EXACT RATE OF DECAY FOR SOLUTIONS TO DAMPED SECOND ORDER ODE’S WITH A DEGENERATE POTENTIAL

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Abstract. We prove exact rate of decay for solutions to a class of second order ordinary differential equations with degenerate potentials, in particular, for potential functions that grow as different powers in different directions in a neigborhood of zero. As a tool we derive some decay estimates for scalar second order equations with non-autonomous damping.

1. Introduction. In this paper we study rate of convergence to equilibrium of solutions to second order ordinary differential equations of the type

\[ \ddot{u} + g(u)\dot{u} + \nabla E(u) = 0, \]  

which describe damped oscillations of a system. We assume that the potential energy \( E : \mathbb{R}^n \rightarrow \mathbb{R}_+ \) has its only local minimum in the origin and \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) is positive (except in the origin), so the term \( g(\dot{u})\dot{u} \) has a damping effect.

The scalar case with \( E(u) = a|u|^p \) and \( g(s) = b|s|^\alpha \) was studied by Haraux [9] and the vector valued case with \( E(u) = \|A^2 u\|^p \) and \( g(s) = (c_1|s|^{\alpha_1}, c_2|s|^{\alpha_2}), A \) being a symmetric positive linear operator on a Hilbert space \( H \) was studied by Abdelli, Anguiano and Haraux [1]. For these cases exact decay rates were derived. Let us mention, that in both cases \( E \) satisfies

\[ c\|\nabla E(u)\| \leq E(u)^{1-\theta} \leq C\|\nabla E(u)\| \]

and \( \|\nabla^2 E(u)\| \leq \|\nabla E(u)\|^{1-2\theta} \) on a neighborhood of zero. The right inequality in (1) is called the Lojasiewicz gradient inequality. Let us mention that the potential functions \( E \) from [9], [1] satisfy (1) and also the condition on \( \nabla^2 E \) with \( \theta = \frac{1}{p} \).

In [5] similar decay estimates as in [9], [1] were derived with the assumptions formulated in terms of the Lojasiewicz gradient inequality, namely for \( E \) satisfying

\[ c\|\nabla E(u)\| \leq E(u)^{1-\theta} \leq C\|\nabla E(u)\| \]

The goal of this paper is to study degenerate cases, where the above assumptions do not hold, e.g. the behavior of \( E \) is not power-like or \( E \) does not satisfy the left

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inequality in (1) with the same \( \theta \) as the right inequality\(^1\). A prototype of such \( E \) is
\[
E(u) = \|u_1\|^{p_1} + \cdots + \|u_k\|^{p_k}
\]  
with \( u = (u_1, \ldots, u_k) \in \mathbb{R}^n \) (\( u_i \in \mathbb{R}^{n_i} \) are not necessarily scalars, \( \sum u_i = n \)) and \( p_1 \geq p_2 \geq \cdots \geq p_k \geq 2 \) are not all equal. We show that in such cases we obtain the same estimates (from above and from below) as for \( E(u) = \|u\|^{p_1} \).

Further, we study the exact decay for the case where \( u_i \) in (2) are scalars. In the case studied in [9] and [1] the authors have shown that if \( \alpha > 1 - \frac{2}{p} \) (i.e. the damping function is smaller than a threshold), then the solutions oscillate and all solutions converge to the origin with the same speed. On the other hand, if \( \alpha < 1 - \frac{2}{p} \) (the damping function is larger than the threshold), then the solutions do not oscillate and there appear solutions with exactly two rates of convergence called fast solutions and slow solutions (see also [2] for existence of slow solutions). We show similar results for the degenerate case, in particular we show that for \( E \) given by (2) with \( u_i \) being scalars, at most \( n + 1 \) speeds of convergence occur (depending on \( p_i \)’s).

While studying the exact decay for solutions to (DP) we look at the equations for single coordinates of \( u \)
\[
\ddot{u}_i + g(\dot{u}) \dot{u}_i + p\|u_i\|^{p_i-2}u_i = 0, \quad i = 1, 2, \ldots, n.
\]
Since we assume \( E \) to be in the special form (2) (a slightly more general case is considered below), these equations are coupled only by the term \( g(\dot{u})u_i \). So, we consider these coordinate equations as non-autonomous problems
\[
\ddot{u}_j + g_j(\dot{u}_j, t) \dot{u}_j + E(u_j) = 0,
\]
where the dependence of \( g \) on other coordinates \( \dot{u}_i, i \neq j \) is hidden in the dependence on \( t \), in particular, \( g_j \) is defined by
\[
g_j(s, t) = g((\dot{u}_1(t), \ldots, \dot{u}_{j-1}(t), s, \dot{u}_{j+1}(t), \ldots, \dot{u}_n(t))).
\]
Therefore, we also give results on decay and oscillations for non-autonomous equations of the type (3) that may be of interest on their own. The results for \( \alpha < 1 - \frac{2}{p} \) are again similar to those in [9], [1]. Decay estimates for another type of non-autonomous damping were derived in [3], [7], [10].

The paper is organized as follows. In Section 2 we present basic definitions and assumptions valid throughout the rest of the paper. Section 3 is devoted to the scalar autonomous problems and Section 4 to scalar non-autonomous problems. The results in this section are based on comparison with the autonomous case. The degenerate vector-valued problem (DP) is studied in Section 5.

2. Basic definitions and preliminaries. In this paper we study three types of equations: the scalar autonomous problem
\[
\ddot{u} + g(\dot{u}) \dot{u} + E'(u) = 0, \quad \text{(AP)}
\]
the scalar non-autonomous problem
\[
\ddot{u} + g(\dot{u}, t) \dot{u} + E'(u) = 0, \quad \text{(NP)}
\]
\(^1\)Some decay estimates for even more general \( E \) satisfying only the Lojasiewicz inequality were obtained in [8], [6] and [4] but these estimates are in many cases not optimal and it is an open question, whether they are optimal at least for some problems.
and the degenerate vector valued problem (DP). The assumption on \( g \in C(\mathbb{R}) \) for (AP), resp. \( g \in C(\mathbb{R}^n) \) for (DP) is

\[
c_g |s|^\alpha \leq g(s) \leq C_g |s|^\alpha
\]  

(G)

(here \(|s|\) denotes the Euclidean norm of \( s \) if \( s \in \mathbb{R}^n \)). In the non-autonomous case we assume only \( g \in C(\mathbb{R} \times \mathbb{R}^+) \),

\[
c_g |s|^\alpha \leq g(s,t)
\]  

(Gn)

for some \( \alpha \in (0,1), c_g, C_g > 0 \) and all \( s \) in any bounded set (with \( c_g, C_g \) depending on the set), and all \( t \geq 0 \) in case of (Gn). The potential function \( E \in C^2(\mathbb{R}) \) in (AP), (NP) is assumed to satisfy

\[
c_E |s|^p \leq E(s) \leq C_E |s|^p, \quad c_E |s|^p \leq E'(s)s \leq C_E |s|^p
\]  

(E)

for some \( p \geq 2, c_E, C_E > 0 \) and all \( s \) in a bounded neighbourhood of the origin. In case of (DP) we assume \( E \in C^2(\mathbb{R}^n) \) is of the form

\[
E(u) = E_1(u_1) + \cdots + E_n(u_n),
\]

where \( u = (u_1, u_2, \ldots, u_n) \), \( E_i \in C^2(\mathbb{R}) \) satisfy (E) with exponents \( p_i \) respectively, and \( p_1 \geq p_2 \geq \cdots \geq p_n \geq 2 \).

