Rotational spreads and rotational parallelisms and oriented parallelisms of $\text{PG}(3, \mathbb{R})$

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Abstract. We introduce topological parallelisms of oriented lines (briefly called oriented parallelisms). Every topological parallelism (of lines) on $\text{PG}(3, \mathbb{R})$ gives rise to a parallelism of oriented lines, but we show that even the most homogeneous parallelisms of oriented lines other than the Clifford parallelism do not necessarily arise in this way. In fact we determine all parallelisms of both types that admit a reducible $\text{SO}_3\mathbb{R}$-action (only the Clifford parallelism admits a larger group (Löwen in Innov Incid Geom. arXiv:1702.03328), and it turns out surprisingly that there are far more oriented parallelisms of this kind than ordinary parallelisms. More specifically, Betten and Riesinger (Aequ Math 81:227–250, 2011) construct ordinary parallelisms by applying $\text{SO}_3\mathbb{R}$ to rotational Betten spreads. We show that these are the only ordinary parallelisms compatible with this group action, but also the ‘acentric’ rotational spreads considered by them yield oriented parallelisms. The automorphism group of the resulting (oriented or non-oriented) parallelisms is always $\text{SO}_3\mathbb{R}$, no matter how large the automorphism group of the non-regular spread is. The isomorphism type of the parallelism depends not only on the isomorphism type of the spread used, but also on the rotation group applied to it. We also study the rotational Betten spreads used in this construction and their automorphisms.

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1. Introduction

In [5], Betten and Riesinger consider the so-called rotational Betten spreads on $\text{PG}(3, \mathbb{R})$, which were discovered by Betten [1]. They admit an action of $\text{SO}_2\mathbb{R}$ which induces axial collineations on the associated 4-dimensional translation plane. Betten and Riesinger observe that the Betten spreads are composed
of a family of reguli with common axis of symmetry and common center (a concentric family, as they call it), and that they are the only spreads that can be described in this way, see [5, Section 4]. They show that each of these spreads gives rise to a parallelism whose equivalence classes are the images of the given spread under a reducible SO$_3\mathbb{R}$-action which extends the given SO$_2\mathbb{R}$-action. They also show that there are other families of reguli with common axis, called acentric, which also define spreads admitting an SO$_2\mathbb{R}$-action of the kind considered.

We shall make the easy observation that the (concentric) Betten spreads are the only compact spreads in PG(3, \mathbb{R}) admitting an axial SO$_2\mathbb{R}$ action that extends to an action of O$_2\mathbb{R}$ = SO$_2\mathbb{R}$ ⋅ Z$_2$, and we deduce that the parallelisms obtained by applying the reducible SO$_3\mathbb{R}$-action to them are the only topological parallelisms that have this group as their automorphism group, see Theorems 4.2 and 5.5 below. Thus, they are among the most homogeneous non-classical topological parallelisms. Indeed, according to [12] the ‘classical’ Clifford parallelism is the only one with a group of dimension ≥ 4, and the only other possible 3-dimensional group is SO$_3\mathbb{R}$ with its action induced by the irreducible Spin$_3$ action on \mathbb{R}^4. This follows from the fact that the automorphism group of a topological parallelism is compact [9,13].

We examine the proof given in [5] that the Betten spreads produce parallelisms with reducible SO$_3\mathbb{R}$-action. It turns out that virtually the same proof can be applied to spreads defined by coaxial but acentric families of reguli. However, this time we do not obtain parallelisms of (non-oriented) lines as before, but parallelisms of oriented lines, see Theorem 5.4. Still their automorphism group is as large as possible for a non-classical parallelism. Again, according to Theorem 5.5, they are characterized by their group.

Therefore, we feel justified to define oriented parallelisms as a new kind of parallelisms. We record their basic properties, which resemble those of ordinary parallelisms except that their set of parallel classes is homeomorphic to the 2-sphere rather than the projective plane.

As a by-product, our approach yields a very short proof of the result from [6] that rotating a rotational spread produces the Clifford parallelism if and only if the spread is the regular (complex) spread and the rotation group fixes the center of the family of reguli, see Theorem 5.6.

Regarding spreads with an axial SO$_2\mathbb{R}$-action, we observe that they are precisely those obtained from coaxial (concentric or acentric) families of reguli. A precise description of the admissible parameters for such families of reguli has been obtained by Harald Löwe (yet unpublished). We consider the automorphism group of such spreads and show that it is 1-dimensional except for the regular spread and a small class of spreads with 2-dimensional groups. The latter spreads belong to some of the 4-dimensional translation planes with 7-dimensional group, which were classified by Betten [2–4]. To be more exact, only certain special instances of the planes described in Satz 1, 2 and 3 of [3]
occur here. In a majority of the cases, we describe these spreads in terms of families of reguli. This description is somewhat simpler than the original one. Finally we study isomorphisms and automorphisms of (oriented) parallelisms with reducible SO$_3\mathbb{R}$-action. We show that the automorphism group is always SO$_3\mathbb{R}$, even though the automorphism groups of the spreads differ in size, see Theorem 5.8. Correspondingly, the same spread rotated by different admissible copies of SO$_3\mathbb{R}$ can yield non-isomorphic (oriented or non-oriented) parallelisms. In fact, typically this is what happens, see Theorem 5.10.

2. Parallelisms versus oriented parallelisms

An oriented line of PG(3, \mathbb{R}) may be defined as a line together with a preferred direction of circuit or as a 2-dimensional subspace of \mathbb{R}^4 together with a vector space orientation. The spaces \mathcal{L} and \mathcal{L}^+$ of lines and of oriented lines are topologized as coset spaces of PGL(4, \mathbb{R}). In this way, \mathcal{L}^+$ becomes a twofold covering space of \mathcal{L}.

