SU(3) symmetry and its breaking effects in semileptonic heavy baryon decays

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We employ the flavor SU(3) symmetry to analyze semileptonic decays of anti-triplet charmed baryons ($\Lambda_c^+\Xi_c^{0,0}$) and find that the experimental data on branching fractions very recently $^4$:

\[ B_{\text{Belle}}(\Xi_c^0 \to \Xi^- e^+ \nu_e) = (1.31 \pm 0.04 \pm 0.07 \pm 0.38)\% , \]
\[ B_{\text{Belle}}(\Xi_c^0 \to \Xi^- \mu^+ \nu_\mu) = (1.27 \pm 0.06 \pm 0.10 \pm 0.37)\% , \]

which is about a factor of 2 more precise than the ALICE result:

\[ B_{\text{ALICE}}(\Xi_c^0 \to \Xi^- e^+ \nu_e) = (2.5 \pm 0.8)\% . \]

This comes from the ALICE measurement of $B(\Xi_c^0 \to \Xi^- e^+ \nu_e)/B(\Xi_c^0 \to \Xi^- \pi^+) = 1.38 \pm 0.14 \pm 0.22\%$ $^3$ and Belle data $B(\Xi_c^0 \to \Xi^- \pi^+) = 1.8 \pm 0.7\%$ $^4$. We anticipate the difference between the above results can be clarified with the improvement of the experimental accuracy and the promising prospects on charmed baryons in the future. The available data on the decays from the anti-triplet heavy baryons to the octet baryons have been collected in Table I while the branching fraction $B(\Xi_c^0 \to \Xi^- e^+ \nu_e) = (1.54 \pm 0.35)\%$ listed is obtained by averaging the Belle and ALICE data.

On the theoretical side, one can apply the SU(3) flavor symmetry to analyze the semileptonic decays and obtain some model-independent relations among different decays $^3, ^10$. For semileptonic charmed baryon decays, we have

\[ \Gamma(\Xi_c^0 \to \Xi^- e^+ \nu_e) = \Gamma(\Xi_c^0 \to \Xi^0 e^+ \nu_e) = \frac{3}{2} \Gamma(\Lambda_c^+ \to \Lambda^0 e^+ \nu_e) . \]
Since the irreducible amplitude can be extracted by fitting data, the SU(3) analysis bridges experimental data and the dynamical approaches like Lattice QCD and model-dependent calculations. We adopt the experimental data on $\Lambda^+_c$ semileptonic decays and the SU(3) relations with the lifetimes $\tau_{\Lambda^+_c} = 2.024 \times 10^{-13}$s, $\tau_{\Xi^+_c} = 1.53 \times 10^{-13}$s, $\tau_{\Sigma^+_c} = 4.56 \times 10^{-13}$s. Then we obtain the branching ratios of $\Xi^+_c$ shown in Table I from which one can find an obvious deviation between experiments and theory.

| channel                     | SU(3) symmetry |
|-----------------------------|-----------------|
| $\Lambda^+_c \to \Lambda^0 e^+ \nu_e$ | $3.6 \pm 0.4$ [33] |
| $\Lambda^+_c \to \Lambda^0 \mu^+ \nu_\mu$ | $3.5 \pm 0.5$ [33] |
| $\Xi^+_c \to \Xi^0 e^+ \nu_e$ | $2.3 \pm 1.5$ [33] |
| $\Xi^+_c \to \Xi^0 \mu^+ \nu_\mu$ | $1.54 \pm 0.35$ [4, 5] |
| $\Xi^+_c \to \Xi^- \mu^+ \nu_\mu$ | $1.27 \pm 0.44$ [4] |

It should be noted that the flavor SU(3) symmetry is an approximate symmetry, since u, d, and s quarks have different masses which breaks SU(3) symmetry. For a more accurate analysis, SU(3) breaking effects should be included, which is the main focus of this work. Compared to the strange quark mass $m_s$, the up and down quark masses $m_u, d$ are much smaller and thus can be neglected. Therefore the s quark mass is the major source for flavor SU(3) symmetry breaking. In this work, we carry out an analysis with the leading-order SU(3) breaking effects on semileptonic anti-triplet charmed baryons decays and explore the scenarios in which recent experimental measurements can be consistently accommodated.

The rest of this paper is organized as follows. In Sec. II, we give the theoretical framework for SU(3) symmetry and study symmetry breaking in semileptonic decays of anti-triplet heavy baryons for the process of $c \to d/s$. In Sec. III, we also obtain numerical results using the SU(3) symmetry term and analyze the SU(3) symmetry breaking effect for the process of $b \to c/u$. A brief conclusion will be presented in the last section.

II. SU(3) SYMMETRY FOR SEMILEPTONIC ANTI-TRIPLET CHARMED BARYON DECAYS

In the flavor SU(3) symmetry limit, hadron multiplets can be classified according to the SU(3) irreducible representation. Baryons with a charm quark and two light quarks can have $3 \otimes 3 = \overline{3} \oplus 6$ representations. The anti-triplet $\bar{3}$ semileptonic baryon ($\Lambda^+_c, \Xi^+_c, \Xi^0_c$) decays are our focus here, whose quark level Feynman diagrams are shown in Fig IIa). In the SM the low-energy effective Hamiltonian for these decays is given as

$$H_{c \to d/s} = \frac{G_F}{\sqrt{2}} [V_{cq} \bar{q} \gamma^\mu (1 - \gamma_5) c \bar{\nu}_\ell \gamma_\mu (1 - \gamma_5) \ell] + h.c.,$$

