HOMOGENIZATION OF STRATIFIED ELASTIC MEDIA WITH HIGH CONTRAST

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Abstract. We determine the asymptotic behavior of the solutions to the linear elastodynamic equations in a stratified medium comprising an alternation of possibly very stiff layers with much softer ones, when the thickness of the layers tends to zero. The limit equations may depend on higher order terms, characterizing bending effects. A part of this work is set in the context of non-periodic homogenization and an extension to stochastic homogenization is presented.

1. Introduction

In this paper, we analyze the asymptotic behavior of the solution to the linear elastodynamic equations in a composite material wherein, at a microscopic scale, possibly very "stiff" layers alternate with a much "softer" medium. Stratified composite media have been intensively investigated over the last decades, especially in the context of diffusion equations [18, 27, 29, 30, 31, 32, 39, 52, 54]. As regards linear elasticity, layered elastic composites have been studied in [26, 28, 33, 38] under assumptions of uniform boundedness and uniform definite positiveness of the elasticity tensor guaranteeing that the effective equation is a standard linear elasticity equation. When these assumptions break down, as for instance in the so-called "high contrast case", the limit equilibrium equation may be of a quite different type: it may correspond, in theory, to the Euler equation associated to the minimization of any lower semi-continuous quadratic form on $L^2$ vanishing on rigid motions [20]. In particular, it may be non-local and depend on higher order derivatives of the displacement. Elastic media with high contrast have been studied under various geometrical assumptions. Composites with stiff grain-like inclusions have been investigated in [7, 8, 45], stiff fibered structures in [8, 12, 13, 46, 50], and stiff media with holes filled with a soft material in [22, 24, 47]. Our aim is to complement this body of work in the context of stratified media. Our approach is based on the two-scale convergence method [3, 5, 19, 23, 40, 41], which yields the convergence to an effective solution. It also yields a first order corrector result in $L^2$ (see Remark 4), but not the rigorous error estimates of higher order with respect to small parameters provided by the asymptotic expansions method [1, 2, 6, 15, 16, 21, 43, 44, 45, 48, 49].

For a given bounded smooth open subset $\Omega$ of $\mathbb{R}^3$, we consider a linear elastodynamic problem like (3.5). We assume that the Lamé coefficients take possibly large values in a subset $B_\epsilon$ of $\Omega$ and much smaller values elsewhere. The set $B_\epsilon$ consists

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of a non-periodic distribution of parallel disjoint homothetic layers of thickness \( r_\varepsilon \), whose median planes are orthogonal to \( e_3 \) and separated by a minimal distance \( \varepsilon \), where \( \varepsilon, r_\varepsilon \) are positive reals converging to zero (see fig. 3.1). The effective volume fraction of the stiff phase is characterized by the parameter \( \theta \) defined by (3.10).

Both cases \( \theta = 0 \) and \( 0 < \theta < 1 \) are investigated. The order of magnitude of the Lamé coefficients in the stiff phase is determined by the parameters \( k \) and \( \kappa \) defined by (3.8).

When the elasticity coefficients in the soft phase are of order 1 and the effective volume fraction of the stiff phase vanishes the limit behavior of the composite is governed, if \( 0 < k < +\infty \), by the equation

\[
(\rho + n\overline{\rho}_1) \frac{\partial^2 u}{\partial t^2} - \text{div}(\sigma) - nk \text{div}\sigma_{\varepsilon'}(u') = (\rho + n\overline{\rho}_1) f \quad \text{in } \Omega \times (0, t_1),
\]

where \( \rho \) denotes the mass density in the softer phase, and \( u', \sigma_{\varepsilon'}, \sigma \) and \( \overline{\rho}_1 \) are defined, respectively, by (2.1), (3.13), and (3.7). The function \( n \) characterizes the rescaled effective number of sections of stiff layers per unit length in the \( e_3 \) direction and is obtained as the weak* limit in \( L^\infty(\Omega) \) of the sequence \( (n_\varepsilon) \) defined by (3.14).

When the order of magnitude of the elasticity coefficients in the stiff layers is larger, that is when \( k = +\infty \), the functions \( u_1 \) and \( u_2 \) vanish on the set \( \{ n > 0 \} \) and the behavior of \( u_3 \) is governed by the equation (3.18), (3.19) or (3.20), depending on the order of magnitude of \( \kappa \). In the case \( 0 < \kappa < +\infty \), this equation involves the 4th partial derivatives of \( u_3 \) with respect to \( x_1, x_2 \):

\[
(\rho + n\overline{\rho}_1) \frac{\partial^2 u_3}{\partial t^2} - (\text{div}\sigma(u))_3 + n\frac{\kappa}{3} \sum_{\alpha,\beta=1}^2 \frac{\partial^4 u_3}{\partial x_\alpha^2 \partial x_\beta^2} = (\rho + n\overline{\rho}_1) f_3 \quad \text{in } \Omega \times (0, t_1),
\]

revealing bending effects. The effective behavior on the set \( \{ n = 0 \} \) is that of a homogeneous material without stiff layers. In Theorem 3, we extend these results to the stochastic case. The set \( B_\varepsilon(\omega) \) then depends on a random element \( \omega \) of some sample space \( \mathcal{D} \subset 2^R \) equipped with a probability \( P \) satisfying (3.22). The limit problem as \( \varepsilon \to 0 \) is deduced from the above equations, \( P \)-almost surely, by substituting for \( n \) the conditional expectation \( E_P n_0(\omega) \) with respect to \( P \) given the \( \sigma \)-algebra \( \mathcal{F} \) of the periodic sets, of the random variable \( n_0 \) defined by (3.23).

If the order of magnitude of the elasticity coefficients in the soft interlayers is strictly smaller than 1 and strictly larger than \( \varepsilon^2 \), the effective equations are deduced from (1.1), (1.2), formally, by removing the term \( \text{div}\sigma(u) \) (see Theorem 4).

When the elastic moduli in the soft phase are of order \( \varepsilon^2 \), the effective behavior of the composite turns sensitive to the slightest geometrical perturbation (see Remark 8). The effective equation can not be expressed simply in terms of the function \( n \) as in the other cases. This characteristic renders the study of non-periodic homogenization a very difficult task: we only treat the case of an \( \varepsilon \)-periodic distribution of stiff layers. The homogenized problem then takes the form of a system of equations coupling some field \( \nu \), characterizing the effective displacement in the stiff layers, with the two-scale limit \( u_0 : \Omega \times (0, t_1) \times (-\frac{1}{2}, \frac{1}{2})^3 \to \mathbb{R}^3 \) of the solution \((u_\varepsilon)\) to (3.5) (see [3, 41]). This field \( \nu \) is obtained as the limit of the sequence \((u_\varepsilon, m_\varepsilon)\), where \( m_\varepsilon \) is the measure supported by the stiff layers defined by (3.35). If \( 0 < k < +\infty \),
the effective behavior of the displacement in the stiff medium is governed by the equation
\[
\frac{\partial^2 \mathbf{v}}{\partial t^2} - k \nabla \nabla \mathbf{v} = \mathbf{p}_1 f + g(u_0) \quad \text{in} \quad \Omega \times (0, T),
\]  
(1.3)
associated with the boundary and initial conditions given in (3.41). This equation displays stretching vibrations with regard to the transversal components \(v_1, v_2\) of \(\mathbf{v}\). It is coupled with the soft phase through the field \(g(u_0)\) which represents the sum of the surface forces applied on each stiff layer by the adjacent soft medium. This field is defined by (3.38), in terms of the restriction of \(u_0\) to \(\Omega \times (0, T) \times Y \setminus A\), which characterizes the effective displacement in the soft interstitial layers. The letters \(Y\) and \(A\) symbolize, respectively, the unit cell and the rescaled stiff layer (see (3.33), (3.34)). The effective displacement in the soft phase is governed by the equation
\[
\rho \frac{\partial^2 \mathbf{u}_0}{\partial t^2} - \nabla \nabla_3 (\sigma_0(u_0)) = \rho f \quad \text{in} \quad \Omega \times (0, T) \times Y \setminus A,
\]
where \(\sigma_0\) is defined by (3.38). This equation is coupled with the variable \(\mathbf{v}\) by the relation (3.37) on \(\Omega \times (0, t_1) \times A\). The weak limit of \((\mathbf{u}_2)\) in \(L^2(\Omega \times (0, T))\) is given by \(\mathbf{u}(x, t) = \int_Y \mathbf{u}_0(x, t, y) dy\).

When the order of magnitude of the elasticity coefficients in the stiff layers is larger, the functions \(v_1\) and \(v_2\) vanish and the effective displacement in the stiff phase is governed by the equation of \(v_3\) given by (3.42), (3.43) or (3.44), depending on the order of magnitude of \(\kappa\). In the case \(0 < \kappa < +\infty\), this equation,
\[
\frac{\partial^2 v_3}{\partial t^2} + \kappa \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^3 v_3}{\partial x_2 \partial x_3^2} = \rho f j_3 + (g(u_0))_3 \quad \text{in} \quad \Omega \times (0, t_1),
\]
involves the 4\textsuperscript{th} partial derivatives of \(v_3\) with respect to \(x_1, x_2\), characterizing bending vibrations. Otherwise, the stiff layers display the behavior of a collection of unstretchable membranes if \((k, \kappa) = (+\infty, 0)\) and that of fixed bodies if \(\kappa = \infty\).

Our results apply as well to the case of equilibrium equations and to multiphase composites (see remarks 6, 7, 10, 11). The paper is organised as follows: in Section 2 we specify the notations and in Section 3 we state our main results. In Section 4, we recall some classical results and introduce a non-periodic variant of the two-scale convergence for which we establish a compactness result. The effective equations are derived in Section 6 by employing apriori estimates demonstrated in Section 5, and a technical lemma proved in the appendix.

2. Notations

In this article, \(\{\mathbf{e}_1, ..., \mathbf{e}_N\}\) stands for the canonical basis of \(\mathbb{R}^N\). Points in \(\mathbb{R}^N\) or in \(\mathbb{Z}^N\) and real-valued functions are represented by symbols beginning by a lightface lowercase (example \(x, i, y, \det \mathbf{A}, ...\)) and vectors and vector-valued functions by symbols beginning by a boldface lowercase (examples: \(\mathbf{x}, \mathbf{i}, \mathbf{u}, \mathbf{f}, \mathbf{g}, \nabla \nabla_3 \mathbf{x}, ...\)). Matrices and matrix-valued functions are represented by symbols beginning by a boldface uppercase (linearized strain tensor). We denote by \(u_i\) or \(u_i(i)\) the components of a vector \(\mathbf{u}\) and by \(A_{ij}\) or \((A)_{ij}\) those of a matrix \(\mathbf{A}\) (that is \(\mathbf{u} = \sum_{i=1}^N u_i \mathbf{e}_i = \sum_{i=1}^N (u_i) \mathbf{e}_i;\) \(\mathbf{A} = \sum_{i,j=1}^N A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \sum_{i,j=1}^N (A)_{ij} \mathbf{e}_i \otimes \mathbf{e}_j\)). We do not employ the usual repeated
index convention for summation. We denote by $A : B = \sum_{i,j=1}^{N} A_{ij} B_{ij}$ the inner product of two matrices, by $\varepsilon_{ijk}$ the three-dimensional alternator, by $u \wedge v = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} u_{i} v_{k} e_{j}$ the exterior product in $\mathbb{R}^{3}$, by $\mathbb{S}^{M}$ ($M \in \mathbb{N}$) the set of all real symmetric matrices of order $M$, by $I_{M}$ the $M \times M$ identity matrix. The symbol $\sharp D$ denotes the cardinality of a finite set $D$. The letter $C$ stands for different constants whose precise values may vary. For any weakly differentiable vector field $\psi : \Omega \subset \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$, we set

$$\psi' := \psi_{1} e_{1} + \psi_{2} e_{2}; \quad e_{\sigma'}(\psi) := \sum_{\alpha,\beta=1}^{2} \frac{1}{2} \left( \frac{\partial \psi_{\alpha}}{\partial x_{\beta}} + \frac{\partial \psi_{\beta}}{\partial x_{\alpha}} \right) e_{\alpha} \otimes e_{\beta} = e_{\sigma'}(\psi') \quad (2.1)$$

We reproduce and modify here some notations from [8]: we denote by $C_{\varepsilon}^{\infty}(Y)$ (resp. $C_{\varepsilon}(Y)$) the set of $Y$-periodic functions of $C^{\infty}(\mathbb{R}^{3})$ (resp. $C(\mathbb{R}^{3})$), by $C_{\varepsilon}^{\infty}(Y \setminus B)$ the set of the restrictions of the elements of $C_{\varepsilon}^{\infty}(Y)$ to $Y \setminus B$, by $H_{1}^{1}(Y)$ (resp. $H_{1}^{1}(Y \setminus B)$) the completion of $C_{\varepsilon}^{\infty}(Y)$ (resp. $C_{\varepsilon}^{\infty}(Y \setminus B)$) with respect to the norm $w \rightarrow (\int_{Y}(|w|^{2} + |\nabla w|^{2})dy)^{\frac{1}{2}}$ (resp. $w \rightarrow (\int_{Y \setminus B}(|w|^{2} + |\nabla w|^{2})dy)^{\frac{1}{2}}$). For any subset $Q$ of the unit cell $Y$, the symbol $Q_{z}$ stands for the periodization on all $\mathbb{R}^{3}$ of $Q$, that is

$$Q_{z} := \bigcup_{z \in \mathbb{Z}^{3}} \{z\} + Q \quad (2.2)$$

3. Setting of the problem and results

We consider a cylindrical domain $\Omega := \Omega' \times (0, L)$, where $\Omega'$ is a bounded smooth domain of $\mathbb{R}^{2}$. Given a small positive real $\varepsilon$, the non-periodic distribution $B_{\varepsilon}$ of disjoint homothetical stiff layers $B_{\varepsilon}^{j}$ will be described in terms of a finite subset $\omega_{\varepsilon}$ of $\mathbb{R}$

$$\omega_{\varepsilon} := \{\omega_{\varepsilon}^{j} : j \in J_{\varepsilon}\} \quad (3.1)$$

satisfying

$$\omega_{\varepsilon} \subset (0, L), \quad \min_{j,j' \in J_{\varepsilon}, j \neq j'} |\omega_{\varepsilon}^{j} - \omega_{\varepsilon}^{j'}| = \varepsilon, \quad \text{dist}(\omega_{\varepsilon}, \{0, L\}) > \frac{\varepsilon}{2} \quad (3.2)$$

and of a small parameters $r_{\varepsilon}$ verifying

$$\varepsilon > r_{\varepsilon}(1 + \delta) \quad \text{for some } \delta > 0, \quad (3.3)$$

by setting (see Fig. 3.1)

$$B_{\varepsilon} := \bigcup_{j \in J_{\varepsilon}} B_{\varepsilon}^{j}; \quad B_{\varepsilon}^{j} := \Omega' \times (\omega_{\varepsilon}^{j} + r_{\varepsilon} I); \quad I := \left(-\frac{1}{2}, \frac{1}{2}\right) \quad (3.4)$$

As in [8], we consider the vibration problem
The Lamé coefficients \( \mu_\varepsilon \), \( \lambda_\varepsilon \) and the mass density \( \rho_\varepsilon \) are assumed to take constant values of possibly different orders of magnitude in the set of layers \( B_\varepsilon \) and in the set of interlayers \( \Omega \setminus B_\varepsilon \). More precisely, we suppose that

\[
\mu_\varepsilon(x) = \mu_{1\varepsilon} 1_{B_\varepsilon}(x) + \mu_0 1_{\Omega \setminus B_\varepsilon}(x), \quad \lambda_\varepsilon(x) = \lambda_{1\varepsilon} 1_{B_\varepsilon}(x) + \lambda_0 1_{\Omega \setminus B_\varepsilon}(x),
\]

\[
\mu_{1\varepsilon} \geq c > 0, \quad l_\varepsilon := \frac{\lambda_{1\varepsilon}}{\mu_{1\varepsilon}}, \quad \lim_{\varepsilon \to 0} l_\varepsilon = l \in [0, +\infty), \quad 0 < \mu_0 < \mu_{1\varepsilon},
\]

and

\[
\rho_\varepsilon(x) = \rho 1_{\Omega \setminus B_\varepsilon} + \frac{\varepsilon}{r_\varepsilon} \rho_1 1_{B_\varepsilon}, \quad \rho, \rho_1 \in (0, +\infty).
\]

We assume and set

\[
k := \lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} \mu_{1\varepsilon} \in (0, +\infty], \quad \kappa := \lim_{\varepsilon \to 0} \frac{r_\varepsilon^3}{\varepsilon} \mu_{1\varepsilon} \in [0, +\infty].
\]

The weak* relative compactness in \( L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \) of the sequence of the solutions to (3.5) is ensured by the following hypothesis:

\[
\sup_{\varepsilon > 0} \int_{\Omega} \left( \rho_\varepsilon |b_\varepsilon|^2 + \sigma_\varepsilon(a_\varepsilon) : e(a_\varepsilon) \right) dx + \int_{\Omega \times (0, t_1)} \rho_\varepsilon |f|^2 dx dt < +\infty.
\]

(3.9)
3.1. Case of interlayers with Lamé coefficients of order 1. We consider the case of extremely thin layers of extremely large stiffness alternating with interlayers of elastic moduli of order 1. The effective volume fraction of the stiff layers is characterized by:

\[ \vartheta := \lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon}. \]  

We assume in this subsection that

\[ \vartheta = 0, \]  

\[ \mu_0 = \mu > 0; \quad \lambda_0 = \lambda \geq 0. \]  

We introduce the operator \( \sigma : H^1(\Omega; \mathbb{R}^3) \to L^2(\Omega; \mathbb{S}^3) \) and \( n_\varepsilon \in L^\infty(\Omega) \) defined by

\[ \sigma(\varphi) := \lambda \text{tr}(\varepsilon(\varphi))I + 2\mu\varepsilon(\varphi) \quad \forall \varphi \in H^1(\Omega; \mathbb{R}^3), \]  

\[ n_\varepsilon(x) := \sum_{i \in Z_\varepsilon} \# (\omega_\varepsilon \cap \left( \varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right] \times (0, L)) \triangleq (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}) \cap (0, L). \]

Assumption (3.2) implies that \( |n_\varepsilon|_{L^\infty(\Omega)} \leq 1 \), therefore, up to a subsequence,

\[ n_\varepsilon \rightharpoonup^* n \text{ weakly* in } L^\infty(\Omega) \text{ for some } n \in L^\infty(\Omega). \]  

The scalar \( \frac{1}{\varepsilon} n_\varepsilon(x) \) is an approximation at \( x \) of the local number of stiff layers per unit length in the \( e_3 \) direction. Under these assumptions, we prove that the solution to (3.5) weakly* converges in \( L^\infty(0; T; H^1_0(\Omega; \mathbb{R}^3)) \) to the unique solution to \( (\mathcal{P}_{n,k,\kappa}^{\text{hom}})) \) defined, in terms of \( k, \kappa, n \) given by (3.8), (3.16), as follows: if \( 0 < k < +\infty \), we get (see (2.1))

\[
\begin{align*}
(\mathcal{P}_{n,k,\kappa}^{\text{hom}}) : & \quad \left\{ \begin{array}{l}
(\rho + n\bar{\rho}_1) \frac{\partial^2 u}{\partial t^2} - \text{div}\sigma(u) - nk\text{div}\sigma_x(u') = (\rho + n\bar{\rho}_1) f \quad \text{in } \Omega \times (0, t_1), \\
 u \in C([0, T]; H^1_0(\Omega; \mathbb{R}^3)) \cap C^1([0, T]; L^2(\Omega; \mathbb{R}^3)), \\
u(0) = a_0, \quad \frac{\partial u}{\partial t}(0) = b_0.
\end{array} \right.
\end{align*}
\]  

If \( k = +\infty \) and \( \kappa = 0 \), the limit problem is deduced from (3.17), formally, by substituting \((0, 0)\) for \((u_1(x), u_2(x))\) when \( n(x) > 0 \):
If $\kappa > 0 \equiv x$ with respect to $x_1, x_2$ in the limit equations, revealing bending effects:

$$
(\mathcal{P}_{(n, +\infty, 0)}^{\text{hom}}) : \begin{cases}
(\rho + n\mathcal{P}_1) \frac{\partial^2 u_3}{\partial t^2} - (\text{div} \sigma(u))_3 = (\rho + n\mathcal{P}_1)f_3 & \text{in } \Omega \times (0, t_1), \\
\frac{\partial^2 u_\alpha}{\partial t^2} - (\text{div} \sigma(u))_\alpha = \rho f_\alpha & \text{in } \{ n = 0 \} \times (0, T), \\
nu_1 = nu_2 = 0, \ u \in C([0, T]; H^1_0(\Omega; \mathbb{R}^3)) \cap C^1([0, T]; L^2(\Omega; \mathbb{R}^3)), \\
u(0) = (a_01 1_{\{n=0\}}, a_02 1_{\{n=0\}}, a_03).
\end{cases}
$$

(3.18)

The case $0 < \kappa < +\infty$ is characterized by the emergence of fourth order derivatives with respect to $x_1, x_2$ in the limit equations, revealing bending effects:

$$
(\mathcal{P}_{(n, +\infty, \kappa)}^{\text{hom}}) : \begin{cases}
(\rho + n\mathcal{P}_1) \frac{\partial^2 u_3}{\partial t^2} - (\text{div} \sigma(u))_3 + n \kappa \frac{1}{3} \frac{1}{l + 2} \sum_{\alpha, \beta=1}^2 \frac{\partial^4 u_3}{\partial x_\alpha^2 \partial x_\beta^2} = (\rho + n\mathcal{P}_1)f_3 & \text{in } \Omega \times (0, t_1), \\
\frac{\partial^2 u_\alpha}{\partial t^2} - (\text{div} \sigma(u))_\alpha = \rho f_\alpha & \text{in } \{ n = 0 \} \times (0, T), \\
nu_1 = nu_2 = 0, \ u \in C([0, T]; H^1_0(\Omega; \mathbb{R}^3)) \cap C^1([0, T]; L^2(\Omega; \mathbb{R}^3)) \\
u(0) = (a_01 1_{\{n=0\}}, a_02 1_{\{n=0\}}, a_03).
\end{cases}
$$

(3.19)

If $\kappa = +\infty$, we get:

$$
(\mathcal{P}_{(n, +\infty, +\infty)}^{\text{hom}}) : \begin{cases}
\frac{\partial^2 u}{\partial t^2} - \text{div} \sigma(u) = \rho f & \text{in } \{ n = 0 \} \times (0, T), \\
nu = 0, \ u \in C([0, T]; H^1_0(\Omega; \mathbb{R}^3)) \cap C^1([0, T]; L^2(\Omega; \mathbb{R}^3)), \\
u(0) = a_0 1_{\{n=0\}}, \ \frac{\partial u}{\partial t}(0) = b_0 1_{\{n=0\}}.
\end{cases}
$$

(3.20)

**Theorem 1.** Assume (3.11), (3.12), (3.16), then the sequence $(u_\epsilon)$ of the solutions to (3.5) weakly* converges in $L^\infty(0, T; H^1_0(\Omega; \mathbb{R}^3))$ to the unique solution of the problem $(\mathcal{P}_{(n, k, \kappa)}^{\text{hom}})$ given by (3.17)-(3.20).

**Remark 1.** (i) When stiff fibers [9, 13] (resp. grain-like inclusions [7]) embedded in a matrix of stiffness of order 1 are considered, the fibers (resp. the inclusions) disappear from the limit problem if $r_\epsilon \ll -\frac{1}{2}$ (resp. $r_\epsilon \ll \varepsilon^3$), where $r_\epsilon$ denotes the diameter of the sections of the fibers (resp. of the inclusions). This never occurs in the stratified case, whatever the choice of $r_\epsilon$. This is related to the fact that the harmonic capacity of a surface in $\Omega$ is always positive, whereas that of a line or a point are equal to zero.

