ON BANACH STRUCTURE OF MULTIVARIATE BV SPACES I

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Abstract. We introduce and study multivariate generalizations of the classical BV spaces of Jordan, F. Riesz and Wiener. The family of the introduced spaces contains or is intimately related to a considerable class of function spaces of modern analysis including BMO, BV, Morrey spaces and those of Sobolev of arbitrary smoothness, Besov and Triebel-Lizorkin spaces. We prove under mild restrictions that the BV spaces of this family are dual and present constructive characterizations of their preduals via atomic decompositions. Moreover, we show that under additional restrictions such a predual space is isometrically isomorphic to the dual space of the separable subspace of the related BV space generated by \( C^\infty \) functions. As a corollary we obtain the “two stars theorem” asserting that the second dual of this separable subspace is isometrically isomorphic to the BV space. An essential role in the proofs play approximation properties of the BV spaces under consideration, in particular, weak* denseness of their subspaces of \( C^\infty \) functions. Our results imply the similar ones (old and new) for the classical function spaces listed above obtained by the unified approach.

1. Introduction

1.1. Important properties of functions of bounded (Jordan) variation and their numerous applications in analysis have been attracting many researchers to define and study their multivariate analogs (Vitaly, Hardy, Lebesgue, Frechet, Tonelli, Kronrod, De Giorgi to name but a few). Each of the proposed definitions was directed to a multivariate generalization of a specific property of univariate BV functions while those introduced in that way possessed (sometimes in disguise) also certain other important properties. For instance, the Hardy variation was introduced initially to generalize the Dirichlet convergence criterion to multivariate Fourier series but later it was discovered that measurable functions of bounded Hardy variation are in addition differentiable almost everywhere.

Another example is the Tonelli variation introduced initially for solving the problem of the characterization of multivariate continuous functions with rectifiable graphs posed by Poincaré. However, at present the modern form of the Tonelli variation given successively by Cesari (1936), Fichera (1954) and De Giorgi (1954) plays an essential role in variational calculus and quasilinear PDEs of the first order, see, e.g., [Gi-84] and [AFP-00] for the results and the corresponding references.

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As in the previous cases, the concept of variation presented below is intimately related to a specific problem of multivariate analysis, the problem of the characterization of images of Sobolev spaces under continuous embeddings in certain spaces of integrable or continuous functions. The connection of the problem with the (Jordan) $BV$ spaces was discovered by Lebesgue [Le-04] and Vitali [Vi-05]. The corresponding result named the fundamental theorem of calculus, see, e.g., [Ru-87, Ch. 7], implies (in fact, is equivalent to) the next assertion.

**Theorem** (Lebesgue). There is a linear isometry of $\dot{W}^1_1(0, 1)$ in $BV[0, 1]$ whose image denoted by $AC[0, 1]$ consists of absolutely continuous functions.

Hereafter $\dot{W}^k_p(\Omega), 1 \leq p \leq \infty, k \in \mathbb{N},$ where $\Omega \subset \mathbb{R}^d$ is a domain, stands for the homogeneous Sobolev space defined by a seminorm given for $f \in L^p(\Omega)$ by

\[
|f|_{W^k_p(\Omega)} := \sum_{|\alpha|=k} \|D^\alpha f\|_{L^p(\Omega)}.
\]

The result was extended to the space $\dot{W}^1_p(0, 1) \hookrightarrow C[0, 1], 1 < p < \infty,$ by F. Riesz [Ri-10]. In this case, the image of the isometry coincides with the space $BV_{\lambda,p}^{1/p'}[0, 1], 1/p + 1/p' = 1,$ of functions of bounded $(\lambda,p)$-variation.

Here $(\lambda,p)$-variation of a function $f \in \ell^\infty[0, 1]$ is given by

\[
\text{var}_{\lambda,p} f := \sup\left\{ \sum_i \left( \frac{|f(x_{i+1}) - f(x_i)|}{|x_{i+1} - x_i|^\lambda} \right)^p \right\}^{1/p},
\]

where $\{x_i\} \subset [0, 1]$ runs over monotone sequences.

Let us note that $BV^0_1$ is Jordan’s space $BV$ and $BV^0_p$ is the Wiener-L. Young space $BV_p$.

The multivariate generalization of the above formulated results requires a new concept of variation that will be presented in the next subsection. The solution of the Sobolev embedding problem (in a sense, sharpening of the Sobolev embedding theorem) was given in [Br-71] and is formulated in Subsection 1.3.4.

The following appropriately reformulated definition (1.2) can be seen as a model case for the presented below concept of variation. Actually, it is readily seen that

\[
\text{var}_{\lambda} f = 2 \sup_{\pi} \left( \sum_{I \in \pi} \left( \frac{E_1(f; I)}{|I|^\lambda} \right)^p \right)^{\frac{1}{p}},
\]

where $\pi$ runs over families of nonoverlapping closed intervals $I \subset [0, 1]$ and

\[
E_1(f; I) := \inf_{c \in \mathbb{R}} \sup_I |f - c| \left( = \frac{1}{2} \text{osc}(f; I) \right).
\]

\footnote{i.e., with pairwise nonintersecting interiors}
Replacing here the underlying space $\ell_\infty[0,1]$ by $L_q([0,1]^d)$, $1 \le q \le \infty$, the families $\pi$ by those of nonoverlapping closed subcubes in $[0,1]^d$ and taking instead of constants polynomials in $x \in \mathbb{R}^d$ of a fixed degree we arrive to the required concept of variation.

The multiparametric family of $BV$ spaces defined by this variation includes or is intimately related to a considerable class of function spaces of modern analysis including $BMO$, $BV$, Morrey spaces and those of Sobolev of arbitrary smoothness, Besov and Triebel-Lizorkin spaces. In turn, the variational representation of the named spaces allows one to study them by a new approach combining tools of geometric analysis and approximation theory. The results obtained in this way for Sobolev type embeddings, pointwise differentiability, Lusin type approximation, the real interpolation and nonlinear $n$-term approximation are presented in the survey [Br-09].

In the present paper, this approach amplified by tools of functional analysis is used to study the Banach structure of the $BV$ spaces introduced (duality, weak $^*$ compactness, two stars theorems etc.). These results imply the similar ones (old and new) for the classical function spaces obtained by the unified approach.

1.2. An important ingredient of the forthcoming definition of the variation is the following notion.

**Definition 1.1.** Local polynomial approximation of a function $f \in L_q^{\text{loc}}(\mathbb{R}^d)$, $1 \le q \le \infty$, is a set function given for a bounded measurable set $S \subset \mathbb{R}^d$ by

$$E_{kq}(f; S) := \inf_{m \in \mathcal{P}_{k-1}^d} \|f - m\|_{L_q(S)},$$

where $\mathcal{P}_k^d$ is the space of polynomials in $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ of degree $k$.

A geometric ingredient of the basic definition is the set of packings in $Q^d$ denoted by $\Pi(Q^d)$. Each packing consists of a finite family of pairwise nonoverlapping subcubes of $Q^d := [0,1]^d$ homothetic to $Q^d$; in what follows, packings are denoted by $\pi, \pi', \pi_i$, etc.

**Definition 1.2.** Let $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $1 \le p, q \le \infty$. A function space $\dot{V}^k_{pq}(Q^d)$ is defined by a seminorm given for $f \in L_q(Q^d)$ by

$$|f|_{\dot{V}^{k\lambda}_{pq}} := \sup_{\pi \in \Pi(Q^d)} \left\{ \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^p \right\}^{\frac{1}{p}};$$

hereafter $|S|$ stands for the $d$-measure of a set $S \subset \mathbb{R}^d$.

It can be easily verified that $\mathcal{P}_{k-1}^d|Q^d$ is the null-space of $\dot{V}^k_{pq}(Q^d)$. Hence, the factor-space

$$V^k_{pq}(Q^d) := \dot{V}^k_{pq}(Q^d)/\mathcal{P}_{k-1}^d|Q^d$$

is normed and (1.6) gives rise to its norm denoted by $\| \cdot \|_{V^k_{pq}}$.

The standard argument proves that $V^k_{pq}(Q^d)$ is a Banach space.
To simplify the notations, we set
\begin{equation}
\kappa := \{k, d, \lambda, p, q\}
\end{equation}
and write
\begin{equation}
\dot{V}_\kappa := \dot{V}^{k\lambda}_{pq}(Q^d), \quad |\cdot|_\kappa := |\cdot|_{V^{k\lambda}_{pq}(Q^d)},
\end{equation}
and similarly write $V_\kappa$ and $\|\cdot\|_\kappa$ for the corresponding factor-space and its norm.

In the sequel, the following separable subspaces of $\dot{V}_\kappa$ and $V_\kappa$ denoted by $\dot{v}_\kappa$ and $v_\kappa$ play an essential role:
\begin{equation}
\dot{v}_\kappa := \text{clos}(C^\infty \cap \dot{V}_\kappa, \dot{V}_\kappa), \quad v_\kappa := \dot{v}_\kappa / P_{d^k-1};
\end{equation}
hereafter $C^\infty$ and $P_{d^k-1}$ denote the following trace-spaces
\begin{equation}
C^\infty := C^\infty(\mathbb{R}^d)|_{Q^d} \quad \text{and} \quad P_{d^k-1} := P_{d^k-1}|_{Q^d}.
\end{equation}
(It will be shown that either $C^\infty \subset \dot{V}_\kappa$ or $\dot{V}_\kappa = P_{d^k-1}$, i.e., $C^\infty \cap \dot{V}_\kappa$ is either $C^\infty$ or $P_{d^k-1}$.)

**Stipulation 1.3.** Throughout the paper we fix the unit cube $Q^d$ and integer $k \geq 1$ removing them from the related symbols. For instance, we write $\kappa := \{\lambda, p, q\}$ instead of that in \[1.8\] and $L_q$ instead of $L_q(Q^d)$. Moreover, we write $\lambda(\kappa), p(\kappa)$, etc. if these belong to $\kappa$. However, these indices will be preserved if they assume other values, for instance, we write $V^{10}_{p\infty}[0, 1]$ instead of $V_\kappa$ with $\kappa = \{1, 1, 0, p, \infty\}$.

Let us note that local approximation $E_{kq}(f; Q)$ is equivalent to the \textit{k-oscillation} of $f$ on $Q$ given by
\begin{equation}
\text{osc}_{kq}(f; Q) := \sup_{h \in \mathbb{R}^d} \|\Delta_k^h f\|_{L_q(Q_{kh})},
\end{equation}
where
\begin{equation}
\Delta_k^h := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \delta_{kh}
\end{equation}
and
\begin{equation}
Q_{kh} := \{x \in Q : x + kh \in Q\}.
\end{equation}
Namely, the next two-sided inequality with the constants of equivalence depending only on $k, d$ is true, see [Br-70],
\begin{equation}
E_{kq}(f; Q) \approx \text{osc}_{kq}(f; Q).
\end{equation}
Hence, as in the one variable theory, $\dot{V}_\kappa$ functions can be equivalently defined by the behaviour of their oscillations.

In the classification of $V_\kappa$ spaces, the following characteristic will be of essence.

**Definition 1.4.** \textit{Smoothness of the space} $V_\kappa$ \textit{denoted by} $s(\kappa)$ \textit{is given by}
\begin{equation}
s(\kappa) := d \left( \lambda + \frac{1}{p} - \frac{1}{q} \right).
\end{equation}
Along with \( p = p(\kappa) \) smoothness is invariant under linear isomorphisms of \( V_\kappa \) spaces. Moreover, it is closely related to the differentiability and approximation characteristics of \( V_\kappa \) functions, see the survey \[Br-09\]. For instance, a function of smoothness \( s \) belongs for almost all \( x \in Q^d \) to the Taylor class \( T^s_q(x) \) if \( 0 < s < k \) and \( t^k_q(x) \) if \( s = k \), see, e.g., \[Zi-87\] Sec. 3.5] for their definitions.

Let us finally note that the classical \( BV \) spaces are defined over the space \( \ell_\infty[0,1] \) of functions bounded on \([0,1]\) while \( V_\kappa \) spaces with \( q = \infty \) are defined over \( L_\infty \) space. To include them and similar spaces in our consideration we use a version of Definition 1.2 with local approximation denoted by \( E_k(\cdot;\cdot) \) that is defined for \( f \in \ell^{\text{loc}}_\infty(\mathbb{R}^d) \) and \( S \subset \mathbb{R}^d \) by

\[
E_k(f;S) := \inf_{m \in \mathcal{P}_{k-1}} \sup_S |f - m|;
\]

the corresponding versions of the spaces \( \dot{V}_\kappa \), \( V_\kappa \), \( \dot{v}_\kappa \) and \( v_\kappa \) with \( \kappa = \{\lambda, p, \infty\} \) based on this definition are denoted by \( \dot{V}^\lambda_p \), \( V^\lambda_p \), \( \dot{v}^\lambda_p \) and \( v^\lambda_p \), respectively (\( k \) and \( Q^d \) are omitted here by Stipulation 1.3).

In more details, \( \dot{V}^\lambda_p \) is defined by a seminorm

\[
|f|_{\dot{V}^\lambda_p} := \sup_{\pi \in \Pi} \left( \sum_{Q \in \pi} (|Q|^{-\lambda}E_k(f;Q))^p \right)^{1/p};
\]

and \( V^\lambda_p := \dot{V}^\lambda_p / \mathcal{P}_{k-1} \).

Moreover, smoothness of this space denoted by \( s(\lambda, p) \) is defined by (1.14) with \( q = \infty \).

**Remark 1.5.** It seems to be natural to identify \( \dot{V}^\lambda_p \) with \( \dot{V}^\{\lambda,p,\infty\}_p \) by choosing for each class \( f \) from the latter space its representative, say, \( \hat{f} \in \ell_\infty(Q^d) \). Unfortunately, this is impossible as the map \( f \mapsto \hat{f} \) is not linear in general and does not preserve local approximation. Nevertheless, as it will be shown in the forthcoming paper such identification is possible for spaces \( \dot{V}^\lambda_p \) and \( \dot{V}^\{\lambda,p,\infty\}_p \) with \( \lambda > 0 \) and \( 1 \leq p \leq \infty \).

1.3. Now we enumerate the classical function spaces that coincide with or are intimately related to the \( V_\kappa \) spaces introduced.

1.3.1. **Consistency with the one variable definitions.**

We begin with the space \( V^\lambda_p [0,1] \) whose associated seminorm is given for \( f \in \ell_\infty[0,1] \) by

\[
|f|_{V^\lambda_p[0,1]} := \sup_{\pi} \left\{ \sum_{I \in \pi} \left( \frac{E_k(f;I)}{|I|^{\lambda}} \right)^p \right\}^{1/p};
\]

here \( \pi \) runs over packings consisting of pairwise nonoverlapping subintervals in \([0,1]\) and local approximation is given by (1.15).

In turn, in the classical definitions, the supremum in (1.17) is taken over coverings of \([0,1]\) by nonoverlapping intervals and \( E_k \) is replaced by the \( k\)-deviation \( \delta_k \) given for \( f \in \ell_\infty[0,1] \).
and \( I := [a, b] \subset [0, 1] \) by
\[
\delta_k(f; I) := |\Delta^k_h f(a)|, \quad \text{where} \quad h := \frac{b-a}{k};
\]
in particular, \( \delta_1(f; I) := |f(b) - f(a)|. \)

Equivalence of \( V^\lambda_p[0, 1] \) with the space obtained by these substitutions follows from the Whitney inequality [Wh-59]
\[
E_k(f; I) \approx \sup_{I'} \delta_k(f; I'),
\]
where the constants of equivalence are independent of \( f \) and \( I \) (note that \( f \) here can be nonmeasurable).

In particular, \( E_1(f; I) = \frac{1}{2} \sup_{x,y \in I} |f(x) - f(y)| \); therefore for \( k = 1 \)
\[
|f|_{V^\lambda_p[0,1]} = 2^{-\frac{1}{p}} \sup_{x} \left( \sum_i \left( \frac{|f(x_{i+1}) - f(x_i)|}{|x_{i+1} - x_i|^\lambda} \right)^p \right)^{\frac{1}{p}},
\]
where \( \{x_i\} \) runs over finite monotone sequences in \([0, 1]\).

The supremum here denoted by \( \var^\lambda_p(f) \), see (1.3), defines a seminormed space of functions on \([0, 1]\) that we denote by \( BV^\lambda_p \).

Thus, seminormed spaces \( BV^\lambda_p \) and \( \dot{V}_p^{1,\lambda} \) are isometrically isomorphic; in particular, the classical spaces of Jordan \((p = 1, \lambda = 0)\), Wiener-L. Young \((1 \le p < \infty, \lambda = 0)\) and F. Riesz \((1 < p < \infty, \lambda = \frac{1}{p'} := 1 - \frac{1}{p})\) are isometrically isomorphic to the corresponding spaces \( \dot{V}_p^{1,\lambda} \).

1.3.2. \( V^\kappa \) spaces of negative smoothness.

The \( V^\kappa \) spaces with \( s(\kappa) < 0 \) are closely related to weighted \( L_p \) spaces with nonintegrable singularities, Morrey spaces and the likes. In particular, Morrey space \( M^q_s(Q^d), 1 \le q < \infty, 0 < s < \frac{d}{q} \), is defined by a norm given for \( f \in L_q(Q^d) \) by
\[
\|f\|_{M^q_s} := \sup_{Q \subset Q^d} |Q|^\frac{s}{q} \left( \frac{1}{|Q|} \int_Q |f|^q \, dx \right)^{\frac{1}{q}}.
\]

In spite of simplicity of the definition, Morrey spaces have numerous applications in PDEs and harmonic analysis, see, e.g., [Ta-92] and [AX-12] and references therein.

The relation to the space \( V^\kappa \) given by the equality
\[
V^\kappa = M^q_s / P^d_{k-1}, \quad \text{where} \quad \kappa := \left\{ \frac{1}{q} - \frac{s}{d}, \infty, q \right\},
\]
follows from the inequality, see [Ca-64],
\[
\|f\|_{M^q_s} \le c \left( \sup_{Q \subset Q^d} |Q|^\frac{s}{q} \frac{1}{q} E_k(f; Q) + \|f\|_q \right).
\]

\(^2\text{Hereafter } X = Y \text{ for (semi-) normed } X, Y \text{ means that they coincide as linear spaces and have equivalent (semi-) norms.}\)
Let us note that here $s(\kappa) = -s < 0$.

1.3.3. $V_\kappa$ spaces of smoothness zero.

Let $p, q, \lambda \in \kappa$ satisfy

$$1 \leq q \leq p < \infty, \quad \lambda = \frac{1}{q} - \frac{1}{p},$$

hence, $s(\kappa) = 0$.

Then for $q < p$

(1.20) \[ L_q/P_{d-1}^d \subsetneq V_\kappa \subsetneq L_q\infty/P_{d-1}^d, \]

and for $q = p$

(1.21) \[ L_q/P_{d-1}^d = V_\kappa \text{ (isometry)}. \]

Further, if $\kappa := \{\lambda, p, q\}$ and $k$ satisfy

$$\lambda = 1 - \frac{1}{p}, \quad 1 < p \leq \infty, \quad q = 1 \quad \text{and} \quad k = 1,$$

i.e., $s(\kappa) = 0$, then the space $\dot{V}_\kappa$ equals up to equivalence of the seminorms to the John-Nirenberg [JN-61] space $BMO_p$ defined by a seminorm given for $f \in L_1$ by

(1.22) \[ |f|_{BMO_p} := \sup_{\pi \in \Pi} \left( \sum_{Q \in \pi} |Q| \left( \frac{1}{|Q|} \int_Q |f - f_Q| \, dx \right)^p \right)^{\frac{1}{p}}; \]

here $f_Q := \frac{1}{|Q|} \int_Q f \, dx$.

