Model-free Nearly Optimal Control of Constrained-Input Nonlinear Systems Based on Synchronous Reinforcement Learning

Han Zhao, Lei Guo
1. School of Artificial Intelligence, Beijing University of Posts and Telecommunications, Beijing 100876, P. R. China
E-mail: han_zhao@bupt.edu.cn

Abstract: In this paper a novel model-free algorithm is proposed. This algorithm can learn the nearly optimal control law of constrained-input systems from online data without requiring any a priori knowledge of system dynamics. Based on the concept of generalized policy iteration method, there are two neural networks (NNs), namely actor and critic NN to approximate the optimal value function and optimal policy. The stability of closed-loop systems and the convergence of weights are also guaranteed by Lyapunov analysis.

Key Words: Optimal Control, Synchronous Reinforcement Learning, Constrained-Input, Actor-Critic Networks

1 Introduction

The control of systems with input constraint has been a widely researched and is a valuable practical problem. In the framework of optimal control, a optimal policy that minimize a user-defined performance index and the corresponding value function can be obtained by solving Hamilton-Jacobi-Bellman (HJB) equation, which is a partial differential equation in non-linear systems and the analytical solution is usually hard to find. Therefore, function approximation using neural networks (NNs) is a useful tool to solve HJB equation. These algorithms belong to the family of adaptive dynamic programming (ADP) or reinforcement learning (RL) methods.

A considerable number of literatures have focused on solving the continuous-time constrained-input optimal control problem based on ADP/RL. In [1], an off-line policy iteration (PI) algorithm that can iteratively solve HJB equation is proposed. The full knowledge of system dynamics and an initial stabilized controller are needed in this algorithm. In these years, many works are intend to relax the restriction of the a priori knowledge of system dynamics. The concept of integral reinforcement learning (IRL, [2]) and generalized PI [3] is brought in [4, 5] to estimate the optimal value function in constrained-input problems. The initial stabilized controller is not needed in these methods but the input gain matrix of the system is still used in policy improvement step.

Another way to solve this problem is to use an identifier NN. Xue et al. [6] proposed an algorithm based on actor-critic-identifier structure to estimate the system dynamics and solve HJB equation online. Comparing with other model-free methods, an extra identifier might brings additional burden on real-time computing.

In terms of IRL, Vamvoudakis [7, 8] proposed a completely model-free algorithm based on the concept of Q-learning [9] to solve continuous-time optimal control problem. Q-function combines the value function and the Hamiltonian on Pontryagin’s minimum principle and provides the information of policy improvement without requiring any a priori. Similarly, the restriction of the system input dynamics can be relaxed by adding exploration signal [10]. Our previous work [11] also present a novel model-free algorithm that combines the exploration signal and synchronous reinforcement learning to solve the adaptive optimal control problem in continuous-time nonlinear systems.

In this paper we present an algorithm to solve the continuous-time constrained-input optimal control problem without knowing any information of system dynamics or a initial stabilized controller. The identifier NN in [6] is also not needed in this method. The remainder of this paper is organized as follows: Section 2 provides the mathematic formulation of the optimal control for continuous-time systems with input constraints. In sention 3 we present the weights tuning law of actor-critic NNs based on exploration-HJB and synchronous reinforcement learning. In the process of learning, policy evaluation and improvement can be done simultaneously and no knowledge of system dynamics is required. The convergence of weights and stability of the closed-loop system is proved through Lyapunov analysis. Finally we design two simulations to verify the effectiveness of our method.

2 Problem Formulation

Consider a continuous-time input-affine system

\[ \dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad x(0) = \xi, \]  

(1)

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) is the measurable state and input vector respectively. \( \xi \) denotes the initial state of the system. \( f(x) \in \mathbb{R}^n \) is called as the system drift dynamics and \( g(x) \in \mathbb{R}^{n \times m} \) is the system input dynamics. The system dynamics \( f(x) + g(x)u \) is assumed to be Lipschitz in a compact set \( \Omega \) and satisfies \( f(0) = 0 \).

The objective of optimal control is to design a controller that minimize a user-defined performance index. The performance index in this paper is defined as

\[ J(x, u) = \int_0^\infty (Q(x(\tau)) + U(u(\tau))) \, d\tau, \]  

(2)
both $Q(x)$ and $U(u)$ are positive definite functions.

In this paper, the constrained-input control problem is considered. The input vector needs to satisfy the constraint $|u_i| \leq \lambda(i = 1, ..., m)$. To deal with this constraint, a non-quadratic integral cost function of input is usually employed \cite{12}:

$$U(u) = 2 \int_0^u \lambda \theta^{-1}(s) R ds,$$  \hspace{1cm} (3)

where $\theta(s) = [\theta_1(s), ..., \theta_m(s)]^T$ and $\theta^{-1}(s) = [\theta_1^{-1}(s), ..., \theta_m^{-1}(s)]^T$. $\theta$ is a bounded one-to-one monotonic odd function. the hyperbolic tangent function $\theta(\cdot) = \tanh(\cdot)$ is often used. $R$ is a positive definite symmetric matrix. For the convenience of computing and analysis, $R$ is assumed to be diagonal, i.e. $R = \text{diag}(r_1, ..., r_m)$.