Suppose that \mathcal{C} \subseteq \mathcal{L} is a compact spread, i.e. a compact set of pairwise disjoint lines covering the point set of PG(3, \mathbb{R}). Then \mathcal{C} is homeomorphic to the 2-sphere \mathbb{S}^2, see [14, 64.4(b)], and hence its inverse image under the covering map \mathcal{L}^+ \to \mathcal{L} is a disjoint union of two 2-spheres, each of which is mapped onto \mathcal{C} and hence is a compact spread consisting of oriented lines. We call each of these spheres an oriented spread. Thus we have proved

**Proposition 2.1.** Every compact spread of PG(3, \mathbb{R}) can be oriented in exactly two ways. \(\square\)

A topological parallelism on PG(3, \mathbb{R}) is a set \Pi of pairwise disjoint compact spreads covering \mathcal{L} which is a compact subset of the hyperspace of all compact subsets of \mathcal{L} (endowed with the Hausdorff topology). Similarly, an oriented topological parallelism is a compact set \Pi^+ of pairwise disjoint compact, oriented spreads covering \mathcal{L}^+. From 2.1 we deduce

**Proposition 2.2.** Every topological parallelism \Pi of PG(3, \mathbb{R}) gives rise to a unique oriented topological parallelism \Pi^+, which consists of all oriented spreads covering members of \Pi. \(\square\)

As we shall see, there is no converse to this proposition.

For the sake of distinction, we shall sometimes refer to non-oriented parallelisms as ordinary parallelisms. The properties of the two kinds of parallelisms are very similar. See, e.g., the section ‘Preliminaries’ of [13] for a brief account of the essentials in the ordinary case. One distinctive feature is the homeomorphism type of a parallelism, considered as a subset of the hyperspace of \mathcal{L} or \mathcal{L}^+:

**Proposition 2.3.** An ordinary topological parallelism \Pi is homeomorphic to the real projective plane. An oriented topological parallelism \Pi^+ is homeomorphic to the 2-sphere.
Proof. Choose a point \( p \) and let \( L_p \subseteq \mathcal{L} \) and \( L_p^+ \subseteq \mathcal{L}^+ \) be the stars of non-oriented lines and of oriented lines passing through \( p \), respectively. The ordinary star is a projective plane, and the oriented one is a 2-sphere. The map sending a parallel class to its unique member in the star is a homeomorphism in both cases. \( \square \)

The last result enables a nice way of proving the well known fact that the space \( \mathcal{L}^+ \) of oriented lines of \( \text{PG}(3, \mathbb{R}) \) is homeomorphic to the product \( S_2 \times S_2 \) of two 2-spheres. One classical proof of this property uses Plücker coordinates, which provide a homeomorphism \( \mathcal{L} \rightarrow Q \) onto the Klein quadric of index 3 in \( \text{PG}(5, \mathbb{R}) \). It is easy to see that \( Q \) is homeomorphic to the quotient of \( S_2 \times S_2 \) with the identification \((x, y) = (−x, −y)\); a detailed proof is given in [8]. From this it follows that the twofold covering space \( \mathcal{L}^+ \) is \( S_2 \times S_2 \).

Another proof is provided in [10, pp. 290–291], and attributed to E. Study; see also [11, 3.4], for yet another proof. Identify \( \mathbb{R}^4 \) with the quaternion skew field \( \mathbb{H} \). Let \( \Lambda \) and \( \Theta \) be the subgroups of \( \text{SO}(4, \mathbb{R}) \) consisting of the maps \( q \mapsto aq \) and \( q \mapsto qb \), respectively, where \( a, b \) are quaternions of norm 1. Then an oriented line \( L \) is mapped to the pair of its stabilizers \((\Lambda_L, \Theta_L)\), both endowed with orientations matching the orientation of \( L \). Now \( \Lambda_L \) and \( \Theta_L \) are both isomorphic to \( \text{SO}_2 \mathbb{R} \). Together with the orientation, each of them defines a unique pure quaternion of norm 1, that is, an element of \( S_2 \). For instance, \( \Lambda_L \) contains exactly two elements of order 4. One of them, say \( \lambda \), comes first when \( \Lambda_L \) is traversed in the sense of orientation, starting from the identity. The pure quaternion \( a \) associated to \( \Lambda_L \) is the one satisfying \( \lambda(q) = aq \).

The following proof also uses the groups \( \Lambda \) and \( \Theta \); the orbits of \( \Lambda \) and \( \Theta \) on \( \mathcal{L}^+ \) form the left and right oriented Clifford parallelism on \( \text{PG}(3, \mathbb{R}) \), respectively.

**Proposition 2.4.** Let \( \Pi_\Lambda^+ \) and \( \Pi_\Theta^+ \) be the left and right oriented Clifford parallelisms of \( \text{PG}(3, \mathbb{R}) \), as above. Then the map

\[
\alpha : \mathcal{L}^+ \rightarrow \Pi_\Lambda^+ \times \Pi_\Theta^+ \approx S_2 \times S_2
\]

sending an oriented line \( L \) to the pair \((\Pi_\Lambda^+(L), \Pi_\Theta^+(L))\) of its parallel classes is a homeomorphism.

**Proof.** The map is surjective because it is equivariant with respect to the transitive \( \text{SO}(4, \mathbb{R}) \)-actions on both spaces. Injectivity of \( \alpha \) follows from transitivity together with the observation that the stabilizer of \( L \) coincides with that of \( \alpha(L) \).

