KERNELS FROM COMPACTIFICATIONS

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Abstract. Associated to any affine scheme with a $\mathbb{G}_m$-action, we provide a Bousfield colocalization on the equivariant derived category $D(\text{Mod}^{\mathbb{G}_m} R)$ by constructing an idempotent integral kernel using homotopical methods. This endows the equivariant derived category with a canonical semi-orthogonal decomposition. As a special case, we demonstrate that grade-restriction windows appear as a consequence of this construction, giving a new proof of wall-crossing equivalences which works over an arbitrary base. The construction globalizes to yield interesting integral transforms associated to $D$-flips.

1. Introduction

A central question in the study of derived categories of coherent sheaves of algebraic varieties is their relationship with birational geometry. Historically, such investigations originated with Orlov’s construction of a semi-orthogonal decomposition associated to a blow-up [Orl92], as well as Bondal and Orlov’s derived equivalences induced by certain elementary flops [BO95]. Much recent effort has since centered around Bondal and Orlov’s conjecture that flops in general induce derived equivalences as well as Kawamata’s related conjecture [Kaw04] that $K$-equivalent varieties have equivalent derived categories. Perhaps the most striking result in this direction is Bridgeland’s construction of a derived equivalence for any threefold flop [Bri02]. A more recent line of inquiry is the description of the equivariant derived categories of geometric invariant theory (GIT) quotients via so-called grade restriction windows, see e.g. [Seg11, BFK12, HHP09, HL15, Bal17, SvdB17, HS16]. These methods sometimes give equivalences or semi-orthogonal decompositions associated to birational maps by viewing the maps as GIT wall-crossings.

Nevertheless, a quick survey of the subject will convince an observer that there is not an agreed upon uniform approach to producing
the functors expected by the $K$-equivalence conjecture. For example, Bridgeland’s techniques require that the flop come from a small contraction over a base of relative dimension one, which limits the applicability to higher dimensional flops. Various families of explicit flops have been considered; notably Namikawa and Kawamata’s study of Mukai flops \cite{Nam03, Nam04, Kaw06} and Cautis, Kamnitzer, and Licata’s study of the the stratified Mukai flop \cite{CKL12}. In these stratified examples a derived equivalence has indeed been observed, but only via fine-tuned choices of explicit Fourier-Mukai kernels. The grade restriction window techniques mentioned above have so far been most effective only for so-called elementary wall-crossings, or when the action is specialized to be quasi-symmetric in the language of \cite{SvdB17}.

In particular, there does not presently appear to be a consensus in the literature for approaching the following problem: given an arbitrary birational map $X \to Y$ of Mori theoretic origin, provide a uniform method of producing a homologically well-behaved functor between $\mathrm{D}^b(\mathrm{coh} X)$ and $\mathrm{D}^b(\mathrm{coh} Y)$. In other words, how to systematically produce a Fourier-Mukai kernel object $P \in \mathrm{D}^b(\mathrm{coh} X \times Y)$ consistent with the expectations of the Bondal-Orlov and Kawamata conjectures? We summarize the main construction of this paper as follows.

**Construction.** Let $Y$ be a scheme with a trivial $\mathbb{G}_m$-action, $\mathcal{A}$ be a quasi-coherent sheaf of $\mathbb{Z}$-graded $\mathcal{O}_Y$-algebras, and set $Z := \mathrm{Spec}_Y \mathcal{A}$.

- Form a $\mathbb{Z}^2$-graded sheaf $Q_{\text{der}}(\mathcal{A})$ of $\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A}$-algebras by deriving a certain partial compactification \cite{Dri13} of the action groupoid.
- Realize $Q_{\text{der}}(\mathcal{A})$ as an object of $\mathrm{D}(\mathcal{Qcoh} \mathbb{G}^2_m \mathbb{Z} \times \mathbb{Z})$.
- Restrict $Q_{\text{der}}(\mathcal{A})$ to open sets to get an equivariant Fourier-Mukai transform

$$\Phi_{Q_{\text{der}}(\mathcal{A})} : \mathrm{D}(\mathcal{Qcoh} \mathbb{G}^2_m U^+) \to \mathrm{D}(\mathcal{Qcoh} \mathbb{G}^2_m U^-).$$

where $U^\pm$ are the corresponding semi-stable loci.

A central case of interest is when $X \to Y$ is a flip relative to a divisor $D$ on $X$, i.e. a $D$-flip. An observation of Reid, see e.g. \cite{Tha96}, allows one to repackage the data of the $D$-flip as a scheme $Z$ affine over the contraction and carrying an action of $\mathbb{G}_m$, with $X$ and $Y$ the respective GIT quotients, so that the above construction applies. We ask in Question \ref{question-D-flop} if $\Phi_{Q_{\text{der}}(\mathcal{A})}$ induces an equivalence for $D$-flops, a potential solution to \cite[Kaw04, Conjecture 5.1]{Kaw04}. However, we presently content ourselves with the following results (see Proposition \ref{proposition-D-flop}, Proposition \ref{proposition-D-flop}, Theorem \ref{theorem-D-flop}, and Corollary \ref{corollary-D-flop} for more precise statements).
Theorem. There is an object $S_{\text{der}} \in D(Q\text{coh}^G Z \times Z)$ and a semi-orthogonal decomposition

$$D(Q\text{coh}^G Z) = \langle \text{Im } \Phi_{Q_{\text{der}}(A)}, \text{Im } \Phi_{S_{\text{der}}(A)} \rangle.$$  

When $Z$ is smooth and affine then the image of $\text{Im } \Phi_{Q_{\text{der}}(A)}$ over $U^+ \times Z$ is equal to the grade restriction window defined in \cite{Seg11, BFK12, HL15}. Hence, when $[U^+/G_m]$ and $[U^-/G_m]$ are $K$-equivalent, $\Phi_{Q_{\text{wc}}(A)}$ is an equivalence.

In the smooth case, we need not derive our construction and simply denote this object by $Q$. Here, the semi-orthogonal decomposition above comes from a certain idempotent property enjoyed by $Q$ which we call Property $P$, see Definition \ref{def:property_P}, which shows that $Q$ induces a Bousfield localization. We remark that when $Z$ is smooth, the proof given here is quite different than those articles as here we produce an explicit geometric kernel, prove functorial identities of that kernel, and deduce these results as corollaries. The proof also works over an arbitrary base. The essential observation here is that the construction of $Q$ behaves well under strongly étale base change (see Proposition \ref{prop:strongly_etale_base_change}) which allows us to reduce to the case of affine space using the Luna Slice Theorem.

When $Z$ is singular, it is not the case that the object $Q$ literally enjoys the idempotent Property $P$ mentioned above. This is problematic as the case of singular affine varieties equipped with a $G_m$-action is quite important for the demands of birational geometry (even for the elementary Mukai flop, the corresponding space $Z$ is singular, for example). This is the reason that we must derive $Q$, i.e. promote $Q$ to an object in derived algebraic geometry.

In Section \ref{sec:smooth_case} we observe that the functor $Q$ extends to a left Quillen functor on the category of graded simplicial rings, i.e. on derived affine schemes equipped with an action of $G_m$. Theorem \ref{thm:derived_variant} shows that this derived variant $Q_{\text{der}}$ does indeed satisfy an analogue of the idempotent property, which we call Property $P_{\text{der}}$, and so we still obtain a semi-orthogonal decomposition in analogy with the smooth case. At an intuitive level, the failure of $Q$ to be well-behaved for singular spaces arises from the non-vanishing of some higher Tor’s (see e.g. Lemma \ref{lem:higher_Tors}), and the homotopical methods mentioned above allow us to bypass this obstruction by encoding the higher Tor’s in an intrinsic derived affine scheme.

Our actual construction of $Q$ comes from the following geometric consideration: if an algebraic group $G$ acts on a scheme $Z$, we consider a space $\tilde{Z}$ which equivariantly extends the action and projection maps;
see Definition 3.1.1. Such a construction for $\mathbb{G}_m$-actions was already considered by Drinfeld [Dri13]. To such data, one can always exhibit an associated faithful functor, see e.g. Proposition 3.2.6. Given Drinfeld’s construction and the central role of $\mathbb{G}_m$-actions via birational cobordisms, we will likewise focus primarily on this case in this article, although the reader will find a discussion of more general group actions in Section 3. We also remark that at a technical level most of the results in the paper are formulated for affine varieties only, but in Section 5.5 we discuss how to associate sheaves to these constructions in order to extend our results to essentially arbitrary $D$-flips, as mentioned above.

The paper is organized as follows.

- In Section 2 we introduce our main object of study, $Q(R)$, and study its basic properties. Subsection 2.2 goes on to show by example how this object arises in the study of flips and flops.
- In Section 3 we discuss the geometric interpretation of $Q$ in terms of compactifications of group actions; here we also introduce the basic criteria for fully-faithfulness and its relation to Bousfield localizations.
- Section 4 treats the case where $\text{Spec } R$ is smooth and exhibits the essential image as a grade restriction window.
- Section 5 then treats the general case by deriving the construction of $Q$ itself. Subsection 5.3 shows that when $\text{Spec } R$ is smooth, these derived replacements effectively trivialize down to the methods of Section 4. Section 5.5 discusses the globalization process for attaching a kernel object to a general $D$-flip.
- In a brief appendix we comment on a small, but important, distinction between our $Q(R)$ and the related constructions in Drinfeld’s article [Dri13].

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2. Affine constructions

2.1. The functor. Let $k$ be a fixed commutative ring. If $R$ is a $\mathbb{Z}$-graded $k$-algebra, we consider the two maps

$$\pi: R \to R \otimes_k k[u, u^{-1}] \cong R[u, u^{-1}] \quad (2.1)$$

$$\sigma: R \to R \otimes_k k[u, u^{-1}] \cong R[u, u^{-1}] \quad (2.2)$$

where $\pi(r) = r$ is the identity and $\sigma$ is the (co-)action map determined by $\sigma(r) = ru^{\deg(r)}$ when $r$ is homogenous. In terms of affine schemes these maps respectively correspond to the projection and action maps

$$\mathbb{G}_m \times_k \text{Spec} R \overset{\pi}{\longrightarrow} \overset{\sigma}{\longrightarrow} \text{Spec} R. \quad (2.3)$$

Throughout this article, we will let $\text{CR}_{\mathbb{G}_m}^k$ denote the category of finitely generated $\mathbb{Z}$-graded $k$-algebras, i.e. the opposite category of the category of affine $k$-schemes which are equipped with $\mathbb{G}_m$-actions. We will often refer to objects of $\text{CR}_{\mathbb{G}_m}^k$ as rings with a $\mathbb{G}_m$-action. Likewise, if an algebraic group $G$ acts on Spec $R$ we will say that $R$ has a $G$-action.

When $R$ is an object of $\text{CR}_{\mathbb{G}_m}^k$ the product space $\mathbb{G}_m \times_k \text{Spec} R$ admits a $\mathbb{G}_m^2$-action whose structure is given in the following lemma, the proof of which is elementary and which we omit.

Lemma 2.1.1. The ring $R[u, u^{-1}]$ carries a natural $\mathbb{G}_m^2$-action

$$\overline{\sigma}: R \to R[u, u^{-1}, v_1, v_1^{-1}, v_2, v_2^{-1}]$$

uniquely determined by

$$\overline{\sigma}(\pi(r)) = \sigma_1(r) \in R[u, u^{-1}, v_1, v_1^{-1}, v_2, v_2^{-1}]$$

$$\overline{\sigma}(\sigma(r)) = \sigma_2(r) \in R[u, u^{-1}, v_1, v_1^{-1}, v_2, v_2^{-1}]$$

where $\sigma_i: R \to R[v_i, v_i^{-1}]$ for $i = 1, 2$ are both identified with $\sigma$ from Equation (2.2).
Moreover, the map \( \hat{\pi} : \mathbb{G}_m \times \text{Spec } R \to \text{Spec } R \) becomes equivariant with respect to the projection \( \pi_1 : \mathbb{G}_m^2 \to \mathbb{G}_m \) onto the first factor, and the map \( \hat{\sigma} \) becomes equivariant with respect to the projection \( \pi_2 : \mathbb{G}_m^2 \to \mathbb{G}_m \) onto the second factor.

**Remark 2.1.2.** The \( \mathbb{G}_m^2 \)-action above is equivalent to saying that \( R[u, u^{-1}] \) has a \( \mathbb{Z}^2 \)-grading with the degree of \( r \in R \in R[u, u^{-1}] \) with \( r \) homogenous being \( (\deg r, 0) \), the degree of \( \sigma(r) \) being \( (0, \deg r) \), and the degree of \( u \) being \( (-1, 1) \).

Given a map \( R \to S \) in \( \text{CR}^{\mathbb{G}_m}_k \), one obtains the obvious induced map \( R[u, u^{-1}] \to S[u, u^{-1}] \). Here we regard \( R[u, u^{-1}] \) and \( S[u, u^{-1}] \) as objects of \( \text{CR}^{\mathbb{G}_m^2}_{k[u]} \).

**Remark 2.1.3.** To stay consistent with Remark 2.1.2, when we write \( \text{CR}^{\mathbb{G}_m^2}_{k[u]} \) we require that \( k[u] \) is equipped with a \( \mathbb{G}_m \)-action such that \( \deg u = (-1, 1) \). Thus objects \( S \) in \( \text{CR}^{\mathbb{G}_m^2}_{k[u]} \) are equipped with a map \( k[u] \to S \) which respects this grading, and likewise morphisms respect this structure. In other words, \( \text{CR}^{\mathbb{G}_m^2}_{k[u]} \) is opposite to the category of affine schemes equipped with \( \mathbb{G}_m^2 \)-actions and an equivariant map to \( \mathbb{A}^1_k \).

**Definition 2.1.4.** Let \( \Delta : \text{CR}^{\mathbb{G}_m}_k \to \text{CR}^{\mathbb{G}_m^2}_{k[u]} \) denote the functor determined by

\[
\Delta(R) := R[u, u^{-1}],
\]

where \( \Delta(R) \) is equipped with the \( \mathbb{G}_m \)-action from Lemma 2.1.1.

We may view \( \Delta(R) \) with its \( \mathbb{G}_m \)-action and its two \( R \)-module structures via \( \pi \) and \( \sigma \) as an object of the derived category \( D^b(\text{mod}^{\mathbb{G}_m} R \otimes_k R) \) \( \mathbb{Z}^2 \)-graded modules by regarding \( \Delta(R) \) as the complex concentrated in homological degree zero. The notation \( \Delta(R) \) introduced above is justified by the following.

**Lemma 2.1.5.** The object \( \Delta(R) \in D^b(\text{mod}^{\mathbb{G}_m} R \otimes_k R) \) is the Fourier-Mukai kernel of the identity functor on \( D^b(\text{mod}^{\mathbb{G}_m} R) \).

**Proof.** The Fourier-Mukai transform associated to \( \Delta(R) \) is given by

\[
\Phi_{\Delta(R)}(M) := \pi_* (\Delta(R) \otimes^L \sigma^* M)^{\mathbb{G}_m} \tag{2.4}
\]

where \( M \in D^b(\text{mod}^{\mathbb{G}_m} R) \) and the functors \( \pi_* \) and \( \sigma^* \) are derived on the left and on the right respectively. In equation (2.4) we recall that taking a derived-push forward in the equivariant setting takes invariants, see e.g. [BKR01], Section 6], which is well-defined since \( \mathbb{G}_m \) is reductive.
and so its functor of invariants is exact. By Remark 2.1.2 the $\mathbb{G}_m$-invariants referred to in Equation (2.4) mean taking invariants on the left factor in the $\mathbb{Z}^2$-grading, i.e. taking terms of degree $(0, \ast)$. Since $\Delta(R)$ is flat (via either module structure), we then compute

$$\left(\Delta(R) \otimes^L \sigma^* M \right)^{\mathbb{G}_m} = \left(\Delta(R) \otimes \sigma^* M \right)^{\mathbb{G}_m} = \left(\Delta(R) \otimes \sigma^* M \right)_{(0, \ast)} \cong \sigma^* M$$

so that $\Phi_{\Delta(R)}(M) = \pi_\ast \sigma^* M$. Likewise, for the inverse Fourier-Mukai transform, we have

$$M \mapsto \sigma_\ast \left(\Delta(R) \otimes \pi^* \sigma^* M \right)^{\mathbb{G}_m} \cong \pi_\ast \sigma^* M.$$ 

To prove the lemma, we thus need to exhibit the two natural isomorphisms

$$M \cong \sigma_\ast \pi^* M$$

and

$$M \cong \pi_\ast \sigma^* M.$$ 

for any $M \in D^b(\text{mod} \mathbb{G}_m R)$. For $M \in \text{mod} \mathbb{G}_m R$ these isomorphisms are respectively given by composing the maps

$$M \to \sigma_\ast \pi^* M$$

and

$$M \to \pi_\ast \sigma^* M$$

with $m \mapsto \sigma_m(m) \in M[u, u^{-1}] \cong \Delta(R) \otimes_R M$ and $m \mapsto m \in M[u, u^{-1}] \cong \Delta(R) \otimes_R M$.

and where $\left(\Delta(R) \otimes M \right)^{\mathbb{G}_m} \cong M$. The result follows for any $M \in D^b(\text{mod} \mathbb{G}_m R)$ since $\Delta(R)$ is again flat via either module structure.

The following definition, though simple, is crucial for this article.

**Definition 2.1.6.** Given an object $R$ of $\text{CR}_{R_{k^m}}$, we define

$$Q(R) := \langle \pi(R), \sigma(R), u \rangle \subseteq R[u, u^{-1}]$$

(2.5)

to be the $k$-subalgebra of $R[u, u^{-1}]$ generated by $u$ and the images of the co-action and projection maps. We regard $Q(R)$ as an object of $\text{CR}_{R_{k^m}}$.

By construction, the maps $\pi$ and $\sigma$ both have images in $Q(R)$. We thus have maps

$$R \xrightarrow{p} Q(R) \xrightarrow{\pi} \text{mod} \mathbb{G}_m R$$

(2.6)

which equal $\pi$ and $\sigma$ respectively (in general $Q(R) \neq R[u, u^{-1}]$, hence we reserve different symbols for these maps). The same construction
as in Lemma 2.1.1 then shows that \( Q(R) \) has \( \mathbb{G}_m^2 \)-action. In terms of the \( \mathbb{Z} \)-grading on \( R \), we may equivalently write
\[
Q(R) = \left( \bigoplus_{i<0} R_i u^i, R[u] \right)
\]
(2.7)
where \( R = \bigoplus_{i \in \mathbb{Z}} R_i \). Notice that when \( r \in R \) is homogeneous with non-negative degree, \( s(r) = ru^\text{deg}(r) \in R[u] \), and thus, in addition to \( R[u] \), only the additional summands \( R_i u^i \) with \( i \) negative are required to generate.

