Critique of Boyu Sima’s Proof that $P \neq \text{NP}$

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Abstract
We review and critique Boyu Sima’s paper, “A solution of the P versus NP problem based on specific property of clique function,” which claims to prove that $P \neq \text{NP}$ by way of removing the gap between the nonmonotone circuit complexity and the monotone circuit complexity of the clique function. We first describe Sima’s argument, and then we describe where and why it fails. Finally, we present a simple example that clearly demonstrates the failure.

1 Introduction

One of the most well-known and long-standing problems in computer science is the question of whether $P = \text{NP}$. A solution to the problem would have wide-ranging implications to everything from economics to cybersecurity. To this end many have claimed to have found a proof that either $P = \text{NP}$ or $P \neq \text{NP}$. However, to this date no such claim has been found to be correct.

There are various methods for attempting such a proof. One such method is by using lower bounds on the complexity of circuits. By showing that a known NP-complete problem has an exponential lower-bounded circuit complexity, you show that $P \neq \text{NP}$. In his paper “A solution of the P versus NP problem based on specific property of clique function” [Sim19], Sima tries to do precisely this. Sima analyzes the clique function and attempts to show that the circuit complexity of the clique function is exponential, thus showing that $P \neq \text{NP}$.

In this paper, we will first present some definitions and some prior work that Sima uses in his argument. We will then present Sima’s argument and describe where Sima’s argument fails due to making an improper generalization and failing to consider the connection between a Boolean variable and its negation. Finally we will provide an example that demonstrates the hole in his algorithm.
2 Preliminaries

The following are the needed definitions and an existing theorem that will be used.

Boolean Functions

A Boolean function of \( k \) variables is a function \( f : \{0, 1\}^k \rightarrow \{0, 1\} \).

Boolean functions (of \( k \) variables) can be expressed as Boolean (propositional) formulas with \( k \) variables and the logical operators (\( \land, \lor, \neg \)).

A Boolean function \( f \) of \( k \) variables is called monotone if

\[
(\forall w, w' \in \{0, 1\}^k : w \leq_{lex} w')[f(w) \leq f(w')].
\]

Note: If a function is monotone, then changing a 0 to a 1 in the input will never cause a decrease in the output, and changing a 1 to a 0 in the input will never cause an increase in the output.

Boolean Circuits

A Boolean circuit is a directed acyclic graph with gate nodes and input nodes. Gate nodes can be one of three types corresponding to the logical operators AND (\( \land \)), OR (\( \lor \)), and NOT (\( \neg \)) and have indegrees of 2, 2 and 1 respectively and unbounded outdegrees. For his purposes, Sima expresses Boolean circuits as Boolean expressions \( f(x_1, x_2, x_3, \ldots) \) where input nodes correspond to the variables, and gate nodes correspond to logical operators. This allows him to work with and modify expressions without working expressly with circuits and their diagrams. It should be noted that the number of logical operators in an expression may not be directly correlated to the complexity of the corresponding circuit as gates within circuits are allowed to output to multiple other gates. However, this does not affect Sima’s argument as he uses these expressions to argue about the correctness (behavior) of the circuits rather than their complexity.

A Boolean circuit with no NOT gates is called monotone.

A Boolean circuit in which only the input nodes are negated (only the input nodes are inputs to NOT gates) is called a standard circuit. Because negations occur only at the input nodes, one can rewrite such a circuit,

\[
f(x_1, x_2, \ldots, x_n),
\]

as a circuit with twice as many input nodes but no negations,

\[
f(x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n).
\]

In this manner one removes the NOT gates, but for any valid assignment of variables, \( (\forall i \in \{1 \ldots n\}) \; [x_i = NOT(\neg x_i)] \).
Circuit Complexity

The circuit complexity of a Boolean function is the size (number of gates) of the smallest Boolean circuit that computes the Boolean function.

The standard circuit complexity of a Boolean function is the size (number of gates) of the smallest standard circuit that computes the Boolean function.

The monotone circuit complexity of a Boolean function is the size (number of gates) of the smallest monotone circuit that computes the Boolean function.

The Clique Function and its Monotone Complexity

For \(1 \leq s \leq m\), let \(\text{CLIQUE}(m, s)\) be the function of \(n = \binom{m}{2}\) variables representing the edges of an undirected graph \(G\) on \(m\) vertices, whose value is 1 if and only if \(G\) contains an \(s\)-clique.

