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Investigation of time-fractional SIQR Covid-19 mathematical model with fractal-fractional Mittage-Leffler kernel

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Abstract In this manuscript, we investigate a nonlinear SIQR pandemic model to study the behavior of covid-19 infectious diseases. The susceptible, infected, quarantine and recovered classes with fractal fractional Atangana-Baleanu-Caputo (ABC) derivative is studied. The non-integer order $\nu$ and fractal dimension $q$ in the proposed system lie between 0 and 1. The existence and uniqueness of the solution for the considered model are studied using fixed point theory, while Ulam-Hyers stability is applied to study the stability analysis of the proposed model. Further, the Adams-Bashforth numerical technique is applied to calculate an approximate solution of the model. It is observed that the analytical and numerical calculations for different fractional-order and fractal dimensions confirm better converging effects of the dynamics as compared to an integer order.

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1. Introduction

The term infection is broadly familiar with the reproduction of microbes like viruses, bacteria, fungi inside the body [1]. In general, infection-related diseases are often quickly recovered, but many infectious diseases have historically adapted to societal spreading in the population. This spreading of infectious diseases such as HIV, Malaria, TB, and contagion (flu and cough) remains to be enormous, causing chaos on human populations [2]. These diseases are typically toxic, which increases their activity in the proper circumstances, such as in the atmosphere, on the surface, within the water, and in the edible food. It is hard to estimate the natural fluctuations of disease having infections across the population, so qualitative and quantitative calculations may be valuable [3].

Currently, numerous pandemic models have extensively been studied. Such as, Ahmad et al. [4] recently used the NSFD scheme to model and calculate the dynamical effects of the SEIQV reaction-diffusion pandemic model. The formulation of the HB epidemic model with saturation occurrence rate is expressed by Khan et al. [5]. Numerical schemes for the solutions of pandemic models for diseases having infections like HIV, dengue fever, and influenza are considered to study the critical points, fixed points and the stability of the models [6]. Moreover, the dynamical effects of a speculative pandemic infectious model include quarantine class, where the existence of the result and uniqueness of the global positive solutions has been investigated [7]. Similarly, the dynamical effects of current coronavirus (2019-nCoV) in terms of the mathematical models have been considered between the unknown hosts and bats, as well as among the people and the accumulation having infections [8]. The calculated central quantity, so, the formation and exploration of mathematical pandemic representation are productive and having obvious methods to investigate the dynamical properties of diseases with harmful infections.

In the current era, fractional calculus (FC) captured a great focus in several fields of mathematics, engineering, economics, control theory, and finance [9–13]. It has revealed that by using the fractional operator, one can describe many physical and biological problems associated with real processes with a higher degree of freedom [14]. The attention of modeling in fractional differential equations (FDEs), the complex real-world phenomenon is increasing to expecting its many properties which are generally absent in integer-order differential equations. It should be noted that the generalization of the integer order calculus of differential and integral equations to sensible or complex numbers is verified by modern calculus [15–20]. Currently, a lot of mathematical models have been proposed in the area of FC, such that SEIR, HIV, TB, population mathematical representation, cancer representation, predator and prey models, etc.

The investigation of fractal-dimension in FC expresses the non-integer order of the independent variable. The concept between fractional and fractal calculus is first presented by Atangana [21,22]. Since then, many applications both conditional and numerical study of fractional-order differential and integral equations for application of the fractal-fractional derivative has been extensively studied [23–32]. The fractal-fractional DEs change the order and as well as dimension of the equation to rational form. This quality generalizes differential equations to any arbitrary order and dimension. The novel coronavirus covid-19 has been discussed by using different techniques as fractional order DEs [33–37].

Here, we investigate a nonlinear SIQR pandemic model to analyze the dynamical effects of disease having infections. In the proposed model quarantine (Q) is a helpful plan to stop and control the spreading of many infectious diseases. Nowadays, investigation on the benefits of isolation and on the flow of infection has extensively discussed [38]. Similarly, the numerical scheme for structure conserving of the dynamics of infectious diseases is studied in [39]. Further, the stochastic behavior of the influenza model with a regular treatment process was introduced [40], where the dynamical behavior of the coronavirus model has been discussed in detail [41].

