A Liouville type theorem for Lane–Emden systems involving the fractional Laplacian

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Abstract
We establish a Liouville type theorem for the fractional Lane–Emden system:

\[
\begin{cases}
(-\Delta)^\alpha u = v^q & \text{in } \mathbb{R}^N, \\
(-\Delta)^\alpha v = u^p & \text{in } \mathbb{R}^N,
\end{cases}
\]

where \( \alpha \in (0, 1), N > 2\alpha \) and \( p, q \) are positive real numbers and in an appropriate new range. To prove our result we will use the local realization of fractional Laplacian, which can be constructed as a Dirichlet-to-Neumann operator of a degenerate elliptic equation in the spirit of Caffarelli and Silvestre (2007 Commun. PDE 32 1245–60). Our proof is based on a monotonicity argument for suitable transformed functions and the method of moving planes in a half infinite cylinder \((\mathbb{R} \times S^N_1, \text{ where } S^N_1 \text{ is the half unit sphere in } \mathbb{R}^{N+1})\) based on maximum principles which are obtained by barrier functions and a coupling argument using a fractional Sobolev trace inequality.

Keywords: non-existence, fractional Laplacian, Lane–Emden system

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1. Introduction
Motivated by their potential applications the study of nonlinear elliptic systems has increased in the last three decades. This is because these types of systems appear as limiting equations of many phenomena, such as pattern formation, population evolution, chemical reaction, etc. Some of these equations are named as Lotka–Volterra, Bose–Einstein, Schrödinger system, Gierer–Meinhardt. The solutions in most of the cases represent concentrations and thus naturally positive solutions of the systems are of particular interest.
There are many well-known results in this field when the diffusion is governed by the Laplacian or more general other local elliptic operators. The correspondence of the diffusion with the fractional Laplacian has been studied recently in [10, 18, 21], [29–32]. Notice that the fractional Laplacian appears in different diffusion models, see for example [1–3, 15, 20, 27] and the references therein.

As far as we know, there are no existence results for systems without variational structure. When the variational structure breaks, the methods developed to prove existence results for locally elliptic systems are obtained by Perron’s Method or topological arguments; for example, the Leray–Schauder degree or Krasnosel’skii’s index theory, where the many assumptions in using these theories are the a priori bounds for solutions. These bounds are obtained in many settings by the classical scaling or blow-up argument due to Gidas and Spruck [14] in the scalar case and [11] for the systems case, see also the references therein. Liouville type theorems are crucial to getting a contradiction for the limiting system or equation. Roughly speaking, better Liouville type theorems give more general existence results. Observe that there are still some problems even in the case ($\alpha = 1$) which is known as the Lane–Emden conjecture, see [23, 25] and below.

The aim of this paper is to establish a new Liouville type theorem for the following Lane–Emden system involving the fractional Laplacian:

\begin{equation}
\begin{cases}
(-\Delta)^\alpha u = v^q & \text{in } \mathbb{R}^N, \\
(-\Delta)^\alpha v = u^p & \text{in } \mathbb{R}^N.
\end{cases}
\end{equation}

where $\alpha \in (0, 1)$ and $N > 2\alpha$.

The fractional Laplacian $(-\Delta)^\alpha$ can be defined, for example, by the Fourier transform. Namely, for a function $u$ in the Schwartz class $S$, we have

\[ (-\Delta)^\alpha u(\xi) = |\xi|^{2\alpha} \hat{u}(\xi). \]

Furthermore, consider the space

\[ \mathcal{L}_\alpha(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^N} \frac{|u(y)|}{1 + |y|^{N+2\alpha}} \, dy < \infty \right\}, \]

endowed with the norm

\[ \|u\|_{\mathcal{L}_\alpha(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(y)|}{1 + |y|^{N+2\alpha}} \, dy < \infty. \]

If $u \in \mathcal{L}_\alpha(\mathbb{R}^N)$ (see [24]), then $(-\Delta)^\alpha u$ can be defined as a distribution, that is, for any $\varphi \in S$,

\[ \int_{\mathbb{R}^N} \varphi(-\Delta)^\alpha u \, dx = \int_{\mathbb{R}^N} u(-\Delta)^\alpha \varphi \, dx. \]

In addition, for some $\sigma > 0$, suppose that $u \in \mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{2\alpha+\sigma}(\mathbb{R}^N)$ if $0 < \alpha < 1/2$ or $u \in \mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{4,2\alpha+\sigma-1}(\mathbb{R}^N)$ if $\alpha \geq 1/2$, that we will simply denote by $u \in \mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{2\alpha+\sigma}(\mathbb{R}^N)$. Then we have

\[ (-\Delta)^\alpha u(x) = C_{N,\alpha} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2\alpha}} \, dy \quad \text{for } x \in \mathbb{R}^N, \]

where $P.V.$ denotes the principal value of the integral and $C_{N,\alpha}$ is a normalization constant.

When $\alpha = 1$, the Lane–Emden system for Laplace operator
\[
\begin{cases}
-\Delta u = v^q \text{ in } \mathbb{R}^N, \\
-\Delta v = u^p \text{ in } \mathbb{R}^N,
\end{cases}
\] (1.2)

has been extensively studied in the literature, see [4, 13, 19, 22, 23]. It has been conjectured that the Sobolev’s hyperbola

\[
\left\{ p > 0, q > 0 : \frac{1}{p + 1} + \frac{1}{q + 1} = 1 - \frac{2}{N} \right\},
\]

is the dividing curve between existence and nonexistence for (1.2). For the radial case, this conjecture was completely solved by [19, 22]. In fact, if the pair \((p, q)\) lies below Sobolev’s hyperbola, that is,

\[
\frac{1}{p + 1} + \frac{1}{q + 1} > 1 - \frac{2}{N},
\] (1.3)

then there is no radial positive solution to system (1.2), see [19] (for \(p > 1, q > 1\)) and [22] (for \(p > 0, q > 0\)). In addition, there are indeed positive radial solutions to system (1.2) if \((p, q)\) lies above Sobolev’s hyperbola (see also [22]).

The conjecture for the more general case, i.e. without the assumption of radial symmetry, has not been completely answered yet. Partial results for nonexistence are known. Define

\[
\gamma_1 = \frac{2(q + 1)}{pq - 1}, \quad \gamma_2 = \frac{2(p + 1)}{pq - 1} \quad \text{if } pq > 1.
\] (1.4)

There are no positive classical supersolutions to (1.2) if

\[
pq \leq 1 \text{ or } pq > 1 \text{ and } \max\{\gamma_1, \gamma_2\} \geq N - 2,
\] (1.5)

see [19, 23, 26]. Moreover, we know that the condition (1.4) is optimal for supersolution. For positive solutions, Felmer and Figueiredo [13] proved that if

\[
0 < p, q \leq \frac{N + 2}{N - 2}, \quad (p, q) = \left( \frac{N + 2}{N - 2}, \frac{N + 2}{N - 2} \right).
\] (1.6)

then problem (1.2) has no classical positive solutions. Notice that for \(N \geq 3\), condition (1.4) and (1.6) are stronger than (1.3). The full conjecture seems difficult for nonradial solutions. As far as we know the full conjecture is true when \(N = 3, 4\), see [23] and [25]. In the high dimensions, apart from (1.5), the conjecture is only known to be true in some subregion of subcritical range:

\[
\min\{\gamma_1, \gamma_2\} \geq \frac{N - 2}{2} \quad \text{and} \quad \left( \gamma_1, \gamma_2 \right) = \left( \frac{N - 2}{2}, \frac{N - 2}{2} \right)
\] (1.7)

by Busca and Manásevich [4]. Note that the condition (1.6) in particular obtains where both exponents are subcritical, that is, the region considered in [13].

The aim of the present paper is to show that the result of Busca and Manásevich [4] can be extended to system (1.1). We prove the following result.

