The Energy of Bianchi Type I and II Universes in Teleparallel Gravity

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Abstract

For certain models, the energy of the universe which includes the energy of both the matter and the gravitational fields is obtained by using the quasilocal energy-momentum in teleparallel gravity. It is shown that in the case of the Bianchi type I and II universes, not only the total energy but also the quasilocal energy-momentum for any region vanishes independently of the three dimensionless coupling constants of teleparallel gravity.

1 Introduction

The localization of energy and momentum of the gravitational field is one of the oldest and most controversial problems of the general theory of relativity [1]. After the energy-momentum pseudotensor of Einstein [2], several other prescriptions have been introduced, leading to a great variety of expressions for the energy-momentum pseudotensor of the gravitational field [3]. These pseudotensors are not covariant objects because they inherently depend on the reference frame, and thus cannot provide a truly physical local gravitational energy-momentum density. Consequently, the pseudotensor approach has been largely questioned, although never abandoned. More recently the idea of quasilocal (i.e. associated with a closed 2-surface) energy-momentum has become popular. A large number of definitions of quasilocal mass have been proposed [4].

It has been shown recently by Chen and Nester [5] that every energy-momentum pseudotensor can be associated with a particular Hamiltonian boundary term, which in turn determines a quasilocal energy-momentum. In this sense, it has been said that the Hamiltonian quasilocal energy-momentum rehabilitates the pseudotensor approach, and dispels the doubts about the physical meaning of these energy-momentum complexes.

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However, Bergqvist [7] studied several different definitions of quasi-local masses for the Kerr and Reissner-Nordström spacetimes and came to the conclusion that not even two of these definitions gave the same result. On the other hand, several authors studied various energy-momentum complexes and obtained stimulating results. Virbhadra and collaborators [8] have demonstrated, with several examples, that for many spacetimes (like the Kerr-Newman, Vaidya, Einstein-Rosen, Bonnor-Vaidya and all Kerr-Schild class spacetimes), different energy-momentum complexes give the same and acceptable energy distribution for a given spacetime.

Despite this difficulty, there has been several attempts that calculated the total energy of the expanding universe. Investigations [9] and [10] proposed that our universe may have arisen as a quantum fluctuation of the vacuum, and mentioned that any conservation law of physics need not have been violated at the time of its creation. Tryon proposed that our universe must have a zero net value for all conserved quantities and presented some arguments, using a Newtonian order of magnitude estimate, favoring the fact that the net energy of our universe may be indeed zero. Their model predicts a universe which is homogeneous, isotropic and closed, and consists equally of matter and anti-matter. The subject of the total energy of the expanding universe was re-opened by [11, 12, 13], using the definitions of the energy-momentum in general relativity. In one of these the Einstein energy-momentum pseudotensor was used to represent the gravitational energy [12], which led to the result that the total energy of a closed Friedman-Robertson-Walker (FRW) universe is zero. In another, the symmetric pseudotensor of Landau-Lifshitz was used [13]. In [14] and [15], the total energy of the anisotropic Bianchi models have been calculated using different pseudotensors, leading to similar results. Recently [16], it has been shown that the open, or critically open FRW universes, as well as Bianchi models evolving into de Sitter spacetimes also have zero total energy. Finally, a similar calculation for the closed FRW universe using the Einstein and Laudau-Lifshitz complexes in teleparallel gravity also led to the same conclusion [17]. There is a well known argument that the energy of a closed universe should vanish. But the argument does not apply to open universes.

By working in the context of teleparallel gravity, it will be shown that not only the total but also the quasilocal energy for any region of the Bianchi type I and II universes vanishes, independently of the three dimensionless coupling constants of teleparallel gravity. It should be remarked that this result is consistent with the results of Banerjee and Sen [14], Xulu, and Radinschi [15] calculated in the framework of general relativity. We will proceed according to the following scheme. In section 2, we review the main features of teleparallel gravity and the expression for the quasilocal energy-momentum in teleparallel gravity. In section 3, we find the tetrad field, the
non-zero components of the Weitzenb"ock connection, the torsion tensor, and then the quasilocal energy and momentum of the Bianchi type I and II universes are calculated. Finally, in section 4, we present a discussion of the result obtained.

