PRE-COURANT ALGEBROIDS

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Abstract. Pre-Courant algebroids are ‘Courant algebroids’ without the Jacobi identity for the Courant–Dorfman bracket. In this paper we examine the corresponding supermanifold description of pre-Courant algebroids and some direct consequences thereof – such as the definition of (sub-)Dirac structures and the notion of the naive quasi-cochain complex. In particular we define symplectic almost Lie 2-algebroids and show how they correspond to pre-Courant algebroids. Moreover, the framework of supermanifolds allows us to economically define and work with pre-Courant algebroids equipped with a compatible non-negative grading. VB-Courant algebroids are natural examples of what we call weighted pre-Courant algebroids, and our approach drastically simplifies working with them. We remark that examples of pre-Courant algebroids are plentiful – natural examples include the cotangent bundle of any almost Lie algebroid.

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Contents

1. Introduction 1
2. Preliminaries 3
  2.1. Pre-Courant algebroids 3
  2.2. Graded bundles 4
  2.3. Pseudo-Euclidean vector bundles 5
  2.4. Almost Lie algebroids 6
3. Symplectic almost Lie 2-algebroids and pre-Courant algebroids 6
  3.1. Symplectic nearly and almost Lie 2-algebroids 6
  3.2. The relation with pre-Courant algebroids 9
  3.3. Examples 9
  3.4. Sub-Dirac structures and almost Lie algebroids 11
  3.5. The standard quasi-cochain complex 12
4. Pre-Courant algebroids with an additional grading 13
  4.1. Weighted pre-Courant algebroids 13
  4.2. VB-Courant algebroids revisited 15
  4.3. Weighted standard cohomology 15
  4.4. Weighted sub-Dirac structures 15
  4.5. Examples 16
5. Concluding remarks 17
Acknowledgements 17
References 17

1. Introduction

There has been a recent drive in the ‘higher categorification’ of mathematics and physics. The categorical approach to ‘higher structures’ is extremely powerful and gives rise, for example, to multisymplectic structures, Courant algebroids, Dirac structures and $L_{\infty}$-algebras, to name just a few. From a physics perspective the categorical way of thinking has proved useful in axiomatising quantum field theory and in particular the path integral approach to conformal and topological field theories. However, this categorical approach to ‘higher structures’ can be difficult to work with, especially in the context of finding concrete examples and applications. Allied to this categorical perspective is the study of supermanifolds equipped with some geometrical structure and a compatible additional grading (see for example [17, 18, 38, 40, 45]). In particular, $Q$-manifolds with additional gradings offer a framework that is much easier to handle than ‘higher categorification’ – for example Lie algebroids are very economically described in these terms following Vaintrob [44].

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Many Q-manifolds arise from ‘classical gauge systems’ – which are also known as QP-manifolds – understood as supermanifolds equipped with an even/odd Poisson bracket and an odd/even homological potential (Hamiltonian) $\theta$, i.e.

$$Q = \{\theta, \cdot\}_\epsilon,$$

where the bracket is the even or odd Poisson bracket (labelled by $\epsilon \in \mathbb{Z}_2$). The condition that $Q^2 = 2[Q, Q] = 0$ follows from the ‘classical master equation’ $\{\theta, \theta\}_\epsilon = 0$. The language here is of course borrowed from the BV-BRST and BFV formalisms of gauge theory. Courant algebroids provide examples of QP-manifolds.

Loosely, a Courant algebroid (cf. [9, 35]) is a vector bundle, whose sections come equipped with a Loday–Leibniz algebra, together with an anchor map and a nondegenerate symmetric bilinear form that satisfy some compatibility conditions. In particular, the bracket on the space of section is not a Lie bracket, but rather a non-skewsymmetric bracket that satisfies a version of the Jacobi identity. For a recent review of the history of Courant algebroids the reader should consult Kosmann-Schwarzbach [29]. Following Roytenberg [38, 39] we know that Courant algebroids can be described in terms of QP-manifolds, and that a class of topological field theories can be associated with them via the AKSZ method (cf. [1]). In addition, Courant algebroids appear in the context of closed string theory with a toroidal target space via double field theory; see for example the work of Deser & Stasheff [10]. Courant algebroids have proved useful in understanding H-flux and R-flux in string theory, see for example [3, 4] Another connection with string theory is the fact that D-branes can be identified with certain Dirac structures [2].

A pre-Courant algebroid is a ‘Courant algebroid’ for which the Jacobi identity for the bracket has been thrown away (cf. Vaisman [43]). We remark that pre-Courant algebroids in the form of Courant algebroids twisted by a closed 4-form naturally appear in the theory of three dimensional sigma models with Wess-Zumino terms [23]. Other natural examples, as we shall show, can be constructed by ‘twisting’ Lie algebroids by a non-closed Lie algebroid 3-form and ‘twisting’ Poisson structures by a 3-vector that is not in the Poisson cohomology. Liu, Sheng & Xu [34] introduce the notion of the naïve quasi-cochain complex of pre-Courant algebroids (also see [11, 42] for the case of Courant algebroids).

In this paper we explain the notion of a pre-Courant algebroid using the language of supermanifolds – as suggested in [34]. We start with a weakened version of a Courant algebroid, understood in ‘super-terms’, where we do not insist on the (odd) function $\theta$ being a homological potential. That is we do not assume that the ‘classical master equation’ is satisfied. Nor do we assume that the violation of the ‘classical master equation’ is controlled in some specific way: though we note that Liu, Sheng & Xu [34] proved that any pre-Courant algebroid is in fact a twisted Courant algebroid, at least for the regular case. We will in due course define symplectic almost Lie 2-algebroids and show that they are in one-to-one correspondence with pre-Courant algebroids.

At the risk of getting slightly ahead of ourselves, a symplectic almost Lie 2-algebroid is essentially a ‘symplectic almost Lie 2-algebroid’ for which $\{\Theta, \Theta\} \neq 0$, but the looser condition $\{\{\Theta, \Theta\}, f\} = 0$ holds for all weight zero functions $f$. Here, $\Theta$ is the degree three Hamiltonian that, together with the weight $-2$ Poisson bracket, encodes the pre-Courant algebroid. Although we have lost the Jacobi identity for the Courant–Dorfman bracket we still have compatibility of the anchor map and the bracket. This is the origin of our nomenclature ‘almost’.

An almost Lie algebroid (see for example [12, 16, 19]) is a ‘Lie algebroid’ for which the Jacobi identity for the bracket on sections has been discarded, but compatibility of the anchor and the bracket remains intact – such structures naturally appear in geometric mechanics with constraints. There is an even looser notion of a skew algebroid where the compatibility of the anchor map and the bracket is lost. However, the ‘almost’ case is much better behaved than the ‘skew’ case for some quite fundamental reasons. First, while for both almost Lie algebroids and skew algebroids one has a clear notion of admissible paths, in order to develop homotopies of admissible paths compatibility of the anchor and the bracket is needed. Secondly, while one cannot fully develop the cohomology theory of almost Lie algebroids, their zeroth and first cohomology groups are perfectly well defined. For general skew algebroids only the zeroth cohomology group can be defined. The situation – as we will show – for pre-Courant, or better put symplectic almost Lie 2-algebroids, is very similar to the case of almost Lie algebroids. Moreover, we will show that the relations between Lie algebroids and Courant algebroids passes over to almost Lie algebroids and pre-Courant algebroids. The general meta-theorem here is that the Jacobi identity can be thrown out, but in order to have interesting and useful structures compatibility of the anchor and bracket must be kept!

It has long been known Courant algebroids are ‘really’ symplectic Lie 2-algebroids (cf. Roytenberg [38]) and that Dirac structures are ‘really’ certain Lagrangian submanifolds thereof (cf. Severa [40]). However, this seems not to be generally appreciated, nor largely utilised. In this paper we will fundamentally understand (sub-)Dirac structures on pre-Courant algebroids in light Severa picture of (isotropic) Lagrangian submanifolds – this sits comfortably with Weinstein’s symplectic creed.

Another direction of categorification is the study certain objects in the category of other objects: double objects in the sense of Ehresmann. For example, double vector bundles are vector bundles in the category of vector bundles. Other examples can then be built by adding further structures on one or both of the vector bundles: for instance VB-algebroids and VB-Courant algebroids are Lie algebroids and Courant algebroids in
the category of vector bundles respectively (or vice versa). The notion of a VB-Courant algebroid is less well known than the notion of a VB-algebroid and were introduced by Li-Bland in his PhD thesis [33] (also see Jotz Lean [24]). We will define VB-pre-Courant algebroids, as well as their higher graded versions\(^1\) in simple terms of pre-Courant algebroids equipped with a compatible homogeneity structure \([17, 18]\). Loosely, a \textit{weighted pre-Courant algebroid} is a pre-Courant algebroid with a compatible \(\mathbb{N}\)-grading. In fact, in [7] we (together with Grabowska) made a very brief incursion into weighted Courant algebroids, but here we will explain the relation with VB-Courant algebroids much more explicitly. We must of course mention that much of the inspiration for working with double and multiple structures comes from Mackenzie’s works on double Lie algebroids, double Lie groupoids and so on in relation to Poisson geometry. Rather than cite all the original literature, we will simply point to Mackenzie’s book [36].

**Main Results:**

- We show that there is a one-to-one correspondence between pre-Courant algebroids and symplectic almost Lie 2-algebroids – Theorem 3.11.
- We prove that sub-Dirac structures on pre-Courant algebroids are almost Lie algebroids – Proposition 3.21.
- We show that there exists a quasi-cochain map between the naïve quasi-cochain complex and the standard quasi-cochain complex – Theorem 3.28.
- We prove that the higher order tangent bundles of a pre-Courant algebroid are canonically weighted pre-Courant algebroids – Theorem 4.3.

Although we generally work with supermanifolds, our main intention is not to generalise pre-Courant algebroids to ‘super pre-Courant algebroids’, but rather to show how the framework of symplectic supermanifolds offers new light on the subject of classical pre-Courant algebroids.

