COMPLEX COOPERATIVE BEHAVIOUR IN RANGE-FREE FRUSTRATED MANY-BODY SYSTEMS

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A brief introduction and overview is given of the complexity that is possible and the challenges its study poses in many-body systems in which spatial dimension is irrelevant and naively one might have expected trivial behaviour.

I. INTRODUCTION

This paper is concerned with many body systems and their cooperative behaviour; in particular when that behaviour is complex and hard to anticipate from the microscopics, even qualitatively and even when the systems are made up of simple individual units with simple inter-unit interactions.

‘Range-free’ (or ‘infinite-ranged’) refers to situations where the interactions are not dependent on the physical separations of individual units, and hence neither on the dimensionality nor on the structure of the embedding space. Such systems are also often referred to as ‘mean-field’, since one can often show (and usually believes) that their behaviour in the thermodynamic limit ($N \rightarrow \infty$ units) is identical to that of an appropriate mean-field approximation to a short-range system.

‘Frustration’ refers to incompatibility between different microscopic ordering tendencies.

Self-consistent mean-field theories do have the ability to describe spontaneous symmetry breaking and phase transitions and they have played an important role in statistical physics. However as pure systems, without quenched Hamiltonian disorder or out-of-equilibrium self-induced disorder, they do not exhibit the interesting non-simple dimension-dependent but details-independent (universal) critical behaviour whose study drove much of the interest of statistical mechanics in the seventies and eighties [1]. For this reason ‘mean-field’ used to be interpreted as fairly trivial.

On the other hand, with quenched disorder and frustration in their interactions range-free many-body systems can, and regularly do, exhibit behaviour that is complex and rich. This paper represents a brief introduction to and partial overview of such systems.

II. GENERAL STRUCTURE AND FEATURES

The general class of systems we consider can be summarized as characterised by schematic ‘control functions’ of the form $H\{\{J_{ij..k}\}, \{S_i\}, X\}$ where

(i) in thermodynamics (statics) the $\{S\}$ are the variables and the $\{J\}$ are quenched (frozen) parameters, or vice-versa,

(ii) in dynamics the $\{S\}$ are the ‘fast’ variables and the $\{J\}$ are ‘slow’ variables, or vice-versa, where ‘fast’ and ‘slow’ refer to the characteristic microscopic time-scales,

(iii) in both cases, the $X$ are intensive control parameters, influencing the system deterministically, quenched-randomly or stochastically, and

(iv) we shall be particularly interested in typical behaviour in situations in which any quenched disorder is drawn independently from identical intensive distributions, enabling (at least in principle) useful thermodynamic-limit measures of the macroscopic behaviour.

The interest arises when the effects of different interactions are ‘frustrated’, in competition with one another. In such cases with detailed balance, at low enough noise the macrostate structure/space is typically fractured (or clustered), in a manner often envisaged in terms of a ‘rugged landscape’ paradigm in which the dynamics is imagined as motion in a very high dimensional landscape of exponentially many hills and valleys, often hierarchically structured, with concomitant confinements, slow dynamics and history dependence. In dynamical systems without detailed balance, strictly there is no such simple Lyapunov ‘landscape’ but the ‘motion’ is analogously complexly hindered, with many effective macroscopic time-scales.

First studied (in physics) in the context of magnetic alloys, such systems are now recognised in many different contexts; in inanimate physical systems, computer science, and information science; in animate biology, economics and social science. In these different systems ‘controllers’ of the ‘control functions’ vary; including the laws of physics, devisors of computer algorithms, human behaviour, governmentally-devised laws etc.
III. THE SHERRINGTON-KIRKPATRICK MODEL

A simply-formulated but richly-behaved canonical model is that of Sherrington and Kirkpatrick (SK) [2], originally introduced as a potentially soluble model corresponding to a novel mean-field theory introduced by Edwards and Anderson (EA) [3] to capture the essential physics of some unusual magnetic alloys, known as spin glasses [4, 5].

