All-loop group-theory constraints
for color-ordered SU($N$) gauge-theory amplitudes

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Abstract

We derive constraints on the color-ordered amplitudes of the $L$-loop four-point function in SU($N$) gauge theories that arise solely from the structure of the gauge group. These constraints generalize well-known group theory relations, such as U(1) decoupling identities, to all loop orders.

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1 Introduction

An exciting recent development in the study of perturbative amplitudes is the discovery of color-kinematic duality of gauge theory amplitudes at both tree and loop level [1, 2]. This duality implies the existence of constraints on tree-level color-ordered amplitudes, which were proven in refs. [3–6]. The BCJ conjecture was also verified through three loops for the $\mathcal{N} = 4$ supersymmetric Yang-Mills four- [2, 7] and five-point [8, 9] amplitudes. (See reviews in refs. [7, 10], which also contain references to related work on the subject.)

The BCJ constraints on tree-level color-ordered amplitudes hold in addition to various well-known SU($N$) group theory relations, such as the U(1) decoupling or dual Ward identity [11, 12] and the Kleiss-Kuijf relations [13]. Group-theory relations also hold for one-loop [14, 15] and two-loop [16] color-ordered amplitudes. They can be elegantly derived by using an alternative color decomposition of the amplitude [17, 18].

The purpose of this note is extend the SU($N$) group theory relations for four-point amplitudes to all loops. We develop a recursive procedure to derive constraints satisfied by any $L$-loop diagram (containing only adjoint fields) obtained by attaching a rung between two external legs of an $(L-1)$-loop diagram. We assume that the most general $L$-loop color factor can be obtained from this subset using Jacobi relations, an assumption that has been proven through $L = 4$. Using this method, we find four independent group-theory constraints for color-ordered four-point amplitudes at each loop level (except for $L = 0$ and $L = 1$, where there are one and three constraints respectively).

The color-ordered amplitudes of a gauge theory are the coefficients of the full amplitude in a basis using traces of generators in the fundamental representation of the gauge group. Color-ordered amplitudes have the advantage of being individually gauge-invariant. Four-point amplitudes of SU($N$) gauge theories can be expressed in terms of single and double traces [14]

\[
\begin{align*}
T_1 &= \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \text{Tr}(T^{a_1} T^{a_4} T^{a_3} T^{a_2}), \\
T_2 &= \text{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) + \text{Tr}(T^{a_1} T^{a_3} T^{a_4} T^{a_2}), \\
T_3 &= \text{Tr}(T^{a_1} T^{a_4} T^{a_2} T^{a_3}) + \text{Tr}(T^{a_1} T^{a_3} T^{a_2} T^{a_4}), \\
T_4 &= \text{Tr}(T^{a_1} T^{a_3}) \text{Tr}(T^{a_2} T^{a_4}), \\
T_5 &= \text{Tr}(T^{a_1} T^{a_4}) \text{Tr}(T^{a_2} T^{a_3}), \\
T_6 &= \text{Tr}(T^{a_1} T^{a_2}) \text{Tr}(T^{a_3} T^{a_4}).
\end{align*}
\]

(1.1)

All other possible trace terms vanish in SU($N$) since Tr($T^a$) = 0. The color-ordered amplitudes can be further decomposed [19] in powers of $N$ as

\[
A^{(L)} = \sum_{\lambda=1}^{3} \left( \sum_{k=0}^{\left\lfloor L/2 \right\rfloor} N^{L-2k} A^{(L,2k)}_{\lambda} \right) T_{\lambda} + \sum_{\lambda=4}^{6} \left( \sum_{k=0}^{\left\lfloor (L-1)/2 \right\rfloor} N^{L-2k-1} A^{(L,2k+1)}_{\lambda} \right) T_{\lambda}
\]

(1.2)

where $A^{(L,0)}_{\lambda}$ are leading-order-in-$N$ (planar) amplitudes, and $A^{(L,k)}_{\lambda}, k = 1, \ldots, L$, are subleading-order, yielding $(3L + 3)$ color-ordered amplitudes at $L$ loops.

