Generalized Laplace Inference in Multiple Change-Points Models

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Abstract

Under the classical long-span asymptotic framework we develop a class of Generalized Laplace (GL) inference methods for the change-point dates in a linear time series regression model with multiple structural changes analyzed in, e.g., Bai and Perron (1998). The GL estimator is defined by an integration rather than optimization-based method and relies on the least-squares criterion function. It is interpreted as a classical (non-Bayesian) estimator and the inference methods proposed retain a frequentist interpretation. This approach provides a better approximation about the uncertainty in the data of the change-points relative to existing methods. On the theoretical side, depending on some input (smoothing) parameter, the class of GL estimators exhibits a dual limiting distribution; namely, the classical shrinkage asymptotic distribution, or a Bayes-type asymptotic distribution. We propose an inference method based on Highest Density Regions using the latter distribution. We show that it has attractive theoretical properties not shared by the other popular alternatives, i.e., it is bet-proof. Simulations confirm that these theoretical properties translate to good finite-sample performance.

JEL Classification: C12, C13, C22

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1 Introduction

In the context of the multiple change-points model analyzed in Bai and Perron (1998), we develop inference methods for the change-point dates for a class of Generalized Laplace (GL) estimators using a classical long-span asymptotic framework. They are defined by an integration rather than an optimization-based method, the latter typically characterizing classical extremum estimators. The idea traces back to Laplace (1774), who first suggested to interpret transformations of a least-squares criterion function as a statistical belief over a parameter of interest. Hence, a Laplace estimator is defined similarly to a Bayesian estimator although the former relies on a statistical criterion function rather than a parametric likelihood function. As a consequence, the GL estimator is interpreted as a classical (non-Bayesian) estimator and the inference methods proposed retain a frequentist interpretation such that the GL estimators are constructed as a function of integral transformations of the least-squares criterion. In a first step, we use the approach of Bai and Perron (1998) to evaluate the least-squares criterion function at all candidate break dates. We then apply a transformation to obtain a proper distribution over the parameters of interest, referred to as the Quasi-posterior. For a given choice of a loss function and (possibly) a prior density, the estimator is then defined either explicitly as, for example, the mean or median of the (weighted) Quasi-posterior or implicitly as the minimizer of a smooth convex optimization problem.

The underlying asymptotic framework considered is the long-span shrinkage asymptotics of Bai (1997), Bai and Perron (1998) and also Perron and Qu (2006) who considerably relaxed some conditions, where the magnitude of the parameter shift is sample-size dependent and approaches zero as the sample size increases. Early contributions to this approach are Hinkley (1971), Bhat- tacharya (1987), and Yao (1987) for estimating break points. For testing for structural breaks, see Hawkins (1977), Picard (1985), Kim and Siegmund (1989), Andrews (1993), Horváth (1993) and Andrews and Ploberger (1994). See also the reviews of Csörgő and Horváth (1997), Perron (2006), Casini and Perron (2019) and references therein.

One of our goals is to develop GL estimates with better small-sample properties compared to least-squares estimates, namely lower Mean Absolute and Root-Mean Squared Errors, and confidence sets with accurate coverage probabilities and relatively short lengths for a wide range of break sizes, whether small or large; existing methods work well for either small or large breaks, but not for both. A second goal is to establish theoretical results that support the reported finite-sample properties about inference.

The asymptotic distribution of the GL estimator is derived via a local parameter related to a normalized deviation from the true fractional break date. The normalization factor corresponds to the rate of convergence of the original (extremum) least-squares estimator as established by Bai and Perron (1998). The asymptotic distribution of the GL estimator then depends on a sample-size dependent smoothing parameter sequence applied to the least-squares criterion function. We
derive two distinct limiting distributions corresponding to different smoothing sequences of the criterion function [cf. Jun, Pinkse, and Wan (2015) for a related application in the context of the cube-root asymptotics of Kim and Pollard (1990)]. In one case, the estimator displays the same limit law as the asymptotic distribution of the least-squares estimator derived in Bai and Perron (1998) [see also Hinkley (1971), Picard (1985) and Yao (1987)]. In a second case, the limiting distribution is characterized by a ratio of integrals over functions of Gaussian processes and resembles the limiting distribution of Bayesian change-point estimators. The latter is exploited for the purpose of constructing confidence sets for the break dates. We use the concept of highest density regions (HDR) introduced by Casini and Perron (2020a) for structural change problems, which best summarizes the properties of the probability distribution of interest. The HDR are common in Bayesian analysis where they are applied to a posterior distribution [see, e.g., Box and Tiao]. Kendall and Stuart discussed the difference between frequentist confidence intervals and Bayesian approaches in relation to the existence of a sufficient statistic. Our procedure yields confidence sets for the break date which, in finite samples, better account for the uncertainty over the parameter space in finite-samples because it effectively incorporates a statistical measure of the uncertainty in the least-squares criterion function. As noted in the literature on likelihood-based inference in some classes of nongranular problems [see e.g., Chernozhukov and Hong (2003), Ghosal, Ghosh, and Samanta (1995), Hirano and Porter (2003) and Ibragimov and Has’minskii (1981)], the Maximum Likelihood Estimator (MLE) is generally not an asymptotically sufficient statistic in these models and so the likelihood contains more information asymptotically than the MLE. Hence, likelihood-based procedures are generally not functions of the MLE even asymptotically. This incompleteness property motivated the study of the entire likelihood rather than just the MLE. Likewise, our method exploits the entire behavior of the objective function.

Laplace’s seminal insight has been applied successfully in many disciplines. In econometrics, Chernozhukov and Hong (2003) introduced Laplace-type estimators as an alternative to classical (regular) extremum estimators in several problems such as censored median regression and non-linear instrumental variable; see also Forneron and Ng (2018) for a review and comparisons. Their main motivation was to solve the curse of dimensionality inherent to the computation of such estimators. In contrast, the class of GL estimators in structural change models serves distinct multiple purposes. First, inference about the break dates presents several challenges, in particular to provide methods with a satisfactory performance uniformly over different data-generating mechanisms and break magnitudes. The GL inference proves to be reliable and accurate in finite-samples. Second, it leads to inference methods that have both frequentist and credibility properties which is not shared by the other popular methods.

Turning to the problem of constructing confidence sets for a single break date, the standard asymptotic method for the linear regression model was proposed in Bai (1997), while Elliott and Müller (2007) proposed to invert the locally best invariant test of Nyblom (1989), and Eo and
Morley (2015) suggested to invert the likelihood-ratio statistic of Qu and Perron (2007). The latter were mainly motivated by finite-sample results indicating that the exact coverage rates of the confidence intervals obtained from Bai’s (1997) method are often below the nominal level when the magnitude of the break is small. It has been shown that the method of Elliott and Müller (2007) delivers the most accurate coverage rates but the average length of the confidence sets is significantly larger than with other methods. The confidence sets for the break dates constructed from the GL inference that we develop result in exact coverage rates close to the nominal level and short length of the confidence sets. This holds true whether the magnitude of the break is small or large. In fact, we show that GL inference is bet-proof, a measure of “reasonableness” of frequentist inference in non-regular problems [see, e.g., Buehler (1959)].

The GL inference developed in this paper has been applied by Casini and Perron (2020b) to achieve finite-sample improvements under the continuous record asymptotic framework of Casini and Perron (2020a). The latter proposed an alternative asymptotic framework to explain the non-standard features of the finite-sample distribution of the least-squares estimator.

The paper is organized as follows. We first focus on the single change-point case. Section 2 presents the statistical setting. We develop the asymptotic theory in Section 3 and the inference methods in Section 4. Results for multiple change-points models are given in Section 5 while Section 6 discusses some theoretical properties of GL inference. Section 7 presents simulation results about the finite-sample performance. Section 8 concludes. All proofs are included in an online supplement [Casini and Perron (2020e)].

2 The Model and the Assumptions

This section introduces the structural change model with a single break, reviews the least-squares estimation method for the break date, and presents the relevant assumptions. We start with introducing the formal setup for our analysis. The following notation is used throughout. We denote the transpose of a matrix $A$ by $A'$. We use $\| \cdot \|$ to denote the Euclidean norm of a linear space, i.e., $\|x\| = (\sum_{i=1}^{p} x_i^2)^{1/2}$ for $x \in \mathbb{R}^p$. For a matrix $A$, we use the vector-induced norm, i.e., $\|A\| = \sup_{x \neq 0} \|Ax\| / \|x\|$. All vectors are column vectors. For two vectors $a$ and $b$, we write $a \leq b$ if the inequality holds component-wise. We use $\lfloor \cdot \rfloor$ to denote the largest smaller integer function. We use $\overset{P}{\rightarrow}$ and $\overset{d}{\rightarrow}$ to denote convergence in probability and convergence in distribution, respectively. $C_b(\mathbb{E})$ is the collection of bounded continuous functions from some specified set $\mathbb{E}$ to $\mathbb{R}$. Weak convergence on either $C_b(\mathbb{E})$ or $D_b(\mathbb{E})$ is denoted by $\Rightarrow$. The symbol “$\triangleq$” stands for definitional equivalence.

We consider a sample of observations $\{(y_t, w_t, z_t) : t = 1, \ldots, T\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which all of the random elements introduced in what follows are defined.
The break date least-squares (LS) estimator \( \hat{S}_\delta \) corresponds to \( D \). A pure structural change model in which all regression parameters are subject to change and \( e \) the model is \( T \) function depending only on \( \arg \min_\theta \). Let \( Y \) where \( X \) is a scalar dependent variable, \( w_t \) and \( z_t \) are regressors of dimensions, \( p \) and \( q \), respectively, and \( e_t \) is an unobserved error term. The true parameter vectors \( \phi^0 \), \( \delta^0_1 \) and \( \delta^0_2 \) are unknown and we define \( \delta^0 \triangleq \delta^0_0 - \delta^0_1 \), with \( \delta^0 \neq 0 \) so that a structural change occurs at date \( T^0_b \). It is useful to re-parametrize the model. Letting \( x_t \triangleq (w_t', z_t')' \) and \( \beta^0 \triangleq ((\phi^0)', (\delta^0_1))' \), we have

\[
y_t = x_t'\beta^0 + e_t, \quad (t = 1, \ldots, T^0_b) \quad y_t = w_t'\phi^0 + z_t'\delta^0_0 + e_t, \quad (t = T^0_b + 1, \ldots, T) \tag{2.1}
\]

where \( y_t \) is a scalar dependent variable, \( w_t \) and \( z_t \) are regressors of dimensions, \( p \) and \( q \), respectively, and \( e_t \) is an unobserved error term. The true parameter vectors \( \phi^0 \), \( \delta^0_1 \) and \( \delta^0_2 \) are unknown and we define \( \delta^0 \triangleq \delta^0_0 - \delta^0_1 \), with \( \delta^0 \neq 0 \) so that a structural change occurs at date \( T^0_b \). It is useful to re-parametrize the model. Letting \( x_t \triangleq (w_t', z_t')' \) and \( \beta^0 \triangleq ((\phi^0)', (\delta^0_1))' \), we have

\[
y_t = x_t'\beta^0 + e_t, \quad (t = 1, \ldots, T^0_b) \quad y_t = x_t'\beta^0 + z_t'\delta^0_0 + e_t, \quad (t = T^0_b + 1, \ldots, T). \tag{2.2}
\]

More generally, we can define \( z_t \triangleq D'x_t \), where \( D \) is a \((p + q) \times q\) matrix with full column rank. A pure structural change model in which all regression parameters are subject to change corresponds to \( D = I_{(p+q)\times(p+q)} \), whereas a partial structural change model arises when \( D = (0_{q \times p}, I_{q \times q})' \). In order to facilitate the derivations, we reformulate model (2.2) in matrix format. Let \( Y = (y_1, \ldots, y_T)' \), \( X = (x_1, \ldots, x_T)' \), \( e = (e_1, \ldots, e_T)' \), \( X_1 = (x_1, \ldots, x_{T_b}, 0, \ldots, 0)' \), \( X_2 = (0, \ldots, 0, x_{T_b+1}, \ldots, x_T)' \) and \( X_0 = (0, \ldots, 0, x_{T^0_b+1}, \ldots, x_T)' \). Further, define \( Z_1, Z_2 \) and \( Z_0 \) in a similar way: \( Z_1 = X_1D, Z_2 = X_2D \) and \( Z_0 = X_0D \). We omit the dependence of the matrices \( X_i \) and \( Z_i \) \((i = 1, 2)\) on \( T_b \). Then, (2.2) is equivalent to

\[
Y = X\beta + Z_0\delta + e. \tag{2.3}
\]

Let \( \theta^0 \triangleq ((\phi^0)', (\delta^0_1)', (\delta^0_2)')' \) denote the true value of the parameter vector \( \theta \triangleq (\phi, \delta_1, \delta) \). The break date least-squares (LS) estimator \( \tilde{T}^LS_b \) is the minimizer of the sum of squared residuals [denoted \( S_T(\theta, T_b) \)] from (2.3). The parameter \( \theta \) can be concentrated out resulting in a criterion function depending only on \( T_b = T\lambda_b \), i.e., \( \tilde{T}^LS_b = \arg \min_{1 \leq T_b \leq T} S_T(\tilde{\theta}^LS(\theta, T_b), T_b) \) where \( \tilde{\theta}^LS(\theta, T_b) = \arg \min_\theta S_T(\theta, T_b) \) with \( S_T(\theta, T_b) = \sum_{t=1}^{T} (y_t - \phi'w_t - \delta_1'z_t)^2 + \sum_{t=T_b+1}^{T} (y_t - \phi'w_t - \delta_1'z_t)^2 \). Also,

\[
\arg \min_{1 \leq T_b \leq T} S_T(\tilde{\theta}^LS(\theta, T_b), T_b) = \arg \max_{T_b} \tilde{\theta}^LS(T_b)(Z'_{M_X}Z_2)\tilde{\theta}^LS(T_b) \tag{2.4}
\]

\[
\triangleq \arg \max_{\lambda_b} Q_T(\tilde{\theta}^LS(\lambda_b), \lambda_b),
\]

where \( M_X \triangleq I - X(X'X)^{-1}X' \), \( \tilde{\theta}^LS(\lambda_b) \) is the least-squares estimator of \( \delta^0 \) obtained by regressing \( Y \) on \( X \) and \( Z_2 \) and the statistic \( Q_T(\tilde{\theta}^LS(\lambda_b), \lambda_b) \) is the numerator of the sup-Wald statistic. The Laplace-type inference builds on the least-squares criterion function \( Q_T(\delta(\lambda_b), \lambda_b) \), where \( \delta(\lambda_b) \) stands for \( \tilde{\theta}^LS(\lambda_b) \) to minimize notational burden.

**Assumption 2.1.** \( T^0_b = [T\lambda^0_b] \), where \( \lambda^0_b \in \Gamma^0 \subset (0, 1) \).
Assumption 2.2. With \( \{ \mathcal{F}_t, \ t = 1, 2, \ldots \} \) a sequence of increasing \( \sigma \)-fields, \( \{ z_t e_t, \mathcal{F}_t \} \) forms an \( L^r \)-mixingale sequence with \( r = 2 + \nu \) for some \( \nu > 0 \). That is, there exist nonnegative constants \( \{ \varrho_1, \} \) and \( \{ \varrho_{2,j}, j \geq 0 \} \) such that \( \varrho_{2,j} \to 0 \) as \( j \to \infty \), and for all \( t \geq 1, j \geq 0 \) and \( r \geq 1 \), (i) \( \| \mathbb{E}(z_t e_t | \mathcal{F}_{t-j}) \|_r \leq \varrho_{1,t} \varrho_{2,j} \), (ii) \( \| z_t e_t - \mathbb{E}(z_t e_t | \mathcal{F}_{t+j}) \|_r \leq \varrho_{1,t} \varrho_{2,j+1} \). In addition, (iii) \( \max_t \varrho_{1,t} < C_1 < \infty \) and (iv) \( \sum_{j=0}^{\infty} j^{1+\nu} \varrho_{2,j} < \infty \) for some \( \nu > 0 \), (v) \( \| z_t \|_r < C_2 < \infty \) and \( \| e_t \|_{2r} < C_3 < \infty \) for some \( C_1, C_2, C_3 > 0 \).

Assumption 2.3. There exists an \( l_0 > 0 \) such that for all \( l > l_0 \), the minimum eigenvalues of \( H_l^* = (1/l) \sum_{T_b^0-T_b^0+t_1}^{T_b^0-1} x_t x_t' \) and \( H_l^{**} = (1/l) \sum_{T_b^0+1}^{T_b^0+l} x_t x_t' \) are bounded away from zero. These matrices are invertible when \( l \geq p + q \) and have stochastically bounded norms uniformly in \( l \).

Assumption 2.4. \( T^{-1} X'X \xrightarrow{p} \Sigma_{XX} \), where \( \Sigma_{XX} \), a positive definite matrix.

These assumptions are standard and similar to those in Perron and Qu (2006). It is well-known that only the fractional break date \( \lambda_b^0 \) (not \( T_b^0 \)) can be consistently estimated, with \( \lambda_b^{LS} \) having a \( T \)-rate of convergence. The corresponding result for the break date estimator \( \hat{T}_b^{LS} \) states that, as \( T \) increases, \( \hat{T}_b^{LS} \) remains within a bounded distance from \( T_b^0 \). However, this does not affect the estimation problem of the regression coefficients \( \theta^0 \), for which \( \hat{\theta}^{LS} \) is a regular estimator; i.e., \( \sqrt{T} \)-consistent and asymptotically normally distributed, since the estimation of the regression parameters is asymptotically independent from the estimation of the change-point. Hence, the regression parameters are essentially estimated as if the change-point was known. More complex is the derivation of the asymptotic distribution of \( \lambda_b^{LS} \); e.g., Hinkley (1971) for an i.i.d. Gaussian process with a mean change. Therefore, to make progress it is necessary to consider a shrinkage asymptotic setting in which the size of the shift converges to zero as \( T \to \infty \); see Picard (1985) and Yao (1987) and extended by Bai (1997) to general linear models.

3 Generalized Laplace Estimation

We define the GL estimator in Section 3.1 and discuss its usefulness in Section 3.2. Section 3.3 describes the asymptotic framework under which we derive the limiting distribution with the results presented in Section 3.4.

3.1 The Class of Laplace Estimators

The class of GL estimators relies on the original least-squares criterion function \( Q_T (\delta(\lambda_b), \lambda_b) \), with the parameter of interest being \( \lambda_b^0 = T_b^0/T \). The Quasi-posterior \( p_T (\lambda_b) \) is defined by the exponential transformation,

\[
p_T (\lambda_b) \triangleq \frac{\exp \left( Q_T (\delta(\lambda_b), \lambda_b) \right) \pi (\lambda_b)}{\int_{\lambda_0}^{\lambda_b} \exp \left( Q_T (\delta(\lambda_b), \lambda_b) \right) \pi (\lambda_b) d\lambda_b}, \tag{3.1}
\]
where $\pi(\cdot)$ is a density function. Note that $p_T(\lambda_b)$ defines a proper distribution over the parameter space $\Gamma^0$. The $\mathcal{L}(\theta, T_b)$-class of estimators are the solutions of smooth convex optimization problems for a given loss function, restricting attention to convex loss functions $l_T(\cdot)$. Examples include (a) $l_T(r) = a_T^m |r|^m$, the polynomial loss function (the squared loss function is obtained when $m = 2$ and the absolute deviation loss function when $m = 1$); (b) $l_T(r) = a_T(\tau - 1(r \leq 0)) r$, the check loss function; where $a_T$ is a divergent sequence. We define the Expected Risk function, under the density $p_T(\cdot)$ and the loss $l_T(\cdot)$ as $R_{l,T}(s) \triangleq \mathbb{E}_{p_T}[l_T(s - \hat{\lambda}_b)]$, where $\hat{\lambda}_b$ is a random variable with distribution $p_T$ and $\mathbb{E}_{p_T}$ denotes expectation taken under $p_T$. Using (3.1) we have,

$$R_{l,T}(s) \triangleq \int_{\Gamma^0} l_T(s - \lambda_b) p_T(\lambda_b) d\lambda_b. \quad (3.2)$$

The Laplace-type estimator $\hat{\lambda}_b^{GL}$ shall be interpreted as a decision rule that, given the information contained in the Quasi-posterior $p_T$, is least unfavorable according to the loss function $l_T$ and the prior density $\pi$. Then $\hat{\lambda}_b^{GL}$ is the minimizer of the expected risk function (3.2), i.e., $\hat{\lambda}_b^{GL} \triangleq \arg\min_{s \in \Gamma^0} [R_{l,T}(s)]$. Observe that the GL estimator $\hat{\lambda}_b^{GL}$ results in the mean (median) of the Quasi-posterior upon choosing the squared (absolute deviation) loss function. The choice of the loss and of the prior density functions hinges on the statistical problem addressed. In the structural change problem, a natural choice for the Quasi-prior $\pi$ is the density of the asymptotic distribution of $\hat{\lambda}_b^{LS}$. This requires to replace the population quantities appearing in that distribution by consistent plug-in estimates—cf. Bai and Perron (1998)—and derive its density via simulations as in Casini and Perron (2020a). The attractiveness of the Quasi-posterior (3.1) is that it provides additional information about the parameter of interest $\lambda_b^0$ beyond what is already included in the point estimate $\hat{\lambda}_b^{LS}$ and its distribution (see Section 3.2). This approach will result in more accurate inference in finite-samples even in cases with high uncertainty in the data as we shall document in Section 7. This is supported in Section 6 showing that the GL inference is bet-proof which is a desirable theoretical property in non-regular problems.

**Assumption 3.1.** Let $l_T(r) \triangleq l(a_T r)$, with $a_T$ a positive divergent sequence. $L$ denotes the set of functions $l : \mathbb{R} \to \mathbb{R}_+$ that satisfy (i) $l(r)$ is defined on $\mathbb{R}$, with $l(r) \geq 0$ and $l(r) = 0$ if and only if $r = 0$; (ii) $l(r)$ is continuous at $r = 0$; (iii) $l(\cdot)$ is convex and $l(r) \leq 1 + |r|^m$ for some $m > 0$.

**Assumption 3.2.** $\pi : \mathbb{R} \to \mathbb{R}_+$ is a continuous, uniformly positive density function satisfying $\pi^0 \triangleq \pi(\lambda_b^0) > 0$, and for some finite $C_\pi < \infty$, $\pi^0 < C_\pi$. Also, $\pi(\lambda_b) = 0$ for all $\lambda_b \notin \Gamma^0$, and $\pi$ is twice continuously differentiable with respect to $\lambda_b$ at $\lambda_b^0$.

Assumption 3.1 is similar to those in Bickel and Yahav (1969), Ibragimov and Has’minskiı (1981) and Chernozhukov and Hong (2003). The convexity assumption on $l_T(\cdot)$ is guided by practical considerations. The dominant restriction in part (iii) is conventional and implicitly assumes
that the loss function has been scaled by some constant. What is important is that the growth of the function \( l_T(r) \) as \( |r| \to \infty \) is slower than \( \exp(\epsilon |r|) \) for any \( \epsilon > 0 \). Assumption 3.2 on the prior is satisfied for any reasonable choice. For priors that have a peak at \( \lambda_0 \) one can apply some basic smoothing techniques to make it differentiable locally [e.g., mean smoothing, Gaussian smoothing and Savitzky-Golay filter]. We did not find any particular difference in the empirical results and so we used the mean smoothing. The assumption on the differentiability of the kernel can be relaxed at the expense of one more step in the proof. Chernozhukov and Hong (2003) assumed differentiability of the prior; we also keep the same assumption and applied the smoothing. The large-sample properties of the \( \mathscr{L}(\theta, T_b) \)-class are studied under the shrinkage asymptotic setting of Bai (1997) and Bai and Perron (1998). Thus, we need the following assumption.

**Assumption 3.3.** Let \( \delta_T \triangleq \delta_0^T \triangleq v_T \delta^0 \) where \( v_T > 0 \) is a scalar satisfying \( v_T \to 0 \) as \( T \to \infty \) and \( T^{1/2-\vartheta} v_T \to \infty \) for some \( \vartheta \in (0, 1/4) \).

We omit the superscript 0 from \( \delta_T^0 \) for notational convenience since it should not cause any confusion. Assumption 3.3 requires the magnitude of the break to shrink to zero at any slower rate than \( T^{-1/2} \). The specific rates allowed differ from those in Bai (1997) and Bai and Perron (1998), since they require \( \vartheta \in (0, 1/2) \). The reason is merely technical; the asymptotics of the Laplace-type estimator involve smoothing the criterion function, and thus one needs to guarantee that \( \hat{\lambda}_b \) approaches \( \lambda_0 \) at a sufficiently fast rate. Under the shrinkage asymptotics, Proposition 1 and Corollary 1 in Bai (1997) state that \( T \|\delta_T\|^2 \left( \hat{\lambda}_b^{LS} - \lambda_0 \right) = O_P(1) \) and \( \hat{\delta}_T^{LS} - \delta_T = o_P(1) \).

### 3.2 Discussion about the GL Approach

We use Figure 1-2 to illustrate the main idea behind the usefulness of the GL method. They present plots of the density of the distribution of \( \hat{T}_b^{LS} \) and \( \hat{T}_b^{GL} \) for the simple model \( y_t = \phi_0 + z_t (\delta_1^0 + \delta^0 \mathbb{1}\{t > T_0^b\}) + e_t \) where \( \{z_t\} \) follows an ARMA(1,1) process and \( e_t \sim i.i.d. \mathcal{N}(0, 1) \). The distributions presented are the exact finite-sample distributions of the LS and GL estimators, Bai’s (1997) classical large-\( N \) limit distribution, and the asymptotic distribution of the GL estimator. Noteworthy are the non-standard features of the finite-sample distribution of the LS estimator when the break magnitude is small, which include multi-modality, fat tails and asymmetry. The central mode is near \( \hat{T}_b^{LS} \) while the other two modes are in the tails near the start and end of the sample period; when the break magnitude is small \( \hat{T}_b^{LS} \) tends to locate the break in the tails since the evidence of a break is weak. It is evident that the classical large-\( N \) asymptotic distribution provides a poor approximation especially for small break sizes. Some of these features have been found in other works [see, e.g., Perron and Zhu (2005), Deng and Perron (2006), Jiang, et al. (2018; 2020), and Casini and Perron (2020a)]. Turning to the densities of the GL estimators, some of the nonstandard features appear also for the GL estimator although to a much lesser extent. In
particular, the densities of the GL estimators are less spread out than the corresponding densities for the LS estimator. For small breaks, the finite-sample distributions of the LS and GL estimators are quite different, which suggests that standard measures of accuracy (e.g., MAE and RMSE) can be expected to differ substantially. For \( \lambda_0 = 0.5 \) the GL estimator exhibits much less variability and more precision. The figures also show that the asymptotic distribution of the GL estimator provides an accurate approximation for large breaks while for small breaks the approximation is less accurate. However, it captures the fat-tails of the finite-sample distribution which suggests that it does not underestimate uncertainty about the break location unlike Bai’s (1997) distribution.

The GL method is useful because it weights the information from the least-squares criterion function with the information from the prior density—which, here, is the density of the asymptotic distribution of \( \hat{T}^{LS}_b \). Note that the least-squares objective function is quite flat when the magnitude of the break is small and so \( \hat{T}^{LS}_b \) is imprecise. The resulting Quasi-posterior, or, e.g., its median, is likely to lead to better estimates in finite-samples, because it takes into account the overall shape of the objective function which weighted by the prior becomes more informative about the uncertainty of the objective function.