2 Teleparallel Gravity

In teleparallel gravity, spacetime is represented by the Weitzenb"ock manifold $W^4$ of distant parallelism \[18\]. This gravitational theory naturally arises within the gauge approach based on the group of the spacetime translations. Denoting the translational gauge potential by $A^a_\mu$, the gauge covariant derivative for a scalar field $\Phi(x^\mu)$ reads \[19\]

$$D_\mu \Phi = h^a_\mu \partial_a \Phi,$$

where

$$h^a_\mu = \partial_\mu x^a + A^a_\mu$$

is the tetrad field, which satisfies the orthogonality condition

$$h^a_\mu h^a_\nu = \delta^\nu_\mu.$$  

The nontrivial tetrad field induces a teleparallel structure on spacetime which is directly related to the presence of the gravitational field; the Riemannian metric arises as

$$g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu.$$  

In this theory, the fundamental field is a nontrivial tetrad, which gives rise to the metric as a by-product. The parallel transport of the tetrad $h^a_\mu$ between two neighbouring points is encoded in the covariant derivative

$$\nabla_\nu h^a_\mu = \partial_\nu h^a_\mu - \Gamma^a_{\mu\nu} h^a_\alpha,$$

where $\Gamma^a_{\mu\nu}$ is the Weitzenb"ock connection, a connection presenting torsion, but no curvature. Imposing the condition that the tetrad be parallel transported in the Weitzenb"ock space-time, we obtain

$$\Gamma^a_{\mu\nu} = h^a_\alpha \partial_\nu h^a_\mu,$$

which gives the explicit form of the Weitzenb"ock connection in terms of the tetrad;

$$T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu}.$$  

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is the torsion of the Weitzenböck connection. As we already remarked, the curvature of the Weitzenböck connection vanishes identically as a consequence of absolute parallelism, or teleparallelism [20].

The action of teleparallel gravity in the presence of matter is given by

\[ S = \frac{1}{16\pi G} \int d^4x \, h \, S_{\lambda \tau \nu} \, T^{\lambda \tau \nu} + \int d^4x \, h \, \mathcal{L}_M, \]  

(8)

where \( h = \det(h^\alpha_\mu) \), \( \mathcal{L}_M \) is the Lagrangian of the matter field, and \( S_{\lambda \tau \nu} \) is the tensor

\[ S_{\lambda \tau \nu} = c_1 T^{\lambda \tau \nu} + \frac{c_2}{2} (T^{\tau \lambda \nu} - T^{\nu \lambda \tau}) + \frac{c_3}{2} (g^{\lambda \nu} T^{\nu \sigma \sigma} - g^{\tau \lambda} T^{\sigma \nu \sigma}), \]  

(9)

with \( c_1, c_2 \) and \( c_3 \) being the three dimensionless coupling constants of teleparallel gravity [20]. For the specific choice

\[ c_1 = \frac{1}{4}, \quad c_2 = \frac{1}{2}, \quad c_3 = -1, \]  

(10)

teleparallel gravity yields the so called teleparallel equivalent of general relativity [23].

By performing the variation in (8) with respect to \( h^\alpha_\mu \), we get the teleparallel field equations,

\[ \partial_\sigma (h S^{\lambda \tau \sigma}) - 4\pi G (ht^{\tau \lambda}) = 4\pi G h T^{\tau \lambda}, \]  

(11)

where

\[ t^{\tau \lambda} = \frac{1}{4\pi G} h \Gamma^{\nu}_{\sigma \lambda} S^{\tau \nu \sigma} - \delta^{\tau \lambda} \mathcal{L}_G \]  

(12)

is the energy-momentum pseudotensor of the gravitational field [21]. Rewriting the teleparallel field equations in the form

\[ h(t^{\tau \lambda} + T^{\tau \lambda}) = \frac{1}{4\pi G} \partial_\sigma (h S^{\tau \nu \sigma}), \]  