Alternatively, amplitudes may be decomposed into a basis of color factors [17, 18]. It is in such a basis that color-kinematic duality is manifest [1, 2]. The number of linearly-independent $L$-loop color factors, however, is less than the number of elements of the $L$-loop trace basis, implying the
existence of constraints among $A_{\lambda}^{(L,k)}$. In this note we show that, for even $L$, the color-ordered amplitudes must satisfy

$$6 \sum_{\lambda=1}^{3} A_{\lambda}^{(L,L-2)} - \sum_{\lambda=4}^{6} A_{\lambda}^{(L,L-1)} = 0,$$

(1.3)

$$A_{\lambda+3}^{(L,L-1)} + A_{\lambda}^{(L,L)} = \text{independent of } \lambda,$$

(1.4)

$$\sum_{\lambda=1}^{3} A_{\lambda}^{(L,L)} = 0,$$

(1.5)

while for odd $L$, the relations are

$$6 \sum_{\lambda=1}^{3} A_{\lambda}^{(L,L-3)} - \sum_{\lambda=4}^{6} A_{\lambda}^{(L,L-2)} + 2 \sum_{\lambda=1}^{3} A_{\lambda}^{(L,L-1)} = 0,$$

(1.6)

$$6 \sum_{\lambda=1}^{3} A_{\lambda}^{(L,L-1)} - \sum_{\lambda=4}^{6} A_{\lambda}^{(L,L)} = 0,$$

(1.7)

$$A_{\lambda}^{(L,L)} = \text{independent of } \lambda.$$  

(1.8)

These constraints generalize known group theory relations at tree-level [11,12], one loop [14], and two loops [16] to all loop orders. In particular, we note that eqs. (1.3), (1.5), (1.7), and (1.8) can alternatively be derived by expanding the amplitude in a $U(N)$ trace basis and requiring that any amplitude containing one or more gauge bosons in the $U(1)$ subgroup vanish. Such $U(1)$ decoupling arguments, however, cannot be used to obtain eqs. (1.4) and (1.6).

Since the space of $L$-loop color factors is by construction at least $(3L - 1)$-dimensional (for $L \geq 2$), eqs. (1.3)-(1.8) are the maximal set of constraints on color-ordered amplitudes that follow from SU($N$) group theory alone.

It is interesting that these constraints only involve the three or four most-subleading-in-$1/N$ color-ordered amplitudes at a given loop order; other amplitudes are not constrained at all by group theory. Of course, color-kinematic duality implies further relations among the amplitudes [1,2]. Other recent work on constraints among loop-level amplitudes includes refs. [20–22].

In sec. 2 we describe the relation between color and trace bases, and how to use this to derive constraints among color-ordered amplitudes. In sec. 3 we apply this to four-point amplitudes through two loops, and then develop and solve all-loop-order recursion relations yielding constraints for four-point color-ordered amplitudes. In the appendix, we provide details about the three- and four-loop cases.

## 2 Color and trace bases

In this section, we schematically outline the approach we use to obtain constraints among color-ordered amplitudes. This approach was used in ref. [23] for tree-level and one-loop five-point...
amplitudes.

The \( n \)-point amplitude in a gauge theory containing only fields in the adjoint representation of \( SU(N) \) (such as pure Yang-Mills or supersymmetric Yang-Mills theory) can be written in a loop expansion, with the \( L \)-loop contribution given by a sum of \( L \)-loop Feynman diagrams. Suppressing \( n \) and \( L \), as well as all momentum and polarization dependence, we can express the \( L \)-loop amplitude in the “parent-graph” decomposition \[24\]

\[ A = \sum_i a_i c_i \] (2.1)

where \( \{ c_i \} \) represents a complete set of color factors of \( L \)-loop \( n \)-point diagrams built from cubic vertices with a factor of the \( SU(N) \) structure constants \( \tilde{f}_{abc} \) at each vertex. Contributions from Feynman diagrams containing quartic vertices with factors of \( \tilde{f}_{ab} \tilde{f}_{cd}, \tilde{f}_{ace} \tilde{f}_{bde}, \) and \( \tilde{f}_{ade} \tilde{f}_{bce} \) can be parceled out among other diagrams containing only cubic vertices. The set of color factors may be overcomplete, in which case they satisfy relations of the form

\[ \sum_i \ell_i c_i = 0. \] (2.2)

In fact, it is often necessary to use an overcomplete basis to make color-kinematic duality manifest \[1,8\]. Although the amplitude (2.1) is gauge invariant, the individual terms in the sum may not be. Any gauge-dependent pieces of the form \( a_i = \ell_i f \) (where \( f \) is independent of \( i \)) will cancel out due to eq. (2.2).