Motivated from the above discussion, the current manuscript describes the dynamical behavior of the proposed model [42] under $\frac{d^\alpha}{d t^\alpha}$ derivative of fractal-fractional in sense of Caputo, to obtain a greater parametric quantity than integer-order. The considered problem has a fractional-order $\alpha$ and fractal dimension $q$ express phenomenon lying between 0 and 1. A good result is obtained, by having the whole density of all compartments converge rapidly for a short interval.

The fractal-fractional sense of the proposed model [42] is given as under

$$\frac{d^\alpha}{d t^\alpha} S(t) = \mathcal{N} - \beta S \int_{0}^{t} S(t) \, dt,$$

$$\frac{d^\alpha}{d t^\alpha} I(t) = \beta S(t) \left( 1 - \frac{I(t)}{C_0} \right) - \gamma I(t) - \nu I(t),$$

$$\frac{d^\alpha}{d t^\alpha} Q(t) = \gamma I(t) - \nu Q(t),$$

$$\frac{d^\alpha}{d t^\alpha} R(t) = \nu Q(t),$$

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad Q(0) = Q_0 \geq 0,$$

$$R(0) = R_0 \geq 0,$$

where the detail of the utilized variables for the separate classes and parameters associated with model (1) are declared below:

- **S(t):** Susceptible class at time $t$(those individuals that may be infected any time),
- **I(t):** Infected class for time $t$(those individuals who has the infection and have the capability to transfer disease to other individuals),
- **Q(t):** Quarantine compartment at time $t$(the isolated individuals, to control the spreading of disease from one to another individual),
- **R(t):** Recovered individuals at time $t$(those people who are healthy now and have develop immunity).

- $\mathcal{N}$: Total population in the affected region called susceptible class, $b$: per capita natural-mortality rate, $\nu$: the average count minimal constants, $\tau$: average removing rate of infected individuals from $I$; $\gamma$: the average removing rate of individuals who may or may not be infected from $Q$, where $\kappa$: coincidence rate of $I$ and $Q$, and $\kappa_2$: the death rate due to disease in $I$ and $\kappa_2$: the death rate due to disease in $Q$.

The manuscript is outlined as follows: In Section 2, we present the definitions and some basic notation from the fractional calculus. Section 3 is devoted to show the existence results and unique solution of the proposed problem by using fixed point theory. In Section 4, the two steps Adam-Bashforth the numerical technique is applied for the numerical solution of the SIQR epidemic model. The graphical representation is presented in the same section. Finally, we conclude our work in Section 5.
2. Preliminaries

Definition 1 [43]. Let \( \mathcal{Y}(t) \) in an open interval \((a, b)\) be a continuous function with fractional-order \(0 < \varphi \leq 1\) and fractal dimension \(0 < q \leq 1\) can be defined in \( \mathcal{A} \mathcal{B} \mathcal{C} \) as

\[
\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) = \frac{d}{dt} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) - \frac{d}{dt} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) \times \int_{0}^{t} (t - s)^{\varphi - 1} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(s)) ds,
\]

where \( \mathcal{A} \mathcal{B} \mathcal{C}(0) = 1 = \mathcal{A} \mathcal{B} \mathcal{C}(1) \) is said to be the normalized constant.

Definition 2 [43]. let us assume that \( \mathcal{Y}(t) \) in an open interval \((a, b)\) be a continuous function with fractional-order \(0 < \varphi \leq 1\) and fractal dimension \(0 < q \leq 1\) in sense of \( \mathcal{A} \mathcal{B} \mathcal{C} \) can be defined as

\[
\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) = \frac{1 - \varphi}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) + \frac{q\varphi}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \times \int_{0}^{t} (t - s)^{\varphi - 1} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(s)) ds.
\]

