On the partial breaking of $\mathcal{N} = 2$ rigid supersymmetry with complex hypermultiplet

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Abstract

We study partial supersymmetry breaking in effective $\mathcal{N} = 2$ $U(1)^n$ gauge theory coupled to complex hypermultiplets by using the method of “arXiv:1501.07842” to which we refer to as ADFT method. We derive the generalisation of the symplectic invariant ADFT formula $\zeta_a = \frac{1}{2} \varepsilon_{abc} (P^{bM} C_{MN} P^{cN})$ capturing information on partial breaking. Our extension of this anomaly is expressed like $d_a = \frac{1}{2} \varepsilon_{abc} P^{bM} C_{MN} P^{cN} + J_a$; the generalized moment maps $P^{aM}$ contain $P^{aM}$ and depend as well on electric/magnetic coupling charges $G_M = (\eta^i, g_i)$, the $J_a$ is an extra contribution induced by Killing isometries in complex hypermatter sector. Using $\text{SP}(2n, \mathbb{R})$ symplectic symmetry, we also give the $\mathcal{N} = 2$ partial breaking condition and derive the model of ”arXiv: 1204.2141” by a particular realisation of $d_a$ anomaly.

Key words: Rigid limit of $\mathcal{N} = 2$ supergravity, Rigid Ward identity, Partial breaking, moment maps.

1 Introduction

The partial breaking of rigid $\mathcal{N} = 2$ supersymmetric field theory was widely studied in the literature [1][2]. To our knowledge, the first model realizing the $\mathcal{N} = 2$ partial breaking was introduced by Antoniadis, Partouche and Taylor (APT) in [1] where the partial breaking, for an $\mathcal{N} = 2$ effective pure abelian gauge theory, was interpreted as resulting from the presence of magnetic Fayet-Ilioupoulos (FI) charges beside the electric ones. The presence of these FI charges allow to evade the no-go theorem which forbids such phenomenon to occur [10][11]. Recently the interpretation of the APT model in terms of triplets of symplectic hyper-Kahler moment maps $(\mathcal{P}^a)^M \equiv \mathcal{P}^{aM}$ was given by Andrianopoli, D’Auria, Ferrara and Trigiante.

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(ADFT) in [7]. There, the authors showed that the partial breaking is induced by a non-vanishing symplectic invariant isotriplet $C_{B}^{A} = \zeta_{c} (\tau^{c})_{B}^{A}$ deforming the $SU(2)_{R}$ invariant APT scalar potential $\delta_{B}^{A} \mathcal{V}_{\text{APT}}$ in the rigid Ward identity. This $SU(2)_{R}$ triplet $\zeta_{c}$ reads in terms of the $\mathcal{P}^{aM}$’s as follows

$$\zeta_{c} = \frac{1}{2} \varepsilon_{abc} \mathcal{P}^{aM} C_{MN} \mathcal{P}^{bN}$$

(1.1)

where $\tau^{c}$ stand for the three Pauli matrices, $\varepsilon_{abc}$ the completely antisymmetric tensor in 3D and the $2n \times 2n$ matrix

$$C_{MN} = \begin{pmatrix} 0_{n} & \mathbf{I}_{n} \\ -\mathbf{I}_{n} & 0_{n} \end{pmatrix}$$

(1.2)

is the $SP(2n, \mathbb{R})$ symplectic metric of the scalar manifold of the Coulomb branch of the $\mathcal{N} = 2$ supersymmetric $U(1)^{n}$ effective gauge theory. The $\mathcal{P}^{aM}$ moment maps are obtained from the gauging of Killing vectors of the hypermultiplet scalar manifold $SO(1, 4)/SO(4)$ in the rigid limit of gauged $\mathcal{N} = 2$ supergravity [8, 9]. In this rigid limit, the observable sector contains only the $\mathcal{N} = 2$ supersymmetric $U(1)^{n}$ vector multiplets while gravitation and hypermultiplet are in the hidden sector; and then the study of the coupling of the APT model, by using the ADFT method [7], to observable hypermatter is still missing.

The aim of the present paper is to fill this gap in the literature by studying the extension of the $\mathcal{N} = 2$ ADFT method to include gauge invariant couplings with observable complex hypermatter. Here, we use the $\mathcal{N} = 1$ language to deal with $\mathcal{N} = 2$ supermultiplets; the complex scalars $q^{1}, q^{2}$ of a hypermultiplet carry opposite gauge charges under $U(1)^{n}$ and are thought of as scalars of two $\mathcal{N} = 1$ multiplets $Q^{u} = \{q^{u}, \chi^{u}\}$ with complex $\chi^{u}$ standing for the fermionic partners of the $q^{u}$ in the complex hypermultiplet. The Weyl fermions $\chi^{1}, \chi^{2}$ carry opposite gauge charge under $U(1)^{n}$, the same charges as for $q^{1}, q^{2}$. By exhibiting the $SP(2n, \mathbb{R})$ symplectic structure of the extended model, and following the ADFT method, we show in this study that the partial breaking of effective $\mathcal{N} = 2$ supersymmetric $U(1)^{n}$ theory coupled to a complex hypermultiplet is due to a generalised moment maps $\mathcal{P}^{aM}$ extending the $\mathcal{P}^{aM}$’s of ADFT appearing in eq(1.1) and having the form $\mathcal{P}^{aM} = \mathcal{P}^{aM} + \delta \mathcal{P}^{aM}$. The moment map deformation $\delta \mathcal{P}^{aM}$ is induced by the coupling of APT theory to the complex hypermultiplet and so depends on the symplectic gauge coupling constants $(g_{i}, \eta_{i})$ and also on $\langle q^{A} \rangle$, the VEVs of the complex fields. The above ADFT deformation $\zeta_{c}$ extends, in presence of observable complex hypermatter, as follows

$$d_{a} = \frac{1}{2} \varepsilon_{abc} \mathcal{P}^{bM} C_{MN} \mathcal{P}^{cN} + \frac{1}{2} \Omega_{uv} \mathcal{N}^{u} r_{a} \mathcal{N}^{v}$$

(1.3)

and is given by the sum of four contributions like $d_{a} = \zeta_{a} + \alpha_{a} + \beta_{a} + \mathcal{J}_{a}$ where $\zeta_{a}$ is as in eq(1.1) and the others coming from hypermatter and its couplings with gauge degrees of freedom. In this relation, the first three contributions $\zeta_{a} + \alpha_{a} + \beta_{a}$ are proportional to the $SP(2n, \mathbb{R})$ metric $C_{MN}$ of the Coulomb branch, while $\mathcal{J}_{a}$ is proportional to the symplectic $\Omega_{uv}$.
in the hypermatter sector. After giving the $\mathcal{N} = 2$ supersymmetry partial breaking condition in terms of above abstract isovector $d_a$, we turn to show that the Antoniadis, Derendinger and Jacot (ADJ) scalar potential $\mathcal{V}_{\text{ADJ}}$ and the partial supersymmetry breaking condition obtained in [6] can be recovered by making a particular choice of the components of $\mathbb{P}^a M$ and $\mathcal{N}_A^u$.

