PHASE TRANSITION FOR THE ONCE-EXCITED RANDOM WALK ON GENERAL TREES

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Abstract. The phase transition of $M$-digging random on a general tree was studied by Collevecchio, Huynh and Kious [4]. In this paper, we study particularly the critical $M$-digging random walk on a superperiodic tree that is proved to be recurrent. We keep using the techniques introduced by Collevecchio, Kious and Sidoravicius [5] with the aim of investigating the phase transition of Once-excited random walk on general trees. In addition, we prove if $T$ is a tree whose branching number is larger than 1, any multi-excited random walk on $T$ moving, after excitation, like a simple random walk is transient.

1. Introduction

In this paper, we study a particular case of multi-excited random walks on trees, introduced by Volkov [11], called the once-excited random walk. Let $M \in \mathbb{N}$, $(\lambda_1, \ldots, \lambda_M) \in (\mathbb{R}_+)^M$ and $\lambda > 0$. Let $T$ be an infinite, locally-finite, tree rooted at $\varrho$. The $(\lambda_1, \ldots, \lambda_M, \lambda)$-ERW on $T$, is a nearest-neighbor random walk $(X_n)$ started at $\varrho$ such that if $X_n$ is on a site for the $i$-th time for $i \leq M$, then the walker takes a random step of a biased random walk with bias $\lambda_i$ (i.e. it jumps on its parent with probability proportional to 1, or jumps on a particular offspring of $\nu$ with probability proportional to $\lambda_i$); and if $i > M$, then $X_n$ takes a random step of a biased random walk with bias $\lambda$. In the case $M = 1$, it is called the once-excited random walk with parameters $(\lambda_1, \lambda)$. We write $(\lambda_1, \lambda)$-OERW for $(\lambda_1, \lambda)$-ERW. The definition of the model and the vocabulary will be made clear in Section 2.3.

Unlike the case of once-reinforced random walk in [5] or digging-random walk in [4], the phase transition of OERW does not depend only on the branching-ruin number and the branching number of tree (see Section 3 for more details). In the case $T$ is a spherically symmetric tree, we give a sharp phase transition recurrence/transience in terms of their branching number and branching-ruin number and others.

In the following, we denote $br(T)$ the branching number of a tree $T$ and $br_r(T)$ the branching-ruin number of a tree $T$, see (2.1) and (2.2) for their definitions. Let us simply emphasize that, for any tree $T$, its branching number is at least one, i.e. $br(T) \geq 1$, whereas the branching-ruin number is nonnegative, i.e. $br_r(T) \geq 0$.

A tree $T$ is said to be spherically symmetric if for every vertex $\nu$, $\deg \nu$ depends only on $|\nu|$, where $|\nu|$ denote its distance from the root and $\deg \nu$ is its number of neighbors. Let $T$ be a spherically symmetric tree. For any $n \geq 0$, let $x_n$ be the number of children of a vertex at level $n$. For any $\lambda_1 \geq 0$

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and \( \lambda > 0 \), we define the following quantities:

\[
\alpha(T, \lambda_1, \lambda) = \liminf_{n \to \infty} \left( \prod_{i=1}^{n} \frac{\lambda^2 + (x_i - 1)\lambda_1 \lambda + \lambda_1}{1 + x_i \lambda_1} \right)^{1/n}.
\]

\[
\beta(T, \lambda_1, \lambda) = \limsup_{n \to \infty} \left( \prod_{i=1}^{n} \frac{\lambda^2 + (x_i - 1)\lambda_1 \lambda + \lambda_1}{1 + x_i \lambda_1} \right)^{1/n}.
\]

\[
\gamma(T, \lambda) = \liminf_{n \to \infty} \frac{-\sum_{i=1}^{n} \ln \left[ 1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i \lambda_1)^n} \right]}{\ln n}.
\]

\[
\eta(T, \lambda) = \limsup_{n \to \infty} \frac{-\sum_{i=1}^{n} \ln \left[ 1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i \lambda_1)^n} \right]}{\ln n}.
\]

**Theorem 1.** Let \( T \) be a spherically symmetric tree, and let \( \lambda_1 \geq 0, \lambda > 0 \). Denote \( X \) the \((\lambda_1, \lambda)\)-OERW on \( T \). Assume that there exists a constant \( M > 0 \) such that \( \sup_{\nu \in V} \deg \nu \leq M \), then we have

1. in the case \( \lambda = 1 \), if \( \eta(T, \lambda_1) < br_r(T) \) then \( X \) is transient and if \( \gamma(T, \lambda_1) > br_r(T) \) then \( X \) is recurrent;
2. assume that \( \lambda_1 \geq 0, \lambda \neq 1 \) and \( br(T) > 1 \), if \( \beta(T, \lambda_1, \lambda) < \frac{1}{br(T)} \) then \( X \) is recurrent and if \( \alpha(T, \lambda_1, \lambda) > \frac{1}{br(T)} \) then \( X \) is transient.

Note that, for a \( b \)-ary tree, we have \( br(T) = b \) and

\[
\alpha(T, \lambda_1, \lambda) = \beta(T, \lambda_1, \lambda) = \frac{\lambda^2 + (b - 1)\lambda_1 \lambda + \lambda_1}{1 + b\lambda_1}
\]

and our result therefore agrees with Corollary 1.6 of [1]. In [1], the authors prove that the walk is recurrent at criticality on regular trees, but this is not expected to be true on any tree. For instance, if \( \lambda_1 = \lambda \), the \((\lambda, \lambda)\)-OERW \( X \) is the biased random walk with parameter \( \lambda \). Therefore \( X \) may be recurrent or transient at criticality (see [2], proposition 22).

Volkov [11] conjectured that, any cookie random walk which moves, after excitation, like a simple random walk (i.e. \( \lambda = 1 \)) is transient on any tree containing the binary tree. This conjecture was proved by Basdevant and Singh [1]. Here, we extend this conjecture to any tree \( T \) whose branching number is larger than 1:

**Theorem 2.** Let \((\lambda_1, ... , \lambda_M) \in (\mathbb{R}_+)^M \) and consider \((\lambda_1, ..., \lambda_M, 1)\)-ERW \( X \) on an infinite, locally finite, rooted tree \( T \). If \( br(T) > 1 \), then \( X \) is transient.

The techniques used in our paper rely on the strategy adopted in [5] or [4]. In particular, for the proof of transience, we here too view the set of edges crossed by \( X \) before returning to \( \varrho \) as the cluster of the root in a particular correlated percolation.

There are two key ingredients that allow us to use the rest of the strategy from [5]. First, we need to define extensions of \( X \), which are a family of coupled continuous-time versions of \( X \) defined on subtrees of \( T \). As in [5], we do this through Rubin’s construction in Section 7. But we will see in Section 7 this...
construction is actually very different to a once-reinforced random walk in [5] or \(M\)-digging random walk in [4].

Second, we need to prove that the correlated percolation mentioned above is in fact a \textit{quasi-independent} percolation, see Lemma [17]. From there, the problem boils down to proving that a certain quasi-independent percolation is supercritical.

We refer to Theorem [1] for the more general result on a general tree.

## 2. The model

First, we review some basic definitions of graph theory and then we define the model of multi-excited random walk on trees which was introduced by Volkov[11] and then made general by Basdevant and Singh[1].

### 2.1. Notation

Let \(T = (V, E)\) be an infinite, locally finite, rooted tree with the root \(\varrho\).

Given two vertices \(\nu, \mu\) of \(T\), we say that \(\nu\) and \(\mu\) are \textit{neighbors}, denoted \(\nu \sim \mu\), if \(\{\nu, \mu\}\) is an edge of \(T\).

Let \(\nu, \mu \in V \setminus \{\varrho\}\), the \textit{distance} between \(\nu\) and \(\mu\), denoted by \(d(\nu, \mu)\), is the minimum number of edges of the unique self-avoiding paths joining \(x\) and \(y\). The distance between \(\nu\) and \(\varrho\) is called \textit{height} of \(\nu\), denoted by \(|\nu|\). The parent of \(\nu\) is the vertex \(\nu^{-1}\) such that \(\nu^{-1} \sim \nu\) and \(|\nu^{-1}| = |\nu| - 1\). We also call \(\nu\) a \textit{child} of \(\nu^{-1}\).

For any \(\nu \in V\), denote by \(\partial(\nu)\) the number of children of \(\nu\) and \(\{\nu_1, ..., \nu_{\partial\nu}\}\) is the set of children of \(\nu\). We define an order on \(T\) by the following way. For all \(\nu\) and \(\mu\), we say that \(\nu \leq \mu\) if the unique self-avoiding path joining \(\varrho\) and \(\mu\) contains \(\nu\), and we say that \(\nu < \mu\) if moreover \(\nu \neq \mu\).

Denote by \(T_n\) the set of vertices of \(T\) at height \(n\). For any \(\nu \in T\), denote by \(T^\nu\) the biggest sub-tree of \(T\) rooted at \(\nu\), i.e. \(T^\nu = T[V^\nu]\), where

\[
V^\nu := \{v \in V(T) : u \leq v\}.
\]

For any edge \(e\) of \(T\), denote by \(e^+\) and \(e^-\) its endpoints with \(|e^+| = |e^-| + 1\), and we define the \textit{height} of \(e\) as \(|e| = |e^+|\).

For two edges \(e\) and \(g\) of \(T\), we write \(g \leq e\) if \(g^+ \leq e^+\) and \(g < e\) if moreover \(g^+ \neq e^+\). For two vertices \(\nu\) and \(\mu\) of \(T\) such that \(\nu < \mu\), we denote by \([\nu, \mu]\) the unique self-avoiding path joining \(\nu\) to \(\mu\). For two neighboring vertices \(\nu\) and \(\mu\), we use the slight abuse of notation \([\nu, \mu]\) to denote the edge with endpoints \(\nu\) and \(\mu\) (note that we allow \(\mu < \nu\)).

For two edges \(e_1\) and \(e_2\) of \(E\), denote by \(e_1 \wedge e_2\) the vertex with maximal distance from the root such that \(e_1 \wedge e_2 \leq e_1^+\) and \(e_1 \wedge e_2 \leq e_2^+\).