### 3.3 Normalized Version of \( \mathcal{R}_{\ell,T} (s) \)

In order to develop the asymptotic results, we introduce a smoothing sequence \( \{\gamma_T\} \) whose properties are specified below and work with a normalized version of \( \mathcal{R}_{\ell,T} (s) \) in order to be able to derive the relevant limit results. We assume that \( \lambda_0^0 \in \Gamma_0 \subset (0, 1) \) is the unknown extremum of \( \bar{Q} (\theta^0, \lambda_0) = \mathbb{E} [Q_T (\theta^0, \lambda_0)] \) and that \( \theta_0 \triangleq \left((\phi^0)'', (\delta^0)'', (\delta^0)''\right)' \in \mathcal{S} \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \). Our analysis is within a vanishing neighborhood of \( \theta_0 \). For any \( \theta \in \mathcal{S} \), let \( \lambda_0^0 (\theta) \) be an arbitrary element of \( \Gamma_0 (\theta) \triangleq \left\{ \lambda_0 \in \Gamma_0 : \bar{Q} (\theta, \lambda_0) = \sup_{\lambda_0 \in \Gamma_0} \bar{Q} (\theta, \lambda_0) \right\} \). Provided a uniqueness condition is assumed (see Assumption 3.6), \( \Gamma_0 (\theta) \) contains a single element, \( \lambda_0^0 \). Further, let \( \bar{Q}_T (\theta, \lambda_0) \triangleq Q_T (\theta, \lambda_0) - Q_T (\theta, \lambda_0^0) \), \( Q_0^T (\theta, \lambda_0) \triangleq \mathbb{E} [Q_T (\theta, \lambda_0) - Q_T (\theta, \lambda_0^0) | X] \), and \( G_T (\theta, \lambda_0) \triangleq \bar{Q}_T (\theta, \lambda_0) - Q_0^T (\theta, \lambda_0) \). These expressions are given by \( G_T (\theta, \lambda_0) = g_e (\theta, \lambda_0), Q_0^T = g_d (\theta, \lambda_0) \) and \( \bar{Q}_T = g_e (\theta, \lambda_0) + g_e (\theta, \lambda_0) \), where

\[
g_d (\theta, \lambda_0) = \delta_T \left\{ (Z_0'MZ_2) (Z_2'MZ_2)^{-1} (Z_2'MZ_0) - Z_0'MZ_0 \right\} \delta_T, \tag{3.3}
\]

and

\[
g_e (\theta, \lambda_0) = 2\delta_T (Z_0'MZ_2) (Z_2'MZ_2)^{-1} Z_2'Me - 2\delta_T (Z_0'Me) + e'MZ_2 (Z_2'MZ_2)^{-1} Z_2'Me - e'MZ_0 (Z_0'MZ_0)^{-1} Z_0'Me. \tag{3.4}
\]
They are derived in Section A.2. For the purpose of developing the asymptotic theory, the GL estimator \( \hat{\lambda}_b^{GL}(\theta) \) is defined as the minimizer of a normalized version of \( R_{l,T}(s) \):

\[
\Psi_{l,T}(s; \theta) = \int_{\Gamma_0} l(s - \lambda_b) \frac{\exp \left( \left( \frac{\gamma_T}{(T \lVert \delta_T \rVert^2)} \right) \bar{Q}_T(\theta, \lambda_b) \right) \pi(\lambda_b) \, d\lambda_b}{\int_{\Gamma_0} \exp \left( \left( \frac{\gamma_T}{(T \lVert \delta_T \rVert^2)} \right) \bar{Q}_T(\theta, \lambda_b) \right) \pi(\lambda_b) \, d\lambda_b} \tag{3.5}
\]

\[
= \int_{\Gamma_0} l(s - \lambda_b) \frac{\exp \left( \left( \frac{\gamma_T}{(T \lVert \delta_T \rVert^2)} \right) (G_T(\theta, \lambda_b) + Q_T^0(\theta, \lambda_b)) \right) \pi(\lambda_b) \, d\lambda_b. 
\]

Note that, under Condition 1 below, this is equivalent to the minimizer of \( R_{l,T}(s) \) since \( \bar{Q}_T(\theta, \lambda_b) \) can always be normalized without affecting its maximization. Different choices of \( \{\gamma_T\} \) give rise to GL estimators with different limiting distributions. Using \( \delta_T \) or any consistent estimate (e.g., \( \hat{\delta}_T^{LS} \)) in the factor \( \gamma_T \) is irrelevant because they are asymptotically equivalent. Our analysis is local in nature and thus we write \( \hat{\lambda}_b^{GL}(\hat{\theta}) \approx \hat{\lambda}_b^{GL}(\theta^0, r_T(\hat{\theta} - \theta^0), r_T(\hat{\theta} - \theta^0)) \), where \( r_T \) is the convergence rate of \( \hat{\theta} - \theta^0 \). Note that \( G_T(\cdot, \cdot) \) and \( Q_T^0(\cdot, \cdot) \) constitute the stochastic and the deterministic part of the objective function, respectively. Both depend on \( r_T(\hat{\theta} - \theta^0) \) and our proof proceeds in conditioning first on the effect of \( r_T(\hat{\theta} - \theta^0) \) on the deterministic part to obtain weak convergence of the stochastic part to a limit process that does not depend on this conditioning. See below for more details. Hence, it is required to introduce two indices \( \tilde{v} \) and \( v \), such that we define \( \hat{\lambda}_b^{GL}(\hat{\theta}) = \hat{\lambda}_b^{GL}(\tilde{v}, v) \) as the minimizer of

\[
\Psi_{l,T}(s; \tilde{v}, v) \triangleq \int_{\Gamma_0} l(s - \lambda_b) \times \frac{\exp \left( \left( \frac{\gamma_T}{(T \lVert \delta_T \rVert^2)} \right) (G_T(\theta^0 + \tilde{v} / r_T, \lambda_b) + Q_T^0(\theta^0 + v / r_T, \lambda_b)) \right) \pi(\lambda_b) \, d\lambda_b}{\int_{\Gamma_0} \exp \left( \left( \frac{\gamma_T}{(T \lVert \delta_T \rVert^2)} \right) (G_T(\theta^0 + \tilde{v} / r_T, \lambda_b) + Q_T^0(\theta^0 + v / r_T, \lambda_b)) \right) \pi(\lambda_b) \, d\lambda_b}. \tag{3.6}
\]

For each \( v \), we show weak convergence as a function of \( \tilde{v} \) to a limit process that does not depend on \( v \). In a second step, we use the monotonicity in \( v \) of \( Q_T^0 \) which, relying on the argument in Jurečková (1977), allows us to achieve weak convergence uniformly in \( v \). We first show the consistency and rate of convergence of \( \hat{\lambda}_b^{GL} \). These results imply that \( \theta^0 \) is estimated as if \( T^0 \) were known. Thus, \( \hat{\theta} \) is \( \sqrt{T} \)-consistent and asymptotically normal so that we set \( r_T = \sqrt{T} \) hereafter. We first show, for each pair \( (v, \tilde{v}) \) with \( v, \tilde{v} \in V \), the convergence of the marginal distributions of the sample function \( \Psi_{l,T}(s; \tilde{v}, v) \) to the marginal distributions of the random function

\[
\Psi^0_l(s) = \int_{\mathbb{R}} l(s - u) \left( \mathcal{V}(u) / \int_{\mathbb{R}} \mathcal{V}(v) \, dv \right) \, du,
\]
where

\[
\mathcal{V}(s) \triangleq \mathcal{V}(s) - \mathcal{A}(s) \triangleq \begin{cases} 
2 \left( (\delta^0)' \Sigma_1 \delta^0 \right)^{1/2} W_1(-s) - |s| (\delta^0)' V_1 \delta^0, & \text{if } s \leq 0 \\
2 \left( (\delta^0)' \Sigma_2 \delta^0 \right)^{1/2} W_2(s) - s (\delta^0)' V_2 \delta^0, & \text{if } s > 0, 
\end{cases}
\]  

and \( W_1, W_2 \) are independent standard Wiener processes defined on \([0, \infty)\). The limit process \( \Psi^0_t(s) \) does not depend on \( v \) nor \( \tilde{v} \). Next, we show that the family of probability measures in \( C_b(K) \), with \( K \triangleq \{s \in \mathbb{R} : |s| \leq K \} \) and \( K < \infty \), generated by the contractions of \( \Psi_{t,T}(s; \tilde{v}, v) \) on \( K \) is dense uniformly in \((v, \tilde{v})\). Finally, we examine the oscillations of the minimizers of the sample criterion \( \Psi_{t,T}(s; v, \tilde{v}) \).

It is important to note that the results derived in this section are more general than what is required for the structural change model. The reason is that the change-point model is recovered as a special case corresponding to \( \Psi_{t,T}(s) = \Psi_{t,T}(s; 0, 0) \). That is, defining the GL estimator in a \( 1/r_T \)-neighborhood of the slope parameter vector \( \theta^0 \) is not strictly necessary and one can essentially develop the same analysis with \( \theta \) fixed at its true value \( \theta^0 \). This relies on the properties of (orthogonal) least-squares projections and would not apply, for example, to the least absolute deviation (LAD) estimator of the break date [cf. Bai (1995)] for which \( \Psi_{t,T}(s; \tilde{v}, v) \) should instead be considered. The same issue is present when estimating structural changes in the quantile regression model [cf. Oka and Qu (2010)] and in using instrumental variables models [cf. Hall, Han, and Boldea (2010) and Perron and Yamamoto (2014; 2015)]. We establish theoretical results under this more general setting since they may be useful for future work.

Let \( \lambda^0_{b,T}(v) \) be the local parameter \( u = \psi_T \left( \lambda_b - \lambda^0_{b,T}(v) \right) \) and let \( \pi_{T,v}(u) \triangleq \pi \left( \lambda^0_{b,T}(v) + u/\psi_T \right) \), \( Q_{T,v}(u) \triangleq Q^0_T(\theta^0 + u/\psi_T, \gamma^0_{b,T} + u/\psi_T) \), and \( G_{T,v}(u, \tilde{v}) \triangleq G_T(\theta^0 + \tilde{v}/\psi_T, \lambda^0_{b,T}(v) + u/\psi_T) \), where the sequence \( \{\psi_T\} \) depends on the results on consistency and rate of convergence of \( \lambda^0_{b,GL} \). In Proposition 3.1, we establish an alternative statement in (3.6) to yield,

\[
\Psi_{t,T}(s; \tilde{v}, v) = \int_{\Gamma_T} l(s - u) \frac{\exp \left( \left( \gamma_T/T \| \delta^0 \|^2 \right) \left( G_{T,v}(u, \tilde{v}) + Q_{T,v}(u) \right) \right) \pi_{T,v}(u) du}{\int_{\Gamma_T} \exp \left( \left( \gamma_T/T \| \delta^0 \|^2 \right) \left( G_{T,v}(w, \tilde{v}) + Q_{T,v}(w) \right) \right) \pi_{T,v}(w) dw},
\]  

(3.8)

where \( \Gamma_T \triangleq \{u \in \mathbb{R} : \lambda^0_b + u/\psi_T \in \Gamma^0\} \).

**Assumption 3.4.** \( \{(z_t, e_t)\} \) is second-order stationary within each regime such that \( \mathbb{E}(z_t z_t') = V_1 \) and \( \mathbb{E}(e_t^2) = \sigma_1^2 \) for \( t \leq T_b^0 \) and \( \mathbb{E}(z_t z_t') = V_2 \) and \( \mathbb{E}(e_t^2) = \sigma_2^2 \) for \( t > T_b^0 \).

**Assumption 3.5.** For \( r \in [0, 1] \), \( (T_b^0)^{-1/2} \sum_{t=1}^{[rT_b^0]} z_t e_t \Rightarrow \mathcal{G}_1(r) \) and \( (T - T_b^0)^{-1/2} \sum_{t=T_b^0+1}^{[r(T - T_b^0)]} z_t e_t \Rightarrow \mathcal{G}_2(r) \), where \( \mathcal{G}_i(\cdot) \) is a multivariate Gaussian process on \([0, 1] \) with zero mean and covariance \( \mathbb{E}[\mathcal{G}_i(u), \mathcal{G}_i(s)] = \min \{u, s\} \Sigma_i \) \( i = 1, 2 \), \( \Sigma_1 \triangleq \lim_{T \to \infty} \mathbb{E}\left[ (T_b^0)^{-1/2} \sum_{t=1}^{T_b^0} z_t e_t \right]^2 \), \( \Sigma_2 \triangleq \lim_{T \to \infty} \mathbb{E}\left[ (T - T_b^0)^{-1/2} \sum_{t=T_b^0+1}^{T} z_t e_t \right]^2 \). Furthermore, for any \( 0 < r_0 < 1 \) with \( r_0 < \lambda_0 \),
\( T^{-1} \sum_{t=[\lambda_0 T]+1}^{[\lambda_0 T]+1} z_t z_t' \overset{P}{\rightarrow} (\lambda_0 - r_0) V_1 \), and with \( \lambda_0 < r_0 \) \( T^{-1} \sum_{t=[\lambda_0 T]+1}^{[\lambda_0 T]+1} z_t z_t' \overset{P}{\rightarrow} (r_0 - \lambda_0) V_2 \) so that \( \lambda_- \) and \( \lambda_+ \) (the minimum and maximum of the eigenvalues of the last two matrices) satisfy \( 0 < \lambda_- \leq \lambda_+ < \infty \).

Assumptions 3.4-3.5 are equivalent to A9 in Bai (1997) and A7 in Bai and Perron (1998). More specifically, Assumption 3.5 requires that, within each regime, an Invariance Principle holds for \( \{z_t e_t\} \). Let \( \zeta_t \overset{D}{=} z_t e_t \). For \( u \leq 0 \) let \( g(\zeta_t; u) \overset{D}{=} (\delta^0)' \sum_{t=T_0^0}^{T_0^0} \zeta_t \) and \( g(\zeta_t; u, \bar{v}, v; \psi_T, r_T) \overset{D}{=} \sqrt{\psi_T} (\delta^0 + \bar{v}/r_T)' \sum_{t=T_0^0}^{T_{T_0^0}^0} \zeta_t \). Define analogously \( g(\zeta_t; u) \) and \( g(\zeta_t; u, \bar{v}, v; \psi_T, r_T) \) for the case \( T_b > T_0^0 \). We now present some technical assumptions that are necessary for the derivation of the asymptotic results for the GL estimate.

**Assumption 3.6.** For some neighborhood \( \Theta^0 \subset S \) of \( \theta^0 \), (i) for all \( \lambda_b \neq \lambda_0^0 \), \( \bar{Q} (\theta^0, \lambda_b) < \bar{Q} (\theta^0, \lambda_0^0) \); (ii) for any \( v, \bar{v}_1, \bar{v}_2 \in V \) and \( u, s \in \mathbb{R} \),

\[
\Sigma (u, s) \overset{D}{=} \lim_{T \to \infty} \mathbb{E} \left[ \bar{g} (\zeta_t; u, \bar{v}_1, v; \psi_T, r_T) \bar{g} (\zeta_t; s, \bar{v}_2, v; \psi_T, r_T)' \right],
\]

does not depend on \( v, \bar{v}_1, \bar{v}_2 \in V \).

Part (i) of Assumption 3.6 is an identification condition. Assumption 3.6-(ii) holds whenever \( \hat{\lambda}_b \) is consistent. With Assumption 3.6-(ii) we fully characterize the Gaussian component of the limit process \( \mathcal{Y} (\cdot) \); it implies that \( \Sigma (\cdot, \cdot) \) is strictly positive and that

\[
\forall u, s \in \mathbb{R} : \begin{cases}
\forall c > 0 : \Sigma (cu, cs) = c\Sigma (u, s), \\
\Sigma (u, u) + \Sigma (s, s) - 2\Sigma (u, s) = \Sigma (u - s, u - s),
\end{cases}
\]

(3.9)

where the second implication requires some simple but tedious manipulations. Finally, the following assumption is automatically satisfied if \( l (\cdot) \) is a convex function with a unique minimum.

**Assumption 3.7.** \( \xi_0^0 \overset{D}{=} \xi (\lambda_0^0) \) is uniquely defined by

\[
\Psi_t (\xi_0^0) \overset{D}{=} \inf \Psi_t (s) = \inf_s \int_{\mathbb{R}} l (s - u) \left( \exp (\mathcal{Y} (u)) / \left( \int_{\mathbb{R}} \exp (\mathcal{Y} (w)) dw \right) \right) du.
\]

### 3.4 Asymptotic Results for the GL Estimate

We first show the consistency and rate of convergence of the GL estimator. The latter allows us to characterize the rate of \( \psi_T \) and proceed with the asymptotic analysis in a neighborhood of \( \lambda_0^0 \). In practice, the squared loss function is often employed. Hence, it is useful to first present in Theorem 3.1 the theoretical results for this case for which the GL estimator is \( \hat{\lambda}_b^{GL} = \int_{\mathbb{R}} \lambda_b \rho_T (\lambda_b) d\lambda_b \), i.e., the Quasi-posterior mean. This allows us to keep the theoretical results tractable and provide
the main intuition without the need of complex notation. This case is also instructive since we can compare our results with corresponding ones for the least-squares and Bayesian change-point estimators. Corresponding results for general loss functions are given in Theorem 3.2.

3.4.1 Consistency and Rate of Convergence

The rate of convergence is similar to that of the LS estimator; the difference being that \( \psi_T = T^{1-2\vartheta} \) with \( \vartheta \in (0, 1/2) \) for the LS estimator and \( \vartheta \in (0, 1/4) \) for the GL estimator.

**Proposition 3.1.** Under Assumptions 2.1-2.4, 3.1-3.3 and 3.6-(i): (i) \( \hat{\lambda}^{\text{GL}}_b = \lambda^0_b + o_P(1) \); (ii) \( \hat{\lambda}^{\text{GL}}_b = \lambda^0_b + O_P \left( \left( T \| \delta_T \| \right)^{-1} \right) \).

3.4.2 The Asymptotic Distribution of the Quasi-posterior Mean

For the squared loss function \( \hat{\lambda}^{\text{GL}*}_b (\bar{\theta}) \triangleq \hat{\lambda}^{\text{GL}*}_b (\bar{v}, v) \), where

\[
\hat{\lambda}^{\text{GL}*}_b (\bar{v}, v) \triangleq \int_{r_0} \lambda_b \exp \left( \frac{\gamma_T}{\left( T \| \delta_T \| \right)^2} \right) \left( G_T \left( \theta^0 + \bar{v} / r_T, \lambda_b \right) + Q_T^b \left( \theta^0 + v / r_T, \lambda_b \right) \right) \pi (\lambda_b) d\lambda_b \bigg/ \int_{r_0} \exp \left( \frac{\gamma_T}{\left( T \| \delta_T \| \right)^2} \right) \left( G_T (\theta^0 + \bar{v} / r_T, \lambda_b) + Q_T^0 (\theta^0 + v / r_T, \lambda_b) \right) \pi (\lambda_b) d\lambda_b,
\]

(3.10)

and \( v, \bar{v} \) each belong to some compact set \( V \subset \mathbb{R}^{p+2v} \). For each \( v \in V \), we consider \( \hat{\lambda}^{\text{GL}*}_b (\cdot, v) \) as a random process with paths in \( \mathcal{D}_b (V) \). We focus on the weak convergence of \( \hat{\lambda}^{\text{GL}*}_b (\cdot, v) \) for fixed \( v \) since the limit process is independent of \( v \) and constant as a function of \( \bar{v} \); we then exploit monotonicity in \( v \). More precisely, we will show that for \( \lambda^0_{b,T} (v) = \lambda^0_{b,T} (\theta^0 + v / r_T) \) and diverging sequences \( \{ \gamma_T \} \) and \( \{ r_T \} \), the sequence \( a_T \left( \hat{\lambda}^{\text{GL}*}_b (\bar{v}, v) - \lambda^0_{b,T} (v) \right) \) converges in distribution in \( \mathcal{D}_b (V) \) for each \( v \) to a limit process not depending on \( v \) nor \( \bar{v} \). Since it is monotonic in \( v \), we do not need to show uniform convergence directly. Introduce the local parameter \( u = \psi_T \left( \lambda_b - \lambda^0_{b,T} (v) \right) \); a simple substitution in (3.10) yields,

\[
\psi_T \left( \hat{\lambda}^{\text{GL}*}_b (\bar{v}, v) - \lambda^0_{b,T} (v) \right) = \frac{\int_{\mathbb{R}} u \exp \left( \frac{\gamma_T}{\left( T \| \delta_T \| \right)^2} \right) \left( \tilde{G}_{T,v} (u, \bar{v}) + Q_{T,v} (u) \right) \pi_{T,v} (u) du}{\int_{\mathbb{R}} \exp \left( \frac{\gamma_T}{\left( T \| \delta_T \| \right)^2} \right) \left( \tilde{G}_{T,v} (u, \bar{v}) + Q_{T,v} (u) \right) \pi_{T,v} (u) du},
\]

(3.11)

where again we have used the notation \( \pi_{T,v} (u) = \pi \left( \lambda^0_{b,T} (v) + u / \psi_T \right) \), \( Q_{T,v} (u) = Q_T^0 \left( \theta^0 + v / r_T \lambda^0_{b,T} (v) + u / \psi_T \right) \) and \( \tilde{G}_{T,v} (u, \bar{v}) = G_T (\theta^0 + \bar{v} / r_T, \lambda^0_{b,T} (v) + u / \psi_T) \). The limit of the GL estimator depends on the limit of the process \( \left( \gamma_T / \left( T \| \delta_T \| \right)^2 \right) \left( \tilde{G}_{T,v} (u, \bar{v}) + Q_{T,v} (u) \right) \). As part of the proof of Theorem 3.1, we show that the sequence of processes \( \left\{ \tilde{G}_{T,v} (u, \bar{v}), T \geq 1 \right\} \) converges weakly in \( \mathcal{D}_b (\mathbb{R} \times V) \) to a Gaussian process \( \mathcal{W}^v \) not varying with \( v \), whereas \( Q_{T,v} (\cdot) \) is approximated by a (deterministic) drift process taking negative values, and is monotonic in \( v \) and flat in \( \bar{v} \). We
show that this implies that $\hat{\lambda}_{b,T}^{\text{GL}}(\tilde{v}, v) - \lambda_{b,T}^0(v)$ is monotonic in $v$ which then leads to uniform convergence in $v$ following the argument of Jurečová (1977).

In anticipation of the results, we make a few comments about the notation for the weak convergence of processes on the space of bounded càdlàg functions $\mathbb{D}_b$. Let $V \subseteq \mathbb{R}^{p+2q}$ be a compact set. Let $W_T(u, \tilde{v}, v)$ denote an arbitrary sample process with bounded càdlàg paths evaluated at the local parameters $u \in \mathbb{R}$, and $v, \tilde{v} \in V$. For each fixed $v \in V$, we shall write $W_T(u, \tilde{v}, v) \Rightarrow \mathcal{W}(u, \tilde{v}, v)$ in $\mathbb{D}_b(\mathbb{R} \times V)$ whenever the process $W_T(\cdot, \cdot, v)$ converges weakly to $\mathcal{W}(\cdot, \cdot, v)$, where $\mathcal{W}(\cdot, \cdot, v)$ also belongs to $\mathbb{D}_b(\mathbb{R} \times V)$. As a shorthand, we shall omit the argument $u(\tilde{v})$ if the limit process does not depend on $u(\tilde{v})$. The same notational conventions are used for the case when $W_T$ is only a function of $(\tilde{v}, v)$. In Theorem 3.1 the convergence holds for every $v \in V$, stated as convergence in $\mathbb{D}_b$.

**Condition 1.** As $T \to \infty$ there exist a positive finite number $\kappa_\gamma$ such that $\gamma_T/T \|\delta_T\|^2 \to \kappa_\gamma$.

**Theorem 3.1.** Assume $l(\cdot)$ is the squared loss function. Under Assumptions 2.1-2.4 and 3.1-3.7, and Condition 1, then in $\mathbb{D}_b$,

$$T \|\delta_T\|^2 \left(\hat{\lambda}_{b}^{\text{GL}} - \lambda_{b}^0\right) \Rightarrow \frac{\int u \exp(\mathcal{W}(u) - \Lambda^0(u)) \, du}{\int \exp(\mathcal{W}(u) - \Lambda^0(u)) \, du} \triangleq \int u p_0^*(u) \, du,$$

(3.12)

where $\mathcal{W}(\cdot)$ and $\Lambda^0(\cdot)$ are defined in (3.7).

Theorem 3.1 states that the asymptotic distribution of the GL estimate is a ratio of integrals of functions of tight Gaussian processes. We shall compare this result with the limiting distribution of the Bayesian change-point estimator of Ibragimov and Has’minskii (1981). They considered a simple diffusion process with a change-point in the deterministic drift [see their eq. (2.17) on pp. 338]. The limiting distribution of the GL estimate from Theorem 3.1 for the case of a break in the mean for model (2.1) is essentially the same (and exactly so in the i.i.d. case with stationary regimes) as theirs. Hence, while the GL estimator has a classical (frequentist) interpretation, it is first-order equivalent in law to a corresponding Bayes-type estimator.

We now present a result about the dual nature of the limiting distribution of the GL estimator. The following proposition shows that, under different conditions on the smoothing sequence parameter $\{\gamma_T\}$, the GL estimator achieves different limiting distributions.

**Condition 2.** As $T \to \infty$, $T \|\delta_T\|^2 / \gamma_T = o(1)$.

**Proposition 3.2.** Assume $l(\cdot)$ is the squared loss function. Under Assumptions 2.1-2.4 and 3.1-3.7, and Condition 2, $T \|\delta_T\|^2 \left(\hat{\lambda}_{b}^{\text{GL}} - \lambda_{b}^0\right) \Rightarrow \arg \max_{s \in \mathbb{R}} \mathcal{V}(s)$ in $\mathbb{D}_b$, with $\mathcal{V}(\cdot)$ defined in (3.7).

**Corollary 3.1.** Define $\Xi = (\delta)^0 \Sigma_2 \delta^0 / (\delta)^0 \Sigma_1 \delta^0$ and $\Xi_Z = (\delta)^0 V_2 \delta^0 / (\delta)^0 V_1 \delta^0$. Under Assumptions 2.1-2.4 and 3.1-3.7, and Condition 2, $(\delta_T^0 V_1 \delta_T^0 / \delta_T^0 \Sigma_1 \delta_T^0) \left(\hat{T}_b^{\text{GL}} - T_{b,T}^0\right) \Rightarrow \arg \max_{s \in \mathbb{R}} \mathcal{V}^*(s)$.
in $\mathbb{D}_b$ where

$$V^*(s) = W_1(-s) - |s|/2$$ if $s \leq 0$;  
$$V^*(s) = \Xi_1^{1/2}W_2(s) - \Xi_2 s/2$$ if $s > 0$.

Corollary 3.1 and Proposition 3.2 show that with enough smoothing applied, the GL estimator is (first-order) asymptotically equivalent to the least-squares or MLE [cf. Bai (1997) and Yao (1987), respectively]. The intuition is that when the criterion function is sufficiently smoothed, the Quasi-posterior probability density converges to the generalized dirac probability measure concentrated at the argmax of the limit criterion function. This is analogous to a well-known result [cf. Corollary 5.11 in Robert and Casella (2004)], stating that in a parametric statistical experiment indexed by a parameter $\theta \in \Theta$, the MLE $\hat{\theta}_{\text{ML}}$ is the limit of a Bayes estimator as the smoothing parameter $\gamma \to \infty$, i.e., using obvious notation:

$$\hat{\theta}_{\text{ML}} = \arg\max_{\theta \in \Theta} L_T(\theta) = \lim_{\gamma \to \infty} \frac{\int_{\Theta} \theta \exp(\gamma L_T(\theta)) \pi(\theta) d\theta}{\int_{\Theta} \exp(\gamma L_T(\theta)) \pi(\theta) d\theta}.$$  

3.4.3 The Asymptotic Distribution for General Loss Functions

For general loss functions satisfying Assumption 3.1, Theorem 3.2 shows that $T \|\delta_T\|^2 (\hat{\lambda}_{b}^{\text{GL}} - \lambda_0^b)$ is (first-order) asymptotically equivalent to $\xi_l^0$ defined by

$$\Psi_l(\xi_l^0) \triangleq \inf_r \Psi_l(r) = \inf_{r \in \mathbb{R}} \left\{ \int_{\mathbb{R}} l(r-u) p_0^*(u) du \right\}. \quad (3.13)$$

**Theorem 3.2.** Under Assumptions 2.1-2.4 and 3.1-3.7, and Condition 1, for $l \in L$, $T \|\delta_T\|^2 (\hat{\lambda}_{b}^{\text{GL}} - \lambda_0^b) \Rightarrow \xi_l^0$ as defined by (3.13).

The existence and uniqueness of $\xi_l^0$ follow from Assumption 3.7. If one interprets $p_0^*(u)$ as a true posterior density function, then $\xi_l^0$ would naturally be viewed as a Bayesian estimator for the loss function $l_T(\cdot)$. In particular, in analogy to the above comparison with the Bayesian estimator of Ibragimov and Has’minskii (1981), one can interpret the GL estimator as a Quasi-Bayesian estimator. While this is by itself a theoretically interesting result, we actually exploit it to construct more reliable inference methods about the date of a structural change. Under the least-absolute deviation loss, the GL estimator converges in distribution to the median of $p_0^*(u)$. We shall use the results in Theorem 3.1-3.2 but not Proposition 3.2 since the latter implies the same confidence intervals as in Bai (1997) and Bai and Perron (1998). GL inference based on the Bayes-type limiting distribution provides a more accurate description of the uncertainty over the parameter space than the inference based on the density of $\arg\max_{s \in \mathbb{R}} V(s)$ which underestimates uncertainty as shown by confidence intervals with empirical coverage rates below the nominal level particularly when the magnitude of the break is small (see Section 7). After some investigation,
we found that both estimation and inference under the least-absolute loss works well and this is what will be used in our simulation study.

4 Confidence Sets Based on the GL Estimator

In this section, we discuss inference procedures for the break date based on the large-sample results of the previous section. Inference under general loss functions based on Theorem 3.2 is what we recommend to use in practice, in particular with an absolute loss function.

Since the limiting distribution from Theorem 3.2 involves certain unknown quantities, we begin by assuming that they can be replaced by consistent estimates. They are easy to construct [cf. Bai (1997) and Bai and Perron (1998); see also Section 7].

Assumption 4.1. There exist sequences of estimators \( \hat{\lambda}_{b,T}, \hat{\delta}_T, \hat{\Xi}_{Z,T}, \) and \( \hat{\Xi}_{e,T} \) such that \( \hat{\lambda}_{b,T} = \lambda_0 + o_p(1), \hat{\delta}_T = \delta + o_P(1), \hat{\Xi}_{Z,T} = \Xi_Z + o_P(1) \) and \( \hat{\Xi}_{e,T} = \Xi_e + o_P(1) \). Furthermore, for all \( u, s \in \mathbb{R} \) and any \( c > 0 \), there exist covariation processes \( \hat{\Sigma}_{i,T}(\cdot) \) (\( i = 1, 2 \)) that satisfy (i) \( \hat{\Sigma}_{i,T}(u, s) = \Sigma^0_i(u, s) + o_P(1) \), (ii) \( \hat{\Sigma}_{i,T}(u - s, u - s) = \hat{\Sigma}_{i,T}(u, u) + \hat{\Sigma}_{i,T}(s, s) - 2 \hat{\Sigma}_{i,T}(s, u) \), (iii) \( \hat{\Sigma}_{i,T}(cu, cu) = c^2 \hat{\Sigma}_{i,T}(u, u) \), (iv) \( \mathbb{E}\left\{ \sup_{\|u\|=1} \hat{\Sigma}^2_{i,T}(u, u) \right\} = O(1) \).

The first part and (i) of the second part follow from consistency of \( \hat{\lambda}_{b,T} \) and from an Invariance Principle [cf. Assumption 3.5]. Part (ii)-(iii) are implied by Assumption 3.6-(ii) and consistency of \( \hat{\lambda}_{b,T} \). Let \( \{ \hat{\mathcal{W}}_T \} \) be a (sample-size dependent) sequence of two-sided zero-mean Gaussian processes with covariance \( \hat{\Sigma}_T \). Construct the process \( \hat{\mathcal{W}}_T \) by replacing the population quantities in \( \mathcal{W} \) by their corresponding estimates from the first part of Assumption 4.1 and further, replace \( \mathcal{W} \) by \( \hat{\mathcal{W}}_T \). Assumption 4.1-(i) basically implies that the finite-dimensional limit law of \( \{ \hat{\mathcal{W}}_T \} \) is the same as the finite-dimensional laws of \( \mathcal{W} \) while parts (ii)-(iii) are needed for the integrability of the transform \( \exp \left( \hat{\mathcal{W}}_T (\cdot) \right) \). Part (iv) is needed for the proof of the asymptotic stochastic equicontinuity of \( \{ \hat{\mathcal{W}}_T \} \).

Let \( \xi_T \) be defined as the sample analogue of \( \xi^0_l \) that uses \( \hat{\mathcal{W}}_T (v) \) in place of \( \mathcal{W}_T (v) \) in (3.13). The distribution of \( \xi_T \) can be evaluated numerically.

Proposition 4.1. Let \( l \in \mathbf{L} \) be continuous. Under Assumption 4.1, \( \hat{\xi}_T \) converges in distribution to the limiting distribution in Theorem 3.2.

The asymptotic distribution theory of the GL estimator may be exploited in several ways to conduct inference about the break date. The finite-sample distribution of the LS and GL estimate of the break date displays significant non-standard features (cf. Figure 1). Hence, a conventional two-sided confidence interval may not result in a confidence set with reliable properties across all break magnitudes and break locations. Thus, as in Casini and Perron (2020a), we use the concept of Highest Quasi-posterior Density (HQPD) regions, defined analogously to the Highest Density Region (HDR); cf. Hyndman (1996). See also Samworth and Wand (2010) and Mason and Polonik.
(2008, 2009) for more recent developments. For an illustrative example on the properties of the HDR see the discussion of Figure 11 in Casini and Perron (2020a).

**Definition 4.1.** Highest Density Region: Let the density function $f_Y(y)$ of a random variable $Y$ defined on a probability space $(\Omega_Y, \mathcal{F}_Y, \mathbb{P}_Y)$ and taking values on the measurable space $(Y, \mathcal{Y})$ be continuous and bounded. The $(1 - \alpha) 100\%$ Highest Density Region is a subset $S(\kappa_\alpha)$ of $Y$ defined as $S(\kappa_\alpha) = \{ y : f_Y(y) \geq \kappa_\alpha \}$ where $\kappa_\alpha$ is the largest constant that satisfies $\mathbb{P}_Y(Y \in S(\kappa_\alpha)) \geq 1 - \alpha$.

For $s = T \| \delta_T \|^2 \left( \hat{\lambda}_b^{LS} - \lambda_b^0 \right)$, the asymptotic distribution theory of Bai (1997) suggests a belief $\pi(s)$ over $s \in \mathbb{R}$. This belief function can be used as a Quasi-prior for $\lambda_b$ in the definition of the Quasi-posterior $p_T(\lambda_b)$.

Let $\mu(\lambda_b)$ denote some density function defined by the Radon-Nikodym equation $\mu(\lambda_b) = dp_T(\lambda_b)/d\lambda_L$, where $\lambda_L$ denotes the Lebesgue measure. The following algorithm describes how to construct a confidence set for $T_b^0$.