The organization of this paper is as follows: In section 2, we give a short review of the ADFT method by focussing on key points; in particular on the derivation of (1.1). In section 3, we couple the APT model to a complex hypermultiplet by using the to ADFT method and develop the study of rigid Ward identity as well as its implication on the structure of the induced scalar potential and the isovector anomaly. In section 4, we first give the condition for $\mathcal{N} = 2$ supersymmetry partial breaking to occur; then we re-derive the ADJ model of [6] by choosing particular values of the components of the moment maps $\mathbb{P}^a M$. Section 5 is devoted to the conclusion;

2 Rigid Ward identity in ADFT method

In this section, we give the main lines of the derivation of the scalar potential $\mathcal{V}_{\text{APT}}^{\mathcal{N}=2}$ and the $2 \times 2$ matrix anomaly $C^A_B$ in the rigid limit of $\mathcal{N} = 2$ supergravity coupled to n abelian vector multiplets and one real hypermultiplet with $SO(4,1)/SO(4)$ quaternionic Kahler geometry by following the ADFT method. In gauged $\mathcal{N} = 2$ supergravity theory, the induced scalar potential $\mathcal{V}_{\text{sugra}}$ is related to the supersymmetric transformations $\delta \psi^A$ of the fermions via the following Ward identity [7],

$$
\delta^A_B \mathcal{V}_{\text{sugra}} = \sum_\tau \alpha_\tau \delta C \psi^\tau A \delta \bar{C} \bar{\psi}^\tau_B
$$

(2.1)

Here, the summation is taken over all the $\psi^{\tau A}$ fermions in the gauged $\mathcal{N} = 2$ supergravity theory; it includes the two gravitini, the gaugini and the hyperini. The $\alpha_\tau$ constants are positive for the spin $\frac{1}{2}$ fermions and negative for the gravitini.

In the rigid limit considered in [8,9], the above supergravity Ward identity (2.1) becomes

$$
\mathcal{G}_{ij} (W^i)^A_C (\bar{W}^j)^C_B = \delta^A_B \mathcal{V}_{\text{APT}}^{\mathcal{N}=2} + C^A_B
$$

(2.2)

where $\mathcal{V}_{\text{APT}}^{\mathcal{N}=2}$ is the APT potential and $C^A_B = \zeta_c C^{(\tau)}_{cA} B$ is an anomalous term which has been shown to have an interpretation in terms of symplectic hyper-Kahler moment maps $\mathbb{P}^a M$ as in (1.1). The other quantities involved in this identity are briefly described below. First, $\mathcal{G}_{ij}$ is the metric of the rigid special Kahler manifold of the Coulomb branch, it is given by $\text{Im} \mathcal{F}_{ij}$ which is the imaginary part of the second derivative of the holomorphic prepotential $\mathcal{F}$ of the effective $\mathcal{N} = 2$ supersymmetric $U(1)^n$ gauge theory [6]. The $n$ quantities $(W^i)^A_B$
are $2 \times 2$ matrices involved in the supersymmetric transformations of the $n$ gaugini doublets $\lambda^{iA}$; they are given by
\[
\delta_B \lambda^{iA} = (W^i)_B^A
\] (2.3)
where $(W^j)^A_B$ matrices read explicitly as follows
\[
(W^j)^A_B = \begin{pmatrix}
   iD^j & -\sqrt{2}F^j \\
   \sqrt{2}F^j + \frac{1}{2}\sqrt{2}m^j & -iD^j
\end{pmatrix}
\] (2.4)

whith $\frac{1}{\sqrt{2}}m^j$ has an interpretation in terms of magnetic FI charges [6]. By substituting the auxiliary fields $(F^j, \bar{F}^j, D^j)$ by their explicit expressions derived from the effective field action of the $\mathcal{N} = 2$ supersymmetric gauge theory, we can bring the above matrices into the following form
\[
(W^i)_B^A = iG^i_{\bar{J}}(\tau_a)_B^A W^i_a, \quad W^i_a = P^a_M \bar{U}^M_i
\] (2.5)
where $P^a_M$ are the moment maps of the 4-dim hyperKahler geometry in the hidden sector of the rigid limit of $\mathcal{N} = 2$ gauged supergravity [8, 9]. An explicit expression of these $P^a_M$'s is given by the values of APT model; its reads in terms of $\mathcal{N} = 2$ electric $e^a_i$ and magnetic $m^{ai}$ Fayet-Iliopoulos charges as follows
\[
P^a_M = \sqrt{2} \begin{pmatrix}
   m^{ai} \\
   e^a_i
\end{pmatrix}
\] (2.6)

In this formulation, the electric $e^a_i$'s and the magnetic $m^{ai}$'s are thought of as real isotriplets; and so the moment maps are also real isotriplets carrying moreover a symplectic quantum number. The complex quantity $U^M_i$ is the gradient of the holomorphic $\text{Sp}(2n, \mathbb{R})$ symplectic section $V^M$ of the rigid special Kahler geometry. In the local coordinate frame where the homogeneous $(X^0, X^I)$ are taken as $(1, \delta^I_i z^i)$, the expressions of $V^M$ and $U^M_i$ read like
\[
V^M = \begin{pmatrix}
   z^i \\
   \mathcal{F}_i
\end{pmatrix}, \quad U^M_i = \frac{\partial}{\partial z^i} V^M
\] (2.7)

The complex $z^i$ are the $n$ scalar fields belonging to the $\mathcal{N} = 2$ supersymmetric $U(1)^n$ vector supermultiplets and $\mathcal{F}_i$ are the symplectic dual of $z^i$; they are given by the gradient of the holomorphic prepotential $\mathcal{F}(z)$ of the effective theory. Putting (2.3) back into (2.2), we can obtain explicit expressions of the APT potential $V_{\text{APT}}^{N=2}$ and the rigid anomaly $C_B^A$ in terms of the following quantities: (i) the symplectic isotriplet moment maps $P^a_M$ of the $\text{SO}(4,1)/\text{SO}(4)$ quaternionic Kahler manifold, (ii) the metric $\mathcal{G}_{ij}$ of the special Kahler manifold; and (iii) the holomorphic sections $U^M_i$; they read as follows
\[
V_{\text{APT}}^{N=2} = \frac{1}{2} P^a_M \mathcal{M}^{MN} P^b_N \\
C_B^A = \frac{1}{2} \varepsilon^{abc} (P^a_M \mathcal{C}^{MN} P^b_N) (\tau^c)_B^A
\] (2.8)
where $\mathcal{C}^{MN}$ is the invariant symplectic metric of $\text{SP}(2n, \mathbb{R})$ and the symplectic coupling matrix $\mathcal{M}^{MN}$ is given by

$$
\mathcal{M}^{MN} = 2U^M_i g^{ij} \tilde{U}^N_j + i\mathcal{C}^{MN}
$$

It is related to the prepotential $\mathcal{F}$ as follows [7, 12]

$$
\mathcal{M}_{MN} = \left( \begin{array}{cc}
\text{Im} \mathcal{F} + \text{Re} \mathcal{F} (\text{Im} \mathcal{F})^{-1} \text{Re} \mathcal{F} & - \text{Re} \mathcal{F} (\text{Im} \mathcal{F})^{-1} \\
-(\text{Im} \mathcal{F})^{-1} \text{Re} \mathcal{F} & (\text{Im} \mathcal{F})^{-1}
\end{array} \right)
$$

Notice that in the left hand of the rigid Ward identity (2.2), there is only a contribution coming from the gauge sector of the theory. In next section, we study the generalisation of this Ward identity by implementing the contribution coming from hypermatter sector. This requires extending the APT model by implementing gauge invariant couplings to complex hypermatter.

## 3 Coupling APT model to hypermatter

In this section, we use the ADFT method introduced above to study the extension of the APT model by implementing gauge invariant couplings between the $n$ vector multiplets and $n_H$ complex hypermultiplets. Here, we will assume that the complex scalar fields of the hypermultiplets parameterise the flat complex hyperKahler manifold $\mathbb{C}^{2n_H}$.

