Temporal-mode continuous-variable 3-dimensional cluster state for topologically-protected measurement-based quantum computation

Kosuke Fukui, Warit Asavanant, and Akira Furusawa

Department of Applied Physics, School of Engineering, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan

Measurement-based quantum computation with continuous variables shows the great promise towards implementation of large-scale quantum computation, where large-scale quantum computation is performed via local measurements on large-scale cluster states. In this paper, we propose the method to generate the large-scale 3-dimensional cluster state which is a platform for topologically protected measurement-based quantum computation. Our method combines a time-domain approach with a divide-and-conquer approach, and has the two advantages for implementing large-scale quantum computation. First, the squeezing level for verification of the entanglement of the 3-dimensional cluster states is experimentally feasible. The second advantage is the robustness against analog errors derived from the finite squeezing of continuous variables during topologically-protected measurement-based quantum computation. Therefore, our method is a promising approach to implement large-scale quantum computation with continuous variables.

I. INTRODUCTION

Quantum computation has a great deal of potential to efficiently solve some hard problems for conventional computers [1, 2]. To realize large-scale quantum computation, measurement-based quantum computation (MBQC) is one of the most promising quantum computation models, where universal quantum computation can be implemented with only adaptive single-qubit measurements on a large-scale cluster state [3, 4]. Among the candidates for quantum states, continuous variables in an optical system have shown a great potential for the generation of large-scale cluster state. In fact, the generation of large-scale 1- and 2-dimensional cluster states has been reported in Refs. [5, 6] and Refs. [7, 8], respectively, where universal MBQC with continuous variables is performed on the 2-dimensional cluster state [9]. This ability to generate a large-scale entanglement generation comes from the fact that squeezed vacuum states can be entangled with only beam-splitter coupling through the time-domain multiplexing approach, which allows us to miniaturize optical circuits [10] and generate unlimited resource regardless of the coherence time of the system. In addition, a frequency-encoded continuous variable in an optical setup is also a promising platform [11-14], where the entangled state composed of more than 60 qumodes has been observed [12].

Regarding fault-tolerant MBQC, the quantum error correction using the GKP qubit [15] will be performed on the large-scale cluster state [16]. In the quantum error correction with the GKP qubit, a standard quantum error correction code such as the Steane’s 7-qubit code [17] is performed on the 2-dimensional cluster state. Alternatively, topologically protected MBQC has attracted much attention due to its high-noise threshold in implementing fault-tolerant MBQC [18, 19]. In topologically protected MBQC, a surface code [20] is performed on a Raussendorf-Harrington-Goyal lattice, which is referred to as the topological cluster state in this work. However, to the best of our knowledge, the specific method for generating the topological cluster state with continuous variables has not been studied so far.

In this paper, we propose a novel method to generate the large-scale topological cluster state, where a time-domain multiplexing approach is combined with a divide-and-conquer approach. Our method has the two advantages for implementing large-scale quantum computation. First, our method shows experimentally feasible squeezing level for verifying the entanglement of the topological cluster state, since the required squeezing level for the topological cluster state is almost the same level with the 2-dimensional cluster state generated by using only a time-domain method. Second, our method provides the noise tolerance against analog errors derived from the finite squeezing during MBQC, where the noise propagation can be reduced thanks to the feature of the generated topological cluster state.

The rest of the paper is organized as follows. In Sec. II, we briefly review the background knowledge regarding the cluster states and measurement-based computation with continuous variables. In Sec. III we propose the method to generate the topological cluster state. In Sec. IV, we analyze the condition of the entanglement of the generated topological cluster state and the error propagation in topologically protected MBQC, showing two advantages of our method for implementing large-scale quantum computation. Sec.V is devoted to discussion and conclusion.

II. CLUSTER STATE WITH CONTINUOUS VARIABLES

In this section, we describe the background regarding the generation of the cluster state with continuous variables by using a time-domain multiplexing. Specifically, we see an example 1D cluster state [5, 6] and the nullifiers [21, 22] to characterize the generated cluster state.

In a continuous variable system, position and momentum operators are defined as

\[ \hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^\dagger), \] (1)

where \( \hat{a} \) and \( \hat{a}^\dagger \) are annihilation and creation operators, and commutator relations are \([\hat{a}, \hat{a}^\dagger] = 1\) and \([\hat{q}, \hat{p}] = i\hbar\) with \( \hbar = 1\).

To describe the cluster states with continuous variables in
an optical setup, we focus on the 1-dimensional cluster generation demonstrated in Refs. [5, 6], which is referred to as the extended EPR state. Fig. 1 shows that a large-scale 1-dimensional cluster state generated by using the time-domain multiplexing approach is composed of the squeezed vacuum states. Temporally localized wave packets from two optical parametric oscillators (OPOs) are used as qumodes for MBQC. In Fig. 1 each colored circle and link represents a qumode and the quantum entanglement, respectively, and the color of the link describes the sign of the edge-weight factor for a weighted continuous-variable cluster state. The arrow represents the phase relationship of the beam-splitter coupling.

