Lattice calculations for B and K mixing

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The bag parameters and the decay constants of neutral $B(s)$ and $K$ mesons were among the non-perturbative hadronic inputs to the classical CKM Unitarity Triangle Analysis. Thanks to the big amount of experimental information collected in the last few years at the $B$-factories and by the CDF collaboration, these matrix elements are now among the outputs of the unitarity fits, once the validity of the Standard Model has been postulated. Lattice calculations of the mixing amplitudes are still needed in order to make a test of the theory, provided that their statistical and systematic errors are under control at the level of a few percent. Here we review some of the recent lattice calculations of these quantities.

I. INTRODUCTION

Lattice QCD calculations of the $B(s)$ and $K$ mixing amplitudes where needed in the past in order to be used as inputs to the Unitarity Triangle Analysis (UTA). After the recent measurements of the angles of the unitarity triangle the $\rho-\eta$ plane is over-constrained and the mixing amplitudes can now be extracted by one of the possible variants of the UTA (see [1] and references there). From the phenomenological point of view it is thus legitimate to ask whether it is still needed a precise lattice calculation of the mixing amplitudes. In answering this question the point to note is that the unitarity fits are over-constrained only within the Standard Model. In this scenario one can make a precision test of the Standard Model and hopefully reveal the presence of new physics.

II. $B_K$

The so-called bag parameter $B_K$ parametrizes the mixing amplitude of the $K_0$ mesons according to

\[
\langle K^0|\hat{O}_1|K^0\rangle = \langle K^0|\hat{O}_{VV+AA}|K^0\rangle = \frac{8}{3} M_0 f_K^2 B_K(\mu)
\]

where the operator $\hat{O}_1 = \bar{s}i\gamma_\mu(1-\gamma_5)d^i\bar{s}i\gamma_\mu(1-\gamma_5)d^i$ is usually conveniently decomposed into a parity-even and a parity-odd part, $\hat{O}_1 = O_{VV+AA} - O_{VA+AV}$.

The breaking of chiral symmetry on the lattice with Wilson fermions complicates considerably the renormalization pattern of the $O_{VV+AA}$ operator that happens to mix with the other 4 operators in the parity-even basis [6, 7, 8]. The issue in such a calculation consists in obtaining the numerical accuracy required in order to keep under control the mixing subtractions or in devising smart strategies to avoid the problem. Two groups have been able to circumvent the mixing problem. The authors of refs. [9, 10] have used the so called twisted mass formulation of lattice QCD (tmQCD) in order to map the matrix element of the parity even operator $O_{VV+AA}$ into the matrix element of the parity odd operator $O_{VA+AV}$ that renormalizes multiplicatively; they have been able to

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2. This formalism has been largely used because it is particularly cheap from the computational point of view but recently it has been proved the feasibility of large scale $N_f = 2$ lattice simulations with pions as light as 300 MeV and physical volumes of the order of 2 fm.
calculate $B_K$ by keeping under control all the sources of errors apart from quenching (non-perturbative renormalization, estimate of $SU(3)$-breaking effects, continuum limit with 5 lattice spacings, estimate of finite volume effects). The authors of refs. [10, 11] avoided the mixing by using a chiral ward identity that again relates the matrix elements of $O_{V+AA}$ to that of $O_{V+AV}$ at the price of computing on the lattice a four-point Green function.

In the case of lattice discretizations that satisfy the so called Ginsparg-Wilson (GW) relation an exact chiral symmetry is preserved also at finite lattice spacing. Practically, the fifth dimension is finite and the lattice chiral symmetry is only approximately preserved. The authors of refs. [21, 22] have performed a calculation of $B_K$ with respectively $N_f = 2$ and $N_f = 2 + 1$ flavours of dynamical domain wall fermions. The $N_f = 2 + 1$ results have been obtained at fixed lattice spacing ($a \simeq 0.12$ fm), with non perturbative renormalization (by neglecting the small mixing due to the “residual mass term”), by interpolating the physical $K$ meson state, on a single volume ($L \simeq 2$ fm); a simulation at the same lattice spacing on a larger volume ($L \simeq 3$ fm) is under way.

