Remarks on restricted Nevanlinna transforms

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Abstract. The Nevanlinna transform $K_{a,\rho}(z)$, of a positive measure $\rho$ and constant $a$, plays an important role in the complex analysis and more recently in the free probability (boolean convolution). It is shown that its restriction to the imaginary axis, $k_{a,\rho}(it)$, can be expressed as the Laplace transform of the Fourier transform (a characteristic function) of $\rho$. Moreover, $k_{a,\rho}$ is sufficient for the unique identification of the measure $\rho$ and the constant $a$. Finally, a relation between the Voiculescu and the boolean convolution is indicated.

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In the complex analysis and in the free probability important roles are played by the Cauchy $G(z)$ and the Nevanlinna $K(z)$ transforms, that is,

$$G_m(z) := \int_{\mathbb{R}} \frac{1}{z - x} m(dx), \quad K_{a, \rho}(z) := a + \int_{\mathbb{R}} \frac{1 + zx}{z - x} \rho(dx), \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad (*)$$

for some finite measures $m$ and $\rho$ and constants $a$. To get the measure $m$ from $G$ usually one uses the following inversion formula

$$m([a, b]) = -\lim_{y \to 0} \frac{1}{\pi} \int_{a}^{b} \Im G_m(x + iy)dx, \quad \text{provided} \quad m([a, b]) = 0;$$

cf. Akhiezer (1965), p. 125 or Lang (1975), p. 380, Bondesson (1992). Thus $G_m$ uniquely determines $m$. However, for the inversion one needs to know the Cauchy transform in strips \( \{x + iy : x \in \mathbb{R}, 0 < y < \epsilon \} \) for some $\epsilon > 0$. In Jurek (2006) it was shown, among others, that the values of $G_m(it), t \neq 0$ are sufficient to identify $m$; also cf. Corollary 2 below. Of course, as holomorphic functions both $G_m$ and $K_{a, \rho}$ are determined by their values on sets having condensation point.

In this note we will consider $G_m$ and $K_{a, \rho}$ only on the imaginary axis without the origin. In particular, we show that the measure $\rho$ can be retrieved from $K_{a, \rho}(it), t \neq 0$, using the classical (standard) Fourier and Laplace transforms; (Theorem 1). Using yet another functional (self-energy functional) we introduce the boolean convolution. Finally, we will indicate rather unexpected relations between the Voiculescu $\oplus$ and the boolean $\uplus$ operations on probability measures; (Proposition 1).

This paper is an exemplification of a general idea that many transforms in the complex analysis and, in particular, in the area of the free probability, are indeed some functionals of the standard Laplace and Fourier transforms when suitably restricted to the imaginary line.

1. Notations, the results and an example. For a real constant $a$ and a finite Borel measure on the real line $\rho$, we define the restricted Nevanlinna transform by

$$k_{a, \rho}(it) := a + \int_{\mathbb{R}} \frac{1 + itx}{it - x} \rho(dx), \quad \text{for} \quad t \neq 0,$$

and similarly the restricted Cauchy transform as follows

$$g_{\rho}(it) := \int_{\mathbb{R}} \frac{1}{it - x} \rho(dx), \quad \text{for} \quad t \neq 0,$$
Let us recall also that the Fourier transform (the characteristic function) $\hat{\mu}$ of a measure $\mu$ is given by

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{itx} \mu(dx), \quad t \in \mathbb{R}. \quad (3)$$

and the Laplace transform of a function $h : (0, \infty) \to \mathbb{C}$ or of a measure $m$ is given

$$\mathcal{L}[h; \lambda] := \int_{0}^{\infty} h(x)e^{-\lambda x} dx, \quad \mathcal{L}[m; \lambda] := \int_{0}^{\infty} e^{-\lambda x} m(dx), \quad \lambda > 0 \quad (4)$$

where $\lambda$ is a such that those integral exist; cf. Gradshteyn and Ryzhik (1994), Chapter 17, for examples of those transforms and their inverses.

