1. Introduction

In this chapter, we review 3-algebras that appear as fundamental properties of string theory. 3-algebra is a generalization of Lie algebra; it is defined by a tri-linear bracket instead of by a bi-linear bracket, and satisfies fundamental identity, which is a generalization of Jacobi identity [1], [2], [3]. We consider 3-algebras equipped with invariant metrics in order to apply them to physics.

It has been expected that there exists M-theory, which unifies string theories. In M-theory, some structures of 3-algebras were found recently. First, it was found that by using \( u(N) \oplus u(N) \) Hermitian 3-algebra, we can describe a low energy effective action of \( N \) coincident supermembranes [4], [5], [6], [7], [8], which are fundamental objects in M-theory.

With this as motivation, 3-algebras with invariant metrics were classified [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22]. Lie 3-algebras are defined in real vector spaces and tri-linear brackets of them are totally anti-symmetric in all the three entries. Lie 3-algebras with invariant metrics are classified into 4 algebra, and Lorentzian Lie 3-algebras, which have metrics with indefinite signatures. On the other hand, Hermitian 3-algebras are defined in Hermitian vector spaces and their tri-linear brackets are complex linear and anti-symmetric in the first two entries, whereas complex anti-linear in the third entry. Hermitian 3-algebras with invariant metrics are classified into \( u(N) \oplus u(M) \) and \( sp(2N) \oplus u(1) \) Hermitian 3-algebras.

Moreover, recent studies have indicated that there also exist structures of 3-algebras in the Green-Schwarz supermembrane action, which defines full perturbative dynamics of a supermembrane. It had not been clear whether the total supermembrane action including fermions has structures of 3-algebras, whereas the bosonic part of the action can be described by using a tri-linear bracket, called Nambu bracket [23], [24], which is a generalization of Poisson bracket. If we fix to a light-cone gauge, the total action can be described by using
Poisson bracket, that is, only structures of Lie algebra are left in this gauge [25]. However, it was shown under an approximation that the total action can be described by Nambu bracket if we fix to a semi-light-cone gauge [26]. In this gauge, the eleven dimensional space-time of M-theory is manifest in the supermembrane action, whereas only ten dimensional part is manifest in the light-cone gauge.

The BFSS matrix theory is conjectured to describe an infinite momentum frame (IMF) limit of M-theory [27] and many evidences were found. The action of the BFSS matrix theory can be obtained by replacing Poisson bracket with a finite dimensional Lie algebra’s bracket in the supermembrane action in the light-cone gauge. Because of this structure, only variables that represent the ten dimensional part of the eleven-dimensional space-time are manifest in the BFSS matrix theory. Recently, 3-algebra models of M-theory were proposed [26], [28], [29], by replacing Nambu bracket with finite dimensional 3-algebras’ brackets in an action that is shown, by using an approximation, to be equivalent to the semi-light-cone supermembrane action. All the variables that represent the eleven dimensional space-time are manifest in these models. It was shown that if the DLCQ limit of the 3-algebra models of M-theory is taken, they reduce to the BFSS matrix theory [26], [28], as they should [30], [31], [32], [33], [34], [35].

2. Definition and classification of metric Hermitian 3-algebra

In this section, we will define and classify the Hermitian 3-algebras equipped with invariant metrics.

2.1. General structure of metric Hermitian 3-algebra

The metric Hermitian 3-algebra is a map \( V \times V \times V \rightarrow V \) defined by \( (x, y, z) \mapsto [x, y; z] \), where the 3-bracket is complex linear in the first two entries, whereas complex anti-linear in the last entry, equipped with a metric \( <x, y> \), satisfying the following properties:

the fundamental identity

\[
[[x, y; z], v; w] = [[x, v; w], y; z] + [x, [y, v; w]; z] - [x, y; [z, w; v]]
\]  \hspace{1cm} \text{(id2)}

the metric invariance

\[
<x, v; w>, y> - <x, [y, w; v]> = 0
\]  \hspace{1cm} \text{(id3)}

and the anti-symmetry

\[
[x, y; z] = -[y, x; z]
\]  \hspace{1cm} \text{(id4)}

for
The Hermitian 3-algebra generates a symmetry, whose generators \( D(x, y) \) are defined by

\[
D(x, y)z = [z, x; y] \tag{6}
\]

From (\( = \)), one can show that \( D(x, y) \) form a Lie algebra,

\[
[D(x, y), D(v, w)] = D(D(x, y)v, w) - D(v, D(y, x)w) \tag{7}
\]

There is an one-to-one correspondence between the metric Hermitian 3-algebra and a class of metric complex super Lie algebras [19]. Such a class satisfies the following conditions among complex super Lie algebras \( S = S_0 \oplus S_1 \), where \( S_0 \) and \( S_1 \) are even and odd parts, respectively. \( S_1 \) is decomposed as \( S_1 = V \oplus \bar{V} \), where \( V \) is an unitary representation of \( S_0 \): for \( a \in S_0, u, v \in V, \)

\[
[a, u] \in V \tag{8}
\]

and

\[
< [a, u], v > + < u, [a^*, v] > = 0 \tag{9}
\]

\( \bar{v} \in \bar{V} \) is defined by

\[
\bar{v} = < , v > \tag{10}
\]

The super Lie bracket satisfies

\[
[V, V] = 0, \quad [\bar{V}, \bar{V}] = 0 \tag{11}
\]

From the metric Hermitian 3-algebra, we obtain the class of the metric complex super Lie algebra in the following way. The elements in \( S_0 \), \( V \), and \( \bar{V} \) are defined by (\( = \)), (\( = \)), and (\( = \)), respectively. The algebra is defined by (\( = \)) and

\[
[D(x, y), z] = D(x, y)z = [z, x; y]
\]

\[
[D(x, y), \bar{z}] = -D(y, x)z = -[z, y; x]
\]

\[
[x, \bar{y}] = D(x, y)
\]

\[
[x, y] = 0
\]

\[
[\bar{x}, \bar{y}] = 0
\]
One can show that this algebra satisfies the super Jacobi identity and (\texttrademark)-(\texttrademark) as in [19].

Inversely, from the class of the metric complex super Lie algebra, we obtain the metric Hermitian 3-algebra by

\[ [x, y; z] = \alpha [[y, \bar{z}], x] \quad \text{(id13)} \]

where \( \alpha \) is an arbitrary constant. One can also show that this algebra satisfies (\texttrademark)-(\texttrademark) for (\texttrademark) as in [19].

2.2. Classification of metric Hermitian 3-algebra

The classical Lie super algebras satisfying (\texttrademark)-(\texttrademark) are \( A(m - 1, n - 1) \) and \( C(n + 1) \). The even parts of \( A(m - 1, n - 1) \) and \( C(n + 1) \) are \( u(m) \oplus u(n) \) and \( sp(2n) \oplus u(1) \), respectively. Because the metric Hermitian 3-algebra one-to-one corresponds to this class of the super Lie algebra, the metric Hermitian 3-algebras are classified into \( u(m) \oplus u(n) \) and \( sp(2n) \oplus u(1) \) Hermitian 3-algebras.

