Oscillation Criteria for Second-Order Nonlinear Neutral Delay Dynamic Equations with Damping on Time Scales

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Abstract. In this paper, we discuss oscillation criteria for second-order nonlinear neutral delay dynamic equations with damping on time scales by using the generalized Riccati transformation and the inequality technique. Under certain conditions, we establish four new oscillation criteria. Our results in this paper are new even for the cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

1. Introduction
In our real life, the movement in the vacuum or ideal state is rare, but the movement with damping and disturbance is widespread. In recent years, the research results relevant to oscillation of second order dynamic equations with damping on time scales are emerging, such as [1-7]. The research results of oscillation for the second order linear, non-linear or semi-linear dynamic equations can be found in reference [8-23]. On the basis of the above work, we will study the oscillatory behavior of all solutions of a more extensive second-order nonlinear neutral delay dynamic equation with damping in this paper, which is given as follows:

$(a(t)z^2(t))^2 + p(t)z^2(t) + q(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, \ t \geq t_0,$

where $z(t) = x(t) + r(t)x(\tau(t))$. In this paper, we give the following hypotheses.

$(H_1)$ $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above and when $t_0 \in \mathbb{T}$ with $t_0 > 0$, we define the time scale interval of the form $[t_0, \infty)$ by $[t_0, \infty) = [t_0, \infty) \cap \mathbb{T}$.

$(H_2)$ $a, r, p, q : \mathbb{T} \to \mathbb{R}$ are positive rd-continuous functions such that $0 \leq r(t) < 1, -p/a \in \mathbb{R}^+$.

$(H_3)$ $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing and differentiable function such that

$\tau(t) \leq t, \quad \lim_{t \to \infty} \tau(t) = \infty, \quad \text{and} \quad \tau(\mathbb{T}) = (\mathbb{T})$.

$(H_4)$ $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for some positive constant $L$,

$f(x)/x \geq L \text{ for all } x \neq 0.$

By a solution of (1.1), we mean a nontrivial real-valued function $x$ satisfying (1.1) for $t \in \mathbb{T}$. A solution $x$ of (1.1) is called oscillatory if it is neither eventually positive nor negative; otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory. Our attention is restricted on those solutions of (1.1) which are not the eventually identically zero.

The purpose of this paper is to establish the oscillation criteria of Philos [24] for equation (1.1).
the two famous results of Philos\cite{24} about oscillation of second order linear differential equations are extended to equation (1.1) in this paper. At the same time, when

\[ \int_0^\infty \left[ \frac{1}{a(t)} e^{-\mu(t)}(t,t_0) \right] \Delta t = \infty, \] (1.2)

we obtain two criteria of equation (1.1) about that each solution is either oscillatory or converges to zero.

2. Some Preliminaries

On the time scale \( T \) we define the forward and backward jump operators by

\[ \sigma(t) = \inf \{ s \in T : s > t \} \quad \text{and} \quad \rho(t) = \sup \{ s \in T : s < t \}. \]

A point \( t \in T \) is said to be left-dense if it satisfies \( \rho(t) = t \), right-dense if it satisfies \( \sigma(t) = t \), left-scattered if it satisfies \( \rho(t) < t \) and right-scattered if it satisfies \( \sigma(t) > t \). The gramine \( \mu \) of the time scale is defined by \( \mu(t) = \sigma(t) - t \). For a function \( f : T \to \mathbb{R} \), the (delta) derivative is defined by

\[ f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \]

if \( f \) is continuous at \( t \) and \( t \) is right-scattered. If \( t \) is right-dense, then the derivative is defined by

\[ f^\Delta(t) = \lim_{s \to t^-} \frac{f(t) - f(s)}{t - s}, \]

provided this limit exists. A function \( f : T \to \mathbb{R} \) is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points. The set of rd-continuous functions \( f : T \to \mathbb{R} \) denoted by \( C_{rd}(T,\mathbb{R}) \) and the set of functions \( f \) is denoted by \( C^1_{rd}(T,\mathbb{R}) \) if the function \( f \) is \( \Delta \)-differentiable and the derivative \( f^\Delta \) is rd-continuous. The derivative \( f^\sigma \) of \( f \), the shift \( f^\sigma \) of \( f \) and the gramine \( \mu \) are related by the formula

\[ f^\sigma = f + \mu f^\Delta \quad \text{where} \quad f^\sigma = f \circ \sigma. \]

We will make use of the following product and quotient rules for the derivative of the product \( f \cdot g \) and the quotient \( f/g \) of two differentiable functions \( f \) and \( g \):

\[ (f \cdot g)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)), \] (2.1)

\[ (f/g)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))} \quad \text{if} \quad gg^\sigma \neq 0. \] (2.2)

For \( b,c \in T \), the Cauchy integral of \( f^\Delta \) is defined by

\[ \int_b^c f^\Delta(t) \Delta t = f(c) - f(b). \]

The integration by parts formula reads

\[ \int_b^c f^\Delta(t)g(t) \Delta t = f(c)g(c) - f(b)g(b) - \int_b^c f^\sigma(t)g^\Delta(t) \Delta t, \] (2.3)
and the infinite integral is defined by
\[ \int_{b}^{\infty} f(s) \Delta s = \lim_{t \to \infty} \int_{b}^{t} f(s) \Delta s. \]
For more details, see [8, 9]

3. Several Lemmas
In this section, we present five lemmas that will be needed in the proofs of our results in Section 4.

Lemma 3.1[8] If \( g \in \mathcal{R}^+ \), i.e., \( g: \mathbb{T} \to \mathbb{R} \) is rd-continuous and such that \( 1 + \mu(t)g(t) > 0 \) for all \( t \in [t_0, \infty)_T \), then the initial value problem \( y^\Delta = g(t)y, \ y(t_0) = y_0 \in \mathbb{R} \) has a unique and positive solution on \([t_0, \infty)_T\), denoted by \( e_g(\cdot, t_0) \). This “exponential function” satisfies the semi group property \( e_g(a, b)e_g(b, c) = e_g(a, c) \).