Alternatively, we show that \( \alpha \) is injective by proving that

\[
\Pi_\Lambda^+(L) \cap \Pi_\Theta^+(L) = \{L\}.
\]

By transitivity, we may assume that \( L \) is the 2-dimensional subspace \( \langle 1, q \rangle^+ \subseteq \mathbb{H} \) with oriented basis \( \{1, q\} \), where \( q \) is a pure quaternion of norm 1. Suppose that there are quaternions \( a, b \) of norm 1 such that the images of \( L \) under \( x \mapsto ax \) and \( x \mapsto xb \) are equal (as oriented lines), that is,

\[
\langle a, aq \rangle^+ = \langle b, qb \rangle^+ = M.
\]
We have to show that $M = L$ as oriented lines. The intersection of $M$ with the sphere of radius 1 is an orbit of a one-parameter group of $\Theta$, hence there is a quaternion $c$ of norm 1 such that the map $\vartheta = (x \rightarrow xc) \in \Theta$ fixes $M$ and sends the vector $b \in M$ to $a \in M$. Then the above equation implies that

$$\langle a, qa \rangle^+ = M = \vartheta(M) = \langle a, qa \rangle^+.$$

Since there is only one oriented orthonormal basis for $M$ containing $a$ as its first element, we conclude that $aq = qa$, i.e., that $a$ belongs to the centralizer of $q$ in $\mathbb{H}^\times$, which equals $L\{0\}$ and is a subgroup of $\mathbb{H}^\times$. It follows that $M = L$. □

Remark. In the sequel, we shall never consider non-topological spreads or parallelisms. Therefore, we shall usually omit the words ‘topological’ or ‘compact’ (in the case of spreads) unless we want to stress that we have just constructed a topological object.

3. Automorphisms

The automorphism group $\Sigma$ of an ordinary or oriented parallelism $\Pi$ or $\Pi^+$ is defined as the group of all collineations of $\text{PG}(3, \mathbb{R})$ that preserve $\Pi$ or $\Pi^+$, respectively. In the ordinary case, we proved that this group is compact [9,13]. The same proof goes through in the oriented case virtually unchanged. Lemma 3.4 of [13] states that every automorphism $\sigma$ induces equivalent actions on the line stars of any two fixed points. Of course, this has to be adapted by using the oriented line stars. The applications of this lemma work just as well. In the proof of compactness, the goal is to rule out all types of unbounded cyclic subgroups of $\Sigma$. The contradiction always arises in the form that certain lines are shown to be parallel by a continuity argument, but they are distinct and have a common point. This contradiction does not depend on the orientations, hence it is not even necessary to keep track of orientations in order to adapt the proof. Thus we have

Theorem 3.1. Like in the ordinary case, the automorphism group of an oriented topological parallelism $\Pi^+$ on $\text{PG}(3, \mathbb{R})$ is compact. □

The automorphism groups of an ordinary parallelism and of its associated oriented parallelism are the same, so by [7] we get part (a) of the next theorem; compare also [13, Corollary 1.2]. The proof of part (b) is again virtually the same as in the oriented case, see [12].

Theorem 3.2. (a) The automorphism group of the oriented Clifford parallelism is the 6-dimensional group $\text{PSO}(4, \mathbb{R}) \cong \text{SO}_3\mathbb{R} \times \text{SO}_3\mathbb{R}$.

(b) A non-classical oriented topological parallelism has an automorphism group of dimension at most 3. □
4. Rotational spreads of $\text{PG}(3, \mathbb{R})$

For general information on compact spreads and their relationship to topological translation planes, we refer the reader to [14, Section 64]. The following considerations are inspired by [5]. We assume in this section that we have a (compact) spread $C$ of $\text{PG}(3, \mathbb{R})$ that admits an axial action of $\text{SO}_2 \mathbb{R}$. When we consider $C$ as a fibration of $\mathbb{R}^4 \backslash \{0\}$ by 2-dimensional subspaces, this means that $\Phi \cong \text{SO}_2 \mathbb{R}$ acts trivially on some 2-space $S \in C$ and induces the ordinary $\text{SO}_2 \mathbb{R}$ on some complementary subspace $W$.

We prefer, however, to represent the spread as a set of lines of $\text{PG}(3, \mathbb{R})$; then $\Phi$ fixes one line $Z$ pointwise and fixes one other line $V$ disjoint from $Z$. The line $Z$ belongs to $C$ by assumption, but in fact this can be proved, see 4.2 below. Moreover, $\Phi$ fixes all hyperplanes spanned by $V$ together with a point of $Z$. We choose one of these hyperplanes, $F$, and consider the affine complement $\mathbb{R}^3$ of $F$, endowed with a $\Phi$-invariant inner product. We may assume that $Z$ is the $z$-axis $Z$, spanned by the third standard basis vector $e_3$, and $V$ is the line at infinity of the plane $\langle e_1, e_2 \rangle$. Thus, the group $\Phi$ consists of the ordinary rotations with axis $Z$. Its orbits on the line space are two fixed lines, $Z$ and $V$, and all reguli carried by $\Phi$-invariant one-sheeted hyperboloids. Since $C$ is a 2-sphere, there must be a second fixed line in $C$, thus $V \in C$. The reguli belonging to $C$ have to be either all right-screwed or all left-screwed, or else $C$ could not be connected. Right-screwed reguli and left-screwed ones are exchanged by the map $(x, y, z) \rightarrow (x, -y, z)$, hence we shall not pay attention to the choice between the two.

We see that $C$ can be described by specifying the one-sheeted hyperboloids carrying the reguli contained in $C$. This can be done by identifying the hyperbola branches obtained by intersecting the hyperboloids with the half plane $E = \{(x, 0, z) \mid x, z \in \mathbb{R}, x > 0\}$. Such a hyperbola is determined by three parameters, namely the coordinates of its vertex $(r, 0, b)$ (the point closest to $Z$), and the slope of its upper asymptote, $a > 0$. By the defining property of a spread, any two of the hyperboloids are disjoint. Hence they have distinct $r$-parameters, and we may label the hyperboloids as $H_r$, $0 < r \in \mathbb{R}$. In fact, every positive $r$ occurs, because the $H_r$ must cover $E$. Thus, $b$ and $a$ become functions of $r$. The function $a$ is decreasing, because the hyperboloids are pairwise disjoint, and in fact strictly decreasing, because two hyperboloids with the same $a$ have the same set of points at infinity. For $r \rightarrow 0$ and $r \rightarrow \infty$, the reguli must converge to $Z$ and to $V$, respectively, which means that $a(r) \rightarrow \infty$ in the first case and $a(r) \rightarrow 0$ in the second.

