Centrifugal Force and Ellipticity behaviour

of a slowly rotating

ultra compact object

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Abstract

Using the optical reference geometry approach, we have derived in the following, a general expression for the ellipticity of a slowly rotating fluid configuration using Newtonian force balance equation in the conformally projected absolute 3-space, in the realm of general relativity. Further with the help of Hartle-Thorne (H-T) metric for a slowly rotating compact object, we have evaluated the centrifugal force acting on a fluid element and also evaluated the ellipticity and found that the centrifugal reversal occurs at around $R/R_s \approx 1.45$, and the ellipticity maximum at around $R/R_s \approx 2.75$. The result has been compared with that of Chandrasekhar and Miller which was obtained in the full 4-spacetime formalism.

1 Introduction

One of the enigmatic problems in the context of pulsars is still the understanding of the internal structure of rotating compact objects. As normally one considers the object to be slowly rotating, most of the model calculations have considered the well known approximate solution of Hartle and Thorne [1], [2] as the basic solution for the fluid configuration in the context
of general relativity. It is generally believed that for any such rotating fluid configuration if one considers the ‘force balance’, in the purely Newtonian physics one encounters no strange behaviour irrespective of the size of the compact object as the two traditional rivals the gravitation vis a vis the centrifugal force acting on a fluid element would always be opposing each other. However, in the context of general relativity as has been shown by Abramowicz and Prasanna [3] for a sufficiently small size object, the centrifugal force acting on a test particle of mass $m_0$ in circular orbit outside a static mass $M$ is given by

$$\tilde{F}_{cfg} = \frac{m_0 \tilde{v}^2}{r} (1 - \frac{3M}{r}),$$

where $\tilde{v}$ is the speed of the particle as seen in the conformally projected 3-space of the optical reference geometry (ACL) [4]. As seen from the above expression the centrifugal force would not oppose gravity if the particle is situated at a distance $r \leq 3M$. As there could exist ultra compact bodies [5] of size $2M \leq R \leq 3M$, it would become relevant to consider the effect of such a centrifugal force reversal on a fluid element of a possible ultra compact rotating configuration.

Another important manifestation of the same result viz, introducing Newtonian forces in general relativity is the explanation of ellipticity maximum for a rotating configuration given by Abramowicz and Miller [6], an effect discovered by Chandrasekhar and Miller [7]. Though the explanation given by Abramowicz and Miller is qualitatively viable, quantitatively there appears a difference in the location of the ellipticity maximum, which perhaps is due to the fact that they considered only the Schwarzschild background geometry, which does not take into account the influence of rotation of the central body inherently.

In the following, we reexamine the scenario by studying the possible centrifugal reversal and the ensuing ellipticity maximum for a slowly rotating fluid configuration by adopting the Hartle - Thorne solution which indeed is better suited to study slowly rotating fluid configuration.
We start with a general axisymmetric, stationary fluid configuration and introduce a formalism to treat the four forces on a fluid element in the 3+1 conformal splitting and then adopting Hartle’s solution as a specific example consider the centrifugal force. Using the Newtonian principle of force balance equation for a rotating spheroid we then derive a general expression for the ellipticity and again study its behaviour for the Hartle solution. We find that the result matches closer to that of Chandrasekhar and Miller result thus validating the more general expression derived.

2 Formalism

The equation of motion for a perfect fluid distribution on a general curved space time

\[ ds^2 = g_{ij}dx^i dx^j, \] (2)

are given by

\[ (\rho + p)(U^i_j U^j) = -h^{ij}p,_{j} \] (3)

where \( \rho \) is the matter density, \( p \) the pressure, \( U^i \) the four velocity and \( h^{ij} \) the projection tensor \( (g^{ij} + U^i U^j) \). This may indeed be expressed as the four force acting on a fluid element

\[ f_i := (\rho + p)(U^i_j U^j) + h^{ij}_i p,_{j} \] (4)

\[ = (\rho + p)[U^j \partial_j U_i - \frac{1}{2} U^m U^j \partial_i g_{jm}] + h^{ij}_i p,_{j}, \] (5)

which when \( p = 0 \) and \( \rho = m_0 \), reduces to the well known four force expression acting on a particle \[ \[ m_0 f_i \equiv P^j \partial_j P_i - \frac{1}{2} P^j P^m \partial_i g_{jm}. \] (6)

