TRANSPORT INDUCED BY ION-IMPURITY FRICTION
IN STRONGLY ROTATING, COLLISIONAL
TOKAMAK PLASMA

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Abstract — A moment approach to the transport theory of impure Tokamak plasmas with strong rotation velocity \( V_r \), in the Pfirsch-Schüller regime with constant temperature is presented. Using the moment approach, the general form of friction and viscosity with effects of large rotation and arbitrary impurity concentration is obtained. In particular, corrections to Braginskii's viscous tensor due to the large rotation are found. When \( V_r \ll v_{th} \), strong in/out and up/down poloidal variations of the impurity density in the flux surface, the first-order poloidal flows, the radial particle flux, and the radial flux of toroidal angular momentum are evaluated. Most importantly, a strong ordering parameter \( \Delta = \delta_0 Z^2 \nu_{in}/2/\omega_i \) is introduced and found to be essentially a measure of up/down density variation; and a large enough \( \Delta \) may lead to a bifurcation of the equilibrium poloidal flows.

1. INTRODUCTION

The problem of neutral beam produced toroidal rotation of Tokamak plasmas has been studied extensively, experimentally (Scott and the TFTR Group, 1988; Suckewer et al., 1984; Isler et al., 1986) and theoretically (Hinton and Wong, 1985; Catto et al., 1987). The surprising experimental observations are a momentum confinement time much shorter than corresponds to Braginskii's (1965) classical perpendicular viscosity, up-down asymmetries of the main ion and impurity density (Smeulders, 1986) and a strong dependence of particle confinement time on co- vs counter-injection.

Theoretically, one approach is a systematic extension of the Larmor radius expansion of the neoclassical drift kinetic equation (Wong, 1987; Catto et al., 1987), the other is the fluid approach (Stacey and Sigmar, 1984; Connor et al., 1987, 1989) based on Braginskii's viscosity tensor. Previously, Stacey and Sigmar (1985) and Stacey and Malik (1989) proposed a gyroviscous damping mechanism in plasmas containing impurity ions [such that \( \alpha \equiv n_i Z^2 / n_i \sim O(1) \)], depending on an \( O(\epsilon) \) up-down asymmetric variation of the particle density in the flux surface and finite poloidal flow velocities for the ions and impurities. Connor et al. (1987), focusing essentially on the zeroth-order (in Larmor radius expansion) fluid momentum balance, have concluded that to this order the poloidal flow velocity and therefore the up/down density variations must vanish. To first order, however, they considered mainly a pure plasma. Also, Wong (1987) has studied an impure plasma system with large rotation and carried out the formal Larmor radius expansion of the transport equations in the banana regime.

With this background in mind, in this work, we study the transport theory of a strongly rotating, impure Tokamak plasma in the Pfirsch-Schüller (P-S) regime, adopting the Larmor radius expansion in conjunction with a moment approach [cf.
The general description of the systematic moment approach to transport theory is given elsewhere (Hsu and Sigmar, 1988). In this approach one (i) derives the set of moment equations for each order in $\delta_{pi} \equiv \rho_{pi}/L$ (where $\rho_{pi}$ is the poloidal ion Larmor radius and $L$ the radial length scale); (ii) solves the set of moment equations including the higher rank moments such as the viscous tensor

$$\Pi \equiv \int dv\, m(vv - \frac{v^2}{3} I) f$$

and the heat flow

$$q \equiv \int dv\, m\frac{v^2}{2} v f,$$

the flow velocity $V$ and the up–down variations of density $n$ and temperature $T$ in the flux surface; and finally (iii) determines the radial transport fluxes. Here we only remark that, according to the well known $H$-theorem, unless there exists an $O(1)$ momentum source, the Coulomb collisions will force the $O(1)$ distribution function to be a purely shifted Maxwellian (Mikhailovskii and Tsypin, 1984)

$$f_M = \frac{n}{\left(\frac{2\pi T}{m}\right)^{3/2}} e^{-\frac{m(\mathbf{u} - \mathbf{v})^2}{2T}},$$

with $\mathbf{u} = \mathbf{v} + \mathbf{V}$, and $\mathbf{v}$ the particle velocity in the moving frame. This implies that, in the absence of an $O(1)$ source, all moments (except for $n, T$ and $V$) must be $\ll O(\delta_{pi})$.

An additional constraint on the zeroth-order flow velocity $V$ in an axisymmetric system is given briefly as follows. Adopting the usual flux coordinate system $(\psi, \theta, \phi)$, (cf. Fig. 1) using the lowest order $O(\delta_p^{-1})$ momentum equation

![Fig. 1.—Illustration of the flux coordinates system $(\psi, \theta, \phi)$.](image-url)
and the $O(1)$ continuity equation

$$ \nabla \cdot (n_j^{(0)} \mathbf{v}_j^{(0)}) = 0, $$

one finds that in an axisymmetric system

$$ \mathbf{v}_j^{(0)} = \frac{K_j^{(0)}(\psi)}{n_j^{(0)}} \mathbf{B} + \omega^{(0)} R^2 \nabla \phi. \quad (2) $$

Here $\phi^{(-1)} = \phi^{(-1)}(\psi)$,

$$ \omega^{(0)} = -c \frac{\partial}{\partial \psi} \phi^{(-1)} \quad (3) $$

is the zeroth-order toroidal rotation frequency which is a species-independent flux quantity, and $(B_p/n^{(0)})K_j^{(0)}(\psi)$ is the zeroth-order poloidal flow. However, as described in the Appendix, a nonvanishing $K_j^{(0)}(\psi)$ will generate an $O(1)$ parallel viscous force which will then damp $K_j^{(0)}$ at a rate $\sim \omega_t^2/v$ (where $\omega_t$ is the transit frequency and $v$ the collision frequency). Therefore, $K_j^{(0)}(\psi)$ must vanish and

$$ \mathbf{v}_j^{(0)} = \omega^{(0)} R^2 \nabla \phi \quad (4) $$

is species independent. That is, the zeroth-order flow $V^{(0)} \sim O(1)v_{th}$ is an axisymmetric system, and if it exists, must be purely in the toroidal direction and must accompany a large radial electric field such that $e\phi(\psi)/T \sim O(\delta_p^{-1})$.

We note that the up–down variations of density and temperature are essential for the neoclassical transport in a toroidally confined plasma because the radial fluxes arise from the radial drift of particles due to the poloidal gradient forces and only up–down symmetric forces (which come from the parallel gradient of up–down asymmetric pieces of $n$ and $T$) can survive the flux surface average. Since the present work is restricted to plasma systems with radially uniform temperature, the remainder of the paper will then concentrate on the derivation of the up–down variations of density, the poloidal flows, and the ensuing radial transport fluxes.

Consider a plasma system with electrons, main ions and one impurity species with the impurity charge $Z \sim m_i/m_e \gg 1$. Due to the largeness of $Z$ and the smallness of $n_i/n_e$, it is useful to write the momentum equations for ions and impurities in the following form

$$ m_i \mathbf{V}_i \cdot \nabla \mathbf{V}_i + \frac{\nabla \cdot \mathbf{\Pi}_i}{n_i} + T_i \nabla (\ln n_i) = -e\mathbf{V} \phi + m_i \Omega_i (\mathbf{V}_i \times \mathbf{b}) + \frac{R_{i}}{n_i} \quad (5) $$

$$ \frac{m_i}{Z} \mathbf{V}_i \cdot \nabla \mathbf{V}_i + \frac{\nabla \cdot \mathbf{\Pi}_i}{n_i Z} + \frac{T_i}{Z} \nabla (\ln n_i) = -e\mathbf{V} \phi + m_i \Omega_i (\mathbf{V}_i \times \mathbf{b}) - \frac{R_{i}}{n_i Z} \quad (6) $$
such that the driving centrifugal force and electric force in both equations appear to be comparable, independent of \( Z \). However, it will become clear in the next section that the centrifugal force can only drive the in–out density variation; and the up–down density variation can only be driven through a nonvanishing parallel friction which, in the P–S regime can be written as (see the Appendix)

\[
R_{\|} \equiv b \cdot R_{\|} = m_i n_i v_{\|} D_{\|} (V_{\|} - V_{\|}),
\]

(7a)

where \( v_{\|} = \sqrt{2} v_{\|} \) is the i–I collision frequency and \( D_{\|} \) is given in equation (A22). Furthermore, from equations (3) and (4), \( V^{(0)} \) is found to be species independent. Thus, the parallel friction scales as

\[
R_{\|} \sim m_i n_i v_{\|} \delta_p v_{\|},
\]

(7b)

and from equation (6), the up/down impurity variation scales as

\[
\frac{n_i - (\theta)}{n_i} \sim \Delta \epsilon.
\]

Here \( n_i - (\theta) \) denotes the up-down asymmetric part of impurity density, overbar denotes the flux surface average and

\[
\Delta \equiv \frac{\delta_p Z^2 \bar{v}_{\|} \sqrt{2}}{\omega_{\|}}.
\]

Consequently, we have two interesting cases, namely (1) \( \Delta \sim \delta_p \) which leads to \( n_i - (\theta)/n_i \sim O(\delta_p \epsilon) \) and will be studied with finite \( \epsilon \) (the inverse aspect ratio) in Section 2; and (2) the case \( \Delta \sim 1 \) which leads to \( n_i - (\theta)/n_i \sim O(\epsilon) \); this strong ordering case will be studied assuming small \( \epsilon \) for simplicity in Section 3. Note that the second case with small rotation has been previously studied (CHANG and HAZELTINE, 1980); however, (i) the importance of the strong ordering parameter \( \Delta \) was not recognized, and (ii) the poloidal flows were not calculated self-consistently.

It will be shown that when \( \Delta \sim O(1) \), poloidal flows for each species should be solved individually and self-consistently; in addition, the poloidal flows are strongly nonlinearly coupled with the density modulations. As a consequence of nonlinearity, one would expect the existence of bifurcated solutions. Indeed, by studying the behavior of the governing equations, we do find the appropriate, but nonstandard, parameter regime which contains multiple roots. In this paper, we will only introduce and briefly discuss the possibility of the bifurcated poloidal equilibrium, since it requires parameters such as very steep radial profiles or large \( \Delta \). However, it is found in another work by the authors (HSU and SIGMAR, 1989) that the bifurcated poloidal equilibrium exists in standard parameter regimes when the effects of temperature gradients are retained.

In Section 4, the radial transport fluxes of particles and of toroidal angular momentum will be evaluated. Section 5 contains a brief comparison with the “gyroviscous transport theory” of Stacey and Sigmar (STACEY and SIGMAR, 1985; STACEY and
MALIK, 1989). In Section 6, conclusions will be given. In the Appendix, we obtain the parallel friction and parallel viscosity including rotation, which contains nontrivial corrections to Braginskii’s viscosity tensor (BRAGINSKII, 1965).

2. USUAL δ₀ ORDERING

In this section, we assume that Δ ∼ δ₀, that is, the friction is omitted in the zeroth-order equations. Therefore, one needs to first solve the O(1) equations for the in–out density variations, and then solve O(δ₀) equations to obtain the up–down density variations and poloidal flows, which in turn, yield the radial transport.

