Hull and White and Alòs type formulas for barrier options in stochastic volatility models with nonzero correlation

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Abstract

Two novel closed-form formulas for the price of barrier options in stochastic volatility models with zero interest rate and dividend yield but nonzero correlation between the asset and its instantaneous volatility are derived. The first is a Hull and White type formula, and the second is a decomposition formula similar in form to the Alòs decomposition for vanilla options. A model-free approximation is also given.

1 Introduction

It is well-known that, when the interest rate and dividend yield are both zero, put-call symmetry (PCS) holds in stochastic volatility (SV) models if the correlation between the price process and its instantaneous volatility is zero. We recall that in this case PCS says

\[ P(S_t, t, K, T) = K S_t C \left( S_t, t, \frac{S_t^2}{K}, T \right) \]  

Although PCS under the Black-Scholes (BS) framework has been known since Bates [4], its application to pricing and hedging barrier options within the BS framework was introduced by Bowie and Carr [5] and Carr and Chou [6]. Their method was later generalised by Andreasen [1] to SV models with zero correlation between the asset and its instantaneous volatility and zero interest rate and dividend yield.

To see how PCS can be used to give the price and hedge of a barrier option in a SV model with zero correlation, consider specifically the up and in (UIP) barrier put option with \( K, S(t) \leq B \). Using PCS it is not difficult to see that the price and hedge of the UIP is given by

\[ UIP(S(t), K, B) = \frac{K}{B} C \left( S(t), \frac{B^2}{K} \right) \]  

Indeed, if at some time in the future the spot price hits the barrier, by application of PCS the call option’s price will be exactly equal to the price of the put option with strike \( K \):

\[ \frac{K}{B} C \left( B, \frac{B^2}{K} \right) = P(B, K) \]
The call option can therefore be liquidated to purchase the put option without generating a profit or loss. If on the other hand the spot never hits the barrier, the UIP will expire worthless as will the call option since then \( B^2/K > S(T) \).

Other barrier options are similarly priced and replicated by application of PCS. The key takeaway is that the specific SV model is irrelevant for the price and the hedge. As long as correlation is zero there is a model free price and (semi-static) hedge for the UIP and other barrier options.

In reality correlation is never zero. Not only in equity markets, but also in FX markets where barrier options are liquid instruments, there will always be some correlation between the price process and its instantaneous volatility. In this case PCS will be broken and the exact price and hedge for barrier options can be found only by numerical methods.

The contribution of this note is to present two novel closed form formulas for the UIP. The first, a generalised Hull and White type formula \([7, 10, 11]\) reduces the dimensionality of the pricing problem by one. This is useful when Monte Carlo simulations are employed for pricing. The reduction of dimension clearly increases computation speed. The second, is an Alôs type decomposition formula \([1]\) which more easily lends itself to finding analytical approximations. This is especially true when the volatility process is assumed to be driven by fractional noise. Although only the UIP is discussed, other single barrier options can be similarly priced.

In the next section the mild model assumptions are stated and notation established. This is followed by a derivation of the barrier Hull and White formula. Next, a decomposition formula for the UIP is given by making use of the anticipating Itô formula. The final section concludes.

2 Assumptions and notation

In this paper we will consider SV models of the form

\[
\log S_T = \log S_t - \frac{1}{2} \int_t^T \sigma_u^2 \, du + \int_t^T \sigma_u \left( \rho dW_u + \sqrt{1 - \rho^2} dW_u^\perp \right)
\]

where \( W_u \) and \( W_u^\perp \) are independent standard Brownian motions, and \( \sigma_u \) is adapted to \( W_u \) and independent of \( S_u \). No other restrictions will be placed on the SV model except that it is well-behaved so that the price of a vanilla call option with strike \( K \) and maturity date \( T \) is

\[
C(S_t, t, K, T) = E_t \left[ (S_T - K)_+ \right]
\]

The SV price can always be expressed in terms of the Black-Scholes (BS) price formula:

\[
C(S_t, t, K, T) = C^{BS}(S_t, t, K, T, I)
\]

Indeed, the above equation is the definition of the implied volatility parameter \( I \) which depends on among other things time to maturity \( T - t \), moneyness \( S_t/K \), instantaneous volatility \( \sigma_t \), and correlation \( \rho \).

