On the structure of strange nonchaotic attractors in pinched skew products

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Abstract

The existence of non-continuous invariant graphs (or strange non-chaotic attractors) in quasiperiodically forced systems has generated great interest, but there are still very few rigorous results about the properties of these objects. In particular it is not known whether the topological closure of such graphs is typically a filled-in set, i.e. consist of a single interval on every fibre, or not. We give a positive answer to this question for the class of so-called pinched skew products, where non-continuous invariant graphs occur generically, provided the rotation number on the base is diophantine and the system satisfies some additional conditions. For typical parameter families these conditions translate to a lower bound on the parameter. On the other hand, we also construct examples where the non-continuous invariant graphs contain isolated points, such that their topological closure cannot be filled in.

1 Introduction

Generally spoken, for a quasiperiodically forced monotone interval map invariant graphs play the same role as fixed points for unperturbed maps. However, in contrast to fixed points there are two different types of invariant graphs. In the simpler case, such a graph is continuous. If in addition it has a negative Lyapunov exponent, a lot of conclusions about its regularity, structural stability etc. can be drawn (see [Sta99, SS00]). More difficult, but also more interesting, is the case of non-continuous invariant graphs. When their Lyapunov exponent is negative, such graphs are often referred to as strange non-chaotic attractors or SNA. Their existence has recently received great attention in theoretical physics, and consequently there are a lot of numerical studies about the topic (PNR01 gives a good overview and further reference). On the other hand, rigorous results are still few. The existence of SNA has only been proved for rather specific systems (see [Her83] for quasiperiodically driven Möbius transformations, Kel96, BO96 for pinched skew products), and even in these cases there are a lot of open questions regarding their further properties. It is one of these questions we want to address here.

Quasiperiodically forced monotone interval maps. In order to give a precise formulation of the problem we want to study, we need a few basic definitions. First of all, a quasiperiodically forced monotone interval map is a continuous map of the form

\( T : \mathbb{T}^1 \times I \to \mathbb{T}^1 \times I , \ (\theta, x) \mapsto (\theta + \omega, T_\theta(x)) \) \hspace{1cm} (1.1)

where \( I \subset \mathbb{R} \) is a compact interval and all the fibre maps \( T_\theta \) are monotonically increasing on \( I \). An invariant graph for such a system \( T \) is a function \( \varphi : \mathbb{T}^1 \to I \) which satisfies

\( T_\theta(\varphi(\theta)) = \varphi(\theta + \omega) \ \forall \theta \in \mathbb{T}^1. \) \hspace{1cm} (1.2)
This equation implies that the corresponding point set $\Phi := \{(\theta, \varphi(\theta)) \mid \theta \in \mathbb{T}^1\}$ is forward invariant, i.e. $T(\Phi) = \Phi$ (not necessarily $T^{-1}(\Phi) = \Phi$, as we did not assume strict monotonicity).

On the other hand, whenever $K$ is a compact and forward invariant set we can define an invariant graph by

$$\varphi^+(\theta) := \max\{x \in I \mid (\theta, x) \in K\}$$

(the invariance being a direct consequence of the monotonicity of the fibre maps). Further, as $K$ is compact $\varphi^+$ will be upper semi-continuous. In the same way a lower semi-continuous invariant graph $\varphi^-$ can be defined via the minimum.

Particularly interesting for our purposes will be the case where $K$ is the global attractor of the system, defined as $K := \bigcap_{n=0}^{\infty} T^n(\mathbb{T}^1 \times I)$. The corresponding graphs $\varphi^+$ and $\varphi^-$ will then be called the upper and lower bounding graphs of the system $T$. (All of this is thoroughly discussed in [Gle02], [Sta03] or [Jäg03].)

The question. Suppose $\varphi$ is an invariant graph which is not continuous. Its topological closure $\overline{\varphi}$ is a compact and invariant set, again bounded from above and below by invariant graphs $\varphi^+$ and $\varphi^-$. If we let $[\varphi^-, \varphi^+]: = \{(\theta, x) \mid \varphi^-(\theta) \leq x \leq \varphi^+(\theta)\}$, then this is a compact invariant set as well and surely $\overline{\varphi}^+ \subseteq [\varphi^-, \varphi^+]$. But is $\overline{\varphi}^+ = [\varphi^-, \varphi^+]$?

This question was already asked by M. Herman in [Her83] (Section 4.14) for certain quasiperiodically forced Moebius-transformations, and then repeated a number of times in different situations (see [Kel96] [Gle02] [Sta03]).

Results. Here, we address the problem in the setting of so-called pinched skew products first introduced in [TOPYS]. Their particular structure allows to prove the existence of non-continuous invariant graphs by a few simple and elegant arguments (see [Kel96] [BO96], a slight generalization can be found in [Gle02]). Often, such systems are given by \( (1.1) \) with $I = [0, L]$, $L > 0$ and fibre maps

$$T_{\theta}(x) = \alpha g(\theta) f(x) \;.$$  

(1.4)

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is monotonically increasing with $f(0) = 0$, $g : \mathbb{T}^1 \to \mathbb{R}^+$ has exactly one zero and $\alpha$ is a positive parameter. A typical example would be $f(x) = \tanh(x)$ and $g(\theta) = |\sin(\pi \theta)|$. Note that $f(0) = 0$ implies that the lower bounding graph is always $\varphi^- \equiv 0$. In short terms, our main result can now be stated as follows (see Cor. 4.2):

Suppose $f$ and $g$ satisfy some mild conditions concerning their geometry and regularity (specified before Cor. 4.2) and the rotation number $\omega$ is of diophantine type.

Then for all sufficiently large parameters $\alpha$ the upper bounding graph $\varphi^+$ of the system $T$ defined by \( (1.1) \) will have the property $\overline{\varphi}^+ = [0, \varphi^+]$.

As mentioned, we also give examples where the topological closure of the upper bounding graph cannot be filled in because the graph contains isolated points. For this we use a certain growth condition on the coefficients $a_n$ of the continued fraction expansion of the rotation number $\omega$ on the base, namely $\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty$. However, this still allows $\omega$ to be diophantine, showing that the additional assumptions on the system we use to derive the above result are indeed crucial, and cannot be neglected.

Overview. Section 2 briefly sketches the arguments used to establish the existence of non-continuous invariant graphs in pinched skew products. In order to gain more insight about their creation and structure, Section 3 introduces the iterated upper boundary lines. This sequence of continuous graphs converges monotonically decreasing to the upper bounding graph, and their shape can be controlled to some extent by putting additional restrictions on the geometry of the system. In Section 4 this is then used to derive the mentioned result, and Section 5 contains the construction of the counterexamples.

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2 Pinched skew products

Suppose a map $T$ as in (1.1) with $I = [0, L]$ for some $L > 0$ additionally satisfies

- $T_\theta(0) = 0$ $\forall \theta \in T^1$ (invariant 0-line) (2.1)
- $T_\theta \equiv 0$ for some $\theta \in T^1$ (pinching) (2.2)

As mentioned, (2.1) ensures that the lower bounding graph of such a system is always $\varphi^- \equiv 0$. Further (2.2) implies that any other invariant graph must equal 0 on a dense subset of $T^1$, namely the forward orbit of $\theta$. Thus, apart from $\varphi^-$ no other invariant graph can be continuous.

