An Algebraic Characterization of Vacuum States in Minkowski Space. III. Reflection Maps

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Abstract

Employing the algebraic framework of local quantum physics, vacuum states in Minkowski space are distinguished by a property of geometric modular action. This property allows one to construct from any locally generated net of observables and corresponding state a continuous unitary representation of the proper Poincaré group which acts covariantly on the net and leaves the state invariant. The present results and methods substantially improve upon previous work. In particular, the continuity properties of the representation are shown to be a consequence of the net structure, and surmised cohomological problems in the construction of the representation are resolved by demonstrating that, for the Poincaré group, continuous reflection maps are restrictions of continuous homomorphisms.

1 Introduction

A basic conceptual problem in local quantum physics\cite{12} is the determination of the spacetime symmetry, causality and stability properties of a theory from the structure of the observable algebras associated with spacetime regions. Within this general setting, it seems unnatural to appeal from the outset to symmetry properties of a theory — such as the action of a spacetime isometry group upon the states and observables — which are absent in generic spacetimes. Instead, the pertinent notions characterizing specific physical systems ought to be based on the states and observables of the system, and possible symmetry and stability properties should be deduced, not posited.

Light was shed on these matters by a condition of geometric modular action (CGMA), proposed in\cite{4,7} and briefly recalled in Section 4, which is designed
to characterize those elements in the state space of a quantum system which admit an interpretation as a “vacuum”. This condition is expressed in terms of the modular conjugations associated to any given family of algebras paired with suitable subregions (wedges) of the underlying space–time and any states by the Tomita–Takesaki modular theory, cf. [2, 13]. It thereby can be applied, in principle, to theories on any space–time manifold. For a motivation of this condition and applications to theories in Minkowski, de–Sitter, anti-de Sitter and a class of Robertson–Walker space–times, we refer the interested reader to [7, 5, 8, 6].

In the present article we revisit the case of Minkowski space theories and resolve some intriguing questions left open by our previous work in [7, 5]. The basic ingredients in that investigation are an isotonous map (henceforth, a net) $W \mapsto \mathcal{R}(W)$ from the family of wedge-shaped regions $W \subset \mathbb{R}^4$, bounded by two characteristic planes, to von Neumann algebras $\mathcal{R}(W)$ on a Hilbert space $\mathcal{H}$, and a state vector $\Omega \in \mathcal{H}$ complying with the CGMA. The modular conjugation associated to any given pair $(\mathcal{R}(W), \Omega)$ was shown to have the geometrical meaning of a reflection $\lambda$ about the edge of $W$. More precisely, denoting this conjugation by $J(\lambda)$, one has for any wedge $W_0$ the equation [7]

$$J(\lambda)\mathcal{R}(W_0)J(\lambda) = \mathcal{R}(\lambda W_0). \quad (1.1)$$

This implies, in particular, that $J(\lambda)$ is also the modular conjugation of the pair $(\mathcal{R}(W'), \Omega)$, where $W' = \lambda W$ denotes the causal complement of $W$; so the net satisfies wedge duality. Moreover, one has

$$J(\lambda)J(\lambda_0)J(\lambda) = J(\lambda\lambda_0\lambda), \quad (1.2)$$

in an obvious notation. As the reflections $\lambda$ generate the proper Poincaré group $\mathcal{P}_+$, these relations lead naturally to the question of whether products of the conjugations $J(\lambda)$ generate an (anti) unitary representation of $\mathcal{P}_+$ which acts covariantly on the net.

This question was answered in the affirmative in [7, 11] under a technical assumption of net continuity. It follows from that assumption that the map

$$\lambda \mapsto J(\lambda) \quad (1.3)$$

from the reflections in $\mathcal{P}_+$ into the group of (anti)unitary operators is continuous. This information, together with relation (1.2), implies that there is a continuous projective representation of $\mathcal{P}_+$ on $\mathcal{H}$ with coefficients in the center of the group $\mathcal{J}$ generated by all conjugations $J(\lambda)$. By an application of Moore cohomology theory, this projective representation lifted to a true representation and the center of $\mathcal{J}$ turned out to be trivial [7].

