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Introduction

Close connection between negative curvature and hyperbolic dynamics is a classical principle. For example, geodesic flow on a compact negatively curved manifold is an important example of an Anosov flow, and symbolic dynamics for Smale’s Axiom A diffeomorphisms \cite{Bow70} is closely related to symbolic dynamics for geodesic flow \cite{Mor21} and to automatic structure on a Gromov hyperbolic group \cite{Can84, Gro87, CP93}.

The aim of this paper is to understand this connection better and to generalize it by combining M. Gromov’s notion of a hyperbolic graph, A. Haefliger’s notion of a compactly generated pseudogroup, and D. Ruelle’s notion of a Smale space.

We define hyperbolic groupoids (or pseudogroups) in some sense generalizing the notion of a Gromov hyperbolic group. Examples of hyperbolic groupoids include actions of Gromov hyperbolic groups on their boundaries, pseudogroups generated by expanding self-coverings (e.g., by restriction of a hyperbolic complex rational function onto its Julia set), natural pseudogroups acting on leaves of stable (or unstable) foliation of an Anosov diffeomorphism, e.t.c.

A new aspect of the theory of hyperbolic groupoids, not present in the case of Gromov hyperbolic groups, is a duality theory, interchanging large-scale and infinitesimal properties of the groupoids. Namely, for every hyperbolic groupoid $\mathcal{G}$ there is a naturally defined dual groupoid $\mathcal{G}^\top$ acting on the Gromov boundary of a Cayley graph of $\mathcal{G}$, which is also hyperbolic and such that $(\mathcal{G}^\top)^\top$ is equivalent to $\mathcal{G}$.

In this way we represent the space on which a hyperbolic groupoid acts as the boundary of a Cayley graph of the dual groupoid. This opens way to new applications of the methods of the theory of Gromov hyperbolic graphs. See, for example, the paper \cite{Nek11}, where the Patterson-Sullivan construction and the visual metric on the boundary of a hyperbolic graph is used to study metrics and measures on spaces on which hyperbolic groupoids act.

A pair of mutually dual hyperbolic groupoids can be put together into one Smale quasi-flow, which is naturally interpreted as the geodesic (quasi-)flow of each of the groupoids. The groupoids are generated by the holonomies of the natural local product structure of the Smale quasi-flow. The duality between hyperbolic groupoids can be interpreted in this sense as duality between past and future behavior of a hyperbolic dynamical system.

A pseudogroup $\mathfrak{G}$ acting on a space $X$ (or the associated groupoid of its
germs $\mathfrak{G}$) is said to be hyperbolic if it is generated by a finite set $S$ of compactly supported contractions such that the Cayley graphs $\mathfrak{G}(x, S)$ of $\tilde{\mathfrak{G}}$ (based at $x \in X$) are Gromov hyperbolic, and the paths going against the orientation (i.e., in the direction inverse to the direction of the generators) are quasi-geodesics all converging to one point $\omega_x$ on the boundary of the Cayley graph. For more details and more precise definition see Definition 0.0.2 in this introduction.

Since generators $S \in S$ are contractions, if two elements $g, h \in \mathfrak{G}$ are such that their targets (ranges) are close enough to each other, then for every product $S_1 S_2 \cdots S_n$ of the generators, the elements $S_1 S_2 \cdots S_n \cdot g$ and $S_1 S_2 \cdots S_n \cdot h$ also have close targets, and we get a map $R^g_h : S_1 S_2 \cdots S_n \cdot g \mapsto S_1 S_2 \cdots S_n \cdot h$ between the positive cones of the corresponding Cayley graphs. Such maps induce homeomorphisms between subsets of the boundaries of the Cayley graphs, which generate the dual pseudogroup $\mathfrak{G}^\dagger$ (and the dual groupoid $\mathfrak{G}^\dagger$ equal to the groupoid of germs of $\mathfrak{G}^\dagger$). This definition of the dual groupoid is described in Section 4.6.

One of the main results of the paper is the Duality Theorem 4.5.7: the dual pseudogroup $\mathfrak{G}^\dagger$ is also hyperbolic and $(\mathfrak{G}^\dagger)^\dagger$ is equivalent to $\mathfrak{G}$.

Another result of the paper is an axiomatic description of the “geodesic quasi-flows” of hyperbolic groupoids. This description is a natural generalization of the D. Ruelle’s notion of a Smale space. We call it Smale quasi-flow.

In particular, we show that the natural projections of a Smale space (e.g., of an Anosov diffeomorphism, or of restriction of an Axiom A diffeomorphism onto the non-wandering set) onto the stable and unstable directions of its local product structure is a pair of mutually dual hyperbolic groupoids. We call them Ruelle groupoids of the Smale space, since the convolution algebras of these groupoids are the Ruelle algebras studied in [Put06, Put96], see also the paper of D. Ruelle [Rue88] where similar algebras and groupoids are considered.

Another class of examples of hyperbolic groupoids are the groupoids of the action of Gromov-hyperbolic groups $G$ on their boundary $\partial G$. Each such a groupoid is self-dual. The corresponding geodesic quasi-flow is equivalent to the diagonal action of $G$ on $\partial G \times \partial G$ minus the diagonal. (For the notion of a geodesic flow associated with a Gromov hyperbolic group see [Gro87, Theorem 8.3.C].)

If $f : \mathcal{M} \to \mathcal{M}$ is an expanding finite self-covering of a compact space, then the groupoid generated by the germs of $f$ is also hyperbolic. Its dual, when $\mathcal{M}$ is connected, is generated by an action of a group (called the iterated monodromy group of $f$) on a full one-sided Bernoulli shift-space and by the shift. The action of the iterated monodromy group has a nice symbolic presentation by finite automata, which makes computations in the dual groupoid very efficient, which in turn gives an efficient description of symbolic dynamics of $f$. This particular case of hyperbolic duality is the main topic of the monograph [Nek05]. See also [BN06, Nek08a, Nek12] for other applications of iterated monodromy groups, and [Nek06] for a slight generalization of iterated monodromy groups, which is still an instance of hyperbolic duality.

A $K$-theoretic duality of the convolution $C^*$-algebras of the Ruelle groupoids
of Smale spaces was proved by J. Kaminker, I. Putnam, and M. Whittaker in their recent paper [KPW10] (see also [KP97]). H. Emerson [Eme03] has proved Poincaré duality for the convolution algebras of the groupoid of the action of a Gromov-hyperbolic group on its boundary. It would be interesting to generalize both results to all pairs of mutually dual hyperbolic groupoids.

A particular case of hyperbolic duality was defined by W. Thurston in [Thu89] in relation with self-similar tilings and numeration systems, see also [Gel97] for a natural generalization.

Relation between Gromov hyperbolicity and hyperbolic dynamics was explored in [Nek03, Pil05, HP09].

Main definitions and results

A **pseudogroup** \( \tilde{\mathcal{G}} \) of local homeomorphisms of a space \( X \) is a set of homeomorphisms between open subsets of \( X \) that is closed under taking inverses, compositions, restrictions onto open subsets, and unions (i.e., if a homeomorphism between open subsets of \( X \) can be covered by elements of the pseudogroup, then it also belongs to the pseudogroup). A **germ** of \( \tilde{\mathcal{G}} \) is an equivalence class of a pair \((F, x)\), where \( F \in \tilde{\mathcal{G}} \) and \( x \) is a point of the domain of \( F \). We identify two germs \((F_1, x)\) and \((F_2, x)\) if there exists a neighborhood \( U \) of \( x \) such that \( F_1|_U = F_2|_U \). The set of germs of elements of a pseudogroup \( \tilde{\mathcal{G}} \) has a natural topology and is a groupoid (i.e., a small category of isomorphisms) with respect to the obvious composition. A pseudogroup is uniquely determined by the groupoid of germs (as a topological groupoid). We will denote by \( \tilde{\mathcal{G}} \) the pseudogroup associated with a groupoid \( \mathcal{G} \) (so that \( \mathcal{G} \) is the groupoid of germs of the pseudogroup \( \tilde{\mathcal{G}} \)).

It is more natural to consider groupoids and pseudogroups up to an equivalence weaker than isomorphism (see [Hae80] and [MRW87]). If \( X_0 \) is an open subset of the space \( X \) on which a pseudogroup \( \tilde{\mathcal{G}} \) acts, then we denote by \( \tilde{\mathcal{G}}|_{X_0} \) the set of elements of \( \tilde{\mathcal{G}} \) such that their range and domain are subsets of \( X_0 \).

**Definition 0.0.1.** Pseudogroups \( \tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2 \) are **equivalent** if there exists a pseudogroup \( \tilde{\mathcal{H}} \) and \( \tilde{\mathcal{H}} \)-transversals \( X_1 \) and \( X_2 \) such that restrictions of \( \tilde{\mathcal{H}} \) onto \( X_1 \) and \( X_2 \) are isomorphic to \( \tilde{\mathcal{G}}_1 \) and \( \tilde{\mathcal{G}}_2 \), respectively.

We say that a subset \( X_0 \) of the space on which a pseudogroup \( \tilde{\mathcal{G}} \) acts is a **\( \tilde{\mathcal{G}} \)-transversal** if it intersects each \( \mathcal{G} \)-orbit.

Pseudogroups and their groupoids of germs appear naturally in the study of **local symmetries** of structures, especially when the structure does not have enough globally defined symmetries, so that group theory does not describe symmetric nature of the structure adequately. Structures of this type are, for example, aperiodic tilings (like the Penrose tiling), Julia sets, attractors of iterated function systems, non-wandering sets of Smale’s Axiom A diffeomorphisms, etc.

In many interesting cases local symmetries include **self-similarities**, i.e., symmetries identifying parts of the object on different scales. Definition of a
hyperbolic groupoid combines the idea of “multiscale” symmetries of a self-similar structure with the Gromov’s idea of a negatively curved Cayley graph. In order to define Cayley graphs of groupoids, we use the following definition due to A. Haefliger [Hae02].

We say that a pseudogroup \( \widetilde{G} \) is compactly generated if there exists an open relatively compact (i.e., having compact closure) \( \widetilde{G} \)-transversal \( X_0 \) and a finite set \( S \) of elements of \( \widetilde{G} \) with range and domain in \( X_0 \) such that

1. every germ \( (F, x) \in \mathcal{G} \) such that \( \{x, F(x)\} \subset X_0 \) can be represented as a germ of a product of elements of \( S \) and their inverses;
2. for every element \( F \in S \) there exists an element \( \hat{F} \) of \( \widetilde{G} \) such that \( F \) is a restriction of \( \hat{F} \), and closures of the domain and range of \( F \) are compact and contained in the domain and range of \( \hat{F} \), respectively.

Suppose that \( X_0 \) and \( S \) are as above. We say then that \( S \) is a compact generating set of \( \widetilde{G}|_{X_0} \). The Cayley graph \( \mathcal{G}(x, S) \) for \( x \in X_0 \) is the oriented graph with the set of vertices equal to the set of germs \( (F, x) \) in \( X_0 \), in which there is an arrow starting at \( (F_1, x) \) and ending in \( (F_2, x) \) if and only if there exists \( F \in S \) such that \( (F_2, x) = (FF_1, x) \).

Similarly to the case of Cayley graphs of groups, the Cayley graph \( \mathcal{G}(x, S) \) does not depend (up to a quasi-isometry) on the choice of the generating set \( S \) and the transversal \( X_0 \). It depends, however, on the choice of the basepoint \( x \).

We will use notation \( |x - y| \) for distance between points \( x \) and \( y \) in a metric space.

**Definition 0.0.2.** A pseudogroup \( \widetilde{G} \) is hyperbolic if there exists an open relatively compact transversal \( X_0 \), a metric \( |\cdot| \) defined on a neighborhood \( \hat{X}_0 \) of the closure of \( X_0 \), and a compact generating set \( S \) of \( \widetilde{G}|_{X_0} \) such that

1. all elements of \( S^{-1} \) are Lipschitz and there exists \( \lambda \in (0, 1) \) such that \( |F(x) - F(y)| < \lambda|x - y| \) for all \( F \in S, x, y \in \text{Dom } F \);
2. there exists \( \delta > 0 \) such that for every \( x \in X_0 \) the Cayley graph \( \mathcal{G}(x, S) \) is \( \delta \)-hyperbolic;
3. for every \( x \in X_0 \) every vertex has at least one incoming and one outgoing arrow in \( \mathcal{G}(x, S^{-1}) \), and all infinite directed paths in \( \mathcal{G}(x, S^{-1}) \) are (in a uniform way) quasi-geodesics converging to one point \( \omega_x \) of \( \partial \mathcal{G}(x, S^{-1}) \).

Note that this definition is a combination of two hyperbolicity conditions: topological (contraction) and large-scale (Gromov hyperbolicity conditions). Interplay between these two conditions and duality between them is the main topic of this paper.

Denote by \( \partial \mathcal{G}_x \) the hyperbolic boundary of the Cayley graph \( \mathcal{G}(x, S) \) minus the special point \( \omega_x \). The space \( \partial \mathcal{G}_x \) does not depend on the choice of \( X_0 \) and \( S \), and hence is defined for every \( x \in X \). Let us denote by \( \partial \mathcal{G} \) the disjoint union (as a set) of the sets \( \partial \mathcal{G}_x \).
Figure 1:

If $S$ is rich enough, then every point $\xi \in \partial G_x$ is the limit as $n \to \infty$ of a sequence of vertices $g_n g_{n-1} \cdots g_1 h$, where $g_i$ are germs of elements $F_i \in S$ and $h$ is a vertex of the Cayley graph $\mathfrak{G}(x, S)$. Then there exists an element $F \in \mathfrak{G}$ such that $h = (F, x)$, and $g_i = (F_i, F_{i-1} F_{i-2} \cdots F_1 F(x))$. Since $F_i$ are contractions, there exists a neighborhood $U$ of $x$ such that all maps $F_n F_{n-1} \cdots F_1 F$ are defined on $U$. Define then the parallel translation of $\xi$ to $y \in U$ as

$$[y, \xi]_F = \lim_{n \to \infty} (F_n F_{n-1} \cdots F_1 F, y) \in \partial \mathfrak{G}_y,$$

see Figure 1.

One can show that the maps $[\cdot, \cdot]_F$ define a topology and a local product structure on $\partial \mathfrak{G}$. Namely, every point $\xi \in \partial \mathfrak{G}_x \subset \partial \mathfrak{G}$ has a neighborhood (called a rectangle) identified by the map $[\cdot, \cdot]_F$ with the direct product of a neighborhood of $x$ in $X$ and a neighborhood of $\xi$ in $\partial \mathfrak{G}_x$. Moreover, the defined direct product decompositions agree on the intersections of the neighborhoods.

For a germ $g = (F, x) \in \mathfrak{G}$ and a point $\xi \in \partial \mathfrak{G}_x$ represented as limit of the sequence $v_n$ of vertices of the Cayley graph $\mathfrak{G}(F(x), S)$ define $\xi \cdot g$ as the limit of the sequence $v_n \cdot g$. Then $\xi \cdot g$ belongs to $\partial \mathfrak{G}_x$, and we get for every $F \in \mathfrak{G}$ a local homeomorphism of $\partial \mathfrak{G}$ mapping $\xi \in \partial \mathfrak{G}_F(x)$ to $\xi \cdot (F, x) \in \partial \mathfrak{G}_x$ for every point $x$ of the domain of $F$. Such local homeomorphisms preserve the local product structure and generate a pseudogroup. Denote by $\partial \mathfrak{G} \rtimes \mathfrak{G}$ the groupoid of germs of this pseudogroup, and call it the geodesic quasi-flow of the hyperbolic groupoid $\mathfrak{G}$.

It is proved in [5,7] that the groupoid $\partial \mathfrak{G} \rtimes \mathfrak{G}$ is compactly generated. It is also proved that it is a quasi-flow, i.e., it has a natural quasi-cocycle $\nu : \partial \mathfrak{G} \rtimes \mathfrak{G} \to \mathbb{R}$ such that its restriction onto any Cayley graph of $\partial \mathfrak{G} \rtimes \mathfrak{G}$ is a quasi-isometry of the Cayley graph with $\mathbb{R}$.

Consider an open relatively compact transversal $W \subset \partial \mathfrak{G}$ and a finite covering of $W$ by open rectangles $R_i = A_i \times B_i$, where $A_i$ are open subsets of $X$ and $B_i$ are open subsets of boundaries $\partial \mathfrak{G}_x$ of Cayley graphs of $\mathfrak{G}$. Since $\partial \mathfrak{G} \rtimes \mathfrak{G}$ preserves the local product structure, every element of the pseudogroup $\partial \mathfrak{G} \rtimes \mathfrak{G}$ can be locally decomposed into a direct product of a homeomorphism $P_+^i(F)$ between open subsets of the sets $A_i$ and a homeomorphism $P_-(F)$ between open subsets of the sets $B_i$. The pseudogroup generated by the local homeomorphisms $P_+^i(F)$ is equivalent to $\mathfrak{G}$. 
**Definition 0.0.3.** The dual groupoid $\mathfrak{G}^\top$ of a hyperbolic groupoid $\mathfrak{G}$ is the groupoid of germs of the pseudogroup generated by projections $P_-(F)$ of elements of the geodesic quasi-flow onto the direction of the hyperbolic boundaries of the Cayley graphs of $\mathfrak{G}$.

One of the main results of the paper is the following duality theorem (see Theorem 4.5.7).

**Theorem 0.0.1.** Suppose that $\mathfrak{G}$ is a hyperbolic groupoid with a locally diagonal geodesic quasi-flow. Then the dual groupoid $\mathfrak{G}^\top$ is also hyperbolic and the groupoid $(\mathfrak{G}^\top)^\top$ is equivalent to $\mathfrak{G}$.

In fact, we prove that the geodesic quasi-flows of $\mathfrak{G}$ and $\mathfrak{G}^\top$ are equivalent. Local diagonality of the geodesic quasi-flow means that the action of $\mathfrak{G}$ on its boundary $\partial \mathfrak{G}$ is in some sense faithful. This condition holds, for instance when the groupoid $\mathfrak{G}$ is minimal, see [Nek11].

In the process of proving the duality theorem we give an axiomatic description of geodesic quasi-flows of hyperbolic groupoids. The corresponding notion is a generalization of the classical notion of a Smale space (see [Rue78]). Informally, a compactly generated pseudogroup $\tilde{\mathfrak{H}}$ acting on a space $X$ is a Smale quasi-flow if the Cayley graphs of $\tilde{\mathfrak{H}}$ are quasi-isometric to $\mathbb{R}$ and elements of $\tilde{\mathfrak{H}}$ expand one and contract the other direction of a local product structure on $X$ preserved under the action of $\tilde{\mathfrak{H}}$. Precise definition has more conditions (see Definition 4.1.1).

Examples of Smale quasi-flows are: Smale spaces (in particular Anosov diffeomorphisms), étale groupoids equivalent to groupoids generated by Anosov flows (in particular, by geodesic flow on a compact negatively curved manifold), diagonal action of a Gromov-hyperbolic group $G$ on $\partial G \times \partial G$ minus the diagonal, e.t.c.

If $\tilde{\mathfrak{H}}$ is a hyperbolic quasi-flow, then we define the groupoids $P_+(\tilde{\mathfrak{H}})$ and $P_-(\tilde{\mathfrak{H}})$ (called Ruelle pseudogroups) generated by the projections of the elements of $\tilde{\mathfrak{H}}$ onto the corresponding directions of the local product structure. We prove the following results (see Theorems 4.3.1, 4.4.1, and 4.5.1).

**Theorem 0.0.2.** If a hyperbolic groupoid $\mathfrak{G}$ has locally diagonal geodesic quasi-flow $\partial \mathfrak{G} \times \mathfrak{G}$ then $\partial \mathfrak{G} \times \mathfrak{G}$ is a Smale quasi-flow.

The Ruelle groupoids of a Smale quasi-flow $\tilde{\mathfrak{H}}$ are hyperbolic, and $\tilde{\mathfrak{H}}$ is equivalent to the geodesic quasi-flows of its Ruelle groupoids.

For example, the groupoid generated by holonomies and the action of an Anosov diffeomorphism on the leaves of the stable foliation is hyperbolic and its dual is the analogous groupoid acting on the leaves of the unstable foliation.

Groupoid generated by the one-sided shift of finite type defined by a finite set of prohibited words $P$ is dual to the groupoid generated by the shift of finite type defined by the set of prohibited words obtained by writing the elements of $P$ in the opposite order.
Theorem 0.0.2 implies that groupoid of the action of a Gromov-hyperbolic group on its boundary is self-dual. For more on these and other examples of hyperbolic duality, see the last chapter of this paper.

Overview of the paper

Chapter 1 is just an overview of some of technical tools used in the paper. In its first section we define a notion which is sometimes a convenient replacement of the notion of a metric, and is especially natural in the setting of hyperbolic dynamics and geometry. Instead of working with a metric we work with a log-scale, which is, up to an additive constant, logarithm (base less than one) of a metric. The triangle inequality is replaced then by the condition

$$\ell(x, z) \geq \min(\ell(x, y), \ell(y, z)) - \delta$$

for some constant $\delta$. It follows from Frink metrization lemma (see [Kel75, Lemma 6.12]) that any function satisfying this inequality (and the conditions $\ell(x, y) = \ell(y, x)$ and $\ell(x, y) = \infty \Leftrightarrow x = y$) is equal, up to an additive constant, to logarithm of a metric.

We define natural notions of Lipschitz and Hölder equivalence of log-scales, and show that log-scales can be “pasted together” from locally defined log-scales (which is one of the reasons for using them in our paper).

Next section of Chapter 1 introduces the necessary notions from the theory of Gromov-hyperbolic metric spaces and graphs. In particular, we prove a hyperbolicity criterion, which will be used later to prove large-scale hyperbolicity of Ruelle groupoids.

The last section of Chapter 1 defines the notion of a local product structure on a topological space. This notion was introduced by Ruelle in [Rue78]. We adapt it to our slightly more general setting.

Chapter 2 is an overview of the theory of pseudogroups and groupoids of germs. In particular, we fix there our notations for groupoids and pseudogroups.

Section 2.3 introduces the notion of a Cayley graph of a groupoid, which is a natural development of A. Haefliger’s notion of a compactly generated groupoid.

In the last section of Chapter 2 we define compatibility conditions between pseudogroups, quasi-cocycles, log-scales, and local product structures.

In Chapter 3 we define the notion of a hyperbolic groupoid $\mathcal{G}$, its boundary $\partial \mathcal{G}$, local product structure on $\partial \mathcal{G}$, and the geodesic quasi-flow, i.e., action of the groupoid $\mathcal{G}$ on $\partial \mathcal{G}$.

Chapter 4 is devoted to the proof of the duality theorem for hyperbolic groupoids and axiomatic description of Smale quasi-flows. In the last two sections we give another definition of the dual groupoid (using partial maps between positive cones of the Cayley graphs), and study irreducibility conditions for hyperbolic groupoids.
In the last chapter of the paper we consider different examples of pairs of dual hyperbolic groupoids and the corresponding hyperbolic quasi-flows: actions of Gromov-hyperbolic groups on their boundaries, groupoids generated by expanding maps, groupoids associated with self-similar groups, Smale spaces, etc.
Chapter 1

Technical preliminaries

Some remarks on notation and terminology

All spaces on which groups, pseudogroups, or groupoids act are assumed to be locally compact and metrizable. A neighborhood $U$ of a point $x$ is any set containing an open set $V$ such that $x \in V$. In particular, neighborhoods are not assumed to be open.

We use notation $|x - y|$ for distance between the points $x$ and $y$ of a metric space.

We write $F \approx G$, where $F$ and $G$ are real functions, if there exists a constant $\Delta > 0$ such that $|F - G| < \Delta$ for all values of the variables.

Notation $x \doteq \lim_{n \to \infty} a_n$ means that there exists a constant $\Delta$ such that for every partial limit $y$ of the sequence $a_n$ we have $|x - y| \leq \Delta$.

Most of our group and pseudogroup actions are right. In particular, in a product of maps $fg$ the map $g$ acts before $f$.

1.1 Logarithmic scales

Definition 1.1.1. A logarithmic scale (a log-scale) on a set $X$ is a function $\ell: X \times X \to \mathbb{R} \cup \{\infty\}$ such that

1. $\ell(x, y) = \ell(y, x)$ for all $x, y \in X$;
2. $\ell(x, y) = +\infty$ if and only if $x = y$;
3. there exists $\delta \geq 0$ such that $\ell(x, z) \geq \min(\ell(x, y), \ell(y, z)) - \delta$, for all $x, y, z \in X$.

Definition 1.1.2. We say that a metric $|x - y|$ on $X$ is associated with the log-scale $\ell$ if there exist constants $\alpha$ and $c$ such that $0 < \alpha < 1$, $c > 1$, and

$$c^{-1}\alpha^{\ell(x,y)} \leq |x - y| \leq c\alpha^{\ell(x,y)}$$

for all $x, y \in X$. 
CHAPTER 1. TECHNICAL PRELIMINARIES

Proposition 1.1.1. For any log-scale $\ell$ on $X$ there exists a metric $|\cdot|$ associated with it.

If $|x - y|$ is a metric on $X$, then the function

$$\ell(x, y) = -\lfloor \ln |x - y| \rfloor$$

is a log-scale associated with it.

Proof. Define for $n \in \mathbb{Z}$

$$E_n = \{(x, y) : \ell(x, y) \geq n\}.$$ 

Note that if $\delta$ is as in Definition 1.1.1, then

$$E_n + 2\delta \cap E_{n+2\delta} \subset E_{n+\delta} \cap E_{n+\delta} \subset E_{n+\delta} \cap E_{n+\delta} \subset E_n.$$

It follows from Frink metrization lemma, see [Kel75, Lemma 6.12], that there exists a metric $|\cdot|$ on $X$ such that

$$E_{2^n} \subset \{(x, y) : |x - y| < 2^{-n}\} \subset E_{2^{n}(n-1)}.$$ 

Suppose that $\ell(x, y) = n$. Then $\ell(x, y) \geq 2\delta[n/(2\delta)]$, hence

$$|x - y| < 2^{-\lfloor n/2 \rfloor} \leq 2^{-\lfloor n/2 \rfloor + 1}.$$ 

On the other hand $\ell(x, y) < 2\delta(|n/(2\delta)| + 1)$, hence

$$|x - y| \geq 2^{-\lfloor n/2 \rfloor + 1} \geq 2^{-\lfloor n/2 \rfloor - 2}.$$ 

We have shown that for $\alpha = 2^{-\lfloor \frac{n}{2} \rfloor}$ we have

$$\frac{1}{4} \cdot \alpha^{\ell(x,y)} \leq |x - y| \leq 2 \cdot \alpha^{\ell(x,y)},$$

which shows that $|\cdot|$ is associated with $\ell$.

The second part of the proposition is obvious. \qed

If a log-scale is defined on a topological space, then we assume that the topology defined by it (i.e., by an associated metric) coincides with the original topology of the space.

Taking into account the described relation between log-scales and metrics, we give the following definitions.

Definition 1.1.3. Let $\ell_1$ and $\ell_2$ be log-scales on the sets $X_1$ and $X_2$, respectively. A map $f : X_1 \rightarrow X_2$ is called Lipschitz if there exists a positive number $n$ such that

$$\ell_2(f(x), f(y)) \geq \ell_1(x, y) - n$$

for all $x, y \in X_1$.

A function $f : X_1 \rightarrow X_2$ is said to be locally Lipschitz if for every $x \in X_1$ there exists a neighborhood $U$ of $x$ such that restriction of $f$ onto $U$ is Lipschitz. Equivalently, $f$ is locally Lipschitz if for every $x \in X_1$ there exist $N$ and $n$ such that $\ell_2(f(x), f(y)) \geq \ell_1(x, y) - n$ for all $y \in X_1$ such that $\ell_1(x, y) \geq N$. 
1.1. LOGARITHMIC SCALES

Definition 1.1.4. Let $\ell$ be a log-scale on $X$. A map $f : X \to X$ is a contraction if there exists $c > 0$ such that

$$\ell(f(x), f(y)) \geq \ell(x, y) + c$$

for all $x, y \in X$.

Note that if $\ell_1$ is a log-scale, and $\ell_2$ is a function such that $\ell_1 - \ell_2$ is uniformly bounded on $X \times X$, then $\ell_2$ is also a log-scale.

Definition 1.1.5. We say that log-scales $\ell_1$ and $\ell_2$ on $X$ are Lipschitz equivalent if there exists $k > 0$ such that

$$|\ell_1(x, y) - \ell_2(x, y)| \leq k$$

for all $x, y \in X$.

They are Hölder equivalent if there exist constants $c > 1$ and $k > 0$ such that

$$c^{-1}\ell_1(x, y) - k \leq \ell_2(x, y) \leq c\ell_1(x, y) + k$$

for all $x, y \in X$.

We say that a log-scale is positive if all its values are positive.

Lemma 1.1.2. Let $\ell_i$ be log-scales on spaces $X_i$ for $i = 1, 2$. Suppose that $X_1$ is compact and $\ell_2$ is positive. If $f : X_1 \to X_2$ is locally Lipschitz, then it is Lipschitz.

In particular, to positive log-scales on a compact space are bi-Lipschitz equivalent if and only if they are locally bi-Lipschitz equivalent.

Proof. The set $X_1$ can be covered by a finite set of $U$ open sets such that restriction of $f$ onto the elements of $U$ are Lipschitz. There exists $n_1$ such that if $\ell_1(x, y) \geq n_1$, then $x, y$ belong to one element of $U$. Then there exists a constant $c > 0$ such that for any pair $x, y \in X_1$ either $\ell_1(x, y) \leq n_1$, or $\ell_2(f(x), f(y)) \geq \ell_1(x, y) - c$. In the first case we have $\ell_2(f(x), f(y)) > 0 \geq \ell_1(x, y) - n_1$. □

Theorem 1.1.3. Let $X$ be a compact metrizable space and let $\{U_i\}_{i \in I}$ be a covering of $X$ by open sets. Suppose that $\ell_i$ is a positive log-scale on $U_i$ such that for every pair $U_i, U_j$ the scales $\ell_i$ and $\ell_j$ are Lipschitz equivalent on the intersection $U_i \cap U_j$. Then there is a log-scale $\ell$ on $X$ such that $\ell$ is Lipschitz equivalent to $\ell_i$ on $U_i$ for every $i \in I$.

Note that the log-scale $\ell$ satisfying the conditions of the theorem is necessarily unique up to Lipschitz equivalence.
Proof. Passing to a finite sub-covering we may assume that \( I \) is finite. By Lebesgue’s number lemma there exists a symmetric neighborhood \( E \) of the diagonal of \( X \times X \) such that for every \( x \in X \) there exists \( i \in I \) such that

\[
\{ y \in X : (x, y) \in E \} \subset U_i.
\]

For every \( i \in I \) the set \( E \cap \overline{U_i} \times \overline{U_i} \) is a symmetric neighborhood of the diagonal of \( \overline{U_i} \times \overline{U_i} \), hence by compactness of \( \overline{U_i} \) there exists \( n_i \) such that for every pair \( x, y \in \overline{U_i} \) such that \( \ell_i(x, y) \geq n_i \) we have \( (x, y) \in E \). Let \( n = \max_{i \in I} n_i \).

Choose for every two-element subset \( \{x, y\} \subset X \) such that \((x, y) \in E\) an element \( U_i \) of the covering such that \( \{x, y\} \subset U_i \) (which exists by the choice of \( E \)). Define then \( \ell_i'(x, y) = \ell_i(x, y) \).

Define then for every pair of points \( x, y \):

\[
\ell(x, y) = \begin{cases} 
\max\{\ell'(x, y), n\}, & \text{if } (x, y) \in E, \\
n, & \text{otherwise}.
\end{cases}
\]

Let us show that \( \ell \) is a log-scale on \( X \) satisfying the conditions of the theorem.

Let \( \Delta \) be such that

\[
\ell_i(x, y) - \Delta \leq \ell_j(x, y) \leq \ell_i(x, y) + \Delta
\]

for all \( i, j \in I \) and \( x, y \in \overline{U_i} \cap \overline{U_j} \). Let \( \delta \) be such that

\[
\ell_i(x, z) \geq \min\{\ell_i(x, y), \ell_i(y, z)\} - \delta
\]

for all \( i \in I \) and \( x, y, z \in \overline{U_i} \). Such numbers exist by conditions of the theorem and finiteness of the set \( I \).

Suppose that \( x, y, z \in X \) are such that \((x, y), (y, z) \in E\). Then there exists \( U_i \) such that \( \{x, y, z\} \subset U_i \). Suppose at first that \((x, y) \notin E\). Then \( \ell_i(x, z) < n \).

If one of the numbers \( \ell'(x, y), \ell'(y, z) \) is less than or equal to \( n \), then we have \( \min\{\ell(x, y), \ell(y, z)\} = n = \ell(x, z) \).

If both numbers \( \ell'(x, y), \ell'(y, z) \) are greater than \( n \), then \( \ell(x, y) = \ell'(x, y), \ell(y, z) = \ell'(y, z) \), and

\[
\ell(x, z) = n > \ell_i(x, z) \geq \min\{\ell_i(x, y), \ell_i(y, z)\} - \delta \\
\geq \min\{\ell'(x, y), \ell'(y, z)\} - \Delta - \delta = \min\{\ell(x, y), \ell(y, z)\} - \Delta - \delta.
\]

Suppose now that \((x, z) \in E\). Then

\[
\ell(x, z) \geq \ell'(x, z) \geq \ell_i(x, z) - \Delta \geq \\
\min\{\ell_i(x, y), \ell_i(y, z)\} - \delta - \Delta \geq \\
\min\{\ell'(x, y), \ell'(y, z)\} - \delta - 2\Delta.
\]

It follows that if \( \ell'(x, y) \) and \( \ell'(y, z) \) are both greater than \( n \), then we have

\[
\ell(x, z) \geq \min\{\ell(x, y), \ell(y, z)\} - \delta - 2\Delta.
\]
1.1. LOGARITHMIC SCALES

Otherwise \( \ell(x, z) \geq n = \min\{\ell(x, y), \ell(y, z)\} \).

Suppose now that at least one of the pairs \((x, y)\) and \((y, z)\) does not belong to \(E\). Then \( \min\{\ell(x, y), \ell(y, z)\} = n, \) and \( \ell(x, z) \geq n = \min\{\ell(x, y), \ell(y, z)\} \).

We have shown that \( \ell \) is a log-scale on \(X\). Let us show that it is Lipschitz equivalent to \(\ell_i\) on every set \(U_i\).

Let \(x, y \in \bigcup U_i\) and suppose that \((x, y) \notin E\). Then \(\ell_i(x, y) \leq n\), hence
\[
\ell(x, y) = n \geq \ell_i(x, y) \geq 1 = \ell(x, y) + (1 - n).
\]

Suppose now that \((x, y) \in E\). Then either \(\ell'(x, y) \leq n\) and \(\ell(x, y) = n\), so that
\[
\ell(x, y) - n - \Delta = -\Delta \leq \ell'(x, y) - \Delta \leq \ell_i(x, y) \leq \ell'(x, y) + \Delta \leq \ell(x, y) + \Delta;
\]
or \(\ell'(x, y) > n\) and
\[
\ell(x, y) - \Delta = \ell'(x, y) - \Delta \leq \ell_i(x, y) \leq \ell'(x, y) + \Delta = \ell(x, y) + \Delta.
\]

It follows that \(\ell_i\) is Lipschitz equivalent to \(\ell\).

Let us reformulate the notions related to completion of a metric space in terms of log-scales.

A sequence \(x_n\) is Cauchy if \(\ell(x_n, x_m) \to \infty\) as \(n, m \to \infty\). Two Cauchy sequences \(x_n\) and \(y_n\) are equivalent if \(\ell(x_n, y_n) \to \infty\). A completion of \(X\) with respect to a log-scale \(\ell\) is the set of equivalence classes of Cauchy sequences in \(X\).

**Proposition 1.1.4.** Suppose that \(x_n\) and \(y_n\) are non-equivalent Cauchy sequences. Then there exists a constant \(\Delta > 0\) such that any two partial limits of the sequence \(\ell(x_n, y_n)\) differ not more than by \(\Delta\) from each other.

**Proof.** We have for all \(n, m\)
\[
\ell(x_m, y_m) \geq \min(\ell(x_n, x_m), \ell(x_n, y_n), \ell(y_n, y_m)) - 2\delta.
\]

There exists a sequence \(n_i\) such that \(\ell(x_n, y_{n_i})\) is bounded from above by some number \(l\). Then for all \(m\) and \(i\) big enough we have \(\ell(x_n, x_m) \geq l \geq \ell(x_n, y_{n_i})\) and \(\ell(y_{n_i}, y_m) \geq l \geq \ell(x_n, y_{n_i})\), hence
\[
\ell(x_m, y_m) \geq \min\{\ell(x_n, x_m), \ell(x_n, y_{n_i}), \ell(y_{n_i}, y_m)\} - 2\delta = \ell(x_n, y_{n_i}) - 2\delta.
\]

In particular, for all \(i\) and \(j\) big enough we have
\[
\ell(x_{n_i}, y_{n_j}) \geq \ell(x_{n_i}, y_{n_i}) - 2\delta.
\]

It follows that there exists \(i_0\) such that all values of \(\ell(x_{n_i}, y_{n_i})\) for \(i \geq i_0\) are less than \(2\delta\) away from each other. Let \(k = \ell(x_{n_{i_0}}, y_{n_{i_0}})\). Then, for all \(m\) and \(i\) big enough we have \(\ell(x_m, y_m) \geq k - 4\delta\) and
\[
k + 2\delta > \ell(x_{n_i}, y_{n_i}) \geq \min(\ell(x_n, x_m), \ell(x_m, y_m), \ell(y_m, y_{n_i})) - \delta = \ell(x_m, y_m) - \delta \geq k - 3\delta,
\]
Figure 1.1: Gromov product

since we may assume that \( \ell(x_m, x_n) \) and \( \ell(y_m, y_m) \) are greater that \( k + 3\delta \).

It follows that the values of \( \ell(x_m, y_m) \) for all \( m \) big enough are at most by \( 3\delta \) away from \( k \).

If \( x = \lim_{n \to \infty} x_n \neq y = \lim_{n \to \infty} y_n \) in the completion of \( X \) with respect to \( \ell \), then we can define \( \ell(x, y) \) as any partial limit of the sequence \( \ell(x_n, y_n) \).

The defined function is a log-scale on the completion and is unique up to a Lipschitz equivalence. We will call this log-scale the natural extension of the log-scale \( \ell \) onto the completion.

1.2 Gromov-hyperbolic metric spaces and graphs

General definitions and main properties

Let \((X, |·|)\) be a metric space. Choose a point \(x_0 \in X\) and define

\[
(x, y)_{x_0} = \frac{1}{2}(|x_0 - x| + |x_0 - y| - |x - y|),
\]

and

\[
\ell_{x_0}(x, y) = \begin{cases} (x, y)_{x_0} & \text{if } x \neq y, \\ \infty & \text{otherwise}. \end{cases}
\]

Note that it follows from the triangle inequality that \((x, y)_{x_0}\) is non-negative and not greater than \(\min(|x_0 - x|, |x_0 - y|)\). See Figure 1.1

Definition 1.2.1. The space \((X, d)\) is called Gromov hyperbolic if \(\ell_{x_0}\) is a log-scale, i.e., if there exists \(\delta > 0\) such that

\[
\ell_{x_0}(x, z) \geq \min(\ell_{x_0}(x, y), \ell_{x_0}(y, z)) - \delta
\]

for all \(x, y, z \in X\).
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Our definition is slightly different from the classical one (which uses \((x, y)_{x_0}\) instead of \(\ell_{x_0}(x, y)\)), but is equivalent to it. Namely, we have the following.

**Lemma 1.2.1.** Inequality (1.1) is equivalent to the inequality

\[
(x, z)_{x_0} \geq \min((x, y)_{x_0}, (y, z)_{x_0}) - \delta
\]  

(1.2)

**Proof.** If \(x = z\), then \(\ell_{x_0}(x, z) = \infty\), hence inequality (1.1) is true. In inequality (1.2) we will have \(\min((x, y)_{x_0}, (y, z)_{x_0}) = (x, y)_{x_0} = (y, z)_{x_0} = 1/2(|x_0 - y| + |x_0 - x| - |x - y|)\) and \((x, z)_{x_0} = |x_0 - x|\). Then inequality (1.2) is

\[
|x_0 - x| > 1/2(|x_0 - y| + |x_0 - x| - |x - y|) - \delta,
\]

and it is equivalent to

\[
|x_0 - x| + |x - y| > |x_0 - y| - 2\delta,
\]

which is always true.

If \(x = y\), then \(\ell_{x_0}(x, y) = \infty\), hence \(\min(\ell_{x_0}(x, y), \ell_{x_0}(y, z)) = \ell_{x_0}(x, z)\), therefore (1.1) is true.

In the case of (1.2) we have \((y, z)_{x_0} = (x, z)_{x_0}\), therefore

\[
(x, z)_{x_0} = (y, z)_{x_0} \geq \min((x, y)_{x_0}, (y, z)_{x_0}) > \min((x, y)_{x_0}, (y, z)_{x_0}) - \delta.
\]

A proof of the following proposition can be found in [Gro87, Corollary 1.1B] and in [GHV91, Proposition 2.2, p. 10].

**Proposition 1.2.2.** If \((X, d)\) is \(\delta\)-hyperbolic with respect to \(x_0\), then it is \(2\delta\)-hyperbolic with respect to any point \(x_1 \in X\).

Note that

\[
(x, y)_{x_0} \leq (x, y)_{x_1} + |x_0 - x_1|,
\]

hence the log-scales \(\ell_{x_0}\) and \(\ell_{x_1}\) are Lipschitz equivalent.

**Boundary and the Busemann function**

**Definition 1.2.2.** Let \((X, d)\) be a Gromov hyperbolic metric space. The boundary \(\partial X\) of \(X\) is the complement of \(X\) in its completion with respect to the log-scale \(\ell_{x_0}\).

Since the Lipschitz class of the log-scale \(\ell_{x_0}\) does not depend on \(x_0\), the boundary \(\partial X\) and the Lipschitz class of the extension of \(\ell_{x_0}\) to \(\partial X\) do not depend on the choice of \(x_0\).

Note that topology defined on \(X\) by the log-scale \(\ell_{x_0}\) is often different from the original topology (defined by the metric \(|x - y|\)). In such cases, the completion of \(X\) with respect to the log-scale \(\ell_{x_0}\) is not homeomorphic to the classical compactification of \(X\) by its boundary.
Let $\xi \in \partial X$ be the limit of an $\ell_{x_0}$-Cauchy sequence $x_n \in X$. For any pair of points $x, y \in X$ consider

$$|x - x_n| - |y - x_n| = 2\ell_{x_0}(y, x_n) - 2\ell_{x_0}(x, x_n) + |x - x_0| - |y - x_0|.$$ 

It follows, that all partial limits of the sequence $|x - x_n| - |y - x_n|$ are, up to an additive constant, equal to $2(\ell_{x_0}(y, \xi) - \ell_{x_0}(x, \xi)) + |x - x_0| - |y - x_0|$.

**Definition 1.2.3.** The function

$$\beta_{\xi}(x, y) \equiv \lim_{x_n \to \xi} (|x - x_n| - |y - x_n|) \equiv 2(\ell_{x_0}(y, \xi) - \ell_{x_0}(x, \xi)) + |x - x_0| - |y - x_0|$$

is called the *Busemann quasi-cocycle* associated with $\xi \in \partial X$.

It follows directly from the definitions that for any three points $x, y, z \in X$ we have

$$\beta_{\xi}(x, z) \equiv \beta_{\xi}(x, y) + \beta_{\xi}(y, z). \quad (1.3)$$

Let us recall the definition of a natural extension of the Gromov product to the boundary and a natural metric on the space $\partial X \setminus \{\omega\}$ (see [GH90, Section 8.1]). Define

$$\ell_{\xi, x_0}(x, y) = \frac{1}{2}(\beta_{\xi}(x, x_0) + \beta_{\xi}(y, x_0) - |x - y|).$$

Note that

$$\ell_{\xi, x_0}(x, y) \equiv \lim_{x_n \to \xi} \left( \frac{1}{2}(|x - x_n| + |y - x_n| - |x - y|) - |x_0 - x_n| \right) \equiv \lim_{x_n \to \xi} (\ell_{x_0}(x, y) - |x_0 - x_n|).$$

It follows that the Lipschitz class of $\ell_{\xi, x_0}$ does not depend on $x_0$.

**Proposition 1.2.3.** Let $X$ be a hyperbolic metric space. Then the function $\ell_{\xi, x_0}$ is a log-scale for all $\xi \in \partial X$ and $x_0 \in X$.

The identity map $X \to X$ extends to a homeomorphism from the completion of $X$ with respect to $\ell_{\xi, x_0}$ to the space $X \cup \partial X \setminus \{\xi\}$.

We will denote $\partial X_\xi = \partial X \setminus \{\xi\}$.