where $q = d, s$ and $G_F$ is the Fermi-constant. $V_{cq}$ is CKM matrix element. With the help of helicity amplitude method, the decay transition amplitude can be written as

$$A(B_c \to B_q \ell^+ \nu_\ell) = \frac{G_F}{\sqrt{2}} V_{cq}^* \langle B_q | \bar{q} \gamma^\mu (1 - \gamma_5) c | B_c \rangle \langle \ell^+ \nu_\ell | \bar{\nu}_\ell \gamma^\nu (1 - \gamma_5) \ell | 0 \rangle g_{\mu \nu},$$

with the decomposition of $g_{\mu \nu}$,

$$g_{\mu \nu} = - \sum_{\lambda=0, \pm 1} c^{\dagger}_\mu(\lambda)c_{\nu}(\lambda) + c^{\dagger}_\mu(t)c_{\nu}(t),$$
\[ \epsilon_\mu(t) = \frac{q^\mu}{\sqrt{q^2}}, \quad (6) \]

where the \( \epsilon_\mu(\lambda) \) is transverse\((\lambda = \pm 1)\) or longitudinal\((\lambda = 0)\) polarization states and \( \epsilon_\mu(t) \) is timelike polarization states.

The above amplitude can be decomposed into the Lorentz invariant hadronic and leptonic matrix elements:

\[ A(B_c \rightarrow B_q \ell^+ \nu) = \frac{G_F}{\sqrt{2}} V_{cq}^* \langle B_q | \bar{q} \gamma^\mu (1 - \gamma_5) c | B_c \rangle \langle \ell^+ \nu | \bar{\nu} \gamma^\nu (1 - \gamma_5) \ell \rangle g_{\mu\nu} \]

\[ = \frac{G_F}{\sqrt{2}} V_{cq}^* \left( - \sum_{\lambda_w = 0, \pm 1} H_{\lambda,\lambda_w} L_{\lambda_w} + H_{\lambda,t} L_t \right), \quad (7) \]

\[ H_{\lambda,\lambda_w} = \langle B_q | \bar{q} \gamma^\mu (1 - \gamma_5) c | B_c \rangle \epsilon_\mu(\lambda_w), \]

\[ L_{\lambda_w} = \langle \ell^+ \nu | \bar{\nu} \gamma^\nu (1 - \gamma_5) \ell | 0 \rangle \epsilon_\nu(\lambda_w), \]

where \( H_{\lambda,\lambda_w}(L_{\lambda_w}) \) is hadronic\( (\text{ leptonic}) \) helicity amplitude, \( \lambda(\lambda)(0, \pm 1, t) \) corresponds to the helicity of the daughter baryon \( (W) \) and the \( \epsilon_\mu(\lambda_W) \) is the polarization vector of \( W \) boson.

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![Feynman Diagram](image-url)

**FIG. 1:** The Feynman diagram of anti-triplet heavy baryons induced by \( c \rightarrow d/s, \quad b \rightarrow u \) (left), and \( b \rightarrow c \) (right).

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In the SM, charmed baryons can decay into octet baryons. The SU(3) anti-triplet and octet matrix are denoted by

\[ T_{c\bar{3}} = \begin{pmatrix} 0 & \Lambda^+_c & \Xi^+_c \\ -\Lambda^+_c & 0 & \Xi^0_c \\ -\Xi^+_c & -\Xi^0_c & 0 \end{pmatrix}, \quad T_8 = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} \\ \Sigma^- - \frac{2\Sigma^0}{\sqrt{6}} \\ \Xi^- - \frac{\Xi^0}{\sqrt{6}} \end{pmatrix} \quad (8) \]

Tree operators of charm quark semileptonic decays into light quarks are categorized into \( c \rightarrow d/s \). Therefore under the flavor SU(3) symmetry, the low-energy effective Hamiltonian can be decomposed in terms of \( H_3 \) shown as:

\[ (H_3)^1 = 0, \quad (H_3)^2 = V_{cd}^*, \quad (H_3)^3 = V_{cs}^*. \quad (9) \]

The corresponding helicity amplitude can be written as:

\[ H_{\lambda,\lambda_w} = \alpha_{1}^{\lambda,\lambda_w} \times (T_{c\bar{3}})^{(ij)}(H_3)^k \epsilon_{ikm}(T_8)^m_j, \quad (10) \]

where the \( \alpha_{1}^{\lambda,\lambda_w} \) represents SU(3) irreducible nonperturbative amplitude. The \( \alpha_{1}^{\lambda,\lambda_w} \) can be expressed by the form factors

\[ \alpha_{1}^{\lambda,\lambda_w} = \bar{u}(\lambda) \left[ f_1 \gamma^\mu + f_2 i\sigma^\mu\nu q^\nu + f_3 q^\mu M_i \right] u(\lambda_i) \epsilon_\mu(\lambda_w) \]
Here $u(\lambda)$ is the spinor of charmed baryons, $u(\lambda)$ is the spinor of the final state, and $f_i$ ($i = 1, 2, 3$), $f'_i$ ($i = 1, 2, 3$) are the form factors. In the heavy quark limit $^{36}$, the $f_2$, $f_3$, $f'_2$, $f'_3$ are suppressed by $1/m_{B_c}$, and only one independent form factor exists if the large-recoil symmetry is adopted $^{37, 38}$. Actually, a previous calculation $^{25}$ also indicates that the form factor of vector parameter $f_1$ and axial-vector parameter $f'_1$ have dominant contributions to the heavy baryon decay processes. Thus we neglect $f_2$, $f'_2$, $f_3$, and $f'_3$ in our later calculations. Expanding Eq. (10), one obtains the relations between the helicity amplitudes of different channels of anti-triplet charmed baryons, which are presented in Table III. In the SU(3) symmetry limit, the branching fractions of $\Xi^+_c \to \Xi^0 \ell^+ \nu_\ell$ and $\Xi^0_c \to \Xi^- \ell^+ \nu_\ell$ can be predicted by using experimental data of $\Lambda^+_c \to \Lambda^0 \ell^+ \nu_\ell$ which are given in Table II. To shed further light on the decay dynamics, we take the pole model as an illustration to access the $q^2$ dependence of form factors $^{39}$.