These latter results suggested that viewing the problem cohomologically was misleading and obscured the presence of an extremely rigid structure encoded in the modular conjugations. It was also desirable to clarify the conceptual status of the technical assumptions underlying the crucial continuity property of the map.
A first step towards the clarification of these points was taken in [5]. There it was shown without any a priori continuity assumptions that a continuous unitary representation of the subgroup of translations acting covariantly on the net can be constructed from the modular conjugations. In the present investigation we want to extend this result to the full proper Poincaré group $P_+$. 

We shall restrict our attention here to nets $W \mapsto \mathcal{R}(W)$ which are locally generated, a case of particular interest being the situation where each $\mathcal{R}(W)$ is the inductive limit of algebras $\mathcal{R}(C)$ associated to double cones $C \subset W$. In fact, this condition is already satisfied if $\Omega$ is cyclic for the algebras $\mathcal{R}(C)$ and satisfies, in addition to the CGMA, a modular stability condition (CMS), recalled in Section 4, which was proposed in [7] for the characterization of stable states. We shall show under these latter two conditions that the map (1.3) provided by the CGMA is continuous.

Our second result clarifies the nature of continuous maps (1.3) of reflections $\lambda \in P_+$ into the topological group $J$, which satisfy the basic relations $J(\lambda)^2 = 1$ and $J(\lambda)J(\lambda_0)J(\lambda) = J(\lambda_0)J(\lambda)$ for any pair of reflections $\lambda, \lambda_0 \in P_+$. We shall show that such reflection maps are restrictions of continuous homomorphisms from $P_+$ into $J$. Phrased differently, any reflection map can be extended uniquely to a true continuous (anti)unitary representation $U$ of $P_+$ which, in view of (1.1), acts covariantly upon the net.

Thus the outcome of the present investigation is the insight that any vector $\Omega$ which is cyclic for the local algebras and complies with the CGMA and the CMS is a vacuum state which is invariant under a continuous (anti)unitary representation of $P_+$ acting covariantly on the net. No further assumptions are needed for the proof of this result. That this arises in the manner shown here provides further evidence that the modular involutions are fundamental objects encoding crucial physical data, which include the causal structure of the theory, its dynamics, and the action of the isometry group upon the observables. Even the space–time itself can be found to be encoded in the modular involutions in certain cases [16].

Our paper is organized as follows. In the next section we consider continuous reflection maps from the proper Lorentz group into an arbitrary topological group and show that they are restrictions of continuous homomorphisms. The continuity of the reflection maps arising in the present context is established in Section 3. These results are combined in Section 4 with theorems we have previously established to yield the desired characterization of Poincaré covariant vacuum states in Minkowski space in terms of the modular objects.

2 Reflection maps and homomorphisms

In this section we study continuous reflection maps from the proper Lorentz group into an arbitrary topological group. We shall show that any such map is the restriction of a continuous homomorphism. Combining this information with results
obtained in [5], this feature can be established also for reflection maps on the proper Poincaré group. In fact, similar results hold for many other groups, suggesting the possibility of a general theorem about reflection maps. This would be of interest in the general context of the CGMA, where reflection maps appear naturally [7], and in the application of the CGMA to other space–times. But we shall not address the general problem here.

2.1 Group theoretical considerations

Let \( \mathbb{L}_+ \) be the proper Lorentz group and \( \mathbb{L}^\uparrow_+ \) be its orthochronous subgroup. Fixing a Lorentz system with proper coordinates \((x_0, \mathbf{x}) \in \mathbb{R}^4\) and metric in diagonal form \(g = \text{diag}(1, -1, -1, -1)\), one can uniquely decompose any \( \Lambda \in \mathbb{L}^\uparrow_+ \) into a rotation \( R \) in the time-zero plane and a boost (velocity transformation) \( B \),

\[
\Lambda = RB. \tag{2.1}
\]

**Remark:** This formula is simply the polar decomposition of \( \Lambda \) in the space \( M(4, \mathbb{R}) \) of real four–by–four matrices. In particular, any Lorentz transformation which is represented by a positive matrix is a boost. This well–known fact recently received some attention again, cf. [14, 17].

