ON A DICHOTOMY OF THE CURVATURE DECAY OF STEADY RICCI SOLITON

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Abstract. We establish a dichotomy on the curvature decay for four dimensional complete noncompact non Ricci flat steady gradient Ricci soliton with linear curvature decay and proper potential function. A similar dichotomy is also shown in higher dimensions under the additional assumption that the Ricci curvature is nonnegative outside a compact subset.

1. Introduction

Let \((M^n, g)\) be a smooth connected Riemannian manifold and \(X\) a smooth vector field on \(M\). The triple \((M, g, X)\) is said to be a Ricci soliton if there is a constant \(\lambda \in \mathbb{R}\) such that
\[
\text{Ric} + \frac{1}{2} L_X g = \lambda g,
\]
where \text{Ric} and \(L_X\) denote the Ricci curvature and Lie derivative with respect to \(X\) respectively. A Ricci soliton is called steady (shrinking, expanding) if \(\lambda = 0 \ (> 0, < 0 \text{ resp.})\). Upon scaling the metric by a constant, we may assume \(\lambda \in \{-\frac{1}{2}, 0, \frac{1}{2}\}\). The soliton is called complete if \((M, g)\) is complete as a Riemannian manifold. It is said to be gradient if \(X\) can be chosen as \(X = \nabla f\) for some smooth function \(f\) on \(M\). In this case, \(f\) is called a potential function and (1) can be rewritten as
\[
\text{Ric} + \nabla^2 f = \lambda g.
\]
Ricci soliton is of great importance as it sometimes arises as a rescaled limit of the Ricci flow near its singularities. When \(f\) in (2) is a constant, the metric becomes Einstein. Hence Ricci soliton can also be viewed as a natural generalization of the Einstein manifold as well.

In view of different examples of steady gradient Ricci solitons, the exponential and linear curvature decays seem to be two generic decays for noncompact steady solitons (see [25, 11] and ref. therein). Munteanu-Sung-Wang [52] raised the following conjecture on the curvature decay of steady solitons:

Conjecture 1.1. [52] If \((M, g, f)\) is a complete non Ricci flat steady gradient Ricci soliton with \(|Rm| \to 0\) as \(x \to \infty\), then either one of the following estimates holds outside a compact set of \(M\):
\[
\begin{align*}
(3) \quad C^{-1}r^{-1} & \leq |Rm| \leq Cr^{-1}; \\
(4) \quad C^{-1}e^{-r} & \leq |Rm| \leq Ce^{-r},
\end{align*}
\]
where \(C\) is a positive constant and \(r\) is the distance function.

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The curvature decays (3) and (4) appear on the Bryant soliton and the Cigar soliton respectively (see [25] and ref. therein). This dichotomy conjecture, once established, will be very useful for the studies of steady soliton due to the classification results of steady solitons under different curvature decay conditions (see [7, 13, 20, 30, 31, 33, 52, 15]). In fact, Deng-Zhu [29] (see also [52]) showed that the above dichotomy conjecture is true under nonnegative sectional curvature and linear scalar curvature decay conditions \( R \leq C(r + 1)^{-1} \). As an application, they classified 3 dimensional steady gradient soliton with linear scalar curvature decay (see [29] and ref. therein).

**Theorem 1.2.** [29] Suppose \((M^n, g, f)\) is a complete nonflat steady gradient Ricci soliton with nonnegative sectional curvature and linear scalar curvature decay, i.e. \( R \leq C(r + 1)^{-1} \). Then either (3) or (4) holds near infinity.

Recently, Lai [49] has resolved a conjecture of Hamilton and constructed a family of 3 dimensional gradient steady solitons that are flying wings. It is natural to examine their curvature properties to see if they fulfill the expectation from Conjecture 1.1. As shown in [49], \(|Rm|\) does not decay uniformly to 0 at infinity on these 3 dimensional examples. Analogous steady solitons were also constructed in dimension \( n \geq 4 \) [49]. Nonetheless, It is unclear at this point if the curvature \(|Rm|\rightarrow 0\) at infinity on these higher dimensional solitons.

It follows from the Hamilton-Ivey pinching estimate that any 3 dimensional complete steady gradient Ricci soliton must have nonnegative sectional curvature (see [20] and ref. therein). However, this significant feature does not hold in higher dimensions. In particular, the examples on some line bundles over the complex projective space constructed by Cao [10] and Appleton [1] don’t have nonnegative sectional curvature on the entire manifold. Hence, a reasonable next step is to consider if the dichotomy holds under linear curvature decay condition in higher dimensions. Under the assumption that the soliton is \( \kappa \) noncollapsed with dimension \( n \geq 4 \), Deng-Zhu [29] obtained the estimate (7) if \(|Rm| \leq C(r + 1)^{-1} \) and \( \text{Ric} \geq 0 \) outside a compact subset (see also [31, 22, 5]). However, there does exist collapsed steady solitons (see [10, 28, 32]). Munteanu-Sung-Wang [52] and the first named author [16] proved that the upper bound of \(|Rm|\) in (4) holds, i.e.

\[ |Rm| \leq Ce^{-r}, \]

if \( r|Rm| \) is uniformly small near infinity and \( f \) is bounded from above. The lower bound in (4) is a result by Chow-Lu-Yang [24] (see also [52]).

In this note, we are aimed at studying the curvature decay without any assumption of non-collapsedness and uniform smallness of \( r|Rm| \) at infinity. In particular, in real dimension 4, if the scalar curvature \( R \) decays at least linearly and the potential function \( f \) is proper, then Conjecture 1.1 will hold. Our main theorem

**Theorem 1.3.** Let \((M^4, g, f)\) be a 4 dimensional, complete, non Ricci flat steady gradient Ricci soliton with proper potential function \( \lim_{x \to \infty} f = -\infty \) and linear scalar curvature decay, i.e. \( R \leq C'(r + 1)^{-1} \). Then there exists a positive constant \( C \) such that either one of the following estimates holds near infinity

\[ C^{-1}r^{-1} \leq |Rm| \leq Cr^{-1}; \]

\[ C^{-1}e^{-r} \leq |Rm| \leq Ce^{-r}, \]

where \( r \) is the distance function.
If the Ricci curvature is nonnegative and goes to 0 uniformly at infinity, then the potential function \( f \) is proper \([12]\). Hence Theorem 1.3 also generalizes Theorem 1.2 and the related results in \([29, 52]\). Instead of using Gromov’s almost flatness theorem \([10]\) or looking at the rescaled flow at infinity as in \([29]\). Here, we apply the parabolic maximum principle to the level set flow of \( f \) as in \([8, 55]\) to obtain the dichotomy estimates. One advantage of this approach is that it doesn’t require any volume noncollapsing condition. Similar argument works as well in higher dimensions under some assumptions on the Ricci curvature.

**Theorem 1.4.** Let \((M^n, g, f)\) be an \(n \geq 5\) dimensional, complete, non Ricci flat steady gradient Ricci soliton with \(\text{Ric} \geq 0\) outside a compact subset and linear Riemann curvature decay, i.e. \(|Rm| \leq C' (r + 1)^{-1}\). Then there exists positive constant \(C\) such that either one of the following estimates holds near infinity

\[
C^{-1}r^{-1} \leq R \leq cn|Rm| \leq Cr^{-1};
\]

\[
C^{-1}e^{-r} \leq R \leq cn|Rm| \leq Ce^{-r},
\]

where \(c_n\) is some dimensional constant.

**Remark 1.5.** Under the linear curvature decay condition, we only need to assume that \(|\text{Ric}| \leq A_0 R\) for some constant \(A_0 > 0\) and \(f\) is proper, instead of \(\text{Ric} \geq 0\) near infinity, then the same conclusion in Theorem 1.4 will also hold. It can be seen from the volume estimate in \([33]\) that when estimate \((8)\) is true, the soliton is collapsed. Hence Theorem 1.4 also recovers an estimate in \([29]\). On the other hand, Theorem 1.4 tells us that

\[
|Rm| \leq cR \text{ on } M.
\]

See \([14]\) and references therein for the case of \(n = 4\). Ancient solution to the Ricci flow and steady soliton satisfying estimate \((9)\) have been recently studied by Ma-Zhang \([50]\) and the first named author-Ma-Zhang \([17]\).

For the level set of potential function under the dichotomy, we have the following two topological classifications of level set for large \(\tau\).

- If we assume that \((5)\) or \((7)\) holds, then by the Gauss equation and Shi’s estimate \([28]\), the level set \(\Sigma_\tau := \{-f = \tau\}\) must have positive scalar curvature for large \(\tau\);
- If we assume that \((6)\) or \((8)\) holds, then it follows from Proposition 4.1 and Shoen-Yau’s trick on minimal surface that the level set is diffeomorphic to a compact flat manifold (see also \([33]\)).

Hence, as a corollary, we obtain that

**Corollary 1.6.** Under the same assumptions as in Theorem 1.3 or Theorem 1.4, there exists a large constant \(\tau_0\) such that either one of the following is true for the compact connected hypersurface \(\Sigma_\tau := \{-f = \tau\}\):

- \((a)\) \(\Sigma_\tau\) is diffeomorphic to a finite quotient of torus for all \(\tau \geq \tau_0\);
- \((b)\) \(\Sigma_\tau\) has positive scalar curvature w.r.t. the intrinsic metric induced by \((M, g)\) for all \(\tau \geq \tau_0\).
Remark 1.7. Under the same conditions as in Theorem 1.4 (with \( n \geq 4 \) instead) and additionally the soliton is \( \kappa \) noncollapsed, Deng-Zhu \([29]\) proved that that the level sets of \( f \) at infinity are diffeomorphic to a compact gradient Ricci shrinker with nonnegative Ricci curvature. In particular for as \( n = 4 \), the level sets at infinity are diffeomorphic to a spherical 3 manifold (see also \([22]\)). However, generally it may not be the case if noncollapsed condition is not assumed. For instance, let \( B_3 \) be the three dimensional Bryant steady soliton, the level sets of \( f \) at infinity of the product soliton \( B_3 \times \mathbb{S}^1 \) are diffeomorphic to \( \mathbb{S}^2 \times \mathbb{S}^1 \) which has infinite fundamental group \( \pi_1 \) and admits no shrinker structure \([38]\). However, it does support a metric of positive scalar curvature as in Corollary 1.6 (b). Here, in the case of positive scalar curvature

- As \( n = 4 \), by the resolutions of the Poincaré Conjecture and the Thurston’s Geometrization Conjecture \([56, 57, 58]\), \( \Sigma_r \) is a connected sum of spherical 3-manifolds and some copies of \( S^1 \times S^2 \);
- As \( n \geq 5 \), by the Ric \( \geq 0 \) near infinity condition, the level sets \( \Sigma_r \) in Corollary 1.6 (b) have almost nonnegative Ricci curvature and hence have first Betti number \( b_1(\Sigma_r) \leq n-2 \) by a result of Cheeger-Colding \([19, \text{ Theorem A.1.13}] \), where \( n \) is the real dimension of the ambient manifold \( M \).

Dimension reduction serves as an important tool in the studies of Ricci flow and has led to a lot of successes in the classification of Ricci solitons (for example, see \([5, 7, 8, 22, 23, 28, 29, 30, 31, 42, 55\] and ref. therein). Under some volume noncollapsing conditions, one may extract different geometric information by looking at the blow down profiles of the soliton at infinity, which usually split like \( \mathbb{R} \times X \), where \( X \) is some ancient solution to the Ricci flow of lower dimension. On 4 dimensional complete noncompact, noncollapsed, non Ricci flat steady gradient soliton with \( R \to 0 \) at \( \infty \) and \( \text{Ric} \geq 0 \) outside compact set, Chow-Deng-Ma \([22]\) studied the possible rescaled limits and showed that the limits at \( \infty \) split off a line. In general, existence of smooth limit at infinity is not expected without the noncollapsedness condition. In such a case, Gromov-Hausdorff topology is one of the most natural notions to work on for the convergence of manifolds. Under the conditions of Theorems 1.3 and 1.4 we apply the level set method in \([29, 30, 31]\) to show that the Gromov-Hausdorff limit of the blow down metrics based at any sequence \( p_i \to \infty \) also splits isometrically. Very recently, Bamler \([2, 3, 4]\) has developed new compactness and partial regularity theories of Ricci flow under relatively weak conditions. Bamler-Chow-Deng-Ma-Zhang \([5]\) also classified the tangent flows at infinity (see \([3]\) for the definition) of 4 dimensional steady soliton singularity models.