Firstly, the two-mode entangled states are generated 1-dimensional cluster state generated by using the time-domain multiplexing approach. In Fig. 1, each colored circle and link represents a qumode, while each link between qumodes represents quantum entanglement. The color of the link denotes the sign of the edge-weight factor for a weighted continuous-variable cluster state. The arrow represents the phase relationship of the beam-splitter coupling.

FIG. 1. The generation of the 1-dimensional cluster state in an optical setup by using a time-domain multiplexing approach. Each colored circle represents a qumode, while each link between qumodes represents quantum entanglement. The color of the link denotes the sign of the edge-weight factor for a weighted continuous-variable cluster state. The arrow represents the phase relationship of the beam-splitter coupling. (i) The generation of the two-mode entangled states by using a beam-splitter coupling between a sequence of modes $A_k$ and $B_k$. (ii) The generation of the 1-dimensional cluster state referred to as the extended EPR state. Fig. 1 shows that a large-scale 1-dimensional cluster states referred to as the extended EPR state. We introduce the nullifiers, where the nullifier corresponds to the stabilizer for cluster states with discrete variables in the case of the infinite squeezing. The nullifiers of mode $k$ for the generated 1-dimensional cluster state in the $q$ and $p$ operators, $\hat{\delta}_q^k$ and $\hat{\delta}_p^k$, are obtained as

$$\hat{\delta}_q^k = \hat{q}_{A,k} + \hat{q}_{B,k} + \hat{q}_{A,k+1} - \hat{q}_{B,k+1},$$

$$\hat{\delta}_p^k = \hat{p}_{A,k} + \hat{p}_{B,k} - \hat{p}_{A,k+1} + \hat{p}_{B,k+1},$$

respectively. From $\hat{a}_{A,k} = \hat{a}^{(ii)}_{A,k}$, we obtain the relation as

$$\hat{\delta}_q^k | 1D \rangle = 0, \quad \hat{\delta}_p^k | 1D \rangle = 0.$$

Thus, nullifiers for the cluster state with the infinite squeezing corresponds to the stabilizer. In the case of the finite squeezing, we can verify the generation of the 1-dimensional cluster state by calculating the inseparable condition for the variance as

$$\langle \langle \hat{\delta}_q^2 \rangle \rangle = e^{-r_q} \hat{q}_{A,k}^2 < \frac{1}{2},$$

$$\langle \langle \hat{\delta}_p^2 \rangle \rangle = e^{-r_p} \hat{p}_{B,k}^2 < \frac{1}{2},$$

where $\langle \hat{\delta}_q^2 \rangle$ and $\langle \hat{\delta}_p^2 \rangle$ are quadratures of the $k$-th squeezed vacuum state with the squeezing parameters $r_q$ and $r_p$ in the modes $A$ and $B$, respectively. In the case of the ideal 1-dimensional cluster state, i.e., the squeezed vacuum state has an infinite squeezing, the nullifiers for the 1D cluster state $| 1D \rangle$ become zero as

$$\langle \langle \hat{\delta}_q^2 \rangle \rangle = 0, \quad \langle \langle \hat{\delta}_p^2 \rangle \rangle = 0.$$

This condition to verify the entanglement generation is called van-Loock-Furusawa criterion [23]. From this criterion, the squeezing level required for the 1-dimensional cluster state is -3.0 dB squeezing of each nullifier, where a squeezing level is equal to $10 \log_{10} e^{-2r}$.

III. GENERATION OF THE TOPOLOGICAL CLUSTER STATE

entanglement is experimentally accessible to date. In our
method, the so-called divide-and-conquer approach [24, 25] is combined with the time-domain method. We note that the purpose of using the divide-and-conquer approach in Refs. [24, 25] is to overcome a problem based on a photon qubit in terms of the probabilistic two-qubit gate for generating the large-scale cluster state, while our purpose is to achieve the feasible squeezing level required for verifying the deterministic entanglement of the large-scale cluster state.

Fig. 2(a) describes the schematic diagram for the experimental setup to generate the large-scale topological cluster state using a miniaturized optical setup. The setup consists of two components. In the first component, the small-scale cluster states are generated without the time-domain multiplexing method, where the size of the basal plane for the topological cluster state is \( V = N \times M \). Time delays are implemented on qumodes \( B_{2,k}, B_{3,k}, B_{4,k}, B_{5,k}, \) and \( B_{6,k} \) by \( 1, N + 1, N + V + 1, N + V, \) and \( V \), respectively, assuming that the \( k \)-th qumode with a time delay \( \Delta t \) is coupled with the qumode in the \( (k + V) \)-th hexagonal cluster state A. (b) Experimental setup for the first component, generating the hexagonal cluster state. Each of the generator of hexagonal cluster A and B in the first component consist of 6 optical parametric oscillators (OPOs) and 6 beam-splitters. (i) The generation of two-mode entangled states. (ii) The generation of the multimode entangled states. (iii) Fourier transformation on three qumodes. (c) The generated hexagonal cluster state A. (d) The beam-splitter coupling between the qumodes A and B in the second component. (e) (left) The generated large-scale topological cluster state, where the size of the basal plane for the space-like direction is \( N \times M \), and the length for the time-like direction is arbitrarily large. The large-scale topological cluster state is generated via the quantum erasure, i.e., the measurement of qumodes B by using the homodyne measurement and the feed-forward depending on the measurement results after (a)(vii). (right) The unit cell of the topological cluster state.

