On the interior motive of certain Shimura varieties: the case of Picard surfaces

Abstract. The purpose of this article is to construct a Hecke-equivariant Chow motive whose realizations equal interior (or intersection) cohomology of Picard surfaces with regular algebraic coefficients. As a consequence, we are able to define Grothendieck motives for Picard modular forms.

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0. Introduction

The purpose of this paper is the construction and analysis of the interior motive of Kuga–Sato families over Picard surfaces. It gives a second example of Shimura varieties where the use of the formalism of weight structures [7] proves to be successful for dealing with a problem, for which explicit geometrical methods seem inefficient, the first example being that of Hilbert–Blumenthal varieties of arbitrary dimension [37].

Let $k$ be a field of characteristic zero, and denote by $DM_{gm}(k)$ the triangulated category of geometrical motives [32]. (We shall actually consider the $F$-linear variant $DM_{gm}(k)_F$ of $DM_{gm}(k)$, for suitable coefficients $F$ of characteristic zero. For the purpose of this introduction, let us agree not to distinguish notation between $DM_{gm}(k)$ and $DM_{gm}(k)_F$.) For a smooth scheme $X$ over $k$, denote by $M_{gm}(X)$ and $M_{gm}^c(X)$ the motive of $X$ and its motive with compact support, respectively [32]. Chow motives form a full sub-category of $DM_{gm}(k)$; indeed they are identified with the category of objects which are pure of weight zero with respect to the motivic weight structure [7]. Both functors $M_{gm}$ and $M_{gm}^c$ agree on smooth and proper $k$-schemes $X$, and yield the Chow motive of $X$. In general, that is, for

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schemes $X$ which are only supposed smooth, they are related by a natural transformation $\iota : M_{gm} \to M_{gm}^c$. The morphism $\iota_X : M_{gm}(X) \to M_{gm}^c(X)$ clearly factors through a Chow motive; indeed, the motive of any smooth compactification of $X$ will provide such a factorization.

The rôle of the interior motive $[36]$ is to give a canonical, and in fact minimal such factorization.

The boundary motive $\partial M_{gm}(X)$ of $X$ $[34]$ is a canonical choice of (the shift by $[-1]$ of) a cone of $\iota_X$; indeed, it fits into a canonical exact triangle

\[(*) \quad \partial M_{gm}(X) \to M_{gm}(X) \to M_{gm}^c(X) \to \partial M_{gm}(X)[1].\]

In order to formulate a reasonable condition sufficient to guarantee the existence of the interior motive, we need to assume in addition that an idempotent endomorphism $e$ of the exact triangle $(*)$ is given. We thus get a direct factor

\[\partial M_{gm}(X)^e \to M_{gm}(X)^e \to M_{gm}^c(X)^e \to \partial M_{gm}(X)^e[1].\]

Now the abstract yoga of triangulated categories shows that isomorphism classes of factorizations $M_{gm}(X)^e \to M_0 \to M_{gm}^c(X)^e$ of $\iota_X$ correspond bijectively to isomorphism classes of morphisms $\partial M_{gm}(X)^e \to N$ (define $N$ as the shift by $[-1]$ of a cone of $M_0 \to M_{gm}^c(X)^e$).

A priori, it thus appears plausible that the existence of a minimal factorization $M_0$, which is pure of weight zero, should be related to a condition on $\partial M_{gm}(X)^e$.

Indeed, a sufficient such condition was identified in $[36]$. It says that the object $\partial M_{gm}(X)^e$ avoids weights $-1$ and $0$ (in practice, this condition is never satisfied for the whole of $\partial M_{gm}(X)$, whence the need to consider direct factors).

Technically speaking, the present paper deals with the verification of that hypothesis in the context of Picard surfaces. Our task is considerably simplified by a recent result of Ancona’s $[2]$, which implies that the analogue of the hypothesis is satisfied for the Hodge structure on boundary cohomology.

Our main task thus consists in proving what could be named the principle of weight conservativity: the absence of certain weights in the realization $R(M)$ of a motive $M$ implies the absence of these weights in $M$ itself. For Artin–Tate motives over a number field, weight conservativity is known $[40]$, and relatively straightforward to establish thanks to the presence of a $t$-structure. However, the boundary motive of a Kuga–Sato family over a Picard surface is not of Artin–Tate type; in fact, motives of certain elliptic curves (with complex multiplication) do occur. While in the general situation, a $t$-structure is expected, its existence has not yet been established—not even on the sub-category of $DM_{gm}(k)$ generated by the motive of a single elliptic curve over a number field!

Our first main result, Theorem 1.13 establishes weight conservativity on the triangulated sub-category $DM_{gm}^{Ab}(k)$ of $DM_{gm}(k)$ generated by Chow motives of Abelian type over a field $k$ which can be embedded into $\mathbb{C}$. Its proof consists of several intermediate steps, each of which may be considered to be of independent interest: (1) first, we show that the realization functor $R$ is conservative on the category of Chow motives of Abelian type (Theorem 1.10), (2) then, we extend conservativity to the whole of $DM_{gm}^{Ab}(k)$ (Theorem 1.12), (3) finally, we show that conservativity actually implies Theorem 1.13. The proof of step (1) uses a result from [4]: $R$ is conservative if its source is semi-primary and pseudo-Abelian, if $R$ itself is radical, and if the only object mapped to zero under $R$ is the zero object. This is where our definition enters: Chow motives of Abelian varieties are finite dimensional $[18]$, hence according to one of the main results from [4], they generate a semi-primary category. The main result from [23], proved for Abelian varieties, can be reformulated to say that $R$ is
radicial on Chow motives of Abelian type. For steps (2) and (3), we make a systematic use of the minimal weight filtration, whose theory was developed in [41].

In Sect. 2, the application of Theorem 1.13 to the boundary motive of a smooth scheme $X$ over $k$ is made explicit (Corollary 2.4). In order to do so, we identify its realizations with boundary cohomology of $X$ (Proposition 2.3).

Section 3 then contains our second main result, Theorem 3.8, which can be seen as the motivic analogue of the main result of [2]. Here, the smooth scheme $X$ is an $N$-th power of the universal Abelian threefold over a smooth Picard surface $S$ associated to a quadratic imaginary number field $F$. Denote by $G$ the reductive group underlying the Shimura variety $S$, and let $V$ be an irreducible algebraic representation of $G$. According to the main result from [1] (valid for arbitrary Shimura varieties of PEL-type), $N$ can be chosen such that the relative Chow motive $h(X/S)$ admits an idempotent $e$ with the following property: the cohomological Hodge theoretic realization of $h(X/S)^e$ coincides with the variation of Hodge structure on $S$ induced by $V$. Theorem 3.8 then implies that the boundary motive $\partial M_{gm}(X)$ lies in $DM_{gm}^{Ab}(F)$; furthermore, its direct factor $\partial M_{gm}(X)^e$ avoids weights $-1$ and $0$ if and only if the representation $V$ is of regular highest weight. Therefore, the interior motive exists. We list its principal properties, using the main results from [36, Sect. 4].

First (Corollary 3.9), we get precise statements on the weights occurring in the motive $M_{gm}(X)^e$ and the motive with compact support $M_{gm}^C(X)^e$. Second (Corollary 3.10), the interior motive is Hecke-equivariant. Corollary 3.10 appears particularly interesting, given the problem of non-existence of equivariant smooth compactifications of $X$ (for $N \geq 1$). Third (Corollary 3.11), the interior motive occurs canonically as a direct factor of the (Chow) motive of any smooth compactification of $X$.

Section 4 is devoted to the verification of the criterion from Theorem 1.13. First, we need to show that in this case, the boundary motive is indeed of Abelian type (Theorem 4.1). This is done using a smooth toroidal compactification. We use co-localization for the boundary motive [34], in order to reduce to showing the statement for the contribution of any of the strata. The latter was identified in the general context of mixed Shimura varieties [35]. For Kuga–Sato families over Picard surfaces, [loc. cit.] shows in particular that these contributions are indeed all of Abelian type over $F$. All that then remains to do is to cite the main result from [2].

In the final Sect. 5, we give the necessary ingredients to perform the construction of the Grothendieck motive associated to a (Picard) automorphic form (Definition 5.5). This is the analogue for Picard surfaces of the main result from [31].

Let us stress that our present approach via weight structures appears necessary because contrary to the situation considered in [31], Hecke-equivariant compactifications of the involved Kuga–Sato families are not known (and possibly not even expected) to exist. One may reasonably expect this approach to generalize. We refer in particular to [9] for the study of Picard varieties of higher dimension.

Notation and conventions. For a perfect field $k$, we denote by $Sch/k$ the category of separated schemes of finite type over $k$, and by $Sm/k \subset Sch/k$ the full sub-category of objects which are smooth over $k$. As far as motives are concerned, the notation of this paper is that of [34–36], which in turn follows that of [32]. We refer to [34, Sect. 1] for a concise review of this notation, and of the definition of the triangulated categories $DM_{gm}^{eff}(k)$ and $DM_{gm}(k)$ of (effective) geometrical motives over $k$. Let $F$ be a commutative $\mathbb{Q}$-algebra. The notation $DM_{gm}^{eff}(k)_F$ and $DM_{gm}(k)_F$ stands for the $F$-linear analogues of these triangulated categories defined in [3, Sect. 16.2.4 and Sect. 17.1.3]. Similarly, let us denote by $CHM^{eff}(k)$ and $CHM(k)$ the categories opposite to the categories of (effective) Chow motives over $k$, and by $CHM^{eff}(k)_F$ and $CHM(k)_F$ the pseudo-Abelian completion of the category.
CHM_{eff}^d (k) \otimes_\mathbb{Z} F$ and $\text{CHM}^d (k) \otimes_\mathbb{Z} F$, respectively. Using [32, Cor. 4.2.6], we canonically identify $\text{CHM}^d (k) F$ and $\text{CHM}^d (k) F$ with a full additive sub-category of $\text{DM}_{gm}^d (k) F$ and $\text{DM}_{gm}^d (k) F$, respectively. For $S \in \mathbb{Sm}/k$, denote by $\text{CHM}^d (S)$ the category opposite to the category of smooth Chow motives over $S$ [11, Sect. 1.3., 1.6], and by $\text{CHM}^d (S)_F$ the pseudo-Abelian completion of the category $\text{CHM}^d (S) \otimes_\mathbb{Z} F$.

1. Motives of Abelian type

Let us start by defining the categories of motives for which we are able to establish conservativity and weight conservativity of realizations. The notations are those fixed at the end of the Introduction; in particular, $k$ denotes a perfect field, and $F$ a commutative $\mathbb{Q}$-algebra.

**Definition 1.1.** Denote by $\bar{k}$ the algebraic closure of $k$.

(a) Define the category of Chow motives of Abelian type over $\bar{k}$ as the strict full dense additive tensor sub-category $\text{CHM}^{Ab}(\bar{k})_F$ of $\text{CHM}(\bar{k})_F$ generated by the following:

1. shifts of Tate motives $\mathbb{Z}(m)[2m]$, for $m \in \mathbb{Z}$,
2. Chow motives of Abelian varieties over $\bar{k}$.

(b) Define the category of Chow motives of Abelian type over $k$ as the (strict) full dense additive tensor sub-category $\text{CHM}^{Ab}(k)_F$ of $\text{CHM}(k)_F$ of Chow motives whose base change to $\bar{k}$ lies in $\text{CHM}^{Ab}(\bar{k})_F \subset \text{CHM}(\bar{k})_F$.

(c) Define the category $\text{DM}_{gm}^{Ab}(k)_F$ as the (strict) full triangulated sub-category of $\text{DM}_{gm}(k)_F$ generated by $\text{CHM}^{Ab}(k)_F$.

Note that both $\text{CHM}^{Ab} (\bar{k})_F$ and $\text{CHM}^{Ab} (k)_F$ are rigid. The following observation will turn out to be vital.

**Proposition 1.2.** The motivic weight structure on $\text{DM}_{gm}^{Ab}(k)_F$ (see [7, Sect. 6]) induces a weight structure on $\text{DM}_{gm}^{Ab}(k)_F$. More precisely, we have

\[ \text{DM}_{gm}^{Ab}(k)_F, w \leq 0 = \text{DM}_{gm}^{Ab}(k)_F \cap \text{DM}_{gm}(k)_F, w \leq 0 \]

and

\[ \text{DM}_{gm}^{Ab}(k)_F, w \geq 0 = \text{DM}_{gm}^{Ab}(k)_F \cap \text{DM}_{gm}(k)_F, w \geq 0, \]

and the heart $\text{DM}_{gm}^{Ab}(k)_F, w = 0$ equals $\text{CHM}^{Ab} (k)_F$.