Now suppose all the fibre maps $T_\theta$ are differentiable and denote the derivative by $DT_\theta$. Then by

$$\lambda(\varphi) := \int_{T^1} \log DT_\theta(\varphi(\theta)) \, d\theta$$

(2.3)

the Lyapunov exponent of an invariant graph $\varphi$ can be defined. It is easy to show that whenever $T$ satisfies some mild conditions (namely when $\theta \mapsto \inf_{x \in I} DT_\theta (x)$ is integrable), the Lyapunov exponent of the upper bounding graph cannot be strictly positive (e.g. Lemma 3.5 in [Jag03]).

For systems of the form (1.4), a simple computation yields

$$\lambda(0) = \log f'(0) + \log \alpha + \int_{T^1} \log g(\theta) \, d\theta$$

(2.4)

whenever $f'(0) > 0$ and $\log g$ is integrable. This is surely positive for sufficiently large $\alpha$. Thus the upper bounding graph $\varphi^+$ cannot be equal to $\varphi^-$ and must therefore be non-continuous in such systems.

Even if they are rather degenerate in some sense, the fact that the existence of strange non-chaotic attractors can be established so easily makes pinched skew products an ideal setting for studying their further properties. The results obtained here may then at least give hints about more general systems. This is further supported by the fact that although invertible systems lack the pinched structure, their minimal sets will still be pinched sets (i.e. consist of only one point on a dense set of fibres, see [Ska03]), and therefore have some similarities with the global attractors of pinched skew products.

![Figure 1: Semi-continuous upper bounding graphs in the parameter family $(\theta, x) \mapsto (\theta + \omega, \tanh(\alpha x) \cdot |\sin(\pi \theta)|)$. Here $\omega$ is the golden mean and $\alpha$ is 5 and 32, respectively.](image)

3 The shape of the iterated upper boundary lines

The argument sketched in the last section gives the non-continuity of the upper bounding graph and ensures that its topological closure contains the 0-line, but apart from that it offers little
further information. To explain the shape of such SNA as depicted in Figure 1, recall that the upper bounding graph was defined via the global attractor $K = \bigcap_{n \in \mathbb{N}} T^n(T^1 \times I)$. The sets $K_n := \bigcap_{i=1}^n T^i(T^1 \times I)$ are bounded above by the iterates of the upper boundary line $T^1 \times \{L\}$. To be more precise, define a monotonically decreasing sequence of continuous graphs by $\varphi_0 \equiv L$ and $\varphi_{n+1}(\theta) := T_{\theta - \omega}(\varphi_n(\theta - \omega))$ (alternatively $\varphi_n(\theta) := T_{\theta - n\omega}(L)$). Then $K_n = [0, \varphi_n]$, and the graphs $\varphi_n$ converge pointwise and monotonically decreasing to $\varphi^+$. It therefore seems reasonable to hope that by understanding the behaviour of the $\varphi_n$ we can gain more insight about their limit object.

Figure 2 shows the first six iterates of the upper boundary line. The pictures suggest a strikingly simple scenario: Suppose $T$ is pinched exactly at $\theta = 0$ (i.e. $T_0 \equiv 0$ and all other fibre maps are strictly monotonically increasing) and let $\tau_n := n\omega \mod 1$. Then in the $n$-th iteration step a new “peak” appears at $\tau_n$. The graphs $\varphi_{n-1}$ and $\varphi_n$ differ on a small interval centered around $\tau_n$, but apart from that they seem to coincide. In addition, the peaks seem to get steeper at an exponential rate, and accordingly the width of the intervals decays exponentially. Figure 3 briefly explains on an heuristic level how such an observation can be used to show that $\Phi^+$ is indeed dense in $[0, \varphi^+]$. A slightly refined version of this will then determine our strategy in the proof of Thm. 4.1 (see beginning of Section 4).

Unfortunately, there are some complications in the general case. When the fibre maps are strictly monotonically increasing, the graphs $\varphi_n$ will be strictly monotonically decreasing and thus two of them can never exactly coincide (except on the orbit of the pinched fibre). Still, it is possible to control the behaviour of these graphs, but instead of concentrating on the peaks as our observations suggest we must rather take care of the regions away from the peaks (see Lemma 3.3). However, this will turn out to be sufficient for our purposes.
Figure 3: Taking the behaviour described above for granted we could argue as follows: (A) Choose any \( x \in [0, \varphi^+(\theta)] \) and any small box \( W \) around \((\theta, x)\) which is below \( \varphi^+(\theta) \). Fix \( n_1 \in \mathbb{N} \), such that the width of the peaks appearing at step \( n_1 \) or later is already small compared to \( W \). As \( \varphi_{n_1}(\theta) \geq \varphi^+(\theta) \) and due to the continuity of \( \varphi_{n_1} \), this graph will be above \( W \) on a whole neighbourhood \( B_r(\theta) \) of \( \theta \). (B) Now let \( n_2 \) be the first time greater than \( n_1 \) a peak comes close to \( B_r(\theta) \) (such that the value of the iterated upper boundary lines changes inside of \( B_r(\theta) \)). The end of that peak which is contained in \( B_r(\theta) \) is still above \( W \) (for now we ignore the case where the peak covers all of \( B_r(\theta) \); this can be treated similarly). On the other hand \( \varphi_{n_2} \) is pinched at \( \tau_{n_2} \). Thus \( \varphi_{n_2} \) crosses \( W \) on an interval \( A \), and as the slope of \( \varphi_{n_2} \) is at most \( \beta \alpha \) for some \( \alpha > 1 \), \( \beta > 0 \) this interval still has a certain length. (C) offers a closer look at what happens in the interval \( A \) after \( n_2 \). As it takes a long time until the next peaks hit this interval and the width of these peaks decays exponentially, they will not cover all of \( A \). (Only the first three of these peaks at times \( n_3, n_4 \) and \( n_5 \) are depicted.) But this means \( \varphi_{n_2} \) already coincides with \( \varphi^+ \) at least for some \( \theta' \in A \). Therefore \( \Phi^+ \cap W \neq \emptyset \), and as this works for any such box \( W \) we can conclude \((\theta, x) \in \Phi^+ \).

In order to obtain the required control we must put several restrictions on the systems we study. As mentioned, for parameter families as in \( \Theta \), these will basically translate to a lower bound on the parameter.