The goal is to find the optimal control law $u^*$ that can stabilize system \ref{1} and minimize performance index \ref{2}. The following value function is used:

$$V^\mu = J(x, u)|_{u=\mu(x)}, V^\mu(0) = 0,$$  \hspace{1cm} (4)

where $\mu(x)$ is a state feedback policy and $\mu(0) = 0$. The value function is well-defined only if the policy is admissible \cite{1}.

**Definition 1** A constrained policy is admissible in $\Omega$, which is denoted as $\mu \in \mathcal{A}(\Omega)$, if:

1) The policy stabilize system \ref{1} in $\Omega$, i.e.

$$\lim_{t \to \infty} \xi + \int_0^t (f + g\mu) d\tau = 0;$$

2) For any state $x \in \Omega$, $V^\mu(x)$ is finite;

3) For any state $x \in \Omega$, $|\mu_i(x)| \leq \lambda(i = 1, ..., m)$.

$\Omega$ is called as admissible domain and $\mathcal{A}(\Omega)$ is the set of admissible policy.

Assume that the set of admissible policy $\mathcal{A}(\Omega)$ of system \ref{1} is not empty and $V^\mu \in C^1(\Omega)$. It is obvious that there exists an optimal policy $\mu^*(x) \in \mathcal{A}(\Omega)$ so that

$$V^{\mu^*}(\xi) = \min_{\mu \in \mathcal{A}(\Omega)} \int_0^\infty (Q(x(\tau) + U(\mu(\tau)))) d\tau, \forall \xi \in \Omega.$$  \hspace{1cm} (5)

The optimal value function satisfies $V^{\mu^*}(x) = V^{\mu^*}(x)$. In the remainder of the paper both of these two symbols may be used to describe the same variable.

According to the definition of the value function \ref{4}, the infinitesimal version of \ref{5} is obtained as

$$0 = Q(x) + U(\mu(x)) + (\nabla V^{\mu^*}(x))^T(f(x) + g(\mu(x))),$$  \hspace{1cm} (6)

where $\nabla V(x) = \partial V(x)/\partial x \in \mathbb{R}^n$. Define the Hamiltonian as

$$H(x, u, \nabla V^{\mu^*}) = Q(x) + U(u) + (\nabla V^{\mu^*})^T(f(x) + g(x)u).$$  \hspace{1cm} (7)

For optimal value function $V^{\mu^*}$ and optimal control law $u^*$, the following Hamilton-Jacobi-Bellman (HJB) equation is satisfied:

$$0 = Q(x) + U(u^*) + \nabla V^{\mu^*T}(f(x) + g(x)u^*),$$  \hspace{1cm} (8)

$$V^{\mu^*}(0) = 0.$$  \hspace{1cm} (9)

The optimal policy can be derived by minimizing the Hamiltonian, for input-affine system \ref{1} and the performance index \ref{2} and \ref{5}, the optimal policy can be directly denoted as

$$u^*(x) = \lambda \theta \left( -\frac{1}{2\lambda} R^{-1} g^T(x) \nabla V^{\mu^*}(x) \right).$$  \hspace{1cm} (10)

Let

$$v^*(x) = -\frac{1}{2} R^{-1} g^T(x) \nabla V^{\mu^*}(x),$$  \hspace{1cm} (11)

the optimal policy can be obtained as

$$u^*(x) = \lambda \theta \left( \frac{v^*(x)}{\lambda} \right).$$  \hspace{1cm} (12)

It is hard to solve the optimal value function and policy in \ref{7} directly and the a priori knowledge of system dynamics is required. Therefore, the online data-driven algorithm is a useful tool to relax the restriction.

### 3 Online Synchronous Reinforcement Learning Algorithm to Solve HJB Equation

In this section a data-driven method that learning the optimal policy is introduced. The framework of integral reinforcement learning (IRL) is proposed in \cite{2}. A completely model-free algorithm based on IRL is developed by Lee et al. \cite{10} to solve the optimal control problem of input-affine systems. However, the initial stabilized controller is still required to start the iteration.