$$f_i(q^2) = \frac{f_i}{1 - \frac{q^2}{m_p^2}},$$  \hspace{1cm} (12)

where $f_i = f_i(q^2 = 0)$ and $m_p = 2.061\text{GeV}$, which is the average mass of $D$ and $D_s$. The differential decay widths can be expressed by these form factors,

$$\frac{d\Gamma}{dq^2} = \frac{(m_\ell^2 - q^2)^2 \sqrt{\lambda} G_{\ell}^2 V_{SU(3)}^2}{284 \pi^3 M^3(q^2)^3} \left[ (f_1(q^2))^2 \times \left( 3s_+m_\ell^2(q^2 + s_-) + s_- (3q^2 + s_+) \right) \left( m_\ell^2 + 2q^2 \right) \right] + \left( f'_1(q^2) \right)^2 \times \left( 3s_-m_\ell^2(q^2 + s_+) + s_+ (3q^2 + s_-) \right) \left( m_\ell^2 + 2q^2 \right).$$  \hspace{1cm} (13)

Here $s_- = (M - M')^2 - q^2, s_+ = (M + M')^2 - q^2$, and $\sqrt{\lambda} = \sqrt{s_- s_+}$. $M$ and $M'$ are the mass of $B_c$ and $B_q$, respectively. $m_\ell$ is the lepton mass. $V_{SU(3)}$ is the SU(3) factor coming from the coefficient of $a_1^{\lambda,\lambda_w}$ in Table III. For instance in the first process in Table III $V_{SU(3)} = -\sqrt{2}/3 V_{cs}^*$. Using the amplitudes, we can fit the parameters $f_1$ and $f'_1$ with experimental data. In the fit, we use the experimental values for the particle masses. The fitted results are shown in Table III. Obviously, the $\chi^2$ in fitting is too large to be considered as a good fit, which implies that the SU(3) symmetry is not a good symmetry for charmed baryon decays. In the previous fit, we have neglected the possible SU(3) breaking effects. Because the light u, d, and s quarks have different masses, the SU(3) symmetry is broken. Neglecting the masses of u and d quark the mass matrix $M$ can be written as:

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \sim m_s \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = m_s \times \omega. \hspace{1cm} (14)$$
TABLE III: Experimental and fit data of anti-triplet charmed baryons decays.

| channel         | branching ratio(%) |
|-----------------|--------------------|
| $Λ_c^+ \to Λ^0 e^+ ν_e$ | $3.60 \pm 0.40$  $1.94 \pm 0.18$ |
| $Λ_c^+ \to Λ^0 μ^+ ν_μ$ | $3.5 \pm 0.5$     $1.87 \pm 0.176$ |
| $Ξ_c^+ \to Ξ^0 e^+ ν_e$ | $2.3 \pm 1.5$     $6.53 \pm 0.60$ |
| $Ξ_c^0 \to Ξ^- e^+ ν_e$ | $1.54 \pm 0.35$  $2.17 \pm 0.20$ |
| $Ξ_c^- \to Ξ^0 -μ^+ ν_μ$ | $1.27 \pm 0.44$  $2.09 \pm 0.19$ |
| $χ^2/d.o.f$ | $14.3$ |
| $f_1$ | $1.05 \pm 0.30$ |
| $f'_1$ | $0.11 \pm 0.95$ |

We can obtain the modified helicity amplitude as

$$H_{λ,λw} = a_1^{λ,λw} × (T_{c3})^{[ij]}(H_3)_j^kέ_{ikm}(T_8)_m^lω^n_l + a_2^{λ,λw} × (T_{c3})^{[im]}(H_3)_j^kέ_{ikm}(T_8)_m^lω^n_l + a_3^{λ,λw} × (T_{c3})^{[in]}(H_3)_j^kέ_{jm}(T_5)_m^lζ^l_j + a_4^{λ,λw} × (T_{c3})^{[in]}(H_3)_j^kέ_{jm}(T_9)_m^lζ^l_j + a_5^{λ,λw} × (T_{c3})^{[ij]}(H_3)_j^kέ_{im}(T_9)_m^lζ^l_j,$$

(15)

The $a_1^{λ,λw}$ is SU(3) symmetric irreducible nonperturbative amplitude and $a_2^{λ,λw}, a_3^{λ,λw}, a_4^{λ,λw}, a_5^{λ,λw}$ are SU(3) symmetry breaking irreducible nonperturbative amplitudes, which are proportional to $m_s$. Furthermore, the SU(3) symmetry breaking terms in Eq. (15) include the contribution of anti-triplet and sextet charmed heavy baryons mixing terms which correspond to the contribution of $a_2^{λ,λw}, a_3^{λ,λw}$ and $a_4^{λ,λw}$.