**Theorem 1.1.** Let \(p, q > 0\) and \(pq > 1\) and set

\[
\beta_1 = \frac{2\alpha(q + 1)}{pq - 1}, \quad \beta_2 = \frac{2\alpha(p + 1)}{pq - 1}.
\] (1.8)
Suppose
\[
\beta_1, \beta_2 \in \left[ \frac{N - 2\alpha}{2}, \frac{N - 2\alpha}{2} \right] \text{ and } (\beta_1, \beta_2) \neq \left( \frac{N - 2\alpha}{2}, \frac{N - 2\alpha}{2} \right).
\] (1.9)

Then, for some \( \sigma > 0 \), there exists no positive solution to system (1.1) in \( \mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{2\alpha + \sigma} \).

Remark 1.1. 
(1) Observe that region (1.9) in particular obtains where both exponents are subcritical, that is
\[
\frac{N}{N - 2\alpha} < p, q \leq \frac{N + 2\alpha}{N - 2\alpha}, \quad \text{with} \quad (p, q) = \left( \frac{N + 2\alpha}{N - 2\alpha}, \frac{N + 2\alpha}{N - 2\alpha} \right).
\] (1.10)

By theorem 3 in [18] and theorem 3 in [34], we know there is no positive solution \( u, v \in \mathcal{L}_\alpha(\mathbb{R}^N) \cap C^{2\alpha + \sigma}(\mathbb{R}^N) \) to system (1.1) if (1.10) holds. Hence, theorem 1.1 is valid for a large region of \((p, q)\) in comparison with theorem 3 in [18].

(2) If we take \( q = 1 \) we can obtain Liouville type results for equation with \((-\Delta)^\alpha \) operator.

Remark 1.2. We note that region (1.9) does not include the point
\[
(\beta_1, \beta_2) = \left( \frac{N - 2\alpha}{2}, \frac{N - 2\alpha}{2} \right).
\]

Indeed, if \( \beta_1 = \beta_2 = \frac{N - 2\alpha}{2} \), then
\[
p = q = \frac{N + 2\alpha}{N - 2\alpha},
\]
and problem
\[
(-\Delta)^\alpha u = u^{N+2\alpha},
\] (1.11)

has nontivial nonnegative solutions called fractional bubble, see Chen et al \([7, 8]\), Jin et al \([16]\) and also Li \([17]\).

In \([6]\), Caffarelli and Silvestre introduced a local realization of the fractional Laplacian \((-\Delta)^\alpha \) in \( \mathbb{R}^N \) through the Dirichlet–Neumann map of an appropriate degenerate elliptic operator in upper half space \( \mathbb{R}^{N+1}_+ \). More precisely, consider an extension of \( u \) to the upper half space \( \mathbb{R}^{N+1}_+ \) so that \( U(x, 0) = u(x) \) and
\[
\Delta_x U + \frac{1 - 2\alpha}{y} U_y + U_{yy} = 0 \quad \text{for} \quad X = (x, y) \in \mathbb{R}^{N+1}_+.
\]

Let \( P_\alpha(x, y) \) denote the corresponding Poisson kernel
\[
P_\alpha(x, y) = c_{N, \alpha} \frac{y^{2\alpha}}{(|x|^2 + y^2)^{N+2\alpha/2}} \quad \text{for} \quad x \in \mathbb{R}^N \text{ and } y > 0,
\]
where \( c_{N, \alpha} \) is a normalization constant (for an explicit value of \( c_{N, \alpha} \) see \([5]\)). If \( u \in \mathcal{L}_\alpha(\mathbb{R}^N) \), we can define
\[
U(x, y) = P_\alpha(\cdot, y) * u = c_{N, \alpha} \int_{\mathbb{R}^N} \frac{y^{2\alpha}}{(|x - \xi|^2 + y^2)^{N+2\alpha/2}} u(\xi) d\xi.
\]
Moreover, for some $\sigma > 0$, suppose as above that $u \in \mathcal{L}_\sigma(\mathbb{R}^N) \cap C^{2\alpha+\sigma}(\mathbb{R}^N)$, then $U \in C^{2}(\mathbb{R}^{N+1}_+) \cap C(\overline{\mathbb{R}^{N+1}_+})$, $y^{1-2\alpha}\partial_y U \in C(\overline{\mathbb{R}^{N+1}_+})$ and

\[
\begin{aligned}
\text{div}(y^{1-2\alpha}\nabla U) &= 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\
U &= u \quad \text{on } \partial\mathbb{R}^{N+1}_+, \\
- \lim_{y \to 0^+} y^{1-2\alpha}U_y &= \kappa_\alpha(-\Delta)^{\alpha}u \quad \text{on } \partial\mathbb{R}^{N+1}_+,
\end{aligned}
\]

where $\text{div}$ and $\nabla$ denote the divergence operator and gradient operator respectively, and

\[
\kappa_\alpha = \frac{\Gamma(1-\alpha)}{2^{\alpha-1}\Gamma(\alpha)}
\]

with $\Gamma$ being the Gamma function, see theorem 1.3 in [9] and also [5, 6, 13, 26].

Using the local formulation established by Caffarelli and Silvestre [6], the above theorem will follow as a corollary of the following Liouville type result for a degenerated system with a coupling with a nonlinear Neumann condition in the upper half space $\mathbb{R}^{N+1}_+$.

**Theorem 1.2.** Let $p, q > 0$, $pq > 1$ and (1.9) hold. Then there exists no positive $C^2(\mathbb{R}^{N+1}_+) \cap C(\overline{\mathbb{R}^{N+1}_+})$ and $y^{1-2\alpha}\partial_y U \in C(\overline{\mathbb{R}^{N+1}_+})$ type solution of

\[
\begin{aligned}
\text{div}(y^{1-2\alpha}\nabla U) &= 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\
- \lim_{y \to 0^+} y^{1-2\alpha}U_y &= V^q \quad \text{on } \partial\mathbb{R}^{N+1}_+, \\
\text{div}(y^{1-2\alpha}\nabla V) &= 0 \quad \text{in } \mathbb{R}^{N+1}_+, \\
- \lim_{y \to 0^+} y^{1-2\alpha}V_y &= U^p \quad \text{on } \partial\mathbb{R}^{N+1}_+.
\end{aligned}
\]

(1.12)

Our proof follows the idea in [4]. Roughly speaking, as in [4], we first transform the elliptic equation (1.12) in upper half space $\mathbb{R}^{N+1}_+$ to an appropriate equation in the upper half infinite cylinder $\mathbb{R} \times S^N_+$ (see (2.3) and (2.5)), where $S^N_+$ is the upper half unit sphere. Then, we study the nonexistence result via a symmetry and monotonicity result (i.e. lemma 3.2) obtained by the method of moving planes. However, some difficulties appear—compare our article with [4]—since in (1.12) the operator is degenerated and the nonlinearity is at the boundary. In particular, we must prove the maximum principle for ‘narrow’ domains, which permits us to get the moving planes started. For this purpose we follow some similar arguments as in [12], which are established for the single equation and a coupling argument by a fractional Sobolev trace inequality (see lemma 3.1). We also need to prove two Hopf’s lemmas where barrier functions need to be constructed, see lemmas 2.1 and 2.2 in section 2.

We end the introduction by mentioning that we can use theorem 1.1 to obtain a priori estimate and existence results for positive solutions of nonlinear elliptic systems involving the fractional Laplacian.

The paper is organized as follows. In section 2, we do a transformation as in [4] to problem (1.12) and present some preliminary results, the Hopf’s lemmas and the strong maximum principle. A monotonicity and symmetry result is shown by the method of moving planes in section 3 and we prove the nonexistence result (theorem 1.2) at the end of section 3.
2. Preliminaries

This section is devoted to introducing some preliminary results, the Hopf’s lemmas and the strong maximum principle. We start this section by transforming (1.12) as in [4] by using polar coordinates and Emden–Fowler variables. We take standard polar coordinates in \( \mathbb{R}^{N+1}_+ : X = (x, y) = r\theta \), where \( r = |X| \) and \( \theta = X/|X| \). Denote \( \theta = (\theta_1, \theta_2, \ldots, \theta_N, \theta_{N+1}) \) and let \( \theta_{N+1} = y/|X| \) denote the component of \( \theta \) in the \( y \)-direction and \( S_+^N = \{ X \in \mathbb{R}^{N+1}_+ : r = 1, \theta_{N+1} > 0 \} \) denote the upper unit half space.

