Hypercyclic Differentiation Operators

Richard Aron and Juan Bès

Abstract. A classical theorem due to G. D. Birkhoff states that there exists an entire function whose translates approximate any given entire function, as accurately as desired, over any ball of the complex plane. We show this result may be generalized to the space $H_{bc}(E)$ of entire functions of compact bounded type defined on a Banach space $E$ with separable dual.

1. Introduction.

The Invariant Subspace Problem asks whether every bounded linear operator on an infinite dimensional Banach space admits a non-trivial, closed invariant subspace, that is, a closed linear manifold which is neither $\{0\}$ nor the whole space and which is mapped into itself by the operator. A related question was raised by M. Edelstein and H. Radjavi: Does every such operator have a non-trivial closed invariant set? (18, p. 1)

P. Enflo created an operator on a (non-Hilbertian) Banach space without non-trivial closed subspaces, to show the answer is negative in general (13). Later, C. Read constructed an operator on $l_1$ with the same property, and also an operator on a Banach space admitting no non-trivial closed invariant subsets (24, 25). The Invariant Subspace Problem remains open for the case in which the Banach space is a Hilbert space.

The notions of cyclic and hypercyclic vectors arise naturally in this study of invariant subspaces and subsets. Suppose $T$ is a continuous linear operator on an $F$-space (i.e., a complete, linear metric space) $X$. A vector $x_0 \in X$ is cyclic for $T$ provided the linear span of the orbit

$$\text{Orb}\{x_0, T\} = \{x_0, Tx_0, T^2x_0, \ldots\}$$

is dense in $X$. If the orbit itself is dense, the operator $T$ and associated vector $x_0$ are called hypercyclic. In this way, an operator $T$ will lack non-trivial, closed invariant subspaces (subsets) if and only if all the non-zero vectors are cyclic (hypercyclic) for $T$.

One of the roots of the modern study of hypercyclicity comes from an intriguing observation of G.D. Birkhoff concerning the orbits of translation operators acting on the space of entire functions (7). Let $H(\mathbb{C})$ denote the space of entire functions of one
complex variable, endowed with the topology of uniform convergence on compact subsets of the plane, and let \( \tau_b : H(\mathbb{C}) \rightarrow H(\mathbb{C}) \) be the operator of “translation by the complex number \( b \)” defined by

\[
\tau_b(f)(z) := f(z + b) \quad (f \in H(\mathbb{C}), z \in \mathbb{C})
\]

Birkhoff’s theorem asserts that if \( b \neq 0 \), then the operator \( \tau_b \) is hypercyclic.

In other words, Birkhoff’s theorem gives a “universal” entire function which, over any compact set, has translates that approximate any entire function as accurately as desired. In the language of dynamics, it states that each translation operator is topologically transitive on \( H(\mathbb{C}) \), and this is a major step in proving that these operators are actually chaotic in one of the commonly accepted senses (cf. [13], section 6).

Birkhoff’s operator may be viewed in several different contexts. As a translation operator, W. Seidel and J. Walsh provided a non-Euclidean version of Birkhoff’s theorem in the space of holomorphic functions on the unit disc, by replacing translation by certain conformal disc automorphisms [27].

It is essentially the only hypercyclic composition operator on \( H(\mathbb{C}) \). Given \( \varphi \in H(\mathbb{C}) \) fixed, the operator \( f \mapsto f \circ \varphi \) on \( H(\mathbb{C}) \) is hypercyclic if and only if \( \varphi(z) = z + b \) for some non-zero complex number \( b \) ([3], Corollary 2.3). However, in other spaces such as the Hardy space \( H^2 \) the situation is quite different. P. Bourdon and J. Shapiro have studied the cyclic and hypercyclic behaviour of composition operators in this space, obtaining complete results for linear fractional maps of the unit disc into itself, and using these maps as “models” to extend this classification to a large class of composition operators ([9], [28]).

Birkhoff’s translation has also been regarded as a differentiation operator. That is, \( \tau_b = e^{bD} \), so that for every \( f \in H(\mathbb{C}) \),

\[
\tau_b(f) = \sum_{n=0}^{\infty} \frac{b^n}{n!} D^n f \quad \text{(convergence in } H(\mathbb{C}))
\]

From this perspective, G. R. MacLane showed in 1952 that the differentiation operator

\[
f \mapsto Df
\]

is also hypercyclic on \( H(\mathbb{C}) \). Thus, MacLane’s theorem assures that there exists an entire function whose sequence of derivatives has every entire function as a limit point ([21], Theorem 6).