A. Symmetry breaking in helicity amplitude

The SU(3) symmetry breaking irreducible nonperturbative amplitudes $a_2^{λ,λw}, a_3^{λ,λw}, a_4^{λ,λw}, a_5^{λ,λw}$ in Eq. (15) can be decomposed in a similar way as that in Eq. (11). Again in our analysis, we only keep the vector and axial vector form factors.

Adding SU(3) symmetry breaking term and expanding the above formula in Eq. (15), one can obtain the amplitudes of different channels which are collected in the “amplitude I” column of Table IV. It can be seen that the parameters $a_1^{λ,λw}$ and $a_2^{λ,λw}$ always appear together in channels $Λ_{c}^{+} \to Λ^{0} ℓ^{+} ν_{ℓ}$, $Ξ_{c}^{+} \to Ξ^{0} ℓ^{+} ν_{ℓ}$ and $Ξ_{c}^{0} \to Ξ^{-} ℓ^{+} ν_{ℓ}$. Therefore the SU(3) symmetry breaking irreducible nonperturbative amplitudes $a_2^{λ,λw}, a_3^{λ,λw}, a_4^{λ,λw}, a_5^{λ,λw}$ can be parametrized as

$$a_1^{λ,λw} + a_5^{λ,λw} = f_1(q^2) × \bar{u}(λ)γ^{μ} u(λ_3)e^{μ}_μ(λ_w) - f'_1(q^2) × \bar{u}(λ)γ^{μ} γ_5 u(λ_3)e^{μ}_μ(λ_w),$$

$$a_2^{λ,λw} - a_4^{λ,λw} = δf_1(q^2) × \bar{u}(λ)γ^{μ} u(λ_3)e^{μ}_μ(λ_w) - δf'_1(q^2) × \bar{u}(λ)γ^{μ} γ_5 u(λ_3)e^{μ}_μ(λ_w),$$

$$a_3^{λ,λw} = Δf_1(q^2) × \bar{u}(λ)γ^{μ} u(λ_3)e^{μ}_μ(λ_w) - Δf'_1(q^2) × \bar{u}(λ)γ^{μ} γ_5 u(λ_3)e^{μ}_μ(λ_w),$$

(16)

where the $a_2^{λ,λw} - a_4^{λ,λw}$ is the combination that appears in helicity amplitude $Ξ_c^{+} \to Ξ^0 ℓ^+ ν_ℓ$ and $Ξ_c^{0} \to Ξ^- ℓ^+ ν_ℓ$.

By using the replacement rule: $f_1^{(1)} \to f_1^{(1)} + δf_1^{(1)}$ of $Ξ_c^{+} \to Ξ^0 ℓ^+ ν_ℓ$ and $Ξ_c^{0} \to Ξ^- ℓ^+ ν_ℓ$, we can directly fit these parameters from the data. In doing the combination of $a_1^{λ,λw} + a_5^{λ,λw}$ together to fit data, the $a_1^{λ,λw}$ in $Λ_{c}^{+} \to nℓ^{+} ν_{ℓ}$, $Ξ_c^{+} \to Σ^0 ℓ^+ ν_ℓ$ and $Ξ_c^{0} \to Σ^- ℓ^+ ν_ℓ$ will need to be treated as $a_1^{λ,λw} - a_5^{λ,λw}$. One can take $a_1^{λ,λw} + a_5^{λ,λw}$, $a_2^{λ,λw} - a_4^{λ,λw}$, $a_2^{λ,λw}, a_3^{λ,λw}$, and $a_5^{λ,λw}$ as independent parameters. The fitted results with two forms to access the $q^2$ dependence in form factors, pole model and constant, are given in Table V with a reasonable $χ^2/d.o.f = 1.6$ and $χ^2/d.o.f = 1.9$, respectively. It suggests that the SU(3) symmetry breaking effects generated by the light quark masses can improve the fit. Results for $δf_1$ and $δf'_1$ characterize the size of SU(3) symmetry breaking. From Table V one can find that SU(3) symmetry breaking effects to the differential decay width for the $Ξ_c^{0} \to Ξ^- e^+ ν_ℓ$ can reach as much as 50%,
TABLE IV: Decay amplitudes of charmed baryons anti-triplet decay into an octet baryon. The amplitudes in column I without $c_{3}^{\lambda=\lambda_{w}}$ term come from Eq. [18]. The effects of $\Xi_0$ and $\Xi_1$ mixing can be obtained by adding terms proportional to $c_{3}^{\lambda=\lambda_{w}}$.

| channel | amplitude I | amplitude II |
|---------|-------------|--------------|
| $\Lambda_{c}^{+} \rightarrow \Lambda^{0} l^{+}\nu$ | $-\sqrt{2}(a_{1}^{\lambda=\lambda_{w}} + a_{5}^{\lambda=\lambda_{w}})V_{cs}$ | $-\sqrt{2}(a_{1}^{\lambda=\lambda_{w}} + a_{5}^{\lambda=\lambda_{w}})V_{cs}$ |
| $\Lambda_{c}^{+} \rightarrow n l^{+}\nu$ | $a_{1}V_{cd}^{*}$ | $a_{1}V_{cd}^{*}$ |
| $\Xi_{c}^{+} \rightarrow \Sigma^{0} l^{+}\nu$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ |
| $\Xi_{c}^{+} \rightarrow \Lambda^{0} l^{+}\nu$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ |
| $\Xi_{c}^{0} \rightarrow \Sigma^{0} l^{+}\nu$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ |
| $\Xi_{c}^{0} \rightarrow \Xi^{-} l^{+}\nu$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ | $(a_{1}^{\lambda=\lambda_{w}} + a_{3}^{\lambda=\lambda_{w}} - a_{5}^{\lambda=\lambda_{w}} - \frac{\lambda_{w}}{\lambda_{w}}\theta)V_{cs}$ |

depending on the kinematics. Compared to the constant fit, the pole model fit results of form factors will be used in Table [V] since a relatively smaller $\chi^2$ is obtained. The inadequacy of the experimental data at this stage prevents a direct analysis of different individual terms especially $a_{2}^{\lambda=\lambda_{w}}$, $a_{3}^{\lambda=\lambda_{w}}$, and $a_{5}^{\lambda=\lambda_{w}}$. We hope that more experimental data can be accumulated to further examine the detailed sources of SU(3) symmetry breaking in the future.

