Space/time noncommutative field theories and causality

H. Bozkaya\textsuperscript{1}, P. Fischer\textsuperscript{2}, H. Grosse\textsuperscript{3}, M. Pitschmann\textsuperscript{4}, V. Putz\textsuperscript{5}, M. Schweda\textsuperscript{6}, R. Wulkenhaar\textsuperscript{7}

\textsuperscript{1,2,4,5,6}Institut für Theoretische Physik, Technische Universität Wien, Wiedner Hauptstraße 8–10, A-1040 Wien, Austria

\textsuperscript{3}Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria

\textsuperscript{5,7}Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstraße 22-26, D-04103 Leipzig, Germany

Abstract

As argued previously, amplitudes of quantum field theories on noncommutative space and time cannot be computed using naïve path integral Feynman rules. One of the proposals is to use the Gell-Mann–Low formula with time-ordering applied before performing the integrations. We point out that the previously given prescription should rather be regarded as an interaction point time-ordering. Causality is explicitly violated inside the region of interaction. It is nevertheless a consistent procedure, which seems to be related to the interaction picture of quantum mechanics. In this framework we compute the one-loop self-energy for a space/time noncommutative $\phi^4$ theory. Although in all intermediate steps only three-momenta play a rôle, the final result is manifestly Lorentz covariant and agrees with the naïve calculation. Deriving the Feynman rules for general graphs, we show, however, that such a picture holds for tadpole lines only.

\textsuperscript{1}work supported by “Fonds zur Förderung der Wissenschaften” (FWF) under contract P15463.
\textsuperscript{2}fischer@hep.itp.tuwien.ac.at, work supported by “Fonds zur Förderung der Wissenschaften” (FWF) under contract P15463.
\textsuperscript{3}grosse@doppler.thp.univie.ac.at
\textsuperscript{4}work supported by “Fonds zur Förderung der Wissenschaften” (FWF) under contract P15463.
\textsuperscript{5}vputz@mis.mpg.de, work supported by “Fonds zur Förderung der Wissenschaften” (FWF) under contract P15015-TPH.
\textsuperscript{6}mschweda@tph.tuwien.ac.at
\textsuperscript{7}raimar.wulkenhaar@mis.mpg.de, Schloßmann fellow.
1 Introduction

Quantum field theories on noncommutative spaces are full of surprises, indicating that a true understanding of quantum field theory is still missing \([1]\). This means, on the other hand, that studying the quantisation of field theories on noncommutative spaces we resolve the degeneracy of various methods developed for commutative geometries: The outcomes of different quantisation schemes ported to noncommutative geometries will no longer coincide.

At the moment we know of two major challenges. First, the evaluation of Feynman graphs as a perturbative solution of the path integral produces a completely new type of infrared-like singularities \([2, 3]\) in non-planar graphs. This can be understood from the power-counting theorem \([4]\) for non-commutative (massive, Euclidian) field theories, which implies the existence of two types (rings and commutants) of non-local divergences.

Second, the case of a Minkowskian signature of the noncommutative geometry ("space/time noncommutativity") turns out to be involved. It was pointed out in \([5]\) that in the Minkowskian (non-degenerate) case the Wick rotation of Euclidian Green’s function does not give a meaningful result, first of all because unitarity would be lost \([6]\). The reason is that the Osterwalder-Schrader theorem \([7]\) does not hold. Already in \([8]\) there was given a proposal for a correct quantisation of field theories on space/time noncommutative geometries: Starting with interaction Hamiltonians on a Fock space

\[
H_I(t) = \int_{x^0 = t} d^3x : (\phi \star \phi \star \cdots \star \phi)(x) :
\]  

(and averaging over the noncommutativity parameter) the contributions to the scattering amplitudes were defined as the Dyson series

\[
G_n(x_1, \ldots, x_k) := \frac{(-i)^n}{n!} \int dt_1 \cdots dt_n \left\langle 0 \left| T\phi(x_1) \cdots \phi(x_k) H_I(t_1) \cdots H_I(t_n) \right| 0 \right\rangle,
\]

where \(T\) denotes the time-ordering with respect to \(\{x_1^0, \ldots, x_k^0, t_1, \ldots, t_n\}\) and \(|0\rangle\) the vacuum state. Unitarity is preserved. In \([5]\) there was added a second proposal, the iterative solution of the (interacting) field equation (Yang-Feldman approach), which has the advantage of being manifestly covariant. Unitarity is preserved as well. We refer, in particular, to \([9]\).

A third approach, the direct application of the Gell-Mann–Low formula for Green’s functions,

\[
G_n(x_1, \ldots, x_k) := \frac{i^n}{n!} \int d^4z_1 \cdots d^4z_n \left\langle 0 \left| T\phi(x_1) \cdots \phi(x_k) \mathcal{L}_I(z_1) \cdots \mathcal{L}_I(z_n) \right| 0 \right\rangle^\text{con},
\]

where \(\mathcal{L}_I\) is the interaction Lagrangian, was elaborated in \([10]\). The superscript \(^\text{con}\) means projection onto the connected part. Unitarity was investigated in \([11]\). That approach was called “time-ordered perturbation theory” in \([10]\), a name which we find ambiguous. The time-ordering in \([10]\) is considered for external vertices and interaction points only, and not with respect to the actual time-order of the fields in the interaction Lagrangian. We give in section \([2]\) a few comments on the two natural ways of time-ordering. The
version used in [10] is an interaction-point time-ordering (IPTO), it is explicitly acausal, and to distinguish from a true causal time-ordering.

Explicit calculations for the proposed quantisation schemes of space/time noncommutative field theories are technically much more cumbersome than Feynman graph computations. It is therefore desirable to extract a powerful calculus out of the general schemes. In a first step one has to familiarise oneself with the computational methods of the new approach.

For that purpose we compute in this paper the one-loop two-point function for a $\phi^4$ theory on noncommutative space-time. The result of the indeed very lengthy but straightforward calculation in interaction point time-ordered perturbation theory agrees with the naive path integral computation of the relevant Feynman graph. Deriving in section 4 the Feynman rules for IPTO, we show, however, that this is true for tadpole lines only (which should be removed anyway by normal ordering).

We may speculate that taking the true causal time-ordering in the Gell-Mann–Low formula one ends up with the usual Feynman rules involving the causal Feynman propagator. It seems, therefore, that causality and unitarity are mutually exclusive properties of space/time noncommutative geometries.