In 1991, a beautiful generalization of Birkhoff’s and MacLane’s theorems was provided by G. Godefroy and J. Shapiro ([13], Thm 5.1):

**Theorem 1.1. (Godefroy-Shapiro)** Let \( \Phi(z) = \sum_{\alpha \geq 0} c_{\alpha} z^{\alpha} \) be a non-constant, entire function on \( \mathbb{C}^N \) of exponential type. Then the operator

\[
f \mapsto \Phi(D) \sum_{\alpha \geq 0} c_{\alpha} D^{\alpha} f \quad f \in H(\mathbb{C}^N)
\]

is hypercyclic.
Moreover, they showed that a continuous linear operator \( T \) on \( H(\mathbb{C}^N) \) commutes with translations if and only if it commutes with the differentiation operators \( \frac{\partial}{\partial z_k} \) \((1 \leq k \leq N)\), and if and only if it is of the form \( T = \Phi(D) \), for some \( \Phi \in H(\mathbb{C}^N) \) of exponential type (\([15]\), Prop. 5.2).

In the next section we will consider Theorem 1.1 for the case \( N = 1 \), and show how it may be generalized by replacing \( H(\mathbb{C}) \) by the space \( H_{bc}(E) \) of complex valued, entire functions on \( E \) of compact type that are bounded on bounded subsets of \( E \). In particular, an extension of Birkhoff’s theorem to all such spaces will be obtained.

Our proof involves a technique that has become standard in this area, the so-called Hypercyclicity Criterion. In 1982, C. Kitai isolated conditions that ensure an operator to be hypercyclic (\([18]\), Thm 1.4). This result was never published, and it was later rediscovered in a broader form by R. Gethner and J. Shapiro (\([14]\), Thm 2.2), who used it to unify the previously mentioned theorems of Birkhoff, MacLane, and Seidel and Walsh, among others. It may be stated as follows:

**Theorem 1.2. (Kitai - Gethner - Shapiro)** Let \( X \) be an \( F \)-space, and \( T : X \to X \) be linear, continuous. Suppose there exist \( X_0, Y_0 \) dense subsets of \( X \), a sequence \((n_k)\) of positive integers, and a sequence of mappings (possibly nonlinear, possibly not continuous) \( S_{n_k} : Y_0 \to X \) so that

\[
i) \quad T^{n_k} \to 0_{k \to \infty} \text{ pointwise on } X_0.
\]
\[
ii) \quad S_{n_k} \to 0_{k \to \infty} \text{ pointwise on } Y_0.
\]
\[
iii) \quad T^{n_k} S_{n_k} = \text{Identity on } Y_0.
\]

Then \( T \) is hypercyclic.

**Note.** It is an open problem whether the above conditions are also necessary (\([19]\), p 544):

(1.1) Does every hypercyclic operator satisfy the Hypercyclicity Criterion?

There are hypercyclic operators, though, that do not satisfy the Criterion for the entire sequence \((n_k) = (k)\) of positive integers. Solving a problem of D. Herrero, H. Salas constructed a weighted shift \( A \) so that both it and its Hilbert Adjoint \( A^* \) were hypercyclic, while their direct sum \( A \oplus A^* \) was not. As a consequence, \( A \) and \( A^* \) could not simultaneously satisfy the Criterion for the sequence \((n_k) = (k)\). For a while, some people were misled into thinking that it was a counterexample to (1.1) (cf. \([17]\), p97), but both \( A \) and \( A^* \) were later noticed in (\([20]\), Section 2) to satisfy the Criterion (cf. also \([5]\), Note 1.22).

If requiring only pointwise convergence in (iii) instead of equality, it is shown in \([6]\) that question (1.1) is equivalent to the following one of D. Herrero (\([17]\), p. 97):

\[\text{Is } T \oplus T \text{ hypercyclic whenever } T \text{ is?}\]

\(A \text{ direct sum } A \oplus B \text{ may fail to be hypercyclic even if each of } A \text{ and } B \text{ is (\([26]\), Corollary 2.6)).}\)

Interest in answering these questions comes also from the study of hypercyclic subspaces, that is, linear subspaces where all the non-zero vectors are hypercyclic. Although every hypercyclic operator admits a dense, invariant hypercyclic subspace (\([17]\), \([8]\), \([4]\), \([2]\)), they do not always admit a closed and infinite dimensional one. The situation is well understood for operators satisfying the Criterion. For instance,
an operator on a Banach space satisfying the Hypercyclicity Criterion will admit a
closed, infinite dimensional hypercyclic subspace if and only if its essential spectrum
intersects the closed unit disk (24, Thm 2.1; cf also 10, 22, 19, 3).