### 3.1 Rigid Ward identity in extended ADFT

The rigid Ward identity of ADFT method coupled to complex hypermatter has two main contributions: a gauge contribution coming from the pure ADFT sector and a matter one coming from complex hypermatter. Before studying these contributions it is interesting to introduce the bosonic and fermionic degrees of freedom of a complex on shell hypermultiplet $H$. The usual four real bosonic degrees are described here by two complex fields $q_1, q_2$ carrying opposite gauge charges under $U(1)^n$ gauge symmetry group of the APT theory. They will be collectively denoted like $q^u$ with $u = 1, 2$. The fermionic degrees of the complex hypermultiplet are given by a pair of complex Weyl $\psi$ and $\xi$ carrying opposite U(1) gauge charges and are collectively denoted by $\chi^u = (\psi, \xi)$. Bosonic and fermionic degrees form two chiral supermultiplets $\{q^u, \chi^u\}$ and can be described by $\mathcal{N} = 1$ chiral superfields $Q^u$ carrying opposite gauge charges. In the case where there are $n_H$ complex hypermultiplets $H^I$, the bosonic and fermionic degrees of freedom are described by adding the extra index $I = 1, ..., n_H$ [13][14]. For convenience, we shall restrict our analysis below to one complex hypermultiplet by dropping out the $I$ index; the generalisation for the particular $\mathbb{C}^{2n_H}$ hyperKahler geometry is straightforward and omitted. In the case $n_H = 1$, the rigid fermionic
transformation generalising (2.3) are given by [6]

\[ \delta \lambda^i_A = (W^i)^A_B \epsilon^B \]
\[ \delta \chi^u = 2 (N^u_A) \epsilon^A \]  

(3.1)

where the \( \lambda^i_A \)'s are the \( n \) gaugini doublets of section 2 and the \( \chi^u \) referring to the 2 hyperini of hypermatter. The scalar matrices \( W^i \) and \( N^u_A \) are given by:

\[ N^u_A = K^u_A G_M \bar{V}^M \]
\[ (W^i)^A_B = i G^i (\tau_a)^A_B (\mathbb{P}^a_M \bar{U}^M_j) \]

(3.2)

with \( \bar{V}^M \) is the antiholomorphic section of the rigid special geometry given by (2.7). The \( G_M \) and \( K^A^u \) stand respectively for the electric- magnetic gauge charges and Killings

\[ G^M = \begin{pmatrix} \eta^i \\ g_i \end{pmatrix} \]
\[ K^A^u = \begin{pmatrix} k^u \\ \bar{k}^u \end{pmatrix} \]

(3.3)

with \( k^u \) giving the Killing vectors of the complex hypermultipllet manifold explicitly given by

\[ k^u = -i \begin{pmatrix} q^1 \\ -q^2 \end{pmatrix} \]
\[ \bar{k}^u = i \begin{pmatrix} \bar{q}^1 \\ -\bar{q}^2 \end{pmatrix} \]

(3.4)

In the second relation of the system of equations (3.2), the \( G^i^j \) and \( \bar{U}^M_j \) are as in (2.5); but the \( \mathbb{P}^a_M \)'s are generalized moment maps. Recall that the real scalars \( (\varphi, \phi^a) \) parametrize the real hyper-Kahler manifold \( SO(4,1) \), which lives in the hidden sector of the rigid limit of \( N = 2 \) gauged supergravity [8,9]; while the complex scalars \( q^u = (q^1, q^2) \) are coordinates of the hyperKahler manifold \( \mathbb{C}^2 \). Thus, one can expect that the \( \mathbb{P}^a_M \) has two main contributions; the previous \( \mathcal{P}^a_M \) coming from the \( SO(4,1)/SO(4) \) factor of the hidden sector and an extra \( \mathcal{R}^a_M \) descending from the complex hypermultipllet parameterising \( \mathbb{C}^2 \). So, we can define \( \mathbb{P}^a_M \) as follows

\[ \mathbb{P}^a_M = \mathcal{P}^a_M + \mathcal{R}^a_M \]
\[ \mathbb{P}^a_M \bigg|_{ADFT} = \mathcal{P}^a_M \]

(3.5)

with

\[ \mathcal{P}^a_M = \sqrt{2} \begin{pmatrix} m^a_i \\ e_i^a \end{pmatrix} \]
\[ \mathcal{R}^a_M = \sqrt{2} \begin{pmatrix} \mathcal{M}^a_i \\ \mathcal{E}^a_i \end{pmatrix} \]

(3.6)

where \( \mathcal{R}^a_M \) is due to hypermatter. The splitting (3.5) implies that \( (W^i)^A_B \) splits as well as the sum of two terms as follows

\[ (W^i)^A_B = (W^i)^A_B + (Y^i)^A_B \]

(3.7)
with \((W^i)^A_B\) given by (2.5) of the ADFT and
\[
(Y^i)^A_B = i\mathcal{G}^{ij} (\tau_a)^A_B (\mathcal{R}_M^a \mathcal{U}_j^M) \tag{3.8}
\]
Moreover, because of the presence of complex hypermatter, the rigid Ward identity (2.2) extends as follows
\[
\mathcal{G}_{ij} \left( \mathcal{W}^i \right)^A_C \left( \mathcal{W}^j \right)_B^C + 2 \left( \mathcal{N}^u_A \right) \left( \mathcal{N}^v_A \right) = \delta^A_B \mathcal{V}^{N=2}_{\text{scal}} + \mathcal{D}^A_B \tag{3.9}
\]
where now \(\mathcal{V}^{N=2}_{\text{scal}}\) is the scalar potential of the deformed APT model which will be shown later on to lead to the ADJ potential \(\mathcal{V}^{N=2}_{\text{ADJ}}\) and a variant of it involving dyonic gauge couplings. The \(\mathcal{D}^A_B\) is an anomalous term that will be determined as well later on. These two \(\mathcal{V}^{N=2}_{\text{scal}}\) and \(\mathcal{D}^A_B\) quantities are functions of the degrees of freedom of the complex hypermultiplets; in particular of the fields \((q^1, q^2)\); they reduce to \(\mathcal{V}^{N=2}_{\text{ADJ}}\) and \(\mathcal{C}^A_B\) of the ADFT if the couplings to complex hypermultiplet are turned off. The left hand side of the rigid Ward identity (3.9) has now two blocks \(G^A_B\) and \(M^A_B\) given by
\[
G^A_B = \mathcal{G}_{ij} \left( \mathcal{W}^i \right)^A_C \left( \mathcal{W}^j \right)_B^C \quad \quad \quad M^A_B = 2 \left( \mathcal{N}^u_A \right) \left( \mathcal{N}^v_A \right) \tag{3.10}
\]
with fermionic field shifts \(\left( \mathcal{W}^i \right)^A_B\) and \(\mathcal{N}^u_A\) as in (3.2). In the next subsection we compute the above \(G^A_B\) and \(M^A_B\) in term of the moment maps (3.5) and compare the obtained expressions with the ones given by ADFT method.

### 3.2 Computing eqs (3.2)

First, we calculate the gauge sector contribution to the rigid Ward identity given by \(G^A_B\); and turn after to the contribution \(M^A_B\) coming from complex hypermatter.

