ON THE CLASSIFICATION OF SCALAR NON-POLYNOMIAL EVOLUTION EQUATIONS: QUASILINEARITY

Ayşe Hüneyra Bilge

Department of Mathematics, Istanbul Technical University, Maslak, Istanbul, Turkey
TUBITAK, Feza Gürsey Institute, Çengelköy, Istanbul, Turkey

Abstract

We prove that, for $m \geq 7$, scalar evolution equations of the form $u_t = F(x, t, u, \ldots, u_m)$ which admit a nontrivial conserved density of order $m + 1$ are linear in $u_m$. The existence of such conserved densities is a necessary condition for integrability in the sense of admitting a formal symmetry, hence integrable scalar evolution equations of order $m \geq 7$ are quasilinear.

1. Introduction

It has been shown that scale invariant polynomial of evolution equations of order greater than seven are symmetries of third and fifth order equations [1] and similar results are obtained in the case where negative powers are involved [2]. The problem of classification of arbitrary evolution equations is thus reduced to proving that such equations have desired polynomiality and scaling properties. As a first step in this direction, we study scalar evolution equation of general type and show that integrable equations of the form

$$u_t = F(x, t, u, \ldots, u_m)$$ (1.1)

where $F$ has partial derivatives of all orders, is linear in $u_m$ for $m \geq 7$. The integrability criteria that we are using is the one given in [3], namely the existence of a “formal symmetry” $R$, which is a formal series in inverse powers of $D = d/dx$, satisfying the operator equation

$$R_t + [R, F_*] = 0,$$ (1.2)

where $F_*$ is the Frechet derivative of $F$. The solvability of the coefficients of $R$ in the class of local functions requires that certain quantities denoted as $\rho^{(i)}$ be conserved densities. These conserved density conditions give overdetermined systems of partial differential equations for $F$ and lead to a classification.
For \( m = 3 \), it is known that there are essentially nonlinear equations that are candidates for being integrable [3] and their study require special techniques as explored in [4]. Our proof based on the existence of the conserved density \( \rho^{(1)} \) is not applicable to third order equations, because for \( m = 3 \), the expression of \( \rho^{(1)} \) given by (2.11) is no longer valid. For \( m \geq 7 \) the requirement that \( \rho_1 \) be a conserved density leads to two equations involving \( \partial F/\partial u_m \) which are are compatible only when \( F \) is linear in \( u_m \). For \( m = 5 \) there are three such equations given by (4.16a-c), but these are compatible and we cannot exclude the possibility of the existence of essentially nonlinear equations at the fifth order.

In Section 2, we give a concise overview of the formal symmetry method and present the expression of the conserved densities \( \rho^{(i)} \), \( i = 1, 2, 3 \). In the proof of our quasilinearity result, we actually do not need the explicit expression of these conserved densities, because the only information we use is the existence of a conserved density of order \( m + 1 \) which is quadratic in \( u_{m+1} \). The computation of the top order terms in derivatives are given in Section 3. Section 4 is devoted to general results and to the discussion of third and fifth order equations. The main result is presented in Section 5.

2. Notation and terminology, formal symmetries, conserved densities

Let \( u = u(x, t) \). A function \( \varphi \) of \( x \), \( t \), \( u \) and the derivatives of \( u \) up to a fixed but finite order will be called a “differential function” [5] and denoted by \( \varphi[u] \). We shall assume that \( \varphi \) has partial derivatives of all orders. For notational convenience, we shall denote indices by subscripts or superscripts in parentheses such as in \( \alpha^{(i)} \) or \( \rho^{(i)} \) and reserve subscripts without parentheses for partial derivatives, i.e.,

\[
\begin{align*}
    u &= u(x, t), \\
    u_k &= \frac{\partial^k u}{\partial x^k}, \\
    u_x &= \frac{\partial u}{\partial x}, \\
    u_t &= \frac{\partial u}{\partial t}, \\
    \varphi &= \varphi(x, t, u, u_1, \ldots, u_n), \\
    \varphi_k &= \frac{\partial \varphi}{\partial u_k}, \\
    \varphi_x &= \frac{\partial \varphi}{\partial x}, \\
    \varphi_t &= \frac{\partial \varphi}{\partial t}.
\end{align*}
\]

(2.1)

In Section 5 only, we shall use the notation \( \rho'' = \rho_{m+1, m+1} \) in order to simplify the presentation.

If \( \varphi \) is a differential function, the total derivative with respect to \( x \) is denoted by \( D\varphi \) and it is given by

\[
D\varphi = \sum_{i=0}^{n} \varphi_i \ u_{i+1} + \varphi_x.
\]

(2.2)

Higher derivatives can be computed by applying the binomial formula as

\[
D^k \varphi = \sum_{i=0}^{n} \sum_{j=0}^{k-1} \binom{k-1}{j} D^j \varphi_i \ u_{i+k-j} + D^{k-1} \varphi_x.
\]

(2.3)

Note that even if \( \varphi \) has an arbitrary functional form, \( D^k \varphi \) is polynomial in \( u_{n+i} \) for \( i \geq 1 \). We shall denote generic functions that depend on at most \( u_n \) by \( O(n) \). More precisely,

\[
\varphi = O(u_n) \quad \text{if and only if} \quad \partial \varphi/\partial u_{n+k} = 0 \quad \text{for} \quad k \geq 1
\]

(2.4)
Clearly, if $\varphi = O(u_n)$, then $D\varphi$ is linear in $u_{n+1}$ and $D^k\varphi$ is polynomial in $u_{n+i}$ for $i \geq 1$. In certain places we shall need to know not only the order of a function but whether it is a polynomial or not. When this distinction is important, we shall use the notation $\varphi = P(u_n)$, i.e.,

$$\varphi = P(u_n) \quad \text{if and only if} \quad \varphi = O(u_n) \quad \text{and} \quad \partial^k \varphi / \partial u_n^k = 0 \quad \text{for some } k. \quad (2.5)$$

For $\varphi$ and $u$ as above, the total derivative with respect to $t$ is denoted by $D_t \varphi$ and is given by

$$D_t \varphi = \sum_{i=0}^{n} \varphi_i D^i F + \varphi_t. \quad (2.6)$$

Equalities up to total derivatives with respect to $x$ will be denoted by $\cong$, i.e.,

$$\varphi \cong \psi \quad \text{if and only if} \quad \varphi = \psi + D\eta, \quad (2.7)$$

for some differential function $\eta$.

