Validity of Emparan-Horowitz-Myers argument in Hawking radiation into massless spin-2 fields

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Abstract

The Hawking radiation for massless spin-2 fields is numerically studied when the spacetime background is \((4 + n)\)-dimensional Schwarzschild black hole phase. In order to check the validity of the Emparan-Horowitz-Myers argument, \textit{black holes radiate mainly on the brane}, we assume that the radial equation for the massless spin-2 fields propagating on the brane obeys the master equation approximately. The transmission coefficient is computed explicitly by making use of the Hawking-Hartle theorem. It is shown that the total emission rates into the bulk are dominant compared to the rates on the visible brane when \(n \geq 3\). However, the bulk-to-brane relative emissivities per degree of freedom always remain \(O(1)\) roughly. The experimental significance of these results in the production of mini black holes in future colliders is briefly discussed.

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One of the most important consequences of the recent brane-world scenarios\cite{1,2} with large or warped extra dimensions is the emergence of low-scale ($\sim 1$ TeV) quantum gravity. This fact opens a possibility for the copious production of the mini black holes in the future colliders such as LHC by high-energy collision experiment\cite{3,4}. In this reason the absorption and Hawking radiation for the higher-dimensional black holes have been extensively explored recently.

Emparan, Horowitz and Myers(EHM) argued in Ref.\cite{5} that the higher-dimensional black holes radiate mainly on the brane via the Hawking radiation. This argument was supported numerically in the case of standard model(SM) field emission by the $(4+n)$-dimensional non-rotating black holes\cite{6,7,8}. EHM argument was also examined in the higher-dimensional rotating black hole background\cite{9}. When black holes have angular momenta, it is well-known that there exist the superradiance modes. It was argued that the existence of the superradiance modes may lead a different conclusion from EHM argument. However, numerical calculation shows that EHM argument still holds in the scalar emission by $5d$ rotating black holes with two different angular momentum parameters\cite{10}.

In this letter we would like to re-examine the EHM argument in Hawking radiation for massless spin-2 fields when the spacetime background is $(4+n)$-dimensional Schwarzschild black hole whose metric is

$$ds^2 = -h dt^2 + h^{-1} dr^2 + r^2 d\Omega_{n+2}^2$$

(1)

where $h = 1 - (r_H/r)^{n+1}$ and the angle part $d\Omega_{n+2}^2$ is a spherically symmetric line element in a form

$$d\Omega_{n+2}^2 = d\theta_{n+1}^2 + \sin^2 \theta_{n+1} \left[ d\theta_n^2 + \sin^2 \theta_n \left( \cdots + \sin^2 \theta_2 \left( d\theta_1^2 + \sin^2 \theta_1 d\varphi^2 \right) \cdots \right) \right].$$

(2)

Since graviton is not localized on the brane unlike the SM particles, its emission spectrum may exhibit different behaviors from spectra for other fields.

The emission of the graviton into bulk was numerically explored in Ref.\cite{11,12,13,14,15}. Following the Regge-Wheeler method, it is well-known that the gravitational perturbations in the spacetime dimensions larger than four consist of three modes according to their tensorial behavior on the spherical section of the background metric: scalar, vector and tensor\cite{16,17,18,19}. Thus the emission spectra for the graviton can be computed via the
Hawking formula \[\textup{(20, 21)}\]

\[
d^2\Gamma_{BL} = \frac{(n+1)(n+4)}{2} \left[ 2^{n+2}\pi^{(n+3)/2}\Gamma\left(\frac{n+3}{2}\right) \right]^{-1} \sum_{A=S,V,T} \frac{\omega^{n+3}\sigma_{BL}^A}{e^{\beta_H\omega} - 1}
\]

where \(\beta_H\) is an inverse Hawking temperature, and S, V, and T denote the corresponding scalar, vector, and tensor modes. Of course, \(\sigma_{BL}^A\) is an total absorption cross section for each mode.