The \( L \)-loop amplitude may alternatively be expressed in terms of a trace basis \( \{ t_\lambda \} \) as

\[ A = \sum_\lambda A_\lambda t_\lambda \] (2.3)

where \( A_\lambda \) are gauge-invariant color-ordered amplitudes. One can convert the amplitude (2.1) into the trace basis by writing

\[ \tilde{f}_{abc} = i\sqrt{2} f_{abc} = \text{Tr}([T^a,T^b]T^c) \] (2.4)

and using the \( SU(N) \) identities

\[ \begin{align*}
\text{Tr}(PT^a)\text{Tr}(QT^a) &= \text{Tr}(PQ) - \frac{1}{N} \text{Tr}(P) \text{Tr}(Q) \\
\text{Tr}(PT^aQT^a) &= \text{Tr}(P) \text{Tr}(Q) - \frac{1}{N} \text{Tr}(PQ)
\end{align*} \] (2.5)

to express the color factor \( c_i \) as a linear combination of traces

\[ c_i = \sum_\lambda M_{i\lambda} t_\lambda. \] (2.6)

The color-ordered amplitudes are then given by

\[ A_\lambda = \sum_i a_i M_{i\lambda}. \] (2.7)
Any constraints \((2.2)\) among the color factors correspond to left null eigenvectors of the transformation matrix
\[
\sum_{i} \ell_i M_{\lambda} = 0. \tag{2.8}
\]
The transformation matrix will also have a set of right null eigenvectors
\[
\sum_{\lambda} M_{\lambda r_{\lambda}} = 0. \tag{2.9}
\]
Each right null eigenvector implies a constraint
\[
\sum_{\lambda} A_{\lambda r_{\lambda}} = 0 \tag{2.10}
\]
on the color-ordered amplitudes.

3 Constraints on color-ordered four-point amplitudes

In eq. \((1.2)\), we decomposed the \(L\)-loop four-point amplitude in terms of the six-dimensional trace basis \(\{T_\lambda\}\) defined in eq. \((1.1)\). The \(1/N\) expansion suggests enlarging the trace basis to the \((3L + 3)\)-dimensional basis \(\{t^{(L)}_\lambda\}\):
\[
\begin{align*}
t^{(L)}_{1+6k} &= N^{L-2k} T_1, \\
t^{(L)}_{2+6k} &= N^{L-2k} T_2, \\
t^{(L)}_{3+6k} &= N^{L-2k} T_3, \\
t^{(L)}_{4+6k} &= N^{L-2k-1} T_4, \\
t^{(L)}_{5+6k} &= N^{L-2k-1} T_5, \\
t^{(L)}_{6+6k} &= N^{L-2k-1} T_6, 
\end{align*}
\tag{3.1}
\]
in terms of which eq. \((1.2)\) becomes
\[
A^{(L)} = \sum_{\lambda=1}^{3L+3} A^{(L)}_\lambda t^{(L)}_\lambda, \quad \text{where} \quad A^{(L)}_\lambda = \begin{cases} A^{(L,2k)}_\lambda, & \lambda = 1, 2, 3, \\
A^{(L,2k+1)}_\lambda, & \lambda = 4, 5, 6. \end{cases} \tag{3.2}
\]
The decomposition \((2.6)\) of color factors \(c_\lambda\) into the trace basis \(\{t^{(L)}_\lambda\}\) shows that the number of independent \(L\)-loop color factors cannot exceed \(3L + 3\). The dimension of the space of color factors is actually less than this, being 2-dimensional at tree level, 3-dimensional at one loop, and \((3L - 1)\)-dimensional for \(L \geq 2\) (only proven for \(L \leq 4\)). As we will illustrate below, this implies the existence of right null eigenvectors \((2.9)\) of the transformation matrix \(M_{\lambda r_{\lambda}}^{(L)}\) and corresponding constraints \((2.10)\) among the color-ordered amplitudes \(A^{(L)}_\lambda\).

