Classes of Analytic Functions Involving a Generalization by the Srivastava-Attiya Operator

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Abstract. In this research article, we present a class defined by a new generalization operator \( \Upsilon_{\mu,\varphi,\kappa,\varepsilon,\zeta,\psi} \) that is studied and investigated recently by many authors. We present several important properties for the function \( f(z) \) which belongs to the class \( \Xi_{\mu,\varphi,\kappa,\varepsilon,\zeta,\psi} \) in the unit disc.

1. Prelude and Definitions

Let \( \Sigma_C \) indicate the class of the following functions:

\[ \Psi(z) = z + \sum_{\kappa=1}^{\infty} \varrho_{\kappa} z^\kappa, \]

which are regular in complex unit disk \( U_C = \{ z : |z| < 1 \} \). Let \( C_V \) be the class of all regular convex functions \( Y \in \Sigma_C \). The operator is defined by Srivastava-Attiya and indicated in article [1] (see also [2, 3]):

\[ \Xi_{\mu,\varphi}(\Psi)(z) = z + \sum_{\kappa=1}^{\infty} \left( \frac{\varphi + 1}{\varphi + \kappa} \right)^\mu \varrho_{\kappa} z^\kappa, \]

where \( z \in U_C, \varphi \in C \setminus \{ 0, 1 \}, \mu \in C \) and \( \Psi \in \Sigma_C \). Indeed, \( \Gamma_{\mu,\varphi}(\Psi) \) is presented as

\[ \Xi_{\mu,\varphi}(\Psi)(z) := X_{\mu,\varphi}(z) \ast \Psi(z) \]

with the stipulation of convolution where \( X_{\mu,\varphi}(z) \) is taken as

\[ X_{\mu,\varphi}(z) := (1 + \varphi)^\mu \left[ E(z, \mu, \varphi) - \varphi^{-\mu} \right], \quad (z \in U_C), \]

where \( E(z, \mu, \varphi) \) is regular function (see, [4, p. 121 \textit{et seq.}], [5, p. 194 \textit{et seq.}])

\[ E(z, \mu, \varphi) := \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{(\kappa + \varphi)^\mu} \]
\[(a \in C \setminus \mathbb{Z}_0^-; s \in C \text{ when } |z| < 1; \Re(\mu) > 1 \text{ when } |z| = 1).\]

Srivastava [6] has proposed a new special function, namely, \(z\)-generalized Hurwitz-Lerch zeta function, (see [7, 8, 9, 10, 11, 12, 13]). Srivastava considered the following function:

\[
E_{\kappa, \ldots, \kappa; \omega_1, \ldots, \omega_{\ell}; \sigma, \nu}(z, \mu, \varphi; \sigma, \nu) = \frac{1}{\kappa} \Gamma(\mu) 
\cdot \sum_{\kappa=0}^{\infty} \prod_{\eta=1}^{\kappa} (\omega_\eta)_{\kappa^\eta} \mathcal{H}_{0,2}^{2,0} \left[ (\varphi + \kappa)\sigma^\frac{1}{z} \right] (\mu, 1, 0, \frac{1}{\kappa}, \kappa) \frac{z^\kappa}{\kappa!} \tag{5}
\]

where

\[
\left( \kappa_\eta \in C \ (\eta = 1, \ldots, \kappa) \right) \text{ and } \omega_\eta \in C \setminus \mathbb{Z}_0^- \ (\eta = 1, \ldots, \ell); \ \tau_\eta > 0 \ (\eta = 1, \ldots, \kappa);
\]

\[
v_\eta > 0 \ (\eta = 1, \ldots, \ell); 1 + \sum_{\eta=1}^{\ell} v_\eta - \sum_{\eta=1}^{\kappa} \tau_\eta \geq 0.
\]

Furthermore, the crucial stipulation is convergence, which is correct for values bounded of \(|z|\), as mentioned:

\[
|z| < U := \left( \prod_{\eta=1}^{\kappa} \tau_\eta^{-\tau_\eta} \right) \left( \prod_{\eta=1}^{\ell} v_\eta \right).
\]

Here, \((\kappa)_\delta\) refer to Pochhammer symbol formulated by:

\[
(\kappa)_\delta := \frac{\Gamma(\kappa + \delta)}{\Gamma(\kappa)} = \begin{cases} \kappa(\kappa + 1) \cdots (\kappa + \kappa - 1) & (\delta = \kappa \in \mathbb{N}; \ \kappa \in C) \\ 1 & (\delta = 0; \ \kappa \in C \setminus \{0\}) \end{cases} \tag{6}
\]

**Definition 1:** Let \(\mathcal{H}\) be the function, the \(\mathcal{H}\)-function included in the part of the right side of (5) is so-called Fox’s \(\mathcal{H}\)-function given as:

\[
\mathcal{H}_{a,b}^{c,d}(z) = \int_{L} \Delta(\mu)z^{-\mu} d\mu \quad (z \in C \setminus \{0\}; \ |\arg(z)| < \pi),
\]

where

\[
\Delta(\mu) = \prod_{\eta=1}^{a} \Gamma(\sigma_\eta + \mathcal{B}_\eta \mu) \cdot \prod_{\eta=1}^{b} \Gamma(1 - \phi_\eta - \mathcal{A}_\eta \mu) 
\cdot \prod_{\eta=1}^{a} \Gamma(\sigma_\eta + \mathcal{B}_\eta \mu) \cdot \prod_{\eta=1}^{b} \Gamma(1 - \phi_\eta - \mathcal{A}_\eta \mu).
\]