Let \( (N, h) \) be a complete Riemannian manifold and \( L \) the cylinder \( \mathbb{R} \times N \) with product metric \( g_L = ds^2 + h \). For any \( \lambda \in \mathbb{R} \), the translation \( \rho_\lambda : L \to L \) is given by \( \rho_\lambda(s, \omega) := (s + \lambda, \omega) \). A Riemannian manifold \( (M, g) \) is said to be smoothly asymptotic to the cylinder \( L \) if there exist a compact set \( K \) of \( M \) and a diffeomorphism \( \Phi : (0, \infty) \times N \to M \setminus K \) such that \( \rho^*_\lambda \Phi^* g \to g_L \) in \( C^k_{\text{loc}} \) sense on \( (0, \infty) \times N, g_L \) as \( \lambda \to \infty \) (see \([55]\)). The asymptotic convergence is at exponential rate if in addition for any integer \( k \geq 0 \), there is a positive constant \( C_k \) such that for all \( s > 0 \),

\[
\sup_{\omega \in N} |\nabla_{g_L} (\Phi^* g - g_L)|_{g_L} (s, \omega) \leq C_k e^{-s}.
\]

Corollary 1.8. Under the same assumptions as in Theorem 1.3 or Theorem 1.4, either one of the following holds:

(a) For any sequence \( p_i \to \infty \) in \( M \), after passing to a subsequence, \( (M, d_{R(p_i)})_i \) converges in pointed Gromov-Hausdorff sense to a cylinder \( (\mathbb{R} \times Y, \sqrt{d_y^2 + d_V^2}, p_\infty) \), where \( d_e \) is the
flat metric on $\mathbb{R}$, $(Y,d_Y)$ denotes a compact Alexandrov space and $\sqrt{d^2_e + d^2_Y}$ indicates the product metric (see also [18]).

(b) For any sequence $p_i \to \infty$ in $M$, $(M,d_{R(p_i),g},p_i)$ converges in pointed Gromov-Hausdorff sense (without passing to subsequence) to the ray $([0,\infty),d_e,0)$, where $d_e$ is the flat metric restricted on $[0,\infty)$. In this case, $(M,g)$ is smoothly asymptotic to the flat cylinder $\mathbb{R} \times (\mathbb{T}^{n-1}/\sim)$ at exponential rate, where $\mathbb{T}^{n-1}/\sim$ is diffeomorphic to the quotient of torus in Corollary 1.6 (a).

Since we assume $\text{Ric} \geq 0$ outside a compact subset when $n \geq 5$, so in this case, the result in Corollary 1.8(a) also follows from the splitting theorem by Cheeger-Colding [18]. Instead, we shall adopt the level set approach by Deng-Zhu [29, 30, 31] which works for all $n \geq 4$ and provides more geometric information of the metric space $(Y,d_Y)$ (see also Proposition 6.1). In particular, it can be seen from the proof that $(Y,d_Y)$ in Corollary 1.8(a) is the Gromov Hausdorff limit of a sequence of level sets as in Corollary 1.6(b) with uniformly bounded curvature and diameter after scaling. In general, due to volume collapsing, the limit $\mathbb{R} \times Y$ in Corollary 1.8(a) can be of lower dimension compared to $M$. On the positively curved Cao steady Kähler soliton on $\mathbb{C}^2$ [10], the corresponding metric space $(Y,d_Y)$ is $\mathbb{C}P^1$ endorsed with a scalar multiple of the Fubini-Study metric (see [28]).

Steady Ricci solitons with fast curvature decay were extensively studied in [13, 15, 27, 29, 33, 52]. As an application of Theorems 1.3 and 1.4, by exploiting the real analyticity of the soliton metric, we prove some classification results on steady gradient Ricci soliton with fast curvature decay under milder conditions. Here, we do not require any global non-negative curvature condition and uniform smallness of the quantity $rR$ at infinity.

**Theorem 1.9.** Suppose $(M^m,g,f)$ is a complete, non Ricci flat steady Kähler gradient Ricci soliton of complex $m$ dimension with nonnegative Ricci curvature outside a compact subset of $M$, where $m \geq 2$, such that

$$\liminf_{x \to \infty} rR = 0.$$  

Further assume that the following conditions are satisfied,

- $R \leq C(r+1)^{-1}$ if $m = 2$;
- $|Rm| \leq C(r+1)^{-1}$ if $m \geq 3$.

Then $M$ is holomorphically isometric to a quotient of $\Sigma \times \mathbb{C}^{m-1}$ and the curvature $Rm$ decays exponentially in $r$, where $\Sigma$ is the Hamilton Cigar soliton.

**Theorem 1.10.** Let $(M^n,g,f)$ be a complete noncompact and nonflat steady gradient Ricci soliton with nonnegative sectional curvature outside a compact subset of $M$ and linear scalar curvature decay, that is, $R \leq C(r+1)^{-1}$. If in addition that

$$\liminf_{x \to \infty} rR = 0,$$

then $M$ is isometric to a quotient of $\Sigma \times \mathbb{R}^{n-2}$, where $\Sigma$ is the Hamilton Cigar soliton.

**Remark 1.11.** If we further assume sectional curvature is nonnegative everywhere on $M$, then Theorem 1.10 also follows from [29] (see also Theorem 7.2 and [13, 52]).
We wrap up the introduction by looking at the analogous dichotomy in other types of Ricci solitons. For complete noncompact and nonflat gradient Ricci shrinker, Munteanu-Wang \cite{54} showed that if $|\text{Ric}|$ is sufficiently small at infinity, then the shrinker is smoothly asymptotic to a cone and hence $|\text{Rm}| \sim r^{-2}$ at infinity by a result of Chow-Lu-Yang \cite{24} (see also \cite{46}). In real dimension 4, it remains an open problem whether shrinker with bounded scalar curvature $R$ must have either $R \geq c$ for some positive constant $c$ or $R \to 0$ at infinity (this holds in the Kähler case [55]). The dichotomy will be extremely useful toward the classification of 4 dimensional shrinker in view of different studies on the asymptotic geometry of the solitons (see [46, 54, 55] and ref. therein).

Before we move on, let us recall the notion of asymptotically conical expanding soliton [35]. Let $X$ be a smooth $n$-1 dimensional closed manifold with Riemannian metric $g_X$, $C(X)$ is defined to be the cone over $X$, i.e. $\{(t, \omega) : t > 0, \omega \in X\}$. $g_C$ and $\nabla_C$ denote the metric $dt^2 + t^2g_X$ on $C(X)$ and its Levi-Civita connection respectively. For any positive constant $S$, $B(o, S) \subseteq C(X)$ is the set given by $\{(t, \omega) : t \geq 0, \omega \in X\}$.

**Definition 1.12.** \cite{34, 35} A complete expanding gradient Ricci soliton $(M, g, f)$ is asymptotically conical (at polynomial rate $r = 2$) with asymptotic cone $(C(X), g_C)$ if there exist constants $S_0 > 0$ and $c_0$, a compact set $K$ in $M$ and a diffeomorphism $\phi : M \setminus K \to C(X) \setminus B(o, S_0)$ such that for any nonnegative integer $k$

\begin{equation}
\sup_{\omega \in X} |\nabla_C \left[ (\phi^{-1})^* g - g_C \right] |_{g_C} (t, \omega) = O(t^{-2-k}) \text{ as } t \to \infty;
\end{equation}

\begin{equation}
f \circ \phi^{-1}(t, \omega) = -\frac{t^2}{4} + c_0 \text{ for all } t > S_0.
\end{equation}

Just like the steady case, there are two generic curvature decays at infinity for nonflat asymptotically conical expander [35]:

\begin{equation}
C^{-1}v^{-1} \leq |\text{Rm}| \leq Cv^{-1};
\end{equation}

\begin{equation}
C^{-1}v^{1-\frac{n}{2}}e^{-v} \leq |\text{Rm}| \leq Cv^{1-\frac{n}{2}}e^{-v},
\end{equation}

where $v = \frac{n}{2} - f$ and $\lim_{x \to \infty} 4r^{-2}v = 1$. The curvature decays (15) and (16) can be found in the Bryant expanding soliton and the Kähler expander constructed by Feldman-Ilmanen-Knopf \cite{37} respectively. It will be interesting to see whether or not (15) and (16) are the only possible curvature decays of expanding soliton. The upper bound in (15) is always satisfied on conical expander. If $\lim_{x \to \infty} r^2|Rm| = 0$, then the upper and lower estimates of $Rm$ in (16) are due to Deruelle [35] and the first named author [16] respectively. Dichotomy result in the expanding case similar to Theorems 1.3 and 1.4 seems to be quite promising. However using the existence and compactness results for Ricci expander by Deruelle [34, 35], one can prove that the above dichotomy fails in 3 dimensional expanding case:

**Theorem 1.13.** \cite{34, 35} There exists a 3 dimensional complete noncompact asymptotically conical expanding gradient Ricci soliton $(M, g, f)$ with nonnegative curvature operator $Rm \geq 0$ and

\begin{equation}
0 = \liminf_{x \to \infty} v|\text{Rm}| < \limsup_{x \to \infty} v|\text{Rm}| < \infty,
\end{equation}

where $v = \frac{n}{2} - f$ and $\lim_{x \to \infty} 4r^{-2}v = 1$. 

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Remark 1.14. Since \( 0 = \liminf_{x \to \infty} v|Rm| \), \( M \) doesn’t satisfy the lower bound in (15). The upper estimate in (16) doesn’t hold as \( \limsup_{x \to \infty} v|Rm| > 0 \).

Though Theorem 1.13 is not explicitly stated in [34, 35], it is essentially due to Deruelle and is a direct consequence of the results in [34, 35]. We shall include its proof in the appendix for the sake of completeness.

The paper is organized as follows. In Section 2, we include the basic preliminaries of steady soliton. In Section 3, we show the exponential curvature decay and prove Theorems 1.3 and 1.4. Sections 4 and 5 will then be devoted to the proofs of Theorems 1.9 and 1.10. In Section 6, we will prove Corollary 1.8. Finally, the proof of Theorem 1.13 will be presented in the appendix.

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2. Preliminaries

Let \((M,g)\) be an \(n\) dimensional connected smooth Riemannian manifold and \(f\) be a smooth function on \(M\). \(\nabla\) is the Levi Civita connection of \(g\). \(Rm\), \(Ric\) and \(R\) denote the Riemann, Ricci and scalar curvatures respectively. \((M,g,f)\) is said to be a gradient steady Ricci soliton with the potential function \(f\) if

\[
Ric + \nabla^2f = 0.
\]

The steady soliton is complete if \((M,g)\) is a complete Riemannian manifold. \((M,g,f)\) is a gradient steady Kähler Ricci soliton if \(M\) satisfies (18) and is a complex manifold with complex structure \(J\) and \(g\) is a Kähler metric compatible with \(J\) on \(M\). Ricci soliton is a self similar solution to the Ricci flow. Given a complete steady gradient Ricci soliton, we consider \(g(t) := \varphi_t^*g\), where \(t \in \mathbb{R}\) and \(\varphi_t\) is the flow of \(\nabla f\) with \(\varphi_0 = \text{id}\). Then \(g(t)\) is an ancient solution to the Ricci flow:

\[
\frac{\partial g(t)}{\partial t} = -2\text{Ric}(g(t)),
\]

\[g(0) = g.\]

It was shown by Chen [20] that any complete ancient solution to the Ricci flow must have nonnegative scalar curvature (see also [61]). Using the strong maximum principle, we see that any complete steady gradient Ricci soliton must have positive scalar curvature \(R > 0\) unless it is Ricci flat. It is also known that compact steady soliton must be Ricci flat [25]. Upon scaling the metric by a constant if necessary, Hamilton [42] showed that for a complete steady gradient Ricci soliton,

\[
|\nabla f|^2 + R = 1 \text{ on } M,
\]

we adopt the above scaling convention throughout this article. By virtue of (18) and the Ricci identity, for any vectors \(X, Y\) and \(Z\)

\[
\nabla_Z\text{Ric}(X, Y) - \nabla_Y\text{Ric}(X, Z) = R(Z, Y, X, \nabla f).
\]
Hereinafter, we fix a point $p_0$ in $M$, then for any $x$ in $M$, $r$, $r(x)$ and $d(x, p_0)$ will be used interchangeably and refer to the distance between $x$ and $p_0$.

