ABSTRACT. We investigate the regularity of weak solutions to the Navier-Stokes equations. Particularly we address the question as to whether the magnitude of 
\[ \left( \int_{\{\eta \geq T^{\beta-\frac{1}{2}}\}} |F[u_0](\eta)|^2 d\eta \right)^{\frac{1}{2}} \]
of the initial data provides a criterion which excludes blow-up solutions at \( T \).
We prove that there is \( \delta > 0 \) such that if the integration is less than \( \delta T^{\frac{1}{4}} \) for some \( \beta \in \left( \frac{1}{3}, \frac{1}{2} \right) \), then the smooth solution on \( \mathbb{R}^3 \times (0, T) \) can be extended smoothly over \( T \). Moreover, by new methods of frequency overlapping and measurement of moving singular set, we prove that the Leray-Hopf solution with \( L^2(\mathbb{R}^3) \) data is regular.

1. Introduction

Consider the Navier-Stokes equations
\[
\begin{aligned}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T) \\
div u &= 0, \quad \text{in } \mathbb{R}^3 \times (0, T) \\
 u(x, 0) &= u_0(x), \quad \text{in } \mathbb{R}^3
\end{aligned}
\] (1.1)
where \( u \) and \( p \) denote the unknown velocity and pressure of incompressible fluid respectively. In this paper, we investigate the regularity of weak solutions to the Navier-Stokes equations. Particularly we address the question as to whether the magnitude of 
\[ \int_{\{\eta \geq T^{\beta-\frac{1}{2}}\}} |F[u_0](\eta)|^2 d\eta \quad (\beta < \frac{1}{2}) \]
of the initial data provides a criterion which excludes blow-up solutions at \( T \).
We call \( u \) is a smooth solution in \( \mathbb{R}^3 \times (0, T) \) to (1.1) if it is a weak solution of (1.1) and \( u \in C^\infty(\mathbb{R}^3 \times (0, T)) \). The main result of the present paper is the following

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Theorem 1.1. There is $\delta > 0$ such that if $u_0 \in L^2(\mathbb{R}^3)$ satisfies

\begin{equation}
\|u_0\|_{L^2} \leq \delta,
\end{equation}

and $u$ is a smooth solution to the problem (1.1) in $\mathbb{R}^3 \times (0, T)$, then $u$ can be extended smoothly to $\mathbb{R}^3 \times (0, T)$.

For all $u_0 \in L^2(\mathbb{R}^3)$ so that $\text{div} u_0 = 0$, Leray (1934) [35] and Hopf (1951) [28] obtained the global existence of weak solution, which now is called the Leray-Hopf solution. Precisely, it is a weak solution of (1.1) satisfying

\begin{enumerate}
  \item $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$,
  \item $\text{div} u = 0$ in $\mathbb{R}^3 \times (0, T)$,
  \item $\int_0^T \int_{\mathbb{R}^3} \{-u \cdot \partial_t \phi + \nabla u \cdot \nabla \phi + (u \cdot \nabla u) \cdot \phi\} dx dt = 0$
\end{enumerate}

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, T))$ with $\text{div} \phi = 0$ in $\mathbb{R}^3 \times (0, T)$.

Notice that for any $t_1 > 0$, there is $t_0 \in (0, t_1)$ such that the Leray-Hopf solution $u(\cdot, t_0) \in H^1(\mathbb{R}^3)$. With the initial data $u(x, t_0)$, the Leray-Hopf solution $u(x, t)$ is smooth at least in a short time interval $(0, T)$ (see [35] [44]). In the present paper we are asserting that the solution can not blow up at the end of this short time interval provided

\begin{equation}
\left(\int_{\{|\eta| \geq (t_2 - t_0)^{\beta} \frac{1}{2}\}} |F[u](\eta, t_0)|^2 d\eta\right)^{\frac{1}{2}} \leq \delta(t_2 - t_0)^{\frac{\beta}{2}} \text{ for some } \beta < \frac{1}{2}.
\end{equation}

Because this condition is not invariant by scaling, we discover that the solution can not blow up provided $\|u(t_0)\|_{L^2}$ is small enough.

Furthermore, from Leray’s theorem (see also [44]), there is a disjoint open interval sequence $\{J_q\}_q$ in $(0, \infty)$ such that the Lebesgue measure of $(0, \infty) \setminus \bigcup q J_q$ is zero, and the Leray-Hopf solution $u$ can be modified on a set of Lebesgue measure zero so that its restriction to each $\mathbb{R}^3 \times J_q$ becomes smooth. The time $T$ in Theorem 1.1 may be considered as the right-side of an open interval $J_q$. From the foregoing argument, we can see that the Leray-Hopf solution $u$ can be extended smoothly over the right-side of $J_q$ provided (1.2) is satisfied. So we get that $u$ is smooth from the left-side of the open interval $J_q$. Since the Lebesgue measure of $(0, \infty) \setminus \bigcup q J_q$ is zero, if (1.2) is satisfied for all $T > 0$ then we get that for all $t > 0$, the Leray-Hopf solution $u(x, t)$ is smooth.

Note that for the initial data $u_0 \in L^p(\mathbb{R}^3)$ with $\text{div} u_0 = 0$ and $p > 3$, the estimates from below near the blow-up time $T$

\[ \|u(t)\|_{L^p} \geq \frac{C}{(T - t)^{\frac{p-3}{2p}}} \]

with constant $C$ independent of $T$ and $t$ was given by Leray [35]. Similar estimates for bounded domain was proved by Giga [19]. Leray also gave the estimate from below for $\|\nabla u(t)\|_{L^2}$ and $\|u(t)\|_{L^\infty}$ near the blow-up. For bounded domain, Foias and Temam [15] gave the corresponding estimate for $\|u(t)\|_{H^1}$. 

2
Regularity of solutions to the Navier-Stokes equations

Since Leray (1934) [35] and Hopf (1951) [28] proved the global existence of weak solutions to the Navier-Stokes equations, the question of the uniqueness and regularity of weak solutions has been a major problem in applied analysis. Over the years there has been intensive work by many authors attacking this problem. In 1984, T. Kato discovered the existence of global smooth solution with small initial data in $L^3$. Its uniqueness was proved by Furioli, Lemarié-Rieusset and Terraneo [17]. The global solution in $\dot{H}^{1/2}$ was investigated by Fujita and Kato [16], which was extended to the Morrey space by Giga and Miyakawa [25], and extended to the Besov space $\dot{B}^{-1/3/q}_{-1/q}(0, \infty)$ by Cannone and Planchon [5], as well as extended to the $BMO^{-1}$ space by Koch and Tataru [31]. Further related works can be founded in [3] [10] [14] [29] [32] [34] [30] [40] [44] [49] and references therein.

This paper is divided into eight sections. In section 2 we will apply the scaling used in Giga and Kohn [21] to (1.1) and introduce the energy inequality for the new unknown velocity and its derivatives. In section 3 the unknown velocity will be decomposed into low- and high-frequency part. The $L^\infty$-norm of the low-frequency part and $L^2$-norm of the high-frequency part will be estimated carefully. The large-time behavior of the weighted norm will be discovered in Proposition 3.2 under the condition (1.3), that will be utilized for proving the regularity of the solution at the blow-up time $T$ in section 4. We first recall a frequency decaying estimate obtained in [2] in section 5, and prove a time-space decaying estimate in good region in section 6. In section 7, we study frequency overlapping arisen in the Navier-Stokes equations and measurement of the moving boundary between good and bad region. In section 8, we will see that the condition (1.3) is satisfied by scaling.

**Motivation 1.** Let $\varphi \in C_0^\infty(\mathbb{R}^3, [0, 1])$ be a radial symmetrical function satisfying

$$
\varphi(\xi) = 1 \quad \forall|\xi| \leq 1, \quad \varphi(\xi) = 0 \quad \forall|\xi| \geq 2, \quad \xi \cdot \nabla \varphi(\xi) \leq 0 \quad \forall \xi.
$$

Let $\alpha \in (0, \frac{1}{8})$ and define

$$
\chi(\xi) = \begin{cases} 
\left( \frac{|\xi|}{\frac{1}{2} + \alpha} \right)^{\frac{1}{2} + 2\alpha}, & \forall|\xi| \leq \frac{1}{2} + \alpha \\
\varphi(\xi), & \forall|\xi| \geq \frac{1}{2} + \alpha.
\end{cases}
$$

Let $\lambda_t := (T - t)^{\frac{1}{2}}$ and define

$$
\Delta^t_{-1} u(x, t) := \mathcal{F}^{-1}[\varphi(\lambda_t \eta)\mathcal{F}[u](\eta, t)], \quad \sqrt{\Delta^t_{-1}} u(x, t) := \mathcal{F}^{-1}[\sqrt{1 - \varphi^2(\lambda_t \eta)}\mathcal{F}[u](\eta, t)]
$$

$$
\tilde{\Delta}^t_{-1} u(x, t) := \mathcal{F}^{-1}[\chi(\lambda_t \eta)\mathcal{F}[u](\eta, t)].
$$
To see that there is some hope to prove a $L^2 - L^\infty$ energy inequality that may be better than the energy inequality in $L^2(\mathcal{R}^3)$, let us consider the integration

\[
\frac{(T-t)}{2} \frac{d}{dt} \left[ \left( (T-t)^{\frac{3}{2}} \| \tilde{\Delta} t u(t) \|_{L^2}^2 + (T-t)^{\frac{1}{2}} \| \sqrt{\Delta_0} u(t) \|_{L^2}^2 \right) \right]
\]

\[
= \frac{1}{4(T-t)^{\frac{3}{2}}} \left( \| \tilde{\Delta} t u(t) \|_{L^2}^2 + \| \sqrt{\Delta_0} u(t) \|_{L^2}^2 \right)
\]

(1.4)

\[
+ \frac{(T-t)^{\frac{1}{2}}}{2} \int_{\mathcal{R}^3} |\mathcal{F}[u](\eta,t)|^2 \frac{d}{dt} \left\{ \chi^2(\lambda t \eta) - \varphi^2(\lambda t \eta) \right\} d\eta
\]

\[= - (T-t)^{\frac{1}{2}} \left( \| \nabla_x \tilde{\Delta} t u(t) \|_{L^2}^2 + \| \nabla_x \sqrt{\Delta_0} u(t) \|_{L^2}^2 \right)
\]

\[
+ (T-t)^{\frac{1}{2}} \int_{\mathcal{R}^3} \left\{ \varphi^2(\lambda t \eta) - \chi^2(\lambda t \eta) \right\} \mathcal{F}[u](\eta,t) \cdot \bar{\mathcal{F}}[u \cdot \nabla_x u](\eta,t) d\eta
\]

where

\[
\int_{\mathcal{R}^3} u(x,t) \cdot (u \cdot \nabla_x) u(x,t) dx = 0
\]

was used. First note that

\[
\frac{1}{4(T-t)^{\frac{3}{2}}} \| \sqrt{\Delta_0} u(t) \|_{L^2}^2 \leq \frac{(T-t)^{\frac{1}{2}}}{4} \| \nabla_x \sqrt{\Delta_0} u(t) \|_{L^2}^2.
\]

The remainder of the 2nd line and 3rd line are estimated in two cases.

Case 1. $\lambda_t |\eta| < \frac{1}{2} + \alpha.$

\[
\frac{1}{4(T-t)^{\frac{3}{2}}} \chi^2(\lambda t \eta) + (T-t)^{\frac{1}{2}} \chi(\lambda t \eta) \chi(\lambda t \eta) \left( -\frac{1}{2} \right) (T-t)^{-\frac{1}{2}} |\eta|
\]

\[
= \frac{1}{4(T-t)^{\frac{3}{2}}} \chi^2(\lambda t \eta) - \frac{1}{2} |\eta| \chi(\lambda t \eta) \left( \frac{1}{2} + 2\alpha \right) \left( \frac{\lambda_t |\eta|}{\frac{1}{2} + \alpha} \right)^{2\alpha - \frac{1}{2}} \frac{1}{2 + \alpha}
\]

\[
= \frac{1}{4(T-t)^{\frac{3}{2}}} \chi^2(\lambda t \eta) - \frac{1 + \alpha}{(T-t)^{\frac{1}{2}}} \chi^2(\lambda t \eta)
\]

\[
= \frac{-\alpha}{(T-t)^{\frac{1}{2}}} \chi^2(\lambda t \eta).
\]

Case 2. $\frac{1}{2} + \alpha \leq \lambda_t |\eta|.$

\[
\frac{1}{4(T-t)^{\frac{3}{2}}} \| \tilde{\Delta} t u(t) \|_{L^2}^2 \leq \frac{(T-t)^{\frac{1}{2}}}{4(\frac{1}{2} + \alpha)^2} \| \nabla_x \tilde{\Delta} t u(t) \|_{L^2}^2.
\]

We find that the 2nd line +3rd line +4th line in the right side of (1.4) can be estimated by

2nd line+3rd line+4th line

\[
\leq \frac{-4(\alpha + \alpha^2)(T-t)^{\frac{1}{2}}}{1 + 4(\alpha + \alpha^2)} \| \nabla_x \tilde{\Delta} t u(t) \|_{L^2}^2 - \frac{3(T-t)^{\frac{1}{2}}}{4} \| \nabla_x \sqrt{\Delta_0} u(t) \|_{L^2}^2.
\]

(1.5)
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The 5th line in the right side of (1.4) can be divided into two terms. First we consider
\[ | < (\Delta_{t-1})^2 u, u \cdot \nabla_x u > =< (\Delta_{t-1})^2 u, u \cdot \nabla_x \sqrt{\Delta_0} u > | \]
\[ = | < (\Delta_{t-1})^2 u, (\Delta_{t-1})^2 u + \sqrt{\Delta_0} u \cdot \nabla_x \sqrt{\Delta_0} u > | \]
\[ \leq \| \nabla_x (\Delta_{t-1})^2 u \|_{L^\infty} \| \sqrt{\Delta_0} u \|_{L^2}^2 + \| \nabla_x (\Delta_{t-1})^2 u \|_{L^4} \| (\Delta_{t-1})^2 u \|_{L^4} \| \sqrt{\Delta_0} u \|_{L^2}. \]

Next we consider
\[ | < (\tilde{\Delta}_{t-1})^2 u, u \cdot \nabla_x u > =< (\tilde{\Delta}_{t-1})^2 u, (\tilde{\Delta}_{t-1})^2 u + \sqrt{\Delta_0} u \cdot \nabla_x (\tilde{\Delta}_{t-1})^2 u + \sqrt{\Delta_0} u > | \]
\[ \leq \| \nabla_x (\tilde{\Delta}_{t-1})^2 u \|_{L^\infty} \| \sqrt{\Delta_0} u \|_{L^2}^2 + 2 \| \nabla_x (\tilde{\Delta}_{t-1})^2 u \|_{L^2} \| (\Delta_{t-1})^2 u \|_{L^\infty} \| \sqrt{\Delta_0} u \|_{L^2} \]
\[ + \| \nabla_x (\tilde{\Delta}_{t-1})^2 u \|_{L^2} \| (\Delta_{t-1})^2 u \|_{L^4}. \]

Note that
\[ \| \nabla_x (\tilde{\Delta}_{t-1})^2 u \|_{L^m} \leq \| \nabla_x \Delta_{t-1}^u u \|_{L^m}. \]

We utilize Sobolev type embedding inequality (see, Lemma 3.1) and the definition of \( \chi \) to find for \( m \in [4, \infty) \),
\[ \| \nabla_x (\Delta_{t-1})^2 u \|_{L^m} \leq \| \nabla_x \Delta_{t-1}^u u \|_{L^m} \]
\[ \leq \frac{C(\alpha)}{(T-t)^\frac{\alpha}{2}} \| \nabla_x \tilde{\Delta}_{t-1}^u u \|_{L^2} \leq \frac{C(\alpha)}{(T-t)^\frac{\alpha}{2}} \| \tilde{\Delta}_{t-1}^u u \|_{L^2}. \]

Then the 5th line in the right side of (1.4) can be estimated by
\[ (1.6) \quad \text{the 5th line} \leq \frac{C(\alpha)}{(T-t)^\frac{\alpha}{2}} \left( \| \tilde{\Delta}_{t-1}^u u(t) \|_{L^2} + \| \sqrt{\Delta_0} u(t) \|_{L^2} \right)^3. \]

From (1.4) (1.5) (1.6) we discover
\[ \frac{(T-t)}{2} \frac{d}{dt} \left( (T-t)^\frac{1}{\alpha} \| \tilde{\Delta}_{t-1}^u u(t) \|_{L^2}^2 + (T-t)^\frac{1}{\alpha} \| \sqrt{\Delta_0} u(t) \|_{L^2}^2 \right) \]
\[ \leq -4(\alpha + \alpha^2)(T-t)^\frac{1}{\alpha} \| \nabla_x \tilde{\Delta}_{t-1}^u u(t) \|_{L^2}^2 - \frac{3(T-t)^\frac{1}{\alpha}}{4} \| \nabla_x \sqrt{\Delta_0} u(t) \|_{L^2}^2 \]
\[ + \frac{C(\alpha)}{(T-t)^\frac{\alpha}{2}} \left( \| \tilde{\Delta}_{t-1}^u u(t) \|_{L^2} + \| \sqrt{\Delta_0} u(t) \|_{L^2} \right)^3. \]

This is just the \( L^2 - L^\infty \) energy inequality (3.11).

**Motivation 2.** The \( L^2 - L^\infty \) energy inequality yields an upper bound on
\[ (T-t)^\frac{1}{\alpha} \| \tilde{\Delta}_{t-1}^u u(t) \|_{L^2} + (T-t)^\frac{1}{\alpha} \| \sqrt{\Delta_0} u(t) \|_{L^2} \]
provided that it is small enough initially. For \( \beta \in (0, \frac{1}{2}) \), we redivide \( u \) as
\[ u(x, t) = u^{\mu_\beta}(x, t) + u^{1-\mu_\beta}(x, t) \]

where
\[ u^{\mu_\beta}(x, t) = \mathcal{F}^{-1} \left[ \varphi \left( |\eta| (T-t)^{\frac{1}{2}-\beta} \right) \mathcal{F}[u](\eta, t) \right](x), \]
\[ u^{1-\mu_\beta}(x, t) = \mathcal{F}^{-1} \left[ (1 - \varphi \left( |\eta| (T-t)^{\frac{1}{2}-\beta} \right) ) \mathcal{F}[u](\eta, t) \right](x). \]
Note that for $\beta < \frac{1}{2}$

$$\lim_{t \to T} \|u^{1-\mu_t^\beta}(t)\|_{L^2} = 0, \quad \text{provided} \quad \lim \sup_{t \to T} \|u(t)\|_{L^2} < \infty.$$ 

We see that for $T - t$ small so that

$$(T - t)^\beta \leq \frac{1}{2} + \alpha,$$

we have

$$\begin{align*}
(T - t)^{\frac{\beta}{2}} \|\hat{\Delta}_{-1} u(t)\|_{L^2} + (T - t)^{\frac{\beta}{2}} \|\sqrt{\Delta_t} u(t)\|_{L^2} \\
\leq (T - t)^{\frac{\beta}{2}} \|\hat{\Delta}_{-1} u^{(\mu_t^\beta)}(t)\|_{L^2} + 2(T - t)^{\frac{\beta}{2}} \|u^{(1-\mu_t^\beta)}(t)\|_{L^2} \\
\lesssim (T - t)^{\frac{\beta+eta}{2} + 2\alpha} \|u^{(\mu_t^\beta)}(t)\|_{L^2} + (T - t)^{\frac{\beta}{2}} \|u^{1-\mu_t^\beta}(t)\|_{L^2}.
\end{align*}$$

(1.8) 

Obviously the low frequency part is small for

$$\beta > \frac{1}{2 + 8\alpha}, \quad T - t < 1,$$

and the high frequency part is small for $T - t > 1$. Because

$$\int_{\{\eta \geq (t_2 - t_0)^{\alpha-\frac{1}{2}}\}} |\mathcal{F}[u](\eta, t_0)|^2 d\eta \leq \delta(t_2 - t_0)^{\frac{1}{2}} \quad \text{for some } \beta < \frac{1}{2},$$

is not invariant under the scaling

$$u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) := \lambda^2 p(\lambda x, \lambda^2 t), \quad \text{for } \lambda \in (0, \infty)$$

while (1.1) is invariant, we find that the high frequency part is always small initially.

**Motivation 3.** It is well-known that nonlinear interactions of the Navier-Stokes equations induce frequency overlapping. Good news is high frequency energy dispersed by low frequency energy, bad news is low frequency part affected by high frequency part. Applying the operator $(1 - \mu_t^\beta)$ to the equations (1.1) for $u(x, t)$, and utilizing $\varphi' \leq 0$ we find

(1.9) 

$$\frac{1}{2} \frac{d}{dt} \|u^{1-\mu_t^\beta}(t)\|_{L^2}^2 \leq (\frac{1}{2}) \|\nabla_x u^{1-\mu_t^\beta}(t)\|_{L^2}^2 < u(t) \partial_t u^{1-(1-\mu_t^\beta)^2}(t), u^{(1-\mu_t^\beta)}(t) >$$

$$\lesssim (\frac{1}{2}) \|\nabla_x u^{1-\mu_t^\beta}(t)\|_{L^2}^2 + (T - t)^{\frac{\beta}{2} + \frac{1}{2}} \|u_0\|_{L^2}^3$$

where

$$f^{(1-\mu_t^\beta)^2} := \left(f^{1-\mu_t^\beta}\right)^{1-\mu_t^\beta}, \quad f^{1-(1-\mu_t^\beta)^2} = \mathcal{F}^{-1}[\left(1 - (1 - \varphi((T - t)^{\frac{\beta}{2}} |\eta|))\right)^2 f(\eta)].$$

According to the estimate from below [35]

$$\|\nabla_x u(t)\|_{L^2} \geq \frac{c_I}{(T - t)^{\frac{\beta}{2}}}, \quad \text{for some } c_I > 0$$

as well as

$$\|\nabla_x u^{\mu_t^\beta}(t)\|_{L^2} \leq (T - t)^{2\beta - 1} \|u^{\mu_t^\beta}(t)\|_{L^2} \leq \frac{c_I}{2(T - t)^{\frac{\beta}{2}}}, \quad \text{for } \beta \geq \frac{1}{4} \text{ and } \|u_0\|_{L^2} \leq \frac{c_I}{2}$$

for some positive constants $c_I$. 

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we have

\[ \| \nabla_x u^{1-\mu_0^2}(t) \|_{L^2}^2 \geq \frac{c_I}{2(T-t)^{\frac{1}{2}}} \geq (T-t)^{\frac{3}{2}(\beta-\frac{1}{2})} \| u_0 \|_{L^2}^3, \quad \text{for } \beta \geq \frac{3}{10} \text{ and } \| u_0 \|_{L^2}^3 \lesssim \frac{c_I}{2}. \]

Returning to (1.9), we find

\[ \frac{d}{dt}\| u^{1-\mu(t)} \|_{L^2}^2 \leq (\frac{-1}{2})\frac{c_I}{2\sqrt{T-t}} \]

that implies

(1.10) \[ \| u^{1-\mu(t)} \|_{L^2}^2 - \frac{c_I}{2}(T-t)^{\frac{1}{2}} \leq \| u^{1-\mu(t_0)} \|_{L^2}^2 - \frac{c_I}{2}(T-t_0)^{\frac{1}{2}}. \]

Consequently the high frequency part in (1.8) is small provided that it is small initially.

2. Energy estimates

As in [20][21][22] where Giga and Kohn introduced similar transformation for the blow-up problem of semi-linear heat equations, we apply

(2.1) \[ y = \frac{1}{\sqrt{T-t}}x, \quad \tau = -\ln(T-t), \quad w(y, \tau) = (T-t)^{1/2}u(x, t), \]

to (1.1) and consider the following new problem

(2.2)

\[
\begin{aligned}
\partial_\tau w &= \Delta_y w - \frac{y}{2} \cdot \nabla_y w - \frac{1}{2} w - \nabla_y w - \nabla_y q, \quad \forall y \in \mathbb{R}^3, \quad \tau > -\ln T \\
\text{div}_y w(y, \tau) &= 0, \quad \text{in } \mathbb{R}^3 \times (-\ln T, \infty) \\
w(y, -\ln T) &= T^{1/2}u_0(T^{1/2}y), \quad \text{in } \mathbb{R}^3
\end{aligned}
\]

where

\[ q(y, \tau) = (T-t)p(x, t). \]

Note that under the transformation (2.1) we have

(2.3)

\[ \| w(\tau) \|_{L^2} = (T-t)^{\frac{1}{4}}\| u(t) \|_{L^2}, \quad \| \nabla_y w(\tau) \|_{L^2} = (T-t)^{\frac{5}{8}}\| \nabla_x u(t) \|_{L^2}, \]

\[ \| \nabla_y^2 w(\tau) \|_{L^2} = (T-t)^{\frac{3}{8}}\| \nabla_x^2 u(t) \|_{L^2}. \]

Multiplying the first one of (2.2) by \( w \) and integrating it over \( \mathbb{R}^3 \), by using the second equation of (2.2) we have

(2.4)

\[
\frac{1}{2} \int_{\mathbb{R}^3} \partial_\tau |w(y, \tau)|^2 dy = (-1) \int_{\mathbb{R}^3} \left( |\nabla_y w(y, \tau)|^2 - \frac{1}{4} |w(y, \tau)|^2 \right) dy \\
- \frac{1}{4} \int_{\mathbb{R}^3} \text{div} (y|w(y, \tau)|^2) dy.
\]

Noting that

(2.5) \[ \int_{\mathbb{R}^3} \text{div} (y|w(y, \tau)|^2) dy = \lim_{R \to \infty} \int_{\partial B_R} |y||w(y, \tau)|^2 d\sigma(y) \geq 0 \]

we obtain
Lemma 2.1. For any $\tau > 0$, we have

\begin{equation}
\frac{1}{2} \frac{d}{dt} \|w(\tau)\|_{L^2(\mathbb{R}^3)}^2 \leq (-1) \{ \|\nabla w(\tau)\|_{L^2(\mathbb{R}^3)}^2 - \frac{1}{4} \|w(\tau)\|_{L^2(\mathbb{R}^3)}^2 \}.
\end{equation}

Furthermore, we take differential in the equations of (2.2) and obtain

\begin{equation}
\partial_t \partial_j w = \Delta \partial_j w - \frac{1}{2} y \cdot \nabla \partial_j w - \partial_j w \\
-(\partial_j w \cdot \nabla) w - (w \cdot \nabla) \partial_j w - \nabla \partial_j q.
\end{equation}

By the same strategy as in the proof of Lemma 2.1, from (2.7) as well as the equation

\begin{align*}
\partial_t \Delta w &= \Delta^2 w - \frac{1}{2} (y \cdot \nabla) \Delta w - \frac{3}{2} \Delta w - \Delta((w \cdot \nabla)w) - \nabla \Delta q
\end{align*}

by taking twice differential in (2.2), we have

Lemma 2.2. For all $\tau > 0$

\begin{equation}
\frac{d}{d\tau} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy \leq -2 \int_{\mathbb{R}^3} |\nabla^2 w(y, \tau)|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy
\end{equation}

and

\begin{equation}
\frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta w(y, \tau)|^2 dy \leq -2 \int_{\mathbb{R}^3} |\nabla \Delta w(y, \tau)|^2 dy - \frac{3}{2} \int_{\mathbb{R}^3} |\Delta w(y, \tau)|^2 dy
\end{equation}

Remark 2.3. (1) As a blow-up argument, we assume that $u(x, t)$ is bounded for $t < T$ and blows up at $t = T$. As a direct corollary, we can prove that $\|u(t)\|_{H^3(\mathbb{R}^3)}$ and $\partial_t\|u(t)\|_{H^m(\mathbb{R}^3)}^2$ ($m = 0, 1, 2$), as well as $\|\partial_t u(t)\|_{L^2(\mathbb{R}^3)}$, $\|\partial_t \nabla_x u(t)\|_{L^2(\mathbb{R}^3)}$ are bounded for $t < T$. So we have the same results for $\|w(\tau)\|_{H^3(\mathbb{R}^3)}$ and $\partial_t\|w(\tau)\|_{H^m(\mathbb{R}^3)}^2$ ($m = 0, 1, 2$) for $\tau < \infty$, as well as the similar results for $q$ by the boundedness of Riesz transformation.