Let \( \tau_n := n \omega \mod 1 \in \mathbb{T}^1 \) as above and suppose \( \omega \in [0, 1] \setminus \mathbb{Q} \) satisfies the diophantine condition

\[
d(\tau_n, 0) \geq c \cdot n^{-d}
\]

for some \( c, d > 0 \). Further assume \( T \) is a pinched skew product with the following additional properties\(^1\):

- \( DT_\theta(x) \leq \alpha \forall (\theta, x) \) and \( DT_\theta(x) \leq \alpha^{-\gamma} \) whenever \( x \geq 1 \) for some \( \alpha > 2, \gamma > 0 \). \( (3.2) \)
- \( \left| \frac{\partial}{\partial \theta} T_\theta(x) \right| \leq \beta \forall (\theta, x) \) for some \( \beta > 0 \). \( (3.3) \)
- Let \( m \in \mathbb{N} \) satisfy \( m \geq 4 + \frac{4}{\gamma} \) and

\[
a \geq (m + 1)^d, \quad b \leq c \quad \text{with} \quad d(\tau_n, 0) \geq b \forall n = 1, \ldots, m - 1. \quad (3.4) (3.5)
\]

Then suppose

\[
T_\theta(x) \geq \min\{1, ax\} \cdot \min\{1, \frac{2}{b} d(\theta, 0)\} \quad \forall (\theta, x) \in \mathbb{T}^1 \times I \quad \text{(reference system)} \quad (3.7)
\]

\(^1\)For the sake of simplicity we state the conditions in terms of derivatives, although the system may not be differentiable everywhere. Then these conditions should always be understood in the way that the mentioned inequalities hold for the liminf and limsup of the respective difference quotients.
Remark 3.1
Here the reference system $R$ with fibre maps $R_{\theta}(x) = \min\{1, ax\} \cdot \min\{1, \frac{2}{\tau} d(\theta, 0)\}$ is only used implicitly as a lower bound for the original system. However, it should be mentioned that this simplified system proved to be very useful in the development of the presented ideas. Actually all results were first derived in this simpler setting, where the heuristics described above can be directly converted into a rigorous proof.

Note also that \((3.2)\) and \((3.7)\) together imply a certain “expansion-contraction-property”: By \((3.7)\) the system is expanding close to the 0-line and by \((3.2)\) it is contracting further away. As we will see, the expansion is responsible for the fact that the peaks become steeper whereas the contraction will be needed to control what happens away from the peaks.

We start with two simple observations:

Lemma 3.2

(a) $d(\tau_n, 0) \leq b \cdot a^{-i} \Rightarrow n \geq a^\frac{i}{i}\\n(b) |\varphi'_n| \leq \beta \cdot \alpha^n.$

Proof:

(a) $d(\tau_n, 0) \leq b \cdot a^{-i} \Rightarrow c \cdot n^{-d} \leq b \cdot a^{-i} \Rightarrow n^{-d} \leq a^{-i} \Rightarrow n \geq a^\frac{i}{i}$

(b) We prove $|\varphi'_n| \leq \beta \cdot (\alpha^n - 1)$ by induction. Note that

$$\varphi'_{n+1}(\theta) = D_{\theta} T_{\theta-\omega}(\varphi_n(\theta - \omega)) \cdot \varphi'_n(\theta - \omega) + \frac{\partial}{\partial \theta} T_{\theta-\omega}(\varphi_n(\theta - \omega)).$$  \((3.8)\)

For $n = 1$ the statement is obvious as $\varphi'_0 = 0$, $\alpha > 2$ and $|\frac{\partial}{\partial \theta} T_{\theta}(x)| \leq \beta$ \((3.3)\). If it is satisfied for some $n \geq 1$ we get (from \((3.2)\) and \((3.3)\))

$$|\varphi'_{n+1}(\theta)| \leq \alpha \cdot |\varphi'_n(\theta - \omega)| + \beta \leq \alpha \cdot \beta \cdot (\alpha^n - 1) + \beta \leq \beta \cdot (\alpha^{n+1} - 1),$$

again using $\alpha \geq 2$.

\[\square\]

Let $J^n_j$ be an interval of length $b \cdot a^{-\frac{n}{n}}$ centered around $\tau_j$, i.e.

$$J^n_j := (\tau_j - \frac{b}{2} \cdot a^{-\frac{n}{n}}, \tau_j + \frac{b}{2} \cdot a^{-\frac{n}{n}}).$$  \((3.9)\)

The following lemma now contains all the information about the iterated upper boundary lines we need. The condition on $\theta$ is probably a little bit surprising at first: One might expect that the difference between $\varphi_{n-1}$ and $\varphi_n$ must be very small outside of a small interval around $\tau_n$. However, this is not entirely true. As we shall see, it is only possible to ensure that $|\varphi_{n-1}(\theta) - \varphi_n(\theta)|$ is small whenever $\theta$ is sufficiently far away from “most” of the $\tau_j$, i.e. from $\tau_q, \tau_{q+1}, \ldots, \tau_n$ at the same time, where $q$ is relatively small compared to $n$. (The reason why we want to be able to omit the first few $\tau_k$ is going to become obvious later, when $q$ is specified in the proof of Thm. \((1)\).) The proof below, together with Figure \((3)\) hopefully clarifies why we have to admit $|\varphi_{n-1}(\theta) - \varphi_n(\theta)|$ to be quite large not only when $\theta$ is close to $\tau_n$, but also when it is close to $\tau_{n-1}, \tau_{n-2}, \ldots$.

Lemma 3.3

(a) Let $\theta \notin \bigcup_{j=q}^{n} J^{n-1}_j$. If $q \leq \frac{n-1}{m}$, then

$$|\varphi_n(\theta) - \varphi_{n-1}(\theta)| \leq L \cdot \alpha^{-(n-1)\lambda}$$  \((3.10)\)

where $\lambda := \gamma(1 - \frac{4}{m}) - \frac{4}{m}$ is positive by \((3.4)\).
(b) If \( \theta \notin \bigcup_{j=q}^{k} J_{\ell_{j}}^{-1} \cup \bigcup_{j=n+1}^{\infty} J_{j}^{-1} \) and \( q \leq \frac{n-1}{m} \), then \( (3.1) \) equally holds for all \( n \geq \ell_{n} \).

(c) Let \( \theta \notin \{ \tau_{j} \mid j \in \mathbb{N} \} \). Then there are infinitely many \( n \in \mathbb{N} \) such that \( \theta \notin \bigcup_{j=1}^{n} J_{j}^{-2} \).

Note that the intervals used in (c) are slightly bigger than those needed for the application of (a) to \( \theta \). In fact, the difference is exactly \( \frac{b}{4} \cdot (a^{-\frac{n-2}{m}} - a^{-\frac{n-1}{m}}) \) to either side. Therefore (a) can be applied to all \( \theta' \) from a small neighborhood of \( \theta \).