**Proof.** Let us show at first that $\ell_{\xi, x_0}$ is a log-scale. For any $x, y, z \in X$ and $x_n \to \xi$ we have

$$\ell_{\xi, x_0}(x, z) \equiv \lim_{n \to \infty} (\ell_{x_n}(x, z) - |x_0 - x_n|) \geq \lim_{n \to \infty} (\min(\ell_{x_n}(x, y), \ell_{x_n}(y, z)) - |x_0 - x_n|) - \delta \equiv \min \left( \lim_{n \to \infty} \ell_{x_n}(x, y) - |x_0 - x_n|, \lim_{n \to \infty} \ell_{x_n}(y, z) - |x_0 - x_n| \right) \equiv \min(\ell_{\xi, x_0}(x, y), \ell_{\xi, x_0}(y, z)).$$
We have
\[2\ell_{x_0}(x, y) - 2\ell_{\xi,x_0}(x, y) =
|x - x_0| + |y - x_0| - \beta_{\xi}(x, x_0) + \beta_{\xi}(y, x_0) =
(|x - x_0| - \beta_{\xi}(x, x_0)) + (|y - x_0| - \beta_{\xi}(y, x_0)),
\]
and
\[|x - x_0| - \beta_{\xi}(x, x_0) = \lim_{n \to \infty} |x - x_0| - |x - x_n| + |x_0 - x_n| =
\lim_{n \to \infty} 2\ell_{x_0}(x, x_n) = 2\ell_{x_0}(x, \xi).
\]
It follows that
\[\ell_{x_0}(x, y) - \ell_{\xi,x_0}(x, y) \doteq \ell_{x_0}(x, \xi) + \ell_{x_0}(y, \xi),
\]
or
\[\ell_{\xi,x_0}(x, y) \doteq \ell_{x_0}(x, y) - \ell_{x_0}(x, \xi) - \ell_{x_0}(y, \xi).
\]
Consider for an arbitrary number \(r\) the set \(C_r\) of points \(\zeta \in X \cup \partial X\) such that \(\ell_{x_0}(\zeta, \xi) < r\). It is complement of the neighborhood \(\{x \in X : \ell_{x_0}(x, \xi) \geq r\}\) of \(\xi \in X \cup \partial X\). Then for any \(x, y \in C_r\) we have
\[\ell_{\xi,x_0}(x, y) \doteq \ell_{x_0}(x, y) - \ell_{x_0}(x, \xi) - \ell_{x_0}(y, \xi) \geq \ell_{x_0}(x, y) - 2r.
\]
We also have
\[\ell_{\xi,x_0}(x, y) \leq \ell_{x_0}(x, y) + k,
\]
for some constant \(k\), since \(\ell_{x_0}\) is non-negative on \(X\).

Consequently, \(\ell_{\xi,x_0}\) and \(\ell_{x_0}\) are Lipschitz equivalent on \(C_r\), hence they are Lipschitz equivalent on complement of any neighborhood of \(\xi\). This finishes the proof of the proposition.

Hyperbolic graphs

Let \(\Gamma\) be a connected graph. Distance between two vertices of \(\Gamma\) is the smallest number of edges in a path connecting them. A geodesic in \(\Gamma\) connecting vertices \(v_1\) and \(v_2\) is a shortest path connecting them. We will consider (and denote) paths in \(\Gamma\) as sequences of vertices.

The graph \(\Gamma\) has bounded degree if there is a number \(k\) such that every vertex of \(\Gamma\) belongs to not more than \(k\) edges.

**Definition 1.2.4.** A subset \(N\) of a metric space \(\Gamma\) is called a net if there exists a constant \(\Delta > 0\) such that for every point \(x \in \Gamma\) there exists a point \(y \in N\) such that \(|x - y| < \Delta\).

A map \(f : \Gamma_1 \to \Gamma_2\) is a quasi-isometry if \(f(\Gamma_1)\) is a net in \(\Gamma_2\) and there exist \(\Lambda > 1\) and \(\Delta > 0\) such that
\[\Lambda^{-1} \cdot |x - y| - \Delta \leq |f(x) - f(y)| \leq \Lambda \cdot |x - y| + \Delta
\]
for all \(x, y \in \Gamma_1\).
**Definition 1.2.5.** A metric space \((X, d)\) is **quasi-geodesic** if there exist positive numbers \(\Lambda\) and \(\Delta\) such that for any two points \(x, y \in X\) there exists a sequence \(x = t_0, t_1, \ldots, t_n = y\) of points of \(X\) such that \(|t_i - t_{i+1}| \leq \Delta\) and \(|i - j| \leq \Lambda \cdot |t_i - t_j|\) for all \(i, j \in \{0, 1, \ldots, n\}\). We call a sequence satisfying these conditions a \((\Lambda, \Delta)\)-**quasi-geodesic**.

A metric space is quasi-geodesic if and only if it is quasi-isometric to a geodesic space, i.e., to a space in which any two points \(x, y\) can be connected by an isometric arc \(\gamma : [0, |x - y|] \to X\). Every quasi-geodesic space is quasi-isometric to a graph.

The following is a classical result of the theory of Gromov hyperbolic spaces (see [GH90, Theorem 5.12] and [GHV91, Proposition 2.1, p. 16]).

**Theorem 1.2.4.** Let \(\Gamma_1\) and \(\Gamma_2\) be quasi-isometric graphs of bounded degree. If \(\Gamma_1\) is Gromov hyperbolic, then so is \(\Gamma_2\).

A graph is Gromov hyperbolic if and only if there exists \(\delta\) such that in any geodesic triangle each side belongs to the \(\delta\)-neighborhood of the union of the other two sides. (One says that such a triangle is \(\delta\)-**thin**, see Figure 1.2).

Another important property of hyperbolic spaces is the following theorem, which can be found, for instance, in [GH90] Theorem 5.11.

**Theorem 1.2.5.** Suppose that \(\Gamma\) is a hyperbolic graph. Then for any pair of positive numbers \(\Lambda, \Delta\) there exists \(\delta\) such that if \(t_0, t_1, \ldots, t_n\) and \(s_0, s_1, \ldots, s_m\) are two \((\Lambda, \Delta)\)-quasi-geodesics connecting \(x = t_0 = s_0\) to \(y = t_n = s_m\), then the sequences \(t_i\) and \(s_i\) are at most \(\delta\) apart (i.e., for every point of one sequence there exists a point of the other on distance less than \(\delta\)).

In particular, if \(\Gamma\) is a hyperbolic graph, then there exists \(\delta\) such that every \((\Lambda, \Delta)\)-quasi-geodesic is at most \(\delta\) away from a geodesic path connecting the same points.
The following proposition is a direct corollary of 1.2.5.

**Proposition 1.2.6.** Let $\Gamma$ be a hyperbolic graph, and let $\xi_1, \xi_2$ be arbitrary points of the space $\Gamma \cup \partial \Gamma$. Then there exists a geodesic path $\gamma$ connecting $\xi_1$ and $\xi_2$, i.e., a geodesic path such that its beginning (resp. end) is either equal to the vertex $\xi_1 \in \Gamma$ (resp. $\xi_2$), or converges to $\xi_1 \in \partial \Gamma$ (resp. to $\xi_2$).

There exists $\delta$ such that for any two points $\xi_1, \xi_2 \in \Gamma \cup \partial \Gamma$ any two geodesic paths $\gamma_1$ and $\gamma_2$ connecting $\xi_1$ and $\xi_2$ are on distance not more than $\delta$ from each other.

**Hyperbolic graphs directed towards a point of the boundary**

**Definition 1.2.6.** Let $\Gamma$ be a graph of bounded degree. A function $\nu : \Gamma \times \Gamma \rightarrow \mathbb{R}$ is called a quasi-cocycle if there are numbers $\Delta > 0$ and $\eta > 0$ such that

$$|
u(v_1, v_2)| \leq \Delta$$

for all pairs adjacent vertices $v_1, v_2 \in \Gamma$, and

$$\nu(v_1, v_2) + \nu(v_2, v_3) - \eta \leq \nu(v_1, v_3) \leq \nu(v_1, v_2) + \nu(v_2, v_3) + \eta,$$

for all $v_1, v_2, v_3 \in \Gamma$.

Note that under the conditions of the definition we have

$$|
u(v_1, v_2)| \leq |v_1 - v_2|(\Delta + \eta),$$

since for any path $u_0 = v_1, u_1, \ldots, u_n = v_2$ we have

$$\nu(v_1, v_2) \leq \nu(u_0, u_1) + \nu(u_1, u_2) + \cdots + \nu(u_{n-1}, u_n) + (n-1)\eta < n\Delta + n\eta$$

and

$$\nu(v_1, v_2) \geq \nu(u_0, u_1) + \nu(u_1, u_2) + \cdots + \nu(u_{n-1}, u_n) - (n-1)\eta \geq -n\Delta - (n-1)\eta \geq -n(\Delta + \eta).$$

If $\nu_1$ is a quasi-cocycle, and $\nu_2 : \Gamma \times \Gamma \rightarrow \mathbb{R}$ is such that $|\nu_1 - \nu_2|$ is bounded, then $\nu_2$ is obviously also a quasi-cocycle.

An example of a quasi-cocycle on a graph is the Busemann quasi-cocycle $\beta_\xi$ associated with a point of the boundary of a hyperbolic graph (see Definition 1.2.3).

**Definition 1.2.7.** Two quasi-cocycles $\nu_1$ and $\nu_2$ are said to be strongly equivalent if $|\nu_1 - \nu_2|$ is uniformly bounded.

They are called (coarsely) equivalent if there exist $\Lambda > 1$ and $k > 0$ such that

$$\Lambda^{-1} \cdot \nu_1(u, v) - k \leq \nu_2(u, v) \leq \Lambda \cdot \nu_1(u, v) + k$$

for all vertices $u, v$.

Let $\Gamma$ be a directed graph and let $\nu$ be a quasi-cocycle. Let $\Delta$ and $\eta$ be as in Definition 1.2.6. Suppose that for any arrow of $\Gamma$ starting at a vertex $v$ and ending in a vertex $u$ we have $\nu(v, u) > 2\eta$. 

Lemma 1.2.7. Any directed path in $\Gamma$ is a $((\Delta + \eta)/\eta, 1)$-quasi-geodesic.

Proof. Let $(\ldots, x_1, x_2, \ldots)$ be an infinite or a finite directed path in $\Gamma$.

We have for any $i < j$:

$$2\eta(j - i) - (j - i - 1)\eta < \nu(x_i, x_{i+1}) + \cdots + \nu(x_{j-1}, x_j) - (j - i - 1)\eta \leq \nu(x_i, x_j) \leq \nu(x_i, x_{i+1}) + \cdots + \nu(x_{j-1}, x_j) + (j - i - 1)\eta < \Delta(j - i) + (j - i - 1)\eta,$$

hence

$$\eta(j - i) < \nu(x_i, x_j) < (\Delta + \eta)(j - i) + \eta.$$

By (1.4) we have for any $i < j$

$$|x_i - x_j| \geq \frac{1}{\Delta + \eta}\nu(x_i, x_j) = \frac{1}{\Delta + \eta}\nu(x_i, x_j) > \frac{\eta}{\Delta + \eta}(j - i),$$

which implies that the path is quasi-geodesic. \qed

Theorem 1.2.8. Let $\Gamma$ be a directed graph such that every vertex of $\Gamma$ has at least one outgoing arrow. Let $\nu$ be a quasi-cocycle on $\Gamma$, and let $\Delta$ and $\eta$ be as in Definition 1.2.6. Suppose that for any arrow of $\Gamma$ with beginning $v$ and end $u$ we have $\nu(v, u) > 2\eta$. Fix a number $\Delta_1 > \Delta + \eta$.

Then the following conditions are equivalent.

1. The graph $\Gamma$ is hyperbolic and $\nu$ is coarsely equivalent to the Busemann quasi-cocycle $\beta_\omega$ associated with a point $\omega \in \partial \Gamma$.

2. There exists $\rho_0 > 0$ such that for any $m > 0$ there exists $k_m > 0$ such that for any directed paths $(u_0, u_1, \ldots)$ and $(v_0, v_1, \ldots)$ with $|u_0 - v_0| \leq m$ we have $|u_i - v_j| < \rho_0$ whenever $|\nu(u_i, v_j)| \leq \Delta_1$, $i > k_m$, and $j > k_m$.

Note that in condition (1) of the theorem together with the condition $\nu(v, u) > 2\eta$ mean that all the arrows of the graph $\Gamma$ are directed towards the point $\omega \in \partial \Gamma$ in the sense that any infinite directed path converges to $\omega$.

The proof of Theorem 1.2.8 is basically the same as the proof of hyperbolicity of the selfsimilarity complex given in [Nek03] (see [Nek05, Theorem 3.9.6]). See also a similar criterion of hyperbolicity of augmented trees in [Kal03].

Proof. Let us prove that (1) implies (2). Let $(u_0, u_1, \ldots)$ and $(v_0, v_1, \ldots)$ be two directed paths such that $|u_0 - v_0| \leq m$. Both paths converge to $\omega$ and are quasi-geodesic. Connect $u_0$ to $v_0$ by a geodesic path $\gamma$, and consider the obtained triangle with vertices $u_0, v_0, \omega$. There exists $\delta$ depending only on $\Gamma$ such that all such triangles are $\delta$-thin (see Theorem 1.2.4 and 1.2.6). In particular, the path $(u_0, u_1, \ldots)$ belongs to the $\delta$-neighborhood of the union of the paths $\gamma$ and $(v_0, v_1, \ldots)$. If $|v_0 - v_i| > m + \delta$, then the distance to $v_i$ from every point of $\gamma$ is greater than $\delta$, hence there exists $u_j$ such that $|v_i - u_j| \leq \delta$. Similarly,
if \(|u_0 - u_i| > m + \delta\), then the distance from \(u_i\) to any point of \(\gamma\) is greater than \(\delta\), hence there exist \(v_j\) such that \(|v_j - u_i| \leq \delta\). The paths \((v_0, v_1, \ldots)\) and \((u_0, u_1, \ldots)\) are quasi-geodesics (see Lemma 1.2.7), hence there exists \(k_m\) not depending on \(v_0\) and \(u_0\) such that for every \(i > k_m\) there exists \(j\) such that \(|v_i - u_j| \leq \delta\) and for every \(i > k_m\) there exists \(j\) such that \(|u_i - v_j| \leq \delta\).

Suppose now that \(i > k_m\) and \(j > k_m\) are such that \(|v_i - u_j| < \Delta_1\). Then there exists \(l\) such that \(|v_i - u_l| \leq \delta\) and there exists \(m\) such that \(|v_m - u_j| \leq \delta\). Then \(|\nu(v_i, u_l)| \leq \delta(\Delta + \eta)\) and \(|\nu(v_m, u_j)| \leq \delta(\Delta + \eta)\). Consequently,

\[
|\nu(v_i, v_m)| \leq |\nu(v_i, u_j)| + |\nu(u_j, v_m)| + \eta \leq \Delta_1 + \delta(\Delta + \eta) + \eta. 
\]

We will get then a uniform bound \(M = (\Delta_1 + \delta(\Delta + \eta) + \eta)/\eta\) for the difference \(|i - m|\). It follows that

\[
|v_i - u_j| \leq |v_i - v_m| + |v_m - u_j| \leq M + \delta,
\]

so that we can take \(\rho_0 = M + \delta\). Note that \(\rho_0\) depends only on \(\Gamma\) and \(\Delta_1\).

Let us prove now that (2) implies (1). Choose a vertex \(x_0 \in \Gamma\), and consider the functions

\[
\lambda(v) = \nu(v, x_0), \quad \nu_1(v, u) = \nu(v, x_0) - \nu(u, x_0) = \lambda(v) - \lambda(u).
\]

Then \(|\nu_1(u, v) - \nu(u, v)| \leq \eta\) for any pair \(u, v \in \Gamma\), \(|\nu_1(u, v)| \leq \Delta + \eta \leq \Delta_1\) for any two adjacent vertices \(u, v\) of \(\Gamma\), and \(\nu_1(u, v) \geq \eta\) for every directed edge \((u, v)\). We also have

\[
\nu_1(u, w) = \nu_1(u, v) + \nu_1(v, w)
\]

for all triples of vertices \(u, v, w\) of \(\Gamma\), and \(\nu_1(u, u) = 0\) for all \(u \in \Gamma\).

Let \((u_0, u_1, \ldots)\) and \((v_0, v_1, \ldots)\) be directed paths in \(\Gamma\). Suppose that \(0 \leq \nu_1(u_i, v_j) \leq \Delta_1\) for some \(i, j\). If \(0 \leq \nu_1(u_i, v_i) \leq \Delta_1 - \eta\), then \(-\eta \leq \nu(u_i, v_i) \leq \Delta_1\). If \(\Delta_1 - \eta > \nu(v_i, v_j) \leq \Delta_1\), then

\[
\nu_1(u_i, v_j) - \Delta_1 \leq \nu_1(u_i, v_j - 1) = \nu_1(u_i, v_j) + \nu_1(v_j, v_j - 1) \leq \nu_1(u_i, v_j) - \eta,
\]

hence \(-\eta \leq \nu_1(u_i, v_j - 1) \leq \Delta_1 - \eta\), so that \(-\Delta_1 < -2\eta \leq \nu(u_i, v_j - 1) \leq \Delta_1\).

We have proved that if \(0 \leq \nu_1(u_i, v_j) \leq \Delta_1\), then either \(|\nu(u_i, v_j)| \leq \Delta_1\), or \(|\nu(u_i, v_j - 1)| \leq \Delta_1\). Similarly, if \(0 \leq \nu_1(v_j, u_i) = -\nu_1(u_i, v_j) \leq \Delta_1\) then either \(|\nu(u_i, v_j)| \leq \Delta_1\), or \(|\nu(u_i - 1, v_i)| \leq \Delta_1\).

It follows that condition (2) of the theorem implies the following condition (when we take \(\rho_1 = \rho_0 + 1\)).

(2)' There exists \(\rho_1 > 0\) such that for any \(m > 0\) there exists \(k_m\) such that for any pair of directed paths \((u_0, u_1, \ldots)\) and \((v_0, v_1, \ldots)\) such that \(|u_0 - v_0| \leq m\) we have \(|u_i - v_j| < \rho_1\) whenever \(|\nu_1(u_i, w_j)| \leq \Delta_1\), \(i > k_m\), and \(j > k_m\).

Denote \(k = k_{2\rho_1}\) and let \(\Delta_2\) and \(r\) be positive integers such that \(r\) is divisible by 4, and

\[
\Delta_2 > (k + 1)\Delta_1, \quad r \geq \frac{4\Delta_2}{\rho_1\eta}.
\]
Consider a new graph $\Gamma_1$ with the set of vertices equal to the set of vertices of $\Gamma$ in which two different vertices $u, v$ are connected by an edge if and only if one of the following conditions holds

- there exists a sequence $(u = w_0, w_1, \ldots, v = w_n)$ of vertices of $\Gamma$ and a number $l \in \mathbb{Z}$ such that $n \leq r$, $|w_i - w_{i+1}| \leq \rho_1$, and $l \Delta_2 \leq \lambda(w_i) < (l + 1) \Delta_2$ for all $i = 0, 1, \ldots, n$;
- there exists a directed path $(u = w_0, w_1, \ldots, v = w_n)$ in $\Gamma$ and a number $l \in \mathbb{Z}$ such that $l \Delta_2 \leq \lambda(u) < (l + 1) \Delta_2$ and $(l - 1) \Delta_2 \leq \lambda(v) \leq l \Delta_2$.

We will imagine the graphs $\Gamma$ and $\Gamma_1$ drawn in such a way that $\lambda(v)$ is the height of the position of the vertex $v$. Then the arrows and directed paths of $\Gamma$ go down.

See Figure 1.3 where the two cases for edges of $\Gamma_1$ are shown.

The edges defined by the first condition are called horizontal. The edges defined by the second conditions are called vertical. For a path $(v_1, \ldots, v_n)$ in $\Gamma_1$, we say that a vertical edge $(v_i, v_{i+1})$ is descending, if $\lambda(v_{i+1}) < \lambda(v_i)$ and ascending otherwise.

Let us call the number $\lfloor \lambda(v)/\Delta_2 \rfloor$ the level of the vertex $v$. It follows from the definition of the edges of $\Gamma_1$ that if $(v, u)$ is a horizontal edge, then $v$ and $u$ belong to the same level, and if $(v, u)$ is a descending vertical edge, then the level of $u$ is one less than the level of $v$.

We will denote by $|u - v|$ and $|u - v|_1$ the distances between the vertices $u$ and $v$ in $\Gamma$ and $\Gamma_1$, respectively.

**Lemma 1.2.9.** The identity map between the sets of vertices of $\Gamma$ and $\Gamma_1$ is a bi-Lipschitz equivalence, i.e., there exists a constant $\Lambda$ such that $\Lambda^{-1} \cdot |u - v| \leq |u - v|_1 \leq \Lambda \cdot |u - v|$ for all $u, v \in \Gamma$.

**Proof.** Suppose that $(u, v)$ is a directed edge of $\Gamma$. Let $l = \lfloor \lambda(u)/\Delta_2 \rfloor$. Then $\eta \leq \nu_1(u, v) = \lambda(u) - \lambda(v) \leq \Delta_1 < \Delta_2$, hence either $\lfloor \lambda(v)/\Delta_2 \rfloor = l$, or $\lfloor \lambda(v)/\Delta_2 \rfloor = l - 1$. In the first case $(u, v)$ is a horizontal edge of $\Gamma_1$, in the case
second case it is a vertical edge of $\Gamma_1$. Consequently, the set of edges of $\Gamma$ is a subset of the set of edges of $\Gamma_1$, hence $|u - v|_1 \leq |u - v|$ for all $u, v \in \Gamma$.

The distance in $\Gamma$ between the vertices connected by a horizontal edge of $\Gamma_1$ is bounded from above by $r\rho_1$. If $(u, v)$ is a vertical edge in $\Gamma_1$, then $\nu_1(u, v) = \lambda(u) - \lambda(v) \leq 2\Delta_2$, and

$$\nu_1(u, v) = \nu_1(w_0, w_1) + \nu_1(w_1, w_2) + \cdots + \nu_1(w_{n-1}, w_n) \geq n\eta,$$

hence $n \leq \frac{2\Delta_2}{\eta} \leq \frac{r\rho_1}{2\eta}$. Consequently, $|u - v| \leq r\rho_1|u - v|_1$ for all $u, v \in \Gamma_1$. □

**Lemma 1.2.10.** Let $(v_0, v_1, \ldots, v_n)$ be a path of horizontal edges in $\Gamma_1$. Let $(u_0, v_0)$ and $(u_n, v_n)$ be ascending vertical edges. Then the vertices $u_0$ and $u_n$ can be connected by a path of horizontal edges of $\Gamma_1$ of length not more than $\lceil (n + 1)/2 \rceil$.

**Proof.** Let $l$ be the level of the vertices $v_i$. Then the vertices $u_0$ and $u_n$ belong to the level $l - 1$.

It follows from the definition of horizontal edges that there exists a sequence $z_0, z_1, \ldots, z_m$ of vertices of $\Gamma$ such that $z_0 = v_0$, $z_m = v_n$, $|z_i - z_{i+1}| \leq 2\rho_1$, $m \leq mn/2$, and $|\lambda(z_i)/\Delta_2| = l$ for all $i = 0, 1, \ldots, m$.

For every $i = 0, 1, \ldots, m$ find an infinite directed path $(w_{0,i}, w_{1,i}, \ldots)$ in $\Gamma$ such that $w_{0,i} = z_i$, and there exist $t_0$ and $t_m$ such that $w_{t_0,0} = u_0$ and $w_{t_m,m} = u_n$. (See Figure 1.4.)

For every $i$ let $s_i$ be the maximal index such that $\lambda(w_{s_i,i}) > (l - 1)\Delta_2$. Note that $s_0 \geq t_0$ and $s_m \geq t_m$. 

![Figure 1.4:](image)
We have \( \lambda(w_{s_i},i) - \lambda(w_{s_{i+1},i}) = \nu_1(w_{s,i},w_{s_{i+1},i}) \leq \Delta_1 \). If \( \lambda(w_{s_i},i) > (l - 1)\Delta_2 + \Delta_1 \), then

\[
\lambda(w_{s_{i+1},i}) \geq \lambda(w_{s_i}) - \Delta_1 > (l - 1)\Delta_2
\]

which contradicts with the choice of \( s_i \). Consequently,

\[
(l - 1)\Delta_2 \leq \lambda(w_{s_i},i) \leq (l - 1)\Delta_2 + \Delta_1.
\]

The length \( s_i \) of the path \( (w_{0,i},w_{1,i},\ldots,w_{s_i,i}) \) is not less than \( \frac{\Delta s - \Delta l}{\nu} > k = k_{2\rho_1} \). We have

\[
|\nu_1(w_{s_i},i,w_{s_{i+1},i+1})| \leq \Delta_1, \quad |w_{0,i} - w_{0,i+1}| \leq 2\rho_1,
\]

hence by condition (2)' we have

\[
|w_{s_i,i} - w_{s_{i+1},i+1}| \leq \rho_1.
\]

The lengths of the \( \Gamma \)-paths \( \gamma_1 = (u_0 = w_{t_0,0},w_{t_0+1,0},\ldots,w_{s_0,0}) \) and \( \gamma_2 = (u_0 = w_{t_m,m},w_{t_m+1,m},\ldots,w_{s_m,m}) \) are not greater than \( \frac{\Delta s - \Delta l}{\nu} \leq \frac{\eta \rho_1}{4} \). We can hence split each of them into segments of length \( \leq \rho_1 \) so that we get not more than \( r/4 \) segments. Putting these two sequences of segments together with the sequence \( w_{s_0,0},w_{s_1,1},\ldots,w_{s_m,m} \) we get a sequence of vertices of \( \Gamma \) consisting of at most \( m + r/2 \leq mn/2 + r/2 = r(n + 1)/2 \) segments of length at most \( \rho_1 \).

All these vertices will belong to the level \( l - 1 \). Consequently, we can find a path in \( \Gamma_1 \) consisting of at most \( \lceil (n + 1)/2 \rceil \) horizontal edges connecting the vertices \( u_0 \) and \( u_n \).

\[ \square \]

**Lemma 1.2.11.** If \( (v_0,v_1,\ldots,v_n) \) is a path in \( \Gamma_1 \) of length at least two such that \( (v_1,v_2) \) is ascending, \( (v_{n-1},v_n) \) is descending and all edges between them are horizontal, then the path is not a geodesic.

**Proof.** By Lemma 1.2.10 the distance between \( v_0 \) and \( v_n \) is not more than \( \lceil (n - 1)/2 \rceil < n \).

\[ \square \]

**Corollary 1.2.12.** In any geodesic path of \( \Gamma_1 \) all descending edges come before all ascending ones.

**Lemma 1.2.13.** A geodesic of \( \Gamma_1 \) can not have more than 6 consecutive horizontal edges.

**Proof.** Let \( v_0,v_1,\ldots,v_7 \) be a horizontal path. Find descending edges \( (v_0,u_0) \) and \( (v_7,u_7) \). Then by Lemma 1.2.10 the distance between \( u_0 \) and \( u_7 \) is not more than 4, hence the \( \Gamma_1 \)-distance between \( v_0 \) and \( v_7 \) is not more than 6.

\[ \square \]

When we talk about distances between paths in \( \Gamma \) or \( \Gamma_1 \) we consider the paths as sets of vertices and use the Hausdorff distance between them, i.e., the smallest number \( R \) such each of the paths (as sets of vertices) belongs to the \( R \)-neighborhood of the other.
Lemma 1.2.14. A geodesic path of $\Gamma_1$ can not have more than 6 horizontal edges, and it is on distance (in $\Gamma_1$) not more than 2 from a path with at most 6 horizontal edges in which all descending edges are at the beginning, and all ascending edges are at the end.

Proof. Let $\gamma = (v_0, \ldots, v_n)$ be a geodesic path in $\Gamma_1$. Suppose that $(v_{i-1}, v_i)$ and $(v_j, v_{j+1})$ for $j > i$ are descending, and all the edges between $v_i$ and $v_j$ are horizontal. Let $(v_i, v)$ be a descending edge. Then, by Lemma 1.2.10 the distance $|v - v_{j+1}|_1$ is not more than $\lceil (j - i + 1)/2 \rceil$. Hence, $j + 1 - i = |v_i - v_{j+1}|_1 \leq \lceil (j - i + 1)/2 \rceil + 1$.

But $\lceil (j - i + 1)/2 \rceil + 1 < j + 1 - i$ for $j - i \geq 3$. It follows that the number $j - i$ of consecutive horizontal edges between descending edges is not more than 2.

If $j - i = 1$ or $j - i = 2$, then $\lceil (j - 1 + 1)/2 \rceil = j - i + 1$, and we can replace the segment $v_i, v_{i+1}, \ldots, v_j, v_{j+1}$ of $\gamma$ by $v_i, v$ followed by a sequence of horizontal edges connecting $v$ to $v_{j+1}$ so that we again get a geodesic path, but the segment of horizontal edges has moved towards the end of $\gamma$.

Similarly, we can not have more than two horizontal edges between two ascending edges, and if we have one or two ascending horizontal edges between ascending ones, we can move the horizontal edges towards the beginning of $\gamma$.

Let us move the first segment of horizontal edges between descending edges forward, as it is described above for as long as we can, i.e., until it will meet another segment of horizontal edges, or an ascending edge. In the first case we will put the horizontal edges together, and continue moving them forward. This may happen at most once (if both segments consist of only one edge). In the same way, move all horizontal edges between ascending edges towards the beginning of the path. At the end we get a geodesic path $\gamma'$ in which all descending edges are at the beginning and all the ascending edges are at the end. It is also easy to see that the Hausdorff distance between $\gamma'$ and $\gamma$ is at most 2. The path $\gamma'$ (and hence the path $\gamma$) can not have more than 6 horizontal paths by Lemma 1.2.14.

Let us say that a path $\gamma$ in $\Gamma_1$ is V-shaped if all its descending edges are at the beginning, all its ascending edges are at the end, and it has not more than 6 horizontal edges. Note that we allow any of the three sets of edges (descending, ascending, and horizontal) to be empty.

We have shown above that every geodesic path in $\Gamma_1$ is on distance not more than 2 from a V-shaped geodesic path connecting the same vertices.

We say that a V-shaped path $\gamma$ has depth $l$ if $l$ is the level of the vertices of its horizontal edges (or the level of the last vertex of the descending part of $\gamma$, or the level of the first vertex of the ascending part of $\gamma$). We say that a V-shaped path $\gamma$ is proper if it has the maximal possible depth among all V-shaped paths connecting the same pair of points as $\gamma$.

Lemma 1.2.15. Let $\gamma_1$ be a geodesic path (in $\Gamma_1$) connecting the vertices $u$ and $v$, and let $\gamma_2$ be any proper V-shaped path connecting the same vertices. Then the distance between $\gamma_1$ and $\gamma_2$ is not more than 9.
Proof. Let \((v_1, v_2, \ldots)\) and \((u_1, u_2, \ldots)\) be descending vertical paths in \(\Gamma_1\) such that \(v_1 = u_1\). Then it follows by induction from Lemma 1.2.10 that \(|v_i - u_i|_1 \leq 1\) for all \(i\).

Let \(\gamma_1\) and \(\gamma_2\) be \(V\)-shaped paths connecting the same pair of vertices, and suppose that \(\gamma_1\) is geodesic and \(\gamma_2\) is proper. Let \(l_2\) be the depth of \(\gamma_2\). Then the depth \(l_1\) of \(\gamma_1\) is not more than \(l_2\). Let \(h_1\) and \(h_2\) be the lengths of the horizontal parts of \(\gamma_1\) and \(\gamma_2\), respectively.

Then the length of the descending part of \(\gamma_1\) minus the length of the descending part of \(\gamma_2\) is equal to \(l_2 - l_1\), and the same is true for the ascending parts.

It follows that the length of \(\gamma_2\) is equal to the length of \(\gamma_1\) minus \(2(l_2 - l_1) + h_1 - h_2\). But \(\gamma_1\) is a geodesic path, hence \(2(l_2 - l_1) + h_1 - h_2 \leq 0\). Consequently, \(l_2 - l_1 \leq (h_2 - h_1)/2 \leq 3\). It follows that the Hausdorff distance between \(\gamma_1\) and \(\gamma_2\) is not more than 7. See Figure 1.5.

If \(\gamma_1\) is any (not necessarily \(V\)-shaped) geodesic, then it is on distance at most 2 from a \(V\)-shaped geodesic connecting the same points, hence it is at distance at most 9 from any proper \(V\)-shaped path connecting the same points.

We are ready now to prove that the graph \(\Gamma_1\) (and hence also \(\Gamma\)) is Gromov-hyperbolic. Let \(a, b, c \in \Gamma_1\) be arbitrary vertices. We have to prove that there exists a constant \(\delta > 0\) (not depending on \(a, b, c\)) such that geodesic triangle with the vertices \(a, b,\) and \(c\) is \(\delta\)-thin. By the previous lemma, it is enough to show that every triangle of proper \(V\)-shaped paths with the vertices \(a, b,\) and \(c\) is \(\delta\)-thin.

Let \(l_{xy}\) for \(xy \in \{ab, bc, ac\}\) be the depth of a proper \(V\)-shaped path \(\gamma_{xy}\) connecting \(x\) to \(y\). Without loss of generality we may assume that \(l_{ab} \geq l_{bc} \geq l_{ac}\) (see Figure 1.6).

Denote by \(u_1\) and \(v_1\) the beginning and the end of the horizontal part of the path \(\gamma_{ab}\). Let \(u_2\) and \(v_2\) be the vertices of level \(l_{ab}\) of the descending parts.
of $\gamma_{ac}$ and $\gamma_{bc}$, respectively. Then $|u_1 - u_2|_1 \leq 1$ and $|v_1 - v_2|_1 \leq 1$. It follows that $|u_2 - v_2|_1 \leq 8$.

Denote now by $v_3$ and $w_1$ the beginning and the end of the horizontal part of $\gamma_{bc}$. Let $u_3$ and $w_2$ be the vertices of the descending and ascending parts of $\gamma_{ac}$ of level $l_{bc}$ (see Figure 1.6).

By Lemma 1.2.10, $|u_3 - v_3|_1 \leq 5$. We also have $|v_3 - w_1|_1 \leq 6$ and $|w_1 - w_2|_1 \leq 1$. It follows that $|u_3 - w_2|_1 \leq 12$. Then by Lemma 1.2.10, $l_{bc} - l_{ac} \leq 2$.

It follows that the segment $[a, u_2]$ of $\gamma_{ac}$ is on distance at most 1 from the segment $[a, u_1]$, segment $[u_2, u_3]$ is on distance at most 8 from $[b, v_3]$, segment $[u_3, w_2]$ is on distance at most 8 from $[c, w_1]$. Consequently, the side $\gamma_{ac}$ belongs to the 8-neighborhood of $\gamma_{ab} \cup \gamma_{bc}$. It is also easy to see that $\gamma_{ab}$ belongs to the 4-neighborhood of $\gamma_{ac} \cup \gamma_{bc}$, and that $\gamma_{bc}$ belongs to the 8-neighborhood of $\gamma_{ac} \cup \gamma_{ab}$. We have proved that the triangle with sides $\gamma_{ab}$, $\gamma_{bc}$, and $\gamma_{ac}$ is 8-thin.

Consequently, by Lemma 1.2.15 every geodesic triangle in $\Gamma_1$ is 26-thin, and the graph $\Gamma$ is Gromov-hyperbolic. It remains to prove that $\nu_1$ is equivalent to the Busemann quasi-cocycle associated with a point of the boundary.

By Lemma 1.2.14 every infinite directed path $(v_1, v_2, \ldots)$ of $\Gamma$ is a quasi-geodesic, hence it converges to a point of $\partial \Gamma$. By condition (2), any two infinite directed paths are eventually on distance not more than $\rho_0$, hence they converge to the same point $\omega \in \partial \Gamma$.

Consequently, every infinite descending path of $\Gamma_1$ converges to $\omega$. Let $\beta(v, u)$ be the Busemann quasi-cocycle on $\Gamma_1$ associated with $\omega$. It is enough to prove that $\beta$ is equivalent to $\nu_1$. Fix an infinite descending path $(w_0, w_1, \ldots)$, where $\lambda(w_0) = 0$.

Then $\beta(v, u) \doteq \lim_{n \to \infty} |v - w_n|_1 - |u - w_n|_1$. Let $(v = v_0, v_1, \ldots)$ and $(u = u_0, u_1, \ldots)$ be infinite descending paths. Let $l_v$ and $l_u$ be the levels of $v$ and $u$, respectively. Then the vertices $v_n$ and $u_n$ will belong to levels $l_v - n$ and $l_u - n$, respectively.
By condition (2)' and Lemma 1.2.10 there exists $k$ such that for all $n \geq k$ the $|\cdot|_1$-diameter of the set $\{w_n, v_{i_n-n}, u_{i_n-n}\}$ is not more than 2. Therefore,

$$|(v - w_n|_1 - |u - w_n|_1) - (l_v - l_u)| = |(|v - w_n|_1 - |u - w_n|_1) - (|v - v_{i_n-n}|_1 - |u - u_{i_n-n}|_1|) \leq 4.$$  

But $l_v \Delta_2 \leq \lambda(v) < (l_v - 1) \Delta_2$ for every $v$, hence $\Delta_2^{-1} \lambda(v) + 1 < l_v \leq \Delta_2^{-1} \lambda(v)$, so that

$$\Delta_2^{-1} \nu_1(v, u) - 1 = \Delta_2^{-1}(\lambda(v) - \lambda(u)) - 1 < l_v - l_u < \Delta_2^{-1}(\lambda(v) - \lambda(u)) + 1 = \Delta_2^{-1} \nu_1(v, u) + 1.$$  

We have shown that $\beta(v, u) \approx \Delta_2^{-1} \nu_1(v, u) \approx \Delta_2^{-1} \nu(v, u)$. The Busemann quasi-cocycle on $\Gamma$ associated with $\omega$ is coarsely equivalent to $\beta$, by Lemma 1.2.9.

**Boundary of a directed hyperbolic graph**

Let $\Gamma$ be a directed graph with a quasi-cocycle $\nu$ such that for every directed edge $(u, v)$ we have $2\eta \leq \nu(u, v) \leq \Delta$, where $\eta$ is as in Definition 1.2.6. We assume that every vertex has at least one outgoing arrow.

Suppose that $\Gamma$ satisfies the equivalent conditions of Theorem 1.2.8. Let $\omega \in \partial \Gamma$ be such that the Busemann quasi-cocycle $\beta_\omega$ is coarsely equivalent to $\nu$. We denote $\partial \Gamma_\omega = \partial \Gamma \setminus \{\omega\}$.

We say that a sequence $(v_1, v_2, \ldots)$ of vertices of $\Gamma$ is an ascending path if $(v_{i+1}, v_i)$ are directed edges of $\Gamma$ for every $i$. It is a descending path if all $(v_i, v_{i+1})$ are directed edges.

Let $v$ and $u$ be vertices of $\Gamma$. By Proposition 1.2.3 the space $\Gamma \cup \partial \Gamma_\omega$ is the completion of the set $\Gamma$ with respect to the log-scale

$$\ell_{\omega, x_0}(v, u) = \frac{1}{2} (\beta_\omega(v, x_0) + \beta_\omega(u, x_0) - |v - u|),$$

where $x_0$ is a fixed vertex of $\Gamma$. Denote

$$\lambda(v) = \beta_\omega(v, x_0).$$

**Proposition 1.2.16.** There exist constants $\Lambda > 1$, $\rho > 0$, and $k > 0$ such that any two vertices $u, v$ of $\Gamma$ can be connected by an $(\Lambda, k)$-quasi-geodesic of the form $\gamma_0 \gamma_1 \gamma_2$, where $\gamma_0$ is a descending path, $\gamma_1$ is a path of length not more than $\rho$, and $\gamma_2$ is an ascending path.

**Proof.** Follows directly from Lemmas 1.2.14 and 1.2.9.

**Proposition 1.2.17.** For a pair of vertices $u, v \in \Gamma$ choose a geodesic path $\gamma_{u,v}$ connecting $u$ to $v$, and denote by $\ell(u, v)$ the minimal value of $\lambda$ at a vertex of $\gamma_{u,v}$. Then there exists a constant $c > 0$ (not depending on $u, v$, and the choice of the geodesics) such that $|\ell(u, v) - \ell_{\omega, x_0}(u, v)| < c$. 

_
\textbf{Proof.} Let \( w_n \in \Gamma \) be a sequence converging to \( \omega \). Let \( \gamma \) be a geodesic path connecting \( v \) to \( u \), and let \( x \) be an arbitrary vertex of \( \gamma \). Then
\[
|v - w_n| \leq |v - x| + |x - w_n|,
\]
\[
|u - w_n| \leq |u - x| + |x - w_n|,
\]
hence
\[
\ell_{w_n}(v, u) \leq \frac{1}{2}(|v - x| + |u - x| + 2|x - w_n| - |v - u|) = |x - w_n|.
\]

On the other hand, there is \( \delta > 0 \) depending only on \( \Gamma \), such that if \( \gamma_v \) and \( \gamma_u \) are geodesics connecting \( w_n \) to \( v \) and \( u \), respectively, and the points \( x \in \gamma \), \( x_v \in \gamma_v \), \( x_u \in \gamma_u \) are such that
\[
|x - v| = |x_v - v|, \quad |x - u| = |x_u - u|, \quad |x_u - w_n| = |x_v - w_n|,
\]
then diameter of \( \{x, x_v, x_u\} \) is less than \( \delta \).

Note that then \( |x_u - w_n| = |x_v - w_n| = \ell_{w_n}(v, u) \), hence \( |x - w_n| \leq |x_u - w_n| = \ell_{w_n}(v, u) + \delta \).

It follows that the minimal value of \( |x - w_n| \) along the geodesic \( \gamma \) belongs to the interval \( [\ell_{w_n}(v, u), \ell_{w_n}(v, u) + \delta] \).

But \( \ell_{\omega, x_0}(v, u) = \lim_{w_n \to \omega} \ell_{w_n}(v, u) - |w_n - w_n| \), and \( \beta_{\omega}(x, x_0) = \lim_{w_n \to \omega} |x - w_n| - |x_0 - w_n| \), hence the minimal value of \( \beta_{\omega}(v, x_0) \) along \( \gamma \) differs from \( \ell_{\omega, x_0}(v, u) \) by a uniformly bounded constant. \( \square \)

\textbf{Corollary 1.2.18.} For every pair of constants \( k > 0 \), \( \Lambda > 1 \) there exists \( c > 0 \) such that if \( \ell(v, u) \) is the minimal value of \( \lambda \) at a vertex of an \( (\Lambda, k) \)-quasi-geodesic connecting \( v \) to \( u \), then \( |\ell(v, u) - \ell(v, u)| < c \).

\textbf{Proof.} It follows from Theorem 1.2.5. \( \square \)

\textbf{Proposition 1.2.19.} Every ascending path in \( \Gamma \) converges to a point of \( \partial \Gamma \omega \). Every point of \( \partial \Gamma \omega \) is a limit of an ascending path.

\textbf{Proof.} By Lemma 1.2.7, every ascending path is a quasi-geodesic, hence it converges to a point of \( \partial \Gamma \omega \). The values of the Busemann quasi-cocycle \( \beta_{\omega}(v_0, v_1) \) increase, hence the limit is different from \( \omega \).

Let \( \xi \in \partial \Gamma \omega \). Let \( u_n \in \Gamma \) be a sequence such that \( \xi = \lim_{n \to \infty} u_n \). For every \( n \) find an infinite descending path \( \gamma_n \) starting at \( u_n \). Then, by Lemma 1.2.7, Proposition 1.2.10 and Proposition 1.2.18, there exist constants \( k_1, k_2 \) such that for any fixed \( t \) and all sufficiently big \( n \) and \( m \) we have \( |s - t| \leq k_1 \) for all \( s \in \gamma_n \) and \( t \in \gamma_m \) such that \( \lambda(s), \lambda(t) \in [l, l + k_2] \). By local finiteness of \( \Gamma \) we can find a sequence \( n_i \) such that the paths \( \gamma_{n_i} \) converge to a bi-infinite directed path connecting \( \omega \) with \( \xi \). \( \square \)
1.3 Local product structures

The notions presented here generalize the classical notions of a local product structure of a manifold with an Anosov diffeomorphism, and more generally the notion of a local product structure of a Smale space (see [Rue78]).

The structure of a direct product decomposition of a topological space can be formalized in the following way.

**Definition 1.3.1.** A rectangle is a topological space $R$ together with a continuous map $\cdot, \cdot : R \times R \to R$ such that

1. $[x, x] = x$ for all $x \in R$;
2. $[x, [y, z]] = [x, z]$ for all $x, y, z \in R$;
3. $[[x, y], z] = [x, z]$ for all $x, y, z \in R$.

For a direct product $R = A \times B$ the map $\cdot, \cdot : R \times R \to R$ obviously satisfies the conditions of the definition.

In the other direction, every rectangle comes with a natural direct product decomposition. Define for $x \in R$

$$P_1(R, x) = \{ y \in R : [x, y] = x \}, \quad P_2(R, x) = \{ y \in R : [x, y] = y \}.$$  

The sets $P_i(R, x)$ are called plaques of the rectangle.

Note that $[x, y] = x$ implies $[y, x] = [y, [x, y]] = y$. Similarly, $[x, y] = y$ implies $[y, x] = [[x, y], x] = x$. On the other hand, $[y, x] = y$ implies $[x, y] = [x, [y, x]] = x$, and $[y, x] = x$ implies $[x, y] = [[y, x], y] = y$. Consequently,

$$P_1(R, x) = \{ y \in R : [y, x] = y \}, \quad P_2(R, x) = \{ y \in R : [y, x] = x \}.$$  

**Proposition 1.3.1.** For every $x \in R$ the map $\cdot, \cdot : P_1(R, x) \times P_2(R, x) \to R$ is a homeomorphism.

In terms of the obtained direct product decomposition $R = P_1(R, x) \times P_2(R, x)$ the map $\cdot, \cdot : R \times R \to R$ is given by the rule

$$([y_1, z_1], [y_2, z_2]) = (y_1, z_2).$$

See Figure 1.7 for the structure of $R$ and the map $\cdot, \cdot$.

