The central limit theorem in
two dimensional oriented percolation* †

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Abstract

We show that the size of supercritical oriented bond percolation converges asymptotically in distribution to the standard normal law when suitably rescaled. This resolves a withstanding for more than thirty years problem by means of an elementary proof whilst disproving the conjectured scaling. As byproduct a random-index central limit theorem for associated random variables is derived. The proof applies to the continuous-time analog of the process, viz. Harris’ basic contact process, equally well.

1. Let \( \mathcal{L} \) be the plane lattice with points \( \{(y, n) \in \mathbb{Z}^2 : y + n \text{ is even}, n \geq 0\} \) and oriented bonds from \( (y, n) \) to \( (y-1, n+1) \) and to \( (y+1, n+1) \), for other transpositions, not necessarily possessing a symmetry axis, cf. [fig.1, p.1001; Durrett [D84]]. We consider bond percolation on \( \mathcal{L} \), that is, each bond is retained independently with probability \( p \). Let \( \xi_n^O \) be the random subset of wet sites in \( \{-n, \ldots, n\} \) provided only the origin, \( O \), is initially wetted, and let \( \Omega_\infty \) denote the event it percolates, i.e. \( \Omega_\infty = \cap_{n \geq 1} \Omega_n, \Omega_n = \{|\xi_n^O| \neq 0\} \). Let also \( d_n = d(\xi_n^O) \) denote the auxiliary diameter process, \( d(A) = \max A - \min A \) and \( A \neq \emptyset \). We will assume that \( p \) is sufficiently large to ensure that \( \rho := P(\Omega_\infty) > 0 \) and hence that also \( \alpha \), the so-called asymptotic velocity of (each of) the endmost points processes, is strictly positive, i.e. \( \lim_{n \to \infty} \frac{d_n}{n} = 2\alpha \) a.s. on \( \Omega_\infty \). For background regarding this extensively studied model, generally viewed as originating in [Broadbent and Hammersley [BH57]] and in [Harris [H74]], in addition to [D84], we refer to [Griffeath [G79, G81]], and to [Liggett [L85]], whereas the interested in newer developments reader is refer to [Bezuidenhout and Grimmett [BG90]], as well as to [D91, D95, L99] and references thereof.

The Central Limit Theorem (CLT) given next, noted as an open problem for instance in [§13; D84] and in [Chpt. 11; D88], with discussions, as well as in [The contact process 1974-1989; D91]], is the main result here.

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†In memoriam \( \mathcal{A}. \mathcal{A}. \mathcal{F}. \) (07/82 - 12/03)
Theorem 1. Let $A_n = \frac{|\xi^O_\infty| - \alpha \rho n}{\sqrt{d_n^2/2}}$. If $G_n(x) = P(A_n \leq x | \Omega_\infty)$, then
$$G_n \xrightarrow{w} N(0, \sigma^2), \text{ as } n \to \infty,$$
where $\sigma^2 = \sum_x \text{Cov}(x \in \check{\xi}, O \in \check{\xi}) < \infty$, for $\check{\xi}$ distributed according to $\nu$.

The somewhat peculiar disproof of the conjectured contribution of the asymptotic variance of the renormalized fluctuations of the endpoints in obtaining the correct scaling is an aftereffect of the dispersal of the expected function of its well-known asymptotic normality in the proof of this desired there result.

The proof of the SLLN associated to the CLT in Theorem 1 is shown in [Durrett and Griffeath [DG83]], see also [Thm. 3.33, Chpt. VI; L85]. This settled a conjectured by Harris [ref. [DG83]] improvement of Harris’ Growth Theorem. Earlier on, the $L^1$-convergence counterpart of this SLLN is shown in [Durrett [D80]].

In [DG83], by means of rigorous rescaling arguments, a number of exponential estimates for the supercritical process, including the important property that $\bar{\nu}$ possesses exponentially decaying correlations, are shown. This property, together with the SLLN for the position of the endpoints, result shown in [Durrett [D80]] and leading to the subadditive theorem of [Liggett L85-2], enable the proof of the SLLN for the size of the process.

The proof of Theorem 1 relies on again exploiting this property in a simple way, along with verifying a specific regularity of deviations of random from deterministic sums condition, also known as the Anscome condition, as well as a particular repetitive coupling due to path intersection properties application that allows us to deal with a non-stopping time involved.