(13)

as a consequence of the antisymmetry of \( S^{\lambda \tau \sigma} \) in the last two indices, we obtain immediately the conservation law

\[ \partial_\tau [h(t^{\tau \lambda} + T^{\tau \lambda})] = 0. \]  

(14)

On the other hand, in the Hamiltonian formalism, the Hamiltonian for a finite region

\[ H(N) = \int_\Sigma N^\mu \mathcal{H}_\mu + \oint_{\partial \Sigma} \mathcal{B}(N) \]  

(15)
generates the spacetime displacement of a finite spacelike hypersurface $\Sigma$ along a vector field $N^\mu$. Noether’s theorems guarantee that $\mathcal{H}_\mu$ is proportional to the field equations. Consequently, the value depends only on the boundary $\mathcal{B}$, which gives the quasilocal energy-momentum [5]. The boundary term $\mathcal{B}(N)$ in teleparallel gravity [6, 21, 23] is given by

$$\mathcal{B}(N) = \frac{1}{8\pi G} N^\mu h S^\alpha_\beta (d\sigma)_{\alpha\beta}. \quad (16)$$

Thus, the quasilocal energy-momentum in teleparallel gravity is

$$P_\mu = \frac{1}{8\pi G} \oint_{\partial \Sigma} h S^\alpha_\beta (d\sigma)_{\alpha\beta}. \quad (17)$$

From this equation, we see that the energy and other quasilocal quantities are given by the integral of the $\mathcal{B}(N)$ over the 2-surface boundary $\partial \Sigma$ of the 3-hypersurface $\Sigma$.

3 Calculation of the total energy

In this section we will calculate the total energy of the Bianchi type I and II spacetimes using the quasilocal energy-momentum in teleparallel gravity given by equation (17).

- **Bianchi type I space-time**

  The homogeneous and anisotropic Bianchi type I spacetimes are expressed by the line element [14]

  $$ds^2 = dt^2 - e^l dx^2 - e^m dy^2 - e^n dz^2, \quad (18)$$

  where $l$, $m$, $n$ are functions of time $t$ only. Using relation (14), we obtain the tetrad components and its inverse

  \[
  h^n_\mu = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & e^l & 0 & 0 \\
  0 & 0 & e^m & 0 \\
  0 & 0 & 0 & e^n \\
  \end{pmatrix}, \quad h^\mu_a = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & e^{-l} & 0 & 0 \\
  0 & 0 & e^{-m} & 0 \\
  0 & 0 & 0 & e^{-n}, \\
  \end{pmatrix}.
  \]

  The corresponding non-vanishing Weitzenbock connection terms are:

  $$\Gamma^1_{10} = \dot{l}, \quad \Gamma^2_{20} = \dot{m} \quad and \quad \Gamma^3_{30} = \dot{n},$$
where dot denotes the derivative with respect to time t. Hence the non-vanishing
torsion components are:

\[
\begin{align*}
T^{1}_{01} &= -T^{1}_{10} = \dot{l}, \\
T^{2}_{02} &= -T^{2}_{20} = \dot{m}, \\
T^{3}_{03} &= -T^{3}_{30} = \dot{n}.
\end{align*}
\]

Then the non-zero components of the tensor \(S_{\nu}^{\sigma \tau}\) read:

\[
\begin{align*}
S^{01}_{11} &= -S^{10}_{11} = \left(c_{1} + \frac{c_{2}}{2}\right) \dot{l} + \frac{c_{3}}{2} \left(\dot{i} + \dot{m} + \dot{n}\right), \\
S^{02}_{22} &= -S^{20}_{22} = \left(c_{1} + \frac{c_{2}}{2}\right) \dot{m} + \frac{c_{3}}{2} \left(\dot{i} + \dot{m} + \dot{n}\right), \\
S^{03}_{33} &= -S^{30}_{33} = \left(c_{1} + \frac{c_{2}}{2}\right) \dot{n} + \frac{c_{3}}{2} \left(\dot{i} + \dot{m} + \dot{n}\right). \\
\end{align*}
\] (19)