On the one hand, there have been so many different calculations of $B_K$ among the years that it is not possible to enter into the details of all of them in this short review\(^3\) (see TABLE I). On the other hand none of this calculations is able to take under control all the sources of systematics at the same time. Since different numbers have been obtained with different actions, techniques, assumptions, etc. we can get an estimate of the systematics by averaging all the results that are “uncorrelated” (in the sense that we neglect results that have been updated by the same collaboration at fixed $N_f$) without the quoted errors (see FIG. I top plot). As a result we get $\hat{B}_K = 0.81(3)$ i.e. a relative error of the order of $4\%$; if instead we take the average of the numbers with $N_f > 0$ by trusting the quoted errors we get $\hat{B}_K = 0.78(2)$ (see FIG. I bottom plot). The previous numbers have to be taken as “provocative” averages: unless a clear statement is made on which lattice results can be trusted and which have to be excluded from phenomenological analysis one should conclude that $B_K$ is presently predicted by the lattice with a few percent error. A conservative estimate of the errors, to be used in phenomenological applications, can be obtained for example by accounting for the dispersion of the results:

$$\hat{B}_K = 0.78(2)$$  \hspace{1cm} (1)

### III. $f_{B_q}$

The decay constants of the $B_q$ mesons, where $q$ stays for either a down or a strange quark, enter in the parametrization of the $B_q - B_q$ mixing amplitudes together with the bag parameters,

$$\langle B_q | O_{V+AA} | B_q \rangle = \frac{8}{3} M_{B_q}^2 f_{B_q} B_{B_q} (\mu)$$

What it is actually needed in order to perform the UTA is the combination $f_{B_q} \sqrt{B_{B_q}}$, that comes out to have a smaller statistical error on the lattice w.r.t. the product of $f_{B_q}$ and $\sqrt{B_{B_q}}$ computed separately. Since there are many more calculations of the decay constants than the bag parameters and since we want to use as much information as possible in taking the averages, we will discuss

\(^3\) we have just mentioned some representative calculations and apologize with the authors whose results have not been covered in greater detail. The same holds also for the following sections.

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**TABLE I: Lattice calculations of the renormalization group invariant (RGI) kaon bag parameter $\hat{B}_K$.**

| Collaboration       | $\hat{B}_K$ | $N_f$ |
|---------------------|-------------|-------|
| JLQCD97 [12]        | 0.868(59)   | 0     |
| Becirevic00 [20]    | 1.01(9)     | 0     |
| CP-PACS01 [13]      | 0.795(29)   | 0     |
| SPQCDR02 [10]       | 0.91(9)     | 0     |
| BosMar03 [14]       | 0.87(8)     | 0     |
| MILC03 [15]         | 0.79(9)     | 0     |
| Babich06 [16]       | 0.79(8)     | 0     |
| ALPHA06 [18]        | 0.735(71)   | 0     |
| RBC03 [21]          | 0.697(33)   | 2     |
| UKQCD04 [19]        | 0.67(18)    | 2     |
| SPQCDR05 [11]       | 1.02(25)    | 2     |
| RBC05 [17]          | 0.78(7)     | 2     |
| RBC-UKQCD06 [22]    | 0.778(36)   | 2+1   |
| HPQCD-UKQCD06 [23]  | 0.85(12)    | 2+1   |
separately the lattice calculation of \( f_{B_s} \) in this section, and of \( B_{B_s} \), in the next section, but first we briefly comment on the issues related to the simulation of the heavy flavours on the lattice.

On currently affordable lattice sizes (at least in unquenched simulations) one has \( a m_b > 1 \) and \( L m_q > 1 \) or \( a m_b < 1 \) and \( L m_q < 1 \), i.e. a relativistic beauty-light meson can be simulated on big volumes with big cutoff effect or on small volumes with big finite volume effects. The different approaches that have been devised in order to solve this problem can be divided into “big volume” and “small volume” strategies.