Proof:

(a) Let \( \theta_{k} := \theta - n\omega + k\omega \), \( x_{k} := \varphi_{k}(\theta_{k}) \) and \( s := \# \{ 1 \leq k < n \mid x_{k} < 1 \} \). Now \( (3.2) \) implies

\[
\varphi_{n-1}(\theta) - \varphi_{n}(\theta) = (\varphi_{0}(\theta_{1}) - \varphi_{1}(\theta_{1})) \cdot \prod_{k=1}^{n-1} \frac{\varphi_{k}(\theta_{k+1}) - \varphi_{k+1}(\theta_{k+1})}{\varphi_{k-1}(\theta_{k}) - \varphi_{k}(\theta_{k})} \leq L \cdot \prod_{k=1}^{n-1} \frac{T_{\theta_{k}}(\varphi_{k-1}(\theta_{k})) - T_{\theta_{k}}(\varphi_{k}(\theta_{k}))}{\varphi_{k-1}(\theta_{k}) - \varphi_{k}(\theta_{k})} \leq L \cdot \alpha^{s} \cdot \gamma(n-1-s)
\]

Thus we are finished if we can show \( s \leq \frac{4(n-1)}{m} \), because then \( s - \gamma(n-1-s) \leq (n-1)(\frac{1}{m} - \gamma(1 - \frac{4}{m})) = -(n-1)\lambda \).

In order to obtain an estimate on \( s \), we first consider blocks of successive times where \( x_{k} \) is smaller than 1. Inside of such a block \( x_{k} \) is multiplied at least by the factor \( a \) in each step, unless \( \theta_{k} \in \left[ -\frac{2}{q}, \frac{2}{q} \right] \). Thus, most of the \( x_{k} \) must even be below \( \frac{1}{a} \). In fact, it is convenient to consider blocks \( l+1, \ldots, p \) \((0 \leq l < p < n)\) of \( p-l \) successive times which satisfy \( x_{l} \geq \frac{1}{a} \), \( x_{k} < \frac{1}{a} \) \( \forall k = l+1, \ldots, p-1 \), \( x_{p} < 1 \) and either \( x_{p} \geq \frac{1}{a} \) or \( x_{p+1} \geq 1 \) or \( p+1 = n \). (In other words, either two blocks are separated by at least one time where \( x_{k} \geq 1 \), or we start a new block when the threshold \( \frac{b}{2} \) is reached.) Then necessarily \( \theta_{l} \in \left[ -\frac{2}{q}, \frac{2}{q} \right] \) (otherwise \( x_{l+1} \geq 1 \)), and the reference system \( (3.7) \) gives \( x_{k+1} \geq a \cdot x_{k} \cdot \min(1, \frac{b}{2} d(\theta_{k}, 0)) \) \( \forall k = l+1, \ldots, p-1 \).

As \( x_{1} \geq \frac{1}{a} \) and \( x_{p} < 1 \) we must have

\[
\prod_{k=l}^{p-1} \min(1, \frac{b}{2} d(\theta_{k}, 0)) \leq a^{-(p-l-1)}.
\]

This means that \( \sum_{1 \leq i < \infty} i \cdot \# \{ l \leq k < p \mid \frac{b}{2} \cdot a^{-i} \leq d(\theta_{k}, 0) < \frac{b}{2} \cdot a^{-i+1} \} \) is an upper bound on the length \( p-l \) of the block, such that every visit \( d(\theta_{k}, 0) \in \left[ \frac{b}{2} \cdot a^{-i}, \frac{b}{2} \cdot a^{-i+1} \right) \) accounts for at most \( i \) times \( j > k \) where \( x_{j} \) can be smaller that 1. Summing up the lengths of all blocks then gives an upper bound on \( s \), namely

\[
s \leq \sum_{1 \leq i < \infty} i \cdot \# \{ 0 \leq k < n-1 \mid \frac{b}{2} \cdot a^{-i} \leq d(\theta_{k}, 0) < \frac{b}{2} \cdot a^{-i+1} \}.
\]

Now \( \theta \notin \bigcup_{j=q}^{n-1} J_{j}^{-1} \) implies

\[
d(\theta_{k}, 0) \geq \frac{b}{2} \cdot a^{-\frac{n-1}{m}} \quad \forall k = 0, \ldots, n-q,
\]

anything else leads to the contradiction \( \theta \in J_{n-k}^{-1} \). In the worst case \( x_{n-q+1}, \ldots, x_{n-1} \) are
all below 1, but this still allows the estimate

\[ s \leq q - 1 + \sum_{1 \leq i \leq \frac{n-1}{a^{n+1}}} i \cdot \# \{ 0 \leq k \leq n - q \mid \frac{b}{2} \cdot a^{-i} \leq d(\theta_k, 0) < \frac{b}{2} \cdot a^{-i+1} \} \leq \]

\[ \leq \frac{n-1}{m} - 1 + \sum_{1 \leq i \leq \frac{n-1}{a^{n+1}}} \# \{ 0 \leq k \leq n - q \mid d(\theta_k, 0) < \frac{b}{2} \cdot a^{-i+1} \} \leq \]

\[ \leq 2(n-1) + \sum_{2 \leq i \leq \frac{n-1}{a^{n+1}}} 1 + \frac{n-1}{a^{\frac{n+1}{2}}} \leq \frac{3(n-1)}{m} + (n-1) \cdot \sum_{i=1}^{\infty} a^{-\frac{n+1}{2}} \leq \frac{4(n-1)}{m} \]

For the step from the third line to the last note that \( \theta_k \) can always visit the interval \((-\frac{b}{2} \cdot a^{-i+1}, \frac{b}{2} \cdot a^{-i+1})\) once, but a second visit at time \( l \) implies \( d(\theta_k, \theta_l) = d(\theta_{l-k}, 0) < \frac{b}{2} \cdot a^{-i+1} \) and thus \( l - k \geq a^{\frac{n+1}{2}} \) by Lemma 3.2(a) (or \( l - k \geq m \) when \( i = 1 \) by \( 3.11 \)).

(b) is a direct consequence of (a).

(c) Let \( n_0 \in \mathbb{N} \) such that

\[ a^{\frac{n+2}{2}} \geq n + 1 \quad \forall n \geq n_0 \quad (3.11) \]

and suppose for some \( k \geq n_0 \) we have \( \theta \in \bigcup_{j=0}^{k} J_{j}^{n-2} \). Let \( \tau_l \) be closest to \( \theta \) of all points \( \tau_1, \ldots, \tau_k \). Then there exists a unique \( n > k \) such that \( d(\theta, \tau_1) \in [b \cdot a^{-\frac{n+2}{2}}, b \cdot a^{-\frac{n+2}{2} + 1}] \). Now assume for some \( j > l \) we have \( d(\tau_j, \theta) < \frac{b}{2} \cdot a^{-\frac{n+2}{2}} \). This implies \( d(\tau_j, \tau_l) = d(\tau_{j-l}, 0) < b \cdot a^{-\frac{n+2}{2}} \) and therefore \( j \geq j - l \geq a^{\frac{n+2}{2}} \geq n + 1 \) by Lemma 3.2(a) and (3.11). Thus \( n \) satisfies our assumption.

