NEGATIVE COHOMOLOGY AND THE ENDO MORPHISM RING OF THE TRIVIAL MODULE

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Abstract. Let \( k \) be a field of characteristic 2 and let \( H \) be a finite group or group scheme. We show that the negative Tate cohomology ring \( \tilde{H}^{\leq 0}(H, k) \) can be realized as the endomorphism ring of the trivial module in a Verdier localization of the stable category of \( kG \)-modules for \( G \) an extension of \( H \). This means in some cases that the endomorphism of the trivial module is a local ring with infinitely generated radical with square zero. This stands in stark contrast to some known calculations in which the endomorphism ring of the trivial module is the degree zero component of a localization of the cohomology ring of the group.

1. Introduction

Let \( G \) be a finite group and \( k \) a field of characteristic \( p > 0 \). Let \( \text{stmod}(kG) \) be the stable category of finitely generated \( kG \)-modules modulo projective modules. It is a tensor triangulated category and its thick tensor ideal subcategories have been classified in terms of support varieties. Given a thick tensor ideal \( \mathcal{M} \), a new category \( \mathcal{C} \) is obtained by localizing \( \text{stmod}(kG) \) at \( \mathcal{M} \) by inverting any map whenever the third object in the triangle of that map is in \( \mathcal{M} \). In such a category, the endomorphism of the trivial module \( \text{End}_\mathcal{C}(k) \) is important because the endomorphism ring of every module is an algebra over it. In favorable cases its action on other modules can be used to define support varieties and other invariants.

In the few cases where \( \text{End}_\mathcal{C}(k) \) has been computed, it turned out to be some sort of homogeneous localization of the cohomology ring of the group (see [12], [8], [5]). However, in all of those examples, the localization is with respect to a thick tensor ideal determined by a subvariety that is a hypersurface or union of hypersurfaces. It was clear that the technique in those examples used for determining \( \text{End}_\mathcal{C}(k) \) does not work even when the determining subvariety is a point in \( \mathbb{P}^2 \). The point of this paper is to show that in such a case the structure of \( \text{End}_\mathcal{C}(k) \) can be very different. In examples, we find that \( \text{End}_\mathcal{C}(k) \) is a local \( k \)-algebra whose maximal ideal is infinitely generated and has square zero.

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We work in the setting that $G = H \times C$ is a finite group scheme where $H$ and $C$ are subgroup schemes defined over $k$ and the group algebra $kC$ is the group algebra of a cyclic group of order $p$. The thick tensor ideal subcategory is collection of all finitely generated $kG$-modules whose variety is the image of the variety of $C$ under the restriction map. Thus a module in $\mathcal{M}$ is projective on restriction to a $kH$-module, but not projective when restricted to $kC$. In this setting we prove that

$$\text{End}_C(k) \cong \hat{H}^{-0}((H, k)),$$

the negative Tate cohomology ring of the subgroup scheme $H$. In the course of the proof we construct the idempotent modules associated to the subcategory $\mathcal{M}$. These are constructed directly from a $kH$-projective resolution of the trivial module for $H$, and it is this resolution that connects us to the Tate cohomology. In the case of group algebras, it has been shown that products in negative cohomology mostly vanish [2] whenever the $p$-rank of the group is at least 2. It is likely that the same holds for general group schemes whenever the Krull dimension of the cohomology ring of $H$ is at least 2.

2. Preliminaries

In this section we establish some notation and recall some known results. For general reference on cohomology see [7] or [4]. For basics on triangulated categories see [11]. We follow that development in [9] for support varieties.

Let $k$ be a field of characteristic $p > 0$, and let $G$ be a finite group scheme defined over $k$. Let $kG$ be its group algebra. Let $\text{mod}(kG)$ denote the category of finitely generated $kG$-modules and $\text{Mod}(kG)$ the category of all $kG$-modules. Recall that $kG$ is a cocommutative Hopf algebra which means that for $M$ and $N$, $kG$-modules, $M \otimes N$ is also a $kG$-module with action given by coalgebra map $kG \to kG \otimes kG$. If $G$ is a finite group, then $g(m \otimes n) = gm \otimes gn$ for $g \in G$, $m \in M$ and $n \in N$. By the symbol $\otimes$ we mean $\otimes_k$ unless otherwise indicated. In addition, $kG$ is self-injective so that projective $kG$-modules coincide with injective $kG$-modules.

For $M$ a $kG$-module, $\Omega^{-1}(M)$ is the cokernel of an injective hull $M \hookrightarrow I$, for $I$ injective. Inductively, we let $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n}(M))$, On the positive side, $\Omega(M)$ is the kernel of a projective cover $P \to M$, for $P$ projective, and $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$.

The stable category $\text{StMod}(kG)$ of $kG$-modules modulo projectives has the same objects as $\text{mod}(kG)$, but the morphisms are given by the formula

$$\text{Hom}_{kG}(M, N) = \text{Hom}_{\text{StMod}(kG)}(M, N) = \text{Hom}_{kG}(M, N)/\text{PHom}_{kG}(M, N)$$

where $\text{PHom}_{kG}$ is the set of homomorphisms that factor through a projective module. The stable category is a tensor triangulated category. Triangles correspond to short exact sequences in $\text{mod}(kG)$. The translation functor is $\Omega^{-1}$. Let $\text{StMod}(kG)$ denote the stable category of all $kG$-modules. It has the same properties.
The cohomology ring $H^*(G, k)$ is a finitely generated $k$-algebra, and every cohomology module $\text{Ext}^r_{kG}(M, N)$ is a finitely generated module over $H^*(G, k)$ for $M$ and $N$ in $\text{mod}(kG)$ [10]. Let $V_G(k) = \text{Proj} H^*(G, k)$ denote the projectivized prime ideal spectrum of $H^*(G, k)$. If $E$ is an elementary abelian $p$-group or rank $r$ (order $p^r$), then modulo its radical $H^*(E, k)/\text{Rad}(H^*(E, k)) \cong k[\zeta_1, \ldots, \zeta_r]$ is a polynomial ring in $r$ variables. Thus when the field is algebraically closed, $V_E(k) \cong \mathbb{P}^{r-1}$ is projective $r - 1$ space.

We define the support variety of a $kG$-module by the method of $\pi$-points [9]. For finite groups this is essentially the same as the development in [3]. A $\pi$-point for $G$ is a flat map $\alpha_K : K[t]/(t^p) \to KG_K$, where $K$ is an extension of $k$, and $\alpha_K$ factors through the group algebra of some unipotent abelian subgroup scheme $C_K \subseteq G_K$ of $G_K$. For $M$ a $kG$-module, let $\alpha_K^*(M_K)$ denote the restriction of $M_K = K \otimes M$ to a $K[t]/(t^p)$-module along $\alpha_K$. Two $\pi$-points $\alpha_K$ and $\beta_L$ are equivalent if for every finite dimensional $kG$-module $M$, $\alpha_K^*(M_K)$ is projective if and only if $\beta_L^*(M_L)$ is projective.

