CURVATURE ESTIMATES FOR \( p \)-CONVEX HYPERSURFACES OF PRESCRIBED CURVATURE

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Abstract. In this paper, we establish curvature estimates for \( p \)-convex hypersurfaces in \( \mathbb{R}^{n+1} \) of prescribed curvature with \( p \geq \frac{n}{2} \). The existence of a star-shaped hypersurface of prescribed curvature is obtained. We also prove a type of interior \( C^2 \) estimates for solutions to the Dirichlet problem of the corresponding equation.

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1. Introduction

Let \( M \subset \mathbb{R}^{n+1} \) be a closed hypersurface and let \( \kappa(X) = (\kappa_1, \cdots, \kappa_n) \) be the principal curvatures of \( M \) at \( X \). Given \( 1 \leq p \leq n \), a \( C^2 \) regular hypersurface \( M \) is called \( p \)-convex if, at each \( X \in M \), \( \kappa(X) \) satisfies
\[
\kappa_{i_1} + \cdots + \kappa_{i_p} \geq 0, \quad \forall \ 1 \leq i_1 < \cdots < i_p \leq n.
\]
In other words, the sum of the \( p \) smallest principal curvatures is nonnegative at each point of \( M \). The notion of \( p \)-convexity goes back to Wu \[36\] and has been studied extensively by Wu \[36\], Sha \[29, 30\] and Harvey-Lawson \[19, 20\].

In this paper, we are interested in finding a \( p \)-convex hypersurface \( M \subset \mathbb{R}^{n+1} \) of prescribed curvature as below
\[
\Pi_{1 \leq i_1 < \cdots < i_p \leq n} (\kappa_{i_1} + \cdots + \kappa_{i_p}) = f(X, \nu(X)), \quad \forall \ X \in M,
\]
where \( \nu(X) \) is the unit outer normal of \( M \) at \( X \), the function \( f(X, \nu) \in C^2(\Gamma) \) is positive and \( \Gamma \) is an open neighborhood of unit normal bundle of \( M \) in \( \mathbb{R}^{n+1} \times S^n \). The Gaussian curvature equation, that corresponds to \( p = 1 \) in (1.1), was studied by Oliker \[24\]. The mean curvature equation corresponding to \( p = n \) in (1.1) was studied by Bakelman-Kantor \[1\] and Treibergs-Wei \[34\]. For general curvature equations, see Caffarelli-Nirenberg-Spruck \[4\] and Gerhardt \[11\]. When \( p = n - 1 \), the equation was studied by Chu-Jiao \[7\] and, in complex settings, it is related to the Gauduchon conjecture which was solved by Székelyhidi-Tosatti-Weinkove \[31\]. For some previous work on this topic, see Tosatti-Weinkove \[32, 33\] and Fu-Wang-Wu \[9, 10\].

It is of great interest in geometry and PDEs to derive a \( C^2 \) estimate for equation (1.1) for general \( f(X, \nu(X)) \). We have the following main result.

**Theorem 1.1.** Suppose \( M \subset \mathbb{R}^{n+1} \) is a closed star-shaped \( p \)-convex hypersurface with \( p \geq \frac{n}{2} \) satisfying the curvature equation (1.1). Then, there is a positive constant \( C \) such that
\[
\sup_{X \in M} \sup_{1 \leq i_1 < \cdots < i_p \leq n} |\kappa_{i_1}(X) + \cdots + \kappa_{i_p}(X)| \leq C.
\]
where $C$ depends on $n$, $p$, $|M|_{C^1}$, $\inf f$ and $|f|_{C^2}$.

We remark that in the theorem only a few conditions are assumed on $f$. Usually, to derive $C^2$ estimate for elliptic equations which are not strictly elliptic there should be some extra assumptions on $f$ due to the dependency on $\nu(X)$. We refer the reader to Ivochkina [21, 22], Guan-Guan [13], Guan-Lin-Ma [16], Guan-Li-Li [15] and Guan-Jiao [14] for more details. Moreover, Guan-Ren-Wang in [17] showed that for the following curvature equation

$$
\sigma_k(\kappa(X)) = f(X, \nu(X)), \quad \forall X \in M,
$$

where $0 < l < k \leq n$ and $\sigma_k$ is the $k$-th elementary symmetric function, estimate (1.2) fails generally, though it may hold for special $f$ as in Guan-Guan [13]. When $l = 0$, some results are known for general $f$. For instance, estimate (1.2) was proved for $k = n$ by Caffarelli-Nirenberg-Spruck [3] and, for $2 \leq k \leq n$, Guan-Ren-Wang [17] obtained the estimate for convex solutions. For $k = n - 1$ and $n - 2$, estimate (1.2) was established by Ren-Wang [25, 26]. They also conjectured that the estimate still holds for $k > \frac{n}{2}$ in [26, 27]. When $f$ is independent of $\nu$, Caffarelli-Nirenberg-Spruck [4] proved the $C^2$ estimate for a general class of fully nonlinear curvature equations. For hypersurfaces of prescribed curvature in Riemannian manifolds and Minkowski space, see [5] and [35, 28]. We also refer the reader to Guan-Zhang [18] and references therein for a class of curvature equations arising from convex geometry.

To obtain the existence of a $p$-convex hypersurface satisfying the prescribed curvature equation (1.1), we assume the following two conditions on $f$. The first one is that there exists two positive constants $r_1 < 1 < r_2$ such that

$$
f\left(X, \frac{X}{|X|}\right) \geq \frac{p^{C^n_{r_1}}}{|X|^{C^n_{r_1}}}, \quad \text{for} \ |X| = r_1;
$$

$$
f\left(X, \frac{X}{|X|}\right) \leq \frac{p^{C^n_{r_2}}}{|X|^{C^n_{r_2}}}, \quad \text{for} \ |X| = r_2.
$$

(1.3)

This condition is used to derive $C^0$ estimates. The second one is that for any fixed unit vector $\nu$,

$$
\frac{\partial}{\partial \rho} \left( \rho^{C^n_{r_2}} f(X, \nu) \right) \leq 0, \quad \text{where} \ \rho = |X|,
$$

(1.4)

and will be used to derive $C^1$ estimates. Actually, with suitable assumptions of $f$, Li [23] proved that the interior gradient estimate holds.

By the continuity method argument as in [2], we can obtain the following result.

**Theorem 1.2.** Let $f \in C^2((B_{r_2} \setminus B_{r_1}) \times S^n)$ be a positive function satisfying (1.3) and (1.4). Then equation (1.1) has a unique $C^{3,\alpha}$ star-shaped $p$-convex solution $M$ in $\{X \in \mathbb{R}^{n+1} : r_1 \leq |X| \leq r_2\}$ for any $\alpha \in (0, 1)$ as long as $p \geq \frac{n}{2}$.