For a given function \( w \) of \( X \in \mathbb{R}^{N+1}_+ \), we write, using the same symbol \( w \) without risk of confusion,

\[
\nabla \cdot (\theta^\alpha \nabla w) + \theta_{N+1} \nabla \cdot (\theta^\alpha \nabla U) = \theta_{N+1} \nabla \cdot (\nu_1 \nabla U) = 0 \quad \text{in } \mathbb{R} \times S_+^N, \\
- \lim_{\theta_{N+1} \to 0^+} \theta_{N+1}^{2\alpha} \partial_{\theta_{N+1}} U = \theta_{N+1}^{2\alpha} \partial_{\theta_{N+1}} U = 0 \quad \text{on } \mathbb{R} \times \partial S_+^N, \\
\nabla \cdot (\theta^\alpha \nabla \nu_1 \nabla \nu_2 \nabla \nu_3) = \theta_{N+1} \nabla \cdot (\nu_1 \nabla \nu_2 \nabla \nu_3) = 0 \quad \text{in } \mathbb{R} \times S_+^N, \\
- \lim_{\theta_{N+1} \to 0^+} \theta_{N+1}^{2\alpha} \partial_{\theta_{N+1}} \nu_1 \nabla \nu_2 \nabla \nu_3 = \theta_{N+1}^{2\alpha} \partial_{\theta_{N+1}} \nu_1 \nabla \nu_2 \nabla \nu_3 = 0 \quad \text{on } \mathbb{R} \times \partial S_+^N, \\
\]

where

\[
\begin{aligned}
\delta_1 &= 2\beta_1 - (N - 2\alpha), \\
\delta_2 &= 2\beta_2 - (N - 2\alpha),
\end{aligned}
\]

Here for \( F \) a vector field define on unite sphere \( \theta \in S^N \) we define \( \text{div}_\theta(f(\theta))) \) by \( \text{div}_\theta(f(X/|X|)) \) in \( \mathbb{R}^{N+1}_+ \setminus \{0\} \) restricted to the unite sphere and \( \nabla \theta(\mu(\theta)) \) is define by \( \nabla (X/|X|) \) in \( \mathbb{R}^{N+1}_+ \setminus \{0\} \) restricted to the unite sphere. For ease of the notation, we define the operators

\[
L_{\alpha}u := \theta_{N+1}^{2\alpha-1} \text{div}_\theta(\theta_{N+1}^{-2\alpha} \nabla u) = \Delta u + \frac{1 - 2\alpha}{\theta_{N+1}} \left( \sum_{i=1}^{N+1} \frac{\partial u}{\partial \theta_i} \theta_{N+1} + \frac{\partial u}{\partial \theta_{N+1}} \right)
\]

Now, if we define \( \beta_1 \) and \( \beta_2 \) as in (1.8), then we write (2.3) as
Here we have used the facts $\beta_1 + 2\alpha - q/\beta_2 = 0$ and $\beta_2 + 2\alpha - p/\beta_1 = 0$ by (1.8). Moreover, with these notations in (2.4), the assumptions (1.9) are equivalent to

\[
\begin{aligned}
&\delta_1, \delta_2 \geq 0, \quad (\delta_1, \delta_2) \neq (0, 0), \\
&\nu_1, \nu_2 > 0, \\
&p, q > 0, \quad pq > 1.
\end{aligned}
\]

In order to prove theorem 1.2, we will use the method of moving planes. The key tools for using the method of moving planes are the Hopf’s lemma and the strong maximum principle. The remainder of the section is devoted to proving these results with respect to the operators we studied. We first show the following weak maximum principle.

**Proposition 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{R} \times S^N_+$ and $w \in C^2(\Omega) \cap C(\Omega)$. Suppose

\[
L_\alpha w + w_{tt} + a(t, \theta)w_t \leq 0 \quad \text{in} \quad \Omega,
\]

where $|a(t, \theta)| \leq a_0 = \text{constant in } \Omega$. Then the nonnegative minimum of $w$ in $\bar{\Omega}$ is achieved on $\partial \Omega$, that is,

\[
\inf_{\Omega} w = \inf_{\Omega} w_{\partial}\Omega.
\]

**Proof.** It is clear that if $L_\alpha w + w_{tt} + a(t, \theta)w_t < 0$ in $\Omega$, then a strong maximum principle holds, that is, $w$ cannot achieve an interior nonnegative minimum in $\bar{\Omega}$. Indeed, if $(t_0, \theta_0) \in \Omega$, then

\[
w_{tt}(t_0, \theta_0) \geq 0, \quad \Delta_{\partial}\Omega w(t_0, \theta_0) \geq 0, \quad w(t_0, \theta_0) = 0, \quad \text{and} \quad \nabla_{\partial}\Omega w(t_0, \theta_0) = 0.
\]

This implies $L_\alpha w(t_0, \theta_0) + w_{tt}(t_0, \theta_0) + a(t_0, \theta_0)w_t(t_0, \theta_0) \geq 0$, which is impossible.

A simple computation, for $\gamma > 0$, gives,

\[
(\varepsilon^\gamma)' + a(\varepsilon^\gamma) = \varepsilon^\gamma(\gamma^2 + a) \geq \varepsilon^\gamma(\gamma^2 - a_0\gamma).
\]

So we can choose $\gamma$ large enough such that $(\varepsilon^\gamma)' + a(\varepsilon^\gamma) > 0$. Hence, for any $\varepsilon > 0$,

\[
L_\alpha (w - \varepsilon \varepsilon^\gamma) + (w - \varepsilon \varepsilon^\gamma)' + a(w - \varepsilon \varepsilon^\gamma) < 0
\]

in $\Omega$ so that

\[
\inf_{\Omega} (w - \varepsilon \varepsilon^\gamma) = \inf_{\partial \Omega} (w - \varepsilon \varepsilon^\gamma).
\]

Letting $\varepsilon \to 0$, we see that

\[
\inf_{\Omega} w = \inf_{\partial \Omega} w,
\]

as asserted in the proposition.
Next we suppose more generally that
\[ L_\omega w + w_t + a(t, \theta)w_t - b(t, \theta)w \leq 0 \quad \text{in} \quad \Omega, \]
where \(|a(t, \theta)| \leq a_0\) and \(b\) is a nonnegative function in \(\Omega\). By considering the subset \(\Omega^- \subset \Omega\), in which \(w < 0\), we can observe that if \(L_\omega w + w_t + a(t, \theta)w_t - b(t, \theta)w \leq 0\) in \(\Omega\), then \(L_\omega w + w_t + a(t, \theta)w_t - b(t, \theta)w \leq 0\) in \(\Omega^-\) and hence the minimum of \(w\) on \(\Omega^-\) must be achieved on \(\partial \Omega^-\) and hence also on \(\partial \Omega\). Thus, writing \(w^- = \min\{w, 0\}\), we obtain:

**Theorem 2.1.** Let \(\Omega\) be an bounded domain in \(\mathbb{R} \times S^N_+\) and \(w \in C^2(\Omega) \cap C(\overline{\Omega})\). Suppose
\[ L_\omega w + w_t + a(t, \theta)w_t - b(t, \theta)w \leq 0 \quad \text{in} \quad \Omega, \]
where \(|a(t, \theta)| \leq a_0\) and \(b\) is a nonnegative function in \(\Omega\). Then
\[ \inf_{\Omega} w \geq \inf_{\partial \Omega} w^- . \]

Next, we prove two Hopf’s lemmas. Let \(D\) be a bounded domain of \(S^N_+\) and \(\bar{t} \in \mathbb{R}\), we define
\[ \Omega_\delta = \{ |t - \bar{t}| \leq \delta \} \times D \subset \mathbb{R} \times S^N_+ \]
for some \(\delta > 0\). The first Hopf’s lemma is

