The structure of graphs with forbidden $C_4$, $\overline{C_4}$, $C_5$, chair and co-chair

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1

Abstract
We find the structure of graphs that have no $C_4$, $\overline{C_4}$, $C_5$, chair and co-chair as induced subgraphs.

1 Introduction
In this paper, graphs are finite and simple. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. Two edges of a graph $G$ are said to be adjacent if they have a common endpoint and two vertices $x$ and $y$ are said to be adjacent if $xy$ is an edge of $G$. The neighborhood of a vertex $v$ in a graph $G$, denoted by $N_G(v)$, is the set of all vertices adjacent to $v$ and its degree is $d_G(v) = |N_G(v)|$. We omit the subscript if the graph is clear from the context. For two set of vertices $U$ and $W$ of a graph $G$, let $E[U,W]$ denote the set of all edges in the graph $G$ that joins a vertex in $U$ to a vertex in $W$. A graph is empty if it has no edges. For $A \subseteq V(G)$, $G[A]$ denotes the sub-graph of $G$ induced by $A$. If $G[A]$ is an empty graph, then $A$ is called a stable. While, if $G[A]$ is a complete graph, then $A$ is called a clique set, that is any two distinct vertices in $A$ are adjacent. The complement graph of $G$ is denoted by $\overline{G}$ and defined as follows: $V(G) = V(\overline{G})$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$.

A graph $H$ is called forbidden subgraph of $G$ if $H$ is not (isomorphic to) an induced subgraph of $G$.

A cycle on $n$ vertices is denoted by $C_n = v_1v_2...v_nv_1$ while a path on $n$ vertices is denoted by $P_n = v_1v_2...v_n$. A chair is any graph on 5 distinct vertices $x,y,z,t,v$ with exactly 5 edges $xy,yz,zt$ and $zv$. The co-chair or $\overline{\text{chair}}$ is the complement of a chair (see the below figure).

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Many graphs encountered in the study of graph theory are characterized by configurations or subgraphs they contain. However, there are occasions where it is easier to characterize graphs by sub-graphs or induced sub-graphs they do not contain. For example, trees are the connected graph without (induced) cycles. Bipartite graphs are those without (induced) odd cycles ([5]). Split graphs are those without induced $C_4$, $\overline{C}_4$ and $C_5$. Line graphs are characterized by the absence of only nine particular graphs as induced sub-graph (see [4]). Perfect graphs are characterized by $C_{2n+1}$ and $\overline{C}_{2n+1}$ being forbidden, for all $n \geq 2$ (see [3]). The purpose of this paper is to find the structure of graphs such that $C_4$, $\overline{C}_4$, $C_5$, chair and co-chair are forbidden subgraphs.

2 Preliminary Definitions and Theorems

**Definition 1.** A graph $G$ is called a split graph if its vertex set is the disjoint union of a stable set $S$ and a clique set $K$. In this case, $G$ is called an $\{S, K\}$-split graph.

If $G$ is an $\{S, K\}$-split graph and $\forall s \in S$, $\forall x \in K$ we have $sx \in E(G)$, then $G$ is called a complete split graph.

If $G$ is an $\{S, K\}$-split graph and $E[S, K]$ forms a perfect matching of $G$, then $G$ is called a perfect split graph.

**Theorem 1.** (Földes and Hammer [1]) $G$ is a split graph if and only if $C_4, \overline{C}_4$ and $C_5$ are forbidden subgraphs of $G$.

**Definition 2.** ([2]) A threshold graph $G$ can be defined as follows:

1) $V(G) = \bigcup_{i=1}^{n+1} (X_i \cup A_{i-1})$, where the $A_i$’s and $X_i$’s are pair-wisely disjoint sets.

2) $K := \bigcup_{i=1}^{n+1} X_i$ is a clique and the $X_i$’s are nonempty, except possibly $X_{n+1}$.
3) $S := \bigcup_{i=0}^{n} A_i$ is a stable set and the $A_i$'s are nonempty, except possibly $A_0$.

4) $\forall 1 \leq j \leq i \leq n, G[A_i \cup X_j]$ is a complete split graph.