**Proof.** The map $\cdot, \cdot : P_1(R, x) \times P_2(R, x) \to R$ is obviously continuous.

For every $z \in R$ we have $[x, [z, x]] = x$, hence $[z, x] \in P_1(R, x)$. We also have $[x, [x, z]] = [x, z]$, hence $[x, z] \in P_2(R, x)$. It follows now from

$$[[z, x], [x, z]] = [[z, x], z] = [z, z] = z$$

that the map

$$z \mapsto ([z, x], [x, z])$$
is continuous and is inverse to $\cdot : P_1(R, x) \times P_2(R, x) \to R$.

In terms of the decomposition $R = P_1(R, x) \times P_2(R, x)$ we have

$$[(y_1, z_1), (y_2, z_2)] = [(y_1, z_1), [y_2, z_2]] = [y_1, z_2] = (y_1, z_2),$$

for all $y_1, y_2 \in P_1(R, x)$ and $z_1, z_2 \in P_2(R, x)$. \qed

The described direct product decomposition $R = P_1(R, x) \times P_2(R, x)$ essentially does not depend on the choice of the point $x$. Namely, we have the following natural homeomorphisms between $P_i(R, x)$ and $P_i(R, y)$ for $x, y \in R$.

**Proposition 1.3.2.** For every $x, y \in R$ the maps

$$H_{1,y} : z \mapsto [z, y] : P_1(R, x) \to P_1(R, y),$$

$$H_{2,y} : z \mapsto [y, z] : P_2(R, x) \to P_2(R, y)$$

are homeomorphisms.

**Proof.** We have $[y, [z, y]] = y$, hence $[z, y] \in P_1(R, y)$ for every $z \in R$. Similarly, $[y, [y, z]] = [y, z]$, hence $[y, z] \in P_2(R, y)$ for every $z \in R$. We have

$$H_{1,x}(H_{1,y}(z)) = [z, y, x] = [z, x] = z$$

for all $z \in P_1(R, x)$ and $H_{2,x}(H_{2,y}(z)) = [x, [y, z]] = [x, z] = z$ for all $z \in P_2(R, x)$. \qed

We can therefore canonically identify the spaces $P_i(R, x)$ with one space $P_i(R)$ and get a direct product decomposition $R = P_1(R) \times P_2(R)$, which we will call the **canonical direct product decomposition** of the rectangle $(R, [\cdot, \cdot])$.

**Definition 1.3.2.** Let $X$ be a topological space. A **local product structure** on $X$ is given by a covering of $X$ by open subsets $R_i$, $i \in I$, together with a
structure of a rectangle \((R_i, [\cdot, \cdot])\) on each set \(R_i\) such that for every pair \(i, j \in I\) and every \(x \in X\) there exists a neighborhood \(U\) of \(x\) such that \([y, z]_i = [y, z]_j\) for all \(y, z \in U \cap R_i \cap R_j\).

Two coverings of \(X\) by open rectangles define the same local product structure on \(X\) if their union satisfies the above compatibility condition.

Note that the condition of the definition is void for pairs \(i, j \in I\) such that \(R_i\) and \(R_j\) are disjoint and for points \(x \in X\) that do not belong to the intersection \(R_i \cap R_j\).

A covering by rectangles \((R_i, [\cdot, \cdot])_i\) satisfying the conditions of the definition is called an \textit{atlas} of the local product structure.

If the space \(X\) is compact, then any local product structure can be defined by one function \([x, y]\) defined for all pairs \((x, y)\) belonging to a neighborhood of the diagonal. It follows that our notion of a local product structure is in this case equivalent to the notion of a local product structure defined in \[Rue78\].

\textbf{Definition 1.3.3.} Let \(X\) be a space with a local product structure on it. An open subset \(R \subset X\) together with a direct product structure \([\cdot, \cdot]\) on \(R\) is called a \textit{rectangle} of \(X\) if the union of an atlas of the local product structure of \(X\) with \(\{(R, [\cdot, \cdot])\}\) is also an atlas of the local product structure, i.e., if it satisfies the compatibility condition of Definition 1.3.2.
Chapter 2

Preliminaries on groupoids and pseudogroups

2.1 Pseudogroups and groupoids

A groupoid is a small category of isomorphisms. More explicitly, it is a set \( G \) with a product \((g_1, g_2) \mapsto G\) defined on a subset \(G(2) \subset G \times G\) and an operation \(g \mapsto g^{-1} : G \rightarrow G\) such that the following conditions hold.

1. if \(g_1g_2\) and \(g_2g_3\) are defined, then \((g_1g_2)g_3 = g_1(g_2g_3)\) and the corresponding products are defined;

2. the products \(gg^{-1}\) and \(g^{-1}g\) are defined for all \(g \in G\);

3. if \(gh\) is defined, then \(ghh^{-1} = g, g^{-1}gh = h\), and the corresponding products are defined.

We denote \(o(g) = g^{-1}g, \quad t(g) = gg^{-1}\)

and call \(o(g)\) and \(t(g)\) origin and target of \(g\), respectively. We imagine \(g\) as an arrow from \(o(g)\) to \(t(g)\). A product \(gh\) is defined if and only if \(t(h) = o(g)\). Then \(o(gh) = o(h)\) and \(t(gh) = t(g)\). We also have \(o(g^{-1}) = t(g)\).

Elements \(g \in G\) such that \(g = o(g)\) (which is equivalent to \(g = t(g)\)) are called units of \(G\). The set of units is denoted \(G^{(0)}\). In terms of category theory, we may identify the units with the objects of the category. Then \(g \in G\) is an isomorphism from the object \(o(g)\) to \(t(g)\).

We denote \(G_x = o^{-1}(x), \quad G^x = t^{-1}(x)\). For any \(A \subset G^{(0)}\) we denote by \(G|_A\) the sub-groupoid of elements \(g \in G\) such that \(o(g), t(g) \in A\), i.e., \(G|_A = G_A \cap G^A\). The groupoid \(G|_A\) is the restriction of \(G\) onto \(A\).

Two units \(x, y \in G^{(0)}\) belong to the same orbit if there exists \(g \in G\) such that \(o(g) = x\) and \(t(g) = y\) (i.e., if the corresponding objects of the category are isomorphic). The relation of belonging to the same orbit is obviously an equivalence. A subset \(A \subset G^{(0)}\) is a \(G\)-transversal if it intersects every orbit.
If \( x \) is a unit of a groupoid \( \mathfrak{G} \), then its isotropy group is the set of elements \( g \in \mathfrak{G} \) such that \( o(g) = t(g) = x \). The groupoid is principal if all its isotropy groups are trivial.

A topological groupoid is a groupoid \( \mathfrak{G} \) with a structure of a topological space such that multiplication and taking inverse are continuous operations. We always assume that every element of \( \mathfrak{G} \) has a compact Hausdorff neighborhood, that the subspace \( \mathfrak{G}^{(0)} \) of units is locally compact and metrizable.

**Definition 2.1.1.** Let \( X \) be a topological space. A pseudogroup \( \tilde{\mathcal{H}} \) acting on \( X \) is a set of homeomorphisms \( F: U \rightarrow V \) between open subsets of \( X \) closed under the following operations:

1. composition;
2. passing to restriction onto an open subset;
3. unions: if \( F: U \rightarrow V \) is a homeomorphism between open subsets such that \( U \) can be covered by a collection of open subsets \( U_i \) such that \( F|_{U_i} \in \tilde{\mathcal{H}} \), then \( F \in \tilde{\mathcal{H}} \).

We always assume that the identical homeomorphism \( X \rightarrow X \) belongs to \( \tilde{\mathcal{H}} \).

Let \( \mathfrak{G} \) be a topological groupoid. If \( U \subset \mathfrak{G} \) is an open set such that \( o : U \rightarrow o(U) \) and \( t : U \rightarrow t(U) \) are homeomorphisms, then the map \( o(g) \rightarrow t(g) \) for \( g \in U \) is a homeomorphism between open subsets of \( \mathfrak{G}^{(0)} \). The set \( U \) is called a bisection. If every element of \( \mathfrak{G} \) has a neighborhood that is a bisection, then the groupoid \( \mathfrak{G} \) is called étale.

The set of local homeomorphisms defined by bisections is a pseudogroup acting on \( \mathfrak{G}^{(0)} \), which will be called the pseudogroup associated with the groupoid and denoted \( \tilde{\mathfrak{G}} \).

Note that we will consider bisections both as maps (elements of the pseudogroup) and subsets of the groupoid of germs. So, for example, if \( F_1, F_2 \in \tilde{\mathfrak{G}} \), then notation \( g \in F_1 \) means that \( g \) is a germ of \( F_1 \), and \( F_1 \subset F_2 \) means that \( F_1 \) is a restriction of \( F_2 \).

In the other direction, for any pseudogroup \( \tilde{\mathcal{H}} \) acting on a space \( X \) a germ of \( \tilde{\mathcal{H}} \) is an equivalence class of a pair \( (F, x) \), where \( F: U \rightarrow V \) is an element of \( \tilde{\mathcal{H}} \) and \( x \in U \). Two pairs \( (F_1, x_1) \) and \( (F_2, x_2) \) are equivalent (define the same germ) if \( x_1 = x_2 \) and there exists a neighborhood \( U \) of \( x_1 \) such that \( F_1|_U = F_2|_U \). We define a topology on the set of all germs of elements of \( \tilde{\mathcal{H}} \) by defining a basis of open sets consisting of all sets of the form \( \{(F, x): x \in U \} \) for \( F: U \rightarrow V \) an element of \( \tilde{\mathcal{H}} \). It is easy to see that the set of all germs of \( \tilde{\mathcal{H}} \) is an étale groupoid with respect to the multiplication

\[
(F_1, x_1)(F_2, x_2) = (F_1F_2, x_2),
\]

where the product is defined if and only if \( F_2(x_2) = x_1 \). Taking inverse is given by the rule \( (F, x)^{-1} = (F^{-1}, F(x)) \).
If $\mathfrak{G}$ is the groupoid of germs of a pseudogroup $\tilde{\mathfrak{H}}$ acting on $X$, then $\tilde{\mathfrak{H}}$ coincides with the pseudogroup $\mathfrak{G}$ associated with $\mathfrak{G}$, and the groupoid of germs of $\mathfrak{G}$ is naturally isomorphic to $\mathfrak{G}$. Here we naturally identify $X$ with the space $\mathfrak{G}(0)$. Note that in general the groupoid of germs of a pseudogroup is not Hausdorff.

We say that $\mathfrak{G}$ is a groupoid of germs, if there exists a pseudogroup such that $\mathfrak{G}$ is its groupoid of germs. In other words, a groupoid $\mathfrak{G}$ is a groupoid of germs if it is étale and coincides with the groupoid of germs of the associated pseudogroup $\tilde{\mathfrak{G}}$.

**Definition 2.1.2.** Let $\mathfrak{G}$ be a groupoid of germs. We say that $U \subset \mathfrak{G}$ is extendable if there is $V \in \tilde{\mathfrak{G}}$ such that $U \subset V$.

Since we assume that the space of units $\mathfrak{G}(0)$ is locally compact and metrizable, every element of $\mathfrak{G}$ has a compact extendable neighborhood $U$. Moreover, we may find such $U$ that there exists an element $\bar{U} \in \mathfrak{G}$ such that $\bar{U} \supset U$ and $\bar{U}$ is extendable.

**Definition 2.1.3.** Suppose that $\mathfrak{G}(0)$ is a metric space. For $U \in \tilde{\mathfrak{G}}$ and $g \in \mathfrak{G}$ we say that $g$ is $\epsilon$-contained in $U$ if the $\epsilon$-neighborhood of $o(g)$ is contained in $o(U)$.

**Lemma 2.1.1.** Let $C \subset \mathfrak{G}$ be a compact set, and let $\mathcal{U} \subset \tilde{\mathfrak{G}}$ be an open covering of $C$. Then there exists $\epsilon > 0$ such that for every $g \in C$ there exists $U \in \mathcal{U}$ such that $g$ is $\epsilon$-contained in $U$.

In conditions of the lemma, we say that $\epsilon$ is a Lebesgue’s number of the covering $\mathcal{U}$ of $C$.

**Proof.** The proof basically repeats the proof of the classical Lebesgue’s number lemma. We may assume that $\mathcal{U}$ is finite, and that no element of $\mathcal{U}$ covers the whole set $C$. For every $g \in C$ and $U \in \mathcal{U}$ denote by $\delta_{g,U}$ supremum of numbers $\epsilon \geq 0$ such that $g$ is $\epsilon$-contained in $U$. Denote $f(g) = \frac{1}{|\mathcal{U}|} \sum_{U \in \mathcal{U}} \delta_{g,U}$. The function $f : C \rightarrow \mathbb{R}$ is continuous, since $\mathfrak{G}$ is étale. It is strictly positive, since $\mathcal{U}$ covers $C$. Let $\epsilon$ be a positive lower bound for the values of $f$ on $C$. Then for every $g \in C$ the average $f(g)$ of numbers $\delta_{g,U}$ is greater than $\epsilon$, hence one of the numbers $\delta_{g,U}$ is greater than $\epsilon$. Consequently, $\epsilon$ satisfies the conditions of the lemma. \hfill \Box

### 2.2 Actions of groupoids and equivalence

#### Actions of groupoids

The following definition appear in [MRW87] and [BH99, III.\mathcal{G} Definition 3.11].

**Definition 2.2.1.** A (right) action of $\mathfrak{G}$ on a space $\mathfrak{B}$ over an open map $P : \mathfrak{B} \rightarrow \mathfrak{G}(0)$ is a continuous map $(x,g) \mapsto x \cdot g$ from the set $\mathfrak{B} \times_P \mathfrak{G} = \{(x,g) : P(x) = t(g)\}$
CHAPTER 2. PRELIMINARIES ON GROUPOIDS AND PSEUDOGROUPS

to $\mathcal{B}$ such that $P(x \cdot g) = o(g)$, and $(x \cdot g_1) \cdot g_2 = x \cdot g_1 g_2$ for all $x \in \mathcal{B}$ and $g_1, g_2 \in \mathcal{G}$ such that $P(x) = t(g_1)$ and $o(g_1) = t(g_2)$.

Similarly, the left action is a map $(g, x) \mapsto g \cdot x$ from $\mathcal{G} \times_p \mathcal{B} = \{(g, x) : P(x) = o(g)\}$ to $\mathcal{B}$ such that $P(g \cdot x) = t(g)$ and $g_1 \cdot (g_2 \cdot x) = g_1 g_2 \cdot x$.

**Definition 2.2.2.** Suppose that we have a right action of $\mathcal{G}$ on $\mathcal{B}$ over a map $P : \mathcal{B} \to \mathcal{G}^{(0)}$. The groupoid of the action $\mathcal{B} \rtimes \mathcal{G}$ is the space $\mathcal{B} \times_p \mathcal{G}$ together with multiplication

$$(x_1, g_1) \cdot (x_2, g_2) = (x_1, g_1 g_2),$$

where the product is defined if and only if $x_2 = x_1 \cdot g_1$.

Note that the space of units of $\mathcal{B} \times \mathcal{G}$ is the set of elements $(x, g) \in \mathcal{B} \times_p \mathcal{G}$ such that $g$ is a unit. The unit $(x, g)$ is uniquely determined by $x$, thus the unit space of $\mathcal{B} \times \mathcal{G}$ is naturally identified with $\mathcal{B}$.

Note also that if $\mathcal{G}$ is a groupoid of germs then the action groupoid $\mathcal{B} \rtimes \mathcal{G}$ is also a groupoid of germs.

A right action is free if $x \cdot g = x$ implies that $g$ is a unit. The action is proper if the map

$$(o, t) : \mathcal{B} \times \mathcal{G} \to \mathcal{B}^2 : (x, g) \mapsto (x, x \cdot g)$$

is proper. In other words, the action is proper, if for every compact set $C \subset \mathcal{B}$ the set of elements $g \in \mathcal{G}$ such that $C \cdot g \cap C \neq \emptyset$ is compact.

Suppose that we have a free proper action of $\mathcal{G}$ on $\mathcal{B}$. Consider the space $\mathcal{B} *_p \mathcal{B} = \{(x, y) \in \mathcal{B}^2 : P(x) = P(y)\}$. We have a natural groupoid structure on $\mathcal{B} *_p \mathcal{B}$ given by

$$(x, y) \cdot (y, z) = (x, z),$$

where $(x, y)$ is a unit if and only if $x = y$ (so that the origin and the target maps are $o(x, y) = y$ and $t(x, y) = x$).

The groupoid $\mathcal{G}$ acts on $\mathcal{B} *_p \mathcal{B}$ by the diagonal action. Consider the space $\mathcal{B} *_p \mathcal{B} / \mathcal{G}$ of orbits of the action (which is Hausdorff, for instance by \cite[Proposition 4.2.1]{Nek05}). Note that the groupoid structure on $\mathcal{B} * \mathcal{B}$ induces a groupoid structure on the quotient, since from $(x_1, y_1) = (x, y) \cdot g_1$ and $(y_1, z_1) = (y, z) \cdot g_2$ follows $y \cdot g_1 = y_1 \cdot g_2$, which implies $g_1 = g_2$ by freeness of the action, so that $(x_1, z_1) = (x, z) \cdot g_1$.

**Equivalence of groupoids and pseudogroups**

The following definition was introduced in \cite{MRW87}.

**Definition 2.2.3.** Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A $(\mathcal{G}, \mathcal{H})$-equivalence is given by a locally compact Hausdorff space $\mathcal{B}$ together with a free proper left $\mathcal{G}$-action and a free proper right $\mathcal{H}$-action over maps $P_1 : \mathcal{B} \to \mathcal{G}^{(0)}$ and $P_2 : \mathcal{B} \to \mathcal{H}^{(0)}$, such that

1. the actions of $\mathcal{G}$ and $\mathcal{H}$ commute, i.e., if $g \cdot x$ and $x \cdot h$ are defined, then $(g \cdot x) \cdot h$ and $g \cdot (x \cdot h)$ are defined and are equal;
2. the maps $P_1$ and $P_2$ induce bijections $\mathcal{B}/\mathcal{H} \to \mathfrak{E}^{(0)}$ and $\mathcal{G}\setminus\mathcal{B} \to \mathcal{H}^{(0)}$.

It is useful to imagine elements $x \in \mathcal{B}$ of an equivalence as arrows from $P_2(x)$ to $P_1(x)$ and to interpret the actions of $\mathcal{G}$ and $\mathcal{H}$ as compositions of these arrows with the arrows of the groupoids.

Note that the first two conditions imply that $P_1$ and $P_2$ induce well defined maps $\mathcal{B}/\mathcal{H} \to \mathfrak{E}^{(0)}$ and $\mathcal{G}\setminus\mathcal{B} \to \mathcal{H}^{(0)}$, respectively. Namely, by the second condition, if $g \cdot x$ and $x \cdot h$ are defined, i.e., if $P_1(x) = o(g)$ and $P_2(x) = t(h)$, then $(g \cdot x) \cdot h$ and $g \cdot (x \cdot h)$ are defined, i.e.,

$$P_2(g \cdot x) = t(h) = P_2(x), \quad P_1(x \cdot h) = o(g) = P_1(x).$$

Condition (2) of the definition implies that for every pair $(x, y) \in \mathcal{B} \ast_{P_1} \mathcal{B}$ there exists $h \in \mathcal{H}$ such that $x \cdot h = y$. The element $h$ is unique, by freeness of the action. The map $(x, y) \mapsto h$ is invariant under the action of $\mathcal{G}$ (since $x \cdot h = y$ is equivalent to $g \cdot x \cdot h = g \cdot y$), and it induces a map from $\mathcal{G}\setminus(\mathcal{B} \ast_{P_1} \mathcal{B})$ to $\mathcal{H}$. It is easy to check that this map is an isomorphism of groupoids.

If $\mathcal{B}_1$ is a $(\mathfrak{E}, \mathcal{H})$-equivalence, and $\mathcal{B}_2$ is a $(\mathcal{H}, \mathfrak{E})$-equivalence, then $\mathcal{B}_1 \otimes_{\mathcal{H}} \mathcal{B}_2$ is a $(\mathfrak{E}, \mathfrak{E})$-equivalence, where $\mathcal{B}_1 \otimes_{\mathcal{H}} \mathcal{B}_2$ is the quotient of the space

$$\{(x, y) : x \in \mathcal{B}_1, y \in \mathcal{B}_2, P_2(x) = P_1(y)\} \subset \mathcal{B}_1 \times \mathcal{B}_2$$

by the equivalence relation $(x, y) = (x \cdot g, g^{-1} \cdot y)$.

**Equivalence of étale groupoids**

**Lemma 2.2.1.** If $\mathcal{B}$ is a right $\mathfrak{E}$-space and $\mathfrak{E}$ is étale, then the action groupoid $\mathcal{B} \times \mathfrak{E}$ is étale.

**Proof.** Suppose that the action is defined over a map $P : \mathcal{B} \to \mathfrak{E}^{(0)}$. The origin and target maps of the action groupoid are given by

$$o(x, g) = (x \cdot g, o(g)), \quad t(x, g) = (x, t(g)).$$

If $U$ is an open neighborhood of $g$ such that $o : U \to \mathfrak{E}^{(0)}$ and $t : U \to \mathfrak{E}^{(0)}$ are homeomorphic embeddings, then $U' = \mathcal{B} \times \mathfrak{E} \cap \mathcal{B} \times U$ is an open set such that restrictions of $o$ and $t$ onto $U'$ are homeomorphic embeddings. Namely, the local inverses of $o$ and $t$ are

$$(x, P(x)) \mapsto (x \cdot (o^{-1}(P(x)))^{-1}, o^{-1}(P(x))), \quad (x, P(x)) \mapsto (x, t^{-1}(P(x))),$$

where $o^{-1}$ and $t^{-1}$ are the inverses of $o : U \to \mathfrak{E}(U)$ and $t : U \to \mathfrak{T}(U)$, respectively. \hfill \Box

Suppose that $\mathcal{B}$ is a $(\mathfrak{E}, \mathcal{H})$-equivalence, where $\mathfrak{E}$ and $\mathcal{H}$ are étale groupoids. Let $P_1 : \mathcal{B} \to \mathfrak{E}^{(0)}$ and $P_2 : \mathcal{B} \to \mathcal{H}^{(0)}$ be the maps over which the actions are defined.

**Lemma 2.2.2.** The maps $P_1$ and $P_2$ are étale.
CHAPTER 2. PRELIMINARIES ON GROUPOIDS AND PSEUDOGROUPS

Proof. Let \( x \in \mathcal{B} \) be an arbitrary point and let \( U \) be a compact neighborhood of \( x \). By properness of the groupoid \( \mathcal{B} \times \mathcal{H} \) the set \( C \) of elements \( (y, h) \in \mathcal{B} \times \mathcal{H} \) such that \( y \in U \) and \( y \cdot h \in U \) is compact. Since \( \mathcal{B} \times \mathcal{H} \) is \'{e}tale, the set of units of \( \mathcal{B} \times \mathcal{H} \) is open, hence \( C' = C \setminus (\mathcal{B} \times \mathcal{H})^{(0)} \) is compact.

The groupoid \( \mathcal{B} \times \mathcal{H} \) is \'{e}tale and principal, hence for \((y, h) \in C' \) there exists a neighborhood \( V \) of \((y, h) \) such that \( V \) is compact and either \( x \not\in \text{o}(V) \cup \text{t}(V) \) (if \( x \neq y \) and \( x \neq y \cdot h \)), or \( x \in \text{o}(V) \) and \( x \not\in \text{t}(V) \) (if \( x = y \)), or \( x \not\in \text{o}(V) \) and \( x \in \text{t}(V) \) (if \( x = y \cdot h \)). It follows that there is a finite set \( A \) of elements \( V \in \mathcal{B} \times \mathcal{H} \) covering \( C' \) and satisfying the above conditions. Then

\[
U' = U \setminus \bigcup_{V \in A, x \not\in \text{o}(V)} \text{o}(V) \cup \bigcup_{V \in A, x \not\in \text{t}(V)} \text{t}(V)
\]

is a neighborhood of \( x \) such that there is no non-unit element \((y, h) \in \mathcal{B} \times \mathcal{H} \) such that \( y \in U' \) and \( y \cdot h \in U' \).

Similarly, we can find a neighborhood \( U'' \subset U \) of \( x \) such that there is no non-unit element \((g, y) \in \mathcal{G} \times \mathcal{B} \) such that \( y \in U'' \) and \( g \cdot y \in U'' \). Consider the intersection \( U' \cap U'' \). It is a neighborhood of \( x \) such that its closure is a compact subset of \( U \) and such that the maps \( P_1 : U' \cap U'' \to \mathcal{G}^{(0)} \) and \( P_2 : U' \cap U'' \to \mathcal{H}^{(0)} \) are injective, since \( P_1(y_1) = P_1(y_2) \) implies existence of \( h \in \mathcal{H} \) such that \( y_1 \cdot h = y_2 \). Since \( P_i \) are continuous and open, we conclude that they are homeomorphic embeddings on \( U' \cap U'' \). \( \square \)

Let \( \mathcal{B} \) be a \((\mathcal{G}, \mathcal{H})\)-equivalence. Consider the disjoint union

\[
\mathcal{G} \vee_{\mathcal{B}} \mathcal{H} := \mathcal{G} \cup \mathcal{B} \cup \mathcal{B}^{-1} \cup \mathcal{H},
\]

where \( \mathcal{B}^{-1} \) is a copy of \( \mathcal{B} \). We denote by \( g^{-1} \) the element of \( \mathcal{B}^{-1} \) corresponding to \( g \in \mathcal{B} \). Denote \( \text{t}(x) = P_1(x), \text{o}(x) = P_2(x), \text{o}(x^{-1}) = P_1(x), \text{t}(x^{-1}) = P_2(x) \) for \( x \in \mathcal{B} \). Define multiplication on \( \mathcal{G} \vee_{\mathcal{B}} \mathcal{H} \) using multiplications inside \( \mathcal{G} \) and \( \mathcal{H} \), actions of \( \mathcal{G} \) and \( \mathcal{H} \) on \( \mathcal{B} \), “flipped” actions:

\[
x^{-1} \cdot g = (g^{-1} \cdot x)^{-1}, \quad h \cdot x^{-1} = (x \cdot h^{-1})^{-1}
\]
on \( \mathcal{B}^{-1} \), and multiplication between elements of \( \mathcal{B} \) and \( \mathcal{B}^{-1} \) given by the rules

\[
x \cdot y^{-1} = g \in \mathcal{G}, \quad \text{if and only if} \quad g \cdot y = x
\]

\[
x^{-1} \cdot y = h \in \mathcal{H}, \quad \text{if and only if} \quad x \cdot h = y
\]

for \( x, y \in \mathcal{B} \). Note that the corresponding elements \( g \) and \( h \) exist and are unique by the definition of equivalences.

It is easy to see that the defined multiplication defines a groupoid structure on \( \mathcal{G} \vee_{\mathcal{B}} \mathcal{H} \).

**Proposition 2.2.3.** If \( \mathcal{G} \) and \( \mathcal{H} \) are \'{e}tale groupoids, then \( \mathcal{G} \vee_{\mathcal{B}} \mathcal{H} \) is also an \'{e}tale groupoid.
Proof. It follows from Lemma 2.2.2 that the operations in the groupoid $G \lor B H$ are continuous and that the obtained groupoid is étale.

Now we can introduce a more convenient definition of equivalence in the case of groupoids of germs.

**Proposition 2.2.4.** Let $G$ and $H$ be groupoids of germs of pseudogroups $\tilde{G}$ and $\tilde{H}$, respectively. The groupoids $G$ and $H$ are equivalent if and only if there exists a pseudogroup $\tilde{G} \lor \tilde{H}$ acting on the disjoint union $G(0) \sqcup H(0)$ such that the restriction of $\tilde{G} \lor \tilde{H}$ onto $G(0)$ (resp. $H(0)$) coincides with $\tilde{G}$ (resp. $\tilde{H}$), and every orbit of $\tilde{G} \lor \tilde{H}$ is a union of one $\tilde{G}$-orbit and one $\tilde{H}$-orbit.

Proof. If $G$ and $H$ are equivalent, then it is easy to see that the pseudogroup associated with the groupoid $G \lor B H$ satisfies the conditions of the proposition.

In the other direction, the set $B$ of germs of elements of $\tilde{G} \lor \tilde{H} \setminus (\tilde{G} \cup \tilde{H})$ with actions of $G$ and $H$ on $B$ defined by composition is an equivalence. The only condition we have to check is properness of the actions. Any compact subset $C$ of $B$ can be covered by compact extendable closures of elements of $\tilde{G} \lor \tilde{H}$. Then for every element $g$ of the action groupoid with origin and target in $C$ there will exist germs $h_1$ and $h_2$ of elements of the covering such that $g = h_1^{-1} h_2$.

**Localization**

Standard approaches to constructing pseudogroups equivalent to a given one are **localization** and **restriction**, which are defined in the following way.

Suppose that $U \subset G(0)$ is an open transversal. Then restriction $\tilde{G}|_U$ is equivalent to $\tilde{G}$ (here $f : U \longrightarrow G(0)$ is the identical embedding).

More generally, we have the following method of constructing a pseudogroup equivalent to a given one.

**Proposition 2.2.5.** Let $G$ be a groupoid, and let $f : Y \longrightarrow G(0)$ be an étale map such that the range of $f$ is a $G$-transversal. Consider the pseudogroup $f^*(\tilde{G})$ generated by local homeomorphisms $F : U \longrightarrow V$ between open subsets of $Y$ such that $f|_U : U \longrightarrow f(U)$, $f|_V : V \longrightarrow f(V)$ are homeomorphisms, and $f|_V \circ F \circ f|_U^{-1}$ is an element of $\tilde{G}$. Then $f^*(\tilde{G})$ is a pseudogroup equivalent to $\tilde{G}$.

Proof. It is easy to check that we can take the pseudogroup $f^*(\tilde{G}) \lor \tilde{G}$ to be equal to the pseudogroup generated by $\tilde{G}$ and restrictions of $f$.

**Example 1.** Consider an atlas of an $n$-dimensional manifold $M$, and let $X$ be the disjoint union of the corresponding open subsets of $\mathbb{R}^n$. Then we have a natural pseudogroup of changes of charts $G$ acting on $X$. It is equal to the lift of the trivial pseudogroup on $M$ by the natural quotient map $X \longrightarrow M$. It follows that the pseudogroup $G$ is equivalent to the trivial pseudogroup (i.e., the pseudogroup consisting of identity maps on open subsets) of the manifold $M$.  
Example 2. Let $G$ be a group acting properly and freely on a space $X$. Then the groupoid of the action $X \rtimes G$ is equivalent to the trivial (i.e., consisting only of units) groupoid of the space of orbits $X/G$. In particular, a manifold is equivalent to the groupoid of the action of the fundamental group on the universal covering.

Another example of application of Proposition 2.2.5 are localizations onto open coverings. If $\{U_i\}_{i \in I}$ is a collection of open subsets of $G^{(0)}$ such that their union is a $G$-transversal, then localization $G|_{\{U_i\}}$ is defined by the map from the disjoint union $\bigsqcup_{i \in I} U_i$ into $G^{(0)}$ equal to the identical embedding on each $U_i$.

We denote by $(U_i, i)$ the copies of $U_i$ in the disjoint union. If $g \in G$ is such that $o(g) \in U_i$ and $t(g) \in U_j$, then we denote by $(g, i, j)$ the corresponding element of the localization such that $o(g, i, j) \in (U_i, i)$ and $t(g, i, j) \in (U_j, j)$.

Equivalence for group actions

Let $G_i$ be a group acting by homeomorphisms on $X_i$, for $i = 1, 2$.

Proposition 2.2.6. The groupoids of the actions $G_1 \rtimes X_1$ and $G_2 \rtimes X_2$ are equivalent if and only if there exists a space $B$ and commuting free proper left and right actions of $G_1$ and $G_2$, respectively, on $B$ such that $B/G_2$ (resp. $G_1\backslash B$) is $G_1$-equivariantly (resp. $G_2$-equivariantly) homeomorphic to $X_1$ (resp. $X_2$).

Proof. Suppose that $B$ is a $(G_1 \rtimes X_1, G_2 \rtimes X_2)$-equivalence, and let $P_i : B \to X_i$ be the corresponding maps. For every $x \in B$ and every $g \in G_1$, $h \in G_2$ we can define

$$g \cdot x = (g, P_1(x)) \cdot x$$

and

$$x \cdot h = x \cdot (h, P_2(x)).$$

It is easy to check that we get actions of $G_i$ on $B$. The actions commute. Freeness of the actions of groupoids is equivalent to the usual freeness of the group actions. The groupoid of the action of $G_i \rtimes X_i$ coincides with the groupoid of the action of $G_i$, hence properness of the actions of groupoids is equivalent to properness of the actions of the groups.

By the remaining condition of the definition of equivalence of groupoids, the map $P_i$ induces a homeomorphism of $B/G_2$ with $X_1$, which is $G_1$-equivariant, since the actions of $G_1$ and $G_2$ commute. The other direction of the proof is similar.

Proposition 2.2.7. If $B$ together with left and right actions of $G_1$ and $G_2$ over maps $P_i : B \to X_i$ is an equivalence between the groupoids $G_1 \rtimes X_1$ and $G_2 \rtimes X_2$, then $P_i : B \to X_i$ are covering maps.
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Let \( \Phi : \mathcal{G} \to \mathcal{B} \) be the universal covering. Then there exists a relatively compact neighborhood \( U \) of \( y_1 \) such that \( P \) is a covering map.

This means that \( \Phi \) is the universal covering. The union of the sets \( \sim \) is a group \( \mathcal{G} \) of homeomorphisms of \( X \). We have a natural epimorphism \( \phi : \mathcal{G} \to \mathcal{B} \). Kernel of \( \phi \) is the fundamental group \( \pi_1(X) \). We call \( \mathcal{G} \) the lift of \( G \) to the universal covering of \( X \).

Proof. The lift of a group action to the universal covering is equivalent as a groupoid to the original action (see Proposition 2.2.6). Choose a point \( x \) containing \( y \). Then the groupoids \( \mathcal{G}_x \) and \( \mathcal{G} \) are equivalent if and only if their lifts to the universal coverings \( \mathcal{X} \) are topologically conjugate.

Recall that two group actions \( (\mathcal{G}_1, X_1) \) and \( (\mathcal{G}_2, X_2) \) are topologically conjugate if there exist a homeomorphism \( F : X_1 \to X_2 \) and an isomorphism \( \phi : \mathcal{G}_1 \to \mathcal{G}_2 \) such that \( F(g(x)) = \phi(g)(F(x)) \) for all \( x \in X_1 \) and \( g \in \mathcal{G}_1 \).

Theorem 2.2.8. Let \( X_1 \) and \( X_2 \) be connected and semi-locally simply connected, and let, for \( i = 1, 2 \), \( G_i \) be a group acting faithfully on \( X_i \) by homeomorphisms.

Then the groupoids \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are equivalent if and only if their lifts to the universal coverings \( \mathcal{X}_i \) are topologically conjugate.

Proof. The lift of a group action to the universal covering is equivalent as a groupoid to the original action (see Proposition 2.2.6). This proves the “if” part of the theorem.

Suppose that the actions \( (\mathcal{G}_1, X_1) \) are equivalent. Let \( \mathcal{B} \) be as in Proposition 2.2.7. Choose a point \( x \) in \( \mathcal{B} \), and let \( X \) be the connected component of \( \mathcal{B} \) containing \( x \). Since \( P_i : \mathcal{B} \to X_i \) are covering maps, \( P_i : X \to X_i \) are also covering maps. In particular, they are onto.

The actions of \( \mathcal{G}_i \) on \( \mathcal{B} \) taken together form an action of \( \mathcal{G}_1 \times \mathcal{G}_2 \) (by the rule \( (g_1, g_2)(x) = g_1 \cdot x \cdot g_2^{-1} \)). Let \( H = \{ g \in \mathcal{G}_1 \times \mathcal{G}_2 : g(X) = X \} \) be the stabilizer of \( X \) in \( \mathcal{G}_1 \times \mathcal{G}_2 \). The action of \( H \) on \( X \) is proper and free, since the actions of \( \mathcal{G}_i \) on \( \mathcal{B} \) are proper and free.

If \( x, y \in X \) are such that \( P_1(x) = P_1(y) \), then there exists a unique element \( g \in G_2 \) such that \( g(x) = y \). Note that then \( g \in H \). It follows that \( G_2 \cap H \) is the group of deck transformations of the covering map \( P_1 : X \to X_1 \), and that the covering is normal.

Let us show that restriction onto \( H \) of the projection \( \mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{G}_1 \) is surjective. Let \( g \in \mathcal{G}_1 \) be an arbitrary element. Since the map \( \mathcal{B} / G_2 \to X_1 \)
induced by $P_1$ is a homeomorphism, and $G_2$ acts by permutations on the set of connected components of $\mathcal{B}$, there exists $g_2 \in G_2$ such that $g \cdot X \cdot g_2 = X$. Then $(g, g_2^{-1}) \in H$.

We have proved that there exists a pair of covering maps $X \rightarrow X_i$ and a group $H$ acting on $X$ such that the lifts of $(G_i, X_i)$ by the covering maps coincide with $(H, X)$. It follows that the lifts of $(G_i, X_i)$ to the universal coverings $\tilde{X}_i$ both are topologically conjugate to the lift of $(H, X)$ to the universal covering of $X$.

\[ \blacksquare \]

**Corollary 2.2.9.** Groupoids of the actions of $G_i$ on simply connected spaces $X_i$ are equivalent if and only if the actions are topologically conjugate. Groupoids of free proper actions $(G_i, X_i)$, $i = 1, 2$, on connected semi-locally simply connected spaces are equivalent if and only if the spaces $X_i/G_i$ are homeomorphic.

### 2.3 Compactly generated groupoids and their Cayley graphs

We say that $X \subset \mathfrak{G}^{(0)}$ is a *topological transversal* if there is an open transversal $X_0 \subset X$.

The following definition is equivalent to a definition by A. Haefliger [Hae02].

**Definition 2.3.1.** Let $\mathfrak{G}$ be an étale groupoid. A *compact generating pair* of $\mathfrak{G}$ is a pair of sets $(S, X)$, where $S \subset \mathfrak{G}$ and $X \subset \mathfrak{G}^{(0)}$ are compact, $X$ is a topological transversal, and for every $g \in \mathfrak{G}|_X$ there exists $n$ such that that $\bigcup_{k \geq 1} (S \cup S^{-1})^k$ is a neighborhood of $g$ in $\mathfrak{G}|_X$. The set $S$ in a generating pair $(S, X)$ is called a *generating set*.

A groupoid $\mathfrak{G}$ is *compactly generated* if it has a compact generating pair.

In other words, $(S, X)$ is a compact generating pair, if $S$ is a generating set of $\mathfrak{G}|_X$ and the word length on $\mathfrak{G}|_X$ defined with respect to $S$ is locally bounded.

**Proposition 2.3.1.** Let $\mathfrak{G}$ be a compactly generated groupoid of germs. Then for every compact topological transversal $X \subset \mathfrak{G}^{(0)}$ there exists a compact set $S \subset \mathfrak{G}|_X$ such that $(S, X)$ is a generating pair.

**Proof.** Let $(S', X')$ be a compact generating pair of $\mathfrak{G}$. Let $X'_0 \subset X'$ and $X_0 \subset X$ be open transversals.

For every $x \in X$ there exists an element $U \in \tilde{\mathfrak{G}}$ such that $o(U) \ni x$, $t(U) \subset X'_0$, and $\overline{U}$ is compact. We get an open covering of $X$ by the sets $o(U)$.

Choose a finite sub-covering $\{o(U_1), o(U_2), \ldots, o(U_r)\}$.

Similarly, there exists a finite set $\{U_{r+1}, U_{r+2}, \ldots, U_{r+s}\}$ of elements of $\tilde{\mathfrak{G}}$ such that $\overline{U_i}$ are compact, $t(U_i)$ cover $X'$, and $o(U_i) \subset X_0$.

Denote $W = \bigcup_{i=1}^{r+s} U_i$, and consider the set $S = (W^{-1} \cdot S' \cdot W) \cap \mathfrak{G}|_X$. The set $S$ has compact closure. For every $g \in \mathfrak{G}|_X$ there exist $i, j \in \{1, \ldots, r\}$ such
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that \( U_i \cdot g \cdot U_j^{-1} \in \mathcal{G}|_{X'_0} \). Since \((S', X')\) is a generating pair, there exists \( m \) such that \( \bigcup_{k=1}^{m} (S' \cup (S')^{-1})^k \) is a neighborhood of \( U_i \cdot g \cdot U_j^{-1} \). Then

\[
W^{-1} \bigcup_{k=1}^{m} (S' \cup (S')^{-1})^k W \supset U_i^{-1} \cdot \bigcup_{k=1}^{m} (S' \cup (S')^{-1})^k \cdot U_j
\]
is a neighborhood of \( g \). Since for every \( x \in X' \) there exists \( U_{r+t} \) such that \( t(U_{r+t}) \ni x \) and \( o(U_{r+t}) \subset X_0 \), we have

\[
(W^{-1} \cdot (S' \cup (S')^{-1})^k \cdot W) \cap \mathcal{G}|_X \subset \big((W^{-1}S'W \cap \mathcal{G}|_X) \cup (W^{-1}S'W \cap \mathcal{G}|_X)^{-1}\big)^k
\]
for all \( k \). It follows that \( \bigcup_{k=1}^{m} (S \cup S^{-1})^k \) is a neighborhood of \( g \) in \( \mathcal{G}|_X \). \( \square \)

**Corollary 2.3.2.** If \( \mathcal{G} \) is a compactly generated groupoid of germs, then every equivalent \( \acute{e}tale \) groupoid is also compactly generated.

**Definition 2.3.2.** Let \((S, X)\) be a compact generating pair of \( \mathcal{G} \). For \( x \in X \) the **Cayley graph** \( \mathcal{G}(x, S) \) is the graph with the set of vertices \( \mathcal{G}_x \) in which two vertices \( g, h \) are connected by an edge if and only if \( gh^{-1} \in S \) or \( hg^{-1} \in S \).

Since a compact subset of \( \mathcal{G} \) can be covered by a finite number of local homeomorphisms, the Cayley graphs \( \mathcal{G}(x, S) \) have uniformly bounded degree.

If \( x \in X \) has trivial isotropy group, then the map \( g \mapsto t(g) \) from \( \mathcal{G}_x \) to the \( \mathcal{G}|_X \)-orbit of \( x \) is a bijection. Then the Cayley graph \( \mathcal{G}(x, S) \) is naturally isomorphic to the **orbital graph**. The set of vertices of the orbital graph is the orbit of \( x \), and two vertices \( y_1, y_2 \) are connected by an arrow from \( y_1 \) to \( y_2 \) if there exists a generator \( s \in S \) such that \( o(s) = y_1 \) and \( t(s) = y_2 \).

It is not hard to show that if \( \mathcal{G} \) is a compactly generated groupoid of germs, then the set of points \( x \in \mathcal{G}^{(0)} \) with trivial isotropy group is co-meager. It follows that generically the Cayley graph \( \mathcal{G}(x, S) \) is naturally isomorphic to the orbital Schreier graph of \( x \).

**Proposition 2.3.3.** If \((S_1, X)\) and \((S_2, X)\) are compact generating pairs of \( \mathcal{G} \), then there exists \( n \) such that \( \bigcup_{k=1}^{n} (S_1 \cup S_1^{-1})^k \) is a neighborhood of \( S_2 \) and \( \bigcup_{k=1}^{n} (S_2 \cup S_2^{-1})^k \) is a neighborhood of \( S_1 \).

**Proof.** For every \( g \in S_1 \) there exists \( n_g \) such that \( \bigcup_{k=1}^{n_g} (S_2 \cup S_2^{-1})^k \) is a neighborhood of \( g \). Then, by compactness of \( S_1 \) there exists \( n \) such that \( \bigcup_{k=1}^{n} (S_2 \cup S_2^{-1})^k \) is a neighborhood of every point \( g \in S_1 \). \( \square \)

**Corollary 2.3.4.** If \((S_1, X)\) and \((S_2, X)\) are compact generating pairs of \( \mathcal{G} \), then the identity map is a quasi-isometry of the Cayley graphs \( \mathcal{G}(x, S_1) \) and \( \mathcal{G}(x, S_2) \).

**Lemma 2.3.5.** Let \( X_1 \subset X_2 \) be compact topological transversals. Let \( x \in X_1 \). Then the set \( \mathcal{G}^{X_1} \) is a net in the Cayley graph \( \mathcal{G}(x, S) \), where \((X_2, S)\) is a compact generating pair.
Proof. We can find a finite collection of local homeomorphisms \( \{U_1, \ldots, U_k\} \) such that \( o(U_i) \) cover \( X_2 \), \( t(U_i) \subset X_0 \), and \( U_i \) are compact. Then there exists \( n \) such that \( \bigcup_{i=1}^k (S \cup S^{-1})^k \). Then for every \( g \in \mathfrak{G}_x \cap \mathfrak{G}|_{X_2} \) there exists \( U_i \) such that \( U_i \cdot g \in \mathfrak{G}|_{X_1} \), hence the set \( \mathfrak{G}_x \cap \mathfrak{G}|_{X_1} \) is an \( n \)-net in \( \mathfrak{G}(x,S) \).

Corollary 2.3.6. Any two Cayley graphs \( \mathfrak{G}(x, S_1) \), \( \mathfrak{G}(x, S_2) \) associated with the same point \( x \in X \) of a compactly generated groupoid are quasi-isometric.