#### 3.2.1 APT sector

The explicit expression of the gauge sector contribution \(G^A_B = \mathcal{G}_{ij} \left( \mathcal{W}^i \right)^A_C \left( \mathcal{W}^j \right)_B^C\) to the rigid Ward identity (3.9) is obtained by substituting \(\left( \mathcal{W}^i \right)^A_C\) by its expression in (3.2). We find after rearranging terms the following expression
\[
G^A_B = \frac{1}{2} \delta^A_B \mathcal{P}^a_M \mathcal{M}^{MN} \mathcal{P}^a_N + \frac{1}{2} \varepsilon_{abc} \left( \mathcal{P}^a_M \mathcal{C}^{MN} \mathcal{P}^b_N \right) (\tau^c)^A_B \tag{3.11}
\]
This \(2 \times 2\) matrix has four terms that can be viewed as the sum of two SU(2)\(_R\) blocks like \(\delta^A_B S_0 + S^A_B\) with \(\text{Tr} S^A_B = 0\). These two blocks are precisely the contributions to the scalar potential \(\mathcal{V}^{N=2}_{\text{scal}}\) and the \(\mathcal{D}^A_B\) anomaly coming from the transformations of gaugini.
Diagonal $S_0$ term

The isosinglet $S_0$ is given by $\frac{1}{2}Tr \left( G_B^A \right)$; it reads in terms of the generalised moment maps $P_M^a$ as follows

$$S_0 = \frac{1}{2}P_M^a M^{MN} P_N^a$$

(3.12)

where $M^{MN}$ is the inverse of (2.10) and where the generalised moment maps $P_M^a$ is given by (3.5); it has an extra $R_M^a$ contribution coming from the gauge invariant complex hypermatter couplings. By substituting $P_M^a$ in (3.12) by $P_M^a + R_M^a$, the above $S_0$ splits as the sum of three terms as follows

$$S_0 = \frac{1}{2}P_M^a M^{MN} P_N^a + \frac{1}{2} R_M^a M^{MN} R_N^a + P_M^a M^{MN} R_N^a$$

(3.13)

where the two first terms depend on $P_M^a$, but the last contribution is independent from the FI charges. So for the limit where the complex hypermatter couplings to gauge sector are turned off; i.e: the limit $R_M^a \to 0$, the isosinglet $S_0$ reduces to the ADFT relation and so the scalar potential coincides with the one of the APT model which in ADFT method is nothing but a particular field realisation of the following expression

$$S_0^{(ADFT)} = \frac{1}{2}P_M^a M^{MN} P_N^a$$

$$= G^{ij} \left[ e_i^a - \mu^k a F_{ki} \right] \left[ e_j^a - \mu^k a F_{kj} \right]$$

(3.14)

expressed in terms of the electric $e_i^a$ and magnetic $\mu^k a$ FI charges. In the limit $P_M^a \to 0$, describing the case where electric and magnetic FI charges are turned off, we have

$$S_0|_{P_M^a \to 0} = \frac{1}{2} R_M^a M^{MN} R_N^a$$

(3.15)

$$= G^{ij} \left[ \mathcal{E}_i^a - \mathcal{M}^{ka} (F)_{ki} \right] \left[ \mathcal{E}_j^a - \mathcal{M}^{ka} (F)_{kj} \right]$$

where the the electric $\mathcal{E}_i^a$ and magnetic $\mathcal{M}^{ka}$ are field dependent couplings. They are functions of the complex matter scalars $q_u$ and the electric $g_i$ and magnetic $\eta^i$ gauge coupling charges. If using the particular realisation given by eq(4.11) namely

$$\mathcal{E}_i^a = g_i F^a, \quad \mathcal{M}^{ia} = \eta^i F^a$$

(3.16)

these two factorised relations can be combined into a symplectic object as follows

$$R_M^a = G_M F^a$$

(3.17)

where $R_M^a$ is as in (3.3). So, we have

$$S_0|_{P_M^a \to 0} = \frac{1}{2} G_M M^{MN} G_N \times |F^a|^2$$

$$= G^{ij} \left[ g_i - \eta^i F_{ii} \right] \left[ g_j - \eta^j F_{jj} \right] \times |F^a|^2$$

(3.18)
By using (3.17), the third cross term in (3.13) reads as follows

\[ \mathcal{P}_M^a \mathcal{M}^{MN} \mathcal{R}_N^a = G_M \mathcal{M}^{MN} \mathcal{P}_N^a \mathcal{F}^a \]  

(3.19)

It depends on the product of the \( \mathcal{P}_N^a \) Fayet-Iliopoulos and the \( G_M \) gauge electric- magnetic coupling charges. The general expression of \( S_0 \) with non vanishing \( \mathcal{P}_M^a \) and \( \mathcal{R}_M^a \) gives the contribution to the scalar potential \( \mathcal{V}_{N=2}^{\text{ADM}} \) in eq(3.9) coming from the gaugino sector.

- **Anomalous term** \( S_B^A \)

The traceless matrix \( S_B^A \) contributes to the \( D_B^A \) anomaly in (3.9); it can be presented as an isotriplet that reads as

\[ S_B^A = \sum_{c=1}^{3} \xi_c (\tau^c)_B \]  

(3.20)

with

\[ \xi_c = \frac{1}{2} \varepsilon_{abc} \mathcal{P}_M^a \mathcal{C}^{MN} \mathcal{P}_N^b \]  

(3.21)

where \( \mathcal{C}^{MN} \) is the \( SP(2n, \mathbb{R}) \) symplectic metric. A non vanishing value of \( \xi_c \) breaks partially \( \mathcal{N} = 2 \) supersymmetry; it is then interesting to study when the norm \( \| \xi \| \) is different from zero. By substituting \( \mathcal{P}_M^a = \mathcal{P}_M^a + \mathcal{R}_M^a \) in (3.21), we learn that the isovector \( \xi_c \) splits like the sum of three isovectors as follows

\[ \xi_c = \zeta_c + \alpha_c + \beta_c \]  

(3.22)

with \( \zeta_c \) the same isotriplet as in the ADFT; but the two extra \( \alpha_c \) and \( \beta_c \) are new isotriplets induced by the presence of complex hypermatter. These three isotriplets are given by

\[ \begin{align*}
\zeta_c &= \frac{1}{2} \varepsilon_{abc} \mathcal{P}_M^a \mathcal{C}^{MN} \mathcal{P}_N^b \\
\beta_c &= \frac{1}{2} \varepsilon_{abc} \mathcal{R}_M^a \mathcal{C}^{MN} \mathcal{R}_N^b \\
\alpha_c &= \varepsilon_{abc} \mathcal{P}_M^a \mathcal{C}^{MN} \mathcal{R}_N^b
\end{align*} \]  

(3.23)

The real isovector \( \zeta_c \) is due to the electric \( e_i^a \) and the magnetic \( m^{ai} \) Fayet-Iliopoulos charges given by (2.6). It reads explicitly as follows

\[ \zeta_c = \varepsilon_{abc} (m^{ai} e_i^b - e_i^a m^{bi}) = 2 \varepsilon_{abc} m^{ai} e_i^b \]  

(3.24)

A non vanishing of the above \( \zeta_c \)'s, which reads in 3- dim vector notation like \( \vec{e}_i \wedge \vec{m}_i \), requires at least the non vanishing of some of the \( e_i^a \) components and some of the \( m^{ai} \)’s. However, in absence of magnetic \( m^{ai} \) FI, the isotriplet \( \zeta_c \) vanishes identically. The term \( \beta_c \) is different from \( \zeta_c \) as it is induced from another source, it is due to the electric \( g_i \) and magnetic \( \eta^i \) gauge coupling constants. Using (3.6), we first have

\[ \beta_c = \varepsilon_{abc} (\mathcal{M}^{ai} \mathcal{E}_i^b - \mathcal{E}_i^a \mathcal{M}^{bi}) = 2 \varepsilon_{abc} \mathcal{M}^{ai} \mathcal{E}_i^b \]  

(3.25)
By following the analysis on Dyonic gauge coupling to complex hypermatter reported in the appendix, we find that electric term $\mathcal{E}_i^a$ and the magnetic $\mathcal{M}^{ai}$ factorise as follows

$$\mathcal{E}_i^a = g_i \mathcal{E}^a \quad , \quad \mathcal{M}^{ai} = \eta^i \mathcal{M}^a \quad (3.26)$$

where $\mathcal{E}^a$ and $\mathcal{M}^a$ triplets with components depending on the scalar fields of the complex matter hypermultiplet. So, the above $\beta_c$ anomaly becomes

$$\beta_c = 2 \left( g_i \eta^i \right) \varepsilon_{abc} \mathcal{M}^a \mathcal{E}^b \quad (3.27)$$