The monotone complexity of the clique function is exponential. Razborov initially showed a superpolynomial lower bound. This was improved by Alon and Boppana \([AB87]\) to be exponential.

3 Summary of Sima’s Argument

Sima builds his argument by attempting to fill in the holes left by Alon and Boppana’s analysis of the lower bounds for the monotone complexity of the clique function. Their paper is only able to establish lower bounds for the monotone complexity of the clique function. The nonmonotone complexity of the clique function is thus left unbounded. Sima argues that the nonmonotone complexity of the clique function is in fact greater than or equal to that of the monotone complexity of the clique function.

In order to show this, Sima attempts to transform a nonmonotone circuit for the clique function into a monotone circuit for the clique function without increasing its size beyond a polynomial factor. He first considers standard circuits (as defined previously). He claims that any circuit can be transformed into a standard circuit by at most doubling the number of gates. Because of this, the difference in complexity between standard and non-standard circuits is at most a factor of 2, thus allowing him to consider only standard circuits. From this point he makes his main argument.

He considers the standard Boolean circuit \(f(x_1, x_2, \ldots, x_n, \neg x_1, \neg x_2, \ldots, \neg x_n)\) that computes the clique function, \(\text{CLIQUE}(m, s)\). At this point he makes the argument that replacing any one of the negated variables (\(\neg x_i\)) with 1 (TRUE) will result in a circuit that computes the same function. To prove this, he first argues that one can “extract” any negated variable in the circuit, moving it to the front of the formula. This results in a formula of the form\(^\dagger\)

\[
f = \neg x_i \land \text{Term}_A \lor \text{Term}_B \lor \text{Term}_C \ldots,
\]

\(^\dagger\)Throughout this paper, we assume the standard operator precedence rules, and in particular that \(\neg\) has higher precedence than \(\land\), which itself has higher precedence than \(\lor\).
where \( x_i \) is the extracted variable and none of TermA, TermB, TermC... include \( \neg x_i \).

He then uses this extracted form of the formula to argue that replacing \( \neg x_1 \) (or some other arbitrary negated variable) with 1 does not change the value of \( f \) when \( f \) calculates the clique function. The argument is as follows.

Sima starts with the extracted form of the standard clique formula where \( \neg x_1 \) is the extracted variable:

\[
f = \neg x_1 \land \text{TermA} \lor \text{TermB} \lor \text{TermC} \ldots .
\]

There are then two cases he considers. Sima refers to \( \neg x_1 \land \text{TermA} \) as Term1, with Term1part1 referring to \( \neg x_1 \) and Term1part2 referring to TermA.

Case 1: Term1part2 has a value of 0.
In this case he argues that because this second part of Term1 has a value of 0, clearly no matter what you set \( \neg x_1 \) to, the value of Term1 will be 0. Thus the output of the circuit will not change.

Case 2: Term1part2 has a value of 1.
In this case, he argues that if Term1part2 has a value of 1 then the value of the entire function is also 1. The reasoning he gives is that if \( \neg x_1 \) is also 1, then \( x_1 \) takes the value 0. As such, by the definition of the clique function, the edge corresponding to \( x_1 \) is disconnected and thus \( \neg x_1 \) (and \( x_1 \)) has no contribution to the size of the clique. Thus as long as the value of Term1part2 is 1, the value of the circuit will be 1 no matter what \( \neg x_1 \) is.

The final step in Sima’s proof is to argue that since this aforementioned process can be done with any of the negated variables (\( \neg x_1, \neg x_2, \ldots , \neg x_n \)), you can (sequentially) replace all the negated variables with the value 1, and the resulting circuit will compute the clique function correctly. Furthermore he states that since this replacement will clearly not increase the complexity of the circuit, and (as you are starting with a standard circuit) the resulting circuit will be monotone, standard circuits for the clique function are no smaller than monotone circuits for the clique function. Since the monotone circuit complexity of the clique function was proven to be exponential, the standard circuit complexity (and thus the complexity overall) of the clique function is also exponential. As such he concludes that \( \text{P} \neq \text{NP} \).

4 Critique of Sima’s Argument

On its surface, Sima’s argument seems sound. He builds off Alon and Boppana’s finding about the monotonic complexity of the clique function by converting a nonmonotone circuit into a monotone one without increasing its complexity. However, the problem lies in his argument about the conversion process. From here let’s follow his argument more closely.