Furthermore, the verification of this particular condition provides with the following random-index extended version of the CLT in [Theorem 1; Cox and Grimmett [CG84]], with a stronger summability condition on the covariances, as byproduct. To state it, we say that an ensemble $\mathbf{X}_I = (X_i; i \in I)$ of random variables is associated whenever, for every finite $J \subset I$, $\text{Cov}(f_1(X_J), f_2(X_J)) \geq 0$, for all coordinate-wise non-decreasing $f_1, f_2$ such that this covariance is defined. We also use $[\cdot]$ to denote integer part.

Theorem 2. Let $X_1, X_2, \ldots$ be a non-degenerate, associated sequence such that $\sup_n E(|X_n|^3) < \infty$ and that $\sum_{n=1}^\infty \text{Cov}(X_n, \sum_{k=1}^{n-1} X_k) < \infty$, also supposed, without loss of generality, to be zero-mean. Let $N_1, N_2, \ldots$ be integer-valued and positive random variables such that $\frac{N_t}{t} \xrightarrow{p} \theta$, as $t \to \infty$, for some $0 < \theta < \infty$. Setting $S_n = \sum_{k=1}^n X_k$, we have that $\frac{S_{N_t}}{\sqrt{\theta_t}} \xrightarrow{w} N(0, \sigma^2)$ and that $\frac{S_{N_t}}{\sqrt{\theta_t}} \xrightarrow{w} N(0, \sigma^2)$, as $t \to \infty$, for $\sigma^2 := \lim_{t \to \infty} \text{Var}(S_{[\theta t]}/\sqrt{[\theta t]}), 0 < \sigma^2 < \infty$.

1[GS81] in reference to the multidimensional result of [H78]
2[Gut [GU09] in respect to [Anscombe [A52]]
3notational convention originating in [Gauss GKF1808].
The proof of Theorem 1 is given immediately below, whereas that of Theorem 2 is deferred to the last section.

2. The starting set of the process $\xi_n$ is notated by $\xi_{n=0}$, where $\xi_{n=0} \subset 2\mathbb{Z}$, $2\mathbb{Z} := \{\ldots, -2, 0, 2, \ldots\}$. Note also that we will use here coordinate-wise notation $\xi(x) := 1(x \in \xi)$ as well.

Lemma 3. Let $\hat{A}_n = \sum_{x=-\infty}^{\infty} \xi_{2\mathbb{Z}}(x) - \alpha \rho_n \sqrt{\alpha n}$, $\xi_{2\mathbb{Z}} = \{\ldots, -2, 0, 2, \ldots\}$. We have that

$$L(\hat{A}_n) \xrightarrow{w} \mathcal{N}(0, \sigma^2), \text{ as } n \to \infty,$$

for $\sigma^2$ as in Theorem 1.

Proof. Let $\hat{\xi}_n$ be such that $\hat{\xi}_{n=0} = \hat{\xi}$ distributed according to $\nu$. Due to the exponential decay of correlations property of $\nu$, general results regarding random fields (for instance [M75], see also [L85]; Thm. 3.23 Chpt. VI) imply

$$\sum_{x=-\infty}^{\infty} \frac{(\hat{\xi}(x) - \rho)}{\sqrt{\alpha n}} \xrightarrow{w} \mathcal{N}(0, \sigma^2), \text{ as } n \to \infty,$$

for $\sigma^2$ by stationarity as above. Let $\tilde{A}_n = \sum_{x=-\infty}^{\infty} \hat{\xi}_n(x) - \alpha \rho_n \sqrt{\alpha n}$, from the display above and stationarity, we have that

$$L(\tilde{A}_n) \xrightarrow{w} \mathcal{N}(0, \sigma^2), \text{ as } n \to \infty,$$

(1)

Monotonicity and applying self-duality twice gives that, there are $C, \gamma$ positive and finite constants, such that, for all $x$,

$$P(\xi_{2\mathbb{Z}}(x) \neq \hat{\xi}_n(x)) = P(\xi_{2\mathbb{Z}}(x) = 1) - P(\hat{\xi}_n(x) = 1) = P(\Omega_n \cap \Omega_{\infty}) \leq Ce^{-\gamma n}, \ n \geq 1$$

where the final inequality is a standard result, see [DG83], or [§12; D84]. From the last display above an application of the Borel-Cantelli lemma gives that $\tilde{A}_n - \bar{A}_n \xrightarrow{a.s.} 0$, as $n \to \infty$, and the proof follows from (1).