(ii) Under (3.12), the case $\vartheta > 0$, $k < +\infty$ has been studied in [27], [38]. In the case $\vartheta > 0$, $k = +\infty$, we came up against technical complications (see Remark 9).
3.2. Stochastic case. Fixing \( d > 0 \) and set
\[
\mathcal{D} := \{ \omega \in 2^\mathbb{R}, \forall (\omega_1, \omega_2) \in \omega^2, \omega_1 \neq \omega_2 \Rightarrow |\omega_1 - \omega_2| \geq d \},
\]
\[
\omega_{\varepsilon}(\omega) := \varepsilon \omega \cap (\varepsilon, L - \varepsilon) \quad \forall \omega \in \mathcal{D}.
\]
(3.21)

Let \( \mathcal{B}_\mathcal{D} \) be the Borel \( \sigma \)-algebra generated by the Hausdorff distance on \( \mathcal{D} \) (see Remark 2), and \( P \) be a probability on \( (\mathcal{D}, \mathcal{B}_\mathcal{D}) \) satisfying
\[
P(A + z) = P(A) \quad \forall z \in \mathbb{Z}, \forall A \in \mathcal{B}_\mathcal{D}.
\]
(3.22)

We consider the random distribution of stiff homothetical layers \( B_{\varepsilon}(\omega_{\varepsilon}(\omega)) \) and the problem \( (P_{\varepsilon}(\omega)) \) obtained by substituting \( \omega_{\varepsilon}(\omega) \) for \( \omega \) in (3.4), (3.5). In what follows, \( \mathcal{F} \) represents the \( \sigma \)-algebra of the \( \mathcal{Y} \)-periodic elements of \( \mathcal{B}_\mathcal{D}, E_P \mathcal{F} \) the conditional expectation of a random variable \( X \) given \( \mathcal{F} \) with respect to \( P \), \( n_{\varepsilon}(\omega) \) the element of \( L^\infty(\Omega) \) defined by substituting \( \omega_{\varepsilon}(\omega) \) for \( \omega_{\varepsilon} \) in (3.14), and \( n_0 : \mathcal{D} \rightarrow \mathbb{N} \) the random variable given by
\[
n_0(\omega) := \# \left( \left( -\frac{1}{2}, \frac{1}{2} \right) \cap \omega \right) \quad \forall \omega \in \mathcal{D}.
\]
(3.23)

The following theorem is proved in [10]:

**Theorem 2.** Under the assumptions stated above, there exists a sequence of reals \( (\varepsilon_k) \) converging to 0 and a \( P \)-negligible subset \( \mathcal{N} \) of \( \mathcal{D} \), such that for all \( \omega \in \mathcal{D} \setminus \mathcal{N} \),
\[
n_{\varepsilon_k}(\omega) \xrightarrow{\ast} E_P n_0(\omega) \quad \text{weakly* in} \quad L^\infty(\Omega).
\]
(3.24)

The following result straightforwardly follows from theorems 1, 2:

**Theorem 3.** Assume (3.11), (3.12), and let \( (\varepsilon_k) \) and \( \mathcal{N} \) be the sequence and the \( P \)-negligible set given by Theorem 2. Then, for all \( \omega \in \mathcal{D} \setminus \mathcal{N} \), the solution to \( (P_{\varepsilon_k}(\omega)) \), weakly\(^*\) converges in \( L^\infty(0,T;H^1_\mathcal{F}(\Omega;\mathbb{R}^3)) \) to the unique solution to the problem \( (P_{\varepsilon_k}^{\text{hom}}(E_P n_0(\omega), k, \kappa)) \) defined by (3.17-3.20).

**Remark 2.** The restriction of the Hausdorff distance \( d_\mathcal{H} \) to \( \mathcal{D} \) is an extended metric on \( \mathcal{D} \), and the mapping \( d_\mathcal{D} : \mathcal{D}^2 \rightarrow [0, 1] \) defined by \( d_\mathcal{D}(\omega, \omega') := \min\{1, d_\mathcal{H}(\omega, \omega')\} \) is a finite metric on \( \mathcal{D} \) which turns \( \mathcal{D} \) into a complete metric space.

3.3. Intermediate case. Under the assumptions
\[
\varepsilon^2 \ll \mu_0 \ll 1, \quad 0 \leq \lambda_0 \leq C \mu_0,
\]
(3.25)
\[
n_{\varepsilon} \rightarrow n \quad \text{strongly in} \quad L^2(\Omega),
\]
(3.26)

we show that the solution to (3.5) weakly\(^*\) converges in \( L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \) to the unique solution to \( (P_{(n,k,\kappa)}^{\text{hom}}(\cdot)) \) defined by
\[
(P_{(n,k,\kappa)}^{\text{hom}}) : \begin{cases}
(\rho(1 - \partial n) + n \mathbf{P}_1) \frac{\partial^2 \mathbf{u}}{\partial t^2} - nk \text{div} \mathbf{\sigma}_x(\mathbf{u}') = (\rho(1 - \partial n) + n \mathbf{P}_1) \mathbf{f} & \text{in} \; \Omega \times (0,t_1), \\

u_1, u_2 \in C([0,T];L^2_n(0,L;H^1_\mathcal{F}(\Omega'))), \\

\mathbf{u} \in C^1([0,T];L^2(\Omega;\mathbb{R}^3)), \quad \mathbf{u}(0) = \mathbf{a}_0, \quad \frac{\partial \mathbf{u}}{\partial t}(0) = \mathbf{b}_0,
\end{cases}
\]
(3.27)
Theorem 4. Under (3.25), (3.26), the solution to (3.5) weakly* converges in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ to the unique solution to $(\mathcal{P}^{\text{hom}}_{(n, +\infty, 0)})$ given by (3.27)-(3.30).

Remark 3. (i) Problems (3.27)-(3.30) are formally deduced from (3.17)-(3.20) by removing the term “$\text{div} \sigma(u)$” (see (3.11)). This indicates that no strain energy is stored in the softer phase.

(ii) Assumption (3.26), stronger than (3.16), precludes the application of Theorem 2 and the extension of Theorem 4 to the setting of stochastic homogenization.

3.4. Case of soft interlayers with Lamé coefficients of order $\varepsilon^2$. We assume that

$$
\mu_0 = \varepsilon^2 \mu_0, \quad \lambda_0 = \varepsilon^2 \lambda_0, \quad \mu_0 > 0, \quad \lambda_0 \geq 0,
$$

and that the stiff layers are periodically distributed (see (3.15)):

$$
B_x := \bigcup_{i \in Z_x} B^i_x, \quad B^i_x := \Omega' \times (\varepsilon i + r_x I).
$$
Under these hypotheses, setting
\[ Y := \left( \frac{1}{2}, \frac{1}{2} \right) ; \quad B := \left( \frac{1}{2}, \frac{1}{2} \right) \times \left( \frac{\vartheta}{2}, \frac{\vartheta}{2} \right) ; \quad \Sigma := \left( \frac{1}{2}, \frac{1}{2} \right) \times \{ 0 \}, \quad (3.33) \]
\[ A := B \text{ if } \vartheta > 0, \quad A := \Sigma \text{ if } \vartheta = 0, \quad (3.34) \]

we show that the solution \( u_\varepsilon \) to (3.5) two-scale converges to \( u_0 \in C([0,T]; L^2(\Omega, H^1_0(Y; \mathbb{R}^3))) \) (see Section 4 for the definition of this convergence), and the sequence \((u_\varepsilon, m_\varepsilon)\), where \( m_\varepsilon \) is the measure defined by

\[ m_\varepsilon := \frac{\varepsilon}{r_\varepsilon} 1_{B_\varepsilon}(x) L^3_\Omega, \quad (3.35) \]

weakly* converges in \( L^\infty(0,T; M(\Omega; \mathbb{R}^3)) \) to \( u \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)) \), where \((u_0, v)\) is the unique solution to the coupled system of equations (comparable in certain respects with \([8, (2.17)])

\[
\begin{cases}
(P_{\text{soft}}^\kappa (\kappa, \vartheta)), \\
(P_{\text{stiff}}^\kappa (k, \kappa)),
\end{cases}
\]

(3.36)

defined below in terms of \( k, \kappa \), and \( \vartheta \) given, respectively, by (3.8) and (3.10). The fields \( u_0 \) and \( v \) are linked by the following relation on \( \Omega \times (0,T) \times A \):

\[
\begin{align*}
u(x, t) &= u_0(x, t, y) \quad \text{in } \Omega \times (0,t_1) \times A \quad \text{if } \vartheta > 0 \quad \text{or} \quad \kappa > 0, \\
v' &= u'_0 \quad \text{on } \Omega \times (0,t_1) \times A \\
v_3 &= \int_A u_{03}(\cdot, y) dH^2(y) \quad \text{if } \vartheta = 0 \quad \text{and} \quad \kappa = 0.
\end{align*}
\]

We introduce the operators \( e_y, \sigma_{0y} : H^1(\Omega; \mathbb{R}^3) \to L^2(\Omega; \mathbb{S}^3) \), \( g : \mathcal{H} \to \mathbb{R}^3 \) defined by

\[
(e_y(w))_{ij} = \frac{1}{2} \left( \frac{\partial w_i}{\partial y_j} + \frac{\partial w_j}{\partial y_i} \right), \quad \sigma_{0y}(w) := \lambda_0 \text{tr}(e_y(w)) I + 2 \mu_0 e_y(w),
\]

\[
g(w) := \begin{cases}
- \int_{\partial(Y \setminus B) \cap \overline{B}} \sigma_{0y}(w) \nu_{Y \setminus B} d\mathcal{H}^2(y), \quad & \text{if } A = B, \\
\int_\Sigma (\sigma_{0y}(w^+) - \sigma_{0y}(w^-)) \cdot e_3 d\mathcal{H}^2(y) \quad & \text{if } A = \Sigma,
\end{cases}
\]

(3.38)

where \( \nu_{Y \setminus B} \) stands for the outward normal to \( \partial(Y \setminus B) \) and

\[
\mathcal{H} := \{ w \in H^1(\Omega \setminus A; \mathbb{R}^3), \quad \text{div}(\sigma_{0y}(w)) \in (H^1(Y \setminus A; \mathbb{R}^3))^* \},
\]

(3.39)

denoting by \( E^* \) the topological dual of a Banach space \( E \), and by \( w^+ \) (resp. \( w^- \)) the restriction of \( w \) to \( \left( \frac{1}{2}, \frac{1}{2} \right) \times (0, \frac{1}{2}) \) (resp. \( \left( \frac{1}{2}, \frac{1}{2} \right) \times \left( \frac{1}{2}, 0 \right) \)). Problem \((P_{\text{soft}}^\kappa (\kappa, \vartheta))\) in (3.36) is the equation of \( u_0 \) in \( \Omega \times (0,T) \times (Y \setminus A) \) coupled with \( v \) (3.37) and given by (denoting by \( v \) the outward normal to \( \partial Y \)): 
and the sequence $(\mathbf{u}_0, \mathbf{v})$ satisfies (3.37),

\[
\begin{align*}
(\mathcal{P}_{\text{soft}}(\kappa, \vartheta)) : & \quad \frac{\partial^2 \mathbf{u}_0}{\partial t^2} - \nabla \cdot (\sigma_{\text{soft}}(\mathbf{u}_0)) = \rho \mathbf{f} \quad \text{in } \Omega \times (0, T) \times Y \setminus A, \\
& \quad (\mathbf{u}_0, \mathbf{v}) \quad \text{on } \Omega \times (0, t_1) \times \partial Y, \\
& \quad \mathbf{u}_0 \in C([0, T]; L^2(\Omega; H^1_0(Y; \mathbb{R}^3))) \cap C^1([0, T]; L^2(\Omega; \mathbb{R}^3)), \\
& \quad \mathbf{u}_0(0) \mathbbm{1}_{Y \setminus A} = \mathbf{a}_0 \mathbbm{1}_{Y \setminus A}, \quad \frac{\partial \mathbf{u}_0}{\partial t}(0) \mathbbm{1}_{Y \setminus A} = \mathbf{b}_0 \mathbbm{1}_{Y \setminus A}.
\end{align*}
\]

Equation (3.40) governs the effective behavior of the displacement in the soft phase. Problem $(\mathcal{P}_{\text{soft}}(\kappa, \vartheta))$ in (3.36) is an equation of $\mathbf{v}$ in $\Omega \times (0, T)$ through the source term $\mathbf{g}(\mathbf{u}_0)$ defined by (3.38). This equation rules the effective behavior of the displacement in the stiff layers. Its form is determined by the order of magnitude of the coefficients $k, \kappa$. If $0 < k < +\infty$, we get (see (2.1))

\[
(\mathcal{P}_{\text{soft}}(k, \vartheta)) : \quad \frac{\partial^2 \mathbf{v}}{\partial t^2} - k \nabla \sigma_{\text{soft}}(\mathbf{v}) = \nabla \cdot \mathbf{f} + \mathbf{g}(\mathbf{u}_0) \quad \text{in } \Omega \times (0, t_1), \\
& \quad \mathbf{v}_1 = \mathbf{v}_2 = 0, \\
& \quad \mathbf{v}_3 \in C^1([0, T]; L^2(\Omega)), \quad \mathbf{v}_3(0) = \mathbf{a}_0, \quad \frac{\partial \mathbf{v}_3}{\partial t}(0) = \mathbf{b}_0.
\]

If $(k, \kappa) = (+\infty, 0)$, we obtain

\[
(\mathcal{P}_{\text{stiff}}(+\infty, 0)) : \quad \mathbf{v}_1 = \mathbf{v}_2 = 0, \\
& \quad \mathbf{v}_3(0) = \mathbf{a}_0, \quad \frac{\partial \mathbf{v}_3}{\partial t}(0) = \mathbf{b}_0.
\]

If $0 < \kappa < +\infty$, the emergence of fourth derivatives of $\mathbf{v}_3$ reveal bending effects:

\[
(\mathcal{P}_{\text{stiff}}(+, \kappa)) : \quad \frac{\partial^2 \mathbf{v}_3}{\partial t^2} + \frac{\kappa}{3} \frac{\partial^4 \mathbf{v}_3}{\partial x_1^2 \partial x_2^2} = \frac{2}{3} \sum_{\alpha, \beta=1}^2 \frac{\partial^4 \mathbf{v}_3}{\partial x_\alpha^2 \partial x_\beta^2} \quad \text{in } \Omega \times (0, t_1), \\
& \quad \mathbf{v}_1 = \mathbf{v}_2 = 0, \\
& \quad \mathbf{v}_3(0) = \mathbf{a}_0, \quad \frac{\partial \mathbf{v}_3}{\partial t}(0) = \mathbf{b}_0.
\]

If $\kappa = +\infty$, the displacement in the stiff layers asymptotically vanishes:

\[
(\mathcal{P}_{\text{stiff}}(+\infty, +\infty)) : \quad \mathbf{v} = 0.
\]

**Theorem 5.** Under (3.31), (3.32), the solution $\mathbf{u}_\varepsilon$ to (3.5) two-scale converges to $\mathbf{u}_0$ with respect to $x$ and weakly* converges in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ to $\mathbf{u} = \int_Y \mathbf{u}_0(., y) \, dy$, and the sequence $(\mathbf{u}_\varepsilon, m_\varepsilon)$, where $m_\varepsilon$ is defined by (3.35), weakly* converges in
$L^\infty(0,T;M(\Omega,\mathbb{R}^3))$ to $vL^3_\Omega$, where $(u_0,v)$ is the unique solution to (3.36). Moreover, $u_\varepsilon(\tau)$ two-scale converges to $u_0(\tau)$ with respect to $x$, for each $\tau \in [0,T]$.

**Remark 4.** One can show (see [8, p.2548] for more details), that if $a_0 = 0$ and if the fields $b_0$, $f$ are sufficiently regular, the following corrector result holds

$$\lim_{\varepsilon \to 0} \left\| u_\varepsilon - u_0 \left( x,t, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega \times (0,t);\mathbb{R}^3)} = 0. \quad (3.45)$$

**Remark 5.** The effective problem (3.36) is non-local in space and time. Non-local effects [3], [5]-[19], [20, 21, 23, 34, 44], [47]-[50], and memory effects [1, 4, 37, 53] are typical of composite media with high contrast.

**Remark 6** (Multiphase stratified elastic media). In the same way as in [8, Section 4], we can extend Theorem 5 to the case of a multiphase medium whereby $m \varepsilon$-periodic disconnected families $B_1^e$, ..., $B_m^e$ of parallel layers are embedded in a soft matrix. The limit problem then takes the form

$$\left\{ \begin{array}{l} (P_{\text{hom}}^{\text{soft}}), \\ (P_{\text{hom}}^{\text{stiff}}[j]), j \in \{1,\ldots,m\}, \end{array} \right.$$ and can be written under the variational form (4.23) for some suitable choice of data $H,V,a,h,\xi_0,\xi_1$. Each family $B_j^m$ is associated to some subset $A_j^m$ of $Y$ like in (3.34). The system $(P_{\text{hom}}^{\text{soft}})$ governs the effective displacement in the soft phase, and only differs from (3.40) by the relation (3.37) which is replaced by a series of relations on each set $\Omega \times (0,T) \times A_j^m$ between $u_0$ and some auxiliary variable $v_j$ characterizing the effective displacement in $B_j^m$. Each problem $(P_{\text{hom}}^{\text{stiff}}[j])$ consists of an equation of $v_j$ of the same form as $(P_{\text{hom}}^{\text{stiff}})$ in (3.36), coupled with $u_0$ through the operator $g_j$ deduced from (3.38) by replacing $A$ by $A_j$. Multiphase composites comprising, besides stiff layers, periodic distributions of fibers and grain-like inclusions, can be considered. Multiphase homogenized models have been studied in [8, 44, 45, 47, 48, 49].

**Remark 7** (Equilibrium equations). One can check that if the solution to

$$-\text{div}(\sigma(\varepsilon)u_\varepsilon) = f \quad \text{in} \quad \Omega, \quad u_\varepsilon \in H^1_0(\Omega,\mathbb{R}^3), \quad f \in L^2(\Omega,\mathbb{R}^3). \quad (3.46)$$

two-scale converges to $u_0 \in L^2(\Omega \times Y;\mathbb{R}^3)$, then

$$u_0 \in V \quad \text{and} \quad a(u_0,w_0) = (f,w_0)_H, \quad \forall w_0 \in V. \quad (3.47)$$

where the Hilbert spaces $V$ and $H$ and the non-negative symmetric bilinear form $a(\cdot,\cdot)$ are those mentioned in Remark 6. The form $a(\cdot,\cdot)$ may fail to be coercive on $V$. One can prove (the proof is similar to that sketched in Remark 11 in the context of Theorem 4) that this coercivity and the convergence of the solution to (3.46) are guaranteed provided that a multiphase stratified composite is considered whereby the set of stiff layers comprises a family $B_j^s$ of parallel layers of thickness $\varepsilon r_j^s$ such that $\kappa_j > 0$, that is with elastic moduli of order larger than $\varepsilon (r_j^s)^{-3}$. Similar results were obtained in [8, Corollary 5.1, Proposition 5.2] for fibers and grain-like inclusions. Note in passing that one should substitute 1 for $\rho_\varepsilon$ in [8, Formula 5.1], otherwise the proof of "(iii) $\Rightarrow$ (iv)" in [8, p. 2552] is false.
Remark 8. Under Assumption (3.31), the slightest perturbation of periodicity leads to a complete change of the form of the effective problem. For instance, if \( m \in \mathbb{N} \) and \( \omega_j^2 = \varepsilon (j + \frac{1}{2}) \) if \( j \) is a multiple of \( m \), and \( \omega_j^2 = \varepsilon j \) otherwise in (3.32), then the limit problem is a system of equations coupling \( u \) and \( \omega \) to a complete change of the form of the effective problem. For instance, if \( m \) is even, simply in terms of the function \( n \) defined by (3.16). The extension of Theorem 5 to the non-periodic case is far beyond the scope of this paper.

4. TWO-SCALE CONVERGENCE AND OTHER ANALYSIS TOOLS

In Section 4.1, we recall some properties of the two-scale convergence of G. Allaire [3] and G. Nguetseng [41] and reproduce some statements of [8] in a suitable form for the present context. In Section 4.2, we introduce a non-periodic notion of two-scale convergence with respect to a sequence of measures and establish a compactness result (Lemma 2). Two classical analysis results are recalled in Section 4.3.

4.1. Two-scale convergence. A sequence \( (f_\varepsilon) \) in \( L^2(0,T; L^2(\Omega)) \) is said to two-scale converge to \( f_0 \in L^2(0,T; L^2(\Omega \times Y)) \) with respect to \( x \) if, for all \( \varphi_0 \in \mathcal{D}(\Omega \times (0,t_1), C^\infty_0(Y)) \),

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} f_\varepsilon(x,t)\varphi_0(x,t,\frac{x}{\varepsilon}) \, dx \, dt = \int_{\Omega \times (0,T) \times Y} f_0 \varphi_0 \, dx \, dt \, dy,
\]

(4.1)

A sequence \( (\varphi_\varepsilon) \subset L^2(0,T; L^2(\Omega)) \) strongly two-scale converges to \( \varphi_0 \in L^2(0,T; L^2(\Omega \times Y)) \) with respect to \( x \) if

\[
\varphi_\varepsilon \rightharpoonup \varphi_0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \| \varphi_\varepsilon \|_{L^2(0,T; L^2(\Omega))} = \| \varphi_0 \|_{L^2(0,T; L^2(\Omega \times Y))},
\]

(4.2)

The symbols \( \rightharpoonup \) and \( \rightharpoonup \) will also denote the two-scale convergence and the strong two-scale convergence of sequences \( (f_\varepsilon) \) in \( L^2(\Omega) \) independent of \( t \), defined by formally considering them as constant in \( t \). Any bounded sequence in \( L^2(0,T; L^2(\Omega)) \) has a two-scale convergent subsequence [41]. An admissible sequence with respect to two-scale convergence is a sequence \( (\varphi_\varepsilon) \subset L^2(0,T; L^2(\Omega)) \) that two-scale converges to some \( \varphi_0 \in L^2(0,T; L^2(\Omega \times Y)) \) and such that, for every two-scale convergent sequence \( (f_\varepsilon) \),

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} f_\varepsilon \varphi_\varepsilon \, dx \, dt = \int_{\Omega \times (0,T) \times Y} f_0 \varphi_0 \, dx \, dt \, dy.
\]

(4.3)

A sequence \( (\varphi_\varepsilon) \) is admissible if and only if it strongly two-scale converges to some \( \varphi_0 \) (see [8, p.2528]). For all \( \psi_0 \in L^2(0,T; L^2(\Omega, C^\infty_0(Y))) \cup L^2(\Omega; C^\infty(\Omega \times (0,t_1))) \), the sequence \( (\psi_\varepsilon(x,t,\frac{x}{\varepsilon}))_{\varepsilon>0} \) strongly two-scale converges to \( \psi_0 \) (see [3], Lemma 5.2, Corollary 5.4). In particular, if \( Q \) is a Borel subset of \( Y \), and \( Q_x \) is its periodization on \( \mathbb{R}^3 \) defined by (2.2), then the sequence \( (1_{Q_x}(\frac{x}{\varepsilon})) \) strongly two-scale converges to \( 1_Q(y) \). Under (3.32), if \( \theta > 0 \), then \( |1_{\Omega \setminus B_x} - 1_{(Y \setminus B)}(\frac{x}{\varepsilon})|_{L^2(\Omega)} \to 0 \), therefore \( 1_{\Omega \setminus B_x} \rightharpoonup 1_{Y \setminus B} \). If \( \theta = 0 \), (1_{\Omega \setminus B_x} strongly converges to 1 in \( L^2(\Omega) \), hence strongly two-scale converges to 1. We deduce that (see (3.34))

\[
1_{\Omega \setminus B_x} \rightharpoonup 1_{Y \setminus A}, \quad 1_{B_x} \rightharpoonup 1_A.
\]

(4.4)
The next Lemma is a straightforward variant of [8, Lemma 6.1].

**Lemma 1.** (i) Let \( (h_\varepsilon) \) be a bounded sequence in \( L^\infty(\Omega \times (0, T) \times Y) \) such that \( h_\varepsilon \to h_0 \). Then, for every sequence \( (\chi_\varepsilon) \subset L^2(0, T; L^2(\Omega)) \), the following implications hold:

\[
\begin{align*}
\chi_\varepsilon & \to \chi_0 \quad \implies \quad \chi_\varepsilon h_\varepsilon \to \chi_0 h_0, \quad \text{(4.5)} \\
\chi_\varepsilon & \to \chi_0 \quad \implies \quad \chi_\varepsilon h_\varepsilon \to \chi_0 h_0. \quad \text{(4.6)}
\end{align*}
\]

(ii) If \( (f_\varepsilon) \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \), then \( (f_\varepsilon) \) two-scale converges, up to a subsequence, to some \( f_0 \in L^\infty(0, T; L^2(\Omega)) \). If in addition \( (f_\varepsilon) \) is bounded in \( W^{1, \infty}(0, T; L^2(\Omega)) \), then \( f_0 \in W^{1, \infty}(0, T; L^2(\Omega)) \) and \( \frac{\partial f_\varepsilon}{\partial t} \) two-scale converges to \( \frac{\partial f_0}{\partial t} \). Besides, if \( f_\varepsilon(0) \to a_0 \), then \( a_0 = f_0(0) \) and \( f_\varepsilon(\tau) \to f_0(\tau), \ \forall \tau \in [0, T] \). Furthermore, if \( \frac{\partial f_\varepsilon}{\partial t} \to \frac{\partial f_0}{\partial t} \) and \( f_\varepsilon(0) \to a_0 \), then \( f_\varepsilon(\tau) \to f_0(\tau), \ \forall \tau \in [0, T] \).