Table V: Experimental data and fit results of anti-triplet charmed baryons decays with symmetry breaking term. The form factors $f_{1}$ and $f_{1}'$ correspond to $a_{1}^{\lambda=\lambda_{w}} + a_{5}^{\lambda=\lambda_{w}}$. The form factors $f_{2}$ and $f_{2}'$ correspond to $a_{2}^{\lambda=\lambda_{w}} - a_{4}^{\lambda=\lambda_{w}}$.

| channel | branching ratio(%) | experimental data | fit data(pole model) | fit data(constant). |
|---------|---------------------|-------------------|---------------------|--------------------|
| $\Lambda_{c}^{+} \rightarrow \Lambda^{0} e^{+}\nu_{e}$ | $3.6 \pm 0.4$ | $3.63 \pm 0.32$ | $3.72 \pm 0.32$ |
| $\Lambda_{c}^{+} \rightarrow \Lambda^{0} \mu^{+}\nu_{\mu}$ | $3.5 \pm 0.5$ | $3.48 \pm 0.30$ | $3.45 \pm 0.30$ |
| $\Xi_{c}^{+} \rightarrow \Xi^{0} e^{+}\nu_{e}$ | $2.9 \pm 1.5$ | $3.89 \pm 0.73$ | $3.92 \pm 0.73$ |
| $\Xi_{c}^{0} \rightarrow \Xi^{-} e^{+}\nu_{e}$ | $1.54 \pm 0.35$ | $1.29 \pm 0.24$ | $1.31 \pm 0.24$ |
| $\Xi_{c}^{0} \rightarrow \Xi^{-} \mu^{+}\nu_{\mu}$ | $1.27 \pm 0.44$ | $1.24 \pm 0.23$ | $1.24 \pm 0.23$ |
| fit parameter (pole model) | $f_{1} = 1.01 \pm 0.87$, $\delta f_{1} = -0.51 \pm 0.92$ | $\chi^2/d.o.f = 1.6$ |
| fit parameter (constant) | $f_{1}' = 0.60 \pm 0.49$, $\delta f_{1}' = -0.23 \pm 0.41$ | $\chi^2/d.o.f = 1.9$ |

In Fig. 2 we plot $dB/dq^{2}$ in $\Xi_{c}^{0/+} \rightarrow \Xi_{c}^{0/+}$ with SU(3) symmetry form factors $f_{1}'(\zeta_{c})$ as constants or parametrized as in Eq. [12] to access the $q^{2}$ distribution. In both cases, the parameters in form factors are independently fitted. This can be tested by an experimental investigation in the future in addition to the branching ratio fitting in Table V.

B. Symmetry breaking caused by the $\Xi_{c}^{0/+}-\Xi_{c}^{0/+}$ mixing

The inclusion of SU(3) breaking effects will lead to $\Xi_{c}^{0/+}$ and $\Xi_{c}^{0/+}$ to mix. Here $\Xi_{c}^{0/+}$ is a component field in the sextet $T_{6}$.

$$T_{6} = \left( \begin{array}{ccc} \Sigma_{c}^{0} & \Sigma_{c}^{+} & \Xi_{c}^{0} \\ \Sigma_{c}^{+} & \Sigma_{c}^{0} & \Xi_{c}^{0} \\ \Xi_{c}^{0} & \Xi_{c}^{0} & \Omega_{c}^{0} \end{array} \right) \right). \quad (17)$$
Fig. 2: The differential decay branching fraction $dB/dq^2$ for the $\Xi_c^0 \rightarrow \Xi^- e^+ \nu_e$. The two curves are obtained by different treatments of form factors.

The mixing between anti-triplet charmed baryons to the sextet states is due to the following term expanding to the first order in $m_s$,

$$H_{\lambda,\lambda_w}(T_c \omega \rightarrow T_s) = d^{\lambda,\lambda_w} \times (T_{c3})^{ij} \omega^k(T_{c6})_{\{ki\}}.$$ \hspace{1cm} (18)

Expanding Eq. (18), one can find the mixing between $\Xi_c^{0(+)}$ and $\Xi_c^{0(-)}$, while other hadrons are not affected. The mixing angle $\theta$ can be introduced to define the mass eigenvalue state $\Xi_c^0$ and $\Xi_c^\pm$,

$$\Xi_c^{0/\pm mass} = \cos \theta \times \Xi_c^{0/+} + \sin \theta \times \Xi_c^{0/\pm'},$$ \hspace{1cm} (19)

where the angle $\theta$ is at the order $O(m_s)$. To the first order in $m_s$, $\cos \theta \sim 1$, $\sin \theta \sim \theta$.