Lemma 1 [44]. Let suppose a solution define for the proposed system in the form of

\[
\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) = \frac{1 - \varphi}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) + \frac{q\varphi}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \times \int_{0}^{t} (t - s)^{\varphi - 1} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(s)) ds
\]

is supplied by

\[
\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) = \frac{1 - \varphi}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) + \frac{q\varphi}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \times \int_{0}^{t} (t - s)^{\varphi - 1} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(s)) ds
\]

Theorem 1 [45]. Let \( \mathcal{Y} \subset \mathcal{Z} \) which is convex, assume that the two operators \( \Phi_{1} \) and \( \Phi_{2} \) with

1. \( \Phi_{1}(\mathcal{Y}) + \Phi_{2}(\mathcal{Y}) \in \mathcal{Y} \) for every \( \mathcal{Y} \in \mathcal{Z} \);
2. \( \Phi_{1} \) has contraction;
3. \( \Phi_{2} \) is compact and continuous.

having the equation \( \Phi_{1}(\mathcal{Y}) + \Phi_{2}(\mathcal{Y}) = \mathcal{Y} \), has one or many solution \( \mathcal{Y} \).

3. Theoretical Approach of the proposed model

Here, we check the existence of result, unique solution and stability analysis via fixed point approach. To prove existence and unique solution for the selected model (1), a Banach-space can be define as \( \mathcal{Y} = \mathcal{G}[0, T] \times \mathcal{R}^{1} \), where \( \mathcal{G} = G[0, T] \) and the space norm is

\[
\| \mathcal{G} \|= ||\mathcal{G}|| = \max_{t \in [0, T]} [a(t)] + ||b(t)|| + ||c(t)|| + ||d(t)||
\]

For this purpose, due to the above discussion we need the integral as differential, here we may rewrite the proposed problem (1) as

\[
\begin{align*}
\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) &= \frac{1}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t)) + \frac{q\varphi}{\mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(t))} \times \int_{0}^{t} (t - s)^{\varphi - 1} \mathcal{A} \mathcal{B} \mathcal{C}(\mathcal{Y}(s)) ds, \\
\text{where } \mathcal{A} \mathcal{B} \mathcal{C}(0) &= 1 = \mathcal{A} \mathcal{B} \mathcal{C}(1) \text{ is said to be the normalized constant.}
\end{align*}
\]
Thus, the operator $C$ is closed, hence $C$ has contraction.

Now we show that $D$ is compact relatively and $D$ is bounded and equicontinuous. Clearly, the whole domain is defined for the operator $D$, so $D$ is continuous, hence $\Psi$ is continuous for arbitrary $\mathcal{Y} \in A$, as below we have

$$
\|D(\mathcal{Y})\| = \max_{t \in [0, \tau]} \left\| \int_0^t \left( (1-s)^{\nu-1} \Psi(s, \mathcal{Y}(s)) \right) ds \right\|
\leq \frac{q^\nu}{\mathcal{B}(p) \Gamma(p)} \int_0^\tau (1-s)^{\nu-1} \left| (1-s)^{\nu-1} \Psi(s, \mathcal{Y}(s)) \right| ds
\leq \frac{q^\nu}{\mathcal{B}(p) \Gamma(p)} \left[ \int_0^\tau (1-s)^{\nu-1} \left| (1-s)^{\nu-1} \Psi(s, \mathcal{Y}(s)) \right| ds \right].
$$

(10)

Hence, Eq. (10) shows that the operator $D$ is bounded. Now for equicontinuity, assume that $t_1 > t_2 \in [0, \tau]$, we have

$$
\|D(\mathcal{Y}(t_2)) - D(\mathcal{Y}(t_1))\|
\leq \frac{q^\nu}{\mathcal{B}(p) \Gamma(p)} \int_0^{t_2} (t_2 - s)^{\nu-1} \left| (1-s)^{\nu-1} \Psi(s, \mathcal{Y}(s)) \right| ds
\leq \frac{q^\nu}{\mathcal{B}(p) \Gamma(p)} \left[ \int_0^{t_2} (t_2 - s)^{\nu-1} \left| (1-s)^{\nu-1} \Psi(s, \mathcal{Y}(s)) \right| ds \right].
$$

Here right hand side of (11) will be zero, when $t_2 \rightarrow t_1$. And by continuity of $D$ so $|D\mathcal{Y}(t_2) - D\mathcal{Y}(t_1)| \rightarrow 0$, as $t_2 \rightarrow t_1$.