is a product interpreted as \(1, a, b, c\) and \(s\) are integers such that \(1 \leq a \leq s, 0 \leq b \leq r, \ \mathcal{A}_\eta > 0 \ (\eta = 1, \cdots, r), \ \mathcal{B}_\eta > 0 \ (\eta = 1, \cdots, s), \ \phi_\eta \in C \ (\eta = 1, \cdots, r), \ \sigma_\eta \in C \ (\eta = 1, \cdots, s)\) and \(L\) is an appropriate type of Mellin-Barnes contour that separates the poles of gamma functions \( (\Gamma(\sigma_\eta + \mathcal{B}_\eta \mu))_{\eta=1}^{a} \).
through function poles
\[ \{ \Gamma(1 - \varphi_\eta + \mathfrak{R}_\eta \mu) \}_{\eta=1}^\nu. \]
It is substantial to point out the fact that [6, p. 1496, see remark 7]

\[
\lim_{\mu \to 0} \left\{ \mathcal{H}_{0,2}^{2,0} \left[ (\psi + \kappa) \sigma^{\frac{1}{2}} \left( \mu, 1, \left( \frac{1}{z} \right) \right) \right] \right\} = \kappa \Gamma(\mu) \quad (\kappa > 0),
\]
and the equation is mentioned by (5) reduces to

\[
E(\zeta, \kappa, \tau, \nu) \quad (\zeta, \mu, \varphi; 0, \kappa) := E(\zeta, \kappa, \tau, \nu) \quad (\zeta, \mu, \varphi)
\]

Definition 2: The function \( E_{\zeta, \kappa, \tau, \nu} \quad (\zeta, \mu, \varphi) \) included in (9) is the multi-parameter new extended Hurwitz-Lerch zeta function \( E(\zeta, \mu, \varphi) \) is given as: (see Srivastava etc. [13, p. 503, Eq. (6.2)])

\[
E(\zeta, \kappa, \tau, \nu) := \sum_{k=0}^{\infty} \frac{\prod_{\eta=1}^{\kappa} (\zeta_\eta)^{k\tau_\eta}}{(\psi + \kappa)^{\mu} \cdot \prod_{\eta=1}^{\ell} (\omega_\eta)^{k\nu_\eta}} \quad (\zeta, \mu, \varphi)
\]

\[
\left( \kappa, \ell \in \mathbb{N}_0; \ z_\eta \in \mathbb{C} (\eta = 1, \cdots, \kappa); \ \psi, \omega_\eta \in \mathbb{C} \setminus \mathbb{Z} (\eta = 1, \cdots, \ell); \right.
\]
\[
\tau_\eta, v_\eta \in \mathbb{R}^+ (\eta = 1, \cdots, \kappa; \ \eta = 1, \cdots, \ell); \quad \Omega > -1 \text{ when } \mu, z \in \mathbb{C};
\]
\[
\Omega = -1 \text{ and } \mu \in \mathbb{C} \text{ when } |z| < \mathbb{U}^*;
\]
\[
\Omega = -1 \text{ and } \Re(\Delta) > \frac{1}{2} \text{ when } |z| = \mathbb{U}^*
\]

with

\[
\mathbb{U}^* := \left( \prod_{\eta=1}^{\kappa} \tau_\eta^{-\tau_\eta} \right) \cdot \left( \prod_{\eta=1}^{\ell} v_\eta^{\nu_\eta} \right),
\]

\[
\Omega := \sum_{\eta=1}^{\ell} \sigma_\eta - \sum_{\eta=1}^{\kappa} \tau_\eta \quad \text{and} \quad \Delta := \omega + \sum_{\eta=1}^{\ell} \omega_\eta - \sum_{\eta=1}^{\kappa} \zeta_\eta + \frac{\kappa - \ell}{2}.
\]

The following operator \( \Xi_{\sigma, \omega}^{\mu, \rho, \kappa} (\mathfrak{g}) \) was presented by Srivastava and Gaboury [9] (see also, [14], [15], and [10]):

\[
\Xi_{\sigma, \omega}^{\mu, \rho, \kappa} (\mathfrak{g}) : \Sigma_C \to \Sigma_C.
\]
It is denoted as follows:

\[ \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z) = \chi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(z) \ast \mathcal{Y}(z), \]  

(13)

where

\[ \chi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(z) := \frac{\kappa}{\prod_{\eta=1}^{\kappa} (\kappa_{\eta})} \cdot \frac{\kappa}{(\varphi + 1, \sigma, \mu)} - \frac{\varphi^{-\mu}}{\kappa} \chi(\varphi, \sigma, \mu) \]

\[ = z + \sum_{k=2}^{\infty} \sum_{k=1}^{\kappa} \frac{(\kappa_{\eta} + 1)}{(\omega_{\eta} + 1)_{k-1}} \left( \frac{\varphi + 1}{\varphi + k} \right)^{\mu} \left( \frac{\Psi(\varphi + k, \sigma, \mu, \chi)}{\Psi(\varphi + 1, \sigma, \mu, \chi)} \right) \frac{z^k}{k!}, \]

(14)

with

\[ \Psi(\varphi, \sigma, \mu, \chi) := \mathcal{H}_{0,2}^{\varphi, \sigma} \left[ \varphi \sigma \pi \right], \]

(15)

By combining between (13) and (14), we have

\[ \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z) = \Psi(\varphi, \sigma, \mu, \chi) \]

\[ = z + \sum_{k=2}^{\infty} \sum_{k=1}^{\kappa} \frac{(\kappa_{\eta} + 1)}{(\omega_{\eta} + 1)_{k-1}} \left( \frac{\varphi + 1}{\varphi + k} \right)^{\mu} \left( \frac{\Psi(\varphi + k, \sigma, \mu, \chi)}{\Psi(\varphi + 1, \sigma, \mu, \chi)} \right) \frac{z^k}{k!}, \]

(15)

\[ \left( \kappa_{\eta} \in C \ (\eta = 1, \ldots, \kappa) \text{ and } \omega_{\eta} \in C \setminus \mathbb{Z}_0^- \ (\eta = 1, \ldots, \ell); \ k \leq \ell + 1; \ z \in \mathbb{C} \right), \]

with

\[ \min\{\Re(\varphi), \Re(\mu)\} > 0; \ \kappa > 0 \text{ if } \Re(\sigma) > 0 \]

and

\[ \mu \in C; \ \varphi \in C \setminus \mathbb{Z}_0^- \text{ if } \sigma = 0. \]