Figure 4: Why do we have to expect \( \varphi_{n-1}(\theta) - \varphi_{n}(\theta) \) to be rather large not only when \( \theta \in J_{n}^{n-1} \), but also when it is in \( J_{j}^{n-1} \) for some \( j < n \). We illustrate this with a simple example where \( n = 6 \) and \( \theta = \theta_6 \) is close to \( \tau_4 \). Ignoring the fact that we are on the circle, the first four peaks of the reference system are drawn in a straight order. They give a lower bound for the points \( y_k := \varphi_{k}(\theta_k) \) and \( x_k := \varphi_{k}(\theta_k) \) \((k = 1, \ldots, 6)\). In the first two steps these points are above 1 and their distance is contracted. But afterwards \( \theta_k \) is inside of the \((k - 2)\)th peak, such that the points can be below 1 and their distance may be expanded again. Thus the assumption on \( \theta \) in Lemma 3.2(a) is needed to exclude that any of the \( \theta_k \) is too close to \( \tau_0 = 0 \), maybe apart from the last \( q - 1 \) steps. The possibility that many different \( \theta_k \) are close to \( \tau_0 \) is then ruled out by the diophantine condition on \( \omega \).
4 The topological closure of the upper bounding graph

Let us first reformulate the result of the last section: Fix some $\theta \in \mathbb{T}^1$ not contained in the forward orbit of the pinching point. When $\theta$ is inside of the $k$-th peak, then the value of $\varphi_n(\theta)$ may change significantly at time $n = k$ and for a certain time afterwards, but then the movement of the iterated upper boundary lines “settles down” in a small neighbourhood of $\theta$, until the next peak visits that neighbourhood. Lemma 3.3(c) guarantees that there will always be such times where the $\varphi_n$ have “settled down” close to $\theta$.

This can now be used to show that indeed $\bar{\Phi}^+ = [0, \varphi^+]$, using a slight modification of the strategy sketched in Figure 3. The peaks correspond to the intervals $J_{n}^{-1}$. $n_2$ will be chosen as before, i.e. the first time after $n_1$ the interval $J_{n_2}^{-1}$ hits the neighbourhood $B_r(\theta)$. Now the main difference to Figure 3 is, that we cannot control $\varphi_{n_1} - \varphi_n$ on $J^{-1}_n$ until $n$ is bigger than $n_3 := m(n_2 + 1) + 1$ (see Lemma 3.3(a), where we will have to choose $q = n_2 + 1$). Thus, instead of looking at the graph $\varphi_{n_2}$ and showing that it coincides with $\varphi^+$ for some $\theta' \in B_r(\theta)$ (compare Fig. 3(B) and (C)), we have to look at $\varphi_{n_3}$ and show that this graph is already very close to $\varphi^+$ for some $\theta' \in B_r(\theta)$ (by controlling $\sum_{n=n_3+1}^\infty (\varphi_{n_1} - \varphi_n)$ via Lemma 3.3(b)).

It may be worth mentioning that the conditions (3.2)–(3.7) we put on our system only had to be used to obtain statement (a) of Lemma 3.3. From now on, this lemma contains everything we need to know about our system (apart from the pinched structure and the diophantine rotation number, of course).

**Theorem 4.1**

Suppose $T$ satisfies all of the assumptions (2.1), (2.2) and (3.1)–(3.7). Then

$$\bar{\Phi}^+ = [0, \varphi^+] .$$

**Proof:**

Let $x \leq \varphi^+(\theta) - 3\epsilon$, $\delta > 0$ and consider boxes $W' := B_3(\theta) \times B_2(x)$, $W := B_3(\theta) \times B_\epsilon(x)$. It suffices to show that $W' \cap \bar{\Phi}^+ \neq \emptyset$ for any such $\epsilon$ and $\delta$.

First of all, we will fix some $n_0 \in \mathbb{N}$ which satisfies certain conditions. Roughly speaking, these imply that for all $n \geq n_0$ the intervals $J^{-1}_n$ are small enough in comparison with $\delta$ and that the time $l$ we have to wait until $J^{-1}_{n+1}$ hits $J^{-1}_n$ again is very large. All the conditions are obviously satisfied when $n_0$ is large enough. Apart from that the reader should be advised not to wonder about these requirements too long in the beginning, but to check each of them only at the time when it is actually used below and the motivation for choosing it becomes apparent.

Let $n_0 \in \mathbb{N}$ satisfy

$$n_0 \geq m + 1$$

$$\frac{b}{2} (a_n - \frac{n}{m} - a_{n+1} - \frac{n+1}{m}) \leq \frac{\delta}{2} \forall n \geq n_0$$

$$|J^{-1}_n| = b \cdot a_n - \frac{n+1}{m} \leq \frac{\delta}{2} \forall n \geq n_0$$

$$\sum_{j \geq a_n - \frac{n+1}{m}} b \cdot a_{j-1} \leq \epsilon \frac{\alpha^{-m(n+1)-1}}{\beta} \forall n \geq n_0$$

$$\sum_{j=n+1}^\infty L \cdot a^{-\lambda(j-1)} \leq \epsilon \forall n \geq n_0$$

For (4.6) note that the left side decays super-exponentially with $n$ whereas the right side only decays exponentially.

Now take some $n_1 \geq n_0$ which satisfies

$$\theta \notin \bigcup_{j=1}^{n_1} J_{n_1}^{n_2} .$$

(4.7)
Such a $n_1$ always exists by Lemma 3.3(c). As pointed out that lemma, we can apply statement (a) of it to all $\theta^\prime \in B_r(\theta)$ where $r := \frac{1}{2} \cdot (a - \frac{a^{-n_1 + 1}}{2} - a^{-n_1}) > 0$, and this is also possible for all $n \geq n_1$ as long as $B_r(\theta) \cap J_j^{n-1} = \emptyset \forall j = n_1+1,\ldots, n$. In addition we can assume, by reducing $r$ further if necessary, that $\varphi_{n_1}(\theta^\prime) > x + 2\epsilon \forall \theta^\prime \in B_r(\theta)$ (note that $\varphi_{n_1}$ is continuous and $\varphi_{n_1}(\theta) \geq \varphi^+(\theta) \geq x + 3\epsilon$).

Let $n_2$ be the first integer greater than $n_1$ such that $J_{n_2}^{n-1}$ hits $B_r(\theta)$ and define $n_3 := m(n_2 + 1) + 1$. 4.3(a) and 4.3(b) ensure that $J_{n_2}^{n-1} \subseteq B_3(\theta)$. Further, 4.3(a) ensures
\[
\theta^\prime \notin \bigcup_{j=n_2+1}^{n_3} J_{n_2}^{j-1}, \quad \forall \theta^\prime \in J_{n_2}^{n-1},
\] (4.8)
because $J_{n_2}^{n-1} \cap J_{n_2}^{j-1} \neq \emptyset$ for some $j > n_2$ implies $d(\theta_j, \theta_{n_2}) \leq b \cdot a^{-n_2}$, and by Lemma 3.2(a) and 4.3(a) this cannot happen if $j - n_2 \leq m(n_2 + 1) + 1 - n_2 = n_3 - n_2$.