Hence, $D$ is bounded and uniformly continuous. By ‘Arzelá-Ascoli statement, a subset $\mathcal{Y} \in A$ of $D$ is compact iff it is closed, bounded, and equi-continuous. As $D$ is compact relatively and completely continuous. Through (1) and (5) we deduce that the proposed model has at least one solution.

The next result is related to uniqueness of the solutions.

**Theorem 3.** Due to assumption (U2), Eq. (5) has one solution, which implies that the considered model has unique solution if

$$
(1-\psi)\mathcal{B}(p) + q^\nu \frac{\mathcal{B}(p) \Gamma(p)}{\mathcal{B}(p) \Gamma(p)} < 1.
$$

**Proof.** Suppose the operator $J : \mathcal{Y} \rightarrow \mathcal{Y}$ by

$$
J\mathcal{Y}(t) = \mathcal{Y}_0(t) + \int_0^t \Psi(t, \mathcal{Y}(t)) - \Psi_0(t) \left( (1-\psi)^{\nu-1} \mathcal{B}(p) \Gamma(p) \right) dt
\leq \frac{q^\nu}{\mathcal{B}(p) \Gamma(p)} \left[ \int_0^t (t-x)^{\nu-1} \left| (1-x)^{\nu-1} \Psi(x, \mathcal{Y}(x)) \right| dx \right].
$$

(12)

Let $\mathcal{Y}, \mathcal{Y}^\prime \in \mathcal{Y}$, then

$$
\|J\mathcal{Y} - J\mathcal{Y}^\prime\| \leq \frac{q^\nu}{\mathcal{B}(p) \Gamma(p)} \left[ \int_0^\tau (t-x)^{\nu-1} \left| (1-x)^{\nu-1} \Psi(x, \mathcal{Y}(x)) \right| dx \right]
\leq \frac{q^\nu}{\mathcal{B}(p) \Gamma(p)} \left[ \int_0^\tau (t-x)^{\nu-1} \left| (1-x)^{\nu-1} \Psi(x, \mathcal{Y}(x)) \right| dx \right].
$$

(13)

and

$$
\Theta = \left[ (1-\psi)\mathcal{B}(p) + q^\nu \frac{\mathcal{B}(p) \Gamma(p)}{\mathcal{B}(p) \Gamma(p)} \right].
$$

(14)

From (13) $J$ has a contraction. Hence (5) has one solution. Thus the problem (1) has one solution.

**3.1. Ulam-Hyers stability**

In this part we define and take some useful consequences on stability for problem (1), the perturb parameter $\psi(t) \in C[0, T]$, which is solution dependent if satisfy $\psi(0) = 0$ as

- $|\psi(t)| \leq \epsilon$ for $\epsilon > 0$;
- $|\psi(t)| \leq \epsilon$ for $\epsilon > 0$.

**Lemma 2.** The perturb problem has a solution

$$
\mathcal{Y}_0(t) + \int_0^t \Psi(t, \mathcal{Y}(t)) + \psi(t),
$$

(15)

fulfill the given relation

$$
\left| \mathcal{Y}(t) - \mathcal{Y}_0(t) \right| \leq \left| \mathcal{Y}(t) - \mathcal{Y}_0(t) \right| + \left| \psi(t) \right| \leq \left| \mathcal{Y}(t) - \mathcal{Y}_0(t) \right| + \left| \psi(t) \right|.
$$

(16)

**Theorem 4.** Utilizing assumption (U2) and Eq. (14), solution of Eq. (5) has UH stability and accordingly, systematic solution of the proposed system is UH-stable if $\Theta < 1$, as given in (14)