Thus, any \( \Lambda \in \mathbb{L}^\uparrow_+ \) generically fixes two spatial directions, the axis of revolution \( \mathbf{r} \) of \( R \) and the boost direction \( \mathbf{b} \) of \( B \). We adopt the convention that \( \mathbf{r}, \mathbf{b} \) are normalized and that rotations are performed by an angle less than or equal to \( \pi \) about \( \mathbf{r} \) in the counterclockwise direction. So \( \mathbf{r} \) is fixed unless \( R = 1 \) or \( R \) is a rotation through the angle \( \pi \), where \( \mathbf{r} \) is fixed only up to a sign. Similarly, unless \( B = 1 \), the direction of \( \mathbf{b} \) is fixed by the condition that the lightlike vector \((1, \mathbf{b})\) is an eigenvector of \( B \) corresponding to its eigenvalue which is larger than 1.

Making use of this convention, we want to show that any \( \Lambda \) can be represented as the product of two reflections about the edges of suitable wedges. Although there are results in the mathematical literature which establish that any element of the Lorentz group can be written as the product of two involutions, the reflections we must employ are restricted to lie in a single conjugacy class of these involutions. We are therefore obliged to provide a proof of this fact here.

We begin by defining for given unit vector \( \mathbf{e} \in \mathbb{R}^3 \) the wedge

\[
W_e = \{ x \in \mathbb{R}^4 \mid x \cdot \mathbf{e} > |x_0| \} \tag{2.2}
\]

and the involution \( \lambda_e \in \mathbb{L}_+ \) inducing the reflection about its edge, i.e.

\[
\lambda_e (1, 0) = -(1, 0), \quad \lambda_e (0, e) = -(0, e), \quad \lambda_e (0, e_\perp) = (0, e_\perp), \tag{2.3}
\]

where the latter equality holds for any \( e_\perp \) which is perpendicular to \( e \). One finds that if \( \mathbf{b} \cdot \mathbf{e} = 0 \) and \( B \) is any boost in the direction of \( \mathbf{b} \), then \( \lambda_e B = B^{-1} \lambda_e \) is a reflection about the edge of the boosted wedge \( B^{-1/2} W_e \). Similarly, if \( \mathbf{r} \cdot \mathbf{e} = 0 \) and \( R \) is any rotation about the direction of \( \mathbf{r} \), then \( R \lambda_e = \lambda_e R^{-1} \) is a reflection about
the edge of $R^{1/2}W_e$, where $R^{1/2}$ is the rotation about $r$ through half the angle of $R$. Thus, choosing for given $R$, $B$ the direction $e$ such that $r \cdot e = b \cdot e = 0$, $R \lambda_e$ as well as $\lambda_e B$ are reflections about the edges of wedges and $R \lambda_e \lambda_e B = R B = \Lambda$.

In the following we shall call the involutions $\lambda$ inducing reflections about the edges of wedges simply reflections, for short, and we shall denote by $R$ the set of all such reflections. We have therefore just proved the following result which has been proven independently by Ellers [10] using a very different argument.

**Lemma 2.1** Every element of the proper orthochronous Lorentz group can be written as a product of two reflections, i.e. for every $\Lambda \in \mathcal{L}_+^1$ there exist two elements $\lambda_1, \lambda_2 \in R$ such that $\Lambda = \lambda_1 \lambda_2$.

This result is crucial in our investigation of reflection maps on the Lorentz group, and a similar result is likely to be just as important in any attempt to generalize our results to other groups. We refer the interested reader to the recent paper of Ellers [10] for a beginning of such a program.

We shall discuss the ambiguities involved in this representation. Let $\Lambda \in \mathcal{L}_+^1$ be given and let $\lambda_1, \lambda_2 \in R$ be reflections such that $\lambda_1 \lambda_2 = \Lambda$. If $\Lambda' \in \mathcal{L}_+^1$ is another reflection satisfying the latter equation, one gets $\lambda_1 \lambda_1' \Lambda = \Lambda \lambda_1 \lambda_1'$, i.e. $\lambda_1' = \lambda_1 \Lambda'$, where $\Lambda'$ commutes with $\Lambda$. Moreover, as $\lambda_1'$ is an involution, $\Lambda'$ must satisfy $\lambda_1 \Lambda' = \Lambda^{-1} \lambda_1$. We therefore consider for given $\Lambda$ and reflection $\lambda_1$ as above the set of Lorentz transformations

$$\Lambda' = \{ \Lambda' \in \mathcal{L}_+^1 \mid \lambda_1 \Lambda' = \Lambda'^{-1} \lambda_1, \ \Lambda' \Lambda = \Lambda \Lambda' \} . \quad (2.4)$$