**Example 2.1.7.** Let \( R \) be a polynomial ring equipped with a \( \mathbb{G}_m \)-action (i.e. a \( \mathbb{Z} \)-grading). Relabelling variables as necessary, write
\[
R = k[x_1^+, \ldots, x_k^+, x_1^-, \ldots, x_l^-]
\]
(2.8)
where each \( x_i^+ \) has non-negative (possibly zero) degree and each \( x_j^- \) has strictly negative degree. Write the degrees as \( a_i = \text{deg}(x_i^+) \leq 0 \) and \( b_j = \text{deg}(x_j^-) < 0 \). As in Equation (2.7), \( Q(R) \) is generated (over \( k \)) by
\[
u, x_1^+, \ldots, x_k^+, x_1^-, \ldots, x_l^-; x_i^- u^{b_i}, \ldots, x_i^- u^{b_l}.
\]
To compress notation, write \( \mathbf{x}^+ \) for the set of \( x_i^+ \)'s and similarly for \( \mathbf{x}^- \). Setting \( y_i^- = x_i^- u^{b_i} \) for \( j = 1, \ldots, l \) and inspecting the relations, one has
\[
Q(R) = k[u, \mathbf{x}^+, \mathbf{x}^-, \mathbf{y}^+, \mathbf{y}^-]/(y_i^- u^{b_i} - x_i^-, \ldots, y_l^- u^{b_l} - x_l^-).
\]
It is convenient to write this more symmetrically by setting \( y_i^+ = x_i^+ u^{a_i} \) for \( i = 1, \ldots, k \) as well so that, in compressed notation,
\[
Q(R) = k[u, \mathbf{x}^+, \mathbf{y}^-]/(\mathbf{x}^+ u^a - \mathbf{y}^+, \mathbf{y}^- u^{-b} - \mathbf{x}^-)
\]
(2.9)
(2.10)
In these variables, the maps \( s, p : R \rightarrow Q(R) \) are given by:
\[
p(x_i^+) = x_i^+ \quad p(x_j^-) = y_j^- u^{b_j}
\]
\[
s(x_i^+) = x_i^+ u^{a_i} \quad s(x_j^-) = y_j^-.
\]
We will interpret these maps geometrically in Example 3.1.3. Also, in Subsection 2.2 we will revisit this example (when \( k = l \)) while studying the Atiyah flop.

**Example 2.1.8.** Suppose that \( R \) is any non-negatively graded ring, i.e. \( \mathbb{G}_m \) acts on \( R \) with non-negative weights. Then \( Q(R) \) is actually generated by \( \pi(R) \) and \( u \) as a \( k \)-algebra. As a module over \( R \) via \( p \), we thus have \( Q(R) \cong R[u] \) or, geometrically, \( \text{Spec} \ Q(R) \cong A_k^1 \times \text{Spec} \ R \).

If \( R \) is instead non-positively graded, then of course \( Q(R) \) is not generated by just \( \pi(R) \) and \( u \). However, we may still construct an isomorphism \( Q(R) \cong R[u] \) where now \( Q(R) \) has the \( R \)-module structure via \( s \). Indeed, one always has the map \( R[u] \rightarrow Q(R) \) given by \( r \mapsto \sigma(r) \)
and \( u \mapsto u \). This map is always injective, and when the weights are all non-positive, it is easily checked to be surjective as well.

For later use we record an elementary property that \( Q \) enjoys with respect to polynomial extensions; in particular, the next lemma shows that \( \text{Spec} \, Q(R[x]) \cong \text{Spec} \, A_k \times \text{Spec} \, Q(R) \).

**Lemma 2.1.9.** Given \( R \in CR_k^{G_m} \), endow \( R[x] \) with a \( G_m \)-action by giving \( x \) degree \( a \). Then

\[
Q(R)[y] \cong Q(R[x])
\]

(2.13)

where \( y \) has degree \((a,0)\) if \( a \geq 0 \) and degree \((0,a)\) if \( a \leq 0 \).

**Proof.** Assume the degree of \( x \) is \( a \leq 0 \). The case \( a \geq 0 \) is analogous.

We have a map

\[
Q(R)[y] \to Q(R[x])
\]

\( y \mapsto u^a x \)

This map is clearly surjective, and has kernel if and only if \( u^a x \) is algebraic over \( Q(R) \). If so, then \( u^a x \) is algebraic over \( R[u,u^{-1}] \) in \( R[x,u,u^{-1}] \) which is impossible, since \( u \) is a unit. The statement about the weight of \( y \) is just the weight of \( u^a x \) via the grading from Remark 2.1.2. \( \square \)

**Remark 2.1.10.** The reader desiring a more geometric understanding of \( Q(R) \) will find such a discussion in Section 3.

**Remark 2.1.11.** Our definition of \( Q(R) \) is actually equivalent to the affine case of a construction of Drinfeld, see [Dri13, Section 2.3]. Namely, if \( R = \bigoplus R_i \) is \( \mathbb{Z} \)-graded, Drinfeld considers the the \( k[t] \)-algebra \( \tilde{R} \) generated by all symbols \( \tilde{r} \) where \( r \in R \), and which are required to be \( k \)-linear and subject to the relations

\[
\tilde{r}_1 \tilde{r}_2 = t^{\mu(n_1,n_2)} \tilde{r}_1 \tilde{r}_2
\]

where each \( r_i \in R_{n_i} \) and where

\[
\mu(m,n) := \min\{|m|,|n|\}
\]

if \( m \) and \( n \) have opposite signs, and \( \mu(m,n) = 0 \) otherwise. It is easy to check that the assignment \( u \mapsto t \), \( r_i \mapsto t^{n_i} r_i \) for \( i < 0 \), and \( r_i \mapsto r_i \) for \( i \geq 0 \) induces an isomorphism \( Q(R) \cong \tilde{R} \). In the Appendix we will discuss in more detail the role of our \( Q(R) \) in the context of Drinfeld’s article.

The study of \( Q(R) \) as a Fourier-Mukai kernel is, in a sense, the fundamental goal of this article. To this end, we first note that formation of \( Q(R) \) is functorial.
Lemma 2.1.12. The assignment of $Q(R)$ to $R$ defines a functor

$$Q : \text{CR}_k^{\mathbb{Z}_m} \to \text{CR}_k^{\mathbb{Z}_m}_{[u]}$$

which preserves surjective morphisms.

Proof. Given $\phi : S \to R$, the corresponding map $S[u, u^{-1}] \to R[u, u^{-1}]$ is surjective when $\phi$ is. Furthermore, one checks that the image of $Q(S)$ under this map is $Q(R)$. \qed

We can relate the functors $Q$ and $\Delta$ by a natural transformation by considering the inclusions

$$\eta_R : Q(R) \hookrightarrow R[u, u^{-1}] = \Delta(R)$$

which come from the definition of $Q(R)$ as a subalgebra of $\Delta(R)$. These inclusions behave predictably; in particular, we have the following commutative diagrams:

$$
\begin{array}{ccc}
Q(R) & \xrightarrow{\eta_R} & R[u, u^{-1}] \\
\downarrow{p} & & \downarrow{\pi} \\
R & & R
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
Q(R) & \xrightarrow{\eta_R} & R[u, u^{-1}] \\
\downarrow{s} & & \downarrow{\sigma} \\
Q(R) & & \Delta(R)
\end{array}
\tag{2.15}
$$

Definition 2.1.13. Let $\eta : Q \to \Delta$ be the natural transformation of functors induced by the inclusions $\eta_R : Q(R) \hookrightarrow \Delta(R)$. Here $Q$ and $\Delta$ are the functors $\text{CR}_k^{\mathbb{Z}_m} \to \text{CR}_k^{\mathbb{Z}_m}_{[u]}$ from Lemma 2.1.12 and Definition 2.1.4 respectively.

We will sometimes abuse notation and denote a map $\eta_R$ by $\eta$ when it is clear from context that we are working with rings and not the natural transformation.

Another way of understanding the relation between $Q$ and $\Delta$ is the following result, which shows in particular that they become identified after localizing by elements of non-zero degree.

Lemma 2.1.14. Let $R$ be an object of $\text{CR}_k^{\mathbb{Z}_m}$. View $Q(R)$ as a right $R$-module via $s$ and a left $R$-module via $p$, and view $\Delta(R)$ as a right $R$-module via $\sigma$ and a left $R$-module via $\pi$. Let $r \in R$ be a homogeneous element, and let $R \to R_r$ be the corresponding homogeneous localization. If $\deg(r) > 0$, then

$$1 \otimes_s \eta : R_r \otimes_s Q(R) \to R_r \otimes_\sigma \Delta(R)$$

is an isomorphism. If $\deg(r) < 0$, then

$$\eta \otimes 1 : Q(R) \otimes R_r \to \Delta(R) \otimes R_r$$
is an isomorphism. If $\deg(r) = 0$, then

$$Q(f)_p \otimes 1 : Q(R)_p \otimes R_r \rightarrow Q(R_r) \otimes R_r \cong Q(R_r)$$

is an isomorphism.

Proof. In all three cases, injectivity of the maps holds because localization is flat and $\eta$ and $Q(f)$ are inclusions. The only non-trivial part of verifying surjectivity in the cases of non-zero degree is to demonstrate that $u^{-1}$ lies in the image. Indeed, in the case where positive degree one has that $(1 \otimes s \eta)(\frac{1}{r} \otimes u^{\deg(r)-1}r) = u^{-1}$ and in the case of negative degree one has that $(\eta_p \otimes 1)(s(r)u^{-1-\deg(r)} \otimes \frac{1}{r}) = u^{-1}$. Surjectivity in the degree zero case is clear.

We conclude this section with some natural loci coming from a $\mathbb{G}_m$-action.

**Definition 2.1.15.** If $R$ is an object of $\text{CR}^{G_m}{_k}$ let $I \pm \subseteq R$ be the ideals generated by elements of positive or respectively negative degree. Set $R^+ := R/I^-$, and $R^- := R/I^+$ (note the swap in signs), and set $R^0 := R/(I^+, I^-)$.

The reason for defining $R^0$ and $R^\pm$ in the above fashion is to match notation with Definition 2.1.16 below, which gives their geometric description.

**Definition 2.1.16.** Let $X$ be any $k$-scheme equipped with a $\mathbb{G}_m$-action. Equip $\text{Spec}(k)$ with the trivial $\mathbb{G}_m$-action. Let $\mathbb{A}^1_+$ denote $\mathbb{A}^1$ equipped with its usual $\mathbb{G}_m$-action by scaling, and let $\mathbb{A}^1_-$ denote $\mathbb{A}^1$ equipped with the inverse action $t \cdot x = t^{-1}x$. Then

\[ X^0 := \text{Hom}^{G_m}(\text{Spec}(k), X) = \{ x \in X \mid t \cdot x = x \text{ for all } t \in \mathbb{A}^1_+ \} \]

is the fixed point locus for the $\mathbb{G}_m$-action,

\[ X^+ := \text{Hom}^{G_m}(\mathbb{A}^1_+, X) = \{ x \in X \mid \lim_{t \to 0} t \cdot x \text{ exists in } X \} \]

is the attracting locus, and

\[ X^- := \text{Hom}^{G_m}(\mathbb{A}^1_-, X) = \{ x \in X \mid \lim_{t \to 0} t^{-1} \cdot x \text{ exists in } X \} \]

is the repelling locus.

We refer to [Dri13, Section 1] for basic properties of these loci. As alluded to above, when $X := \text{Spec} R$ is affine and equipped with a $\mathbb{G}_m$-action, we have

\[ X^0 = \text{Spec } R^0 \]

and

\[ X^+ = \text{Spec } R^+ = V(I^+) \]
using the notation of Definition \[2.1.15\] By construction there are inclusions \( R^0 \subset R^\pm \) which give maps

\[ q^\pm : X^\pm \to X^0. \tag{2.16} \]

In terms of Definition \[2.1.16\] the maps \( q^\pm \) can equivalently be understood as being induced by the \( \mathbb{G}_m \)-equivariant inclusions \( \text{Spec}(k) \hookrightarrow \mathbb{A}_k^1 \).

Geometrically, this means that the maps \( q^\pm \) correspond to taking a point to its limit along either \( \mathbb{G}_m \)-action or the inverse \( \mathbb{G}_m \)-action.

**Remark 2.1.17.** The ideals \( I^\pm \) can be recovered directly from \( Q(R) \) equipped with its maps \( p \) and \( s \). Namely,

\[
I^+ = s^{-1}(uQ(R)) \subseteq R \\
I^- = p^{-1}(uQ(R)) \subseteq R.
\]

### 2.2. Flop equivalences via \( Q(R) \)

We now observe that \( Q(R) \) provides the derived equivalence constructed by Bondal and Orlov for the standard Atiyah flop [BO95, Section 3]. For \( n \geq 2 \), let

\[
R = k[x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-] \tag{2.17}
\]

with the \( \mathbb{G}_m \)-action given by taking \( \deg(x_i^+) = 1 \) and \( \deg(x_i^-) = -1 \) for each \( i \). The ideals \( I^\pm \subset R \) from Definition \[2.1.15\] are \( I^+ = (x_1^+, \ldots, x_n^+) = (x^+) \) and \( I^- = (x_1^-, \ldots, x_n^-) = (x^-) \). Let \( X = \text{Spec } R = \mathbb{A}^{2n} \) and set

\[
U^\pm := X \setminus V(I^\pm) \\
X/\pm := U^\pm /\mathbb{G}_m \\
X/0 := \text{Spec } R^{\mathbb{G}_m}.
\]

**Remark 2.2.1.** The invariant subring \( R^{\mathbb{G}_m} \) should not be confused with \( R^0 \) from Definition \[2.1.15\] where the defining ideal is \((I^+, I^-)\); in other words, the invariant theory quotient \( X/0 \) is not the same thing as the fixed locus \( X^0 \).

For the standard Atiyah flop, \( X/\pm \mathbb{G}_m \) are both isomorphic to the total space of \( O(-1)^{\oplus n} \) on \( \mathbb{P}^{n-1} \) and \( X/0 \mathbb{G}_m \) is a singular affine quadric. Let

\[
p^\pm : X/\pm \to X/0
\]

be the corresponding birational contractions.

The diagram

\[
\begin{array}{ccc}
X/\pm & \xrightarrow{\text{----------------}} & X/- \\
\downarrow & & \downarrow \\
X/0 & & 
\end{array}
\]
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is the prototypical example of a flop. [BO95, Theorem 3.9] proves that
the functor
\[ p^{-}p^{+} : D^{b}(\text{coh } X/+/) \to D^{b}(\text{coh } X/\)\] (2.18)
is an equivalence of derived categories of coherent sheaves (where the
functors \( p^{-} \) and \( p^{+} \) are, of course, derived on the right and left, respectively). For this example, the actions of \( \mathbb{G}_{m} \) on \( U^{\pm} \) are free, and so we
may identify
\[ D^{b}(\text{coh } X/+/) = D^{b}(\text{coh } \mathbb{G}_{m} U^{\pm}) \] (2.19)
and recognize the Bondal-Orlov flop equivalence as an equivalence of
equivariant derived categories.

Remark 2.2.2. Taking the functor to be \( p^{-}p^{+} \) is equivalent to taking
the fiber product along \( p^{\pm} \) as the Fourier-Mukai kernel of the functor.
Instead of the fiber product, one could also take as a Fourier-Mukai
kernel a common blow-up resolving the birational map. For the Atiyah
flop, though, the fiber product and the resolution are isomorphic, see
[Kaw04, Proposition 5.5].

We now observe that the fiber product agrees with an appropriate
restriction of \( Q(R) \).

Proposition 2.2.3. With \( Q(R) \) as in Equation (2.17), let
\[ Y^{uc} := \text{Spec } Q(R) \times_{X \times X} (U^{+} \times U^{-}) \] (2.20)
denote the restriction of \( Q(R) \) to the open subset \( U^{-} \times U^{+} \subset X \times X \). (The
“uc” stands for “wall-crossing”.) Then there is a natural isomorphism
\[ Y^{uc} \\sim U^{+} \times X/\)\] (2.21)
In other words, if we regard \( Q^{uc} = \Gamma(Y^{uc}, \mathcal{O}) \) as an object of \( D^{b}(\text{coh } \mathbb{G}_{m} U^{-} \times U^{+}) \), then the Fourier-Mukai functor
\[ \Phi_{Q^{uc}} : D^{b}(\text{coh } \mathbb{G}_{m} U^{-}) \to D^{b}(\text{coh } \mathbb{G}_{m} U^{+}) \]
is an equivalence and \( \Phi_{Q^{uc}} \) is naturally isomorphic to to \( p^{-}p^{+} \) under
the identification in Equation (2.19).

Proof. The universal property of the fiber product gives a map
\[ \phi : R \otimes_{\mathbb{G}_{m}} R \to Q(R) \] (2.22)
which is, explicitly, given by \( r_{1} \otimes r_{2} \mapsto r_{1}s(r_{2}) \), i.e. \( \phi = p \otimes_{\mathbb{G}_{m}} s \). Taking
\( k = n \) and \( l = n \) in Example 2.1.7 to describe \( Q(R) \), we have:
\[ \phi : k[x^{+}, x^{-}, y^{+}, y^{-}] \to k[u, x^{+}, y^{-}, u^{-1}y_{1}, \ldots, u^{-1}y_{n}] \]
We can work on a cover of \( U^{+} \times U^{-} \) given by inverting the monomials in
\( x_{1} \) and \( y_{2} \). One can check that inverting any such pair reduces \( \phi \) to an
isomorphism. The statement about $\Phi_{\text{Qec}}$ being an equivalence is then just a rephrasing of the Bondal-Orlov equivalence, i.e. Equation (2.18).

Example 2.2.4. Let

$$R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1y_1 + \ldots + x_ny_n)$$

with $n \geq 2$ and with each $\deg(x_i) = 1$ and each $\deg(y_j) = -1$. Here Spec $R/+ \to$ Spec $R/-$ is the elementary Mukai flop. Note that here Spec $R$ is singular. In Example 3.2.11 we will see that such singularities lead to a poorly behaved Fourier-Mukai functor associated to $Q(R)$, and we will correct this behavior in Example 5.2.3 by replacing $Q(R)$ with a suitable affine derived scheme $Q_{\text{der}}(R)$. It is not difficult to show that a suitable generalization of Proposition 2.2.3 holds once this correction is made.

### 3. Functors from compactifications

#### 3.1. Partial compactifications of group actions.

We now address the geometric interpretation of $Q(R)$. In this section, in order to provide context for the connection between $Q(R)$ and geometric invariant theory (GIT), we will give definitions and results for varieties which are not necessarily affine and actions by algebraic groups other than $\mathbb{G}_m$. The reader only concerned with the level of generality required for the later portions of this paper may well assume that $G = \mathbb{G}_m$ and $X = \text{Spec } R$ is affine throughout this section as well; under these assumptions Proposition 3.1.2 and Proposition 3.1.6 summarize the relevant statements needed from this subsection.

Let $G$ be an algebraic group acting on a variety $X$. Let $\bar{\sigma} : G \times X \to X$ be the action map and $\bar{\pi} : G \times X \to X$ the projection. The space $G \times X$ itself admits a $G \times G$ action given by

$$(g_1, g_2) \cdot (g, x) = (g_2gg_1^{-1}, \bar{\sigma}(g_1, x))$$  

which makes $\bar{\pi}$ and $\bar{\sigma}$ equivariant. With this action, the map $G \times X \to X \times X$ given by $(g, x) \mapsto (x, \bar{\sigma}(g, x))$ becomes equivariant, where $X \times X$ has the obvious $G \times G$ action.

**Definition 3.1.1.** Let $\tilde{X}$ be an algebraic variety together with an action of $G \times G$ which is equipped with a $G \times G$-equivariant open immersion

$$i : G \times X \to \tilde{X},$$

and a $G \times G$-equivariant morphism

$$(\bar{\sigma}, \bar{s}) : \tilde{X} \to X \times X$$

such that the following diagram commutes:
i.e. $\bar{p} \circ i = \bar{\pi}$ and $\bar{s} \circ i = \bar{\sigma}$. In the above situation, we say that $\tilde{X}$ equipped with the maps $\bar{p}, \bar{s}, i$, is a partial compactification of the action of $G$ on $X$.

The following result shows that $Q(R)$ encodes exactly this kind of structure when $G = \mathbb{G}_m$ and $X$ is affine.

**Proposition 3.1.2.** Let $\text{Spec } R$ be an affine variety with a $\mathbb{G}_m$-action. Consider the maps

$$
\text{Spec } Q(R) \xrightarrow{\bar{\rho}} \text{Spec } R.
$$

corresponding to the ring maps $p, s : R \to Q(R)$ and let

$$
\bar{\eta} : \text{Spec } Q(R) \to \text{Spec } R[u, u^{-1}] = \mathbb{G}_m \times \text{Spec } R
$$

be the open immersion corresponding to $\eta : Q(R) \to R[u, u^{-1}]$. Then $\text{Spec } Q(R)$ together with the maps $\bar{p}, \bar{s}, \bar{\eta}$ form a partial compactification of the action of $\mathbb{G}_m$ on $\text{Spec } R$.