By a solution to (AP), (NP), (DP) we always mean a classical solution defined on \( \mathbb{R}^+ \). If \( u \) (resp. \( (u_1, \ldots, u_n) \)) is a solution to one of these equations, then \( v \) (resp. \( (v_1, \ldots, v_n) \)) always denotes its velocity, i.e. \( v = \dot{u} \) (resp. \( v_i = \dot{u}_i, i = 1, \ldots, n \)). We denote

\[
\mathcal{E}(u, v) = \frac{1}{2} \|v\|^2 + E(u).
\]

This function is non-increasing along solutions since

\[
\frac{d}{dt} \mathcal{E}(u(t), v(t)) = \langle v(t), \dot{v}(t) \rangle + \langle \nabla E(u(t)), v(t) \rangle = -g(v(t)) \|v(t)\|^2 \leq 0,
\]

whenever \( u \) is a solution to any of the studied equations. Sometimes, we write \( \mathcal{E}(t) \) instead of \( \mathcal{E}(u(t), v(t)) \).

If \( \alpha \geq 1 - \frac{2}{p} \) (\( p, \alpha \) from (E), (Gn), (G)), we speak about the oscillatory case, otherwise we speak about the non-oscillatory case. In the non-oscillatory case, we say that the solution \( u \) is a fast solution if it converges to zero and \( \lim_{t \to +\infty} \|v(t)\|_{\mathcal{E}(u(t))} = +\infty \) (i.e. the kinetic energy is much bigger than the potential energy of \( u \) as \( t \) tends to infinity). On the other hand, \( u \) is called a slow solution if it converges to zero and \( \lim_{t \to +\infty} \|v(t)\|_{\mathcal{E}(u(t))} = 0 \).

Let us now present two easy lemmas that show that the fast solutions converge to zero faster than slow solutions and how the speed of convergence depend on the trajectory in the uv plane, i.e. on the ratio of \( \|u(t)\| \) and \( \|v(t)\| \). Let \( X(a, b) = \{u \in C^2((a, b)) : u > 0 \text{ on } (a, b)\} \). By trajectory of \( u \) we mean the function \( V_u : u(t) \mapsto v(t) \), i.e. \( V_u(x) = v(u^{-1}(x)), x \in (u(a), u(b)) \), where \( v = \dot{u} \).

**Lemma 2.1.** Let \( a < x < y < b \) and let \( u_1, u_2 \in X(a, b) \) with \( V_{u_1} \geq V_{u_2} \) on \( [x, y] \). Then \( u_2 \) needs more time than \( u_1 \) to get from \( x \) to \( y \), i.e. if \( u_1(t_1) = x = u_2(t_2) \) and \( u_1(s_1) = y = u_2(s_2) \), then \( s_1 - t_1 \leq s_2 - t_2 \). Moreover, \( u_2 \leq u_1 \) on \( (t_1, s_1) \) if \( t_1 = t_2 \). If, moreover, \( V_{u_1}(x) > V_{u_2}(x) \), then \( s_1 - t_1 < s_2 - t_2 \) and if \( t_1 = t_2 \), then \( u_2 < u_1 \) on \( (t_1, s_1) \).

**Proof.** We have for \( i = 1, 2 \)

\[
s_i - t_i = \int_{t_i}^{s_i} 1 dt = \int_{t_i}^{s_i} \frac{\dot{u}_i(t)}{v_i(t)} dt = \int_{t_i}^{s_i} \frac{\dot{u}_i(t)}{V_{u_i}(u(t))} dt = \int_x^y \frac{1}{V_{u_i}(u)} du.
\]
The assertion now follows easily from $V_{u_1} \geq V_{u_2}$ (resp. $V_{u_1} > V_{u_2}$ on a neighborhood of $x$).

\begin{proof}
Let $u \in X(0, +\infty)$ with $\lim_{t \to +\infty} (u(t), v(t)) = 0$. If $V_u(x) \geq cx^a$ on $(-\varepsilon, 0)$ for some $a > 1$, $\varepsilon > 0$, then $u(t) \leq c t^{-\frac{1}{a-1}}$ for some $c$ and all $t$ large enough.

If $V_u(x) \leq cx^a$ on $(-\varepsilon, 0)$ for some $a > 1$, $\varepsilon > 0$, then $u(t) \geq c t^{-\frac{1}{a-1}}$ for some $c$ and all $t$ large enough.

Proof. $V_u(x) \geq c|x|^a$ means $v(t) = \dot{u}(t) \geq c|u(t)|^a$. Dividing by $|u(t)|^a$ and integrating from $t_0$ to $t$ we get
\[ \frac{1}{1-a} \left( |u(t_0)|^{1-a} - |u(t)|^{1-a} \right) \geq c(t - t_0), \]

i.e.
\[ |u(t)| \leq \left( (a-1)c(t-t_0) + |u(t_0)|^{1-a} \right)^\frac{1}{1-a} \leq c t^{-\frac{1}{a-1}}. \]

The opposite estimate follows similarly. \qed

Finally, $f(t) \sim h(t)$ means that there exist $T$, $c$, $C > 0$ such that $cf(t) \leq h(t) \leq Cf(t)$ for all $t \geq T$. By
\[ |u(t)| \leq Cf(t), \quad |u(t)| \geq Cf(t), \quad |v(t)| \leq Cf(t), \quad |v(t)| \geq Cf(t) \]
in the theorems and lemmas below we mean that there exist $C > 0$, $T > 0$ such that the inequality holds for all $t \geq T$.