**Proof.** The category $\text{DM}_{gm}^{Ab}(k)_F$ is generated by $\text{CHM}^{Ab} (k)_F$ as a triangulated category. The latter is contained in $\text{CHM}(k)_F$, therefore it is negative, meaning that

\[ \text{Hom}_{\text{DM}_{gm}^{Ab}(k)_F} (M_1, M_2[i]) = \text{Hom}_{\text{DM}_{gm}(k)_F} (M_1, M_2[i]) = 0 \]

for any two objects $M_1, M_2$ of $\text{CHM}^{Ab} (k)_F$, and any integer $i > 0$ [32, Cor. 4.2.6]. Therefore, [7, Thm. 4.3.2 II 1] can be applied to ensure the existence of a weight structure $v$ on $\text{DM}_{gm}^{Ab}(k)_F$, uniquely characterized by the property of containing $\text{CHM}^{Ab} (k)_F$ in its heart. Furthermore [7, Thm. 4.3.2 II 2], the heart of $v$ is equal to the category of retracts of $\text{CHM}^{Ab} (k)_F$. But by definition, $\text{CHM}^{Ab} (k)_F$ is pseudo-Abelian, hence $\text{DM}_{gm}^{Ab}(k)_F, v = 0 = \text{CHM}^{Ab} (k)_F$. In particular,

\[ \text{DM}_{gm}^{Ab}(k)_F, v = 0 \subset \text{DM}_{gm}(k)_F, w = 0. \]
The existence of weight filtrations for the weight structure $v$ then formally implies that

$$DM^{Ab}_{gm}(k)_{F,v \leq 0} \subset DM_{gm}(k)_{F,w \leq 0},$$

and that

$$DM^{Ab}_{gm}(k)_{F,v \geq 0} \subset DM_{gm}(k)_{F,w \geq 0}.$$  

We leave it to the reader to prove from this (cmp. [7, Lemma 1.3.8]) that in fact

$$DM^{Ab}_{gm}(k)_{F,v \leq 0} = DM^{Ab}_{gm}(k) \cap DM_{gm}(k)_{F,w \leq 0}$$

and

$$DM^{Ab}_{gm}(k)_{F,v \geq 0} = DM^{Ab}_{gm}(k) \cap DM_{gm}(k)_{F,w \geq 0}.$$  

\[ \square \]

**Corollary 1.3.** The category $DM^{Ab}_{gm}(k)_{F}$ is pseudo-Abelian.

**Proof.** The weight structure $w$ on $DM^{Ab}_{gm}(k)_{F}$ is bounded in the sense that its heart $CHM^{Ab}_{gm}(k)_{F}$ generates the triangulated category $DM^{Ab}_{gm}(k)_{F}$. Furthermore, the category $CHM^{Ab}_{gm}(k)_{F}$ is pseudo-Abelian. Our claim thus follows from [7, Lemma 5.2.1].  

\[ \square \]

Next, let us consider realizations ([15, Sect. 2.3 and Corrigendum]; see [10, Sect. 1.5] for a simplification of this approach). We shall concentrate on two realizations (Theorem 1.13 then formally generalizes to any of the other realizations “with weights” considered in [15]):

(i) when $k$ is embeddable into $\mathbb{C}$, the Hodge theoretic realization

$$R_{\sigma} : DM_{gm}(k)_{F} \rightarrow D$$

associated to a fixed embedding $\sigma$ of $k$ into the field $\mathbb{C}$ of complex numbers. Here, $D$ is the bounded derived category of mixed graded-polarizable $\mathbb{Q}$-Hodge structures [5, Def. 3.9, Lemma 3.11], tensored with $F$,

(ii) the $\ell$-adic realization

$$R_{\ell} : DM_{gm}(k)_{F} \rightarrow D$$

for a prime $\ell$ not dividing $\text{char}(k)$. Here, $D$ is the bounded “derived category” of constructible $\mathbb{Q}_{\ell}$-sheaves on $\text{Spec}(k)$ [12, Sect. 6], tensored with $F$.

Choose and fix one of these two, denote it by $R$, and recall that it is a contravariant tensor functor mapping the pure Tate motive $\mathbb{Z}(m)$ to the pure Hodge structure $\mathbb{Q}(-m)$ (when $R = R_{\sigma}$) and to the pure $\mathbb{Q}_{\ell}$-sheaf $\mathbb{Q}_{\ell}(-m)$ (when $R = R_{\ell}$), respectively [15, Thm. 2.3.3]. The category $D$ is equipped with a $t$-structure; write $D^{t=0}$ for its heart, and $H^{n} : D \rightarrow D^{t=0}, n \in \mathbb{Z}$, for the cohomology functors.

In order to analyze the functor $R$, let us start by recalling some terminology from category theory. Following [4, Sect. 1.1], we refer to a category $\mathcal{A}$ as an $F$-category ($F = \text{our fixed commutative } \mathbb{Q}\text{-algebra}$), if the morphisms in $\mathcal{A}$ are equipped with the structure of an $F$-module, and if composition of morphisms in $\mathcal{A}$ is $F$-linear. Similarly, a functor $T : \mathcal{A} \rightarrow \mathcal{B}$ is an $F$-functor if it is $F$-linear on morphisms.
Definition 1.4. ([17]) Let $\mathfrak{A}$ be an $F$-category. The radical of $\mathfrak{A}$ is the ideal $\text{rad}_\mathfrak{A}$ which associates to each pair of objects $A, B$ of $\mathfrak{A}$ the subset

$$\text{rad}_\mathfrak{A}(A, B) := \{ f \in \text{Hom}_\mathfrak{A}(A, B), \forall g \in \text{Hom}_\mathfrak{A}(B, A), \text{id}_A - gf \text{ invertible} \}$$

of $\text{Hom}_\mathfrak{A}(A, B)$.

In [4, Déf. 1.4.1], the radical is referred to as the Kelly radical of $\mathfrak{A}$. It can be checked that $\text{rad}_\mathfrak{A}$ is indeed a two-sided ideal of $\mathfrak{A}$ in the sense of [4, Sect. 1.3], i.e., for each pair of objects $A, B$, $\text{rad}_\mathfrak{A}(A, B)$ is an $F$-submodule of $\text{Hom}_\mathfrak{A}(A, B)$, and for each pair of morphisms $h : A' \to A$ and $k : B \to B'$ in $\mathfrak{A}$,

$$k \text{rad}_\mathfrak{A}(A, B) h \subset \text{rad}_\mathfrak{A}(A', B').$$

Definition 1.5. ([4, Déf. 1.4.6]) An $F$-functor $T : \mathfrak{A} \to \mathfrak{B}$ between $F$-categories is said to be radicial if $T(\text{rad}_\mathfrak{A}) \subset \text{rad}_\mathfrak{B}$.

Definition 1.6. ([4, Déf. 2.3.1]) An $F$-category $\mathfrak{A}$ is called semi-primary if

1. for all objects $A$ of $\mathfrak{A}$, the radical $\text{rad}_\mathfrak{A}(A, A)$ is nilpotent, i.e., there exists a positive integer $N$ such that $\text{rad}_\mathfrak{A}(A, A)^N = 0 \subset \text{End}_\mathfrak{A}(A)$,

2. the $F$-linear quotient category $\overline{\mathfrak{A}} := \mathfrak{A}/\text{rad}_\mathfrak{A}$ is semi-simple.

We then have the following result.

Proposition 1.7. ([4, Prop. 2.3.4 f]) Assume $F$ to be a finite direct product of fields of characteristic zero. Let $T : \mathfrak{A} \to \mathfrak{B}$ an $F$-functor of $F$-categories. Assume the following:

1. the category $\mathfrak{A}$ is semi-primary and pseudo-Abelian,
2. the functor is radicial,
3. whenever $A$ is an object of $\mathfrak{A}$, and $T(A) = 0$, then $A = 0$.

Then $T$ is conservative.

The first step of our analysis consists in applying Proposition 1.7 to the restriction of the functor

$$H^n R := (H^n R)_n \in \mathbb{Z} : D^A_{\text{gr}}(k)_F \to \text{Gr}_{\mathbb{Z}} D^{t=0}$$

to $\text{CHM}^A_F(k)_F \subset D^A_{\text{gr}}(k)_F$. Here, we denote by $\text{Gr}_{\mathbb{Z}} D^{t=0}$ the $\mathbb{Z}$-graded category associated to the heart $D^{t=0}$ of the $t$-structure on the target $D$ of the realization $R$. Let us check the hypotheses from Proposition 1.7.

Proposition 1.8. Assume $F$ to be a finite direct product of fields of characteristic zero. All motives in $\text{CHM}^A_F(k)_F$ are finite dimensional [18, Def. 3.7]. The $F$-category $\text{CHM}^A_F(k)_F$ is semi-primary (and pseudo-Abelian).

Proof. According to [25, pp. 54–55], finite dimensionality can be checked after base change to the algebraic closure of $k$. Thus, the first claim follows from our definition, and from [19, Thm. (3.3.1)].

The second claim follows from the first, and from [4, Thm. 9.2.2] (see also [24, Lemma 4.1]).
This shows hypothesis 1.7 (1). By [18, Cor. 7.3], whenever $M$ is an object of $CHM^{Ab}(k)_F$, and $H^nR(M) = 0$ for all $n \in \mathbb{Z}$, then $M = 0$, whence hypothesis 1.7 (3). It remains to check hypothesis 1.7 (2).

For the rest of the section, we assume $k$ to be a field embeddable into $\mathbb{C}$, and $F$ a finite direct product of fields of characteristic zero. The following is a reformulation of the main result from [23].

**Theorem 1.9.** The restriction of the functor

$$H^*R : DM^{Ab}_{gm}(k)_F \rightarrow Gr_{\mathbb{Z}} D^{t=0}$$

to $CHM^{Ab}(k)_F$ maps the radical $rad_{CHM^{Ab}(k)_F}$ to zero. In particular, it is radicial.

**Proof.** By comparison, the statement for the $\ell$-adic realization follows from the one for the Hodge theoretic realization, hence we may assume that $R$ equals the latter. Base change from to $k$ to $\mathbb{C}$ is a radicial functor on finite dimensional Chow motives ([4, Thm. 9.2.2], [25, p. 55]; see also [41, Cor. 6.2]). We may therefore work over $\mathbb{C}$.

Let $B$ be a proper, smooth variety over $\mathbb{C}$, $r = 1, 2$, such that $M_{gm}(B)$ is finite dimensional. The radical

$$rad_{CHM(\mathbb{C})_F}(M_{gm}(B), M_{gm}(B))$$

then coincides with the maximal tensor ideal

$$N(M_{gm}(B), M_{gm}(B))$$

[4, Thm. 9.2.2]. It thus consists of the classes of numerically trivial cycles on $B \times_{\mathbb{C}} B$ [4, Ex. 7.1.2]. But according to [23, Thm. 4], numerical and homological equivalence coincide on Abelian varieties over $\mathbb{C}$. In other words,

$$H^*R(rad_{CHM(\mathbb{C})_F}(M_{gm}(B), M_{gm}(B))) = 0$$

if $B$ is an Abelian variety over $\mathbb{C}$. We leave it to the reader to generalize this statement, and show that

$$H^*R(rad_{CHM(\mathbb{C})_F}(M_{gm}(B)(m)[2m], M_{gm}(B)(m)[2m])) = 0$$

for any Abelian variety $B$ over $\mathbb{C}$ and any $m \in \mathbb{Z}$.

Our claim then follows from [4, Lemme 1.3.2].

**Theorem 1.10.** Assume $k$ to be a field embeddable into $\mathbb{C}$, and $F$ a finite direct product of fields of characteristic zero. The restriction of the functor

$$H^*R : DM^{Ab}_{gm}(k)_F \rightarrow Gr_{\mathbb{Z}} D^{t=0}$$

to $CHM^{Ab}(k)_F$ is conservative.

**Proof.** All the hypotheses of Proposition 1.7 are indeed verified.

The second step of our analysis aims at conservativity of $H^*R$ on the whole of $DM^{Ab}_{gm}(k)_F$. We shall need the following technical result.
Lemma 1.11. Let $M$ and $N$ be objects of $\text{DM}_{\text{gm}}^\text{Ab}(k)_F$, and assume that $M$ is of weights $\geq 0$, and that $N$ is of weights $\leq 0$. Then the radical

$$\text{rad}_{\text{DM}_{\text{gm}}^\text{Ab}(k)_F}(M, N)$$

is contained in the kernel of $H^n R$.