Finally, we define a particular class of trees, which is called \textit{superperiodic tree}. Let \(T_1 = (V_1, E_1)\) and \(T_2 = (V_2, E_2)\) be two trees. A \textit{morphism} of \(T_1\) to \(T_2\) is a map \(f : T_1 \rightarrow T_2\) such that whenever \(\nu\) and \(\mu\) are incident in \(T_1\), then so are \(f(\nu)\) and \(f(\mu)\) in \(T_2\).

Let \(N \geq 0\). An infinite, locally finite and rooted tree \(T\) with the root \(\varrho\), is said to be \(N\)-\textit{superperiodic} if for every \(\nu \in V(T)\), there exists an injective morphism \(f : T \rightarrow T^{f(\nu)}\) with \(f(\varrho) \in T^\nu\) and \(|f(\varrho)| - |\nu| \leq N\). A tree \(T\) is called \textit{superperiodic} if there exists \(N \geq 0\) such that it is \(N\)-superperiodic.
2.2. Some quantities on trees. In this section, we review the definitions of branching number, growth rate and branching-ruin number. We refer the reader to ([6], [8]) for more details on the branching number and growth rate and [5] for more details on the branching-ruin number.

In order to define the branching number and the branching-ruin number of a tree, we will need the notion of cutsets. Let \( T \) be an infinite, locally finite and rooted tree. A cutset in \( T \) is a set \( \pi \) of edges such that every infinite simple path from \( a \) must include an edge in \( \pi \). The set of cutsets is denoted by \( \Pi \).

The branching number of \( T \) is defined as
\[
br(T) = \sup \left\{ \gamma > 0 : \inf_{\pi \in \Pi} \sum_{e \in \pi} \gamma^{-|e|} > 0 \right\} \in [1, \infty].
\]

The branching-ruin number of \( T \) is defined as
\[
br_r(T) = \sup \left\{ \gamma > 0 : \inf_{\pi \in \Pi} \sum_{e \in \pi} |e|^{-\gamma} > 0 \right\} \in [0, \infty].
\]

These quantities depend on the structure of the tree. If \( T \) is spherically symmetric, then there is really no information in the tree than that contained in the sequence \(|T_n|, n \geq 0\). Therefore, a tree which is spherically symmetric and whose \( n \) generation grows like \( b^n \) (resp. \( n^b \)), for \( b \geq 1 \), has a branching number (resp. branching-ruin number) equal to \( b \). For more general trees, this becomes more complicated. In the other word, there exists a tree whose \( n \) generation grows like \( b^n \) (resp. \( n^b \)), for \( b \geq 1 \), but its branching number (resp. branching-ruin number) is not equal to \( b \). For instance, the tree 1-3 in ([8], page 4) is an example.

Finally, we review the definition of growth rate of an infinite, locally finite and rooted tree \( T \). Define the lower growth rate of \( T \) by
\[
gr(T) = \liminf |T_n|^{\frac{1}{n}}.
\]

Similarly, we can define upper growth rate of \( T \) by
\[
\bar{gr}(T) = \limsup |T_n|^{\frac{1}{n}}.
\]

In the case \( gr(T) = \bar{gr}(T) \), we define the growth rate of \( T \), denoted by \( gr(T) \), by taking the common value of \( gr(T) \) and \( \bar{gr}(T) \).

Now, we state a relationship between the branching number and growth rate of a superperiodic tree.

**Theorem 3** (see [8]). Let \( T \) be a \( N \)-superperiodic tree with \( \bar{gr}(T) < \infty \). Then the growth rate of \( T \) exists and \( gr(T) = \br(T) \). Moreover, we have \( |T_n| \leq gr(T)^{n+N} \).

2.3. Definition of the model. Now, we define the model of multi-excited random walk on trees. Let \( C = (\lambda_1, ..., \lambda_M; \lambda) \in (\mathbb{R}_+)^M \times \mathbb{R}_+^* \) and \( T = (V, E) \) be an infinite, locally finite and rooted tree with the root \( \varrho \). A \( C \) multi-excited random walk is a stochastic process \( X := (X_n)_{n \geq 0} \) defined on some probability space, taking the values in \( T \) with the transition probability defined by:

\[
P(X_0 = \varrho) = 1,
\]
\[ P(X_{n+1} = (X_n)_i | X_0, \ldots, X_n) = \begin{cases} \frac{\lambda_j}{1 + \partial(X_n) \lambda_j} & \text{if } j \leq M \\ \frac{1}{1 + \partial(X_n) \lambda_j} & \text{if } j > M \end{cases} \]

\[ P(X_{n+1} = X_n^{-1} | X_0, \ldots, X_n) = \begin{cases} \frac{1}{1 + \partial(X_n) \lambda_j} & \text{if } j \leq M \\ \frac{1}{1 + \partial(X_n) \lambda_j} & \text{if } j > M \end{cases} \]

where \( i \in \{1, \ldots, k\} \) and \( j = |\{0 \leq k \leq n : X_k = X_n\}| \).

We have some particular cases:

- If \( C = (0, \ldots, 0; \lambda) \), then \( C \) multi-excited random walk is \( M \)-digging random walk with parameter \( \lambda \) (\( M \)-DRW\( \lambda \)), which was studied in [4].
- If \( M = 0 \), then \( C \) multi-excited random walk is the biased random walk with parameter \( \lambda \), which was studied in [7].
- If \( C = (\lambda_1; \lambda) \), then \( C \) multi-excited random walk is \( (\lambda_1, \lambda) \)-OERW.

The return time of \( X \) to a vertex \( \nu \) is defined by:

\[ T(\nu) := \inf\{n \geq 1 : X_n = \nu\}. \]  

We say that \( X \) is transient if

\[ P(T(\rho) = \infty) > 0. \]  

Otherwise, we say that \( X \) is recurrent.

3. Main results

3.1. Main results about Once-excited random walk. Let \( \lambda_1 \geq 0 \) and \( \lambda > 0 \) and we consider the model \( (\lambda_1, \lambda) \)-OERW on an infinite, locally finite and rooted tree \( T \). First, we define the following functions. For any \( e \in E \), we set \( \psi(e, \lambda) = 1 \) and \( \phi(e, \lambda_1, \lambda) = 1 \) if \( |e| = 1 \) and, for any \( e \in E \) with \( |e| > 1 \), we set

\[ \psi(e, \lambda) = \frac{\lambda |e|-1 - 1}{\lambda |e| - 1} \quad \text{if } \lambda \neq 1, \]

\[ \psi(e, \lambda) = \frac{|e| - 1}{|e|} \quad \text{if } \lambda = 1. \]

\[ \phi(e, \lambda_1, \lambda) = \frac{\lambda_1}{1 + \partial(e^-) \lambda_1} + \frac{1}{1 + \partial(e^-) \lambda_1} \psi(e, \lambda)\psi(e^{-1}, \lambda) + \frac{(\partial(e^-) - 1) \lambda_1}{1 + \partial(e^-) \lambda_1} \psi(e, \lambda) \]

Finally, for any \( e \in E \), we define:

\[ \Psi(e, \lambda_1, \lambda) = \prod_{g \leq e} \phi(g, \lambda_1, \lambda). \]

We refer the reader to Lemma [14] for the probabilistic interpretation of these functions.

In the following, we assume that

\[ \exists M \in \mathbb{N} \text{ such that } \sup\{\deg \nu : \nu \in V\} \leq M. \]
Let us define the quantity $RT(T, X)$ which was introduced in [3]:
\begin{equation}
RT(T, X) = \sup\{\gamma > 0 : \inf_{\pi \in \Pi} \sum_{e \in \pi} (\Psi(e))^{\gamma} > 0\}.
\end{equation}

**Theorem 4.** Consider an $(\lambda_1, \lambda)$-OERW on an infinite, locally finite, rooted tree $T$, with parameters $\lambda_1 \geq 0$ and $\lambda > 0$. If $RT(T, X) < 1$ then $X$ is recurrent. If $RT(T, X) > 1$ and if (3.4) holds, then $X$ is transient.

In the following, we consider the case $T$ is spherically symmetric.

**Lemma 5.** Consider a $(\lambda_1, \lambda)$-OERW $X$ on a spherically symmetric $T$, with parameters $\lambda_1 \geq 0$ and $\lambda > 0$. Assume that there exists a constant $M > 0$ such that $\sup_{\nu \in V} \deg \nu \leq M$. We have that
\begin{enumerate}
  \item in the case $\lambda = 1$, if $\eta(T, \lambda_1) < br_r(T)$ then $RT(T, X) > 1$ and if $\gamma(T, \lambda_1) > br_r(T)$ then $RT(T, X) < 1$;
  \item assume that $\lambda_1 \geq 0$, $\lambda \neq 1$ and $br_r(T) > 1$, if $\beta(T, \lambda_1, \lambda) < \frac{1}{br_r(T)}$ then $RT(T, X) < 1$ and if $\alpha(T, \lambda_1, \lambda) > \frac{1}{br_r(T)}$ then $RT(T, X) > 1$.
\end{enumerate}

Note that Theorem 1 is a consequence of Theorem 4 and Lemma 5.

### 3.2. Main results about critical $M$-Digging random walk.

Let $M \in \mathbb{N}^*$, $\lambda > 0$ and we consider the model $M$-DRW$_\lambda$ on an infinite, locally finite and rooted tree $T$. In [4], Collevecchio-Huynh-Kious was proved that there is a phase transition with respect to the parameter $\lambda$, i.e there exists a critical parameter $\lambda_c$. A natural question that arises: what happens if $\lambda = \lambda_c$? As we said in the introduction, there is no a good answer for this question.

In [1], Basdevant-Singh proved the critical $M$-digging random walk is recurrent on the regular trees. In this paper, we prove the critical $M$-digging random walk is still recurrent on a particular class of trees which contains the regular trees.

**Theorem 6.** Let $M \in \mathbb{N}^*$ and $T$ be a superperiodic tree whose upper-growth rate is finite. Then the critical $M$-digging random walk on $T$ is recurrent.