We say that a $\pi$-point $\alpha_K$ specializes to $\beta_L$ if, for any finitely generated $kG$-module $M$, the projectivity of $\alpha_K^*(M)$ implies the projectivity of $\beta_L^*(M)$. So two $\pi$-points are equivalent if each specializes to the other.

Let $V_G(k)$ denote the set of all equivalence classes of $\pi$-points. Then $V_G(k)$ is a scheme and is isomorphic as a scheme to $V_G(k)$. Essentially, the class of a $\pi$-point $\alpha_K$ corresponds to the homogeneous prime ideal that is the kernel of the restriction map $H^*(G, K) \to H^*(K[t]/(t^p), K)/\text{Rad}(H(K[t]/(t^p)))$ along $\alpha_K$. Thus, since we extend the field $k$, a $\pi$-point may correspond to generic point of a homogeneous irreducible subvariety of $H^*(G, k)$. The support variety $V_G(M)$ of a $kG$-module $M$ is the set of all equivalence classes of $\pi$-points $\alpha_K$ such that $\alpha_K^*(M_K)$ is not projective. If $M$ is a finitely generated module then $V_G(M)$ is a closed subvariety of $V_G(k)$. Otherwise, it is just a subset.

A subcategory of a tensor triangulated category is thick if it is closed under taking of direct summands. It is a thick tensor ideal if, in addition, the tensor product of an object in the subcategory with any other object is again in the subcategory. There is a complete classification of the thick tensor ideals of $\text{stmod}(kG)$. Suppose that $\mathcal{V}$ is a subset of $V_G(k)$ that is closed under specializations, meaning that if $\alpha_K$ specializes to $\beta_L$ and if $\alpha_K$ is in $\mathcal{V}$ then so is $\beta_L$. Let $\mathcal{M}_\mathcal{V}$ be the full subcategory of $\text{stmod}(kG)$ generated by all $kG$-modules $M$ such that $V_G(M) \subseteq \mathcal{V}$. The properties of the support variety are sufficient to insure that any $\mathcal{M}_\mathcal{V}$ is a thick tensor ideal in $\text{stmod}(kG)$.

**Theorem 2.1.** [4] [9] Every thick tensor ideal in $\text{stmod}(kG)$ is equal to $\mathcal{M}_\mathcal{V}$ for some some subset $\mathcal{V} \subseteq V_G(k)$ which is closed under specialization.
If \( \mathcal{M} \) is a thick subcategory of a triangulated category \( \mathcal{C} \), then the Verdier localization of \( \mathcal{C} \) at \( \mathcal{M} \) is the category whose objects are the same as those of \( \mathcal{C} \) and whose morphisms are obtained by inverting a morphism if the third object in the triangle of that morphism is in the subcategory \( \mathcal{M} \). Thus, a morphism from \( L \) to \( N \) in the localized category has the form

\[
N \xrightarrow{\theta} M \xrightarrow{\gamma} L
\]

where the third object in the triangle of the map \( \theta \) is in \( \mathcal{M} \). So in the localized category \( \theta^{-1}\gamma \) is a morphism.

### 3. Resolution modules

Assume that \( k \) is a field of characteristic \( p \), and consider a finite group scheme of the form \( G = H \times C \) where \( C \) is the group scheme of a cyclic group of order \( p \) and \( H \) is finite group scheme defined over \( k \). Let \( z \) be a generator for \( C \), and \( Z = z - 1 \), so that \( kC \cong k[Z]/(Z^p) \) and \( kG \cong kH \otimes k[Z]/(Z^p) \).

Suppose that the complex

\[
\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0
\]

is a complex of \( kH \)-module, with all \( C_i = \{0\} \) for \( i < 0 \). We use the complex to define a sequence of \( kG \)-modules which we call resolution modules. For any \( n > 0 \), let \( M(P_*, n) \) be the \( kG \)-module whose restriction to \( kH \) is the infinite direct sum

\[
C_0 \oplus C_1^{p-1} \oplus C_2 \oplus C_3^{p-1} \oplus \cdots \oplus C_{2n-1}^{p-1}.
\]

For \( i \) odd, define the action of \( Z \) on \( (m_1, \ldots, m_{p-1}) \in C_i^{p-1} \) to be

\[
Z(m_1, \ldots, m_{p-1}) = (0, m_1, \ldots, m_{p-2}) + \partial(m_{p-1}) \in C_i^{p-1} \oplus C_{i-1}
\]

while for \( i = 2j \) and \( m \in C_i \), let

\[
Zm = \begin{cases} 
0 & \text{if } m \in C_0 \\
(\partial(m), 0, \ldots, 0) \in C_{2j-1}^{p-1} & \text{if } m \in C_{2j}, \ j > 0.
\end{cases}
\]

Note that the action of \( Z \) commutes with that of \( kH \), since the boundary maps \( \partial \) are \( kH \)-homomorphisms. Moreover, \( Z^p M = \{0\} \) because \( C_* \) a complex. Consequently, the relations define a \( kG \)-module.

We have a nested sequence of modules

\[
M(C_*, 1) \subseteq M(C_*, 2) \subseteq M(C_*, 3) \subseteq \cdots \subseteq M(C_*, \infty).
\]

where \( M(C_*, \infty) \) is the limit. That is \( M(C_*, \infty) \) is the module whose restriction to \( H \) is \( \oplus_{i \geq 0} (C_{2i} \oplus C_{2i+1}^{p-1}) \) with the action by \( Z \) defined as above.

Note that if \( C_* \) is a complex of finitely generated modules, then every \( M(C_*, n) \) is finitely generated, though \( M(C_*, \infty) \) may not be.
Lemma 3.1. Suppose that \( C_* \) and \( D_* \) are chain complexes of \( kH \)-modules in non-negative degrees. We have the following.

1. Any chain map \( \sigma : C_* \to D_* \) in even degrees induces a homomorphism \( M(\sigma) : M(C_*, n) \to M(D_*, n) \) for every \( n \geq 0 \) and \( n = \infty \).
2. For every \( n \geq 0 \) and for \( n = \infty \), \( M(C_* \oplus D_*, n) \cong M(C_*, n) \oplus M(D_*, n) \).
3. If \( J \) is a subgroup scheme of \( H \), then the restriction of \( M(C_*, n) \) to \( J \times C \) is isomorphic to \( M((C_*)_i, kJ, n) \) where \( (C_*)_i \) is the restriction of \( C_* \) to a complex of \( kJ \)-modules.
4. If \( C_* \) is an exact complex of projective modules, then in the stable module category \( M(C_*, \infty) \) is zero.
5. If \( C_* \) and \( D_* \) are projective resolutions of the same module \( N \), then in the stable category \( M(C_*, \infty) \cong M(D_*, \infty) \).