The method of proving Theorem 1.1 can be applied to obtain an interior $C^2$ estimate for the Dirichlet problem of the corresponding equation in the Euclidean space. Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$. For a function $u \in C^2(\Omega)$, denote by $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$ the eigenvalues of the Hessian $D^2u$. We say that $u \in C^2(\Omega)$ is $p$-plurisubharmonic if the eigenvalues of $D^2u$ satisfy $\lambda_1 + \cdots + \lambda_p \geq 0$, where
for all $1 \leq i_1 < \cdots < i_p \leq n$ (see [24]). Given a $C^2$ $p$-plurisubharmonic function $v$ on $\bar{\Omega}$, consider the following Dirichlet problem,

$$\Pi_{1 \leq i_1 < \cdots < i_p \leq n} (\lambda_{i_1} + \cdots + \lambda_{i_p}) = f(x, u, Du), \text{ in } \Omega$$

with boundary data

$$u = v, \text{ on } \partial \Omega,$$

where $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a positive function. By the same argument as that of Theorem 1.1, we can prove the following interior estimate.

**Theorem 1.3.** Suppose that a $p$-plurisubharmonic function $u \in C^4(\Omega) \cap C^1(\bar{\Omega})$ is a solution to the Dirichlet problem of equation (1.5) and satisfies $u < v$ in $\Omega$. Then, there exist constants $C$ and $\beta$ depending only on $n, p$, $\|u\|_{C^4}$, $\|v\|_{C^1}$, $\inf f$, $\|f\|_{C^2}$ and $\Omega$ such that

$$\sup_{\Omega} (v - u)^{\beta} \Delta u \leq C$$

as long as $p \geq \frac{n}{2}$.

**Remark 1.4.** As a byproduct of the proof of the above theorem, one can conclude the following global $C^2$ estimate for equation (1.5)

$$\sup_{\Omega} |D^2u| \leq C(1 + \sup_{\partial \Omega} |D^2u|),$$

where $C$ is a constant as in Theorem 1.3.

We shall only give an outline for the proof of Theorem 1.3 as it is almost the same as that of Theorem 1.1. The estimate (1.6) can also be seen in some sense as a generalization of Theorem 0.4 in [8], since there the right hand side function $f$ does not depend on $Du$. Such an estimate for the $k$-Hessian equation $\sigma_k(\lambda) = f(x, u)$ has been proved by Chou-Wang [6]. The function $f$ depending on $\nu$ or $Du$ creates substantial difficulties to derive a $C^2$ estimate, as the bad term $-Ch_{11}$ or $-Cu_{11}$ appears when one applies the maximum principle to the test function. One way to overcome this is like in [17, 25, 26] to control the bad third order terms, which is very hard even for the $k$-Hessian equation. Another way is as in [1] to control $-Ch_{11}$ firstly by good terms (see Lemma 3.2), and then the bad third order terms can be eliminated easily by Lemma 3.4. Thanks to Dinew [8], where many properties of the operator have been proved, we can follow the argument in [7] to prove the estimate.

The paper is organized as follows. In Section 2, we recall some properties of the operator from [8]. In Section 3, we prove the curvature estimate. In Section 4 we derive the gradient estimate and in Section 5 we apply the continuity method to prove Theorem 1.2. Finally, we give an outline of the proof of Theorem 1.3 in Section 6.

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2. Preliminaries

Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and let $\nabla$ be the connection on it. Assume that $M$ is star-shaped with respect to the origin, i.e. the position vector $X$ of $M$
can be written as $X(x) = \rho(x)x$, where $x \in \mathbb{S}^n$. Then the unit outer normal of $M$ is given by
\[
\nu = \frac{\rho x - \nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}.
\]

Let $\{e_1, \cdots, e_n\}$ be a smooth local orthonormal frame on $\mathbb{S}^n$. Then the metric of $M$ is given by $g_{ij} = \rho^2 \delta_{ij} + \rho_i \rho_j$, and the second fundamental form of $M$ is
\[
(2.1) \quad h_{ij} = \frac{\rho^2 \delta_{ij} + 2\rho_i \rho_j - \rho \rho_{ij}}{\sqrt{\rho^2 + |\nabla \rho|^2}}.
\]

The principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_n)$ are the eigenvalues of $h_{ij}$ with respect to $g_{ij}$.

At a point $X_0$ in $M$, choose a local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$. The following geometric formulas are well known:
\[
(2.2) \quad X_{ij} = -h_{ij} \nu \quad \text{(Gauss formula)},
\]
\[
(\nu)_i = h_{ij} e_j \quad \text{(Weingarten equation)},
\]
\[
S_{ijk} = S_{ikj} \quad \text{(Codazzi formula)},
\]
\[
R_{ijkl} = e_i h_{jk} - e_j h_{ik} \quad \text{(Gauss equation)},
\]
where $R_{ijkl}$ is the $(4,0)$-Riemannian curvature tensor and the formula
\[
(2.3) \quad h_{ij,kl} = h_{klij} + h_{mkj}(h_{mj} h_{il} - h_{ml} h_{ij}) + h_{mli}(h_{mj} h_{kl} - h_{ml} h_{kj}).
\]

We recall the $p$-convex cones introduced by Harvey and Lawson [20].

**Definition 2.1.** Let $p \in \{1, \cdots, n\}$. The cone $\mathcal{P}_p$ is defined by
\[
\mathcal{P}_p = \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n | \forall 1 \leq i_1 < i_2 < \cdots < i_p \leq n, \lambda_{i_1} + \cdots + \lambda_{i_p} > 0\}.
\]

Associated to $\mathcal{P}_p$ is the cone of symmetric $n \times n$ matrices defined by
\[
\mathcal{P}_p = \{A | \forall 1 \leq i_1 < i_2 < \cdots < i_p \leq n, \lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) > 0\}.
\]

We call $A$ is $p$-positive if $A \in \mathcal{P}_p$.

For convenience, we introduce the following notations
\[
F(h_{ij}) := F(\kappa) = \prod_{1 \leq i_1 < \cdots < i_p \leq n}(\kappa_{i_1} + \cdots + \kappa_{i_p}) \quad \text{and} \quad \tilde{F} = F \frac{1}{C_n^p},
\]
where $C_n^p = \frac{n!}{p!(n-p)!}$. Equation (4.4) then can be written as
\[
(4.4) \quad \tilde{F}(h_{ij}) := \tilde{F}(\kappa) = \tilde{f}(X, \nu(X)),
\]
where $\kappa = (\kappa_1, \cdots, \kappa_n)$ and $\tilde{f} = f \frac{1}{C_n^p}$. Denote
\[
F_{ij} = \frac{\partial F}{\partial h_{ij}}, \quad F_{ijkl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}}, \quad \text{and} \quad F = \sum F_{ii}.
\]

Direct calculations show that
\[
\tilde{F}^{ij} = \frac{1}{C_n^p} F \frac{1}{C_n^p}^{-1} F_{ij}
\]
and
\[
\tilde{F}^{ij,kl} = \frac{1}{C_n^p} F \frac{1}{C_n^p}^{-1} F^{ij,kl} + \frac{1}{C_n^p} \left(\frac{1}{C_n^p} - 1\right) F \frac{1}{C_n^p}^{-2} F^{ij} F^{kl}.
\]
We remark that \( \tilde{F} \) is concave with respect to \( h_{ij} \) by Lemma 1.13 and Corollary 1.14 in [8]. And the equation is elliptic as the matrix \( \left( \frac{\partial \tilde{F}}{\partial h_{ij}} \right) \) is positive definite for \( \{h_{ij}\} \in P_p \).