2.1. The zeroth-order solution

The zeroth-order of equations (5), (6), (A1) can be written as

\[ m_j n_j^{(0)} V_{jk}^{(0)} \cdot \nabla V_j^{(0)} + \nabla n_j^{(0)} T_j^{(0)} = -n_j^{(0)} Z_j e \nabla \phi^{(0)} + m_j n_j^{(0)} \Omega_j V_j^{(1)} \times \mathbf{b}, \tag{9} \]

\[ n_j^{(0)} T_j^{(0)} W_2[\nabla V_j^{(0)}] = \Omega_j K_2[\Pi_j^{(1)}] \equiv \Omega_j (\Pi_j^{(1)} + Z_2). \tag{10} \]

Here tensor operators W_2[A] and K_2[A] are defined in equations (A2) and (A3), and \( \mathbf{b} \) means transpose of a second-rank tensor. The parallel projection of equation (9) yields

\[ T_j \mathbf{B} \cdot \nabla \ln n_j^{(0)} = \frac{m_j}{2} \omega_j^{(0)^2} \mathbf{B} \cdot \nabla R^2 - Z_j e \mathbf{B} \cdot \nabla \phi^{(0)}, \tag{11} \]

and thus

\[ n_j^{(0)} = N_j(\psi) \exp \left( \frac{m_j}{2} \omega_j^{(0)^2} R^2 - Z_j e \phi^{(0)} \right), \tag{12} \]

By using the small-mass ratio \( m_e/m_i \) and quasineutrality, the electrostatic potential satisfies

\[ \mathbf{B} \cdot \nabla \phi^{(0)} = \frac{T_e}{e} \mathbf{B} \cdot \nabla \ln n_i^{(0)} \left( 1 + \frac{Z_i^{(0)}}{Z} \right), \tag{13} \]

where

\[ Z_i^{(0)} = \frac{n_i^{(0)} Z^2}{n_i^{(0)}}. \]

Then, for a two-ion species plasma, equations (11)–(13) yield
\[ \chi(0) \left( 1 + \chi(0) \left( \frac{Z}{T_i} - 1 \right) \right) = \chi(\psi) \exp \left[ \frac{m_i \omega_i^2 R^2}{2 T_i} \left( \frac{T_e + \frac{T_i}{Z}}{T_i + T_e} \right) \right] \]

where \( \mu = m_i/Zm_i \) which equals 2 for a fully ionized impurity. The densities are thus determined by

\[ n_i^{(0)} = n(\psi) \chi(0) \frac{T_i}{2T_i - t_i} \exp \left( \frac{m_i (1 - \mu) \omega_i^2 R^2}{2 \left( T_i - \frac{T_1}{Z} \right)} \right), \quad (15) \]

\[ n_i^{(1)} = \frac{n(\psi)}{Z^2} \chi(0) \frac{ZT_i}{Z^2} \exp \left( \frac{m_i (1 - \mu) \omega_i^2 R^2}{2 \left( T_i - \frac{T_1}{Z} \right)} \right). \quad (16) \]

The flux functions \( n(\psi) \) and \( \chi(\psi) \) can be determined by taking the flux average of equations (15) and (16). Denoting the flux surface average by

\[ \bar{f} \equiv \langle f \rangle_\phi \]

and the impurity strength parameter by

\[ \chi_0 \equiv \frac{n_i Z^2}{n_i}, \]

one finds that \( \chi(\psi) \) is determined implicitly through

\[ \chi_0 = \frac{\left\langle \chi(0) \frac{ZT_i}{ZT_i - t_i} \exp \left( \frac{m_i (1 - \mu) \omega_i^2 R^2}{2 \left( T_i - \frac{T_1}{Z} \right)} \right) \right\rangle_\phi}{\left\langle \chi(0) \frac{T_i}{ZT_i - t_i} \exp \left( \frac{m_i (1 - \mu) \omega_i^2 R^2}{2 \left( T_i - \frac{T_1}{Z} \right)} \right) \right\rangle_\phi} \quad (17) \]

and \( n(\psi) \) is determined by
Here, $\omega^{(0)}(\psi), \tilde{n}_i(\psi),$ and therefore $\alpha_0,$ are assumed to be prescribed. Henceforth, we shall drop the subscript $\psi$ on all flux surface averages, for simplicity.

It is important to note that equations (14)–(18) give the exact solution of the zeroth-order moment equations without assumptions on the inverse aspect ratio $\varepsilon$ or the impurity concentration. When the impurity species is supersonic, i.e. $V \sim v_{\text{thi}} \gg v_{\text{thi}},$ one finds, as shown in Figs 2a–f, that

$$\max(n^{(0)}_i) - \tilde{n}_i \simeq \tilde{n}_i.$$ 

Therefore, the small $\varepsilon$ expansion

\[ n(\psi) = \frac{\tilde{n}_i}{\left< a^{(0)} \frac{\tau_i}{T_i - T_0} \exp \left( \frac{m_i (1 - \mu) \omega^{(0)^2} R^2}{2 \left( \frac{T_i}{Z} \right)} \right) \right>_{\psi}} \]  

(18)

![Graph](image_url)

**Fig. 2.** $O(1)$ solutions for $z = n_i Z^2/n_\psi, n_i,$ and $n_\psi$ (with $\varepsilon = 1/6, Z = 6, x_0 = 1, \mu = 16/6$) vs the poloidal angle $\theta$ (cf. equations (14)–(18)). In Figs 2a–c $\omega R/v_{\text{thi}} = 0.7,$ and in Figs 2d–f, $\omega R/v_{\text{thi}} = 0.9.$
is no longer appropriate. Note that Figs 2a–f refers to a numerical solution of equations (14)–(18), with

$$h_i^{(0)} = \tilde{h}_i + \varepsilon \tilde{h}_i \cos \theta$$

The significance of the above result, as shown in Figs 2, is that the supersonic impurity is strongly pushed out (in the flux surface toward larger $R$) and becomes very dilute for a large domain in the surface. The ion density, on the other hand, varies much more slowly in the surface. Therefore, the friction force will not be able to vanish everywhere on the surface no matter how strong the $i$–$I$ collisional coupling is. Hence, we expect substantial up-down asymmetric density variations, to be shown in the next section.

It is interesting to discuss the hollow profile of the main ion density near the outboard side where $\theta = 0$ (cf. Figs 1–2). By using equations (11) and (13), we obtain
which shows that (i) when \( \tilde{x}^{(0)} \) peaks strongly near \( \theta = 0 \), the term in the large parentheses of (19) becomes negative, and (ii) the degree of the hollowness strongly depends on \( \mu \). This can be understood from the fact that the density distributions of both species are due to the centrifugal force and ambipolar field. The heavier species has the larger centrifugal force, which is proportional to the mass, and therefore has a larger chance to peak near the outboard side. When this peak becomes so high that it induces an ambipolar field stronger than the centrifugal force on the lighter species, the lighter species will be pushed away from the outboard side and peak at the position where the ambipolar field is balanced with the centrifugal force.

So far, the derivation is general. In particular, radial, and thus parallel temperature gradients which may induce thermal friction, and therefore strongly alter radial transport, have not been ruled out. However presently, in the \( O(6,\mu) \) equation, we will neglect these effects and concentrate on the impurity-induced transport in a strongly rotating Tokamak plasma without temperature gradient effects which will be included in a separate work.

2.2. The first-order solution: poloidal flow and up-down asymmetry

First, from equations (9)-(10), the \( O(6,\mu) \) perpendicular moments, the diamagnetic flow and the gyroviscous tensor becomes

\[
\mathbf{V}_{\perp j}^{(1)} = \frac{1}{m_j \Omega_j} \mathbf{b} \times \left( T_j \nabla \ln n_j^{(0)} - \frac{m_j}{2} \omega_j^{(0)} \nabla R^2 + Z_j e \nabla \phi^{(0)} \right), \quad (20)
\]

\[
\Pi_{\perp j}^{(1)} = K_2^{-1} \left[ \frac{n_j^{(0)} T_j}{\omega_j} \right] \mathbf{W}_2 [\mathbf{V}_{\perp j}^{(0)}] \]

\[
= \frac{n_j^{(0)} T_j}{4 \Omega_j} \frac{\partial \omega_j^{(0)}}{\partial \psi} \left[ (\mathbf{Jb} - \mathbf{Be}_\phi) \left( \mathbf{e}_\phi + \frac{3 \mathbf{I}}{B} \mathbf{b} \right) + \frac{1}{B} \nabla \phi \nabla \psi \right] + \mathbf{\Pi}_{\perp j}^{(1)}, \quad (21a)
\]

where \( \mathbf{e}_\phi \equiv R^2 \nabla \phi \) and \( \mathbf{W}_2 \) is defined in equation (A2). Here, \( \Pi_j \) is decomposed into parallel and perpendicular components

\[
\Pi_{|| j} \equiv \mathbf{b} \cdot \Pi_j \cdot \mathbf{b}, \quad (21b)
\]

\[
\Pi_{\perp j} \equiv \Pi_j - \frac{1}{2} \Pi_{|| j} (\mathbf{bb} - \frac{1}{3} \mathbf{I}). \quad (21c)
\]

Moreover,

\[
K_2^{-1}[\mathbf{A}] = \frac{1}{4} [(\mathbf{b} \times \mathbf{A} \cdot (\mathbf{I} + 3 \mathbf{bb})) + \mathbf{\Pi}_{|| j}] \quad (21d)
\]

is the inverse tensor operator (Hsu et al., 1986) of \( K_2 \) defined in equation (A3). We
note that the rank-3 inverse tensor operator $K_3^{(-1)}$ of

$$K_3[A] \equiv A \times b + \tfrac{1}{3}$$

has also been derived in Hsu (1987). By taking the $\nabla \psi$ projection of equation (20) and adopting equation (11), one finds

$$V_j \cdot \nabla \psi \simeq O(\delta_p^2).$$

As mentioned, the general solution of the continuity equation

$$\nabla \cdot n_j V_j = 0$$

in an axisymmetric system has the form, to $O(\delta_p)$,

$$V_j^{(1)} = \frac{K_j^{(1)}(\psi)}{n_j^{(0)}} B + \omega_j^{(1)} R^2 \nabla \phi,$$  \hspace{1cm} (22)

which, by using equations (12) and (20), leads to

$$\omega_j^{(1)}(\psi, \theta) = -\frac{B}{m_j \Omega_j} \left( T_j \frac{\partial}{\partial \psi} \ln N_j(\psi) + \frac{m_j}{2} R^2 \frac{\partial}{\partial \psi} \phi^{(0)} \right).$$  \hspace{1cm} (23)

Here, $K_j^{(1)}(\psi)$ corresponds to the first-order poloidal flow

$$V_{p,j}^{(1)} = \frac{K_j^{(1)}(\psi)}{n_j^{(0)}} B_p$$

and is to be determined.

It will be shown that this species dependent $\omega_j^{(1)}$, which induces the i–I friction, generates the $O(\delta_p)$ poloidal flow and the up–down asymmetric density variations. On the other hand, near the quasi-static state, $\partial / \partial t \simeq O(\delta_p^2)$, the supersonic-impurity toroidal rotation leads to an impurity poloidal flow which is much smaller than the ion poloidal flow, in contrast to the small rotation case. This is due to the fact that $(K_j/n_j)B$ cannot be much larger than its driving term $\omega_j^{(1)} R$ anywhere in the surface, while the extremely large variation of $n_i$, compared with that of $n_i$ and $R$, keeps $(K_i/n_i)B$ very small.

We now proceed to study the $O(\delta_p)$ terms of the parallel momentum equation

$$-m_j n_j^{(0)} \omega_j^{(0)} \omega_j^{(1)} B \cdot \nabla R^2 - \tfrac{1}{2} m_j n_j^{(1)} \omega_j^{(0)} B \cdot \nabla R^2 = B \cdot \nabla \Pi_j^{(1)} + B \cdot \nabla \Pi_j^{(1)} T_j = -Z_j e(n_j^{(0)} B \cdot \nabla \phi^{(1)} + n_j^{(1)} B \cdot \nabla \phi^{(0)}) + BR_{ij}. \hspace{1cm} (24)$$

Here, similar to $\phi^{(0)}$, one has
\[ \mathbf{B} \cdot \nabla \phi^{(1)} = \frac{T_e}{e} \mathbf{B} \cdot \nabla (\ln (n_i + Zn_i))^{(1)} \]

and from equation (21a)

\[ \mathbf{B} \cdot \nabla \cdot \Pi_{ij}^{(1)} = \frac{B}{\Omega_j} T_j I^2 \frac{\partial \omega^{(0)}_{ij}}{\partial \psi} \left( B^2 \mathbf{B} \cdot \nabla \frac{n^{(0)}_{ij}}{B^2} \right) + O \left( \frac{B_p^2}{B^2} \right). \]  

(25)

By using equations (4), (21), (A12) and (A13), the parallel viscosity, in the P–S regime, becomes

\[ \Pi_{||}^{(1)} = -2D_{3j} \frac{n^{(0)}_{jj}}{\nu_{jj}} T_j K_j^{(1)} \left( \frac{\mathbf{b} \cdot \nabla B}{n^{(0)}_{jj}} + \frac{2}{3} \mathbf{B} \cdot \nabla \frac{1}{n^{(0)}_{jj}} \right) \]

(26)

and the parallel friction becomes

\[ BR_{||} = -R_{||} B \simeq +m_i \tilde{v}_{||} D_{||} \left( n^{(0)}_i K_i - n^{(0)}_i K_i \right) B^2 + n^{(0)}_i n^{(0)}_i (\alpha^{(1)}_i - \alpha^{(1)}_i) I. \]

(27)

Note that \( \Pi_{||}^{(1)} \) is different from Braginskii’s result not only in including the i–I collisions but also in carefully including corrections due to the large rotation.