Then, for any smooth function $\omega$, we define the weighted Laplacian w.r.t. $\omega$ by $\Delta_\omega := \Delta - \nabla \omega \cdot \nabla$, where $\Delta$ is Beltrami Laplacian operator. The following identities are well known for steady gradient Ricci soliton (see [25])

\begin{align}
\Delta f v &= 1 \\
\Delta f + R &= 0 \\
\nabla R &= 2\text{Ric}(\nabla f) \\
\Delta f R &= -2|\text{Ric}|^2,
\end{align}

where $v = -f$. When $f$ is proper, i.e.

$$\lim_{x \to \infty} f(x) = -\infty,$$

by adding a constant to $f$ if necessary, we may assume that $v = -f \geq 10$ on $M$. Furthermore by [14, Lemma 2], $f$ and $v$ must be distance-like and satisfy

$$\lim_{x \to \infty} \frac{v}{r} = -\lim_{x \to \infty} \frac{f}{r} = 1.$$ 

Since we assume that the scalar curvature $R \to 0$ as $x \to \infty$, $\lim_{x \to \infty} |\nabla f|^2 = \lim_{x \to \infty} 1 - R = 1$ and the level sets $\Sigma_\tau := \{-f = \tau\}$ are smooth compact hypersurfaces for all large $\tau$. Moreover, $\Sigma_\tau$ are connected by a result of Munteanu-Wang [53].

3. Curvature decay

The goal of this section is to prove the following key proposition

**Proposition 3.1.** Let $(M^n, g, f)$ be a complete noncompact non Ricci flat steady gradient Ricci soliton of real dimension $n$ with proper potential function. Further suppose that the following conditions are satisfied:

- $|\text{Ric}| \leq A_0 R$ on $M$ for some constant $A_0 > 0$;
- $|\text{Rm}| \leq A_1 (r + 1)^{-1}$ on $M$ for some constant $A_1 > 0$;
- $\lim_{x \to \infty} rR = 0$.

Then for some constant $C > 0$

$$|\text{Rm}| \leq Ce^{-r}.$$  

We will make use of the diameter and curvature estimates by Deng-Zhu [29, 31]. By viewing the soliton as a solution to the Ricci flow and our assumption $|\text{Rm}| \leq C(r + 1)^{-1}$ implies, by the Shi’s estimate [23], that for any nonnegative integer $k$, there exists positive constant $C_k$ such that on $M$

$$|\nabla^k \text{Rm}| \leq C_k (r + 1)^{-(k+2)}.$$

Using (28), we may argue in the same way as in [29, 31] to get the following lemma

**Lemma 3.2.** [29, 31] If $(M^n, g, f)$ is a complete noncompact steady gradient Ricci soliton of real dimension $n$ with proper potential function $f$ and linear Riemann curvature decay $|\text{Rm}| \leq A_1 (r + 1)^{-1}$, then there exists constant $C > 0$ such that for all sufficiently large $\tau$

$$\text{diam}(\Sigma_\tau) \leq C \sqrt{\tau},$$
where \( \text{diam}(\Sigma_\tau) \) denote the intrinsic diameter of the level set \( \Sigma_\tau = \{ -f = \tau \} \) w.r.t. induced metric by \( g \).

With the diameter estimate on the level sets, we may proceed as in [29] to show the following lemma:

**Lemma 3.3.** [29] Let \( (M^n, g, f) \) be a complete noncompact steady gradient Ricci soliton of real dimension \( n \) with proper potential function \( f \). Further suppose that the curvature tensor \( Rm \) decays at least linearly i.e.

\[
|Rm| \leq A_1(r + 1)^{-1}. \tag{30}
\]

If in addition, \( r(p_i) R(p_i) \to 0 \) for a sequence of points \( p_i \to \infty \), then there exists a positive constant \( \varepsilon(n, A_1) \) such that

\[
\sup_{B(p_i, \varepsilon; r(p_i)^{-1}g)} rR \to 0 \text{ as } i \to \infty, \tag{31}
\]

where \( B(p_i, \varepsilon; r(p_i)^{-1}g) \) is the geodesic ball on \( M \) centered at \( p_i \) with radius \( \varepsilon \) w.r.t. metric \( r(p_i)^{-1}g \).

**Proof.** The argument is due to Deng-Zhu [29] and we include the proof for the sake of completeness. Let \( r_i = r(p_i) \) and \( g_i = r_i^{-1}g \). We consider the Ricci flow associated to the steady soliton, i.e. \( g(t) := \phi^*_t g, \phi_t \) is the flow of the vector field \( \nabla f \) with \( \phi_0 = \text{id} \) and \( t \in \mathbb{R} \). For any \( z \in B(p_i, 1; g_i) = B(p_i, \sqrt{r_i}; g) \), by the triangle inequality

\[
r_i + \sqrt{r_i} \geq r(z) \geq r_i - \sqrt{r_i}. \tag{32}
\]

Moreover, using \( |\nabla f| \leq 1 \) (see (20)), for \( t \in \left[ -\frac{1}{10}, \frac{1}{10} \right] \), we have \( d_g(\phi_{r_it}(z), z) \leq \frac{r_i}{10} \) and

\[
r(\phi_{r_it}(z)) \geq r(z) - \frac{r_i}{10} \geq \frac{r_i}{2}. \tag{33}
\]

Let \( g_i(t) := \frac{1}{r_i^2} g(r_i t) \), then \( g_i(0) = g_i \) and by (30), on \( B(p_i, 1; g_i) \times \left[ -\frac{1}{10}, \frac{1}{10} \right] \),

\[
|\text{Rm}(g_i(t))|(z) = r_i |\text{Rm}(g(r_i t))|(z) = r_i |\text{Rm}(g)|(\phi_{r_it}(z)) \leq \frac{A_1 r_i}{r(\phi_{r_it}(z))} \leq 2A_1. \tag{34}
\]

Thank to the Shi’s estimates (28), for all positive integer \( k \)

\[
|\nabla^k \text{Rm}(g_i(t))|(z) = \frac{r_i^{-k+2}}{2} |\nabla^k \text{Rm}(g(r_i t))|(z) = \frac{r_i^{-k+2}}{2} |\nabla^k \text{Rm}(g)|(\phi_{r_it}(z)) \leq C_k 2^{\frac{k+4}{2}}. \tag{35}
\]

We are about to take subsequential limit of \( g_i(t) \). Let \( \eta_i : (\mathbb{R}^n, g_e) \to (T_{p_i} M, g_i(p_i)) \) be a linear isometry with \( \eta_i(0) = 0 \). By the Rauch Comparison theorem, the map \( F_i := \exp_{p_i}^{g_i(0)} \circ \eta_i \) is a local diffeomorphism on \( \{ x \in \mathbb{R}^n : |x| < \min \left( \frac{1}{2}, \frac{\pi}{\sqrt{2A_1}} \right) \} \), where \( \exp_{p_i}^{g_i(0)} \) is the exponential map at \( p_i \) w.r.t. \( g_i(0) \). By the Hamilton compactness theorem, (34) and (35) (see also [23], [25], [29]
Lemma 4.4] and [16, Theorem 12]), there exist positive constant \( \delta = \delta(n, A_1) < \min \left\{ \frac{1}{2}, \frac{\pi}{\sqrt{2A_1}} \right\} \) and a subsequence \( i_l \) such that as \( l \to \infty \)

\[
\left( B_\delta(0), F_{i_l}^* g_{i_l}(t) \right) \to \left( B_\delta(0), g_\infty(t) \right)
\]

in \( C^\infty_{\text{loc}} \) sense on \( B_\delta(0) \times (-\frac{1}{10}, \frac{1}{10}) \), where \( B_\delta(0) := \{ x \in \mathbb{R}^n : |x| < \delta \} \) and \( g_\infty(t) \) is a solution to the Ricci flow. Note that \( \delta \) is independent on the subsequence taken.

\( F_{i_l}^* g_{i_l}(t) \) has nonnegative scalar curvature, so does \( g_\infty(t) \). Furthermore by the assumption and smooth convergence

\[
R_{g_\infty(0)}(0) = \lim_{l \to \infty} R_{F_{i_l}^* g_{i_l}(0)}(0) = \lim_{l \to \infty} r_{i_l} R_{g_{i_l}(0)} = \lim_{l \to \infty} r_{i_l} R_{p_{i_l}} = 0.
\]

Due to the strong minimum principle, \( R_{g_\infty(t)} \equiv 0 \) on \( B_\delta(0) \times (-\frac{1}{10}, 0] \). For the above \( \delta \), we want to show that the original sequence \( p_i \) satisfies

\[
\lim_{i \to \infty} \sup_{B(p_i, \delta/2; g_i)} r R = 0,
\]

Suppose on the contrary, by passing to a subsequence if necessary, we may assume there is constant \( \epsilon_0 > 0 \) such that for all \( i \)

\[
\sup_{B(p_i, \delta/2; g_i)} r R \geq \epsilon_0.
\]

By the locally uniform convergence \([36]\) and distance estimate \([32]\)

\[
o(1) = \sup_{z \in B_\frac{\delta}{2}(0)} R_{F_{i_l}^* g_{i_l}(0)}(z) \geq \sup_{y \in B(p_{i_l}, \frac{\delta}{2}; g_{i_l})} R_{g_{i_l}(0)}(y) = \sup_{y \in B(p_{i_l}, \frac{\delta}{2}; g_{i_l})} r_{i_l} R_{g_{i_l}(y)} \geq (1 - o(1)) \sup_{y \in B(p_{i_l}, \frac{\delta}{2}; g_{i_l})} r(y) R_{g_{i_l}(y)} \geq (1 - o(1)) \epsilon_0,
\]

which is impossible. This completes the proof of the lemma. \( \square \)

Using Lemma 3.3, Deng-Zhu [29] can show a better convergence result.