### 4. An example

In this section, we give an example to prove that the phase transition of once-excited random walk $(\lambda_1, \lambda)$-OERW on a tree $T$ does not depend only on the branching-ruin number and the branching number of $T$.

If $T$ is a spherically symmetric tree, recall that $x_n(T)$ is the number of children of a vertex at level $n$.

Let $T$ (resp. $\tilde{T}$) be a spherically symmetric such that for any $n \geq 0$, we have $x_n(T) = 2$ (resp. $x_n(\tilde{T}) = 1$ if $n$ is odd and $x_n(\tilde{T}) = 4$ if not). Then we obtain:

\begin{align*}
  (4.1) & \quad br(T) = br(\tilde{T}) = 2. \\
  (4.2) & \quad br_r(T) = br_r(\tilde{T}) = \infty.
\end{align*}

**Lemma 7.** Consider a $(1, (\sqrt{3} - 1)/2)$-OERW $X$ (resp. $\tilde{X}$) on $T$ (resp. $\tilde{T}$). Then $X$ is recurrent, but $\tilde{X}$ is transient.
Proof. Note that \( T \) is a binary tree, then we can apply Corollary 1.6 of [1] to imply that \( X \) is recurrent. On the other hand, by a simple computation we have
\[
\alpha \left( \tilde{T}, 1, \frac{\sqrt{3} - 1}{2} \right) = \beta \left( \tilde{T}, 1, \frac{\sqrt{3} - 1}{2} \right) > \frac{1}{2}.
\]
By Theorem 1 and 4.3, we obtain \( \tilde{X} \) is transient. \( \square \)

5. PROOF OF THEOREM 2

Lemma 8. Let \( T \) be an infinite, locally finite and rooted tree. If \( br(T) > 1 \) then \( br_r(T) = +\infty \).
Proof. See ([4], proof of Lemma 8, Case V). \( \square \)

Lemma 9. Let \( (\lambda_1, ..., \lambda_M) \in (\mathbb{R}^+)^M \) and \( T \) be an infinite, locally finite and rooted tree. If \( M\text{-DRW}_1 \) is transient, then \( (\lambda_1, ..., \lambda_M, 1)\text{-ERW} \) is transient.
Proof. See ([1], Section 3). \( \square \)

Remark 10. Let \( T(\varrho) \) (resp. \( S(\varrho) \)) the return of \( M\text{-DRW}_1 \) (resp. \( (\lambda_1, ..., \lambda_M, 1)\text{-ERW} \)) to the root \( \varrho \) of \( T \). It is simple to see that
\[
P(T(\varrho) < \infty) \leq P(S(\varrho) < \infty).
\]

Proposition 11. Let \( (\lambda_1, ..., \lambda_M) \in (\mathbb{R}^+)^M \) and consider \( (\lambda_1, ..., \lambda_M, 1)\text{-ERW} X \) on an infinite, locally finite, rooted tree \( T \). If \( br(T) > 1 \), then \( X \) is transient.
Proof. Note that if \( \lambda_i = 0 \) for all \( 1 \leq i \leq M \) and \( \lambda = 1 \), then \( X \) is a \( M \)-digging random walk with parameter \( 1 \) \( (M\text{-DRW}_1) \). On the other hand, we have \( (\lambda_1, ..., \lambda_M, 1)\text{-ERW} \) is more transient than \( M\text{-DRW}_1 \), i.e if \( M\text{-DRW}_1 \) is transient then \( (\lambda_1, ..., \lambda_M, 1)\text{-ERW} \) is transient. We complete the proof by using Lemma (8) and Theorem 2 in [4]. \( \square \)

6. PROOF OF LEMMA 5 AND THEOREM 1

In this section, we prove Lemma 5. Theorem 1 then trivially follows from Theorem 4.

Lemma 12. Recall the definition of \( \Psi(e, \lambda_1, \lambda) \) as in (3.3). We have that, if \( \lambda \neq 1 \), for any \( |e| > 1 \),
\[
\Psi(e, \lambda_1, \lambda) = \prod_{g \leq e, |g| > 1} \frac{\lambda^2 + (\partial(g^-) - 1)\lambda_1\lambda + \lambda_1}{1 + \partial(g^-)\lambda_1} \prod_{g \leq e, |g| > 1} \left( 1 - \lambda|g| \frac{1 + \partial(g^-)\lambda_1}{\lambda^2 + (\partial(g^-) - 1)\lambda_1\lambda + \lambda_1} \right).
\]
and if \( \lambda = 1 \), for any \( |e| > 1 \),
\[
\Psi(e, \lambda_1, \lambda) = \prod_{g \leq e, |g| > 1} \left( 1 - \frac{(\partial(g^-) - 1)\lambda_1 + 2}{|g| (1 + \partial(g^-)\lambda_1)} \right).
\]
Proof. We compute the quantity \( \Psi(e, \lambda_1, \lambda) \) by using (3.1), (3.2) and (3.3). We will proceed by distinguishing two cases.
Case 1: \( \lambda \neq 1 \).

By (3.1), (3.2) and (3.3), we have
\[
\Psi(e, \lambda_1, \lambda) = \prod_{g \leq e, |g| > 1} \phi(g, \lambda_1, \lambda)
\]
By (3.1), 3.2 and (3.3), we have

\[
\begin{align*}
&= \prod_{g \leq \varepsilon, |g| > 1} \left( \frac{\lambda_1}{1 + \vartheta(g^-)\lambda_1} + \frac{1}{1 + \vartheta(g^-)\lambda_1} \psi(e, \lambda) \psi(e^{-1}, \lambda) + \frac{(\vartheta(g^-) - 1)\lambda_1}{1 + \vartheta(g^-)\lambda_1} \psi(e, \lambda) \right) \\
&= \left( \prod_{g \leq \varepsilon, |g| > 1} \frac{1}{1 + \vartheta(g^-)\lambda_1} \right) \prod_{g \leq \varepsilon, |g| > 1} \left( \lambda_1 + \psi(e, \lambda) \psi(e^{-1}, \lambda) + (\vartheta(g^-) - 1)\lambda_1 \psi(e, \lambda) \right)
\end{align*}
\]

By 3.1 we have:

\[
\begin{align*}
\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\vartheta(g^-) - 1)\lambda_1 \psi(g, \lambda) \\
= \lambda_1 + \left( \frac{1 - (1/\lambda)|g|^{-2}}{1 - (1/\lambda)|g|^{-1}} \right) + \left( (\vartheta(g^-) - 1)\lambda_1 \frac{1 - (1/\lambda)|g|^{-1}}{1 - (1/\lambda)|g|^{-1}} \right) \\
= \lambda_1 + \left( \frac{\lambda |g| - \lambda^2}{\lambda |g| - 1} \right) + (\vartheta(g^-) - 1)\lambda_1 \left( \frac{\lambda |g| - \lambda}{\lambda |g| - 1} \right) \\
= \frac{\lambda^2 + (\vartheta(g^-) - 1)\lambda_1 \lambda + \lambda_1 - \lambda |g| (1 + \vartheta(g^-)\lambda_1)}{1 - \lambda |g|} \\
= \left( \lambda^2 + (\vartheta(g^-) - 1)\lambda_1 \lambda + \lambda_1 \right) \left( \frac{1 - \lambda |g| \left( \frac{1 + \vartheta(g^-)\lambda_1}{\lambda^2 + (\vartheta(g^-) - 1)\lambda_1 \lambda + \lambda_1} \right)}{1 - \lambda |g|} \right).
\end{align*}
\]

Therefore we obtain 6.1

**Case II**: \( \lambda = 1 \).

By (3.1), 3.2 and (3.3), we have

\[
\Psi(e, \lambda_1, \lambda) = \prod_{g \leq \varepsilon, |g| > 1} \phi(g, \lambda_1, \lambda)
\]

\[
= \prod_{g \leq \varepsilon, |g| > 1} \left( \frac{\lambda_1}{1 + \vartheta(g^-)\lambda_1} + \frac{1}{1 + \vartheta(g^-)\lambda_1} \psi(e, \lambda) \psi(e^{-1}, \lambda) + \frac{(\vartheta(g^-) - 1)\lambda_1}{1 + \vartheta(g^-)\lambda_1} \psi(e, \lambda) \right) \\
= \left( \prod_{g \leq \varepsilon, |g| > 1} \frac{1}{1 + \vartheta(g^-)\lambda_1} \right) \prod_{g \leq \varepsilon, |g| > 1} \left( \lambda_1 + \psi(e, \lambda) \psi(e^{-1}, \lambda) + (\vartheta(g^-) - 1)\lambda_1 \psi(e, \lambda) \right)
\]

By 3.1 we have:

\[
\begin{align*}
\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\vartheta(g^-) - 1)\lambda_1 \psi(g, \lambda) \\
= \lambda_1 + \frac{|g| - 2}{|g|} + (\vartheta(g^-) - 1)\lambda_1 \frac{|g| - 1}{|g|} \\
= \frac{\lambda_1 |g| + |g| - 2 + (\vartheta(g^-) - 1)\lambda_1 (|g| - 1)}{g} \\
= 1 + \vartheta(g^-)\lambda_1 - \frac{(\vartheta(g^-) - 1)\lambda_1 + 2}{|g|}
\end{align*}
\]

Therefore we obtain 6.2 \( \square \)
Proof of Lemma 5. We will proceed by distinguishing a few cases.

**Case I:** $\lambda \neq 1$, $br(T) > 1$ and $\beta(T, \lambda_1, \lambda) < \frac{1}{br(T)}$.

By (2.1), there exists $\delta \in (0, 1)$ such that

$$\inf_{\pi \in \Pi} \sum_{e \in \Pi} \beta^{(1-\delta)^2|e|} = 0. \quad (6.5)$$

As $\beta < \beta^{(1-\delta)}$, there exists $c > 0$, for any $n > 0$,

$$\prod_{i=1}^{n} \frac{\lambda^2 + (x_i - 1)\lambda_1 \lambda + \lambda_1}{1 + x_i \lambda_1} \leq c \beta^{(1-\delta)n}. \quad (6.6)$$

By (6.1) and (6.6), there exists $C > 0$ such that for any $\pi \in \Pi$,

$$\sum_{e \in \pi} \Psi(e)^{1-\delta} \leq C \sum_{e \in \Pi} \beta^{(1-\delta)^2|e|}. \quad (6.7)$$

Therefore, by (6.5),

$$\inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1-\delta} = 0, \quad (6.8)$$

which implies that $RT(T, X) < 1$.