Proof. The first item is clear since any chain map commutes with the boundary map and hence the \( kH \)-map \( M(\sigma) : M(C_*, n) \to M(D_*, n) \), which is defined on the direct sum of the terms of the complex, is a \( kG \)-homomorphism. The proofs of Items 2 and 3 are straightforward.

Suppose that \( C_* \) is an exact complex of projective modules. Then \( C_* \) is a direct sum of complexes having the form \( 0 \to D_{i+1} \to D_i \to 0 \). For \( n > i \), either \( M(D_*, n) \cong D_{i+1}^{p-1} \oplus D_i \) or \( M(D_*, n) \cong D_{i+1} \oplus D_i^{p-1} \) as \( kH \)-modules. Because \( \partial \) maps \( D_{i+1} \) isomorphically onto \( D_i \) and \( D_i \) is projective as a \( kH \)-module, we conclude that \( M(D_*, n) \cong D_i \otimes k[Z]/(Z^p) \) is a projective module.

Suppose that \( C_* \) and \( D_* \) are projective resolutions of the same module \( N \). Then there are chain maps \( \sigma : C_* \to D_* \) and \( \tau : D_* \to C_* \) that lift the identity of \( N \). That is, the compositions \( \sigma \tau \) and \( \tau \sigma \) are homotopic to the identity maps on \( D_* \) and \( C_* \), respectively. It follows that there are exact complexes \( P_* \) and \( Q_* \) of projective \( kH \)-modules such that \( C_* \oplus P_* \cong D_* \oplus Q_* \). Thus part (4) follows from parts (2) and (3). \( \square \)

From the above we see that there is a functor \( \Gamma : \text{stmod}(kH) \to \text{StMod}(kG) \) that takes a \( kH \)-module \( N \) to \( M(P_*, \infty) \) where \( P_* \) is a \( kH \)-projective resolution of \( N \). Moreover, it can be checked that this is a functor of triangulated categories, since for any map between objects in \( \text{stmod}(kH) \), the mapping cone of the induce map on projective resolutions is a projective resolution of the third object in the triangle of that map.

What is interesting, is that if we assume that \( p = 2 \) and adjust the Hopf algebra structure on \( kG \), then \( \Gamma \) is also a functor of tensor triangulated categories. That is, the normal coalgebra structure on the group algebra \( kC \) is the diagonal which takes a group element \( g \) to \( g \otimes g \). Because \( kG \) is a product of algebras \( kG \cong \ldots \).
is the direct sum $Z \otimes C$ with the augmentation and boundary maps negated. That is, we set $kH \otimes Z$.

Now note that if $P_\ast$ and $Q_\ast$ are projective resolutions of $kH$-modules $L$ and $N$, then in $M(P_\ast \otimes Q_\ast, \infty)$ we have that $Z(p \otimes q) = p \otimes Zq + Zp \otimes q = p \otimes \partial(q) + \partial(p) \otimes q = \partial(p \otimes q)$. Thus we have, using the Lie coalgebra structure, that $\Gamma(L \otimes M) = M(P_\ast \otimes Q_\ast, \infty) \cong M(P_\ast, \infty) \otimes M(Q_\ast, \infty) = \Gamma(L) \otimes \Gamma(N)$.

In addition, it implies that, with the Lie coalgebra structure,

$$\Gamma(k) \otimes \Gamma(k) \cong \Gamma(k \otimes k) = \Gamma(k)$$

so that $\Gamma(k)$ is an idempotent module. In the Section 6 we show that $\Gamma(k)$ is idempotent even without the change in the Hopf structure.

4. An exact sequence

Assume that $k$ and $G = H \times C$ are as before. The purpose of this section is to construct a triangle in $\text{StMod}(kG)$ that in Section 6 is shown to be the canonical triangle of idempotent modules associated to the thick subcategory of $\text{stmod}(kG)$ consisting of modules whose support varieties are in the image of $V_G(k)$ in $V_G(k)$. Let $k$ denote the complex of $kH$-modules that has $k$ in degree 0 and the zero module in all other degrees.

Suppose that $P_\ast$ is a projective resolution of the trivial $kH$-module $k$. Let $C_\ast$ be the augmented complex in nonnegative degrees:

$$
\ldots \longrightarrow P_2 \overset{-\partial}{\longrightarrow} P_1 \overset{-\partial}{\longrightarrow} P_0 \overset{-\varepsilon}{\longrightarrow} k \longrightarrow 0
$$

with the augmentation and boundary maps negated. That is, we set $C_0 = k$ and $C_i = P_{i-1}$, for $i > 0$. Let $N(P_\ast, n)$ be the module whose restriction to a $kH$-module is the direct sum

$$
k \oplus P_0^{p-1} \oplus P_1 \oplus P_2^{p-1} \oplus \cdots \oplus P_{n-1}
$$

Multiplication by $Z$ annihilates the direct summand $k$. For $m = (m_1, \ldots, m_{p-1}) \in P_2^{p-1}$, define

$$
Zm = \begin{cases}
-\varepsilon(m_{p-1}) + (0, m_1, \ldots, m_{p-2}) & \text{if } i = 0 \\
-\partial(m_{p-1}) + (0, m_1, \ldots, m_{p-2}) & \text{if } i > 0
\end{cases}
$$

For $m \in P_{2i-1}$, let $Zm = -(\partial(m), 0, \ldots, 0) \in P_{2i-2}^{p-1}$. Thus, $N(P_\ast, n)$ looks like $N(C_\ast, n)$ except that it has an odd rather than even number of $kH$-summand. In the limit, $N(P_\ast, \infty) \cong M(C_\ast, \infty)$. 
It is easy to see that Lemma 3.1 holds for this construction. The main result of this section is the construction of an exact sequence with the form given in the next proposition.

**Proposition 4.1.** For the projective resolution $P_*$ as above and any $n > 0$, including $n = \infty$, there is a projective $kG$-module $Q$ and an exact sequence

$$0 \longrightarrow M(P_*, n) \xrightarrow{\theta} k \oplus Q \xrightarrow{\mu} N(P_*, n) \longrightarrow 0$$

where the class of the map $\theta$ in $\text{Hom}_{kG}(M(P_*, n), k)$ is the class of the map induced by the augmentation $\varepsilon : P_* \rightarrow k$ and the class of the map $\mu$ in $\text{Hom}_{kG}(k, N(P_*, n))$ is the class of the map induced by the degree zero inclusion of $k$ into the augmented projective resolution $(P_*, \varepsilon)$.