### 4.2. Two-scale convergence with respect to \((m_\varepsilon)\). One can easily check that the sequence \((m_\varepsilon)\) defined by (3.35) is bounded in \( \mathcal{M}(\overline{\Omega}) \) and satisfies

\[
m_\varepsilon \overset{\ast}{\to} nL^3_{\Omega} \quad \text{weakly* in} \quad \mathcal{M}(\overline{\Omega}),
\]

where \( n \) is defined by (3.16). Notice that

\[
n = 1 \quad \text{under (3.32)}.
\]

In what follows, the symbol \( L^2_n(\Omega; \mathbb{R}^3) \) stands for the set of all Borel fields \( w : \Omega \to \mathbb{R}^3 \) such that \( \int_\Omega |w|^2 \, dx < +\infty \). Similarly, for any Hilbert space \( H \), we denote by \( L^2_n(0, L; H) \) the set of all Borel fields \( w : (0, L) \to H \) such that \( \int_0^L |w|^2_H \, dx < +\infty \). We set (see (3.3), (3.4))

\[
y_\varepsilon(z) := \sum_{j \in J_\varepsilon} \left( z - \omega^j_\varepsilon \right) \chi_{I_{\varepsilon}[r_\varepsilon, (1+\delta)r_\varepsilon]}(z).
\]

We say that a sequence \( (f_\varepsilon) \) in \( L^2(0, T; L^2(\Omega)) \) two-scale converges to \( f_0 \in L^2(0, T; L^2(\Omega \times I)) \) with respect to the sequence of measures \((m_\varepsilon)\) if for each \( \psi \in \mathcal{D}(\Omega \times (0, t_1); C^\infty_c(I)) \), the following holds

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0, t_1)} f_\varepsilon(x, t) \psi(x, t, y_\varepsilon(x, t)) \, dm_\varepsilon dt = \int_{\Omega \times (0, t_1) \times I} f_0 \psi \, ndx \, dt \, dy_3.
\]

(Note: \( f_\varepsilon \overset{m_\varepsilon}{\Rightarrow} f_0 \).)

**Lemma 2.** Let \((f_\varepsilon)\) be a sequence in \( L^2(0, T; L^2(\Omega)) \) satisfying

\[
\sup_{\varepsilon > 0} \int_{\Omega \times (0, T)} |f_\varepsilon|^2 \, dm_\varepsilon dt < +\infty.
\]

Then \((f_\varepsilon)\) two-scale converges with respect to \((m_\varepsilon)\), up to a subsequence, to some \( f_0 \in L^2_n(0, T; L^2(\Omega \times I)) \). In addition, if

\[
\sup_{\varepsilon > 0, \tau > 0} \int_\Omega |f_\varepsilon| |(\tau)\, dm_\varepsilon < +\infty,
\]

then \( f_0 \in L^\infty(0, T; L^2_n(\Omega \times I)) \).
Proof. By Cauchy-Schwarz Inequality and by (4.11), we have
\[
\left| \int_{\Omega \times (0,t_1)} f_\epsilon(x,t) \psi \left( x, t, \frac{y_\epsilon(x)}{r_\epsilon} \right) \, dm_\epsilon \, dt \right| \\
\leq C \left( \int_{\Omega \times (0,t_1)} |f_\epsilon|^2 \, dm_\epsilon \, dt \right)^{\frac{1}{2}} \left( \int_{\Omega \times (0,t_1)} |\psi|^{2} \, dm_\epsilon \, dt \right)^{\frac{1}{2}} \\
\leq C \|\psi\|_{L^\infty(\Omega \times (0,t_1) \times I)} \quad \forall \psi \in C(\Omega \times (0,t_1) \times I).
\]  
(4.13)
Hence, by the Riesz representation theorem, for each \( \epsilon > 0 \) there exists a finite Radon measure \( \theta_\epsilon \in \mathcal{M}(\Omega \times (0,t_1) \times I) \) such that
\[
\int \psi \, d\theta_\epsilon = \int_{\Omega \times (0,t_1)} f_\epsilon(x,t) \psi \left( x, t, \frac{y_\epsilon(x)}{r_\epsilon} \right) \, dm_\epsilon \, dt \quad \forall \psi \in C(\Omega \times (0,t_1) \times I).
\]  
(4.14)
By (4.13) and (4.14), the sequence \( \theta_\epsilon \) is bounded in \( \mathcal{M}(\Omega \times (0,t_1) \times I) \), thus weakly* converges, up to a subsequence, to some \( \theta \in \mathcal{M}(\Omega \times (0,t_1) \times I) \). By Cauchy-Schwarz Inequality, we have
\[
\left| \int \psi \, d\theta_\epsilon \right| \leq \left( \int_{\Omega \times (0,t_1)} |f_\epsilon|^2 \, dm_\epsilon \, dt \right)^{\frac{1}{2}} \left( \int_{\Omega \times (0,t_1)} |\psi|^{2} \, dm_\epsilon \, dt \right)^{\frac{1}{2}}.
\]  
(4.15)
The proof of the next statement is similar to that of [3, Lemma 1.3]:
\[
\lim_{\epsilon \to 0} \int_{\Omega \times I} \left[ \varphi \left( x, \frac{y_\epsilon(x)}{r_\epsilon} \right) \right]^{2} \, dm_\epsilon = \int_{\Omega \times I} |\varphi|^{2} \, ndxdy_3 \quad \forall \varphi \in C(\Omega \times I).
\]  
(4.16)
We deduce from (4.11), (4.14), (4.15), (4.16), and from the weak* convergence in \( \mathcal{M}(\Omega \times (0,t_1) \times I) \) of \( \theta_\epsilon \) to \( \theta \), that
\[
\left| \int \psi \, d\theta \right| = \lim_{\epsilon \to 0} \left| \int \psi \, d\theta_\epsilon \right| \leq C \|\psi\|_{L^\infty(\Omega \times (0,t_1) \times I)} \quad \forall \psi \in C(\Omega \times (0,t_1) \times I).
\]
Thus, the linear form \( \psi \to \int \psi \, d\theta \) is continuous on \( C(\Omega \times (0,t_1) \times I) \) with respect to the strong topology of \( L^2_\infty(\Omega \times (0,t_1) \times I) \). By a density argument, this linear form can be extended to a continuous linear form on \( L^2_\infty(\Omega \times (0,t_1) \times I) \) which, by the Riesz representation theorem, takes the form \( \psi \to \int_{\Omega \times (0,t_1) \times I} \psi f_0 \, ndxdtdy \) for some \( f_0 \in L^2_\infty(\Omega \times (0,t_1) \times I) \). We infer that \( \theta = nf_0L^2_{\epsilon,\infty}(\Omega \times (0,t_1) \times I) \), and then, taking (4.14) and the weak* convergence of \( \theta_\epsilon \) to \( \theta \) into account, deduce (4.10). Under (4.12), by Fubini’s Theorem and Cauchy-Schwarz inequality, we have
\[
\int_{\Omega \times (0,t_1)} f_\epsilon(x,t) \psi \left( x, t, \frac{y_\epsilon(x)}{r_\epsilon} \right) \, dm_\epsilon \, dt \leq C \int_{(0,T)} \left( \int_{\Omega \times I} \psi \left( x, t, \frac{y_\epsilon(x)}{r_\epsilon} \right) \, dm_\epsilon \right)^{\frac{1}{2}} \, dt.
\]
By passing to the limit as \( \epsilon \to 0 \) in the last inequality, thanks to (4.16) and to the Dominated Convergence Theorem, we get \( \int_{\Omega \times (0,t_1)} \int_{(0,T)} \psi |f_\epsilon| \, ndxdtdy \leq C \|\psi\|_{L^1(0,T;L^2_\epsilon(\Omega \times I))} \) and deduce, by the arbitrary choice of \( \psi \), that \( f_0 \in L^\infty(0,T;L^2_\epsilon(\Omega \times I)) \).  \( \square \)
4.3. **Two classical results.** For the reader’s convenience, we reproduce below Lemma A2 of [11] (see also [17] for a more general version) and Theorem 6.2 of [8], which collects some abstract results proved in [25, 35, 36]. The lemma will be employed to identify the limit of the sequence \((u_\varepsilon, m_\varepsilon)\), where \(u_\varepsilon\) is the solution to (3.5). The theorem will be applied to check the existence, the uniqueness, and some regularity properties of the solution to Problem (3.5) and of the associated limit problems.

**Lemma 3.** Let \(K\) be a compact subset of \(\mathbb{R}^N\) and \((\theta_\varepsilon)\) a bounded sequence of positive Radon measures on \(K\), weakly* converging in \(M(K)\) to some \(\theta \in M(K)\). Let \((f_\varepsilon)\) be a sequence of \(\theta_\varepsilon\)-measurable functions such that \(\sup_\varepsilon \int |f_\varepsilon|^2 d\theta_\varepsilon < +\infty\). Then the sequence \((f_\varepsilon \theta_\varepsilon)\) is sequentially relatively compact in the weak* topology \(\sigma(M(K), C(K))\) and every cluster point is of the form \(f\theta\), with \(f \in L^2(\theta)\). Moreover, if \(f_\varepsilon \theta_\varepsilon \rightharpoonup f\theta\), then

\[
\liminf_{\varepsilon \to 0} \int |f_\varepsilon|^2 d\theta_\varepsilon \geq \int |f|^2 d\theta. \tag{4.17}
\]

**Theorem 6.** Let \(V\) and \(H\) be separable Hilbert spaces such that \(V \subset H = H' \subset V'\), with continuous and dense imbeddings. Let \(||\cdot||_V, ||\cdot||_H, (\cdot, \cdot)_V, (\cdot, \cdot)_H\) denote their respective norm and inner product. Let \(a: V \times V \to \mathbb{R}\) be a continuous bilinear symmetric form on \(V\). Let \(A \in L(V, V')\) be defined by \(a(\xi, \tilde{\xi}) = (A\xi, \tilde{\xi})_{(V', V)}, \forall (\xi, \tilde{\xi}) \in V^2\). Assume that

\[
\exists (\lambda, \alpha) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad a(\xi, \xi) + \lambda ||\xi||^2_H \geq \alpha ||\xi||^2_V, \quad \forall \xi \in V. \tag{4.18}
\]

Let \(h \in L^2(0, T; H), \xi_0 \in V, \xi_1 \in H\). Then there exists a unique solution \(\xi\) to

\[
A\xi(t) + \frac{\partial^2 \xi}{\partial t^2}(t) = h(t), \quad \xi \in L^2(0, T; V),
\]

\[
\frac{\partial \xi}{\partial t} \in L^2(0, T; H), \quad \xi(0) = \xi_0, \quad \frac{\partial \xi}{\partial t}(0) = \xi_1. \tag{4.19}
\]

Furthermore, we have

\[
\xi \in C([0, T]; V) \cap C^1([0, T]; H), \quad \frac{\partial \xi}{\partial t} \in L^2(0, T; V), \quad \frac{\partial^2 \xi}{\partial t^2} \in L^2(0, T; V'). \tag{4.20}
\]

Besides, setting

\[
e(\tau) := \frac{1}{2} \left[ \left( \frac{\partial \xi}{\partial t}(\tau), \frac{\partial \xi}{\partial t}(\tau) \right)_H + a(\xi(\tau), \xi(\tau)) \right], \quad \forall \tau \in [0, T], \tag{4.21}
\]

the following holds

\[
e(\tau) = e(0) + \int_0^\tau \left( h, \frac{\partial \xi}{\partial t} \right)_H \, dt, \quad \forall \tau \in [0, T]. \tag{4.22}
\]

Problem (4.19) is equivalent to
\[
\int_0^T \left( a(\xi(t), \tilde{\xi}(t)) \eta(t) + (\xi(t), \tilde{\xi})_H \frac{\partial^2 \eta}{\partial t^2}(t) \right) dt + (\xi_0, \tilde{\xi})_H \frac{\partial \eta}{\partial t}(0) \\
- (\xi_1, \tilde{\xi})_H \eta(0) = \int_0^T (h, \tilde{\xi})_H \eta(t) dt,
\]
(4.23)

\[\forall \tilde{\xi} \in V, \quad \forall \eta \in \mathcal{D}(-\infty, T[); \quad \xi \in L^2(0, T; V), \quad \frac{\partial \xi}{\partial t} \in L^2(0, T; H).\]

5. Asymptotic behavior of the solution to (3.5)

In this section, we establish a series of estimates satisfied by the solution \( u \) to (3.5) (see Proposition 1), and investigate in lemmas 5, 6, and 7, the asymptotic behavior of sequences satisfying such estimates. These results are synthetized in Corollary 1. We start with a key inequality.

**Lemma 4.** We have

\[
\int \left( \frac{\varphi_1}{r^2} + \frac{\varphi_2}{r^2} + |\varphi_3|^2 \right) dm \leq C \int |e(\varphi)|^2 dm \quad \forall \varphi \in H^1_0(\Omega; \mathbb{R}^3). \quad (5.1)
\]

**Proof.** By (3.4) and (3.35), it is sufficient to show that for all \( j \in J \), and all \( \varphi \in H^1(B_{\frac{1}{2}}; \mathbb{R}^3) \) such that \( \varphi = 0 \) on \( \partial B_{\frac{1}{2}} \cap \partial \Omega \),

\[
\int_{B_{\frac{1}{2}}} \left( \frac{\varphi_1}{r^2} + \frac{\varphi_2}{r^2} + |\varphi_3|^2 \right) dx \leq C \int_{B_{\frac{1}{2}}} |e(\varphi)|^2 dx. \quad (5.2)
\]

By Korn’s inequality, we have

\[
\int_{\Omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)} |\psi|^2 dy \leq C \int_{\Omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)} |e(\psi)|^2 dy \quad \forall \psi \in W,
\]

\[W := \left\{ \psi \in H^1 \left( \Omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right) ; \mathbb{R}^3 \right), \quad \psi = 0 \text{ on } \partial \Omega' \times \left(-\frac{1}{2}, \frac{1}{2}\right) \right\}.
\]

By making the change of variable \( y = (x_1, x_2, \frac{x_3 - \omega_i}{r_e}) \), we get, for all \( \psi \in W \),

\[
\int_{B_{\frac{1}{2}}} |\psi|^2 \left( x_1, x_2, \frac{x_3 - \omega_i}{r_e} \right) dx \leq C \int_{B_{\frac{1}{2}}} |e(\psi)|^2 \left( x_1, x_2, \frac{x_3 - \xi}{r_e} \right) dx. \quad (5.3)
\]

Setting \( \varphi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \frac{1}{r_e} \psi_3 \end{pmatrix} \), a straightforward computation yields

\[
\int_{B_{\frac{1}{2}}} \left( |\varphi_1|^2 + |\varphi_2|^2 + r_e^2 |\varphi_3|^2 \right) (x) dx = \int_{B_{\frac{1}{2}}} |\psi|^2 \left( x_1, x_2, \frac{x_3 - \omega_j}{r_e} \right) dx
\]

\[
\leq C \int_{B_{\frac{1}{2}}} |e(\psi)|^2 \left( x_1, x_2, \frac{x_3 - \omega_j}{r_e} \right) dx \leq C \int_{B_{\frac{1}{2}}} |e(\varphi)|^2 (x) dx.
\]

The inequality (5.2) is proved. □
We infer from (3.5), (3.6), (3.35), and (5.8), that there exists a unique solution \(u_\varepsilon\) to (3.5). Moreover,

\[
\frac{\partial u_\varepsilon}{\partial t} \in L^2(0, T; H^1_0(\Omega; \mathbb{R}^3)), \quad \frac{\partial^2 u_\varepsilon}{\partial t^2} \in L^2(0, T; H^{-1}(\Omega; \mathbb{R}^3)).
\]  

(5.4)

Under (3.6), (3.9), there exists a constant \(C > 0\) such that

\[
\int_\Omega \rho_\varepsilon |e(u_\varepsilon)|^2(\tau) dx + \int_\Omega \left( \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 + |u_\varepsilon|^2 \right)(\tau) dx \leq C \quad \forall \tau \in [0, T],
\]

\[
\int |e(u_\varepsilon)|^2(\tau) + |u_\varepsilon|^2(\tau) dm_\varepsilon \leq C \left( \frac{\rho_\varepsilon}{\varepsilon \mu_1} \right)^{-1} \quad \forall \tau \in [0, T],
\]

(5.5)

\[
\int |u_{\varepsilon 3}|^2(\tau) dm_\varepsilon \leq C \left( \frac{\rho_\varepsilon^3}{\varepsilon \mu_1} \right)^{-1}, \quad \int |u_\varepsilon|^2(\tau) dm_\varepsilon \leq C \quad \forall \tau \in [0, T].
\]

(5.6)

**Proof.** Problem (3.5) is equivalent to (4.23), where \(H := L^2(\Omega; \mathbb{R}^3), (\xi, \tilde{\xi})_H := \int_\Omega \rho_\varepsilon \xi \cdot \tilde{\xi} dx, V := H^1_0(\Omega; \mathbb{R}^3)(V' = H^{-1}(\Omega; \mathbb{R}^3)), a(\xi, \tilde{\xi}) := \int_\Omega \sigma_\varepsilon(\xi) : e(\xi) dx\), and \((\xi_0, \tilde{\xi}_0, \mathbf{h}) = (a_0, b_0, f)\). By (3.7), \((H, (\cdot, \cdot)_H)\) is a Hilbert space and the assumptions of Theorem 6 are satisfied. Therefore, Problem (3.5) has a unique solution. Assertion (5.4) follows from (4.20). By (4.22) we have, for all \(\tau \in [0, T]\),

\[
\frac{1}{2} \int_\Omega \left( \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 + \sigma_\varepsilon(u_\varepsilon) : e(u_\varepsilon) \right)(\tau) dx
\]

\[
= \frac{1}{2} \int_\Omega \left( \rho_\varepsilon |b_0|^2 + \sigma_\varepsilon(a_0) : e(a_0) \right) dx + \int_{\Omega \times (0, \tau)} \rho_\varepsilon f \cdot \frac{\partial u_\varepsilon}{\partial t} dx dt.
\]

We deduce from Cauchy-Schwarz Inequality that

\[
\left| \int_{\Omega \times (0, \tau)} \rho_\varepsilon f \cdot \frac{\partial u_\varepsilon}{\partial t} dx dt \right| \leq \sqrt{\int_{\Omega \times (0, t_1)} \rho_\varepsilon f^2 dx dt} \sqrt{\int_{\Omega \times (0, t_1)} \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt}.
\]

Taking (3.9) into account, we infer

\[
\int_\Omega \left( \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 + \sigma_\varepsilon(u_\varepsilon) : e(u_\varepsilon) \right)(\tau) dx
\]

\[
\leq C \left( 1 + \sqrt{\int_{\Omega \times (0, t_1)} \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt} \right) \quad \forall \tau \in [0, T].
\]

(5.7)

By integrating (5.7) with respect to \(\tau\) over \((0, T)\), we deduce that \(\int_{\Omega \times (0, t_1)} \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 dx dt \leq C\) and then, coming back to (5.7), that

\[
\int_\Omega \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2(\tau) dx + \int_\Omega \sigma_\varepsilon(u_\varepsilon) : e(u_\varepsilon)(\tau) dx \leq C \quad \forall \tau \in [0, T].
\]

(5.8)

We infer from (3.5), (3.6), (3.35), and (5.8), that
Lemma 5. Let (\( u_\varepsilon \)) be a sequence in \( L^\infty(0,T;H^1_0(\Omega;\mathbb{R}^3)) \) satisfying

\[
\sup_{\varepsilon > 0, \tau \in (0,T)} \int \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 (\tau) + \mu_0 \varepsilon |e(u_\varepsilon)|^2(\tau)dx \leq C \quad \forall \tau \in [0,T],
\]

(5.9)

By (3.7), (3.35), and (5.9), and by the continuity of \( a_0 \) (see (3.5)), we have

\[
\int |u_\varepsilon|^2(\tau)dx + \int |u_\varepsilon|^2(\tau)d\varepsilon \leq \int \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 (\tau)d\varepsilon \leq C \quad \forall \tau \in [0,T].
\]

(5.10)

By (5.1) and (5.9), we have

\[
\int \int \frac{u_{\varepsilon 1}^2}{\varepsilon^2} + \frac{u_{\varepsilon 2}^2}{\varepsilon^2} + |u_{\varepsilon 3}|^2 (\tau)d\varepsilon \leq \frac{C}{\varepsilon^2} \int |e(u_\varepsilon)|^2(\tau)d\varepsilon \leq \frac{C}{\varepsilon^2 \mu_\varepsilon} \quad \forall \tau \in [0,T],
\]

which, joined with (5.9), (5.10) completes the proof of (5.5).

By (3.7), (3.35), and (5.9), and by the continuity of \( a_0 \) (see (3.5)), we have

\[
\int |u_\varepsilon|^2(\tau)dx + \int |u_\varepsilon|^2(\tau)d\varepsilon = \int \left| a_0 + \int_0^\tau \frac{\partial u_\varepsilon}{\partial t}(t)d\tau \right|^2 (\mathcal{L}^3 + m_\varepsilon)(x)
\]

\[
+ \int \rho_\varepsilon \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2 (\tau)d\varepsilon \leq C \quad \forall \tau \in [0,T].
\]

(5.11)

Then there exists \( v \in L^\infty(0,T;L^2_0(\Omega;\mathbb{R}^3)) \) such that, up to a subsequence,

\[
u_\varepsilon, m_\varepsilon \overset{\text{weakly*}}{\rightarrow} n v, \quad v_1, v_2 \in L^\infty(0,T;L^2_0(\Omega;H^1_0(\Omega'))),
\]

where \( e_\varepsilon \to e_\varepsilon' \) \( \text{weakly in } L^\infty(0,T;M(\overline{\Omega};\mathbb{R}^3)) \).

Furthermore,

\[
nv_1 = nv_2 = 0 \quad \text{if} \quad \liminf_{\varepsilon \to 0} \sup_{\tau \in (0,T)} \int |e(u_\varepsilon)|^2(\tau)d\varepsilon = 0,
\]

(5.13)

Moreover, if

\[
\sup_{\varepsilon > 0, \tau \in (0,T)} \int \mu_\varepsilon |e(u_\varepsilon)|^2(\tau)dx < +\infty; \quad \varepsilon^2 \ll \mu_\varepsilon,
\]

(5.14)

\( u_\varepsilon \to n \) strongly in \( L^2(\Omega) \) and \( u_\varepsilon \overset{\text{weakly*}}{\rightarrow} u \) weakly in \( L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \),

or if

\[
\sup_{\varepsilon > 0, \tau \in (0,T)} \int |e(u_\varepsilon)|^2(\tau)dx < +\infty,
\]

(5.15)

\( u_\varepsilon \to u \) strongly in \( L^2(0,T;L^2(\Omega;\mathbb{R}^3)) \).
then \(n\mathbf{u} = n\mathbf{v}\).

Proof. By applying Lemma 3 to \(\theta_\varepsilon := L_1((0,T) \otimes \mathbb{R}^3)\), taking (4.7) and (5.11) into account, we obtain the convergences

\[
\mathbf{u}_\varepsilon m_\varepsilon \rightharpoonup n\mathbf{v} \quad \text{weakly* in} \quad \mathcal{M}((0,T) \times \Omega; \mathbb{R}^3)),
\]

\[
(\mathbf{e}_\varepsilon(\mathbf{u}_\varepsilon'))m_\varepsilon \rightharpoonup n\Xi \quad \text{weakly* in} \quad \mathcal{M}((0,T) \times \Omega; \mathbb{S}_3)),
\]

(5.16)

up to a subsequence, for some suitable \(\mathbf{v} \in L^2(0, T; L^2(\Omega; \mathbb{R}^3)), \Xi \in L^2(0, T; L^2(\Omega; \mathbb{S}^3))\) such that \(\Xi_{ij} = 0\) if \(i \neq j\). By (5.11), the sequences \((\mathbf{u}_\varepsilon m_\varepsilon)\) and \((\mathbf{e}_\varepsilon(\mathbf{u}_\varepsilon')m_\varepsilon)\) are bounded in \(L^\infty(0, T; \mathcal{M}(\Omega))\), therefore the convergences (5.16) also hold with respect to the weak* topology of \(L^\infty(0, T; \mathcal{M}(\Omega))\). Let us fix \(\Psi \in C^\infty(\Omega \times (0, t_1); \mathbb{S}^2)\).