Example 3. Let \( \theta \in \mathbb{R} \) be an irrational number. Consider the group \( G \cong \mathbb{Z}^2 \) acting on \( \mathbb{R} \) and generated by the transformations \( x \mapsto x+1 \) and \( x \mapsto x+\theta \). Let \( \mathfrak{G} \) be the corresponding groupoid of germs. Note that all \( G \)-orbits are dense, hence any open subset of \( \mathbb{R} \) is a \( G \)-transversal.

Let us represent the action of \( G \) on \( \mathbb{R} \) by projecting onto the first coordinate its action on \( \mathbb{R}^2 \) generated by the maps \( a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 1 \\ y \end{pmatrix} \) and \( b : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + \theta \\ y + 1 \end{pmatrix} \).

If \( I = [x_1, x_2] \subset \mathbb{R} \) is a finite closed interval and \( t \in (x_1, x_2) \), then \( \mathfrak{G}_t \) can be represented by the part \( \hat{I} \) of the \( G \)-orbit of \( \begin{pmatrix} t \\ 0 \end{pmatrix} \) that is projected to \( [x_1, x_2] \).

Each point \( \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \in \hat{I} \) represents the germ of the translation \( x \mapsto x + (r_1 - t) \) at \( t \). See Figure 2.1.

Fix a sufficiently big set \( R \subset G \), and consider the set \( S_R \) of germs at points of \( I \) of elements of \( R \). Then the Cayley graph \( \mathfrak{G}(t, S_R) \) is isomorphic to the graph with the set of vertices \( I \) in which two vertices are connected by an edge if and only if is the image of the other under the action of an element of \( R \).
It is easy to see that the Cayley graphs of $G$ are quasi-isometric to $\mathbb{R}$. Increasing $I$ increases the set of vertices of the Cayley graph, but the smaller Cayley graph is a net inside the larger one.

Note that groupoid $G$ is equivalent to the groupoid generated by rotation $x \mapsto x + \theta \pmod{1}$ of the circle $\mathbb{R}/\mathbb{Z}$. Orbits of the latter can be naturally identified with $\mathbb{Z}$, and the Cayley graphs of this groupoid coincide with the Cayley graphs of $\mathbb{Z}$.

**Example 4.** Consider the group $G$ acting on the Cantor set $\{0,1\}^\infty$ and generated by transformations $a$ and $b$ defined inductively by the rules

$$a(0w) = 1w, \quad a(1w) = 0b(w), \quad b(0w) = 0w, \quad b(1w) = 1a(w)$$

for all $w \in \{0,1\}^\infty$. This group is the *iterated monodromy group* of the complex polynomial $z^2 - 1$ (see [5.2]). It is often called the *basilica group*.

We can take the whole $\{0,1\}^\infty$ as our transversal, and take the set of all germs of transformations $a$ and $b$ as the generating set $S$. It is not hard to show that the groupoid of germs $G(x,S)$ is principal ($G$ is *regular* in terminology of [Nek09]). It follows that the Cayley graphs $G(x,S)$ coincide with the orbital graphs of the action of $G$ on $\{0,1\}^\infty$ (orbital graphs are also called *Schreier graphs*, in the case of group actions).

The Schreier graphs of the Basilica group where extensively studied in the paper [DDMSN10]. In particular, it was shown that depending on the basepoint $x$ the graphs $G(x,S)$ can have one, two, or four ends. In particular, the quasi-isometry class of $G(x,S)$ depends on $x$. In fact, it follows from their results that there are uncountably many different quasi-isometry classes of the Cayley graphs $G(x,S)$.

### 2.4 Relations in Hausdorff groupoids

Let $G$ be a Hausdorff groupoid of germs. Let $S$ be a finite set of extendable compact subsets $F \subset G$. Let $\hat{F} \supset K$ be extensions of the sets $F$ to elements of $\hat{G}$. We will denote the set of extensions $\hat{F}$ by $\hat{S}$. Denote $S = \bigcup_{F \in S} F$.

Suppose that $g_1 \cdot g_2 \cdots g_n = o(g_n)$ for some $g_i \in S$. Denote by $F_i$ the element of $S$ such that $g_i \in F_i$.

Then $g_1 \cdot g_2 \cdots g_n = o(g_n)$ is a germ of the composition $\hat{F}_1 \cdot \hat{F}_2 \cdots \hat{F}_n$. Denote by $E$ the set of points $x \in o(\hat{F}_n)$ such that the germ of $\hat{F}_1 \cdot \hat{F}_2 \cdots \hat{F}_n$ at $x$ is defined and is trivial (i.e., equal to the germ of the identity).

**Lemma 2.4.1.** The set $E$ is relatively closed and open in $o(\hat{F}_1 \cdot \hat{F}_2 \cdots \hat{F}_n)$.

**Proof.** The set $E$ is obviously open. Suppose that $x$ belongs to the closure of $E$ in $o(\hat{F}_1 \cdot \hat{F}_2 \cdots \hat{F}_n)$. Consider the germ $g$ of the homeomorphism $\hat{F}_1 \cdot \hat{F}_2 \cdots \hat{F}_n$ at the point $x$. Intersection of every open neighborhood $V$ of $x$ with $E$ is a non-empty open set, hence every neighborhood of $g$ has a non-empty intersection with the identical homeomorphism, i.e., $g$ and the germ of the identity at $x$ do
not have disjoint neighborhoods. It follows that \( g \) is a germ of identity, hence \( x \) belongs to \( E \).

**Corollary 2.4.2.** For any finite set \( R \) of sequences \((F_1, F_2, \ldots, F_n)\) of elements of \( S \) there exists \( \epsilon > 0 \) such that for all \( x, y \in \mathbb{R} \) such that \( |x - y| < \epsilon \), and for every sequence \((F_1, F_2, \ldots, F_n) \in R\), if the \((F_1 \cdot F_2 \cdots F_n, x)\) is trivial, then so is the germ \((\tilde{F}_1 \cdot \tilde{F}_2 \cdots \tilde{F}_n, y)\).

**Proof.** For every sequence \( r = (F_1, F_2, \ldots, F_n) \in R \) denote by \( E_r \) the set of points \( x \) for which the germ of \( \tilde{F}_1 \cdot \tilde{F}_2 \cdots \tilde{F}_n \) at \( x \) is trivial. The set \( E_r \) is open and relatively closed in \( o(\tilde{F}_1 \cdot \tilde{F}_2 \cdots \tilde{F}_n) \), by Lemma 2.4.1. Domain of \( F_1 \cdot F_2 \cdots F_n \) is contained in \( o(\tilde{F}_1 \cdot \tilde{F}_2 \cdots \tilde{F}_n) \) and is compact. It follows that the intersection \( E_r \cap o(F_1 \cdot F_2 \cdots F_n) \) is compact, and that \( E_r \) is an open neighborhood of \( E_r \cap o(F_1 \cdot F_2 \cdots F_n) \). Consequently, there exists \( \epsilon_r > 0 \) such that for every \( x \in E_r \cap o(F_1 \cdot F_2 \cdots F_n) \) the \( \epsilon_r \)-neighborhood of \( x \) is contained in \( E_r \). We can take \( \epsilon \) equal to minimum of \( \epsilon_r \) for all \( r \in R \).

### 2.5 Groupoids with additional structure

#### Lipschitz structure

**Lemma 2.5.1.** Let \( \mathcal{G} \) be a groupoid of germs, and let \( X_1 \) and \( X_2 \) be compact topological \( \mathcal{G} \)-transversals. Let \( \ell_1 \) be a positive log-scale on \( X_1 \) such that the elements of \( \mathcal{G} \mid X_1 \) are locally Lipschitz with respect to \( \ell_1 \). Then there exists a unique, up to a Lipschitz equivalence, positive log-scale \( \ell_2 \) on \( X_2 \) such that the elements of \( \mathcal{G} \mid X_1 \cup X_2 \) are locally Lipschitz with respect to \( \ell_1 \) and \( \ell_2 \).

**Proof.** There exists a finite set \( F_i, i \in I \), of relatively compact extendable elements of \( \mathcal{G} \) such that \( o(F_i) \) cover \( X_2 \) and \( t(F_i) \) are subsets of the interior of \( X_1 \). Define \( \ell_i(x, y) = \ell_1(F_i(x), F_i(y)) \) for \( x, y \in o(F_i) \). We get a covering of the compact set \( X_2 \) by open subsets \( U_i = o(F_i) \) and a collection of log-scales \( \ell_i \) defined on \( U_i \) such that \( \ell_i \) and \( \ell_j \) are Lipschitz equivalent to each other on the intersection \( \overline{U_i} \cap \overline{U_j} \) (since an extension of \( \overline{U_i} (U_i)^{-1} \) is locally Lipschitz and \( \ell_1 \) is positive). Hence, by Theorem 1.1.3 there exists a log-scale \( \ell_2 \) on \( X_2 \) which is Lipschitz equivalent to every log-scale \( \ell_i, i \in I \). It is easy to see that this log-scale satisfies the conditions of the lemma.

In view of Lemma 2.5.1 we adopt the following definition.

**Definition 2.5.1.** A **Lipschitz structure** on \( \mathcal{G} \) is given by a positive log-scale \( \ell \) on a compact topological transversal \( X \) such that all elements of \( \mathcal{G} \) act by locally Lipschitz transformations with respect to \( \ell \). If \( \ell' \) is a log-scale on a compact topological transversal \( X' \), then \( \ell \) and \( \ell' \) define the same Lipschitz structure if the elements of \( \mathcal{G} \mid X \cup X' \) are locally Lipschitz with respect to \( \ell \) and \( \ell' \). (In particular, \( \ell \) and \( \ell' \) are locally Lipschitz equivalent on \( X \cap X' \).)
Two equivalent groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ have equivalent Lipschitz structures (are equivalent as groupoids with Lipschitz structure) if their Lipschitz structures define a Lipschitz structure on $\mathcal{G}_1 \vee \mathcal{G}_2$.

Local product structure

**Definition 2.5.2.** Let $X$ be a topological space with a local product structure. A homeomorphism $F : U \rightarrow V$ between open subsets of $X$ preserves the local product structure if for every point $x \in U$ and a pair of rectangles $(R_i, [\cdot, \cdot])$, $(R_j, [\cdot, \cdot])$ such that $x \in R_i$ and $F(x) \in R_j$ there exists a rectangular neighborhood $W$ of $x$ such that

$$F([y, z]) = [F(y), F(z)]_j$$

for all $y, z \in W$.

Let $X$ be a space with a local product structure. A pseudogroup $\tilde{\mathcal{G}}$ acting on $X$ (and its groupoid of germs) is said to preserve the local product structure if every element $F : U \rightarrow V$ of $\tilde{\mathcal{G}}$ preserves the local product structure.

Note that if a groupoid $\mathcal{G}$ preserves a local product structure on $\mathcal{G}^{(0)}$ then $\mathcal{G}$ itself has a natural local product structure. Namely, for every germ $(F, x)$ we can find a rectangular neighborhood $R$ of $x$ such $F(R)$ is also a rectangle and $F([y, z]) = [F(y), F(z)]$ for all $y, z \in R$. We transform then the set of germs $\{(F, x) : x \in R\}$ into a rectangle by setting $[(F, y), (F, z)] = (F, [y, z])$.

**Definition 2.5.3.** Let $\mathcal{G}$ be a groupoid preserving a local product structure and let $\mathcal{R}$ be a covering of a $\mathcal{G}$-transversal by open rectangles. An element $U \in \mathcal{G}$ is called a rectangle subordinate to $\mathcal{R}$ if there exist rectangles $R_i, R_j \in \mathcal{R}$ such that $o(U) \subset R_i$, $t(U) \subset R_j$ and $U([x, y]_{R_i}) = [U(x), U(y)]_{R_j}$.

For a germ $g \in \mathcal{G}$, a rectangle $U \in \mathcal{G}$ such that $g \in U$, and an index $i = 1, 2$, the germ of the projection $P_i(U)$ at $P_i(o(g))$ is denoted $P_i(g)$.

Next proposition is a direct corollary of the definitions.

**Proposition 2.5.2.** Let $\mathcal{G}$ be a groupoid of germs and suppose that there exists a local product structure preserved by $\mathcal{G}$ on an open transversal $X_0$. Then the local product structure on $X_0$ can be extended in a unique way to a local product structure on $\mathcal{G}^{(0)}$ preserved by $\mathcal{G}$.

**Corollary 2.5.3.** Let $\tilde{\mathcal{G}}_1$ be a pseudogroup preserving a local product structure of the space $\mathcal{G}_1^{(0)}$ and let $\tilde{\mathcal{G}}_2$ be an equivalent pseudogroup. Then there exists a unique local product structure on $\mathcal{G}_2^{(0)}$ such that the equivalence pseudogroup $\tilde{\mathcal{G}}_1 \vee \tilde{\mathcal{G}}_2$ preserves the local product structure of the disjoint union $\mathcal{G}_1^{(0)} \sqcup \mathcal{G}_2^{(0)}$.

Let $\mathcal{G}$ be a pseudogroup preserving a local product structure of $\mathcal{G}^{(0)}$. Let $\mathcal{R} = \{R_i\}_{i \in I}$ be a covering of $\mathcal{G}^{(0)}$ by rectangles. Let $F : U \rightarrow V$ be a rectangular element of $\mathcal{G}$ such that $U$ and $V$ are a sub-rectangles of $R_i$ and
CHAPTER 2. PRELIMINARIES ON GROUPOIDS AND PSEUDOOGROUPOIDS

$R_j \in \mathcal{R}$, respectively. Then there exist homeomorphisms $P_1(F) : P_1(U) \rightarrow P_1(V)$ and $P_2(F) : P_2(U) \rightarrow P_2(V)$ such that $F(x, y) = (P_1(F)(x), P_2(F)(y))$ with respect to the canonical decompositions $U = P_1(U) \times P_2(U)$ and $V = P_1(V) \times P_2(V)$. We call the homeomorphisms $P_1(F)$ and $P_2(F)$ projections of the rectangle $F$.

Definition 2.5.4. Let $k = 1$ or $2$. Projection $P_k(\bar{\mathcal{G}}, \mathcal{R})$ of $\bar{\mathcal{G}}$ (with respect to the covering $\mathcal{R}$) is the pseudogroup of local homeomorphisms of the space $P_k(\bar{\mathcal{G}}^{(0)}) = \bigsqcup_{k \in I} P_k(R_i)$ generated by projections $P_k(F)$ of rectangular elements $F$ of $\mathcal{G}_\mathcal{R}$.

We will denote by $P_k(\mathcal{G}, \mathcal{R})$ the groupoid of germs of the pseudogroup $P_k(\mathcal{G}, \mathcal{R})$.

Let $\mathcal{G}$ be a pseudogroup preserving a local product structure on $\mathcal{G}^{(0)}$. Let $X$ be a compact topological transversal. Let $|\cdot|$ be a metric on a neighborhood $\tilde{X}$ of $X$.

Definition 2.5.5. We say that a finite covering $\mathcal{R}$ of $X$ by relatively compact rectangles $R_i \subset \tilde{X}$ has compressible first direction if there exists a constant $\lambda \in (0, 1)$ such that for every $R_i \in \mathcal{R}$ and every $x \in R_i$ there exists a rectangular element $F \in \mathcal{G}$ and a rectangle $R_j \in \mathcal{R}$ such that $P_1(R_i, x) \subset o(F)$, $F(\tilde{P}_1(R_i, x)) \subset R_j \cap X$, and

$$|F(y_1) - F(y_2)| < \lambda |y_1 - y_2|$$

for all $y_1, y_2 \in P_1(R_i, x)$.

We say that a covering by rectangles is compressible if it is compressible in both directions.

Proposition 2.5.4. If $\mathcal{R}$ and $\mathcal{R}'$ are coverings that are compressible in the first direction, then the second projections $P_2(\bar{\mathcal{G}}, \mathcal{R})$ and $P_2(\bar{\mathcal{G}}, \mathcal{R}')$ are equivalent.

Note that conclusion of Proposition 2.5.4 holds in many other cases, not only when the covering is compressible (e.g., when the space is locally connected).

Proof. Let $\mathcal{R}$ be a compressible covering of $X$. It is enough to prove the proposition for every covering $\mathcal{R}'$ of $X$ by sub-rectangles of elements of $\mathcal{R}$, since for any two coverings there exists a covering by rectangles that are sub-rectangles of both coverings.

For $R' \in \mathcal{R}'$ and $R \in \mathcal{R}$ such that $R' \subset R$ the projection $P_2(R')$ is naturally identified with a subset of $P_2(R)$. In this way the projections $P_2(R')$ for $R' \in \mathcal{R}'$ cover a transversal of the space $\bigsqcup_{R \in \mathcal{R}} P_2(R)$ on which $P_2(\mathcal{G}, \mathcal{R})$ acts. Let us show that $P_2(\bar{\mathcal{G}}, \mathcal{R}')$ is equivalent to the localization $P_2(\mathcal{G}, \mathcal{R})|_{P_2(R')}$, hence it is equivalent to $P_2(\mathcal{G}, \mathcal{R})$. See [24] for the definition of a localization.

The generators of the pseudogroup $P_2(\bar{\mathcal{G}}, \mathcal{R}')$ are naturally identified with elements of the localization $P_2(\mathcal{G}, \mathcal{R})|_{P_2(R')}$. Namely, the projection of an element $F \in \mathcal{G}$ onto $P_2(R')$ for $R' \in \mathcal{R}'$ is naturally identified with the restrictions
2.5. Locally diagonal groupoids

Let \( \mathfrak{G} \) be a groupoid of germs preserving a local product structure on \( \mathfrak{G}^{(0)} \). We say that the groupoid \( \mathfrak{G} \) is locally diagonal (with respect to the local product structure) if there exists a covering \( \mathcal{R} \) of a topological transversal by open rectangles such that if \( g \in \mathfrak{G} \) is such that \( o(g) \in R, t(g) \in R \) for some \( R \in \mathcal{R} \) and one of the projections \( P_i(g), i = 1, 2, \) is trivial, then \( g \) is a unit.

It is clear that if a covering \( \mathcal{R} \) satisfies the conditions of Definition 2.5.6, then any covering by subrectangles of elements of \( \mathcal{R} \) also satisfies the conditions of the definition.

**Definition 2.5.6.** Let \( \mathfrak{G} \) be a groupoid of germs preserving a local product structure on \( \mathfrak{G}^{(0)} \). We say that the groupoid \( \mathfrak{G} \) is locally diagonal (with respect to the local product structure) if there exists a covering \( \mathcal{R} \) of a topological transversal by open rectangles such that if \( g \in \mathfrak{G} \) is such that \( o(g) \in R, t(g) \in R \) for some \( R \in \mathcal{R} \) and one of the projections \( P_i(g), i = 1, 2, \) is trivial, then \( g \) is a unit.

**Proposition 2.5.5.** Suppose that a covering \( \mathcal{R} \) of a topological transversal \( X \subset \mathfrak{G}^{(0)} \) satisfies the conditions of Definition 2.5.6. Then for any topological transversal \( X' \) there exists a covering \( \mathcal{R}' \) of \( X' \) satisfying the conditions of Definition 2.5.6.

**Proof.** We can find a covering \( \mathcal{R}' \) of \( X' \) by open rectangles such that for every \( R' \in \mathcal{R}' \) there exists a rectangle \( U \in \mathfrak{G} \) such that \( o(U) = R', t(U) \) is a sub-rectangle of a rectangle \( R \in \mathcal{R} \), and \( U \) agrees with the product decomposition of \( R \) and \( R' \). Then for any element \( g \in \mathfrak{G} \) such that \( o(g), t(g) \in R' \) we have the corresponding conjugate \( h = U g U^{-1} \) such that \( o(h), t(h) \in R \). A projection \( P_i(g) \) is trivial if and only if the projection \( P_i(h) \) is trivial. The element \( g \) is
a unit if and only if $h$ is a unit. It follows that $\mathcal{R}'$ satisfies the conditions of Definition 2.5.6.

**Corollary 2.5.6.** If a groupoid $\mathcal{G}$ preserving a local product structure on $\mathcal{G}^{(0)}$ is locally diagonal, then every equivalent groupoid is also locally diagonal.

**Quasi-cocycles**

**Definition 2.5.7.** A quasi-cocycle on a groupoid $\mathcal{G}$ is a map $\nu : \mathcal{G}|_X \to \mathbb{R}$, where $X$ is a compact topological transversal, such that there exists a constant $\eta \geq 0$ for which the following conditions hold:

1. For every $g \in \mathcal{G}|_X$ there exists a neighborhood $U$ of $g$ such that $|\nu(g) - \nu(h)| \leq \eta$ for all $h \in U$;

2. For all $(g_1, g_2) \in \mathcal{G}^{(2)}|_X$

   $$\nu(g_1) + \nu(g_2) - \eta \leq \nu(g_1 g_2) \leq \nu(g_1) + \nu(g_2) + \eta.$$  

If the above inequalities hold for $\nu$ and $\eta$, then we say that $\nu$ is an $\eta$-quasi-cocycle.

**Definition 2.5.8.** We say that two quasi-cocycles $\nu_1$ and $\nu_2$ (defined for the same transversal $X$) are strongly equivalent if $|\nu_1(g) - \nu_2(g)|$ is uniformly bounded. They are (coarsely) equivalent if there exist $\Lambda > 1$ and $k > 0$ such that $\Lambda^{-1} \cdot \nu_1(g) - k \leq \nu_2(g) \leq \Lambda \cdot \nu_1(g) + k$ for all $g \in \mathcal{G}|_X$.

**Lemma 2.5.7.** Let $X_1$ and $X_2$ be compact topological transversals of a groupoid $\mathcal{G}$. Let $\nu_1$ be a quasi-cocycle on $\mathcal{G}|_{X_1}$. Then there exists a unique, up to strong equivalence, quasi-cocycle $\nu$ on $\mathcal{G}|_{X_1 \cup X_2}$ such that $\nu$ and $\nu_1$ are strongly equivalent on $\mathcal{G}|_{X_1}$.

**Proof.** Find a finite set $F_i$ of relatively compact elements of $\mathcal{G}$ such that $o(F_i)$ cover $X_2$ and $t(F_i) \subseteq X_1$. For every $g \in \mathcal{G}|_{X_1 \cup X_2}$ we either have $g' = g \in \mathcal{G}_{X_1}$, or $g' = h_2 h_1^{-1} \in \mathcal{G}_{X_1}$, or $g' = g h_1^{-1} \in \mathcal{G}_{X_1}$, or $g' = g_2 h \in \mathcal{G}_{X_1}$ for some $h_1, h_2 \in F = \bigcup F_i$. If we define $\nu(g) = \nu_1(g')$, then $\nu$ will be a quasi-cocycle satisfying the conditions of the lemma (it is defined only up to a strong equivalence, since $h_i$ and $g$ are not unique).

For every quasi-cocycle $\nu$ on $\mathcal{G}|_{X_1 \cup X_2}$ the set of values of $\nu$ on $F \cap \mathcal{G}|_{X_1 \cup X_2}$ is bounded. For every element $g \in \mathcal{G}|_{X_2}$ there exist elements $h_1, h_2 \in F \cap \mathcal{G}|_{X_1 \cup X_2}$ such that $h_2 h_1^{-1}$ is an element of $\mathcal{G}|_{X_1}$. It follows that $\nu$ is unique up to a strong equivalence.

We give then, in view of the previous lemma, the following definition.
Definition 2.5.9. A graded groupoid is a groupoid of germs $\mathcal{G}$ together with a quasi-cocycle $\nu : \mathcal{G}|_X \rightarrow \mathbb{R}$.

Two quasi-cocycles $\nu_1 : \mathcal{G}|_{X_1} \rightarrow \mathbb{R}$ and $\nu_2 : \mathcal{G}|_{X_2} \rightarrow \mathbb{R}$ define the same grading of $\mathcal{G}$ (or are strongly equivalent) if there exists a quasi-cocycle $\nu : \mathcal{G}|_{X_1 \cup X_2} \rightarrow \mathbb{R}$ such that the restrictions of $\nu$ onto $\mathcal{G}|_{X_i} \subset \mathcal{G}|_{X_1 \cup X_2}$ are strongly equivalent to $\nu_i$.

Two graded groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent as graded groupoids if the corresponding quasi-cocycles defined the same grading of $\mathcal{G}_1 \cup \mathcal{G}_2$.

If a graded groupoid $\mathcal{G}$ is compactly generated, then the corresponding quasi-cocycle $\nu$ defines a quasi-cocycle $\tilde{\nu}$ in the sense of Definition 1.2.6 on each of its Cayley graphs $\mathcal{G}(x, S)$ by $\tilde{\nu}(g, h) = \nu(gh^{-1})$.

Compatibility with the local product structure

If $\ell_1$ and $\ell_2$ are log-scales on spaces $A$ and $B$, respectively, then we denote by $\ell_1 \times \ell_2$ the log-scale on $A \times B$ given by

$$((a_1, b_1), (a_2, b_2)) = \min\{\ell_1(a_1, a_2), \ell_2(b_1, b_2)\}. \quad (2.1)$$

It is easy to check that it is a log-scale and that it defines the product topology on the space $A \times B$.

Definition 2.5.10. Let $\ell$ be a log-scale on a compact space $X$ with a local product structure. We say that the log-scale is compatible with the local product structure if there exists a covering of $X$ by open rectangles $\mathcal{R} = \{R_i\}$ of $X$ by open rectangles and log-scales $\ell_{i,1}$ and $\ell_{i,2}$ on $P_1(R_i)$ and $P_2(R_i)$ such that $\ell$ is Lipschitz equivalent to $\ell_{i,1} \times \ell_{i,2}$ on $R_i$.

Definition 2.5.11. Let $\nu : \mathcal{G}|_X \rightarrow \mathbb{R}$ be a quasi-cocycle. We say that the corresponding grading is compatible with the local product structure if there exist a covering of $X$ by open rectangles $\mathcal{R} = \{R_i\}$ and a constant $c > 0$ such that if $g_1, g_2 \in \mathcal{G}$ are such that $\{o(g_1), o(g_2)\} \subset R_i$, $\{t(g_1), t(g_2)\} \subset R_j$ for some $R_i, R_j \in \mathcal{R}$, and for some $i \in \{1, 2\}$ we have $P_i(g_1) = P_i(g_2)$, then $|\nu(g_1) - \nu(g_2)| \leq c$. 

Chapter 3

Hyperbolic groupoids

3.1 Main definition

Definition 3.1.1. We say that a Hausdorff groupoid of germs $G$ is hyperbolic if there exists a compact generating pair $(S, X)$ of $G$, a metric $|\cdot|$ defined on an open neighborhood of $X$, and numbers $\lambda \in (0, 1), \delta, \Lambda, \Delta > 0$ such that

1. elements of $\tilde{G}$ are locally Lipschitz with respect to $|\cdot|$;
2. every element $g \in S$ is a germ of an element $U \in \tilde{G}$ which is a $\lambda$-contraction with respect to $|\cdot|$, i.e., $|U(x) - U(y)| \leq \lambda|x - y|$ for all $x, y \in o(U)$;
3. for every $x \in X$ the Cayley graph $G(x, S)$ is $\delta$-hyperbolic;
4. $o(S) = t(S) = X$;
5. for every $x \in X$ there exists a point $\omega_x$ of the boundary of $G(x, S)$ such that every infinite directed path in $G(x, S^{-1})$ is a $(\Lambda, \Delta)$-quasi-geodesic converging to $\omega_x$.

Definition 3.1.1 is a combination of two contraction conditions: the elements of $S$ are germs of contracting maps on $G(0)$, while the elements of $S^{-1}$ are large-scale contracting on the Cayley graphs (see Theorem 1.2.8).

Example 5. As a simple example of a hyperbolic groupoid, consider the groupoid $\mathfrak{G}$ of germs of the pseudogroup of transformations of the circle $\mathbb{R}/\mathbb{Z}$ generated by the expanding self-covering $f : x \mapsto 2x$. Consider the generating pair $(S, \mathbb{R}/\mathbb{Z})$, where $S$ is the set of all germs of $f^{-1}$. The set of vertices of the Cayley graph $\mathfrak{G}(x, S)$ is the set of germs of the form $(f^{n_1}, y)^{-1}(f^{n_2}, x)$, where $y \in \mathbb{R}/\mathbb{Z}$ and $n_1, n_2 \geq 0$ are such that $f^{n_2}(x) = f^{n_1}(y)$. Every point of $\mathbb{R}/\mathbb{Z}$ has exactly two $f$-preimages and one $f$-image, and the Cayley graph $\mathfrak{G}(x, S)$ is a regular tree of degree three such that every vertex has one incoming and two outgoing arrows. See Figure 3.1 (Recall that the edges of the Cayley graph $\mathfrak{G}(x, S)$ are oriented according to the action of the elements of $S$, i.e., according
to the action of $f^{-1}$.) If the isotropy group of $x$ in $\mathfrak{F}$ is trivial (i.e., if $x$ is not eventually periodic with respect to the action of $f$), then the vertices of $\mathfrak{F}(x, S)$ are in a bijection with the grand orbit of $x$ (i.e., with the $\mathfrak{F}$-orbit of $x$), and the graph $\mathfrak{F}(x, S)$ is the graph of the action of $f$ on the grand orbit.

It is easy to see now that $\mathfrak{F}$ and $S$ satisfy the conditions of Definition 3.1.1. Elements of $S$ are germs of contractions (since $f$ is expanding), the Cayley graphs $\mathfrak{F}(x, S)$ are Gromov-hyperbolic (are 0-hyperbolic, in fact), and the paths going against the orientation in the Cayley graphs are geodesics converging to one point of the boundary (to the limit of the forward $f$-orbit of $x$).

**Example 6.** Let $\mathbb{Z}_2$ be the ring of dyadic integers, i.e., formal expressions

$$\sum_{n=0}^{\infty} a_n 2^n,$$

where $a_n \in \{0, 1\}$. The distance between $\sum_{n=0}^{\infty} a_n 2^n$ and $\sum_{n=0}^{\infty} b_n 2^n$ is $1/2^k$, where $k$ is the smallest index such that $a_n \neq b_n$.

Then the map $a : x \mapsto x + 1$ is an isometry, and the map $s : x \mapsto 2x$ is a homeomorphism onto its range, contracting the distance twice. Let $\mathfrak{G}$ be the pseudogroup generated by these transformations, and let $\mathfrak{G}$ be the corresponding groupoid of germs.

Consider then the generating set $S$ consisting of germs of the transformations $s$, $as$, and $a^{-1}s$. Note that the set of germs of $a$ is equal to the disjoint union of the sets of germs of $as \cdot s^{-1}$ and $s \cdot (a^{-1}s)^{-1}$, therefore $S$ is a generating set.

The Cayley graph of $\mathfrak{G}$ with respect to this generating set is shown on Figure 3.2. Here the edges are directed upward. It is easy to check that the groupoid $\mathfrak{G}$ and the generating set $S$ satisfy the conditions of Definition 3.1.1.
3.1. MAIN DEFINITION

Figure 3.2: Cayley graph of $\mathfrak{G}$

**Definition 3.1.2.** We say that a subset $S \subset \mathfrak{G}$ is *contracting* (with respect to a metric on a subset of $\mathfrak{G}^{(0)}$) if every germ $g \in S$ belongs to a contraction $U \in \tilde{\mathfrak{G}}$.

**Proposition 3.1.1.** Suppose that $\mathfrak{G}$ is a hyperbolic groupoid. Let $Y_1$ be a compact topological $\mathfrak{G}$-transversal.

Then there exists a generating pair $(S, Y_1)$ of $\mathfrak{G}$ satisfying the conditions of Definition 3.1.1.

**Proof.** Let $X$, $\tilde{X}$, $| \cdot |$, and $S$ be as in Definition 3.1.1. Let $X_0 \subset X$ be an open transversal. Denote by $\ell$ a log-scale on $\tilde{X}$ such that $| \cdot |$ is associated with $\ell$.

Let $\tilde{Y}$ be a relatively compact open neighborhood of $Y_1$. Let $Y_0 \subset Y_1$ be an open transversal. Then, by Lemma 2.5.1 there exists a unique, up to a Lipschitz equivalence, log-scale $\ell'$ on $\tilde{Y}$ such that all elements of $\mathfrak{G}$ are locally Lipschitz with respect to $\ell$ and $\ell'$.

Cover the closure of $\tilde{Y}$ by a finite number of sets of the form $t(U_i)$, where $U_i \in \tilde{\mathfrak{G}}$ are relatively compact, extendable, and such that $o(U_i) \subset X_0$. Similarly, cover $X$ by sets of the form $o(V_i)$ where $V_i \in \tilde{\mathfrak{G}}$ are relatively compact, extendable, and $t(V_i) \subset Y_0$.

For every $n$ consider then the restriction $R_n$ of the set

$$A_n = \left( \bigcup U_i \cup \bigcup V_i \right) \cdot S^n \cdot \left( \bigcup U_i \cup \bigcup V_i \right)^{-1}$$

onto $Y_1$. Let $S'_n = R_n \cup R_{n+1}$. Then $(S'_n, Y_1)$ is a generating pair of $\mathfrak{G}$ for every $n$ (see the proof of Proposition 2.3.1). We have $o(S'_n) = t(S'_n) = Y_1$. Since $V_i$
and \( U_i \) are locally Lipschitz, if \( n \) is big enough, all elements of \( S'_n \) will have contracting neighborhoods. Moreover, if \( | \cdot |' \) is a metric associated with \( \ell' \), for all \( n \) big enough, all elements of \( S'_n \) have contracting neighborhoods with respect to \( | \cdot |' \).

Let \( y \in Y_1 \) be an arbitrary point, and consider the Cayley graph \( G(y, S'_n) \). Let \( U_i \) be such that \( y \in t(U_i) \). For every \( g \in G_y \), find \( U_{j(g)} \) such that \( t(g) \in t(U_{j(g)}) \), and define \( F(g) = U_{j(g)}^{-1} g U_i \). Then for any choice of indices \( j(g) \), the map \( F : G_y \rightarrow G_x \) is a quasi-isometry of the Cayley graphs \( G(y, S'_n) \) and \( G(y, S) \). Moreover, there is a uniform estimate on the coefficients of the quasi-isometry. It follows then that condition (3) of Definition 3.1.1 is satisfied for \((S'_n, Y_1)\). The directed paths in the Cayley graph \( G(y, (S'_n)^{-1}) \) are quasi-geodesics converging to \( F^{-1}(\omega_x) \), since their \( F \)-images are on uniformly bounded distance from quasi-geodesics converging to \( \omega_x \) obtained from the oriented paths in \( G(x, S^{-1}) \) by making jumps of length \( n \) or \( n + 1 \).

\[ \square \]

**Corollary 3.1.2.** A groupoid of germs equivalent to a hyperbolic groupoid is hyperbolic.

### 3.2 Busemann quasi-cocycle

Let \( G \) be a hyperbolic groupoid. Consider its Cayley graph \( G(x, S) \). Let \( \omega_x \in \partial G(x, S) \) be as in Definition 3.1.1. Consider the associated Busemann quasi-cocycle \( \beta_{\omega_x}(g, h) \).

**Proposition 3.2.1.** The map \( \beta : G|_X \rightarrow \mathbb{R} \) given by \( \beta(gh^{-1}) = \beta(\omega_x)(g, h) \) is a well defined (up to a strong equivalence) quasi-cocycle.

Note that the strong equivalence class of \( \beta \) depends on the generating pair \((X, S)\).

**Proof.** If we have two pairs \( g_1, h_1 \) and \( g_2, h_2 \) and two points \( x_1, x_2 \) such that \( g_1 h_1^{-1} = g_2 h_2^{-1} \) and \( o(g_i) = o(h_i) = x_i \), then \( g_1^{-1} g_2 = h_1^{-1} h_2 \), and the map \( F : G(x_1, S) \rightarrow G(x_2, S) \) given by \( F(g) = g \cdot g_1^{-1} g_2 \)

is an isomorphism of directed graphs mapping \( \omega_{x_1} \) to \( \omega_{x_2} \). It follows that \( \beta_{\omega_{x_1}}(g_1, h_1) = \beta_{\omega_{x_2}}(g_2, h_2) \), i.e., that \( \beta \) is well defined.

It satisfies the condition \( \beta(gh) = \beta(g) + \beta(h) \), since Busemann quasi-cocycle is a quasi-cocycle on the Cayley graph. It remains to prove that there exists a constant \( c > 0 \) such that for every \( g \in G|_X \) there exists a neighborhood \( U \) of \( g \) such that \( |\beta(g) - \beta(h)| < c \) for all \( h \in U \). Since \( \omega_x \) is the limit of every directed path in \( G(x, S^{-1}) \), we can compute \( \beta_{\omega_x}(g, h) \) by following arbitrary directed paths in \( G(x, S^{-1}) \) starting in \( g \) and \( h \), and finding a moment when they are on some bounded distance \( \Delta \) (depending only on the constant \( \delta \) of hyperbolicity) from each other. It follows that the value of \( \beta_{\omega_x}(g, h) \) depends only on a finite subgraph of the Cayley graph, so using Corollary 2.4.2 we conclude that \( \beta \) is a locally bounded.

\[ \square \]
Definition 3.2.1. If $G$ is a hyperbolic groupoid, then any quasi-cocycle $\nu : G|_Y \to \mathbb{R}$ coarsely equivalent to the quasi-cocycle $\beta(gh^{-1}) = \beta_{\omega_\epsilon}(g, h)$ is called a Busemann quasi-cocycle of the hyperbolic groupoid.

Example 7. Let $f$ be a complex rational function seen as a self-map of the Riemann sphere. Suppose that it is hyperbolic, i.e., that it is expanding on a neighborhood of its Julia set $J_f$. Then the groupoid of germs $\mathfrak{F}$ generated by the restriction of $f$ onto $J_f$ is hyperbolic (see Example 5). Note that a natural Busemann cocycle is given just by the degree:

$$\beta((f^n, x)^{-1}(f^m, y)) = n - m.$$  

This is precisely the Busemann cocycle on the Cayley graph (described in Example 5).

On the other hand, it is easy to see that $\nu((F, x)^{-1}(F, y)) = -\ln |F'(x)|,$

for $(F, x) \in \mathfrak{F},$ is a cocycle coarsely equivalent to $\beta,$ hence it is also a Busemann cocycle on the hyperbolic groupoid generated by $f.$

3.3 Complete generating sets

This section is purely technical. We need to prove existence generating sets of hyperbolic groupoids with special properties that will be convenient later.

Proposition 3.3.1. Let $G$ be a hyperbolic groupoid graded by a Busemann quasi-cocycle $\beta$. Let $X$ be a compact topological transversal, and let $X_0 \subset X$ be an open transversal.

Then there exist a compact generating set $S$ of $G|_X$, a metric $| \cdot |$ on a neighborhood $\tilde{X}$ of $X$, and an $\eta$-quasi-cocycle $\nu : G|_{\tilde{X}} \to \mathbb{R}$ strongly equivalent to $\beta$, such that

1. for every $g \in S$ we have $\nu(g) > 3\eta$;
2. $o(S) = t(S) = X$, and for every $x \in X$ there exists $g \in S$ such that $t(g) = x$ and $o(g) \in X_0$;
3. there exists $\lambda \in (0, 1)$ such that for every $g \in S$ there exists an element $U \in G|_{\tilde{X}}$ containing $g$ and such that $|U(x) - U(y)| < \lambda |x - y|$ for all $x, y \in o(U)$;
4. every element $g \in G|_X$ is equal to a product of the form $g_n \cdots g_1 \cdot (h_m \cdots h_1)^{-1}$ for some $g_i, h_i \in S$.

Definition 3.3.1. We say that a generating set $S$ is complete if it satisfies the conditions of Proposition 3.3.1.
We will denote paths in the Cayley graph \( \mathfrak{G}(x, S) \) as sequences \((g_1, g_2, \ldots, g_n)\) of elements of \( S \) corresponding to its edges, when the initial vertex of the path is clear from the context. If \( h \) is the initial vertex of the path, then it passes through the vertices \( h, g_1 h, g_2 g_1 h, \ldots, g_n \cdot g_2 g_1 h \).

**Proof.** Let \((S, X)\) be a generating pair satisfying the conditions of Definition \ref{def:3.1.1}.

Since \( t(S) = X \), for every \( h \in \mathfrak{G}_{|X} \) there exist infinite paths \( \gamma' = (\ldots, g'_2, g'_1) \) and \( \gamma'' = (\ldots, g''_2, g''_1) \) in \( \mathfrak{G}(o(h), S) \) such that \( t(g'_1) = o(h) \) and \( t(g''_n) = t(h) \). The paths \( \gamma' \) and \( \gamma'' \) are quasi-geodesics converging to the point \( \omega \) such that \( \nu \) is coarsely equivalent to the Busemann quasi-cocycle \( \beta_\omega \). By hyperbolicity of \( \mathfrak{G}(o(h), S) \) (see Theorem \ref{thm:1.2.8}), there exists a constant \( \Delta \) depending only on \( \mathfrak{G}(o(h), S) \) such that some truncated paths

\[
(\ldots, g'_{n+2}, g'_{n+1}, g'_1), \quad (\ldots, g''_{m+2}, g''_{m+1}, g''_m)
\]

are on distance at most \( \Delta \) from each other.

It follows that there exists a compact set \( Q_0 \subset \mathfrak{G}_{|X} \) such that every element of \( \mathfrak{G}_{|X} \) is equal to a product \( g_n \cdot g_1 \cdot s \cdot (h_m \cdots h_1)^{-1} \) for some \( g_i, h_i \in S \) and \( s \in Q_0 \).

There exists \( n_0 \) such that \( S \cup S^{n_0} Q_0 \) is a generating set satisfying the conditions of Definition \ref{def:3.1.1}. Then every element \( h \) can be written as a product \( g_n \cdot g_1 \cdot (h_1 \cdots h_m)^{-1} \).

Since the set \( X_0 \cap \mathfrak{G}_x \) is a net in the Cayley graph \( \mathfrak{G}(x, S) \), there exists a compact set \( A \subset \mathfrak{G}_{|X} \) such that for every \( y \in X \) there exists \( g \in A \) such that \( o(g) \in X_0 \) and \( t(g) = y \). Replacing \( S \) by \( S \cup AS^{n_1} \) for some \( n_1 \), we will get a generating set satisfy the condition (2) of the proposition (we use the fact that \( o(S) = t(S) = X \)). The remaining conditions of our proposition follow directly from the conditions of Definition \ref{def:3.1.1}.

**Proposition 3.3.2.** Suppose that \( S \) is complete. There exists a constant \( \Delta_1 \) such that every geodesic path \( \gamma \) in the Cayley graph is \( \Delta_1 \)-close to a path of the form \( (g_n, \ldots, g_1, h_1^{-1}, \ldots, h_m^{-1}) \) for \( g_i, h_i \in S \).

**Proof.** The path \((g_n, \ldots, g_1, h_1^{-1}, \ldots, h_m^{-1})\) is the union of two sides of a quasi-geodesic triangle. Therefore, the statement of the proposition follows from Theorem \ref{thm:1.2.8} and Proposition \ref{prop:1.2.10}.

### 3.4 Boundaries of the Cayley graphs

Let \( \mathfrak{G} \) be a hyperbolic groupoid, and let \( X, X_0 \subset X \) and \( S \) satisfy the conditions of Proposition \ref{prop:3.3.1}.

Denote by \( \partial \mathfrak{G}_x \) for \( x \in X \) the boundary of the hyperbolic graph \( \mathfrak{G}(x, S) \) minus the point \( \omega_x \). Since the quasi-isometry type of the Cayley graph \( \mathfrak{G}(x, S) \) and the strong equivalence class of \( \nu \) do not depend on the choice of the generating pair \((S, X)\), the boundary \( \partial \mathfrak{G}_x \) does not depend on \((S, X)\) and is well defined for all \( x \in \mathfrak{G}^{(0)} \).
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Example 8. Let $\mathcal{G}$ be the groupoid from Example 5. Then $\partial \mathcal{G}_x$ is the boundary of the tree of the grand orbit of $x$ without the limit of the forward orbit of $x$.

We adopt the following definition.

Definition 3.4.1. For a map $F : X \to X$ and a point $t \in X$ the tree of preimages $T_{(F,t)}$ of the point $t$ is the tree with the set of vertices equal to the disjoint union $\bigsqcup_{n \geq 0} F^{-n}(t)$ in which two vertices $v \in F^{-n}(t)$ and $u \in F^{-(n-1)}(t)$ are connected by an edge if $f(v) = u$.

Every point $\xi \in \partial \mathcal{G}_x$ is the limit of a path of germs of the form $g_n = (f^n, x_n)^{-1}(f^k, x)$, where $k$ is fixed, $x_n$ is a sequence of points such that $f(x_{n+1}) = x_n$ for all $n$, and $f^n(x_n) = f^k(x)$.

For a fixed value of $k$ the set of limits of the sequences $g_n$ is naturally identified with the boundary of the tree of preimages $T_{(f,y_k)}$, where $y_k = f^k(x)$. The boundary $B_k = \partial T_{(f,y_k)}$ of the tree of preimages of $y_k = f^k(x)$ is naturally a subset of the boundary $B_{k+1}$ of the tree of preimages of $y_{k+1} = f^{k+1}(x)$. The boundary $\partial \mathcal{G}_x$ is then the inductive limit of the boundaries $B_k$. See Figure 3.3.

Example 9. Let $\mathcal{G}$ be the groupoid from Example 6. Let $x \in \mathbb{Z}_2 = \mathcal{G}^{(0)}$. Then the boundary $\partial \mathcal{G}_x$ can be naturally identified with the set of formal series

$$\sum_{n=-\infty}^{\infty} a_n 2^n$$

where $a_n \in \{0, 1\}$ and $x - \sum_{n=0}^{\infty} a_n 2^n \in \mathbb{Z}$. Here two series $\sum_{n=-\infty}^{\infty} a_n 2^n$ and $\sum_{n=-\infty}^{\infty} b_n 2^n$ are considered to represent the same point of the boundary if
both differences
\[ \sum_{n=-\infty}^{-1} a_n 2^n - \sum_{n=0}^{\infty} b_n 2^n, \quad \sum_{n=0}^{\infty} b_n 2^n - \sum_{n=0}^{\infty} a_n 2^n \]
belong to \( \mathbb{Z} \) and are equal. Here the first difference is a difference of real numbers, and the second difference is a difference of elements of \( \mathbb{Z}_2 \).