**Lemma 2.1.** Suppose that \((t_0, \theta_0) \in \{ |t - \bar{t}| = \delta \} \times D\) and \(w \in C^2(\Omega_\delta) \cap C(\Omega_\delta \cup (t_0, \theta_0))\) is a solution of
\[ L_\omega w + w_t + a(t, \theta)w_t - b(t, \theta)w \leq 0 \quad \text{in} \quad \Omega_\delta, \]
where \(a\) and \(b\) are bounded functions and \(b\) is nonnegative. Assume in addition that \(w(t, \theta) > 0\) for every \((t, \theta) \in \Omega_\delta\) and \(t \neq t_0\). Moreover, \(w(t_0, \theta) = 0\) if \((t_0, \theta) \in \overline{\Omega_\delta}\). Then
\[ \limsup_{t \to t_0} \frac{w(t, \theta) - w(t_0, \theta)}{t - t_0} < 0. \]

**Proof.** For \(0 < \rho < \delta\), we define an auxiliary function \(\phi\) as
\[ \phi(t) = e^{-\beta|t-\bar{t}|^2} = e^{-\beta t^2}, \]
where \(|t - \bar{t}| > \rho\) and \(\beta\) is a positive constant to be determined later. We notice that \(0 < \phi(t) < 1\). A direct calculation gives
\[ \phi_t + a\phi_t - b\phi = e^{-\beta|t-\bar{t}|^2}(4\beta^2|t-\bar{t}|^2 + 2\beta(a|t-\bar{t}| - 1) - b). \]
Hence we can chose \(\beta\) large enough such that
\[ \phi_t + a\phi_t - b\phi \geq 0 \]
in \(\Omega_\delta \setminus \Omega_{\delta/2}\). Since \(w > 0\) in \((t, \theta) \in \overline{\Omega_\delta}\) and \(t \neq t_0\) and \(\phi(t_0) = 0\), there is a \(\varepsilon > 0\) such that \(w - \varepsilon \phi \geq 0\) on \(\partial \Omega_\delta \cup \partial \Omega_{\delta/2}\). Moreover, we have
\[ L_{\alpha}(w - \varepsilon \phi) + (w - \varepsilon \phi)_{tt} + c_1(w - \varepsilon \phi) - c_2(w - \varepsilon \phi) \leq 0 \]

in \( \Omega_\delta \setminus \Omega_{\varepsilon/2} \). Hence, the weak maximum principle (see theorem 2.1) implies that

\[ w - \varepsilon \phi \geq 0 \quad \text{in} \quad \Omega_\delta \setminus \Omega_{\varepsilon/2} \, . \]

Taking the outer normal derivative at \((t_0, \theta_0)\), we obtain

\[ \lim_{t \to t_0} \frac{w(t, \theta) - w(t_0, \theta_0)}{t - t_0} \leq \varepsilon \lim_{t \to t_0} \frac{\phi(t) - \phi(t_0)}{t - t_0} = -2\varepsilon \beta e^{-\beta t^2} < 0, \]

as required. \( \square \)

**Remark 2.1.** If in addition \( w \) is \( C^1 \), then we have

\[ \frac{\partial w}{\partial t}(t_0, \theta_0) < 0. \]

Next we establish the second Hopf’s lemma on the boundary \( \mathbb{R} \times \partial S^N_{\varepsilon+} \). We denote as before that \( \theta = (\tilde{\theta}, \theta_{N+1}) \in S^N_{\varepsilon+} \) with \( \theta_{N+1} > 0 \). For \( r_\varepsilon \in \mathbb{R} \) and \( \delta > 0 \) we define

\[ S_0 = \{ \theta \in S^N_{\varepsilon+}, \ | \ \theta_{N+1} < \delta \}, \]

\[ t_0^\varepsilon := t_\varepsilon - \delta, t_1^\varepsilon := t_\varepsilon + \delta, \]

\[ C_\delta := S_0 \times (t_0^\varepsilon, t_1^\varepsilon), \]

\[ \Gamma_\delta := \partial C_\delta = A_1^\varepsilon \cup A_2^\varepsilon \cup A_3^\varepsilon \cup A_4^\varepsilon, \]

where

\[ A_1^\varepsilon = \{ \theta \in S^N_{\varepsilon+}, \ | \ \theta_{N+1} = 0 \} \times [t_0^\varepsilon, t_1^\varepsilon] \]

\[ A_2^\varepsilon = \{ \theta \in S^N_{\varepsilon+}, \ | \ \theta_{N+1} = \delta \} \times [t_0^\varepsilon, t_1^\varepsilon], \]

\[ A_3^\varepsilon = \cap S_0 \times [t_0^\varepsilon] \] and \( A_4^\varepsilon = S_0 \times \{ t_1^\varepsilon \} \).

**Lemma 2.2.** Let \( w \in C^2(\overline{C_\delta}) \cap C(\overline{C_\delta}) \) satisfy

\[
\begin{align*}
L_\alpha w + w_{tt} + aw_t - bw & \leq 0 \quad \text{in} \quad C_\varepsilon, \\
w & > 0 \quad \text{in} \quad C_\varepsilon, \\
w(t_0, \tilde{\theta}_0, 0) & = 0,
\end{align*}
\]

for some \((t_0, \tilde{\theta}_0, 0) \in A_1^\varepsilon \) and where \( a \) and \( b \) are constants.

Then,

\[ -\lim_{\theta_{N+1} \to 0} \sup_{\theta_{N+1} \to 0} \frac{\partial w(t_0, \tilde{\theta}_0, \theta_{N+1})}{\theta_{N+1}} < 0. \]

**Proof.** Here we follow the argument in [5]. Let \( \phi \) be the first eigenfunction of the Sturm–Liouville problem

\[
\begin{align*}
(\phi e^{\alpha t})_t & = -\lambda e^{\alpha t} \phi \quad \text{in} \quad (t_0^{\beta/2}, t_0^{\beta/2}), \\
\phi & > 0 \quad \text{in} \quad (t_0^{\beta/2}, t_0^{\beta/2}), \\
\phi(t_0^{\beta/2}) & = \phi(t_0^{\beta/2}) = 0,
\end{align*}
\]
with \( \| \phi \|_{L^\infty} \leq C \). Now we define \( g(\theta_{N+1}) = \theta_{N+1}^{2\alpha - 1}(\theta_{N+1} + A\theta_{N+1}^2) \). By direct computation we see that

\[
L_0 g(\theta_{N+1}) = (\theta_{N+1})^{2\alpha - 1} \left[ (2\alpha + 1)A - \left(N(2\alpha + (2\alpha + 1)\theta_{N+1})\theta_{N+1}\right) \right].
\]

Now we define \( w_\alpha(t, \theta_{N+1}) = \phi(t)g(\theta_{N+1}) \), then

\[
\tilde{L}(w_\alpha) := L_0 w_\alpha + g(\theta_{N+1})(\phi_\theta + a\phi - b\phi) = \phi(t) L_0 g(\theta_{N+1}) + g(\theta_{N+1})(-\lambda_1 - \beta)\phi(t),
\]

therefore if \( A \) is large we get \( \tilde{L}(w_\alpha) \geq 0 \) in \( C_{N/2} \).

Since \( w_\alpha \equiv 0 \) in \( A_2 \setminus \Omega \) and \( w \) is strictly positive in \( \Omega \), we find that there exists \( \varepsilon > 0 \) such that such that \( w \geq \varepsilon w_\alpha \) in \( \Gamma_{N/2} \). By the weak maximum principle (see theorem 2.1), we have

\[
w \leq \varepsilon w_\alpha \text{ in } C_{N/2}
\]

Consequently, this leads to

\[
\limsup_{\theta_{N+1} \to 0^+} \theta_{N+1}^{1-2\alpha} w(t_0, \theta_{N+1}) \leq -\varepsilon \phi(t_0) \leq 0,
\]

as claimed in the lemma.

Finally, by the above two Hopf’s lemmas and an argument similar to corollary 4.12 in [5], we obtain the following version of the strong maximum principle.