(2) Since $u(x, t), \partial_t u(x, t) \in L^2(\mathbb{R}^3)$ for $t < T$,

\begin{equation}
\int_0^t \int_{\mathbb{R}^3} |\partial_h u(x, h)||u(x, h)| dx dh < \infty,
\end{equation}

we can use Fubini theorem to obtain

\begin{equation}
2 \int_{\mathbb{R}^3} \partial_t u(x, t) \cdot u(x, t) dx = \frac{d}{dt} \int_0^t \int_{\mathbb{R}^3} \partial_h |u(x, h)|^2 dx dh
\end{equation}

\begin{equation}
= \frac{d}{dt} \int_{\mathbb{R}^3} \int_0^t \partial_h |u(x, h)|^2 dh dx = \frac{d}{dt} \int_{\mathbb{R}^3} |u(x, t)|^2 dx.
\end{equation}
Regularity of solutions to the Navier-Stokes equations

Noting that
\[ \partial_t u(x, t) = (T-t)^{-\frac{3}{2}} \left\{ \partial_\tau w(\frac{x}{(T-t)^{1/2}}, \tau) + \frac{x}{2(T-t)^{1/2}} \cdot \nabla_y w(\frac{x}{(T-t)^{1/2}}, \tau) + \frac{1}{2} w(\frac{x}{(T-t)^{1/2}}, \tau) \right\} \]
where \( \tau = (-) \ln(T-t) \), from

(2.11) \[ \int_{\mathbb{R}^3} |\partial_t u(x, t)|^2 \, dx = (T-t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} |\partial_\tau w(y, \tau) + \frac{y}{2} \cdot \nabla_y w(y, \tau) + \frac{1}{2} w(y, \tau)|^2 \, dy \]
and

(2.12) \[ \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx = (T-t)^{1/2} \int_{\mathbb{R}^3} |w(y, \tau)|^2 \, dy \]
we have for \( t < T \)

(2.13) \[ \int_{\mathbb{R}^3} |\partial_\tau w(y, \tau) + \frac{y}{2} \cdot \nabla_y w(y, \tau)|^2 \, dy < \infty. \]

Moreover, from (2.10), we get

(2.14) \[ (T-t)^{-\frac{3}{2}} \left\{ \partial_\tau \int_{\mathbb{R}^3} |w(y, \tau)|^2 \, dy - \frac{1}{2} \int_{\mathbb{R}^3} |w(y, \tau)|^2 \, dy \right\} \]
\[ = \frac{d}{dt} \left( (T-t)^{\frac{1}{2}} \int_{\mathbb{R}^3} |w(y, \tau)|^2 \, dy \right) = \frac{d}{dt} \left( \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx \right) = 2 \int_{\mathbb{R}^3} \partial_t u(x, t) \cdot u(x, t) \, dx \]
\[ = 2 \int_{\mathbb{R}^3} (T-t)^{-\frac{3}{2}} \left\{ \partial_\tau w(\frac{x}{(T-t)^{1/2}}, \tau) + \frac{x}{2(T-t)^{1/2}} \cdot \nabla_y w(\frac{x}{(T-t)^{1/2}}, \tau) \right. \]
\[ + \frac{1}{2} w(\frac{x}{(T-t)^{1/2}}, \tau) \right\} \cdot (T-t)^{-\frac{1}{2}} w(\frac{x}{(T-t)^{1/2}}, \tau) \, dx \]
\[ = (T-t)^{-\frac{1}{2}} \int_{\mathbb{R}^3} \left\{ 2 \partial_\tau w(y, \tau) \cdot w(y, \tau) + (y \cdot \nabla_y w(y, \tau)) \cdot w(y, \tau) + |w(y, \tau)|^2 \right\} \, dy. \]
By using (2.14), from (2.4) we get (2.6) again.

3. \((L^\infty, L^2)\)-DECOMPOSITION OF \( w \)

In this section we will prove that \( w \) can be decomposed as the sum of a \( L^\infty(0, \infty; L^m(\mathbb{R}^3)) \) \( (m \in [4, \infty]) \) part and a \( L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3)) \) part.

Let \( \varphi \in C_0^\infty(\mathbb{R}^3, [0, 1]) \) be a radial symmetrical function satisfying

(3.1) \[ \varphi(\xi) = 1 \quad \forall |\xi| \leq 1, \quad \varphi(\xi) = 0 \quad \forall |\xi| \geq 2, \quad \xi \cdot \nabla \varphi(\xi) \leq 0 \quad \forall \xi. \]
Like the Littlewood-Paley analysis, we define the operators

\[ \Delta_{-1} w(y, \tau) = \mathcal{F}^{-1} [\varphi] \ast w(y, \tau), \]

(3.2) \[ \Delta_0 w(y, \tau) = \mathcal{F}^{-1} [1 - \varphi] \ast w(y, \tau), \]
\[ \sqrt{\Delta_0} w(y, \tau) = \mathcal{F}^{-1} [\sqrt{1 - \varphi^2}] \ast w(y, \tau). \]
Notice that
\[ w(y, \tau) = \Delta_{-1}w(y, \tau) + \Delta_0w(y, \tau), \]
\[ \|w(\tau)\|_{L^2(\mathbb{R}^3)}^2 = \|\Delta_{-1}w(\tau)\|_{L^2(\mathbb{R}^3)}^2 + \|\sqrt{\Delta_0w(\tau)}\|_{L^2(\mathbb{R}^3)}^2, \]
\[ \|\Delta_0w(\tau)\|_{L^2} \leq \|\sqrt{\Delta_0w(\tau)}\|_{L^2}. \]
So we write (2.6) as
\[ \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\sqrt{\Delta_0w(y, \tau)}|^2 dy \leq - \int_{\mathbb{R}^3} |\nabla \sqrt{\Delta_0w(y, \tau)}|^2 - \frac{1}{4} |\sqrt{\Delta_0w(y, \tau)}|^2 \]
\[ = - \int_{\mathbb{R}^3} |\nabla \Delta_{-1}w(y, \tau)|^2 - \frac{1}{4} |\Delta_{-1}w(y, \tau)|^2 \]
\[ - \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_{-1}w(y, \tau)|^2 dy. \]

Applying the operator \( \Delta_{-1} \) to the first equation of (2.2), we have
\[ \partial_\tau \Delta_{-1}w = \Delta \Delta_{-1}w = \frac{1}{2} \Delta_{-1}(y \cdot \nabla w) - \frac{1}{2} \Delta_{-1}w - \Delta_{-1}(w \nabla \nabla w) - \nabla \Delta_{-1}q. \]

Multiplying (3.4) by \( \Delta_{-1}w \) and integrating over \( \mathbb{R}^3 \) we get
\[ \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_{-1}w|^2 dy = - \int_{\mathbb{R}^3} |\nabla \Delta_{-1}w|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |\Delta_{-1}w|^2 dy \]
\[ - \frac{1}{2} \int_{\mathbb{R}^3} \Delta_{-1}(y \cdot \nabla w) \cdot \Delta_{-1}wdy - \int_{\mathbb{R}^3} \Delta_{-1}(w \cdot \nabla w) \cdot \Delta_{-1}wdy, \]
where \( \text{div } w = 0 \) is used to cancel the term including \( q \).

Because
\[ \int_{\mathbb{R}^3} y \cdot \nabla |\Delta_{-1}w|^2 dy = 2 \int_{\mathbb{R}^3} y_j \Delta_{-1}w \cdot \partial_j \Delta_{-1}wdy \]
\[ = 2 \int_{\mathbb{R}^3} \xi_j \varphi \mathcal{F}[w] \cdot \partial_j (\varphi \mathcal{F}[w])d\xi \]
\[ = -3 \int_{\mathbb{R}^3} \varphi^2 |\mathcal{F}[w]|^2 dy, \]
we have
\[ \int_{\mathbb{R}^3} \partial_j \{y_j |\Delta_{-1}w|^2\} dy = 0. \]
So
\[ \int_{\mathbb{R}^3} \partial_j (\Delta_{-1}(y_j w) \cdot \Delta_{-1}w)dy \]
\[ = \int_{\mathbb{R}^3} \partial_j \{\mathcal{F}^{-1}[\varphi] \ast (y_j w) \cdot \mathcal{F}^{-1}[\varphi] \ast w\} dy \]
\[ = (-1) \int_{\mathbb{R}^3} \partial_j \{\varphi_j \ast w \cdot \mathcal{F}^{-1}[\varphi] \ast w\} dy + \int_{\mathbb{R}^3} \partial_j \{y_j |\Delta_{-1}w|^2\} dy \]
\[ = 0 \]
where \( \varphi_j(y) = y_j \mathcal{F}^{-1}[\varphi](y) \).
Regularity of solutions to the Navier-Stokes equations

Since
\[ \int \Delta_{-1}(y \cdot \nabla w) \cdot \Delta_{-1} w dy \]
\[ = - \sum_{j=1}^{3} \int \Delta_{-1}(y_j w) \cdot \Delta_{-1} \partial_j w dy - 3 \int |\Delta_{-1} w|^2 dy \]
\[ = - \sum_{j=1}^{3} \int \varphi(\xi) F[y_j w] \cdot \varphi(\xi) F[\partial_j w] d\xi - 3 \int |\Delta_{-1} w|^2 dy \]
and
\[ F[y_j w] = i \frac{\partial}{\partial \xi_j} F[w], \quad F[\partial_j w] = i \xi_j F[w], \]
we have
\[ (3.6) \]
\[ \int \Delta_{-1}(y \cdot \nabla w) \cdot \Delta_{-1} w dy = - \sum_{j=1}^{3} \int \varphi^2(\xi) \xi_j \frac{\partial}{\partial \xi_j} F[w] \cdot F[w] d\xi - 3 \int |\Delta_{-1} w|^2 dy \]
\[ = - \sum_{j=1}^{3} \frac{1}{2} \int \varphi^2(\xi) \xi_j \frac{\partial}{\partial \xi_j} |F[w]|^2 d\xi - 3 \int |\Delta_{-1} w|^2 dy \]
\[ = \sum_{j=1}^{3} \frac{1}{2} \int \xi_j \frac{\partial}{\partial \xi_j} \varphi^2(\xi) |F[w]|^2 d\xi + \frac{3}{2} \int \varphi^2(\xi) |F[w]|^2 d\xi - 3 \int |\Delta_{-1} w|^2 dy \]
\[ = \frac{1}{2} \int \xi \cdot \nabla \varphi^2(\xi) |F[w]|^2 d\xi - \frac{3}{2} \int |\Delta_{-1} w|^2 dy. \]

From (3.3) and (3.6), we get
\[ (3.7) \]
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\Delta_{-1} w|^2 dy = - \int_{\mathbb{R}^3} |\nabla \Delta_{-1} w|^2 dy + \frac{1}{4} \int_{\mathbb{R}^3} |\Delta_{-1} w|^2 dy \]
\[ - \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \varphi^2(\xi) |F[w]|^2 d\xi - \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla) w) \cdot \Delta_{-1} w dy. \]

From (3.3) and (3.7), we have
\[ (3.8) \]
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\sqrt{\Delta_0} w(y, \tau)|^2 dy \leq - \int_{\mathbb{R}^3} |\nabla \sqrt{\Delta_0} w(y, \tau)|^2 - \frac{1}{4} |\sqrt{\Delta_0} w(y, \tau)|^2 dy \]
\[ + \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \varphi^2(\xi) |F[w]|^2 d\xi + \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla) w) \cdot \Delta_{-1} w dy \]
\[ \leq - \frac{3}{4} \int_{\mathbb{R}^3} |\nabla \sqrt{\Delta_0} w(y, \tau)|^2 dy \]
\[ + \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \varphi^2(\xi) |F[w]|^2 d\xi + \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla) w) \cdot \Delta_{-1} w dy \]
where \(|\xi|^2 |F[\sqrt{\Delta_0} w]|^2 \geq |F[\sqrt{\Delta_0} w]|^2\) was used in the last step.
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Let $\alpha \in (0, \frac{1}{8})$ and define

$$
\chi(\xi) = \begin{cases} 
\left( \frac{|\xi|}{\frac{1}{2} + \alpha} \right)^{\frac{1}{2} + 2\alpha}, & \forall |\xi| \leq \frac{1}{2} + \alpha \\
\varphi(\xi), & \forall |\xi| \geq \frac{1}{2} + \alpha.
\end{cases}
$$

Instead of $\varphi$ by $\chi$, we define the operator

$$
\tilde{\Delta}_{-1} f = \mathcal{F}^{-1}[\chi(\xi)\mathcal{F}[f](\xi)].
$$

Applying $\tilde{\Delta}_{-1}$ to (2.2), in almost exactly the same way that we derived (3.7) we have

$$
\frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 dy = - \int_{\mathbb{R}^3} |\nabla \tilde{\Delta}_{-1} w|^2 dy + \frac{1}{4} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 dy
$$

$$
- \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \chi^2(\xi)[\mathcal{F}[w](\xi, \tau)]^2 d\xi - \int_{\mathbb{R}^3} \tilde{\Delta}_{-1}((w \cdot \nabla)w) \cdot \tilde{\Delta}_{-1} w dy.
$$

Combining it with (3.8), we have

$$
\frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 + |\sqrt{\Delta_0} w|^2 dy
$$

$$
\leq - \frac{3}{4} \int_{\mathbb{R}^3} |\nabla \sqrt{\Delta_0} w|^2 dy - \alpha \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 dy
$$

$$
+ \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1} w - \tilde{\Delta}_{-1}((w \cdot \nabla)w) \cdot \tilde{\Delta}_{-1} w dy
$$

$$
+ \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla (\varphi^2 - \chi^2)[\mathcal{F}[w]]^2 d\xi - \int_{\mathbb{R}^3} |\nabla \tilde{\Delta}_{-1} w|^2 dy + \left( \frac{1}{4} + \alpha \right) \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 dy.
$$

For $|\xi| \leq 1$, the last term is written as

$$
A = \int \left\{ \frac{1}{4} \xi \cdot \nabla (-\chi^2) - |\xi|^2 \chi^2 + \left( \frac{1}{4} + \alpha \right) \chi^2 \right\} [\mathcal{F}[w]]^2 d\xi
$$

and noting that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ as well as the definition of $\chi$, $A \leq 0$. For $|\xi| \in [1, 2]$, the last term is written as

$$
B = \int \left\{ \frac{1}{4} (1 - \left( \frac{1}{2} + \alpha \right)^{1+4\alpha}) \xi \cdot \nabla \varphi^2 - (|\xi|^2 - (\frac{1}{4} + \alpha))(\frac{1}{2} + \alpha)^{1+4\alpha} \varphi^2 \right\} [\mathcal{F}[w]]^2 d\xi,
$$

and $B \leq 0$. So we get

(3.9)

$$
\frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 + |\sqrt{\Delta_0} w|^2 dy \leq - \frac{3}{4} \int_{\mathbb{R}^3} |\nabla \sqrt{\Delta_0} w|^2 dy - \alpha \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 dy
$$

$$
+ \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1} w - \tilde{\Delta}_{-1}((w \cdot \nabla)w) \cdot \tilde{\Delta}_{-1} w dy.
$$
Lemma 3.1. (1) For any \( m \in [4, \infty] \),
\[
\|\Delta_{-1}f\|_{L^m(\mathbb{R}^3)} \leq C(\alpha)\|\tilde{\Delta}_{-1}f\|_{L^2(\mathbb{R}^3)}, \quad \forall f \in L^2(\mathbb{R}^3)
\]
where the constant \( C(\alpha) < \infty \) depends only on \( \alpha \).

(2) For all \( \beta = (\beta_1, \beta_2, \beta_3) \) (\( \beta_j \in \mathbb{N}, j = 1, 2, 3 \))
\[
\|D^\beta \Delta_0 w(\cdot, \tau)\|_{L^2(\mathbb{R}^3)} \leq \|D^\beta \sqrt{\Delta_0} w(\cdot, \tau)\|_{L^2(\mathbb{R}^3)}
\]
where \( D^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3} \).

Proof. From Hausdorff-Young inequality
\[
\|\Delta_{-1}f\|_{L^{m}(\mathbb{R}^3)} \leq (2\pi)^{3/m'} \left( \int_{\mathbb{R}^3} |\varphi(\xi)\mathcal{F}[f](\xi)|^{m'} d\xi \right)^{1/m'} \\
\leq (2\pi)^{3/m'} \left( \int_{\mathbb{R}^3} |\xi|^{\frac{4}{2}+2\alpha} \varphi(\xi)\mathcal{F}[f](\xi)\right)^{1/2} \left( \int_{|\xi| \leq 2} |\xi|^{-\left(\frac{4}{2}+2\alpha\right)\frac{2m'}{2m'}} d\xi \right)^{\frac{2-m'}{2m'}} \\
\leq C(\alpha) \left( \int_{\mathbb{R}^3} |\chi(\xi)\mathcal{F}[f](\xi)|^2 \right)^{1/2}
\]
for \( \alpha \in (0, \frac{1}{8}) \). So we have (1).

To prove (2), we only need to consider the case \( |\beta| = \sum_{1 \leq j \leq 3} \beta_j = 0 \). Since \( 0 \leq \varphi \leq 1 \) and \( 1 - \varphi^2 = (1 - \varphi)(1 + \varphi) \geq (1 - \varphi)^2 \), in this case we have
\[
\int_{\mathbb{R}^3} |\Delta_0 w(y, \tau)|^2 dy = \int_{\mathbb{R}^3} (1 - \varphi(\xi))^2 |\mathcal{F}[w](\xi, \tau)|^2 d\xi \\
\leq \int_{\mathbb{R}^3} (1 - \varphi^2(\xi)) |\mathcal{F}[w](\xi, \tau)|^2 d\xi = \int_{\mathbb{R}^3} |\sqrt{\Delta_0} w(y, \tau)|^2 dy. \quad \Box
\]

Now we estimate the last term in the right of (3.9). We only need to consider the integration for \( \varphi \) in the last term, because for \( \chi \) the proof is same. Notice that
\[
\int \Delta_{-1}(w_j \partial_j w) \cdot \Delta_{-1} w dy = - \int (w_j w) \cdot (\Delta_{-1})^2(\partial_j w) dy \\
= - \int w_j \sqrt{\Delta_0} w \cdot (\Delta_{-1})^2(\partial_j w) dy \\
= - \int \Delta_{-1}w_j \sqrt{\Delta_0} w \cdot (\Delta_{-1})^2(\partial_j w) dy - \int \Delta_0 w_j \sqrt{\Delta_0} w \cdot (\Delta_{-1})^2(\partial_j w) dy.
\]
Because
\[
\left| \int \Delta_{-1}w_j \sqrt{\Delta_0} w \cdot (\Delta_{-1})^2(\partial_j w) dy \right| \\
\leq (\int |\Delta_{-1}w|^4 dy)^{1/2} (\int |\sqrt{\Delta_0} w|^2 dy)^{1/2} \\
\leq C \|\tilde{\Delta}_{-1}w\|_{L^4(\mathbb{R}^3)}^2 \|\sqrt{\Delta_0} w\|_{L^2(\mathbb{R}^3)} (\text{by Lemma 3.1 (1)-(2)})
\]
There is Proposition 3.2.

Moreover, for \( \tau > \tau_0 \)

\[
\int \Delta_0 w \sqrt{\Delta_0 w \cdot (\Delta_1)^2 (\partial_j w)} dy \leq C \| \Delta_1 (\partial_j w) \|_{L^\infty(\mathbb{R}^3)} \| \sqrt{\Delta_0 w} \|_{L^2(\mathbb{R}^3)}
\]

the last term in the right of (3.9) can be estimated by

\[
C \{ \| \Delta_1 w \|_{L^2(\mathbb{R}^3)}^3 + \| \Delta_1 w \|_{L^2(\mathbb{R}^3)}^2 \| \sqrt{\Delta_0 w} \|_{L^2(\mathbb{R}^3)} + \| \Delta_1 w \|_{L^2(\mathbb{R}^3)} \| \sqrt{\Delta_0 w} \|_{L^2(\mathbb{R}^3)} \}
\]

So we get

\[
\frac{1}{2} d \int_{\mathbb{R}^3} \left( |\Delta_1 w(y, \tau)|^2 + |\sqrt{\Delta_0 w(y, \tau)}|^2 \right) dy + \left( \frac{3}{4} - \alpha \right) \int_{\mathbb{R}^3} |\nabla \sqrt{\Delta_0 w(y, \tau)}|^2 dy
\]

\[
\leq -\alpha \int_{\mathbb{R}^3} |\Delta_1 w(y, \tau)|^2 + |\sqrt{\Delta_0 w(y, \tau)}|^2 dy + C \int_{\mathbb{R}^3} |\Delta_1 w(y, \tau)|^2 + |\sqrt{\Delta_0 w(y, \tau)}|^2 dy)^{3/2}
\]

Proposition 3.2. There is \( \delta = \delta(\alpha) > 0 \) such that if

\[
\left( \int_{\mathbb{R}^3} |\Delta_1 w(y, \tau_0)|^2 + |\sqrt{\Delta_0 w(y, \tau_0)}|^2 dy \right)^{\frac{3}{2}} \leq \delta^\frac{3}{2}, \quad \text{(Assumption 1)}
\]

then for all \( \tau > \tau_0 \)

(3.12)

\[
\frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_1 w(y, \tau)|^2 + |\sqrt{\Delta_0 w(y, \tau)}|^2 dy \leq -\alpha \int_{\mathbb{R}^3} |\Delta_1 w(y, \tau)|^2 + |\sqrt{\Delta_0 w(y, \tau)}|^2 dy.
\]

Moreover, for \( w(y, \tau) = \Delta_1 w(y, \tau) + \Delta_0 w \), and for all \( m \in [4, \infty] \),

\[
\| D^\beta \Delta_1 w(\tau) \|_{L^m(\mathbb{R}^3)} \leq C(\beta) \delta, \quad \forall \tau > \tau_0, \quad \forall \beta,
\]

\[
\lim_{\tau \to \infty} \| \Delta_1 w(\tau) \|_{L^m(\mathbb{R}^3)} = 0,
\]

(3.13)

\[
\sup_{\tau \geq \tau_0} \int_{\mathbb{R}^3} |\Delta_0 w|^2 dy + \int_{\tau_0}^\infty d\tau \int_{\mathbb{R}^3} |\nabla \Delta_0 w|^2 dy \leq C\delta,
\]

(3.14)

\[
\lim_{\tau \to \infty} \int_{\mathbb{R}^3} |\Delta_0 w|^2 dy = 0.
\]

Remark 3.3. Suppose \( \psi \) is a radial function satisfying

\[
\psi \in C(\mathbb{R}^3, [0, 1]), \quad \xi \cdot \nabla_\xi \psi(\xi) \in L^\infty(\mathbb{R}^3).
\]

(3.15)
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Since \( \psi(\xi)F[w](\xi, \tau) \in L^2(\mathbb{R}^3) \), we have \( F^{-1}[\psi] * w = F^{-1}[\psi F[w]] \in L^2(\mathbb{R}^3) \) and

\[
\int_{\mathbb{R}^3} |F^{-1}[\psi] * w(y, \tau)|^2 dy (T-t)^{-\frac{3}{2}} = \int_{\mathbb{R}^3} |F^{-1}[\psi] * w(\frac{\bar{x}}{(T-t)^{1/2}}, \tau)|^2 d\bar{x}
\]

(3.16)

\[
= \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} F^{-1}[\psi] \left( \frac{\bar{x}}{(T-t)^{1/2}} - z \right) w(z, \tau) dz \right|^2 d\bar{x}
\]

\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^{-1}[\psi] \left( \frac{\bar{x} - x}{(T-t)^{1/2}} - z \right) (T-t)^{1/2} u((T-t)^{1/2}z, t) dz^2 d\bar{x}
\]

where \( \tau = (-) \ln(T-t) \). Performing further calculations, we also have

(3.17)

\[
\partial_t \{(T-t)^{3/2} \psi((T-t)^{1/2}x)F[u](\xi, t)\} \]

\[
= \partial_t F\left[ \int_{\mathbb{R}^3} F^{-1}[\psi] \left( \frac{\bar{x} - x}{(T-t)^{1/2}} \right) u(x, t) dx \right]
\]

\[
= F\left[ \int_{\mathbb{R}^3} F^{-1}[\psi] \left( \frac{\bar{x} - x}{(T-t)^{1/2}} \right) \partial_t u(x, t) + \left\{ \frac{\bar{x} - x}{2(T-t)^{3/2}} \right\} F^{-1}[\psi] \left( \frac{\bar{x} - x}{(T-t)^{1/2}} \right) u(x, t) dx \right]
\]

(3.18)

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} F^{-1}[\psi] \left( \frac{\bar{x} - x}{(T-t)^{1/2}} \right) u(x, t) dx^2 d\bar{x}
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \left| (T-t)^{3/2} \psi((T-t)^{1/2}x)F[u](\xi, t) \right|^2 d\xi
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \int_{0}^{T} \partial_t |(T-h)^{3/2} \psi((T-h)^{1/2}x)F[u](\xi, h)|^2 dh d\xi
\]

\[
= \frac{d}{dt} \int_{\mathbb{R}^3} \int_{0}^{T} \partial_t |(T-h)^{3/2} \psi((T-h)^{1/2}x)F[u](\xi, h)|^2 d\xi dh
\]

\[
= \int_{\mathbb{R}^3} \partial_t |(T-t)^{3/2} \psi((T-t)^{1/2}x)F[u](\xi, t)|^2 d\xi
\]

\[
= 2Re \int_{\mathbb{R}^3} \left\{ (-) \frac{3}{2} (T-t)^{1/2} \psi((T-t)^{1/2}x) F[u](\xi, t) - (T-t) \xi \cdot \psi'(T-t)^{1/2} \right\} \partial_t F[u](\xi, t) \cdot (T-t)^{3/2} \psi((T-t)^{1/2}x) \overline{F[u](\xi, t)} d\xi,
\]
where because $\mathcal{F}[u](\xi, t), \partial_t \mathcal{F}[u](\xi, t) \in L^2(\mathbb{R}^3)$ for $t < T$ and $\psi$ satisfies (3.15), we have

$$
\int_{\mathbb{R}^3} |\partial_t|(T-t)^{3/2} \psi((T-t)^{1/2} \xi) \mathcal{F}[u](\xi, t)|^2 d\xi < \infty
$$

and Fubini theorem can be used.