Notice that we can also write

\[
\log S_T = \log \tilde{S}_T + \log M_{t, T}
\]

with

\[
\log \tilde{S}_T := \log S_t - \frac{1}{2} \int_t^T \sigma_u^2 \, du + \sqrt{1 - \rho^2} \int_t^T \sigma_u \, dW_u^\perp
\]

and

\[
\log M_{t, T} := -\frac{\rho^2}{2} \int_t^T \sigma_u^2 \, du + \rho \int_t^T \sigma_u \, dW_u
\]
In other words, \( \log \tilde{S}_t = \log S_t \) and \( \log M_{t,t} = 0 \). With the above notation, the generalised Hull and White (HW) formula reads
\[
C(S_t, t, K, T) = E_t \left[ C^{BS} \left( S_{M_t, t}, t, K, T, \sqrt{1 - \rho^2} \sigma_{t,T} \right) \right] \tag{10}
\]
where
\[
\sigma_{t,T}^2 := \frac{1}{T-t} \int_t^T \sigma_u^2 du \tag{11}
\]
Because the HW formula is essentially an average over BS prices with adjusted spot price \( S_{M_t, t} \) and implied volatility \( \sqrt{1 - \rho^2} \sigma_{t,T} \), it is also known as the 'mixing formula'.

3 Barrier mixing formula

Our first result is the following proposition for an up-and-in (UIP) barrier option. Results for other type of barrier options can be similarly derived.

**Proposition 1.** The price of an UIP with \( K, S_t \leq B \) at any time \( t \) before the barrier is hit for the first time, or when the barrier is hit for the first time, is
\[
UIP(t) = E_t \left[ \frac{K}{BM_{t,T}} \left( S_T - \frac{B^2 M_{t,T}^2}{K} \right)_+ \right] \tag{12}
\]

**Proof.** Define the process \( F(t) \) as
\[
F(t) := E_t \left[ \frac{K}{BM_{t,T}} \left( S_T - \frac{B^2 M_{t,T}^2}{K} \right)_+ \right]
\]
Then, by conditioning,
\[
E_t \left[ \frac{K}{BM_{t,T}} \left( S_T - \frac{B^2 M_{t,T}^2}{K} \right)_+ \right] = E_t \left[ E_t \left[ \frac{K}{BM_{t,T}} \left( S_T - \frac{B^2 M_{t,T}^2}{K} \right)_+ \bigg| \mathcal{F}_t \right] \right] = E_t \left[ \frac{K}{BM_{t,T}} C^{BS} \left( S_{M_t, t}, t, \frac{B^2 M_{t,T}^2}{K}, T, \sqrt{1 - \rho^2} \sigma_{t,T} \right) \right]
\]
Suppose now at some time \( t^* \) the barrier is hit for the first time. Then \( S_{t^*} = B \), and by applying PCS to the BS price,
\[
E_{t^*} \left[ \frac{K}{BM_{t^*, T}} C^{BS} \left( S_{M_{t^*}, t^*}, \frac{B^2 M_{t^*, T}^2}{K}, T, \sqrt{1 - \rho^2} \sigma_{t^*, T} \right) \right] = E_{t^*} \left[ \frac{K}{BM_{t^*, T}} C^{BS} \left( BM_{t^*, t^*}, \frac{B^2 M_{t^*, T}^2}{K}, T, \sqrt{1 - \rho^2} \sigma_{t^*, T} \right) \right] = E_{t^*} \left[ P^{BS} \left( BM_{t^*, t^*}, K, T, \sqrt{1 - \rho^2} \sigma_{t^*, T} \right) \right] = P(S_{t^*}, K, T)|_{S_{t^*}=B}
\]
The derivative \( F(t^*) \) can thus be liquidated to purchase a put option with strike \( K \) without generating a profit or loss. If on the other hand the asset price never hits the barrier, then \( F(T) = 0 \) is zero since \( M_{T,T} = 1 \) and \( B^2 / K > S_T \). Thus, under all possible scenarios the process \( F(t) \) gives has a payoff equal to that of an UIP and by no arbitrage it must be the price of an UIP.
Since the price of an UIP is given by the Hull and White type formula