2 Comments on time-ordering and causality

By “noncommutative $\mathbb{R}^4$” one understands the algebra $\mathbb{R}_\theta^4$ of Schwartz class functions on ordinary four-dimensional space, equipped with the multiplication rule

\[(f \ast g)(x) = \int d^4 s \int \frac{d^4 l}{(2\pi)^4} f(x - \frac{1}{2} l^\nu) g(x + s) e^{i l^\nu s}, \quad \tilde{l}^\nu := l_\mu \theta^{\mu \nu}.\] (4)

The product (4) characterised by a real skew-symmetric translation-invariant tensor $\theta^{\mu \nu} = -\theta^{\nu \mu}$ of dimension $[\text{length}]^2$ is associative and noncommutative, it is a non-local product on rapidly decreasing functions.

We consider a scalar field theory on $\mathbb{R}_\theta^4$ given by the classical action

\[\Sigma = \int d^4 z \left( \frac{1}{2} g^{\mu \nu} (\partial_\mu \phi \ast \partial_\nu \phi)(z) - \frac{1}{2} m^2 (\phi \ast \phi)(z) + \frac{g}{4!} (\phi \ast \phi \ast \phi \ast \phi)(z) \right),\] (5)

with $\phi \in \mathbb{R}_\theta^4$. By definition (4) we have

\[(\phi \ast \phi \ast \phi \ast \phi)(z) = \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{i l_i s_i} \right) \times \phi(z - \frac{1}{2} l_1) \phi(z + s_1 - \frac{1}{2} l_2) \phi(z + s_1 + s_2 - \frac{1}{2} l_3) \phi(z + s_1 + s_2 + s_3).\] (6)

If $g^{\mu \nu}$ is the Minkowskian metric $g^{\mu \nu} = \text{diag}(1, -1, -1, -1)$, we cannot simply Wick-rotate the Euclidian Green’s functions obtained by evaluation of the path integral, see [5]. Here we shall follow the proposal of [10] and use the Gell-Mann–Low formula [3] to define the quantum field theory. However, one has to be more careful with the definition of the time-ordering. Let us consider the simplest case of the two-point function at first order in $g$,

\[G(x, y) = \frac{g}{4!} \int d^4 z \langle 0 \left| T(\phi(x) \phi(y) (\phi \ast \phi \ast \phi \ast \phi)(z)) \right| 0 \rangle.\] (7)
Let us discuss this nuance for the geometrical situation relevant for (7):

In the same manner as on commutative space-time, the integration over the interaction point is performed after taking the time-ordered product. Since the \(\ast\)-product for \(\phi^0 \neq 0\) is non-local in time, one has to say clearly what one understands under time-ordering. Let us discuss this nuance for the geometrical situation relevant for (7):

This arrangement of fields corresponds to the following non-vanishing contribution to the true time-ordering of (7):

\[
G^S(x, y) = \int d^4 z \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{i s_i} \right) \tau(s^0_1 + s^0_2 + s^0_3 + \frac{l^0_1}{2}) \tau(z^0 - \frac{l^0_1}{2} - x^0) \times \tau(x^0 - z^0 - s^0_1 + \frac{l^0_2}{2}) \tau(z^0 + s^0_2 - \frac{l^0_2}{2} - y^0) \tau(y^0 - z^0 - s^0_3 + \frac{l^0_3}{2}) \times \left\langle 0 \left| \phi(z + s_1 + s_2 + s_3) \phi(z - \frac{l_1}{2}) \phi(x) \phi(z + s_1 - \frac{l_2}{2}) \phi(y) \phi(z + s_1 + s_2 - \frac{l_3}{2}) \right| 0 \right\rangle.
\]

(9)

Here, \(\tau(t)\) denotes the step function \(\tau(t) = 1\) for \(t > 0\) and \(\tau(t) = 0\) for \(t < 0\). There are \(6! = 720\) different contributions to (7) when interpreting the time-ordering in the Gell-Mann–Low formula as the name suggests. The time-ordering guarantees that causal processes only contribute to the \(S\)-matrix. Positive energy solutions propagate forward in time and negative energy solutions backward.

There exists a modification of (7), where the time-ordering is defined with respect to the interaction point:

\[
G'(x, y) = \int d^4 z \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{i s_i} \right) \tau(x^0 - z^0) \tau(z^0 - y^0) \times \left\langle 0 \left| \phi(x) \phi(z - \frac{l_1}{2}) \phi(z + s_1 - \frac{l_2}{2}) \phi(z + s_1 + s_2 - \frac{l_3}{2}) \phi(y) \right| 0 \right\rangle.
\]

(10)

There are now only \(3! = 6\) different contributions of this type. Since the individual fields are now (in most of the cases) at the wrong place with respect to the time-order, the interpretation (10) of the Gell-Mann–Low formula violates causality. Now both energy solutions propagate in any direction of time. There is, however, an argument in favour of (10): Contributions (2) to the Dyson series are precisely ordered with respect to the time
stamp of the interaction Hamiltonians. It does not matter how the time-dependence of
the interaction Hamiltonian is produced from the time-dependence of the constituents.

Since it is completely unclear how to derive the Gell-Mann–Low formula in the non-
commutative setting, we have no guidance so far whether (9) or (10) (or none of the two)
is the correct one. The authors of [10] do not mention (9). They use the exponential
form of the ⋆-product, which is a formal translation of a correct formula in momentum
space, but which might be dangerous in position space. See also the discussion in [12].

Apart from avoiding subtleties with generalised derivatives, the use of (9) instead of the
exponential form simplifies the calculations considerably.

3 The one-loop two-point function in “interaction point time-ordered perturbation theory”

Since the calculation of the sum of terms (10) is (at least) by a factor of 120 simpler than
the calculation of the sum of terms (9), we evaluate in this paper the one-loop two-point
function interpreted according to (10). The name “time-ordered perturbation theory”
used in [10] does not seem appropriate to us, because the previous discussion shows that
this approach is precisely not based on time-ordering. We should better call it “interaction
point time-ordered perturbation theory”, and use the symbol \( T_I \) instead of the true causal
time-ordering \( T \). The calculation can be shortened considerably when starting directly
from the Feynman rule (39) derived in section 4. But without computing at least one
example one has little understanding for the starting point (34) of the general derivation.