Noticing that \( \omega^{(1)}_i = \omega^{(1)}_i (R, \psi) \), \( n^{(0)}_i = n^{(0)}_i (R, \psi) \) are both up–down symmetric, we can decompose the first-order density \( n^{(1)}_j \) into up–down symmetric (even) and up–down asymmetric (odd) portions

\[ n^{(1)}_j = n^{(1)}_{j+} + n^{(1)}_{j-}. \]

The even part of equation (24), by using equations (19)–(22), yields

\[ T_j n^{(0)}_j \mathbf{B} \cdot \nabla \frac{n^{(1)}_{jj}}{n^{(0)}_{jj}} + \mathbf{B} \cdot \nabla \Pi^{(1)}_{||} - \frac{2}{3} \Pi^{(1)}_{||} \left( \mathbf{b} \cdot \nabla B \right) = -Z_j n^{(0)}_j \mathbf{B} \cdot \nabla \phi^{(1)} + BR^{(1)}_{||}. \]

(28)

In the P–S regime, by using equations (26)–(27) and performing the flux-surface average,

\[ \frac{\tilde{n}_j}{m_i \tilde{v}_{||} B_0} \left\{ \frac{1}{n^{(0)}_j} \text{ equation (28)} \right\}, \]

one obtains the coupled equations for the poloidal flow quantities \( K^{(1)}_j \)

\[ \left( \left< Q_i \right> + \left< \frac{x^{(0)}_i A_i}{x_0} \right> \right) U_i - \left< A_i \right> U_i = \left< E_i \right>, \]  

(29a)

and
\[
\left( \langle Q_i \rangle + \frac{\alpha_0}{\alpha_{\theta}} A_i \right) U_i - \langle A_i \rangle U_i = - \langle E_i \rangle. \tag{29b}
\]

Here,

\[
\begin{align*}
U_j &\equiv K_j^{(1)} B_0 \frac{1}{n_j \rho_{\text{plvhi}}}, \\
A_i &\equiv B_0^2 D_{ii}, \\
E_j &\equiv I \frac{n_i^{(0)}}{B_0} \frac{\rho_{\text{plvhi}}}{\rho_i} D_{ii} (\omega_i^{(1)} - \omega_i^{(1)}) / (\delta_{\text{plvhi}}), \\
Q_j &\equiv \frac{3 n_j T_j}{m_i n_i \rho_{\text{plvhi}}} D_{jj} \left( \frac{n_j}{n_i^{(0)}} B \cdot \nabla \frac{B}{B_0} + \frac{2}{3} B_0^{-1} B \cdot \nabla \frac{n_j}{n_i^{(0)}} \right)^2,
\end{align*}
\tag{29c}
\]

where the coefficients \( D_{ij} \) are defined in the Appendix. \( I(\psi) = R B_0 \) is the usual measure of the toroidal magnetic field, defined through the representation

\[
B = I(\psi) \nabla \varphi + \nabla \varphi \times \nabla \psi.
\]

The flux-surface average is

\[
\langle \cdots \rangle \equiv \frac{1}{V'} \int d\theta J(\cdots),
\]

\( V' \equiv \frac{1}{V} \int d\theta J \), and \( J \equiv |\nabla \varphi \times \nabla \psi \cdot \nabla \theta|^{-1} \) is the Jacobian.

It is now clear that the nonvanishing term \( \omega_i^{(1)} - \omega_i^{(1)} \), which can be written as

\[
\omega_i^{(1)} - \omega_i^{(1)} = - \frac{B}{m_i \Omega_i} \left( T_i - T_j \right) \frac{\partial}{\partial \psi} \ln n(\psi) + \frac{m_i (\mu - 1)}{2} R^2 \frac{\partial}{\partial \psi} \ln n(\psi), \tag{30}
\]

drives the first order nonzero poloidal flow even in an up–down symmetric magnetic configuration, in contrast to the pure plasma case (Connor et al., 1987). \( n(\psi) \) was determined in equation (18).

It is interesting to note that, by summing equation (28) over species and then flux averaging, we obtain

\[
\sum_j T_j \left( n_j^{(0)} B \cdot \nabla \frac{n_j^{(1)}}{n_j^{(0)}} \right) = \frac{3}{2} \sum_j \langle \Pi_{ij}^{(0)} (b \cdot \nabla B) \rangle. \tag{31a}
\]

For a slowly rotating, impure plasma, this reduces to the well-known relation (Hirshman and Sigmar, 1981).
Before proceeding to solve for equation (28), it is noticed in this equation that for each species there exists an odd function \( G_j(\psi, \theta) \) which satisfies

\[
\mathbf{B} \cdot \nabla G_j = \frac{1}{Z_j n_j^{(0)}} (BR_j^{(1)} - B \cdot \nabla \Pi_j^{(1)} + \frac{3}{2} \Pi_j^{(1)} (\mathbf{b} \cdot \nabla B)).
\]  

(32)

Because it is an odd periodic function, we expand

\[
G_j = \sum_{k=1}^{\infty} G_{jk} \sin k\theta,
\]

whence

\[
G_{jk} = \frac{2V'}{k} \left( \frac{BR_j^{(1)} - B \cdot \nabla \Pi_j^{(1)} + \frac{3}{2} \Pi_j^{(1)} (\mathbf{b} \cdot \nabla B)}{Z_j n_j^{(0)}} \right) \cos k\theta.
\]  

(33)

In the P–S regime, using equations (26)–(30), this becomes

\[
G_{ik} = \frac{2V' m \bar{v}_{th i} B_0}{\bar{v}_{th i}} \delta_{pi} \left\{ \left( E_i \cos k\theta \right) + \left( A_i \cos k\theta \right) U_i + \left( \frac{Q_{ik} \sin k\theta - Q_{ik} \cos k\theta}{\chi^{(0)} \alpha_0 A_i \cos k\theta} \right) U_i \right\},
\]  

(34a)

and

\[
G_{ik} = \frac{2V' m \bar{v}_{th i} B_0}{\bar{v}_{th i}} \left\{ - \left( E_i \cos k\theta \right) + \left( A_i \cos k\theta \right) U_i + \left( \frac{Q_{ik} \sin k\theta - Q_{ik} \cos k\theta}{\chi^{(0)} \alpha_0 A_i \cos k\theta} \right) U_i \right\},
\]  

(34b)

where

\[
Q_{ij} = \frac{2\bar{\eta}_j T_j}{m \bar{v}_{th i} \bar{v}_{th j}} D_{3f} J \left( \frac{\bar{\eta}_j}{n_j^{(0)}} b \cdot \nabla \frac{B}{B_0} + \frac{2}{3B_0} \mathbf{B} \cdot \nabla \frac{\bar{\eta}_j}{n_j^{(0)}} \right) \frac{\bar{\eta}_j}{n_j^{(0)}}.
\]  

(34c)

Now, by defining

\[
\bar{\eta}_j = \frac{n_j^{(1)}}{n_j^{(0)}},
\]

equations (28) and (32), after some manipulations, yield
It is then straightforward to obtain the solutions for the odd density components

\[ n_i = \left( \frac{1}{1 + \frac{\alpha(0)}{Z}} \right) \nabla \cdot \mathbf{B} \cdot \nabla n_i = \left( \frac{1}{1 + \frac{\alpha(0)}{Z}} \right) \mathbf{B} \cdot \nabla (G_i - G_1), \]

(35a)

\[ \sum_j (n_j^{(0)} T_j \mathbf{B} \cdot \nabla n_j) + (n_i^{(0)} + Z n_i^{(0)}) T_e \mathbf{B} \cdot \nabla \left( \frac{\nabla n_i^{(0)} + Z n_i^{(0)}}{n_i^{(0)} + Z n_i^{(0)}} \right) = \sum_j (Z n_j^{(0)} \mathbf{B} \cdot \nabla G_j). \]

(35b)

It is then straightforward to obtain the solutions for the odd density components

\[ \dot{n}_i = \left( \frac{1 + \frac{\alpha(0)}{Z}}{1 + \frac{\alpha(0)}{Z}} \right) G_i - \frac{T_e}{T_i} \alpha(0) G_1, \]

(36a)

\[ \dot{n}_1 = Z \left( \frac{1 + \frac{\alpha(0)}{Z}}{1 + \frac{\alpha(0)}{Z}} \right) \frac{T_i}{T_1} + \frac{T_e}{T_1} \frac{T_i}{T_1} \frac{T_1}{T_1} G_1 - \frac{T_e}{T_i} \frac{T_i}{T_1} G_1. \]

(36b)

Summarizing then, we have obtained the solution for \( K_j \) from equations (29) and \( n_j^{(i)} \) from equations (36), namely, the poloidal flows and up–down asymmetric portions of the densities, which have been shown to be essential for the radial transport (Connor et al., 1987; Stacey and Sigmar, 1985). To determine them from given values of \( \omega(0)(\psi), n_j(\psi) \), one needs \( n_j^{(0)} \) whose analytic form is highly nonlinear [cf. equations (14)–(16)]. Nevertheless, it is numerically straightforward to calculate \( n_j^{(0)} \) from equations (14)–(16), and therefore \( \omega_j^{(1)} - \omega_j^{(1)}, K_j, n_j^{(1)} \). The results are shown in Figs 3–4 with the same parameters as given before. The results shown in Figs 3 agree with those of Zanino et al. (1989), in which a particle and momentum conserving time-dependent code is used to describe the poloidal equilibrium evolution.

The ion poloidal flows for both cases are \( O(\delta_p) \) and the impurity poloidal flows are too small to account for. This difference from the rotationless case, in which the ion poloidal flow are usually much smaller than the impurity poloidal flow, can be understood from equations (31). The small increase of the ion poloidal flow for larger rotation is due to the inertial contribution to the diamagnetic flows as shown in equation (30).

From Figs 3b and 3d, one also notices that the odd portion of the impurity density increases with the rotation as expected from equations (36). Moreover, Figs 3 show that not only the first but also the second poloidal harmonic (sin 2θ) dominates the up–down density modulations. This is a result of the nonlinearity arising from the finite value of \( (\omega(0)^2 R^2/v_{th})_e \).

To estimate the magnitude of these results analytically, we assume a strong supersonic impurity and subsonic main ion, i.e.
whence, from equation (15)

$$\frac{n_i^{(0)} - \bar{n}_i}{\bar{n}_i} \leq O(\varepsilon).$$

One finds, from equations (14)-(16),

$$\omega_i^{(1)} - \omega_i^{(1)} \approx - \frac{B}{m_i \Omega_i} \left( \frac{T_i}{Z} - T_i \right) \frac{\partial}{\partial \psi} \ln \bar{n}_i + \frac{m_i (\mu - 1)}{2} (R^2 - \langle R^2 \rangle) \frac{\partial}{\partial \psi} \omega_i^{(0)^2}. \tag{37}$$

The supersonic impurity toroidal rotation makes the coefficient

**FIRST ORDER SOLUTION**

![FIRST ORDER SOLUTION](image)

**Fig. 3.** $O(\delta_p)$ up-down asymmetric modulation of $n_i$ and $n_i$ vs $\theta$ [cf. equations (33)-(36)]. for $\omega R/V_{th} = 0.7$ (Figs 3a-b) and $\omega R/V_{th} = 0.9$ (Figs 3c-d). Also, the magnitudes of the normalized poloidal flows $U_j$ are given [cf. equations (29)]. Other parameters are the same as in Figs 2.
very large. The parallel viscous term, which can be smaller than the friction force in the P–S regime, i.e.

\[ \nabla_v \gg \frac{\nu^2 \vartheta^2}{\vartheta_{ii}} \gg \frac{\nu^2 \vartheta^2}{Z^2 \vartheta_{ii}} \]

can be neglected for simplicity. We then find, neglecting \( O(\epsilon) \) terms, for \( U_i \) defined in (29c),

\[
U_i \sim \frac{BI}{m_i B_0 \Omega_i \delta_{pi} \nu_{th}} \left[ \left(T_i - \frac{T_1}{Z} \right) \frac{\langle n_i^{(0)} \rangle}{n_i} \frac{D_{ii}}{\vartheta_{ii}} \frac{\partial}{\partial \vartheta_{ii}} \ln n_i \right] \sim O(1). \tag{38}
\]

Note that unless the toroidal velocity profile is much steeper than the density profile
or unless

\[ \omega^{(0)} R \gg v_{thi}, \]

equation (38) gives a fair estimate of the normalized ion poloidal flow.