**Remark 1:** It follows by (13) and (15), then \( \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y}) \) (15) can be determined for \( \varphi \in C \setminus \mathbb{Z}^- \) by the limit relationship as follows:

\[ \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z) = \lim_{\varphi \to 0} \left[ \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z) \right]. \]

(16)

For \(-1 \leq \varepsilon < 1, \xi \geq 0 \text{ and } \psi \in C \setminus \{0\}, \) we find \( \mathcal{Y} \in \Sigma_C \) is in the class \( \Gamma^{\mu,\varphi,\kappa,\xi,\xi,\psi}_{(\omega_x),\omega_r,\sigma} \) if

\[ \Re \left( 1 + \frac{1}{\psi} \left( z \frac{\left( \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z) \right)'}{\Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z)} - \varepsilon \right) \right) > \xi \left( 1 + \frac{1}{\psi} \left( z \frac{\left( \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z) \right)'}{\Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y})(z)} - 1 \right) \right), \]

(17)

where \( z \in \mathbb{U}_C \) and \( \Xi^{\mu,\varphi,\kappa}_{(\omega_x),\omega_r,\sigma}(\mathcal{Y}) \) is given by (15).

The essential purpose of the article is to study the foremost features, such as upper bounds bound, extreme points, radii of close to convexity, star-likeness and convexity for the class \( \Gamma^{\mu,\varphi,\kappa,\xi,\xi,\psi}_{(\omega_x),\omega_r,\sigma}. \) In addition, neighbourhood and integral mean outcomes for \( \Gamma^{\mu,\varphi,\kappa,\xi,\xi,\psi}_{(\omega_x),\omega_r,\sigma} \) are discussed.
2. Coefficient inequalities
The results of the first section give an essential stipulation for \( \Psi \), mentioned by (1), in \( \Psi_{(\mu,\varphi,\sigma,\nu,\xi,\mu,\nu,\xi,\nu)} \).

**Theorem 1:** et \( \Psi \) be a function given by (1), then \( \Psi \in \Psi_{(\mu,\varphi,\sigma,\nu,\xi,\mu,\nu,\xi,\nu)} \) if and only if

\[
\sum_{k=2}^{\infty} \frac{(k + \psi)(1 - \zeta) - (e - \zeta)}{k!} \prod_{\eta=1}^{k} (\omega_{\eta} + 1) \frac{\mu}{(\varphi + k)} \left( \frac{\Psi(\varphi + \kappa, \sigma, \mu, \xi)}{\Psi(\varphi + 1, \sigma, \mu, \xi)} \right) \xi_k \leq (1 - e) + \psi (1 - \zeta) \tag{18}
\]

where \(-1 \leq e < 1, \zeta \geq 0\) and \( \psi \in C \setminus \{0\} \).

**Proof:** Let \( \Psi \in \Psi_{(\mu,\varphi,\sigma,\nu,\xi,\mu,\nu,\xi,\nu)} \). Therefore, from (17) we have the following

\[
\Re \left\{ 1 + \frac{1}{\psi} \left( z \left( 1 - e + \sum_{k=2}^{\infty} \prod_{\eta=1}^{k} (\omega_{\eta} + 1) \frac{\mu}{(\varphi + k)} \left( \frac{\Psi(\varphi + \kappa, \sigma, \mu, \xi)}{\Psi(\varphi + 1, \sigma, \mu, \xi)} \right) \xi_k \right) - \zeta \right) \right\}.
\]

If we let \(|z| \to 1\) along the x-axis, we acquire

\[
\Re \left\{ 1 + \frac{1}{\psi} \left( \sum_{k=2}^{\infty} \prod_{\eta=1}^{k} (\omega_{\eta} + 1) \frac{\mu}{(\varphi + k)} \left( \frac{\Psi(\varphi + \kappa, \sigma, \mu, \xi)}{\Psi(\varphi + 1, \sigma, \mu, \xi)} \right) \xi_k \right) \right\}.
\]
which is equivalent to (18)

$$
\sum_{k=2}^{\infty} \frac{(k + \psi) (1 - \zeta) - (\epsilon - \zeta)}{k!} \prod_{\eta=1}^{\kappa} (\omega_\eta + 1)_{\kappa-1} \left( \frac{\phi + 1}{\phi + k} \right)^{\mu} \left( \Psi (\phi + k, \sigma, \mu, \lambda) \right) \mathcal{Q}_k 
\leq (1 - \epsilon) + \psi (1 - \zeta).
$$

Conversely, if we consider that inequality (18) to be true, we gain

$$\Re \left\{ 1 + \frac{1}{\psi} \left( \frac{z (\mathcal{B}^{\mu, \psi, \kappa, \lambda}, (\omega_\eta), (\sigma, \phi, \mu, \lambda))(z)}{\mathcal{B}^{\mu, \psi, \kappa, \lambda}} - \epsilon \right) \right\} - \zeta \left| 1 + \frac{1}{\psi} \left( \frac{z (\mathcal{B}^{\mu, \psi, \kappa, \lambda}, (\omega_\eta), (\sigma, \phi, \mu, \lambda))(z)}{\mathcal{B}^{\mu, \psi, \kappa, \lambda}} - 1 \right) \right|^2 > 0,$$

if

$$\Re \left\{ 1 + \frac{1}{\psi} \left( \frac{\left( \sum_{k=2}^{\infty} \frac{(k + \psi) (1 - \zeta) - (\epsilon - \zeta)}{k!} \prod_{\eta=1}^{\kappa} (\omega_\eta + 1)_{\kappa-1} \left( \frac{\phi + 1}{\phi + k} \right)^{\mu} \left( \Psi (\phi + k, \sigma, \mu, \lambda) \right) \mathcal{Q}_k \right)^{\frac{1}{\psi}}}{\left( \sum_{k=2}^{\infty} \frac{(k + \psi) (1 - \zeta) - (\epsilon - \zeta)}{k!} \prod_{\eta=1}^{\kappa} (\omega_\eta + 1)_{\kappa-1} \left( \frac{\phi + 1}{\phi + k} \right)^{\mu} \left( \Psi (\phi + k, \sigma, \mu, \lambda) \right) \mathcal{Q}_k \right)^{\frac{1}{\psi}} - \zeta} \right\} \geq 0,$$