A symmetry $\sigma$ of $u_t = F[u]$ is a differential function which satisfies the linearized equation $\sigma_t = F_* \sigma$ where $F_*$ is the Frechet derivative of $F$ while a conserved density $\rho$ is a differential function satisfying $\rho_t = D\eta$ for some differential function $\eta$. The order of a symmetry or of a conserved density is the order of the highest derivative of $u$ in $\sigma$ or $\rho$. The recursion operator $R$ is defined to be a linear operator that sends symmetries to symmetries, i.e. $R\sigma$ is a symmetry whenever $\sigma$ is a symmetry [5] and satisfies Eq.(1.2). A formal series which satisfies Eq.(1.2) up to a finite order is called a formal symmetry.

In $1 + 1$ dimensions the integral terms in the recursion operator can be expanded in inverse powers of $D$ and one can look for a formal series satisfying Eq.(1.2). It is known that whenever $R$ is a solution, $R^{k/n}$ is also a solution of Eq.(1.2), hence the order of $R$ is irrelevant as long as we deal with recursion operators expressed as a formal series in $D^{-1}$.

It is known that the quantities

$$\rho^{(-1)} = F_{-1/m}, \quad \rho^{(0)} = F_{m-1}/F_m, \quad (2.8)$$

where

$$F_m = \frac{\partial F}{\partial u_m}, \quad F_{m-1} = \frac{\partial F}{u_{m-1}} \quad (2.9)$$

are conserved densities for equations of any order [3].

For the computation of the conserved densities for arbitrary $m$, one can start with a formal series for $R$, substitute in Eq.(1.2) and solve the operator equation recursively, obtaining at each step linear first order differential equations for the coefficients of $R$. The solvability of these equations in the class of local functions gives the conserved density conditions, $\rho^{(i)} = D\eta^{(i)}$, including the ones above.
Alternatively, one can use the fact that the coefficient of $D^{-1}$ in the commutator of any two formal series is a total derivative [3]. If we start with a formal series $R$ of order 1 satisfying (1.2), the coefficient of $D^{-1}$ will be the conserved density $\rho^{(1)}$. By the remark above, $R^k$ also satisfies (1.2), hence the coefficient of $D^{-1}$ in $R^k$ will also be a conserved density that we denote by $\rho^{(k)}$. For $k = m$ the top $m - 2$ terms in $R^m$ should coincide with $F_*$, and the coefficients of $R$ are expressed in terms of the partial derivatives of $F$ for each $m$.

For third order equations, the expression of $\rho^{(1)}$ involves $\eta^{(-1)}$ defined by $D_i \rho^{(-1)} = D \eta^{(-1)}$. But for equations of order 5 or higher, $\rho^{(1)}$ and $\rho^{(2)}$ are independent of $\eta^{(-1)}$ and $\eta^{(0)}$. Similarly for 5th order equations, $\rho^{(3)}$ depends on $\eta^{(1)}$, but for $m \geq 7$, $\rho^{(3)}$ is independent of $\eta^{(i)}$ with $i \leq 2$. We have computed the conserved densities $\rho^{(1)}$ and $\rho^{(2)}$ for $m \geq 5$ and $\rho^{(3)}$ for $m \geq 7$ with the method described above and compared with the results of direct integration of the first order equations obtained recursively from Eq.(1.2).

The formulas below were obtained analytically for $m \geq 13$ and proved by induction. We have computed the conserved densities at lower orders directly using REDUCE and a symbolic integration package [6] and checked that they obey the general formula. The results are given in Eqs.(2.11-13) where we use the notation

$$a = F_m^{1/m}, \quad \alpha_i = F_{m-i}/F_m, \quad i = 1, 2, 3, 4.$$  \hspace{1cm} (2.10)

Then for $m \geq 5$ we have

$$\rho^{(1)} = a^{-1} (D^2 a)^2 - \frac{12}{m(m+1)} a^2 D a \alpha_1 + a \left[ \frac{12}{m(m+1)} \alpha_1^2 - \frac{24}{m(m+1)} \alpha_2 \right], \hspace{1cm} (2.11)$$

$$\rho^{(2)} = a(Da) \left[ D \alpha_1 + \frac{3}{m} \alpha_2^2 - \frac{6}{m^2} \alpha_1 \right] + 2a^2 \left[ -\frac{1}{m^2} \alpha_1 \alpha_2 + \frac{3}{m(m-1)} \alpha_1 \alpha_2 - \frac{3}{m(m-1)(m-2)} \alpha_3 \right], \hspace{1cm} (2.12)$$

while for $m \geq 7$,

$$\rho^{(3)} = a \left( D^2 a \right)^2 - \frac{60}{m(m+1)(m+3)} a^2 D^2 a \alpha_1 (1) + \frac{1}{m^2} \alpha_1^2 (1) - \frac{2}{m^3} \alpha_1 (2)$$

$$+ 30a \left( D a \right)^2 \left[ \frac{(m-1)}{m(m+1)(m+3)} D \alpha_1 (1) + \frac{1}{m^2} \alpha_1 (2) - \frac{2}{m^3} \alpha_2 \right]$$

$$+ \frac{120}{m(m-1)(m+3)} a^2 Da \left[ -\frac{(m-1)(m-3)}{m} \alpha_1 (1) D \alpha_1 (3) + (m-3) D \alpha_2 (1) - \frac{(m-1)(2m-3)}{m^2} \alpha_1 (2) + \frac{6(m-2)}{m^3} \alpha_1 (3) + 6 \alpha_3 \right]$$

$$+ \frac{60}{m(m-1)(m+3)} a^3 \left[ \frac{(m-1)}{m} (D \alpha_1 (1))^2 - \frac{4}{m} D \alpha_1 (1) \alpha_2 (2) + \frac{(m-1)(2m-3)}{m^3} \alpha_1 (1) \alpha_3 (4) \right.$$

$$- 4 \frac{(2m-3)}{m} \alpha_1 (1) \alpha_2 (2) + \frac{8}{m} \alpha_1 (1) \alpha_3 (3) + \frac{8}{m} \alpha_1 (2) - \frac{8}{m^3} \alpha_4 (4) \left]. \hspace{1cm} (2.13)$$
3. Top four terms in the derivatives of differential functions