**Arrangement:** This paper is arranged as follows. In Section 2 we recap the basic notions needed in this paper. In Section 3 we present our main definitions of symplectic almost Lie 2-algebroids and how they are related to pre-Courant algebroids. In Section 4 we provide a definition of a pre-Courant algebroid carrying an additional compatible graded structure and show how these generalise VB-Courant algebroids. We finish this paper in Section 5 with a few concluding remarks.

2. Preliminaries

In this section we sketch some background material needed for the rest of this paper. The informed reader may safely skip this section.

2.1. Pre-Courant algebroids. For the moment we will work exclusively in the category of smooth manifolds. Let \((E, \langle \cdot, \cdot \rangle)\) be a pseudo-Euclidean vector bundle over \(M\). The metric induces an isomorphism between \(E\) and its dual \(E^*\) in the standard way.

**Definition 2.1** (Vaisman [43]). A Courant vector bundle is a pseudo-Euclidean vector bundle equipped with an anchor map

\[ \rho: E \to TM, \]

such that \(\ker(\rho)\) is a coisotropic distribution in \(E\).

On any Courant vector bundle we have the following operator

\[ D : C^\infty(M) \to \text{Sec}(E) \]

given by \(\langle Df, e \rangle := \rho(e)f\).

**Definition 2.2** (Vaisman [43]). A pre-Courant algebroid is a Courant vector bundle \((E, \langle \cdot, \cdot \rangle, \rho)\) equipped with a binary operation on \(\text{Sec}(E)\), which is denoted “\(\circ\)”, that satisfies:

1. \(\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)];\)
2. \(e_1 \circ e_2 + e_1 \circ e_2 = D(e_1, e_2);\)
3. \(\rho(e_1)(e_2, e_3) = \langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle;\)

for all \(e_1, e_2, e_3 \in \text{Sec}(E)\).

Note that we do not insist that the Jacobi identity be satisfied for the binary operation, which we will refer to as a \textit{pre-bracket}. However, if the Jacobi identity in Loday–Leibniz form is satisfied then we speak of a Courant algebroid.

**Example 2.3.** It is well known that quadratic Lie algebras are precisely Courant algebroids over a point – the anchor is trivially zero and defining properties of a Courant algebroid reduce to the definition of a Lie algebra with equipped with a invariant quadratic polynomial. Similarly, if we replace a quadratic Lie algebra with a quadratic skew algebra (i.e., relax the Jacobi identity) then we obtain an algebraic example of a pre-Courant

\(^1\)See [7] for the notion of weighted Lie algebroids and weighted Lie groupoids.
algebroid. Interestingly, we have non-trivial examples of such skew algebras, principally the seven dimensional cross product:
\[
\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7
\]
\[
(a, b) \mapsto a \times b
\]
which we define (following Cayley) using a basis viz \( e_i \times e_j = \sum_k \epsilon_{ijk} e_k \), where \( \epsilon_{ijk} \) is a totally antisymmetric tensor with value +1 for \( ijk = 123, 145, 176, 246, 257, 365 \). From this definition (among other identities) we have that
\[
\begin{align*}
(1) & \quad a \times b = -b \times a, \\
(2) & \quad a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b),
\end{align*}
\]
here \( \cdot \) is of course the standard Euclidean dot product. However, in contrast to the three dimensional cross product, the Jacobi identity is not satisfied – this is tightly related to the fact that the seven dimensional cross product can be expressed in terms of the octonions. Clearly, we obtain in this way a pre-Courant algebroid over a point.

We will present other examples of pre-Courant algebroids in Subsection 3.3 using the language of supermanifolds.

Following Stiénon & Xu [42] and Liu, Sheng & Xu [34] we have a quasi-cochain complex associated with any pre-Courant algebroid. The subspaces of \( \text{Sec}(\wedge^k E^*) \) defined as
\[
C^k(E) = \{ \Psi \in \text{Sec}(\wedge^k E^*) \mid i_D f \Psi = 0 \text{ for all } f \in C^\infty \},
\]
posses a covariant derivative \( D : C^k(E) \to C^{k+1}(E) \) defined as
\[
D \Psi(e_1, \cdots, e_k) = \sum_{i=1}^{k+1} (-1)^{i+1} \rho(e_i) \Psi(e_1, \cdots, \hat{e}_i, \cdots, e_{k+1}) + \sum_{i<j} (-1)^{i+j} \Psi(e_i \circ e_j, e_1, \cdots, \hat{e}_i, \cdots, \hat{e}_j, \cdots, e_{k+1}),
\]
which is the extension on the operator (2.1).

**Definition 2.4** (Liu, Sheng & Xu [34]). The *naïve quasi-cochain complex* of a pre-Courant algebroid is the quasi-chain complex
\[
(C^*(E), D).
\]

It is important to note that in general we do not have \( D^2 = 0 \) due to the non-trivial Jacobiator. Thus, we have only a quasi-cochain complex.

### 2.2 Graded bundles.

Our general understanding of a ‘graded supermanifold’ is in the spirit of Th. Voronov [45], who defines *graded manifolds* as supermanifolds equipped with a privileged class of atlases in which coordinates are assigned weights in \( \mathbb{Z} \), which in general is independent of the Grassmann parity. Moreover, the coordinate changes are decreed to be polynomial in non-zero weight coordinates and respect the weight. An additional condition is that all the non-zero weight coordinates that are Grassmann even must be ‘cylindrical’.

In precise terms, it means that the associated *weight vector field is complete*.

An important class of graded manifolds are those that carry non-negative grading. For the moment we will consider only purely even manifolds explicitly, although the statements here generalise to the supercase (cf. [25]). We will furthermore require that this grading is associated with a smooth action \( h : \mathbb{R} \times F \to F \) of the monoid \( (\mathbb{R}, \cdot) \) of multiplicative reals on a manifold \( F \) – a *homogeneity structure* in the terminology of [18]. This action reduced to \( \mathbb{R}_{\geq 0} \) is the one-parameter group of diffeomorphism integrating the *weight vector field*, thus the weight vector field is in this case *h-complete* [14] and only *non-negative integer weights* are allowed. Thus the algebra \( \mathcal{A}(F) \subset C^\infty(F) \) spanned by homogeneous functions is \( \mathcal{A}(F) = \bigoplus_{i \in \mathbb{N}} \mathcal{A}^i(F) \), where \( \mathcal{A}^i(F) \) consists of homogeneous function of degree \( i \).

Importantly, we have that for \( t \neq 0 \) the action \( h_t \) is a diffeomorphism of \( F \) and, when \( t = 0 \), it is a smooth surjection \( \tau = h_0 \) onto \( F_0 = M \), with the fibres being diffeomorphic to \( \mathbb{R}^N \) (or in the supercase \( \mathbb{R}^N[\mathcal{P}] \)). Thus, the objects obtained are particular kinds of *polynomial bundles* \( \tau : F \to M \), i.e. fibrations which locally look like \( U \times \mathbb{R}^N \) and the change of coordinates (for a certain choice of an atlas) are polynomial in \( \mathbb{R}^N \). For this reason graded manifolds with non-negative weights and *h-complete* weight vector fields \( \Delta \) are also known as *graded bundles* [18].

On a general graded bundle, one can always pick an atlas of \( F \) consisting of charts for which we have homogeneous local coordinates \( (x^a, y^I_w) \), where \( w(x^a) = 0 \) and \( w(y^I_w) = w \) with \( 1 \leq w \leq k \), for some \( k \in \mathbb{N} \) known as the *degree* of the graded bundle. Note that, according to this definition, a graded bundle of degree \( k \) is automatically a graded bundle of degree \( l \) for \( l \geq k \). However, there is always a *minimal degree*. A little more explicitly, the changes of local coordinates are of the form
\[
x'^a = x^a(x),
\]
\[
y'_w = \sum \frac{1}{w!} y^I_w y_1 J_1 y_2 J_2 \cdots y_n J_n x^J_1 \cdots x^J_n (x),
\]
where \( w = w_1 + \ldots + w_n \) and we assume the tensors \( T_{j_k \ldots j_1}^{\prime} \) to be (super)symmetric in lower indices. Naturally, changes of coordinates are invertible and so, automatically, \( \left( T_j^{\prime}(x) \right) \) is an invertible matrix. Importantly, a graded bundle of degree \( k \) admits a sequence of surjections

\[
F = F_k \xrightarrow{\tau_k^{k-1}} F_{k-1} \xrightarrow{\tau_{k-2}^{k-1}} \ldots \xrightarrow{\tau_2^1} F_2 \xrightarrow{\tau_1^2} F_1 \xrightarrow{\tau_1^1} F_0 = M,
\]

where \( F_l \) itself is a graded bundle over \( M \) of degree \( l \) obtained from the atlas of \( F_k \) by removing all coordinates of degree greater than \( l \).

A homogeneity structure is said to be regular if

\[
\frac{d}{dt}\Big|_{t=0} h_t(p) = 0 \iff p = h_0(p),
\]

for all points \( p \in F \). Moreover, if homogeneity structure is regular then the graded bundle is of degree 1 and we have a vector bundle structure. The converse is also true and \( \tau^1 : F_1 \to M \) is a vector bundle.

If the Grassmann parity of the homogeneous local coordinates is given by the weight mod 2, then we speak of an \( N \)-manifold (cf. [38, 40]) – this means that \( h_{-1} \) acts as the parity operator, i.e., it flips sign of any Grassmann odd function.

In general we will not assume that the Grassmann parity has any direct relation with the assignment of weight – we only assume that the weight(s) and the Grassmann parity are compatible in the sense that the action of the homogeneity structure and the parity operator commute, that is, we can find local coordinates that have both a well defined weight and Grassmann parity.

Morphisms between graded bundles necessarily preserve the weight; in other words morphisms relate the respective homogeneity structures, or equivalently morphisms relate the respective weight vector fields. Evidently, morphisms of graded bundles can be composed as standard maps between smooth manifolds and so we obtain the category of graded bundles.

Following [17, 18], the notion of double and \( n \)-fold graded bundles is clear: we have a (super)manifold equipped with \( n \) homogeneity structures that commute. For example, if we have a double graded bundle then we have two homogeneity structures, \( h^1 \) and \( h^2 \) (say), such that

\[
h^1 \circ h^2 = h^2 \circ h^1,
\]

for all \( s \) and \( t \in \mathbb{R} \). If both the homogeneity structures are regular, then we have a double vector bundle. The obvious statements hold for any number of regular homogeneity structures. For further details the reader is encouraged to consult [14, 18, 38, 40, 45].