**Proof.** Consider $\mathcal{Y} \in \mathcal{Y}$ be solution and $\mathcal{T} \in \mathcal{Y}^\prime$ be a unique solution of Eq. (5), then

$$
\left| \mathcal{T}(t) - \mathcal{T}(t) \right| = \left| \mathcal{T}(t) - \mathcal{T}_0(t) \right| + \left| \Psi(t, \mathcal{T}(t)) - \Psi_0(t) \right|
\leq \frac{(1-\psi)^{\nu-1} \mathcal{B}(p) \Gamma(p)}{\mathcal{B}(p) \Gamma(p)} \left[ \int_0^\tau (t-x)^{\nu-1} \left| (1-x)^{\nu-1} \Psi(x, \mathcal{Y}(x)) \right| dx \right].
$$

(14)
\[
\begin{align*}
&+ \frac{q^\varphi}{\mathcal{B}(\varphi)\Gamma(\varphi)} \int_0^t (t-x)^{\nu-1} x^{\alpha-1} \Psi(x, \mathcal{S}(x))dx, \\
&\leq \nu_{\varphi, q} \left\| \frac{(1 - \varphi)^{1/\varphi}}{\mathcal{B}(\varphi)} (\mathcal{S}_{\mathcal{A}}(\varphi)) \right\| \mathcal{S}(x) \mathcal{T}(\mathcal{S}(x))dx, \\
&+ \frac{q^\varphi}{\mathcal{B}(\varphi)\Gamma(\varphi)} \mathcal{B}(\varphi)\Gamma(\varphi) \| \mathcal{S}(x) \mathcal{T}(\mathcal{S}(x))dx, \\
&\leq \nu_{\varphi, q} \Theta \mathcal{S}(x) \mathcal{T}(\mathcal{S}(x))dx.
\end{align*}
\]

(17)

From (17), maybe written as
\[
\mathcal{S}(x) \mathcal{T}(\mathcal{S}(x))dx \leq \nu_{\varphi, q} \Theta \mathcal{S}(x) \mathcal{T}(\mathcal{S}(x))dx.
\]

(18)

Hence which is the result of the determined stability. □

4. Qualitative analysis

In this section of the manuscript we have to determine the numerical approximate solution for fractal-fractional order of the proposed model (1), using \( \mathcal{A} \mathcal{B} \) derivative in sense of Caputo with famous fractal-fractional Adam-Bashforth iterative approximate scheme for approximate solution. Moreover, we apply the fractal-fractional \( \mathcal{A} \mathcal{B} \) approach to Interpreter the obtain approximate solution [46] of the system (1). According (3), the proposed system maybe written as:

\[
\begin{align*}
\mathcal{A} \mathcal{B} \mathcal{D}^\varphi(\mathcal{S}(t)) &= q^{\alpha-1} \mathcal{I}_1^\varphi(\mathcal{S}(t), t), \\
\mathcal{A} \mathcal{B} \mathcal{D}^\varphi(\mathcal{I}(t)) &= q^{\alpha-1} \mathcal{I}_2^\varphi(\mathcal{I}(t), t), \\
\mathcal{A} \mathcal{B} \mathcal{D}^\varphi(\mathcal{Q}(t)) &= q^{\alpha-1} \mathcal{I}_3^\varphi(\mathcal{Q}(t), t), \\
\mathcal{A} \mathcal{B} \mathcal{D}^\varphi(\mathcal{R}(t)) &= q^{\alpha-1} \mathcal{I}_4^\varphi(\mathcal{R}(t), t),
\end{align*}
\]

where \( \mathcal{I}_i, i = 1, 2, 3, 4 \) are discussed in (3), apply anti-derivative of random non-integer order \( \varphi \) and fractal-dimension \( q \) for the \( \alpha^\varphi \) equation of (3) using \( \mathcal{A} \mathcal{B} \mathcal{D} \), we get

\[
\mathcal{S}(t) - \mathcal{S}(0) = (1 - \varphi)^{1/\varphi} \mathcal{I}_1^\varphi(\mathcal{S}(t), t) + \frac{q^\varphi}{\mathcal{B}(\varphi)\Gamma(\varphi)} \int_0^t \frac{(t-x)^{\nu-1} x^{\alpha-1} \mathcal{I}_1^\varphi(\mathcal{S}(x), x)}{dx}.
\]