3. Scalar autonomous problem. In this section we study the autonomous problem (AP). We assume that $g$ satisfies (G) and $E$ satisfies (E) with $\alpha < 1 - \frac{2}{p}$, which is the non-oscillatory case. We first formulate the main result, Theorem 3.1. In fact, it is a minor generalization of results proved by Haraux in [9]. However, important are the lemmas below leading to the proof of the theorem. They are needed in the next section for investigation of the non-autonomous problem.

\begin{theorem}
Let $\alpha < 1 - \frac{2}{p}$, Then all solutions converge to zero and do not oscillate (e.g. $u$, $v$ change sign only finitely many times). Further, any solution to (AP) is either fast or slow. Moreover, every fast solution satisfies
\[ u(t) \sim t^{-\frac{1}{a-1}}, \quad v(t) \sim t^{-\frac{1}{p-1}} \]

and every slow solution satisfies
\[ u(t) \sim t^{-\frac{a+1}{p+2-\alpha}}, \quad v(t) \sim t^{-\frac{a+1}{p+2-\alpha} - 1} = t^{-\frac{p-1}{p-2+\alpha}}. \]

\end{theorem}

We first show that some sets are positively invariant for solutions of (AP), namely the sets $O_{\varepsilon, K}$, $N_{\varepsilon, K}$, $P_{\delta, \eta}$ defined below.

\begin{lemma}
Denote $\kappa_0 = \frac{g}{C_{\varepsilon}}$ and $\kappa = \frac{g}{C_K}$. Let $K \in (0, \kappa)$ and $\varepsilon > 0$ satisfy
\[ \varepsilon^{p-2-\alpha \kappa} \leq \left( \frac{C_{\kappa}(\alpha + 1)}{p-1} \right)^{1+\alpha} (\kappa - K)^{1+\alpha} K^{1-\alpha}. \]

Then the sets
\[ O_{\varepsilon, K} = \left\{ (u, v) \in \mathbb{R}^2 : -\varepsilon \leq u \leq 0, \ \kappa_0^{-1} |u|^{\frac{a+1}{a-1}} \leq v \leq K^{-\frac{1}{a-1}} |u|^{\frac{a+1}{a-1}} \right\} \]
\[ N_{\varepsilon, K} = \left\{ (u, v) \in \mathbb{R}^2 : -\varepsilon \leq u \leq 0, \ 0 \leq v \leq K^{-\frac{1}{a-1}} |u|^{\frac{a+1}{a-1}} \right\} \]
\end{lemma}
are positively invariant for solutions \( u \) of (AP). Moreover, any solution in \( O_{\varepsilon,K} \) is a slow solution, it enters the set \( N_{\varepsilon,K} \), and satisfies (5).

**Proof.** We show that the vectors \((\hat{u}, \hat{v})\) point into \( N \) resp. \( O \) (we omit the subscripts) if \((u, v) \in \partial N \) resp. \( \partial O \). For \( u = -\varepsilon, v \geq 0 \) and \( v = 0, u < 0 \) this is obvious. For \( v = \kappa_0 \frac{1}{\varepsilon} |u|^\frac{\alpha}{\alpha+1}, u < 0 \) this follows from
\[
\hat{v} = -g(v)v - E'(u) \geq -C_v |u|^{\alpha+1} + C_E |u|^{p-1} = 0 > \frac{d}{du} \kappa_0 \frac{1}{\varepsilon} |u|^\frac{\alpha}{\alpha+1}.
\]
It remains to investigate the upper part of the boundary, i.e. \( v(t) = K^{-\frac{1}{\alpha+1}} |u|^{\frac{\alpha}{\alpha+1}}, u < 0 \). Here we have
\[
\hat{v} \leq -c_g |v|^{\alpha+1} + C_E |u|^{p-1} \leq C_E \left( K - \frac{c_g}{C_E} \right) v^\alpha = C_E \left( K - \frac{c_g}{C_E} \right) K^{-\frac{1}{\alpha+1}} |u|^{\frac{\alpha(p-1)}{\alpha+1}}
\]
and
\[
\frac{d}{du} K^{-\frac{1}{\alpha+1}} |u|^{\frac{\alpha}{\alpha+1}} = -K^{-\frac{1}{\alpha+1}} \frac{p-1}{\alpha+1} |u|^{\frac{p-2-\alpha}{\alpha+1}}.
\]
Therefore, \( \hat{v} < \frac{d}{du} K^{-\frac{1}{\alpha+1}} |u|^{\frac{\alpha}{\alpha+1}} < 0 \) if and only if \( K \in (0, \kappa) \) and
\[
|u|^{\frac{p-2-\alpha}{p-1}} \leq \frac{\alpha+1}{p-1} K^{\frac{1}{p-1}} C_E \left( \frac{c_g}{C_E} - K \right)
\]
and the positive invariance is proved.

Since \( \frac{p-2-\alpha}{p-1} > \frac{\alpha}{\alpha+1} \) we have \( |u|^{\frac{\alpha}{\alpha+1}} < C |u|^{\frac{\alpha}{\alpha+1}} \) (for all \( u \) in a bounded set), and therefore any solution in \( O \) is slow. Moreover, if a solution \((u, v)\) belongs to \( O \setminus N \), then the functions \( u \) and \( v \) are increasing, and therefore the solution enters \( N \). By Lemma 2.2, \( u(t) \sim t^{-\frac{1}{p+\alpha}} \) and due to \( \kappa_0^{-\frac{1}{\alpha+1}} |u|^{\frac{\alpha}{\alpha+1}} \leq v \leq K^{-\frac{1}{\alpha+1}} |u|^{\frac{\alpha}{\alpha+1}} \) we have \( v(t) \sim t^{-\frac{1}{p+\alpha}} \) .

**Lemma 3.3.** There exist \( \delta, \eta > 0 \) such that the set
\[
P_{\delta,\eta} = \{(u,v) \in \mathbb{R}^2 : -\delta \leq u < 0, 0 \leq v \leq \eta |u|^{\frac{1}{p-1}}\}
\]
is positively invariant for (AP) and any solution to (AP) with \((u(t_0), v(t_0)) \in P_{\delta,\eta}\) for some \( t_0 > 0 \) is a slow solution and satisfies (5).