Denoting the expression under supremum by $\gamma(\pi; f)$ we define a subspace of $BMO_p$ denoted by $VMO_p$ by the condition

(1.23) \[ \lim_{\varepsilon \to \infty} \sup_{|\pi| \leq \varepsilon} \gamma(\pi; f) = 0, \]

where $|\pi| := \sup_{Q \in \pi} |Q|$.

The mostly used spaces with $p = \infty$ are denoted by $BMO, VMO$; the latter was introduced and studied for $d = 1$ in [Sa-75]. Numerous applications of these spaces in analysis are summarized in the book [St-93].

In general, we have for $k \geq 1$ and $1 \leq q < p \leq \infty$, $\lambda = \frac{1}{q} - \frac{1}{p}$ the equality

(1.24) \[ V_\kappa = BMO_p/P_{d-1}^d \quad \text{and} \quad V_\kappa = VMO_p/P_{d-1}^d. \]

1.3.4. $V_\kappa$ spaces of positive smoothness.

(a) First, let $p, q, \lambda \in \kappa$ and $s(\kappa)$ satisfy

(1.25) \[ 1 \leq p < q < \infty, \quad \lambda = 0, \quad s(\kappa) = k. \]

Then it is true that

(1.26) \[ \dot{W}_p^k = \dot{V}_\kappa. \]

If $\lambda > 0$, this equality holds also for $q = \infty$. 

Now let $d, p, q \in \kappa$ and $s(\kappa)$ be such that
\[(1.27) \quad d \geq 2, \quad 1 = p \leq q \leq \frac{d}{d-k} < \infty \quad \text{and} \quad s(\kappa) = k.\]
Then it is true that
\[(1.28) \quad BV^k = \dot{V}_k,\]
where $BV^k$ consists of $L_1$ functions whose $k$-th distributional derivatives are finite Borel measures on $Q^d$, see [Br-71, §4, Thm. 12].

Hence, a seminorm of $BV^k$ is given for $f \in L_1$ by
\[(1.29) \quad |f|_{BV^k} := \sum_{|\alpha|=k} \text{var} D^\alpha f \left( := \sum_{|\alpha|=k} \|D^\alpha f\|_M \right).\]

For $k = 1$, this gives the seminorm of the classical space $BV(Q^d)$, see, e.g., the books [Gi-84], [AFP-00] for properties and numerous applications of this space in analysis.

Remark 1.6. (1) The case $s(\kappa) = k$ is maximal, since $\dot{V}_k = \mathcal{P}_{d-1}^d$ if $s(\kappa) > k$, see Lemma 3.1 below.

(2) Conditions (1.25), (1.27) imply continuous embeddings of the corresponding spaces in $L_q$. Moreover, if $\lambda > 0$ these embeddings are compact.

(b) Finally, we consider relations of $V_k$ spaces of smoothness
\[0 < s := s(\kappa) < k,\]
to the homogeneous Besov (Lipschitz) spaces $\dot{B}_{p}^{s,\theta}$ and $\dot{B}_{p}^{s,\infty}$. The various applications of these spaces are surveyed in [Tr-92].

Let us recall that the space $\dot{B}_{p}^{s,\theta}$, $1 \leq p, \theta \leq \infty$, is defined by one of equivalent seminorms given for $f \in L_p(Q^d)$, an integer $0 \leq \ell < s$ and $k = k(s) := \min\{n \in \mathbb{N} : n > s\}$ by
\[(1.30) \quad |f|_{B(p)} := \sup_{|\alpha|=\ell} \left\{ \int_0^1 \left( \frac{\omega_{k-\ell,p}(D^\alpha f; t)}{t^{s-\ell}} \right)^\theta \frac{dt}{t} \right\}^{\frac{1}{\theta}};\]
here $\omega_{k,p}(f; \cdot)$ is the $k$-th modulus of continuity of $f \in L_p$, given by, cf. (1.12),
\[(1.31) \quad \omega_{k,p}(f; t) := \sup_{\|h\|_\infty \leq \ell} \left\{ \|\Delta_h f\|_{L_p(Q^d_{kh})} \right\},\]
where $Q_{kh}^d := \{x \in Q^d : x + kh \in Q^d\}$, $\|h\|_\infty := \max_{1 \leq i \leq d} |h_i|$. 


Now under the conditions
\[(1.32) \quad k = k(s), \quad 1 \leq p < q < \infty \quad \text{and} \quad s > 0\]
on \(k, p, q \in \kappa\) and \(s := s(\kappa)\) the following continuous embeddings are true
\[(1.33) \quad \dot{B}_p^s := \dot{B}_p^{sp} \subset \dot{\nu}_\kappa \subset \dot{V}_p \subset \dot{B}_p^{s\infty}.\]
For \(q = \infty\) the right-hand side embedding remains to be true but that of the left-hand side is true for \(\dot{B}_p^s\) replaced by \(\dot{B}_p^{s1}\).
Moreover, for \(p = \infty, q \leq \infty\)
\[(1.34) \quad \dot{V}_\kappa = \dot{B}_p^s.\]

**Remark 1.7.** (1) For \(1 < p \leq 2\), the left embedding \((1.33)\) can be sharpen by replacing \(\dot{B}_p^s\) by the larger space \(F_p^{s2}/P_{k(s)}\); here \(F_p^{s}\) is the Triebel-Lizorkin space, see, e.g., \([Tr-92]\) for its definition.
(2) Using the real interpolation, see, e.g., \([BL-76], \text{Thm. 6.4.3(1)}\) one can represent the space \(\dot{B}_p^{s\infty}\) as an interpolating space of the couple \((\dot{\nu}_{\kappa_0}, \dot{\nu}_{\kappa_1})\), where \(\kappa_i := \{s_i, p, q_i\}, 1 \leq p < q_i, i = 0, 1, \) and \(s(\kappa_0) = s(\kappa_1)\). Under this conditions we have
\[(1.34) \quad \dot{B}_p^{s\infty} = (\dot{\nu}_{\kappa_0}, \dot{\nu}_{\kappa_1})_\theta\infty,\]
where \(s = s(1 - \theta) + s\theta, 0 < \theta < 1\).

The paper is organized as follows.

In Section 2, we define the predual to the space \(V_\kappa\) denoted by \(U_\kappa\) and that to \(V_\kappa^\lambda\) denoted by \(U_p^\lambda\). Then we formulate the main results of the paper and some directly following applications to the classical spaces described in Subsections 1.3.1–1.3.4.

In Section 3, we prove two results on \(C^\infty\) approximation of \(V_\kappa\) functions formulated in Subsection 2.2. The first one plays an essential role in the proofs of our duality results while the second one provides an important characterization of functions of the space \(\dot{V}_\kappa\).

In Sections 4 and 5, we prove Theorem 2.5 describing the basic properties of the space \(U_\kappa\) and Theorem 2.6 asserting that under mild restrictions on the parameters the spaces \(U_\kappa^*\) and \(V_\kappa\) are isometrically isomorphic.

Finally, in Section 6, we prove Theorem 2.7 asserting that under some additional restrictions the spaces \(V_\kappa^*\) and \(U_\kappa\) are isometrically isomorphic. Passing to duals in the obtained relation we get the “two stars theorem” stating that \(V_\kappa^{**}\) and \(V_\kappa\) are isometrically isomorphic.

2. FORMULATION OF MAIN RESULTS

2.1. Duality. In the first part, we define and describe the basic properties of a Banach space that under mild restrictions is a predual to the space \(V_\kappa\) (recall that \(\kappa = \{\lambda, p, q\}\), see Stipulation [1.3]). We also briefly discuss here similar results for the spaces \(V_\lambda^\lambda, \lambda \geq 0\), see [1.16], leaving the detail account to a forthcoming paper.
In the second part, we present two approximation results for functions of $V^\kappa$ spaces. The first one is essentially used in the proofs of the duality theorems while the second one in the applications concerning the function spaces presented in Subsections 1.3.1–1.3.4.

Finally, we formulate the applications and refer to known before special cases of the presented results.

In the forthcoming formulations, we use the following:

**Notation 2.1.** We write for linear (semi-) normed vector spaces
\begin{equation}
X \hookrightarrow Y
\end{equation}
if there is a linear continuous injection of $X$ into $Y$, and replace $\hookrightarrow$ by $\subset$ if the injection embeds $X$ into $Y$ as a linear subspace.

Further, we say that these spaces are *isomorphic* and write
\begin{equation}
X \cong Y
\end{equation}
if $X \hookrightarrow Y$ and $Y \hookrightarrow X$, and
\begin{equation}
X = Y
\end{equation}
if, in addition, they coincide as linear spaces, hence, have equivalent (semi-) norms.

Finally, spaces $X$ and $Y$ are said to be *isometrically isomorphic* if the injections in (2.2) are of norm 1. We write in this case
\begin{equation}
X \equiv Y.
\end{equation}

2.1.1. *Space predual to $V^\kappa$.* The space under consideration denoted by $U^\kappa$ is constructed by using the following building blocks.

**Definition 2.2 (k-atom).** A function $a \in L_q'$ is said to be a $\kappa$-atom on a subcube $Q \subset Q^d$ if it satisfies the conditions
\begin{enumerate}
  \item $\text{supp } a \subset Q$;
  \item $\|a\|_{q'} \leq |Q|^{-\lambda}$;
  \item $\int_{Q^d} x^\alpha a(x) \, dx = 0$ for all $|\alpha| \leq k - 1$.
\end{enumerate}

As above, $\kappa := \{\lambda, p, q\}$ and $\frac{1}{q} + \frac{1}{q'} = 1$ for $1 \leq q \leq \infty$.

The subject of the definition is denoted by $a_Q$.

Let us recall, see Stipulation 1.3, that $k$ and $Q^d$ are fixed and removed from almost all notations, e.g., $\|a\|_{q'} := \|a\|_{L_q'(Q^d)}$.

**Definition 2.3 (k-chain).** A function $b \in L_{q'}$ is said to be a $\kappa$-chain subordinate to a packing $\pi \in \Pi$ if $b$ belong to the linear span of the family of $\kappa$-atoms $\{a_Q\}_{Q \in \pi}$.

The subject of this definition is denoted by $b_\pi$.

Moreover, we write
\begin{equation}
[b_\pi]_{p'} := \|\{c_Q\}_{Q \in \pi}\|_{p'} := \left\{ \sum_{Q \in \pi} |c_Q|^{p'} \right\}^{\frac{1}{p'}}
\end{equation}
whenever
\[ b_\pi = \sum_{Q \in \pi} c_Q a_Q. \]
This clearly defines a norm on the linear span of the family \( \{a_Q\}_{Q \in \pi}. \)

Further, let \( U^0_\kappa \subset L_{q'} \) denote the linear span of the set \( A_\kappa \) of all \( \kappa \)-atoms, i.e.,
\[ U^0_\kappa := \text{linspan}\{a_Q \in A_\kappa\}. \]
Every \( f \in U^0_\kappa \) can be represented (in infinitely many ways) as a finite sum of \( \kappa \)-chains by
\[ f = \sum_\pi b_\pi. \]
The space \( U^0_\kappa \) is equipped with the seminorm
\[ \|f\|_{U^0_\kappa} := \inf \sum_\pi [b_\pi]_{p'}, \]
where infimum is taken over all representations \( \text{(2.7).} \)

**Definition 2.4.** The space \( U_\kappa \) is the completion of the seminormed space \( (U^0_\kappa, \| \cdot \|_{U^0_\kappa}) \).

The next result describes the basic properties of the space \( U_\kappa \).

**Theorem 2.5.** (a) The closed unit ball of \( U_\kappa \) denoted by \( B(U_\kappa) \) is the closure of the symmetric convex hull of the set \( B_\kappa := \{b_\pi \in U^0_\kappa : [b_\pi]_{p'} \leq 1\} \).

(b) If \( p, q \in \kappa \) satisfy the conditions
\[ 1 < q \leq \infty \quad \text{and} \quad 1 \leq p \leq \infty, \]
then \( U_\kappa \) is separable.

(c) If \( p, q \in \kappa \) and \( s := s(\kappa) \) satisfy the conditions
\[ 1 < q \leq \infty, \quad 1 \leq p \leq \infty, \quad s \leq k, \]
then \( U_\kappa \) is Banach.

Now we present a duality theorem for the space \( V_\kappa \).

**Theorem 2.6.** If \( p, q \in \kappa \) and \( s := s(\kappa) \) satisfy conditions \( \text{(2.9)} \), then
\[ U^*_\kappa \equiv V_\kappa. \]
More precisely, each continuous linear functional on \( U_\kappa \) has the form
\[ f(u) = \int_{Q^d} fu \, dx \quad \text{for all} \quad u \in U^0_\kappa (\subset L_{q'}), \]
where \( f \in V_\kappa (\subset L_q) \) and \( \|f\|_{V_\kappa} \) is equal to the linear functional norm.

\[ ^3 \text{in fact, we show that under mild restrictions on} \ \kappa \ \text{\( \text{(2.8)} \) is a norm.} \]
2.1.2. **Two Stars Theorem.** Using the properties of \( U_\kappa \) and \( V_\kappa \) presented here and in the next subsection and some basic facts of the Banach space theory we prove the following:

**Theorem 2.7.** Let \( p, q \in \kappa \) and \( s := s(\kappa) \) satisfy the conditions

\[
1 < p \leq \infty, \quad 1 < q < \infty \quad \text{and} \quad s < k.
\]

Then

\[
V_\kappa^* \equiv U_\kappa.
\]

Specifically, the result asserts that under the identification of \( \tau_\kappa \) with a subspace of \( U_\kappa^* \) by Theorem 2.6 each continuous linear functional on \( \tau_\kappa \) has the form \( T_u(f) = f(u) \) for all \( f \in \tau_\kappa \), where \( u \in U_\kappa \) and \( \|u\|_{U_\kappa} \) is equal to the linear functional norm.

From here and Theorem 2.6 we obtain:

**Corollary 2.8.** Under conditions (2.10)

\[
V_{\kappa}^{**} \equiv V_{\kappa}.
\]

**Remark 2.9.** The restriction \( q > 1 \) is necessary. In fact, the space \( V_\kappa \) with \( \kappa = \{0, 1, 1\} \), hence, \( s(\kappa) = 0 < k \), is isometrically isomorphic to the space \( L_1/P_{d-1} \cong L_1 \), see (1.21), that is not dual.

The restriction \( s(\kappa) < k \) is necessary as well, see Remark 2.15 below.

2.1.3. **Space predual to \( V_\lambda^p \).** Let us recall that \( V_\lambda^p \) is defined by seminorm (1.16),

\[
V_\lambda^p := \hat{V}_\lambda^p/P_{d-1} \quad \text{and} \quad V_\lambda^p := \text{clos}(C^\infty /P_{d-1}, V_\lambda^p).
\]

To define a predual to \( V_\lambda^p \) denoted by \( U_\lambda^p \) we follow the scheme of Subsection 2.1.1 that begins with the definition of atoms. The required version is motivated by the equality \( \ell_1^* = \ell_\infty \), where \( \ell_1 \) is defined by a norm given for \( f : Q^d \rightarrow \mathbb{R} \) by

\[
\|f\|_{\ell_1} := \sum_{x \in Q^d} |f(x)|.
\]

The space \( \ell_1 \) is nonseparable but every \( f \in \ell_1 \) has at most countable support. This leads to the following:

**Definition 2.10.** A function \( a_Q : Q^d \rightarrow \mathbb{R} \) is said to be a \((\lambda, p)\)-atom on \( Q \subset Q^d \) if \( a_Q \) is supported by \( Q \) and satisfies the conditions

(i) \[
\|a_Q\|_{\ell_1} \leq |Q|^{-\lambda};
\]

(ii) \[
\sum_{x \in Q^d} m(x)a_Q(x) = 0 \quad \text{for every} \quad m \in P_{d-1}.
\]

Having this we repeat word-for-word definitions of Subsection 2.1.1 to introduce \((\lambda, p)\)-chains and their norms, see (2.4) and (2.5), preserving the very same notations.
Further, \((U_p^\lambda)^0 \subset \ell_1\) is the linear span of the set \(\mathcal{A}_{\{\lambda,p\}}\) of all \((\lambda,p)\)-atoms, i.e.,

\[
(U_p^\lambda)^0 := \text{linspan}\{a_Q \in \mathcal{A}_{\{\lambda,p\}}\}.
\]

As above, see (2.8), this space is equipped with the seminorm \(\| \cdot \|_{(U_p^\lambda)^0}\).

Finally, the completion of \((U_p^\lambda)^0\) under this seminorm gives the required space \(U_p^\lambda\).

The basic result for this case asserts:

**Theorem 2.11.** (a) Let the parameters of the space \(V_p^\lambda\) and its smoothness \(s := d(\lambda + \frac{1}{p})\) satisfy the conditions

\[
\lambda \geq 0, \quad 1 < p \leq \infty \quad \text{and} \quad s \leq k.
\]

Then \((U_p^\lambda)^* \equiv V_p^\lambda\).

(b) If \(\lambda, p\) satisfy (2.14) and \(s < k\), then \((V_p^\lambda)^* \equiv U_p^\lambda\) if \(\lambda > 0\) and \((V_p^0)^* \equiv U_p^0\).

As a corollary we obtain the corresponding two stars theorem: \((V_p^\lambda)^{**} \equiv V_p^\lambda\) if \(\lambda > 0\) and \((V_p^0)^{**} \equiv V_p^0\).

### 2.2. Approximation Theorems.

We present here two results on \(C^\infty\) approximation of \(V_\kappa\) functions.

The first result plays an essential role in the proofs of the duality theorems.

**Theorem 2.12.** (a) For each function \(f \in \dot{V}_\kappa, \kappa := \{\lambda, p, q\}\), there is a sequence \(\{f_n\}_{n \in \mathbb{N}} \subset C^\infty\) linearly depending on \(f\) such that

\[
\lim_{n \to \infty} |f_n|_{V_\kappa} = |f|_{V_\kappa}.
\]

(b) Moreover,

\[
\lim_{n \to \infty} \|f - f_n\|_q = 0 \quad \text{if} \quad 1 \leq q < \infty
\]

and

\[
\lim_{n \to \infty} \int_{Q^d} (f - f_n)g \, dx = 0 \quad \text{for each} \quad g \in L_1 \quad \text{if} \quad q = \infty.
\]

The second result characterizes \(\dot{V}_\kappa\) functions admitting \(C^\infty\) approximation, i.e., functions of the subspace \(\dot{V}_\kappa\).

**Theorem 2.13.** Let \(f \in \dot{V}_\kappa, \kappa := \{\lambda, p, q\}\) satisfies one of the conditions

\[
1 \leq p \leq \infty, \quad 1 \leq q < \infty \quad \text{and} \quad s(\kappa) < k,
\]

or

\[
1 \leq p \leq \infty, \quad q = \infty, \quad \lambda \geq 0 \quad \text{and} \quad s(\kappa) < k.
\]

Then \(f \in \dot{V}_\kappa\) if and only if

\[
\lim_{\varepsilon \to 0} \sup_{|\pi| \leq \varepsilon} \left( \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^p \right)^{\frac{1}{p}} = 0;
\]
hereafter $|\pi| := \sup_{Q \in \pi} |Q|$.

**Remark 2.14.** The restriction $s(\kappa) < k$ is necessary. In fact, for $s(\kappa) = s$ the subspace of functions of $\dot{V}_\kappa$ satisfying condition (2.20) is (algebraically) isomorphic to $P^d_{k-1}$, see the argument of the proof of Lemma 3.1, while $\dot{V}_\kappa$ contains all functions from $C^\infty(\subset L_q)$, see the proof of Theorem 2.5 (c).

2.3. **Applications.** We begin with the result describing duality properties of the “classical” spaces $\dot{V}^\lambda_p[0,1]$, $\lambda \geq 0$, $1 \leq p \leq \infty$. It is a corollary of the above formulated Theorem 2.11.