### 2.3. Pseudo-Euclidean vector bundles

Given a pseudo-Euclidean vector bundle \( (E, (,)) \) (in the category of manifolds) we know (see [38]) that it has a minimal symplectic realisation as a symplectic N-manifold of degree 2. The supermanifold \( T^*\Pi E \) – which is canonically fibred over \( \Pi(E \oplus E^*) \) – is a bi-graded bundle, where we use the phase-lift (cf. [14]) of the homogeneity structure on \( \Pi E \). By passing to total weight we obtain an N-manifold of degree 2 which carries a canonical symplectic structure. The minimal symplectic realisation of the pseudo-Euclidean vector bundle, which we denote as \( F_2 \) is given by the pullback of \( T^*\Pi E \) with respect to the embedding \( E \hookrightarrow E \oplus E^* \) given by \( X \mapsto (X, (\frac{1}{2} X, \cdot)) \). That is the symplectic N-manifold completes the following diagram.

\[
\begin{array}{ccc}
F_2 & \longrightarrow & T^*\Pi E \\
\downarrow & & \downarrow \\
\Pi E & \longrightarrow & \Pi(E \oplus E^*)
\end{array}
\]

It is well known that we can always find affine Darboux charts consisting of local coordinates \( (x^a, \xi^i, p_a) \) of weight 0, 1 and 2 respectively such that

\[
\omega = dp_a dx^a + \frac{1}{2} d\xi^i d\xi^j g_{ij},
\]

where \( g_{ij} \) corresponds to the pseudo-Euclidean structure on \( E \). In particular, the local coordinates \( (\xi^i) \) corresponds to a local basis \( (e_i) \) of \( \text{Sec}(E) \) can be chosen such that \( \langle e_i, e_j \rangle = g_{ij} = \text{constant} \).

In affine Darboux coordinates the associated non-degenerate Poisson bracket is given by

\[
\{X, Y\} = \frac{\partial X}{\partial p_a} \frac{\partial Y}{\partial x^a} - \frac{\partial X}{\partial x^a} \frac{\partial Y}{\partial p_a} + (-1)^{\frac{|X|+1}{2}} g^{ij} \frac{\partial X}{\partial \xi^j} \frac{\partial Y}{\partial \xi^i},
\]

for \( X \) and \( Y \in \mathcal{A}^*(F_2) \). It is clear that this Poisson bracket is of weight \(-2\).

Functions of weight one on \( F_2 \) are identified with sections of \( E^* \), up to a shift in Grassmann parity. As we have a pseudo-Euclidean structure we can then make the canonical identification

\[
\mathcal{A}^1(F_2) \simeq \text{Sec}(E^*) \simeq \text{Sec}(E).
\]
In terms of local coordinates this identification is no more that $e_i = \xi^1 g_{i1}$.

**Remark 2.5.** If we start with a vector bundle in the category of supermanifolds then we have a more general symplectic graded super bundle of degree 2, however the (global) constructions are identical to the purely even case.

### 2.4. Almost Lie algebroids

As we will encounter almost Lie algebroids in this paper, it is worth recalling the definition.

**Definition 2.6.** An almost Lie algebroid is a triple $(E, \rho, [ , ])$ where

1. $\pi : E \to M$ is a vector bundle (in the category of supermanifolds);
2. $\rho : E \to TM$ is a vector bundle morphism covering the identity on $M$, known the anchor;
3. $[ , ] : \text{Sec}(E) \times \text{Sec}(E) \to \text{Sec}(E)$ is a (graded) skewsymmetric bi-linear map, i.e.,

$$[u, v] = (-1)^{w(u)v} [v, u],$$

such that the quasi-derivation rule

$$[u, f v] = \rho(u)(f) v + (-1)^{\tilde{f}u} f [u, v],$$

holds for all $u$ and $v \in \text{Sec}(E)$ and $f \in C^\infty(M)$. Moreover the bracket and the anchor are compatible in the following sense

$$[\rho(u), \rho(v)] = \rho([u, v]).$$

Note that we do not insist that the bracket on sections is a Lie bracket, i.e., we do not in general have the Jacobi identity. When the Jacobi identity holds, we have the standard notion of a Lie algebroid.

Almost Lie algebroids can also be formulated in terms of an odd vector field on a supermanifold in more-or-less the same way as Lie algebroids can be formulated in terms of Q-manifolds following Vaintrob [44]. The reader can easily check the following proposition.

**Proposition 2.7.** An almost Lie algebroid $(E, \rho, [ , ]) \equiv$ an odd vector field $d_E$ of weight one on the supermanifold $\Pi E$, such that

$$d_E^2 f = 0,$$

for all $f \in C^\infty(M)$.

The bracket and anchor can be recovered using the derived bracket formalism [28].

We will also encounter almost Lie algebroids that carry a compatible non-negative grading (cf. [7, 8]). For the purposes of this paper it will be convenient to use the following definition.

**Definition 2.8.** A weighted almost Lie algebroid of degree $k$ is a double graded bundle $E_{(k-1,1)}$ such that the vector bundle $\pi : E_{(k-1,1)} \to M_{(k-1,0)}$ is an almost Lie algebroid such that the bracket is of weight $-k$, i.e., for two homogeneous sections of weight $r_1$ and $r_2$ the resulting bracket is a homogeneous section of weight $r_1 + r_2 - k$.

Let us employ local coordinates $(x^a_w, Y_i^\Sigma)$ on $E_{(k-1,1)}$ - here we have split the coordinates base and fibre coordinates with respect to the vector bundle structure. Let us denote some chosen basis of the sections as $(e_k^w)$. Then a homogeneous section of weight $r$ is locally of the form

$$u[r] = u^\Sigma[r - 1 - w](x, y) e_k^w,$$

as we consider a basis element to carry weight $(w, 1)$. Here the notation $u^\Sigma[r - 1 - w](x, y)$ means the homogeneous part of the component of weight $r - 1 - w$. If this component is seemingly of negative weight we understand it to be zero.

### 3. Symplectic almost Lie 2-algebroids and pre-Courant algebroids

#### 3.1. Symplectic nearly and almost Lie 2-algebroids

Taking our cue from Roytenberg [38] we propose the following (preliminary) definition.

**Definition 3.1.** A symplectic nearly Lie 2-algebroid consist of the following data

1. A degree 2 graded super bundle $(F_2, h)$.
2. A nondegenerate Poisson bracket $\{ , \}$ of weight $-2$.
3. An odd function $\Theta \in \mathcal{A}^1(F_2)$. 


Note that we do not require any further condition on the function $\Theta$, in particular we do not insist on the ‘classical master equation’ $\{\Theta, \Theta\} = 0$. This means that the Hamiltonian vector field
\[ Q_\Theta = \{\Theta, \bullet\} \in \text{Vect}(F_2), \]
is not in general homological. Thus, we will refer to $\Theta$ as a pre-homological potential. Moreover, take note that we do not insist that $F_2$ be an $N$-manifold and so Grassmann parity and weight may be quite independent.

**Remark 3.2.** We understand a Lie $n$-algebroid to be a graded superbundle of degree $n$ equipped with a Grassmann odd homological vector field of weight 1 – as we are not restricting ourselves to N-manifolds these supermanifolds also appear under the name non-linear Lie algebroids in the works of Voronov [45, 46]. Understanding Lie $n$-algebroids in terms of brackets and anchors is harder work than the classical case of Lie algebras, see Bonavolontà & Poncin [5] for details. A symplectic Lie $n$-algebroid is then a Lie $n$-algebroid equipped with a symplectic structure such that the homological vector field is a Hamiltonian vector field, see Ševera [40]. The $n = 2$ case was first discussed by Roytenberg [38].

**Remark 3.3.** Recall that multivector fields on a manifold $M$ can be identified with functions on the supermanifold $\Pi^*M$ – all functions are automatically polynomial in the fibre coordinates as these are Grassmann odd. In particular, vector fields are identified with linear functions on $\Pi^*M$ which – assuming $M$ is a manifold and not a supermanifold – are necessarily Grassmann odd. The odd symplectic bracket on $\Pi^*M$ is identified, up to a shift in Grassmann parity, with the Schouten–Nijenhuis bracket on multivector fields. Restriction of the odd symplectic bracket to linear functions is the standard Lie bracket on vector fields. One should keep the description of the Lie bracket on vector fields in terms of an odd symplectic bracket in mind when reading the rest of this paper – it will help explain some of our notation and the appearance of certain minus signs.

From the data of a symplectic nearly Lie 2-algebroid we ‘cook-up’ the following structures on $\text{Sec}(E)$ using the derived bracket formalism (cf. [27, 28])

1. A Courant–Dorfman pre-bracket
   \[ \{\sigma, \psi\}_\Theta := (-1)^{\bar{\sigma}+1}\{\Theta, \sigma\}, \psi\}. \]
2. An anchor map $\rho_\Theta : A^1(F_2) \rightarrow \text{Vect}(M)$
   \[ \rho_\Theta(\sigma)f = (-1)^{\bar{\sigma}+1}\{\Theta, \sigma\}, f\}. \]
3. A nondegenerate pairing
   \[ \langle \sigma, \psi \rangle := \{\sigma, \psi\} \in C^\infty(M) \]

Note that the Courant–Dorfman pre-bracket is not a Lie bracket. However, it is a derived bracket, and as such we have the following well known results. First, we do not have the skew-antisymmetry property, but rather

\[ \{\sigma, \psi\}_\Theta = (-1)^{(\bar{\sigma}+1)(\bar{\psi}+1)}\{\psi, \sigma\} + (-1)^{\bar{\sigma}}\{\Theta, \langle \sigma, \psi \rangle\}. \]

Thus the Courant–Dorfman pre-bracket is closer to a Loday–Leibniz bracket than a Lie bracket in this respect. However, we also lose the Jacobi identity in general. The Jacobiator is given by

\[ J_\Theta(\sigma, \psi, \lambda) := \llbracket \sigma, \llbracket \psi, \lambda \rrbracket_\Theta \rrbracket_\Theta - \llbracket \llbracket \sigma, \psi \rrbracket_\Theta, \lambda \rrbracket_\Theta - (-1)^{(\bar{\sigma}+1)(\bar{\psi}+1)}\{\psi, \llbracket \sigma, \lambda \rrbracket_\Theta \rrbracket_\Theta \}
\]

\[ = (-1)^{\bar{\psi}+1}\frac{1}{21}\{\{\Theta, \Theta\}, \sigma\}, \psi, \lambda\}. \]

That is, the violation of the Jacobi identity is measured by the third order higher derived bracket which is generated by $\{\Theta, \Theta\}$.