**Proof.** Let us define \( K(u) = C\varepsilon^{\frac{1}{p+\alpha}} |u|^{\frac{p-2-\alpha}{p-1}} \), where \( C = \left( \frac{c_g(1+\alpha)}{2(1-p)} \right)^{1+\alpha} \). Let \( \delta > 0 \) be such that \( K(u) \leq \delta |\varepsilon| \) for all \( u \in [-\delta, 0] \). Then for any \( u \in [-\delta, 0] \), inequality (6) holds with \((\varepsilon, K) = (u, K(u))\) and therefore (by Lemma 3.2) the set \( O_{-u,K(u)} \) is positively invariant. We have
\[
K(u)^{-\frac{1}{p+\alpha}} |u|^{\frac{1}{p-1}} = C^{-\frac{1}{p-1}} |u|^{-\frac{2-\alpha}{p-1}} |u|^{\frac{1}{p-1}} = C^{-\frac{1}{p-1}} |u|^{\frac{1}{p-1}}.
\]
Set \( \eta = C^{-\frac{1}{p-1}} \). If \( 0 \leq v(t_0) \leq \eta |u(t_0)|^{\frac{1}{p-1}} \), then \( 0 \leq v(t) \leq K(u(t_0))^{-\frac{1}{p+\alpha}} |u(t_0)|^{\frac{1}{p-1}} \)
\[i.e. \( (u(t_0), v(t_0)) \in O_{-u(t_0),K(u(t_0))} \). Then \((u(t), v(t)) \in O_{-u(t_0),K(u(t_0))}\) for all \( t \geq t_0 \) and by Lemma 3.2 it is a slow solution and satisfies (5).
Proof. Let $\delta, \eta$ be the constants form Lemma 3.3, then obviously $u(t_n) \geq -\delta$ for all $n$ large enough and
\[
v(t_n) \leq M|u(t_n)|^{\frac{p}{q}} = \eta|u(t_n)|^{\frac{1}{1-p}} \leq \frac{M}{\eta} |u(t_n)|^{\frac{p-2+\alpha}{\alpha}} \leq \eta|u(t_n)|^{\frac{1}{1-p}}\]
for large $n$ since $\frac{p-2+\alpha}{\alpha} > 0$.

**Proposition 3.5.** Let $u$ be a solution to (AP) satisfying $u < 0$, $v = \dot{u} > 0$ on $(T_0, +\infty)$ and $(u(t), v(t)) \to (0, 0)$. Then $u$ is either fast solution or slow solution. In the latter case, $u$ satisfies (5).

Proof. If $u$ is not a fast solution, then there exists $M > 0$ such that $\frac{v(t_n)}{|u(t_n)|^2} \leq M$ for a sequence $t_n \nearrow +\infty$. By Lemma 3.4 we have $(u(t_n), v(t_n)) \in P_{\delta, \eta}$ for large $n$. Hence $u$ is a slow solution by Lemma 3.3 and (5) holds. \hfill \Box

**Lemma 3.6.** Any fast solution $u$ of (AP) with $u < 0$, $v > 0$ on $(T, +\infty)$ satisfies (4).

Proof. By Lemma 3.3, any fast solution satisfies $v(t) > \eta|u(t)|^{\frac{1}{1-p}}$ for all $t$ sufficiently large. By Lemma 2.2, $u(t) \leq ct^{-\frac{1}{\alpha-p}}$. It follows that
\[
\dot{v} \leq -c_0 v^{\alpha+1} + C\varepsilon|u|^{p-1} \leq -kv^{\alpha+1} + C|v|^{(p-1)(1-\alpha)} \leq (-\kappa + \varepsilon)|v|^{1+\alpha}
\]
since $(p-1)(1-\alpha) = p - \alpha p - 1 + \alpha = p - 2\alpha p + (1 + \alpha) > 1 + \alpha$. By Lemma 2.2 we have $v(t) \leq Ct^{-\frac{\alpha}{p}}$. Since $u$ is a fast solution, we have $\mathcal{E}(t) \sim v(t)^2$ and due to $\mathcal{E}(t) \geq ct^{-\frac{\alpha}{p}}$ we have $v(t) \geq ct^{-\frac{\alpha}{p}}$. Now $v(t) \sim t^{-\frac{\alpha}{p}}$ and by integration we have $u(t) \sim t^{-\frac{1}{\alpha-p}}$. \hfill \Box

**Proof of Theorem 3.1.** Convergence to zero follows from Theorem 4.1 below and absence of oscillations follows from Proposition 4.4 below. Then any solution satisfies $u < 0$, $v > 0$ on $(T, +\infty)$ or symmetrically $u > 0$, $v < 0$. By Proposition 3.5, any solution is slow or fast and slow solutions satisfy (5). By Lemma 3.6, fast solutions satisfy (4). \hfill \Box

4. Nonautonomous damping. In this section we study the non-autonomous problem (NP). We keep the assumption (E) and assume that $g$ satisfies (Gn). We show that for $g$ bounded all solutions converge to zero and that they do not oscillate if $\alpha < 1 - \frac{2}{p}$. Then we study decay of the non-oscillatory solutions.

**Theorem 4.1.** Let $g$ satisfy (Gn) for some $\alpha \geq 0$ and $g(s, t) \leq M$ for all $s$ from a bounded set and all $t \geq 0$. Then any solution to (NP) converges to zero as $t \to +\infty$.

Proof. Let $u$ be a solution to (NP). Since $\frac{d}{dt}\mathcal{E}(u(t), v(t)) = -g(v(t), t)v^2(t) \leq 0$, it follows (due to $E(s) \geq c_E|s|^p$) that $(u(t), v(t))$ is bounded and the omega-limit set
\[
\omega(u, v) = \{ (\phi, \psi) \in \mathbb{R}^2 : \exists n \nearrow +\infty, u(t_n) \to \phi, v(t_n) \to \psi \}
\]
is nonempty. Let $(\phi, \psi) \in \omega(u, v)$. If $\psi \neq 0$, then $\frac{d}{dt}\mathcal{E}(u(t), v(t)) < -c_g|v(t)|^{\alpha+2} \leq -\varepsilon < 0$ for all $t$ such that $(u(t), v(t))$ belongs to a small neighborhood $N$ of $(\phi, \psi)$. Due to boundedness of $\dot{u}$, $\dot{v}$ the solution $(u, v)$ spends infinite time in $N$, which is a contradiction with boundedness of $\mathcal{E}(u(t), v(t))$ from below. So, $\psi = 0$. Since $\omega$ is connected, it is an interval $[a, b] \times \{0\}$. However, $\mathcal{E}$ is constant on $\omega$, hence $\omega$ is a singleton, i.e. $\lim u(t) = \phi$. 

Since $g(v(t), t)$ is bounded we have for $t \to +\infty$ $g(v(t), t)v(t) \to 0$. Since $E'(u(t)) \to E'(\phi)$ we get from (NP) $\dot{v}(t) \to -E'(\phi)$. Therefore, $E'(\phi) = 0$ (otherwise, we have a contradiction with $v(t) \to 0$). It follows by (E) that $\phi = 0$.