At tree level, the space of color factors is spanned by the \(t\)-channel exchange diagram
\[
C_{st}^{(0)} = \tilde{f}_{a_1a_2b} f_{a_3a_2b} = t^{(0)}_1 - t^{(0)}_3 \tag{3.3}
\]
and the corresponding \(s\)-channel exchange diagram
\[
C_{ts}^{(0)} = \tilde{f}_{a_1a_2b} f_{a_3a_4b} = t^{(0)}_1 - t^{(0)}_2. \tag{3.4}
\]
The $u$-channel diagram is related to these by the Jacobi identity. With $\{c_1,c_2\} = \{C_{st}^{(0)}, C_{ts}^{(0)}\}$, the transformation matrix (2.6) and its right null eigenvector (2.9) are

$$M^{(0)}_{i\lambda} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad r^{(0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$ (3.5)

which implies the U(1) decoupling identity among color-ordered tree amplitudes [11, 12]

$$A_1^{(0)} + A_2^{(0)} + A_3^{(0)} = 0.$$ (3.6)

This is eq. (1.5) for $L = 0$.

The color factor of the one-loop box diagram

$$C_{st}^{(1)} = C_{ts}^{(1)} = \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{ca_3d} \tilde{f}^{da_4e} = t_1^{(1)} + 2(t_4^{(1)} + t_5^{(1)} + t_6^{(1)})$$ (3.7)

and its independent permutations $C_{us}^{(1)}$ and $C_{tu}^{(1)}$ span the space of one-loop color factors, giving

$$M^{(1)}_{i\lambda} = \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}.$$ (3.8)

Alternatively, we can choose\(^3\) for our basis $NC_{st}^{(0)}$ and $NC_{ts}^{(0)}$, together with $C_{st}^{(1)}$, to give

$$M^{(1)}_{i\lambda} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}.$$ (3.9)

In either case, the transformation matrix has three independent right null eigenvectors

$$r^{(1)} = \begin{pmatrix} 6u \\ -u \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad \text{where} \quad u \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad y \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ (3.10)

implying three relations among the one-loop color-ordered amplitudes [14]

$$A_4^{(1)} = A_5^{(1)} = A_6^{(1)} = 2(A_1^{(1)} + A_2^{(1)} + A_3^{(1)}).$$ (3.11)

These are eqs. (1.7) and (1.8) for $L = 1$.

At two loops, the ladder and non-planar diagrams\(^4\) yield the color factors

$$C_{st}^{(2L)} = \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{cgd} \tilde{f}^{def} \tilde{f}^{ga_3h} \tilde{f}^{habf} = t_1^{(2)} + 6t_6^{(2)} + 2t_7^{(2)} + 2t_8^{(2)} - 4t_9^{(2)},$$ (3.12)

$$C_{st}^{(2NP)} = \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{cgd} \tilde{f}^{hfe} \tilde{f}^{gash} \tilde{f}^{da_4f} = -2t_4^{(2)} - 2t_5^{(2)} + 4t_6^{(2)} + 2t_7^{(2)} + 2t_8^{(2)} - 4t_9^{(2)}.$$ (3.13)

\(^3\)This makes sense since we can use the Jacobi identity to replace the one-loop box diagram with another box diagram with permutated legs plus a tree diagram with one of the vertices replaced by a triangle diagram. The latter is proportional to a tree diagram since $\tilde{f}^{da_1b} \tilde{f}^{ba_2c} \tilde{f}^{ca_3d} = N \tilde{f}_{a_1a_2a_3}$.