Lemma 3.2[8] Assume that \( v: \mathbb{T} \to \mathbb{R} \) is strictly increasing and \( \tilde{v} := v(T) \) is a time scale. Let \( w: \tilde{T} \to \mathbb{R} \). If \( v^\Delta(t) \) and \( w^\Delta(v(t)) \) exist on \( \tilde{T} \), where
\[ \tilde{T} = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases} \]
Then
\[ (w \circ v)^\Delta = (w^\Delta \circ v)^\Delta. \] (3.1)

Lemma 3.3[25] Assume that \( X \) and \( Y \) are nonnegative real numbers, then
\[ \lambda X T^{\lambda-1} - X^\lambda \leq (\lambda - 1) Y^\lambda \text{ for all } \lambda > 1. \] (3.2)
Where the equality holds if and only if \( X = Y \).

Lemma 3.4[26] Let \( a, b \in \mathbb{T} \) and \( a < b \), for positive rd-continuous functions \( f, g: [a, b] \to \mathbb{R} \) we have
\[ \int_{a}^{b} \left| f(s)g(s) \right| \Delta s \leq \left( \int_{a}^{b} \left| f(s) \right|^{\rho} \Delta s \right)^{1/p} \left( \int_{a}^{b} \left| g(s) \right|^{q} \Delta s \right)^{1/q}, \] (3.3)
where \( p > 1 \) and \( 1/p + 1/q = 1 \).

Lemma 3.5 Assume that (H1)-(H4) and (1.2) hold. Let \( x(t) \) be an eventually positive solution of (1.1). Then there exists \( t_i \in [t_0, \infty)_T \) such that
\[ z^\Delta(t) > 0 \text{ and } [a(t)z^\Delta(t)]^\Delta < 0 \text{ for all } t \in [t_i, \infty)_T. \] (3.4)

Proof Suppose that \( x(t) \) is an eventually positive solution of (1.1). There exists \( t_i \in [t_0, \infty)_T \) such that \( x(t) > 0 \) and \( x(\tau(t)) > 0 \) for \( t \in [t_i, \infty)_T \). From the definition of \( z(t) \), we get \( z(t) > 0 \) for \( t \in [t_i, \infty)_T \), at the same time for \( t \in [t_i, \infty)_T \), from equation (1.1) we get
\[ (a(t)z^\Delta(t))^\Delta + p(t)z^\Delta(t) > 0. \] (3.5)
Hence, from Lemma 3.1 and (2.2) we obtain
\[ \left[ \frac{az^\Delta}{e_{-p/a}(\cdot, t_0)} \right]^\Delta = \left[ \frac{az^\Delta}{e_{-p/a}(\cdot, t_0)} \right] e_{-p/a}(\cdot, t_0) - e_{-p/a}(\cdot, t_0)az^\Delta = \frac{az^\Delta}{e_{-p/a}(\cdot, t_0)} + pe_{-p/a}(\cdot, t_0) < 0 \] (3.6)
for \([t_i, \infty)_T\). So \( az^\Delta / e_{-p/a}(\cdot, t_0) \) is decreasing. By Lemma 3.1, \( z^\Delta(t) \) is either eventually positive or
eventually negative. Therefore, for arbitrary \( t \in [t, \infty)_T \), we have

\[
z^3(t) > 0. \tag{3.7}
\]

Otherwise, we assume that (3.7) is not satisfied, then there exists \( t_2 \in [t, \infty)_T \) such that \( z^3(t) < 0 \) for all \( t \in [t_2, \infty)_T \). Because (3.6) is decreasing, from Lemma 3.1 we have

\[
a(t)z^3(t) \leq a(t_2)z^3(t_2) = -\frac{M}{e_{-p/a}(t_2, t_0)} \tag{3.8}
\]

for \( t \in [t_2, \infty)_T \), where \( M = a(t_2)|z^3(t_2)| > 0 \). By (3.8) and Lemma 3.1, we get

\[
-z^3(t) \geq Me_{-p/a}(t_2, t_2)/a(t), \quad t \in [t_2, \infty)_T,
\]

i.e.

\[
z^3(t) \leq -Me_{-p/a}(t_2, t_2)/a(t), \quad t \in [t_2, \infty)_T. \tag{3.9}
\]

After integrating the two side of inequality (3.9) from \( t_2 \) to \( t \in [t, \infty)_T \), we have

\[
z(t) \leq z(t_2) - M \int_{t_2}^{t} \left( e_{-p/a}(s, t_2) / a(s) \right) \Delta s, \quad t \in [t_2, \infty)_T. \tag{3.10}
\]

Neatly, we find the limits of the two sides of (3.10) when \( t \to \infty \). From (1.2), we get \( \lim_{t \to \infty} z(t) = -\infty \). Therefore; \( z(t) \) is eventually negative, which is contradictory to \( z(t) > 0 \). So the inequality (3.7) holds. From (3.7) and (3.5), it is obvious that the second inequality of (3.4) holds. This completes the proof.

4. Main Results

Firstly, the two famous results of Philos\cite{24} about oscillation of second order linear differential equations are extended to equation (1.1) when condition (1.2) is satisfied.