By integration by parts, we have

\[
\sum_{\alpha, \beta = 1}^2 \int_{\Omega \times (0,t_1)} (\mathbf{e}_\varepsilon(\mathbf{u}_\varepsilon'))_{\alpha\beta} (\mathbf{u}_\varepsilon m_\varepsilon) (x,t) \, dm_\varepsilon dt = - \int_{\Omega \times (0,t_1)} (\mathbf{u}_\varepsilon \mathbf{e}_1 + \mathbf{u}_\varepsilon \mathbf{e}_2) \cdot \mathbf{div}_\varepsilon \Psi (x,t) \, dm_\varepsilon dt.
\]

By passing to the limit as \(\varepsilon \to 0\), taking (5.16) into account, we obtain

\[
\sum_{\alpha, \beta = 1}^2 \int_{\Omega \times (0,t_1)} \Xi_{\alpha\beta} \Psi_{\alpha\beta} n \, dxdt = - \int_{\Omega \times (0,t_1)} (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) \cdot \mathbf{div}_\varepsilon \Psi \, dxdt.
\]

(5.17)

By making \(\Psi\) vary in \(\mathcal{D}(\Omega \times (0, t_1); \mathbb{S}^2)\), we deduce that \(n \mathbf{e}_\varepsilon(\mathbf{v}') (:= \mathbf{e}_\varepsilon(n\mathbf{v}')) = n\Xi\) in the sense of distributions on \(\Omega \times (0, t_1)\), and then infer from Korn inequality in \(H^1(\Omega')\) that \(v_{11}, v_{22} \in L^\infty(0, T; L^2(0, L; H^1(\Omega'))),\) that is \(v_1, v_2 \in L^\infty(0, T; L^2(0, L; H^1(\Omega'))).\)

By integrating (5.17) by parts for \(\Psi \in C^\infty(\Omega \times (0, t_1); \mathbb{S}^2)\), we get \(\int_0^T \int_0^L \int_{\partial\varepsilon} n \mathbf{v}' \cdot \Psi \mathbf{v} \, dH^d \, dx \, dt = 0\) and infer from the arbitrariness of \(\Psi\) that \(v_1, v_2 \in L^\infty(0, T; L^2(0, L; H^1(\Omega'))))\). Assertion (5.13) is a consequence of the next inequalities (holding for \(\alpha \in \{1, 2\}\), deduced from (4.17), (5.1), (5.16))

\[
\int_{\Omega \times (0,t_1)} |v_{\alpha \alpha}|^2 n \, dxdt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} |u_{\alpha\alpha}|^2 dm_\varepsilon (x) \, dt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} |\mathbf{e}(\mathbf{u}_\varepsilon)|^2 dm_\varepsilon (x) \, dt,
\]

\[
\int_{\Omega \times (0,t_1)} |v|^2 n \, dxdt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} |u_{\alpha\alpha}|^2 dm_\varepsilon (x) \, dt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \left| \frac{1}{r_\varepsilon} \mathbf{e}(\mathbf{u}_\varepsilon) \right|^2 dm_\varepsilon (x) \, dt.
\]

It remains to show that under (5.14) or (5.15), \(n\mathbf{u} = n\mathbf{v}\). To that aim, we set

\[
\mathbf{v}_\varepsilon (x,t) := \sum_{j \in J_\varepsilon} \mathbf{u}_\varepsilon (x_1, x_2, \omega_\varepsilon^j, t) \mathbf{1}_{\left(\omega_\varepsilon^j - \varepsilon^j, \omega_\varepsilon^j + \varepsilon^j\right)} (x_3).
\]

(5.18)

By Fubini’s Theorem, Jensen’s inequality and Korn’s inequality in \(H^1_0(\Omega; \mathbb{R}^3)\), we have
\[ \int |\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon|^2(\tau)dm_\varepsilon = \varepsilon \sum_{j \in J_\varepsilon} \int_{\Omega} |\mathbf{u}_\varepsilon(x', x_3, \tau) - \mathbf{v}_\varepsilon(x', \omega^j_\varepsilon, \tau)|^2dx_3 \]

\[ \leq \varepsilon \sum_{j \in J_\varepsilon} \int_{\Omega} \left( \int_{\omega^j_\varepsilon - \frac{\varepsilon}{2}}^{\omega^j_\varepsilon + \frac{\varepsilon}{2}} |\mathbf{u}_\varepsilon(x', \omega^j_\varepsilon, \tau)|^2dx_3 \right)^2dx_3 \]

\[ \leq \varepsilon \int_{\Omega} \left( \sum_{j \in J_\varepsilon} \int_{\Omega} |\mathbf{u}_\varepsilon(x', \omega^j_\varepsilon, \tau)|^2dx_3 \right)^2dx_3 \] \hspace{1cm} \text{(5.19)}

Therefore, under (5.14) or (5.15), \( \lim_{\varepsilon \to 0} \int |\mathbf{u}_\varepsilon - \mathbf{v}_\varepsilon|^2(\tau)dm_\varepsilon = 0 \). Hence, by (5.16),

\[ \mathbf{v}_\varepsilon \Rightarrow \mathbf{v} \quad \text{weakly* in} \quad L^\infty(0, T; \mathcal{M}(\Omega; \mathbb{R}^3)). \] \hspace{1cm} (5.20)

We define (see (3.1, 3.15))

\[ \mathbf{v}_\varepsilon(x, t) := \sum_{i \in Z_\varepsilon} \left( \sum_{\omega_i \in \omega_\varepsilon \cap (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})} \mathbf{u}_\varepsilon(x', \omega_i^j, t) \right) \mathbb{1}_{(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})}(x_3). \] \hspace{1cm} (5.21)

Noticing that by (3.2) and (3.15),

\[ \omega_\varepsilon \subset \bigcup_{i \in Z_\varepsilon} \left( \varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right), \] \hspace{1cm} (5.22)

we infer from (3.35) and (5.18) that

\[ \int_{\Omega} |\mathbf{v}_\varepsilon|^2(\tau)dx \leq C \sum_{i \in Z_\varepsilon} \sum_{\omega_i \in \omega_\varepsilon \cap (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})} \varepsilon \int_{\Omega'} |\mathbf{u}_\varepsilon(x', \omega_i^j, \tau)|^2dx' \]

\[ = C \varepsilon \sum_{i \in Z_\varepsilon} r_\varepsilon \int_{\Omega'} |\mathbf{u}_\varepsilon(x', \omega_i^j, \tau)|^2dx' = C \int |\mathbf{v}_\varepsilon|^2(\tau)dm_\varepsilon. \] \hspace{1cm} (5.23)

Therefore, by (5.11) and (5.19), under (5.14) or (5.15), the sequence \((\mathbf{v}_\varepsilon)\) is bounded in \(L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))\). Thus the following convergence holds, up to a subsequence

\[ \mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \quad \text{weakly* in} \quad L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)). \] \hspace{1cm} (5.24)

To identify \(\mathbf{w}\), we fix \(\varphi \in \mathcal{D}(\Omega \times (0, t_1); \mathbb{R}^3)\) and set \(\varphi_\varepsilon(x, t) := \sum_{i \in Z_\varepsilon} \varphi(x', \varepsilon i, t) \mathbb{1}_{(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})}(x_3)\). Noticing that \(\|\varphi_\varepsilon - \varphi\|_{L^\infty(\Omega \times (0, t_1); \mathbb{R}^3)} \leq C\varepsilon\), we infer

\[ \int_{\Omega \times (0, t_1)} \mathbf{w} \cdot \varphi dxdt = \lim_{\varepsilon \to 0} \int_{\Omega \times (0, t_1)} \mathbf{v}_\varepsilon \cdot \varphi_\varepsilon dxdt. \] \hspace{1cm} (5.25)

On the other hand, by (3.35), (5.18) and (5.21), we have
\[ 
\int_{\Omega} \mathbf{v}_\varepsilon \cdot \dot{\varphi}_\varepsilon(t) dx = \sum_{i \in \mathcal{Z}_\varepsilon} \sum_{\omega_i \in \Omega \cap \left( \varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right]} \int_{\Omega' \times (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})} \mathbf{u}_\varepsilon(x', \omega'_2, t) \cdot \varphi(x', \varepsilon i, t) dx \\
= \sum_{i \in \mathcal{Z}_\varepsilon} \sum_{\omega_i \in \Omega \cap \left( \varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right]} \int_{\Omega' \times (\omega'_2 + \frac{\varepsilon}{2}, \omega'_2 + \frac{\varepsilon}{2})} \dot{\mathbf{v}}_\varepsilon(x, t) \cdot \varphi(x', \varepsilon i, t) dm_{\varepsilon}. 
\]

For all \( i \in \mathcal{Z}_\varepsilon \) and all \( \omega'_2 \in \omega_2 \cap \left( \varepsilon \Omega - \varepsilon, \varepsilon \right] \), we have \(|\varphi(x_1, x_2, \varepsilon, t) - \varphi(x, t)| \leq C\varepsilon \) in \( \Omega' \times \left( \omega'_2 + \frac{\varepsilon}{2}, \omega'_2 + \frac{\varepsilon}{2} \right] \times (t, T) \). Taking (5.22) into account, we deduce

\[ 
\left| \sum_{i \in \mathcal{Z}_\varepsilon} \sum_{\omega_i \in \Omega \cap \left( \varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right]} \int_{\Omega' \times (\omega'_2 + \frac{\varepsilon}{2}, \omega'_2 + \frac{\varepsilon}{2})} \dot{\mathbf{v}}_\varepsilon(x, t) \cdot \varphi(x', \varepsilon i, t) dm_{\varepsilon} - \int_{\Omega' \times (\omega'_2 + \frac{\varepsilon}{2}, \omega'_2 + \frac{\varepsilon}{2})} \dot{\varphi}_\varepsilon(t) dm_{\varepsilon} \right| 
\leq C\varepsilon \int_{\Omega} \left| \dot{\mathbf{v}}_\varepsilon \right| dm_{\varepsilon}. 
\]

Therefore, by (5.20) and (5.26), the following holds

\[ 
\lim_{\varepsilon \to 0} \int_{\Omega \times (0, t_1)} \dot{\mathbf{v}}_\varepsilon \cdot \dot{\varphi}_\varepsilon dxdt = \lim_{\varepsilon \to 0} \int_{\Omega \times (0, t_1)} \dot{\mathbf{v}}_\varepsilon \cdot \varphi dm_{\varepsilon} dt = \int_{\Omega \times (0, t_1)} \mathbf{n} \varepsilon \cdot \varphi dxdt. 
\] (5.27)

Joining (5.25) and (5.27) we deduce, by the arbitrary choice of \( \varphi \),

\[ 
\mathbf{w} = \mathbf{n} \varepsilon. 
\] (5.28)

On the other hand, by (3.14), (5.21), (5.22), Jensen’s inequality and Korn’s inequality in \( H_0^1(\Omega) \), we have

\[ 
\int_{\Omega} |n_\varepsilon \mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2(\tau) dx 
\]

\[ 
= \sum_{i \in \mathcal{Z}_\varepsilon} \int_{\Omega' \times (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})} \left| \sum_{\omega_i \in \Omega \cap \left( \varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right]} \mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon(x', \omega'_2, \tau) \right|^2 dx 
\]

\[ 
\leq C \sum_{i \in \mathcal{Z}_\varepsilon} \sum_{\omega_i \in \Omega \cap \left( \varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right]} \int_{\Omega' \times (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})} |\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon(x', \omega'_2, \tau)|^2 dx 
\]

\[ 
\leq C\varepsilon^2 \sum_{i \in \mathcal{Z}_\varepsilon} n_\varepsilon(\varepsilon i) \int_{\Omega' \times (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2})} \left| \frac{\partial \mathbf{u}_\varepsilon}{\partial x_3} \right|^2(\tau) dx \leq C\varepsilon^2 \int_{\Omega} |\mathbf{w}_\varepsilon|^2(\tau) dx. 
\] (5.29)

We infer that, under either (5.14) or (5.15), the sequence \( \int_{\Omega \times (0, t_1)} |n_\varepsilon \mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2(\tau) dxdt \) converges to \( 0 \). Therefore, by (5.24) and (5.28),

\[ 
n_\varepsilon \mathbf{u}_\varepsilon \xrightarrow{w} \mathbf{n} \varepsilon \text{ weakly* in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)). 
\] (5.30)

Under (5.15), it easily follows from the strong convergence of \( \mathbf{u}_\varepsilon \) to \( \mathbf{u} \) in \( L^2(0, T; L^2(\Omega; \mathbb{R}^3)) \) and the weak* convergence of \( n_\varepsilon \) to \( n \in L^\infty(\Omega) \) (see (3.16)), that \( n_\varepsilon \mathbf{u}_\varepsilon \)
Lemma 6. Let \((u_\varepsilon)\) be a sequence in \(L^\infty(0, T; H^1_0(\Omega; \mathbb{R}^3))\) satisfying

\[
\sup_{\varepsilon > 0, \tau \in (0, T)} \int \left| \frac{1}{r_\varepsilon} e(u_\varepsilon) \right|^2 (\tau) dm_\varepsilon < +\infty. \tag{5.31}
\]

Then, up to a subsequence, the convergences (5.12) take place. Moreover,

\[
v_3 \in L^\infty(0, T; L^2_n(0, L; H^2_0(\Omega'))). \tag{5.32}
\]

Besides, up to a subsequence, the following convergences hold (see (4.10)):

\[
\frac{u_\varepsilon}{r_\varepsilon} \xrightarrow{\varepsilon \to 0} \xi_\alpha(x, t) - \frac{\partial v_3}{\partial x_\alpha}(x, t) y_3 \quad (\alpha \in \{1, 2\}),
\]

\[
\left( \frac{1}{r_\varepsilon} e(u_\varepsilon) \right)_{\alpha\beta} \xrightarrow{\varepsilon \to 0} \frac{1}{2} \left( \frac{\partial \xi_\alpha}{\partial x_\beta} + \frac{\partial \xi_\beta}{\partial x_\alpha} \right)(x, t) - \frac{\partial^2 v_3}{\partial x_\alpha \partial x_\beta}(x, t) y_3 \quad (\alpha, \beta \in \{1, 2\}),
\]

\[
\xi_1, \xi_2 \in L^\infty(0, T; L^2_n(0, L; H^3_0(\Omega'))). \tag{5.33}
\]

Proof. By (5.1) and (5.31), we have

\[
\sup_{\varepsilon > 0, \tau \in (0, T)} \int \left( \left| \frac{u_{\varepsilon 1}}{r_\varepsilon} \right|^2 + \left| \frac{u_{\varepsilon 2}}{r_\varepsilon} \right|^2 + |u_{\varepsilon 3}|^2 \right) dm_\varepsilon < +\infty. \tag{5.34}
\]

By (5.31) and (5.34), Assumption (5.11) of Lemma 5 is verified, hence, up to a subsequence, the convergences (5.12) take place. By Lemma 2, (5.31) and (5.34), there exists \(v \in L^\infty(0, T; L^2_n(\Omega; \mathbb{R}^3))\), \(\zeta_1, \zeta_2, \zeta_3 \in L^\infty(0, T; L^2_n(\Omega \times I))\), and \(\Xi^b \in L^\infty(0, T; L^2_n(\Omega \times I; \mathbb{S}^3))\), such that

\[
u_1 = \nu_2 = 0;
\]

\[
u_3 = m_\varepsilon \zeta_{03};
\]

\[
\frac{u_{\varepsilon 1}}{r_\varepsilon} \xrightarrow{r_\varepsilon \to 0} m_\varepsilon \zeta_{\alpha \alpha} \quad (\alpha \in \{1, 2\});
\]

\[
\left( \frac{1}{r_\varepsilon} e(u_\varepsilon) \right)_{\alpha\beta} \xrightarrow{\varepsilon \to 0} m_\varepsilon \Xi^b. \tag{5.35}
\]

We establish below that

\[
n(x) \zeta_{03}(x, t, y_3) = n(x)v_3(x, t) \quad \text{a.e. in } \Omega \times (0, t_1) \times I. \tag{5.36}
\]

Then, we fix a matrix field \(\Psi\) satisfying

\[
\Psi \in C^\infty \left( \overline{\Omega \times (0, T)}; \mathcal{D}_2(I; \mathbb{S}^3) \right), \quad \Psi_{33} = 0. \tag{5.37}
\]

Noticing that \(x \to \Psi \left( x, t, \frac{u_{\varepsilon}(x)}{r_\varepsilon} \right)\) vanishes on the complement of the support of \(m_\varepsilon\), by integration by parts we get
\[ \int_{\Omega \times (0,T)} e(\mathbf{u}_\varepsilon) : \mathbf{f}(x,t, \frac{y(x_3)}{\varepsilon}) \, dm_xdt = - \int_{\Omega \times (0,T)} \mathbf{u}_\varepsilon \cdot \nabla_x \mathbf{f}(x,t, \frac{y(x_3)}{\varepsilon}) \, dm_xdt \]

\[ - \sum_{\alpha=1}^2 \int_{\Omega \times (0,T)} u_{\alpha \alpha} \frac{\partial \Psi_{\alpha \beta}}{\partial y_3} \left( x,t, \frac{y(x_3)}{\varepsilon} \right) \, dm_xdt. \]  

(5.38)

By passing to the limit as \( \varepsilon \to 0 \) in (5.38), taking (5.31), (5.35) and (5.36) into account, we infer

\[ 0 = - \int_{\Omega \times (0,t_1) \times I} v_3(\nabla_x \Psi)_{3} ndxdtdy_3 - \sum_{\alpha=1}^2 \int_{\Omega \times (0,t_1) \times I} \zeta_{0 \alpha} \frac{\partial \Psi_{\alpha \beta}}{\partial y_3} ndxdtdy_3. \]  

(5.39)

Fixing \( \alpha \in \{1, 2\}, \varphi \in C^\infty(\Omega \times (0,T)), \psi \in \mathcal{D}_1(I), \) and selecting in (5.39) a field of the form \( \Psi(x,t,y_3) := \varphi(x) \psi(y_3)(\mathbf{e}_\alpha \otimes e_3 + e_3 \otimes \mathbf{e}_\alpha) \), we get

\[ 0 = - \int_{\Omega \times (0,t_1)} v_3(x,t) \frac{\partial \varphi}{\partial x_\alpha}(x,t) ndxdtdy_3 \left( \int_I \psi(y_3)dy_3 \right) \]

\[ - \int_{\Omega \times (0,t_1)} \left( \int_I \zeta_{0 \alpha}(x,t,y_3) \frac{\partial \psi}{\partial y_3}(y_3)dy_3 \right) \varphi(x,t) ndxdtdy_3. \]

Choosing \( \psi \) such that \( \left( \int_I \psi(y_3)dy_3 \right) \neq 0 \), and making \( \varphi \) vary in \( C^\infty(\Omega \times (0,T)) \), we deduce that

\[ v_3 \in L^\infty(0,T; L^2(0,L; H^1_0(\Omega'))), \]  

(5.40)

then, by integration by parts with respect to \( x_\alpha \), infer

\[ 0 = \int_{\Omega \times (0,t_1) \times I} \varphi(x,t) \psi(y_3) \frac{\partial v_3}{\partial x_\alpha}(x,t) ndxdtdy_3 - \int_{\Omega \times (0,t_1) \times I} \zeta_{0 \alpha}(x,t,y_3) \frac{\partial \psi}{\partial y_3}(y_3) \varphi(x,t) ndxdtdy_3. \]

We deduce, from the arbitrary choice of \( \varphi \) and \( \psi \), that

\[ \zeta_{0 \alpha} \in L^\infty (0,T; L^2_n(\Omega; H^1(I))): n \frac{\partial \zeta_{0 \alpha}}{\partial y_3}(x,t,y_3) = -n \frac{\partial v_3}{\partial x_\alpha}(x,t) \quad \text{in } \Omega \times (0,t_1) \times I, \]

and then that

\[ n\zeta_{0 \alpha}(x,t,y_3) = n\xi_{0}(x,t) - n \frac{\partial v_3}{\partial x_\alpha}(x,t)y_3 \quad \text{in } \Omega \times (0,t_1) \times I, \]  

(5.41)

for some suitable \( \xi_{0} \in L^\infty(0,T; L^2_n(\Omega)) \). Next, we choose a matrix field \( \Psi \) satisfying (5.37) and \( \Psi_{3k} = 0 \ \forall \ k \in \{1, 2, 3\} \). By multiplying (5.38) by \( \frac{1}{\varepsilon} \), we get

\[ \sum_{\alpha,\beta=1}^2 \int_{\Omega \times (0,t_1)} e_{\alpha \beta}(u_\varepsilon) \Psi_{\alpha \beta} \left( x,t, \frac{y(x_3)}{\varepsilon} \right) \, dm_xdt = \]

\[ - \sum_{\alpha,\beta=1}^2 \int_{\Omega \times (0,t_1)} u_{\alpha \alpha} \frac{\partial \Psi_{\alpha \beta}}{\partial x_\beta} \left( x,t, \frac{y(x_3)}{\varepsilon} \right) \, dm_xdt. \]

By passing to the limit as \( \varepsilon \to 0 \), thanks to (5.35) and (5.41), we find
\[
\sum_{\alpha, \beta = 1}^{2} \int_{\Omega \times (0, t_1) \times I} \Xi_{\alpha \beta} \Psi_{\alpha \beta} n dx dt dy
\]

\[
= - \sum_{\alpha, \beta = 1}^{2} \int_{\Omega \times (0, t_1) \times I} \left( \xi_\alpha (x, t) - \frac{\partial v_3}{\partial x_\alpha} (x, t) y_3 \right) \frac{\partial \Psi_{\alpha \beta}}{\partial x_\beta} (x, t, y_3) n dx dt dy_3.
\]

By the arbitrary choice of the functions \( \Psi_{\alpha \beta} (= \Psi_{\beta \alpha}) \) in \( C^\infty \left( \overline{\Omega} \times (0, T); \mathcal{D}(I) \right) \) and (5.40), we deduce that for \( \alpha, \beta \in \{1, 2\} \), the following holds

\[
x_\alpha \in L^\infty \left( 0, T; L^2 \left( 0, L; H^1_0 (\Omega') \right) \right), \quad v_3 \in L^\infty \left( 0, T; L^2 \left( 0, L; H^2_0 (\Omega') \right) \right),
\]

\[
n \Xi_{\alpha \beta} (x, t, y_3) = \frac{1}{2} \left( \frac{\partial \xi_\alpha}{\partial x_\beta} + \frac{\partial \xi_\beta}{\partial x_\alpha} \right) (x, t) - n \frac{\partial^2 v_3}{\partial x_\alpha \partial x_\beta} (x, t) y_3 \quad \text{in} \quad \Omega \times (0, t_1) \times I.
\]

The proof of Lemma 6 is achieved.

**Proof of (5.36).** We set \( \overline{v}_{\varepsilon 3}(x, t) := \sum_{j \in J_\varepsilon} u_\varepsilon(x_1, x_2, \omega_\varepsilon^j, t) 1_{\left( \omega_\varepsilon^j - \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2} \right)} (x_3) \) (see (3.4)). By (5.31), we have

\[
\int |u_{\varepsilon 3} - \overline{v}_{\varepsilon 3}(\tau)|^2 dm_x \leq \varepsilon \sum_{j \in J_\varepsilon} \int_{\Omega'} dx' \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} |u_{\varepsilon 3}(x', \tau) - u_{\varepsilon 3}(x', \omega_\varepsilon^j, \tau)|^2 dx_3
\]

\[
\leq \varepsilon \sum_{j \in J_\varepsilon} \int_{\Omega'} dx' \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} \left| \frac{\partial u_{\varepsilon 3}}{\partial x_3} \right|^2 (x', \omega_\varepsilon^j, \tau) dx_3 \leq C r_\varepsilon^2 \int |\epsilon(u_\varepsilon)|^2 (x) dm_x \leq C r_\varepsilon^4.
\]

We easily deduce from (5.35) and (5.42) that

\[
\overline{v}_{\varepsilon 3} \overset{\text{weakly}}{\rightharpoonup} n v_3 \quad \text{in} \quad L^\infty \left( 0, T; \mathcal{M}(\Omega) \right); \quad \overline{v}_{\varepsilon 3} \overset{\text{max}}{\rightharpoonup} \zeta_{03}.
\]

Fixing \( \psi \in \mathcal{D}(\Omega \times (0, t_1) \times I) \), we set (see (4.9))

\[
\overline{\psi}_\varepsilon \left( x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) := \sum_{j \in J_\varepsilon} \psi \left( x', \omega_\varepsilon^j, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) 1_{\left( \omega_\varepsilon^j - \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2} \right)} (x_3).
\]

We have

\[
\left| \overline{\psi}_\varepsilon \left( x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) - \psi \left( x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) \right| \leq C r_\varepsilon \quad \text{in} \quad B_\varepsilon.
\]

By making the change of variables \( y = \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon} \), we get

\[
\int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} \psi \left( x', \omega_\varepsilon^j, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dx_3 = r_\varepsilon \int \psi \left( x', \omega_\varepsilon^j, t, y_3 \right) dy_3.
\]

We infer
\[
\int_{\Omega \times (0, T)} \bar{v}_3 \bar{\psi}_e \left( x, t, \frac{y_e(x_3)}{r_e} \right) d\mu_{\varepsilon} dt = \frac{\varepsilon}{r_e} \sum_{j \in J_e} \int_{\Omega \times (0, T)} \frac{\omega^3_{j+} + \omega^3_{j-}}{2} u_{e,3}(x', \omega^3_{j}, t) \psi \left( x', \frac{y_e(x_3)}{r_e} \right) dx_3 \\
= \frac{\varepsilon}{r_e} \sum_{j \in J_e} \int_{\Omega \times (0, T)} r_e u_{e,3}(x', \omega^3_{j}, t) \left( \int_{J} \psi \left( x', \omega^3_{j}, t, y_3 \right) dy_3 \right) dx_3 \\
= \frac{\varepsilon}{r_e} \sum_{j \in J_e} \int_{\Omega \times (0, T)} \omega^3_{j+} + \omega^3_{j-} u_{e,3}(x', \omega^3_{j}, t) \left( \int_{J} \psi \left( x', \omega^3_{j}, t, y_3 \right) dy_3 \right) dx_3 \\
= \int_{\Omega \times (0, T)} \bar{v}_3(x, t) \bar{\psi}_e(x, t) d\mu_{\varepsilon} dt,
\]

where \( \bar{\psi}_e(x, t) := \sum_{j \in J_e} \left( \int_{J} \psi \left( x', \omega^3_{j}, t, y_3 \right) dy_3 \right) 1_{\left(\omega^3_{j-}, \omega^3_{j+}\right)}(x_3) \). Noticing that 
\[
\left| \bar{\psi}_e(x, t) - \left( \int_{J} \psi \left( x, t, y_3 \right) dy_3 \right) \right| \leq C r_e \text{ in } B_e,
\]
we deduce successively from (5.43), (5.46), (5.45), and again (5.43) that

\[
\int_{\Omega \times (0, T) \times I} v_3(x, t) \left( \int_{J} \psi \left( x, t, y_3 \right) dy_3 \right) ndx dt = \lim_{\varepsilon \to 0} \int_{\Omega \times (0, T)} \bar{v}_3(x, t) \left( \int_{J} \psi \left( x, t, y_3 \right) dy_3 \right) d\mu_{\varepsilon} dt \\
= \lim_{\varepsilon \to 0} \int_{\Omega \times (0, T)} \bar{v}_3(x, t) \bar{\psi}_e(x, t) d\mu_{\varepsilon} dt = \lim_{\varepsilon \to 0} \int_{\Omega \times (0, T)} \bar{v}_3(x, t) \bar{\psi}_e \left( x, t, \frac{y_e(x_3)}{r_e} \right) d\mu_{\varepsilon} dt \\
= \lim_{\varepsilon \to 0} \int_{\Omega \times (0, T)} \bar{v}_3(x, t) \psi \left( x, t, \frac{y_e(x_3)}{r_e} \right) d\mu_{\varepsilon} dt = \int_{\Omega \times (0, T) \times I} \zeta_{03}(x, t, y_3) \psi(x, t, y_3) ndx dt.
\]

By the arbitrary choice of \( \psi \), Assertion (5.36) is proved. \( \square \)

The next Lemma is specific to the periodic case. Given a sequence \((u_\varepsilon)\) satisfying (5.11) and (5.48) (and possibly (5.31)), we establish some relations satisfied by its two-scale limit \( u_0 \) and by the field \( v \) introduced in Lemma 5.