that is,

$$\sum_{k=2}^{\infty} \frac{(k + \psi) (1 - \zeta) - (\epsilon - \zeta)}{k!} \prod_{\eta=1}^{\kappa} (\omega_\eta + 1)_{\kappa-1} \left( \frac{\phi + 1}{\phi + k} \right)^{\mu} \left( \Psi (\phi + k, \sigma, \mu, \lambda) \right) \mathcal{Q}_k 
\leq (1 - \epsilon) + \psi (1 - \zeta),$$

which implies that $\mathcal{B}^{\mu, \psi, \kappa, \lambda, e, \zeta, \psi}$. This proves the required stipulation.

**Corollary 1:** Let $\mathcal{B}$ be mentioned by (1). If $\mathcal{B}^{\mu, \psi, \kappa, \lambda, e, \zeta, \psi}$, then

$$\mathcal{Q}_k \leq \frac{k! \left( (1 - \epsilon) + \psi (1 - \zeta) \right)}{(k + \psi) (1 - \zeta) - (\epsilon - \zeta)} \prod_{\eta=1}^{\kappa} (\omega_\eta + 1)_{\kappa-1} \left( \frac{\phi + 1}{\phi + k} \right)^{\mu} \left( \Psi (\phi + k, \sigma, \mu, \lambda) \right), \quad (19)$$

where $-1 \leq \epsilon < 1, \zeta \geq 0$ and $\psi \in \mathbb{C} \setminus \{0\}$, with equality

$$\mathcal{B}(z) = z + \frac{k! \left( (1 - \epsilon) + \psi (1 - \zeta) \right)}{(k + \psi) (1 - \zeta) - (\epsilon - \zeta)} \prod_{\eta=1}^{\kappa} (\omega_\eta + 1)_{\kappa-1} \left( \frac{\phi + 1}{\phi + k} \right)^{\mu} \left( \Psi (\phi + k, \sigma, \mu, \lambda) \right) z^k. \quad (20)$$
3. Extreme points
The results of the second section define extreme points for the class \( \gamma_{\mu, \varphi, \kappa, \varepsilon, \zeta, \psi}^{(\kappa, \varepsilon, \omega)_{(\kappa, \varepsilon, \omega)}} \).

**Theorem 2:** Let \( \Psi_1(z) = z \) and

\[
\Psi_k(z) = z + \frac{k! ((1 - \varepsilon) + \psi (1 - \zeta))}{(k + \psi) (1 - \zeta) - (\varepsilon - \zeta)} \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} \left( \frac{\varphi + \kappa}{\varphi + 1} \right)^{\mu} \left( \frac{\Psi (\varphi + 1, \sigma_\mu, \mu)_{1}}{\Psi (\varphi + \kappa, \sigma_\mu, \mu)} \right) \xi_k \xi^k
\]

for \( k = 2, 3, \ldots \). Then \( \Psi \in \gamma_{\mu, \varphi, \kappa, \varepsilon, \zeta, \psi}^{(\kappa, \varepsilon, \omega)_{(\kappa, \varepsilon, \omega)}} \) if and only if

\[
\Psi(z) = \sum_{k=1}^{\infty} \xi_k \Psi_k(z),
\]

where \( \xi_k \geq 0 \) and \( \sum_{k=1}^{\infty} \xi_k = 1 \).

**Proof:** Let

\[
\Psi(z) = \sum_{k=1}^{\infty} \xi_k \Psi_k(z)
\]

Then, it follows that

\[
\sum_{k=2}^{\infty} \frac{(k + \psi) (1 - \zeta) - (\varepsilon - \zeta)}{k! ((1 - \varepsilon) + \psi (1 - \zeta))} \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} \left( \frac{\varphi + \kappa}{\varphi + 1} \right)^{\mu} \left( \frac{\Psi (\varphi + 1, \sigma_\mu, \mu)_{1}}{\Psi (\varphi + \kappa, \sigma_\mu, \mu)} \right) \xi_k
\]

\[
= \sum_{k=2}^{\infty} \xi_k = 1 - \xi_k \leq 1.
\]

Therefore, by Theorem 1, \( \Psi \in \gamma_{\mu, \varphi, \kappa, \varepsilon, \zeta, \psi}^{(\kappa, \varepsilon, \omega)_{(\kappa, \varepsilon, \omega)}} \).

Conversely, let \( \Psi \in \gamma_{\mu, \varphi, \kappa, \varepsilon, \zeta, \psi}^{(\kappa, \varepsilon, \omega)_{(\kappa, \varepsilon, \omega)}} \). Using Corollary 1, we have

\[
\xi_k \leq \frac{k! ((1 - \varepsilon) + \psi (1 - \zeta))}{(k + \psi) (1 - \zeta) - (\varepsilon - \zeta)} \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} \left( \frac{\varphi + \kappa}{\varphi + 1} \right)^{\mu} \left( \frac{\Psi (\varphi + 1, \sigma_\mu, \mu)_{1}}{\Psi (\varphi + \kappa, \sigma_\mu, \mu)} \right). 
\]
We notice that (22), is true if

\[ Y = \text{close-to-convexity of order } \pi \]

Utilizing Theorem 1

Proof: This can be shown as follows

\[ \Re \{Y\} > \pi, (0 < \pi < 1) \]

\[ \text{where } Y \text{ is close-to-convexity of order } \pi \]

4. Close-to-convexity, radii of starlikeness and convexity

The outcomes of the third section discuss the radii of close-to-convexity, star-likeness and convexity for the class \( \gamma_{\eta,\omega,\phi,\rho}^{\mu} \) mentioned as follow:

**Theorem 3:** Let \( \gamma(z) \) be in \( \gamma_{\eta,\omega,\phi,\rho}^{\mu} \), then \( \gamma \) is close-to-convexity of order \( \pi \) \( (0 \leq \pi < 1) \) in \( |z| < r_1 \), that is \( \Re \{\gamma'(z)\} \geq \pi \), \( (0 \leq \pi < 1) \) where

\[ r_1 = \inf_{\kappa \geq 2} \left\{ \frac{(1 - \pi)(k + \psi)(1 - \zeta) - (e - \zeta)}{\kappa! ((1 - e) + \psi(1 - \zeta))} \prod_{\eta=1}^{\kappa} (\omega_{\eta} + 1)_{k-1} \left( \frac{\varphi + 1}{\varphi + k} \right)^{\mu} \left( \Psi (\varphi + \kappa, \sigma, \mu, n) \right) \right\}^{1/\pi} \]

The outcome is sharp for \( \gamma(z) \) mentioned by Theorem 3.

**Proof:** This can be shown as follows

\[ |\gamma'(z) - 1| \leq 1 - \pi \]

we are taken the left-side. We get

\[ |\gamma'(z) - 1| \leq \sum_{k=2}^{\infty} \kappa \varrho_k |z|^{k-1} \]

this is less than \( 1 - \pi \) if

\[ \sum_{k=2}^{\infty} \frac{\kappa}{1 - \pi} \varrho_k |z|^{k-1} < 1 \]

Utilizing Theorem 1 \( \gamma(z) \in \gamma_{\eta,\omega,\phi,\rho}^{\mu,\nu,\xi,\zeta} \) if and only if

\[ \sum_{k=2}^{\infty} \frac{(k + \psi)(1 - \zeta) - (e - \zeta)}{\kappa! ((1 - e) + \psi(1 - \zeta))} \prod_{\eta=1}^{\kappa} (\omega_{\eta} + 1)_{k-1} \left( \frac{\varphi + 1}{\varphi + k} \right)^{\mu} \left( \Psi (\varphi + \kappa, \sigma, \mu, n) \right) \varrho_k < 1 \]

We notice that (22), is true if

\[ \frac{\kappa}{1 - \pi} |z|^{k-1} \leq \sum_{k=2}^{\infty} \frac{(k + \psi)(1 - \zeta) - (e - \zeta)}{\kappa! ((1 - e) + \psi(1 - \zeta))} \prod_{\eta=1}^{\kappa} (\omega_{\eta} + 1)_{k-1} \left( \frac{\varphi + 1}{\varphi + k} \right)^{\mu} \left( \Psi (\varphi + \kappa, \sigma, \mu, n) \right) \varrho_k \]
We notice that (23), is true if
for the left side. We yield

\[ |z| \leq \left\{ \frac{(1 - \pi) (k + \psi) (1 - \zeta) - (e - \zeta)}{k} \right\} \prod_{\eta=1}^{\kappa} (\alpha_{\eta} + 1)_{k-1} \sum_{\ell=1}^{\kappa} \frac{(\varphi + 1)^{\mu}}{(\varphi + \kappa)} \Psi (\varphi + 1, \sigma, \mu, \kappa) \right\}^{\frac{1}{1 - \pi}}, \]

which proves the theorem.

**Theorem 4:** If \( \Psi(z) \) is in \( \mathbb{Y}^{\mu, \varphi, \psi, \kappa, \ell, \mu} \), then \( \Psi(z) \) is starlike of order \( \pi \ (0 \leq \pi < 1) \) in \( |z| < r_2 \), where

\[
    r_2 = \inf_{\kappa \in \mathbb{Z}} \left\{ \prod_{\eta=1}^{\kappa} (\alpha_{\eta} + 1)_{k-1} \sum_{\ell=1}^{\kappa} \frac{(\varphi + 1)^{\mu}}{(\varphi + \kappa)} \Psi (\varphi + 1, \sigma, \mu, \kappa) \right\}^{\frac{1}{1 - \pi}},
\]

It is a sharp result which we have for the extremal function \( \Psi(z) \) given by Theorem 3.

**Proof:** This can be proven as follows

\[
    \left| \frac{\Psi'(z)}{\Psi(z)} - 1 \right| \leq 1 - \pi,
\]

for the left side. We yield

\[
    \left| \frac{\Psi'(z)}{\Psi(z)} - 1 \right| \leq \frac{\sum_{\kappa=2}^{\infty} (\kappa - 1) \varphi_k z^{k-1}}{1 + \sum_{\kappa=2}^{\infty} \varphi_k |z|^{k-1}}.
\]

The last equation is less than \( 1 - \pi \) if

\[
    \sum_{\kappa=2}^{\infty} \frac{(1 - \pi)}{(\kappa - \pi)} \varphi_k |z|^{k-1} \leq 1.
\]

Utilizing Theorem 1 \( \Psi(z) \in \mathbb{Y}^{\mu, \varphi, \psi, \kappa, \ell, \mu} \) if and only if

\[
    \sum_{\kappa=2}^{\infty} (k + \psi) (1 - \zeta) - (e - \zeta) \prod_{\eta=1}^{\kappa} (\alpha_{\eta} + 1)_{k-1} \sum_{\ell=1}^{\kappa} \frac{(\varphi + 1)^{\mu}}{(\varphi + \kappa)} \Psi (\varphi + 1, \sigma, \mu, \kappa) \varphi_k < 1.
\]

We notice that (23), is true if

\[
    \frac{\kappa - \pi}{1 - \pi} |z|^{k-1} \leq \frac{(k + \psi) (1 - \zeta) - (e - \zeta)}{k! ((1 - e) + \psi (1 - \zeta))} \prod_{\eta=1}^{\kappa} (\alpha_{\eta} + 1)_{k-1} \sum_{\ell=1}^{\kappa} \frac{(\varphi + 1)^{\mu}}{(\varphi + \kappa)} \Psi (\varphi + 1, \sigma, \mu, \kappa) \varphi_k.
\]
or, equivalently,

\[
|z| \leq \left\{ \frac{(1-\pi)(k+\psi)(1-\zeta)-(\varepsilon-\zeta)}{(k-\pi)k'((1-\varepsilon)+\psi(1-\zeta))} \prod_{\eta=1}^{k} (\omega_{\eta}+1)_{k-1} \left( \frac{\psi+1}{\varphi+k} \right)^{\mu} \left( \Psi(\psi+1,\sigma,\mu,\kappa) \right) \right\}^{1/\omega-1},
\]

which proves the theorem.