In this section we shall obtain the expression of \( D^k \varphi \) up to top four highest derivatives. Let \( \varphi \) and \( u \) be as in Eq.(2.1), and \( D^k \varphi \) be given by (2.3). We start by writing the expressions of \( D^k \varphi \) for \( k \leq 6 \) up to top four derivatives. As there is a truncation in the summations, for \( k \geq 4 \) we should assume that \( n \geq 3 \).

\[
D\varphi = \sum_{i=0}^{n} \varphi_i u_{i+1} + \varphi_x, \quad (3.1a)
\]

\[
D^2\varphi = \sum_{i=0}^{n} \varphi_i u_{i+2} + \sum_{i=0}^{n} D\varphi_i u_{i+1} + D\varphi_x, \quad (3.1b)
\]

\[
D^3\varphi = \sum_{i=0}^{n} \varphi_i u_{i+3} + 2 \sum_{i=0}^{n} D\varphi_i u_{i+2} + \sum_{i=0}^{n} D^2\varphi_i u_{i+1} + D^2\varphi_x, \quad (3.1c)
\]

\[
D^4\varphi = \sum_{i=n-3}^{n} \varphi_i u_{i+4} + 3 \sum_{i=0}^{n} D\varphi_i u_{i+3} + 3 \sum_{i=0}^{n} D^2\varphi_i u_{i+2} + \sum_{i=0}^{n} D^3\varphi_i u_{i+1} + D^3\varphi_x + O(u_{n}), \quad (3.1d)
\]

\[
D^5\varphi = \sum_{i=n-3}^{n} \varphi_i u_{i+5} + 4 \sum_{i=0}^{n} D\varphi_i u_{i+4} + 6 \sum_{i=0}^{n} D^2\varphi_i u_{i+3} + 4 \sum_{i=0}^{n} D^3\varphi_i u_{i+2} \\
+ \sum_{i=0}^{n} D^4\varphi_i u_{i+1} + D^4\varphi_x + O(u_{n+1}), \quad (3.1e)
\]

\[
D^6\varphi = \sum_{i=n-3}^{n} \varphi_i u_{i+6} + 5 \sum_{i=0}^{n} D\varphi_i u_{i+5} + 10 \sum_{i=0}^{n} D^2\varphi_i u_{i+4} + 10 \sum_{i=0}^{n} D^3\varphi_i u_{i+3} \\
+ 5 \sum_{i=0}^{n} D^4\varphi_i u_{i+2} + \sum_{i=0}^{n} D^5\varphi_i u_{i+1} + D^5\varphi_x + O(u_{n+2}). \quad (3.1f)
\]

For \( k \geq 7 \), we keep terms of orders \( u_{n+k}, u_{n+k-1}, u_{n+k-2} \) and \( u_{n+k-3} \) and write the general formula as below.

\[
D^k \varphi = \varphi_n u_{n+k} + \varphi_{n-1} u_{n+k-1} + \varphi_{n-2} u_{n+k-2} + \varphi_{n-3} u_{n+k-3} + \ldots \\
+ (k-1) [D \varphi_n u_{n+k-1} + D \varphi_{n-1} u_{n+k-2} + D \varphi_{n-2} u_{n+k-3} + \ldots ] \\
+ \binom{k-1}{2} [D^2 \varphi_n u_{n+k-2} + D^2 \varphi_{n-1} u_{n+k-3} + \ldots ] \\
+ \binom{k-1}{3} [D^3 \varphi_n u_{n+k-3} + \ldots ] \\
+ \ldots \\
+ \binom{k-1}{2} [D^{k-3} \varphi_n u_{n+k-3} + \ldots ] \\
+ (k-1) [D^{k-2} \varphi_n u_{n+k-2} + D^{k-2} \varphi_{n-1} u_{n+k-3} + \ldots ] \\
+ [D^{k-1} \varphi_n u_{n+k-1} + D^{k-1} \varphi_{n-1} u_{n+k-2} + \ldots ] \\
+ D^{k-1} \varphi_x + O(u_{n+k-4}). \quad (3.2)
\]

Note that, for \( k \geq 7 \), only the indicated terms of the first four lines contribute to the top four terms. In the last three lines, there are contributions from top three terms in \( D^{k-1} \varphi_i \), top two terms in \( D^{k-2} \varphi_i \) and from
the top term in $D^{k-3}\varphi_i$. For $k \geq 7$ these different types of contributions do not mix up and we can obtain a
general formula by inspection, obtaining recursively the top first, second and third terms of any derivative.
Once the general formula is “guessed” it can be proved easily by induction.

**Proposition 3.1.** Let $\varphi$ and $u$ be as in Eq. (2.1) and assume that $n \geq 3$. Then

\[
D^k \varphi = \varphi_{n+1} + O(u_n), \\
\quad \quad \quad \quad (3.4a)
\]

\[
D^2 \varphi = \varphi_{n+2} + [\varphi_{n+1} + 2D\varphi_n]u_{n+1} - \varphi_{nn} u^2_{n+1} + O(u_n), \\
\quad \quad \quad \quad (3.4b)
\]

\[
D^3 \varphi = \varphi_{n+3} + [\varphi_{n+2} + 3D\varphi_n]u_{n+2} + [\varphi_{n+1} + 3D\varphi_n - 3D^2\varphi_n]u_{n+1} \\
\quad \quad \quad \quad - 3\varphi_{nn} u_{n+1} u_{n+2} - 3D\varphi_{nn} u^2_{n+1} + \varphi_{nnn} u^3_{n+1} \\
\quad \quad \quad \quad - 3\varphi_{nnn} u^2_{n+1} u_{n+3} + O(u_n), \\
\quad \quad \quad \quad (3.4c)
\]