In our previous work \cite{11}, a algorithm called synchronous integral Q-learning is proposed, which can continuously update the weights of both two NNs to estimate the optimal value function and policy. In this paper we extend the algorithm to solve the constrained-input problem. By adding a bounded piece-wise continous non-zero exploration signal $e$ into input, the system \ref{1} is derived as

$$\dot{x} = f(x) + g(x)(u + e).$$  \hspace{1cm} (13)

Substituting \ref{11} into HJB equation \ref{7}. For any time $t > T$ and time interval $T > 0$, integrating \ref{7} from $[t - T, t]$ along the trajectory of system \ref{11}, the following integral temporal difference equation is obtained as

$$V^*(x(t)) - V^*(x(t - T)) - \int_{t-T}^t \nabla V^{\mu^*T}(x)g(x)e(\tau)d\tau$$

$$= -\int_{t-T}^t (Q(x) + U(u^*)) d\tau.$$  \hspace{1cm} (14)

The following HJB equation with signal $e$ is derived by substituting \ref{9}:

$$V^*(x(t)) - V^*(x(t - T)) + 2 \int_{t-T}^t e^T Re(\tau)d\tau$$

$$= -\int_{t-T}^t (Q(x) + U(u^*)) d\tau.$$  \hspace{1cm} (15)
Both $V^*(x)$ and $v^*(x)$ can be solved in integral exploration-HJB equation \[13\] and the optimal policy is computed by \[10\]. Note that the knowledge of system dynamics is not needed in these two equations.

Since the analytical solution of $V^*(x)$ and $v^*(x)$ cannot be determined. We can approximate them by choosing a proper structure of neural networks (NNs). Assume $V^*(x)$ is a smooth function. Based on the assumption of system dynamics, $v^*(x)$ is also smooth. Therefore, there exists two single-layer NNs, namely actor and critic NN, so that:

1) $V^*(x)$ and its gradient can be universal approximated:

$$V^*(x) = w_c^T \phi_c(x) + \varepsilon_c(x),$$

$$\nabla V^*(x) = \nabla \phi_c^T(x) w_c + \nabla \varepsilon_c(x),$$

2) $v^*(x)$ can be universal approximated:

$$v^*(x) = -\frac{1}{2} R^{-1} g^T(\nabla \phi_c^T(x) w_c + \nabla \varepsilon_c(x))$$

$$= w_a^T \phi_a(x) + \varepsilon_a(x),$$

where $\phi_c(x) : \mathbb{R}^n \to \mathbb{R}^{N_c}, w_c, \varepsilon_c$ is the activation function, weights and the reconstruction error of the critic NN, respectively. $N_c$ is the number of neurons in the hidden layer of the critic NN. Similarly, in actor NN, we denote $\phi_a(x) : \mathbb{R}^n \to \mathbb{R}^{N_a}, w_a, \varepsilon_a$ is the activation function, weights and the reconstruction error, respectively. $N_a$ is the number of neurons in the hidden layer of the actor NN. $\varepsilon_c(x)$ and $\varepsilon_a(x)$ are bounded on a compact set $\Omega$. According to Weierstrass approximation theorem, when $N_c, N_a \to \infty$, we have $\varepsilon_c, \nabla \varepsilon_c, \varepsilon_a \to 0$.

We define the Bellman error $\varepsilon_B$ based on \[13\] as

$$w_c^T \Delta \phi_c + 2 \int_{T-T}^t \phi_a^T(x(\tau)) w_c R e(\tau) d\tau$$

$$+ \int_{T-T}^t (Q(x(\tau)) + U(u^*(x(\tau))) d\tau = \varepsilon_B,$$

where $\Delta \phi_c = \phi_c(x(t)) - \phi_c(x(t - T))$. Define the integral reinforcement as

$$\rho(x,u) = \int_{T-T}^t (Q(x(\tau)) + U(u(x(\tau))) d\tau.$$

Using the characteristic of Kronecker product, the following equation can be derived from \[17\]:

$$\varepsilon_B - \rho(x,u^*) = W^T \delta,$$

where

$$W = [w_c^T, \text{col}\{w_a\}^T]^T,$$

$\text{col}\{w_a\}$ is the reshaped column vector of $w_a$, and

$$\delta = [\Delta \phi_c^T, (\int_{T-T}^t 2\phi_a \otimes (Re(\tau))d\tau)^T]^T.$$

Before introducing the weights tuning law, it is necessary to make some assumptions of NNs.

**Assumption 1** The system dynamics and NNs satisfy:

a. $f(x)$ is Lipschitz and $g(x)$ is bounded.

$$\|f(x)\| \leq b_f \|x\|, \|g(x)\| \leq b_g.$$

b. The reconstruction errors and the gradient of critic NN’s error is bounded.

$$\|\varepsilon_c\| \leq b_{\varepsilon_c}, \|\varepsilon_a\| \leq b_{\varepsilon_a}, \|\nabla \varepsilon_c\| \leq b_{\varepsilon_{ccx}}.$$

c. The activation functions and the gradient of critic NN’s activation function is bounded.

$$\|\phi_c(x)\| \leq b_{\phi_c}, \|\phi_a(x)\| \leq b_{\phi_a}, \|\nabla \phi_c(x)\| \leq b_{\phi_{ccx}}.$$

d. The optimal weights of NNs $\|w_c\|, \|w_a\|$ are bounded, and the amplitude of exploration signal is bounded by a constant $\nu_c$.