**Proof.** This follows easily from previous observations: the equivariant structure on $Q(R)$ is determined by the $\mathbb{Z}^2$-grading given in Remark 2.1.2. The fact that $\bar{\eta}$ is an open immersion follows from the fact that it is the localization along $u$. That Figure 3.1.1 commutes is exactly dual to the commutativity of the diagrams in (2.15). \qed

**Example 3.1.3.** Let us revisit Example 2.1.7 where we considered the case of a $\mathbb{G}_m$-action on a polynomial ring, i.e. we now consider $\text{Spec } (R) = \mathbb{A}^{k+l}$ where the first $k$-coordinates are acted on with non-negative weights and the last $l$-coordinates with strictly negative weights. Following Example 2.1.7 write

$$
\mathbb{A}^{k+l+1} = \text{Spec } R[u, x^+, y^-] = \text{Spec } Q(R).
$$

The $p$ and $s$ module structures computed in Equations (2.11) and (2.12) determine the maps $\bar{p}$ and $\bar{s}$ in the commutative diagram
where $\hat{\sigma}$ and $\hat{s}$ are the action and projection maps, and $i$ is an inclusion chosen to make the diagram commute. Explicitly:

\[
i(u, x_1^+, \ldots, x_k^+, x_{-1}^-, \ldots, x_{-l}^-) = (u, x_1^+, \ldots, x_k^+, u^{b_1}x_{-1}^-, \ldots, u^{b_l}x_{-l}^-)
\]

\[
\hat{p}(u, x_1^+, \ldots, x_k^+, y_{-1}^-, \ldots, y_{-l}^-) = (x_1^+, \ldots, x_k^+, u^{-b_1}y_{-1}^-, \ldots, u^{-b_l}y_{-l}^-)
\]

\[
\hat{s}(u, x_1^+, \ldots, x_k^+, y_{-1}^-, \ldots, y_{-l}^-) = (u^{a_1}x_1^+, \ldots, u^{a_k}x_k^+, y_{-1}^-, \ldots, y_{-l}^-).
\]

Note that the map $i$ is not the “naive” inclusion in these coordinates.

The data of the partial compactification of an action of $G$ on $X$ automatically encodes a notion of a boundary on $\tilde{X}$ as well as distinguished “unstable” and “semistable” loci in $X$, as in the following definition.

**Definition 3.1.4.** If $\tilde{X}$ is a partial compactification of an action $\sigma : G \times X \to X$, we define the boundary of $\tilde{X}$ to be

\[
\partial_{\tilde{X}} := \tilde{X} \setminus i(G \times X),
\]

the $\tilde{s}$-unstable locus to be

\[
X^{us} = \tilde{s}(\partial_{\tilde{X}}),
\]

and the $\tilde{s}$-semistable locus to be

\[
X^{ss} = X \setminus X^{us}.
\]

Note that $X^{ss}$ itself admits a $G$-action as the $G \times G$ action on $\tilde{X}$ extends the action on $G \times X$ and $\tilde{s}$ was assumed equivariant. In this article we are primarily concerned with the affine group scheme $\mathbb{G}_m$ and the case where $X = \text{Spec } R$ is affine, in which case $\partial_{\tilde{X}}$ is a automatically a divisor in $\tilde{X}$ (although $X^{us} = \tilde{s}(\partial_{\tilde{X}})$ will rarely be a divisor in $X$).

**Remark 3.1.5.** The terminology in Definition 3.1.4 is chosen to emphasize the connection with geometric invariant theory (GIT). The next lemma demonstrates the precise relationship between our notion of $\tilde{s}$-semistability and semistability in GIT. We will also explain the relationship for actions by higher rank tori in Example 3.1.7 and Proposition 3.1.8.
Proposition 3.1.6. Let $\mathbb{G}_m$ act on $X = \text{Spec } R$ and let $\text{Spec } Q(R)$ partially compactify the action as in Proposition 3.1.2. Then the $\bar{s}$-unstable locus is the repelling locus $X^-$ as defined in Definition 2.1.16. In particular, $X^{ss} = X \setminus X^-$. 

Proof. Since $Q(R) = \langle u, \pi(R), \sigma(R) \rangle$, the ideal which defines the boundary $\partial \subseteq \text{Spec } Q(R)$ is simply $\langle u \rangle \subseteq Q(R)$. Hence, $\bar{s}(\partial)$ is given by the ideal $\langle u \rangle \cap s(R)$. By the explicit gradings in Remark 2.1.2, it is easy to see that $\langle u \rangle \cap s(R) = I^+$ is the ideal generated by all homogeneous elements of positive degree, which in turn is the ideal defining $X^-$. \[\square\]

It follows immediately from the above proposition that, in the case of a $\mathbb{G}_m$-action on $X = \text{Spec } R$ equipped with the partial compactification of the action given by $\text{Spec } Q(R)$, one has

$$X^{ss} = X \setminus X^- = \text{Spec } R \setminus \text{Spec } (R/I^+) = \bigcup_{r \in I^+} \text{Spec } R_r.$$  

(3.2)

We conclude this subsection with two brief but instructive examples of partial compactifications of actions which go beyond the case of a $\mathbb{G}_m$-action (and thus are not logically necessary for the remainder of this article). In the first example, $G$ is a higher rank torus acting on any affine space. In the second example, $G$ is non-abelian.

Example 3.1.7. Let $T = \mathbb{G}_m^n$ be a torus and let $\chi_*(T) = \text{Hom}(T, \mathbb{G}_m)$ denote its character lattice. Suppose that $T$ acts on a ring $R$ with co-action map

$$\sigma : R \to R \otimes_k k[T] \cong R \otimes_k k[\chi_*(T)] \cong R[x_1^\pm, \ldots, x_n^\pm].$$

Now, let $C \subset \chi_*(T)$ be any finitely generated submonoid. Let

$$Q^C_T(R) = \langle R[C], \sigma(R) \rangle \subseteq R[\chi_*(T)]$$

be the subalgebra generated by the image of the action and the monoid ring. A trivial generalization of Proposition 3.1.2 shows that $\text{Spec } Q^C_T(R)$ is a partial compactification of the action of $T$ on $X = \text{Spec } R$.

Now, suppose Pic(\text{Spec } R) \otimes \mathbb{Q}$ is trivial. Then, following [Tha96 Section 2] or [DH98 Section 3], the cone of ample $T$-linearized $\mathbb{Q}$-divisors on $X = \text{Spec } R$ is a cone in $\chi_*(T) \otimes \mathbb{Q}$. This cone admits a chamber decomposition such that the relative interiors of the respective chambers correspond exactly to GIT quotients. That is, two characters $\chi$ and $\chi'$ lie in the relative interior of the same chamber exactly when the GIT semistable loci $X^{ss}(\chi)$ and $X^{ss}(\chi')$ are equal.

Of particular interest is the case where the monoid $C$ is itself the (integral points of) of a GIT chamber. Although not required for the remainder of the article, it seems worthwhile to observe that a generalization of Proposition 3.1.6 holds in this setting which relates the
notion of semistability given in Definition 3.1.4 with the usual notion of semistability in Geometric Invariant Theory (GIT).

**Proposition 3.1.8.** As above, let a torus $T$ act on a ring $R$ and suppose $\text{Pic}(\text{Spec } R) \otimes \mathbb{Q} = 0$. Let $C$ be the monoid of integral points of the closure of a GIT chamber, and let $L$ be any line bundle which lies in the relative interior of the same GIT chamber.

Then the $\hat{s}$-semistable locus in the sense of Definition 3.1.4 equals the semistable locus for the action of $T$ on $X$ equipped with the linearization $L$ in the sense of GIT.

**Proof.** The proof is essentially the same as that of Proposition 3.1.6. Indeed, the ideal defining the boundary of $\text{Spec } Q_C^T(R)$ is $\langle \chi_1 \otimes 1, \ldots, \chi_k \otimes 1 \rangle$ where the $\chi_i$ are any set of characters minimally generating the monoid $C$. Thus, ideal defining the $\hat{s}$-unstable locus in $X$ is $I := \langle \chi_1 \otimes 1, \ldots, \chi_k \otimes 1 \rangle \cap s(R)$. However, localizing $R$ at $I$ is the same as localizing $R$ at $\chi \otimes 1$ where $\chi \in C$ is any character in the relative interior of $C$, since we have $\chi = \sum a_i \chi_i$ with each $a_i > 0$. \hfill $\square$

**Remark 3.1.9.** As the reader may have already guessed, Example 3.1.7 in particular shows how to realize toric varieties via partial compactifications of torus actions on affine space.

Namely, if one specializes to the case where $R = k[x_1, \ldots, x_n]$ and so $\text{Spec } R = \mathbb{A}^n$, the above discussion reduces to torus actions on affine space. Here the GIT chambers agree with the cones in the GKZ-fan (see e.g. [CLS11, Section 14.4]), and the GIT quotients are the realization of toric varieties via the Cox construction. It is not difficult to give an explicit combinatorial description of a rational polyhedral cone describing the affine toric variety $Q_C^T$ where $C$ is any chamber of the GKZ-fan, although doing so would be too much of a digression, so we leave this as an exercise to the combinatorially minded reader.

**Example 3.1.10.** Let $W$ be a vector space of dimension $k$ and $V$ be a vector space of dimension $n$ with $k \leq n$. Set $G = \text{Gl}(W)$ and $X = \text{Hom}(W, V)$ so that $G$ acts on $X$ by right multiplication. Then $\tilde{X} = \text{End}(W) \times X$ is a partial compactification of $G \times X$. The projection and multiplication maps extend naturally to maps $\tilde{X} \to X$, and here $X^{ss}$ is identified with the set of matrices of full rank, so that $X^{ss}/G$ is the Grassmannian $\text{Gr}(n, k)$.

### 3.2. Functors from partial compactifications.

We now show how to associate functors between derived categories given the data of a partial compactification of an action. This is essentially done by taking the partial compactification itself as a Fourier-Mukai kernel.
Definition 3.2.1. Given \( \tilde{X} \) a partial compactification of a \( G \)-action on \( X \) in the sense of Definition 3.1.1, define

\[
Q_{X,G} := (\tilde{p} \times \tilde{s})_* \mathcal{O}_{\tilde{X}} \in D^b(\text{Qcoh}^{G\times G} X \times X),
\]

(3.3)

which is simply the derived push-forward of the structure sheaf under the extended action and projection maps.

Remark 3.2.2. If \( X \) is affine and \( G = \mathbb{G}_m \) then, since the functor \((\tilde{p} \times \tilde{s})_*\) is exact, Proposition 3.1.2 shows that \( Q(R) \) with its \( p\)-s-bimodule structure corresponds to the equivariant sheaf \( Q_{\text{Spec} R, \mathbb{G}_m} \).

However, as suggested by the examples of flops from Section 2.2, we are not literally interested in endofunctors \( D(\text{Qcoh}^G X) \to D(\text{Qcoh}^G X) \) as would obtained by taking \( Q_{X,G} \) as a kernel object. Rather, we are interested in functors \( D(\text{Qcoh}^{G\times G} X^{ss}) \to D(\text{Qcoh}^G X) \) where \( X^{ss} \) is the semistable locus from Definition 3.1.4.

Definition 3.2.3. Let \( \tilde{X} \) be a partial compactification of a \( G \)-action on \( X \). Then \( Q_{X,G}^{ss} \) denotes the quasi-coherent sheaf on \( X^{ss} \times X \) obtained by restricting \( Q_{X,G} \) from \( X \times X \). That is,

\[
Q_{X,G}^{ss} = (j \times \text{Id})^* Q
\]

(3.4)

where \( j : X^{ss} \to X \) is the inclusion.

For the purposes of the paper, we are primarily concerned with the case of a \( \mathbb{G}_m \)-action on an affine variety, for which we reserve different notation as follows.

Definition 3.2.4. If \( X = \text{Spec} R \) has a \( \mathbb{G}_m \)-action, then \( Q_+(R) \) denotes the quasi-coherent sheaf obtained by restricting the sheaf associated to \( Q(R) \) to the quasi-affine variety \( X^{ss} \times X = X \setminus X^- \times X \).

We will frequently abuse notation and drop the implicit reference to \( R \) and just write \( Q_+ \), and will also use

\[
Q_+ \in D^b(\text{Qcoh}^{\mathbb{G}_m \times \mathbb{G}_m} X^{ss} \times X)
\]

(3.5)

to denote the corresponding object of the derived category. Taking \( Q_+ \) as a Fourier-Mukai kernel, we have the functor

\[
\Phi_{Q_+} : D(\text{Qcoh}^{\mathbb{G}_m} X^{ss}) \to D(\text{Qcoh}^{\mathbb{G}_m} X).
\]

(3.6)

Remark 3.2.5. For an arbitrary ring \( R \), \( Q \) is a module, hence a bounded complex. Since it may not be perfect in general, tensor product with \( Q \) may not preserve boundedness unless \( R \) has finite Tor dimension. When \( X = \text{Spec} R \) is smooth, we will establish a “cohomological properness” result in Proposition 4.1.5 which in particular implies that the essential image lands in the derived category of complexes
with bounded and coherent coherent sheaves. We leave cohomological properness of $\Phi_Q$ beyond the smooth case for a fuller discussion.

We now show that this functor is automatically faithful.

**Proposition 3.2.6.** Let $\mathbb{G}_m$ act on $X = \text{Spec } R$. Let $Q_+$ be as in Equation (3.5). Then the Fourier-Mukai functor $\Phi_{Q_+}$ from Equation (3.6) is faithful.

**Proof.** This essentially follows from Lemma 2.1.14. In more detail, by the open affine cover of $X^{ss}$ given in Equation (3.2), one obtains the obvious cover of $X^{ss} \times X$. Lemma 2.1.14 says exactly that $Q$ restricts to $\Delta$ on each open affine subset of this cover, which by Lemma 2.1.5 is the Fourier-Mukai kernel of the identity functor. If $j : X^{ss} \to X$ denotes the inclusion and $j^* : D(\text{Qcoh } X) \to D(\text{Qcoh } X^{ss})$ the restriction functor, we thus have that $j^* \circ \Phi_{Q_+} = \text{Id}$, and the result follows. □

**Remark 3.2.7.** It is not much more difficult to prove a strengthened version of Proposition 3.2.6 valid for any kernel $Q_{X,G}^{ss}$ from Definition 3.2.3. Indeed, the other main result from this Section, Proposition 3.3.9, also admits such a generalization as well. Since we do not require such generality for the remainder of the article, though, we omit these generalizations for the sake of brevity.

Fullness of the functor $\Phi_{Q_+}$ is more subtle and addressing this is, in a sense, the main technical content of Section 3.3 (when $\text{Spec } R$ is smooth) and Section 5.1 (in general). We will see in Lemma 3.3.6 that fullness of the functor $\Phi_{Q_+}$ is intimately related to properties of the tensor product $Q(R) \otimes_p Q(R)$, which we now study.

The tensor product $Q(R) \otimes_p Q(R)$ inherits a $\mathbb{G}_m^3$-action from the $\mathbb{G}_m^4$-action on $Q(R) \otimes_k Q(R)$. Explicitly, let $(a, b) \in \mathbb{Z}^2$ denote the weight of a homogenous element of $r \in Q(R)$. Then we have the following weights on homogenous elements of $Q(R) \otimes_p Q(R)$:

$$\deg (r \otimes 1) = (a, b, 0)$$

$$\deg (1 \otimes r) = (0, a, b).$$

The $\mathbb{G}_m^3$-action is such that the map

$$\text{Spec } Q(R) \times_{\text{Spec } R} \text{Spec } Q(R) \to \text{Spec } R \times \text{Spec } R$$

corresponding to $p \otimes s$ is equivariant for the projection $\mathbb{G}_m^3 \to \mathbb{G}_m^2$ onto the first and third factor. The next lemma relates $Q(R) \otimes_p Q(R)$ to $Q(R)$ directly after taking suitable invariants. In what follows, let
$Z \subset \mathbb{Z}^3$ be the inclusion into the middle factor. If $M$ is a $\mathbb{Z}^3$-graded module, we let $(M)_0$ denote the $\mathbb{Z}^2$-graded sub-module obtained by taking degree zero in the middle factor. Likewise, if $f : M \to N$ is a map of $\mathbb{Z}^3$-graded modules, $(f_0) : (M)_0 \to (N)_0$ is the corresponding restriction.

**Lemma 3.2.8.** We have a commutative diagram

$$
\begin{array}{ccc}
(Q(R) \otimes \Delta(R))_0 & \xrightarrow{(1 \otimes \eta)_0} & (Q(R) \otimes \pi \Delta(R))_0 \\
\downarrow{(\eta \otimes 1)_0} & & \downarrow{\sim} \\
(\Delta(R) \otimes \delta \Delta(R))_0 & \xrightarrow{\sim} & Q(R)
\end{array}
$$

**Proof.** Both the top and bottom map take $a \otimes b$ to the degree zero piece inside $\Delta(R) \otimes \Delta(R)$ which happens to land in $Q(R)$. □

**Definition 3.2.9.** Following Lemma 3.2.8, we set $\rho_R$ to be the map

$$
\rho_R : (Q(R) \otimes \Delta(R))_0 \to Q(R)
$$

given by either $(1 \otimes \eta)_0$ or equivalently $(\eta \otimes 1)_0$.

In very specific situations, the map $\rho_R$ may be an isomorphism.

**Lemma 3.2.10.** Let $R$ be an object of $\mathbb{CR}^\text{gm}_k$ and assume that either the weights of $R$ are all non-positive or all non-negative. Then $\rho_R$ is an isomorphism.

**Proof.** Assume for simplicity that all the weights are non-negative; the proof for non-positive weights is similar. Using Lemma 3.2.8, it suffices to show that $(\eta \otimes 1)_0$ is an isomorphism. By Example 2.1.8, $Q(R) \cong R[u]$ so that $Q(R)$ is flat over $R$ via $p$. We first note that injectivity of $(\eta \otimes 1)_0$ then follows from flatness and because the functor of invariants is exact. To demonstrate surjectivity, we have to exhibit elements of $Q(R) \otimes \Delta(R)$ of middle degree zero that map to $\sigma(r), \pi(r)$, and $u$. They are, respectively, $1 \otimes \sigma(r), \pi(r) \otimes 1$, and $u \otimes u$. □

Note that in the above lemma, one has $Q(R) \otimes \Delta(R) \cong R[u,v]$, and the gradings from Remark 2.1.2 immediately imply that $(Q(R) \otimes \Delta(R))_0 \cong Q(R)$ as $R$-bimodules. The above lemma shows that $\rho_R$ indeed implements this isomorphism.

**Example 3.2.11.** However, the map $\rho_R$ is not an isomorphism in general. An elementary counterexample is $R = k[x,y]/(xy)$ where $[x] = 1$ and $[y] = -1$. Letting $z = yu^{-1}$, one has

$$
Q(R) \cong k[x,z,u]/(xz)
$$
and
\[ Q(R) \otimes_R Q(R) \cong k[x', z', u, u'] / (xz'u'u''). \]
The element \( x \otimes z' \) has middle degree 0 in \( Q(R) \otimes_R Q(R) \) and is sent to 0 in \( Q(R) \) under \( \rho_R \). In Section 5.2 we will remedy such a failure by deriving \( Q \), and will revisit this particular example again in Example 5.2.3.

**Remark 3.2.12.** Note that in the above example \( \text{Spec } R \) is singular. We will see in Lemma 4.2.5 that \( \rho_R \) is an isomorphism whenever \( \text{Spec } R \) is smooth.