From (3.16)-(3.18), we find

(3.19)

$$
(T - t)^{5/2} \frac{d}{dt} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy - \frac{7}{2} (T - t)^{5/2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy
$$

$$
= \frac{d}{dt} \{(T - t)^{7/2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy \}
$$

$$
= \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi](\frac{\bar{x} - x}{(T - t)^{1/2}}) u(x, t) dx \frac{d\bar{x}}{d\bar{\bar{x}}}
$$

$$
= 2(T - t)^{5/2} \int_{\mathbb{R}^3} \{(\frac{3}{2}) \psi(\xi) \mathcal{F}[w](\xi, \tau) - \frac{\xi}{2} \cdot \psi'(\xi) \mathcal{F}[w](\xi, \tau) \} \cdot \psi(\xi) \mathcal{F}[w](\xi, \tau) d\xi
$$

$$
+ 2 \int_{\mathbb{R}^3} \{ \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi](\frac{\bar{x} - x}{(T - t)^{1/2}}) \partial_\tau u(x, t) dx \} \cdot \{ \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi](\frac{\bar{x} - x}{(T - t)^{1/2}}) u(x, t) dx \} \frac{d\bar{x}}{d\bar{\bar{x}}}
$$

$$
= (-3)(T - t)^{5/2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy
$$

$$
- \frac{(T - t)^{5/2}}{2} \int_{\mathbb{R}^3} (\xi \cdot \nabla_\xi \psi^2(\xi)) |\mathcal{F}[w](\xi, \tau)|^2 d\xi
$$

$$
+ 2(T - t)^{5/2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi](y - z) \{ \partial_\tau w(z, \tau) + \frac{z}{2} \cdot \nabla_\tau w(z, \tau) + \frac{1}{2} w(z, \tau) \} dz \cdot \{ \mathcal{F}^{-1}[\psi] * w(y, \tau) \} dy
$$

So we have

(3.20)

$$
\frac{2}{d\tau} \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] * \{ \partial_\tau w + \frac{y}{2} \cdot \nabla_y w \} (y, \tau) \cdot \mathcal{F}^{-1}[\psi] * w(y, \tau) dy
$$

$$
\frac{d}{d\tau} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy - \frac{3}{2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^3} (\xi \cdot \nabla_\xi \psi^2(\xi)) |\mathcal{F}[w](\xi, \tau)|^2 d\xi.
$$

Note that $\varphi$ and $\chi$ satisfy (3.15), and we can use (3.20) to obtain (3.7) for $\varphi$ and $\chi$ again. Furthermore, since $1 - \varphi$ satisfies (3.15) and $\|\partial_t \nabla_x u(t)\|_{L^2(\mathbb{R}^3)}$ is bounded for $t < T$, we can prove the same equation as (3.20) for $(1 - \varphi)$ and $\nabla_y w$ instead of $\psi$ and $w$, which is another proof to obtain (4.4) from (4.1) in next section.
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4. $L^\infty$-estimate of $\Delta_0 w$

Applying the operator $\Delta_0$ (see (3.2)) to (2.7), and integrating over $\mathbb{R}^3$ we have

$$
\frac{1}{2} \int_{\mathbb{R}^3} \frac{\partial}{\partial \tau} |\Delta_0 \nabla w|^2 dy = - \int_{\mathbb{R}^3} |\nabla^2 \Delta_0 w|^2 dy - \int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy
$$

(4.1)

$$
- \sum_{j=1}^{3} \frac{1}{2} \int_{\mathbb{R}^3} \Delta_0 (y \cdot \nabla \partial_j w) \cdot \Delta_0 \partial_j w dy
$$

$$
- \sum_{j=1}^{3} \int_{\mathbb{R}^3} \Delta_0 ((\partial_j w \cdot \nabla) w) \cdot \Delta_0 \partial_j w + \Delta_0 ((w \cdot \nabla) \partial_j w) \cdot \Delta_0 \partial_j w dy.
$$

Since the support of $1 - \varphi$ is not compact, we can not do the same thing as in (3.6) for the 3rd term in the right side of (4.1). But with more patience, by using $\Delta_0 f = f - \Delta_{-1} f$, we find

$$
\int \Delta_0 ((y \cdot \nabla) \partial_j w) \cdot \Delta_0 \partial_j w dy
$$

(4.2)

$$
= \int ((y \cdot \nabla) \partial_j w) \cdot \partial_j w dy - \int ((y \cdot \nabla) \partial_j w) \cdot \Delta_{-1} \partial_j w dy
$$

$$
- \int \Delta_{-1} ((y \cdot \nabla) \partial_j w) \cdot \partial_j w dy + \int \Delta_{-1} ((y \cdot \nabla) \partial_j w) \cdot \Delta_{-1} (\partial_j w) dy.
$$

Recall from (2.5) that

$$
\int ((y \cdot \nabla) \partial_j w) \cdot \partial_j w dy \geq - \frac{3}{2} \int |\partial_j w|^2 dy.
$$

On the other hand, as in (3.6) we have

$$
\int \Delta_{-1} ((y \cdot \nabla) \partial_j w) \cdot \Delta_{-1} (\partial_j w) dy = \frac{1}{2} \int \xi \cdot \nabla \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi - \frac{3}{2} \int \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi.
$$

The remainder two terms in the right side of (4.2) is written as

$$
2 \int \varphi \partial_k (\xi_k \mathcal{F}[\partial_j w]) \cdot \mathcal{F}[\partial_j w] d\xi
$$

$$
= - \int 2(\xi \cdot \nabla \varphi)|\mathcal{F}[\partial_j w]|^2 + \varphi \cdot \nabla |\mathcal{F}[\partial_j w]|^2 d\xi
$$

$$
= - \int (\xi \cdot \nabla \varphi)|\mathcal{F}[\partial_j w]|^2 d\xi + 3 \int \varphi|\mathcal{F}[\partial_j w]|^2 d\xi.
$$

Then the right of (4.2) is larger than

$$
- \frac{3}{2} \int |\partial_j w|^2 dy - \frac{3}{2} \int \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi + \frac{1}{2} \int \xi \cdot \nabla \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi
$$

(4.3)

$$
- \int (\xi \cdot \nabla \varphi)|\mathcal{F}[\partial_j w]|^2 d\xi + 3 \int \varphi|\mathcal{F}[\partial_j w]|^2 d\xi
$$

$$
= \frac{1}{2} \int \xi \cdot \nabla (1 - \varphi(\xi))^2 |\mathcal{F}[\partial_j w]|^2 d\xi - \frac{3}{2} \int |\Delta_0 \partial_j w|^2 dy.
$$
Recall from (3.1) that
\[ \xi \cdot \nabla (1 - \varphi(\xi))^2 = |\xi| \frac{d}{d\xi} (1 - \varphi(\xi))^2 \geq 0. \]

Instead of the 3rd term in the right side of (4.1) by (4.2)-(4.3), we get
\[ \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy \leq - \int_{\mathbb{R}^3} |\nabla^2 \Delta_0 w|^2 dy - \frac{1}{4} \int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy \]
(4.4)
\[ - \sum_{j=1}^{3} \int_{\mathbb{R}^3} \Delta_0((\partial_j w \cdot \nabla) w) \cdot \Delta_0 \partial_j w + \Delta_0((w \cdot \nabla) \partial_j w) \cdot \Delta_0 \partial_j w dy. \]

Decompose the last integration of the right side of (4.4) by \( w = \Delta w + \Delta_0 w \) and note that
\[ |\int ((\Delta_0 w \cdot \nabla) \partial_j \Delta_0 w) \cdot \partial_j \Delta_0 w dy| \leq \|\nabla^2 \Delta_0 w\|_{L^2(\mathbb{R}^3)} (\int |\Delta_0 w|^2 |\nabla \Delta_0 w|^2 dy)^{1/2} \]
\[ \leq C \|\nabla^2 \Delta_0 w\|_{L^2(\mathbb{R}^3)}^{3/2} \|\nabla \Delta_0 w\|_{L^2(\mathbb{R}^3)}^{3/2}, \]
\[ |\int ((\Delta_{-1} w \cdot \nabla) \partial_j \Delta_0 w) \cdot \partial_j \Delta_0 w dy| \leq \|\Delta_{-1} w\|_{L^\infty(\mathbb{R}^3)} \|\nabla^2 \Delta_0 w\|_{L^2(\mathbb{R}^3)} \|\nabla \Delta_0 w\|_{L^2(\mathbb{R}^3)}, \]
\[ |\int ((\Delta_0 w \cdot \nabla) \partial_j \Delta_{-1} w) \cdot \partial_j \Delta_0 w dy| \leq C \|\Delta_{-1} w\|_{L^\infty(\mathbb{R}^3)} \|\Delta_0 w\|_{L^2(\mathbb{R}^3)} \|\nabla \Delta_0 w\|_{L^2(\mathbb{R}^3)}, \]
and
\[ |\int ((\Delta_{-1} w \cdot \nabla) \partial_j \Delta_{-1} w) \cdot \partial_j \Delta_0 w dy| \leq C (\int |\Delta_{-1} w|^4 dy)^{1/2} \|\nabla \Delta_0 w\|_{L^2(\mathbb{R}^3)} \]
as well as the same estimates for another term in the last integration of the right side of (4.4). In present section, we always assume assumption 1. Then by Proposition 3.2 we find
\[ \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\nabla \Delta_0 w(y, \tau)|^2 dy \leq - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla^2 \Delta_0 w|^2 dy - \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \Delta_0 w(y, \tau)|^2 dy \]
(4.5)
\[ + C \|\nabla \nabla \Delta_0 w(\tau)\|_{L^2(\mathbb{R}^3)} \{ C \delta - \|\nabla \nabla \Delta_0 w(\tau)\|_{L^2(\mathbb{R}^3)} + C \|\nabla \Delta_0 w(\tau)\|_{L^2(\mathbb{R}^3)}^5 \} \]
Note that (see Remark 4.4) there is \( \delta_1 > 0 \) such that if for some \( \tau_1 \geq 0 \)
\[ \|\nabla \Delta_0 w(\tau_1)\|_{L^2(\mathbb{R}^3)} \leq \delta_1 \]
(4.6)
then
\[ \|\nabla \Delta_0 w(\tau)\|_{L^2(\mathbb{R}^3)} \leq \delta_1, \quad \forall \tau \geq \tau_1. \]
From (3.14), (4.6) can be satisfied provided that (3.11) is satisfied. So we have

**Lemma 4.1.** Suppose (3.14) is satisfied. Then there is \( \delta_1 > 0 \) \((\delta_1 \downarrow 0 \text{ as } \delta \downarrow 0)\) and \( \tau_1 > 0 \) such that
\[ \|\nabla \Delta_0 w(\tau)\|_{L^2(\mathbb{R}^3)} \leq \delta_1, \quad \forall \tau \geq \tau_1. \]
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Decompose the last term in the right side of (2.8) by \( w = \Delta_{-1}w + \Delta_0w \), and note that
\[
|\int \partial_j \Delta_{-1}w_k \partial_j \Delta_{-1}w_l \partial_l \Delta_{-1}w_k dy| \leq \|\nabla \Delta_{-1}w\|_{L^\infty(\mathbb{R}^3)} \int |\nabla \Delta_{-1}w|^2 dy \\
\leq C\delta \int |\nabla \Delta_{-1}w|^2 dy,
\]
\[
|\int \partial_j \Delta_{-1}w_k \partial_j \Delta_{-1}w_l \partial_l \Delta_0w_k dy| \leq (\int |\nabla \Delta_{-1}w|^4 dy)^{1/2}(\int |\nabla \Delta_0w|^2 dy)^{1/2} \\
\leq C\delta (\int |\nabla \Delta_0w|^2 dy)^{1/2},
\]
and
\[
|\int \partial_j \Delta_{-1}w_k \partial_j \Delta_0w_l \partial_l \Delta_0w_k dy| \leq \|\nabla \Delta_{-1}w\|_{L^\infty(\mathbb{R}^3)} \int |\nabla \Delta_0w|^2 dy \\
\leq C\delta \int |\nabla \Delta_0w|^2 dy,
\]
as well as
\[
|\int \partial_j \Delta_0w_k \partial_j \Delta_0w_l \partial_l \Delta_0w_k dy| \leq C\|\nabla \Delta_0w\|^3_{L^2(\mathbb{R}^3)} \|\nabla^2 \Delta_0w\|^{3/2}_{L^2(\mathbb{R}^3)} \\
\leq \|\nabla^2 \Delta_0w\|^2_{L^2(\mathbb{R}^3)} + C\delta_1.
\]

Finally we find
\[
(4.7) \quad \frac{d}{d\tau} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy \leq -\int_{\mathbb{R}^3} |\nabla^2 w(y, \tau)|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy + C\delta_1.
\]

**Lemma 4.2.** Suppose (3.11) is satisfied. Then there is \( \delta_1 > 0 \) (\( \delta_1 \downarrow 0 \) as \( \delta \downarrow 0 \)) and \( \tau_1 > 0 \) such that for all \( \tau \geq \tau_1 \),
\[
(4.8) \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy \leq e^{-\frac{1}{2}(\tau-\tau_1)} \int_{\mathbb{R}^3} |\nabla w(y, \tau_1)|^2 dy + 2C\delta_1 (1 - e^{-\frac{1}{2}(\tau-\tau_1)}).
\]

Returning to (2.9), and noting that \( \text{div} \Delta w = 0 \) implies
\[
\int (\Delta w) \cdot (w \cdot \nabla) \Delta w dy = 0,
\]
the last term in the right side of (2.9) can be written as the sum of the following type
\[
\int |\nabla^2 w|^2 |\nabla w| dy.
\]
Applying the estimates
\[
(\int |\nabla w|^2 dy)^{1/2}(\int |\nabla^2 w|^4 dy)^{1/2} \\
\leq C(\int |\nabla w|^2 dy)^{1/2}(\int |\Delta w|^2 dy)^{1/4}(\int |\nabla \Delta w|^2 dy)^{3/4}
\]

Lemma 4.3. Suppose (3.11) is satisfied. Then there is \( \delta_1 > 0 \) (\( \delta_1 \downarrow 0 \) as \( \delta \downarrow 0 \)) and \( \tau_1 > 0 \) such that for all \( \tau \geq \tau_1 \),

\[
\int_{\mathbb{R}^3} |\Delta w(y, \tau)|^2 dy \leq e^{-(\frac{3}{2} - C\delta_1)(\tau - \tau_1)} \int_{\mathbb{R}^3} |\Delta w(y, \tau_1)|^2 dy,
\]

that is

\[
\int_{\mathbb{R}^3} |\Delta_x u(x, t)|^2 dx \leq \left( \frac{T - t_0}{T - t} \right)^{C\delta_1} \int_{\mathbb{R}^3} |\Delta_x u(x, t_0)|^2 dx.
\]

Remark 4.4. Suppose a nonnegative continuous function \( h(\tau) \) satisfies

\[
\frac{d}{d\tau} h(\tau) \leq F(h(\tau)) := C\delta - Bh(\tau) + h^5(\tau), \quad \forall \tau > 0,
\]

where \( C, B \) and \( \delta \) are positive constants. If \( \delta \) is small enough so that

\[
h_- := \frac{1}{2}(B - \sqrt{B^2 - 4C\delta}) \in (0, 1),
\]

and if \( h(0) < h_- \), then for all \( \tau > 0 \), \( F(h(\tau)) \leq C\delta - Bh(\tau) + h^2(\tau) \) and \( h(\tau) \in [0, h_-] \).

5. Frequency Decaying Estimates

In this section, \( M_j \) (\( j = 1, 2, 3 \)) will be used to denote universal constants. Utilizing the localization properties discovered in [2], we investigate the frequency decay rate of the vorticity \( \omega = \text{curl} \ u \). From (1.1) we have

\[
\partial_t \omega = \Delta \omega - (u \cdot \nabla) \omega + (\omega \cdot \nabla) u, \quad \text{in} \ \mathbb{R}^3 \times (0, T)
\]

\[
\omega(x, 0) = \omega_0, \quad \text{in} \ \mathbb{R}^3
\]

\[
u(x, t) = (K * \omega)(x, t) := \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \times \omega(y, t) dy.
\]

Since \( \omega \in L^2(0, T; L^2(\mathbb{R}^3)) \), there is \( t_0 \in (0, T) \) such that \( \omega(t_0) \in L^2(\mathbb{R}^3) \). The integral formulation of (5.1) with the initial data \( \omega(t_0) \) is written as follows

\[
\omega(t) = S(t - t_0)\omega(t_0) + A_{t_0}(\omega, u)(t)
\]

\[
u(t) = (K * \omega)(t)
\]

where

\[
A_{t_0}(\omega, u)(t) := \int_{t_0}^t S(t - s)((\omega \cdot \nabla) u - (u \cdot \nabla) \omega)(s) ds
\]
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and so on. We will use the standard iteration scheme \[26\]
\[
\omega^{(j+1)}(t) = S(t - t_0)\omega(t_0) + A_0(\omega^{(j)}, u^{(j)})(t),
\]
(5.3)
\[
\omega^{(0)}(t) = S(t - t_0)\omega(t_0),
\]
\[
u^{(j)}(t) = K * \omega^{(j)}(t), \quad j = 0, 1, 2, \ldots,
\]
to solve (5.2) in \((t_0, T^*)\).

Suppose \(F[\omega](t_0) \in L^2(\mathbb{R}^3)\). Let us note carefully
\[
\|e^{\sqrt{t-t_0}\xi} |F[A_0(\omega^{(j)}, u^{(j)})]\|_{C_t L^2_x} := \sup_{t \in (t_0, T^*)} \left( \int e^{2\sqrt{t-t_0}|\xi|} |F[A_0(\omega^{(j)}, u^{(j)})]|(\xi, t) |d\xi \right)^{1/2}
\]
\[
\leq \sup_{t \in (t_0, T^*)} \int_{t_0}^{t} \left( \int e^{2\sqrt{t-t_0}|\xi|} |F[\omega^{(j)}] \cdot \nabla u^{(j)} - u^{(j)} \cdot \nabla \omega^{(j)}| |(\xi, s)| d\xi \right)^{1/2} d\xi 
\]
\[
\leq e^2 \sup_{t \in (t_0, T^*)} \int_{t_0}^{t} \left( \int e^{-(t-s)|\xi|^2} |F[\omega^{(j)}] \cdot \nabla u^{(j)} - u^{(j)} \cdot \nabla \omega^{(j)}| |(\xi, s)| d\xi \right)^{1/2} d\xi 
\]
\[
\leq 2e^2 \rho^{3/4} (T^* - t_0)^{1/4} \| \sqrt{\tau-t_0} |F[\omega^{(j)}]|(\xi, t) \|_{C_t L^2_x}^2.
\]
Here the third line is derived from
\[
e^{-(t-s)|\xi|^2} e^{\sqrt{t-t_0}|\xi|} \leq e^2 e^{-(t-s)|\xi|^2/2} e^{\sqrt{s-t_0}|\xi-s|} e^{\sqrt{s-t_0}|\xi|}.
\]
The 4th line is derived from
\[
\left( \int e^{-(t-s)|\xi|^2} |\xi|^2 f^2(\xi, s) d\xi \right)^{1/2} \leq \left( \int e^{-3(t-s)|\xi|^2} |\xi|^6 d\xi \right)^{1/6} \left( \int f^3(\xi, s) d\xi \right)^{1/3},
\]
with \(f = \int e^{\sqrt{s-t_0}|\eta|} |F[u^{(j)}]|(\eta, s) e^{\sqrt{s-t_0}|\xi-\eta|} |F[\omega^{(j)}]|(\xi-\eta, s) d\eta\), and from the Hausdorff-Young inequality,
\[
\left( \int f^3(\xi, s) d\xi \right)^{1/3} \leq \left( \int |e^{\sqrt{s-t_0}|\eta|} |F[u^{(j)}]|(\eta, s) |^{6/5} d\eta \right)^{5/6} \left( \int |e^{\sqrt{s-t_0}|\eta|} |F[\omega^{(j)}]|(\eta, s) |^{2} d\eta \right)^{1/2}
\]
\[
\leq \|1 \|_{L^{\infty}} \left( \int |e^{\sqrt{s-t_0}|\eta|} |F[\nabla u^{(j)}]|(\eta, s) |^{2} d\eta \right)^{1/2} \left( \int |e^{\sqrt{s-t_0}|\eta|} |F[\omega^{(j)}]|(\eta, s) |^{2} d\eta \right)^{1/2}.
\]
In the 5th line
\[
\|e^{\sqrt{t-t_0}|\xi|} |F[\nabla u^{(j)}]|(\xi, t) \|_{C_t L^2_x} \leq 3 \|e^{\sqrt{t-t_0}|\xi|} |F[\omega^{(j)}]|(\xi, t) \|_{C_t L^2_x}
\]
is derived from
\[
\nabla u_i = (-1)^3 \sum_{k=1}^{3} \nabla (-\Delta)^{-1} \partial_k (\partial_k u_i - \partial_i u_k)
\]
and the \(L^\infty\)-boundedness of the Fourier transform \(\sqrt{-1} \xi_j/|\xi|\) of the Riesz transform \(R_j = \partial_j/\sqrt{-\Delta}\).
Observer additionally that
\[ \|e^{\sqrt{t-t_0}\xi}\mathcal{F}[\omega(0)](\xi,t)\|_{C_t L^2_\xi} = \sup_{t \in (t_0,T^*)} \left( \int e^{2\sqrt{t-t_0}\xi - 2(t-t_0)|\xi|^2}|\mathcal{F}[\omega](\xi,t)|^2 d\xi \right)^{1/2} \leq e^2 \|\omega(t_0)\|_{L^2}. \]

Utilizing the notations \( K_0 = M_1 \|\omega(t_0)\|_{L^2} \) and
\[ K_j = \|e^{\sqrt{t-t_0}\xi}\mathcal{F}[\omega^{(j)}](\xi,t)\|_{C_t L^2_\xi}, \quad \forall j \geq 1, \]
we find that, for \( j = 0, 1, 2, \ldots \),
\[ K_{j+1} \leq K_0 + M_2 (T^* - t_0)^{1/4} K_j^2. \]

Claim 1. Suppose
\[ (5.4) \quad 4K_0 M_2 (T^* - t_0)^{1/4} < 1. \]
Then \( \{K_j\}_j \) is increasing and
\[ K_j < K_\infty := \frac{1 - \sqrt{1 - 4K_0 M_2 (T^* - t_0)^{1/4}}}{2M_2 (T^* - t_0)^{1/4}}. \]

Claim 2. For \( j = 1, 2, \ldots \),
\[ \|e^{\sqrt{t-t_0}\xi}\mathcal{F}[\omega^{(j)}](\xi,t) - e^{\sqrt{t-t_0}\xi}\mathcal{F}[\omega^{(j-1)}](\xi,t)\|_{C_t L^2_\xi} \leq \frac{1}{2} \|e^{\sqrt{t-t_0}\xi}\mathcal{F}[\omega^{(j-1)}](\xi,t) - e^{\sqrt{t-t_0}\xi}\mathcal{F}[\omega^{(j-1)}](\xi,t)\|_{C_t L^2_\xi} \]
provided
\[ 2M_2 (T^* - t_0)^{1/4} K_\infty \leq \frac{1}{4}. \]

Since the last inequality is valid provided
\[ (5.5) \quad 4K_0 M_2 (T^* - t_0)^{1/4} \leq \frac{7}{16}, \]
we proved

**Proposition 5.1.** Suppose \( \omega(t_0) \in L^2(\mathbb{R}^3) \). Then for \( T^* \) satisfying (5.5), there is a unique solution \( \omega(t) \) of (5.4) such that
\[ (5.6) \quad \sup_{t \in (t_0,T^*)} \left( \int e^{2\sqrt{t-t_0}\xi} |\mathcal{F}[\omega](\xi,t)|^2 d\xi \right)^{1/2} \leq \frac{1 - \sqrt{1 - 4K_0 M_2 (T^* - t_0)^{1/4}}}{2M_2 (T^* - t_0)^{1/4}}. \]

Recall from the energy inequality for the Leray-Hopf solutions
\[ \|u(T)\|_{L^2}^2 + \frac{1}{2} \int_t^T \|\nabla_x u(h)\|_{L^2}^2 dh \leq \|u(t)\|_{L^2}^2. \]

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that there is \( h_j \uparrow T \) such that
\[
(T - h_j)^{\frac{1}{2}} \| \nabla_x u(h_j) \|_{L^2} \to 0, \quad \text{as} \quad j \to \infty.
\]

So we have

**Corollary 5.2.** There is \( h_j \uparrow T \) such that \( \| \omega(h_j) \|_{L^2} \leq \frac{M_3}{(T - h_j)^{1/2}} \) and (5.5) are satisfied. Moreover at \( \bar{t} = h_j + (T - h_j)^2 \),
\[
\| \mathcal{F}[\omega](\bar{t}) \|_{L^2(\mu \geq 1/(T - \bar{t})^{1+\frac{\beta}{2}})} \leq \frac{1}{2M_2} \exp\left[\frac{-1}{2(T - \bar{t})^{1+\frac{\beta}{2}}}\right]
\]
for all \( \epsilon > 0 \). Let \( t := T - (T - \bar{t})^{\frac{1+\beta}{2}} \). Equivalently, at \( \bar{t} = T - (T - t)^{\frac{1+\beta}{2}} \),
\[
\| \mathcal{F}[\omega](\bar{t}) \|_{L^2(\mu \geq (T - t)^{\frac{\beta}{2}})} \leq \frac{1}{2M_2} (T - t)^{\frac{1}{2-\beta}} \exp\left[\frac{-1}{2(T - t)^{1+\frac{\beta}{2}}}\right].
\]

6. **TIME-SPACE DECAYING ESTIMATES**

In this section, we estimate the time decaying of the \( L^2 \)-norm of vorticity in the region

\[(6.1) \quad |u(x, t)| \leq \left(\frac{a}{T - t}\right)^{1/2}.\]

First we assume \( Q \equiv \Omega \times (t_0, T) \subset \mathbb{R}^3 \times (0, T) \) and assume (6.1) in \( Q \). Let us consider the local vorticity equations

\[\partial_\mu \omega = \Delta \omega - \text{div}(u \otimes \omega - \omega \otimes u), \quad \text{in} \ Q\]
\[u(x, t) = (K * \omega)(x, t),\]

and take a cut-off function
\[
\psi \in C^\infty(Q), \ \text{spt}(\psi) \subset Q, \ \psi \equiv 1 \ \text{in} \ Q_{\frac{1}{n}} := \Omega_{\frac{1}{n}} \times (t_0 + \frac{1}{n}, T),
\]
\[
|\partial_\psi \psi| \leq 2n, \ |\nabla \psi| \leq 2n, \ |\Delta \psi| \leq 4n^2.
\]

where for \( n = 1, 2, 3, \ldots \)
\[\Omega_{\frac{1}{n}} := \{x \in \Omega, \ \text{dist}(x, \mathbb{R}^3 \setminus \Omega) > \frac{1}{n}\}.
\]

Then \( \omega_\psi(x, t) := \psi(x, t)\omega(x, t) \) satisfies

\[\begin{align*}
\partial_t \omega_\psi &= \Delta \omega_\psi - \text{div}(u \otimes \omega_\psi - \omega_\psi \otimes u) + f, \quad \text{in} \ \mathbb{R}^3 \times (t_0, T) \\
\omega_\psi(x, t_0) &= 0, \quad \text{on} \ \mathbb{R}^3
\end{align*}\]

where
\[f = (\partial_\psi + \Delta \psi)\omega - 2\partial_\psi ((\partial_\psi \psi)\omega) + (u \cdot \nabla \psi) \omega - (\omega \cdot \nabla \psi) u.\]
Multiple (6.3) by $\omega_\psi$ and integrate over $\mathbb{R}^3$

$$\frac{d}{dt} \|\omega_\psi(t)\|_{L^2}^2 + \|
abla \omega_\psi(t)\|_{L^2}^2 \leq \frac{C_1}{T-t} \|\omega_\psi(t)\|_{L^2}^2 + C_2 n^2 \|\omega(t)\|_{L^2}^2$$

where $C_j (j = 1, 2)$ are universal positive constants. Now multiply by $(T-t)^{C_1a}$:

$$\frac{d}{dt} \left((T-t)^{C_1a}\|\omega_\psi(t)\|_{L^2}^2\right) + (T-t)^{C_1a} \|
abla \omega_\psi(t)\|_{L^2}^2 \leq C_2 n^2 (T-t)^{C_1a} \|\omega(t)\|_{L^2}^2,$$

and integrate from $t_0$ to $t < T$:

$$\|\omega_\psi(t)\|_{L^2}^2 + \int_{t_0}^t \left(\frac{T-h}{T-t}\right)^{C_1a} \|
abla \omega_\psi(h)\|_{L^2}^2 dh \leq C_2 n^2 \int_{t_0}^t \left(\frac{T-h}{T-t}\right)^{C_1a} \|\omega(h)\|_{L^2}^2 dh \leq C_2 n^2 \|u_0\|_{L^2}^2 \left(\frac{T-t_0}{T-t}\right)^{C_1a}.$$

Similarly, we can take a cut-off function

$$\psi \in C^\infty(Q_\frac{1}{n}), \text{spt}(\psi) \subset Q_\frac{1}{n}, \psi \equiv 1 \text{ in } Q_\frac{1}{n} := \Omega_{-\frac{1}{n}} \times (t_0 + \frac{2}{n}, T),$$

$$|\partial_t \psi| \leq 2n, \quad |\nabla \psi| \leq 2n, \quad |\Delta \psi| \leq 4n^2$$

where for $n = 2, 3, ...$

$$\Omega_{-\frac{1}{n}} := \{x \in \Omega, \text{ dist}(x, \mathbb{R}^3 \setminus \Omega) > \frac{2}{n}\},$$

and to assert for $t \in [t_0 + \frac{2}{n}, T)$

$$\|\omega(t)\|_{L^2(\Omega_{-\frac{1}{n}})}^2 + \int_{t_0 + \frac{2}{n}}^t \left(\frac{T-h}{T-t}\right)^{C_1a} \|
abla \omega(h)\|_{L^2(\Omega_{-\frac{1}{n}})}^2 dh \leq C_2 n^2 \int_{t_0 + \frac{2}{n}}^t \left(\frac{T-h}{T-t}\right)^{C_1a} \|\omega(h)\|_{L^2(\Omega_{-\frac{1}{n}})}^2 dh \leq (C_2 n^2)^2 \|u_0\|_{L^2}^2 \left(\frac{T-t_0}{T-t}\right)^{C_1a} (t-t_0 + \frac{1}{n})$$

where $\|\omega(h)\|_{L^2(\Omega_{-\frac{1}{n}})}^2$ is estimated by (6.4). Iterating the argument above, thereby deduce for $m = 1, 2, 3, ..., n$, for $t \in [t_0 + \frac{m}{n}, T)$

$$\|\omega(t)\|_{L^2(\Omega_{-\frac{m}{n}})}^2 + \int_{t_0 + \frac{m}{n}}^t \left(\frac{T-h}{T-t}\right)^{C_1a} \|
abla \omega(h)\|_{L^2(\Omega_{-\frac{m}{n}})}^2 dh \leq (C_2 n^2)^m \|u_0\|_{L^2}^2 \left(\frac{T-t_0}{T-t}\right)^{C_1a} (t-t_0)^{m-1}.$$
Regularity of solutions to the Navier-Stokes equations

Next we assume \( \text{(6.1)} \) in \( Q := \Omega \times (t_0, t) \) \( (t < T) \) and apply \( \text{(6.6)} \) to the dilation scaling \( \omega_{\lambda_1} (z, s) = \lambda_1^2 \omega (\lambda_1 z, \lambda_1^2 s) \) with

\[
(6.7) \quad \frac{t - t_0}{\lambda_1^2} > 1.
\]

We find

**Proposition 6.1.** Assume \( \text{(6.1)} \) in \( Q := \Omega \times (t_0, t) \) \( (t < T) \) and \( \lambda_1 < \sqrt{t - t_0}. \) Then for \( h \in [t_0 + \lambda_1^2, t], \)

\[
(6.8) \quad \| \omega (h) \|^2_{L^2(\Omega - \lambda_1)} \leq (C_2 n^2)^n \| u_0 \|^2_{L^2} \left( \frac{T - t_0}{T - h} \right)^{C_1 n} (h - t_0)^{-n - 1} \lambda_1^{-2n}
\]

where \( n = 1, 2, 3, \ldots, C_1 \) and \( C_2 \) are universal constants and

\[
\Omega_{-\lambda_1} := \{ x \in \Omega, \ dist(x, \mathbb{R}^3 \setminus \Omega) > \lambda_1 \}.
\]

On the other hand, by taking a cut-off function \( \psi \) only cutting off in space, we can prove

