Self-consistent gravitational lens reconstruction

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ABSTRACT

We present a new method for directly determining accurate, self-consistent cluster lens mass and shear maps in the strong lensing regime from the magnification bias of background galaxies. The method relies upon pixellization of the surface mass density distribution which allows us to write down a simple, solvable set of equations. We also show how pixellization can be applied to methods of mass determination from measurements of shear and present a simplified method of application. The method is demonstrated with cluster models and applied to magnification data from the lensing cluster Abell 1689.

Key words: galaxies; clusters: general – cosmology: theory – gravitational lensing – large-scale structure of Universe.

1 INTRODUCTION

The possibility of reconstructing cluster lens mass distributions from the magnification bias of background galaxies was first suggested by Broadhurst, Taylor & Peacock (1995) and first demonstrated by Taylor et al. (1998, T98 hereafter). They showed how a direct, local measure of the lens convergence, \( \kappa \), and the shear were related to the lens magnification by

\[
\frac{\partial^2}{\partial x^2} \kappa + \frac{\partial^2}{\partial y^2} \kappa = 2 \frac{k^2}{S} \kappa.
\]

They showed that one could place quite stringent bounds on \( \kappa \) and \( \gamma \). In addition, T98 found an exact solution for the profile of axisymmetric lenses, although not for more general 2D cases.

Inverse reconstruction methods based on maximum likelihood (Bartelmann et al. 1996) and maximum entropy (Seitz & Schneider 1995; Bridle et al. 1998) have gone some way towards providing a unification of both shear and magnification information. Until now, however, no direct method using only magnification has existed.

In this letter, we show how to directly compute an accurate, self-consistent 2D distribution of \( \kappa \) and \( \gamma \) in the strong lensing regime from magnification. This direct approach has the advantage over indirect alternatives that uncertainties can easily be determined and the application is much quicker. The method is based on pixellization of the \( \kappa \) distribution, suggested by AbdelSalam, Saha & Williams (1998), who used it to estimate the mass of Abell 370 from multiple images. We generalize the method further and also derive a simplified solution to the problem of estimating mass from shear, based on the approach of Kaiser & Squires (1993).

2 RECONSTRUCTION OF \( \kappa \) AND \( \gamma \)

T98 showed how to estimate cluster surface mass using the magnification measured from the distortion in background galaxy number counts. Here our problem is to find an accurate method for reconstructing the surface mass density, given the magnification by an arbitrary lens. The inverse magnification factor at a given position in the lens plane is

\[
A^{-1} = |(1 - \kappa)^2 - \gamma^2|,
\]

where \( \kappa \) is the lens convergence and \( \gamma \) is the shear. The shear can be decomposed into two orthogonal polarization states, \( \gamma_1 \) and \( \gamma_2 \), which are related to the lens convergence by

\[
\gamma_1 = \frac{1}{2} \frac{\partial^2}{\partial \theta_1^2} \kappa, \quad \gamma_2 = \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \kappa.
\]

To find a stable solution to equation (1), we first pixellize the image. Following AbdelSalam et al. (1998), we can now write

\[
\gamma_i^m = D_i^m \kappa_m, \quad i = 1, 2
\]

with summation implied over index \( m \) and \( \kappa_m \) and \( \gamma_i^m \) are the pixellized convergence and shear distributions respectively. The
transformation matrices, $\mathbf{D}^{mn}_1$, are

$$\mathbf{D}^{mn}_1 = \frac{1}{2} (\partial_x^2 - \partial_y^2) \int_m d^2 \theta \ln |\theta_n - \theta'|$$

$$= \frac{1}{\pi} \tan^{-1} \left( \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2 - 1/4} \right),$$

and

$$\mathbf{D}^{mn}_2 = \partial_x \partial_y \int_m d^2 \theta \ln |\theta_n - \theta'|$$

$$= \frac{1}{2\pi} \ln \left( \frac{(x_1^2 + x_2^2)^2 - 2x_1x_2 + 1/4}{(1/2 + x_1^2 + x_2^2)^2 - (x_1 - x_2)^2} \right),$$

with the integration acting over the $m^{th}$ pixel. $x = \theta_n - \theta_m$ is the difference between pixels $m$ and $n$ which are assumed to be square in calculating these analytic expressions. Equation (1) can now be written as the vector equation,

$$\mathbf{1} - 2\mathbf{x} + \kappa \mathbf{G}\mathbf{x} = -\mathbf{P}\mathbf{A}^{-1} = 0$$