Denote by \( \partial \mathcal{G} \) the disjoint union (as a set) of the spaces \( \partial \mathcal{G}_x \) for all \( x \in \mathcal{G}^{(0)} \). We will introduce a natural topology on \( \partial \mathcal{G} \) later.

The following proposition follows from Proposition 1.2.19.

**Proposition 3.4.1.** For all \( h \in \mathcal{G}_x \) and \( g_i \in S, i \geq 1 \), such that \( \tau(h) = o(g_1) \) and \( \tau(g_i) = o(g_{i+1}) \) the sequence \( g_n \cdots g_1 \cdot h \) converges to a point of \( \partial \mathcal{G}_x \).

For every point \( \xi \in \partial \mathcal{G}_x \) there exist \( h \in \mathcal{G}_x \) and a sequence \( g_i \in S \) such that
\[ \xi = \lim_{n \to \infty} g_n \cdots g_1 \cdot h. \]
We will use the following notation.
\[ \cdots g_2 g_1 \cdot g = \lim_{n \to \infty} g_n \cdots g_2 g_1 \cdot g \in \partial \mathcal{G}_{o(g)}. \]
Similarly, for \( U_1 \subset \mathcal{G} \) and \( g \in \mathcal{G} \), we denote
\[ \cdots U_2 U_1 \cdot g = \lim_{n \to \infty} U_n \cdots U_2 U_1 \cdot g. \]

Recall that a log-scale on \( \partial \mathcal{G}_x \) is defined by the function
\[ \ell_{\omega_x} (\xi_1, \xi_2) = \lim_{g \to \omega_x, g_1 \to \xi_1, g_2 \to \xi_2} \ell_g (g_1, g_2) - |x - g|, \]
where the limit is taken with respect to any convergent sub-sequence (see 1.2), \( \ell_g \) denotes the Gromov product in the Cayley graph (see 1.2), and \(|x - g|\) is combinatorial distance between vertices of the Cayley graph.

The log-scale \( \ell_{\omega_x} (\xi_1, \xi_2) \) is Hölder equivalent to the minimum value of \( \nu \) along a geodesic connecting \( \xi_1 \) to \( \xi_2 \) in the Cayley graph \( \mathcal{G}(x, S) \), by Corollary 1.2.18.

We get the following fact from Proposition 3.3.2 and Theorem 1.2.5.

**Proposition 3.4.2.** Denote by \( \ell(\xi_1, \xi_2) \) the maximum value of \( n \) for which there exist representations
\[ \xi_1 = \cdots g_n \cdots g_1 \cdot g, \quad \xi_2 = \cdots h_n \cdots h_1 \cdot g, \]
where \( g_i, h_i \in S, g \in \mathcal{G}_x, \) and \( \nu(g) \geq n \).

Then the function \( \ell \) is a log-scale Hölder equivalent to \( \ell_{\omega_x} (\xi_1, \xi_2) \). The Lipschitz class of \( \ell \) does not depend on the choice of the generating set \( S \) and the transversal \( X \).
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We call the log-scale \( \ell \) defined in Proposition 3.4.2 (more precisely, its Lipschitz class) the natural log-scale on \( \partial \mathfrak{G}_x \) defined by the quasi-cocycle \( \nu \).

Denote by \( \mathcal{T}_g \) the set of the limits in \( \partial \Omega \) of the form \( \lim_{n \to \infty} g_n \cdots g_1 \cdot g \) for \( g \in S \). In particular, \( \mathcal{T}_x \) for \( x \in \mathfrak{G}^{(0)} \) is the set of the limits in \( \partial \mathfrak{G}_x \) of the form \( \lim_{n \to \infty} g_n \cdots g_1 \), where \( g_n \in S \) and \( o(g_1) = x \).

**Proposition 3.4.3.** The sets \( \mathcal{T}_g \) are compact.

**Proof.** For every \( x \in X \) the set \( \Sigma_x \) of sequences \( (g_1, g_2, \ldots) \in S^\infty \) such that \( o(g_1) = x \) and \( t(g_n) = o(g_{n+1}) \) is closed and hence compact in the product topology of \( S^\infty \). It follows from Proposition 3.4.2 that for every \( g \in \mathfrak{G}|_X \) the map

\[ \Sigma_{o(g)} \hookrightarrow \mathcal{T}_g : (g_1, g_2, \ldots) \mapsto g_2 g_1 \cdot g \]

is continuous. It is surjective by definition. Consequently, \( \mathcal{T}_g \) is a continuous image of a compact set. \( \square \)

**Proposition 3.4.4.** There exists a compact set \( A \subset \mathfrak{G}|_X \) such that for every \( h \in \mathfrak{G}_X \) there exists \( a \in A \) such that \( T_{ah} \) is a neighborhood of \( T_h \) and \( t(a) \in X_0 \).

**Proof.** It is enough to prove the proposition for the cases when \( h \) is a unit \( x \in X \). By Proposition 3.4.2 a neighborhood of a point \( \xi \in \mathcal{T}_x \) is the set of points \( \zeta \in \partial \mathfrak{G}_x \) such that \( \xi = \cdots g_2 g_1 \cdot g \) and \( \zeta = \cdots h_2 h_1 \cdot g \) for some \( g \in \mathfrak{G}_x \) such that \( \nu(g) \geq 0 \) and \( g_i, h_i \in S \).

Since every vertex of \( \mathfrak{G}(x, S) \) is the end of a path \( (\ldots, f_2, f_1) \) of edges corresponding to elements of \( S \), and \( \nu \) is bounded on \( S \), there exists \( \Delta > 0 \) such that every \( g \in \mathfrak{G}_x^S \) such that \( \nu(g) \geq 0 \) can be represented in the form \( g = f_k \cdots f_1 \cdot g' \) for \( f_i \in S \) and \( 0 \leq \nu(g') \leq \Delta \).

Consequently, a neighborhood of \( \xi \) is the union of the sets \( \bigcup_{g \in B_\xi} \mathcal{T}_g \), where \( B_\xi \) is the set of elements \( g \in \mathfrak{G}_x^S \) such that \( 0 \leq \nu(g) \leq \Delta \) and \( \xi \in \mathcal{T}_g \).

Let \( g \in B_\xi \), so that \( \xi = \cdots s_2 s_1 \cdot g \) for some \( s_i \in S \). We know that \( \xi = \cdots g_2 g_1 \cdot g \) for some \( g_i \in S \), since \( \xi \in \mathcal{T}_g \). It follows from Theorem 1.2.8 that there exists a uniform constant \( \Delta_1 \) (not depending on \( x \) and \( \xi \)) such that the paths \( \cdots g_2 g_1 \cdot g \) and \( \cdots s_2 s_1 \cdot g \) are at most \( \Delta_1 \) apart. It follow that there is a uniform upper bound on the length of elements of \( B_\xi \), hence there exists a compact set \( B \) such that \( B_\xi \subset B \) for all \( \xi \). Then \( \bigcup_{a \in B \cap \mathfrak{G}_x} T_a \) is a neighborhood of \( T_x \).

Let \( \{a_1, a_2, \ldots, a_n\} \subset \mathfrak{G}_x^S \) be an arbitrary set. Let us prove by induction on \( n \) that there exists a product \( g = h_m \cdots h_1 \in \mathfrak{G}_x^S \) of elements of \( S \) such that for every \( a_i \) the element \( a_i g \) can be written as a product \( f_k \cdots f_1 \) of elements of \( S \) and \( m \) is bounded from above by a function of \( n \) and the lengths of representations of \( a_i \) as products of elements of \( S \) and \( S^{-1} \). It is true for \( n = 1 \), by condition (4) of Proposition 3.3.1 and Proposition 3.3.2. If there is such an element \( g' \) for \( \{a_1, a_2, \ldots, a_{n-1}\} \), then, again by Propositions 3.3.1 and 3.3.2 there exists a product \( l_1 \cdots l_1 \) of elements of \( S \) such that \( a_n g l_1 \cdots l_1 \) is equal to a product of elements of \( S \). Then we can take \( g = g' l_1 \cdots l_1 \) for the set \( \{a_1, \ldots, a_n\} \), since then every element \( a_i g = a_i g' l_1 \cdots l_1 \) is equal to a product of elements of \( S \).
The size of the sets \( B \cap \mathfrak{G}_x \) and the lengths of elements of \( B \) are uniformly bounded. Consequently, there exists a compact set \( A' \) such that for every \( x \in X \) there exists an element \( a \in A' \) that can be written as a product of elements of \( S \) and is such that \( ba \) is as a product of elements of \( S \) for all \( b \in B \cap \mathfrak{G}_x \). Then \( T_{\alpha(a)} \supset T_{ba} \), hence \( T_{\alpha^{-1}} \supset T_{b} \) for all \( b \in B \cap \mathfrak{G}_x \). It follows that \( T_{\alpha^{-1}} \) is a neighborhood of \( T_x \), hence we can take \( A = (A')^{-1} \). We can assume that \( t(a^{-1}) \in X_0 \) by condition (2) of Proposition 3.3.1.

**Proposition 3.4.5.** If \( \xi = \ldots g_2 g_1 \) for \( g_i \in S \) belongs to the interior of \( T_x \), then for every compact set \( Q \) there exists \( n \) such that every point of the form \( \ldots h_2 h_1 \cdot f \cdot g_n \cdots g_1 \), where \( h_i, g_i \in S \) and \( f \in Q \), belongs to the interior of \( T_x \).

**Proof.** Let \( \zeta = \ldots h_2 h_1 \cdot f \cdot g_n \cdots g_1 \in \partial \mathfrak{G}_x \). Then \( \ldots h_2 h_1 \cdot f \cdot g_{n+1} g_{n+2} \cdots \) is a quasi-geodesic path connecting \( \zeta \) to \( \xi \). It follows that for some uniform constant \( \Delta > 0 \) we have \( |\ell(\zeta, \xi) - \nu(g_n \cdots g_1)| < \Delta \). Since \( \xi \) is an internal point of \( T_x \), for \( n \) big enough, the point \( \zeta \) will belong to the interior of \( T_x \).

### 3.5 Coarse uniqueness of the Busemann quasi-cocycle

**Proposition 3.5.1.** The Busemann quasi-cocycle on a hyperbolic groupoid \( \mathfrak{G} \) is unique up to a coarse equivalence.

**Proof.** It is enough to show that the points \( \omega_x \in \partial \mathfrak{G}(x, S) \) are uniquely determined by \( \mathfrak{G} \) and \( x \). Let \( (S, X) \) be a complete compact generating pair. Note that for any two generating pairs \( (S_i, Y_i) \) we can find one generating pair \( (S, X) \) such that \( S_1 \cup S_2 \subset S \) and \( Y_1 \cup Y_2 \subset X \).

Let \( X_2 \) be a compact neighborhood of \( X \). Let \( \xi \in \partial \mathfrak{G}(x, S) \) be an arbitrary point different from \( \omega_x \). If \( \ldots g_2 g_1 \) is a geodesic path in \( \mathfrak{G}(x, S) \) covering to \( \xi \), then it is on a finite distance from a path \( \ldots h_2 h_1 \) in the Cayley graph \( \mathfrak{G}(x, S) \) where \( h_i \in S \). It follows that there exist neighborhoods \( U_i \) of \( \mathfrak{G} \) of the elements \( g_i \) such that for every \( n \) the composition \( F_n = U_n \cdots U_2 U_1 \) is defined on \( \mathfrak{G}(U_1) \) and \( |F_n(y_1) - F_n(y_2)| \to 0 \) as \( n \to \infty \) for all \( y_1, y_2 \in \mathfrak{G}(U_1) \). Note that since \( X_2 \) is compact, the last condition is purely topological and does not depend on the choice of the metric \( |\cdot| \).

On the other hand, if \( \ldots g_2 g_1 \) converges to \( \omega_x \), then it is on a bounded distance in the Cayley graph \( \mathfrak{G}(x, S) \) to a path of the form \( \ldots h_{2^{-1}} h_{1^{-1}} \), where \( h_i \in S \). For any neighborhoods \( W_i \) of the elements \( h_i^{-1} \) the map \( W_n \cdots W_2 W_1 \) is \( \lambda^n \)-expanding on a neighborhood of \( x \). It follows that for any neighborhoods \( U_i \) of \( \mathfrak{G} \) of the elements \( g_i \) and any \( y_1, y_2 \in \mathfrak{G}(U_1) \) such that \( y_1 \neq y_2 \) the sequence \( |F_n(y_1) - F_n(y_2)| \) does not converge to 0, if it is defined for all \( n \).

We see that the point \( \omega_x \) is uniquely determined in \( \partial \mathfrak{G}(x, S) \) by a condition that uses only purely topological properties of \( \mathfrak{G} \) (and does not use the grading and the metric).
3.6. THE BOUNDARY $\partial \mathfrak{G}$

A corollary of Proposition 3.5.1 is the fact that the Hölder class of the natural log-scale on $\partial \mathfrak{G}_x$ is uniquely determined, i.e., does not depend on the choice of the generating pair and Busemann quasi-cocycle.

In some cases the choice of a strong equivalence class of a Busemann quasi-cocycle may be important (see Example 7). We adopt therefore the following definition.

**Definition 3.5.1.** A graded hyperbolic groupoid is a hyperbolic groupoid together with a choice of (a strong equivalence class of) a Busemann quasi-cocycle.

Note that a strong equivalence class of a Busemann quasi-cocycle determines a Lipschitz class of the natural log-scale on the boundaries $\partial \mathfrak{G}_x$. More on the natural metrics and log-scales on $\partial \mathfrak{G}_x$, see [Nek11].

3.6 The boundary $\partial \mathfrak{G}$

For $\xi \in \partial \mathfrak{G}$ define $P(\xi) \in \mathfrak{G}^{(0)}$ by the condition $\xi \in \partial \mathfrak{G}_{P(\xi)}$.

**Theorem 3.6.1.** Let $\mathfrak{G}$ be a hyperbolic groupoid. There exist unique local product structure and topology on $\partial \mathfrak{G}$ satisfying the following condition.

Let $S$ be a complete generating set of $\mathfrak{G}$, and let $\hat{S}$ be a finite covering of $S$ by contracting positive elements of $\mathfrak{G}$. Then there exist a covering $\mathcal{R}$ of $\partial \mathfrak{G}$ by open rectangles and local homeomorphisms $U_R \in \mathfrak{G}$ for every $R \in \mathcal{R}$ such that $o(U_R) = P(R)$, and for every $\xi \in R$ there exists a sequence $F_i \in \mathcal{S}$ such that $o(F_n \cdots F_1 U_R) = o(U_R)$ for all $n$ and

$$[\zeta, \xi]_R = \ldots F_2 F_1 (U_R, P(\zeta))$$

for all $\zeta \in R$.

**Example 10.** Let $\mathfrak{F}$ be as in Example 5. Let $U$ be an arbitrary open arc of the circle $\mathbb{R}/\mathbb{Z} = \mathfrak{F}^{(0)}$ (not coinciding with the whole circle). Then $U$ is evenly by any iteration $f^n$ of $f$.

For every $x \in U \subset \mathbb{R}/\mathbb{Z}$ the boundary $B_x$ of the tree of preimages orbit of $x$ is an open subset of $\partial \mathfrak{F}_x$ (see Example 5). For every $\xi \in B_x$ there exists a unique sequences $U_0 = U, U_1, U_2, \ldots$ of arcs of $\mathbb{R}/\mathbb{Z}$ such that $f(U_{n+1}) = U_n$ and $\xi$ is the limit of the germs at $x$ of the local homeomorphisms $f^{-n} : U \to U_n$. Then for every $y \in U$, we get the corresponding point of $T_y$ equal to the limit of the germs at $y$ of the same maps $f^{-n} : U \to U_n$. We see that there is a natural identification of the union of the boundaries $B_y$ of the tree of preimages of points of $U$ with the direct product $U \times T_x$.

**Proof.** Let $X$ be a compact topological $\mathfrak{G}$-transversal, and let $\hat{X}$ be its open neighborhood. Fix a metric $| \cdot |$ on $\hat{X}$ and a quasi-cocycle $\nu : \mathfrak{G} |\hat{X} \to \mathbb{R}$ satisfying the conditions of Definition 3.1.1.

Let us start with some technical lemmas.
Lemma 3.6.2. Let $S \subset \mathcal{G}|_{\hat{X}}$ be a compact subset such that $\nu(g) > 2\eta$ for every $g \in S$.

Then there exists a compact set $Q \subset \mathcal{G}|_{\hat{X}}$ and a number $\Delta > 0$ such that for all sequences $g_n, h_n \in S$ such that $o(g_1) = o(h_1), o(g_{n+1}) = t(g_n), o(h_{n+1}) = t(h_n)$ the following conditions are equivalent

1. the sequences $g_n \cdot g_{n-1} \cdots g_1$ and $h_n \cdot h_{n-1} \cdots h_1$ converge to the same point of $\partial \mathcal{G}_x$, where $x = o(g_1) = o(h_1)$;

2. there exist a sequence $f_k \in Q$, and strictly increasing sequences $n_k$ and $m_k$ such that $n_k - n_{k-1} \leq \Delta$ and $m_k - m_{k-1} \leq \Delta$ for all $k \geq 1$,

$$f_k \cdot g_{n_k} \cdot g_{n_k-1} \cdots g_{n_k-1+1} = h_{m_k} \cdot h_{m_k-1} \cdots h_{m_k-1+1} \cdot f_{k-1}$$

for all $k \geq 1$, and $f_0 = o(g_1)$.

See the left-hand side of Figure 3.4 for an illustration for the lemma.

Proof. Embed $S$ into a compact generating set $S_1$ of $\mathcal{G}|_{X'}$ for some compact topological transversal $X' \subset \hat{X}$. There exists $n$ such that the set $S^n_1 \cup S^{n+1}_1$ can be embedded into a complete generating set $S_2$ (see the proof of Proposition 3.3.1). It is enough to prove our proposition for any compact set containing $S^n$. Therefore, we assume that $S$ satisfies the conditions of Proposition 3.3.1 for a compact topological transversal $X'$.

The quasi-geodesics $(\ldots, g_n, \ldots, g_1)$ and $(\ldots, h_n, \ldots, h_1)$ converge to the same point of $\partial \mathcal{G}_x$ if and only if they are on a finite distance from each other. If $\Delta$ is as in condition (2) and $r$ is the maximal length of elements of $Q$, then every point of one of the paths $(\ldots, g_2, g_1)$ and $(\ldots, h_2, h_1)$ is on distance not more than $\Delta + r$ from the other, hence the quasi-geodesics $(\ldots, g_n, \ldots, g_1)$ and $(\ldots, h_n, \ldots, h_1)$ converge to the same point of the boundary.

In the other direction, there exists $\Delta_1$ such that if quasi-geodesic paths $(\ldots, g_2, g_1)$ and $(\ldots, h_2, h_1)$, $g_i, h_i \in S$, starting at a common vertex converge
to the same point of the boundary, then they are on a distance not more than $\Delta_1$ from each other.

Let $Q$ be the set of elements of $\mathfrak{G}$ which can be represented as products of not more than $\Delta_1$ elements of $S \cup S^{-1}$. Then $Q$ is a compact set.

Define $n_0 = m_0 = 0$. By induction, if $m_k$ and $n_k$ are defined, define $n_{k+1}$ to be such that distance from $(g_{n_k}, \ldots, g_1)$ to $(g_{n_{k+1}}, \ldots, g_1)$ in $\mathfrak{G}(x, S)$ is greater than $2\Delta_1$, but less than some fixed constant $\epsilon$ (which we can find since $(\ldots, g_2, g_1)$ is quasi-geodesic). Then define $m_{k+1}$ so that the distance between $(g_{n_{k+1}}, \ldots, g_1)$ to $(h_{m_{k+1}}, \ldots, h_1)$ is not more than $\Delta_1$. Then $m_{k+1} > m_k$ and the differences $m_{k+1} - m_k$ and $n_{k+1} - n_k$ are uniformly bounded. \hfill \Box

**Lemma 3.6.3.** Let $S$ and $B$ be finite sets of elements of $\mathfrak{G}|_X$ such that all elements of $S$ are positive contractions, and all elements of $B$ are bi-Lipschitz. Let $\epsilon > 0$. There exists a number $\delta_0 > 0$ for which the following statements hold.

Suppose that the sequences $U_n, V_n \in S \cup B$ and $g_n, h_n \in \mathfrak{G}$ are such that each sequence $(U_n), (V_n)$ contains at most one element of $B$, $g_n$ and $h_n$ are $\epsilon$-contained in $U_n, V_n$, respectively, $\mathfrak{o}(g_0) = \mathfrak{o}(h_0)$, and the sequences $g_n, g_{n-1} \cdots g_0$ and $h_n h_{n-1} \cdots h_0$ are defined and converge to the same point of $\partial \mathfrak{G} \mathfrak{o}(g_0)$.

Then for every $y \in \hat{X}$ such that $|y - \mathfrak{o}(g_0)| < \delta_0$ the sequences

$$U_n \cdots U_{n-1} \cdots (U_0, y), \quad V_n \cdots V_{n-1} \cdots (V_0, y)$$

are defined and converge to the same point.

See the right-hand side of Figure 3.4.

**Proof.** There exists $k$ such that for every $F \in B$ and for every sequence $F_1, \ldots, F_k$ of elements of $S$ the map $F_1 \cdots F_k : F$ is either empty or is a contraction such that $\nu(g) > 2\eta$ for every its germ $g$. Consequently, it is enough to prove our lemma for the case when $B$ is empty.

Let $S$ be the closure of the sets of elements $g \in \mathfrak{G}$ such that there exists $U \in S$ such that $g$ is $\epsilon$-contained in $U$.

Let $Q$ and $\Delta$ be as in Lemma 3.6.2 for the set $S$. The set $Q$ can be covered by a finite set of relatively compact extendable elements of $\mathfrak{G}|_X$. Therefore, there exists a finite collection $R$ of elements $\hat{R}_i \in \mathfrak{G}|_X$ and open subsets $R_i \subset \hat{R}_i$ such that $R_i$ cover $Q$, $\overline{R_i}$ are compact and are contained in $\hat{R}_i$.

Let us apply Corollary 2.4.2 for the set $S \cup R$, and some extensions $\hat{F}$ of its elements.

The number of possible sequences of the form

$$(R_k, G_{n_k}, \ldots, G_{n_{k-1}+1}, H_{m_k}, \ldots, H_{m_{k-1}+1}, R_{k-1}),$$

where $n_k - n_{k-1}, m_k - m_{k-1} \leq \Delta$, $G_i, H_i \in S$, and $R_i \in R$ is finite. Consequently, by Corollary 2.4.2 there exists $\delta_0 > 0$ such that for every relation of the form

$$f_k \cdot g_k \cdots g_{n_k+1} = h_m \cdots h_{m+1} \cdot f_{k-1}$$
for \( f_k \in R_k, f_{k-1} \in R_{k-1} \) and \( g_i \in G_i, h_i \in H_i \), for \( R_i \in \mathcal{R}, G_i, H_i \in \mathcal{S} \) the maps
\[
\hat{R}_k \cdot \hat{G}_{n_k} \cdots \hat{G}_{n_{k-1}+1}, \quad \hat{H}_{m_k} \cdots \hat{H}_{m_{k-1}+1} \cdot \hat{R}_{k-1}
\]
are equal on the \( \delta_0 \)-neighborhood of \( o(g_{n_{k-1}+1}) \).

Taking \( \delta_0 \) sufficiently small (in particular, \( \delta_0 < \epsilon \)), we may assume that \( o(\hat{R}_k), t(\hat{R}_k), o(\hat{G}_i), o(\hat{H}_i), o(\hat{R}_{k-1}), t(\hat{R}_{k-1}) \) always contain the \( \delta_0 \)-neighborhoods of \( o(f_k), t(f_k), o(g_i), o(h_i), o(f_{k-1}), t(f_{k-1}) \), respectively. Since the maps \( G_i \) and \( H_i \) are contractions, this will imply that the maps
\[
\hat{R}_k \cdot \hat{G}_{n_k} \cdots \hat{G}_{n_{k-1}+1}, \quad \hat{H}_{m_k} \cdots \hat{H}_{m_{k-1}+1} \cdot \hat{R}_{k-1}
\]
are defined on the \( \delta_0 \)-neighborhood of \( t(g_{n_{k-1}}) = o(g_{n_{k-1}+1}) \). This implies part (1) of the lemma (since \( \hat{F} \) and \( F \) for \( F \in \mathcal{S} \) coincide on the \( \epsilon \)-neighborhood of \( o(g) \) for all \( g \) that are \( \epsilon \)-contained in \( F \)).

We may assume that \( X \) contains open transversals \( X_0 \) and \( X'_0 \) such that \( X'_0 \subset X_0 \). Let \( S \) be a generating set of \( \mathfrak{G} \) satisfying the conditions of Proposition 3.3.1 for \( X_0 \). Let \( S \) be a finite covering of \( S \) by relatively compact extendable elements of \( \mathfrak{G} \). We assume that every \( F \in \mathcal{S} \) is a \( \lambda \)-contraction and \( \nu(g) > 2\eta \) for every \( g \in F \).

Let \( A \) be a compact set satisfying the conditions of Proposition 3.3.1. Let \( \Delta_2 \) be such that any two paths \((\ldots, g_2, g_1), (\ldots, h_2, h_1), g_i, h_i \in \mathcal{S} \), converging to one point of \( \partial \mathfrak{G} \) are eventually on distance not more than \( \Delta_2 \) from each other. Let \( Q \) be the set of elements of \( \mathfrak{G} \) which can be represented as a product of length at most \( \Delta_2 \) of elements of \( \mathcal{S} \) (in particular, \( Q \) contains \( X \)). Denote \( B = A \cdot Q \).

Find a finite covering \( \mathcal{B} = \{ U_i \} \) of \( B \) by open relatively compact extendable bi-Lipschitz elements of \( \mathfrak{G}|_{\mathcal{B}} \). Denote \( \widehat{B} = \bigcup_{U_i \in \mathcal{B}} U_i \).

Let \( \epsilon > 0 \) be such that \( 2\epsilon \) is a common Lebesgue’s number of the coverings \( \mathcal{S} \) and \( B \) of \( S \) and \( B \), respectively.

Let us apply Lemma 3.6.3 to \( \mathcal{B}, \mathcal{S} \), and \( \epsilon \). Let us fix a number \( \delta_0 < \epsilon \) satisfying the conditions of the lemma. We assume that the \( \delta_0 \)-neighborhood of \( o(F) \) is contained in \( o(\hat{F}) \) for every \( F \in \mathcal{S} \), and that the \( \delta_0 \)-neighborhood of \( X_0 \) is contained in \( X'_0 \).

**Definition 3.6.1.** We say that \( U \in \mathfrak{G} \) is *admissible* if it is relatively compact, extendable, and the closure of \( t(U) \) is a subset of \( X'_0 \) of diameter less than \( \delta_0 \).

Let \( U \in \mathfrak{G} \) be admissible and consider an arbitrary point \( \zeta = \cdots g_2 g_1 g \in \mathcal{T}_g \), where \( g_i \in \mathcal{S} \) and \( g \in U \). Let \( F_i \in \mathcal{S} \) be such that \( g_i \) is \( \epsilon \)-contained in \( F_i \) for every \( i \). Then the map \( F_n \cdots F_1 U \) is defined on \( o(U) \).

Define then for \( y \in o(U) \)
\[
[y, \zeta]_U = \cdots F_2 F_1 \cdot (U, y), \tag{3.1}
\]
where \( (U, y) \), as usual, denotes the germ of \( U \) at \( y \).
Lemma 3.6.4. The map \([\cdot, \cdot]_U: o(U) \times T_g \rightarrow \partial \mathcal{G}\) is well defined and injective for every \(g \in U\).

Proof. Suppose that \(\xi = \cdots g_2 g_1 \cdot g = \cdots h_2 h_1 \cdot g \) for \(g, h_i \in S\) and \(g \in U\). Let \(U_i\) and \(V_i \in S\) be such that \(g, h_i \in S\) and \(g \in U\). Let \(U_i\) and \(V_i \in S\) be such that \(g, h_i \in S\) and \(g \in U\). It follows then directly from part (1) of Lemma 3.6.3 that for every \(y \in o(U)\) we have
\[
\cdots U_2 U_1 \cdot (U, y) = \cdots V_2 V_1 \cdot (U, y),
\]
which proves that the map \([\cdot, \cdot]_U\) is well defined. Injectivity is proved in the same way.

\[\square\]

Proposition 3.6.5. Let \(U\) be admissible. For fixed \(y \in o(U)\) and \(g \in U\) the map \([y, \cdot]_U: T_g \rightarrow \partial \mathcal{G}_y\) is Lipschitz with respect to the natural log-scales on \(\partial \mathcal{G}_{o(g)}\) and \(\partial \mathcal{G}_y\).

Proof. If \(\xi_1, \xi_2 \in T_g\) can be represented as \(\xi_1 = \cdots g_n g_{n+1} \cdot g\) and \(\xi_2 = \cdots h_n h_{n+1} \cdot g\) for some \(n \geq 1\) and \(g, h_i \in S\), then \([y, \xi_1]_U = [y, g_n g_{n+1} \cdot g\] for some germs \(g'\) and \(h'\) of elements of \(S\). It follows from the definition of the log-scale on \(\partial \mathcal{G}_x\), Theorem 1.2.5, and Lemma 1.2.7 that there exists \(\Delta_0\) (depending only on \((S, X)\)) such that
\[
\ell_g([y, \xi_1]_U, [y, \xi_2]_U) \geq \ell_{o(g)}(\xi_1, \xi_2) - \Delta_0,
\]
where \(\ell_y\) and \(\ell_{o(g)}\) are the natural log-scales on \(\partial \mathcal{G}_y\) and \(\partial \mathcal{G}_{o(g)}\), respectively. We used the fact that \(\ell_y\) and \(\ell_{o(g)}\) do not depend, up to a Lipschitz equivalence, on the choice of the generating set.

\[\square\]

Denote by \(T_g^o\) the interior of \(T_g\) as a subset of the space \(\partial \mathcal{G}_{o(g)}\). Denote for an admissible \(U \in \mathcal{G}\) and \(g \in U\)
\[
R_{g,U} = \{[y, \xi]_U : \xi \in T_g^o, y \in o(U)\} \subset \partial \mathcal{G}.
\]

By Lemma 3.6.4 the map \([y, \xi]_U: o(U) \times T_g^o \rightarrow R_{g,U}\) is a bijection. We consider then \(R_{g,U}\) as a rectangle with respect to the direct product decomposition \(R_{g,U} \approx o(U) \times T_g^o\). At this moment we do not have any topology on \(\partial \mathcal{G}\) and \(R_{g,U}\) yet.

Lemma 3.6.6. Let \(\xi \in R_{g_1,U_1} \cap R_{g_2,U_2}\). Then there exist \(g\) and \(U\) such that \(R_{g,U}\) is defined, \(\xi \in R_{g,U} \subset R_{g_1,U_1} \cap R_{g_2,U_2}\) and the rectangle \(R_{g,U}\) is compatible with \(R_{g_1,U_1}\) and \(R_{g_2,U_2}\).

Proof. Let \(x = P(\xi)\). Denote \(x_i = t(g_i)\). Since \(\xi\) belongs to the intersection \(R_{g_1,U_1} \cap R_{g_2,U_2}\), there exist points \(\xi_1 \in T_{g_1}^o\) and \(\xi_2 \in T_{g_2}^o\) such that \(\xi = [x, \xi_1]_{U_1} = [x, \xi_2]_{U_2}\). Denote that \(x \in o(U_1) \cap o(U_2)\). Let \(\xi_1 = \cdots g_2 g_1\) and \(\xi_2 = \cdots r_{2} r_{1} g_2\) where \(s_i, r_i \in S\). Let \(S_i\) and \(R_i\) be elements of \(\mathcal{G}\) such that \(s_i\) and \(r_i\) are \(2\varepsilon\)-contained in \(S_i\) and \(R_i\), respectively. Then
\[
\xi = \cdots S_2 S_1 U_1, x = \cdots R_2 R_1 U_2, x.
\]
CHAPTER 3. HYPERBOLIC GROUPOIDS

Figure 3.5:

Since the points $\xi_1$ and $\xi_2$ belong to the interior of the sets $T_{g_1}$ and $T_{g_2}$, there exists $n_0$ such that for all $k \geq n_0$, for all sequences $H_1, H_2, \ldots \in S$, and for all $b \in \hat{B}$ we have

$$
\cdots H_2 H_1 \cdot b \cdot S_k S_{k-1} \cdots S_1 g_1 \in T_{g_1}^\circ, \quad \cdots H_2 H_1 \cdot b \cdot R_k R_{k-1} \cdots R_1 g_2 \in T_{g_2}^\circ,
$$

whenever the corresponding products are defined, see Proposition 3.4.5.

There exist indices $k_1$ and $k_2$ both greater than $n_0$ such that $q = (R_{k_2} \cdots R_1 (U_2, x)) \cdot (S_{k_1} \cdots S_1 (U_1, x))^{-1} \in Q$.

There exists $f_1 \in A$ such that $o(f_1) = S_{k_1} \cdots S_1 U_1 (x)$, $t(f_1) \in X_0$, and $\xi$ is an internal point of $T_{f_1, S_{k_1} \cdots S_1 (U_1, x)}$, see Figure 3.5. Then $f_2 = f_1 q$ belongs to $B$ and

$$(f_1 S_{k_1} \cdots S_1 U_1, x) = (f_2 R_{k_2} \cdots R_1 U_2, x).$$

Let $F_1, F_2 \in B$ such that $f_1$ and $f_2$ are $2\varepsilon$-contained in $F_1$ and $F_2$, respectively. Then the germs $(F_1 S_{k_1} \cdots S_1 U_1, x)$ and $(F_2 R_{k_2} \cdots R_1 U_2, x)$ coincide. Let $U$ be a common restriction of $F_1 S_{k_1} \cdots S_1 U_1$ and $F_2 R_{k_2} \cdots R_1 U_2$ onto an open neighborhood of $x$. Note that $o(U) \subset o(U_1) \cap o(U_2)$. We may assume that $U$ is admissible, since $t(f_1) = t(f_2) \in X_0$. Denote $g = (U, x)$, and consider the rectangle $R_{g, U}$. By the choice of $f_1$, $\xi$ belongs to interior of $T_g$, hence $\xi \in R_{g, U}$.

It remains to show for every $i = 1, 2$ that $R_{g, U} \subset R_{g_i, U_i}$ and that the direct product structure on $R_{g, U}$ agrees with the direct product structure on $R_{g_i, U_i}$. It is enough to prove the statements for $i = 1$ (since we will use only that $f_i \in B$, and not the way they were defined, so that both $i = 1$ and $i = 2$ will be equivalent).

In order to show that $R_{g, U} \subset R_{g_1, U_1}$, we have to show that for every $y \in o(U)$ and $\zeta \in T_{g_1}^\circ$ we have $[y, \zeta]_U \in R_{g_1, U_1}$, i.e., that there exists $\zeta_1 \in T_{g_1}^\circ$.
such that \([y, \zeta]_U = [y, \zeta_1]_{U_1}\). Then, in order to show that the direct product structures of \(R_{g,U}\) and \(R_{g_1,U_1}\) agree, it will be enough to show that for every \(z \in \omega(U)\) we have \([z, \zeta]_U = [z, \zeta_1]_{U_1}\).

Let \(\xi = \ldots h_2 h_1 \cdot g\), where \(h_i \in S\). Let \(H_i \in S\) be such that \(h_i\) is \(2\epsilon\)-contained in \(H_i\). Then

\[ [y, \zeta]_U = \ldots H_2 H_1 (U, y) = \ldots H_2 H_1 F_1 S_{k_1} \ldots S_1 (U_1, y). \]

By the choice of \(k_1\), the point

\[ \zeta_1 = \ldots H_2 H_1 F_1 S_{k_1} \ldots S_1 \cdot g_1 \]

belongs to \(T_{g_1}^\circ\). In particular, it can be represented in the form \(\ldots t_2 t_1 g_1\) for \(t_i \in S\). If \(T_i \in S\) are such that \(t_i\) is \(2\epsilon\)-contained in \(T_i\), then

\[ [y, \zeta_1]_{U_1} = \ldots T_2 T_1 (U_1, y) = \ldots H_2 H_1 F_1 S_{k_1} \ldots S_1 (U_1, y) = [y, \zeta]_U, \]

where the last equality follows by Lemma \[3.6.3\] from the equality

\[ \ldots T_2 T_1 (U_1, \omega(g_1)) = \ldots H_2 H_1 F_1 S_{k_1} \ldots S_1 (U_1, \omega(g_1)). \]

In fact, by Lemma \[3.6.3\] we have

\[ [z, \zeta_1]_{U_1} = \ldots T_2 T_1 (U_1, z) = \ldots H_2 H_1 F_1 S_{k_1} \ldots S_1 (U_1, z) = [z, \zeta]_U, \]

for all \(z \in \omega(U)\), which finishes the proof.

**Lemma 3.6.7.** The set of rectangles \(R_{g,U}\) form a base of topology on \(\partial G\). The map \([\cdot, \cdot]_U : \omega(U) \times T_{g}^\circ \rightarrow R_{g,U}\) is a homeomorphism.

**Proof.** The first statement follows directly from Lemma \[3.6.6\]. Let us prove the second statement. We know that the map is a bijection (see Lemma \[3.6.4\]). Let \(A \subset \omega(U)\) and \(B \subset T_{g}^\circ\) be open sets, and let \(y \in A\), \(\zeta \in B\) be arbitrary points. Using Proposition \[3.4.3\] (as in the proof of Lemma \[3.6.6\]) we show that there exist \(g_1 \in G\) and admissible \(U_1 \in \mathcal{S}\) such that \(T_{g_1} \subset B\) and \(\zeta \in T_{g_1}^\circ\), \(g_1 \in U_1\), and \(R_{g_1,U_1} \subset R_{g,U}\).

Then, applying Lemma \[3.6.6\] to \(R_{g_1,U_1}\) and \(\xi = [y, \zeta]_U\), we see that there exist \(U' \in \mathcal{S}\) and \(g' \in G\) such that \([y, \zeta]_{U'} \in R_{g',U'} \subset [A, B]_U\). This shows that image of an open subset of \(\omega(U) \times T_{g}^\circ\) is open, i.e., that the map inverse to \([\cdot, \cdot]_U : \omega(U) \times T_{g}^\circ \rightarrow R_{g,U}\) is continuous.

It remains to prove that the map \([\cdot, \cdot]_U\) is continuous. Continuity on the first argument follows directly from the definition, while continuity on the second argument follows from Lemma \[3.6.3\].

**Lemma 3.6.8.** If \(\delta_0\) is small enough, then the rectangles \(R_{g,U}\) form an atlas of a local product structure on the topological space \(\partial G\).
Proof. Let $R_{g_1,u_1}$ and $R_{g_2,u_2}$ be two rectangles, and let $\xi \in R_{g_1,u_1} \cup R_{g_2,u_2}$. We have to show that there exists a rectangle $R_{g,u}$ containing $\xi$ and such that the direct product structure on $R_{g,u}$ agrees with the direct product structures on both rectangles $R_{g_i,u_i}$. Note that since the rectangles form a basis of topology of $\partial G$, the condition obviously holds when $\xi$ does not belong to the intersection of the closures of $R_{g_i,u_i}$.

Therefore, we suppose that $\xi \in \overline{R_{g_1,u_1}} \cap \overline{R_{g_2,u_2}}$. We can embed $S$ into a bigger complete generating set $S'$ so that for every $g \in \mathcal{G}$ the new set $\mathcal{T}_g = \{ \cdots g_2 g_1 \cdot g : g_i \in S' \}$ is a neighborhood of $\mathcal{T}_g = \{ \cdots g_2 g_1 : g_i \in S \}$. Let $S'$ be the corresponding covering (we assume that $S \subset S'$). If $R_{h,W}$ is a rectangle defined using $S$, such that $t(W)$ is sufficiently small (i.e., has diameter less than the value of $\delta_0$ corresponding to $S'$ and $S'$), then it is a sub-rectangle of the rectangle $R'_{h,W}$ defined using $S'$ and $S'$, by Lemma 3.6.4.

We can always extend the admissible elements $U_i$ to admissible elements $\hat{U}_i$ such that $\overline{U}_i \subset \hat{U}_i$.

It follows that if $\delta_0$ is small enough (i.e., if it satisfies the conditions of Lemma 3.6.3 for $S'$ and $S'$), then closure of every rectangle $R_{g_i,u_i}$ can be embedded into a rectangle defined using $S'$. Then Lemma 3.6.6 (applied to $S'$ and $S'$) implies that rectangles $R_{g,u}$ form an atlas of a local product structure on $\partial G$.

Independence of topology and the local product structures from the choices of the Lipschitz structure, grading, and the sets $S$, $S$, $X$, $X$, $\mathcal{X}$, etc., made in their construction follow directly from Lemma 3.6.4 since for any two choices of the sets we can find one including the first two as subsets (possibly after replacing $S$ and $S$ by $S^n \cup S^{n+1}$ and $S^n \cup S^{n+1}$, if we change the Lipschitz structures and grading). Independence on the choice of $\delta_0$ follows from Lemma 3.6.6.

This finishes the proof of the theorem, since the constructed local product and topology satisfy its conditions, and every local product structure satisfying the conditions of the theorem coincides with the one we have constructed (by Lemma 3.6.4).

\section{Geodesic quasi-flow}

Denote by $P : \partial \mathcal{G} \to \mathcal{G}^{(0)}$ the map defined by the condition $\xi \in \partial \mathcal{G} P(\xi)$.

It is easy to see from the formula for the local product in Theorem 3.6.1 that the map $P$ is compatible with the local product structure on $\partial \mathcal{G}$, i.e., for every point $\xi \in \partial \mathcal{G}$ there exists an open rectangle $R$ containing $\xi$ such that $P : R \to \mathcal{G}^{(0)}$ induces homeomorphisms of the plaques $P_2(R,x)$ with $P(R)$.

Let $\xi = \cdots h_2 h_1 \cdot h \in \partial \mathcal{G}$ for $h_i \in S$ and $h \in \mathcal{G}$. Define then for $g \in \mathcal{G}$ such that $t(g) = o(h) = P(\xi)$

$$\xi \cdot g = \cdots h_2 h_1 \cdot hg.$$

We get in this way a right action of $\mathcal{G}$ on $\partial \mathcal{G}$ over the projection $P : \partial \mathcal{G} \to \mathcal{G}^{(0)}$. 

\footnotesize

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\normalsize
3.7. GEODESIC QUASI-FLOW

Consider the groupoid of this action, i.e., the set
\[ \partial \mathcal{G} \times \mathcal{G} = \{ (\xi, g) \in \partial \mathcal{G} \times \mathcal{G} : P(\xi) = t(g) \} \]
with topology induced from the direct product topology on \( \partial \mathcal{G} \times \mathcal{G} \) and groupoid structure given by the multiplication rule
\[ (\xi_2, g_2) \cdot (\xi_1, g_1) = (\xi_1, g_2g_1), \]
where the product is defined if and only if \( \xi_2 = \xi_1 \cdot g_1 \) (see Definition 2.2.2).

We naturally identify the space of units of \( \partial \mathcal{G} \times \mathcal{G} \) with \( \partial \mathcal{G} \).

We call the groupoid \( \partial \mathcal{G} \times \mathcal{G} \) the geodesic quasi-flow of \( \mathcal{G} \). It is easy to see that \( \partial \mathcal{G} \times \mathcal{G} \) is a Hausdorff groupoid of germs preserving the local direct product structure on \( \partial \mathcal{G} \).

**Proposition 3.7.1.** If \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are equivalent hyperbolic groupoids, then the groupoids \( \partial \mathcal{G}_1 \times \mathcal{G}_2 \) and \( \partial \mathcal{G}_2 \times \mathcal{G}_1 \) are equivalent.

**Proof.** It follows from the definition of the local product structure on \( \partial \mathcal{G} \) that the space \( \partial(\mathcal{G}_1 \vee \mathcal{G}_2) \) is disjoint union of the spaces \( \partial \mathcal{G}_1 \) and \( \partial \mathcal{G}_2 \) and that the restriction of the action of \( \mathcal{G}_1 \vee \mathcal{G}_2 \) onto \( \partial \mathcal{G}_1 \) coincides with the action of \( \mathcal{G}_1 \) on \( \partial \mathcal{G}_1 \). It follows that \( \partial(\mathcal{G}_1 \vee \mathcal{G}_2) \times (\mathcal{G}_1 \vee \mathcal{G}_2) \) is an equivalence groupoid between \( \partial \mathcal{G}_1 \times \mathcal{G}_2 \) and \( \partial \mathcal{G}_2 \times \mathcal{G}_1 \). \( \square \)

**Proposition 3.7.2.** The groupoid \( \partial \mathcal{G} \times \mathcal{G} \) is compactly generated.

**Proof.** We will use notations from the proof of Theorem 3.6.1. Denote by \( T \subset \partial \mathcal{G} \) the closure of the set of points of the form \( \cdots g_2g_1 \) for \( g_i \in F \subset S \) and \( o(g_1) \in X \).