Set \( t = \mathcal{I}_i, \) for \( i = 0, 1, 2, 3 \ldots \),

\[
\begin{align*}
\mathcal{S}(t_{\mathcal{I}_i}) - \mathcal{S}(0) &= (1 - \varphi)^{1/\varphi} \mathcal{I}_1^\varphi(\mathcal{S}(t_{\mathcal{I}_i}), t_{\mathcal{I}_i}) + \frac{q^\varphi}{\mathcal{B}(\varphi)\Gamma(\varphi)} \int_0^{t_{\mathcal{I}_i}} \frac{(t_{\mathcal{I}_i}-x)^{\nu-1} x^{\alpha-1} \mathcal{I}_1^\varphi(\mathcal{S}(x), x)}{dx}, \\
&= (1 - \varphi)^{1/\varphi} \mathcal{I}_1^\varphi (t_{\mathcal{I}_i}) + \frac{q^\varphi}{\mathcal{B}(\varphi)\Gamma(\varphi)} \int_0^{t_{\mathcal{I}_i}} (t_{\mathcal{I}_i}-x)^{\nu-1} x^{\alpha-1} \mathcal{I}_1^\varphi(\mathcal{S}(x), x)dx.
\end{align*}
\]

The estimated function \( \mathcal{I}_1 \) in \( [t_{\mathcal{I}_i}, t_{\mathcal{I}_{i+1}}] \) by the interpolation polynomial as follows

\[
\mathcal{I}_1 \approx \frac{\mathcal{I}_1}{h} (t - t_{\mathcal{I}_i}) - \frac{\mathcal{I}_1}{h} (t - t_{\mathcal{I}_{i+1}})
\]

and

\[
\mathcal{S}(t_{\mathcal{I}_i+1}) = \mathcal{S}(0) + \left( 1 - \frac{\varphi}{\mathcal{B}(\varphi)\Gamma(\varphi)} \right) \frac{t_{\mathcal{I}_i}^{\nu-1}}{h} \left[ \mathcal{I}_1 \left( \mathcal{S}(t_{\mathcal{I}_i}), t_{\mathcal{I}_i} \right) + \frac{q^\varphi}{\mathcal{B}(\varphi)\Gamma(\varphi)} \sum_{i=0}^\beta \frac{t_{\mathcal{I}_i}^{\nu-1}}{h} \mathcal{I}_1 \left( \mathcal{S}(t_{\mathcal{I}_i}), t_{\mathcal{I}_i} \right) \right].
\]

(20)

Calculate \( I_{t_{\mathcal{I}_i+1}}, \) \( I_{t_{\mathcal{I}_i}} \), and \( I_{t_{\mathcal{I}_i+1}} \) we obtain

\[
\begin{align*}
\mathcal{S}(t_{\mathcal{I}_i}) &= \mathcal{S}(0) + \frac{1}{\mathcal{B}(\varphi)\Gamma(\varphi)} \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right) \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right)^{\nu-1}, \\
&= \frac{1}{\mathcal{B}(\varphi)\Gamma(\varphi)} \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right) \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right)^{\nu-1} - \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right)^\varphi, \\
&= \frac{1}{\mathcal{B}(\varphi)\Gamma(\varphi)} \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right) \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right)^{\nu-1} - \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right)^\varphi, \\
&= \frac{1}{\mathcal{B}(\varphi)\Gamma(\varphi)} \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right) \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right)^{\nu-1} - \left( t_{\mathcal{I}_i} - t_{\mathcal{I}_{i-1}} \right)^\varphi.
\end{align*}
\]

(21)
In parallel, we calculate same numerical scheme for the other classes. The dynamical behavior of the susceptible compartment $S$ for the domain $t$, where $t = 50$ and $t = 5$ respectively are presented in Fig. 1a and b. Fig. 1b is the magnified form of Fig. 1a. It is obvious that initially the susceptible class $S$ shows the rise with the passage of time and then goes to a stable state, while the other three compartments Figs. 2, 3, a show decay and then converges to their equilibrium points at various fractional-order $\varphi$ and fractal dimension $q$ for $\eta = 0.5$.