**Proof.** For $i \geq 0$, let $Q_i = P_i \otimes k[Z]/(Z^p)$ which is a projective module over $kG = kH \otimes k[Z]/(Z^p)$. Let $Q = Q_0 \oplus \cdots \oplus Q_{2n-1}$ or let $Q = Q_0 \oplus Q_1 \oplus \cdots$ in the case that $n = \infty$. We define the maps $\theta$ and $\mu$ as follows. Let $\ell_Q$ and $\ell_N$ be a generator for the $kH$-summand isomorphic to $k$ in $k \oplus Q$ and in $N(P_*, n)$, respectively. Then

$$\theta(m) = \varepsilon(m)\ell_Q \otimes 1 \oplus m \otimes Z^{p-1} \in k \oplus Q_0 \quad \text{for } m \in P_0.$$  

For $0 < i < n$, let

$$\theta(m) = \partial(m) \otimes 1 \oplus m \otimes Z^{p-1} \in Q_{2i-1} \oplus Q_{2i} \quad \text{for } m \in P_{2i}.$$  

For $1 \leq i \leq n$ and $(m_1, \ldots, m_{p-1}) \in P_{2i-1}^{p-1}$, let

$$\theta(m_1, \ldots, m_{p-1}) = \sum_{j=1}^{p-1} \partial(m_j) \otimes Z^{j-1} \oplus \sum_{j=1}^{p-1} m_j \otimes Z^i \in Q_{2i-1} \oplus Q_{2i}.$$  

The map $\mu$ is given by the following rules. First, let $\mu(\ell_Q) = \ell_N$. For an element

$$\sum_{j=0}^{p-1} m_j \otimes Z^j \in Q_1,$$  

$$\mu(\sum_{j=0}^{p-1} m_j \otimes Z^j) = \begin{cases} -\varepsilon(m_{p-1}) \oplus (m_0, \ldots, m_{p-2}) \in k \oplus P_0^{p-1} & \text{if } i = 0 \\ -\partial(m_{p-1}) \oplus (m_0, \ldots, m_{p-2}) \in P_{i-1} \oplus P_i^{p-1} & \text{if } i \text{ is even} \\ -(\partial m_1, \ldots, \partial m_{p-1}) \oplus m_0 \in P_{i-1}^{p-1} \oplus P_i & \text{if } i \text{ is odd} \end{cases}$$

It is easy to see that $\theta$ and $\mu$ are $kH$-homomorphisms. Hence to see that they are $kG$-homomorphisms, it is only necessary to show that the maps commute with the action of $Z$. We leave it also to the reader to check that $\mu \theta = 0$. Once this is done, the exactness of the sequence can be demonstrated by noting that there is an obvious filtration on the sequence itself such that the successive quotients have the form either $0 \rightarrow P_i \rightarrow k \oplus Q_i \rightarrow P_i^{p-1} \rightarrow 0$ or $0 \rightarrow P_i^{p-1} \rightarrow k \oplus Q_i \rightarrow P_i \rightarrow 0$ where the maps are induced by $\theta$ and $\mu$. One can show that these quotient sequences are exact. Finally, the maps to and from the summand $k$ in the middle term of the sequence can be determined to be as asserted from the construction. \[\square\]
5. Idempotent Modules

In this section, we review some information that we require on idempotent modules. We also give a brief description of a calculation of the endomorphism ring of the trivial module in the stable category localized at a thick subcategory defined by the subvariety of an ideal generated by a single element in $H^*(G, k)$ (see Theorem 2.1). This material is taken mostly from Rickard’s paper [12]. Variations on the theme and other accounts, can be found in [8], [6] and the last section of [5])

In general, associated to a thick tensor ideal $\mathcal{M}$ in $\text{stmod}(kG)$, for any $X$ in $\text{StMod}(kG)$, there is a distinguished triangle in $\text{StMod}(kG)$ having the form

$$\begin{align*}
\mathcal{E}_M(X) & \xrightarrow{\theta_X} X \xrightarrow{\mu_X} F_M(X) \xrightarrow{} \Omega^{-1}(\mathcal{E}_M(X))
\end{align*}$$

and having certain universal properties [12]. Let $\mathcal{M}^\oplus$ denote the closure of $\mathcal{M}$ in $\text{StMod}(kG)$ under arbitrary direct sums. The map $\theta_X$ is universal for maps from objects in $\mathcal{M}^\oplus$ to $X$, meaning that if $Y$ is in $\mathcal{M}^\oplus$ then any map $Y \to X$ factors through $\theta_X$. The map $\mu_X$ is universal for maps from $X$ to $\mathcal{M}$-local objects. An object $Y$ is $\mathcal{M}$-local if $\text{Hom}_{kG}(M, Y) = \{0\}$ for all $M$ in $\mathcal{M}$. The universal property says that for a module $Y$ that is $\mathcal{M}$-local, any map $X \to Y$ factors through $\mu_X$.

For a module $X$, the canonical triangle for $X$ is the tensor product of $X$ with the canonical triangle for $k$. The modules $\mathcal{E}_M(k)$ and $F_M(k)$ are idempotent module in that $\mathcal{E}_M(k) \otimes \mathcal{E}_M(k) \cong \mathcal{E}_M(k)$ and $F_M(k) \otimes F_M(k) \cong F_M(k)$ in the stable category. In addition, the two are orthogonal meaning that $\mathcal{E}_M(k) \otimes F_M(k)$ is projective, i. e. zero in the stable category.

It is helpful to know the support varieties of these modules. The following is well known, but we sketch a proof.

**Proposition 5.2.** Suppose that $\mathcal{V}$ is a collection of subvarieties of $\mathcal{V}_G(k)$ that is closed under specialization. Let $\mathcal{M} = \mathcal{M}_\mathcal{V}$, the thick tensor ideal of all finitely generated $kG$-modules $M$ such that $\mathcal{V}_G(M)$ is in $\mathcal{V}$. Then $\mathcal{V}_G(\mathcal{E}_M(k)) = \mathcal{V}$ and $\mathcal{V}_G(\mathcal{F}_M(k)) = \mathcal{V}_G(k) \setminus \mathcal{V}$.

**Proof.** The fact that $\mathcal{E}_M(k) \otimes F_M(k)$ is projective, implies that their support varieties are disjoint. The fact that the trivial module is the third object in a triangle involving the two implies that $\mathcal{V}_G(\mathcal{E}_M(k)) \cup \mathcal{V}_G(\mathcal{F}_M(k)) = \mathcal{V}_G(k)$. If $V$ a closed subvariety of $\mathcal{V}$, then the universal property says that that the identity homomorphism of a finitely generated module $M$ with $\mathcal{V}_G(M) = V$ factors through $M \otimes \mathcal{E}_M(k)$. Thus $V \in \mathcal{V}_G(\mathcal{E}_M(k))$. On the other hand, if $V \in \mathcal{V}_G(k)$ is not in $\mathcal{V}$, then there is a module $M$ in $\text{StMod}(kG)$ whose support variety is the one (generic) point $V$. Then $M$ is $\mathcal{M}$-local, and the universal property implies that $M$ is a direct summand of $M \otimes F_M(k)$. So $V$ is not in $\mathcal{V}_G(\mathcal{E}_M(k))$. \qed
Choose a nonnilpotent element \( \zeta \in H^n(G, k) \) for some \( n > 0 \) and let \( V = V_G(\zeta) \) be the variety of the ideal generated by \( \zeta \). Note that \( \zeta \) is represented by a cocycle \( \zeta : k \to \Omega^{-n}(k) \). For convenience, we denote the shifts of this map \( \Omega^t(k) \to \Omega^{t-n}(k) \), also by \( \zeta \). Let \( \mathcal{M}_V \) be the thick tensor ideal consisting of all finitely generated \( kG \)-modules \( M \) with \( V_G(M) \subseteq V \). The cohomology of any element in \( \mathcal{M} \) is annihilated by a power of \( \zeta \). As a consequence, for \( X \) in \( \text{stmod}(kG) \), any map \( \tau : k \to X \), whose third object in its triangle is in \( \mathcal{M} \), has the property that it is a factor of \( \zeta^t : k \to \Omega^{-tn}(k) \) for some \( t \), sufficiently large. That is, there is a map \( \beta : X \to \Omega^{-tn}(k) \) such that \( \zeta^t = \beta \tau \).