Here are the main results, in particular, the formula how to obtain the measure $\rho$ knowing only restricted Nevanlinna transforms. (Below, $\Re z$, $\Im z$, $z$ denote the real part, the imaginary part and the conjugate of a complex $z \in \mathbb{C}$, respectively.)

**Theorem 1.** (An inversion formula.) For the restricted Nevanlinna functional $k_{a, \rho}$ we have that: $a = \Re k_{a, \rho}(i), \quad \rho(\mathbb{R}) = -\Im k_{a, \rho}(i)$; and the identity

$$\mathcal{L}[\hat{\rho}; w] = \int_{0}^{\infty} \hat{\rho}(r)e^{-wr} dr = \frac{ik_{a, \rho}(-iw) - i\Re k_{a, \rho}(i) - w\Im k_{a, \rho}(i)}{w^2 - 1}$$

holds for $w > 0$. In particular, the constant $a$ and the measure $\rho$ are uniquely determined by the functional $k_{a, \rho}$.

Since part of the right-hand side formula can be viewed as Laplace transform of some exponential functions we get

**Corollary 1.** For the restricted Nevanlinna functional $k_{a, \rho}$ and $w > 1$ we have

$$\int_{0}^{\infty} [\hat{\rho}(r) - \frac{1}{2}(ik_{a, \rho}(i)e^{-r} + \overline{k_{a, \rho}(i)e^{r}})] e^{-wr} dr = \frac{ik_{a, \rho}(-iw)}{w^2 - 1}.$$

In particular, if $a = 0$ and $\nu$ is a probability measure then for $k_{0, \nu}$ we get

$$\int_{0}^{\infty} (\hat{\nu}(r) - \cosh r) e^{-wr} dr = \frac{ik_{0, \nu}(-iw)}{w^2 - 1}, \quad w > 1.$$

**Corollary 2.** For a finite measure $\rho$ and its restricted Cauchy transform $g_{\rho}$ we have

$$\mathcal{L}[\hat{\rho}; w] = i g_{\rho}(iw), \quad w \neq 0.$$
In the following example we show explicitly that shifted reciprocals of restricted Cauchy transforms of discrete measures are, indeed, restricted Nevalinna transforms; (see the formula (5) below).

**Example.** For a set \( b = \{b_1, b_2, ..., b_m\} \) of distinct real numbers let us define a discrete probability measure \( \mu_b := \frac{1}{m} \sum_{j=1}^{m} \delta_{b_j} \) and the canonical polynomial \( P_b(z) = \prod_{j=1}^{m} (z - b_j) \). If \( \{\xi_1, \xi_2, ..., \xi_{m-1}\} \) is the set of zeros of the polynomial \( P'_b(z) \) (the derivative of \( P_b(z) \)) then we have

\[
\begin{align*}
&it - \frac{1}{G_{\mu_b}(it)} = it - \sum_{j=1}^{m} \frac{1}{it - b_j} = a_b + \int_{\mathbb{R}} \frac{1 + itx}{it - x} \rho_b(dx), \; t \neq 0, \\
&\text{where}
\end{align*}
\]

\[
\begin{align*}
&\alpha_k := -m P(\xi_k) P''(\xi_k) = m \left[ \sum_{j=1}^{m} \frac{1}{(\xi_k - b_j)^2} - \left( \sum_{j=1}^{m} \frac{1}{\xi_k - b_j} \right)^2 \right]^{-1} \geq 0 \\
&\beta_k := \frac{b_1 + b_2 + ... + b_m}{m} - \sum_{j=1}^{m-1} \alpha_j \xi_j, \; \rho_b(dx) := \sum_{j=1}^{m-1} \alpha_j \delta_{\xi_j}(dx).
\end{align*}
\]

Note that the procedure described in the Example can be iterated. Namely, in the second step we may start with the probability measure concentrated on the roots \( \xi_j, \; j = 1, 2, ..., m - 1 \), and so on.