**Algorithm 1.** GL HQDR-based Confidence Sets for $T_b^0$:
1. Estimate by least-squares the break date and the regression coefficients from model (2.3);
2. Set the Quasi-prior $\pi(\lambda_b)$ equal to the probability density of the limiting distribution from Corollary 3.1;
3. Construct the Quasi-posterior given in (3.1);
4. Obtain numerically the density $\mu(\lambda_b)$ as explained above and label it by $\hat{\mu}(\lambda_b)$;
5. Compute the Highest Quasi-Posterior Density (HQPD) region of the probability distribution $\hat{p}_T(\lambda_b)$ and include the point $T_b$ in the level $(1 - \alpha)\%$ confidence set $C_{HQPD}(cv_\alpha)$ if $T_b$ satisfies Definition 4.1.

If a general Quasi-prior $\pi(\lambda_b)$ is used, one begins directly with step 3.

In principle, any Quasi-prior $\pi(\lambda_b)$ satisfying Assumption 3.2 can be used. Note that $C_{HQPD}(cv_\alpha)$ retains a frequentist interpretation, since no parametric likelihood function of the data is required.

### 5 Models with Multiple Change-Points

Following Bai and Perron (1998), the multiple linear regression model with $m$ change-points is

$$y_t = w_t'\phi^0 + z_t'\delta_j^0 + e_t, \quad (t = T^0_{j-1} + 1, \ldots, T^0_j)$$

for $j = 1, \ldots, m + 1$, where by convention $T^0_0 = 0$ and $T^0_{m+1} = T$. There are $m$ unknown break points $(T^0_1, \ldots, T^0_m)$ and consequently $m + 1$ regimes each corresponding to a distinct parameter value $\delta^0_j$. The purpose is to estimate the unknown regression coefficients together with the break points when $T$ observations on $(y_t, w_t, z_t)$ are available. Many of the theoretical results follow
directly from the single break case; the break points are asymptotically distinct and thus, given the mixing conditions, our results for the single break date extend readily to multiple breaks. More complicated is the computation of the estimates of the break dates which has been addressed by Bai and Perron (2003) who proposed an efficient algorithm based on the principle of dynamic programming; see also Hawkins (1976).

Let \( T_i \triangleq \lfloor T \lambda_i \rfloor \) and \( \theta \triangleq (\phi', \delta', \Delta_1', \ldots, \Delta_m')' \) where \( \Delta_i = \delta_{i+1} - \delta_i, i = 1, \ldots, m \). The class \( \mathcal{L} (\theta, T_i; 1 \leq i \leq m) \) of GL estimators in multiple change-points models relies on the least-squares criterion function \( Q_T (\delta (\lambda_b), \lambda_b) = \sum_{i=1}^{m+1} \sum_{T_{i-1}}^{T_i} (y_t - w_i' \phi - z_i' \delta_i)^2 \), with \( \lambda_b \triangleq (\lambda_i; 1 \leq i \leq m) \). In order to state the large-sample properties, we need to introduce the shrinkage theoretical framework of Bai and Perron (1998).

**Assumption 5.1.** (i) Let \( x_t = (w_t', z_t')' \), \( X = (x_1, \ldots, x_T)' \) and \( \Xi_\theta = \text{diag} (X_1, \ldots, X_{m+1}) \) be the diagonal partition of \( X \) at \( (T_1, \ldots, T_m) \). For each \( i = 1, \ldots, m+1 \), \( X_i' X_i / (T_i - T_{i-1}) \) converges to a non-random positive definite matrix not necessarily the same for all \( i \). (ii) Assumption 2.3 holds. (iii) The matrix \( \sum t=k z_t z_t' \) is invertible for \( l - k \geq q \). (iv) \( T_i^0 = \lfloor T \lambda_i^0 \rfloor \), where \( 0 < \lambda_1^0 < \cdots < \lambda_m^0 < 1 \). (v) Let \( \Delta_{T,i} = v_T^0 \Delta_i^0 \) where \( v_T > 0 \) is a scalar satisfying \( v_T \to 0 \) and \( T^{1/2 - \theta} v_T \to \infty \) for some \( \theta \in (0, 1/4) \), and \( E \| z_t \|^2 < C, E \| e_t \|^{2 \theta} < C \) for some \( C < \infty \) and all \( t \).

**Assumption 5.2.** Let \( \Delta T_i^0 = T_i^0 - T_{i-1}^0 \). For \( i = 1, \ldots, m+1 \), uniformly in \( s \in [0, 1] \), (a) \( (\Delta T_i^0)^{-1} \sum_{T_{i-1}^0}^{T_i^0} [s \Delta T_i^0] z_t z_t' \to \mathbb{P} s V_i, (\Delta T_i^0)^{-1} \sum_{T_{i-1}^0}^{T_i^0} [s \Delta T_i^0] e_i' \to \mathbb{P} s \sigma_i^2 \), and

\[
\left( \Delta T_i^0 \right)^{-1} \sum_{t=T_{i-1}^0}^{T_i^0} [s \Delta T_i^0] z_t u_t' \to \mathbb{P} s \Sigma_i;
\]

(b) \( (\Delta T_i^0)^{-1/2} \sum_{T_{i-1}^0}^{T_i^0} [s \Delta T_i^0] z_t u_t \to \mathbb{P} \mathcal{G}_i (s) \) where \( \mathcal{G}_i (s) \) is a multivariate Gaussian process on \( [0, 1] \) with mean zero and covariance \( E [\mathcal{G}_i (s) \mathcal{G}_i (u)] = \min \{ s, u \} \Sigma_i \).

Next, for \( i = 1, \ldots, m \), define \( \Xi_{Z,i} = (\Delta_i^0)' V_{i+1} \Delta_i^0 / (\Delta_i^0)' V_i \Delta_i^0, \Xi_{e,i}^2 = (\Delta_i^0)' \Sigma_{i+1} \Delta_i^0 / (\Delta_i^0)' \Sigma_i \Delta_i^0 \), and let \( W_1^{(i)} (s) \) and \( W_2^{(i)} (s) \) be independent Wiener processes defined on \( [0, \infty) \), starting at \( 0 \) when \( s = 0 \); \( W_1^{(i)} (s) \) and \( W_2^{(i)} (s) \) are also independent over \( i \). Finally, define

\[
\gamma^{(i)} (s) = \gamma^{(i)} (s) - A_i^0 (s) = \begin{cases} 2 \left( (\Delta_i^0)' \Sigma_i \Delta_i \right)^{1/2} W_1^{(i)} (-s) - |s| (\Delta_i^0)' V_i \Delta_i, & \text{if } s \leq 0 \\ 2 \left( (\Delta_i^0)' \Sigma_{i+1} \delta_0 \right)^{1/2} W_2^{(i)} (s) - s (\Delta_i^0)' V_{i+1} \Delta_i, & \text{if } s > 0. \end{cases}
\] (5.1)

We now extend the notation of Section 3 to the present context. By redefining the Quasi-posterior \( p (\lambda_b) \) in terms of \( \lambda_b \), the GL estimator as the minimizer of the associated risk function [recall (3.2)], \( \lambda_b^{GL} = \arg\min_{s \in I^0} \{ R (\lambda_b, t) (s) \} \), where now \( I^0 = B_1 \times \cdots \times B_m, \) with \( B_i \) a compact subset of \( (0, 1) \). The sets \( B_i \) are disjoint and satisfy \( \sup_{\lambda \in B_i} < \inf_{\lambda \in B_{i+1}} \) for all \( i \).
Assumption 5.3. Assumptions 3.1-3.2 hold with obvious modifications to allow for the multidimensional parameter \( \lambda_b \in \Gamma^0 \). Assumption 3.6 holds where now in part (i) \( \lambda_b \) replaces \( \lambda_b \), and in part (ii) \( \Sigma^{(i)} (\cdot, \cdot) \) replaces \( \Sigma (\cdot, \cdot) \) and is defined analogously for each regime.

Assumption 3.7 implies that \( \xi^0_{l,i} \triangleq \xi (\lambda^0) \) is uniquely defined by \( \Psi (\xi^0_{l,i}) \triangleq \inf_s \Psi_{l,i} (s) = \inf_s \int_{\mathbb{R}} l (s - u) \left( \exp \left( \mathcal{Y}^{(i)} (u) \right) / \left( \int_{\mathbb{R}} \exp \left( \mathcal{Y}^{(i)} (w) \right) dw \right) \right) du \). The GL estimator is defined as the minimizer of

\[
\mathcal{R}_{l,T} \triangleq \int_{\mathbb{R}} l (s - \lambda_b) \frac{\exp (-Q_T (\delta (\lambda_b), \lambda_b)) \pi (\lambda_b)}{\int_{\mathbb{R}} \exp (-Q_T (\delta (\lambda_b), \lambda_b)) \pi (\lambda_b) d\lambda_b} d\lambda_b.
\]

The analysis is now in terms of the \( m \times 1 \) local parameter \( u \) with components \( u_i = T \| \Delta_{T,i} \|^2 (\lambda_i - \lambda_{T0,i} (v)) \), with \( \lambda_{T0,i} (v) = \lambda_{i,T} (\theta^0 + v / r_T) \).

Theorem 5.1-5.2 extend corresponding results from Theorem 3.1-3.2, respectively, to multiple change-points. The fast rate of convergence implies that asymptotically the behavior of the GL estimator only matters in a small neighborhood of each \( T^0_i \). Since each such neighborhood increases at rate \( 1 / v_T \) while \( T \to \infty \) at a faster rate, given the mixing conditions, these are asymptotically distinct and the limiting distribution is then similar to that in the single break case. This is the same argument underlying the analysis of Bai and Perron (1998) and of Ibragimov and Has’minskii (1981). The same comments as those in Section 3 apply.

Condition 3. For \( 1 \leq i \leq m \) there exist positive finite numbers \( \kappa_{\gamma,i} \) such that \( \gamma_T / T \| \Delta_{T,i} \|^2 \to \kappa_{\gamma,i} \).

Theorem 5.1. Assume \( l (\cdot) \) is the squared loss function. Under Assumption 5.1-5.3 and Condition 3, we have in \( \mathbb{D}_b \),

\[
T \| \Delta_{T,i} \|^2 \left( \hat{\lambda}^{GL}_i - \lambda^0_i \right) \Rightarrow \frac{\int_{\mathbb{R}} u \exp \left( \mathcal{Y}^{(i)} (u) - A^0_i (u) \right) du}{\int \exp \left( \mathcal{Y}^{(i)} (u) - A^0_i (u) \right) du}.
\]

Turning to the general case of loss functions satisfying Assumption 3.1, Theorem 5.2 shows that the random quantity \( T \| \delta_T \|^2 \left( \hat{\lambda}^{GL}_i - \lambda^0_i \right) \) is (first-order) asymptotically equivalent to the random variable \( \xi^0_{l,i} \) determined by

\[
\Psi (\xi^0_{l,i}) \triangleq \inf_r \Psi_{l,i} (r) = \inf_{\mathbb{R}} \left\{ \int_{\mathbb{R}} l (r - u) \frac{\exp \left( \mathcal{Y}^{(i)} (u) - A^0_i (u) \right)}{\int \exp \left( \mathcal{Y}^{(i)} (u) - A^0_i (u) \right) du} du \right\}.
\]

Theorem 5.2. Under Assumptions 5.1-5.3 and Condition 3, for \( l \in \mathcal{L} \), \( T \| \Delta_{T,i} \|^2 \left( \hat{\lambda}^{GL}_i - \lambda^0_i \right) \Rightarrow \xi^0_{l,i} \), as defined by (5.3).

A direct consequence of the results of this section is that statistical inference for the break dates \( T^0_i \) (i = 1, ..., m) can be carried out using the same methods for the single break case.
6 Theoretical Properties of GL Inference

This section shows that the GL-HPDR confidence sets are bet-proof. The betting framework and the notion of bet-proofness are useful to study the properties of frequentist inference in non-regular problems. The literature concluded that frequentist confidence sets may exhibit undesirable properties in non-regular problems [e.g., Buehler (1959), Cornfield (1969), Cox (1958), Müller and Norest (2016) Pierce (1973), Robinson (1977) and Wallace (1959)]. For example, the confidence sets can be too short or empty with positive probability. This arises because frequentist procedures often have the property that, conditional on a sample point lying in some subset of the sample space, the conditional confidence level is less than the unconditional confidence level uniformly in the parameters.

We use the same betting framework as in Buehler (1959). Let $P (\cdot | \lambda_b)$ denote the likelihood of the data $Y \in \mathcal{Y}$ conditional on $\lambda_b \in \Gamma^0$. Assume $P (\cdot | \lambda_b)$ has density $p (\cdot | \lambda_b)$ with respect to a finite measure $\zeta$. We define a $1 - \alpha$ confidence set by a rejection probability rule $\varphi : \Gamma^0 \times \mathcal{Y} \mapsto [0, 1]$ satisfying $\int [1 - \varphi (\lambda_b, y)] p (y | \lambda_b) d\zeta (y) \geq 1 - \alpha$, with $\varphi (\lambda_b, y)$ the probability that $\lambda_b$ is not included in the set when $y$ is observed. For any realization of the data $Y = y$, an inspector can choose to object to the confidence set $\varphi$. The inspector’s objection $\tilde{b} : \mathcal{Y} \mapsto [0, 1]$ takes value 1 if there is an objection. Denote by $B$ the set of all measurable strategies $\tilde{b}$. When $\tilde{b} = 1$ the inspector receives 1 if $\varphi$ does not contain $\lambda_b$, and she loses $\alpha / (1 - \alpha)$ otherwise. For a given parameter $\lambda_b$ and betting strategy $\tilde{b}$, the inspector’s expected loss is,

$$L_\alpha (\varphi, \tilde{b}, \lambda_b) = \frac{1}{1 - \alpha} \int [\alpha - \varphi (\lambda_b, y)] \tilde{b} (y) p (y | \lambda_b) d\zeta (y).$$

A confidence set $\varphi$ is said to be bet-proof at level $1 - \alpha$ if for each $\tilde{b} \in B$, $L_\alpha (\varphi, \tilde{b}, \lambda_b) \geq 0$ for some $\lambda_b \in \Gamma^0$. If there exists a strategy $\tilde{b}$ such that $L_\alpha (\varphi, \tilde{b}, \lambda_b) < 0$ for all $\lambda_b \in \Gamma^0$, then the inspector would be right on average and would make positive expected profits. Hence, such $\varphi$ would be an “unreasonable” confidence set. Without loss of substance, we restrict our attention to a change in the mean of a sequence of i.i.d. Gaussian variables. Let $y_t = \delta_T 1 \{t > T^0_b\} + e_t$, where $e_t \sim i.i.d. \mathcal{N} (0, 1)$. The result below can also be shown to hold for fixed shifts $\delta_T = \delta^0$. For ease of exposition, we assume $\delta^0$ known. The general case leads to similar results, with more lengthy derivations without any gain in intuition.

Recall that $\varphi$ is such that the Quasi-posterior probability $p_T (\lambda_b | y) = p_T (\lambda_b)$ of excluding $\lambda_b$ is less than or equal to $\alpha$,

$$\int \varphi (\lambda_b, y) p_T (\lambda_b | y) d\lambda_b \leq \alpha \quad \text{for all } y \in \mathcal{Y}. \quad (6.1)$$

Proposition 6.1. Under Assumptions 2.1-2.4 and 3.1-3.7, and Condition 1, for $l \in L : (i) \varphi$ is...
bet-proof at level $1 - \alpha$; (ii) If (6.1) holds with equality, then $\varphi$ is the shortest confidence set in the class of level $1 - \alpha$ confidence sets, i.e., there cannot exist a level $1 - \alpha$ confidence set $\varphi'$ with the property that, for all $y \in \mathcal{Y}$ \( \int \varphi' (\lambda_b, y) \, d\lambda_b \geq \int \varphi (\lambda_b, y) \, d\lambda_b \), and for all $y \in \mathcal{Y}_0$ with $\zeta (\mathcal{Y}_0) > 0$, $\int \varphi' (\lambda_b, y) \, d\lambda_b > \int \varphi (\lambda_b, y) \, d\lambda_b$.

Part of the proof shows that the Quasi-posterior is asymptotically equivalent (in total variation distance) to the Bayesian posterior. Given the conservativeness allowed by Definition 4.1, the GL confidence interval is asymptotically a superset of a Bayesian credible interval. Bet-proofness is a useful criterion in change-point models where popular inference methods face some difficulties, as shown in the next section. Proposition 6.1 suggests that GL inference should not suffer from these issues; the simulations in the next section will confirm that this is indeed the case.

7 Finite-Sample Evaluations

The purpose of this section is twofold. Section 7.1 assesses the accuracy of the GL estimate of the change-point while Section 7.2 evaluates the small-sample properties of the proposed method to construct confidence sets. We consider DGPs that take the form:

$$y_t = D_t \alpha^0 + Z_t \beta^0 + Z_t \delta^0 1 \{ t > T_b^0 \} + e_t, \quad t = 1, \ldots, T, \quad (7.1)$$

with a sample size $T = 100$. Three versions of (7.1) are investigated: M1 involves a break in mean: $Z_t = 1$, $D_t$ absent, and $e_t \sim \text{i.i.d. } \mathcal{N} (0, 1)$; M2 is similar to M1 but with $e_t = 0.3 e_{t-1} + u_t$, $u_t \sim \mathcal{N} (0, 1)$; M3 is a dynamic model with $D_t = y_{t-1}$, $Z_t = 1$, $e_t \sim \text{i.i.d. } \mathcal{N} (0, 0.5)$ and $\alpha^0 = 0.6$. We set $\beta^0 = 1$ in M1-M2 and $\beta^0 = 0$ in M3. We consider $\lambda_0 = 0.3$ and 0.5, and break magnitudes $\delta^0 = 0.3, 0.4, 0.6$ and 1. Additional simulations are presented in the supplement.

7.1 Precision of the Change-point Estimate

We consider the following estimators of $T_b^0$: the least-squares estimator (OLS), the GL estimator under a least-absolute loss function (GL-LN); the GL estimator under a least-absolute loss function with a uniform prior (GL-Uni). We compare the mean absolute error (MAE), standard deviation (Std), root-mean-squared error (RMSE), and the 25% and 75% quantiles. We set the trimming parameter $\epsilon$ equal to 0.05. As explained in Casini and Perron (2020b), the trimming $\epsilon$ should not be chosen too high because otherwise the estimate might tend to overestimate (resp. underestimate) the break date if it is in the first (resp. second) half of the sample. They found that $\epsilon = 0.05$ performs well for different locations of the break date and this is also confirmed in the simulations in this section. See Section 1 and 5 in Casini and Perron (2020b) for more discussion.
Tables 1-3 present the results. When the magnitude of the break is small, the OLS estimator displays quite large MAE, which increases as the change-point point moves toward the tails. In contrast, the GL estimator shows substantially lower MAE uniformly over break magnitudes and break locations. In addition, the GL estimator has smaller variance as well as lower RMSE compared to the OLS estimator. Notably, the distribution of GL-LN concentrates a higher fraction of the mass around the mid-sample relative to the finite-sample distribution of the OLS estimate. This is mainly due to the fact that the Quasi-posterior essentially does not share the marked trimodality of the finite-sample distribution [cf. Casini and Perron (2020a)]. When the break magnitude is small, the objective function is quite flat with a small peak at the OLS estimate. The Quasi-posterior has higher mass close to the OLS estimate—which corresponds to the middle mode—and accordingly lower mass in the tails. The GL estimator that uses the uniform prior (GL-Uni) is also more precise than the OLS estimator, though the margin is smaller. The latter is due to the fact that the GL estimate uses information only from the OLS objective function. We have not reported the bias. However, here is a summary of its behavior which can also be learned from Figures 1-2. When $\lambda_0^b = 0.5$, the bias is small and close to zero because the finite-sample distributions of the estimators are symmetric. When $\lambda_0^b < 0.5$, the bias is positive which means that the break date estimators tend to be on the right of $\lambda_0^b$. The opposite hold for $\lambda_0^b > 0.5$.

7.2 Properties of the GL Confidence Sets

We now assess the performance of the suggested inference procedure for the break date. We compare it with the following existing methods: Bai’s (1997) approach, Elliott and Müller’s (2007) approach based on inverting a sequence of locally best invariant tests using Nyblom’s (1989) statistic, the inverted likelihood-ratio (ILR) method of Eo and Morley (2015) which inverts the likelihood-ratio test of Qu and Perron (2007) and the HDR method proposed in Casini and Perron (2020a) based on continuous record asymptotics, labelled OLS-CR. These methods have been discussed in detail in Casini and Perron (2020a) and in Chang and Perron (2018). We can summarize their properties as follows. The confidence intervals obtained from Bai’s (1997) method display empirical coverage rates often below the nominal level when the size of the break is small. In general, Elliott and Müller’s (2007) approach achieves the most accurate coverage rates but the average length of the confidence sets is always substantially larger relative to other methods.\(^1\) In addition, this approach breaks down in models with serially correlated errors or lagged dependent variables, whereby the length of the confidence set approaches the whole sample as the magnitude of the

\(^1\)This problem is more severe when the errors are serially correlated or the model includes lagged dependent variables. Regarding the former, this in part may be due to issues with Newey and West HAC-type estimators when there are breaks [see Casini (2018, 2019), Casini, Deng, and Perron (2020), Casini and Perron (2019), Chang and Perron (2018), Crainiceanu and Vogelsang (2007), Deng and Perron (2006), Fossati (2018), Juhl and Xiao (2009), Kim and Perron (2009), Martins and Perron (2016), Perron and Yamamoto (2021) and Vogelsang (1999)].
break increases. The ILR has coverage rates often above the nominal level and an average length significantly longer than with the OLS-CR method when the magnitude of the shift is small. Here, we shall show that the GL inference performs well in terms of coverage probability compared with the other methods and is characterized by shorter lengths of the confidence sets.

When the errors are uncorrelated (i.e., M1 and M3) we simply estimate variances rather than long-run variances. The least-squares estimation method is employed with a trimming parameter $\epsilon = 0.15$ and we use the required degrees of freedom adjustment for the statistic $\hat{U}_T$ of Elliott and Müller (2007). To construct the OLS-CR method, we follow the steps outlined in Casini and Perron (2020a). To implement Bai’s (1997) method we use the usual steps described in Bai (1997) and Bai and Perron (1998). We implement the GL estimator using a least-absolute loss with the prior from Corollary 3.1. For model M2, the estimate of the long-run variance is the pre-whitened heteroskedasticity and autocorrelation (HAC) estimator of Andrews and Monahan (1992). We consider the version $\hat{U}_T$ proposed by Elliott and Müller (2007) that allows for heterogeneity across regimes; using the restricted version when applicable leads to similar results. Finally, the last row of each panel includes the rejection probability of the 5%-level sup-Wald test using the asymptotic critical value of Andrews (1993); it serves as a statistical measure of the magnitude of the break.

Overall, the results in Table 4-6 confirm previous findings about the performance of existing methods. Bai’s (1997) method has a coverage rate below the nominal level when the size of the break is small. For example, in model M2, with $\lambda_0 = 0.5$ and $\delta^0_0 = 0.8$, it has a coverage probability below 82% even though the Sup-Wald test rejects roughly for 70% of the samples. With smaller break sizes, it systematically fails to cover the true break date with correct probability. In contrast, the method of Elliott and Müller (2007) yields very accurate empirical coverage rates. However, the average length of the confidence intervals obtained is systematically much larger than those from all other methods across all DGPs, break sizes and break locations. For large break sizes, Bai’s (1997) method delivers good coverage rates and the shortest average length among all methods.

The GL method displays good coverage rates across different break magnitudes and tends to have the shortest lengths among all methods for all break magnitudes, except for $\delta^0 = 1.6$ in model M2 for which Bai’s (1997) confidence interval is slightly shorter. In Model M3, the coverage rates of OLS-CR are more accurate than those with the GL method although the difference is not large. Thus the GL method strikes a good balance between adequate coverage probability and short average lengths, thus confirming the theoretical results on bet-proofness. This is also consistent with Figures 1-2 which show that the asymptotic distribution of the GL estimator does not underestimate uncertainty about the break location even when the break magnitude is small thereby yielding good coverage rates also in this case. In model M1 the GL method leads to shorter lengths than Bai’s even for large breaks. This is not in contradiction with Figure 2 because model M1 is a simpler model than that reported in the figure which shows that the density of the asymptotic distribution of the GL estimator is more spread out than that from Bai (1997).
Non-reported simulations show that the GL method is robust to heteroskedastic errors $e_t = \vert z_t \vert u_t$ and non-normal errors. The case of multiple breaks is not considered since they are expected to be similar as in the single break case by virtue of the assumption that the break dates are sufficiently separated. Finally, in the supplement we compare the GL method above with its continuous record counterparts developed in Casini and Perron (2020b). Overall, we find that both estimation and confidence intervals based on GL-LN perform well relative to the continuous record counterparts, where significant gains appear to occur when there is high serial correlation in the errors. See the additional results reported in the supplement.

8 Conclusions

We developed large-sample results for a class of Generalized Laplace estimators in multiple change-points models where popular methods face some challenges due to the non-regularities of the problem. The GL method exploits the insight of Laplace who proposed to generate a density from taking an exponential transformation of a least-squares criterion. The class of GL estimators exhibits a dual limiting distribution; namely, the classical shrinkage asymptotic distribution of Bai and Perron (1998), or a Bayes-type asymptotic distribution [cf. Ibragimov and Has’minskii (1981)]. Simulations show that the GL estimator is more accurate than OLS. Similarly, inference has superior finite-sample properties relative to popular methods and these properties are shown to be supported by theoretical results. Since the issues about the finite-sample performance of OLS especially for small breaks continue to hold in more complex structural change models, we believe that our method can be usefully extended to those models. For example, the GL approach can be immediately applied to nonlinear models (e.g., instrumental variable models, linear model with restrictions, nonlinear regression models, etc.) even though particular attention to the appropriate choice of the prior should be given in each context. We believe that our approach can also be relevant for high-dimensional regression with structural changes although this would require a careful consideration of additional aspects related to the growing number of regressors.
9 Supplementary Material

Casini, A. and P. Perron (2020c). Supplement to “Generalized Laplace Inference in Multiple Change-points Models”, Econometric Theory Supplementary Material. To view, please visit: [[doi will be inserted here by typesetter]]

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Figure 1: The probability density of the LS estimator for the model \( y_t = \mu^0 + Z_t \delta^0_1 + Z_t \delta^0_2 \{ t > \lfloor T \lambda_0 \rfloor \} + e_t, \) \( Z_t = 0.3Z_{t-1} + u_t - 0.1u_{t-1}, \) \( u_t \sim \text{i.i.d.} \mathcal{N} (0, 1) , \) \( e_t \sim \text{i.i.d.} \mathcal{N} (0, 1), \) \( \{ u_t \} \) independent from \( \{ e_t \}, \) \( T = 100 \) with \( \delta^0_0 = 0.3 \) and \( \lambda_0 = 0.25 \) and 0.5 (the left and right panel, respectively). The black dotted line is the density of the asymptotic distribution from Bai (1997), the red broken line break is the density of the finite-sample distribution of the LS estimator, the green broken line is the density of the finite-sample distribution of the GL estimator, and the blue broken line is the density of the asymptotic distribution of the GL estimator.

Figure 2: The descriptions and comments given in Figure 1 apply but with a break magnitude \( \delta^0 = 1.5. \)
Table 1: Small-sample accuracy of the estimate of the break point $T_0^b$ for model M1

| $\delta^0$ | MAE  | Std  | RMSE | $Q_{0.25}$ | $Q_{0.75}$ | MAE  | Std  | RMSE | $Q_{0.25}$ | $Q_{0.75}$ |
|------------|------|------|------|------------|------------|------|------|------|------------|------------|
|            |      |      |      |            |            |      |      |      |            |            |
| $\lambda_0 = 0.3$ |      |      |      |            |            |      |      |      |            |            |
| $\delta^0 = 0.3$ | OLS  | 21.99| 27.51| 30.53      | 24         | 66    | 21.51| 26.85| 26.79      | 34         | 71    |
| GL-LN      | 13.44| 15.03| 18.99| 28         | 54         | 11.85| 14.51| 14.93      | 38         | 60    |
| GL-Uni     | 17.56| 22.88| 25.51| 26         | 56         | 16.90| 22.03| 22.13      | 38         | 61    |
| $\delta^0 = 0.4$ | OLS  | 20.48| 26.30| 28.51      | 23         | 57    | 15.64| 21.79| 21.23      | 40         | 61    |
| GL-LN      | 13.02| 15.52| 18.29| 29         | 51         | 9.46 | 11.84| 12.30      | 44         | 56    |
| GL-Uni     | 17.68| 22.30| 24.64| 27         | 54         | 12.38| 17.69| 17.15      | 42         | 57    |
| $\delta^0 = 0.6$ | OLS  | 13.04| 20.82| 15.92      | 28         | 41    | 11.06| 16.05| 16.89      | 45         | 55    |
| GL-LN      | 9.20 | 13.67| 13.67| 29         | 40         | 7.04 | 9.92 | 10.46      | 47         | 53    |
| GL-Uni     | 11.49| 18.59| 14.23| 27         | 39         | 9.11 | 13.92| 13.48      | 45         | 55    |
| $\delta^0 = 1$ | OLS  | 3.49 | 4.61 | 4.61       | 28         | 32    | 2.92 | 5.24 | 5.23       | 48         | 52    |
| GL-LN      | 3.41 | 4.53 | 4.52 | 28         | 32         | 2.89 | 5.44 | 5.20       | 49         | 51    |
| GL-Uni     | 3.63 | 4.56 | 4.61 | 28         | 32         | 2.90 | 5.21 | 5.22       | 48         | 52    |

The model is $y_t = \delta^0 1\{t > [T_0^b]\} + e_t, e_t \sim i.i.d. \mathcal{N}(0, 1), T = 100$. The columns refer to Mean Absolute Error (MAE), standard deviation (Std), Root Mean Squared Error (RMSE) and the 25% and 75% empirical quantiles. OLS is the least-squares estimator; GL-LN is the GL estimator under a least-absolute loss function with the density of the long-span asymptotic distribution as the prior; GL-Uni is the GL estimator under a least-absolute loss function with a uniform prior. The number of simulations is 3,000.