In Ref.\[11\] the bulk graviton emission rate is compared to those for the SM fields propagating on the brane and concluded that the bulk graviton emissivities are highly enhanced with increasing \(n\). However, the bulk-to-brane ratio for the graviton emissivities was not computed in the paper. Thus, the result of the paper does not lead any conclusion whether EHM argument is valid or not in the problem of the spin-2 field emission. In Ref.\[12\] the ratio of bulk graviton emissivity to bulk scalar was computed, which is summarized in Table I. Table I indicates that the emission of the graviton fields become dominant more and more with increasing \(n\). In this Letter we will compute the brane decay rates for massless spin-2 fields numerically and as a result, we will show that the bulk-to-brane ratio of the total emissivity becomes 0.76, 0.66, 1.59, 4.25 and 23.93 when \(n = 1, 2, 3, 4\) and 6 respectively. This indicates that the total emissivities into the bulk become dominant when \(n \geq 3\). Since, however, the bulk spin-2 fields have \((n+4)(n+1)/2\) polarization states while brane fields have only two helicities, the emission rates per degree of freedom into the bulk become roughly same order with those on the brane, which strongly support the EHM argument. In spite of \(O(1)\) roughly in the bulk-to-brane relative emissivities the dominance of the bulk emission rates indicates that we cannot ignore the missing energy portion in the future experiments relating to the brane-world black holes.

Table I: Graviton-to-Scalar Ratio in Bulk Emission

| \(n\) | 0 | 1 | 2 | 3 | 4 | 6 |
|------|---|---|---|---|---|---|
| Graviton / Scalar | 0.052 | 1.48 | 5.95 | 12.1 | 18.8 | 34.1 |

Now we would like to discuss the emission of the spin-2 fields on the brane, whose metric is projected from the \((4 + n)\)-dimensional Schwarzschild spacetime \[\textup{(1)}\]:

\[
ds_4^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]
In fact, the spin-2 graviton is a particle living in the bulk. Thus, it seems to be ridiculous to consider the graviton propagating on the brane. We think this bulk-nature of the graviton is a main reason why the radial equations for the axial- and polar-perturbation are not uniquely determined[22, 23] when the spacetime is a projected metric [4]. Since the purpose of this paper is to check the validity of the EHM argument in the spin-2 field, we will consider the hypothetical spin-2 field whose radial equation is assumed to be obeyed by the master equation as follows.

In Ref.[6, 7, 8, 24] the perturbations for the scalar($s = 0$), fermion($s = 1/2$) and vector($s = 1$) fields were discussed in this background by employing Newman-Penrose formalism and the following radial master equation was derived:

$$\Lambda^2 Y + P \Lambda_Y - Q Y = 0 \quad (5)$$

where $\Lambda_{\pm} = d/dr_{\pm} \pm i\omega$, $\Lambda^2 = \Lambda_+ \Lambda_-$, $d/dr_s = h d/dr$ and

$$P = \frac{d}{dr_s} \ln \left( \frac{r^2}{h} \right)^{-s}, \quad Q = \frac{h}{r^2} \left[ A_{\ell s} + (2s + n + 1)(ns + s + 1)(1 - h) \right] \quad (6)$$

with $A_{\ell s} = \ell (\ell + 1) - s(s + 1)$.

Although Eq.(5) was derived without considering the graviton($s = 2$), one can easily show Eq.(5) is valid for the graviton when $n = 0$[25]. In Ref.[26], furthermore, Eq.(5) is assumed to be valid for arbitrary positive $n$ for the graviton and derived the physically relevant quasinormal frequencies. However, one can show by directly applying the Newman-Penrose formalism that the master equation (5) is not valid for the spin-2 graviton fields propagating on the brane. As commented above, we guess the proper radial equation for the graviton confined on the brane does not exist due to the bulk-nature of the graviton field. Since, however, the original purpose of the present paper is to check the validity of the EHM argument of the spin-2 field, we consider the hypothetical spin-2 field whose radial equation is assumed to be Eq.(5) with $s = 2$. In this Letter, therefore, we will use Eq.(5) for the computation of the emission spectra for the spin-2 fields propagating on the brane\(^1\).

Defining $R = f^{-1} Y$ where $f$ is defined as $(1/f)df/dr_s = -P/2$, one can transform Eq.(5) into the Schrödinger-like expression $\Lambda^2 R = V_{br} R$ where the effective potential $V_{br}$ is in general

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\(^1\) Another reason for the choice of Eq.(5) as a radial equation is as follows. Since the quasinormal frequencies computed in Ref.[26] are physically reasonable, we can assume that Eq.(5) is to some extent valid approximately to describe the graviton propagating on the brane. In this reason Eq.(5) can be used for the approximate computation of the Hawking radiation into the graviton confined on the brane.
complex in the following expression

\[ V_{br} = i\omega P + \frac{P^2}{4} + \frac{1}{2} \frac{dP}{dr_x} + Q. \] (7)