\[
D^4 \varphi = \varphi_{n+4} + [\varphi_{n+3} + 4D\varphi_n]u_{n+3} + [\varphi_{n+2} + 4D\varphi_n - 6D^2\varphi_n]u_{n+2} \\
\quad \quad \quad \quad + [\varphi_{n+1} + 4D\varphi_n - 2]u_{n+1} + 4D^3\varphi_n]u_{n+1} \\
\quad \quad \quad \quad - 4\varphi_{nn} u_{n+1} u_{n+3} - 6D^2\varphi_{nn} u^2_{n+1} - 12D\varphi_{nn} u_{n+1} u_{n+2} - 3\varphi_{nn} u^2_{n+2} \\
\quad \quad \quad \quad + 6\varphi_{nnn} u^2_{n+1} u_{n+2} - 10\varphi_{nnn} u_{n+1} u_{n+2} - 12D\varphi_{nnn} u_{n+1} u_{n+2} \\
\quad \quad \quad \quad + 4D\varphi_{nnn} u^3_{n+1} - \varphi_{nnnn} u^4_{n+1} + 6\varphi_{nnn,1} u_{n+1}^3 \\
\quad \quad \quad \quad - 4\varphi_{nnn,2} u^2_{n+1} u_{n+3} + 9\varphi_{nnn,1} u_{n+1}^2 + O(u_n), \\
\quad \quad \quad \quad (3.4d)
\]

\[
D^5 \varphi = \varphi_{n+5} + [\varphi_{n+4} + 5D\varphi_n]u_{n+4} + [\varphi_{n+3} + 5D\varphi_n - 10D^2\varphi_n]u_{n+3} \\
\quad \quad \quad \quad + [\varphi_{n+2} + 5D\varphi_n - 2]u_{n+2} + 10D^3\varphi_n]u_{n+2} \\
\quad \quad \quad \quad - 10\varphi_{nn} u_{n+2} u_{n+3} - 15D\varphi_{nn} u^2_{n+2} - 10\varphi_{nnn} u^2_{n+2} + P(u_{n+1}), \\
\quad \quad \quad \quad (3.4e)
\]

\[
D^6 \varphi = \varphi_{n+6} + [\varphi_{n+5} + 6D\varphi_n]u_{n+5} + [\varphi_{n+4} + 6D\varphi_n - 15D^2\varphi_n]u_{n+4} \\
\quad \quad \quad \quad + [\varphi_{n+3} + 6D\varphi_n - 2]u_{n+3} + 15D^3\varphi_n]u_{n+3} \\
\quad \quad \quad \quad + [\varphi_{n+2} + 6D\varphi_n - 2]u_{n+2} + 15D^2\varphi_n]u_{n+2} + 20D^3\varphi_n]u_{n+3}
\quad \quad \quad \quad (3.4f)
\]
Proof. The expressions (3.4a-f) are obtained iteratively by expanding the derivatives in (3.1a-f) and keeping the relevant terms, which is a tedious but straightforward computation.

Similarly, to prove (3.3d) by induction, one has to check that it holds for $k = 7$, by replacing $k = 7$ in (3.2). Then assuming (3.3d) holds for $k \geq 7$, we compute

$$D^{k+1} \varphi = \varphi_n u_{n+k+1} + D \varphi_n u_{n+k} + [\varphi_{n-1} + k D \varphi_n] u_{n+k} + [D \varphi_{n-1} + k D^2 \varphi_n] u_{n+k-1} + [\varphi_{n-2} + k D \varphi_{n-1} + \left(\frac{k}{2}\right) D^2 \varphi_n] u_{n+k-1} + [D \varphi_{n-2} + k D^2 \varphi_{n-1} + \left(\frac{k}{2}\right) D^3 \varphi_n] u_{n+k-2} + [\varphi_{n-3} + k D \varphi_{n-2} + \left(\frac{k}{2}\right) D^2 \varphi_{n-1} + \left(\frac{k}{3}\right) D^3 \varphi_n] u_{n+k-2} + P(u_{n+k-3}), \quad (3.5a)$$

and

$$\varphi_n u_{n+k} + [\varphi_{n-1} + (k+1) D \varphi_n] u_{n+k} + \left[\varphi_{n-2} + (k+1) D \varphi_{n-1} + \left(\frac{k}{2}\right) D^2 \varphi_n\right] u_{n+k} \quad (3.5b)$$

in (3.5b), we obtain Eq.(3.3d). The validity of the remaining expressions in (3.3a-d) for the indicated values of $k$ can be checked from Eqs.(3.4a-f).

4. General Results on Classification

In this section we shall obtain certain general results on classification. We start by stating the following well known result which is proved easily from the conserved density condition.

**Proposition 4.1.** Let $F = F(x, t, u, \ldots, u_{2k})$. If $\rho = \rho(x, t, u, \ldots, u_n)$ is a conserved density of the evolution equation $u_t = F$, then $\rho_{nn} = 0$, for $n \geq 2$ and $k \geq 1$.

**Proof.** Let $n = 2k + l$. Then $D_t \rho$ is the sum of terms $\rho_{2k+l-j} D^{2k+l-j} F$. Integrating this $k - j$ times we obtain $D^{k-j} \rho_{2k+l-j} D^{k+l} F$ which is quadratic in $u_{3k+l-j}$. Thus the contribution to the highest order nonlinearity comes from the first term corresponding to $l = 0$ only. It follows that $\rho_{nn} = 0$.