Table 2: Small-sample accuracy of the estimates of the break point $T_0^b$ for model M2

| $\delta^0$ | MAE  | Std  | RMSE | $Q_{0.25}$ | $Q_{0.75}$ | MAE  | Std  | RMSE | $Q_{0.25}$ | $Q_{0.75}$ |
|------------|------|------|------|------------|------------|------|------|------|------------|------------|
|            |      |      |      |            |            |      |      |      |            |            |
| $\lambda_0 = 0.3$ |      |      |      |            |            |      |      |      |            |            |
| $\delta^0 = 0.3$ | OLS  | 26.61| 22.85| 33.03      | 23         | 76    | 24.09| 28.29| 28.08      | 23         | 73    |
| GL-LN      | 19.33| 10.17| 24.87| 29         | 61         | 16.01| 18.78| 19.81      | 29         | 62    |
| GL-Uni     | 24.76| 21.05| 31.34| 26         | 70         | 20.93| 25.37| 25.39      | 28         | 65    |
| $\delta^0 = 0.4$ | OLS  | 23.10| 27.99| 30.85      | 21         | 68    | 20.47| 25.55| 25.54      | 33         | 70    |
| GL-LN      | 16.59| 18.59| 22.75| 29         | 60         | 13.68| 17.06| 17.12      | 38         | 61    |
| GL-Uni     | 21.51| 25.87| 28.83| 24         | 61         | 17.91| 22.94| 22.91      | 37         | 62    |
| $\delta^0 = 0.6$ | OLS  | 17.64| 23.51| 25.01      | 24         | 50    | 15.51| 20.93| 20.91      | 41         | 59    |
| GL-LN      | 13.42| 16.63| 18.63| 28         | 47         | 11.06| 14.90| 14.38      | 46         | 54    |
| GL-Uni     | 16.01| 21.54| 22.75| 25         | 47         | 13.92| 19.11| 19.91      | 40         | 58    |
| $\delta^0 = 1$ | OLS  | 8.71 | 15.87| 15.79      | 27         | 34    | 7.24 | 10.73| 10.72      | 47         | 54    |
| GL-LN      | 8.25 | 15.27| 15.61| 27         | 34         | 6.88 | 9.21 | 9.19       | 47         | 52    |
| GL-Uni     | 8.65 | 14.96| 15.21| 27         | 33         | 6.96 | 10.44| 10.45      | 46         | 53    |

The model is $y_t = \delta^0 1\{t > [T_0^b]\} + e_t, e_t = 0.3e_{t-1} + u_t, u_t \sim i.i.d. \mathcal{N}(0, 1), T = 100$. The notes of Table 1 apply.
Table 3: Small-sample accuracy of the estimates of the break point $T^0_\lambda$ for model M3

| | MAE | Std | RMSE | $Q_{0.25}$ | $Q_{0.75}$ | MAE | Std | RMSE | $Q_{0.25}$ | $Q_{0.75}$ |
|---|-----|-----|------|---------|---------|-----|-----|------|---------|---------|
| $\delta^0 = 0.3$ | OLS | 23.66 | 28.14 | 31.32 | 22 | 69 | 22.01 | 26.61 | 26.59 | 33 | 72 |
| | GL-LN | 19.31 | 19.22 | 26.28 | 30 | 57 | 14.89 | 18.18 | 19.08 | 39 | 61 |
| | GL-Uni | 21.38 | 24.12 | 28.08 | 24 | 64 | 18.76 | 22.88 | 22.01 | 31 | 66 |
| $\delta^0 = 0.4$ | OLS | 19.31 | 25.76 | 27.71 | 23 | 57 | 18.14 | 23.43 | 23.44 | 38 | 60 |
| | GL-LN | 15.04 | 17.64 | 21.29 | 29 | 51 | 12.36 | 16.43 | 16.52 | 40 | 60 |
| | GL-Uni | 18.46 | 22.74 | 25.18 | 25 | 58 | 15.91 | 20.42 | 20.42 | 37 | 62 |
| $\delta^0 = 0.6$ | OLS | 12.02 | 19.02 | 19.82 | 25 | 37 | 10.28 | 15.51 | 15.58 | 45 | 55 |
| | GL-LN | 9.29 | 12.86 | 14.61 | 29 | 40 | 8.46 | 11.84 | 11.86 | 45 | 55 |
| | GL-Uni | 12.33 | 18.43 | 19.54 | 27 | 41 | 8.90 | 14.54 | 14.53 | 45 | 55 |
| $\delta^0 = 1$ | OLS | 3.72 | 6.88 | 6.89 | 28 | 32 | 3.85 | 6.98 | 6.98 | 48 | 52 |
| | GL-LN | 3.49 | 6.44 | 6.57 | 28 | 32 | 3.45 | 6.09 | 6.10 | 48 | 52 |
| | GL-Uni | 4.37 | 8.12 | 8.24 | 28 | 32 | 3.86 | 6.97 | 6.96 | 48 | 52 |

The model is $y_t = \delta^0 1\{t > [T^0_\lambda]\} + \alpha^0 y_{t-1} + \epsilon_t$, $\epsilon_t \sim i.i.d. \mathcal{N}(0, 0.5)$, $\delta^0 = 0.6$, $T = 100$. The notes of Table 1 apply.

Table 4: Small-sample coverage rates and lengths of the confidence sets for model M1

| | Cov. | Lgth. | Cov. | Lgth. | Cov. | Lgth. | Cov. | Lgth. |
|---|------|------|------|------|------|------|------|------|
| $\delta^0 = 0.4$ | OLS-CR | 0.922 | 77.52 | 0.934 | 49.46 | 0.946 | 22.51 | 0.938 | 10.48 |
| | Bai (1997) | 0.812 | 58.12 | 0.862 | 28.75 | 0.928 | 13.78 | 0.928 | 8.16 |
| | $\hat{U}_T(T_m).neq$ | 0.950 | 75.45 | 0.950 | 41.68 | 0.950 | 21.78 | 0.950 | 14.79 |
| | ILR | 0.959 | 76.14 | 0.973 | 35.79 | 0.976 | 14.44 | 0.977 | 7.15 |
| | GL-LN | 0.942 | 49.76 | 0.948 | 22.45 | 0.958 | 10.47 | 0.965 | 5.15 |
| | sup-W | 0.384 | 0.916 | 1.000 | 1.000 |
| $\delta^0 = 0.8$ | OLS-CR | 0.928 | 74.95 | 0.928 | 46.68 | 0.930 | 21.47 | 0.958 | 10.22 |
| | Bai (1997) | 0.830 | 56.64 | 0.870 | 28.72 | 0.904 | 13.89 | 0.962 | 8.27 |
| | $\hat{U}_T(T_m).neq$ | 0.952 | 77.51 | 0.952 | 44.72 | 0.952 | 22.51 | 0.952 | 14.21 |
| | ILR | 0.952 | 78.28 | 0.966 | 39.78 | 0.969 | 31.29 | 0.968 | 18.23 |
| | GL-LN | 0.942 | 49.60 | 0.948 | 23.89 | 0.958 | 11.14 | 0.980 | 5.60 |
| | sup-W | 0.316 | 0.866 | 0.992 | 1.000 |

The model is $y_t = \delta^0 1\{t > [T^0_\lambda]\} + \epsilon_t$, $\epsilon_t \sim i.i.d. \mathcal{N}(0, 1)$, $T = 100$. Cov. and Lgth. refer to the coverage probability and the average length of the confidence set (i.e., the average number of dates in the confidence set). sup-W refers to the rejection probability of the sup-Wald test using a 5% asymptotic critical value. The number of simulations is 3,000.
Table 5: Small-sample coverage rates and lengths of the confidence sets for model M2

| λ₀ = 0.5 | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. |
|----------|--------|------|-------|--------|------|-------|--------|------|-------|
|          |        | 0.952| 80.29 | 0.954  | 57.70| 0.957  | 30.04  | 0.963  | 15.10 |
|          | Bai (1997) | 0.804| 64.64 | 0.824  | 43.53| 0.907  | 13.03  | 0.930  | 7.81  |
|          | \( \hat{U}_T (T_m) \) \( \neq 0 \) | 0.967| 87.30 | 0.967  | 72.70| 0.957  | 36.70  | 0.957  | 30.20 |
|          | ILR    | 0.937| 81.88 | 0.945  | 57.43| 0.972  | 21.99  | 0.972  | 18.96 |
|          | GL-LN  | 0.933| 55.13 | 0.912  | 32.97| 0.935  | 20.03  | 0.961  | 10.62 |
|          | sup-W  | 0.316| 0.699 | 1.00   | 1.00 | 1.00   |        | 1.00   |        |

| λ₀ = 0.3 | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. |
|----------|--------|------|-------|--------|------|-------|--------|------|-------|
|          |        | 0.945| 79.25 | 0.957  | 54.93| 0.962  | 29.91  | 0.970  | 15.37 |
|          | Bai (1997) | 0.823| 63.79 | 0.851  | 26.33| 0.895  | 13.07  | 0.946  | 7.87  |
|          | \( \hat{U}_T (T_m) \) \( \neq 0 \) | 0.966| 88.23 | 0.953  | 59.66| 0.950  | 39.65  | 0.951  | 32.39 |
|          | ILR    | 0.945| 84.37 | 0.945  | 62.97| 0.971  | 33.74  | 0.987  | 17.92 |
|          | GL-LN  | 0.945| 53.79 | 0.923  | 34.75| 0.934  | 19.92  | 0.944  | 10.04 |
|          | sup-W  | 0.314| 0.699 | 1.00   | 1.00 | 1.00   |        | 1.00   |        |

The model is \( y_t = \delta^0 \{ t > \lceil T_{\lambda_0} \rceil \} + \varepsilon_t, \varepsilon_t \sim i.i.d. N(0, 1), \) \( T = 100. \) The notes of Table 4 apply.

Table 6: Small-sample coverage rates and lengths of the confidence sets for model M3

| λ₀ = 0.5 | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. |
|----------|--------|------|-------|--------|------|-------|--------|------|-------|
|          |        | 0.954| 80.29 | 0.952  | 57.23| 0.957  | 30.21  | 0.963  | 15.20 |
|          | Bai (1997) | 0.781| 55.85 | 0.845  | 26.23| 0.902  | 13.03  | 0.932  | 7.81  |
|          | \( \hat{U}_T (T_m) \) \( \neq 0 \) | 0.958| 81.28 | 0.959  | 55.34| 0.957  | 36.71  | 0.957  | 30.20 |
|          | ILR    | 0.934| 65.96 | 0.956  | 33.73| 0.975  | 21.96  | 0.984  | 17.45 |
|          | GL-LN  | 0.912| 60.90 | 0.925  | 32.93| 0.964  | 19.23  | 0.971  | 9.23  |
|          | sup-W  | 0.407| 0.931 | 1.00   | 1.00 | 1.00   |        | 1.00   |        |

| λ₀ = 0.3 | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. | OLS-CR | Cov. | Lgth. |
|----------|--------|------|-------|--------|------|-------|--------|------|-------|
|          |        | 0.968| 83.69 | 0.951  | 54.13| 0.962  | 29.31  | 0.970  | 15.37 |
|          | Bai (1997) | 0.795| 64.06 | 0.853  | 26.33| 0.896  | 13.07  | 0.946  | 7.85  |
|          | \( \hat{U}_T (T_m) \) \( \neq 0 \) | 0.960| 86.42 | 0.953  | 59.13| 0.950  | 39.65  | 0.951  | 32.28 |
|          | ILR    | 0.934| 67.73 | 0.964  | 35.30| 0.971  | 33.74  | 0.987  | 17.92 |
|          | GL-LN  | 0.912| 60.28 | 0.945  | 36.08| 0.974  | 22.72  | 0.975  | 12.71 |
|          | sup-W  | 0.232| 0.884 | 0.999  | 1.00 | 1.00   |        | 1.00   |        |

The model is \( y_t = \delta^0 \{ t > \lceil T_{\lambda_0} \rceil \} + \alpha^0 y_{t-1} + \varepsilon_t, \varepsilon_t \sim i.i.d. N(0, 0.5), \alpha^0 = 0.6, \) \( T = 100. \) The notes of Table 4 apply.
Supplemental Material to
Generalized Laplace Inference in Multiple Change-Points Models

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Abstract

This supplemental material is structured as follows. Section A contains the Mathematical Appendix which includes all proofs of the results in the paper. Section B includes further simulation results comparing the GL-LN method to the GL estimators proposed in Casini and Perron (2020b).
A  Mathematical Appendix

The mathematical appendix is structured as follows. Section A.2 presents some preliminary lemmas which will be used in the sequel. The proofs of the theoretical results in the paper are in Section A.3-A.5.

A.1  Additional Notation

The \((i, j)\) element of \(A\) is denoted by \(A^{(i,j)}\). For a matrix \(A\), the orthogonal projection matrices \(P_A, M_A\) are defined as \(P_A = A (A'A)^{-1} A'\) and \(M_A = I - P_A\), respectively. Also, for a projection matrix \(P\), \(\|PA\| \leq \|A\|\). We denote the \(d\)-dimensional identity matrix by \(I_d\). When the context is clear we omit the subscript notation in the projection matrices. We denote the \(i \times j\) upper-left (resp., lower-right) sub-block of \(A\) as \([A]_{i \times j}\) (resp., \([A]_{i \times j}^r\)). Note that the norm of \(A\) is equal to the square root of the maximum eigenvalue of \(A'A\), and thus, \(\|A\| \leq \|tr(A'A)\|^{1/2}\). For a sequence of matrices \(\{A_T\}\), we write \(A_T = o_P(1)\) if each of its elements is \(o_P(1)\) and likewise for \(O_P(1)\). For a random variable \(\xi\) and a number \(r \geq 1\), \(\|\xi\|_r = (E\|\xi\|_r^r)^{1/r}\). \(K\) is a generic constant that may vary from line to line; we may sometime write \(K_r\) to emphasize the dependence of \(K\) on a number \(r\). For two scalars \(a\) and \(b\), \(a \wedge b = \inf \{a, b\}\). We may use \(\sum k\) when the limits of the summation are clear from the context. Unless otherwise stated \(A^c\) denotes the complementary set of \(A\).

A.2  Preliminary Lemmas

We first present results related to the extremum criterion function \(Q_T(\delta(T_b), T_b)\) under the following assumption (Assumptions 3.1-3.2 are not needed in this section).

Assumption A.1. We consider model (2.3) with Assumptions 2.1-2.4 and 3.3-3.5.

Lemma A.1. The following inequalities hold \(\mathbb{P}\text{-a.s.}:

\[
\begin{align*}
\text{(A.1)} & \quad (Z_0'MZ_0) - (Z_0'MZ_2) (Z_2'MZ_2)^{-1} (Z_2'MZ_0) \geq D' (X'X_\Delta X_\Delta) (X_2'X_2)^{-1} (X_0'X_0) D, \quad T_b < T_0^0 \\
\text{(A.2)} & \quad (Z_0'MZ_0) - (Z_0'MZ_2) (Z_2'MZ_2)^{-1} (Z_2'MZ_0) \geq D' (X'X_\Delta X_\Delta) (X'X - X_2'X_2)^{-1} (X'X - X_0'X_0) D, \quad T_b \geq T_0^0
\end{align*}
\]

Proof. See Lemma A.1 in Bai (1997). \(\square\)

Recall that \(Q_T(\delta(\lambda_0), \lambda_0) = \delta(T_b) (Z_2'MZ_2) \delta(T_b)\). We decompose \(Q_T(\delta(\lambda_0), \lambda_0) - Q_T(\delta(\lambda_0^0), \lambda_0^0)\) into a “deterministic” and a “stochastic” component. It follows by definition that,

\[
\delta(\lambda_0) = (Z_2'MZ_2)^{-1} (Z_2'MY) = (Z_2'MZ_0)^{-1} (Z_2'MZ_0) \delta_T + (Z_2'MZ_2)^{-1} Z_2 Me,
\]

and

\[
\delta\left(\lambda_0^0\right) = (Z_0'MZ_0)^{-1} (Z_0'MY) = \delta_T + (Z_0'MZ_0)^{-1} Z_0'Me.
\]

Therefore

\[
Q_T(\delta(\lambda_0), \lambda_0) - Q_T\left(\delta\left(\lambda_0^0\right), \lambda_0^0\right) = \delta(\lambda_0) - \delta\left(\lambda_0^0\right) = \delta\left(\lambda_0\right)' (Z_2'MZ_2) \delta(\lambda_0) - \delta\left(\lambda_0^0\right)' (Z_2'MZ_0) \delta\left(\lambda_0^0\right) \delta_T + \delta\left(\lambda_0\right)' (Z_2'MZ_2) \delta(\lambda_0) - \delta\left(\lambda_0^0\right)' (Z_0'MZ_0) \delta\left(\lambda_0^0\right) \delta_T
\]

\[
\triangleq g_d(\delta_T, \lambda_0) + g_e(\delta_T, \lambda_0), \tag{A.3}
\]

where

\[
g_d(\delta_T, \lambda_0) = \delta_T \left( (Z_0'MZ_2) (Z_2'MZ_2)^{-1} (Z_2'MZ_0) - Z_0'MZ_0 \right) \delta_T, \tag{A.5}
\]
Lemma A.4. Under Assumption A.1, for any $\epsilon > 0$ there exists a $C < \infty$ and a positive sequence $\{\nu_T\}$, with $\nu_T \to \infty$ as $T \to \infty$, such that

$$
\liminf_{T \to \infty} \mathbb{P} \left[ \sup_{K \leq |u| \leq \eta T \|\delta_T\|^2} Q_T (\delta (\lambda_b ), \lambda_b) - Q_T (\delta (\lambda^0_b ), \lambda^0_b) < -C \nu_T \right] \geq 1 - \epsilon,
$$

for all sufficiently large $K$ and a sufficiently small $\eta > 0$.

Proof. Note that on $\{K \leq |u| \leq \eta T \|\delta_T\|^2\}$ we have $K/ \|\delta_T\|^2 \leq |T_b - T^0_b| \leq \eta T$. In view of (A.8), the statement $Q_T (\delta (\lambda_b ), \lambda_b) - Q_T (\delta (\lambda^0_b ), \lambda^0_b) < -C \nu_T$ follows from showing that as $T \to \infty$,

$$
\mathbb{P} \left( \sup_{T_b \in \mathcal{B}_{K,T}^c} g_e (\delta_T, \lambda_b) \geq \inf_{T_b \in \mathcal{B}_{K,T}^c} \left| T_b - T^0_b \right|^K \mathcal{F}_d (\delta_T, \lambda_b) \right) < \epsilon,
$$

We arbitrarily define $\mathcal{F}_d (\delta^0, \lambda_b) = \delta^T \delta_T$ when $\lambda_b = \lambda^0_b$. Observe that $\mathcal{F}_d (\delta_T, \lambda_b)$ is non-negative because the matrix inside the braces in (A.5) is negative semidefinite. (A.3) can be written as

$$
Q_T (\delta (\lambda_b ), \lambda_b) - Q_T (\delta (\lambda^0_b ), \lambda^0_b) = -\left| T_b - T^0_b \right| \mathcal{F}_d (\delta_T, \lambda_b) + g_e (\delta_T, \lambda_b), \quad \text{for all } \lambda_b. \tag{A.9}
$$

We use the notation $u = T \|\delta_T\|^2 (\lambda_b - \lambda^0_b)$. For $\eta > 0$, let $B_{T,\eta} \triangleq \{T_b : |T_b - T^0_b| \leq T \eta \}$, $B_{T,K} \triangleq \{T_b : |T_b - T^0_b| \leq K/ \|\delta_T\|^2 \}$ and $B_{T,K}^c \triangleq \{T_b : T \eta \geq |T_b - T^0_b| > K/ \|\delta_T\|^2 \}$, with $K > 0$. Note that $B_{T,\eta} = B_{T,K} \cup B_{T,K}^c$. Further, let $B_{T,\eta}^c \triangleq \{T_b : |T_b - T^0_b| > T \eta \}$.

Lemma A.2. Under Assumption A.1, $Q_T (\delta (\lambda_b ), \lambda_b) - Q_T (\delta (\lambda^0_b ), \lambda^0_b) = -\delta^T Z^T \delta_T \delta_T = o_P (1)$, uniformly on $B_{T,K}$ for $K$ large enough.

Proof. It follows from Lemma A.5 in Bai (1997).

Lemma A.3. Under Assumption A.1, for $T_b = T^0_b + |u|/ \|\delta_T\|^2$, we have $\delta^T Z^T \delta_T = \delta^T \sum_{t=T_b+1}^{T_b} z_t' \delta_T = |u| \|\delta^0\| \nabla \delta^0 + o_P (1)$, where $\nabla = V_1$ if $u \leq 0$ and $\nabla = V_2$ if $u > 0$.

Proof. It follows from basic arguments (cf. Assumptions 3.4-3.5).

Lemma A.4. Under Assumption A.1, for any $\epsilon > 0$ there exists a $C < \infty$ and a positive sequence $\{\nu_T\}$, with $\nu_T \to \infty$ as $T \to \infty$, such that

$$
\liminf_{T \to \infty} \mathbb{P} \left[ \sup_{K \leq |u| \leq \eta T \|\delta_T\|^2} Q_T (\delta (\lambda_b ), \lambda_b) - Q_T (\delta (\lambda^0_b ), \lambda^0_b) < -C \nu_T \right] \geq 1 - \epsilon,
$$

for all sufficiently large $K$ and a sufficiently small $\eta > 0$.

Proof. Note that on $\{K \leq |u| \leq \eta T \|\delta_T\|^2\}$ we have $K/ \|\delta_T\|^2 \leq |T_b - T^0_b| \leq \eta T$. In view of (A.8), the statement $Q_T (\delta (\lambda_b ), \lambda_b) - Q_T (\delta (\lambda^0_b ), \lambda^0_b) < -C \nu_T$ follows from showing that as $T \to \infty$,
where \( \kappa \in (1/2, 1) \). Suppose \( T_b < T_b^0 \). We show that
\[
\mathbb{P} \left( \sup_{T \lambda_b \in B_{T,K}^c} \frac{\| \delta_T \|}{K} g_e (\delta_T, \lambda_b) \geq \frac{1}{\| \delta_T \|^{2\kappa - 1}} \left( \frac{1}{K} \right)^{1-\kappa} \inf_{T \lambda_b \in B_{T,K}^c} \| \overline{g}_d (\delta_T, \lambda_b) \| < \epsilon \right) < \epsilon. \tag{A.10}
\]

Lemma A.5-(ii) stated below implies that \( \inf_{T \lambda_b \in B_{T,K}^c} \| \overline{g}_d (\delta_T, \lambda_b) \| \) is bounded away from zero as \( T \to \infty \) for large \( K \) and small \( \eta \). Next, we show that
\[
\sup_{T \lambda_b \in B_{T,K}^c} K^{-1} \| \delta_T \| g_e (\delta_T, \lambda_b) = o_\mathbb{P} (1). \tag{A.11}
\]

Consider the first term of (A.6),
\[
2 \delta_T (Z_0' M Z_2) (Z_2' M Z_2)^{-1} Z_2 M e = 2 \delta_T (Z_0' M Z_2 / T) (Z_2' M Z_2 / T)^{-1} Z_2 M e = 2 C \| \delta_T \| O_\mathbb{P} (1) O_\mathbb{P} (1) O_\mathbb{P} (T^{1/2}) = CO_\mathbb{P} \left( \| \delta_T \| T^{1/2} \right).
\]

When multiplied by \( \| \delta_T \| \mathbb{K} \), this term is \( O_\mathbb{P} \left( \| \delta_T \|^2 T^{1/2} / K \right) \) which goes to zero for large \( K \). The second term in (A.6), when multiplied by \( \| \delta_T \| / K \), is
\[
2 K^{-1} \| \delta_T \| \delta_T' (Z_0' M e) = K^{-1} \| \delta_T \| O_\mathbb{P} \left( \| \delta_T \| T^{1/2} \right) = K^{-1} O_\mathbb{P} \left( \| \delta_T \|^2 T^{1/2} \right),
\]

which converges to zero using the same argument as for the first term. Consider now the first term of (A.7), \( T^{-1/2} e' M Z_2 (Z_2' M Z_2 / T)^{-1} T^{-1/2} Z_2 M e = O_\mathbb{P} (1) \). A similar argument can be used for the second term which is also \( O_\mathbb{P} (1) \). The latter two terms multiplied by \( \| \delta_T \| / K \) is \( O_\mathbb{P} (\| \delta_T \| / K) = o_\mathbb{P} (1) \). This proves (A.11) and thus (A.10). To conclude the proof, note that \( \kappa \in (1/2, 1) \) implies \( \| \delta_T \|^{-(2\kappa - 1)} \to \infty \), so that we can choose \( \nu_T = \left( \| \delta_T \|^2 / K \right)^{-(1-\kappa)}. \)

Lemma A.5. Let \( \overline{g}_d \triangleq \inf_{T \lambda_b - \lambda_b'} > K \| \delta_T \|^2 \overline{g}_d (\delta_T, \lambda_b) \). Under Assumption A.1,
(i) for any \( \epsilon > 0 \) there exists some \( C > 0 \) such that \( \lim_{T \to \infty} \mathbb{P} \left( \overline{g}_d > C \| \delta_T \|^2 \right) \leq 1 - \epsilon; \)
(ii) with \( B_{T,K}^c = \left\{ T : T \eta \geq |T_b - T_b^0| \geq K / \| \delta_T \|^2 \right\} \), for any \( \epsilon > 0 \) there exists a \( C > 0 \) such that \( \lim_{T \to \infty} \mathbb{P} \left( \inf_{T \lambda_b \in B_{T,K}^c} \| \overline{g}_d (\delta_T, \lambda_b) \| > C \right) \leq 1 - \epsilon. \)

Proof. Part (i) was proved in Lemma A.2 of Bai (1997). As for part (ii), by Lemma A.1,
\[
\overline{g}_d (\delta, \lambda_b) \geq \delta_T D' \frac{X'A'X}{T_b^0 - T_b} (X_0'X_0)^{-1} (X_0'X_0) D \delta_T \geq \lambda_{J,T_b},
\]

where \( \lambda_{J,T_b} \) is the minimum eigenvalue of \( D'J (T_b) D \), with \( J (T_b) \triangleq || \delta_T ||^2 (T_b^0 - \lambda_b)^{-1} X_0'A'X_0 (X_0'X_0)^{-1} (X_0'X_0) \).

It is sufficient to show that, for \( T_b \in B_{T,K}^c, \lambda_{J,T_b} \) is bounded away from zero with large probability for large \( K \) and small \( \eta \). We have \( \left\| J (T_b)^{-1} \right\| \leq \left\| \left\| \| \delta_T \|^2 (T_b^0 - \lambda_b)^{-1} X_0'A'X_0 \right\| \right\| \left\| (X_0'X_0)^{-1} \right\| \) and by Assumptions 2.3-2.4 \( \left\| (X_0'X_0)^{-1} \right\| \leq \left\| X_0'X_0 \right\| \left\| (X_0'X_0)^{-1} \right\| \) is bounded. Next, note that \( (T_b^0 - T_b)^{-1} X_0'A'X_0 = (T_b^0 - T_b)^{-1} \sum_{t=T_b^0+1}^{T_b^0} x_t'x_t' \) is larger than \( (T_b^{-1}) \sum_{t=T_b^0+1}^{T_b^0} \left[ K / \| \delta_T \|^2 \right] x_t'x_t \) on \( B_{T,K}^c \), and for all \( K \), \( \left( \| \delta_T \|^2 / K \right) \sum_{t=T_b^0+1}^{T_b^0+1} \left[ K / \| \delta_T \|^2 \right] x_t'x_t \) is positive definite with large probability as \( T \to \infty \) by Assumption 2.3. Now, \( (K / \eta T) \left( \| \delta_T \|^2 / K \right) \sum_{t=T_b^0+1}^{T_b^0+1} \left[ K / \| \delta_T \|^2 \right] x_t'x_t = O_\mathbb{P} (1) \), by choosing sufficiently large \( K \) and small \( \eta \). Thus, \( \left\| \left\| \delta_T \|^2 (T_b^0 - T_b)^{-1} X_0'A'X_0 \right\| \right\| \) is bounded with large probability for such large \( K \).
and small \( \eta \), which in turn implies that \( \| J ( T_b )^{-1} \| \) is bounded. Since \( D \) has full column rank, \( \lambda_{J,T_b} \) is bounded away from zero for sufficiently large \( K \) and small \( \eta \).

**Lemma A.6.** Under Assumption A.1, for any \( \epsilon > 0 \) there exists a \( C > 0 \) such that

\[
\liminf_{T \to \infty} \mathbb{P} \left[ \sup_{|u| \geq T \| \delta_T \|^2 \eta} Q_T ( \delta ( \lambda_b ), \lambda_b ) - Q_T ( \delta ( \lambda_b^0 ), \lambda_b^0 ) \geq - C \nu_T \right] \geq 1 - \epsilon,
\]

for every \( \eta > 0 \), where \( \nu_T \to \infty \).

**Proof.** Fix any \( \eta > 0 \). Note that on \( \{|u| \geq T \| \delta_T \|^2 \eta\} \) we have \( |T_b - T_b^0| \geq T \eta \). We proceed in a similar manner to Lemma A.4. Let \( B_{T,\eta}^c \triangleq \{ T_b : |T_b - T_b^0| \geq T \eta \} \) and recall (A.8). First, as in Lemma A.5-(i), we have \( \inf_{T,\lambda_b \in B_{T,\eta}^c} \mathcal{F}_d ( \delta_T, \lambda_b ) \geq C \| \delta_T \|^2 \) with large probability for some \( C > 0 \). Noting that \( T \eta \inf_{T,\lambda_b \in B_{T,\eta}^c} \mathcal{F}_d ( \delta_T, \lambda_b ) \) diverges at rate \( \tau_T = T \| \delta_T \|^2 \), the claim follows if we can show that \( g_c ( \delta_T, \lambda_b ) = O_\mathbb{P} ( \tau_T^{1/2} ) \), with \( 0 \leq \omega < 1 \) uniformly on \( B_{T,\eta}^c \). This is shown in Lemma A.7 below, which suggests setting \( \omega \in (1/2, 1) \). Then, choose \( \nu_T = \left( T \| \delta_T \|^2 \right)^{1-\omega} \).

**Lemma A.7.** Under Assumption A.1, uniformly on \( B_{T,\eta}^c \), \( |g_c ( \delta_T, \lambda_b )| = O_\mathbb{P} \left( \| \delta_T \|^2 \log T \right) \).

**Proof.** We show that \( T^{-1} |g_c ( \delta^0, \lambda_b )| = O_\mathbb{P} \left( \| \delta_T \| T^{-1/2} \log T \right) \) uniformly on \( B_{T,\eta}^c \). Note that

\[
\sup_{T,\lambda_b \in B_{T,\eta}^c} |g_c ( \delta_T, \lambda_b )| \leq \sup_{q \leq T, \lambda_b \leq T-q} |g_c ( \delta_T, \lambda_b )|,
\]

and recall that \( q = \dim ( z_t ) \) is needed for identification. Observe that

\[
\sup_{q \leq T_b \leq T-q} \left\| (Z_0'MZ_2)^{-1/2} Z_2'Me \right\| = O_\mathbb{P} \left( \log T \right), \tag{A.12}
\]

by the law of iterated logarithms [cf. Billingsley (1995), Ch. 1, Theorem 9.5]. Next,

\[
\sup_{q \leq T_b \leq T-q} T^{-1/2} (Z_0'MZ_2) (Z_2'MZ_2)^{-1/2} = O_\mathbb{P} \left( 1 \right), \tag{A.13}
\]

which can be proved using the inequality \( (Z_0'MZ_2) (Z_0'MZ_2) (Z_0'MZ_2) \leq Z_0'MZ_0 = O_\mathbb{P} \left( T \right) \) (valid for all \( T_b \)). Thus, by (A.12) and (A.13), the first term on the right-hand side of (A.6) multiplied by \( T^{-1} \) is such that

\[
\sup_{q \leq T_b \leq T-q} 2\delta_T T^{-1} (Z_0'MZ_2) (Z_2'MZ_2)^{-1} Z_2'Me = O_\mathbb{P} \left( \| \delta_T \| T^{-1/2} \log T \right). \tag{A.14}
\]

The second term on the right-hand side of (A.6) is \( 2 \delta_T Z_0'Me = O_\mathbb{P} \left( \| \delta_T \| T^{1/2} \right) \). Using (A.12), and dividing by \( T \), the first term of (A.7) is \( O_\mathbb{P} \left( (\log T)^2 / T \right) \) while the last term is \( O_\mathbb{P} \left( T^{-1} \right) \). When divided by \( T \), they are of order \( O_\mathbb{P} \left( (\log T)^2 / T \right) \) and \( O_\mathbb{P} \left( T^{-1} \right) \), respectively. Therefore, \( |g_c ( \delta^0, \lambda_b )| = O_\mathbb{P} \left( \| \delta_T \| T^{1/2} \log T \right) \), uniformly on \( B_{T,\eta}^c \).