**Remark 3.2.13.** Intuitively, the property of \( \rho_R \) being an isomorphism is similar to the characterization of derived open immersions as being finitely-presented ring maps \( f : A \to B \) such that \( B \otimes_A B \cong B \), see e.g. [TV08, Lemma 2.1.6]. In particular, this suggests we should derive the tensor product in \( Q(R) \otimes_R Q(R) \) to obtain a fully-faithful functor. When \( \text{Spec } R \) is smooth, we will see in Proposition 4.2.6 that this tensor product is automatically derived (i.e. its higher Tor’s vanish). For singular cases we will implicitly derive this tensor product during the course of Section 5.1 by deriving the \( Q \) functor itself.

### 3.3. Bousfield localizations

In this section we address the fullness of the functor \( \Phi_{Q_*} \) from Equation (3.6). Indeed, we will show more and in Proposition 3.3.9 exhibit a semi-orthogonal decomposition of \( D(Q\text{coh}^G_{\text{Spec } R}) \) such that \( \Phi_{Q_*} \) gives the inclusion of one of the factors. This semi-orthogonal decomposition will come from a Bousfield localization.

**Definition 3.3.1.** Let \( \mathcal{T} \) be a triangulated category. A Bousfield localization is an exact endofunctor \( L : \mathcal{T} \to \mathcal{T} \) equipped with a natural transformation \( \delta : \text{Id}_\mathcal{T} \to L \) such that:

a) \( L\delta = \delta L \) and

b) \( L\delta : L \to L^2 \) is invertible.

If instead we have an endofunctor \( C : \mathcal{T} \to \mathcal{T} \) equipped with a natural transformation \( \epsilon : C \to 1 \) such that

a) \( C\epsilon = \epsilon C \) and

b) \( C\epsilon : C^2 \to C \) is invertible,
then one calls \( C \) a Bousfield colocalization.

We refer to [Kra12, Section 4] for background on Bousfield localizations and colocalizations for triangulated categories.
Remark 3.3.2. If \( P, P' \in D^b(\text{mod}^G \mathbb{Z}_n R \otimes_k R) \) and \( \delta : P \to P' \) is any map, then one can easily check that \( \Phi_P(\delta)(A) = \delta(\Phi_P(A)) \) for any \( A \in D(\text{Mod}^G \mathbb{Z}_n R) \). This means that the first condition for being a Bousfield localization or colocalization is automatically satisfied in the setting of Fourier-Mukai functors, provided has a morphism \( \delta : \Delta \to P \) or \( \epsilon : P \to \Delta \) (recall \( \Delta \) is the kernel of the equivariant identity functor).

Definition 3.3.3. Suppose we have maps of endofunctors

\[
C \xrightarrow{\epsilon} \text{Id}_T \xrightarrow{\delta} L
\]

of a triangulated category \( T \) such that

\[
C_x \xrightarrow{\epsilon_x} x \xrightarrow{\delta_x} Lx
\]

is an exact triangle for any object \( x \). Then \( C \to \text{Id}_T \to L \) is called a a Bousfield triangle for \( T \) if any of the following equivalent conditions are satisfied:

(i) \( L \) is Bousfield localization and \( C(\epsilon_x) = \epsilon_{Cx} \),

(ii) \( C \) is a Bousfield colocalization and \( L(\delta_x) = \delta_{Lx} \),

(iii) \( L \) is Bousfield localization and \( C \) is a Bousfield colocalization.

Let us prove the equivalence of the three conditions in the definition.

Proof. i) \( \Rightarrow \) ii): Suppose \( L \) is Bousfield localization. Set \( x = Ly \). Then we get a triangle

\[
CLy \xrightarrow{\epsilon_{Ly}} Ly \xrightarrow{\delta_{Ly}} L^2y
\]

and the map \( Ly \to L^2y \) is an isomorphism. Therefore, \( CLy = 0 \). Now, consider the triangle

\[
C^2y \xrightarrow{C(\epsilon_y)} Cy \xrightarrow{C(\delta_y)} CLy.
\]

Since \( CLy = 0 \) the first map is an isomorphism as desired. ii) \( \Rightarrow \) i) is by symmetry. As i) is equivalent to ii), it is obvious that they are both equivalent to iii). \( \square \)

Lemma 3.3.4. Let \( C \to \text{Id}_T \to L \) be a Bousfield triangle for a triangulated category \( T \). Then there is a weak semi-orthogonal decomposition

\[
T = \langle \text{Im} L, \text{Im} C \rangle.
\]

Here \( \text{Im} \) denotes the essential image.

Proof. By definition, for any object \( x \) of \( T \), we have a triangle

\[
Cx \to x \to Lx
\]

in \( T \). Let \( f : Cx \to Ly \) be any map. We have a commutative diagram

...
with the left vertical map an isomorphism since $C$ is a Bousfield colocalization. Since $L y \to L^2 y$ is also an isomorphism, its (co)cone $C Ly$ must be 0. Thus, up to composition with an isomorphism, $f = 0$. □

Lemma 3.3.5. Let $C_1 \xrightarrow{c_1} 1 \xrightarrow{\delta_1} L_1$ and $C_2 \xrightarrow{c_2} 1 \xrightarrow{\delta_2} L_2$ be Bousfield triangles for a triangulated category $\mathcal{T}$ such that $L_1 C_2 \xrightarrow{L_1(c_2)} L_1$ is an isomorphism. Then there is a weak semi-orthogonal decomposition

$$\mathcal{T} = \langle \text{Im } C_2 \circ L_1, \text{Im } C_2 \circ C_1, \text{Im } L_2 \rangle.$$ 

This induces a fully-faithful functor

$$F : \mathcal{T} / \text{Im } C_1 \to \mathcal{T}.$$ 

Proof. For the first statement, by Lemma 3.3.4 we only need to further decompose $\text{Im } Q$ in Equation (3.11). From Lemma 3.3.4 and Lemma 3.3.8, we know that for any object $M \in \mathcal{T}$ there is exact triangle

$$C_2(C_1(M)) \to C_2(M) \to C_2(L_1(M))$$

so we only need to check semi-orthogonality. Let $N \in \mathcal{T}$ be another object and

$$f : C_2(C_1(M)) \to C_2(L_1(N))$$

be any morphism. Using the semi-orthogonality of $\text{Im } C_2$ and $\text{Im } L_2$, $f$ corresponds uniquely to a map

$$g : C_2(C_1(M)) \to L_1(N).$$

We have a commutative diagram

$$\begin{array}{ccc}
C_2(C_1(M)) & \xrightarrow{g} & L_1(N) \\
\downarrow & & \downarrow \\
L_1(C_2(C_1(M))) & \xrightarrow{L_1 g} & L_1^2(N)
\end{array}$$

Next, we note that

$$L_1(C_2(C_1(M))) \cong L_1(C_1(M))$$

by assumption

$$\cong 0$$

since $L_1 C_1 = 0$. 


Hence \( g = 0 \) and therefore \( f = 0 \).

For the second part of the statement, apply [Orl09, Lemma 1.4] (see also Proposition 4.9.1 of [Kra12]) to the case \( D = \text{Im} C_2, \mathcal{N} = \text{Im} C_2 C_1 \). This gives an equivalence

\[
\text{Im} C_2 L_1 \cong \text{Im} C_2 / \text{Im} C_2 C_1.
\]

Now apply [Orl09, Lemma 1.1] with \( D = \text{Im} C_2, D' = \mathcal{T}, \mathcal{N} = \text{Im} C_2 C_1 \), and \( \mathcal{N}' = \text{Im} C_1 \). This gives an equivalence

\[
\text{Im} C_2 / \text{Im} C_2 C_1 \cong \mathcal{T} / \text{Im} C_1.
\]

Tracing through these equivalences, we have an equivalence

\[
\mathcal{T} / \text{Im} C_1 \cong \text{Im} C_2 L_1.
\]

which induces the fully-faithful functor \( F \).

We now show that, under certain conditions, the exact triangle

\[
Q \xrightarrow{\eta_R} \Delta \rightarrow \text{cone} \eta_R \rightarrow Q[1]
\]

in \( D^b(\text{mod}^{G_m} R \otimes_k R) \) yields a Bousfield triangle for the associated Fourier-Mukai transforms.

**Lemma 3.3.6.** Let \( R \) be an object of \( \mathcal{C}R^G_m \). Then the triangle of functors

\[
\Phi_Q \xrightarrow{\eta} \text{Id} \rightarrow \Phi_{\text{cone} \eta_R}
\]

is a Bousfield triangle if

a) The map \( \rho_R : (Q(R)_s \otimes_p Q(R))_0 \rightarrow Q(R) \) is an isomorphism, and

b) \( \text{Tor}^R_i(Q(R)_p, Q(R)_s) = 0 \) for all \( i > 0 \), where the subscripts on \( Q(R) \) denote the \( R \)-module structures given by \( p \) or \( s \) respectively.

**Proof.** By Remark 3.3.2, the functors form a Bousfield triangle if and only if

\[
(Q_s \otimes_p Q)_0 \xrightarrow{Q(\eta)} Q
\]

is an isomorphism. Since \( \text{Tor}^R_i(Q(R)_p, Q(R)_s) = 0 \) for all \( i > 0 \), \( (Q_s \otimes_p Q)_0 = (Q_s \otimes_p Q)_0 \). Therefore, the map \( Q(\eta) \) is just \( \rho_R \), which is an isomorphism by assumption.

**Definition 3.3.7.** If \( R \) is an object of \( \mathcal{C}R^G_m \) is such that conditions a) and b) appearing in Lemma 3.3.6 are both satisfied, we say that \( R \) has Property \( P \).

In particular, Lemma 3.3.4 says that if \( R \) has Property \( P \), then there is a semi-orthogonal decomposition

\[
D(\text{Qcoh}^{G_m} \text{Spec} R) = \langle \text{Im} \Phi_Q, \text{Im} \Phi_{\text{cone} \eta} \rangle.
\]
We can further refine this semi-orthogonal decomposition using Lemma 3.3.5 by considering Bousfield (co)localizations coming from local cohomology. Let \( \Gamma^{+} \) denote the local cohomology of \( X = \text{Spec} \, R \) along \( V(I^+) = X^- \).

**Lemma 3.3.8.** There is a Bousfield triangle

\[
\Gamma^{+} \to \text{Id} \to J^{+} \tag{3.14}
\]

for \( D^b(\text{Qcoh}^{\mathbb{G}_m} \text{Spec} \, R) \) where

\[
J^{+} := j_* \circ j^* \tag{3.15}
\]

and \( j : \text{Spec} \, R \setminus V(I^+) \to \text{Spec} \, R \) is the inclusion.

**Proof.** This is standard. See Example 1.2 of [HR17] which applies to algebraic stacks and, in particular, our situation. \( \square \)

We then refine our semi-orthogonal decomposition from Equation (3.13) as follows.

**Proposition 3.3.9.** Let \( R \) be an object of \( \mathcal{CR}^{\mathbb{G}_m}_k \) which has Property \( P \). Then there is a semi-orthogonal decomposition

\[
D(\text{Qcoh}^{\mathbb{G}_m} \text{Spec} \, R) = \langle \text{Im} \, \Phi_Q \circ J_+, \text{Im} \, \Phi_Q \circ \Gamma^+, \text{Im} \, \Phi_{\text{cone}} \rangle.
\]

Furthermore, the functor

\[
\Phi_{Q_*} : D(\text{Qcoh}^{\mathbb{G}_m} \text{Spec} \, R \setminus V(I^+)) \to D(\text{Qcoh}^{\mathbb{G}_m} \text{Spec} \, R)
\]

is fully-faithful.

**Proof.** We note that \( J_* \circ \Phi_Q = J_* \) since inverting any \( r \in R \) of positive weight trivializes \( Q \) by Lemma 2.1.14. Therefore, we may apply Lemma 3.3.5 to obtain the result, noting that the map \( F \) in that lemma is exactly \( \Phi_{Q_*} \) in this case. \( \square \)

### 4. The smooth case

We now study the faithful functor

\[
\Phi_{Q_*} : D^b(\text{coh}^{\mathbb{G}_m} X^-) \to D^b(\text{Qcoh}^{\mathbb{G}_m} X) \tag{4.1}
\]

from Equation (3.6) in the case where \( X = \text{Spec} \, R \) is a smooth affine scheme with \( \mathbb{G}_m \)-action. As mentioned in the introduction, the smoothness hypothesis is unreasonably restrictive for the demands of birational geometry, although it does subsume many simple examples and, not surprisingly, permits some dramatic simplifications compared to the general case that we will pursue in Section 5.2.
4.1. Affine space. We first consider the case where $X = \mathbb{A}^n$; here we
study $\Phi_Q$ via direct calculations. We begin by showing in two lemmas
that $X = \mathbb{A}^n$ (equipped with any weights) has Property $P$, i.e. satisfies
the hypotheses of Lemma 3.3.6.

Lemma 4.1.1. Let $R = k[x_1, \ldots, x_n]$ where the $x_i$ are equipped with any
degrees. Then the map $\rho_R$ from Definition 3.2.9 is an isomorphism.

Proof. This can be verified by an explicit calculation using the gradings
on $Q(R) \otimes_p Q(R)$ from Equations (3.8) and (3.9). However, let us
instead give a quick proof more in the spirit of arguments we will use
later in Section 5.2.

We induct on $n$. First, if $n = 1$, i.e. $R = k[x_1]$, then the statement of the Lemma follows directly from Lemma 3.2.10 (Alternately, note
that the statement for $n = 0$ is trivial). We then need to show that
for any object $S$ of $\mathbb{C}^{\text{Grm}}_k$, if $\rho_S$ is an isomorphism then $\rho_{S[x]}$ is also an
isomorphism (where $x$ is given any weight). Indeed, by Lemma 2.1.9
\[ Q(S[x]) \otimes Q(S[x]) \cong Q(S)[y] \otimes Q(S)[y] \]
and it’s easily verified that
\[ (Q(S)[y] \otimes Q(S)[y])_0 \cong (Q(S) \otimes Q(S))_0[y]. \]
Then, using Lemma 2.1.9 again,
\[ (Q(S) \otimes Q(S))_0[y] = Q(S)[y] = Q(S[x]). \]

Lemma 4.1.2. Let $R = k[x_1, \ldots, x_n]$ where the $x_i$ are equipped with
any degrees. Then $\text{Tor}^R_i(Q(R)_p, Q(R)_s) = 0$ for $i > 0$.

Proof. In the notation of Example 2.1.7, we have
\[ Q(R) = k[x^+, x^-, y^+, y^-]/(y^+ - u^a x^+, x^- - u^b y^-) \]
which exhibits $Q(R)$ as a complete intersection ring inside $R \otimes_k R[u]$ where
\[ R \otimes_k R = k[x^+, x^-, y^+, y^-]. \]
Since $R \otimes_k R[u]$ is a flat $R \otimes_k R$-module, the Koszul resolution
\[ K^\bullet_{R \otimes_k R[u]}(y^+ - u^a x^+, x^- - u^b y^-). \]
of $Q(R)$ as $R \otimes_k R[u]$-module gives a flat resolution of $Q(R)_s$ as an
$R$-module. Hence,
\[ Q(R)_p \otimes_R Q(R)_s = Q(R)_p \otimes_R K^\bullet_{R \otimes_k R[u]}(y^+ - u^a x^+, x^- - u^b y^-) \]
\[ = K^\bullet_{Q(R) \otimes_k R[u]}(y^+ - u^a x^+, x^- - u^b y^-). \]
where we identify
\[ Q(R) \otimes_k R[u] = k[s^+, s^-, y^+, y^-, u, v]. \]

Since \( y_i^+ = u^a_i v_i^a_i s_i^+, s_i^- = u^{-b_i} y_i^- \) is a regular sequence (it just solves out certain variables), this complex has no higher homology. \( \square \)

Combining Lemmas \[4.1.1\] and \[4.1.2\] with Proposition \[3.3.9\] immediately gives the following.

**Proposition 4.1.3.** Let \( \mathbb{G}_m \) act on \( \mathbb{A}^n = \text{Spec} R \). Then
\[ \Phi_{Q_*} : D^b(\text{coh} \mathbb{G}_m \mathbb{A}^n \setminus V(I^+)) \to D^b(\text{Qcoh} \mathbb{G}_m \mathbb{A}^n) \]
is fully-faithful.

We now study the essential image of \( \Phi_{Q_*} \) in the affine space case and show that it coincides with the subcategory generated by weights in the interval \((\mu, 0]\) where \(-\mu\) is the sum of the positive weights of the \( \mathbb{G}_m \)-action. To this end, we first establish a useful set of related generating objects.

**Lemma 4.1.4.** Equip \( R = k[x_1, \ldots, x_n] \) with any \( \mathbb{G}_m \)-action and set
\[ \mu = - \sum_{\deg x_j > 0} \deg x_j. \] (4.2)

Then \( D^b(\text{coh} \mathbb{G}_m \mathbb{A}^n \setminus V(I^+)) \) is generated by objects of the form \( j^* \mathcal{O}(i) \) where \( i \in \mathbb{N} \) is in \((\mu, 0]\). Here \( j : \mathbb{A}^n \setminus V(I^+) \to \mathbb{A}^n \) is the inclusion and \( R(i) \) denotes structure sheaf of \( \mathbb{A}^n = \text{Spec} R \) equipped with weight \( i \) with respect to the \( \mathbb{G}_m \)-action.

**Proof.** The Koszul complex on \( \{ x_j \mid \deg x_j > 0 \} \) is acyclic on \( \mathbb{A}^n \setminus V(I^+) \), so the object \( j^* \mathcal{O}(\mu) \) is generated by such objects. Similarly, tensoring the Koszul complex by some \( j^* \mathcal{O}(t) \), we get that \( j^* \mathcal{O}(\mu + t) \) is generated by those \( \mathcal{O}(i) \) with \( \mu + t < i \leq t \). By induction, we can thus generate any \( j^* \mathcal{O}(k) \) with \( k \leq 0 \) by the objects \( j^* \mathcal{O}(i) \) with \( i \in (\mu, 0]\). Similarly, we can generate an object of the form \( j^* \mathcal{O}(i) \) with \( i > 0 \) by twisting the Koszul complex by \( j^* \mathcal{O}(i) \) and peeling off the top term. \( \square \)

**Proposition 4.1.5.** With notation as above, the essential image of the equivariant Fourier-Mukai functor
\[ \Phi_{Q_*} : D^b(\text{coh} \mathbb{G}_m \mathbb{A}^n \setminus V(I^+)) \to D^b(\text{Qcoh} \mathbb{G}_m \mathbb{A}^n) \]
is the full subcategory generated by those \( R(i) \) such that \( i \in (\mu, 0]\).
Proof. Since $\Phi_{Q_s}$ is fully-faithful by Proposition 4.1.3 Lemma 4.1.4 tells us that the essential image of $\Phi_{Q_s}$ is equivalent to the full subcategory of $\mathbb{D}^b\text{Qcoh}(A^n)$ generated by the objects $\Phi_{Q_s}(j^*\mathcal{O}(i))$ with $i \in (\mu, 0]$. Indeed, we will show that $\Phi_{Q_s}(j^*\mathcal{O}(i)) = R(i)$ for such objects, thus proving the Proposition.

With notation as in Example 2.1.7 write $R = k[x^+, x^-]$ and
\[ Q(R) = R \otimes_k R[u]/(y^+ - u^a x^+, x^- - u^{-b} y^-) = k[x^+, y^-, u]. \]

To push forward an object from $A^n \setminus V(x^+) \times A^n$ to $A^n$, we let $C$ be the Čech resolution of $A^n \setminus V(x^+) \times A^n$ given by inverting the $x_i^+$'s. We then compute:
\[
\Phi_{Q_s}(\mathcal{O}(i)) = (C \otimes_{R \otimes_k R} Q(R)_s \otimes_R R(i, 0))(0,*) \\
= C \otimes_{R \otimes_k R} (Q(R)_s)(i,*) \\
= C \otimes_{R \otimes_k R} R \otimes_k R[u]/(y^+_i - u^{a_i} x^+_i, x^-_i - u^{-b_i} y^-_i).
\]

Here, the notation $(i, *)$ refers to the $\mathbb{Z}^2$ grading $C \otimes_R M \otimes_R Q(R)$ inherits as a $R \otimes_k R$-module, and $(i, *)$ means the sum over all degree $(i, d)$ pieces for all $d \in \mathbb{Z}$ (this is the degree restriction which corresponds to taking $\mathbb{G}_m$-invariants in the equivariant Fourier-Mukai transform).