\(^4\)It can be easily shown that any other two-loop diagram is related to these ones by Jacobi relations.
The non-planar color factors can be expressed in terms of the planar ones,

$$3C^{(2NP)}_{st} = C^{(2L)}_{st} - C^{(2L)}_{ts} - C^{(2L)}_{us} + C^{(2L)}_{su},$$  \hfill (3.14)

and a linear relation exists among the planar color factor and its permutations,

$$0 = C^{(2L)}_{st} - C^{(2L)}_{ts} + C^{(2L)}_{us} - C^{(2L)}_{su} + C^{(2L)}_{tu} - C^{(2L)}_{ut}. \hfill (3.15)$$

We could therefore choose five of the six permutations of the ladder diagram to span the space of two-loop color factors; alternatively, we can use $N^2C^{(0)}_{st}$, $N^2C^{(0)}_{ts}$, and $NC^{(1)}_{st}$, together with $C^{(2L)}_{st}$ and $C^{(2L)}_{ts}$, to obtain

$$M^{(2)}_{i\lambda} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 \\ 1 & 0 & 0 & 0 & 6 & 2 \\ 1 & 0 & 0 & 0 & 6 & 2 -4 \end{pmatrix}.$$  \hfill (3.16)

The two-loop transformation matrix has four independent right null eigenvectors

$$r^{(2)} = \begin{pmatrix} 6u \\ -u \\ 0 \\ x \\ y \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x \\ y \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  \hfill (3.17)

implying four two-loop group-theory relations \[16\]

$$0 = A_4^{(2)} + A_7^{(2)} - 2(A_1^{(2)} + A_2^{(2)} + A_3^{(2)})$$
$$0 = A_5^{(2)} + A_8^{(2)} - 2(A_1^{(2)} + A_2^{(2)} + A_3^{(2)})$$
$$0 = A_6^{(2)} + A_9^{(2)} - 2(A_1^{(2)} + A_2^{(2)} + A_3^{(2)})$$
$$0 = A_7^{(2)} + A_8^{(2)} + A_9^{(2)}.$$  \hfill (3.18)

equivalent to eqs. \[13\]-\[15\] for $L = 2$.

We now employ a recursive procedure to obtain null eigenvectors for higher-loop color factors. An $(L + 1)$-loop diagram may be obtained from an $L$-loop diagram by attaching a rung between two of its external legs, $i$ and $j$. This corresponds to contracting its color factor with $f^a_{\alpha a'} b f^{b a'}_{a j}$.

Note that if $i$ and $j$ are not adjacent, this will convert a planar diagram into a nonplanar diagram. First consider the effect of this procedure \[25\] on the trace basis \[11\]

$$T_{\lambda} \rightarrow \sum_{\kappa=1}^{6} G_{\lambda\kappa} T_{\kappa}, \quad \text{where} \quad G = \begin{pmatrix} NA & B \\ C & ND \end{pmatrix}. \hfill (3.19)$$

with

$$A = \begin{pmatrix} e_{12} + e_{14} & 0 & 0 \\ 0 & e_{12} + e_{13} & 0 \\ 0 & 0 & e_{13} + e_{14} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2e_{14} - 2e_{13} & 2e_{12} - 2e_{13} \\ 2e_{13} - 2e_{14} & 0 & 2e_{12} - 2e_{14} \\ 2e_{12} - 2e_{13} & 2e_{14} - 2e_{12} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} e_{12} - e_{14} & e_{14} - e_{12} & 0 \\ e_{12} - e_{13} & 0 & e_{13} - e_{12} \\ e_{14} - e_{13} & e_{13} - e_{14} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2e_{13} & 0 & 0 \\ 0 & 2e_{14} & 0 \\ 0 & 0 & 2e_{12} \end{pmatrix}. \hfill (3.20)$$
where the coefficient of $e_{1j}$ corresponds to connecting legs $1$ and $j$. On the expanded basis (3.1), the same procedure yields yields

$$t^{(L)}_{\lambda} \rightarrow \sum_{\kappa=1}^{3L+6} g_{\lambda\kappa} t^{(L+1)}_{\kappa}$$

(3.21)

where $g$ is the $(3L + 3) \times (3L + 6)$ matrix

$$g = \begin{pmatrix}
A & B & 0 & 0 & 0 & \ldots \\
0 & D & C & 0 & 0 & \ldots \\
0 & 0 & A & B & 0 & \ldots \\
0 & 0 & 0 & D & C & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$  

(3.22)