**Proposition 3.4.** The anchor map satisfies the following

\[ \llbracket \sigma, f \psi \rrbracket_\Theta = \rho_\Theta(\sigma)f \psi + (-1)^{(\bar{\sigma}+1)}f \llbracket \sigma, \psi \rrbracket_\Theta, \]

\[ \rho_\Theta(\sigma)(\psi, \lambda) = \llbracket \llbracket \sigma, \psi \rrbracket_\Theta, \lambda \rrbracket_\Theta + (-1)^{(\bar{\psi}+1)}\{\psi, \llbracket \sigma, \lambda \rrbracket_\Theta \rrbracket_\Theta \}. \]

**Proof.** The first equation follows directly from the Leibniz rule for the Poisson bracket. The second equations follows from the Jacobi identity for the Poisson bracket. \qed

We however, in general, lose compatibility of the anchor map with the pre-bracket structure. Via direct computation, following what is known about standard Lie algebroids, we see that

\[ J_\Theta(\sigma, \psi, f \lambda) - (-1)^{(\bar{\psi}+1)}f J_\Theta(\sigma, \psi, \lambda) = [\rho_\Theta(\sigma), \rho_\Theta(\sigma)]f \lambda - \rho_\Theta(\llbracket \sigma, \psi \rrbracket_\Theta) f \lambda. \]

This implies that we have compatibility of the anchor map and the Courant–Dorfman pre-bracket if the Jacobi identity is satisfied. However for the current situation we have the following result.
Proposition 3.5. The violation of the compatibility of the anchor map with the Courant–Dorfman pre-bracket is measured by \( \{\{\Theta, \Theta\}, f\} \). More specifically we have
\[
\rho_\Theta(\sigma, \rho_\Theta(\psi)) f - \rho_\Theta([[\sigma, \psi]_\Theta]) f = \frac{1}{3!}(\sigma + \psi) + \tilde{\psi} \{\{\Theta, \Theta\}, f\}, \sigma, \psi \}.
\]
In particular, the anchor and the Courant–Dorfman pre-bracket are compatible if and only if \( \{\{\Theta, \Theta\}, f\} = 0 \) for all \( f \in C^\infty(M) \).

Proof. Equation (3.4) follows from direct evaluation of the left hand side of (3.3) in terms of the Poisson bracket and using the Jacobi identity. The calculation is not illuminating and so we omit details. It is clear that if \( \{\{\Theta, \Theta\}, f\} = 0 \) then the right hand side of (3.4) is zero.

In the other direction we can use affine Darboux coordinates \((x, \xi, p)\). In affine Darboux coordinates we have
\[
\Theta = \xi^i Q_i^j(x) p_a + \frac{1}{3!} \xi^i \xi^j \xi^k Q_{kji}(x),
\]
and then using (2.2) we arrive at the observation that \( \{\{\Theta, \Theta\}, f\} \) is independent of the coordinate \( p \) and is of weight 2 – we have a monomial of order two in \( \xi \) with coefficients from \( C^\infty(M) \). Thus it is sufficient to consider
\[
\{\{\{\Theta, \Theta\}, f\}, e_i, e_j\} = 0,
\]
where we have \( e_i = \xi^l g_{li} \) etc. Non-degeneracy of the Poisson bracket – specifically the fact that we can choose coordinates such that \( \{e_i, e_j\} = \pm \delta_{ij} \) – means that (3.5) implies that \( \{\{\Theta, \Theta\}, f\} = 0 \).

Remark 3.6. Via the Jacobi identity for the Poisson bracket it is clear that \( \{\{\Theta, \Theta\}, f\} = 0 \Leftrightarrow \{\Theta, \{\Theta, f\}\} = 0 \).

The considerations above suggest the following refinement of the notion of a symplectic nearly Lie 2-algebroid.

Definition 3.7. A symplectic almost Lie 2-algebroid is a symplectic nearly Lie 2-algebroid (Definition 3.1) with the additional condition that
\[
\{\{\Theta, \Theta\}, f\} = 0,
\]
for any and all \( f \in C^\infty(M) \).

Proposition 3.8. The Jacobiator of the Courant–Dorfman pre-bracket associated with a symplectic almost Lie 2-algebroid is (graded) skew-antisymmetric.

Proof. Directly from the definitions and application of the Jacobi identity for the Poisson bracket we have
\[
J_\Theta(\sigma, \psi, \lambda) + (\sigma + \psi) J_\Theta(\psi, \sigma, \lambda) = \pm \{\Sigma, \{\psi, \sigma\}\}, \lambda \},
\]
where we use the shorthand \( \Sigma = \frac{1}{2} \{\Theta, \Theta\} \). As \( \{\psi, \sigma\} = \{\psi, \sigma\} \) is a function on \( M \) the right hand side of the above vanishes. Again directly from the Jacobi identity we have
\[
J_\Theta(\sigma, \psi, \lambda) + (\sigma + \psi) J_\Theta(\psi, \sigma, \lambda) = \pm \{\Sigma, \{\psi, \sigma\}, \lambda, \psi\} \}
\]
Using the skewsymmetry of the Poisson bracket and the Jacobi identity again we arrive at
\[
J_\Theta(\sigma, \psi, \lambda) + (\sigma + \psi) J_\Theta(\psi, \sigma, \lambda) = \pm \{\Sigma, \{\psi, \lambda\}, \sigma\} \pm \{\Sigma, \{\sigma, \{\psi, \lambda\}\} \} = 0,
\]
as \( \{\sigma, \{\psi, \lambda\}\} = 0 \) on weight grounds and the other term vanishes via the reasoning applied earlier.

Using Proposition 3.5 and Proposition 3.8 we arrive at the following results.

Proposition 3.9. For any symplectic almost Lie 2-algebroid the associated Jacobiator \( J_\Theta(\sigma, \psi, \lambda) \)

1. is linear over \( C^\infty(M) \).
2. vanishes if any of the sections, say \( \sigma \), it is evaluated on are \( Q_\Theta \) closed, i.e., if any of the sections is such that \( \{\Theta, \sigma\} = 0 \). In particular, the Jacobiator vanishes when any of the sections it is evaluated on are \( Q_\Theta \)-exact, i.e., any section of the form \( \{\Theta, f\} \) for some \( f \in C^\infty(M) \).

Remark 3.10. Proposition 3.8 and part (1) of Proposition 3.9 were first proved by Liu, Sheng & Xu [34], but in the ‘traditional framework’. We remark, that using supermanifolds and the derived bracket construction simplifies the proofs of these statements significantly.
3.2. **The relation with pre-Courant algebroids.** The notion of symplectic almost Lie 2-algebroid is closely related to the notion of a pre-Courant algebroid as defined by Vaisman [43]. Indeed we have the following theorem.

**Theorem 3.11.** There is a one-to-one correspondence between symplectic almost Lie 2-algebroids for which $F_2$ is an N-manifold and pre-Courant algebroids.

**Proof.** In one direction the correspondence is clear. From a symplectic almost Lie 2-algebroid we construct a pre-Courant algebroid structure on $E = HF^*_F \simeq E^*$. This can easily be seen taking care with the shift in Grassmann parity – we describe pre-Courant algebroids in the category of manifolds in this way. In particular, Proposition 3.5 corresponds to Definition 2.2 (1); (3.1) corresponds to Definition 2.2 (2); and the second equation of Proposition 3.4 corresponds to Definition 2.2 (3).

In the other direction we can follow more-or-less the arguments of Roytenberg [38, Theorem 4.5] and so we will not repeat them in full. The key points are that there is a one-to-one correspondence between pseudo-Euclidean vector bundles and symplectic graded bundles of degree 2, see Section 2.3. The non-degeneracy of the Poisson bracket means that given a Courant–Dorfmann pre-bracket, as a derived (pre-)bracket, it uniquely defines the almost homological potential $\Theta$. Again taking care with the Grassmann parity we can construct from any pre-Courant algebroid bracket, via a shift in parity an Courant–Dorfmann pre-bracket and thus define the almost homological potential that derives this pre-bracket. \hfill \Box

In affine Darboux coordinates $(x^a, p_{\xi}, \xi^i)$ any weight 3 pre-homological potential must be of the form

$$\Theta = \xi^i Q^a_i(x)p_a + \frac{1}{3!} \xi^i \xi^j \xi^k Q_{kji}(x).$$

Then using the Poisson bracket (2.2) one can directly deduce the structure equations for a symplectic almost Lie 2-algebroid (assuming $F_2$ is an N-manifold)

$$Q^b_i \frac{\partial Q^a_j}{\partial x^b} - Q^b_j \frac{\partial Q^a_i}{\partial x^b} = g^{lm} Q_{mij} Q^l_i = 0, \quad g^{ij} Q^a_i Q^b_j = 0.$$

The above is only part of the conditions that one obtains for a Courant algebroid (symplectic Lie 2-algebroid) – see [38, Theorem 4.5]. Just for completeness, the pre-bracket and pairing are directly related to the pre-homological potential viz

$$\langle [e_i, e_j], e_k \rangle = Q_{ijk}, \quad \rho_\Theta(e_i) = Q^a_i \frac{\partial}{\partial x^a}.$$  

removing that $e_i = \xi^i g_{ji}$ and that $g_{ij} = \text{constant}$ in affine Darboux coordinates. These local formula are of course no different to the standard ones for Courant algebroids following Roytenberg [38].

**Warning.** We take symplectic almost Lie 2-algebroids (where $F_2$ is not necessarily an N-manifold) to define pre-Courant algebroids in the category of (real) supermanifolds – because of this and Theorem 3.11 we will at times use the terms pre-Courant algebroid and symplectic almost Lie 2-algebroid interchangeably.

