UNEXPECTED DISTRIBUTION PHENOMENON RESULTING FROM CANTOR SERIES EXPANSIONS

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Abstract. We explore in depth the number theoretic and statistical properties of certain sets of numbers arising from their Cantor series expansions. As a direct consequence of our main theorem we deduce numerous new results as well as strengthen known ones.

1. Introduction

We will prove a general result that will have six seemingly unrelated number theoretic applications. Unfortunately, it will take several pages to state this result. After this we describe the applications and then prove our theorem.

The $Q$-Cantor series expansion, first studied by G. Cantor in [9], is a natural generalization of the $b$-ary expansion. Let $N_k := \mathbb{Z} \cap [k, \infty)$. If $Q \in N_k^2$, then we say that $Q$ is a basic sequence. If $\lim_{n \to \infty} q_n = \infty$, then we say that $Q$ is infinite in limit. Given a basic sequence $Q = \{q_n\}_{n=1}^\infty$, the $Q$-Cantor series expansion of a real $x$ in $\mathbb{R}$ is the (unique) expansion of the form

$$x = E_0(x) + \sum_{n=1}^\infty \frac{E_n(x)}{q_1 q_2 \ldots q_n},$$

(1.1)

where $E_0(x) = \lfloor x \rfloor$ and $E_n(x)$ is in $\{0, 1, \ldots, q_n - 1\}$ for $n \geq 1$ with $E_n(x) \neq q_n - 1$ infinitely often. We will write $E_n$ in place of $E_n(x)$ when there is no room for confusion. Moreover, we will abbreviate (1.1) with the notation $x = E_0.E_1E_2E_3\ldots$ w.r.t. $Q$. Clearly, the $b$-ary expansion is a special case of (1.1) where $q_n = b$ for all $n$. If one thinks of a $b$-ary expansion as representing an outcome of repeatedly rolling a fair $b$-sided die, then a $Q$-Cantor series expansion may be thought of as representing an outcome of rolling a fair $q_1$-sided die, followed by a fair $q_2$-sided die and so on.

The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was first studied by P. Erdős, A. Rényi and P. Turán. This early work was done by P. Erdős and A. Rényi in [15] and [16] and by A. Rényi in [33], [34], and [35] and by P. Turán in [38].

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1G. Cantor’s motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number $e = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of J. Galambos [19].

2Uniqueness can be proven in the same way as for the $b$-ary expansions.
We recall the following standard definitions (see [23]). An asymptotic distribution function \( f : [0, 1] \to [0, 1] \) is a non-decreasing function such that \( f(0) = 0 \) and \( f(1) = 1 \). For a sequence of real numbers \( \omega = \{x_n\} \) with \( x_n \in [0, 1) \) and an interval \( I \subseteq [0, 1) \), define \( A_n(I, \omega) := \#\{i \leq n : x_i \in I\} \). A sequence of real numbers \( \omega = \{x_n\} \) has asymptotic distribution function \( f \) if

\[
\lim_{n \to \infty} \frac{A_n([0, x), \omega)}{n} = f(x).
\]

For the rest of this paper we will abbreviate asymptotic distribution function as adf. We say that a sequence \( \omega \) is uniformly distributed mod 1 if \( \omega \) has \( f(x) = x \) as its adf. For the rest of this paper we will abbreviate uniformly distributed mod 1 as u.d. mod 1. Clearly, not all sequences have an adf. A sequence of real numbers \( \omega = \{x_n\} \) has upper asymptotic distribution function \( \overline{f} \) if

\[
\limsup_{n \to \infty} \frac{A_n([0, x), \omega)}{n} = \overline{f}(x).
\]

The sequence \( \omega \) has lower asymptotic distribution function \( \underline{f} \) if

\[
\liminf_{n \to \infty} \frac{A_n([0, x), \omega)}{n} = \underline{f}(x).
\]

Every sequence of real numbers \( \omega \) has an upper and a lower adf. We note that the sequence \( \omega \) has adf \( f(x) \) if and only if \( f = \overline{f} = \underline{f} \).

A great deal of information about the \( b \)-ary expansion of a real number \( x \) may be obtained by studying the distributional properties of the sequence \( \Omega_b(x) := \{b^n x\}_{n=0}^\infty \). For example, it is well known that a real number \( x \) is normal in base \( b \) if and only if the sequence \( \Omega_b(x) \) is uniformly distributed mod 1.

Thus, we are motivated to make the following definitions for the Cantor series expansions. For every basic sequence \( Q \), define \( T_{Q,n}(x) := q_n q_{n-1} \cdots q_1 x \) (mod 1) and \( \Omega_Q(x) := \{T_{Q,n}(x)\}_{n=0}^\infty \). For integers \( m \) and \( r \), we define \( \Omega_{Q,m,r}(x) := \{T_{Q,mn+r}(x)\}_{n=0}^\infty \). For any eventually increasing function \( f : \mathbb{N} \to \mathbb{N} \), we define \( \Omega_{Q,f,m,r}(x) := \{T_{Q,f(mn+r)}(x)\}_{n=0}^\infty \). Furthermore, set \( \Omega'(x) := \left\{ \frac{x}{q_n} \right\}_{n=1}^\infty \). The sequences \( \Omega'_{Q,m,r}(x) \) and \( \Omega'_{Q,f,m,r}(x) \) are defined similarly.

We should note that the relationship between the digits of the \( Q \)-Cantor series expansion of a real number \( x \) and the sequence \( \Omega_Q(x) \) is far more complex than the analogous relationship for \( b \)-ary expansions. The most current results can be found in [26]. Thus, when generalizing problems involving digits in some \( b \)-ary expansion, we can consider either a problem involving digits in a \( Q \)-Cantor series expansion or a problem involving the distributional properties of the sequence \( \Omega_Q(x) \). Often the theory will be different. For this paper we will always choose the latter option.

The sequence \( \Omega_Q(x) \) was studied by J. Galambos in [20] and by T. Šalát in [40] and several other papers. The main focus of this paper is to study sets of reals numbers \( x \) so that \( \Omega_Q(x) \) and various subsequences of \( \Omega_Q(x) \) have specific upper and lower adfs. This will allow us to attack a wide range of problems.

**Definition 1.1.** A set of functions \( \{f_{m,r}\}_{m \in \mathbb{N}, 0 \leq r < m} \) is called a linear family if for all \( m, r, \) and \( d \)

\[
f_{m,r} = \frac{1}{d} \sum_{i=0}^{d-1} f_{m,d,i+r}.
\]
The set of functions \( \{f_{m,r}\} \) where \( f_{m,r}(x) = x \) for all \( m \) and \( r \) gives an example of a linear family of adfs. A non-trivial example is given in the proof of Theorem 1.13 in Section 5.2.

For \( p, q \in \mathbb{Z}[X] \), set
\[
d_{p,q,s,n} := \frac{#(p(N) \cap q(N) \cap \{s, s + 1, \ldots, n\})}{#(q(N) \cap \{s, s + 1, \ldots, n\})}.
\]
Define a relation \( \preceq \) among polynomials \( p, q \in \mathbb{Z}[X] \) so
\[
q \preceq p \text{ if } \limsup_{n-s \to \infty} d_{p,q,s,n} > 0.
\]
If \( p \preceq q \) and \( q \preceq p \), then we write \( p \approx q \).

**Definition 1.2.** A set of polynomials \( P \subseteq \mathbb{Z}[X] \) is **saturated** if for any \( f \in \mathbb{Z}[X] \) there exists a polynomial \( p \in P \) and a linear polynomial \( \mu \in \mathbb{Z}[X] \) such that \( f = p \circ \mu \). The set \( P \) is **sparsely intersecting** if for each \( i \) and \( j \) we have \( p_i \preceq p_j \) or \( p_j \preceq p_i \).

We will prove the following lemma in Section 2.1.

**Lemma 1.3.** There is a saturated sparsely intersecting set of polynomials.

A real number \( x \) is **computable** if there exists \( b \in \mathbb{N} \) with \( b \geq 2 \) and a total recursive function \( f : \mathbb{N} \to \mathbb{N} \) that calculates the digits of \( x \) in base \( b \). A sequence of real numbers \( \{x_n\} \) is **computable** if there exists a total recursive function \( f : \mathbb{N}^2 \to \mathbb{N} \) such that for all \( m, n \) we have that \( f(m, n)^{-1} < x_n < f(m, n)^{-1} \). A sequence of functions \( \{f_n\} \) from a metric space \( X \) to \( \mathbb{R} \) is **uniformly computable** if the double sequence \( \{f_n(x_m)\} \) is computable for any computable sequence \( \{x_m\} \) and if there is a recursive function \( \gamma(n, k) \) such that for all \( n, k \) and \( x, y \in X \), we have that \( d(x, y) \leq \frac{1}{\gamma(n, k)} \) implies \( |f_n(x) - f_n(y)| \leq \frac{1}{\gamma(n, k)} \). A function \( f \) is **uniformly computable** if the sequence \( \{f, f, f, \ldots\} \) is uniformly computable. A sequence \( \{x_n\} \) is **uniformly computable** if there is a uniformly computable function \( f : \mathbb{N} \to \mathbb{R} \) such that \( f(n) = x_n \).

**Definition 1.4.** A basic sequence \( Q = \{q_n\} \) is a **computably growing basic sequence** if it is infinite in limit and the sequence \( \{\inf\{i : \forall j \geq i \ (q_j \geq n)\}\}_{n=1}^\infty \) is computable.

**Definition 1.5.** A linear family of adfs \( \{f_{m,r}\}_{m \in \mathbb{N}, 0 \leq r < m} \) is an **explicit linear family of adfs** if for each \( m, r \in \mathbb{N} \) with \( 0 \leq r < m \) the following hold.

1. The real numbers \( f_{m,r}(q) \) and \( \inf f_{m,r}^{-1}(q) \) are computable for every \( q \in \mathbb{Q} \).
2. If \( f_{m,r} \) is discontinuous at \( t \), then \( t \) is a computable real number and \( f_{m,r}(t) \) is a computable real number.
3. The function \( f_{m,r} \) is either continuous, has only finitely many discontinuities, or the set of its discontinuities may be written in the form \( \{t_n : n \in \mathbb{N}\} \), where \( \{t_n\} \) is a uniformly computable sequence.

**Definition 1.6.** A sparsely intersecting set of polynomials \( P = \{p_i\} \) is an **explicit sparsely intersecting set of polynomials** if for all \( p, q \in P \) there exists a computable sequence \( \{N(m)\} \) such that if \( n - s > N(m) \), then \( d_{p,q,s,n} < \frac{1}{m} \) or \( d_{p,q,s,n} < \frac{1}{m} \).

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\(^3\)It is easy to verify that \( \preceq \) is a preorder but not a partial order. Similarly, we can show that \( \approx \) is an equivalence relation.
Given a basic sequence $Q$, a set of sparsely intersecting polynomials $P$, and a set of linear families $F$ of adfs, define

$$
\Phi_{Q,P,F} = \left\{ x \in (0,1) : \text{Q}_{Q,p,m,r}(x) \text{ has upper and lower adfs } f_{p,m,r, f_{p,m,r}} \forall p \in P, m \in \mathbb{N}, 0 \leq r < m \right\}.
$$

We may now state the main theorem of this paper.