**Case II:** $\lambda \neq 1$, $br(T) > 1$ and $\alpha(T, \lambda_1, \lambda) > \frac{1}{br(T)}$.

First, note that if $\lambda > 1$ and $br(T) > 1$ then $X$ is transient. Now, assume that $\lambda < 1$, $br(T) > 1$ and $\alpha(T, \lambda_1, \lambda) > \frac{1}{br(T)}$. We have that there exists $\delta > 0$ and $\varepsilon > 0$ such that

$$\inf_{\pi \in \Pi} \sum_{e \in \Pi} \alpha^{(1+\delta)^2|e|} > \varepsilon. \quad (6.9)$$

By (6.1) and $\lambda < 1$, we obtain $\alpha < 1$, therefore $\alpha^{1+\delta} < \alpha$. We have that there exists $c > 0$, for any $n > 0$,

$$\prod_{i=1}^{n} \frac{\lambda^2 + (x_i - 1)\lambda_1 \lambda + \lambda_1}{1 + x_i \lambda_1} \geq c \alpha^{(1+\delta)n}. \quad (6.10)$$

By (6.1) and (6.10), there exists $C > 0$ such that for any $\pi \in \Pi$,

$$\sum_{e \in \pi} \Psi(e)^{1+\delta} \geq C \sum_{e \in \Pi} \alpha^{(1+\delta)^2|e|}. \quad (6.11)$$

Therefore, by (6.9),

$$\inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1+\delta} > 0, \quad (6.12)$$

which implies that $RT(T, X) > 1$.

**Case III:** $\lambda = 1$ and $\eta(T, \lambda_1) < br(T)$.

We have that there exists $\delta > 0$ and $\varepsilon > 0$ such that

$$\inf_{\pi \in \Pi} \sum_{e \in \pi} |e|^{-(1+\delta)^2}\eta > \varepsilon. \quad (6.13)$$
As \( \eta < (1 + \delta)\eta \), by (\ref{eq:2.4}) there exists \( c > 0 \), for any \( n > 0 \),
\begin{equation}
\prod_{i=1}^{n} \left[ 1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i\lambda_1)^2} \right] \geq cn^{-(1+\delta)\eta}.
\end{equation}
By (\ref{eq:6.2}) and (\ref{eq:6.14}) there exists \( C > 0 \) such that for any \( \pi \in \Pi \),
\begin{equation}
\sum_{e \in \pi} \Psi(e)^{1+\delta} \geq C \sum_{e \in \Pi} |e|^{-(1+\delta)2\eta}.
\end{equation}
Therefore, by (\ref{eq:6.13}),
\begin{equation}
\inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1+\delta} > 0,
\end{equation}
which implies that \( RT(T, X) > 1 \).

**Case IV:** \( \lambda = 1 \) and \( \gamma(T, \lambda_1) > br_r(T) \)

We have that there exists \( \delta > 0 \) such that
\begin{equation}
\inf_{\pi \in \Pi} \sum_{e \in \pi} |e|^{-(1-\delta)2\eta} = 0.
\end{equation}
As \( \eta > (1 - \delta)\eta \), by (\ref{eq:2.4}) there exists \( c > 0 \), for any \( n > 0 \),
\begin{equation}
\prod_{i=1}^{n} \left[ 1 - \frac{(x_i - 1)\lambda_1 + 2}{(1 + x_i\lambda_1)^2} \right] \leq cn^{-(1-\delta)\eta}.
\end{equation}
By (\ref{eq:6.2}) and (\ref{eq:6.18}) there exists \( C > 0 \) such that for any \( \pi \in \Pi \),
\begin{equation}
\sum_{e \in \pi} \Psi(e)^{1-\delta} \leq C \sum_{e \in \Pi} |e|^{-(1-\delta)2\eta}.
\end{equation}
Therefore, by (\ref{eq:6.17}),
\begin{equation}
\inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e)^{1-\delta} > 0,
\end{equation}
which implies that \( RT(T, X) < 1 \).

7. Extensions

First of all, let us describe the dynamic of this model. If \( X \) visits a vertex \( \nu \) for the first time, three cases can occur for visiting \( \nu_1 \) (see Figure 1):
- It eats the cookie at \( \nu \) and returns to the parent of \( \nu \) (i.e. \( \nu^{-1} \)) with probability \( \frac{1}{1+\phi(\nu)\lambda_1} \). It then visits \( \nu \) for the second time, and goes to \( \nu_1 \) with probability \( \frac{\lambda}{1+\phi(\nu)\lambda_1} \).
- It goes directly to \( \nu_1 \) with probability \( \frac{\lambda_1}{1+\phi(\nu)\lambda_1} \).
- It goes to one of the children of \( \nu \) except for \( \nu_1 \), with probability \( \frac{(\phi(\nu)-1)\lambda_1}{1+\phi(\nu)\lambda_1} \). It then visits \( \nu \) for the second time, and goes to \( \nu_1 \) with probability \( \frac{\lambda}{1+\phi(\nu)\lambda_1} \).

Now, we introduce a construction of once-excited random walk by using the Rubin’s construction. Let \( (\Omega, \mathcal{F}, P) \) denote a probability space on which
\begin{equation}
Y = (Y(\nu, \mu, k) : (\nu, \mu) \in V^2, \text{ with } \nu \sim \mu, \text{ and } k \in \mathbb{N})
\end{equation}
\begin{equation}
Z = (Z(\nu, \mu) : (\nu, \mu) \in V^2, \text{ with } \nu \sim \mu)
\end{equation}
are two families of independent mean 1 exponential random variables, where \((\nu, \mu)\) denotes an ordered pair of vertices. Let

\[ U = (U_\nu : \nu \in V) \]

is a family of independent uniformly random variables on \([0, 1]\) which is independent to \(Y\) and \(Z\).

For any pair vertices \(\nu, \mu \in V\) with \(\nu \sim \mu\), we define the following quantities

\[ r(\nu, \mu) = \begin{cases} 
\lambda|\nu|^{-1}, & \text{if } \mu < \nu, \\
\lambda|\mu|^{-1}, & \text{if } \nu < \mu.
\end{cases} \]

Let \(T'\) be a sub-tree of \(T\), we define the extension \(X^{(T')} = (V', E')\) on \(T'\) in the following way. Denote by \(g'\) the root of \(T'\) which be defined as the vertex of \(V'\) with smallest distance to the root of \(T\). For any family of nonnegative integers \(k = (k_\mu)_{\mu: [\nu, \mu] \in E'}\), we let

\[ A_{k, n, \nu}^{(T')} := \{X_n^{(T')} = \nu\} \cap \bigcap_{\mu: [\nu, \mu] \in E'} \{\#1 \leq j \leq n: (X_j^{(T')}, X_j^{(T')}) = (\nu, \mu)\} = k_\mu. \]

\[ t_\nu(n) := \#\{1 \leq j \leq n : X_j^{(T')} = \nu\}. \]

\[ h_\nu := \inf\{i \geq 1 : t_\nu(i) = 2\}. \]

\[ \overline{A}_{k, n, \nu}^{(T')} := \{X_n^{(T')} = \nu\} \cap \bigcap_{\mu: [\nu, \mu] \in E'} \{\#h_\nu \leq j \leq n: (X_j^{(T')}, X_j^{(T')}) = (\nu, \mu)\} = k_\mu. \]

\[ \mathcal{I}^T(\nu) := \#\{i \in \{1, 2, \ldots, \partial(\nu)\} : \nu_i \in V(T')\}. \]

Set \(X_0^{(T')} = g'\) and on the event \(A_{k, n, \nu}^{(T')} \cap \{t_\nu(n) \leq 1\}:\)
\begin{itemize}
  \item If $U_\nu < \frac{1}{1+\delta(\nu)\lambda_1}$, then we set $X^{\nu}_{n+1} = \nu^{-1}$.
  \item If $U_\nu \in \left[\frac{1+(j-1)\lambda_1}{1+\delta(\nu)\lambda_1} \quad \frac{1+j\lambda_1}{1+\delta(\nu)\lambda_1}\right]$ and $j \in \mathcal{I}(\nu)$, then we set $X^{\nu}_{n+1} = v_j$.
  \item If $U_\nu \in \left[\frac{1+(j-1)\lambda_1}{1+\delta(\nu)\lambda_1} \quad \frac{1+j\lambda_1}{1+\delta(\nu)\lambda_1}\right]$ for some $j \notin \mathcal{I}(\nu)$ and
    \[
    \left\{ \nu' = \arg \min_{\mu : [\nu,\mu] \in E^c} \left\{ Z(\nu,\mu) \right\} \right\},
    \]
    \begin{equation}
    \tag{7.10}
    \mathcal{A}_{k,n,\nu}^{\nu'} \cap \left\{ t_\nu(n) \geq 2 \right\} \cap \left\{ \nu' = \arg \min_{\mu : [\nu,\mu] \in E^c} \left\{ \sum_{i=0}^{k_n} \frac{Y(\nu,\mu,i)}{r(\nu,\mu)} \right\} \right\},
    \end{equation}
    we set $X^{\nu'}_{n+1} = \nu'$.
  \end{itemize}

On the event

\begin{equation}
\tag{7.10}
\mathcal{A}_{k,n,\nu}^{\nu'} \cap \left\{ t_\nu(n) \geq 2 \right\} \cap \left\{ \nu' = \arg \min_{\mu : [\nu,\mu] \in E^c} \left\{ \sum_{i=0}^{k_n} \frac{Y(\nu,\mu,i)}{r(\nu,\mu)} \right\} \right\},
\end{equation}

we set $X^{\nu'}_{n+1} = \nu'$, where the function $r$ is defined in (7.4) and the clocks $Y$’s are from the same collection $\mathcal{Y}$ fixed in (7.1).