Note that there is an interesting identity: \(S^{\alpha \beta \gamma} + S^{\beta \gamma \alpha} + S^{\gamma \alpha \beta} = 0\). Using Eq. (17), the quasilocal energy within any region \(\Sigma\) is

\[
P_{0} = \frac{1}{8\pi G} \oint_{\partial \Sigma} hS_{0}^{\alpha \beta} (d\sigma)_{\alpha \beta} = 0,
\] (20)

since \(S_{0}^{\alpha \beta} = 0\) for all \(\alpha, \beta\). Our calculation is consistent with the Radinschi
and Xulu results [15]. From Eq. (19) it is also readily follow that

\[
P_{i} = \frac{1}{4\pi G} \oint_{\partial \Sigma} hS_{i}^{0j} (d\sigma)_{0j} = 0.
\] (21)

• **Bianchi type II space-time**

The Bianchi type II line element [22] is

\[
ds^{2} = dt^{2} - f^{2}dx^{2} - g^{2}dy^{2} + 2xg^{2}dydz - (x^{2}g^{2} + f^{2})dz^{2},
\] (22)

where \(f(t)\) and \(g(t)\) depend only on time.

Using again the relation (14), we obtain the tetrad components and its inverse

\[
h^{\alpha}_{\mu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & g & -xg \\
0 & 0 & 0 & f
\end{pmatrix},
\]

\[
h_{\alpha}^{\mu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & f^{-1} & 0 & 0 \\
0 & 0 & g^{-1} & 0 \\
0 & 0 & x^{-1} & f^{-1}
\end{pmatrix}.
\]

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The non-vanishing connections are:

\[
\Gamma_{10} = \Gamma_{30} = \frac{\dot{f}}{f}, \quad \Gamma_{20} = \frac{\dot{g}}{g},
\]

\[
\Gamma_{20} = x \left( \frac{\dot{f}}{f} - \frac{\dot{g}}{g} \right), \quad \Gamma_{31} = -1.
\]

Modulo anti-symmetry the non-vanishing torsion components are:

\[
T_{01} = T_{03} = \frac{\dot{f}}{f}, \quad T_{02} = \frac{\dot{g}}{g},
\]

\[
T_{03} = x \left( \frac{\dot{f}}{f} - \frac{\dot{g}}{g} \right), \quad T_{13} = -1.
\]

Modulo anti-symmetry the non-vanishing components of \( S \) are:

\[
S_{01} = S_{03} = \left( c_1 + \frac{c_2}{2} + c_3 \right) \frac{\dot{f}}{f} + \frac{c_3 \dot{g}}{2 g},
\]

\[
S_{12} = S_{23} = -c_1 x \frac{g^2}{f^4},
\]

\[
S_{20} = \left( c_1 + \frac{c_2}{2} + \frac{c_3}{2} \right) \frac{\dot{g}}{g} + c_3 \frac{\dot{f}}{f}, \quad S_{21} = c_1 \frac{g^2}{f^4},
\]

\[
S_{30} = \left( c_1 + \frac{c_2}{2} \right) \left( \frac{\dot{f}}{f} - \frac{\dot{g}}{g} \right) x, \quad S_{31} = -c_1 x^2 \frac{g^2}{f^4} + \frac{c_2}{2} \frac{1}{f^2}.
\]  \( (23) \)

Therefore the quasilocal energy within any region is

\[
P_0 = \frac{1}{8\pi G} \oint_{\partial \Sigma} h S_0^{\alpha \beta} (d\sigma)_{\alpha \beta} = 0,
\]

\( (24) \)

since \( S_0^{\alpha \beta} = 0 \) for all \( \alpha \) and \( \beta \), a result consistent with that of Banerjee and Sen [14]. Also, using Eq. (23), a short calculation gives the quasilocal momentum

\[
P_i = \frac{1}{8\pi G} \oint_{\partial \Sigma} h S_i^{\alpha \beta} (d\sigma)_{\alpha \beta} = 0.
\]

\( (25) \)
4 Final Remarks

In order to compute the gravitational energy, we have considered the teleparallel quasilocal energy-momentum. Working in the context of teleparallel gravity, we have calculated the quasilocal energy-momentum of the Bianchi type I and II universes. Our basic result is that the quasilocal energy and momentum, which includes the energy of both the matter and the gravitational fields vanishes for all regions, not just for the total space. It is also independent of the three teleparallel dimensionless coupling constants, which means that it is valid not only in the teleparallel equivalent of general relativity, but also in any teleparallel model. According to these calculations, the total energy vanishes everywhere because the energy contributions from the matter and gravitational fields inside an arbitrary two-surface boundary $\partial \Sigma$ of the 3-hypersurface $\Sigma$ cancel each other.