We thus have reduced the problem to computing the Čech cohomology of $Q(R) = k[x^+, y^-, u]$ with deg $x_i^+ = (a_i, 0)$, deg $y_i^- = (0, b_i)$, and deg $u = (-1, 1)$. This is a well known computation. Namely, one can check that the cohomology vanishes when the cohomological degree is in $(\mu, 0)$, and in degree zero is given by $u^{-1}k[u^a x^+, y^-] = R(i)$.

$$\square$$

Remark 4.1.6. The above proof also exhibits an isomorphism of functors
\[ \Phi_{Q_s} \circ j^* = \text{Id} \]
when restricted to the full subcategory generated by those $R(i)$ such that $i \in (\mu, 0]$. Another consequence is that the essential image of $\Phi_{Q_s}$ actually lies in $\mathbb{D}^b\text{coh}(A^n)$, as promised in Remark 3.2.5.

4.2. The smooth affine case in general. We now consider the case where a ring $R$ in $\mathbb{CR}_{k}^{\mathbb{G}_m}$ is such that $\text{Spec } R$ is smooth. To make reductions to affine space case, we will repeatedly make use of the Luna Slice Theorem, to which we refer to [Dre04] for expository background or [Lun73] for the original result. Accordingly, from this point on we assume $k$ is a field. Let us state a version at the level of generality we require; in particular we will only require the theorem near points on the fixed locus, which simplifies the statement.
Proposition 4.2.1. Let $\mathbb{G}_m$ act on a smooth affine variety $X = \text{Spec } R$, and let $x \in X$ be a fixed point for the action. Then there exists a $\mathbb{G}_m$-invariant affine subvariety $V \subseteq X$ containing $x$, called the slice at $x$, and a diagram

$$
\begin{array}{c}
V \\
\downarrow g \quad \downarrow f \\
T_x X & \to & X
\end{array}
$$

where the maps $f, g$ are strongly étale (see Remark 4.2.2 below for a reminder of this definition), and $T_x X$ denotes the tangent space to $X$ at $x$.

Moreover, it can be arranged so that $V = \text{Spec } R_r$ where $\deg r = 0$ and that the image of $V$ under $g$ is $\text{Spec } T_t$ where $\deg t = 0$, where we have written $T_x X := \text{Spec } T$.

Let us prove that our version indeed does follow from the version in [Dre04, Theorems 5.3 and 5.4].

Proof. The only non-trivial differences between 4.2.1 and [Dre04, Theorems 5.3 and 5.4] are that we require $g$ to be strongly étale and not just strongly étale onto its image, and that we may take $V = \text{Spec } R_r$ as claimed.

Taking $V$ from [Dre04, Theorems 5.3 and 5.4], we may cover $V$ by open subsets of the form $\text{Spec } R_a$. If some $\text{Spec } R_a$ contains the fixed point $x$, we must have $\deg r = 0$. Since strongly étale morphisms base change under localization by degree zero elements we may replace $V$ by $\text{Spec } R_a$ with $\deg a = 0$ and we still know that $g$ is strongly étale onto its image.

Now, similarly cover $\text{im } g$ by open subsets of the form $\text{Spec } T_t$. Once again, in order to contain the origin (the image of the fixed point), we must have $\deg t = 0$. Hence, if we replace $\text{Spec } R_a$ by $\text{Spec } R_{\text{deg } t}$, then $g$ is a strongly étale map to $\text{Spec } T_t$. Furthermore as $\deg t = 0$, the inclusion into $\text{Spec } T$ is strongly étale as well. Therefore, $g$, as a composition, is strongly étale.

Remark 4.2.2. Let $G$ be any reductive group. Recall that if $\phi : R \to S$ is a map of rings, such that $\hat{\phi} : \text{Spec } S \to \text{Spec } R$ is equivariant, one says that $\phi$ is strongly étale if the induced map on invariant subrings

$$
\phi_G : R^G \to S^G
$$

is étale and there is an isomorphism $S \cong R \otimes_{R^G} S^G$ in $\text{CR}_G$ where the tensor product is taken with respect to $\phi$ and the inclusion $R^G \subseteq R$. 

We now show that the functor $Q$ satisfies base change for strongly étale ring maps.

**Proposition 4.2.3.** Let $f : R \to S$ be a strongly étale map of rings with $\mathbb{G}_m$-actions. Then,

- a) there is an isomorphism of bimodules
  $$S_f \otimes_S Q(R) \cong Q(R) \otimes_f S \cong Q(S),$$
  
- b) and an isomorphism of functors,
  $$f^* \circ \Phi_{Q(R)} \cong \Phi_{Q(S)} \circ f^*.$$

**Proof.** We first prove a). We have the obvious map

$$Q(R) \otimes_R S \to Q(S)$$

given by $Q(f) \otimes 1$; we claim this map is an isomorphism. This map sits as the top arrow in the commutative diagram

$$
\begin{array}{ccc}
Q(R) \otimes f S & \longrightarrow & Q(S) \\
\downarrow & & \downarrow \\
R[u, u^{-1}] \otimes f S & \longrightarrow & S[u, u^{-1}].
\end{array}
$$

and the left vertical arrow is injective since $f : R \to S$ is étale and thus, in particular, is flat. It follows that $Q(f) \otimes 1$ is also injective.

To demonstrate surjectivity, we use the isomorphism $S \cong R \otimes_{R^G} S^G$ afforded by the strongly étale condition. Namely, given $s \in S$, we write $s = \sum_i r_i \otimes s_i$ where each $r_i \in R$ and $s_i \in S^{G_m}$. Let $\sigma : S \to S[u, u^{-1}]$ be the co-action ring map for the $\mathbb{G}_m$-action on $S$. Then

$$\sigma(s) = \sigma(\sum_i r_i \otimes s_i) = \sum_i \sigma(r_i) \otimes s_i.$$

Now, write each $r_i \in R$ as a sum of homogenous elements: $r_i = \sum r_i^j$ where $\deg(r_i^j) := n_{ij}$. Then

$$\sigma(r_i) = \sum_j r_i^j u^{n_{ij}}$$

and so

$$\sigma(r_i) \otimes s_i = \sum_j r_i^j u^{n_{ij}} \otimes s_i = \sum_j u^{n_{ij}} \otimes f(r_i^j) s_i$$

since the tensor product is over $R$ via $f$ on the right. To show that $\sigma(s)$ is in the image of $Q(f) \otimes 1$, it suffices to show that if $u^k \in Q(R)$ for some $k \in \mathbb{Z}$, then $u^k \in Q(S)$. But this is clear as $Q(f) : Q(R) \to Q(S)$ is obtained, by definition, by restricting the map $R[u, u^{-1}] \to S[u, u^{-1}]$ induced by $f$. It is trivial that any $s \in S$ is in the image of $Q(f) \otimes 1$, and so we have shown that any element of the form $s$, $\sigma(s)$, and also the element $u$ are all in the image, but such elements generate $Q(S)$ by definition.
Statement b) is the following chain of equalities:

\[
f^*(\Phi_Q(R)(M)) = S \otimes_R (Q(R) \otimes_R M)_{(s,0)} \\
= S_0 \otimes_{R_0} (Q(R) \otimes_R M)_{(s,0)} \quad \text{since } f \text{ is strongly étale} \\
= (S \otimes_R Q(R) \otimes_R M)_{(s,0)} \quad \text{by part a)} \\
= (Q(S) \otimes_R M)_{(s,0)} \\
= \Phi_{Q(S)}(f^*M).
\]

\[\square\]

**Remark 4.2.4.** In the above proposition, it is necessary that the map \( f : R \to S \) is strongly étale and not just étale. For example, \( k[x] \to k[x, x^{-1}] \) with \( \deg x > 0 \) is an open immersion, but it is easy to verify that base change for \( Q \) does not hold.

**Lemma 4.2.5.** Let \( R \) be an object of \( \text{CR}^{G_m}_k \) such that \( \text{Spec} R \) is smooth. Then the map \( \rho_R \) from Definition 3.2.9 is an isomorphism.

**Proof.** It suffices to prove the map is locally an isomorphism. We know that \( Q(R) \cong \Delta(R) \) away from the contracting locus, hence \( \rho_R \) is an isomorphism over \( \text{Spec} R \backslash V(I^*) \). Now, for each point in the fixed locus one obtains \( \mathbb{G}_m \)-invariant affine open neighborhood produced by the Luna Slice theorem \[4.2.1\] These neighborhoods cover the contracting locus so it remains to check that \( \rho_R \) is an isomorphism upon restriction to each such neighborhood. But \( \rho_R \) respects base change for strongly étale morphisms by Proposition \[4.2.3\], so the Luna Slice Theorem reduces us to the case of affine space, which was Lemma \[4.1.1\]. \[\square\]

**Proposition 4.2.6.** Let \( R \) be an object of \( \text{CR}^{G_m}_k \) such that \( \text{Spec} R \) is smooth. Then \( Q(R) \) with its \( R \)-module structure induced by \( p \) is Tor independent of \( Q(R) \) with its \( R \)-module structure induced by \( s \). That is,

\[
\text{Tor}^R_p(Q(R)_p, Q(R)_s) = 0
\]

for all \( p > 0 \).

**Proof.** The proof is similar to that of the previous lemma. Namely, by Lemma \[2.1.14\] \( \Phi_{Q(R)} = \text{Id} \) away from \( X^- \), and since \( Q \) base changes under all maps in the Luna Slice Theorem by Proposition \[4.2.3\], we are reduced to the case where \( \text{Spec} R = \mathbb{A}^n \), where the statement was proved in Lemma \[4.1.2\]. \[\square\]
Definition 4.2.7. Let $R$ be an object of $\mathcal{CR}_{k}^{\mathbb{G}_{m}}$ and set $\mu$ to be the sum of the weights of the conormal bundle of $\text{Spec } R/I^{+} = X^{-}$ in $X = \text{Spec } R$. (Note that $X^{-}$ is smooth since $X = \text{Spec } R$ was assumed smooth, see e.g. [Dri13, Proposition 1.4.20].) Assume that the fixed locus is connected. The grade restriction window, denoted by $\mathcal{W}$, is the full subcategory of $D^b(\text{coh}^{\mathbb{G}_{m}} \text{Spec } R)$ consisting of objects $A$ such that for any fixed point $x$ and some affine étale slice $V = \text{Spec } S$ at $x$, the restriction

$$A \otimes_R S \in D^b(\text{coh}^{\mathbb{G}_{m}} \text{Spec } S)$$

is generated by $S(i)$ for $i \in (\mu, 0]$.

Lemma 4.2.8. Let $R$ be an object of $\mathcal{CR}_{k}^{\mathbb{G}_{m}}$ such that $X = \text{Spec } R$ is smooth. Assume that the fixed locus is connected. Then the essential image of Fourier-Mukai functor

$$\Phi_{Q_*} : D^b(\text{coh}^{\mathbb{G}_{m}} X \setminus V(I^+)) \to D^b(Q\text{coh}^{\mathbb{G}_{m}} X)$$

lies in $\mathcal{W} \subseteq D^b(\text{coh}^{\mathbb{G}_{m}} X)$. Furthermore, on $\mathcal{W}$ there is an isomorphism of functors

$$(\Phi_{Q_*} \circ j^*)|_{\mathcal{W}} = \text{Id}_{\mathcal{W}}$$

where $j : X \setminus V(I^+) \to X$ is the inclusion.

Proof. We know that $\Phi_{Q_*}$ satisfies base change for strongly étale morphisms by Proposition 4.2.3. Furthermore, since semi-stable loci are preserved under strongly étale morphisms ([SvdB16, Lemma 3.2.1]), $\Phi_{Q_*}$ also satisfies base change. Furthermore, since $\mathcal{W}$ is defined locally, to show that $\text{Im } \Phi_{Q_*} \subseteq \mathcal{W}$, it suffices to verify that the essential image of $\Phi_{Q_*}$ lands in $\mathcal{W}$ locally.

Cover $\text{Spec } R$ by open affine $\mathbb{G}_{m}$-invariant neighborhoods $V = \text{Spec } S$ of the fixed locus produced by the Luna Slice Theorem, and let $g : V \to T_x X := \text{Spec } T$ be the strongly étale map. If $S$ is any $\mathbb{G}_{m}$-invariant subvariety of $\text{Spec } R$, let $\mathcal{W}_L$ be the grade restriction window on Spec $L$ and $j_L : \text{Spec } L \setminus \text{Spec } L^{-} \to \text{Spec } L$ be the inclusion.

We know that $\Phi_{Q_*(T)}$ lands in $\mathcal{W}_T$ by Proposition 4.1.3. Furthermore, $D^b(\text{coh}^{\mathbb{G}_{m}} V \setminus V^{-})$ is generated by $j^*_S S(i) = g^* j^*_T T(i)$ for all $i \in \mathbb{Z}$. Hence, $\Phi_{Q_*(S)}$ lies in $\mathcal{W}_S$. Thus $\Phi_{Q_*}$ lands in $\mathcal{W}$ locally, as desired.

To prove the latter statement of the lemma, suppose $M \in \mathcal{W}$, then we can cover $\text{Spec } R$ by open affine $\mathbb{G}_{m}$-invariant neighborhoods $V = \text{Spec } S$ of the fixed locus produced by the Luna Slice Theorem where $M|_V$ is generated by $S(i)$ for $\mu < i \leq 0$, as above. We have that

$$g^*|_{\mathcal{W}_T} = g^* \circ (\Phi_{Q_*(T)} \circ j^*_T)|_{\mathcal{W}_T} = (\Phi_{Q_*(S)} \circ j^*_S)|_{\mathcal{W}_S} \circ g^*$$

by Proposition 4.1.3 by strongly étale base change.
Since the generators of \( W \) lie in the essential image of \( g^* \), this implies
\[
(\Phi_{Q*}(S) \circ j^*)|_{W} = \text{Id}_{W}
\]
Hence, we have shown this isomorphism locally on \( \text{Spec } R \).

Now, we know that \( j^*, j_* \) and \( \Phi_Q \) satisfy base change for strongly étale morphisms by Proposition 4.2.3. Furthermore the projection formula gives an isomorphism of functors
\[
\Phi_Q \circ j^* \cong \Phi_Q \circ j_* \circ j^*.
\]
Hence \( \Phi_Q \circ j^* \) satisfies base change for strongly étale morphisms. There are two natural morphisms
\[
\Phi_Q \longrightarrow \text{Id} \\
\Phi_Q \circ j_* \circ j^*
\]
The vertical arrow is the unit of the adjunction and the horizontal arrow comes from the map \( Q \rightarrow \Delta \). We have checked that both maps are (locally) isomorphisms on \( W \) and the result follows. \( \square \)

**Theorem 4.2.9.** Let \( \text{Spec } R \) be a smooth affine variety with a \( \mathbb{G}_m \)-action and connected fixed locus. The functor
\[
\Phi_{Q*} : D^b(\text{coh}^{\mathbb{G}_m} \text{Spec } R \setminus V(I^*)) \rightarrow W
\]
is an equivalence of categories.

**Proof.** Combining Lemmas 4.2.5 and 4.2.6 with Proposition 3.3.9 gives that \( \Phi_{Q*} \) is fully-faithful. Essential surjectivity follows immediately from the isomorphism
\[
(\Phi_{Q*} \circ j^*)|_{W} = \text{Id}_{W}
\]
which is part of Lemma 4.2.8. \( \square \)

**Remark 4.2.10.** The assumption of a connected fixed locus can be removed by putting more care into the definition of \( W \). Namely, one needs to keep track of the parameter \( \mu \) for each connected component of the fixed locus.

**Corollary 4.2.11.** Let \( \text{Spec } R \) be a smooth affine variety with a \( \mathbb{G}_m \)-action and connected fixed locus. Let \( \mu_* \) be the sum of the weights of the conormal bundle of \( \text{Spec } R/I^* \) in \( X = \text{Spec } R \). Define
\[
\Phi^\text{wc} := j^* \circ (- \otimes \mathcal{O}(-\mu_+ - 1)) \circ \Phi_{Q*}
\]
where \( j_* : \text{Spec } R \setminus V(I^-) \rightarrow \text{Spec } R \) is the inclusion. If \( \mu_* + \mu_- = 0 \) then
\[
\Phi^\text{wc} : D^b(\text{coh}^{\mathbb{G}_m} \text{Spec } R \setminus V(I^*)) \rightarrow D^b(\text{coh}^{\mathbb{G}_m} \text{Spec } R \setminus V(I^-))
\]
is an equivalence of categories.

Proof. By the previous theorem, $\Phi_Q^+$ gives an equivalence

$$D^b(\text{coh}\ G_m \text{Spec } R \setminus V(I^+)) \cong W^+.$$ 

On the other hand, we may regard $\text{Spec } R$ with the inverse $G_m$-action to produce an isomorphic stack. This exchanges $I^+$ with $I^-$ so we get an equivalence

$$D^b(\text{coh}\ G_m \text{Spec } R \setminus V(I^-)) \cong W^-.$$ 

Under these identifications, the assumption that $\mu_+ = \mu_-$ ensures that

$$W^+ \otimes \mathcal{O}(-\mu_+ - 1) = W^-.$$ 

The result follows as $j^*$ is the inverse to $\Phi_{Q_-}$ on $W^-$. \qed

5. The general case: homotopical methods

5.1. Deriving $Q$. We let $\text{sCR}_k^{G_m}$ denote the category of simplicial commutative rings with $G_m$-actions over $k$, i.e. the category of simplicial objects in the category of $\mathbb{Z}$-graded commutative rings. We may refer to a ring $R$ in $\text{CR}_k^{G_m}$ as an ordinary ring when it becomes necessary to emphasize that it is not a simplicial ring. For simplicial sets, we will denote $n$-simplices by $\Delta[n]$, the union of their faces by $\partial \Delta[n]$, and let $\Lambda^i[n]$ denote the the simplicial horn, which we recall is the union of all of the faces of $\Delta[n]$ except for the $i$-th face.

Remark 5.1.1. We will denote elements of $\text{sCR}_k^{G_m}$ by $R_*$, and for $n \in \mathbb{Z}$ we denote the ordinary ring structure at level $n$ with respect to the underlying simplicial set by $R_n$. The reader is allowed to be concerned about a potential clash in notation with the grading on a $\mathbb{Z}$-graded ring (for example, each $R_n$ as above is itself $\mathbb{Z}$-graded). We have made efforts to ensure that no explicit $\mathbb{Z}$-gradings are referred to in this section, and so the reader should henceforth assume that all subscripts refer to a simplicial level, unless told otherwise.

Recall that $\text{sCR}_k^{G_m}$ has distinguished objects which play the role of free objects. Namely, let

$$F : \text{sSet} \to \text{sCR}_k$$

be the left adjoint to the forgetful functor from simplicial rings to simplicial sets (here $F$ stands for “free” and not “forget”). Given a simplicial set $X$ and a weight $a \in \mathbb{Z}$, we likewise have an object of $\text{sCR}_k^{G_m}$ denoted $F(X)^a$, where we declare the degrees of the generators (with respect to the $\mathbb{Z}$-grading) of $F(X_n)^a$ to all be $a$. 
Proposition 5.1.2. The category $\text{sCR}^\text{Gm}_k$ possesses a simplicial cofibrantly generated model structure where:

- the generating cofibrations are the maps $F(\partial \Delta[n])^a \to F(\Delta[n])^a$ for $a \in \mathbb{Z}$ and $n \geq 0$.
- The generating trivial cofibrations are $F(\Lambda^r[n])^a \to F(\Delta[n])^a$ for $a \in \mathbb{Z}$, $n \geq 0$, and $0 \leq r \leq n$.
- The weak equivalences are those of the underlying simplicial sets, i.e. a map is a weak equivalence if and only if it is a weak equivalence after applying the forgetful functor to simplicial sets.