**Theorem 4.1** Assume that (H1)-(H4) and (1.2) hold. Let \( H : D_T = \{(t, s) : t \geq s \geq t_0, t, s \in [t_0, \infty)_T \} \to \mathbb{R} \) be rd-continuous function, such that \( H(t, t) = 0, t \geq t_0; \) \( H(t, s) > 0, t > s \geq t_0; \) \( t, s \in [t_0, \infty)_T \), and \( H \) has a non-positive continuous \( \Delta \)-partial derivative \( H^\Delta(t, s) \) with respect to the second variable, let \( h : D_T \to \mathbb{R} \) be a rd-continuous function, and satisfies

\[
-H^\Delta(t, s) = h(t, s)\sqrt{H(t, s), \quad t, s \in D_T.} \tag{4.1}
\]

We have

\[
0 < \inf_{t \to \infty} \left[ \liminf_{t \to \infty} \left( H(t, s) / H(t, t_0) \right) \right] \leq \infty, \quad T_0 \in [t_0, \infty)_T. \tag{4.2}
\]

If there exist a positive and differentiable function \( \delta : T \to \mathbb{R} \) such that \( \delta^\Delta(t) \geq 0 \) for \( t \in [t_0, \infty)_T \), and a real rd-continuous function \( \Psi : [t_0, \infty)_T \to \mathbb{R} \) such that

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{a(\tau(s))}{\delta(s)\tau^\Delta(s)} G^2(\tau(s)) \Delta s < \infty, \tag{4.3}
\]

\[
\int_{t_0}^{\infty} \frac{\tau(t)}{\delta(s)} \tau^\Delta(s) \left( \frac{\Psi^\Delta(s)}{\delta(s)} \right)^2 \Delta s = \infty, \tag{4.4}
\]
\[
\limsup_{t \to \infty} \frac{1}{H(t,T)} \int_t^{\infty} \left[ LH(t,s)\varphi(s)(1-r(t,s))) - \frac{a(t(s))}{4\varphi(s)\tau^2(s)} G^2(t,s) \right] ds \geq \Psi(T),
\]

where \( T \in [T_0, \infty)_T \), \( G(t,s) = (\delta(s) - \delta(s))p(s)/a(s) \sqrt{H(t,s) - \delta(s)h(t,s)} \), \( G(t,s) = \max\{0,G(t,s)\} \), \( \Psi_x(t) = \max\{0,\Psi(t)\} \). Then Equation (1.1) is oscillatory on \([t_0, \infty)_T \).

**Proof** Assume that (1.1) has a nonoscillatory solution \( x(t) \) on \([t_0, \infty)_T \). Without loss generality we may assume that there exists a \( t_1 \in [t_0, \infty)_T \), such that \( x(t) > 0 \) and \( x(\tau(t)) > 0 \) for all \( t \in [t_0, \infty)_T \). By the definition of \( x(t) \), it follows

\[
x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) - r(t)z(\tau(t)) \geq (1 - r(t))z(t), \quad t \in [t_1, \infty)_T.
\]

Since it satisfies \( \lim_{t \to \infty} \tau(t) = \infty \), there exists \( T_0 \in [t_0, \infty)_T \) such that \( \tau(t) \geq t_1 \) for all \( t \in [T_0, \infty)_T \). Then if it satisfies \( t \in [T_0, \infty)_T \), we have

\[
x(\tau(t)) \geq (1 - r(\tau(t)))z(\tau(t)).
\]

By Lemma 3.5 and (H3), we obtain that

\[
z \circ \tau^\sigma \geq z \circ \tau, \quad az^\lambda \geq a^\sigma z^\lambda^\sigma
\]

on \([T_0, \infty)_T \) (where \( (z^\lambda)^\sigma \) is short hand for \( z^{\lambda \sigma} \)), and

\[
z^\lambda \circ \tau \geq \frac{az^\lambda}{(a \circ \tau)}
\]

holds. In Lemma 3.2, let \( v = \tau, w = z \), and \( T \) is unbounded above by (H1), so \( T^T = T \), and \( \bar{T} = v(T) = \tau(T) = T \), by (H1), using Lemma 3.2, we get \((z \circ \tau)^\lambda = (z^\lambda \circ \tau)^\lambda\).

By the first inequality in (4.7), we obtain that

\[
\frac{(z \circ \tau)^\lambda}{(z \circ \tau)} \geq \frac{(z^\lambda \circ \tau)^\lambda}{(z \circ \tau^\sigma)}
\]

holds on \([T_0, \infty)_T \). Now we define the function \( W \) by

\[
W = \delta az^\lambda / (z \circ \tau).
\]

Then we have \( W > 0 \) on \([T_0, \infty)_T \), and

\[
W^{(1)}(z \circ \tau) \leq \delta(az^\lambda)^\lambda / (z \circ \tau) + a^\sigma z^\lambda^\sigma \left((z \circ \tau)^\lambda - \delta(z \circ \tau)^\lambda\right) / \left((z \circ \tau)(z \circ \tau^\sigma)\right)
\]

\[
\leq \frac{-Lq\delta(x \circ \tau)}{(z \circ \tau)} - p\delta z^\lambda / (z \circ \tau) + a^\sigma z^\lambda^\sigma \left((z \circ \tau)^\lambda - \delta(z \circ \tau)^\lambda\right) / \left((z \circ \tau)(z \circ \tau^\sigma)\right)
\]

\[
\leq \frac{-Lq\delta(1 - r \circ \tau)}{(z \circ \tau)} - p\delta z^\lambda / (z \circ \tau) + a^\sigma z^\lambda^\sigma \left((z \circ \tau)^\lambda - \delta(z \circ \tau)^\lambda\right) / \left((z \circ \tau)(z \circ \tau^\sigma)\right)
\]

\[
\leq \frac{-Lq\delta(1 - r \circ \tau)}{(z \circ \tau)} - (p/\delta)(a \circ \tau)^\sigma W^\sigma + (\delta^\lambda / \delta^\sigma) W^{(1)} - \left(\delta a^\sigma z^\lambda^\sigma (z \circ \tau)^\lambda\right) / \left((z \circ \tau)(z \circ \tau^\sigma)\right)
\]