The SK model is characterized by a Hamiltonian

$$H = - \sum_{ij} J_{ij} \sigma_i \sigma_j; \sigma = \pm 1$$

(1)

where the i, j label spins \(\sigma\), taken for simplicity as Ising, and the interactions \(\{J_{ij}\}\) are chosen randomly and independently from a distribution \(P_{exch}(J_{ij})\). Dynamically the system can be considered to follow any standard single-spin-flip dynamics corresponding to a temperature \(T\). Were normal equilibration to occur it would be characterized by Boltzmann-Gibbs statistics, \(p(\sigma) \sim \exp(-H(\sigma)/T)\).

However, if the distribution \(P_{exch}(J)\) has sufficient variance compared with its mean and the temperature is sufficiently low, normal equilibration does not occur and complex macro-behaviour results beneath a transition temperature. The interesting regime, known as the ‘spin glass phase’, occurs at intensive \(T\) if the variance of \(P_{exch}(J)\) scales with \(N\) as \(J^2/N\), the mean as \(J_0/N\). As Parisi showed, in a series of papers (e.g. [6–8]) which involved amazing insight and highly original conceptualization and methodology, this glassy state is characterized by a hierarchy of ‘metastable’ macrostates, differences between restricted and Gibbsian thermodynamic averages, as well as non-self-averaging; see also e.g. [9].

These features can be characterized by the macrostate overlap distribution functions \(P(q) = \sum_{S,S'} \delta(q - |\langle (O)_S \rangle_S \langle (O)_{S'} \rangle_{S'}|)\) where \((O)_S\) denotes a thermodynamic average of \(O\) over the macrostate \(S\). For a conventional system, with a single macrostate, \(P(q)\) has a single delta function, while for a system with entropically extensively many macrostates \(P(q)\) has more structure. When the state structure is continuously hierarchical, as it is for SK, there is a continuum of weight in the disorder-averaged overlap distribution function \(P(q) = \int D\{\sigma\} P_{exch}(\{\sigma\}) P(\{\sigma\}) dq\). Non-self-averaging arises in different \(P(\{\sigma\})\) for different realizations of quenched disorder even when that disorder is chosen i.i.d. from an intensive \(P_{exch}(J)\). Ultrametricity [8] is a feature of the hierarchical order.

Later studies [10] have further exposed the existence of slow dynamics and aging and a remarkable non-trivial quantitative relationship to the thermodynamics [11, 12]. One feature of this is a modification of the normal fluctuation-dissipation relationship to \(-dR/dC = \beta X(C)\) where \(R\) is a response function and \(C\) a related correlation function, parenthetically connected through the measurement time \(t\) in the limits of large initial waiting time/field application time \(t_w\) and \(t\) itself, and \(X(C)\) is related to the average overlap distribution \(\bar{P}(q)\) by \(X(C) = \int_0^C \bar{P}(q) dq\). In the normal fluctuation-dissipation theorem \(X(C)\) is replaced by unity.

The original exposure of the subtleties of the SK model utilised unusual and non-rigorous mathematics and ansätze, together with unconventional physical conceptualization, going far beyond the conventional realms of rigorous mathematical physics and probability theory. The predictions have however long been shown to be in accord with computer simulations of the model (e.g. [13]) and consequently were believed by physicists. Their rigorous demonstration, though, has been a non-trivial challenge which has required deep analysis and led to new rigorous mathematical methodologies in recent years [14–17], finally completely vindicating Parisi’s theory [18]. Thus there is no doubt that there is deep complexity in the SK model.

On the other hand, while it is generally believed that real spin glass transitions do occur also in the short-ranged EA model for dimensions three or more and also in the experimental magnetic alloys that first stimulated its study, it remains controversial as to whether or to what extent all the subtle predictions of the SK model apply to these systems (see e.g. [11][19]).

Hence it becomes appropriate to ask whether range-free frustrated and disordered systems have a wider relevance beyond as over-idealized models of real magnetic alloys and as challenges for mathematicians. Possibly remarkably, it turns out that the answer is a resounding “Yes”; they turn out to be rather ubiquitous in many areas of science.