Now we do some basic calculations which will be used in the next section. Our calculations are carried out at a point \( X_0 \) on the hypersurface \( M \), and we use coordinates such that at this point \( \{h_{ij}\} \) is diagonal and its eigenvalues with respect to \( g_{ij} \) are ordered as \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n \). Note that \( F_{ij} \) is also diagonal at \( X_0 \) and we have the following formulas

\[
F_{kk} = \frac{\partial F}{\partial \kappa_k} = \sum_{k \in \{i_1, \ldots, i_p\}} \frac{F(\kappa)}{\kappa_{i_1} + \cdots + \kappa_{i_p}},
\]

for which we refer to Lemma 1.10 in [8]. We also have formulas for the second order derivatives of \( F \) at \( X_0 \) as below

\[
F_{kk,kl} = \frac{\partial^2 F}{\partial \kappa_k \partial \kappa_l} = \sum_{k \in \{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_p\} \neq \emptyset} \frac{F(\kappa)}{(\kappa_{i_1} + \cdots + \kappa_{i_p})(\kappa_{j_1} + \cdots + \kappa_{j_p})},
\]

and, for \( k \neq r \),

\[
F_{kr,rk} = \frac{F_{kk} - F_{rr}}{\kappa_k - \kappa_r} = - \sum_{k \notin \{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_p\} \neq \emptyset} \frac{F(\kappa)}{(\kappa_{i_1} + \cdots + \kappa_{i_p})(\kappa_{j_1} + \cdots + \kappa_{j_p})}.
\]

Otherwise, we have \( F_{ij,kl} = 0 \). See Lemma 1.12 in [8] for the above formulas. These formulas can also be easily obtained from Theorem 5.5 in [2]. The following properties of the function \( F \) which are very similar to the properties of \( \sigma_k \) were proved by Dinew [8].

**Lemma 2.2** ([8]). Suppose that the diagonal matrix \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \) belongs to \( P_p \) and that \( \lambda_1 \geq \cdots \geq \lambda_n \). Then,

1. \( \tilde{F}^{11}(A)\lambda_1 \geq \frac{1}{4} \tilde{F}(A) \);
2. \( \sum_{k=1}^{n} \tilde{F}^{kk}(A) \geq p \);
3. \( \sum_{k=1}^{n} F^{kk}(A)\lambda_k = C_p \tilde{F}(A) \);
4. there is a constant \( \theta = \theta(n, p) \) such that \( F^{jj}(A) \geq \theta \sum F^{ii} \) for all \( j \geq n - p + 1 \).

For the reader’s convenience, we provide a short proof of the above lemma in the appendix.

### 3. Curvature Estimates

Set \( u = \langle X, \nu \rangle \), which is the support function of the hypersurface \( M \). Clearly, we have

\[
u = \frac{\rho^2}{\sqrt{\rho^2 + |\nabla \rho|^2}}.
\]

There exists a positive constant \( C \) depending on \( \inf_M \rho \) and \( |\rho|_{C^1} \) such that

\[
\frac{1}{C} \leq \inf_M u \leq u \leq \sup_M u \leq C.
\]
In order to prove Theorem 1.1, we consider the following auxiliary function

\[ G = \log \kappa_{max} - \log(u - a) + \frac{A}{2}|X|^2 \]

where \( \kappa_{max} \) is the largest principal curvature, \( a = \frac{1}{2} \inf_M u > 0 \) and \( A \geq 1 \) is a large constant to be determined. Suppose the maximum of \( G \) is achieved at a point \( X_0 \in M \). Choose a local orthonormal frame \( \{e_1, \cdots, e_n\} \) around \( X_0 \) such that

\[ h_{ij} = \delta_{ij} h_{ii} \text{ and } h_{11} \geq h_{22} \geq \cdots \geq h_{nn} \text{ at } X_0. \]

Since \( \kappa_{max} \) may not be differentiable, we define a new function \( \hat{G} \) near \( X_0 \) by

\[ \hat{G} = \log h_{11} - \log(u - a) + \frac{A}{2}|X|^2. \]

It is easy to see \( \hat{G} \) achieves a maximum at \( X_0 \). Now, differentiating \( \hat{G} \) at \( X_0 \) twice yields that

\[ 0 = \frac{h_{11i}}{h_{11}} - \frac{u_i}{u - a} + A(X,e_i) \tag{3.1} \]

and

\[ 0 \geq \frac{h_{11ii}}{h_{11}} - \left( \frac{h_{11i}}{h_{11}} \right)^2 - \frac{u_{ii}}{u - a} + \left( \frac{u_i}{u - a} \right)^2 + A(1 + \langle X, X_{ii} \rangle). \tag{3.2} \]

Contracting (3.2) with \( F_{ii} \), we get

\[ 0 \geq \frac{\hat{F}_{ii} h_{11ii}}{h_{11}} - \frac{\hat{F}_{ii} h_{11i}^2}{h_{11}^2} - \frac{\hat{F}_{ii} u_{ii}}{u - a} + \frac{\hat{F}_{ii} u_i^2}{(u - a)^2} + A\hat{F}_{ii}(1 + \langle X, X_{ii} \rangle). \tag{3.3} \]

**Lemma 3.1.** We have

\[ 0 \geq -2 \sum_{i \geq 2} \hat{F}_{ii} h_{11i} h_{11i}^2 - \frac{\hat{F}_{ii} h_{11i}^2}{h_{11}^2} - C h_{11} \tag{3.4} \]

\[ + a \frac{\hat{F}_{ii} h_{ii}^2}{u - a} + \frac{\hat{F}_{ii} u_i^2}{(u - a)^2} + A \sum_i \hat{F}_{ii} - CA. \]

**Proof.** From the formula (2.3), we have

\[ \hat{F}_{ii} h_{11ii} = \hat{F}_{ii} h_{ii} h_{11}^2 - \hat{F}_{ii} h_{ii} h_{11} + \hat{F}_{ii} h_{11}. \tag{3.5} \]