Thus, this defines $\mathbf{X}^{\mathcal{I}}$ as the extension on the whole tree. By using the properties of independent exponential random variables, it is easy to check that this construction is a construction of $(\lambda_1, \lambda)$-OERW on the tree $\mathcal{T}$. We refer the reader to ([4], section 7) for more discussions on this construction.

In the case $\mathcal{T}' = [\varrho, \nu]$ for some vertex $\nu$ of $\mathcal{T}$, we write $\mathbf{X}^{(\nu)}$ instead of $\mathbf{X}^{([\varrho, \nu])}$ and we denote $T^{(\nu)}(\cdot)$ the return times associated to $\mathbf{X}^{(\nu)}$. For simplicity, we will also write $\mathbf{X}^{(\nu)}$ and $T^{(\nu)}(\cdot)$ instead of $\mathbf{X}^{(\varrho)}$ and $T^{(\varrho)}(\cdot)$ for $\nu \in E$.

**Remark 13.** Let $\mathcal{T}'$ be a proper subtree of $\mathcal{T}$. Note that $\mathbf{X}^{\mathcal{T}'}$ is not $(\lambda_1, \lambda)$-OERW on $\mathcal{T}'$, that is different with $M$-digging random walk (see [1], section 7) and once-reinforced random walk (see [5], section 5).

Finally, we give a probabilistic interpretation of the functions $\phi$ and $\Psi$:

**Lemma 14.** For any $\nu \in E$ and any $\varrho \leq \nu$, we have

\begin{equation}
\phi(\varrho, \lambda_1, \lambda) = \mathbb{P} \left( T^{(\nu)}(\varrho^+) \circ \theta_T(\varrho^{-}) < T^{(\nu)}(\varrho) \circ \theta_T(\varrho^{-}) \right),
\end{equation}

\begin{equation}
\Psi(\varrho, \lambda_1, \lambda) = \mathbb{P} \left( T^{(\nu)}(\varrho^+) < T^{(\nu)}(\varrho) \right),
\end{equation}

where $\theta$ is the canonical shift on the trajectories.

**Proof.** Let $\nu \in E$ and $\varrho \leq \nu$. For simplicity, we set

\begin{align*}
\mathcal{A} &:= \{ T^{(\nu)}(\varrho^+) \circ \theta_T(\varrho^{-}) < T^{(\nu)}(\varrho) \circ \theta_T(\varrho^{-}) \}, \\
\mathcal{I}_1 &:= \left[ \frac{1+(j-1)\lambda_1}{1+\varphi(\varrho^{-})\lambda_1} \quad \frac{1+j\lambda_1}{1+\varphi(\varrho^{-})\lambda_1} \right], \\
\mathcal{I}_2 &:= [0,1] \setminus \left( \left[ \frac{1+(j-1)\lambda_1}{1+\varphi(\varrho^{-})\lambda_1} \quad \frac{1+j\lambda_1}{1+\varphi(\varrho^{-})\lambda_1} \right] \cup \left[ \frac{1}{1+\varphi(\varrho^{-})\lambda_1} \right] \right),
\end{align*}

where $j \in \{1, ..., \varphi(\varrho^{-})\}$ such that $(\varrho^{-})_j = \varrho^+$. We have that

\begin{equation}
\tag{7.13}
\mathbb{P} \left( \mathcal{A} \right) = \mathbb{P} \left( \left| U_{\varrho^{-}} < \frac{1}{1+\varphi(\varrho^{-})\lambda_1} \right| \right) \times \mathbb{P} \left( U_{\varrho^+} < \frac{1}{1+\varphi(\varrho^{-})\lambda_1} \right) + \mathbb{P} \left( \mathcal{A} \mid \mathcal{I}_1 \right) \times \mathbb{P} \left( U_{\varrho^{-}} \in \mathcal{I}_1 \right) + \mathbb{P} \left( \mathcal{A} \mid \mathcal{I}_2 \right) \times \mathbb{P} \left( U_{\varrho^{-}} \in \mathcal{I}_2 \right).
\end{equation}
On the other hand, we have the following equalities:

\[(7.14) \quad \mathbb{P}(A \Big| U_{g^-} < \frac{1}{1 + \partial(g^-)\lambda_1}) \times \mathbb{P}(U_{g^-} < \frac{1}{1 + \partial(g^-)\lambda_1}) = \frac{1}{1 + \partial(g^-)\lambda_1} \psi(g, \lambda)\psi(g^{-1}, \lambda)\]

\[(7.15) \quad \mathbb{P}(A \Big| \mathcal{I}_1) \times \mathbb{P}(U_{g^-} \in \mathcal{I}_1) = \frac{\lambda_1}{1 + \partial(g^-)\lambda_1}.

\[(7.16) \quad \mathbb{P}(A \Big| \mathcal{I}_2) \times \mathbb{P}(U_{g^-} \in \mathcal{I}_2) = \frac{(\partial(g^-) - 1)\lambda_1}{1 + \partial(g^-)\lambda_1} \psi(g, \lambda).

We use (7.13), (7.14), (7.15) and (7.16) to obtain the results. \qed

8. Recurrence in Theorem 4: The case \(RT(\mathcal{T}, X) < 1\)

**Proposition 15.** If

\[(8.1) \quad \inf_{\pi \in \Pi} \sum_{e \in \pi} \Psi(e) = 0,

then \(X\) is recurrent.

**Proof.** The proof is identical to the proof of Proposition 10 of [5]. \qed

9. Transience in Theorem 4: The case \(RT(\mathcal{T}, X) > 1\)

In order to prove transience, we use the relationship between the walk \(X\) and its associated percolation.

9.1. **Link with percolation.** Denote by \(C(g)\) the set of edges which are crossed by \(X\) before returning to \(g\), that is:

\[(9.1) \quad C(g) = \{e \in E : T(e^+) < T(g)\}.

We define another percolation which will be more easy to study. In order to do this, we use the Rubin’s construction and the extensions introduced in Section 7. We define

\[(9.2) \quad C_{CP}(g) = \{e \in E : T^{(e)}(e^+) < T^{(e)}(g)\}.

We say that an edge \(e \in E\) is open if and only if \(e \in C_{CP}(g)\).

**Lemma 16.** We have that

\[(9.3) \quad \mathbb{P}(T(g) = \infty) = \mathbb{P}(|C(g)| = \infty) = \mathbb{P}(|C_{CP}(g)| = \infty).

**Proof.** We can follow line by line the proof of Lemma 11 in [3]. \qed

For simplicity, for a vertex \(v \in V\), we write \(v \in C_{CP}(g)\) if one of the edges incident to \(v\) is in \(C_{CP}(g)\). Besides, recall that for two edges \(e_1\) and \(e_2\), their common ancestor with highest generation is the vertex denoted \(e_1 \wedge e_2\).

**Lemma 17.** Let \(\lambda_1 \geq 0\), \(\lambda > 0\) and \(\mathcal{T}\) be an infinite, locally finite and rooted tree with the root \(g\). Assume that the condition (3.4) holds with some constant \(M\). Then the correlated percolation induced by \(C_{CP}\) is quasi-independent, i.e. there exists a constant \(C_Q \in (0, +\infty)\) such that, for any two edges \(e_1, e_2\), we have that

\[(9.4) \quad \mathbb{P}(e_1, e_2 \in C_{CP}(g)| e_1 \wedge e_2 \in C_{CP}(g)) \leq C_Q \mathbb{P}(e_1 \in C_{CP}(g)| e_1 \wedge e_2 \in C_{CP}(g)) \times \mathbb{P}(e_2 \in C_{CP}(g)| e_1 \wedge e_2 \in C_{CP}(g)).\]
Proof. Recall the construction of Section [7]. Note that if \( e_1 \land e_2 = \varrho \), then the extensions on \( \{ \varrho, e_1 \} \) and \( \{ \varrho, e_2 \} \) are independent, then the conclusion of Lemma holds with \( C = 1 \). Assume that \( e_1 \land e_2 \neq \varrho \), and note that the extensions on \( \{ \varrho, e_1 \} \) and \( \{ \varrho, e_2 \} \) are dependent since they use the same clocks on \( \{ \varrho, e_1 \land e_2 \} \). Denote by \( e \) the unique edge of \( T \) such that \( e^+ = e_1 \land e_2 \). For \( i \in \{ 1, 2 \} \), let \( v_i \) be the vertex which is the offspring of \( e^+ \) lying the path from \( \varrho \) to \( e_i \). Note that \( v_i \) could be equal to \( e_i^+ \). Let \( i_1 \) (resp. \( i_2 \) ) be an element of \( \{ 1, \ldots, \partial(e^+) \} \) such that \((e^+)^{i_1} = v_1 \) (resp. \((e^+)^{i_2} = v_2 \).