In the following discussion, we use the classical definition of the $\dot{V}^\lambda_p[0,1]$ seminorm given for $f \in \ell_\infty[0,1]$ by

$$|f|_{\dot{V}^\lambda_p} := \sup_{\{x_i\}} \left( \sum_i \left( \frac{\delta_k(f;x_i,x_{i+1})}{|x_{i+1} - x_i|^{1/p}} \right)^p \right)^{1/p},$$

where $\{x_i\}$ runs over all monotone finite sequences in $[0,1]$ and

$$\delta_k(f;a,b) := |\Delta^k(f;a)|, \quad \text{where} \quad h := \frac{b-a}{k}.$$

Let us recall that

$$V^\lambda_p := \dot{V}^\lambda_p/P^1_{k-1} \quad \text{and} \quad v^\lambda_p := \text{clos}(C^\infty/P^1_{k-1}, V^\lambda_p).$$

**A1. Duality Properties of $\dot{V}^\lambda_p[0,1]$:**

Assume that

$$1 < p \leq \infty \quad \text{and} \quad 0 \leq \lambda < k - \frac{1}{p}.$$  

Then

$$\left( v^\lambda_p \right)^{**} \equiv V^\lambda_p \quad \text{if} \quad \lambda > 0 \quad \text{and} \quad \left( v^0_p \right)^{**} \equiv V^0_p.$$

Let us show that the restrictions on $\lambda, p$ in (2.22) are necessary. In fact, the space $\dot{V}^\lambda_p[0,1]$ with $k = p = 1$, $\lambda = 0$, hence, $\lambda = k - \frac{1}{p}$, coincides with the Jordan space $BV[0,1]$. By the (modernized form of) Lebesgue theorem, see Subsection 1.1,

$$W^1_1[0,1] \equiv AC[0,1].$$

Since this isometry preserves $C^\infty$ functions and the subspace $C^\infty$ is dense in the Sobolev space,

$$\dot{V}^0_1 := \text{clos}(C^\infty, BV) = AC \equiv \dot{W}^1_1.$$  

Factorizing by constants we obtain

$$v^0_1 \equiv W^1_1 \equiv \dot{W}^1_1/\mathbb{R} \equiv L_1;$$

this, in turn, implies that

$$\left( v^0_1 \right)^{**} \equiv L^*_{\infty}.$$
Assuming that (2.23) is true in this case we get

\[ BV \cong (V^1_0)^{**} \cong L^*_\infty. \]

However, this is false as the cardinality of \( BV \) is strictly less than that of \( L^*_\infty \).

**Remark 2.15.** This also shows that the restriction on \( \kappa \) in Theorem 2.7 is necessary as well. In fact, it will be proved in a forthcoming paper that for \( \kappa = \{\lambda, p, \infty\} \) with \( \lambda \geq 0, 1 \leq p \leq \infty \)

\[ V_\kappa \hookrightarrow V^\lambda_p \quad \text{and} \quad V_\kappa \equiv V^\lambda_p. \]

If Theorem 2.7 is valid for \( k = d = p = 1, \lambda = 0 \), hence, for \( s(\kappa) = k \), then we have a contradiction

\[ L^*_\infty \cong (V_\kappa)^{**} \cong V_\kappa \hookrightarrow V^0_1 = BV/\mathbb{R}. \]

We complete this discussion by referring to the papers [DeL-61] and [Ki-84]. The first one contains the two stars theorem (2.23) for the space \( Lip^\lambda, 0 < \lambda < 1 \), that coincides with \( V^\lambda_\infty[0,1] \) while the second one proves (2.23) for the Wiener-L. Young space \( BV^p_p[0,1], 1 < p < \infty \), that coincides with \( V^p_0[0,1] \).

The subsequent applications present new results concerning approximation and duality properties of Morrey, John-Nirenberg, Sobolev and Lipschitz spaces. Their proofs directly follow from the formulated above results for \( V_\kappa \) spaces via the corresponding isomorphisms between them and the spaces under consideration.

Our first result uses the isomorphism for Morrey space \( M^s_q \), see (1.19), where we take \( k = 1 \); hence, we have

\[ (2.24) \quad M^s_q/\mathbb{R} \cong V_\kappa, \quad s > 0, \]

where \( \kappa := \{\lambda, p, q\} \) and \( k \) satisfy

\[ (2.25) \quad k := 1, \quad \lambda := \frac{1}{q} - \frac{s}{d} > 0, \quad p = \infty, \quad 1 < q < \infty. \]

**A2. Properties of Morrey Space:**

(a) Let \( f \in M^s_q \), where \( s > 0 \) and \( q \) satisfy (2.25).

There exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset C^\infty \) linearly depending on \( f \) such that

\[ \lim_{n \to \infty} f_n = f \quad \text{in} \quad L^q \quad \text{and} \quad \lim_{n \to \infty} \|f_n\|_{M^s_q} \cong \|f\|_{M^s_q} \]

with constants of equivalence depending on \( d \) and \( \lambda \) only.

Moreover, this sequence converges to \( f \) in \( M^s_q \) if and only if

\[ (2.26) \quad \lim_{|Q| \to 0} \frac{1}{|Q|} \int_Q |f(x) - f_Q|^q \, dx = 0. \]

(b) Let the parameters \( k, \lambda, p \) of \( \kappa \) satisfy condition (2.25) while \( 1 < q < \infty \). Then it is true that

\[ U^*_\kappa \cong M^s_q/\mathbb{R}. \]
Moreover, denoting by \( m^s_q \) the space of functions \( f \in L^q \) satisfying \( \text{(2.26)} \) we have
\[
(\text{2.27}) \quad (m^s_q / \mathbb{R})^* \cong U^*_\kappa.
\]
In particular,
\[
(\text{2.28}) \quad (m^s_q / \mathbb{R})^{**} \cong M^s_q / \mathbb{R}.
\]

**Remark 2.16.** Some other preduals to the space \( M^s_q \) see in \( \text{[Zo-86]} \) and \( \text{[AX-12]} \).

To derive the next result we use isomorphism \( \text{(1.22)} \) with \( k = 1 \). Hence, in this case we have
\[
BMO_p / \mathbb{R} \cong V^*_\kappa,
\]
where \( \kappa := \{ \lambda, p, q \} \) and \( k \) satisfy
\[
(\text{2.29}) \quad k := 1, \quad \lambda := \frac{1}{q} - \frac{1}{p}, \quad 1 \leq q < p \leq \infty.
\]

**A3. Properties of BMO Space:**

(a) Let \( f \in BMO_p \) and \( k, p, q \) satisfy \( \text{(2.29)} \). There exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset C^\infty \) linearly depending on \( f \) such that
\[
\lim_{n \to \infty} f_n = f \quad \text{in} \quad L^q \quad \text{and} \quad \lim_{n \to \infty} |f_n|_{BMO_p} \approx |f|_{BMO_p}
\]
with constants of equivalence depending on \( d \) and \( \lambda \) only.
Moreover, this sequence converges to \( f \) in \( BMO_p \) if and only if \( f \in VMO_p \), i.e.,
\[
\lim_{\varepsilon \to \infty} \sup_{|\pi| \leq \varepsilon} \left( \sum_{Q \in \pi} |Q| \left( \frac{1}{|Q|} \int_Q |f - f_Q| \, dx \right)^p \right)^{\frac{1}{p}} = 0.
\]

(b) Let the parameters \( k, \lambda, p \) of \( \kappa \) satisfy condition \( \text{(2.29)} \) but \( 1 < q < \infty \). Then it is true that
\[
(\text{2.30}) \quad U^*_\kappa \cong BMO_p / \mathbb{R}
\]
and, moreover,
\[
(\text{2.31}) \quad VMO^*_p \cong U^*_\kappa.
\]

In particular,
\[
(\text{2.32}) \quad VMO^*_p \cong BMO_p.
\]

**Remark 2.17.**

1. The special case \( p = \infty \) of (b), i.e., the relations \( U^*_\kappa \cong BMO / \mathbb{R} \) and \( VMO^* = U_\kappa, \kappa = \{ \frac{1}{q}, \infty, q \} \), was proved in \( \text{[CW-77]} \) in a more general setting of functions on homogeneous metric spaces. The space \( U_\kappa \) is denoted by \( H^1 \) in \( \text{[CW-77]} \) to emphasize its connection with the atomic decomposition of the classical Hardy space \( H^1(\mathbb{R}^d) \) and the celebrated C. Fefferman duality theorem \( \text{[F-71]} \) asserting that \( BMO(\mathbb{R}^d) = (H^1(\mathbb{R}^d))^* \).

2. Relation \( \text{(2.30)} \) presents a family of preduals to the space \( BMO_p \) which are pairwise isomorphic due to \( \text{(2.31)} \).
The third result is derived from isomorphism (1.28)

$$BV^k = \dot{V}_\kappa,$$

where $\kappa = \{\lambda, p, q\}$ satisfies

$$\lambda = \frac{k}{d} - \frac{1}{q'}, \quad p = 1, \quad 1 \leq q \leq \infty,$$

hence, $s(\kappa) = k$.

**A4. Properties of $BV^k$ Space:**

(a) Let $f \in BV^k$ and $\kappa$ satisfy (2.33). There exists a sequence $\{f_n\}_{n \in \mathbb{N}} \subset C^\infty$ linearly depending on $f$ such that

$$\lim_{n \to \infty} f_n = f \quad \text{in} \quad L_q \quad \text{and} \quad \lim_{n \to \infty} |f_n|_{BV^k} \approx |f|_{BV^k}$$

with constants of equivalence depending on $k, q$ and $\lambda$. Moreover, the sequence converges to $f$ in $BV^k$ if and only if

$$\lim_{\varepsilon \to \infty} \sup_{|\pi| \leq \varepsilon} \left( \sum_{Q \in \pi} \frac{E_{k,q}(f;Q)}{|Q|^\lambda} \right) = 0.$$  \hspace{1cm} (2.35)

(b) If $\lambda, p \in \kappa$ satisfy (2.33) and $1 < q \leq \infty$, then it is true that

$$U^*_\kappa \approx BV^k / K_{k-1}.$$  \hspace{1cm} (2.36)

**Remark 2.18.** For $k = 1$, i.e., for the space $BV(Q^d)$, statement (2.34) is known, see, e.g., [Zi-87, Thm. 5.3.3] while that of (2.35) asserting that $BV(Q^d)$ has a predual space admitting an atomic decomposition is new. Duality of $BV(Q^d)$ was mentioned without the proof in [AFP-00, Remark 3.12].

Finally, we use isomorphism (1.34) asserting that

$$\dot{V}_\kappa = \dot{B}^s_{\infty}$$

under the conditions on $k$ and $\kappa := \{\lambda, p, q\}$ given by

$$k = \min\{n \in \mathbb{N} : n > s\}, \quad \lambda := \frac{s}{d} + \frac{1}{q}, \quad p = \infty, \quad 1 \leq q \leq \infty \quad \text{and} \quad s > 0.$$  \hspace{1cm} (2.37)

Since $\dot{B}^s_{\infty}$ consists of continuous up to Lebesgue measure zero functions, it is naturally identified with the space $\dot{\Lambda}^s(Q^d)$ of functions $f \in C(Q^d)$ satisfying the condition

$$\sup_{x, h} \{ |\Delta^k_{h}(f; x)| : x, x + kh \in Q^d \} \leq C \|h\|^s, \hspace{1cm} (2.38)$$

where $k := k(s)$.

This is equipped with a (Banach) seminorm given for $f$ satisfying (2.38) by

$$|f|_{\Lambda^*} := \inf C.$$
Further, replacing the right-hand side in (2.38) by \( o(\|h\|_{\infty}^s) \), \( h \to 0 \), we define the subspace of \( \dot{\Lambda}^s \) denoted by \( \dot{\lambda}^s \).

As usual, we also set
\[
\Lambda^s := \dot{\Lambda}^s / \mathcal{P}^d_{k(s) - 1}, \quad \lambda^s := \dot{\lambda}^s / \mathcal{P}^d_{k - 1}.
\]

**A5. Properties of \( \Lambda^s \):**

(a) Let \( f \in \dot{\Lambda}^s \) and \( k \) and \( \kappa \) satisfy (2.37). There exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset C^\infty \) linearly depending on \( f \) such that
\[
\lim_{n \to \infty} f_n = f \quad \text{in} \quad C \quad \text{and} \quad \lim_{n \to \infty} |f_n|_{\Lambda^s} \approx |f|_{\Lambda^s}
\]
with constants of equivalence depending on \( d \) and \( s \) and tending to \( \infty \) if \( s \to k \). Moreover, the sequence converges to \( f \) in \( \dot{\Lambda}^s \) if and only if it belongs to \( \dot{\lambda}^s \).

(b) If \( k \) and \( \kappa \) satisfy (2.37) but with \( 1 < q < \infty \), then it is true that
\[
U^s_\kappa \cong \Lambda^s
\]
and, moreover,
\[
(\lambda^s)^* \cong U^s_\kappa.
\]
In particular,
\[
(\lambda^s)^{**} \cong \Lambda^s.
\]

As it was mentioned above, relation (2.42) with \( s \in (0, 1) \) and \( d = 1 \) was originally proved in [DeL-61]. Similar to (2.40)–(2.42) statements for analogous spaces of functions on \( \mathbb{R}^d \) follow from the main results of [Ha-97].

**2.4. Comments.** (a) Using Definitions 1.2 and 2.2 with \( \mathbb{R}^d \) substituted for \( Q^d \) and \( f \in L_q(\mathbb{R}^d) \) we define the family of spaces \( V_\kappa(\mathbb{R}^d) \) and \( U_\kappa(\mathbb{R}^d) \). The corresponding space \( v_\kappa(\mathbb{R}^d) \) is defined as \( \text{clos}(C^\infty_0(\mathbb{R}^d), V_\kappa(\mathbb{R}^d)) \), where \( C^\infty_0(\mathbb{R}^d) \) consists of \( C^\infty \) functions with compact supports. Then under the restriction \( \lambda(\kappa) > 0 \) all of the main results of Subsection 2.2 hold with the very same proofs.

For \( k(\kappa) = 1 \), the same is true for the family of spaces \( \{V_\kappa\} \) of functions on complete doubling metric spaces, see, e.g., [BB-11, sec. 4.3] for their definition and properties. In this case, cubes are replaced by metric balls.

(b) Let \( \omega : [0, 1] \to \mathbb{R}_+ \) be a continuous nondecreasing function satisfying the conditions
\[
\lim_{\lambda \to 1^+} \left( \sup_{t \in (0, \lambda^{-1}]} \frac{\omega(\lambda t)}{\omega(t)} \right) = 1 \quad \text{and} \quad \lim_{t \to 0} \frac{t^\sigma}{\omega(t)} < \infty,
\]
where \( \sigma := \frac{1}{q} - \frac{1}{p} + \frac{1}{q} \).

Replacing in Definitions 1.2 and 2.2 \( |Q|^\lambda \) by \( \omega(|Q|) \) we introduce the spaces denoted by \( V^\omega_\kappa \) and \( U^\omega_\kappa \), where \( \kappa := (\kappa, p, q) \). The analogs of Theorems 2.5, 2.6 and 2.12 hold for these spaces with minor changes in the proofs.
The same is true for Theorems 2.7, 2.13 and Corollary 2.8 under one more condition on $\omega$:

(2.44) \[ \lim_{t \to 0} \frac{t^\sigma}{\omega(t)} = 0. \]

### 3. Proofs of Approximation Theorems

#### 3.1. BV spaces of large smoothness.

The next result shows that $\dot{V}_\kappa = P^d_{k-1}$ if $s(\kappa) > k$. In particular, Theorem 2.12 trivially holds in this case.

**Lemma 3.1.** Let $f \in \dot{V}_\kappa$, where $\kappa := \{\lambda, p, q\}$ and $s := s(\kappa) > k$. Then $f$ equals a polynomial of degree $\leq k - 1$ a.e. on $Q^d$.

**Proof.** Setting $r := \min\{p, q\}$ and using the Hölder inequality we have

\[
E_{kr}(f; Q) \leq |Q|^\frac{1}{r} - \frac{1}{q} E_{kq}(f; Q).
\]

Since $\frac{s}{q} := \lambda + \frac{1}{p} - \frac{1}{q}$, this and the Hölder inequality imply

\[
\left\{ \sum_{Q \in \pi} \left( |Q|^{-\frac{s}{q}} E_{kr}(f; Q) \right)^r \right\}^{\frac{1}{r}} \leq \left\{ \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^r |Q|^{1 - \frac{1}{p}} \right\}^{\frac{1}{r}}
\]

\[
\leq \left\{ \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^r \right\}^{\frac{1}{rp}} \left\{ \sum_{Q \in \pi} |Q| \right\}^{\frac{1}{pq} - \frac{1}{p}} \leq |f|_{V_\kappa}.
\]

Now, let $\Pi(t) \subset \Pi$ consist of packings formed by congruent cubes of sidelength $t \leq 1$. Taking in the previous inequality supremum over all $\pi \in \Pi(t)$ we obtain

\[
(3.1) \quad t^{-s} \sup_{\pi \in \Pi(t)} \left\{ \sum_{Q \in \pi} \left( E_{kr}(f; Q) \right)^r \right\}^{\frac{1}{r}} \leq |f|_{V_\kappa}.
\]

Due to Theorem 4 of [Br-71, §2] supremum here is bounded from below by $c \omega_{kr}(f; t)$ with $c = c(k, d) > 0$. This and (3.1) imply that

\[
\lim_{t \to \infty} t^{-k} \omega_{kr}(f; t) \leq c |f|_{V_\kappa} \lim_{t \to 0} t^{s - k} = 0.
\]

In turn, this gives

\[
\omega_{kr}(f; \cdot) = 0,
\]

see, e.g., [Ti-63, sec. III.3.3].

However, Theorem 2 of [Br-70] asserts that

\[
E_{kr}(f; Q^d) \leq c(k, d) \omega_{kr} \left( f; \frac{1}{k} \right),
\]

i.e., $E_{kr}(f; Q^d) = 0$ in this case and $f \in P^d_{k-1}$ up to Lebesgue measure zero. \( \square \)

In the subsequent proofs we assume that $V_\kappa$ spaces are of smoothness $\leq k$. 
3.2. **Proof of Theorem 2.12 (a).** First we define a sequence of linear operators \( \{ T_n \}_{n \in \mathbb{N}} \) acting from \( L_1 \) to \( C^\infty \).

Let \( \varphi \in C^\infty(\mathbb{R}^d) \) be a function supported by the \( \ell^d_\infty \) unit cube \([-1, 1]^d\) such that

\[
0 \leq \varphi \leq 1 \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi \, dx = 1.
\]

We denote by \( f^0 \) the extension of \( f \in L_1 \) by 0 outside \( Q^d \) and then define \( f^0_n, n \in \mathbb{N}, \) by the formula

\[
(3.2) \quad f^0_n(x) := \int_{\|y\|_\infty \leq 1} f^0(x - \frac{y}{n+1}) \varphi(y) \, dy, \quad x \in \mathbb{R}^d;
\]

hereafter \( \|y\|_\infty := \max_{1 \leq i \leq d} |y_i| \) is the \( \ell^d_\infty \) norm.

Next, let \( a_n : \mathbb{R}^d \to \mathbb{R}^d \) be the \( \lambda_n \)-dilation with center \( c \), where \( \lambda_n := \frac{n}{n+1} \) and \( c = c_{Q^d} \) is the center of \( Q^d \).