To take into account the mixing effects for physical $\Xi_c$ states, one needs to work out the sextet semileptonic decay amplitudes which are given to the first order in $\omega$

$$H_{\lambda,\lambda_w} = c_1^{\lambda,\lambda_w} \times (T_{c6})^{ij} (H_3)^k \epsilon_{km}(T_s)_m^l \omega_j^\ell + c_2^{\lambda,\lambda_w} (T_{c6})^{in} (H_3)^k \epsilon_{km}(T_s)_m^l \omega_j^n + c_3^{\lambda,\lambda_w} \times (T_{c6})^{in} (H_3)^k \epsilon_{jm}(T_s)_m^l \omega_j^n + c_4^{\lambda,\lambda_w} \times (T_{c6})^{in} (H_3)^k \epsilon_{jm}(T_s)_m^l \omega_j^n.$$ \hspace{1cm} (20)

The helicity amplitude for anti-triplet charmed baryon with mass eigenvalue state $\Xi_c^{mass}$ can be obtained by using Eq. (18), Eq. (19) and Eq. (20). At the leading order, the helicity amplitudes for the decay channel of mass eigenvalue states $\Xi_c^{0mass} \rightarrow \Xi^- e^+ \nu_e$ and $\Xi_c^{+mass} \rightarrow \Xi^0 e^+ \nu_e$ become

$$H_{\lambda,\lambda_w}^{0mass} \times V_{cs}^* (a_1^{\lambda,\lambda_w} + a_2^{\lambda,\lambda_w} - a_3^{\lambda,\lambda_w} + a_4^{\lambda,\lambda_w} + \frac{c_1^{\lambda,\lambda_w}}{\sqrt{2}} \theta),$$ \hspace{1cm} (21)

where we have neglected the $O(m_s^2)$ and higher order corrections. The helicity amplitudes of other channels are listed in the “amplitude I” column of Table IV. In the table, the states in the first column are understood to be the mass eigenstates for the case with the mixing effect.

It is clear that the existing experimental data is insufficient to determine all these parameters. But one can see that by introducing the effective amplitude $a_4^{\lambda,\lambda_w} = a_4^{\lambda,\lambda_w} + c_1^{\lambda,\lambda_w}/\sqrt{2}$ and $a_2^{\lambda,\lambda_w} = a_2^{\lambda,\lambda_w} + \sqrt{2} c_1^{\lambda,\lambda_w}$, the effect of $\theta$ and $c_1^{\lambda,\lambda_w}$ can be absorbed into $a_2^{\lambda,\lambda_w}$ and $a_4^{\lambda,\lambda_w}$. The helicity amplitudes with $a_2^{\lambda,\lambda_w}$, $a_4^{\lambda,\lambda_w}$ are listed in the “
amplitude II column of Table IV. Therefore, our fit results for the case without mixing effects are still valid, but the form factors $\delta f_1$ and $\delta f'_1$ correspond to the new effective amplitudes $g_{2\lambda,\lambda'}$, $a_{4\lambda,\lambda'}$.

Although several other form factors such as $\Delta f_1$ and $\Delta f'_1$ cannot be constrained due to the lack of experimental data, in some scenarios, we still estimate the branching fractions of some processes from the results in Table IV. We can estimate the branching fractions of $\Lambda_c^+ \to n e^+ \nu_e$ and $\Lambda_c^+ \to n \mu^+ \nu_\mu$:

$$B(\Lambda_c^+ \to n e^+ \nu_e) = (0.520 \pm 0.046)\%, \quad B(\Lambda_c^+ \to n \mu^+ \nu_\mu) = (0.506 \pm 0.045)\%,$$

by assuming $a_{5\lambda,\lambda'}$ giving no contribution. If the process of $\Lambda_c^+ \to n \ell^+ \nu_\ell$ is measured by experiments, the contributions of $a_{5\lambda,\lambda'}$ will be obtained. The branching fractions of $\Xi_c^+ \to \Sigma^0 \ell^+ \nu_\ell$, $\Xi_c^+ \to \Lambda^0 \ell^+ \nu_\ell$, and $\Xi_c^0 \to \Sigma^- \ell^+ \nu_\ell$ can also be estimated by assuming $a_{2\lambda,\lambda'}$, $a_{3\lambda,\lambda'}$, and $a_{5\lambda,\lambda'}$ giving no contributions.

$$B(\Xi_c^+ \to \Sigma^0 \ell^+ \nu_\ell) = (0.496 \pm 0.046)\%, \quad B(\Xi_c^+ \to \Lambda^0 \ell^+ \nu_\ell) = (0.481 \pm 0.044)\%,$$
$$B(\Xi_c^+ \to \Sigma^- \ell^+ \nu_\ell) = (0.333 \pm 0.031)\%, \quad B(\Xi_c^0 \to \Sigma^- \ell^+ \nu_\ell) = (0.323 \pm 0.029)\%.$$ (23)

For the processes of $\Xi_c^+ \to \Sigma^0 \ell^+ \nu_\ell$, $\Xi_c^+ \to \Lambda^0 \ell^+ \nu_\ell$, and $\Xi_c^0 \to \Sigma^- \ell^+ \nu_\ell$, once some of the processes are established in future experiments, we can fit the form factor $\Delta f_1^{(\ell)}$ which reflects the contribution of $a_{3\lambda,\lambda'}$. Then the branching fractions for the other processes depending on $a_{3\lambda,\lambda'}$ can also be established.