**Lemma 3.4.** [29] Under the same notations and assumptions as in Lemma 3.3, we have

\[
\lim_{i \to \infty} \sup_{\Sigma_{-f(p_i)}} r R \to 0 \text{ as } i \to \infty.
\]

**Proof.** The proof is due to Deng-Zhu [29] and we provide a sketch of the argument for reader’s convenience. We argue by contradiction, by passing to subsequence if necessary, there exists constant \( \epsilon_0 > 0 \) such that for all \( i \), we can find \( q_i \in \Sigma_{-f(p_i)} \) such that

\[
r(q_i) R(q_i) \geq \epsilon_0.
\]
Let \( \gamma_i \) be a normalized minimizing intrinsic geodesic joining \( p_i \) to \( q_i \) on \( \Sigma_{-f(p_i)} \) with respect to induced metric \( g_i = r(p_i)^{-1} g \), \( l(\gamma_i) \) the length of \( \gamma_i \) with respect to \( g_i \). Then \( \frac{N_i}{8} \leq l(\gamma_i) < \frac{(N_i+1)c}{8} \) for some unique nonnegative integer \( N_i \), where \( c \) is the constant as in Lemma 3.3. Moreover by \( (26) \), for all large \( i \) and \( a \in \Sigma_{-f(p_i)} \)

\[
\frac{1}{2} r(a) \leq -f(a) \leq 2r(a).
\]

In view of Lemma 3.2 we have the following upper bound for \( N_i \)

\[
N_i \leq 16C \epsilon^{-1}.
\]

Hence by taking further subsequence, we may assume \( N_i \equiv N_1 \) for all \( i \). For each \( 0 \leq j \leq N_1 \), we define the sequence \( p_i^j := \gamma_i \left( \frac{4^j}{8} \right) \). Using \( (43) \), We see that \( p_i^0 = p_i \) and the distance function on \( M \) satisfies

\[
d_{r(p_i^j) \sim g}(p_i^j, p_i^{j+1}) = \sqrt{ \frac{r(p_i)}{r(p_i^j)} } d_{g_i}(p_i^j, p_i^{j+1}) \leq 2d_{g_i}(p_i^j, p_i^{j+1}) \leq \frac{\epsilon}{4}
\]

and similarly

\[
d_{r(p_i^{N_1}) \sim g}(p_i^{N_1}, q_i) \leq \frac{\epsilon}{4}.
\]

By \( (45) \) and Lemma 3.3 \( r(p_i)R(p_i) \rightarrow 0 \) would imply \( r(p_i^1)R(p_i^1) \rightarrow 0 \). Applying Lemma 3.3 again with \( p_i \) replaced by \( p_i^1 \), we see that \( r(p_i^2)R(p_i^2) \rightarrow 0 \). Similarly by \( (46) \) and repeating the same procedure finitely many times, we conclude that

\[
0 = \lim_{i \to \infty} r(q_i)R(q_i)
\]

\[
\geq \epsilon_0,
\]

which is absurd. Result follows.

\[ \square \]

To apply the parabolic maximum principle as in \( [8, 55] \), we need the evolution equation of \( vR \) along the level set flow:

**Lemma 3.5.** Suppose that \( (M^n, g, f) \) is a complete non Ricci flat steady gradient Ricci soliton with proper potential function \( \lim_{x \to \infty} f = -\infty \) and linear Riemann curvature decay \( |Rm| \leq A_1(r + 1)^{-1} \) for some constant \( A_1 \). Then there exist positive constants \( a_0 \) and \( C_0 \) such that on \( \{ x \in M : v(x) \geq a_0 \} \), \( |\nabla f|^2 \geq \frac{1}{2} \) and

\[
\Delta_{\Sigma_\tau}(vR) - \langle \nabla f, \nabla (vR) \rangle \geq -2v|Ric|^2 + R - C_0 v^{-\frac{3}{2}},
\]

where \( v := -f \) and \( \Delta_{\Sigma_\tau} \) denotes the intrinsic Laplacian of the level set \( \Sigma_\tau := \{-f = \tau\} \) with respect to the induced metric by \( g \).

**Proof.** By \( (20) \), \( |\nabla f|^2 = 1 - R = 1 - o(1) \geq \frac{1}{2} \) outside a compact set. Let \( \{e_i\}_{i=1}^n \) be an orthonormal frame near infinity such that \( e_n = \frac{\nabla f}{|\nabla f|} \). It is known that the Laplace operators on \( M \) and \( \Sigma_\tau \) are related by the following formula: for any smooth function \( \omega \),

\[
\Delta_\omega = \nabla^2 \omega(e_n, e_n) + H_\tau e_n \omega + \Delta_{\Sigma_\tau} \omega.
\]
where $\Delta$ and $\Delta_{\Sigma_{\tau}}$ are the Laplacian operators on $M$ and $\Sigma_{\tau}$ respectively, $H_{\tau}$ is the mean curvature of $\Sigma_{\tau}$. From the soliton equation, we see that $\langle e_{i}, \nabla e_{i} \rangle = -\text{Ric}(e_{i}, e_{i})|\nabla f|^{-1}$ for $1 \leq i \leq n - 1$ and $H_{\tau} = \sum_{i=1}^{n} \langle \nabla_{e_{i}} e_{n}, e_{i} \rangle$. Hence

$$\Delta \omega = \Delta_{\Sigma_{\tau}} \omega + \frac{\nabla^{2} \omega(\nabla f, \nabla f)}{|\nabla f|^{2}} - \frac{R - \text{Ric}(\nabla f, \nabla f)}{|\nabla f|^{2}} \langle \nabla f, \nabla \omega \rangle. \quad (50)$$

We substitute $\omega = v R$ in (50) and estimate the terms on the R.H.S. of (50) one by one. Using (28), (26) and

$$2\text{Ric}(\nabla f, \nabla f) = \langle \nabla R, \nabla f \rangle = \Delta R + 2|\text{Ric}|^{2} = O(v^{-2}), \quad (51)$$

we get

$$-\frac{R - \text{Ric}(\nabla f, \nabla f)}{|\nabla f|^{2}} \langle \nabla f, \nabla (vR) \rangle = O(v^{-1}) (-|\nabla f|^{2}R + v\langle \nabla f, \nabla R \rangle) = O(v^{-2}). \quad (52)$$

For the Hessian term in the normal direction in (50), direct calculation and (51) yield

$$\nabla^{2}(vR)(\nabla f, \nabla f) = (vR)_{ij} f_{i} f_{j}$$

$$= vR_{ij} f_{i} f_{j} + 2v_{i} R_{i j} f_{j} + R_{ij} f_{i} f_{j}$$

$$= v\nabla^{2} R(\nabla f, \nabla f) - 2|\nabla f|^{2}(\nabla R, \nabla f) + R\text{Ric}(\nabla f, \nabla f)$$

$$= v\nabla^{2} R(\nabla f, \nabla f) + O(v^{-2}) + O(v^{-3})$$

$$= v\nabla^{2} R(\nabla f, \nabla f) + O(v^{-2}). \quad (53)$$

By (24) $\nabla R = 2\text{Ric}(\nabla f)$, the second Bianchi identity and Shi’s estimate (28),

$$v\nabla^{2} R(\nabla f, \nabla f) = vR_{ilklt, ijl f_{j}}$$

$$= 2v R_{kli k,j f_{i} f_{j}}$$

$$= 2v (R_{kli k, j} f_{j}) f_{i} f_{j} - 2v R_{kli k} f_{i} f_{j} f_{j}$$

$$= 2v ([R_{kli k, i} f_{j}] + |\text{Ric}|^{2}) f_{j} + \frac{v|\nabla R|^{2}}{2}$$

$$= v\langle \nabla \Delta R, \nabla f \rangle + 2v\nabla_{f} |\text{Ric}|^{2} + \frac{v|\nabla R|^{2}}{2}$$

$$= O(v^{-\frac{\delta}{2}}) + O(v^{-2})$$

$$= O(v^{-\frac{\delta}{2}}). \quad (54)$$

Hence by (22) and (25), there exists a constant $C_{0} > 0$ such that

$$\Delta_{\Sigma_{\tau}}(vR) - \langle \nabla f, \nabla (vR) \rangle \geq \Delta (vR) - \langle \nabla f, \nabla (vR) \rangle - O(v^{-\frac{\delta}{2}})$$

$$= v\Delta f R + R\Delta f v + 2\langle \nabla R, \nabla v \rangle - O(v^{-\frac{\delta}{2}}) \geq -2v|\text{Ric}|^{2} + R - C_{0}v^{-\frac{\delta}{2}}$$

near infinity. \hfill \Box

As an intermediate step toward the estimate on the curvature tensor $|Rm|$, we use the maximum principle to establish the exponential decay of the norm of Ricci tensor $|\text{Ric}|$. 


Lemma 3.6. Under the same assumptions as Proposition 3.1, the Ricci curvature decays exponentially near infinity,

\[ |\text{Ric}| \leq Ce^{-r}. \]  

Proof. For all small \( \varepsilon > 0 \), we can find a large \( a_1 > 1 + a_0 \) such that

\[ \frac{8C_0A_0^2}{\sqrt{a_1}} \leq \frac{1}{2} \quad \text{and} \quad \frac{2C_0}{\sqrt{a_1}} \leq \frac{\varepsilon}{1000}, \]

where \( a_0, C_0 \) and \( A_0 \) are the constants in Lemma 3.5 and Proposition 3.1 respectively.

Claim: For any \( y \) in the set of \( \{ v \geq a_1 \} \),

\[ v(y)R(y) \leq \varepsilon. \]

We first assume the claim and prove the lemma. By (26) (see also [14, Lemma 2]), \( \lim_{x \to \infty} r^{-1}v = 1 \) and hence from (58) \( \lim_{x \to \infty} rR = 0 \). Thanks to the assumption \( |\text{Ric}| \leq A_0R \),

\[ \Delta_f R = -2|\text{Ric}|^2 \geq -2A_0^2R^2. \]

We may then apply Proposition 1 and Lemma 3 in [14] to conclude that

\[ |\text{Ric}| \leq A_0R \]

\[ \leq Ce^{-r}. \]

Now it remains to justify the claim, i.e. (58). Suppose the claim does not hold. Then there exists \( y_0 \) in the set of \( \{ v \geq a_1 \} \) such that \( v(y_0)R(y_0) > \varepsilon \). By \( \liminf_{x \to \infty} rR = 0 \) and Lemma 3.4, we can find sequences of \( p_i \to \infty \) with \( \tau_i := v(p_i) > v(y_0) \) and \( \varepsilon_i := \sup_{\Sigma_{\tau_i}} vR \to 0 \) as \( i \to \infty \). Moreover for all large \( i, \varepsilon_i < \varepsilon \). Therefore, we can always choose a \( z_0 \in \{ \tau_i \geq v \geq a_1 \} \) with largest possible \( v(z_0) \) such that \( v(z_0)R(z_0) = \varepsilon \). By the choices of \( \varepsilon_i \) and \( z_0 \), \( v(z_0) < \tau_i \) and \( vR < \varepsilon \) on \( \{ \tau_i \geq v > v(z_0) \} \). We may invoke the parabolic maximum principle to conclude that at \( z_0 \)

\[ \langle \nabla f, \nabla (vR) \rangle \geq 0; \]

\[ \Delta_{\Sigma_{v(z_0)}} (vR) \leq 0. \]

By \( |\text{Ric}| \leq A_0R \) and Lemma 3.5

\[ 0 \geq -2A_0^2v^2R^2 + vR - C_0v^{-1/2}. \]

From the above inequality, we have two possible cases:

Case 1:

\[ \varepsilon = v(z_0)R(z_0) \]

\[ \geq 1 + \sqrt{1 - \frac{8C_0A_0^2}{v^{1/2}}} \]

\[ \geq \frac{3}{8A_0^2}, \]

which is impossible for all small \( \varepsilon \). We also used \( v(z_0) \geq a_1 \) and (57) in the last inequality.