Proof. By comparison, we may assume that $R = R_\sigma$ equals the Hodge theoretic realization. Let $f \in \text{rad}_{\text{DM}_{\text{gm}}^\text{Ab}(k)_F}(M, N)$, $n \in \mathbb{Z}$, and consider the morphism

$$H^n R(f) : H^n R(N) \longrightarrow H^n R(M).$$

It is a morphism of graded-polarizable mixed $\mathbb{Q}$-Hodge structures. Its source $H^n R(N)$ is of weights at least $n$, and its target $H^n R(M)$, of weights at most $n$. Furthermore, the morphism $H^n R(f)$ is strict with respect to the weight filtrations. Therefore, it is sufficient to show that the morphism

$$\text{Gr}_n^W H^n R(f) : \text{Gr}_n^W H^n R(N) \longrightarrow \text{Gr}_n^W H^n R(M)$$

induced by $H^n R(f)$ is zero.

Applying Proposition 1.2, we choose weight filtrations

$$M_0 \xrightarrow{\iota_0} M \longrightarrow M_{\geq 1} \longrightarrow M_0[1]$$

and

$$N_{\leq -1} \longrightarrow N \xrightarrow{\pi_0} N_0 \longrightarrow N_{\leq -1}[1]$$

of $M$ and $N$, with $M_0, N_0 \in \text{CHM}_{\text{gm}}^\text{Ab}(k)_F$, $M_{\geq 1} \in \text{DM}_{\text{gm}}^\text{Ab}(k)_{F, w \geq 1}$ and $N_{\leq -1} \in \text{DM}_{\text{gm}}^\text{Ab}(k)_{F, w \leq -1}$. The composition $\pi_0 \circ f \circ \iota_0$ belongs to

$$\text{rad}_{\text{DM}_{\text{gm}}^\text{Ab}(k)_F}(M_0, N_0) = \text{rad}_{\text{CHM}_{\text{gm}}^\text{Ab}(k)_F}(M_0, N_0);$$

therefore (Theorem 1.9) it belongs to the kernel of $H^n R$:

$$H^n R(\pi_0 \circ f \circ \iota_0) = 0.$$ 

But $H^n R(\pi_0) : H^n R(N_0) \rightarrow H^n R(N)$ factors through an epimorphism

$$H^n R(N_0) \rightarrow \text{Gr}_n^W H^n R(N),$$

while $H^n R(\iota_0) : H^n R(M) \rightarrow H^n R(M_0)$ factors through a monomorphism

$$\text{Gr}_n^W H^n R(M) \hookrightarrow H^n R(M_0),$$

and the composition

$$H^n R(N_0) \rightarrow \text{Gr}_n^W H^n R(N) \xrightarrow{\text{Gr}_n^W H^n R(f)} \text{Gr}_n^W H^n R(M) \hookrightarrow H^n R(M_0)$$

equals $H^n R(\pi_0 \circ f \circ \iota_0)$ [7, Prop. 2.1.2 2]. This composition being zero means that $\text{Gr}_n^W H^n R(f)$ is zero. \qed
**Theorem 1.12.** Assume $k$ to be a field embeddable into $\mathbb{C}$, and $F$ a finite direct product of fields of characteristic zero. Then the functor

$$H^* R : DM_{gm}^{Ab}(k)_{F} \to \text{Gr}_2 D^{f=0}$$

is conservative.

**Proof.** Given that $DM_{gm}^{Ab}(k)_{F}$ is triangulated, and that $H^* R$ is cohomological, it suffices to show that whenever $M \in DM_{gm}^{Ab}(k)_{F}$, and $H^* R(M) = 0$, then $M = 0$. Recall that the heart $CHM^{Ab}(k)_{F}$ of the weight structure on $DM_{gm}^{Ab}(k)_{F}$ is semi-primary (Proposition 1.8). Take the minimal weight filtration

$$M_{\leq 0} \to M \to M_{\geq 0} \to \delta \to M_{\leq 0}[1]$$

of $M$ [41, Thm. 2.2] ($M_{\leq 0} \in DM_{gm}^{Ab}(k)_{F,w \leq 0}$, $M_{\geq 1} \in DM_{gm}^{Ab}(k)_{F,w \geq 1}$), i.e., the weight filtration of $M$ whose isomorphism class is determined by the condition

$$\delta \in \text{rad}_{DM_{gm}^{Ab}(k)_{F}}(M_{\geq 1}, M_{\leq 0}[1]).$$

Applying Lemma 1.11 to $\delta[-1]$, we see that $H^* R(\delta) = 0$. In other words, the Bockstein morphisms in the long exact cohomology sequence for $R(M)$ are all zero. Now $H^* R(M) = 0$, therefore we see that $H^* R(M_{\leq 0}) = 0$ and $H^* R(M_{\geq 1}) = 0$. Since the motivic weight structure is bounded, we are thus reduced to the case where $M$ is pure, say of weight $m$. But then $M[-m]$ is in the heart of $w$, i.e. (Proposition 1.2), in $CHM^{Ab}(k)_{F}$. Our claim thus follows from Proposition 1.8, and from [18, Cor. 7.3].

Here is the main result of this section; it generalizes [40, Thm. 3.11].

**Theorem 1.13.** Let $k$ be a field embeddable into $\mathbb{C}$, $F$ a finite direct product of fields of characteristic zero, and $R$ one of the two realizations considered above (Hodge-theoretic or $\ell$-adic). Then $R$ respects and detects the weight structure $w$ on $DM_{gm}^{Ab}(k)_{F}$. More precisely, let $M \in DM_{gm}^{Ab}(k)_{F}$, and $\alpha \leq \beta$ two integers.

(a) $M$ lies in the heart $CHM^{Ab}(k)_{F}$ of $w$ if and only if the $n$-th cohomology object $H^n R(M) \in D^{f=0}$ of $R(M)$ is pure of weight $n$, for all $n \in \mathbb{Z}$.

(b) $M$ lies in $DM_{gm}^{Ab}(k)_{F,w \leq 0}$ if and only if $H^n R(M)$ is of weights $\geq n - \alpha$, for all $n \in \mathbb{Z}$.

(c) $M$ lies in $DM_{gm}^{Ab}(k)_{F,w \geq 1}$ if and only if $H^n R(M)$ is of weights $\leq n - \beta$, for all $n \in \mathbb{Z}$.

(d) $M$ is without weights $\alpha, \alpha + 1, \ldots , \beta$ if and only if $H^n R(M)$ is without weights $n - \beta, \ldots , n - (\alpha + 1), n - \alpha$, for all $n \in \mathbb{Z}$.

Here, absence of weights is in the sense of [36, Def. 1.10]. In the situation of Theorem 1.13 (d), it means that there is an exact triangle

$$M_{\leq \alpha-1} \to M \to M_{\geq \beta+1} \to M_{\leq \alpha-1}[1]$$

in $DM_{gm}^{eff}(k)_{F}$, with $M_{\leq \alpha-1} \in DM_{gm}^{Ab}(k)_{F,w \leq \alpha-1}$, $M_{\geq \beta+1} \in DM_{gm}^{Ab}(k)_{F,w \geq \beta+1}$.

**Proof of Theorem 1.13.** The motivic weight structure is bounded, and the “only if” parts of statements (a)–(d) are true. It therefore suffices to prove the “if” part of statement (d).

Consider the minimal weight filtrations of $M$, concentrated at weight $\alpha$ and at weight $\beta + 1$, respectively [41, Thm. 2.2]:

$$M_{\leq \alpha-1} \to M \to M_{\geq \alpha} \to M_{\leq \alpha-1}[1],$$

$$M_{\leq \beta} \to M \to M_{\geq \beta+1} \to M_{\leq \beta}[1].$$
Here, \( M_{\leq \alpha - 1} \in D^A_{gm}(k)_F \) etc., and both \( \delta_\alpha \) and \( \delta_{\beta + 1} \) belong to the radical. By orthogonality, the identity on \( M \) extends to a morphism of exact triangles

\[
\begin{array}{cccc}
M_{\leq \alpha - 1} & \rightarrow & M & \rightarrow & M_{\geq \alpha} \\
\downarrow m & & \downarrow & & \downarrow m[1] \\
M_{\leq \beta} & \rightarrow & M & \rightarrow & M_{\geq \beta + 1}
\end{array}
\]

By Lemma 1.11, both \( H^*R(\delta_\alpha) \) and \( H^*R(\delta_{\beta + 1}) \) are zero. Thus, the above morphism of exact triangles induces a morphism of exact sequences

\[
\begin{array}{cccc}
0 & \rightarrow & H^*R(M_{\leq \alpha - 1}) & \rightarrow & H^*R(M_{\geq \alpha}) & \rightarrow & 0 \\
H^*R(m) & \downarrow & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^*R(M_{\leq \beta}) & \rightarrow & H^*R(M_{\geq \beta + 1}) & \rightarrow & 0
\end{array}
\]

Our hypothesis on weights avoided in \( H^*R(M) \) implies that the monomorphism \( H^*R(m) \) is in fact an isomorphism. But then (Theorem 1.12) so is \( m \) itself. This yields a weight filtration

\[
M_{\leq \alpha - 1} \rightarrow M \rightarrow M_{\geq \beta + 1} \rightarrow M_{\leq \alpha - 1}[1]
\]

of \( M \) avoiding weights \( \alpha, \alpha + 1, \ldots, \beta \).

**Corollary 1.14.** Let \( k \) be a field embeddable into \( \mathbb{C} \), and \( F \) a finite direct product of fields of characteristic zero. Denote by \( \text{DMAT}(k)_F \) the triangulated category of Artin–Tate motives over \( k \) [40, Def. 1.3]. Then \( R \) respects and detects the weight structure on \( \text{DMAT}(k)_F \).

Note that [40, Thm. 3.11] only concerns the case where \( k \) is an algebraic number field.

**Proof of Corollary 1.14.** Indeed, \( \text{DMAT}(k)_F \) is a full sub-category of \( \text{DM}^A_{gm}(k)_F \), \( \square \)

**Corollary 1.15.** Let \( k \) be a field embeddable into \( \mathbb{C} \), \( F \) a finite direct product of fields of characteristic zero, and \( \alpha \leq \beta \) be two integers. Assume that \( N \in \text{DM}^{ef}_{gm}(k)_F \) is a successive extension of objects \( M \) of \( \text{DM}^{ef}_{gm}(k)_F \), each satisfying one of the following properties.

(i) \( M \) is without weights \( \alpha, \alpha + 1, \ldots, \beta \).

(ii) \( M \) lies in \( \text{DM}^A_{gm}(k)_F \), and the cohomology object \( H^nR(M) \) of its image \( R(M) \) under \( R \) is without weights \( n - \beta, \ldots, n - (\alpha + 1), n - \alpha \), for all \( n \in \mathbb{Z} \).

Then \( N \) is without weights \( \alpha, \alpha + 1, \ldots, \beta \).

**Proof.** Apply [40, Prop. 2.11] and Theorem 1.13 (d), \( \square \)

2. A criterion on the existence of the interior motive

Let \( k \) be a field embeddable into \( \mathbb{C} \), and \( F \) a finite direct product of fields of characteristic zero.

Fix \( X \in Sm/k \), and consider the exact triangle

\[
(*) \quad \partial M_{gm}(X) \rightarrow M_{gm}(X) \rightarrow M_{gm}^C(X) \rightarrow \partial M_{gm}(X)[1]
\]
in $DM_{gm}^{\text{eff}}(k)$ [34, Prop. 2.2]. Fix an idempotent endomorphism $e$ of the image of the exact triangle in the category $DM_{gm}^{\text{eff}}(k)_F$. Denote by $M_{gm}(X)^e$, $M_{gm}^c(X)^e$ and $\partial M_{gm}(X)^e$ the images of $e$ on $M_{gm}(X)$, $M_{gm}^c(X)$ and $\partial M_{gm}(X)$, respectively, considered as objects of $DM_{gm}^{\text{eff}}(k)_F$. Recall the following assumption.

**Assumption 2.1.** ([36, Asp. 4.2]) The object $\partial M_{gm}(X)^e$ is without weights $-1$ and $0$.

In order to apply the results from [36, Sect. 4], allowing in particular to construct the *interior motive*, one needs to verify Assumption 2.1. The results obtained in Sect. 1 yield the following criterion.