Then, non vanishing $\beta_c$ requires non vanishing $g_i \eta^i$ and non vanishing $\vec{E} \wedge \vec{M}$. However, from the particular realisation eq(4.11) we learn that $\mathcal{E}^a = \mathcal{M}^a = \vec{\mathcal{E}}$; then $\beta_c$ vanishes identically.

$$\beta_c = 0 \quad (3.28)$$

Concerning the anomaly $\alpha_c$, it is due to both the $(e_i^a, m^{ai})$ electric- magnetic FI and the $(g_i, \eta^i)$ electric-magnetic gauge charges. By using (3.6) and (3.26) as well as setting $g_i m^{ai} = \kappa^a$, $\eta^i e_i^a = \pi^a \quad (3.29)$

we can put the $\alpha_c$ triplet into the form

$$\alpha_c = \varepsilon_{abc} \left[ \kappa^a \mathcal{E}^b - \pi^a \mathcal{M}^b \right] \quad (3.30)$$

Substituting $\mathcal{E}^a = \mathcal{M}^a = \mathcal{F}^a$, we end with

$$\alpha_c = \varepsilon_{abc} \left( \kappa^a - \pi^a \right) \mathcal{F}^b \quad (3.31)$$

### 3.2.2 More on scalar potential and anomaly $D_B^A$

The contribution $M_B^A = 2N_B^u \bar{N}_u^A$ of the hypermatter sector to the rigid Ward identity \([3.9]\) comes from the coupling between the $n$ vector $V_N^{N=2}$ multiplets and the matter hypermultiplet $H^{N=2}$. Expanding this matrix $M_B^A$ on Pauli matrices basis like $J_0 \delta_B^A + J_a (\tau^a)_B^A$ and equating with $2N_B^u \bar{N}_u^A$, it follows that

$$J_0 = \mathcal{N}_A^u \bar{N}_u^A \quad , \quad J_a = \mathcal{N}_B^u \left( \tau_a \right)_B^A \bar{N}_u^A \quad (3.32)$$

which read also like

$$J_0 = \frac{1}{2} \varepsilon_{AB} \Omega_{uv} \mathcal{N}_A^u \bar{N}_B^v \quad , \quad J_a = \frac{1}{2} \Omega_{uv} \mathcal{N}_u^a \tau_v \bar{N}_v^u \quad (3.33)$$

By substituting $\mathcal{N}_A^u$ by its expression in terms of the degrees of freedom of the complex hypermultiplet namely,

$$\mathcal{N}_A^u = \left( \eta^i \bar{F}_i - g_i \bar{z}^i \right) K_A^u \quad , \quad \bar{N}_u^A = \left( \eta^i \bar{F}_i - g_i z^i \right) \bar{K}_u^A \quad (3.34)$$
with $K^u_A$ given by (3.3),

$$K^u_A = \begin{pmatrix} -\bar{k}_u \\ k^u \end{pmatrix}, \quad \bar{K}^u_A = \begin{pmatrix} -k^u \\ \bar{k}_u \end{pmatrix}$$  \hspace{1cm} (3.35)

we can write down the explicit expression of $M^A_B$ as follows

$$M^A_B = M^2 \left( K^u_A \bar{K}^B_B \right)$$  \hspace{1cm} (3.36)

with

$$M^2 = 2 \left| \eta^j \bar{F} - g_i z^i \right|^2$$  \hspace{1cm} (3.37)

and

$$K^u_A \bar{K}^B_B = \frac{1}{2} (K^u_A \bar{K}^A_A) \delta^A_B + \frac{1}{2} (K^u_{\tau a} \bar{K}^A_u) (\tau^a)^A_B$$ \hspace{1cm} (3.38)

reading explicitly like

$$K^u_A \bar{K}^B_B = \begin{pmatrix} k^u \bar{k}_u & - (\bar{k}_u)^2 \\ - (k^u)^2 & k^u \bar{k}_u \end{pmatrix}$$  \hspace{1cm} (3.39)

Then, using (3.4), we can determine the expression of $K^u_A \bar{K}^B_B$ in terms of the complex scalars. We find

$$+ \bar{k}_u k^u = |q_1|^2 + |q_2|^2$$
$$- (k^u)^2 = (q_1)^2 + (q_2)^2$$
$$- (\bar{k}_u)^2 = (\bar{q}_1)^2 + (\bar{q}_2)^2$$  \hspace{1cm} (3.40)

Adding the obtained $M^A_B$ with the $G^A_B$ contribution (3.11) coming from the gauge sector we can compute the contribution to the $D^A_B$ anomaly and the $V^{\mathcal{N}=2}_{\text{adj}}$ scalar potential by using the rigid Ward identity (3.9) that we rewrite like

$$G^A_B + M^A_B = D^A_B + \delta B^A \psi^{\mathcal{N}=2}_{\text{scal}}$$  \hspace{1cm} (3.41)

The contribution of $M^A_B$ reads explicitly as

$$M^A_B = \mathcal{J}_0 \delta^A_B + \mathcal{J}_a (\tau^a)^A_B$$ with

$$\mathcal{J}_0 = \frac{M^2}{2} (K^u_A \bar{K}^u_B)$$
$$\mathcal{J}_a = \frac{M^2}{2} (K^u_{\tau a} \bar{K}^u_B)$$ \hspace{1cm} (3.42)

and

$$\mathcal{J}_0 = M^2 \left( |q_1|^2 + |q_2|^2 \right)$$
$$\mathcal{J}_a \tau^a = M^2 \begin{pmatrix} 0 & (\bar{q}_1)^2 + (\bar{q}_2)^2 \\ (q_1)^2 + (q_2)^2 & 0 \end{pmatrix}$$  \hspace{1cm} (3.43)

where the effective mass $M^2$ is given by (3.36). We also have

$$\mathcal{J}_a = M^2 \begin{pmatrix} \text{Re} \left[ (q_1)^2 + (q_2)^2 \right] \\ \text{Im} \left[ (q_1)^2 + (q_2)^2 \right] \\ 0 \end{pmatrix}$$  \hspace{1cm} (3.44)
Notice moreover that the $\eta^i F_i$ term in the expression of $M^2$ generalize the hypermultiplet mass parameter in the ADJ model [6]. This hypermultiplet mass term can be viewed as $M^2 = |Z|^2$ describing a BPS saturation condition [15] with complex $Z$ standing for the central charge of the $\mathcal{N} = 2$ supersymmetry theory given by

$$Z = \sqrt{2} (g_i z^i - \eta^i F_i) = \sqrt{2} G_M V^M$$

(3.45)

where the $g_i$’s are the electric charges and the $\eta^i$’s their magnetic partners. Notice as well that using (3.3) and $M^2 = |Z|^2$, we can express $J_0$ and $J_a$ in terms of the symplectic gauge coupling constants $G_M$, the holomorphic section $V^M$ and its complex conjugate $\bar{V}^N$ as follows

$$J_0 = G_M G_N \left(K_u \bar{K}_u\right) V^M \bar{V}^N$$

$$J_a = G_M G_N \left(K_u \tau_a \bar{K}_u\right) V^M \bar{V}^N$$

(3.46)

showing that they are proportional to $(G_M)^2$, the square of the symplectic gauge coupling constants with coefficients like $g_i g_j$, $g_i \eta^j$ and $\eta^i \eta^j$. From the rigid Ward identity (3.9), we obtain the following scalar potential

$$\mathcal{V}_{\text{scal}}^{\mathcal{N}=2} = S_0 + J_0$$

(3.47)

where $S_0$ is as in (3.13). We also obtain the matrix anomaly $D^A_B$ which reads as follows

$$D^A_B = S^A_B + J^A_B$$

(3.48)

with $S^A_B$ given by eqs (3.20-3.23). Moreover, being traceless, this $D^A_B$ matrix can be also expressed like

$$D^A_B = \sum d_a (\tau^a)^A_B$$

(3.49)

with

$$d_a = \zeta_a + \alpha_a + \beta_a + J_a$$

(3.50)

A non vanishing norm of the isotriplet $d_a$ can break partially supersymmetry as discussed in next section.