Proof. This seems to be well-known. For example, it is a special case of [DHK97, Theorem 9.8] (see the discussion above the theorem for how their result specializes to $\text{sCR}^\text{Gm}_k$). It is also [GJ99, Example 5.10].

Remark 5.1.3. A similar statement holds for $\text{sCR}^\text{Gm}_k[u]$.

The main property we will need is the following.

Corollary 5.1.4. Any cofibrant object in $\text{sCR}^\text{Gm}_k$ is a retract of a sequential colimit of pushouts along the generating cofibrations $F(\partial \Delta[n])^a \to F(\Delta[n])^a$.

Proof. For cofibrantly generated model categories in general, this fact is known as the small object argument, see e.g. [Hov01, Theorem, 2.1.14 and Corollary 2.1.15].

By applying the functor $Q$ from Section 2.1 level-wise and to all face and degeneracy maps, we obtain a functor which we also denote by $Q$. More precisely, by viewing $\text{sCR}^\text{Gm}_k$ as the functor category from the opposite of the simplex category to $\text{CR}^\text{Gm}_k$ (and likewise for $\text{CR}^\text{Gm}_k[u]$), we obtain an induced functor

$$Q : \text{sCR}^\text{Gm}_k \to \text{sCR}^\text{Gm}_k[u],$$

which by abuse of notation we also denote $Q$. For any $R_\bullet$ an object of $\text{sCR}^\text{Gm}_k$, the object $Q(R_\bullet)$ comes equipped with action and projection simplicial ring maps

$$p, s : R_\bullet \to Q(R_\bullet)$$

which agree level-wise with the ordinary action and projection ring maps.

Lemma 5.1.5. The functor $Q : \text{sCR}^\text{Gm}_k \to \text{sCR}^\text{Gm}_k[u]$ is left Quillen.

Proof. We first show that $Q$ preserves cofibrations and trivial cofibrations. Let $f\text{CR}_k^a$ denote the full subcategory of $\text{CR}^\text{Gm}_k$ consisting of free commutative $k$-algebras whose generating set of elements has weight
a. By Example 2.1.8, \( Q(R_n) \cong R_n[u] \) for each \( n \) with respect to one of the two module structures depending on the sign of the weights, i.e.

\[
Q(R_\bullet) \cong R_\bullet[u] \tag{5.3}
\]

for any object \( R_\bullet \in \text{fCR}_k^n \) where \([u] = (a,0)\) if \( a \geq 0 \) and \([u] = (0,a)\) of \( a < 0 \). Given this, we see that

\[
QF_k(X)^a \cong \begin{cases} F_k[u](X)^{(a,0)} & \text{if } a \geq 0 \\ F_k[u](X)^{(0,a)} & \text{if } a < 0 \end{cases}
\]

were \( F_k, F_k[u] \) denote the free functors for simplicial commutative \( k \) and \( k[u] \)-algebras respectively. In otherwords there is an isomorphism of functors

\[
QF_k \cong F_k[u].
\]

It follows that \( Q \) preserves the generating cofibrations and the generating trivial cofibrations from 5.1.2. By [Hov01, Lemma 2.1.20], this implies that that \( Q \) preserves all cofibrations and trivial cofibrations.

It remains to show that \( Q \) admits a right adjoint. In the Appendix in Equation (6.2) we will show that \( Q : \text{CR}^{G_m}_k \rightarrow \text{CR}^{G^2}_k[u] \)

has a right adjoint. As taking simplicial objects is a 2-functor, we have a corresponding adjoint for \( Q \) regarded as the induced functor on simplicial objects.

The above result allows us to define our promised derived replacement of the functor \( Q \).

**Definition 5.1.6.** Let

\[
LQ : \text{Ho}(\text{sCR}^{G_m}_k) \rightarrow \text{Ho}(\text{sCR}^{G^2}_k[u]) \tag{5.4}
\]

be the total left derived functor of \( Q \). Here \( \text{Ho} \) denotes the homotopy category, i.e. the localization of the category \( \text{sCR}^{G_m}_k \) (resp. \( \text{sCR}^{G^2}_k[u] \)) at weak equivalences.

In other words, if \( R_\bullet \) is an object of \( \text{sCR}^{G_m}_k \), we have

\[
LQ(R_\bullet) = Q(S_\bullet) \tag{5.5}
\]

where \( S_\bullet \rightarrow R_\bullet \) is any cofibrant replacement, which is well-defined since \( Q \) is a left Quillen functor. That is, if Cofib denotes a cofibrant replacement functor in \( \text{sCR}^{G_m}_k \) then

\[
LQ := Q \circ \text{Cofib}. \tag{5.6}
\]
5.2. **Property** $P_{\text{der}}$. We now introduce the analogue of the map $\rho_R$ from Definition 3.2.9 and the analogue of Property $P$ from Definition 3.3.7. The main result of this subsection will be Theorem 5.2.7, which will show that cofibrant objects have Property $P_{\text{der}}$. This result has the effect of bypassing the Tor vanishing assumptions in Lemma 3.3.6, which was our original criterion for fully-faithfulness of the functor on derived categories associated to $Q_+$.

**Definition 5.2.1.** We say that an object $R_\bullet$ of $s\text{CR}_G^m_k$ has Property $P_{\text{der}}$ if the map

$$\beta_{R_\bullet} : (p \otimes_k s)_* Q(R_\bullet) \overset{L}{\otimes}_p Q(R_\bullet) \to Q(R_\bullet)$$

given by the composition

$$(p \otimes_k s)_* Q(R_\bullet) \overset{L}{\otimes}_p Q(R_\bullet) \to (p \otimes_k s)_* Q(R_\bullet) \otimes_p Q(R_\bullet) \xrightarrow{\rho_{R_\bullet}} Q(R_\bullet)$$

is a weak equivalence. Here the first map in the composition is the truncation of the derived tensor product after application of $(p \otimes s)_*$ (with takes middle invariants with respect to the $G^p_\mu$-action on the tensor product) and $\rho_{R_\bullet}$ is the map from Definition 3.2.9 applied level-wise.

We use the notation $\beta$ (with no subscript) to denote the natural transformation of the two functors $s\text{CR}_G^m_k \to s\text{CR}_G^p_k$ determined by the left and right side of Equation (5.7). In particular, $R_\bullet$ has Property $P_{\text{der}}$ exactly when $\beta(R_\bullet) = \beta_{R_\bullet}$ is a weak equivalence.

**Example 5.2.2.** Let $R_\bullet$ be an object of $s\text{CR}_G^m_k$ such that, at each level $n$, $R_n$ has only non-negative weights (resp. at each level has only non-positive weights) Then $Q(R_\bullet)$ is level-wise flat over $R_\bullet$ via one of either $s$ or $p$, and so the map

$$Q(R_\bullet) \overset{L}{\otimes}_p Q(R_\bullet) \to Q(R_\bullet) \otimes_p Q(R_\bullet)$$

is a weak equivalence, see e.g. [Qui67, Corollary II.6.10]. Also $\rho_{R_\bullet}$ is a weak equivalence, indeed it is actually an isomorphism level-wise by Lemma 3.2.10. It follows that $R_\bullet$ has Property $P_{\text{der}}$.

In particular, the objects $F(\partial \Delta[n])^a$ and $F(\Delta[n])^a$ of $\text{CR}_G^m_k$ have Property $P_{\text{der}}$ for any $a \in \mathbb{Z}$ and $n \geq 0$.

For more explicit examples of $Q(R_\bullet)$ and $\beta_{R_\bullet}$, it is typically more convenient to work with dg-algebras instead of simplicial rings (which one may do via the Dold-Kan correspondence, at least in characteristic zero).
Example 5.2.3. Consider the example $R = k[x, y]/xy$ with $\deg x = 1$ and $\deg y = -1$. Assume that $k$ is a field of characteristic zero. Example 3.2.11 showed that without deriving this example, $\beta_R$ (i.e. $\rho_R$ from Definition 3.2.9) is not an isomorphism. However, $R$ has a cofibrant replacement by the dg-algebra $S = k[x, y, e]$ with $d(e) = xy$ where $e$ has homological degree 1 and weight 0. To compute $\beta_S$ we take the degree zero part of

$$Q(S) \otimes S Q(S) = k[x, yu^{-1}, e, u] \otimes_{k[x, y, e]} k[x', y'u'^{-1}, e', u']$$

The degree zero part is $k[x, y'u'^{-1}, e, uu']$ which is the realization of the isomorphism $Q(S) = (Q(S) \otimes S [u, u^{-1}])_0$. Hence, we have corrected the failure of $\beta_R$ to be an isomorphism. That is, Property $P_{der}$ holds.

We now prove a general lemma which gives conditions for a natural transformation between model categories to assign cofibrant objects to weak equivalences.

Lemma 5.2.4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between model categories and $\eta : F \to G$ be a natural transformation. Assume that

a) $\mathcal{C}$ is cofibrantly generated;

b) $\mathcal{D}$ is a combinatorial model category in the sense of [Dug01];

c) there is an initial cofibrant object $c_0 \in \mathcal{C}$ and $\eta(c_0)$ is a weak-equivalence;

d) if $\eta(c)$ is a weak equivalence for some object $c \in \mathcal{C}$, then any pushout of $\eta(c)$ along a generating cofibration is a weak equivalence;

e) $\eta$ commutes with sequential colimits.

Then, $\eta(c)$ is a weak equivalence for any cofibrant object $c \in \mathcal{C}$.

Proof. By assumption, any cofibrant object is a retract of a sequential colimit of pushouts along generating cofibrations. Since any natural transformation respects retracts, it suffices to prove that any sequential colimit of pushouts along generating cofibrations is a weak equivalence. This follows from transfinite induction and the assumptions since in a combinatorial model category being a filtered colimit of weak equivalences, is itself a weak equivalence by [Dug01, Proposition 7.3].

We now begin the process of verifying that the hypotheses of Lemma 5.2.4 are satisfied by the natural transformation $\beta$. Only the hypotheses d) and e) are non-trivial to verify. We first show that $\beta$ satisfies condition d).
Lemma 5.2.5. Assume that $R_\bullet$ has Property $P_{\text{der}}$ and that we have a map $f : F(\partial \Delta[n])^a \to R_\bullet$. Then the pushout along the natural map $F(\partial \Delta[n])^a \to F(\Delta[n])^a$ has Property $P_{\text{der}}$.

Proof. For notational simplicity let $S := F(\partial \Delta[n])^a$ and $T := F(\Delta[n])^a$. We want to check that

$$\beta_{T \otimes S} R_\bullet : (p \otimes s) \circ (Q(T \otimes S \otimes R_\bullet) \otimes Q(T \otimes S \otimes R_\bullet)) \to Q(T \otimes S \otimes R_\bullet)$$

is a weak equivalence.

The map $S \to T$ is a cofibration so level-wise it is a retract of a free commutative extension. By Lemma 5.1.5, the map $Q(S) \to Q(T)$ is then also a cofibration. In particular, it is level-wise flat. Therefore, the natural map

$$Q(T) \otimes_{Q(S)} Q(R_\bullet) \to Q(T) \otimes_{Q(S)} Q(R_\bullet)$$

(5.8)

is a weak equivalence.

Now, we have the following chain of weak equivalences

$$\left(Q(T) \otimes_{Q(S)} Q(R_\bullet)\right) \otimes_{Q(S)} Q(R_\bullet) = (Q(T) \otimes_{Q(S)} Q(R_\bullet)) \otimes_{Q(S)} (Q(T) \otimes_{Q(S)} Q(R_\bullet))$$

$$\geq (Q(T) \otimes_{Q(S)} Q(R_\bullet)) \otimes_{Q(S)} (Q(T) \otimes_{Q(S)} Q(R_\bullet))$$

$$\geq Q(T \otimes S \otimes R_\bullet) \otimes_{Q(S)} Q(T \otimes S \otimes R_\bullet)$$

The first weak equivalence above holds because it is an isomorphism for tensor products or ordinary rings, and a well chosen cofibrant replacement functor commutes with taking tensor products/coproducts (see Proposition 2.3 of [Dug01] which we may apply due to Proposition 5.1.2). The second weak equivalence follows from Equation (5.8).

For the last equivalence above, we will show in the appendix in Corollary 6.0.2 that $Q$ preserves arbitrary colimits; in particular, it preserves tensor products.

Denote the above chain of weak equivalences by $\phi$. Since $G_m$ is linearly reductive, $(p \otimes_k s)_\bullet$ preserves weak equivalences, and so

$$\beta_{T \otimes S} R_\bullet \circ (p \otimes s)_\bullet \phi = \beta_T \otimes \beta_S \beta_{R_\bullet}.$$ 

(5.9)

Since $\beta_T, \beta_S$ are weak equivalences by Example 5.2.2 and $\beta_{R_\bullet}$ is a weak equivalence by assumption, the right hand side is a colimit of weak equivalences. Hence, by the 2 out of 3 condition for weak equivalences, $\beta_{T \otimes S} R_\bullet$ is a weak equivalence, as desired. \qed
We now verify that $\beta$ satisfies condition e).

**Lemma 5.2.6.** The natural transformation $\beta$ from Definition 5.2.1 commutes with sequential colimits.

**Proof.** The functor $Q$ preserves colimits by Corollary 6.0.2 (this applies to simplicial objects since taking simplicial objects is a 2-functor). Furthermore, colimits commute with coproducts. The result follows. □

**Theorem 5.2.7.** Let $R_\bullet$ be a cofibrant object of $sCR_k^{G_m}$. Then $R_\bullet$ has Property $P_{der}$.

**Proof.** This is a direct application of Lemma 5.2.4. Namely, we set $C = sCR_k^{G_m}$, $D = sCR_k^{G_m}$, and $\eta = \beta$. The initial object is $k$ which is cofibrant and trivially satisfies Property $P_{der}$. The remaining conditions are verified by Proposition 5.1.2, Corollary 5.1.4, Lemma 5.2.5, and Lemma 5.2.6. □

5.3. **Base change and recovery of the smooth case.** In this section we address the difference between $Q(R)$ and $LQ(R)$ when $R$ is an object of $CR_k^{G_m}$, i.e. an ordinary $\mathbb{Z}$-graded commutative ring. In particular, in Proposition 5.3.4 we will exhibit a weak equivalence between them when $\text{Spec } R$ is smooth, so that the general approach of Section 5 effectively reduces to the results of Section 4 under this assumption. We first prove a strongly étale base change result which is a derived version of Proposition 4.2.3.

**Proposition 5.3.1.** Let $R$ and $S$ be objects of $CR_k^{G_m}$ and let $f : R \to S$ be a strongly étale graded homomorphism of ordinary rings. Regard $R$ and $S$ as objects of $sCR_k^{G_m}$ by viewing them as constant simplicial rings. Then the base change map

$$LQ(R) \overset{1}{\otimes} f S \to LQ(S)$$

is a weak equivalence.

**Proof.** By definition of strongly étale, there is an isomorphism of rings

$$S = R \otimes_{R_0} S_0$$

where the subscripts here refer to degree with respect to the $\mathbb{Z}$-gradings and not the level (as these rings are not simplicial). Now, let $R_\bullet \to S_\bullet$ be the image of $R \to S$ under the cofibrant replacement functor on $sCR_k^{G_m}$. Since taking the degree 0 piece at each level preserves weak equivalences, takes generating cofibrations to generating cofibrations,
and commutes with sequential colimits, we have a cofibrant replacement \((R_\bullet)_0 \to (S_\bullet)_0\) of \(R_0 \to S_0\) in \(sCR_k\). Since \(R_0 \to S_0\) is flat, this gives a weak equivalence of coproducts

\[
S = R \otimes S_0 = R_\bullet \otimes (S_\bullet)_0
\]

Now apply \(Q\) to get weak equivalences

\[
LQ(S) = Q(R_\bullet \otimes (S_\bullet)_0) = LQ(R) \otimes (S_\bullet)_0
\]

\[
= LQ(R) \otimes S_0 = LQ(R) L S.
\]

**Corollary 5.3.2.** If \(R \to S\) is strongly étale, then there is an isomorphism of \(S\)-modules

\[
\pi_i(LQ(S)) = \pi_i(LQ(R)) \otimes_R S.
\]

**Proof.** This follows from a Quillen spectral sequence ([Qui67, Theorem 6 (c), Section 6.8]). Namely, we have an \(E^2\) page

\[
E^2_{pq} = \pi_p(\pi_q(LQ(R)) \otimes_R S) \Rightarrow \pi_{p+q}(LQ(R) \otimes_R S) = \pi_{p+q}(LQ(S)).
\]

Since \(R, S\) are constant simplicial rings and \(S\) is in particular flat over \(R\), this forces \(p = 0\) and the spectral sequence degenerates. \(\square\)

Recall from Lemma 2.1.14 that, for ordinary rings with a \(\mathbb{G}_m\)-action, \(Q(R)\) and \(\Delta(R)\) become identified away from the contracting locus, i.e. after localizing by elements of positive degree. This intuitively suggests that \(Q\) does not require deriving after taking such a localization. We formulate this intuition more precisely as follows.

**Lemma 5.3.3.** If \(R\) is an object of \(CR^m_k\) and \(r \in R\) has positive degree, then there is a weak equivalence

\[
LQ(R)_s \otimes_R R_r = Q(R)_s \otimes_R R_r = \Delta(R)_s \otimes_R R_r.
\]

**Proof.** The second equality is an isomorphism and is just Lemma 2.1.14. For the first weak equivalence, notice that the inclusion \(Q(R) \to \Delta(R)\) gives to a short exact sequence of simplicial \(R[u]\)-modules:

\[
0 \to LQ(R) \to L\Delta(R) \to L(\Delta(R)/Q(R)) \to 0.
\]

Applying \((- \otimes_{k[u]} k[u, u^{-1}]\)) annihilates \(L(\Delta(R)/Q(R))\) giving an isomorphism

\[
LQ(R) \otimes_{k[u]} k[u, u^{-1}] \to L\Delta(R) \otimes_{k[u]} k[u, u^{-1}].
\]
In particular, we have an isomorphism of $R[u]$-modules
\[
\pi_i(LQ(R) \otimes_{k[u]} k[u, u^{-1}]) = \pi_i(L\Delta(R) \otimes_{k[u]} k[u, u^{-1}]) = \pi_i(\Delta(R)) = \begin{cases} 
R[u, u^{-1}] & \text{if } i = 0 \\
0 & \text{if } i > 0.
\end{cases}
\]

Now, under the isomorphism
\[
\pi_0(LQ(R) \otimes_R R_r) = \pi_0(LQ(R)) \otimes_R R_r,
\]
the element $u \otimes 1$ becomes a unit since $r$ has positive degree (recall that $s(r) = ru^{\deg r}$). Hence,
\[
\pi_i(LQ(R)) \otimes_R R_r = \begin{cases} 
R_r[u, u^{-1}] & \text{if } i = 0 \\
0 & \text{if } i > 0.
\end{cases}
\]

**Proposition 5.3.4.** Suppose that $R$ is an object of $CR_k^{sm}$ such that Spec $R$ is smooth. Then there is a weak equivalence
\[
LQ(R) = Q(R).
\]

**Proof.** We have a map $LQ(R) \to Q(R)$; we must show that the induced map
\[
\pi_*LQ(R) \to Q(R)
\]
of $R$-modules is an isomorphism, i.e. that $\pi_i(LQ(R)) = 0$ for $i > 0$ and $\pi_0(LQ(R)) = Q(R)$. This can be done locally on $R$. In particular, by taking open sets coming from étale slices, we must exhibit isomorphisms
\[
\pi_i(LQ(R)) \otimes_R R_r = Q(R) \otimes_R R_r
\]
when $r$ has degree zero (to cover the fixed locus) and when $r$ has strictly positive degree (to cover the contracting locus). On the fixed locus, Proposition 4.2.1 gives in particular a strongly étale map $f : T \to R_r$ where $T$ is a free object of $CR_k^{sm}$ and $r$ has degree zero. Now $LQ(T) = Q(T)$ since $T$ is cofibrant when regarded as a constant simplicial ring. Proposition 5.3.1 and Proposition 4.2.3 then give
\[
LQ(R_r) = Q(T) \otimes_T R_r = Q(R_r).
\]
In particular,
\[
\pi_i(LQ(R_r)) = \begin{cases} 
Q(R_r) & \text{if } i = 0 \\
0 & \text{if } i > 0.
\end{cases}
\]
On the other hand, Corollary 5.3.2 says that
\[
\pi_i(LQ(R_r)) = \pi_i(LQ(R)) \otimes_R R_r.
\]
It follows that

\[ \pi_i(LQ(R)) \otimes_R R_r = \begin{cases} Q(R_r) = Q(R) \otimes_R R_r & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \]

Similarly, if \( \deg r > 0 \), Lemma 5.3.3. gives

\[ \pi_i(LQ(R)) \otimes_R R_r = \begin{cases} Q(R) \otimes_R R_r & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \]

\[ \square \]

5.4. Localizations and Semi-orthogonal Decompositions in the General Case. This section develops localizations and semi-orthogonal decompositions associated to objects of \( sCR_{G^m} \) in analogy with the case of smooth commutative rings which we considered in Section 3.3.