Differentiating equation (3.1) twice at \( X_0 \), we obtain

\[ \hat{F}_{ii} h_{ii} = (d_X \hat{f})(e_k) + h_{kk}(d_v \hat{f})(e_k) \tag{3.6} \]

and

\[ \hat{F}_{ii} h_{ikk} + \hat{F}_{pq,r} h_{pqk} h_{rsk} \geq -C - C h_{11}^2 + \sum_l h_{lkk}(d_v \hat{f})(e_l). \tag{3.7} \]

By the concavity of \( \hat{F} \) and the Codazzi formula, we have

\[ -\hat{F}_{pq,r} h_{pqk} h_{rsk} \geq -2 \sum_{i \geq 2} \hat{F}_{ii,i} h_{11i}^2. \tag{3.8} \]

Note that by Lemma 2.2 (3) we have \( \hat{F}_{ii} h_{ii} = \hat{f} \). Hence, we see that

\[ \hat{F}_{ii} h_{11ii} \geq -2 \sum_{i \geq 2} \hat{F}_{ii,i} h_{11i}^2 - \hat{F}_{ii} h_{ii} h_{11} + \sum_l h_{l11}(d_v \hat{f})(e_l) - C - C h_{11}^2. \tag{3.9} \]
We now compute the term \( \tilde{F}_{ii}u_{ii} \). By (2.2), we have

\[
u_i = h_{ii}\langle X, e_i \rangle \text{ and } u_{ii} = \sum_k h_{ii \kappa} \langle X, e_k \rangle - u_{ii}^2 + h_{ii}.
\]

Hence, we obtain

\[
\tilde{F}_{ii}u_{ii} = \sum_k \tilde{F}_{ii}h_{ii \kappa} \langle X, e_k \rangle - u\tilde{F}_{ii}h_{ii}^2 + \tilde{f}
\]

(3.10)

By the Gauss formula, we have

\[
\langle X, X_{ii} \rangle = -h_{ii}\langle X, \nu \rangle = -h_{ii}u.
\]

Substituting (3.9), (3.10) and (3.11) in (3.3), we obtain that

\[
0 \geq -\frac{2}{h_{11}} \sum_{i \geq 2} \tilde{F}_{1i,i1}h_{1i}^2 - \tilde{F}_{ii}h_{ii}^2 + \frac{1}{h_{11}} \sum_i h_{ii11}(d_{ij}\tilde{f})(e_i) - Ch_{11}
\]

(3.12)

\[
-\frac{\tilde{F}_{ii}h_{1i11}^2}{h_{1i1}^2} - \sum_k h_{kk}(d_{ij}\tilde{f})(e_k)\langle X, e_k \rangle + \frac{u\tilde{F}_{ii}h_{ii}^2}{u-a} - \frac{C}{u-a} + \frac{\tilde{F}_{ii}u_i^2}{(u-a)^2} + A \sum \tilde{F}_{ii} - CA.
\]

By the Codazzi formula, we have

\[
\langle X, X_{ii} \rangle = -h_{ii}\langle X, \nu \rangle = -h_{ii}u.
\]

Therefore, we arrive at

\[
0 \geq -\frac{2}{h_{11}} \sum_{i \geq 2} \tilde{F}_{1i,i1}h_{1i}^2 - \tilde{F}_{ii}h_{ii}^2 - Ch_{11} - \frac{C}{u-a}
\]

(3.13)

\[
-\frac{\tilde{F}_{ii}h_{1i11}^2}{h_{1i1}^2} + \frac{u\tilde{F}_{ii}h_{ii}^2}{u-a} + \frac{\tilde{F}_{ii}u_i^2}{(u-a)^2} + A \sum \tilde{F}_{ii} - CA,
\]

which is just the inequality (3.3).

\[\square\]

Next, we deal with the bad term \(-Ch_{11}\).

**Lemma 3.2.** Suppose \( p \geq \frac{n}{2} \). If \( h_{11} \) is large enough, we have

\[
Ch_{11} \leq \frac{A\tilde{F}_{ii}h_{ii}^2}{2(u-a)} + A \sum \tilde{F}_{ii}
\]

for sufficiently large \( A \).

**Proof.** Note that

\[
\kappa_{n-p+1} + \kappa_{n-p+2} + \cdots + \kappa_n > 0.
\]

We divide the proof into two cases.

Case 1. Suppose \( \kappa_n \leq -\delta \kappa_1 \), where \( \delta > 0 \) is a small constant to be determined later. By Lemma (2.2) we see \( F_{nn} \geq \theta \sum \tilde{F}_{ii} \geq \theta p \). We then obtain that

\[
\tilde{F}_{nn}h_{nn}^2 \geq \delta^2 \kappa_1 \tilde{F}_{nn} \geq \theta p \delta^2 \kappa_1^2.
\]
Therefore, for sufficiently large $\kappa_1$, we have

$$C h_{11} \leq \frac{a \tilde{F}^{nn} h_{nn}}{2(u - a)}$$

Case 2. Now $\kappa_n \geq -\delta \kappa_1$. We further divide this case into two cases. Subcase 2.1. Suppose $\kappa_{n-p+1} + \kappa_{n-p+2} + \cdots + \kappa_n < \frac{\delta}{\kappa_1}$. Since

$$F^{nn} \geq \frac{F(\kappa)}{\kappa_{n-p+1} + \kappa_{n-p+2} + \cdots + \kappa_n},$$

we see that

$$\tilde{F}^{nn} = \frac{1}{C_n} F^{\frac{1}{C_n} - 1} F^{nn} \geq \frac{F^{\frac{1}{C_n} \kappa_1}}{C_n} = \frac{\tilde{f} \kappa_1}{C_n \delta}.$$ 

Choosing $\delta$ sufficiently small, we obtain that

$$C \kappa_1 \leq \tilde{F}^{nn}.$$ 

Subcase 2.2. Suppose $\kappa_{n-p+1} + \kappa_{n-p+2} + \cdots + \kappa_n \geq \frac{\delta}{\kappa_1}$. For a fixed $(p-1)$-tuple $2 \leq i_1 < \cdots < i_{p-1} \leq n$, we have

$$\kappa_1 + \kappa_{i_1} + \cdots + \kappa_{i_{p-1}} \geq (1 - (p - 1)\delta) \kappa_1.$$ 

Hence, we have

$$F^{nn} \geq \prod_{2 \leq i_1 < \cdots < i_{p-1} \leq n} (\kappa_1 + \kappa_{i_1} + \cdots + \kappa_{i_{p-1}}) \times \prod_{2 \leq i_1 < \cdots < i_p \leq n} (\kappa_1 + \kappa_{i_1} + \cdots + \kappa_{i_p})$$

$$\geq [(1 - (p - 1)\delta) \kappa_1]^{n_p - 1} \frac{\delta}{C_p^{\frac{1}{n_p}}}. 