### 4 Partial breaking and ADJ model

In this section, we study the partial breaking of $\mathcal{N} = 2$ supersymmetry in the generalised APT model constructed in section 3. This generalized APT model contains also the model of ADJ as a particular choice of the dyonic FI charges and couplings.
4.1 Partial breaking

As in the ADFT method, the partial breaking of extended supersymmetry can occur whenever the \( D_B^A \) matrix deformation given by eq (3.48) has a non vanishing VEV. This condition can be stated as

\[
\langle D_B^A \rangle \neq 0 \quad \Leftrightarrow \quad \langle \zeta_a \rangle + \langle \alpha_a \rangle + \langle \beta_a \rangle + \langle J_a \rangle \neq 0
\]  

(4.1)

and it is fulfilled if one of the four isotriplets \( \langle \zeta_a \rangle, \langle \alpha_a \rangle, \langle \beta_a \rangle \) and \( \langle J_a \rangle \) is different from zero. If two of these VEVs or more are different from zero, one has to ensure that their sum is non zero. This feature can be established by starting from the rigid Ward identity (3.9) that we rewrite it as follows

\[
G_B^A + M_B^A = \delta_B^A \mathcal{V}_{\text{scal}}^{\mathcal{N}=2} + D_B^A
\]

(4.2)

with scalar potential \( \mathcal{V}_{\text{scal}}^{\mathcal{N}=2} \) given by eqs (3.47). By performing a similarity transformation on (4.2) by multiplying its members on right by the matrix transformation \( U \) and on left by its inverse \( U^{-1} \), we can diagonalise the traceless matrix anomaly \( D_B^A \) like

\[
\tilde{D}_B^A = \begin{pmatrix} \|d\| & 0 \\ 0 & -\|d\| \end{pmatrix}, \quad \tilde{D} = U^{-1} D U
\]

(4.3)

where \( \|d\| \) stands for the norm of the isovector \( d^a \) in eq (3.49). Under this transformation, the rigid Ward identity becomes

\[
\tilde{G}_B^A + \tilde{M}_B^A = \begin{pmatrix} \mathcal{V}_{\text{scal}}^{\mathcal{N}=2} + \|d\| & 0 \\ 0 & \mathcal{V}_{\text{scal}}^{\mathcal{N}=2} - \|d\| \end{pmatrix}
\]

(4.4)

So the partial breaking occurs when

\[
\mathcal{V}_{\text{scal}}^{\mathcal{N}=2} = \pm \|d\|, \quad d \neq 0
\]

(4.5)

In what follows, we give an illustration of this condition and its solution by first showing how the ADJ model can be recovered from this construction; and how (4.5) is realized in terms of the electric and magnetic FI charges.

4.2 Deriving the ADJ model

To recover the ADJ model, we give particular values to the moment maps in (3.6). This is obtained by making appropriate choices of the charges namely: (1) the values of the \( (e_i^a, m^ia) \) electric/magnetic FI charges; and (2) the values of the \( (g_i, \eta^i) \) gauge coupling charges. For the case of the electric \( e_i^a \) and magnetic \( m^ia \) FI charges, which can be written as follows

\[
\mathcal{L}_{FI}^{(\text{elec})} = e_i^a Y^{ai}
\]

(4.6)
where
\[ e_a^i = \frac{1}{4} \begin{pmatrix} -\text{Im} e_i \\ \text{Re} e_i \\ \sqrt{2} E_i \end{pmatrix} \quad (4.7) \]
are the electric FI charges vector, while the \( SU(2)_R \) triplets \( Y^{ai} \) are the \( \mathcal{N} = 2 \) auxiliary fields,
\[ Y^{ai} = \begin{pmatrix} -2 \text{Im} F^i \\ -2 \text{Re} F^i \\ \sqrt{2} D^i \end{pmatrix} \quad (4.8) \]
Using the above \( SO(3) \sim SU(2)_R \) vector \( Y^{ai} \), we can write the magnetic FI term of \([6]\) as follows
\[ \mathcal{L}_{FI}^{(mag)} = \text{Im} \mathcal{F}_{ij} m^{ia} Y^{aj}, \]
where the electric FI charges vector \( m^{ia} \) reads explicitly as follows
\[ m^{ia} = \frac{1}{4} \begin{pmatrix} m^i \\ 0 \\ 0 \end{pmatrix} \quad (4.9) \]
Moreover, as was shown in \([6]\), the complex hypermultiplet contributions modify the above electric FI charges as follows:
\[ \mathcal{L}_{hyp} + \mathcal{L}_{int} \supset \int d^4 \theta \left[ Q_1 e^{-2gY} Q_1 + Q_2 e^{2gY} Q_2 \right] + \int d^2 \theta \left[ i\sqrt{2} \left( \sum_{l=1}^{n} g_l X^l \right) Q_1 Q_2 \right] + h.c \\
= \mathcal{E}_i^{a} Y^{ia} + ... \]
with \( Y^{ia} \) are the auxiliary field vectors \([4.8]\) and
\[ \mathcal{E}_i^{a} = g_i F^a \quad (4.10) \]
where
\[ F^a = \sqrt{2} \begin{pmatrix} \text{Im} (i q_1 q_2) \\ \text{Re} (i \bar{q}_1 \bar{q}_2) \\ -\frac{1}{2} \left[ |q_1|^2 - |q_2|^2 \right] \end{pmatrix} \quad (4.11) \]
Thus, from the above discussion we conclude that the ADJ model is obtained by choosing the FI and coupling charges \( e_a^i, m^{ia} \) and \( \mathcal{E}_i^{a} \) in \([3.6]\) as respectively in the equations \((4.7), (4.9)\) and \((4.10)\) while setting \( M^{ia} = 0 \).
We note that thanks to the magnetic coupling, in the appendix, one can have the following magnetic term
\[ \mathcal{L}_{CS} \supset \int d^2 \theta \sum_{l=1}^{n} \eta^l \mathcal{F}_l \left( i\sqrt{2} Q_1 Q_2 \right) + h.c \\
= \text{Im} \mathcal{F}_{ij} \mathcal{M}^{ia} Y^{aj} \]
with $\mathcal{M}^{a}$ being the following magnetic coupling charge vectors

$$
\mathcal{M}^{ai} = \eta^{i} \mathcal{M}^{a} = \eta^{i} \begin{pmatrix}
\sqrt{2} \text{Im} (i q_1 q_2) \\
\sqrt{2} \text{Re} (i \bar{q}_1 \bar{q}_2) \\
0
\end{pmatrix} \quad (4.12)
$$

By substituting the above choices back into eq(3.47), the effective scalar potential $V_{scal} = S_0 + \mathcal{J}_0$ is given by

$$
V_{scal} = 2 |\eta^i \mathcal{F}_i - g_i z^i|^2 \bigg|_{\eta=0} (|q_1|^2 + |q_2|^2) \\
+ \frac{1}{16} G^{ij} \left( e_i + i m^k \mathcal{F}_{ik} - 4i \sqrt{2} g_i \right) \left(e_j - i m^l \mathcal{F}_{jl} + 4i \sqrt{2} g_j \bar{q}_1 \bar{q}_2 \right) \\
+ \frac{1}{8} G^{ij} \left[ \xi_i - 2g_i (|q_1|^2 - |q_2|^2) \right] \left[ \xi_j - 2g_j (|q_1|^2 - |q_2|^2) \right] \quad (4.13)
$$