**Lemma 7.** Assume that \( B_e \) is the \( \varepsilon \)-periodic set defined by (3.32) and let \((u_\varepsilon)\) be a sequence in \( L^\infty(0, T; H^1_0(\Omega; \mathbb{R}^3)) \) satisfying (5.11) and

\[
\sup_{\tau \in [0, T], \varepsilon > 0} \int_{\Omega} \left| u_\varepsilon \right|^2 + \varepsilon^2 |e(u_\varepsilon)|^2(\tau) dx < +\infty. \tag{5.48}
\]

Then, up to a subsequence, the convergences (5.12) take place with \( n = 1 \). Moreover

\[
u_\varepsilon \rightharpoonup u_0 \text{ and } \varepsilon e(u_\varepsilon) \rightharpoonup e_\rho(u_0) \text{ in accordance with (4.1),}
\]

\[
u_0 \in L^\infty(0, T; L^2(\Omega; H^1_0(\mathcal{Y}; \mathbb{R}^3))). \tag{5.49}
\]

If, in addition,

\[
\sup_{\tau \in [0, T], \varepsilon > 0} \rho \left| \frac{\partial u_\varepsilon}{\partial t} \right|^2(\tau) dx < +\infty, \tag{5.50}
\]

then
\( u_0 \in W^{1,\infty}(0, T; L^2(\Omega_s Y; \mathbb{R}^3)) , \)

\[
\frac{\partial u_\varepsilon}{\partial t} \longrightarrow \frac{\partial u_0}{\partial t}, \quad u_\varepsilon(\tau) \longrightarrow u_0(\tau) \quad \forall \tau \in [0, T].
\]

Moreover,

(i) If \( \vartheta > 0 \), then

\[
u_0(x, t, y) = v(x, t) \quad \text{in} \quad \Omega \times (0, T) \times B.
\]

(ii) If \( \vartheta = 0 \), then

\[
u'_0(x, t, y) = v'(x, t) \quad \text{in} \quad \Omega \times (0, t_1) \times \Sigma \quad \text{and} \quad \nu_3(.) = \int_\Sigma u_03(., y)dH^2(y)
\]

If, in addition, the estimate (5.31) is satisfied, then

\[
u_0(x, t, y) = v(x, t) \quad \text{on} \quad \Omega \times (0, T) \times \Sigma.
\]

**Proof.** The convergences (5.12) are deduced from (5.11) in Lemma 5. Under (5.48), by Lemma 1 (ii), the sequence \((u_\varepsilon)\) (resp. \((\varepsilon e(u_\varepsilon))\)) two-scale converges, up to a subsequence, to some \(u_0 \in L^\infty(0, T; L^2(\Omega_s Y; \mathbb{R}^3))\) (resp. \(\Xi^m \in L^\infty(0, T; L^2(\Omega_s Y; \mathbb{S}^3))\)). Choosing \(\Psi \in D(\Omega \times (0, t_1); C^2_1(Y; \mathbb{S}^3))\) and passing to the limit as \(\varepsilon \to 0\) in the equation

\[
\int_{\Omega \times (0, t_1)} \varepsilon e(u_\varepsilon) : \Psi \left(x, t, \frac{x}{\varepsilon}\right) dxdt = -\int_{\Omega \times (0, t_1)} \varepsilon u_\varepsilon \cdot \text{div}_x \Psi \left(x, t, \frac{x}{\varepsilon}\right) dxdt - \int_{\Omega \times (0, t_1)} u_\varepsilon \cdot \text{div}_y \Psi \left(x, t, \frac{x}{\varepsilon}\right) dxdt,
\]

we infer \(\int_{\Omega \times (0, T) \times y} \Xi^m : \Psi dxdtdy = -\int_{\Omega \times (0, T) \times y} u_0 \cdot \text{div}_y \Psi dxdtdy\) and deduce, by the arbitrary choice of \(\Psi\), that \(u_0 \in L^\infty(0, T; L^2(\Omega; H^1_0(Y; \mathbb{R}^3)))\) and \(e_y(u_0) = \Xi^m\). Assertion (5.49) is proved. Under (5.50), the convergences (5.51) are a straightforward consequence of Lemma 1 (ii).

If \( \vartheta > 0 \), by (5.11) and (5.48), the sequence \((\varepsilon e(u_\varepsilon)1_B)\) strongly converges to 0 in \(L^2(\Omega \times (0, T); \mathbb{R}^3)\). On the other hand, by (4.4), (5.49) and Lemma 1, \((\varepsilon e(u_\varepsilon)1_B)\) two-scale converges to \(\Xi^m 1_B\). We deduce that \(\Xi^m = 0\) in \(\Omega \times (0, T) \times B\). Let us fix \(\Psi \in D(\Omega \times (0, t_1); D_2(B; \mathbb{S}^3))\). Then for \(\varepsilon\) small enough, the support of \(\Psi \left(x, t, \frac{x}{\varepsilon}\right)\) is included in \(B_\varepsilon \times (0, T)\). Passing to the limit as \(\varepsilon \to 0\) in (5.55), we find 0 = \(-\int_{\Omega \times (0, T) \times B} u_0 \cdot \text{div}_y \Psi dxdtdy\) and infer from the arbitrariness of \(\Psi\) that \(e_y(u_0) = 0\) in \(\Omega \times (0, T) \times B\). Hence, for a. e. \((x, t) \in \Omega \times (0, t_1)\), the restriction of \(u_0(x, t, \cdot)\) to \(B\) is a rigid displacement. Since \(u_0\) is \(Y\)-periodic, we deduce that

\[
u_0 = a \quad \text{in} \quad \Omega \times (0, T) \times B,
\]

for some \(a \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))\). By (4.4), (5.49) and Lemma 1 (i), the sequence \((u_\varepsilon 1_B)\) two-scale converges to \(u_0(x, t, y)1_B(y)\). Fixing \(\varphi \in D(\Omega \times (0, t_1); \mathbb{R}^3)\), taking (5.12), (5.48) and (5.56) into account, and noticing that \(\frac{x}{\varepsilon} \to \frac{1}{|B|}\), we deduce
Let us fix $\vartheta = 0$ (i.e. that $r_\varepsilon \ll \varepsilon$). Since the stiff layers are periodicy distributed, by (3.32) the field $\mathbf{v}_\varepsilon$ defined by (5.21) takes the form

$$\mathbf{v}_\varepsilon(x, t) := \sum_{i \in Z_\varepsilon} \mathbf{u}_\varepsilon(x_1, x_2, \varepsilon i, t) \mathbb{I}_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]}(x_3), \quad (5.57)$$

and coincides in $B_\varepsilon$ with the field $\mathbf{v}_\varepsilon$ given by (5.18). Therefore, by (5.19),

$$\int |\mathbf{v}_\varepsilon - \mathbf{u}_\varepsilon|^2(\tau)dm_\varepsilon \leq C \frac{r_\varepsilon}{\varepsilon} \int_\Omega \varepsilon^2 |e(\mathbf{u}_\varepsilon)|^2(\tau)dx \quad \forall \tau \in [0, T]. \quad (5.58)$$

Since $r_\varepsilon \ll \varepsilon$, we deduce from (5.48) and (5.58) that $\int |\mathbf{v}_\varepsilon|^2(\tau)dm_\varepsilon \leq C$. On the other hand, taking (3.35), (3.32) and (5.57) into account, it is easy to check that $\int |\mathbf{v}_\varepsilon|^2(\tau)dm_\varepsilon = \int_\Omega |\mathbf{v}_\varepsilon|^2(\tau)dx$, therefore the sequence $(\mathbf{v}_\varepsilon)$ is bounded in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$. It then follows from Lemma 1 (ii) that

$$\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v}_0. \quad (5.59)$$

up to a subsequence, for some $\mathbf{v}_0 \in L^\infty(0, T; L^2(\Omega \times Y; \mathbb{R}^3))$. We establish below that

$$\frac{\partial \mathbf{v}_0}{\partial y_3} = 0 \ \text{a.e. in } \Omega \times (0, T) \times Y, \quad \mathbf{u}_0 = \mathbf{v}_0 \ \text{on } \Omega \times (0, T) \times \Sigma, \quad (5.60)$$

and that (see (2.1))

$$\mathbf{v}'(x, t) = \mathbf{v}_0'(x, t, y) \ \text{a.e. in } \Omega \times (0, T) \times Y,$$

$$v_3(x, t) = \int \left(-\frac{s_2}{2} \right)^2 \pi_{03}(x, t, s_1, s_2, y_3)ds_1ds_2, \quad (5.61)$$

yielding (5.53). The next equation (proved below)

$$\frac{\partial \pi_{03}}{\partial y_\alpha} = 0 \ \forall \alpha \in \{1, 2\}, \quad \text{if } (5.31) \text{ holds true}, \quad (5.62)$$

joined with (5.53), yields (5.54). It remains to prove (5.60), (5.61) and (5.62).

**Proof of (5.60).** Let us fix $\psi \in D(\Omega \times (0, t_1); D_\psi(Y; \mathbb{R}^3))$. By (5.57) we have

$$\int_{\Omega \times (0, t_1)} \mathbf{v}_\varepsilon \cdot \frac{\partial \psi}{\partial y_3} \left(x, t, \frac{x}{\varepsilon} \right) dxdt$$

$$= \sum_{i \in Z_\varepsilon} \int_{\Omega \times (0, T)} \mathbf{u}_\varepsilon(x_1, x_2, \varepsilon i, t) \cdot \left(\int_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]} \frac{\partial \psi}{\partial y_3} \left(x, t, \frac{x'}{\varepsilon}, \frac{x_3}{\varepsilon} \right) dx_3 \right) dx'. dt.$$

Since $\psi(x, t, \cdot) \in D_\psi(Y; \mathbb{R}^3)$, the following holds
\[
\int \left( e^{-\frac{t}{\varepsilon}} \right) \frac{\partial \psi}{\partial y_3} \left( x, t, \frac{x'}{\varepsilon}, x_3 \right) \, dx_3 = \frac{1}{\varepsilon} \int \left( e^{-\frac{t}{\varepsilon}} \right) \frac{\partial \psi}{\partial y_3} \left( x, t, \frac{x'}{\varepsilon}, y_3 \right) \, dy_3 = 0,
\]
therefore \( \int_{\Omega \times (0, t_1)} \mathbf{v}_\varepsilon \cdot \frac{\partial \psi}{\partial y_3} (x, t, y) \, dx \, dt = 0. \) By passing to the limit as \( \varepsilon \to 0, \) we infer \( \int_{\Omega \times (0, T) \times Y} \mathbf{v}_0 \cdot \frac{\partial \psi}{\partial y_3} (x, t, y) \, dx \, dt \, dy = 0, \) and deduce from the arbitrary choice of \( \psi \)

\[
\frac{\partial \mathbf{n}_0}{\partial y_3} = 0 \quad \text{in} \quad \Omega \times (0, t_1) \times Y. \tag{5.63}
\]

We set \( Y^+ := \left( -\frac{1}{2}, \frac{1}{2} \right] \times (0, \frac{1}{2}) \) and fix \( \Psi \in \mathcal{D}(\Omega \times (0, t_1); \mathcal{D}_2(Y; \mathbb{S})) \). Then, for each \( i \in \mathbb{Z}_\varepsilon \) (defined by (3.15)), the field \( \Psi(x, t, \frac{y}{\varepsilon}) \) vanishes on \( \partial (\Omega' \times (\varepsilon i - \frac{1}{2}, \varepsilon i + \frac{1}{2})) \times (0, T) \), and, for \( \varepsilon \) small enough, the support of \( \Psi(x, t, \frac{y}{\varepsilon}) \) is included in \( \bigcup_{\varepsilon \in \mathbb{Z}_\varepsilon} \Omega' \times [\varepsilon i - \frac{1}{2}, \varepsilon i + \frac{1}{2}] \times (0, T) \). Hence, by integration by parts, we get

\[
\int_{\Omega \times (0, t_1)} \varepsilon \mathbf{e}(\mathbf{u}_\varepsilon) : \Psi \left( x, t, \frac{x'}{\varepsilon} \right) 1_{Y^+} \left( \frac{x'}{\varepsilon} \right) \, dx \, dt = \sum_{i \in \mathbb{Z}_\varepsilon} \int_{\Omega' \times (\varepsilon i - \frac{1}{2}, \varepsilon i + \frac{1}{2}) \times (0, T)} \varepsilon \mathbf{e}(\mathbf{u}_\varepsilon) : \Psi \left( x, t, \frac{x'}{\varepsilon} \right) \, dx \, dt
\]

\[
= -\sum_{i \in \mathbb{Z}_\varepsilon} \varepsilon \int_{\Omega' \times (\varepsilon i - \frac{1}{2}, \varepsilon i + \frac{1}{2}) \times (0, T)} \mathbf{u}_\varepsilon \cdot \Psi \left( x, t, \frac{x'}{\varepsilon}, 0 \right) e_3 \, d\mathcal{H}^2(x) \, dt
\]

\[
- \int_{\Omega \times (0, t_1)} \varepsilon \mathbf{e}(\nabla_y \Psi) \left( x, t, \frac{x'}{\varepsilon}, 0 \right) 1_{Y^+} \left( \frac{x'}{\varepsilon} \right) + \mathbf{u}_\varepsilon \cdot \nabla_y \Psi \left( x, t, \frac{x'}{\varepsilon}, 0 \right) 1_{Y^+} \left( \frac{x'}{\varepsilon} \right) \, dx \, dt. \tag{5.64}
\]

We set \( \mathbf{v}_\varepsilon(x, t, \frac{x'}{\varepsilon}, 0) := \sum_{i \in \mathbb{Z}_\varepsilon} \Psi \left( x_1, x_2, \varepsilon i, t, \frac{x'}{\varepsilon}, 0 \right) 1_{(\varepsilon i - \frac{1}{2}, \varepsilon i + \frac{1}{2})} (x_3) \). Notice that

\[
\left| \frac{\partial}{\partial y_3} \mathbf{v}_\varepsilon \left( x, t, \frac{x'}{\varepsilon}, 0 \right) - \mathbf{v}_\varepsilon \left( x, t, \frac{x'}{\varepsilon}, 0 \right) \right| \leq C\varepsilon,
\]

\[
\left| \frac{\partial}{\partial y_3} \mathbf{v}_\varepsilon \left( x, t, \frac{x'}{\varepsilon}, 0 \right) - \mathbf{v}_\varepsilon \left( x, t, \frac{x'}{\varepsilon}, 0 \right) \right| \leq C\varepsilon \quad \text{for} \quad \alpha \in \{1, 2\}. \tag{5.65}
\]

By the definitions \( \mathbf{v}_\varepsilon \) and \( \mathbf{v}_\varepsilon \) (see (5.57)), there holds

\[
-\sum_{i \in \mathbb{Z}_\varepsilon} \varepsilon \int_{\Omega' \times (\varepsilon i - \frac{1}{2}, \varepsilon i + \frac{1}{2}) \times (0, T)} \mathbf{u}_\varepsilon \cdot \Psi \left( x, t, \frac{x'}{\varepsilon}, 0 \right) e_3 \, d\mathcal{H}^2(x) \, dt = -\int_{\Omega \times (0, t_1)} \mathbf{v}_\varepsilon \cdot \nabla_y \mathbf{e}_3 \, dx \, dt. \tag{5.66}
\]

Taking (5.59) and (5.65) into account, and noticing that by (5.63) there holds \( \mathbf{v}_0(x, t, y) = \mathbf{v}_0(x, t, y, 0) \) in \( \Omega \times (0, t_1) \times Y \), we obtain

\[
\lim_{\varepsilon \to 0} -\int_{\Omega \times (0, t_1)} \mathbf{v}_\varepsilon \cdot \nabla_y \mathbf{e}_3 \, dx \, dt = -\int_{\Omega \times (0, t_1) \times Y} \mathbf{v}_0 \cdot \Psi \left( x, t, y', 0 \right) e_3 \, dx \, dy \, dt
\]

\[
= -\int_{\Omega \times (0, t_1) \times \Sigma} \mathbf{v}_0 \cdot \mathbf{e}_3 \, dx \, dy \, dt \tag{5.67}
\]

By passing to the limit as \( \varepsilon \to 0 \) in (5.64), applying Lemma 1 (i) with \( h_\varepsilon = 1_{Y^+} \left( \frac{x}{\varepsilon} \right) \) and taking (5.49), (5.66) and (5.67) into account, we get

\[
\int_{\Omega \times (0, t_1) \times Y^+} \mathbf{e}_y(u_0) \cdot \Psi \, dx \, dy = \int_{\Omega \times (0, t_1) \times \Sigma} \mathbf{v}_0 \cdot \mathbf{e}_3 \, dx \, dy \, dt - \int_{\Omega \times (0, t_1) \times Y^+} \mathbf{u}_0 \cdot \mathbf{div}_y \Psi \, dx \, dy \, dt.
\]
By integration by parts, we have

$$-\int_{\Omega \times (0,t_1) \times Y} \mathbf{u}_0 \cdot \text{div}_y \Psi \, dx \, dt \, dy = \int_{\Omega \times (0,t_1) \times \Sigma} \mathbf{u}_0 \cdot \Psi \mathbf{e}_3 \, dx \, dt \, dy + \int_{\Omega \times (0,t_1) \times Y} \mathbf{e}_y(u_0) : \Psi \, dx \, dt \, dy.$$ 

Joining the last two equations, we infer that

$$\int_{\Omega \times (0,t_1) \times \Sigma} \mathbf{u}_0 \cdot \Psi \mathbf{e}_3 \, dx \, dt \, dy = \int_{\Omega \times (0,t_1) \times \Sigma} \mathbf{v}_0 \cdot \Psi \mathbf{e}_3 \, dx \, dt \, dy.$$ 

By the arbitrary choice of $\Psi$ (and by (5.63)), we deduce that (5.60) holds.

**Proof of (5.61).** Let us fix $\psi \in D(\Omega \times (0,t_1); \mathbb{R}^3)$ and set

$$\psi_\varepsilon(x,t) := \sum_{i \in \mathbb{Z}} \psi(x_1, x_2, \varepsilon i, \varepsilon i, t) \mathbf{1}_{[\varepsilon i - \varepsilon i + \frac{1}{2}]}(x_3).$$

(5.68)

By (3.35) and (5.57) we have

$$\int_{\Omega \times (0,t_1)} \mathbf{u}_\varepsilon \cdot \psi_\varepsilon \, dm_\varepsilon = \varepsilon \sum_{i \in \mathbb{Z}} \int_{\Omega \times (0,T)} \mathbf{u}_\varepsilon(x', \varepsilon i, t) \cdot \psi(x', \varepsilon i, t) \, dx \, dt$$

$$= \sum_{i \in \mathbb{Z}} \int_{\Omega \times (0,T)} \mathbf{u}_\varepsilon(x', \varepsilon i, t) \cdot \psi(x', \varepsilon i, t) \, dx \, dt = \int_{\Omega \times (0,t_1)} \mathbf{v}_\varepsilon \cdot \psi_\varepsilon \, dx \, dt.$$ 

(5.69)

We infer from (5.59) and from the estimate

$$|\psi - \psi_\varepsilon|_{L^\infty(\Omega \times (0,t_1); \mathbb{R}^3)} \leq C \varepsilon,$$ 

(5.70)

that

$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \mathbf{v}_\varepsilon \cdot \psi_\varepsilon \, dx \, dt = \int_{\Omega \times (0,t_1)} \left( \int_Y \mathbf{v}_0(x,t,y) \, dy \right) \psi(x,t) \, dx \, dt.$$ 

(5.71)

By (5.58) and (5.70), the following holds

$$\lim_{\varepsilon \to 0} \left| \int_{\Omega \times (0,t_1)} \mathbf{u}_\varepsilon \cdot \psi \, dm_\varepsilon - \int_{\Omega \times (0,t_1)} \mathbf{v}_\varepsilon \cdot \psi_\varepsilon \, dm_\varepsilon \right| = 0.$$ 

(5.72)

The weak* convergence of $(\mathbf{u}_\varepsilon \varepsilon)$ to $\mathbf{v}$ and (5.69), (5.71), (5.72), imply

$$\int_{\Omega \times (0,t_1)} \mathbf{v} \cdot \psi \, dx \, dt = \lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \mathbf{u}_\varepsilon \cdot \psi \, dm_\varepsilon = \int_{\Omega \times (0,t_1)} \left( \int_Y \mathbf{v}_0(x,t,y) \, dy \right) \cdot \psi(x,t) \, dx \, dt,$$

yielding, by the arbitrary choice of $\psi$,

$$\mathbf{v}(x,t) = \int_Y \mathbf{v}_0(x,t,y) \, dy \quad \text{in} \quad \Omega \times (0,t_1).$$ 

(5.73)

By (5.63) and (5.73), the proof of (5.61) is achieved provided that we establish that

$$\frac{\partial \mathbf{v}_0}{\partial y_\beta} = 0 \quad \forall \alpha, \beta \in \{1,2\}.$$ 

(5.74)
To that aim, let us fix $\Psi \in D\left(\Omega \times (0, t_1); C^\infty_0((-\frac{1}{2}, \frac{1}{2})^2; \mathbb{S}^3)\right)$. Since $u_\varepsilon$ vanishes on $\partial \Omega \times (0, T)$, by integrating by parts with respect to $x_1$ and $x_2$, we get (see (2.1))

$$\int_{\Omega \times (0, t_1)} e_{x'}(u_\varepsilon') : \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt = -\int_{\Omega \times (0, t_1)} u_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt$$

$$-\frac{1}{\varepsilon} \int_{\Omega \times (0, t_1)} u_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt.$$  

By (5.48), the term of the left hand side and the first term of the right hand side of the above equation are bounded, therefore

$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0, t_1)} u_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt = 0.$$  

(5.75)

On the other hand, by (5.58) and (5.65), there holds

$$\lim_{\varepsilon \to 0} \left| \int_{\Omega \times (0, t_1)} u_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt - \int_{\Omega \times (0, t_1)} v_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt \right| = 0.$$  

(5.76)

A computation analogous to (5.69) yields

$$\int_{\Omega \times (0, t_1)} v_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt = \int_{\Omega \times (0, t_1)} v_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dx dt.$$  

(5.77)

By (5.65) and by the two-scale convergence of $v_\varepsilon$ to $v_0$ (see (5.59)), there holds

$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0, t_1)} v_\varepsilon' \cdot \text{div}' \Psi \left(x, t, \frac{x'}{\varepsilon}\right) dx dt = \int_{\Omega \times (0, t_1)} v_0' \cdot \text{div}' \Psi \left(x, t, y', y_3\right) dx dt dy.$$  

(5.78)

Joining (5.75), (5.76), (5.77), and (5.78), we get

$$\int_{\Omega \times (0, t_1) \times Y} v_0' \cdot \text{div}' \Psi \left(x, t, y'\right) dx dt dy = 0.$$  

(5.79)

hence $e_{y'}(v_0') = 0$, in the sense of distributions. We deduce that $y' \to v_0'(x, t, y', y_3)$ is a rigid displacement. By integrating (5.79) by parts, we infer

$$\int_{\Omega \times (0, t_1) \times \partial (-\frac{1}{2}, \frac{1}{2})^2} \nabla v_0'(x, t, y) \cdot \nabla \Psi \left(x, t, y'\right) \nu dx dt dH^1(y') = 0,$$

and infer from the arbitrary choice of $\Psi \in D\left(\Omega \times (0, t_1); C^\infty_0((-\frac{1}{2}, \frac{1}{2})^2; \mathbb{S}^3)\right)$, that $v_0' \in L^2(\Omega \times (0, t_1), H^1((-\frac{1}{2}, \frac{1}{2})^2; \mathbb{S}^3))$. The periodicity of $v_0'$ with respect to $y'$ and the fact that $y' \to v_0'(x, t, y', y_3)$ is a rigid displacement imply that $y' \to v_0'(x, t, y')$ is a constant field. Assertion (5.74) is proved. The proof of (5.61) is achieved.