With these remarks, the entire contribution to the one-loop two-point function in noncommutative \( \phi^4 \) theory reads

\[
G(x, y) = \frac{g}{4!} \int d^4 z \left< 0 \left| T_I \left( \phi(x) \phi(y) \left( \phi \star \phi \star \phi \star \phi \right)(z) \right) \right| 0 \right>
\]

\[
= \frac{g}{4!} \int d^4 z \left( \tau(x^0 - y^0) \tau(y^0 - z^0) \left< 0 \left| \phi(x) \phi(y) \left( \phi \star \phi \star \phi \star \phi \right)(z) \right| 0 \right>
+ \tau(x^0 - z^0) \tau(z^0 - y^0) \left< 0 \left| \phi(x) \left( \phi \star \phi \star \phi \star \phi \right)(z) \phi(y) \right| 0 \right>
+ \tau(y^0 - x^0) \tau(x^0 - z^0) \left< 0 \left| \phi(y) \left( \phi \star \phi \star \phi \star \phi \right)(z) \phi(x) \right| 0 \right>
+ \tau(z^0 - x^0) \tau(y^0 - x^0) \left< 0 \left| \left( \phi \star \phi \star \phi \star \phi \right)(z) \phi(x) \phi(y) \right| 0 \right>
+ \tau(z^0 - y^0) \tau(y^0 - x^0) \left< 0 \left| \left( \phi \star \phi \star \phi \star \phi \right)(z) \phi(y) \phi(x) \right| 0 \right> \right),
\]

(11)

with the ⋆-product given in (9). We follow the usual strategy to obtain in the end the
amputated on-shell momentum-space one-loop two-point function. We insert (9) into (11)
and split each field (at given position \( x \)) \( \phi(x) = \phi^+(x) + \phi^-(x) \) into negative and positive

\footnote{The derivatives in the exponential form of the ⋆-product are generalised derivatives in the sense of
distribution theory, not ordinary derivatives. As such one cannot apply the naïve rules of differential
calculus. To make this transparent, write \( \phi(x + a) \phi(y) = \exp(a^\mu \partial^\mu) \phi(x) \phi(y) \), and hide the exponential
of the derivatives in the definition of the product. It would be completely wrong to use the step function
\( \tau(x^0 - y^0) \) or \( \tau(y^0 - x^0) \) for the product \( \phi(x + a) \phi(y) \). One of the authors (R.W.) is grateful to Edwin
Langmann for explaining this matter to him.}
frequency parts, which have the property
\[ \phi^-(x)|0\rangle = 0 , \quad \langle 0|\phi^+(x) = 0 . \] (12)

Our conventions are listed in the Appendix, they are opposite to [10]. It is convenient now to commute the $\phi^-$ to the right and the $\phi^+$ to the left, using the commutation rule
\[ [\phi^-(x_1), \phi^+(x_2)] = D^+(x_1 - x_2) , \] (13)
where $D^+(x_1 - x_2)$ is the positive frequency propagator
\[ D^+(x_1 - x_2) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-ik(x_1 - x_2)} , \quad \omega_k = \sqrt{k^2 + m^2} , \] (14)
and $k^\mu_\pm = (+\omega_k, -\vec{k})$ the positive energy on-shell four-momentum. A lengthy but completely standard computation yields
\[
G(x, y) = G^{\text{con}}(x, y) + G^{\text{disc}}(x, y) ,
\] (15)
\[
G^{\text{disc}}(x, y) = \frac{g}{4!} \int d^4z \prod_{i=1}^3 \left( \frac{d^4l_i}{(2\pi)^4} e^{il_i s_i} \right) \left\{ \left( \tau(x^0 - y^0)\tau(y^0 - z^0)D^+(x-y) + \tau(x^0 - z^0)\tau(z^0 - y^0)D^+(x-y) \right) \right. \\
+ (x \leftrightarrow y) \left. \left\{ \left( D^+(-\frac{1}{2}\vec{l}_2-s_2+\frac{1}{2}\vec{l}_3)D^+(-\frac{1}{2}\vec{l}_1-s_1-s_2-s_3) \\
+ D^+(-\frac{1}{2}\vec{l}_1-s_1-s_2+\frac{1}{2}\vec{l}_3)D^+(-\frac{1}{2}\vec{l}_2-s_2-s_3) \\
+ D^+(-\frac{1}{2}\vec{l}_2-s_2-s_3)D^+(x-z+\frac{1}{2}\vec{l}_1)D^+(y-z-s_1-s_2+\frac{1}{2}\vec{l}_3) \\
+ D^+(-\frac{1}{2}\vec{l}_3-s_3)D^+(x-z+\frac{1}{2}\vec{l}_1)D^+(y-z-s_1-s_2-\frac{1}{2}\vec{l}_2) \right) \right. \\
+ (x \leftrightarrow y) \left. \right\} \right\}
\] (16)
\[
G^{\text{con}}(x, y) = \frac{g}{4!} \int d^4z \prod_{i=1}^3 \left( \frac{d^4l_i}{(2\pi)^4} e^{il_i s_i} \right) \left\{ \left( \tau(x^0 - y^0)\tau(y^0 - z^0) \right. \right.
\times \left. \left\{ \left( D^+(-\frac{1}{2}\vec{l}_1-s_1-s_2-s_3)D^+(x-z-s_1+\frac{1}{2}\vec{l}_2)D^+(y-z-s_1-s_2+\frac{1}{2}\vec{l}_3) \\
+ D^+(-\frac{1}{2}\vec{l}_1-s_1-s_2+\frac{1}{2}\vec{l}_3)D^+(x-z-s_1+\frac{1}{2}\vec{l}_2)D^+(y-z-s_1-s_2-s_3) \\
+ D^+(-\frac{1}{2}\vec{l}_2-s_2-s_3)D^+(x-z+\frac{1}{2}\vec{l}_1)D^+(y-z-s_1-s_2+\frac{1}{2}\vec{l}_3) \\
+ D^+(-\frac{1}{2}\vec{l}_3-s_3)D^+(x-z+\frac{1}{2}\vec{l}_1)D^+(y-z-s_1-s_2-\frac{1}{2}\vec{l}_2) \right) \right.
+ (x \leftrightarrow y) \left. \right\} \right\}
\]
We have to take the connected part $G^\text{con}(x, y)$ only. Inserting (14) we can perform the $s_i$-integrations, which result in $\delta$-distributions in $l_i$, so that the $l_i$ integration can be performed as well. The result has a remarkably compact form:

\[
G^\text{con}(x, y) = \frac{g}{12} \int d^4z \int \frac{d^3k_1}{(2\pi)^32\omega_{k_1}} \int \frac{d^3k_2}{(2\pi)^32\omega_{k_2}} \cos(\frac{l_1^+ k_1^+}{2} + \frac{l_2^+ k_2^+}{2})
\times \left\{ \tau(x^0 - y^0)\tau(y^0 - z^0)e^{-ik_1^+(z-x)}e^{-ik_2^+(y-z)} I^{++}(k_1^+, k_2^+) + \tau(y^0 - x^0)\tau(x^0 - z^0)e^{-ik_1^+(z-x)}e^{-ik_2^+(y-z)} I^{--}(k_1^+, k_2^+) + \tau(x^0 - z^0)\tau(z^0 - y^0)e^{-ik_1^+(z-x)}e^{-ik_2^+(y-z)} I^{--}(k_1^+, k_2^+) + \tau(y^0 - z^0)\tau(z^0 - x^0)e^{-ik_1^+(z-x)}e^{-ik_2^+(y-z)} I^{--}(k_1^+, k_2^+) + \tau(z^0 - x^0)\tau(x^0 - y^0)e^{-ik_1^+(z-x)}e^{-ik_2^+(y-z)} I^{--}(k_1^+, k_2^+) + \tau(z^0 - y^0)\tau(y^0 - x^0)e^{-ik_1^+(z-x)}e^{-ik_2^+(y-z)} I^{--}(k_1^+, k_2^+) \right\},
\]

where $(\tilde{\kappa}^+) = (\kappa^+)\theta^{\mu\nu}$ and

\[
I^{\kappa\lambda}(k_1^+, k_2^+) = \int \frac{d^3k}{(2\pi)^32\omega_k} \left( 3 + e^{ik_1^+\tilde{\kappa}^+ + ik_2^+\tilde{\kappa}^+} + e^{ik_1^+\tilde{\kappa}^+} + e^{ik_2^+\tilde{\kappa}^+} \right), \quad \kappa, \lambda = \pm 1.
\]
Next we pass to the Fourier-transformed Green’s function

\[ G^{\text{con}}(p, q) = \int d^4x \, d^4y \, e^{ipx+igy} \, G^{\text{con}}(x, y). \]  

We insert the identity (use the residue theorem)

\[ \tau(x^0 - y^0) = \lim_{\delta \to 0} \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \, e^{-it(x^0-y^0)} \]

and perform the integrations over \( x, y, z \). The result is a host of \( \delta \)-distributions, which allow us to integrate over \( \vec{k}_1, \vec{k}_2, t_1, t_2 \):

\[
G^{\text{con}}(p, q) = \lim_{\delta_1, \delta_2 \to 0} \frac{g}{12} \frac{\left( \frac{i}{2\pi} \right)^2}{(2\pi)^3 2\omega_{k_1}^2} \int d^3k_1 \int d^3k_2 \cos(\frac{1}{2}p^+ \vec{k}^+) \times \\
\left( e^{i(x^0(p_0 - t_1 - \omega_{k_1}) + y^0(q_0 - t_1 - \omega_{k_2}) + \vec{x}(t_1 - \vec{k} - \vec{p} - \vec{y}(\vec{k}_2 - \vec{q}) - \vec{z}(\vec{k}_1 + \vec{k}_2)))} \right) I^{++}(p^+, k^+) \\
+ e^{i(x^0(p_0 + t_2 - \omega_{k_2}) + y^0(q_0 + t_2 - \omega_{k_1}) + \vec{x}(t_2 - \vec{k} - \vec{p} - \vec{y}(\vec{k}_2 - \vec{q}) - \vec{z}(\vec{k}_1 + \vec{k}_2)))} I^{++}(p^+, k^+) \\
+ e^{i(x^0(p_0 - t_1 - \omega_{k_1}) + y^0(q_0 + t_2 + \omega_{k_2}) + \vec{x}(t_1 - \vec{k} - \vec{p} - \vec{y}(\vec{k}_2 - \vec{q}) + \vec{z}(\vec{k}_1 - \vec{k}_2)))} I^{++}(p^-, k^-) \\
+ e^{i(x^0(p_0 + t_2 + \omega_{k_1}) + y^0(q_0 - t_1 + \omega_{k_2}) + \vec{x}(t_2 - \vec{k} - \vec{p} + \vec{y}(\vec{k}_2 - \vec{q}) + \vec{z}(\vec{k}_1 - \vec{k}_2)))} I^{++}(p^-, k^-) \\
+ e^{i(x^0(p_0 - t_1 - \omega_{k_1}) + y^0(q_0 + t_2 + \omega_{k_2}) - \vec{x}(t_1 + \omega_{k_1} + \omega_{k_2}) - \vec{z}(\vec{k}_1 + \vec{p} - \vec{y}(\vec{k}_2 + \vec{q}) - \vec{z}(\vec{k}_1 + \vec{k}_2)))} I^{--}(p^+, k^+) \\
+ e^{i(x^0(p_0 + t_2 + \omega_{k_1}) + y^0(q_0 - t_1 + \omega_{k_2}) - \vec{x}(t_2 + \omega_{k_1} + \omega_{k_2}) - \vec{z}(\vec{k}_1 + \vec{p} - \vec{y}(\vec{k}_2 + \vec{q}) - \vec{z}(\vec{k}_1 + \vec{k}_2)))} I^{--}(p^+, k^+) \\
+ e^{i(x^0(p_0 + t_2 - \omega_{k_1}) + y^0(q_0 + t_2 + \omega_{k_2}) - \vec{x}(t_1 + \omega_{k_1} + \omega_{k_2}) - \vec{z}(\vec{k}_1 + \vec{p} - \vec{y}(\vec{k}_2 + \vec{q}) - \vec{z}(\vec{k}_1 + \vec{k}_2)))} I^{--}(p^-, k^-) \\
+ e^{i(x^0(p_0 - t_1 + \omega_{k_1}) + y^0(q_0 + t_2 + \omega_{k_2}) - \vec{x}(t_2 + \omega_{k_1} + \omega_{k_2}) - \vec{z}(\vec{k}_1 + \vec{p} - \vec{y}(\vec{k}_2 + \vec{q}) - \vec{z}(\vec{k}_1 + \vec{k}_2)))} I^{--}(p^-, k^-) \\
= \lim_{\delta_1, \delta_2 \to 0} \frac{g}{12} (2\pi)^4 \delta(p + q) \times \\
\left( \frac{1}{p_0 - \omega_p + i\delta_1} \frac{1}{\omega_p + \omega_q - i\delta_2} \cos(\frac{1}{2}p^+ \vec{q}^+) I^{++}(p^+, q^+) \\
+ \frac{1}{q_0 - \omega_q + i\delta_1} \frac{1}{\omega_p + \omega_q - i\delta_2} \cos(\frac{1}{2}p^+ \vec{q}^+) I^{++}(p^+, -q^+) \\
+ \frac{1}{p_0 - \omega_p + i\delta_1} \frac{1}{q_0 + \omega_q - i\delta_2} \cos(\frac{1}{2}p^+ \vec{q}^+) I^{--}(p^+, q^+) \\
+ \frac{1}{q_0 - \omega_q + i\delta_1} \frac{1}{p_0 + \omega_p - i\delta_2} \cos(\frac{1}{2}p^+ \vec{q}^+) I^{--}(p^+, -q^+) \\
+ \frac{1}{\omega_p + \omega_q - i\delta_1} \frac{1}{q_0 - \omega_q + i\delta_2} \cos(\frac{1}{2}p^+ \vec{q}^+) I^{--}(p^+, q^+) \\
+ \frac{1}{\omega_p + \omega_q - i\delta_1} \frac{1}{-q_0 - \omega_q + i\delta_2} \cos(\frac{1}{2}p^+ \vec{q}^+) I^{--}(p^+, -q^+) \right). \]  