**Theorem 5:** If \( \mathcal{Y}(z) \) is in \( T^{\mu,\varphi,\psi,\varepsilon,\xi,\Theta}_{(\omega_{\alpha}),\sigma,\omega} \), then \( \mathcal{Y}(z) \) is convex of order \( \pi \) (0 \( \leq \pi < 1 \)) in \( |z| < r_{3} \), where

\[
r_{3} = \inf_{k \geq 2} \left\{ \frac{(1-\pi)(k+\psi)(1-\zeta)-(\varepsilon-\zeta)}{k(k-\pi)k'((1-\varepsilon)+\psi(1-\zeta))} \prod_{\eta=1}^{k} (\omega_{\eta}+1)_{k-1} \left( \frac{\psi+1}{\varphi+k} \right)^{\mu} \left( \Psi(\psi+1,\sigma,\mu,\kappa) \right) \right\}^{1/\omega-1},
\]

**Proof:** By using the \( \mathcal{Y}(z) \) which is convex if and only if \( z\mathcal{Y}'(z) \) is starlike, prove it by following the same steps for Theorem 4.

5. **Integral means**

This section discusses integral means for \( \mathcal{Y}(z) \) which includes \( T^{\mu,\varphi,\psi,\varepsilon,\xi,\Theta}_{(\omega_{\alpha}),\sigma,\omega} \), motivated by authors [16] and [17], by following the outcome of the subordination, the outcome is required for the investigation.

**Lemma 1:** [18]. Let \( \mathcal{Y}(z) \) and \( \Psi(z) \) are regular in \( \mathbb{U} \) with \( \Psi(z) < \mathcal{Y}(z) \), then

\[
\int_{0}^{2\pi} |\Psi(re^{i\theta})|^{\varrho} d\theta \leq \int_{0}^{2\pi} |\mathcal{Y}(re^{i\theta})|^{\varrho} d\theta, \quad \rho > 0, \quad z = re^{i\theta} \quad \text{and} \quad 0 < r < 1.
\]

By employing Lemma 1, we present the following theorem.

**Theorem 6:** Let \( \rho > 0 \). If \( \mathcal{Y}(z) \) mentioned by (1) is in \( T^{\mu,\varphi,\psi,\varepsilon,\xi,\Theta}_{(\omega_{\alpha}),\sigma,\omega} \) and \( (\Theta^{\mu,\varphi,\psi,\varepsilon,\xi,\Theta,\kappa}_{(\omega_{\alpha}),\sigma,\omega})_{k=2}^{\infty} \) is a non-decreasing sequence, then for \( z = re^{i\theta} \), \( 0 < r < 1 \), we yield

\[
\int_{0}^{2\pi} |\mathcal{Y}(re^{i\theta})|^{\rho} d\theta \leq \int_{0}^{2\pi} |\mathcal{Y}_{2}(re^{i\theta})|^{\rho} d\theta,
\]

where

\[
\mathcal{Y}_{2}(z) = z + \frac{k_1((1-\varepsilon)+\psi(1-\zeta))}{(\Theta^{\mu,\varphi,\psi,\varepsilon,\xi,\Theta,\kappa}_{(\omega_{\alpha}),\sigma,\omega})} \prod_{\eta=1}^{k} (\omega_{\eta}+1)_{k-1} \left( \frac{\psi+1}{\varphi+k} \right)^{\mu} \left( \Psi(\psi+1,\sigma,\mu,\kappa) \right) z^2,
\]

and \( T^{\mu,\varphi,\psi,\varepsilon,\xi,\Theta}_{(\omega_{\alpha}),\sigma,\omega} = (k + \psi)(1-\zeta) - (\varepsilon-\zeta) \).

**Proof:** Let \( \mathcal{Y}(z) \) and \( \mathcal{Y}_{2}(z) \) be given by (26), respectively, then we have to show that

\[
\int_{0}^{2\pi} |1 + \sum_{k=2}^{\infty} \kappa_k z^{k-1}|^{\rho} d\theta \leq \int_{0}^{2\pi} |1 + \frac{k_1((1-\varepsilon)+\psi(1-\zeta))}{(\Theta^{\mu,\varphi,\psi,\varepsilon,\xi,\Theta,\kappa}_{(\omega_{\alpha}),\sigma,\omega})} z|^{\rho} d\theta.
\]
By using Lemma 1, it is enough to demonstrate that

\[ 1 + \sum_{k=2}^{\infty} q_k z^{k-1} < 1 + \frac{\kappa !((1 - \varepsilon) + \psi (1 - \zeta))}{(\Theta^{(\kappa, \pi)}_{w(z)}, \omega, \sigma)} z. \]

Setting

\[ 1 + \sum_{k=2}^{\infty} q_k z^{k-1} < 1 + \frac{\kappa !((1 - \varepsilon) + \psi (1 - \zeta))}{(\Theta^{(\kappa, \pi)}_{w(z)}, \omega, \sigma)} w z \]  

From (28) and Lemma 1, we acquire

\[ |w(z)| = \left| \sum_{k=2}^{\infty} \frac{\kappa !((1 - \varepsilon) + \psi (1 - \zeta))}{(\Theta^{(\kappa, \pi)}_{w(z)}, \omega, \sigma)} q_k z^{k-1} \right| \leq |z| \sum_{k=2}^{\infty} \frac{\kappa !((1 - \varepsilon) + \psi (1 - \zeta))}{(\Theta^{(\kappa, \pi)}_{w(z)}, \omega, \sigma)} q_k \leq |z| < 1. \]

This is sufficient to demonstrate Theorem 6 clearly.