**Main Theorem 1.7.** If $Q$ is infinite in limit, $P$ is a set of sparsely intersecting polynomials, and $F$ is a set of linear families of upper and lower adfs given by (1.2), then

$$
dim_H \Phi_{Q,P,F} = 1.
$$

Furthermore, if $Q$ is computable and computably growing, $P$ is explicit, and $F$ is explicit, then there is a subset $\Phi'_{Q,P,F}$ of $\Phi_{Q,P,F}$ such that the following hold.

1. The set $\Phi'_{Q,P,F}$ has full Hausdorff dimension.
2. There exist computable sequences $\{\alpha(n)\}$ and $\{\beta(n)\}$ such that

$$
\Phi'_{P,Q,F} = \left\{ x \in (0,1) : \alpha(n) \leq E_n(x) \leq \beta(n) \right\}.
$$

The set $\Phi'_{Q,P,F}$ and computable sequences $\{\alpha(n)\}$ and $\{\beta(n)\}$ are constructed in Section 2.4.

Main Theorem 1.7 is proven in Section 2.4.

1.1. **Application I: Equivalent definitions of normality.** We recall the modern definition of a normal number.

**Definition 1.8.** A real number $x$ is normal of order $k$ in base $b$ if all blocks of digits of length $k$ in base $b$ occur with relative frequency $b^{-k}$ in the $b$-ary expansion of $x$. Moreover, $x$ is simply normal in base $b$ if it is normal of order 1 in base $b$ and $x$ is normal in base $b$ if it is normal of order $k$ in base $b$ for all natural numbers $k$.

It is well known that É. Borel [4] was the first mathematician to study normal numbers. In 1909 he gave the following definition.

**Definition 1.9 (É. Borel).** A real number $x$ is normal in base $b$ if each of the numbers $x, bx, b^2x, \cdots$ is simply normal (in the sense of Definition 1.8), in each of the bases $b, b^2, b^3, \cdots$.

É. Borel proved that Lebesgue almost every real number is normal, in the sense of Definition 1.9 in all bases. In 1940, S. S. Pillai [32] simplified Definition 1.9 by proving that

**Theorem 1.10 (S. S. Pillai).** For $b \geq 2$, a real number $x$ is normal in base $b$ if and only if it is simply normal in each of the bases $b, b^2, b^3, \cdots$.

Theorem 1.10 was improved in 1951 by I. Niven and H. S. Zuckerman [29] who proved

**Theorem 1.11 (I. Niven and H. S. Zuckerman).** Definition 1.8 and Definition 1.9 are equivalent.
It should be noted that both of these results require some work to establish, but were assumed without proof by several authors. For example, M. W. Sierpinski assumed Theorem 1.10 in [36] without proof. Moreover, D. G. Champernowne [10], A. H. Copeland and P. Erdős [12], and other authors took Definition 1.8 as the definition of a normal number before it was proven that Definition 1.8 and Definition 1.9 are equivalent. More information can be found in Chapter 4 of the book of Y. Bugeaud [8].

The following theorem was proven by H. Furstenberg in his seminal paper “Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation” [18] on page 23 as an application of disjointness to stochastic sequences.

**Theorem 1.12** (H. Furstenberg). Suppose that $x = d_0.d_1d_2\ldots$ is the $b$-ary expansion of $x$. Then $x$ is normal in base $b$ if and only if for all natural numbers $m$ and $r$ the real number $0.d_rd_{m+r}d_{2m+r}d_{3m+r}\ldots$ is normal in base $b$.

It is interesting to note that although Furstenberg did not provide an alternate proof of Theorem 1.11, he showed that an entirely different definition of normality is equivalent to Definition 1.8 (See appendix for more details) We will say that $x$ is *AP normal of type I in base $b$* if $x$ satisfies Definition 1.9 and *AP normal of type II in base $b$* if $x$ satisfies the notion introduced in Theorem 1.12. Thus, for numbers expressed in base $b$

\[(1.3)\text{ normality } \iff \text{AP normality of type I } \iff \text{AP normality of type II}.\]

The authors feel that the equivalence of Definition 1.8 and Definition 1.9 and other similar ones is a far more delicate topic than is typically assumed. The core of E. Borel’s definition is that a number is normal in base $b$ if blocks of digits occur with the desired relative frequency along all infinite arithmetic progressions. We say that a real number $x$ is *Q-distribution normal* if $\Omega_Q(x)$ is u.d. mod 1. A real number $x$ is *AP Q-distribution normal of type I* if for all $m \in \mathbb{N}$ and $0 \leq r < m$ we have that $\Omega_{Q,m,r}(x)$ is u.d. mod 1. If $x = E_0,E_1E_2\ldots$ w.r.t. $Q$, then we say that $x$ is *AP Q-distribution normal of type II* if the real number $0.E_mE_{m+r}E_{2m+r}\ldots$ is $\Omega_{Q,m,r}(x)$ is u.d. mod 1 for any $m > 1$. We will prove in Section 3.2 that Q-distribution normality is not equivalent to AP Q-distribution normality in a particularly strong way. The following theorem describes exactly how much (1.3) may be extended to Q-Cantor series expansions when $Q$ is infinite in limit.

**Theorem 1.13.** Let $Q$ be a basic sequence that is infinite in limit. Then

1. AP Q-distribution normality of type I is equivalent to AP Q-distribution normality of type II.
2. The set of real numbers that are Q-distribution normal and AP Q-distribution abnormal is a meagre set with zero measure and full Hausdorff dimension.

Furthermore, if $Q$ is computable and computably growing, then the proof of Theorem 1.13 provides a computable example of a real number that is Q-distribution normal and AP Q-distribution abnormal. See Section 1.4 for further discussion. However, Main Theorem 1.7 is far stronger since it allows us to specify upper and lower adfs along polynomially indexed subsequences of $\Omega_Q(x)$. Theorem 1.13 only requires knowledge of $\Omega_Q(x)$ along infinite arithmetic progressions.
We note that far less is known if we extend (1.3) to analogous definitions involving digits. The problem is discussed in [24] and partial results are given. One substantial difference is that the analogous generalizations of the definitions of AP normality of types I and II are no longer equivalent. However, these definitions are technical, so we choose not to state any of these results here.

1.2. Application II: Computing the Hausdorff dimension of sets of real numbers whose digits have specified frequencies.

The following well known result was proven for $b = 2$ by A. S. Besicovitch in [4] and for all other $b$ by H. Eggleston in [14].

**Theorem 1.14** (H. Eggleston). Let $b \in \mathbb{N}_2$ and $\vec{p} = (p_0, p_1, \ldots, p_{b-1})$ be a probability vector. Then the Hausdorff dimension of the set of all real numbers $x$ where the digit $i$ occurs in the $b$-ary expansion of $x$ with relative frequency $p_i$ for all $i = 0, 1, \ldots, b - 1$ is equal to

$$-\frac{\sum_{i=1}^{b-1} p_i \log p_i}{\log b}.$$ 

There have been numerous improvements of Theorem 1.14. Moreover, Theorem 1.14 has been extended to certain classes of Cantor series expansions. Early work was done by J. Peyrière in [31] and Y. Kifer in [22]. We mention a similar result proven by Y. Xiong in [43].

**Theorem 1.15** (Y. Xiong). Suppose that $Q$ is infinite in limit and that $\vec{p} = (p_n)$ is an infinite probability vector. For $m > 0$, let

$$B_m(\vec{p}) = \left\{ x \in [0, 1) : \lim_{n\to\infty} \frac{N_Q((k), x)}{n} = p_k, \text{ for } 0 \leq k \leq m \right\};$$

$$B(\vec{p}) = \left\{ x \in [0, 1) : \lim_{n\to\infty} \frac{N_Q((k), x)}{n} = p_k, \text{ for } k \geq 0 \right\}.$$ 

Then the following hold.

1. If $\lim_{n\to\infty} \frac{\log q_n}{\sum_{j=1}^{\infty} \log q_i} = 0$, then

$$\dim_H(B_m(\vec{p})) = \sup_{t \in T_m} \liminf_{n \to \infty} \frac{\sum_{j=1}^{n} t_j \log q_j}{\sum_{j=1}^{\infty} \log q_j},$$

where

$$T_m = \left\{ t \in \{0, 1\}^\mathbb{N} : \sum_{j=m}^{\infty} p_j = \sum_{j=1}^{n} t_j \right\}.$$ 

2. If $Q$ is increasing and the sequence $\left\{ \frac{\log q_n}{\sum_{j=1}^{\infty} \log q_j} \right\}$ is bounded, then $\dim_H(B(\vec{p})) = 0$.

C. M. Colebrook [11] proved a similar result to Theorem 1.14 about the Hausdorff dimension of the set of real numbers $x$ where the sequence $O_b(x)$ has a given adf. A special case of Main Theorem 1.7 extends C. M. Colebrook’s result in a surprising way to a large class of Cantor series expansions.

**Theorem 1.16.** Suppose that $Q$ is infinite in limit and that $f$ is an adf. Then the set of real numbers $x$ such that $O_Q(x)$ has adf of $f$ has full Hausdorff dimension.

We note that the sets considered in Theorem 1.15 have much smaller Hausdorff dimension than those considered in Theorem 1.16. This is in sharp contrast to the case of the $b$-ary expansions.
1.3. Application III: Analyzing the Hausdorff dimension of sets of numbers without digit frequencies. It is difficult in general to analyze the set of real numbers whose frequencies of digits do not exist. This is discussed in L. Olsen’s paper [30]. S. Albeverio, M. Pratsiovytyi, and G. Torbin proved in [1] that the set of real numbers whose frequencies of digits in base $b$ do not exist has zero measure and full Hausdorff dimension. We extend their result to the following theorem.

**Theorem 1.17.** If $Q$ is infinite in limit, then the set of real numbers $x$ such that $O_Q(x)$ has no adf has zero measure and full Hausdorff dimension.

Theorem 1.17 is proven in Section 3.2.

1.4. Application IV: Constructing examples of normal numbers. The most well known construction of a normal number in base 10 is due to Champernowne. The number

$$0.123456789101112\cdots,$$

formed by concatenating the digits of every natural number written in increasing order in base 10, is normal in base 10. The number formed by concatenating the digits of the natural numbers in base $b$ in order is normal in base $b$. A. H. Copeland and P. Erdős in [12] showed that the number

$$0.2357111317192329\cdots,$$

formed by concatenating the digits of all prime numbers is normal in base $b$. H. Davenport and P. Erdős in [13] showed that the number formed by concatenating the value of a positive integer valued polynomial at each natural number yields a normal number in base $b$. Many similar and more sophisticated results have been proven since then. For example, J. Vandehey [21] and M. Madritsch and R. Tichy [25] have given similar constructions. A more extensive list of results can be found in Y. Bugeaud’s book [8].

A real number is **absolutely normal** if it is normal in base $b$ for all $b \in \mathbb{N}_2$. M. W. Sierpiński gave an example of an absolutely normal number that is not computable in [36]. The authors feel that examples such as M. W. Sierpiński’s are not fully explicit since they are not computable real numbers, unlike Champernowne’s number. A. M. Turing gave the first example of a computable absolutely normal number in an unpublished manuscript. This paper may be found in his collected works [39]. See [2] by V. Becher, S. Figueira, and R. Picchi for further discussion.