**Theorem 2.2.** Let \( w \in C^2(\bar{C}_3) \cap C(\bar{C}_3) \) and \( \theta_{N+1}^{1-2\alpha} w_{\theta_{N+1}} \in C(\bar{C}_3) \) satisfy

\[
\begin{cases}
L_0 w + w_t + a(t, \theta) w_t - b(t, \theta) w \leq 0 & \text{in } C_3,
\
- \lim_{\theta_{N+1} \to 0^+} \theta_{N+1}^{1-2\alpha} w_{\theta_{N+1}} \geq 0 & \text{on } A_1^\delta,
\
w \geq 0, \quad w \not\equiv 0 & \text{on } \Omega,
\end{cases}
\]

for some constants \( a \) and \( b \). Here \( A_1^\delta := \{ \theta \in S_\mu, \theta_{N+1} = 0 \} \times (0^+, t_i^\delta) \). Then \( w > 0 \) in \( C_3 \cup A_1^\delta \).

### 3. Proofs

We prove our main result in this section via the method of moving planes, for which we give some preliminary notations. We define

\[
\Sigma_\mu = \{ (t, \theta) : t \in (-\infty, \mu), \theta \in S_\mu \}
\]

\[
T_\mu = \{ (t, \theta) : t = \mu, \theta \in S_\mu \}
\]

\[
w^\mu(t, \theta) = U(2\mu - t, \theta) - U(t, \theta),
\]

\[
z^\mu(t, \theta) = V(2\mu - t, \theta) - V(t, \theta).
\]

A direct calculation shows the comparison functions \( w^\mu \) and \( z^\mu \) satisfy...
\[ \begin{aligned}
L_{\omega}w^\mu + w^\mu_t + \delta_1w^\mu_t - \nabla w^\mu &= -2\delta_1 \nabla_U \quad \text{in } \Sigma_{t_0}, \\
\lim_{\theta_{N+1} \to 0^-} \partial_{\theta_{N+1}} w^\mu &= \epsilon w^\mu \quad \text{on } \partial \Sigma_{t_0}, \\
L_{\omega}z^\mu + z^\mu_t + \delta_2 z^\mu_t - \nabla z^\mu &= -2\delta_2 \nabla_U \quad \text{in } \Sigma_{t_0}, \\
\lim_{\theta_{N+1} \to 0^-} \partial_{\theta_{N+1}} z^\mu &= d^\mu w^\mu \quad \text{on } \partial \Sigma_{t_0},
\end{aligned} \tag{3.1} \]

where \( \partial \Sigma_{t_0} := (\mathbb{R} \times \partial S^N_{t_0}) \cap \Sigma_{t_0} \).

\[
c^\mu(t, \theta) = \begin{cases} 
\frac{(\nabla (2\mu - t, \theta))^p - (\nabla (t, \theta))^p}{\nabla (2\mu - t, \theta) - \nabla (t, \theta)} & \text{if } \nabla (2\mu - t, \theta) \neq \nabla (t, \theta), \\
0 & \text{if } \nabla (2\mu - t, \theta) = \nabla (t, \theta),
\end{cases} \tag{3.2}
\]

and

\[
d^\mu(t, \theta) = \begin{cases} 
\frac{(\nabla (2\mu - t, \theta))^p - (\nabla (t, \theta))^p}{\nabla (2\mu - t, \theta) - \nabla (t, \theta)} & \text{if } \nabla (2\mu - t, \theta) \neq \nabla (t, \theta), \\
0 & \text{if } \nabla (2\mu - t, \theta) = \nabla (t, \theta).
\end{cases} \tag{3.3}
\]

From (3.2) and (3.3), we have that

\[ c^\mu > 0, \quad d^\mu > 0 \quad \text{in } \mathbb{R} \times \partial S^N_{t_0}. \tag{3.4} \]

Moreover, the definitions of \( w^\mu \) and \( z^\mu \) imply

\[ w^\mu \equiv z^\mu \equiv 0 \quad \text{on } T_{t_0}. \tag{3.5} \]

Next, we show that \( \nabla_U \) and \( \nabla_U \) decay monotonically near \(-\infty\). In fact, by differentiating (2.2), we find that

\[ U_\ell = r^{\beta_1}(\beta_1 U + rU_\ell) \quad \text{and} \quad V_\ell = r^{\beta_2}(\beta_2 V + rV_\ell). \tag{3.6} \]

So taking into account \( \beta_1, \beta_2 > 0, r = e^t, U > 0 \) and \( V > 0 \), we can obtain \( t_0 \) for which

\[ U_\ell > 0 \quad \text{and} \quad V_\ell > 0 \quad \text{in } \Sigma_{t_0}, \tag{3.7} \]

and

\[ 0 < U(t, \theta), V(t, \theta) < \varepsilon_0 \quad \text{in } \Sigma_{t_{0}}, \tag{3.8} \]

where \( 0 < \varepsilon_0 \ll 1 \).

Now, we fix \( t_0 \) such that (3.7) and (3.8) holds. The following maximum principle for system (3.1) near \(-\infty\) is needed, which permits us to get the moving planes method started.

**Lemma 3.1.**

1. For all \( \mu \in (-\infty, t_0] \) one has \( w^\mu \geq 0 \) and \( z^\mu \geq 0 \) in \( \Sigma_{t_{0}} \).
2. Suppose that for \( \mu \in (t_0, +\infty) \), we have \( w^\mu \geq 0 \) and \( z^\mu \geq 0 \) on \( T_{t_{0}} \). Then \( w^\mu \geq 0 \) and \( z^\mu \geq 0 \) in \( \Sigma_{t_{0}} \).

**Proof.** Observe that in both cases (1) and (2) we have \( w^\mu \geq 0 \) and \( z^\mu \geq 0 \) on \( T_{t_{0}}, \mu \) by (3.5), where \( t_0 \land \mu = \min \{t_0, \mu\} \). Therefore, we treat both cases at the same time by a contradiction argument, assuming that
\[
\min \left\{ \inf_{\Sigma_{\nu'}} w^\mu, \inf_{\Sigma_{\nu'}} z^\mu \right\} < 0.
\] (3.9)

Up to symmetry in the argument, there are two cases to rule out.

Case I: \(\inf_{\Sigma_{\nu'}} w^\mu < 0\) and \(\inf_{\Sigma_{\nu'}} z^\mu \geq 0\).

We consider the function

\[
W(x) = \begin{cases} 
\max\{-w^\mu(x), 0\}, & x \in \Sigma_{\nu'}, \\
0, & x \in (\Sigma_{\nu'})^c.
\end{cases}
\] (3.10)

Therefore, using the equation (3.1), we have

\[
0 \leq \int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} |\nabla W|^2 e^{\delta^\mu} d\theta d\tau = -\int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} \nabla_\nu \nabla_\mu W \cdot e^{\delta^\mu} d\theta d\tau
\]

\[
= \int_{\Sigma_{\nu'}} \operatorname{div}(\theta^{1-2\alpha}_{N+1} \nabla_\nu w^\mu) W e^{\delta^\mu} d\theta d\tau - \int_{\partial_\nu \Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} \frac{\partial w^\mu}{\partial \nu} W e^{\delta^\mu} d\theta d\tau
\]

\[
= -\int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} (w^\mu_\delta + \delta_1 w^\mu_\tau) W e^{\delta^\mu} d\theta d\tau + \int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} \nu^\mu_w W e^{\delta^\mu} d\theta d\tau
\]

\[
- \int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} 2 \delta_1 U_\tau W e^{\delta^\mu} d\theta d\tau - \int_{\partial_\nu \Sigma_{\nu'}} \nu^\mu_w W e^{\delta^\mu} d\theta d\tau
\]

\[
\leq -\int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} (w^\mu_\delta + \delta_1 w^\mu_\tau) W e^{\delta^\mu} d\theta d\tau,
\] (3.11)

since (3.7), \(\delta_1 \geq 0, \nu_1 > 0\) and \(z^\mu \geq 0\) on \(\partial_\nu \Sigma_{\nu'}\) by the continuity of \(z^\mu\).