For $m$ odd, top 2 terms in the expansion of the derivatives contribute to the highest order nonlinearity and we have a nontrivial result.
Proposition 4.2. Let \( \rho \) be a conserved density of order \( n \geq m \) for an evolution of order \( m = 2k + 1 \). Then for \( k \geq 2 \),

\[
\frac{m}{2} F_m D\rho_{nn} - (n - \frac{1}{2} m) DF_m \rho_{nn} = \rho_{nn} F_{m-1} \quad m \geq 7, \quad n \geq m \tag{4.1}
\]

Proof. From the proof of Proposition 5.1 it will be seen that only top terms in \( D_t \rho \) will contribute to the highest order nonlinearity. Let \( n = 2k + l + 1 \). Then the highest nonlinearity will be of order \( 3k + l + 1 \) and

\[
D_t \rho \geq \rho_n D^{2k+l+1} F + \rho_{n-1} D^{2k+l} F + O(u_{3k+l}^2)
\]

\[
\cong (-1)^{k+1} [D^{k+1} \rho_m - D^k \rho_{n-1}] D^{k+l} F + O(u_{3k+l}^2).
\]

Furthermore, only top two derivatives of each term will contribute to the highest order nonlinearity. Hence we can use the formula (3.3b) which is valid beyond the third derivative. We note that (3.3b) can be used for \( k = 2 \) also, because only the top derivative in \( D^k \rho_{n-1} \) is needed. Thus

\[
(-1)^{k+1} D_t \rho \cong \left[ \rho_{nn} u_{3k+l+2} + (k + 1) D\rho_{nn} u_{3k+l+1} \right]
\]

\[
\times \left[ F_m u_{3k+l+1} + [F_{m-1} + (k + l) DF_m] u_{3k+l} \right] u_{3k+l+1}^2 + O(u_{3k+l}^2)
\]

\[
\cong \left[ -\frac{1}{2} D(\rho_{nn} F_m) - \rho_{nn} [F_{m-1} + (k + l) DF_m] + (k + 1) F_m D\rho_{nn} \right] u_{3k+l+1}^2 + O(u_{3k+l}^2)
\]

\[
\cong \left[ (k + \frac{1}{2}) D\rho_{nn} F_m - (k + l + \frac{1}{2}) \rho_{nn} DF_m - \rho_{nn} F_{m-1} \right] u_{3k+l+1}^2 + O(u_{3k+l}^2).
\]

The coefficient of \( u_{3k+l+1}^2 \) gives (4.1) and the proposition is proved.

The implications of (4.1) are listed below. Conserved densities of order \( n > m \) are necessarily quadratic in the highest derivative and there is essentially only one conserved density at each order. The corollary can be proved easily by replacing the indicated values for \( n \) in (4.1).

Corollary 4.3. Let \( \rho \) be a conserved density of order \( n \geq m \) for an evolution equation of order \( m = 2k + 1 \geq 5 \). Then

i. For \( n = m \),

\[
F_m \rho_{mm} = F_{mm} \rho_{mm} \tag{4.2a}
\]

ii. If \( \rho \) and \( \eta \) are two conserved densities of order \( n \) with \( \rho_{nn} \neq 0, \eta_{nn} \neq 0 \), then

\[
\frac{D\rho_{nn}}{\rho_{nn}} = \frac{D\eta_{nn}}{\eta_{nn}} \tag{4.2b}
\]

iii. For \( n > m \),

\[
\rho_{nnn} = 0 \tag{4.2c}
\]

Applying (4.2a) to \( \rho^{(-1)} \), we will show that there are three classes for \( F \), the first one being the quasilinear equations. For \( m = 3 \), both the essentially nonlinear classes are well known candidates for integrable equations.
Proposition 4.4. Let $\rho = F^{-1/m}_m$ be a conserved density of the evolution equation $u_t = F$. Then $F$ belongs to one of the classes below.

\begin{enumerate}
  \item $\rho_m = 0$ : \quad $F = A_m + B$, \quad $A_m = B_m = 0$. \hfill (4.3a)
  \item $\rho_{mm} = 0$ : \quad $F = (A_m + B)^{1-m} + C$, \quad $A_m = B_m = C_m = 0$. \hfill (4.3b)
  \item $\rho_{mm} \neq 0$ : \quad $F_m = \left[c^{(1)}_m F^2 + c^{(2)}_m F + c^{(3)}_m\right]^{m/(m-1)}$, \quad $c^{(1)}_m = c^{(2)}_m = c^{(3)}_m = 0$. \hfill (4.3c)
\end{enumerate}

**Proof.** First note that $\rho_m = 0$ corresponds to $F_{mm} = 0$, in which case the conserved density condition (4.2a) is identically satisfied and gives the class (4.3a). Then we compute

$$
\rho_{mm} = \frac{1}{m} (\frac{1}{m} + 1) F_m^{\frac{1}{m}-2} F_{mm}^2 - \frac{1}{m} F_m^{\frac{1}{m}-1} F_{mmm} \tag{4.4}
$$

and $\rho_{mm} = 0$ gives

$$
(\frac{1}{m} + 1) F_{mm}^2 + F_m F_{mmm} = 0 \tag{4.5}
$$

which can be integrated to give the class of equations (4.3b). For $\rho_{mm} \neq 0$, Eq.(4.2a) gives

$$
\frac{F_{mm}}{F_m} = \frac{\rho_{mmm}}{\rho_{mm}}, \tag{4.6}
$$

which implies that $F_m$ is proportional to $\rho_{mm}$ and using Eq.(4.4) we obtain a third order ordinary differential equation. Integrating this twice we obtain

$$
\rho_m = \mu F + \nu, \quad \mu_m = \nu_m = 0. \tag{4.7}
$$

Substituting $\rho = F^{-1/m}_m$ in (4.7), integrating once more and renaming integration constants we have Eq.(4.3c). We also note that the class (4.3c) with $c^{(1)}_m = 0$ corresponds to the class (4.3b).

The equation (4.3c) can be integrated for $m = 3$ but it has no antiderivative for $m \geq 5$. Thus for $m = 3$ we have the following well known result.