**Proof of (5.62).** We assume (5.31), fix $\psi \in D\left(\Omega \times (0, t_1); D_\alpha((-\frac{1}{2}, \frac{1}{2})^2)\right)$, $\eta \in D(\mathbb{S})$, and $\alpha \in \{1, 2\}$. Noticing that the mapping $x \to \psi \left(x, t, \frac{x'}{r_\varepsilon}\right) \eta \left(\frac{x-x_3}{r_\varepsilon}\right)$ is compactly supported in $B_\varepsilon$, by integration by parts we obtain
By (5.1) we have

\[
\int_{\Omega \times (0,t_1)} \left( \frac{\partial u_{e\alpha}}{\partial x_3} + \frac{\partial u_{e3}}{\partial x_3} \right) \psi \left( x, t, \frac{x'}{\varepsilon} \right) \eta \left( \frac{y_e(x_3)}{r_e} \right) \, dm_\varepsilon \, dt = - \int_{\Omega \times (0,t_1)} \frac{u_{e\alpha}}{r_e} \frac{\partial \psi}{\partial x_3} \left( x, t, \frac{x'}{\varepsilon} \right) + u_{e3} \frac{\partial \psi}{\partial x_3} \left( x, t, \frac{x'}{\varepsilon} \right) \eta \left( \frac{y_e(x_3)}{r_e} \right) \, dm_\varepsilon \, dt
\]

where

\[
u_{3}(y) = \frac{1}{3} \int_{\Omega \times (0,t_1)} \frac{\partial \overline{\psi}_e}{\partial y_\alpha} \left( x, t, \frac{x'}{\varepsilon} \right) \eta \left( \frac{y_e(x_3)}{r_e} \right) \, dm_\varepsilon \, dt
\]

and then deduce from (5.58) and from an estimate analogous to (5.65) that

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \frac{u_{e\alpha}}{r_e} \frac{\partial \psi}{\partial x_3} \left( x, t, \frac{x'}{\varepsilon} \right) + u_{e3} \frac{\partial \psi}{\partial x_3} \left( x, t, \frac{x'}{\varepsilon} \right) \eta \left( \frac{y_e(x_3)}{r_e} \right) \, dm_\varepsilon \, dt = 0,
\]

hence, by (5.31), all terms of the three first lines of (5.80) are bounded. We infer

\[
\int_{\Omega \times (0,t_1)} \left( \frac{\partial u_{e\alpha}}{\partial x_3} + \frac{\partial u_{e3}}{\partial x_3} \right) \psi \left( x, t, \frac{x'}{\varepsilon} \right) \eta \left( \frac{y_e(x_3)}{r_e} \right) \, dm_\varepsilon \, dt = 0.
\]

By the arbitraryness of \( \varepsilon \), assertion (5.62) is proved.
In the next Corollary, we derive from Proposition 1 and lemmas 5, 6, 7, a series of convergences and identification relations for various sequences associated with the solution to (3.5).

**Corollary 1.** Let \( \mathbf{u}_\varepsilon \) be the solution to (3.5).

(i) Up to a subsequence, the convergences (5.12) hold and

\[
 n v_1 = n v_2 = 0 \quad \text{if} \quad k = +\infty, \quad n v = 0 \quad \text{if} \quad \kappa = +\infty. \tag{5.84}
\]

Under (5.14) or (5.15), \( n u = n v \).

(ii) If \( \kappa > 0 \), the relation (5.32) and convergences (5.33) hold.

(iii) In the periodic case, that is under (3.31) and (3.32), the convergences and relations (5.49), (5.51) hold. If \( \vartheta > 0 \) (resp. \( \vartheta = 0 \)), the relations (5.52) (resp. (5.53)) are verified. If in addition \( \kappa > 0 \), then (5.54) holds.

**Proof.** Noticing that by (3.8) and (5.5), the estimate (5.11) holds, Assertion (i) follows from Lemma 5 (Assertion (5.84) is a consequence of (3.8), (5.5), and (5.13)). If \( \kappa > 0 \), by (3.8) and (5.5) the estimate (5.31) holds, and Assertion (ii) follows from Lemma 6. In the periodic case, by (3.31), (3.32), and (5.5), \( \mathbf{u}_\varepsilon \) satisfies (5.48) and (5.50), hence (iii) results from Lemma 7. \( \square \)

6. **Proof of Theorems 1, 4, 5**

In the spirit of Tartar [53], we will multiply (3.5) by an appropriate test field \( \phi_\varepsilon \), integrate by parts, and, passing to the limit as \( \varepsilon \to 0 \) by means of the convergences derived in Corollary 1, obtain a variational problem equivalent to the announced limit problem, and also to (4.23) for some suitable \( H, V, a, h, \xi, \xi_0, \xi_1 \). Theorem 6 will yield existence, uniqueness, and regularity of the effective displacement. Uniqueness implies that the convergences obtained in Corollary 1 for subsequences hold for the complete sequences.

6.1. **Proof of Theorem 5.** We set

\[
 H := \left\{ (\mathbf{w}_0, \psi) \in L^2(\Omega \times Y; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3), \begin{cases}
 \mathbf{w}_0 = \psi \quad \text{in} \quad \Omega \times B \\
 \int_{\Omega \times Y} \rho \mathbf{w}_0 \cdot \bar{\mathbf{w}}_0 \, dxdy,
\end{cases} \right\}
\]

if \( \vartheta > 0 \),

\[
 H := L^2(\Omega \times Y; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3),
\]

\[
 ((\mathbf{w}_0, \psi), (\mathbf{w}_0, \tilde{\psi}))_H := \int_{\Omega \times Y} \rho \mathbf{w}_0 \cdot \bar{\mathbf{w}}_0 \, dxdy + \int_{\Omega} \bar{\rho}_1 \psi \cdot \tilde{\psi} \, dx,
\]

\[
 H := L^2(\Omega \times Y; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3),
\]

\[
 ((\mathbf{w}_0, \psi), (\mathbf{w}_0, \tilde{\psi}))_H := \int_{\Omega \times Y} \rho \mathbf{w}_0 \cdot \bar{\mathbf{w}}_0 \, dxdy + \int_{\Omega} \bar{\rho}_1 \psi \cdot \tilde{\psi} \, dx,
\]

if \( \vartheta = 0 \).

We easily deduce from the positiveness of \( \rho \) and \( \bar{\rho}_1 \) (see (3.7)) that \( H \) is a Hilbert space. We fix a couple \( (\mathbf{w}_0, \psi) \in L^2(0, T; H) \) satisfying (see (3.34))
\[ w_0 \in C^\infty([0, T]; D(\Omega; C^\infty(Y; \mathbb{R}^3))), \quad \psi \in C^\infty([0, T]; D(\Omega; \mathbb{R}^3)), \]  
\[ w_0(T) = \frac{\partial w_0}{\partial t}(T) = \psi(T) = \frac{\partial \psi}{\partial t}(T) = 0, \] 
\[ \psi_1 = \psi_2 = 0 \text{ if } k = +\infty, \quad \psi = 0 \text{ if } \kappa = +\infty, \] 
\[ w_0(x, t, y) = \psi(x, t) \quad \text{in } \Omega \times (0, t_1) \times A. \] 

We choose a sequence \((\alpha_\varepsilon)\) of positive reals such that

\[ \varepsilon r_\varepsilon \ll \alpha_\varepsilon \ll 1, \] 

and set

\[ C_\varepsilon := \{ x \in \Omega, \text{dist}(x, B_\varepsilon) < \alpha_\varepsilon r_\varepsilon \}. \]

It is useful to notice that

\[ L^3(C_\varepsilon \setminus B_\varepsilon) \leq C_{\alpha_\varepsilon r_\varepsilon}, \] 

and that, by (6.5), the following estimate holds for \( m \in \{1, 2\} \):

\[ |\psi(x, t) - w_0(x, t, x_\varepsilon) + \frac{\partial^m \psi_\varepsilon}{\partial t^m}(x, t) - \frac{\partial^m w_0}{\partial t^m}(x, t, x_\varepsilon)| \leq C_{\varepsilon r_\varepsilon} \] 

in \( C_\varepsilon \times (0, T). \)

By (6.7), we can fix a sequence \((\eta_\varepsilon)\) in \( C^\infty(\overline{\Omega})\) satisfying

\[ 0 \leq \eta_\varepsilon \leq 1, \quad \eta_\varepsilon = 1 \text{ in } B_\varepsilon, \quad \eta_\varepsilon = 0 \text{ in } \Omega \setminus C_\varepsilon, \quad |\nabla \eta_\varepsilon| < \frac{C}{r_\varepsilon \alpha_\varepsilon}. \]

The sequence of test fields \((\phi_\varepsilon)\) mentioned above will be defined by

\[ \phi_\varepsilon(x, t) := \eta_\varepsilon(x) \hat{\phi}_\varepsilon(x, t) + (1 - \eta_\varepsilon(x))w_0(x, t, x_\varepsilon), \]

where \(\hat{\phi}_\varepsilon\) is described in Section 7. As \(\phi_\varepsilon(x, t) = w_0(x, t, x_\varepsilon)\) in \( \Omega \setminus C_\varepsilon \times (0, T) \), we deduce from (6.5), (6.9), (6.10), (6.11), and (7.4) that the following estimates hold in \( \Omega \times (0, T) \) for \( m \in \{1, 2\} \):

\[ |\phi_\varepsilon(x, t) - w_0(x, t, x_\varepsilon)| + \left| \frac{\partial^m \phi_\varepsilon}{\partial t^m}(x, t) - \frac{\partial^m w_0}{\partial t^m}(x, t, x_\varepsilon) \right| \leq C \left( r_\varepsilon + \frac{\alpha_\varepsilon r_\varepsilon}{\varepsilon} \right). \]

It is also interesting to notice that by (6.10), (6.11), and (7.4),

\[ |\phi_\varepsilon(x, t) - \psi(x, t)| + \left| \frac{\partial^m \phi_\varepsilon}{\partial t^m}(x, t) - \frac{\partial^m \psi}{\partial t^m}(x, t) \right| \leq C_{r_\varepsilon} \text{ in } B_\varepsilon \times (0, T). \]

By (3.5) and (3.6) we have \(|\sigma_\varepsilon(\phi_\varepsilon)| \leq C_{\mu_0\varepsilon} |\nabla \phi_\varepsilon|\) in \( \Omega \setminus B_\varepsilon \times (0, T) \), therefore by (6.10), (6.12), (7.4), the next estimates are satisfied in \( C_\varepsilon \setminus B_\varepsilon \times (0, T) \):

\[ |\sigma_\varepsilon(\phi_\varepsilon)| \leq C_{\mu_0\varepsilon} \left( |\nabla \eta_\varepsilon| \left| \phi_\varepsilon(x, t) - w_0(x, t, x_\varepsilon) \right| + \left| \nabla \hat{\phi}_\varepsilon(x, t) \right| + \left| \nabla \phi_\varepsilon(x, t) \right| \right), \]

\[ \leq C_{\mu_0\varepsilon} \left( \frac{1}{\alpha_\varepsilon r_\varepsilon} \left( r_\varepsilon + \frac{\alpha_\varepsilon r_\varepsilon}{\varepsilon} \right) + \frac{C}{\varepsilon} \right) \leq C_{\mu_0\varepsilon} \left( \frac{1}{\varepsilon} + \frac{1}{\alpha_\varepsilon} \right), \]
yielding

\[ \mathbf{\sigma}(\phi_\varepsilon) \leq C\mu_0 \left( \frac{1}{\varepsilon} + \frac{1}{\alpha_\varepsilon} \right) \quad \text{in} \ (C_\varepsilon \setminus B_\varepsilon) \times (0, T). \]  

(6.15)

Applying (4.5) to \( \chi_\varepsilon = \rho_\varepsilon \mathbf{1}_{\Omega \setminus B_\varepsilon}, \ h_0 \in \{ \mathbf{w}_0, \frac{\partial^m \mathbf{w}_0}{\partial t^m}, \mathbf{w}_0(0) \}, \) we deduce from (3.7), (4.4), (6.6), and (6.12), that the following convergences hold for \( m \in \{ 1, 2 \} \)

\[
\rho_\varepsilon \phi_\varepsilon \mathbf{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbf{1}_{Y \setminus A}(y) \mathbf{w}_0, \quad \rho_\varepsilon \frac{\partial^m \phi_\varepsilon}{\partial t^m} \mathbf{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbf{1}_{Y \setminus A}(y) \frac{\partial^m \mathbf{w}_0}{\partial t^m}, \\
\rho_\varepsilon \phi_\varepsilon(0) \mathbf{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbf{1}_{Y \setminus A}(y) \mathbf{w}_0(0), \quad \rho_\varepsilon \frac{\partial \phi_\varepsilon}{\partial t}(0) \mathbf{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbf{1}_{Y \setminus A}(y) \frac{\partial \mathbf{w}_0}{\partial t}(0). 
\]

(6.16)

By multiplying (3.5) by \( \phi_\varepsilon, \) after integrations by parts we obtain (see (6.3))

\[
\int_{\Omega \times (0,t_1)} \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial^2 \mathbf{u}_\varepsilon \, dx \, dt + \int_{\Omega \times (0,t_1)} \rho_\varepsilon \mathbf{a}_0 \cdot \frac{\partial \mathbf{u}_\varepsilon}{\partial t}(0) \, dx - \int_{\Omega \times (0,t_1)} \rho_\varepsilon \mathbf{b}_0 \cdot \phi_\varepsilon(0) \, dx \\
+ \int_{\Omega \times (0,t_1)} \mathbf{e}(\mathbf{u}_\varepsilon) : \mathbf{\sigma}(\phi_\varepsilon) \, dx \, dt = \int_{\Omega \times (0,t_1)} \rho_\varepsilon \mathbf{f} \cdot \phi_\varepsilon \, dx \, dt. 
\]

(6.17)

By (3.35) and (3.7), we have

\[
\int_{\Omega \times (0,t_1)} \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial^2 \mathbf{u}_\varepsilon \, dx \, dt = \int_{\Omega \times (0,t_1)} \rho \mathbf{1}_{\Omega \setminus B_\varepsilon} \mathbf{u}_\varepsilon \cdot \partial^2 \mathbf{u}_\varepsilon \, dx \, dt \\
+ \int_{\Omega \times (0,t_1)} \frac{r_\varepsilon}{\varepsilon} \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial^2 \mathbf{u}_\varepsilon \, dm_\varepsilon(x) \, dt. 
\]

(6.18)

We deduce from (5.12), (5.49) (see Corollary 1), and (6.16) that

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \rho_\varepsilon \mathbf{1}_{\Omega \setminus B_\varepsilon} \mathbf{u}_\varepsilon \cdot \partial^2 \mathbf{u}_\varepsilon \, dx \, dt = \int_{\Omega \times (0,T) \setminus Y} \rho \mathbf{1}_{Y \setminus A}(y) \mathbf{w}_0 \cdot \partial^2 \mathbf{w}_0 \, dx \, dt \, dy. 
\]

By (3.7), (5.12), and (6.13), we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \frac{r_\varepsilon}{\varepsilon} \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial^2 \mathbf{u}_\varepsilon \, dm_\varepsilon(x) \, dt = \int_{\Omega \times (0,T)} \rho \mathbf{1}_Y \mathbf{v} \cdot \partial^2 \mathbf{v} \, dx \, dt. 
\]

The last two equations imply

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial^2 \mathbf{u}_\varepsilon \, dx \, dt = \int_{\Omega \times (0,t_1) \setminus Y \setminus A} \rho \mathbf{u}_0 \cdot \partial^2 \mathbf{w}_0 \, dx \, dt \, dy + \int_{\Omega \times (0,T)} \rho \mathbf{1}_Y \mathbf{v} \cdot \partial^2 \mathbf{v} \, dx \, dt. 
\]

(6.19)
By (6.10) and (6.11), we have

\[
\lim_{\epsilon \to 0} \int_{\Omega} \rho \mathbf{a}_0 \cdot \frac{\partial \phi_\epsilon}{\partial t}(0) dx = \int_{\Omega} \rho \mathbf{a}_0 \cdot \frac{\partial \mathbf{w}_0}{\partial t} dxdy + \int_{\Omega} \overline{\mathbf{p}}_1 \mathbf{a}_0 \cdot \frac{\partial^2 \psi}{\partial t^2} dxdt,
\]

\[
\text{in terms of } k
\]

Finally, the limit of the sequence \((I_n)\) is defined by (6.21) is computed in Lemma 8 and (3.31) into account, we deduce that

\[
\left| \frac{1}{\epsilon} \sigma_\epsilon(\phi_\epsilon) - \sigma_{0y}(w_0) \left( x, t, \frac{y}{\epsilon} \right) \right|_{\Omega \setminus C_\epsilon} \leq C \epsilon,
\]

where the operator \(\sigma_{0y}\) is defined by (3.38). The following convergence

\[
\mathbf{I}_{\Omega \setminus C_\epsilon} \to \mathbf{I}_{Y \setminus A},
\]

(6.23)

follows from (4.4) and from the strong convergence of \(\mathbf{I}_{C_\epsilon \setminus B_\epsilon}\) to 0 in \(L^2(\Omega)\), which results from (6.6) and (6.8). By applying Assertion (4.5) of Lemma 1 to \(h_0 := \sigma_{0y}(w_0)\) and \(\chi_\epsilon := \mathbf{I}_{\Omega \setminus C_\epsilon}\), taking (6.22) into account, we infer

\[
\frac{1}{\epsilon} \sigma_\epsilon(\phi_\epsilon) \mathbf{I}_{\Omega \setminus C_\epsilon} \to \sigma_{0y}(w_0) \mathbf{I}_{Y \setminus A}(y).
\]

(6.24)

We deduce from (5.49), (6.21), and (6.24) that

\[
\lim_{\epsilon \to 0} I_{1\epsilon} = \int_{\Omega \times (0,t_1) \times Y \setminus A} e_y(\mathbf{u}_0) : \sigma_{0y}(w_0) dxdtdy.
\]

(6.25)

By (3.31), (6.15) and (6.8), we have

\[
\int_{C_\epsilon \setminus B_\epsilon \times (0,T)} \left| \frac{1}{\epsilon} \sigma_\epsilon(\phi_\epsilon) \right|^2 dxdt \leq C \left( \frac{\alpha_\epsilon^2}{\epsilon^2} + \frac{\alpha_\epsilon}{\epsilon} \right),
\]

therefore, by (6.6), the sequence \(\left( \frac{1}{\epsilon} \sigma_\epsilon(\phi_\epsilon) \mathbf{I}_{C_\epsilon \setminus B_\epsilon} \right)\) strongly converges to 0 in \(L^2(C_\epsilon \setminus B_\epsilon \times (0,T); S^3)\). Accordingly, we infer from (5.48) and (6.21) that

\[
\lim_{\epsilon \to 0} I_{2\epsilon} = 0.
\]

(6.26)

Finally, the limit of the sequence \((I_{3\epsilon})\) defined by (6.21) is computed in Lemma 8 in terms of \(k\) and \(\kappa\). Passing to the limit as \(\epsilon \to 0\) in (6.17), collecting (6.4), (6.19), (6.20), (6.21), (6.25), (6.26), and (7.5), we obtain the variational formulation given, according to the order of magnitude of \(k\) and \(\kappa\), by (6.27), (6.40), (6.42), or (6.33). We distinguish 4 cases:

Case 0 < \(k < +\infty\). We find
Thus by a density argument the variational formulation (6.27) is equivalent to (4.23). By (4.8), (5.12), (5.49), and (5.51) we have

\[ e_y(u_0) : \sigma_{0y}(w_0) dxdt + k \int \Omega_x'(v') : \sigma_x'(\psi') dxdt \]

yielding (4.18). We deduce from Theorem 6 that \( \xi = (u_0, v) \) is the unique solution to (6.27). By (4.19), (4.20), (6.28), the following holds

\[ \xi \in C([0, T]; V) \cap C^1([0, T]; H), \quad \xi(0) = (a_0, a_0), \quad \frac{\partial \xi}{\partial t}(0) = (b_0, b_0). \]
It follows from (6.30), from the next inequalities (deduced from (6.1), (6.28))
\[
\|w_0\|_{L^2(\Omega; L^2(\Omega; \mathbb{R}^3))} + \|\psi\|_{L^2(\Omega; \mathbb{R}^3)} + \|\psi_1\|_{L^2(0, L, H^1_0(\Omega'))} \\
+ \|\psi_2\|_{L^2(0, L, H^1_0(\Omega'))} \leq C\|\{w_0, \psi\}\|_V \quad \forall (w_0, \psi) \in V,
\]
and the next implication, holding for all couple \((E_1, E_2)\) of Banach spaces
\[
A \in \mathcal{L}(E_1, E_2), \quad B \in C^k([0, T]; E_1) \quad \Rightarrow \quad A \circ B \in C^k([0, T]; E_2); \quad \frac{ds}{dt}(A \circ B) = A \circ \frac{ds}{dt}B \quad \forall s \leq k
\]
applied with \(B = \xi = (u_0, v), E_1 \in \{H, V\}, A(\xi, E_2) \in \left\{ \left( u_0, L^2(\Omega; H^1_0(Y; \mathbb{R}^3)) \right), \left( \nu, L^2(\Omega; \mathbb{R}^3) \right), \left( v_0, L^2(0, L; H^1_0(\Omega')) \right), \left( u_0, L^2(\Omega; \mathbb{R}^3) \right), \left( \nu, L^2(\Omega) \right) \right\}; \) that
\[
u \in C^1([0, T]; L^2(\Omega; \mathbb{R}^3)), \quad v(0) = a_0, \quad \tau \frac{\partial v}{\partial t}(0) = b_0,
\]
and, letting \(w_0\) vary over \(D(\Omega \times (0, T) \times Y \setminus A; \mathbb{R}^3)\), deduce
\[
\rho \frac{\partial^2 u_0}{\partial t^2} - \text{div}_y(\sigma_{0y}(u_0)) = \rho f \quad \text{in} \quad \Omega \times (0, T) \times Y \setminus A.
\]
By integrating (6.33) by parts with respect to \((t, y)\) for \(u_0\) satisfying (6.2), (6.3), (6.5), we infer from (6.34) that \(\int_{\Omega \times (0, t_1) \times Y} \sigma_{0y}(u_0) \nu \cdot w_0 dx dt dy = 0\) (\(\nu := \text{outward pointing normal to } \partial Y\)). Noticing that \(\sigma_{0y}(u_0) \nu = 0 \mathcal{H}^2\) a.e. on \(\partial Y \cap A\) (because if \(v > 0\), then \(A = B\) and, by (5.52), \(\sigma_{0y}(u_0) = 0\) in \(B\), whereas if \(r_e < \varepsilon\), then \(A = \Sigma\) and \(\mathcal{H}^2(\partial Y \cap \Sigma) = 0\)), we deduce
\[
\sigma_{0y}(u_0) \nu(x, t, y) = -\sigma_{0y}(u_0) \nu(x, t, -y) \quad \text{on} \quad \Omega \times (0, t_1) \times \partial Y.
\]
Fixing \((w_0, \psi) \in L^2(0, T; H)\) satisfying (6.2), (6.3), we infer from the \(Y\)-periodicity of \(u_0\), (6.5), and (6.35), that (see (3.38))
\[
\int_{\Omega \times (0, t_1) \times \partial (Y \setminus A)} \sigma_{0y}(u_0) \nu \cdot w_0 dx dt dy = -\int_{\Omega \times (0, t_1) \times \partial (Y \setminus A) \cap Y} \sigma_{0y}(u_0) \nu \cdot \psi dx dt dy
\]
\[
= \int_{\Omega \times (0, t_1)} g(u_0) \cdot \psi dx dt.
\]
By subtracting (6.37) from (6.27), we find

\[ \int_{\Omega \times (0,T)} \rho \mathbf{u}_0 \cdot \frac{\partial^2 \mathbf{w}_0}{\partial t^2} \, dx \, dt + \int_{\Omega \times (0,t_1)} \rho \mathbf{a}_0 \cdot \frac{\partial \mathbf{w}_0}{\partial t} (0) \, dx \, dt - \int_{\Omega \times Y \setminus A} \rho \mathbf{b}_0 \cdot \mathbf{w}_0 (0) \, dx \, dy \\
+ \int_{\Omega \times (0,T) \times Y \setminus A} \mathbf{e}_y (\mathbf{u}_0) : \sigma_0 \mathbf{y} (\mathbf{w}_0) \, dx \, dt \, dy + \int_{\Omega \times (0,t_1)} \mathbf{g}_0 (\mathbf{u}_0) \cdot \mathbf{v} \, dx \, dt = \int_{\Omega \times (0,T) \times Y \setminus A} \rho \mathbf{f} \cdot \mathbf{w}_0 \, dx \, dt \, dy. \] (6.37)

By subtracting (6.37) from (6.27), we find

\[ \int_{\Omega \times (0,t_1)} \mathbf{p}_1 \cdot \frac{\partial^2 \mathbf{v}}{\partial t^2} \, dx \, dt - \int_{\Omega \times (0,t_1)} \mathbf{g}(\mathbf{u}_0) \cdot \mathbf{v} \, dx \, dt + k \int_{\Omega \times (0,t_1)} \mathbf{e}_y (\mathbf{v}^\prime) : \mathbf{\sigma}_y (\mathbf{v}^\prime) \, dx \, dt \\
+ \int_{\Omega} \mathbf{p}_1 \mathbf{a}_0 \cdot \frac{\partial \mathbf{v}}{\partial t} (0) \, dx - \int_{\Omega} \mathbf{p}_1 \mathbf{b}_0 \cdot \mathbf{v} (0) \, dx = \int_{\Omega \times (0,t_1)} \mathbf{p}_1 \mathbf{f} \cdot \mathbf{v} \, dx \, dt. \] (6.38)

Making \( \mathbf{v} \) vary in \( \mathcal{D}(\Omega \times (0,t_1); \mathbb{R}^3) \), we infer

\[ \mathbf{p}_1 \frac{\partial^2 \mathbf{v}}{\partial t^2} - k \text{div}\mathbf{\sigma}_y (\mathbf{v}^\prime) = \mathbf{p}_1 \mathbf{f} + \mathbf{g}(\mathbf{u}_0) \quad \text{in} \quad \Omega \times (0,t_1). \] (6.39)

By (6.32), (6.34), (6.35), (6.39), and Lemma 7, the couple \((\mathbf{u}_0, \mathbf{v})\) is a solution to (3.36), (3.41). Conversely, any solution to (3.36), (3.41) satisfies (6.27).