Now first assume $\theta \in J_{n_2}^{n-1}$. As $\varphi_{n_3}$ is pinched at $\theta_{n_2}$ and at the same time $\varphi_{n_3}(\theta) \geq \varphi^+(\theta) \geq x + 3\epsilon$ this graph will cross the box $W$ from below to above when going from $\tau_{n_2}$ to $\theta$. Thus, if we define
\[
A := \{\theta^\prime \in J_{n_2}^{n-1} \mid \varphi_{n_3}(\theta^\prime) \in B_r(x)\},
\] we can use Lemma 3.2(b) to obtain that
\[
|A| \geq \frac{2\epsilon}{\beta} \cdot \alpha^{-n_3}. \tag{4.9}
\]
Let
\[
B := A \setminus \bigcup_{j=n_3+1}^{\infty} J_{n_2}^{j-1} \tag{4.10}
\]
$B$ is still a set of positive measure: Let $n_4$ be the first time where $J_{n_4}^{n-1}$ hits $J_{n_2}^{n-1}$ again. Then $n_4 \geq a^{-n_2}$ (again Lemma 3.2(a)) and therefore
\[
|A \cap \bigcup_{j=n_3+1}^{\infty} J_{n_2}^{j-1}| \leq \sum_{j=n_4}^{\infty} |J_{n_2}^{j-1}| \leq \sum_{j \geq a^{-n_2}} b \cdot a^{-j} \leq \sum_{j \geq a^{-n_2}} b \cdot a^{-j} \leq \frac{\epsilon}{\beta} \cdot \alpha^{-n_3}
\]
where we used $n_3 = m(n_2 + 1) + 1$ in the last step. Thus we have $|B| \geq \frac{\epsilon}{\beta} \cdot \alpha^{-n_3}$.

Combining 4.3(b) and 4.10 we see that it is possible to apply Lemma 3.3(b) with $q = n_2 + 1$ and $\tilde{n} = n_3$ for every $\theta^\prime \in B$. Thus the values of the graphs $\varphi_n (n \geq n_3)$ do not differ to much anymore on $B$, and by using $\varphi_n \setminus \varphi^+$ we get
\[
\varphi_{n_3}(\theta^\prime) - \varphi^+(\theta^\prime) = \sum_{n=n_3+1}^{\infty} \varphi_{n-1}(\theta^\prime) - \varphi_n(\theta^\prime) \leq \sum_{n=n_3+1}^{\infty} L \cdot a^{-n} \leq \epsilon \forall \theta^\prime \in B.
\]
As $\varphi_{n_3}(\theta^\prime) \in B_3(x) \forall \theta^\prime \in B$ and $B \subseteq B_3(\theta)$ this proves $\varphi^+_n \cap W^r \neq \emptyset$.

When $\theta$ is not contained in $J_{n_2}^{n-1}$ we can equally conclude that $\varphi_{n_3}$ crosses $W$ inside of the interval $J_{n_2}^{n-1}$, such that the above argument applies in exactly the same way. To see this, note that when $\theta \notin J_{n_2}^{n-1}$, then at least one endpoint of $J_{n_2}^{n-1}$ is contained in $B_r(\theta)$. Denote it by $\tilde{\theta}$. As $\tilde{\theta} \notin J_{n_2}^{n-1}$ (recall that these intervals are open) we can combine 4.7, the definition of $n_2$ and 4.3(b) (which also applies to $\tilde{\theta}$ as the endpoint of $J_{n_2}^{n-1}$) to obtain that $\tilde{\theta} \notin \bigcup_{j=1}^{n_2} J_{n_2}^{j-1} \forall n = n_1, \ldots, n_3$ and as $n_1 \geq m_1 + 1$ by 4.1(a) we can apply Lemma 3.3(a) with $q = 1$ to obtain
\[
\varphi_{n_3}(\tilde{\theta}) \geq \varphi_{n_1}(\tilde{\theta}) - \sum_{j=n_1+1}^{n_2} \varphi_{j-1}(\tilde{\theta}) - \varphi_j(\tilde{\theta}) \geq x + 2\epsilon - \sum_{j=n_1+1}^{n_2} L \cdot a^{-(n-1)\lambda} \geq x + \epsilon
\]
Again $\varphi_{n_3}(\theta_{n_2}) = 0$ and thus the graph has to cross $W$ between $\theta_{n_2}$ and $\tilde{\theta}$. This completes the proof of the theorem.
We now want to apply this to parameter families \( T = T_\alpha \) with fibre maps given by (1.4). To that end suppose \( g : T^1 \rightarrow T^1 \) and \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) are continuous functions which satisfy the following assumptions:

- \( g \) is differentiable and strictly positive on \( T^1 \setminus \{0\} \).
- \( g(\theta_0) = 0 \).
- \( \lim_{\theta \nearrow \theta_0} -g'(\theta) > 0 \) and \( \lim_{\theta \searrow \theta_0} g'(\theta) > 0 \).
- \( f \) is differentiable and monotonically increasing.
- \( f(0) = 0 \), \( f'(0) > 0 \).
- \( \alpha_1 + \gamma f'(\alpha) \rightarrow 0 (\alpha \rightarrow \infty) \) for some \( \gamma > 0 \).

Corollary 4.2
Suppose \( \omega \) satisfies a diophantine condition (3.11) and \( T \) is given by

\[
(\theta, x) \mapsto (\theta + \omega, \alpha g(\theta) f(x))
\]

with maps \( f \) and \( g \) satisfying the above assumptions (3.11), (3.16). Then there exists an \( \alpha_0 > 0 \) such that

\[
[0, \omega^+]
\]

wherever \( \alpha \geq \alpha_0 \).

Proof:
Choose \( m \geq 4 + \frac{2}{\gamma} \). W.l.o.g. we can assume \( \theta_0 = 0 \). First note that the system

\[
(\theta, x) \mapsto (\theta + \omega, \alpha_1 \alpha_2 g(\theta) f(x))
\]

is topologically conjugated to the system

\[
(\theta, x) \mapsto (\theta + \omega, \alpha_1 g(\theta) f(\alpha_2 x))
\]

by \( h : (\theta, x) \mapsto (\theta, \alpha_2 x) \). (3.11), (3.13) and \( f'(0) > 0 \) imply that this system is bounded from below by a suitable reference system (3.7) if \( \alpha_1 \) and \( \alpha_2 \) are large enough. Once \( \alpha_1 \) is fixed, (3.13) implies that the expansion-contraction-condition (3.2) is satisfied when \( \alpha_2 \) is sufficiently large. It is also easy to see that the system is pinched, has an invariant 0-line, and that \( \partial_\theta T_\theta(x) \) is bounded for any \( \alpha_1 \) and \( \alpha_2 \). Thus all the assumptions of Thm. 4.1 are satisfied, and the conjugacy \( h \) of course preserves the property we are interested in. \( \square \)

Example 4.3
Consider \( g(\theta) = |\sin(\pi \theta)| \), \( f(x) = \tanh(x) \) and let \( \omega \) be the golden mean. The smallest possible value for \( m \) in (3.3) is \( m = 5 \) (if we can take \( \gamma > 4 \), which we verify below). The closest return up to time \( m - 1 = 4 \) is \( \omega^3 \approx 0.236 \ldots \), which we take for \( b \). Now we just try if \( \alpha = 8 \) works. To that end note that \( 3|\sin(\pi \theta)| > \min\{1, \frac{1}{\alpha} d(0, \theta)\} \) and \( \frac{1}{4} \tanh(1) > 1 \). Thus \( T_\theta(x) = 4\cos(\pi \theta)| \tanh(8x) \) satisfies (3.7) with \( a = 8 \) and \( b = \omega^3 \). As \( \tanh'(x) = \cosh^{-2}(x) < 2e^{-2x} \), it is also easily checked that (3.2) holds for some \( \gamma > 4 \) \((2e^{-16} < 32^{-4}) \). Thus we can choose \( \alpha_0 = 32 \) for this particular parameter family.