Since the method for the computation of the reflection and transmission coefficients is explained for the case of the complex potential in Ref. [25], we will adopt the procedure of Chandrasekhar for the computation of the absorption and emission spectra. The solution convergent in the near-horizon and asymptotic regimes are respectively

\[ R_{NH} = C_{NH} r_H^{1-s} \left( \frac{x_H}{n+1} \right)^{-\rho_n} \sum_{N=0}^{\infty} d_{\ell,N} (x-x_H)^{N+\rho_n} \] (8)

\[ R_\infty = \omega^{-s} x^{s+1} e^{-ix} \sum_{N=0}^{\infty} \tau_{N(+)} x^{-N+1} + C_\infty \omega^s x^{-s+1} e^{ix} \sum_{N=0}^{\infty} \tau_{N(-)} x^{-N+1} \]

with \( x = \omega r, x_H = \omega r_H, \) \( d_{\ell,0} = \tau_{0(\pm)} = 1 \) and

\[ \rho_n = -\frac{s}{2} - i \frac{x_H}{n+1}. \] (9)

The sign for \( \rho_n \) is chosen by \( \text{Im} \rho_n < 0 \) to ensure the incoming behavior of the spin-2 wave in the near-horizon regime. The recursion relations for the coefficients \( d_{\ell,N} \) and \( \tau_{N(\pm)} \) can be explicitly derived by inserting Eq. (8) into the wave equation, i.e. \( \Lambda^2 R = V_{br} R. \)

The transmission coefficients \( T_{BR} \) for the complex potential (7) can be derived as follows. Firstly, we relates changes of the black hole mass \( dM \) to changes of the horizon area \( d\Sigma \).

Although this relation is simply

\[ d\Sigma = \frac{4G_n}{T_H} dM \] (10)

for the bulk metric (11), where \( T_H \) and \( G_n \) are the Hawking temperature and \( (4+n) \)-dimensional Newton constant, it becomes slightly complicated form for our case as follows;

\[ d\Sigma = \frac{\xi_n}{(\pi/\beta_H)^{1-n}} dM \] (11)

where \( \beta_H \equiv 1/T_H, \)

\[ \xi_n = \frac{2^{5+2n} \pi^2 G_n}{(n+1)^n (n+2) \Omega_{n+2}} \] (12)

and \( \Omega_{n+2} \) is an area of a unit \( (n+2) \)-sphere.

Employing the Hawking-Hartle theorem [27, 28, 29] we express the variation in the area in terms of the variations in the spin coefficients. Finally the variations in the spin coefficients are identified with the perturbation in the Weyl tensor, using Ricci identities. Assuming \( Y \equiv \)
$r h^2 \bar{R}$ satisfies the master equation \([5]\) where the Weyl tensor $\Psi_0$ is $\Psi_0 = \bar{R}(r) S(\theta)e^{im\phi-i\omega t}$, one can derive the transmission coefficient in the following expression

$$T_{BR} = \frac{(n + 2)\Omega_{n+2}^r r_H^{n-6} \omega^2}{8\pi G_n \left[ \omega^2 + (2\pi / \beta H)^2 \right]}|C_{NH}|^2.$$  \hspace{1cm} (13)

Since we adopt the unit $G_0 = 1$, we should know the relation between $G_n$ and $G_0$. Let $\gamma_n \equiv G_n / G_0$. Then $\gamma_n$ can be numerically computed by noting that $T_{BR}$ in Eq. (13) should be saturated to unity in $\omega \rightarrow \infty$ limit. The numerical result strongly suggests that $C_{NH}$ is dependent on neither $\omega$ nor $n$, which implies $\gamma_n = (n + 2)\Omega_{n+2}/8\pi$.

FIG. 1: The $\omega$-dependence of $T_{BR}$ when $\ell = 2$ (a) and $\ell = 3$ (b). The increasing rate of $T_{BR}$ tends to reduce with increasing $n$, which indicates that the barrier heights of the real effective potentials become higher with increasing $n$.

Fig. 1 is a plot of $T_{BR}$ as a function of the energy $\omega$ for $\ell = 2$ (Fig. 1(a)) and $\ell = 3$ (Fig. 1(b)). As expected $T_{BR}$ is saturated to unity with increasing $\omega$. The increasing rate of $T_{BR}$ tends to decrease with increasing $n$, which implies that the barrier heights of the real effective potentials become higher with increasing $n$ although we do not know the exact expression of the real effective potential. However, the low-energy increasing rate of $T_{BR}$ when $\ell = n = 2$ (see Fig. 1(a)) seems to be extra-ordinarily large, which enables us to guess that the width of the potential barrier in this case may be narrower compared to the other cases.