He starts with a(n) (arbitrary) standard circuit that computes the clique function. This is then expressed in the form \( f = f(x_1, x_2, \ldots , x_n, \neg x_1, \neg x_2, \ldots , \neg x_n) \). His first claim is that any of the negated variables (\( \neg x_i \)) can be extracted, putting the circuit into the form \( f = \neg x_i \land \text{TermA} \lor \text{TermB} \lor \text{TermC} \ldots , \) where each of the terms TermA, TermB, ... does not contain \( \neg x_i \). This is in fact trivial. By first expanding the Boolean formula into Sum of Products form and then reorganizing and factoring out \( \neg x_i \), this is easily achievable.
He then considers extracting one of the negated variables (using $\neg x_1$ as his example, but extending the argument to any negated variable) and makes arguments for two cases concerning the value of Term1 (again where Term1 = $\neg x_1 \land $ TermA, Term1part1 = $\neg x_1$, and Term1part2 = TermA).

In the first case he considers when Term1part2 has a value of 0. His argument in this case is sound. When Term1part2 has a value of 0, the contribution from Term1 to the overall function will be 0 no matter the value of $\neg x_1$.

However, in his second case he runs into trouble. When the value of Term1part2 is 1, he again tries to present an argument for why setting the value of $\neg x_1$ to 1 will not change the value of the function. But here he misunderstands what he is actually arguing and neglects to fully comprehend the connection between $x_1$ and $\neg x_1$. His initial argument is true. If Term1part1 is 1, then clearly the value of $f$ will be 1. Due to the clique function being monotone, this does in fact mean that the $x_1$ edge has no contribution to the value of the clique function as adding an edge to the graph (changing the value of $\neg x_1$ from 1 to 0 and vice versa for $x_1$) will not cause a clique in the graph to disappear. As such any assignment of $\neg x_1$ will result in the same value of the function. A more formal description of what Sima proves follows below.

**Definition 4.1** Let $A$ be an assignment of the Boolean variables $x_1, x_2, x_3, \ldots, x_n$. Define $A'$ to be the same assignment as $A$ except with the value of $x_1$ reversed (changed from 0 to 1 or vice versa).

**Claim 4.2** If both Term1part1 and Term1part2 evaluate to 1 given an assignment $A$, then both $f(A) = 1$ and $f(A') = 1$. The same is true swapping $A$ and $A'$, as $A = A''$

By his argument Sima is able to prove the claim above: that if a variable assignment $A$, or its corresponding assignment $A'$, in which the assigned value of $x_1$ is reversed, causes the value of Term1 to be 1, then BOTH $f(A) = 1$ and $f(A') = 1$. However, Sima mistakenly assumes that all assignments where Term1part2 = 1 fall into the form of $A$ or $A'$. This appears, at first, to be true as Term1part1 = $\neg x_1$. So by flipping the value of $\neg x_1$ (i.e. changing from $A$ to $A'$) you can always make Term1 evaluate to 1. However, this ignores the connection between $x_1$ and $\neg x_1$. Because $x_1$ must be equal to the negation of $\neg x_1$ in any valid assignment, it is possible for there to be variable assignments $B$ for which the value of Term1part2 is 1, while neither $B$ nor $B'$ result in both Term1part1 AND Term1part2 having a value of 1. We will describe one such case below.

### 4.1 An Illustrative Example

Consider the following example: Let $f(x_1, x_2, \ldots, x_n)$ be a monotone circuit that computes the clique function, $\text{CLIQUE}(m, s)$, for some $s \geq 3$. Now append to the circuit (via the logical OR operator) the term $\neg x_1 \land x_1$. 