First, we will construct the \( u(m) \oplus u(n) \) Hermitian 3-algebra from \( A(m - 1, n - 1) \), according to the relation in the previous subsection. \( A(m - 1, n - 1) \) is simple and is obtained by dividing \( sl(m, n) \) by its ideal. That is, \( A(m - 1, n - 1) = sl(m, n) \) when \( m \neq n \) and \( A(n - 1, n - 1) = sl(n, n) / \Lambda_{12n} \).

Real \( sl(m, n) \) is defined by

\[
\begin{pmatrix}
  h_1 & c \\
  ic^+ & h_2
\end{pmatrix}
\]

(id15)

where \( h_1 \) and \( h_2 \) are \( m \times m \) and \( n \times n \) anti-Hermite matrices and \( c \) is an \( n \times m \) arbitrary complex matrix. Complex \( sl(m, n) \) is a complexification of real \( sl(m, n) \), given by

\[
\begin{pmatrix}
  \alpha & \beta \\
  \gamma & \delta
\end{pmatrix}
\]

(id16)

where \( \alpha, \beta, \gamma, \) and \( \delta \) are \( m \times m, n \times m, m \times n, \) and \( n \times n \) complex matrices that satisfy

\[ \text{tr} \alpha = \text{tr} \delta \quad \text{(id17)} \]

Complex \( A(m - 1, n - 1) \) is decomposed as \( A(m - 1, n - 1) = S_0 \oplus V \oplus \bar{V} \), where
\[
\begin{pmatrix}
\alpha & 0 \\
0 & \delta
\end{pmatrix} \in S_0
\]
\[
\begin{pmatrix}
0 & \beta \\
0 & 0
\end{pmatrix} \in V
\]
\[
\begin{pmatrix}
0 & 0 \\
\gamma & 0
\end{pmatrix} \in \overline{V}
\]

\((=)\) is rewritten as \(V \rightarrow \overline{V}\) defined by
\[
B = \begin{pmatrix}
0 & \beta \\
0 & 0
\end{pmatrix} \mapsto B^\dagger = \begin{pmatrix}
0 & 0 \\
\beta^\dagger & 0
\end{pmatrix}
\]

where \(B \in V\) and \(B^\dagger \in \overline{V}\). \((=)\) is rewritten as
\[
\begin{pmatrix}
X, Y; Z
\end{pmatrix} = \alpha \begin{pmatrix}
Y, Z^\dagger
\end{pmatrix},
X = \alpha \begin{pmatrix}
yz^\dagger x - xz^\dagger y \\
0
\end{pmatrix}
\]

for
\[
X = \begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix} \in V
\]
\[
Y = \begin{pmatrix}
0 & y \\
0 & 0
\end{pmatrix} \in V
\]
\[
Z = \begin{pmatrix}
0 & z \\
0 & 0
\end{pmatrix} \in V
\]

As a result, we obtain the \(u(m) \oplus u(n)\) Hermitian 3-algebra,
\[
[x, y; z] = \alpha (yz^\dagger x - xz^\dagger y)
\]

where \(x, y,\) and \(z\) are arbitrary \(n \times m\) complex matrices. This algebra was originally constructed in [8].

Inversely, from \((=)\), we can construct complex \(A(m-1, n-1)\). \((=)\) is rewritten as
\[
D(x, y) = (xy^\dagger, y^\dagger x) \in S_0
\]

\((=)\) and \((=)\) are rewritten as
\[
\begin{align*}
[(xy^+, y^+x), (x'y^+, y^+x')] &= [(xy^+, x'y^+), (y^+x, y^+x')] \\
[(xy^+, y^+x), z] &= xy^+z - y^+x \\
[(xy^+, y^+x), w^+] &= y^+xw^+ - w^+xy^+ \\
[x, y^+] &= (xy^+, y^+x) \\
[x, y] &= 0 \\
[x^+, y^+] &= 0
\end{align*}
\]  
(id24)

This algebra is summarized as

\[
\begin{pmatrix}
xy^+ & z \\
0 & y^+x^\prime
\end{pmatrix}
\begin{pmatrix}
x'y^+ & z' \\
0 & y^+x^\prime
\end{pmatrix}
\]  
(id25)

which forms complex \(A(m - 1, n - 1)\).

Next, we will construct the \(sp(2n) \oplus u(1)\) Hermitian 3-algebra from \(C(n + 1)\). Complex \(C(n + 1)\) is decomposed as \(C(n + 1) = S_0 \oplus V \oplus \overline{V}\). The elements are given by

\[
\begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & c & -a^T
\end{pmatrix}
\in S_0
\]

\[
\begin{pmatrix}
0 & 0 & x_1 & x_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\in V
\]  
(id26)

where \(\alpha\) is a complex number, \(a\) is an arbitrary \(n \times n\) complex matrix, \(b\) and \(c\) are \(n \times n\) complex symmetric matrices, and \(x_1, x_2, y_1\) and \(y_2\) are \(n \times 1\) complex matrices. \((\Rightarrow)\) is rewritten as \(V \rightarrow \overline{V}\) defined by \(B \mapsto \overline{B} = UB^*U^{-1}\), where \(B \in V, \overline{B} \in \overline{V}\) and
\[
U = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]  
(id27)

Explicitly,

\[
B = \begin{pmatrix}
0 & 0 & x_1 & x_2 \\
0 & 0 & 0 & 0 \\
x_2^T & 0 & 0 \\
0 & -x_1^T & 0 & 0
\end{pmatrix} \rightarrow \bar{B} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & x_2^* & -x_1^* \\
-x_1^T & 0 & 0 & 0 \\
-x_2^T & 0 & 0 & 0
\end{pmatrix}
\]  
(id28)

(=) is rewritten as

\[
[X, Y; Z] = \alpha[[Y, Z], X]
\]

\[
\begin{pmatrix}
0 & 0 & y_1 & y_2 \\
0 & 0 & 0 & 0 \\
y_2^T & 0 & 0 & 0 \\
0 & -y_1^T & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & z_2^* & -z_1^* \\
-z_1^T & 0 & 0 & 0 \\
-z_2^T & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & x_1 & x_2 \\
0 & 0 & 0 & 0 \\
x_2^T & 0 & 0 & 0 \\
-x_1^T & 0 & 0 & 0
\end{pmatrix}
\]  
(id29)

\[
\begin{pmatrix}
0 & 0 & w_1 & w_2 \\
0 & 0 & 0 & 0 \\
w_2^T & 0 & 0 & 0 \\
0 & -w_1^T & 0 & 0
\end{pmatrix}
\]

for
where \( w_1 \) and \( w_2 \) are given by

\[
(w_1, w_2) = - (y_1 z_1^* + y_2 z_2^*) (x_1, x_2) + (x_1 z_1^* + x_2 z_2^*) (y_1, y_2) + (x_2 y_1^T - x_1 y_2^T) (z_2^*, - z_1^*)
\]