**A.3 Proofs of Results in Section 3**

We denote by \( P \) the class of polynomial functions \( p : \mathbb{R} \to \mathbb{R} \). Let \( \mathbf{U}_T \triangleq \{ u \in \mathbb{R} : \lambda_b^0 + u / \psi_T \in \Gamma^0 \} \), \( \Gamma_{T,\psi} \triangleq \{ u \in \mathbb{R} : |u| \leq \psi_T \} \), \( \Gamma_{T,\psi}^c \triangleq \mathbb{R} - \Gamma_{T,\psi} \), and \( \tilde{U}_{T,\psi} \triangleq \mathbf{U}_T - \Gamma_{T,\psi} \). For \( u \in \mathbb{R} \), let \( R_{T,v} ( u ) \triangleq Q_{T,v} ( u ) - \).
\( A^0 (u) \) and \( \overline{G}_{T,v} (u) \triangleq \sup_{v \in \mathcal{V}} \overline{G}_{T,v} (u, \tilde{v}) \). The generic constant \( 0 < C < \infty \) used below may change from line to line. Finally, let \( \overline{\gamma}_T \triangleq \gamma_T / \| \delta_T \|^2 \).

### A.3.1 Proof of Proposition 3.1

We begin with the proof for the case of a fixed shift.

**Lemma A.8.** Under Assumptions 2.1-2.4, 3.1-3.3 (except that \( \delta_T = \delta^0 \)) and 3.6-(i), \( \lambda^\text{GL}_b = \lambda^0_b + o_P(1) \).

**Proof.** Let \( \overline{S}_T (\delta (\lambda_b), \lambda_b) \triangleq Q_T (\delta (\lambda_b), \lambda_b) - Q_T (\delta (\lambda^0_b), \lambda^0_b) \). From (A.9),

\[
\overline{S}_T (\hat{\delta} (\lambda_b), \lambda_b) = - \left| T - T^0 \right| \overline{g}_d (\delta^0, T_b) + g_e (\delta^0, T_b),
\]

where \( g_e (\delta^0, T_b) \) and \( \overline{g}_d (\delta^0, T_b) \) are defined in (A.6)-(A.8). By Lemma A.24 in Bai (1997), \( \lim_{T \to \infty} \overline{g}_d (\delta^0, T_b) > 0 \) and \( T^{-1} \sup_{T_b} |g_e (\delta^0, T_b)| = O_P \left( T^{-1/2} \log T \right) \). Thus, for any \( B > 0 \) if \( |\lambda^\text{GL}_b - \lambda^0_b| > B \) we have that,

\[
-\overline{S}_T (\hat{\delta} (\lambda_b), \lambda_b) \to \infty \text{ at rate } TB. \tag{A.15}
\]

Let \( p_T (u) \triangleq p_{1,T} (u) / \overline{p}_T \) with \( p_{1,T} (u) = \exp (Q_T (\delta (u), u)) \) and \( \overline{p}_T \triangleq \int_{U_T} p_{1,T} (w) \, dw \). By definition, \( \lambda^\text{GL}_b \) is the minimum of the function \( \int_{T_0} l (s - u) p_{1,T} (u) \pi (u) \, du \) with \( s \in I^0 \). Using a change in variables,

\[
\int_{T_0} l (s - u) p_{1,T} (u) \pi (u) \, du = T^{-1} \overline{p}_T \int_{U_T} l (T (s - \lambda^0_b) - u) \frac{p_T (\lambda^0_b + T^{-1} u) \pi (\lambda^0_b + T^{-1} u)}{p_T (\lambda^0_b + T^{-1} u) \pi (\lambda^0_b + T^{-1} u)} \, du,
\]

where \( U_T \triangleq \{ u \in \mathbb{R} : \lambda^0_b + T^{-1} u \in I^0 \} \). Thus, \( \lambda_{\delta,T} \triangleq T \left( \lambda^\text{GL}_b - \lambda^0_b \right) \) is the minimum of the function,

\[
\overline{S}_T (s) \triangleq \int_{U_T} l (s - u) \frac{p_T (\lambda^0_b + T^{-1} u) \pi (\lambda^0_b + T^{-1} u)}{p_T (\lambda^0_b + T^{-1} u) \pi (\lambda^0_b + T^{-1} u)} \, du,
\]

where the optimization is over \( U_T \). We shall show that for any \( B > 0 \),

\[
\mathbb{P} \left[ \left| \lambda^\text{GL}_b - \lambda^0_b \right| > B \right] \leq \mathbb{P} \left[ \inf_{|s| > TB} \overline{S}_T (s) \leq \overline{S}_T (0) \right] \to 0. \tag{A.16}
\]

By assumption the prior is bounded and so we can proceed the proof for the case \( \pi (u) = 1 \) for all \( u \). By the properties of the family \( L \) of loss functions, we can find \( \overline{\pi}_1, \overline{\pi}_2 \in \mathbb{R} \), with \( 0 < \overline{\pi}_1 < \overline{\pi}_2 \) such that as \( T \) increases,

\[
\overline{I}_{1,T} \triangleq \sup \{ l (u) : u \in \Gamma_{1,T} \} < \overline{I}_{2,T} \triangleq \inf \{ l (u) : u \in \Gamma_{2,T} \},
\]

where \( \Gamma_{1,T} \triangleq U_T \cap \{ |u| \leq \overline{\pi}_1 \} \) and \( \Gamma_{2,T} \triangleq U_T \cap \{ |u| > \overline{\pi}_2 \} \). With this notation,

\[
\overline{S}_T (0) \leq \overline{I}_{1,T} \int_{\Gamma_{1,T}} p_T (u) \, du + \int_{U_T \cap \{ |u| > \overline{\pi}_1 \}} l (u) p_T (u) \, du.
\]

If \( l \in L \) then for a sufficiently large \( T \) the following relationship holds: \( l (u) - \inf_{|u| > TB/2} l (v) \leq 0, \) \( |u| \leq (TB/2)^\vartheta \) for some \( \vartheta > 0 \). It also follows that for large \( T \) we have \( TB > 2\overline{\pi}_2 \) and \( (TB/2)^\vartheta > \overline{\pi}_2 \). Let
\( \Gamma_{T,B} \triangleq \{ u : (|u| > TB/2) \cap U_T \} \). Then, whenever \(|s| > TB \) and \(|u| \leq TB/2 \), we have,

\[
|u - s| > TB/2 > \bar{\nu}_2 \quad \text{and} \quad \inf_{u \in \Gamma_{T,B}} l(u) \geq \bar{l}_{2,T}. \tag{A.17}
\]

With this notation,

\[
\inf_{|s| > TB} S_T(s) \geq \inf_{u \in \Gamma_{T,B}} l_T(u) \int_{(|w| \leq TB/2) \cap U_T} p_T(w) \, dw \\
\geq \bar{l}_{2,T} \int_{(|w| \leq TB/2) \cap U_T} p_T(w) \, dw,
\]

from which it follows that

\[
S_T(0) - \inf_{|s| > TB} S_T(s) \leq -\varpi \int_{\Gamma_{1,T}} p_T(u) \, du \\
+ \int_{U_T \cap ((TB/2)^0 \geq |u| \geq \bar{\nu}_1)} \left( l(u) - \inf_{|s| > TB/2} l_T(s) \right) p_T(u) \, du \\
+ \int_{U_T \cap (|u| > (TB/2)^0)} l(u) p_T(u) \, du,
\]

where \( \varpi \triangleq \bar{l}_{2,T} - \bar{l}_{1,T} \). The last inequality can be manipulated further using (A.17),

\[
S_T(0) - \inf_{|s| > TB} S_T(s) \leq -\varpi \int_{\Gamma_{1,T}} p_T(u) \, du \\
+ \int_{U_T \cap (|u| > (TB/2)^0)} l_T(u) p_T(u) \, du. \tag{A.18}
\]

Since \( l \in L \), we have \( l(u) \leq |u|^a \), \( a > 0 \) when \( u \) is large enough. Thus, given (A.15), the second term of (A.18) converges to zero. Since \( \int_{\Gamma_{1,T}} p_T(u) \, du > 0 \), the first term of (A.18) is negative which then leads to \( S_T(0) - \inf_{|s| > TB} S_T(s) < 0 \) or \( S_T(0) < \inf_{|s| > TB} S_T(s) \). Thus, we have (A.16). \( \square \)

**Lemma A.9.** Under Assumptions 2.1-2.4, 3.1-3.3 and 3.6-(i), for \( l \in L \) and any \( B > 0 \) and \( \varepsilon > 0 \), we have for all large \( T \), \( \mathbb{P} \left[ \left| \hat{\lambda}_{b,0}^G - \lambda_{b,0}^0 \right| > B \right] < \varepsilon. \)

**Proof.** The structure of the proof is similar to that of Lemma A.8. By Proposition 1 in Bai (1997), eq. (A.15) holds with \( O_P \left( T \| \delta_T \| \right) \) in place of \( O_P (TB) \), \( B > 0 \). One can then follow the same steps as in the previous lemma to yield the result. \( \square \)

**Lemma A.10.** Under Assumptions 2.1-2.4, 3.1-3.3 and 3.6-(i), for \( l \in L \) and for every \( \varepsilon > 0 \) there exists a \( B < \infty \) such that for all large \( T \), \( \mathbb{P} \left[ T v_T^2 \left| \hat{\lambda}_{b,0}^G - \lambda_{b,0}^0 \right| > B \right] < \varepsilon. \)

**Proof.** See Lemma A.29 which proves a stronger result needed for Theorem 3.2. \( \square \)

Parts (i) and (ii) of Proposition 3.1 follow from Lemma A.9 and Lemma A.10, respectively.

### A.3.2 Proof of Theorem 3.1

We start with the following lemmas.

**Lemma A.11.** For any \( a \in \mathbb{R}, |c| \leq 1, \) and integer \( i \geq 0, \left| \exp(a) - \sum_{j=0}^i (ca)^j / j! \right| \leq |c|^{i+1} \exp(|a|). \)
Proof. The proof is immediate and the same as the one in Jun et al. (2015). Using simple manipulations,

\[ \left| \exp(ca) - \sum_{j=0}^{i} \left( ca \right)^{j} / j! \right| \leq \left| \sum_{j=i+1}^{\infty} \left( ca \right)^{j} / j! \right| \leq \left| c \right|^{i+1} \left| \sum_{j=i+1}^{\infty} \left( a \right)^{j} / j! \right| \leq \left| c \right|^{i+1} \exp \left( \left| a \right| \right). \]

Lemma A.12. $\bar{G}_{T,v}(u, \bar{v}) \Rightarrow \mathcal{W}(u)$ in $\mathbb{D}_{b}(C \times V)$, where $C \subset \mathbb{R}$ and $V \subset \mathbb{R}^{p+2q}$ are both compact sets, and

\[ \mathcal{W}(u) \triangleq \begin{cases} 2 \left( (\delta^{0})^{\prime} \Sigma_{1} \delta^{0} \right)^{1/2} W_{1}(-u), & \text{if } u < 0 \\ 2 \left( (\delta^{0})^{\prime} \Sigma_{2} \delta^{0} \right)^{1/2} W_{2}(u), & \text{if } u \geq 0. \end{cases} \]

Proof. Consider $u < 0$. According to the expansion of the criterion function given in Lemma A.2, for any $(u, \bar{v}) \in C \times V$, $\bar{G}_{T,v}(u, \bar{v})$ satisfies $2\text{sgn} \left( T_{b} - T_{b}(u) \right) \delta_{T} Z_{e}^{\prime} e = \delta_{T} \delta_{T}^{\prime} \Sigma_{e} Z_{e}^{\prime} e \Rightarrow \mathcal{G}_{1}(-u)$, where $\mathcal{G}_{1}$ is a multivariate Gaussian process. In particular, $(\delta^{0})^{\prime} \mathcal{G}_{1}(-u)$ is equivalent in law to $\left( (\delta^{0})^{\prime} \Sigma_{1} \delta^{0} \right)^{1/2} W_{1}(-u)$, where $W_{1}(\cdot)$ is a standard Wiener process on $[0, \infty)$. Similarly, for $u \geq 0$, $\delta_{T} Z_{e}^{\prime} e \Rightarrow \left( (\delta^{0})^{\prime} \Sigma_{2} \delta^{0} \right)^{1/2} W_{2}(u)$, where $W_{2}(\cdot)$ is another standard Wiener process on $[0, \infty)$ which is independent of $W_{1}$. Hence, $\bar{G}_{T,v}(u, \bar{v}) \Rightarrow \mathcal{W}(u)$ in $\mathbb{D}_{b}(C \times V)$. \qed

Lemma A.13. Fix any $a > 0$ and let $\varpi \in (1/2, 1]$. (i) For any $\nu > 0$ and any $\varepsilon > 0$,

\[ \lim_{T \to \infty} \sup_{u \in \Gamma_{T,v}} \mathbb{P} \left[ \sup_{u \in \Gamma_{T,v}} \left\{ \bar{G}_{T,v}(u) - a \left\| \delta^{0} \right\|^{2} |u|^{\varpi} \right\} > \nu \right] < \varepsilon. \]

(ii) For $\bar{u} \in \mathbb{R}_{+}$ let $\bar{\Gamma} \triangleq \{ u \in \mathbb{R} : |u| > \bar{u} \}$. Then, for every $\varepsilon > 0$,

\[ \lim_{\bar{u} \to \infty} \lim_{T \to \infty} \mathbb{P} \left[ \sup_{u \in \bar{\Gamma}} \left\{ \bar{G}_{T,v}(u) - a \left\| \delta^{0} \right\|^{2} |u|^{\varpi} \right\} > \varepsilon \right] = 0. \]

Proof. We begin with part (i). Upon using Lemma A.12 and the continuous mapping theorem, with any nonnegative integer $i$,

\[ \lim_{T \to \infty} \mathbb{P} \left[ \sup_{u \in \Gamma_{T,v}} \left\{ \bar{G}_{T,v}(u) - a \left\| \delta^{0} \right\|^{2} |u|^{\varpi} \right\} > \nu \right] \leq \lim_{T \to \infty} \mathbb{P} \left[ \sup_{u > \Gamma_{T,v}} \left\{ \bar{G}_{T,v}(u) - a \left\| \delta^{0} \right\| |u|^{\varpi} \right\} > \nu \right] \leq \lim_{T \to \infty} \mathbb{P} \left[ \sup_{u \geq 1} \left\{ \bar{G}_{T,v}(u) - a \left\| \delta^{0} \right\| |u|^{\varpi} \right\} > \nu \right] \leq \mathbb{P} \left[ \sup_{u \geq 1} \left\{ \mathcal{W}(u) - a \left\| \delta^{0} \right\| |u|^{\varpi} \right\} > \nu \right] \leq \sum_{r=1}^{\infty} \mathbb{P} \left[ \sup_{r-1 \leq |u| < r} \left\{ \mathcal{W}(u) - a \left\| \delta^{0} \right\| |u|^{\varpi} \right\} > \nu \right]. \]

Then,

\[ \sum_{r=1}^{\infty} \mathbb{P} \left[ \sup_{r-1 \leq |u| < r} \frac{1}{\sqrt{T}} \mathcal{W}(u) > \inf_{r-1 \leq |u| < r} \frac{1}{\sqrt{T}} \left\| \delta^{0} \right\| |u|^{\varpi} \right] \]
where $0 < c \leq 1$. By Markov’s inequality,

$$
\sum_{r=i+1}^{\infty} P \left[ \sup_{c<s \leq 1} |\mathcal{W}(s)|^4 > C^4 \left\| \mathbf{d}^0 \right\|^4 r^{4(\varpi-1/2)} c^4 \right] \leq \frac{C^4}{\left\| \mathbf{d}^0 \right\|^4} \sum_{r=i+1}^{\infty} r^{-4(\varpi-2)}. \quad (A.20)
$$

By Proposition A.2.4 in van der Vaart and Wellner (1996), $E(\sup_{s \leq 1} |\mathcal{W}(s)|^4) \leq C E(\sup_{s \leq 1} |\mathcal{W}(s)|)^4$ for some $C < \infty$, which is finite by Corollary 2.2.8 in van der Vaart and Wellner (1996). Choose $K$ (thus $\varpi$) large enough such that the right-hand side in (A.20) can be made arbitrarily smaller than $\varepsilon > 0$. The proof of the second part is similar and omitted.

Lemma A.14. Fix any $a > 0$. For any $\varepsilon > 0$ there exists a $C < \infty$ such that

$$
P \left[ \sup_{u \in \mathbb{R}} \left\{ \overline{G}_{T,v}(u) - a \left\| \mathbf{d}^0 \right\|^2 \mid u \right\} > C \right] < \varepsilon, \quad \text{for all } T.
$$

Proof. For any finite $T$, $\overline{G}_{T,v}(u) \in \mathbb{D}_b$ by definition. As for the limiting case, fix any $0 < \varpi < \infty$,

$$
\limsup_{T \to \infty} P \left[ \sup_{u \in \mathbb{R}} \left\{ \overline{G}_{T,v}(u) - a \left\| \mathbf{d}^0 \right\|^2 \mid u \right\} > C \right] \leq \limsup_{T \to \infty} P \left[ \sup_{u \leq \varpi} \overline{G}_{T,v}(u) > C \right] + \limsup_{T \to \infty} P \left[ \sup_{u > \varpi} \overline{G}_{T,v}(u) > a \left\| \mathbf{d}^0 \right\|^2 \right].
$$

The second term converges to zero letting $\varpi \to \infty$ from Lemma A.13-(ii). For the first term, let $C \to \infty$, use the continuous mapping theorem and Lemma A.12 to deduce that it converges to zero by the properties of $\mathcal{W} \in \mathbb{D}_b$.

Lemma A.15. Let

$$
A_1(u, \nu) = u^m \pi_{T,v}(u) \exp \left( \tilde{\gamma}_T \overline{G}_{T,v}(u, \nu) + Q_{T,v}(u) \right), \quad (A.21)
$$

$$
A_2(u, \nu) = u^m \pi^0 \exp \left( \tilde{\gamma}_T \overline{G}_{T,v}(u, \nu) - \Lambda_0(u) \right).
$$

For $m \geq 0$,

$$
\liminf_{T \to \infty} P \left[ \sup_{\nu \in \mathbb{V}} \left| \int_{\Gamma^c_{T,v}} (A_1(u, \nu) - A_2(u, \nu)) \right| < \varepsilon \right] \geq 1 - \varepsilon.
$$

Proof. We consider each integrand $A_i(u, \nu)$ ($i = 1, 2$) separately on $\Gamma^c_{T,v}$. Let us consider $A_1$ first. Lemma A.4 yields that whenever $\tilde{\gamma}_T \to \kappa_\gamma < \infty$, $A_1(u, \nu) \leq C_1 \exp (-C_2 \nu_T)$ where $0 < C_1, C_2 < \infty$ and $\nu_T$ is a divergent sequence. Note that the number $C_1$ follows from Assumption 3.2 (cf. $\pi(\cdot) < \infty$). The argument for $A_2(u, \nu)$ relies on Lemma A.13-(i), which shows that $G_{T,v}(u, \nu)$ is always less than $C |u|^{\varpi}$ uniformly on $\Gamma^c_{T,v}$, with $C > 0$ and $\varpi \in (1/2, 1)$. Thus, $A_2(u, \nu) = o_T(1)$ uniformly on $\mathbb{V}$.
Let $\Gamma_{T,K} \triangleq \{ u \in \mathbb{R} : |u| < K, K > 0 \}$, and $\Gamma_{T,\eta} \triangleq \{ u \in \mathbb{R} : K \leq |u| \leq \eta \psi_T, K, \eta > 0 \}$.  

**Lemma A.16.** For any polynomial function $p \in \mathcal{P}$ and any $C < \infty$, let

$$D_T \triangleq \sup_{\bar{v} \in \mathcal{W}} \int_{\Gamma_{T,K}} |p(u)| \exp \left\{ C \tilde{G}_{T,v}(u, \bar{v}) \right\} \left| \exp (R_{T,v}(u)) - 1 \right| \exp (-A^0(u)) \, du = o_{\mathbb{P}}(1).$$

Proof. Let $0 < \epsilon < 1$. We shall use Lemma A.11 with $i = 0$, $a = R_{T,v}(u)/c$, and $c = \epsilon$ to deduce that $D_T = O_{\mathbb{P}}(\epsilon)$ and then let $\epsilon \rightarrow 0$. Note that

$$\epsilon^{-1} D_T \leq C \int_{\Gamma_{T,K}} |p(u)| \exp \left\{ C \tilde{G}_{T,v}(u, \bar{v}) \right\} \left| \exp (-A^0(u)) \right| \left| \pi_{T,v}(u) - \pi^0 \right| \, du.$$

By definition, $K \geq u = \| \delta_T \|_2 (T_b - T^0_b)$ on $\Gamma_{T,K}$. By Lemma A.2-A.3, on $\Gamma_{T,K}$ we have $R_{T,v}(u) = O_{\mathbb{P}}(\| \delta_T \|_2^2)$ for each $u$. Thus, for large enough $T$, the right-hand side above is $O_{\mathbb{P}}(1)$ and does not depend on $\epsilon$. Thus, $D_T = \epsilon O_{\mathbb{P}}(1)$. The claim of the lemma follows by letting $\epsilon$ approach zero. 

**Lemma A.17.** For $p \in \mathcal{P}$,

$$D_{2,T} \triangleq \sup_{\bar{v} \in \mathcal{V}} \int_{\Gamma_{T,\eta}} |p(u)| \exp \{ \tilde{\gamma}_T \tilde{G}_{T,v}(u, \bar{v}) \} \exp (-A^0(u)) \left| \pi_{T,v}(u) - \pi^0 \right| \, du = o_{\mathbb{P}}(1).$$

Proof. By the differentiability of $\pi(\cdot)$ at $\lambda^0_b$ (cf. Assumption 3.2), for any $u \in \mathbb{R}$ $|\pi_{T,v}(u) - \pi^0| \leq \left| \lambda^0_b \tilde{T}(v) - \pi^0 \right| + C \psi_T^{-1} |u|$, with $C > 0$. The first term on the right-hand side is $o(1)$ and does not depend on $u$. Recalling that $\tilde{G}_{T,v}(u, \bar{v}) = \sup_{\bar{v} \in \mathcal{V}} |\tilde{G}_{T,v}(u, \bar{v})|$,

$$D_{2,T} \leq K \left[ o(1) \int_{\Gamma_{T,\eta}} d_T(u) \, du + \psi_T^{-1} \int_{\Gamma_{T,\eta}} |u| d_T(u) \, du \right] \leq K \left[ o(1) O_{\mathbb{P}}(1) + \psi_T^{-1} O_{\mathbb{P}}(1) \right],$$

where $d_T(u) \triangleq |p(u)| \exp \{ \tilde{\gamma}_T \tilde{G}_{T,v}(u, \bar{v}) \} \exp (-A^0(u))$ and the $O_{\mathbb{P}}(1)$ terms follows from Lemma A.14 and $\tilde{\gamma}_T \rightarrow \kappa_\gamma < \infty$. Since $\psi_T \rightarrow \infty$, we have $D_{2,T} = o_{\mathbb{P}}(1)$. 

**Lemma A.18.** For any $p \in \mathcal{P}$ and constants $C_1, C_2 > 0$, $\int_{\Gamma_{T,\psi}} |p(u)| \exp \left\{ \tilde{\gamma}_T \left\{ \tilde{C}_{1,T}(u) - C_2 |u| \right\} \right\} \, du = o_{\mathbb{P}}(1).$

Proof. It follows from Lemma A.13. 

**Lemma A.19.** For $p \in \mathcal{P}$ and constants $a_1, a_2, a_3 \geq 0$, with $a_2 + a_3 > 0$, let

$$D_{3,T} \triangleq \int_{U_2} |p(u)| \exp \left\{ \tilde{\gamma}_T \left\{ a_1 \tilde{G}_{T,v}(u) + a_2 Q_{T,v}(u) - a_3 A^0(u) \right\} \right\} \, du = o_{\mathbb{P}}(1).$$

Proof. It follows from Lemma A.6. 

**Lemma A.20.** For any integer $m \geq 0$,

$$\sup_{\bar{v} \in \mathcal{V}} \left| \int_{\mathbb{R}} u^m \exp \left\{ \tilde{\gamma}_T \tilde{G}_{T,v}(u, \bar{v}) \right\} \left[ \pi_{T,v}(u) \exp (Q_{T,v}(u)) - \pi^0 \exp (-A^0(u)) \right] \, du \right|$$

$$= \sup_{\bar{v} \in \mathcal{V}} \left| \int_{\mathbb{R}} (A_1(u, \bar{v}) - A_2(u, \bar{v})) \, du \right| = o_{\mathbb{P}}(1).$$
Proof. By Assumption 3.2, $A_1(u, \tilde{v}) = 0$ for $u \in \Gamma_{T,\psi}^c - \tilde{U}_T$. Then, omitting arguments, we can write,

$$
\sup \left| \int_{\mathbb{R}} (A_1 - A_2) \right| \leq \sup \left| \int_{\Gamma_{T,\psi}} (A_1 - A_2) \right| + \sup \left| \int_{\Gamma_{T,\psi}} A_2 \right| + \sup \left| \int_{\mathbb{R}} A_1 \right|.
$$

(A.22)

The first right-hand side term above converges in probability to zero by Lemma A.16-A.17. The second and the last term are each $o_P(1)$ by, respectively, Lemma A.18 and Lemma A.19.

We are now in a position to conclude the proof of Theorem 3.1.

Proof. Let $\mathbf{V} \subset \mathbb{R}^{p+2q}$ be a compact set. From (3.11),

$$
\psi_T \left( \tilde{\lambda}_b^{GL,*}(\tilde{v}, v) - \lambda_{b,T}^0(v) \right) = \frac{\int_{\mathbb{R}} u \exp \left( \tilde{\gamma}_T \left( \tilde{G}_{T,v}(u, \tilde{v}) + Q_{T,v}(u) \right) \right) \pi_{T,v}(u) du}{\int_{\mathbb{R}} \exp \left( \tilde{\gamma}_T \left( \tilde{G}_{T,v}(u, \tilde{v}) + Q_{T,v}(u) \right) \right) \pi_{T,v}(u) du}.
$$

For a large enough $T$, by Lemma A.20 the right-hand is uniformly in $\tilde{v} \in \mathbf{V}$ equal to

$$
\frac{\int_{\mathbb{R}} u \exp \left( \tilde{\gamma}_T \tilde{G}_{T,v}(u, \tilde{v}) \right) \exp \left( -\Lambda^0(u) \right) du}{\int_{\mathbb{R}} \exp \left( \tilde{\gamma}_T \tilde{G}_{T,v}(u, \tilde{v}) \right) \exp \left( -\Lambda^0(u) \right) du} + o_P(1).
$$

The first term is integrable with large probability by Lemma A.13-A.14. Thus, by Lemma A.12 and the continuous mapping theorem, we have for each $v \in \mathbf{V}$,

$$
T \left| \delta_T \right|^2 \left( \tilde{\lambda}_b^{GL,*}(\tilde{v}, v) - \lambda_{b,T}^0(v) \right) \Rightarrow \frac{\int_{\mathbb{R}} u \exp \left( \Psi(u) \right) \exp \left( -\Lambda^0(u) \right) du}{\int_{\mathbb{R}} \exp \left( \Psi(u) \right) \exp \left( -\Lambda^0(u) \right) du}.
$$

(A.23)

Note that $\partial_{\theta} Q_T^0(\theta, \cdot)$ is monotonic and bounded for all $\theta \in \mathbf{S}$. The argument of Theorem 4.1 in Jurečová (1977) can be used in (A.23) to achieve uniformity in $v$.

A.3.3 Proof of Proposition 3.2

We first need to introduce further notation. For a scalar $\overline{u} > 0$ define $\Gamma_{\overline{u}} \triangleq \{ u \in \mathbb{R} : |u| \leq \overline{u} \}$. Note that $\gamma_T^{-1} = o(1)$. We shall be concerned with the asymptotic properties of the following statistic:

$$
\xi_T(\tilde{v}) = \frac{\int_{\Gamma_{\overline{u}}} u \exp \left( \tilde{\gamma}_T \left( \tilde{G}_{T,v}(u, \tilde{v}) + Q_{T,v}(u) \right) \right) \pi_{T,v}(u) du}{\int_{\Gamma_{\overline{u}}} \exp \left( \tilde{\gamma}_T \left( \tilde{G}_{T,v}(u, \tilde{v}) + Q_{T,v}(u) \right) \right) \pi_{T,v}(u) du}.
$$

Furthermore, for every $\tilde{v} \in \mathbf{V}$, let $\xi_0(\tilde{v}) = \arg \max_{u \in \Gamma_{\overline{u}}} \Psi(u)$. It turns out that $\xi_0(\tilde{v})$ is flat in $\tilde{v}$ and thus we write $\xi_0 = \xi_0(\tilde{v})$. Finally, recall that $u = T \left| \delta_T \right|^2 \left( \lambda_b - \lambda_{b,T}^0(v) \right)$.