These assumptions are trivial and can be easily satisfied except the assumption of $g(x)$. With the assumption that system dynamics is Lipschitz, Bellman error $\varepsilon_B$ is bounded on a compact set. When $N_c, N_a \to \infty$, $\varepsilon_B \to 0$ universally \[1\].

In this paper we use $\varepsilon(x) = \tanh(\cdot)$. The non-quadratic cost $U(u)$ is derived as

$$U(u) = 2 \int_0^\mu \lambda \tanh^{-1}(\frac{u}{\lambda}) R d\sigma$$

$$= 2\lambda u^T R \tanh^{-1}(\frac{u}{\lambda}) + \lambda^2 R \ln \left(1 - (\frac{u}{\lambda})^2\right),$$

where $\overline{R} = [r_1, \ldots, r_m], 1_m$ is the $m$-dimensional vector with all elements are one. According to \[17\] and \[19\], the approximate exploration-HJB equation is obtained as

$$\int_{T-T}^t \left(-Q - 2\lambda \tanh^T(\frac{w_c^T \phi_a}{\lambda}) R w_a^T \phi_a \right.$$

$$-\lambda^2 R \ln \left(1_m - \tanh^2(\frac{w_a^T \phi_a}{\lambda}) + \varepsilon_{HJB}\right) d\tau = W^T \delta,$$

in which $\varepsilon_{HJB}(x)$ is the error raised from the reconstruction errors of two NNs. Since the ideal weights $w_b$ and $w_a$ are unknown, they are approximated in real time as

$$\hat{V}(x) = w_c^T \phi_c(x),$$

$$\hat{v}_1(x) = w_a^T \phi_a(x).$$

The optimal policy is obtained as

$$u(x) = \lambda \tanh \left(\frac{w_a^T \phi_a(x) + \varepsilon_a(x)}{\lambda}\right).$$

We use an actor NN to approximate the optimal policy as

$$\hat{u}(x) = \lambda \tanh \left(\frac{\hat{v}_2(x)}{\lambda}\right),$$

where $\hat{v}_2(x) = w_a^T \phi_a(x)$. Note that $\hat{v}_1$ and $\hat{v}_2$ have the same approximate structure. However, the estimated weights in $\hat{v}_2$ will not be directly implemented on the controller for Lyapunov stability. Define $\hat{W} = [w_c^T, \text{col}\{w_a\}]^T$, the approximate Bellman error of the critic NN is obtained as

$$E = \hat{W}^T \delta + R(x, \hat{u}).$$
In order to minimize the error, we define the objective function of critic NN as
\[ K = \frac{1}{2} E^T E. \] (25)

Based on the objective function, the modified gradient-descent law is obtained as
\[ \dot{\hat{w}} = -\alpha_1 \frac{\delta}{1 + \delta^T \delta} E, \] (26)
where \( \alpha_1 \) is the learning rate that determines the speed of convergence. In order to guarantee the stability of the closed-loop system and the convergence. Define the approximate error of the actor NN as
\[ E_u = \lambda R \left( \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) - \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) \right) \] (27)
The gradient-descent tuning law of \( \hat{w}_a \) is set as
\[ \text{col}\{\hat{w}_a\} = -\alpha_2 \left( Y \text{col}\{\hat{w}_a\} + E_u \otimes \phi_a - \left( E_u \tanh^2(\frac{\bar{w}_a^T \phi_a}{\lambda}) \right) \otimes \phi_a \right), \] (28)
where \( Y \) is a designed parameter to guarantee stability. Define \( m_\delta = 1 + \delta^T \delta \) and \( \overline{\delta} = \delta / m_\delta \), the following theorem is derived.

**Theorem 1** If all terms in Assumption 1 is satisfied and signal \( \overline{\delta}(t) \) is persistently excited [13], i.e., there exists two constant \( \beta_1 < \beta_2 \) such that \( \beta_1 I \leq \int_{-T}^{t-T} \overline{\delta} \, d\tau \leq \beta_2 I, \forall t > T. \) There exists a positive integer \( N_\delta \) and a sufficiently small reinforcement interval \( T > 0 \) such that, when the number of neurons \( N_c, N_a > N_\delta \), the states in closed-loop system, the estimated error of \( \hat{V}(x), \hat{v}_1(x) \) and \( \hat{u}(x) \) are uniformly ultimately bounded.