We explain the first component to generate the small-scale cluster states referred to as the hexagonal cluster state in this work. Fig. 2(b) shows a schematic picture of generation of the hexagonal cluster state A. Each of generators of the hexagonal cluster state consists of six OPOs and six 50:50 beam splitters. The transmittances of beam splitters are obtained from the decomposition technique for the beam splitter network [27]. Here we describe the transformation of annihilation and creation operators in the generator labeled with A. The hexagonal cluster state B is generated in the same way as the hexagonal cluster state A. In Fig. 2(b)(i), the generation of the two-mode entangled states by a beam-splitter coupling between a sequence of modes \( i \) and \( j \) is described. This beam-
splitter coupling transforms the operators as

\[ \hat{U}_{BS} \left( \hat{a}_{A_{i,k}}, \hat{a}_{A_{i,k}}^\dagger \right) \hat{U}_{BS}^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{A_{i,k}} \\ \hat{a}_{A_{i,k}}^\dagger \end{pmatrix} \]

\[ = \begin{pmatrix} \hat{a}_{A_{i,k}} \\ \hat{a}_{A_{i,k}}^\dagger \end{pmatrix}, \tag{11} \]

where the sets of indices \((i, j)\) are \((1,6), (5,4), \) and \((3,2)\). In Fig. 2(b)(ii), the multimode entangled state are generated by a beam-splitter coupling between a sequence of modes. After this beam-splitter coupling, the operators become

\[ \hat{U}_{BS} \left( \hat{a}_{A_{i,k}}^{(i)}, \hat{a}_{A_{i,k}}^{(ii)} \right) \hat{U}_{BS}^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{A_{i,k}}^{(i)} \\ \hat{a}_{A_{i,k}}^{(ii)} \end{pmatrix} \]

\[ = \begin{pmatrix} \hat{a}_{A_{i,k}}^{(i)} \\ \hat{a}_{A_{i,k}}^{(ii)} \end{pmatrix}, \tag{12} \]

where the sets of indices \((i, j)\) are \((1,4), (5,2), \) and \((3,6)\). After the Fourier transformation on modes \(i \) \((i = 1, 3, 5)\) described in Fig. 2(b), the hexagonal cluster state is generated, as shown in Fig. 2(c). The operators for the hexagonal cluster state become

\[ \hat{U}_F \left( \hat{a}_{A_{i,k}}^{(i)}, \hat{a}_{A_{i,k}}^{(ii)} \right) \hat{U}_F^\dagger = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_{A_{i,k}}^{(i)} \\ \hat{a}_{A_{i,k}}^{(ii)} \end{pmatrix} \]

\[ = \begin{pmatrix} \hat{a}_{A_{i,k}}^{(i)} \\ -\hat{a}_{A_{i,k}}^{(ii)} \end{pmatrix}, \tag{13} \]

where operators for qumodes \(i \) \((i = 2, 4, 6)\) are \(\hat{a}_{A_{i,k}}^{(i)} = \hat{a}_{A_{i,k}}^{(ii)}\).

In the same way as the generation of the hexagonal cluster state \(A\) in the first component, the hexagonal cluster state \(B\) is obtained at the same time with the same configuration of optical elements for the hexagonal cluster state \(A\).

In the second component, the large-scale topological cluster state is generated by a beam-splitter coupling between qumodes \(A\) and \(B\), and by the measurement of qumodes belonging to the hexagonal cluster state \(B\). In this component, the beam-splitter coupling for the first row transforms the operator as

\[ \hat{U}_{BS} \left( \hat{a}_{A_{i,k}}^{(iii)}, \hat{a}_{B_{i,k}}^{(iii)} \right) \hat{U}_{BS}^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{A_{i,k}}^{(iii)} \\ \hat{a}_{B_{i,k}}^{(iii)} \end{pmatrix} \]

\[ = \begin{pmatrix} \hat{a}_{A_{i,k}}^{(iii)} \\ \hat{a}_{B_{i,k}}^{(iii)} \end{pmatrix}. \tag{15} \]

Other operators are transformed in the same way as the first row described in Eq. (14). After the beam-splitter coupling between the qumodes \(A\) and \(B\), the large-scale entangled state, not the topological cluster state, is generated in Fig. 2(a)(vii). To obtain the large-scale topological cluster state, the qumodes \(B\) need to be removed from the large-scale entangled state by using the so-called quantum erasure the time-domain multiplexing approach is applied to hexagonal cluster states \(A\) and \(B\) in Fig. 2(a)(v)-(vii). Each of modes composed of the hexagonal cluster \(A\) is coupled with the mode of the hexagonal cluster \(B\) after time delays as shown in Fig. 2(d). In this work, we assume the generation of the topological cluster states with the lattice size \(N \times M\) after generating hexagonal cluster states \(A\) and \(B\) in Fig. 2(a)(iv), time delays are implemented to qumodes \(B_{2,k}, B_{3,k}, B_{4,k}, B_{5,k},\) and \(B_{6,k}\) by \(1, N + 1, N + 1 + V, N + V, \) and \(V, \) respectively, whereas the qumodes in the hexagonal cluster states \(A\) do not have a time delay, as shown in Fig. 2(a)(v). The optical delay lines \(1, N, M, V\) are used to implement time delays \(\Delta t, N \times \Delta t, M \times \Delta t,\) and \(V \times \Delta t = N \times M \times \Delta t.\) In Fig. 2(a), unit of time delay, \(\Delta t,\) is omitted for brevity. After the time delays, the qumodes of hexagonal clusters \(A\) and \(B\) are coupled by 50:50 beam splitters in Fig. 2(a)(vi). The following equation is a list for the pairs of two modes coupled by a beam splitter in terms of the \(k\)-th hexagonal cluster \(B\) as