The elements of a spread $C$ as considered here (except $Z$ and $V$) are determined by two parameters $r > 0$ and $\varphi \in \Phi$, hence we see that $C$ is compact (in fact, a 2-sphere).

We now distinguish two cases;

*Case 1 (the concentric case)* the function $b$ is constant.

*Case 2 (the acentric case)* the function $b$ is not constant.
In Case 1, the conditions on the function $a$ stated above are sufficient to ensure that the hyperbolae $H_r$ are pairwise disjoint and cover $E$. The points $p \in F$ at infinity are also simply covered by $C$, and we obtain a spread, which is called a rotational Betten spread in [5]. These spreads include the classical (regular, complex) spread, which is obtained precisely when $a(r) = a(1)^r$, see Theorem 4.3. In Case 2, extra conditions on the function $b$ are needed to ensure the disjointness of the hyperbola branches. The conditions have been determined by Harald L"owe (yet unpublished). Examples of spreads of this type are given in [5]. Definitely, the class of spreads in Case 2 is much larger than the class defined by Case 1.

It is obvious that Case 1 occurs if and only if the spread $C$ is invariant under the orthogonal reflection $\sigma$ in some line perpendicular to $Z$, so that $C$ admits the group $\Psi = \Phi \cdot \langle \sigma \rangle \cong O_2R$.

Definition 4.1. We shall call the spreads described here (as well as all isomorphic copies) $SO_2R$-rotational spreads, and the special ones arising in Case 1 will be called $O_2R$-rotational spreads. The names are justified by the following

Theorem 4.2. Let $\Gamma$ be either $O_2R$ or $SO_2R$. A (compact) spread $C$ of $PG(3,R)$ is a $\Gamma$-rotational spread if and only if it is invariant under an action of the group $\Gamma$ on $PG(3,R)$ that is equivalent to the projective extension of the standard action of $\Gamma \leq SO_3R$ on $R^3$.

Proof. The ‘if’ part being obvious, we merely consider the ‘only if’ assertion. It suffices to show that $C$ satisfies the basic assumption of the preceding considerations, namely, that the action of the identity component $\Phi \leq \Gamma$ is axial. By the hypothesis of the theorem, $\Phi$ fixes some line $Z$ pointwise; we have to show that $Z \in C$. For the corresponding fibration of $R^4$ and the lifted action of $\Phi$, we see that some 2-space $S$ is fixed elementwise. It follows that $S$ belongs to the fibration, or else $S$ would be a Baer subplane of the translation plane defined by the fibration, and there would exist at most one non-trivial automorphism fixing $S$ pointwise, see [14, 55.21]. \hfill \Box

Theorem 4.3. Let $C$ be a $\Gamma$-rotational spread defined by the functions $a(r)$ and $b(r)$. The spread $C$ is regular if and only if $b(r)$ is constant and $a(r) = a(1)^r$. In this case, the group Aut$C$ of all collineations preserving $C$ is known to be 7-dimensional.

Proof. If $b$ is not constant, then $C$ is not an $O_2R$-spread and certainly not classical. If $b$ is constant, we may assume that $b \equiv 0$, and then the assertion is proved in [1]; note that Betten’s parameter $t$ corresponds to our $\frac{1}{r}$, and his function $F(t)$ is our $a(\frac{1}{r})$; compare [5]. The automorphism group of the regular spread is $(GL_2 \mathbb{C})/\mathbb{R}^\times$. \hfill \Box

Now we consider the non-regular cases.

Proposition 4.4. Let $C$ be a non-regular $\Gamma$-rotational spread. Then the automorphism group $\Sigma = \text{Aut}C \leq \text{PGL}_4R$ is at most 2-dimensional, and the identity
component $\Phi = \Gamma^1$ is the only subgroup isomorphic to $SO_2\mathbb{R}$ in $\Sigma$. In particular, $\Phi$ is a characteristic subgroup of $\Sigma$.

**Proof.** If $\Sigma$ contains a 2-torus $\Delta$, then $\mathcal{C}$ is regular. Indeed, all 2-tori in $\text{PGL}_4\mathbb{R}$ are conjugate, hence we may assume that the 2-torus is induced by the group of complex diagonal matrices with entries of norm one. This group is ineffective on the fibration of $\mathbb{R}^4$ corresponding to $\mathcal{C}$. The one-dimensional kernel of ineffectivity contains no elements with eigenvalue 1 except the identity because the line at infinity of the associated translation plane is fixed, hence it acts like the group of complex scalar matrices, and the claim follows.

If $\Sigma$ contains several copies of $SO_2\mathbb{R}$ but not a 2-torus, then $\dim \Sigma \geq 3$, and the translation plane defined by $\mathcal{C}$ has an automorphism group of dimension at least $4 + 3 + 1 = 8$. In fact, the latter dimension equals 8 since we assumed that $\mathcal{C}$ is non-regular, see [14, 73.1]. According to [14, 73.10, 73.11], there is a unique 4-dimensional translation plane with a solvable 8-dimensional automorphism group $\Xi$. The stabilizer $\Xi_0$ of the origin acts on $\mathbb{R}^4$ by lower triangular matrices, and does not contain a non-trivial torus group. Similarly, the translation planes with a non-solvable automorphism group do not admit an axial torus group, see [14, 73.13 and 73.19 combined with 73.18(e)]. Thus we end up with a contradiction and the theorem is proved. \hfill \Box

**Corollary 4.5.** If $\mathcal{C}$ and $\mathcal{C}'$ are non-regular $\Gamma$-rotational spreads and $\Phi \cong SO_2\mathbb{R}$ is contained in the automorphism groups of both spreads, then every isomorphism $\xi : \mathcal{C} \rightarrow \mathcal{C}'$ belongs to the normalizer $N_s \Phi$ in $\text{PGL}_4\mathbb{R}$. \hfill \Box

The next theorem shows that rotational spreads with $\dim \Sigma = 2$ are determined by finitely many real parameters. Thus, a generic $\Gamma$-rotational spread has a one-dimensional automorphism group.