**Remark 3.12.** An immediate consequence of Theorem 3.11 is that the modular class of any pre-Courant algebroid is zero. The modular class of a pre-Courant algebroid is essentially defined as a measure of the existence of a Berezin volume on $F_2$ that is preserved with respect to the Lie derivative of $Q_\Theta$ (see [13, 15] for the skew algebroid case). However, as $F_2$ is symplectic we can always use the symplectic form to build a Berezin volume, and – just as in the classical case – any Hamiltonian vector field preserves this volume. Thus, all pre-Courant algebroids – and so all Courant algebroids – are unimodular.

3.3. **Examples.** In this subsection we present a few simple examples of symplectic almost Lie 2-algebroids by slightly weakening some of the well-known constructions of symplectic Lie 2-algebroids.

**Example 3.13.** Continuing Example 2.3, consider $T^*\mathbb{H}R^7$ which we equip with coordinates $(\xi^i, \pi_j)$ of bi-weight $(1,0)$ and $(0,1)$ respectively. By passing to total weight it is clear that the canonical Poisson bracket is of total weight $-2$. Moreover, we have $[\xi^i, \pi_j] = \delta^i_j$ and so we recover the standard Euclidean metric on $\mathbb{R}^7$. The pre-homological potential here is

$$\Theta = \frac{1}{2} \xi^i \xi^j \xi^k \delta^{ij} \pi_l.$$  

A quick calculation shows that

$$[\pi_i, \pi_j]_\Theta = \sum_k e_{ijk} \pi_k,$$

then being careful with the shift in Grassmann parity we recover the seven dimensional cross product. As the seven dimensional cross product does not satisfy the Jacobi identity $\{\Theta, \Theta\} \neq 0$. As there are no non-zero function of weight zero on $T^*\mathbb{H}R^7$ it is obvious that we have a ‘symplectic almost Lie 2-algebra’.
The degree 2 Poisson bracket is just the canonical Poisson bracket on $F_2 \simeq T^* (M \times \Pi \mathfrak{g})$. The pre-homological potential is given by

$$\Theta = -\xi^i Q^i_\ell(x)p_\ell + \frac{1}{2!} \epsilon^i \xi^j Q^k_{ji}(x)\pi_k,$$

where $Q^i_\ell$ describes the action and $Q^k_{ji}$ is the structure constant of the skew algebra. The minus sign is to ensure we have a right action. We can view $\psi = \psi^i \pi_i$ as an element of $\mathfrak{g}$ (once we shift the parity) and write the action as

$$\rho_\Omega (\psi) = \psi^i Q^i_\ell(x) \frac{\partial}{\partial x^\ell}.$$

A simple calculation shows $\{\Theta, \{\Theta, f\}\} = 0$ for all $f \in C^\infty (M)$ and so we have a pre-Courant algebroid. The compatibility of the anchor and the pre-bracket means

$$[\rho_\Omega (\psi_1), \rho_\Omega (\psi_2)] = \rho_\Omega ([\psi_1, \psi_2]_\Omega).$$

That is, the action still provides a representation although we have lost the Jacobi identity for the ‘bracket algebra’.
Example 3.17. In [3] it was shown how a contravariant version of generalised geometry can be constructed starting from a Poisson manifold: their motivation was to construct a ‘stringy background’ with R-flux. We can weaken the constructions slightly and obtain a ‘contravariant pre-Courant algebroid’. Let \((M, \mathcal{P})\) be a Poisson manifold – here we understand the Poisson structure as a function \(\mathcal{P} \in \mathcal{A}^2(\Pi^* T^* M)\) such that \([\mathcal{P}, \mathcal{P}]_{SN} = 0\). The bracket here is the canonical Schouten–Nijenhuis bracket. We then pass to the Hamiltonian vector field associated with the Poisson structure and take the symbol. In short, have
\[
P = \frac{1}{2} \Lambda^{ab}(x) x^*_b x^*_a \quad \Rightarrow \quad \theta := \Lambda^{ab}(x) x^*_b p_a - \frac{1}{2} \pi^c \frac{\partial \Lambda^{ab}}{\partial x^c}(x) x^*_b x^*_a \in \mathcal{A}^3(\Pi^* T^* M),
\]

where we have employed local coordinates \((x^a, x^*_a, p_a, \pi^a)\) of total weight 0, 1, 2 and 1 respectively on \(\Pi^* T^* M\).

A direct calculation shows that \([\theta, \theta] = 0\) is equivalent to \(\mathcal{P}\) being a Poisson structure, i.e., \([\mathcal{P}, \mathcal{P}]_{SN} = 0\) (see Counter Example 3.18). Thus, we have a symplectic Lie 2-algebroid and a genuine Courant algebroid.

Now we add ‘R-flux’ \(\mathcal{R} \in \mathcal{A}^3(\Pi^* T^* M)\) to the mix. We do not assume any compatibility condition between this ‘R-flux’ and the Poisson structure – usually it is assumed that \([\mathcal{P}, \mathcal{R}]\) \(\equiv 0\) (see Example 3.19). We can now use a graded subbundle \(\Theta \subseteq \mathcal{R}\) with the trivial (almost) Lie algebroid structure and take the symbol. In short, have
\[
\Theta := \Lambda^{ab}(x) x^*_b p_a - \frac{1}{2} \pi^c \frac{\partial \Lambda^{ab}}{\partial x^c}(x) x^*_b x^*_a + \frac{1}{3!} \mathcal{R}^a_{bc}(x) x^*_b x^*_c x^*_a.
\]

Note that
\[
\{\theta, \mathcal{R}\}, f) = 0,
\]

for all \(f \in C^\infty(M)\) even if we do not have \([\theta, \mathcal{R}] = 0\), as there are no conjugate variables present in the final expression. The similarity with Example 3.14 is clear.

Counter Example 3.18. Modifying Example 3.17 slightly, let is start with a quasi-Poisson structure – that is a function \(\mathcal{P} \in \mathcal{A}^2(\Pi^* T^* M)\), but now \([\mathcal{P}, \mathcal{P}]_{SN} \neq 0\). As before we have
\[
\Theta := \Lambda^{ab}(x) x^*_b p_a - \frac{1}{2} \pi^c \frac{\partial \Lambda^{ab}}{\partial x^c}(x) x^*_b x^*_a \in \mathcal{A}^3(\Pi^* T^* M).
\]

Clearly, we do not have a Courant algebroid, but do we have a pre-Courant algebroid? The answer is no. A direct calculation shows that
\[
\{\theta, \{\theta, f\}\} = \frac{1}{2} \left( \Lambda^{bc} \frac{\partial \Lambda^{ad}}{\partial x^c} - \Lambda^{dc} \frac{\partial \Lambda^{ab}}{\partial x^c} - \Lambda^{ac} \frac{\partial \Lambda^{bd}}{\partial x^c} \right) x^*_d x^*_b \frac{\partial f}{\partial x^a}.
\]

for any \(f \in C^\infty(M)\). Thus, we require \(\mathcal{P} \in \mathcal{A}^2(\Pi^* T^* M)\) to be Poisson if the above is to vanish, and so \(\{\theta, \theta\} = 0\). That is, given a quasi-Poisson structure we can build a symplectic nearly Lie 2-algebroid – a ‘non-homological Courant algebroid’ – but not a pre-Courant algebroid.

3.4. Sub-Dirac structures and almost Lie algebroids. Example 3.15 leads us to the following observation.

Proposition 3.19. The cotangent bundle of any almost Lie algebroid \((\Pi E, d_E)\) is canonically a pre-Courant algebroid.

Proof. This follows from Example 3.15 simply by equipping the dual \(\Pi E^*\) with the trivial (almost) Lie algebroid structure, that is, \(d_E^* = 0 \iff \nu = 0\).

The above proposition and the previous examples show that pre-Courant algebroids are to almost Lie algebroids what Courant algebroids are to Lie algebroids. The converse – via Dirac structures – is also true. We take our understanding of Dirac structures from Severa [40].

Definition 3.20. Let \((F_2, h, \{\cdot, \cdot\}, \Theta)\) be a pre-Courant algebroid. A sub-Dirac structure is a graded subbundle \(\mathcal{D} \hookrightarrow F_2\) over \(M_0 := b_0(\mathcal{D})\) such that
\[
(1) \quad \mathcal{D} \text{ is isotropic.}
(2) \quad \mathcal{Q}_0 \text{ is tangent to } \mathcal{D}
\]

Moreover, is the isotropic subbundle is maximal then we speak of a Dirac structure.

Proposition 3.21. The vector bundle defined by \(\text{Sec}(L) := \mathcal{A}^1(\mathcal{D})\) comes with the structure of an almost Lie algebroid.

Proof. This follows directly from Definition 3.20 as the violation of the symmetry of the pre-bracket vanishes on sections of any isotropic subbundle, see (3.1). Moreover, the condition that \(\mathcal{Q}_0\) is tangent to \(\mathcal{D}\) ensures that the Courant–Dorfman pre-bracket closes on \(\mathcal{A}^1(\mathcal{D})\). Then comparison with Definition 2.6, taking care with the shift in Grassmann parity, completes the proof.

Remark 3.22. Proposition 3.21 is known to (at least) Gritzmann & Xu [22, page 4] in the context of sub-Dirac structure supported over all of \(M\). Sheng & Liu [41] showed that for H-twisted Courant algebroids (cf. [23]), Dirac structures are H-twisted Lie algebroids (cf. [21]).
complex algebroids we have the standard cochain complex as defined by Roytenberg \[ \text{but we will restrict our attention to pre-Courant algebroids. When we are dealing with genuine Courant algebroids we have the standard quasi-cochain complex as defined by Roytenberg.}\]

Clearly, \( \Pi_{E}^{1} \) is a Dirac structure and in particular \((\Pi_{E}, d_{E} := Q_{\Theta}|_{\Pi_{E}})\) is an almost Lie algebroid. It is obvious that \( \Theta = 0 \) on \( \Pi_{E} \), as it must.