An important step in this direction is understanding how to interpret \( LQ(R_\bullet) \) as a Fourier-Mukai kernel object. That is, associated to \( LQ(R_\bullet) \) we wish to construct a corresponding object in the homotopy category of simplicial modules

\[ \operatorname{Ho}(s\text{Mod}_{G^m}^R(L_\bullet \otimes_k R_\bullet)), \]

which we will do in Definition 5.4.4. This requires some attention to the model structure on the category of simplicial modules. A reader unconcerned with these details may wish to simply inspect Definition 5.4.4 and the corresponding semi-orthogonal decomposition in Proposition 5.4.7 and bypass the remainder of the section.

For the model structure on the category of simplicial modules, we refer directly to [Qui67, Chapter II.6]. To endow a triangulated structure on the respective homotopy categories we use categories of spectra which are a model for the derived category, see e.g. [Jar15, Section 8.2] or [Sch97, Sections 2 and 3]; we will follow the exposition of the latter article very closely. In particular, the next definition is just [Sch97, Definition 2.1.1 and Definition 2.1.2] applied to simplicial rings with a \( G_m \)-action.

**Definition 5.4.1.** Let \( R_\bullet \) be an object of \( s\text{Mod}_{G^m}^R R_\bullet \). A **spectrum** \( M \) in \( s\text{Mod}_{G^m}^R R_\bullet \) is a collection of objects \( M_n \) and maps \( \Sigma M_n \rightarrow M_{n+1} \) for each degree \( n \in \mathbb{N} \) where \( \Sigma M_n \) is the suspension of \( M_n \). Maps of spectra \( M \rightarrow N \) are defined to be collections of maps \( M_n \rightarrow N_n \) such that the obvious squares commute. The category of spectra in \( s\text{Mod}_{G^m}^R R_\bullet \) is denoted by \( (s\text{Mod}_{G^m}^R R_\bullet) \).

In the above \( \Sigma \) denotes the suspension endofunctor on \( s\text{Mod}_{G^m}^R R_\bullet \), which is defined since \( s\text{Mod}_{G^m}^R R_\bullet \) is a simplicial model category with
a zero object. For later use, we recall that the right adjoint of $\Sigma$ is the loop functor $\Omega$. [Sch97, Corollary 3.1.4] describes the model category structure on the category of spectra; we do not reproduce the details here for the sake of brevity, and because we only require certain structural properties. However, we remind the reader that a map of spectra is said to be a strict fibration if it is degree-wise a fibration with respect to the model structure on $\text{sMod}^{G_m}_R$. Likewise, a map of spectra is said to be a strict weak equivalence if it is degree-wise a weak equivalence. The fibrations (sometimes called stable fibrations) and weak equivalences (sometimes called stable weak equivalences) require additional properties that we will delegate to the proofs below.

Given any object $X$ of $\text{sMod}^{G_m}_R$, we obtain a suspension spectrum, denoted $\Sigma^\infty X$, by setting $\Sigma^X_n = \Sigma^n X$ and taking the identity maps in each degree $n$. This induces a suspension functor

$$\Sigma^\infty : \text{sMod}^{G_m}_R \to (\text{sMod}^{G_m}_R)^\infty.$$  

(5.10)

One defines an $\Omega$-spectrum to be a spectrum $M$ such that each map $M_n \to \Omega M_{n+1}$ is a weak homotopy equivalence, and a spectrum is said to be connective if, for each $n \geq 1$, the higher homotopies of $\Omega M_n$ vanish.

The following result, which is [Sch97, Lemma 2.2.2], allows us to use spectra to understand the homotopy category of the category of simplicial modules.

**Proposition 5.4.2.** The total left derived functor of the suspension functor $\Sigma^\infty$ gives an equivalence between the homotopy category of graded simplicial modules and of the homotopy category of connective spectra.

**Remark 5.4.3.** The triangulated category $\text{Ho}(\text{sMod}^{G_m}_R)$ is actually equivalent to a perhaps more familiar triangulated category via the Dold-Kan correspondence. For simplicity, assume $R = R_\bullet$ is discrete, i.e. all higher homotopies vanish. Then the Dold-Kan correspondence gives a Quillen equivalence between $\text{sMod}^{G_m} R$ and $\text{Mod}^{G_m}_{\leq 0} R$, the category of chain complexes of $R$-modules vanishing in positive degrees. This induces a Quillen equivalence of spectra

$$(\text{sMod}^{G_m} R)^\infty \cong (\text{Mod}^{G_m}_{\leq 0} R)^\infty.$$  

One can easily identify $\text{Ho}(\text{Mod}^{G_m}_{\leq 0} R)^\infty$ with connective spectra becoming complexes with homology concentrated in non-positive degrees, see e.g. [Jar15, Section 8.2]. Hence, if one prefers, one can always in practice work with differential graded modules over a graded commutative dg-algebra for the purposes of this section. We have not done so, though, because our definition of $LQ$ via simplicial rings (instead
of connective chain complexes) sits more naturally in the category of spectra of modules.

Up to one important detail we will discuss immediately below, we now have enough structure in place to give a repackaging of $LQ(R_\bullet)$ as a kernel object.

**Definition 5.4.4.** Let $R_\bullet$ be an object of $sCR^G_{k^m}$ and $S_\bullet \to R_\bullet$ a cofibrant replacement. Then $LQ(R_\bullet) = Q(S_\bullet)$ is naturally a $\mathbb{Z}^2$-graded simplicial $S_\bullet \otimes_k S_\bullet$-module, and hence gives an object

$$Q_{\text{der}}(R_\bullet) := \Sigma^\infty LQ(R_\bullet) = \Sigma^\infty Q(S_\bullet)$$

(5.11)
of $\text{Ho}(s\text{Mod}^{G^m} R_\bullet \otimes_k R_\bullet)^\infty$ under the identification of homotopy categories in Proposition 5.4.2.

As hinted above, there is a subtlety in the above Definition, in that $Q_{\text{der}}(R_\bullet)$ is not actually well-defined unless the weak equivalence $S_\bullet \to R_\bullet$ induces an equivalence

$$(s\text{Mod}^{G^m} S_\bullet \otimes_k S_\bullet)^\infty \to (s\text{Mod}^{G^m} R_\bullet \otimes_k R_\bullet)^\infty.$$ 

Indeed, we will show in Proposition 5.4.6 that this is always the case.

To this end, we first record a simple lemma.

**Lemma 5.4.5.** Any strict fibration $M \to N$ of $\Omega$-spectra in $(s\text{Mod}^{G^m} R_\bullet)^\infty$ is a fibration.

**Proof.** Let $S$ denote the endofunctor of $s\text{Mod}^{G^m} R_\bullet$ which is defined before [Sch97, Definition 2.1.4] (and note that Schwede uses the notation $Q$ for this functor, which is totally different than our use of $Q$). Roughly, $S(M)$ is a weakly equivalent $\Omega$-spectrum associated to $M$. By [Sch97, Proposition 2.1.5], $M \to N$ being a fibration is equivalent to it being a strict fibration and the natural map

$$f : M \to S(M) \times_{S(N)} N$$

being a weak equivalence. Since $S(A)$ and $S(N)$ are $\Omega$-spectra, the maps $M \to S(M)$ and $N \to S(N)$ are strict weak equivalences. So we have a diagram

$$
\begin{CD}
S(M) \times_{S(N)} N @>f>> S(M) = S(M) \times_{S(B)} S(N) \\
M @>h>> S(M) \\
\end{CD}
$$

where $g, h$ are weak equivalences. Hence, so is $f$, as desired. \qed
Proposition 5.4.6. If \( f : R^\prime_\bullet \to R_\bullet \) is a weak equivalence in \( sCR_k^{G_m} \), then restriction of scalars induces a Quillen equivalence

\[
\text{Res} : (s\text{Mod}_{G_m} R_\bullet)^\infty \to (s\text{Mod}_{G_m} R^\prime_\bullet)^\infty.
\]

Likewise, restriction of scalars induces a Quillen equivalence

\[
(s\text{Mod}_{G^2_m} R_\bullet L \otimes_k R_\bullet)^\infty \to (s\text{Mod}_{G^2_m} R^\prime_\bullet L \otimes_k R^\prime_\bullet)^\infty.
\]

Proof. First, we show that \( \text{Res} \) gives a Quillen adjunction. Extension of scalars is a left adjoint, so we must show that \( \text{Res} \) preserves (stable) weak equivalences and (stable) fibrations. The forgetful functor \( s\text{Mod}_{G_m} R_\bullet \to s\text{Set} \) satisfies the remark given after [Sch97, Corollary 2.1.6]. Namely, the weak equivalences are those inducing isomorphisms on homotopy groups of the underlying spectra. Since the underlying spectra do not change under restriction of scalars, we see that \( \text{Res} \) preserves weak equivalences.

We now show that \( \text{Res} \) preserves fibrations. Restriction of scalars preserves strict fibrations of spectra and \( \Omega \)-spectra, as all objects of \( s\text{Mod}_{G_m} R_\bullet \) are fibrant, and so \( \text{Res} M \to \text{Res} N \) is a strict fibration if \( M \to N \) is a fibration. Again using [Sch97, Proposition 2.1.5], it remains to show that

\[
\text{Res} M \to S^\prime \text{Res} M \times_{S^\prime \text{Res} N} \text{Res} N
\]

is a weak equivalence, where \( S \) and \( S^\prime \) denote the endofunctors of \( s\text{Mod}_{G_m} R_\bullet \) and \( s\text{Mod}_{G_m} R^\prime_\bullet \) used in the proof of Lemma 5.4.5.

Now consider the following diagram.

\[
\begin{array}{ccc}
\text{Res} M & \longrightarrow & \text{Res} SM \\
\downarrow & & \downarrow \\
\text{Res} N & \longrightarrow & \text{Res} SN
\end{array}
\quad
\begin{array}{ccc}
S^\prime \text{Res} M & \longrightarrow & S^\prime \text{Res} SM \\
\downarrow & & \downarrow \\
S^\prime \text{Res} N & \longrightarrow & S^\prime \text{Res} SN
\end{array}
\]

The left square is homotopy cartesian by assumption. Lemma 5.4.5 shows that the right square is homotopy cartesian. Hence the full rectangle is homotopy cartesian. We likewise have a diagram

\[
\begin{array}{ccc}
\text{Res} M & \longrightarrow & S^\prime \text{Res} M \\
\downarrow & & \downarrow \\
\text{Res} N & \longrightarrow & S^\prime \text{Res} N
\end{array}
\quad
\begin{array}{ccc}
S^\prime \text{Res} M & \longrightarrow & S^\prime \text{Res} SM \\
\downarrow & & \downarrow \\
S^\prime \text{Res} N & \longrightarrow & S^\prime \text{Res} SN
\end{array}
\]

From above the full rectangle is homotopy cartesian and the right square is homotopy cartesian by [Sch97, Proposition 2.1.3(e)]. Hence, the left square is homotopy cartesian, so \( \text{Res} M \to \text{Res} N \) is a fibration, as claimed, and so \( \text{Res} \) gives a Quillen adjunction.
To see Res is a Quillen equivalence, it remains to check that we have an equivalence on the homotopy categories. By \[SS03, \text{Theorem A.1.1}\], the objects $\Sigma^l R_\bullet(a)$ for $l, a \in \mathbb{Z}$ form a compact set of generators for the homotopy category $\text{Ho}(\text{sMod}^{G_m} R_\bullet)^\infty$. Hence, it suffices to show that, in the homotopy category, restriction of scalars is fully-faithful on these objects. But we assumed $R_\bullet \rightarrow R'_\bullet$ is a weak equivalence, and restriction of scalars commutes with suspension and shifts, so we see that restriction of scalars is indeed fully-faithful on this category. We thus have that Res gives a Quillen equivalence, as claimed.

For the second part of the statement regarding bimodules, the proof is entirely the same once one observes that
\[
R'_\bullet \otimes_k R'_\bullet \rightarrow R_\bullet \otimes_k R_\bullet
\]
is also a weak equivalence since derived tensor products preserve weak equivalences.

Thus, $Q_{\text{der}}(R_\bullet)$ as given in Definition 5.4.4 is indeed well-defined. Let $\Delta_{\text{der}}(R_\bullet)$ be the object of $\text{Ho}((\text{sMod}^{G_m} R'_\bullet \otimes_k R'_\bullet)^\infty)$ determined by the diagonal object $\Delta(R_\bullet)$, and let $S_{\text{der}}(R_\bullet)$ denote the cone in $\text{Ho}((\text{sMod}^{G_m} R'_\bullet \otimes_k R'_\bullet)^\infty)$ which fits into the exact triangle
\[
Q_{\text{der}}(R_\bullet) \rightarrow \Delta(R_\bullet) \rightarrow S_{\text{der}}(R_\bullet).
\]
So we get
\[
\Phi_{Q_{\text{der}}} : \text{Ho}(\text{sMod}^{G_m} R_\bullet)^\infty \rightarrow \text{Ho}(\text{sMod}^{G_m} R_\bullet)^\infty
\]
\[
M \mapsto (M \otimes \eta Q_{\text{der}})_0
\]
the corresponding equivariant integral transform on the homotopy category of spectra. We define $\Phi_{S_{\text{der}}}$ similarly.

**Proposition 5.4.7.** For any object $R_\bullet$ of $\text{sCR}^{G_m}_k$, there is a semi-orthogonal decomposition
\[
\text{Ho}(\text{sMod}^{G_m} R_\bullet)^\infty = \langle \text{Im } \Phi_{S_{\text{der}}}, \text{Im } \Phi_{Q_{\text{der}}} \rangle
\]
which preserves connective spectra.

**Proof.** By Lemma 3.3.4, it is enough to show that
\[
\Phi_{Q_{\text{der}}} \eta \rightarrow \text{Id} \xrightarrow{\text{cone}(\eta)} \Phi_{S_{\text{der}}}
\]
is a Bousfield triangle, where we recall that $\eta$ is map on functors induced from $Q_{\text{der}} \rightarrow \Delta$. Furthermore, by Remark 3.3.2, it is enough to show that the map
\[
Q(\eta) : (Q_{\text{der}} \otimes R_\bullet, Q_{\text{der}})_0 \rightarrow Q_{\text{der}}
\]
is an isomorphism. This is equivalent to showing that the map
\[ \beta_R : (p \otimes_k s)_L LQ \rightarrow LQ \]
is an isomorphism. This follows immediately from Theorem 5.2.7. \(\square\)

**Example 5.4.8.** This is a continuation of Examples 3.2.11 and 5.2.3. Recall that in these examples, \(R = \mathbb{k}[x, y]/(xy)\) where \(\deg x = 1\) and \(\deg y = -1\). Assume that \(\mathbb{k}\) is a field of characteristic zero. We saw that
\[ Q(\mathbb{R}) = \mathbb{k}[x, y, u^{-1} y, u]/(xy) \cong k[x, z, u]/(xz). \]
and, as a dg-algebra,
\[ Q_{\text{der}}(\mathbb{R}) = \mathbb{k}[x, z, u, e] \]
with \(d(e) = xzu\). Regarding this as an \(R\)-module, we have
\[ Q_{\text{der}}(\mathbb{R}) = k[x, z, u]/xzu \]
where \(y\) acts by \(zu\). It follows that
\[ S_{\text{der}} = (k[x, y, u, u^{-1}]/xy)/(k[x, z, u]/xzu). \]
Now, we can regard \(Q_{\text{der}}(\mathbb{R})\) as a quotient of \(k[x, y, z, u]/xy\) by the regular element \(uz - y\). This allows one to compute
\[ (Q_{\text{der}}(\mathbb{R})_L \otimes_p Q_{\text{der}}(\mathbb{R}))_0 = Q_{\text{der}}(\mathbb{R}). \]
In other words
\[ \Phi_{Q_{\text{der}}(\mathbb{R})} \circ \Phi_{Q_{\text{der}}(\mathbb{R})} = \Phi_{Q_{\text{der}}(\mathbb{R})}, \]
which is exactly the property that grants the existence of a semi-orthogonal decomposition. By checking on the set of generators \(R(i), R/y(i), R/x(i)\) for all \(i\), it follows that \(\text{Im } \Phi_{Q_{\text{der}}} = \text{full subcategory of } D^b(\text{mod}^{G_m} k[x, y]/(xy))\) generated by \(R(i)\) for \(i \leq 0\), which we denote by \(\text{Perf}^{\perp}_{\leq 0}\). Hence, we get a semi-orthogonal decomposition,
\[ D^b(\text{mod}^{G_m} k[x, y]/(xy)) = (\text{Perf}^i_{\leq 0}, \text{Perf}_{\geq 0}). \]
It is worthwhile to observe here that \(\text{Im } \Phi_{Q_{\text{der}}} \) lies in perfect complexes.

5.5. **The global case and D-flips.** We now undertake the task of globalizing \(Q_{\text{der}}\) beyond the affine case and applying this to diagrams coming from flips. The aim of this subsection is to show that, with this globalization, one can use \(Q_{\text{der}}\) to give a “wall-crossing” functor associated to any \(D\)-flip. As this will require synthesizing many of the previous constructions in a new context of sheaves of modules, let us give a quick overview of this subsection to guide the reader.

- We first recall a well-known construction of Reid and Thaddeus which associates a sheaf of graded \(\mathcal{O}\)-modules \(\mathcal{A}\) to a \(D\)-flip.
As a first step to building a wall-crossing functor associated to the flip, we consider $Q(A)$, essentially a sheafy version of $Q(R)$ from the affine case. Without difficulties we can likewise consider a functor $Q$ on simplicial sheaves of graded $\mathcal{O}$-modules.

Via Blander’s model structure on the category of simplicial sheaves, we can likewise consider derived variants $LQ$ and $Q_{\text{der}}$ just as in the affine case.

We now recall the definition of a $D$-flip. This is a small variation on the definition of a flip in the sense of Mori theory, but which is well adapted to the setting of $\mathbb{G}_m$-actions.

**Definition 5.5.1.** Let $X^-, X_0$ and $X^+$ be varieties over a field $k$ such that:

a) There is a small contraction $f : X^- \to X_0$, i.e. $f$ is a proper birational morphism whose exceptional locus has codimension at least two (in particular, $X_0$ is not $\mathbb{Q}$-factorial).

b) $X^-$ is equipped with a $\mathbb{Q}$-Cartier divisor $D$ such that $\mathcal{O}(-D)$ is $f$-ample.

c) $X^+$ admits a small contraction $g : X \to X_0$, and the induced birational map $h : X^- \to X^+$ is such that $\mathcal{O}(h_*D)$ is $\mathbb{Q}$-Cartier and $g$-ample.