\[ UIP(t) = E_t \left[ \frac{K}{BM_{t,T}} C^{BS} \left( S_t, t, \frac{B^2 M_{t,T}}{K}, T, \sqrt{1 - \rho^2 \sigma_{t,T}} \right) \right] \]  \hspace{1cm} (13)

the dimensionality of the pricing problem has been reduced by one. This also reduces the computation time for pricing of barrier options using Monte Carlo simulations. Notice that the above equation can be loosely interpreted as saying that the price of an UIP is the average over BS UIP prices with adjusted spot price \( S_t \), implied volatility \( \sqrt{1 - \rho^2 \sigma_{t,T}} \), and adjusted barrier \( BM_{t,T} \). So the main difference with the vanilla HW formula (10) is that in equation (13) the barrier level is also ‘mixed’.

### 4 Barrier decomposition formula

In this section techniques from Malliavin calculus will be made use of. The reader is referred to [2] for further background and details on Malliavin calculus and its applications in derivatives pricing. First of all we state without proof the anticipating Itô formula.

**Proposition 2.** Let

\[
X_T = X_t + \int_t^T \alpha_u \, du + \int_t^T \beta_u \, dW_u + \int_t^T \beta^+_u \, dW^+_u
\]

\[
Y_T = Y_t + \int_t^T \omega_u \, du + \int_t^T \nu_u \, dW_u + \int_t^T \nu^+_u \, dW^+_u
\]

and

\[
Z_T = \int_t^T \theta_u \, du
\]

where \( \theta_u \) is adapted to \( W_u \). Then

\[
F(t, X_T, Y_T, Z_T) = F(t, X_t, Y_t, Z_t) + \int_t^T \frac{\partial F}{\partial u}(u, X_u, Y_u, Z_u) \, du
\]

\[ + \int_t^T \frac{\partial^2 F}{\partial X_u^2}(u, X_u, Y_u, Z_u) \, dX_u + \int_t^T \frac{\partial^2 F}{\partial Y_u^2}(u, X_u, Y_u, Z_u) \, dY_u
\]

\[ + \int_t^T \frac{\partial^2 F}{\partial Z_u^2}(u, X_u, Y_u, Z_u) \, dZ_u + \frac{1}{2} \int_t^T \frac{\partial^2 F}{\partial (X_u)^2}(u, X_u, Y_u, Z_u) \, (dX_u)^2
\]

\[ + \frac{1}{2} \int_t^T \frac{\partial^2 F}{\partial (Y_u)^2}(u, X_u, Y_u, Z_u) \, (dY_u)^2 + \frac{1}{2} \int_t^T \frac{\partial^2 F}{\partial (Z_u)^2}(u, X_u, Y_u, Z_u) \, (dZ_u)^2
\]

\[ + \int_t^T \frac{\partial^2 F}{\partial X_u \partial Y_u}(u, X_u, Y_u, Z_u) \, dX_u \, dY_u + \int_t^T \frac{\partial^2 F}{\partial X_u \partial Z_u}(u, X_u, Y_u, Z_u) \, \left( D^W_u Z_u \right) \beta_u \, du
\]

\[ + \int_t^T \frac{\partial^2 F}{\partial Y_u \partial Z_u}(u, X_u, Y_u, Z_u) \, \left( D^W_u Z_u \right) \nu_u \, du \]  \hspace{1cm} (14)

where \( D^W_u \) denotes the Malliavin derivative with respect to \( W_u \).

We now have the necessary ingredients to prove the following theorem.