(6)

where $\mathbf{A}^{-1}$ is the $N$-dimensional vector of pixellized inverse magnification values, $\kappa$ is the transpose of the vector $\kappa$ of pixellized convergence values and $\mathbf{1}$ is the vector $(1, 1, 1, \ldots)$. The matrix $\mathbf{G}$ is the $N \times N \times N$ matrix

$$\mathbf{G}^{pqm} = \delta_{pm} \delta_{qm} - D^{mn}_1 D^{mn}_2 - D^{mn}_1 D^{mn}_2,$$

(7)

where $\delta_{ij}$ is the Kronecker delta, and summation is only over indices $p$ and $q$. The parity of the measured inverse amplification $A^{-1}(\theta)$ is handled by $\mathbf{P}$ which flips from being $+1$ outside regions bounded by critical lines to $-1$ within such regions.

The amplification equation in the form of equation (6) is the first main result of this letter. We can now solve for $\kappa$ numerically (see Section 4) given a measured inverse amplification. Having solved for $\kappa$, the corresponding shear distribution can then be calculated from equation (3).

### 3 APPLICATION TO CLUSTER MODELS

We apply the method to two types of idealized cluster models. Starting with a predetermined cluster mass density distribution, the corresponding shear distribution is derived using Fourier methods (see, for example, Bartelmann & Weiss 1994). From these, the resulting magnification is calculated from equation (1) and then windowed to remove boundary effects. Using equation (6), we solve for $\kappa, \gamma$ is then solved using equation (3). A grid of 32 by 32 pixels is used in both models.

#### 3.1 Truncated isothermal sphere model

We first test the method with a simple truncated isothermal lens model. The pixellated mass distribution is laid down using $\kappa \propto (r + r_0)^{-1}$, where $r$ is the radial distance from the centre of the sphere and $r_0$ is a constant.

Fig. 1 shows the $\kappa$ and $\gamma$ distribution from which the magnification distribution was calculated, the solved $\kappa$ and $\gamma$ distribution, and the difference between them. The plotted distributions are smoothed from the underlying grid and the white dashes highlight the critical line of the lens. The residuals are shown as percentage deviations from the true distribution. These are less than one per cent for $\kappa$ over most of the grid, which is negligible in comparison to the errors typically found in practice from background clustering, shot noise (T98) and the uncertainties resulting from the use of local $\kappa$ estimators (see van Kampen 1998). The recovered shear distribution is more affected, although it still fares better than $\gamma$ calculated from uncorrected Fourier techniques. The main contribution to these residuals is from boundary effects arising from trying to recover a non-local shear in a finite area. Since much work has been carried out in the removal of such effects (see Squires & Kaiser 1996 and Seitz & Schneider 1995, for example) which have little impact on the recovered $\kappa$, we shall address the problem elsewhere.

#### 3.2 Dumb-bell mass model

The method was also tested with a more general dumb-bell model. Magnification was determined in the same fashion as for the isothermal model, setting a negative parity inside the critical lines, shown by the white dashes in Fig. 2. Once again the residuals between the initial and solved $\kappa$ are typically less than one per cent, while those for $\gamma$ are typically 10 per cent and again come mainly from boundary effects.

### 4 PRACTICAL CONSIDERATIONS

We solve equation (6) with the hybrid Powell method (NAG routine C05PCF). The number of equations needed to solve for $\kappa$ is equal to the total number of grid pixels, which can prove computationally intensive for especially fine grids. We find that this is not a problem for grid resolutions used to measure magnification bias in practice. The $32 \times 32$ grid of pixels used for the models in Section 3 was solved in approximately one minute on an average workstation. The residuals exhibit no noticeable dependence on grid size.