By Proposition 3.4.1 for every \( \xi \in \partial \mathcal{G} \) there exists \( g \in \mathcal{G} \) such that \( \xi \) is an internal point of \( T_g \) and \( t(g) \in X_0 \). Let \( \xi = \cdots g_2g_1 \cdot g \) for \( g_i \in S \). Then \( \xi \cdot g^{-1} = \cdots g_2g_1 \) is an internal point of \( T_{t(g)} \). Let \( U \subset X_0 \) be an open neighborhood of \( o(g_1) \) of diameter less than \( \delta_0 \). Then \( \xi \cdot g^{-1} \) belongs to the open set \( R_{t(g), U} \subset T \). It follows that \( T \) is a topological transversal.

Let us show that \( T \) is compact. Every point of \( T \) can be represented as \( \cdots g_2g_1 \), where \( g_i \) belong to the elements of the set \( S \). Consider the compact space \( \tilde{S} = \bigcup_{F \in S} \tilde{F} \). The set of all composable sequences \( (\cdots, g_2, g_1) \) of elements of \( \tilde{S} \) is a closed subset of the Cartesian product \( (\tilde{S})^\infty \). Hence the space of all composable sequences is compact. It follows from Proposition 3.4.2 and Theorem 3.6.1 that the map \( (\cdots, g_2, g_1) \mapsto \cdots g_2g_1 \) from the space of composable sequences to \( \partial \mathcal{G} \) is continuous. Hence, \( T \) is contained in a continuous image of a compact space, hence \( T \) is compact (since we assumed that it is closed).

Consider the set
\[ T \times_p S = \{ (\xi, g) \in \partial \mathcal{G} \times \mathcal{G} : \xi \in T, g \in S, P(\xi) = t(g) \} \]
of germs of the action of elements of \( S \) on \( T \). It is compact, since \( T \) and \( S \) are compact, and \( P \) and \( o \) are continuous.
Let us show that $T \times _P S$ is a generating set of the restriction of $\partial \mathfrak{G} \times \mathfrak{G}$ onto $T$. Let $(\xi, g) \in \partial \mathfrak{G} \times \mathfrak{G}$, be an arbitrary element of $(\partial \mathfrak{G} \times \mathfrak{G})|_T$. Then $t(g) = P(\xi), \xi \in T$, and $\xi \cdot g \in T$. In particular, $g \in \mathfrak{G}|_X$. By Proposition 3.3.1 the element $g \in \mathfrak{G}$ can be written in the form $r_1 \cdots r_k \cdot (s_1 \cdots s_l)^{-1}$ for $r_i, s_i \in S$. Every point $\xi \cdot r_1 \cdots r_m$ and $\xi \cdot g \cdot s_1 \cdots s_{n-1} = \xi \cdot r_1 \cdots r_k \cdot s_i \cdots s_{n-1}$, for $1 \leq m \leq k$ and $1 \leq n \leq l$, belongs to $T$. It follows that $(\xi, g)$ is a product of $k + l$ elements of $T \times _P S \cup (T \times _P S)^{-1}$.

The set $S$ is a generating set of $\mathfrak{G}|_X$, and the length of a representation of an element $g \in \mathfrak{G}|_X$ as a product of the form $r_1 \cdots r_k \cdot (s_1 \cdots s_l)^{-1}$ is bounded from above by a function of the $S$-length of $g$ (see Proposition 3.3.2). It follows that there exists a neighborhood $U$ of $g$ in $\mathfrak{G}$ and a number $N$ such that every element $h \in U \cap \mathfrak{G}|_X$ can be written as a product $r_1 \cdots r_k \cdot (s_1 \cdots s_l)^{-1}$ for $k + l \leq N$. Consequently every element of $T \times _P U$ is written as a product of length at most $N$ of elements of $T \times _P S$, which shows that $T \times _P S$ is a generating set of $(\partial \mathfrak{G} \times \mathfrak{G})|_T$. \qed
Chapter 4

Smale quasi-flows and duality

4.1 Definitions

Definition 4.1.1. Let $\mathcal{H}$ be a graded Hausdorff groupoid of germs with a Lipschitz structure and a local product structure on $\mathcal{H}^{(0)}$ preserved by $\mathcal{H}$. We assume that the Lipschitz structure and the grading agree with the local product structure (see Definition 2.5).

Let $X$ be a compact topological transversal, let $| \cdot |$ be a metric defined on a compact neighborhood $\tilde{X}$ of $X$ and compatible with the Lipschitz structure. Let $\nu : \mathcal{G}|_{\tilde{X}} \to \mathbb{R}$ be an $\eta$-quasi-cocycle compatible with the grading.

We say that the graded groupoid $\mathcal{H}$ is a Smale quasi-flow if there exists a compact generating set $S$ of $\mathcal{H}|_X$ such that

1. $\nu(g) > 3\eta$ for every $g \in S$;
2. $o(S) = t(S) = X$;
3. there exists $\lambda$ such that $0 < \lambda < 1$ and every element $g \in S$ has a rectangular neighborhood $F \in \mathcal{H}|_{\tilde{X}}$ such that
   \[ |F(x) - F(y)| \leq \lambda |x - y| \]
   for all $x, y \in P_1(o(F), o(g))$, and
   \[ |F(x) - F(y)| \geq \lambda^{-1} |x - y| \]
   for all $x, y \in P_2(o(F), o(g))$;
4. the groupoid $\mathcal{H}$ is locally diagonal with respect to the local product structure (see Definition 2.5.6);
5. for every $r > 0$ and every compact subset $A \subset \tilde{X}$ the closure of the set
   \[ \{ g \in \mathcal{G} : o(g) \in A, t(g) \in A, |\nu(g)| \leq r \} \]
   is compact.
We will call the first and second directions of the local product structure of a Smale space stable and unstable, and denote projections $P_1$ and $P_2$ by $P_+$ and $P_-$, respectively.

The last condition of Definition 4.1.1 implies that $\nu$ is a quasi-isometry of the Cayley graph of $\mathcal{H}$ with $\mathbb{R}$, which is the reason why we call such groupoids quasi-flows.

**Definition 4.1.2.** We say that a covering $\mathcal{R}$ of a transversal of a Smale quasi-flow $\mathcal{H}$ by open rectangles is fine if it satisfies the conditions of Definitions 2.5.10, 2.5.11, and 2.5.6.

It is easy to see that if a covering $\mathcal{R}$ is fine then any covering by subrectangles of $\mathcal{R}$ is also fine.

The following statements are straightforward (compare with Proposition 3.1.1 and Corollary 3.1.2).

**Proposition 4.1.1.** Let $\mathcal{H}$ be a Smale quasi-flow. Then for any compact topological transversal $X$ of $\mathcal{H}$, a metric on a neighborhood of $X$ compatible with the Lipschitz structure, and a quasi-cocycle $\nu : \mathcal{H} \to \mathbb{R}$ compatible with the grading of $\mathcal{H}$ there exists a compact generating set $S$ of $\mathcal{H}|_X$ satisfying the conditions of Definition 4.1.1.

**Corollary 4.1.2.** Every groupoid of germs equivalent to a Smale quasi-flow is also a Smale quasi-flow.

**Example 11.** Suppose that $\nu : \mathcal{H} \to \mathbb{R}$ is a cocycle, i.e., that $\nu(g_1 g_2) = \nu(g_1) + \nu(g_2)$, and $\nu$ is continuous, and suppose that it satisfies condition (5) of Definition 4.1.1. Consider then the action of $G$ on $G(0) \times \mathbb{R}$ given by the formula

$$h \cdot (g, t) = (h g, \nu(h) + t).$$

Note that the natural action of $\mathbb{R}$ on $G(0) \times \mathbb{R}$ is free, proper, and commutes with the defined action of $G$. It follows from condition (5) of Definition 4.1.1 that the action of $G$ on $G(0) \times \mathbb{R}$ is proper. Suppose that it is free, i.e., that $\nu(g) = 0$ and $o(g) = t(g)$ for $g \in G$ implies that $g$ is a unit. Then (see 2.2) the groupoid of the action of $\mathbb{R}$ on the space $M = G \setminus (G(0) \times \mathbb{R})$ is equivalent to the groupoid $\mathcal{G}$. It follows from compact generation that $M$ is a compact. It is Hausdorff by properness of the action of $G$ on $G(0) \times \mathbb{R}$. We see that in this case the groupoid $\mathcal{G}$ is equivalent to a flow on a compact metric space. The space $M$ has a natural local decomposition into a direct product of three spaces $P_r \times P_+ \times I$, where $I$ is an interval of $\mathbb{R}$, so that the plaques in the direction of $I$ are orbits of the flow, plaques in the direction of $P_+$ are contracted, and the plaques in the direction of $P_-$ are expanded by the flow. In this sense the action of $\mathbb{R}$ on $M$ is a Smale flow.

If the action of $G$ on $G(0) \times \mathbb{R}$ is not free, then $M = G \setminus (G(0) \times \mathbb{R})$ is an orbispace and $G$ is equivalent to a Smale orbispace flow. It is defined by the action of $\mathbb{R}$ on the orbispace atlas $G(0) \times \mathbb{R}$. For more on orbispaces see [Nek05].
4.2 Special generating sets and Ruelle groupoids

Similarly to hyperbolic groupoids (see Proposition 3.3.1) we will need generating sets with special properties.

If \((A, | \cdot |_A)\) and \((B, | \cdot |_B)\) are metric spaces, then the direct product of the metrics \(| \cdot |_A\) and \(| \cdot |_B\) is the metric on \(A \times B\) given by

\[
|(a_1, b_1) - (a_2, b_2)|_{A \times B} = \max(|a_1 - a_2|_A, |b_1 - b_2|_B).
\]

**Proposition 4.2.1.** Every Smale quasi-flow is equivalent to a groupoid \(\mathcal{H}\) satisfying the following properties.

The space of units \(\mathcal{H}^{(0)}\) is a disjoint union of a finite number of rectangles \(W_1 = A_1 \times B_1, \ldots, W_n = A_m \times B_m\). The covering \(\mathcal{R} = \{W_i\}\) is fine, and the metric on \(\mathcal{H}^{(0)}\) is locally isometric to the direct product of metrics on \(A_i\) and \(B_i\).

There exists an open transversal \(X_0\) equal to a union of open sub-rectangles \(W_i^0 = A_i^0 \times B_i^0 \subset W_i\) such that the closure of \(W_i^0\) is compact. Denote by \(X\) the union of closures of the rectangles \(W_i^0\).

There exists a finite set \(S\) of elements of the pseudogroup \(\mathcal{H}\) such that

1. every \(F \in S\) is a rectangle \(A_F \times B_F\);
2. for every \(F \in S\) there exist \(i, j \in 1, \ldots, n\) such that \(o(F) \subset W_i, t(F) \subset W_j, o(A_F) = A_i, t(B_F) = B_j\);
3. intersections of \(o(F)\) and \(t(F)\) with \(X\) are non-empty;
4. \(A_F\) and \(B_{F}^{-1}\) are \(\lambda\)-contracting for some \(\lambda \in (0, 1)\);
5. \(S = \{(F, x) : x, F(x) \in X\}\) is a generating set of \(\mathcal{H}|_X\);
6. \(o(S) = t(S) = X\);
7. \(\nu(g) > 2\eta\) for all germs of elements of \(S\);

See Figure 4.1 illustrating Proposition 4.2.1.

**Proof.** Let \(\mathcal{H}'\) be a Smale quasi-flow. Let \(X'\) be a compact topological transversal of \(\mathcal{H}'\). We can choose \(X'\) equal to a union of a finite set of rectangles \(\bar{R}\) equal to the closure of an open rectangle \(R\) and is contained in an open rectangle \(\bar{R}\). Let us denote by \(\mathcal{R}\) the set of the rectangles \(R_i\) and let \(\bar{\mathcal{R}}\) be the set of the rectangles \(\bar{R}\). Then the union of the rectangles \(R \in \mathcal{R}\) is a \(\mathcal{H}'\)-transversal, which we will denote \(X'_0\). We assume that \(\bar{\mathcal{R}}\) is fine.

According to Definition 2.5.10 we may assume that the metric \(| \cdot |\) on \(X'\) is uniformly Lipschitz-equivalent to metrics \(| \cdot |_R\) on \(\bar{R}\) for \(R \in \mathcal{R}\) such that

\[
|[x_1, y_1]|_R - |x_2, y_2|_R| = \max\{|x_1 - x_2|_R, |y_1 - y_2|_R\}
\]

for all \(x_1, y_1, x_2, y_2 \in \bar{R}\).
Passing, if necessary, to subrectangles and using Definition 1.3.2, we may assume that for any two rectangles $R_i, R_j \in \mathcal{R}$ the products structures $[\cdot, \cdot]_i$ and $[\cdot, \cdot]_j$ of the rectangles $\hat{R}_i$ and $\hat{R}_j$ agree on the intersection $\hat{R}_i \cap \hat{R}_j$.

Let $S'$ be a compact generating set of $\mathcal{H}'|_{X'}$ satisfying the conditions of Definition 4.1.1. We may assume (as usual, passing to $(S')_n \cup (S')_{n+1}$) that $S'$ satisfies the contraction-expansion property with respect to the metrics $| \cdot |_{R}$.

Let us find a covering $\mathcal{V}$ of $S'$ by a finite number of open rectangles $V$ such that the closure of each rectangle $V \in \mathcal{V}$ is compact and extendable to a rectangle $\hat{V} \in \hat{\mathcal{H}}'$; the rectangles $\hat{V}$ satisfy the expansion-contraction condition with a coefficient $\lambda \in (0, 1)$, and are subordinate to the covering $\{ \hat{R} : R \in \mathcal{R} \}$.

Let $\delta$ be such that for all $V \in \mathcal{V}$ and $R_o, R_t \in \mathcal{R}$ such that $o(V) \cap R_o \neq \emptyset$ and $t(V) \cap R_t \neq \emptyset$, the $\delta$-neighborhood of $o(V)$ in $(\hat{R}_o, | \cdot |_{R_o})$ is contained in $o(\hat{V})$, and the $\delta$-neighborhood of $t(V)$ in $(\hat{R}_t, | \cdot |_{R_t})$ is contained in $t(\hat{V})$.

Such $\delta$ exists, since the metrics $| \cdot |_{R}$ are pairwise Lipschitz on intersections of their domains, and $\mathcal{V}$ is subordinate to $\mathcal{R}$. Let now $\epsilon > 0$ be such that $\epsilon < \min(\delta(\lambda^{-1} - 1), \delta)$.

We can find a finite set $\mathcal{W} = \{ W_1, \ldots, W_n \}$ of open sub-rectangles of $X'$ such that each closure $\overline{W}$ of each $W \in \mathcal{W}$ is a subrectangle of a rectangle $R(W) \in \mathcal{R}$ and is of diameter less than $\epsilon$ (with respect to $| \cdot |_{R(W)}$). $X'_0 = \bigcup_{W \in \mathcal{W}} W$, and $X' = \bigcup_{W \in \mathcal{W}} \overline{W}$.

For every $W \in \mathcal{W}$ let $\hat{W}$ be the $(\delta - \epsilon)$-neighborhood of $\overline{W}$ in the space $(\hat{R(W)}, | \cdot |_{R(W)})$. Then, by the properties of $| \cdot |_{R(W)}$, the set $\hat{W}$ is an open sub-rectangle of $\hat{R(W)}$.

Let $\mathcal{H}$ be localization of $\mathcal{H}'$ onto the covering $\{ \hat{W}_1, \ldots, \hat{W}_n \}$. We will denote the elements of the localization $\mathcal{H}$ by $(g, i, j)$ for $i, j \in \{1, \ldots, n\}$ (see definition of localization in 2.2).

Let $X_0$ be the union of the copies of the rectangles $W$, and let $X$ be the closure of $X_0$ (i.e., the union of the copies of $\overline{W}$). Since every point of $X'_0$ is an interior point of a rectangle $W$, the set $X_0$ is an open transversal of $\mathcal{H}$.

Define a metric on $\mathcal{H} = \bigsqcup_{W \in \mathcal{W}} \hat{W}$ equal to $| \cdot |_{R(W)}$ on every subset $\hat{W}$ (and, for example, equal to one for any two points belonging to different elements of
Lemma 4.2.2. The set \( S \) of copies \( (g, i, j) \) of elements \( g \in S' \) such that \( o(g), t(g) \in X \) is a generating set of \( \mathcal{S} | X \) satisfying condition (8).

Proof. Let \( (g, s, r) \in \mathcal{S} | X \), where \( g \in \mathcal{S}' \) and \( s \) and \( r \in \{1, \ldots, n\} \) are such that \( o(g) \in W_o \) and \( t(g) \in W_t \). Note that by definition of \( X \), we in fact have \( o(g) \in W_o = W_o \cap X \) and \( t(g) \in W_t = W_t \cap X \). It follows that \( g \in \mathcal{S}' | X \).

Consequently, there exists a neighborhood \( U \in \mathcal{S}' \) of \( g \) and \( n \) such that every element of \( U \cap \mathcal{S}' | X \) can be represented as a product of not more than \( n \) elements of \( S' \cup (S')^{-1} \). The set of elements of the form \( (h, s, r) \), where \( h \in U \) is a neighborhood of \( (g, s, r) \).

If \( (h, s, r) \) belongs to \( \mathcal{S} | X \), then \( o(h) \in W_o \) and \( t(h) \in W_t \). For any representation \( h = g_1 \cdots g_k \) of \( h \) as a product of elements of \( S' \cup (S')^{-1} \) we can then find indices \( i_k = s, \ldots, i_1, i_0 = r \) such that \( o(g_m) \in W_{i_m} \) and \( t(g_m) \in W_{i_{m-1}} \)

for all \( m = 1, 2, \ldots, m \). Then \( (h, s, r) = (g_1, i_1, i_0)(g_2, i_2, i_1) \cdots (g_m, i_m, i_{m-1}) \) is a representation of \( (h, s, r) \) as a product of elements of \( S \). It follows that \( S \) is a generating set of \( \mathcal{S} | X \).

Let \( (g, s, r) \in S \) for some \( s, r \in \{1, \ldots, n\} \) and \( g \in S' \). Denote \( R_o = R(W_o) \) and \( R_t = R(W_t) \).

Suppose that \( g \in V \in \mathcal{V} \). The sets \( o(V) \) and \( t(V) \) contain the \( \delta \)-neighborhoods of \( o(V) \) and \( t(V) \) in the metric spaces \( (R_o, | \cdot |_{R_o}) \) and \( (R_t, | \cdot |_{R_t}) \), respectively. The sets \( W_o \) and \( W_t \) are the \( (\delta - \epsilon) \)-neighborhoods of \( W_o \) and \( W_t \), which in turn are contained in the \( \epsilon \)-neighborhoods of the points \( o(g) \) and \( t(g) \), respectively.

It follows that \( W_o \) and \( W_t \) are contained in the \( \delta \)-neighborhoods of \( o(g) \in R_o \) and \( t(g) \in R_t \), respectively, which implies that \( o(V) \supset W_o \) and \( t(V) \supset W_t \).

For every \( x \in W_o \) the plaque \( P_+(W_o, x) \) has diameter less than \( \delta \) and is mapped by \( \hat{V} \) into the intersection of the \( \lambda \delta \)-neighborhood of \( \hat{V}(x) \) with \( P_+(W_t, \hat{V}(x)) \). But since \( \lambda \delta < \delta - \epsilon \), the image \( \hat{V}(P_+(W_o, x)) \) is contained \( P_+(W_t, \hat{V}(x)) \). The same is true for the inverse of \( \hat{V} \) and the plaques \( P_-(W_o, x) \), \( P_-(W_t, \hat{V}(x)) \), which implies that the set of maps of the form

\[
\hat{V} : W_o \cap (\hat{V})^{-1}(W_t) \rightarrow W_t \cap \hat{V}(W_o)
\]

such that \( V \cap S' \neq \emptyset \) satisfy the conditions of the proposition.

Definition 4.2.1. Let \( \mathfrak{G} \) be a Smale quasi-flow. Its Ruelle groupoids \( \mathcal{P}_i(\mathfrak{G}) \) for \( i = +, - \) are the groupoids \( \mathcal{P}_i(\mathfrak{G}, \mathcal{R}) \), where \( \mathcal{R} \) is a compressible covering of a topological \( \mathfrak{G} \)-transversal by rectangles.

By Proposition 14.2.3, compressible coverings exist. By Proposition 25.3, Ruelle groupoids \( \mathcal{P}_i(\mathfrak{G}) \) are unique, up to an equivalence of groupoids.

Since the Lipschitz structure on \( \mathfrak{G}^{(0)} \) is compatible with the local product structure, we get naturally defined Lipschitz structures on the Ruelle groupoids.
4.3 Geodesic quasi-flow of a hyperbolic groupoid as a Smale quasi-flow

**Theorem 4.3.1.** Let \( \mathfrak{G} \) be a hyperbolic groupoid and suppose that the geodesic quasi-flow \( \partial \mathfrak{G} \times \mathfrak{G} \) is locally diagonal (with respect to the local product structure on \( \partial \mathfrak{G} \)). Then \( \partial \mathfrak{G} \times \mathfrak{G} \) is a Smale quasi-flow.

**Proof.** By Proposition 3.7.2, the geodesic quasi-flow \( \partial \mathfrak{G} \times \mathfrak{G} \) is compactly generated. Let \((S,X)\) be a generating pair of \( \mathfrak{G} \) satisfying the conditions of Proposition 3.3.1 and consider the atlas of the local product structure on \( \partial \mathfrak{G} \) defined in the proof of Theorem 3.6.1.

Denote by \( \ell_{\alpha(h)}(\xi_1,\xi_2) \) the natural log-scale on \( \partial \mathfrak{G}_{\alpha(h)} \) (as defined in Proposition 3.4.2). Let us define then on each rectangle \( R_{h,U} \) the log-scale
\[
\ell_{h,U}(x_1,\xi_1,\xi_2) = \min\{\ell_{\alpha(h)}(\xi_1,\xi_2),\ell(x_1,\xi_2)\},
\]
where \( \ell \) is the log-scale defining the Lipschitz structure of \( \mathfrak{G} \).

Note that by Proposition 3.6.5 there exists a uniform constant \( \Delta \) such that
\[
|\ell_y([y,\xi_1], [y,\xi_2]) - \ell_{\alpha(h)}(\xi_1,\xi_2)| \leq \Delta
\]
for all \( U, \xi_1, \xi_2 \), and \( y \) for which the corresponding expressions are defined.

Suppose that \( g \in \mathfrak{G} \) is such that \( \tau(g) \in \alpha(U_i) \) and \( \alpha(g) \in \alpha(U_j) \) so that \( g \) moves the fiber \( \tau(g), T^\alpha_{\tau(g)}|_{U_i} \subset \partial \mathfrak{G}_{\alpha(g)} \) of \( R_{h_i},U_i \) to the fiber \( \alpha(g), T^\alpha_{\alpha(g)}|_{U_j} \subset \partial \mathfrak{G}_{\alpha(g)} \) of \( R_{h_j},U_j \). For all \( \xi_1,\xi_2 \in \partial \mathfrak{G}_{t(g)} \) we have
\[
\ell_{\alpha(g)}(\xi_1 \cdot g,\xi_2 \cdot g) = \ell_{\alpha(g)}(\xi_1,\xi_2) + \nu(g), \tag{4.1}
\]
by definition of the natural log-scale.

It follows that for all \( \xi_1,\xi_2 \in R_{h_i},U_i \), such that \( \mathcal{P}_+(\xi_1) = \mathcal{P}_+(\xi_2) = \mathcal{P}_+(\xi_1 \cdot g,\xi_2 \cdot g) \in R_{h_j},U_j \) we have
\[
\ell_{h_j,U_j}(\xi_1 \cdot g,\xi_2 \cdot g) = \ell_{h_i,U_i}(\xi_1,\xi_2) + \nu(g). \tag{4.2}
\]

It follows that the log-scales \( \ell_{h_i,U_i} \) define a Lipschitz structure on \( \partial \mathfrak{G} \times \mathfrak{G} \).

Let us define a quasi-cocycle \( \nu \) on \( \partial \mathfrak{G} \times \mathfrak{G} \) just by transferring it from the quasi-cocycle \( \nu \) on \( \mathfrak{G} \):
\[
\nu(\xi,g) = \nu(g). \tag{4.3}
\]
Let us show that this quasi-cocycle is compatible with the local product structure. We will use the covering by the rectangles \( R_{h_i},U_i \) again. If two elements of \( \partial \mathfrak{G} \times \mathfrak{G} \) have equal projections onto the second coordinate of the local product structure, then they are of the form \((\xi_1,g)\) and \((\xi_2,g)\), hence the values of \( \nu \) on them are equal.

If two elements have equal projections on the first coordinate, then the corresponding elements \( g_1 \) and \( g_2 \) of \( \mathfrak{G} \) act on the corresponding factors \( T^\alpha_{h_i} \to T^\alpha_{h_i} \) of the rectangles by transformations coinciding on some non-empty open sets. But then using Proposition 3.6.5 and (4.1) we get a universal upper
bound on the difference $|\nu(g_1) - \nu(g_2)|$, which shows that the quasi-cocycle is compatible with the local product structure.

It follows from \cite{1.2} that after passing, if necessary, to $S^n \cup S^{n+1}$, we will obtain a generating set of $\mathcal{G}$ acting by expanding maps on the first direction, and contracting maps on the second direction of rectangles $R_{h_i, U_i}$.

It remains to show that the last condition of Definition \ref{4.1.1} is satisfied. Let $A \subset \partial \mathcal{G}$ be a compact set, and let $r > 0$. We can cover the set $A$ by a finite set of rectangles $R_{h_i, U_i}$ for $h_i \in \mathcal{G}$ and $U_i \in \mathcal{G}$. Suppose that $\xi_1 \cdot g = \xi_2$ for some $\xi_1, \xi_2 \in A$ and $g \in \mathcal{G}$ such that $|\nu(g)| < r$. Let $\xi_1 = \ldots g_2'g_1' \cdot h_i$ and $\xi_2 = \ldots g_2''g_1'' \cdot h_j$ for some $g_k', g_k'' \in S$. Then $\ldots g_2'g_1' \cdot h_i g = \ldots g_2''g_1'' \cdot h_j$. Let $M$ be the maximal value of $\nu$ on an element of $S$ and suppose that $\nu(g)$ is non-negative. There exists a uniform constant $\Delta_1$ and an index $k$ such that

$$0 \leq \nu(h_i g) - \nu(g_k' \cdots g_1') \leq 2M,$$

and

$$(k - 1)\eta \leq \nu(g_k' \cdots g_1'') \leq \nu(g) + \nu(h_i) - \nu(h_j) + 2M + 3\eta,$$

where $| \cdot |$ denotes the length of the shortest representation as a product of elements of $S \cup S^{-1}$. But then

$$|g| \leq |h_j| + k + \Delta_1 + |h_i|,$$

and hence we get a uniform bound on $k$ and $|g|$, which implies condition (5) of Definition \ref{4.1.1}.

\subsection{4.4 Hyperbolicity of Ruelle groupoids}

\textbf{Theorem 4.4.1.} Let $\mathcal{F}$ be a Smale quasi-flow and let $\nu : \mathcal{F}|_X \rightarrow \mathbb{R}$ be a quasi-cocycle compatible the grading, where $X$ is a compact topological transversal. Let $\mathcal{R}$ be a finite compressible covering of $X$ by open rectangles. Then there exists a quasi-cocycle $\nu_+ : P_+(\mathcal{F}, \mathcal{R}) \rightarrow \mathbb{R}$ such that the groupoid $P_+(\mathcal{F}, \mathcal{R})$ is hyperbolic, $\nu_+$ is a Busemann cocycle, and $|\nu_+(P_+(g)) - \nu(g)|$ is uniformly bounded for all $g \in \mathcal{F}|_X$.

\textbf{Proof.} Let us pass to an equivalent groupoid satisfying the conditions of Proposition \ref{4.2.4} and use its notations. The fact that the generating set $P_+(\mathcal{S})$ is contracting follows directly from the definitions.

Let $\Delta_1$ be such that $\nu(g) \leq \Delta_1 - \eta$ for all $g \in F \in \mathcal{S}$. Let $X'$ be a compact neighborhood of $X$, and let $Q$ be the closure of the set of elements $g \in \mathcal{F}$ such that $|\nu(g)| < \Delta_1$ and $o(g), t(g) \in X'$. There exists $\delta_0 > 0$ and a finite collection $\mathcal{U} \subset \mathcal{F}$ such that for every $g \in Q$ there exists $U \in \mathcal{U}$ such that $g \in U$, and $o(U)$ and $t(U)$ contain the $\delta_0$-neighborhoods of $o(g)$ and $t(g)$, respectively. We assume that the elements $U \in \mathcal{U}$ are $\Lambda$-Lipschitz for some common constant $\Lambda$.

Let $P_+(s_1) \cdots P_+(s_n)$ be any product of elements of $P_+(\mathcal{S}) \cup P_+(\mathcal{S}^{-1})$. Let $F_i \in \mathcal{S} \cup \mathcal{S}^{-1}$ be such that $s_i \in F_i$. Let $R_1, \ldots, R_{n+1} \in \mathcal{R}$ be such that $o(F_n) \subset R_{n+1}$, $t(F_n) \cup o(F_{n-1}) \subset R_n$, $\ldots$, $t(F_1) \subset R_1$ (see Figure \ref{4.2}).
For every $i = 1, \ldots, n + 1$ consider a sequence $G_{ik} \in S^{-1}$ such that \( \{t(s_i), o(s_{i-1})\} \subset o(G_{ik} \cdots G_{i2} G_{i1}) \) and $G_{ik} \cdots G_{i1}(t(s_i)) \in X$. Such sequences exist by conditions (2), (5), (6) of Proposition 4.2.1. Then $P_-(G_{ik} \cdots G_{i1})$ are $\lambda^k$-contractions, which are defined on whole side $P_-(R_i)$ of the rectangle $R_i$.

We will use notation

$$\prod_{l}^{n} G_{i_l} = G_{i,n} G_{i,n-1} \cdots G_{i,1}. $$

We can find a sequence $k_1, k_2, \ldots, k_{n+1}$ such that the values of $\nu$ on the set

$$\left( \prod_{1}^{k_i} G_{i_l} \right) F_{i} \left( \prod_{1}^{k_{i+1}} G_{i+1} \right)^{-1}$$

are not more than $\Delta_1$ in absolute value.

Moreover, we can make the indices $k_i$ as large as we wish. We may hence assume that diameters of $P_-(t \left( \prod_{1}^{k_i} G_{i_l} \right))$ are less than $\epsilon = \delta_0 \frac{1}{\lambda^{k+1} - \lambda}$. If $k_i$ and $k_{i+1}$ are big enough, then the germs

$$t_i = \left( \prod_{1}^{k_i} G_{i_l} \right) s_i \left( \prod_{1}^{k_{i+1}} G_{i+1} \right)^{-1}$$
are such that \( o(t_i), t(t_i) \in X' \), hence \( t_i \in Q \). Consequently, there exist \( \Lambda \)-Lipschitz elements \( H_i \in \mathcal{U} \) such that \( t_i \in H_i \) and \( o(H_i) \) and \( t(H_i) \) contain the \( \delta_0 \) neighborhoods of \( o(t_i) \) and \( t(t_i) \), respectively.

The distance between \( t(t_{i+1}) \) and \( o(t_i) \) is less than \( \epsilon \). It follows that

\[
|H_n(o(t_n)) - o(t_{n-1})| < \epsilon < \delta_0,
\]
\[
|H_{n-1}H_n(o(t_n)) - o(t_{n-2})| < \Lambda \epsilon + \epsilon < \delta_0,
\]
\[
\vdots
\]
\[
|H_2H_3 \cdots H_n(o(t_n)) - o(t_1)| < \Lambda^{n-2} \epsilon + \cdots + \Lambda \epsilon + \epsilon < \delta_0,
\]

hence \( H_1 \cdots H_n(o(t_n)) \) is defined, and

\[
|H_1H_2 \cdots H_n(o(t_n)) - t(t_n)| < \Lambda^{n-1} \epsilon + \Lambda^{n-2} \epsilon + \cdots + \Lambda \epsilon = \delta_0.
\]

Since the maps \( F_i \) and \( G_{i,j} \) are rectangles, i.e., agree with the product structure on the rectangles of \( \mathcal{Y}^{(0)} \), we get that

\[
P_+(\prod_{i=1}^{k_1} G_i) P_+(s_1) \cdots P_+(s_n) P_+\left(\prod_{i=1}^{k_{n+1}} G_{n+1}\right)^{-1} = P_+(r_1 \cdots r_n)
\]

for some germs \( r_i \) of \( H_i \) such that \( o(r_n) = o(t_n) \).

By the same argument,

\[
P_+\left(\prod_{i=1}^{k_1} G_i\right) P_+(s_1) \cdots P_+(s_n) P_+\left(\prod_{i=1}^{k_{n+1}} G_{n+1}\right)^{-1} = P_+(r'_1 \cdots r'_n),
\]

for some germs \( r'_i \) of \( H_i \) such that \( t(t'_1) = t(t_1) \).

We can find \( l_1 > k_1 \) and \( l_{n+1} > k_{n+1} \) such that

\[
\left| \mu \left( \prod_{i=1}^{l_1} G_{l_1+1} \cdots r_n \cdot \prod_{i=1}^{l_{n+1}} G_{n+1} \right)^{-1} \right| \leq \Delta_1,
\]

and hence

\[
\prod_{i=1}^{l_1} G_{l_1+1} \cdots r_n \cdot \prod_{i=1}^{l_{n+1}} G_{n+1} \in Q.
\]

We have proved therefore the following result.

**Lemma 4.4.2.** For every \( g \in P_+(\mathcal{S}) \), and every pair of sequences \( G'_i, G''_i \in S^{-1} \) such that \( o(g) \in o(G'_i \cdots G'_1) \), \( t(g) \in o(G''_i \cdots G''_1) \), \( G''_i \cdots G''_1(t(g)) \in X \), there exist \( l, k \geq 1 \) and \( r \in Q \) such that

\[
g = P_+(G''_i \cdots G''_1)^{-1} P_+(rG'_i \cdots G'_1).
\]

Moreover, \( r \) and \( k \) exist for all sufficiently big \( l \), and \( r \) and \( l \) exist for all sufficiently big \( k \).
Suppose that the groupoid \( P_+ (\mathcal{F}) \) is not Hausdorff. Then there exist two elements \( h_1, h_2 \) that can not be separated by neighborhoods, hence there exists an element \( g = h_1^{-1} h_2 \in P_+ (\mathcal{F}) \) such that \( o(g) = t(g) \), \( g \) is not a unit, but \( g \) can not be separated by disjoint neighborhoods from the unit \( o(g) \). This means that for every neighborhood \( U \in P_+ (\mathcal{F}) \) of \( g \) there exists an open subset \( W \subset U \) which is an identical homeomorphism.

By Lemma 4.4.2 if \( G_1, G_2, \ldots \) is a sequence of elements of \( S \) such that \( o(g) = t(g) \in o(G_k G_{k-1} \cdots G_1) \) and \( G_k G_{k-1} \cdots G_1 (o(g)) \in X \) for all \( k \), then there exist \( k \) and \( l \), and \( r, r' \in Q \) such that

\[
g = P_+ (G_k \cdots G_1)^{-1} P_+ (r) P_+ (G_l \cdots G_1), \quad o(r) \in t(G_l \cdots G_1)
\]

and

\[
g = P_+ (G_k \cdots G_1)^{-1} P_+ (r') P_+ (G_l \cdots G_1), \quad t(r') \in t(G_k \cdots G_1).
\]

For any neighborhood \( W \in \mathcal{F} \) of \( r \) the map

\[
W_1 = P_+ (G_k \cdots G_1)^{-1} P_+ (W) P_+ (G_l \cdots G_1)
\]

is a neighborhood of \( g \) belonging to \( P_+ (\mathcal{F}) \). Hence, \( o(g) \) is a limit of a sequence of trivial germs of \( W_1 \).

If \( l > k \), then

\[
W_1 = P_+ (G_k \cdots G_1)^{-1} P_+ (W) P_+ (G_l \cdots G_1) = P_+ (G_k \cdots G_1)^{-1} (P_+ (WG_l \cdots G_{k+1})) P_+ (G_k \cdots G_1),
\]

hence trivial germs of the transformation \( P_+ (WG_l \cdots G_{k+1}) \) accumulate on \( x_k = P_+ (G_k \cdots G_1) (o(g)) \). But then, by condition (4) of Definition 4.1.1, trivial germs of the transformation \( W G_l \cdots G_{k+1} \) accumulate on the points of \( P_+^{-1} (x_k) \cap o(W G_l \cdots G_{k+1}) \).

Consider the point \( z = o((G_l \cdots G_1)^{-1} r) \). Then \( z \in o(G_k \cdots G_1) \cap P_+^{-1} (o(g)) \). The point \( G_k \cdots G_1 (z) \) belongs to \( o(W G_l \cdots G_{k+1}) \) and

\[
W G_l \cdots G_1 (z) = G_k \cdots G_1 (z),
\]

since the trivial germs of \( W G_l \cdots G_{k+1} \) accumulate on \( G_k \cdots G_1 (z) \). Then the local homeomorphism \( (G_k \cdots G_1)^{-1} W G_l \cdots G_1 (z) \) is defined (i.e., is not empty), hence \( g = P_+ ((G_k \cdots G_1)^{-1} r G_l \cdots G_1) \). It follows that every neighborhood of \( (G_k \cdots G_1)^{-1} r G_l \cdots G_1 \) contains trivial germs, which by Hausdorffness of \( \mathcal{F} \) implies that \( (G_k \cdots G_1)^{-1} r G_l \cdots G_1 \) is trivial, hence \( g \) is also trivial, which is a contradiction. The case \( k < l \) is treated in the same way, but using \( r' \). Thus, \( P_+ (\mathcal{F}) \) is Hausdorff.

Let us define a grading on \( P_+ (\mathcal{F}) \). Consider an arbitrary element \( g \in P_+ (\mathcal{F}) \). According to Lemma 4.4.2 for any two sequences \( G'_1, G''_2, \ldots \) and \( G'_1, G''_2, \ldots \) of elements of \( S^{-1} \) such that \( o(g) \in o(G''_1 \cdots G'_{l+1}) \), \( t(g) \in o(G''_{l+1} \cdots G'_1) \), and \( G''_1 \cdots G''_{l+1} (t(g)) \in X \) there exist \( k, l \geq 1 \) and \( r \in Q \) such that

\[
g = P_+ (G''_1 \cdots G''_{l+1})^{-1} P_+ (r G''_l \cdots G'_1).
\]
Define
\[ \nu_+(g) = \nu(G'_k \cdots G'_1, z_1) - \nu(G''_l \cdots G''_1, z_2), \]
where \( z_1 \) and \( z_2 \) are arbitrary points in the domains of the corresponding local homeomorphisms. Since we assume that the covering \( \mathcal{R} \) by rectangles of \( \mathcal{S} \) (see Proposition 4.2.1) is fine, the value of \( \nu_+(g) \) up to a uniformly bounded constant does not depend on the choice of the points \( z_1 \) and \( z_2 \).

Let us show that \( \nu_+ \) is well defined, up to strong equivalence. Suppose that \( H'_1, H'_2, \ldots, H''_1, H''_2, \ldots, \) is another pair of sequences of elements of \( \mathcal{S} \), and let \( m, n \in \mathbb{N} \), and \( t \in Q \) are such that
\[ g = P_+(H''_n \cdots H'''_{t_1})^{-1}P_+(tH'_m \cdots H'_1). \]

We can find indices \( k_1 > k, l_1 > l, m_1 > m, \) and \( n_1 > n \) such that all the differences
\[
\begin{align*}
|\nu(G'_{k_1} \cdots G'_{k+1}, y_1) - \nu(G''_{l_1} \cdots G''_{l+1}, y_2)|, \\
|\nu(H''_{m_1} \cdots H''_{m+1}, y_3) - \nu(H'''_{n_1} \cdots H'''_{n+1}, y_4)|,
|\nu(G'_{k_1} \cdots G'_{k}, y_5) - \nu(H''_{l_1} \cdots H''_{l+1}, y_6)|
\end{align*}
\]
are less than \( \Delta_1 + 2c \), where \( y_i \) are arbitrary points in the domains of the corresponding maps, and \( c \) is as in Definition 2.5.11.

Then the elements
\[
r' = G'_{k_1} \cdots G'_{k+1}r^{-1}(G''_{l_1} \cdots G''_{l+1})^{-1}, \quad u = H''_{m_1} \cdots H''_{m+1}z(G'_{k_1} \cdots G'_{k})^{-1},
\]
\[
t' = H''_{n_1} \cdots H''_{n+1}t(H''_{m_1} \cdots H''_{m+1})^{-1}
\]
belong to a fixed compact set \( Q' \), where \( z \in P_+^{-1}(o(g)) \) is such that \( G'_{k_1} \cdots G'_{k}(z) = o(r) \) and \( H''_{m_1} \cdots H''_{m+1}(z) = o(t) \) (see Figure 4.3). Here we identify \( z \) with the corresponding unit of the groupoid. In particular, for \( F \in \mathcal{S} \) the product \( Fz \) coincides with the germ \( (F, z) \).

Note that then the product \( t'ur' \) is defined and belongs to \( (Q')^3 \).

It follows from
\[ g = P_+(G''_l \cdots G''_1)^{-1}P_+(r)P_+(G'_{k_1} \cdots G'_{k}) = P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_m \cdots H''_1) \]
that
\[
\begin{align*}
P_+((Q')^3) & \ni P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H'_{m_1} \cdots H'_{m+1})^{-1}\cdot P_+(H'_{m_1} \cdots H'_{m+1})^{-1}\cdot P_+(r)P_+(G'_{k_1} \cdots G'_{k})^{-1} = \\
P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H'_{m_1} \cdots H'_{m+1})\cdot P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H'_{m_1} \cdots H'_{m+1})^{-1} = \\
P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1} = \\
(P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1} = \\
P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1} = \\
P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1} = \\
P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1} = \\
P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1}P_+(t)P_+(H''_n \cdots H''_1)^{-1}.
\end{align*}
\]
Let $M$ be the upper bound on the value of $|\nu|$ on elements of $(Q')^3$. Then the values of $|\nu|$ on $H''_{n_1} \cdots H''_1 z_2 (G''_{l_1} \cdots G''_1)^{-1}$ is bounded above by $M + C$ for every $z_2 \in \mathbb{P}^{-1}_+ (t(g))$, where $c$ is as in Definition 2.5.11. Consequently,

$$|\nu(H''_{n_1} \cdots H''_1, z_2) - \nu(G''_{l_1} \cdots G''_1, z_2)| < M + C + \eta.$$ 

By the same arguments,

$$|\nu(H'_{m_1} \cdots H'_{1}, z) - \nu(G'_{k_1} \cdots G'_{1}, z)| < M + C + \eta.$$ 

We have

$$\nu(G'_{k_1} \cdots G'_{k+1}, \bar{z}) - \eta \leq \nu(G'_{k_1} \cdots G'_{1}, z) - \nu(G'_{k_1} \cdots G'_{1}, z) \leq \nu(G'_{k_1} \cdots G'_{k+1}, \bar{z}) + \eta,$$

where $\bar{z} = G'_{k} \cdots G'_{1} (z)$; and

$$\nu(G''_{l_1} \cdots G''_{l+1}, \bar{z}_2) - \eta \leq \nu(G''_{l_1} \cdots G''_{1}, z_2) - \nu(G''_{l_1} \cdots G''_{1}, z_2) \leq \nu(G''_{l_1} \cdots G''_{l+1}, \bar{z}_2) + \eta,$$

where $\bar{z}_2 = G''_{l} \cdots G''_{1} (z_2)$. Therefore,

$$\|\nu(G'_{k_1} \cdots G'_{1}, z) - \nu(G'_{k_1} \cdots G'_{1}, z) - (\nu(G''_{l_1} \cdots G''_{1}, z_2) - \nu(G''_{l_1} \cdots G''_{1}, z_2))\| \leq \|\nu(G'_{k_1} \cdots G'_{k+1}, \bar{z}) - \nu(G''_{l_1} \cdots G''_{l+1}, \bar{z}_2)\| + 2\eta \leq \Delta_1 + 2C + 2\eta.$$
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Similarly,

\[ |\nu(H_{n_1} \cdots H'_1, z) - \nu(H_m \cdots H'_1, z) - \nu(H_{n_1} \cdots H'_1, z_2) + \nu(H_m' \cdots H'_1, z_2)| \leq \Delta_1 + 2C + 2\eta. \]

It follows that

\[ |(\nu(G''_1 \cdots G''_1, z_2) - \nu(G'_k \cdots G'_1, z)) - (\nu(H''_1 \cdots H'_1, z_2) - \nu(H''_1 \cdots H'_1, z))| \leq |\nu(H''_1 \cdots H'_1, z_2)| + |\nu(H''_1 \cdots H'_1, z)| + |\nu(G''_1 \cdots G''_1, z)| \]

\[ + |\nu(G'_k \cdots G'_1, z) - \nu(G'_k \cdots G'_1, z)| + |\nu(H''_1 \cdots H'_1, z_2) + \nu(H''_1 \cdots H'_1, z)| \leq 2M + 2C + 2\eta + 2\Delta_1 + 4\eta, \]

hence the map \( \nu_+ \) is well defined, up to an additive constant. It also implies that \( \nu_+ \) is a quasi-cocycle.

Theorem 1.2.8 and Lemma 4.4.2 imply that the directed Cayley graph of \( P_+(S) \), where oriented edges correspond to the elements of \( P_+(S^{-1}) \), is Gromov hyperbolic, so that the quasi-cocycle \( \nu_+ \) is coarsely equivalent to the Busemann quasi-cocycle associated with the limit of the directed paths.

