Local Approximability of Minimum Dominating Set on Planar Graphs

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Abstract. We show that there is no deterministic local algorithm (constant-time distributed graph algorithm) that finds a \((7 - \epsilon)\)-approximation of a minimum dominating set on planar graphs, for any positive constant \(\epsilon\). In prior work, the best lower bound on the approximation ratio has been \(5 - \epsilon\); there is also an upper bound of 52.

1 Introduction

This work studies one of the last uncharted corners in the area of deterministic local algorithms: planar graphs.

A local algorithm is a distributed graph algorithm that runs in \(O(1)\) communication rounds, independently of the size of the network. While the theory of randomised local algorithms is still in its infancy, we have nowadays a good understanding of the capabilities of deterministic local algorithms.

For many classical graph problems, there are exactly matching upper and lower bounds on the best possible approximation ratio that can be achieved by a deterministic local algorithm [6]. In many cases, we can apply a straightforward two-step procedure to derive tight lower bounds:

1. Prove tight bounds for anonymous networks (without unique identifiers).
2. Apply a simulation argument [2] to show that unique identifiers do not help.

However, there are some isolated examples of natural questions in which the above two-step procedure fails badly. Perhaps the most intriguing example is dominating sets on planar graphs:

1. We do not have tight bounds for this problem in anonymous networks.
2. Planar graphs are not closed under lifts, and therefore the simulation argument [2] cannot be applied.

In this work we are interested in the smallest \(\alpha\) such that there is a deterministic local algorithm that finds an \(\alpha\)-approximation of a minimum dominating set in any planar graph. The current bounds are very far from being tight:

- \(5 - \epsilon < \alpha \leq 636\) for anonymous networks [1, 7],
- \(5 - \epsilon < \alpha \leq 52\) in the \(LOCAL\) model [1, 3, 4, 8].

In this work we give the first improvement on the lower bounds in six years: we prove a lower bound \(\alpha > 7 - \epsilon\) for both models, for any positive constant \(\epsilon\).

2 Proof Overview

Let \(A\) be a deterministic distributed algorithm with running time \(T = O(1)\) in the \(LOCAL\) model. Assume that \(A\) finds a dominating set \(D = A(G)\) in any planar graph \(G\).

Pick sufficiently large \(m \gg T\) and \(r\). Let \(m' = m - 2T\). We will construct a planar graph \(G\) with \(n = m^2r\) nodes as shown in Figure 1a. There are \(r\) blocks with \(m \times m\) nodes in each
The nodes of each block are partitioned to \textit{internal nodes} and \textit{boundary nodes}: there are \( m' \times m' \) internal nodes, and they are surrounded by boundary areas of width \( T \). Let \( B_i \) be the set of nodes in block \( i \), and let \( I_i \subseteq B_i \) be the set of internal nodes in block \( B_i \). We will prove the following lemma.

\textbf{Lemma 1.} For any \( m \) and any sufficiently large \( r \), we can assign unique identifiers in \( G \) so that \( I_i \subseteq \mathcal{A}(G)\) for all \( 1, 2, \ldots, r - \ell \), for some \( \ell = o(r) \).

In other words, all internal nodes of blocks \( 1, 2, \ldots, r - \ell \) are in the dominating set \( D = \mathcal{A}(G) \) produced by algorithm \( \mathcal{A} \). Now if we choose large enough \( m \) and \( r \), we can make the contributions of the boundary nodes and the contributions of the remaining \( o(r) \) blocks arbitrarily small. In particular, for any positive constant \( \epsilon' \), we can pick \( m \) and \( r \) such that \( |D| \geq (1 - \epsilon')n \).

On the other hand, there is a dominating set \( D^* \) which contains only a fraction \( 1/7 \) of the internal nodes; see Figure 1b. Therefore \( |D^*| \leq (1/7 + \epsilon')n \), and the claim follows: for any positive constant \( \epsilon \) we can show that algorithm \( \mathcal{A} \) cannot find a factor \( 7 - \epsilon \) approximation of a minimum dominating set on planar graphs.

