The rainbow vertex connection number of edge corona product graphs

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Abstract. Let $G_1, G_2$ be a special graphs with vertices of $G_1$ 1, 2, ..., $n$ and edges of $G_1$ 1, 2, ..., $m$. The generalized edge corona product of graphs $G_1$ and $G_2$, denoted by $G_1 \circ G_2$ is obtained by taking one copy of graph $G_1$ and $m$ copy of $G_2$, thus for each edge $e_k = ij$ of $G$, joining edge between the two end-vertices $i$, $j$ of $e_k$ and each vertex of the $k$-copy of $G_2$. A rainbow vertex-coloring graph $G$ where adjacent vertices $u-v$ and its internal vertices have distinct colors. A path is called a rainbow path if no two verticess of the path have the same color. A rainbow vertex-connection number of graph $G$ is minimum number of colors in graph $G$ to connected every two distinct internal vertices $u$ and $v$ such that a graph $G$ naturally rainbow vertex-connected, denoted by $rvc (G)$. In this paper, we determine minimum integer for rainbow vertex coloring of edge corona product on cycle and path such as $P_n \circ P_m, P_n \circ C_m, C_n \circ P_m, and C_n \circ C_m$.

1. Introduction

The concept of rainbow connection in graphs as follows at the first time was introduced in 2008 by Chartrand et al [2]. Let $G = (V, E)$ be a non trivial and connected, the rainbow connection number of $G$ is the minimum number of colors in a rainbow connected edge-coloring of $G$, denoted by $rc(G)$. The graph $G$ is rainbow-connected if $G$ has a rainbow $u-v$ path for every two vertices $u$ and $v$ of $G$. The graph $G$ with size $m$ and diameter $diam(G)$ be a connected graph, then

$$m \geq rc(G) \geq diam(G). \quad (1)$$

Krivelevich and Yuster [5] was proposes a new concept of a rainbow connection and called by rainbow vertex-connected. A rainbow vertex-coloring graph $G$ where adjacent vertices $u-v$ and its internal vertices have distinct colors. A path is called a rainbow path if no two verticess of the path have the same color. A path is called a rainbow path if no two verticess of the path have the same color [8]. A vertex-colored graph $G$ is rainbow connected if any two vertices are connected by a rainbow path. A rainbow vertex-connection number of graph $G$ is minimum number of colors in graph $G$ to connected every two distinct internal vertices $u$ and $v$ such that a graph $G$ naturally rainbow vertex-connected, denoted by $rvc (G)$ [1].Several results on rainbow vertex-coloring of some families of graphs such as:
• Cycle $C_n$ of order $n \geq 3$:

$$rvc(C_n) = \begin{cases} 
0, & n = 3 \\
1, & n = 4, 5 \\
3, & n = 9 \\
\left\lceil \frac{n}{2} \right\rceil - 1, & n = 6, 7, 8, 10, 11, 12, 13 \text{ or } 15 \\
\left\lceil \frac{n}{2} \right\rceil, & n \geq 16 \text{ or } 14 [3]
\end{cases}$$

• Pencil graph $P_n$ for $n \geq 2$, $rvc(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$ if $n \leq 7$ and $rvc(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1$ otherwise [9]

• Path $P_n$ of order $n \geq 3$, $rvc(P_n) = n - 2$ [10]

Furthermore, X. Li and Y. Shi [7] studied the following theorem and gave the lower bound for $rvc(G)$.

**Theorem A.** Let $G$ be a nontrivial connected graph of order $n$, then

$$rvc(G) \geq diam(G) - 1.$$  \hfill (2)

Hou Yaoping and Shiu Wai-Chee [4] observed edge corona product on simple graph. Let graph $G_1$ is a special graph with vertices $1, 2, ..., n$ and edges $e_1, e_2, ..., e_m$ and $G_2$ is special graphs too. The generalized edge corona product of graphs $G_1$ and $G_2$ denoted by $G_1 \circ G_2$, is obtained by taking one copy of graphs $G_1$ and $m$ copy of $G_2$, thus for each edge $e_k = ij$ of $G$, joining edges between the two end-vertices $i, j$ of $e_k$ and each vertex of the $k$-copy of $G_2$[11].