Case 2:
\[ \varepsilon = v(z_0) R(z_0) \leq 1 - \sqrt{1 - \frac{8C_0A_0^2}{v^{1/2}}} \]
\[
= \frac{8C_0A_0^2v^{-1/2}}{(1 + \sqrt{1 - 8C_0A_0^2v^{-1/2}})4A_0^2} \leq 2C_0v^{-1/2} \leq 2C_0a_1^{-1/2}. \tag{65}
\]

It follows from (57) that \(\varepsilon \leq 2C_0a_1^{-1/2} \leq \frac{\varepsilon}{1000}.\) We again arrive at a contradiction. This justifies our claim and thus completes the proof. \(\Box\)

**Proof of Proposition 3.1:** Recall that our goal is to show \(|Rm| \leq C e^{-r}\). Deruelle [35, Lemma 2.8] proved a local derivative estimate for tensor \(T\) on an expanding gradient soliton satisfying an elliptic equation of the form
\[ \Delta f T = -\lambda T + Rm \ast T. \tag{67} \]
As pointed out by him in [35], the same argument also works for steady gradient Ricci soliton. Hence we apply his result with \(T = \text{Ric}\) and \(\lambda = 0\) to get
\[ |\nabla^2 \text{Ric}| \leq C e^{-r} \leq C' e^f. \tag{68} \]
Let \(\psi_s\) be the flow of the vector field \(-\nabla f/|\nabla f|^2\) with \(\psi_0 = \text{id}\). For all \(q\) near infinity, \(\psi_s(q)\) is well defined for all \(s \geq 0\) and
\[ f(\psi_s(q)) = f(q) - s. \tag{69} \]
Hence \(\psi_s(q) \to \infty\) as \(s \to \infty\). It follows from the Ricci identity and the soliton equation (see also (21)) that
\[ R_{kl,i} - R_{ki,l} = R_{ikj}f_j. \]
By the second Bianchi identity and (68), we differentiate the quantity \(|\text{Rm}|^2\) along the flow
\[
\frac{\partial}{\partial s}|\text{Rm}|^2(\psi_s(q)) = -\frac{\langle \nabla|\text{Rm}|^2, \nabla f \rangle}{|\nabla f|^2}
= -2|\nabla f|^{-2}R_{ijkl}R_{ijkl,\alpha}f_\alpha
= -2|\nabla f|^{-2}R_{ijkl}[ - (R_{ijka}f_\alpha)_k + (R_{ijka}f_\alpha)t - R_{ijla}R_{ak} + R_{ijka}R_{al}]
\geq -Ce^{f(q)-s}|\text{Rm}| - Ce^{f(q)-s}|\text{Rm}|^2
\geq -2Ce^{f(q)-s} - 2Ce^{f(q)-s}|\text{Rm}|^2. \tag{70}
\]
We may integrate the above inequality and get
\[ \ln \left(1 + |\text{Rm}|^2(\psi_s(q)) \right) \geq 2Ce^{f(q)-s} - 2Ce^{f(q)}. \tag{71} \]
Letting $s \to \infty$, together with the fact that $\lim_{x \to \infty} |Rm| = 0$, one has

$$|Rm|^2(q) \leq e^{2Ce^f(q)} - 1$$

(72)

$$\leq 2Ce^{2Ce^f(q)}e^{f(q)}$$

$$\leq C''e^f(q).$$

We also used $f \leq -10$ on $M$ in the last inequality. Thank to (26) and (72), we see that $\lim_{x \to \infty} r|Rm| = 0$. We then apply [14, Theorem 2] (see also [52]) and conclude that

$$|Rm| \leq Ce^{-r}.$$  

(73)

$\square$

**Proof of Theorem 1.3:** By our assumption $R \to 0$ as $x \to \infty$. It then follows from [14] that

$$|\text{Ric}| \leq c_1 |\text{Rm}| \leq A_0 R \text{ on } M.$$  

(74)

- If $\lim \inf_{x \to \infty} rR > 0$, then $R \sim r^{-1}$ near infinity and estimate (5) on $\text{Rm}$ is a consequence of (74).
- If $\lim \inf_{x \to \infty} rR = 0$, we apply Proposition 3.1 to see that

$$|\text{Rm}| \leq Ce^{-r}.$$  

(75)

By the lower bound for $R$ established by Chow-Lu-Yang [24] (see also [52]), $|\text{Rm}|$ is bounded below by a constant multiple of $e^{-r}$.  

$\square$

**Proof of Theorem 1.4:** Since $\text{Ric} \geq 0$ outside a compact subset, there exists a large positive constant $A_0$ such that

$$|\text{Ric}| \leq A_0 R.$$  

(76)

Using $|\text{Rm}| \to 0$ as $x \to \infty$ and an estimate on the potential function $f$ in [15, Proposition 1] (see also [22]), we have $\lim_{x \to \infty} f = -\infty$. Theorem 1.4 then follows similarly from Proposition 3.1 as in the proof of Theorem 1.3.  

$\square$

4. **Proof of Theorem 1.9**

We first investigate the topology of the level sets of $f$ and show that they are diffeomorphic to a quotient of torus at infinity if the curvature decays sufficiently fast.

**Proposition 4.1.** Let $(M^n, g, f)$ be a complete noncompact non Ricci flat steady gradient Ricci soliton of real dimension $n$ with proper potential function. Further suppose that the following conditions are satisfied:

- $|\text{Ric}| \leq A_0 R$ on $M$ for some constant $A_0 > 0$;
- $|\text{Rm}| \leq A_1(r + 1)^{-1}$ on $M$ for some constant $A_1 > 0$;
- $\lim_{r \to \infty} rR = 0$.

Then for all sufficiently large $\tau$, the level sets $\Sigma_\tau := \{x : -f(x) = \tau\}$ are diffeomorphic to a finite quotient of torus.
Proof. By Proposition 3.1 and a volume estimate by Munteanu-Sesum [51], we see that $|Rm|$ is integrable, i.e. in $L^1(M)$. Thanks to the properness of $f$ and a result of Munteanu-Wang [53], $M$ is connected at infinity and hence the level sets $\{-f = \tau\}$ are connected smooth compact hypersurfaces in $M$ for all large $\tau$. Moreover, there exists a large $\tau_0$ such that $\nabla f \neq 0$ on $\{-f \geq \tau_0\}$ and $\Sigma_\tau = \{-f = \tau\}$ are diffeomorphic to each other for all $\tau \geq \tau_0$. The same argument used by Deruelle [33, Proposition 2.3] shows that the level sets are diffeomorphic to a compact flat manifold. Indeed, if $\psi_s$ is the flow of $-\nabla f$ with $\psi_0 = \text{id}$. Then $\psi_s : \Sigma_{\tau_0} \to \Sigma_{\tau_0+s}$ are diffeomorphisms for all $s \geq 0$ and when restricted on the tangent space of the level set $T\Sigma_{\tau_0}$, the pull back metric satisfies

$$\frac{\partial}{\partial s}\psi_s^* g = -\psi_s^* \nabla f \frac{\nabla f}{|\nabla f|^2} g$$

$$= \psi_s^* \left( \frac{2 \text{Ric}}{|\nabla f|^2} \right)$$

$$= O(e^{-\tau_0-s}) \psi_s^* g.$$  

Proposition 3.1 was used in the last equation. Hence $\psi_s^* g$ are uniformly equivalent to each other for all $s \geq 0$ and we can find a constant $C > 0$ such that for all $\tau \geq \tau_0$

$$\text{Diam}(\Sigma_\tau) \leq C \text{Diam}(\Sigma_{\tau_0});$$

$$\text{Vol}(\Sigma_\tau) \leq C \text{Vol}(\Sigma_{\tau}).$$

(78)

Similar argument by Deruelle [33, Lemma 2.5] shows that for all nonnegative integer $k$, there exists $C_k$ independent of $\tau$ such that

$$|\nabla^k_{\Sigma_\tau} Rm_{\Sigma_\tau}| \leq C_k,$$

where $Rm_{\Sigma_\tau}$ and $\nabla_{\Sigma_\tau}$ denote the curvature tensor and the Riemannian connection of $\Sigma_\tau$ with respect to the induced metric by $g$. It follows from $\lim_{\tau \to \infty} \sup_{\Sigma_\tau} |Rm_{\Sigma_\tau}| = 0$ and the Hamilton Compactness Theorem [33] that for any sequence $\tau_i \to \infty$, by passing to subsequence if necessary, $\Sigma_{\tau_i}$ converges smoothly to a compact flat manifold as $i \to \infty$. Hence, for large $\tau$, $\Sigma_\tau$ is diffeomorphic to a finite quotient of the torus $\mathbb{T}^{n-1}$ by the Bieberbach’s Theorem (see [33] and reference therein).

With the topological restriction on the level sets in Proposition 4.1, we will prove that the level sets $\Sigma_\tau$ are flat with respect to the induced metric for all large $\tau$. The key observation is that torus does not admit any nontrivial metric with nonnegative scalar curvature. This approach was used by Deruelle [33] to study steady soliton with integrable curvature.

Lemma 4.2. Under the assumptions in Theorem 1.9 for all sufficiently large $\tau$, the level sets $\Sigma_\tau := \{x : -f(x) = \tau\}$ with induced metrics $g|_{\Sigma_\tau}$ from $M$ are flat.

Proof. By Remark 1.5 $|Rm| \leq cR$ for some positive constant $c > 0$ (when $m = 2$, the estimate follows from [14], where $m$ is the complex dimension). It can be seen from [15] Proposition 1] (see also [22]) that $f$ is proper. We may then invoke Proposition 4.1 to see that the level sets $\Sigma_\tau := \{x : -f(x) = \tau\}$ are diffeomorphic to a quotient of torus.

We are going to show that they have nonnegative scalar curvature. Using the properness of $f$, we may choose larger $\tau$ such that $\text{Ric} \geq 0$ on $\{x : -f(x) \geq \tau\}$. The second fundamental form of $\Sigma_\tau$ (w.r.t. the normal $-\nabla f$) is given by $-\frac{\nabla^2 f}{|\nabla f|^2} = \frac{\text{Ric}}{|\nabla f|^2} \geq 0$. Let $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{2m-1}$
be the eigenvalues of the second fundamental form of $\Sigma$. Then we must have
\[
H^2_{\Sigma_{\tau}} - |A_{\Sigma_{\tau}}|^2 = \left( \sum_{k=1}^{2m-1} \mu_k \right)^2 - \sum_{k=1}^{2m-1} \mu_k^2
\]
(80)
\[
= \sum_{1 \leq k < l} 2\mu_k \mu_l \\
\geq 0,
\]
where $H_{\Sigma_{\tau}}$ and $A_{\Sigma_{\tau}}$ refer to the mean curvature and the second fundamental form of $\Sigma_{\tau}$ respectively. By the Gauss equation, the intrinsic scalar curvature of $\Sigma_{\tau}$, denoted by $R_{\Sigma_{\tau}}$ satisfies
\[
R_{\Sigma_{\tau}} = R - \frac{2\text{Ric}(\nabla f, \nabla f)}{|
abla f|^2} + H^2_{\Sigma_{\tau}} - |A_{\Sigma_{\tau}}|^2
\]
(81)
\[
\geq R - \frac{2\text{Ric}(\nabla f, \nabla f)}{|
abla f|^2} - \frac{\text{Ric}(J\nabla f, J\nabla f)}{|
abla f|^2}
\]
\[
\geq 0,
\]
we also used the Kählerity of ambient metric $g$ in the last equality. Thus, the metric on $\Sigma_{\tau}$ is of nonnegative scalar curvature and hence is flat since there exists no nonflat metric on torus with nonnegative scalar curvature (See [41, 60]).

Now, we are in a position to prove Theorem 1.9.

**Proof of Theorem 1.9:** By Lemma 4.2, we have $\Sigma_{\tau}$ are flat for all large $\tau$. Hence equalities hold in (81) and we have the following identities outside a compact subset of $M$
\[
R(\nabla f)^2 = 2\text{Ric}(\nabla f, \nabla f)
\]
(82)
\[
R^2 = 2|\text{Ric}|^2
\]
(83)
Using a result of [45] (see also [36, 44]): Ricci solitons are real-analytic. Hence, the soliton metric $g_{ij}$ is real analytic in its geodesic normal coordinates. Moreover $f$ satisfies
\[
\Delta f = -R
\]
(84)
and by the elliptic regularity theory (see [39, p.110] and ref. therein), $f$ is also real analytic in the geodesic normal coordinates.

Thanks to the analytic continuation, we see that (82) and (83) indeed hold globally on $M$. Since the set of critical points $\{\nabla f = 0\}$ is nowhere dense in $M$, it follows from (82) and (83) that $\text{Ric} \geq 0$ on $M$ and the kernel of $\text{Ric}$ is a smooth subbundle of the tangent bundle of $M$ with real rank $2m - 2$. By the Kählerity of $g$, $\text{Ric}$ only has two distinct eigenvalues, namely 0 with real multiplicity $2m - 2$ and $\frac{R}{2}$ with real multiplicity 2. Moreover, $\nabla f$ and $J\nabla f$ span the eigenspace of the eigenvalue $\frac{R}{2}$ wherever $\nabla f \neq 0$.