Recall that the self-energy functional \( E_\mu \), of the probability measure \( \mu \), is defined as follows

\[ E_\mu(z) = z - \frac{1}{G_\mu(z)}, \; z \in \mathbb{C} \setminus \mathbb{R}. \]  

(7)

And similarly as above to \( e_\mu(it) := E_\mu(it), t \neq 0 \), we will refer to as a restricted self-energy functional.

Here is how to express \( a \) and \( \rho \) in terms of \( \mu \) using only the restricted functionals.

**Corollary 3.** For a probability measure \( \mu \) let

\[ z_\mu := -g_\mu(i) = c_\mu + i d_\mu = \int_{\mathbb{R}} \frac{x}{1 + x^2} \mu(dx) + i \int_{\mathbb{R}} \frac{1}{1 + x^2} \mu(dx) \in \mathbb{C}. \]

(8)

If \( e_\mu(it) = k_{a, \rho}(it) \), for \( t \neq 0 \), then the constants \( a \) and \( \rho(\mathbb{R}) \) are given by formulae

\[ a = \frac{c_\mu}{|z_\mu|^2} \quad \text{and} \quad \rho(\mathbb{R}) = 1 - \frac{d_\mu}{|z_\mu|^2} > 0, \]

(9)

and the Fourier transform \( \hat{\rho} \) satisfies the equation

\[ \mathcal{L}[|z_\mu|^2 \hat{\rho} - \frac{1}{2}(z_\mu e^w + z_\mu e^{-w})]; \; w = \frac{1}{(w^2 - 1) i g_\mu(-iw)}, \; w > 1. \]

(10)
Since for any probability measures \( \mu, \nu \) there exists a unique probability measure \( \gamma \) such that
\[
E_\mu(z) + E_\nu(z) = E_\gamma(z),
\]
we call it the boolean convolution and denote by \( \gamma = \mu \uplus \nu \); for more details cf. Speicher - Woroudi (1997).

**Remark 1.** Boolean convolution has the property that all probability measures are \( \uplus \)-infinitely divisible. That feature has also the max-convolution because for each distribution function \( F \), \( F^{1/n} \) (the \( n \)-th root) is also distribution function and taking independent identically distributed (as \( F^{1/n} \)) r.v. \( X_{n,1}, X_{n,2}, \ldots, X_{n,n} \) then \( \max\{X_{n,1}, \ldots, X_{n,n}\} \) has distribution function \( F \).

For a probability measure \( \mu \), let
\[
F_\mu(z) := \frac{1}{G_\mu(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \quad \text{and} \quad V_\mu(z) := F^{-1}_\mu(z) - z, \quad z \in D \subset \mathbb{C},
\]
where \( D \) is so called Stolz angle in which the inverse \( F^{-1}_\mu \) exists; cf. Bercovici-Voiculescu (1993) and references therein. Since for any probability measures \( \mu, \nu \) there exists a unique probability measure \( \gamma \) such that
\[
V_\mu(z) + V_\nu(z) = V_\gamma(z),
\]
we call it the Voiculescu convolution and denote by \( \gamma = \mu \oplus \nu \). A relation between \( \oplus \)-infinite divisibility and some random integrals with respect to classical Lévy processes is given in Jurek (2007), Corollary 6.

Here are seemingly unexpected relations between the Voiculescu \( \oplus \) and the boolean \( \uplus \) operations on probability measures.

**Proposition 1.** For probability measures \( \mu_1 \) and \( \mu_2 \) there exist unique probability measures \( \nu_1, \nu_2 \) such that
\[
F_{\mu_1}(F_{\nu_1}(z)) = F_{\mu_2}(F_{\nu_2}(z)) = F_{\mu_1 \uplus \mu_2}(z), \quad z \in \mathbb{C}^+.
\]
Furthermore, the above measures satisfy the equation \( \nu_1 \uplus \nu_2 = \mu_1 \oplus \mu_2 \).