\[ (A, 1, k) \iff (B, 1, k), \]

\[ (A, 2, k + 1) \iff (B, 2, k), \]

\[ (A, 3, k + N + 1) \iff (B, 3, k), \]

\[ (A, 4, k + N + 1 + V) \iff (B, 4, k), \]

\[ (A, 5, k + N + V) \iff (B, 5, k), \]

\[ (A, 6, k + V) \iff (B, 6, k). \tag{14} \]
arbitrarily large. We note that during the MBQC the qumodes 4, 5, and 6 in the first $V$ hexagonal clusters A will be measured in the $q$ quadrature, since those do not couple with any other qumodes, and do not compose the topological cluster state.

To get a more intuitive understanding of using the time-domain multiplexing method, we describe the schematic view of generating entanglement between neighboring hexagonal cluster states A in Fig. 3. We here focus on two hexagonal cluster states A whose time delay is $N \times \Delta t$, and see the entanglement generation between them via two hexagonal cluster states B with a time delays. Figs. 3(a) and (b) show four hexagonal cluster states before and after time delays, respectively. Then, beam-splitter coupling between qumodes A and B with the same temporal mode index is implemented, as shown in Fig. 3(c), where the beam-splitter coupling is depicted by dotted lines. The entangled state is generated with four hexagonal cluster states after the beam-splitter coupling, as shown in Fig. 3(d). Then, we implement the quantum erasure; namely, the qumodes B are measured in the $q$ quadrature and the feed-forward operation depending on the measurement results is implemented on qumodes A. As a results, the cluster state, which is a part of the topological cluster state, is generated, as shown in Fig. 3(e).

IV. ANALYSIS

In this section, we firstly analyze the nullifiers of the qumodes composed of generated hexagonal and topological cluster states generated by the proposed method. We then describe the verification of the generated topological cluster state by using the nullifiers, and obtain the required squeezing level for the verification. We finally show a robustness against analog errors in generated states by describing the fact that errors in the $q(p)$ quadrature, which are derived from the finite squeezing, do not propagate on the basis in the $p(q)$ quadrature between qumodes.

A. Nullifier of the topological cluster state

We firstly describe the nullifier of the generated hexagonal cluster state, which obeys the transformations described in Eqs. (11)-(13). In the following, we see the generation of the hexagonal cluster state A. The initial nullifiers for the 6 modes in the temporal mode index $k$ are described as

$$\{\hat{p}_{A,2n-1,k}, \hat{q}_{A,2n,k}\},$$  \hspace{1cm} (16)

where $n = 1, 2, 3$. For sake of simplicity, we omit labels A and $k$ in Eq. (16) as $\{\hat{p}_{2n-1}, \hat{q}_{2n}\}$. The nullifiers for the entangled states after the first beam-splitter coupling become

$$\{\hat{p}_{2n} + \hat{p}_{2n+1 \mod 6}, \hat{q}_{2n} - \hat{q}_{2n+1 \mod 6}\}.$$

(17)

We then perform the second beam-splitter coupling in Fig. 2(b)(ii), and Fourier transformations on modes 1, 3, and 5 in Fig. 2(b)(iii). The nullifiers are transformed as

$$\{\hat{q}_{2n-1} - \hat{q}_{2n+1 \mod 6} - \hat{p}_{2n+2 \mod 6} - \hat{p}_{2n+4 \mod 6},$$

$$\hat{q}_{2n} - \hat{q}_{2n+4 \mod 6} - \hat{p}_{2n+1 \mod 6} - \hat{p}_{2n+3 \mod 6}\}.$$  \hspace{1cm} (18)

By taking linear combinations, the nullifiers become

$$\{\hat{p}_{2n-1} + \hat{q}_{2n} - \hat{q}_{2n+4 \mod 6},$$

$$\hat{p}_{2n} - \hat{q}_{2n+1 \mod 6} + \hat{q}_{2n+5 \mod 6}\},$$

(19)

which corresponds to the nullifiers for the hexagonal cluster state described in Fig. 2(c). In the same way as the hexagonal cluster A, the nullifiers for the hexagonal cluster B are obtained.