To prove the splitting of $M$, we proceed as in [15] to show that the kernel of $\text{Ric}$ is invariant under parallel translation. Let $E$ be the kernel of $\text{Ric}$, it is a smooth subbundle of tangent bundle of real rank $2m - 2$ (see also [52]). Suppose at $p$, $\nabla f \neq 0$, by the orthogonal decomposition, the tangent space at $p$ can be splitted into $T_pM = E_p \oplus \text{span}\{\nabla f, J\nabla f\}$. Let $X$ be a smooth
section of \(E\) defined locally near \(p\) and \(Y\) be any smooth vector field defined near \(p\), then \(JX\) is also a smooth section of \(E\). At \(p\)
\[
\langle \nabla_Y X, \nabla f \rangle = Y(\langle X, \nabla f \rangle - \langle X, \nabla_y \nabla f \rangle) = \text{Ric}(X,Y) = 0.
\]
Similarly, \(\nabla_Y JX \perp \nabla f\), thus \(\nabla_Y X(p)\) is in \(E_p\). If \(\nabla f = 0\) at \(p\), by the real analyticity of \(g\) (see \[36, 44, 45\]), \(\{\nabla f = 0\}\) has no interior point in \(M\), we may find a sequence \(p_k \to p\) as \(k \to \infty\) with \(\nabla f(p_k) \neq 0\),
\[
\text{Ric}(\nabla_Y X)(p) = \lim_{k \to \infty} \text{Ric}(\nabla_Y X)(p_k) = 0.
\]
Hence, we conclude that \(E\) is invariant under parallel translation.

By the De Rham Splitting Theorem \[47, 48\], the universal covering space of \(M\) splits isometrically and holomorphically as \(M_1 \times M_2\), where \(M_1, M_2\) are complex \(m-1\) and \(1\) dimensional Kähler manifolds respectively. Moreover, the tangent bundle of \(M_1\) is given by the kernel of \(\text{Ric}\) and the tangent bundle of \(M_2\) coincides with the nonzero eigenspace of \(\text{Ric}\). From this we conclude that \(M_1\) is Ricci flat and \(M_2\) is nonflat. \(M\) also induces a steady Kähler gradient soliton structure on \(M_2\) and by the classification of complex \(1\) dimensional nontrivial complete steady Kähler gradient Ricci soliton \[23\], \(M_2\) is holomorphically isometric to the Cigar soliton \(\Sigma\). When \(m = 2\), \(M_1\) is flat. If \(m \geq 3\), since \(|R_m| \leq cR\) (see \[9\]) also holds on \(M_1 \times \Sigma\) and \(M_1\) is Ricci flat, for any \((a,b) \in M_1 \times \Sigma\),
\[
|R_m M_1|(a) \leq |R_m M_1 \times \Sigma|(a,b) \leq cR_{\Sigma}(b) \to 0\quad\text{as } b \to \infty,
\]
we used the fact that the curvature of \(\Sigma\) decays at infinity. Hence \(M_1\) is also flat for \(m \geq 3\). The exponential curvature decay is a consequence of the conditions \(\lim \inf_{x \to \infty} R = 0\), Theorems \[1.3\] and \[1.4\] This completes the proof of the theorem.

\[\square\]

5. Proof of Theorem 1.10

We begin with an elementary lemma on the regularity of the component functions of a parallel vector field on real analytic manifolds.

**Lemma 5.1.** Let \((M^n, g)\) be a complete Riemannian manifold such that in any geodesic normal coordinates, the corresponding metric coefficients \(g_{ij}\) are real analytic functions. Suppose that \(\gamma : [a,b] \to M\) is a geodesic and \(V(t)\) is a parallel vector field along \(\gamma(t)\). For any \(t_0 \in (a,b)\) and \(\{x_i\}\) a geodesic normal coordinate centered at \(\gamma(t_0)\), the component functions \(V^i(t)\) of \(V\) are real analytic functions in \(t\).

**Proof.** Let \(t_0\) and \(\{x_i\}\) be the number and geodesic coordinate as in the statement of the lemma. In the local coordinate, \(\gamma(t) = (t - t_0)a\) for some \(a \in \mathbb{R}^n\), \(V(t) = V^i(t) \frac{\partial}{\partial x^i}(\gamma(t))\) and \(\frac{dV}{dt} = 0\) can be rewritten as
\[
\frac{dV^i(t)}{dt} + a^jV^k(t)\Gamma^i_{jk}(t - t_0)a = 0,
\]
where $\Gamma^i_{jk}$ are the Christoffel symbols. Since $g_{ij}(x)$ are real analytic, $\Gamma^i_{jk}(x)$ are also real analytic in $x$ and hence $\Gamma^i_{jk}(t-t_0)a$ are real analytic in $t$. By [21, Theorem 1.3], $V(t)$, as the solution to the above ODE, is real analytic on $(t_0-\delta,t_0+\delta)$ for some $\delta > 0$. \qed

With the above preparations, we shall prove Theorem 1.10.

**Proof of Theorem 1.10:** Since the sectional curvature of $M$ is nonnegative near infinity and $R \to 0$ as $x \to \infty$, $f$ is proper ([15, 22]) and $|\text{Ric}| \leq A_0 R$ on $M$ for some positive constant $A_0$. Using Proposition 4.1, the level sets $\Sigma_\tau = \{f = \tau\}$ are diffeomorphic to a quotient of torus $T^{n-1}$ for all sufficiently large $\tau$. If $M$ has nonnegative sectional curvature at $p$, then for all unit tangent vector $v \in T_pM$,

(86) \[ R \geq 2\text{Ric}(v,v). \]

Therefore, we can apply the Gauss equation as in (81) to see that the intrinsic scalar curvature $R_{\Sigma_\tau}$ is nonnegative for all large $\tau$. Since it is well-known that on $T^{n-1}, n \geq 2$, any metric with nonnegative scalar curvature is flat (See [41, 60]). Hence, the induced metric on $\Sigma_\tau$ is flat and the equality in (80) holds and

(87) \[ R = \frac{2\text{Ric}(\nabla f, \nabla f)}{|
abla f|^2}. \]

It is thanks to the above equation and (86) that $\nabla f$ is an eigenvector of Ric and $\text{Ric}(\nabla f, v) = 0$ for all $v \in T\Sigma_\tau$. Hence

\begin{align*}
2|\text{Ric}|^2 &= 2R_{aa}^2 + 2|
abla f|^2|\text{A}_{\Sigma_\tau}|^2 \\
&= 2R_{aa}^2 + 2|
abla f|^2H_{\Sigma_\tau}^2 \\
&= 2R_{aa}^2 + 2(R - R_{aa})^2 \\
&= R^2,
\end{align*}

where $a = \frac{\nabla f}{|\nabla f|}$. We also used (87) in the last equality. It can be seen from (88) and an argument by Munteanu-Sung-Wang [52, Proposition 5.4] that wherever $M$ has nonnegative sectional curvature, Ric has two distinct eigenvalues, 0 with multiplicity $n-2$ and $R/2$ with multiplicity 2.

Let $K$ be a compact set such that $M$ has nonnegative sectional curvature and $\nabla f \neq 0$ on $M \setminus K$. By the strong maximum principle [26, Theorem 12.50], the kernel of the Ricci tensor Ric is invariant under parallel translation on $M \setminus K$. Due to the De Rham Splitting Theorem [59, Theorem 10.3.1], for all $p \in M \setminus K$, there are open neighborhood $U$ of $p$ in $M \setminus K$, manifolds $(U_1^{n-2}, g_1)$ and $(U_2^2, g_2)$ such that the following isometric splitting is true

(89) \[ (U, g|_U) \cong (U_1 \times U_2, g_1 + g_2); \]

$TU_1 = \text{null}(\text{Ric})$; $TU_2 = \text{null}(\text{Ric})^\perp$, where null(Ric) denotes the nullspace of the Ricci curvature. It can be seen from the splitting that $(U_1^{n-2}, g_1)$ is flat.

We are going to show that $M$ has nonnegative sectional curvature everywhere. Once it is established, Theorem 1.10 will be a consequence of the result by Deng-Zhu [29] (see also [52]). Fix $p \in M \setminus K$ and open set $U$ which splits isometrically as in (89), for any $q \in K$, let $\gamma : [0, d] \to M$ be a normalized geodesic joining $p$ to $q$, i.e. $\gamma(0) = p$ and $\gamma(d) = q$. Let $\{e_i\}_{i=1}^{n-2}$
and \( \{\mu_i\}_{i=1}^2 \) be orthonormal bases for \( \text{null}(\text{Ric}) \) and \( \text{null}(\text{Ric})^\perp \) at \( p \) respectively. Their parallel translations along \( \gamma \) are denoted by \( \{e_i(t)\}_{i=1}^{n-2} \) and \( \{\mu_i(t)\}_{i=1}^2 \). For any parallel vector fields \( A(t) \) and \( B(t) \) along \( \gamma \), we consider the following identities

\[
R(e_i(t), e_j(t), e_k(t), e_l(t))(\gamma(t)) = 0 \text{ for } i, j, k, l = 1, \cdots, n - 2;
\]

(90)

\[
R(e_i(t), \mu_j(t), A(t), B(t))(\gamma(t)) = 0 \text{ for } i = 1, \cdots, n - 2, j = 1, 2;
\]

(91)

\[
R(e_i(t), A(t), \mu_j(t), B(t))(\gamma(t)) = 0 \text{ for } i = 1, \cdots, n - 2, j = 1, 2;
\]

(92)

\[
2R(\mu_1(t), \mu_2(t), \mu_2(t), \mu_1(t))(\gamma(t)) - R(\gamma(t)) = 0;
\]

(93)

\[
L := \sup\{s \in [0, d] : (90), (91), (92) \text{ and } (93) \text{ hold for all } t \in [0, s]\}.
\]

By the isometric splitting of \( U \) in (89) and the invariance of \( \text{null}(\text{Ric}) \) and \( \text{null}(\text{Ric})^\perp \) under parallel translation on \( U \), we see that \( L > 0 \).

We claim that \( L = d \). Suppose on the contrary, it follows from a result of Kotschwar [45] that the metric coefficients \( g_{ij} \) of a Ricci soliton are real analytic in the geodesic normal coordinates (see also [36, 44]). By Lemma 5.1 the component functions of \( \{e_i(t)\}_{i=1}^{n-2} \), \( \{\mu_i(t)\}_{i=1}^2 \), \( A(t) \) and \( B(t) \) are real analytic in \( t \) near \( L \) in the geodesic normal coordinate of \( (M, g) \) centered at \( \gamma(L) \). Hence, the L.H.S. of (90), (91), (92) and (93) are real analytic functions in \( t \) on \( (L - \delta, L + \delta) \) and vanish on \( (L - \delta, L) \) for some \( \delta > 0 \). Thanks to the analytic continuation, the L.H.S. of (90), (91), (92) and (93) are identically zero on \( (L - \delta, L + \delta) \) and thus \( L \geq L + \delta \), which is absurd. This justifies our claim.

Finally. For any \( A, B \in T_q M \), we may write

\[
A = \sum_{i=1}^{n-2} A'_i e_i(d) + \sum_{\alpha=1}^{2} A''_\alpha \mu_\alpha(d);
\]

(95)

\[
B = \sum_{i=1}^{n-2} B'_i e_i(d) + \sum_{\alpha=1}^{2} B''_\alpha \mu_\alpha(d),
\]

where \( A'_i \), \( A''_\alpha \), \( B'_i \) and \( B''_\alpha \) are some constants independent on \( t \). Hence by (91), (92), (93) and \( L = d \),

\[
R(A, B, A)(q) = A'_i B'_j B''_{k} A''_l R(e_i(d), e_j(d), e_k(d), e_l(d)) + A''_\alpha B''_{\beta} B''_{\gamma} A''_\delta R(\mu_\alpha(d), \mu_\beta(d), \mu_\gamma(d), \mu_\delta(d))
\]

\[
= \frac{1}{2} (A''_1 B''_1 - A''_2 B''_2)^2 R(\gamma(d)) \geq 0.
\]

Thus \( M \) has nonnegative sectional curvature everywhere. Result then follows from the strong maximum principle argument as in [29, 52].

\[ \Box \]
6. GROMOV-HAUSSDORFF LIMIT AT INFINITY

In this section, we prove the following proposition which implies Corollary 1.8. Moreover, \( \text{Ric} \geq 0 \) near infinity is not needed in the proposition.