The resulting circuit \( f' \) is now
\[
f'(x_1, x_2, \ldots, x_n) = x_1 \land \neg x_1 \lor f(x_1, x_2, \ldots, x_n),
\]
or in standard form
\[
f'(x_1, x_2, \ldots, x_n, \neg x_1) = x_1 \land \neg x_1 \lor f(x_1, x_2, \ldots, x_n).
\]
Since the appended term is NEVER satisfied and is adjoined to \( f \) via an OR operator, the resulting \( f' \) will have the same behavior as \( f \) and will also calculate \( CLIQUE(m, s) \) correctly. Now following Sima’s process and extracting the negated \( \neg x_1 \), the result is
\[
f'(x_1, x_2, \ldots, x_n, \neg x_1) = x_1 \land \neg x_1 \lor f(x_1, x_2, \ldots, x_n),
\]
where \( TermB, TermC, \ldots \) are terms (without negation) containing any of the variables except \( \neg x_1 \). Note that since \( f \) is a monotone circuit, the only negated variable will be the \( \neg x_1 \) that was just introduced. What Sima refers to as \( Term1 \) in this case is \( \neg x_1 \land x_1 \), with \( Term1part1 \) being \( \neg x_1 \) and \( Term1part2 \) being \( x_1 \). Now, per Sima’s algorithm, set \( \neg x_1 \) to 1, resulting in the following circuit \( f'' \):
\[
f''(x_1, x_2, \ldots, x_n, 1) = 1 \land (x_1) \lor TermB \ldots
\]
From here it is clear to see that as long as \( x_1 \) has a value of 1, the value of \( f'' \) will be 1. Looking at the equation from another perspective, even if ONLY \( x_1 \) has a value of 1 and all the other variables are set to 0, the circuit will still output 1. However, bringing this back to our definition of the clique function, if there is a graph with only one edge, it is impossible to have a clique of size anything greater than 2. Thus this new \( f'' \) clearly does not compute the same function as \( f \).

The reason Sima’s argument fails is that the set of all variable assignments \( A \) for which both \( Term1part1 \) and \( Term1part2 \) evaluate to 1, and their corresponding assignments \( A' \), does not equal the set of all assignments \( S \) for which \( Term1part2 \) evaluates to be 1. This is stated more formally below. Let \( Term1part2(A) \) be the value of \( Term1part2 \) given the assignment \( A \) and similarly for \( Term1part1(A) \). Then
\[
\{ A \mid Term1part2(A) = 1, Term1part1(A) = 1 \} \cup \{ A \mid Term1part2(A') = 1, Term1part1(A') = 1 \} \neq \{ A \mid Term1part2(A) = 1 \}.
\]
In fact, there are no valid assignments \( A \) (and thus no corresponding \( A' \)) for which both \( Term1part1 \) and \( Term1part2 \), in the example presented in this section, evaluate to 1. However, any assignment in which \( x_1 \) is 1 causes \( Term1part2 \) to evaluate to 1. Because \( x_1 \) and \( \neg x_1 \), although treated as different variables in the standard formula, must be negations of each other in any valid assignment, it is possible to create cases that Sima’s argument fails to cover such as the one presented above.

By failing to consider this connection between \( \neg x_1 \) and \( x_1 \), Sima makes an intuitive generalization that, although on the surface may seem reasonable, leaves loopholes that can change the behavior of the function. The above example illustrates the error in his logic and provides a specific counterexample to his process.
5 Conclusion

Sima’s argument attempts to build off of the findings of Alon and Boppana \[AB87\] in order to extend the exponential lower bound for the monotone circuit complexity of the clique function, to the nonmonotone circuit complexity of the clique function. He describes a clever attempt to convert standard (nonmonotone) circuits into monotone circuits and attempts to prove that this conversion holds in the case of the clique function. However, in doing so he makes an unfounded generalization in his argument. This results in specific cases that can be exploited by using the connection between variables and their negations.

By describing his flaw and presenting a counterexample to his process, we have demonstrated that his method is not satisfactory. It is possible that through using a more specific description of what the minimal nonmonotone circuit of the clique function looks like, one could sidestep the problems that we have described in this paper and establish an exponential lower bound on the nonmonotone complexity of the clique function. In fact, in 2005 Amano and Maruoka were able to show that the lower bound for the complexity of nonmonotone circuits for the clique function with at most \(1/6\log\log(n)\) negation gates is in fact superpolynomial \[AM05\]. However, until such time as we have a better understanding of the clique function and its properties, a proof such as presented in Sima’s paper is not possible.

References

[AB87] Noga Alon and Ravi B. Boppana. The monotone circuit complexity of Boolean functions. *Combinatorica*, 7(1):1–22, 1987.

[AM05] Kazuyuki Amano and Akira Maruoka. A superpolynomial lower bound for a circuit computing the clique function with at most \((1/6)\log\log N\) negation gates. *SIAM Journal on Computing*, 35(1):201–216, 2005.

[Sim19] Boyu Sima. A solution of the P versus NP problem based on specific property of clique function. Technical Report arXiv:1911.00722 [cs.CC], Computing Research Repository, arXiv.org/corr/, November 2019.