**Corollary 4.5 (Third order equations).** Let $u_t = F(x,t,u,\ldots,u_3)$. If $\rho = F_3^{-1/3}$ is a conserved density then $F$ belongs to one of the classes below.

\begin{enumerate}
  \item $F = A_3 + B$, \quad $A_3 = B_3 = 0$. \hfill (4.8a)
  \item $F = (A_3 + B)^{-2} + C$, \quad $A_3 = B_3 = C_3 = 0$. \hfill (4.8b)
  \item $F = \frac{2A_3 + B}{(A_3^2 + B_3 + C)^{1/2}} + E$, \quad $A_3 = B_3 = C_3 = E_3 = 0$. \hfill (4.8c)
\end{enumerate}
We now arrive at the discussion of equations of order \( m \geq 5 \). Recall that the expression of \( \rho^{(1)} \) given by Eq.(2.11) which is valid for \( m \geq 5 \) is of the form (4.2c) with

\[
\rho_{nn} = 2a^{-1}a_m^2,
\]

(4.9)

where \( n = m + 1 \). From (4.1) we obtain

\[
\frac{a_{mm}}{a_m} - \frac{m + 3}{2} \frac{a_m}{a} = 0
\]

(4.10)

which gives

\[
a = (Au_m + B)^{-\frac{2}{m+1}}
\]

(4.11)

where \( A \) and \( B \) are independent of \( u_m \). It follows that

\[
F = \frac{1}{A} \frac{1+m}{1-m} (Au_m + B)^{(1-m)/(1+m)} + C, \quad A_m = B_m = C_m = 0.
\]

(4.12)

It can be checked that this corresponds to

\[
c_1 = \left(\frac{1+m}{1-m}\right)^2 A^2, \quad c_2 = -2c_1C, \quad c_3 = c_1C^2
\]

(4.13)

in (4.3c) and it is inconsistent with (4.3b). As we shall see that for \( m \geq 7 \) only quasilinear equations are allowed, we state the Proposition below for \( m = 5 \).

**Proposition 4.6 (Fifth order equations).** Let \( \rho = \rho^{(1)} \) given by Eq.(2.11) be a conserved density of the evolution equation \( u_t = F(x,t,u,\ldots,u_5) \). Then integrable equations consist of two classes

\[
F = Au_5 + B, \quad A_5 = B_5 = 0,
\]

(4.14a)

\[
F = \frac{3}{2A} (Au_1 + B)^{-2/3} + C, \quad A_5 = B_5 = C_5 = 0, \quad A_4 = C_4 = 0.
\]

(4.14b)

**Proof.** Substituting (4.12) in the conserved density condition and taking logarithmic derivatives we obtain

\[
F_{m-1}/F_m = mDA/A.
\]

(4.15)

Taking the derivative of \( F \) with respect to \( u_{m-1} \) and substituting, we see that the expression above should be linear in \( u_m \). The second derivative with respect to \( u_m \) gives \( C_{m-1} = 0 \), while the coefficient of the linear term leads to \( A_{m-1} = 0 \). Substituting \( m = 5 \), we have (4.14b).

**Remark 4.7.** Let \( \rho \) be a conserved density of order 6 for an evolution equation of order 5. In the expression \( D_t \rho \), the coefficients of \( u_8^2 u_6, u_7^2 u_6 \) and \( u_7^2 u_5^2 \) depend only on \( \rho_{66} \). These expressions give respectively

\[
7F_{55} \rho_{66} - 5\rho_{6655} = 0,
\]

\[
-42F_{555}\rho_{66} - 2F_{55} \rho_{665} + 25F_5 \rho_{6655} = 0,
\]

\[
-28F_{555}\rho_{66} + 24F_{555} \rho_{665} - 23F_{55} \rho_{6655} + 15F_5 \rho_{6655} = 0.
\]
After eliminations we obtain

\[ F_{55} \rho_{66} = \frac{5}{7} F_5 \rho_{65} = 0, \]  
\[ F_{55} \rho_{665} = \frac{1}{7} F_5 \rho_{655} = 0, \]  
\[ F_{55} \rho_{6655} = \frac{5}{11} F_5 \rho_{6555} = 0. \]  

(4.16a)

(4.16b)

(4.16c)

It can be checked that the equations above are consistent with

\[ F_5 = a_5^5, \quad \rho_{66} = a_5^{-1} a_5^2, \quad a_{55} = 4 a_5^{-1} a_5^2, \]

where the last equation corresponds to Eq.(4.10) with \( m = 5 \). Thus we cannot exclude the possibility of the existence of essentially nonlinear equations at the 5th order

5. Equations of order greater than seven.

We consider now evolution equations of order \( m \geq 7 \) admitting a conserved density \( \rho \) of order \( n = m + 1 \), i.e.,

\[ m = 2k + 1, \quad n = m + 1 = 2k + 2. \]

From Corollary 4.3, we know that \( \rho \) is quadratic in \( u_n \), i.e,

\[ \rho = \alpha u_n^2 + \beta u_n + \gamma, \quad \alpha_n = \beta_n = \gamma_n = 0. \]  

(5.1)

For convenience, in the proof of Lemma 5.3 and afterwards, we shall use the notation

\[ \rho' = \rho_{m+1}, \quad \rho'' = \rho_{m+1,m+1}, \quad \rho'_{j} = \rho_{m+1,j}, \quad \rho''_{j} = \rho_{m+1,m+1,j}, \quad j = 0, \ldots m. \]

The following relation can be easily deduced from Eq.(5.1).

Lemma 5.1. Let \( \rho = \rho(x,t,u,\ldots,u_{m+1}) \) be a quadratic function of \( u_{m+1} \). Then

\[ D\rho' = \rho_{m}' u_{m+1} + O(u_m), \quad D^2 \rho'' = \rho_{m}'' u_{m+2} + P(u_{m+1}), \quad D^3 \rho''' = \rho_{m}''' u_{m+3} + P(u_{m+2}), \]

\[ D\rho'_{j} = \rho_{m}'_{j} u_{m+2} + P(u_{m+1}), \quad D^2 \rho''_{j} = \rho_{m}''_{j} u_{m+3} + P(u_{m+2}). \]  

(5.2)

We shall now show that, in the integral of \( D_t \rho \), the contribution to top two nonlinear terms \( u_{3k+2}^2 \) and \( u_{3k+1}^2 \) will come from top four derivatives.