Proof of Theorem 1. Let $\bar{A}_n = \sum_{x=I_n}^{r_n} \xi_{2\mathbb{Z}}(x) - \alpha \rho_n \sqrt{\alpha n}$, where $r_n = \sup \xi_{2\mathbb{Z}}, \xi_{2\mathbb{Z}} = \{\ldots, -2, 0, 2, \ldots\}$, and $I_n = \inf \xi_{2\mathbb{Z}}, \xi_{2\mathbb{Z}} = \{0, 2, \ldots\}$, and also let $\rho_n := P(\Omega_n)$. We have that $\{\xi_{2\mathbb{Z}}(x) - \rho_n; \ x \in \mathbb{Z}\}$ are associated by a corollary to Harris’ correlation inequality [H77], see [Thm. 2.14, Chpt. II; L85], and have exponentially decaying correlations [DG83] or [D84]; §13], hence, as also by self-duality, $E|\xi_{2\mathbb{Z}}(x)|^2 \leq 1$, the argument in Proposition 4 below gives that

$$\sum_{x=I_n}^{r_n} \xi_{2\mathbb{Z}}(x) - \sum_{x=-\infty}^{\infty} \xi_{2\mathbb{Z}}(x) \xrightarrow{p} 0, \text{ as } n \to \infty,$$
and hence an application of Slutsky’s theorem by Lemma 3 implies that

\[ \mathcal{L}(\bar{A}_n) \xrightarrow{w} \mathcal{N}(0, \sigma^2), \quad n \to \infty. \]  

(2)

Let \( \tau = \inf\{n \geq 0 : r_n^+ = l_n^- \text{ and } r_m^- \geq l_n^- \forall m > n, \} \), i.e. \( \tau \) is the first time such that \( \bar{r}_n = \bar{l}_n = x \) and \((x, n)\) percolates. Note that

\[ \{\tau \geq n\} \subseteq \{\exists m \geq n : r_m^- < 0\} \cup \{\exists m \geq n : l_m^+ > 0\}, \]

and hence we have that there are \( C, \gamma \) positive and finite constants, such that

\[ \mathbb{P}(\tau \geq n) \leq C e^{-\gamma n}, \quad n \geq 1, \]  

(3)

by employing that \( \mathbb{P}(\exists m \geq n : r_m^- < 0) \leq C e^{-\gamma n} \), which is a simple consequence of large deviation results for the endpoints, see [DG83], or [§11; D84].

Let \( A'_n = |\xi_{O_n}| - \alpha \rho n \sqrt{\alpha n} \) and note that by Slutsky’s theorem it suffices to show this result for \( A'_n \) instead of \( A_n \) by the existence of the asymptotic velocity result of [D80]. We will also need to employ the following coupling from [D80] below.

\[ \xi_{O_n} = \xi_{2n}^{2\mathbb{Z}} \cap [r_n^-, l_n^+] \text{ on } \Omega_{\infty}. \]  

(4)

Letting \( p_k = \mathbb{P}(\tau = k), \ k = 0, 1, \ldots, \) we have that

\[
\lim_{n \to \infty} \mathbb{E}(e^{i\theta \bar{A}_n}) = \lim_{n \to \infty} \sum_{k=0}^{\infty} \mathbb{E}(e^{i\theta \bar{A}_n} | \tau = k)p_k
= \sum_{k=0}^{\infty} p_k \lim_{n \to \infty} \mathbb{E}(e^{i\theta \bar{A}_n} | \tau = k)
= \lim_{n \to \infty} \mathbb{E}(e^{i\theta A'_n} | \Omega_{\infty}),
\]

(5)

where in the second line above we used (3) and dominated convergence, whereas in the third one we used that

\[
\lim_{n \to \infty} \mathbb{E}(e^{i\theta \bar{A}_n} | \tau = k) = \lim_{n \to \infty} \mathbb{E}(e^{i\theta \bar{A}_{n+k}} | \tau = k) = \lim_{n \to \infty} \mathbb{E}(e^{i\theta A'_n} | \Omega_{\infty}),
\]

which comes from applying (4) by translation invariance, and recomposing the event \( \{\tau = k, r_k^- = l_k^+ = x\} \) in the form \( \Omega^{(k,x)}_{\infty} \cap F \) for \( \Omega^{(k,x)}_{\infty} \) denoting the event that \((k, x)\) percolates and \( F \) being a measurable with respect to the \( \sigma \)-algebra associated with the construction up until time \( k \) event, and then employing independence of events measurable with respect to disjoint parts of \( \mathcal{L} \), property inherited by the independence of the Bernoulli r.v. in the construction. By (2) and (5) the proof is complete by evoking Levy’s convergence theorem.

\[ \square \]

3. We give the proof of Theorem 2. Note that it suffices to only show the first conclusion of Theorem 1 for the second one follows from this by an application of
Slutsky’s theorem. Note also that there is no loss of generality in assuming \( \theta = 1 \).