**Case** \( k = +\infty, \ \kappa = 0 \). We obtain

\[ \int_{\Omega \times (0,t_1) \times Y \setminus A} \rho \mathbf{a}_0 \cdot \frac{\partial^2 \mathbf{w}_0}{\partial t^2} \, dx \, dt \, dy + \int_{\Omega \times Y \setminus A} \rho \mathbf{a}_0 \cdot \frac{\partial \mathbf{w}_0}{\partial t} (0) - \rho \mathbf{b}_0 \cdot \mathbf{w}_0 (0) \, dx \, dy \\
+ \int_{\Omega \times (0,t_1)} \mathbf{p}_1 \mathbf{v}_0 \cdot \frac{\partial^2 \mathbf{\psi}_0}{\partial t^2} \, dx \, dt + \int_{\Omega} \mathbf{p}_1 \mathbf{a}_0 \cdot \frac{\partial \mathbf{\psi}_0}{\partial t} (0) - \mathbf{p}_1 \mathbf{b}_0 \mathbf{\psi}_0 (0) \, dx = \int_{\Omega \times (0,t_1) \times Y \setminus A} \mathbf{f} \cdot \mathbf{w}_0 \, dx \, dt \, dy + \int_{\Omega \times Y \setminus A} \mathbf{f}_0 \mathbf{\psi}_0 \, dx \, dy. \] (6.40)

This variational formulation is satisfied for all \((\mathbf{w}_0, \mathbf{\psi}) \in L^2(0,T; H)\) verifying (6.2), (6.3), and (6.4). We set \((H \text{ and } V \text{ being given by} (6.1), (6.28))

\[ \xi = (\mathbf{u}_0, \mathbf{v}), \quad H^{(2)} := \{ (\mathbf{w}, \mathbf{\psi}) \in H, \ \psi_1 = \psi_2 = 0 \}, \]

\[ (\mathbf{.,.})_{H^{(2)}} := (\mathbf{.,.})_H, \quad V^{(2)} := V \cap H^{(2)}, \quad ((\mathbf{.,.}))_{V^{(2)}} := ((\mathbf{.,.}))_V, \]

\[ h^{(2)} := (f \mathbf{1}_{Y \setminus A} + f_3 \mathbf{e}_3 \mathbf{1}_A, f_3 \mathbf{e}_3), \]

\[ a^{(2)}((\mathbf{u}_0, \mathbf{v}), (\mathbf{w}_0, \mathbf{\psi})) := \int_{\Omega \times Y \setminus A} \mathbf{e}_y (\mathbf{u}_0) : \sigma_0 (\mathbf{w}_0) \, dx \, dy, \]

\[ \xi_0^{(2)} := (\mathbf{a}_0 \mathbf{1}_{Y \setminus A} + a_0 \mathbf{e}_3 \mathbf{1}_A, a_0 \mathbf{e}_3), \quad \xi_1^{(2)} := (b_0 \mathbf{1}_{Y \setminus A} + b_0 \mathbf{e}_3 \mathbf{1}_A, b_0 \mathbf{e}_3). \] (6.41)

By (4.8), (5.51) and (5.84), we have \( \xi \in L^2(0,T; V^{(2)}) \) and \( \xi' \in L^2(0,T; H^{(2)}) \). Therefore, by a density argument, the variational problem (6.40) is equivalent to (4.23). By (6.29), (6.41), the estimate (4.18) is satisfied. We deduce from Theorem 6 that \( \xi = (\mathbf{u}_0, \mathbf{v}) \) is the unique solution to (6.40) and that \( \xi \in C([0,T]; V^{(2)}) \cap C^1([0,T]; H^{(2)}) \), \( \xi (0) = \xi_0^{(2)} \), \( \frac{\partial \xi}{\partial t} (0) = \xi_1^{(2)} \). Then, repeating the argument employed to prove (6.32), we infer from (6.31) and (6.41) that the initial-boundary conditions
and regularity properties stated in (3.40), (3.42) are satisfied. Setting \( \psi_3 = 0 \) in (6.40), we get (6.33) and deduce (6.34), (6.35), (6.36), (6.37). Then, subtracting (6.37) from (6.40), taking (6.4) into account, we find

\[
\int_{\Omega \times (0, t_1)} \rho v_3 \frac{\partial^2 \psi_3}{\partial t^2} dxdt - \int_{\Omega \times (0, t_1)} (g(u_0))_3 \psi_3 dxdt + \int_{\Omega} \rho_{\partial t} a_{\partial t} \frac{\partial \psi_3}{\partial t} (0) dx
\]

\[
- \int_{\Omega} \rho_{\partial t} b_{\partial t} \psi_3 (0) dx = \int_{\Omega \times (0, t_1)} \overline{\rho}_1 f_3 \psi_3 dxdt.
\]

Making \( \psi_3 \) vary in \( \mathcal{D}(\Omega \times (0, t_1)) \), we deduce that \( \overline{\rho}_1 \frac{\partial^2 \psi_3}{\partial t^2} = \overline{\rho}_1 f_3 + (g(u_0))_3 \) in \( \Omega \times (0, t_1) \) and infer that \( (u_0, v) \) is solution to (3.36), (3.42).

**Case** 0 < \( \kappa < +\infty \). Passing to the limit as \( \varepsilon \to 0 \) in (6.17), we obtain

\[
\int_{\Omega \times (0, t_1) \times \mathbb{Y} \setminus A} \rho \mu_y \cdot \frac{\partial^2 \omega_0}{\partial t^2} dxdt + \int_{\Omega \times Y \setminus A} \rho \mu_y \cdot \frac{\partial \omega_0}{\partial t} \omega_0 (0) - \rho \mu_y \cdot \omega_0 (0) dx.
\]

\[
H^{(3)} := H^{(2)},
\]

\[
V^{(3)} := \left\{ \begin{array}{l}
(w_0, \psi_3, e_3) \in V^{(2)} \bigg| \psi_3 \in L^2(0, L; L^2(\Omega)) \\
 w_0 (x, y) = \psi_3 (x) e_3 \text{ on } \Omega \times \Sigma \end{array} \right\},
\]

(((u_0, v_3 e_3), (w_0, \psi_3 e_3))_{V^{(3)}} := ((u_0, v_3 e_3), (w_0, \psi_3 e_3))_{V^{(3)}}
\]

\[
+ \int_{\Omega} \left( \frac{\partial^2 v_3}{\partial x_1^2} \frac{\partial^2 \psi_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_2^2} \frac{\partial^2 \psi_3}{\partial x_1^2} - \partial^2 v_3 \frac{\partial^2 \psi_3}{\partial x_1^2} \partial x_1^1 \right) dx,
\]

\[
\alpha^{(3)}((u_0, v), (w_0, \psi)) := \int_{\Omega \times Y \setminus B} e_y (u_0) : \sigma_y (w_0) dxdt + \pi^{(3)}(v, \psi),
\]

\[
\xi^{(3)} := \xi^{(2)}, \quad \xi^{(3)} := \xi^{(2)}, \quad h^{(3)} := h^{(2)}.
\]

By Corollary 1 (ii) and assertions (4.8), (5.51) and (5.84), there holds \( \xi = (u_0, v) \in L^2(0, T; V^{(3)}) \) and \( \xi' \in L^2(0, T; H^{(3)}) \) hence, by a density argument, the variational formulation (6.42) is equivalent to (4.23). By (6.28), (6.29), (6.41), (6.43), (6.44), and (7.6), for all \( \xi = (w_0, \psi) \in V^{(3)} \), we have

\[
||\tilde{\xi}||^2_{V^{(3)}} \leq ||\xi||^2_{V^{(3)}} + C\alpha^{(3)}(\psi, \psi) \leq C(||\tilde{\xi}||_{H^2} + a^{(3)}(\tilde{\xi}, \tilde{\xi})),
\]

\[
\xi^{(3)} := \xi^{(2)}, \quad \xi^{(3)} := \xi^{(2)}, \quad h^{(3)} := h^{(2)}.
\]
hence Assumption (4.18) is satisfied. We deduce from Theorem 6 that \( \xi = (u_0, v) \) is the unique solution to (6.42) and that \( \xi \in C([0, T]; V^{(3)}) \cap C^1([0, T]; H^{(3)}) \), \( \xi(0) = \xi_0^{(3)}, \frac{\partial \xi}{\partial t}(0) = \xi_1^{(3)} \), yielding, by the inequality (6.31) joined with

\[
\|\psi_3\|_{L^2(0, T; H_0^{(3)}(\Omega'))} \leq C(\|u_0, \psi\|_{H^{(3)}}, \forall (u_0, \psi) \in V^{(3)},
\]

the initial-boundary conditions and regularity properties stated in (3.40), (3.43). Repeating the argument of the case \( 0 < k < +\infty \), we set \( \psi_3 = 0 \) in (6.42), obtain (6.33), deduce (6.34), (6.35), (6.36), (6.37), substract (6.37) from (6.42), and get

\[
\int_{\Omega \times (0, t_1)} p_3 \frac{\partial^2 \psi_3}{\partial t^2} \, dx dt + \frac{\kappa}{6} \int_{\Omega \times (0, t_1)} H(v_3) : H^\sigma(\psi_3) \, dx dt
\]

\[
- \int_{\Omega \times (0, t_1)} (g(u_0))_3 \psi_3 \, dx dt - \int_{\Omega} \bar{p}_1 (b_0)_3 \psi_3(x, 0) \, dx = \int_{\Omega \times (0, t_1)} \bar{p}_1 f_3 \psi_3 \, dx dt.
\]

By (7.6), the following equation holds in the sense of distributions in \( D'(\Omega \times (0, t_1)) \)

\[
\frac{\kappa}{6} H(v_3) : H^\sigma(\psi_3)_{D', D} = \frac{\kappa}{3} l + l + 1 \left\langle \sum_{\alpha, \beta = 1}^{2} \frac{\partial^4 v_3}{\partial x^\alpha_1 \partial x^\beta_2}, \psi_3 \right\rangle_{D', D}.
\]

Making \( \psi_3 \) vary in \( D(\Omega \times (0, t_1)) \) in (6.45), we infer

\[
\bar{p}_1 \frac{\partial^2 v_3}{\partial t^2} + \frac{\kappa}{3} l + l + 1 \sum_{\alpha, \beta = 1}^{2} \frac{\partial^4 v_3}{\partial x^\alpha_1 \partial x^\beta_2} = \bar{p}_1 f_3 + (g(u_0))_3, \quad \text{in} \ \Omega \times (0, t_1),
\]

and deduce that \( (u_0, v) \) satisfies (3.36), (3.43).

**Case** \( k = +\infty \). By (7.5) we have \( I_{3e} = 0 \). By passing to the limit as \( \varepsilon \to 0 \) in (6.17), we obtain (6.33) and, taking (5.84) into account, deduce in a similar manner that \( (u_0, v) \) satisfies (3.36), (3.44). The proof of Theorem 5 is achieved.

### 6.2. Proofs of Theorems 1 and 4.

Under the assumptions of Theorem 1, by (3.12) and (5.5), the sequence \( (u_\varepsilon) \) (resp. \( \left( \frac{\partial u_\varepsilon}{\partial t} \right) \)) is bounded in \( L^\infty(0, T; H_0^{(3)}(\Omega; \mathbb{R}^3)) \) (resp. \( L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \)), therefore by the Aubin-Lions-Simon lemma (see [51, Corollary 6]), \( (u_\varepsilon) \) strongly converges in \( L^2(0, T; L^2(\Omega; \mathbb{R}^3)) \) and weakly* converges in \( L^\infty(0, T; H_0^{(3)}(\Omega; \mathbb{R}^3)) \), up to a subsequence, to some \( u \in L^\infty(0, T; H_0^{(3)}(\Omega; \mathbb{R}^3)) \). In particular, assumption (5.15) of Lemma 5 is satisfied, hence \( nu = nv \).

Under the assumptions of Theorem 4, by the apriori estimates (5.5), the sequence \( (u_\varepsilon) \) is bounded in \( L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \), hence weakly* converges in \( L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \), up to a subsequence, to some \( u \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \). By (3.25) and (3.26), assumption (5.14) of Lemma 5 is satisfied, thus we also get \( nu = nv \). Applying Corollary 1, we deduce in both cases from (5.12), (5.32), (5.84), and (7.5), that
We fix a field \( \psi(0) \)

We multiply Equation (3.5) by \( n \)

We deduce from (6.8) and from the estimate

\( L \) is bounded in \( L^\infty(0, T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^3)) \),

\( u_1, u_2 \in L^\infty(0, T; L^2(0, L; H^1_0(\Omega))) \),

\( u_3 \in L^\infty(0, T; L^2_n(0, L; H^2(\Omega'))) \),

\( u = 0 \) \( k = +\infty \),

\( \kappa > 0 \),

\( \kappa = +\infty \),

\( I_{n,k,\kappa}(v, \psi) = I_{n,k,\kappa}(u, \psi) \).

Let us check that

\[ I_{\Omega, B_\varepsilon} \rightarrow 1 - \vartheta n \text{ weakly* in } L^\infty(\Omega), \]  

(6.47)

where \( \vartheta \) is defined by (3.10). If \( \vartheta = 0 \), (6.47) follows from the fact that \(|B_\varepsilon| \rightarrow 0\). Otherwise, if \( \vartheta > 0 \), then the sequence \((\frac{\varepsilon}{r} I_{B_\varepsilon})\) is bounded in \( L^\infty(\Omega) \) and, by (4.7), weakly* converges in \( L^\infty(\Omega) \) to \( n \). It then follows from (3.10) that \((I_{B_\varepsilon})\) weakly* converges in \( L^\infty(\Omega) \) to \( \vartheta n \), yielding (6.47). Next, we check that

\[ u_\varepsilon I_{\Omega, B_\varepsilon} \rightarrow u(1 - \vartheta n) \text{ weakly in } L^2(\Omega \times (0, T); \mathbb{R}^3). \]  

(6.48)

If \( \vartheta = 0 \), Assertion (6.48) follows from the weak convergence of \((u_\varepsilon)\) to \( u \) in \( L^2(\Omega \times (0, T)) \) and the convergence of \( L^3(B_{\varepsilon}) \) to \( 0 \). Otherwise, \( \vartheta > 0 \), then \((\frac{\varepsilon}{r} u_\varepsilon I_{B_\varepsilon})\) is bounded in \( L^2(\Omega \times (0, T)) \), and weakly converges, by (6.46), to \( nu \). Hence, by (3.10), \((u_\varepsilon I_{B_\varepsilon})\) weakly converges to \( n \vartheta \), yielding (6.48).

We fix a field \( \psi \) verifying (6.2), (6.3), and

\[ n\psi_1 = n\psi_2 = 0 \text{ if } k = +\infty; \quad n\psi = 0 \text{ if } \kappa = +\infty. \]  

(6.49)

The sequence of test fields \((\phi_\varepsilon)\) defined by substituting \( \psi \) for \( u_0 \) in (6.11), that is

\[ \phi_\varepsilon(x, t) := \eta_\varepsilon(x) \hat{\psi}_\varepsilon(x, t) + (1 - \eta_\varepsilon(x))\psi(x, t), \]  

(6.50)

where \( \hat{\psi}_\varepsilon(x, t) \) is described in Section 7, and \( \eta_\varepsilon \) satisfies (6.10), now with respect to the non-periodic sets \( B_\varepsilon, C_\varepsilon \) given by (3.4), (6.7). We assume that (see Remark 9)

\[ \frac{r_\varepsilon}{\varepsilon} \ll \alpha_\varepsilon \ll 1 \quad \text{under the assumptions of Theorem 1}, \]  

(6.51)

\[ \mu_\varepsilon \ll \alpha_\varepsilon \ll 1 \quad \text{under the assumptions of Theorem 4}. \]

By (6.10), (6.50), and (7.4), the following estimates hold in \( \Omega \times (0, T) \) for \( m \in \{1, 2\} \):

\[ \left| \phi_\varepsilon(x, t) - \psi \left(x, t, \frac{x}{\varepsilon}\right) \right| + \left| \frac{\partial^m \phi_\varepsilon(x, t) - \partial^m \psi}{\partial t^m} \left(x, t, \frac{x}{\varepsilon}\right) \right| \leq C r_\varepsilon. \]  

(6.52)

We deduce from (6.8) and from the estimate \( \sigma_\varepsilon(\phi_\varepsilon(x, t)) \leq C \frac{\mu_\varepsilon}{\alpha_\varepsilon} \) in \( C_\varepsilon \setminus B_\varepsilon \times (0, T) \), obtained in a similar manner as (6.14), that

\[ \int_{C_\varepsilon \setminus B_\varepsilon \times (0, T)} |\sigma_\varepsilon(\phi_\varepsilon(x, t))|^2 \, dx \, dt \leq C \frac{\mu_\varepsilon^2}{\alpha_\varepsilon} r_\varepsilon. \]  

(6.53)

We multiply Equation (3.5) by \( \phi_\varepsilon \) and integrate it by parts to get (see (6.17), (6.21))
\[
\int_{\Omega \times (0,t_1)} \rho_\varepsilon u_\varepsilon \cdot \frac{\partial^2 \phi_\varepsilon}{\partial t^2} \, dx \, dt + \int_{\Omega} \rho_\varepsilon a_0 \cdot \frac{\partial \phi_\varepsilon}{\partial t}(0) \, dx - \int_{\Omega} \rho_\varepsilon b_0 \cdot \phi_\varepsilon(0) \, dx \\
+ I_{1\varepsilon} + I_{2\varepsilon} + I_{3\varepsilon} = \int_{\Omega \times (0,t_1)} \rho_\varepsilon f \cdot \phi_\varepsilon \, dx \, dt.
\]

(6.54)

By the same argument as the one used to get (6.19), (6.20), splitting each term as in (6.18) and taking into account (3.7), (4.7), (6.47), (6.48), (6.52), we obtain

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \rho_\varepsilon u_\varepsilon \cdot \frac{\partial^2 \phi_\varepsilon}{\partial t^2} \, dx \, dt & = \int_{\Omega \times (0,t_1)} \rho(1 - \vartheta n) u \cdot \frac{\partial^2 \psi}{\partial t^2} \, dx \, dt + \int_{\Omega \times (0,T)} \rho \cdot \frac{\partial^2 \psi}{\partial t^2} \, ndx \, dt, \\
\lim_{\varepsilon \to 0} \int_{\Omega} \rho_\varepsilon b_0 \cdot \frac{\partial \phi_\varepsilon}{\partial t}(0) \, dx & = \int_{\Omega} \rho(1 - \vartheta n) b_0 \cdot \psi(0) \, dx + \int_{\Omega} \rho \cdot \frac{\partial \psi}{\partial t}(0) \, ndx \, dt, \\
\lim_{\varepsilon \to 0} \int_{\Omega} \rho_\varepsilon \cdot \phi_\varepsilon(0) \, dx & = \int_{\Omega} \rho(1 - \vartheta n) \cdot \psi(0) \, dx + \int_{\Omega} \rho \cdot \psi(0) \, ndx \, dt, \\
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,t_1)} \rho_\varepsilon f \cdot \phi_\varepsilon \, dx \, dt & = \int_{\Omega \times (0,t_1)} \rho(1 - \vartheta n) f \cdot \psi \, dx \, dt + \int_{\Omega \times (0,T)} \rho \cdot \frac{\partial \psi}{\partial t} \, ndx \, dt.
\end{align*}
\]

(6.55)

Under the assumptions of Theorem 1, by (3.12), (3.13) and (6.50) we have \( \sigma_\varepsilon(\phi_\varepsilon) = \sigma(\psi) \) in \( \Omega \setminus C_\varepsilon \times (0,T) \), and by (3.11) and (6.7), \( \lim_{\varepsilon \to 0} |C_\varepsilon| = 0 \), therefore the sequence \( \sigma_\varepsilon(\phi_\varepsilon) I_{\Omega\setminus C_\varepsilon} \) strongly converges to \( \sigma(\psi) \) in \( L^2(\Omega \times (0,T); S^3) \). We deduce from the weak* convergence of \( (u_\varepsilon) \) to \( u \) in \( L^\infty(0,T; H^1_b(\Omega; \mathbb{R}^3)) \) that

\[
\lim_{\varepsilon \to 0} I_{1\varepsilon} = \int_{\Omega \times (0,t_1)} e(u) : \sigma(\psi) \, dx \, dt. \tag{6.56}
\]

Under the assumptions of Theorem 4, noticing that \( |\sigma_\varepsilon(\phi_\varepsilon) I_{\Omega\setminus C_\varepsilon}| = |\sigma_\varepsilon(\psi) I_{\Omega\setminus C_\varepsilon}| \leq C\mu_\varepsilon \) and taking (3.25), (5.5), (6.21) into account, we get

\[
\limsup_{\varepsilon \to 0} I_{1\varepsilon} \leq \limsup_{\varepsilon \to 0} C\mu_\varepsilon \left( \int_{\Omega \times (0,t_1)} \mu_\varepsilon |e(u_\varepsilon)(\tau)|^2 \, dx \, dt \right)^{\frac{1}{2}} = 0. \tag{6.57}
\]

By (5.5), and (6.53), we have

\[
I_{2\varepsilon} \leq \left( \int_{\Omega \times (0,t_1)} |e(u_\varepsilon)|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_{C_\varepsilon \setminus B_\varepsilon \times (0,T)} |\sigma_\varepsilon(\phi_\varepsilon(x,t))|^2 \, dx \, dt \right)^{\frac{1}{2}} \leq C \left( \frac{\mu_\varepsilon}{\alpha_\varepsilon} \right)^{\frac{1}{2}}.
\]

Under the assumptions of Theorem 1 (resp. Theorem 4), we deduce from (3.12) and (6.51) that

\[
\lim_{\varepsilon \to 0} I_{2\varepsilon} = 0. \tag{6.58}
\]

Collecting (6.55), (6.56), (6.57), (6.58), and (7.5), by passing to the limit as \( \varepsilon \to 0 \) in (6.17), we obtain, under the assumptions of Theorem 1,
and, under the assumptions of Theorem 4,

\[
\int_{\Omega \times (0, t_1)} (\rho + \overline{p}_1 n) u \cdot \frac{\partial^2 \psi}{\partial t^2} \, dx \, dt + \int_{\Omega \times (0, t_1)} (\rho + \overline{p}_1 n) \left( a_0 \cdot \frac{\partial \psi}{\partial t} (0) - b_0 \cdot \psi (0) \right) \, dx + \mathcal{I}_{a, n, k, \kappa} (u, \psi) = \int_{\Omega \times (0, t_1)} (\rho + \overline{p}_1 n) f \cdot u \, dx dt.
\]

(6.59)

The variational formulation (6.60), joined with (6.46), is equivalent to (4.23), where

\[
H_{n, k, \kappa} = \left\{ \psi \in L^2(\Omega; \mathbb{R}^3) \mid \begin{array}{c}
\psi_1 = \psi_2 = 0 \quad \text{if} \ k = +\infty; \\
\psi = 0 \quad \text{if} \ \kappa = +\infty
\end{array} \right\};
\]

\[
V_{n, k, \kappa} = \left\{ \psi \in H_{n, k, \kappa} \mid \begin{array}{c}
\psi_1, \psi_2 \in L^2_n(0, L; H^1_0(\Omega')) \quad \text{if} \ 0 < k < +\infty; \\
\psi_3 \in L^2_n(0, L; H^1_0(\Omega')) \quad \text{if} \ 0 < \kappa < +\infty
\end{array} \right\};
\]

\[
(u, \psi)_{H_{n, k, \kappa}} = \int_{\Omega} (\rho (1 - \vartheta n) + \overline{p}_1 n) u \cdot \psi \, dx;
\]

\[
(u, \psi)_{V_{n, k, \kappa}} = (u, \psi)_{H_{n, k, \kappa}} + \mathcal{I}_{a, n, k, \kappa} (u, \psi); \quad a_{n, k, \kappa} (u, \psi) = \mathcal{I}_{n, k, \kappa} (u, \psi);
\]

\[
h_{n, k, \kappa} = \mathcal{H}_{n, k, \kappa} (f), \quad \xi_{0, n, k, \kappa} = \mathcal{H}_{n, k, \kappa} (a_0), \quad \xi_{1, n, k, \kappa} = \mathcal{H}_{n, k, \kappa} (b_0), \quad \mathcal{H}_{n, k, \kappa} (g) := \begin{cases}
g & \text{if} \ 0 < k < +\infty, \\
(g_1 \mathbb{1}_{(n=0)}, g_2 \mathbb{1}_{(n=0)}, g_3) & \text{if} \ \kappa = +\infty.
\end{cases}
\]

(6.61)

(6.62)

The variational formulation (6.59), joined with (6.46), is equivalent to (4.23), with data deduced from (6.61), (6.62) by substituting \( \overline{V}_{n, k, \kappa} \) and \( \overline{a}_{n, k, \kappa} \) for \( V_{n, k, \kappa} \) and \( a_{n, k, \kappa} \), where

\[
\overline{V}_{n, k, \kappa} := V_{n, k, \kappa} \cap H^1_0(\Omega; \mathbb{R}^3); \quad (u, \psi)_{\overline{V}_{n, k, \kappa}} = ((u, \psi))_{V_{n, k, \kappa}} + \int_{\Omega} \nabla u \cdot \nabla \psi \, dx,
\]

\[
\overline{a}_{n, k, \kappa} (u, \psi) := a_{n, k, \kappa} (u, \psi) + \int_{\Omega} e (u) : \sigma (\psi) \, dx.
\]

(6.63)

The assumptions of Theorem 6 are satisfied in both cases, guaranteeing existence, uniqueness and regularity properties of the solution. Finally, by integrations by parts, it is easy to check that the variational problems (6.59), (6.60), associated with (6.46), are equivalent to the problems announced in theorems 1, 4.