Therefore, we have obtained the poloidal flows and up-down density modulations driven by the relative diamagnetic flow between main ions and impurities (i-I), using the \( \delta_{pi} \) ordering scheme while keeping important nonlinear effects. However, from equations (29) and (34), it is found that

\[ G_i \sim \frac{m_i \bar{n}_i \bar{v}_{ti} B_0 \delta_{pi} v_{thi} q R_0}{\bar{n}_i Z} \varepsilon, \]

which, together with equation (36b), yields

\[ \frac{n_j^{(1)}}{n_j^{(0)}} \sim \frac{\delta_{pi} Z^2 \bar{v}_{ti} \sqrt{2}}{\omega_{ti}} \varepsilon = \Delta \varepsilon \]

as predicted in Section 1. This implies that in the strong ordering regime where

![Fig. 4.—Normalized density modulation and poloidal flow [cf. equations (40)-(44)] for \( Z = 8, A_d = 1, \alpha_0 = 1 \) plotted vs \( \omega_0^2 \) (with \( \xi_i = 1.5, \delta_{pi} = 0.02 \), in Fig. 4a and with \( \xi_i = 0.2, \delta_{pi} = 0.01 \) in Fig. 4c), vs \( \Delta \) (with \( \xi_i = 1.5, \delta_{pi} = 0.02 \), in Fig. 4b).](image)
FIG. 4.—continued
\( \Delta \sim O(1) \), the density modulation will be \( \sim O(\varepsilon) \) and the conventional ordering scheme (Rutherford, 1974), which treats the density to be constant on the surface in the parallel friction, will break down. On the other hand, for the case \( \Delta \sim O(\delta_{pl}) \), one finds \( n_{i} / n_{e} \sim O(\delta_{pl} \varepsilon) \), which will not lead to very strong transport. Furthermore, note that although the strong ordering parameter \( \Delta \) involves a small parameter \( \delta_{pl} \), with the help of \( Z^{2} \), it can be easily of order one in the outer region of the Tokamak plasma. Hence, the strong ordering scheme calculation to be presented in the next section is of particular interest for studies of impurity transport.

### 3. STRONG IMPURITY ORDERING \( \Delta \sim 1 \)

In this section, we consider the strong ordering case \( \Delta \sim 1 \) [where \( \Delta \) is defined in equation (8)]; therefore, the parallel friction term is kept in the zeroth-order of the impurity equation (6) but neglected in the \( O(1) \) ion equation (5). This is because of the smallness of \( n_{i} / n_{e} = \varepsilon / Z^{2} \ll 1 \). Note that, even for a light impurity such as \( C^{-6} \) or \( O^{+8} \), the ordering \( \Delta \sim 1 \) in the P–S regime is still appropriate. Now, taking the parallel projection of the \( O(1) \) terms, equations (5) and (13) yield

\[
\mathbf{B} \cdot \nabla \ln n_{i} = \frac{m_{i} \omega^{0}\mathbf{R}^{2}}{2(T_{i} + T_{e})} \mathbf{B} \cdot \nabla \ln \left( \frac{1 + \alpha^{0}}{Z} \right),
\]

and the subtraction of equation (6) from (5) yields

\[
\frac{m_{i}}{2} (\mu - 1) \omega^{0} \mathbf{R}^{2} + \left( T_{i} - \frac{T_{i}}{Z} \right) \mathbf{B} \cdot \nabla \ln n^{(0)} - \frac{T_{i}}{Z} \mathbf{B} \cdot \nabla \ln \alpha^{(0)} = \sqrt{2m_{i}Z_{ii}D_{ii}} \left\{ \mathbf{B}^{2} \left( \frac{K_{i}^{(1)}}{n_{i}^{(0)}} - \frac{K_{i}}{n_{i}} \right) - \frac{BI}{m_{i} \Omega} n_{i}^{(0)} \right\} + \frac{m_{i}}{2} (\mu - 1) \omega^{0} \frac{\partial \mathbf{R}^{2}}{\partial \psi} + \left( T_{i} - \frac{T_{i}}{Z} \right) \frac{\partial}{\partial \psi} \ln n^{(0)} - \frac{T_{i}}{Z} \frac{\partial}{\partial \psi} \ln \alpha^{(0)}. \]

Here, the right-hand side of equation (39b) is \( \mathbf{B} \cdot \mathbf{R}_{ii} / n_{i} Z \). It is also useful to combine equations (39a) and (39b) into

\[
\frac{m_{i}}{2} \left( \mu - 1 + \frac{ZT_{i} - T_{i}}{Z(T_{i} + T_{e})} \right) \omega^{0} \mathbf{R}^{2} = \frac{T_{i}}{Z} \mathbf{B} \cdot \nabla \left( \frac{1}{1 + \frac{T_{i}}{ZT_{i}}} \ln \left( 1 + \frac{\alpha^{(0)}}{Z} \right) + \frac{T_{i}}{T_{i}} \ln \alpha^{(0)} \right) = \frac{\mathbf{B} \cdot \mathbf{R}_{ii}}{n_{i} Z}. \]

Note that equations (39) form a closed 2-D nonlinear system, which can be reduced to a 1-D system by assuming that \( (\partial / \partial \theta) f(\psi, \theta) \sim r_{\psi} (\partial / \partial r) f(\psi, \theta) \), that is,
where \( r_n \) is the radial length scale which, for the example of a parabolic profile, can be taken as

\[
r_n = \frac{a^2}{2r}.
\]

Therefore, \( \frac{\partial}{\partial \psi} \ln \tilde{n}_i^{(0)} \) and \( \frac{\partial}{\partial \psi} \ln \rho_i^{(0)} \) on the right-hand side of equation (39) can be replaced by \( \frac{\partial}{\partial \psi} \ln \tilde{n}_i \) and \( \frac{\partial}{\partial \psi} \ln \rho_i \) which become pure driving terms. Then the resulting more tractable 1-D equation can be solved by using a similar procedure as in Section 2.2 to include the nonlinear effects due to finite \( \varepsilon \) and strong rotation via higher harmonics as indicated in Section 2.2.

It is important to note here that in addition to the effects of higher harmonics, there is another nonlinear mechanism involved in equation (39), that is, the nonlinear coupling between density modulation and poloidal flows due to strong ordering. This can be understood by observing on the right-hand side of equation (39b) that the density can no longer be treated as constant on the surface when \( \Delta \sim O(1) \) as discussed before. Consequently, the poloidal flows \( K_j B_p/E_i \) for both species are strongly coupled to the density modulations, and are needed to be calculated self-consistently and individually, in contrast to the conventional ordering scheme (RUTHERFORD, 1974).

In this section, we will not consider the effects of higher harmonics, but rather concentrate on the nonlinear effects due to strong ordering.

### 3.1. Solutions

To further proceed with the analytic study, we suppress the higher harmonic terms from the equations by assuming \( \varepsilon \ll 1 \) and neglecting terms which are obviously algebraically \( O(1/Z) \) smaller than others, for simplicity. Equations (39) thus yield, to \( O(1) \)

\[
U_i - U_i = A_d \equiv -\frac{r_n}{2} \left\{ \frac{\partial}{\partial r} \ln \tilde{n}_i - \frac{T_i}{Z T_i} \frac{\partial}{\partial r} \ln \tilde{n}_i \right\},
\]

and to \( O(\varepsilon) \)

\[
\frac{\partial}{\partial \theta} y - a_d \sin \theta + a_d \left( 2A_d - \frac{r_n}{Z} \frac{r (\mu - 1) \omega_0^2}{Z} \right) \cos \theta.
\]

Here, the normalized coefficients are defined by

\[
\omega_0^2 \equiv \frac{\omega(0)^2 R_0^2}{v_{thi}^2}, \quad y \equiv \frac{\rho(0) - \rho_0}{\rho_0 \varepsilon}, \quad U_j \equiv \frac{K_j^{(1)} B_0}{\tilde{n}_j \delta_{pi} v_{thi}}, \quad \delta_{pi} \equiv \frac{m_i c v_{thi}}{e B_p r_n},
\]

\[
K_j^{(1)} B_0 \equiv \frac{m_i c v_{thi}}{e B_p r_n},
\]

where \( r_n \) is the radial length scale which, for the example of a parabolic profile, can be taken as

\[
r_n = \frac{a^2}{2r}.
\]
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\[ a_\Delta \equiv \frac{2\Delta D_{iL}}{\left(1 - \frac{T_i}{ZT_i}\right) \frac{T_e}{T_i + T_e} \alpha_0 + \frac{T_i}{T_i}}, \quad (40d) \]

and

\[ b_{\varphi} \equiv \frac{2\left(\mu - 1 + \frac{ZT_i - T_i}{Z(T_i + T_e)}\right)\omega_0^2}{\left(1 - \frac{T_i}{ZT_i}\right) \frac{T_e}{T_i + T_e} \alpha_0 + \frac{T_i}{T_i}}, \quad (40e) \]

such that the toroidal flow will be of the order of the impurity thermal velocity for \( \omega_0 \approx 1 \), the density modulation will be \( \approx O(\varepsilon) \) for \( y \approx O(1) \), and poloidal flows will be \( \approx O(\rho/a) \) if \( U_j \approx O(1) \). Note that the retaining of \( 1/Z \) terms in equations (40) is to allow for (i) arbitrary radial gradients for each species, and (ii) large toroidal rotation especially near the center of the plasma. In particular, the radial density profile of impurity can evolve toward \( Z \) times sharper than that of main ion; in addition, the term \( (r_n/r)[(\mu - 1)\omega_0^2/Z] \) can become dominant near the magnetic axis where \( r_n/r \gg 1 \), even for moderate rotation.

The solution for the impurity strength parameter \( y \) defined in (40c), equation (40b), is thus

\[ y = y_c \cos \theta + y_s \sin \theta \quad (41a) \]

with

\[ \begin{align*}
y_c &= y_{\varphi} - a_\Delta^2 U_i \left(2\Delta D_{iL} \frac{r_n}{r} \frac{(\mu - 1)\omega_0^2}{Z} \right) \left(1 + a_\Delta^2 U_i^2 \right), \quad (41b) \\
y_s &= a_\Delta \left[ b_{\varphi} U_i + 2\Delta D_{iL} \frac{r_n}{r} \frac{(\mu - 1)\omega_0^2}{Z} \right] \left(1 + a_\Delta^2 U_i^2 \right). \quad (41c) \end{align*} \]

Here, \( y_c \) measures the in–out density modulation and \( y_s \) measures the up–down density modulation. The ion density modulation can then be derived from equations (39a) and (41), and the impurity density modulation is obtained from

\[ \frac{\tilde{n}_I(\theta)}{\varepsilon \tilde{n}_i} = y + \frac{\tilde{n}_I(\theta)}{\varepsilon \tilde{n}_i} \approx y + O\left(\frac{1}{Z}\right). \quad (41d) \]

The importance of the parameter \( \Delta \), measuring the strength of the parallel friction, can be seen from the dependence of \( y \) on \( a_\Delta \). That is, if \( \Delta \sim \delta_{pi} \ll 1 \), then the \( O(1) \) density modulation, which reduces to the \( O(1) \) solution in Section 2 for small \( \varepsilon \), is
up-down symmetric and is driven purely by the centrifugal force via \( b_\omega \). On the other hand, for finite \( \Delta \), \( y_c \) and \( y_s \) are strongly coupled to each other. It is interesting to note here that the strong ordering \( \Delta \sim 1 \) does not explicitly depend on the impurity concentration, and therefore equations (41) are still valid and useful for impurity species with \( \alpha \ll 1 \); however, the ion dynamics, e.g. \( U_i, n_i \), reduce to that of a pure plasma system. In this case, one may expect that both \( y_c \) and \( y_s \) increase with rotation through \( b_\omega \), as predicted from a simple argument: since \( y_c \) is basically driven by centrifugal force and \( y_s \) basically by parallel friction, increasing rotation increases \( y_c \), and increasing \( y_c \) induces an increasing parallel friction which then drives an increasing \( y_s \). However, this simple concept will not be relevant for impurities of strength \( \alpha \sim 1 \), because the ion poloidal flow will be strongly coupled to the impurity flow through friction. In other words, simply assuming the ion poloidal flow to be driven by the temperature gradient as in a pure plasma system (Chang and Hazeltine, 1980) is totally inadequate for an impure plasma. In fact, it will be shown later that by self-consistently evaluating \( U_i \), one finds that \( y_s \) decreases with increasing rotation for \( \alpha \sim 1 \) and \( \omega_0 > 1 \). Moreover, by taking the limit that \( \Delta \ll 1 \) and \( \omega_0 \ll 1 \), the expression for \( y_s \) in equation (41c) becomes independent of normalized poloidal flow \( U_i \), as in the theory using conventional ordering (Rutherford, 1974).