**Proposition 6.1.** Let \((M^n, g, f)\) be a complete non Ricci flat steady gradient Ricci soliton with dimension \( n \geq 4 \) and proper potential function \( f \).

(a) If \((5)\) or \((7)\) holds, then for any \( p_i \to \infty \) in \( M \), after passing to a subsequence, \((M, d_{R(p_i)g}, p_i)\) converges in pointed Gromov-Hausdorff sense to a cylinder \((\mathbb{R} \times Y, \sqrt{d_e^2 + d_Y^2}, p_\infty)\), where \( d_e \) is the flat metric on \( \mathbb{R} \), \((Y, d_Y)\) denotes a compact Alexandrov space and \( \sqrt{d_e^2 + d_Y^2} \) indicates the product metric.

(b) If instead \((6)\) or \((8)\) is true, then for any \( p_i \to \infty \) in \( M \), \((M, d_{R(p_i)g}, p_i)\) converges in pointed Gromov-Hausdorff sense (without passing to subsequence) to the ray \([0, \infty), d_e, 0)\), where \( d_e \) is the flat metric restricted on \([0, \infty)\). In this case, \((M, g)\) is smoothly asymptotic to the cylinder \( \mathbb{R} \times (\mathbb{T}^{n-1}/\sim) \) with flat product metric at exponential rate, where \( \mathbb{T}^{n-1}/\sim \) is diffeomorphic to the quotient of torus in Corollary 1.6.(a).

**Proof of Proposition 6.1.** By the properness of \( f \) \((26)\), there is a large \( \tau_0 \) such that on \( \{-f \geq \tau_0\} \)

\[
2^{-1}r \leq -f \leq 2r.
\]

Let \( R_i = R(p_i), \tau_i = -f(p_i) \to \infty, h_i = R(p_i)g \) and \( \tilde{h}_i = h_i|_{\Sigma_{\tau_i}} \). It follows from Lemma 3.2 that the intrinsic diameter of \( \Sigma_{\tau_i} = \{-f = \tau_i\} \) with respect to the scaled metric \( \tilde{h}_i = h_i|_{\Sigma_{\tau_i}} \) is uniformly bounded from above

\[
\text{diam} (\Sigma_{\tau_i}, \tilde{h}_i) \leq C \sqrt{\tau_i R_i} \\
\quad \leq C' \sqrt{r(p_i)^{-1} \tau_i} \\
\quad \leq C' \sqrt{2}.
\]

We will separate the argument into two cases, namely linear and exponential curvature decays.

**Case (a): Linear curvature decay**

We shall apply the level set method by Deng-Zhu \([29, 30, 31] \) to show the convergence. By the Gauss equation, \((5)\) or \((7)\) and \((96)\), we see that \( c^{-1} \tau_i^{-1} \leq R_i \leq c \tau_i \) and

\[
|Rm_{\Sigma_{\tau_i}}(\tilde{h}_i)| \leq CR_i^{-1}\left(|Rm(g)| + \frac{|\text{Ric}(g)|^2}{|\nabla g f|^2}\right) \\
\quad \leq C' \tau_i \left(|Rm(g)| + \frac{|\text{Ric}(g)|^2}{|\nabla g f|^2}\right) \\
\quad \leq C'' \tau_i (\tau_i^{-1} + \tau_i^{-2}) \\
\quad \leq C'''.
\]

By Gromov Compactness theorem \([9, \text{Theorem 10.7.2}] \) (see also \([59] \)), after passing to a subsequence,

\[
(\Sigma_{\tau_i}, d_{\tilde{h}_i}, p_i) \longrightarrow (Y, d_Y, p_\infty)
\]
converges in pointed Gromov-Hausdorff topology as \( i \to \infty \), where \((Y, d_Y, p_\infty)\) is a compact Alexandrov space.

Next we consider a type of sets introduced by Deng-Zhu [29, 30, 31], for any \( p \in M \) and \( k > 0 \)

\[
M_{p,k} := \left\{ y : \left| f(y) - f(p) \right| \leq \frac{k}{\sqrt{R(p)}} \right\}.
\]

Moreover, they [30, Lemma 3.1] showed that for all \( k > 0 \), there is a large \( I \) such that for all \( i \geq I \)

\[
B_{h_i}(p_i, k) \subseteq M_{p_i,k}.
\]

Let \( \psi_{s} \) be the flow of the vector field \(-\frac{\nabla g}{|\nabla g|^2}\) with \( \psi_0 = \text{id} \). Then for all large \( i \), \( \psi_{s} : \Sigma_{\tau_i} \to \Sigma_{\tau_i+s} \) are diffeomorphisms for \( s \in [-\frac{k}{R_{\tau_i}}, \frac{k}{R_{\tau_i}}] \). Hence we can define the following diffeomorphism \( \Gamma_i : [-k, k] \times \Sigma_{\tau_i} \to M_{p_i,k} \), where

\[
\Gamma_i(s, q) := \psi_{\sqrt{R_i}}(q).
\]

We compare the pull back metrics as in [31, Lemma 4.2]. When restricting on \( T\Sigma_{\tau_i} \),

\[
\frac{\partial}{\partial s} \psi_{s} \frac{\psi^*_{s}}{\sqrt{R_i}} h_i = \sqrt{R_i} \psi^*_{s} \left( \frac{2\text{Ric}}{|\nabla g|^2} \right) \leq C \sqrt{R_{i}^2} \psi_{s}^{2} h_i \leq \frac{2C \psi_{s}^{2} h_i}{s + \tau_i \sqrt{R_i}}.
\]

Similarly,

\[
\frac{\partial}{\partial s} \psi_{s} \frac{\psi^*_{s}}{\sqrt{R_i}} h_i \geq -\frac{2C \psi_{s}^{2} h_i}{s + \tau_i \sqrt{R_i}}.
\]

By integrating the above differential inequalities, we have for all large \( i \) and \( s \in [-k, k] \)

\[
\left(1 - \frac{2k}{\tau_i \sqrt{R_i}}\right)^{2C} h_i \leq \psi_{s} \frac{\psi^*}{\sqrt{R_i}} h_i \leq \left(1 + \frac{2k}{\tau_i \sqrt{R_i}}\right)^{2C} h_i \quad \text{on } T\Sigma_{\tau_i}.
\]

Since \( R \sim (-f)^{-1} \), we also have

\[
1 \leq \left| \frac{\partial}{\partial s} \psi_{s} \frac{\psi^*_{s}}{\sqrt{R_i}} \right|^{2} \frac{1}{|\nabla g|^2} = \left(1 - R\right)^{-1} \leq \left(1 - \frac{2C'}{\tau_i - c\sqrt{R_i}}\right)^{-1},
\]

Hence by \( R_i \sim \tau_i^{-1} \), we conclude that on \([-k, k] \times \Sigma_{\tau_i} \), for all large \( i \) (fixing \( k > 0 \))

\[
\left(1 - o(1)\right) \left(ds^2 + \tilde{h}_i\right) \leq \Gamma_i^* h_i \leq \left(1 + o(1)\right) \left(ds^2 + \tilde{h}_i\right),
\]

where \( \tilde{h}_i = h_i|_{\Sigma_{\tau_i}} \). In view of [32], for any \( \varepsilon > 0 \), we can consider a sequence of Gromov Hausdorff approximations \( F_i : (\Sigma_{\tau_i}, d_{\tilde{h}_i}, p_i) \to (Y, d_Y, p_\infty) \). Using [32], (100) and (106), one may check that \((\text{id} \circ F_i) \circ \Gamma_i^{-1}\) is an \( \varepsilon \) isometry from \( M_{p_i,k} \) to \([-k, k] \times Y \) for all large \( i \). This implies the pointed Gromov-Hausdorff convergence to the product space \( \left(\mathbb{R} \times Y, \sqrt{d_{\varepsilon}^2 + d_Y^2}, (0, p_\infty)\right) \) and finishes the proof for Proposition 6.1 in Case (a).
Remark 6.2. It can be seen from the Gauss equation, (21) and Shi’s estimate (28) that \((\Sigma_{\tau_i}, g_{\tilde{h}_i})\) has uniformly positive scalar curvature,

\[
R_{\tilde{h}_i} \geq R_i^{-1} R_g - 2R_i^{-1} \text{Ric}_g \left( \frac{\nabla f}{|\nabla f|} \left( \frac{\nabla f}{|\nabla f|} \right) \right) - cR_i^{-1} |\text{Ric}(g)|^2 \\
\geq c - c' \tau_i^{-1}.
\]

Case (b): Exponential curvature decay

Again by Lemma 3.2 and \(R \leq Ce^{-r} \leq C'e^f\),

\[
\text{diam}(\Sigma_{\tau_i}, \tilde{h}_i) \leq C\sqrt{\tau_i R_i} \leq C'\sqrt{\tau_i e^{-r_i}} \rightarrow 0 \text{ as } i \rightarrow \infty.
\]

Hence we have the following convergence in Gromov-Hausdorff sense (without taking subsequence)

\[
(\Sigma_{\tau_i}, d_{\tilde{h}_i}, p_i) \rightarrow (\{0\}, d_0, 0) \text{ as } i \rightarrow \infty,
\]

where \(d_0\) is the discrete metric on the singleton \(\{0\}\).

To proceed, we define another type of sets similar to \(M_{p,k}\) in (99), namely

\[
N_{p,k} := \{y : f(y) \geq f(p) - \frac{k}{\sqrt{R(p)}}\}.
\]

We first show an analog to (100): for any \(k > 0\)

\[
B_{h_i}(p_i, k) \subseteq N_{p_i,k}.
\]

Suppose on the contrary, we can find a point \(z \in B_{h_i}(p_i, k) \setminus N_{p_i,k}\) and a distance minimizing geodesic \(\gamma : [0, T] \rightarrow M\) with respect to \(h_i\) joining \(p_i\) to \(z\). By restricting \(\gamma\) on a smaller interval if necessary, we may further assume that for all \(t\), \(-f(\gamma(t)) \leq \tau_i + kR_i^{-1/2}\), \(f(\gamma(0)) = f(p_i) = -\tau_i\) and \(f(\gamma(T)) = -\tau_i - kR_i^{-1/2}\). Hence by \(|\nabla_g f|_g \leq 1\),

\[
k > d_{h_i}(p_i, z) \geq l_{h_i}(\gamma) = \sqrt{R_i} \int_0^T |\dot{\gamma}|_g(s) \, ds \\
\geq -\sqrt{R_i} \int_0^T \langle \dot{\gamma}, \nabla_g f \rangle_g(s) \, ds \\
= k,
\]

which is impossible. Therefore, we must have (111). To show the pointed Gromov-Hausdorff convergence, for any \(\varepsilon > 0\), we construct an \(\varepsilon\) isometry from \(F_i : N_{p_i,k} \rightarrow [0, \infty)\) for all large \(i\),

\[
F_i(x) := \begin{cases} 
0 & \text{if } f(x) > -\tau_i \\
-\sqrt{R_i} (f(x) + \tau_i) & \text{if } f(x) \leq -\tau_i.
\end{cases}
\]