Then \( a_n(Q^d) \) is a subcube of \( Q^d \) centered at \( c_{Q^d} \) and

\[
(3.3) \quad \text{dist}_{\ell^d_\infty}(\partial Q^d, a_n(Q^d)) = \frac{1}{n+1}.
\]

Now we define \( f_n := T_n(f) \in C^\infty \) by setting

\[
(3.4) \quad f_n = f^0_n \circ a_n.
\]

Let \( \kappa := \{ \lambda, p, q \} \) be such that

\[
(3.5) \quad 1 \leq p, q \leq \infty \quad \text{and} \quad s(\kappa) \leq k.
\]

We show that under these assumptions the sequence \( \{ f_n \}_{n \in \mathbb{N}}, f \in V_\kappa, \) satisfies the inequality

\[
(3.6) \quad \lim_{n \to \infty} |f_n|_{V_\kappa} \leq |f|_{V_\kappa}.
\]

To prove (3.6) we need the following duality result.

**Lemma 3.2.** Let \( f \in L_r(Q), 1 \leq r < \infty. \) There is a function \( g_Q(f) \in L_r(Q) \) such that

\[
(3.7) \quad E_{kr}(f; Q) = \int_Q f g_Q(f) \, dx;
\]

\[
(3.8) \quad \|g_Q(f)\|_{L_r(Q)} = 1 \quad \text{and} \quad \int_Q x^\alpha g_Q(f) \, dx = 0, \quad |\alpha| \leq k - 1.
\]

**Proof.** Since the best approximation in (3.7) is the distance in \( L_r(Q) \) from \( f \) to \( P^d_{k-1}|Q \), the Hahn-Banach theorem implies, see, e.g., [DSch-58, Lemma II.3.12], existence of a linear functional \( \ell \in L_r(Q)^* \) of norm 1 orthogonal to \( P^d_{k-1}|Q \) and such that \( \ell(f) = E_{kr}(f; Q) \). Identifying \( \ell \) with the function \( g_Q(f) \in L_r(Q) \) (= \( L_r(Q)^* \)) and representing \( \ell(f) \) as the integral we obtain the result. \( \square \)
Using this we first prove the following inequality for $1 \leq q < \infty$ that will be then extended to $q = \infty$.

\begin{equation}
E_{kq}(f_n; Q) \leq \lambda_n^{-\frac{d}{q}} \int_{\|y\|_\infty \leq 1} E_{kq}(f^0; Q_n(y)) \varphi(y) \, dy,
\end{equation}

where the cube $Q_n(y) := a_n(Q) - (n + 1)^{-1}y$ is containing in $Q^d$ by (3.3).

For its proof we use Lemma 3.2 to write

\begin{equation}
E_{kq}(f_n; Q) = \int_Q f_n g_Q \, dx,
\end{equation}

where $g_Q \in L_{q'}$, $1 < q' \leq \infty$, is such that

\begin{equation}
\|g_Q\|_{L_{q'}(Q)} = 1 \quad \text{and} \quad \int_Q g_Q(x) x^\alpha = 0, \quad |\alpha| \leq k - 1.
\end{equation}

Using the definition of $f_n$, see (3.4), changing the order of integration in the inequality (3.10) and using the orthogonality of $g_Q$ to $P^{d}_{k-1}$ we have for any $m \in P^{d}_{k-1}$

\begin{equation}
E_{kq}(f_n; Q) \leq \int_{\|y\|_\infty \leq 1} \varphi(y) dy \int_Q (f^0 - m)(a_n(x) - (n + 1)^{-1}y) g_Q(x) \, dx.
\end{equation}

Changing variable $x$ in the inner integral by $z := a_n(x) - (n + 1)^{-1}y$, hence, changing $Q$ by $Q_n(y)$ and $dx$ by $\lambda_n^{-d} \, dz$ and then estimating the obtained integral by the Hölder inequality we have

\begin{equation}
E_{kq}(f_n; Q) \leq \lambda_n^{-\frac{d}{q}} \int_{\|y\|_\infty \leq 1} \varphi(y) dy \left( \int_{Q_n(y)} |f^0 - m|^q \, dx \right)^{\frac{1}{q}}.
\end{equation}

Taking here infimum over $m \in P^{d}_{k-1}$ and noting that $f^0 = f$ on $Q_n(y) \subset Q^d$ we prove (3.9) for $1 \leq q < \infty$.

Let us extend the just proved inequality to $q = \infty$. To this end we need the following:

**Lemma 3.3.** If $g \in L_{\infty}$ and $Q \subset Q^d$, then

\begin{equation}
\lim_{q \to \infty} E_{kq}(g; Q) = E_{k\infty}(g; Q).
\end{equation}

**Proof.** Let $m \in P^{d}_{k-1}$ be such that

\[ \|g - m\|_{L_{\infty}(Q)} = E_{k\infty}(g; Q). \]

Then for each $q \in [1, \infty)$

\[ E_{kq}(g; Q) \leq \|g - m\|_{L_q(Q)} \leq \|g - m\|_{L_{\infty}(Q)} = E_{k\infty}(g; Q). \]

This shows that

\begin{equation}
\lim_{q \to \infty} E_{kq}(g; Q) \leq E_{kq}(g; Q).
\end{equation}
Conversely, let \( \{q_i\}_{i \in \mathbb{N}} \subset [1, \infty) \) and \( m_{q_i} \in \mathcal{P}^d_{k-1} \) be such that
\[
\lim_{q \to \infty} E_{kq}(g; Q) = \lim_{i \to \infty} E_{kq_i}(g; Q) \quad \text{and} \quad \|g - m_{q_i}\|_{L_{q_i}(Q)} = E_{kq_i}(g; Q).
\]
Since \( \dim \mathcal{P}^d_{k-1} < \infty \), for some \( c = c(k, d, Q) > 0 \)
\[
\|m_{q_i}\|_{L_\infty(Q)} \leq c\|m_{q_i}\|_{L_{q_i}(Q)} \leq 2c\|g\|_{L_{q_i}(Q)}.
\]
Hence, there are a subsequence \( \{m_{q_j}\}_{j \in J} \) of \( \{m_{q_i}\}_{i \in \mathbb{N}} \) and a polynomial \( \tilde{m}_\infty \in \mathcal{P}^d_{k-1} \) such that \( \lim_{j \to \infty} m_{q_j} = \tilde{m}_\infty \) uniformly on \( Q \).
Therefore
\[
\lim_{j \to \infty} \|g - m_{q_j}\|_{L_{q_j}(Q)} = \|g - \tilde{m}_\infty\|_{L_\infty(Q)}
\]
and we have
\[
\lim_{q \to \infty} E_{kq}(g; Q) = \lim_{j \to \infty} E_{kq_j}(g; Q) = \|g - \tilde{m}_\infty\|_{L_\infty(Q)} \geq E_{k\infty}(g; Q).
\]
Inequalities (3.14) and (3.15) imply the required statement. □

Now if \( f \in L_\infty \), then for each \( q \in [1, \infty) \), \( y \in Q^d \)
\[
E_{kq}(f^0; Q_n(y)) \leq \|f\|_{L_\infty}
\]
so that using Lemma 3.3 and the Lebesgue dominated convergence theorem we derive from (3.9) letting \( q \to \infty \)
\[
E_{k\infty}(f_n; Q) = \lim_{q \to \infty} E_{kq}(f_n; Q) \leq \lim_{q \to \infty} \lambda_n^{-\frac{d}{q}} \int_{\|y\| \leq 1} E_{kq}(f^0; Q_n(y)) \varphi(y) dy
\]
\[
\int_{\|y\| \leq 1} \lim_{q \to \infty} E_{kq}(f^0; Q_n(y)) \varphi(y) dy = \int_{\|y\| \leq 1} E_{k\infty}(f^0; Q_n(y)) \varphi(y) dy.
\]
This proves inequality (3.9) for \( q = \infty \).

Using (3.9) let us prove inequality (3.6).
Let \( \pi \in \Pi \) and \( 1 \leq p, q \leq \infty \). Applying (3.9) and noting that
\[
|Q_n(y)| = \lambda_n^d |Q|, \quad Q \in \pi,
\]
we obtain
\[
\gamma(\pi; f_n) := \left( \sum_{Q \in \pi} (|Q|^{-\lambda} E_{kq}(f_n; Q))^p \right)^{\frac{1}{p}}
\]
(3.16)
\[
\leq \lambda_n^{d(\lambda - \frac{1}{q})} \left( \sum_{Q \in \pi} \left( |Q_n(y)|^{-\lambda} \int_{\|y\| \leq 1} E_{kq}(f_n; Q_n(y)) \varphi(y) dy \right)^p \right)^{\frac{1}{p}}.
\]
Applying to the right-hand side the Minkowski inequality and noting that \( \pi(y) := \{Q_n(y)\}_{Q \in \pi} \) is also a packing we obtain
\[
\gamma(\pi; f_n) \leq \lambda_n^{d\left(\frac{1}{\lambda} - \frac{1}{q}\right)} \int_{\|y\| \leq 1} \gamma(\pi(y); f) \varphi(y) dy
\]
\[
\leq \lambda_n^{d\left(\frac{1}{\lambda} - \frac{1}{q}\right)} \sup_{\pi} \gamma(\pi; f) \int_{\|y\| \leq 1} \varphi(y) dy =: \lambda_n^{d\left(\frac{1}{\lambda} - \frac{1}{q}\right)} |f|_{V_\kappa}.
\]
Thus, we have from here
\[
|f_n|_{V_\kappa} := \sup_{\pi} \gamma(\pi; f_n) \leq \lambda_n^{d\left(\frac{1}{\lambda} - \frac{1}{q}\right)} |f|_{V_\kappa}.
\]
Passing to \( n \to \infty \) and noting that \( \lambda_n := \frac{n}{n+1} \to 1 \) we prove inequality \((3.6)\).

To complete the proof of assertion (a) it remains to show that conversely
\[
(3.17) \quad |f|_{V_\kappa} \leq \lim_{n \to \infty} |f_n|_{V_\kappa}.
\]
This will be done using assertion (b) of the theorem that is proved now.

3.3. Proof of Theorem 2.12 (b). First let \( q < \infty \). Then given \( \varepsilon > 0 \) for every \( f \in \dot{V}_\kappa \subset L_q \) there is a \( C^\infty \) function \( g \) supported by the open cube \((0,1)^d\) such that
\[
(3.18) \quad \|f - v\|_q < \varepsilon
\]
Further, as in \((3.2)\) we define the regularizer \( v_n, n \in \mathbb{N} \), by setting
\[
v_n(x) := \int_{\|y\| \leq 1} v(a_n(x) - \frac{y}{n+1}) \varphi(y) dy, \quad x \in Q^d.
\]
Then we estimate \( f - f_n \) by
\[
(3.19) \quad \|f - f_n\|_q \leq \|f - v\|_q + \|v - v_n\|_q + \|f_n - v_n\|_q.
\]
To estimate the second term we write
\[
(v - v_n)(x) = \int_{\|y\| \leq 1} (v(x) - v(a_n(x) - (n+1)^{-1}y)) \varphi(y) dy.
\]
By the mean value theorem the absolute value of the integral is bounded from above by
\[
c(d)|v|_{C^1(Q^d)} \max\{\|x - a_n(x) - (n+1)^{-1}y\|_\infty : x \in Q^d, \|y\|_\infty \leq 1\}.
\]
Since \( a_n(x) := \frac{n}{n+1}(x - c) + c \), where \( c := c_{Q^d} \), the above maximum is bounded by
\[
\frac{1}{2(n+1)} + \frac{1}{n+1}. \quad \text{Thus}, \quad \|v - v_n\|_q \to 0 \quad \text{as} \quad n \to \infty \quad \text{and}
\]
\[
(3.20) \quad \lim_{n \to \infty} \|f - f_n\|_q \leq \varepsilon + \lim_{n \to \infty} \|f_n - v_n\|_q.
\]
Finally, by the Minkowski inequality following by the change of variables, cf. \((3.12)\), we have
\[
\|f_n - v_n\|_q := \left\{ \int_{Q^d} dx \right\| \|y\|_\infty \leq 1 (f - v) (a_n(x) - (n + 1)^{-1} y) \varphi(y) dy \right\}^{\frac{1}{q}} \leq \lambda_n^{-\frac{d}{q}} \int_{\|y\|_\infty \leq 1} \varphi(y) \left( \int_{\hat{Q}_n^d(y)} |f - v|^q dx \right)^{\frac{1}{q}} dy,
\]
where \(Q_n^d(y) := a_n(Q^d) - (n + 1)^{-1} y \subset Q^d\).

Replacing \(Q_n^d(y)\) by \(Q^d\) and letting \(n\) to \(\infty\) we derive from here and \((3.18)\) the inequality
\[
(3.21) \quad \lim_{n \to \infty} \|f - f_n\|_q \leq 2 \varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
This proves assertion (b) for \(1 < q < \infty\).

Now let \(q = \infty\). We should prove that for every \(f \in \dot{V}_\kappa \subset L_\infty\)
\[
(3.22) \quad \lim_{n \to \infty} \int_{Q^d} (f - f_n) g dx = 0 \quad \text{for each} \quad g \in L_1.
\]

Since the space \(C_0^\infty(Q^d)\) of \(C^\infty\) functions compactly supported by the interior \(\hat{Q}^d\) of \(Q^d\) is dense in \(L_1\) and the sequence \(\{f_n\}_{n \in \mathbb{N}}\) is bounded in \(L_\infty\) by \((3.6)\), it suffices to prove that
\[
\lim_{n \to \infty} \int_{Q^d} f_n g dx = \int_{Q^d} f g dx \quad \text{for each} \quad g \in C_0^\infty(Q^d).
\]
Inserting the integral representation of \(f_n\) given by \((3.3)\) in the left-hand side of the above equation, changing the order of integration and then making change of variables we obtain
\[
(3.23) \quad \lambda_n^{-d} \int_{\|y\|_\infty \leq 1} \varphi(y) \left( \int_{\hat{Q}_n^d(y)} f^0(z) g\left(a_n^*(z) + \frac{y}{n}\right) dz \right) dy;
\]

here \(a_n^*\) is the \(\lambda_n^{-1}\) dilation of \(\mathbb{R}^d\) with respect to \(c = c_{Q^d}\) and \(\hat{Q}_n^d(y)\) is the cube \(a_n^*(Q^d) + \frac{y}{n}\) containing \(Q^d\).

However, \(\hat{Q}_n^d(y)\) in \((3.23)\) can be replaced by \(Q^d\), since \(\text{supp} f^0 = Q^d\).

Thus, we obtain from \((3.23)\)
\[
\int_{Q^d} f_n g dx = \int_{Q^d} f \hat{g}_n dx,
\]
where we set
\[
\hat{g}_n(x) := \lambda_n^{-d} \int_{\|y\|_\infty \leq 1} g\left(a_n^*(x) + \frac{y}{n}\right) \varphi(y) dy.
\]
Replacing here $a_n^*(x)$ by $x$ we obtain a regularizer of $g$ denoted by $g_n := \hat{g}_n \circ a_n$. Since $g \in C_0^\infty(Q^d)$, the sequence $\{g_n\}_{n \in \mathbb{N}}$ converges to $g$ uniformly in $x \in Q^d$.

This implies that
\begin{equation}
\lim_{n \to \infty} \int_{Q^d} f \hat{g}_n \, dx = \int_{Q^d} fg \, dx + \lim_{n \to \infty} \lambda_n^{-d} \int_{Q^d} (\hat{g}_n - g_n) f \, dx.
\end{equation}

The integral in the second summand is bounded by
\[ \|f\|_\infty \max \left\{ \left| g(a_n^*(x) + \frac{y}{n}) - g(x + \frac{y}{n}) \right| : x \in Q^d, \|y\|_\infty \leq 1 \right\} \]
\[ \leq \|f\|_\infty c(d) \max_{x \in Q^d} \|a_n^*(x) - x\|_\infty. \]

Since maximum here tends to 0 and $\lambda_n^{-d} \to 1$ as $n \to \infty$, the second summand in (3.24) is 0.

This proves (3.22) and assertion (b) of the theorem.

3.4. Proof of Theorem 2.12 (a) (conclusion). Now we prove inequality (3.17).

First, let $q < \infty$. Since the function $f \mapsto E_{kq}(f; Q)$, $f \in L_q$, satisfies
\[ |E_{kq}(f; Q) - E_{kq}(f_n; Q)| \leq \|f - f_n\|_q, \]
we conclude from here and assertion (b) of the theorem that
\begin{equation}
\lim_{n \to \infty} E_{kq}(f_n; Q) = E_{kq}(f; Q).
\end{equation}

Now let $\pi \in \Pi$ be a packing. Then (3.25) implies that
\[ \gamma(\pi; f) := \left( \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^p \right)^{\frac{1}{p}} = \lim_{n \to \infty} \gamma(\pi; f_n) \leq \lim_{n \to \infty} |f_n|_{V_\kappa}. \]

Taking here supremum over all $\pi \in \Pi$ we obtain the required inequality
\[ |f|_{V_\kappa} \leq \lim_{n \to \infty} |f_n|_{V_\kappa}. \]

Now let $q = \infty$. Let $J \subset \mathbb{N}$ be an infinite subset such that
\begin{equation}
\lim_{J \ni J_j \to \infty} |f_j|_{V_\kappa} = \lim_{n \to \infty} |f_n|_{V_\kappa}.
\end{equation}

Since for every $Q \subset Q^d$, $j \in J$
\[ E_{k\infty}(f_j; Q) \leq |Q|^\lambda |f_j|_{V_\kappa} \leq \lambda_{j}^{d(\lambda - \frac{1}{q})} |Q|^\lambda |f|_{V_\kappa}, \]
see the inequality before (3.17), polynomials $m_j^Q \in P_{k-1}^d$ satisfying
\[ E_{k\infty}(f_j; Q) = \|f_j - m_j^Q\|_{L_\infty(Q)} \]
are uniformly bounded in \( L_\infty(Q) \), see the proof of Lemma 3.3. Hence, there is an infinite subset \( J^Q \subset J \) and a polynomial \( m^Q \in \mathcal{P}_{k-1}^d \) such that uniformly in \( x \in Q \)

\[
\lim_{J^Q \ni j \to \infty} m^Q_j(x) = m^Q(x).
\]

This and (3.22) imply that the sequence \( \{(f_j - m^Q_j)|_Q\}_{j \in J^Q} \) weak* converges in \( L_\infty(Q) \) to \((f - m^Q)|_Q\).

Since the norm of a dual Banach space is countably lower semicontinuous in the weak* topology, we then have

\[
(3.27) \quad E_{k\infty}(f; Q) \leq \|f - m^Q\|_{L_\infty(Q)} \leq \lim_{J^Q \ni j \to \infty} \|f_j - m^Q_j\|_{L_\infty(Q)} = \lim_{J^Q \ni j \to \infty} E_{k\infty}(f_j; Q).
\]

Now let \( \pi \in \Pi \) be a packing. Using the Cantor diagonal procedure we find an infinite subset \( \tilde{J}^Q \subset \bigcap_{Q \in \pi} J^Q \subset J \) such that (3.27) holds for every \( Q \in \pi \) with \( \tilde{J} \) instead of \( J^Q \).

This and (3.26), in turn, imply that

\[
\left\{ \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{k\infty}(f; Q) \right)^p \right\}^{\frac{1}{p}} \leq \left\{ \sum_{Q \in \pi} \lim_{J^Q \ni j \to \infty} \left( |Q|^{-\lambda} E_{k\infty}(f_j; Q) \right)^p \right\}^{\frac{1}{p}} \leq \lim_{J^Q \ni j \to \infty} \left( |Q|^{-\lambda} E_{k\infty}(f_j; Q) \right)^p \leq \lim_{J^Q \ni j \to \infty} |f_j|_{V_\kappa} = \lim_{n \to \infty} |f_n|_{V_\kappa}.
\]

Taking here supremum over all \( \pi \in \Pi \) we obtain (3.17) for \( q = \infty \) as well. It remains to combine (3.17) and (3.6) to obtain

\[
\lim_{n \to \infty} |f_n|_{V_\kappa} = |f|_{V_\kappa}.
\]

This completes the proof of assertion (a) of Theorem 2.12 for \( s(\kappa) \leq k \), see (3.5). The proof of Theorem 2.12 is complete.