IV. BEYOND MAGNETIC ALLOYS

Range-free frustrated and disordered many-body problems occur in many scenarios outside of physics, for example in many of the hard optimization problems studied by computer scientists, and in situations in which correlation between individuals occurs through the transfer of information available to all, irrespective of physical separation, as epitomised by modern interaction through the internet and telephones, or commonly available through the world-wide web, newspapers, radio and television. These systems are usually different in detail from the SK model but share some of the same conceptual and technical challenges, as well as providing further challenges of their own.
A. The SK model as an optimization exercise

A simple illustration of the possibilities of extension comes from viewing the SK model as an optimization problem which is describable in everyday terms as “The Dean’s Problem” [20]. One imagines a University Dean who has to place $N$ students in two dormitories, but with the challenge that every pair of students $(i,j)$ either likes or dislikes one another to an extent $J_{ij}$ [53]. His problem is to choose to which dorm to allocate each student so as to ensure the greatest satisfaction overall. Labelling the dorm choices by $\sigma_i = \pm 1$ the SK Hamiltonian becomes the ‘cost function’ that the Dean should minimise, with the $\{\sigma_i\}$ in its ground state the optimal choice [54]. Allowing the Dean a degree of uncertainty in his decision-making provides an analogue of ‘temperature’.

B. Simulated annealling

There are many other combinatorial optimization problems that can be viewed as finding the minimum of a cost function of the form $H(\{J_{ij,k}\}, \{S_i\}, X)$. In a ‘simulated annealing’ [21] ‘spin-flip’ computer algorithm to minimise the cost function, an annealing ‘temperature’ $T_A$ is introduced artificially via a stochastic probability measure determined by $\exp(-\delta H/T_A)$, where $\delta H$ is the change in the cost function engendered by the flip, so that equilibration at $T_A$ would yield the corresponding Boltzmann distribution, and $T_A$ is gradually reduced to the value of interest (zero for a minimum of $H$, or $T$ if that is the stochastic noise of actual uncertainty). Correspondingly, $T_A$ can be introduced into an effective equilibrium statistical mechanics, with Boltzmann weighting $\exp(-H\{S\}/T_A)$, examined analytically and the minimum found from $H_{min} = \lim_{T_A \to 0} \{-T_A \ln \sum_{\{S\}} \exp(-H\{S\}/T_A)\}$

C. $p$-spin spin glass, satisfiability and error-correction

In the $p$-spin glass model one replaces the binary interaction of SK by one involving $p$ spins:

$$H = - \sum_{\{i_1i_2...i_p\}} J_{i_1i_2...i_p} \sigma_{i_1}\sigma_{i_2}...\sigma_{i_p}, \quad (2)$$

with the $J$ again drawn randomly and independently, all from the same distribution, and then quenched [55]. This apparently innocuous extension of the SK model yields new behaviour in several ways. Firstly, instead of a continuous onset of a hierarchy of different levels of metastability, with a growing continuous range of state overlaps, there is a discontinuous onset of many orthogonal but otherwise equivalent metastates of finite overlap order parameter [56][25, 26]. Secondly, the dynamical transition is no longer at the same temperature as the thermodynamic transition but is higher [30]. Thirdly, there is another lower temperature (continuous) thermodynamic transition to a state with a continuous range of overlap distributions [31].

This type of behaviour, of a dynamical transition pre-empting a thermodynamic one, both with discontinuous onset of overlap, turns out to be rather common in frustrated many-body systems[57]. So too is the lower temperature thermodynamic transition to a continuous range of overlap distributions [58]. Consequently the determination of the minimum achievable cost function often has the difficulties associated with the full hierarchical character of the SK model at $T = 0$.