We will use Main Theorem 1.7 to construct a computable $Q$-distribution normal number when $Q$ is computable and computably growing.

**Theorem 1.18.** Suppose that $Q$ is computable and computably growing, $P = \{X\}$, and $\mathcal{F}_{1,1,1}(x) = \mathcal{F}_{1,1,1}(x) = x$. If $\{\alpha(n)\}$ is the sequence given in Main Theorem 1.7, then the real number $\sum_{n=1}^{\infty} \frac{\alpha(n)}{q_1^{\alpha(n)}q_2^{\alpha(n)}}$ is computable and $Q$-distribution normal.

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4The $n$’th digit of A. M. Turing’s number may be computed with an algorithm that is doubly exponential in $n$. V. Becher, P. A. Heiber, and T. A. Slaman constructed an absolutely normal number in [10] whose digits may be computed in polynomial time.
1.5. **Application V: Constructing examples of real numbers with different digital frequencies.** There is substantial literature in pertaining to the explicit construction of numbers with different digital frequencies. The notes of section 1.8 in [23] provide a good list of papers on the subject. We solve an analogous problem for Cantor series expansions with $Q$ infinite in limit: constructing a computable real number $x$ so that $O_Q(x)$ has a given adf $\phi$. This follows immediately from Main Theorem 1.7 and is a more general version of Theorem 1.18.

**Theorem 1.19.** Suppose that the singleton $\{\phi\}$ is an explicit set of adfs, $Q$ is computable and computably growing, $P = \{X\}$, and $T_{1,1,1} = T_{1,1,1} = \phi$. If $\{\alpha(n)\}$ is the sequence given in Main Theorem 1.7 then the real number $\xi = \sum_{n=1}^{\infty} \frac{\alpha(n)}{q_1 \cdots q_n}$ is computable and $O_Q(\xi)$ has adf $\phi$.

1.6. **Application VI: Sharpening known theorems.** We mention two results from other papers that will follow as corollaries of our main theorem. In fact the immediate corollaries will be stronger than the results stated in this section. The following theorem was proven by J. Peyrière in [31].

**Theorem 1.20 (J. Peyrière).** Suppose that $Q$ is infinite in limit. Then for all $l \in (0, 1)$

$$\dim_H \left\{ x = 0.E_1E_2 \cdots \text{w.r.t. } Q : \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} E_j = l \right\} = 1. $$

For any sequence $X = \{x_n\}$ of real numbers, let $A(X)$ denote the set of accumulation points of $X$. Given a set $D \subseteq [0, 1]$, let

$$E_D(Q) := \{ x = 0.E_1E_2 \cdots \text{w.r.t. } Q : A(O_Q'(x)) = D \}. $$

The following results are proven by Y. Wang, Z. Wen, and L. Xi in [41].

**Theorem 1.21 (Y. Wang, Z. Wen, and L. Xi).** If $Q$ is infinite in limit, then $\dim_H(E_D(Q)) = 1$ for every closed set $D$.

Let

$$E_{D,m,r}(Q) := \{ x = 0.E_1E_2 \cdots \text{w.r.t. } Q : A(O_{Q,m,r}'(x)) = D_{m,r} \}. $$

We will prove the following theorem as a corollary of Main Theorem 1.7 in Section 3.1.

**Theorem 1.22.** Suppose that $k \in \mathbb{N}$ and closed sets $D_{m,r}$ are contained in $[0, 1]$ for $1 \leq m \leq k$, and $0 \leq r < m$. If $\bigcap_{m=1}^{k} \bigcap_{r=0}^{m-1} D_{m,r} \neq \emptyset$, then

$$\dim_H \left( \bigcap_{m=1}^{k} \bigcap_{r=0}^{m-1} E_{D_{m,r,m,r}}(Q) \right) = 1. $$

Note that Theorem 1.21 follows immediately from Theorem 1.22 by setting $k = 1$. Similarly, we may obtain Theorem 1.20 by setting $k = 1$ and $D_{1,1} = \{l\}$.\footnote{Note that Theorem 1.20 follows a direct consequence of Theorem 1.21 by considering the closed set $D = \{l\}$ for $0 \leq l \leq 1$.}
2. Main theorem and construction

2.1. Intersection of Polynomial Sequences. We will need the following theorem by Y. Bilu and R. Tichy [5].

**Theorem 2.1** (Y. Bilu and R. Tichy). Let \( p, q \in \mathbb{Z}[X] \). If the equation \( p(x) = q(y) \) has infinitely many integer solutions, then there exist a polynomial \( \phi \in \mathbb{Q}[X] \), linear polynomials \( u, v \in \mathbb{Q}[X] \) and polynomials \( f, g \in \mathbb{Z}[X] \) where \( (f, g) \) is a standard pair so that \( \phi \circ p \circ u = \phi \circ q \circ g \circ v \). The standard pairs are

1. \((x^m, ax^r p(x)^m)\) with \( 0 \leq r < m, (r, m) = 1 \) and \( r + \deg(p) > 0 \);
2. \((x^2, (ax^2 + b)p(x)^2)\);
3. \((D_m(x, a^n), D_n(x, a^m))\), where \( D_m(x, a) \) is the \( m \)th Dickson polynomial and \((m, n) = 1\);
4. \((a^{-m/2} D_m(x, a), -b^{-n/2} D_n(x, b))\) with \((m, n) = 2\);
5. \(((ax^2 - 1)^3, 3x^4 - 4x^3)\).

With this theorem we can prove the following.

**Lemma 2.2.** If \( p, q \in \mathbb{Z}[X] \), then \( p \approx q \) if and only if there exist linear polynomials \( \mu, \lambda \in \mathbb{Z}[X] \) such that \( p \circ \mu = q \circ \lambda \).

**Proof.** For the forward direction, suppose that two linear polynomials \( \mu(n) = mn + r \) and \( \lambda(n) = mn' + r' \) exist so that \( p \circ \mu = q \circ \lambda \). Then

\[
\limsup_{s \to \infty} \frac{\#(p(n) \cap q(n) \cap \{s, s + 1, \ldots, n\})}{\#(q(n) \cap \{s, s + 1, \ldots, n\})} \geq \frac{1}{m'} > 0.
\]

Similarly, \( \limsup_{s \to \infty} \frac{\#(q(n) \cap p(n) \cap \{s, s + 1, \ldots, n\})}{\#(p(n) \cap \{s, s + 1, \ldots, n\})} \geq \frac{1}{m} > 0 \).

For the reverse direction we look at two cases: when \( \deg(p) = \deg(q) \) and when \( \deg(p) \neq \deg(q) \). Suppose first that \( \deg(p) = \deg(q) = 1 \). Write \( p(n) = a_k n^k + \ldots + a_1 n + a_0 \) and \( q(n) = b_l n^l + \ldots + b_1 n + b_0 \). Then \( \limsup_{s \to \infty} \frac{\#(p(n) \cap q(n) \cap \{s, s + 1, \ldots, n\})}{\#(p(n) \cap \{s, s + 1, \ldots, n\})} = 1 \).

But \( \limsup_{s \to \infty} \frac{\#(p(n) \cap q(n) \cap \{s, s + 1, \ldots, n\})}{\#(p(n) \cap \{s, s + 1, \ldots, n\})} = 1 \). Since

\[
\#(p(n) \cap q(n) \cap \{1, 2, \ldots, n\}) \leq \#(p(n) \cap \{1, 2, \ldots, n\}),
\]

we have that

\[
\limsup_{n \to \infty} \frac{\#(p(n) \cap q(n) \cap \{s, s + 1, \ldots, n\})}{\#(q(n) \cap \{s, s + 1, \ldots, n\})} \leq \limsup_{n \to \infty} \frac{b_l n^l + b_1 s^l}{a_k n^k + a_1 s^k} = 0.
\]

But \( p \) and \( q \) have different degrees, and so they cannot be equal when composed with linear polynomials. Thus the theorem holds for this case.

Now suppose that \( \deg(p) = \deg(q) = k \). In order to have that \( p \approx q \), there must be infinitely many integer solutions to the equation \( p(x) = q(y) \). By Theorem 2.1 we must have that there exists a polynomial \( \phi \in \mathbb{Q}[X] \), linear polynomials \( \mu, \lambda \in \mathbb{Q}[X] \), and a standard pair of polynomials \( f, g \in \mathbb{Z}[X] \).

Since \( \deg(p) = \deg(q) \), we must have that \( \deg(f) = \deg(g) \). But there are only a few cases where a standard pair of polynomials can have the same degree. For standard pairs of the first and third type, we must have that \( f \) and \( g \) are linear. For
standard pairs of the second and fourth type, we must have that \( f \) and \( g \) are of
the form \( ar^2 + b \) with \( a, b \in \mathbb{Q} \). Standard pairs of the fifth kind cannot have equal
degrees. If \( f \) and \( g \) are linear polynomials, then the proof of the previous direction
suffices.

We only need to prove the claim when \( f \) and \( g \) are quadratic with zero linear
term, or equivalently when \( f(n) = n^2 \) and \( g(n) = an^2 + b \). In this case, note that
the equation \( x^2 = dy^2 + c \) is equivalent to \( ax^2 + by^2 = c \) with integers \( a, b, \) and
\( c \). The solutions of this equation are distributed according to a number of different
relations between \( a, b, \) and \( c \). The only scenario where this equation has infinitely
many solutions is when \(-ab\) is not square and positive. In that case, the equation
can be rewritten as \( x_1^2 - dy^2 = N \), with \( x_1 = ax \), \( d = -ab \) and \( N = ac \). Integer
solutions to this equation give an upper bound to solutions of the original equation.
This is the generalized Pell equation, whose solutions are well known. They are
of the form \( r_iu_i^2 \) for \( 1 \leq i \leq m \) where there are finitely many base solutions \( r_i \),
and all further solutions are generated by multiplying by units \( u_i \) in \( \mathbb{Z}[^{\sqrt{d}}] \). Thus
\( \#(p(N) \cap q(N) \cap \{s, s + 1, \ldots, n\}) \leq \sum_{i=1}^m \frac{\log(n)}{\log(n)} \) so
\[
\limsup_{n \to \infty} \frac{\#(p(N) \cap q(N) \cap \{s, s + 1, \ldots, n\})}{\#(q(N) \cap \{s, s + 1, \ldots, n\})} \leq \limsup_{n \to \infty} \frac{\sum_{i=1}^m \frac{\log(n)}{\log(n)}}{n^\pi - s^\pi} = 0.
\]

Proof of Lemma 2.3. Start by ordering \( \mathbb{Z}[X] \) as follows. First, list all polynomials
of degree less than or equal to 1 with coefficients whose absolute values are less
than or equal to 1. Then list all polynomials of degree at most 2 and coefficients
with absolute values at most 2 ordered lexicographically, removing any repeated
polynomials. At step \( n \), list all polynomials of degree at most \( n \) and coefficients
with absolute values at most \( n \) ordered lexicographically. In this way, we create a
bijection between the natural numbers and \( \mathbb{Z}[X] \). If \( p_i \approx p_j \) for any \( j < i \), then
there exist \( q \) and \( \mu, \lambda \) such that \( p_j = q \circ \mu \) and \( p_i = q \circ \lambda \). Replace \( p_j \) by \( q \),
remove any other instances of \( q \) in the ordering, and remove \( p_i \). Let \( P_i \) be the result of
this operation completed for \( p_1, p_2, \ldots, p_i \). Then \( P = \bigcap P_i \) is our desired indexed
set. \( \square \)

2.2. Explicit asymptotic distribution functions and polynomials.

Lemma 2.3. If \( \{f_{n,m}\} \) is an explicit linear family of adfs, then there exists a
sequence of explicit linear families of continuous adfs \( \{g_{n,m,r}\} \) so that \( g_{n,m,r} \) converges
to \( f_{n,m} \) pointwise.