As the events \( \{ e \in C_{CP} \} \) and \( U_{e_1 \land e_2} \) are independent, therefore:

\[
\mathbb{P}(e_1, e_2 \in C_{CP}(\varrho) | e \in C_{CP}(\varrho)) = A + B + C + D,
\]

where

\[
A = \mathbb{P}\left( e_1, e_2 \in C_{CP}(\varrho) | e \in C_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) \mathbb{P}\left( U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right)
\]

\[
B = \mathbb{P}\left( e_1, e_2 \in C_{CP}(\varrho) | e \in C_{CP}(\varrho), U_{e^+} \in \left[ \frac{1 + (i_1 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_1\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right) \times \mathbb{P}\left( U_{e^+} \in \left[ \frac{1 + (i_1 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_1\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right).
\]

\[
C = \mathbb{P}\left( e_1, e_2 \in C_{CP}(\varrho) | e \in C_{CP}(\varrho), U_{e^+} \in \left[ \frac{1 + (i_2 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_2\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right) \times \mathbb{P}\left( U_{e^+} \in \left[ \frac{1 + (i_2 - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_2\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right).
\]

\[
D = \mathbb{P}\left( e_1, e_2 \in C_{CP}(\varrho) | e \in C_{CP}(\varrho), U_{e^+} \in \bigcup_{i \in \{ 1, \ldots, \partial(e^+) \} \setminus \{ i_1, i_2 \}} \left[ \frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right) \times \mathbb{P}\left( U_{e^+} \in \bigcup_{i \in \{ 1, \ldots, \partial(e^+) \} \setminus \{ i_1, i_2 \}} \left[ \frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right).
\]

In the same way, for any \( j \in \{ 1, 2 \} \), we have:

\[
\mathbb{P}(e_j \in C_{CP}(\varrho) | e \in C_{CP}(\varrho)) = E_j + F_j + G_j,
\]

where

\[
E_j = \mathbb{P}\left( e_j \in C_{CP}(\varrho) | e \in C_{CP}(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) \mathbb{P}\left( U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right)
\]

\[
F_j = \mathbb{P}\left( e_j \in C_{CP}(\varrho) | e \in C_{CP}(\varrho), U_{e^+} \in \left[ \frac{1 + (i_j - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_j\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right) \times \mathbb{P}\left( U_{e^+} \in \left[ \frac{1 + (i_j - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i_j\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right).
\]

\[
G_j = \mathbb{P}\left( e_j \in C_{CP}(\varrho) | e \in C_{CP}(\varrho), U_{e^+} \in \bigcup_{i \in \{ 1, \ldots, \partial(e^+) \} \setminus \{ i_1 \}} \left[ \frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right) \times \mathbb{P}\left( U_{e^+} \in \bigcup_{i \in \{ 1, \ldots, \partial(e^+) \} \setminus \{ i_1 \}} \left[ \frac{1 + (i - 1)\lambda_1}{1 + \partial(e^+)\lambda_1}, \frac{1 + i\lambda_1}{1 + \partial(e^+)\lambda_1} \right] \right).
\]
Lemma 18. There exists four constants \((\alpha_1, \alpha_2, \alpha_3, \alpha)\) depend on \(T\), \(\lambda\) and \(\lambda_1\) such that:

\[
A \leq \alpha_1 E_1 E_2.
\]

\[
B \leq \alpha_2 F_1 E_2.
\]

\[
C \leq \alpha_3 F_2 E_1.
\]

\[
D \leq \alpha_4 G_1 G_2.
\]

We deduce from Lemma 18 that

\[
A + B + C + D \leq \alpha(E_1 + F_1 + G_1)(E_2 + F_2 + G_2),
\]

where \(\alpha = \max_{i \in \{1,2,3,4\}} \alpha_i\). The latter inequality concludes the proof of Proposition.

\[
\Box
\]

**Proof of Lemma 18** Now, we will adapt the argument from the proof of Lemma 12 in [5]. We prove that there exists \(\alpha_1\) such that \(A \leq \alpha_1 E_1 E_2\) and we use the same argument for the other inequalities.

First, by using condition 3.4, note that,

\[
\mathbb{P} \left( U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) = \frac{1}{1 + \partial(e^+)\lambda_1} \geq \frac{1}{1 + M\lambda_1},
\]

we then obtain:

\[
\mathbb{P} \left( U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) \leq (1 + M\lambda_1) \left[ \mathbb{P} \left( U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) \right]^2.
\]

On the event \(\{e \in C_P(\varrho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \}\) we have \(X^{(e)}_{T^{(e)}(e^+)+1} = e^-\). We then define \(T^{(e)}(e^+) := \inf \{n \geq T^{(e)}(e^+) + 1 : X^{(e)}_n = e^+\}\). We define the following quantities:

\[
N(e) = \left| \left\{ \bar{T}^{(e)}(e^+) \leq n \leq T^{(e)}(\varrho) \circ \theta_{\bar{T}^{(e)}(e^+)} : (X^{(e)}_n, X^{(e)}_{n+1}) = (e^+, e^-) \right\} \right|,
\]

\[
L(e) = \sum_{j=0}^{N(e)-1} Y(e^+, e^-, j) \frac{r(e^+, e^-)}{r(e^+, e^-)},
\]

where \(|A|\) denotes the cardinality of a set \(A\) and \(\theta\) is the canonical shift on trajectories. Note that \(L(e)\) is the time consumed by the clocks attached to the oriented edge \((e^+, e^-)\) before \(X^{(e)}, X^{(e_1)}\) or \(X^{(e_2)}\) goes back to \(\varrho\) once it has returned \(e^+\) after the time \(T^{(e)}(e^+)\). Recall that these three extensions are coupled and thus the time \(L(e)\) is the same for the three of them.

For \(i \in \{1,2\}\), recall that \(v_i\) is the vertex which is the offspring of \(e^+\) lying the path from \(\varrho\) to \(e_i\). Note that \(v_i\) could be equal to \(e_i^+\). We define for \(i \in \{1,2\}\):

\[
N^*(e_i) = \left| \left\{ \bar{T}^{(e)}(e^+) \leq n \leq T^{(e)}(e_i^+) : (X^{(e)}_n, X^{(e)}_{n+1}) = (e^+, v_i) \right\} \right|,
\]

\[
L^*(e_i) = \sum_{j=0}^{N^*(e_i)-1} Y(e^+, e^-, j) \frac{r(e^+, e^-)}{r(e^+, e^-)}.
\]
Here, \( L^*(e_i), i \in \{1, 2\} \), is the time consumed by the clocks attached to the oriented edge \((e^+, v_i)\) before \(X^{(e_i)}\), or \(X^{[e^+, e_i^+]}\), hits \(e_i^+\). Notice that the three quantities \(L(e), L^*(e_1)\) and \(L^*(e_2)\) are independent, and we also have:

\[
\begin{align*}
\mathbb{P}\left( e_1, e_2 \in C_{CP}(\rho) \big| e \in C_{CP}(\rho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) &= \psi(e, \lambda) \mathbb{P}\left( L(e) > L^*(e_1) \lor L^*(e_2) \right). \\
\mathbb{P}\left( e_1 \in C_{CP}(\rho) \big| e \in C_{CP}(\rho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) &= \psi(e, \lambda) \mathbb{P}\left( L(e) > L^*(e_1) \right). \\
\mathbb{P}\left( e_2 \in C_{CP}(\rho) \big| e \in C_{CP}(\rho), U_{e^+} < \frac{1}{1 + \partial(e^+)\lambda_1} \right) &= \psi(e, \lambda) \mathbb{P}\left( L(e) > L^*(e_2) \right).
\end{align*}
\]

Now, the random variable \(N(e)\) is simply a geometric random variable (counting the number of trials) with success probability \(\lambda^{1-|e|}/\sum_{g \leq e} \lambda^{1-|g|}\). The random variable \(N(e)\) is independent of the family \(Y(e^+, e^-, j)\). As \(Y(e^+, e^-, j)\) are independent exponential random variable for \(j \geq 0\), we then have that \(L(e)\) is an exponential random variables with parameter

\[
p := \frac{\lambda^{1-|e|}}{\sum_{g \leq e} \lambda^{1-|g|}} \times \lambda^{|e|-1} = \frac{1}{\sum_{g \leq e} \lambda^{1-|g|}}.
\]

A priori, \(L^*(e_1)\) and \(L^*(e_2)\) are not exponential random variable, but they have a continuous distribution. Denote \(f_1\) and \(f_2\) respectively the densities of \(L^*(e_1)\) and \(L^*(e_2)\). Then, we have that

\[
\begin{align*}
\mathbb{P}\left( L(e) > L^*(e_1) \lor L^*(e_2) \right) &= \int_{0}^{+\infty} \int_{0}^{+\infty} \int_{x_1 \lor x_2}^{+\infty} p e^{-pt} f_1(x_1) f_2(x_2) dx_1 dx_2 dx_2 \\
&= \int_{0}^{+\infty} \int_{0}^{+\infty} e^{-p(x_1 \lor x_2)} f_1(x_1) f_2(x_2) dx_1 dx_2 \\
&\leq \int_{0}^{+\infty} \int_{0}^{+\infty} e^{-p(x_1 + x_2)} f_1(x_1) f_2(x_2) dx_1 dx_2.
\end{align*}
\]

Thus, one can write

\[
\begin{align*}
\mathbb{P}\left( L(e) > L^*(e_1) \lor L^*(e_2) \right) &\leq \left( \int_{0}^{+\infty} e^{-px_1/2} f_1(x_1) dx_1 \right) \cdot \left( \int_{0}^{+\infty} e^{-px_2/2} f_2(x_2) dx_2 \right).
\end{align*}
\]

Note that:

\[
\int_{0}^{+\infty} e^{-px_1/2} f_1(x_1) dx_1 = \mathbb{P}\left( \overline{L}(e) > L^*(e_1) \right),
\]

where \(\overline{L}(e)\) is an exponential variable with parameter \(p/2\). Note that, in view of (9.22), \(\overline{L}(e)\) has the same law as \(L(e)\) when we replace the weight of an edge \(g'\) by \(\lambda^{1-|g'|+1}/2\) for \(g' \leq e\) only, and keep the other weights the same.