As a result, we obtain the \( sp(2n) \oplus u(1) \) Hermitian 3-algebra,

\[
[x, y; z] = \alpha ((y \odot \tilde{z}) x + (\tilde{z} \odot x) y - (x \odot y) \tilde{z})
\]

for \( x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \), where \( x_1, x_2, y_1, y_2, z_1, \) and \( z_2 \) are n-vectors and

\[
\tilde{z} = (z_2^*, - z_1^*)
\]

\[
a \odot b = a_1 \cdot b_2 - a_2 \cdot b_1
\]

3. 3-algebra model of M-theory

In this section, we review the fact that the supermembrane action in a semi-light-cone gauge can be described by Nambu bracket, where structures of 3-algebra are manifest. The 3-algebra Models of M-theory are defined based on the semi-light-cone supermembrane action. We also review that the models reduce to the BFSS matrix theory in the DLCQ limit.

3.1. Supermembrane and 3-algebra model of M-theory

The fundamental degrees of freedom in M-theory are supermembranes. The action of the covariant supermembrane action in M-theory [36] is given by
\[
S_{M2} = \int d^3 \sigma \left( \sqrt{-G} + \frac{i}{4} \bar{\Psi} \Gamma_{MN} \partial_\sigma \Psi (\Pi^M_\beta \Pi^N_\gamma + \frac{i}{2} \Pi^M_\beta \bar{\Psi} \Gamma^N_\gamma \partial_\gamma \Psi - \frac{1}{12} \bar{\Psi} \Gamma^M_\beta \partial_\beta \Psi \bar{\Psi} \Gamma^N_\gamma \partial_\gamma \Psi) \right)
\]

where \(M, N = 0, \ldots, 10, \alpha, \beta, \gamma = 0, 1, 2\), \(G_{\alpha \beta} = \Pi^M_\alpha \Pi^M_\beta\) and \(\Pi^M_\alpha = \partial_\alpha X^M - i \frac{1}{2} \bar{\Psi} \Gamma^M_\alpha \partial_\alpha \Psi\). \(\Psi\) is a SO(1, 10) Majorana fermion.

This action is invariant under dynamical supertransformations,

\[
\begin{align*}
\delta \Psi &= \\
\delta X^M &= -i \bar{\Psi} \Gamma^M
\end{align*}
\]

These transformations form the \(= 1\) supersymmetry algebra in eleven dimensions,

\[
\begin{align*}
[\delta_1, \delta_2] X^M &= -2i \Gamma^M \quad \text{(id 37)} \\
[\delta_1, \delta_2] \Psi &= 0 \quad \text{(id 38)}
\end{align*}
\]

The action is also invariant under the \(\kappa\)-symmetry transformations,

\[
\begin{align*}
\delta \Psi &= (1 + \Gamma) \kappa (\sigma) \\
\delta X^M &= i \bar{\Psi} \Gamma^M (1 + \Gamma) \kappa (\sigma) \quad \text{(id 39)}
\end{align*}
\]

where

\[
\Gamma = \frac{1}{3! \sqrt{-G}} \epsilon^{\alpha \beta \gamma} \Pi^L_\alpha \Pi^M_\beta \Pi^N_\gamma \Gamma_{LMN}
\]

If we fix the \(\kappa\)-symmetry (=) of the action by taking a semi-light-cone gauge [26] Advantages of a semi-light-cone gauge against a light-cone gauge are shown in [37], [38], [39]

\[
\Gamma^{012} \Psi = - \Psi
\]

we obtain a semi-light-cone supermembrane action,

\[
S_{M2} = \int d^3 \sigma \left( \sqrt{-G} + \frac{i}{4} \bar{\Psi} \Gamma_{\mu \nu} \partial_\sigma \Psi \left( \Pi^\mu_\beta \Pi^\nu_\gamma + \frac{i}{2} \Pi^\mu_\beta \bar{\Psi} \Gamma^\nu_\gamma \partial_\gamma \Psi - \frac{1}{12} \bar{\Psi} \Gamma^\mu_\beta \partial_\beta \Psi \bar{\Psi} \Gamma^\nu_\gamma \partial_\gamma \Psi \right) + \bar{\Psi} \Gamma_{ij} \partial_\sigma \Psi \partial_\beta X^I \partial_\gamma X^J \right)
\]
where $G_{\alpha \beta} = h_{\alpha \beta} + \Pi_+^{\mu} \Pi_+^{\nu}$, $\Pi_+^{\mu} = \partial_+ X^\mu - \frac{i}{2} \bar{\psi} \Gamma^\mu \partial_+ \psi$, and $h_{\alpha \beta} = \partial_+ X^I \partial_+ X_I$.

In [26], it is shown under an approximation up to the quadratic order in $\partial_+ X^\mu$ and $\partial_+ \psi$ but exactly in $X^I$, that this action is equivalent to the continuum action of the 3-algebra model of M-theory,

$$S_{cl} = \int d^3 \sqrt{-g} \left( \frac{1}{2} \{ X^I, X^J, X^K \}^2 - \frac{1}{2} \{ A_{\mu \nu} \{ \phi^a, \phi^b, X^I \} \}^2 - \frac{1}{3} E^{\mu \nu \lambda} A_{\mu \nu} \overline{A}_{\nu \rho} A_{\lambda \rho} \{ \phi^a, \phi^c, \phi^d \} \{ \phi^b, \phi^e, \phi^f \} + \frac{1}{2} \Lambda \right)$$

where $I, J, K = 3, \ldots, 10$ and $\{ \phi^a, \phi^b, \phi^c \} = \xi^a \partial_+ \phi^a \partial_+ \phi^b \partial_+ \phi^c$ is the Nambu-Poisson bracket. An invariant symmetric bilinear form is defined by $\int d^3 \sqrt{-g} \phi^a \phi^b$ for complete basis $\phi^a$ in three dimensions. Thus, this action is manifestly VPD covariant even when the world-volume metric is flat. $X^I$ is a scalar and $\Psi$ is a $SO(1, 2) \times SO(8)$ Majorana-Weyl fermion satisfying (=). $E^{\mu \nu \lambda}$ is a Levi-Civita symbol in three dimensions and $\Lambda$ is a cosmological constant.