Given any $\Lambda' \in \Lambda'$, the elements $\lambda_1' = \lambda_1 \Lambda'$ and $\lambda_2' = \lambda_1 \Lambda'$ are involutions, and their product is equal to $\Lambda$. But these involutions are not always reflections. Nonetheless, if $\Lambda$ is such that for each $\Lambda' \in \Lambda'$ there is some $\Lambda'^{-1/2} \in \Lambda'$ whose square is $\Lambda'$, one has $\lambda_1' = \Lambda'^{-1/2} \lambda_1 \Lambda'^{-1/2}$ and $\lambda_2' = \Lambda'^{-1/2} \lambda_1 \Lambda'^{-1/2}$. So, in this case, $\lambda_1'$ and $\lambda_2'$ are both reflections of the form given above.

In the following we shall focus our attention on certain specific elements $\Lambda \in \mathcal{L}_+^i$ which are of the form $\Lambda = \Lambda_1 \Lambda_0 \Lambda_1^{-1}$, where $\Lambda_1 \in \mathcal{L}_+^1$ is arbitrary and $\Lambda_0 \ (\Lambda_0^2 \neq 1)$ is an element of the stability group $\mathcal{L}_0 \subset \mathcal{L}_+^1$ of some fixed wedge $W_e_0$. We recall that $\mathcal{L}_0$ is the abelian subgroup of $\mathcal{L}_+^1$ generated by all rotations $R_0$ about $e_0$ and all boosts $B_0$ in the direction of $e_0$.

**Remark:** Disregarding three special cases, all conjugacy classes of $\mathcal{L}_+^1$ are of this form. This can be seen by proceeding to the covering group $SL(2, \mathbb{C})$ of $\mathcal{L}_+^1$ and making use of the Jordan normal form of two-by-two matrices.

One finds by explicit computation (most conveniently in the covering group) that the commutant of $\Lambda$ is equal to the abelian group $\Lambda_1 \mathcal{L}_0 \Lambda_1^{-1}$. Hence if $e$ is
such that $e \cdot e_0 = 0$ and if $\lambda$ is the reflection about the edge of $W_e$, one obtains for the reflection $\lambda_1 = \Lambda_1 \lambda \Lambda_1^{-1}$ about the edge of $\Lambda_1 W_e$ the equality

$$\lambda_1 \Lambda' = \Lambda' \lambda_1 \quad \Lambda' \in \Lambda_1 \mathcal{L}_0 \Lambda_1^{-1}.$$ 

(2.5)

Hence $\Lambda' = \Lambda_1 \mathcal{L}_0 \Lambda_1^{-1}$ in this case. Moreover, as $\mathcal{L}_0$ is stable under taking square roots, there is also for each $\Lambda' \in \Lambda_1 \mathcal{L}_0 \Lambda_1^{-1}$ a square root $\Lambda'^{1/2} \in \Lambda_1 \mathcal{L}_0 \Lambda_1^{-1}$. Hence for these special elements $\Lambda$ we have complete control of their representation in terms of products of reflections. We summarize these results in the following lemma.

Lemma 2.2 Let $\Lambda \in \Lambda' \mathcal{L}_0 \Lambda'^{-1}$, $\Lambda^2 \neq 1$, and let $\lambda_1$ be the reflection about the edge of $\Lambda' W_e$, where $e$ is orthogonal to $e_0$. Then $\lambda_2 = \lambda_1 \Lambda$ is a reflection and $\lambda_1 \lambda_2 = \Lambda$. Moreover, any pair of reflections $\lambda'_1, \lambda'_2$ with product $\Lambda$ arises from $\lambda_1, \lambda_2$ by the adjoint action of some element of $\Lambda' \mathcal{L}_0 \Lambda'^{-1}$.

2.2 Reflection maps

Let $J$ be a topological group. We consider maps $\lambda \mapsto J(\lambda)$ from the set of reflections $\mathcal{R} \subset \mathcal{L}_+$ into $J$. There is no loss of generality to assume that the subgroup generated by the set of elements $\{J(\lambda) \mid \lambda \in \mathcal{R}\}$ is dense in $J$.

Definition 2.3 A map $J : \mathcal{R} \to J$ is a reflection map if for every $\lambda \in \mathcal{R}$ the element $J(\lambda) \in J$ is an involution and

$$J(\lambda_1)J(\lambda_2)J(\lambda_1) = J(\lambda_1 \lambda_2 \lambda_1),$$

(2.6)

for all $\lambda_1, \lambda_2 \in \mathcal{R}$.