2. Birkhoff’s theorem for the space $H_{bc}(E)$.

Given a Banach Space $E$, let $H_{bc}(E)$ be the Fréchet Algebra generated by the
elements of the dual space $E^*$, endowed with the topology of uniform convergence
on balls of $E$. The space $H_{bc}(E)$ consists of entire functions $f : E \to \mathbb{C}$ of so-called
compact type that are bounded on bounded subsets of $E$. That is, $H_{bc}(E)$ consists
of all functions of the form

$$f = \sum_{n=0}^{\infty} P_n,$$

where

$$P_n \in \overline{\text{span}} \{ \varphi^n : \varphi \in E^* \} \quad n = 0, 1, 2, \ldots$$

$$\|P_n\|^\frac{1}{n} = \left( \sup_{\|x\| \leq 1} |P_n(x)| \right)^{\frac{1}{n}} \to 0 \quad n \to \infty.$$

Parenthetically, we remark that the terminology compact type apparently comes
from the fact that if $E^*$ has the approximation property (see, e.g., 23, p. 195), then
any compact linear mapping $E \to E^*$ is a uniform limit of finite rank mappings,
which are of the form $\sum_{j=1}^{k} \phi_j \psi_j$. The associated 2–homogeneous polynomial, $x \in E \to \sum_{j=1}^{k} \phi_j(x) \psi_j(x)$ is of finite type, and limits of such finite type polynomials are called 2–homogeneous polynomials of compact type. Of course, the situation
generalizes to $n$–homogeneous polynomials. Also, the space $H_{bc}(E)$ is not as arcane
as may first appear to the non-specialist. Indeed, if $E^*$ has the approximation
property, then $H_{bc}(E)$ has a useful alternative description, as the space of entire
functions which are uniformly continuous on bounded subsets of $E$, with respect to
the weak topology (1, 12). Moreover if the Michael problem, which asks whether
every $\mathbb{C}$–valued homomorphism on a complex Fréchet algebra is automatically
continuous, has an affirmative solution for the Fréchet algebra $H_{bc}(c_0)$, then it has
an affirmative solution for every Fréchet algebra (23).

Given $a \in E$ fixed and $\Phi = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{C})$ of exponential type, let
$\Phi_a(D) : H_{bc}(E) \to H_{bc}(E)$ be defined as

$$\Phi_a(D)(f) = \sum_{n=0}^{\infty} c_n d^n f(a).$$

That is,

$$\Phi_a(D)(f)(x) = \sum_{n=0}^{\infty} c_n d^n f(x)(a) \quad \text{for all } x \in E,$$

where $d^n f(x)$ is the $n$–homogeneous polynomial associated with the $n^{th}$-Fréchet
derivative of $f$ at the point $x \in E$. The operator $\Phi_a(D)$ is continuous on $H_{bc}(E)$,
and when $\Phi(z) = e^z$ the induced operator is “translation by $a$” (14, Lemma 8). i.e.,

$$\tau_a(f)(x) = f(x + a) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n f(x)(a) = \Phi_a(D)(f)(x) \quad (2.1)$$
for all \( x \in E \) and all \( f \in H_{bc}(E) \). We are ready now to state the main result:

**Theorem 2.1.** Let \( E \) be a Banach space with separable dual \( E^* \), and \( 0 \neq a \in E \). Let \( \Phi(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{C}) \) be non-constant, of exponential type. Then the operator

\[
\Phi_a(D) : H_{bc}(E) \rightarrow H_{bc}(E)
\]

\[
f \mapsto \sum_{n=0}^{\infty} c_n df(a)
\]

is hypercyclic.

**Note.** If \( E^* \) fails to be separable, so does \( H_{bc}(E) \), and this prevents \( \Phi_a(D) \) from being hypercyclic. Thus, from now on we assume \( E^* \) to be separable.

In particular, Birkhoff’s theorem for the spaces \( H_{bc}(E) \) follows:

**Corollary 2.2.** Let \( E \) be a Banach space with separable dual \( E^* \), and \( 0 \neq a \in E \). Then the operator “translation by ‘a’ ”

\[
\tau_a(f)(x) = f(x+a) \quad (x \in E)
\]

is hypercyclic on \( H_{bc}(E) \).