Since \(\beta_1 > 0\), the definitions of \(U\) (see (2.2)) and \(\bar{w}^\mu\) imply

\[
\lim_{t \to -\infty} \inf_{\nu \in \Sigma_{\nu'}} \inf_{\theta \in S^N} w^\mu(t, \theta) \geq 0.
\]

This implies that \(W \equiv 0\) as \(t \to -\infty\) and \(\theta \in S^N\). Therefore, we can estimate that

\[
\int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} (w^\mu_\delta + \delta_1 w^\mu_\tau) W e^{\delta^\mu} d\theta d\tau = \int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} (e^{\delta^\mu} w^\mu_\delta) W d\theta d\tau
\]

\[
= -\int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} e^{\delta^\mu} w^\mu_\delta W d\theta d\tau
\]

\[
= \int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} e^{\delta^\mu} |W|^2 d\theta d\tau
\]

\[
\geq 0.
\] (3.12)

Together (3.11) and (3.12), we have

\[
\int_{\Sigma_{\nu'}} \theta^{1-2\alpha}_{N+1} |\nabla_\nu W|^2 e^{\delta^\mu} d\theta d\tau \equiv 0.
\]
This is impossible unless $W \equiv 0$ in $\Sigma_{\theta, \mu}$ and therefore $w^\mu \geq 0$ in $\Sigma_{\theta, \mu}$. This contradicts $\inf_{\Sigma_{\theta, \mu}^\mu} w^\mu < 0$.

Case II:

$$\inf_{\Sigma_{\theta, \mu}^\mu} w^\mu < 0 \text{ and } \inf_{\Sigma_{\theta, \mu}^\mu} z^\mu < 0.$$ 

Since $\overline{w^\mu} \geq 0$ on $T_{\theta, \mu}$ and $\lim_{t \to -\infty} \inf_{\Sigma_{\theta, \mu}^\mu} w^\mu(t, \bar{\theta}) \geq 0$, there exists a point $(\bar{t}, \bar{\theta}) \in \Sigma_{\theta, \mu}^\mu \cup \partial_t \Sigma_{\theta, \mu}^\mu$ such that the negative infimum of $w^\mu$ is achieved, that is,

$$w^\mu(\bar{t}, \bar{\theta}) = \inf_{\Sigma_{\theta, \mu}^\mu} w^\mu < 0.$$

If $(\bar{t}, \bar{\theta}) \in \Sigma_{\theta, \mu}^\mu$, then

$$w^\mu(\bar{t}, \bar{\theta}) = 0, \quad \Delta \mu w^\mu(\bar{t}, \bar{\theta}) \geq 0, \quad w^\mu_\bar{t}(\bar{t}, \bar{\theta}) = 0, \quad \text{and} \quad \nabla w^\mu(\bar{t}, \bar{\theta}) = 0.$$

Thus, from (2.6), (3.7) and the first equation of (3.1), we have that

$$0 < -\nu_1 w^\mu(\bar{t}, \bar{\theta}) \leq -2\delta_1 \nabla \mu(\bar{t}, \bar{\theta}) \leq 0,$$

since $\nu_1 > 0$ and this is a contradiction. So $(\bar{t}, \bar{\theta}) \in \partial_t \Sigma_{\theta, \mu}$, which implies that $\partial_{\theta, \mu} \overline{w^\mu}(\bar{t}, \bar{\theta}) \geq 0$. Therefore, $c^\mu(\bar{t}, \bar{\theta}) z^\mu(\bar{t}, \bar{\theta}) \leq 0$ and thus $z^\mu(\bar{t}, \bar{\theta}) \leq 0$ since $c^\mu \geq 0$. By the continuity of $z^\mu$, we know $z^\mu(t, \theta) \leq 0$ on $\partial_t \Sigma_{\theta, \mu}$. Then

$$0 < V(2\mu - t, \theta) \leq V(t, \theta) < \varepsilon_0 \text{ on } \partial_t \Sigma_{\theta, \mu}.$$

By the mean value principle, we have $c^\mu(t, \theta) \leq \rho \varepsilon_0^{p-1}$ on $\partial_t \Sigma_{\theta, \mu}$. Similarly, we have $d^\mu(t, \theta) \leq \rho \varepsilon_0^{-1}$ on $\partial_t \Sigma_{\theta, \mu}$.

We define function

$$Z(x) = \begin{cases} \max\{-z^\mu(x), 0\}, & x \in \Sigma_{\theta, \mu}^\mu, \\ 0, & x \in (\Sigma_{\theta, \mu}^\mu)^c. \end{cases} \quad (3.13)$$

Without of loss generality, we suppose $\delta_1 \leq \delta_2$. By a similar estimate to that in Case I, we have

$$\int_{\Sigma_{\theta, \mu}^\mu} \theta_{N+1}^{-2\alpha} \nabla Z^2 e^{\delta t} d\theta dt \leq \int_{\Sigma_{\theta, \mu}^\mu} \theta_{N+1}^{-2\alpha} \nabla Z^2 e^{\delta t} d\theta dt \leq -\int_{\partial_t \Sigma_{\theta, \mu}^\mu} d^\mu W z_e^{\delta} d\theta dt$$

$$= \int_{\partial_t \Sigma_{\theta, \mu}^\mu} d^\mu W z_e^{\delta} d\theta dt$$

$$= \int_{\partial_t \Sigma_{\theta, \mu}^\mu} d^\mu W e^{\delta z - \delta z_0} d\theta dt$$

$$\leq \rho \varepsilon_0 p^{-1} e^{\delta z - \delta z_0} \int_{\partial_t \Sigma_{\theta, \mu}^\mu} W e^{\delta t} d\theta dt.$$
By the H"older and fractional Sobolev trace inequalities as in [28] (see also [33] for the Sobolev trace inequality in all $\mathbb{R}^n$), we know
\[
\int_{\partial_2 \Sigma_{\nu, \mu}} WZ e^{\delta \theta} d\theta dt \leq \left( \int_{\partial_2 \Sigma_{\nu, \mu}} |e^{\delta \theta/2} Z|^2 d\theta dt \right)^{1/2} \left( \int_{\partial_2 \Sigma_{\nu, \mu}} |e^{\delta \theta/2} W|^2 d\theta dt \right)^{1/2} \\
\leq C_{N, \alpha} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla Z|^2 e^{\delta \theta} d\theta dt \right)^{1/2} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2},
\]
where $\theta = (\vartheta, \vartheta_{N+1}) \in S_{\nu}^N$ and $C_{N, \alpha}$ is a positive constant depending only on $N$ and $\alpha$. Hence,
\[
\left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla Z|^2 e^{\delta \theta} d\theta dt \right)^{1/2} \leq p \varepsilon_0^{-1} C_{N, \alpha} e^{\delta \theta_{N+1}} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2}, \tag{3.14}
\]
Moreover, as per the argument in Case I, we have
\[
\int_{\partial_2 \Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \leq - \int_{\partial_2 \Sigma_{\nu, \mu}} \varepsilon_0^{-1} C_{N, \alpha} e^{\delta \theta_{N+1}} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2}.
\]
Similarly, by the H"older and fractional Sobolev trace inequalities, we have
\[
\int_{\partial_2 \Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \leq q \varepsilon_0^{-1} C_{N, \alpha} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2}, \tag{3.15}
\]
Therefore,
\[
\left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2} \leq q \varepsilon_0^{-1} C_{N, \alpha} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla Z|^2 e^{\delta \theta} d\theta dt \right)^{1/2},
\]
Then, combining (3.14) and (3.15), we have
\[
\left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2} \leq p q C_{N, \alpha} d^{{\delta_{\vartheta N}} - 2\alpha} \varepsilon_0^{-1} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla W|^2 e^{\delta \theta} d\theta dt \right)^{1/2}
\]
and
\[
\left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla Z|^2 e^{\delta \theta} d\theta dt \right)^{1/2} \leq p q C_{N, \alpha} d^{{\delta_{\vartheta N}} - 2\alpha} \varepsilon_0^{-1} \left( \int_{\Sigma_{\nu, \mu}} \theta_{N+1}^{-2\alpha} |\nabla Z|^2 e^{\delta \theta} d\theta dt \right)^{1/2}.
\]
These are impossible unless \( W \equiv 0 \) and \( Z \equiv 0 \) in \( \Sigma_{h, \mu} \), since \( \varepsilon_0 \ll 1 \) and \( pq > 1 \) implies that \( p + q > 2 \). Thus \( w^\mu \geq 0 \) and \( z^\mu \geq 0 \) in \( \Sigma_{h, \mu} \), which contradict our assumptions. We thus complete the proof of lemma 3.1.