For $p \geq \frac{n}{2}$, a direct calculation shows that

$$C_p^{\frac{1}{n_p}} - C_{p - 1}^{\frac{1}{n_p}} = \frac{(n - 1) \cdots (n - p + 1)}{(p - 1)!} \left(1 - \frac{n - p}{p}\right) \geq 0.$$ 

Therefore, we obtain

$$F^{nn} \geq c_3 \kappa_1,$$

where $c_3 = [(1 - (p - 1)\delta)]^{n_p - 1} \delta C_{n_p - 1}^{\frac{1}{n_p}}$. It then follows that, for sufficiently large $A$,

$$C \kappa_1 \leq \frac{A}{2} \tilde{F}^{nn}. \quad \square$$

By the above Lemma, (3.3) becomes

$$0 \geq -\frac{2}{h_{11}} \sum_{i \geq 2} \tilde{F}^{i, i} h_{11}^2 - \frac{\tilde{F}^{i, i} h_{11}^2}{h_{11}^2} + \frac{a \tilde{F}^{i, i} h_{11}^2}{2(u - a)},$$

$$+ \frac{\tilde{F}^{i, i} u_i^2}{(u - a)^2} + \frac{A}{2} \sum \tilde{F}^{ii} - CA.$$ 

Lemma 3.3. For $\kappa_1$ sufficiently large, we have

$$|\kappa_{n-p+1}|, \cdots, |\kappa_n| \leq CA.$$
Proof. By the critical equation (3.1) and the Cauchy-Schwarz inequality, we have
\[(3.16) \quad \tilde{F}_{ii} h_{i1}^2 \geq - (1 + \varepsilon) \tilde{F}_{ii} u_i^2 - \left(1 + \frac{1}{\varepsilon}\right) A^2 \tilde{F}_{ii} \langle X, e_i \rangle^2. \]

From (3.15) and \(-\tilde{F}_{ii,i1} \geq 0\), we see that
\[(3.17) \quad 0 \geq \frac{a \tilde{F}_{ii} h_{i1}^2}{2(u-a)} - \frac{C A^2}{\varepsilon} \sum \tilde{F}_{ii} - CA. \]

Using \(u_i = h_{ii} \langle X, e_i \rangle\) and choosing \(\varepsilon\) sufficiently small, we obtain from (3.17) that
\[(3.18) \quad 0 \geq \frac{a \tilde{F}_{ii} h_{i1}^2}{4(u-a)} - \frac{C A^2}{\varepsilon} \sum \tilde{F}_{ii}, \]

where we also used \(\sum \tilde{F}_{ii} \geq p\). By Lemma 2.2, we now arrive at
\[(3.19) \quad 0 \geq \frac{a \theta}{4(u-a)} \left( \sum \tilde{F}_{ii} \right) \left( \sum \tilde{h}_{ii}^2 \right) - \frac{C A^2}{\varepsilon} \sum \tilde{F}_{ii}, \]

which implies that
\[\sum_{i \geq n-p+1} \tilde{h}_{ii}^2 \leq C A^2. \]

Lemma 3.4. Given \(1 > \delta > 0\), there is an \(\varepsilon = \varepsilon(p, \delta) > 0\) such that,
\[-2F_{i1,i1} + 2 \frac{F_{11}}{\kappa_1} \geq (1 + \delta) \frac{F_{ii}}{\kappa_1}, \quad i = 2, 3, \ldots, n, \]
for \(\kappa_1\) sufficiently large.

Proof. Recall that \(\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n\). By the formula \(F_{11} = \frac{F_{11} - F_{ii}}{\kappa_1 - \kappa_i}\), we see that
\[\frac{F_{ii}}{\kappa_1} = \frac{\kappa_1 - \kappa_i}{\kappa_1} \left( -F_{i1,i1} \right) + \frac{F_{11}}{\kappa_1}. \]

Since \(\kappa_i > 0\) for \(i \leq n-p+1\), we obtain
\[\frac{F_{ii}}{\kappa_1} \leq -F_{i1,i1} + \frac{F_{11}}{\kappa_1}, \quad \text{for} \quad i = 2, \cdots, n-p+1. \]

By Lemma 3.3 we can assume that \(|\kappa_i| \leq \varepsilon \kappa_1\) for \(i \geq n-p+2\) for sufficiently small \(\epsilon\) and large \(\kappa_1\). Hence, we have
\[\frac{F_{ii}}{\kappa_1} \leq - (1 + \varepsilon) F_{i1,i1} + \frac{F_{11}}{\kappa_1}, \quad \text{for} \quad i = n-p+2, \cdots, n. \]

By the above two inequalities, we get the desired inequality. \(\square\)

By Lemma 3.4, (3.15) becomes
\[(3.20) \quad 0 \geq \sum_{i \geq 2} \tilde{F}_{ii} h_{i1}^2 \tilde{h}_{i1}^2 - \frac{\tilde{F}_{ii} h_{i1}^2}{h_{i1}^2} - 2 \sum_{i \geq 2} \tilde{F}_{i1} h_{11}^2 - \frac{\tilde{F}_{ii} h_{i1}^2}{h_{i1}^2} + \frac{a \tilde{F}_{ii} h_{ii}^2}{2(u-a)}
+ \frac{\tilde{F}_{ii} u_i^2}{(u-a)^2} + \frac{A}{2} \sum \tilde{F}_{ii} - CA. \]
By the critical equation (3.1), the Cauchy-Schwarz inequality and $u_i = h_{ii} \langle X, e_i \rangle$, we see that

$$\begin{align*}
\frac{\tilde{F}^{11} h_{11}^2}{h_{11}^2} &\geq - (1 + \varepsilon) \frac{\tilde{F}^{11} u_{i1}^2}{(u - a)^2} - \left(1 + \frac{1}{\varepsilon}\right) A^2 \tilde{F}^{11} \langle X, e_1 \rangle^2 \\
&\geq - \frac{\tilde{F}^{11} u_{i1}^2}{(u - a)^2} - C \varepsilon \frac{\tilde{F}^{11} h_{11}^2}{(u - a)^2} - CA^2 \tilde{F}^{11}
\end{align*}$$

(3.21)

and

$$2 \sum_{i \geq 2} \frac{\tilde{F}^{11} h_{11 i}^2}{h_{11 i}^2} \leq C \sum_{i \geq 2} \tilde{F}^{11} h_{11 i}^2 + CA^2 F^{11}.$$

Note that

$$F^{11} = \sum_{1 \notin \{i_1, \ldots, i_p\}} \frac{F(\kappa)}{\kappa_{n-p+1} + \cdots + \kappa_n} \leq \frac{C}{\kappa_1}.$$

Hence, by Lemma 3.2, we obtain

$$2 \sum_{i \geq 2} \frac{\tilde{F}^{11} h_{11 i}^2}{h_{11 i}^2} \leq \frac{a \tilde{F}^{11} h_{11 i}^2}{4(u - a)} + \frac{A}{4} \sum \tilde{F}^{11 i} + CA^2 \tilde{F}^{11}.$$