We next explain the nullifier of the generated topological cluster state, which obeys the transformations described in Eqs. (14) and (15). As shown in Sec. III, the topological cluster state is generated from hexagonal clusters A and B by using the time-domain multiplexing approach, which leads to reduction of the requirement for an experimental setup to generate large-scale cluster states. After time delays described in...
For the necessary condition of an inseparability between i
boring modes \( K \) generated cluster state by using the van Loock-Furusawa in-
tors of momentum and position operators, respectively, and We then consider the multimode cluster states M and N in Fig.

\[
\begin{align*}
\hat{\rho}_{B,1,k} + \frac{1}{2}(\hat{q}_{A,2,k} + \hat{q}_{B,2,k} + \hat{q}_{A,2,k+1} - \hat{q}_{B,2,k+1} - \hat{q}_{A,6,k} - \hat{q}_{B,6,k} + \hat{q}_{B,6,k} + \hat{q}_{A,6,k} + \hat{q}_{B,6,k}), \\
\hat{\rho}_{A,1,k} + \frac{1}{2}(\hat{q}_{A,1,k} + \hat{q}_{B,1,k} + \hat{q}_{A,1,k-1} - \hat{q}_{B,1,k-1} - \hat{q}_{B,3,k} - \hat{q}_{A,3,k} + \hat{q}_{B,3,k} + \hat{q}_{B,3,k}).
\end{align*}
\]

In a similar manner to the nullifiers for qumodes \( A_{1,k} \) and \( A_{2,k} \), we can obtain those for other qumodes.

**B. Verification of the generated topological cluster state**

We discuss sufficient conditions of entanglement for the generated cluster state by using the van Loock-Furusawa inseparability criteria in order to verify the generated topological cluster state. Here we consider the \( K \)-mode cluster state for the general case. The nullifiers for the general cluster state are given by \( \delta = \hat{p} - \hat{C}_\hat{k} \), where \( \hat{p} \) and \( \hat{C}_\hat{k} \) are column vec-
tors of momentum and position operators, respectively, and \( C \) is an \( K \times K \) adjacency matrix. The nullifiers for neighboring modes \( i \) and \( j \) are described as

\[
\begin{align*}
\hat{\delta}_i &= \hat{p}_i - C_{ij}\hat{q}_j - \sum_{m \in M} C_{im}\hat{q}_m - \sum_{l \in L} C_{ij}\hat{q}_l, \\
\hat{\delta}_j &= \hat{p}_j - C_{ji}\hat{q}_i - \sum_{n \in N} C_{jn}\hat{q}_n - \sum_{l \in L} C_{ji}\hat{q}_l,
\end{align*}
\]

where \( m, n, \) and \( l \) are the label for qumodes belonging to the multimode cluster states M, N, and L, as shown in Fig. 4a). We then consider the multimode cluster states M and N in Fig.

\[
\begin{align*}
\hat{\delta}_{A,1,k} &= \hat{p}_{A,1,k} + \frac{1}{2}(\hat{q}_{A,2,k} + \hat{q}_{B,2,k} + \hat{q}_{A,2,k+1} - \hat{q}_{B,2,k+1} - \hat{q}_{A,6,k} - \hat{q}_{B,6,k} + \hat{q}_{B,6,k} + \hat{q}_{A,6,k} + \hat{q}_{B,6,k}), \\
\hat{\delta}_{A,2,k} &= \hat{p}_{A,2,k} + \frac{1}{2}(\hat{q}_{A,1,k} + \hat{q}_{B,1,k} + \hat{q}_{A,1,k-1} - \hat{q}_{B,1,k-1} - \hat{q}_{B,3,k} - \hat{q}_{A,3,k} + \hat{q}_{B,3,k} + \hat{q}_{B,3,k}).
\end{align*}
\]

respectively. We apply the generated cluster state with our method to Eqs. \( 25 \)\( -27 \) as

\[
\langle \Delta^2 \hat{\delta}_{A,1,k} \rangle + \langle \Delta^2 \hat{\delta}_{A,2,k} \rangle = 3\hbar e^{-2\tau} < \hbar.
\]

Thus, we can verify the generation of the topological cluster state, if the inequality

\[
e^{-2\tau} < \frac{1}{3}
\]

is satisfied. From the van-Loock-Furusawa criterion, the required squeezing level to satisfy the above inequality is \(~4.77\text{dB}~). Consequently, our method provides almost the same required squeezing level, \(-4.5\text{ dB}~), to show sufficient conditions of entanglement for the 2-dimensional cluster state which has been demonstrated in Ref. \( 7 \).

Here we mention that this benefit of the feasible squeezing of the generated cluster state comes from the economical use
of a beam-splitter coupling. Generally, a beam-splitter coupling leads to a decrease in the amplitude of the edge-weight factor $|p|$, without the aid of the decomposition technique in Ref. [27]. Besides, the smaller the amplitude of the edge-weight factor, the more the required squeezing level to show sufficient conditions is $|F|$. In our method, we firstly generate appropriate small-scale building blocks, i.e., hexagonal cluster states by using the decomposition technique. Then the topological cluster state is constructed from building blocks by using the only one beam-splitter coupling per node of the topological cluster state. In the conventional method, on the other hand, a topological cluster state will be generated from the building blocks, which is two-mode entangled states, by using the more than three beam-splitter couplings per node. Hence, our method can provide the feasible squeezing to verify the generated cluster state.