By Case I of lemma 3.1, we have \( w^\mu \geq 0 \) and \( z^\mu \geq 0 \) in \( \Sigma_\mu \) for all \( \mu \in (-\infty, t_0] \). This enables us to define a maximal value of \( \mu \) up to which the positivity of these functions holds. This is the purpose of the following lemma.

**Lemma 3.2.** We have either

1. \( \mu \in (-\infty, t_0] \) \( \forall \mu \in (-\infty, \Lambda) \), there exist \( w, z \) such that \( w^\mu \geq 0 \) and \( z^\mu \geq 0 \) in \( \Sigma_\mu \).

2. \( \mu \in (t_0, +\infty) \) for any \( \mu \in (t_0, +\infty) \), we have \( w^\mu > 0 \) and \( z^\mu > 0 \) in \( \Sigma_\mu \).

Moreover, in the latter case one has

\[
\begin{align*}
\mathcal{U}_{\mu} > 0 \quad \text{and} \quad \mathcal{V}_{\mu} > 0 \quad \text{in} \quad \mathbb{R} \times \mathcal{S}_N.
\end{align*}
\]

**Proof.** Define

\[
\Lambda = \sup \{ \mu \in \mathbb{R} : \forall \lambda \in (-\infty, \mu), \quad w^\lambda \geq 0 \quad \text{and} \quad z^\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda \}.
\]

By lemma 3.1, it is clear that \( \Lambda > -\infty \). Next, we prove that either \( \Lambda < +\infty \), in which case (1) holds with \( \bar{\mu} = \Lambda \), or \( \Lambda = +\infty \) in which case we have (2) together with (3.16).

If \( \Lambda = +\infty \), then case (2) is trivially satisfied. Moreover, in this case (3.16) is a consequence of the following argument. Since (3.5), we have

\[
\frac{\partial w^\mu}{\partial t} \leq 0 \quad \text{and} \quad \frac{\partial z^\mu}{\partial t} \leq 0 \quad \text{in} \quad \Sigma_\mu.
\]

By the definitions of \( w^\mu \) and \( z^\mu \), we know

\[
\frac{\partial w^\mu}{\partial t} = -2\mathcal{U}_{\mu} \quad \text{and} \quad \frac{\partial z^\mu}{\partial t} = -2\mathcal{V}_{\mu} \quad \text{on} \quad T_\mu.
\]

Therefore,

\[
\mathcal{U}_{\mu} \geq 0 \quad \text{and} \quad \mathcal{V}_{\mu} \geq 0 \quad \text{on} \quad T_\mu
\]

for all \( \mu \in \mathbb{R} \) and thus throughout \( \mathbb{R} \times \mathcal{S}_N \). Then, by (3.1), we have

\[
\begin{align*}
L_{w^\mu} w^\mu + \nu_0 w^\mu - \nu w^\mu &\leq 0 \quad \text{in} \quad \Sigma_\mu, \\
L_{z^\mu} z^\mu + \nu_0 z^\mu - \nu z^\mu &\leq 0 \quad \text{in} \quad \Sigma_\mu.
\end{align*}
\]

Applying the Hopf’s lemma (lemma 2.1) to each equation in (3.19) yields

\[
\frac{\partial w^\mu}{\partial t} < 0 \quad \text{and} \quad \frac{\partial z^\mu}{\partial t} < 0 \quad \text{on} \quad T_\mu,
\]

and thus we have (3.16) thanks to (3.18).

Suppose that \( \Lambda < +\infty \). We prove case (1) by contradiction and assume that \( w^\Lambda \not\equiv 0 \) or \( z^\Lambda \not\equiv 0 \) in \( \Sigma_{\Lambda} \). For all \( -\infty < \mu \leq \Lambda \), by the strong maximum principle, we know that \( w^\mu > 0 \) and \( z^\mu > 0 \) in \( \Sigma_\mu \). The above arguments imply that \( \mathcal{U}_{\mu} > 0 \) and \( \mathcal{V}_{\mu} > 0 \) on \( T_\mu \) for \( -\infty < \mu < \Lambda \). Hence, \( \mathcal{U}_{\mu} > 0 \) and \( \mathcal{V}_{\mu} > 0 \) in \( \Sigma_{\Lambda} \).
Therefore, by (3.1) we have that, for $-\infty < \mu < \Lambda$,
\[
\begin{align*}
L_\omega w^\mu + w^\mu_\theta + \delta_1 w^\mu_\theta - \nu_1 w^\mu & = -2\delta_1 \bar{U}_\ell \leq 0 \quad \text{in } \Sigma_\mu, \\
L_\omega z^\mu + z^\mu_\theta + \delta_2 z^\mu_\theta - \nu_2 z^\mu & = -2\delta_2 \bar{V}_\ell \leq 0 \quad \text{in } \Sigma_\mu.
\end{align*}
\] (3.20)

Now, evaluating (3.20) at $\mu = \Lambda$ by continuity, we obtain
\[
L_\omega w^\Lambda + w^\Lambda_\theta + \delta_1 w^\Lambda_\theta - \nu_1 w^\Lambda \leq 0 \quad \text{in } \Sigma_\Lambda.
\] (3.21)

and
\[
L_\omega z^\Lambda + z^\Lambda_\theta + \delta_2 z^\Lambda_\theta - \nu_2 z^\Lambda \leq 0 \quad \text{in } \Sigma_\Lambda.
\] (3.22)

An application of the strong maximum principle (see theorem 2.2) to (3.21) and (3.22) implies that either $w^\Lambda > 0$ or $w^\Lambda \equiv 0$ in $\Sigma_\Lambda$ on the one hand $z^\Lambda > 0$ or $z^\Lambda \equiv 0$ in $\Sigma_\Lambda$ on the other hand.

It is easy to check that (3.1) and theorem 2.2 rules out the cases $w^\Lambda > 0$ and $z^\Lambda \equiv 0$ in $\Sigma_\Lambda$ as well as $w^\Lambda \equiv 0$ and $z^\Lambda > 0$ in $\Sigma_\Lambda$. Here we only show the case $w^\Lambda > 0$ and $z^\Lambda \equiv 0$ in $\Sigma_\Lambda$ is impossible. In fact, since $z^\Lambda \equiv 0$ in $\Sigma_\Lambda$, by the continuity up to the boundary, we have $z^\Lambda = 0$ and $\partial_{\theta_b} z^\Lambda \equiv 0$ on $\partial \Sigma_\Lambda$. If $w^\Lambda > 0$ in $\Sigma_\Lambda$, then applying theorem 2.2 to (3.1) we know $w^\Lambda > 0$ on $\partial \Sigma_\Lambda$ and thus $d^\Lambda > 0$. So by (3.1) we have
\[
0 = - \lim_{\theta_b \to 0^+} \partial_{\theta_b} \left( \partial_{\theta_b} w^\Lambda \right) = d^\Lambda > 0
\] (3.23)
on $\partial \Sigma_\Lambda$. This is a contradiction.

Hence in the following we may assume that both $w^\Lambda$ and $z^\Lambda$ are strictly positive in $\Sigma_\Lambda$. By the Hopf’s lemma (lemma 2.1) we have
\[
\frac{\partial w^\Lambda}{\partial t} < 0 \quad \text{and} \quad \frac{\partial z^\Lambda}{\partial t} < 0 \quad \text{on } T_\Lambda,
\] (3.24)
since $w^\Lambda = 0$ and $z^\Lambda = 0$ on $T_\Lambda$.

Next, we claim that there exists $\varepsilon > 0$ such that $w^\mu \geq 0$ and $z^\mu \geq 0$ in $\Sigma_\mu$ for all $\mu \in (\Lambda, \Lambda + \varepsilon)$. This will result in a contradiction with the definition of $\Lambda$; hence the lemma will be proved.