\[
\leq \frac{-Lq\delta(1 - r \circ \tau) + \delta / \delta^\sigma W^{(1)} - \left(\delta a^\sigma z^\lambda^\sigma (z \circ \tau)^\lambda\right) / (z \circ \tau^\sigma)^2}{2}
\]

\[
\leq \frac{-Lq\delta(1 - r \circ \tau) + \delta / \delta^\sigma W^{(1)} - \left(\delta a^\sigma z^\lambda^\sigma (z \circ \tau)^\lambda\right) / (z \circ \tau^\sigma)^2}{2}
\]

\[
\leq \frac{-Lq\delta(1 - r \circ \tau) + \delta / \delta^\sigma W^{(1)} - \left(\delta a^\sigma z^\lambda^\sigma (z \circ \tau)^\lambda\right) / (z \circ \tau^\sigma)^2}{2}
\]

\[
\leq \frac{-Lq\delta(1 - r \circ \tau) + \delta / \delta^\sigma W^{(1)} - \left(\delta a^\sigma z^\lambda^\sigma (z \circ \tau)^\lambda\right) / (z \circ \tau^\sigma)^2}{2}
\]
then we obtain
\[
W^2 \leq -Lq(t)\delta(t)(1-r(\tau(t))) + \left[\frac{\partial \delta(t)}{\partial (\sigma(t))}\right] W(\sigma(t)) - \left[\frac{\partial \delta(t)}{\partial (\tau(\sigma)))}\right] W^2(\sigma(t))
\]
on \left[ T_0, \infty \right) \text{,}
where \( \delta(t) = \delta^2(t) - \delta(t)p(t) / a(t) \). Thus, for every \( t, T \in \left[ T_0, \infty \right) \text{ with } t \geq T \geq T_0, \) by (2.3), we get
\[
\int_T^t L H(t,s) q(s) \delta(s)(1-r(\tau(s))) \Delta s
\]
\[
\leq H(t,T)W(T) - \int_T^t (-H^2(t,s)) W(\sigma(s)) \Delta s + \int_T^t \left[ H(t,s) \delta(s) / \delta(\sigma(s)) \right] W(\sigma(s)) \Delta s
\]
\[
- \int_T^t H(t,s) \left[ \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right] W^2(\sigma(s)) \Delta s
\]
\[
= H(t,T)W(T) + \int_T^t \left[ \delta(s)H(t,s) - \delta(\sigma(s))h(t,s)\sqrt{H(t,s)} / \delta(\sigma(s)) \right] W(\sigma(s)) \Delta s
\]
\[
- \int_T^t H(t,s) \left[ \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right] W^2(\sigma(s)) \Delta s
\]
\[
\leq H(t,T)W(T) + \int_T^t \left[ H(t,s) - \delta(\sigma(s))h(t,s) \sqrt{H(t,s)} - \delta(\sigma(s))h(t,s) \sqrt{H(t,s)} \delta^2(\sigma(s)) \right] W(\sigma(s)) \Delta s
\]
\[
- \int_T^t H(t,s) \left[ \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right] W^2(\sigma(s)) \Delta s
\]
\[
\leq H(t,T)W(T) + \int_T^t \left[ G_s(t,s) / \delta(\sigma(s)) \right] \sqrt{H(t,s)} W(\sigma(s)) \Delta s
\]
\[
- \int_T^t H(t,s) \left[ \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right] W^2(\sigma(s)) \Delta s
\]
\[
, \qquad (4.11)
\]
where \( G_s(t,s) = \delta(s)\sqrt{H(t,s)} - \delta(\sigma(s))h(t,s) = \left[ \delta^2(s) - \delta(\sigma(s))p(s) / a(s) \right] \sqrt{H(t,s)} - \delta(\sigma(s))h(t,s) \), \( G_s(t,s) = \max\{0, G_s(t,s)\} \).

So using Lemma 3.3, let
\[
X = \left[ H(t,s) \left( \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right) \right] W(\sigma(s))
\]
\[
Y = \left( G_s(t,s) / \left( 2\delta(\sigma(s)) \right) \right) \left[ \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right] W^2(\sigma(s))
\]
Using the inequality (3.2), let \( \lambda = 2 \), we have
\[
\left[ G_s(t,s) / \delta(\sigma(s)) \right] \sqrt{H(t,s)} W(\sigma(s)) - H(t,s) \left[ \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right] W^2(\sigma(s))
\]
\[
\leq \left[ G_s(t,s) / \left( 2\delta(\sigma(s)) \right) \right] \left[ \delta(s) \tau^2(s) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right] W^2(\sigma(s))
\]
\[
\leq \left[ G_s(t,s) / \left( 4\delta(\sigma(s)) \tau^2(\sigma(s)) \right) \right] G_s^2(t,s)
\]
\[
, \quad (4.12)
\]
From (4.11) and (4.12), we obtain
\[
\int_T^t \left[ LH(t,s) \delta(s)q(s)(1-r(\tau(s))) - \left( a(\tau(s)) / \left( 4\delta(s) \tau^2(\sigma(s)) \right) \right) \right] G_s^2(t,s) \Delta s \leq H(t,T)W(T)
\]
i.e.,
\[
(H(t,T))^{-1} \int_T^t \left[ LH(t,s) \delta(s)q(s)(1-r(\tau(s))) - \left( a(\tau(s)) / \left( 4\delta(s) \tau^2(\sigma(s)) \right) \right) \right] G_s^2(t,s) \Delta s \leq W(T)
\]
From condition (4.5), we have
\[
\Psi(T) \leq W(T), \quad T \in \left[ T_0, \infty \right) \text{ and }
\]
\[
\limsup_{T \to \infty} \Psi(T) \leq W(T)
\]
By (4.11), we have
\[
(H(t,T))^{-1} \int_T^t LH(t,s) \delta(s)q(s)(1-r(\tau(s))) \Delta s
\]
\[
\leq W(T) + (H(t,T))^{-1} \int_T^t \left[ G_s(t,s) / \delta(\sigma(s)) \right] \sqrt{H(t,s)} W(\sigma(s)) \Delta s
\]
\[-(H(t,T))^{-1} \int_{T}^{t} H(t,s) \left( \delta(s) \tau^\lambda(s) \big/ a(\tau(s)) \delta^2(\sigma(s)) \right) W^2(\sigma(s)) \Delta s,\]