The Powell algorithm is an iterative process and therefore requires an initial estimate of the solution to start from. The choice of the initial estimate turns out to be irrelevant. We have tried a wide range of initial distributions, and even starting from a uniform distribution we arrived at the same final solution.

We have found, however, that the correct choice of pixel parity (especially for low grid resolutions) is essential in order to achieve a sensible result. Inappropriate assignment of parities to pixels manifest themselves, as one would expect, by $\kappa$ being overestimated when a pixel is wrongly assumed to lie inside a critical line, and underestimated in the reverse situation. This provides a means of checking whether critical line positions have been properly defined by looking for large discontinuities in the $\kappa$ distribution. Models with dual critical lines requiring dual parity flips have also been tested and we find that $\kappa$ can be recovered just as well.

Finally, to ensure that the method does not break down with noisy data, we introduced a random noise term to the amplification. Errors in $\kappa$ resulting from noise in the inverse amplification propagate as one would expect from equation (6). For an isothermal lens we recovered the expected result, $\delta \kappa = \delta A/2A^2$, indicating that pixellization does not lead to spurious noise properties.

### 5 APPLICATION TO ABELL 1689

We apply the method to the magnification data presented in T98 for the lensing cluster A1689. A $12 \times 12$ grid is used as the best compromise between shot noise in galaxy counts per bin and the resolution of the derived $\kappa$ map. Identification of the critical line was achieved by locating giant arc positions in the observed image.

Fig. 3 shows the solved mass density and shear distribution. Comparison with the mass density map illustrated in T98 (their fig. 6), which was produced with the sheet $\kappa$ estimator, shows very similar structure. We find that the value of $\kappa$ at the peak calculated here is approximately 10 per cent lower than the peak value in T98, since the sheet estimator over-estimates $\kappa$ inside critical line regions. This has little effect on the total integrated mass of
A1689 found in T98. The $\gamma$ distribution is shown for completeness, although it undoubtedly suffers from boundary effects typically found in the models.

6 SHEAR ANALYSIS

Having shown that pixellization allows us to accurately reconstruct surface mass densities from magnification data, we now apply it to shear analysis. Shear analysis exploits the idea that a given distribution of images of galaxies lying behind a lensing cluster will, in the statistical mean, have regions of lens-induced correlations in image orientation and ellipticity. Measuring the quadrupole moments of individual galaxy images enables the construction of a map of the ellipticity parameters, $e_{ij}$ (Valdes, Tyson & Jarvis 1983). The ellipticity parameters relate to the surface mass density and shear via

\[ \text{Residuals} = 100 \times \frac{\kappa_{\text{init}} - \kappa_{\text{solved}}}{\kappa_{\text{init}}} \]

\[ \text{Residuals} = 100 \times \frac{\gamma_{\text{init}} - \gamma_{\text{solved}}}{\gamma_{\text{init}}} \]

![Truncated isothermal sphere model](image)

Figure 1. Truncated isothermal sphere model. The initial $\kappa$ and $\gamma$ used to form the magnification distribution from which the solved $\kappa$ and $\gamma$ are derived. Underlying grid dimensions are $32 \times 32$. White dashes show the position of the critical line. Contours are linearly spaced and are set at the same levels in both $\kappa$ plots and in both $\gamma$ plots. Residuals are expressed as percentages of $(\kappa_{\text{init}} - \kappa_{\text{solved}})/\kappa_{\text{init}}$. 
One way of solving this for $k$ in the weak lensing regime is to follow the approach of Kaiser & Squires (1993). Generalizations of this to the strong regime have been made by K95. One would have hoped that an alternative to such approaches would be to pixellize equation (8) and use equation (3) to solve it by matrix inversion. However, the resulting matrix equation is ill-conditioned, since the matrix $D_{mn}^{\kappa}$ is singular and $D_{mn}^{\gamma}$ is itself ill-conditioned. Instead, we show a new, simplified expression for the solution to Kaiser’s ellipticity equation and then pixellize it.