We want to show that any continuous reflection map is the restriction of a continuous homomorphism $V : \mathcal{L}_+ \to J$.

For this to be true, it would be necessary to define for given $\Lambda \in \mathcal{L}_+^\uparrow$ and corresponding pair of reflections $\lambda_1, \lambda_2$ with $\lambda_1 \lambda_2 = \Lambda$ the element

$$V(\Lambda) \doteq J(\lambda_1)J(\lambda_2).$$

(2.7)

Yet it is a priori not clear whether this element (a) is independent of the choice of the pair of reflections into which $\Lambda$ is decomposed, (b) has the right continuity properties and (c) defines a homomorphism. We shall start by making specific choices of reflections for special $\Lambda$ and establish, step by step, properties (a)–(c) of $V$. In this discussion we make use of arguments and results in [7], which we shall recall here in somewhat modified form for the convenience of the reader.

Let us consider first the action of $V$ on rotations and boosts, $R, B$. To this end we choose a vector $e$ which is orthogonal to the axis of revolution of $R$, respectively
the boost direction of \( B \); such vectors are called admissible in the following. Let \( \lambda_e, R\lambda_e \) and \( B\lambda_e \) be the reflections defined above. We then set
\[
V_e(R) = J(R\lambda_e)J(\lambda_e), \quad V_e(B) = J(B\lambda_e)J(\lambda_e)
\]
(2.8)
and observe that \( V_e(R)^{-1} = J(\lambda_e)J(R\lambda_e) \) and \( V_e(B)^{-1} = J(\lambda_e)J(B\lambda_e) \). Note that because of relation (2.6), we also have
\[
V_e(R)J(\lambda)V_e(R)^{-1} = J(R\lambda_e)J(\lambda_e)J(\lambda)J(R\lambda_e) = J(R\lambda R^{-1})
\]
(2.9)
for every \( \lambda \in \mathcal{R} \), where we have used \( \lambda_e R\lambda_e = R^{-1} \). Similarly, we also have
\[
V_e(B)J(\lambda)V_e(B)^{-1} = J(B\lambda B^{-1})
\]
(2.10)

**Lemma 2.4** The elements \( V_e(B), V_e(R) \) defined above do not depend on the choice of the vector \( e \) within the above-stated limitations.

**Proof.** Consider first the case of boosts. If \( B = 1 \), there is nothing to prove. So let \( B \neq 1, e \) be one of the admissible vectors for this boost, and let \( B_1 \) be any boost in the same direction as that of \( B \). Note that \( V_e(B_1)J(\lambda_e) = J(B_1\lambda_e)J(\lambda_e)^2 = J(B_1\lambda)e = J(\lambda_e)^2J(B_1\lambda_e) = J(\lambda)eV_e(B_1)^{-1} \). Hence, for any \( n \in \mathbb{N} \)
\[
V_e(B_1)^{2n}\lambda = V_e(B_1)^n\lambda V_e(B_1)^{-n} = J(B_1^n\lambda_e B_1^{-n}) = J(B_1^{2n}\lambda_e)
\]
using (2.10). Consequently, one has
\[
V_e(B_1)^{2n} = V_e(B_1)^{2n}\lambda (\lambda_e)^2 = J(B_1^{2n}\lambda_e)J(\lambda_e) = V_e(B_1^{2n})
\]
Similarly, one sees that
\[
V_e(B_1)J(B_1\lambda_e) = J(B_1\lambda_e)J(\lambda_e)J(B_1\lambda_e) = J(B_1\lambda_e)V_e(B_1)^{-1}
\]
and therefore
\[
V_e(B_1)^{2n+1} = V_e(B_1)^{2n}J(B_1\lambda_e)J(\lambda_e) = V_e(B_1)^n\lambda J(B_1\lambda_e)V_e(B_1)^{-n}J(\lambda_e)
\]
\[
= J(B_1^{2n+1}\lambda_e)J(\lambda_e) = V_e(B_1^{2n+1})
\]
Thus, one has \( V_e(B_1)^n = V_e(B_1^n) \), for all \( n \in \mathbb{N} \).