**Theorem 2.2.** Let $k$ be a field embeddable into $\mathbb{C}$, and $F$ a finite direct product of fields of characteristic zero. Let $X \in Sm/k$, and $\alpha \leq \beta$ two integers such that $\alpha \leq -1$ and $\beta \geq 0$. Assume that $\partial M_{gm}(X)^e$ is a successive extension of objects $M$ of $DM_{gm}^{\text{eff}}(k)_F$, each satisfying one of the following properties.

(i) $M$ is without weights $\alpha, \alpha+1, \ldots, \beta$.

(ii) $M$ lies in $DM_{gm}^{Ab}(k)_F$, and the cohomology object $H^nR(M)$ of its image $R(M)$ under $R$ is without weights $n-\beta, \ldots, n-(\alpha+1)$, $n-\alpha$, for all $n \in \mathbb{Z}$.

Then Assumption 2.1 holds.

**Proof.** This is Corollary 1.15 for $N = \partial M_{gm}(X)^e$. \qed

Let $X \in Sm/k$. Recall from [36, Def. 4.1 (a)] that $c(X, X)$ contains a canonical sub-algebra $c_{1,2}(X, X)$ (of “bi-finite correspondences”) acting on the exact triangle

$$(*) \quad \partial M_{gm}(X) \longrightarrow M_{gm}(X) \longrightarrow M_{gm}^c(X) \longrightarrow \partial M_{gm}(X)[1].$$

Denote by $\tilde{c}_{1,2}(X, X)$ the quotient of $c_{1,2}(X, X)$ by the kernel of this action. The algebra $\tilde{c}_{1,2}(X, X) \otimes F$ is a canonical source of idempotent endomorphisms $e$ of ($*$), and it is for such choices that the results in [36] were formulated. However, they remain valid in the present, more general context.

Assuming $e \in \tilde{c}_{1,2}(X, X) \otimes F$, the group $H^nR(\partial M_{gm}(X)^e)$ equals the $e$-part of the boundary cohomology of $X$ with respect to the natural action (via cycles) of the algebra $\tilde{c}_{1,2}(X, X) \otimes F$ [37, Prop. 2.5]. We shall need a variant of this result. In order to explain it, recall that boundary cohomology is defined via a compactification $j : X \hookrightarrow \overline{X}$; writing $i : \overline{X} \hookrightarrow \overline{X}$ for the complementary immersion, one defines $\partial H^n(\bullet)$ as cohomology of $\overline{X}$ with coefficients in $i^* R_{j_*}(\bullet)$. Thanks to proper base change, this definition is independent of the choice of $j$, as is the long exact cohomology sequence

$$\cdots \rightarrow H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F \rightarrow \partial H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F \rightarrow \cdots$$

(in the Hodge theoretic setting) resp.

$$\cdots \rightarrow H^n(X(\bar{k}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} F \rightarrow \partial H^n(X(\bar{k}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} F \rightarrow \cdots$$

(in the $\ell$-adic setting).

Let us now assume in addition that an object $S \in Sm/k$ is given, together with a factorization of the structure morphism of $X$, defining on $X$ the structure of a proper, smooth scheme over $S$. In that situation, the Chow group $CH_{dX}(X \times S X)$ ($d_X :=$ the absolute dimension of $X$) acts contravariantly on the above exact cohomology sequences. By [38, Thm. 2.2 (a)], it also acts on the exact triangle ($*$).
Proposition 2.3. Let $X$ be proper and smooth over $S \in Sm/k$, and $e \in \text{CH}_{dX}(X \times_S X) \otimes_{\mathbb{Z}} F$ an idempotent. Then $H^nR(\partial M_{gm}(X)^e)$ is isomorphic to $(\partial H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F)^e$ (in the Hodge theoretic setting) resp. $(\partial H^n(X_{\overline{k}}, \mathbb{Q}_{\ell}) \otimes_{\mathbb{Q}_{\ell}} F)^e$ (in the $\ell$-adic setting), for all $n$.

This result follows from [37, Prop. 2.5], if the class $e$ can be represented by a cycle in $c_{1,2}(X, X) \otimes_{\mathbb{Z}} F$.

Proof of Proposition 2.3. We imitate the proof of [37, Prop. 2.5], and show that the image under $R$ of the canonical morphism

$$t : M_{gm}(X) \longrightarrow M_{gm}^c(X)$$

can be $\text{CH}_{dX}(X \times_S X)$-equivariantly identified with the canonical morphism

$$R\Gamma_c(X) \longrightarrow R\Gamma(X)$$

in the target $D$ of $R$ of classes of complexes $R\Gamma_c(X)$ and $R\Gamma(X)$ computing cohomology with resp. without support.

The first half of the proof of [37, Prop. 2.5] contains the identification of $R(t)$ with $R\Gamma_c(X) \hookrightarrow R\Gamma(X)$. It thus remains to show that this identification is compatible with the action of $\text{CH}_{dX}(X \times_S X)$.

Thus, let $[\mathfrak{Z}] \in \text{CH}_{dX}(X \times_S X)$ be the class of a cycle on $X \times_S X$. By [22, Lemma 5.18], it can be represented by a cycle $\mathfrak{Z}$ belonging to the group $c_S(X, X)$ of finite correspondences, i.e., the projection onto the first factor $X \times_S X \to X$ is finite on $\mathfrak{Z}$. Similarly, the transposed class $t^* [\mathfrak{Z}]$ can be represented by $\mathfrak{Z}^t \in c_S(X, X)$. By [38, Thm. 2.2 (a)], the class $[\mathfrak{Z}]$ acts on $M_{gm}(X)$ via the finite correspondence $\mathfrak{Z}$. By [38, Thm. 2.2 (b), (c), Rem. 2.14], it acts on $M_{gm}^c(X)$ via the endomorphism dual to the one induced by $\mathfrak{Z}^t(d_X)[2d_X]$ under the duality isomorphism

$$M_{gm}(X)^* (d_X)[2d_X] \xrightarrow{\sim} M_{gm}^c(X)$$

[32, proof of Thm. 4.3.7 3]. We thus identify the commutative diagram

$$\begin{array}{ccc}
M_{gm}(X) & \xrightarrow{t} & M_{gm}^c(X) \\
\downarrow{[\mathfrak{Z}]} & & \downarrow{[\mathfrak{Z}]}
\end{array}$$

with

$$\begin{array}{ccc}
M_{gm}(X) & \xrightarrow{t} & M_{gm}(X)^*(d_X)[2d_X] \\
\downarrow{\mathfrak{Z}} & & \downarrow{\mathfrak{Z}^t \mathfrak{Z}}
\end{array}$$

By [10, pp. 6–7], $R$ sends $\mathfrak{Z}$ to $\mathfrak{Z}^* : R\Gamma(W) \to R\Gamma(V)$, for any finite correspondence $\mathfrak{Z}$ on the product $V \times_k W$ of two smooth $k$-schemes. It follows that $R$ sends the latter commutative diagram to the commutative diagram

$$\begin{array}{ccc}
R\Gamma(X) & \xrightarrow{\sim} & R\Gamma(X)^* (-d_X)[-2d_X] \\
\mathfrak{Z}^* \downarrow & & \uparrow{(\mathfrak{Z}^t)^*} \mathfrak{Z} \\
R\Gamma(X) & \xrightarrow{\sim} & R\Gamma(X)^* (-d_X)[-2d_X]
\end{array}$$
Now the endomorphism \(((\partial X)^\ast)(-d_X)[-2d_X]\) of \(R \Gamma(X)^\ast(-d_X)[-2d_X]\) corresponds to the endomorphism \(I(\partial X)^\ast = I[\partial X]^\ast = [\partial X]^\ast\) of \(R \Gamma_c(X)\). But this means precisely that our identification of the image under \(R\) of \(M_{gm}(X) \to M_{\text{Mgm}}(X)\) with the canonical morphism \(R \Gamma_c(X) \to R \Gamma(X)\) is compatible with the action of \(\text{CH}_{d_X}(X \times_S X)\).

Here is how the theory developed so far will be used in the sequel.

**Corollary 2.4.** Let \(k\) be a field embeddable into \(\mathbb{C}\), and \(F\) a finite direct product of fields of characteristic zero. Let \(X\) be proper and smooth over \(S \in \text{CH}_{d_X}(X \times_S X) \otimes \mathbb{Z} F\) an idempotent. Assume that the motive \(\partial M_{gm}(X)^e\) lies in \(DM_{gm}^a(k, F)\), and that \((\partial H^n(X_{\mathbb{C}}, \mathbb{Q}) \otimes \mathbb{Q} F)^e\) (in the Hodge theoretic setting) resp. \((\partial H^n(X_{\mathbb{C}}, \mathbb{Q}_l) \otimes \mathbb{Q} F)^e\) (in the \(\ell\)-adic setting) is without weights \(n - \beta, \ldots, n - (\alpha + 1), n - \alpha\), for all \(n \in \mathbb{Z}\). Then \(\partial M_{gm}(X)^e\) is without weights \(\alpha, \alpha + 1, \ldots, \beta\). If furthermore \(\alpha \leq -1\) and \(\beta \geq 0\), then Assumption 2.1 holds.

**Proof.** Apply Theorem 2.2 and Proposition 2.3. \(\square\)

### 3. Statement of the main result

In order to state our main result (Theorem 3.8), let us introduce the geometrical situation we are going to consider from now on. \(F\) is now a quadratic imaginary number field, and the base \(k\) equals \(F\). The scheme \(X\) is a power of the universal Abelian threefold over a Picard surface, and \(e\) is associated to a dominant weight \((k_1, k_2, c, d) \in \mathbb{Z}^2, k_1 \geq k_2 \geq 0\) (see below for the precise definition). Theorem 3.8 implies that in this context, Assumption 2.1 is satisfied if and only if \((k_1, k_2, c, d)\) is regular: \(k_1 > k_2 > 0\). More precisely, \(\partial M_{gm}(X)\) lies in \(DM_{gm}^a(k, F)\), and setting \(k := \min(k_1 - k_2, k_2)\), the \(e\)-part of \(\partial M_{gm}(X)\) is without weights \(-k, -(k - 1), \ldots, k - 1\). We then list the main consequences of this result (Corollaries 3.9–3.11), applying the theory developed in [36, Sect. 4]. The proof of Theorem 3.8 will be given in Sect. 4. It is an application of Corollary 2.4; in order to verify the hypotheses of the latter, we heavily rely on the main result of [2].

Denote by \(f \mapsto \tilde{f}\) the non-trivial element of the Galois group of \(F\) over \(\mathbb{Q}\). Note that there is an isomorphism of \(\mathbb{Q}\)-algebras

\[
F \otimes_{\mathbb{Q}} F \xrightarrow{\sim} F \times F, \quad q \otimes f \mapsto (q \cdot f, \tilde{q} \cdot f);
\]

it is \(F\)-linear with respect to multiplication on the right on \(F \otimes_{\mathbb{Q}} F\). For any \(F\)-vector space \(V\), this isomorphism induces a decomposition

\[
V' \otimes_{\mathbb{Q}} F = V'_+ \oplus V'_-;
\]

of the \(F\)-vector space \(V' \otimes_{\mathbb{Q}} F\) into two sub-spaces, where

\[
V'_+ := \{ v \in V' \otimes_{\mathbb{Q}} F \mid (f \otimes 1)v = (1 \otimes f)v \},
\]

\[
V'_- := \{ v \in V' \otimes_{\mathbb{Q}} F \mid (f \otimes 1)v = (1 \otimes \tilde{f})v \}.
\]

The projections \(\pi_+\) and \(\pi_-\) onto \(V'_+\) and \(V'_-\), respectively, are induced by scalars in \(F \otimes_{\mathbb{Q}} F\): for any non-zero element \(x \in F\) satisfying \(\tilde{x} = -x\), we have

\[
\pi_+ = \frac{1}{2} \left( 1 \otimes 1 + x \otimes \frac{1}{x} \right), \quad \pi_- = \frac{1}{2} \left( 1 \otimes 1 - x \otimes \frac{1}{x} \right).
\]
Conjugation \( v \otimes f \mapsto v \otimes \bar{f} \) on \( V' \otimes_{\mathbb{Q}} F \) exchanges \( V'_+ \) and \( V'_- \). The restriction of \( \pi_+ \) to \( V' = V' \otimes_{\mathbb{Q}} \mathbb{Q} \subset V' \otimes_{\mathbb{Q}} F \) induces an isomorphism \( V' \sim V'_+ \) of \( \mathbb{Q} \)-vector spaces, which is \( F \)-linear (for the given \( F \)-structure on \( V' \)). The analogous statement holds for \( \pi_- \), except that the induced isomorphism \( V' \sim V'_- \) is \( F \)-antilinear. Under these identifications, conjugation \( V'_+ \to V'_- \) corresponds to the identity on \( V' \).