The dynamical representation of the infected class $I$ shows in Fig. 2a and b for the domain $t = 50$ to $t = 5$ respectively.

The magnified portion of Fig. 2a is the Fig. 2b. Initially, this class decay rapidly and then goes to stable and converges to the equilibrium points at different fractional-order $\varphi$ and $q$ for $\eta = 0.5$. Accordingly, the infection is controlled or tends to zero.

Fig. 3a and b shows the dynamical effect of Quarantine compartment $Q$ for the domain $t = 50$ to $t = 5$ respectively. Fig. 3b is the magnified portion of Fig. 3a. This compartment shows rise initially for a short period of time and then converges to their equilibrium points at various fractional orders of $\varphi$ and fractal dimension $q$ for $\eta = 0.5$. It is concluded that with the passage of time the infection decreases due to the quarantine compartment, the said class rise and then goes to stable and converges to the equilibrium points at various fractional-order $\varphi$ and fractal dimension $q$ for $\eta = 0.5$.

In parallel, we calculate same numerical scheme for the other classes $I$, $Q$, and $R$ respectively as

$$\mathbb{S}(t_{\varphi+1}) = \left\{ \begin{array}{l}
\mathbb{S}(0) + \frac{1}{\varphi \Gamma(\varphi)} \left( \varphi_{\varphi+1} \right) \left[ t_1(\mathbb{S}(t), t_0) \right] + \sum_{n=0}^{\varphi} \frac{\varphi_{\varphi+1} t_1(\mathbb{S}(t), t_0)}{n!} \\
\quad \times \left[ \varphi_{\varphi+1} \left[ (\beta + 1 - \alpha)\varphi(\beta - \alpha + 2 + \varphi) - (\beta - \alpha)\varphi(\beta - \alpha + 2 + 2\varphi) \right] \right] \\
- \frac{1}{\varphi \Gamma(\varphi)} \left( \varphi_{\varphi+1} \right) \left[ t_1(\mathbb{S}(t), t_0) \right] \left[ (\beta + 1 - \alpha)^{\varphi+1} - (\beta - \alpha)^{\varphi}(\beta - \alpha + 1 + \varphi) \right] \right\},
\right.\nonumber$$

(23)

The dynamical representation of the infected class $I$ shows in Fig. 2a and b for the domain $t = 50$ to $t = 5$ respectively.

The magnified portion of Fig. 2a is the Fig. 2b. Initially, this class decay rapidly and then goes to stable and converges to the equilibrium points at different fractional-order $\varphi$ and $q$ for $\eta = 0.5$. Accordingly, the infection is controlled or tends to zero.

$$\mathbb{I}(t_{\varphi+1}) = \left\{ \begin{array}{l}
\mathbb{I}(0) + \frac{1}{\varphi \Gamma(\varphi)} \left( \varphi_{\varphi+1} \right) \left[ t_2(\mathbb{I}(t), t_0) \right] + \sum_{n=0}^{\varphi} \frac{\varphi_{\varphi+1} t_2(\mathbb{I}(t), t_0)}{n!} \\
\quad \times \left[ \varphi_{\varphi+1} \left[ (\beta + 1 - \alpha)\varphi(\beta - \alpha + 2 + \varphi) - (\beta - \alpha)\varphi(\beta - \alpha + 2 + 2\varphi) \right] \right] \\
- \frac{1}{\varphi \Gamma(\varphi)} \left( \varphi_{\varphi+1} \right) \left[ t_2(\mathbb{I}(t), t_0) \right] \left[ (\beta + 1 - \alpha)^{\varphi+1} - (\beta - \alpha)^{\varphi}(\beta - \alpha + 1 + \varphi) \right] \right\},
\right.\nonumber$$

(24)

the equilibrium points at different fractional-order $\varphi$ and $q$ for $\eta = 0.5$. Accordingly, the infection is controlled or tends to zero.