**Corollary 4.** For \( n \geq 2 \) and probability measures \( \mu_1, \mu_2, \ldots, \mu_n \) there exist unique probability measures \( \nu_1, \nu_2, \ldots, \nu_n \) such that \( F_{\mu_1}(F_{\nu_1}(z)) = F_{\mu_2}(F_{\nu_2}(z)) = \ldots = F_{\mu_n}(F_{\nu_n}(z)) = F_{\mu_1 \uplus \mu_2 \uplus \ldots \uplus \mu_n}(z), \quad z \in \mathbb{C}^+ \). Furthermore, the above measures satisfy the equation \( \nu_1 \uplus \nu_2 \uplus \ldots \uplus \nu_n)^{1/(n-1)} = \mu_1 \oplus \mu_2 \oplus \ldots \oplus \mu_n \).

**Remark 2.** Let us note that the operations \( \uplus \) and \( \oplus \) give identities that might be interest in themselves. Moreover, finding a real analysis proofs of them seems to be very challenging.
(a) For probability measures $\mu$ and $\nu$ there exists unique measure $\mu \oplus \nu$ such that
\[
\int_{\mathbb{R}} \frac{1}{1-itx} \mu(dx) - 1 + \int_{\mathbb{R}} \frac{1}{1-itx} \nu(dx) - 1 = \int_{\mathbb{R}} \frac{1}{1-itx} \mu \oplus \nu(dx) - 1,
\]
for $t \in \mathbb{R}$; cf. Theorem 2 and Remark 1.1.1 in Jurek (2006) for other forms of the above formula and some comments.

(b) For measures $\mu_1$ and $\mu_2$ there exist unique measures $\nu_1$, $\nu_2$ and $\mu_1 \oplus \mu_2$ such that for their restricted Cauchy transforms we have
\[
g_{\nu_1}(it) \int_{\mathbb{R}} \frac{1}{1-xg_{\nu_1}(it)} \mu_1(dx) = g_{\mu_1 \oplus \mu_2}(it) = g_{\nu_2}(it) \int_{\mathbb{R}} \frac{1}{1-xg_{\nu_2}(it)} \mu_2(dx),
\]
for all $t \neq 0$. (Using Corollary 2 we may express the above identity in terms of classical Laplace and Fourier transforms.)

2. Auxiliary results and proofs. Note that
\[
g_m(it) = g_m(-it), \quad k_\rho(it) = k_{a,\rho}(-it), \quad e_\mu(it) = e_\mu(-it), \quad t \neq 0,
\]
and therefore we may consider those functions only on the positive half-line.

Proof of Theorem 1. From (1) we get
\[
k_{a,\rho}(i) = a - i\rho(\mathbb{R}). \quad (14)
\]
Further, since
\[
\frac{1 + itx}{it - x} = \frac{1 - t^2}{it - x} - it
\]
therefore from (1) and (2) we infer that
\[
k_{a,\rho}(it) = a + (1 - t^2)g_\rho(it) - it\rho(\mathbb{R}), \quad g_\rho(it) = \frac{k_{a,\rho}(it) - a + it\rho(\mathbb{R})}{1 - t^2}. \quad (15)
\]
On the other hand, in Jurek (2006) on p. 189, it was noticed that
\[
\int_0^\infty \hat{\rho}(ts)e^{-s}ds = \frac{1}{it}g_\rho\left(\frac{1}{it}\right), \quad t \neq 0, \quad \text{and} \quad \lim_{t \to 0} \frac{1}{it}g_\rho\left(\frac{1}{it}\right) = \rho(\mathbb{R}).
\]
This with (11) and (12) give
\[
\int_0^\infty \hat{\rho}(ts)e^{-s}ds = \frac{1}{it} \frac{(k_{a,\rho}(\frac{1}{it}) - \Re k_{a,\rho}(i) - \frac{1}{it} \Im k_{a,\rho}(i))}{1 + (\frac{1}{it})^2}
\]
\[
= \frac{k_{a,\rho}(\frac{1}{it}) - \Re k_{a,\rho}(i) - \frac{1}{it} \Im k_{a,\rho}(i)}{it + (\frac{1}{it})}.
\]
By putting $t = \frac{1}{w} > 0$ one gets
\[
\int_0^\infty \hat{\phi}(\frac{s}{w}) e^{-s} ds = \frac{k_{a,\rho}(-iw) - \Re k_{a,\rho}(i) + iw\Im k_{a,\rho}(i)}{\frac{1}{w} - iw} = \frac{iwk_{a,\rho}(-iw) - iw\Re k_{a,\rho}(i) - w^2\Im k_{a,\rho}(i)}{w^2 - 1}
\]
and substituting once again $\frac{s}{w} = r > 0$, one arrives at
\[
\int_0^\infty \hat{\phi}(r) e^{-wr} dr = \frac{iwk_{a,\rho}(-iw) - iw\Re k_{a,\rho}(i) - w^2\Im k_{a,\rho}(i)}{w(w^2 - 1)}.
\]
which gives the formula from Theorem 1. Finally, since the left-hand side is the Laplace transform of the Fourier transform $\hat{\rho}$ of the measure $\rho$, therefore it is uniquely determined by $k_{a,\rho}$. This completes the proof.