from the above inequality, let \( T = T_0 \), and denote
\[
A(t) = (H(t,T_0))^{-1} \int_{T_0}^{t} G_{t,s} \delta(\sigma(s)) \sqrt{H(t,s)} W(\sigma(s)) \Delta s, \]
\[
B(t) = (H(t,T_0))^{-1} \int_{T_0}^{t} H_{t,s} \left( \delta(s) \tau^\lambda(s) / a(\tau(s)) \delta^2(\sigma(s)) \right) W^2(\sigma(s)) \Delta s, \]

meanwhile noting that (4.13), we obtain
\[
\liminf_{t \to \infty} [B(t) - A(t)] \leq W(T_0) - \limsup_{t \to \infty} (H(t,T_0))^{-1} \int_{T_0}^{t} LH(t,s) \delta(s)q(s)(1 - r(\tau(s))) \Delta s \leq W(T_0) - \Psi(T_0) < \infty. \]

Now we assert that
\[
\int_{T_0}^{\infty} \left( \delta(s) \tau^\lambda(s) / a(\tau(s)) \delta^2(\sigma(s)) \right) W^2(\sigma(s)) \Delta s < \infty \quad (4.14)
\]
holds. Suppose to the contrary that
\[
\int_{T_0}^{\infty} \left( \delta(s) \tau^\lambda(s) / a(\tau(s)) \delta^2(\sigma(s)) \right) W^2(\sigma(s)) \Delta s = \infty, \quad (4.15)
\]
by (4.2), there exists a constant \( \varepsilon > 0 \) such that
\[
\inf_{t \in [T_{0}, \infty)} \left( \inf_{t \in [T_{0}, \infty)} \left( H(t,s) / H(t,T_0) \right) \right) > \varepsilon > 0. \quad (4.16)
\]
From (4.15), there exists a \( T \in [T_{0}, \infty) \), for arbitrary real number \( M > 0 \) such that
\[
\int_{T_0}^{t} \left( \delta(s) \tau^\lambda(s) / a(\tau(s)) \delta^2(\sigma(s)) \right) W^2(\sigma(s)) \Delta s \geq M / \varepsilon, \quad (4.17)
\]
for \( t \in [T, \infty) \). By (2.3), we have
\[
B(t) = (H(t,T_0))^{-1} \int_{T_0}^{t} H(t,s) \left[ \int_{T_0}^{s} \left( \delta(u) \tau^\lambda(u) / a(\tau(u)) \delta^2(\sigma(u)) \right) W^2(\sigma(u)) \Delta u \right] \Delta s
\]
\[
= (H(t,T_0))^{-1} \int_{T_0}^{t} \left[ -H^\lambda(t,s) \left[ \int_{T_0}^{\sigma(s)} \left( \delta(u) \tau^\lambda(u) / a(\tau(u)) \delta^2(\sigma(u)) \right) W^2(\sigma(u)) \Delta u \right] \right] \Delta s
\]
\[
\geq (H(t,T_0))^{-1} \int_{T_0}^{t} \left[ -H^\lambda(t,s) \left[ \int_{T_0}^{\sigma(s)} \left( \delta(u) \tau^\lambda(u) / a(\tau(u)) \delta^2(\sigma(u)) \right) W^2(\sigma(u)) \Delta u \right] \right] \Delta s
\]
From (4.16), there exists a \( t_{2} \in [T_{0}, \infty) \) such that \( H(t,T) / H(t,T_0) \geq \varepsilon \) for \( t \in [t_{2}, \infty) \). So \( B(t) \geq M \).
Since \( M \) is arbitrary, we have
\[
\lim_{t \to \infty} B(t) = \infty, \quad (4.17)
\]
Selecting a sequence \( \{T_{n} \}_{n=1}^{\infty} : T_{n} \in [T_{0}, \infty) \) with \( \lim_{n \to \infty} T_{n} = \infty \) satisfying
\[
\lim_{n \to \infty} [B(T_{n}) - A(T_{n})] = \liminf_{n \to \infty} [B(t) - A(t)] < \infty,
\]
then there exists a constant \( M_0 > 0 \) such that
\[
B(T_{n}) - A(T_{n}) \leq M_0 \quad (4.18)
\]
for sufficiently large positive integer \( n \). From (4.17), we can easily see
\[
\lim_{n \to \infty} B(T_{n}) = \infty, \quad (4.19)
\]
(4.18) implies that
\[
\lim_{n \to \infty} A(T_n) = \infty, \quad (4.20)
\]
From (4.18) and (4.19), we have
\[
A(T_n) / B(T_n) = -1 \geq -M_0 / B(T_n) > -M_0 / (2M_0) = -1 / 2,
\]
i.e.,
\[
A(T_n) / B(T_n) > 1 / 2
\]
for sufficiently large positive integer \( n \), which together with (4.20) implies
\[
\lim_{n \to \infty} A^2(T_n) / B(T_n) = \lim_{n \to \infty} A(T_n)(A(T_n) / B(T_n)) = \infty. \quad (4.21)
\]
On the other hand, by Lemma 3.4, we obtain
\[
\begin{align*}
A(T_n) &= \left( H(T_n, T_0) \right)^{-1} \int_{T_0}^{T_n} \left[ G_1(T_n, s) / \delta(\sigma(s)) \right] \sqrt{H(T_n, s) W(\sigma(s))} \Delta s \\
&= \int_{T_0}^{T_n} \left[ H(T_n, s) \delta(\sigma(s)) \tau^A(s) / H(T_n, T_0) \right]^{1/2} \left[ W(\sigma(s)) / \left( \sqrt{a(\tau(s))} \delta(\sigma(s)) \right) \right]^{1/2} \times \\
&\quad \times \left[ \sqrt{a(\tau(s))} G_1(T_n, s) / H(T_n, T_0) \right] \sqrt{H(T_n, s) H(T_n, s) \delta(\sigma(s)) \tau^A(s) / H(T_n, T_0)}^{1/2} \Delta s \\
&\leq \left\{ \int_{T_0}^{T_n} \left[ H(T_n, s) \delta(\sigma(s)) \tau^A(s) / H(T_n, T_0) \right] W(\sigma(s)) / \left( \sqrt{a(\tau(s))} \delta(\sigma(s)) \right) \Delta s \right\}^{1/2} \times \\
&\quad \times \left\{ \int_{T_0}^{T_n} \left[ a(\tau(s)) G_1^2(T_n, s) / H^2(T_n, T_0) \right] H(T_n, s) H(T_n, s) \delta(\sigma(s)) \tau^A(s) / H(T_n, T_0) \right\}^{1/2} \Delta s \\
&= \left[ B(T_n) \right]^{1/2} \left\{ \left( H(T_n, T_0) \right)^{-1} \int_{T_0}^{T_n} \left[ a(\tau(s)) G_1^2(T_n, s) \right] \delta(\sigma(s)) \tau^A(s) \right\}^{-1} \Delta s.
\end{align*}
\]
The above inequality show that
\[
\left[ A(T_n) \right]^2 / B(T_n) \leq \left( H(T_n, T_0) \right)^{-1} \int_{T_0}^{T_n} \left[ a(\tau(s)) G_1^2(T_n, s) \right] \delta(\sigma(s)) \tau^A(s) \right\}^{-1} \Delta s.
\]
Hence, (4.21) implies
\[
\lim_{n \to \infty} \left[ H(T_n, T_0) \right]^{1/2} \int_{T_0}^{T_n} \left[ a(\tau(s)) G_1^2(T_n, s) \right] \delta(\sigma(s)) \tau^A(s) \right\}^{-1} \Delta s = \infty,
\]
which contradicts (4.3). Therefore (4.14) holds. Noting \( \Psi(T) \leq W(T) \) for \( T \in [T_0, \infty) \), by using (4.14), we obtain
\[
\int_{T_0}^{\infty} \left( \delta(\sigma(s)) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right) \Psi^2(\sigma(s)) \Delta s < \int_{T_0}^{\infty} \left( \delta(\sigma(s)) / \left( a(\tau(s)) \delta^2(\sigma(s)) \right) \right) W^2(\sigma(s)) \Delta s < \infty,
\]
which is contradicting with (4.4). This completes the proof.