5) The only edges of $G$ are the edges of the subgraphs mentioned above.

In this case, $G$ is called an $\{S, K\}$-threshold graph.

**Theorem 2.** (Hammer and Chvátal [2]) $G$ is a threshold graph if and only if $C_4$, $\overline{C_4}$ and $P_4$ are forbidden subgraphs of $G$.

### 3 Main Results

**Lemma 1.** Suppose that $C_4$, $\overline{C_4}$, $C_5$, chair and co-chair are forbidden subgraphs of $G$. If the path $mbb'm'$ is an induced subgraph of $G$, then:

\[
N(m) - \{b\} = N(m') - \{b'\}
\]

and

\[
N(b) - \{m\} = N(b') - \{m'\}.
\]

**Proof.** Since $C_4$, $\overline{C_4}$ and $C_5$ are forbidden, then $G$ is an $\{S, K\}$-split graph for some stable set $S$ and a clique set $K$. Since $mbb'm'$ is an induced subgraph of $G$, then $m, m' \in S$ and $b, b' \in K$.

Assume that there is $x \in N(m) - \{b\}$ but $x \notin N(m') - \{b'\}$. Since $xm$ is an edge of $G$ and $S$ is stable, then we must have $x \in K$. But $K$ is a clique, then $x$ is adjacent to $b$ and $b'$. Thus $G[\{x, m, b, b', m'\}]$ is a co-chair. Contradiction. So $N(m) - \{b\} \subseteq N(m') - \{b'\}$. By symmetry, $N(m') - \{b'\} \subseteq N(m) - \{b\}$. Thus $N(m) - \{b\} = N(m') - \{b'\}$.

Assume that there is $x \in N(b) - \{m\}$ but $x \notin N(b') - \{m'\}$. Suppose that $x \in S$. Then $G[\{x, m, b, b', m'\}]$ is a chair. Contradiction. Thus $x \in K$. But $K$ is a clique. Whence $x \in N(b')\{m'\}$. Thus $N(b) - \{m\} \subseteq N(b') - \{m'\}$. By symmetry, $N(b') - \{m'\} \subseteq N(b) - \{m\}$. Therefore $N(b) - \{m\} = N(b') - \{m'\}$.

\[\square\]

**Proposition 1.** If $P_4$ is a forbidden subgraph of an $\{S, K\}$-split graph $G$, then $G$ is an $\{S, K\}$-threshold graph.

**Proof.** We prove this by induction on the number of vertices of $G$. This is clearly true for small graphs. Suppose that $P_4$ is a forbidden subgraph of an $\{S, K\}$-split graph $G$. It is clear that $G$ is a threshold graph. We have to prove that $G$ is a $\{S, K\}$-threshold graph. Let $x \in K$ be a vertex with minimum degree in $G$, that is $d_G(x) = \min\{d_G(y); y \in K\}$ and $G' := G - x$ be the graph induced by the vertices of $G$ except $x$ (If $K = \phi$, then the statement is true).

Then $P_4$ is a forbidden subgraph of the $\{S, K - \{x\}\}$-split graph $G'$. By the induction hypothesis, $G'$ is an $\{S, K - \{x\}\}$-threshold graph. We follow the notations in Definition 2. Assume that $\exists a \in S - A_n$ such that $ax \in E(G)$. Let $x_n \in X_n$. Since $d(x_n) \geq d(x)$, then there is $a_n \in A_n$ such that $a_nx_n \in E(G)$ but $a_nx \notin E(G)$. Then $ax_nx_n a_n$ is an induced $P_4$ in $G$. Contradiction. Thus we
may suppose that $N(x) \cap S \subseteq A_1$. If $N(x) \cap A_n = \emptyset$, then we add $x$ to $X_{n+1}$. If $N(x) \cap A_n = A_n$, then we add $x_n$ to $X_n$. Otherwise $\emptyset \subseteq N(x) \cap A_n \subseteq A_n$. In this case we do the following: remove from $A_n$ the element of $N(x) \cap A_n$, remove the elements of $X_{n+1}$ to the new set $X_{n+2}$ and add $x$ to $X_{n+1}$ (so that the new $X_{n+1} = \{x\}$). Then $G$ is $\{S, K\}$-threshold graph.