Now fix a three-dimensional \( F \)-vector space \( V \), together with an \( F \)-valued Hermitian form \( J \) on \( V \) of signature \((2, 1)\).

**Definition 3.1.** ([13, 2.1]) The group scheme \( G \) over \( \mathbb{Q} \) is defined as the group of unitary similitudes

\[
G := GU(V, J) \subset \text{Res}_{F/\mathbb{Q}} \text{GL}_F(V).
\]

Thus, for any \( \mathbb{Q} \)-algebra \( R \), the group \( G(R) \) equals

\[
\{ g \in \text{GL}_F \otimes_{\mathbb{Q}} R(V \otimes_{\mathbb{Q}} R), \exists \lambda(g) \in R^\times, \; J(g \bullet, g \bullet) = \lambda(g) \cdot J(\bullet, \bullet) \}.
\]

In particular, the similitude norm \( \lambda(g) \) defines a canonical morphism

\[\lambda : G \longrightarrow \mathbb{G}_m.\]

The decomposition \( V \otimes_{\mathbb{Q}} F = V_+ \oplus V_- \), together with the morphism \( \lambda \), induces an isomorphism

\[
\phi : GF := G \times_{\mathbb{Q}} F \sim \text{GL}_F(V) \times_F \mathbb{G}_m, \; g \mapsto (g|_{V_+}, \lambda(g))
\]

[13, (2.1.1)]; here we use the above identification of \( V \) and \( V_+ \). One deduces that \( \lambda \) is an epimorphism of algebraic groups. The reader will check that under \( \phi \), conjugation \( f \mapsto \bar{f} \) on \( GF \) corresponds to the involution \((h, t) \mapsto (\bar{h}^*, \bar{t})\). Here, the map \( h \mapsto h^* \) associates to \( h \in \text{GL}_F(V) \) the adjoint automorphism \( h^* \), characterized by the formula

\[J(v_1, h^*v_2) = J(h^{-1}v_1, v_2), \; \forall v_1, v_2 \in V.\]

Under \( \phi \), the group of \( \mathbb{Q} \)-rational points of \( G \) is identified with

\[
\{(h, q) \in GL_F(V) \times_{\mathbb{Q}}^\times, \; h = qh^*\}
\]

[13, (2.1.2)]. From the definition, and from the isomorphism \( \phi \), one also deduces the following statement.

**Proposition 3.2.**

(a) The group \( G \) is split over \( F \), but not over \( \mathbb{Q} \).

(b) The center \( Z(G) \) of \( G \) equals \( \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m, F \subset \text{Res}_{F/\mathbb{Q}} \text{GL}_F(V) \) (inclusion of scalar automorphisms). In particular, it is isogeneous to the product of \( \mathbb{G}_m \) and a torus of compact type.

Thus, the irreducible algebraic representations of \( GF \) are indexed by the dominant weights of \( \text{GL}_F(V) \times_F \mathbb{G}_m, F \), which we describe now. \( F \)-split tori, together with an inclusion into a Borel subgroup of \( \text{GL}_F(V) \), are in bijection with \( F \)-bases of \( V \); fix one such basis, use it to identify \( \text{GL}_F(V) \) with \( \text{GL}_3, F \); the split torus with the subgroup \( T \) of diagonal matrices, and the Borel subgroup with the subgroup of upper triangular matrices in \( \text{GL}_3, F \). The Weyl group is the symmetric group \( \mathfrak{S}_3 \); it acts by permuting the base vectors, hence
the coordinate functions of \( T \). Let us normalize things as follows: we consider quadruples \((k_1, k_2, c, r) \in \mathbb{Z}^4\) satisfying two congruence relations:
\[
c \equiv k_1 + k_2 \mod 2, \quad r \equiv \frac{c + k_1 + k_2}{2} \mod 2.
\]
To such a quadruple, let us associate the (representation-theoretic) weight
\[
\alpha(k_1, k_2, c, r) : T \times F \rightarrow \mathbb{G}_m, \quad (\text{diag}(a, a^{-1}b, b^{-1}v), f) \mapsto a^{k_1-k_2b_k2v \frac{c(k_1+k_2)}{2} + \frac{1}{2}(r + \frac{3c-(k_1+k_2)}{2})}.
\]
Note that restriction of \( \alpha(k_1, k_2, c, r) \) to \((T \cap \text{SL}_3, F) \times F \{1\}\) corresponds to the projection onto \((k_1, k_2)\). In particular, the weight \( \alpha(k_1, k_2, c, r) \) is dominant if and only if \( k_1 \geq k_2 \geq 0 \); it is regular if and only if \( k_1 > k_2 > 0 \). Note also that the composition of \( \alpha(k_1, k_2, c, r) \) with the cocharacter
\[
\mathbb{G}_m \rightarrow T \times F \mathbb{G}_m, \quad x \mapsto (\text{diag}(x, x, x), x^2)
\]
equals
\[
\mathbb{G}_m, \quad x \mapsto x^{-r}.
\]
The determinant on \( T \) equals \( \alpha(0, 0, 2, -3) \), and \( \lambda = \alpha(0, 0, 0, -2) \).

**Definition 3.3.** The analytic space \( \mathcal{H} \) is defined as the complex open 2-ball,
\[
\mathcal{H} := \{(z_1, z_2) \in \mathbb{C}, \ |z_1|^2 + |z_2|^2 < 1\}.
\]

In order to identify the points of \( \mathcal{H} \) with certain morphisms of algebraic groups from the Deligne torus \( S \) to \( G_{\mathbb{R}} \) [2, Def. 2.5], one needs to choose an embedding of \( F \) into \( \mathbb{C} \). From now on, we thus assume such a choice of embedding to be fixed. Therefore, we get a canonical action of \( G(\mathbb{R}) \) on \( \mathcal{H} \) (by conjugation). This action is by analytical automorphisms, and it is transitive [13, (1.3.3)]. In fact, \( (G, \mathcal{H}) \) are pure Shimura data [26, Def. 2.1]. Their reflex field [26, Sect. 11.1] equals \( F \) [13, Lemma 4.2]. According to Proposition 3.2 (b), the Shimura data \((G, \mathcal{H})\) satisfy condition (+) from [35, Sect. 5].

Let us now fix additional data: (A) an open compact subgroup \( K \) of \( G(\mathbb{A}_F) \) which is neat [26, Sect. 0.6], (B) a quadruple \((k_1, k_2, c, r) \in \mathbb{Z}^4\) satisfying the above congruences
\[
c \equiv k_1 + k_2 \mod 2 \quad \text{and} \quad r \equiv \frac{c + k_1 + k_2}{2} \mod 2,
\]
and in addition,
\[
k_1 \geq k_2 \geq 0.
\]
In other words, the character \( \alpha := \alpha(k_1, k_2, c, r) \) is dominant. We also impose the condition
\[
r \geq 0.
\]
These data (A)–(B) are used as follows. The *Shimura variety* \( S := S^K(G, \mathcal{H}) \) is an object of \( \text{Sm}/F \). This is the *Picard surface* of level \( K \) associated to \( F \). According to Miyake (see [13, Prop. 4.6, Thm. 4.9]), it admits an interpretation as modular space of Abelian threefolds with additional structures, among which a complex multiplication by the ring of integers \( \mathfrak{o}_F \). In particular, there is a universal family \( A \) of Abelian threefolds over \( S \).

Denote by \( \text{Rep}(G_F) \) the Tannakian category of algebraic representations of \( G_F \) in finite dimensional \( F \)-vector spaces. The following result holds in the general context of Shimura varieties of \( PEL \)-type.
Theorem 3.4. ([1, Thm. 8.6]) Let $R \in \{\mathbb{Q}, F\}$. There is an $R$-linear tensor functor
\[
\mu = \mu_R : \text{Rep}(G_R) \longrightarrow \text{CHM}^R(S)_R.
\]
It has the following properties.
(i) The composition of $\mu$ with the cohomological Hodge theoretic realization is isomorphic to the canonical construction functor $\mu$ (e.g. [33, Thm. 2.2]) to the category of admissible graded-polarizable variations of Hodge structure on $S$.
(ii) The functor $\mu$ commutes with Tate twists.
(iii) The functor $\mu$ maps the representation $V$ to the dual of the relative Chow motive $h^1(A/\mathbb{S})$.

Given that the representation on $V$ is faithful, it follows that any relative Chow motive in the image of $\mu$ is isomorphic to a direct sum of direct factors of Tate twists of $h(A^n/\mathbb{S})$, for suitable $r_i \in \mathbb{N}$, where $A^n$ denotes the $r_i$-fold fibre product of $A$ over $S$. From the construction [1, Thm. 8.6], it follows that up to isomorphism, $\mu_F$ is obtained from $\mu_{\mathbb{Q}}$ by formally tensoring with $F$ over $\mathbb{Q}$, and passing to the pseudo-Abelian completions.

Definition 3.5. (a) Denote by $V_{\alpha} \in \text{Rep}(GF)$ the irreducible representation of highest weight $\alpha$.
(b) Define $\mathbb{E}V \in \text{CHM}^F(S)_F$ as
\[
\mathbb{E}V := \mu_F(V_{\alpha}).
\]
It will be useful to compare our parametrization of highest weights to that of [2, Sect. 4]. There, the standard basis of characters of the split torus $T \times F \mathbb{G}_{m,F}$ is used. First step: write $(a, b, \gamma, d)$ instead of $(a, b, c, d)$ as in [loc. cit.], and $\lambda(a, b, \gamma, d)$ for the associated character. Second step: identify the change of parameters $(a, b, \gamma, d) \leftrightarrow (k_1, k_2, c, r)$.

Lemma 3.6. The character $\alpha(k_1, k_2, c, r)$ of $T \times F \mathbb{G}_{m,F}$ equals
\[
\lambda\left(\frac{c + k_1 - k_2}{2}, \frac{c - k_1 + k_2}{2}, \frac{c - (k_1 + k_2)}{2}, -\frac{1}{2}\left(r + \frac{3c - (k_1 + k_2)}{2}\right)\right).
\]
The character $\alpha(a, b, \gamma, d)$ equals
\[
\alpha(a - \gamma, b - \gamma, a + b, -(a + b + \gamma + 2d)).
\]

Proof. This follows directly from the definitions. We leave the computation to the reader. $\square$

According to [2, Lemma 4.13], the Chow motive $\mathbb{E}V$ is a direct factor of $h^r(A^N/\mathbb{S})$, for a suitable large enough positive integer $N$. More precisely, the representation $V_{\alpha}$ is a direct factor of the exterior algebra $\Lambda^*(V^N)^\vee$ of the dual of the representation $V^N$. Therefore, there is an idempotent $e_{\alpha}$ acting on $\Lambda^*(V^N)^\vee$, whose image equals $V_{\alpha}$. Applying $\mu_F$, we get an idempotent acting on the relative Chow motive $h(A^N/\mathbb{S})$, equally denoted by $e_{\alpha}$, and whose image equals $\mathbb{E}V$. According to [38, Thm. 2.2 (a)], the relative Chow motive $\mathbb{E}V$ gives rise to an exact triangle
\[
\partial M_{gm}(\mathbb{E}V) \longrightarrow M_{gm}(\mathbb{E}V) \xrightarrow{\mu} M^c_{gm}(\mathbb{E}V) \longrightarrow \partial M_{gm}(\mathbb{E}V)[1]
\]
in $DM_{gm}(F)_F$. By functoriality, and by [38, Thm. 2.2 (a1)], this triangle coincides with the $e_{\alpha}$-part of the triangle
\[
\partial M_{gm}(A^N) \longrightarrow M_{gm}(A^N) \longrightarrow M^c_{gm}(A^N) \longrightarrow \partial M_{gm}(A^N)[1]
\]
denoted $(\ast)$ in Sect. 2.
Remark 3.7. The restriction \( r \geq 0 \) on our character \( \alpha = \alpha(k_1, k_2, c, r) \) is not very serious. For negative \( r \), the Chow motive \( \mathbb{Z} \mathcal{V} \) is defined the same way, the only difference being that it is not effective. Indeed, it is then a direct factor of \( h^{-r}(A^N/S)(-r) \), for some \( N \gg 0 \).

Here is our main result.