Note the appearance of \( \delta(p+q) \) implementing conservation of the four-momentum (translation invariance). We have used \( \omega_{\pm k} = \omega_k \).
Following [10] we amputate the external legs by multiplying (22) by the inverse propagators $-i(p_0^2 - \omega_p^2)$ and $-i(q_0^2 - \omega_q^2)$. Using $(\pm k)^+ = \pm k^\pm$, in particular the identity

$$
\mathcal{I}^{\pm\pm}((\pm p)^+, (\pm q)^+) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( 3 + e^{i\hat{p}^+ k_+^+ + i\hat{q}^+ k_+^+} + e^{i\hat{p}^- k_-^+ + i\hat{q}^- k_-^+} \right) \equiv \mathcal{I}(p^+, q^+) \; ,
$$

we obtain

$$(2\pi)^4 \delta(p + q) \Gamma(p, q) = -(p_0^2 - \omega_p^2)(q_0^2 - \omega_q^2)G(p, q)$$

$$= - \lim_{\delta_1, \delta_2 \to 0} \frac{g}{12} (2\pi)^4 \delta(p + q) (p_0^2 - \omega_p^2)(q_0^2 - \omega_q^2)$$

$$\times \left[ \frac{1}{p_0 - \omega_p + i\delta_1 - \omega_p + \omega_q - i\delta_2} \cos\left( \frac{1}{2} p^+ (\hat{q}^+) \right) \mathcal{I}(p^+, q^+) ight. + \frac{1}{q_0 - \omega_q + i\delta_1 - \omega_p + \omega_q - \omega_p} \cos\left( \frac{1}{2} p^+ (\hat{q}^+) \right) \mathcal{I}(p^+, q^-) + \frac{1}{p_0 - \omega_p + i\delta_1 - q_0 + \omega_q - i\omega_q} \cos\left( \frac{1}{2} p^+ (\hat{q}^+) \right) \mathcal{I}(p^- , q^+)$$

$$+ \frac{1}{q_0 - \omega_q + i\delta_1 - \omega_p + i\omega_q} \cos\left( \frac{1}{2} p^+ (\hat{q}^+) \right) \mathcal{I}(p^- , q^-) + \left. \frac{1}{\omega_p + \omega_q - i\delta_1 - q_0 - \omega_q + i\omega_q} \cos\left( \frac{1}{2} p^+ (\hat{q}^+) \right) \mathcal{I}(p^- , q^-) \right) .$$

Taking on-shell external momenta $p_0 = \omega_p$ and $q_0 = -\omega_q$ there survives a single term (the third one):

$$
\Gamma(p^+, q^-) = \lim_{p_0 \to \omega_p , q_0 \to -\omega_q} \Gamma(p, q) = \frac{g}{12} \cos\left( \frac{1}{2} p^+(\hat{q}^-) \right) \mathcal{I}(p^+, q^-)$$

$$= \frac{g}{12} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( 4 + 2 \cos(k^+ \hat{p}^+) \right) .$$

In the last line we have used momentum conservation $p^+ = -q^-$ and the skew-symmetry of $\theta$. The remaining integral over $k$ consists of a planar $\theta$-independent part and a non-planar $\theta$-dependent part (the cosine). The planar part coincides (up to a factor $\frac{2}{3}$) with the commutative result, it is divergent and to be renormalised as usual by multiplicative renormalisation (or better completely removed by normal ordering).

To compute the non-planar part, first note that

$$
\cos(k^+ \hat{p}^+) = \cos(\omega_k \hat{p}_0 - \hat{k} \hat{p}) = \cos(\omega_k \hat{p}_0) \cos(\hat{k} \hat{p}) + \sin(\omega_k \hat{p}_0) \sin(\hat{k} \hat{p}) \; ,
$$

where $\hat{p}_0 := (\hat{p}^+)_0$ and $\hat{p} = \hat{p}^+$. The uneven sine-term will drop under the integral. Using
the residue theorem we have

\[
e^{i \omega_0 \tilde{p}_0} \overline{2 \omega_k} = \begin{cases} \lim_{\epsilon \to 0} \frac{1}{2 \pi i} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k + i\epsilon)(k_0 - \omega_k + i\epsilon)} & \text{for } \tilde{p}_0 > 0, \\ \lim_{\epsilon \to 0} \frac{1}{2 \pi i} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k - i\epsilon)(k_0 - \omega_k + i\epsilon)} & \text{for } \tilde{p}_0 < 0, \end{cases} \tag{27}
\]

\[
e^{-i \omega_0 \tilde{p}_0} \overline{2 \omega_k} = \begin{cases} \lim_{\epsilon \to 0} \frac{1}{2 \pi i} \int_{-\infty}^{\infty} dk_0 \frac{-e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k - i\epsilon)(k_0 - \omega_k + i\epsilon)} & \text{for } \tilde{p}_0 > 0, \\ \lim_{\epsilon \to 0} \frac{1}{2 \pi i} \int_{-\infty}^{\infty} dk_0 \frac{-e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k + i\epsilon)(k_0 - \omega_k - i\epsilon)} & \text{for } \tilde{p}_0 < 0. \end{cases} \tag{28}
\]