**Remark 4.2.** It can be seen from the proof of Theorem 4.1 that if we omit the assumption on boundedness of $g$, then we would still have $(u(t), v(t)) \to (\phi, 0)$. However, $\phi$ is not necessarily zero. In fact, we show that $u(t) = 1 + t^{-1}$ solves for $t$ large enough (NP) with

$$g(s, t) = |s|^{\alpha} + \max \left\{ 0, 2t^{-1} + t^{2}E(1 + t^{-1}) - t^{-2\alpha} \right\}.$$ 

Since $v(t) = -t^{-2}$ we have for large $t$ (such that $2t^{-1} + t^{2}E(1 + t^{-1}) - t^{-2\alpha} > 0$)

$$g(v(t), t)v(t) = (t^{-2\alpha} + 2t^{-1} + t^{2}E(1 + t^{-1}) - t^{-2\alpha})(-t^{-2}) = -2t^{-3} - E(1 + t^{-1}),$$

which is exactly $-\dot{v}(t) - E(u(t))$. Let us prove the following comparison lemma.

**Lemma 4.3.** Let $0 \leq g_{1}(s, t) < g_{2}(s, t)$ for any $s \in \mathbb{R}$, $t, \tilde{t} \geq 0$ and let $E'(s)s > 0$ for all $s \neq 0$. Let $u_{i}, i = 1, 2$ be, respectively, solutions to

$$\ddot{u}_{i} + g_{i}(\dot{u}_{i}, t)\dot{u}_{i} + E'(u_{i}) = 0, \quad i = 1, 2 \tag{8}$$

with $u_{1}(t_{0}) = u_{2}(t_{0}) < 0, v_{1}(t_{0}) = v_{2}(t_{0}) > 0$ for some $t_{0} \geq 0$. Let $t_{1} > t_{0}$ be such that $u_{1} < 0, v_{1} > 0$ on $(t_{0}, t_{1})$. Then $u_2 > 0$ and $u_2 < u_1$ on $(t_0, t_1)$ and the trajectories $V_{1}(x) = u_{1}(u_{1}^{-1}(x))$ satisfy $V_{2}(x) < V_{1}(x)$ on $(u_{1}(t_{0}), u_{2}(t_{1}))$.

**Proof.** Obviously, the solution $(u_{2}, v_{2})$ cannot cross the halfline $\{ u < 0, v = 0 \}$ since $v_{2}(t) = -E(u_{2}(t)) > 0$ on this halfline. So, $v_{2} = v_{2}$ stays positive as long as $u_{2} < 0$. Let $t_{2} = \sup \{ t \in [t_{0}, t_{1}] : u_{2} < 0 \text{ on } [t_{0}, t] \}$. Then either $t_{2} = t_{1}$ or $u(t_{2}) = 0$. Then trajectories $V_{1}$, resp. $V_{2}$ are well defined on $(u_{1}(t_{0}), u_{1}(t_{1}))$, resp. $(u_{2}(t_{0}), u_{2}(t_{2}))$. For $t \in [t_{0}, t_{1}]$ it holds that

$$V_{i}(u_{i}(t)) = \frac{\dot{u}_{i}(t)}{\dot{u}_{i}(t)} = \frac{-g_{i}(v_{i}(t), t)v_{i}(t) - E'(u_{i}(t))}{v_{i}(t)}. \tag{9}$$

So, if for any $s, t \in (t_{0}, t_{2})$ we have $u_{1}(t) = u_{2}(s), v_{1}(t) = v_{2}(s)$, then $V'_{2}(u_{2}(s)) < V'_{1}(u_{2}(s))$ (since $g_{1} < g_{2}$ and other terms in (9) are equal for $i = 1$ and $i = 2$). This leads to a contradiction (take the infimum of such $s$), and therefore $V_{2} < V_{1}$ on $(u_{1}(t_{0}), \min \{ u_{1}(t_{1}), u_{2}(t_{2}) \})$. It follows from Lemma 2.1 that $u_{2} < u_{1}$ on $(t_{0}, t_{2})$ and $t_{2} = t_{1}$.

**Proposition 4.4.** Let $g$ satisfy (Gn) for some $\alpha < 1 - \frac{2}{p}$ and let $u$ be a solution to (NP) such that $\lim_{t \to +\infty} u(t) = 0$. Then $u$ does not oscillate, i.e. $u, \dot{u}$ do not change sign on $(t_{0}, +\infty)$ for some $t_{0} \geq 0$.

**Proof.** Let us assume for contradiction that a solution $u$ to (NP) oscillates, i.e. there exists a sequence $t_{n} \nearrow +\infty$ such that $u(t_{n}) = 0$ or $v(t_{n}) = 0$. We show that for every $\varepsilon > 0$ there exists $T_{\varepsilon}$ such that $v(T_{\varepsilon}) = 0$ and $|u(T_{\varepsilon})| \leq \varepsilon$. In fact, if $v(t) \neq 0$ on some $(T_{\varepsilon}, +\infty)$, then $u$ would be monotone on $(T_{\varepsilon}, +\infty)$, and it would be a contradiction with existence of $t_{n}$. So, there exists a sequence $s_{n} \nearrow +\infty$ with $v(s_{n}) = 0$ and since any solution converges to zero, for large $n$ we have $|u(s_{n})| \leq \varepsilon$.

Assume (without loss of generality) that $u(s_{n}) < 0$. Then $\dot{v}(s_{n}) = -E'(u(s_{n})) > 0$, so the solution enters the set $P_{\delta,n}$ defined in Lemma 3.3. We show that $P_{\delta,n}$ is positively invariant for solutions of (NP). Obviously, for $u = -\delta, v > 0$ we have $\dot{u} = 0, v > 0$ we have $\dot{v} = -E(u) > 0$ and for the remaining part of the
boundary \( v = \eta|u|^{\frac{1}{1-\alpha}} \) we have \( \dot{u} = \dot{u}_1, \dot{v} \leq \dot{v}_1 \), where \( u_1 \) is the solution to (AP) with \( g(s) = c_d|s|^\alpha \) going through the same point of the boundary.

In the following we consider only solutions satisfying \( u < 0, v = \dot{u} > 0 \) on \((T, +\infty)\). We now formulate and prove two main theorems of this section. Theorem 4.5 is applied in the next section. In fact, it says that any fast solution converges faster than any slow solution, even for solutions to different problems with the same \( \alpha \) (and possibly different \( p \)'s). Theorem 4.6 says that if the non-autonomous part of the damping is smaller than the natural damping given by the velocity of slow solutions to the corresponding autonomous problem, then the non-autonomous part does not influence the decay.