The continuum action of 3-algebra model of M-theory (=) is invariant under 16 dynamical supersymmetry transformations,

$$\delta \frac{X^I}{\phi^a} = -i \Gamma^I \psi$$
$$\delta A_{\mu} = \{ \psi^a, \phi^b, X^I \} \Gamma_\mu \Gamma^I - \frac{1}{6} \{ X^I, X^J, X^K \} \Gamma_{IJK}$$

where $\Gamma^\mu \Gamma_\mu = -1$. These supersymmetries close into gauge transformations on-shell,

$$[\delta \nu, \delta \sigma] X^I = \Lambda_{\nu \sigma} \{ \phi^a, \phi^b, X^I \}$$
$$[\delta \nu, \delta \sigma] A_{\mu \nu} = \Lambda_{\nu \sigma} \{ \phi^a, \phi^b, A_{\mu \nu} \{ \phi^c, \phi^d, \phi^e \} \}$$

where gauge parameters are given by $\Lambda_{\nu \sigma} = 2i \Gamma^\mu \Gamma_{\mu \nu} - i \frac{i}{2} \Gamma_{IJK} \Gamma^J X^I X^K$. $O^A_{\nu \nu} = 0$ and $O^\Psi = 0$ are equations of motions of $A_{\mu \nu}$ and $\Psi$, respectively, where
\[
O^A_{\mu\nu} = A_{\mu a b} \{ \phi^a, \phi^b, A_{\nu c d} \{ \phi^c, \phi^d, \} \} - A_{\nu a b} \{ \phi^a, \phi^b, A_{\mu c d} \{ \phi^c, \phi^d, \} \}
+ E_{\mu\nu\lambda} \left( - \{ X_I, A_{\nu a b} \{ \phi^a, \phi^b, X_I \} \} + \frac{i}{2} \{ \bar{\Psi}, \Gamma^\lambda \Psi, \} \right)
\]

(\Rightarrow) implies that a commutation relation between the dynamical supersymmetry transformations is

\[
\delta_2 \delta_1 - \delta_1 \delta_2 = 0
\]

up to the equations of motions and the gauge transformations.

This action is invariant under a translation,

\[
\delta X^I(\sigma) = \eta^I, \quad \delta A^\mu(\sigma, \sigma') = \eta^\mu(\sigma) - \eta^\mu(\sigma')
\]

where \(\eta^I\) are constants.

The action is also invariant under 16 kinematical supersymmetry transformations

\[
\delta \Psi = \tilde{\eta}^I
\]

and the other fields are not transformed. \(\tilde{\eta}^I\) is a constant and satisfy \(\Gamma_{012} \tilde{\eta}^I = \tilde{\eta}^I\) and should come from sixteen components of thirty-two \(= 1\) supersymmetry parameters in eleven dimensions, corresponding to eigen values \(\pm 1\) of \(\Gamma_{012}\) respectively. This \(= 1\) supersymmetry consists of remaining 16 target-space supersymmetries and transmuted 16 \(\kappa\)-symmetries in the semi-light-cone gauge \([26], [25], [40]\).

A commutation relation between the kinematical supersymmetry transformations is given by

\[
\delta_2 \delta_1 - \delta_1 \delta_2 = 0
\]

A commutator of dynamical supersymmetry transformations and kinematical ones acts as

\[
(\delta_2 \delta_1 - \delta_1 \delta_2) X^I(\sigma) = i \bar{\eta}^I, \quad (\delta_2 \delta_1 - \delta_1 \delta_2) A^\mu(\sigma, \sigma') = \frac{i}{2} \bar{\eta}^\mu(\sigma) - \bar{\eta}^\mu(\sigma')
\]

where the commutator that acts on the other fields vanishes. Thus, the commutation relation is given by
\[ \delta_2 \delta_1 - \delta_1 \delta_2 = \delta_\eta \]  
(id53)

where \( \delta_\eta \) is a translation.

If we change a basis of the supersymmetry transformations as

\[ \delta' = \delta + \delta \]
\[ \delta' = i(\delta - \delta) \]  
(id54)

we obtain

\[ \delta_2' \delta_1' - \delta_1' \delta_2' = \delta_\eta \]
\[ \delta_2' \delta_1 - \delta_1' \delta_2' = \delta_\eta \]
\[ \delta_2' \delta_1 - \delta_1' \delta_2 = 0 \]  
(id55)

These thirty-two supersymmetry transformations are summarised as \( \Delta = (\delta', \delta') \) and (=) implies the \( = 1 \) supersymmetry algebra in eleven dimensions,

\[ \Delta_2 \Delta_1 - \Delta_1 \Delta_2 = \delta_\eta \]  
(id56)

3.2. Lie 3-algebra models of M-theory

In this and next subsection, we perform the second quantization on the continuum action of the 3-algebra model of M-theory: By replacing the Nambu-Poisson bracket in the action (=) with brackets of finite-dimensional 3-algebras, Lie and Hermitian 3-algebras, we obtain the Lie and Hermitian 3-algebra models of M-theory [26], [28], respectively. In this section, we review the Lie 3-algebra model.

If we replace the Nambu-Poisson bracket in the action (=) with a completely antisymmetric real 3-algebra’s bracket [21], [22],

\[ \{ \phi^a, \phi^b, \phi^c \} \rightarrow [T^a, T^b, T^c] \]  
(id58)

we obtain the Lie 3-algebra model of M-theory [26], [28],
\[ S_0 = -\frac{1}{12} [X^I, X^J, X^K]^2 - \frac{1}{2} (A_{\mu ab} [T^a, T^b, X^I])^2 \]

\[ - \frac{1}{3} E^{\mu \nu \lambda} A_{\mu ab} A_{\nu cd} A_{\lambda ef} [T^a, T^c, T^d] [T^b, T^e, T^f] \]

\[ - \frac{i}{2} \bar{\Psi} T^\mu A_{\mu ab} [T^a, T^b, \Psi] + \frac{i}{4} \bar{\Psi} T_{ij} [X^I, X^J, \Psi] \]

We have deleted the cosmological constant \( \Lambda \), which corresponds to an operator ordering ambiguity, as usual as in the case of other matrix models [27], [41].

This model can be obtained formally by a dimensional reduction of the \( = 8 \) BLG model [4], [5], [6],

\[ S_{BLG} = \int d^3 x \left< -\frac{1}{12} [X^I, X^J, X^K]^2 - \frac{1}{2} (D_\mu X^I)^2 - E^{\mu \nu \lambda} (\frac{1}{2} A_{\mu ab} \partial_\nu A_{\lambda cd} T^a [T^b, T^c, T^d] \right. \]

\[ + \frac{1}{3} A_{\mu ab} A_{\nu cd} A_{\lambda ef} [T^a, T^c, T^d] [T^b, T^e, T^f] \left. \right) \]

\[ + \frac{i}{2} \bar{\Psi} T^\mu D_\mu \Psi + \frac{i}{4} \bar{\Psi} T_{ij} [X^I, X^J, \Psi] \]

The formal relations between the Lie (Hermitian) 3-algebra models of M-theory and the \( = 8 \) \( = 8 \) BLG models are analogous to the relation among the \( = 4 \) super Yang-Mills in four dimensions, the BFSS matrix theory [27], and the IIB matrix model [41]. They are completely different theories although they are related to each others by dimensional reductions. In the same way, the 3-algebra models of M-theory and the BLG models are completely different theories.