Letting \( u(j) = \sum_{k=j+1}^{\infty} \text{Cov}(X_j, X_k) \), we have by assumption that \( \sum_j u(j) < \infty \) so that \( u(j) < \infty \), \( \forall j \). We thus have that the hypotheses of Theorem 1 in \([CG84]\) are met, so that

\[
\frac{S_t}{\sqrt{[t]}} \xrightarrow{w} \mathcal{N}(0, \sigma^2), \text{ as } t \to \infty,
\]

and \( \sigma^2 := \lim_{t \to \infty} \text{Var}(S_t/\sqrt{[t]}) < \infty \), also by the assumptions in \([CG84]\). Thus, the proof is complete by combining the last display above with the following statement and another application of Slutsky’s theorem.

**Proposition 4.** \( \frac{S_{N_t} - S_t}{\sqrt{[t]}} \xrightarrow{p} 0 \) as \( t \to \infty \).

Remark: The conclusion in Proposition 4 remains valid if hypothesis \( \sup_i \mathbb{E}(|X_i|^3) < \infty \) is relax to \( \sup_i \mathbb{E}(|X_i|^2) < \infty \).

**Proof.** Let \( \epsilon > 0 \) and also let \( m(t) = [t(1-\epsilon)^3] + 1 \) and \( n(t) = [t(1+\epsilon)^3] \). Because \( \sum_{k=1}^{\infty} \text{Cov}(X_k, S_{k-1}) < \infty \) and we have assumed that the sequence \( X_1, X_2, \ldots \) is associated, we may apply [Theorem 2.1; \( S\mathcal{O}\mathcal{S} \)] to derive that there is a \( C \) independent of \( k \) and \( \epsilon \) such that

\[
P \left( \max_{k=m(t), \ldots, t} |S_k - S_t| \geq \epsilon \sqrt{[t]} \right) \leq \frac{8}{\epsilon^2 [t]} \left( \sum_{k=m(t)}^{[t]} \mathbb{E}(X_k)^2 + 2 \sum_{k=m(t)}^{[t]} \text{Cov}(X_k, S_{k-1}) \right) \leq C \left( \frac{[t] - m(t)}{\epsilon^2 [t]} \right).
\]

Partitioning according to \( N_t \in [m(t), n(t)] \) gives

\[
P(S_{N_t} - S_t \geq \epsilon \sqrt{[t]}) \leq P \left( \max_{k=m(t), \ldots, t} |S_k - S_t| \geq \epsilon \sqrt{[t]} \right) + P \left( \max_{k=[t], \ldots, n(t)} |S_k - S_t| \geq \epsilon \sqrt{[t]} \right) + P (N_t \not\in [m(t), n(t)]) \leq C \left( \frac{[t] - [t(1-\epsilon)^3] - 1}{\epsilon^2 [t]} + \frac{[t(1+\epsilon)^3] - [t]}{\epsilon^2 [t]} \right) + P (N_t \not\in [m(t), n(t)]),
\]

where for the second inequality we used (6) and then repeated it for the second term. However, \( \limsup_{t \to \infty} P (N_t \not\in [m(t), n(t)]) = 0 \), hence, the last display gives that \( \limsup_{t \to \infty} P(S_{N_t} - S_t \geq \epsilon \sqrt{[t]}) \leq 2C\epsilon \), which due to the arbitrariness of \( \epsilon \) completes the proof.

\[\square\]

Remark: The proof of Theorem 1 here is elementary, does not necessitate invoking any general CLTs for associated random variables, and relies on Proposition 4.
The latter follows Rényi’s \cite{R60} direct proof approach for checking the Anscombe condition via Kolmogorov’s inequality (in the case i.i.d. summands). However, invoking the Invariance Principle (IP) in \cite[Theorem 3; Newman and Wright \cite{NW81}] and relying on \cite[Theorem 17.1; Billingsley \cite{B68}], or also \cite[§5.2; \cite{GU09}], the main parts in the proof of Theorem 1 may be emulated for obtaining the corresponding arcsine laws, limit laws for the maxima, etc., extensions. This approach circumvents Proposition 4, but hence does not lead to Theorem 2.

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