Now let \( R_\phi \) be a rotation by \( \phi \) about the axis established by the direction of the boost \( B \). Since \( R_\phi \) and \( B_1 \) commute, one obtains from relation (2.9)
\[
V_e(R_\phi)V_e(B_1)V_e(R_\phi)^{-1} = V_e(R_\phi)J(B_1\lambda_e)J(\lambda_e)V_e(R_\phi)^{-1}
\]
\[
= J(B_1R_\phi\lambda_e R_\phi^{-1})J(R_\phi\lambda_e R_\phi^{-1}) = J(B_1\lambda_{R_\phi\lambda_e})J(\lambda_{R_\phi\lambda_e}) = V_{R_\phi\lambda_e}(B_1)
\]
On the other hand, according to (2.10), the element \( V_{R_\phi\lambda_e}(B_1)V_e(B_1)^{-1} \) must commute with \( J(\lambda) \), for every \( \lambda \in \mathcal{R} \). Since \( J(\mathcal{R}) \) generates \( \mathcal{J} \), this implies that there exists some element \( Z_\phi \) in the center of \( \mathcal{J} \) such that
\[
V_{R_\phi\lambda_e}(B_1) = Z_\phi V_e(B_1)
\]

Setting $\phi = 2m\pi/n$, for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, one sees from the preceding two relations that

$$V_e(B_1) = V_{e^{2m\pi/n}}e(B_1) = V_e(R_{2m\pi/n})^n V_e(B_1)V_e(R_{2m\pi/n})^{-n} = Z_{2m\pi/n}^n V_e(B_1),$$

and consequently $Z_{2m\pi/n}^n = 1$. Hence,

$$V_{R_{2m\pi/n}e}(B_1^n) = V_{R_{2m\pi/n}e}(B_1)^n = Z_{2m\pi/n}^n V_e(B_1)^n = V_e(B_1)^n = V_e(B_1^n),$$

and setting $B_1 = B^{1/n}$ one obtains

$$V_{R_{2m\pi/n}e}(B) = V_e(B),$$

for all $n \in \mathbb{N}$, $m \in \mathbb{Z}$. By hypothesis, the reflection map is continuous, so the element $V_e(R_\phi)$ depends continuously on $\phi$ for any admissible $e$, and the same is thus also true of $V_{R_\phi e}(B)$. It therefore follows from the preceding relation that $V_{R_\phi e}(B) = V_e(B)$ for any rotation $R_\phi$, proving the assertion for the case of the boosts.

For the rotations $R$, one proceeds in exactly the same way. The role of $R_\phi$ is now to be played by the rotations about the axis of revolution fixed by $R$. □

In light of this result, we may omit the index $e$ and set

$$V(B) \doteq V_e(B), \quad V(R) \doteq V_e(R). \quad (2.11)$$

**Lemma 2.5** The elements $V(B)$ and $V(R)$ depend continuously on the boosts $B$ and rotations $R$, respectively.

**Proof.** Let $\{B_n\}_{n \in \mathbb{N}}$ be a sequence of boosts converging to $B$. If $B \neq 1$ the distance between the unit disks parameterizing the corresponding orthogonal admissible vectors converges to 0. In particular, there exists a sequence of unit vectors $e_n$, admissible for $B_n$, converging to the unit vector $e$, admissible for $B$. Hence, the sequence $\{B_n e_n\}$ converges to $Be$. By the assumed continuity of the reflection map, one concludes that $V(B_n) = J(B_n \lambda e_n)J(\lambda e_n)$ converges to $J(B \lambda e)J(\lambda e) = V(B)$ as $n \to \infty$.

If, on the other hand, the sequence $\{B_n\}$ converges to 1, the corresponding unit disks need not converge. Nonetheless, due to the compactness of the unit ball in $\mathbb{R}^3$, for any sequence of unit vectors $e_n \in \mathbb{R}^3$ there exists a subsequence $\{e_{\sigma(n)}\}$ which converges to some unit vector $e_\sigma$. Since $\{B_{\sigma(n)}\}$ converges to 1, the corresponding sequence $\{B_{\sigma(n)} e_{\sigma(n)}\}$ converges to $e_\sigma$. One therefore has

$$V(B_{\sigma(n)}) = J(B_{\sigma(n)} \lambda e_{\sigma(n)})J(\lambda e_{\sigma(n)}) \to J(\lambda e_\sigma)J(\lambda e_\sigma) = 1.$$

Since the choice of sequence $\{e_n\}$ was arbitrary, the proof of the continuity of $V(B)$ with respect to the boosts $B$ is complete. The argument for the rotations is analogous after the boost direction is replaced by the axis of revolution of the respective rotation. □
Lemma 2.6  With the above definitions, one has the following.