III. SU(3) SYMMETRY ANALYSIS IN ANTI-TRIPLET BEAUTY BARYONS SEMILEPTONIC DECAYS

The anti-triplet beauty baryon semileptonic decays are governed by the Hamiltonian:

$$\mathcal{H}_{b\to u/c} = \frac{G_F}{\sqrt{2}} \left[V_{qb}^* V_{qc} (1 - \gamma_5) b \bar{\ell} \gamma^\mu (1 - \gamma_5) \nu_\ell \right] + h.c.,$$ (24)

where $q = u, c$.

The $b \to c$ transition is an SU(3) singlet, while the $b \to u$ transition forms an SU(3) triplet $H_3'$ with $(H_3')^1 = 1$ and $(H_3')^{2,3} = 0$. The SU(3) matrix representation of anti-triplet beauty baryons are given as

$$T_{b\bar{c}} = \begin{pmatrix} 0 & \Lambda_b^0 & \Xi_b^0 \\ -\Lambda_b^0 & 0 & \Xi_b^- \\ -\Xi_b^0 & -\Xi_b^- & 0 \end{pmatrix}. \quad (25)$$

We write the helicity amplitude in SU(3) analysis in a similar fashion as what has been done for semileptonic charmed anti-triplet decays, as

$$H_{\lambda,\lambda'} = b_{1\lambda,\lambda'} \times (T_{b\bar{c}})_{ij} [h] (H_3')^{jk} \epsilon_{ikm} (T_{b\bar{c}})^m_{[ij]} + e_{1\lambda,\lambda'} \times (T_{b\bar{c}})^{[ij]} (T_{b\bar{c}})_{ij}, \quad (26)$$

where $b_{1\lambda,\lambda'}$ and $e_{1\lambda,\lambda'}$ are respectively similar to $a_{1\lambda,\lambda'}$ in the previous section. The Feynman diagrams for the two term in $H_{\lambda,\lambda'}$ are shown in (a) and (b) of Fig II respectively.

Expanding the $H_{\lambda,\lambda'}$, one can obtain SU(3) amplitudes are listed in Table IV and the SU(3) relations can be given as follows:

$$\Gamma(\Lambda_b^0 \to p \ell^- \bar{\nu}_\ell) = \Gamma(\Xi_b^0 \to \Sigma^+ \ell^- \bar{\nu}_\ell) = 2 \Gamma(\Xi_b^- \to \Sigma^0 \ell^- \bar{\nu}_\ell) = 6 \Gamma(\Xi_b^- \to \Lambda^0 \ell^- \bar{\nu}_\ell);$$
$$\Gamma(\Lambda_b^0 \to \Lambda_c^+ \ell^- \bar{\nu}_\ell) = \Gamma(\Xi_b^0 \to \Xi_c^+ \ell^- \bar{\nu}_\ell) = \Gamma(\Xi_b^- \to \Xi_c^0 \ell^- \bar{\nu}_\ell). \quad (27)$$

Using the experimental data $B(\Lambda_b^0 \to \Lambda_c^+ \ell^- \bar{\nu}_\ell) = (6.2_{-1.3}^{+1.1})\%$ and $B(\Lambda_b^0 \to p \mu^- \bar{\nu}_\mu) = (4.1 \pm 1.0)\%$, we give the prediction in third column of Table IV.
TABLE VI: Amplitudes beauty baryons $\Xi_b$ and $\Lambda_b$ decays into octet and anti-triplet baryons.

| channel | amplitude | branching fraction (%) |
|---------|-----------|------------------------|
| $\Lambda_b^0 \to p\ell^\nu_\ell$ | $b_1^{\lambda\lambda_w}$ | 4.1 ± 1.0(input)$[32]$ |
| $\Xi_b^0 \to \Sigma^0\ell^\nu_\ell$ | $-b_1^{\lambda\lambda_w} + b_2^{\lambda\lambda_w}$ | 4.1 ± 1.0 |
| $\Xi_b^- \to \Sigma^0\ell^-\bar{\nu}_\ell$ | $b_1^{\lambda\lambda_w} \sqrt{3}$ | 2.2 ± 0.5 |
| $\Xi_b^- \to \Lambda^0\ell^-\bar{\nu}_\ell$ | $b_2^{\lambda\lambda_w} \sqrt{2}$ | 0.7 ± 0.2 |
| $\Lambda_b^0 \to \Lambda^+_\ell^-\bar{\nu}_\ell$ | $2e_1^{\lambda\lambda_w}$ | 6.2$^{+1.4}_{-1.3}$ |
| $\Xi_b^0 \to \Xi^0\ell^-\bar{\nu}_\ell$ | $2e_1^{\lambda\lambda_w}$ | 6.2$^{+1.4}_{-1.3}$ |
| $\Xi_b^- \to \Xi^0\ell^-\bar{\nu}_\ell$ | $2e_1^{\lambda\lambda_w}$ | 6.6$^{+1.5}_{-1.4}$ |