Obviously, \(F_i(p_i) = 0\). As before, \(\psi_s\) denote the flow of the vector field \(-\frac{\nabla f}{|\nabla f|_g}\) with \(\psi_0 = \text{id}\). For any \(a, b \in N_{p_i,k}\) satisfying \(-\tau_i \geq f(a) \geq f(b)\), let \(\beta := f(a) - f(b) \geq 0\), by the proof of (78) in Proposition 3.1.
Hence

\[ d_{h_i}(a, b) \leq d_{h_i}(a, \psi_\beta(a)) + d_{h_i}(\psi_\beta(a), b) \]

(112)

\[ \leq \sqrt{R_i} \int_0^\beta \frac{1}{||\nabla f||_g} \, ds + \sqrt{R_i} \, \text{diam} \, (\Sigma_{-f(b)}, g) \]

\[ \leq \sqrt{R_i}(1 - ce^{-\tau_i})^{-1}\beta + C \, \text{diam} \, (\Sigma_{\tau_i}, \tilde{h}_i) \]

Using a similar argument as in the proof of (111), we also have

(113)

\[ d_{h_i}(a, b) \geq \sqrt{R_i}\beta. \]

Hence by (108), (112) and fixing \( k > 0 \), we may take \( i \) to be sufficiently large such that

(114)

\[ |F_i(a) - F_i(b) - d_{h_i}(a, b)| = |\sqrt{R_i}\beta - d_{h_i}(a, b)| \]

\[ \leq 2ce^{-\tau_i} + C \, \text{diam} \, (\Sigma_{\tau_i}, \tilde{h}_i) \rightarrow 0. \]

When \( f(a) > -\tau_i \geq f(b) \), it follows from (96) that there is a positive constant \( c_0 \) such that for all large \( i \)

(115)

\[ \{ x : f(x) \geq -\tau_i \} \subseteq B_g(p_0, 2\tau_i + c_0) \]

and thus for all \( y, z \in \{ x : f(x) \geq -\tau_i \} \),

\[ d_{h_i}(y, z) \leq (4\tau_i + 2c_0)\sqrt{R_i} \leq C(4\tau_i + 2c_0)e^{-\tau_i/2}. \]

By (108) and (113), we see that

\[ |F_i(a) - F_i(b)| - d_{h_i}(a, b) \leq |F_i(p_i) - F_i(b)| - d_{h_i}(p_i, b) + d_{h_i}(a, p_i) \]

\[ \leq 2ce^{-\tau_i} + C \, \text{diam} \, (\Sigma_{\tau_i}, \tilde{h}_i) \]

\[ +C(4\tau_i + 2c_0)e^{-\tau_i/2} \rightarrow 0. \]

It remains to verify that \( F_i \) is almost surjective, i.e. \( [0, k - \varepsilon] \subseteq F_i(B_{h_i}(p_i, k)) \). From the construction of \( F_i \), we have for any \( s \in [0, k - \varepsilon] \), \( s = F_i\left(\Sigma_{\tau_i + s/\sqrt{R_i}}\right) \). Thanks to (112), for all large \( i \) (fixing \( k \)),

\[ d_{h_i}(p_i, \Sigma_{\tau_i + s/\sqrt{R_i}}) \leq (1 - ce^{-\tau_i})^{-1}s + C \, \text{diam} \, (\Sigma_{\tau_i}, \tilde{h}_i) \]

\[ < k. \]

Hence \( F_i \) is an \( \varepsilon \) isotopy and we get the pointed Gromov-Hausdorff convergence to the ray. For the smooth convergence to a cylinder, by the local Shi derivative estimates [35 Lemma 2.6], \( \nabla R = 2\text{Ric}(\nabla f) \) and (2), we have for all integer \( k \geq 0 \), there is a positive constant \( C_k \) such that

(116)

\[ |\nabla^k \text{Rm}(g)| \leq C_k R \leq C'_k e^{-r} \text{ on } M. \]

When restricted on \( \Sigma_{\tau_0} \), \( \psi_s : \Sigma_{\tau_0} \rightarrow \Sigma_{\tau_0 + s} \) are diffeomorphisms for all \( s \geq 0 \) and we denote the pull back metric on \( \Sigma_{\tau_0} = \{ -f = \tau_0 \} \) by \( g_s := \psi_s^* g \). Since the second fundamental form of \( \Sigma_{\tau_0 + s} \) is \( -\frac{\text{Ric}}{\nabla f|_g} \), we may apply (116) and the computation in (77) to conclude that

(117)

\[ \left| \nabla^k g_s \left( \frac{\partial}{\partial s} \right)_g \right|_{g_s} \leq C_k e^{-\tau_0 - s} \]
for all $s \geq 0$ and integer $k \geq 0$. By [6, Proposition A.5], $g_s$ converges in $C^\infty$ sense to a smooth metric $g_\infty$ on $\Sigma_{\tau_0}$ as $s \to \infty$. The limit $g_\infty$ agrees with the subsequential limit in the proof of Proposition 4.1 and thus is flat. Moreover, $C^{-1}g_s \leq g_\infty \leq Cg_s$ for all $s \geq 0$. By the compactness of $\Sigma_{\tau_0}$ and the smooth convergence, for all $k \geq 0$, $|\nabla^k_{g_\infty}g_s|_{g_\infty} \leq C'_k$ for all $s \geq 0$. One may then argue by induction as in [6, Lemma A.4] to see that

$$|\nabla^k_{g_\infty} \frac{\partial}{\partial s} g_s|_{g_\infty} \leq C_k e^{-\tau_0 - s}.$$  

Hence by integrating the above estimates, we have for all $k \geq 0$,

$$|\nabla^k_{g_\infty} (g_s - g_\infty)|_{g_\infty} \leq C_k e^{-\tau_0 - s}.$$  

We define the asymptotic cylinder $L$ for the steady soliton $(M, g)$ as follows. Let $L := \mathbb{R} \times \Sigma_{\tau_0}$ with the product metric $g_L := ds^2 + g_\infty$. $\Phi : (0, \infty) \times \Sigma_{\tau_0} \to \{-f > \tau_0\}$ is the diffeomorphism given by $\Phi(s, \omega) := \psi_s(\omega)$. It can be seen that

$$\Phi^* g = |\nabla f|_{g}^{-2} ds^2 + g_s.$$  

Then by (116), (119) and a direct (though tedious) induction argument, we have for any integers $k, p \geq 0$, and vectors $w_i \in T\Sigma_{\tau_0}$ with $|w_i|_{g_\infty} = 1$,

$$|\nabla^p_{g_\infty} \frac{\partial^k}{\partial s^k} \left( \Phi^*_{s, \omega} g - g_L \right)(w_1, \cdots, w_{p+2})| \leq C_{k,p} e^{-\tau_0 - s},$$

where $C_{k,p}$ is some positive constant. Estimate (121), together with the fact that $\nabla_{g_L} \frac{\partial}{\partial s} \equiv 0$ on $L$, implies the asymptotic convergence to $(L, g_L)$ at exponential rate. This completes the proof of Proposition 6.1. □
Appendix A. Dichotomy in the expanding case

We shall give a proof of Theorem 1.13 which is a direct consequence of the results due to Deruelle [34, 35]. The key ingredient of the proof is the application of the existence and compactness results of conical expander in [34, 35].

Proof of Theorem 1.13: We pick a smooth metric $h$ on $X = \mathbb{S}^2$ with positive but nonconstant Gauss curvature, for instance, the one induced by an ellipsoid embedded in $\mathbb{R}^3$. By scaling the metric $h$ if necessary, we may assume that the curvature operator of $h$ satisfies

$$Rm(h) \geq id_{\Lambda^2T^*X} \text{ on } X$$

with equality holds somewhere at $\omega_0 \in X$ (this is possible since $X$ is compact and of real dimension 2). By the existence result of conical expander [34, Theorem 1.3], there exists an asymptotically conical gradient expander $(M^3, g, f)$ with $Rm(g) \geq 0$ and asymptotic cone given by $(C(X), dt^2 + t^2 h)$. Indeed, let $\{c_i\}_{i=1}^\infty$ be a strictly increasing sequence of positive numbers with $\lim_{i \to \infty} c_i = 1$. We denote the metric $c_i h$ by $h_i$ and by (122)

$$Rm(h_i) > id_{\Lambda^2T^*X} \text{ on } X.$$ 

By [34, Theorem 1.3], for each $i$, there exists an asymptotically conical gradient expander $(M^3_i, g_i, f_i)$ with $Rm(g_i) > 0$ and asymptotic cone given by $(C(X), dt^2 + t^2 h_i)$. Since for all large $i$, $(X, h_i)$ satisfies

$$|\nabla^k Rm(h_i)|(|\omega|) = \frac{k_{i/2}^{k+2}}{c_i^{k+2}} \leq 2 \sup_X |\nabla^k Rm(h)|.$$ 

It follows from [34, Remark 4.11] and (13) that we can find a sequence of positive numbers $\{\Lambda_k\}_{k=1}^\infty$ independent of $i$ such that

$$\limsup_{x \to \infty} r_i^{2+k} |\nabla^k Rm(g_i)| \leq \Lambda_k$$

and the asymptotic volume ratio is bounded from below

$$\lim_{r \to \infty} \frac{Vol_{g_i} (B_{g_i}(p,r))}{r^3} \geq \frac{Vol_{g}(X)}{6}.$$ 

where $r_i$ is the distance function w.r.t. $g_i$ on $M_i$. Shifting the potential $f_i$ by constants if necessary, we apply the compactness result for conical expander by Deruelle [35, Theorem 4.9]) and conclude that $(M^3_i, g_i, f_i)$ converges smoothly and subsequentially as $i \to \infty$ to an asymptotically conical expander $(M^3, g, f)$ with $Rm(g) \geq 0$ and asymptotic cone given by $(C(X), dt^2 + t^2 h)$. Hence the existence of conical expander asserted follows. Using $Rm(g) \geq 0$ and [35, Proposition 2.4], we have $\lim_{x \to \infty} 4 r^{-2} v = 1$, where $v := n/2 - f$. It remains to justify (17). By the virtue of (13), we see that $\limsup_{x \to \infty} v |Rm| < \infty$ and for any $\omega \in X$

$$\lim_{t \to \infty} 4 v \circ \phi^{-1}(t,\omega)|Rm(g)| \circ \phi^{-1}(t,\omega) = |Rm(g_C)|(1,\omega),$$

26
where $g_C = dt^2 + t^2 h$ and $\phi$ is the diffeomorphism as in Definition 1.12. Since $g_C$ is a warped product with warping function $t$, its curvature tensor satisfies:

(128) \[ \text{Rm}(g_C) \left( \frac{\partial}{\partial t}, \cdot, \cdot, \cdot \right) = 0; \]

(129) \[ \text{Rm}(g_C)(A, B, C, D)(t, \omega) = t^2 \left[ \left( \text{Rm}(h)(A, B, C, D)(\omega) - (h(A, D)h(B, C) - h(A, C)h(B, D))(\omega) \right \right] \]

for any $A, B, C, D \in T_\omega X$. Hence $\forall \omega \in X$,

(130) \[ |\text{Rm}(g_C)|(1, \omega) = \left| \text{Rm}(h) \right| - \frac{h \bigcirc h}{2} (\omega), \]

where $h \bigcirc h_{\alpha\beta\gamma\delta} := 2h_{\alpha\delta}h_{\beta\gamma} - 2h_{\alpha\gamma}h_{\beta\delta}$. By the construction of $h$ (122), $\text{Rm}(h)(\omega_0) = \text{id}_{\Lambda^2 T_\omega X}(\omega_0)$ and (127), it can be seen that

$$0 = |\text{Rm}(g_C)|(1, \omega_0) = \liminf_{x \to \infty} 4v|\text{Rm}|.$$

As $(X, h)$ is not of constant curvature and satisfies (122), there exists $\omega_1 \in X$ such that $\text{Rm}(h)(\omega_1) > \text{id}_{\Lambda^2 T_\omega X}(\omega_1)$ and thus both $|\text{Rm}(g_C)|(1, \omega_1)$ and $\limsup_{x \to \infty} v|\text{Rm}|$ are positive. This justifies (17). \qed
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