Now let us look for another Dirac structure here. Consider a ‘bi-vector field’ \( \Lambda = \frac{1}{2} \Lambda^{ij}(x) \pi_{j} \pi_{i} \) which we view as a function on the supermanifold \( \Pi_{E}^{*} \). Associated with any bi-vector is the Lagrangian submanifold \( \iota : \mathcal{D}_{\Lambda} \hookrightarrow T^{*} \Pi_{E} \) given in local coordinates as

\[ \xi^{i} \circ \iota = \{ \Lambda, \xi^{i} \}, \quad p_{a} \circ \iota = \{ \Lambda, p_{a} \}. \]

Direct calculation (which is not illuminating) gives

\[ \Theta \circ \iota = \frac{1}{2} \{ \Lambda, \{ \Lambda, \Theta(2,1) \} \} + \frac{1}{3!} \{ \Lambda, \{ \Lambda, \{ \Lambda, \Theta(3,0) \} \} \}. \]

Thus as \( \mathcal{D}_{\Lambda} \) is a Dirac structure if and only if \( \Theta \circ \iota = 0 \), we see that \( \Lambda \) has to be a twisted Poisson structure on the almost Lie algebroid \( (\Pi_{E}, d_{E}) \).

Example 3.24. In relation to the previous example, let us now consider the graph of a ‘two from’ \( \alpha = \frac{1}{2} \xi^{ij} \alpha_{ij}(x) \), which we consider as a function on \( \Pi_{E} \). Associated with any two form is the Lagrangian submanifold \( \iota : \mathcal{D}_{\alpha} \hookrightarrow T^{*} \Pi_{E} \) given in local coordinates as

\[ p_{a} \circ \iota = \frac{\partial \alpha}{\partial x^{a}}, \quad \pi_{i} \circ \iota = \frac{\partial \alpha}{\partial \xi^{i}}. \]

Thus we see that

\[ \Theta \circ \iota = \xi^{i} Q_{a}^{k} \frac{\partial \alpha}{\partial x^{a}} + \frac{1}{2!} \xi^{i} \xi^{j} Q_{ji}^{k} \frac{\partial \alpha}{\partial \xi^{k}} + \frac{1}{3!} \xi^{i} \xi^{j} \xi^{k} Q_{kji}^{l} = 0 \]

simply means \( d_{E} \alpha + \Theta(3,0) = 0 \).

3.5. The standard quasi-cochain complex. In light of Theorem 3.11, we see that given any pre-Courant algebroid we can construct a quasi-cochain complex.

Definition 3.25. The standard quasi-cochain complex of a pre-Courant algebroid is defined as the quasi-cochain complex

\[ (\mathcal{A}^{*}(F_{2}), \mathcal{Q}_{\Theta} = \{ \Theta, \bullet \}). \]

The standard quasi-cochain complex is not in general a cochain complex as \( \mathcal{Q}_{\Theta}^{2} \neq 0 \). This definition also holds equally as well for more general ‘non-homological Courant algebroids’ (i.e., symplectic nearly Lie 2-algebroids), but we will restrict our attention to pre-Courant algebroids. When we are dealing with genuine Courant algebroids we have the standard cochain complex as defined by Roytenberg \[38\].

However, pre-Courant algebroids do have a well defined zeroth and first standard cohomology groups: this follows from the fact that \( \mathcal{Q}_{\Theta}^{2}(f) = 0 \) for any and all \( f \in \mathcal{A}^{0}(F_{2}) = C^{\infty}(M) \). These cohomology groups have the same interpretation as the standard cohomology of a Courant algebroid. The zeroth standard cohomology group \( H_{0}^{0}(E) \) consists of function on \( M \) that are constant along the leaves of the characteristic foliation defined as the image of the anchor. The first standard cohomology group \( H_{1}^{0}(E) \) consists of sections of \( E \) in the kernel of the anchor modulo sections of the form \( Q_{\Theta}(f) \) for functions \( f \in C^{\infty}(M) \). The higher cohomology groups are not defined.

Following Stiénon & Xu \[42\] and Liu, Sheng & Xu \[34\] we also have a quasi-cochain complex known as the naïve quasi-cochain complex associated with any pre-Courant algebroid, Definition 2.4. We proceed to describe the naïve quasi-cochain complex in the super-framework. Given any symplectic nearly Lie 2-algebroid we define a map

\[ \iota : C^{\infty}(M) \times \mathcal{A}^{k}(F_{2}) \to \mathcal{A}^{k-1}(F_{2}), \quad (f, X) \mapsto \iota_{f}X := (-1)^{f} \{ \{ \Theta, f \}, X \}. \]

Then we define

\[ \mathcal{L}^{k}(F_{2}) := \{ \alpha \in \mathcal{A}^{k}(F_{1}) \subset \mathcal{A}^{k}(F_{2}) \mid \iota_{f} \alpha = \theta \quad \text{for all} \quad f \in C^{\infty}(M) \}. \]
Note that $L^k(F_2)$ consists of all $\kappa \in A^1(F_1)$ such that $\{\Theta, \kappa, f\} = 0$ for all $f$. Thus we see that for $k = 1$ we simply have the kernel of the anchor.

**Proposition 3.26.** Given any pre-Courant algebroid $(L^\bullet(F_2), Q_\Theta)$ is a quasi-cochain complex.

**Proof.** We just need to show that $L^\bullet(F_2)$ is closed with respect to $Q_\Theta$. Via direct application of the Jacobi identity it is a straightforward exercise to show that
\[
i_f(Q_\Theta \alpha) + (-1)^i Q_\Theta(i_f \alpha) = \{\Theta, \{\Theta, f\}, \alpha\}.
\]
Then we see that $i$ and $Q_\Theta$ commute if and only if we have a pre-Courant algebroid. Thus, if $\alpha$ is in $L^\bullet(F_2)$ then so is $Q_\Theta \alpha$.

**Remark 3.27.** It is important to note that we do not obtain a quasi-cochain complex in this way for symplectic nearly Lie 2-algebroids.

Let once again consider only pre-Courant algebroids in the category of purely even manifolds – this just reduces the need to keep track of certain sign factors. Much like the case of standard Courant algebroids we have the following theorem.

**Theorem 3.28.** There exists a quasi-cochain map between the naïve quasi-cochain complex and the standard quasi-cochain complex associated with any pre-Courant algebroid.

**Proof.** Clearly we have a module homomorphism between $\text{Sec}(\wedge^k E^*)$ and $A^k(F_1)$. Moreover the conditions $i_D^f \Psi = 0$ and $i_f \alpha = 0$ are equivalent and so we obtain a module homomorphism
\[
\phi : C^k(E) \to L^k(F_2).
\]

The only thing to prove is that $D$ and $Q_\Theta$ are equivalent. However, this follows from the same arguments given by Stiénon & Xu [42] with only minor changes due to the fact that we use a non-skewsymmetric (pre)-bracket. It is sufficient to consider the k=1 case and identify $\Psi \in \text{Sec}(E^*)$ as an element of $\text{Sec}(E)$ using the pseudo-Euclidean structure. The general case follows from the Leibniz rule and linearity. With these comments in mind consider, it is a simple exercise to show that
\[
\langle \Psi \circ e_1, e_2 \rangle - D\Psi(e_1, e_2) = \rho(e_2)\langle \Psi, e_1 \rangle - \langle e_2 \circ \Psi, e_1 \rangle - \langle \Psi, e_2 \circ e_1 \rangle + \langle \Psi, D(e_1, e_2) \rangle,
\]
where we have used $e_1 \circ e_2 + e_2 \circ e_1 = D(e_1, e_2)$. Then using the definition of $i_D^f \Psi$ we obtain
\[
\langle \Psi \circ e_1, e_2 \rangle - D\Psi(e_1, e_2) = \rho(e_2)\langle \Psi, e_1 \rangle - \langle e_2 \circ \Psi, e_1 \rangle - \langle \Psi, e_2 \circ e_1 \rangle + \rho(\Psi)\langle e_1, e_2 \rangle.
\]
As $\Psi \in C^1(E)$ the final term of the above vanishes. Note that under the module homomorphism $\phi$ we have the identification (with minor abuse of notation)
\[
\langle \Psi \circ e_1, e_2 \rangle \leftrightarrow \{\{\Theta, \Psi\}, e_1\}, e_2\}.
\]
This we can identify the two quasi-differentials, i.e.,
\[
D\Psi \leftrightarrow \{\Theta, \Psi\},
\]
and thus we have a quasi-cochain map. \qed

From the expression for the Jacobiorator (3.2) it is clear that if
\[
J_\Theta(\kappa, \bullet, \bullet) = 0,
\]
for all $\kappa \in L^1(F_2) \subset A^1(F_1)$ then $(L^\bullet(F_2), Q_\Theta)$ is a cochain complex. The resulting cohomology, following [34], we refer to as the naïve cohomology. Note that we do not in general require that $Q_\Theta^2 = 0$ on the whole of $A^\bullet(F_2)$ and so we are not forced to consider Courant algebroids.

The condition on the Jacobiorator (3.6) is quite restrictive, but not prohibitively so. In particular, twisted Courant algebroids (cf. [23]) are natural examples of pre-Courant algebroids that have a well defined naïve cohomology.

### 4. Pre-Courant Algebroids with an Additional Grading

#### 4.1. Weighted pre-Courant algebroids

Following [7], the notion of a weighted pre-Courant algebroid is clear – or really the notion of a weighted symplectic almost Lie 2-algebroid is clear.

**Definition 4.1.** A weighted pre-Courant algebroid of degree $k$ is a symplectic almost Lie 2-algebroid equipped with an additional compatible homogeneity structure of degree $k - 1$.

To be more explicit we have:

1. A double graded (super) bundle $(F := F(k-1, 2), l, h)$, where the homogeneity structures are of degree $k - 1$ and 2 respectively.
(2) A nondegenerate Poisson bracket on the double graded bundle of bi-weight \((1 - k, -2)\).

(3) A pre-homological potential \(\Theta \in A^{(k-1,3)}(F)\), such that \(\{\Theta, \Theta\}, f\} = 0\) for all \(f \in A^{(0,0)}(F)\).