Comparing this expression with the ADJ model, we learn that $V_{scal}$ is precisely the scalar potential $V_{ADJ}^{N=2}$ given by eq(3.1) of [6] where the scalar field $z_k$ is shifted by a constant like

$$
z_k \rightarrow z_k - \frac{im g_k}{\sqrt{2} g^2}, \quad g^2 = \sum_{i=1}^{2} g_i \quad (4.14)
$$

These shifts of $z_k$ correspond to replace the electric central charge $Z = i \sqrt{2} (\eta^i \mathcal{F}_i - g_i z^i)|_{\eta=0}$ of our construction by

$$
Z = m + ig_i z^i \sqrt{2} \quad (4.15)
$$

Notice that VEV of $V_{scal}$ depends on the coupling constants and the VEVs of the scalar fields of the theory namely the complex doublet $(q_1, q_2)$ and the symplectic $(z^i, F_i)$. If for instance the condition $\frac{\partial V_{scal}}{\partial q_0} = 0$ is solved by $\langle q_0 \rangle = 0$, we have

$$
\langle V_{scal} \rangle = \frac{1}{16} G^{ij} \left( e_i + i m^k \langle F_{ik} \rangle \right) \left(e_j - i m^l \langle F_{jl} \rangle \right) + \frac{1}{8} G^{ij} \xi_i \xi_j \geq 0 \quad (4.16)
$$

with $\langle F_{ik} \rangle = \mathcal{F}_{ik} \langle \langle z \rangle \rangle$ where the $\langle z^i \rangle$'s solve the condition $\frac{\partial V_{scal}}{\partial z^i} = 0$. For these values the anomaly vector (3.51) reads as

$$
\langle d_i \rangle = 2 \epsilon_{abc} m^{ai} e^{b}_c = 2 (\bar{m}^i \wedge \bar{e}^i) \quad (4.17)
$$

where $e_i^a$ and $m^b$ are the electric and magnetic charges eqs(4.17, 4.19). Thus, the norm $\langle |d| \rangle$ characterising the partial breaking of $\mathcal{N} = 2$ supersymmetry reads as follows

$$
\langle |d| \rangle = \frac{1}{8} \sqrt{m^i m^j \left( \text{Re} e_i \text{Re} e_j + 2 \xi_i \xi_j \right)} \quad (4.18)
$$

If we further choose the complex electric FI charges $e_i$ to be pure imaginary, the VEVs of the scalar potential (4.16) and the norm (4.18) become

$$
\langle V_{scal} \rangle = \frac{1}{16} G^{ij} \left( \text{Im} e_i + m^k \mathcal{F}_{ik} \right) \left( \text{Im} e_j + m^l \bar{F}_{jl} \right) + \frac{1}{8} G^{ij} \xi_i \xi_j \\
\langle |d| \rangle = \frac{1}{4 \sqrt{2}} m^i \xi_i \quad (4.19)
$$
Hence, the partial breaking condition (4.5) becomes

\[ G^\bar{j} \left( \text{Im} e_i + m^k \mathcal{F}_{ik} \right) \left( \text{Im} e_j + m^l \mathcal{F}_{jl} \right) + 2G^\bar{j} \xi_i \xi_j = \pm 2\sqrt{2} m^i \xi_i \quad (4.20) \]

which can be rewritten as follows

\[ \left( \text{Im} e_j + m^k \text{Re} \mathcal{F}_{kj} \right)^2 + \left( G^\bar{j} m^j \mp \sqrt{2} \xi_i \right)^2 = 0 \quad (4.21) \]

and so has the following solutions

\[ \xi_i = \pm \frac{1}{\sqrt{2}} G^\bar{j} m^j , \quad \text{Im} e_j = -m^k \text{Re} \mathcal{F}_{kj} \quad (4.22) \]

which coincides with the partial breaking condition given in eq(4.9) of the ADJ model [6].

5 Conclusion

In this paper we have studied the partial breaking of rigid \( \mathcal{N} = 2 \) supersymmetric gauge theory of \( n \) vector multiplets coupled to complex hypermultiplets by using the ADFT method given in [7]. To that purpose, we have first reviewed the ADFT method where partial breaking of rigid supersymmetry is induced by an anomalous isotriplet vector \( \zeta_a \) originating from the hidden sector in the rigid limit of gauged \( \mathcal{N} = 2 \) supergravity. This isovector has the form \( \zeta_a \sim \varepsilon_{abc} m^a e^b \) with the isotriplet \( e^a \) and \( m^a \) standing for the electric and magnetic Fayet- Iliopoulos terms. In our construction, which may be viewed as a generalisation of ADFT methods by adding complex hypermatter coupled to gauge degrees of freedom, we showed that the anomaly \( \zeta_a \) gets three extra contributions given by (4.1) namely \( \langle d_a \rangle = \langle \zeta_a \rangle + \langle \alpha_a \rangle + \langle \beta_a \rangle + \langle J_a \rangle \). The \( \zeta_a \) is as in ADFT methods, and the three other contributions \( \langle \alpha_a \rangle, \langle \beta_a \rangle \) and \( \langle J_a \rangle \) are induced by the presence of complex hypermatter. The \( \langle \alpha_a \rangle \) is induced by the coupling between FI constant and the electric-magnetic coupling charges. The \( \langle \beta_a \rangle \) is due to the coupling between electric and magnetic charges and \( \langle J_a \rangle \) to local dyonic mass. Non zero contribution of these anomalies are dependent on non-zero VEVS of the hypermatter fields \( \langle q^u \rangle \) determined by minimizing the scalar potential. Their expressions have been explicitly studied in subsection 3.2.

In the extension of the ADFT method developed in this paper, we also gave the rigid Ward identity and the induced scalar potential of as well as the general condition for partial breaking. By choosing a particular values of the components of our generalized moment maps, we derived as well the scalar potential of the ADJ model and their partial breaking condition. It would be interesting to obtain the extended APT method, studied in the present paper, as an observable sector in the rigid limit of \( \mathcal{N} = 2 \) gauged supergravity coupled to complex hypermatter. Progress in this direction will be reported in a future occasion.
A Dyonic couplings

First, we recall that in $\mathcal{N} = 2$ supersymmetry one can distinguish two multiplets:

- Vector multiplet: which has the following $\tilde{\theta}$- expansion
  $W(x, \theta, \tilde{\theta}) = X(x, \theta) + i\sqrt{2}\tilde{\theta}W(x, \theta) + \tilde{\theta}^2\left(-\frac{1}{4}\bar{D}^2\bar{X}^i\right)$ (A.1)

  where $X(x, \theta)$ and $W(x, \theta)$ are $\mathcal{N} = 1$ chiral superfields where the fermionic $W(x, \theta)$’s are precisely the $\mathcal{N} = 1$ superfield strengths living inside the $\mathcal{N} = 2$ ones.

- Hypermultiplet: which can be described by two $\mathcal{N} = 1$ chiral superfields $Q^u = (Q^1, Q^2)$ having opposite gauge charges and $\theta$- expansions as follows
  $Q^u = q^u + \sqrt{2}\theta\chi^u + \theta\theta G^u$ (A.2)

  where $G^u$ are auxiliary fields.

The aim of this appendix is to give the extension of ADJ model by allowing dyonic gauge couplings. To that purpose, we are interested only in the non-kinetic terms of the action, namely the quadratic complex mass term

$$L_{\text{mass}} = m \int d^2\theta Q_1 Q_2 + h.c$$ (A.3)

where $Q_1$ and $Q_2$ are the two $\mathcal{N} = 1$ chiral superfields constituting the $\mathcal{N} = 2$ hypermultiplet, and the couplings to the gauge to the $X^i$ chiral superfields, of the $\mathcal{N} = 2$ vector multiplet, given by the following tri-superfield interactions,

$$L_{\text{int}} = \int d^2\theta \left[i\sqrt{2}\left(\sum_{l=1}^{n_v} g_l X^l\right)Q_1 Q_2\right] + h.c,$$ (A.4)

where $X^l$, with $l = 1, ..., n_v$ are the first components of the $\mathcal{N} = 2$ vector superfields $W^l$.