**Proof.** It follows immediately from (2.1) and Theorem 2.1.

We will make use of the following two lemmas.

**Lemma 2.3.** \( B = \{ e^\varphi : \varphi \in E^* \} \) is a linearly independent subset of \( H_{bc}(E) \).

**Proof.** Let \( \{ e^{\varphi_i} \}_{i \in I} \) be a maximal linearly independent subset of \( B \). Fix \( \varphi \in E^* \), and assume there exist non-zero constants \( c_{i_1}, \ldots, c_{i_r} \in \mathbb{C} \) so that

\[
c_{i_1} e^{\varphi_{i_1}} + \cdots + c_{i_r} e^{\varphi_{i_r}} = e^\varphi
\]

Let \( a \in E \) be arbitrary. Applying the operator \( f \mapsto df(.)a \) in (2.2), it follows that

\[
c_{i_1} \varphi_{i_1}(a) e^{\varphi_{i_1}} + \cdots + c_{i_r} \varphi_{i_r}(a) e^{\varphi_{i_r}} = \varphi(a) e^\varphi.
\]

Since \( \{ e^{\varphi_i} \}_{i \in I} \) is linearly independent and \( c_{i_1}, \ldots, c_{i_r} \) are non-zero, by (2.2) and (2.3) we have

\[
\varphi_{i_1}(a) = \cdots = \varphi_{i_r}(a) = \varphi(a).
\]

Since \( a \in E \) is arbitrary,

\[
\varphi_{i_1} = \cdots = \varphi_{i_r} = \varphi.
\]

Hence, \( \{ \varphi_i \}_{i \in I} = E^* \), and Lemma 2.3 follows.

Our next lemma generalizes a result of Gupta, which was needed in [16] to obtain information about the range and kernel of convolution operators.

**Lemma 2.4.** Let \( U \) be a non-empty open subset of \( E^* \). Then

\[
S = \text{span} \{ e^\varphi : \varphi \in U \}
\]

is dense in \( H_{bc}(E) \).
Proof. Given \( \phi_0 \in E^* \), the map

\[
H_{bc}(E) \rightarrow H_{bc}(E)
\]

g \mapsto \frac{e^{t\phi_0}}{t^k} \in \mathbb{C},

is a homeomorphism. So

\[
\text{span} \{ e^{t\phi_0} : \phi \in U \} = H_{bc}(E) \iff \text{span} \{ e^\phi : \phi \in U \} = H_{bc}(E).
\]

Hence, we may assume that 0 \in U. Reducing U if necessary, we may also assume that for some \( \delta > 0 \),

\[
U = \{ \phi \in E^* : ||\phi|| < \delta \}.
\]

In particular, 1 \in \overline{U}, for 0 \in U. It will then suffice to show the following:

Claim: For every \( \phi \in U \) and \( n \geq 1 \), \( \phi^n \in \overline{U} \).

To prove the claim, let \( \phi \in U \) and suppose the claim is true for \( n \leq k - 1 \).

Then for each \( 0 < t < 1 \) we have

\[
g_t = \frac{e^{t\phi} - 1 - t\phi - \frac{[t\phi]^2}{2!} - \cdots - \frac{[t\phi]^{k-1}}{(k-1)!}}{t^k} \in \overline{U},
\]

since \( t\phi \in U \). So given \( x \in E \),

\[
\left| \left( g_t - \frac{\phi^n}{k!} \right)(x) \right| = \left| \frac{1}{t^k} \left[ e^{t\phi} - 1 - t\phi - \frac{[t\phi]^2}{2!} - \cdots - \frac{[t\phi]^{k-1}}{(k-1)!} \right](x) \right|
\[
\leq \left| \frac{1}{t^k} \sum_{n \geq k+1} \frac{[t\phi]^n}{n!} (x) \right|
\[
\leq t \sum_{n \geq k+1} t^{n-k-1} \frac{||\phi||^n}{n!} \leq t e^{||\phi||}.
\]

Thus, \( g_t \xrightarrow{t \to 0} \frac{\phi^n}{k!} \) in \( H_{bc}(E) \), and \( \frac{\phi^n}{k!} \in \overline{U} \). So the claim holds.

\[ \Box \]

Proof of Theorem 2.6. Consider the function \( g : E^* \rightarrow \mathbb{C} \) defined by

\[
g(\phi) = \sum_{n=0}^{\infty} P_n(\phi),
\]

where each \( P_n : E^* \rightarrow \mathbb{C} \) is the \( n \)-homogeneous polynomial defined by

\[
P_n(\phi) = c_n \phi^n(a) \quad (n \geq 0).
\]