To evaluate the ion poloidal flow self-consistently, one needs to utilize the \( O(\delta_m) \) ion momentum equation (5) which yields

\[
\left\langle \frac{\mathbf{B} \cdot \nabla \cdot \Pi_{11}}{n_i^{(0)}} \right\rangle = \left\langle \frac{\mathbf{B} \cdot \mathbf{R}_{1i}^{(1)}}{n_i^{(0)}} \right\rangle,
\]

where, using equation (39c),

\[
\left\langle \frac{\mathbf{B} \cdot \mathbf{R}_{1i}^{(1)}}{n_i} \right\rangle = \frac{m_i \omega_i^{(0)2}}{2Z} \left( \mu - 1 + \frac{Z T_i - T_e}{Z(T_i + T_e)} \right) \left\langle \alpha_i \mathbf{B} \cdot \nabla R^2 \right\rangle. \tag{43}
\]

Note that, using the fact that equation (23) is still valid for ions in the strong ordering case, \( \left\langle \mathbf{B} \cdot (\mathbf{V}_i \cdot \nabla \mathbf{V}_i) \right\rangle \) has been neglected from equation (42). Hence, by using equations (21), (25), (26) and (43), equation (42) yields

\[
\xi_i a_\Delta (U_i - A_d) \left\{ \left( 1 + \frac{4 \omega_i^2 T_i}{3Z(T_i + T_e)} - \frac{2 \alpha_0 T_e}{3Z(T_i + T_e)} \right) y_c^2 + \left( \frac{2 \alpha_0 T_e}{3Z(T_i + T_e)} \right)^2 y_s^2 \right\} = -\alpha_0 (g_\omega + b_\omega) y_s, \tag{44}
\]

where

\[
\xi_i \equiv \frac{3}{2\sqrt{2}} \frac{D_{ii}(\alpha_0) \omega_i^2}{D_{ii}(\alpha_0) \bar{\nu}_{ii}^2}
\]

measures the parallel viscosity [e.g. \( \xi_i \sim (\omega_i/\bar{\nu}_{ii})^2 \) for \( \alpha = 1 \), and
represents the gyroviscosity which can become important only when the toroidal rotation is of $O(\delta_{pl})$.

Therefore, the normalized ion poloidal flow $U_i$ and the impurity density modulation $y$ can be determined straightforwardly by solving equations (41) and (44). The results are given in Figs 4a–c as functions of parameters $\omega_0$, and $\Delta$ defined after equations (40) and (44), for $Z = 8$, $x = 1$ and $A_d = 1$. From equation (44) and Fig. 4c, it is noticed that the results are sensitive to the parameter $\xi$, arising from parallel viscosity. Since our derivation of the parallel viscosity (cf. the Appendix) is based upon the assumption of high collisionality, it is adequate to set $\xi_1 = 0.2$ (with $\delta_{pl} = 0.01$) as in Fig. 4c. However, the main ion in a typical Tokamak plasma is mainly in the plateau regime. By analogy with neoclassical transport theory in the small rotation case (Hirshman and Sigmar, 1981), we conjecture that taking $\xi_1 \approx O(1)$ will give a reasonable estimate for the main ion plateau regime with $\nu_i/\nu_o \approx O(1)$. The results for $\xi_1 = 1.5$ with $\delta_{pl} = 0.02$ are given in Figs 4a–b.

3.2. Limiting cases

Although there is no direct validity restriction on the value of rotation in equations (41b, c) and (44), the linearization leading to equations (40) comes from a large aspect ratio assumption relying on the smallness of $(\omega_0^2 R^2/\nu_0^2)(\delta_{pl}) = (\mu_0 \delta_{pl})$. This implies that if $\mu_0 \delta_{pl}$ becomes finite, equations (41) and (44) are no longer valid, and as indicated in Section 2 the nonlinear effects via higher poloidal harmonics of $\ln n_{i(0)}$ and $\ln \lambda_{i(0)}$ in equations (39) become significant. In that case, one needs to solve the nonlinear equations numerically. However, this case will not be included in the present work.

To get the limiting solution for very large rotation ($\omega_0 \gg 1$) and finite $\alpha_0$, equations (40), (41) and (44) yield

$$U_i \approx - A_d + \frac{r_n}{r} \frac{(\mu - 1) \omega_0^3}{b_o Z} + O\left(\frac{1}{\omega_0^2}\right),$$

$$y_s \approx \frac{\xi_1 a_\Delta (A_d - \frac{r_n}{r} \frac{(\mu - 1) \omega_0^3}{b_o Z})}{\alpha_0 (g_{\omega} + b_o)} \left(1 + \frac{4 \omega_0^3}{3} \frac{T_i}{Z (T_i + T_e)} \left(1 - \frac{T_1}{T_i} \right) \frac{T_e}{T_i} \frac{\alpha_0 + T_e}{T_i} \frac{T_1}{T_i} \right)^2.$$
That is, $y_s$ increases with $\omega_0^2$ for small rotation. The turning point at which $y_s$ starts to decrease with rotation occurs near the point

$$b_\omega U_1 - \frac{r_n}{r} \frac{(\mu - 1)}{Z} \omega_0^2 = 0.$$  

For $A_d \sim 1$ and $Z \gg 1$, this occurs when $U_1$ changes sign, i.e. when the impurity poloidal flow reverses direction (cf. Fig. 4a).

Moreover, it is of particular interest to study the case of a heavy test impurity (i.e. $\alpha_0 = f_{\text{He}} Z^2 / \tilde{n}_i < 1$) using our strong ordering scheme. This is because the ordering parameter $\Delta = \delta_{\text{He}} Z^2 \sqrt{2 \bar{v}_i} / \omega_{\text{He}}$ is independent of $\alpha_0$ but is proportional to $Z^2$. It is worth mentioning that although equations (39a) and (44) do not give an adequate description of the ion dynamics for $\alpha < \delta_{\text{He}}$, they nonetheless provide the correct description to $O(\delta_{\text{He}})$ in the limit $\alpha_0 = 0$. One obtains

$$U_1 = 0 \quad \text{and} \quad n_i \ll e^{[m_{\text{He}}/m_i] \frac{R^2}{2 (1 + T_i)}, \quad (47a)$$

as predicted by other authors (HINTON and WONG, 1985; CONNER et al., 1987). Also, the impurity density modulation can be determined by taking $\alpha_0 = 0$ from equation (41), which yields

$$y_s = \frac{a_\Delta \left[ A_d(2 + b_\omega) - \frac{r_n}{r} \frac{(\mu - 1)}{Z} \omega_0^2 \right]}{1 + a_\Delta^2 A_d^2}. \quad (47b)$$

With $A_d \sim 1$ and $Z \gg 1$, this implies that $y_s$ increases with $\omega_0^2$ as shown in Fig. 5.

### 3.3. Bifurcated poloidal equilibrium

As has been indicated in the Introduction and as will be shown explicitly in the next section, neoclassical transport is directly driven by the (up/down) poloidal asymmetry of the density profiles. Poloidal flow equilibrium is established in the transit time scale. This equilibrium is described by the solutions of poloidal density modulations and poloidal flows studied in the preceding and the present sections. Since the density modulations and poloidal flows are found strongly coupled in the strong ordering regime, one would expect that there may exist multiple solutions in this regime.

Indeed, equations (41) and (44) appear to form a cubic algebraic equation for $U_1$. In Fig. 6a, we take $\zeta_i = 1, A_d = 1, \alpha_0 = 1, \Delta = 1.5, \omega_0^2 = 0.5$, and plot both sides of equations (44) vs $U_1$. Note that the left-hand side corresponds to the parallel viscous force and the right-hand side corresponds to the parallel frictional force, respectively.
It is important to point out two crucial quantities in Fig. 6a: (1) the intercept with the horizontal axis at vanishing parallel viscous force, i.e. $A_d$; (2) the width of the peak for the curve of the parallel frictional force, i.e. $1/a_\Delta$. To have multiple roots from balancing these two forces requires that the intercept be much larger than the width; i.e. $A_d a_\Delta \gg 1$. In Fig. 6b, we take $\zeta_i = 1, A_d = 1, \alpha_0 = 1, \Delta = 6, \omega_0^2 = 0.5$, and indeed find multiple roots.

The other important control parameter of course is the toroidal $E \times B$ rotation measured by $\omega_0$, since it obviously controls the height of the peak. The asymptotic behavior of the solutions for small and large rotation is described in equations (45) and (46). In Fig. 6c, we present the impurity poloidal equilibrium rotation vs $\omega_0^2$ obtained by choosing the same $\zeta_i, A_d, \alpha$ and $\Delta$ as in Fig. 6b to further illustrate the existence of multiple-root regions. Note that the unstable root in between the other two root branches is omitted in Fig. 6c.

The significance of Fig. 6 is the bifurcation of the poloidal equilibria. One observes that (1) in the multiple-root region, there exist two stable poloidal equilibria, and after a deviation from one equilibrium state due to some disturbance, the system can choose to go to either one of the stable equilibrium states depending on the frequency and amplitude of the disturbance; (2) near the boundaries between regions of single root and multiple roots, a moderate change in the control parameters such as $A_d$ (corresponding to radial profile scale), $a_\Delta$ [defined in equation (40d)], and $\omega_0$ (corresponding to toroidal rotation due to radial electric field), can force the system to jump from one poloidal equilibrium to another within the transit time scale. In the present calculation, the existence of bifurcation requires a large $A_d a_\Delta$, i.e. very steep
FIG. 6.—In Figs 6a and b, normalized $||$ viscous and frictional forces are plotted vs normalized impurity poloidal flow $U_i$, for $\omega_i = 0.5$. In Fig. 6c, equilibrium impurity poloidal flow $U_i$ plotted vs $\omega_i$. Here, $Z = 8$, $A_d = 1$, $a_0 = 1$, $\xi = 1$. Also, $\Delta = 1.5$ in Fig. 6a; and $\Delta = 6$ in Figs 6b and c.
Concerning the radial transport, not only can the neoclassical transport but also the turbulence-driven anomalous transport be strongly affected by the poloidal equilibrium. That is, if the time scale for establishing turbulence is much longer than that for establishing poloidal equilibrium, the turbulence structure will be established upon the poloidal equilibrium. More explicitly, if the equilibrium contains a large poloidal variation, to calculate the turbulence structure, one should not consider the equilibrium quantities (such as densities and/or temperatures) to be constant on the flux surface, nor should one neglect the equilibrium poloidal flows. Thus the bifurcation of the poloidal equilibrium is likely to alter the transport behavior of the plasma promptly. In this work, however, we restrict ourselves to the development of the nonlinear neoclassical transport.

4. NEOCLASSICAL TRANSPORT

In this section, we evaluate the radial transport flux using the moment approach and the strong ordering results obtained in Section 3. The detailed formalism of the moment approach to the transport theory of magnetized plasma is given elsewhere (Hsu and Sigmar, 1988); here, we only briefly describe the approach and present the results.
Due to the neglect of temperature variations, only the particle diffusion and toroidal momentum damping are considered. We start with the moment equations

\[
\frac{\partial}{\partial t} n_j + \nabla \cdot n_j \mathbf{v}_j = N_j, \quad (48a)
\]

\[
\frac{\partial}{\partial t} m_j n_j \mathbf{v}_j + \nabla \cdot (m_j n_j \mathbf{v}_j \mathbf{v}_j + \mathbf{p}_j) = n_j \mathbf{e} Z_j \mathbf{e} + m_j n_j \Omega_j \mathbf{v}_j \times \mathbf{b} + \mathbf{r}_j + \mathbf{m}_{ij}, \quad (48b)
\]

\[
\frac{\partial}{\partial t} (\mathbf{p}_j + m_j n_j \mathbf{v}_j \mathbf{v}_j) + \nabla \cdot \mathbf{q}_j = n_j Z_j \mathbf{e} (\mathbf{e} \mathbf{v}_j + \mathbf{v}_j \mathbf{e}) + (\mathbf{r}_j \mathbf{v}_j + \mathbf{v}_j \mathbf{r}_j)
\]

\[+ \Omega_j[(\mathbf{p}_j + m_j n_j \mathbf{v}_j \mathbf{v}_j) \times \mathbf{b} + \text{transpose}] + \mathbf{r}_{ij} + \frac{1}{2} \mathbf{q}_{ij}^2 \mathbf{i} + \mathbf{p}_{ij}. \quad (48c)
\]

Here, \(N_{ij}, \mathbf{M}_{ij}\) and \(\mathbf{P}_{ij}\) are the source input to particle, momentum, and stress tensor. We define

\[
\mathbf{q}_j = \int \mathbf{v} m_j (\mathbf{v} + \mathbf{v}_j)(\mathbf{v} + \mathbf{v}_j) f_j \quad (48d)
\]

(with \(\mathbf{v}\) the particle velocity in the rest frame of the fluid moving with velocity \(\mathbf{v}_j\))

\[
\mathbf{P}_{ij} = \int \mathbf{v} m_j \left(\mathbf{v} \mathbf{v} - \frac{\mathbf{v}^2}{3} \mathbf{i}\right) C_j(f_j), \quad (48e)
\]

and

\[
\mathbf{Q}_{ij} = \int \mathbf{v} \frac{m_j v^2}{2} C_j(f_j). \quad (48f)
\]

\(C_j\) is the collision operator, \(\mathbf{R}_j\) is the collisional interspecies friction, and \(\mathbf{P}\) is the pressure tensor. Hence, the particle conservation for species \(j\) is, upon flux averaging,

\[
\frac{\partial}{\partial t} \langle n_j \rangle + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \Gamma_j = \langle N_{ij} \rangle, \quad (49a)
\]

and the flux surfaced toroidal momentum conservation is

\[
\sum_j \left(\frac{\partial}{\partial t} (m_j n_j \mathbf{v}_j \cdot \mathbf{e}_\phi) \right) + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \sum_j \mathbf{P}_{ij} = \sum_j \langle \mathbf{M}_{ij} \cdot \mathbf{e}_\phi \rangle + \frac{1}{c} \langle \mathbf{J} \cdot \nabla \psi \rangle. \quad (49b)
\]

Here, we have introduced the contravariant quantities \(\mathbf{e}_\phi \equiv R^2 \nabla \phi\),

\[
\Gamma_j \equiv \langle n_j \mathbf{v}_j \cdot \nabla \psi \rangle \quad (50)
\]

the particle flux, and
the toroidal angular momentum flux. Concerning the radial current on the right-hand side of equation (49b), it has been shown by many authors (Hsu and Sigmar, 1988; Connors et al., 1987) that

\[ \frac{1}{c} \langle \mathbf{J} \cdot \nabla \psi \rangle \approx \frac{v_{A}^{2}}{c^{2}} \frac{\partial}{\partial t} \langle m_{j} n_{j} \mathbf{V}_{j} \cdot \mathbf{e}_{\varphi} \rangle \]

which is therefore negligible in low beta plasmas.