**Proposition 3.** The price of an UIP can be decomposed as follows:

\[
UIP(t) = K \left[ E_t \left[ C^{BS} \left( S_t, t, \frac{B^2 M_{t,T}}{K}, T, \sqrt{1 - \rho^2 \sigma_{t,T}} \right) \right] + \rho \mathbb{E} \left[ \int_t^T H(X_u, Y_u, Z_u) \, du \right] \right]
\]

\[ + \mathbb{E} \left[ \int_t^T \left( \int_u^T \frac{\partial^2 F}{\partial X_u \partial Z_u}(u, X_u, Y_u, Z_u) \, dX_u \right) \, dY_u \right]
\]

\[ + \mathbb{E} \left[ \int_t^T \left( \int_u^T \frac{\partial^2 F}{\partial Y_u \partial Z_u}(u, X_u, Y_u, Z_u) \, dY_u \right) \, dZ_u \right] \]  \hspace{1cm} (15)
with

\[ H \left( \tilde{S}_u, u; \frac{B^2}{K} M_{t,u}, T, \sigma_{u,T} \right) := \frac{1}{2} \frac{\partial}{\partial \log M_{t,u}} \left( \frac{\partial^2}{(\partial \log \tilde{S}_u)^2} - \frac{\partial}{\partial \log \tilde{S}_u} \right) C^{BS} \left( \tilde{S}_u, u; \frac{B^2}{K} M_{t,u}, T, \sigma_{u,T} \right) \]  

(16)

Proof. The Itô anticipating formula will be used with

\[ X_u := \log \tilde{S}_u, \quad Y_u := \log M_{t,u}, \quad u \in [t, T] \]

Notice that this implies that

\[ X_t = \log S_t, \quad \alpha_u = -\frac{(1 - \rho^2)}{2} \sigma_u^2, \quad \beta_u = 0, \quad \beta_u^+ = \sqrt{1 - \rho^2} \sigma_u \]

and

\[ Y_t = \log M_{t,t} = 0, \quad \omega_u = -\frac{\rho^2}{2} \sigma_u^2, \quad \nu_u = \rho \sigma_u, \quad \nu_u^+ = 0 \]

Define also

\[ Z_u := \int_u^T \sigma_u^2 \, du \]

Notice that

\[ \sigma_{u,T} = \sqrt{\frac{Z_u}{T-u}} \]

Next, let

\[ F(u, X_u, Y_u, Z_u) := \frac{K}{B} C^{BS} \left( e^{X_u}, u; \frac{B^2}{K} e^{Y_u}, T, \sigma_{u,T} \right) \]

which is the Black-Scholes-Margrabe formula. In particular this means that

\[ F(T, X_T, Y_T, Z_T) = \frac{K}{B} \left( e^{X_T} - \frac{B^2}{K} e^{Y_T} \right)_+ \]

\[ = \frac{K}{B} \left( \tilde{S}_T - \frac{B^2}{K} M_{t,T} \right)_+ \]

\[ = \frac{K}{B M_{t,T}} \left( S_T - \frac{B^2 M^2_{t,T}}{K} \right)_+ \]

and

\[ F(t, X_t, Y_t, Z_t) = \frac{K}{B} c^{BS} \left( S_t, t; \frac{B^2}{K} T, \sigma_{t,T} \right) \]

The remainder of the proof is a straightforward application of the anticipating Itô formula, and makes use of the BS pde, the relationship between BS gamma and BS vega, and the duality between BS gamma and BS 'shadow gamma'. The latter is the BS probability density. After the simplifications due to the aforementioned identities and relationships, the last step is to take the expectation of both sides of the equality in (14) to arrive at formula (15). \hfill \Box
5 Model-independent approximation

In what follows the volatility swap strike $\kappa_-(t)$ and the dual volatility swap strike $\kappa_+(t)$, defined by

$$
\kappa_-(t) := E_t[\sigma_{t,\tau}], \quad \kappa_+(t) := E_t\left[\frac{S(T)}{S(t)}\sigma_{t,T}\right],
$$

will be of interest.

In [8, 9] a useful approximate relationship between volatility swaps strikes and the implied volatility smile is derived. It reads

$$
\kappa_+(t) - \kappa_-(t) \approx \rho E_t\left[\sigma_{t,\tau}\int_t^T \sigma(u)dW_u\right] \approx I_+(t) - I_-(t).
$$

The implied volatilities $I_{\pm}(t)$ and the corresponding strikes $K_{\pm}(t)$ satisfy the condition $d_{\pm}(t) = 0$, where

$$
d_{\pm}(t) = \log\left(\frac{S(t)/K_\pm(t)}{I_\pm(t)\sqrt{T-t}}\right) \pm \frac{I_\pm(t)\sqrt{T-t}}{2} = 0.
$$

Following [9], $I_-(t)$ is called the zero vanna implied volatility (ZVIV) and $I_+(t)$ the dual zero vanna implied volatility (DZVIV).