Proof. We mimic the proof in [23] that for any adf \( f \), there exists a sequence of
continuous adfs converging to \( f \). Define increasing sequences \( a^n = (a^n_i)_{i=1}^\infty \) such
that \( a^n \) contains \( \frac{i}{n+1} \) for \( 0 \leq i \leq n \) and all \( t \) so that
\[
(2.1) \quad \lim_{x \to t^+} f_{n,m}(x) - \lim_{x \to t^-} f_{n,m}(x) > \frac{1}{n}.
\]
There are finitely many \( t \) that satisfy (2.1), so \( a^n \) is a finite sequence. Each element
of \( a^n \) is a computable real number as well. Note that \( a^n_{i+1} - a^n_i < \frac{1}{n} \).

Let \( g_{n,m,r} (a^n_i) = f_{n,m}(a^n_i) \) and piecewise linear between \( a^n_i \) and \( a^n_{i+1} \). Then
\( g_{n,m,r} \) is continuous, non-decreasing, \( g_{n,m,r}(0) = 0 \), and \( g_{n,m,r}(1) = 1 \). Note that
for any $a^n_i$, since $\{f_{m,r}\}$ is a linear family of adfs, we have that
\[
g_{n,m,r}(a^n_i) = \frac{1}{d} \sum_{i=0}^{d-1} g_{n,md,m+i+r}(a^n_i)
\]
for all $d$. As $g_{n,m,r}$ is piecewise linear, we have that this equality holds for all $x \in [0,1]$. So $\{g_{n,m,r}\}$ is a linear family of adfs. As $g_{n,m,r}$ is continuous, we only need to check that $g_{n,m,r}(q)$ and $\inf(g_{n,m,r}(q))$ are computable real numbers for all $q \in \mathbb{Q}$. We have that $a^n_i \leq q < a^n_{i+1}$ for some $i$, so
\[
g_{n,m,r}(q) = \frac{f_{m,r}(a^n_{i+1}) - f_{m,r}(a^n_i)}{a^n_{i+1} - a^n_i}(q - a^n_i) + f_{m,r}(a^n_i)
\]
since $g_{n,m,r}$ is piecewise linear. But the set of computable real numbers is closed under the usual field operations of the reals (see [12]), so the real number $g_{n,m,r}(q)$ is computable. If $g^{-1}_{n,m,r}(q)$ consists of a single point, say $r$, then we have that $a^n_i \leq r \leq a^n_{i+1}$ and
\[
r = (q - f_{m,r}(a^n_i)) \frac{a^n_{i+1} - a^n_i}{f_{m,r}(a^n_{i+1}) - f_{m,r}(a^n_i)} + a^n_i,
\]
which is a computable real number. If $g^{-1}_{n,m,r}(q)$ does not consist of a single point, we have that there are maximum and minimum integers $i$ and $j$ such that $g_{n,m,r}(a^n_i) = g_{n,m,r}(a^n_j) = q$. Since $g_{n,m,r}(a^n_{i+1}) \neq q$, for any $a^n_{i+1} \leq x < a^n_i$ we have that $g_{n,m,r}(x) < q$. So the real number $\inf\{g^{-1}_{n,m,r}(q)\} = a^n_i$ is computable. Thus $g_{n,m,r}$ is an explicit linear family of adfs.

To see that $g_{n,m,r}$ converges pointwise to $f_{m,r}$, let $t \in [0,1]$. If $t$ is a discontinuity of $f_{m,r}$, then $\lim_{x \to t^-} f_{m,r}(x) = \lim_{x \to t^+} f_{m,r}(x) > 0$, which implies that for some $n_0$ we have that $\lim_{x \to t^-} f_{m,r}(x) - \lim_{x \to t^+} f_{m,r}(x) > \frac{1}{n_0}$. Then $g_{n,m,r}(t) = f_{m,r}(t)$ for $n > n_0$. Now suppose $t$ is not a discontinuity of $f_{m,r}$. Let $\epsilon > 0$. Then for some $n_0$, if $x \in \left(t - \frac{1}{n_0}, t + \frac{1}{n_0}\right)$, then $f_{m,r}(x) \in (f_{m,r}(t) - \epsilon, f_{m,r}(t) + \epsilon)$. For some $i$, we have that $a^n_{i+1} \leq t \leq a^n_{i+1}$. As $a^n_{i+1} - a^n_i < \frac{1}{n_0}$, we know that $a^n_i, a^n_{i+1} \in \left(t - \frac{1}{n_0}, t + \frac{1}{n_0}\right)$. Thus $f_{m,r}(a_i^n) > f_{m,r}(t) - \epsilon$ and $f_{m,r}(a_{i+1}^n) < f_{m,r}(t) + \epsilon$. But since $g_{n,m,r}(a^n_i) = f_{m,r}(a_i^n)$ for all $i$ and $g_{n,m,r}(a^n_i) \leq g_{n,m,r}(t) \leq g_{n,m,r}(a^n_{i+1})$, then we have that $f_{m,r}(t) - \epsilon < g_{n,m,r}(t) < f_{m,r}(t) + \epsilon$. Thus $g_{n,m,r}$ converges pointwise to $f_{m,r}$.

The following result by S. Tengely [37] is useful for constructing an explicit set of polynomials and proving Main Theorem 1.7.

**Theorem 2.4** (S. Tengely). Let $p, q \in \mathbb{Z}[X]$ be monic polynomials with deg $p = n \leq \deg q = m$ such that $p(X) - q(Y)$ is irreducible in $\mathbb{Q}[X,Y]$ and $\gcd(n,m) > 1$. Let $d > 1$ be a divisor of $\gcd(n,m)$. If $(x, y) \in \mathbb{Z}^2$ is a solution of the Diophantine equation $p(x) = q(y)$, then
\[
\max\{|x|, |y|\} \leq d^\frac{m^2 - m - 2}{2m} m^{\frac{3m}{2}} (m/d + 1)^{\frac{3m}{2}} (h+1)^{\frac{m^2 + mn + m}{2m}} + 2m,
\]
where $h = \max\{H(p), H(q)\}$ and $H(\cdot)$ denotes the maximum of the absolute values of the coefficients.

We prove the following theorem to have that the second part of Main Theorem 1.7 is not vacuous.
Lemma 2.5. There is a sparsely intersecting explicit set of polynomials that contains \( p(X) = X \).

Proof. We proceed with the construction exactly as before, creating an ordering of all polynomials. Note that by construction, \( p_1 \) is the identity polynomial. As before, at step \( i > 1 \) we check if \( p_i \) and \( p_j \) satisfy the properties of Theorem 2.4 for \( 1 < j < i \). If they do, then there is a computable bound \( M \) on the absolute values of solutions to \( p_i(x) = p_j(y) \). Thus if we set \( N(m) = \frac{M}{m} \), we have that if \( n > N(m) \), then \( d_{p_i, p_j, n} < \frac{M}{n} \).

This ensures sparse intersection and explicitness when we do not consider \( p_1 \). Suppose \( j = 1 < i \) and \( p_i(n) = m \). Writing \( p_i(n) = a_k n^k + \ldots + a_0 \), set \( a^* = \max\{a_k, \ldots, a_0\} \). Note that if \( n > \frac{2ka^*}{a_k} \), then \( p_i(n) > \frac{1}{2} a_k n^k \).

Thus

\[
d_{p_i, p_j, n} \leq \frac{2ka^*}{a_k n} + \sqrt{\frac{2}{a_k n}}
\]

for \( n > \frac{2ka^*}{a_k} \). Set \( N(m) = \left[ \max\left\{ \frac{4ka^* m}{a_k}, \frac{8m^2}{a_k} \right\} \right] \). Therefore \( N(m) \) is a computable sequence. Moreover, if \( n > N(m) \), then \( d_{p_i, p_j, n} < \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m} \).

If \( p_i \) and \( p_j \) do not satisfy the conditions of Theorem 2.4 for any \( 1 < j < i \), we remove \( p_i \) from \( P \) and relabel \( p_{i+1} \) to \( p_i \) and so on. Let \( P_1 \) be the resulting set of this procedure conducted for \( 1 \leq j \leq i \). Set \( P = \bigcap P_i \). \( \square \)

2.3. Homogeneous Moran set structure. We will construct a subset \( \Phi'_{Q, P, F} \) of \( \Phi_{Q, P, F} \) so that \( \Phi'_{Q, P, F} \) has the structure of a homogeneous Moran set. Let \( \{n_k\} \) be a sequence of positive integers and \( \{c_k\} \) be a sequence of positive numbers such that \( n_k \geq 2 \), \( 0 < c_k < 1 \), \( n_1 c_1 \leq \delta \), and \( n_k c_k \leq 1 \), where \( \delta \) is a positive real number. For any \( k \), let \( D_k = \{(i_1, \ldots, i_k) : 1 \leq i_j \leq n_j, 1 \leq j \leq k\} \), and \( D = \bigcup D_k \), where \( D_0 = \emptyset \). If \( \sigma = (\sigma_1, \ldots, \sigma_k) \in D_k \), \( \tau = (\tau_1, \ldots, \tau_m) \in D_m \), put \( \sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m) \).

Definition 2.6. Suppose \( J \) is a closed interval of length \( \delta \). The collection of closed subintervals \( \mathcal{F} = \{J_\sigma : \sigma \in D\} \) of \( J \) has homogeneous Moran structure if:

1. \( J_\emptyset = J \);
2. \( \forall k \geq 0, \sigma \in D_k, J_{\sigma+1}, \ldots, J_{\sigma+n_k+1} \) are subintervals of \( J_\sigma \) and \( J_{\sigma+1} \cap J_{\sigma+2} = \emptyset \) for \( i \neq j \);
3. \( \forall k \geq 1, \forall \sigma \in D_k, 1 \leq j \leq n_k, c_k = \frac{\lambda(J_{\sigma+j})}{\lambda(J_\sigma)} \).

Suppose that \( \mathcal{F} \) is a collection of closed subintervals of \( J \) having homogeneous Moran structure. Let \( E(\mathcal{F}) = \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma \). We say \( E(\mathcal{F}) \) is a homogeneous Moran set determined by \( \mathcal{F} \), or it is a homogeneous Moran set determined by \( J \), \( \{n_k\} \), \( \{c_k\} \). We will need the following theorem of D. Feng, Z. Wen, and J. Wu from [17].