Now, since \( \Phi \) is of exponential type, there exists \( R > 0 \) so that

\[
|c_n| \leq \frac{R^n}{n!} \quad (n \geq 1).
\]

Given \( \phi \in E^* \) with \( ||\phi|| \leq 1 \) and \( n \geq 1 \),

\[
|P_n(\phi)| = |c_n| |\phi^n(a)| \leq \frac{R^n}{n!} ||\phi||^n ||a||^n \leq \left( \frac{e ||a|| R}{n} \right)^n,
\]

and so

\[
||P_n||^\frac{1}{n} \xrightarrow{n \to \infty} 0.
\]
Thus, \( g : E^* \to \mathbb{C} \) is entire (moreover, it is of bounded type), and non-constant, since \( \Phi \) is non-constant. So the sets

\[
U := \left\{ \varphi \in E^* : \sum_{n=0}^{\infty} c_n \varphi^n(a) = |g(\varphi)| < 1 \right\}
\]

\[
V := \left\{ \varphi \in E^* : \sum_{n=0}^{\infty} c_n \varphi^n(a) = |g(\varphi)| > 1 \right\}
\]

are both open, non-empty. Hence, according to Lemma 2.4,

\[
X_0 := \text{span} \{ e^\varphi : \varphi \in U \}
\]

\[
Y_0 := \text{span} \{ e^\varphi : \varphi \in V \}
\]

are both dense subspaces of \( H_{bc}(E) \). Next, notice that if \( T = \Phi_a(D) \), given \( \varphi \in E^* \)

\[
T(e^\varphi) = \sum_{n=0}^{\infty} c_n d^n(e^\varphi)a = \sum_{n=0}^{\infty} c_n \varphi^n(a) e^\varphi = g(\varphi) e^\varphi.
\]

By \( (2.4) \),

\[
T^n \xrightarrow{n \to \infty} 0 \quad \text{pointwise on } X_0.
\]

Also, by Lemma 2.3 there exists a (possibly discontinuous) linear map \( S : Y_0 \to Y_0 \) determined by

\[
S(e^\varphi) = [g(\varphi)]^{-1} e^\varphi \quad (\varphi \in E^*)
\]

which by \( (2.5) \) and \( (2.6) \) satisfies

\[
\begin{cases}
S^n \xrightarrow{n \to \infty} 0 & \text{pointwise on } Y_0. \\
TS = \text{id}_{Y_0} & \text{on } Y_0.
\end{cases}
\]

By Theorem 1.2, \( T = \Phi_a(D) \) is hypercyclic. \( \square \)

**Remarks:**

i) For \( E = \mathbb{C} \) and \( a = 1 \), Theorem 2.1 yields Theorem 1.1 (for the case \( N = 1 \)), since \( H_{bc}(\mathbb{C}) = H(\mathbb{C}) \), and given any \( \Phi \in H(\mathbb{C}) \) of exponential type, \( \Phi(D) = \Phi_1(D) \).

ii) Using similar arguments, one may obtain the following two theorems (cf. [5], Chapter 2):

**Theorem 2.5.** Let \( E \) be a Banach space with separable dual \( E^* \). Let \( \{b_1, b_2, \ldots, b_r\} \) be a linearly independent subset of \( E \), and let \( \Phi^1, \ldots, \Phi^r \) in \( H(\mathbb{C}) \) be non-constant, of exponential type. Then

\[
A = \Phi^1_{b_1}(D) + \Phi^2_{b_2}(D) + \cdots + \Phi^r_{b_r}(D)
\]

is hypercyclic on \( H_{bc}(E) \).
Theorem 2.6. Let $E$ be a Banach space with separable dual $E^*$, and let $\{b_n\}_{n \geq 0} \subset E \setminus \{0\}$ be bounded. Then the operator

$$A : H_{bc}(E) \rightarrow H_{bc}(E)$$

$$f \mapsto \sum_{n=0}^{\infty} c_n d^n f(.) b_n$$

is hypercyclic, where $\Phi(z) = \sum_{n=0}^{\infty} c_n z^n \in H(\mathbb{C})$ is entire, non-constant and of exponential type.

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\textbf{Department of Mathematics and Computer Science, Kent State University, Kent, OH 44242. U.S.A.}  
\textit{E-mail address:} aron@mcs.kent.edu

\textbf{Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403. U.S.A.}  
\textit{E-mail address:} jbes@bgnet.bgsu.edu