Remark 4.1 From Theorem 4.1, we can obtain different conditions for oscillation of all solutions of Equation (1.1) with different choices of \( \delta(t) \) and \( H(t, s) \). For example, \( H(t, s) = (t - s)^n \) or \( H(t, s) = \ln^n ((t + 1) / (s + 1)) \).

Theorem 4.2 Assume that (H1)-(H4), (1.2), (4.1)-(4.2) and (4.4) hold, Where \( H, h, \delta \) and \( \Psi \) are defined in Theorem 4.1. Assume that
\[
\liminf_{t \to \infty} \left( H(t, T_0) \right)^{-1} \int_{T_0}^{t} LH(t, s) \delta(s) q(s) (1 - r(\tau(s))) \Delta s < \infty \quad (4.22)
\]
and
lim inf \(\lim_{s \to \infty} (H(t,T))^{-1} \int_0^s \left[ LH(t,s) \delta(s) q(s)(1 - r(\tau(s))) - \left( a(\tau(s)) / (4 \delta(s) \tau^\Delta(s)) \right) G^2_\tau(t,s) \right] d\tau \geq \Psi(T) \) \quad (4.23)

holds, where \( T \in [T_0, \infty) \), \( G(t,s) = \left( \delta^\Delta(s) - \delta(\tau(s)) \rho(s) / a(s) \right) \sqrt{H(t,s) - \delta(s) h(t,s)} \), \( G(t,s) = \max \{ 0, G(t,s) \} \).

Then Equation (1.1) is oscillatory on \( [T_0, \infty) \).

**Proof** Assume that (1.1) has a nonoscillatory solution \( x(t) \) on \( [T_0, \infty) \). Without loss generality we may assume that there exists a \( t_i \in [T_0, \infty) \), such that \( x(t_i) > 0 \) and \( x(\tau(t_i)) > 0 \) for all \( t \in [t_i, \infty) \). So \( z(t) > 0 \) and there exists a \( T_0 \in [t_i, \infty) \), such that

\[ z(t) > 0 \] for all \( t \in [T_0, \infty) \).