Next, given some $L$-loop diagram with color factor $c^{(L)}_i$, we can connect two of its external legs with a rung to obtain an $(L + 1)$-loop diagram with color factor

$$c^{(L+1)}_i = \sum_{\kappa=1}^{3L+6} M^{(L+1)}_{i\kappa} t^{(L)}_{\kappa}$$

(3.23)

where

$$M^{(L+1)}_{i\kappa} = \sum_{\lambda=1}^{3L+3} M^{(L)}_{i\lambda} g_{\lambda\kappa}, \quad \text{with} \quad c^{(L)}_i = \sum_{\lambda=1}^{3L+3} M^{(L)}_{i\lambda} t^{(L)}_{\lambda}.$$  

(3.24)

Now, suppose we possess a complete set of $L$-loop color factors $\{c^{(L)}_i\}$ and a maximal set of right null eigenvectors $\{r^{(L)}_\lambda\}$:

$$\sum_{\lambda=1}^{3L+3} M^{(L)}_{i\lambda} r^{(L)}_{\lambda} = 0.$$  

(3.25)

Then the color factors of all $(L + 1)$-loop diagrams obtained by connecting two external legs of any $L$-loop diagram will have a right null eigenvector

$$\sum_{\kappa=1}^{3L+6} M^{(L+1)}_{i\kappa} r^{(L+1)}_{\kappa} = 0$$

(3.26)

provided that $r^{(L+1)}_{\kappa}$ satisfies

$$\sum_{\kappa=1}^{3L+6} g_{\lambda\kappa} r^{(L+1)}_{\kappa} = \text{linear combination of} \ \{r^{(L)}_{\lambda}\}.$$  

(3.27)

We can now solve eq. (3.27) recursively, beginning with the set of $L = 2$ right null eigenvectors (3.17), the first case with four independent eigenvectors. The maximal set of right null
eigenvectors satisfying eq. (3.27) is

\[
\{r^{(2\ell+1)}\} = \begin{pmatrix}
\vdots \\
0 \\
6u \\
-u \\
2u \\
0 \\
\end{pmatrix}, \quad \begin{pmatrix}
\vdots \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}, \quad \begin{pmatrix}
\vdots \\
0 \\
x \\
y \\
0 \\
\end{pmatrix}, \quad \{r^{(2\ell)}\} = \begin{pmatrix}
\vdots \\
0 \\
6u \\
-u \\
x \\
y \\
\end{pmatrix}.
\]

The constraints on color-ordered amplitudes

\[
\sum_\lambda A^{(L)}_\lambda r^{(L)}_\lambda = 0
\]

that follow from the set of right null eigenvectors (3.28) can be written in terms of eq. (3.27) to yield the constraints (1.3)-(1.8) given in the introduction.

Since there are generally four linearly-independent null eigenvectors in a \((3L+3)\)-dimensional trace space, the space of \(L\)-loop color factors satisfying eq. (3.27) is generally \((3L-1)\)-dimensional.\(^5\)

Since there are no further independent solutions of eq. (3.27), we have shown that the full space of \(L\)-loop color factors is at least \((3L-1)\)-dimensional.

We have not strictly shown that eq. (3.28) are null eigenvectors for any possible color factor associated with an \(L\)-loop diagram, but rather only for those that can be obtained from an \((L-1)\)-loop diagram by attaching a rung between two external legs. It is therefore conceivable (but we think unlikely) that the space of all \(L\)-loop color factors could be greater than \((3L-1)\)-dimensional. However, for \(L = 3\) and \(L = 4\), it has been shown \(^{24}\) that, despite the fact that many diagrams cannot be obtained by attaching a rung to the external legs of lower-loop diagrams, all color factors can be related to these using Jacobi relations (see the appendix for further discussion of \(L = 3\) and \(L = 4\)). It would be nice to have a proof of this for all \(L\), however.