### 4 Proof of Theorem 1

Define the error of weights as \( \hat{w}_c = w_c - \hat{w}_c, \hat{w}_a = w_a - \hat{w}_a, \) and \( \hat{w}_a = w_a - \hat{w}_a. \) Consider the following Lyapunov function
\[ L = V^*(x(t)) + \frac{1}{2} \hat{W}^T(t) \alpha_1^{-1} \hat{W}(t) + \frac{1}{2} \hat{w}_a^T(t) \alpha_2^{-1} \hat{w}_a(t). \] (29)
The derivative of (29) to time is
\[ \dot{L} = \dot{V}^*(x(t)) + \hat{W}^T(t) \alpha_1^{-1} \dot{\hat{W}}(t) + \hat{w}_a^T(t) \alpha_2^{-1} \dot{\hat{w}_a}(t). \] (30)
The first term in (30) is obtained as
\[ \dot{V}^* = w_c^T \nabla \phi_c(f + g \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) + ge) + \nabla \varepsilon_c(f + g \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) + ge) \] (31)
From (16), we have
\[ w_c^T \nabla \phi_c g \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) = -2 \phi_a^T w_a R \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) - 2 \varepsilon_c^T g \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}). \] (32)
Using \( w_a = \bar{w}_a + \tilde{w}_a \) and the fact that \( x^T \tanh(x) \geq 0, \forall x \)
for the first term on the right side of (32), it can be obtained as
\[-2 \phi_a^T w_a R \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) \leq -2 \phi_a^T \tilde{w}_a R \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}). \] (33)
Therefore (31) turns into
\[ \dot{V}^* \leq -Q - U \left( \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) \right) + \varepsilon_{HJB} + \varepsilon_1 \] (34)
where \( \varepsilon_1 = \nabla \varepsilon_c^T (f + ge) - 2 \varepsilon_a^T R \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}). \) According to assumption II it satisfies
\[ \varepsilon_1 \leq b_{exc} b_f \| x \| + b_{exc} b_e b_e + 2 \lambda b_{exc} \sigma_{min}(R), \] (35)
where \( \sigma_{min}(R) \) denotes the minimum singular value of matrix \( R. \) Substituting the derivative of the approximate HJB equation (20) into (34), we have
\[ \dot{V}^* \leq -Q - U \left( \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) \right) + \varepsilon_{HJB} + \varepsilon_1 \] (36)
The last term in (36) satisfies
\[ -w_c^T \nabla \phi_c g \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) \leq \lambda b_e b_{exc} \| w_c \|. \] (37)
Since \( Q \) and \( U \) are positive definite, there exists a \( q > 0 \) such that \( x^T q x < Q + U. \) Therefore (36) can be derived as
\[ \dot{V}^* \leq -x^T q x + k_1 \| x \| + k_2 - 2 \phi_a^T \tilde{w}_a R \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}), \] (38)
where
\[ k_1 = b_{exc} b_f, \]
\[ k_2 = b_{exc} b_e b_e + 2 \lambda b_{exc} \sigma_{min}(R) + \lambda b_e b_{exc} \| w_c + \varepsilon_h, \]
\( \varepsilon_h \) is the upper bound of \( \varepsilon_{HJB}. \)

Now analyzing the second term in (30). According to the tuning law (26), we have the following error dynamics
\[ \hat{W}(t) = \alpha_1 \overline{\delta} / m_\delta E(t) \] (39)
\[ \gamma(t) = \overline{\delta}^T \hat{W}(t). \]
Substituting HJB equation (20) into (39), it can obtained as
\[ E(t) = \hat{W}^T(t) \delta(t) + \int_{t-T}^{t} (Q + \bar{U}) d\tau \]
\[ = \int_{t-T}^{t} \left( Q + U(\hat{u}) - Q - U(u^*) + \varepsilon_{HJB} \right) d\tau \]
\[ + \hat{w}_a^T(t) \nabla \phi_c(f + g \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) + ge) \]
\[ - w_c^T \nabla \phi_c (f + g \lambda \tanh(\frac{\bar{w}_a^T \phi_a}{\lambda}) + ge) \] (40)
\[ - 2 \text{col}\{\hat{w}_a\} \int_{t-T}^{t} \phi_a \otimes R \, d\tau, \]
According to the definition of $U$, we have
\[
U(\dot{u}) - U(u^*) = \\
2\lambda^2 R \left( \lambda \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) - \tanh(\frac{w_{a2}^T \phi_a}{\lambda}) \right) \\
+ \lambda^2 R \left( \ln \left( 1 - \tanh^2(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) \right) \right) \\
- \lambda^2 R \left( \ln \left( 1 - \tanh^2(\frac{w_{a2}^T \phi_a}{\lambda}) \right) \right)
\] (41)