**Theorem 3.8.** The boundary motive \( \partial M_{gm}(A^N) \) lies in the triangulated sub-category \( \text{DM}^{Ab}_{gm}(F) \subset \text{DM}^{Ab}_{gm}(F)_{\mathbb{Q}} \). Its direct factor \( \partial M_{gm}(\mathbb{Z} \mathcal{V}) \) is without weights

\[-k, -(k-1), \ldots, k-1,\]

where \( k := \min(k_1 - k_2, k_2) \). Both weights \(-k+1\) and \( k \) do occur in \( \partial M_{gm}(\mathbb{Z} \mathcal{V}) \). In particular, Assumption 2.1 holds for \( \partial M_{gm}(\mathbb{Z} \mathcal{V}) \) if and only if \( k \geq 1 \), i.e., if and only if \( \alpha \) is regular.

The category \( \text{DM}^{Ab}_{gm}(F) \) being pseudo-Abelian (Corollary 1.3), the motive \( \partial M_{gm}(\mathbb{Z} \mathcal{V}) \) belongs to \( \text{DM}^{Ab}_{gm}(F) \). Theorem 3.8 should be compared to [37, Thm. 3.5], which treats the case of Hilbert-Blumenthal varieties. As here, regularity of the character is sufficient for the corresponding direct factor of the boundary motive to avoid weights \(-1\) and \( 0 \). However, as soon as the dimension of the Hilbert–Blumenthal variety is strictly greater than one, there are irregular dominant characters whose associated direct factors nonetheless avoid these weights.

Theorem 3.8 will be proved in Sect. 4. Let us give its main corollaries, assuming that \( k = \min(k_1 - k_2, k_2) \geq 1 \), i.e., \( k_1 > k_2 > 0 \). Consider the weight filtration

\[ C_{\leq -(k+1)} \rightarrow \partial M_{gm}(\mathbb{Z} \mathcal{V}) \rightarrow C_{\geq k} \rightarrow C_{\leq -(k+1)}[1] \]

avoiding weights \(-k, \ldots, k-1\) [36, Def. 1.6, Cor. 1.9]. Thus,

\[ C_{\leq -(k+1)} \in \text{DM}^{Ab}_{gm}(F, w \leq -(k+1)), \ C_{\geq k} \in \text{DM}^{Ab}_{gm}(F, w \geq k). \]

Furthermore, according to Theorem 3.8,

\[ C_{\leq -(k+1)} \notin \text{DM}^{Ab}_{gm}(F, w \leq -(k+2)), \ C_{\geq k} \notin \text{DM}^{Ab}_{gm}(F, w \geq k+1). \]

**Corollary 3.9.** ([36, Thm. 4.3]) Assume \( k_1 > k_2 > 0 \), i.e., \( k \geq 1 \).

(a) The motive \( M_{gm}(\mathbb{Z} \mathcal{V}) \) is without weights \(-k, -(k-1), \ldots, -1 \), and the motive \( M^c_{gm}(\mathbb{Z} \mathcal{V}) \) is without weights \( 1, 2, \ldots, k \). The Chow motives \( \text{Gr}_0 \, M_{gm}(\mathbb{Z} \mathcal{V}) \) and \( \text{Gr}_0 \, M^c_{gm}(\mathbb{Z} \mathcal{V}) \) [36, Prop. 2.2] are defined, and they behave functorially with respect to \( M_{gm}(\mathbb{Z} \mathcal{V}) \) and \( M^c_{gm}(\mathbb{Z} \mathcal{V}) \). In particular, any endomorphism of \( M_{gm}(\mathbb{Z} \mathcal{V}) \) induces an endomorphism of \( \text{Gr}_0 \, M_{gm}(\mathbb{Z} \mathcal{V}) \), and any endomorphism of \( M^c_{gm}(\mathbb{Z} \mathcal{V}) \) induces an endomorphism of \( \text{Gr}_0 \, M^c_{gm}(\mathbb{Z} \mathcal{V}) \).

(b) There are canonical exact triangles

\[ C_{\leq -(k+1)} \rightarrow M_{gm}(\mathbb{Z} \mathcal{V}) \xrightarrow{\pi_0} \text{Gr}_0 \, M_{gm}(\mathbb{Z} \mathcal{V}) \rightarrow C_{\leq -(k+1)}[1] \]

and

\[ C_{\geq k} \rightarrow \text{Gr}_0 \, M^c_{gm}(\mathbb{Z} \mathcal{V}) \xrightarrow{i_0} M^c_{gm}(\mathbb{Z} \mathcal{V}) \rightarrow C_{\geq k}[1], \]

which are stable under the natural action of the endomorphism rings of \( M_{gm}(\mathbb{Z} \mathcal{V}) \) and \( M^c_{gm}(\mathbb{Z} \mathcal{V}) \), respectively. In particular, weight \(-k+1\) occurs in \( M_{gm}(\mathbb{Z} \mathcal{V}) \), and weight \( k+1 \) occurs in \( M^c_{gm}(\mathbb{Z} \mathcal{V}) \).
(c) There is a canonical isomorphism $Gr_0 \text{Mgm}(\mathbb{A}V) \xrightarrow{\sim} Gr_0 M^c_{gm}(\mathbb{A}V)$ in $\text{CHM}(F)_F$. As a morphism, it is uniquely determined by the property of making the diagram

$$
\begin{array}{c}
M_{gm}(\mathbb{A}V) \xrightarrow{\mu} M^c_{gm}(\mathbb{A}V) \\
\pi_0 \downarrow \downarrow i_0 \\
Gr_0 M_{gm}(\mathbb{A}V) \xrightarrow{} Gr_0 M^c_{gm}(\mathbb{A}V)
\end{array}
$$

commute.

(d) Let $N \in \text{CHM}(F)_F$ be a Chow motive. Then $\pi_0$ and $i_0$ induce isomorphisms

$$
\text{Hom}_{\text{CHM}(F)_F}(Gr_0 M_{gm}(\mathbb{A}V), N) \xrightarrow{\sim} \text{Hom}_{DM_{gm}(F)_F}(M_{gm}(\mathbb{A}V), N)
$$

and

$$
\text{Hom}_{\text{CHM}(F)_F}(N, Gr_0 M^c_{gm}(\mathbb{A}V)) \xrightarrow{\sim} \text{Hom}_{DM_{gm}(F)_F}(N, M^c_{gm}(\mathbb{A}V)).
$$

(e) Let $M_{gm}(\mathbb{A}V) \rightarrow N \rightarrow M^c_{gm}(\mathbb{A}V)$ be a factorization of the canonical morphism $u : M_{gm}(\mathbb{A}V) \rightarrow M^c_{gm}(\mathbb{A}V)$ through a Chow motive $N \in \text{CHM}(F)_F$. Then $Gr_0 M_{gm}(\mathbb{A}V) = Gr_0 M^c_{gm}(\mathbb{A}V)$ is canonically a direct factor of $N$, with a canonical direct complement.

Proof. This is [36, Thm. 4.3]. The functoriality statements of parts (a) and (b) are more general than [loc. cit.], and follow from [36, Prop. 1.7]. \hfill \Box

Henceforth, we identify $Gr_0 M_{gm}(\mathbb{A}V)$ and $Gr_0 M^c_{gm}(\mathbb{A}V)$ via the canonical isomorphism of Corollary 3.9 (c). The equivariance statements from Corollary 3.9 (a), (b) apply in particular to cycles coming from the Hecke algebra associated to the Shimura variety $S$.

**Corollary 3.10.** Assume $k \geq 1$. Then $Gr_0 M_{gm}(\mathbb{A}V)$ carries a natural action of the Hecke algebra $\mathcal{H}(K, G(\mathbb{A}_f))$ associated to the neat open compact subgroup $K$ of $G(\mathbb{A}_f)$.

Proof. This is an application of [38, Ex. 2.16]. We refer to the proof of [37, Cor. 3.8], where the details of the construction are spelled out for Hilbert–Blumenthal varieties. The reader will have no difficulties to translate them to the present context. \hfill \Box

**Corollary 3.11.** ([36, Cor. 4.6]) Assume $k \geq 1$, and let $\mathcal{A}^N$ be any smooth compactification of $A^N$. Then $Gr_0 M_{gm}(\mathbb{A}V)$ is canonically a direct factor of the Chow motive $M_{gm}(\mathcal{A}^N)$, with a canonical direct complement.

Furthermore, [36, Thm. 4.7, Thm. 4.8] on the Hodge theoretic and $\ell$-adic realizations [15, Cor. 2.3.5, Cor. 2.3.4 and Corrigendum] apply, and tell us in particular that $Gr_0 M_{gm}(\mathbb{A}V)$ is mapped to the part of interior cohomology of $A^N$ fixed by $e_\mathcal{A}$. In particular, the $L$-function of the Chow motive $Gr_0 M_{gm}(\mathbb{A}V)$ is computed via the $e_\mathcal{A}$-part of) interior cohomology of $A^N$.

**Definition 3.12.** ([36, Def. 4.9]) Let $k \geq 1$. We call $Gr_0 M_{gm}(\mathbb{A}V)$ the $e_\mathcal{A}$-part of the interior motive of $A^N$. 

Remark 3.13. By [36, Thm. 4.14], control of the reduction of some compactification of $A^N$ implies control of certain properties of the $\ell$-adic realization of $\text{Gr}_0 \, M_{gm}(E\mathcal{V})$. To the best of the author’s knowledge, the sharpest result known about reduction of compactifications of $A^N$ is stated in [29, Sect. 1.2]: there exists a smooth compactification of $A^N$ having good reduction at each prime ideal $p$ dividing neither the level $n$ of $K$ nor the absolute discriminant $d$ of $F$. Since the argument is somewhat involved, let us reproduce it: first, according to [21, Theorems on p. 34, Sect. 8], the Picard surface $S$ and its canonical smooth toroidal compactification $\tilde{S}$ both admit models $S$ and $\tilde{S}$, respectively, which are smooth over the ring $\sigma_F[1/(nd)]$; furthermore, the $\sigma_F[1/(nd)]$-scheme $\tilde{S}$ is proper. In addition, the complement $D$ of $S$ in $\tilde{S}$ continues to be a smooth divisor, and the universal Abelian scheme $A$ over $S$ admits canonical extensions, first, to an Abelian scheme $A$ over $S$, and then, to a semi-Abelian scheme $G$ over $\tilde{S}$. The reader may find it useful to consult [6, Chap. I], in particular [6, Thm. I.3.2.4, Thm. I.5.1.1, Prop. I.5.3.4] for the proofs of these statements. Second, one applies [28, Thm. 1]: for any $N \geq 0$, there exists a compactification $\tilde{G}^N$ of $G^N$, which is regular, and proper and flat over $\tilde{S}$. In fact, looking at the construction of [loc. cit.], which uses toric compactifications, one sees that the morphism $\tilde{G}^N \to \tilde{S}$ is in fact semi-stable. Writing étale-local equations (and using that $D$ is smooth over $\sigma_F[1/(nd)]$!), one sees that $\tilde{G}^N$ is smooth over $\sigma_F[1/(nd)]$.

[36, Thm. 4.14] then yields the following: (a) for all prime ideals $0 \neq p$ not dividing $nd$, the $p$-adic realization of $\text{Gr}_0 \, M_{gm}(E\mathcal{V})$ is crystalline, (b) if furthermore $p$ and $\ell$ are coprime, then the $\ell$-adic realization of $\text{Gr}_0 \, M_{gm}(E\mathcal{V})$ is unramified at $p$.

Corollary 3.14. Let $0 \neq p$ be a prime ideal of $\sigma_F$ dividing neither the level of $K$ nor the discriminant of $F$. Let $\ell$ be coprime to $p$. Then the characteristic polynomials of the following coincide: (1) the action of Frobenius $\phi$ on the $\phi$-filtered module associated to the (crystalline) $p$-adic realization of $\text{Gr}_0 \, M_{gm}(E\mathcal{V})$, (2) the action of a geometric Frobenius automorphism at $p$ on the (unramified) $\ell$-adic realization of $\text{Gr}_0 \, M_{gm}(E\mathcal{V})$.

Proof. According to our construction, and what was recalled in Remark 3.13, there is a smooth and proper scheme $\tilde{G}^N_{F_p}$ over the residue field $\mathbb{F}_p$ of $p$, and an endomorphism $e_{\mathbb{F}_p,p}$ of the Chow motive associated to $\tilde{G}^N_{\mathbb{F}_p}$, whose images in the endomorphism rings of the realizations of $\tilde{G}^N_{\mathbb{F}_p}$ become idempotent; furthermore, the latter are projectors onto the realizations of $\text{Gr}_0 \, M_{gm}(E\mathcal{V})$. The claim thus follows from [16, Thm. 2.2]).