Replacing \( S \) by the inverted Smale quasi-flow (i.e., the quasi-flow graded by \( -\nu \) in which the names of projections \( P_+ \) and \( P_- \) are swapped) we get the following corollary.

Corollary 4.4.3. Let \( S \) be a Smale quasi-flow and let \( \nu : S \rightarrow R \) be the corresponding quasi-cocycle. Then there exists a quasi-cocycle \( \nu_- : P_-(S) \rightarrow R \) such that the groupoid \( P_-(S) \) graded by \( \nu_- \) is hyperbolic and \( |\nu_- (P_-(g)) + \nu(g)| \) is uniformly bounded for all \( g \in S \).

4.5 Duality theorem

Theorem 4.5.1. Let \( S \) be a Smale quasi-flow. Then \( S \) is equivalent (as a graded groupoid with a local product structure) to the groupoid \( \partial P_+(S) \times P_+(S) \).

Proof. We assume that \( S \), \( X \), \( X_0 \), and \( S \) satisfy the conditions of Proposition 1.2.1 and use its notations. Let us denote \( \mathfrak{G} = P_+ (S) \).

Let \( \eta \) be such that \( \nu \) is an \( \eta \)-quasi-cocycle. Let \( \Delta_1 - \eta \) be an upper bound on the value of \( |\nu| \) on elements of \( S \), and let \( Q \) be a compact set containing all elements of \( S \) such that \( |\nu(g)| \leq \Delta_1 \) and \( o(g), t(g) \in X' \), where \( X' \) is a compact neighborhood of \( X \). Let us cover \( Q \) by a finite set \( U \) of Lipschitz rectangles \( U \in S \). Let \( \delta > 0 \) be such that for every element \( r \in Q \) there exists \( U \in U \) such that \( U \owns r \) and the \( \delta \)-neighborhoods of \( o(r) \) and \( t(r) \) are contained in \( o(U) \) and \( t(U) \), respectively.

Let \( \cdots g_2 g_1 \) be a composable sequence of germs of elements of \( P_+(S) \). Let \( G_k \in S \) be such that \( g_k \in P_+(G_k) \). Then the homeomorphism \( F_n = G_n \cdots G_1 \)
is a rectangle such that $P_+(o(F_n)) = o(A_{G_1})$ and $P_-(t(F_n)) = t(B_{G_n})$ (here $A_F = P_+(F)$ and $B_F = P_-(F)$ and $o(A_F)$ and $t(B_F)$ are sides of the rectangles $W_i$ from Proposition 4.2.3, see Figure 4.4.

Diameter of $P_-(o(F_n))$ is not more than $\lambda^n \cdot p$, where $p$ is an upper bound on diameters of $P_-(R)$ for $R \in \mathcal{R}$. The set $P_-(o(G_n \cdot \cdots \cdot G_1))$ contains the closure of the set $P_-(o(G_{n-1} \cdot \cdots \cdot G_1))$. It follows that for every infinite composable sequence $\cdots g_2 g_1$ the intersection

$$\bigcap_n P_-(o(G_n \cdot \cdots \cdot G_1))$$

is just one point $\xi$. Denote by $\Phi(\ldots, g_2, g_1)$ the point of $\mathcal{F}(0)$ such that

$P_+(\Phi(\ldots, g_2, g_1)) = o(g_1), \quad P_-(\Phi(\ldots, g_2, g_1)) = \xi$.

**Lemma 4.5.2.** Let $\ldots g_2 g_1, \ldots h_3 h_1$ be points of $\partial \mathfrak{G}$, where $g_i, h_i$ are germs of elements of $P_+(\mathcal{S})$. Suppose that $g \in \mathfrak{G}$ is such that $\ldots g_2 g_1 g = \ldots h_3 h_1$. Then there exists a unique element $\tilde{g} \in \mathcal{F}$ such that $P_+(\tilde{g}) = g$, $o(\tilde{g}) = \Phi(\ldots, g_2, g_1)$, and $t(\tilde{g}) = \Phi(\ldots, h_2, h_1)$.

**Proof.** By Lemma 4.4.2 there exist sequences $G_i', G_i'' \in \mathcal{S}$ and an element $r \in \mathcal{Q}$ such that $q = P_+(G_i'' \cdot \cdots \cdot G_i' r)P_+(G_i' \cdot \cdots \cdot G_i' r)^{-1}$. Let $G_i, H_i \in \mathcal{S}$ be such that $g_i \in P_+(G_i)$ and $h_i \in P_+(H_i)$. Let $U \in \mathcal{U}$ be such that $r \in U$.

By Lemma 4.6.2 and Lemma 4.4.2 there exists a sequence of elements $r_i \in \mathcal{Q}$ such that $P_+(r_i) \cdot g_n \cdot \cdots \cdot g_1 \cdot g = h_m \cdot \cdots \cdot h_1$ for some increasing sequences $n_i$ and $m_i$. Let $U_i \in \mathcal{F}$ be such that $r_i \in U_i$.

We have

$$P_+(r_i) \cdot g_n \cdot \cdots \cdot g_1 \cdot P_+(G_i'' \cdot \cdots \cdot G_i' r) = h_m \cdot \cdots \cdot h_1 \cdot P_+(G_k' \cdot \cdots \cdot G_1')$$

Consider the products $U_i \cdot G_{n_i} \cdots G_1 G_i'' \cdots G_i' U$ and $H_{m_i} \cdots H_1 G_k' \cdots G_1'$. Denote

$$z_1 = (G_{n_i} \cdots G_1 G_i'' \cdots G_i' U)^{-1}(o(r_i)),$$

$$z_2 = (H_{m_i} \cdots H_1 G_k' \cdots G_1')^{-1}(t(r_i)).$$
Then $P_+(z_1) = P_+(z_2)$, and the germ of
\[ P_+((G_1')^{-1} \cdots (G_k')^{-1} H_1^{-1} \cdots H_n^{-1} U_i G_{n_i} \cdots G_k G_k' \cdots G_1' U) \]
at $P_+(z_1)$ is trivial. It follows, by condition (4) of Definition 4.4.1, that the germ of
\[ (G_1')^{-1} \cdots (G_k')^{-1} H_1^{-1} \cdots H_n^{-1} U_i G_{n_i} \cdots G_k G_k' \cdots G_1' U \]
is trivial at $z_1 = z_2$. Consequently,
\[ o(U_i G_{n_i} \cdots G_1 G_k' \cdots G_1') \cap o(H_n \cdots H_k G_k' \cdots G_1') \neq \emptyset. \]
Passing to the limit as $i \to \infty$ and using the fact that $P_-(G_i)$ and $P_-(H_i)$ are expanding, we conclude that there exists $y$ such that $P_+(y) = P_+(o(r))$ and
\[ y \in o(H_n \cdots H_k G_k' \cdots G_1') \]
for all $n$. Taking $r' = (U, y)$, we get an element $\tilde{g} = G_1 \cdots G_k' r'(G_k' \cdots G_1')^{-1}$ such that $o(\tilde{g}) = \Phi(\ldots, g_2, g_1)$, $t(\tilde{g}) = \Phi(\ldots, h_2, h_1)$ and $P_+(\tilde{g}) = g$. Uniqueness of $g$ follows from the fact that $\tilde{f}$ is locally diagonal and the covering $R$ is fine.

Denote by $T$ the set of points of $\partial \mathfrak{G}$ that can be represented as $\ldots g_2 g_1$ for $g_i \in P_+(G_i)$ for some $G_i \in S$.

Applying Lemma 4.5.2 to the case when $g$ is a unit, we conclude that $\Phi(\ldots, g_2, g_1)$ depends only on $\ldots g_2 g_1 \in T$, and we get a well defined map from $T$ to $\tilde{f}^{(0)}$.

Moreover, Lemma 4.5.2 implies that $(\ldots g_2 g_1, g) \to \tilde{g}$ is a well defined map from $(\partial \mathfrak{G} \times \mathfrak{G})|_T$ to $\tilde{f}$. We will denote this map also by $\Phi$, so that $\Phi(\ldots, g_2 g_1) = \Phi(\ldots, g_2, g_1)$ and, in conditions of Lemma 4.5.2, $\tilde{g} = \Phi(\ldots, g_2 g_1, g)$. It is easy to see (using uniqueness of $\tilde{g}$) that $\Phi : (\partial \mathfrak{G} \times \mathfrak{G})|_T \to \tilde{f}$ is a homomorphism of groupoids (i.e., a functor of the corresponding small categories).

Note also that the homomorphism $\Phi$ is uniquely determined by its restriction onto $T$, since $\tilde{g} = \Phi(\xi, g)$ is uniquely determined by the condition $o(\tilde{g}) = \Phi(\xi)$, $t(\tilde{g}) = \Phi(\xi \cdot g)$ and $P_+(\tilde{g}) = g$.

The definition of $P_-(\Phi(\ldots, g_2 g_1))$ depends only on the sequence $G_k \in S$ such that $g_k \in P_+(G_k)$. It follows that the homomorphism $\Phi$ agrees with the local product structures on the geodesic quasi-flow and $\tilde{f}$.

**Lemma 4.5.3.** Suppose that $h \in \tilde{f}$ is such that $o(h) = \Phi(\ldots, g_2 g_1)$ and $t(h) = \Phi(\ldots, h_2 h_1)$. Then $g_2 g_1 = \ldots h_2 h_1 \cdot P_+(h)$.

**Proof.** Let $G_i, H_i \in S$ be such that $g_i \in P_+(G_i)$ and $h_i \in P_+(H_i)$. Denote $z = o(h)$. There exist increasing sequences $m_k$ and $n_k$ such that
\[ |\nu((H_m \cdots H_1 h) \cdot (G_n \cdots G_1)^{-1})| \leq \Delta_1. \]
Then $(H_m \cdots H_1) h (G_n \cdots G_1)^{-1} \in Q$, and for
\[ r_k = P_+(H_m \cdots H_1) h (G_n \cdots G_1)^{-1} \in P_+(Q) \]
we have $r_k \cdot g_{n_k} \cdots g_1 = h_{m_k} \cdots h_1 P_+(h)$, hence $g_2 g_1 = \ldots h_2 h_1 P_+(h)$. \(\square\)
Lemma 4.5.4. The homomorphism $\Phi$ is continuous.

Proof. It is enough to show that $\Phi$ is continuous on $T$, by the above remark on uniqueness of definition of $\bar{g}$. Moreover, it is enough to show that it is continuous on each $\partial S_x$, since $\Phi$ agrees with the local product structure and is obviously continuous on the first direction of the local product structure of $\partial S$.

Then the proof of continuity of $\Phi$ becomes essentially the same as the proof of Lemma 4.5.2. Suppose that $\ldots g_{2}g_{1}$ and $\ldots h_{2}h_{1}$ are close in $\partial S_x$, where $x = o(g_1) = o(h_1)$. Then there exists $r \in Q$ and large indices $m$ and $n$ such that $P_+(r)g_m \cdots g_1g = h_m \cdots h_1$. Then, as in the proof of Lemma 4.5.2, we conclude that the domains of $G_n \cdots G_1$ and $H_m \cdots H_1$ are close to each other, hence $\Phi(\ldots g_{2}g_{1})$ and $\Phi(\ldots h_{2}h_{1})$ are close.

Lemma 4.5.5. The homomorphism $\Phi$ is injective, and the inverse partial map is a continuous homomorphism.

Proof. Again, it is enough to show that it is injective on $T$ and that the map inverse to $\Phi|_T$ is continuous.

Injectivity and functoriality of the inverse follow directly from Lemma 4.5.3. Continuity of $\Phi^{-1}|_T$ is proved in a way similar to the proof of Lemma 4.5.3. Namely, suppose that distance between $z_1 = \Phi(\ldots g_{2}g_{1})$ and $z_2 = \Phi(\ldots h_{2}h_{1})$ is less that $\Lambda^{-n}\delta$, where $\Lambda$ is such that the maps $P_+(G)$ for $G \in S$ are $\Lambda$-Lipschitz. Then there exist $G_i, H_i \in S$ such that $g_i \in G_i$, $h_i \in H_i$, and

$$\{z_1, z_2\} \subset o(G_k \cdots G_1) \cap o(H_k \cdots H_1)$$

for all $k \leq n$. Then in the same way as in the proof of injectivity of $\Phi|_T$, we show that for $n$ big enough there exist big indices $m_k$ and $n_k$ and $r_k \in P_+(Q)$ such that $r_k : g_{m_k} \cdots g_1 = h_{m_k} \cdots h_1$, which implies that $\ldots g_{2}g_{1}$ and $\ldots h_{2}h_{1}$ are close to each other.

Lemma 4.5.6. The set $X_0$ belongs to the range of $\Phi$, and for every $x \in X_0$ the point $\Phi^{-1}(x)$ belongs to the interior of $T$.

Proof. By Proposition 4.4.1 for every point $x \in X$ there exists $h \in S$ such that $o(h) = x$ and $t(h) \in X$. Applying this fact infinitely many times we will get a composable sequence $\ldots h_2h_1$ of elements of $S$ such that $o(h_1) = x$. Then $\Phi(\ldots P_+(h_2)P_+(h_1)) = x$, hence $X_0$ belong to the range of $\Phi$.

Let $\xi = \ldots g_{2}g_{1}$, and let $G_i \in S$ be such that $g_i \in P_+(G_i)$. Suppose that $\Phi(\xi) \in X_0$.

There exists a compact set $Q_1 \subset S|_X$ such that the set $N_n$ of points $\zeta \in \partial S_{o(g_1)}$ such that $\ell_{o(g_1)}(\zeta, \xi) \geq n$ is contained in the set of points representable in the form $\ldots g_n'g_{n+1}a g_{n} \cdots g_1$, where $a \in Q_1$ and $g_i' \in P_+(S)$. It follows then from Lemma 4.4.2 that there exists a constant $k_0$ such that the set $N_n$ is contained in the set of points of the form

$$\zeta = \ldots P_+(h_2)P_+(h_1)P_+(r(g_{n-k_0} \cdots g_1))$$
for some composable sequence \( \ldots h_2 h_1 r \) of elements \( h_i \in S \) and \( r \in Q \). We can find a sequence \( H_0, H_1, \ldots, H_m \) of elements of \( S \) such that \( t(r) \in t(H_0 \cdots H_m) \) and \( r' = (H_0 \cdots H_m)^{-1} r G_{n-k_0} \cdots G_1 \in Q \). Let \( h_0, h_1, \ldots, h_m \) be the germs of \( H_0, H_1, \ldots, H_m \) such that \( h_0 \cdots h_m \) is defined and we have \( t(h_0 \cdots h_m) = t(r) \). Then \( t(r') = \Phi(\ldots P_+(h_2)P_+(h_1)P_+(h_0) \cdots P_+(h_{-m})) \) and the distance between \( o(r') \) and \( \Phi(\zeta) \) is not more than \( \lambda^{n-k_0} p \). It follows that \( o(r') \) belongs to \( X_0 \subset \Phi(T) \), for all \( n \) big enough. Then \( o(r') = \Phi(\ldots h'_2 h'_1) \) for some \( h'_i \in P_+(S) \),

\[
\zeta = \ldots P_+(h_2)P_+(h_1)P_+(r)g_{n-k_0} \cdots g_1 = \\
\ldots P_+(h_2)P_+(h_1)P_+(h_0) \cdots P_+(h_{-m})P_+(r'),
\]

and

\[
t(r') = \Phi(\ldots P_+(h_2)P_+(h_1)P_+(h_0) \cdots P_+(h_{-m})).
\]

By Lemma 4.5.3 we have then \( \zeta = \ldots h'_2 h'_1 \), hence \( \zeta \) belongs to \( T \). We have shown that all points of \( N_n \) belong to \( T \) for all \( n \) big enough. It follows then that \( \xi \) is an internal point of \( T \).

We have shown that \( \Phi \) is an isomorphism of the restrictions of groupoids \( \partial G \times G \) and \( \mathcal{H} \) onto \( T \) and \( \Phi(T) \). We have also shown that \( \Phi^{-1} \) maps \( X_0 \) to an open subset of \( T \). Consequently, \( \Phi \) implements an isomorphism of restrictions of \( \mathcal{H} \) and \( \partial G \times G \) onto open transversals, which implies that \( \mathcal{H} \) and \( \partial G \times G \) are equivalent.

Suppose that \( G \) is a hyperbolic groupoid such that \( \partial G \times G \) is locally diagonal. Then \( P_+(\partial G \times G) \) is equivalent to \( G \) (which follows directly from the definition of the local product structure on \( \partial G \)), see Theorem 3.6.1.

**Definition 4.5.1.** Let \( G \) be a hyperbolic groupoid such that \( \partial G \times G \) is locally diagonal. Then the dual groupoid \( G^\top \) is the projection \( P_-(\partial G \times G) \).

Note that the grading of \( P_-(\partial G \times G) \) is projection of the quasi-cocycle \( -\tilde{\nu} \), where \( \tilde{\nu} \) is the lift of \( \nu \) to \( \partial G \times G \) given by (4.3).

The following theorem is then a direct corollary of Theorems 4.3.1, 4.4.1 and 4.5.1.

**Theorem 4.5.7.** Let \( G \) be a hyperbolic groupoid with locally diagonal geodesic quasi-flow. Then its dual \( G^\top \) is also a hyperbolic groupoid and the groupoid \( (G^\top)^\top \) is equivalent to \( G \).

Note also that for a graded hyperbolic groupoid \( (G, \nu) \) there is a well defined, up to strong equivalence of quasi-cocycles, quasi-cocycle \( \nu^\top \) on \( G^\top \). Namely, it is shown in the proof of Theorem 4.3.1 that the lift \( \tilde{\nu} \) of \( \nu \) to the geodesic flow \( \partial G \times G \) is a quasi-cocycle satisfying the definitions of the quasi-cocycle on a Smale quasi-flow. Theorem 4.4.1 shows that there are unique, up to strong equivalence, quasi-cocycles \( \nu_+ \) and \( \nu_- \) which are projections of \( \tilde{\nu} \) onto the Ruelle groupoids of \( \partial G \times G \). Moreover, \( \nu_+ \) is strongly equivalent to \( \nu \). We
can take then $\nu^T = -\nu_-$. Theorem 4.5.1 shows then that this construction is a duality of graded groupoids, i.e., that applying it twice we get back the original graded groupoid $(\mathcal{G}, \nu)$.

### 4.6 Another definition of the dual groupoid

Let $\mathcal{G}$ be a hyperbolic groupoid. Let $(S, X)$ be a complete generating pair of $\mathcal{G}$ (see Proposition 3.3.1). Let $\mathcal{S}$ be a finite covering of $S$ by contracting positive elements of $\mathcal{G}$.

We denote by $\mathcal{G}^\omega_x$ the union $\mathcal{G}^\omega_x = \mathcal{G}(x, S) \cup \partial \mathcal{G}(x, S) \setminus \{\omega_x\}$ of the set of vertices $\mathcal{G}^\omega_x$ of the Cayley graph $\mathcal{G}(x, S)$ with the boundary $\partial \mathcal{G}_x$. The set $\mathcal{G}^\omega_x$ comes with the topology defined by the natural log-scale on $\mathcal{G}^\omega_x$ defined as in Proposition 3.4.2.

Let $A \subset \mathcal{G}$ be a compact set satisfying the conditions of Proposition 3.4.4. Suppose also that for any two sequences $g_i, h_i$ of germs of elements of $\mathcal{S}$ an equality $g_2 g_1 \cdot q = \ldots h_2 h_1 \cdot h$ for some $g, h, o(g) = o(h) \in X$ implies that for all sufficiently big $n$ there exists $m$ and $a \in A$ such that $ag_n \cdots g_1 g = h_m \cdots h_1 h$. Existence of such a set $A$ follows from hyperbolicity of the Cayley graphs of $\mathcal{G}$ and the fact that all directed paths in $\mathcal{G}(x, S)$ are quasi-geodesics.

Find then a finite covering $\mathcal{A} = \{U\}$ of $A$ by bi-Lipschitz elements of $\mathcal{G}$. Let $\hat{A}$ be the set of germs of the elements of $\mathcal{A}$.

The following lemma is proved in the same way as 3.6.3.

**Lemma 4.6.1.** Let $\epsilon$ be a common Lebesgue’s number of the coverings $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{A}^{-1}$ of $S$, $\mathcal{A}$, and $\mathcal{A}^{-1}$, respectively. There exists $0 < \delta_0 < \epsilon$ such that the following condition is satisfied.

Let $U_i, V_i, i = 1, 2, \ldots$ be finite or infinite sequences of elements of the set $S \cup \mathcal{A}$ in which at most one element belongs to $\mathcal{A}$. Let $|x - y| < \delta_0$ for $x, y \in X$, the $\epsilon$-neighborhoods of $U_i \cdots U_1(x)$ and $V_i \cdots V_1(x)$ are contained in $o(U_{i+1})$ and $o(V_{i+1})$, respectively. Then an equality

$$\ldots U_2 U_1, x) = (\ldots V_2 V_1, x)$$

of finite or infinite products of germs implies

$$\ldots U_2 U_1, y) = (\ldots V_2 V_1, y).$$

Fix $\delta_0$ satisfying the conditions of Lemma 4.6.1. Suppose that $g \in \mathcal{G}|_X$ and $h \in \mathcal{G}$ are such that $|t(g) - t(h)| < \delta_0$. For a finite or infinite product $\xi = \ldots g_2 g_1 g \in T_g$, where $g_i \in S$, find elements $U_i \in S$ such that $g_i$ is $\epsilon$-contained in $U_i$. Define then

$$R^h_g(\xi) = \ldots U_2 U_1 \cdot h. \quad (4.4)$$

By Lemma 4.6.1 $R^h_g(\xi)$ depends only on $g$, $h$, and $\xi$ (and does not depend on the choice of the generators $g_i$ or the choice of the elements $U_i$). Note that $R^h_g(\xi) \notin \mathcal{G}|_X$ in general (even for $\xi \in \mathcal{G}_x^\omega$).
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Theorem 4.6.2. The space $\mathcal{dG}_X$ of germs of restrictions of the maps $R^h_g$, for $g \in \mathcal{G}|_X$, $h \in \mathcal{G}$ onto open subsets of the disjoint union $\bigsqcup_{x \in X} \partial \mathcal{G}_x$ is a groupoid (i.e., is closed under taking compositions and inverses), and depends only on $\mathcal{G}$ and $X$. Restriction of $\mathcal{dG}_X$ onto any set $\bigsqcup_{x \in Y} \partial \mathcal{G}_x$ for $Y \subset X$ does not depend on $X$.

Proof. If $x, y$ are units of $\mathcal{G}$ such that $R^y_x$ is defined, then restriction of the transformation $R^y_x$ onto $\partial \mathcal{G}_x$ is equal to the transformation $\xi \mapsto [y, \xi] V$, where $V$ is a neighborhood of points $x, y$ of diameter less than $\delta_0$. See (3.1) on page 58.

The groupoid $\mathcal{H}$ of germs of the pseudogroup acting on the disjoint union $\bigsqcup_{x \in X} \partial \mathcal{G}_x$ and generated by the transformations $\cdot : \mathcal{G}$ and $\xi \mapsto \xi \cdot g$ is equivalent to the projection $P_{\partial \mathcal{G} \times \mathcal{G}}$ of the geodesic quasi-flow onto the direction of the boundaries of the Cayley graphs. Every element of the groupoid $\mathcal{H}$ is, by Lemma 4.4.2, equal to a composition of $\xi \mapsto \xi \cdot g$, followed by $\xi \mapsto [x, \xi] U$, and then by $\xi \mapsto \xi \cdot h$ for some $g, h \in \mathcal{G}$.

Restriction of the transformation $R^h_g$ onto an open subset of $\partial \mathcal{G} \circ (g)$ is equal to composition of the transformations

$$\xi \mapsto \xi \cdot g^{-1} \mapsto [t(h), \xi \cdot g^{-1}] V \mapsto [t(h), \xi \cdot g^{-1}] V \cdot h,$$

where $V$ is a neighborhood of diameter less than $\delta_0$ of $t(g)$ in $\mathcal{G}(0)$.

It follows that the set of germs of $R^h_g$ is equal to the groupoid $\mathcal{H}$. The last statement of the theorem is straightforward.

Proposition 4.6.3. Let $\mathcal{G}$ be a hyperbolic groupoid. The dual groupoid $\mathcal{G}^\top$ is equivalent to $\mathcal{dG}_X$.

Proof. Follows directly from the proof of Theorem 4.6.2.

Below is a reformulation of the last proposition, which is less explicit, but is probably more elegant and self-contained.

Let $\mathcal{G}$ be a hyperbolic groupoid. Let $(S, X)$ be any compact generating pair of $\mathcal{G}$. Let $S'$ and $X'$ be compact neighborhoods of $S$ and $X'$, respectively. Let $| \cdot |$ be a metric on $S'$.

In the following definition we write $g_n \to \infty$ if $g_n$ eventually leaves every finite set.

Definition 4.6.1. We say that a map $F : T_1 \to T_2$, where $T_1 \subset \mathcal{G}_x$ and $T_2 \subset \mathcal{G}_y$ are compact neighborhoods of points $\xi_1 \in \partial \mathcal{G}_x, \xi_2 \in \partial \mathcal{G}_y$, is a local automorphism of $\mathcal{G}$, if the following condition holds. If sequence $g_n, h_n \in T_1$ are such that $g_n \to \infty, h_n \to \infty$, and $g_n^{-1} h_n \in S$, then $F(g_n) F(h_n)^{-1}$ eventually belong to $S'$, and

$$|g_n h_n^{-1} - F(g_n) F(h_n)^{-1}| \to 0$$

as $n \to \infty$.

Note that the condition of the definition does not depend on the choice of the metric.
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Proposition 4.6.4. The set of germs of maps $\partial G \rightarrow \partial G$ by all local automorphisms of $G$ coincides with $dG_X$ and is hence equivalent to $G^\top$.

Proof. It is easy to check that the set of local automorphisms does not depend on the generating sets $S$ and $S'$, hence we may assume that they satisfy the conditions of Proposition 3.3.1 (it depends, however, on $X$ and $X'$).

Let $S$ be, as in the proof of Theorem 3.6.1, a finite covering of $S$ by elements of $\mathfrak{G}$. Let $\delta$ be a number smaller than the Lebesgue’s number of the covering $S$. Then $|g_n h_n^{-1} - F(g_n) F(h_n)^{-1}| \rightarrow 0$ implies that for all $n$ big enough we have $F(g_n) F(h_n)^{-1} \in U_n$, where $U_n \in S$ are such that $g_n h_n^{-1}$ is $\delta$-contained in $U_n$. It follows that every local automorphism, up to a finite set can be covered by transformations of the form $R_h^g$.

On the other hand, it is easy to see that every transformation $R_h^g$ is a local automorphism. Consequently, the set of germs of local automorphisms on the boundaries of the Cayley graphs coincides with the set of germs of the maps $R_h^g$, i.e., with $dG_X$. \qed

4.7 Minimal hyperbolic groupoids

Let $\mathfrak{G}$ be a hyperbolic groupoid and let $(S, X)$ be its generating pair. We say that a Cayley graph $G(x, S)$ is topologically mixing if for every point $\xi \in \partial G_x$ and every neighborhood $U$ of $\xi$ in $\mathfrak{G}_x$ the set of accumulation points of $t(U \cap \mathfrak{G}_x)$ contains the interior of $X$.

Proposition 4.7.1. Let $\mathfrak{G}$ be a hyperbolic groupoid. Then the following conditions are equivalent.

1. Some Cayley graph of $\mathfrak{G}$ is topologically mixing.

2. Every Cayley graph of $\mathfrak{G}$ is topologically mixing.

3. Every $\mathfrak{G}$-orbit is dense in $\mathfrak{G}^{(0)}$.

Proof. Note that (2) obviously implies (3). It remains to prove that (1) implies (2) and that (3) implies (1).

Let us show that (1) implies (2). We will split the proof into four lemmas.

Lemma 4.7.2. If $\mathfrak{G}(x, S)$ is topologically mixing, then $\mathfrak{G}(x, S')$ is topologically mixing for any generating set $S'$ of $\mathfrak{G}|_X$.

Proof. The Lipschitz class of the log-scale on $\mathfrak{G}_x^X$ does not depend on the choice of the generating set, hence the set of open neighborhoods of points of $\mathfrak{G}_x^X$ does not depend on $S$. \qed

We may assume now that all generating pairs in our proof satisfy the conditions of Proposition 3.3.1.
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Lemma 4.7.3. Let \((X', S')\) be a compact generating pair of \(\mathcal{G}\) such that \(\mathcal{G}(x, S')\) is topologically mixing. Let \((X, S)\) be a generating pair such that \(X \subset X'\). Then \(\mathcal{G}(x, S)\) is topologically mixing.

Proof. For every neighborhood \(W \subset \mathcal{G}^{X}\) of a point \(\xi \in \partial \mathcal{G}_{x}\) and every point \(y\) of the interior of \(X\), the point \(y\) is a limit of a sequence of \(t(g_{n})\) for pairwise different \(g_{n} \in W\). Since \(y\) is an internal point of \(X\), for all sufficiently big \(n\) we will have \(t(g_{n}) \in X\), hence \(g_{n} \in \mathcal{G}|_{X}\). Consequently, the set of accumulation points of \(t(W \cap \mathcal{G}|_{X})\) contains the interior of \(X\), which shows that \(\mathcal{G}(x, S)\) is topologically mixing. \(\square\)

Lemma 4.7.4. Let \((X, S)\) and \((X', S')\) be compact generating pairs of \(\mathcal{G}\) such that \(X \subset X'\) and \(\mathcal{G}(x, S)\) is topologically mixing. Then \(\mathcal{G}(x, S')\) is topologically mixing.

Proof. The identical embedding \(\mathcal{G}^{X} \rightarrow \mathcal{G}^{X'}\) is a quasi-isometry of the Cayley graphs \(\mathcal{G}(x, S) \rightarrow \mathcal{G}(x, S')\).

There exists a finite collection \(\mathcal{U}\) of elements \(U \in \mathcal{G}\) such that \(o(U)\) cover \(X'\) and \(t(U) \subset X\). Denote \(A = \bigcup_{U \in \mathcal{U}} U \cap \mathcal{G}|_{X'}\). It follows from elementary properties of Gromov hyperbolic graphs that for every neighborhood \(W'\) of \(\xi \in \partial \mathcal{G}_{x}\) in \(\mathcal{G}^{X'}\) there exists a neighborhood \(W\) of \(\xi\) in \(\mathcal{G}^{X}\) such that for every \(g \in W'\) and \(h \in A\) we have \(h^{-1}g \in W'\). Let \(y\) be an arbitrary internal point of \(X'\). Let \(U \in \mathcal{U}\) be such that \(y \in o(U)\). Then \(U(y)\) is an internal point of \(X\), and hence it is a limit of a sequence of points of the form \(t(g_{n})\) for a sequence of different elements \(g_{n} \in W\). Then \(t(g_{n})\) belongs to \(t(U)\) and \(U^{-1}(t(g_{n}))\) belongs to \(X'\) for all sufficiently big \(n\). If \(h\) is the germ of \(U\) at \(U^{-1}(t(g_{n}))\), then \(h \in A\), \(h^{-1}g_{n}\) belongs to \(W'\), and \(y\) is equal to the limit of the sequence \(t(h^{-1}g_{n})\). Consequently, the Cayley graph \(\mathcal{G}(x, S')\) is also topologically mixing. \(\square\)

The following lemma will finish the proof that (1) implies (2).

Lemma 4.7.5. If a Cayley graph \(\mathcal{G}(x, S)\) is topologically mixing for some \(x \in X\), then the Cayley graph \(\mathcal{G}(x', S)\) is topologically mixing for every \(x' \in X\).

Proof. Since the property of being topologically mixing does not depend on the choice of the generating set \(S\), we may assume that \(S\) is complete. Let \(S\) be a finite open covering of \(X\) by positive contracting elements of \(\mathcal{G}\).

Note that if \(\mathcal{G}(x, S)\) is topologically mixing, then for any point \(y \in X\) in the orbit of \(x\) the Cayley graph \(\mathcal{G}(y, S)\) is topologically mixing. It follows that we may assume that \(x'\) belongs to an open transversal \(X_{0} \subset X\) and that \(x\) is arbitrarily close to \(x'\) (since the orbit of \(x\) is dense in the interior of \(X\)).

Let \(\xi' \in \partial \mathcal{G}_{x'}\) be an arbitrary point. Let \(X' \supset X\) be a compact set such that \(t(F)\) is contained in \(X'\) for every \(F \in S\). Let \(W\) be a neighborhood of \(\xi'\) in \(\mathcal{G}^{X'}\).

Note that then the set of values of \(R_{x'}\) belongs to \(\mathcal{G}^{X'}\). Denote \(\xi = R_{x'}(\xi')\). We have \(R_{x'}(\xi') = \xi'\). There exists \(g \in \mathcal{G}_{x}\) such that \(T_{g}\) is a neighborhood of \(\xi\) in \(\partial \mathcal{G}_{x}\) and \(R_{x}(T_{g}) \subset W \cap \partial \mathcal{G}_{x'}\). The set \(T_{g}\) is a neighborhood of \(\xi\) in \(\mathcal{G}^{X'}\), hence \(t(T_{g})\) is dense in the interior of \(X\). Denote \(T'_{g} = R_{x'}(T_{g})\). The set
of accumulation points of $T_g$ in $\overline{G^X}$ is equal to $T_g$. The set of accumulation points of $T'_g$ in $\overline{G^X}$ is $R'_g$ ($T_g$). Consequently, all but finitely many points of $T'_g$ belong to $W$. But the set of accumulation points of $t(T_g)$ is equal to the set of accumulation points of $t(T'_g)$, due to the fact that the elements of $S$ are contracting. Consequently, the set of accumulation points of $t(W \cap \overline{G^X})$ contains the set of accumulation points of $t(T_g)$, hence it contains the interior of $X$. We have shown that for every neighborhood $W$ of $\xi'$ in $\overline{G^X}$ the set of accumulation points of $t(W \cap \overline{G^X})$ contains the interior of $X$. Repeating the argument from the proof of Lemma 4.7.4 we conclude that the set of accumulation points of $t(W \cap \overline{G^X})$ contains the interior of $X'$. \hfill \Box

Finally, let us show that (3) implies (1). Let $\mathfrak{G}(x, S)$ be a Cayley graph, where $(S, X)$ is a generating pair. We have to show that $t(\mathfrak{G}^X)$ is dense in the interior of $X$. Let $U \subset X$ be an arbitrary open subset. Since every orbit of $\mathfrak{G}$ is dense, $U$ is a transversal. Consequently, the set of elements $g \in \mathfrak{G}^X$ such that $t(g) \in U$ is a net in the Cayley graph $\mathfrak{G}(x, S)$. This implies that for every neighborhood $W$ in $\overline{\mathfrak{G}^X}$ the set $t(W)$ intersects $U$. \hfill \Box

**Definition 4.7.1.** We say that a hyperbolic groupoid $\mathfrak{G}$ is **minimal** if it satisfies the equivalent conditions of Proposition 4.7.1.

**Proposition 4.7.6.** If $\mathfrak{G}$ is minimal, then the groupoid $d\mathfrak{G}_x$ equal to restriction of $d\mathfrak{G}^X$ onto $\partial \mathfrak{G}_x$ is equivalent to the groupoid $d\mathfrak{G}^X$.

**Proof.** The set $\partial \mathfrak{G}_x$ is an open subset of the space of units of the groupoid $d\mathfrak{G}^X$. It follows from minimality that for every $y \in \mathfrak{G}^{(0)}$ there exist elements $g \in \mathfrak{G}$ and $x \in X$ such that $o(g) = y$ and $|t(g) - x| < \delta_0$. Consequently, $\partial \mathfrak{G}_x$ is a $d\mathfrak{G}^X$-transversal. \hfill \Box

Hence, we can define the dual groupoid $\mathfrak{G}^T$ as the groupoid of germs of restrictions of the maps $R^h_g$ onto open subsets of $\partial \mathfrak{G}_x$.

**Proposition 4.7.7.** If $\mathfrak{G}$ is minimal, then $\mathfrak{G}^T$ is also minimal.

**Proof.** It is enough to show that the groupoid $d\mathfrak{G}_x$ is minimal. By definition of topologically mixing Cayley graphs, for any $g, h \in \mathfrak{G}^{X_0}$ and any neighborhood $U$ of $t(g)$ there exists $f \in T_h$ such that $t(f) \in U$ and $T_f \subset T_h$. Then restriction of $R^h_f : T_g \to T_h$ onto the interior of $T_g$ is an element of $d\mathfrak{G}^X$. It follows that orbit of any internal point of $T_g$ is dense. \hfill \Box

For definition of a locally diagonal groupoid, see 2.5.6

**Proposition 4.7.8.** If $\mathfrak{G}$ is minimal, then the geodesic quasi-flow $\partial \mathfrak{G} \times \mathfrak{G}$ is locally diagonal.
Proof. Consider the covering of $\partial G \times G$ by the rectangles $R_{U,g}$, as defined in the proof of Theorem 3.6.1. Suppose that this covering does not satisfy the conditions of 2.5.6. Then there exists a rectangle $R_{U,g}$ and a non-unit element $h \in G$ such that $[o(h), \xi]_{U} = \xi \cdot h$ for all $\xi \in T_{g}^{0}$.

Let $\xi = \cdots g_{2}g_{1} \cdot g$ for $g_{i} \in S$ be an arbitrary point of $T_{g}^{0}$, and let $G_{i} \in S$ be such that $g_{i}$ is $\epsilon$-contained in $G_{i}$. Then $o(h), t(h) \in o(U)$, and $[o(h), \xi]_{U}$ is represented by the product $G_{2}G_{1} \cdot (U, o(h))$. If $g' = (U, o(h))$ and $g'_{i}$ are the germs of $G_{i}$ such that $G_{2}G_{1} \cdot (U, o(h)) = g_{2}g_{1} \cdot g'$, then $\cdots g_{2}g_{1} \cdot gh = \cdots g_{2}g_{1} \cdot g'$.

There exists a constant $\Delta$ such that for any element $h \in G$ which has a trivial projection on the second direction of the geodesic quasi-flow we have $|\nu(h)| \leq \Delta$ (see (4.1) on page 70). Let $Q$ be the set of elements $(\xi, h)$ of the geodesic quasi-flow such that $\xi \in T_{U(h)}$, $t \cdot h \in T_{O(h)}$, and $|\nu(h)| \leq \Delta$. Then $Q$ has compact closure.

Consider the sequence
\[
\begin{align*}
h_{0} &= gh(g')^{-1} = UhU^{-1}, \\
h_{1} &= g_{2}gh(g'_{1}g')^{-1} = G_{1}Uh(G_{1}U)^{-1}, \\
&\vdots \\
h_{n} &= g_{n} \cdots g_{1}gh(g'_{n} \cdots g'_{1}g')^{-1} = G_{n} \cdots G_{1}Uh(G_{n} \cdots G_{1}U)^{-1}.
\end{align*}
\]

For each element $h_{n}$ we have
\[
\cdots g_{n+2}g_{n+1} \cdot h_{n} = \cdots g'_{n+2}g'_{n+1}
\]
and
\[
[o(h_{n}), \cdots g_{n+2}g_{n+1}]_{G_{n} \cdots G_{1}U} = \cdots g_{n+2}g_{n+1}.
\]

There exists $n_{0}$ such that $T_{g} \cdots g_{2}g_{1} \subset T_{g}^{0}$ for all $n \geq n_{0}$. It follows that all elements $h_{n}$ have trivial projections onto the second coordinate of the geodesic quasi-flow. Note that all elements $h_{n}$ are non-units and belong to $Q$ for $n \geq 1$.

Let $H$ be the union of the sequences $h_{n}$ for all possible choices of $\xi \in T_{g}^{0}$, representations $\xi = \cdots g_{2}g_{1} \cdot g$, and $G_{i} \in S$.

Then by definition of minimality, the sets $o(H)$ and $t(H)$ are dense in the interior of $X$. Note also that $|o(h_{n}) - t(h_{n})| \to 0$ as $n \to \infty$.

The set $Q$ can be covered by a finite set $U$ of extendable local homeomorphisms $U \in G$. Choose for each $U \in U$ an extension $\overline{U} \in G$ such that $\overline{U} \subset \hat{U}$. For every $U \in U$ the set of points $x \in o(U)$ such that the germ $(\overline{U}, x)$ is non-trivial and $\overline{U}(x) = x$, is nowhere dense. It follows that there exists an internal point $x$ of $X$ such that for every $U \in U$ either $(U, x)$ is trivial (hence $x$ belongs to an open subset of the set of fixed points of $U$), or $U(x) \neq x$. Then there exist numbers $r > 0$ and $\epsilon_{0} > 0$ such that for every $g \in Q$ such that $|x - o(g)| < r$ either $g$ is trivial, or $|o(g) - t(g)| > \epsilon_{0}$.

But then we get a contradiction, since there will exist $h_{n} \in H$ such that $|x - o(h_{n})| < r$ and $|o(h_{n}) - t(h_{n})| < \epsilon_{0}$. \qed
Chapter 5

Examples of hyperbolic groupoids and their duals

5.1 Gromov hyperbolic groups

Let $G$ be a Gromov hyperbolic group, i.e., a finitely generated group such that its Cayley graph is hyperbolic. The group $G$ acts from right on its left Cayley graph by isomorphisms, hence it acts on the boundary $\partial G$ of the Cayley graph by homeomorphisms. Suppose that for any non-trivial element $g \in G$ and for any $\xi \in \partial G$ the germ $(\xi, g)$ is non-trivial (we denote the germ of $g$ at $\xi$ by $(\xi, g)$, since the action is right). This is true, for instance, when $G$ is torsion-free.

Let $G$ be the groupoid of germs of the action, which will coincide with the groupoid $G \ltimes \partial G$ of the action. It is generated by the compact set of germs of generators of $G$. In order to maintain correct order of multiplication, we will denote by $(g, \xi)$ the germ of the transformation $\zeta \rightarrow \zeta \cdot g$ at $\xi \in \partial G$.

Proposition 5.1.1. The groupoid $\mathcal{G}$ of the action of $G$ on $\partial G$ is hyperbolic.

Proof. Fix a finite generating set $A$ of $G$ and let $|\cdot|$ be the word length function on $G$ defined by $A$. We assume that $A = A^{-1}$ and $1 \in A$. Then distance between $g$ and $h \in G$ is equal to $|gh^{-1}| = |hg^{-1}|$.

Let us use the standard log-scale on $\partial G$ equal to the Gromov product $\ell(\xi_1, \xi_2)$ computed with respect to the basepoint 1. Recall, that it is given by

$$\ell(\xi_1, \xi_2) = \lim_{n \rightarrow \infty} \frac{1}{2}(|g_n| + |h_n| - |g_nh_n^{-1}|),$$

where $g_n \rightarrow \xi_1$ and $h_n \rightarrow \xi_2$ as $n \rightarrow \infty$.

Define $\nu(g, \xi)$ as the value of the Busemann quasi-cocycle $\beta_\xi(g, 1)$, i.e., as any partial limit of the sequence $|g_ng^{-1}| - |g_n|$, for $g_n \rightarrow \xi$ (see Figure 5.1).

We have then

$$\nu(g, \xi) = -2\ell(g, \xi) + |g|.$$

Let $\delta_2$ be a constant such that for any triple $z_1, z_2, z_3 \in G \cup \partial G$ we have

$$\ell(z_1, z_3) > \min(\ell(z_1, z_2), \ell(z_2, z_3)) - \delta_2.$$
Suppose that $\zeta$ is sufficiently close to $\xi$, so that
\[ \ell(\zeta, \xi) > \ell(g, \xi) + \delta_2. \] (5.1)
Then
\[ \ell(g, \zeta) > \min(\ell(g, \xi), \ell(\zeta, \xi)) - \delta_2, \]
and
\[ \ell(\xi, g) > \min(\ell(g, \zeta), \ell(\zeta, \xi)) - \delta_2, \]
which implies that
\[ \ell(\xi, g) > \ell(g, \zeta) - \delta_2, \]
since the other case $\ell(\xi, g) > \ell(\zeta, \xi) - \delta_2$ is not possible, due to (5.1). Consequently,
\[ |\nu(\xi, gh) - \nu(\xi, g)| = 2|\ell(\xi, g) - \ell(g, \xi)| < 2\delta_2, \]
hence there exists a constant $\delta_3$ such that
\[ |\nu(\xi, gh) - \nu(\xi, g)| < \delta_3 \]
for all $\xi, \zeta \in \partial G$ such that $\ell(\xi, \zeta) > \ell(g^{-1}, \xi) + \delta_2$. In particular, we conclude that $\nu : \mathcal{G} \to \mathbb{R}$ is locally bounded.

Suppose that $(g, \xi) \cdot (h, \zeta)$ is a composable pair of elements of $\partial G \times G$, i.e., that $\zeta \cdot h^{-1} = \xi$. Then $(g, \xi) \cdot (h, \zeta) = (gh, \zeta)$, and
\[
\nu(\xi, gh) = \lim_{h_n \to \zeta} |h_n h^{-1} g^{-1} - |h_n| = \lim_{h_n \to \zeta} \left( |h_n h^{-1} g^{-1} - |h_n h^{-1}| \right) + \lim_{h_n \to \zeta} \left( |h_n h^{-1} - |h_n| \right) = \lim_{g_n \to \zeta} \left( |g_n g^{-1} - |g_n| \right) + \lim_{h_n \to \zeta} \left( |h_n h^{-1} - |h_n| \right) = \nu(\xi, g) + \nu(\zeta, h),
\]
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i.e., \( \nu \) is a quasi-cocycle.