**Big volume strategies.** Lattice simulations of heavy quarks on physical volumes are performed by recurring to effective field theories. One possibility is to simulate the lattice static action and, eventually, the relativistic theory with heavy quark masses in the charm region in order to interpolate the bottom region. Another possibility is the so called Fermilab approach \([24, 25, 26]\) that consists in improving the heavy quark action with mass dependent coefficients, provided that \( |\alpha_s| \ll 1 \); the procedure smoothly interpolates between heavy and light quarks and the continuum limit can be taken although the mass dependence of the improving coefficients makes the procedure highly non trivial. Still another possibility is to simulate on the lattice a discretized form of the non relativistic heavy quark action expanded to a given order in \( v^2 \) and \( \alpha_s \) (see for example \([27]\)); as a consequence of the non-renormalizability of the theory, the continuum limit cannot be taken and the matching with full QCD can be done only perturbatively; furthermore the expansion is expected to work for onium systems but it has been widely used also to study heavy–light mesons.

**Small volume strategies.** The step scaling method (SSM) has been proposed in \([28]\) in order to deal with two-scale, \( E_l \ll E_h \), problems in lattice QCD. The method starts from the following identity

\[
O(E_h, E_l, \infty) = \frac{O(E_h, E_l, 2L_0)}{\sigma(E_l, E_l, L_0)} \frac{O(E_h, E_l, 4L_0)}{\sigma(E_h, E_h, 2L_0)} \ldots
\]

The idea is to start the computation on a small volume where the high energy scale can be set to its physical value and to correct for the finite volume effects by computing the step scaling functions, \( \sigma(E_h, E_l, L) \), stopping the recursion when the last factor is equal to one within the required precision. The strength of the method can be better understood by specializing the previous identity to the case of a heavy–light meson observable, say \( f_B \). In this case the dependence of the step scaling functions upon the \( b \) quark mass can be predicted by using

\[
\sigma(m_k, m_d, L) = \frac{f_B^0(m_d, 2L)}{f_B^0(m_d, L)} = \frac{1 + \frac{f_B^1(m_d, 2L)}{m_h} + \ldots}{1 + \frac{f_B^1(m_d, L)}{m_h} + \ldots}
\]

The maximum number of points of the lattice discretization is fixed during the recursion so that, in order to have \( a m_b \ll 1 \) at each step, the step scaling functions have to be computed at heavy quark masses smaller than the physical beauty mass. Nevertheless, from the previous equation, it is clear that the step scaling functions depend mildly upon the high energy scale thanks to the cancellation \( f_B^1(m_d, 2L) - f_B^1(m_d, L) \) that gets stronger and stronger as the volume is increased. Another useful application of finite volume techniques has been used, originally in ref. \([29]\) to implement the renormalization procedure of HQET fully non-perturbatively. The idea is to match the effective theory with full QCD on a small volume at the scale \( \mu = m_b \). Then the running of the matching factors is computed through a step scaling recursion in the effective theory. This method can be applied in conjunction with the SSM to compute, for example, \( \sigma_{\text{stat}}(m_d, L) \) (see ref. \([30]\)).

We now come to the lattice results for \( f_{B_s} \) and \( r_B = f_{B_s}/f_B \). Among the quenched results those of \([30, 47]\) have been obtained through the SSM by keeping under control all the systematics apart from the quenching. Among the \( N_f = 2 \) results, the ones of \([48, 54]\) have been obtained within the static approximation giving a particularly high value of \( r_B \); also the result of \([52]\) is static but non-perturbatively renormalized. The \( N_f = 3 \) results \([49, 53]\) have been obtained by using

![FIG. 2: Top. Average of the uncorrelated \( f_{B_s} \) results without the quoted errors. Bottom. Average of the \( f_{B_s} \) results with \( N_f > 0 \) with the quoted errors (red lines).](image-url)
TABLE II: Lattice calculations of $f_{B_s}$ (MeV) and $r_B = f_{B_s}/f_B$.

| collaboration     | $f_{B_s}$   | $r_B$ | $N_f$ |
|-------------------|-------------|-------|-------|
| Fermilab99 [38]   | 185(16)     | 0     | 0     |
| MILC08 [39]       | 171(14)     | 0     | 0     |
| JLQCD99 [40]      | 191(18)     | 0     | 0     |
| UKQCDO0 [32]      | 204(27)     | 0     | 0     |
| APE00 [33]        | 235(20)     | 0     | 0     |
| ALPHA03 [45]      | 206(10)     | 0     | 0     |
| ROEMIE03 [47]     | 192(7)      | 0     | 0     |
| ROMEII-ALPHA06 [30] | 191(6) | 0     | 0     |
| CP-PACS00 [41]    | 250(18)     | 1.20(64) | 2    |
| CP-PACS01 [42]    | 242(52)     | 1.17(29) | 2    |
| MILC02 [43]       | 217(36)     | 1.16(5)  | 2    |
| JLQCD03 [36]      | 251(17)     | 1.13(12) | 2   |
| UKQCDO4 [48]      | 256(45)     | 1.38(15) | 2   |
| Gadiyak05 [50]    | 341(32)     | 1.38(15) | 2   |
| ALPHA06 [52]      | 297(14)     | 1.38(15) | 2   |
| HPQCD05 [49]      | 259(32)     | 2+    | 2+   |
| Fermilab-MILC-HPQCD06 [53] | 253(42) | 1.27(6) | 2+   |