**Theorem 4.6.** Let $\mathcal{C}$ be a non-regular $\Gamma$-rotational spread with a 2-dimensional automorphism group. Then the translation plane defined by $\mathcal{C}$ is one of the following planes described in [3]:

1. If $\mathcal{C}$ is $O_2\mathbb{R}$-rotational, then the plane is one of the planes of Satz 1, and the parameters $p, c, w$ satisfy $p = c = 0$, with no further restriction on $w \in ]0, 1[.$

2. If $\mathcal{C}$ is not $O_2\mathbb{R}$-rotational, then the plane is either
   1a) any of the remaining planes of Satz 1 with parameters $p = 0 \neq c$, or
   1b) any of the planes of Satz 2 with $p = 0$ and $|d| \geq \frac{1}{2}$, or
   1c) any of the planes of Satz 3 with parameters $p, q, c, d$ satisfying $p = 0 < q$ and $d > 0$, as well as the conditions (2) stated in [3].

The properties of the planes are described exhaustively in [3]. See also [11, p. 77], where some minor details about the isomorphism types of the planes given in [3], Satz 3 are corrected.

**Proof.** The 4-dimensional translation planes with a 7-dimensional automorphism group have been classified by Betten [2–4], and complete information
on the spreads and the automorphisms is given there. The planes of [2,4] do not admit any torus action. Inspection of the planes of [3] shows that only the planes identified in the theorem admit an axial \(\text{SO}_2\mathbb{R}\)-action, and only those in (1) admit \(\text{O}_2\mathbb{R}\). □

We close this section by describing some of the \(\Gamma\)-rotational spreads with 2-dimensional automorphism group in terms of the functions \(a(r)\) and \(b(r)\) that define the hyperboloids whose right-screwed reguli make up the spread. We do this in the cases (1), (2a) and (2b) of Theorem 4.6. In principle, it can also be done in case (2c), but the result is not too pleasant.

Following the notation of Betten [3], we let \(\mathbb{R}^4 = W \oplus S\), and all spreads are given as fibrations of \(\mathbb{R}^4\{0\}\) by 2-dimensional subspaces. All fibrations shall contain \(W\) and \(S\) as elements, and \(S\) is the axis of rotation. The remaining 2-dimensional subspaces are given in [3] in the form \(\{(v, Av) \mid v \in W\}\), where \(A\) is a regular \(2 \times 2\) matrix. We shall select some of these matrices so that we get a cross section to the \(\text{SO}_2\mathbb{R}\)-orbits in the fibration. This is achieved by setting the rotation parameter \(\varphi\) equal to zero. Then we pass to affine coordinates in \(\mathbb{R}^3\) by setting the third coordinate equal to 1, and compute the distance \(r\) of the resulting line from the \(z\)-axis as well as the \(z\)-coordinate \(b(r)\) of the closest point and the slope \(a(r)\) of the given line.

For the planes of [3], Satz 1 with \(p = 0\), the cross section is given by the matrices

\[
A(s) = \begin{pmatrix}
s & 0 \\
swc & sw \\
\end{pmatrix}, \quad s > 0.
\]

Here, \(w \in ]0,1[\) and \(c \in \mathbb{R}\) are parameters defining the fibration, and \(s > 0\) parametrizes the cross section. Taking points of the 2-dimensional subspace defined by \(A(s)\) and dividing by the third coordinate, we obtain the points of the corresponding line in \(\mathbb{R}^3\) as

\[
\left(\frac{1}{s}, \frac{y}{sx}, sw^{-1}\left(c + \frac{y}{x}\right)\right).
\]

The point closest to the \(z\)-axis is obtained by setting \(y = 0\), hence \(r = s^{-1}\), and the spreads arising from [3], Satz 1 with \(p = 0\) are described by

\[
a(r) = r^{-w}, \quad b(r) = r^{1-w}c.
\]

In the special case \(c = 0\) we obtain the \(\text{O}_2\mathbb{R}\)-rotational spreads with 2-dimensional automorphism group.

The planes of [3], Satz 2 with parameter \(p = 0\) are given by

\[
A(t) = \begin{pmatrix}
et & 0 \\
ten & de^t \\
\end{pmatrix}, \quad t \in \mathbb{R}.
\]

The parameter \(d\) satisfying \(|d| \geq \frac{1}{2}\) defines the fibration. Proceeding as before, for fixed \(t\) we obtain a line in \(\mathbb{R}^3\) with points

\[
\left(e^{-t}, e^{-t}\frac{y}{x}, t + d\frac{y}{x}\right).
\]
We see that $r = e^{-t}$, and the planes of [3], Satz 2 with $p = 0$ are described by

$$a(r) = \frac{d}{r}, \quad b(r) = -\ln r.$$  

Recall from Theorem 4.3 that $a(r) = \frac{d}{r}, b(r) = 0$ defines the regular complex spread.

5. $SO_3\mathbb{R}$-invariant parallelisms of $\text{PG}(3, \mathbb{R})$ obtained from rotational spreads

The group $SO_3\mathbb{R}$ can act on $\text{PG}(3, \mathbb{R})$ either irreducibly or reducibly. The lifted action on $\mathbb{R}^4$ is equivalent to the action of the group $\text{Spin}_3$ of quaternions of norm 1 on the quaternion skew field $\mathbb{H}$ given by $q \to aq$ or $q \to aqa^{-1}$, respectively, where $q \in \mathbb{H}$ and $a \in \text{Spin}_3$. The latter, reducible action has an affine description on $\mathbb{R}^3$ as the ordinary rotation group $\Omega$.