Thus it can be shown \([12]\) that the module \( \mathcal{F}_\mathcal{M}(k) \) can be taken to be the direct limit (to be precise, we take a homotopy colimit) of the system

\[
k \xrightarrow{\zeta} \Omega^{-n}(k) \xrightarrow{\zeta} \Omega^{-2n}(k) \xrightarrow{\zeta} \Omega^{-3n}(k) \xrightarrow{\zeta} \cdots
\]

Likewise, \( \mathcal{E}_\mathcal{M}(k) \) can be taken to be the homotopy colimit of the third objects in the triangles of the maps \( \zeta^t : k \to \Omega^{-tn}(k) \). It is the third object in the triangle \( k \to \mathcal{F}_\mathcal{M}(k) \). For \( X \) in \( \text{StMod}(kG) \), the canonical triangle \([5.1]\) involving \( X \) is the tensor product of \( X \) with this one. Also \( \mathcal{F}_\mathcal{M}(X) \) is \( \mathcal{M} \)-local and the universal properties are satisfied.

From all of this, it is routine to show the following (see \([12]\)).

**Proposition 5.3.** Let \( \mathcal{C} \) be the Verdier localization of the category \( \text{stmod}(kG) \) at the thick tensor ideal \( \mathcal{M}_V \) for \( V = V_G(\zeta) \) as above. Then the endomorphism ring \( \text{Hom}_\mathcal{C}(k, k) \) of the trivial module \( k \) is the degree zero part of the localized cohomology ring \( H^*(G, k)[\zeta^{-1}] \).

**Proof.** Recall that there is a natural identification \( \text{Hom}(k, \Omega^{-m}(k)) \cong H^m(G, k) \) for any \( m \geq 0 \). Choose an element in \( \text{Hom}_\mathcal{C}(k, k) \). It must have the form \( \tau^{-1}\gamma \) where \( \gamma : k \to X \) and \( \tau : k \to X \) has the property that the third object in the triangle of \( \tau \) is in \( \mathcal{M}_V \). Then as above, for some \( t \) there exist \( \zeta : X \to \Omega^{-tn}(k) \) such that \( \zeta^t = \tau \beta \). So we have a diagram

\[
k \xrightarrow{\tau} X \xrightarrow{\gamma} k \xrightarrow{\beta} \Omega^{-tn}(k)
\]

and \( \tau^{-1}\gamma = (\beta \tau)^{-1}(\beta \gamma) = \zeta^{-t}\beta \gamma \) where \( \beta \gamma \) is an element of \( H^t(G, k) \). \( \square \)

We end this section with a straightforward calculation that is needed later.

**Proposition 5.4.** Suppose that \( G = \langle y, z \rangle \) is an elementary abelian group of order \( p^2 \), and \( H = \langle y \rangle \). Let \( k \) be a field of characteristic \( p \). Let \( P_\ast \) be a minimal \( kH \)-projective resolution of the trivial module \( k \). Let \( V \) be the variety corresponding

\[
\text{ENDOMORPHISM RING OF THE TRIVIAL MODULE 9}
\]
to the point defined by the subgroup \( \langle z \rangle \), and let \( \mathcal{M}_V \) be the thick tensor ideal of finitely generated module \( M \) such that \( V_G(M) = V \). Then \( \mathcal{E}_M(k) \cong M(P_*, \infty) \), \( \mathcal{F}_M(k) \cong N(P_*, \infty) \) and the canonical triangle of \( k \) as in [5, 7] is the triangle as given in Proposition 4.1, defined by the augmentation map \( \varepsilon : P_* \to k \).

Proof. First notice that in the minimal resolution \( P_* \), every \( P_i \cong kH \), a \( p \) dimensional module with basis consisting of \( 1, Y, \ldots, Y^{p-1} \) where \( Y = 1 + y \). The map \( \partial : P_{2i+1} \to P_{2i} \) takes \( 1 \) to \( Y \) and \( \partial : P_{2i} \to P_{2i-1} \) takes \( 1 \) to \( Y^{p-1} \). Let \( Z = 1 + z \). Hence, \( N(P_*, n) \) has a basis \( u_0, \ldots, u_{2n-1} \) where \( u_{2i} = (1, 0, \ldots, 0) \in P_{2i}^{p-1} \), and \( u_{2i+1} = 1 \in P_{2i+1} \) for \( i = 0, \ldots, n-1 \). Thus, we can see from the definition of \( N(P_*, n) \) that

\[ Y u_{2i} = -Z u_{2i+1} \] for \( 0 \leq i \leq n - 1 \) and \( Y^{p-1} u_{2i-1} = -Z^{p-1} u_{2i} \) for \( 1 \leq i \leq n - 1 \).

The map \( \varepsilon : k \to N(P_*, n) \) induced by the augmentation \( P_* \to k \) takes \( 1 \to Z^{p-1} u_0 \).