1. \( V(R)V(B)V(R)^{-1} = V(RBR^{-1}) \) for all boosts \( B \) and rotations \( R \).
2. \( V(\cdot) \) defines a true representation of every continuous one-parameter subgroup of boosts or rotations.

Proof. Statement (1) follows from relation (2.9) and Lemma 2.4, which imply (\( e \) being admissible for both the rotation \( R \) and the boost \( B \))

\[
V(R)V(B)V(R)^{-1} = V(R)J(B\lambda_e)J(\lambda_e)V(R)^{-1} \\
= J(RBR^{-1}R\lambda_eR^{-1})J(R\lambda_eR^{-1}) = J(RBR^{-1}\lambda_{Re})J(\lambda_{Re}) \\
= V(RBR^{-1}) .
\]

The last equality follows from the fact that \( RBR^{-1} \) is again a boost whose direction is orthogonal to \( Re \).

Now let \( G : \mathbb{R} \to \mathcal{L}_+^\uparrow \) be a continuous one-parameter group of boosts or rotations. As in the proof of Lemma 2.4 one shows by an elementary computation on the basis of relation (2.6) that \( V(G(u))^n = V(G(u)^n) = V(G(nu)) \). Consequently, one finds that, for \( m_1, m_2, n \in \mathbb{N} \),

\[
V(G(m_1/n))V(G(m_2/n)) = V(G(1/n))^{m_1}V(G(1/n))^{m_2} \\
= V(G(1/n))^{m_1+m_2} = V(G((m_1 + m_2)/n)) .
\]

As \( V(G(-u)) = V(G(u)^{-1}) = V(G(u))^{-1} \) by (2.8) and Lemma 2.4, this relation extends to arbitrary \( m_1, m_2 \in \mathbb{Z} \) and \( n \in \mathbb{N} \). The stated assertion (2) thus follows once again from the continuity properties of \( V(\cdot) \) established so far. \( \square \)

Given any \( \Lambda \in \mathcal{L}_+^\uparrow \), we make use of its unique polar decomposition \( \Lambda = RB \) and choose a direction \( e \) which is orthogonal to both the axis of revolution of \( R \) and the boost direction of \( B \). We recall that the corresponding reflection \( \lambda_e \) satisfies both \( R\lambda_e = \lambda_e R^{-1} \) and \( B\lambda_e = \lambda_e B^{-1} \). Hence \( R\lambda_e = RB\lambda_e B \) and \( \lambda_e B \) are reflections, and we can define

\[
V(\Lambda) \doteq J(RB\lambda_e B)J(\lambda_e B) = J(R\lambda_e)J(\lambda_e)^2J(\lambda_e B)J(\lambda_e)^2 \\
= J(R\lambda_e)J(\lambda_e)J(B\lambda_e)J(\lambda_e) = V(R)V(B) ,
\]

where we made use of the properties of reflection maps. Since \( R, B \) depend continuously on \( \Lambda, V(\Lambda) \) is continuous in \( \Lambda \) as well. Moreover,

\[
V(\Lambda)J(\lambda)V(\Lambda)^{-1} = J(\Lambda\lambda\Lambda^{-1}) ,
\]

for any \( \lambda \in \mathcal{R} \) and \( \Lambda \in \mathcal{L}_+^\uparrow \).

Let \( \mathcal{L}_0 \subset \mathcal{L}_+^\uparrow \) be the abelian stability group of any given wedge \( W_{e_0} \). It is generated by two one-parameter subgroups: the rotations about the axis fixed by
\( e_0 \) and the boosts in the corresponding direction. It follows from the properties of \( V(\cdot) \) established so far that for any \( \Lambda = RB \in \mathcal{L}_0 \) one has
\[
V(R)V(B) = V(R)V(B)V(R)^{-1}V(R) = V(RBR^{-1})V(R) = V(B)V(R). \tag{2.14}
\]
Hence for any \( \Lambda_0 = R_0B_0 \in \mathcal{L}_0 \) one obtains
\[
V(\Lambda_0)V(\Lambda)V(\Lambda_0)^{-1} = V(R_0)V(B_0)V(R)V(B_0)^{-1}V(R_0)^{-1} = V(\Lambda). \tag{2.15}
\]
This fact puts us into the position of being able to prove that for a large set of Lorentz transformations \( \Lambda \in \mathcal{L}_0^\uparrow \) the corresponding \( V(\Lambda) \) do not depend on the choice of reflections in the decomposition of \( \Lambda \).

