $s_{j_1j_2}$ are different and overlapping, we show that $f(i_1,i_2) \neq f(j_1,j_2)$. Suppose otherwise, then

$$c(s_{i_1i_2}) = f(i_1,i_2) = f(j_1,j_2) = c(s_{j_1j_2}).$$

It follows that since $c$ is a coloring, $s_{i_1i_2}$ and $s_{j_1j_2}$ are not adjacent, and hence not overlapping, a contradiction. Thus, $f$ is a distribution with $\chi(G_S)$ players and the theorem is proved.

The previous theorem seems to suggest that $\chi(G_S)$ is the least $k$ such that $C = (S,r)$ is $k$-playable. Before seeing this is not always the case, consider the clique submatrices of $S$ that have more different nonzero notes than the other clique submatrices. Call this maximum number $\omega(C)$ and let $A$ be one such submatrix. Observe that $A$ induces a clique in $G_S$ since notes of $A$ are all overlapping. Hence, $\omega(C) \leq \omega(G_S)$, where $\omega(G_S)$ is the clique number of $G_S$, which leads to $\omega(C) \leq \chi(G_S)$ since clearly we have $\omega(G_S) \leq \chi(G_S)$ . Thus, the following theorem says that $\chi(G_S)$ is not always the minimum number of players needed to play $C$. It can be fewer.
**Theorem 2.** Every composition $C = (S, r)$ is playable by $\omega'(C)$ players but not less.

**Proof.** We have to find a distribution $f$ of $C$ with $\omega'(C)$ players. Consider the $p$-th clique submatrix $A_p$ that has $\omega'(C)$ different nonzero notes. If $(i_1, i_2)$ and $(j_1, j_2)$ are different indices of two of the nonzero notes of $A_p$ in $S$ and $s_{i_1 i_2} \neq s_{j_1 j_2}$, set

$$f(i_1, i_2) \neq f(j_1, j_2),$$

where both are in $\{1, 2, \ldots, \omega'(C)\}$. If $s_{i_1 i_2} = s_{j_1 j_2}$, set the $f$ values of the indices to be the same. Hence, indices of nonzero entries of $A_p$ in $S$ are mapped onto $\{1, 2, \ldots, \omega'(C)\}$ under $f$.

For the next step, consider the $q$-th clique submatrix $A_q$, where $q \geq p$. Assume that the indices of nonzero notes of $A_q$ are already mapped through $f$ into $\{1, 2, \ldots, \omega'(C)\}$ in the same way as the previous paragraph, that is by setting the $f$ values of the different indices to be different if the respective notes are different, otherwise the same. This local mapping needs not be surjective, since the number of notes of $A_q$ can be less than $\omega'(C)$. Consider the next clique submatrix $A_{q+1}$. All but the indices of the last column of $A_{q+1}$ in $S$ are not mapped yet. Hence, we set their $f$ values the same way. Note that if $A_{q+1}$ has strictly more different nonzero notes than $A_q$ has, the mapping into $\{1, 2, \ldots, \omega'(C)\}$ can still be constructed since the number of different nonzero notes of $A_{q+1}$ does not exceed $\omega'(C)$. If the last column of $A_{q+1}$ are zeros, then there is no need to map them.

Recursively by the second paragraph, we have set the $f$ values of indices of all columns after $A_p$ in $S$ onto $\{1, 2, \ldots, \omega'(C)\}$. By working backward to consider $A_{p-1}, A_{p-2}, \ldots, A_1$ in the similar way, we have a surjection $f: i_1 \rightarrow \{1, 2, \ldots, \omega'(C)\}$. It is also a distribution by the fact that we recursively mapped indices of nonzero notes of other clique submatrices the same way as we mapped indices of nonzero notes of $A_p$. Hence, $f$ is a distribution of $C$ with $\omega'(C)$ players, and thus $\omega'(C)$-playable.

Note that $C$ is never $z$-playable if $z < \omega'(C)$. To see this, assume that $C$ is indeed $z$-playable and consider $A_p$, which has more than $z$ different nonzero notes. By the pigeonhole principle, at least two different indices of different respective nonzero notes of $A_p$ in $S$ have the same image under the corresponding distribution. This contradicts the definition of a distribution. Hence, $C$ is not $z$-playable. Thus, the theorem is proved. 

We now apply the results to Figure 2. In this example, $C$ is the song “Happy Birthday”. We need to construct a matrix $S$ that represents the score of the song. The number of rows of $S$ is the number of lines the song has. The number of columns/beats of $S$ is the number of the smallest beats. For example in Figure 2, the matrix $S$ has just one row since the song has just one line. The song appears to have 15 beats, but the smallest beat, as seen in the two note “5”s on the third beat, is a half beat. Hence, there are 30 columns of $S$. Let us set the range $r = 2$ beat/column. Thus, we have generated a graph $G_S$ as in Figure 3 below. Note that the clique submatrices are 1-by-3 matrices.

![Figure 3](image)

**Figure 3.** The matrix $S$ and the graph $G_S$. 


By Theorem 1 and using the graph coloring of $G_5$, we know that we need three players to play the song. However, we can see that $\omega(C) = 2$ since we can easily find a clique submatrix with two nonzero different entries but no clique submatrix has three such entries. For example, as in the proof of Theorem 2, set $p = 5$. Then, $A_p = [5 \ 5 \ 6]$ is a clique submatrix that we want. To distribute this song with $f:I_5 \rightarrow \{1,2\}$, we can set $f(1, g) = 1$ if $g$ is odd and $f(1, g) = 2$ if $g$ is even. This $f$ is easily seen to be a distribution with 2 players. Figure 4 below shows how the song can be played.

![Figure 4](image)

Figure 4. How the song is played using Theorem 2.

In the above figure, two players A and B play the song together. The arrows are pointing to the part where each of them should play. As seen in the figure, the note/angklung “5” is held by both A and B but played in different parts. This shows that while Theorem 2 may give a distribution with a fewer number of players than the chromatic number given in Theorem 1, it may also result in a situation where players hold too many angklungs as sometimes copies of angklungs that represent the same notes are needed. If not too many angklungs are held by the players, then Theorem 2 can be an option for an angklung ensemble to use, especially when only very few people are available.

Acknowledgement

I would like to acknowledge the help provided by my fellow angklung players in Keluarga Paduan Angklung ITB during the completion of this research. Their supports and my experience in playing angklung served as great motivation to conduct this research.

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