Another bijection for 021-avoiding ascent sequences

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Abstract

Chen and collaborators give a recursively defined bijection from 021-avoiding ascent sequences to 021-avoiding (aka 132-avoiding) permutations. Here we give an algorithmic bijection from 021-avoiding ascent sequences to Dyck paths. Our bijection does not appear to be closely related to the Chen bijection but, like the Chen bijection, it preserves several interesting statistics.

1 Introduction

Several recent papers treat pattern avoidance in ascent sequences [1, 2, 3, 4]. A striking result [2] is that 021-avoiding ascent sequences are counted by the Catalan numbers. William Chen and his collaborators [1] give an elegant recursively defined bijection from 021-avoiding ascent sequences to 021-avoiding (aka 132-avoiding) permutations. In this paper we give a bijection from 021-avoiding ascent sequences to Dyck paths that builds up the Dyck path iteratively. In Section 2 we recall the relevant definitions for ascent sequences and terminology for Dyck paths. Section 3 presents the bijection and Section 4 its inverse. Lastly, Section 5 mentions some statistics preserved by the bijection.

2 Ascent sequences and Dyck paths

An ascent in a sequence of integers is a pair of consecutive entries with the first smaller than the second. An ascent sequence is a sequence \((u_1, u_2, \ldots, u_n)\) of nonnegative integers such that \(u_0 = 0\) and \(u_i \leq 1 + \#\text{ ascents in } (u_1, u_2, \ldots, u_{i-1})\) for \(i \geq 2\). Due to the initial 0, it is clear that 021-avoiding ascent sequences can be characterized as ascent sequences in which the nonzero entries are weakly increasing.
A Dyck path is a lattice path of upsteps $U = (1, 1)$ and downsteps $D = (1, -1)$, the same number of each, that stays weakly above the horizontal line, called ground level, that joins its initial and terminal points (vertices). We have the usual notions of size (number of upsteps), ascent (maximal sequence of contiguous upsteps), descent, peak ($UD$), peak vertex (the vertex between the $U$ and $D$), valley ($DU$), and valley vertex. An ascent is short if it has length 1, otherwise long.

The height of a vertex in a Dyck path is its vertical height above ground level. A return downstep is one that returns the path to ground level. An elevated Dyck path is one with exactly one return (necessarily at the end). The degree of elevation of a Dyck path is the height of its lowest valley vertex (undefined for pyramid Dyck paths—$U^n D^n$—which have no valleys). Thus the degree of elevation is 0 precisely for non-elevated Dyck paths. Upsteps and downsteps come in matching pairs: travel due east from an upstep to the first downstep encountered. More precisely, $D_0$ is the matching downstep for upstep $U_0$ if $D_0$ terminates the shortest Dyck subpath that starts with $U_0$. It is convenient to define a key downstep in a Dyck path to be a downstep on the terminal descent whose matching upstep is the middle $U$ of a $DUU$.

### 3 The bijection

Suppose $(u_i)_{i=1}^n$ is a 021-avoiding ascent sequence. Start with $UD$ as the current path. For $i = 2, 3, \ldots, n$ in turn, successively increment by 1 the size of the current path $P$ as follows.

**Case 1.** If $u_i = 0$, insert $UD$ at the last peak vertex of $P$, so that the resulting path has a long last ascent. This is the only case that results in a long last ascent.

**Henceforth, suppose $u_i \neq 0$.**

**Case 2.** If $u_i = u_{i-1}$, elevate $P$ (prepend $U$ and append $D$).
Let \( a \) and \( m \) denote respectively the number of ascents and the maximum entry in \((u_1, u_2, \ldots, u_{i-1})\).