3.5. **Proof of Theorem 2.13** Let \( \hat{V}^0_\kappa \) denote a linear space of functions \( f \in L_q \) satisfying

\[
(3.28) \quad \limsup_{\varepsilon \to 0} \sup_{|\pi| \leq \varepsilon} \left( \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^p \right)^{\frac{1}{p}} = 0,
\]

where \( |\pi| := \sup_{Q \in \pi} |Q| \), and \( V^0_\kappa := \hat{V}^0_\kappa / \mathcal{P}_{k-1}^d \).

We begin with the following assertion.

\( V^0_\kappa \) is a closed subspace of the space \( V_\kappa \).

First we show that \( V^0_\kappa \) is a linear subspace of \( V_\kappa \).
Let $f \in \dot{V}^0_\kappa$, and $\varepsilon_0 > 0$ and $c := c(\varepsilon_0)$ be such that
\begin{equation}
(3.29) \quad \gamma(\pi; f) := \left( \sum_{Q \in \pi} (|Q|^{-\lambda} E_{kq}(f; Q))^p \right)^{\frac{1}{p}} \leq c < \infty
\end{equation}
for every packing $\pi$ with $|\pi| \leq \varepsilon_0$.

Now, an arbitrary $\pi$ is decomposed into packings $\pi_1, \pi_2$ such that $|\pi_1| \leq \varepsilon_0$ and $|Q| > \varepsilon_0$ for every $Q \in \pi_2$.

By the second condition $\text{card } \pi_2 \leq \frac{|Q_0|}{\varepsilon_0} = \varepsilon_{-1}$, hence, we have
\begin{equation}
(3.30) \quad \gamma(\pi; f) \leq \left( \gamma(\pi_1; f)^p + \gamma(\pi_2; f)^p \right)^{\frac{1}{p}} \leq \left( c(\varepsilon_0)^p + \varepsilon_0^{1-\lambda_p} \max_{Q \in \pi_2} E_{kq}(f; Q)^p \right)^{\frac{1}{p}}
\end{equation}
\leq c(\varepsilon_0, \lambda, p)(1 + \|f\|_q).

We conclude that 
\begin{equation}
|f|_{V^0_\kappa} := \sup_{\pi} \gamma(\pi; f) < \infty \quad \text{for every } f \in \dot{V}^0_\kappa,
\end{equation}
i.e., $V^0_\kappa$ is a linear subspace of $V_\kappa$.

It remains to prove closedness of $V^0_\kappa$ in $V_\kappa$. To this end we define a seminorm $T : V_\kappa \to \mathbb{R}_+$ given for $\hat{f} := \{f\} + \mathcal{P}^d_{k-1} \in V_\kappa$ by
\begin{equation}
T(\hat{f}) := \lim_{\varepsilon \to 0} \sup_{|\pi| \leq \varepsilon} \left( \sum_{Q \in \pi} (|Q|^{-\lambda} E_{kq}(f; Q))^p \right)^{\frac{1}{p}}.
\end{equation}
Since $T(\hat{f}) \leq \|\hat{f}\|_{V_\kappa}$ for all $\hat{f} \in V_\kappa$, seminorm $T$ is continuous on $V_\kappa$. This implies closedness of the preimage $T^{-1}(\{0\}) = V^0_\kappa$ in $V_\kappa$.

Further, we prove that under the assumptions for $\kappa := \{\lambda, p, q\}$
\begin{equation}
(3.31) \quad 1 \leq p \leq \infty, \quad 1 \leq q < \infty \quad \text{and} \quad s(\kappa) < k
\end{equation}
the subspaces $V_\kappa$ and $V^0_\kappa$ coincide.

First, we show that $V_\kappa$ is a closed subspace of $V^0_\kappa$. Since $V_\kappa = \text{clos}(C^\infty / \mathcal{P}^d_{k-1}, V_\kappa)$ and $V^0_\kappa$ is closed in $V_\kappa$, it suffices to prove that $C^\infty \subset \dot{V}^0_\kappa$.

To this end we estimate $E_{kq}(f; Q)$ with $f \in C^\infty$ by the Taylor formula as follows
\begin{equation}
E_{kq}(f; Q) \leq c(k, d)|Q|^\frac{k+d}{2} \max_{|\alpha|=k} \max_Q |D^\alpha f|.
\end{equation}
This implies that, see (1.14),

\[
\gamma(\pi; f) := \left( \sum_{Q \in \pi} \left| Q \right|^\lambda E_{k_q}(f; Q) \right)^{\frac{1}{p}} \leq c \left( \sum_{Q \in \pi} |Q|^{-\lambda + \frac{k_q}{q}} \right)^{\frac{1}{p}}.
\]

(3.32)

\[
= c \left( \sum_{Q \in \pi} |Q|^{\frac{k_q}{d} - 1} \right)^{\frac{1}{p}} \leq c \max_{Q \in \pi} |Q|^{\frac{k_q}{d}} \left( \sum_{Q \in \pi} |Q| \right)^{\frac{1}{p}};
\]

where \( c = c(k, d, f) := c(k, d) \max_{|\alpha| = k} \max_Q |D^\alpha f| \).

Since \( s(\kappa) < k \) and the last sum in (3.32) is \( \leq 1 \), we obtain that

\[
\sup_{|\pi| \leq \varepsilon} \gamma(\pi; f) \leq c \varepsilon^{\frac{k_q}{d} - 1} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0,
\]

i.e., \( f \in V^0_\kappa \) as required.

**Remark 3.4.** In the proof of the embedding \( V_\kappa \subset V^0_\kappa \), the restriction \( q < \infty \) is not used.

The converse embedding follows from the next result.

**Lemma 3.5.** Let \( f \in \dot{V}^0_\kappa \). There is a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset C^\infty \) such that

(3.33)

\[
|f - f_n|_{V_\kappa} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

**Proof.** Let us show that the sequence \( \{f_n\}_{n \in \mathbb{N}} \) given by (3.34) is the required one.

Let \( \varepsilon \in (0, 1) \) and \( \pi \in \Pi \); we write \( \pi = \pi_1 \cup \pi_2 \), where \( \pi_1 \) consists of all cubes \( Q \in \pi \) of sidelengths \( < \varepsilon \) and \( \pi_2 := \pi \setminus \pi_1 \). Then

(3.34)

\[
\text{card} \pi_2 \leq \frac{|Q|^d}{\varepsilon^d} = \varepsilon^{-d}.
\]

Now we write using notation of (3.29)

(3.35)

\[
\gamma(\pi; f - f_n) \leq \sum_{i=1}^{2} \gamma(\pi_i; f - f_n)
\]

and estimate each term of the right-hand side.

Since cubes in \( \pi_2 \) have sidelengths \( \geq \varepsilon \) we obtain, as in the derivation of (3.30), the inequality

(3.36)

\[
\gamma(\pi_2; f - f_n) \leq \varepsilon^{-(\lambda + \frac{k_q}{q})d} \|f - f_n\|_q.
\]

Next, we prove that

(3.37)

\[
\gamma(\pi_1; f - f_n) \leq \left( 1 + \lambda_n^{d(\lambda - \frac{1}{q})} \right) \sup_{|\pi| \leq \varepsilon^d} \gamma(\pi; f).
\]

To this end, first we write

\[
\gamma(\pi_1; f - f_n) \leq \gamma(\pi_1; f) + \gamma(\pi_1; f_n)
\]
and note that by the definition of $\pi_1$

\begin{equation}
\gamma(\pi_1; f) \leq \sup_{|x| \leq \varepsilon} \gamma(\pi; f).
\end{equation}

Moreover, by (3.16) following the Minkowski inequality we also have

\begin{equation}
\gamma(\pi_1; f) \leq \lambda_n d(\lambda - \frac{1}{p}) \int_{\|y\| \leq 1} \gamma(\pi_1(y); f) \varphi(y) dy,
\end{equation}

where $\pi_1(y) := \{Q_n(y)\}_{Q \in \pi_1} =: a_n(\pi_1) - (n + 1)^{-1} y$ is also a packing of subcubes of $Q^d$ whose sidelengths $\leq \lambda_n \varepsilon < \varepsilon$. Hence, we conclude that

\begin{equation}
\gamma(\pi_1; f) \leq \lambda_n d(\lambda - \frac{1}{p}) \left( \sup_{|x| \leq \varepsilon} \gamma(\pi; f) \right) \int_{\|y\| \leq 1} \varphi dy = \lambda_n d(\lambda - \frac{1}{p}) \sup_{|x| \leq \varepsilon} \gamma(\pi; f).
\end{equation}

This and (3.38) complete the proof of inequality (3.37).

Now inequalities (3.35)–(3.37) imply

\begin{equation}
\gamma(\pi; f - f_n) \leq \left( 1 + \lambda_n d(\lambda - \frac{1}{p}) \right) \sup_{|x| \leq \varepsilon} \gamma(\pi; f) + \varepsilon^{-d(\lambda + \frac{1}{p})} \|f - f_n\|_q.
\end{equation}

Taking here supremum over all packings $\pi$ and then letting $n$ to $\infty$ we have

\begin{equation}
\sup_{n \to \infty} \|f - f_n\|_{V_\kappa} = \sup_{n \to \infty} \sup_{\pi} \gamma(\pi; f - f_n) \leq 2 \sup_{|x| \leq \varepsilon} \gamma(\pi; f) + \lim_{n \to \infty} \|f - f_n\|_q.
\end{equation}

Since $f \in \hat{V}_\kappa^0$ and $q < \infty$, the first term on the right-hand side tends to 0 as $\varepsilon \to 0$ by the definition of $\hat{V}_\kappa^0$ and the second term tends to 0 as $n \to \infty$ by Theorem 2.12(b).

Thus, (3.33) is proved. \hfill $\square$

The lemma implies that $V_\kappa^0 \subset \text{clos}(C^\infty/P_{k-1}^d; V_\kappa) =: v_\kappa$. Together with the converse embedding this proves coincidence of $v_\kappa$ and $V_\kappa^0$ under conditions (3.34).

To complete the proof of Theorem 2.13 we must show that the spaces $V_\kappa^0$ and $v_\kappa$ coincide for $\kappa := \{\lambda, p, q\}$ satisfying the conditions

\begin{equation}
1 \leq p \leq \infty, \quad \lambda \geq 0 \quad \text{and} \quad s(\kappa) < k.
\end{equation}

To this end, note that $v_\kappa \subset V_\kappa^0$ for $q = \infty$, see Remark 3.4, and inequality (3.39) is proved for all $1 \leq q \leq \infty$. Hence, for $q = \infty$ and $f_n$ defined by (3.4) with $f \in \hat{V}_\kappa^0$ we have

\begin{equation}
\gamma(\pi; f - f_n) \leq 2 \sup_{|x| \leq \varepsilon} \gamma(\pi; f) + \varepsilon^{-d(\lambda + \frac{1}{p})} \|f - f_n\|_\infty.
\end{equation}

Thus, as before the converse embedding $V_\kappa^0 \subset v_\kappa$ will be proved if we show that

\begin{equation}
\lim_{n \to \infty} \|f - f_n\|_\infty = 0.
\end{equation}

As in the proof of Theorem 2.12(b), see Subsection 3.3, this is derived from the following:

**Lemma 3.6.** Under conditions (3.40) for every $f \in \hat{V}_\kappa^0$ and $\eta \in (0, 1)$ there is a function $v \in C^\infty$ such that

\begin{equation}
\|f - v\|_\infty < \eta.
\end{equation}
Proof. Since \( \lambda \geq 0 \), we have for \( f \in \dot{V}_\kappa^0 \)

\[
\sup_{|Q| \leq \varepsilon^d} E_{k\infty}(f; Q) \leq \sup_{|x| \leq \varepsilon^d} \left( \sum_{Q \in \pi} \left( |Q| \lambda^{-1} E_{k\infty}(f; Q) \right)^p \right)^{\frac{1}{p}} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

Moreover, since the \( k \)-th difference \( \Delta^k_h \), see (1.12), annihilates polynomials from \( P_{d-1} \) and is supported by a cube of sidelength at most \( k\|h\|_{\infty} \), the \( k \)-modulus of continuity, see (1.31) for its definition, satisfies

\[
(3.43) \quad \omega_{k\infty}(f; t) \leq 2^k \sup_{|Q| \leq (kt)^d} E_{k\infty}(f; Q) \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.
\]

Due to Marchaud’s inequality estimating \( \omega_1 \) via \( \omega_k \), see, e.g., [DVL-96], this implies that

\[
(3.44) \quad \lim_{t \rightarrow 0} \omega_{1\infty}(f; t) = 0.
\]

To proceed we need the next result.

**Theorem** (Jackson, see, e.g., [DVL-96]). For every \( n \in \mathbb{N} \) there is a linear operator \( T_n : C_\infty(0, 1) \rightarrow P_{4n}^1 \) such that

\[
(3.45) \quad \|f - T_n\|_{C[0,1]} \leq c_0 \sup_{|x-y| \leq n^{-1}} |f(x) - f(y)| := \omega_1(f; n^{-1}),
\]

where \( c_0 \) is a numerical constant.

It is essential that \( T_n \) is an integral operator; hence, (3.45) can be extended to functions \( f \in L_\infty(0, 1) \).

Further, let \( T^i_n f, f \in L_\infty(Q^d) \), be the result of the application of \( T_n \) to the function \( x_i \mapsto f(x), x = (x_1, \ldots, x_d), 0 \leq x_i \leq 1 \). By the Fubini theorem the operators \( T^i_n \), \( 1 \leq i \leq d \), are pairwise commute, hence, \( T_n f := \left( \prod_{i=1}^d T^i_n \right) f \) is a polynomial of degree \( 4n \) in each variable \( x_i \). Moreover, by (3.45)

\[
(3.46) \quad \|f - T_n f\|_{\infty} \leq \sum_{i=1}^d \|f - T^i_n f\|_{\infty} \leq dc_0 \omega_1(f; n^{-1}).
\]

This and (3.44) complete the proof of (3.42). \( \square \)

Now (3.42) gives, cf. (3.21),

\[
\lim_{n \rightarrow \infty} \|f - f_n\|_{\infty} \leq 2\eta
\]

which along with (3.41) implies for \( f \in \dot{V}_\kappa^0 \) that

\[
\lim_{n \rightarrow \infty} |f - f|_{V_\kappa} \leq 2 \sup_{|\pi| \leq \varepsilon^d} \gamma(\pi; f) + \varepsilon^{-d(\lambda + \frac{1}{p})/2}\eta.
\]

Letting here \( \eta \rightarrow 0 \) and then \( \varepsilon \rightarrow 0 \) we conclude that \( f \in \text{clos}(C_\infty, \dot{V}_\kappa) \).

This proves the converse inequality

\[
V_\kappa^0 \subset \text{clos}(C_\infty / P_{d-1}^d, V_\kappa) =: v_\kappa.
\]
and therefore coincidence of $V_0^0$ and $v_\kappa$ under conditions (3.40).

The proof of Theorem 2.13 is complete.

4. Proof of Theorems 2.5

(a) For the set
\[ B_\kappa := \{ b_\pi \in U_\kappa^0 : [b_\pi]_{p'} \leq 1 \} \]
we denote by $\hat{B}_\kappa \subset U_\kappa$ the closure of its convex symmetric hull, i.e.,
\[ \hat{B}_\kappa := \text{clos} \left( \left\{ \sum_\pi \lambda_\pi b_\pi : \sum_\pi |\lambda_\pi| \leq 1, \{b_\pi\} \subset B_\kappa \right\}, U_\kappa \right). \]
Since
\[ \left\| \sum_\pi \lambda_\pi b_\pi \right\|_{U_\kappa} \leq \sum_\pi |\lambda_\pi| [b_\pi]_{p'} \leq 1, \]
see (2.8), we conclude that $\hat{B}_\kappa \subset B(U_\kappa)$. We should prove that these sets coincide.

Let on the contrary $B(U_\kappa) \setminus \hat{B}_\kappa \neq \emptyset$ and $h \in B(U_\kappa) \setminus \hat{B}_\kappa$ is of norm 1. Then by the Hahn-Banach theorem there is a functional $F \in U_\kappa^*$ of norm 1 strictly separating $\hat{B}_\kappa$ and $\{h\}$, that is,
\[ F(h) = 1 \quad \text{and} \quad \sup_{b_\pi \in B_\kappa} |F(b_\pi)| \leq 1 - \varepsilon \]
for some $\varepsilon \in (0,1)$.

Further, $B(U_\kappa^0) = B(U_\kappa) \cap U_\kappa^0$ is dense and $B(U_\kappa) \setminus \hat{B}_\kappa$ is open in $B(U_\kappa)$. Hence, there is a sufficiently close to $h$ element $g \in B(U_\kappa^0) \setminus \hat{B}_\kappa$ such that
\[ \|g\|_{U_\kappa} < 1 \quad \text{and} \quad 1 = F(h) \geq F(g) \geq 1 - \frac{\varepsilon}{2}. \]
By (2.8) there is a representation $g = \sum_\pi b_\pi$ such that $\sum_\pi [b_\pi]_{p'} \leq 1$.

Now let
\[ b_\pi^* := \frac{b_\pi}{[b_\pi]_{p'}}. \]
Using again (2.8) we have $\|b_\pi^*\|_{U_\kappa} \leq 1$; hence, $|F(b_\pi^*)| \leq 1 - \varepsilon$ by (4.2). This and (4.3) then imply
\[ 1 - \frac{\varepsilon}{2} \leq |F(g)| \leq \sum_\pi |b_\pi| [b_\pi]_{p'} |F(b_\pi^*)| \leq 1 - \varepsilon, \]
a contradiction.

Thus, $B(U_\kappa) = \hat{B}_\kappa$.

(b) We should prove separability of the space $(U_\kappa, \| \cdot \|_{U_\kappa})$ under the assumption
\[ 1 < q := q(\kappa) \leq \infty, \quad 1 \leq p := p(\kappa) \leq \infty. \]
To this end we define a closed linear subspace $\hat{L}_{q'}$ of $L_{q'}$ by the conditions
\begin{equation}
\int_{Q^d} f(x)x^\alpha \, dx = 0, \quad |\alpha| \leq k - 1, \quad f \in L_{q'}.
\end{equation}

By our definition, $U^0_{\kappa} \subset \hat{L}_{q'}$. Since $q' < \infty$ by (4.4), the metric space $L_{q'}$ and hence $(U^0_{\kappa}, \| \cdot \|_{q'})$ are separable.

Further, every function $f \in \hat{L}_{q'}$ is a $\kappa$-chain subordinate to the packing $\pi = \{Q^d\}$.

In fact, a function $f = c_{Q^d}a_{Q^d}$, where $c_{Q^d} := \|f\|_{q'}$ and $a_{Q^d} := f/\|f\|_{q'}$, vanishes on $P_{d}^{k-1}$ and, moreover, $\|a_{Q^d}\|_{q'} = 1 (= |Q^d|^{-\lambda})$. Hence, by the definition of the seminorm of $U_{\kappa}$,
\begin{equation}
\|f\|_{U_{\kappa}} \leq |c_{Q^d}| = \|f\|_{q'}.
\end{equation}

In other words, the linear embedding
\begin{equation}
\hat{L}_{q'} \subset U_{\kappa}
\end{equation}
holds with the embedding constant 1. In particular, if $S$ is a dense countable subset of $\hat{L}_{q'}$ with respect to the topology defined by norm $\| \cdot \|_{q'}$, then it is dense with respect to the topology defined by seminorm $\| \cdot \|_{U_{\kappa}}$. Since $U^0_{\kappa}$ is a dense subspace of $U_{\kappa}$, the set $S$ is dense in $U_{\kappa}$ as well.