Substituting (3.21) and (3.22) into (3.20), we have

$$0 \geq \frac{a \tilde{F}^{11 i} h_{11 i}^2}{4(u - a)} - \frac{\tilde{F}^{11} u_{i1}^2}{(u - a)^2} - C \varepsilon \frac{\tilde{F}^{11} h_{11 i}^2}{(u - a)^2} - \frac{CA^2}{\varepsilon} \tilde{F}^{11} + \frac{\tilde{F}^{11 i} u_{i1}^2}{(u - a)^2} + \frac{A}{4} \sum \tilde{F}^{11 i} - CA^2 \tilde{F}^{11} - CA.$$

Choosing $\varepsilon$ sufficiently small and assuming $h_{11}$ sufficiently large, we derive that

$$0 \geq \frac{a \tilde{F}^{11 i} h_{11 i}^2}{8(u - a)} + \frac{A}{4} \sum \tilde{F}^{11 i} - CA.$$

It then follows that

$$\sum \tilde{F}^{11 i} \leq C.$$

Next we prove that under this condition, one have

$$\tilde{F}^{11} \geq \frac{1}{C}.$$

Since $\sum F^{11} \leq C$, in particular we have

$$\frac{F(\kappa)}{\kappa_{n-p+1} + \cdots + \kappa_n} \leq C.$$

This implies that

$$\kappa_{n-p+1} + \cdots + \kappa_n \geq \frac{1}{C},$$

where $C$ also depends on $\inf f$. This yields

$$\tilde{F}^{11 i} \geq \frac{1}{C}, \quad \forall 1 \leq i \leq n.$$

Substituting the above inequality into (3.24), we obtain

$$0 \geq \frac{h_{11}^2}{C} - CA,$$

from which we can derive an upper bound for $h_{11}$. Theorem 1.1 is proved.
4. Gradient Estimates

Before we apply the continuity method to obtain a solution to equation (1.1), we need to derive a \( C^1 \) estimate for the equation. We show that there exists a positive constant \( C \) depending on \( n, p, \inf \rho, \sup \rho, \inf f \) and \( |f|_{C^1} \) such that

\[
|\nabla \rho| \leq C,
\]

where \( \nabla \) denotes the connection on \( S^n \). Note that

\[
u = \rho^2 \sqrt{\rho^2 + |\nabla \rho|^2}.
\]

We only need to derive a positive lower bound of \( u \). As in [16] we consider the following quantity

\[
w = - \log u + \gamma(|X|^2),
\]

where the function \( \gamma(\cdot) \) will be determined later. Suppose the maximum of \( w \) is achieved at \( X_0 \in M \). If at \( X_0 \), \( X \) is parallel to \( \nu \), we have

\[
u = \langle X, \nu \rangle = \rho \geq \inf M \rho,
\]

which gives a lower bound since \( \rho \) is assumed to have a positive lower bound. If at \( X_0 \), \( X \) is not parallel to \( \nu \), we can choose a local orthonormal frame \( \{e_1, e_2, \cdots, e_n\} \) such that

\[
\langle X, e_1 \rangle \neq 0 \quad \text{and} \quad \langle X, e_i \rangle = 0 \quad \text{for} \quad i \geq 2.
\]

Differentiate \( w \) at \( X_0 \) to obtain that

\[
0 = w_i = - \frac{u_{ii}}{u} + 2\gamma' \langle X, e_i \rangle = - \frac{h_{i1}\langle X, e_1 \rangle}{u} + 2\gamma' \langle X, e_i \rangle,
\]

where in the last equality we used the Weingarten equation. Hence, we have

\[
h_{11} = 2\gamma' u \quad \text{and} \quad h_{ii} = 0 \quad \text{for} \quad i \geq 2.
\]

Without loss of generality, we can assume \( \{h_{ij}\} \) is diagonal at \( X_0 \). Differentiating \( w \) at \( X_0 \) a second time and contracting with \( \{\tilde{F}^{ij}\} \), we obtain that

\[
0 \geq \tilde{F}^{ii} \left( - \frac{u_{ii}}{u} + \frac{u_i^2}{u^2} + \gamma''(|X|^2)_i^2 + \gamma'(|X|^2)_i \right).
\]

Combining (4.1) with the above inequality, we arrive at

\[
0 \geq - \frac{\tilde{F}^{ii} u_{ii}}{u} + 4(\gamma'^2 + \gamma'')F^{11} \langle X, e_1 \rangle^2 + \gamma' \tilde{F}^{ii} (|X|^2)_i.
\]

By (3.10), we have

\[
\tilde{F}^{ii} u_{ii} = \langle X, e_1 \rangle \langle (d_X \hat{f})(e_1) + h_{11}(d_{\nu} \hat{f})(e_1) \rangle - u \tilde{F}^{ii} h_{ii} + \hat{f}.
\]

Also, we have

\[
\tilde{F}^{ii} (|X|^2)_i = 2 \sum \tilde{F}^{ii} - 2u \hat{f},
\]

where we used \( \tilde{F}^{ij} h_{ij} = \hat{f} \). Recall that \( h_{11} = 2\gamma' u \). Substituting the above two equalities into (4.3) we get

\[
0 \geq - \frac{1}{u} \left( \langle X, e_1 \rangle \langle d_X \hat{f} \rangle(e_1) + \hat{f} \right) - 2 \langle X, e_1 \rangle \gamma' (d_{\nu} \hat{f})(e_1) + \tilde{F}^{ii} h_{ii}^2 + 4(\gamma'^2 + \gamma'')F^{11} \langle X, e_1 \rangle^2 + 2\gamma' \left( \sum \tilde{F}^{ii} - u \hat{f} \right).
\]
At $X_0$, we see that $X = \langle X, e_1 \rangle e_1 + \langle X, \nu \rangle \nu$. It then follows that

$$(dX \tilde{f})(X) = \langle X, e_1 \rangle (dX \tilde{f})(e_1) + \langle X, \nu \rangle (dX \tilde{f})(\nu).$$

From (4.4), we see that

$$0 \geq \frac{\partial}{\partial \rho}\left( \rho^{cn} f(X, \nu) \right) = \frac{\partial}{\partial \rho}\left( \rho^{cn} \tilde{f}^{cn}(X, \nu) \right)$$

$$= C_n^\nu (\rho \tilde{f})^{cn-1} \left( \tilde{f} + (dX \tilde{f})(X) \right)$$

$$= C_n^\nu (\rho \tilde{f})^{cn-1} \left( \tilde{f} + \langle X, e_1 \rangle (dX \tilde{f})(e_1) + \langle X, \nu \rangle (dX \tilde{f})(\nu) \right).$$

We therefore obtain

$$-(\tilde{f} + \langle X, e_1 \rangle (dX \tilde{f})(e_1)) \geq \langle X, \nu \rangle (dX \tilde{f})(\nu) = u(dX \tilde{f})(\nu).$$