C. Robustness against analog errors

In QC with squeezed vacuum states, the displacement errors derived from a finite squeezing generally propagate between qumodes by two-qubit gates, and are accumulated due to the quantum-teleportation-based gate in MBQC. Thus, the quantum error correction is needed to correct them for implementing large-scale quantum computation by using an appropriate code such as the GKP qubit [15]. Nevertheless, the large displacement error occurs the qubit-level error, i.e., bit- and phase-flip errors in the code word of the GKP qubit. Thus, the accumulation of displacement errors should be reduced to improve the noise tolerance against analog errors. In this subsection, we show the second advantage of our approach, i.e., a desirable noise tolerance against analog errors during MBQC.

In the following, let us look the noise propagation between squeezed vacuum states, since the detailed analysis of the quantum error correction with the GKP qubit is out of the scope of the present work. For simplicity, we focus on the propagation of the displacement error from the qumode 1 to the qumode 4, as shown in Fig. 5(a), assuming that the qumodes 1 and 4 are measured in the $p$ quadrature for the $Z$ and $X$ stabilizers, respectively. Fig. 5(b) shows an equivalent circuit for MBQC on the cluster state. We here introduce the CZ gate which corresponds to the operator $\exp(-i g q_j q_k)$ for qumodes $j$ and $k$ with the factor $g$ corresponding to the sign of interaction strength of the CZ gate, i.e., $g = \mp 1$. The CZ gate transforms the interaction strength as

$$g \rightarrow g' = g \Delta p_{j,k},$$

where $\Delta p_{j,k}$ is the displacement error for qumode $j$ and $k$. The displacement error for qumodes except for the qumode 1 are zero. Taking into account the CZ gate, the deviation errors of qumodes 2 and 6 in the $p$ quadrature are transformed to those of the qumodes 3 and 5 in the $q$ quadrature as

$$\Delta p_{2,6} = g \Delta q_{1,1}, \quad \Delta p_{5,3} = -g \Delta q_{1,1}. \quad (31)$$

After the measurement on qumodes 2 and 6 in the $p$ quadrature, displacement errors of the qumodes 2 and 6 in the $p$ quadrature are transformed to those of the qumodes 3 and 5 in the $q$ quadrature as

$$\Delta q_{3,5} = \Delta q_{1,1}, \quad \Delta q_{3,5} = \Delta q_{1,1}. \quad (32)$$

FIG. 4. Verification of the cluster state. (a) Separability for a general cluster state. $C_{ij}$ represents the edge-weight factor for qumodes $i$ and $j$. (b) Separability for a particular cluster state. (c) Separability for the topological cluster state generated by using our method, focusing on qumodes $A_{1,k}$ and $A_{2,k}$.

FIG. 5. Error propagation in the generated topological cluster state. (a) Error propagation from the qumode 1 to the qumode 4, where qumodes 2 and 6 are input states, and qumodes 1 and 4 are used for the syndrome measurement of $Z$ and $X$ stabilizers. (b) An equivalent quantum circuit for MBQC on the cluster state within the framework for a circuit-based model. $\hat{F}$ denotes the Fourier transformation and is implemented by the measurement of the qumode in the $p$ quadrature. $\pm$ denotes the sign of interaction strength of the CZ gate, i.e., the sign of the edge-weight factor.
We note that the displacement errors are amplified by $g$, according to the procedure of MBQC. Those of deviation errors of qumodes 3 and 5 eventually propagate on the qumode 4 in the $p$ quadrature by the CZ gates. This transformation corresponds to the Fourier transformation on the inputs A and B in Fig. 5(b) within the framework for a circuit-based model. After the CZ gates between qumodes 3 and 4, and 5 and 4, the deviation errors of the qumode 4 is

$$\Delta_{q,4} = g\Delta_{q,1} - g\Delta_{q,1} = 0,$$

where the edge-weight factors with respect to the qumodes 3 and 5 are $+g$ and $-g$, respectively. We can see that the analog error derived from the qumode 1 is canceled out in the qumode 4, and therefore the generated topological cluster state has a robustness against displacement errors during topologically protected MBQC [28]. Since this feature is obtained thanks to the sign of the edge-weight factors of the generated topological cluster state, our method is practical to realize fault-tolerant MBQC with the robustness of analog errors, in addition to a reasonable squeezing level for the verification of the entanglement.

In the end of this section, we note the effect of the edge-weight factor on the quantum error correction with the GKP qubit. To perform the quantum error correction with the GKP qubit, the amplitude of edge-weight factors should be set to 1, since the amplitude of the interaction of the two-qubit gate between GKP qubits should be 1 in the code word of the GKP qubit. Therefore, the strength of the entanglement of the topological cluster state will be recovered to adjust the amplitude of the edge-weight factor to 1 [30, 31]. As a result, this entanglement recovery increases the noise derived from a finite squeezing of the squeezed vacuum states by the inverse of the edge-weight factor. For example, the amplitude of the edge-weight factor of the 3-dimensional cluster state by using only the time-domain multiplexing approach is $1/4\sqrt{2}$ [32]. Thus, our method with the amplitude of the edge-weight factor 1/2 has an advantage for performing quantum error correction with the GKP qubit [33].