**Theorem 4.5.** Let \( g \) satisfy (Gn) with \( \alpha < 1 - \frac{2}{p} \). Then any solution to (NP) which converges to zero is either fast or slow. Further, slow solutions satisfy \( |v(t)| \leq Ct^{-\frac{\alpha+1}{\alpha}} \) and fast solutions satisfy \( |u(t)| \leq ct^{-\frac{1-\alpha}{\alpha}}, c|u(t)|^{\frac{1}{1-\alpha}} \leq v(t) \leq Ct^{-\frac{1}{\alpha}}. \)

**Proof.** Let \( u \) be a solution that is not fast. Then there exists \( M > 0 \) such that \( \frac{v(t_n)}{|u(t_n)|^\alpha} \leq M \) for a sequence \( t_n, n \to +\infty \). By Lemma 3.4, there exists \( n \in \mathbb{N} \) such that \( (u(t_n), v(t_n)) \in P_{\delta,n} \). Let us consider the solution \( u_1 \) of the autonomous problem (AP) with \( u_1(t_n) = u(t_n), v_1(t_n) = v(t_n) \). By Lemma 3.3, \( u_1 \) is a slow solution to (AP) and it satisfies (5) by Theorem 3.1. By the comparison Lemma 4.3 and (Gn), the trajectories satisfy \( V(x) < V_1(x) \) and \( u(x) \geq u_1(x) = t (1 - \frac{1}{\alpha}) \) (and \( (u(t), v(t)) \) belongs to \( O_{e,K} \) for some \( e, K \), what we use in the next Theorem).

Let \( u \) be a fast solution. Then \( v(t) > \eta|u(t)|^{\frac{1}{1-\alpha}} \) on some interval \((T, +\infty)\) (otherwise, we would proceed as in the first paragraph of this proof and obtain that \( u \) is a slow solution). By Lemma 2.2 we have \( u(t) \leq Ct^{-\frac{1-\alpha}{\alpha}} \). Further, we have

\[
\dot{v} = -g(v, t)v + E'(u) \leq -cg^{\alpha}v^{1+\alpha} + CE^{1-p}v^{(1-p)(1-\alpha)} \leq (-c + \varepsilon)v^{1+\alpha}
\]

since \( 1 + \alpha > (p - 1)(1 - \alpha) \). Therefore, (again by Lemma (2.2)) we obtain \( v(t) \leq Ct^{-\frac{1}{\alpha}}. \)

**Theorem 4.6.** Let \( g \) satisfy (Gn) with \( \alpha < 1 - \frac{2}{p} \) and

\[
g(s, t) \leq D_g(|s| + t^{-\frac{p-1}{p-2\alpha}})^\alpha.
\]

Then any slow solution satisfies (5).

**Proof.** In the proof of Theorem 4.5 we have shown that any slow solution \((u(t), v(t))\) belongs to \( O_{e,K} \) for some \( e, K \) and all \( t \geq t_n \). Let us set \( T = t_n \) and take \( \varepsilon > 0 \) such that \( v(T) > \varepsilon T^{-\frac{p-1}{p-2\alpha}} \). We show that \( v(t) > \varepsilon t^{-\frac{p-1}{p-2\alpha}} \) for all \( t > T \). In fact, let \( t_0 = \inf\{t > T : v(t) \leq \varepsilon t^{-\frac{p-1}{p-2\alpha}}\} \). Then \( v(t_0) = \varepsilon t_0^{-\frac{p-1}{p-2\alpha}} \) and by Theorem 4.5 we have \( |u(t_0)| \geq \varepsilon t_0^{-\frac{p-1}{p-2\alpha}} \), and therefore

\[
\dot{v}(t_0) = -g(v(t_0))v + E'(u) \geq -D_g(1 + \varepsilon)^\alpha t_0^{-\frac{p-1}{p-2\alpha}} - \varepsilon t_0^{-\frac{p-1}{p-2\alpha}} + cE^{p-1}t_0^{-\frac{(p+1)(p-1)}{p-2\alpha}} = (c^{p-1}E - D_g(1 + \varepsilon)^\alpha t_0^{-\frac{p+1}{p-2\alpha}}(p-1)) > 0
\]

if \( \varepsilon \) is small enough. It follows that for \( t \in (t_0 - \delta, t_0) \) we have

\[
v(t) < v(t_0) = \varepsilon t_0^{-\frac{p-1}{p-2\alpha}} < \varepsilon t^{-\frac{p-1}{p-2\alpha}},
\]
Now, if we compare the solution \( u \) with the solution \( u_2 \) of \( u + C(\varepsilon)\dot{u}^{\alpha+1} - p|u|^{p-1} = 0 \), \( u_2(T) = u(T) \), \( v_2(T) = v(T) \), the comparison Lemma 4.3 yields \( |u(t)| \leq |u_2(t)| \sim Ct^{-\frac{\alpha+1}{\alpha}} \). Now we have \( u(t) \sim t^{-\frac{\alpha+1}{\alpha}} \) and due to \( v \leq Cu^{-1} \) (since \( (u(t), v(t)) \in O_{\varepsilon,K} \)) we have \( v(t) \sim t^{-\frac{\alpha+1}{\alpha}} - 1 \).

5. Degenerate potential. In this section we investigate the problem (DP). We assume that \( g \) satisfies (G) and \( E \in C^2(\mathbb{R}^n) \) is of the form

\[
E(u) = E_1(u_1) + \cdots + E_n(u_n),
\]

where \( u = (u_1, u_2, \ldots, u_n) \), \( E_i \in C^2(\mathbb{R}) \) satisfy (E) with exponents \( p_i \) respectively, and \( p_1 \geq p_2 \geq \cdots \geq p_n \geq 2 \). Then (DP) can be written as the following system of equations for \( u = (u_1, \ldots, u_n) \)

\[
\ddot{u}_i + g(u)\dot{u}_i + E_i'(u_i) = 0, \quad i = 1, 2, \ldots, n.
\]  

The equations are coupled only by the term \( g(u) \). Below, we often write \( E(t) \) instead of \( E(u(t), \dot{u}(t)) \). Let us start with the decay estimates for solutions of (DP).

**Theorem 5.1.** Let \( u \) be a solution to (DP). If \( \alpha \geq 1 - \frac{2}{p_i} \), then

\[
E(t) \sim C_2t^{-\frac{\alpha}{2}}.
\]  

If \( \alpha < 1 - \frac{2}{p_i} \), then

\[
C_1t^{-\frac{(1+\alpha)p_i}{p_i - 2\theta}} \geq E(t) \geq C_2t^{-\frac{\alpha}{2}}.
\]  

**Remark 5.2.** Let us remark that Theorem 5.1 remains valid (with the same proof) if \( u_i \) are vector valued functions with values in \( \mathbb{R}^{n_i} \), \( g : \mathbb{R}^{\sum n_i} \to \mathbb{R} \), and \( E_i : \mathbb{R}^{n_i} \to \mathbb{R} \). In this case, \( |s| \) in conditions (E), (G) denotes the Euclidean norm of \( s \). We can also assume that \( E_i \) satisfy (1) and \( \|\nabla^2 E(u)\| \leq \|
abla E(u)\|^{\frac{2p_i}{2p_i - \theta}} \) with \( \theta = \frac{1}{p_i} \) instead of (E). Then Theorem 5.1 remains valid with a similar proof where we define \( H_j(t) = E_j(t) + \varepsilon\|\nabla E_j(u_j(t))\|^{\beta_j}(\nabla E_j(u_j), v_j) \) with appropriate \( \beta_j \)'s, cf. [5].