Using the identities

\[ \langle n_{j} \mathbf{V}_{j} \cdot \nabla \psi \rangle = \langle n_{j} \mathbf{V}_{j} \times \mathbf{B} \cdot \mathbf{e}_{\varphi} \rangle \]

\[ \langle \mathbf{e}_{\varphi} \cdot \mathbf{\Pi} \cdot \nabla \psi \rangle = \frac{1}{2} \langle \mathbf{e}_{\varphi} \mathbf{e}_{\varphi} \cdot (\mathbf{\Pi} \times \mathbf{B} + \text{transpose}) \rangle, \]

and the largeness of \( \Omega \) in a magnetized plasma, \( (B/m_j \Omega_j)\langle e_\varphi \cdot \text{equation (48a)} \rangle \) yields

\[ \Gamma_j = - \frac{B}{m_j \Omega_j} \langle \mathbf{R}_{j} \cdot \mathbf{e}_{\varphi} \rangle = - \frac{IB}{m_j \Omega_j} \left( \frac{R_{||}}{B} \right) - \frac{B}{m_j \Omega_j} \langle \mathbf{R}_{\perp j} \cdot \mathbf{e}_{\varphi} \rangle \]

and similarly \( (B/2\Omega_j)\langle e_\varphi \cdot \text{equation (48c)} \rangle \) yields

\[ \Pi_j = - \frac{B}{\Omega_j} \left\{ \omega^{(0)} \left( \frac{R^2}{B} R_{||} \right) + \left( \frac{3I^2 - B^2 R^2}{4B^2} \Pi_{||j} \right) \right\} - \frac{B}{\Omega_j} \left\{ \omega^{(0)} \langle R^2 \mathbf{R}_{\perp j} \cdot \mathbf{e}_{\varphi} \rangle + \frac{1}{2} \langle \mathbf{e}_{\varphi} \cdot \mathbf{\Pi}_{\perp j} \cdot \mathbf{e}_{\varphi} \rangle \right\}. \]

Here, the terms involving the parallel collisional moments, such as \( R_{||} \) and \( \Pi_{||j} \), correspond to the neoclassical fluxes, and terms such as \( \mathbf{R}_{\perp j} \) and \( \Pi_{\perp j} \), correspond to the classical fluxes. Here, we defined

\[ \Pi_{||j} \equiv \mathbf{b} \cdot \Pi_{ej} \cdot \mathbf{b}, \]

\[ \Pi_{\perp j} \equiv \Pi_{ej} - \Pi_{||j} \left( \mathbf{b} \mathbf{b} - \frac{1}{3} \mathbf{I} \right). \]

The classical fluxes, which are smaller than the neoclassical terms by a factor \( B^2 / B^2 \), can be determined straightforwardly using equations (52)–(53), (A8), (A15), (22) and (21a); that is

\[ \Gamma^{\text{class}}_j = \frac{1}{Z_j} \frac{B}{\Omega_i} \left( n_{i} v_{i} \frac{|\nabla \psi|^2}{B^2} (\omega^{(1)} - \omega^{(1)}) \right), \]
\[ \Pi^{\text{class}} = \sum_{j=1}^{\infty} \Pi_j^{\text{class}} = (\mu - 1) \frac{B}{\Omega_i} \omega_i^{(0)} \left\{ m_i n_i v_i R^2 \frac{|\nabla \psi|^2}{B^2} (\omega_i^{(1)} - \omega_i^{(1)}) \right\} \]
\[ - \frac{3}{5} \left( \frac{B}{\Omega_i} \right)^2 \frac{\partial \omega_i^{(0)}}{\partial \psi} T_i \left\{ n_i (v_i + v_{ii}) R^2 \frac{|\nabla \psi|^2}{B^2} \right\}, \quad (55) \]

where \( m_i/m_n, Z \gg 1 \) has been used. It is also worth mentioning here that equations (52) and (53) agree with results obtained in the standard \( \delta_p \) ordering scheme, and all the terms omitted are carefully justified as negligible when considering the strong ordering \( \Delta \sim 1 \) by noticing that \( Z \gg 1 \). For instance, in deriving equation (52),

\[ \left| \frac{1}{V'} \frac{\partial}{\partial \psi} \left\langle V' (e_\nu \cdot (m_j n_j v_j + P_j) \cdot \nabla \psi) \right\rangle \right| \sim \delta_{pi} \frac{\omega_i^{(0)} R_0}{v_{thi}} \ll 1 \]

has been used to drop the nonambipolar piece of the particle flux.

For neoclassical fluxes, using the parallel collisional moments derived in the Appendix, the small \( \varepsilon \) expansion and equation (39c), equations (52), (53) yield the neoclassical particle flux

\[ \Gamma_i^{\text{neo}} = -Z \Gamma_i^{\text{neo}} = \frac{I_{V_{thi}}}{2 \Omega_\nu} \omega_\nu \tilde{n}_i e^2 \left[ \left( 1 - \frac{T_1}{ZT_i} \right) \frac{T_e \alpha_0}{T_i + T_e} + \frac{T_1}{T_i} \right] \left( 1 + \frac{b_{\omega}}{2} \right) y_i, \quad (56) \]

and the neoclassical momentum flux

\[ \Pi^{\text{neo}} = \Pi_i^{\text{neo}} + \Pi_{e|i}^{\text{neo}} = \frac{I_{V_{thi}}}{\Omega_\nu} \omega_\nu \tilde{n}_i \omega_i^{(0)} R_0^2 \varepsilon^2 \]
\[ \times \left\{ (\mu - 1) \left[ \left( 1 - \frac{T_1}{ZT_i} \right) \frac{T_e \alpha_0}{T_i + T_e} + \frac{T_1}{T_i} \right] \left( 1 + \frac{b_{\omega}}{4} \right) + \frac{1}{3} \frac{T_e}{T_i + T_e} \delta_{pi} \frac{Z^{-3/2} U_i}{\omega_\nu} \right\} y_i. \quad (57) \]

Thus, the neoclassical fluxes appear to be driven explicitly by the up-down density modulation \( y_i \), as expected from the heuristic prediction in the Introduction. That is, neoclassical transport arises from the radial drift of particles due to the poloidal gradient forces and only the contribution from the up-down variations of density (and/or temperature, if temperature variations are included) survive the flux-surface average. Note that the second term in equation (57), coming from the term involving \( \Pi_{e|i} \) in equation (53), contains the ion gyroviscosity [cf. the term involving \( n_j T_j V^2 \) on the left-hand side of equation (A9)]. On the other hand, the contribution from \( \Pi_{e|i} \), which contains the impurity gyroviscosity, is found to be \( 1/Z \) smaller than that from the ion gyroviscosity, and is thus neglected to yield equation (57). The significance of this gyroviscous contribution as studied in STACEY and SIGMAR (1985) will be discussed further in Section 5.
One can then estimate the particle confinement time $\tau_n$ and angular momentum confinement time $\tau_\phi$ by

$$\frac{1}{\tau_n} \equiv \left| \frac{1}{V'} \frac{\partial}{\partial \psi} V' \Gamma_j \right| \approx \frac{n_i}{Z_j n_j} \frac{\delta p_i e^2 L_n \omega_n}{\omega_n} \left( 1 - \frac{T_i}{ZT_i} \right) \frac{T_e \gamma_0}{T_i + T_e + T_i} \left( 1 + \frac{b_\omega}{2} \right) \gamma_j, \quad (58)$$

and

$$\frac{1}{\tau_\phi} \equiv \left| \frac{1}{V'} \frac{\partial}{\partial \psi} V' \Pi \right| \approx \frac{n_i}{m_i n_i} \frac{\delta p_i e^2 L_o \omega_o}{\omega_o} \left( \mu - 1 \right) \left( 1 - \frac{T_i}{ZT_i} \right) \frac{T_e \gamma_0}{T_i + T_e + T_i} \left( 2 + \frac{b_\omega}{2} \right)$$

$$+ \frac{2}{3} \frac{T_e}{T_i + T_e + T_i} \delta p L_x \frac{U_i}{\omega_o} \gamma_j, \quad (59)$$

with the profile factors

$$L_{nj} \equiv \left| \frac{r_n}{2r} \frac{\partial}{\partial r} r \ln (V' \Gamma_j) \right| \quad \text{and} \quad L_\phi \equiv \left| \frac{r_n}{2r} \frac{\partial}{\partial r} r \ln (V' \Pi) \right|.$$

Note that the classical contributions are neglected from equations (58) and (59) for simplicity.

By noticing that both $1/\tau_n$ and $1/\tau_\phi$ scale as $(n_i/n_j) \delta p_i e^2 \omega_j$, one finds two interesting features: (i) for finite $\omega_0$, $\Delta$ and $\xi_i$ [defined in equations (40) and (44)], the momentum damping rate is of the same order of magnitude as the ion particle diffusion rate which agrees with the experimental observation; (ii) the damping rate scaling $(n_i/n_j) \delta p_i e^2 \omega_j$ coincides approximately with that of the “gyroviscous theory” (STACEY and SIGMAR, 1985; STACEY and MALIK, 1989) (if it was assumed that impurity gyroviscosity prevails over ion gyroviscosity owing to the strong impurity density modulation). However, it is obvious that the physical origins in the present work are of collisional nature. (This will be discussed in more detail in Section 5.)

In Figs 7a–b, the normalized damping rates $(\tau_n \omega_i (n_i/n_j) \delta p_i e^2 L_m)^{-1}$ and $(\tau_\phi \omega_i (n_i/n_j) \delta p_i e^2 L_o)^{-1}$ are presented as functions of parameters $\omega_0$ and $\Delta$. It is also interesting to note here, from the observation of Fig. 6b, that $(\tau_n \omega_i (n_i/n_j) \delta p_i e^2 L_m)^{-1} \approx 4\Delta$ which leads to a damping rate scaling $1/\tau_\phi \approx 4\Delta^2 \delta p_i e^2 L_o \gamma_j$. Thus, if one takes $\gamma_j = \sqrt{2} \omega_0 \delta j \approx \gamma_i$ and $q^2 \approx 10$, then an enhancement of the momentum damping rate by two orders of magnitude over the previously derived classical damping rate (CONNOR et al., 1987) (using Braginskii’s perpendicular viscosity tensor in a pure plasma), is not difficult to find.

Since the radial fluxes are explicitly proportional to $\gamma_j$, their limiting behavior at large or small rotation is similar to that of $\gamma_j$, discussed in Section 3. In addition, the particle diffusion flux for a test impurity also behaves quite similarly to the behavior of $\gamma_j$ vs $\omega_0$, as shown in Fig. 5.