V. CONCLUSION

In this work, we have proposed the method to generate the topological cluster state for implementing topologically protected MBQC with the linear optics. In our method, the squeezing level required for verifying the generated cluster state is an experimentally feasible value, which is almost the same level with the 2-dimensional cluster state generated by using the conventional method. Moreover, in the generated cluster state, analog errors are canceled out and prevented from propagating between the qumodes thanks to the feature of a sign of an edge-weight factor. For the quantum error correction with the GKP qubit, the generated cluster state has an advantage due to the smaller amplitude of the edge-weight factor, compared to that by using only the time-domain multiplexing approach. These features are compatible with the analog quantum error correction [34] and high-threshold topologically protected MBQC with the GKP qubit [35, 36]. High-threshold topologically protected MBQC on the topological cluster state generated by our method will provide a new approach to implement large-scale MBQC with an experimentally feasible squeezing level. In addition, although we apply our approach to the topological cluster state in this paper, our method can be applied to a variety of entangled states such as the 3-dimensional lattice for a color code [37, 38], the 2-dimensional honeycomb state [39], and so on. Furthermore, our method can be applied to several promising architectures for a scalable quantum circuit proposed recently in Refs. [40–42]. Hence, we believe this work will provide a new way to generate the large-scale resource state to implement fault-tolerant MBQC with continuous variables.

ACKNOWLEDGEMENTS

This work was partly supported by Japan Society for the Promotion of Science (JSPS) KAKENHI (grant 18H05207), CREST (Grant No. JP- MJC15N5), UTokyo Foundation, and donations from Nichia Corporation. W. A. acknowledges financial support from the Japan Society for the Promotion of Science (JSPS).

[1] P. W. Shor, Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer, SIAM J. Comp. 26, 1484 (1997).
[2] L. K. Grover, Quantum Mechanics Helps in Searching for a Needle in a Haystack, Phys. Rev. Lett. 79, 325 (1997).
[3] R. Raussendorf and H. J. Briegel, A One-Way Quantum Computer, Phys. Rev. Lett. 86, 5188 (2001).
[4] H. J. Briegel and R. Raussendorf, Persistent Entanglement in Arrays of Interacting Particles, Phys. Rev. Lett. 86, 910 (2001).
[5] S. Yokoyama, R. Ukaï, S. C. Armstrong, C. Sornphiphatphong, T. Kaji, S. Suzuki, J. Yoshikawa, H. Yonezawa, N. C. Menicucci, and A. Furusawa, Ultra-Large-Scale Continuous-Variable Cluster States Multiplexed in the Time Domain, Nat. Photonics 7, 982 (2013).
[6] J. Yoshikawa, S. Yokoyama, T. Kaji, C. Sornphiphatphong, Y. Shiozawa, K. Makino, and A. Furusawa, Generation of one-million-mode continuous-variable cluster state by unlimited time-domain multiplexing, APL Photonics 1 060801 (2016).
[7] W. Asavanant, Y. Shiozawa, S. Yokoyama, B. Charoensombutamon, H. Emura, R. N. Alexander, S. Takeda, J. Yoshikawa, N. C. Menicucci, H. Yonezawa, and A. Furusawa, Generation of Time-Domain-Multiplexed Two-Dimensional Cluster State, Science 366, 373 (2019).
[8] M. V. Larsen, X. Guo, C. R. Breum, J. S. Neergaard-Nielsen, and U. L. Andersen, Deterministic Generation of a Two-Dimensional Cluster State, Science 366, 369 (2019).
[9] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Universal Quantum Computation
with Continuous-Variable Cluster States, Phys. Rev. Lett. 97, 110501 (2006).

[10] N. C. Menicucci, Temporal-Mode Continuous-Variable Cluster States Using Linear Optics, Phys. Rev. A 83, 062314 (2011).

[11] N. C. Menicucci, S. T. Flammia, and O. Pfister, One-Way Quantum Computing in the Optical Frequency Comb, Phys. Rev. Lett. 101, 130501 (2008).

[12] M. Pysher, Y. Miwa, R. Shahrokhshahi, R. Bloomer, and O. Pfister, Parallel Generation of Quadrupartite Cluster Entanglement in the Optical Frequency Comb, Phys. Rev. Lett. 107, 030505 (2011).

[13] M. Chen, N. C. Menicucci, and O. Pfister, Experimental Realization of Multiparticle Entanglement of 60 Modes of a Quantum Optical Frequency Comb, Phys. Rev. Lett. 112, 120505 (2014).

[14] J. Roslund, R. M. Araújo, S. Jiang, C. Fabre, and N. Treps, Wavelength-Multiplexed Quantum Networks with Ultrafast Frequency Combs, Nature Photonics, 8, 109-112 (2014).

[15] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Phys. Rev. A 64, 012310 (2001).

[16] N. C. Menicucci, Fault-Tolerant Measurement-Based Quantum Computing with Continuous-Variable Cluster States, Phys. Rev. Lett. 112, 120504 (2014).