Using the ACL formalism of 3+1 conformal slicing of the space time

\[ ds^2 = dl^2 - g_{00}(dt + 2\omega^{\alpha} dx^{\alpha})^2, \] (7)
with $dl^2$ representing the positive definite metric of the absolute 3-space $\bar{g}_{\mu\nu} dx^\mu dx^\nu$, equation (5) may be rewritten as

$$ f_0 = \Phi^{-1}(\rho + p) \bar{U}_\mu \partial_\mu U_0 + h_0^\mu p_\mu $$

(8)

$$ f_\alpha = \Phi^{-1}(\rho + p) \left[ \bar{U}_\mu \bar{\nabla}_\mu \bar{U}_\alpha + 2 U^0 \bar{U}_\mu \omega_\mu \alpha + \frac{M_0^2}{2\Phi} \partial_\alpha \Phi \right] $$

+ $2\omega_\alpha f_0 + (h_\alpha^\mu - 2\omega_\alpha h_0^\mu) p_\mu$

(9)

with

$$ M_0^2 = U_0^2 - \bar{g}_{\mu\nu} \bar{U}_\mu \bar{U}_\nu $$

$$ \bar{U}_\mu = \Phi U^\mu $$

$$ \bar{U}_\alpha = \bar{g}_{\alpha\beta} \bar{U}_\beta $$

$$ \omega_{\mu\nu} = \partial_\mu \omega_\alpha - \partial_\alpha \omega_\mu $$

(10)

$$ g_{00} = -\Phi $$

$$ g_{\mu0} = -2\Phi \omega_\mu $$

$$ g_{\mu\nu} = \Phi \bar{g}_{\mu\nu} - 4\Phi \omega_\mu \omega_\nu $$

and $\bar{\nabla}_\mu$ denotes the covariant derivative in the absolute 3-space.

As we are interested in slowly rotating fluid configurations, we shall consider the most general stationary axisymmetric space time metric as given by

$$ ds^2 = -e^{2\nu} dt^2 + e^{2\psi} d\phi^2 + e^{2\mu_1} dr^2 + e^{2\mu_2} d\theta^2 $$

(11)

where $\nu, \psi, \mu_1, \mu_2$ and $\omega$ are functions of $r$ and $\theta$ and $d\dot{\phi} = d\phi - \omega dt$. For this metric when we have only $V^\phi \equiv \frac{d\phi}{dt} \neq 0$, the normalisation condition yields

$$ U^t = \left[ e^{2\nu} - e^{2\psi} (\Omega - \omega)^2 \right]^{-1/2} $$

(12)

where $\Omega = \frac{d\phi}{dt}$ is the angular velocity of the fluid as measured by the stationary observer, and $\omega$ is the angular velocity acquired by an observer falling freely from infinity. As this frame $(t, \phi, r, \theta)$ is static, the equations (8) and
(9) simplify considerably and then one can calculate the 3-component of the force as given by

\[ f_\alpha = \Phi^{-1}(\rho + p) \left[ \bar{U}^\mu \nabla_\mu \bar{U}_\alpha + \frac{M_0^2}{2\Phi} \partial_\alpha \Phi \right] + h^{\mu}_{\alpha} p_{,\mu} \]  

which in fact, when zero, gives the equation of hydrodynamic equilibrium for a rotating fluid configuration. Now given a metric (approximate or post-Newtonian solution) one can calculate the ‘centrifugal acceleration’ term \( \bar{U}^\mu \nabla_\mu \bar{U}_\alpha \) and ‘gravitational acceleration’ term \( \frac{M_0^2}{2\Phi} \partial_\alpha \Phi \), which for the metric (11) yields

\[ F_{cf} = e^{2\nu} \frac{2\psi - (2\psi - \omega)^2}{e^{2\nu} - e^{2\psi}} \]  

\[ F_g = e^{2\nu} \]  

where prime denotes differentiation with respect to \( r \).