**Definition 5.1.** Let $\text{AGL}_3\mathbb{R}$ be the group of affine transformations of $\mathbb{R}^3$, and let $\Omega \leq \text{AGL}_3\mathbb{R}$ be any affine group isomorphic to $SO_3\mathbb{R}$, and let $C$ be a spread invariant under the natural action of $\Gamma$ (so $C$ is a $\Gamma$-rotational spread by Theorem 4.2). Then we say that $\Omega$ is $\Gamma$-admissible for $C$. Note that for $\Gamma = O_2\mathbb{R}$ this implies that $b(r)$ is constant and that $SO_3\mathbb{R}$ fixes the common center of the reguli. Up to conjugacy in $\text{AGL}_3\mathbb{R}$, we may always assume that $\Phi = \Gamma^1$ consists of the ordinary rotations about the $z$-axis $Z$.

**Lemma 5.2.** Fix a group $\Gamma \leq \text{GL}_3\mathbb{R}$ isomorphic to $SO_2\mathbb{R}$ or to $O_2\mathbb{R}$ such that $\Phi = \Gamma^1$ consists of the ordinary rotations about the $z$-axis $Z$.

If a rotation group $\Omega \leq \text{AGL}_3\mathbb{R}$ is $\Gamma$-admissible, then all other $\Gamma$-admissible rotation groups $\Omega' \leq \text{AGL}_3\mathbb{R}$ are obtained as conjugates $\Omega^\xi$, where $\xi(x, y, z) = (x, y, sz + t)$ with $s, t \in \mathbb{R}$ and $s > 0$; if $\Gamma = O_2\mathbb{R}$, then only $t = 0$ is allowed.

**Proof.** All rotation groups are conjugate in $\text{AGL}_3\mathbb{R}$, and all one-parameter groups of $\Omega'$ are conjugate in $\Omega'$. Therefore, $\Omega' = \Omega^\alpha$ for some affine map $\alpha$ normalizing $\Phi = \Gamma^1$. Thus $\alpha$ fixes the rotation axis $Z$ of $\Phi$ and the orthogonal space $Z^\perp$ that is invariant under $\Phi$. The map induced by $\alpha$ on $Z$ must be of the form $z \mapsto sz + t$, and on $Z^\perp$ a homothety is induced. Now the homotheties of $\mathbb{R}^3$ as well as the map $(x, y, s) \mapsto (x, y, -z)$ normalize $\Omega$, hence we may assume that $s > 0$ and that the identity is induced on $Z^\perp$. From this he assertion for $\Gamma = \Phi$ follows. If $\Gamma = O_2\mathbb{R}$, then both $\Omega$ and $\Omega'$ fix the unique fixed point of $\Gamma$, whence $t = 0$. \hfill \Box

**Definition 5.3.** If $C$ is $\Gamma$-rotational and $\Omega$ is $\Gamma$-admissible, we define a set of spreads by

$$\Omega(C) = \{\omega(C) \mid \omega \in \Omega\}.$$  

Furthermore, let $C^+$ be one of the two oriented spreads obtained from $C$ and define $\Omega(C^+)$ in the same manner.
Our first main result is the following theorem. Assertion (1) is contained in [5] under the hypothesis that $C$ is a rotational Betten spread as defined earlier.

**Theorem 5.4.** Let $C$ be a $\Gamma$-rotational spread and let $\Omega$ be a $\Gamma$-admissible affine rotation group.

1. If $\Gamma = O_2^R$, then the set $\Omega(C)$ of spreads is a topological parallelism of $PG(3, R)$.
2. If $C$ is not an $O_2^R$-spread or $\Omega$ is not $O_2^R$-admissible, then $\Omega(C)$ is not a parallelism, but $\Omega(C^+)$ is an oriented topological parallelism.
3. The oriented parallelism in assertion (2) does not arise from any ordinary parallelism.

**Proof.** Assertion (3) is an immediate consequence of (2), because the image of $\Omega(C)$ under the covering map $L^+ \to L$ is $\Omega(C)$. The negative part of assertion (2) follows from the fact that a topological parallelism $\Pi$ is homeomorphic to the real projective plane; therefore, the stabilizer of a transitive $\Omega$-action on $\Pi$ is $O_2^R$. Hence, the spreads $C \in \Pi$ are $O_2^R$-spreads, and $\Omega$ is $O_2^R$-admissible. We shall now prove (1) and the positive part of (2) simultaneously. We shall use $L^*$ as a shorthand for $L$ or $L^+\oplus$, whichever is appropriate, and similarly for other sets of lines. Note that, in general, $M^+$ denotes the full inverse image of $M \subseteq L$ under the covering map $L^+ \to L$, whereas $C^+$ denotes just one of the two connected components of the inverse image.

We may assume that $\Omega$ fixes the origin $0 \in \mathbb{R}^3$ and that $C$ is constructed as in Sect. 4, with respect to the group $\Phi$ of rotations about the $z$-axis $Z$; compare Theorem 4.2. First we describe the line orbits of $\Omega$. Clearly, $\Omega$ is transitive on the sets $L_0^*$ and $L_F^*$, where the subscript 0 refers to lines passing through the origin and the subscript $F$ indicates lines contained in the hyperplane $F$ at infinity. The remaining orbits are the sets $T_d^*$ consisting of the lines or oriented lines tangent to the sphere of radius $d > 0$ centered at 0. We have to show two things, namely

(i) $C^*$ contains lines or oriented lines from every $\Omega$-orbit, and
(ii) If a line $L \in C^*$ has an image $\omega(L) \in C^*$ where $\omega \in \Omega$, then $\omega \in \Gamma$.