Inserting (26), (27) and (28) into (25) we obtain for the non-planar graph

\[
\Gamma_{\text{non-planar}}(p^+, q^-) = \frac{2}{6} \int g \frac{d^3k}{(2\pi)^3} \cos(k_+ \tilde{p}) \cos(k_- \tilde{p}) = \lim_{\epsilon \to 0} \frac{g}{6} \int \frac{d^4k}{(2\pi)^4} \Re\left(\frac{i}{k_0^2 - (\tilde{k}^2 + m^2) + i\epsilon}\right) e^{-ik\tilde{p}} \cos(k_+ \tilde{p}) \cos(k_- \tilde{p}), \tag{29}
\]

independent of the sign of \(\tilde{p}_0\). The result (29) can obviously be obtained by Feynman rules, with the prescription that in non-planar tadpoles the propagator to use is the real part of the Feynman propagator. That real part is arithmetic mean of causal and acausal propagators. The observed acausality is no surprise, because according to (10) the interaction time-ordering \(T\) explicitly violates causality. As we shall see in section 4 the just given Feynman rule is true for tadpole lines only.

Apart from taking the real part, the evaluation of (29) coincides with the computation in the “naïve” Feynman graph approach. Let us nevertheless repeat the steps. We employ Zimmermann’s \(\epsilon\)-trick

\[
\frac{1}{k_0^2 - m^2 + i\epsilon} \mapsto \frac{1}{k_0^2 + \omega_k^2(\epsilon-\epsilon') - \epsilon'} = e^{-i\epsilon'} (\epsilon' - i)(k_0^2 + \omega_k^2(\epsilon - \epsilon' + i\epsilon'))^{-1}, \tag{30}
\]

the denominator of which has for \(\epsilon' < \epsilon\) a positive real part, which allows us to introduce a Schwinger parameter:

\[
\Gamma_{\text{non-planar}}(p^+, q^-) = \Re\left(\lim_{\epsilon \to 0, \epsilon' < \epsilon} \frac{i g}{6} \int \frac{d^4k}{(2\pi)^4} \int_0^{\infty} d\alpha \ e^{-\alpha((\epsilon'-i)k_0^2 + (\tilde{k}^2 + m^2)(\epsilon - \epsilon' + i\epsilon'))} e^{-ik_0 \tilde{p} + ik\tilde{p}}\right) = \Re\left(\lim_{\epsilon \to 0, \epsilon' < \epsilon} \frac{2ig}{3(4\pi)^2} \frac{1}{(\epsilon - \epsilon')\sqrt{2}} \int_0^{\infty} d\alpha \ e^{\frac{-\alpha^2}{4\alpha(\epsilon' - \epsilon) - 4\alpha(\epsilon - \epsilon' + i\epsilon')}} \alpha m^2 (\epsilon - \epsilon' + i\epsilon') \right) = -\Re\left(\frac{2g}{3(4\pi)^2} \sqrt{-\frac{m^2}{p_0^2}} K_1\left(\sqrt{\frac{m^2}{p_0^2} + (\epsilon - \epsilon')^2}\right)\right). \tag{31}
\]
We have used \( \int_0^\infty \frac{dz}{\alpha^*} \exp(-u\alpha - v/(4\alpha)) = 4\sqrt{(u/v)K_1(\sqrt{uv})} \) for \( \Re u > 0 \) and \( \Re v > 0 \).

In the particular case where the external momentum \( p \) is put on-shell, we have

\[
-p^2 = \tilde{p}^2 - p_0^2 = (\theta_{00} \sqrt{\tilde{p}^2 + m^2} + \theta_{ij} p^i)^2 - (\theta_{00} p^0)^2 \geq 0 ,
\]

because \( \tilde{p}^\mu \) has to be space-like or null as a vector which is orthogonal to the time-like vector \( p^\mu \). Thus, the projection onto the real part in (31) is superfluous, and (31) agrees exactly with the naïve Feynman rule computation of the sum of graphs

\[
P \quad \quad + \quad \quad P
\]

However, if these graphs appear as subgraphs in a bigger graph, the momentum \( p \) will be the off-shell momentum through a propagator, and the projection to the real part makes a difference.

4 The general case

The graph we have computed (for off-shell external momenta!) is very often made responsible for the so-called UV/IR mixing. In fact the situation is more complex, as it is very well described in [4]. The ultimate goal must be to derive the power-counting theorem for interaction point time-ordered perturbation theory (for noncommutative space and time). In a first step one has to derive graphical rules to assign an integral to a given graph.

Let us therefore consider the momentum integral for a general Feynman graph for a noncommutative \( \phi^4 \) theory. A given connected contribution to the \( E \)-point function at order \( V \) in the coupling constant has after performing the Wick contractions, insertion of the \( D^+ \) according to (14), integration over \( s_i \) and \( l_i \) appearing in (9) and insertion of step functions (21) the form

\[
G(x_1, \ldots, x_E) = \lim_{\epsilon \to 0} \int \prod_{v=1}^V \frac{g^4 d^4 z_v}{4!} \int \prod_{e=1}^E \frac{d^3 p_e}{(2\pi)^3 2\omega_{p_e}} \int \prod_{i=1}^I \frac{d^3 k_i}{(2\pi)^3 2\omega_{k_i}} \int \prod_{s=1}^{E+V-1} \frac{i dt_s}{(2\pi)(t_s+i\epsilon)}
\
\times \exp \left( -i \sum_{v=1}^V \sum_{s=1}^{E+V-1} V_{vs} z_v^0 t_s - i \sum_{e=1}^E \sum_{s=1}^{E+V-1} T_{es} x_e^0 t_s \right)
\
\times \exp \left( -i \sum_{v=1}^V z_v \left( \sum_{i=1}^I J_{vi} k_i^+ + \sum_{e=1}^E J_{ve} p_e^+ \right) \right) \exp \left( -i \sum_{e=1}^E \sigma_e p_e^+ x_e \right)
\
\times \exp \left( i g^{\mu\nu} \left( \sum_{i,j=1}^I I_{ij} k_i^{+\mu} k_j^{+\nu} + \sum_{i=1}^I \sum_{e=1}^E I_{ie} k_i^{+\mu} p_e^{+\nu} + \sum_{e,f=1}^E I_{ef} p_e^{+\mu} p_f^{+\nu} \right) \right) .
\]