In fact, many of the problems of interest in computer science are effectively range-free but on random graphs of finite connectivity, in contrast to the full connectivity of the SK- and $p$- spin models of eqns. (1) and (2) above. The conceptual ideas extend, albeit made more complicated (and currently incompletely solved) by the need for higher order overlap functions within the full (replica) theory of [2, 3, 6] [59]. Dilution, however, has also brought to the fore an alternative and highly successful computational methodology in the form of ‘survey propagation’ [24].

One example of the conceptual transfer of these ideas is to random satisfiability problems in computer science, both in explaining the existence of satisfiable-unsatisfiable (SAT-UNSAT) phase transitions [32] and in leading to the recognition that for random $K(> 2)$-SAT, where $K$ refers to the length of the individual clauses to be satisfied simultaneously, there should be a region of ‘HARD-SAT’ separating practically satisfiable SAT problems from UNSAT as the constraint density, the ratio of the number of constrained clauses to the number of variables, is increased [28]; this ratio can be considered as playing a role reminiscent of that of the inverse of temperature in the $p$-spin model with the transitions analogues of the $p$-spin dynamical and thermodynamic transitions [60]. In fact, on closer examination, random $K$-SAT exhibits an even richer sequence of phase transitions; see e.g. [29].

It is possible to interpolate between the type of behaviour of the $p \geq 3$ model and that of the $p = 2$ SK model. One way is to add a magnetic field $h$ to the $p$-spin glass. This leads to a sequence of behaviours as $h$ is increased; for small
it is qualitatively as described above for the zero-field $P$-spin model, followed at a first critical field by the coming together of the dynamical and thermodynamic spin glass transitions and replacement of the discontinuous onset of non-trivial $P(0)$ by a continuous one [30], but still with a single delta function onset at non-zero $q$, in addition to that at $q = 0$, and then at a higher critical field by a transition to a continuously distributed hierarchy of metastable states and overlaps[61]. This suggests a possible utility in adding an extra ‘effective field’ in the computer algorithmic optimization, to avoid the dynamical pre-emption of a thermodynamic transition.

D. Interacting agents

Another interesting class of range-free problems is of systems where many ‘agents’, each with individual characteristics but with no direct interactions between them, behave in a cooperatively complex fashion by all reacting to common ‘information’. This common information acts as an effectuator for correlation between the agents. Frustration and complexity arise when the goals are such that not all can ‘win’.

1. The Minority Game

A minimalist model that illustrates this class is the so-called ‘Minority Game’(MG)[35, 36], introduced to emulate some features of a stockmarket in which players make profits by buying when the price is low and selling when the price is high. In a simple version of this model $N$ agents at each time-step $t$ simultaneously make one of two choices, which we shall denote $\pm 1$. Their ‘objectives’ are to make the minority choice. They make their choices on the basis of (i) some ‘information’ $I(t)$ commonly available to all, (ii) the operation on that information by each agent $i$ of one of a pair of individual strategy operators $S_i^\alpha; \alpha = +, -$ with the output determining the ‘choice’ made, (iii) individual ‘point-scores’ $p_i(t)$ that enable the agents to ‘decide’ which of their two strategies to employ at each step. The strategy pairs are chosen randomly and independently at the outset and thereafter fixed. The information $I(t)$ varies at each time-step and hence so does the outcome of the strategies acting upon it. The space of the strategies spans the two possible outputs equally. In the simplest deterministic version of the game the strategy $S_i^\alpha$ employed by agent $i$ at time $t$ is that labelled by the same sign as $p_i(t)$. The points are updated according to

$$p_i(t + 1) = p_i(t) - [\hat{S}_i^\alpha(I(t))] A(t)$$

where $[\hat{S}_i^\alpha(I)] = \pm 1$ is the action choice of the strategy $\hat{S}_i^\alpha$ acting on the information $I$ and and $A(t)$ is the average ‘choice’ over the strategies actually employed,

$$A(t) = N^{-1} \sum_j [\hat{S}_j^\alpha(I(t))];$$

i.e. by increasing the point-score bias for strategies leading to minority behaviour. In the original formulation [37] the information used was the Boolean string indicating the minority choice in the previous $m$ time-steps of play and the $S$ were Boolean operators. However, essentially similar behaviour is obtained for a system in which $I(t)$ is randomly generated at each time $t$, equally probably from the whole space of $m$ binaries [38].