For the processes we predicted, we expect them to be measured by Belle II and LHCb. The SU(3) symmetry of these processes will probably be tested. Due to the lack of experimental data at this stage, we can not explore the SU(3) symmetry breaking effects by fitting the form factors. We have also worked out how to include SU(3) symmetry breaking effects. The helicity amplitude including SU(3) symmetry breaking about $b$ quark decays is given as:

$$
H_{\lambda,\lambda,\omega} = b_1^{\lambda\lambda_w} \times (T_{b3})^{[ij]}(H_3')^k \epsilon_{ikm}(T_8)_j^m + b_2^{\lambda\lambda_w} \times (T_{b3})^{[im]}(H_3')^k \epsilon_{ikm}(T_8)_j^m \omega_i^j
$$

$$
+ b_3^{\lambda\lambda_w} \times (T_{c3})^{[im]}(H_3')^k \epsilon_{ikm}(T_8)_j^m \omega_i^j + b_4^{\lambda\lambda_w} \times (T_{c3})^{[im]}(H_3')^k \epsilon_{ikm}(T_8)_j^m \omega_i^j
$$

$$
+ b_5^{\lambda\lambda_w} \times (T_{c3})^{[ij]}(H_3')^k \epsilon_{ikm}(T_8)_j^m \omega_i^j + e_1^{\lambda\lambda_w} \times (T_{b3})^{[ij]}(H_3')^k \epsilon_{ikm}(T_8)_j^m \omega_i^j,
$$

where the $b_{1-5}^{\lambda\lambda_w}, e_{1-2}^{\lambda\lambda_w}$ are SU(3) symmetry irreducible nonperturbative amplitude and $b_2^{\lambda\lambda_w}, b_3^{\lambda\lambda_w}, b_4^{\lambda\lambda_w}, b_5^{\lambda\lambda_w}, e_2^{\lambda\lambda_w}$ are SU(3) symmetry breaking irreducible nonperturbative amplitudes. Here we have written the $b_{1-5}^{\lambda\lambda_w}$ terms in a similar fashion as that for $a_{i}^{\lambda\lambda_w}$ terms. But the $b_5^{\lambda\lambda_w}$ term has no contribution, because $(H_3')^k \omega_i^m$ is equal to zero. Expanding the formula above, we collected the SU(3) amplitudes in Table VII.

TABLE VII: SU(3) symmetry breaking amplitudes of beauty baryons $\Xi_b$ and $\Lambda_b$ decay into octet baryons and anti-triplet charmed baryons, respectively.

| channel | amplitude |
|---------|-----------|
| $\Lambda_b^0 \to p\ell^-\bar{\nu}_\ell$ | $b_1^{\lambda\lambda_w}$ |
| $\Xi_b^0 \to \Sigma^0\ell^-\bar{\nu}_\ell$ | $-b_1^{\lambda\lambda_w} + b_2^{\lambda\lambda_w} - b_3^{\lambda\lambda_w}$ |
| $\Xi_b^- \to \Sigma^0\ell^-\bar{\nu}_\ell$ | $b_1^{\lambda\lambda_w} - b_2^{\lambda\lambda_w} - b_3^{\lambda\lambda_w}$ |
| $\Xi_b^- \to \Lambda^0\ell^-\bar{\nu}_\ell$ | $b_2^{\lambda\lambda_w} + 2b_3^{\lambda\lambda_w} - b_4^{\lambda\lambda_w} - b_5^{\lambda\lambda_w}$ |
| $\Lambda_b^0 \to \Lambda^+_\ell^-\bar{\nu}_\ell$ | $2e_1^{\lambda\lambda_w}$ |
| $\Xi_b^0 \to \Xi^+\ell^-\bar{\nu}_\ell$ | $2e_1^{\lambda\lambda_w} + e_2^{\lambda\lambda_w}$ |
| $\Xi_b^- \to \Xi^0\ell^-\bar{\nu}_\ell$ | $2e_1^{\lambda\lambda_w} + e_2^{\lambda\lambda_w}$ |

A number of relations for decay widths can be readily deduced from Table VII

$$
\Gamma(\Xi_b^- \to \Sigma^0\ell^-\bar{\nu}_\ell) = \frac{1}{2} \Gamma(\Xi_b^- \to \Sigma^+\ell^-\bar{\nu}_\ell),
$$

$$
\Gamma(\Xi_b^0 \to \Xi^+\ell^-\bar{\nu}_\ell) = \Gamma(\Xi_b^- \to \Xi^0\ell^-\bar{\nu}_\ell).
$$

It can be seen from Eq. (29) that though the SU(3) symmetry breaking effects caused by the light quark mass are taken into account, there are still relations in these processes. These relations result from isospin symmetry which can only be broken if non-zero u and d quark masses with different values are included. We strongly suggest our experimental colleagues carry out measurements for these decays.
IV. CONCLUSION

We have investigated the semileptonic decay of anti-triplet heavy baryons using SU(3) symmetry based on the latest experimental data. In the SU(3) symmetry limit, when fitting the available experimental data to the SU(3) symmetry analysis, we can only obtain a fit with a least \( \chi^2/d.o.f = 14.3 \) which means SU(3) symmetry is not a good symmetry for semileptonic charmed anti-baryon decays. We have then carried out detailed analyses with SU(3) symmetry breaking effect due to mass difference between \( s \) quark and \( u/d \) quark mass. In one scenario, we obtain a reasonable description of all relevant data with \( \chi^2/d.o.f = 1.6 \). As an estimation, we give the branching ratios for

\[
\Lambda_c^+ \to n \ell^+ \nu_\ell, \quad \Xi_c^+ \to \Sigma^0 \ell^+ \nu_\ell, \quad \Xi_c^0 \to \Lambda^0 \ell^+ \nu_\ell \text{ and } \Xi_c^0 \to \Sigma^- \ell^+ \nu_\ell \text{ in some scenarios.}
\]

We have also extended the analysis to the semileptonic decays of anti-triplet beauty baryons. However, the lack of experimental data prevents us from an in-depth study. Instead, we find a set of SU(3) relations in Eq. (27) and isospin relation in Eq. (29) between the decay widths of such processes. Our results will help to explore the physics behind these SU(3) symmetry breaking experimental data with more experimental data available in the future.

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