Lemma A.21. Let $\Gamma_{T,\overline{u}} = U_T - \Gamma_{\overline{u}}$. Then for any $\epsilon > 0$ and $m = 0, 1$,

$$
\lim_{\overline{u} \to \infty} \lim_{T \to \infty} \mathbb{P} \left( \sup_{u \in \mathbf{V}} \int_{\Gamma_{T,\overline{u}}} |u|^m \exp \left( \tilde{\gamma}_T \left( \tilde{G}_{T,v}(u, \tilde{v}) + Q_{T,v}(u) \right) \right) \pi_{T,v}(u) du > \epsilon \right) = 0.
$$

Proof. Let $J_1$ and $J_2$ denote the numerator and denominator, respectively, in the display of the lemma. Then,

$$
\mathbb{P} \left( J_1/J_2 > \epsilon \right) \leq \mathbb{P} \left( J_2 \leq \exp \left( -\overline{\gamma} \right) \right) + \mathbb{P} \left( J_1 > \epsilon \exp \left( -\overline{\gamma} \right) \right),
$$

(A.24)
for any constant $\overline{a} > 0$. Let us consider the second term in (A.24). For an arbitrary $a > 0$, let $H(\overline{a}, a) = \left\{ u \in \Gamma_{\overline{a}, a} : \sup_{v \in V} \left| \tilde{G}_{T,v}(u, \overline{v}) \right| \leq a |u| \right\}$. Let $\lambda = 2 \sup_{\lambda_i \in T_0} |\lambda_i|$. Note that $\lambda < 2$ and $
abla_u \sup_{u \in H(\overline{a}, a)} |u| \leq \overline{T} \|\delta_T\|^2$. By Assumptions 2.4 and 3.4, and Lemma A.6, $Q_{T,v}(u) \leq -\min \left( A^0(u)/2, \eta \overline{\lambda} \|\delta_T\|^2 T \right)$ uniformly for all large $T$ where $\eta > 0$. Thus,

$$\sup_{u \in H(\overline{a}, a)} \sup_{v \in V} \exp \left( \tilde{\gamma}_T \left[ \tilde{G}_{T,v}(u, \overline{v}) + Q_{T,v}(u) \right] \right) \leq \sup_{u \in H(\overline{a}, a)} \sup_{v \in V} \exp \left( \tilde{\gamma}_T \left[ a |u| - A^0(u)/4 + \left( A^0(u)/2 + Q_{T,v}(u) \right) \right] \right) \leq \sup_{u \in \Gamma_{\overline{a}, a}} \exp \left( \tilde{\gamma}_T \left[ a |u| - A^0(u) - \min \left( A^0(u)/4, A^0(u)/4 + \eta \|\delta_T\|^2 T \right) \right] \right) \leq \sup_{u \in \Gamma_{\overline{a}, a}} \exp \left( \gamma_T \left[ a |u| - C_2 |u| \right] \right) + \exp \left( \gamma_T \left[ a \lambda - \eta C \right] \right) \leq \sup_{u \in \Gamma_{\overline{a}, a}} \exp \left( \gamma_T \left[ a - C_2 \right] \right) + \exp \left( \gamma_T \left[ a \lambda - \eta C \right] \right) = o \left( \exp \left( -\gamma_T \overline{\pi} \right) \right),$$

when $a > 0$ is chosen sufficiently small and for some $\overline{\pi}_1 > 0$. Furthermore, by Lemma A.13-(ii) below with $\overline{\pi} = 1$,

$$\lim_{\overline{\pi} \to \infty} \lim_{T \to \infty} \mathbb{P} \left( u \in \Gamma_{\overline{a}, \overline{\pi}, \overline{\pi} - H(\overline{a}, a), c} \right) \leq \lim_{\overline{\pi} \to \infty} \lim_{T \to \infty} \mathbb{P} \left( \sup_{|u| > \overline{\pi}} \left| \tilde{G}_{T,v}(u, \overline{v}) \right| > a \right) = 0. \quad (A.26)$$

By combining (A.25)-(A.26), $\mathbb{P} \left( J_1 > \epsilon \exp \left( -\overline{\pi} \gamma_T \right) \right) \to 0$ as $T \to \infty$. Next, we consider the first right-hand side term in (A.24). Recall the definition of $\lambda_+$ from Assumption 3.5 and let $0 < b \leq \overline{\pi}/4 \lambda_+$. Note that for $G_{T,v}(b) \triangleq \sup_{|u| \leq b} \sup_{v \in V} \left| \tilde{G}_{T,v}(u, \overline{v}) \right|$, $\mathbb{P} \left( J_2 \leq \exp \left( -\overline{\pi} \gamma_T \right) \right) \leq \mathbb{P} \left( G_{T,v}(b) \leq \overline{\pi}, J_2 \leq \exp \left( -\overline{\pi} \gamma_T \right) \right) + \mathbb{P} \left( G_{T,v}(b) > \overline{\pi} \right). \quad (A.27)$

Under Assumption 3.2 and the second part of Assumption 3.5, using the definition of $b$,

$$\mathbb{P} \left( G_{T,v}(b) \leq \overline{\pi}, J_2 \leq \exp \left( -\overline{\pi} \gamma_T \right) \right) \leq \mathbb{P} \left( C_{\pi} \int_{|u| \leq b} \exp \left( \tilde{\gamma}_T \left( -\overline{\pi}/2 - \lambda_+ b \right) \right) \text{d}u \leq \exp \left( -\overline{\pi} \gamma_T \right) \right) \leq \mathbb{P} \left( C_{\pi} b \exp \left( \overline{\pi} \gamma_T / 2 \right) \leq 1 \right) \to 0,$$

as $T \to \infty$. We shall use the uniform convergence in Lemma A.12 for the second right-hand side term in (A.27) to deduce that (recall that $\overline{\pi}$ was chosen sufficiently small and $b \leq \overline{\pi}/4 \lambda_+$),

$$\lim_{b \to 0} \lim_{T \to \infty} \mathbb{P} \left( | \tilde{Q}_{T,v}(u, \overline{v}) - \bar{V}(u) | > \overline{\pi} \right) = 0.$$

**Lemma A.22.** As $T \to \infty$, $\xi_T(\overline{v}) \Rightarrow \xi_0$ in $\mathcal{D}_b(V)$.

**Proof.** Let $B = \Gamma_{\overline{\pi}} \times V$. For any fixed $\overline{u}$, Lemma A.12 and the result $\sup_{(u, \overline{v}) \in B} \left| \tilde{Q}_{T,v}(u, \overline{v}) - A^0(u) \right| = o_{\mathbb{P}}(1)$ (cf. Lemma A.3), imply that $\tilde{Q}_{T,v} \Rightarrow \bar{V}$ in $\mathcal{D}_b(B)$. By the Skorokhod representation theorem [cf. Theorem 6.4 in Billingsley (1999)] we can find a probability space $\left( \Omega, \mathcal{F}, \mathbb{P} \right)$ on which there exist processes $\tilde{Q}_T(u, \overline{v})$ and $\bar{V}(u)$ which have the same law as $\tilde{Q}_T(u, \overline{v})$ and $\bar{V}(u)$, respectively, and with the property that

$$\sup_{(u, \overline{v}) \in B} \left| \tilde{Q}_T(u, \overline{v}) - \bar{V}(u) \right| \to 0 \quad \mathbb{P} - \text{a.s.} \quad (A.28)$$

S-11
Let
\[\xi_T (\bar{v}) = \frac{\int_{\Gamma_T} u \exp \left( \tau_T \tilde{Q}^\omega_{T,v} (u, \bar{v}) \right) \pi_{T,v} (u) \, du}{\int_{\Gamma_T} \exp \left( \tau_T \tilde{Q}^\omega_{T,v} (u, \bar{v}) \right) \pi_{T,v} (u) \, du},\]
and \(\tilde{\xi}_0 \triangleq \arg\max_{u \in \Gamma_T} \tilde{\gamma}^\omega (u)\). We shall rely on (A.28) to establish that
\[\sup_{\bar{v} \in \mathcal{V}} |\xi_T (\bar{v}) - \tilde{\xi}_0| \to 0 \quad \text{\(\bar{v}\)-a.s..} \tag{A.29}\]

Let us indicate any pair of sample paths of \(\tilde{Q}_T (u, \bar{v})\) and \(\tilde{\gamma}\), for which (A.28) holds with a superscript \(\omega\), by \(\tilde{Q}^\omega_{T,v}\) and \(\tilde{\gamma}^\omega\), respectively. For arbitrary sets \(S_1, S_2 \subset B\), let \(\bar{\rho} (S_1, S_2) \triangleq \text{Leb} (S_1 - S_2) + \text{Leb} (S_2 - S_1)\) where \(\text{Leb} (A)\) is the Lebesgue measure of the set \(A\). Further, for an arbitrary scalar \(c > 0\) and function \(T : B \to \mathbb{R}\), define \(S (T, c) \triangleq \{(u, \bar{v}) \in B : |T (u, \bar{v}) - \tilde{\gamma}^\omega (u)| \leq c\}\) where \(\tilde{\gamma}_M \triangleq \max_{u \in \Gamma_T} \tilde{\gamma}^\omega (u)\). The first step is to show that
\[\bar{\rho} \left( S \left( \tilde{Q}^\omega_{T,v}, c \right), S \left( \tilde{\gamma}^\omega, c \right) \right) = o (1). \tag{A.30}\]

Let \(S_{1,T} (c) = S \left( \tilde{Q}^\omega_{T,v}, c \right) - S \left( \tilde{\gamma}^\omega, c \right)\) and \(S_{2,T} (c) = S \left( \tilde{\gamma}^\omega, c \right) - S \left( \tilde{Q}^\omega_{T,v}, c \right)\). We first establish that \(\text{Leb} (S_{2,T} (c)) = o (1)\). For an arbitrary \(\tau > 0\), define the set \(S_T (\tau) \triangleq \{(u, \bar{v}) \in B : |\tilde{Q}^\omega_{T,v} (u, \bar{v}) - \tilde{\gamma}^\omega (u)| \leq \tau\}\) and its complement (relative to \(B\)) \(\tilde{S}_T (\tau) \triangleq \{(u, \bar{v}) \in B : |\tilde{Q}^\omega_{T,v} (u, \bar{v}) - \tilde{\gamma}^\omega (u)| > \tau\}\). We have
\[\text{Leb} (S_{2,T} (c)) = \text{Leb} \left( S_{2,T} (c) \cap \tilde{S}_T (\tau) \right) + \text{Leb} \left( S_{2,T} (c) \cap \tilde{S}_T (\tau) \right) \leq \text{Leb} \left( S_{2,T} (c) \cap \tilde{S}_T (\tau) \right) + \text{Leb} \left( \tilde{S}_T (\tau) \right) \cdot \]

Note that \(\text{Leb} \left( \tilde{S}_T (\tau) \right) = o (1)\) since the path \(\omega\) satisfies (A.28). Furthermore, \(S_{2,T} (c) \cap \tilde{S}_T (\tau) \subset C_T (c, \tau)\) where \(C_T (c, \tau) \triangleq \{(u, \bar{v}) \in B : c \leq |\tilde{Q}^\omega_{T,v} (u, \bar{v}) - \tilde{\gamma}_M| \leq c + \tau\}\). In view of (A.28),
\[\lim_{\tau \downarrow 0} \lim_{\tau \to \infty} \text{Leb} \left( C_T (c, \tau) \right) = \lim_{\tau \downarrow 0} \text{Leb} \left( \{(u, \bar{v}) \in B : c \leq |\tilde{\gamma}^\omega (u) - \tilde{\gamma}_M| \leq c + \tau\} \right)
= \text{Leb} \left( \{(u, \bar{v}) \in B : |\tilde{\gamma}^\omega (u) - \tilde{\gamma}_M| = c\} \right) = 0,
\]
by the path properties of \(\tilde{\gamma}^\omega\). Since \(\text{Leb} (S_{1,T} (c)) = o (1)\) can be proven in a similar fashion, (A.30) holds. For \(m = 0, 1, C_1 < \infty\) and by Assumption 3.2 we know there exists some \(C_2 < \infty\) such that
\[\sup_{\bar{v} \in \mathcal{V}} \int_{S \left( \tilde{Q}^\omega_{T,v} (u, \bar{v}), c \right)} |u|^m \exp \left( \tau_T \left( \tilde{Q}^\omega_{T,v} (u, \bar{v}) - \tilde{\gamma}_M \right) \right) \pi_{T,v} (u) \, du \leq C_1 \exp (-c \tau_T) C_2 \int_{\Gamma_T} |u|^m \, du = o (1),\]
since \(\{u \leq \bar{u}\}\) on \(\Gamma_T\) and recalling that \(\tau_T \to \infty\). This gives an upper bound to the same function where \(u\) replaces \(|u|\). Then,
\[\sup_{\bar{v} \in \mathcal{V}} \int_{\Gamma_T} u \exp \left( \tau_T \tilde{Q}^\omega_{T,v} (u, \bar{v}) \right) \pi_{T,v} (u) \, du \leq \text{ess sup} S \left( \tilde{Q}^\omega_{T,v}, c \right) + o (1).\]
By (A.28) we deduce \( \text{ess sup}_S \left( \tilde{Q}_{T,v}^c, c \right) + o(1) = \text{ess sup}_S \left( \tilde{\varphi}^c, c \right) + o(1) \). The same argument yields
\[
\inf_{v \in \mathbf{V}} \int_{\Gamma_{1997}} u \exp \left( \tilde{\gamma}_T \tilde{Q}_{T,v}^c (u, \overline{v}) \right) \pi_{T,v} (u) \, du \\
\geq \text{ess inf}_u \left\{ \int_{\Gamma_{1997}} \tilde{\varphi}^c (u, \overline{v}) \pi_{T,v} (u) \, du \right\} + o(1).
\]
Since almost every path \( \omega \) of the Gaussian process \( \tilde{\varphi} \) achieves its maximum at a unique point on compact sets [cf. Bai (1997) and Lemma 2.6 in Kim and Pollard (1990)], we have
\[
\lim_{c \downarrow 0} \inf \text{ess sup}_S \left( \tilde{\varphi}^c, c \right) = \lim_{c \downarrow 0} \text{ess sup}_u \left\{ \int_{\Gamma_{1997}} \tilde{\varphi}^c (u, \overline{v}) \pi_{T,v} (u) \, du \right\} = \arg \max_u \tilde{\varphi}^c (u).
\]
Hence, we have proved (A.29) which by the dominated convergence theorem then implies the weak convergence of \( \tilde{\xi}_T \) toward \( \xi_0 \). Since the law of \( \tilde{\xi}_T \) \( (\xi_0) \) under \( \mathbb{P} \) is the same as the law of \( \xi_T \) \( (\xi_0) \) under \( \mathbb{P} \), the claim of the Lemma follows.

We are now in a position to conclude the proof of Proposition 3.2. For a set \( \mathbf{T} \subset \mathbb{R} \) and \( m = 0, 1 \) we define \( J_m (\mathbf{T}) \triangleq \int_{\mathbf{T}} u^m \exp \left( \gamma_T (G_{T,v} (u, \overline{v}) + Q_{T,v} (u)) \right) \pi_{T,v} (u) \, du \). Hence, with this notation equation (3.11) can be rewritten as \( T \| \delta_T \|^2 \left( \lambda_{b,T} \tilde{v} - \lambda_{b,T} (v) \right) = J_1 (\mathbb{R}) / J_0 (\mathbb{R}) \). Applying simple manipulations, we obtain,
\[
J_1 (\mathbb{R}) / J_0 (\mathbb{R}) = \frac{J_1 (\Gamma_{1997}) + J_1 \left( \Gamma_{c,2} \right)}{J_0 (\Gamma_{1997}) + J_0 \left( \Gamma_{c,2} \right)} = \frac{J_1 (\Gamma_{1997})}{J_0 (\Gamma_{1997})} \left[ 1 - \frac{J_0 (\Gamma_{1997})}{J_0 (\mathbb{R})} \right] + \frac{J_1 (\Gamma_{c,2})}{J_0 (\mathbb{R})}.
\]
By Lemma A.21, \( J_m \left( \Gamma_{c,2} \right) / J_0 (\mathbb{R}) = o_p (1) \) \( (m = 0, 1) \) uniformly in \( \overline{v} \in \mathbf{V} \). By Lemma A.22, with \( \xi_T (\overline{v}) = J_1 (\Gamma_{1997}) / J_0 (\Gamma_{1997}) \), the first right-hand side term in (A.31) converges weakly to \( \arg \max_{u \in \mathbb{R}} \varphi (u) \) in \( D_b (\mathbf{V}) \).

### A.3.4 Proof of Corollary 3.1

The proof involves a simple change in variable. We refer to Proposition 3 in Bai (1997).

### A.3.5 Proof of Theorem 3.2

We begin by introducing some notation. Since \( l \in \mathbf{L} \), for all real numbers \( B \) sufficiently large and \( \vartheta \) sufficiently small the following relationship holds
\[
\inf_{|u| > B} l \left( u \right) \leq \sup_{|u| \leq B^\vartheta} l \left( u \right) \geq 0.
\]
Let \( \zeta_{T,v} (u, \overline{v}) = \exp \left( G_{T,v} (u, \overline{v}) - A^0 (u) \right) \), \( \Gamma_T \triangleq \{ u \in \mathbb{R} : \lambda_b \in \varGamma_0 \} \) and
\[
\Gamma_M = \{ u \in \mathbb{R} : u \leq M \leq |u| < M + 1 \} \cap \varGamma_T,
\]
and define
\[
J_{1,M} \triangleq \int_{\Gamma_M} \zeta_{T,v} (u, \overline{v}) \pi_{T,v} (u) \, du, \quad J_2 \triangleq \int_{\Gamma_T} \zeta_{T,v} (u, \overline{v}) \pi_{T,v} (u) \, du.
\]
In some steps in the proof we shall be working with elements of the following families of functions. A function \( f_T : \mathbb{R} \rightarrow \mathbb{R} \) is said to belong to the family \( \mathbf{F} \) if it satisfies the following properties: (1) For fixed
Proof. The random variable \( T \| \delta_T \|^2 (\hat{\lambda}_b^\text{GL} - \lambda_0) = \bar{T}_T \) is a minimizer of the function

\[
\Psi_{l,T}(s) = \int_{\Gamma_T} l(s-u) \frac{\exp \left( G_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right) \pi_{T,v}(u)}{\int_{\Gamma_T} \exp \left( G_{T,v}(w, \bar{v}) + Q_{T,v}(w) \right) \pi_{T,v}(w) \, dw} \, du.
\]

Observe that Lemma A.16-A.20 apply to any polynomial \( p \in P \); therefore, they are still valid for \( l \in L \). We then have that the asymptotic behavior of \( \Psi_{l,T}(s) \) only matters when \( u \) (and thus \( s \)) varies on \( \Gamma_K = \{ u \in \mathbb{R} : u \leq K \} \). By Lemma A.27-A.28, for any \( \theta > 0 \), there exists a \( \bar{T} \) such that for all \( T > \bar{T} \),

\[
\mathbb{E} \left[ \int_{\Gamma_K} \frac{\exp \left( G_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right)}{\int_{\Gamma_T} \exp \left( G_{T,v}(w, \bar{v}) + Q_{T,v}(w) \right) \, dw} \, dw \right] \leq \frac{c_\theta}{K^\theta}, \quad (A.34)
\]

Therefore, for all \( T > \bar{T} \),

\[
\Psi_{l,T}(s) = \frac{\int_{|u| \leq K} l(s-u) \exp \left( G_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right) \, du}{\int_{|u| \leq K} \exp \left( G_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right) \, du} + o_P(1), \quad (A.35)
\]

where the \( o_P(1) \) term is uniform in \( T > \bar{T} \) as \( K \) increases to infinity. By Assumption (3.2), \( |\pi_{T,v}(u) - \pi^0| \leq |\pi \left( \hat{\lambda}_b^{T}(v) \right) - \pi^0| + C \psi_T^{-1} |u| \), with \( C > 0 \). On \( \{ |u| \leq K \} \), the first term on the right-hand side is \( o(1) \) and does not depend on \( u \). The second term is negligible when \( T \) is large. Thus, without loss of generality we set \( \pi_{T,v}(u) = 1 \) for all \( u \) in what follows.

Next, we show the convergence of the marginal distributions of the estimate \( \Psi_{l,T}(s) \) to the marginals of the random function \( \Psi_l(s) \), where the region of integration in the definition of both the numerator and denominator of \( \Psi_{l,T}(s) \) and \( \Psi_l(s) \) is restricted to \( \{ |u| \leq K \} \) only, in view of (A.35). For a finite integer \( n \), choose arbitrary real numbers \( a_j \) \( (j = 0, \ldots, n) \) and introduce the following estimate:

\[
\sum_{j=1}^{n} a_j \int_{|u| \leq K} l(s_j - u) \zeta_{T,v}(u, \bar{v}) \, du + a_0 \int_{|u| \leq K} l(s_0 - u) \zeta_{T,v}(u, \bar{v}) \, du. \quad (A.36)
\]

By Lemma A.24 and A.30, we can invoke Theorem I.A.22 in Ibragimov and Has’minskii (1981) which gives the convergence in distribution of the estimate in (A.36) towards the distribution of the following random variable:

\[
\sum_{j=1}^{n} a_j \int_{|u| \leq K} l(s_j - u) \exp (\mathcal{V}(u)) \, du + a_0 \int_{|u| \leq K} l(s_0 - u) \exp (\mathcal{V}(u)) \, du.
\]

By the Cramer-Wold Theorem [cf. Theorem 29.4 in Billingsley (1995)] this suffices for the convergence in distribution of the vector

\[
\int_{|u| \leq K} l(s_i - u) \zeta_{T,v}(u, \bar{v}) \, du, \ldots, \int_{|u| \leq K} l(s_n - u) \zeta_{T,v}(u, \bar{v}) \, du, \int_{|u| \leq K} l(s_0 - u) \zeta_{T,v}(u, \bar{v}) \, du,
\]

to the distribution of the vector

\[
\int_{|u| \leq K} l(s_i - u) \exp (\mathcal{V}(u)) \, du, \ldots, \int_{|u| \leq K} l(s_n - u) \exp (\mathcal{V}(u)) \, du, \int_{|u| \leq K} l(s_0 - u) \exp (\mathcal{V}(u)) \, du.
\]
As a consequence, for any $K_1, K_2 < \infty$, the marginal distributions of
\[
\int_{|u| \leq K_1} l(s-u) \exp \left( G_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right) du
\]

converge to the marginals of $\int_{|u| \leq K_1} l(s-u) \exp (\mathcal{Y}(u)) du / \left( \int_{|u| \leq K_2} \exp (\mathcal{Y}(w)) dw \right)$. The same convergence result extends to the distribution of
\[
\int_{M \leq |u| < M+1} \exp \left( G_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right) du,
\]
towards the distribution of $\int_{M \leq |u| < M+1} (\exp (\mathcal{Y}(u)) du / \int_{|u| \leq K^2} \exp (\mathcal{Y}(w)) dw)$. By choosing $K_2 > M+1$ we deduce
\[
\mathbb{E} \left[ \int_{M \leq |u| < M+1} \exp (\mathcal{Y}(u)) du \right] \leq \lim_{T \to \infty} \mathbb{E} \left[ \int_{\Gamma_M} \exp \left( G_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right) du \right] \leq c_0 M^{-\theta},
\]
in view of (A.34). This leads to
\[
\Psi_{l,t}(s) = \int_{|u| \leq K} l(s-u) \frac{\exp (\mathcal{Y}(u)) du}{\int_{|u| \leq K} \exp (\mathcal{Y}(w)) dw} + o_p(1),
\]
where the $o_p(1)$ term is uniform as $K$ increases to infinity. We then have the convergence of the finite-dimensional distributions of $\Psi_{l,t}(s)$ toward $\Psi_l(s)$. Next, we need to prove the tightness of the sequence $\{\Psi_{l,t}(s), T \geq 1\}$. More specifically, we shall show that the family of distributions on the space of continuous functions $\mathbb{C}_b(K)$ generated by the contractions of $\Psi_{l,t}(s)$ on $\{|s| \leq K\}$ are dense. For any $l \in \mathbb{L}$ the inequality $l(u) \leq 2^r \left(1 + |u|^2\right)^r$ holds for some $r$. Let
\[
\Upsilon_K(\varpi) = \int \sup_{|s| \leq K, |y| \leq \varpi} |l(s+y-u) - l(s-u)| (1 + |u|^2)^{-r-1} du.
\]
Fix $K < \infty$. We show $\lim_{\varpi \to 0} \Upsilon_K(\varpi) = 0$. Note that for any $\kappa > 0$, we can choose a $M$ such that
\[
\int_{|s| > M} \sup_{|y| \leq \varpi} |l(s+y-u) - l(s-u)| (1 + |u|^2)^{-r-1} du < \kappa.
\]
We now use Lusin’s Theorem [cf. Section 3.3 in Royden and Fitzpatrick (2010)]. Since $l(\cdot)$ is measurable, there exists a continuous function $g(u)$ in the interval $\{u \in \mathbb{R} : |u| \leq K + 2M\}$ which agrees with $l(u)$ except on a set whose measure does not exceed $\kappa \left(2\Upsilon\right)^{-1}$, where $\Upsilon$ is the upper bound of $l(\cdot)$ on $\{u \in \mathbb{R} : |u| \leq K + 2M\}$. Denote the modulus of continuity of $g(\cdot)$ by $w_g(\varpi)$. Without loss of generality assume $|g(u)| \leq \Upsilon$ for all $u$ satisfying $|u| \leq K + 2M$. Then,
\[
\int_{|u| > M, |s| \leq K, |y| \leq \varpi} |l(s+y-u) - l(s-u)| (1 + |u|^2)^{-r-1} du
\leq \int_{|s| \leq K, |y| \leq \varpi} |l(s+y-u) - l(s-u)| (1 + |u|^2)^{-r-1} du
\leq w_g(\varpi) \int_{|s| \leq K, |y| \leq \varpi} (1 + |u|^2)^{-r-k} du + 2\Upsilon \operatorname{Leb}\{u \in \mathbb{R} : |u| \leq K + 2M, l \neq g\},
\]
S-15
and \( T \leq C w_g(\varpi) + \kappa \) for some \( C \). Hence, \( \Upsilon_K(\varpi) \leq C w_g(\varpi) + 2\kappa \) since \( \kappa \) can be chosen arbitrarily small and (for each fixed \( \kappa \) ) \( w_g(\varpi) \rightarrow 0 \) as \( \varpi \downarrow 0 \) by definition. By Assumption 3.7, there exists a number \( C < \infty \) such that

\[
E \left[ \sup_{|s| \leq K, |y| \leq \varpi} |\Psi_{l,T}(s + y) - \Psi_{l,T}(s)| \right] 
\leq \int_{\mathbb{R}} \sup_{|s| \leq K, |y| \leq \varpi} |l(s + y - u) - l(s - u)| E \left( \frac{\exp \left( \tilde{G}_{T,v}(u, \tilde{v}) + Q_{T,v}(u) \right)}{\int_{\mathbb{U}_T} \exp \left( \tilde{G}_{T,v}(w, \tilde{v}) + Q_{T,v}(w) \right) dw} \right) du
\leq C \Upsilon_K(\varpi).
\]

Markov’s inequality together with the above bound establish that the family of distributions generated by the contractions of \( \Psi_{T,l} \) is dense in \( C_b(K) \). Since the finite-dimensional convergence in distribution was demonstrated above, we can deduce the weak convergence \( \Psi_{l,T} \Rightarrow \Psi_l \) in \( \mathbb{D}_b(V) \) uniformly in \( \lambda_0^0 \in K \).

Finally, we examine the oscillations of the minimum points of the sample criterion \( \Psi_{l,T} \). Consider an open bounded interval \( A \) that satisfies \( P \{ \xi_0^0 \in b(A) \} = 0 \), where \( b(A) \) denotes the boundary of the set \( A \). Choose a real number \( K \) sufficiently large such that \( A \subset \{ s : |s| \leq K \} \) and define for \( |s| \leq K \) the functionals \( H_A(\Psi) = \inf_{s \in A} \Psi_l(s) \) and \( H_{A^c}(\Psi) = \inf_{s \notin A^c} \Psi_l(s) \). Let \( M_T \) denote the set of minimum points of \( \Psi_{l,T} \). We have

\[
P[M_T \subset A] = P[H_A(\Psi) < H_{A^c}(\Psi), M_T \subset \{ s : |s| \leq K \}]
\geq P[H_A(\Psi) < H_{A^c}(\Psi)] - P[M_T \notin \{ s : |s| \leq K \}].
\]

Therefore,

\[
\lim \inf_{T \to \infty} P[M_T \subset A] \geq P[H_A(\Psi) < H_{A^c}(\Psi)] - \sup_T P[M_T \notin \{ s : |s| \leq K \}],
\]

and \( \limsup_{T \to \infty} P[M_T \subset A] \leq P[H_A(\Psi) < H_{A^c}(\Psi)] \). Moreover, the minimum of the population criterion \( \Psi_l(\cdot) \) satisfies \( P[\xi_0^0 \in A] \leq P[H_A(\Psi) < H_{A^c}(\Psi)] \) and \( P[\xi_0^0 \in A] + P[|\xi_0^0| > K] \geq P[H_A(\Psi) \leq H_{A^c}(\Psi)] \).

Lemma A.29 shall be used to deduce that the following relationship holds,

\[
\limsup_{T \to \infty} E \left[ l(T \|\delta_T\|^2 (\tilde{\lambda}_b^{GL} - \lambda_b^0)) \right] < \infty,
\]

for any loss function \( l \in L \). Hence, the set \( M_T \) of absolute minimum points of the function \( \Psi_{l,T}(s) \) are uniformly stochastically bounded for all \( T \) large enough: \( \lim_{K \to \infty} P[M_T \notin \{ s : |s| \leq K \}] = 0 \). The latter result together with the uniqueness assumption (cf. Assumption 3.7) yield

\[
\lim_{K \to \infty} \left\{ \sup_T P[M_T \notin \{ s : |s| \leq K \}] + P[|\xi_0^0| > K] \right\} = 0.
\]

Hence, we have

\[
\lim_{T \to \infty} P[M_T \subset A] = P[\xi_0^0 \in A] \quad \text{(A.38)}.
\]

The last step involves showing that the length of the set \( M_T \) approaches zero in probability as \( T \to \infty \). Let \( A_d \) denote an interval in \( \mathbb{R} \) centered at the origin and of length \( d < \infty \). Equation (A.38) guarantees that \( \lim_{d \to \infty} \sup_{T \to \infty} P[M_T \notin A_d] = 0 \). Choose any \( \epsilon > 0 \) and divide \( A_d \) into admissible subintervals
whose lengths do not exceed $\epsilon/2$. Then,

$$
P \left[ \sup_{s_i, s_j \in M_T} |s_i - s_j| > \epsilon \right] \leq P \left[ M_T \not\subseteq A_d \right] + (1 + 2d/\epsilon) \sup \{ \mathbb{P} [H_A (\Psi_{l,T}) = H_{A^c} (\Psi_{l,T})] \},
$$

where the term $1 + 2d/\epsilon$ is an upper bound on the admissible number of subintervals and the supremum in the second term is over all possible open bounded subintervals $A \subset A_d$. The weak convergence result implies $P [H_A (\Psi_{l,T}) = H_{A^c} (\Psi_{l,T})] \to P [H_A (\Psi_T) = H_{A^c} (\Psi_T)]$ as $T \to \infty$. Since $P [H_A (\Psi_T) = H_{A^c} (\Psi_T)] = 0$ and $P [M_T \not\subseteq A_d] \to 0$ for large $d$, then $P \left[ \sup_{s_i, s_j \in M_T} |s_i - s_j| > \epsilon \right] = o(1)$. Since $\epsilon > 0$ can be chosen arbitrarily small we deduce that the distribution of $T \|\delta_T\|^2 \left( \hat{\lambda}_{GL}^0 - \lambda_0^0 \right)$ converges to the distribution of $\xi_i^0$. 

\[ \square \]

**Lemma A.23.** Let $u_1, u_2 \in \mathbb{R}$ be of the same sign with $0 < |u_1| < |u_2|$. For any integer $r > 0$ and some constants $c_r$ and $C_r$ which depend on $r$ only, we have uniformly in $\bar{v} \in V$,

$$
\mathbb{E} \left[ \left( \zeta_{T,v}^{1/2r} (u_2, \bar{v}) - \zeta_{T,v}^{1/2r} (u_1, \bar{v}) \right)^{2r} \right] \leq c_r \left( \delta^0 \right)^r \left( |u_2 - u_1| \Sigma_i \delta^0 \right)^r \leq C_r |u_2 - u_1|^r,
$$

where $\Sigma_i$ is defined in Assumption 3.5 and $i = 1$ if $u_1 > 0$ and $i = 2$ if $u_1 > 0$.