Theorem 2.7 (D. Feng, Z. Wen, and J. Wu). If \( S \) is a homogeneous Moran set determined by \( J \), \( \{n_k\} \), \( \{c_k\} \), then

\[
\dim_H(S) \geq \liminf_{k \to \infty} \frac{\log(n_1 n_2 \ldots n_k)}{-\log(c_1 c_2 \ldots c_k n_k+1)}.
\]
2.4. The construction. The construction given in this section will be related to the constructions given by the second author in [27, 28]. Suppose that we are given a computable and computably growing basic sequence $Q = \{q_n\}$, an explicit set of sparsely intersecting polynomials $P = \{p_n\}$, and a set of explicit linear families of upper and lower adfs $F$ defined by (1.2). Let
\[
\left\{ \{g_{n,p,m,r}\}_{m \in \mathbb{N}, 0 \leq r < m}\right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{ \{g_{n,p,m,r}\}_{m \in \mathbb{N}, 0 \leq r < m}\right\}_{n=1}^{\infty}
\]
be sequences of explicit linear families of continuous adfs where $\{g_{n,p,m,r}\}_{n=1}^{\infty}$ and $\{g_{n,p,m,r}\}_{n=1}^{\infty}$ converge pointwise to $f_{p,m,r}$ and $f_{p,m,r}$, respectively. We define
\[
\Delta_k = \min_{1 \leq j \leq k} \left\{ \min_{0 \leq l < k!} \left( \lambda \left( g_{k,p_1,k!}^{-(j/k)} \left( \frac{j}{k} \right) \right) \right) \right\}
\]
with $\lambda$ the Lebesgue measure. Note that
\[
\lambda \left( g_{k,p_1,k!}^{-(j/k)} \left( \frac{j}{k} \right) \right) = \sup \left\{ g_{k,p_1,k!}^{-(j/k)} \left( \frac{j}{k} \right) \right\} - \inf \left\{ g_{k,p_1,k!}^{-(j/k)} \left( \frac{j}{k} \right) \right\},
\]
and so is a computable real number since it is the difference of two computable real numbers. This means that $\Delta_k$ is a computable real number since we are taking finitely many maximums and minimums of computable real numbers. Define
\[
\epsilon_k = \min \left\{ \log(q_k)^\frac{1}{2}, \log(q_1 \ldots q_{k-1})^\frac{1}{2} \right\};
\]
\[
\nu_{j,1} = \min \left\{ t \in \mathbb{N} : \min \left\{ \log(q_k)^\frac{1}{2}, \log(q_1 \ldots q_{k-1})^\frac{1}{2} \right\} \geq \log(4) - \log(\Delta_j), \forall k \geq t \right\};
\]
\[
\nu_{j,2} = \min \left\{ t \in \mathbb{N} : \frac{\#(p_k(\mathbb{N}) \cap p_1(\mathbb{N}) \cap \{1, 2, \ldots, n\})}{\#(p_1(\mathbb{N}) \cap \{1, 2, \ldots, n\})} < \frac{1}{(j!)(j+1)^3}, \forall l < k \leq j, \forall n > t \right\}.
\]
We have that $\{\nu_{j,1}\}$ is a computable sequence as $\Delta_k$ is a computable real number and $Q$ is a computably growing basic sequence. We also have that $\{\nu_{j,2}\}$ is a computable sequence since $P$ is an explicit sparsely intersecting set of polynomials and by Theorem 2.4. Finally, set
\[
\nu_j = \max \{\nu_{j,1}, \nu_{j,2}\}
\]
We will define sequences of integers $\{l_j\}$ and $\{L_j\}$ inductively. Set
\[
l_1 = \max\{\nu_2 - 1, 1\};
\]
\[
\psi_j = \min \left\{ t \in \mathbb{N} : \frac{\#(p_k(\mathbb{N}) \cap p_1(\mathbb{N}) \cap \{L_{j-1} + j^ljt, L_{j-1} + j^ljt + 1, \ldots L_{j-1} + j^ljt + n\})}{\#(p_1(\mathbb{N}) \cap \{L_{j-1} + j^ljt, L_{j-1} + j^ljt + 1, \ldots L_{j-1} + j^ljt + n\})} < \frac{1}{(j+1)^{l+2}(j+1)^3} \right\}
\]
for all $n \geq (j-1)!t$, and $k, l < j + 1$;
\[
l_j = \max \left\{ \min \{ t \in \mathbb{N} : L_{j-1} + j^ljt \geq \nu_j - 1 \}, \psi_j, j^2 \right\};
\]
\[
L_j = \sum_{i=1}^{j} \nu_j l_j.
\]
Clearly, the sequence $\{\nu_j\}$ is computable since the sequences $\{\nu_{j,1}\}$ and $\{\nu_{j,2}\}$ are computable. The sequence $\{\psi_j\}$ is computable since $P$ is explicit. Thus the sequences $\{l_j\}$ and $\{L_j\}$ are also computable.
Let $U = \{(i, b, c, d) \in \mathbb{N}^4 : b \leq l_i, c \leq i, d \leq i!\}$. Define $\Xi_l : U \to p_l(\mathbb{N})$ by $\Xi_l(i, b, c, d) = p_l(L_i - 1 + bli + cl + d)$. It is easy to show that $\Xi_l$ is a bijection.

Put $P'_i = \{p_{\sigma(j)} : j = 1, \ldots, n\}$, where $\sigma$ is a permutation of $\{1, \ldots, n\}$ such that if $\sigma(i) < \sigma(j)$, then $p_j < p_i$. We can find such a $\sigma$ since $P$ is sparsely intersecting. Reorder $P_i$ so that it is equal to $P'_i$. Define $i(n) = \max\{i : n > p_i(L_i), \forall l \leq i\}$, $\rho(n) = \max\{j : n \in p_j(\mathbb{N}), p_j \in P_i\}$, and $(i(n), b(n), c(n), d(n)) = \Xi^{-1}_l(n)$. These functions are defined as $\Xi_l$ is a bijection. Define the sets

$$V_{i,n}^* = \begin{cases} [0, q_n) & \text{if } l < i \\ q_n \mathcal{F}_{i(n),l,i(n)}d(n)\left(\frac{c(n)}{n^{1/d(n)}}, \frac{c(n)+1}{n^{1/d(n)}}\right) & \text{if } i(n) \equiv 0 \text{ (mod 2)}, l \geq i(n) > 1 \\ q_n \mathcal{R}_{i(n),l,i(n)}d(n)\left(\frac{c(n)}{n^{1/d(n)}}, \frac{c(n)+1}{n^{1/d(n)}}\right) & \text{if } i(n) \equiv 1 \text{ (mod 2)}, l \geq i(n) > 1 \end{cases}$$

Thus

$$V_{i,n} = \left[\inf(V_{i,n}^*), \inf(V_{i,n}^*) + q_n\Delta_i(n) - q_n - 1\right] \cap \mathbb{Z}.$$  

Note that we can write $V_{\rho(n),n} = \{\alpha(n), \alpha(n) + 1, \ldots, \beta(n)\}$, where $\{\alpha(n)\}$ and $\{\beta(n)\}$ are sequences of integers. Moreover, $\alpha(n) = \left[\inf(V_{\rho(n),n}^*)\right]$ and $\beta(n) = \left[\inf(V_{\rho(n),n}^*) + q_n\Delta_i(n) - q_n - 1\right]$. We will discuss the computability of these sequences in the proof of Main Theorem 17.

**Set**

$$\Phi'_{Q,P,F} := \{x \in [0,1) : E_n \in V_{\rho(n),n}\}.$$  

**Lemma 2.8.** A sequence $\{x_n\}$ has a continuous adf $f$ if and only if the sequence $\{f(x_n)\}$ is u.d. mod 1.

**Proof.** Since $f$ is non-decreasing, $x_n \leq \gamma$ if and only if $f(x_n) \leq f(\gamma)$. Thus $\frac{A_n([0,\gamma],x_n)}{n} = \frac{A_n([0,f(\gamma)],f(x_n))}{n}$. But $\{x_n\}$ has continuous adf $f$, so

$$\lim_{n \to \infty} A_n([\gamma,\gamma], x_n) = f(\gamma^+ - f(\gamma^-) = 0.$$  

Thus

$$\lim_{n \to \infty} \frac{A_n([0,\gamma],x_n)}{n} = \lim_{n \to \infty} \frac{A_n([0,f(\gamma)],f(x_n))}{n} = \lim_{n \to \infty} \frac{A_n([0,f(\gamma)],f(x_n))}{n}$$

since $\lambda([\gamma,\gamma]) = 0$. Therefore

$$\lim_{n \to \infty} A_n([0,\gamma], x_n) - f(\gamma) = \lim_{n \to \infty} A_n([0,f(\gamma)], f(x_n)) - f(\gamma).$$

As $f$ is continuous, it must map $[0,1]$ onto $[0,1]$, so this second limit satisfies the definition for uniform distribution mod 1. These limits converge to 0 if and only if the other does, and we are done. \qed

**Definition 2.9.** Let $\omega = \{x_i\}$ be a sequence of real numbers. The **upper discrepancy** with respect to adf $f$ of $\omega$ is

$$\overline{D}'_n(\omega) := \max \left\{\sup_{\gamma} \left\{\frac{A_n([0,\gamma], x_i)}{n} - f(\gamma)\right\}, 0\right\}.$$  

The **lower discrepancy** of $\omega$ is

$$\underline{D}'_n(\omega) := \min \left\{\inf_{\gamma} \left\{\frac{A_n([0,\gamma], x_i)}{n} - f(\gamma)\right\}, 0\right\}.$$
Proof.

(1) The inequality \( \overline{D}_n^f(\omega) \leq \underline{D}_n^f(\omega) \) holds.

(2) If \( n \), a sequence of real numbers of length \( |\omega| \), then

\[
\overline{D}_n^f(\omega_1 \omega_2 \ldots \omega_i) \leq \frac{\sum_{i=1}^j |\omega| D_{\omega_i}^f(\omega_i)}{\sum_{i=1}^j |\omega_i|}.
\]

(3) If \( \lim_{n \to \infty} \overline{D}_n^f(\omega) = 0 \), then the upper adf \( \overline{f} \) of \( \omega \) is bounded above by \( f \).

(4) If \( \{f_{m,r}\} \) is a linear family of adfs, then for all \( d, m, \) and \( r \), we have that

\[
\overline{D}_n^{f_{m,r}}(\omega) \leq \max_{0 \leq i \leq m} \left\{ D_n^{f_{d,m,i+r}}(\omega) \right\} + \frac{md(d+1)}{n-md}.
\]

(5) If \( \overline{f} \) and \( f \) are the upper and lower adfs of \( \omega \), respectively, then \( \overline{D}_n^f(\omega) \leq \overline{D}_n^\omega(\omega) \).

(6) If \( \omega \) is non-decreasing, then \( \overline{D}_n^f(\omega) \leq \max \left\{ |f(z_i) - \frac{1}{n}|, |f(z_i) - \frac{i+1}{n}| \right\} \).

Proof.

(1) We have that

\[
\max \left\{ \sup_{\gamma} \left\{ \frac{A_n([0, \gamma), z_i]}{n} - f(\gamma) \right\}, 0 \right\} \leq \sup_{\gamma} \left\{ \frac{A_n([0, \gamma), z_i]}{n} - f(\gamma) \right\} \leq \sup_{\gamma} \left\{ \frac{A_n([0, \gamma), z_i]}{n} - f(\gamma) \right\} = 0,
\]

which implies that \( \overline{f}(\gamma) \leq f(\gamma) \) for all \( \gamma \).