**Proposition 4.** The price of an UIP can be approximated as follows:

$$
UIP(t) \approx \frac{K}{B} C\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right) - \frac{K}{B} S(t)\sqrt{T-t} (I_+(t) - I_-(t)).
$$

Proof. The first step is to Taylor expand equation (13) around $\rho = 0$ to give

$$
UIP(t) \approx \frac{K}{B} E_t\left[C^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right)\right] + \frac{K}{B} \rho E_t\left[\int_t^T \sigma(u)dW(u)\right]\frac{B^2}{K} \frac{\partial}{\partial \left(\frac{B^2}{K}\right)} C^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right)\right].
$$

Using the property that $C^{BS} = S(t) \left(\partial C^{BS}/\partial S(t)\right) + K \left(\partial C^{BS}/\partial K\right)$,

$$
UIP(t) \approx \frac{K}{B} E_t\left[C^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right)\right] + \frac{K}{B} \rho E_t\left[\int_t^T \sigma(u)dW(u)\right]C^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right) - 2S(t)\Delta^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right)\right]\right]
$$

$$
\approx \frac{K}{B} E_t\left[C^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right)\right] + \frac{K}{B} \rho E_t\left[\int_t^T \sigma(u)dW(u)\right]C^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right) - 2S(t)\Delta^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right)\right]\right]
$$

$$
= \frac{K}{B} E_t\left[C^{BS}\left(S(t), \frac{B^2}{K}\right)\right] + \frac{K}{B} \rho E_t\left[\int_t^T \sigma(u)dW(u)\right]C^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right) - 2S(t)\Delta^{BS}\left(S(t), \frac{B^2}{K}, \sigma_{t,T}\right)\right].
$$

where $\Delta^{BS} := \partial C^{BS}/\partial S(t)$. 

The next step is to notice that since $B^2/K > S(t)$, for an option with strike $B^2/K$ there always exists a ‘fake’ DZIV $I_+$ (which is not the DZIV $I_+$). Expanding now around $I_+$ gives

$$C^{BS} \left( S(t), \frac{B^2}{K}, \sigma_{t,T} \right) \approx C^{BS} \left( S(t), \frac{B^2}{K}, I_+ \right) + \frac{S(t)\sqrt{T-t}}{\sqrt{2\pi}} (\sigma_{t,T} - I_+)$$

and

$$S(t) \Delta^{BS} \left( S(t), \frac{B^2}{K}, \sigma_{t,T} \right) \approx S(t) \Delta^{BS} \left( S(t), \frac{B^2}{K}, I_+ \right) + \frac{S(t)\sqrt{T-t}}{\sqrt{2\pi}} (\sigma_{t,T} - I_+) .$$

Hence,

$$UIP(t) \approx \frac{K}{B} C \left( S(t), \frac{B^2}{K} \right) - \frac{K}{B} \frac{S(t)\sqrt{T-t}}{\sqrt{2\pi}} \rho E_1 \left[ \left( \sigma_{t,T} \int_t^T \sigma(u) dW(u) \right) \right].$$

Using the approximation (15) gives the desired result.

As $C \left( S(t), \frac{B^2}{K} \right)$ is a vanilla option, it is observable. The ZVIV $I_-(t)$ and DZVIV $I_+(t)$ are market observables as well. Hence the approximate price (20) of a UIP can be deduced from only the vanilla options volatility skew.

6 Conclusion

Two novel closed form formulas for the price of an UIP has been derived for general SV models with nonzero correlation between the price process and its instantaneous volatility. The formulas are valid for both ‘classical’ SV models as well as those driven by fractional noise. The practical relevance of the first, which is a generalised Hull and White type formula, is to reduce the computation time in Monte Carlo simulations. The second, an Alós type decomposition formula, facilitates deriving analytical approximations in various rough volatility models.

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