For simplicity, for any \(g \in E\), we set \(w(g) = \lambda^{1-|g|}.\) For \(g \in E\) such that \(e < g\), define the functions \(\overline{\psi}\) and \(\overline{\phi}\) in a similar way as \(\psi\) and \(\phi\), except that we replace the weight of an edge \(g'\) by \(\lambda^{1-|g'|+1}/2\) for \(g' \leq e\) only, and keep the other weights the same, that is, for \(g \in E, e < g,\)
Lemma 19. There exists a constant $c = c(\lambda_1, \lambda)$ which do not depend on $e$, $e_1$ and $e_2$, such that:

$$\sum_{e < g \leq e_1} \left( \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \right) \leq c.$$  

(9.29)

On the other hand, by using Lemma [19] for any $e$ and $e_1$ we have that

$$\sum_{e < g \leq e_1} \left( \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) \right) \leq 2c.$$  

(9.30)

By using $[19]$ Lemma 19 and condition (3.4), we obtain:

$$\prod_{e < g \leq e_1} \left( 1 + \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 \psi(g, \lambda)} \right)^{\phi(g, \lambda, \lambda)} \prod_{e < g \leq e_1} \phi(g, \lambda, \lambda) \prod_{e < g \leq e_1} \left( \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 \psi(g, \lambda)} \right).$$  

(9.28)

Now, we compute the product:

$$\prod_{e < g \leq e_1} \left( 1 + \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 \psi(g, \lambda)} \right)^{\phi(g, \lambda, \lambda)} \prod_{e < g \leq e_1} \phi(g, \lambda, \lambda) \prod_{e < g \leq e_1} \left( \frac{\tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) - \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 (\tilde{\psi}(g, \lambda) - \psi(g, \lambda))}{\lambda_1 + \psi(g, \lambda) \psi(g^{-1}, \lambda) + (\partial(g^{-1}) - 1) \lambda_1 \psi(g, \lambda)} \right).$$  

(9.27)

We obtain:

$$\tilde{\psi}(g, \lambda, \lambda) = \frac{\lambda_1}{1 + \partial(g^{-1}) \lambda_1} + \frac{1}{1 + \partial(g^{-1}) \lambda_1} \tilde{\psi}(g, \lambda) \tilde{\psi}(g^{-1}, \lambda) + \frac{(\partial(g^{-1}) - 1) \lambda_1}{1 + \partial(g^{-1}) \lambda_1} \tilde{\psi}(g, \lambda).$$  

(9.26)

$$\tilde{\phi}(g, \lambda, \lambda) = \frac{\lambda_1}{1 + \partial(g^{-1}) \lambda_1} + \frac{1}{1 + \partial(g^{-1}) \lambda_1} \tilde{\phi}(g, \lambda) \tilde{\phi}(g^{-1}, \lambda) + \frac{(\partial(g^{-1}) - 1) \lambda_1}{1 + \partial(g^{-1}) \lambda_1} \tilde{\phi}(g, \lambda).$$  

(9.25)
We have just proved that
\[ (9.32) \int_{0}^{+\infty} e^{-px/2} f_{1}(x_{1}) dx_{1} \leq \exp \left( Mc + \frac{2c}{\lambda_{1}} \right) \times \mathbb{P}(e_{1} \in C_{CP}(g) | e_{1} \land e_{2} \in C_{CP}(g)). \]
By doing a very similar computation, one can prove that
\[ (9.33) \int_{0}^{+\infty} e^{-px/2} f_{1}(x_{2}) dx_{2} \leq \exp \left( Mc + \frac{2c}{\lambda_{1}} \right) \times \mathbb{P}(e_{2} \in C_{CP}(g) | e_{1} \land e_{2} \in C_{CP}(g)). \]
Moreover, we have
\[ (9.34) \psi(e, \lambda) \geq \frac{\lambda}{1 + \lambda}. \]
The conclusion (9.4) follows by using (9.16), (9.24), (9.34), together with (9.32) and (9.33). □

It remains to prove Lemma 19.

**Proof of Lemma 19** By a simple computation, for any \( e < g \leq e_{1} \),
\[ (9.35) \tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{\sum_{g' \leq e} w(g')^{-1}}{\sum_{g' \leq g} w(g')^{-1} + \sum_{g' \leq e} w(g')^{-1}} \times \left( 1 - \frac{1}{\lambda} \right). \]
We will proceed by distinguishing three cases.

**Case I:** \( \lambda < 1.\)
By (9.35), we have that
\[ (9.36) \tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{\left( 1 - \frac{1}{\lambda|e|} \right)}{\left( 1 - \frac{1}{\lambda|e|} + 1 - \frac{1}{\lambda|e|} \right)} \times \left( 1 - \frac{1}{\lambda} \right). \]
Hence, there exists a constant \( c_{1} \) such that
\[ (9.37) 0 \leq \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \leq c_{1} |g|^{-|e|}. \]
Therefore we obtain
\[ (9.38) \sum_{e < g \leq e_{1}} \left( \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \right) \leq c_{1} \sum_{e < g \leq e_{1}} |g|^{-|e|} \leq c_{1} \sum_{i \geq 0} \lambda^{i} < \infty. \]

**Case II:** \( \lambda = 1.\)
By (9.35), we have that
\[ (9.39) \tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{|e|}{|g|(|g| + |e|)}. \]
Therefore we obtain
\[ \sum_{e < g \leq e_{1}} \left( \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \right) \leq \sum_{n \geq |e|} \left( \frac{|e|}{n(n + |e|)} \right) \leq \sum_{n \geq |e|} \left( \frac{1}{n} - \frac{1}{n + |e|} \right) \leq \sum_{n = |e|}^{2|e| - 1} \frac{1}{n}. \]
On the other hand, we have:
\( \lim_{n \to \infty} \left( \sum_{k=n}^{2n-1} \frac{1}{k} \right) = \lim_{k \to \infty} \left( \sum_{k=0}^{n-1} \frac{1}{n+k} \right) = \lim_{k \to \infty} \left( \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1+k/n} \right) = \int_0^1 \frac{dx}{1+x}. \)

We use (9.40) and (9.41) to obtain the result.

**Case III: \( \lambda > 1 \).**

By (9.35), we have that
\[
\tilde{\psi}(g, \lambda) - \psi(g, \lambda) = \frac{1 - \frac{1}{X|g|}}{1 - \frac{1}{X|g|} + 1 - \frac{1}{X|g|}} \times \left( 1 - \frac{1}{\lambda} \right).
\]

Hence, there exists a constants \( c_2 \) such that
\[
0 \leq \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \leq \frac{c_2}{\lambda|g|}.
\]

Therefore we obtain
\[
\sum_{e < g \leq e_1} \left( \tilde{\psi}(g, \lambda) - \psi(g, \lambda) \right) \leq c_2 \sum_{e < g \leq e_1} \frac{1}{\lambda|g|} \leq c_2 \sum_{i \geq 0} \left( \frac{1}{\lambda} \right)^i < \infty.
\]

\( \square \)

9.2. Transience in Theorem 4: The case \( RT(\mathcal{T}, X) > 1 \).

**Proposition 20.** If \( RT(\mathcal{T}, X) > 1 \) and if (3.4) is satisfied then \( X \) is transient.

**Proof.** The proof is now easy, we can follow line by line the Appendix A.2 of [4]. \( \square \)

10. PROOF OF THEOREM 6

This section is independent with the previous sections. In this section, we prove a criterion which can apply to the critical \( M \)-digging random walk on superperiodic trees. We will use the Rubin’s construction (resp. the definition of \( C(\varrho) \), \( C_{CP}(\varrho) \) from section 7 (resp. section 8.1) of [4]. We will allow ourselves to omit these definitions and refer the readers to [4] for more details.

The main idea for the proof of Theorem 6 is that the number of surviving rays of the percolation \( C_{CP}(\varrho) \) almost surely is either zero or infinite. This property was proved in the case of Bernoulli percolation (see [8] proposition 5.27) or target percolation (see [10], lemma 4.2). The main difficulty that we have to face is that the FKG inequality is not true for our percolation.

10.1. Some definitions. Let \( \lambda > 0 \), \( M \in \mathbb{N} \) and \( \mathcal{T} \) be an infinite, locally finite and rooted tree. For each \( v \in V(\mathcal{T}) \), recall the definition of subtree \( \mathcal{T}^v \) of \( \mathcal{T} \) from Section 2.1. Let \( X^{v,\lambda} \) be the \( M \)-digging random walk on \( \mathcal{T}^v \). We say that \( \mathcal{T} \) is uniformly transient if for any \( \lambda \) such that the \( M \)-digging random walk on \( \mathcal{T} \) with parameter \( \lambda \) is transient (i.e. \( X^{v,\lambda} \) is transient),
\[
\exists \alpha_\lambda > 0, \forall v \in V(\mathcal{T}), \mathbb{P}(\forall n > 0, X^{v,\lambda}_n \neq v) \geq \alpha_\lambda.
\]

It is called weakly uniformly transient if there exists a sequence of finite pairwise disjoint \( \pi_n \) such that
\[
\exists \alpha_\lambda > 0, \forall v \in \bigcup_n V(\pi_n), \mathbb{P}(\forall n > 0, X^{v,\lambda}_n \neq v) \geq \alpha_\lambda.
\]
where \( V(\pi_n) = \{ e^- : e \in \pi_n \} \).

**Remark 21.**
- If \( T \) is uniformly transient, then \( T \) is also weakly uniformly transient, but the reverse is not always true.
- The superperiodic trees are uniformly transient.

An infinite self-avoiding path starting at \( \varrho \) is called a ray. The set of all rays, denoted by \( \partial T \), is called the boundary of \( T \). Let \( \phi : \mathbb{Z}^+ \to \mathbb{R} \) be a decreasing positive function with \( \phi(n) \to 0 \) as \( n \to \infty \). The Hausdorff measure of \( T \) in gauge \( \phi \) is

\[
\liminf_{\Pi} \sum_{v \in \Pi} \phi(|v|),
\]

where the \( \liminf \) is taken over \( \Pi \) such that the distance from \( \varrho \) to the nearest vertex in \( \Pi \) goes to infinity. We say that \( T \) has \( \sigma \)-finite Hausdorff measure in gauge \( \phi \) if \( \partial T \) is the union of countably many subsets with finite Hausdorff measure in gauge \( \phi \).

Finally, if \( \lambda \) is such that the \( M \)-digging random walk \( X \) with parameter \( \lambda \) on \( T \) is transient, on the event \( \{ T(\varrho) = \infty \} \), its path determines an infinite branch in \( T \), which can be seen as a random ray \( \omega^\infty \), and call it the limit walk of \( X \). Equivalently, on the event \( \{ T(\varrho) = \infty \} \), we define the limit walk as follows: For any \( k \geq 1 \),

\[
(10.3) \quad \omega^\infty(k) = v \iff v \in T_k \text{ and } \exists n_0, \forall n > n_0 : X_n \in T^v.
\]

Note that \( \mathbb{P}(\omega^\infty(0) = \varrho) = 1 \). For any \( k \geq 1 \), we call the \( k \)-first steps of \( \omega^\infty \) is \((\omega^\infty(0), \cdots, \omega^\infty(k))\), denoted by \( \omega_{[0,n]}^\infty \).