(2) The proof is identical to the proof of Theorem 2.6 in Chapter 2 of [23].

(3) We have that

\[
\lim_{n \to \infty} \max_{\gamma} \left\{ \sup_{\gamma} \left\{ \frac{A_n([0, \gamma), \omega]}{n} - f(\gamma) \right\} \right\} = 0
\]

which implies that

\[
\lim_{n \to \infty} \sup_{\gamma} \left\{ \frac{A_n([0, \gamma), \omega]}{n} - f(\gamma) \right\} \leq 0 = \lim_{n \to \infty} \sup_{\gamma} \left\{ \frac{A_n([0, \gamma), \omega]}{n} - \overline{f}(\gamma) \right\}.
\]

But this implies that \( \overline{f}(\gamma) \leq f(\gamma) \) for all \( \gamma \).

(4) Note that

\[
d \sup_{\gamma} \frac{A_{n/m, i]}([0, \gamma), \{z_{m+i+r}\})}{\frac{n}{m}} - f_{m,r}(\gamma) \leq \sum_{j=0}^{d-1} A_{\frac{n}{m}}\left([0, \gamma), \left\{z_{d+m+i+j+r}\right\}\right) + 1 + d
\]

\[
= \sum_{j=0}^{d-1} \frac{A_{\frac{n}{m}}\left([0, \gamma), \left\{z_{d+m+i+j+r}\right\}\right) - \frac{d}{md}}{d -md} - f_{m,d,m+i+j+r}(\gamma) + \frac{md(d+1)}{n-md}
\]

\[
= \frac{md(d+1)}{n-md} + \sum_{i=0}^{d-1} \overline{D}_{n,m,i+j+r}(\omega) \leq \frac{md(d+1)}{n-md} + d \max_{0 \leq i < m} \overline{D}_{n,m,i+j+r}(\omega).
\]

Dividing by \( d \) yields the result.
(5) Since \( f \leq 7 \), we have that \( \sup \left\{ \frac{A_n([0,\gamma);z]}{n} - \gamma(\gamma) \right\} \leq \sup \left\{ \frac{A_n([0,\gamma);z]}{n} - f(\gamma) \right\} \).

(6) The proof is identical to the proof of Theorem 1.4 in Chapter 2 of [23].

\( \square \)

Define \( v(n) := \#V_{p(n),n} = \alpha(n) - \beta(n) + 1 \).

**Lemma 2.11.** For all natural number \( n \) we have \( v(n) > 4q_k^{1-\epsilon_k} - 2 \geq 2 \).

**Proof.** The interval \( V_{p(n),n} \) is of length \( q_n \Delta_i(n) - 1 \). Thus \( v(n) \geq q_n \Delta_i(n) - 2 \). But \( q_k > 4\Delta_k^{-1} \), so \( 4q_k^{1-\epsilon_k} < \Delta_k \). Thus \( v(n) > 4q_k^{1-\epsilon_k} - 2 \). But \( \epsilon_k \leq 1 \), so \( 4q_k^{1-\epsilon_k} - 2 \geq 4 - 2 = 2 \). \( \square \)

**Lemma 2.12.** Suppose that \( Q \) is a basic sequence that is infinite in limit, \( f : \mathbb{N} \to \mathbb{N} \) is an eventually increasing function, and both \( \mathcal{O}_{Q,f,m,r}(x) \) and \( \left\{ \left( E_{f(mn+r)} + 1 \right)/q_{f(mn+r)} \right\} \) have the same upper and lower adfs. Then \( \mathcal{O}_{Q,f,m,r}(x) \) has the same upper and lower adfs as \( \mathcal{O}_{Q,f,m,r}(x) \).

**Proof.** Suppose that \( g \) is the upper adf of the sequence \( \mathcal{O}_{Q,f,m,r}(x) \). Let \( Q_f = \{q_f(n)\}_{n=0}^{\infty} \). As \( f \) is eventually increasing, \( Q_f \) is a basic sequence that is infinite in limit. It is clear from the definition of \( T_{Q,n}(x) \) that

\[
\frac{E_{f(n)}}{q_f(n)} \leq T_{Q,f(n)}(x) \leq \frac{E_{f(n)} + 1}{q_n}.
\]

Hence

\[
A_n \left[ [0,\gamma), \left\{ \frac{E_{f(n)}}{q_f(n)} + 1 \right\} \right] - g(\gamma) \leq \frac{A_n(\gamma, \mathcal{O}_{Q,f}(x))}{n} - g(\gamma) \leq \frac{A_n \left[ [0,\gamma), \mathcal{O}_{Q,f}(x) \right]}{n} - g(\gamma).
\]

We also have that \( \frac{E_{f(n)} + 1}{q_f(n)} - \frac{E_{f(n)}}{q_f(n)} = \frac{1}{q_f(n)} \) which goes to 0 as \( n \) goes to \( \infty \). But \( \frac{E_{f(n)} + 1}{q_f(n)} \) and \( \mathcal{O}_{Q,f}(x) \) have the same upper adf by assumption. Thus,

\[
\limsup_{n \to \infty} A_n \left[ [0,\gamma), \mathcal{O}_{Q,f}(x) \right] - g(\gamma) = 0
\]

for all \( \gamma \), so \( \mathcal{O}_{Q,f}(x) \) has the same upper adf as \( \mathcal{O}_{Q,f}(x) \). The proof for the lower adf is identical.

We will use the following basic lemma.

**Lemma 2.13.** Let \( L \) be a real number and \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \) be two sequences of positive real numbers such that \( \sum_{n=1}^{\infty} b_n = \infty \) and \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \). Then

\[
\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} = L.
\]

We now prove the second part of Main Theorem 1.7. The proof that the set \( \Phi_{Q,P,F} \) has full Hausdorff dimension is done similarly.

**Proof of Main Theorem 1.7.** Let \( x \in \Phi_{Q,P,F}, \ p \in P \), and \( m \in \mathbb{N}, \ 0 \leq r < m \). The sequence \( \mathcal{O}_{Q,p,m,r}(x) \) can then be written as

\[
\mathcal{O}_{Q,p,m,r}(x) = Y_{1,0} \ldots Y_{1,l_1-1} Y_{2,0} Y_{2,1} \ldots Y_{2,l_2-1} \ldots
\]
where each $Y_{i,k}$ is a block of digits with length $\frac{d_i}{m}$ when $i \geq m$. When $m < i$, the length of $Y_{i,k}$ is less than $\frac{d_i}{m}$. We will look at the discrepancies of the blocks $Y_{i,k}$, and apply Theorem 2.10 to show the claim.

Put $X_j = Y_{j,0} \ldots Y_{j,l_j-1}$. By the definition of the upper discrepancy, we have that

$$\overline{D}_{Y_{i,k}}(Y_{i,k}) < \frac{1}{i},$$

unless $k \in \{0, l_i-1\}$, in which case the upper discrepancy is bounded by 1. We must also consider when our $X_j$’s intersect other polynomials in $\{p(n)\}$. For $L_{i-1} \leq n \leq L_i$, there are at most $d_{L_i}$ polynomials that could intersect $p$, and since max $d_{p_k, p_j, n} < \frac{1}{m}$ at stage $i$, we have that the number of terms that increase the discrepancy is at most $\frac{d_{L_i}}{i} \leq \frac{1}{i}$. Because $\overline{D}_{n,p,m,r}$ does not converge to $\overline{D}_{p,m,r}$ uniformly, we cannot measure the discrepancy of the sequence with respect to $\overline{D}_{p,m,r}$ directly. Instead, we use the fact of pointwise convergence and discrepancy with respect to $\overline{D}_{n,p,m,r}$.

Defining $k(n) = n - L_{i(n)}$, we can write $k(n) = a(n)i(n)i(n) + b(n)$, and

$$\limsup_{n \to \infty} \left( \frac{A_n([0, \gamma), O'_Q,p,m,r(x))}{n} - \overline{D}_{p,m,r}(\gamma) \right)$$

$$= \limsup_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} A_{i,j}([0, \gamma), X_j) + A_k([0, \gamma), X_i(n)) - \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + k(n)\overline{D}_{p,m,r}(\gamma)}{\frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + k(n)\overline{D}_{p,m,r}(\gamma)}$$

$$+ \limsup_{n \to \infty} \left( \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + k(n)\overline{D}_{p,m,r}(\gamma) \right)$$

$$\leq \limsup_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + k(n)\overline{D}_{p,m,r}(\gamma)$$

For this first sum $S_1$, we find that

$$\limsup_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + k(n)\overline{D}_{p,m,r}(\gamma)$$

$$= \limsup_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + \frac{\sum_{j=1}^{i(n)-1} \frac{d_{j_i}}{m} \overline{D}_{j_{p,m,r}}(\gamma)}{j_i} + k(n)\overline{D}_{p,m,r}(\gamma)$$

$$= S_1 + S_2 + S_3.$$
\[ \limsup_{n \to \infty} \frac{m(i(n) - 1)![(i(n) - 1) + 1]}{i(n)!} \leq \limsup_{n \to \infty} \frac{m}{i(n)} + \frac{1}{i(n)} = 0. \]

For the second term \( S_2 \) we have that
\[
\limsup_{n \to \infty} \frac{\sum_{j=1}^{i(n)-1} j^2 j^j}{m} \left[ \frac{j^j}{m} \right] \leq \limsup_{n \to \infty} \frac{m}{i(n)} + \frac{1}{i(n)} = 0.
\]

For the final term \( S_3 \) we get
\[
\limsup_{n \to \infty} \sum_{j=1}^{i(n)-1} \frac{j^j \gamma_{ij} \text{p.m.r}(\gamma)}{m} \leq \limsup_{n \to \infty} \frac{m}{i(n)} + \frac{1}{i(n)} = 0.
\]

Thus the sequence \( \mathcal{O}_{Q,p,m,r}(x) \) has adf bounded above by \( \mathcal{T}_{p,m,r} \).

This calculation only shows that the upper adf of the sequence \( \mathcal{O}_{Q,p,m,r}(x) \) is bounded above by \( \mathcal{T}_{k,m,r} \). To show that this is the actual upper adf, we must find a sequence \( a_n \) along which
\[
\lim_{n \to \infty} \frac{A_{a_n}([0, \gamma), \mathcal{O}_{Q,p,m,r}(x))}{a_n} - \mathcal{T}_{p,m,r}(\gamma) = 0.
\]

Let \( a_n = L_{2n} \). Note that for the second term in the previous sum, we have that
\[
\lim_{n \to \infty} \left( \mathcal{G}_{i(a_n),p,m,r}(\gamma) - \mathcal{T}_{p,m,r}(\gamma) \right) = 0. \]

So we need only check that the first term \( S_1 \) goes to 0, as \( a(n) \) and \( b(n) \) must be 0 for all \( n \). Thus by Lemma 2.13
\[
\lim_{n \to \infty} \left( \frac{A_{a_n}([0, \gamma), \mathcal{O}_{Q,p,m,r}(x))}{a_n} - \mathcal{T}_{p,m,r}(\gamma) \right) = 0.
\]

This implies that \( \mathcal{O}_{Q,p,m,r}(x) \) has upper adf \( \mathcal{T}_{p,m,r} \). Note that throughout the proof we only used that \( \alpha(n) \leq E_n \leq \beta(n) + 1 \), so \( \{ E_{(m+n)+1} \} \) has the same upper adf
as $O'_{Q,m,r}(x)$. Therefore by Lemma 2.12 the upper adf of $O_{Q,m,r}(x)$ is $f'_{m,r}$. The proof that $O'_{Q,m,r}(x)$ has lower adf $f'_{m,r}$ follows similarly.