Remark 3.15. The case $\mathfrak{g} = \alpha(0, 0, 0, 0)$ is not covered by Corollaries 3.9–3.11. It concerns the motive and the motive with compact support of the base scheme $S$. In this situation, the best replacement of the interior motive is the intersection motive $M_{I*}(S)$ of $S$ with respect to its Baily–Borel compactification [8, 39].

4. Proof of the main result

We keep the notation of the preceding section. In order to prove Theorem 3.8, the idea is to apply the criterion from Corollary 2.4.
Theorem 4.1. For any integer $N \geq 0$, the boundary motive $\partial M_{gm}(A^N)$ lies in the triangulated sub-category $DM_{gm}^{Ab}(F)_{\mathbb{Q}}$ of $DM_{gm}(F)_{\mathbb{Q}}$.

Proof. The variety $A^N$ is a mixed Shimura variety over $S = S^K(G, \mathcal{H})$. More precisely, using the description of the morphisms $h : S \to G_{\mathbb{R}}$ from [2, Def. 2.5], the representation $V$ of $G$ is seen to be of Hodge type $((-1, 0), (0, -1))$ in the sense of [26, Sect. 2.16]. The same statement is then true for the $r$-th power $V^N$ of $V$. By [26, Prop. 2.17], this allows for the construction of the unipotent extension $(P^N, \tilde{X}^N)$ of $(G, \mathcal{H})$ by $V^N$.

The pair $(P^N, \tilde{X}^N)$ constitute mixed Shimura data [26, Def. 2.1]. By construction, they come endowed with a morphism $\pi^N : (P^N, \tilde{X}^N) \to (G, \mathcal{H})$ of Shimura data, identifying $(G, \mathcal{H})$ with the pure Shimura data underlying $(P^N, \tilde{X}^N)$. In particular, $(P^N, \tilde{X}^N)$ also satisfy condition $(+)$ from [35, Sect. 5].

Now there is an open compact neat subgroup $K^N$ of $P^N(\mathbb{A}_f^\infty)$, whose image under $\pi^N$ equals $K$, and such that $A^N$ is identified with the mixed Shimura variety $S^{K^N}(P^N, \tilde{X}^N)$ [26, Sect. 3.22, Thm. 11.18 and 11.16]. Furthermore, the morphism $\pi^N$ of Shimura data induces a morphism $S^{K^N}(P^N, \tilde{X}^N) \to S^K(G, \mathcal{H})$, which is identified with the structure morphism of $A^N$.

In order to obtain control on the boundary motive of $A^N$, we choose a smooth toroidal compactification $A^N\tilde{}$. It is associated to a $K^N$-admissible complete smooth cone decomposition $\mathcal{G}$, i.e., a collection of subsets of

$$C(P^N, \tilde{X}^N) \times P^N(\mathbb{A}_f^\infty)$$

satisfying the axioms of [26, Sect. 6.4]. Here, $C(P^N, \tilde{X}^N)$ denotes the conical complex associated to $(P^N, \tilde{X}^N)$ [26, Sect. 4.24].

We refer to [26, 9.27, 9.28] for criteria sufficient to guarantee the existence of the associated compactification $A^N\tilde{} := S^{K^N}(P^N, \tilde{X}^N, \mathcal{G})$. It comes equipped with a natural (finite) stratification into locally closed strata. The unique open stratum is $A^N$. Any stratum $A^N\tilde{}_\sigma$ different from the generic one is associated to a rational boundary component $(P_1, \tilde{X}_1)$ of $(P^N, \tilde{X}^N)$ [26, Sect. 4.11] which is proper, i.e., unequal to $(P^N, \tilde{X}^N)$.

First, co-localization for the boundary motive [34, Cor. 3.5] tells us that $\partial M_{gm}(A^N)$ is a successive extension of (shifts of) objects of the form

$$M_{gm}(A^N\tilde{}_\sigma, i^!_\sigma j^*_\mathbb{Z}).$$

Here, $j$ denotes the open immersion of $A^N$ into $A^N\tilde{}$, $i_\sigma$ runs through the immersions of the strata $A^N\tilde{}_\sigma$ different from $A^N$, and $M_{gm}(A^N\tilde{}_\sigma, i^!_\sigma j^*_\mathbb{Z})$ is the motive of $A^N\tilde{}_\sigma$ with coefficients in $i^!_\sigma j^*_\mathbb{Z}$ defined in [34, Def. 3.1].

Next, by [35, Thm. 6.1], there is an isomorphism

$$M_{gm}(A^N\tilde{}_\sigma, i^!_\sigma j^*_\mathbb{Z}) \simto \text{Hom}(\mathbb{Z}(\sigma), M_{gm}(S^{K_1}(P_1, \tilde{X}_1)))[\text{dim } \sigma].$$

Recall [35, p. 971] that the group of orientations $\mathbb{Z}(\sigma)$ is (non-canonically) isomorphic to $\mathbb{Z}$, hence

$$\text{Hom}(\mathbb{Z}(\sigma), M_{gm}(S^{K_1}(P_1, \tilde{X}_1))) \cong M_{gm}(S^{K_1}(P_1, \tilde{X}_1)).$$

$S^{K_1}(P_1, \tilde{X}_1)$ is a Shimura variety associated to the data $(P_1, \tilde{X}_1)$ and an open compact neat subgroup $K_1$ of $P_1(\mathbb{A}_f^\infty)$. In order to show our claim, we are thus reduced to showing...
that $M_{gm}(S^{K_1})$ is an object of $DM_{gm}^{Ab}(F)_{Q}$, for any Shimura variety $S^{K_1} = S^{K_1}(P_1, X_1)$ associated to a proper rational boundary component $(P_1, X_1)$ of $(P^N, X^N)$, and any open compact neat subgroup $K_1$ of $P_1(\mathbb{A}_f)$.

Given that $P^N$ is a unipotent extension of $G$, the pure Shimura data underlying $(P_1, X_1)$ coincides with the pure Shimura data underlying some proper rational boundary component of $(G, \mathcal{H})$. These boundary components are determined in [2, Sect. 3]. In particular [2, Lemma 3.8, Prop. 3.12], the pure Shimura data $(G_1, \mathcal{H}_1)$ underlying any such component all equal $(\text{Res}_{F/Q} \mathbb{G}_{m, F}, \{\ast\})$, where $\{\ast\}$ is a single point. Altogether, we see that $(P_1, X_1)$ is a unipotent extension of $(\text{Res}_{F/Q} \mathbb{G}_{m, F}, \{\ast\})$.

We are ready to conclude. As follows directly from the definition of the canonical model (emp. [26, Sect. 11.3, 11.4]), the pure Shimura variety $S^{\pi N}(K_1)(\text{Res}_{F/Q} \mathbb{G}_{m, F}, \{\ast\})$ underlying $S^{K_1}$ equals the spectrum of a finite (Abelian) field extension $C$ of $F$. Consider the factorization of $\pi^N : (P_1, X_1) \to (G_1, \mathcal{H}_1)$ corresponding to the weight filtration $1 \subset U \subset W$ of the unipotent radical $W$ of $P_1$. It gives the following:

$$(P_1, X_1) \xrightarrow{\pi_1} (P_1', X_1') := (P_1, X_1)/U \xrightarrow{\pi_a} (G_1, \mathcal{H}_1)$$

On the level of Shimura varieties, we get:

$$S^{K_1} \xrightarrow{\pi_t} S^{\pi_1(K_1)} = S^{\pi_1(K_1)}(P_1', X_1') \xrightarrow{\pi_a} S^{\pi N}(K_1)(\text{Res}_{F/Q} \mathbb{G}_{m, F}, \{\ast\})$$

By [26, 3.12–3.22 (a), Prop. 11.10], $\pi_a$ is in a natural way a torsor under an Abelian variety, while $\pi_t$ is a torsor under a split torus. After base change over $S^{\pi N}(K_1) = S^{\pi N}(K_1)(\text{Res}_{F/Q} \mathbb{G}_{m, F}, \{\ast\})$ to the algebraic closure of $C$, the morphism $\pi_a$ thus becomes isomorphic to an Abelian variety; in particular,

$$M_{gm}(S^{\pi_t(K_1)}) \in CHM^{Ab}(F)_{Q}.$$ 

As for $S^{K_1}$, it is the fibre product over $S^{\pi_i(K_1)}$ of a finite number $m$ of $S^{\pi_i(K_1)}$-schemes, all of which are isomorphic to complements $\mathfrak{L}^*_i$ of the zero section in line bundles $\mathfrak{L}_i$ over $S^{\pi_i(K_1)}$, $i = 1, \ldots, m$.

Let $Y$ be an $S^{\pi_i(K_1)}$-scheme, and $i \in \{1, \ldots, m\}$. Homotopy invariance and the Mayer-Vietoris axiom [32, Sect. 2.2] together imply that the canonical morphism

$$M_{gm}(\mathfrak{L}_i \times_{S^{\pi_i(K_1)}} Y) \to M_{gm}(Y)$$

is an isomorphism. Furthermore, the Gysin triangle

$$M_{gm}(Y)(1)[1] \to M_{gm}(\mathfrak{L}_i^* \times_{S^{\pi_i(K_1)}} Y) \to M_{gm}(\mathfrak{L}_i \times_{S^{\pi_i(K_1)}} Y) \to M_{gm}(Y)(1)[2]$$

associated to the zero section of $\mathfrak{L}_i$ is exact [32, Prop. 3.5.4]. Together, the two statements imply that if $M_{gm}(Y)$ belongs to $DM_{gm}^{Ab}(F)_{Q}$, then so does $M_{gm}(\mathfrak{L}_i^* \times_{S^{\pi_i(K_1)}} Y)$. □

**Remark 4.2.** A more detailed analysis of the proof reveals that the boundary motive $\partial M_{gm}(A^N)$ actually lies in the full triangulated sub-category of $DM_{gm}^{Ab}(F)_{Q}$ generated by Tate twists and Chow motives $M_{gm}(X)$ associated to (proper) schemes $X \in \text{Sm}/F$, whose base change to $\bar{F}$ is isomorphic to a finite disjoint union of products of elliptic curves with complex multiplication by $F$. 
In order to apply the results from Sect. 2, we need to identify the weights occurring in the Hodge structure on the $e_{\underline{a}}$-part of the boundary cohomology of $A^N$,

$$\left(\partial H^n(A^N(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F\right)^{e_{\underline{a}}} = 0,$$

for all integers $n$.

**Proposition 4.3.** There is a canonical isomorphism of Hodge structures

$$\left(\partial H^n(A^N(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F\right)^{e_{\underline{a}}} \cong \partial H^{n-r}(S(\mathbb{C}), \mu(V_{\underline{a}}))$$

for all integers $n$.

**Proof.** Recall that the representation $V_{\underline{a}}$ is a direct factor, via the idempotent $e_{\underline{a}}$, of the exterior algebra $\Lambda^*(V^N)^{\vee}$ of the dual of the representation $V^N$. Applying the cohomological Hodge theoretic realization $\mu$, we get an idempotent acting on all relative cohomology objects $\mathcal{H}^q(A^N/S)$, whose image equals $\mu(V_{\underline{a}})$ if $q = r$, and zero otherwise.