An injective \( kH \)-resolution of \( k \) has the form

\[ 0 \longrightarrow k \overset{\varepsilon}{\longrightarrow} R_0 \longrightarrow R_1 \longrightarrow \ldots \]

where every \( R_i \cong kH \), \( \varepsilon(1) \in Y^{p-1} R_0 \), and the boundary maps alternate between multiplication by \( Y \) and by \( Y^{p-1} \). Similarly, for \( A = kZ/(Z^p) \) an \( A \)-injective resolution of \( k \) has the form \( 0 \to k \to Q_0 \to Q_1 \ldots \) where every \( Q_i \cong A \) and the maps are as above with \( Z \) substituted for \( Y \). Thus, a \( kG \)-injective resolution of the \( k \) is the tensor product \( Q_* \otimes R_* \). Now, \( \Omega^{-2n}(k) \) is the quotient

\[ \Omega^{-2n}(k) = (Q_* \otimes R_*)_{2n-1}/\partial((Q_* \otimes R_*)_{2n-2}). \]

This module is generated by the classes of the element \( v_i = 1 \otimes 1 \) in \( Q_i \otimes R_{2n-i-1} \) for \( i = 0, \ldots, 2n - 1 \). We have that for \( 1 \otimes 1 \in Q_{2i} \otimes R_{2(n-i)-2} \),

\[ \partial(1 \otimes 1) = Z \otimes 1 + Y \otimes 1 \in (Q_{2i+1} \otimes R_{2(n-i)-1}) \oplus (Q_{2i} \otimes R_{2(n-i)-1}), \]

while for \( 1 \otimes 1 \) in \( Q_{2i-1} \otimes R_{2(n-i)-1} \)

\[ \partial(1 \otimes 1) = Z^{p-1} \otimes 1 - 1 \otimes Y^{p-1} \in (Q_{2i} \otimes R_{2(n-i)-1}) \oplus (Q_{2i-1} \otimes R_{2(n-i)}). \]

Thus we have relations \( Z u_{2i+1} = -Y u_{2i} \) and \( Z^{p-1} u_{2i} = Y^{p-1} u_{2i-1} \). These are (except for signs) the same relations as for \( N(P_*, n) \), and hence \( \Omega^{-2n}(k) \cong N(P_*, n) \).

Next notice that, with the above identification, the map \( k \to N(P_*, n) \) takes \( 1 \in k \) to the class of \( Z^{p-1} u_0 \) which is contained in \( Q_0 \otimes R_{2n-1} \). In the case that \( n = 1 \), this is the cohomology class of the inflation to \( G \) of the polynomial generator \( \zeta \) in degree 2 of \( H^*(H, k) \). For \( n > 1 \) it represents the class of \( \zeta^n \). Thus we have a sequence of maps

\[ k \overset{\zeta}{\longrightarrow} \Omega^{-2}(k) \overset{\zeta}{\longrightarrow} \Omega^{-4}(k) \overset{\zeta}{\longrightarrow} \Omega^{-6}(k) \longrightarrow \ldots \]

By the construction of [12] that is summarized at the beginning of this section, we have that the canonical triangle associated to \( \mathcal{M}_V \) is the triangle of the map \( k \to N(P_*, \infty) \). This proves the proposition. \qed
6. Canonical triangles

Assume as before that \( kG = kH \otimes kC \) where \( kH \) and \( kC \) are Hopf subalgebras and \( kC \cong k[Z]/(Z^p) \). Let \( \mathcal{W} \) be the point in \( V_G(k) \) which is the image of the restriction map \( \text{res}_{G,C}^* : V_C(k) \rightarrow V_G(k) \). That is, \( \mathcal{W} \) is the equivalence class of the inclusion \( k[Z]/(Z^p) \rightarrow kG \) viewed as a \( \pi \)-point. Let \( \mathcal{M} = \mathcal{M}_\mathcal{W} \) be the thick tensor ideal of \( \text{stmod}(kG) \) consisting of all modules whose variety is in \( \mathcal{W} \). Thus a module in \( \mathcal{M} \) is a projective module on restriction to a \( kH \)-module and is not projective when restricted to \( kC \).

Let \( P_* \) be a \( kH \)-projective resolution of the trivial \( kH \)-module \( k \). As in the last section, let \( \mathcal{E} = \Gamma(k) = M(P_*, \infty) \) and \( \mathcal{F} = N(P_*, \infty) \). Let

\[
\mathcal{S} : \quad \mathcal{E} \xrightarrow{\mathcal{S}} k \xrightarrow{\mu} \mathcal{F} \xrightarrow{} \Omega^{-1}(\mathcal{E}).
\]

be the triangle of the exact sequence in Proposition [4.1].

For any \( X \in \text{StMod}(kG) \), tensoring with \( \mathcal{S} \) we obtain a triangle

\[
X \otimes \mathcal{S} : \quad \mathcal{E}(X) \xrightarrow{} X \xrightarrow{} \mathcal{F}(X) \xrightarrow{} \Omega^{-1}(\mathcal{E}(X)).
\]

Our objective is to show that the triangles \( X \otimes \mathcal{S} \) satisfy certain universal properties. This allows us to calculate the endomorphisms in the Verdier localization at the subcategory \( \mathcal{C} \). The first step is to establish the support varieties of the modules \( \mathcal{E} \) and \( \mathcal{F} \).

**Proposition 6.1.** The variety of \( \mathcal{E} \) is \( V_G(\mathcal{E}) = \mathcal{W} \), the set consisting of the single equivalence class or \( \pi \)-points as above. The variety of \( \mathcal{F} \) is \( V_G(k) \setminus \mathcal{W} \).

**Proof.** Suppose that \( K \) is an extension of \( k \) and let \( \alpha_K : K[t]/(t^p) \rightarrow KG \) be a \( \pi \)-point. Let \( \beta : k[t]/(t^p) \rightarrow KG \) be the \( \pi \) point given by \( \beta(t) = \mathcal{Z} \). Our objective is to show that if \( \alpha_K \) is not equivalent to \( \beta \) then the class of \( \alpha_K \) is not in the variety of \( \mathcal{E} \). We know that the class of \( \beta \) is in \( V_G(\mathcal{E}) \). So assume that \( \alpha_K \) is not equivalent to \( \beta \). Recall that \( V_G(k) \cong \text{Proj} \ H^*(G, k) \). The equivalence send the class of \( \alpha_K \) to the variety of the kernel of the induced map from the cohomology of \( G \) to that of the \( A_K = K[t]/(t^p) \). Recall that \( H^*(A, K)/\text{Rad}(H^*(A, K)) \cong K[T] \), a polynomial ring, and \( H^*(G, K) \cong H(H, K) \otimes H^*(C, K) \). Moreover, since \( C \) is a cyclic group of order \( p \), \( H^*(C, K)/\text{Rad}(H^*(C, K)) \cong K[\zeta] \) is a polynomial ring in the degree two element \( \zeta \). In particular, two \( \pi \)-points are in the same class if varieties of the corresponding kernels are the same.

For the \( \pi \)-point \( \alpha_K \), let \( \varphi_\alpha : H^*(G, K) \rightarrow H^*(A, K) \cong K[T] \) be the map induced by the restriction. Let \( \varphi : H^*(H, K) \rightarrow K[T] \) be the restriction of \( \varphi_\alpha \) to \( kH \) and then inflated to \( kG \). That is, we restrict \( \varphi_\alpha \) to the subring \( H(H, K) \otimes 1 \) of \( H^*(G, K) \) and compose this with the map \( H^*(G, K) \rightarrow H^*(H, K) \). Notice that if the image of \( \varphi \), which is a subring of \( K[T] \), is only the field \( K \), then \( \alpha_K \) is equivalent to \( \beta \). That
is, in such a case, if \( \eta \otimes \zeta^n \in H(H, K) \otimes H^*(C, K) \) and the degree of \( \eta \) is greater than zero, then \( \varphi(\eta \otimes \zeta^n) = \varphi(\eta \otimes 1)\varphi(1 \otimes \zeta^n) = 0 \). Thus \( \alpha_K \) and \( \beta \) correspond to the same element of \( \text{Proj} H^*(G, k) \). Hence, we may assume that the kernel of \( \varphi \) is a nonzero prime ideal in \( H^*(H, K) \), which is not the ideal of all positive elements.