The construction of the Courant–Dorfman pre-bracket, the anchor map and the pairing remains formally the same as before. However, we now understand the Courant–Dorfman pre-bracket as closing on \(A^{(k,1)}(F)\). Note that \(A^{(k,1)}(F)\) has a module structure over \(A^{(0,0)}(F)\), and thus we can identify \(A^{(k,1)}(F)\) with the sections of the graded vector bundle \(\pi : F_{(k-1,1)} \to F_{(k-1,0)}\); or really the dual vector bundle, but we can then use the pseudo-Euclidean structure. Just by counting the weights we see that

\[
[A^{(p,1)}(F), A^{(q,1)}(F)]_\Theta \subset A^{(m,1)}(F),
\]

where \(m = p + q - k + 1\). The anchor can be considered as a map

\[
\rho_\Theta : A^{(p,1)}(F) \to \text{Vect}(F_{(k-1,0)}),
\]

where the vector field – for a given section – that maps to is homogeneous and of weight \(p + 1 - k\). The pseudo-Euclidean structure can be considered as a map

\[
\{\, \}, : A^{(p,1)}(F) \times A^{(q,1)}(F) \to A^{(m,0)}(F),
\]

where again \(m = p + q - k + 1\). Note that if \(p + q - k < 1\) then both the Courant–Dorfman pre-bracket and the pairing take the value zero. Extension to inhomogeneous functions is via linearity.

We can always employ affine Darboux coordinates \((x^A_i, p_B^a, \zeta^I_w)\) on \(F_{(k-1,2)}\). Notationally, we understand the additional label to represent the weight with respect to the homogeneity structure \(1 : \mathbb{R} \times F \to F\), thus \(0 \leq w < k\). Because we have employed the phase lift \(x^A_i\) and \(p_B^k\) are conjugate coordinates. Then, it is clear that the pre-homological potential must be of the form

\[
\Theta = \xi^I \xi^J \Theta_I^A[w' - w](x) p^k_{(k-w-1)} + \frac{1}{3!} \xi^I \xi^J \xi^K \Theta_{KJI} \Theta_I^A k - 1 - w - w' \Theta_{I}^A(x).
\]

Here \(Q_I^A\) and \(Q_{KJI}\) are the homogeneous pieces of the structure functions. Any expression that is seemingly negative in weight is understood to be zero. Via inspection we arrive at the following proposition.

**Proposition 4.2.** The anchor map of a weighted Pre-Courant algebroid can be considered as a morphisms in the category of double graded (super) bundles

\[
\rho_\Theta : F_{(k-1,1)} \to \Pi F_{(k-1,0)},
\]

where the homogeneity structures on the antitangent bundle arise from the tangent lift the homogeneity on \(F_{(k-1,0)}\), and the natural regular homogeneity structure associated with any tangent bundle.

The degree 1 case is just the theory of symplectic almost Lie 2-algebroids, and so via Theorem 3.11 covers the standard theory of pre-Courant algebroids. The degree 2 case covers vector bundle or VB-Courant algebroids.

**Theorem 4.3.** Let \((F_2, h, \{\, , \}, \Theta)\) be a pre-Courant algebroid. Then the higher tangent bundle \(T^{k-1}F_2\) is canonically a weighted pre-Courant algebroid of degree \(k\).

**Proof.** Clearly \(T^{k-1}F_2\) is a double graded bundle where the homogeneity structures are the natural one on a higher tangent bundle and the prolongation of the homogeneity structure on \(F_2\). Thus we can employ natural local coordinates

\[
\left( x^A_i, \xi^I_w, p^k_B \right),
\]

on \(T^{k-1}F_2\). Here \(0 \leq \epsilon < k\) denotes the weight which we understand as coming from the ‘number of derivatives’ of the original coordinates on \(F_2\). The Poisson bracket also lifts (see [31]) with the conjugate variables being \((x^A_i, p^k_B)\) and \((\xi^I_w, \xi^J_w)\), where \(\delta = \epsilon - k + 1\). Thus we have a non-degenerate Poisson bracket of bi-weight \((1 - k, -2)\). The higher tangent prolongation (or complete lift cf. [37]) of \(\Theta\) to a function on \(T^{k-1}F_2\), which we denote as \(\Theta^{(c)}\), is formally constructed in the same way as the classical case on purely even manifolds by taking derivatives. It is not hard to see that \(\Theta^{(c)}\) is of bi-weight \((k - 1, 3)\). Moreover, it follows that (again see [31])

\[
\{\Theta^{(c)}, \Theta^{(c)}\} |_{T^{k-1}F_2} = \{\Theta, \Theta\}^{(c)}.
\]

Note that \(\{\Theta, \Theta\}, f_0 = 0\), for any \(f_0 \in \mathcal{C}^\infty(M)\) implies that \(\{\Theta, \Theta\}\) is independent of the momenta: this is equivalent to the structure equations of the pre-Courant algebroid. Thus, we deduce that \(\{\Theta^{(c)}, \Theta^{(c)}\} |_{T^{k-1}F_2}\) is similarly independent of the coordinates \(p_c\) – that is we have a function on \(T^{k-1}F_1\). We then see that

\[
\{\Theta^{(c)}, \Theta^{(c)}\} |_{T^{k-1}F_2}, f |_{T^{k-1}F_2} = 0,
\]

for all \(f \in \mathcal{A}(T^{k-1}M)\), due to the lack of conjugate variables and conclude that we have the structure of a weighted pre-Courant algebroid.

**Corollary 4.4.** Two immediate consequences of Theorem 4.3 are:
(1) The higher tangent bundle of order $k - 1$ of a Courant algebroid is a weighted Courant algebroid of degree $k$.

(2) The tangent bundle of a (pre-)Courant algebroid is a weighted (pre-)Courant algebroid of degree 2.

**Remark 4.5.** As far as we are aware, the first proof that the tangent bundle of a Courant algebroid canonically comes with the structure of a VB-Courant algebroid was given by Bouaizaa & Zaalani [6] in the standard framework. Li-Bland in his PhD thesis [33] sketched the super-proof of this.

4.2. VB-Courant algebroids revisited. As a degree 1 (and so regular) homogeneity structure encodes a vector bundle structure (cf. [17]) the natural definition of a VB-pre-Courant algebroid is evident.

**Definition 4.6.** A VB-pre-Courant algebroid is a weighted pre-Courant algebroid (Definition 4.1) of degree $k = 2$.

Within $\mathcal{A}^{(\bullet, 1)}(F)$ there are two privileged modules over $\mathcal{A}^{(0, 0)}(F)$, $\mathcal{A}^{(1, 1)}(F)$ and $\mathcal{A}^{(0, 1)}(F)$. In the standard formulation of double vector bundles $\mathcal{A}^{(1, 1)}(F)$ corresponds to linear sections and $\mathcal{A}^{(0, 1)}(F)$ corresponds to core sections of the dual of the double vector bundle $\mathcal{F}(1, 1)$ over $\mathcal{F}(1, 0)$. By (4.1) we see that

$$\llbracket \mathcal{A}^{(1, 1)}(F), \mathcal{A}^{(1, 1)}(F) \rrbracket \in \mathcal{A}^{(1, 1)}(F), \quad \llbracket \mathcal{A}^{(1, 1)}(F), \mathcal{A}^{(0, 1)}(F) \rrbracket \in \mathcal{A}^{(0, 1)}(F), \quad \llbracket \mathcal{A}^{(0, 1)}(F), \mathcal{A}^{(0, 1)}(F) \rrbracket \in \mathcal{A}^{0, 0}(F) = 0.
$$

Thus, the Courant–Dorfman pre-bracket, in the traditional language is linear. Moreover, Proposition 4.2 implies that the anchor, thought of as a map $\mathcal{F}(1, 1) \to \mathcal{T} \mathcal{F}(1, 0)$ is also linear.

One can also directly observe that

$$\llbracket \mathcal{A}^{(1, 1)}(F), \mathcal{A}^{(1, 1)}(F) \rrbracket \subset \mathcal{A}^{(1)}(F), \quad \llbracket \mathcal{A}^{(1, 1)}(F), \mathcal{A}^{(0, 1)}(F) \rrbracket \subset \mathcal{A}^{(0, 1)}(F), \quad \llbracket \mathcal{A}^{(0, 1)}(F), \mathcal{A}^{(0, 1)}(F) \rrbracket = 0,
$$

which by Gracia-Saz & Mehta [20, Lemma 2.8] implies that the pairing is also linear.

Thus we see that our definition of a VB-pre-Courant algebroid is what one would expect in a more ‘traditional language’. Moreover, Definition 4.6 covers the notion of a VB-Courant algebroid as given by Li-Bland [33]. In particular, we have a simpler notion of a VB-Courant algebroid in terms of a Courant algebroid equipped with a regular homogeneity structure. This should be compared with [33, Proposition 3.2.1], which is essentially equivalent to our definition of a VB-Courant algebroid.

4.3. Weighted standard cohomology. It clear that the graded structure on $\mathcal{A}^{(\bullet, \bullet)}(F)$ and the fact that $Q_A$ is of bi-weight $(0, 1)$ means that the zero and first cohomology groups of weighted pre-Courant algebroid inherit a graded structure. We have

$$H_{st}^{0, \bullet}(F) = \bigoplus_{r=0} H_{st}^{0, r}(F) \subset \mathcal{A}^{(0, \bullet)}(F),$$

and

$$H_{st}^{1, \bullet}(F) = \bigoplus_{r=0} H_{st}^{1, r}(F) \subset \mathcal{A}^{(1, \bullet)}(F),$$

for the respective standard cohomology groups. That is it makes sense to speak of homogeneous cochains and coboundaries in relation to the zeroth and first standard cohomology.

It is obvious that $H_{st}^{0, \bullet}(F)$ is a graded algebra with addition and multiplication inherited from that on $\mathcal{A}^{(0, \bullet)}(F)$. We will denote an element of the first standard cohomology $[\gamma] \in H_{st}^{1}(F)$. Then we can define a multiplication

$$f \cdot [\gamma] = f \ast [\gamma] \in H_{st}^{0, p+q}(F),$$

where $f \in H_{st}^{0, q}(F)$. It is clear that this multiplication does not depend on the representative of $[\gamma]$. The addition of elements in $H_{st}^{1, \bullet}(F)$ is inherited from the addition in $\mathcal{A}^{(1, \bullet)}(F)$. We thus have the following result.