TABLE III: Lattice calculations of $B_{B_s}(m_b)$ and $B_B(m_b)$.

| collaboration         | $B_{B_s}(m_b)$ | $B_B(m_b)$ | $N_f$ |
|-----------------------|----------------|------------|-------|
| UKQCDO0 [32]          | 0.90(4)        | 0.91(6)    | 0     |
| APE00 [33]            | 0.92(7)        | 0.93(10)   | 0     |
| SPQCDR01 [34]         | 0.87(5)        | 0.87(6)    | 0     |
| JLQCD02 [35]          | 0.86(5)        | 0.84(6)    | 0     |
| JLQCD03 [36]          | 0.85(64)       | 0.836(68)  | 2     |
| Gadiyak05 [50]        | 0.864(76)      | 0.812(82)  | 2     |
| HPQCD06 [49]          | 0.76(11)       | 0         | 3     |

rooted staggered fermions for the dynamical light quarks and NRQCD and Fermilab respectively for the heavy. By looking at TABLE II it emerges that quenched lattice calculations, though compatible within themselves, are systematically smaller than unquenched results; the quoted errors are still large but unquenching seems to have a significant effect on this observable (this is not the case of $B_K$ within the quoted errors) and the “provocative” average ($f_{B_s} = 245(13)$ MeV and $r_B = 1.24(4)$) prefers unquenched results. The unquenched average, static points included, with an error that takes into account the spread of the results is

$$f_{B_s} = 268(17)(20) \text{ MeV}, \quad \frac{f_{B_s}}{f_B} = 1.20(2)(5) \quad (2)$$

IV. $B_{B_q}$

In order to calculate the bag parameters of the $B_q$ mesons one has to face at the same time the problem of the mixing, as for $B_K$, and the problem related to the presence of a heavy and a light quark, as for $f_{B_s}$. For this reason the number of lattice calculations of $B_{B_q}$ is much smaller than in the case of $f_{B_s}$ or $B_K$.

Nevertheless, by looking at TABLE III it emerges that, within the quoted errors, $B_{B_q}$ does not seem to depend upon the number of dynamical flavours, the renormalization systematics (the quenched result of ref. [34] has been non-perturbatively renormalized), the strategy used to handle with heavy quarks and even the light quark mass. Actually, the matrix element $\langle B_q|O_{V+AA}|B_q\rangle$ does show a sizable dependence upon all these variables but through the vacuum saturation approximation, i.e. $8M_{B_q}^2 f_{B_q}^2/3$. The average of the $N_f > 0$ calculations, with an error that takes into account the spread of the results, is

$$B_{B_s}(m_b) = 0.84(3)(5) \quad B_B(m_b) = 0.83(1)(6) \quad (3)$$

V. A CALCULATION OF $G(\omega)$

FIG. 3: Comparison of lattice data and experimental determinations of $G(\omega)$: $V_{cb} = 41(5) \times 10^{-3}$ is extracted by using the experimental points at $\omega = 1.2$.

We now change subject to put up to the results of a preliminary quenched calculation [54] of the form factor $G(\omega)$, needed in order to extract $V_{cb}$ from the exclusive semileptonic decay $B_s \to D_s \ell \nu$ ($\omega = p_B \cdot p_\ell / M_B M_D$). The calculation has been carried on by using the SSM and by defining the form factor and the kinematical factors in terms of ratios of three point correlation functions. The results have been obtained with a relative error of about 4% for values of $1 \leq \omega \leq 1.2$ were experimental data do not need to be extrapolated (FIG. 3).

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