The most obviously relevant macroscopic measure in the MG is the volatility, the variance of the choices. Computer simulations demonstrated that it has scaling behaviour, the volatility per agent versus the information dimension per agent $d = D/N = 2^m/N$ approaching independence of $N$ as the latter is increased, and also has a cusplike minimum at a critical $d_c$ with behaviour ergodic for $d > d_c$ but non-ergodic for $d < d_c$ [62]. Fig 1 shows this behaviour for a slightly different variant of the model in which the strategies are taken as $D = dN$-dimensional binary strings $S_i^\alpha = S_i^{\alpha,1}, S_i^{\alpha,2}, \ldots, S_i^{\alpha,D}; i = 1, \ldots, N, \alpha = \pm$, with each component $S_i^{\alpha,\mu}; \mu = 1, \ldots, D$ chosen randomly and independently at the outset and thereafter fixed (quenched), and the stochastic ‘information’ consists in randomly choosing $\mu(t)$ at each time-step and then using the corresponding strategy elements. This is reminiscent of the behaviour of the susceptibility of the SK spin glass, shown in Fig 2, if one compares the volatility with the inverse susceptibility and the information dimension with the temperature. Hence one is tempted to analyze the MG using methodology developed for spin glasses.

Updating the point-score only after $M$ steps where $M \geq O(N)$ leads to an averaging over the random information to produce an effective interaction between the agents and yield the so-called ‘batch’ game (with temporally-rescaled update dynamics)

$$p_i(t + 1) = p_i(t) - \sum_j J_{ij} \text{sgn}(p_j(t)) - h_i \equiv p_i(t) - \partial H/\partial s_i \bigg|_{s_i = \text{sgn}(p_i(t))},$$

with $H$ the Hamiltonian of the system. The optimization, to avoid the dynamical pre-emption of a thermodynamic transition.

\[ \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \]
FIG. 1: Volatilities in Minority Games with 2 strategies per agent; Shown are (i) different biases of initial point asymmetries between each agent’s 2 strategies: $p_i(0) = 0.0$ (circles), 0.5 (squares) and 1.0 (diamonds), (ii) a comparison between the results of simulation of the deterministic many-agent dynamics (open symbols) and the numerical evaluation of the analytically-derived stochastic single-agent ensemble dynamics. From [39].

FIG. 2: Schematic susceptibility of the SK spin glass in an applied field $H$, as predicted by Parisi theory. The upper curve shows the full Gibbs average, obtained from the full $q(x)$ and interpreted as the field-cooled (FC) susceptibility. The lower curve shows the result of restricting to one thermodynamic state, as obtained from $q(1)$ and interpreted as the zero-field-cooled susceptibility. From [40].

where $H$ is an effective ‘Hamiltonian’

$$H = \sum_{ij} J_{ij} s_i s_j + h_i s_i$$

(6)

and $J_{ij}$ and $h_i$ are effective ‘exchange’ and ‘field’ terms given by

$$J_{ij} = N^{-1} \sum_{\mu=1}^{D} \xi^\mu_i \xi^\mu_j, \quad h_i = N^{-1/2} \sum_{\mu=1}^{D} \omega^\mu_i \xi^\mu_i,$$

(7)

where $\omega_i = (S_i^1 + S_i^2)/2$, $\xi_i = (S_i^1 - S_i^2)/2$. Since the $S$ are random so are the exchange and field terms. Hence $H$ is a disordered and frustrated control function. The expression for the $\{J_{ij}\}$ is very reminiscent of the Hebbian-inspired synapses of the Hopfield neural network model [41], where the $\{\xi_i^\mu\}$ are the stored memories, but crucially with the opposite sign ensuring that here the $\{\xi_i^\mu\}$ are now repellors rather attractors[63].