Substituting this into (4.5) we obtain

$$0 \geq (dX \tilde{f})(\nu) - 2\langle X, e_1 \rangle \gamma (dX \tilde{f})(e_1) + \tilde{F}^{\nu} h_i^2$$

$$+ 4(\gamma^2 + \gamma') F^{11} \langle X, e_1 \rangle^2 + 2\gamma' \sum \tilde{F}^{\nu} - u\tilde{f}).$$

(4.6)

Now we choose $\gamma(t) = \frac{\gamma}{t}$, where $\alpha$ is a large constant to be determined later. Recall that $h_{11} = 2\gamma u$ at $X_0$, which implies that $h_{11}(X_0) < 0$. This means that $h_{11} \in \{ \kappa_{n-p+2}, \kappa_{n-p+3}, \ldots, \kappa_0 \}$ and therefore by Lemma 2.2

$$F^{11} \geq -\sum F^{\nu}. $$

Similar to (7), we can assume $\langle X, e_1 \rangle^2 \geq \frac{1}{4} \inf M \rho^2$. Now we arrive at

(4.7)

$$0 \geq \left( \frac{\alpha^2}{C} - C\alpha \right) \sum \tilde{F}^{\nu} - C\alpha. $$

Choosing $\alpha$ sufficiently large, we obtain a contradiction. Therefore, $X_0$ is parallel to $\nu$, and $u$ has a positive lower bound.

5. Existence of a solution

We use the continuity method as in (4) to prove Theorem 1.2. Consider the following family of functions

$$f^t(X, \nu) = tf(X, \nu) + (1 - t)p^{cn}\left[ \frac{1}{|X|^{c_n^\nu}} + \varepsilon \left( \frac{1}{|X|^{c_n^\nu}} - 1 \right) \right],$$

where $\varepsilon$ is a small positive constant such that

$$\min_{r_1 \leq r \leq r_2} \left[ \frac{1}{\rho^{c_n^\nu}} + \varepsilon \left( \frac{1}{\rho^{c_n^\nu}} - 1 \right) \right] \geq c_0 > 0,$$

for some positive constant $c_0$. It is easy to see that $f^t(X, \nu)$ satisfies (1.3) and (1.4) with strict inequalities for $0 \leq t < 1$.

Let $M_t$ be the solution of the equation

$$F(\kappa) = f^t(X_t, \nu_t),$$

where $X_t$ and $\nu_t$ are position vector and unit outer normal of $M_t$ respectively. Clearly, when $t = 0$, we have $M_0 = S^n$ and $X_0 = x$. For $t \in (0, 1)$, suppose $\rho_t = |X_t|$ attains its maximum at the point $x_0$. At this point, by (2.1), we have

$$g_{ij} = \rho_t^2 \delta_{ij} \text{ and } h_{ij} = -\rho_t^2 \delta_{ij} \geq \rho_t^2 \delta_{ij}.$$
under a smooth local orthonormal frame on $\mathbb{S}^n$. Then, we have

$$F(\kappa) \geq F\left(\frac{1}{\rho_t}(1, \ldots, 1)\right) = \frac{p^{C_p}_{\kappa}}{\rho_t^n}.$$ 

On the other hand, at $x_0$, the unit outer normal $\nu_t$ is parallel to $X_t$. If $\rho_t(x_0) = r_2$, we obtain

$$\frac{p^{C_p}_{\kappa}}{r_2 C_n} \leq F(\kappa) = f^t(X_t, \nu_t) \leq \frac{p^{C_p}_{\kappa}}{r_2 C_n},$$

which is a contradiction. So we have $\sup_M \rho_t \leq r_2$. Similarly argument at the minimum point of $\rho_t$ gives that $\inf_M \rho_t = \rho_1$. Hence, $C^0$ estimate follows.

Combining our $C^1$ estimate, our $C^2$ estimate, the Evans-Krylov theorem with the argument in [4], we get the existence and uniqueness of solution to equation (1.1). Theorem 1.2 is proved.

6. Proof of Theorem 1.3

By Lemmas 3.1, 3.2, 3.3 and 3.4 one can prove Theorem 1.3. For completeness, we include an outline here.

**Proof.** We consider the following function

$$G(x, \xi) = \log u_{\xi \xi} + \frac{a}{2} |\nabla u|^2 + \frac{A}{2} |x|^2 + \beta \log(v - u),$$

where $a$, $A$, and $\beta$ are constants to be determined later. Suppose that $G$ achieves its maximum at $(x_0, \xi_0)$. Around $x_0$, we choose coordinate system such that $\xi_0 = e_1$ and $u_{ij}(x_0)$ is diagonal such that

$$u_{11} \geq u_{22} \geq \cdots \geq u_{nn} \text{ at } x_0.$$ 

This can be done as in [12]. Thus, the new function defined by

$$\hat{G}(x) = \log u_{11} + \frac{a}{2} |\nabla u|^2 + \frac{A}{2} |x|^2 + \beta \log(v - u)$$

also attains its maximum at $x_0$. Differentiate it once to obtain

$$0 = \frac{u_{111}}{u_{11}} + au_{i}u_{ii} + Ax + \frac{\beta(v - u)_i}{v - u}. \tag{6.1}$$

Differentiating it twice and by similar computations as Lemma 3.1 and Lemma 3.2 we can arrive at

$$0 \geq -\frac{2}{u_{11}} \sum_{i \geq 2} \hat{F}^{1i, 1i} u_{11i}^2 - \frac{\hat{F}^{ii} u_{11i}^2}{u_{11}^2} + \frac{a \hat{F}^{ii} u_{ii}^2}{2} \tag{6.2}$$

$$+ \frac{A}{2} \sum \hat{F}^{ii} - CA \frac{\beta \hat{F}^{ii}(v - u)_i^2}{(v - u)^2} - C\beta.$$

We remark that in the above inequality we used

$$\sum_k F^{kk} v_{kk} = \sum_k \sum_{k \in \{i_1, \ldots, i_p\}} F(\lambda) \frac{v_{kk}}{\lambda_{i_1} + \cdots + \lambda_{i_p}}$$

$$= \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} F(\lambda) \left(\sum_{i_1} \frac{v_{i_1 i_1}}{\lambda_{i_1} + \cdots + \lambda_{i_p}} + \cdots + \sum_{i_p} \frac{v_{i_p i_p}}{\lambda_{i_1} + \cdots + \lambda_{i_p}}\right)$$
which is nonnegative since \( v \) is \( p \)-plurisubharmonic. Using the same argument as in Lemma 3.3, we obtain

\[
|u_{ii}| \leq \frac{C}{v - u}
\]

for \( i \geq n - p + 1 \), where \( C \) depends on \( a, A \) and \( \beta \).