**Proof of Theorem 5.1.** Let \( u = (u_1, \ldots, u_n) \) be a solution to (DP). Let us define

\[
E_j(t) = \frac{1}{2n}\|v(t)\|^2 + E_j(u_j(t))
\]

and

\[
H_j(t) = E_j(t) + \varepsilon|u_j(t)|^{\beta_j}u_jv_j
\]

with \( \beta_j = \frac{\alpha p_j}{2} \) if \( \alpha \geq 1 - \frac{2}{p_j} \) and \( \beta_j = \frac{p_j - 2\theta}{p_j} \) otherwise.

The last term in the definition of \( H_j \) is estimated by (we write \( u_j \) instead of \( u_j(t) \))

\[
\varepsilon \left( |u_j|^{2(\beta_j+1)} + |v_j|^2 \right) \leq C\varepsilon \left( E_j(u_j)\frac{2}{\beta_j} |v_j|^{\beta_j+1} + \|v\|^2 \right) \leq C\varepsilon \left( E_j(u_j) + \|v\|^2 \right),
\]

where we applied the Young inequality, then \( E_j(u) \sim u^{p_j} \) and finally \( 2(\beta_j+1) \geq p_j \) and boundedness of \( E_j(u_j(t)) \). It follows that \( H_j(t) \sim E_j(t) \). Further, we have (we
write $v$, $u_j$ instead of $v(t)$, $u_j(t)$ for every $t \geq 0$

$$H'_j(t) = -(g(v)v_j, v_j) - \varepsilon|u_j|^{\beta_j}u_j E'_j(u_j)$$
$$+ \varepsilon \beta_j |u_j|^{\beta_j} v_j^2$$
$$+ \varepsilon |u_j|^{\beta_j} v_j^2$$
$$- \varepsilon |u_j|^{\beta_j} u_j g(v)v_j.$$  \hspace{1cm} (14)

Here the first line satisfies (due to (G), (E))

$$-\langle g(v)v_j, v_j \rangle - \varepsilon|u_j|^{\beta_j}u_j E'_j(u_j) \sim -|v_j|^{\alpha+2} - |v_j|^2 \sum_{k \neq j} |v_k|^\alpha - \varepsilon|u_j|^{\beta_j+p_j}.$$  

The second and third lines in (14) are by the Young inequality estimated by

$$C\varepsilon|v_j|^{\alpha+2} + \frac{\varepsilon}{4} |u_j|^q \leq C\varepsilon|v_j|^{\alpha+2} + \frac{\varepsilon}{4} |u_j|^{\beta_j+p_j}$$

since $q = \frac{\alpha+2}{\alpha} \beta_j \geq \beta_j + p_j$. The last line in (14) is estimated by

$$\varepsilon|u_j|^{\beta_j+1}|v_j| \sum_{k=1}^n |v_k|^\alpha \leq \frac{\varepsilon}{4} |u_j|^{\beta_j+p_j} + C\varepsilon|v_j|^{\frac{\beta_j+p_j}{\beta_j+1}} \left( \sum_{k=1}^n |v_k|^\alpha \right)^{\frac{\beta_j+p_j}{\beta_j+1}}.$$  

Since $\frac{\beta_j+p_j}{\beta_j+1} \geq \frac{\alpha+2}{\alpha}$, the last expression is estimated by

$$\frac{\varepsilon}{4} |u_j|^{\beta_j+p_j} + C\varepsilon|v_j|^{\alpha+2} + C\varepsilon \sum_{k \neq j} |v_k|^{\alpha+2}.$$  

This term cannot be absorbed into the first line of (14) but after summing over $j$ it can be absorbed and we obtain

$$H' = \sum H'_j \sim -\sum |v_j|^{\alpha+2} - \varepsilon \sum |u_j|^{\beta_j+p_j},$$

i.e.,

$$H' \sim - \left( \|v\|^{\alpha+2} + \sum |u_j|^{\beta_j+p_j} \right).$$  

It follows that

$$-\frac{d}{dt} H' \sim \frac{\|v\|^{\alpha+2} + \sum |u_j|^{\beta_j+p_j}}{\left( \|v\|^2 + \sum |u_j|^{p_j} \right)^\frac{\beta_j+p_j}{\beta_j+1}}.$$  \hspace{1cm} (15)

The right-hand side is bounded from below by a positive constant if $2B \geq \alpha + 2$, $Bp_j \geq \beta_j + p_j$ for all $j$. For oscillatory coordinates, i.e. if $\alpha \geq 1 - \frac{2}{p_j}$, these inequalities hold if $B \geq \frac{\alpha+2}{2}$. So, if $\alpha \geq 1 - \frac{2}{p_j}$ (all coordinates are oscillatory), we have

$$\mathcal{E}(t) \sim H(t) \leq C t^{-\frac{4}{\alpha+2}} = C t^{-\frac{\beta}{2}}.$$  

For the non-oscillatory coordinates $\alpha < 1 - \frac{2}{p_j}$ we need to take a larger $B$, in particular $B \geq (1 - \frac{1}{p_j}) \frac{\alpha+2}{\alpha+1}$. Since $p_j$ is the largest among the non-oscillatory coordinates, the number $B = (1 - \frac{1}{p_j}) \frac{\alpha+2}{\alpha+1}$ is the least suitable and we obtain

$$\mathcal{E}(t) \sim H(t) \leq C t^{-\frac{1}{p_j-2-\alpha}} = C t^{-\frac{(1+\alpha)p_j}{p_j-2-\alpha}}.$$
On the other hand, the right-hand side of (15) is bounded from above if $2B \leq \alpha + 2$, $Bp_j \leq \beta_i + p_i$ for all $j$. Here, the best choice (largest possible $B$) is always $B = \frac{\alpha + 2}{2}$ (for both oscillatory and non-oscillatory coordinates) and we obtain

$$\mathcal{E} \sim H(t) \geq Ct^{-\frac{2}{\alpha}},$$

which completes the proof.