Finally, it is of considerable interest to discuss the difference of radial transport due
Fig. 7.—Normalized diffusion rates [cf. equations (58)–(59)] plotted vs $\omega_0^2$ and $\Delta$. Other parameters are the same as in Figs 4a–c.
to the direction of momentum input occurring in Neutral Beam Injection (NBI) experiments; namely, the co- vs counter-injection. It is observed (Suckewer et al., 1986; Isler et al., 1986) that the toroidal rotation $V_{\phi i}$ changes sign when the beam direction is changed from co- to counter-going with respect to the plasma current. However, noting that

$$V_i = \frac{K_i}{n_i} B + \omega_1 R^2 \nabla \phi + \omega_0 R \nabla \phi$$

one has

$$V_{\phi i} \approx \frac{K_i}{n_i} B_0 + \omega_1 R_0 + \omega_0 R_0 + O(\varepsilon).$$

That is,

$$V_{\phi i} \approx \left[ \delta_{\phi i} \left( U_i + A_d - \frac{1}{2} \frac{T_i}{Z T_i} \frac{\partial}{\partial r} \ln n_i \right) + \frac{\omega_0}{\sqrt{Z}} \right] v_{thi}. \quad (60)$$

In Figs 8a–b, we present the radial fluxes vs $\sqrt{Z}(|V_{\phi i}|/v_{thi})$ for co-injection ($V_{\phi} > 0$) and counter-injection ($V_{\phi i} < 0$), using equations (60), (41) and (44).

To give an analytic estimate of the difference between radial fluxes due to co- and counter-injection, we take the case of a test impurity ($\alpha_0 \to 0$). Assuming the radial profile remains the same for co- and counter-injection and using equations (47a), (47b), (60) and (56), one finds

$$(\Gamma_{\phi i}^{\text{co}})_{\text{count}} - (\Gamma_{\phi i}^{\text{co}})_{\text{co}} = -\frac{\Omega_{\phi i}}{\Omega_{\phi i}} \omega_0 \nabla_{\phi} e^2 \frac{T_i}{Z T_i} \frac{\partial}{\partial r} \ln n_i \frac{\delta_{\phi i}}{Z A_d} \sqrt{Z} A_d \nabla_{\phi} v_{\phi}
\times \left\{ 4 A_d b_0 - \frac{\mu - 1}{Z} + 2 b_0 (\hat{\nabla}_{\phi}^2 + \delta_{\phi i}^2 Z A_d^2) \left( 2 A_d b_0 - \frac{\mu - 1}{Z} \right) \right\}. \quad (61)$$

Here

$$A_d \equiv -\frac{1}{2} r_n \frac{\partial}{\partial r} \ln n_i, \quad \hat{\nabla}_{\phi} \equiv \sqrt{Z} \frac{|V_{\phi i}|}{v_{thi}}$$

and

$$b_0 \equiv b_d (\alpha_0 = 0) = \frac{T_i}{2 \omega_0^2} = \frac{T_i}{T_e} \left( \mu - 1 + \frac{T_i - T_e}{T_i + T_e} \right).$$

That is, for both co- and counter-injection the impurity has an inward flux, but much less for co-injection than for counter-injection. The difference is given in equation (61) and is increasing with increasing $\delta_{\phi i}$. 
Fig. 8.—Normalized diffusion rates plotted vs $\sqrt{Z(V_{th}/v_{th})}$ [cf. equation (60)] for co- and counter-injection. In Fig. 8a, $\alpha_0 = 1$; and in Fig. 8b, $\alpha_0 = 0$. 
Based on the observation that Braginskii’s viscosity tensor contains the gyroviscous piece (Stacey and Sigmar, 1985)

\[ \langle R^2 \nabla \phi \cdot \Pi_{3,4} \cdot \nabla \phi \rangle = - \left\langle \eta_4 \frac{B_p R^3}{r} \frac{\partial}{\partial \theta} \frac{V_\phi}{R} \right\rangle \tag{62} \]

where from (22), (23)

\[ \frac{V_\phi}{R} \equiv \omega(\psi, \theta) = \frac{K(\psi)I(\psi)}{R^3 n} - \frac{p'(\psi)}{en} - c\tilde{\Phi}'(\psi, \theta) - c\Phi'(\psi). \tag{63} \]

Stacey and Sigmar (1984, 1985) developed a semi-heuristic transport theory for ionic particle and toroidal momentum transport in impure plasmas. From equations (62), (63) and involving charge neutrality to express \( (\partial/\partial \theta)\Phi \) in terms of \( \partial n_j/\partial \theta \), this transport (62) is driven entirely by the magnitude of \( \partial n_j/\partial \theta \) and \( K_j(\psi) \equiv n_j V_p/B_p \), i.e. the poloidal flow velocity. Indeed, defining the momentum confinement time through

\[ \tau_\phi^{-1} = \left| \frac{1}{V_\phi} \frac{\partial V_\phi}{\partial t} \right| \]

and using our definition of \( y_s \) in equations (40) and (41), equations (62) and (63) yield the scaling

\[ \tau_{ii} = \frac{r_u}{r_a} \left( \frac{V_p}{V_\phi} \right) \left( \frac{B_\phi}{B_p} \right) \left( \frac{u_{thi} \tau_{ii}}{R_0 q} \right) \epsilon^2 \frac{\alpha}{Z} y_s. \tag{64} \]

Here one additional power of \( \epsilon \) comes from taking the flux-surface average in (62), and \( \alpha/Z \) comes from the fact that

\[ \frac{\bar{n}_i(\sin \theta)}{\epsilon \langle n_i \rangle} \sim \frac{\alpha}{Z} \frac{\bar{n}_i(\sin \theta)}{\epsilon \langle n_i \rangle}, \]

as implied by equation (39). Note that equation (64) can also be obtained using the second term of equation (57) involving \( U_i/\omega_0 \).

In Stacey and Sigmar (1984, 1985), the magnitudes of \( y_s \) and \( V_p \) were determined semi-heuristically as

\[ y_s = O(1), \quad \frac{V_p}{v_{thi}} \frac{B_\phi}{B_p} = O(1) \]

and fixed as constants which leads to the scaling \( \tau_\phi \sim R^2 Z eB/T \) used consequently (Stacey and Malik, 1989) to compare with several experiments, quite successfully in some cases.
From the rigorous, self-consistent calculation of the present paper the quantities \( y_s \) and \( V_p \) and their scaling with plasma parameters have been determined. Most importantly, the magnitude \( y_s \sim O(1) \) is confirmed but \( (V_p/v_{thi})(B_q/B_p) = O(\delta_p) \), i.e. down by one order! When inserted in the heuristic formula (Stacey and Sigmar, 1985) of equation (64), the momentum confinement becomes too long compared to the experiments. Another shortcoming of Stacey and Sigmar (1985) is the straightforward use of the tensor element of equation (62) derived by Braginskii in the limit of small rotation. (The extension to large rotation \( V_p \sim v_{thi} \) is given in the present work.)

Summarizing, when the ad hoc values for \( \dot{n}_i/\langle n_i \rangle \) and \( (V_p/v_{thi})(B_q/B_p) \) claimed in Stacey and Sigmar (1984, 1985) are replaced by the self-consistent rigorous values derived in the present paper, the gyroviscous theory yields too low transport rates. Conversely, if different physical circumstances should exist allowing \( (V_p/v_{thi})(B_q/B_p) \) to be zeroth-order in \( \delta_p \) and the ordering \( \dot{n}_i/\langle n_i \rangle \sim 1 \) would still pertain, then the gyroviscous theory (also using the large rotation extension of the viscosity tensor derived in this paper) may become applicable. A zero-order \( V_p \) usually implies a zeroth-order parallel viscous force and, thus, a zeroth-order dissipation which would have to be balanced by a zeroth-order momentum source. (Usual NBI is ordered smaller than that.) Another possibility stems from a recent theoretical suggestion (Shang and Crume, 1989) that \( \Pi \) will strongly decrease with increasing poloidal velocity for large \( V_p \). Then, a zeroth-order \( V_p \) could provide a parallel viscous force which is small, e.g. of \( O(\delta_p) \).

We close this section pointing out that the calculation presented in this paper contains the gyroviscous part of \( \Pi \) naturally [via terms marked by \( \Pi_{,1} \) in equation (53) and \( nTVV''(1) \) in equation (A9)] but the toroidal momentum flux \( \Pi \) of equation (53) [which enters our result for the confinement time \( \tau_\varphi \) in equation (59)] is driven by collisional terms, most notable the parallel friction \( R_{1,i} \), thus describing a collisional mechanism. Incidentally, the "gyroviscous transport" of equation (64) can also be collisional if \( y_s \) is shown to be caused by collisions. The rigorous result for \( \tau_\varphi \) derived here and shown in Figs 7 in the strong ordering appropriate for the Pfirsch–Schluter regime strongly exceeds the original perpendicular viscosity driven scaling \( \tau_\varphi \sim a^2/\eta_i \) [cf. Connor et al., 1987 and equation (36) and the expression below], but we suggest undertaking comparisons with experiments using the extension of the present calculation (Hsu, 1990) which includes the thermal friction and the coupling to the energy balance equation (but in its present form does not yet include extension into the banana regime for the main ion).

6. SUMMARY AND CONCLUSIONS

The transport phenomena induced by ion-impurity friction in strongly rotating Tokamak plasma have been studied, assuming a constant temperature profile. First, using the standard ordering scheme (Rutherford, 1974) (in which the parallel friction is considered first order in \( \delta_p \), up–down density variations which drive the neoclassical transport have been found driven by parallel i–I friction. These variations are of order \( \delta_{pi} \). Next, by recognizing that \( \delta_{pi}Z^2 \) is of lower order than \( \delta_{pi} \) itself and therefore implementing a strong ordering

\[
\Delta \equiv \frac{\delta_{pi}Z^2}{\omega_{ci}} \sqrt{2n_i} \sim 1,
\]
strong up–down density variations of order $\varepsilon$ have been found in equations (41). One also finds that for plasmas with $z_0 \equiv \mathcal{N}Z^2/\mathcal{N}_i \gg \delta$, the main ion dynamics is strongly coupled with the impurities through $\text{-I-I}$ friction. Hence, the ion poloidal flows are determined self-consistently via equation (44). Moreover, the transport fluxes driven by the strong up–down density variations of order $\varepsilon$ derived in Section 3 are calculated.

The strong ordering calculation presented in this work allows for arbitrary rotation as long as $(\mathcal{O}' \approx R' \approx \mathcal{C} \approx 1) \ll 1$ and $\varepsilon \ll 1$ is satisfied. We then find that while for small rotations, the up/down density asymmetry increases with rotation (as one would expect), for large rotations, the up–down density variation (and thus the radial transport) decreases with rotation [cf. equation (45b)]. The onset of the decrease of up/down variation with respect to rotation occurs near the point where the impurity poloidal flow reverses its direction.

In addition, due to the strong coupling between density variations and poloidal flows in the strong ordering regime, the possibility of the existence of bifurcated poloidal equilibria is discovered. It is found that for very steep radial gradients or large $\Delta$, there exists a multiple-root region [cf. Figs (6a–c)]. The possible impact of bifurcated poloidal equilibria on the transport is briefly discussed. (A possible connection with observed bifurcations is postponed, Hsu, in preparation.)

Furthermore, a brief discussion of the effects of co- and counter-injection on radial transport is given. One concludes that for both injection directions the impurity species diffuses inwardly, but with co-injection the inward diffusion flux is smaller than with counter-injection.

Two remarks can be made on the two restrictions imposed on the present work: (i) uniform temperature profile; and (ii) Pfirsch–Schlüter regime for main ions and impurities.

First, since the radial transport obtained here is induced by the parallel friction, when temperature variations are included, a "thermal friction" term can enter which is expected to strongly modify the up–down variations and thus the radial transport. [A calculation has been done (Hsu and Sigmar, 1989; Hsu, in preparation) extending the present work to include the temperature variations, and discussing the bifurcated poloidal equilibria extensively.] Second, although the present work is developed in the collisional regime, by analogy from the neoclassical transport theory in the small rotation case (Hirshman and Sigmar, 1981), one can expect the present calculation to be qualitatively adequate for the main ion plateau regime except for the Pfirsch–Schlüter definition of $\xi_i < 1$ in equation (44) due to the parallel viscosity. In the plateau regime one can take $\xi_i \approx O(1)$ to roughly estimate the radial transport. The results for $\xi_i = 1.5$ are shown in Figs 4a, 4b, 7a and 7b.

Finally, the relationship of the present work to the "gyroviscous transport theory" (Stacey and Sigmar, 1984, 1985; Stacey and Malik, 1989) was discussed in Section 5. It is found that the $O(\varepsilon)$ up/down asymmetry postulated to exist in (Stacey and Sigmar, 1984, 1985) was calculated in the present work driven by collisional friction (i.e. having a dissipative origin). But as long as the self-consistent value of the impurity poloidal flow velocity remains within the ordering $\mathcal{V}_p \sim (B_p/B)\delta_{pi}v_{thi}$ found here the gyroviscous part of the stress tensor will play a subdominant role, even for $\mathcal{N}_i/\langle n_i \rangle \sim O(\varepsilon)$.
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APPENDIX

In this Appendix we present the derivation of the first-order moments $\Pi^{(1)}$ and $R^{(1)}$ in terms of measurable quantities such as $n$, $T$, and $V$, using the moment approach developed elsewhere (Hsu and Sigmar, 1988). In particular the effects of strong rotation and i-I collision have been included in the parallel viscosity $\Pi$. As a consequence, corrections to Braginskii's parallel viscosity tensor owing to the inclusion of zeroth-order rotation are found. In addition, a general form of parallel friction for arbitrary $\alpha = \frac{n_2 Z_2^2}{n_1} = \frac{v_{th}^2}{v_{th}}\frac{1}{v_{th}}$ is obtained and agrees with the result in Table I of Rutherford (1974). At the end, a brief discussion of $\Pi^{(1)}$ in a system with fast time evolution in the P-S regime is given for instructive purposes.