\section{Proof of Lemma 1}

The proof uses the strategy of repeated applications of Ramsey’s theorem; cf. Czygrinow et al. [1, Lemma 4]. We will use the notation \( \mathcal{A}(G, v) \in \{0, 1\} \) to refer the \textit{local output} of node \( v \) when we apply algorithm \( \mathcal{A} \) to graph \( G \); we have \( \mathcal{A}(G, v) = 1 \) if node \( v \) is in the dominating set computed by algorithm \( \mathcal{A} \). By definition, \( \mathcal{A}(G, v) \) only depends on the radius-\( T \) neighbourhood of \( v \) in \( G \).

Let \( k = 2T + 1 \), \( K = k^2 \), and \( M = m^2 \). Consider any internal node \( v \in I_i \) of any block \( B_i \). The structure of graph \( G \) in the radius-\( T \) neighbourhood does not depend on the choice of \( v \). Hence the local output of node \( v \) only depends on the unique identifiers in the local neighbourhood. The local neighbourhood is contained within a rectangular \( k \times k \) region \( R_v \) around \( v \); see Figure 1c.

Let \( V = \{1, 2, \ldots, n\} \) be the set of unique identifiers. Consider any \( K \)-subset of identifiers \( X \subseteq V \), \( |X| = K \). We will associate a \textit{colour} \( c(X) \in \{0, 1\} \) with each such set, as follows:

1. Pick an internal node \( v \).
2. Assign the identifiers from \( X \) to region \( R_v \) in an increasing order by rows: the smallest \( k \) identifiers to the bottom row from left to right, etc. Assign the identifiers from \( V \setminus X \) to the remaining nodes arbitrarily.
3. Apply algorithm \( \mathcal{A} \), and set \( c(X) = \mathcal{A}(G, v) \).
Now we have defined a colouring of all $K$-subsets of $V$; by restriction, we also have a colouring of all $K$-subsets of any $V' \subseteq V$. We say that $Y \subseteq V$ is monochromatic if $c(X_1) = c(X_2)$ for any $K$-subsets $X_1$ and $X_2$ of $Y$. By Ramsey’s theorem [5] there exists an integer $N = N(K, M)$ such that the following holds: if $V'$ is any $N$-subset of $V$, then there always exists a monochromatic subset $Y \subseteq V'$ of size $M$.

Now we will pick $r$ and $\ell$ so that $\ell M > N$ and $\ell = o(r)$. Let $V_1 = V$. For each $i = 1, 2, \ldots, r-\ell$, we define the identifiers of block $i$ as follows.

1. As $|V_i| \geq N$, we can find a monochromatic subset $Y_i \subseteq V_i$ of size $M$.
2. Assign the identifiers from $Y_i$ to block $B_i$ in an increasing order by rows: the smallest $m$ identifiers to the bottom row from left to right, etc.
3. Set $V_{i+1} = V_i \setminus Y_i$.

Finally, assign the remaining $\ell M$ identifiers from $V_{r-\ell+1}$ to blocks $r - \ell + 1, \ldots, r$ arbitrarily.

To complete the proof, consider a block $i$, where $1 \leq i \leq r - \ell$. Let $v \in I_i$ be an internal node of the block. Consider the $k \times k$ region $R_v$ around $v$, and let $X_v$ be the set of unique identifiers assigned to region $R_v$. Observe that the identifiers of $X_v$ are assigned in an increasing order by rows. It follows that $A(G, v) = c(X_v)$, i.e., the local output of the internal node $v$ is simply the colour of subset $X_v$. Furthermore, $X_v \subseteq Y_i$ and $Y_i$ was monochromatic. Hence all internal nodes of block $i$ produce the same output. The common output cannot be 0; otherwise there would be nodes that are not dominated. Hence $I_i \subseteq A(G)$.

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