We proceed as in the proof of Theorem 4.1 to obtain (4.11) and (4.12), so that

\[ (H(t,T))^{-1} \int_0^s \left[ LH(t,s) \delta(s) q(s)(1 - r(\tau(s))) - \left( a(\tau(s)) / (4 \delta(s) \tau^\Delta(s)) \right) G^2_\tau(t,s) \right] d\tau \leq W(T) \] Hence, (4.23) implies \( \Psi(T) \leq W(T) \), \( T \in [T_0, \infty) \);

\[ \lim_{s \to \infty} \left( H(t,T) \right)^{-1} \int_0^s LH(t,s) \delta(s) q(s)(1 - r(\tau(s))) - \left( a(\tau(s)) / (4 \delta(s) \tau^\Delta(s)) \right) G^2_\tau(t,s) \right] d\tau \leq \Psi(T) \] \quad (4.24)

From the above inequality and (4.22), we have

\[ \lim_{s \to \infty} \left( H(T_n,t) \right)^{-1} \int_0^s \left( a(\tau(s)) / (4 \delta(s) \tau^\Delta(s)) \right) G^2_\tau(t,s) d\tau < \infty \]

Therefore, there exists a sequence \( \{ T_n \}_{n=1}^\infty : T_n \in [T_0, \infty) \) with \( \lim_{n \to \infty} T_n = \infty \) such that

\[ \lim_{n \to \infty} \left( H(T_n,n) \right)^{-1} \int_0^s \left( a(\tau(s)) / (4 \delta(s) \tau^\Delta(s)) \right) G^2_\tau(t,s) d\tau < \infty \]

Definitions of \( A(t) \) and \( B(t) \) are as in Theorem 4.1, from (4.11), and noting (4.24), we have

\[ \lim sup_{t \to \infty} [B(t) - A(t)] \leq W(T_0) - \lim inf_{t \to \infty} \left( H(t,T_0) \right)^{-1} \int_0^s LH(t,s) \delta(s) q(s)(1 - r(\tau(s))) d\tau \leq W(T_0) - \Psi(T_0) < \infty \]

For the above sequence \( \{ T_n \}_{n=1}^\infty \),

\[ \lim_{n \to \infty} [B(T_n) - A(T_n)] \leq \lim sup_{t \to \infty} [B(t) - A(t)] < \infty \]

We proceed by reduction to absurdity to obtain (4.14). The rest proof is similar to that of Theorem 4.1 and hence is omitted. This completes the proof.

If (1.2) is not satisfied, i.e., if the condition (1.3) holds, we can obtain the following results.

**Theorem 4.3** Assume that \((H_1)-(H_4), (1.3) \) and (4.1)-(4.5) hold, Where \( H, h, \delta \) and \( \Psi \) are defined in Theorem 4.1. Assume that

\[ \int_0^\infty \left( a(\tau(s)) / (4 \delta(s) \tau^\Delta(s)) \right) G^2_\tau(t,s) d\tau = \infty \] \quad (4.25)

holds. Then every solution \( x(t) \) of Equation (1.1) is either oscillatory or converges to zero on \( [T_0, \infty) \).

**Proof** As the proof of Theorem 4.1, assume that (1.1) has a nonoscillatory solution \( x(t) \) on \( [T_0, \infty) \).

Without loss generality we may assume that there exists \( t_i \in [T_0, \infty) \), such that \( x(t_i) > 0 \) and \( x(\tau(t_i)) > 0 \) for all \( t \in [t_i, \infty) \). So \( z(t) > 0 \) and there exists \( t_2 \in [t_1, \infty) \), such that
\(x(t) \geq (1 - r(t))z(t)\) for \(t \in [t,\infty)_T\). In the proof of Lemma 3.5, we find that \(z^\Delta(t)\) is either eventually positive or eventually negative. Thus, we shall distinguish the following two cases:

(I) \(z^\Delta(t) > 0\) for \(t \in [t,\infty)_T\); (II) \(z^\Delta(t) < 0\) for \(t \in [t,\infty)_T\).

Case (I). When \(z^\Delta(t)\) is an eventually positive, the proof is similar to that of the proof of Theorem 4.1, we can obtain Equation (1.1) is oscillatory.

Case (II). When \(z^\Delta(t)\) is an eventually negative, \(z(t)\) is decreasing and \(\lim_{t \to \infty} z(t) = b \geq 0\) exists.

Therefore, there exists \(T_0 \in [t,\infty)_T\), such that

\[
\int_{t_0}^t z^\Delta(t)dt < -b\int_{t_0}^t e^{-p/a}(t,\sigma(s))q(s)(1-r(\tau(s)))\Delta s
\]

for all \(t \in [T_0,\infty)_T\), and thus

\[
\int_{t_0}^t z^\Delta(t)dt < -b\int_{t_0}^t e^{-p/a}(t,\sigma(s))q(s)(1-r(\tau(s)))\Delta s
\]

for all \(t \in [T_0,\infty)_T\). Assuming \(b > 0\) and using (4.25) in (4.28), we can get \(\lim_{t \to \infty} z(t) = -\infty\), and this is a contradiction to the fact that \(z(t) > 0\) for \(t \in [t,\infty)_T\). Thus \(b = 0\), i.e. \(\lim_{t \to \infty} z(t) = 0\). Then, it follows from \((1-r(t))z(t) \leq x(\Delta t) \leq z(t)\) that \(\lim_{t \to \infty} x(t) = 0\) holds. This completes the proof.

Using the same method as in the proofs of Theorem 4.2 and 4.3, we can easily obtain the following results.

**Theorem 4.4** Assume that (H1)-(H4), (1.3), (4.1)-(4.2), (4.4), (4.22)-(4.23) and (4.25) hold, where \(H, h, \delta\) and \(\Psi\) are defined in Theorem 4.2. Then every solution \(x(t)\) of Equation (1.1) is either oscillatory or converges to zero on \([t,\infty)_T\).