The fields in the action (=) are spanned by the Lie 3-algebra \( T^a \) as \( X^I = X^I_a T^a \), \( \Psi = \Psi_a T^a \) and \( A^\mu = A_{\mu ab} T^a \otimes T^b \), where \( I = 3, \ldots, 10 \) and \( \mu = 0, 1, 2 \). \( < > \) represents a metric for the 3-algebra. \( \Psi \) is a Majorana spinor of \( SO(1,10) \) that satisfies \( \Gamma_{012} \Psi = \Psi \). \( E^{\mu \nu \lambda} \) is a Levi-Civita symbol in three-dimensions.

Finite dimensional Lie 3-algebras with an invariant metric is classified into four-dimensional Euclidean 4-algebra and the Lie 3-algebras with indefinite metrics in [9], [10], [11], [21], [22]. We do not choose 4 algebra because its degrees of freedom are just four. We need an algebra with arbitrary dimensions \( N \), which is taken to infinity to define M-theory. Here we choose the most simple indefinite metric Lie 3-algebra, so called the Lorentzian Lie 3-algebra associated with \( \mathfrak{u}(N) \) Lie algebra,

\[ [T^{-1}, T^a, T^b] = 0 \]

\[ [T^0, T^i, T^j] = [T^i, T^j] = f^{ij}_k T^k \]

\[ [T^i, T^j, T^k] = f^{ijk} T^{-1} \]
where $a = -1, 0, i (i = 1, \ldots, N^2)$. $T^i$ are generators of $u(N)$. A metric is defined by a symmetric bilinear form,

$$
<T^{-1}, T^0> = -1
$$

$$
<T^i, T^j> = \eta^{ij}
$$

(id62)

and the other components are 0. The action is decomposed as

$$
S = \text{Tr} \left( -\frac{1}{4} (x^0)^2 [x^I, x^J]^2 + \frac{1}{2} (x^0 [x^I, x^J])^2 - \frac{1}{2} (x^I b_\mu + [a_\mu, x^I])^2 - \frac{1}{2} E^{\mu\lambda\nu} b_\mu [a_\nu, a_\lambda] + i \bar{\psi}_0 \Gamma^\mu b_\mu \psi - \frac{i}{2} \bar{\psi} \Gamma^\mu [a_\mu, \psi] + \frac{i}{2} x^0 \bar{\psi} \Gamma_{ij} [x^I, \psi] - \frac{i}{2} \bar{\psi} \Gamma_{ij} [x^I, x^J] \psi \right)
$$

(id63)

where we have renamed $X^0 I \rightarrow x^0 I, X^i T^i \rightarrow x^I, \Psi^0 \rightarrow \psi^0, \Psi^I T^i \rightarrow \psi, 2A_{\mu 0} T^i = a_\mu$ and $A_{\mu \nu} [T^i, T^j] \rightarrow b_\mu a_\nu$ correspond to the target coordinate matrices $X^\mu$, whereas $b_\mu$ are auxiliary fields.

In this action, $T^{-1}$ mode; $X^0 I, \Psi^0$ or $A^0_{\mu 0}$ does not appear, that is they are unphysical modes. Therefore, the indefinite part of the metric (=) does not exist in the action and the Lie 3-algebra model of M-theory is ghost-free like a model in [42]. This action can be obtained by a dimensional reduction of the three-dimensional $= 8$ BLG model [4], [5], [6] with the same 3-algebra. The BLG model possesses a ghost mode because of its kinetic terms with indefinite signature. On the other hand, the Lie 3-algebra model of M-theory does not possess a kinetic term because it is defined as a zero-dimensional field theory like the IIB matrix model [41].

This action is invariant under the translation

$$
\delta x^I = \eta^I, \quad \delta a^\mu = \eta^\mu
$$

(id64)

where $\eta^I$ and $\eta^\mu$ belong to $u(1)$. This implies that eigen values of $x^I$ and $a^\mu$ represent an eleven-dimensional space-time.

The action is also invariant under 16 kinematical supersymmetry transformations

$$
\delta \psi = \cdot
$$

(id65)

and the other fields are not transformed. $\cdot$ belong to $u(1)$ and satisfy $\Gamma_{012} \cdot = \cdot$ and should come from sixteen components of thirty-two = 1 supersymmetry parameters in eleven dimensions, corresponding to eigen values $\pm 1$ of $\Gamma_{012}$ respectively, as in the previous subsection.

A commutation relation between the kinematical supersymmetry transformations is given by
\[
\delta_2 \delta_1 - \delta_1 \delta_2 = 0
\]  (id66)

The action is invariant under 16 dynamical supersymmetry transformations,

\[
\delta X^I = i \Gamma^I \Psi
\]

\[
\delta A_{\mu ab}[T^a, T^b, T_{c d}] = i \Gamma_\mu [X^I, \Psi, X^J, X^K] \Gamma_{IJK}
\]

\[
\delta \Psi = - A_{\mu ab}[T^a, T^b, X^I] \Gamma^\mu \Gamma_0 \frac{1}{6}[X^I, X^J, X^K] \Gamma_{IJK}
\]

where \(\Gamma_{012} = -1\). These supersymmetries close into gauge transformations on-shell,

\[
[\delta_1, \delta_2] X^I = \Lambda_{cd}[T^c, T^d, X^I]
\]

\[
[\delta_1, \delta_2] A_{\mu ab}[T^a, T^b, T^c, T^d] = A_{\mu ab}[T^a, T^b, T^c, T^d] - A_{\mu ab}[T^a, T^b, A_{\mu cd}[T^c, T^d, X^I] + 2 i \sqrt{2} \Gamma^\nu_1 O_\mu^A + i \bar{\psi} \Gamma^I \psi, X^I, X^J, X^K]
\]

where gauge parameters are given by \(\Lambda_{ab} = 2 i \sqrt{2} \Gamma^A_{1 A} X^a X^b\). \(O_{\mu \nu}^A = 0\) and \(O_\mu^\Psi = 0\) are equations of motions of \(A_{\mu \nu}\) and \(\Psi\), respectively, where

\[
O_{\mu \nu}^A = A_{\mu ab}[T^a, T^b, T^c, T^d] - A_{\nu ab}[T^a, T^b, T^c, T^d] + E_{\mu \nu \lambda} \left( -[X^I, A_{\lambda cd}[T^a, T^b, T^c, T^d]] + \frac{i}{2} \bar{\psi} \Gamma^\lambda \psi X^I, X^J, X^K \right)
\]

\[
O_\mu^\Psi = -\Gamma^\mu A_{\mu ab}[T^a, T^b, \Psi] + \frac{1}{2} \Gamma_{IJK} [X^I, X^J, \Psi]
\]

(\(=\)) implies that a commutation relation between the dynamical supersymmetry transformations is

\[
\delta_2 \delta_1 - \delta_1 \delta_2 = 0
\]  (id70)

up to the equations of motions and the gauge transformations.