6. Inclusion relations

In the last section, the \((\kappa, \pi)\) neighborhood of \(\Psi(z) \in \Sigma_{C}\) is examined. In addition, inclusion relations including \(D_{\kappa, \pi}(\widehat{\Psi})\), which was defined by many authors see for example [19], [20], [21], [22] and [23], is discussed.

**Definition 3:** For any function \(\Psi(z) \in \Sigma_{C}\). If the function \(\Psi \in \Upsilon^{(\kappa, \pi)}_{\omega, \sigma}\) and \(\pi \geq 0\), we define

\[ D_{\kappa, \pi}(\Psi) = \left\{ \Psi \in \Upsilon^{(\kappa, \pi)}_{\omega, \sigma} = \right. \right. z + \sum_{k=2}^{\infty} q_k z^{k} \mbox{ and } \sum_{k=2}^{\infty} k q_k = \pi \left. \left. \leq \pi \right\} \right. \]  

(29)

In specific, for function \(\widehat{\Psi}(z) = z\)

(30)

we promptly gain

\[ D_{\kappa, \pi}(\Psi) = \left\{ \Psi \in \Upsilon^{(\kappa, \pi)}_{\omega, \sigma} = \right. \right. z + \sum_{k=2}^{\infty} q_k z^{k} \mbox{ and } \sum_{k=2}^{\infty} k |q_k| \leq \delta \} \right. \]  

(31)

**Theorem 7:** Let

\[ \pi = \frac{4 \prod_{\eta=1}^{\ell} (\omega_{\eta} + 1)^{k_{-1}} (\varphi + 2)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \kappa) ((1 - \varepsilon) + \psi (1 - \zeta))}{((2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\omega_{\eta} + 1)^{k_{-1}} (\varphi + 1)^{\mu} \Psi (\varphi + 2, \sigma, \mu, \kappa)} , \]

then \(\Upsilon^{(\kappa, \pi)}_{\omega, \sigma} \subset D_{\kappa, \pi}(\widehat{\Psi})\).

**Proof:** For \(\Psi \in \Upsilon^{(\kappa, \pi)}_{\omega, \sigma}\), Theorem 1 yields

\[ \frac{(2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\omega_{\eta} + 1)^{k_{-1}} (\varphi + 1)^{\mu} \Psi (\varphi + 2, \sigma, \mu, \kappa)}{2 \prod_{\eta=1}^{\ell} (\omega_{\eta} + 1)^{k_{-1}} (\varphi + 2)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \kappa)} \sum_{k=2}^{\infty} |q_k| \leq (1 - \varepsilon) + \psi (1 - \zeta) \]
so that
\[
\sum_{k=2}^{\infty} |\varrho_k| \leq \frac{2 \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} (\varphi + 2)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \chi) ((1 - \varepsilon) + \psi (1 - \zeta))}{((2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\kappa_\eta + 1)_{k-1} (\varphi + 1)^{\mu} \Psi (\varphi + 2, \sigma, \mu, \chi)}.
\] (32)

On the other hand, from (18) and (32), we have
\[
((2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\kappa_\eta + 1)_{k-1} (\varphi + 2)^{\mu} \Psi (\varphi + 2, \sigma, \mu, \chi) \sum_{k=2}^{\infty} |\varrho_k| \leq 2 ((1 - \varepsilon) + \psi (1 - \zeta)) \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} (\varphi + 2)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \chi).
\]

Hence,
\[
\sum_{k=2}^{\infty} \kappa |\varrho_k| \leq \frac{4 \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} (\varphi + 2)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \chi) ((1 - \varepsilon) + \psi (1 - \zeta))}{((2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\kappa_\eta + 1)_{k-1} (\varphi + 1)^{\mu} \Psi (\varphi + 2, \sigma, \mu, \chi)} = \pi.
\] (33)

Then we indicate the neighborhood for the class \( \Sigma_{e,\alpha,\alpha,\beta,\gamma} \), which we define in the following definition:

**Definition 4:** \( \mathfrak{U}(z) \in \Sigma_{C} \) is said in \( \Gamma_{(\kappa_\alpha), (\omega_\alpha), \alpha, \beta} \) if there exists \( \Phi(z) \in \Gamma_{(\kappa_\alpha), (\omega_\alpha), \alpha, \beta} \) that
\[
\left| \frac{\mathfrak{U}(z)}{\Phi(z)} - 1 \right| < 1 - \Sigma, \, (z \in \mathbb{U}_C, 0 \leq \Sigma < 1).
\] (34)

**Theorem 8:** If \( \Phi(z) \in \Gamma_{(\kappa_\alpha), (\omega_\alpha), \alpha, \beta, \gamma} \) and
\[
\Sigma = 1 - \frac{((2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} (\varphi + 1)^{\mu} \Psi (\varphi + 2, \sigma, \mu, \chi)}{2 ((2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\kappa_\eta + 1)_{k-1} (\varphi + 1)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \chi) - \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} (\varphi + 2)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \chi) (1 - \varepsilon) + \psi (1 - \zeta))},
\] (35)

then
\[
N_{k,\pi}(\Phi) \subset \Gamma_{(\kappa_\alpha), (\omega_\alpha), \alpha, \beta, \gamma, \zeta}.
\]

**Proof:** If we assume \( \mathfrak{U}(z) \in D_{k,\pi}(\Phi) \), from (33) we have
\[
\sum_{k=2}^{\infty} \kappa |\varrho_k| \leq \pi,
\] (36)
which clearly implies that
\[
\sum_{k=2}^{\infty} |\varrho_k| \leq \frac{\delta}{2}.
\]