2. Result

In this section, we determined our result of rainbow vertex coloring of edge corona product on cycle and path as follows. For every $n$ and $m$ are elements of natural number with $n \geq 2$ and $m \geq 3$.

2.1. The rainbow vertex-coloring of edge corona product on $P_n \circ P_m$

We start with a rainbow vertex-connection number of edge corona product of a two path on $n$ vertices $P_n$ and $m$ vertices $P_m$. Edge corona product between two of path graph denoted by $P_n \circ P_m$ with $n \geq 2$ and $m \geq 2$. We present the rainbow vertex-coloring of edge corona product on $P_n \circ P_m$ graph as follows.

**Theorem 2.1.** Let $G$ be a edge corona product of path graph $P_n$ and $P_m$, the rainbow vertex-connection number of $P_n \circ P_m$ is $rvc(P_n \circ P_m)= n - 2$.

**Proof.** Path graph $P_n$ with $n \geq 2$ has a vertex set $V(P_n) = \{u_i; 1 \leq i \leq n\}$ and an edge set $E(P_n) = \{u_iu_{i+1}; 1 \leq i \leq n - 1\}$. Edge corona product between two of path graph is denoted by $P_n \circ P_m$ with $n \geq 2$ and $m \geq 2$ has a vertex set $V(P_n \circ P_m) = \{u_i; 1 \leq i \leq n\} \cup \{v^k_j; 1 \leq j \leq m; 1 \leq k \leq n - 1\}$ and an edge set $E(P_n \circ P_m) = \{u_iu_{i+1}; 1 \leq i \leq n - 1\} \cup \{v^k_jv^k_{j+1}; 1 \leq j \leq m - 1; 1 \leq k \leq n - 1\}$.

Begin with $rvc(P_n \circ P_m) = n - 2$. Since $diam(P_n \circ P_m) = n - 1$, by using Theorem A, we have $rvc(P_n \circ P_m) \geq n - 2$. Furthermore to prove that $(P_n \circ P_m) \leq n - 2$, by vertex coloring $v$ according to the following formula as follow:

$$f(v) = \begin{cases} 
1, & \text{for } v = u_i; i = 1, n \\
i - 1, & \text{for } v = u_i; 2 \leq i \leq n - 1 \\
1, & \text{for } v = v^k_j; 1 \leq j \leq m; 1 \leq k \leq n - 1
\end{cases}$$
It easily to see that the colors of vertices are $n-1$, that is, $c: V(P_n \ast P_m) \rightarrow \{1,2,3,...,n-2\}$. Then $rvc(P_n \ast P_m) \leq n-2$. So, if we combining both of them, we have the vertices minimum colors is $rvc(P_n \ast P_m) = n-2$.

The following theorem we determine the rainbow vertex connection number of edge corona of a path $P_n$ and a cycle graph $C_m$. Edge corona product of path and cycle graph denoted by $P_n \ast C_m$ such that a rainbow connection-number as follows

2.2. The rainbow vertex-coloring of edge corona product on $P_n \ast C_m$

**Theorem 2.2.** Let $G$ be edge corona product of path $P_n$ and cycle $C_m$ graph, the rainbow vertex connection number of $P_n \ast C_m$ is $rvc(P_n \ast C_m) = n-2$.