The 16 dynamical supersymmetry transformations (\(=\)) are decomposed as
\[ \delta x^I = i \Gamma^I \psi \]
\[ \delta x_0^I = i \Gamma^I \psi_0 \]
\[ \delta x_{-1}^I = i \Gamma^I \psi_{-1} \]
\[ \delta \psi = - \left( b_\mu x_0^I + [a_\mu', x^I] \right) \Gamma^\mu \Gamma_1 - \frac{1}{2} x_0^I \left[ x^I, x^K \right] \Gamma_{ijk} \]
\[ \delta \psi_0 = 0 \]
\[ \delta \psi_{-1} = - \text{Tr} \left( b_\mu x^I \right) \Gamma^\mu \Gamma_1 - \frac{1}{6} \text{Tr} \left( \left[ x^I, x^J \right] x^K \right) \Gamma_{ijk} \]
\[ \delta a_\mu = i \Gamma^\mu \Gamma_1 \left( \psi_0 x_0^I - \psi x^I \right) \]
\[ \delta b_\mu = i \Gamma^\mu \Gamma_1 \left( x^I, \psi \right) \]
\[ \delta A_{\mu-1} = i \Gamma^\mu \Gamma_1 \frac{1}{2} \left( x_{-1}^I \psi_0 - \psi_{-1} x_0^I \right) \]
\[ \delta A_{\mu-10} = i \Gamma^\mu \Gamma_1 \frac{1}{2} \left( x_{-1}^I \psi - \psi_{-1} x_0^I \right) \]

and thus a commutator of dynamical supersymmetry transformations and kinematical ones acts as

\[ (\delta_2 \delta_1 - \delta_1 \delta_2) x^I = i \Gamma^I \gamma_2 = \eta^I \]
\[ (\delta_2 \delta_1 - \delta_1 \delta_2) a^\mu = i \Gamma^\mu \Gamma_1 x_0^I \gamma_2 = \eta^\mu \]
\[ (\delta_2 \delta_1 - \delta_1 \delta_2) A_{\mu-1}^I T^i = \frac{1}{2} i \Gamma^\mu \Gamma_1 x_{-1}^i \gamma_2 \]

where the commutator that acts on the other fields vanishes. Thus, the commutation relation for physical modes is given by

\[ \delta_2 \delta_1 - \delta_1 \delta_2 = \delta_\eta \]

where \( \delta_\eta \) is a translation.

\( (=), (\Rightarrow), \) and \( (\Rightarrow) \) imply the \( = 1 \) supersymmetry algebra in eleven dimensions as in the previous subsection.

### 3.3. Hermitian 3-algebra model of M-theory

In this subsection, we study the Hermitian 3-algebra models of M-theory [26]. Especially, we study mostly the model with the \( u(N) \oplus u(N) \) Hermitian 3-algebra \( (=) \).

The continuum action \( (=) \) can be rewritten by using the triality of \( SO(8) \) and the \( SU(4) \times U(1) \) decomposition [8], [43], [44] as
\[
S_{cl} = \int d^3 \sqrt{-g} \left( - V - A_{\mu ab \{Z^A, T^a, T^b\} A_{\delta \rho}^{\lambda \mu} \{Z_{\lambda}, T^c, T^d\} + \frac{1}{3} E^{\mu \nu \lambda} A_{\mu ab \nu \delta} A_{\lambda \rho}^{\lambda \mu} \{T^a, T^c, T^d\} \{T^b, T^f, T^e\} \right. \\
+ i \bar{\psi}^A \Gamma^\mu A_{\mu ab \{\psi_{aT}, T^a, T^b\} + i E^{\mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D}} \mathcal{Z}_D \{\bar{\psi}_{aT}, \psi_{bT}, \mathcal{Z}_{\mathcal{A}}\} \\
\left. - i \bar{\psi}^A \{\psi_{aT}, Z^b, Z_B\} + 2i \bar{\psi}^A \{\psi_{bT}, Z^A, Z_A\} \right)
\]

where fields with a raised \( A \) index transform in the 4 of SU(4), whereas those with lowered one transform in the \( \bar{4} \). \( A_{\mu ab} (\mu = 0, 1, 2) \) is an anti-Hermitian gauge field, \( Z^A \) and \( Z_A \) are a complex scalar field and its complex conjugate, respectively. \( \bar{\psi}_A \) is a fermion field that satisfies

\[
\Gamma^{012} \bar{\psi}_A = - \psi_A
\]

and \( \psi^A \) is its complex conjugate. \( E^{\mu \nu \lambda} \) and \( E^{\mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D}} \) are Levi-Civita symbols in three dimensions and four dimensions, respectively. The potential terms are given by

\[
V = \frac{2}{3} \gamma^C \gamma_D \gamma_B
\]

\[
\gamma^C \gamma_D = [Z^C, Z^D, Z_B] \cdot \frac{1}{2} \delta^C_B [Z^E, Z^D, Z_E] + \frac{1}{2} \delta^D_B [Z^E, Z^C, Z_E]
\]

If we replace the Nambu-Poisson bracket with a Hermitian 3-algebra's bracket [19], [20],

\[
\int d^3 \sqrt{-g} \rightarrow < > \\
\{ \phi^a, \phi^b, \phi^c \} \rightarrow [T^a, T^b, \bar{T}^c]
\]

we obtain the Hermitian 3-algebra model of M-theory [26],

\[
S = \int d^3 \sqrt{-g} \left( - V - A_{\mu ab \{Z^A, T^a, T^b\} A_{\delta \rho}^{\lambda \mu} \{Z_{\lambda}, T^c, T^d\} + \frac{1}{3} E^{\mu \nu \lambda} A_{\mu ab \nu \delta} A_{\lambda \rho}^{\lambda \mu} \{T^a, T^c, T^d\} \{T^b, T^f, T^e\} \right. \\
+ i \bar{\psi}^A \Gamma^\mu A_{\mu ab \{\psi_{aT}, T^a, T^b\} + i E^{\mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D}} \mathcal{Z}_D \{\bar{\psi}_{aT}, \psi_{bT}, \mathcal{Z}_{\mathcal{A}}\} \\
\left. - i \bar{\psi}^A \{\psi_{aT}, Z^b, Z_B\} + 2i \bar{\psi}^A \{\psi_{bT}, Z^A, Z_A\} \right)
\]

where the cosmological constant has been deleted for the same reason as before. The potential terms are given by
This matrix model can be obtained formally by a dimensional reduction of the $=6$ BLG action \[8\], which is equivalent to ABJ(M) action \[7\], \[45\]. The authors of \[46\], \[47\], \[48\], \[49\] studied matrix models that can be obtained by a dimensional reduction of the ABJM and ABJ gauge theories on $S^3$. They showed that the models reproduce the original gauge theories on $S^3$ in planar limits.