Next, since \( \Phi(z) \in \Gamma_{(\kappa_\alpha), (\omega_\alpha), \alpha, \beta, \gamma} \), we have
\[
\sum_{k=2}^{\infty} \lambda_k \leq \frac{4 \prod_{\eta=1}^{\ell} (\omega_\eta + 1)_{k-1} (\varphi + 2)^{\mu} \Psi (\varphi + 1, \sigma, \mu, \chi) ((1 - \varepsilon) + \psi (1 - \zeta))}{((2 + \psi) (1 - \zeta) - (\varepsilon - \zeta)) \prod_{\eta=1}^{\ell} (\kappa_\eta + 1)_{k-1} (\varphi + 1)^{\mu} \Psi (\varphi + 2, \sigma, \mu, \chi)}.
\]
that is,

\[
\frac{\mathcal{Y}(z)}{\Psi(z)} - 1 \leq \sum_{k=2}^{\infty} \left| \zeta_k - \lambda_k \right| \leq \frac{\sum_{k=2}^{\infty} \left| \zeta_k - \lambda_k \right|}{1 - \sum_{k=2}^{\infty} \left| \lambda_k \right|} \leq 1 - \Sigma.
\]

As long as we find \( \Sigma \) is mentioned by (35) and by the last definition, \( \mathcal{Y}(z) \in \Gamma_{(\kappa, \sigma, \mu, \zeta, \psi, \Sigma)}^{(\lambda \kappa, \nu, \kappa, \zeta, \psi, \Sigma)} \).

References
[1] H. M. Srivastava and A. A. Attiya 2007 An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination, *Integral Transforms Spec. Funct.* 18 207–216
[2] N. E. Cho, I. H. Kim and H. M. Srivastava 2010 Sandwich-type theorems for multivalent functions associated with the Srivastava-Attiya operator *Appl. Math. Comput.* 217 918–928
[3] D. Răducanu and H. M. Srivastava 2007 A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch zeta function *Integral Transforms Spec. Funct.* 18 933–943
[4] H. M. Srivastava and J. Choi 2001 Series Associated with Zeta and Related Functions Kluwer Academic Publishers, Dordrecht, Boston and London
[5] H. M. Srivastava and J. Choi 2012 *Zeta and q-Zeta Functions and Associated Series and Integrals* Elsevier Science Publishers, Amsterdam, London and New York
[6] H. M. Srivastava 2014 A new family of the \( \lambda \)-generalized Hurwitz-Lerch zeta functions with applications *Appl. Math. Inform. Sci.* 8 1485–1500
[7] F. Ghanim 2017 Certain Properties of Classes of Meromorphic Functions Defined by a Linear Operator and Associated with the Hurwitz-Lerch Zeta Function *Advanced Studies in Contemporary Mathematics (The Jangjeon Mathematical Society (JMS))* 27 (2) 175–180
[8] F. Ghanim and Hiba F. Al-Janay 2019 A Certain Subclass of Univalent Meromorphic Functions Defined by a Linear Operator Associated with the Hurwitz-Lerch-Zeta Function, Rad HAZU Matematicke znanosti (Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.) 23 71–83
[9] H. M. Srivastava and S. Gaboury 2015 A new class of analytic functions defined by means of a generalization of the Srivastava-Attiya operator *J. Inequal. Appl.* Article ID 39 1–15
[10] H. M. Srivastava, S. Gaboury and F. Ghanim 2015 A unified class of analytic functions involving a generalization of the Srivastava–Attiya operator *Applied Mathematics and Computation* 251 35–45
[11] H. M. Srivastava, S. Gaboury and F. Ghanim 2015 Certain subclasses of meromorphically univalent functions defined by a linear operator associated with the \( \lambda \)-generalized Hurwitz-Lerch zeta function *Integral Transforms Spec. Funct.* 26 (4) 258–272
[12] H. M. Srivastava, S. Gaboury and F. Ghanim 2015 Some Further Properties of a Linear Operator Associated with the \( \lambda \)-Generalized Hurwitz-Lerch Zeta Function Related to the Class of Meromorphically Univalent Functions *Applied Mathematics and Computation* 259 1019–1029
[13] H. M. Srivastava, R. K. Saxena, T. K. Pogány and R. Saxena 2011 Integral and computational representations of the extended Hurwitz-Lerch zeta function *Integral Transforms Spec. Funct.* 22 487–506
[14] K. A. Challab, M. Darus and F. Ghanim 2017 Certain Problems Related to Generalized Srivastava-Attiya Operator *Asian European Journal of Mathematics* 10 (1) 1–21
[15] K. A. Challab, M. Darus and F. Ghanim 2019 Some Application on Hurwitz Lerch Zeta Function Defined by a Generalization of the Srivastava Attiya Operator *Kragujevac Journal of Mathematics* 43 (2) 201–217
[16] Hiba F. Al-Janay, F. Ghanim and M. Darus 2019 Some Geometric Properties of Integral Operators Proposed by Hurwitz-Lerch Zeta Function *Journal of Physics: Conference Series* 1212 (1) 1–6
[17] H. Silverman 1997 Integral means for univalent functions with negative coefficients *Houston J. Math* 23 169–174
[18] J. Littlewood 1925 On inequalities in the theory of functions, *Proceedings of the London Mathematical Society* 2 481–519
[19] O. Altintaş and Ö. Özkan, and HM. Srivastava 2000 Neighborhoods of a class of analytic functions with negative coefficients *Applied Mathematics Letters* 13 63–67
[20] Goodman and W. Adolph 1957 Univalent functions and nonanalytic curves *Proceedings of the American Mathematical Society* 8 598–601
[21] B. Frasin and M. Darus 2004 Integral means and neighborhoods for analytic univalent functions with negative coefficients *Soochow Journal of Mathematics* 30 217–224
[22] G. Murugusundaramoorthy and H.M. Srivastava 2004 Neighborhoods of certain classes of analytic functions of complex order *J. Inequal. Pure Appl. Math* 5 1–8
[23] S. Ruscheweyh 1981 Neighborhoods of univalent functions *J. Inequal. Pure Appl. Math* 81 521–527