**Proof of Corollary 1.** Simple note that
\[
\mathbb{L}[\frac{1}{2}(e^x - e^{-x})]; w] = \frac{1}{w^2 - 1},
\]
\[
\mathbb{L}[\frac{1}{2}(e^x + e^{-x})]; w] = \frac{w}{w^2 - 1}, \quad w > 1,
\]
which with Theorem 1 give the corollary.

**Proof of Corollary 2.** Using the definitions (3) and (4) we have
\[
\mathbb{L}[\hat{\rho}; w] = \int \int e^{-x(w - i\xi)} dr \rho(\xi) dx = \int \frac{1}{w - i\xi} \rho(\xi) dx = -i g_{\rho}(-iw),
\]
which completes the proof.

Here is an auxiliary lemma that might be of an independent interest. It is a key argument for the example.

**Lemma 1.** (a) If $P(z) := \prod_{j=1}^{m} (z - b_j)$, $z \in \mathbb{C}$, for some complex numbers $b_j$, $j = 1, 2, ..., m$, and $P'(z)$ is its derivative then
\[
\frac{P'(z)}{P(z)} = \sum_{j=1}^{m} \frac{1}{z - b_j}; \quad \frac{P''(z)}{P(z)} = \left(\sum_{j=1}^{m} \frac{1}{z - b_j}\right)^2 - \sum_{j=1}^{m} \frac{1}{(z - b_j)^2}.
\]
(b) If $b_j$’s are distinct complex numbers and $\xi_1, ..., \xi_{m-1}$ denote the zeros of the equation $P'(z) = 0$ then $\xi_j$ are different from $b_1, b_2, ..., b_m$. Furthermore,
\[
W_m(z) := \frac{z P'(z) - m P(z)}{P'(z)} = \frac{b_1 + b_2 + ... + b_m}{m} + \sum_{j=1}^{m-1} \frac{\alpha_j}{z - \xi_j} \quad (16)
\]
where
\[
\alpha_k := -m \frac{P(\xi_k)}{P'(\xi_k)} = m \left[ \sum_{j=1}^{m} \frac{1}{(\xi_k - b_j)^2} - \left( \sum_{j=1}^{m} \frac{1}{\xi_k - b_j} \right)^2 \right]^{-1},
\]
for \( k = 1, 2, \ldots, m - 1 \).

(c) If \( b_j \)'s are distinct real numbers, for \( j = 1, 2, \ldots, m \), then \( \alpha_k > 0 \), for \( k = 1, 2, \ldots, m - 1 \).

Proof. (a) Since \( P'(z) = \sum_{j=1}^{m} \prod_{k \neq j, k=1}^{m} (z - b_k) \) we get the first part of (a). Differentiating both sides of the identity \( P'(z) = P(z) \sum_{j=1}^{m} \frac{1}{z - b_j} \), we get the second part of (a).

(b) Assume that \( P \) and \( P' \) have a common root. Without loss of generality, let say that \( \xi_1 = b_1 \). Then \( P'(b_1) = \prod_{k=2}^{m} (b_1 - b_k) = 0 \), which contradicts the assumption that all \( b_j \) are distinct.