Compared to this, the lower bound for the existence of an SNA obtained from (2.34) is \( \log \alpha > - \int_{T^1} \log |\cos(\pi \theta)| d\theta \) or \( \alpha > \exp(- \int_{T^1} \log |\cos(\pi \theta)| d\theta) = 2 \), still leaving a certain gap between the two conditions. Of course, by suitable modifications of the assumptions and in the proofs above, this gap could be closed a little bit further (especially in the case of the reference system mentioned in Rem. 3.1), but closing it completely that way seems to be out of reach. However, as the approach used here requires a rather strong control about the behaviour of the systems, it does not seem very surprising that “something gets lost along the way”. Thus
the existence of this gap does not necessarily have to mean that counterexamples should be expected for lower parameters \(\alpha\). After all, there does not seem to be a qualitative change in the behaviour of the iterated upper boundary lines for smaller \(\alpha\) (consider Fig. 2 for example), as long as the upper bounding graph is not equal to 0 anyway.

**Remark 4.4**
When looking at pictures from numerical simulations as in Figure 2, the result we obtained may seem a little bit surprising at first. There, the larger we choose the parameter \(\alpha\), the “thinner” the SNA seems to be around the 0-line. However, our approach offers a perfect explanation for this. If \(\alpha\) is large, the peaks will become steeper and narrower very fast. Thus they will be too small to be detected numerically quite soon, and the SNA seems to coincide with the iterated upper boundary line after a few steps already. On the other hand, as we have seen it is exactly this fast decay of the width of the peaks which enabled us to prove our result.

The following remark contains a rather informal discussion of two possible slight generalizations of Thm. 4.1 and its corollary. We refrained from including them in the statement of the theorem, because although the basic idea does not change at all, this would have made the proof far less readable. The most prominent examples are already covered anyway.

**Remark 4.5** 
(a) First, the diophantine condition on \(\omega\) can be weakened to some extent. It was used to ensure that the return time to an interval of length \(b \cdot a^{-i}\) grows exponentially in \(i\) (see Lemma 3.2(a)), but actually all we need is that this quantity grows faster than linearly. This is the case as long as the rotation number \(\omega\) satisfies
\[
d(\omega, \tau_n) \geq s \cdot e^{-d\sqrt{n}}
\]
for some \(s > 0, d > 1\) (then the return times will asymptotically grow at least like like \(i^d\)). The requirements on the other parameters will certainly have to be stronger in this case, but apart from that the proof above needs only the slightest modifications.

(b) Second, instead of working with the reference system (3.7), it is also possible to use a system of the kind \(R_\theta(x) = \min\{1, ax\} \cdot \min\{1, (\frac{1}{2} d(\theta, 0))^p\}\), i.e. to replace the “sharp peak” at \(\theta = 0\) with a critical point of finite order \(p\). Then, instead of counting the visits in the intervals \((-\frac{a}{2} a^{-i}, \frac{a}{2} a^{-i})\) one would have to count the visits in \((-\frac{a}{2} a^{-i}, \frac{a}{2} a^{-i})\) (compare the proof of Lemma 3.3(a)). The diophantine condition (or the one mentioned in (a)) still guarantees that the resulting sums converge, only the conditions on the parameters must certainly be altered again.

## 5 Upper bounding graphs which contain isolated points

For the construction of counterexamples, we need some facts about the combinatorics of irrational rotations, as they can be found in [dMvS93] (for example). Let \((a_n)_{n \in \mathbb{N}}\) be the coefficients of the continued fraction expansion of \(\omega\), \((q_n)_{n \in \mathbb{N}}\) the closest return times and \((\sigma_n)_{n \in \mathbb{N}}\) the closest returns. Then the following equations hold
\[
\sigma_n = q_n \omega \mod 1 = \tau_{q_n}
\]
\[
q_0 = 1, \quad q_1 = a_1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}
\]
\[
\sigma_0 = \omega, \quad \sigma_1 = 1 - a_1\omega, \quad \sigma_{n-1} = a_{n+1}\sigma_n + \sigma_{n+1}
\]
\[
\frac{1}{q_{n+2}} \leq \sigma_n \leq \frac{1}{q_{n+1}}
\]

For the remainder of this section we will assume that
\[
\sum_{i=0}^{\infty} \frac{q_i}{q_{i+1}} < \infty.
\]
As \( \frac{1}{a_{n+1}} \leq \frac{\alpha}{q_{n+1}} \leq \frac{1}{a_{n+1}} \) (see (5.2)) this is equivalent to
\[
\sum_{i=1}^{\infty} \frac{1}{a_i} < \infty . \tag{5.6}
\]

**Construction of a suitable function** \( g \): Choose \( n_1 \in \mathbb{N} \) such that \( \sum_{i=n_1}^{\infty} \frac{\alpha}{q_{i+1}} < \frac{1}{2} \) and, for the sake of simplicity, \( \sigma_{n_1} \) is to the right of 0 (in a local sense). Then all points \( \sigma_{n_1+2i} \) \((i \in \mathbb{N})\) will be to the right of 0, whereas all points \( \sigma_{n_1+2i+1} \) \((i \in \mathbb{N})\) will be to the left. Let
\[
I_n := \begin{cases} 
[-\sigma_n + \sigma_{n+2}, -\sigma_{n+2}] & \text{if } n-n_1 \text{ is even} \\
[-\sigma_{n+2}, -\sigma_n + \sigma_{n+2}] & \text{if } n-n_1 \text{ is odd} 
\end{cases} \tag{5.7}
\]
and \( I' := \bigcup_{n=n_1}^{\infty} I'_n \cup \{0\} = [-\sigma_{n_1}, -\sigma_{n_1+1}] \). Now choose any \( a > 2 \) and a function \( g \) with the property that \( g|_{I_n} \equiv a^{-n_1} \) and \( g|_{I'_n \setminus I_n} \) joins the two levels \( a^{-q_n} \) on \( I_n \) and \( a^{-q_{n-2}} \) on \( I_{n-2} \) in a continuous and monotone way (such that \( g(I'_n) = [a^{-q_n}, a^{-q_{n-2}}] \)).