Indeed, (i) expresses that $\Omega(C^*)$ covers $L^*$, and (ii) expresses that any two $\omega$-images of $C^*$ are either disjoint or equal. We shall use the notation of Sect. 4. Condition (i) is obtained as follows. For the two special orbits the assertion is obvious, since $Z \in L_0^*$ and $V \in L_F^*$, no matter which orientation we choose for $Z$ and $V$. Now the half plane $E$ is simply covered by the hyperbola branches $H_r$, and every $H_r$ contains a unique point closest to the origin, say at distance $d = d(r)$. Then the regulus $R_r \subseteq C$ carried by the hyperboloid corresponding to $H_r$ is contained in $T_d$. Since the hyperbolae are pairwise disjoint and cover $E$, every value $d$ occurs exactly once. This proves (i).
Condition (ii) is easily checked for the two $\Gamma$-invariant lines $Z$ and $V$ (with or without orientation). Now suppose that $L \in R^*_r \subseteq C^*$ and $\omega(L) \in C^*$. Then $\omega(L) \in T^*_d$ because rotations leave $T^*_d$ invariant, and $\omega(L) \in R^*_r$ by injectivity of the map $r \to d(r)$. Therefore, there is a rotation $\varphi \in \Phi \leq \Gamma$ such that $\varphi \omega(L) = L$ as nonoriented lines. If $\Gamma = O_2^R$, then it follows that $\varphi \omega \in \Gamma$ and, hence, $\omega \in \Gamma$. If $\Gamma = SO_2^R$, then $\varphi \omega(L) \in C^+$ implies that $\varphi \omega(L) = L$ as oriented lines, and then $\varphi \omega = \text{id}$. This proves (ii).

Finally, compactness of the group $\Omega$ implies that $\Omega(C^*)$ is compact and, hence, is a topological parallelism or oriented parallelism.

Theorem 5.5. Every topological parallelism or oriented parallelism of $\text{PG}(3, \mathbb{R})$ admitting a reducible $SO_3^R$-action is isomorphic to one of the examples given by Theorem 5.4.

Proof. Up to conjugacy in $\text{PGL}_4^R$, there is only one reducible action of $SO_3^R$ on $\text{PG}(3, \mathbb{R})$. Thus we may assume that $\Omega$ is $SO_3^R$ acting on the affine space $\mathbb{R}^3$ in the ordinary way. Now let $\Pi$ be a parallelism invariant under this action, possibly an oriented one. Then $\Omega$ does not act trivially on $\Pi$, because the line orbits of $\Omega$ are not contained in spreads. Thus $\Omega$ is transitive on $\Pi$, because $\Omega$ does not contain any 2-dimensional closed subgroups. We know that $\Pi$ is a 2-sphere or a projective plane according as $\Pi$ is oriented or not, see Proposition 2.3, and hence the stabilizer $\Gamma$ of a spread $C^* \in \Pi$ is $SO_2^R$ or $O_2^R$, respectively. According to Theorem 4.2, $C$ is a $\Gamma$-rotational spread. Moreover, $\Omega$ is $\Gamma$-admissible for $C$, and the theorem follows.

It is tempting to call the parallelisms characterized by Theorem 5.5 rotational parallelisms. However, this name has been given to a different class of parallelisms (the 3-dimensional regular parallelisms with 2-dimensional automorphism group), see [8].

Let us now consider the regular case. The following result has been proved in [6], Theorem 8.1b in two distinct, very explicit ways, but the possible effect of changing the group $\Omega$ is not considered. Here we give a short conceptual proof.

Theorem 5.6. Suppose that $C$ is the complex, regular spread and that $\Omega = SO_3^R$ is $O_2^R$-admissible for $C$. Then $\Omega(C)$ is the Clifford parallelism.

Corollary 5.7. The Clifford parallelism is the only regular topological parallelism admitting a reducible $SO_3^R$-action.

Proof. Assume the hypothesis of 5.6. According to Theorem 5.5, the Clifford parallelism is of the form $\Omega'(C')$. There are axial subgroups $\Phi \leq \Omega$ and $\Phi' \leq \Omega'$ isomorphic to $SO_2^R$. Let $Z^{(i)} \in C'(i)$ be the axis of $\Phi^{(i)}$ and $V^{(i)} \in C'(i)$ the second fixed line of $\Phi^{(i)}$. The group $\Sigma = \text{Aut} C$ is doubly transitive on $C$, hence there is a collineation $\alpha$ of $\text{PG}(3, \mathbb{R})$ that sends $C'$ to $C$ and $(Z', V')$ to $(Z, V)$. Replacing $\Omega'$ with the conjugate $\Omega'^\alpha$, we obtain that $C = C'$, $Z = Z'$ and $V = V'$. Now $\Omega^{(i)}$ fixes some hyperplane $F^{(i)}$ containing $V$, and since $N_{S_2^C} \Phi$ is transitive on the set of such hyperplanes we may assume that $F = F'$. Finally, the subgroups $O_2^R \leq \Omega^{(i)}$ containing $\Phi$ can be adjusted to be equal using an affine translation in the direction of $Z$. 

We have now achieved that the groups $\Omega$ and $\Omega'$ are both affine with respect to the same affine space and are both $O_2(R)$-admissible for $C$. By Lemma 5.2, this implies that $\Omega' = \Omega^\xi = \xi^{-1}\Omega\xi$, where the linear map $\xi$ is given in suitable coordinates by $(x, y, z) \mapsto (sx, sy, sz)$, for some $s > 0$. Now $\rho : (x, y, z) \mapsto (sx, sy, sz)$ is an automorphism of $C$, and $\xi\rho : (x, y, z) \mapsto (sx, sy, sz)$ centralizes $\Omega$. Thus we have

$$\Omega(C) = \Omega^{\xi\rho}(C) = \rho^{-1}\Omega^{\xi}(\rho C) = \rho^{-1}\Omega'(C) \cong \Omega'(C).$$

This proves Theorem 5.6, and the corollary follows in view of Theorem 5.5.

\[\square\]

Note, however, that the classical spread can yield different regular oriented parallelisms, if a rotation group is applied which does not fix the common center of the reguli [Theorem 5.4(2)]. Next, we look at the non-regular case, where in contrast to the previous result it will turn out that the choice of an admissible group $\Omega$ does matter, see Theorem 5.10.