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References

[1] Misner C. W., Thorne K. S. and Wheeler J. A.. (1973). Gravitation (Freeman, San Francisco).

[2] See Trautman A. (1958). in Gravitation: an Introduction to current research, ed. by L. Witten (Wiley, New York).

[3] Some examples are: Papapetrou A. (1951). Proc. R. Ir. Acad. A 25, 11; Bergmann P. G. and Thompson R. (1953). Phys. Rev. 89, 400; Landau L. D. and Lifshitz E. M. (1962). The Classical Theory of Fields (Addison-Wesley, Reading, MA), 2nd ed.; Møller C. (1958). Ann. Phys. 4, 347; Weinberg S. (1972). Gravitation and Cosmology (Wiley, New York).

[4] See Brown J. D. and York J. W. Jr. (1993). Phys. Rev. D 47, 1407; Lau S. (1993). Class. Quant. Grav. 10, 2379; Szabados L. B. (1994). Class. Quant. Grav. 11, 1847; Hayward S. A. (1994) Phys. Rev. D 49, 831.
[5] Chen C. M. and Nester J. M. (1999). *Class. Quant. Grav.* **16**, 1279.

[6] Nester J. M. (1989). *Int. J. Mod. Phys.* A**4**, 1755.

[7] Bergqvist G. (1992). *Class. Quant. Grav.* **9**, 1917.

[8] Virbhadra K. S. (1990). *Phys. Rev.* D **41**, 1086; *ibid.* **42**, 1066; *ibid.* **42**, 2919; Virbhadra K. S. (1992). *Pramana J. Phys.* **38**, 31; Aguirregabiria J. M., Chamorro A. and Virbhadra K. S. (1996). *Gen. Rel. Grav.* **28**, 1393.

[9] Albrow M. G. (1973). *Nature* **241**, 56.

[10] Tryon E. P. (1973). *Nature* **246**, 396.

[11] Cooperstock F. I. and Israelit M. (1994). *Gen. Rel. Grav.* **26**, 323.

[12] Rosen N. (1994). *Gen. Rel. Grav.* **26**, 319.

[13] Garecki J. (1995). *Gen. Rel. Grav.* **27**, 55; Johri V. B., Kalligas D., Singh P. G. and Everitt C. W. F. (1995). *ibid.* **27**, 313; Feng S. and Duan Y. (1996). *Chin. Phys. Lett.* **13**, 409.

[14] Banerjee N. and Sen S. (1999). *Pramana J. Phys.* **49**, 609.

[15] Radinschi I. (1999). *Acta Phys. Slov.* **49**, 789; Xulu S. (2000) *Int. J. Theor. Phys.* **39**, 1153.

[16] Cooperstock F. I. and Faraoni V. (2003). *Astrophys. J.* **587**, 483.

[17] Vargas T. to appear in *Gen. Rel. Grav.* (2004)

[18] Weitzenböck R. (1923). *Invariantentheorie* (Groningen: Noordhoff)

[19] de Andrade V. C. and Pereira J. G. (1997). *Phys. Rev.* D **56**, 4689.

[20] Hayashi K. and Shirafuji T. (1979). *Phys. Rev.* D **19**, 3524.

[21] de Andrade V. C., Guillen L. C. T. and Pereira J. G. (2000). *Phys. Rev. Lett.* **84**, 4533.

[22] Alex Harvey. (1983). *Phys. Rev.* D **28**, 2121.

[23] Maluf J. W. (1995). *J. Math. Phys.* **36**, 4242.