From now on, let us assume that $u_i$ are scalar valued. For a solution $u = (u_1, \ldots, u_n)$ and any fixed $i \in \{1, 2, \ldots, n\}$ let us denote $f_j(t) = \sum_{i \neq j} \hat{u}_i^2(t) \geq 0$. Then $u_j$ solves the nonautonomous problem (3) with

$$g_j(\hat{u}_i, t) = g\left(\sqrt{\hat{u}_i^2 + f_j(t)}\right) \geq c_g (\hat{u}_i^2 + f_j(t))^{\frac{2}{\alpha}} \geq c_g |\hat{u}_j|^{\alpha},$$

so $(Gn)$ is satisfied. Moreover, by Theorem 5.1 we know that every solution converges to zero. Now, we can apply the results from the previous section to obtain more gentle properties of solutions. In particular, we show that each solution to (DP) has one of (at most) $n + 1$ speeds of convergence to the origin that are given by fast and slow solutions of the equations (11). First of all, by Theorem 4.5 we have the following:

**Corollary 5.3.** Let $u = (u_1, \ldots, u_n)$ be a solution to (DP). If $i \in \{1, \ldots, n\}$ is such that $\alpha < 1 - \frac{2}{p_i}$, then $u_i$ does not oscillate. Moreover, for such $i$, the function $u_i$ (as a solution of (3)) is either fast and satisfies

$$|u(t)| \leq Ct^{-\frac{1}{2}} - \frac{\alpha}{\alpha}, \quad |v(t)| \leq Ct^{-\frac{1}{2}}$$

or slow and satisfies

$$|u(t)| \geq Ct^{-\frac{1}{2}} - \frac{\alpha}{\alpha}.$$

So, we speak about a non-oscillatory coordinate if $\alpha < 1 - \frac{2}{p_i}$ and about oscillatory coordinate if $\alpha \geq 1 - \frac{2}{p_i}$ (we do not know whether the oscillatory coordinates really oscillate) and a non-oscillatory coordinate of a particular solution can be called slow coordinate or fast coordinate. We now show that there appear at most $n + 1$ different rates of convergence of solutions to (DP), in particular, if there are $k$ non-oscillatory coordinates, then each solution has one of the $k + 1$ possible decay rates.

**Theorem 5.4.** Let $m$ be such that $1 - \frac{2}{p_{m+1}} \leq \alpha < 1 - \frac{2}{p_m}$ (set $m = 0$ if $1 - \frac{2}{p_i} \leq \alpha$ for all $j$ and $m = n$ if $\alpha < 1 - \frac{2}{p_j}$ for all $j$). Then for any solution to (DP) its energy satisfies

$$\mathcal{E}(t) \sim t^{-\frac{2}{\alpha}} \quad \text{or} \quad \mathcal{E}(t) \sim t^{-\frac{(1+\alpha)p_i}{p_j-2-\alpha}}$$

for some $j \in \{1, \ldots, m\}$. Moreover, to each of the $m + 1$ decay rates there exists a solution with this decay.

**Proof.** The moreover part is easy. If all coordinates except $u_j$ are zero, then $u_j$ satisfies (AP). Hence, by [9], the function $u_j$ decays as $t^{-\frac{2}{\alpha}}$ if it is an oscillatory coordinate and if it is a non-oscillatory coordinate, then it is a slow solution with $\mathcal{E}_j(t) \sim t^{-\frac{(1+\alpha)p_i}{p_j-2-\alpha}}$ or a fast solution with $\mathcal{E}_j(t) \sim t^{-\frac{2}{\alpha}}$. Existence of slow solutions follows from Lemma 3.2, existence of fast solutions was proved in [9, Theorem 3.4] for $g(s) = c|s|^{\alpha}$, $E(u) = |u|^p$ and the general case can be proved by modifying that proof. It remains to show that no other speeds of convergence appear.
By Theorem 4.5 we have $E_i(u_j(t)) = \theta_j(t)$, and therefore we can sum over $i \geq j$ only and obtain

$$E_i(u_i(t)) \sim \sum_{i=1}^{n} E_i(u_i) \quad \text{and} \quad \sum_{i=1}^{n} |u_i|^{\beta_i+p_i} \sim \sum_{i=1}^{n} |u_i|^{\beta_i+p_i}$$

we can sum over $i \geq j$ only and obtain

$$\frac{d}{dt} H' \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} |u_i|^{\beta_i+p_i} \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} E_i(u_i)$$

$$\frac{d}{dt} H' \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} |u_i|^{\beta_i+p_i} \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} E_i(u_i) \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} |u_i|^{\beta_i+p_i}$$

We can proceed as in the proof of Theorem 5.1, take $B = (1-\alpha)\frac{n+2}{\alpha}$ and obtain

$$ct^{-\frac{B}{2}} \leq E(t) \sim H(t) \leq Ct^{-\frac{(1+\alpha)p_j}{\beta_j-2-\alpha}}$$

By Theorem 4.5 we have $|u_j(t)| \geq ct^{-\frac{1+\alpha}{\beta_j-2-\alpha}}$. Hence, $|E_j(t)| \geq ct^{-\frac{(1+\alpha)p_j}{\beta_j-2-\alpha}}$ and therefore $E(t) \sim t^{-\frac{(1+\alpha)p_j}{\beta_j-2-\alpha}}$.

It remains to discuss the case when all non-oscillatory coordinates are fast. We show that in this case $E(t) \sim t^{-\frac{B}{2}}$. If there are no oscillatory coordinates, we are done, since fast coordinates satisfy $E_i(t) \sim v_i(t)^2 \leq C t^{-\frac{B}{2}}$ by Theorem 4.5.

Let us now assume that coordinates $1, \ldots, j-1$ are fast non-oscillatory and coordinates $j, \ldots, n$ are oscillatory. We show that the fast coordinates $i = 1, \ldots, j-1$ satisfy

$$E_i(u_i(t)) \leq C v_i^2(t) \quad \text{and} \quad |u_i(t)|^{\beta_i+p_i} \leq C v_i(t)^{2+\alpha},$$

and therefore we can sum over $i \geq j$ again, i.e.

$$\frac{d}{dt} H' \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} |u_i|^{\beta_i+p_i} \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} E_i(u_i) \sim \frac{n+2}{\alpha} \sum_{i=1}^{n} |u_i|^{\beta_i+p_i}$$

In fact, the first inequality in (17) follows immediately from the definition of fast solutions and the second inequality in (17) follows from

$$|u_i|^{\beta_i+p_i} = |u_i(t)|^{2+\alpha}(p_i-1) \leq C v_i(t)^{2+\alpha}$$
since $\frac{1-\alpha}{1+\alpha}(p_1 - 1) \geq 1$. Now, we can again proceed as in the proof of Theorem 5.1, take $B = \frac{2+\alpha}{2}$ and obtain that the right-hand side in (18) is larger than a positive constant, which yields $\mathcal{E}(t) \sim H(t) \leq Ct^{-\frac{\alpha}{2}}$. Hence, $\mathcal{E}(t) \sim t^{-\frac{\alpha}{2}}$.

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