There are \( E+V-1 \) step functions according to the time differences of the \( E \) external points \( x_e \) and the \( V \) interaction points \( z_v \). For each \( s \) there are two non-vanishing \( T_{ss} \), where these two indices * are either two indices \( e \), one index \( e \) and one index \( v \), or two indices \( v \). The \( T_{ss} \) for which the vertex * (\( z_v \) or \( x_e \)) is later equals +1, the other one −1.
This gives the second line in (34). An external point \( x_e \) is linked via the external line with momentum \( p_e \) to exactly one vertex \( z_v \), i.e. for given \( e \) there is a single non-vanishing \( J_{ve} \). For our \( \phi^4 \) theory there are \( I = 2V - \frac{1}{2}E \) internal lines (\( E \) is even) with momentum \( k_i \) which link a vertex \( z_v \) to another vertex \( z_{v'} \). Thus, if \( v \neq v' \) (no tadpoles) for given \( i \) there are two non-vanishing \( J_{vi} \), whereas for \( v = v' \) we have \( J_{vi}k_i = 0 \). We orient the internal and external lines forward in time. Then, the incidence matrices \( J_{ni}, J_{ve} \) equal \(-1\) if the line leaves \( v \) and \(+1\) if the line arrives at \( v \). Similarly, \( \sigma_e = -1 \) if the line \( e \) leaves \( x_e \) and \( \sigma_e = +1 \) if the line \( e \) arrives at \( x_e \). The matrices \( I_{ij}, I_{ie}, I_{ef} \) are the intersection matrices \([13, 14]\), which instead of the Euclidian rosette construction are in IPTO obtained as follows: According to the definition (6) of the \( \ast \)-product, write at each vertex \( v \) the four fields in (6) as a time-sequence where \( z_v - \frac{1}{2}\tilde{t}_1 \) is the latest point and \( z_v + s_1 + s_2 + s_3 \) the earliest point\(^2\), irrespective of the actual time-order of these four points. Connect these points with vertices \( y_1, y_2, y_3, v_4 \) according to the following picture:

![Diagram](image)

The phase factor produced by the \( s_n \) and \( l_n \) variables is then given by

\[
\int \prod_{n=1}^{3} \left( d^4s_n \frac{d^4l_n}{(2\pi)^4} \exp(is_n l_n) \right) \times \exp \left( -ik_1^+(s_1+s_2+s_3)J_{v1} - ik_2^+(s_1+s_2-\frac{1}{2}\tilde{t}_3)J_{v2} - ik_3^+(s_1-\frac{1}{2}\tilde{t}_2)J_{v3} - ik_4^+(-\frac{1}{2}\tilde{t}_1)J_{v4} \right)
\]

\[
= \exp \left( \frac{i}{2} \sum_{j=2}^{4} \sum_{i=1}^{j-1} k_{i,j}^+ J_{vi} k_{j}^+ J_{vj} \right) \equiv \exp \left( \frac{i}{2} \sum_{i,j=1}^{4} \tau_{ij}^v k_{i}^+ J_{vi} k_{j}^+ J_{vj} \right). \tag{36}
\]

We have to define \( \tau_{ij}^v = +1 \) if the line \( i \) is connected to an “earlier” field \( \phi \) in the vertex \( v \) than the line \( j \), otherwise \( \tau_{ij}^v = 0 \). Summing over all vertices and distinguishing external and internal lines, we are led to the following identification in (34):

\[
I_{ij} = \frac{1}{2} \sum_{v=1}^{V} \tau_{ij}^v J_{vi} J_{vj}, \quad I_{ie} = \frac{1}{2} \sum_{v=1}^{V} (\tau_{ie}^v - \tau_{ei}^v) J_{vi} J_{ve}, \quad I_{ef} = \frac{1}{2} \sum_{v=1}^{V} \tau_{ef}^v J_{ve} J_{vf}. \tag{37}
\]

Once more we notice the enormous computational advantage of using the \( \ast \)-product in the form \([4]\).

\(^2\)By the way, this defines the time-orientation of tadpole lines.
The vectors $\vec{q}_e$ are always outgoing from internal vertices. There are now $E+V$ time-component $\delta$-functions involving the $E+V-1$ integration variables $t_s$, after integration over which there is one remaining $\delta$-function for the energy conservation $\delta(\sum_{e=1}^{E} q^0_e)$. We multiply (38) by the inverse propagators $\prod_{e=1}^{E} (-i)(q^2_e-\omega^2_q)$, remove $(2\pi)^3\delta^4(\sum_{e=1}^{E} q_e)$ by convention and put $q^0_e = \sigma_e \omega_q e$. There is a non-vanishing contribution only if the external vertices $x_e$ are either before or after the internal vertices $z_i$. Defining a time-order of vertices $v' < v$ if $z^0_{v'} < z^0_v$, we finally get

$$
\Gamma(q^\sigma_1, \ldots, q^\sigma_E) = \lim_{\epsilon \to 0} \frac{g^V}{4!V} \int \prod_{i=1}^{I} \frac{d^3k_i}{(2\pi)^32\omega_q_e} \prod_{e=1}^{E} \frac{i(2\pi)^3\delta^3(\sum_{i=1}^{I} J_{vi,k_i} + \sum_{e=1}^{E} J_{ve}\sigma_e \vec{q}_e)}{\sum_{v' < v} \left( \sum_{i=1}^{I} J_{vi}\omega_k_i + \sum_{e=1}^{E} J_{ve}\omega_q_e \right) + i\epsilon} \times \exp \left( i\theta^{\mu\nu} \left( \sum_{i,j=1}^{I} I_{ij} k^+_{i,j,k_{j,v}} + \sum_{i=1}^{I} \sum_{e=1}^{E} I_{ie}\sigma_e k^+_{i,\mu,q^\mu_{e,\nu}} + \sum_{e,f=1}^{E} I_{ef}\sigma_f q^\sigma_{e,\mu}\vec{q}_{f,v} \right) \right).
$$

(39)
We now see that the graph we have computed was very special. Because of \( V = 1 \) the denominator in (39) was absent so that the integration over the propagator momentum \( k_1 \) was identical to the naïve Feynman graph computation. This remains true for all tadpole lines \( i \), because for them \( J_{vi} k_1^+ = 0 \) for all \( v \). For internal lines connecting points in different vertices we need new techniques to perform the integrations.