**Proposition 6.2.** Assume \( \text{(6.7)} \) in \( Q := \Omega \times (t_0, t) \) \( (t < T) \). Then for any \( \lambda_1 > 0, \) for \( h \in [t_0, t], \) we have

\[
(6.9) \quad \| \omega (h) \|^2_{L^2(\Omega - \lambda_1)} \leq \| \omega (t_0) \|^2_{L^2(\Omega - \lambda_1)} + (C_2 n^2)^n \| \mu_0 \|^2_{L^2} \left( \frac{T - t_0}{T - h} \right)^{C_1 n} (h - t_0)^{-n - 1} \lambda_1^{-2n}
\]

where \( n = 1, 2, 3, \ldots, C_1 \) and \( C_2 \) are universal constants and

\[
\Omega_{-\lambda_1} := \{ x \in \Omega, \ dist(x, \mathbb{R}^3 \setminus \Omega) > \lambda_1 \}.
\]

7. FREQUENCY OVERLAPPING AND MEASUREMENT OF MOVING BOUNDARY OF SINGULAR SET

Let \( \varphi \in C_0^\infty (\mathbb{R}^3, [0, 1]) \) be a radial symmetrical function satisfying

\[
\varphi (\xi) = 1 \quad \forall |\xi| \leq 1, \quad \varphi (\xi) = 0 \quad \forall |\xi| \geq 2, \quad \xi \cdot \nabla \varphi (\xi) \leq 0 \quad \forall \xi.
\]

For \( \beta \in (0, \frac{1}{2}), \) let \( \rho_\beta (t) := (T - t)^\beta, \lambda_t := (T - t)^{\frac{1}{2}} \) and

\[
f^{\mu_\beta}_t := \mathcal{F}^{-1}[\varphi (\frac{\lambda_t}{\rho_\beta (t)} |\eta|) f(\eta)], \quad f^{1 - \mu_\beta}_t := \mathcal{F}^{-1}[(1 - \varphi (\frac{\lambda_t}{\rho_\beta (t)} |\eta|)) f(\eta)].
\]

We now refine the operator \( \mu_t^\beta \) and \( 1 - \mu_t^\beta. \) Let \( \chi \in C_0^\infty (B_{4/3}(0)) \) and \( \phi \in C_0^\infty (B_{8/3}(0) \setminus B_{3/4}(0)) \) be the Littlewood-Paley dyadic decomposition that satisfy \( \text{(6)} \):

\[
\chi (\xi) + \sum_{q \geq 0} \phi (2^{-q} \xi) = 1, \quad \frac{1}{3} \leq \chi^2 (\xi) + \sum_{q \geq 0} \phi^2 (2^{-q} \xi) \leq 1, \quad \forall \xi \in \mathbb{R}^3.
\]
Define
\[ \Delta_{-1}^\phi u = F^{-1}[\chi(\xi)F[u](\xi)], \quad \Delta_q^\phi u = F^{-1}[\phi(2^{-q}\xi)F[u](\xi)], \quad \forall q \geq 0, \]
and
\[ S_j u = \sum_{-1 \leq k \leq j-1} \Delta^\phi_k u, \quad \Delta_j^\phi(u) = S_{j+1}(u) - S_j(u). \]

For the product \( uv \) of \( u \) and \( v \), we use Bony’s paraproduct to decompose it as the sum
\[ uv = T_u v + T_v u + R(u, v) \]
of the paraproducts
\[ T_u v := \sum_{j \geq 1} S_{j-1} u \Delta_j^\phi v, \quad T_v u := \sum_{j \geq 1} \Delta_j^\phi u S_{j-1} v, \]
and the remainder
\[ R(u, v) := \sum_{j \geq -1} \sum_{j-1 \leq k \leq j+1} \Delta^\phi_k u \Delta^\phi_{j-k} v, \]
where
\[ S_j v = \sum_{-1 \leq k \leq j-1} \Delta^\phi_k v, \quad S_0 v = \Delta_{-1}^\phi v, \quad S_{-1} v = 0. \]

By direct calculation we discover, for all \( q \geq -1 \),
\[ \Delta^\phi_q(T_v u) = \Delta^\phi_q \left( \sum_{q-2 \leq j \leq q+4} \Delta^\phi_j u S_{j-1} v \right). \]

\[ \Delta^\phi_q(R(u, v)) = \sum_{j \geq q-3} \sum_{k=j-1}^{j+1} \Delta^\phi_k u \Delta^\phi_{j-k} v. \]

Let
\[ q^\beta_i \in \mathbb{N} \]
with
\[ \log_2 \frac{\rho^\beta(t)}{\lambda_t} - \frac{1}{2} < q^\beta_i \leq \log_2 \frac{\rho^\beta(t)}{\lambda_t} + \frac{1}{2}. \]

It is easy to check
\[ u^{\mu^\beta_i} = \sum_{j \leq q^\beta_i} \Delta^\phi_j(u^{\mu^\beta_i}), \quad u^{1-\mu^\beta_i} = \sum_{j \geq q^\beta_i-2} \Delta^\phi_j(u^{1-\mu^\beta_i}). \]

Applying the operator \((1 - \mu^\beta_i)\) to the equations (1.1) for \( u(x, h) \) we find (7.1)
\[ \frac{1}{2} \frac{d}{dh} \|u^{1-\mu^\beta_i}(h)\|_{L^2}^2 = (-) \|\nabla_x u^{1-\mu^\beta_i}(h)\|_{L^2}^2 - < u_j(h) \partial_j u(h), u^{1-\mu^\beta_i} > \]
\[ = (-) \|\nabla_x u^{1-\mu^\beta_i}(h)\|_{L^2}^2 < u^{1-\mu^\beta_i}(h) \partial_j u^{1-\mu^\beta_i} > \]
\[ - < u_j^{\mu^\beta_i \cap (1-\mu^\beta_i)} \partial_j u^{1-(1-\mu^\beta_i)^2}(h), u^{(1-\mu^\beta_i)^2}(h) > - < u_j^{\mu^\beta_i} \partial_j u^{1-(1-\mu^\beta_i)^2 \cap (1-\mu^\beta_i)}(h), u^{(1-\mu^\beta_i)^2}(h) > \]
Regularity of solutions to the Navier-Stokes equations

where

\[ f^{(1-\mu^\beta)^2} := \left( f^{1-\mu^\beta} \right)^{1-\mu^\beta}, \quad f^{1-(1-\mu^\beta)^2} = \mathcal{F}^{-1} \left[ \left( 1 - \varphi \left( \frac{\lambda_t}{\rho_\beta(t)} | \eta | \right) \right)^2 f(\eta) \right], \]

\[ f^{\mu^\beta \cap (1-\mu^\beta)} := \sum_{q=\mu^\beta-4}^{\mu^\beta+1} \Delta^\phi_q(f). \]

Notice that

\[ \| f^{(1-\mu^\beta)^2} \|_{L^2} = \| \mathcal{F}^{-1} \left[ \left( 1 - \varphi \left( \frac{\lambda_t}{\rho_\beta(t)} | \eta | \right) \right)^2 \mathcal{F}[f](\eta) \right] \|_{L^2} \leq \| f^{1-\mu^\beta} \|_{L^2} \]

and

\[ \| f^{1-(1-\mu^\beta)^2} \|_{L^2} = \| \varphi \left( \frac{\lambda_t}{\rho_\beta(t)} | \eta | \right) \left( 2 - \varphi \left( \frac{\lambda_t}{\rho_\beta(t)} | \eta | \right) \right) \mathcal{F}[f](\eta) \|_{L^2} \leq 2 \| f^{\mu^\beta} \|_{L^2}. \]

Finally we discover

\[
\frac{d}{dh} \| u^{1-\mu^\beta} (h) \|_{L^2}^2 = (-2) \| \nabla u^{1-\mu^\beta} (h) \|_{L^2}^2 - 2 < u^{(1-\mu^\beta)^2}, u_j \partial_j u^{1-(1-\mu^\beta)^2} > \\
\leq -2 \| \nabla u^{1-\mu^\beta} (h) \|_{L^2}^2 + c_{III} \| \nabla \times u^{\mu^\beta} (h) \|_{L^\infty} \| u^{1-\mu^\beta} (h) \|_{L^2} (\| u^{1-\mu^\beta} (h) \|_{L^2} + \| u^{\mu^\beta \cap (1-\mu^\beta)} (h) \|_{L^2}) \\
+ c_{II} \| u^{\mu^\beta} (h) \|_{L^\infty} \| u^{1-\mu^\beta} (h) \|_{L^2} \| \nabla \times u^{\mu^\beta \cap (1-\mu^\beta)} (h) \|_{L^2}
\]

where at first glance, we have the estimate

\[ \| \nabla \times u^{\mu^\beta} (h) \|_{L^\infty} \leq C \int_{\{ \eta \leq (T-t)^{1-\frac{1}{2}} \}} | \eta | | \mathcal{F}[u](\eta, h) | d\eta \leq C \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^{\frac{1}{2}} \| u^{\mu^\beta} \|_{L^2}. \]

Remark that the terms including \( u^{\mu^\beta \cap (1-\mu^\beta)} \) reflect the frequency overlapping phenomena in Navier-Stokes equations induced by nonlinearity.

Furthermore apply the Leray projection operator \( \mathcal{P} \), gradient operator \( \nabla^m \) and the operator \( (1-\mu^\beta) \) to the Navier-Stokes equations (1.1), we have

(7.2)

\[ \partial_t \nabla^m u^{1-\mu^\beta} = \mathcal{P} \Delta \nabla^m u^{1-\mu^\beta} - \mathcal{P} \nabla^m (\partial_j (u_j u))^{1-\mu^\beta}. \]

Multiplying it with \( m = 2 \) by \( \nabla^2 u^{1-\mu^\beta} \), and integrating over \( \mathbb{R}^3 \), we have

\[
\frac{d}{dh} \| \nabla^2 u^{1-\mu^\beta} (h) \|_{L^2}^2 = (-2) \| \nabla^3 u^{1-\mu^\beta} (h) \|_{L^2}^2 \\
-2 < \nabla^2 u^{(1-\mu^\beta)^2}, \nabla^2 u_j \partial_j u + 2 \nabla u_j \cdot \partial_j \nabla u + u_j \partial_j \nabla^2 u^{1-(1-\mu^\beta)^2} > (h).
\]

First for the nonlinear part we have the estimates

\[
\begin{align*}
| < \nabla^2 u_j^{(1-\mu^\beta)^2}, \nabla^2 u_j^{1-\mu^\beta} \partial_j u^{1-\mu^\beta} | & \leq \| \nabla^2 u^{1-\mu^\beta} \|_{L^2} \| \nabla^2 u^{1-\mu^\beta} \|_{L^3} \| \nabla u^{1-\mu^\beta} \|_{L^6} \\
& \leq C \| \nabla^2 u^{1-\mu^\beta} \|_{L^2} \| \nabla^3 u^{1-\mu^\beta} \|_{L^2} \leq C \| \nabla^2 u^{1-\mu^\beta} \|_{L^2}^2 + C \| \nabla^2 u^{1-\mu^\beta} \|_{L^2}^{\frac{10}{3}},
\end{align*}
\]

for any \( \epsilon > 0 \), and

\[
| < \nabla^2 u_j^{(1-\mu^\beta)^2}, u_j^{\mu^\beta} \partial_j \nabla^2 u^{\mu^\beta} | \leq C \| \nabla^2 u^{1-\mu^\beta} \|_{L^2} + C \| u^{\mu^\beta} \|_{L^2} \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^{\frac{3}{2} + \frac{4}{7}} \| \nabla u^{\mu^\beta} \|_{L^2},
\]

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as well as
\[
| < \nabla^2 u_j^{(1-\mu_j^\beta)^2}, \nabla^2 u_j^{1-\mu_j^\beta} \partial_j u_j^{\mu_j^\beta} + \nabla u_j^{1-\mu_j^\beta} \partial_j \nabla u_j^{\mu_j^\beta} + u_j^{1-\mu_j^\beta} \partial_j \nabla^2 u_j^{\mu_j^\beta} | \leq \|
abla^2 u_j^{1-\mu_j^\beta} \|_{L^2}^2 \|
abla u_j^{\mu_j^\beta} \|_{L^2} + \|
abla^2 u_j^{1-\mu_j^\beta} \|_{L^2} \|
abla u_j^{1-\mu_j^\beta} \|_{L^6} \|
abla^2 u_j^{\mu_j^\beta} \|_{L^3} \leq C \|
abla^2 u_j^{1-\mu_j^\beta} \|_{L^2}^{10} \|
abla u_j^{\mu_j^\beta} \|_{L^2} \|
abla u_j^{\mu_j^\beta} \|_{L^2}^2.
\]

Take \( \epsilon = \frac{1}{8} \) and note also
\[
\|
abla^2 u_j^{1-\mu_j^\beta} \|_{L^2}^2 \leq \|
abla^3 u_j^{1-\mu_j^\beta} \|_{L^2} \|
abla u_j^{1-\mu_j^\beta} \|_{L^2} \leq \|
abla^3 u_j^{1-\mu_j^\beta} \|_{L^2} \|
abla u_j^{1-\mu_j^\beta} \|_{L^2}^{1/2} \|
abla^2 u_j^{1-\mu_j^\beta} \|_{L^2}^{1/2} \|
abla u_j^{1-\mu_j^\beta} \|_{L^2}^{1/2},
\]
and
\[
\|
abla^2 u_j^{1-\mu_j^\beta} \|_{L^2}^{3/2} \leq \| [Re u_j^{1-\mu_j^\beta}(h)]^{3/2} \| \nabla u_j^{1-\mu_j^\beta} \|_{L^2},
\]
as well as
\[
\frac{\|
abla^2 u_j^{1-\mu_j^\beta} \|_{L^2}^{3/2}}{\|
abla u_j^{1-\mu_j^\beta} \|_{L^2}^{1/2}} \leq \|
abla^3 u_j^{1-\mu_j^\beta} \|_{L^2}.
\]

Consequently
\[
(7.3)
\frac{d}{dh} \|
abla^2 u_j^{1-\mu_j^\beta}(h) \|_{L^2}^2 \leq -\frac{3}{4} \|
abla^3 u_j^{1-\mu_j^\beta}(h) \|_{L^2}^2 - \frac{\|
abla^2 u_j^{1-\mu_j^\beta}(h) \|_{L^2}^{3/2}}{\|
abla u_j^{1-\mu_j^\beta}(h) \|_{L^2}} + c_1 \|
abla^2 u_j^{1-\mu_j^\beta}(h) \|_{L^2}^{10} + c_2 \left( \frac{\rho(t)}{\lambda_t} \right)^{5+10} \|
abla u_j^{\mu_j^\beta}(h) \|_{L^2}^2,
\]
and
\[
(7.4)
\frac{d}{dh} \|
abla^2 u_j^{1-\mu_j^\beta}(h) \|_{L^2}^2 \leq -\left( \frac{3}{4} - c_1 [Re u_j^{1-\mu_j^\beta}(h)]^{3/2} \right) \|
abla^3 u_j^{1-\mu_j^\beta}(h) \|_{L^2}^2 - \frac{\|
abla^2 u_j^{1-\mu_j^\beta}(h) \|_{L^2}^{3/2}}{\|
abla u_j^{1-\mu_j^\beta}(h) \|_{L^2}} + \tilde{c}_2 \left( \frac{\rho(t)}{\lambda_t} \right)^{3+10} \|
abla u_j^{\mu_j^\beta}(h) \|_{L^2}^2,
\]
where \( c_1 \) is a universal constant and
\[
\tilde{c}_2 = C \max_h \left( \|
abla u_j^{\mu_j^\beta}(h) \|_{L^2}^{6} + \|
abla u_j^{\mu_j^\beta}(h) \|_{L^2} \left( \frac{\rho(t)}{\lambda_t} \right)^{4-\delta} \right),
\]
\[
c_2 = C \max_h \left( \|
abla u_j^{\mu_j^\beta}(h) \|_{L^2}^{20} + \|
abla u_j^{\mu_j^\beta}(h) \|_{L^2} \left( \frac{\rho(t)}{\lambda_t} \right)^{4-\delta} \right).
\]
Regularity of solutions to the Navier-Stokes equations

Similarly we can deduce
\begin{equation}
\frac{d}{dh} \| \nabla_x^3 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 \\
\leq - \left( 1 - Re^{1-\mu_i^\beta} (h) - [Re^{1-\mu_i^\beta} (h)]^{\frac{1}{2}} - [Re^{1-\mu_i^\beta} (h)]^{\frac{3}{2}} - [Re^{1-\mu_i^\beta} (h)]^{2} \right) \| \nabla_x^4 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 \\
+ C \left( \frac{\rho_t(t)}{\lambda_t} \right)^7 \| u^{\mu_i^\beta} (h) \|_{L^2}^2 \| \nabla u^{\mu_i^\beta} (h) \|_{L^2}^2 \\
+ C \left( \| \nabla u^{\mu_i^\beta} (h) \|_{L^\infty}^7 + \| \nabla^3 u^{\mu_i^\beta} (h) \|_{L^6}^2 + \| \nabla^2 u^{\mu_i^\beta} (h) \|_{L^\infty}^7 + \| \nabla^3 u^{\mu_i^\beta} (h) \|_{L^3}^2 \right) \\
\leq - \left( 1 - Re^{1-\mu_i^\beta} (h) - [Re^{1-\mu_i^\beta} (h)]^{\frac{1}{2}} - [Re^{1-\mu_i^\beta} (h)]^{\frac{3}{2}} - [Re^{1-\mu_i^\beta} (h)]^{2} \right) \| \nabla_x^4 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 \\
+ c_6 \left( \frac{\rho_t(t)}{\lambda_t} \right)^7 \| u^{\mu_i^\beta} (h) \|_{L^2}^2 \| \nabla u^{\mu_i^\beta} (h) \|_{L^2}^2 \\
and
\end{equation}
\begin{equation}
\frac{d}{dh} \| \nabla_x^4 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 \\
\leq - \left( 1 - [Re^{1-\mu_i^\beta} (h)]^{\frac{7}{4}} - [Re^{1-\mu_i^\beta} (h)]^{\frac{3}{2}} - [Re^{1-\mu_i^\beta} (h)]^{2} \right) \| \nabla_x^5 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 \\
+ c_7 \left( \frac{\rho_t(t)}{\lambda_t} \right)^{\frac{23}{4}} \| u^{\mu_i^\beta} (h) \|_{L^2}^2 \| \nabla u^{\mu_i^\beta} (h) \|_{L^2}^2
\end{equation}

where $c_7$ is a universal constant provided $\alpha_1 > 4\beta$.

**Lemma 7.1.** Suppose for $\bar{t} < t < T$ there is $t_1 \in (\bar{t}, T)$ such that
\[ \| u^{1-\mu_i^\beta} (h) \|_{L^2} < B := \frac{1}{(2c_1)^{\frac{a}{2}}(2c_2)^{\frac{a}{2}} \left( \frac{\lambda_t}{\rho_t(t)} \right)^{\frac{a+1}{4}}}, \quad \bar{t} \leq \forall h \leq t_1 \]
and suppose
\[ \| \nabla^2 u^{1-\mu_i^\beta} (\bar{t}) \|_{L^2}^2 \leq \frac{1}{(2c_1 B)^6}. \]
Then we have $\bar{t} \leq \forall h \leq t_1$
\[ \| \nabla^2 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 \leq \frac{1}{(2c_1 B)^6}, \quad Re^{1-\mu_i^\beta} (h) := \| u^{1-\mu_i^\beta} (h) \|_{L^2}^\frac{1}{2} \| \nabla u^{1-\mu_i^\beta} (h) \|_{L^2}^\frac{1}{2} \leq \frac{1}{(2c_1 B)^{\frac{3}{4}}}. \]
Moreover, for $h \in (\bar{t}, t_1)$
\[ \| \nabla_x^2 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 + \frac{1}{4} \int_{\bar{t}}^h \| \nabla_x^3 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 dh \leq \| \nabla_x^2 u^{1-\mu_i^\beta} (\bar{t}) \|_{L^2}^2 + c_2 \left( \frac{\rho_t(t)}{\lambda_t} \right)^{3 + \frac{18}{4}} ; \]
\[ \| \nabla_x^3 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 + \frac{1}{4} \int_{\bar{t}}^h \| \nabla_x^4 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 dh \leq \| \nabla_x^3 u^{1-\mu_i^\beta} (\bar{t}) \|_{L^2}^2 + c_6 \| u(\bar{t}) \|_{L^2}^4 \left( \frac{\rho_t(t)}{\lambda_t} \right)^7 ; \]
\[ \| \nabla_x^4 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 + \frac{1}{4} \int_{\bar{t}}^h \| \nabla_x^5 u^{1-\mu_i^\beta} (h) \|_{L^2}^2 dh \leq \| \nabla_x^4 u^{1-\mu_i^\beta} (\bar{t}) \|_{L^2}^2 + c_7 \| u(\bar{t}) \|_{L^2}^9 \left( \frac{\rho_t(t)}{\lambda_t} \right)^\frac{23}{4} . \]
Jian Zhai

Proof. The solution to the super-equation

\[
\begin{aligned}
\frac{d}{dh}Z(h) &= -\frac{Z^{3/2}(h)}{B} + c_1Z^{5/3}(h) + c_2 \left( \frac{\rho(t)}{\lambda_t} \right)^{5+\frac{10}{\beta_t}}, \quad \forall h \in (\bar{t}, t_1] \\
Z(\bar{t}) &= \|\nabla^2 u^{1-\mu_1^\beta}(\bar{t})\|_{L^2}^2
\end{aligned}
\]

is less than \( Z_0 \) because the right side at \( Z = Z_0 \) is minus and

\[
\|\nabla^2 u^{1-\mu_1^\beta}(\bar{t})\|_{L^2}^2 < Z_0.
\]

Utilizing (7.3) we have \( \|\nabla^2 u^{1-\mu_1^\beta}(h)\|_{L^2}^2 \leq Z_0 \). On the other hand,

\[
[Re^{1-\mu_1^\beta}(h)]^8 = \frac{\|\nabla u^{1-\mu_1^\beta}(h)\|_{L^2}^4 \|u^{1-\mu_1^\beta}(h)\|_{L^2}^6}{\|u^{1-\mu_1^\beta}(h)\|_{L^2}^2} \leq \|\nabla^2 u^{1-\mu_1^\beta}(h)\|_{L^2}^2 \|u^{1-\mu_1^\beta}(h)\|_{L^2}^6
\]

\[
\leq \frac{\|u^{1-\mu_1^\beta}(h)\|_{L^2}^6}{(2c_1 B)^6} \leq \frac{1}{(2c_1)^6}.
\]

Furthermore, utilizing (7.4) for \( h \in (\bar{t}, t_1) \)

\[
\|\nabla^2 u^{1-\mu_1^\beta}(h)\|_{L^2}^2 + \frac{1}{4} \int_{\bar{t}}^h \|\nabla^2 u^{1-\mu_1^\beta}(\bar{h})\|_{L^2}^2 d\bar{h}
\]

\[
\leq \|\nabla^2 u^{1-\mu_1^\beta}(\bar{t})\|_{L^2}^2 + c_2 \left( \frac{\rho(t)}{\lambda_t} \right)^{3+\frac{10}{\beta_t}} \int_{\bar{t}}^h \|\nabla u^{\mu_1^\beta}(\bar{h})\|_{L^2}^2 d\bar{h}
\]

\[
\leq \|\nabla^2 u^{1-\mu_1^\beta}(\bar{t})\|_{L^2}^2 + c_2 \left( \frac{\rho(t)}{\lambda_t} \right)^{3+\frac{10}{\beta_t}}.
\]

In light of (7.5), we have another one. \( \square \)

Multiply (7.2) with \( m = 1 \) by \( \nabla u^{(1-\mu_1^\beta)^2} \), and control the nonlinear part by

\[
| < \nabla^2 u^{(1-\mu_1^\beta)^2}, u_j^{(1-\mu_1^\beta)^2} \partial_j u^{(1-\mu_1^\beta)^2} > | \leq c_3 \|\nabla^2 u^{1-\mu_1^\beta}\|_{L^2}^2 \|\nabla u^{1-\mu_1^\beta}\|_{L^2}^2 \|u^{1-\mu_1^\beta}\|_{L^2}^2
\]

and

\[
| < u^{(1-\mu_1^\beta)^2}, u_j^{\mu_1^\beta} \partial_j \nabla^2 u^{1-\mu_1^\beta} > | \leq \|u^{1-\mu_1^\beta}\|_{L^2}^2 \|u^{\mu_1^\beta}\|_{L^2}^2 \|\nabla^3 u^{\mu_1^\beta}\|_{L^2}^2
\]

\[
\leq C \left( \frac{\rho(t)}{\lambda_t} \right)^{2+\frac{1}{\beta_1}} \|u^{1-\mu_1^\beta}\|_{L^2}^2 \|\nabla u^{\mu_1^\beta}\|_{L^2}^2
\]

as well as

\[
| < \nabla u^{(1-\mu_1^\beta)^2}, \nabla u_j^{1-\mu_1^\beta} \partial_j u^{1-\mu_1^\beta} + u_j^{1-\mu_1^\beta} \partial_j u^{1-\mu_1^\beta} > | \leq 2 \|\nabla u^{1-\mu_1^\beta}\|_{L^6} \|\nabla u^{1-\mu_1^\beta}\|_{L^3} \|\nabla u^{\mu_1^\beta}\|_{L^3} + \|\nabla^2 u^{1-\mu_1^\beta}\|_{L^2} \|u^{1-\mu_1^\beta}\|_{L^6} \|\nabla u^{\mu_1^\beta}\|_{L^3}
\]

\[
\leq C \|\nabla^2 u^{1-\mu_1^\beta}\|_{L^2}^4 \|\nabla u^{1-\mu_1^\beta}\|_{L^2}^4 \|u^{1-\mu_1^\beta}\|_{L^2}^4 \|\nabla u^{\mu_1^\beta}\|_{L^2}^4 \left( \frac{\rho(t)}{\lambda_t} \right)^{\frac{1}{2}}.
\]
In light of the proof of the foregoing lemma then we see

Then we have

\[ \frac{d}{dh} \| \nabla u^{1-\mu_\beta}(h) \|_{L^2}^2 = (-2) \| \nabla^2 u^{1-\mu_\beta}(h) \|_{L^2}^2 - 2 < \nabla u^{1-\mu_\beta}, \nabla u_j \partial_j u + u_j \partial_j \nabla u^{\mu_\beta} > \]

\[ \leq \left( -2 + 2(c_3 + \epsilon_2) \Re^{1-\mu_\beta}(h) \right) \| \nabla^2 u^{1-\mu_\beta}(h) \|_{L^2}^2 + \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^\frac{3}{2} \left( C_{c_2} \| u^{\mu_\beta}(h) \|_{L^2} \| \nabla u^{\mu_\beta}(h) \|_{L^2}^2 + C \| u^{1-\mu_\beta} \|_{L^2} \| \nabla u^{\mu_\beta} \|_{L^2}^2 \right). \]

for any \( \epsilon_2 > 0 \). Let \( c_5 = 2(c_3 + \epsilon_2) \). Note that

\[ \| \nabla^2 u^{1-\mu_\beta}(h) \|_{L^2}^2 \geq \frac{\| \nabla u^{1-\mu_\beta}(h) \|_{L^2}^4}{\| u^{1-\mu_\beta}(h) \|_{L^2}^2} = \frac{\| \nabla u^{1-\mu_\beta}(h) \|_{L^2}^6}{\| \nabla u^{1-\mu_\beta}(h) \|_{L^2}^2 \| u^{1-\mu_\beta}(h) \|_{L^2}^2} = \frac{\| \nabla u^{1-\mu_\beta}(h) \|_{L^2}^6}{\Re^{1-\mu_\beta}(h)^4} \]

and

\[ \left( C_{c_2} \| u^{\mu_\beta}(h) \|_{L^2} \| \nabla u^{\mu_\beta}(h) \|_{L^2}^2 + C \| u^{1-\mu_\beta} \|_{L^2} \| \nabla u^{\mu_\beta} \|_{L^2}^2 \right) \leq c_4 \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^2 \]

where

\[ c_4 := \max_{\tilde{t} \leq h \leq t_1} \left( C_{c_2} \| u^{\mu_\beta}(h) \|_{L^2}^3 + C \| u^{1-\mu_\beta} \|_{L^2} \| u^{\mu_\beta} \|_{L^2}^2 \right). \]

In light of the proof of the foregoing lemma then we see

**Lemma 7.2.** Suppose there is \( t_1 \in (\tilde{t}, t] \) such that

\[ \Re^{1-\mu_\beta}(h) \leq \frac{1}{c_5}, \quad \tilde{t} \leq \forall h \leq t_1, \]

and \( T - t \) small enough so that

\[ \| \nabla u^{1-\mu_\beta}(h) \|_{L^2} \leq \frac{c_4}{c_5^{2/3}} \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^{3/4}. \]

Then we have \( \tilde{t} \leq \forall h \leq t_1 \)

\[ \| \nabla u^{1-\mu_\beta}(h) \|_{L^2} \leq \frac{c_4}{c_5^{2/3}} \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^{3/4}. \]

**Remark 7.3.** Following inequalities will be used

\[ \| \nabla u^{1-\mu_\beta}(h) \|_{L^2} \leq \| \nabla^2 u^{1-\mu_\beta}(h) \|_{L^2}^{1/3} \Re^{1-\mu_\beta}(h)^{4/3}, \]

\[ \| \nabla^2 u^{1-\mu_\beta}(h) \|_{L^2} \leq \| \nabla^3 u^{1-\mu_\beta}(h) \|_{L^2}^{5/6} \Re^{1-\mu_\beta}(h)^{1/6}, \]

\[ \| \nabla^3 u^{1-\mu_\beta}(h) \|_{L^2} \leq \| \nabla^4 u^{1-\mu_\beta}(h) \|_{L^2}^{7/8} \Re^{1-\mu_\beta}(h)^{1/8}, \]

\[ \| \nabla^4 u^{1-\mu_\beta}(h) \|_{L^2} \leq \| \nabla^5 u^{1-\mu_\beta}(h) \|_{L^2}^{9/10} \Re^{1-\mu_\beta}(h)^{1/10}, \]

\[ \Re^{1-\mu_\beta}(h) \leq \| u^{1-\mu_\beta}(h) \|_{L^2}^{2/3} \| \nabla^2 u^{1-\mu_\beta}(h) \|_{L^2}^{1/3} \leq \| u^{1-\mu_\beta}(h) \|_{L^2}^{2/3} \| (T - t)^{(-\beta)} \|_{L^2}^{\frac{2}{3}(-\beta)}. \]
7.1. **Refined Estimate of** $\| \nabla_x u^\mu_\beta (h) \|_{L^\infty}$. For $t_0 \in [0, T)$, for $t \in (t_0, T)$ and for $a > 0$ let

$$G^a(h) := \{ x \in \mathbb{R}^3 : |u(x, h)| > \frac{a}{(T-h)^{\frac{1}{2}}} \}, \quad \forall h \in (t_0, T)$$

(7.9)

$$G^a(t_0, t] := \cup_{t_0 < h \leq t} G^a(h), \quad G^{a,c}(t_0, t] := \mathbb{R}^3 \setminus G^a(t_0, t]$$

$$g^a(t_0, t] := \cap_{t_0 < h \leq t} G^a(h).$$

Sometimes we also use $G^a$ and $G^{a,c}$ as well as $g^a$ instead of $G^a(t_0, t]$ and $G^{a,c}(t_0, t]$ as well as $g^a(t_0, t]$.