This completes the proof of part (b) of the theorem.

**Remark 4.1.** The argument of the proof and our definition of $U^0_{\kappa}$ show that under the assumptions of part (b) of the theorem $U^0_{\kappa} = \hat{L}_{q'}$. Thus $U_{\kappa}$ is the completion of $\hat{L}_{q'}$ with respect to the seminorm $\| \cdot \|_{U^0_{\kappa}}$.

(c) We should prove under the assumptions
\begin{equation}
1 < q := q(\kappa) \leq \infty, \quad 1 \leq p := p(\kappa) \leq \infty, \quad s(\kappa) \leq k,
\end{equation}
that the space $U_{\kappa}$ is Banach.

Since $U_{\kappa}$ is the completion of $(U^0_{\kappa}, \| \cdot \|_{U^0_{\kappa}})$, it suffices to prove that $\| \cdot \|_{U^0_{\kappa}}$ is a norm, i.e., that if $\|g\|_{U^0_{\kappa}} = 0$ for some $g \in U^0_{\kappa}$, then $g = 0$. To prove this, we first show that for every $f \in \hat{V}_{\kappa}$ and $g \in U^0_{\kappa}$,
\begin{equation}
\left| \int_{Q^d} fg \, dx \right| \leq \|f\|_{\hat{V}_{\kappa}} \|g\|_{U^0_{\kappa}};
\end{equation}
the integral exists, since $\hat{V}_{\kappa} \subset L_q$ and $U^0_{\kappa} \subset L_{q'}$.

Let $m_{Q} \in \mathcal{P}_{k-1}^d$ be such that
\[ \|f - m_{Q}\|_{L_q(Q)} = E_{kq}(f; Q). \]
Then for $b_\pi := \sum_{Q \in \pi} c_{Q} a_{Q}$ we obtain by the definition of $\kappa$-atoms
\[ \int_{Q^d} f b_\pi \, dx = \sum_{Q \in \pi} c_{Q} \int_{Q} (f - m_{Q}) a_{Q} \, dx. \]
Applying twice the Hölder inequality we derive from here

\[
\left| \int_{Q^d} f b_{\pi} \, dx \right| \leq \left( \sum_{Q \in \pi} |c_Q|^p \right)^{\frac{1}{p'}} \left( \sum_{Q \in \pi} \left( \|f - m_Q\|_{L_q(Q)} \|a_Q\|_{q'} \right)^p \right)^{\frac{1}{p'}}
\]

\[
\leq [b_{\pi}]_{p'} \left( \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^p \right)^{\frac{1}{p'}} \leq [b_{\pi}]_{p'} \|f\|_{V^\kappa}.
\]

Applying this estimate to \( g \in U^0 \kappa \) that can be represented as a finite sum of \( \kappa \)-chains \( b_{\pi} \), and then taking infimum over all such representations we obtain that

\[
\left| \int_{Q^d} f g \, dx \right| \leq \left( \inf_{\pi} \sum_{Q \in \pi} [b_{\pi}]_{p'} \right) \|f\|_{V^\kappa} = \|g\|_{U^\kappa} \|f\|_{V^\kappa},
\]

as required.

Then from (4.9) and the equality \( \|g\|_{U^\kappa} = 0 \) we obtain that

(4.10) \( \int_{Q^d} f g \, dx = 0 \) for all \( f \in \hat{V}^\kappa \).

Next, we show that \( C^\infty := C^\infty(\mathbb{R}^d)|_{Q^d} \subset \hat{V}^\kappa \) (in particular, (4.10) is valid for all \( f \in C^\infty \)).

Let \( \varphi \in C^\infty \). By the Taylor formula we have for \( Q \subset Q^d \),

\[
E_{kq}(\varphi; Q) \leq c(k, d) |Q|^\frac{k}{d} \max_{|\alpha| = k} |D^\alpha \varphi| \leq c(k, d, \varphi) |Q|^\frac{k}{d} + 1.
\]

This implies that

(4.11) \( |\varphi|_{V^\kappa} := \sup_{\pi} \left( \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(\varphi; Q) \right)^p \right)^{\frac{1}{p'}} \leq c(k, d, \varphi) \sup_{\pi} \left( \sum_{Q \in \pi} |Q|^{\frac{k}{d} + 1} \right)^{\frac{1}{p'}} \leq c(k, d, \varphi) |Q|^\frac{k}{d} + 1.
\]

Here the power of \( |Q| \) equals \( \frac{k-s(\kappa)}{d} + 1 \), see (1.1), and by (1.8) \( s(\kappa) \leq k \). Hence, the sum in the right-hand side is bounded from above by \( \left( \sum_{Q \in \pi} |Q| \right)^{\frac{1}{p'}} \leq |Q^d|^{\frac{1}{p'}} = 1 \). Therefore \( |\varphi|_{V^\kappa} < \infty \) for all \( \varphi \in C^\infty \), that is, \( C^\infty \subset \hat{V}^\kappa \).

Hence equality (4.10) is valid for every \( f \in C^\infty \).

Now let first \( q < \infty \), hence, \( C^\infty \) is dense in \( L_q \). Then (4.10) is true also for all \( f \in L_q \) and this implies \( g = 0 \), as required.

Similarly, we derive that \( g = 0 \) for \( q = \infty \) using the following

**Lemma 4.2.** For every \( f \in L_\infty \) there exists a sequence \( \{f_n\} \subset C^\infty \) such that

(4.12) \( \lim_{i \to \infty} \int_{Q^d} (f - f_n) g \, dx = 0 \) for all \( g \in L_1 \).
Proof. Let $\varphi$ be a nonnegative even $C^\infty$ function on $\mathbb{R}^d$ supported by the unit Euclidean ball and the $L_1$-norm 1. Extending $f, g$ by zero to $\mathbb{R}^d$ we define $f_\varepsilon, g_\varepsilon, \varepsilon > 0$, to be convolutions of the extensions (denoted by $f_0, g_0$) with the function $\varphi_\varepsilon : x \mapsto \varepsilon^{-d} \varphi(\frac{x}{\varepsilon})$, $x \in \mathbb{R}^d$. Then $f_\varepsilon, g_\varepsilon \in C^\infty(\mathbb{R}^d)$ and $\|g_0 - g_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$ for every $g \in L_1$. Moreover,

$$\int_{\mathbb{R}^d} (f_0 - f_\varepsilon)g_0 \, dx = \int_{\mathbb{R}^d} f_0(g_0 - g_\varepsilon) \, dx$$

for all $f \in L_\infty$ and $g \in L_1$. The absolute value of the right-hand side is bounded from above by $\|f\|_\infty \|g_0 - g_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$. This clearly implies (4.12). □

Remark 4.3. Since $L_1^* = L_\infty$, this lemma in other terms means that the set $C^\infty$ is dense in $L_\infty$ in the weak* topology.

The proof of part (c) of the theorem is complete.

5. Proof of Theorem 2.6

We prove that under the conditions

$$1 \leq p := p(\kappa) \leq \infty, \quad 1 < q := q(\kappa) \leq \infty, \quad s := s(\kappa) \leq k,$$

the dual to the Banach space $U_\kappa$, see Theorem 2.5, is isometrically isomorphic to the Banach space $V_\kappa$.

Let us recall that the space $V_\kappa = \dot{V}_\kappa/\mathcal{P}_{k-1}^d$, see Definition 1.2 i.e., its elements are factor-classes $\{f\} + \mathcal{P}_{k-1}^d$, where functions $f \in L_q$ satisfy

$$|f|_{\dot{V}_\kappa} := \sup_{\pi \in \Pi(Q^d)} \left\{ \sum_{Q \in \pi} \left( |Q|^{-\lambda} E_{kq}(f; Q) \right)^p \right\}^{\frac{1}{p}} < \infty.$$ 

Since $\mathcal{P}_{k-1}^d$ is the null space of $\dot{V}_\kappa$, the norm of a class $\dot{f} \in V_\kappa$ satisfies

$$\|\dot{f}\|_{\dot{V}_\kappa} := \inf\{|g|_{V_\kappa} : g \in \dot{f}\} = |g|_{V_\kappa}$$

for every $g \in \dot{f}$. Moreover, $\int_{Q^d} fh \, dx = \int_{Q^d} gh \, dx$ for functions $f, g$ of the same class and every $h \in U_\kappa^0$, since $\mathcal{P}_{k-1}^d$ is orthogonal to $U_\kappa^0$. By this reason we will use in the forthcoming proof functions in $\dot{V}_\kappa$ instead of their related classes in $V_\kappa$.

Now we prove the isometry $(U_\kappa)^* \equiv V_\kappa$.

Due to (4.6) and Remark 4.1 the natural embedding $E : \dot{L}_{q'} \hookrightarrow U_\kappa$ is of norm $\leq 1$ and has dense image. Passing to the conjugate map we obtain that

$$E^* : U_\kappa^* \hookrightarrow \dot{L}_{q'}^* \equiv L_q/\mathcal{P}_{k-1}^d$$

is a linear injection of norm $\leq 1$. On the other hand, $V_\kappa$ is contained in $L_q/\mathcal{P}_{k-1}^d$. Let us check that $\text{range}(E^*)$ is in $V_\kappa$ and that the linear map $E^* : U_\kappa^* \to V_\kappa$ is of norm $\leq 1$. 

To this end, for \( \ell \in U^*_k \) we denote by \( f_\ell \in L_q \) an element whose image in \( L_q/P^d_{k-1} \) coincides with \( E^*(\ell) \). Then we take for every \( Q \subset Q^d \) a \( \kappa \)-atom denoted by \( \hat{a}_Q \) such that

\[
(5.2) \quad \int_Q f_\ell \hat{a}_Q \, dx = |Q|^{-\lambda} E_{kq}(f_\ell; Q);
\]

its existence directly follows from Lemma 3.2 and the definition of \( \kappa \)-atoms.

Then for a \( \kappa \)-chain \( \hat{b}_\pi \) given by \( \hat{b}_\pi := \sum_{Q \in \pi} c_Q \hat{a}_Q \) we get from (5.2)

\[
[E^*(\ell)](\hat{b}_\pi) = \int_{Q^d} \hat{b}_\pi f_\ell \, dx = \sum_{Q \in \pi} c_Q |Q|^{-\lambda} E_{kq}(f_\ell; Q).
\]

This, in turn, implies

\[
\sum_{Q \in \pi} c_Q |Q|^{-\lambda} E_{kq}(f_\ell; Q) \leq \|E^*(\ell)\|_{L_q} \|\hat{b}_\pi\|_{U_\kappa} \leq \|\ell\|_{U^*_\kappa} \|\hat{b}_\pi\|_{U_\kappa} \leq \left( \sum_{Q \in \pi} |c_Q| \right)^{\frac{1}{p'}} \|\ell\|_{U^*_\kappa}.
\]

Taking here supremum over all \( (c_Q)_{Q \in \pi} \) of the \( \ell_{p'}(\pi) \) norm 1 and then supremum over all \( \pi \) we conclude that

\[
|f_\ell|_{V_\kappa} := \sup_{\pi} \left( \sum_{Q \in \pi} \left( \frac{E_{kq}(f_\ell; Q)}{|Q|^{\lambda}} \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \leq \|\ell\|_{U^*_\kappa}.
\]

Hence, \( E^*(\ell) \in V_\kappa \) for every \( \ell \in U^*_k \) and \( E^* : U^*_k \hookrightarrow V_\kappa \) is a linear injection of norm \( \leq 1 \).

Next, let us show that there is a linear injection of norm \( \leq 1 \)

\[
(5.3) \quad F : V_\kappa \hookrightarrow U^*_k
\]

such that

\[
(5.4) \quad FE^* = \text{id}|_{U^*_k}.
\]

Actually, let \( f \in V_\kappa \) and \( \ell_f : U^*_k \rightarrow \mathbb{R} \) be a linear functional given for \( g \in U^0_\kappa \) by

\[
(5.5) \quad \ell_f(g) := \int_{Q^d} fg \, dx.
\]

Due to (4.9)

\[
|\ell_f(g)| \leq |f|_{V_\kappa} \|g\|_{U_\kappa}.
\]

Thus, \( \ell_f \) continuously extends to a linear functional from \( U^*_k \) (denoted by the same symbol) and the linear map \( F : V_\kappa \rightarrow U^*_k, \{f\} + P^d_{k-1} \mapsto \ell_f \), is of norm \( \leq 1 \). Moreover, \( F \) is an injection. Indeed, let \( \ell_f = 0 \) for some \( f \in V_\kappa \). Since \( U^0_\kappa = \hat{L}_q \) and \( V_\kappa \subset L_q \), equality (5.5) implies that \( \ell_f|_{U^0_\kappa} \) determines the trivial functional on \( \hat{L}^*_q = L_q/P^d_{k-1} \). Hence, \( f \in P^d_{k-1} \), i.e., \( f \) determines the zero element of \( V_\kappa \), as required.

Further, by the definitions of \( E^* \) and \( F \) we have for each \( h \in U^*_k \) and \( g \in U^0_\kappa \)

\[
[FE^*(h)](g) = \ell_{E^*(h)}(g) = \int_{Q^d} E^*(h)g \, dx = h(E(g)) = h(g).
\]
The proof of (5.3) and (5.4) is complete. In turn, the established results mean that \( \text{range}(E^*) = V_\kappa \), \( \text{range}(F) = U_\kappa^* \) and \( E^* \) are isometries.

Theorem 2.6 is proved.

6. Proof of Theorem 2.7

Let \( p, q, \lambda \in \kappa \) and \( s(\kappa) \) be such that

\[
1 < p \leq \infty, \quad 1 < q < \infty \quad \text{and} \quad s(\kappa) < k.
\]

We prove that under these assumptions the Banach spaces \( V_\kappa^* \) and \( U_\kappa \) are isometrically isomorphic. Along with Theorem 2.6 this directly implies the two stars theorem asserting that \( V_\kappa^{**} \) and \( V_\kappa \) are isometrically isomorphic (see Corollary 2.8).

The proof of the theorem is based on main results of Subsections 6.1 and 6.2: Propositions 6.1, 6.3 and Lemma 6.2. Subsection 6.3 contains the concluding part of the proof.

6.1. In the subsequent text we identify \( U_\kappa \) with its image under the natural embedding \( U_\kappa \hookrightarrow U_\kappa^{**} \). Moreover, identifying \( V_\kappa \) and \( U_\kappa^* \), see Theorem 2.6, we regard \( U_\kappa \) as a linear subspace of \( V_\kappa^* (= (U_\kappa^*)) \).

Further, \( i : v_\kappa \hookrightarrow V_\kappa \) is the natural embedding, cf. (1.10), and \( i^* : V_\kappa^* \to v_\kappa^* \) is its adjoint.

Proposition 6.1. (a) \( i^* : V_\kappa^* \to v_\kappa^* \) is a surjective linear map of norm one such that \( i^*|_{U_\kappa} : U_\kappa \to v_\kappa^* \) is an isometry.

(b) The image \( i^*(B(U_\kappa)) \) of the closed unit ball of \( U_\kappa \) is a dense subset of the closed unit ball \( B(v_\kappa^*) \) in the weak* topology of \( V_\kappa^* \).

Proof. (a) We need the following

Lemma 6.2. The subspace \( v_\kappa \) is weak* dense in the space \( V_\kappa (= U_\kappa^*) \).

Proof. It suffices for each \( f \in V_\kappa \) to find a bounded sequence \( \{f_n\}_{n \in \mathbb{N}} \subset v_\kappa \) such that

\[
\lim_{n \to \infty} (f - f_n)(u) = 0 \quad \text{for all} \quad u \in U_\kappa.
\]

Let \( \tilde{f} \in \tilde{V}_\kappa \) be such that \( f = \{\tilde{f}\} + \mathcal{P}^d_{k-1} \in V_\kappa (= \tilde{V}_\kappa / \mathcal{P}^d_{k-1}) \). We choose \( \tilde{f}_n \in C^\infty \subset \tilde{V}_\kappa \) to be the approximation of \( \tilde{f} \) given by Theorem 2.12 and then define

\[
f_n := \{\tilde{f}_n\} + \mathcal{P}^d_{k-1} \in v_\kappa.
\]

Applying to \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \) and \( \tilde{f} \in V_\kappa \) inequality (2.15) we obtain

\[
\lim_{n \to \infty} |\tilde{f}_n|_{V_\kappa} = |\tilde{f}|_{V_\kappa}.
\]

This gives boundedness of the sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( v_\kappa \).

Since \( U_\kappa^0 \) is dense in \( U_\kappa \), the latter implies that it suffices to prove (6.2) for \( u \) being a \( \kappa \)-atom, say \( a_Q \).

In this case, we have for any polynomial \( m \in \mathcal{P}^d_{k-1} \)

\[
(f - f_n)(a_Q) = \int_Q (\tilde{f} - \tilde{f}_n)a_Q \, dx = \int_Q (\tilde{f} - \tilde{f}_n - m)a_Q \, dx.
\]
We choose \( m \) here such that
\[
E_{kq}(\tilde{f} - \tilde{f}_n; Q) = \| \tilde{f} - \tilde{f}_n - m \|_{L_q(\Omega)}
\]
and estimate the integral in \( (6.4) \) by the Hölder inequality. This gives
\[
\left| \int_Q (\tilde{f} - \tilde{f}_n)a_Q \, dx \right| \leq |Q|^{-\lambda} E_{kq}(\tilde{f} - \tilde{f}_n; Q) \leq |Q|^{-\lambda} \| \tilde{f} - \tilde{f}_n \|_q \to 0 \quad \text{as} \quad n \to \infty
\]
by Theorem 2.12(b).

Hence, we conclude that for the sequence \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \subset C^\infty \) and every \( \kappa \)-atom \( a_Q \)
\[
(f - f_n)(a_Q) = \int_Q (\tilde{f} - \tilde{f}_n)a_Q \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]
This completes the proof of the lemma.

Now we finish the proof of assertion (a).
By definition, \( i^* \) maps \( V_{\kappa}^* \) linearly to \( V_{\kappa}^* \) by
\[
i^*(f^*) := f^*|_{V_{\kappa}}, \quad f^* \in V_{\kappa}^*.
\]
Moreover, every \( f^* \in V_{\kappa} \) by the Hahn-Banach theorem is extended to some element of \( V_{\kappa}^* \) with the same norm. Hence, \( i^* \) is a linear surjection of norm one.

To prove that \( i^*|_{U_{\kappa}} \) is an isometry, we have to show that \( \|i^*(v)\|_{V_{\kappa}^*} = \|v\|_{U_{\kappa}} \) for all \( v \in U_{\kappa} \).

In fact, let \( u \in U_{\kappa} \setminus \{0\} \). By the Hahn-Banach theorem there exists \( f \in V_{\kappa} \) such that \( \|f\|_{V_{\kappa}} = 1 \) and \( f(u) = \|u\|_{U_{\kappa}} \). By Lemma 6.2 and 6.3 there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset V_{\kappa} \) weak* converging to \( f \) such that
\[
\lim_{n \to \infty} \|f_n\|_{V_{\kappa}} = \|f\|_{V_{\kappa}} = 1.
\]
These imply that
\[
\|v\|_{U_{\kappa}} = |f(v)| = \lim_{n \to \infty} |f_n(v)| \leq \sup_{g \in B(V_{\kappa})} |g(v)| \leq \sup_{h \in B(V_{\kappa})} |f(v)| \leq \|v\|_{U_{\kappa}}.
\]
Hence,
\[
\|v\|_{U_{\kappa}} = \sup_{g \in B(V_{\kappa})} |g(v)| = \sup_{g \in B(V_{\kappa})} |(i^*(v))(g)| := \|i^*(v)\|_{V_{\kappa}^*},
\]
as required.

This proves that \( i^*|_{U_{\kappa}} \) is an isometry and completes the proof of assertion (a) of the proposition.