This is done in the following way.

We split the domain into three disjoint subsets:
\[
\Sigma_\mu = \Sigma_\mu \cup (\Sigma_{\Lambda - \delta} \setminus \Sigma_\mu) \cup (\Sigma_\mu \setminus \Sigma_{\Lambda - \delta}),
\]
for some small $\delta > 0$ to be defined.

We start by checking the second set. For a given $\delta > 0$, we know $w^\Lambda > 0$ and $z^\Lambda > 0$ in the compact set $\Sigma_{\Lambda - \delta} \setminus \Sigma_\mu$. Therefore, a straightforward continuity argument implies that there exists $\varepsilon_2 = \varepsilon_2(\delta) > 0$ such that for all $\mu \in (\Lambda, \Lambda + \varepsilon_2),$
\[
\min \left\{ \inf_{\Sigma_{\Lambda - \delta} \setminus \Sigma_\mu} w^\mu, \inf_{\Sigma_{\Lambda - \delta} \setminus \Sigma_\mu} z^\mu \right\} > 0.
\]

We carry on the analysis by examining the first part of the domain. By the above consideration, for all $\mu \in (\Lambda, \Lambda + \varepsilon_2)$, we have $w^\mu \geq 0$ and $z^\mu \geq 0$ on $T_\mu \subset (\Sigma_{\Lambda - \delta} \setminus \Sigma_\mu)$. An application of case (2) of lemma 3.1, we have $w^\mu > 0$ and $z^\mu > 0$ in $\Sigma_\mu$.

Finally, we do the analysis on the third part of the domain, namely $\Sigma_\mu \setminus \Sigma_{\Lambda - \delta}$. A simple continuity argument shows that (3.24) remains valid if $\Lambda$ is replaced by any $\mu$ in a small right neighborhood of $\Lambda$, that is, there exists $\varepsilon_3 > 0$ such that for all $\mu \in (\Lambda, \Lambda + \varepsilon_3),$

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\[ \frac{\partial w^\mu}{\partial t} < 0 \quad \text{and} \quad \frac{\partial z^\mu}{\partial t} < 0 \quad \text{on} \ T_\mu. \]

Elliptic estimates give locally uniform \( C^2 \) bounds for \( w^\mu \) and \( z^\mu \) in \((t, \theta)\) as well as in \( \mu \). Hence

\[ \inf_{\mu - \varepsilon < \xi < \mu} \left( -\frac{\partial w^\mu}{\partial t} \right) \geq \inf_{T_\mu} \left( -\frac{\partial w^\mu}{\partial t} \right) - C\varepsilon, \]

for any \( \varepsilon \in (0, 1) \) and \( \theta \in S^N_\Lambda \). Similarly for \( z^\mu \). This means that we can choose \( \varepsilon_3' \in (0, \varepsilon_3) \) such that \( \partial w^\mu/\partial t < 0 \) and \( \partial z^\mu/\partial t < 0 \) in the set \( \{(t, \theta) : \mu - \varepsilon_3' < t < \mu, \theta \in S^N_\Lambda \} \) for all \( \mu \in [\Lambda, \Lambda + \varepsilon_3'/2] \). Hence, by \( w^\mu = 0 \) and \( z^\mu = 0 \) on \( T_\mu \), we know \( w^\mu > 0 \) and \( z^\mu > 0 \) in the set \( \{(t, \theta) : \mu - \varepsilon_3' < t < \mu, \theta \in S^N_\Lambda \} \) for all \( \lambda \in [\Lambda, \Lambda + \varepsilon_3'/2] \).

Fixing now \( \delta = \varepsilon_3'/2 \) and \( \varepsilon = \min\{\varepsilon_3(\delta), \varepsilon_3\} \), summing up the above results, we have \( w^\mu > 0 \) and \( z^\mu > 0 \) in \( \Sigma_\mu \) for all \( \mu \in (\Lambda, \Lambda + \varepsilon) \). This contradicts the definition of \( \Lambda \) (see (3.17)). \( \square \)

**Proof of theorem 1.2.** Suppose that \((U, V)\) are positive solutions to (1.12). Then the comparison functions \( w^\mu \) and \( z^\mu \) satisfy the alternative in lemma 3.1. Next, we prove both cases (1) and (2) in lemma 3.2 cannot happen.

First, we show that case (1) in lemma 3.2 provides a contradiction. In order to get the contradiction, we translate the origin to \( \bar{\mu} \), that is, define \( \bar{U}(t, \theta) = \bar{U}(t + \bar{\mu}, \bar{\theta}) \) and \( \bar{V}(t, \theta) = \bar{V}(t + \bar{\mu}, \bar{\theta}) \). Since \( w^\mu = 0 \) and \( z^\mu = 0 \) in \( \Sigma_\mu \), then those two functions are even in the variable \( t \), i.e.

\[ \bar{U}(-t, \theta) = \bar{U}(t, \theta) \quad \text{and} \quad \bar{V}(-t, \theta) = \bar{V}(t, \theta), \quad (3.25) \]

This implies \( \bar{U} \) and \( \bar{V} \) are odd functions. On the other hand, by the first and third equations of (3.1) and (3.2), \( \varepsilon_1, \varepsilon_2 = (0, 0) \) (see 2.6), we know \( \bar{U} \) or \( \bar{V} \) are even in variable \( t \). So we can conclude that \( \bar{U} \) or \( \bar{V} \) must be constant in the whole domain \( \mathbb{R} \times \Delta^N_\Lambda \). This contradicts the regularity of \( \bar{U} \) or \( \bar{V} \) at the origin, see (2.2).

We complete the proof of theorem 1.2 by showing that case (2) in lemma 3.2 is also impossible. Observe that (1.12) is translation invariant in the \( x \) direction for \( X = (x, y) \in \mathbb{R}^N \times \mathbb{R}^+ \). Hence we can change the initial function, i.e. we define \( U^0(x, y) = U(x - x_0, y), V^0(x, y) = V(x - x_0, y) \) (and corresponding \( \bar{U}^0, \bar{V}^0 \)) for any \( x_0 \in \mathbb{R}^N \). Repeating the whole discussion for these new functions, only two cases can arise. First, there exists an origin \( x_0 \) such that case (1) holds. As we have shown, this is impossible. Another case is, for any origin \( x_0 \in \mathbb{R}^N \), (2) holds for the transformed functions \( \bar{U}^0 \) and \( \bar{V}^0 \). So by (3.6), we have in particular

\[ \beta \lambda U(X) + \nabla U(X) \cdot (x - x_0, y) \geq 0 \]

for any \( X = (x, y) \in \mathbb{R}^N \times \mathbb{R}^+ \) and \( x_0 \in \mathbb{R}^N \). Hence we can obtain

\[ \nabla U(X) \cdot \left( \frac{x - x_0}{|x - x_0, y|}, \frac{y}{|x - x_0, y|} \right) \geq -\frac{\beta \lambda U(X)}{|x - x_0, y|} \quad (3.26) \]

for all \( X \in (\mathbb{R}^N \setminus x_0) \times \mathbb{R}^+ \). Now we let \( e = (e_1, e_2, \ldots, e_{N+1}) \in S^N_\Lambda \) with \( e_{N+1} > 0 \) fixed and choose \( x_0 = x - \sigma(e_1, \ldots, e_N) \) and \( y = \sigma e_{N+1} \) for \( \sigma > 0 \). Thus we rewrite (3.26) as

\[ \nabla U(X) \cdot e \geq -\frac{\beta \lambda U(X)}{\sigma}. \]
Letting $\sigma \rightarrow +\infty$ yields
\[ \nabla U(X) \cdot e \geq 0. \] (3.27)

Since (3.27) holds for any $e \in S^N_+$ and $X \in \mathbb{R}^{N+1}$, then we deduce that
\[ \nabla U(X) \equiv 0. \]

This is impossible by the second equation of (1.12) since the solution we are dealing with is nontrivial.

\[ \square \]

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