[17] A. M. Steane, Overhead and Noise Threshold of Fault-Tolerant Quantum Error Correction, Phys. Rev. A 68, 042322 (2003).

[18] R. Raussendorf, J. Harrington, and K. Goyal, Topological Fault-Tolerance in Cluster State Quantum Computation, New J. Phys. 9, 199 (2007).

[19] R. Raussendorf, J. Harrington, and K. Goyal, A Fault-Tolerant One-Way Quantum Computer, Ann. Phys. (Amsterdam) 321, 2242 (2006).

[20] A. Y. Kitaev, Fault-Tolerant Quantum Computation by Anyons, Ann. Phys. (Amsterdam) 303, 2 (2003).

[21] M. Gu, C. Weedbrook, N. C. Menicucci, T. C. Ralph, and P. van Loock, Quantum Computing with Continuous-Variable Clusters, Phys. Rev. A 79, 062318 (2009).

[22] N. C. Menicucci, S. T. Flammia, and P. van Loock, Graphical Calculus for Gaussian Pure States, Phys. Rev. A, 83, 042335 (2011).

[23] P. van Loock, and A. Furusawa, Detecting Genuine Multiparticle Continuous-Variable Entanglement, Phys. Rev. A 67, 052315 (2003).

[24] M. A. Nielsen, Optical Quantum Computation Using Cluster States, Phys. Rev. Lett. 93, 040503 (2004).

[25] C. M. Dawson, H. L. Haselgrove, and M. A. Nielsen, Noise Thresholds for Optical Quantum Computers, Phys. Rev. Lett. 96, 020501 (2006).

[26] Y. Miwa, R. Ukai, J. Yoshikawa, R. Filip, P van Loock, and A. Furusawa, Demonstration of Cluster-State Shaping and Quantum Erasure for Continuous Variables, Phys. Rev. A 82, 032305 (2010).

[27] P. van Loock, C. Weedbrook, and M. Gu, Building Gaussian Cluster States by Linear Optics, Phys. Rev. A 76, 032321 (2007).

[28] We may point out that the reduction of the noise propagation in the generated cluster state will be compatible with the technique introduced in Ref. [29]. The technique in Ref. [29] is implemented by intentionally using the SUM and inverse-SUM gates, while in our case the robustness against analog errors is inherent in our generated cluster state.

[29] K. Noh and C. Chamberland, Fault-Tolerant Bosonic Quantum Error Correction with the Surface-Gottesman-Kitaev-Preskill Code, Phys. Rev. A 101, 012316 (2020).

[30] S. Glancy and E. Knill, Error Analysis for Encoding a Qubit in an Oscillator, Phys. Rev. A 73, 012325 (2006).

[31] K. H. Wan, A. Neville, and S. Koltzhammer, A Memory-Assisted Decoder for Approximate Gottesman-Kitaev-Preskill Codes [arXiv:1912.00829]

[32] Bo-Han Wu, R. N. Alexander, S. Liu, Z. Zhang, Quantum Computing Architecture based on Large-Scale Multi-Dimensional Continuous-Variable Cluster States in a Scalable Photonic Platform, [arXiv:1909.05455]

[33] We should note that the edge-weight factor $1/\sqrt{2} [32]$ is obtained by just applying the method introduced in Ref. [10], and thus there may be a more efficient protocol using only a time-domain multiplexing method. In this work, we just use the edge-weight factor to simply compare our work with the conventional method.

[34] K. Fukui and A. Tomita and A. Okamoto, Analog Quantum Error Correction with Encoding a Qubit into an Oscillator, Phys. Rev. Lett. 119, 180507 (2017).

[35] K. Fukui, A. Tomita, A. Okamoto, and K. Fujii, High-Threshold Fault-Tolerant Quantum Computation with Analog Quantum Error Correction, Phys. Rev. X 8, 021054 (2018).

[36] K. Fukui, High-Threshold Fault-Tolerant Quantum Computation with the GKP Qubit and Realistically Noisy Devices, [arXiv:1906.09767]

[37] H. Bombin and M. A. Martin-Delgado, Topological Quantum Distillation, Phys. Rev. Lett. 97, 180501 (2006).

[38] B. J. Brown, N. H. Nickerson, and D. E. Browne, Fault-Tolerant Error Correction with the Gauge Color Code, Nat. Commun. 7, 12302 (2016).

[39] M. Van den Nest, A. Miyake, W. Dür, and H. J. Briegel, Universal Resources for Measurement-Based Quantum Computation, Phys. Rev. Lett. 97, 150504 (2006).

[40] S. Takeda and A. Furusawa, Universal Computing with Measurement-Induced Continuous-Variable Gate Sequence in a Loop-Based Architecture, Phys. Rev. Lett. 119, 120504 (2017).

[41] R. N. Alexander, S. Yokoyama, A. Furusawa, and N. C. Menicucci, Universal Quantum Computation with Temporal-Mode Bilayer Square Lattices, Phys. Rev. A 97, 032302 (2018).

[42] S. Takeda, K. Takase, and A. Furusawa, On-Demand Photonic Entanglement Synthesizer, Sci. Adv. 5, eaaw4530 (2019).