Fig. 3a and b shows the dynamical effect of Quarantine compartment $Q$ for the domain $t = 50$ to $t = 5$ respectively. Fig. 3b is the magnified portion of Fig. 3a. This compartment shows rise initially for a short period of time and then converges to their equilibrium points at various fractional orders of $\varphi$ and fractal dimension $q$ for $\eta = 0.5$. It is concluded that with the passage of time the infection decreases due to the quarantine compartment, the said class rise and then goes to stable and converges to the equilibrium points at various fractional-order $\varphi$ and fractal dimension $q$.
Fig. 1 Numerical simulation for the initial value $S_0 = 0.5$ of the susceptible class $S$ for the proposed model (1) at four different arbitrary orders of $\psi$ and $q$ for $\eta = 0.5$.

Fig. 2 Numerical simulation for the initial value $I_0 = 0.1$ of the susceptible class $I$ for the proposed model (1) at four different arbitrary orders of $\psi$ and $q$ for $\eta = 0.5$.

Fig. 3 Numerical simulation for the initial value $Q_0 = 0.00$ of the susceptible class $Q$ for the proposed model (1) at four different arbitrary orders of $\psi$ and $q$ for $\eta = 0.5$.

Fig. 4 Numerical simulation for the initial value $R_0 = 0.00$ of the susceptible class $R$ for the proposed model (1) at four different arbitrary orders of $\psi$ and $q$ for $\eta = 0.5$. 
The dynamical behavior of the recovered class $R$ for the domain $t = 50$ to $t = 5$ respectively are in Fig. 4a and b. where Fig. 4b is the magnified portion of Fig. 4a. The recovered class $R$ get a rise and become stable when the infected class $I$ and quarantine class $Q$ decreases at various fractional orders $\varphi$ and $q$ for $\eta = 0.5$. As infected $I$ and quarantine $Q$ classes decrease the recovery class $R$ rises and then become decreases. The stability occurs quickly at low fractional order of $\varphi$ and $q$.

5. Conclusion

We have studied the nonlinear dynamics of the SIQR model with the fractal-fractional derivative. The existence and uniqueness of the solution of the proposed model are investigated by fixed point theory. For the stability analysis, the Ulam-Hyers stability approach is applied. To provide the analytical estimated solution we apply the Adams-Bashforth numerical scheme. Using values of the parameters from Table 1 we interpret the numerical solution and its behavior for the different transmission parameters for several arbitrary orders with fractal dimensions. The Quarantine class $Q$ plays a vital role in the proposed model (1) to control the spreading of pandemic disease. It is noted that when the transmission rate decreases the infection also decreases the speed of spreading of Covid-19 infection. As a conclusion from the numerical results that quarantine individuals have a great influence on the transmission of Covid-19 infection. Since transmission of Covid-19 increase due to humans interactions though the first source of the disease was an animal or unknown host. Therefore, the minimization of the infection is subjected to the application of a quarantine policy. The investigation herein suggests that, that if the quarantine strategy is implemented in a true spirit, the infection will be certainly reduced.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table 1 Calculated numerical value of the parameters [42] utilized in proposed model (1).

| Cases | $\gamma$ | $b$ | $\lambda$ | $\tau$ | $\kappa$ | $\kappa_1$ | $\kappa_2$ | $\eta$ |
|-------|---------|-----|-----------|-------|---------|---------|---------|------|
| 1     | 3       | 2.29| 1.5       | 0.5   | 3.1     | 0.01    | 0.03    | 0.02 | 0.5 |
| 2     | 3       | 2.29| 1.5       | 0.5   | 3.1     | 0.01    | 0.03    | 0.02 | 0.6 |
| 3     | 3       | 2.29| 1.5       | 0.5   | 3.1     | 0.01    | 0.03    | 0.02 | 0.7 |
| 4     | 3       | 2.29| 1.5       | 0.5   | 3.1     | 0.01    | 0.03    | 0.02 | 0.8 |
| 5     | 3       | 2.29| 1.5       | 0.5   | 3.1     | 0.01    | 0.03    | 0.02 | 0.9 |

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