According to \([4]\), term $\ln \left( 1 - \tanh^2(\frac{w_{a2}^T \phi_a}{\lambda}) \right)$ can be written as
\[
\ln \left( 1 - \tanh^2(\frac{w_{a2}^T \phi_a}{\lambda}) \right) = \ln 4 - 2\frac{w_{a2}^T \phi_a}{\lambda} \text{sgn}(\frac{w_{a2}^T \phi_a}{\lambda}) + \varepsilon_D,
\] (42)
where $||\varepsilon_D|| \leq \ln 4$. Similarly, we have
\[
\ln \left( 1 - \tanh^2(\frac{w_{a2}^T \phi_a}{\lambda}) \right) = \ln 4 - 2\frac{w_{a2}^T \phi_a}{\lambda} \text{sgn}(\frac{w_{a2}^T \phi_a}{\lambda}) + \varepsilon_D,
\] (43)
where $||\varepsilon_D|| \leq \ln 4$. The following inequality is derived from \([41]\):
\[
U(\dot{u}) - U(u^*) = \lambda^2 R (\varepsilon_D - \varepsilon_D) \\
2\lambda^2 R \left( \lambda \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) - \tanh(\frac{w_{a2}^T \phi_a}{\lambda}) \right) \\
+ \lambda^2 R \left( \ln \left( 1 - \tanh^2(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) \right) \right) \\
- \lambda^2 R \left( \ln \left( 1 - \tanh^2(\frac{w_{a2}^T \phi_a}{\lambda}) \right) \right).
\] (44)

In \([4]\), a continuous approximation of $x \text{sgn}(x)$ is provided:
\[
0 \leq x \text{sgn}(x) - x \tanh(\kappa x) \leq \frac{3.59}{\kappa}.
\] (45)
Combining \((44)\) and \((45)\), the approximate HJB error can be obtained as
\[
E(t) = \int_{t-T}^{t} \left( 2\lambda^2 R \left( \frac{\tilde{w}_{a2}^T \phi_a}{\lambda} \right) \\
- 2\lambda^2 R \left( \frac{w_{a2}^T \phi_a}{\lambda} \right) + \varepsilon_D - \varepsilon_D \right) + \varepsilon_D + \tilde{\varepsilon}_H J B \\
+ 2\lambda^2 R \left( \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) - \tanh(\frac{w_{a2}^T \phi_a}{\lambda}) \right) \\
+ 2\lambda^2 R \left( \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) - \tanh(\frac{w_{a2}^T \phi_a}{\lambda}) \right) + \lambda^2 R (\varepsilon_D - \varepsilon_D) \\
+ \tilde{w}_{c}^T (t) \nabla \phi_c(f + g) + \tilde{w}_{c}^T (t) \nabla \phi_c g \lambda \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) \\
- \tilde{w}_{c}^T \nabla \phi_c(f + g) - \tilde{w}_{c}^T \nabla \phi_c g \lambda \tanh(\frac{w_{a2}^T \phi_a}{\lambda}) d\tau \\
- 2\text{col}(w_{a1}(t))^T \int_{t-T}^{t} \phi_a \otimes \text{Red} \tau,
\] (46)
in which the approximate error satisfies $0 \leq \varepsilon_D \leq 7.18/\kappa$.

Substituting \((42)\) into \((40)\), we can get
\[
E(t) = -\delta^T \tilde{W}(t) + \int_{t-T}^{t} \text{col}(\tilde{w}_{a2}(\tau))^T M d\tau + N(t),
\] (47)
where
\[
M = 2\lambda \phi_a \otimes R \left( \lambda \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) - \tanh(\frac{w_{a2}^T \phi_a}{\lambda}) \right),
\]
\[
N(t) = \int_{t-T}^{t} (\varepsilon_H J B + \lambda^2 R (\varepsilon_D - \varepsilon_D) + \varepsilon_D + \varepsilon_2 + 2\lambda^2 R \left( \text{sgn}(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) - \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) \right) ) d\tau,
\]
and
\[
\varepsilon_2 = (2\varepsilon_a R + \nabla \varepsilon_c g) \lambda \left( \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) - \tanh(\frac{w_{a2}^T \phi_a}{\lambda}) \right).
\]

Substituting \((47)\) into \((39)\), the error dynamics can be rewritten as
\[
\dot{W}(t) = \alpha_1 \left( -\delta^T \tilde{W}(t) \\
+ \frac{\delta}{m_s} \int_{t-T}^{t} \text{col}(\tilde{w}_{a2})^T M d\tau + \frac{\gamma}{m_s} N(t) \right).
\] (48)

The second term in \((50)\) is obtained as
\[
\tilde{W}(t)^{\dagger} \tilde{W}(t) = \tilde{W}(t)^{\dagger} \delta \tilde{W}(t) \\
+ \tilde{W}(t)^{\dagger} \tilde{W}(t) \delta \int_{t-T}^{t} \text{col}(\tilde{w}_{a2})^T M d\tau + \tilde{W}(t)^{\dagger} \delta \int_{t-T}^{t} N(t) d\tau.
\] (49)
For a sufficiently small reinforcement interval, the integral term in \((49)\) can be approximated as
\[
\int_{t-T}^{t} \text{col}(\tilde{w}_{a2}(\tau))^T M d\tau \approx TM^T \text{col}(\tilde{w}_{a2}(t)).
\] (50)