We observe from section 6 with $\Omega = G^{a,c}(t_0, t]$ and scaling constant $\lambda_1 < \sqrt{t-t_0}$ that for $h \in [t_0 + \lambda_1^2, t]$

(7.10) $\| \omega(h) \|_{L^2(G_{-\lambda_1}^{-}(t_0, t])}^2 \leq (2^2 n^2) \| u(t_0) \|_{L^2}^2 \left( \frac{T-t_0}{T-h} \right)^{c_1 a}$$

where $n = 1, 2, 3, \ldots$ and

$$G_{-\lambda_1}^{-}(t_0, t] := \{ x \in G^{a,c}(t_0, t], \text{ dist}(x, G^a(t_0, t]) > \lambda_1 \} = \mathbb{R}^3 \setminus G^a_{\lambda_1}(t_0, t].$$

On the other hand, suppose

(7.11) $\frac{2}{T-h} \| u^{\mu_\beta}(h) \|_{L^2(\mathbb{R}^3)} \leq \frac{a}{2(T-h)^{\frac{1}{2}}}, \quad \forall h \in (t_0, T).$

Then

$$|u^{\mu_\beta}(x, h)| \leq (T-t)^{\frac{3}{2}(\beta-\frac{1}{2})} \| u^{\mu_\beta}(h) \|_{L^2(\mathbb{R}^3)} \leq \frac{a}{2(T-h)^{\frac{1}{2}}}, \quad \forall h \in (t_0, t], \quad \forall x \in \mathbb{R}^3$$

We find for $0 < b \leq a$

$$\| \nabla_x u^{\mu_\beta}(h) \|_{L^2(G_{-\lambda_1}^{-}(t_0, t])} \leq \left( \int_{G^a_{\lambda_1}(t_0, t]} |\nabla_x u^{\mu_\beta}(x, h)|^6 dx \right)^{\frac{1}{6}} \left( \int_{g^b(t_0, t]} dx + |G^a_{\lambda_1}(t_0, t] \setminus g^b(t_0, t)] \right)^{\frac{1}{2}}$$

$$\leq \| \nabla_x u^{\mu_\beta}(h) \|_{L^p(G_{-\lambda_1}^{-}(t_0, t])} \left( \left( \frac{2}{b} \right)^2 (T-h) \int_{g^b(t_0, t]} |u^{1-\mu_\beta}(x, h)|^2 dx + |G^a_{\lambda_1}(t_0, t] \setminus g^b(t_0, t)] \right)^{\frac{1}{2}}.$$

Finally we discover

**Lemma 7.4.** Let $0 \leq t_0 < t < T$, $a > 0$ and $\beta \in (0, \frac{1}{2})$. Suppose (7.11). Then

(1) for $\lambda_1 < \sqrt{t-t_0}$, for $h \in [t_0 + \lambda_1^2, t]$, we have for $0 < b \leq a$

$$\| \nabla_x u^{\mu_\beta}(h) \|_{L^\infty(\mathbb{R}^3)} \leq (T-t)^{\frac{3}{2}(\beta-\frac{1}{2})} \left( C^2 \frac{n}{2} \| u(t_0) \|_{L^2(\mathbb{R}^3)} \left( \frac{T-t_0}{T-h} \right)^{\frac{c_1 a}{2}} (h-t_0)^{a-\frac{b}{2}} \lambda_1^{-n} \right)$$

$$+ \| \nabla_x^{2} u^{\mu_\beta}(h) \|_{L^2(\mathbb{R}^3)} \left( \left( \frac{2}{b} \right)^2 (T-h) \| u^{1-\mu_\beta}(h) \|_{L^2(\mathbb{R}^3)}^2 + |G^a_{\lambda_1}(t_0, t] \setminus g^b(t_0, t)] \right)^{\frac{1}{2}}.$$

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where \( n = 1, 2, 3, \ldots \), \( C_1 \) and \( C_2 \) are universal constants.

(2) for \( \lambda_1 > 0 \), for \( h \in [t_0, t] \), we have for \( 0 < b \leq a \)

\[
\| \nabla_x u^\mu (h) \|_{L^\infty(\mathbb{R}^3)}
\leq (T - t)^{\frac{1}{2}(\beta - \frac{1}{2})} \left\{ \| \nabla_x u(t_0) \|_{L^2(\mathbb{R}^3)} + C_n^2 n \| u(t_0) \|_{L^2(\mathbb{R}^3)} \left( \frac{T - t_0}{T - h} \right)^{\frac{C_n}{2}} (h - t_0)^{\frac{n-1}{2} \lambda_1^{-n}} \right. \\
+ \left. \| \nabla^2 u^\mu (h) \|_{L^2(\mathbb{R}^3)} \left( \left( \frac{2}{b} \right)^2 (T - h) \| u^1 \|_{L^2(\mathbb{R}^3)} \| \frac{\partial u^\mu(t)}{\partial x} \|_{L^2(\mathbb{R}^3)} + |G_\lambda^a (t_0, t) \cap g^b(t_0, t)| \right)^{\frac{1}{2}} \right\}
\]

where \( n = 1, 2, 3, \ldots \), \( C_1 \) and \( C_2 \) are universal constants.

Notice that by dividing the inner product in (7.1) into two parts we have the improved inequality (7.12)

\[
\frac{d}{dh} \| u^1 - \mu^\beta (h) \|_{L^2} \leq -2 \| \nabla u^1 - \mu^\beta (h) \|_{L^2} + c_{II} \left\| \nabla_x u^\mu (h) \|_{L^\infty(\mathbb{R}^3)} + \| u^\mu (h) \|_{L^2(\mathbb{R}^3)} \right\| L^2(\mathbb{R}^3) \left( \| u^1 - \mu^\beta (h) \|_{L^2(\mathbb{R}^3)} \right)
\]

for \( h \in [t_0 + \lambda_1^2, t] \), where

\[
(7.13)
\]

\[
\left\| \nabla_x u^\mu (h) \|_{L^\infty(\mathbb{R}^3)} \left( \| u^1 - \mu^\beta (h) \|_{L^2(\mathbb{R}^3)} \right)^2 \right. \\
+ \left. \| u^\mu (h) \|_{L^2(\mathbb{R}^3)} \| \nabla_x u^\mu (h) \|_{L^2(\mathbb{R}^3)} \right\| L^2(\mathbb{R}^3) \left( \| u^1 - \mu^\beta (h) \|_{L^2(\mathbb{R}^3)} \right)
\]

can be controlled by \( \| \nabla u^1 - \mu^\beta (h) \|_{L^2} \) provided \( \| u_0 \|_{L^2} \) is small enough, and

\[
\| u^\mu (h) \|_{L^2(\mathbb{R}^3)} \| \nabla_x u^\mu (h) \|_{L^2(\mathbb{R}^3)} \| \nabla_x u^\mu (h) \|_{L^2(\mathbb{R}^3)} \left( \| u^1 - \mu^\beta (h) \|_{L^2(\mathbb{R}^3)} \right)^2
\]

will be estimated by following lemma.
Lemma 7.5. For any \( \varphi_1 \in C^1_0(\mathbb{R}^3) \) and \( \psi_1 \in C^1_0(\mathbb{R}^3) \), there is \( \zeta \in C(\mathbb{R}^3) \) such that

\[
F^{-1} \left[ \varphi_1 \left( \frac{\lambda_t}{\rho_\beta(t)} \right) \right] F^{-1} \left[ \psi_1 u(\cdot) \right] (x) - \psi_1(x) F^{-1} \left[ \varphi_1 \left( \frac{\lambda_t}{\rho_\beta(t)} \right) \right] F^{-1} \left[ \psi_1 u(\cdot) \right] (x)
\]

(7.14)

\[
= \sum_{j=1}^{3} i \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^2 \int_{\mathbb{R}^3} F^{-1}[\partial_j \varphi_1] \left( \frac{\rho_\beta(t)}{\lambda_t} \right) (x-\bar{x}) \left( \partial_j \psi_1 \circ \zeta(x, \bar{x}) \right) u(\bar{x}) d\bar{x}
\]

\[
= \sum_{j=1}^{3} i \left( \frac{\lambda_t}{\rho_\beta(t)} \right) F^{-1} \left[ \partial_j \varphi_1(\xi) \right] \left( \frac{\lambda_t}{\rho_\beta(t)} \right) (x-\bar{x}) \left( \partial_j \psi_1 \circ \zeta(x, \bar{x}) \right) u(\bar{x}) d\bar{x}
\]

Proof. For any \( x, \bar{x} \), there is \( \theta \in [0, 1] \) such that \( \bar{x} = \theta x + (1-\theta) \bar{x} \)

\[
\psi_1(x) - \psi_1(\bar{x}) = (x-\bar{x}) \cdot \nabla \psi_1(\bar{x})
\]

where \( \theta = \theta(x, \bar{x}) \) continuously depends on \( x \) and \( \bar{x} \). Take \( \zeta = \theta x + (1-\theta) \bar{x} \) and note that

\[
F \left[ \frac{\rho_\beta(t)}{\lambda_t} (x_j - \bar{x}_j) \right] \left( \frac{\rho_\beta(t)}{\lambda_t} \right)^3 F^{-1}[\varphi_1] \left( \frac{\rho_\beta(t)}{\lambda_t} \right) (x-\bar{x}) (\eta) = \frac{i \eta_j}{\lambda_t} \varphi_1^* \left( \frac{\lambda_t}{\rho_\beta(t)} \right) |\eta|
\]

we get the conclusion.

\[
\square
\]

Take

\[
\varphi_1 = (1 - \varphi(\frac{\rho_\beta(t)}{\lambda_t} |\eta|)) \varphi(\frac{\rho_\beta(t)}{\lambda_t} |\eta|)
\]

and \( \psi_1 \in C^1_0 \) satisfying \( \psi_1 \geq 0, |\nabla x \psi_1| \lesssim \lambda_t^{-1} \) and

\[
\psi_1(x) = \begin{cases} 
1, & \forall x \in G^{a,c}_{-\lambda_t}(t_0, t] \\
0, & \forall x \in G^a(t_0, t].
\end{cases}
\]

and note that

\[
(T-t)^{\beta-\frac{1}{2}} \|u^{1-\mu_\beta}(h)\|_{L^2(G^{a,c}_{-\lambda_t}(t_0, t])} \leq (T-t)^{\beta-\frac{1}{2}} \|\psi_1 u^{1-\mu_\beta}(h)\|_{L^2},
\]

\[
(T-t)^{\beta-\frac{1}{2}} \|(|\nabla x \psi_1| \times u)^{1-\mu_\beta}(h)\|_{L^2} \leq \|\nabla x \psi_1 u^{1-\mu_\beta}(h)\|_{L^2} \leq \|\nabla x \psi_1 u^{1-\mu_\beta}(h)\|_{L^2} + \|\psi_1 u^{1-\mu_\beta}(h)\|_{L^2} + \|u \cdot \nabla x \psi_1 u^{1-\mu_\beta}(h)\|_{L^2}
\]

Applying Lemma 7.5, we discover

(7.15)

\[
\|u^{\frac{2-\beta}{2-\beta} \cap (1-\mu_\beta)}(h)\|_{L^2(G^{a,c}_{-\lambda_t}(t_0, t])}
\]

\[
\leq (T-t)^{\frac{1}{2} - \beta} \left( \|\omega_{\psi_1}^{1-\mu_\beta}(h)\|_{L^2} + \|(|\nabla x \psi_1| \times u)^{1-\mu_\beta}(h)\|_{L^2} + \|\psi_1 \nabla x \psi_1 u^{1-\mu_\beta}(h)\|_{L^2} + \|u \cdot \nabla x \psi_1 u^{1-\mu_\beta}(h)\|_{L^2} \right)
\]

\[
+ (T-t)^{\frac{1}{2} - \beta} \|F^{-1} \left[ \partial_j \varphi(\xi) \right] \left( \frac{\lambda_t}{\rho_\beta(t)} \right) (x-\bar{x}) \left( \partial_j \psi_1 \circ \zeta(x, \bar{x}) \right) u(\bar{x}) d\bar{x}
\]

\[
\leq (T-t)^{\frac{1}{2} - \beta} \left( \|\omega_{\psi_1}^{1-\mu_\beta}(h)\|_{L^2} + \|(|\nabla x \psi_1| \times u)^{1-\mu_\beta}(h)\|_{L^2} + \|\psi_1 \nabla x \psi_1 u^{1-\mu_\beta}(h)\|_{L^2} + \|u \cdot \nabla x \psi_1 u^{1-\mu_\beta}(h)\|_{L^2} \right)
\]

\[
+ (T-t)^{\frac{1}{2} - \beta} \lambda_t^{-1} \|u(h)\|_{L^2}
\]

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where $\|\omega_{\psi_t}^{1-\mu^\beta}(h)\|_{L^2}$ can be estimated by (7.10). We have similar estimate for $\|u^\beta_t \cap (1-\mu^\beta)(h)\|_{L^2(G_{\alpha-s}(t_0,t))}$.

7.2. frequency decay is maintained.

Lemma 7.6. There is $t_0 \in (0, T)$ and $c_I > 0$ such that for $t \in [t_0, T)$,

$$(7.16) \quad \|\nabla u(t)\|^2_{L^2} \geq \frac{c_I}{(T-t)^\frac{3}{2}}.$$  

Moreover, for each $\beta \geq \frac{3}{10}$, there is $t_1 = t_1(\beta, \|u_0\|_{L^2}) \in [t_0, T)$ such that for $t \in [t_1, T)$,

$$(7.17) \quad \|\nabla u^{1-\mu^\beta}(t)\|^2_{L^2} \geq \frac{c_I}{2(T-t)^\frac{3}{2}}.$$  

and

$$(7.18) \quad \|u^{1-\mu^\beta}(t)\|_{L^2} - c_I(T-t)^\frac{3}{4} \leq \|u^{1-\mu^\beta_1}(t_1)\|^2_{L^2} - c_I(T-t_1)^\frac{3}{4}.$$  

Proof. (7.16) was given by Leray [35]. Recall from the definition of $\mu^\beta_1$ that

$$\|\nabla u(t)\|^2_{L^2} = \|\nabla u^{\mu^\beta_1}(t)\|^2_{L^2} + \|\nabla u^{1-\mu^\beta_1}(t)\|^2_{L^2} \leq (T-t)^{2\beta-1}\|\nabla u^{\mu^\beta_1}(t)\|^2_{L^2} + \|\nabla u^{1-\mu^\beta_1}(t)\|^2_{L^2}.$$  

Observe that for $\beta > \frac{1}{4}$

$$(T-t)^{2\beta-1} < (T-t)^\frac{3}{4},$$

and for $\beta \geq \frac{3}{10}$

$$\frac{c_I}{2(T-t)^\frac{3}{2}} \geq c_{II}(T-t)^{-(\frac{3}{4}+\beta)}\|u^{1-\mu^\beta_1}(t)\|^2_{L^2} \|u_0\|^2_{L^2}$$

provided $T-t$ is small enough and $\|u_0\|_{L^2} \leq c_I/2c_{II}$. Consequently, for $\beta \geq \frac{3}{10}$, there is $t_1 = t_1(\beta, \|u_0\|_{L^2}) \in (0, T)$ such that for $t \in [t_1, T)$

$$\|\nabla u^{\mu^\beta_1}(t)\|^2_{L^2} \leq \frac{c_I}{(T-t)^\frac{3}{2}},$$

$$\|\nabla u^{1-\mu^\beta_1}(t)\|^2_{L^2} \geq \frac{c_I}{2(T-t)^\frac{3}{2}} \geq c_{II}(T-t)^{-(\frac{3}{4}+\beta)}\|u^{1-\mu^\beta_1}(t)\|^2_{L^2} \|u_0\|^2_{L^2}.$$  

Applying these observation to

$$\frac{d}{dt}\|u^{1-\mu^\beta}(t)\|^2_{L^2} = \frac{d}{dt}\|h_{-t}||u^{1-\mu^\beta}(h)\|^2_{L^2}$$

$$+ \frac{1-2\beta}{T-t} \int_{\mathbb{R}^3} (1 - \varphi(\frac{\lambda_t}{\rho(t)}|\eta|)) \varphi'(\frac{\lambda_t}{\rho(t)}|\eta|) \frac{\lambda_t}{\rho(t)} |\eta||\mathcal{F}[u](\eta,t)|^2 d\eta$$

$$\leq -2\|\nabla u^{1-\mu^\beta}(h)\|^2_{L^2} + c_{II} \left( \frac{\beta(t)}{\lambda_t} \right)^\frac{3}{4} \|u^{\mu^\beta}(h)\|_{L^2} \|u^{1-\mu^\beta}(h)\|_{L^2} \|u^{1-\mu^\beta}(h)\|_{L^2} + \|u^{\mu^\beta \cap (1-\mu^\beta)}(h)\|_{L^2}.$$
where \( \varphi' \leq 0 \), we find
\[
\frac{d}{dt} \| u^{1-\mu_t^2}(t) \|_{L^2}^2 \leq \frac{-ct}{2(T-t)^{3/2}}.
\]

Then we proved this lemma. \( \square \)

7.3. **estimate of** \( \| u(t) - S(t)u_0 \|_{L^2} \).

Let
\[
(7.19) \quad v(x, t) = u(x, t) - S(t)u_0.
\]

Here \( S(h) = \exp\{ h\Delta \} \) denotes the heat kernel operator. Note that \( v \) is a solution of the equation
\[
(7.20) \quad \partial_t v = \mathcal{P} \Delta v - \mathcal{P}\partial_j(u_j u), \quad \forall t > 0, \quad \forall x \in \mathbb{R}^3
\]

with zero initial data. Here \( \mathcal{P} \) is the Leray projection operator.

For \( \beta \in (0, \frac{1}{2}) \), recall the definition of \( f^{1-\mu_t^2}, f^{\mu_t^2} \) and \( f^{\mu_t^2 \cap (1-\mu_t^2)} \) at the beginning of this section and let \( \beta' = \beta'(<\beta) \in (0, \frac{1}{2}) \) satisfy
\[
(7.21) \quad q_t^{\beta'} = q_t^\beta + 2.
\]

Note that
\[
2^{-3}(T-t)^{\beta'-\frac{3}{2}} \leq 2^{q_t^{\beta'} - \frac{3}{2}} \leq (T-t)^{\beta'-\frac{3}{2}} \leq 2^{q_t^{\beta'} + \frac{3}{2}} \leq 2^{2(T-t)^{\beta'-\frac{3}{2}}}
\]

and
\[
\mu_t^{\beta'} \subset \mu_t^{\beta}, \quad \mu_t^{\beta'} \cap (1 - \mu_t^{\beta}) \subset 1 - \mu_t^{\beta}, \quad 1 - \mu_t^{\beta'} \subset 1 - \mu_t^{\beta}.
\]

**Lemma 7.7.** For all \( 0 < h < T \)
\[
\| \nabla_x v(h) \|_{L^1(\mathbb{R}^3)} \leq C(\| u_0 \|_{L^2}^2 + T\| u_0 \|_{L^2} \| \nabla^2_x u_0 \|_{L^2}).
\]

**Proof.** For \( \sigma \in (0, 1), \) \( j, k \in \{1, 2, 3\} \) fixed, \( V_\sigma(x, y, t) := \frac{1}{\sigma}(v_j(x+\sigma e_k, t)-v_j(x, t)) \).

We have
\[
\partial_t V_\sigma + u \cdot \nabla_x V_\sigma - \Delta_x V_\sigma = \frac{1}{\sigma}(\partial_j p - \partial_j p(\cdot + \sigma e_k)) + \frac{1}{\sigma}(u(\cdot + \sigma e_k) - u) \cdot \nabla_x u_j(\cdot + \sigma e_k)
\]
\[
- \frac{1}{\sigma} u \cdot \nabla_x (S(t)u_0(\cdot + \sigma e_k) - S(t)u_0)_j
\]

We deduce that the right-hand side, denoted by \( m_\sigma, \) is bounded in \( L^1(\mathbb{R}^3 \times (0, T)) \)
uniformly in \( \sigma \in (0, 1] \) and
\[
\lim_{\sigma \to 0} \| m_\sigma \|_{L^1(\mathbb{R}^3 \times (0, T))} \lesssim \| u_0 \|_{L^2}^2 + T\| u_0 \|_{L^2} \| \nabla^2_x u_0 \|_{L^2}.
\]

Since \( u \in L^2(0, T; H^1(\mathbb{R}^3)) \) and \( V_\sigma \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; L^6(\mathbb{R}^3)), \) we deduce from [29] Lemma 2.3 that we have
\[
\partial_t (V_\sigma(\cdot)) + u \cdot \nabla_x (V_\sigma(\cdot)) - \Delta_x (V_\sigma(\cdot)) = (m_\sigma(\cdot)) + r_\sigma(\cdot)
\]

where \( (V_\sigma(\cdot)) \) denotes the \( \epsilon \)-smoothing of \( V_\sigma \) and
\[
r_\sigma(\cdot) \to 0 \quad \text{as} \quad \epsilon \downarrow 0 \quad \text{in} \quad L^2(0, T; L^1(\mathbb{R}^3)) \cap L^1(0, T; L^2_\sigma(\mathbb{R}^3))
\]
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for each $\sigma > 0$ fixed. Letting $\epsilon \downarrow 0$ we recover
\[
\partial_t |V_\sigma| + (u \cdot \nabla_x)|V_\sigma| - \Delta_x |V_\sigma| \leq |m_\sigma|, \quad \text{in } \mathbb{R}^3 \times (0, T)
\]
\[|V_\sigma|(x, 0) = 0.\]

Multiply by $\varphi(|x|/n)$ where $\varphi \in C_0^\infty(\mathbb{R}^3)$
\[
\varphi(x) = \begin{cases}
1, & \forall |x| \leq 1, \\
\in [0, 1], \forall 1 < |x| < 2, \\
0, & \forall |x| \geq 2,
\end{cases}
\]
and integrate over $\mathbb{R}^3 \times [0, t]$
\[
\int |V_\sigma(x, t)|\varphi(|x|/n)dx \leq C \left( \|u_0\|_{L^2_x}^2 + T\|u_0\|_{L^2_x}\|\nabla_x^2 u_0\|_{L^2_x} \right) + \frac{1}{n^2} \int_0^t \int |V_\sigma|(-\Delta \varphi)(|x|/n)dxds
\]
\[
+ \frac{1}{n} \int_0^t \int |u||V_\sigma||\nabla \varphi(|x|/n)|dxds.
\]

First letting $n \to \infty$, next letting $\sigma \downarrow 0$, we deduce $\|\nabla_x v(t)\|_{L^1(\mathbb{R}^3)} \leq C(\|u_0\|_{L^2_x}^2 + T\|u_0\|_{L^2_x}\|\nabla_x^2 u_0\|_{L^2_x})$.

Another proof. Let $\bar{\omega} = \nabla_x \times v$. We have
\[
\partial_t \bar{\omega} - \Delta_x \bar{\omega} = (\omega \cdot \nabla_x)u - (u \cdot \nabla_x)\omega.
\]

We find
\[
\int |\bar{\omega}(x, h)|dh + \int_{t_0}^h \int \frac{|\bar{\omega}(x, h')||\nabla_x \bar{\omega}(x, h')|^2}{|\omega(x, h')|}dxdh'
\]
\[
= -\int_{t_0}^h \int \left( (u(x, h') \cdot \nabla_x)\bar{\omega}(x, h') \cdot \frac{\bar{\omega}(x, h')}{|\omega(x, h')|} \right) dxdh'
\]
\[
+ \int_{t_0}^h \int \left( (\omega(x, h') \cdot \nabla_x)u(x, h') - (u(x, h') \cdot \nabla_x)S(h' - t_0)\omega(x, t_0) \right) \frac{\bar{\omega}(x, h')}{|\omega(x, h')|} dxdh'
\]
\[
= \int_{t_0}^h \int \left( (\omega(x, h') \cdot \nabla_x)u(x, h') - (u(x, h') \cdot \nabla_x)S(h' - t_0)\omega(x, t_0) \right) \frac{\bar{\omega}(x, h')}{|\omega(x, h')|} dxdh'
\]
\[
\leq C(\|u_0\|_{L^2_x}^2 + T\|u_0\|_{L^2_x}\|\nabla_x^2 u_0\|_{L^2_x}).
\]
\[
\square
\]

7.4. measurement of $G^a(t_0, t] \setminus g^b(t_0, t]$. Suppose (7.11) is satisfied at $t_0$, that is
\[
(T - t)^{\frac{\alpha}{2}(\beta - \frac{1}{2})}\|u^{\mu^a}(t_0)\|_{L^2(\mathbb{R}^3)} \leq \frac{a}{2(T - t_0)^{\frac{\alpha}{2}}},
\]
which implies
\[
|u^{\mu^a}(x, t_0)| \leq (T - t)^{\frac{\alpha}{2}(\beta - \frac{1}{2})}\|u^{\mu^a}(t_0)\|_{L^2(\mathbb{R}^3)} \leq \frac{a}{2(T - t_0)^{\frac{\alpha}{2}}}, \forall x \in \mathbb{R}^3.
\]

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Furthermore suppose
\[(7.22) \quad |u^{1-\mu_t^3}(x, t_0)| < \frac{a}{4(T-t_0)^{\frac{3}{2}}}, \quad \forall x \in \mathbb{R}^3.\]

Then for \(t_0 < h \leq t,\)
\[|v^{1-\mu_t^3}(x, h)| = |u^{1-\mu_t^3}(x, h) - S(h - t_0)u^{1-\mu_t^3}(t_0)| \geq \frac{a}{4(T-h)^{\frac{3}{2}}}, \quad \forall x \in G^a(h).\]

Without loss generality, when \[(7.11) \text{ and } (7.22)\] are satisfied, we will use \(u^{1-\mu_t^3}\) (even \(v^{1-\mu_t^3}\)) instead of \(u\) in the definition of \(G^a(h)\) (see \[(7.9)\]).