Suppose that \( \xi_1 \) and its complex conjugate \( \bar{\xi}_1 \) are two complex roots of \( P'(z) = 0 \). Then from (a) we have
\[
P'(\xi_1) = P(\xi_1) \sum_{j=1}^{m-1} \frac{1}{\xi_1 - b_j} = 0 = P(\bar{\xi}_1) \sum_{j=1}^{m-1} \frac{1}{\xi_1 - b_j}.
\]
Since \( P(\xi_1) \neq 0 \) and \( P(\bar{\xi}_1) \neq 0 \) therefore
\[
\sum_{j=1}^{m-1} \left[ \frac{1}{\xi_1 - b_j} - \frac{1}{\xi_1 - b_j} \right] = i 2(\Re \xi_1) \sum_{j=1}^{m} \frac{1}{|\xi_1 - b_j|^2} = 0,
\]
and hence \( \Re \xi_1 = \Re \xi_2 = \Re \xi_3 = \ldots = \Re \xi_{m-1} \), that is, all roots of \( P'(z) = 0 \) are real.

Let us note that
\[
P(z) = \prod_{k=1}^{m} (z - b_k) = z^m + (-b_1 - b_2 - \ldots - b_m)z^{m-1} + Q_{m-2}(z),
\]
for some polynomial \( Q_{m-2} \) of degree \( m-2 \). Then \( z P'(z) - mP(z) = (b_1 + \ldots + b_m)z^{m-1} + \tilde{Q}_{m-2}(z) \) is a polynomial of degree \( m-1 \), (for another polynomial of degree \( m-2 \). Consequently, \( W_m(z) \) is a rational function (a ratio of two polynomials of degree \( m-1 \)). Since \( \xi_1, \ldots, \xi_{m-1} \) are zeros of \( P'(z) = 0 \), i.e., simple poles of \( W_m(z) \), then invoking the theorem on the decomposition of rational function into a sum of simple fractions
\[
W_m(z) = z - \frac{m P(z)}{P'(z)} = \frac{(b_1 + \ldots + b_m)z^{m-1} + \tilde{Q}_{m-2}(z)}{mz^{m-1} + (m - 1)(-b_1 - b_2 - \ldots - b_m)z^{m-2} + \tilde{Q}'_{m-2}(z)} = \frac{b_1 + b_2 + \ldots + b_m}{m} + \sum_{j=1}^{m-1} \frac{\alpha_j}{z - \xi_j}. \tag{17}
\]
Putting $\bar{b} := (b_1 + \ldots + b_m)/m$ and multiplying both sides by $z - \xi_k$, from the above we have

$$\alpha_k + (z - \xi_k) \sum_{j \neq k, j = 1}^{m-1} \frac{\alpha_j}{z - \xi_j} = (z - \xi_k)(z - \bar{b}) - m P(z) \left( \frac{P'(z) - P'({\zeta}_k)}{z - \xi_k} \right)^{-1},$$

and then letting $z \to \xi_k$ we get explicitly that

$$\alpha_k := -m \frac{P({\zeta}_k)}{P''({\zeta}_k)} = m \left[ \sum_{j=1}^{m} \frac{1}{(\xi_k - b_j)^2} - \left( \sum_{j=1}^{m} \frac{1}{\xi_k - b_j} \right)^2 \right].$$

(c) Since $P(x)$ is a polynomial of m-th degree for $x \in \mathbb{R}$ and $P(b_k) = P(b_{k+1}) = 0$ (for $b_j \in \mathbb{R}$) then, by the Mean Value Theorem, there exists exactly one $\xi_j$ (in that interval) such that $P'({\zeta}_k) = 0$. If $P({\zeta}_k) > 0$ then $P$ must be concave on that interval and therefore $P''({\zeta}_k) < 0$. Consequently, $\alpha_j > 0$. In the opposite case we have convex function that also leads to positivity of the parameter $\alpha_k$. This completes the proof of Lemma 1.