Once $T_{BR}$ is computed, it is straightforward to compute the emission spectrum \([20, 21]\)

$$\frac{d^2 \Gamma_{BR}}{d\omega dt} = \frac{\omega^3 \sigma_{BR}^{BR}}{\pi^2 (e^{\beta H \omega} - 1)}.$$  \hspace{1cm} (14)
where $\sigma^{BR}$ is a total absorption cross section defined $\sigma_T = \sum_\ell \pi (2\ell + 1) T_{BR}/\omega^2$. We compute $\sigma^{BR}$ numerically by making use of the quantum mechanical scattering theories with numerical analytic continuation, which was introduced in detail in Ref.\cite{10, 30}.

\[
\sigma_T = \sum_\ell \pi (2\ell + 1) T_{BR}/\omega^2.
\]

FIG. 2: The $\omega$-dependence of the spin-2 field emission spectra for $n = 1$ (Fig. 2(a)) and $n = 4$ (Fig. 2(b)). The decay rates on the brane are plotted by red color. The blue line are bulk emission spectra for each mode. This figure indicates that the bulk emission rates highly increase with increasing $n$ compared to the brane emission rates.

In Fig. 2 the $\omega$-dependence of the emission spectra is plotted when $n = 1$ (Fig. 2(a)) and $n = 4$ (Fig. 2(b)). The decay rates on the brane are plotted by red color. For the comparison the bulk emission spectra for each mode are plotted together by blue color. Fig. 2 shows that the bulk emissivities are in general dominant in the high-energy domain while the brane decay is dominant in the opposite domain. This is mainly due to the power difference of $\omega$ in the emission spectra formula defined in Eq.(3) and (14). Fig. 2 also indicates that the bulk decay rates are comparatively larger than the brane decay when $n = 4$.

For the precise comparison we consider the total emission rate defined

\[
\Gamma_{tot} \equiv \int_0^\infty d\omega \, \frac{\partial^2 \Gamma}{\partial \omega dt}.
\]  

\textbf{Table II:} Brane versus Bulk in $\Gamma_{tot}/\Gamma_{tot}^S$
The relative total emissivities for spin-0, spin-1, and spin-2 fields are summarized in Table II. Each total emission rate is divided by the four-dimensional scalar rate $\Gamma_{S_{tot}}^{S} = 2.98 \times 10^{-4}$. The abbreviations S, V and T denote the scalar, vector and tensor modes for the bulk fields respectively.

Table II shows several interesting features. Firstly, the total bulk emissivities for the spin-2 fields become dominant when $n \geq 3$ compared to the emission rate for the spin-2 fields propagating on the brane. Secondly, the emission of the spin-2 fields into the bulk becomes dominant in the presence of the extra dimensions compared to other bulk SM fields. In the brane case, however, the emission rates for the spin-2 fields are not dominant. This seems to be due to the fact that spin-2 graviton, in general, is not confined on the brane unlike the SM particles.

Table III: Bulk versus Brane per degree of freedom

| $n$ | 0  | 1  | 2  | 3  | 4  | 6  |
|-----|----|----|----|----|----|----|
| spin-0 | 1.0 | 0.40 | 0.24 | 0.22 | 0.24 | 0.52 |
| spin-1 | 1.0 | 0.33 | 0.22 | 0.20 | 0.23 | 0.50 |
| spin-2 | 1.0 | 0.31 | 0.15 | 0.23 | 0.42 | 1.37 |

Table III shows the spin-dependence of the bulk-to-brane emissivities per degree of freedom (d.o.f.). Since the massless spin-1 and spin-2 fields have $n + 2$ and $(n + 4)(n + 1)/2$ polarization states respectively, the relative emissivities per d.o.f. can be read directly from...
Table II. Table III indicates that the relative emissivities per d.o.f. are always less than unity except spin-2 case with $n = 6$, which supports the EHM argument.

In this paper the emission rates for the spin-2 fields on the brane and in the bulk are explicitly computed. It is found that although the total bulk emissivities becomes dominant when $n \geq 3$, the bulk-to-brane relative emissivities per degree of freedom remains $O(1)$, which strongly supports the EHM argument. However, as indicated by Table II, the total missing energy arising due to the bulk emissivities is not negligible. Thus we should carefully consider the missing energy portion in the future experiment relating to the blane-world black holes.

It is of interest to derive a real effective potentials from the master equation \[25\] by employing the transformation theory and interpret the results of the present letter in terms of the potentials. It is of greatly interest also to explore the Hawking radiation for the graviton in the rotating black hole background which is a still open problem. We would like to study these issues in the future.

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