**Proof.** The proof is given for the case $u_2 > u_1 > 0$. The other case is similar and thus omitted. We follow closely the proof of Lemma III.5.2 in Ibragimov and Has’minskii (1981). Let $\mathcal{V} (u_i) = \exp (\mathcal{V} (u_i))$, $i = 1, 2$. We have

$$
\mathbb{E} \left[ (V^{1/2r} (u_2) - V^{1/2r} (u_1))^{2r} \right] = \Sigma_{j=0}^{2r} (\frac{j}{r}) (-1)^j \mathbb{E}_u \left[ V^{j/2r} (u_2) \right],
$$

where $V_{u_1} (u_2) \triangleq \exp (\mathcal{V} (u_2) - \mathcal{V} (u_1))$. Using the Gaussian property of $\mathcal{V} (u)$, for each $u \in \mathbb{R}$, we have

$$
\mathbb{E}_u \left[ V^{j/2r} (u_2) \right] = \exp \left( \frac{1}{2} \left( \frac{j}{2r} \right)^2 4 \left( \delta^0 \right)^r |u_2 - u_1| |\Sigma_2| \delta^0 - \frac{j}{2r} |A^0 (u_2) - A^0 (u_1)| \right). \tag{A.39}
$$

Then,

$$
\mathbb{E} \left[ (V^{1/2r} (u_2) - V^{1/2r} (u_1))^{2r} \right] = \Sigma_{j=0}^{2r} (\frac{j}{r}) (-1)^j d^{j/2r} with
$$

$$
d \triangleq \exp \left( \frac{j}{2r} - \frac{2r - j}{r} \left( \delta^0 \right)^r |u_2 - u_1| |\Sigma_2| \delta^0 - |A^0 (u_2) - A^0 (u_1)| \right).
$$

Let $B \triangleq 2 (\delta^0)^r (|u_2 - u_1| |\Sigma_2| \delta^0 - |A^0 (u_2) - A^0 (u_1)|)$. There are different cases to be considered:

1. $B < 0$. Note that

$$
d = \exp \left( \frac{j}{2r} - \left( \delta^0 \right)^r |u_2 - u_1| |\Sigma_2| \delta^0 - \left( \delta^0 \right)^r |u_2 - u_1| |\Sigma_2| \delta^0 + B \right)$$

$$
= \exp \left( - \frac{2r - j}{r} \left( \delta^0 \right)^r |u_2 - u_1| |\Sigma_2| \delta^0 \right) e^B,
$$

which then results in

$$
\mathbb{E} \left[ (V^{1/2r} (u_2) - V^{1/2r} (u_1))^{2r} \right] \leq p_r (a), \tag{A.40}
$$

where $p_r (a) \triangleq \Sigma_{j=0}^{2r} (\frac{j}{r}) (-1)^j a^{(2r-j)}$ and $a = e^{B/2r} \exp \left(-r^{-1} (\delta^0)^r |u_2 - u_1| |\Sigma_2| \delta^0 \right)$.

2. $(\delta^0)^r (|u_2 - u_1| |\Sigma_2| \delta^0) = |A^0 (u_2) - A^0 (u_1)|$. This case is the same as the previous one but with $a = \exp \left(-r^{-1} (\delta^0)^r (|u_2 - u_1| |\Sigma_2| \delta^0) \right)$.
(3) \( B > 0 \). Upon simple manipulations, \( \mathbb{E} \left[ \left( \mathcal{Y}^{1/2r} (u_2) - \mathcal{Y}^{1/2r} (u_1) \right)^{2r} \right] \leq p_r (a) \), where

\[
p_r (a) = e^{-B/2r} \sum_{j=0}^{2r} \binom{2r}{j} (-1)^j a^{(2r-j)},
\]

with \( a = \exp \left( -r^{-1} (\delta^0)' \left( |u_2 - u_1| \Sigma_2 \right) \delta^0 \right) \). We can thus proceed with the same proof for all the above cases. Let us consider the first case. We show that at the point \( a = 1 \), the polynomial \( p_r (a) \) admits a root of multiplicity \( r \). This can be established by verifying the equalities \( p_r (1) = p_r^{(1)} (1) = \cdots = p_r^{(r-1)} (1) = 0 \).

One then recognizes that \( p_r^{(i)} (a) \) is a linear combination of summations \( S_k (k = 0, 1, \ldots, 2i) \) given by \( S_k = e^B \sum_{j=0}^{2r} \binom{2r}{j} r^k \). Thus, one only needs to verify that \( S_k = 0 \) for \( k = 0, 1, \ldots, 2r - 2 \). This follows because the expression for \( S_k \) is found by applying the operator \( e^B a (d/da) \) to the function \( (1 - a^2)^{2r} \) and evaluating it at \( a = 1 \). Consequently, \( S_k = 0 \) for \( k = 0, 1, \ldots, 2r - 1 \). Using this result into (A.40) we find, with \( \tilde{p}_r (a) \) being a polynomial of degree \( r^2 - r \),

\[
\mathbb{E} \left[ \left( \mathcal{Y}^{1/2r} (u_2) - \mathcal{Y}^{1/2r} (u_1) \right)^{2r} \right] = (1 - a)^r \tilde{p}_r (a) \leq \left( r^{-1} (\delta^0)' \left( |u_2 - u_1| \Sigma_2 \right) \delta^0 \right)^r \tilde{p}_r (a), \quad (A.41)
\]

where the last inequality follows from \( 1 - e^{-c} \leq c \), for \( c > 0 \). Next, let \( \zeta_{T,v}^{1/2r} (u_2, u_1) = \zeta_{T,v}^{1/2r} (u_2) - \zeta_{T,v}^{1/2r} (u_1) \).

By Lemma A.3 and A.12, the continuous mapping theorem and (A.41), \( \lim_{T \to \infty} \mathbb{E} \left[ \zeta_{T,v}^{1/2r} (u_2, u_1) \right] \leq (1 - a)^r \tilde{p}_r (a) \), uniformly in \( \tilde{v} \in \mathbf{V} \). Noting that \( j \leq 2r \), we can set \( C_r = \max_{0 \leq a \leq 1} e^B \tilde{p}_r (a) / r^r \) to prove the lemma.

\[ \square \]

**Lemma A.24.** For \( u_1, u_2 \in \mathbb{R} \) being of the same sign and satisfying \( 0 < |u_1| < |u_2| < K < \infty \). Then, for all \( T \) sufficiently large, we have

\[
\mathbb{E} \left[ \left( \zeta_{T,v}^{1/4} (u_2, \tilde{v}) - \zeta_{T,v}^{1/4} (u_1, \tilde{v}) \right)^4 \right] \leq C_1 |u_2 - u_1|^2, \quad (A.42)
\]

where \( 0 < C_1 < \infty \). Furthermore, for the constant \( C_1 \) from Lemma A.23, we have

\[
\mathbb{P} \left[ \zeta_{T,v} (u, \tilde{v}) > \exp (-3C_1 |u| / 2) \right] \leq \exp (-C_1 |u| / 4). \quad (A.43)
\]

Both relationships are valid uniformly in \( \tilde{v} \in \mathbf{V} \).

\[ \square \]

**Proof.** Suppose \( u > 0 \). The relationship in (A.42) follows from Lemma A.23 with \( r = 2 \). By Markov’s inequality and Lemma A.23,

\[
\mathbb{P} \left[ \zeta_{T,v} (u, \tilde{v}) > \exp (-3C_1 |u| / 2) \right] \leq \exp (3C_1 |u| / 4) \mathbb{E} \left[ \zeta_{T,v}^{1/2} (u, \tilde{v}) \right] \leq \exp \left( 3C_1 |u| / 4 - \left( \delta^0 \right)' (|u| \Sigma_2) \delta^0 \right) \leq \exp (-C_1 |u| / 4). \quad \square
\]

**Lemma A.25.** Under the conditions of Lemma A.24, for any \( \vartheta > 0 \) there exists a finite real number \( c_\vartheta \) and a \( \overline{T} \) such that for all \( T > \overline{T} \), \( \sup_{\tilde{v} \in \mathbf{V}} \mathbb{P} \left[ \sup_{|u| > M} \zeta_{T,v} (u, \tilde{v}) > M^{-\vartheta} \right] \leq c_\vartheta M^{-\vartheta} \).

\[ \square \]

**Proof.** It can be shown using Lemma A.23-A.24.
Lemma A.26. For every sufficiently small $\epsilon \leq \bar{\epsilon}$, where $\bar{\epsilon}$ depends on the smoothness of $\pi(\cdot)$, there exists $0 < C < \infty$ such that
\[
P \left[ \int_0^\epsilon \zeta_{T,v}(u, \bar{v}) \pi \left( \lambda_b^0 + u/\psi_T \right) du < \epsilon \pi \left( \lambda_b^0 \right) \right] < Ce^{1/2}. \tag{A.44}
\]

Proof. Since $E(\zeta_{T,v}(0, \bar{v})) = 1$ and $E(\zeta_{T,v}(u, \bar{v})) \leq 1$ for sufficiently large $T$, we have
\[
E|\zeta_{T,v}(u, \bar{v}) - \zeta_{T,v}(0, \bar{v})| \leq \left( E \left| \zeta_{T,v}^{1/2}(u, \bar{v}) + \zeta_{T,v}^{1/2}(0, \bar{v}) \right|^2 \right)^{1/2} \leq C |u|^{1/2}, \tag{A.45}
\]
by Lemma A.23 with $r = 1$. By Assumption 3.2, $|\pi_{T,v}(u) - \pi_0| \leq |\pi \left( \lambda_b^0 \psi(v) \right) - \pi_0| + C\psi^{-1}|u|$, with $C > 0$. The first term on the right-hand side is $o(1)$ (and independent of $u$) while the second is asymptotically negligible for small $u$. Thus, for a sufficiently small $\tau > 0$,
\[
\int_0^\epsilon \zeta_{T,v}(u, \bar{v}) \pi_{T,v}(u) du > \frac{\tau^0}{2} \int_0^\epsilon \zeta_{T,v}(u, \bar{v}) du.
\]
Next, using $\zeta_{T,v}(0, \bar{v}) = 1$,
\[
P \left[ \int_0^\epsilon \zeta_{T,v}(u, \bar{v}) \pi_{T,v}(u) du < \epsilon/2 \right] \leq P \left[ \int_0^\epsilon (\zeta_{T,v}(u, \bar{v}) - \zeta_{T,v}(0, \bar{v})) du < -\epsilon/2 \right] \\
\leq P \left[ \int_0^\epsilon |\zeta_{T,v}(u, \bar{v}) - \zeta_{T,v}(0, \bar{v})| du > \epsilon/2 \right],
\]
and by Markov’s inequality together with (A.45) the last expression is less than or equal to
\[
(2/\epsilon) \int_0^\epsilon E|\zeta_{T,v}(u, \bar{v}) - \zeta_{T,v}(0, \bar{v})| du < 2Ce^{1/2}.
\]
\[
\square
\]

Lemma A.27. For $f_T \in F$, and $M$ sufficiently large, there exist constants $c, C > 0$ such that
\[
P \left[ J_{1,M} > \exp(-cf_T(M)) \right] \leq C \left( 1 + M^C \right) \exp(-cf_T(M)), \tag{A.46}
\]
uniformly in $\bar{v} \in V$.

Proof. In view of the smootheness property of $\pi(\cdot)$, without loss of generality we consider the case of the uniform prior (i.e., $\pi_{T,v}(u) = 1$ for all $u$). We begin by dividing the open interval $\{u : M \leq |u| < M + 1\}$ into $I$ disjoint segments denoting the $i$-th one by $\Pi_i$. For each segment $\Pi_i$ choose a point $u_i$ and define $J_{1,M}^\Pi = \sup_{\bar{v} \in V} \sum_{i \in I} \zeta_{T,v}(u_i, \bar{v}) \Leb(\Pi_i) = \sup_{\bar{v} \in V} \sum_{i \in I} \int_{\Pi_i} \zeta_{T,v}(u_i, \bar{v}) du$. Then,
\[
P \left[ J_{1,M}^\Pi > (1/4) \exp(-cf_T(M)) \right] \leq P \left[ \max_{i \in I} \sup_{\bar{v} \in V} \zeta_{T,v}^{1/2}(u_i, \bar{v}) \Leb(\Pi_i) \right]^{1/2} \left( 1/2 \right) \exp(-f_T(M)/2) \\
\leq \sum_{i \in I} P \left[ \zeta_{T,v}^{1/2}(u_i, \bar{v}) > (1/2) \Leb(\Pi_i) \right]^{1/2} \exp(-f_T(M)/2) \\
\leq 2I \left( \Leb(\Pi_i) \right)^{1/2} \exp(-f_T(M)/2), \tag{A.47}
\]
where the last inequality follows from applying Lemma A.24 to each summand. Upon using the inequality

S-19
\[ \exp \left( -f_T(M)/2 \right) < 1/2 \] (which is valid for sufficiently large \( M \)), we have
\[ \mathbb{P} \left[ J_{1,M} > \exp \left( -f_T(M)/2 \right) \right] \leq \mathbb{P} \left[ \left| J_{1,M} - J_{1,M}^\Pi \right| > (1/2) \exp \left( -f_T(M)/2 \right) \right] + \mathbb{P} \left[ J_{1,M}^\Pi > \exp \left( -f_T(M) \right) \right]. \]

Focusing on the first term,
\[ \mathbb{E} \left[ J_{1,M} - J_{1,M}^\Pi \right] \leq \sum_{i \in I} \int_{\Pi_i} \mathbb{E} \left| \xi_{T,v}^{1/2}(u, \bar{v}) - \xi_{T,v}^{1/2}(u_i, \bar{v}) \right| du \]
\[ \leq \sum_{i \in I} \int_{\Pi_i} \left( \mathbb{E} \left| \xi_{T,v}^{1/2}(u, \bar{v}) + \xi_{T,v}^{1/2}(u_i, \bar{v}) \right| \mathbb{E} \left| \xi_{T,v}^{1/2}(u, \bar{v}) - \xi_{T,v}^{1/2}(u_i, \bar{v}) \right| \right)^{1/2} du \]
\[ \leq C (1 + M)^C \sum_{i \in I} \int_{\Pi_i} |u_i - u|^{1/2} du, \]

where for the last inequality we have used Lemma A.24 since we can always choose the partition of the segments such that each \( \Pi_i \) contains either positive or negative \( u_i \). Since each summand on the right-hand side above is less than \( C (MI^{-1})^{3/2} \) there exist numbers \( C_1 \) and \( C_2 \) such that
\[ \mathbb{E} \left[ J_{1,M} - J_{1,M}^\Pi \right] \leq C_1 \left( 1 + M^{C_2} \right) I^{-1/2}. \] (A.48)

Using (A.47) and (A.48) we have
\[ \mathbb{P} \left[ J_{1,M} > \exp \left( -f_T(M)/2 \right) \right] \leq C_1 \left( 1 + M^{C_2} \right) I^{-1/2} + 2I \left( \text{Leb} (\Gamma_M) \right)^{1/2} \exp \left( -f_T(M)/12 \right). \]

The relationship in the last display leads to the claim of the lemma if we choose \( I \) satisfying
\[ 1 \leq I^{3/2} \exp \left( -f_T(M)/4 \right) \leq 2. \]

**Lemma A.28.** For \( f_T \in F \), and \( M \) sufficiently large, there exist constants \( c, C > 0 \) such that
\[ \mathbb{E} \left[ J_{1,M}/J_2 \right] \leq C \left( 1 + M^C \right) \exp \left( -cf_T(M) \right), \] (A.49)
uniformly in \( \bar{v} \in V \).

**Proof.** Note that \( J_{1,M}/J_2 \leq 1 \). Thus, for any \( \epsilon > 0 \),
\[ \mathbb{E} \left[ J_{1,M}/J_2 \right] \leq \mathbb{P} \left[ J_{1,M} > \exp \left( -cf_T(M)/2 \right) \right] + (4/\epsilon) \exp \left( -cf_T(M) \right) + \mathbb{P} \left[ \int_{\Gamma_T} \xi_{T,v}(u, \bar{v}) du < \epsilon/4 \right]. \]

By Lemma A.27, the first term is bounded by \( C \left( 1 + M^C \right) \exp \left( -cf_T(M)/4 \right) \) while for the last term we can use (A.44) to deduce
\[ \mathbb{E} \left[ J_{1,M}/J_2 \right] \leq C \left( 1 + M^C \right) \exp \left( -cf_T(M) \right) + (4/\epsilon) \exp \left( -cf_T(M) \right) + C \epsilon^{1/2}. \]

Finally, choose \( \epsilon = \exp \left( (1-2c/3)f_T(M) \right) \) to complete the proof of the lemma. \( \square \)

**Lemma A.29.** For \( l \in L \) and any \( \vartheta > 0 \), \( \lim_{B \to \infty} \lim_{T \to \infty} B^{\vartheta} \mathbb{P} \left[ \psi_T \left( \hat{\lambda}_b^{\text{GL}} - \lambda_b^0 \right) > B \right] = 0. \)

**Proof.** Let \( p_T(u) \triangleq p_{1,T}(u)/\mathfrak{p}_T \) where \( p_{1,T}(u) = \exp \left( \hat{G}_{T,v}(u, \bar{v}) + Q_{T,v}(u) \right) \) and \( \mathfrak{p}_T \triangleq \int_{U_T} p_{1,T}(w) dw. \)

By definition, \( \hat{\lambda}_b^{\text{GL}} \) is the minimum of the function \( \int_{\Gamma^0} l \left( T \| \delta_T \|^2 (s - u) \right) p_{1,T}(u) \pi_{T,v}(u) du \) with \( s \in \Gamma^0 \).

Upon using a change in variables,
\[ \int_{\Gamma^0} l \left( T \| \delta_T \|^2 (s - u) \right) p_{1,T}(u) \pi_{T,v}(u) du \]
\[
= \left( T \| \delta_T \|^2 \right)^{-1} \mathbb{P}_T \int_{U_T} l \left( T \| \delta_T \|^2 \left( s - \lambda_0^b \right) - u \right) p_T \left( \lambda_0^b + \left( T \| \delta_T \|^2 \right)^{-1} u \right) du.
\]

Thus, \( \lambda_{\delta,T} \triangleq T \| \delta_T \|^2 \left( \hat{\lambda}_b^{GL} - \lambda_0^b \right) \) is the minimum of the function

\[
S_T(s) \triangleq \int_{U_T} l\left(s-u\right) \frac{p_T \left( \lambda_0^b + \left( T \| \delta_T \|^2 \right)^{-1} u \right)}{\int_{U_T} p_T \left( \lambda_0^b + \left( T \| \delta_T \|^2 \right)^{-1} w \right)} \frac{\pi_{T,v} \left( \lambda_0^b + \left( T \| \delta_T \|^2 \right)^{-1} w \right)}{\pi_{T,v} \left( \lambda_0^b + \left( T \| \delta_T \|^2 \right)^{-1} w \right)} dw,
\]

where the optimization is over \( U_T \). The random function \( S_T(\cdot) \) converges with probability one in view of Lemma A.27-A.28 together with the properties of the loss function \( l \) [cf. (A.35) and the discussion surrounding it]. Therefore, we shall show that the random function \( S_T(s) \) is strictly larger than \( S_T(0) \) on \( \{|s| > B\} \) with high probability as \( T \rightarrow \infty \). This reflects that

\[
\mathbb{P} \left[ |T \| \delta_T \|^2 \left( \hat{\lambda}_b^{GL} - \lambda_0^b \right) | > B \right] \leq \mathbb{P} \left[ \inf_{|s| > B} S_T(s) \leq S_T(0) \right]. \tag{A.50}
\]

We present the proof for the case \( \pi_{T,v}(u) = 1 \) for all \( u \). The general case follows with no additional difficulties due to the assumptions satisfied by the prior \( \pi(\cdot) \). By the properties of the family \( L \) of loss functions, we can find \( \varpi_1, \varpi_2 \in \mathbb{R} \), with \( 0 < \varpi_1 < \varpi_2 \) such that as \( T \) increases,

\[
\bar{l}_{1,T} \triangleq \sup \{ l(u) : u \in \Gamma_{1,T} \} < \bar{l}_{2,T} \triangleq \inf \{ l(u) : u \in \Gamma_{2,T} \},
\]

where \( \Gamma_{1,T} \triangleq U_T \cap \{|u| \leq \varpi_1\} \) and \( \Gamma_{2,T} \triangleq U_T \cap \{|u| > \varpi_2\} \). With this notation,

\[
S_T(0) \leq \bar{l}_{1,T} \int_{\Gamma_{1,T}} p_T(u) \, du + \int_{U_T \cap \{|u| \leq \varpi_1\}} l(u) \, p_T(u) \, du.
\]

Furthermore, if \( l \in L \), then for sufficiently large \( B \) the following relationships hold: (i) \( l(u) - \inf_{|u| > B/2} l(v) \leq 0 \); (ii) \( |u| \leq (B/2)^\theta \), \( \theta > 0 \). We shall assume that \( B \) is chosen so that \( B > 2\varpi_2 \) and \( (B/2)^\theta > \varpi_2 \) hold. Let \( \Gamma_{T,B} \triangleq \{ u : (|u| > B/2) \cap U_T \} \). Then, whenever \( |s| > B \) and \( |u| \leq B/2 \), we have

\[
|u - s| > B/2 > \varpi_2 \quad \text{and} \quad \inf_{u \in \Gamma_{T,B}} l(u) \geq \bar{l}_{2,T}. \tag{A.51}
\]

With this notation,

\[
\inf_{|s| > B} S_T(s) \geq \inf_{u \in \Gamma_{T,B}} l_T(u) \int_{\{|u| \leq B/2\} \cap U_T} p_T(w) \, dw
\]

\[
\geq \bar{l}_{2,T} \int_{\{|u| \leq B/2\} \cap U_T} p_T(w) \, dw,
\]

from which it follows that

\[
S_T(0) - \inf_{|s| > B} S_T(s) \leq -\bar{\omega} \int_{\Gamma_{1,T}} p_T(u) \, du + \int_{U_T \cap \{(B/2)^\theta \geq |u| \geq \varpi_1\}} \left( l(u) - \inf_{|s| > B/2} l_T(s) \right) p_T(u) \, du
\]

\[
+ \int_{U_T \cap \{|u| > (B/2)^\theta\}} l(u) \, p_T(u) \, du,
\]

S-21
where \( \varpi \triangleq \bar{t}_{2,T} - \bar{t}_{1,T} \). The last inequality can be manipulated further using (A.51), so that

\[
S_T (0) - \inf_{|s| > B} S_T (s) \leq -\varpi \int_{\Gamma_{1,T}} p_T (u) \, du + \int_{U_T \cap \{ |u| > (B/2)^0 \}} l_T (u) \, p_T (u) \, du. \tag{A.52}
\]

Let \( B_0 \triangleq (B/2)^0 \) and fix an arbitrary number \( \varpi > 0 \). For the first term of (A.52), Lemma A.26 implies that for sufficiently large \( T \), we have

\[
\mathbb{P} \left[ \int_{\Gamma_{1,T}} p_T (u) \, du < 2 \left( \varpi B^\varpi \right)^{-1} \right] \leq c \left( \varpi B^\varpi \right)^{-1/2}, \tag{A.53}
\]

where \( 0 < c < \infty \). Next, let us consider the second term of (A.52). We show that for large enough \( T \), an arbitrary number \( \varpi > 0 \),

\[
\mathbb{P} \left[ \int_{U_T \cap \{ |u| > B_0 \}} l(u) p_T (u) \, du > B^{-\varpi} \right] \leq c B^{-\varpi}. \tag{A.54}
\]

Since \( l \in L \), we have \( l(u) \leq |u|^a, a > 0 \) when \( u \) is large enough. Choosing \( B \) large leads to

\[
\mathbb{E} \left[ \int_{U_T \cap \{ |u| > B_0 \}} l(u) p_T (u) \, du \right] \leq \sum_{i=0}^\infty (B_0 + i + 1)^a \mathbb{E} (J_{1,B_0+i}/J_2),
\]

where \( J_{1,B_0+i}, J_2 \) are defined as in (A.33). By Lemma A.28,

\[
\mathbb{E} (J_{1,B_0+i}/J_2) \leq c (1 + (B_0 + i)^a) \exp (-bf_T (B_0 + i)),
\]

where \( f_T \in F \) and thus for some for some \( b, 0 < c < \infty \),

\[
\mathbb{E} \left[ \int_{U_T \cap \{ |u| > B_0 \}} l(u) p_T (u) \, du \right] \leq c \int_0^\infty (1 + v^a) \exp (-bf_T (v)) \, dv \leq c \exp (-bf_T (B_0)).
\]

By property (ii) of the function \( f_T \) in the class \( F \), for any \( d \in \mathbb{R}, \lim_{v \to \infty} \lim_{T \to \infty} v^d e^{-bf_T(v)} = 0 \). Thus, we know that for \( T \) large enough and some \( 0 < c < \infty \),

\[
\mathbb{E} \left[ \int_{U_T \cap \{ |u| > B_0 \}} l(u) p_T (u) \, du \right] \leq c B^{-\varpi},
\]

from which we deduce (A.54) after applying Markov’s inequality. Therefore, for sufficiently large \( T \) and large \( B \), combining equation (A.50), and (A.53)-(A.54), we have

\[
\mathbb{P} \left[ T \| \delta_T \|^2 (\lambda^G_0 - \lambda^0_0) > B \right] \leq \mathbb{P} \left[ -\varpi \int_{\Gamma_{1,T}} p_T (u) \, du + \int_{U_T \cap \{ |u| > B_0 \}} l_T (u) \, p_T (u) \, du \leq 0 \right] \leq \mathbb{P} \left[ \int_{\Gamma_{1,T}} p_T (u) \, du < 2 \left( \varpi B^\varpi \right)^{-1} \right] + \mathbb{P} \left[ \int_{U_T \cap \{ |u| > B_0 \}} l(u) \, p_T (u) \, du > B^{-\varpi} \right] \leq c \left( B^{-\varpi/2} + B^{-\varpi} \right),
\]

which can be made arbitrarily small choosing \( B \) large enough.

\[\square\]

Lemma A.30. As \( T \to \infty \), the marginal distributions of \( \zeta_{T,v} (u, \tilde{v}) \) converge to the marginal distributions
of \( \exp(\mathcal{V}(u)) \).

**Proof.** The results follows from Lemma A.3, Lemma A.12 and the continuous mapping theorem. \( \square \)

### A.4 Proofs of Section 4

#### A.4.1 Proof of Proposition 4.1

The preliminary lemmas below consider the Gaussian process \( \mathcal{W} \) on the positive half-line with \( s > 0 \). The case \( s \leq 0 \) is similar and omitted. The generic constant \( C > 0 \) used in the proofs of this section may change from line to line.

**Lemma A.31.** For \( \omega > 3/4 \), we have \( \lim_{T \to \infty} \limsup_{|s| \to \infty} \left| \hat{\mathcal{W}}_T(s) \right| / |s|^\omega = 0, \mathbb{P}\text{-}a.s. \)

**Proof.** For any \( \epsilon > 0 \), if we can show that

\[
\sum_{i=1}^{\infty} \mathbb{P} \left[ \sup_{|s| \leq 1, 1 \leq i \leq s} \left| \hat{\mathcal{W}}_T(s) \right| / |s|^\omega > \epsilon \right] < \infty, \tag{A.55}
\]

then by the Borel-Cantelli lemma, \( \mathbb{P} \left[ \limsup_{|s| \to \infty} \left| \hat{\mathcal{W}}_T(s) \right| / |s|^\omega > \epsilon \right] = 0. \) Proceeding as in the proof of Lemma A.13,

\[
\mathbb{P} \left[ \sup_{|s| \leq 1, 1 \leq i \leq s} \left| \hat{\mathcal{W}}_T(s) \right| / |s|^\omega > \epsilon \right] \leq \mathbb{P} \left[ \sup_{|s| \leq 1} \left| \hat{\mathcal{W}}_T(s) \right| > \epsilon |s|^{-1/2} \right]
\]

\[
\leq \frac{1}{\epsilon^4} \mathbb{E} \left[ \mathbb{E} \left( \sup_{|s| \leq 1} \left| \hat{\mathcal{W}}_T(s) \right|^4 \middle| \hat{\Sigma}_T \right) \right]^{1/(4\omega - 2)}.
\]

The series \( \sum_{i=1}^{\infty} i^{-p} \) is a Riemann’s zeta function and satisfies \( \sum_{i=1}^{\infty} i^{-p} < \infty \) if \( p > 1 \). Then,

\[
\sum_{i=1}^{\infty} \mathbb{P} \left[ \sup_{|s| \leq 1, 1 \leq i \leq s} \left| \hat{\mathcal{W}}_T(s) \right| / |s|^\omega > \epsilon \right] \leq \left( C/\epsilon^4 \right) \mathbb{E} \left[ \mathbb{E} \left( \sup_{|s| \leq 1} \left| \hat{\mathcal{W}}_T(s) \right|^4 \middle| \hat{\Sigma}_T \right) \right]^{1/(4\omega - 2)} \]

\[
\leq \left( C/\epsilon^4 \right) \mathbb{E} \left[ \mathbb{E} \left( \sup_{|s| \leq 1} \left| \hat{\mathcal{W}}_T(s) \right| \middle| \hat{\Sigma}_T \right) \right]^{4}, \tag{A.56}
\]

where \( C > 0 \) and the last inequality follows from Proposition A.2.4 in van der Vaart and Wellner (1996). The process \( \hat{\mathcal{W}}_T \), conditional on \( \hat{\Sigma}_T \), is sub-Gaussian with respect to the semimetric \( d_{TV}^2(t, s) = \hat{\mathcal{S}}_T(t, t) + \hat{\mathcal{S}}_T(s, s) \), which by invoking Assumption 4.1-(ii,iii) is bounded by

\[
\hat{\mathcal{S}}_T(t - s, t - s) \leq |t - s| \sup_{|s| = 1} \hat{\mathcal{S}}_T(s, s).
\]

Theorem 2.2.8 in van der Vaart and Wellner (1996) then implies

\[
\mathbb{E} \left( \sup_{|s| \leq 1} \left| \hat{\mathcal{W}}_T(s) \right| \middle| \hat{\Sigma}_T \right) \leq C \sup_{|s| = 1} \hat{\mathcal{S}}_T^{1/2} (s, s).
\]

The above inequality can be used into the right-hand side of (A.56) to deduce that the latter is bounded by

\( C \mathbb{E} \left( \sup_{|s| = 1} \hat{\mathcal{S}}_T (s, s) \right) \). By Assumption 4.1-(iv) \( C \mathbb{E} \left( \sup_{|s| = 1} \hat{\mathcal{S}}_T^2 (s, s) \right) < \infty \), and the proof is concluded. \( \square \)

**Lemma A.32.** \( \{ \hat{\mathcal{W}}_T \} \) converges weakly toward \( \mathcal{W} \) on compact subsets of \( \mathbb{D}_b \).
Proof. By the definition of $\hat{\mathcal{W}}_T (\cdot)$, we have the finite-dimensional convergence in distribution of $\hat{\mathcal{W}}_T$ toward $\mathcal{W}$. Hence, it remains to show the (asymptotic) stochastic equicontinuity of the sequence of processes $\{\hat{\mathcal{W}}_T, T \geq 1\}$. Let $C \subseteq \mathbb{R}^+$ be any compact set. Fix any $\eta > 0$ and $\epsilon > 0$. We show that for any positive sequence $\{d_T\}$, with $d_T \downarrow 0$, and for every $t, s \in C$,

$$\limsup_{T \to \infty} \mathbb{P} \left( \sup_{|t-s| < d_T} \left| \hat{\mathcal{W}}_T (t) - \hat{\mathcal{W}}_T (s) \right| > \eta \right) < \epsilon.$$  \hfill (A.57)

By Markov’s inequality, $\mathbb{P} \left( \sup_{|t-s| < d_T} \left| \hat{\mathcal{W}}_T (t) - \hat{\mathcal{W}}_T (s) \right| > \eta \right) \leq \mathbb{E} \left( \sup_{|t-s| < d_T} \left| \hat{\mathcal{W}}_T (t) - \hat{\mathcal{W}}_T (s) \right| \right) / \eta$. Let $\hat{\Sigma}_T (t, s)$ denote the covariance matrix of $\left( \hat{\mathcal{W}}_T (t), \hat{\mathcal{W}}_T (s) \right)'$ and $\mathcal{N}$ be a two-dimensional standard normal vector. Letting $t \triangleq [1 \ -1]'$, we have

$$\left[ \mathbb{E} \sup_{|t-s| < d_T} \left| \hat{\mathcal{W}}_T (t) - \hat{\mathcal{W}}_T (s) \right| \right]^2 = \left[ \mathbb{E} \sup_{|t-s| < d_T} \left| t' \tilde{\Sigma}_T (t, s) t \right| \right]^2 \leq \mathbb{E} \left[ \sup_{|t-s| < d_T} \tilde{\Sigma}_T (t - s, t - s) \right],$$

and so $\mathbb{E} \left[ \sup_{|t-s| < d_T} \tilde{\Sigma}_T (t - s, t - s) \right] \leq 2d_T \mathbb{E} \left[ \sup_{|s| = 1} \tilde{\Sigma}_T (s, s) \right]$ where we have used Assumption 4.1-(iii) in the last step. As $d_T \downarrow 0$ the right-hand side goes to zero since $\mathbb{E} \left[ \sup_{|s| = 1} \tilde{\Sigma}_T (s, s) \right] = O (1)$ by Assumption 4.1-(iv). \hfill \Box

**Lemma A.33.** Fix $0 < a < \infty$. For any $p \in \mathbb{P}$ and for any positive sequence $\{a_T\}$ satisfying $a_T \overset{p}{\to} a$,

$$\int_{\mathbb{R}} |p(\xi)| \exp \left( \mathcal{W}_T (\xi) \right) \exp (-a_T |\xi|) \, d\xi \overset{d}{\to} \int_{\mathbb{R}} |p(\xi)| \exp \left( \mathcal{W} (\xi) \right) \exp (-a |\xi|) \, d\xi.$$

**Proof.** Let $B_+ \subseteq \mathbb{R}^+$ be any non-empty compact subset of $\mathbb{R}^+$. Let

$$G = \left\{ (W, a_T) \in D_b (\mathbb{R}, \mathcal{B}, \mathbb{P}) \times B_+ : \limsup_{|s| \to \infty} |W (s)| / |s|^\omega = 0, \omega > 3/4, a_T = a + o_p (1) \right\},$$

and denote by $f : G \to \mathbb{R}$ the functional given by $f (G) = \int |p(\xi)| \exp (W (\xi)) \exp (-a_T |\xi|) \, d\xi$. In view of the continuity of $f (\cdot)$ and $a_T \overset{p}{\to} a$, the claim of the lemma follows by Lemma A.31-A.32 and the continuous mapping theorem. \hfill \Box

We are now in a position to conclude the proof of Proposition 4.1. Suppose $\gamma_T = C T \| \hat{\mathcal{W}}_T \|_2^2$ for some $C > 0$. Under mean-squared loss function, $\xi_T$ admits a closed form:

$$\hat{\xi}_T = \frac{\int u \exp \left( \mathcal{W}_T (u) - \Lambda_T (u) \right) \, du}{\int \exp \left( \mathcal{W}_T (u) - \Lambda_T (u) \right) \, du}.$$

By Lemma A.33, we deduce that $\hat{\xi}_T$ converges in law to the distribution stated in (3.12). For general loss functions, a result corresponding to Lemma A.33 can be shown to hold since $l (\cdot)$ is assumed to be continuous.
A.5 Proofs of Section 5

Rewrite the GL estimator $\lambda^\text{GL}_b$ as the minimizer of

$$R_{t,T} \triangleq \int_{T^0} l (s - \lambda_b) \frac{\exp (-Q_T (\delta (\lambda_b), \lambda_b)) \pi (\lambda_b)}{\int_{T^0} \exp (-Q_T (\delta (\lambda_b), \lambda_b)) \pi (\lambda_b)} d\lambda_b. \quad (A.58)$$

We show with the following lemma that, for each $i$, $\hat{\lambda}^\text{GL}_i \xrightarrow{p} \lambda^0_i$ no matter whether the magnitude of the shifts is fixed or not. Then, the proof of Theorem 3.2 can be repeated for each $i = 1, \ldots, m$ separately.