**Theorem 5.8.** Let $C^*$ be a (possibly regular) $\Gamma$-rotational spread (with orientation if $\Gamma = SO_2(R)$), and let $\Omega = SO_3(R)$ be $\Gamma$-admissible for $C$. Suppose that the (oriented or non-oriented) parallelism $\Pi = \Omega(C^*)$ is not Clifford. Then its full automorphism group $\Sigma = Aut \Pi$ is equal to the group $\Omega = SO_3(R)$ that was used to construct it.

Note that the theorem applies to the oriented parallelisms obtained from the complex spread by rotating it about a point other than the center of the family of reguli.

**Proof.** We know that $\Sigma$ is compact [13] and contains the 3-dimensional group $\Omega$, and $\Sigma$ is at most 3-dimensional by [12] because $\Pi$ is not Clifford by assumption, hence $\Sigma^1 = \Omega$. Now $\Sigma$ is contained in a maximal compact subgroup $PO(4, R) = Z_2 \cdot SO_4(R)$ of $PGL(4, R)$. The group $\Omega$ is the diagonal of $SO_4(R) \cong SO_3(R) \times SO_3(R)$ and is normalized by $\Sigma$ since $\Omega = \Sigma^1$. This implies that either $\Sigma$ equals $\Omega$ itself or $\Sigma$ is the extension of $\Omega$ by the reflection $\varrho$ in a plane perpendicular to $Z$. However, $\varrho$ sends every right screwed regulus in $C$ to a left screwed regulus, and this regulus is not contained in $C$ (no matter what the orientation is). This proves that $\Sigma = \Omega$. As an alternative argument, one could use Lemma 3.4 of [13] in order to rule out the possibility $\varrho \in \Sigma$. \[\square\]

**Proposition 5.9.** Let two $SO_2(R)$-rotational spreads $C$, $C'$ and $\Gamma$-admissible copies $\Omega$ and $\Omega'$ of $SO_3(R)$ be given. If the parallelisms or oriented parallelisms $\Omega(C)$ and $\Omega'(C')$ are isomorphic, then the spreads $C$ and $C'$ are isomorphic.

**Proof.** Let $\xi : \Omega(C) \rightarrow \Omega'(C')$ be an isomorphism. Since $\Omega'$ is transitive on $\Omega'(C')$, we may assume that $\xi(C) = C'$. \[\square\]

The converse of this implication, however plausible, is far from being true. The first indication for this is the fact that the additional automorphisms of the spreads discussed in Theorem 4.6 do not lead to a larger automorphism group of $\Omega(C)$, according to Theorem 5.8. In fact, we know that in the generic case, the automorphism group of a $\Gamma$-rotational spread is 1-dimensional, see Theorem 4.6, and then we have the following
Theorem 5.10. Let $C$ be a $\Gamma$-rotational spread such that $\Sigma = \text{Aut } C \leq \text{PGL}_4 \mathbb{R}$ is 1-dimensional, and let $\Omega \cong \text{SO}_3 \mathbb{R}$ be $\Gamma$-admissible (in particular, $\Omega \leq \text{AGL}_3 \mathbb{R}$ contains $\Phi = \Gamma^1$). Then there are uncountably many conjugates $\Omega^\xi = \xi^{-1} \Omega \xi \leq \text{AGL}_3 \mathbb{R}$ that define pairwise non-isomorphic parallelisms or oriented parallelisms $\Omega^\xi(C)$.

Proof. Let $\Omega$ and $\Omega'$ both be $\Gamma$-admissible for $C$ and let $\Pi = \Omega(C)$ and $\Pi' = \Omega'(C)$. First we note that if there is an isomorphism $\rho : \Pi \rightarrow \Pi'$, then there is an isomorphism $\sigma : \Pi \rightarrow \Pi'$ contained in $\Sigma$. This follows from transitivity of $\Omega = \text{Aut } \Pi$. Indeed, there is $C' \in \Pi$ such that $\rho(C') = C \in \Pi \cap \Pi'$, and there is $\omega \in \Omega$ such that $\omega(C) = C'$. Then $\sigma := \rho \omega$ maps $C$ to itself. Now $\Sigma/\Phi$ is at most countable and $\Phi \leq \Omega$ maps $\Pi$ to itself, hence $\Pi$ is isomorphic to at most countably many parallelisms $\Pi'$. The same holds for every $\Omega^\xi(C)$ if $\xi$ normalizes $\Phi$.

On the other hand, according to Lemma 5.2, the normalizer of $\Phi$ in $\text{AGL}_3 \mathbb{R}$ contains a 2-parameter subgroup $\Xi$ such that the conjugates $\Omega^\xi$ for $\xi \in \Xi$ are all $\Gamma$-admissible and pairwise distinct, namely the group $\Xi$ consisting of all affine maps $(x, y, z) \mapsto (x, y, sz + t)$ with $s, t \in \mathbb{R}$ and $s > 0$. (If $\Gamma = O_2 \mathbb{R}$ and we want ordinary parallelisms, we have to set $t = 0$.) Then the parallelisms or oriented parallelisms $\Omega^\xi(C)$ are also pairwise distinct, because every $\Omega^\xi$ is the automorphism group of $\Omega^\xi(C)$, by Theorem 5.8. This proves the theorem. $\square$

We remark that in the last proof we have not exhausted all possibilities of modifying $\Omega$ since we kept the hyperplane at infinity fixed.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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