**Definition 3.** A graph $G$ is called a comb if:

1) $V(G)$ is disjoint union of sets $A_0, ..., A_n, M_1, ..., M_l, X_1, ..., X_{n+1}, Y_2, ..., Y_{l+2}$. Let $Y_1 = X_1$ (These sets are called the sets of the comb $G$).

2) $S := A \cup M$ is a stable set, where $M = \bigcup_{i=1}^{l} M_i$ and $A = \bigcup_{i=0}^{n} A_i$.

3) $K := X \cup Y$ is a clique, where $X = \bigcup_{i=1}^{n+1} X_i$ and $Y = \bigcup_{i=1}^{l+2} Y_i$.

4) $\forall 1 \leq j \leq i \leq n$, $G[A_i \cup X_j]$ is a complete split graph.

5) $G[A \cup Y]$ is a complete split graph.

6) $\forall 1 \leq i \leq l$, $G[Y_i \cup M_i]$ is a perfect split graph.

7) $\forall 1 \leq i < j \leq l$, $G[Y_i \cup M_i]$ is a complete split graph.

8) $\exists 1 \leq k_0 \leq l$, $\forall i \leq k_0$, $G[Y_i + M_i]$ is a complete split graph.

9) $X_{n+1}, Y_{l+2}, Y_{l+1}, M_l$ and $A_0$ are the only possibly empty sets.

10) The only edges of $G$ are the edges of the subgraphs mentioned above.

In this case, we say that $G$ is an $\{S, K\}$-comb.

**Lemma 2.** Every $\{S, K\}$-threshold graph is an $\{S, K\}$-comb.

**Proof.** Let $G$ be an $\{S, K\}$-threshold graph defined as in Definition 2. Following the notations in Definition 3, we take $l = 1$ and $M_1 = Y_{l+1} = Y_{l+2} = \emptyset$. This shows that $G$ is an $\{S, K\}$-comb.

**Theorem 3.** If chair and co-chair are forbidden subgraphs of an $\{S, K\}$-split graph $G$, then $G$ is an $\{S, K\}$-comb.

**Proof.** We prove the statement by induction on the number of vertices. The statement is true for small graphs. Suppose that chair and co-chair are forbidden subgraphs of an $\{S, K\}$-split graph $G$. If $P_4$ is also a forbidden subgraph of $G$, then $G$ is an $\{S, K\}$-threshold graph, and hence, $G$ is an $\{S, K\}$-comb. So we may suppose that $G$ contains an induced path $abba'$. Then $N(a) - \{b\} = N(a') - \{b'\}$ and $N(b) - \{a\} = N(b') - \{a'\}$. Let $S' = S - a', K' = K - b'$ and $G' = G[S' \cup K']$. Then chair and co-chair are forbidden subgraphs of the $\{S', K'\}$-split graph $G'$. Then $G'$ is an $\{S', K'\}$-comb with $S' = A \cup M$ and $K' = X \cup Y$ (we follow the notations as in Definition 3).
If \( a \in S' \) and \( b \in K' \), then we add \( a' \) to the set of the comb \( G' \) that contains \( a \) and \( b' \) to the set of the comb \( G' \) that contains \( b \). Thus \( G \) is \( \{ S, K \} \)-comb.

Otherwise, \( a \in K \) while \( b \in S \). First we suppose that \( n \geq 1 \). Then there is \( x \in A_1 \) because \( A_1 \neq \phi \). We have the following cases:

- case 1: assume that \( a \in Y \) and \( b \in M \). Then \( xabb'a'x \) is an induced \( C_5 \) in \( G \). Contradiction.