First, the equation of moment $\Pi = \int dv m(\mathbf{v} - \langle \mathbf{v}^2 \rangle) \mathbf{f}$

\[
\frac{d}{dt} \Pi_j + \Pi_j(\mathbf{V} \cdot \mathbf{V}_j) + \mathbf{V} \cdot \mathbf{\Theta}_j + W_j[V_0, T, \mathbf{V}_j] + \Pi_j(\mathbf{V} \cdot \mathbf{V}_j) + \frac{3}{2} q_j = \Omega_j K_j [\Pi_j] + \Pi_j + \Pi_j
\]

(A1)

is obtained by taking $\int dv m(\mathbf{v} - \langle \mathbf{v}^2 \rangle) \mathbf{f}$ over the kinetic equation. Here, $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}_j \cdot \nabla$.

\[
\mathbf{\Theta} = \int \mathbf{v} m(v^2 - \frac{v^2}{2} (\mathbf{V} + \mathbf{V}_j)) f(v),
\]

where $\mathbf{V}_j$ denotes the cyclic permutation components (without repeating) that make the rank-k tensor $(\mathbf{A} + \mathbf{V}_j)$ cyclically symmetric. Note that the rank-3 tensor $\mathbf{\Theta}$ corresponds to the order-3 Legendre polynomial of pitch angle and is important only with the inclusion of particle trapping effects which is outside the scope of the present work. Moreover, for any rank-2 tensor $\mathbf{A}$, define

\[
W_j[A] \equiv (\mathbf{A} + \mathbf{V}_j) - \frac{3}{2} (\mathbf{A} : \mathbf{I})
\]

(A2)

and

\[
K_j[A] \equiv \mathbf{A} \times \mathbf{b} + \mathbf{V}_j.
\]

(A3)

For two ion species, the main ion collisional moment $\Pi_{ij}$ defined in equation (48e) can be expressed as
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\[ \Pi_i = -v_i \int dv \, H(x)m \left( \mathbf{v} - \frac{v^2}{3} \mathbf{I} \right)f_i, \]  
(A4a)

where

\[ \phi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dy \, e^{-y^2} \]  
(A4b)

is the error function, and \( x = v/v_{th} \). Note that equations (A4a) and (A4b) contain the effects of both like collision and unlike collisions. Actually, the term involving \( x \) in equation (A4b) comes from unlike collisions. The unlike collision operator (Hinton and HAZELTINE, 1976)

\[ C_{\parallel} \approx \frac{3\sqrt{\pi} v_{thi} v_i}{8} \left\{ \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{U}(\mathbf{v}) \cdot \frac{\partial}{\partial \mathbf{v}} f_i + \left( \frac{2v \cdot (\mathbf{V}_i - \mathbf{V})}{v^2} + \frac{m}{m_i} (T_i - T) \right) \frac{2}{v^2} f_{th} \right\} \]

(deduced by assuming both species are nearly Maxwellian, \( m_i/m \sim Z \gg 1, T_i \sim T, \) and \( |\mathbf{V}_i - \mathbf{V}| \ll v_{th} \)), and the linearized like collision operator \( C_{\parallel} \) are used. Here, \( \mathbf{U}(\mathbf{v}) \equiv (c^2 \mathbf{I} - \mathbf{v} \mathbf{v}^T)/v^2 \). The fact that the integrand in equation (A4) is proportional to \((v - v^2/3) \mathbf{I} \) corresponds to the rotational symmetry property of the Fokker–Planck collision operator (HINTON and HAZELTINE, 1976).

Expanding \( H \) in Sonine polynomials

\[ H(x) = \sum_{l=0}^{\infty} c_{\omega l} L^{l+2}(x^2) \]  
(A4c)

and defining

\[ \Pi_i \equiv \frac{(2l)!}{(l+2)!} \int dv \, m \left( \mathbf{v} - \frac{v^2}{3} \mathbf{I} \right)L^{l+2}(x^2)f_i(\mathbf{v}), \]  
(A5)

equation (A4) becomes

\[ \Pi_i = -v_i \sum_{l=0}^{\infty} c_{\omega l} \Pi_i. \]  
(A6)

Here \( L^{l+2}(x^2) \) is the Sonine polynomial (BRAGINSKII, 1965) of order \( 5/2 \). Using the orthogonality of Sonine polynomials, \( c_{\omega l} \) can be determined from

\[ c_{\omega l} = \frac{2}{(2l)!} \int_0^\infty dx \, x^4 H(x)L^{l+2}(x^2) e^{-x^2}. \]  
(A7)

Moreover, from the fact that the distribution function \( f \) is reasonably smooth in speed \( v \) owing to the energy diffusion piece of the collision operator, only the first two terms in \( l \) are worth keeping. Actually, as also pointed out by BRAGINSKI (1965), a very small correction but of high complexity will arise from keeping higher order terms in \( l \). Hence, equations (A6) and (A7) yield

\[ \Pi_i \approx -\frac{3}{2}(v_i + v_{th})\Pi_i + \frac{3}{2}(v_i + 2v_{th})\Pi_i, \]  
(A8)

where \( \Pi_i^* = \Pi_i^{l+2} \) defined in equation (A5).

By assuming transport ordering (whence both the time evolution term and the source term are negligible) and high collisionality \( (v \gg \omega) \), and neglecting the effects of particle trapping and temperature variations, to \( O(1) \), equation (A1) reduces to equation (10), while to \( O(\delta) \) its parallel component becomes

\[ \mathbf{bb} \cdot \{ \mathbf{P}^{(0)} + \mathbf{P}^{(1)} \nabla \} + \mathbf{P}^{(1)} (\nabla \cdot \mathbf{P}^{(0)}) + \mathbf{W}_2 \left[ 3 \mathbf{V} \mathbf{q}_2 + n^{(0)} T \mathbf{V} \mathbf{V}^{(1)} + \mathbf{P}^{(1)} \nabla \mathbf{V}^{(0)} \right] = \mathbf{P}_1. \]  
(A9)
It is important to point out that terms with $\Pi_{ij}^{(1)}$, deduced in equation (21a) from (10), must be kept to retain the effects of large rotation.

To evaluate the parallel viscosity from equations (A8)–(A9), the equation for the moment $\Pi_{i}^{\parallel}$ is needed. To $O(\delta_{ei})$, this equation has the form

$$bb : W_{2} [\hat{z} \mathbf{v} q^{*}] = \Pi_{i}^{(1)}_{\parallel},$$

where

$$\Pi_{i}^{(1)}_{\parallel} = \Pi_{i}^{(1)}_{\parallel}^{-1},$$

and

$$\Pi_{i}^{(1)}_{\parallel} = \left( \frac{3}{l+\frac{3}{2}} \right) ! \int d \nu \, m \left( \frac{v_{\nu}^{2}}{3} \right) L_{i}^{(2)}(x^{2}) C_{i}(f).$$

Using the same procedures yielding $\Pi_{i}$, except for an extra weighting function $L_{i}^{(2)}(x^{2})$ in the integrand of equation (A4), $\Pi_{i}^{\parallel}$ is obtained as

$$\Pi_{i}^{\parallel} \approx - \frac{9}{35} \left[ (v_{\nu} + 2v_{i}) \Pi_{i} + \frac{205}{36} \left( v_{\nu} + \frac{204}{205} v_{i} \right) \Pi^{\parallel} \right].$$

Equations (A8)–(A11) thus yield

$$\Pi_{i}^{(1)} = - \frac{D_{II}}{v_{II}} bb : \left\{ V_{i}^{(0)} \cdot \nabla T_{ii}^{(0)} + \Pi_{i}^{(0)} (V_{i}^{(0)} \cdot V_{i}^{(0)}) + W_{2} \left[ \hat{z} \mathbf{v} q_{i} + n^{(0)} T_{i} V_{i}^{(1)} + \Pi_{i}^{(1)} (V_{i} \cdot V_{i}^{(0)}) \right] \right\} + \frac{D_{II}}{v_{II}} bb : W_{2} [\hat{z} \mathbf{v} q^{*}],$$

where

$$D_{II} = \frac{1025 + 1020 \sqrt{2a}}{1068 + 1806 \sqrt{2a} + 1152 a^{2}}$$

and

$$D_{II} = \frac{630 (1 + 2 \sqrt{2a})}{1068 + 1806 \sqrt{2a} + 1152 a^{2}}.$$
Using a procedure similar to that for deriving equations (A7)-(A8) (except that order-$\frac{1}{2}$ Sonine polynomials $L_{k/2}$ are used to generate the coefficients), the parallel friction can be expressed as

$$R_p = m_0 n_i v_i \left( (V_i - V^*) + \frac{3}{5n_i T_i} (q_i + \frac{4}{5} q_i^*) \right), \quad (A15)$$

where $q_i = q_{i+1}^{-1}$, $q_i^* = q_{i+1}^{-2}$, and

$$q_i^* = -T_i \frac{(\frac{3}{5})! \cdot !}{(\frac{3}{5} + \frac{3}{5})!} \int dv v^2 L^{2/3}(x^2) f_i(v). \quad (A16)$$

Thus two more moment equations for $q_i$ and $q_i^*$ to $O(\delta_e)$ are needed to determine the parallel friction, i.e.

$$0 = q_{i+1}^{(3)} \quad \text{and} \quad 0 = q_{i+1}^{(4)}. \quad (A17)$$

Here, the collisional moments are defined, similar to equations (16), as $q_{i+1} = q_{i+1}^{-1}$, $q_{i+1}^* = q_{i+1}^{-2}$, and

$$q_{i+1}^{(3)} = -T_i \frac{(\frac{3}{5})! \cdot !}{(\frac{3}{5} + \frac{3}{5})!} \int dv v^2 L^{2/3}(x^2) C_i(f_i). \quad (A18)$$

Following the same derivation leading to equation (A15), equation (16) yields

$$q_{i+1} = \frac{3}{5} v_i n_i T_i (V_i - V^*) - \frac{3}{5} (v_n + \frac{3}{5} v_i) q_i - \frac{3}{5} (v_n + \frac{3}{5} v_i) q_i^*, \quad (A19)$$

$$q_{i+1}^* = \frac{3}{5} v_i n_i T_i (V_i - V^*) - \frac{3}{5} (v_n + \frac{3}{5} v_i) q_i - \frac{3}{5} (v_n + \frac{3}{5} v_i) q_i^*. \quad (A20)$$

It is then straightforward to obtain

$$R_{i+1}^{(0)} = m_0 n_i v_i D_{i+1} (V_i^{(0)} - V^*_{i+1}^{(0)}), \quad (A21)$$

where

$$D_{i+1} = \frac{576 + 488 \sqrt{2} x + 128 x^2}{576 + 1208 \sqrt{2} x + 434 x^2}. \quad (A22)$$

Note that equation (A22) is general for arbitrary $x$ and agrees with the results given in Table I of Hirshman and Sigmar (1981).

It is also of interest to study the relaxation of poloidal flow due to the parallel viscosity in the P-S regime. Assuming that there exists a zeroth-order poloidal flow, the $O(1)$ parallel projection of equation (A1) becomes

$$\frac{\partial}{\partial t} \Pi^{(0)}_{ii} - \Pi^{(0)}_{ij} = -bb : W_{ij} [n_i^{(0)} T_j V^{(0)}], \quad (A23)$$

Equations (A8), (A23) and (2) then imply that for a given $R^{(0)}$ the parallel viscosity will relax to be constant with respect to time in a short time of $O(1/v)$. This relaxed zeroth-order parallel viscosity is thus obtained as

$$\Pi^{(0)}_{ii} = -\frac{D_{i+1}}{v_{ij}} \cdot bb : W_{ij} [n_i^{(0)} T_j V^{(0)}], \quad (A24)$$

by further imposing $\partial/\partial t \ll v$ and using equations (A8), (A11) and (A23). Equations (A24) and (48b) indicate that the poloidal flow will damp due to the parallel viscosity at a rate $\sim v_{ij}/v$ for a Tokamak plasma in the P-S regime. A recent calculation (Shawing and Hirshman, 1989) shows that this relaxation rate for poloidal flow is still basically true in the banana-plateau regime.