To compute the Hausdorff dimension of $\Phi'_{Q,P,F}$, we note that $\Phi'_{Q,P,F}$ is a homogeneous Moran set with $J = [0, 1]$, $n_k = v(k) > 4q_k^{1-\epsilon_k} - 2 \geq 2$ by Lemma 2.11 and $c_k = q_k^{-1}$. So

$$\dim_H (\Phi'_{Q,P,F}) \geq \liminf_{n \to \infty} \frac{\log(q_1^{1-\epsilon_1} \cdots q_n^{1-\epsilon_n})}{-\log(1/q_1 \cdots 1/q_n^{1-\epsilon_n})} = \liminf_{n \to \infty} \frac{\log(q_1 \cdots q_n)}{\log(q_1 \cdots q_n) - \epsilon_n \log(q_n)}$$

$$= \liminf_{n \to \infty} \frac{1}{1 - \epsilon_n \log(q_n) / \log(q_1 \cdots q_n)} = 1.$$

We will now prove that $\{\alpha(n)\}$ and $\{\beta(n)\}$ are computable sequences. Recall that $\alpha(n) = \left[\inf(V'_n)\right]$ and $\beta(n) = \left[\inf(V'_n) + q_n \Delta_i(n) - q_n - 1\right]$. To see these sequences are computable, note that $\{c(n)\}$ and $\{i(n)\}$ are computable sequences as $\{L_n\}$ and $\{p_i(n)\}$ are computable sequences for any polynomial $p_i$. Thus $\{c(n)/i(n)\}$ is a computable sequence. By the explicitness of $\left\{\overline{g}_{q_i(n), l, i(n), l, i(n)}\right\}$ for each $n$ and uniform computability of the discontinuities of $f_{p,m,r}$, we have that $\left\{\inf \overline{g}_{q_i(n), l, i(n), l, i(n)} \left(\frac{c(n)}{i(n)}\right)\right\}_{n=1}^{\infty}$ is a computable sequence. Thus we have that $\{\alpha(n)\}$ is the ceiling of the product of two computable sequences as $Q$ is a computably growing basic sequence. Hence $\{\alpha(n)\}$ is a computable sequence. Similarly, it can be shown that $\{\beta(n)\}$ is a computable sequence. $\square$

3. Applications

3.1. Accumulation points along arithmetic progressions. In the same vein as [41], we would like to prove results about the accumulation points of $O'_{Q}(x)$ and prove Theorem 1.22. It is difficult to say anything about accumulation points along polynomially indexed subsequences of $O'_{Q}(x)$. Inequivalent polynomials can intersect infinitely often, so the accumulation points of $O'_{Q,f,m,r}(x)$ could be difficult to find. However, if we only sample along arithmetic progressions, a classification of the accumulation points of $O'_{Q,m,r}(x)$ can be given.

For notational convenience, we define

**Definition 3.1.** For $f : [0, 1] \to [0, 1]$, set $I(f) := \{x \in [0, 1] : f \text{ is increasing at } x\}$

After establishing lemmas, we will prove the following theorem.

**Theorem 3.2.** Given a basic sequence $Q$ that is infinite in limit and a linear family of upper and lower adfs $\{f_{m,r}\}$ and $\{f_{m,r}\}$, the set of real numbers $x$ such that $O_{Q,m,r}(x)$ has upper and lower adfs $f_{m,r}$ and $f_{m,r}$ and such that $O'_{Q,m,r}(x)$ has accumulation points equal to $I(f_{m,r}) \cup I(f_{m,r})$ has Hausdorff dimension 1.

**Lemma 3.3.** If $x \in I(f_{m,r}) \cup I(f_{m,r})$, then $x$ is an accumulation point of $O'_{Q,m,r}(x)$.

**Proof.** Suppose that $x \in I(f_{m,r}) \cup I(f_{m,r})$. If $x \in I(f_{m,r})$, then either $f(x - \epsilon) > f(x)$ or $f(x) < f(x + \epsilon)$ for all $\epsilon > 0$. The proof for both of these cases is identical,
so suppose that \( \overline{f}(x - \epsilon) < \overline{f}(x) \). Then
\[
\limsup_{n \to \infty} \frac{A_n([x - \epsilon, x), x_i)}{n} = \overline{f}(x) - \overline{f}(x - \epsilon) > 0.
\]
This implies that there are infinitely many \( x_i \in [x - \epsilon, x) \) for all \( \epsilon > 0 \). Let \( \epsilon_n = \frac{1}{n} \). Construct a sequence \( x_n \) by choosing
\( x_n \in [\overline{f}(x - \epsilon_n), \overline{f}(x)]. \) Thus \( x \) is an accumulation point of \( \mathcal{O}_{Q,m,r}(x) \). The case where \( x \in I(\underline{f}_{m,r}) \) is identical, replacing the lim sup with lim inf. \( \square \)

**Lemma 3.4.** If \( f : [0,1] \to [0,1] \) is non-decreasing and \( x = \inf\{f^{-1}(y)\} \), then \( x \in I(f) \).

**Proof.** Since \( f \) is non-decreasing, if \( z < x \), then \( f(z) \leq f(x) \). Set \( x = \inf\{f^{-1}(y)\} \).

Then if \( f(z) = f(x) \), we have that \( f(z) = y \) and \( z \in f^{-1}(y) \). This contradicts \( x \) being the infimum of \( f^{-1}(y) \). Thus \( f(z) < f(x) \). So we have that for all \( z < x \), \( f(z) < f(x) \), and \( x \in I(f) \). \( \square \)

**Proof of Theorem 3.2.** Let \( x = E_0 E_1 E_2 \cdots \) w.r.t. \( Q \). To see that the set of accumulation points of \( \mathcal{O}_{Q,m,r}(x) \) is exactly \( I(\overline{f}_{m,r}) \cup I(\underline{f}_{m,r}) \), note that by Lemma 3.3 the set of accumulation points of \( \mathcal{O}_{Q,m,r}(x) \) contains \( I(\overline{f}_{m,r}) \cup I(\underline{f}_{m,r}) \).

For the converse direction, note that by the construction, if \( i(n) \) is even, then
\[
\frac{E_n}{q_n} \in \left[ \inf_{i(n) \equiv 0 \mod (i(n)!)} \frac{c(i(n))}{i(n)!}, \inf_{i(n) \equiv 1 \mod (i(n)!)} \frac{c(i(n))}{i(n)!} + \Delta_{i(n)} - \frac{1}{q_n} \right].
\]
If \( i(n) \) is odd, then
\[
\frac{E_n}{q_n} \in \left[ \inf_{i(n) \equiv 0 \mod (i(n)!)} \frac{c(i(n))}{i(n)!} + \Delta_{i(n)} - \frac{1}{q_n}, \inf_{i(n) \equiv 1 \mod (i(n)!)} \frac{c(i(n))}{i(n)!} \right].
\]
The term \( \Delta_{i(n)} - \frac{1}{q_n} \) goes to 0. Define a sequence \( \{\overline{y}_n\} \) with
\[
\overline{y}_n = \inf_{i(n) \equiv 0 \mod (i(n)!)} \frac{c(i(n))}{i(n)!} + \Delta_{i(n)} - \frac{1}{q_n}.
\]
Define the sequence \( \{y_n\} \) similarly. Then
\[
\liminf_{n \to \infty} \frac{E_{m,r} + \Delta_{i(n)}}{q_{m,r} - \overline{y}_n} = 0.
\]
But \( \overline{y}_n \) is not contained in \( I(\overline{f}_{m,r}) \). Since \( \overline{f}_{m,r} \) is a linear family of adfs, we have that this set is a subset of \( I(\overline{f}_{m,r}) \), so \( \overline{y}_n \) is not in \( I(\overline{f}_{m,r}) \). As \( I(\overline{f}_{m,r}) \) is closed, all accumulation points of \( \overline{y}_n \) must lie in \( I(\overline{f}_{m,r}) \). The same statements for \( \underline{f}_{m,r} \) are true. Furthermore, we have that by construction \( \frac{E_{m,r} + \Delta_{i(n)}}{q_{m,r}} \) is either arbitrarily close to the sequence \( \{\overline{y}_n\} \) or \( \{y_n\} \), so the set of accumulation points of \( \mathcal{O}_{Q,m,r}(x) \) is contained in the set of accumulation points of \( \{\overline{y}_n\} \cup \{y_n\} \). Thus it is a subset of \( I(\overline{f}_{m,r}) \cup I(\underline{f}_{m,r}) \). \( \square \)

**Definition 3.5.** Let \( (X, T) \) be a topological space and \( (X, \mathcal{B}(X), \mu) \) be the measure space of \( X \) with the Borel \( \sigma \)-algebra. Then
\[
\text{supp}(\mu) := \{x \in X : \text{for every neighborhood } N \text{ of } x, \mu(N) > 0\}.
\]
Lemma 3.6. For every closed set $D \subseteq [0, 1]$, there is an adf $f_D$ so that $I(f_D) = D$.

Proof. First we construct a measure $\mu_D$ so that supp($\mu_D$) = $D$. First consider the Borel $\sigma$–algebra of $D$ with the subspace topology. Define

\[
\mu_D^*(S) = \sup \{b - a : (a, b) \cap D = (a, b) \cap S\}.
\]

Extend this measure to the Borel $\sigma$–algebra on $[0,1]$ by $\mu_D(S) = \mu_D^*(S \cap D)$. Then supp($\mu_D$) = $D$, $\mu_D(D) = 1$. If $S \cap D = \emptyset$, then $\mu_D(S) = 0$. Let $f_D(0) = 0$ and $f_D(x) = \mu_D([0, x])$ if $x > 0$. Then $f_D(0) = 0$, $f_D(1) = \mu_D([0,1] \cap D) = 1,$ and $f_D$ is non-decreasing since $\mu_D([0,x]) \leq \mu_D([0,y])$ if $x \leq y$. If $S$ is a set that does not intersect $D$, then $\mu_D(S) = 0$ and $f_D(x) = f_D(y)$ for all $x, y \in S$. Thus, we see that $f_D(x) - f_D(y) \geq y - x > 0$.

Proof of Theorem 1.13. We will define adfs $f_{q,s}$ such that if $q \leq k$ we will have $I(f_{q,s}) = D_{q,s}$ and if $q > k$ we will have that $I(f_{q,s}) \subseteq D_{m,r}$ for some $m \leq k$. Applying Theorem 3.2 will yield the desired result.

For $q \leq k$, let $A_{q,s} = D_{q,s}$. For $q > k$, let

\[
A_{q,r} = \bigcap_{d,q} D_{d,r} \quad \text{(mod $d$)} \cap \bigcap_{d,q} D_{d,j}.
\]

Define $f_{q,s} = f_{A_{q,s}}$, where $f_D$ is as defined in Lemma 3.6. Then by Theorem 3.2 the sequence $\{f_{q,m,r}(x)\}$ has accumulation points $D_{m,r}$ for $m \leq k$.