**Proof.** Path graph $P_n$ with $n \geq 2$ has a vertex set $V(P_n) = \{u_i ; 1 \leq i \leq n\}$ and an edge set $E(P_n) = \{u_i u_{i+1} ; 1 \leq i \leq n-1\}$. A cycle graph $C_m$ with $m \geq 3$ has a vertices set $V(C_m) = \{u_i ; 1 \leq i \leq m\}$ and an edges set $E(C_m) = \{u_i u_{i+1} ; 1 \leq i \leq m-1\} \cup \{u_1 u_m\}$. Then $P_n \ast C_m$ has a vertex set $V(P_n \ast C_m) = \{u_i u_{i+1} ; 1 \leq i \leq n-1\} \cup \{v^k_j ; 1 \leq j \leq m; 1 \leq k \leq n-1\}$ and an edge set $E(P_n \ast C_m) = \{u_i u_{i+1} ; 1 \leq i \leq n-1\} \cup \{v^k_j u^k_{j+1} ; 1 \leq j \leq m-1; 1 \leq k \leq n-1\} \cup \{v^k_j v^k_{j+1}\}$

Begin with $rvc(P_n \ast C_m) = n-2$. Since $diam(P_n \ast C_m) = n-1$, by Theorem A, we have $rvc(P_n \ast C_m) \geq n-2$. Furthermore to prove that $(P_n \ast C_m) \leq n-2$, by vertex coloring $v$ according to the following formula as follow:

$$f(v) = \begin{cases} 
1, & \text{for } v = u_i; \ i = 1, n \\
 i-1, & \text{for } v = u_i; \ 2 \leq i \leq n-1 \\
 1, & \text{for } v = v^k_j; \ 1 \leq j \leq m; \ 1 \leq k \leq n-1 
\end{cases}$$

It easily to see that the color of vertices are $n-1$, that is, $c: V(P_n \ast C_m) \rightarrow \{1,2,3,...,n-2\}$. Thus $rvc(P_n \ast C_m) \leq n-2$. So, if we combining both of them, we have the vertices minimum colors is $rvc(P_n \ast P_m) = n-2$.

Now we present the rainbow vertex connection number of edge corona product of a cycle graph on $n$ vertices $C_n$ and a path on $m$ vertices $P_m$. Edge corona product of cycle graph and path is denoted by $C_n \ast P_m$ with $n \geq 3$ and $m \geq 2$, as follows

2.3. The rainbow vertex-coloring of edge corona product on $C_n \ast P_m$

**Theorem 2.3.** Let $G$ be edge corona product of cycle $C_n$ and path $P_m$, the rainbow vertex connection number of $C_n \ast P_m$, is

$$rvc(C_n \ast P_m) = \begin{cases} 
\left\lceil \frac{n}{2} \right\rceil, & n \leq 4 \\
\left\lfloor \frac{n}{2} \right\rfloor, & n \geq 5 
\end{cases}$$

**Proof.** Cycle graph $C_m$ with $m \geq 3$ has a vertex set $V(C_m) = \{x_i ; 1 \leq i \leq m\}$ and an edge set $E(C_m) = \{u_i u_{i+1} ; 1 \leq i \leq m\} \cup \{u_1 u_m\}$. A path graph $P_n$ with $n \geq 2$ has a vertex set $V(P_n) = \{u_i ; 1 \leq i \leq n\}$ and an edge set $E(P_n) = \{u_i u_{i+1} ; 1 \leq i \leq n-1\}$. Edge corona product of cycle graph and path is denoted by $C_n \ast P_m$ with $n \geq 3$ and $m \geq 2$ has a vertices set $V(C_n \ast P_m) = \{u_i ; 1 \leq i \leq n\} \cup \{v^k_j ; 1 \leq j \leq m; 1 \leq k \leq n\}$ and an edges set $E(C_n \ast P_m) = \{u_i u_{i+1} ; 1 \leq i \leq n-1\} \cup \{u_i u_{n} \cup \{u_i v^k_j ; 1 \leq j \leq m; 1 \leq k \leq n\} \cup \{v^k_j v^k_{j+1} ; 1 \leq j \leq m-1; 1 \leq k \leq n-1\} \cup \{v^k_j v^k_{j+1}\}$
Begin with $rvc(C_n \circ P_m) = \left\lfloor \frac{n}{2} \right\rfloor$, $2 \leq n \leq 4$. Since $diam(P_n \circ C_m) = \left\lceil \frac{n}{2} \right\rceil + 1$ by Theorem A, we have $rvc(C_n \circ P_m) \geq \left\lceil \frac{n}{2} \right\rceil$. Furthermore to prove that $rvc(C_n \circ P_m) \geq \left\lfloor \frac{n}{2} \right\rfloor$, by vertex coloring $v$ according to the following formula as follows:

$$f(v) = \begin{cases} 
1, & \text{for } v = u_i; \ i = 1, n \\
1 - i, & \text{for } v = u_i; \ 2 \leq i \leq n - 1 \\
1, & \text{for } v = v_j^k; \ 1 \leq j \leq m; \ 1 \leq k \leq n - 1 
\end{cases}$$

It is easy to see that the color of vertices are $n-1$, that is, $c: V(C_n \circ P_m) \rightarrow \left\{1, 2, 3, ..., \left\lfloor \frac{n}{2} \right\rfloor \right\}$. Thus

$rvc(C_n \circ P_m) \leq \left\lfloor \frac{n}{2} \right\rfloor$. So if we combining both of them, we have the vertices minimum colors is $rvc(C_n \circ P_m) = \left\lfloor \frac{n}{2} \right\rfloor$.

On second step we show that $rvc(C_n \circ P_m) = \left\lceil \frac{n}{2} \right\rceil; n \geq 5$. Since $diam(P_n \circ C_m) = \left\lceil \frac{n}{2} \right\rceil + 1$ by Theorem A, we have $rvc(C_n \circ P_m) \geq \left\lceil \frac{n}{2} \right\rceil$. Furthermore to prove that $(C_n \circ P_m) \leq \left\lfloor \frac{n}{2} \right\rfloor$, by vertex coloring $v$ according to the following formula as follows:

$$f(v) = \begin{cases} 
1, & \text{for } v = u_i; \ i = 1, n \\
1 - i, & \text{for } v = u_i; \ 2 \leq i \leq n - 1 \\
1, & \text{for } v = v_j^k; \ 1 \leq j \leq m; \ 1 \leq k \leq n - 1 
\end{cases}$$

It is easy to see that the color of vertices are $n-1$, that is, $c: V(C_n \circ P_m) \rightarrow \left\{1, 2, 3, ..., \left\lceil \frac{n}{2} \right\rceil \right\}$. Thus

$rvc(C_n \circ P_m) \leq \left\lceil \frac{n}{2} \right\rceil$. So if we combining both of them, we have the vertices minimum colors is $rvc(C_n \circ P_m) = \left\lfloor \frac{n}{2} \right\rfloor$.

The following theorem determine the rainbow vertex connection number of edge corona of a denoted by $C_n \circ C_m$ as follows.

2.4. The rainbow vertex-coloring of edge corona product on $C_n \circ C_m$

Theorem 2.4. Let $G$ be edge corona product of cycle graph where $n$ and $m$ are a natural number with $n \geq 3$ and $m \geq 3$, the rainbow vertex coloring number of $C_n \circ C_m$ is

$rvc(C_n \circ C_m) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor, & n \leq 4 \\
\left\lfloor \frac{n}{2} \right\rceil, & n \geq 5 
\end{cases}$

Proof. Edge corona product between of two cycle graph and path is denoted by $C_n \circ C_m$ with $n \geq 3$ and $m \geq 3$ has a vertex set $V(C_n \circ C_m) = \{u_i; 1 \leq i \leq n\} \cup \{v_j^k; 1 \leq j \leq m; 1 \leq k \leq n\}$ and an edge set $E(C_n \circ C_m) = \{u_i u_{i+1}; 1 \leq i \leq n - 1\} \cup \{u_i v_j^k; 1 \leq i \leq n; 1 \leq j \leq m; 1 \leq k \leq n - 1\} \cup \{v_j^k v_{j+1}^k; 1 \leq j \leq m - 1; 1 \leq k \leq n - 1\}$