Methodologies

There are two main methodologies employed to study statics, the replica procedure and the cavity method (see e.g. [9]). The most common method for the cooperative dynamics is the generating functional method [42, 43].
In the replica method one studies the disorder-averaged free energy

$$\int D\{J\} P_{\text{exch}}(\{J\})(-T \ln \text{Tr}_{\{\sigma\}} \exp(-H_{\{J\}}(\{\sigma\})/T)),$$

using the identity $\ln Z = \text{Lim}_{n \to 0} \{ Z^n - 1 \}/n$, identifying the power $n$ as describing $n$ replicas, $\alpha = 1,..n$; with $n$ eventually taken to 0. Macroscopic order parameters are introduced through multiplication by unity of the form

$$1 = \int \prod_{(\alpha\beta)} D^{\alpha\beta} q^{\alpha\beta} \delta(q^{\alpha\beta} - N^{-1} \sum (\sigma^\alpha_i \sigma^\beta_i) \mathcal{H}_{\alpha\beta}),$$

where the $\alpha, \beta$ label replicas and $\mathcal{H}_{\alpha\beta}$ is the effective Hamiltonian after disorder averaging. The microscopic variables $\{\sigma^\alpha_i\}$ are integrated out and the dominant extremum [64] with respect to the $q^{\alpha\beta}$ is taken in the limit $N \to \infty$. In the most natural ansatz, replica symmetry among $q^{\alpha\beta}; \alpha \neq \beta$ was assumed [2, 3], but this proved to be too naive. The correct solution for the SK model requires Parisi’s much more subtle ansatz of replica symmetry breaking [6]. This ansatz introduces a hierarchy of spontaneous replica symmetry breaking (RSB) with a sequence of $q_i, x_i; i = 1,..K$ that in the limit of $K \to \infty$ yields a continuous order function $q(x): 0 \leq x \leq 1$, later shown [7] to be related to the average overlap distribution through $\tilde{P}(q) = \int dx \delta(q - g(x))$.

The dynamical functional method for the SK model is discussed in [10, 19]. Here we describe instead its use for the Minority Game [36]. A generating functional can be defined by

$$Z = \int \prod_t dp(t) W(p(t+1) | p(t)) P_0(p(0)),$$

where $p(t) = (p_1(t), . . . , p_N(t))$, $W(p(t+1) | p(t))$ denotes the transformation operation of eqn. (5) and $P_0(p(0))$ denotes the probability distribution of the initial score differences.

Averaging over the specific choices of quenched strategies, introducing macroscopic two-time correlation order functions via

$$1 = \int \prod_{t, t'} DC(t, t') \delta(C(t, t') - N^{-1} \sum \text{sgn} p_i(t) \text{sgn} p_i(t'))$$

and similar expressions for response functions $G(t, t')$ and derivative-variable correlators $K(t, t')$, and integrating out the microscopic variables, the averaged generating functional may then be transformed exactly into a form

$$Z = \int DCD\tilde{C}DG\tilde{D}DK\tilde{K} \exp \left( N\Phi(C, \tilde{C}, G, \tilde{G}, K, \tilde{K}) \right),$$

where $\Phi$ is $N$-independent, the bold-face notation denotes matrices in time and the tilded variables are complementary ones introduced to exponentiate the delta functions in eqn. (IV D 1) and its partners. Being extremally dominated, in the large-$N$ limit this yields the effective single agent stochastic dynamics

$$p(t+1) = p(t) - \alpha \sum_{t' \leq t} (1 + G)_{tt'}^{-1} \text{sgn} p(t') + \sqrt{\eta(t)},$$

where $\eta(t)$ is coloured noise determined self-consistently over the corresponding ensemble by

$$\langle \eta(t)\eta(t') \rangle = [(1 + G)^{-1}(1 + C)(1 + G^T)^{-1}]_{tt'}.$$

Fig. 1 demonstrates the veracity of this result in a comparison of the results of computer simulation of the original deterministic many-body problem eqn. (11) and the numerical evaluation of the self-consistently noisy single-agent ensemble of eqn. (13).