- case 2: assume that \( a \in X_i \) and \( b \in A_j \). Then by definition of comb, we have \( i \leq j \). Then \( xabb'a'x \) is an induced \( C_5 \) in \( G \). Contradiction. So \( i = j \). Assume that there is \( y \in \bigcup_{t=1}^n A_t - \{ b \} \). Then \( yaba'b'y \) is an induced \( C_5 \) in \( G \). Contradiction. Thus we must have \( i = n \) and \( A_i = A_n = \{ b \} \). Assume that there is \( y \in X_{n+1} \). Then \( yaba'b'y \) is an induced \( C_5 \) in \( G \). Contradiction. Thus we must have \( X_{n+1} = \phi \). In this case, we do the following: remove \( a \) from \( X_n \) and add it to \( A_n \), remove \( b \) from \( A_n \) and add it to \( X_n \), add \( b' \) to \( X_{n+1} \), create \( A_{n+1} = \{ a' \} \) and \( X_{n+2} = \phi \). Thus \( G \) is an \( \{ S, K \} \)-comb.

- case 3: assume that \( a \in X_i \) and \( b \in M_j \). Then by the definition of a comb, we must have \( i = 1 = j \). But this is already discussed in case 1, because \( X_1 = Y_1 \).

- case 4: Assume that \( a \in Y_i \) and \( b \in A_j \). The case when \( i = 1 \) is already discussed in case 2. So we may assume that \( i > 1 \). Let \( y \in M_1 \). Then \( yaba'b'y \) is an induced \( C_5 \) in \( G \). Contradiction.

Second, suppose that \( n = 0 \). That is \( A = A_0 \) and so there is no \( A_1 \) and no \( X_2 \). We have the following cases:

- case 1: Assume that \( a \in Y_1 \) and \( b \in M_1 \). If \( i > 1 \) or \( Y_i \neq \{ b \} \), then there is \( c \in \bigcup_{t=1}^i A_t - \{ a \} \). Then \( cabb'a'c \) is an induced \( C_5 \) in \( G \). Contradiction. Thus \( i = 1 \) and \( Y_1 = \{ a \} \). Hence \( M_1 = \{ b \} \). We can do the following: remove \( a \) from \( Y_1 \) and add it to \( M_1 \), remove \( b \) from \( M_1 \) and add it \( Y_1 \), add \( b' \) to \( Y_1 \) and add \( a' \) to \( M_1 \). Thus \( G \) is an \( \{ S, K \} \)-comb.

- case 2: Assume that \( a \in Y_i \) and \( b \in M_j \) with \( i > j \). There exist \( c \in Y_j \) such that \( cb \) is an edge of \( G \). If there is \( y \in N_{G'}(a) - N_{G'}(b) \), then \( yabb'a'y \) is an induced \( C_5 \) in \( G \). Contradiction. Thus, we must have \( j = 1 \), \( Y_1 = \{ c \} \), \( M_1 = \{ b \} \), \( i = 2 \) and \( X_2 = \phi \). We can do the following: remove \( a \) from \( Y_2 \) and add it \( M_1 \), remove \( b \) from \( M_1 \) and add it \( Y_1 \) and remove \( c \) from \( Y_1 \) and add it \( Y_2 \). Thus \( G \) is an \( \{ S, K \} \)-comb.

- case 3: \( a \in Y_i \) and \( b \in M_j \) with \( i < j \). This case is impossible by the definition of the comb.

\[ \square \]

**Corollary 1.** \( G \) is a comb if and only if \( C_4, \overline{C_4}, C_5 \), chair and co-chair are forbidden subgraphs of \( G \).
Proof. The necessary condition is obvious by the definition of a comb. For the sufficient condition it is enough to note that the statement \( C_4, \overline{C}_4, C_5 \), chair and co-chair are forbidden subgraphs of \( G \) is equivalent to the statement that \( G \) is a split graph and chair and co-chair are forbidden subgraphs of \( G \).

\[ \square \]

**Corollary 2.** \( G \) is a comb if and only if \( \overline{G} \) is a comb.

**Proof.** Enough to note that the complement of \( C_4, \overline{C}_4, C_5 \), chair and co-chair are \( \overline{C}_4, C_4, C_5 \), co-chair and chair.

\[ \square \]

**Corollary 3.** \( G \) is a comb if and only if every induced subgraph of \( G \) is a comb.

**References**

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