Proposition 5.2. Let \( \rho = \rho(x,t,u,\ldots,u_n) \) and \( u_t = F(x,t,u,\ldots,u_m) \), where \( n = m + 1 \) and \( m = 2k + 1 \). Then

\[ (-1)^{k+1} D_t \rho \cong \left[ D^{k+1} \rho_n - D^k \rho_{n-1} \right] D^{k+1} F - \left[ D^k \rho_{n-2} D^k F - D^k \rho_{n-3} \right] D^k F + O(u_{3k}). \]  

(5.3)
Proof. We write

\[ D_t \rho = \rho_n D^n F + \rho_{n-1} D^{n-1} F + \rho_{n-2} D^{n-2} F + \rho_{n-3} D^{n-3} F + \rho_{n-4} D^{n-4} F + \ldots + \rho_0 F + \rho_t, \]

\[ = \rho_{2k+2} D^{2k+2} F + \rho_{2k+1} D^{2k+1} F + \rho_{2k} D^{2k} F + \rho_{2k-1} D^{2k-1} F + \rho_{2k-2} D^{2k-2} F + \ldots + \rho_0 F + \rho_t, \]

We integrate the first term \( k + 1 \) times, the second and the third \( k \) times etc., so that each term is a product of terms of orders equal or differing at most by one. Hence we have

\[ (-1)^{k+1} D_t \rho \geq \left[ D^{k+1} \rho_{2k+2} - D^k \rho_{2k+1} \right] D^{k+1} F - \left[ D^k \rho_{2k} - D^{k-1} \rho_{2k-1} \right] D^k F \]

\[- \left[ D^{k-1} \rho_{2k-2} - D^{k-2} \rho_{2k-3} \right] D^{k-1} F + \ldots + \rho_0 F + \rho_t. \]

It will be sufficient to show that third term is of order \( u_{3k} \), because by a similar reasoning it can be seen that the remaining terms are of lower orders. As \( \rho_{2k-2} \) is of polynomial in \( u_{2k+2} \) the term in the brackets is linear in \( u_{3k+1} \) and polynomial in other derivatives of order larger than \( m \). Similarly \( D^{k-1} F \) is polynomial in \( u_{3k} \), hence after integration by parts, their product is of order \( u_{3k} \). It follows that the contribution to top two nonlinear terms comes from top four terms in the expansion of \( D_t \rho \).

The contribution to the coefficient of \( u_{3k+2}^2 \) comes from the top two derivatives of the first term of (5.3) and it has already been given in (4.1). We shall need now the coefficients of

\[ u_{3k+2}^2 u_{2k+2} \quad \text{and} \quad u_{3k+1}^2 u_{2k+4}. \]

We shall now show that at most top four terms of each derivative contribute to the coefficient of \( u_{3k+1}^2 \) and actually the contribution to the coefficient of \( u_{3k+1}^2 u_{2k+4} \) comes only from the top two terms in each product in

\[ (D^{k+1} \rho_{n} - D^k \rho_{n-1}) D^{k+1} F. \]

For this we shall compute the relevant terms in each factor of \( D_t \rho \) separately.

Lemma 5.3. Let \( \rho \rho(x, t, u, \ldots, u_n) \) be quadratic in \( u_n = u_{m+1} \), \( F = F(x, t, u, \ldots, u_m) \) and let \( k \geq 6 \). Then

\[ \begin{align*}
D^{k+1} \rho - D^k \rho_m &= \rho'' u_{m+k+2} + [r_1 \rho'' u_{m+1} + O(u_m)] u_{m+k+1} \\
&\quad + [r_2 \rho'' u_{m+2} + P(u_{m+1})] u_{m+k} \\
&\quad + [r_3 \rho'' u_{m+3} + P(u_{m+2})] u_{m+k-1} \\
&\quad + P(u_{m+k-2})
\end{align*} \tag{5.4a} \]
\[ D^k \rho_{m-1} - D^{k-1} \rho_{m-2} = P(u_{m+1}) u_{m+k+1} + P(u_{m+2}) u_{m+k} + P(u_{m+k-1}) \]  
\[ D^{k+1} F = F_m u_{m+k+1} + [f_1 F_{mm} u_{m+1} + O(u_m)] u_{m+k} \]
\[ + [f_2 F_{mm} u_{m+2} + P(u_{m+1})] u_{m+k-1} \]
\[ + [f_3 F_{mm} u_{m+3} + P(u_{m+2})] u_{m+k-2} \]
\[ + P(u_{m+k-3}) \]  
\[ D^k F = O(u_m) u_{m+k} + P(u_{m+1}) u_{m+k-1} + P(u_{m+k-2}) \]  
where
\[ r_1 = k + 1, \quad r_2 = 1 + \binom{k+1}{2}, \quad r_3 = \binom{k+1}{2} + \binom{k+1}{3} - \binom{k}{2} \]  
\[ f_1 = k + 1, \quad f_2 = \binom{k+1}{2}, \quad f_3 = \binom{k+1}{3} \]  

**Proof.** We substitute \( \varphi = \rho' \) in (3.3d) which is valid for \( k + 1 \geq 7 \) and get
\[ D^{k+1} \rho' = \rho'' u_{m+k+2} + [\rho'_m + (k + 1)D\rho'']_m u_{m+k+1} \]
\[ + [\rho'_{m-1} + (k + 1)D\rho'_m + (\frac{k+1}{2})D^2\rho'']_m u_{m+k} \]
\[ + [\rho'_{m-2} + (k + 1)D\rho'_{m-1} + (\frac{k+1}{2})D^2\rho'_m + (\frac{k+1}{3})D^3\rho'']_m u_{m+k-1} \]
\[ + P(u_{m+k-2}) \]
\[ = \rho'' u_{m+k+2} + [\rho'_m + (k + 1)\rho''_m u_{m+1} + O(u_m)] u_{m+k+1} \]
\[ + \left[ [(k + 1) + \binom{k+1}{2}]\rho''_m u_{m+2} + P(u_{m+1}) \right] u_{m+k} \]
\[ + \left[ [(\frac{k+1}{2}) + \binom{k+1}{3}]\rho''_m u_{m+3} + P(u_{m+2}) \right] u_{m+k-1} \]
\[ + P(u_{m+k-2}). \]  