\[
S_{BLG} = \int d^3x < -V - D_\mu Z^A D^\mu Z^A + E^{\mu\nu\lambda} \left( \frac{1}{2} A_{\mu\nu\delta} A_{\lambda\delta\beta} \bar{T}^{\beta} \left[ T^a, T^b; \bar{T}^{c} \right] \right) \\
+ \frac{1}{3} A_{\mu\lambda a} A_{\nu\gamma c} A_{\lambda\delta d} \bar{T}^{d} \left[ T^a, T^c; \bar{T}^{b} \right] \left[ T^b, T^f; \bar{T}^{e} \right] \right) \\
- i \bar{\psi}^A \Gamma^\mu D_\mu \psi + \frac{i}{2} E_{ABCD} \bar{\psi}^A[Z^C, Z^D; \psi^B] - i \frac{1}{2} E_{ABCD} \bar{\psi}^A[Z^B, Z^C; \psi^D] \\
- i \bar{\psi}^A[Z^B, Z^C; \psi^D] >
\]

The Hermitian 3-algebra models of M-theory are classified into the models with $u(m) \oplus u(n)$ Hermitian 3-algebra (\=) and $sp(2n) \oplus u(1)$ Hermitian 3-algebra (\=). In the following, we study the $u(N) \oplus u(N)$ Hermitian 3-algebra model. By substituting the $u(N) \oplus u(N)$ Hermitian 3-algebra (\=) to the action (\=), we obtain

\[
S = \text{Tr} \left( -\frac{(2\pi)^2}{k^2} V - (Z^A A_{\mu}^R - A_{\mu}^L Z^A)(Z^A A_{R\mu} - A_{L\mu} Z^A) \right) - \frac{k}{2\pi} \bar{\psi}^A (A_{\mu}^R A_{\mu}^R - A_{\mu}^L A_{\mu}^L) - \frac{k}{2\pi} E^{\mu\nu\lambda} (A_{\mu}^R A_{\nu}^R A_{\lambda}^R - A_{\mu}^L A_{\nu}^L A_{\lambda}^L)
\]

where $A_{\mu}^R = -\frac{k}{2\pi} i A_{\mu a} T^a T^a$ and $A_{\mu}^L = -\frac{k}{2\pi} i A_{\mu a} T^a T^a$ are $N \times N$ Hermitian matrices. In the algebra, we have set $\alpha = \frac{2\pi}{k}$, where $k$ is an integer representing the Chern-Simons level. We choose $k = 1$ in order to obtain 16 dynamical supersymmetries. $V$ is given by

\[
V = \frac{1}{3} Z^A B^C Z^B C^D Z^C D^E + \frac{1}{3} Z^A C^D Z^B B^E Z^C D^F + \frac{4}{3} Z^A Z^B Z^C D^F + \frac{4}{3} Z^A Z^B Z^C D^F + \frac{4}{3} Z^A Z^B Z^C D^F
\]

By redefining fields as
we obtain an action that is independent of Chern-Simons level:

\[
S = \text{Tr} \left( - V - (Z^A A^R_{\mu} - A^L_{\mu} Z^A)(Z^A A^R_{\mu} - A^L_{\mu} Z^A)^\dagger - \frac{i}{3} E^{\mu\nu\lambda}(A^R_{\mu} A^R_{\nu} A^L_{\lambda} - A^L_{\mu} A^L_{\nu} A^L_{\lambda}) \right) \\
- \bar{\psi}^A \Gamma^\mu (\psi_A A^R_{\mu} - A^L_{\mu} \psi_A) + i E_{ABCD} \bar{\psi}^A Z^B \psi_B Z^D - i E_{ABCD} Z_D^{\dagger} \psi_B Z_C^{\dagger} A^L_{\mu} A^L_{\nu} A^L_{\lambda} \\
- i \bar{\psi}^A Z_B^{\dagger} Z_B + i \bar{\psi}^A Z_B^{\dagger} Z_B \psi_A + 2 i \bar{\psi}^A Z_B^{\dagger} Z_B \psi_A - 2 i \bar{\psi}^A Z_B^{\dagger} Z_B \psi_A \\
\text{as opposed to three-dimensional Chern-Simons actions.}
\]

If we rewrite the gauge fields in the action as \( A^L_{\mu} = A^L_{\mu} + b_{\mu} \) and \( A^R_{\mu} = A^R_{\mu} - b_{\mu} \), we obtain

\[
S = \text{Tr} \left( - V - ([A^L_{\mu}, Z^A] + [b_{\mu}, Z^A])([A^L_{\mu}, Z_A] - [b_{\mu}, Z_A]) + i E^{\mu\nu\lambda}(2 b_{\mu} b_{\nu} b_{\lambda} + 2 A^L_{\mu} A^L_{\nu} A^L_{\lambda}) \right) \\
+ \bar{\psi}^A \Gamma^\mu (A^L_{\mu} \psi_A + [b_{\mu}, \psi_A]) + i E_{ABCD} \bar{\psi}^A Z^B \psi_B Z^D - i E_{ABCD} Z_D^{\dagger} \psi_B Z_C^{\dagger} A^L_{\mu} A^L_{\nu} A^L_{\lambda} \\
- i \bar{\psi}^A \psi_A Z_B^{\dagger} Z_B + i \bar{\psi}^A Z_B^{\dagger} Z_B \psi_A + 2 i \bar{\psi}^A Z_B^{\dagger} Z_B \psi_A - 2 i \bar{\psi}^A Z_B^{\dagger} Z_B \psi_A \\
\]

where \([, , ]\) and \{, , \} are the ordinary commutator and anticommutator, respectively. The \(u(1)\) parts of \( A^\mu \) decouple because \( A^\mu \) appear only in commutators in the action, \( b^\mu \) can be regarded as auxiliary fields, and thus \( A^\mu \) correspond to matrices \( X^\mu \) that represents three space-time coordinates in M-theory. Among \( N \times N \) arbitrary complex matrices \( Z^A \), we need to identify matrices \( X^I \) \((I = 3, \ldots, 10)\) representing the other space coordinates in M-theory, because the model possesses not \( SO(8) \) but \( SU(4) \times U(1) \) symmetry. Our identification is

\[
Z^A = - i X^{A2} - X^{A6}, \\
X^I = \hat{X}^I + i x^I \\
\]

where \( \hat{X}^I \) and \( x^I \) are \( su(N) \) Hermitian matrices and real scalars, respectively. This is analogous to the identification when we compactify ABJM action, which describes N M2 branes, and obtain the action of N D2 branes [50], [7], [51]. We will see that this identification works also in our case. We should note that while the \( su(N) \) part is Hermitian, the \( u(1) \) part is anti-Hermitian. That is, an eigen-value distribution of \( X^\mu \), \( Z^A \), and not \( X^I \) determine the space-
time in the Hermitian model. In order to define light-cone coordinates, we need to perform Wick rotation: \( a^0 \rightarrow -i a^0 \). After the Wick rotation, we obtain

\[
\hat{A}^0 = A^0 - i a^0 \quad \text{(id89)}
\]

where \( A^0 \) is a \( su(N) \) Hermitian matrix.