Substituting \((50)\) into \((49)\) and applying Young’s inequality, then \((49)\) is derived as
\[
\tilde{W}(t)^{\dagger} \tilde{W}(t) \leq -d\tilde{W}(t)^{\dagger} \delta \tilde{W}(t) + \tilde{W}(t)^{\dagger} \tilde{W}(t) \delta \int_{t-T}^{t} N(t) d\tau \\
+ \frac{\varepsilon}{2m_s} \text{col}(\tilde{w}_{a2})^T MM^T \text{col}(\tilde{w}_{a2}).
\] (51)
where $d = 1 - T^2/2\varepsilon$. Eq. \((50)\) can be obtained as
\[
\dot{L} \leq -x^T qx + k_1 \|x\| + k_2 \\
- d\tilde{W}(t)^{\dagger} \delta \tilde{W}(t) + \tilde{W}(t)^{\dagger} \delta \int_{t-T}^{t} N(t) d\tau \\
+ \frac{\varepsilon}{2m_s} \text{col}(\tilde{w}_{a2})^T MM^T \text{col}(\tilde{w}_{a2}) \\
- 2\phi_a \tilde{w}_{a2} R \lambda \tanh(\frac{\tilde{w}_{a2}^T \phi_a}{\lambda}) \\
+ \text{col}(\tilde{w}_{a2}) \otimes \alpha_1 \text{col}(\tilde{w}_{a2}).
\] (52)
Substituting the tuning law of $\dot{w}_{a2}$ and $\dot{w}_a = w_a - \dot{w}_{a2}$, the last two terms of (52) can be written as
\[
\begin{align*}
\text{col}(\dot{w}_{a2})^T \alpha_2^{-1} \text{col}(\dot{w}_{a2}) - 2\phi_a^T \dot{w}_{a2} R \lambda \tanh(\frac{\dot{w}_{a2}^T \phi_a}{\lambda}) \\
= -\text{col}(\dot{w}_{a2})^T Y \text{col}(\dot{w}_{a2}) + \text{col}(\dot{w}_{a2})^T k_3,
\end{align*}
\]
where
\[
k_3 = Y \text{col}(w_a) - \left( E_u \tanh^2(\frac{\dot{w}_{a2} \phi_a}{\lambda}) \right) \otimes \phi_a \\
- \lambda R \tanh(\frac{\dot{w}_{a2}^T \phi_a}{\lambda}) \otimes \phi_a
\]
Eq. (52) becomes
\[
\dot{L} \leq -x^T q x + k_1 \|x\|^2 + k_2 \\
- dW^T(t) \alpha \dot{W}(t) + \dot{W}^T(t) N(t) \\
- \text{col}(\dot{w}_{a2})^T B \text{col}(\dot{w}_{a2}) + \text{col}(\dot{w}_{a2})^T k_3,
\]
where $B = Y - \varepsilon MM^T / 2m^2$. Choose $T$ and $Y$ properly such that both $d$ and $B$ are positive. Therefore, $\dot{L}$ is negative if
\[
\|x\| > \frac{k_1}{2\sigma_{\min}(q)} + \sqrt{\frac{k_1^2}{4\sigma_{\min}(q)^2} + \frac{k_2}{\sigma_{\min}(q)}},
\]
\[
\|\Phi(t) W\| > N, \quad \|\dot{w}_{a2}\| > \frac{k_3}{\sigma_{\min}(B)}.
\]
The inequalities above shows that the states of the closed-loop system, the output of the error dynamics, and the error $\dot{w}_{a2}$ are UUB. Since the signal $\Phi(t)$ is persistently excited. The weights of critic NN are also UUB.

5 Simulation

In this section we design two experiments to vary the effectiveness of our method. Since there are no analytic solutions to nonlinear optimal constrained-input control problem, we choose our first case as a linear system with a large enough upper bound that the input will not be near to the saturation. The optimal solution in this case should be same as the standard linear quadratic regulator (LQR) problem.

The system in the first case is chosen as
\[
\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u.
\]
The cost function is defined as $Q = x_1^2 + x_2^2$ and $R = 1$. The input constraint is $\lambda = 30$. It becomes a LQR problem near the origin and the optimal value function is quadratic in the states and the optimal control law is linear. Therefore, we choose the activation functions as $\phi_c = [x_1^2, x_1 x_2, x_2^2]^T$ and $\phi_a = [x_1, x_2]^T$. By solving algebraic Riccati equation, the optimal weights are obtained as $W^* = \begin{bmatrix} 0.8779, -0.1904, 0.2492, -1.6601, -0.0577 \end{bmatrix}^T$. As for the hyperparameters, we set reinforcement interval as $T = 0.01 s$ and the learning rate of two NNs is set as $\alpha_1 = 10^3$ and $\alpha_2 = 20$, respectively. The exploration signal we add into the input is $e = \sum_{k=1}^{100} \sin(\omega_k t)$, where $\omega_k$ is uniformly sampled from $[-50, 50]$.