Then we repeat the same computations for \( D^k \rho_m \) up to top three terms using (3.3c) which is valid for \( k \geq 5 \).
\[ D^k \rho_m = \rho'_m u_{m+k+1} + [\rho_{mm} + kD\rho'_m] u_{m+k} \]
\[ + [\rho_{m,m-1} + kD\rho_{mm} + \binom{k}{2}D^2\rho'_m] u_{m+k-1} \]
\[ + P(u_{m+k-2}) \]
\[ = \rho'_m u_{m+k+1} + [k\rho''_m u_{m+2} + P(u_{m+1})] u_{m+k} \]
\[ + \left[ \binom{k}{2}\rho''_m u_{m+3} + P(u_{m+2}) \right] u_{m+k-1} \]
\[ + P(u_{m+k-2}). \]  

Subtracting (5.7) from (5.6) we obtain (5.4a). The derivation of Eqs. (5.4b-d) is straightforward.  

In the products of (5.4a) with (5.4c) and (5.4b) with (5.4d) most of the terms will not contribute to the top two nonlinear terms, as indicated below.
Lemma 5.4. For $k \geq 6$, the following relations hold.

\[ u_{m+k+2} P(u_{m+k-3}) \cong u_{m+k+1} P(u_{m+k-2}) \cong u_{m+k} P(u_{m+k-1}) \cong P(u_{m+k-1}) \quad (5.8a) \]
\[ u_{m+k+2} u_{m+k-1} P(u_{m+1}) \cong u_{m+k+2} u_{m+k-2} P(u_{m+2}) \]
\[ \cong u_{m+k+1} u_{m+k} P(u_{m+1}) \cong u_{m+k+1} u_{m+k-1} P(u_{m+2}) \cong u_{m+k}^2 P(u_{m+2}) \quad (5.8b) \]
\[ u_{m+k+2} u_{m+k} \vee (u_m) \cong u_{m+k+1}^2 O(u_m) \quad (5.8c) \]

It follows that the contribution to the top two nonlinear terms of $D_t \rho$ comes from

\[
\left[ \rho'' u_{m+k+2} + \rho''_m [r_1 u_{m+1} u_{m+k+1} + r_2 u_{m+2} u_{m+k} + r_3 u_{m+3} u_{m+k-1}] \right] \\
\times \left[ F_m u_{m+k+1} + F_{mm} [f_1 u_{m+1} u_{m+k} + f_2 u_{m+2} u_{m+k-1} + f_3 u_{m+3} u_{m+k-2}] \right], \quad (5.9)
\]

where $r_i$'s and $f_i$'s are given by Eqs.(5.5a-b). Substituting these values and integrating by parts we obtain the following result.

Proposition 5.5. Let $u_t = F(x, t, u, \ldots, u_m)$ be an evolution equation of order $m$ and $\rho = \rho(x, t, u, \ldots, u_n)$ be a conserved density of order $n = m+1$ satisfying $\rho_{nnn} = 0$. The coefficients of $u_{3k+2}^2 u_{2k+2}$ and $u_{3k+1}^2 u_{2k+4}$ give respectively

\[
(2k+1)\rho'' F_m = (2k+3)\rho'' F_{mm}, \quad (5.10a) \\
(2k+1)(k^2 + k + 6)\rho'' F_m = (2k+3)(k+1)(k+2)\rho'' F_{mm} \quad (5.10b)
\]

where $\rho'' = \rho_{nn}$

Proof. The general formulas can be used for $k \geq 6$, i.e., $m \geq 13$ and Eqs.(5.10a-b) can be obtained from Eq.(5.9). For $m = 5$, straightforward computation of the conserved density condition is possible with REDUCE and the results have already been given by Eqs.(4.161-c). For $m = 7$ and $m = 8$ it is necessary to use a combination of symbolic and analytic computations. For $m = 11$, there is little discrepancy from the general formulas and the required coefficient scan be obtained easily with analytical.

It can be easily checked that the equations (5.10a) and (5.10b) are inconsistent except for $k = 2$. For $k = 2$, (5.10b) is the coefficient of $u_2^2 u_8$, and leads to Eq.(4.16b) after integration. Thus for $k \geq 3$ Thus

\[ \alpha_m F_m = \alpha F_{mm} = 0. \]

It follows that for $\alpha \neq 0$, $F_{mm} = 0$, and we have the following corollary.
Corollary 5.6. Let \( u_t = F(x,t,u,\ldots,u_m) \) be an evolution equation of order \( m \geq 7 \) and \( \rho = \rho(x,t,u,\ldots,u_n) \) be a conserved density of order \( n = m + 1 \), with \( \rho_{nn} \neq 0 \). Then \( F_{mm} = 0 \).

Note that in order to prove the quasilinearity result, the explicit form of \( \rho^{(1)} \) is not needed. We have only used here the fact that it is indeed quadratic in \( u_{m+1} \).

References.

[1] J.A. Sanders and J.P. Wang, “On the integrability of homogeneous scalar evolution equations”, Journal of Differential Equations, vol. 147,(2), pp.410-434, (1998).

[2] J.A. Sanders and J.P. Wang, “On the integrability of non-polynomial scalar evolution equations”, Journal of Differential Equations, vol. 166,(1), pp.132-150, (2000).

[3] A.V. Mikhalov, A.B. Shabat and V.V Sokolov. “The symmetry approach to the classification of integrable equations” in ‘What is Integrability?’ edited by V.E. Zakharov (Springer-Verlag, Berlin 1991).

[4] R.H. Heredero, V.V. Sokolov and S.I. Svinolupov, “Classification of 3rd order integrable evolution equations”, Physica D, vol.87 (1-4), pp.32-36, (1995).

[5] P.J. Olver, Lie Application of Lie Groups to Differential Equations (Springer-Verlag, Berlin 1993)

[6] A.H. Bilge, “A REDUCE program for the integration of differential polynomials”, Computer Physics Communications, vol. 71, p.263, (1992).