It follows from the definitions and right-invariance of the metric \(| \cdot |\) that

\[
\ell(\xi \cdot g^{-1}, \zeta \cdot g^{-1}) - \ell(\xi, \zeta) \doteq \frac{1}{2}(\nu(g, \xi) + \nu(g, \zeta)).
\]

Consequently, if \( \xi \) and \( \zeta \) are close enough, so that \( \ell(\xi, \zeta) > \ell(\xi, g) + \delta_2 \); then we have

\[
\ell(\xi \cdot g^{-1}, \zeta \cdot g^{-1}) \doteq \ell(\xi, \zeta) + \nu(g, \xi),
\]

since \( |\nu(g, \xi) - \nu(g, \zeta)| < \delta_3 \).

It follows that \( g \) is contracting on neighborhoods of points \( \xi \) for which \( \nu(g, \xi) \) is big enough.

It is sufficient now to find for every \( c > 0 \) a compact generating set of \( \mathfrak{G} \) consisting of germs \((g, \xi)\) such that \( \nu(g, \xi) > c \). It will satisfy then conditions of Definitions 3.1.1 which will show that \( \mathfrak{G} \) is hyperbolic.

Let \( \delta \) be such that any geodesic triangle with vertices in \( G \cup \partial G \) is \( \delta \)-thin. Let \( \delta_4 \) be such that for any \( \xi \in \partial G \), \( g \in G \), and a sequence \( g_n \in G \) such that \( g_n \rightarrow \xi \) difference between any two partial limits of \(|g_n g^{-1}| - |g_n|\) is less than \( \delta_4 \).

We say that an infinite product \( \ldots g_2 g_1 \) of elements of \( A \) is a geodesic path converging to \( \xi \) if the sequence \( 1, g_1, g_2 g_1, g_3 g_2 g_1, \ldots \) is a geodesic path converging to \( \xi \). For every point \( \xi \in \partial G \) there exists a geodesic path \( \ldots g_2 g_1 \) converging to \( \xi \).

Let \( S_n \) be the set of germs \((g, \xi)\) such that there exists a geodesic path \( \ldots g_2 g_1 \) converging to \( \xi \cdot g^{-1} \) where \( g^{-1} = g_n \cdot g_1 \). Then for every \((g, \xi) \in S_n\) we have \( \nu(g, \xi) > n - \delta_4 \).

For every geodesic path \( \ldots g_2 g_1 = \xi \) and for every \( k \geq 1 \) the germ of the transformation \((g_k \cdot g_{k+1})^{-1} \) at the point \( \ldots g_k g_{k+1} \) can be written as a product of elements of \( S_n \cup S_{n+1} \) and their inverses. Note also that since the length of elements of \( S_n \) (with respect to the generating set \( A \)) is bounded above by \( n \), the set \( S_n \cup S_{n+1} \) has compact closure.

Let \((g, \xi)\) be an arbitrary element of \( \mathfrak{G} \). Let \( \ldots g_2 g_1 = \xi \) and \( h_2 = \xi \cdot g^{-1} \) be geodesic paths. There exist indices \( n_1 \) and \( n_2 \) both greater than \( m \) and such that distance from \( g_{n_1} \cdot g_{n_1} \cdot g \) to \( n_2 \cdot h_1 \) is less than \( \delta \). Denote

\[ r = h_{n_2} \cdot h_1 \cdot g \cdot (g_{n_1} \cdot g_{n_1})^{-1}. \]

Then \( g = (h_{n_2} \cdot h_1)^{-1} r g_{n_1} \cdot g_{n_1} \), and \((g, \xi) = (g, \ldots g_2 g_1)\) is decomposed into the product of the germs

\[ (g_{n_1} \cdot g_{n_1}, \ldots g_{n_2} g_{n_1}), (r, \ldots g_{n_1+2} g_{n_1+1}), (h_{n_2} \cdot h_1, \ldots g_{n_1+1} g_{n_1+2} r^{-1}). \]

The first and the last germs are products of elements of \( S_n \cup S_{n+1} \) and their inverses. It follows that every element of \( \mathfrak{G} \) can be written as a product of elements of \( Z_n = (S_n \cup S_{n+1}) \cup A^4 \cdot (S_n \cup S_{n+1}) \). The length of the representation of \((g, \xi)\) as a product of elements of \( Z_n \) is bounded from above by a function of \(|g|\). Consequently, \( Z_n \) is a generating set of the groupoid \( \mathfrak{G} \).
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Note that there exists $R > 0$ such that $|\nu(g, \xi)| < R$ for all $g \in A^\delta$. It follows that taking sufficiently big $n$ we can make the values of $\nu$ on $Z_n$ arbitrarily big. □

Denote by $\partial^2 G$ the direct product $\partial G \times \partial G$ minus the diagonal. The space $\partial^2 G$ has a natural local product structure as an open subset of the direct product $\partial G \times \partial G$. The action $(\xi_1, \xi_2) \cdot g = (\xi_1 \cdot g, \xi_2 \cdot g)$ of $G$ on $\partial^2 G$ obviously preserves the local product structure.

**Proposition 5.1.2.** There is an isomorphism of the geodesic quasi-flow $\partial^2 \mathfrak{G} \times \mathfrak{G}$ with the groupoid of the action of $G$ on $\partial^2 G$ identifying the natural projection $P : \partial^2 \mathfrak{G} \longrightarrow \partial^2 G = \mathfrak{G}^{(0)}$ with the projection of $\partial^2 G$ onto the first coordinate $\partial G$ of the direct product $\partial G \times \partial G \supset \partial^2 G$.

**Proof.** For every point $\xi \in \partial G$ the Cayley graph $\mathfrak{G}(\xi, S)$ is isomorphic to the Cayley graph of the group $G$, where the isomorphism is the map $\Gamma_\xi : (\xi, g) \mapsto g$. The quasi-cocycle $\nu$ restricted to $\mathfrak{G}_\xi$ is then identified by $\Gamma_\xi$ with the Busemann quasi-cocycle $\beta_\xi$. Consequently, the isomorphism $\Gamma_\xi$ induces a homeomorphism $\Gamma_\xi : \partial \mathfrak{G}_\xi \longrightarrow \partial G \setminus \{\xi\}$. Denote by $\Gamma(\zeta)$ for $\zeta \in \partial \mathfrak{G}_\xi$ the point $(\xi, \Gamma_\xi(\zeta))$ of $\partial^2 G$. We get then a bijection $\Gamma : \partial \mathfrak{G} \longrightarrow \partial^2 G$. The fact that it is continuous and agrees with the local product structure follows directly from the definitions and Theorem 3.6.1. It is also checked directly that the map induces an isomorphism of groupoids. □

**Corollary 5.1.3.** The groupoid $\mathfrak{G}^\top$ is equivalent to $\mathfrak{G}$.

5.2 Expanding self-coverings

**General definitions.**

Example 5 is naturally generalized in the following way. Let $f : X \longrightarrow X$ be a finite degree self-covering map of a compact metric space $X$, and suppose that it is expanding, i.e., that there exists a metric on $X$ such that $f$ locally expands the distances by a factor greater than one.

Since $f$ is a covering, it is a local homeomorphism, so that it generates a pseudogroup $\mathfrak{F}$ and the associated groupoid of germs $\mathfrak{G}$. A natural compact generator of $\mathfrak{G}$ is the set $S_f$ of all germs of $f$. Every element $g \in \mathfrak{F}$ can be written as $(f, y)^{-n_1} (f, x)^{n_2}$, where $x = o(g)$, $y = t(g)$, and $n_1, n_2 \in \mathbb{N}$ are such that $f^{n_1}(y) = f^{n_2}(x)$. Note that there exists a pair $(n_1, n_2)$ such that if $(m_1, m_2)$ is such that $g = (f, y)^{-m_1} (f, x)^{m_2}$, then $(m_1, m_2) = (n_1 + k, m_2 + k)$ for some $k \geq 0$. It follows that $g$ is uniquely determined by the triple $(y, n_2 - n_1, x)$. Multiplication is then given by the formula

$$(z, n_1, y) \cdot (y, n_2, x) = (z, n_1 + n_2, x).$$

For every $x \in X$ the $\mathfrak{F}$-orbit of $x$ is the grand orbit of $x$, i.e., the set

$$\bigcup_{n \geq 0} \bigcup_{k = 0}^{\infty} f^{-k}(f^n(x)).$$
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Since $f$ is a covering map, the Cayley graph $\mathfrak{G}(x, S_f)$ is a tree. It has a special point of the boundary corresponding to the forward orbit of $x$. The map $\nu(y, n, x) = -n$ coincides with the Busemann cocycle on the tree associated with this point.

It is easy to see now that the groupoid $\mathfrak{G}$ is hyperbolic (graded by $\nu = -\deg$).

Let us describe its dual. Denote by $\hat{X}$ the inverse limit of the constant sequences of spaces $X$ with respect to the maps $f : X \to X$. The map $f$ naturally induces a homeomorphism $\hat{f} : \hat{X} \to \hat{X}$ called the natural extension of $f$.

Points of the space $\hat{X}$ are sequences $(x_1, x_2, \ldots)$ of points of $X$ such that $f(x_n) = x_{n-1}$. Denote by $P : \hat{X} \to X$ the projection $P(x_1, x_2, \ldots) = x_1$.

Let $r > 0$ be such that for every open subset $U \subset X$ of diameter less than $r$ the set $f^{-1}(U)$ can be decomposed into a disjoint union of a finite number of subsets $U_i$ such that $f : U_i \to U$ is an expanding homeomorphism. Then diameters of $U_i$ are also less than $r$, and we conclude that $P^{-1}(U)$ is the direct product of $U$ with a Cantor set $C^x$ naturally identified with the boundary of the tree $\bigsqcup_{n \geq 0} f^{-n}(x)$ of inverse images of a point $x \in U$ (where a vertex $z \in f^{-n}(x)$ is connected to the vertex $f(x) \in f^{-(n-1)}(x)$).

In particular, the space $\hat{X}$ is a fiber bundle over $P : \hat{X} \to X$ and we get a natural local product structure defined by the covering of $\hat{X}$ by the sets $P^{-1}(U) \approx U \times C^x$. The local product structure is obviously preserved by $\hat{f}$. The homeomorphism $\hat{f} : \hat{X} \to \hat{X}$ is expanding in the direction of the factors $U \subset X$ and contracting in the direction of the factors $C^x$ of the local product structure.

It is easy to check now that the groupoid generated by the action of the homeomorphism $\hat{f}$ on $\hat{X}$ is a Smale quasi-flow and that its projection onto the direction of $X$ is equivalent to $\mathfrak{G}$. The dual groupoid $\mathfrak{G}^\top$ acts on the disjoint union of the Cantor sets $C^x$ and is generated by the holonomies of the local product structure (by the partial homeomorphisms between the Cantor sets $C^x$ coming from identifications of the common parts of the covering $\{U \times C^x\}$ of $\hat{X}$) and by the action of $\hat{f}$. The sets $P^{-1}(U_1)$ and $P^{-1}(U_2)$ intersect if and only if $U_1$ and $U_2$ intersect, and the corresponding holonomy is a homeomorphism $C_{x_1} \to C_{x_2}$ for $x_1 \in U_1$ and $x_2 \in U_2$.

If $X$ is connected, then each set $C^x$ is an open transversal of $\mathfrak{G}^\top$ and the groupoid $\mathfrak{G}^\top$ is generated by the holonomy group of the fiber bundle $P : \hat{X} \to X$ and by projection of the action of $\hat{f} : \hat{X} \to \hat{X}$. The holonomy group is called the iterated monodromy group of the map $f : X \to X$.

**Example 12.** Let $X = \{1, 2, \ldots, d\}$ be a finite alphabet, and let $T \subset X \times X$ be a set of words of length 2. The set $T$ can be described by the matrix $A = (a_{ij})_{1 \leq i, j \leq d}$ where $a_{ij} = 0$ if $ij \notin T$ and $a_{ij} = 1$ if $ij \in T$. Consider the space $F$ of sequences $x_1x_2\ldots \in X^\mathbb{N}$ such that $x_ix_{i+1} \in T$ for all $i \geq 1$. Then the space $F$ is invariant under the shift map $f(x_1x_2\ldots) = x_2x_3\ldots$ and the map $f : F \to F$ is an expanding self-covering.
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The groupoid generated by the shift $f$ is a well known object, called the Cuntz-Krieger groupoid $\mathcal{D}_A$, see [CK80].

The natural extension of $f : F \to F$ is the space of two-sided sequences $\ldots x_{-1}x_0x_1 \ldots$ such that $x_i x_{i+1} \in T$ for all $i \in \mathbb{Z}$ together with the shift map. It is easy to see now that the groupoid dual to $\mathcal{D}_A$ is the Cuntz-Krieger groupoid $\mathcal{D}_A^\tau$ defined by the transposed matrix $A^\top$.

Example 13. Let $F_2$ be the free group generated by $a$ and $b$. Then the boundary $\partial F_2$ is naturally identified with the one-sided shift $F$ of finite type over the alphabet $X = \{a, b, a^{-1}, b^{-1}\}$ defined by the set $T = X^2\{(aa^{-1}, a^{-1}a, bb^{-1}, b^{-1}b)\}$. It is easy to see that the groupoid of the action of $F_2$ on its boundary is isomorphic to the groupoid generated by the one-sided shift $F$.

Iterated monodromy groups

Suppose that $f : X \to X$ is an expanding self-covering and the space $X$ is path connected and locally path connected. Let, as above, $\mathfrak{F}$ be the groupoid generated by $f$. We can realize then the dual groupoid $\mathfrak{F}^\tau$ as a groupoid with the space of units equal to the boundary of the tree of preimages $T_{(f,x_0)}$ of a point $x_0 \in X$, see Definition 3.4.1. The boundary of the tree $T_{(f,x_0)}$ is naturally identified with the fiber $P^{-1}(x_0)$ of the natural extension $\hat{X}$, i.e., with the set of sequences $(x_0, x_1, x_2, \ldots)$ such that $f(x_{i+1}) = x_i$ for all $i = 0, 1, \ldots$.

The pseudogroup associated with the dual groupoid $\mathfrak{F}^\tau$ is generated by the holonomy group and the action of $f$. More explicitly, for any element $\gamma$ of the fundamental group $\pi_1(X, x_0)$ the corresponding homeomorphism $M_\gamma$ of $C_{x_0} = \partial T_{(f,x_0)}$ is the limit of the monodromy actions of $\gamma$ on the levels $f^{-n}(x_0)$ of the tree $T_{(f,x_0)}$. Namely, for every $z \in f^{-n}(x_0)$ there exists a unique lift of $\gamma$ that starts at $z$; denote the end of this lift by $\gamma(z)$; then $M_\gamma(x_0, x_1, \ldots) = (x_0, \gamma(x_1), \gamma(x_2), \ldots)$ for every $(x_0, x_1, \ldots) \in C_{x_0}$.

The homeomorphism $f^{-1}$ maps $(x_0, x_1, \ldots)$ to $(x_1, x_2, \ldots)$. If $\alpha$ is a path starting at $x_1$ and ending in $x_0$ then we get the associated element of the pseudogroup $\mathfrak{F}^\tau$ equal to the map $L_\alpha : (x_0, x_1, \ldots) \to (y_0, y_1, \ldots)$ where $y_n$ is the end of the lift of $\alpha$ by $f^n$ starting at $x_{n+1}$ (note that then $y_0 = x_0$).

We have then the following description of $\mathfrak{F}^\tau$.

Proposition 5.2.1. Choose a generating set $S$ of $\pi_1(X, x_0)$ and a collection $\alpha_x$ of paths connecting $x \in f^{-1}(x_0)$ to $x_0$. Then the pseudogroup $\mathfrak{F}^\tau$ is generated by the local homeomorphisms $L_\alpha$, and $M_\gamma$ for $\gamma \in S$.

Note that

$$L_{\alpha_x}^{-1} M_\gamma L_{\alpha_x} = M_{\alpha_x^{-1} \gamma_{x} \alpha_x}$$  \hspace{1cm} (5.2)

where $\gamma_{x}$ is the lift of $\gamma$ by $f$ starting at $x$ and $y$ is the end of $\gamma_{x}$, see Figure 5.2.2. Here we multiply paths in the same way as functions, i.e., in the product $\gamma_{x} \alpha_x$ the path $\alpha_x$ is passed before $\gamma_{x}$.

We can find a homeomorphism of $C_{x_0}$ with the space of infinite sequences $a_1 a_2 \ldots$ over the alphabet $X = f^{-1}(x_0)$ conjugating the maps $L_{\alpha_x}$ with the
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Figure 5.2:

map \( a_1a_2\ldots \mapsto xa_1a_2\ldots \). Then equality (5.2) becomes a recurrent formula for computing the action of the generators of \( \pi_1(X, x_0) \) on the space of infinite sequences.

Note that the obtained formulae defining the groupoid \( \mathcal{F}^T \) do not use the fact that \( f \) is expanding. The group of homeomorphisms of \( C_{x_0} \) generated by \( M_\gamma \) for \( \gamma \in \pi_1(X) \) is called the \textit{iterated monodromy group} of the self-covering \( f \).

**Example 14.** Let us revisit Example 5. Consider the circle \( \mathbb{R}/\mathbb{Z} \) and its self-covering \( f: x \mapsto 2x \). Let \( \mathcal{F} \) be the groupoid generated by \( f \). Note that it is equivalent to the groupoid generated by the maps \( x \mapsto x+1 \) and \( x \mapsto 2x \) acting on \( \mathbb{R} \).

The natural extension of \( f \) is topologically conjugate to the Smale solenoid (see [BS02]).

Let us compute \( \mathcal{F}^T \) using (5.2). Take \( x_0 = 0 \) and let \( \gamma \) be the generator of the fundamental group \( \pi_1(\mathbb{R}/\mathbb{Z}, x_0) \) equal to the image of the segment \([0,1]\) (where 0 is the beginning). The point \( x_0 \) has two preimages 0 and 1/2. Let \( \alpha_0 \) be the trivial path at 0, and let \( \alpha_1 \) be the path from 1/2 to 0 equal to the image of the segment \([0,1/2]\). Consider the space \( \{0,1\}^\omega \) of binary infinite sequences, and identify the transformations \( L_{\alpha_0} \) and \( L_{\alpha_1} \) with the maps \( a_1a_2\ldots \mapsto 0a_1a_2\ldots \) and \( a_1a_2\ldots \mapsto 1a_1a_2\ldots \), respectively. Denote \( M_\gamma = \tau \). Then it follows from (5.2) that the transformation \( \tau \) of the space of binary sequences is defined by the recurrent rule

\[
\tau(0w) = 1w, \quad \tau(1w) = 0\tau(w).
\]

Note that this transformation coincides with the rule of adding 1 to a dyadic integer, so that \( \tau(a_0a_1\ldots) = b_0b_1\ldots \) is equivalent to \( 1+\sum_{n=0}^{\infty} a_n2^n = \sum_{n=0}^{\infty} b_n2^n \) in the ring of dyadic integers. The transformation \( \tau \) is known as the adding machine, or the odometer.

Note that transformations \( L_{\alpha_0} \) and \( L_{\alpha_1} \) are identified then with the maps \( x \mapsto 2x \) and \( x \mapsto 2x + 1 \) on the ring of dyadic integers.

As a corollary we get the following description of \( \mathcal{F}^T \), connecting Examples 5 and 6.
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Proposition 5.2.2. Let $\mathcal{F}$ be the groupoid generated by the maps $x \mapsto x+1$ and $x \mapsto 2x$ on $\mathbb{R}$. Then it is hyperbolic and the dual groupoid $\mathcal{F}^\triangledown$ is the groupoid generated by the maps $x \mapsto x+1$ and $x \mapsto 2x$ on the ring $\mathbb{Z}_2$ of dyadic integers.

Hyperbolic rational functions

In many cases of expanding self-coverings $f : X \rightarrow X$ the fundamental group of $X$ is too complicated, but we can embed $X$ into a space $\mathcal{M}$ with a finitely generated fundamental group so that $f$ can be extended to a covering $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ of $\mathcal{M}$ by a subset $\mathcal{M}_1 \subset \mathcal{M}$. Let $f^n : \mathcal{M}_n \rightarrow \mathcal{M}$ be the $n$th iteration of the partial map $f$. If $X = \bigcap_{n \geq 1} \mathcal{M}_n$, then the iterated monodromy group of $f : X \rightarrow X$ can be computed directly on $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ using the recurrent relation (5.2). This is a particular case of a more general method of approximation of expanding self-coverings described in [Nek08a]. For more on iterated monodromy groups see the monograph [Nek05] and [BGN03, Nek08b].

A particular class of examples comes from holomorphic dynamics. Let $f \in \mathbb{C}(z)$ be a rational function seen as a self-map of the Riemann sphere $\hat{\mathbb{C}}$. Let $C_f$ be the set of critical values of $f$, and let $P_f = \bigcup_{n \geq 1} f^n(C_f)$ be the post-critical set of $f$.

Suppose that $f$ is hyperbolic, i.e., is expanding on a neighborhood of its Julia set in some metric. Then the post-critical set $P_f$ accumulates on a union of a finite number of cycles, which are disjoint with the Julia set. We get hence a covering map $f : \mathcal{M}_1 \rightarrow \mathcal{M}$, where $\mathcal{M} = \hat{\mathbb{C}} \setminus P_f$ and $\mathcal{M}_1 = f^{-1}(\mathcal{M}) \subset \mathcal{M}$ are open neighborhoods of the Julia set. The iterated monodromy group of the action of $f$ on the Julia set of $f$ can be computed then from the covering $f : \mathcal{M}_1 \rightarrow \mathcal{M}$.

For example, direct computation (see [Nek05, Subsection 5.2.2]) show that the iterated monodromy group of the polynomial $z^2 - 1$ is generated by the transformations $a$ and $b$ of the space of infinite binary sequences $\{0, 1\}^\omega$ given by the recurrent rules:

$$a(0w) = 1w, \quad a(1w) = 0b(w), \quad b(0w) = 0w, \quad b(1w) = 1a(w).$$

It follows that the groupoid dual to the groupoid generated by the action of $z^2 - 1$ on its Julia set is the groupoid generated by the transformations $a$, $b$, and by the one-sided shift $x_1x_2\ldots \mapsto x_2x_3\ldots$. The Julia set is shown on Figure 5.3.

Iterated monodromy groups have interesting group theoretic properties and are effective tools in the study of symbolic dynamics of expanding self-coverings, see [Nek05, Nek08b, BN06, Nek08a, Nek12].

5.3 Contracting self-similar groups

We have seen above that dual groupoids of groupoids generated by expanding self-coverings of connected topological spaces are generated by the shift map.
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and by a group $G$ of homeomorphisms $g : X^\omega \to X^\omega$ satisfying recurrent rules of the form

$$g(xw) = yg_x(w)$$

for $x, y \in X$ and $g, g_x \in G$.

**Definition 5.3.1.** A group $G$ acting faithfully on $X^\omega$ is said to be **self-similar** if for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that $g(xw) = yh(w)$ for all $w \in X^\omega$.

It follows from the definition that for every $g \in G$ and for every finite word $v \in X^*$ there exists an element $h \in G$ such that $g(vw) = uh(w)$ for some word $u$ of length equal to the length of $v$. We denote then $h = g|_v$.

Denote by $S_x$ the map $S_x(w) = xw$. Then $S_x^{-1}$ is restriction of the shift onto the set of sequences starting with $x$. We will denote $S_{x_1, x_2, \ldots, x_n} = S_{x_1}S_{x_2} \cdots S_{x_n}$.

Then condition $g(vw) = uh(w)$ is equivalent to the equality

$$gS_v = S_u h.$$  

In particular, $g|_v = S_u^{-1}gS_v$.

**Proposition 5.3.1.** Let $G$ be a self-similar group acting on $X^\omega$ and suppose that it is contracting, i.e., that there exists a finite set $Q \subset G$ such that for every $g \in G$ there exists a positive integer $n$ such that $g|_v \in Q$ for all words of length at least $n$.

If the groupoid of germs of $G$ is Hausdorff, then the groupoid $\mathcal{G}$ of germs of the pseudogroup generated by the shift and $G$ is hyperbolic.

**Proof.** Every element of the inverse semigroup generated by the transformations $g \in G$ and $S_x$, $x \in X$, is equal to a product of the form $S_y g S_u^{-1}$, since every product $gS_x$ can be rewritten as $S_y g|_x$, the product $S_x^{-1} S_y$ is equal to the identical homeomorphism, and the product $S_x^{-1} S_y$ is empty for $x \neq y$.

Moreover, since every homeomorphism $g \in G$ is written as a disjoint union of local homeomorphisms $S_y g|_x S_x^{-1}$, any element of $\mathcal{G}$ is a germ of a local homeomorphism $S_y g S_x^{-1}$ for $g \in Q$. We conclude that $\mathcal{G}$ is compactly generated by $Q$ and $S_x$ for $x \in X$.

Define the degree $\nu$ of any germ of a local homeomorphism $S_y g S_x^{-1}$ as length of $v$ minus length of $u$. It is a well defined continuous cocycle $\nu : \mathcal{G} \to \mathbb{Z}$. It
is easy to see that every element of $\mathcal{G}$ of positive degree has a neighborhood which is a contracting map. Hence, the groupoid $\mathcal{G}$ has a compact contracting generating set. Hyperbolicity follows then from Theorem 1.2.8.

The groupoid dual to the groupoid associated with a self-similar contracting group is generated by the limit dynamical system. Consider the space of left-infinite sequences $X^{-\omega}$ with the direct product topology. We say that sequences $\ldots x_2x_1, \ldots y_2y_1$ are asymptotically equivalent (with respect to the action of $G$) if there exists a finite subset $N \subset G$ and a sequence $g_k \in N$ such that $g_k(x_k \ldots x_1) = y_k \ldots y_1$ for all $k$. The quotient of $X^{-\omega}$ by the asymptotic equivalence relation is called the limit space of $G$. The shift $\ldots x_2x_1 \mapsto \ldots x_3x_2$ agrees with the asymptotic equivalence relation, and hence it induces a continuous self-map of the limit space. This self-map is the limit dynamical system of the contracting self-similar group $G$. If the groupoid of germs of $G$ is principal, then the limit dynamical system is a self-covering (see [Nek09]).

If the groupoid of germs of $G$ is Hausdorff, then the limit dynamical system is a self-covering of an orbispace, and the groupoid dual to $\mathcal{G}$ is still generated by the self-covering (but the definitions should be modified to include the orbispace case).

Since the groupoid $\mathcal{G}^\top$ acts on the limit space of $G$, the boundaries of the Cayley graphs of $\mathcal{G}$ are locally homeomorphic to the limit space. The fact that the limit space is the boundary of a naturally defined Gromov hyperbolic graph $\Gamma(G)$ was noted for the first time in [Nek03]. Definition of this graph is very similar to the definition of the Cayley graphs of $\mathcal{G}$, in fact $\Gamma(G)$ is locally isomorphic to the Cayley graphs of $\mathcal{G}$ (in particular, positive cones of the Cayley graphs are isomorphic to positive cones of $\Gamma(G)$). It is defined in the following way. The set of vertices of $\Gamma(G)$ is the set $X^*$ of all finite words over the alphabet $X$. Two vertices are connected by an edge if and only if either they are of the form $v, s(v)$ for a generator $s$ of $G$ and a word $v \in X^*$, or they are of the form $v, S_x(v) = xv$ for $v \in X^*$ and $x \in X$. If $G$ is contracting, then $\Gamma(G)$ is Gromov hyperbolic and its boundary is naturally homeomorphic to the limit space of $G$.

See Figure 5.4 where a part of the graph $\Gamma(G)$ for $G$ equal to the iterated monodromy group of $z^2 - 1$ is shown. Compare it with the Julia set of $z^2 - 1$ on Figure 5.3. Here we draw only edges corresponding to generators $a, b$, and $S_1$, since $S_0 = a^{-1}S_1$.

There are many examples of contracting self-similar groups for which the groupoid of germs of the action on $X^\omega$ is not Hausdorff. For such groups the limit dynamical system is a covering of an orbispace by its sub-orbispace, and it does not generate any étale groupoid. However, theory of iterated monodromy groups for such partial self-coverings of orbispaces is well developed (see [Nek05, Nek08a]), which suggests that a more general duality theory for hyperbolic groupoids should be developed to include non-Hausdorff (and non-étale) groupoids, e.g., sub-hyperbolic rational functions.
5.4 Smale spaces

Definitions

Smale spaces were defined by D. Ruelle (see [Rue78]) as synthetic models of hyperbolic dynamical systems, and are generalizations of axiom A diffeomorphisms restricted to the non-wandering set.

Definition 5.4.1. A Smale space is a compact metric space $X$ together with a local product structure and a homeomorphism $f : X \rightarrow X$ which preserves the local product structure and is locally contracting in the first direction and locally expanding in the second direction of the local product structure (see condition (3) of Definition 4.1.1).

Just by definition, the groupoid of germs of the homeomorphism $f : X \rightarrow X$ is compactly generated and has a natural continuous cocycle $\nu$ equal to the degree of the iteration of $f$. We see immediately that this groupoid satisfies almost all conditions of Definition 4.1.1. It is not assumed, however, in the standard definition of a Smale space that $f$ is locally Lipschitz and that the local product structure agrees with the metric in the sense of Definition 2.5.10.

But any Smale space carries a natural log-scale, defined by D. Fried [Fri83], which satisfies all the compatibility conditions.

Let $U \subset X \times X$ be a neighborhood of the diagonal such that for $(f^k(x), f^k(y)) \in U$ for all $k \in \mathbb{Z}$ implies that $x = y$. Such a neighborhood exists for every Smale space.
Define then \( \ell(x, y) \) to be the maximal value of \( n \) such that \( (f^k(x), f^k(y)) \in U \) for all \( k \in \mathbb{Z} \) such that \( |k| < n \). If such \( n \) does not exist, we set \( \ell(x, y) = 0 \).

Denote by \( U_n \) the set of pairs \((x, y)\) such that \( \ell(x, y) \geq n \). Then \( U_n = \bigcap_{|k|<n} f^k(U) \), where \( f^k(x, y) = (f^k(x), f^k(y)) \). The sets \( U_n \) are neighborhoods of the diagonal, and \( \bigcap_0 U_n \) is equal to the diagonal.

**Lemma 5.4.1.** The defined function \( \ell \) is a log-scale compatible with the topology on \( X \). The log-scale \( \ell \) does not depend, up to a Lipschitz equivalence, on the choice of \( U \).

**Proof.** Since \( X \) is compact and intersection of the sets \( U_n \) is equal to the diagonal, there exists \( n_0 \) such that \( U_{n_0} \cap U_{n_0} \subset U \). Then \( f^k(U_{n_0}) \cap f^k(U_{n_0}) \subset f^k(U) \) for every \( k \), hence \( U_{n_0+n_0} \cap U_{n_0+n_0} \subset U_n \) for every \( n \). It follows that for every positive integer \( n \) and for all \( x, y, z \in X \) inequalities \( \ell(x, y) \geq n + n_0 \) and \( \ell(y, z) \geq n + n_0 \) imply \( \ell(x, z) \geq n \). Consequently, \( \ell(x, z) \geq \min(\ell(x, y), \ell(y, z)) - n_0 \) for all \( x, y, z \in X \).

If \( U' \) is another neighborhood of the diagonal, and \( \ell' \) is the corresponding log-scale, then there exists \( n_1 \) such that \( U_{n_1} \subset U' \), and then \( f^k(U_{n_1}) \subset f^k(U') \), hence \( U_{n_1+n_1} \subset U'_n \), so that \( \ell'(x, y) \geq \ell(x, y) - n_1 \).

We call the log-scale \( \ell \) the natural log-scale on the Smale space. It is obvious that \( f \) is Lipschitz with respect to the natural log-scale. In fact, \( |\ell(f(x), f(y)) - \ell(x, y)| \leq 1 \).

**Lemma 5.4.2.** The natural log-scale is compatible with the local product structure.

**Proof.** Let \( \epsilon > 0 \) be such that inequalities \( |f^k(x) - f^k(y)| < 2\epsilon \) for all \( k \in \mathbb{Z} \) imply that \( x = y \). Let \( U \) be the set of pairs \((x, y)\) such that \( |x - y| < \epsilon \) and let \( U' \) be the set of pairs such that \( |x - y| < 2\epsilon \). Let \( \ell \) and \( \ell' \) be the corresponding log-scales. We know that they are Lipschitz equivalent.

Let \( R \subset X \) be a rectangle of diameter less than \( \epsilon \). Let us show that there exists a constant \( D \) such that

\[
|\ell(x, y) - \min(\ell(x, [x, y]), \ell(y, [x, y]))| < D
\]

for all \( x, y \in R \).

Suppose that \( k \) is a positive integer such that \( (f^k(x), f^k(y)) \notin U' \), i.e., \( |f^k(x) - f^k(y)| \geq 2\epsilon \). Since \( f \) is contracting in the first direction of the local product structure, \( |f^k([x, y]) - f^k(y)| < \epsilon \), therefore \( |f^k([x, y]) - f^k(x)| > \epsilon \), by the triangle inequality.

If \( k \) is a positive integer such that \( |f^k([x, y]) - f^k(x)| > 2\epsilon \), then by the same argument, \( |f^k(x) - f^k(y)| > \epsilon \).

Similarly, if \( k \) is negative and \( |f^k(x) - f^k(y)| \geq 2\epsilon \), then \( |f^k([x, y]) - f^k(y)| > \epsilon \); and if \( |f^k([x, y]) - f^k(x)| > 2\epsilon \), then \( |f^k(x) - f^k(y)| > \epsilon \).

It follows that

\[
\ell'(x, y) \geq \min(\ell([x, y], x), \ell([x, y], y)), \quad \min(\ell'(x, y], x), \ell'(x, y], y)) \geq \ell(x, y).
\]
But the difference $|\ell' - \ell|$ is uniformly bounded, hence the difference $|\ell(x, y) - \min(\ell([x, y], x), \ell([x, y], y))|$ is uniformly bounded too.

Similar arguments show that there exists a constant $D$ such that if $x_1, x_2$ are such that $x_2 \in P_+(R, x_1)$, then for every $y \in R$ we have

$$|\ell(x_1, x_2) - \ell([x_1, y], [x_2, y])| < D, \quad |\ell(x_1, [x_1, y]) - \ell(x_2, [x_2, y])| < D.$$ 

This finishes the proof of the lemma. \hfill \square

We have thus proved the following proposition.

**Proposition 5.4.3.** The groupoid of germs generated by the homeomorphism $f : X \to X$ of a Smale space is a Smale quasi-flow.

**Corollary 5.4.4.** Ruelle groupoids of the groupoid of germs of a Smale space are mutually dual hyperbolic groupoids.

All examples above, except for Gromov hyperbolic groups, are Ruelle groupoids of some Smale spaces.

The Smale space associated with an expanding self-covering $f : X \to X$ is the natural extension $\tilde{f} : \tilde{X} \to \tilde{X}$. The Smale space associated with a contracting group is called the limit solenoid and is defined in a way similar to the definition of the limit space, but starting from bi-infinite sequences (see [Nek05, Section 5.7]). Some other examples are considered below.

**Quadratic irrational rotation**

Consider the classical Anosov diffeomorphism of the two-torus $\mathbb{R}^2/\mathbb{Z}^2$ defined by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Let $\mathcal{H}$ be the groupoid generated by it. Lifting $\mathcal{H}$ to the universal covering $\tilde{\mathcal{H}}$ of the torus we get a groupoid equivalent to $\mathcal{H}$ generated by the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ and the linear map $A$. In other words, it is the groupoid of germs of the group of affine transformations of the form $\tilde{v} \mapsto A^n \tilde{v} + \tilde{a}$, where $n \in \mathbb{Z}$ and $\tilde{a} \in \mathbb{Z}^2$.

The eigenvalues of $A$ are $\frac{3+\sqrt{5}}{2}$. The eigenvectors $\begin{pmatrix} 1 \\ -1+\sqrt{5} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1-\sqrt{5} \end{pmatrix}$ are orthogonal. Ratio of lengths of projections of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ onto an eigenspace is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. Note that $\varphi^2$ and $\varphi^{-2}$ are the eigenvalues of $A$.

It follows that Ruelle groupoids of the Smale quasi-flow $\mathcal{H}$ are equivalent to the groupoid of germs of the group of affine transformations of $\mathbb{R}$ generated by $x \mapsto x + 1$, $x \mapsto x + \varphi$, and $x \mapsto \varphi^2 x$. It is the groupoid of germs of the group of affine transformations of the form $x \mapsto \varphi^{2n} x + \alpha$ for $n \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}[\varphi]$.

In general, if $\theta$ is a root of the polynomial $x^2 + bx + 1$ for $b \in \mathbb{Z}$, $|b| > 2$, then the groupoid generated by $x \mapsto x + 1$, $x \mapsto x + \theta$, and $x \mapsto \theta x$ is hyperbolic and self-dual.
Vershik transformations, Williams solenoids, and aperiodic tilings

A Bratteli diagram is defined by two sequences \((V_0, V_1, \ldots)\) and \((E_0, E_1, \ldots)\) of finite sets and sequences \(o_n : E_n \to V_n, t_n : E_n \to V_{n+1}\) of maps. We interpret \(\bigcup_{i \geq 0} V_i\) and \(\bigcup_{i \geq 0} E_i\) as the sets of vertices and edges, respectively. An edge \(e \in E_n\) connects the vertex \(o_n(e) \in V_n\) with \(t_n(e) \in V_{n+1}\).

A Bratteli diagram is called stationary if the sequences \(V_n = V, E_n = E, o_n = o, t_n = t\) are constant. If we then identify \(V\) with \(\{1, 2, \ldots, d\}\) for \(d = |V|\), we can describe the stationary Bratteli diagram by the matrix \(A = (a_{ij})\), where \(a_{ij}\) is the number of edges \(e \in E\) such that \(o(e) = i\) and \(t(e) = j\).

The space of paths in a Bratteli diagram is the space of all sequences \((e_0, e_1, \ldots)\) such that \(t(e_n) = o(e_{n+1})\) for all \(n \geq 0\) with the topology of a subset of the direct product \(E_0 \times E_1 \times \cdots\).

A Vershik-Bratteli diagram is a Bratteli diagram in which for every \(v \in V_n, n \geq 1\), there is a linear ordering of the set of edges \(t_{n-1}^{-1}(v)\). A stationary Vershik-Bratteli diagram is a stationary Bratteli diagram with the same orderings on each level.

Suppose that \((e_0, e_1, \ldots)\) is a path in a Vershik-Bratteli diagram consisting not only of maximal edges. Find the first non-maximal edge \(e_n\). Define then

\[
\tau(e_0, e_1, \ldots) = (f_0, f_1, \ldots, f_n, e_{n+1}, e_{n+2}, \ldots),
\]

where \(f_n\) is the next after \(e_n\) edge in the linear order of \(t_{n-1}^{-1}(t_n(e_n))\), and all \(f_i\) for \(i < n\) are minimal. This condition uniquely determines a continuous map \(\tau\) from the set of all non-maximal paths to the set of all non-minimal paths of the diagram. If the diagram has only one minimal path (i.e., a path consisting of minimal edges and only) and one maximal path, then we set the image of the maximal path under \(\tau\) to be the minimal path. In this way we get a homeomorphism of the space of paths of the diagram, called the Vershik transformation.

For example, the binary adding machine transformation is conjugate to the Vershik transformation defined by the diagram consisting of single vertices and two edges on each level.

Consider a stationary Vershik-Bratteli diagram containing only one minimal and only one maximal paths. Let \(\tau : F \to F\) be the corresponding Vershik transformation. Since the diagram is stationary, \(F\) is shift-invariant. Let \(\mathfrak{G}\) be the groupoid of germs of the pseudogroup generated by \(\tau\) and the shift. It is easy to check using Theorem 1.2.8 (in the same way as in Proposition 5.3.1) that the groupoid \(\mathfrak{G}\) is hyperbolic. Its geodesic quasi-flow is an example of a Williams solenoid, see [Wil74]. Instead of describing the general situation, let us just analyze one example.

Consider the stationary Bratteli diagram defined by the matrix

\[
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\]

Let us label the edges of the diagram as it is shown on Figure 5.5. We order the edges by the relation \(1 < 2 < 3\) and \(4 < 5\).
5.4. SMALE SPACES

Figure 5.5:

The space $F$ of infinite paths is the subset of the space $\{1, 2, 3, 4, 5\}^\omega$ consisting of sequences $a_0a_1\ldots$ such that

$$a_ia_{i+1} \in \{11, 12, 14, 21, 22, 24, 31, 32, 34, 43, 45, 53, 55\}$$

for all $i$. The corresponding Vershik transformation is given by the recurrent rules

$$\tau(1w) = 2w, \quad \tau(2w) = 3w,$$
$$\tau(31w) = 1\tau(1w), \quad \tau(32w) = 4\tau(2w), \quad \tau(34w) = 4\tau(4w),$$
$$\tau(4w) = 5w, \quad \tau(5w) = 1\tau(w).$$

Let $\mathcal{G}$ be the groupoid of germs of the pseudogroup generated by the shift and $\tau$. We will see later that it is hyperbolic. The corresponding Smale space $\partial \mathcal{G}$ can be described in the following way. Consider the mapping torus of the homeomorphism $\tau$, i.e., direct product $I \times F$, where $I = [0, 1]$ is the unit interval, modulo the equivalence identifying every point $(1, w)$ with $(0, \tau(w))$.

Let us denote it by $T$. We will denote by $I_w$ the subset $I \times \{w\}$ of $T$. The space $T$ has a natural local product structure inherited from the direct product $I \times F$.

Denote by $A_w$ the union $I_{11w} \cup I_{21w} \cup I_{31w} \cup I_{12w} \cup I_{22w} \cup I_{32w} \cup I_{43w} \cup I_{53w} = A_{1w} \cup A_{2w} \cup B_{3w}$ and by $B_w$ the union $I_{14w} \cup I_{24w} \cup I_{34w} \cup I_{45w} \cup I_{55w} = A_{4w} \cup B_{5w}$

are intervals. Let us change the metric on $T$ so that all intervals $A_w$ have length 1 and the intervals $B_w$ have length $\frac{\sqrt{5} - 1}{2}$. Then length of the interval $A_{1w} \cup A_{2w} \cup B_{3w}$ is equal to $2 + \frac{\sqrt{5} - 1}{2} = \frac{3 + \sqrt{5}}{2}$, and length of $A_{4w} \cup B_{5w}$ is equal to $1 + \frac{\sqrt{5} - 1}{2} = \frac{1 + \sqrt{5}}{2} = \frac{3 + \sqrt{5}}{2}$.

Consider now the homeomorphism $f : T \to T$ mapping $A_{1w} \cup A_{2w} \cup B_{3w}$ by an affine orientation-preserving map onto $A_w$, and $A_{4w} \cup B_{5w}$ onto $B_w$. Then
$f$ contracts the distances inside the intervals by $\frac{3+\sqrt{5}}{2}$, and acts as the shift on the fibers $F$ of the local product structure. It follows that $(T, f)$ is a Smale space. It follows directly from the definition that the holonomy pseudogroup of the stable lamination is generated by the homeomorphism $\tau : F \rightarrow F$, and that the unstable Ruelle groupoid of $(T, f)$ is equivalent to $\mathcal{G}$.

Let us describe the dual groupoid $\mathcal{G}^\perp$, which is the Ruelle groupoid acting on the stable direction. The leaves of the stable lamination are the path connected components of $T$, are homeomorphic to $\mathbb{R}$ and are unions of intervals $A_w$ and $B_w$. Each leaf is tiled by the intervals $A_w$ and $B_w$ in some order. The map $f^{-1} : T \rightarrow T$ will map each tile $A_w$ into the union of the tiles $A_{1w} \cup A_{2w} \cup B_{3w}$, and will map each tile $B_w$ into the union of the tiles $A_{4w} \cup B_{5w}$. We get a self-similarity, or inflation rule for the obtained class of tilings of $\mathbb{R}$. In fact this inflation rule determines the class of tilings in a unique way. Let us write for every leaf of the stable foliation a sequence of letters $A$ and $B$ according to the types ($A_w$ or $B_w$) of the corresponding intervals of the tiling. Then it follows from the inflation rule that the obtained set of bi-infinite sequences is the substitution shift: it is the space of all bi-infinite sequences $w$ such that every finite subword of $w$ is a subword of an element of the sequence $A, AAB, AABAABAB, \ldots$ obtained from $A$ by iterations of the endomorphism $A \mapsto AAB, B \mapsto AB$ of the free semigroup generated by $A$ and $B$.

All the tilings of leaves of $T$ are aperiodic, i.e., have no symmetries. On the other hand, they have many local symmetries: every finite portion of the tiling appears infinitely often in it (and in any other tiled leaf of $T$). The Ruelle pseudogroup of the stable direction of $T$ is the pseudogroup generated by all such local symmetries and the self-similarity given by the inflation rule.

Note that the stable Ruelle groupoid $\mathcal{G}^\perp$ of the Smale space $(T, f)$ is equivalent to a sub-groupoid of the groupoid generated by the transformations $x \mapsto x + 1, x \mapsto x + \frac{1+\sqrt{5}}{2},$ and $x \mapsto \frac{3+\sqrt{5}}{2}x$, which is the Ruelle groupoid of the Anosov diffeomorphism defined by
\[
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix},
\]
described above.

Another classical example of a self-similar aperiodic tilings is the Penrose tiling (see [Pen84, Gar77]). See [Nek00] for description of the pseudogroup dual to the pseudogroup generated by local symmetries and self-similarity of the Penrose tiling.

Paper [Nek06] studies a general situation of a self-similar inverse semigroup, in particular contracting self-similar inverse semigroups (in the spirit of Definition 5.3.1). Groupoid of germs of the inverse semigroup generated by a contracting self-similar inverse semigroup and the shift is hyperbolic (if it is Hausdorff).
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