**Ellipticity**

It is well known that a rotating fluid configuration deviates from spherical symmetry and depending upon the degree of rotation the equatorial diameter expands whereas the polar diameter contracts thereby producing a change in shape. The various equilibrium configurations of rotating fluids have been well discussed in the literature, and the sequence goes through Maclaurin spheroids to Jacobi ellipsoids [9]. For slowly rotating configuration one can consider the Maclaurin spheroid with the ellipticity defined through the usual definition of

\[ \epsilon = \frac{1 - (1 - e^2)^{1/2}}{(1 - e^2)^{1/6}}, \]

\( e \) being the eccentricity defined as \( e = (1 - b^2/a^2)^{1/2} \), where \( b \) and \( a \) are polar and equatorial radii respectively.
Maclaurin had shown that the acceleration due to gravity at the equator and pole has the values

\[
g_{\text{equator}} = 2\pi G \rho a \left( \frac{1-e^2}{e^2} \right)^{1/2} \left[ \sin^{-1}e - e(1-e^2)^{1/2} \right]
\]

\[
g_{\text{pole}} = 4\pi G \rho a \left( \frac{1-e^2}{e^2} \right)^{1/2} \left[ e - (1-e^2)^{1/2} \right] \sin^{-1}e,
\]

wherein he had considered the possible effects that could arise due to the internal stresses in the body. However, as we are looking for a solution in general relativity, wherein the gravitational field inside the body is described through a metric which is a solution of Einstein’s equations for a perfect fluid distribution, the gravitational potentials \( g_{ij} \) would be incorporating the effects of all characteristics of the fluid distribution. With this proviso, in the new language of optical reference geometry it is sufficient to consider the modified expression for the gravitational and centrifugal accelerations as given by (14) and use the Newtonian force balance equation to relate the eccentricity with the acceleration. Thus generalising the Newtonian equation

\[
g_{\text{equator}} - a\Omega^2 = g_{\text{pole}}(1-e^2)^{1/2}
\]

(17)

to

\[
F_{ge} - F_{cf} = F_{gp}(1-e^2)^{1/2}
\]

(18)

and using the force expression as given by

\[
\theta = \pi/2 : F_{ge} = e^{2\nu_0(r,\pi/2)}\nu_0'(r,\pi/2),
\]

(19)

\[
\theta = 0 : F_{gp} = e^{2\nu_0(r,0)}\nu_0'(r,0),
\]

(20)

and \( F_{cf} \) as in (14), the eccentricity of the configuration would be given by

\[
e^2 = \left( 1 - \left[ \frac{F_{cf} - F_{ge}}{F_{gp}} \right]^2 \right)
\]

(21)

and the ellipticity in the limit of slow rotation \( e << 1 \),

\[
\epsilon = \frac{1}{2} e^2.
\]

(22)
Hartle’s Solution

For a slowly rotating perfect fluid configuration Hartle has obtained an approximate solution of Einstein’s equations given as

\[ ds^2 = \left[ -e^{\nu_0} (1 + 2(h_0 + h_2 P_2)) \right] dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} \left[ 1 + 2\frac{(m_0 + m_2 P_2)}{(r - 2M)} \right] dr^2 + r^2 [1 + 2(\nu_2 - h_2) P_2] \]
\[ d\theta^2 + \sin^2 \theta (d\phi - \omega dt)^2 \] + \mathcal{O}(\Omega^3), \tag{23}

which represents the rotation as perturbation over the non-rotating metric

\[ ds^2 = -e^{\nu_0}(r)dt^2 + [1 - 2M(r)]^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{24} \]