![Figure 5: Construction of the function g (with \( n_1 = 1 \)).](image)

Note that \( |\log g| \) is integrable:
\[
\int_{\mathbb{T}} |\log g(\theta)| \, d\theta \leq \log a \cdot \sum_{i=n_1}^{\infty} q_i \cdot |\sigma_i| \leq \log a \cdot \sum_{i=n_1}^{\infty} \frac{q_i}{q_{i+1}} \leq \frac{1}{2} \log a .
\]

Let further \( f(x) := \min\{1, ax\} \) and consider the system \( T \) given by
\[
T : (\theta, x) \mapsto (\theta + \omega, g(\theta)f(x)) . \tag{5.9}
\]

**Claim 1:**
\[
\varphi^+(0) = 1
\]

**Proof:**
Note that \( \varphi_n(\theta) \neq \varphi_{n-1}(\theta) \) requires \( \varphi_{n-1}(\theta - \omega) \neq \varphi_{n-2}(\theta - \omega) \), and in our particular example
also \( \varphi_{n-1}(\theta - \omega) < \frac{1}{a} \). Let \( n \) be the first time that \( \varphi_n(0) < 1 \). Then necessarily \( \varphi_{n-\tau_j}(\tau_j) < \frac{1}{a} \) \( \forall j = 1, \ldots, n-1 \). Thus for the first \( n-1 \) steps the backwards orbit of \((0, \varphi_n(0))\) is always in the expanding region and therefore
\[
\varphi_n(0) = T_{\tau_n}^n(1) = a^{n-1} \prod_{j=1}^{n} g(\tau_j).
\]
As \( \varphi_n(0) < 1 \) this implies
\[
\prod_{j=1}^{n} g(\tau_j) \leq a^{-(n-1)}.
\]
However, we can estimate \(- \log_a \prod_{j=1}^{n} g(\tau_j)\) simply by counting how often \((\tau_j)_{j=1}^{n} \) visits the intervals \( I'_{n_j} \). As the time between two such visits must be at least \( q_{n+1} \left( |I'_{n_j}| < |\sigma_n| \right) \), this gives
\[
- \log_a \prod_{j=1}^{n} g(\tau_j) \leq \sum_{i=n_2}^{\infty} q_i \cdot \frac{n}{q_{n+1}} \leq \frac{n}{2}.
\]
W.l.o.g. we can assume that \( n \geq 2 \) \((\tau_{-1} \notin I)\) and thus arrive at a contradiction. This proves claim 1.

**Claim 2:**
\[
\varphi^+(\theta) \leq \frac{1}{a} \quad \forall \theta \in I \setminus \{0\}
\]

**Proof:**
This follows directly from the properties of \( g \): When \( n - n_1 \) is even we have
\[
\varphi_{q_n} \leq \frac{1}{a} \quad \text{on} \quad [-\sigma_n + \sigma_{n+2}, 0] + q_n \omega = [\sigma_{n+2}, \sigma_n]
\]
and \( \bigcup_{i=0}^{\infty} [\sigma_{n_1+2i+2}, \sigma_{n_1+2i+1}] = (0, \sigma_{n_1}] \). The same argument applies to the left side, which proves claim 2.

The two claims together imply that \((0, \varphi^+(0)) = (0, 1)\) is isolated in \( \Phi^+ \), and by continuity of \( T \) the same is true for all points in the backwards orbit of this point.

We close with some final remarks concerning the regularity of the examples just constructed. First of all, it is possible to replace the rather degenerate function \( f \) by a differentiable and strictly monotonically increasing function \( \tilde{f} \). To that end, suppose \( \tilde{f} : [0, 2] \to [0, 2] \) satisfies \( \tilde{f}(0, \frac{1}{2}) = f(0, \frac{1}{2}) \), \( \tilde{f}' > 0 \) and \( \tilde{f}(2) \leq 2 \). Then the system
\[
\tilde{T} : (\theta, x) \mapsto (\theta + \omega, g(\theta) \tilde{f}(x))
\]
is bounded from below by \((-\frac{2}{a})\), which ensures \( \varphi^+(0) \geq 1 \). On the other hand, the same arguments as in the proof of claim 2 show that \( \varphi^+(\theta) \leq \frac{2}{a} < 1 \quad \forall \theta \in [\sigma_{n_1+1}, \sigma_{n_1}] \). As \( \log g \) is integrable (see above) \( \tilde{T} \) also fits perfectly into the framework of Section 2.

Finally, although \((-\frac{2}{a})\) certainly rules out the golden mean or any rotation number with bounded coefficients \( a_n \), it still allows \( \omega \) to be of diophantine type. Then it is possible to show, by combining the diophantine condition \((-\frac{2}{a})\) with some elementary estimates obtained from \((-\frac{2}{a})\), that \( a_{n+2}^{-n-2} \) converges to zero as \( n \) goes to infinity. Thus the connecting pieces of \( g \) on \( I'_{n} \setminus I_n \) can be chosen such that \( g'(\theta) \to 0 \) as \( \theta \to 0 \), in which case \( g \) is even differentiable. However, more or less the same estimates yield that \( g \) will then be infinitely flat at \( \theta = 0 \), such that it cannot be bounded from below by any suitable reference system (compare Rem. [4.10 b]).

**References**

[BO96] Zina I. Bezhaeva and Valery I. Oseledets. An example of a strange non-chaotic attractor. *Functional analysis and its applications*, 30(4):223–229, 1996.
[dMvS93] W. de Melo and S. van Strien. *One-dimensional dynamics*. Springer, 1993.

[Gle02] Paul Glendinning. Global attractors of pinched skew products. *Dynamical Systems*, 17:287–294, 2002.

[GOPY84] Celso Grebogi, Edward Ott, Steven Pelikan, and James A. Yorke. Strange attractors that are not chaotic. *Physica D*, 13:261–268, 1984.

[Her83] Michael R. Herman. Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d’un théorème d’Arnold et de Moser sur le tore de dimension 2. *Commentarii Mathematici Helvetici*, 58:453–502, 1983.

[Jäg03] T. Jäger. Quasiperiodically forced interval maps with negative schwarzian derivative. *Nonlinearity*, 16(4):1239–1255, 2003.

[Kel96] Gerhard Keller. A note on strange nonchaotic attractors. *Fundamenta Mathematicae*, 151(2):139–148, 1996.

[PNR01] Awadhesh Prasad, Surendra Singh Negi, and Ramakrishna Ramaswamy. Strange nonchaotic attractors. *International Journal of Bifurcation and Chaos*, 11(2):291–309, 2001.

[SS00] J. Stark and R. Sturman. Semi-uniform ergodic theorems and applications to forced systems. *Nonlinearity*, 13(1):113–143, 2000.

[Sta99] J. Stark. Regularity of invariant graphs for forced systems. *Ergodic theory and dynamical systems*, 19(1):155–199, 1999.

[Sta03] J. Stark. Transitive sets for quasiperiodically forced monotone maps. *Dynamical Systems*, 18(4):351–364, 2003.