Consider the spectral sequence of Hodge structures

$$E_2^{p,q} = \partial H^p(S(\mathbb{C}), \mathcal{H}^q(A^N/S)) \Longrightarrow \partial H^{p+q}(A^N(\mathbb{C}), \mathbb{Q}).$$

It is compatible with the action of the Chow group

$$\text{CH}_{3N+2}(A^N \times_S A^N) \otimes_{\mathbb{Q}} F = \text{End}_{CHM^*(S)}(h(A^N/S)).$$

Note that on the $E_2$-terms of the spectral sequence, the Chow group acts only on $\mathcal{H}^*(A^N/S)$. Thus, for the fixed part under $e_{\underline{a}}$, we obtain a spectral sequence

$$\partial H^p(S(\mathbb{C}), \mathcal{H}^q(A^N/S)^{e_{\underline{a}}}) \Longrightarrow \left(\partial H^{p+q}(A^N(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F\right)^{e_{\underline{a}}}.$$

But $\mathcal{H}^q(A^N/S)^{e_{\underline{a}}} = 0$ for $q \neq r$, while $\mathcal{H}^r(A^N/S)^{e_{\underline{a}}} = \mu(V_{\underline{a}})$. $\square$

**Remark 4.4.** The same proof show that

$$\left(H^n(A^N(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F\right)^{e_{\underline{a}}} \cong H^{n-r}(S(\mathbb{C}), \mu(V_{\underline{a}}))$$

for all integers $n$, and likewise for cohomology with compact support. Saper’s vanishing theorem [30, Thm. 5] says that if $\underline{a}$ is regular, then the groups $H^{n-r}(S(\mathbb{C}), \mu(V_{\underline{a}}))$ vanish for $n < r + 2$. By duality, one obtains that $H^{n-r}_c(S(\mathbb{C}), \mu(V_{\underline{a}})) = 0$ for $n > r + 2$. It follows that interior cohomology

$$\left(H^n_1(A^N(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} F\right)^{e_{\underline{a}}} \cong H^{r+n-2}_1(S(\mathbb{C}), \mu(V_{\underline{a}}))$$

vanishes unless $n = r + 2$. By comparison, the analogous statement is true for $\ell$-adic cohomology. In other words, the realizations of $\text{Gr}_0 M_{gm}(\mathbb{Z})$ are concentrated in the single cohomological degree $r + 2$ if $\underline{a}$ is regular, and they take the values

$$\left(H^{r+n}_1(A_F^N, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} F\right)^{e_{\underline{a}}} \cong H^{r}_2(S_F, \mu_\ell(V_{\underline{a}}))$$

(in the Hodge theoretic setting) resp.

$$\left(H^{r+n}_1(A_F^N, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} F\right)^{e_{\underline{a}}} \cong H^{r}_2(S_F, \mu_\ell(V_{\underline{a}}))$$

(in the $\ell$-adic setting); see e.g. [27, Sect. (4.1)] for the $\ell$-adic version $\mu_\ell$ of the canonical construction functor.
Proof of Theorem 3.8. According to Proposition 4.3, we need to control $\partial H^{n-r}(S(\mathbb{C}), \mu(V_{\mathfrak{A}}))$. We use the Baily–Borel compactification $S^\ast$ of $S$. The complement of $S$ consists of finitely many cusps; the boundary cohomology of $S(\mathbb{C})$ therefore coincides with the direct sum over the cusps of the degeneration of the coefficients to the boundary of $S^\ast$.

The first claim of Theorem 3.8 is Theorem 4.1. Fix a cusp of $S^\ast$, denote by $j$ the open immersion of $S$, and by $i$ the closed immersion of the cusp into $S^\ast$. We need to know the weights occurring in

$$i^* R^{n-r} j_*(\mu(V_{\mathfrak{A}})).$$

Recall (Lemma 3.6) that

$$\mathfrak{a} = \lambda \left( \frac{c + k_1 - k_2}{2}, \frac{c - k_1 + k_2}{2}, \frac{c - (k_1 + k_2)}{2}, -\frac{1}{2} \left( r + \frac{3c - (k_1 + k_2)}{2} \right) \right)$$

in the parametrization of [2, Sect. 4]. The main result of [2] implies that

1. $0 \neq i^* R^0 j_*(\mu(V_{\mathfrak{A}}))$ is of weight $r - k_1$,
2. $0 \neq i^* R^1 j_*(\mu(V_{\mathfrak{A}}))$ is of weights $(r + 1) - k_2$ and $(r + 1) - (k_1 - k_2)$,
3. $0 \neq i^* R^2 j_*(\mu(V_{\mathfrak{A}}))$ is of weights $(r + 2) + k_2 + 1$ and $(r + 2) + (k_1 - k_2) + 1$,
4. $0 \neq i^* R^3 j_*(\mu(V_{\mathfrak{A}}))$ is of weight $(r + 3) + k_1 + 1$,

and that $i^* R^m j_*(\mu(V_{\mathfrak{A}})) = 0$ whenever $m < 0$ or $m > 3$ [2, Thm. 1.2].

Therefore, $\partial H^{n-r}(S(\mathbb{C}), \mu(V_{\mathfrak{A}}))$ is without weights $n - (k_1 - 1), \ldots, n + k - 1, n + k$, where $k = \min(k_1 - k_2, k_2)$, while weight $n - k$ occurs for $n = r + 1$, and weight $n + k + 1$ for $n = r + 2$. Our claim thus follows from Proposition 4.3 and Corollary 2.4. 

5. The motive for an automorphic form

This final section contains the analogues for Picard surfaces of the main results from [31]. Since we shall not restrict ourselves to the case of Hecke eigenforms, our notation becomes a little more technical than in [loc. cit.]; we chose however to keep it compatible with the one used in [20].

We continue to consider the situation of Sects. 3 and 4. In particular, we fix a dominant $\mathfrak{a} = \alpha(k_1, k_2, c, r)$, which we assume to be regular, i.e., $k_1 > k_2 > 0$. Consider the Chow motive $\text{Gr}_0 M_{gm}(\mathbb{Z} \mathfrak{V})$. According to Remark 4.4, its Hodge theoretic realization equals

$$R \left( \text{Gr}_0 M_{gm}(\mathbb{Z} \mathfrak{V}) \right) = H^2_! \left( S(\mathbb{C}), \mu(V_{\mathfrak{A}}) \right) \left[ -(r + 2) \right].$$

By Corollary 3.10, the Hecke algebra $\mathcal{H}(K, G(\mathfrak{A}_f))$ acts on $\text{Gr}_0 M_{gm}(\mathbb{Z} \mathfrak{V})$.

**Theorem 5.1.** ([14, Chap. 2, Thm. 2, p. 50]) Let $L$ be any field extension of $F$. Then the $\mathcal{H}(K, G(\mathfrak{A}_f)) \otimes_F L$-module $H^2_! \left( S(\mathbb{C}), \mu(V_{\mathfrak{A}}) \right) \otimes_F L$ is semi-simple.

Note that [14, Chap. 3, Sect. 4.3.5] gives a proof of Theorem 5.1, while the statement in [14, Chap. 2, Thm. 2, p. 50] is “non-adelic”. Denote by $R(\mathfrak{S}) := R(\mathcal{H}(K, G(\mathfrak{A}_f)))$ the image of the Hecke algebra in the endomorphism algebra of $H^2_! \left( S(\mathbb{C}), \mu(V_{\mathfrak{A}}) \right)$.

**Corollary 5.2.** Let $L$ be any field extension of $F$. Then the $L$-algebra $R(\mathfrak{S}) \otimes_F L$ is semi-simple.
In particular, the isomorphism classes of simple right $R(\mathfrak{S}) \otimes_F L$-modules correspond bijectively to isomorphism classes of minimal right ideals.

Fix $L$, and let $Y_{\pi f}$ be such a minimal right ideal of $R(\mathfrak{S}) \otimes_F L$. There is a (primitive) idempotent $e_{\pi f} \in R(\mathfrak{S}) \otimes_F L$ generating $Y_{\pi f}$. Recall the following definition.

**Definition 5.3.** ([20, p. 291]) The Hodge structure $W(\pi f)$ associated to $Y_{\pi f}$ is defined as

$$W(\pi f) := \text{Hom}_{R(\mathfrak{S}) \otimes_F L} \left( Y_{\pi f}, H^2_1(S(\mathbb{C}), \mu(V_\underline{\alpha}) \otimes_F L) \right).$$

The Galois module $W(\pi f)_\ell$ associated to $Y_{\pi f}$ is defined analogously. In order to define a motivic object whose realizations equal $W(\pi f)$ and $W(\pi f)_\ell$, respectively, one uses the idempotent generator $e_{\pi f}$ of $Y_{\pi f}$.

**Proposition 5.4.** There is a canonical isomorphism of Hodge structures

$$W(\pi f) \xrightarrow{\sim} (H^2_1(S(\mathbb{C}), \mu(V_\underline{\alpha}) \otimes_F L) \cdot e_{\pi f}).$$

**Proof.** Obviously,

$$\text{Hom}_{R(\mathfrak{S}) \otimes_F L} \left( R(\mathfrak{S}) \otimes_F L, H^2_1(S(\mathbb{C}), \mu(V_\underline{\alpha}) \otimes_F L) \right)$$

is canonically identified with

$$H^2_1(S(\mathbb{C}), \mu(V_\underline{\alpha}) \otimes_F L)$$

by mapping an morphism $g$ to the image of $1 = 1_{R(\mathfrak{S})}$ under $g$. Inside

$$\text{Hom}_{R(\mathfrak{S}) \otimes_F L} \left( R(\mathfrak{S}) \otimes_F L, H^2_1(S(\mathbb{C}), \mu(V_\underline{\alpha}) \otimes_F L) \right),$$

the object $W(\pi f)$ contains precisely those morphisms $g$ vanishing on $1 - e_{\pi f}$, in other words, satisfying the relation

$$g(1) = g(e_{\pi f}) = g(1) \cdot e_{\pi f}.$$ 

\[\square\]

An analogous statement holds in the context of Galois modules.

Since we do not know whether the Chow motive $\text{Gr}_0 M_{gm}(\mathcal{E} \mathcal{V})$ is finite dimensional, we cannot apply [18, Cor. 7.8], and therefore do not know whether $e_{\pi f}$ can be lifted idempotently to the Hecke algebra $\mathfrak{S}(K, G(\mathcal{A}, f))$. This is why we need to descend to the level of Grothendieck motives. Denote by $\text{Gr}_0 M_{gm}(\mathcal{E} \mathcal{V})'$ the Grothendieck motive underlying $\text{Gr}_0 M_{gm}(\mathcal{E} \mathcal{V})$.

**Definition 5.5.** Assume $\underline{\alpha} = \alpha(k_1, k_2, c, r)$ to be regular. Let $L$ be a field extension of $F$, and $Y_{\pi f}$ a minimal right ideal of $R(\mathfrak{S}) \otimes_F L$. The motive associated to $Y_{\pi f}$ is defined as

$$\mathcal{W}(\pi f) := e_{\pi f} \cdot \text{Gr}_0 M_{gm}(\mathcal{E} \mathcal{V})'.$$

Definition 5.5 should be compared to [31, Sect. 4.2.0]. Given our construction, the following is obvious (recall that the realizations are contravariant).
**Theorem 5.6.** Assume \( \alpha = \alpha(k_1, k_2, c, r) \) to be regular, i.e., \( k_1 > k_2 > 0 \). Let \( L \) be a field extension of \( F \), and \( Y_{\pi_f} \) a minimal right ideal of \( R(\mathfrak{H}) \otimes_F L \). The realizations of the motive \( \mathcal{W}(\pi_f) \) associated to \( Y_{\pi_f} \) are concentrated in the single cohomological degree \( r+2 \), and they take the values \( \mathcal{W}(\pi_f) \) (in the Hodge theoretic setting) resp. \( \mathcal{W}(\pi_f)_\ell \) (in the \( \ell \)-adic setting).

A special case occurs when \( Y_{\pi_f} \) is of dimension one over \( L \), i.e., corresponds to a non-trivial character of \( R(\mathfrak{H}) \) with values in \( L \). The automorphic form is then an eigenform for the Hecke algebra. This is the analogue of the situation considered in [31] for elliptic cusp forms.

The motive \( \mathcal{W}(\pi_f) \) being a direct factor of \( \text{Gr}_{0} M_{gm}(\mathbb{A}^\vee)' \), our results on the latter from Sect. 3 have obvious consequences for the realizations of \( \mathcal{W}(\pi_f) \).

**Corollary 5.7.** Assume \( \alpha = \alpha(k_1, k_2, c, r) \) to be regular. Let \( L \) be a field extension of \( F \), and \( Y_{\pi_f} \) a minimal right ideal of \( R(\mathfrak{H}) \otimes_F L \). Let \( 0 \neq \mathfrak{p} \) be a prime ideal of \( \mathfrak{o}_F \) dividing neither the level of \( K \) nor the discriminant of \( F \). Let \( \ell \) be coprime to \( \mathfrak{p} \).

(a) The \( \mathfrak{p} \)-adic realization \( \mathcal{W}(\pi_f)_\mathfrak{p} \) of \( \mathcal{W}(\pi_f) \) is crystalline.

(b) The \( \ell \)-adic realization \( \mathcal{W}(\pi_f)_\ell \) of \( \mathcal{W}(\pi_f) \) is unramified at \( \mathfrak{p} \).

(c) The characteristic polynomials of the following coincide: (1) the action of Frobenius \( \phi \) on the \( \phi \)-filtered module associated to \( \mathcal{W}(\pi_f)_\mathfrak{p} \), (2) the action of a geometric Frobenius automorphism at \( \mathfrak{p} \) on \( \mathcal{W}(\pi_f)_\ell \).

**Proof.** As for (c), in order to apply [16, Thm. 2.2)], use that both realizations are cut out by the same cycle from the cohomology of a smooth and proper scheme over the residue field \( \mathbb{F}_\mathfrak{p} \) of \( \mathfrak{p} \) (cmp. the proof of Corollary 3.14).

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