Let \( \gamma_K : k[t]/(t^p) \to KH \otimes 1 \subseteq kG \) be a \( \pi \) point corresponding to \( \varphi \). Let \( KE = K[u, v]/(u^p, v^p) \) and let \( \mu : KE \to KG \) be defined by \( \mu(t) = \gamma_K(u) \) and \( \mu(u) = \beta_K(v) = 1 \otimes Z \). Note that \( \mu(u) \) and \( \mu(v) \) commute so that the given conditions on \( \mu \) define a homomorphism. Moreover, \( \mu \) is a flat embedding since \( \gamma_K \) is a flat embedding and \( \beta_K(K[t]/(t^p)) = 1 \otimes KC \). Note especially that the kernel of \( \varphi_\alpha \) is contained in the kernel of \( \gamma_K^* : H^*(G, K) \to H^*(E, K) \). Thus there is some \( \pi \) point \( \hat{\alpha}_K : K[t]/(t^p) \to KE \) such that the composition \( \gamma_K \hat{\alpha}_K \) is equivalent to \( \alpha_K \).

We consider the restriction \( M = \mu^*(E) \) to a KE-module. The projective resolution \( K \otimes P_\ast \) restricted to \( KE \) is KE-projective resolution of \( K \). Hence, we have that \( M \cong M(K \otimes P_\ast, \infty) \). Hence, from Lemma 3.1, we know the variety of \( M \) and know also that the third object in the triangle of \( \theta : M \to K \) is \( N = N(K \otimes P_\ast, \infty) \). The variety of consists only of the class of the \( \pi \)-point \( \hat{\beta} : K[t]/(t^p) \to KE \) given by \( t \mapsto v \). It follows that \( \hat{\alpha}_K \) is not in the variety of \( N \) and that \( \alpha_K \) is not in the variety of \( \mathcal{E} \). Since the restriction to \( kE \) takes triangles to triangles, we have also that \( \alpha_K \) is in the variety of \( \mathcal{F} \). This proves the theorem.

We can now establish the universal properties.

**Theorem 6.2.** Assume the hypotheses and notation of this section. For any \( kG \)-module \( X \), the triangle \( X \otimes \mathcal{S} \) is the canonical triangle as in 5.1 for \( X \) relative to \( \mathcal{M} \). In particular, \( 1_X \otimes \theta \) is universal with respect to maps from objects in \( \mathcal{M}^\otimes \) to \( X \) and \( 1_X \otimes \mu \) is universal with respect to maps from \( X \) to \( \mathcal{M} \)-local objects.

**Proof.** First note that \( \mathcal{E} \otimes \mathcal{F} \) is a projective module, zero in the stable category, because the intersection of the varieties of the two modules is empty. Thus tensoring \( \mathcal{S} \) with \( \mathcal{E} \) or \( \mathcal{F} \), we see that \( \mathcal{E} \) and \( \mathcal{F} \) are idempotent modules.

Suppose next that \( M \) is in \( \mathcal{M} \). Then

\[
\text{Hom}_{kG}(M, X \otimes \mathcal{F}) = \text{Hom}_{kG}(k, M^\ast \otimes X \otimes \mathcal{F}) - \{0\}.
\]

since \( M \) is finitely generated and the dual \( M^\ast \) of \( M \) and \( \mathcal{F} \) have disjoint varieties. Thus, \( X \otimes \mathcal{F} \) is \( \mathcal{M} \)-local, and by Lemma 5.2 of 12, \( X \otimes \mathcal{F} \) is \( \mathcal{M}^\otimes \)-local. We claim that the map from \( X \) to \( X \otimes \mathcal{F} \) is universal map from \( X \) to \( \mathcal{M} \)-local objects. The reason is that if \( N \) is \( \mathcal{M} \)-local then \( \text{Hom}_{kG}(X \otimes \mathcal{E}, N) = \{0\} \) and so \( \text{Hom}_{kG}(X \otimes \mathcal{F}, N) \cong \text{Hom}(X, N) \), the isomorphism being induced by the map \( X \to X \otimes \mathcal{F} \). So any map from \( X \) to \( N \) factors through \( 1_X \otimes \mu \).

Likewise, we can see that the map \( X \otimes \mathcal{E} \to X \) is universal with respect to map from an object \( Y \) in \( \mathcal{M}^\otimes \) to \( X \). That is, if \( M \) is in \( \mathcal{M}^\otimes \), then because \( \mathcal{F} \) is \( \mathcal{M}^\otimes \)-local, the canonical triangle for \( X \) yields an isomorphism \( \text{Hom}_{kG}(Y, X \otimes \mathcal{E}) \cong \text{Hom}_{kG}(Y, X \otimes \mathcal{F}) \).
\( \text{Hom}_{kG}(Y, X) \). Thus any map from \( Y \) to \( X \) factors through \( 1_X \otimes \theta \). This proves the theorem. \( \square \)

7. The endomorphism ring of the trivial module

In this section we assume the hypotheses and notation of Section 6, just previous. It is well known that the endomorphism ring of the trivial module in the localized category is associated to the structure of the module \( F \). Here is where we use our development of the structure of \( F \).

Let \( G = H \times C \) as before. Let \( \mathcal{M} \) denote the thick tensor ideal in \( \text{stmod}(kG) \) consisting of all \( kG \)-modules whose variety is the point in \( V_G(k) \) which is the image of the restriction map \( \text{res}_{G,C}^* : V_C(k) \to V_G(k) \). Let \( \mathcal{C} \) be the Verdier localization of \( \text{stmod}(kG) \) at \( \mathcal{M} \). We are interested in the ring \( \text{Hom}_{kG}(k, k) = \text{Hom}_{\mathcal{C}}(k, k) \). Given a morphism \( k \to \sigma M \leftarrow \tau k \) with the third object of the triangle of \( \sigma \) in \( \mathcal{M} \), we get a diagram

\[
\begin{array}{ccc}
U & \downarrow \nu & \mathcal{F} \\
\mathcal{F} & \xrightarrow{k} & \mathcal{F} \\
\downarrow \sigma & & \downarrow F \\
M & &
\end{array}
\]

with \( U \) in \( \mathcal{M} \). Because \( \mathcal{F} \) is \( \mathcal{M} \)-local the composition \( \mu \nu \) is the zero map. Thus, there is a map \( \varphi : M \to \mathcal{F} \) such that \( \varphi \sigma = \mu \). The morphism \( \sigma^{-1} \tau \) is equal to one of the form \( \mu^{-1} \alpha \) for \( \alpha = \varphi \tau \). Thus every endomorphism of \( k \) is factors through \( \mathcal{F} \).