**Proposition 4.7.** Fix some weighted pre-Courant algebroid $F$. The associated first standard cohomology group $H_{st}^{1}(F)$ is a graded algebra module over $H_{st}^{0, \bullet}(F)$ via the proceeding constructions.

4.4. Weighted sub-Dirac structures. Following Definition 3.20 the notion of a weighted sub-Dirac structure is intuitively clear – we have a sub-Dirac structure with an additional compatible homogeneity structure. More formally we have the following definition.

**Definition 4.8.** Let $(F_{(k-1,2)}, \mathfrak{h}, \{ , \}, \Theta)$ be a weighted pre-Courant algebroid of degree $k$. A weighted sub-Dirac structure of degree $k$ is a double graded subbundle $i[k] : \mathcal{D}_{(k-1,2)} \hookrightarrow F_{(k-1,2)}$, possibly supported over $F_{0(k-1,0)} := \mathfrak{h}_0(D_{(k-1,2)})$ and $F_{0(0,2)} := i_0(D_{(k-1,2)})$.

\[
\begin{array}{cccc}
\mathcal{D}_{(k-1,2)} & \mathfrak{h}_0(D_{(k-1,2)}) & F_{(k-1,2)} & F_{0(0,2)} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
F_{0(k-1,0)} & M_0 & F_{(k-1,0)} & M \\
\end{array}
\]

such that
(1) $\mathcal{D}_{(k-1,2)}$ is isotropic.
(2) $Q_\Theta$ is tangent to $\mathcal{D}_{(k-1,2)}$.

If $\mathcal{D}_{(k-1,2)}$ is Lagrangian then we have a weighted Dirac structure.

**Remark 4.9.** The notion of a sub-VB-Dirac structure is clear: we are considering substructures of weighted pre-Courant algebroids of degree $k = 2$. VB-Dirac structures, to our knowledge, first appear in [33] in the more traditional framework of double vector bundles.

**Proposition 4.10.** The graded vector bundle defined by $\text{Sec}(L_{(k-1,1)}) := \mathcal{A}^{(k)}(\mathcal{D}_{(k-1,2)})$ comes with the structure of a weighted almost Lie algebroid of degree $k$.

**Proof.** It is clear from Proposition 3.21 that by forgetting the additional graded structure that we have an underlyng almost Lie algebroid structure. The only question is the weight of almost Lie bracket. A homogeneous section of weight $r$ of $L$ can be viewed as an element of $\mathcal{A}^{(r-1)}(\mathcal{D}_{(k-1,2)})$, and thus by (4.1) the almost Lie bracket carries weight $-k$. Then via definition (2.8) we have the structure of a weighted almost Lie algebroid. $\square$

**Proposition 4.11.** Let $\iota : \mathcal{D} \hookrightarrow F_2$ be a sub-Dirac structure. Then $T^{k-1} \mathcal{D} \hookrightarrow T^{k-1} F_2$ is a weighted sub-Dirac structure of degree $k$.

**Proof.** Due to the functorial properties of taking the higher tangent bundle it is clear that $T^{k-1} \mathcal{D}$ is a graded double subbundle of $T^{k-1} F_2$. As $\mathcal{D}$ is isotropic then so is $T^{k-1} \mathcal{D}$ with respect to the lifted Poisson structure. Similarly, as $Q_{\Theta(c)} = (Q_\Theta)^c$ (see [31]) we see that the $Q_{\Theta(c)}$ is tangent to $T^{k-1} \mathcal{D}$. $\square$

**Remark 4.12.** Higher order tangent lifts of Dirac structures in $TM \times_M T^* M$ were first explored by Kouotchop Wamba, Ntyam & Wouafo Kamga [30] in the classical language. Also see Kouotchop Wamba & Ntyam [32].

4.5. **Examples.** In this subsection we present a few simple examples of weighted pre-Courant algebroids and sub-Dirac structures.

**Example 4.13.** Using Theorem 4.3 and Proposition 4.11 any pre-Courant algebroid and a sub-Dirac structure thereof gives rise to a weighted pre-Courant algebroid and a weighted sub-Dirac structure via the higher order tangent functor. As a specific example, $\mathcal{T}^* ITM$, where for simplicity we assume $M$ is a manifold, comes canonically with the structure of a Courant algebroid – the homological potential is just the symbol of de Rham differential on $M$ and the Poisson structure is just the canonical Poisson structure on the cotangent bundle, remembering that we assign weight 2 to the momenta. Obviously, ITM is a Dirac structure with the canonical Lie algebroid structure. Then $T^{k-1}(\mathcal{T}^* ITM) \cong \mathcal{T}^* IT(\mathcal{T}^{k-1} M)$ is canonically a weighted Courant algebroid of degree $k$, and $IT(\mathcal{T}^{k-1} M)$ is a weighted Dirac structure thereof.

**Example 4.14.** Consider $\mathcal{T}^* F_{k-1}$, where $F_{k-1}$ is any degree $k-1$ graded bundle and we assign weight 2 to the ‘momenta’. That is we employ local coordinates of the form

$$\left( \left( x^a, y_w, \eta_{w}, \xi, \eta_{(k-1),2}, \eta_{(w,2)} \right) \right),$$

where $(0 < w < k)$. Note that the conjugate pairs here are $(x, p^{k-1})$ and $(y_w, q^{k-1-w})$ and so the canonical Poisson bracket on the cotangent bundle carries bi-weight $(1-k, -2)$. However, notice that there are no function of bi-weight $(\bullet, \text{odd})$ and so the only possible choice of a (pre-)homological potential is just $\Theta = 0$ – which leads to a not so interesting structure. Any isotropic subbundle is a sub-Dirac structure of degree $k$.

**Example 4.15.** As an example of a VB-structure, consider $F_{(1,2)} := \mathcal{T}^* I T E^*$, where $E \rightarrow M$ is a vector bundle which we will for simplicity assume to be in the category of real manifolds. We equip $F_{(1,2)}$ with local coordinates of the form

$$\left( \left( x^a, \eta_{(0,0)}, y^b, \eta_{(1,0)}, dx_j, p_c, \delta^k, p_{\delta^k}, \eta_{(0,1)}, \delta_{(1,1)}, \delta_{(2,2)}, \delta_{(1,1)}, \delta_{(0,1)} \right) \right).$$

With this assignment of bi-weight it is clear that the canonical Poisson bracket carried bi-weight $(-1, -2)$ as desired.Canonically

$$\theta = dx^a p_a + dq_i y^i$$

p provides a homological potential and we have the structure of a VB-Courant algebroid – in our language a weighted Courant algebroid of degree $2$.

Now, let us ‘twist’ the homological potential by adding any linear 3-form

$$\beta = \frac{1}{3!} dx^a dx^b dx^c \beta_{abc}(x) q_i + \frac{1}{2!} dx^a dx^b \beta_{abc}(x) dq_i,$$

which we understand as a function of bi-weight $(1, 3)$ on $IT E^*$. We then claim that

$$\Theta := \theta + \beta.$$
is a pre-homological potential and that we have the structure of a weighted pre-Courant algebroid of degree 2. To see this note that we have

$$\{\Theta, \Theta\} = 2\{\theta, \beta\} = 2d\beta \neq 0$$

where $d$ is the de Rham differential on $\Pi T E^*$. We make no assumption about the closure of $\beta$. It is not hard to see that

$$\{\{\Theta, \Theta\}, f\} = 2\{d\beta, f\} = 0$$

for all $f \in A^*(E^*)$ due to the lack of conjugate variables in the final Poisson bracket.

An obvious example of a weighted Dirac structure here is $\Pi T E^*$ which comes canonically with a de Rham differential. Let us find a less obvious example of weighted Dirac structure by considering a linear two form

$$\alpha = \frac{1}{2} dx^a dx^b \alpha^{ab}_a(x) q_i + dx^a \alpha^i_a(x)d q_i,$$

which we understand as a function of bi-weight $(1,2)$ on $\Pi T E^*$. Associated with any such two form is the Lagrangian subbundle

$$\iota : D_\alpha \hookrightarrow T^*\Pi T E^*$$

defined locally viz

$$p_a \circ \iota = \frac{\partial \alpha}{\partial x^a}, \quad y^i \circ \iota = \frac{\partial \alpha}{\partial q_i}, \quad \pi_b \circ \iota = \frac{\partial \alpha}{\partial dx^b}, \quad \chi^j \circ \iota = \frac{\partial \alpha}{\partial dq_j}.$$  

Thus, to ensure that $Q\Theta$ is tangent to $D_\alpha$ we require $\Theta \circ \iota = 0$, and so we arrive at

$$d\alpha + \beta = 0.$$  

This then give the consistency condition that $\beta$ has to be closed – this in terms implies that we are dealing with a genuine Courant algebroid. In particular, given a closed linear two form we can construct a weighted Courant algebroid of degree 2 via $\beta = -d\alpha$ such that $D_\alpha$ is a weighted Dirac structure.

5. Concluding remarks

We have shown that pre-Courant algebroids are ‘really’ symplectic almost Lie 2-algebroids, just as Courant algebroids are ‘really’ symplectic Lie 2-algebroids. This shift in the fundamental starting place allows for many of the known facts about pre-Courant algebroids to be neatly reformulated, and in many cases the proofs are significantly simpler. For example, the notion of (sub-)Dirac structures on pre-Courant algebroids as certain isotropic submanifolds is conceptually neat and clear.

Moreover, working with symplectic almost Lie 2-algebroids allows for a very economic and powerful framework to understand pre-Courant algebroids with a compatible non-negative grading – weighted pre-Courant algebroids in our language. Our framework in terms of graded super bundles generalises VB-Courant algebroids, while at the same time drastically simplifying working with them. In particular, the notion of weighted sub-Dirac structures is almost obvious and covers the case of VB-Dirac structures.

Pre-Courant algebroids and especially weighted pre-Courant algebroids have a rich and interesting super-geometrical structure which deserves further study. We are convinced that the understanding of pre-Courant algebroids as symplectic nearly Lie 2-algebroids will allow for further mathematical developments and applications in theoretical physics.

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