**Remark 4.2** The theorems in this paper are new even for the cases of \(T = \mathbb{R}\) and \(T = \mathbb{Z}\).

**Example** Consider the second-order delay 2-difference equation with damping

\[
\left[ t^2 z^\Delta(t) \right]^2 + t^4 z^\Delta(t) + t^3 x(t/2) = 0, \quad t \geq t_0 = 2,
\]

where \(z(t) = x(t) + t^2 x(t/2)\). Here \(a(t) = t^2, r(t) = 1/2, p(t) = t^4, q(t) = t^3, f(u) = u, \tau(t) = t/2\). Then \(T = \mathbb{Z}\) is unbounded above, \(\sigma(t) = 2t\) and \(\mu(t) = t\). Conditions (H1) and (H3) are clearly satisfied, (H4) holds with \(L = 1\), and (H2) is satisfied as

\[
1 + \mu(t)\left( -p(t)/a(t) \right) = 1 - t^{-1} > 0 \quad \text{for all} \quad t \geq 2.
\]

Next, by [27, Lemma 2] and (H2), we obtain

\[
e^{-p/a}(t,2) \geq 1 - \int_2^t (p(s)/a(s))\Delta s = 1 - \int_2^t s^{-2} \Delta s = 2t^{-1} \quad \text{for all} \quad t \geq 2,
\]

so

\[
\int_2^t \left( e^{-p/a}(s,2)/a(s) \right) \Delta s \geq \int_2^\infty \left( 2s^{-1}/s^2 \right) \Delta s = \int_2^\infty 2s \Delta s \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty.
\]
Hence (1.2) is satisfied. Now let \( H(t,s) = (t-s)^2 \), then
\[
H^3(t,s) = \left( (t-2s)^2 - (t-s)^2 \right) / s = -(2t-3s) < 0 \quad \text{for all} \quad t > s \geq t_0 = 2.
\]

Since
\[-H^3(t,s) = 2t-3s = \left[ (2t-3s) / (t-s) \right] \sqrt{(t-s)^2} = \left[ (2t-3s) / (t-s) \right] \sqrt{H(t,s)},
\]
let \( h(t,s) = (2t-3s) / (t-s) \), then condition (4.1) holds. We have
\[
0 < \inf_{s \in [t_0, \infty)} \liminf_{t \to \infty} \left[ H(t,s) / H(t_0,s) \right] = \inf_{s \in [t_0, \infty)} \liminf_{t \to \infty} \left( (t-s)^2 / (t-t_0)^2 \right) = 1 < \infty, \quad \text{for all} \quad t_0 \in [t_0, \infty),
\]
so condition (4.2) holds. Let \( \delta(t) = t \) as \( t \geq 2 \), then \( \delta^3(t) = 1 \) for all \( t \in [t_0, \infty) \), and
\[
G(t,s) = \left( \delta^3(s) - \delta(s) p(s) / a(s) \right) \sqrt{H(t,s) - \delta(s) h(t,s)}
\]
\[
= \left( 1 - s^{-1} \right) \sqrt{H(t,s) - s(2t-3s) / \sqrt{H(t,s) < \sqrt{H(t,s)}}}
\]
for all \( t > s \geq t_0 = 2 \). Hence
\[
\int_{t_0}^T \left( a(\delta(s)) / (\delta(s) \delta^3(s)) \right) G^2(t,s) \Delta s < \int_{t_0}^T \left( \delta^3(s) / (\delta(s) \delta^3(s)) \right) \Delta s = 8 \int_{t_0}^T \left( (t-s)^2 / s \right) \Delta s
\]
\[
= 8 \left[ t^2 s^{-2} / (2^{-2} - 1) - 2t s^{-1} / (2^{-1} - 1) + \ln s / \ln 2 \right] = 8 \left[ (8 / 3 + \ln t / \ln 2) - 8 \left[ 4t^2 / (3T_0^2) + 4t / T_0 + \ln T_0 / \ln 2 \right] \right].
\]
We get
\[
\limsup_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right)
\]
\[
\leq \limsup_{t \to \infty} \left[ 8 \left[ (8 / 3 + \ln t / \ln 2) - 8 \left[ 4t^2 / (3T_0^2) + 4t / T_0 + \ln T_0 / \ln 2 \right] \right] \right] / (t - T_0)^2 = 32 / (3T_0^2) < \infty, \quad (4.30)
\]
thus condition (4.3) holds. Let \( \Psi(t) = 1 / (2t) \), then
\[
\int_{t_0}^T \left( \delta(s) \delta^3(s) / a(\delta(s)) \right) \left( \Psi(\sigma(s)) / \delta(\sigma(s)) \right) \Delta s = \int_{t_0}^T \left( 1 / (8s^2) \right) \Delta s = \ln s / (512 \ln 2)^{t_0} = \infty,
\]
i.e., condition (4.4) holds. Since
\[
\int_{t_0}^T \left( LH(t,s) \delta(s) q(s) (1 - r(\tau(s))) \right) \Delta s = \int_{t_0}^T \left( (t-s)^2 s / (2s^3) \right) \Delta s = 2^{-1} \int_{t_0}^T \left[ t^2 s^{-2} - 2t s + 1 \right] \Delta s
\]
\[
= 2^{-1} \left[ -2t^2 / s - 2t \ln s / \ln 2 + s \right] = 2^{-1} \left[ -2t / s + \ln 2 + t \right] - 2^{-1} \left[ -2t^2 / T - 2t \ln T / \ln 2 + T \right],
\]
then
\[
\limsup_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right)
\]
\[
\limsup_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right)
\]
\[
\leq 8 / (3T_0^2).
\]
Moreover, (4.30) implies
\[
\limsup_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right)
\]
Thus, when \( T \) is enough large, we have
\[
\limsup_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right) \liminf_{t \to \infty} \left( (t \ln t / \ln 2) / (t \ln T_0 / \ln 2) \right)
\]
\[
\geq T^{-1} - 8 / (3T_0^2) \geq \Psi(T),
\]
so (4.5) is satisfied. By Theorem 4.1, Equation (4.29) is oscillatory on \([t_0, \infty)\). Similarly, conditions (4.22) and (4.23) are satisfied as well. By Theorem 4.2, we can also obtain that Equation (4.29) is oscillatory. But the other known results cannot be applied in Equation (4.29).

5. Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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