(b) By the Goldstine theorem, see, e.g., [DSch-58, Thm. 5.5.1], \( B(U_{\kappa}) \) is weak* dense in \( B(U_{\kappa}^*) (= B(V_{\kappa}^*)) \). Since \( i^* \) is a bounded surjective linear map of norm one, \( i^*(B(V_{\kappa}^*)) \) coincides with the closed unit ball \( B(V_{\kappa}^*) \) of \( V_{\kappa}^* \). Moreover, \( i^* \) is weak* continuous; hence, density of \( B(U_{\kappa}) \) in \( B(V_{\kappa}^*) \) implies that the weak* closure of \( i^*(B(U_{\kappa})) \) coincides with \( B(V_{\kappa}^*) \).

Proposition 6.1 is proved. 

6.2. In the next result, $\mathcal{A}_\kappa$ denotes the set of $\kappa$-atoms and $\overline{\mathcal{B}}_\kappa$ the closure in $U_\kappa$ of the set of $\kappa$-chains $\mathcal{B}_\kappa := \{b_\pi \in U_\kappa^0 : [b_\pi]_{p'} \leq 1\}$, see Definitions 2.2 and 2.3.

**Proposition 6.3.** (a) If $1 < p := p(\kappa) < \infty$, then $i^*(\overline{\mathcal{B}}_\kappa)$ is a subset of $B(V^*_\kappa)$ compact in the weak* topology of $V^*_\kappa$.

(b) If $1 \leq p := p(\kappa) \leq \infty$, then the same is true for the set $i^*(\mathcal{A}_\kappa)$.

**Proof.** (a) Since for $b_\pi \in \mathcal{B}_\kappa$
\[\|b_\pi\|_{U_\kappa^0} \leq [b_\pi]_{p'} \leq 1,\]
$\mathcal{B}_\kappa \subset B(U_\kappa)$. Since $i^*$ maps $B(U_\kappa)$ in $B(V^*_\kappa)$, cf. Proposition 6.1(b), this gives the embedding
\[i^*(\mathcal{B}_\kappa) \subset B(V^*_\kappa).\]

Further, by the Banach-Alaoglu theorem $B(V^*_\kappa)$ is compact in the weak* topology of $V^*_\kappa$. Moreover, by separability of $V_\kappa$ the ball $B(V^*_\kappa)$ equipped with this topology is metrizable. Hence, to establish assertion (a) it suffices to prove that the limit of every sequence of $i^*(\mathcal{B}_\kappa)$ converging in the weak* topology of $B(V^*_\kappa)$ belongs to $i^*(\mathcal{B}_\kappa)$. In turn, since $\mathcal{B}_\kappa$ is dense in $\overline{\mathcal{B}}_\kappa$ in the norm topology of $U_\kappa (\subset V^*_\kappa)$ and $\|i^*\|_{V^*_\kappa \rightarrow V^*_\kappa} = 1$, it suffices to prove this statement for sequences from the set $i^*(\overline{\mathcal{B}}_\kappa)$.

Hence, we should prove the following:

**Statement 6.4.** If $\{b^n\}_{n \in \mathbb{N}} \subset \mathcal{B}_\kappa$ is such that the sequence $\{i^*(b^n)\}_{n \in \mathbb{N}}$ weak* converges in $B(V^*_\kappa)$, then its limit belongs to $i^*(\overline{\mathcal{B}}_\kappa)$.

**Proof.** Let $b^n$ has the form
\[b^n := \sum_{i=1}^{N(n)} c^n_i a Q^n_i, \quad n \in \mathbb{N},\]
where $\pi_n := \{Q^n_i : 1 \leq i \leq N(n)\}$ is a packing.

Without loss of generality we assume that
\[|Q^n_i| \leq |Q^n_{i+1}|, \quad 1 \leq i < N(n).\]

Further, we extend sequences $\pi_n$, $\{c^n_i\}$ and $\{a Q^n_i\}$ by setting
\[Q^n_i := \{0\}, \quad c^n_i := 0, \quad a Q^n_i := 0 \quad \text{for} \quad i > N(n).\]

Hence, we write
\[b^n := \sum_{i=1}^{\infty} c^n_i a Q^n_i, \quad n \in \mathbb{N}.\]

Since Statement 6.4 suffices to prove for any infinite subsequence of $\{b^n\}$, we use several times the Cantor diagonal method to construct in the next lemma a suitable for the consequent proof subsequence.

**Lemma 6.5.** There is an infinite subsequence $\{b^n\}_{n \in J}, \ J \subset \mathbb{N}$, such that for every $i \in \mathbb{N}$ the following is true:
Since by the definition of a $\kappa$-atom

(a) $\{Q^n_i\}_{n \in J}$ converges in the Hausdorff metric to a closed subcube of $Q^d$ denoted by $Q_i$;

(b) $\{i^*(a^n_{Q^n_i})\}_{n \in \mathbb{N}} \subset B(V^*_\kappa)$ converges in the weak$^*$ topology of $B(V^*_\kappa)$;

(c) if the limiting cube $Q_i$ has a nonempty interior, then the sequence $\{a^n_{Q^n_i}\}_{n \in J}$ converges in the weak topology of $L_q$ (regarded as the dual space of $L_q$, $q \in (1, \infty)$);

(d) the sequence $\{c^n := (c^n_i)_{i \in \mathbb{N}}\}_{n \in J}$ of vectors from $B(\ell_p'(\mathbb{N}))$ converges in the weak topology of $\ell_p'(\mathbb{N}) = \ell_p(\mathbb{N})^*$, $p \in (1, \infty)$) to a vector denoted by $c \in B(\ell_p'(\mathbb{N}))$.

Proof. (a) Parameterizing the set of closed subcubes of $Q^d$ by their centers and radii and using the Bolzano-Weierstrass theorem we conclude that $\{Q^n_{1_i}\}_{n \in \mathbb{N}}$ contains a converging in the Hausdorff metric subsequence, say, $\{Q^n_{1_i}\}_{n \in J_1}$. In turn, $\{Q^n_{2_i}\}_{n \in J_1}$ contains a converging in this metric subsequence, say, $\{Q^n_{2_i}\}_{n \in J_2}, J_2 \subset J_1$, etc. Setting then $n_i := \min J_i$ and $J^a := \{n_i\}_{i \in \mathbb{N}}$, we obtain the required subsequence $\{Q^n_{1_i}\}_{n \in J^a}$ converging to some closed cube $Q_i \subset Q^d$, $i \in \mathbb{N}$.

(b) Since $\|a^n_{Q^n_i}\|_{L_q} \leq 1$ for all $i, n \in \mathbb{N}$, the sequences $\{i^*(a^n_{Q^n_i})\}_{n \in J^a} \subset B(V^*_\kappa)$, $i \in \mathbb{N}$, while this ball is compact in the (metrizable) weak$^*$ topology of $V^*_\kappa$. Hence, these sequences contain converging in the weak$^*$ topology subsequences and therefore applying as in (a) the Cantor diagonal process to sequences $\{i^*(a^n_{Q^n_i})\}_{n \in J^a} \subset B(V^*_\kappa)$, $i \in \mathbb{N}$, we find an infinite subset $J^b \subset J^a$ such that each sequence $\{i^*(a^n_{Q^n_i})\}_{n \in J^b}, i \in \mathbb{N}$, converges in the weak$^*$ topology of $B(V^*_\kappa)$.

Hence, the subsequence $\{b^n\}_{n \in J^b}$ of $\{b^n\}_{n \in \mathbb{N}}$ satisfies conditions (a) and (b) of the lemma.

(c) Now, let $I \subset \mathbb{N}$ be such that for each $i \in I$,

$$\lim_{J^b \ni n \to \infty} |Q^n_{1_i}| = |Q_i| > 0.$$ 

Since by the definition of a $\kappa$-atom

$$\lim_{J^b \ni n \to \infty} \|a^n_{Q^n_i}\|_{L_q} \leq \lim_{J^b \ni n \to \infty} |Q^n_{1_i}|^{-\lambda} = |Q_i|^{-\lambda} < \infty,$$

each sequence $\{a^n_{Q^n_i}\}_{n \in J^b}, i \in I$, is bounded in the reflexive (as $1 < q < \infty$) space $L_q$. By the Banach-Alaoglu theorem each such a sequence contains a weak converging in $L_q$ subsequence. Applying then the Cantor diagonal process to the family of sequences $\{a^n_{Q^n_i}\}_{n \in J^b}, i \in I$, we find an infinite subset $J^c \subset J^b$ such that all sequences $\{a^n_{Q^n_i}\}_{n \in J^c}, i \in I$, converge in the weak topology of $L_q$.

Thus the subsequence $\{b^n\}_{n \in J^c}$ of $\{b^n\}_{n \in \mathbb{N}}$ satisfies conditions (a)–(c) of the lemma.

(d) Since $\|c^n\|_{\ell'_p} := \|(c^n_i)_{i \in \mathbb{N}}\|_{\ell'_p} = |b^n|_{\ell'_p} \leq 1$, $n \in \mathbb{N}$, see (6.3), the sequence $\{c^n\}_{n \in J^c} \subset B(\ell_p'(\mathbb{N}))$. Moreover, $\ell_p'$ is reflexive as $1 < p < \infty$ and therefore by the Banach-Alaoglu theorem there exists an infinite subset $J^d \subset J^c$ such that the sequence $\{c^n\}_{n \in J^d}$ weak converges to a vector, say, $c \in B(\ell_p'(\mathbb{N}))$.

We set $J := J^d$. Then the subsequence $\{b^n\}_{n \in J}$ of $\{b^n\}_{n \in \mathbb{N}}$ satisfies the required conditions (a)–(d).
Thus, from now on without loss of generality we assume that the sequence \( \{b^n\} \subset B_\kappa \) of \( \kappa \)-chains, see (6.9), satisfies the assertions of Lemma 6.5. In particular, there are closed cubes \( Q_i \subset Q^d, i \in \mathbb{N} \), such that in the Hausdorff metric
\[
Q_i = \lim_{n \to \infty} Q^n_i.
\]
Since for each \( n \in \mathbb{N} \) the cubes \( Q^n_i, i \in \mathbb{N} \), are nonoverlapping and their volumes form a nonincreasing sequence, see (6.7), the same is true for the family of cubes \( \{Q_i\}_{i \in \mathbb{N}} \). Thus, for every \( i \in \mathbb{N} \)
\[
Q_i \cap \hat{Q}_{i+1} = \emptyset \quad \text{and} \quad |Q_i| \geq |Q_{i+1}|.
\]
Here \( \hat{S} \) stands for the interior of \( S \subset \mathbb{R}^d \).

Now we let \( N = \infty \) if \( |Q_i| \neq 0 \) for all \( i \in \mathbb{N} \), otherwise, \( N \) be the minimal element of the set of integers \( n \in \mathbb{Z}_+ \) such that
\[
|Q_i| = 0 \quad \text{for} \quad i > n.
\]
Then due to our assumptions, see Lemma 6.5 (c), for \( N \neq 0 \) there are functions \( a_i \in L_{q'} \), \( 1 \leq i < N + 1 \), such that in the weak topology of \( L_{q'} \)
\[
a_i = \lim_{n \to \infty} a^n_{Q_i}.
\]
The properties of these functions are presented in the next result.

**Lemma 6.6.** (1) If \( N \neq \infty \) and \( i > N \), then in the weak* topology of \( B(v^*_\kappa) \)
\[
\lim_{n \to \infty} \hat{a}^n_{Q^n_i}(v) = 0 \quad \text{for every} \quad v \in v_\kappa.
\]
(2) If \( N \neq 0 \) and \( 1 \leq i < N + 1 \), then the function \( a_i \) is a \( \kappa \)-atom subordinate to \( Q_i \).

**Proof.** (1) We have to prove that for each \( i > N \)
\[
\lim_{n \to \infty} \hat{a}^n_{Q^n_i}(v) = 0 \quad \text{for every} \quad v \in v_\kappa,
\]
where \( \hat{a}^n_{Q^n_i} \) is the image of \( a^n_{Q^n_i} \) under the natural embedding \( U_\kappa \hookrightarrow U_\kappa^{**} = V_\kappa^* \), see (6.5). Since \( C^\infty / P^d_{k-1} \) is dense in \( v_\kappa \), we can take \( v \in C^\infty / P^d_{k-1} \) in which case
\[
|\hat{a}^n_{Q^n_i}(v)| = \left| \int_{Q^n_i} \hat{v} a^n_{Q^n_i} dx \right| \leq \|a^n_{Q^n_i}\|_{L_{q'}(Q^n_i)} E_{kq}(\hat{v}; Q^n_i),
\]
here \( \hat{v} \in C^\infty \) is a representative of the factor-class \( v \in C^\infty / P^d_{k-1} \).

Due to Definition 2.2 and the Taylor formula the right-hand side is bounded from above by
\[
|Q^n_i|^{-\lambda} c(k, d)|\hat{v}|_{W_k^q(Q^d)}|Q^n_i|^k + \frac{1}{q} = c(k, d, \hat{v})|Q^n_i|^{-\lambda + \frac{k}{q} + \frac{1}{q}}.
\]
By the definition of \( s(\kappa) \), see (1.14), and (6.1)
\[
-k + \frac{1}{q} + \frac{k}{d} \geq \left( \lambda - \frac{1}{q} + \frac{1}{p} \right) = \frac{k-s(\kappa)}{d} > 0.
\]
Hence, since $|Q^n_i| \to 0$ as $n \to \infty$ for each $i > N$, the right-hand side in (6.16) tends to 0 as $n$ tends to $\infty$.

This proves (6.14).

(2) For $N \neq 0$ and each $1 \leq i < N + 1$ the sequence $\{Q^n_i\}_{n \in \mathbb{N}}$ converges to a compact cube $Q_i \subset Q^d$ with $|Q_i| > 0$ and the sequence $\{a_{Q^n_i}\}_{n \in \mathbb{N}}$ converges to $a_i \in L_{q'}$ in the weak topology of $L_{q'}$, see Lemma 6.2. Hence, by the Fatou lemma we have

$$\|a_i\|_{q'} \leq \lim_{n \to \infty} \|a_{Q^n_i}\|_{q'} \leq \lim_{n \to \infty} |Q^n_i|^{-\lambda} = |Q_i|^{-\lambda}.$$ 

In other words, $a_i$ satisfies the inequality

$$\|a_i\|_{q'} \leq |Q_i|^{-\lambda}.$$ 

It remains to prove that

(6.17) $\supp a_i \subset Q_i$ and $a_i \perp P_{k-1}^d$.

If, on the contrary, $|\supp a_i \setminus Q_i| > 0$, then there is a nontrivial closed cube $\hat{Q} \subset Q^d \setminus Q_i$ such that

(6.18) $\int_Q |a_i|^q dx > 0$.

However, $Q^n_i \to Q_i$ in the Hausdorff metric as $n \to \infty$ and therefore $\hat{Q} \cap Q = \emptyset$ for all sufficiently large $n$. This and condition (c) of Lemma 6.2 imply that

$$0 = \lim_{n \to \infty} \int_{Q^d} (f \cdot 1_Q)a_{Q^n_i} dx = \int_{Q^d} f a_i dx;$$

here $f \in L_q$ and $1_S$ stands for the indicator of a set $S \subset \mathbb{R}^d$.

Since $f$ is arbitrary, $a_i|_Q$ is zero in $L_{q'}(Q)$ in contradiction to (6.18).

To prove the second assertion of (6.17) we use the fact that $a_{Q^n_i} \perp P_{k-1}^d$ for all $i$ and $n$. Hence, due to Lemma 6.2(c) for every polynomial $m \in P_{k-1}^d$

$$0 = \lim_{n \to \infty} \int_{Q^d} ma_{Q^n_i} dx = \int_{Q^d} ma_i dx.$$

Thus, $a_i$ is a $\kappa$-atom subordinate to $Q_i$ (in the sequel denoted by $a_{Q_i}$).

Now we show that for $1 \leq N < \infty$

(6.19) $v_N := \sum_{i=1}^N c_i a_{Q_i} \in \mathcal{B}_\kappa$.

In fact, by Lemma 6.6(2) and (6.11) $a_{Q_i}$ are $\kappa$-atoms and $\{Q_i\}_{1 \leq i \leq N}$ is a packing. Moreover, by Lemma 6.5(d) the $\kappa$-atom $v_N$ satisfies

$$[v_N]_{\mu'} = \|c\|_{\mu'} \leq 1,$$

as required.
Further, for $N = \infty$

$$v_\infty := \sum_{i=1}^{\infty} c_i a_{Q_i} \in \bar{B}_\kappa. \quad (6.20)$$

Indeed, by Lemmas 6.5(d) and 6.6(2)

$$\left\| \sum_{i=\ell}^{m} c_i a_{Q_i} \right\|_{U_\kappa} \leq \left( \sum_{i=\ell}^{m} |c_i|^{p'} \right)^{\frac{1}{p'}} \to 0$$

as $\ell, m \to \infty$, i.e., the series in (6.20) converges in $U_\kappa$. Moreover, its partial sums belong to $\bar{B}_\kappa$, cf. (6.19), hence, $v_\infty$ belongs to the closure of $\bar{B}_\kappa$.

Setting for $N = 0$

$$v_0 := 0 \quad (6.21)$$

we complete the proof of Statement 6.4 by showing that in the weak* topology of $B(v_\kappa^*)$

$$\lim_{n \to \infty} i^*(b^n) = i^*(v_N). \quad (6.22)$$

As in Lemma 6.6(1) it suffices to prove that for every $f \in C^\infty/P_{d}^{k-1}$

$$\lim_{n \to \infty} f(b^n) = f(v_N). \quad (6.23)$$

To prove (6.23), we fix $\varepsilon \in (0, 1)$ and denote by $\pi^n \subset \{Q^n_i\}_{i \in \mathbb{N}}$ the packing containing all cubes of nonzero volumes. Further, we represent $\pi^n$ as the union of two packings (one of which is possibly empty)

$$\pi_1^n := \{Q^n_i \in \pi^n : |Q^n_i| \leq \varepsilon\} \quad \text{and} \quad \pi_2^n := \pi^n \setminus \pi_1^n. \quad (6.24)$$

If $\pi_1^n \neq \emptyset$, then due to inequality (3.32) for every $S \subset \pi_1^n$ and $f \in C^\infty/P_{d}^{k-1}$

$$\left| \sum_{Q^n_i \in S} c^n_i f(a_{Q^n_i}) \right| \leq c(k, d, f)\varepsilon^{\frac{k-\alpha(s)}{d}}. \quad (6.25)$$

Also, if $\pi_2^n \neq \emptyset$, then for some natural number $\ell_\varepsilon(n)$ we have

$$\pi_2^n = \{Q_1^n, \ldots, Q_{\ell_\varepsilon(n)}^n\}, \quad \text{see (6.7)}; \quad \text{moreover, comparing volumes of } Q^d \text{ and of the union of cubes of } \pi_2^n \text{ we have}$$

$$\ell_\varepsilon(n) \leq \frac{1}{\varepsilon}, \quad n \in \mathbb{N}. \quad (6.26)$$

For $\pi_2^n = \emptyset$ we write

$$\ell_\varepsilon(n) := 0. \quad (6.27)$$

Now, we set

$$\ell_\varepsilon := \lim_{n \to \infty} \min\{\ell_\varepsilon(n), N\} \in [0, \min\{\frac{1}{\varepsilon}, N\}]$$

and choose an infinite subsequence

$$J := \{n \in \mathbb{N} : \ell_\varepsilon(n) = \ell_\varepsilon\}.$$
Next, if $N \neq 0$, then by the definition of $v_N$, see (6.19)–(6.21) there is some $\ell := \ell(f, \varepsilon) \in \mathbb{N}$ satisfying $\ell \varepsilon \leq \ell < N + 1$ such that

\begin{equation}
|f(v_N) - f\left(\sum_{i=1}^{\ell} c_i a_{Q_i}\right)| < \varepsilon.
\end{equation}

In this case, for $n \in \mathbb{N}$ and $m \in J$ we have

\begin{equation}
|f(v_N) - f(b^n)| \leq |f(v_N) - f(b^m)| + |f(b^m) - f(b^n)|
\end{equation}

\begin{equation}
\leq \left|f\left(v_N - \sum_{i=1}^{\ell} c_i a_{Q_i}\right)\right| + \left|\sum_{i=1}^{\ell} (c_i - c_i^m) f(a_{Q_i^m})\right| + \left|\sum_{i=1}^{\ell} c_i f(a_{Q_i^m} - a_{Q_i})\right|
\end{equation}

\begin{equation}
+ \left|f\left(b^m - \sum_{i=1}^{\ell} c_i^m a_{Q_i^m}\right)\right| + |f(b^m) - f(b^n)| =: I + II_m + III_m + IV_m + V_{m,n}.
\end{equation}

Here $I < \varepsilon$ due to (6.27), $II_m$ and $III_m$ tend to 0 as $m \to \infty$, since by Lemma 6.5(b),(d)

\[ \lim_{n \to \infty} f(a_{Q_i^m} - a_i) = 0 \quad \text{and} \quad \lim_{n \to \infty} (c_i - c_i^n) = 0, \]

and, moreover, due to (3.32)

\[ \sup_i |f(a_{Q_i^m})| \leq \gamma(k, d, f) \sup_i |Q_i^m|^{\kappa - \delta(k)} < \infty. \]

Further, $IV_m < c(k, d, f) \varepsilon^{\kappa - \delta(k)}$ by (6.24) because $b^m - \sum_{i=1}^{\ell} c_i^m a_{Q_i^m} = \sum_{i=\ell+1}^{\infty} c_i a_{Q_i^m}$ and all cubes $Q_i^m$ with $i > \ell$ belong to $\pi_1^n$.