4 Conclusions

In this note, we have extended known group theory identities for four-point color-ordered amplitudes in \(SU(N)\) gauge theories to all loop orders. We have shown that color-ordered amplitude generally must satisfy four independent relations at each loop order (except for \(L = 0\) and \(L = 1\), where there are one and three constraints respectively). This was achieved via a recursive procedure that derives the constraints on \(L\)-loop color factors generated by attaching a rung between two external legs of an \((L-1)\)-loop color factor. Assuming that all \(L\)-loop color factors are linear combinations of those just described (i.e., via Jacobi relations), then the constraints derived apply to all \(L\)-loop color-ordered amplitudes. Although this has been established through four loops, it would clearly be desirable to have an all-orders proof of this assumption.

\(^5\)One for \(L = 0\) and three for \(L = 1\).

\(^6\)Two-dimensional for \(L = 0\) and three-dimensional for \(L = 1\).
The recursive method employed in this note can also be extended to \( n \)-point functions with \( n > 4 \) to yield constraints on the color-ordered amplitudes beyond those already known at tree-\cite{13} and one-loop \cite{14,15} level, although the size of the color basis grows quickly with \( n \).

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**A Appendix**

In appendix B of ref. \cite{24}, bases for the space of all three- and four-loop color factors were identified. In this appendix, we explicitly check that the right null eigenvectors of these spaces coincide with our recursive solution \((3.28)\), and therefore that all three- and four-loop color-ordered amplitudes indeed satisfy the group theory constraints eqs. \((1.3)-(1.8)\).

The basis for three-loop color factors can be chosen as \( N^3C_{st}^{(0)} \), \( N^3C_{ts}^{(0)} \), \( N^2C_{st}^{(1)} \), \( NC_{st}^{(2)} \), and \( NC_{ts}^{(2)} \), plus the color factor for the three-loop ladder diagram

\[
C_{st}^{(3L)} = i_1^{(3)} + 14t_6^{(3)} + 2t_7^{(3)} + 2t_8^{(3)} + 8t_{10}^{(3)} + 8t_{11}^{(3)} + 8t_{12}^{(3)}
\]  
(A.1)

and two of its permutations \( C_{ts}^{(3L)} \) and \( C_{us}^{(3L)} \), yielding the transformation matrix

\[
M_{i\lambda}^{(3)} = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 6 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 14 & 2 & 2 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 14 & 0 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0
\end{pmatrix} . \tag{A.2}
\]

The four independent right null eigenvectors of this matrix

\[
r^{(3)} = \begin{pmatrix}
6u \\
-u \\
2u \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
6u \\
-2u
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
x
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
y
\end{pmatrix} \tag{A.3}
\]

agree with those in eq. \((3.28)\), and imply the four constraints among the color-ordered amplitudes given by eqs. \((1.6)-(1.8)\) with \( L = 3 \).

\footnote{7 Only \( C_{st}^{(3L)} \) is used in ref. \cite{23}, but the authors also include \( NC_{st}^{(0)} \) and \( NC_{ts}^{(0)} \) in their basis, which in our approach are independent of \( N^3C_{st}^{(0)} \) and \( N^3C_{ts}^{(0)} \).}
The four-loop color basis can be chosen as $(N \times)$ the three-loop basis plus three color factors from the four-loop ladder diagram and two\footnote{Only $C_{st}^{(4L)}$ and $C_{ts}^{(4L)}$ are used in ref. \cite{24}, but the authors also include $NC_{st}^{(1)}$, which in our approach counts as independent from $N^3C_{st}^{(1)}$.} permutations, $C_{st}^{(4L)}$, $C_{ts}^{(4L)}$, and $C_{us}^{(4L)}$, yielding

$$M_{i\lambda}^{(4)} = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 6 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 14 & 2 & 2 & 0 & 8 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 14 & 0 & 2 & 0 & 2 & 8 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 8 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 30 & 2 & 2 & 0 & 24 & 0 & 8 & 16 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 30 & 0 & 0 & 0 & 2 & 2 & 24 & 0 & 0 & 16 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix}. \quad (A.4)$$

The four independent right null eigenvectors of this matrix

$$r^{(4)} = \begin{pmatrix}
0 \\
0 \\
6u \\
-u \\
0 \\
ym \\
x \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
x \\
y \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}, \quad (A.5)$$

agree with those in eq. (3.28). The right null eigenvalues imply the four relations among color-ordered amplitudes given by eqs. (1.3)-(1.5) for $L = 4$.

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