Remark 9. The assumption stated in the first line of (6.51) is employed to derive (6.62) and requires \( (3.11) \). The case \( \mu_0 = \mu > 0, \vartheta > 0, k = +\infty \) is open.
Remark 10 (Multiphase case). Theorems 1, 4 can be extended to the case of m distributions $B^s_\varepsilon$ (s ∈ {1, ..., m}) of parallel disjoint homothetical layers of thickness $r^s_\varepsilon$, Lamé coefficients $\lambda^s_\varepsilon, \mu^s_\varepsilon$, and mass density $\frac{1}{r^s_\varepsilon} p^s_1$, defined in terms of a finite subset $\omega^s_\varepsilon$ of (0, L) and $r^s_\varepsilon$ by a formula like (3.4). The sets $\omega^s_\varepsilon$ are disjoint and their union $\omega_\varepsilon := \bigcup_{s=1}^m \omega^s_\varepsilon$ satisfies (3.2), which implies that the minimal distance between two distinct points of $\omega_\varepsilon$ is equal to $\varepsilon$. We suppose that $\varepsilon > r^s_\varepsilon(1+\delta), \forall s \in \{1, ..., m\}$ for some $\delta > 0$ and set $\vartheta^{[s]} := \lim_{\varepsilon \to 0} \frac{r^s_\varepsilon}{\varepsilon}$. The Lamé coefficients in $\Omega \setminus \bigcup_{s=1}^m B^s_\varepsilon$, are assumed to be constant and denoted by $\lambda_{0\varepsilon}, \mu_{0\varepsilon}$.

When $\lambda_{0\varepsilon}, \mu_{0\varepsilon}$ satisfy (3.25) and each sequence $(n^s_\varepsilon)$ strongly converges to $n^{[s]}$ in $L^1(\Omega)$, the solution to (3.5) weakly* converges in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ to the unique solution to the problem (4.23), where the data are deduced from (6.61) as follows:

$$H := \bigcap_{s=1}^m H^s_{\varepsilon}, k^s_{\varepsilon}, \kappa^s_{\varepsilon}; (u, \psi)_H = \int_\Omega (\rho(1 - \sum_{s=1}^m \vartheta^{[s]} n^{[s]}) + \overline{p}^s_1 n^{[s]} u \cdot \psi dx$$

$$V := \bigcap_{s=1}^m V^{s}_{\varepsilon}; (u, \psi)_V = (u, \psi)_H + \sum_{s=1}^m I^{s}_{\varepsilon} n^{[s]} k^{[s]} (u, \psi),$$

$$a(u, \psi) = \sum_{s=1}^m a^{s}_{\varepsilon} n^{[s]} k^{[s]} (u, \psi),$$

$$h = H(f), \xi_0 = H(a_0), \xi_1 = H(b_0),$$

$$(H(g)(x))_a = \begin{cases} 0 & \text{if } \exists s \in \{1, ..., m\}, n^{[s]}(x) > 0 \text{ and } k^{[s]} = +\infty, \\
g_\alpha & \text{otherwise}, \quad (\alpha \in \{1, 2\}), \end{cases}$$

$$(H(g)(x))_b = \begin{cases} 0 & \text{if } \exists s \in \{1, ..., m\}, n^{[s]}(x) > 0 \text{ and } k^{[s]} = +\infty, \\
g_3(x) & \text{otherwise}. \end{cases}$$

When $\lambda_{0\varepsilon}, \mu_{0\varepsilon}$ satisfy (3.12), and when $\vartheta^{[s]} = 0$ for each $s \in \{1, ..., m\}$, the solution to (3.5) weakly* converges in $L^\infty(0, T; H^1_0(\Omega; \mathbb{R}^3))$ to the unique solution to (4.23), with data $\tilde{H}, \tilde{V}, \tilde{a}, \ldots$ deduced from $H, V, a, \ldots$ defined in (6.64) as follows:

$$\tilde{H} := H; \quad \tilde{V} = V \cap H^1_0(\Omega; \mathbb{R}^3); \quad ((u, \psi)_V) := (u, \psi)_V + \int_\Omega \nabla u \cdot \nabla \psi dx;$$

$$\tilde{a}(u, \psi) = a(u, \psi) + \int_\Omega e(u) : \sigma(\psi) dx; \quad (h, \xi_0, \xi_1) := (h, \xi_0, \xi_1).$$

Remark 11 (Elliptic case). When $\lambda_{0\varepsilon}, \mu_{0\varepsilon}$ satisfy (3.12), and when $\vartheta^{[s]} = 0$ for each $s \in \{1, ..., m\}$, the solution $u_\varepsilon$ to the equilibrium problem

$$- \text{div}(\sigma(\varepsilon u_\varepsilon)) = f \quad \text{in } \Omega, \quad u_\varepsilon \in H^1_0(\Omega, \mathbb{R}^3), \quad f \in L^2(\Omega, \mathbb{R}^3),$$

is bounded in $H^1_0(\Omega; \mathbb{R}^3)$ and weakly converges to the unique field $u \in \tilde{V}$ satisfying $a(u, \vartheta) = (f, \vartheta)_H$, $\forall \vartheta \in \tilde{V},$ where $\tilde{V}$ is the Hilbert space and $\tilde{a}(\ldots)$ the continuous coercive bilinear form on $\tilde{V}$ given by (6.65).

If $\lambda_{0\varepsilon}, \mu_{0\varepsilon}$ satisfy (3.25), each sequence $(n^s_\varepsilon)$ strongly converges to $n^{[s]}$ in $L^2(\Omega; \mathbb{R}^3)$, and $u_\varepsilon$ is bounded in $L^2(\Omega; \mathbb{R}^3)$, then $u_\varepsilon$ weakly converges, up to a subsequence, to
some \( u \in V \) verifying \( a(u, ψ) = (f, ψ)_H \) \( \forall ψ \in V \), with \( H, V, a(\cdot, \cdot) \) defined by (6.64). In this case, the non-negative bilinear form \( a(\cdot, \cdot) \) may fail to be coercive on \( L^2(Ω; \mathbb{R}^3) \) and the sequence \( u_ε \) to be bounded in \( L^2(Ω; \mathbb{R}^3) \). These coercivity and boundedness are guaranteed by the existence of \( s \in \{1, \ldots, m\} \) and \( c > 0 \) such that \( κ[s] > 0 \) and \( n_ε[s] \geq c \) a.e. in \( Ω_ε := \bigcup_{i \in \mathbb{Z}^3} (εi - \frac{ε}{2}, εi + \frac{ε}{2}] \) (see (3.15)). (Notice that if the second assumption in (3.2) is replaced by \( \min_{i,j,e,i,j \neq e} |ω_e - ω_i| = ηε \) for some arbitrarily fixed \( η \in (0, \frac{1}{2}) \), our proofs are unchanged and \( n_ε[s] \geq c_1Ω_ε \) does not imply that \( B_ε[s] \) is \( ε \)-periodic).

Sketch of the proof. Let \( s \) be such that \( κ[s] > 0 \). The bilinear form associated with (6.66), namely \( a_ε(φ, ψ) = \int_Ω e(φ) : σ_ε(ψ) dx \) \( \forall (φ, ψ) \in (H^1(Ω; \mathbb{R}^3))^2 \), satisfies, by (3.5), (3.8), and (3.25)

\[
a_ε(φ, ψ) \geq C \int_Ω ε^2 |e(φ)|^2 dx + C \int_Ω \left| \frac{1}{|e_ε^1|} e(φ) \right|^2 dm_ε[s].
\]

(6.67)

Let \( u_ε \) be a sequence in \( H^1(Ω; \mathbb{R}^3) \), and let \( m_ε[s], \bar{u}_ε[s], \bar{v}_ε[s] \) be defined by substituting \( ω_ε[s] \) for \( ω_ε \) in (3.35), (5.18), (5.21). We have, since \( n_ε[s] \geq c_1Ω_ε \),

\[
\int_Ω |u_ε|^2 dx \leq \int_Ω |u_ε|^2 dx + C \int_Ω |n_ε[s]u_ε - \bar{v}_ε[s]|^2 dx + C \int_Ω |\bar{v}_ε[s]|^2 dx.
\]

Looking back at (5.29), and using the fact that \( u_ε \) vanishes on \( ∂Ω \), we obtain

\[
\int_{Ω \setminus Ω_ε} |u_ε|^2 dx + C \int_Ω |n_ε[s]u_ε - \bar{v}_ε[s]|^2 dx
\]

\[
\leq Cε^2 \int_{Ω \setminus Ω_ε} \left( \nabla u_ε \right)^2 (\tau) dx + Cε^2 \sum_{i \in \mathbb{Z}^3} \int_{Ω \setminus Ω_ε} |\nabla \bar{u}_ε|^2 (\tau) dx
\]

\[
\leq Cε^2 \int_Ω |\nabla u_ε|^2 dx \leq Cε^2 \int_Ω |e(u_ε)|^2 dx,
\]

yielding \( \int_Ω |u_ε|^2 dx \leq Cε^2 \int_Ω |e(u_ε)|^2 dx + C \int_Ω |v_ε[s]|^2 dx \). On the other hand, by (5.23), (5.19), (5.1),

\[
\int_Ω |v_ε[s]|^2 dx \leq \int_Ω |v_ε[s]|^2 dm_ε[s] \leq C \int_Ω |u_ε - \bar{v}_ε[s]|^2 dm_ε[s] + C \int_Ω |u_ε|^2 dm_ε[s]
\]

\[
\leq Cεr_ε[s] \int_Ω |e(u_ε)|^2 dm_ε[s] + C \int_Ω \left| \frac{1}{r_ε} e(u_ε) \right|^2 dm_ε[s],
\]

therefore, for all \( u_ε \in H^1_0(Ω; \mathbb{R}^3) \), \( \int_Ω |u_ε|^2 dx \leq Ca_ε(u_ε, u_ε) \). In the particular case when \( u_ε \) is the solution to (6.66), we infer \( \int_Ω |u_ε|^2 dx \leq C \int_Ω f \cdot u_ε dx \leq C \left( \int_Ω |u_ε|^2 dx \right)^{\frac{1}{2}} \), hence \( u_ε \) is bounded in \( L^2(Ω; \mathbb{R}^3) \). We choose a smooth field \( ψ \in V \) and consider the associated sequence of test field \( φ_ε \) used for the proof of the multiphase case, whose construction is similar to (6.50). Repeating the argument of [8, p. 40, (iii) ⇒ (i)], we find that \( a(ψ, ψ) = lim_{ε \to 0} a_ε(φ_ε, φ_ε) \geq \)

$c \lim_{\varepsilon \to 0} \int_{\Omega} |\phi|^{2} \, dx = \int_{\Omega} |\psi|^{2} \, dx$. By a density argument, we deduce $\int_{\Omega} |\psi|^{2} \, dx \leq Ca(\psi, \psi) \quad \forall \psi \in V$.

7. Appendix

A common step in the proofs of theorems 1, 4, and 5 lies in the computation of the limit of the sequence $(I_{3\varepsilon})$ defined by (see (6.11), (6.21))

$$I_{3\varepsilon} := \int_{B_{\varepsilon} \times (0, T)} e(u_{\varepsilon}) : \sigma(\hat{\psi}_{\varepsilon}) \, dx \, dt,$$

where $u_{\varepsilon}$ is the solution to (3.5) and the oscillating test fields $\hat{\psi}_{\varepsilon}$ is defined below, in terms of $\psi \in C^{\infty}([0, T]; D(\Omega; \mathbb{R}^{3}))$ satisfying (6.4), of $\delta$ given by (3.3), and of the order of magnitude of the parameters $k$ and $\kappa$. We introduce the field $\overline{\psi}_{\varepsilon}$ given by

$$\overline{\psi}_{\varepsilon}(x, t) := \sum_{j \in J_{\varepsilon}} \left( \int_{(\omega_{j}^{-\frac{\varepsilon}{2}} - \omega_{j}^{-\frac{\varepsilon}{2}} + \frac{\alpha}{\varepsilon})} \psi_{j} (x_{1}, x_{2}, s_{3}, t) \, ds_{3} \right) \mathbf{1} (\omega_{j}^{-\frac{\varepsilon}{2}} - \omega_{j}^{(1+4)}(x_{3}) \right) \mathbf{1}_{(\omega_{j}^{-\frac{\varepsilon}{2}} + \omega_{j}^{(1+4)}(x_{3})}) (x_{3}).$$

(i) If $0 < k \leq +\infty$ and $\kappa = 0$, we set

$$\hat{\psi}_{\varepsilon}(x, t) := \overline{\psi}_{\varepsilon}(x, t) + \varepsilon_{\varepsilon} W_{1\varepsilon} \left( x, t, \frac{y_{\varepsilon}(x_{3})}{r_{\varepsilon}} \right),$$

$$W_{1\varepsilon}(x, t, y_{3}) := \left( \begin{array}{c} -\frac{\partial \overline{\psi}_{\varepsilon}}{\partial x_{1}} y_{3} \\ -\frac{\partial \overline{\psi}_{\varepsilon}}{\partial x_{2}} y_{3} \\ \frac{l_{\varepsilon}}{l_{\varepsilon} + 2} \left( \frac{\partial^{2} \overline{\psi}_{\varepsilon}}{\partial x_{1}^{2}} + \frac{\partial^{2} \overline{\psi}_{\varepsilon}}{\partial x_{2}^{2}} \right) y_{3} \end{array} \right).$$

where the function $y_{\varepsilon}(\cdot)$ is defined by (4.9).

(ii) If $0 < \kappa \leq +\infty$, we set

$$\hat{\psi}_{\varepsilon}(x, t) := \overline{\psi}_{\varepsilon}(x, t) + \varepsilon_{\varepsilon} W_{1\varepsilon} \left( x, t, \frac{y_{\varepsilon}(x_{3})}{r_{\varepsilon}} \right) + \varepsilon_{\varepsilon} W_{2\varepsilon} \left( x, t, \frac{y_{\varepsilon}(x_{3})}{r_{\varepsilon}} \right),$$

$$W_{1\varepsilon}(x, t, y_{3}) := \left( \begin{array}{c} -\frac{\partial \overline{\psi}_{\varepsilon}}{\partial x_{1}} y_{3} \\ -\frac{\partial \overline{\psi}_{\varepsilon}}{\partial x_{2}} y_{3} \\ \frac{l_{\varepsilon}}{2(l_{\varepsilon} + 2)} \left( \frac{\partial^{2} \overline{\psi}_{\varepsilon}}{\partial x_{1}^{2}} + \frac{\partial^{2} \overline{\psi}_{\varepsilon}}{\partial x_{2}^{2}} \right) y_{3} \end{array} \right).$$

It is usefull to notice that $\hat{\psi}_{\varepsilon}$ is continuously differentiable in $C_{\varepsilon} \times (0, T)$ (see (6.7)), that $\hat{\psi}_{\varepsilon} = 0$ if $\kappa = +\infty$ and that for $m \in \{1, 2\}$,

$$\left( \frac{\partial^{m} \hat{\psi}_{\varepsilon}(x, t)}{\partial t^{m}} - \frac{\partial^{m} \psi_{\varepsilon}(x, t)}{\partial t^{m}} \right) \mathbf{1}_{C_{\varepsilon}}(x) \leq C_{\varepsilon},$$

$$\left| \nabla \hat{\psi}_{\varepsilon}(x, t) \right| \mathbf{1}_{C_{\varepsilon}} \leq C.$$
the operators $\mathbf{H}$, $\mathbf{H}^{\sigma}$ being defined by

$$
\mathbf{H}(\psi) := \begin{pmatrix}
\frac{\partial^2 \psi}{\partial x_1^2} & \frac{\partial^2 \psi}{\partial x_1 \partial x_2} & 0 \\
\frac{\partial^2 \psi}{\partial x_2 \partial x_1} & \frac{\partial^2 \psi}{\partial x_2^2} & 0 \\
0 & 0 & 0
\end{pmatrix};
$$

$$
\mathbf{H}^{\sigma}(\psi) := \begin{pmatrix}
\frac{\partial^2 \psi}{\partial x_1^2} + \frac{1}{l+2} \frac{\partial^2 \psi}{\partial x_2^2} & \frac{1}{l+2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2} & \frac{1}{l+2} \frac{\partial^2 \psi}{\partial x_2 \partial x_1} & 0 \\
\frac{\partial^2 \psi}{\partial x_1 \partial x_2} & \frac{\partial^2 \psi}{\partial x_2^2} + \frac{1}{l+2} \frac{\partial^2 \psi}{\partial x_2 \partial x_1} & \frac{1}{l+2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.  \tag{7.6}
$$

Proof. Case $0 < k < +\infty$. We easily check that

$$
\| \psi - \overline{\psi}_\varepsilon \|_{L^\infty(B_\varepsilon \times (0,T))} + \left\| \frac{\partial (\psi - \overline{\psi}_\varepsilon)}{\partial x_\alpha} \right\|_{L^\infty(B_\varepsilon \times (0,T))} \leq C r_\varepsilon \quad (\alpha \in \{1, 2\}),
$$

$$
\left\| \frac{\partial^2 (\psi - \overline{\psi}_\varepsilon)}{\partial x_\alpha \partial x_\beta} \right\|_{L^\infty(B_\varepsilon \times (0,T))} \leq C r_\varepsilon \quad (\alpha, \beta \in \{1, 2\}).  \tag{7.7}
$$

A straightforward computation yields (see (3.5), (3.6))

$$
\mathbf{\sigma}_\varepsilon(\overline{\psi}_\varepsilon) = k \int_{\Omega \times (0,t_1)} e_{x'}(\mathbf{u}') : \mathbf{\sigma}_{x'}(\psi') n \, dx \, dt
$$

$$
\mu_\varepsilon \left( 2 \frac{\partial v_1}{\partial x_1} + \frac{2l+1}{l+2} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) + \frac{\partial v_2}{\partial x_2} \right) + \frac{\partial v_1}{\partial x_2} + \frac{2l+1}{l+2} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) + \frac{\partial v_2}{\partial x_2} = 0
\right) I_{B_\varepsilon}.  \tag{7.8}
$$

Case $(k, \kappa) = (+\infty, 0)$. By (6.4), (7.8), we have $|\mathbf{\sigma}_\varepsilon(\overline{\psi}_\varepsilon) I_{B_\varepsilon}| \leq C \mu_\varepsilon r_\varepsilon$, thus, by (3.8), the second line of (5.5), and (7.1), there holds
By (3.35) and (7.1), we have
\[ I_{3c} \leq C \mu_1 \varepsilon \int_{B_2(0,T)} |e(u_\varepsilon)| dx dt = C \mu_1 \frac{r_\varepsilon^2}{\varepsilon} \int_{B_2(0,T)} |e(u_\varepsilon)| dm_\varepsilon dt \]
\[ \leq C \mu_1 \frac{r_\varepsilon^2}{\varepsilon} \sqrt{\int_{J_{B_2(0,T)}} |e(u_\varepsilon)|^2 dm_\varepsilon dt} \leq C \mu_1 \frac{r_\varepsilon^2}{\varepsilon} \sqrt{\frac{\varepsilon}{r_\varepsilon \mu_1}} \leq C \sqrt{\frac{r_\varepsilon^3}{\varepsilon^2} \mu_1} = o(1). \]

Case $0 < \kappa < +\infty$. A straightforward computation gives
\[
\frac{r_\varepsilon^2}{\varepsilon} \sigma_\varepsilon(\psi_\varepsilon) \mathbf{1}_{B_2} = \frac{2}{\varepsilon^2} M_1 + 2 \kappa \mathbf{H}(\psi_\varepsilon) \frac{y_\varepsilon(x_3)}{r_\varepsilon} \mathbf{1}_{B_2} \left( \begin{array}{ccc} \frac{2(l+1)}{l+2} & \frac{l_1}{l+2} & 0 \\ \frac{l_1}{l+2} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + r_\varepsilon O \left( \frac{r_\varepsilon^3}{\varepsilon^2} u_\varepsilon \right).
\]

We deduce from (3.6), (3.8), (7.6), and (7.7), that
\[
\lim_{\varepsilon \to 0} \left. \frac{r_\varepsilon^2}{\varepsilon} \sigma_\varepsilon(\psi_\varepsilon) \mathbf{1}_{B_2} \right|_{L^\infty(B_2(0,T) ; \mathbb{R}^3)} = 0. \tag{7.9}
\]

By (3.35) and (7.1), we have
\[
I_{3c} = \frac{r_\varepsilon}{\varepsilon} \int_{\Omega \times (0,t_1)} e(u_\varepsilon) : \sigma_\varepsilon(\psi_\varepsilon) dm_\varepsilon dt = \int_{\Omega \times (0,t_1)} \frac{1}{r_\varepsilon} e(u_\varepsilon) : \frac{r_\varepsilon^2}{\varepsilon} \sigma_\varepsilon(\psi_\varepsilon) dm_\varepsilon dt. \tag{7.10}
\]

Taking Corollary 1 (ii) into account, we infer from (5.31), (5.33), (7.9), (7.10), that
\[
\lim_{\varepsilon \to 0} I_{3c} = \lim_{\varepsilon \to 0} - \int_{\Omega \times (0,t_1)} \frac{1}{r_\varepsilon} e_{x_\varepsilon}(u_\varepsilon) : \kappa \mathbf{H}(\psi_\varepsilon) \frac{y_\varepsilon(x_3)}{r_\varepsilon} dm_\varepsilon dt
\]
\[
= -2 \kappa \sum_{\alpha, \beta} \int_{\Omega \times (0,t_1) \times I} \mathbf{H}(\psi_\varepsilon) \frac{y_\varepsilon(x_3)}{r_\varepsilon} \left( \frac{1}{2} \frac{\partial \xi_\alpha}{\partial x_\beta} + \frac{\partial \xi_\beta}{\partial x_\alpha} \right)(x,t) - \frac{\partial^2 v_3}{\partial x_\alpha \partial x_\beta}(x,t)y_3 ndxdtdy_3
\]
\[
= \frac{\kappa}{6} \int_{\Omega \times (0,t_1)} \mathbf{H}(v_3) : \mathbf{H}(\psi_\varepsilon) ndxdt.
\]

\[\square\]

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