Finally, $V_{m,n} \to 0$ as $m, n \to \infty$ because due to the condition of Statement 6.4 the sequence $\{f(b^n)\}_{n \in \mathbb{N}}$ converges.

Applying these facts to (6.28) we obtain that there is some $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$

\[ |f(v_N) - f(b^n)| < 2\varepsilon + c(d, k, f) \varepsilon^{\kappa - \delta(k) - \delta(k)/d}. \]

This implies that

\[ \lim_{n \to \infty} f(b^n) = f(v_N) \]

and completes the proof of Statement 6.4 for $N \neq 0$, see (6.22), (6.23).

Finally, if $N = 0$, then $v_N = 0$ and for all sufficiently large $n \in \mathbb{N}$ the packing $\pi_2^n = \emptyset$. Thus, for such $n$ we have by (6.24)

\[ |f(v_N) - f(b^n)| \leq c(d, k, f) \varepsilon^{\kappa - \delta(k) - \delta(k)/d}, \]

which implies that $\lim_{n \to \infty} f(b^n) = 0 = f(v_N)$ and completes the proof of Statement 6.4 in this case as well.

The proof of part (a) of Proposition 6.3 is complete. Hence, if $1 < p := p(\kappa) < \infty$, then the injection $t^*|_{U_n} : U_n \to v^*_n$ maps the set $B_n$ in a subset of $B(v_n^*)$ compact in the weak* topology of $v_n^*$. 
In this case, we should prove compactness in the weak∗ topology of $v^*_κ$ of the image under $i^*$ of the set of $κ$-atoms $A_κ ⊂ U^0_κ$. Similarly to Statement [6.44] this is equivalent to the following statement:

If $\{b^n\}_{n ∈ N} ⊂ A_κ$ is such that the sequence $\{i^*(b^n)\}_{n ∈ N}$ weak∗ converges in $B(v^*_κ)$, then its limit belongs to $i^*(A_κ)$.

As before, we may assume without loss of generality that $\{b^n\}_{n ∈ N}$ satisfies conditions (a), (b), (c) of Lemma 6.5. Moreover, condition (d) of the lemma is trivially fulfilled for all $1 ≤ p' ≤ ∞$. Therefore as in the proof of part (a) of the proposition the sequence $\{i^*(b^n)\}_{n ∈ N}$ weak∗ converges to $i^*(v_N)$, see (6.19)–(6.21). Since here $N ⊂ \{0, 1\}$, $v_N ∈ A_κ$.

The proof of the proposition is complete. □

6.3. Now, we complete the proof of Theorem 2.7.

First, we consider the case of $p := p(κ) ∈ (1, ∞)$.

We begin with the following:

**Lemma 6.7.** The space $v_κ$ isometrically embeds in the space $C(i^*(\bar{B}_κ))$ of continuous functions on the (metrizable) compact space $i^*(\bar{B}_κ)$.

**Proof.** Due to Theorem 2.5 (a) the symmetric convex hull of $B_κ$ denoted by $sc(B_κ)$ is dense in the closed unit ball $B(U_κ)$ in the norm topology; hence the same is true for $sc(\bar{B}_κ)$. Moreover, the image $i^*(B(U_κ))$ of this ball is dense in $B(v^*_κ)$ in the weak∗ topology of the latter closed ball. Hence, the set

\[
i^*(sc(\bar{B}_κ)) = sc(i^*(\bar{B}_κ))
\]

is weak∗ dense in $B(v^*_κ)$.

This implies for every element $v ∈ v_κ$ regarded as a bounded linear functional on $v^*_κ$ the equality

\[
\|v\|_{v_κ} = \sup_{v^* ∈ i^*(\bar{B}_κ)} |v(v^*)|.
\]

In fact, by the Hahn-Banach theorem

\[
\|v\|_{v_κ} = \sup_{v^* ∈ B(v^*_κ)} |v(v^*)|.
\]

Since every such $v$ is continuous in the weak∗ topology of $v^*_κ$, we can replace $B(v^*_κ)$ by its weak∗ dense subset $sc(i^*(\bar{B}_κ))$. In turn, the latter set can be replaced by the smaller set $i^*(\bar{B}_κ)$ because, by definition,

\[
i^*(\bar{B}_κ) ⊂ sc(i^*(\bar{B}_κ)) := \left\{ \sum_i \lambda_i v^*_i : \{v^*_i\} ⊂ i^*(\bar{B}_κ), \{\lambda_i\} ⊂ \mathbb{R}, \sum_i |\lambda_i| ≤ 1 \right\}
\]

and the supremum of $|v(v^*)|$ over the latter set is bounded from above by

\[
\sup \left( \sum_i |\lambda_i| \max_i |v(v^*_i)| \right) ≤ \max_{v^* ∈ i^*(\bar{B}_κ)} |v(v^*)|,
\]

as required.
Finally, since $v|_{i^*(B_κ)}$ is a continuous function on $i^*(B_κ)$ in the weak* topology induced from $B(v_κ^*)$ and its supremum norm equals $\|v\|_{v_κ}$, see (6.30), the map

$$v_κ \ni v \mapsto v(v^*) \quad v^* \in i^*(B_κ),$$

is a linear isometric embedding of $v_κ$ in $C(i^*(B_κ))$. □

Now let $v^*$ be a linear continuous functional on the space $v_κ$ regarded as the closed subspace of $C(i^*(B_κ))$. By the Hahn-Banach theorem $v^*$ can be extended to a linear continuous functional, say, $\hat{v}^*$ on the latter space with the same norm. In turn, by the Riesz representation theorem there is a regular finite Borel measure $\hat{v}$ on the compact space $i^*(B_κ)$ denoted by $\mu_{v^*}$ that represents $\hat{v}^*$.

This implies that

$$v(v^*) = \int_{i^*(B_κ)} v \, d\mu_{v^*}, \quad v \in v_κ.$$

At the next stage we exploit this measure to find a similar representation for elements of $v_κ$.

To this end, we use the weak* density of the subspace $v_κ$ in the space $V_κ$, see Lemma 6.2. According to this lemma, for every $v \in V_κ$ there is a bounded in the $v_κ$ norm sequence $\{v_j\}_{j \in \mathbb{N}} \subset v_κ$ such that

$$\lim_{j \to \infty} v_j(u) = v(u), \quad u \in U_κ.$$

Now let $\tau : i^*(U_κ) \to U_κ$ be the inverse to the injection $i^*|_{U_κ} : U_κ \to v_κ^*$, see Proposition 6.1(a). Making the change of variable $u \to \tau(v^*)$ we derive from (6.33)

$$\lim_{j \to \infty} v_j(v^*) = (v \circ \tau)(v^*), \quad v^* \in i^*(B_κ).$$

Since linear functionals $v_j : v_κ \to \mathbb{R}$ are continuous in the weak* topology defined by $v_κ$ their traces to $i^*(B_κ)$ are continuous functions in the weak* topology induced from $B(v_κ^*)$. This implies the following:

**Lemma 6.8.** The function $(v \circ \tau)|_{i^*(B_κ)}$ is $\mu_{v^*}$-integrable and bounded.

Moreover, a function $φ_{v^*} : V_κ \to \mathbb{R}$ given by

$$φ_{v^*}(v) := \int_{i^*(B_κ)} v \circ \tau \, d\mu_{v^*}$$

belongs to $V_κ^*$.

**Proof.** Since $\mu_{v^*}$ is a regular Borel measure, every continuous function on $i^*(B_κ)$ is $\mu_{v^*}$-measurable. Moreover, pointwise limits of sequences of such functions are $\mu_{v^*}$-measurable as well. In particular, $(v \circ \tau)|_{i^*(B_κ)}$ being the pointwise limit of a sequence of continuous functions, see (6.34), is $\mu_{v^*}$-measurable.

Further, the sequence $\{v_j\}_{j \in \mathbb{N}}$ is bounded in the $v_κ$-norm and $i^*(B_κ) \subset B(v_κ^*)$. Therefore, for every $v^* \in i^*(B_κ)$

$$|v^*| \leq \sup_j |v_j(v^*)| \leq \|v^*\|_{v_κ} \cdot \sup_j \|v_j\|_{v_κ} \leq \sup_j \|v_j\|_{v_κ} < \infty.$$
This implies boundedness of $(v \circ \tau)|_{\tau^*(B_\kappa)}$.

Since, in turn, the measure $\mu_{\nu^*}$ is finite, we conclude that the integral in (6.35) is well-defined and the function $\phi_{\nu^*}$ satisfies

$$|\phi_{\nu^*}(v)| \leq \left(\sup_{\tau^*(B_\kappa)} |v \circ \tau| \right) |\mu_{\nu^*}|, \quad v \in V_\kappa.$$ 

Further, since $\tau := (i^*|_{U_\kappa})^{-1}$, the supremum here is bounded from above by

$$\|v\|_{V_\kappa} \cdot \sup_{b \in B_\kappa} \|b\|_{U_\kappa} \leq \|v\|_{V_\kappa}.$$ 

Thus, $\phi_{\nu^*}$ is a linear continuous functional on $V_\kappa$ of norm $\leq |\mu_{\nu^*}|$. \hfill \Box

At the next stage we establish weak* continuity of $\phi_{\nu^*}$ on $V_\kappa$ regarded as the dual space of $U^\kappa_\nu$, see Theorem 2.6.

To this end it suffices to show that $\phi_{\nu^*}^{-1}(R) \subset V_\kappa$ is weak* closed for every closed subinterval $R \subset \mathbb{R}$. Since this preimage is convex, we can use the Krein-Smulian weak* closedness criterion, see, e.g., [DSch-58] Thm. V.5.7. In our case, it asserts that $\phi_{\nu^*}^{-1}(R)$ is weak* closed iff $B_r(0) \cap \phi_{\nu^*}^{-1}(R)$ is for every $r > 0$; here $B_r(0) := \{v \in V_\kappa : \|v\|_{V_\kappa} \leq r\}$.

In turn, since $V_\kappa = U^\kappa_\nu$ and $U_\kappa$ is separable, see Theorem 2.5(b), every bounded subset of $V_\kappa$ equipped with the induced weak* topology is metrizable. Hence, weak* closedness of $B_r(0) \cap \phi_{\nu^*}^{-1}(R)$ is a consequence of the following:

**Lemma 6.9.** If a sequence $\{v_j\}_{j \in \mathbb{N}} \subset B_r(0) \cap \phi_{\nu^*}^{-1}(R)$ converges to some $v \in V_\kappa$, then $v \in B_r(0) \cap \phi_{\nu^*}^{-1}(R)$.

**Proof.** Weak* convergence of $\{v_j\}_{j \in \mathbb{N}}$ to $v$ implies pointwise convergence of the sequence of functions $\{v_j \circ \tau|_{\tau^*(B_\rho)}\}_{j \in \mathbb{N}}$ to the function $v \circ \tau|_{\tau^*(B_\rho)}$. Further, the functions of this sequence are $\mu_{\nu^*}$-measurable and bounded by $\sup_j \|v_j\|_{V_\kappa}$, see Lemma 6.8. Moreover, by the assumption of Lemma 6.9

$$(6.36) \quad \sup_j \|v_j\|_{V_\kappa} \leq r \quad \text{and} \quad \phi_{\nu^*}(v_j) \in R, \quad j \in \mathbb{N}.$$ 

Therefore, the Lebesgue pointwise convergence theorem implies

$$\lim_{j \to \infty} \phi_{\nu^*}(v_j) = \int_{\tau^*(B_\rho)} \left(\lim_{j \to \infty} v_j \circ \tau\right) d\mu_{\nu^*} = \int_{\tau^*(B_\rho)} v \circ \tau d\mu_{\nu^*} = \phi_{\nu^*}(v).$$

Since $R \subset \mathbb{R}$ is closed, the limit on the left-hand side belongs to $R$, hence, the limit point $v \in B_r(0) \cap \phi_{\nu^*}^{-1}(R)$ as required. \hfill \Box

Thus $\phi_{\nu^*}$ is a weak* continuous linear functional from $V_\kappa^*$. By the definition of the weak* topology on $V_\kappa = U^\kappa_\nu$ every such functional is uniquely determined by an element of $U_\kappa$, i.e., for some $u_{\nu^*} \in U_\kappa$

$$\phi_{\nu^*}(v) = v(i^* (u_{\nu^*})), \quad v \in V_\kappa.$$ 

On the other hand, see (6.32), for all $v \in V_\kappa$

$$\phi_{\nu^*}(v) = v^*(v).$$
Since $i^*|_{U_\kappa} : U_\kappa \to v_\kappa$ is an isometry, see Proposition 6.1(a), $v_\kappa$ separates points of $U_\kappa$. Hence, these two equalities imply that
\[ v^* = i^*(u_\kappa^*). \]
Thus, every point $v^* \in v_\kappa^*$ is the image under $i^*$ of some point of $U_\kappa$, i.e., $i^* : U_\kappa \to v_\kappa^*$ is a surjection. Moreover, $i^*|_{U_\kappa}$ is also an isometry. Hence $i^*$ is a linear isometric isomorphism of the Banach spaces $U_\kappa$ and $v_\kappa^*$.

This completes the proof of Theorem 2.7 for $1 < p < \infty$.

Now, we consider the case of $p = \infty$.

The proof repeats line by line the proof of the previous case with the related set $B_\kappa$ replaced by the set $A_\kappa$ of all $\kappa$-atoms. In this derivation we take into account that $i^*(A_\kappa)$ is a weak* compact subset of $B(v_\kappa^*)$, see Proposition 6.3(b), and that sc($i^*(A_\kappa)$) is a weak* dense subset of $B(v_\kappa^*)$ because sc($A_\kappa$) is dense in $B(U_\kappa)$. We leave the details to the readers.

The proof of the theorem is complete. \hfill \Box

REFERENCES

[AFP-00] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variations and Discontinuity Problems, Oxford Sci. Publ., 2000.

[AX-12] D. R. Adams, J. Xiao, Morrey spaces in harmonic analysis, Ark. Mat. 50 2 (2012), 201–230.

[BB-11] A. Brudnyi, Yu. Brudnyi, Methods of Geometric Analysis in Extension and Trace Problems, Monographs in Mathematics 102, Birkhäuser, 2011.

[BL-76] J. Bergh, J. Löfström, Interpolation Spaces - An Introduction, Grundlehren der mathematischen Wissenschaften 223, Springer, 1976.

[Br-70] Yu. Brudnyi, A multivariate analog of a theorem of Whitney, Mat. Sb. 82 (1970), 175–191; English transl. in: Math. USSR Sbornik 11 (1970), 157–170.

[Br-71] Yu. Brudnyi, Spaces defined by local polynomial approximation, Trudy Mosc. Mat. Ob-va 24 (1971), 69–132; English transl. in Proc. Moscow Math. Soc. 24 (1971), 73–139.

[Br-94] Yu. Brudnyi, Adaptive approximation of functions with singularities, Transl. of Proc. Moscow Math. Soc. (1994), 123–186.

[Br-09] Yu. Brudnyi, Sobolev spaces and their relatives: local polynomial approximation approach. In: Sobolev Spaces in Mathematics II, Springer, 2009, 31–68.

[Br-17] Yu. Brudnyi, Nonlinear piecewise polynomial approximation and multivariate $BV$ functions of a Wiener-L. Young type, J. Approx. Theory 218 (2017), 9–41.

[BP-74] J. Bergh, J. Peetre, On the $V_p$ spaces ($0 < p \leq \infty$), Bull Unione Mat. Ital. 10 (1974), 632–648.

[Ca-64] S. Campanato, Proprieta di rena famiglia di spazi funzionali, Ann. Scuola Norm. Sup. Piza 18 (1964), 137–160.

[CW-77] R. R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. AMS 83 4 (1977), 569–645.

[DeL-61] K. De Leeuw, Banach spaces of Lipschitz functions, Studia Math. XXI (1961), 55–66.

[DSch-58] N. Dunford and J. T. Schwartz, Linear Operators. Part 1, Intersci. Publ., 1958.

[DVL-96] R. DeVore, G. G. Lorentz, Constructive Approximation, Springer, 1993.

[F-71] C. Fefferman, Characterizations of bounded mean oscillation, Bull. Amer. Math. Soc. 77 (1971), 587–588.

[Gi-84] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkhäuser. 1984.

[Ha-97] L. G. Hanin, Duality for general Lipschitz classes and applications, Proc. London Math. Soc. 75 (3) (1997), 134–156.
[JN-61] F. John, L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (1961), 415–426.

[Ki-84] S. Kisliakov, A remark on the spaces of bounded p-variation, Math. Nachr. 119 (1984), 137–140.

[Le-04] H. Lebesgue, Leçons sur l’intégration et la recherche des fonctions primitives, Gauthier-Villar. 1904.

[Ri-10] F. Riesz, Untersuchungen über systeme integrierbarer Funktionen, Math. Ann. 69 (1910), 449–497.

[Ru-87] W. Rudin, Real and Complex Analysis, McGraw-Hill Book Comp. 1987.

[Sa-75] D. Sarason, Functions of vanishing mean oscillation, Trans. Amer. Math. Soc. 207 (1975), 391–405.

[St-93] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press. 1993.

[Ta-92] M. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, Comm. Partial Diff. Eq. 17 (1992), 1407–1456.

[Ti-63] A. F. Timan, Theory of Approximation of Functions of a Real Variable, Pergamon Press. 1963.

[Tr-92] H. Triebel, Theory of Function Spaces II, Birkhäuser. 1992.

[Vi-05] G. Vitali, Sulle funzioni ad integrale null., Rend. Circ. Mat. Palermo. 1905.

[Wh-59] H. Whitney, On bounded functions with bounded nth differences, Proc. AMS 10 (1959), 480–481.

[Zi-87] W. P. Ziemer, Weakly Differentiable Functions, Springer. 1987.

[Zo-86] C. T. Zorko, Morrey space, Proc. Amer. Math. Soc. 98 (1986), 586–592.

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