Recall now the coarea formula \[18\]
\[
\int |D\chi_{b,a}((T-h)^{\frac{1}{2}}|v(x, h)|)|dx = \int_{-\infty}^{+\infty} \mathcal{H}^2 \left(\left\{ \chi_{b,a}((T-h)^{\frac{1}{2}}|v(x, h)|) = \sigma \right\} \right) d\sigma
\]
\[
= \int_{b}^{a} \mathcal{H}^2 \left(\left\{ (T-h)^{\frac{1}{2}}|v(x, h)| = \sigma \right\} \right) d\sigma
\]
\[
= (a-b)\mathcal{H}^2 \left(\left\{ (T-h)^{\frac{1}{2}}|v(x, h)| = \sigma^*(h) \right\} \right), \quad \text{for some } \sigma^*(h) \in (b, a),
\]
where \(0 < b < a\) and
\[
\chi_{b,a}(r) := \begin{cases} b, & \forall r \leq b \\ r, & \forall b < r < a \\ a, & \forall r \geq a. \end{cases}
\]

On the other hand, employing Lemma \[7.7\]
\[
\int |D\chi_{b,a}((T-h)^{\frac{1}{2}}|v(x, h)|)|dx \leq (T-h)^{\frac{1}{2}} \int_{b \leq (T-h)^{\frac{1}{2}}|v(x, h)| \leq a} |\nabla_x v(x, h)|dx
\]
\[
\leq (T-h)^{\frac{1}{2}} \int_{\mathbb{R}^3} |\nabla_x v(x, h)|dx \leq C(\|u_0\|^2_{L^2} + T\|u_0\|^2_{L^2}\|\nabla^2_x u_0\|_{L^2})(T-h)^{\frac{1}{2}},
\]
we deduce
\[(7.23) \quad \mathcal{H}^2 \left(\left\{ (T-h)^{\frac{1}{2}}|v(x, h)| = \sigma^*(h) \right\} \right) \leq C(a-b)^{-1}(\|u_0\|^2_{L^2} + T\|u_0\|^2_{L^2}\|\nabla^2_x u_0\|_{L^2})(T-h)^{\frac{1}{2}},
\]
and
\[
|G^a(t_0, t)| \leq \mathcal{H}^3 \left(\bigcup_{t_0 \leq h \leq t} \left\{ (T-h)^{\frac{1}{2}}|v(x, h)| \geq \sigma^*(h) \right\} \right)
\]
\[
\leq \sup_{t_0 \leq h \leq t} |G^b(h)| \leq b^{-2}\|u_0\|^2_{L^2} (T-t_0).
\]

Let \(G_{\lambda_1}^a(h) := \text{the } \lambda_1-\text{neighborhood of } G^a(h).\)

**Lemma 7.8.** Suppose \[(7.11) \text{ and } (7.22)\]. Then for \(0 < b < a, \) we have
\[(7.24) \quad |G_{\lambda_1}^{\sigma^*(h)}(h) \setminus G^a(h)| \leq C(a-b)^{-1}(\|u_0\|^2_{L^2} + T\|u_0\|^2_{L^2}\|\nabla^2_x u_0\|_{L^2})(T-t_0)^{\frac{1}{2}+\alpha_1},
\]
\[
|G^a(t_0, t) \setminus g^b(t_0, t)| \leq \int_{t_0}^t \frac{a}{2(a-b)(T-h)}|G^b(h) \setminus G^a(h)| dh + \int_{t_0}^t \frac{(T-h)^{\frac{1}{2}}}{a-b} dh \int_{G^a(h) \setminus G^b(h)} |\partial_t u^{1-\mu_t^3}(x, h)||dx.
\]
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**Proof.** To deduce (7.24), we only need to measure the $\lambda_1$-neighborhood of $
abla \{ (T-h)^{\frac{1}{2}} | v(x, h) | = \sigma^*(h) \}$ (t_0 \leq h \leq t), that is $| \bigcup_{h=t_0} \{ (T-h)^{\frac{1}{2}} | v(x, h) | = \sigma^*(h) \} |$. Since at each time $h$, the volume of $\lambda_1$-neighborhood of $
abla \{ (T-h)^{\frac{1}{2}} | v(x, h) | = \sigma^*(h) \}$ ($t_0 \leq h \leq t$) is estimated by

$$ C(a - b)^{-1}(\| u_0 \|_L^2 + T \| u_0 \|_L^2 \| \nabla^2 u_0 \|_L^2)(T - h)^{\frac{3}{2}} \lambda_1, $$

the initial volume is bounded by (7.25).

On the other hand, for $\sigma \in (b, a)$

$$ G^a(t_0, t) \subset \bigcup_{h=t_0} \{ (T-h)^{\frac{1}{2}} | u^{1-\mu_1^\beta} (x, h) | \geq \sigma \}, $$

$$ g^b(t_0, t) \supset \cap_{h=t_0} \{ (T-h)^{\frac{1}{2}} | u^{1-\mu_1^\beta} (x, h) | \geq \sigma \} $$

and from

$$ | u^{1-\mu_1^\beta} (x, h) | = \frac{\sigma}{(T-h)^{\frac{3}{2}}}, \quad \text{for } x \in \partial G^a(h), $$

the normal velocity

$$ \nu_\sigma(x, h) := \frac{dx}{dh} \frac{\nabla_x | u^{1-\mu_1^\beta} (x, h) |}{| \nabla_x | u^{1-\mu_1^\beta} (x, h) |} = \frac{\sigma}{2(T-h)^{\frac{3}{2}} | \nabla_x | u^{1-\mu_1^\beta} (x, h) |} - \frac{\partial_h | u^{1-\mu_1^\beta} (x, h) |}{| \nabla_x | u^{1-\mu_1^\beta} (x, h) |} $$

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Due to the motion of the boundary $\partial G^\sigma(h)$, the new increasing volume in unit time equals to

$$\int_{\partial G^\sigma(h)} |\nu_\sigma(x,h)| d\mathcal{H}^2(x).$$

Let

$$\nu(x,h) = \frac{a}{2(T-h)^\frac{3}{2}|\nabla_x|u_1^{1-\mu_1^2}(x,h)||} + \frac{|\partial_h|u_1^{1-\mu_1^2}(x,h)||}{|\nabla_x|u_1^{1-\mu_1^2}(x,h)||} \geq \sup_{b<\sigma<a} |\nu_\sigma(x,h)|.$$

By applying the general coarea formula to $f(x,h) := (T-h)^{\frac{3}{2}}|u_1^{1-\mu_1^2}(x,h)|$ [K] (pp.103 Th.3) we have

$$(T-h)^{\frac{3}{2}} \int_{G^b(h) \setminus G^a(h)} \nu(x,h)|\nabla_x|u_1^{1-\mu_1^2}(x,h)||dx = \int_b^a d\sigma \int_{\partial G^\sigma(h)} \nu(x,h)d\mathcal{H}^2(x).$$
Integrate over \([t_0, t]\]

\[
\int_{t_0}^{t} (T - h)^{\frac{1}{2}} \int_{G^a(h) \setminus G^b(h)} \nu(x, h) |\nabla_x u| \, dx \, dh = \int_{t_0}^{t} \int_{\partial G^a(h)} \nu(x, h) \mathcal{H}^2(x) \, d\sigma dh
\]

\[
= \int_{t_0}^{t} \int_{\partial G^a(h)} \nu(x, h) \mathcal{H}^2(x) \, dhd\sigma
\]

\[
= (a - b) \int_{t_0}^{t} \int_{\partial G^{**}(h)} \nu(x, h) \mathcal{H}^2(x) \, dh
\]

for some \(\sigma^{**} \in (b, a)\). We find

\[
|G^a(t_0, t) \setminus g^b(t_0, t)| \\
\leq \mathcal{H}^3 \left( \bigcup_{h = t_0}^{t} \{ (T - h)^{\frac{1}{2}} |u_1 - \mu^2_i(x, h)| \geq \sigma^{**} \} \right) \setminus \bigcap_{h = t_0}^{t} \{ (T - h)^{\frac{1}{2}} |u_1 - \mu^2_i(x, h)| \geq \sigma^{**} \}
\]

\[
\leq \int_{t_0}^{t} dh \int_{\partial G^{**}(h)} \nu(x, h) \mathcal{H}^2(x)
\]

\[
= \int_{t_0}^{t} \frac{(T - h)^{\frac{1}{2}}}{a - b} \int_{G^a(h) \setminus G^b(h)} |\nabla_x u| u_1 - \mu^2_i (x, h) ||\nu(x, h)dx dh
\]

So the new increasing volume due to the motion of boundary is bounded by

\[
|G^a(t_0, t) \setminus g^b(t_0, t)| \leq \int_{t_0}^{t} dh \frac{a}{2(a - b)(T - h)^{\frac{1}{2}}} \int_{G^a(h) \setminus G^b(h)} \left( \frac{a}{2(T - h)^{\frac{1}{2}}} + |\partial_h u_1 - \mu^2_i(x, h)| \right) dx
\]

\[
\leq \int_{t_0}^{t} \frac{a}{2(a - b)(T - h)^{\frac{1}{2}}} |G^a(h) \setminus G^b(h)| dh + \int_{t_0}^{t} \frac{(T - h)^{\frac{1}{2}}}{a - b} dh \int_{G^b(h) \setminus G^a(h)} |\partial_h u_1 - \mu^2_i(x, h)| \, dx.
\]

Furthermore, from

\[
|G^b(h)| = |\{ b \leq (T - h)^{\frac{1}{2}} |u_1 - \mu^2_i(x, h)| \} | \leq \frac{T - h}{b^2} \|u_1 - \mu^2_i(h)\|_{L^2}^2
\]

we have

\[
\int |D\chi_{b,u}(T - h)^{\frac{1}{2}} |u_1 - \mu^2_i(x, h)| \, dx \leq (T - h)^{\frac{1}{2}} \int_{b \leq (T - h)^{\frac{1}{2}} |u_1 - \mu^2_i(x, h)| \leq a} |\nabla_x u_1 - \mu^2_i(x, h)| \, dx
\]

\[
\leq (T - h)^{\frac{1}{2}} \left| \{ b \leq (T - h)^{\frac{1}{2}} |u_1 - \mu^2_i(x, h)| \leq a \} \right| \left( \int_{G^b(h) \setminus G^a(h)} |\nabla_x u_1 - \mu^2_i(x, h)|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq (T - h)^{\frac{1}{2}} \frac{(T - h)^{\frac{1}{2}}}{b} \|u_1 - \mu^2_i(h)\|_{L^2(G^b(h) \setminus G^a(h))} \|\nabla_x u_1 - \mu^2_i(h)\|_{L^2(G^b(h) \setminus G^a(h))},
\]

we deduce

\[
\mathcal{H}^2 \left( \left\{ (T - h)^{\frac{1}{2}} |u_1 - \mu^2_i(x, h)| = \sigma^*(h) \right\} \right) \leq \frac{T - h}{b(a - b)} \|u_1 - \mu^2_i(h)\|_{L^2} \|\nabla_x u_1 - \mu^2_i(h)\|_{L^2}.
\]
Notice that
\[(7.32)\]
\[
\int_{t_0}^{t} (T - h)^{\frac{1}{2}} dh \int_{G^h(h) \setminus G^a(h)} |\partial_{h} u_1^{1-\mu_t^2}(x, h)| dx
\]
\[
= \int_{t_0}^{t} (T - h)^{\frac{1}{2}} dh \int_{G^h(h) \setminus G^a(h)} \left( \frac{u_1^{1-\mu_t^2}}{|u_1^{1-\mu_t^2}|} \cdot \left( \Delta_x u_1^{1-\mu_t^2} - (u \cdot \nabla_x u + \nabla_x p)^{-1-\mu_t^2} \right) \right) (x, h) dx
\]
\[
\leq (T - t_0)^{\frac{1}{2}} \left( \int_{t_0}^{t} \Delta_x u_1^{1-\mu_t^2}(h) \|L^2 dh \|^{\frac{1}{2}} \max_h |G^b(h) \setminus G^a(h)|^{\frac{1}{2}} + (T - t_0)^{\frac{1}{2}} \left( |G^b(h) \setminus G^a(h)|^{\frac{1}{2}} \|u_1^{1-\mu_t^2}(h)\|_{L^2} \right) dh
\]
where \((7.31)\) (see Lemma 7.2) and
\[
\partial_{\mu} p^{1-\mu_t^2} = R_l \sum_{k=1}^{3} R_k \sum_{j=1}^{3} (u_j \partial_{j} u_k)^{1-\mu_t^2}
\]
\[
= R_l \sum_{k=1}^{3} R_k \sum_{j=1}^{3} (u_j^{\mu_t^2} \partial_{j} u_k^{\mu_t^2} (1-\mu_t^2) + \partial_{j} u_k^{\mu_t^2} u_j^{\mu_t^2 (1-\mu_t^2)} + u_j^{\mu_t^2} \partial_{j} u_k^{1-\mu_t^2} + u_j^{1-\mu_t^2} \partial_{j} u_k^{1-\mu_t^2} + \partial_{j} u_k^{\mu_t^2} u_j^{1-\mu_t^2})
\]
\[
\sum_{l=1}^{3} \int_{G^h(h) \setminus G^a(h)} \frac{u_1^{1-\mu_t^2}}{|u_1^{1-\mu_t^2}|} R_l \sum_{k=1}^{3} R_k \sum_{j=1}^{3} u_j^{\mu_t^2} \partial_{j} u_k dx
\]
\[
\leq |G^b(h) \setminus G^a(h)|^{\frac{1}{2}} \sum_{l=1}^{3} \left( |R_l \sum_{k=1}^{3} R_k \sum_{j=1}^{3} u_j^{\mu_t^2} \partial_{j} u_k|_{L^2(\mathbb{R}^3)} \right)
\]
\[
\leq |G^b(h) \setminus G^a(h)|^{\frac{1}{2}} \sum_{k=1}^{3} \sum_{j=1}^{3} \|u_j^{\mu_t^2} \partial_{j} u_k\|_{L^2(\mathbb{R}^3)}
\]
\[
\lesssim |G^b(h) \setminus G^a(h)|^{\frac{1}{2}} \|u_j^{\mu_t^2} (h)\|_{L^\infty} \|\nabla_x u(h)\|_{L^2}.
\]
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\[
\sum_{l=1}^{3} \left| \int_{|
abla h| G^{\sigma}(h) \cap G^{\sigma}(h) \frac{1-\mu^0}{u^1-\mu^0} R_{l} \sum_{k=1}^{3} R_{k} \sum_{j=1}^{3} (u_{j}^{1-\mu_{t}} \partial_{j} u_{k}^{1-\mu_{t}} + \partial_{j} u_{k}^{\mu_{t}} u_{j}^{1-\mu_{t}}) dx \right|
\leq \sum_{l=1}^{3} \left\| \frac{1-\mu^0}{u^1-\mu^0} (h) \right\|_{L^\infty(\mathbb{R}^3)} \left\| R_{l} \sum_{k=1}^{3} R_{k} \sum_{j=1}^{3} (u_{j}^{1-\mu_{t}} \partial_{j} u_{k}^{1-\mu_{t}} + \partial_{j} u_{k}^{\mu_{t}} u_{j}^{1-\mu_{t}}) (h) \right\|_{L^1(\mathbb{R}^3)}
\lesssim \sum_{k=1}^{3} \sum_{j=1}^{3} (u_{j}^{1-\mu_{t}} \partial_{j} u_{k}^{1-\mu_{t}} + \partial_{j} u_{k}^{\mu_{t}} u_{j}^{1-\mu_{t}}) (h) \right\|_{\mathcal{H}^1(\mathbb{R}^3)}
\lesssim \left\| u^{1-\mu_{t}} (h) \right\|_{L^2} \left\| \nabla u (h) \right\|_{L^2}
\]

as well as similar estimates for \((u \cdot \nabla u)^{1-\mu_{t}}\) were used. Utilizing Lemma \ref{lem:7.1} we find

**Lemma 7.9.** Suppose \ref{7.11} and for \(0 < t < T\) there is \(t_1 \in (t_0, T)\) such that

\[
\left\| u^{1-\mu_{t}} (h) \right\|_{L^2} < B := \frac{1}{(2c_1)^{\frac{3}{9}} (2c_2)^{\frac{1}{9}}} \left( \lambda_t \right)^{\frac{1}{2} + \frac{1}{2}} , \quad t_0 \leq \forall h \leq t_1
\]

as well as suppose

\[
\left\| \nabla^2 u^{1-\mu_{t}} (t_0) \right\|_{L^2} < Z_0 := \frac{1}{(2c_1 B)^6}.
\]

Then for \(0 < b < a\), we have

\[
|G_{\lambda_1}^{\sigma}(h) \setminus G^{\sigma}(h)| \leq \frac{c_2^2}{b(a-b)(c_1 + h)} (T-t)^{\frac{1}{2} - \frac{6}{9} (\frac{1}{2} - \beta)} (T-t_0)^{1+a_1}, \quad \forall h \in (t_0, t_1),
\]

\[
|G^{\sigma}(t_0, t_1)| \setminus g^b(t_0, t_1) \leq \frac{a(t-t_0)}{2(a-b)b^2} \max_h \left\| u^{1-\mu_{t}} (h) \right\|_{L^2}^2
\]

\[
+ \frac{(T-t_0)^{\frac{1}{2}}}{a-b} \left( \left\| \nabla u^{1-\mu_{t}} (t_0) \right\|_{L^2} + (T-t)^{\frac{2}{3} - \frac{2}{9} (\frac{1}{2} - \beta)} \max_h \left\| G^{\sigma}(h) \right\|_{\mathcal{H}^1} \right)^{\frac{1}{2}} \max_h \left\| G^{\sigma}(h) \right\|_{\mathcal{H}^1}^{\frac{1}{2}}
\]

\[
+ \frac{(T-t_0)^{\frac{1}{2}}}{a-b} \left( \max_h \left\| G^{\sigma}(h) \right\|_{\mathcal{H}^1} \right)^{\frac{1}{2}} \max_h \left\| u^{1-\mu_{t}} (h) \right\|_{L^2}^2 \cdot \max_h \left\| G^{\sigma}(h) \right\|_{\mathcal{H}^1}^{\frac{1}{2}} + \frac{1}{2} \max_h \left\| u^{1-\mu_{t}} (h) \right\|_{L^2}^2 \cdot \max_h \left\| G^{\sigma}(h) \right\|_{\mathcal{H}^1}^{\frac{1}{2}}.
\]

**Proof.** The new increasing volume is derived from \ref{7.29} \ref{7.30} \ref{7.32}. Applying \ref{7.31} the initial volume is bounded by

\[
\lambda_1 \mathcal{H}^2 \left( \left\{(T-t)^{\frac{1}{2}} \left| u^{1-\mu_{t}} (x, t_0) \right| = \sigma^{\sigma}(t_0) \right\} \right) \leq \frac{(T-t_0)^{1+a_1}}{b(a-b)} \left\| u^{1-\mu_{t}} (t_0) \right\|_{L^2} \left\| \nabla u^{1-\mu_{t}} (t_0) \right\|_{L^2}
\]

\[
\leq \frac{(T-t_0)^{1+a_1}}{b(a-b)} \left\| u^{1-\mu_{t}} (t_0) \right\|_{L^2} \left\| \nabla u^{1-\mu_{t}} (t_0) \right\|_{L^2} \leq \frac{(T-t_0)^{1+a_1}}{b(a-b)} (T-t)^{\frac{1}{2}} \frac{1}{(2c_1 B)^{\frac{1}{2}}}
\]

\[
= \frac{(T-t_0)^{1+a_1}}{b(a-b)c_1^{\frac{3}{2}}} (T-t)^{\frac{1}{2} - \frac{6}{9} (\frac{1}{2} - \beta)} . \quad \Box
\]

For \(x \in \partial G^{\sigma}(h)\), let \(H(x, h)\) be the mean curvature of \(\partial G^{\sigma}(h)\). Observe from

\[
\text{d} \mathcal{H}^2 (\partial G^{\sigma}_{\lambda_1}(h)(y)) = (1 + \lambda_1 H(x, h))^2 \text{d} \mathcal{H}^2 (\partial G^{\sigma}(h))
\]
that due to the motion of the boundary \( \partial G^\sigma_{\lambda_1}(h) \), the new increasing volume in unit time equals to

\[
\int_{\partial G^\sigma_{\lambda_1}(h)} |\nu_\sigma(x, h)| d\mathcal{H}^2(x) = \int_{\partial G^\sigma(h)} |\nu_\sigma(x, h)|(1 + \lambda_1 H(x, h))^2 d\mathcal{H}^2(x).
\]

By applying the general coarea formula to \( f(x, h) := (T - h)^{1/2}|u^{1-\mu_1^\beta}(x, h)| \) again we have

\[
(T - h)^{1/2} \int_{G^\beta(h) \setminus G^\sigma(h)} (1 + \lambda_1 H(x, h))^2 \nu(x, h)|\nabla_x|u^{1-\mu_1^\beta}(x, h)||dx dh
\]

\[
= \int_a^b d\sigma \int_{\partial G^\sigma(h)} (1 + \lambda_1 H(x, h))^2 \nu(x, h) d\mathcal{H}^2(x).
\]

Integrate over \([t_0, t]\)

\[
\int_{t_0}^t (T - h)^{1/2} \int_{G^\beta(h) \setminus G^\sigma(h)} (1 + \lambda_1 H(x, h))^2 \nu(x, h)|\nabla_x|u^{1-\mu_1^\beta}(x, h)||dx dh
\]

\[
= \int_{t_0}^t \int_b^a \int_{\partial G^\sigma(h)} (1 + \lambda_1 H(x, h))^2 \nu(x, h) d\mathcal{H}^2(x) d\sigma dh
\]

\[
= \int_b^a \int_{t_0}^t \int_{\partial G^\sigma(h)} (1 + \lambda_1 H(x, h))^2 \nu(x, h) d\mathcal{H}^2(x) dh d\sigma
\]

\[
= (a - b) \int_{t_0}^t \int_{\partial G^{**}(h)} (1 + \lambda_1 H(x, h))^2 \nu(x, h) d\mathcal{H}^2(x) dh
\]

for some \( \sigma^{**} \in (b, a) \). We find

\[
|G^\sigma_{\lambda_1}(t_0, t) \setminus g^b_{\lambda_1}(t_0, t)|
\]

\[
\leq \mathcal{H}^3 \left( \bigcup_{h_0=t_0}^t \lambda_1 - \text{n.b. of } \{(T - h)^{1/2}|u^{1-\mu_1^\beta}(x, h)| \geq \sigma^{**}\} \right)
\]

\[
\leq \int_{t_0}^t dh \int_{\partial G^{**}(h)} (1 + \lambda_1 H(x, h))^2 \nu(x, h) d\mathcal{H}^2(x)
\]

\[
= \int_{t_0}^t \frac{(T - h)^{1/2}}{(a - b)} \int_{G^\beta(h) \setminus G^\sigma(h)} |\nabla_x|u^{1-\mu_1^\beta}(x, h)||\nu(x, h)\|(1 + \lambda_1 H(x, h))^2 \nu(x, h) dx dh
\]
Regularity of solutions to the Navier-Stokes equations

So the new increasing volume due to the motion of boundary is bounded by (7.36)

\[ |G^a_{0i}(t_0, t) \setminus g^b_{0i}(t_0, t)| \leq \]

\[ \int_{t_0}^{t} \frac{a}{2(T - h)^2} \int_{G^b(h) \setminus G^a(h)} (1 + \lambda_1 H(x, h))^2 \left( \frac{a}{2(T - h)^2} + |\partial_h|u^{1-\mu_i}(x, h)| \right) dx \]

\[ \leq \int_{t_0}^{t} \frac{a}{2(a - b)(T - h)} |G^b(h) \setminus G^a(h)| (1 + \lambda_1 H(x, h))^2 dh \]

\[ + \int_{t_0}^{t} \frac{(T - h)}{a - b} dh \int_{G^b(h) \setminus G^a(h)} |\partial_h|u^{1-\mu_i}(x, h)| (1 + \lambda_1 H(x, h))^2 dx. \]

Furthermore, for \( x \in \partial G^a(h) \), the level set expression of the mean curvature is (7.37)

\[ H(x, h) = \frac{\Delta_x |u^{1-\mu_i}(x, h)|}{|\nabla_x |u^{1-\mu_i}(x, h)||} - \frac{\nabla_x |u^{1-\mu_i}(x, h)| |\nabla_x^2 |u^{1-\mu_i}(x, h)| |\nabla_x |u^{1-\mu_i}(x, h)|}{|\nabla_x |u^{1-\mu_i}(x, h)||^3} \]

Let

\[ \Omega(h) := \{ x \in G^b(h) : |\nabla_x |u^{1-\mu_i}(x, h)|| \geq \frac{b}{(T - h)^{\frac{1}{2} + a_1}} \}. \]

Clearly in \( \Omega(h) \) (7.38)

\[ |H(x, h)| \leq \frac{4(T - h)^{\frac{1}{2} + a_1}}{b} |\nabla_x^2 |u^{1-\mu_i}(x, h)|| \leq \frac{4(T - h)^{\frac{1}{2} + a_1}}{b} \left( |\nabla_x^2 |u^{1-\mu_i}(x, h)|| + \frac{|\nabla_x |u^{1-\mu_i}(x, h)|^2}{|u^{1-\mu_i}(x, h)||} \right) \]

\[ \leq \frac{4(T - h)^{\frac{1}{2} + a_1}}{b} \left( |\nabla_x^2 |u^{1-\mu_i}(x, h)|| + \frac{(T - h)^{\frac{1}{2}}}{b} |\nabla_x |u^{1-\mu_i}(x, h)|^2 \right). \]

On the other hand, for \( x \in \partial G^{a+}(h) \setminus \Omega(h) \),

\[ |u^{1-\mu_i}(y, h)| = |u^{1-\mu_i}(x, h)| - \lambda_1 |\nabla |u^{1-\mu_i}(x, h)|| + |x - y|^2 \max_{x' \in G^{b/2}(h) \setminus G^a(h)} |\nabla^2 |u^{1-\mu_i}(x', h)|| \]

\[ \geq \frac{b}{(T - h)^{\frac{1}{2}}}, \quad \forall |y - x| \leq \lambda_1 \]

provided (7.39)

\[ \max_{x' \in G^{b/2}(h) \setminus G^a(h)} |\nabla^2 |u^{1-\mu_i}(x', h)|| \leq \frac{b}{4(T - h)^{\frac{1}{2} + 2a_1}}. \]

So we have (7.40)

\[ \partial G^{a+}_{\lambda_1}(h) \setminus \Omega(h) \subset G^{b/2}(h), \]

\[ \cup_{t_0 \leq t \leq \epsilon} \partial G^{a+}_{\lambda_1}(h) \setminus \Omega(h) \subset (\cup_{t_0 \leq t \leq \epsilon} G^{b/2}(h)) \subset (G^{b/2}(t_0, t_1) \setminus g^{b/4}(t_0, t_1)) \cup G^{b/4}(t_0). \]

We discover
Lemma 7.10. Suppose \((7.11)\)\((7.39)\) and for \(t_0 < t < T\) there is \(t_1 \in (t_0, T)\) such that
\[
\|u^{1-\mu^\alpha}(h)\|_{L^2} < B := \frac{1}{(2c_1)^\frac{9}{2} (2c_2)^{\frac{1}{4}}} \left(\frac{\lambda t}{\rho_\beta(t)}\right)^{\frac{3}{2} + \frac{1}{\beta}}, \quad t_0 \leq h \leq t_1
\]
as well as suppose
\[
\|\nabla^2 u^{1-\mu^\alpha}(t_0)\|_{L^2} < Z_0 := \frac{1}{(2c_1 B)^6}.
\]
Then for \(0 < b < a\), we have
\[
|G_{\lambda_1}(t_0, t_1) \setminus g^b(t_0, t_1)| \leq \frac{c_2^{\frac{3}{2}} (T-t)^{\frac{3}{2} - \frac{88}{20} (\frac{1}{2} - \beta)}}{b(a-b)(c_1)^{\frac{1}{2}}} (T-t_0)^{1+a_1}
\]
for \(a < 2\).

Moreover
\[
|G_{\lambda_1}(t_0, t_1) \setminus g^b(t_0, t_1)| \leq 2|G_{\lambda_1}(t_0, t_1) \setminus g^b(t_0, t_1)| + \frac{c_2^{\frac{3}{2}} (T-t)^{\frac{3}{2} - \frac{88}{20} (\frac{1}{2} - \beta)}}{b(a-b)(c_1)^{\frac{1}{2}}} (T-t_0)^{1+a_1}
\]

\[
+ 2 \int_{t_0}^{t} \frac{a}{2(a-b)(T-t)} \int_{G^b(h) \setminus G^a(h)} (1 + \lambda_1 H(x, h))^2 dx dh
\]
\[
+ \int_{t_0}^{t} \frac{(T-t)}{a-b} dh \int_{G^b(h) \setminus G^a(h)} |\partial_h |u^{1-\mu^\alpha}(x, h)||(1 + \lambda_1 H(x, h))^2 dx.
\]

8. Assumption 1 is satisfied by scaling

We see that for \(T - t\) small so that
\[
(T-t)^\beta \leq \frac{1}{2} + \alpha, \quad \text{(Assumption 2)}
\]
we have
\[
(T - t)^{\frac{3}{2}} \int_{\{\eta \leq \frac{1}{(T-t)^{\frac{1}{2}} + 2\alpha}\}} ((T-t)^{\frac{1}{2}} |\eta| )^{2(1+2\alpha)} |\mathcal{F}[u](\eta, t)|^2 d\eta + 2\|u^{1-\mu^\alpha}(t)\|_{L^2} \leq (T-t)^{\frac{3}{2} + 2\alpha} \frac{1}{46} ||u^{1-\mu^\alpha}(t)||_{L^2}.
\]
Regularity of solutions to the Navier-Stokes equations

For fixed \( \alpha \in (0, \frac{1}{8}) \), if we take \( \beta \in (0, \frac{1}{2}) \) such that
\[
(T - t)^{\beta(\frac{1}{2} + 2\alpha) - \frac{1}{2}} \|u^\beta(t)\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{2}, \quad \text{(Assumption 3)}
\]
and
\[
(T - t)^{-\frac{1}{4}} \|u^{1 - \mu}(t)\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{2}, \quad \text{(Assumption 4)}
\]
then, (3.11) is satisfied at \( t \).