Begin with $rvc(C_n \circ P_m) = \left\lfloor \frac{n}{2} \right\rfloor; 2 \leq n \leq 4$. Since $diam(C_n \circ C_m) = \left\lceil \frac{n}{2} \right\rceil + 1$, by Theorem A, we have $rvc(C_n \circ C_m) \geq \left\lceil \frac{n}{2} \right\rceil$. Furthermore to prove that $(C_n \circ C_m) \leq \left\lfloor \frac{n}{2} \right\rceil$, by vertex coloring $v$ according to the following formula as follows:

$$f(v) = \begin{cases} 
1, & \text{for } v = u_i; \ i = 1, n \\
1 - i, & \text{for } v = u_i; \ 2 \leq i \leq n - 1 \\
1, & \text{for } v = v_j^k; \ 1 \leq j \leq m; \ 1 \leq k \leq n - 1 
\end{cases}$$
It is easily to see that the color of vertices are \( \left\lfloor \frac{n}{2} \right\rfloor \) for \( n \leq 4 \), that is, \( c : V(C_n \circ P_m) \rightarrow \{1, 2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \} \). Thus \( rvc(C_n \circ P_m) \leq \left\lfloor \frac{n}{2} \right\rfloor \). So if we combining both of them, we have the vertices minimum colors is \( rvc(C_n \circ P_m) = \left\lfloor \frac{n}{2} \right\rfloor \).

On second step we show that \( rvc(C_n \circ P_m) = \left\lceil \frac{n}{2} \right\rceil; \ n \geq 5 \). Since \( diam(P_n \circ C_m) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) by Theorem A, we have \( rvc(C_n \circ P_m) \geq \left\lfloor \frac{n}{2} \right\rfloor \). Furthermore to prove that \( (C_n \circ P_m) \leq \left\lceil \frac{n}{2} \right\rceil \), by vertex coloring \( v \) according to the following formula as follows:

\[
f(v) = \begin{cases} 
1, & \text{for } v = u_i; \ i = 1, n \\
1, & \text{for } v = v_i; \ 2 \leq i \leq n - 1 \\
i - 1, & \text{for } v = u_i; \ 2 \leq i \leq n - 1 \\
1, & \text{for } v = v_j; \ 1 \leq j \leq m; \ 1 \leq k \leq n - 1
\end{cases}
\]

It is easily to see that the color of vertices are \( \left\lceil \frac{n}{2} \right\rceil \) for \( n \geq 5 \), that is, \( c : V(C_n \circ P_m) \rightarrow \{1, 2, 3, \ldots, \left\lceil \frac{n}{2} \right\rceil \} \). Thus \( rvc(C_n \circ P_m) \leq \left\lceil \frac{n}{2} \right\rceil \). So if we combining both of them, we have the vertices minimum colors is \( rvc(C_n \circ P_m) = \left\lceil \frac{n}{2} \right\rceil \).

3. Conclusion
In this paper, we have determined the exact values of total rainbow connection number of edge corona product of cycle and path on such as \( P_n \circ P_m, P_n \circ C_m, C_n \circ P_m, \) dan \( C_n \circ C_m \) depend on \( rvc(G_1) \) as follows \( rvc(P_n \circ P_m) \) and \( rvc(P_n \circ C_m) \) is \( n - 2 \), \( C_n \circ P_m \) dan \( C_n \circ C_m \) is \( \left\lceil \frac{n}{2} \right\rceil \), \( n \geq 5 \). As we did this proof, it was difficult to get a minimum rainbow vertex connection number. Thus, it still gives the following open problem.

Open Problem 1. Let \( G \) be edge corona product of cycle and any graph, determine sharper lower bound of \( rvc \ (G) \)

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