We begin with the case of fixed shifts.

Lemma A.34. Under Assumptions 5.1-5.2, except that $\Delta_T, i = \Delta^0_i$ for all $i$, for $l \in L$ and any $B > 0$ and $\varepsilon > 0$, we have for all large $T$, $\mathbb{P} \left[ |\hat{\lambda}^\text{GL}_i - \lambda^0_i| > B \right] < \varepsilon$ for each $i$.

Proof. Let $S_T (\delta (\lambda_b), \lambda_b) \triangleq Q_T (\delta (\lambda_b), \lambda_b) - Q_T (\delta (\lambda^0_b), \lambda^0_b)$. Without loss of generality, we assume there are only three change-points and provide a proof by contradiction for the consistency result. In particular, we suppose that all but the second change-point are consistently estimated. That is, consider the case $T_2 < T^0_2$ and for some finite $C > 0$ assume that $|\lambda_2 - \lambda^0_2| > C$. $Q_T (\delta (\lambda_b), \lambda_b)$ can be decomposed as,

$$Q_T (\delta (\lambda_b), \lambda_b) = \sum_{t=1}^{T} c^2_t + \sum_{t=1}^{T} d^2_t - 2 \sum_{t=1}^{T} c_t d_t,$$

where $d_t = u_t (\hat{\phi} - \phi^0) + \delta_t (\hat{\delta} - \delta^0)$, for $t \in [\hat{T}_{j-1} + 1, \hat{T}_j] \cap [T^0_{j-1} + 1, T^0_j]$ (for $i = 1, \ldots, m$ and $j = 1, \ldots, m+1$) where $\hat{\phi}$ and $\hat{\delta}$ are asymptotically equivalent to the corresponding least-squares estimates. Bai and Perron (1998) showed that

$$T^{-1} \sum_{t=1}^{T} d^2_t \xrightarrow{p} K > 0 \quad \text{and} \quad T^{-1} \sum_{t=1}^{T} c_t d_t = o_p (1).$$

Note that $Q_T (\delta (\lambda^0_b), \lambda^0_b) = S_T (T^0_1, T^0_2, T^0_3)$, where $S_T (T^0_1, T^0_2, T^0_3)$ denotes the sum of squared residuals evaluated at $(T^0_1, T^0_2, T^0_3)$. Since $T^{-1} S_T (T^0_1, T^0_2, T^0_3)$ is asymptotically equivalent to $T^{-1} \sum_{t=1}^{T} c^2_t$, this implies that $T^{-1} S_T (\delta (\lambda_b), \lambda_b) > 0$ for all large $T$. For some finite $K > 0$, this implies

$$S_T (\delta (\lambda_b), \lambda_b) \geq TK. \quad (A.59)$$

Let

$$U_T \triangleq \left\{ u \in \mathbb{R} : \lambda^0_b + T^{-1} u \in T^0 \right\}.$$ 

Define $p_T (u) \triangleq p_{1,T} (u) / \overline{p}_T$ where $p_{1,T} (u) = \exp (-Q_T (\delta (u), u))$ and $\overline{p}_T \triangleq \int_{U_T} p_{1,T} (u) du$. By definition, $\lambda^\text{GL}_b$ is the minimum of the function $\int_{T^0} l (s - \lambda_b) p_{1,T} (u) \pi (u) du$ with $s \in T^0$. Upon using a change in variables,

$$\int_{T^0} l (s - u) p_{1,T} (u) \pi (u) du = T^{-1} \overline{p}_T \int_{U_T} l \left( T (s - \lambda^0_b) - u \right) p_T (\lambda^0_b + T^{-1} u) \pi (\lambda^0_b + T^{-1} u) du.$$ 

Thus, $\lambda_{\delta,T} \triangleq T \left( \lambda^\text{GL}_b - \lambda^0_b \right)$ is the minimum of the function,

$$S_T (s) \triangleq \int_{U_T} l (s - u) \frac{p_T (\lambda^0_b + T^{-1} u) \pi (\lambda^0_b + T^{-1} u)}{\int_{U_T} p_T (\lambda^0_b + T^{-1} w) \pi (\lambda^0_b + T^{-1} w) dw} du.$$ 

S-25
where the optimization is over $U_T$. As in the proof of Lemma A.8, we exploit the following relationship,

$$
P\left[\frac{X_{\lambda_0}^{GL} - \lambda_0^0}{\lambda_0^0} > B\right] \leq P\left[\inf_{|s|>TB} S_T(s) \leq S_T(0)\right]. \quad (A.60)
$$

Thus, we need to show that the random function $S_T(s)$ is strictly larger than $S_T(0)$ on $\{|s|>TB\}$ with high probability as $T \to \infty$. The same steps as in Lemma A.8 lead to,

$$
S_T(0) - \inf_{|s|>TB} S_T(s) \leq -\omega \int_{\Gamma_{1,T}} p_T(u) \, du + \int_{U_T \cap \{|u|>(TB/2)^a\}} l_T(u) p_T(u) \, du.
$$

We can use the relationship (A.59) in place of (A.15) in Lemma A.8 to show that the second term above converges to zero. The first term is negative using the same argument as in Lemma A.8. Thus, $S_T(0) - \inf_{|s|>TB} S_T(s) < 0$. This gives a contradiction to the fact that $\hat{\lambda}^{GL}$ minimizes $\int_{\Gamma_0} l(s-u) p_{1,T}(u) \pi(u) \, du$.

Hence, each change-point is consistently estimated.

**Lemma A.35.** Under Assumptions 5.1-5.2, for $l \in L$ and any $B > 0$ and $\varepsilon > 0$, we have for all large $T$,

$$
P\left[\hat{\lambda}_i^{GL} - \lambda_i^0 \geq B\right] < \varepsilon \quad \text{for each } i.
$$

**Proof.** The structure of the proof is similar to that of Lemma A.34. The difference consists on the fact that now $T^{-1} \sum_{i=1}^{T} d_t^2 \overset{P}{\to} 0$ even when a break is not consistently estimated. However, Bai and Perron (1998) showed that $T^{-1} \sum_{i=1}^{T} d_t^2 > 2T^{-1} \sum_{i=1}^{T} \varepsilon_i d_t$ and thus one can proceed as in the aforementioned proof to complete the proof.

**Lemma A.36.** Under Assumptions 5.1-5.2, for $l \in L$ and for every $\varepsilon > 0$ there exists a $B < \infty$ such that for all large $T$, $P\left[\left|\hat{\lambda}_i^{GL} - \lambda_i^0\right| > B\right] < \varepsilon$ for each $i$.

**Proof.** Let $S_T(\delta(\lambda_i), \lambda_i) \triangleq Q_T(\delta(\lambda_i), \lambda_i) - Q_T(\delta(\lambda_i^0), \lambda_i^0)$. Without loss of generality, we assume there are only three change-points and provide an explicit proof only for $\lambda_2$. We use the same notation as in Bai and Perron (1998), pp. 69-70. Note that their results concerning the estimates of the regression parameters can be used in our context because once we have the consistency of the fractional change-points the estimates of the regression parameters are asymptotically equivalent to the corresponding least-squares estimates. For each $\varepsilon > 0$, let $V_\varepsilon = \{(T_1, T_2, T_3) \colon |\hat{T}_i - T_i^0| \leq \varepsilon T, 1 \leq i \leq 3\}$. By the consistency result, for each $\varepsilon > 0$ and large $T$, we have $|\hat{T}_i - T_i^0| \leq \varepsilon T$, where $\hat{T}_i = \hat{T}_i^{GL} = T\hat{\lambda}_i^{GL}$. Hence, $P\{\{(\hat{T}_1, \hat{T}_2, \hat{T}_3) \in V_\varepsilon\} \to 1$ with high probability. Therefore we only need to examine the behavior of $S_T(\delta(\lambda_i), \lambda_i)$ for those $T_i$ that are close to the true break dates such that $|T_i - T_i^0| < \varepsilon T$ for all $i$. By symmetry, we can, without loss of generality, consider the case $T_2 < T_2^0$. For $C > 0$, define

$$
V_\varepsilon^*(C) = \{(T_1, T_2, T_3) \colon |\hat{T}_i - T_i^0| < \varepsilon T, 1 \leq i \leq 3 \mbox{, } T_2 - T_2^0 < -C/v_T^2\}.
$$

Define the sum of squared residuals evaluated at $(T_1, T_2, T_3)$ by $S_T(T_1, T_2, T_3)$, $SSR_2 = S_T(T_1, T_2^0, T_3) = S_T(T_1, T_2, T_3) - SSR_1 = S_T(T_1, T_2, T_3) - S_T(T_1, T_2^0, T_3)$ and $SSR_3 = S_T(T_1, T_2^0, T_3)$. We have omitted the dependence on $\delta$. With this notation, we have $S_T(\delta(\lambda_i), \lambda_i) = S_T(T_1, T_2, T_3) - S_T(T_1, T_2^0, T_3)$ which can be decomposed as

$$
S_T(\delta(\lambda_i), \lambda_i) = [(SSR_1 - SSR_3) - (SSR_2 - SSR_3)] + (SSR_2 - S_T(T_1^0, T_2^0, T_3^0)) .
$$

(A.62)
In their Proposition 4-(ii), Bai and Perron (1998) showed that the first term on the right-hand side above satisfies the following: for every \( \varepsilon > 0 \), there exists \( B > 0 \) and \( \epsilon > 0 \) such that for large \( T \),
\[
\mathbb{P} \left[ \min \left\{ \left[ S_T (T_1, T_2, T_3) - S_T (T_1, T_2, T_3) \right] / \left( T_2^0 - T_2 \right) \right\} \leq 0 \right] < \varepsilon,
\]
where the minimum is taken over \( V^*_C (C) \). The second term of (A.62) divided by \( T_2^0 - T_2 \) can be shown to be negligible for \( \{T_1, T_2, T_3\} \in V^*_C (C) \) and \( C \) large enough because on \( V^*_C (C) \) the consistency result guarantees that \( \tilde{\lambda}_i \) can be made arbitrary close to \( \lambda_i^0 \). This leads to a result similar to (A.59) where \( T \) is replaced by \( v_T^{-2} \). Then one can continue with the same argument used in the second part of the proof of Lemma A.34. \( \square \)

### A.6 Proofs of Section 6

#### A.6.1 Proof of Proposition 6.1

Let
\[
p_{1,T} (y | \lambda_b^0 + \psi_T^{-1} u) \triangleq \exp \left( \left( \tilde{G}_{T,0} (u, 0) + Q_{T,0} (u) \right) / 2 \right),
\]
where \( \tilde{G}_{T,0} (u, 0) \) and \( Q_{T,0} (u) \) were defined in equation (3.8). Let \( p_1 (y | \lambda_b) \triangleq \exp \left( \left( L^2 (\lambda_b) - L^2 (\lambda) \right) / 2 \right) \) where \( L (\lambda_b) = (T_b (T - T_b))^{1/2} \left( \mathbb{Y}_{T_b} - \bar{\mathbb{Y}}_{T_b} \right) \) with \( \bar{\mathbb{Y}}_{T_b} = T_b^{-2} \sum_{t=1}^{T_b} y_t \) and \( \mathbb{Y}_{T_b} = (T - T_b)^{-1} \sum_{t=T_b+1}^{T} y_t \).

Following Bhattacharya (1994) we use a prior \( \tilde{\pi} (\cdot) \) on the random variable \( \bar{\lambda}_b \). The posterior distribution of \( \tilde{\lambda}_b = \lambda_b \) is given by \( p (\lambda_b | y) = h (\lambda_b) / \int_{\mathbb{R}} h (s) \, ds \) where \( h (\lambda_b) = p_1 (y | \lambda_b) \tilde{\pi} (\lambda_b) \). The total variation distance between two probability measures \( \nu_1 \) and \( \nu_2 \) defined on some probability space \( S \in \mathbb{R} \) is denoted as \( |\nu_1 - \nu_2|_{TV} \triangleq \int_{S} |\nu_1 (u) - \nu_2 (u)| \, du \). Given the local parameter \( \lambda_b = \lambda_b^0 + (Tv_T)^{-1} u \) with \( u \in [-M, M] \) for a given \( M > 0 \), the posterior for \( u \) is equal to \( p^* (u | y) = (Tv_T)^{-1} p \left( (Tv_T)^{-1} u + \lambda_b^0 | y \right) \) while the quasi-posterior is given by \( p_T^* (u | y) = (Tv_T)^{-1} p_T \left( (Tv_T)^{-1} u + \lambda_b^0 | y \right) \).

**Lemma A.37.** Let Assumptions 3.2-3.3 and 3.6-(i) hold and \( \tilde{\pi} (\cdot) \) satisfy Assumption 3.2. Then,
\[
\left| p_T^* \left( Tv_T \left( \bar{\lambda}_b - \lambda_b^0 \right) | y \right) - p^* \left( Tv_T \left( \bar{\lambda}_b - \lambda_b^0 \right) | y \right) \right|_{TV} \xrightarrow{P} 0.
\]

**Proof.** By assumption 3.2, \( \pi (\cdot) \) and \( \tilde{\pi} (\cdot) \) are bounded, and
\[
\sup_{|u| \leq M} \left| \pi \left( (Tv_T)^{-1} u + \lambda_b^0 \right) - \pi \left( \lambda_b^0 \right) \right| \xrightarrow{P} 0,
\]
\[
\sup_{|u| \leq M} \left| \tilde{\pi} \left( (Tv_T)^{-1} u + \lambda_b^0 \right) - \tilde{\pi} \left( \lambda_b^0 \right) \right| \xrightarrow{P} 0.
\]
Since \( \pi (\cdot) [\tilde{\pi} (\cdot)] \) appears in both the numerator and denominator of \( p_T^* (\cdot | y) [p^* (\cdot | y)] \), it cancels from that expression asymptotically. Turning to the Laplace estimator, the results of Section 3 (see Lemma A.2 and A.4) imply that for \( u \leq 0 \), using \( Q (\delta (\lambda_b), \lambda_b) / 2 \) in place of \( Q (\delta (\lambda_b), \lambda_b) \),
\[
\exp \left( \left( \tilde{G}_{T,0} (u, 0) + Q_{T,0} (u) \right) / 2 \right) \xrightarrow{P} 0 \quad \text{for large } T,
\]
where \( A_T = o_T (1) \) is uniform in the region \( u \leq \eta Tv_T^2 \) for small \( \eta > 0 \). By symmetry, the case \( u > 0 \) results
in the same relationship as (A.63) with $e_{T_b^o - t}$ replaced by $e_{T_b^o + t}$. The results in the proof of Theorem 1 in Bai (1994) combined with the arguments referenced for the derivation of (A.63) suggest that for $u \leq 0,$

$$\exp \left( \left( L^2 \left( T v_1^2 \right)^{-1} u + \lambda_0^b \right) - L^2 \left( \lambda_0^b \right) \right) / 2 \right)$$

(A.64)

$$= \exp \left( \delta_T \sum_{t=0}^\infty e_{T_b^o - t} - |u| \delta_0^2 / 2 \right) (1 + B_T),$$

where $B_T = o_P(1)$ is uniform in the region $u \leq \eta T v_1^2$ for small $\eta > 0$. By symmetry, the case $u > 0$ results in the same relationship as (A.64) with $e_{T_b^o - t}$ replaced by $e_{T_b^o + t}$. By Lemma A.6 and the results in Bai (1994), $p_T(u|y)$ and $p(u|y)$ are negligible uniformly in $u$ for $u > \eta T v_1^2$ for every $\eta$. Thus, (A.63)-(A.64) yield,

$$|p_T^* \left( T v_1^2 \left( \bar{\lambda}_b - \lambda_0^b \right), y \right) - p^* \left( T v_1^2 \left( \bar{\lambda}_b - \lambda_0^b \right), y \right)|_{TV} \leq |A_T| + |B_T| \xrightarrow{p} 0.$$

Continuing with the proof of Proposition 6.1, we begin with part (i). Note that $\varphi(\lambda_b, y)$ is defined by

$$\int (1 - \varphi(\lambda_b, y)) p_T(y|\lambda_b) d\Pi(\lambda_b) \geq 1 - \alpha$$

for all $y$, where $\Pi(\cdot)$ is a probability measure on $\Gamma^0$ such that $\Pi(\lambda_b) = \pi(\lambda_b) d\lambda_b$. The fact that $|1 - \varphi(\lambda_b, y)| \leq 1$ and Lemma A.37 lead to,

$$\int (1 - \varphi(\lambda_b, y)) p_T(y|\lambda_b) d\Pi(\lambda_b) \quad (A.65)$$

= $\int (1 - \varphi(\lambda_b, y)) p(y|\lambda_b) d\Pi(\lambda_b) + o_P(1).$

Given that Definition 4.1 of the GL confidence interval involves an inequality that explicitly allows for conservativeness, (A.65) implies the following relationship,

$$\int \varphi(\lambda_b, y) p_T(y|\lambda_b) d\Pi(\lambda_b) = \int \varphi(\lambda_b, y) p(y|\lambda_b) d\Pi(\lambda_b) + \varepsilon_T \leq \alpha \int p(y|\lambda_b) d\Pi(\lambda_b),$$

where $\varepsilon_T = \int \varphi(\lambda_b, y) (p_T(y|\lambda_b) - p(y|\lambda_b)) d\Pi(\lambda_b)$. Rearranging, we have,

$$\int (\alpha - \varphi(\lambda_b, y)) p(y|\lambda_b) d\Pi(\lambda_b) - \varepsilon_T \geq 0,$$

for all $y$. Now multiply both sides by $\bar{b}(y) \geq 0$ and integrating with respect to $\zeta(y)$ yields,

$$\int \int (\alpha - \varphi(\lambda_b, y)) \bar{b}(y) p(y|\lambda_b) d\zeta(y) d\Pi(\lambda_b) - \varepsilon_T \int \bar{b}(y) d\zeta(y) \geq 0,$$

or

$$(1 - \alpha) \int L_\alpha \left( \varphi, \bar{b}, \lambda_b \right) d\Pi(\lambda_b) - \varepsilon_T \int \bar{b}(y) d\zeta(y) \geq 0.$$
Taking the limit as $T \to \infty$,

$$(1 - \alpha) \int L_{\alpha} \left( \varphi, \tilde{b}, \lambda_b \right) d\Pi (\lambda_b) \geq 0.$$ 

The latter implies that $L_{\alpha} \left( \varphi, \tilde{b}, \lambda_b \right) \geq 0$ for some $\lambda_b$. Thus, $\varphi$ is box-proof at level $1 - \alpha$.

We now prove part (ii). We use a proof by contradiction. If $\int \varphi' (\lambda_b, y) d\lambda_b \geq \int \varphi (\lambda_b, y) d\lambda_b$ for all $y \in \mathcal{Y}$ and $\int \varphi' (\lambda_b, y) d\lambda_b > \int \varphi (\lambda_b, y) d\lambda_b$ for all $y \in \mathcal{Y}_0$ with $\zeta (\mathcal{Y}_0) > 0$, then we show that $\int \varphi' (\lambda_b, y) p (y \mid \lambda_b) d\zeta (y) > \alpha$ for some $\lambda_b \in \Gamma^0$. By Lemma A.37 and (6.1) holding with equality,

$$
\int \varphi (\lambda_b, y) p_T (y \mid \lambda_b) d\Pi (\lambda_b) = \alpha \int p_T (y \mid \lambda_b) d\Pi (\lambda_b)
= \alpha \int p (y \mid \lambda_b) d\Pi (\lambda_b) + o_P (1).
$$

Integrating both sides with respect to $\zeta (y)$ yields,

$$
\int \left( \int \varphi (\lambda_b, y) p (y \mid \lambda_b) d\zeta (y) \right) d\Pi (\lambda_b) = \alpha + o_\mathbb{P} (1). \tag{A.66}
$$

By Assumption 3.2, $\pi (\lambda_b) > 0$ for all $\lambda_b \in \Gamma^0$. Taking the limit as $T \to \infty$ of both sides of (A.66) yields

$$
\int \left( \int \varphi (\lambda_b, y) p (y \mid \lambda_b) d\zeta (y) \right) d\Pi (\lambda_b) \leq \alpha + o_\mathbb{P} (1),
$$

for all $y \in \mathcal{Y}$ and

$$
\int \varphi (\lambda_b, y) p (y \mid \lambda_b) d\Pi (\lambda_b) < \int \varphi' (\lambda_b, y) p (y \mid \lambda_b) d\Pi (\lambda_b),
$$

for all $y \in \mathcal{Y}_0$. Integrating both sides with respect to $\zeta$ yields

$$
\int \left( \int \varphi (\lambda_b, y) p (y \mid \lambda_b) d\zeta (y) \right) d\Pi (\lambda_b) < \int \left( \int \varphi' (\lambda_b, y) p (y \mid \lambda_b) d\zeta (y) \right) d\Pi (\lambda_b),
$$

or

$$
\int \left( \int (\varphi (\lambda_b, y) - \varphi' (\lambda_b, y)) p (y \mid \lambda_b) d\zeta (y) \right) d\Pi (\lambda_b) < 0.
$$

Since $\varphi (\lambda_b, y)$ is similar, there exists a $\lambda_b$ such that $\int \varphi' (\lambda_b, y) p (y \mid \lambda_b) d\zeta (y) > \alpha$. Thus, $\varphi'$ is not of level $1 - \alpha$. \square
Comparison to Casini and Perron (2020b)

In this section we compare the GL-LN method to the GL estimators/confidence intervals proposed in Casini and Perron (2020b). Table S-1-S-2 report the results. We have considered a data-generating mechanism with higher serial dependence in the errors. In terms of the empirical performance of the estimators, Table S-1 shows that overall the estimator that does better is $\hat{\lambda}_b^{GL-LN}$. $\hat{\lambda}_b^{GL-CR-Iter}$ is the one that does best when $\lambda_0 = 0.5$ but it does worse in relative terms when the break is in the tails. The performance of $\hat{\lambda}_b^{GL-LN}$ is in general superior to $\hat{\lambda}_b^{GL-CR}$ especially for medium to large breaks both in terms of MAE and RMSE. From other simulations (not reported), we conclude that GL-LN does in general better for moderate to large breaks. $\hat{\lambda}_b^{GL-CR-Iter}$ is the one that does best when the break is in the middle but its precision deteriorates as the break moves to the tails. In addition, $\hat{\lambda}_b^{GL-LN}$ is valid for models with multiple breaks and models with trending regressors that are not covered in Casini and Perron (2020b). So overall we believe that the estimators $\hat{\lambda}_b^{GL-LN}$, $\hat{\lambda}_b^{GL-CR}$ and $\hat{\lambda}_b^{GL-CR-Iter}$ can be seen as complementary.

Turning to the finite-sample performance of the confidence intervals, Table S-2 clearly shows that when there is higher serial dependence in the errors, the method that dominates is GL-LN. The gain in terms of coverage accuracy and lengths can be substantial relative to the GL-CR and GL-CR-Iter. When the serial dependence in the errors is low (not reported), the difference in performance of the three confidence intervals becomes smaller.

Overall, we find that both estimation and confidence intervals based on GL-LN perform well relative to the continuous record counterparts, where major gains appear to occur when there is high serial correlation in the errors.
Table S-1: Small-sample accuracy of the estimates of the break point $T^0_b$

| $\delta^0$ | Method   | MAE  | Std  | RMSE | $Q_{0.25}$ | $Q_{0.75}$ |
|-----------|----------|------|------|------|------------|------------|
|           | $\lambda_0 = 0.3$ |      |      |      |            |            |
| 0.3       | OLS      | 26.84| 28.12| 33.00| 21         | 76         |
|           | GL-LN    | 13.63| 14.07| 17.25| 27         | 56         |
|           | GL-CR    | 12.79| 13.13| 18.46| 29         | 57         |
|           | GL-CR-Iter | 14.47| 10.29| 20.21| 28         | 58         |
|           | GL-Uni   | 21.78| 21.73| 27.71| 28         | 66         |
| 0.4       | OLS      | 23.62| 26.99| 30.23| 21         | 70         |
|           | GL-LN    | 11.53| 13.66| 15.44| 29         | 51         |
|           | GL-CR    | 16.36| 13.86| 21.49| 29         | 61         |
|           | GL-CR-Iter | 17.19| 10.81| 20.35| 28         | 57         |
|           | GL-Uni   | 20.18| 21.25| 26.30| 28         | 64         |
| 0.6       | OLS      | 19.80| 24.62| 26.25| 21         | 57         |
|           | GL-LN    | 8.86 | 11.63| 12.77| 29         | 42         |
|           | GL-CR    | 12.84| 13.66| 15.44| 30         | 56         |
|           | GL-CR-Iter | 14.85| 11.52| 17.56| 29         | 52         |
| 1.0       | OLS      | 16.04| 20.05| 21.77| 26         | 56         |
|           | GL-LN    | 11.69| 18.43| 19.26| 27         | 40         |
|           | GL-CR    | 6.82 | 10.85| 12.81| 27         | 38         |
|           | GL-CR-Iter | 10.67| 7.54 | 13.02| 30         | 39         |
|           | GL-Uni   | 9.44 | 14.60| 15.15| 27         | 37         |

For $\delta^0 = 0.4$, $\lambda_0 = 0.5$: $y_t = \delta^0_1 + \delta^0_1 \mathbb{I}_{\{t > \lfloor T \lambda_0 \rfloor\}} + e_t$, $e_t = 0.6 e_{t-1} + u_t$, $u_t \sim i.i.d. \mathcal{N}(0, 0.49)$, $T = 100$.

The model is $y_t = \delta^0_1 + \delta^0_1 \mathbb{I}_{\{t > \lfloor T \lambda_0 \rfloor\}} + e_t$, $e_t = 0.6 e_{t-1} + u_t$, $u_t \sim i.i.d. \mathcal{N}(0, 0.49)$, $T = 100$.

Table S-2: Small-sample coverage rates and lengths of the confidence sets

| $\delta^0$ | Method | Cov. | Lgth. | Cov. | Lgth. |
|-----------|--------|------|-------|------|-------|
|           | $\lambda_0 = 0.5$ |      |       |      |       |
| 0.4       | OLS-CR   | 0.910| 67.57 | 0.911| 68.87 |
|           | Bai (1997) | 0.808| 67.57 | 0.811| 50.22 |
|           | GL-LN    | 0.925| 57.43 | 0.965| 37.35 |
|           | GL-CR    | 0.885| 60.05 | 0.884| 52.63 |
|           | GL-CR-Iter | 0.911| 76.72 | 0.911| 69.06 |
| 0.8       | OLS-CR   | 0.927| 75.58 | 0.910| 66.20 |
|           | Bai (1997) | 0.838| 66.86 | 0.821| 49.34 |
|           | GL-LN    | 0.965| 54.57 | 0.974| 32.88 |
|           | GL-CR    | 0.898| 57.32 | 0.888| 50.29 |
|           | GL-CR-Iter | 0.930| 75.87 | 0.913| 66.13 |
| 1.6       | OLS-CR   | 0.910| 75.24 | 0.917| 64.17 |
|           | Bai (1997) | 0.808| 67.03 | 0.852| 50.40 |
|           | GL-LN    | 0.921| 57.96 | 0.962| 39.63 |
|           | GL-CR    | 0.912| 56.74 | 0.909| 48.68 |
|           | GL-CR-Iter | 0.894| 75.15 | 0.923| 64.14 |

The model is $y_t = \delta^0_1 + \delta^0_1 \mathbb{I}_{\{t > \lfloor T \lambda_0 \rfloor\}} + e_t$, $e_t = 0.6 e_{t-1} + u_t$, $u_t \sim i.i.d. \mathcal{N}(0, 0.49)$, $T = 100$.

S-T-1