8.1. initial conditions. Suppose the constant \( \delta > 0 \) (see Proposition 3.2) is small enough and is fixed from now on, and suppose the initial data
\[
\|u_0\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{2}.
\]
Recall from the energy inequality for the Leray-Hopf solutions
\[
\|u(T)\|_{L^2}^2 + \frac{1}{2} \int_t^T \|\nabla_x u(h)\|_{L^2}^2 dh \leq \|u(t)\|_{L^2}^2,
\]
that there is \( h_j \uparrow T \) such that
\[
(T - h_j)^{\frac{1}{2}} \|\nabla_x u(h_j)\|_{L^2} \to 0, \quad \text{as } j \to \infty.
\]
We find
\[
\|u(h_j)\|_{L^2} \leq \|u_0\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{2}, \quad \|\nabla u(h_j)\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{(T - h_j)^{\frac{1}{2}}}.
\]
Simply we assume
\[
\|u_0\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{2}, \quad \|\nabla u_0\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{T^{\frac{1}{3}}} \quad \text{(Assumption 5.0)}
\]
Notice that (1.11) is invariant under the scaling
\[
u_\lambda(z, s) := \lambda u(\lambda z, \lambda^2 s), \quad p_\lambda(z, s) := \lambda^2 p(\lambda z, \lambda^2 s)
\]
for all \( \lambda \in (0, \infty) \).
Observe that if (3.11) failed then
\[
T^{\frac{1}{4}} < \frac{\|u_0\|_{L^2}}{\delta^\frac{1}{2}} \leq \frac{1}{2}.
\]
For all \( \gamma_1 \in (0, \frac{1}{4}) \) we have \( \lambda_\gamma := \left(\frac{1}{T^{\frac{1}{4}} + \gamma_1}\right)^{\frac{1}{\gamma_1}} = \frac{1}{(T^{\frac{1}{4}} + \gamma_1)^{\frac{1}{\gamma_1}}} \quad (> 2^{\frac{1}{\gamma_1}} - 2 > 1) \) and
\[
T^{\frac{1}{4} + \gamma_1}_{\lambda_\gamma} \|u_\lambda(0)\|_{L^2} = \lambda^{-2\gamma_1} T^{\frac{1}{4} + \gamma_1} \|u_0\|_{L^2} \leq \frac{\delta^\frac{1}{2}}{2}
\]
where
\[
T_{\lambda_\gamma} = \lambda^{-2} T \quad \text{(< 2^{\frac{1}{\gamma_1}} \quad \to 0 \quad \text{as } \gamma_1 \to 0)}
\]
is the new blow-up time for \( u_{\lambda_t} \). On the other hand,

\[
\|u_{\lambda}(0)\|_{L^2} = \lambda_{\gamma}^{-1/2}\|u_0\|_{L^2} \leq \lambda_{\gamma}^{-1/2}\frac{\delta^2}{2} \leq \left( \frac{1}{2} \right)^{1/\gamma} \frac{\delta^2}{2}, \quad (\to 0 \text{ as } \gamma_1 \to 0)
\]

\[
\|\nabla u_{\lambda}(0)\|_{L^2} = \lambda_{\gamma}^{1/2}\|\nabla u_0\|_{L^2} \leq \frac{\delta^2}{2} = \frac{\delta^2}{\lambda_{\gamma}^{1/2}T^{1/\gamma}} = \frac{\delta^2}{T_{\lambda_{\gamma}}^{1/\gamma_1}}.
\]

Without loss of generality we assume

\[
\begin{cases}
T < \left( \frac{1}{2} \right)^{1/\gamma_1}, \\
\|u_0\|_{L^2} \leq \left( \frac{1}{2} \right)^{1/\gamma_1} \frac{\delta^2}{2}, \quad \|\nabla u_0\|_{L^2} \leq \frac{\delta^2}{T_{\lambda_{\gamma}}^{1/\gamma_1}}, \quad (\text{Assumption 5})
\end{cases}
\]

\[
T^{1/\gamma_1} \|u_0\|_{L^2} \leq \frac{\delta^2}{2} \quad \text{for all } \gamma_1 \in (0, \frac{1}{100}).
\]

8.2. scaling. For \( t \in (0, T) \), for \( \beta \in (0, \frac{1}{2}) \), note that at \( t_\lambda := \lambda^{-2}t \)

\[
\|u_{\lambda(t_\lambda)}\|_{L^2} = \lambda^{1/\beta} \left( \int_{\{\eta\geq 1\}} \left( 1 - \varphi(2^{1/\beta}(T-t)^{1/\beta} |\eta|) \right)^2 |\mathcal{F}[u](\eta, t)|^2 d\eta \right)^{1/2}.
\]

From now on we take

\[
\lambda = (T-t)^{\frac{1}{2} - \frac{\alpha_1}{\alpha}} \quad \alpha_1 \in (4\beta, 2).
\]

Observe that

\[
\begin{align*}
\text{for } \alpha_1 > 4\beta: & \quad |\eta| \geq \frac{1}{\lambda^{2\beta}(T-t)^{1/\beta}} = \frac{1}{(T-t)^{\frac{1}{2} - \frac{2\beta}{\alpha}} - \text{as } T-t \to 0,} \\
& \quad (T_{\lambda} - t_{\lambda}) = (T-t)^{\frac{1}{\gamma_1}} \to 0 \quad \text{as } T-t \to 0, \\
\text{for } \alpha_1 < 2: & \quad \lambda = (T-t)^{\frac{1}{2} - \frac{1}{\alpha_1}} = (T_{\lambda} - t_{\lambda})^{\frac{1}{1 - \beta}} \to \infty \quad \text{as } T-t \to 0.
\end{align*}
\]

Lemma 8.1. Suppose assumption 5, and take \( a, \alpha, \alpha_1 \) and \( \beta \) as well as \( \beta \) such that

\[
\begin{cases}
(1) \quad a \in \left( \frac{1}{2}, \frac{1}{\alpha_1} - \frac{\delta^2}{2}, \frac{3}{1250C_1} \right), \quad \alpha \in (0, \frac{1}{8}), \quad \beta \in \left[ \max\left\{ \frac{\alpha_1}{4(1+4\alpha)}, \frac{1}{9} \right\}, \frac{1}{2} \right), \\
(2) \quad \alpha_1 \in \left( \frac{400}{99}, \min\{\frac{100}{23}, \frac{1306}{323}\}\beta \right), \\
(3) \quad \beta \in \left( \frac{-1 - \alpha_1}{4 - 2\alpha_1}, \frac{-1(\alpha_1 - 4\beta)}{8(1 - \beta) - 2\alpha_1} \right),
\end{cases}
\]

Let \( T^* = T^{1+4\gamma_1} \) and

\[
t = T - (T - T^*)^\frac{1-2\beta}{\alpha_1(\beta - \beta)}.
\]
Regularity of solutions to the Navier-Stokes equations

Then $T^* < t < T$ and at $t_\lambda = t/\lambda^2$, assumption 2, 3 and 4 are satisfied for $u_\lambda(z,t_\lambda)$ provided $\gamma_1 \in (0, \frac{1}{16})$ is small enough. That is, assumption 1 in \(3.11\) is satisfied at $\tau^\beta = -\ln(T_\lambda - t_\lambda)$ where

$$w_\lambda(y, \tau) := (T_\lambda - t_\lambda)^{\frac{1}{2}} u_\lambda(z, t_\lambda), \quad y := \frac{1}{(T_\lambda - t_\lambda)^{\frac{1}{2}}} z, \quad \tau := -\ln(T_\lambda - t_\lambda).$$

**Remark 8.2.** For example, we may take

$$\alpha = \frac{1}{16}, \quad \alpha_1 = \frac{4002}{990}, \quad \beta = \frac{100}{21}, \quad \bar{\beta} \in \left(\frac{-21}{8868}, \frac{-21}{17778}\right).$$

**Proof.** Step 1. Recall from Proposition 3.2 if assumption 1 failed then

$$T^{\frac{1}{\bar{\gamma}_1}} \|u_0\|_{L^2} > \delta^{\frac{1}{2}}.$$

Furthermore from assumption 5

$$T < \left(\frac{1}{2}\right)^{\frac{1}{\bar{\gamma}_1}}. \quad \tag{8.5}$$

On the other hand for $\frac{-(\alpha_1 - 4\beta)}{4 - 2\alpha_1} < \bar{\beta} < \beta$, we have

$$1 - 2\bar{\beta} - \frac{4}{\alpha_1}(\beta - \bar{\beta}) > 0, \quad \frac{1 - 2\bar{\beta}}{1 - 2\beta - 4} > 1.$$

We find

$$T^* < t < T$$

and

$$T - t = (T - T^*)^{\frac{1 - 2\beta}{\alpha_1}(\beta - \bar{\beta})} \geq \left(1 - \frac{1}{2^4}\right)^{\frac{1 - 2\beta}{\alpha_1}(\beta - \bar{\beta})} \|u_0\|_{L^2} \left(\frac{1 - 2\bar{\beta}}{1 - 2\beta - 4}\right)^{\frac{1 - 2\beta}{\alpha_1}(\beta - \bar{\beta})} \quad \tag{8.6}$$

$$\geq \|u_0\|_{L^2} \left(\frac{1 - 2\bar{\beta}}{1 - 2\beta - 4}\right)^{\frac{1 - 2\beta}{\alpha_1}(\beta - \bar{\beta})}\left(1 - \frac{1}{2^4}\right)^\frac{2}{\delta^{\frac{1}{2}}} \geq \|u_0\|_{L^2} \left(\frac{1 - 2\bar{\beta}}{1 - 2\beta - 4}\right)^{\frac{1 - 2\beta}{\alpha_1}(\beta - \bar{\beta})} \left(1 - \frac{1}{2^4}\right)^\frac{2}{\delta^{\frac{1}{2}}} \geq 1.$$ 

Step 2. Recall from assumption 5: $\|\nabla u_0\|_{L^2} \leq \frac{\delta^{\frac{1}{2}}}{T^{\frac{1}{2} + \gamma_1}}$, and Proposition 5.1 with $T^* = T^{1+4\gamma_1}$ that

$$||\mathcal{F}[\omega](T^*)||_{L^2(|n| \geq 1/(T-t)^{\frac{1}{2} - 2\beta})} \leq \frac{1}{2M(T^*)^{\frac{1}{2}}} \text{exp}[-\frac{\sqrt{T^*}}{(T-t)^{\frac{1}{2} - 2\beta}}] \quad \tag{8.7}$$

$$= \frac{1}{2M(T^*)^{\frac{1}{2}+\gamma_1}} \text{exp}\left[-\frac{T^{\frac{1}{2}+2\gamma_1}}{(T-t)^{\frac{1}{2} - 2\beta}}\right].$$
which exponentially decays as $T \downarrow 0$ provided
\[
\frac{(\frac{1}{2} - \frac{2\alpha}{\alpha_1})(1 - 2\beta)}{1 - 2\beta - \frac{1}{\alpha_1}(\beta - \beta)} > \frac{1}{2} + 2\gamma_1,
\]
or equivalently
\[
\beta \in \left( \frac{-(\alpha_1 - 4\beta)}{4 - 2\alpha_1}, \frac{-(\gamma_1)(\alpha_1 - 4\beta)}{(1 - 2\beta) + 2\gamma_1(2 - \alpha_1)} \right). 
\]
Note that (8.8) follows from assumption 6(3).

We discover
\[
\|u_{\lambda}^{1-\mu\beta}(T^*_{\lambda})\|_{L^2} \leq (T_{\lambda} - t_{\lambda})^\frac{1}{2} - \beta \|\nabla_z u_{\lambda}^{1-\mu\beta}(T^*_{\lambda})\|_{L^2} = (T_{\lambda} - t_{\lambda})^\frac{1}{2} - \beta \frac{2}{\lambda} \|F[\omega](T^*)\|_{L^2(\eta \geq 1/(T - T^*)^{\frac{1}{2} - \frac{2\beta}{\lambda}})}
\]
\[
\leq \frac{(T_{\lambda} - t_{\lambda})^\frac{1}{2} - \beta \frac{2}{\lambda}}{2M_2 T^* + \gamma_1} \exp[\frac{-T_{\lambda}^\frac{1}{2} + 2\gamma_1}{(T - T^*)^{\frac{1}{2} - \frac{2\beta}{\lambda}} - \frac{1}{\alpha_1}(1 - 2\beta)}]
\]
\[
\leq \frac{\delta^\frac{1}{2}}{4}(T_{\lambda} - t_{\lambda})^\frac{1}{2}
\]
provided $\gamma_1 > 0$ is small enough (so $T$ is small enough), where (8.2) (8.4) (8.5) (8.7) (8.8) and assumption 5-6 were used.

Step 3. (8.2) (8.4) imply
\[
(T_{\lambda} - t_{\lambda})^{1-2\beta} = (T_{\lambda} - T^*_{\lambda})^{1-2\beta}
\]
and
\[
t_{\lambda} - T^*_{\lambda} = t_{\lambda} - T_{\lambda} + (T_{\lambda} - t_{\lambda})^{1-2\beta} = A(t)(T_{\lambda} - t_{\lambda})^{1-2\beta} \leq (T_{\lambda} - t_{\lambda})^{1-2\beta}
\]
where
\[
A(t) := 1 - (T_{\lambda} - t_{\lambda})^{2(\frac{\beta - \beta}{1-2\beta})} \in (0, 1).
\]

During $s \in [T^*_{\lambda}, T_{\lambda})$, we recall from the $u_{\lambda}$ version of (7.1) that
\[
\frac{1}{2} \frac{d}{ds}\|u_{\lambda}^{1-\mu\beta}(s)\|_{L^2}^2 = (-)\|\nabla_z u_{\lambda}^{1-\mu\beta}(s)\|_{L^2}^2 - < u_{\lambda}^{1-\mu\beta}(s) \partial_j u_{\lambda}^{1-\mu\beta}(s) > u_{\lambda}^{1-\mu\beta}(s) + < \nabla_z u_{\lambda}^{1-\mu\beta}(s) >
\]
\[
- < \nabla_z \left( u_{\lambda}^{1-\mu\beta} \partial_j u_{\lambda}^{1-\mu\beta} \right) > (s), \nabla_z^{-1} u_{\lambda}^{1-\mu\beta}(s) >
\]
\[
- < \nabla_z \left( u_{\lambda}^{1-\mu\beta} \partial_j u_{\lambda}^{1-\mu\beta} \right) > (s), \nabla_z^{-1} u_{\lambda}^{1-\mu\beta}(s) >
\]
\[
50
\]
We find

\begin{equation}
\frac{d}{ds} \|u_\lambda^{1-\mu} (s)\|_{L^2}^2 \leq (-2) \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2}^2 \\
+ c_{II} \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))} \|u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))} \|u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))}
\end{equation}

\begin{equation}
+ c_{II} \sum_{m_1+m_2=1} \|\nabla_z u_\lambda^{1-\mu} \cap (1-\mu) (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))}
\times \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))} \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))}
\end{equation}

\begin{equation}
+ c_{II} \sum_{m_1+m_2=2} \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))} \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))} \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))}
\times \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))} \|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2 (G_{c-L_1}^\alpha (t_0, t_\lambda))}
\end{equation}

where \( m_1 = 0, 1 \). Note

\begin{equation}
\|\nabla_z u_\lambda^{1-\mu} (s)\|_{L^2} \geq (T_\lambda - t_\lambda)^{\beta-\frac{1}{2}} \|u_\lambda^{1-\mu} (s)\|_{L^2}.
\end{equation}

In view of (8.9), the right side of (8.12) at \( T_\lambda^* \) is negative, and we have this lemma provided the right side of (8.12) for \( s \in (T_\lambda^*, t_\lambda) \) is negative.

Step 4. Take

\begin{equation}
\lambda_1 = (T_\lambda - T_\lambda^*)^{a_1}, \quad \text{for } \frac{1}{2} \leq a_1 < 1.
\end{equation}

Recall that

\begin{equation}
(T_\lambda - t_\lambda)^{\frac{3}{2} (\beta-\frac{1}{2})} \|u_\lambda^{1-\mu} (s)\|_{L^2} \leq (T_\lambda - t_\lambda)^{\frac{3}{2} (\beta-\frac{1}{2})} \lambda^{-\frac{1}{2}} \|u_0\|_{L^2} \leq \frac{a}{2(T_\lambda - T_\lambda^*)^{\frac{3}{2}}}, \quad \forall s \in (T_\lambda^*, t_\lambda)
\end{equation}

provided \((s_0 = T_\lambda^*)\)

\begin{equation}
\left\{ \begin{array}{c}
\left(\frac{1}{2}\right)^{\frac{1}{\alpha_1}} - \delta^{\frac{1}{2}} \leq a, \\
\beta \geq \frac{(-)(\alpha_1 - 4\beta)}{24\beta - 8 - 2\alpha_1}.
\end{array} \right.
\end{equation}

The second condition in (8.15) follows from (8.8) since for \( \beta \in (0, \frac{1}{2}) \)

\begin{equation}
\frac{(-)(\alpha_1 - 4\beta)}{4 - 2\alpha_1} > \frac{(-)(\alpha_1 - 4\beta)}{24\beta - 8 - 2\alpha_1}.
\end{equation}

Note also that (8.15) follows from assumption 6(1)(3).
In light of Lemma [7.4] (2) then we see for \( s \in [T^*_\lambda, t_\lambda] \)
\[
\| \nabla_z u_{\lambda_{\lambda}}^{\mu_{\lambda}}(s) \|_{L^2(G_{\lambda_{\lambda}}^{a_{\lambda}}(T^*_\lambda, t_\lambda))}
\]
\[
\leq \| \nabla_z u_{\lambda}(T^*_\lambda) \|_{L^2(G_{\lambda_1}^{a_{\lambda}}(T^*_\lambda, t_\lambda))} + C_2 \| u_{\lambda}(T^*_\lambda) \|_{L^2} \left( \frac{T^*_\lambda - T^*_\lambda}{T^*_\lambda - s} \right)^{\frac{C_1 a}{2}} \lambda_1^{-1}
\]
\[
\leq \frac{1}{2M_2} \lambda^{-2\gamma} T^{-\frac{4}{3} + \gamma} + C_2 \| u_{\lambda}(T^*_\lambda) \|_{L^2} \left( \frac{T^*_\lambda - T^*_\lambda}{T^*_\lambda - t_\lambda} \right)^{\frac{C_1 a}{2}} \lambda_1^{-1}
\]
\[
\leq \left( \frac{1}{2M_2} + C_2^2 \right) (T^*_\lambda - t_\lambda)^{\frac{1}{2} \left( 1 - \frac{a}{2} \right) - \frac{a}{4} (a_1 - \frac{a}{2}) \left( \frac{1 - 2b}{1 - 2\gamma} \right) + (\frac{1}{2} - \gamma) \left( \frac{a}{2} + 1 \right) \frac{1 - 2b}{1 - 2\gamma} ) =: K_1(s)
\]
and
\[
\| \nabla_z u_{\lambda}^{\mu_{\lambda}}(s) \|_{L^2(G_{\lambda_{\lambda}}^{a_{\lambda}}(T^*_\lambda, t_\lambda))} \leq \| \nabla_z u_{\lambda}^{\mu_{\lambda}}(s) \|_{L^2} |G_{\lambda_1}^{a_{\lambda}}(T^*_\lambda, t_\lambda)|^{\frac{1}{2}}
\]
\[
\leq \left( \frac{2}{b} \right) (T^*_\lambda - T^*_\lambda)^{\frac{1}{2}} \| \nabla_z u_{\lambda}^{\mu_{\lambda}}(s) \|_{L^2} \| u_{\lambda}^{1 - \mu_{\lambda}}(s) \|_{L^2} |G_{\lambda_1}^{a_{\lambda}}(T^*_\lambda, t_\lambda)|^{\frac{1}{2}} \| \nabla_z u_{\lambda}^{\mu_{\lambda}}(s) \|_{L^2}
\]
\[
=: K_2(s) + K_3(s)
\]
provided (8.15) are satisfied, where \( C_1 \) and \( C_2 \) are universal constants, and

\[
G^a(s) := \{ z \in \mathbb{R}^3 : |u_{\lambda}(z, s)| > \frac{a}{(T^*_\lambda - s)^{\frac{1}{2}}} \}, \quad \forall s \in (T^*_\lambda, t_\lambda)
\]

\[
G^a(T^*_\lambda, t_\lambda) := \bigcup_{T^*_\lambda < s \leq t_\lambda} G^a(s), \quad G^{a_{\lambda}}(T^*_\lambda, t_\lambda) := \mathbb{R}^3 \setminus G^a(T^*_\lambda, t_\lambda)
\]

\[
G_{\lambda_{\lambda}}^{a_{\lambda}}(T^*_\lambda, t_\lambda) := \{ z \in G^{a_{\lambda}}(T^*_\lambda, t_\lambda), \quad \text{dist}(z, G^a(T^*_\lambda, t_\lambda)) > \lambda_1 \} = \mathbb{R}^3 \setminus \bigcup_{T^*_\lambda < s \leq t_\lambda} G^a(s).
\]

Sometimes we also use \( G^a \) and \( G^{a_{\lambda}} \) as well as \( g^a \) instead of \( G^a(t_0, t] \) and \( G^{a_{\lambda}}(t_0, t] \) as well as \( g^a(t_0, t] \).

Suppose (8.8). Then \( \| \nabla_z u_{\lambda}^{1 - \mu_{\lambda}}(T^*_\lambda) \|_{L^2}^2 \) decays exponentially as \( (T^*_\lambda - t_\lambda) \downarrow 0 \). Furthermore suppose (8.15). Then

\[
\| \nabla_z u_{\lambda}^{1 - \mu_{\lambda}}(T^*_\lambda) \|_{L^2}^2 < Z_0, \quad G^b(T^*_\lambda) = \emptyset.
\]

Suppose for \( s \in (T^*_\lambda, s_1) \)
\[
\| u_{\lambda}^{1 - \mu_{\lambda}}(s) \|_{L^2} \leq \frac{\delta^2}{4} (T^*_\lambda - t_\lambda)^{\frac{1}{2}}.
\]

Then for \( s \in (T^*_\lambda, s_1) \)
\[
\| u_{\lambda}^{1 - \mu_{\lambda}}(s) \|_{L^2} < B
\]
provided

\[
(8.16) \quad \beta \geq \frac{1}{9}.
\]
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On the other hand, utilize (7.8) and Lemma (7.1)
\[
\|\nabla_z^2 u_\lambda^{1-\mu_\lambda} (s)\|_{L^\infty} \leq \|\nabla_z^{3+\frac{3}{2}} u_\lambda^{1-\mu_\lambda} (s)\|_{L^2} \leq [\text{Re}^{1-\mu_\lambda} (s)]^{\frac{3}{2}} \|\nabla_z^4 u_\lambda^{1-\mu_\lambda} (s)\|_{L^2}^\frac{5}{2}
\]
\[
\leq (T_\lambda - t_\lambda)^{\frac{3}{2} \left(\frac{1}{2} - \frac{\mu_\lambda}{\lambda} + \frac{\mu_\lambda}{\lambda} - \frac{\beta}{\text{Re}^{\alpha_1}}\right)}.
\]
Similarly
\[
\|\nabla_z u_\lambda^{1-\mu_\lambda} (s)\|_{L^\infty} \leq \|\nabla_z^2 u_\lambda (s)\|_{L^2}^{\frac{3}{2}} [\text{Re}^{1-\mu_\lambda} (s)]^{\frac{3}{2}}
\]
\[
\leq (T_\lambda - t_\lambda)^{\frac{3}{2} \left(\frac{1}{2} - \frac{\mu_\lambda}{\lambda} + \frac{\mu_\lambda}{\lambda} - \frac{\beta}{\text{Re}^{\alpha_1}}\right)}.
\]
We can check that (7.39) is satisfied provided
\[
(8.17) \quad \alpha_1 < \frac{1306}{323} \beta.
\]
Furthermore,
\[
(8.18) \quad \lambda_1 |H(z, s)| \lesssim 1
\]
provided (8.17) and
\[
(8.17) \quad \alpha_1 < \frac{1902}{465} \beta.
\]
Clearly (8.17) implies (8.18).

Recall Lemma (7.9) and Lemma (7.10) and note that the 3rd and 4th term in the right side of (7.41) (the terms where \(H\) is involved) are less than \(|G^a_{\lambda_1} \setminus g^b|\) provided (8.17). We discover
\[
|G^a_{\lambda_1} (T^*_{\lambda_1}, s_1) \setminus g^b(T^*_{\lambda_1}, s_1)| \leq \frac{c_2 (T_\lambda - t_\lambda)^{\frac{3}{2} \left(\frac{1}{2} - \frac{\mu_\lambda}{\lambda} + \frac{\mu_\lambda}{\lambda} - \frac{\beta}{\text{Re}^{\alpha_1}}\right)}}{b(a-b)(\alpha_1)\frac{3}{2}} (T_\lambda - T^*_{\lambda_1})^{1+\alpha_1} + \frac{a(t_\lambda - T^*_{\lambda_1})}{2(a-b)b^2} (T_\lambda - t_\lambda)^{\frac{5}{2}}
\]
\[
+ \frac{(T_\lambda - T^*_{\lambda_1})^{\frac{5}{2}}}{b(a-b)} (T_\lambda - t_\lambda)^{-\frac{1}{2} + \frac{\alpha_1}{2}} (u_\lambda (0))^{\frac{3}{2}} (T_\lambda - t_\lambda)^{\frac{3}{2}} (T_\lambda - t_\lambda)^{\frac{5}{2}} + \frac{(T_\lambda - T^*_{\lambda_1})^{\frac{1}{2}}}{a-b} (u_\lambda (0))^{\frac{3}{2}} (T_\lambda - t_\lambda)^{\frac{3}{2}} (T_\lambda - t_\lambda)^{\frac{5}{2}}
\]
\[
\lesssim (T_\lambda - t_\lambda)^{\frac{3}{2} \left(\frac{1}{2} - \frac{\mu_\lambda}{\lambda} + \frac{\mu_\lambda}{\lambda} - \frac{\beta}{\text{Re}^{\alpha_1}}\right) + (1+\alpha_1)\frac{1-\beta}{1-2\beta} + \frac{3}{4} \left(\frac{1-\alpha_1}{1-2\beta}\right) + (T_\lambda - t_\lambda)^{\frac{5}{2} + \frac{1-2\beta}{1-2\beta}} + (T_\lambda - t_\lambda)^{\frac{5}{2} + \frac{\alpha_1}{4} + \frac{3}{2}}
\]
\[
+ (T_\lambda - t_\lambda)^{\frac{1}{2} + \frac{\alpha_1}{4} + \frac{1}{2} + \frac{3}{4} \left(\frac{1-\alpha_1}{1-2\beta}\right) + \frac{3}{2} \left(\frac{1-\alpha_1}{1-2\beta}\right)} =: |G^a_{\lambda_1} \setminus g^b|
\]
where we utilize
\[
c_2 \leq C \|u_\lambda (0)\|_{L^2}^{\frac{5}{2}}, \quad (\text{see Lemma (7.1)}
\]
\[
0 < \frac{3}{8} - \frac{27}{28} \left(\frac{1}{2} - \beta\right) + (1+\alpha_1) \frac{1-2\beta}{1-2\beta} + \frac{3}{16} \left(\frac{1}{2} - \frac{\alpha_1}{4}\right)
\]
\[
- \left(\frac{1}{4} - \frac{5}{4} \left(\frac{1}{2} - \beta\right) + \frac{3}{2} \left(\frac{1}{2} - \frac{2\beta}{1-2\beta}\right) + \frac{3}{4} \left(\frac{1}{2} - \frac{\alpha_1}{4}\right)\right)
\]
\[
\approx -\frac{3}{224} - \frac{2}{7} \beta + \frac{9}{64} \alpha_1, \quad \Leftarrow (\alpha_1 > 4\beta, \quad \beta > \frac{3}{62}) \Leftarrow (8.16),
\]
and
\[ 0 < \frac{1}{2} + \frac{\beta}{2} - \frac{\alpha_1}{4} + \frac{1 - 2\beta}{1 - 2\beta} - \left( \frac{1}{4} - \frac{5}{4} \left( \frac{\beta}{2} - \beta \right) + \frac{3}{2} \left( \frac{1 - 2\beta}{1 - 2\beta} \right) + \frac{3}{4} \left( \frac{\beta}{2} - \frac{\alpha_1}{4} \right) \right) \]
\[ \approx \frac{3}{4} \beta - \frac{3}{16} \alpha_1, \quad \Leftrightarrow (4\beta < \alpha_1). \]