Starting with the equation (K95),

$$\partial_j k = \partial_j \gamma_j$$

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(Kaiser 1995, hereafter K95):

$$\epsilon_j = \frac{\gamma_j}{1 - \kappa}, \quad \gamma_{ij} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}.$$ 

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(8)
and using equation (8), one can show that
\[ \partial_i \ln(1 - \kappa) = -\partial_j \ln(\delta_{ij} + \epsilon_{ij}). \]  

The term on the right hand side is obtained from the definition,
\[ \ln(I + B) = B + \frac{1}{2}B^2 + \frac{1}{3}B^3 + \ldots \]  

where \( I \) is the identity matrix and \( B \) is an arbitrary square matrix. Using this expansion and collecting even and odd terms

**Figure 3.** Abell 1689 solved convergence and shear distributions. Darker areas represent a higher distribution density. White dashes show the observed critical line. The plots are smoothed from a 12 × 12 grid with north up and east to the left.

**Figure 4.** Reconstruction of \( \kappa \) from the ellipticity parameters. Contours are at the same levels in both \( \kappa \) plots. The distortion field is illustrated by plotting the apparent shape of an intrinsically circular background object.

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we find
\[ \ln(\delta_i + e_i) = -\frac{1}{2}\ln(1 - e^2)\delta_i + \frac{1}{2}\ln \left( \frac{1 + e}{1 - e} \right) \frac{e_i}{e}, \] (12)
where \( e^2 = e_1^2 + e_2^2 \) and \( e_i = \gamma_i/(1 - \kappa) \). This result requires that \( e < 1 \). Inserting equation (8) into the magnification equation (1) we find
\[ A^{-1} = [(1 - \kappa)^2(1 - e^2)] \] (13)
Hence the parity changes when \( e > 1 \). Since \( e_{ij} \) and \( e_{ij}^{-1} \) are observationally indistinguishable and flip from one to another whenever there is a parity change, we can satisfy the criterion \( e < 1 \) just by noting the critical line positions and inverting the ellipticity matrix when one is crossed.

Finally, inserting equation (12) into equation (10), and solving for \( \kappa \) we find the pixellized solution is
\[ \kappa_i = 1 - (1 - e_{ij}^2)^{1/2} \exp \left[ -\frac{1}{2}(D_{1m}^m m + D_{2m}^m m) \right] \] (14)
where
\[ s_i = \frac{e_i}{e} \ln \left( \frac{1 + e}{1 - e} \right), \quad i = 1, 2. \] (15)
Equation (14) is the second main result of this letter. We can directly calculate \( \kappa \) given a measured ellipticity field. Fig. 4 shows the results of reconstructing \( \kappa \) using equation (14) for the dumb-bell model. The ellipticity parameters are calculated from equation (8) using the \( \kappa \) and \( \gamma \) distribution. We normalize the reconstructed \( \kappa \) to both peaks in the initial \( \kappa \) distribution. The residuals, again being dominated by boundary effects, show that reconstruction is possible to within approximately 10 per cent across the field of view. This can be alleviated by a larger field of view.

7 SUMMARY
We have outlined a method for directly calculating accurate, self-consistent surface mass density and shear distributions from the lens amplification and critical line positions. The method has been demonstrated with the isothermal sphere and dumb-bell cluster models. We find that it reconstructs the surface density to within one per cent over most of the field of view. The reconstruction of the shear pattern only has a fractional accuracy of a few tenths because of boundary effects. We have applied the method to magnification data from Abell 1689, and have reconstructed its surface mass and shear distribution.

We have also found a simplified solution to the problem of estimating surface mass density from galaxy ellipticities. This approach puts the calculation of surface mass from shear and from magnification on an equal footing, and we shall investigate the combined analysis elsewhere.

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