After 180 s, the exploration signal is set to 0 and the simulation is ended at $t_f = 200 s$. The weights are converge to $\dot{W}(t_f) = [0.8777, -0.1876, 0.2461, -1.6618, -0.0589]^T$ and $\dot{w}_{a2}(t_f) = [-1.6618, -0.0596]^T$, which are close to the optimal weights.

The second case is a nonlinear system with $f(x) = [-x_1 + x_2, -0.5(x_1 + x_2) + 0.5x_2 \cos(2x_1) + 2]^T$ and $g(x) = [0, \cos(2x_1) + 2]^T$. In this case the upper bound of input is set as $\lambda = 0.5$, we choose $\phi_c = [x_1^2, x_2^2, x_1 x_2, x_1^3, x_2^3, x_1 x_2^2, x_1 x_2^3]^T$ and $\phi_a = [x_1, x_2, x_1^2, x_2^2, x_1 x_2, x_1^3, x_2^3, x_1 x_2^2, x_1 x_2^3]^T$. The learning rates are set as $\alpha_1 = 10^3, \alpha_2 = 10$. Due to the input constraint, the exploration signal is chosen as
\[
e(t) = \lambda \tanh(\frac{\sum_{k=1}^{100} \sin(\omega_k t) + \dot{w}_{a2}(t) \phi_a}{\lambda}) - u(t).
\]
After learning, the weights converge to
\[
\dot{w}_c(t_f) = [0.0601, 1.0401, -0.1214, -0.0755, 0.1327, -0.0816, -0.0551, 0.0048]^T, \\
\dot{w}_{a1}(t_f) = [-0.1876, -3.1325, -0.0278, -0.2763, -0.2390, -2.0274, -0.0607, 1.9738, 0.6572]^T, \\
\dot{w}_{a2}(t_f) = [0.1826, -3.1356, -0.0164, -0.5104, -0.2490, -2.0156, -0.0420, 1.9659, 0.6565]^T.
\]

6 Conclusion

This paper presents a novel adaptive optimal control method to solve constrained-input problem in a completely model-free way. By adding exploration signal, actor and critic NNs can simultaneously update their weights and no a priori knowledge of system dynamics and initial stabilized controller is required during the learning phase. The efficacy of the proposed method is also shown in simulation results.

References

[1] M. Abu-Khalaf and F. Lewis, “Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach,” *Automatica*, vol. 41, pp. 779–791, 2005.
[2] D. Vrabie, O. Pastravanu, F. Lewis, and M. Abu-Khalaf, “Adaptive optimal control for continuous-time linear systems based on policy iteration,” *Automatica*, vol. 45, pp. 477–484, 2009.
[3] R. Sutton and A. Barto, *Reinforcement learning: an introduction*. Cambridge University Press, 1998.
[4] H. Modares, F. Lewis, and M.-B. Naghibi-Sistani, “Integral reinforcement learning and experience replay for adaptive optimal control of partially-unknown constrained-input continuous-time systems,” *Automatica*, vol. 50, no. 1, pp. 193–202, 2014.
[5] H. Modares and F. Lewis, “Optimal tracking control of nonlinear partially-unknown constrained-input systems using integral reinforcement learning,” *Automatica*, vol. 50, no. 7, pp. 1780–1792, 2014.
[6] S. Xue, B. Luo, D. Liu, and Y. Yang, “Constrained event-triggered $H_\infty$ control based on adaptive dynamic programming with concurrent learning,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, pp. 1–13, 2020.

[7] K. Vanvoudakis, “Non-zero sum nash q-learning for unknown deterministic continuous-time linear systems,” *Automatica*, vol. 61, pp. 274–281, 2015.

[8] K. Vanvoudakis and F. Lewis, “Q-learning for continuous-time linear systems: a model-free infinite horizon optimal control approach,” *Systems & Control Letters*, vol. 100, pp. 14–20, 2017.

[9] C. Watkins, *Learning from delayed rewards*. PhD thesis, King’s College of Cambridge, 1989.

[10] J. Lee, J. Park, and Y. Choi, “Integral reinforcement learning for continuous-time input-affine nonlinear systems with simultaneous invariant explorations,” *IEEE Transactions on Neural Networks and Learning Systems*, vol. 26, pp. 916–932, 2015.

[11] L. Guo and H. Zhao, “Online adaptive optimal control algorithm based on synchronous integral reinforcement learning with explorations,” *arXiv e-prints*, 2021. arxiv.org/abs/2105.09006.

[12] S. Lyshevski, “Optimal control of nonlinear continuous-time systems: design of bounded controllers via generalized non-quadratic functionals,” in *Proceedings of the 1998 American Control Conference*, vol. 1, pp. 205–209, 1998.

[13] P. Ioannou and B. Fidan, *Adaptive control tutorial*. SIAM, 2006.