Expressing the forces as obtained above in terms of the H-T potentials one gets

\[ F_{cf} = r^2\bar{\omega}^2(1/r - \nu'_0/2), \tag{25} \]
\[ F_{ge} = \frac{1}{2}e^{\nu_0}[\nu'_0(1 + 2h_0 - h_2) + 2h'_0 - h'_2], \tag{26} \]
\[ F_{gp} = \frac{1}{2}e^{\nu_0}[\nu'_0(1 + 2h_0 + 2h_2) + 2h'_0 + 2h'_2], \tag{27} \]
yielding for the ellipticity

\[ \epsilon = 3(h_2 + h'_2/\nu'_0) + \frac{r^2\bar{\omega}^2}{e^{\nu_0}} (2/r\nu'_0 - 1). \tag{28} \]

3 Results and discussion

In the present work we have derived the general expressions for the centrifugal acceleration \( F_{cf} \) (using ACL formalism) and ellipticity \( \epsilon \) (replacing Newtonian acceleration by relativistic counterparts in the force balance equation
Using H-T solution we obtain the values of the centrifugal force and the ellipticity for a slowly rotating configuration. For the sake of comparison we have also calculated the ellipticity as given by Chandrasekhar and Miller

\[ \epsilon_{H-T}(r) = -\frac{3}{2} \left[ \frac{\xi_2(r)}{r} + v_2(r) - h_2(r) \right]. \]

(29)

where

\[ \xi_2 = 2p^*_2/(d\nu_0/dr), \quad p^*_2 = -h_2 - \frac{1}{3}r^2e^{-\nu_0}\tilde{\omega}^2. \]

(30)

Writing the expressions in dimensionless units:

\[ \tilde{F}_{ef} = \frac{F_{ef}}{(G^2J^2/c^4R_s^5)}, \quad \tilde{\epsilon} = \frac{\epsilon}{(G^2J^2/c^6R_s^4)}, \]

(31)

where \( J \) is the angular momentum and \( R_s(=2M) \) is the Schwarzschild radius, we have evaluated the quantities for a series of homogeneous configurations with decreasing radii keeping \( M \) and \( J \) conserved, \( \tilde{\epsilon} \) denotes our calculations (equation (28)) whereas \( \tilde{\epsilon}_{H-T} \) denotes the values for corresponding configurations as obtained using equation (29) in table 1.

Comparing our present result with that of Abramowicz and Miller, who had obtained the maximum at \( R/R_s = 3 \), using pure Schwarzschild geometry, we see that incorporating the effects of rotation in the geometry (even approximately) improves the result as the maximum \( R/R_s \approx 2.75 \) shifts closer to that obtained by Chandrasekhar and Miller \( R/R_s \approx 2.3 \) (fig 2).

Regarding the centrifugal force the general expression obtained above does show the reversal at \( R/R_s \approx 1.45 \) and a maximum at \( R/R_s \approx 2.1 \) (fig 1). It is interesting to note that even after including the effects of fluid distribution in the space time geometry, the centrifugal force reversal seems to occur closer to the value as was known in the Schwarzschild geometry. However, as the ellipticity maximum indicates a possible change in shape of the configuration, it is to be noted that our expression shows that for a collapsing configuration, the change occurs earlier \( (R/R_s \approx 2.75) \) than what had been obtained by Chandrasekhar and Miller \( (R/R_s \approx 2.3) \). As the shape of the body does
depend upon the ellipticity as its value starts decreasing after reaching a maximum, the body would in principle tend towards a different shape from that of a spheroid. This could in principle introduce non axisymmetric deformation in the structure of the body which might generate gravitational radiation. However a full significance of the result obtained would become clear only after a more detailed analysis which takes into account inhomogeneity in the fluid configuration as well as from more realistic equations of state, than that of constant density used in the above analysis.