We will start from the Lagrangian density (A.4) and show that it has an interpretation in terms of $\mathcal{N} = 2$ Chern-Simon interaction using the dual tensorial description of the hypermultiplet $(Q_1, Q_2)$. Then, we turn to study the $\mathcal{N} = 2$ dyonic gauge invariant couplings

A.1 $\mathcal{N} = 2$ Chern-Simons action

In this description, the $L_{\text{int}}$ can be derived by starting from the $\mathcal{N} = 2$ chiral superspace action

$$L_{\text{CS}} = -2i \int d^2\theta d^2\bar{\theta} \left(\sum_{l=1}^{n_v} g_l W^l\right) T^{N=2} + h.c$$ (A.5)
where $T^{N=2}$ is the $\mathcal{N} = 2$ tensor superfield with expansion as follows

$$T^{N=2} = Y(x, \theta) + \sqrt{2} \bar{\theta} \Upsilon(x, \theta) - \bar{\theta}^2 \left( \frac{i}{2} \Phi + \frac{1}{4} \bar{D}^2 \bar{Y} \right)$$  \hspace{1cm} (A.6)

In this expansion $Y(x, \theta)$ and $\Phi$ are bosonic $\mathcal{N} = 1$ chiral superfields while $\Upsilon_\alpha$ is a spinor $\mathcal{N} = 1$ chiral superfield related to the $\mathcal{N} = 1$ linear superfield $L$ like

$$L = D \Upsilon + \bar{D} \bar{\Upsilon} \quad , \quad DDL = DDL = 0$$  \hspace{1cm} (A.7)

The degrees of freedom are carried by the superfields $\Phi$ and $L$ as the $Y$ can be gauged out. Indeed, notice that in the above $\bar{\theta}$- expansion of $T^{N=2}$, the spinor superfield $\Upsilon_\alpha$ is defined up to the following gauge transformation

$$\delta_{\text{gauge}} \Upsilon_\alpha = -i W'_\alpha \quad , \quad W'_\alpha = -\frac{1}{4} \bar{D} \bar{D} D_\alpha \Delta'$$  \hspace{1cm} (A.8)

where $\Delta'$ is a general $\mathcal{N} = 1$ real superfield and $W'_\alpha$ is its field strength. The $\mathcal{N} = 2$ superspace version of (A.8) is given by

$$\delta_{\text{gauge}} T^{N=2} = -W'$$  \hspace{1cm} (A.9)

where $W'$ is a $\mathcal{N} = 2$ vector superfield strength. By $\bar{\theta}$- expansion of both sides of this transformation, we obtain

$$\delta_{\text{gauge}} Y = \frac{1}{2} \bar{D} \bar{D} \Delta'$$

$$\delta_{\text{gauge}} \Upsilon_\alpha = -i W'_\alpha$$

$$\delta_{\text{gauge}} \Phi = 0$$  \hspace{1cm} (A.10)

Notice that the gauge transformation (A.9) is just the $\mathcal{N} = 2$ superspace version of the gauge transformation $\delta_{\text{gauge}} b^{\rho\sigma} = \partial^\rho \Lambda^\sigma$, where $b^{\rho\sigma}$ is a 2-form field whose field strength 3-form is unchanged under this gauge transformation [17]. The gauge transformation (A.9) allows us to eliminate the superfield $Y'$ but one can instead choose a gauge in which has it has only a non vanishing imaginary part of the auxiliary field,

$$Y^{\text{gauged}} = \frac{i}{4!} \Theta^2 \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}$$  \hspace{1cm} (A.11)

where $F^{\mu\nu\rho\sigma}$ is a 4-form. Substituting (A.1) and (A.6) back into (A.5), we obtain the following CS lagrangian density

$$\mathcal{L}_{CS} = -\int d^2 \theta \left[ \sum_{l=1}^n g_l \left( X^l \Phi \right) + \sum_{l=1}^n g_l \left( W'_l \Upsilon^\alpha \right) + i \sum_{l=1}^n g_l \left( m^l Y^l \right) \right] + hc$$  \hspace{1cm} (A.12)

Comparing this action with (A.3-A.4), we learn that the gauge invariant $\Phi$ can be realised as the product of two chiral superchamps as follows

$$\Phi = i \sqrt{2} Q_1 Q_2$$  \hspace{1cm} (A.13)

The mass term $mQ_1 Q_2$ can be generated by using the gauge invariant shift $X^l \to X^l - \frac{i}{\sqrt{2}} \mu^l$ and setting $m = \sum_{l=1}^n g_l \mu^l$.  

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A.2 \( \mathcal{N} = 2 \) dyonic gauge couplings

Here, we want to comment on a property of the \( \mathcal{N} = 2 \) superfield lagrangian density (A.5). This superdensity has a remarkable dependence on the \( \mathcal{N} = 2 \) prepotentials namely on the quantity

\[
\Theta = \left( \sum_{l=1}^{n} g_l \mathcal{W}^l \right) \mathcal{T}^{N=2} \tag{A.14}
\]

which leads, after integration with respect to \( \tilde{\theta} \), to

\[
\int d^2 \tilde{\theta} \Theta = \sum_{l=1}^{n} (g_l X^l) \Phi + \sum_{l=1}^{n} (g_l W_l^\alpha) \Upsilon^\alpha \tag{A.15}
\]

As the \( \mathcal{N} = 1 \) chiral superfield combination \( gX = \sum_{l=1}^{n} g_l X^l \) is just a half part of a symplectic invariant \( \mathcal{N} = 1 \) chiral superfield namely

\[
\sum_{l=1}^{n} \left( g_l X^l - i \eta^l \frac{\partial \mathcal{F}}{\partial X^l} \right) = G^M C_{MN} V^M \tag{A.16}
\]

with

\[
G^M = \begin{pmatrix} i \eta^l \\ g_l \end{pmatrix}, \quad V^M = \begin{pmatrix} X^l \\ \frac{\partial \mathcal{F}}{\partial X^l} \end{pmatrix} \tag{A.17}
\]

where \( \eta^l \) are real parameters. One may think about (A.14) as just a part of the following symplectic invariant quantity

\[
\Xi = \left( \sum_{l=1}^{n} g_l \mathcal{W}^l - i \eta^l \frac{\partial \mathcal{F}}{\partial \mathcal{W}^l} \right) \mathcal{T}^{N=2} \tag{A.18}
\]

These relations suggest that the \( \mathcal{N} = 2 \) Chern-Simons action (A.5) can be made symplectic invariant as follows

\[
\mathcal{L}_{CS} = \int d^2 \theta d^2 \tilde{\theta} \sum_{l=1}^{n} \left( g_l \mathcal{W}^l - i \eta^l \frac{\partial \mathcal{F}}{\partial \mathcal{W}^l} \right) \mathcal{T}^{N=2} + \hbar c \tag{A.19}
\]

which, by integration with respect to \( \tilde{\theta} \), leads to

\[
\mathcal{L}_{CS} = \int d^2 \theta \sum_{l=1}^{n} (g_l X^l - i \eta^l \mathcal{F}_i) \Phi + \sum_{l=1}^{n} (g_l - i \eta^l \mathcal{F}_{jl}) W_l^\alpha \Upsilon^\alpha \tag{A.20}
\]

with

\[
\mathcal{F}_i = \frac{\partial \mathcal{F}(X)}{\partial X^i}, \quad \mathcal{F}_{ij} = \frac{\partial^2 \mathcal{F}(X)}{\partial X^i \partial X^j} \tag{A.21}
\]
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