**Case 3.** If \( u_i = a + 1 \), append \( UD \) to \( P \).

**Case 4.** If we’re not in one of the three previous cases, then \( u_i \in A_i \) (list of allowable \( u_i \)'s), where \( A_i := (m, m+1, \ldots, a) \) if \( u_{i-1} = 0 \), and \( := (m+1, \ldots, a) \) if \( u_{i-1} > 0 \) (which implies that \( u_{i-1} = m \), and so \( u_i = m \) was covered in Case 2). This assertion about \( A_i \) holds because the nonzero entries of \((u_1, u_2, \ldots, u_i)\) are weakly increasing and \( u_i \) is bounded above by \( a + 1 \). Let \( j \) denote the position of \( u_i \) in the list \( A_i \), and \( e \geq 0 \) the degree of elevation of \( P \); \( e \) is defined because \( P \) will not be a pyramid path. Insert \( UD \) at the top vertex of the \( j \)th key downstep \( D_j \) of \( P \) and transfer \( e \) upsteps from the start of the path to the ascent containing the matching upstep of \( D_j \).

The specified insertion is always possible (and reversible) because \( |A_i| \) is always equal to the number of key downsteps in \( P \), as can be verified by a straightforward induction considering the various cases. Note that the resulting path has a short last ascent, is not elevated, and does not end with \( UD \).

As an example, the 021-avoiding ascent sequence 01012203 produces the following sequence of Dyck paths (key downsteps encountered en route are in blue).

The inverse mapping is given in the next section.
4 The inverse bijection

To reverse the mapping proceed as follows. Start with a “current path” taken as the given Dyck path. Each step of the algorithm produces an entry of the ascent sequence and modifies the current path $P$ to a one-size-smaller path according to which of the following four mutually exclusive cases $P$ lies in (which match the four cases in Section 3). Proceed until $P = UD$ and then set $u_1 = 0$.

In all cases, $i$ denotes the size of the current path $P$, and $Q$ denotes the new one-size-smaller path that replaces $P$. An example accompanies the description in each case.

**Case 1.** The last ascent of $P$ is long. Set $u_i = 0$. Then delete the last peak.

**Case 2.** The last ascent of $P$ is short and $P$ is elevated. Set $u_i = u_{i-1}$. (Thus the actual determination of $u_i$ is delayed to a later step in the algorithm.) Then lower the path, that is, delete the first and last steps.

**Case 3.** The last ascent of $P$ is short and $P$ ends with $UD$. (This case is distinct from Case 2 because $P$ has size $\geq 2$.) Set $u(i) =$ number of valleys in $P$. Then delete the last peak (= last two steps).

**Case 4.** The last ascent of $P$ is short and $P$ is neither elevated nor ends $UD$. Here we use both the current path $P$ and its successor path $Q$ to determine $u_i$. Mark the second downstep on the terminal descent of $P$ and then delete the last peak. The marked
downstep remains. We need to ensure that it is a key downstep in \( Q \). To do so, locate its matching upstep, then transfer all upsteps \textit{preceding} this matching upstep in its ascent to the start of the path to get \( Q \). Note that the marked downstep is now indeed a key downstep in \( Q \). Set \( u_i = \text{number of valleys in } P \) minus the position of the marked downstep among all key downsteps of \( Q \) when scanned from right to left. This step works because, under the bijection, \# ascents in the sequence equals \# valleys in the path (proof by induction), and, as noted in Case 4 of the bijection, the allowable range \(|A_i|\) is always equal to the number of key downsteps in \( P \).

\[
u_i = 3 - 2 = 1
\]

5 Equidistributions

The bijection of Sections 3 and 4 preserves several statistics as tabulated below. In the case of the all-zero ascent sequence of length \( n \) (which corresponds to the pyramid path \( U^n D^n \)), a little hiccup arises and the number of terminal 0s must be interpreted as \( n - 1 \).

| 021-avoiding ascent sequence | \( \leftrightarrow \) | Dyck path |
|------------------------------|-----------------|------------|
| \# initial 0s                | \( \leftrightarrow \) | length first descent |
| \# terminal 0s               | \( \leftrightarrow \) | length last ascent \(-1\) |
| \# ascents                   | \( \leftrightarrow \) | \# \( DU \)s (valleys) |
| \# descents                  | \( \leftrightarrow \) | \# \( DUU \)s |
| \# entries immediately preceding last nonzero entry and equal to it | \( \leftrightarrow \) | degree of elevation |
References

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