**10.2. Proof of Theorem 6**

We begin with the following proposition:

**Proposition 22.** Let \( X \) be a \( M \)-digging random walk with parameter \( \lambda_M \) on an uniformly transient tree \( T \) and recall the definition of \( C_{CP} \) from \( X \) as in (4), Section 7). Consider the percolation induced by \( C_{CP} \) and let \( \phi(n) = \mathbb{P}(\varrho \leftrightarrow v) \) for \( v \in T_n \).

1. Almost surely, the number of surviving rays is either zero or infinite.
2. If \( \partial T \) has \( \sigma \)-finite Hausdorff measure in the gauge \( \{ \phi(n) \} \), then \( \mathbb{P}(\varrho \leftrightarrow \infty) = 0 \). In particular, \( X \) is recurrent.

The overall strategy for the proof of Proposition 22 is as follows. First, if \( X \) is recurrent, then the percolation induced by \( C_{CP} \) almost surely have no surviving ray. Next, assume that \( X \) is transient. On the event \( \{ T(\varrho) = \infty \} \), the limit walk \( \omega^\infty \) is a surviving ray of \( C_{CP}(\varrho) \). Given \( n \in \mathbb{N} \) and conditioning on \( \omega^\infty_{[0,n]} \), by using the Rubin’s construction and the definition of uniformly transient, we prove that there exists a surviving ray in \( T_{\omega^\infty} \) with probability larger than a constant \( c \) which do not depend on \( i \) and \( \omega^\infty \) (see Figure 2).

The following basic lemma is necessary:

**Lemma 23.** Let \( \lambda > 0 \) and \( T \) be an infinite, locally finite and rooted tree. Let \( \overline{M} := (m_v, v \in V(T)) \) be a family of non-negative integers. Denote by \( X \) the \( M \)-digging random walk with parameter \( \lambda_M \) and \( Y \) the \( \overline{M} \)-digging random walk associated with the inhomogeneous initial number of cookies \( \overline{M} \) with parameter \( \lambda \) (see 4, section 2.3.2 for more details on the definition of \( \overline{M} \)-digging random walk). Denote by \( T_X(\varrho) \) (resp. \( T_Y(\varrho) \)) the return time of \( X \) (resp. \( Y \)) to \( \varrho \). Assume that \( m(v) \leq M \) for all \( v \in V(T) \), we then have

\[
(10.4) \quad \mathbb{P}(T_X(\varrho) < \infty) \leq \mathbb{P}(T_Y(\varrho) < \infty).
\]
ONCE-EXCITED RANDOM WALK

\[ \varrho \]

\[ v_1 \]

\[ v_2 \]

\[ v_4 \]

\[ v_3 \]

\[ v_5 \]

\[ v_6 \]

\[ \omega^\infty \]

\[ T v_1 \]

\[ T v_2 \]

\[ T v_4 \]

\[ T v_5 \]

\[ T v_6 \]

Figure 2. The proof’s idea of Proposition 22. The limit walk \( \omega^\infty \) is in red. Conditioning on the event \( \{ \omega^\infty(0) = \varrho, \omega^\infty(1) = v_1, \ldots, \omega^\infty(6) = v_6 \} \) and denote by \( \ell \) the last time the critical \( M \)-digging random walk \( X \) on \( T \) visits \( v_6 \). For each \( 1 \leq i \leq 6 \), running the walk \( X^v_i, \lambda_c \) on \( T^{v_i} \). The property of uniformly transient implies that there exists a surviving ray (in green) in \( T^{v_i} \) with probability is larger than a constant which do not depend on \( i \).

Proof. The proof is simple, therefore it is omitted. \( \square \)

Proof of Proposition 22. Let \( A_k \) denote the event that exactly \( k \) rays survive and assume that

\[ \mathbb{P}(A_k) > 0, \]

Hence,

\[ \mathbb{P}(|C_{CP}(\varrho)| = \infty) > 0. \]

By (10.6) and Lemma 22 in [4], we have that:

\[ \mathbb{P}(T(\varrho) = \infty) > 0, \]

and therefore \( X \) is transient.

On the event \( \{ T(\varrho) = \infty \} \), the limit walk \( \omega^\infty \) of \( X \) is well defined and it is a surviving ray. Let \( n \) be a positive integer and \( \gamma := (\gamma_0 = \varrho, \gamma_1 = v_1, \ldots, \gamma_n = v_n) \) be a path of length \( n \) of \( T \). Denote by \( B_{n, \gamma} \) the following event:

\[ B_{n, \gamma} := \{ \omega^\infty_{[0,n]} = \gamma \}. \]

For any \( 1 \leq k \leq n \), define a sub-tree \( T^{v_i} \) of \( T \) in the following way (see Figure 2).
• The root of $T^{v_i}$ is the vertex $v_i$.
• If $\partial(v_i) < 2$ then $T^{v_i}$ is a tree with a single vertex $v_i$: for example, $T^{v_3}$ in Figure 2.
• If $\partial(\gamma_i) \geq 2$, choose one of its children which is different to $v_{i+1}$, denoted by $v$. We then set:

$$
\begin{align*}
\begin{cases}
(T^{v_i})_1 = \{v\} \\
(T^{v_i})_v = T^v
\end{cases}
\end{align*}
$$

Note that for every pair $(i,j) \in [1,n]^2$, we have $V(T^{v_i}) \cap V(T^{v_j}) = \emptyset$.

Now, conditioning on the event $B_{n,\gamma}$. Let $\ell$ be the last time $X$ visits $v_n$, i.e.

$$
\ell := \sup\{k > 0 : X_k = v_n\}.
$$

By the definition of limit walk, $\ell$ is finite on the event $B_{n,\gamma}$. For each $i \in [1,n]$ and for all $v \in V(T^{v_i})$, denote by $m^i(v)$ the remaining number of cookies at $v$ after time $\ell$, i.e.

$$
(10.10)\quad m^i(v) := M - \#\{k \leq \ell : X_k = v\}.
$$

By using the extensions introduced in ([4], Section 7), the next steps on the tree $T^{v_i}$ are given by the digging random walk associated with the inhomogeneous initial number of cookies $(m^i(v), v \in V(T^{v_i}))$ and the same parameter $\lambda_c$ as $X$, denoted by $X^{v_i,m^i,\lambda_c}$ (see [4], section 2.3.2 for more details on the definition of $X^{v_i,m^i,\lambda_c}$). Denote by $T^{v_i,m^i,\lambda_c}$ the return time of $X^{v_i,m^i,\lambda_c}$ to the root $v_i$ of $T^{v_i}$. By the definition of uniformly transient and Lemma 23, there exists a constant $c > 0$ which do not depend on $n$ and $\gamma$ such that for any $i$,

$$
(10.11)\quad \mathbb{P}(T^{v_i,m^i,\lambda_c} < \infty) > c.
$$

On the event $\{T^{v_i,m^i,\lambda_c} < \infty\}$, note that $C_{CP}$ contains a surviving ray in $T^{v_i}$. By (10.11), we have

$$
(10.12)\quad \mathbb{P}(A_k | B_{n,\gamma}) \leq \binom{n}{k}(1 - c)^{n-k}
$$

On the other hand, we have $A_k \subset \bigcup_{\gamma:|\gamma|=n} B_{n,\gamma}$, therefore by (10.12) we obtain:

$$
(10.13)\quad \mathbb{P}(A_k) = \sum_{\gamma:|\gamma|=n} \mathbb{P}(A_k | B_{n,\gamma}) \times \mathbb{P}(B_{n,\gamma}) \leq \left(\sum_{i=1}^{k} \binom{n}{i}(1 - c)^{n} \sum_{\gamma:|\gamma|=n} \mathbb{P}(B_{n,\gamma}) \right) \leq \left(\sum_{i=1}^{k} \binom{n}{i}(1 - c)^{n} \right) \leq 1.
$$

Since (10.13) holds for any $n$ then we obtain the following contradiction

$$
(10.14)\quad \mathbb{P}(A_k) = 0.
$$

For part (2), the proof is similar to part (ii), Lemma 4.2 in [10].

In the same method as in the proof of Proposition 22, we can prove the slightly stronger result (the proof of which we omit):

**Proposition 24.** Let $X$ be a $M$-digging random walk with parameter $\lambda_c$ on a weakly uniformly transient tree $T$ and recall the definition of $C_{CP}$ from $X$ as in ([4], Section 7). Consider the percolation induced by $C_{CP}$ and let $\phi(n) = \mathbb{P}(\emptyset \leftrightarrow v)$ for $v \in T_n$.

1. With probability one, the number of surviving rays is either zero or infinite.
(2) If $\partial T$ has $\sigma$-finite Hausdorff measure in the gauge $\{\phi(n)\}$, then $P(\varrho \leftrightarrow \infty) = 0$. In particular, $X$ is recurrent.

The following corollary is an immediate consequence of Proposition 24.

**Corollary 25.** Let $M \in \mathbb{N}$ and $T$ be a weakly uniformly transient tree such that $\partial T$ has $\sigma$-finite Hausdorff measure in the gauge $\{\phi(n)\} = \left(\frac{1}{br(T)}\right)^n$ if $br(T) > 1$ and $\{\phi(n)\} = \frac{1}{n^{1+\varepsilon}}$ if $br(T) = 1$. Then the critical $M$-digging random walk on $T$ is recurrent.

**Proposition 26.** Let $M \in \mathbb{N}^*$ and $T$ be a superperiodic tree whose upper-growth rate is finite. The critical $M$-digging random walk on $T$ is recurrent.

**Proof.** This is a consequence of Corollary 25 and Theorem 3. $\square$

**Remark 27.** If $M = 0$, then $M$-DRW$_\lambda$ is the biased random walk with parameter $\lambda$. The recurrence of critical biased random walk on $T$ is a consequence of Theorem 3 and Nash-Williams criterion (see [5] or [8]).

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