**Proof of the Example.** From Lemma 1 we have that the measure $\rho_b$ is finite and positive. Furthermore, for $a_b$ given by (6) and using (13) (in Lemma 1) we get

$$\int_{\mathbb{R}} \frac{1 + zx}{z-x} d\rho_b(x) = \sum_{j=1}^{m-1} \frac{1 + \xi_j^2 + z\xi_j - \xi_j^2}{z - \xi_j} \frac{\alpha_j}{1 + \xi_j^2} = \sum_{j=1}^{m-1} \frac{\alpha_j}{z - \xi_j} + \sum_{j=1}^{m-1} \frac{\alpha_j \xi_j}{1 + \xi_j^2} = W_m(z) - a_b = z - \frac{m P_b(z)}{P''(z)} - a_b = z - \frac{1}{G_{\rho_b}(z)} = F_{\rho_b}(z) - a_b.$$

Substituting in the above it for $z$, one gets equality (5) in the Example.

**Proof of Corollary 3.** From (2) and we get immediately the expression (8) for $-g_\mu(i)$. From (11), $e_\mu(i) = a - i\rho(\mathbb{R})$ and hence we infer equalities in (9).

In view of the assumption, $k_{a, \rho}$ in Corollary 1 may be replaced by $e_\mu$ then, using (7) and (9), one gets

$$ik_{a, \rho}(-iw) - i \Re k_{a, \rho}(i) - w \Im k_{a, \rho}(i) = w - \frac{i}{g_\mu(-iw)} - \frac{e_\mu}{|z_\mu|^2} - w(1 - \frac{d_\mu}{|z_\mu|^2}).$$

Consequently from Corollary 1 we get the required identity.

**Proof of Proposition 1.** From Theorem 2.1 in Chistyakov and Goetze (2005), (cf. also Biane (1998)) for measures $\nu_1$ and $\nu_2$ there exists uniquely determined probability measures $\nu_1, \nu_2$ and $\mu$ such that

$$z = F_{\nu_1}(z) + F_{\nu_2}(z) - F_{\mu_1}(F_{\nu_1}(z)) \quad \text{and} \quad F_{\mu_2}(F_{\nu_1}(z)) = F_{\mu_2}(F_{\nu_2}(z)) = F_{\mu}(z),$$

9
where \( \mu = \mu_1 \oplus \mu_2 \) (Voiculescu convolution). Hence

\[
E_{\nu_1 \oplus \nu_2}(z) = E_{\nu_1}(z) + E_{\nu_2}(z) = z - F_{\nu_1}(z) + z - F_{\nu_2}(z) = z - F_{\mu_1 \oplus \mu_2}(z).
\]

From the uniqueness of the self-energy functional we get \( \mu_1 \oplus \mu_2 = \nu_1 \oplus \nu_2 \), which completes the proof.

Proof of Corollary 4. From Corollary 2.2 in Chistyakov and Goetze (2005), for measures \( \mu_1, ..., \mu_n \) there exists uniquely determined probability measures \( \nu_1, ..., \nu_n \) and \( \mu \) such that

\[
z = F_{\nu_1}(z) + ... + F_{\nu_n}(z) - (n - 1)F_{\mu_1}(F_{\nu_1}(z))
\]

and \( F_{\mu_1}(F_{\nu_1}(z)) = ... = F_{\mu_n}(F_{\nu_n}(z)) = F_{\mu}(z) \),

where \( \mu = \mu_1 \oplus ... \oplus \mu_n \) (the Voiculescu convolution). Thus

\[
E_{\nu_1 \oplus ... \oplus \nu_2}(z) = E_{\nu_1}(z) + ... + E_{\nu_n}(z) = z - F_{\nu_1}(z) + ... + z - F_{\nu_n}(z)
\]

\[
= (n - 1)(z - F_{\mu_1}(F_{\nu_1}(z)) = (n - 1)(z - F_{\mu_1 \oplus ... \oplus \mu_n}(z) = (n - 1)E_{\mu_1 \oplus ... \oplus \mu_n},
\]

which completes the proof.

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