3.2. Proof of Theorem 1.13 and Theorem 1.17

Proof of Theorem 1.13. The first assertion follows directly from Lemma 2.12. For the second, the proof that the set of $Q$-distribution normal numbers is meagre follows identically to the case of the $b$-ary expansion. The fact that the set of $Q$-distribution normal numbers and the set of numbers $x$ where $\mathcal{O}_{Q,m,r}(x)$ is u.d. mod 1 have full measure for all $m > 1$ follows by a well known result of H. Weyl. Thus their difference set has measure zero.

Let $d(n) = \frac{\text{lcm}(1,2,\ldots,n)}{\text{lcm}(1,2,\ldots,n-1)}$, $D(n) = \prod_{i=1}^n d(i)$, and $\Gamma(n, r) = \{(r_1, r_2, \ldots, r_{n-1}) \in \mathbb{N}^{n-1} : 3x \in \mathbb{N} \text{ where } x \equiv r_t \text{ (mod } t) \text{ } \forall t \in [1, n] \text{ and } x \equiv r \text{ (mod } n)\}$. Set $S_{1,0} = [0, 1]$. For all $n$ and $0 \leq r \leq n - 1$, set

\[
S_{n,r} = \bigcup_{r \in \Gamma(n, r)} \left( \frac{1}{d(n)} \prod_{i=1}^{n-1} S_i r_i \right) + \frac{r \text{ mod } d(n)}{D(n)}.
\]

For $r \neq r'$, we have that $S_{n,r} \cap S_{n,r'} = \emptyset$ and $\lambda(S_{n,r}) = \lambda(S_{n,r'})$. Furthermore, for all $d$, we have that $S_{m,r} = \bigcup_{i=0}^{d} S_{nd,mi+r}$. Using notation from Lemma 3.6 define $f_{m,r} = f_{S_{m,r}}$. Thus, we see that $\{f_{m,r}\}$ is a linear family of adfs. Moreover, $f_{n,r}(x) = x$ for all $x$ only if $n = 1$ and $r = 0$. Thus the set of real numbers $x$ where $\mathcal{O}_{Q,m,r}(x)$ has the adf $f_{m,r}$ for all $m$ and $r$ is a subset of the set of numbers that are $Q$-distribution normal but AP $Q$-distribution abnormals and has full Hausdorff dimension by Main Theorem 1.17.

Proof of Theorem 1.17. Let $P$ be a set of sparsely intersecting polynomials and let $f_{p,m,r}(x) = x$ and $\tilde{f}_{p,m,r}(x) = x^2$ for every $x \in [0, 1]$ and $p \in P$. Then $\Phi_{Q,P,F}$ has full Hausdorff dimension. Moreover, for every $x \in \Phi_{Q,P,F}$, the upper and lower adfs of $\mathcal{O}_{Q,p,m,r}(x)$ are not equal. Thus $\mathcal{O}_{Q,p,m,r}(x)$, and in particular $\mathcal{O}_{Q}(x)$, does

\[\text{UNEXPECTED DISTRIBUTION PHENOMENON 21} \]


not have an adf for all $x$ on a set of full Hausdorff dimension. The set $\Phi_{Q,P,F}$ is of zero measure since for almost every $x$ the sequence $\mathcal{O}_Q(x)$ is u.d. mod 1. \qed

\section*{Appendix}

I. Niven and H. S. Zuckerman wrote in \[29\]:

Let $R$ be a real number with fractional part $x_1x_2x_3\cdots$ when written to scale $r$. Let $N(b, n)$ denote the number of occurrences of the digit $b$ in the first $n$ places. The number $R$ is said to be \textbf{simply normal} to scale $r$ if $\lim_{n\to\infty} \frac{N(b, n)}{n} = \frac{1}{r^v}$ for each of the $r$ possible values of $b$; $R$ is said to be \textbf{normal} to scale $r$ if all the numbers $R, rR, r^2R, \cdots$ are simply normal to all the scales $r, r^2, r^3, \cdots$. These definitions, for $r = 10$, we introduced by Émile Borel \[7\], who stated (p. 261) that “la propriété caractéristique” of a normal number is the following: that for any sequence $B$ whatsoever of $v$ specified digits, we have

\begin{equation}
\lim_{n\to\infty} \frac{N(B, n)}{n} = \frac{1}{r^v},
\end{equation}

where $N(B, n)$ stands for the number of occurrences of the sequence $B$ in the first $n$ decimal places ... If the number $R$ has the property (1) then any sequence of digits $B = b_1b_2\cdots b_v$ appears with the appropriate frequency, but will the frequencies all be the same for $i = 1, 2, \cdots, v$ if we count only those occurrences of $B$ such that $b_1$ is an $i, i+v, i+2v, \cdots - th$ digit? It is the purpose of this note to show that this is so, and thus to prove the equivalence of property (1) and the definition of normal number.

It is not difficult to see how the equivalent definition of normality introduced in Theorem 1.12 may be confused with the notion discussed in Theorem 1.12.

\section*{References}

1. S. Albeverio, M. Pratsiovytyi, and G. Torbin, Topological and fractal properties of real numbers which are not normal, Bull. Sci. Math. 129 (2005), no. 8, 615–630.
2. V. Becher, S. Figueira, and R. Picchi, Turing’s unpublished algorithm for normal numbers, Theoret. Comput. Sci. 377 (2007), no. 1–3, 126–138.
3. V. Becher, P. A. Heiber, and T. A. Slaman, A polynomial-time algorithm for computing absolutely normal numbers, Inform. and Comput. 232 (2013), 1–9.
4. A. S. Besicovitch, On the sum of digits of real numbers represented in the dyadic system, Math. Ann. 110 (1935), no. 1, 321–330.
5. Y. Bilu and R. Tichy, The diophantine equation $f(x) = g(y)$, Acta Arithmetica 45 (2000), 261–288.
6. Jens Blanck, Vasco Brattka, and Peter Hertling (eds.), Computability and complexity in analysis, 4th international workshop, cca 2000, swansea, uk, september 17-19, 2000, selected papers, Lecture Notes in Computer Science, vol. 2064, Springer, 2001.
7. E. Borel, Les probabilités dénombrables et leurs applications arithmétiques, Rend. Circ. Mat. Palermo 27 (1909), 247–271.
8. Y. Bugeaud, Distribution modulo one and Diophantine approximation, Cambridge University Press, Cambridge, 2012.
9. G. Cantor, Über die einfachen Zahlensysteme, Zeitschrift für Math. und Physik 14 (1869), 121–128.
10. D. G. Champernowne, The construction of decimals normal in the scale of ten, Journal of the London Mathematical Society 8 (1933), 254–260.
11. C. M. Colebrook, *The Hausdorff dimension of certain sets of nonnormal numbers*, Michigan Math. J. 17 (1970), 103–116.
12. A. H. Copeland and P. Erdős, *Note on normal numbers*, Bulletin of the American Mathematical Society 52 (1946), 857–860.
13. H. Davenport and P. Erdős, *Note on normal decimals*, Can. J. Math. 4 (1952), 58–63.
14. H. Eggleston, *The fractional dimension of a set defined by decimal properties*, Quart. J. Math., Oxford Ser. 20 (1949), 31–36.
15. P. Erdős and A. Rényi, *On Cantor’s series with convergent $\sum 1/q_n$*, Annales Universitatis L. Eötvös de Budapest, Sect. Math. (1959), 93–109.
16. P. Erdős and A. Rényi, *Some further statistical properties of the digits in Cantor’s series*, Acta Math. Acad. Sci. Hungar 10 (1959), 21–29.
17. D. Feng, Z. Wen, and J. Wu, *Some dimensional results for homogeneous Moran sets*, Sci. China Ser. A 40 (1997), no. 5, 475–482.
18. H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1 (1967), 1–49.
19. J. Galambos, *Representations of real numbers by infinite series*, Lecture Notes in Math., vol. 502, Springer-Verlag, Berlin, Hiedelberg, New York, 1976.
20. _____, *Uniformly distributed sequences mod 1 and Cantor’s series representation*, Czechoslovak Mathematical Journal 26 (1976), 636–641.
21. Vandehey J., *The normality of digits in almost constant additive functions*, Monatsh. Math. 171 (2013), no. 3–4, 481–497.
22. Y. Kifer, *Fractal dimensions and random transformations*, Trans. Amer. Math. Soc. 5 (1996), 2003–2038.
23. L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Dover, Mineola, NY, 2006.
24. B. Li and B. Mance, *Number theoretic applications of a class of Cantor series fractal functions part II*, arXiv:1310.2379.
25. M. Madritsch and Tichy R., *Construction of normal numbers via generalized prime power sequences*, J. Integer Seq. 16 (2013), no. 2, 13.2.12.
26. B. Mance, *Number theoretic applications of a class of Cantor series fractal functions part I*, arXiv:1310.2377.
27. _____, *On the Hausdorff dimension of countable intersections of certain sets of normal numbers*, arXiv:1302.7064.
28. _____, *Cantor series constructions of sets of normal numbers*, Acta Arith. 156 (2012), 223–245.
29. I. Niven and H. S. Zuckerman, *On the definition of normal numbers*, Pacific J. Math. 1 (1951), 103–109.
30. L. Olsen, *Applications of multifractal divergence points to sets of numbers defined through their N-adic expansion*, Math. Proc. Cambridge Philos. Soc. 136 (2004), no. 1, 139–165.
31. J. Pöyri, *Calculs de dimensions de Hausdorff*, Duke Math. J. 44 (1977), no. 3, 591–601.
32. S. S. Pillai, *On normal numbers*, Proc. Indian Acad. Sci. 12 (1940), 179–184.
33. A. Rényi, *On a new axiomatic theory of probability*, Acta Math. Acad. Sci. Hungar. 6 (1955), 329–332.
34. _____, *On the distribution of the digits in Cantor’s series*, Mat. Lapok 7 (1956), 77–100.
35. _____, *Probabilistic methods in number theory*, Shuxue Jinzhan 4 (1958), 465–510.
36. M. W. Sierpiński, *Démonstration élémentaire du théorème de M. Borel sur les nombres absolument normaux et détermination effective d’un tel nombre*, Bull. Soc. Math. France 45 (1917), 125–153.
37. S. Tengely, *Effective methods for Diophantine equations*, Ph.D. thesis, Thomas Stieltjes Institute for Mathematics, 2005.
38. P. Turán, *On the distribution of “digits” in Cantor systems*, Mat. Lapok 7 (1956), 71–76.
39. A. M. Turing, *Collected Works of A. M. Turing*, North-Holland Publishing Co., Amsterdam, 1992.
40. T. Šalát, *Zu einigen Fragen der Gleichverteilung (mod 1)*, Czech. Math. J. 18 (93) (1968), 476–488.
41. Yi Wang, Zhixiong Wen, and Lifeng Xi, *Some fractals associated with Cantor expansions*, J. Math. Anal. Appl. 354 (2009), no. 2, 445–450.
42. K. Weihrauch, *A Simple Introduction to Computable Analysis*, Electronic Colloquium on Computational Complexity.
43. Y. Xiong, Besicovitch-eggleston subsets in the Cantor expansion, J. Math. (Wuhan) 30 (2010), 827–831.

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