Claim 1. (1) Suppose (8.8) (8.15) (8.16) and for \( s \in (T^*_\lambda, s_1) \)
\[ \| u^{1-\mu^\beta}_\lambda (s) \|_{L^2} \leq \frac{\delta^{\frac{1}{2}}}{4} (T_\lambda - t_\lambda)^{\frac{1}{4}}. \]
Then
\[ |G^a_\lambda (T^*_\lambda, s_1) \setminus g^b (T^*_\lambda, s_1) | \lesssim (T_\lambda - t_\lambda)^{\frac{1}{4} - \frac{\beta}{4}} (\frac{1 - 2\beta}{1 - 2\beta} + \frac{3}{4} (\frac{1 - 2\beta}{1 - 2\beta} + \frac{3}{4} (\frac{\beta}{2} - \frac{\alpha_1}{4})) = |G^a_\lambda \setminus g^b|. \]
(2) Assumption 6(3) implies
\[ 1 - \frac{\alpha_1}{2} < \frac{1 - 2\beta}{1 - 2\beta} = 1 - \frac{\alpha_1}{2} + \theta (\frac{\alpha_1}{4} - \beta), \quad \text{for some } \theta \in (0, 1). \]
(3) For \( b = a/2, a_1 \downarrow \frac{1}{2} \) and \( \gamma_1 \) small enough, as well as
\[ (8.19) \quad 4\beta < \alpha_1 \leq \left( \frac{100}{23} \right) \beta, \]
we have
\[ K_2 \leq K_1, \quad K_3 \leq K_1. \]
Moreover
\[ |G^a_\lambda \setminus g^b| \leq (T_\lambda - t_\lambda)^{\left( \frac{1 - \alpha_1}{4} + \frac{1}{4} (1 + a_1)^{\frac{1 - 2\beta}{2}} \right)}. \]
(4) Suppose
\[ \| \nabla u^{1-\mu^\beta}_\lambda (s_1) \|_{L^2} < K_1 \]
at some \( s_1 \in (T^*_\lambda, t_\lambda) \). Then for
\[ T^*_\lambda := s_1 + K_1^{-4} \]
we have
\[ \| \nabla u_\lambda (T^*_\lambda) \|_{L^2([\xi \leq (T_\lambda - t_\lambda)^{\frac{1}{4} - \beta}])} \leq \frac{1}{2M_2 (T^*_\lambda - s_1)^{\frac{1}{4}}} \exp \left[ \frac{(1 - 2a_1 - \beta + (a_1 - \frac{1}{2}) \frac{3}{10} \alpha_1 - (2a_1 - C_1 a - 2(\frac{1}{4} - \gamma_1)) \theta (\frac{\alpha_1}{4} - \beta)}{2M_2 (T^*_\lambda - s_1)^{\frac{1}{4}}} \right] \]
which decays to zero exponentially as \( T_\lambda - t_\lambda \to 0 \), provided
\[ (8.20) \quad 0 < (1 - 2a_1) - \beta + \left( a_1 - \frac{1}{4} - \frac{C_1 a}{2} \right) \alpha_1 - (2a_1 - C_1 a - 2(\frac{1}{4} - \gamma_1)) \theta (\frac{\alpha_1}{4} - \beta), \]
\[ (\alpha_1 > \frac{24\beta + 50C_1 a}{54} - \frac{2}{3} \frac{a_1}{C_1 a} \quad \text{for } a_1 \downarrow \frac{1}{2}, \quad \gamma_1 \in (0, \frac{1}{100}). \]
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Claim 1(4) follows from Proposition 5.1 where \( T_\lambda^{**} < T_\lambda \) because \( T_\lambda \) is supposed as the blow-up time, and Claim 1(3) follows from

\[
0 < (-) \frac{5}{2} \left( \frac{1}{2} - \beta \right) + \frac{1}{4} + \left( \frac{1}{2} + a_1 \right) \frac{1 - 2 \beta}{1 - 2 \beta} \\
\approx \frac{5}{2} \beta - \frac{a_1}{2}, \quad \Leftarrow \ (8.19)
\]

and

\[
0 < (-) \frac{5}{2} \left( \frac{1}{2} - \beta \right) + \frac{3}{2} \left( \frac{1}{2} - \frac{1}{4} \right) + \frac{3}{2} \left( \frac{1 - 2 \beta}{1 - 2 \beta} \right) + \frac{3}{2} \left( \frac{1}{2} - \frac{a_1}{4} \right) \\
+ \frac{25}{8} \beta - \left( \frac{23}{32} - \frac{C_1 a}{4} \right) \alpha_1, \quad \Leftarrow \ (8.19)
\]

as well as

\[
0 < \frac{1}{4} - \frac{5}{4} \left( \frac{1}{2} - \beta \right) + \frac{3}{2} \left( \frac{1 - 2 \beta}{1 - 2 \beta} \right) + \frac{3}{4} \left( \frac{1 - a_1}{4} \right) - \frac{1}{2} \left( \frac{1 - \alpha_1}{4} \right) - \frac{1}{2} + \frac{1 - 2 \beta}{1 - 2 \beta} \\
\approx \frac{1}{4} - \frac{5}{4} \left( \frac{1}{2} - \beta \right) - \frac{1}{4} \left( \frac{1}{2} - \frac{a_1}{4} \right) + \frac{1}{2} \left( 1 - \frac{a_1}{4} \right) = \frac{5}{4} \beta - \frac{3}{10} \alpha_1, \quad \Leftarrow \ (8.19).
\]

Step 5. In light of Claim 1(4), we suppose

\[
\| \nabla_z u_\lambda^{1-\mu_\lambda^\beta}(s_1) \|_{L^2} \geq K_1.
\]

Otherwise, for \( T_\lambda^{**} < t_\lambda \) we can replace \( T_\lambda^{**} \) by \( T_\lambda^{**} \), and for \( T_\lambda^{**} \geq t_\lambda \) we can get Lemma 5.1 by replacing \( t_\lambda \) by \( T_\lambda^{**} \) and noting that

\[
T_\lambda - t_\lambda \geq T_\lambda - T_\lambda^{**} = T_\lambda - s_1 - K_1^{-4} > T_\lambda - t_\lambda - K_1^{-4},
\]

\[
\| u_\lambda^{1-\mu_\lambda^\beta}(T_\lambda^{**}) \|_{L^2} \leq \| u_\lambda^{1-\mu_\lambda^\beta}(T_\lambda^{**}) \|_{L^2} \leq \frac{\delta_1}{4} (T_\lambda - T_\lambda^{**})^{\frac{1}{2}}.
\]

In view of (7.13), (7.15), similarly we have

\[
\| u_\lambda^{\mu_\lambda^\beta}(s) \|_{L^\infty(G_{\alpha_\lambda}^\beta(t^* \lambda,T_\lambda),t^* \lambda))} \leq \| \psi_1 u_\lambda^{\mu_\lambda^\beta}(s) \|_{L^\infty} \\
\leq \| \psi_1 u_\lambda^{\mu_\lambda^\beta}(s) \|_{L^\infty} + (T_\lambda - t_\lambda)^{\frac{1}{2}} \phi^{1-\beta}(F_{\alpha_\lambda^\beta}^{-1}[\varphi',(\frac{\lambda_1}{\rho_\beta(t_\lambda)})] \ast (\psi_1 u) \|_{L^\infty} \\
\leq (T_\lambda - t_\lambda)^{\frac{1}{2}} \| \psi_1 u_\lambda^{\mu_\lambda^\beta}(s) \|_{L^2} + C(T_\lambda - t_\lambda)^{\frac{1}{2}} \lambda_1^{-1} \| u_\lambda(s) \|_{L^2} \\
\leq (T_\lambda - t_\lambda)^{\frac{1}{2}} \frac{1}{4} (4-\beta) K_1
\]

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and
\[(8.22)\]
\[
\|\nabla_z \mu^\beta_{\Lambda} (s)\|_{L^\infty(G^{a, c}_{\Lambda_1}(T^*_\lambda, t_\lambda))} \leq \|\psi_1 \nabla_z \mu^\beta_{\Lambda} (s)\|_{L^\infty}
\]
\[
\leq \|\nabla_z (\psi_1 u_{\Lambda})^{\alpha}_{\mu^\beta_{\Lambda}} (s)\|_{L^\infty} + \|F_{\eta}^{-1} \left[ \frac{\lambda_{t_{\Lambda}}}{\rho_{\beta}(t_{\Lambda})} \eta \right] \| (\psi_1 u_{\Lambda})\|_{L^\infty}
\]
\[
\lesssim (T_{\lambda} - t_{\lambda})^{-\frac{1}{2} - \beta} \left( \|\psi_1 \nabla_z \mu^\beta_{\Lambda} (s)\|_{L^2} + \lambda_{1}^{-1} \|\mu^\beta_{\Lambda} (s)\|_{L^2(G^{a, c}(T^*_\lambda, t_\lambda))}\right)
\]
\[
\lesssim (T_{\lambda} - t_{\lambda})^{-\frac{3}{2} - \beta} K_1,
\]
as well as
\[
\|\nabla_z \mu^{\alpha}_{\Lambda} (1-\mu^\beta_{\Lambda}) (s)\|_{L^2(G^{a, c}_{\Lambda_1}(T^*_\lambda, t_\lambda))} \leq \|\psi_1 \nabla_z \mu^{\alpha}_{\Lambda} (1-\mu^\beta_{\Lambda}) (s)\|_{L^2}
\]
\[
\leq \|\nabla_z (\psi_1 u_{\Lambda})^{\alpha}_{\mu^{\alpha}_{\Lambda}} (1-\mu^\beta_{\Lambda}) (s)\|_{L^2} + (T_{\lambda} - t_{\lambda})^{\frac{1}{2} - \beta} \|F_{\eta}^{-1} \left[ \frac{\lambda_{t_{\Lambda}}}{\rho_{\beta}(t_{\Lambda})} \eta \right] \| (\psi_1 \nabla_z u_{\Lambda})\|_{L^2}
\]
\[
\lesssim (T_{\lambda} - t_{\lambda})^{-(m-1)\left(\frac{1}{2} - \beta\right)} \|\psi_1 \nabla_z \mu^{\alpha}_{\Lambda} (1-\mu^\beta_{\Lambda}) (s)\|_{L^2} + (T_{\lambda} - t_{\lambda})^{-(m-1)\left(\frac{1}{2} - \beta\right)} \lambda_{1}^{-1} \|u_0\|_{L^2}
\]
\[
\lesssim (T_{\lambda} - t_{\lambda})^{-(m-1)\left(\frac{1}{2} - \beta\right)} K_1,
\]
and
\[
\|\nabla_z \mu^{\alpha}_{\Lambda} (s)\|_{L^3(G^{a, c}_{\Lambda_1}(T^*_\lambda, t_\lambda))} \leq \|\psi_1 \nabla_z \mu^{\alpha}_{\Lambda} (s)\|_{L^3}
\]
\[
\leq \|\nabla_z (\psi_1 u_{\Lambda})^{\alpha}_{\mu^{\alpha}_{\Lambda}} (s)\|_{L^3} + \|T_{\lambda} - t_{\lambda}\|^{\frac{1}{2} - \beta} \lambda_{1}^{-1} \|\nabla_z \mu^{\alpha}_{\Lambda} (1-\mu^\beta_{\Lambda}) (s)\|_{L^3}
\]
\[
\lesssim (T_{\lambda} - t_{\lambda})^{-\frac{m}{2} + \beta} \|\psi_1 \nabla_z \mu^{\alpha}_{\Lambda} (s)\|_{L^2} + (T_{\lambda} - t_{\lambda})^{-(m-\beta)\left(\frac{1}{2} - \beta\right)} \lambda_{1}^{-1} \|u_0\|_{L^2}
\]
\[
\lesssim (T_{\lambda} - t_{\lambda})^{-\frac{m}{2} + \beta} K_1.
\]

In view of (8.9) (8.14) (8.21) (8.22), the right of (8.12) is negative provided
\[(8.23)\]
\[
\|\nabla_z \mu^{\alpha}_{\Lambda} (s)\|_{L^2} \lesssim (T_{\lambda} - t_{\lambda})^{-\frac{3}{2} - \beta} \left( \|\psi_1 \nabla u_{\Lambda}\|_{L^2} \right)
\]
\[
+ \lambda_{1}^{-1} \|\mu^{\alpha}_{\Lambda} (s)\|_{L^2(G^{a, c}(T^*_\lambda, t_\lambda))} \right) \|\mu^{\alpha}_{\Lambda} (s)\|_{L^2(G^{a, c}_{\Lambda_1}(t_{\lambda}, t_\lambda))} \right) \|\mu^{\alpha}_{\Lambda} (s)\|_{L^2(G^{a, c}_{\Lambda_1}(t_{\lambda}, t_\lambda))} \right)
\]
\[
=: J_1(s) + J_2(s)
\]
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and

\[(8.24)\]
\[\|\nabla_z u_\lambda^{1-\mu_\lambda}(s)\|^2_{L^2} \gtrsim\]
\[(T_\lambda - t_\lambda)^{-2(1/2-\beta)}\|u_\mu_\lambda(s)\|_{L^\infty(G_{\lambda_1}^a)} \|\nabla_z^{-1} u_\lambda^{1-\mu_\lambda}(s)\|_{L^6(G_{\lambda_1}^a(t_0, t_\lambda))} \|u_\lambda^{\mu_\lambda \wedge (1-\mu_\lambda)}(s)\|_{L^\infty(G_{\lambda_1}^a(t_0, t_\lambda))}\]
\[\left(|G_{\lambda_1}^a| + |G_{\lambda_1}^a \setminus g^b|\right) \]
\[=: J_3(s) + J_4(s)\]

where

\[
\left(\int_{G_{\lambda_1}^a} dz\right)^{2/6} \leq \left(\int_{y_\mu_\lambda} u_\lambda^{1-\mu_\lambda}(z, s)^2 dz\right)^{2/6} \left(\frac{T_\lambda - T_\lambda^*}{b^2}\right)^{2/6} + |G_{\lambda_1}^a \setminus g^b|^{2/6} \]
\[=: L_1 + L_2,\]

and

\[
\left(\int_{G_{\lambda_1}^a} dz\right)^{2/6} \leq \left(\int_{y_\mu_\lambda} u_\lambda^{1-\mu_\lambda}(z, s)^2 dz\right)^{2/6} \left(\frac{T_\lambda - T_\lambda^*}{b^2}\right)^{2/6} + |G_{\lambda_1}^a \setminus g^b|^{2/6} \]
\[=: L_3 + L_4.\]

First note

\[
J_1 \lesssim (T_\lambda - t_\lambda)^{-(1/2-\beta)} K_1\|u_\mu_\lambda(s)\|^2_{L^2} \leq K_1(T_\lambda - t_\lambda)^{-1(1/2-\beta)}\|u_\lambda^{1-\mu_\lambda}(s)\|_{L^2} \lesssim \|\nabla_z u_\lambda^{1-\mu_\lambda}(s)\|^2_{L^2} \]
provided

\[(8.25)\]
\[\beta > 0.\]

Next note that \(J_2 \lesssim K_1^2\) provided

\[0 < -\frac{1}{2} \left(\frac{1}{2} - \beta\right) + \frac{1}{4} = \frac{\beta}{2},\]

that is \((8.25)\) again.

On the other hand,

\[(8.26)\]
\[J_3 L_1 := (T_\lambda - t_\lambda)^{-2(1/2-\beta)}\|u_\mu_\lambda(s)\|_{L^\infty(G_{\lambda_1}^a)} \|\nabla_z^{-1} u_\lambda^{1-\mu_\lambda}(s)\|_{L^6(G_{\lambda_1}^a(t_0, t_\lambda))} \|u_\lambda^{\mu_\lambda \wedge (1-\mu_\lambda)}(s)\|_{L^\infty(G_{\lambda_1}^a(t_0, t_\lambda))}\]
\[\times \left(\int_{y_\mu_\lambda} u_\lambda^{1-\mu_\lambda}(z, s)^2 dz\right)^{2/6} \left(\frac{T_\lambda - T_\lambda^*}{b^2}\right)^{2/6} \]
\[\lesssim (T_\lambda - t_\lambda)^{-5(1/2-\beta)}\|u_\lambda(s)\|_{L^2} \|u_\lambda^{1-\mu_\lambda}(s)\|_{L^2}^{1+\frac{2}{3}} \left(\frac{T_\lambda - T_\lambda^*}{b^2}\right)^{5/6}
\]
\[J_3 L_1 \text{ is controlled by } \|\nabla_z u_\lambda^{1-\mu_\lambda}(s)\|^2_{L^2} > K_1^2 \text{ provided}
\]
\[0 < -5\left(\frac{1}{2} - \beta\right) + \frac{1}{4} \left(1 + \frac{5}{3}\right) + \left(\frac{5}{6} + 2a_1 - C_1 a\right) \left(\frac{1 - 2\beta}{1 - 2\beta}\right) + C_1 a
\]
\[\approx 5\beta - \left(\frac{11}{12} - \frac{c_1 a}{2}\right)\alpha_1.\]
That is $J_3 L_1$ is controlled by $\|\nabla_z u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2_z}$ provided

$$(8.27) \quad \alpha_1 < \frac{5\beta}{\frac{9}{2} - \frac{C_1 a}{2}}.$$ 

On the other hand,

$$J_4 L_3 = (T_\lambda - t_\lambda)^{-\left(\frac{1}{2} - \beta\right)} \|\mu_1^\lambda(s)\|_{L^\infty(G_1^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2(G_2^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^6(G_3^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^6(G_4^\lambda)}$$

$$\times \left(\int_{G^\alpha} |u_{\lambda}^{1-\mu_1^\lambda}(z, s)|^2 dz\right)^{\frac{1}{2}} \left(\frac{T_\lambda - T_\lambda^*}{a^2}\right)^{\frac{1}{2}}$$

$$\lesssim (T_\lambda - t_\lambda)^{-\frac{5}{2}(\frac{1}{2} - \beta)} \|\mu_1^\lambda(s)\|_{L^2} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2(G_3^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2} \left(\frac{T_\lambda - T_\lambda^*}{a^2}\right)^{\frac{1}{2}}.$$ 

$J_4 L_3$ is controlled by $\|\nabla_z u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2_z} > K_1 \|\nabla_z u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2_z}$ provided

$$0 < -\frac{5}{2} \left(\frac{1}{2} - \beta\right) + \frac{1}{4} \left(1 + \frac{2}{3}\right) + \frac{1}{3} \left(\frac{1}{2} - 2\beta\right) + C_1 a + \left(a_1 - \frac{C_1 a}{2}\right) \left(\frac{1}{2} - 2\beta\right)$$

$$\approx \frac{5}{2} \beta - (\frac{1}{6} + \frac{1}{4} - \frac{C_1 a}{2}) \alpha_1$$

as $a_1 \downarrow \frac{1}{2}$. That is

$$(8.28) \quad \alpha_1 < \frac{5\beta}{\frac{9}{2} - \frac{C_1 a}{2}}.$$ 

Step 6. On the other hand,

$$J_3 L_2 \lesssim (T_\lambda - t_\lambda)^{-5(\frac{1}{2} - \beta)} \|\mu_1^\lambda(s)\|_{L^2} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2(G_1^\lambda \setminus g^b)^{\frac{1}{6}})},$$

and $J_3 L_2$ is controlled by $\|\nabla_z u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2_z} > K_2$ provided

$$0 < \frac{1}{4} - 5\left(\frac{1}{2} - \beta\right) + \frac{5}{6} \left(\frac{1}{2} + a_1\right) \left(\frac{1}{2} - 2\beta\right) + \frac{C_1 a}{2} + 2(\alpha_1 - \frac{C_1 a}{2}) \left(\frac{1}{2} - 2\beta\right)$$

$$\approx 5\beta - \left(\frac{5}{12} + \frac{5}{24} + \frac{1}{2} - \frac{C_1 a}{2}\right) \alpha_1$$

as $a_1 \downarrow \frac{1}{2}$. That is

$$(8.29) \quad \alpha_1 < \frac{5\beta}{\frac{9}{2} - \frac{C_1 a}{2}}.$$ 

Furthermore

$$J_4 L_4 \lesssim (T_\lambda - t_\lambda)^{-\left(\frac{1}{2} - \beta\right)} \|\mu_1^\lambda(s)\|_{L^\infty(G_1^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^2(G_2^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^6(G_3^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^6(G_4^\lambda)} \|u_{\lambda}^{1-\mu_1^\lambda}(s)\|_{L^6(G_5^\lambda)} \|G_{\lambda_1}^a \setminus g^b\|^{|\beta|}.$$
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and \( J_4 L_4 \) is controlled by \( \| \nabla z u^{1-\mu_1}(s) \|_{L^2}^2 > K_1 \| \nabla z u^{1-\mu_1}(s) \|_{L^2} \) provided

\[
0 < -\frac{5}{2} \left( \frac{1}{2} - \beta \right) + \frac{1}{4} + \frac{1}{3} \left( \frac{1}{2} + a_1 \right) \left( \frac{1}{2} - \beta \right) + \frac{1}{3}  \left( \frac{1}{2} - \frac{\alpha_1}{4} \right) + \frac{C_1 a}{2} + (a_1 - \frac{C_1 a}{2}) \left( 1 - \frac{2\beta}{1 - 2\beta} \right)
\]

\[
\approx \frac{5\beta}{2} - \left( \frac{1}{2} - \frac{C_1 a}{4} \right) \alpha_1
\]
as \( a_1 \downarrow \frac{1}{2} \). That is

\[
(8.30) \quad \alpha_1 < \frac{5\beta}{1 - C_1 a}
\]

For \( C_1 a < \frac{3}{250} \) we have \( (8.20) (8.27) (8.28) (8.29) (8.30) \). For \( \gamma_1 \) small enough, Assumption 6 implies \( (8.8) (8.15) (8.16) (8.17) (8.19) (8.20) (8.27) (8.28) (8.29) (8.30) \).

\[\square\]

8.3. proof of theorem 1.1. Because (1.1) as well as \((T-t)^{-\frac{1}{2}} \| u(t) \|_{L^2} \) are invariant under the scaling

\[
u_\lambda (x,t) := \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda (x,t) := \lambda^2 p(\lambda x, \lambda^2 t)
\]

for all \( \lambda \in (0, \infty) \), for any \( u_0 \in L^2(\mathbb{R}^3) \) satisfies assumption 5, there is \( \lambda \in (0, \infty) \) such that for \( u_\lambda \), assumption 1 in Proposition 3.2 is satisfied at some \( \tau_0 \). From Lemma 4.1-4.3 and Proposition 3.2 as well as former results on the estimate from below for \( \| \nabla z u_\lambda(s) \|_{L^2} \) near the blow-up or the Serrin regularity condition of Leray-Hopf solutions (see, for example, [19] [35] [45] [52]), we discover that \( u_\lambda \) can be extended smoothly over \( T_\lambda \). Finally we proved Theorem 1.1.

References

[1] J. Bergh, J. Lofstrom, Interpolation spaces-an introduction, Springer-Verlag (1976).
[2] L. Brandolese, Atomic decomposition for the vorticity of a viscous flow in the whole space, Math. Nachr., 273, 28-42 (2004).
[3] L. Caffarelli, R. Kohn, L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math. 35, 771-831 (1982).
[4] M. Cannone, Ondelettes, paraproduits et Navier-Stokes, Diderot Editeur (1995).
[5] M. Cannone, F. Planchon, On the regularity of the bilinear term for solutions to the incompressible Navier-Stokes equations, Rev. Mat. Iberoamericana, 16, N.1, 1-16 (2000).
[6] J-Y. Chemin, Perfect incompressible fluids, Oxford Science Publications (1998).
[7] Z.M. Chen, W.G. Price, Blow-up rate estimates for weak solutions of the Navier-Stokes equations, Proc. R. Soc. Lond. A, 457, 2625-2642 (2001).
[8] P. Constantin, Geometric statistics in turbulence, SIAM Review, 36, 73-73 (1994).
[9] P. Constantin, C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, Indiana Univ. Math. J., 42(3), 775-789 (1993).
[10] P. Constantin, C. Foias, Navier-Stokes equations, Univ. of Chicago Press (1988).
[11] H. Dong, D. Du, The Navier-Stokes equations in the critical Lebesgue space, Commun. Math. Phys., (2009).
[12] L. Escauriaza, G. Seregin, V.Šverák, $L_{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness, Uspekhi Mat. Nauk., 58, 3-44(2003), Russian Math. Survays, 58:2, 211-250(2003).
[13] E.B. Fabes, B.F. Jones, N.M. Rivere, The initial value problem for the Navier-Stokes equations with data in $L^p$, Arch.Rat.Mech.Anal. 45, 222-240(1972).
[14] R. Farwig, H. Sohr, Optimal initial value conditions for the existence of local strong solutions of the Navier-Stokes equations , Math. Ann., 345, 631-642(2009).
[15] C.Foias, R.Temam, Some analytic and geometric properties of the solutions of the Navier-Stokes equations, J.Math.Pure Appl. 58, 339-368(1979).
[16] H. Fujita, T. Kato, On the Navier-Stokes initial value problem I, Arch.Rational Mech. Anal. 16,269-315(1964).
[17] G. Furioli, P-G. Lemarié-Rieusset, E. Terraneo, Sur l'unicité dans $L^3(\mathbb{R}^3)$ des solutions ‘mild’ des équations de Navier-Stokes, C.R.Acad.Sciences Paris, Série 1, 1253-1256(1997).
[18] M. Giaquinta, G. Modica, J. Souček, Cartesian currents in the calculus of variations I , Springer(1998)
[19] Y. Giga, Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system, J.Diff.Eqs., 62, 186-212(1986).
[20] Y. Giga, R. Kohn, Asymptotically self-similar blowup of semilinear heat equation, Comm. Pure Appl. Math. 38, 297-319(1985).
[21] Y. Giga, R. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math J. 36, 1-40(1987).
[22] Y. Giga, R. Kohn, Nondegeneracy of blowup for semilinear heat equations, Comm. Pure Appl. Math. 42, 845-884(1989).
[23] Y. Giga, S. Matsui, S. Sasayama, Blow up rate for semilinear heat equation with subcritical nonlinearity, Preprint(2003), Indiana Univ. Math. J. to appear.
[24] Y. Giga, S. Matsui, S. Sasayama, On blow up rate for sign-changing solutions in a convex domain, preprint(2003).
[25] Y.Giga, T. Miyakawa, Solutions in $L_r$ of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal, 89, 267-281(1985).
[26] Y. Giga, T. Miyakawa, Navier-Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, Comm. Part. Diff. Equus. 14, 577-618(1989).
[27] Y. Giga, H. Sohr, Abstract $L^p$ estimates for the cauchy problem with applications to the Navier-Stokes equations in exterior domains, J. Funct. Anal. 102, 72-94(1991).
[28] E. Hopf, Über die anfangsrufeaufgabe für die hydrodynamischen grundgleichungen, Math. Nachr., 4, 213-231(1951).
[29] T.Y. Hou, Blow-up or no blow-up? a unified computational and analytic approach to 3D incompressible Euler and Navier-Stokes equations, Acta Numerica, 1-70(2009)
[30] T. Kato, Strong $L^p$-solutions of the Navier-Stokes equation in $\mathbb{R}^m$, with applications to weak solutions, Math. Z. 187, 471-480(1984).
[31] H.Koch, D.Tataru, Well-posedns for the Navier-Stokes equations, Advances in Mathematics, 157, 22-35(2001).
[32] G. Koch, N. Nadirashvili, G.A. Seregin, V. Šverák, Liouville theorems for the Navier-Stokes equations and applications, Acta. Math. 203, 83-105(2009).
[33] H. Kim, H. Kozono, Interior regularity criteria in weak spaces for the Navier-Stokes equations, manuscripta math. 115, 85-100(2004).
[34] H. Kozono, Y. Taniuchi, Bilinear estimates in BMO and the Navier-Stokes equations, Math.Z, 235, 173-194(2000).
[35] J. Leray, Essai sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Math., 63, 193-248(1934).
[36] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem, Chapman & Hall(2002).
Regularity of solutions to the Navier-Stokes equations

[37] O.A. Ladyzenskaja, On the uniqueness and smoothing of generalized solutions to the Navier-Stokes equations, Sem.Math.V.A.Steklov.Math.Inst.Leningrad, 5, 60-66(1969).

[38] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural’ceva, Linear and quasi-linear equations of parabolic type, Vol.23, Trans.Math.Mono. AMS(1968).

[39] P.L. Lions, Mathematical Topics in Fluid Mechanics Vol.1. Clarendon Press.Oxford(1996).

[40] F. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, Comm. Pure Applied Math., 51, 241-257(1998).

[41] Y. Meyer, Wavelets, paraproducts, and Navier-Stokes equations, International Press, Cambridge, Massachusetts(1996).

[42] J.R. Miller, M. O’Leary, M. Schonbek, Nonexistence of singular pseudo-self-similar solutions of the Navier-Stokes system, Math. Ann. 319, 809-815(2001).

[43] G. Prodi, Un teorema di unicità per le equazioni di Navier-Stokes, Ann. Mat. Pura Appl., 48,173-182(1959).

[44] V. Scheffer, Partial regularity of solutions to the Navier-Stokes equations, Pacific J. Math. 66, 535-552(1976).

[45] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, Arch. Rational Mech. Anal., 9, 187-195(1962).

[46] H. Sohr, A regularity class for the Navier-Stokes equations in Lorentz spaces, J. Evol. Eq., 1, 441-467(2001).

[47] M. Struwe, On partial regularity results for the Navier-Stokes equations, Comm.Pure Appl.Math.41, 437-458(1988).

[48] H. Triebel, Interpolation theory, Function spaces, Differential operators, Johann Ambrosins Barth Verlag Heidelberg,Leipzig(1995).

[49] T.P. Tsai, On Leray’s self-similar solutions of the Navier-Stokes equations satisfying local energy estimates, Arch. Rational Mech. Anal. 143, 29-51(1998).

[50] W.von Wahl, The equations of Navier-Stokes and abstract parabolic equations, Braunschweig(1985).

[51] J. Zhai, Some estimates for the blowing up solutions of semilinear heat equations, Adv. Math. Sci. Appl. 6, 217-225(1996).

[52] J. Zhai, Regularity of Leray-Hopf solutions to Navier-Stokes equations (I)-Critical interior regularity in weak spaces, preprint(2006), arXiv:math.AP/0611958.