By (6.1) and the Cauchy-Schwarz inequality, we have

\[
-\tilde{F}_{11} u_{111}^2 \geq -C a^2 \tilde{F}_{11} u_{111}^2 - CA^2 \tilde{F}_{11} - \frac{C \beta^2 \tilde{F}_{11}}{(v - u)^2}
\]

and

\[
-\sum_{i \geq 2} \frac{\beta \tilde{F}_{ii} (v - u)^2}{u_{11i}} \geq -\frac{3}{\beta} \sum_{i \geq 2} \frac{\tilde{F}_{ii} u_{11i}^2}{u_{11i}} - \frac{C \beta^2 \tilde{F}_{11}}{(v - u)^2}u_{111}^2 - \frac{C \beta^2 \tilde{F}_{11}}{(v - u)^2}u_{111}^2.
\]

Substituting the above two inequalities into (6.2) we have

\[
0 \geq \frac{2}{8} \sum_{i \geq 2} \tilde{F}_{11} u_{11i}^2 + \frac{4}{8} \sum_{i \geq 2} \tilde{F}_{ii} u_{11i}^2 + CA^2 \tilde{F}_{11} + \frac{C \beta^2 \tilde{F}_{11}}{(v - u)^2}u_{111}^2
\]

for sufficiently large \((v - u)u_{111}\) and \(A\). Combining Lemma 3.4 with (6.4) and choosing \(a\) sufficiently small and \(\beta\) sufficiently large such that \(\delta \geq \frac{4}{\beta}\), we get from (6.3) that

\[
0 \geq \frac{a}{4} \tilde{F}_{11} u_{111}^2 - \frac{C}{(v - u)^2} \tilde{F}_{11} - CA - \frac{C}{v - u}
\]

where in the second inequality we assumed \((v - u)u_{111}\) is large enough.

By Lemma 2.2 (1) we have that \(\tilde{F}_{11} u_{111} \geq c_0\), where \(c_0 > 0\) depends on \(\inf \tilde{f}\). From (6.5), we then obtain

\[
(v - u)u_{111} \leq C
\]

which implies the estimate (1.6).

\[\square\]

7. Appendix

In this appendix, we include a proof of Lemma 2.2. For \(A \in P_p\), recall the notations

\[
F(A) := F(\lambda(A)) = \Pi_{1 \leq i_1 < \cdots < i_p \leq n} (\lambda_{i_1} + \cdots + \lambda_{i_p}) \quad \text{and} \quad \tilde{F} = F_{\frac{v}{\lambda}}.
\]
where \( \lambda(A) = (\lambda_1, \cdots, \lambda_n) \) are the eigenvalues of \( A \) and \( P_p \) is defined in Definition 2.1. Suppose that the diagonal matrix \( A = \text{diag}(\lambda_1, \cdots, \lambda_n) \) belongs to \( P_p \) and \( \lambda_1 \geq \cdots \geq \lambda_n \).

**Lemma 1.** \( \tilde{F}^{11}(A) \lambda_1 \geq \frac{1}{n} \tilde{F}(A) \)

**Proof.** We have that
\[
\tilde{F}^{11}(A) = \frac{1}{C_p n} [F(A)]_{c_k}^{-1} \sum_{1 \in \{i_1, \ldots, i_p\}} \frac{F(A)}{\lambda_{i_1} + \cdots + \lambda_{i_p}}
\]
\[
\geq \frac{1}{C_p n} [F(A)]_{c_k}^{-1} C_p^{p-1} \frac{F(A)}{p \lambda_1}
\]
\[
= \frac{1}{n \lambda_1} [F(A)]_{c_k}^{-1},
\]
where in the inequality we used \( \lambda_{i_1} + \cdots + \lambda_{i_p} \leq p \lambda_1 \). \( \Box \)

**Lemma 2.** \( \sum_{k=1}^n \tilde{F}^{kk}(A) \geq p. \)

**Proof.** We have that
\[
\sum_{k=1}^n \tilde{F}^{kk}(A) = \frac{1}{C_p n} [F(A)]_{c_k}^{-1} \sum_{k=1}^n \sum_{k \in \{i_1, \ldots, i_p\}} \frac{F(A)}{\lambda_{i_1} + \cdots + \lambda_{i_p}}
\]
\[
= \frac{n C_p^{p-1}}{[C_p n]^2} [F(A)]_{c_k}^{-1} \sum_{1 \leq i_1 < \ldots < i_p \leq n} \frac{1}{\lambda_{i_1} + \cdots + \lambda_{i_p}}
\]
\[
\geq \frac{n C_p^{p-1}}{[C_p n]^2} [F(A)]_{c_k}^{-1} \frac{C_p n}{[F(A)]_{c_k}^{-1}} = p,
\]
where the inequality of arithmetic and geometric means was used in the inequality. \( \Box \)

**Lemma 3.** \( \sum_{k=1}^n F^{kk}(A) \lambda_k = C_p F(A) \).

**Proof.** Observe that
\[
\sum_{k=1}^n F^{kk}(A) \lambda_k = F(A) \sum_{k=1}^n \sum_{k \in \{i_1, \ldots, i_p\}} \frac{\lambda_k}{\lambda_{i_1} + \cdots + \lambda_{i_p}}
\]
\[
= F(A) \frac{n C_p^{p-1}}{p} = C_p F(A).
\]
\( \Box \)

**Lemma 4.** \( \sum_{k=1}^n \tilde{F}^{kk}(A) \lambda_k = \tilde{F}(A) \).

**Proof.** Observe that
\[
\sum_{k=1}^n \tilde{F}^{kk}(A) \lambda_k = \frac{1}{C_p n} [F(A)]_{c_k}^{-1} \sum_{k=1}^n F^{kk}(A) \lambda_k = \tilde{F}(A).
\]
\( \Box \)
**Lemma 5.** There is a constant $\theta = \theta(n, p)$ such that, for all $j \geq n - p + 1$,

$$F^{jj}(A) \geq \theta \sum_{i=1}^{n} F^{ii}(A).$$

**Proof.** Note that, for $j \geq n - p + 1$,

$$F^{jj}(A) \geq F(A) \frac{1}{\lambda_{n-p+1} + \cdots + \lambda_n} \sum_{1 \leq i_1 < \cdots < i_p \leq n} \frac{F(A)}{\lambda_{i_1} + \cdots + \lambda_{i_p}},$$

and

$$\sum_{k=1}^{n} F^{kk}(A) = \sum_{k=1}^{n} \sum_{k \in \{i_1, \cdots, i_p\}} \frac{F(A)}{\lambda_{i_1} + \cdots + \lambda_{i_p}} \geq \frac{n C_{n-1}^{p-1}}{C_n^p} \sum_{1 \leq i_1 < \cdots < i_p \leq n} \frac{F(A)}{\lambda_{i_1} + \cdots + \lambda_{i_p}}$$

Thus the desired inequality is proved. \hfill \Box

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