Note that we need not worry that \( \mathcal{F} \) is infinitely generated. In the above, we could have replaced \( \mathcal{F} \) with the submodule whose restriction to \( kH \) is \( N(P_s, n)_{\downarrow H} \cong k \oplus P_0^{p-1} \oplus \cdots \oplus P_{2n-1} \) for \( n \) sufficiently large. By the same argument as above, this is the third object in the triangle of the map \( M(P_s, n) \to k \). The point of using \( \mathcal{F} \) is that we have a context to compute compositions of morphisms.

So suppose we have two endomorphisms of \( k \) in \( \mathcal{C} \), \( \mu^{-1} \alpha \) and \( \mu^{-1} \beta \). For the purposes of computing the product we may assume that there exist \( m \) and \( n \) such that \( \alpha(k) \subseteq P_m \) and \( \beta(k) \subseteq P_n \). That is, otherwise \( \alpha \) and \( \beta \) can be written as sums of such elements. The product is the map \( \mu^{-1} \alpha \beta \) in the diagram

\[
\begin{array}{ccc}
k & \xrightarrow{\mu} & \mathcal{F} \\
\alpha & \downarrow & \beta \\
\mathcal{F} & \xleftarrow{\alpha'} & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{F} & \xleftarrow{\text{Id}} & \mathcal{F} \\
\end{array}
\]
The construction of $\alpha'$ is through a chain map on complexes

\[
\begin{array}{ccccccc}
\ldots & P_2 & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & 0 \\
\downarrow & & & & & & & \\
\ldots & P_{m+2} & \rightarrow & P_{m+1} & \rightarrow & P_m & \rightarrow & 0 \\
\downarrow & & & & & & & \\
\ldots & P_{m+2} & \rightarrow & P_{m+1} & \rightarrow & P_m & \rightarrow & 0 \\
\downarrow & & & & & & & \\
\ldots & P_{m+2} & \rightarrow & P_{m+1} & \rightarrow & P_m & \rightarrow & 0 \\
\end{array}
\]

obtained by lifting the map $\alpha$. Then the chain map induces the map $\alpha' : \mathcal{F} \rightarrow \mathcal{F}$ as in Lemma 3.1. It is clear that the diagram commutes, i.e. $\alpha' \mu = \alpha$.

All of this sets up the proof of our main theorem.

**Theorem 7.1.** In the localized category $\mathcal{C}$ the endomorphism ring of $k$ is isomorphic to the negative Tate cohomology ring of $H$:

$$\text{Hom}_{kG}(k, k) \cong \hat{\text{H}}^{-0}(H, k).$$

**Proof.** Because the projective resolution $P_*$ is minimal, we know that $\text{Hom}_{kG}(k, P_n) \cong \hat{\text{H}}^{-n-1}(H, k)$. The Tate cohomology group $\hat{\text{H}}^{-n-1}(H, k)$ is also isomorphic to homotopy classes of chain maps of degree $-n - 1$ from the augmented complex $(P_*, \varepsilon)$ to itself as in the diagram. Indeed that diagram defines the correspondence. The product of two elements is defined (see [2]) as the composition of the chain maps. Thus the element represented by $\alpha' \beta$ is the product of the elements represented by $\alpha$ and $\beta$ in the Tate cohomology.

There are a few cases in which we can be very specific about the structure of the endomorphism ring of the trivial module in these localized categories. We end the section with two examples and a remark on the role of Hopf structures. The reader should recall that a group has periodic cohomology (in characteristic $p$) if and only if its Sylow $p$-subgroup is either cyclic or quaternion (with $p = 2$).

**Proposition 7.2.** Suppose that $H$ has periodic cohomology meaning that the cohomology ring $H^*(H, k)$ has Krull dimension one. Then the endomorphism ring of $k$ is the degree zero part of a localization of the cohomology ring of $G$. Specifically,

$$\text{Hom}_{kG}(k, k) \cong \text{Hom}_{\mathcal{C}}(k, k) \cong \sum_{n \geq 0} H_{nm}(G, k) \zeta^{-n}$$

where $\zeta$ is an element is a regular element of degree $m$ in $H^*(H, k)$.

**Proof.** Because $G = H \times C_p$ is a direct product, $H^*(G, k) \cong H^*(H, k) \otimes H^*(C_p, k)$. If $\zeta \in H^m(H, k)$ is a regular element, the variety $V$ is $V = V_G(\zeta)$, where we identify $\zeta$ with $\zeta \otimes 1$. So the proposition follows from the discussion in Section 5. We note that this result is compatible with Theorem 7.1 since $\sum_{n \leq 0} \hat{\text{H}}^{-n}(H, k) \cong \sum_{n \leq 0} \hat{\text{H}}^{-n}(H, k) \gamma^n$ where $\gamma$ is a degree two generator for $H^*(C_p, k)$. 

\[\square\]
On the other hand, suppose that $H$ has 2-rank at least 2. Then we get a very different result.

**Proposition 7.3.** Suppose that $H$ is an elementary abelian $p$-group of order at least $p^2$. Then $\text{Hom}_C(k, k)$ is a local $k$-algebra whose radical is infinitely generated and has square zero.

**Proof.** By [2], the product of any two elements in negative cohomology for $H$ zero. So by Theorem 7.1, the radical of $\text{Hom}_C(k, k)$ which consist of all elements in negative degrees, has square zero. At the same time, $\hat{H}^{-n}(H,k)$ is Tate dual to $H^{-n-1}(H,k)$ and hence its dimension grows as $n$ becomes large. □

**Remark 7.4.** The assumption throughout this article has been that $G = H \times C$ where $H$ and $C$ are subschemes, i.e. that $kH$ and $kC$ are cocommutative sub-Hopf algebras of $kG$. Even though the coalgebra structure played a role in the proofs, it has no role in the statement of the main theorem. Neither the endomorphism ring $\text{End}_C(k)$ nor the structure of the negative cohomology ring depend on even the existence of a coalgebra structure. Hence, if we have two algebras $kH$ and $kC$ on which we can create coalgebra maps of the right sort, then we come to the same result. For example, suppose that $G$ is an elementary abelian $p$-group and that $\alpha : k[t]/(t^p)$ is a $\pi$-point defined over $k$. Let $kC$ be the image of $\alpha$. Then $kC$ has a complement $kH$ such that $kH$ is the group algebra of an elementary abelian $p$-group of one smaller rank than $G$ and $kG \cong kH \otimes kC$. Let $\mathcal{M}$ denote the thick tensor ideal of modules whose support varieties are either empty or consist only of the class of $\alpha$, and let $C$ be the localization of $\text{stmod}(kG)$ at $\mathcal{M}$. Then the main theorem applies.

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