### 3.4. DLCQ Limit of 3-algebra model of M-theory

It was shown that M-theory in a DLCQ limit reduces to the BFSS matrix theory with matrices of finite size [30], [31], [32], [33], [34], [35]. This fact is a strong criterion for a model of M-theory. In [26], [28], it was shown that the Lie and Hermitian 3-algebra models of M-theory reduce to the BFSS matrix theory with matrices of finite size in the DLCQ limit. In this subsection, we show an outline of the mechanism.

DLCQ limit of M-theory consists of a light-cone compactification, \( x^- \approx x^- + 2\pi R \), where \( x^\pm = \frac{1}{\sqrt{2}}(x^{10} \pm x^0) \), and Lorentz boost in \( x^{10} \) direction with an infinite momentum. After appropriate scalings of fields [26], [28], we define light-cone coordinate matrices as

\[
X^0 = \frac{1}{\sqrt{2}}(X^+ - X^-) \\
X^{10} = \frac{1}{\sqrt{2}}(X^+ + X^-) \quad \text{(id91)}
\]

We integrate out \( b^\mu \) by using their equations of motion.

A matrix compactification [52] on a circle with a radius \( R \) imposes the following conditions on \( X^- \) and the other matrices \( Y \):

\[
X^- (2\pi R) 1 = U^\dagger X^- U \\
Y = U^\dagger Y U \quad \text{(id92)}
\]

where \( U \) is a unitary matrix. In order to obtain a solution to (\( = \)), we need to take \( N \rightarrow \infty \) and consider matrices of infinite size [52]. A solution to (\( = \)) is given by \( X^- = \bar{X}^+ + \bar{X}^- \), \( Y = \bar{Y} \) and

\[
U = \begin{pmatrix}
\ddots & \ddots & \ddots \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & \ddots
\end{pmatrix} \otimes 1_{\mu \times \mu} \in U(N) \quad \text{(id93)}
\]
Backgrounds $\bar{X}^-$ are

$$
\bar{X}^- = - T^3 \bar{X}_0 T^0 - (2\pi R) \text{diag}(\ldots, s - 1, s, s + 1, \ldots) \otimes 1_{n \times n}
$$

(id94)

in the Lie 3-algebra case, whereas

$$
\bar{X}^- = -i(T^3 \bar{X}) 1 - i(2\pi R) \text{diag}(\ldots, s - 1, s, s + 1, \ldots) \otimes 1_{n \times n}
$$

(id95)

in the Hermitian 3-algebra case. A fluctuation $\bar{x}$ that represents $u(N)$ parts of $\bar{X}^-$ and $\bar{Y}$ is

$$
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\bar{x}(0) & \bar{x}(1) & \bar{x}(2) \\
\bar{x}(-1) & \bar{x}(0) & \bar{x}(2) \\
\bar{x}(-2) & \bar{x}(-1) & \bar{x}(2) \\
\bar{x}(-2) & \bar{x}(-1) & \bar{x}(2) \\
\vdots & \vdots & \vdots \\
\end{pmatrix}
$$

(id96)

Each $\bar{x}(s)$ is a $n \times n$ matrix, where $s$ is an integer. That is, the $(s, t)$-th block is given by $\bar{x}_{s,t} = \bar{x}(s - t)$.

We make a Fourier transformation,

$$
\bar{x}(s) = \frac{1}{2\pi R} \int_0^{2\pi R} d\tau x(\tau) e^{i(\tau-s)\frac{\gamma}{2}}
$$

(id97)

where $x(\tau)$ is a $n \times n$ matrix in one-dimension and $R \bar{R} = 2\pi$. From $(\tau)-=(\tau)$, the following identities hold:

$$
\sum_{s} \bar{x}_{s,t} \bar{x}_{t,u} = \frac{1}{2\pi R} \int_0^{2\pi R} d\tau x(\tau) x^{\dagger}(\tau) e^{i(s-u)\frac{\gamma}{2}}
$$

$$
\text{tr} \left( \sum_{s,t} \bar{x}_{s,t} \bar{x}_{t,u} \right) = V \frac{1}{2\pi R} \int_0^{2\pi R} d\tau \text{tr} (x(\tau) x^{\dagger}(\tau))
$$

(id98)

$$
\left[ \bar{x}^-, \bar{x} \right]_{s,t} = \frac{1}{2\pi R} \int_0^{2\pi R} d\tau \partial_\tau x(\tau) e^{i(s-t)\frac{\gamma}{2}}
$$

where $\text{tr}$ is a trace over $n \times n$ matrices and $V = \sum_n 1$.

Next, we boost the system in $x^{10}$ direction:
The DLCQ limit is achieved when \( T \to \infty \), where the "novel Higgs mechanism" [50] is realized. In \( T \to \infty \), the actions of the 3-algebra models of M-theory reduce to that of the BFSS matrix theory [27] with matrices of finite size,

\[
S = \frac{1}{g^2} \int \! \! \! \! d\tau \text{tr} \left( \frac{1}{2} (D_0 x^P)^2 - \frac{1}{4} [x^P, x^Q]^2 + \frac{1}{2} \overline{\psi} \Gamma^0 D_0 \psi - \frac{i}{2} \overline{\psi} \Gamma^P [x_P, \psi] \right)
\]

where \( P, Q = 1, 2, \ldots, 9 \).

### 3.5. Supersymmetric deformation of Lie 3-algebra model of M-theory

A supersymmetric deformation of the Lie 3-algebra Model of M-theory was studied in [53] (see also [54], [55], [56]). If we add mass terms and a flux term,

\[
S_m = \left\{ -\frac{1}{2} \mu^2 (x^I)^2 - \frac{i}{2} \mu \overline{\Psi} \Gamma_{3456} \Psi + H_{ijkl} [x^i, x^j, x^k, x^l] \right\}
\]

such that

\[
H_{ijkl} = \begin{cases} 
-\frac{\mu}{6} \delta_{ijkl} & (i, j, k, l = 3, 4, 5, 6 or 7, 8, 9, 10) \\
0 & (\text{otherwise})
\end{cases}
\]

From this action, we obtain various interesting solutions, including fuzzy sphere solutions [53].
4. Conclusion

The metric Hermitian 3-algebra corresponds to a class of the super Lie algebra. By using this relation, the metric Hermitian 3-algebras are classified into $u(m) \oplus u(n)$ and $sp(2n) \oplus u(1)$ Hermitian 3-algebras.

The Lie and Hermitian 3-algebra models of M-theory are obtained by second quantizations of the supermembrane action in a semi-light-cone gauge. The Lie 3-algebra model possesses manifest $= 1$ supersymmetry in eleven dimensions. In the DLCQ limit, both the models reduce to the BFSS matrix theory with matrices of finite size as they should.

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