Then $X^+$ is the $D$-flip of $X^-$ over $X_0$.

It is easy to check that a $D$-flip is unique if it exists. The following construction is recalled from [Tha96, Proposition 1.7] and relates $D$-flips to graded rings.

**Proposition 5.5.2.** Let $X^+$ be the $D$-flip of $X^-$ over $X_0$. Fix $n \in \mathbb{N}$ and set

$$\mathcal{A} := \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_{X_0}(kD).$$

For $n$ sufficiently large there are isomorphisms

$$X^- = \left[ \text{Spec}_{X_0} \mathcal{A}^{-}/\mathbb{G}_m \right],$$

$$X^+ = \left[ \text{Spec}_{X_0} \mathcal{A}^{+}/\mathbb{G}_m \right],$$

i.e. these quotient stacks are represented by $X^-, X^+$ respectively.

**Proof.** We choose $n$ such that $\mathcal{A}_{\geq 0}$ is generated in degree one. We will show in Lemmas 5.5.3 and 5.5.4 below that it follows that $\left[ \text{Spec}_{X_0} \mathcal{A}^{+}/\mathbb{G}_m \right]$ is represented by $\text{Proj}_{X_0} \mathcal{A}_{\geq 0}$. Since $X^- \to X_0$ is a small contraction, $\mathcal{A}_{\geq 0} = \bigoplus_{k \in \mathbb{N}} \mathcal{O}_{X_0}(kD) = \bigoplus_{k \in \mathbb{N}} \mathcal{O}_{X_-}(kD)$. But $\text{Proj}_{X_0} \mathcal{A}_{\geq 0}$ is the relative projectivization of the relatively ample line bundle $\mathcal{O}_{X_-}(D)$, and is
thus isomorphic to $X^-$, as claimed. The other equality holds by symmetry.

We now formulate the two lemmas promised in the above proof.

**Lemma 5.5.3.** Let $R$ be an object of $CR^G_k$ and suppose that $I^+$ is generated in degree one, i.e. by elements in $R_1$. Then the global quotient stack

$$[\text{Spec } R \setminus V(I^+)/G_m]$$

is represented by the scheme $Y$ that is obtained by gluing the open affine varieties $\text{Spec}(R_{f_\alpha})_0$ along $\text{Spec}(R_{f_\alpha f_\beta})_0$ for a set of generators $\{f_\alpha\}$ in $R_1$.

**Proof.** This is the same as showing that $\text{Spec } R \setminus V(I^+)$ is a $G_m$-torsor over $Y$. Now, by assumption, $\text{Spec } R \setminus V(I^+)$ is covered by $\text{Spec } R_f$ with $f \in R_1$, so it is enough to check that $\text{Spec } R_f$ is itself a $G_m$-torsor over $\text{Spec}(R_f)_0$. Indeed, it is the trivial torsor. Namely, there is an isomorphism of rings

$$R_f \rightarrow (R_f)_0[u, u^{-1}]$$

$$g \mapsto gf^{-1}u^i$$

for $g \in (R_f)_i$ with the inverse map determined by $u \mapsto f$ and $h \mapsto h$ for $h \in (R_f)_0$.

The space $Y$ as constructed in Lemma 5.5.3 is a quotient of $\text{Spec } R \setminus V(I^+)$ by $G_m$, hence is equipped with a line bundle $O_Y(1)$ coming from the pullback of the map to $[pt/G_m]$. Notice that if $V(I^+)$ has codimension at least 2 then there is an isomorphism $\Gamma(Y, O_Y(i)) = R_i$.

**Lemma 5.5.4.** Let $R$ be an object of $CR^G_k$ such that $I^+$ is generated in degree one and that $V(I^+)$ has codimension at least two. Then the line bundle $O_Y(1)$ defined above is ample.

**Proof.** Ampleness is equivalent to the complements of sections forming a cover $Y$ such that the natural map to the coordinate ring is an open immersion. By assumption, the complements of global sections of $O_Y(1)$ given by elements $f \in R_1$ cover $Y$. Hence, it suffices to show that the map

$$Y \rightarrow \text{Proj}(\bigoplus_{i \in \mathbb{N}} \Gamma(Y, O_Y(i))) = \text{Proj } R_{\geq 0}$$

is an open immersion. This is so because an open subset $\text{Spec}(R_f)_0 \subseteq Y$ is homeomorphic to its image $\text{Spec}((R_{\geq 0})_f)_0$, as the natural inclusion of rings $(R_{\geq 0})_f \rightarrow R_f$ is an isomorphism since $f$ is a unit in $R_f$. Namely, the inclusion is surjective since any element $gf^i$ is the image of $gf^Nf^{i-N}$ for $N \gg 0$. 

□
We now want to use $Q_{\text{der}}$ from Equation (5.11) to define a wall-crossing functor for $D$-flips. Of course, $Q_{\text{der}}$ was only defined in the affine case, and so we must sheafify. Let us begin this process first with our original functor

$$Q : \text{CR}^\mathbb{G}_m \to \text{CR}^\mathbb{G}_m[u]$$

from Definition 2.1.3. This $Q$ automatically gives a presheaf of $\mathcal{O}_Y$-algebras on the affine site over Spec $k$. Denote the category of $\mathbb{Z}$-graded $\mathcal{O}_Y$-algebras by $\text{CR}^\mathbb{G}_m \mathcal{O}_Y$. Sheafifying gives a functor

$$Q : \text{CR}^\mathbb{G}_m \mathcal{O}_Y \to \text{CR}^\mathbb{G}_m \mathcal{O}_Y[u]$$

which, again, by abuse we also denote $Q$.

**Remark 5.5.5.** An object $Q(A)$ with $A$ an object of $\text{CR}^\mathbb{G}_m \mathcal{O}_Y$ is not necessarily quasi-coherent. This is a minor complication for our main goal, which is to define Fourier-Mukai functor between suitable homotopy categories of quasi-coherent sheaves, and not sheaves of arbitrary $\mathcal{O}$-modules. The following lemma will help us alleviate these concerns.

**Lemma 5.5.6.** Let $Y$ be a scheme with trivial $\mathbb{G}_m$-action and $A$ an object of $\text{CR}^\mathbb{G}_m \mathcal{O}_Y$. Then $Q(A)$ is quasi-coherent.

In particular, if $Y$ is affine, then $Q(A)$ is the sheaf associated to $Q(A(Y))$.

**Proof.** Let $U \subseteq V \subseteq Y$ be affine open subsets. Then quasi-coherence of $Q(A)$ is the same as having

$$Q(A(U)) = Q(A(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y(U))$$

$$= Q(A(V)) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_Y(U).$$

Indeed, the first equality holds because of quasi-coherence of $A$. The second equality follows since the $\mathbb{G}_m$-action on $Y$ is trivial. \qed

Since the space $X_0$ in a $D$-flip diagram is in practice usually singular, we must derive

$$Q : \text{CR}^\mathbb{G}_m \mathcal{O}_Y \to \text{CR}^\mathbb{G}_m \mathcal{O}_Y[u]$$

in this sheaf-theoretic setting, analogous to how we derived $Q$ in the affine setting during the course of Section 5. Fortunately, we will be able to directly transfer results from Section 5 with only minor difficulties, once we understand a suitable model structure on simplicial sheaves. To this end, let $\text{sCR}^\mathbb{G}_m \mathcal{O}_Y$ denote the category of simplicial objects in $\text{CR}^\mathbb{G}_m \mathcal{O}_Y$. The following result is essentially [Bla01, Theorem 2.1].
Proposition 5.5.7. The category $sCR^\mathbb{G}_m_{O_Y}$ admits a cofibrantly generated simplicial model structure where the weak equivalences are the maps that induce isomorphisms on the homotopy sheaves, and a set of generating cofibrations are

$$\text{Sym}_{O_Y} i_! O_U \otimes \partial \Delta [n]^a \to \text{Sym}_{O_Y} i_! O_U \otimes \Delta [n]^a,$$

where $U \to Y$ is any open immersion and $a \in \mathbb{Z}$ is any weight. (Here $i_! O_U$ denotes the extension by zero sheaf.)

Proof. From [Bla01, Theorem 2.1], and in particular the final remark in the proof, we know that the category $sSh(Y)$ of simplicial sheaves of sets on $Y$ has a model structure with the weak equivalences as above and a set of generating cofibrations given by

$$i_! O_U \otimes \partial \Delta [n] \to i_! O_U \otimes \Delta [n]$$

We then have a family of free, forgetful adjunctions between $sCR^\mathbb{G}_m_{O_Y}$ and $sSh(Y)$ which induce the claimed model category structure (see e.g., [DHK97, Lemma 9.1] or [GJ99, Theorem 5.8]).

The functor $Q$ from equation 5.12 gives a functor (which, as usual, we also denote by $Q$) on simplicial categories

$$Q : sCR^\mathbb{G}_m_{O_Y} \to sCR^\mathbb{G}_m_{O_Y}[u]$$

in the obvious way. Using the model category structure in Proposition 5.5.7, it is easy to verify that this functor $Q$ is still left Quillen, so we can define a derived functor $LQ$ in total analogy with the affine case from Definition 5.1.6.

Definition 5.5.8. Let

$$LQ : \text{Ho} \left( sCR^\mathbb{G}_m_{O_Y} \right) \to \text{Ho} \left( sCR^\mathbb{G}_m_{O_Y][u] \right)$$

be the total left derived functor of the functor $Q$ in Equation 5.13

In particular, any $LQ(A)$ is an object of

$$sMod^\mathbb{G}_m(A \otimes_{O_Y} A).$$

Likewise, given $A$ an object of $sCR^\mathbb{G}_m_{O_Y}$, we may consider the category of spectra of sheaves of simplicial $A$-modules $(sMod^\mathbb{G}_m A)^\infty$, and define an object $Q_{der}(A)$ of $\text{Ho}(sMod^\mathbb{G}_m (A \otimes_{O_Y} A))$ and a corresponding Fourier-Mukai functor just as in Definition 5.4.4. Let us verify that this process does indeed result in desirable properties; in particular, we want that the kernel object $Q_{der}(A)$ preserves quasi-coherence, and is itself quasi-coherent (at least, up to weak equivalence).
Proposition 5.5.9. Let \( Y \) be a scheme with a trivial \( \mathbb{G}_m \)-action and \( \mathcal{A} \) an object of \( \text{sCR}_{\mathcal{O}_Y}^{\mathbb{G}_m} \) which is quasi-coherent as a \( \mathcal{O}_Y \)-module. Then the object \( LQ(\mathcal{A}) \) of \( \text{sMod}^{\mathbb{G}_m}(\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A}) \) is locally weakly equivalent to a simplicial quasi-coherent sheaf. Hence the functor

\[ \Phi_{Q_{\text{der}}} : \text{Ho(\text{sMod}^{\mathbb{G}_m} \mathcal{A})}^\infty \to \text{Ho(\text{sMod}^{\mathbb{G}_m} \mathcal{A})}^\infty \]

preserves the full subcategories with quasi-coherent homotopy sheaves.

Proof. Let \( U \subseteq Y \) be an affine open subset and write \( \text{Spec}_U, \mathcal{A}(U) = \text{Spec} R \). Let \( \mathcal{A}|_U \) be the sheaf associated to the \( k \)-algebra \( R \). We need to verify that the sheaf associated to \( LQ(\mathcal{A}|_U) \) is weakly equivalent to \( \mathcal{A}|_U \).

We first verify this for the case where \( U = Y \) (so that \( Y \) itself is affine). Inspecting Proposition 5.5.7, we see that the sheaves associated to the generating cofibrations in \( \text{sCR}^{\mathbb{G}_m}_k \) are a subset of generating cofibrations in \( \text{sCR}^{\mathbb{G}_m}_{\mathcal{O}_Y} \). Similarly, a weak equivalence in \( \text{sCR}^{\mathbb{G}_m}_k \) sheafifies to a weak equivalence in \( \text{sCR}^{\mathbb{G}_m}_{\mathcal{O}_Y} \). Thus, if we take a cofibrant replacement \( S_\bullet \to R \) and sheafify, we still get a cofibrant replacement. Using this resolution to compute \( LQ(\mathcal{A}) \) yields the conclusion by Proposition 5.5.6.

In general, if \( U \subset Y \) is an affine open subset, let \( S_\bullet \) be a cofibrant replacement of \( \mathcal{A} \) in \( \text{sCR}^{\mathbb{G}_m}_{\mathcal{O}_Y} \). Then \( (S_\bullet)|_U \) is a cofibrant replacement of \( \mathcal{A}|_U \). Hence

\[ LQ(\mathcal{A}|_U) = Q(S_\bullet)|_U = Q((S_\bullet)|_U) = LQ(\mathcal{A}|_U), \]

which we have already argued is weakly equivalent to the sheaf associated to \( LQ(R) \).

The Dold-Kan correspondence gives an equivalence between the full subcategory of \( \text{Ho(\text{sMod}^{\mathbb{G}_m} \mathcal{A})}^\infty \) with quasi-coherent homotopy sheaves and

\[ \text{D(\text{Qcoh}^{\mathbb{G}_m} \text{Spec}_Y \mathcal{A})}, \]

where the latter is defined to be the full subcategory of

\[ \text{D(\text{Mod}^{\mathbb{G}_m} \mathcal{O}_{\text{Spec}_Y \mathcal{A}})} \]

consisting of complexes with quasi-coherent cohomology.

Let \( Q_{\text{vec}}^{\text{der}} \) be the induced object on \( X^- \times X^+ \) via restriction and the isomorphism in Proposition 5.5.2. This is a generalization of the kernel object \( Q_{\text{vec}} \) that was introduced in Equation (2.20) for the Bondal-Orlov flop equivalence, or the functor \( j^* \circ \Phi_Q \), from Corollary 4.2.11.

Question 5.5.10. Suppose \( X^- \to X^+ \) is a D-flip which is a K-equivalence between two smooth projective \( k \)-varieties, i.e. a flop. Is the functor \( \Phi_{Q_{\text{vec}}^{\text{der}}} \) an equivalence?
Remark 5.5.11. We know that $\Phi_{Q^{\text{proj}}}$ preserves quasi-coherence, so that we can view this as a question about quasi-coherent unbounded complexes. Moreover, the compact objects of this category are precisely the perfect objects [Nee96, Corollary 2.3]. Since $X^+,X^-$ are smooth by assumption, these coincide with bounded coherent complexes so a positive answer to this question should resolve [Kaw04, Conjecture 5.1] (see [Kaw08]).

Remark 5.5.12. To answer the question affirmatively (at least in the Gorenstein case), it should suffice to prove it in the case where $Y$ is affine (see [RMdS07, Theorem 1.22]). Furthermore, note that when $Y$ is affine, one only requires the machinery up through Section 5.2 to study $\Phi_{Q^{\text{proj}}}$.

6. Appendix: Relations with Drinfeld’s space

We expand on Remark 2.1.11. There we observed that $Q(R)$ agrees with the affine case of the main construction of [Dri13], although we will soon record one subtle distinction.

Let $k$ be a field and $Z$ a (not necessarily affine) $k$-scheme of finite type over a field $k$ and possessing an action by $G_m$ and an open cover which is $G_m$-stable, i.e. $G_m$ acts locally linearly. Drinfeld defines a fpqc sheaf on the category of $k$-schemes over $A_1^k$ as follows: for an arbitrary scheme $T$ over $A_1^k$ assign the set

$$\text{Hom}_{A_1^k}(X \times_{A_1^k} T, Z)$$

where $X := A_1^1 \times A_1^1$ is equipped with the product map $X \to A_1^1$ and $X$ has the “hyperbolic” $G_m$-action $t \cdot (x,y) = (tx,t^{-1}y)$. Amongst other results, [Dri13, Section 2.4] proves that this functor is representable, and so there exists a scheme $\tilde{Z}$ over $A_1^k$ such that

$$\text{Hom}_{A_1^k}(T, \tilde{Z}) = \text{Hom}_{A_1^1}(X \times_{A_1^1} T, Z). \quad (6.1)$$

Equivalently, this means that we have an adjunction

$$F: \{k\text{-schemes over } A_1^1\} \rightleftarrows \{G_m\text{-schemes}\} : G$$

where $F(T) := X \times_{A_1^1} T$, $G(Z) := \tilde{Z}$ and the actions are locally linear on the category on the right. As in Remark 2.1.11, if $Z = \text{Spec} R$ is affine then so is $\tilde{Z}$, and $\tilde{Z} = \text{Spec} Q(R)$. Therefore both $F$ above and its adjoint $G$ preserve affine schemes. Restricting the adjunction above to affine schemes and then taking opposite categories, we get an adjunction

$$Q: CR_{k}^{G_m} \rightleftarrows CR_{k[u]}: Q_{\text{Drin}}^{\text{ad}}$$

where $Q_{\text{Drin}}^{\text{ad}}(S) := k[x,y] \otimes_{k[u]} S$. 


However, the reader should be careful here. Recall that in Lemma 2.1.12 we defined $Q$ with the following target

$$Q : CR^G_m \to CR^{G_2}_m$$

not with target $CR_k[u]$ as above. This is no inherent contradiction, as [Dri13, Section 2.1.17] constructs a $G_2$-action on any $\tilde{Z}$. However, this does mean that $Q^\text{ad}_{\text{Drin}}$ as defined above is not the correct adjoint of our functor $Q$. We will prove below that $Q$ has the following adjoint.

Given a $\mathbb{Z}^2$-graded $k$-algebra $S$ over $k[u]$ where $\deg u = (-1, 1)$, let $Q^\text{ad}(S)$ denote the subalgebra of $S \otimes_{k[u]} k[x, y]/(xy)$ generated by

$$S(i, 0)x^i \text{ for } i \geq 0 \text{ and }$$

$$S(0, -i)y^{-i} \text{ for } i < 0$$

Here, $u$ maps to $xy$ (so it is zero), $\deg x = (0, 1)$, and $\deg y = (-1, 0)$. Since $Q^\text{ad}(S)$ has non-zero degree only in the $(i, i)$ summands, we may regard it as a $\mathbb{Z}$-graded $k$-algebra.

**Proposition 6.0.1.** There is an adjunction

$$Q : CR^G_m \rightleftarrows CR^{G_2}_m : Q^\text{ad}.$$  \hfill (6.2)

**Proof.** Let $S, T \in CR^{G_2}_m$ and let $R \in CR^G_m$ with maps

$$p : R \to Q(R)$$

$$s : R \to Q(R).$$

Given any map $f : Q(R) \to S$ in $CR^{G_2}_m$, define

$$\phi_f : R \to Q^\text{ad}(S) \subseteq S \otimes_{k[u]} k[x, y]/(xy)$$

by

$$r \mapsto \begin{cases} f(p(r)) \otimes x^\deg r & \text{when } \deg r \geq 0 \\ f(s(r)) \otimes y^{-\deg r} & \text{when } \deg r \leq 0. \end{cases}$$

Conversely, given a map $g : R \to Q^\text{ad}(S)$ in $CR^G_m$ define

$$\psi_g : Q(R) \to S$$

by defining its values on the generating elements of $Q(R)$ as follows:

$$r \mapsto g(r)_x \text{ if } \deg r \geq 0$$

$$s(r) \mapsto g(r)_y \text{ if } \deg r < 0$$
where \( g(r)_x \) (resp. \( g(r)_y \)) is the \( x \)-component (resp. \( y \)-component) of \( g(r) \) under the decomposition as abelian groups

\[
S \otimes_{k[u]} k[x, y]/(xy) \cong S/u[x] \oplus S/u[y].
\]

The maps \( f \mapsto \phi_f \) and \( g \mapsto \psi_g \) are inverse isomorphisms which give the adjunction. \( \Box \)

Since any functor with a right adjoint preserves colimits, we have the following.

**Corollary 6.0.2.** The functor \( Q : \text{CR}_{k}^{G_m} \to \text{CR}_{k[u]}^{G_m} \) preserves colimits.

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