References

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Figure Captions

Fig. 1 Plots for centrifugal force $\bar{F}_{cf}$ in units ($G^2 J^2 / c^4 R_s^5$) for decreasing values of radius $R$ in terms of Schwarzschild radius $R_s$.

Fig. 2 This shows two curves of ellipticity. The solid line corresponds to our calculation $\bar{\epsilon}$ and the dotted line represents the $\bar{\epsilon}_{H-T}$ as used by Chandrasekhar and Miller. Both the quantities are in units of ($G^2 J^2 / c^8 R_s^4$).

Table Captions

Table 1 Shows the ellipticity $\bar{\epsilon}$ (equation 28), $\bar{\epsilon}_{H-T}$ (equation 29) and the centrifugal force $\bar{F}_{cf}$ (equation 25) (units of these quantities are described in equation (31)) for a sequence of decreasing radius with conserved mass and angular momentum for a homogeneous distribution.
| $R/R_s$ | $\bar{\epsilon}_{H-T}$ | $\bar{\epsilon}$ | $\bar{F}_{cJ}$ |
|--------|----------------|----------------|---------------|
| 1.125  | 5.604732E+0   | 9.135673E+0   | -1.105476E+0 |
| 1.150  | 6.090158E+0   | 9.419599E+0   | -1.111393E+0 |
| 1.200  | 6.728176E+0   | 9.973936E+0   | -1.106851E+0 |
| 1.300  | 7.970848E+0   | 1.121079E+1   | -6.367954E-1 |
| 1.400  | 9.033281E+0   | 1.249040E+1   | -2.781407E-1 |
| 1.500  | 9.893101E+0   | 1.370501E+1   | 1.036231E-4  |
| 1.600  | 1.056893E+1   | 1.479904E+1   | 1.982318E-1  |
| 1.700  | 1.108810E+1   | 1.575131E+1   | 3.318464E-1  |
| 1.800  | 1.147746E+1   | 1.656041E+1   | 4.172701E-1  |
| 1.900  | 1.176069E+1   | 1.723471E+1   | 4.679588E-1  |
| 2.000  | 1.195771E+1   | 1.778676E+1   | 4.941394E-1  |
| 2.100  | 1.208496E+1   | 1.823045E+1   | 5.033122E-1  |
| 2.200  | 1.215588E+1   | 1.857943E+1   | 5.008806E-1  |
| 2.300  | 1.218139E+1   | 1.884639E+1   | 4.906997E-1  |
| 2.400  | 1.217034E+1   | 1.904277E+1   | 4.755023E-1  |
| 2.500  | 1.212995E+1   | 1.917867E+1   | 4.572160E-1  |
| 2.600  | 1.206606E+1   | 1.926295E+1   | 4.371926E-1  |
| 2.700  | 1.198343E+1   | 1.930324E+1   | 4.163726E-1  |
| 2.750  | 1.193634E+1   | 1.930894E+1   | 4.058754E-1  |
| 2.800  | 1.188595E+1   | 1.930607E+1   | 3.954040E-1  |
| 2.900  | 1.177679E+1   | 1.927725E+1   | 3.747254E-1  |
| 3.000  | 1.165855E+1   | 1.922168E+1   | 3.546271E-1  |
| 4.000  | 1.029998E+1   | 1.788030E+1   | 2.021015E-1  |
| 5.000  | 9.029724E+0   | 1.615380E+1   | 1.209157E-1  |
| 10.000 | 5.351696E+0   | 1.014123E+1   | 1.982923E-2  |
| 20.000 | 2.896627E+0   | 5.641617E+0   | 2.794844E-3  |
| 35.000 | 1.710614E+0   | 3.370030E+0   | 5.475721E-4  |
| 50.000 | 1.213098E+0   | 2.400532E+0   | 1.914418E-4  |
| 80.000 | 7.668072E-1   | 1.523602E+0   | 4.751822E-5  |
| 100.000| 6.157617E-1   | 1.224927E+0   | 2.446291E-5  |
FIG. 1
FIG. 2