5 Summary

As a warm-up for the general treatment we have computed the one-loop two-point function for a \( \phi^4 \) theory on noncommutative space and time in the framework of “interaction point time-ordered perturbation theory” . The calculation is based on free fields (on the mass shell), but at the end the loop momenta become general four-momenta. Our final result (for that graph) agrees with a Feynman graph computation, provided that \textit{one assigns to the internal line the real part of the Feynman propagator}. This can be understood as the inclusion of acausal processes in the \( S \)-matrix, because IPTO explicitly violates causality. One may speculate that the true time-ordering of the \(*\)-product (9) will produce the naïve Feynman rules involving the standard causal Feynman propagator in non-planar graphs. This approach was shown to violate unitarity of the \( S \)-matrix. We have thus to decide whether we prefer to give up (micro-) causality or unitarity in noncommutative field theories\(^3\).

Next we have derived the Feynman rules (39) for general Green’s functions. Power-counting tells us that (39) is expected to diverge if there are subgraphs with \( E \leq 4 \) external lines. If there are non-planar divergent graphs, it is not possible to absorb the divergences by local (hence planar) counterterms. One has therefore to analyse whether the oscillating phases render the power-counting divergent integral finite. This requires to develop techniques for the computation of (39) in analogy to the treatment of the Euclidian case in [4]. Of urgent interest are the evaluations of the two-loop two-point function and the one-loop four-point function.

Acknowledgement

We would like to thank the Erwin Schrödinger Institute in Vienna, where this paper was finished, for hospitality and for support by the programme “Noncommutative geometry and quantum field theory”.

A Conventions for Fock space and propagators

To fix our notation and for convenience we list our conventions for free fields and propagators \( D^\pm(x – y) \) and \( \Delta_F(x – y) \).

The free fields (solutions of the homogeneous Klein-Gordon equation) are mode-

\(^3\) Assuming space-time noncommutativity to be a model of quantum-gravitational background effects \( \theta \sim l^2_{\text{Planck}} \), one can view this abandonment of causality in the \(*\)-product as its breakdown at the Planck scale.
decomposed into negative ($\phi^+$) and positive ($\phi^-$) frequency parts $\phi(x) = \phi^+(x) + \phi^-(x)$,

$$\phi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} a_k^- e^{-ix\cdot k^+} , \quad \phi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} a_k^+ e^{ix\cdot k^+} , \quad (40)$$

with the ladder operators $a^-, a^+$ obeying

$$a_k^- |0\rangle = 0 , \quad (0|a_k^+ = 0 , \quad [a_k^-, a_k^+] = \delta^3(\vec{p} - \vec{q}) . \quad (41)$$

With these definitions we obtain for the two-point vacuum expectation values and the commutators of positive and negative frequency parts

$$\langle 0|\phi(x)\phi(y)|0\rangle = [\phi^-(x), \phi^+(y)] = D^+(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-i(x-y)\cdot k^+} ,$$

$$\langle 0|\phi(y)\phi(x)|0\rangle = -[\phi^+(x), \phi^-(y)] = D^-(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{i(x-y)\cdot k^+} , \quad (42)$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and $(k^+) = (\pm \omega_k, \vec{k})$. For the Feynman propagator we hence find

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-i(x-y)k}}{k^2 - m^2 + i\varepsilon} , \quad (43)$$

and for its complex conjugate

$$\langle 0|\tau(y^0 - x^0)\phi(x)\phi(y) + \tau(x^0 - y^0)\phi(y)\phi(x)|0\rangle = \Delta_F^*(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{-i(x-y)k}}{k^2 - m^2 - i\varepsilon} . \quad (44)$$

These propagators are solutions of the homogeneous and inhomogeneous wave equation, respectively:

$$(\partial_\mu \partial^\mu - m^2)\Delta_F(x-y) = 0 , \quad (\partial_\mu \partial^\mu - m^2)\Delta^*_F(x-y) = -i\delta^4(x-y) . \quad (45)$$

References

[1] K. G. Wilson and J. B. Kogut, “The Renormalization Group And The Epsilon Expansion,” Phys. Rept. 12 (1974) 75.

[2] S. Minwalla, M. Van Raamsdonk and N. Seiberg, “Noncommutative perturbative dynamics,” JHEP 0002 (2000) 020 [arXiv:hep-th/9912072].

[3] A. Matusis, L. Susskind and N. Toumbas, “The IR/UV connection in the non-commutative gauge theories,” JHEP 0012 (2000) 002 [arXiv:hep-th/0002075].

[4] I. Chepelev and R. Roiban, “Convergence theorem for non-commutative Feynman graphs and renormalization,” JHEP 0103 (2001) 001 [arXiv:hep-th/0008090].

[5] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, “On the unitarity problem in space/time noncommutative theories,” Phys. Lett. B 533 (2002) 178 [arXiv:hep-th/0201222].
[6] J. Gomis and T. Mehen, “Space-time noncommutative field theories and unitarity,” Nucl. Phys. B 591 (2000) 265 [arXiv:hep-th/0005129].

[7] K. Osterwalder and R. Schrader, “Axioms For Euclidean Green’s Functions,” Commun. Math. Phys. 31 (1973) 83.

[8] S. Doplicher, K. Fredenhagen and J. E. Roberts, “The Quantum Structure Of Space-Time At The Planck Scale And Quantum Fields,” Commun. Math. Phys. 172 (1995) 187.

[9] C. h. Rim and J. H. Yee, “Unitarity in space-time noncommutative field theories,” [arXiv:hep-th/0205193]

[10] Y. Liao and K. Sibold, “Time-ordered perturbation theory on noncommutative spacetime: Basic rules,” [arXiv:hep-th/0205269]

[11] Y. Liao and K. Sibold, “Time-ordered perturbation theory on noncommutative spacetime. II. Unitarity,” [arXiv:hep-th/0206011]

[12] R. Wulkenhaar, “Quantum field theories on noncommutative $\mathbb{R}^4$ versus $\theta$-expanded quantum field theories,” [arXiv:hep-th/0206018]

[13] T. Filk, “Divergencies In A Field Theory On Quantum Space,” Phys. Lett. B 376 (1996) 53.