The analogous equations for the $p$-spin spherical spin glass formed the basis for recognition of the dynamical transitions mentioned earlier and the existence of aging solutions and modifications to conventional fluctuation-dissipation relations.

V. CRITICAL BEHAVIOUR AND CORRELATION LENGTH

Having commented earlier that standard non-frustrated non-disordered infinite-ranged systems do not have interesting critical behaviour, it is relevant to note that again frustrated disordered systems are different [46, 47, 49, 50], having interesting critical behaviour at low temperature and applied magnetic field, even though mean-field.
Parisi replica symmetry breaking involves an infinite sequence of hierarchies. \(K\)-RSB has \(K\) step-breaks in the order function \(q(x) : 0 \leq x \leq 1\). The exact free energy is formally obtained by finding the supremum with respect to the break and plateau values \(q_i, x_i\) and taking \(K \to \infty\). The continuum limit was given as a set of implicit equations already in Parisi’s early work. Most (but not all) of the subsequent analysis has been perturbative near the transition temperature for spin glass onset. Numerical evaluations have until a couple of years ago been restricted to just the first few steps of RSB, but very recently very high accuracy numerical extremizations for high orders of RSB have been performed at zero and low temperatures and have shown interesting features.

At low temperatures the steps \(x_i\) scale as \(x_i \sim a_i T\), with the \(a_i\) having non-zero limits as \(T \to 0\) and exposing critical points at both \(a = 0\) and \(a = \infty\). As \(T \to \infty\) the \(K\)-step approximation of \(q_i\) against \(a_i\) approaches a fixed-point function \(q^*(a)\) of form close to \(q^*(a) = (\sqrt{\pi}/2a)\text{erf}(\xi/a)\) with \(\xi\) a ‘correlation function’ in \(a\)-space given by \(\xi \approx 2/\sqrt{\pi}\). The degree of RSB can be viewed as an effective one-dimensional lattice of size \(K\), with \(K \to \infty\) the analogue of the infinite-length lattice (or thermodynamic) limit. Similarly, finite-\(K\) approximation yields an analogue of finite-size effects, including finite-size scaling. Note however that this new type of finite-size scaling is for a mean-field problem in the thermodynamic limit and is in a space of degree of approximation. There are also finite \(K\)-size scalings when the system is perturbed away from the \(T = 0\) critical point (at \(a = 0\)) and for finite applied field \(h\) near \(a = \infty\). Correspondingly there are further ‘correlation lengths’ in temperature-deviation and in field-deviation, which of course also determine the extent of RSB needed to get a good approximation as temperature or field become non-zero.

Conclusions

In this short paper it has only been possible to present a brief and non-detailed vignette of the complexity that can and does exist in disordered and frustrated many-body systems, even within a dimension-free mean-field situation. The puzzles, intrigues and challenges have developed and been a source of intense study for over 30 years. Finite-range systems have also been a great source of interest, again with significant progress but still subject to some controversy.

The case of systems with variables having different fundamental timescales, such as fast neurons and slow synapses or evolutionary models with different timescales for phenotypes and genotypes, have not been discussed. Nor has the problem of dynamical sticking in effectively self-determined disordered states of some systems without quenched disorder in their control functions but started far from equilibrium.

Also, in this brief review, only some of the simplest models have been described. It is however clear that many extensions and more realistic/complete scenarios exist that are still effectively range-free, yet complex, interesting and challenging.

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Note also that the usual spin glass or neural network dynamics is different in detail from that of the minority game, e.g.
random sequential rather than parallel.

[64] The correct extremum is actually the maximum [9, 15, 16].

[65] $q(x)$ is related to the averaged overlap distribution $P(q)$ by $P(q) = \int dx \delta(q - q(x))$.

[66] See also [51].

[67] For the most complete perturbative study the reader is referred to [52].

[68] Strictly the behaviour is found to deviate slightly but subtly - see Refs. [47, 49].