**Lemma 2.7** Let \( \Lambda \in \mathcal{L}_0 \), \( \Lambda^2 \neq 1 \), \( \Lambda' \in \mathcal{L}_0^\uparrow \). Then \( V(\Lambda) = J(\lambda_1)J(\lambda_2) \) for any pair of reflections \( \lambda_1, \lambda_2 \) satisfying \( \lambda_1\lambda_2 = \Lambda \).

**Proof.** In view of relation \((2.13)\) it suffices to establish the statement for \( \Lambda \in \mathcal{L}_0 \), \( \Lambda^2 \neq 1 \). Let \( \lambda_1, \lambda_2 \) be reflections as in definition \((2.12)\) such that \( \Lambda = \lambda_1\lambda_2 \) and \( V(\Lambda) = J(\lambda_1)J(\lambda_2) \). If \( \lambda_3, \lambda_4 \) are reflections such that \( \lambda_3\lambda_4 = \lambda_1\lambda_2 \), there is by Lemma \((2.2)\) a \( \Lambda_0 \in \mathcal{L}_0 \) such that \( \lambda_3 = \Lambda_0\lambda_1\Lambda_0^{-1}, \lambda_4 = \Lambda_0\lambda_2\Lambda_0^{-1} \). Hence by relations \((2.13)\) and \((2.15)\) one obtains
\[
J(\lambda_3)J(\lambda_4) = V(\Lambda_0)J(\lambda_1)J(\lambda_2)V(\Lambda_0)^{-1} = V(\Lambda_0)V(\Lambda)V(\Lambda_0)^{-1} = V(\Lambda),
\]
proving the statement. \(\square\)

This result will greatly simplify the computations which will show that \( V(\cdot) \) is a homomorphism. Let \( R_1, R_2 \) be arbitrary rotations such that \((R_1R_2)^2 \neq 1\) and let \( e \) be orthogonal to the axes of revolution of \( R_1, R_2 \). Taking into account the fact that any rotation is an element of the stability group \( \mathcal{L}_0 \) of some suitable wedge \( W_{e_0} \), we obtain from the preceding lemma the equalities
\[
V(R_1)V(R_2) = J(R_1\lambda_e)J(\lambda_e)J(R_2\lambda_e)J(\lambda_e) = J(R_1\lambda_e)J(\lambda_eR_2) = V(R_1R_2), \tag{2.16}
\]
and this equation extends by continuity to arbitrary pairs of rotations. Next, let \( B_1, B_2 \) be arbitrary boosts. Then \( B_1B_2 = B_1^{1/2}(B_1^{1/2}B_2B_1^{1/2})B_1^{-1/2} \), where the expression in brackets is a positive matrix and hence a boost. So \( B_1B_2 \) belongs to the class of Lorentz transformations covered by the preceding lemma. Choosing \( e \) orthogonal to the boost directions of \( B_1, B_2 \), we therefore have
\[
V(B_1B_2) = J(B_1\lambda_e)J(\lambda_eB_2) = J(B_1\lambda_e)J(\lambda_e)^2J(\lambda_eB_2) = V(B_1)V(B_2). \tag{2.17}
\]
On the other hand, proceeding to the polar decomposition \( B_1B_2 = RB \) we get by definition \( V(B_1B_2) = V(R)V(B) \) and hence
\[
V(B_1)V(B_2) = V(R)V(B). \tag{2.18}
\]
Now let $\Lambda_1 = R_1 B_1$, $\Lambda_2 = R_2 B_2$ be arbitrary proper orthochronous Lorentz transformations. Introducing the boost $B_3 = R_2^{-1} B_1 R_2$ and making use of the polar decomposition $B_3 B_2 = RB$, we obtain from the preceding results the chain of equalities

$$V(\Lambda_1) V(\Lambda_2) = V(R_1) V(B_1) V(R_2) V(B_2) = V(R_1) V(R_2) V(B_3) V(B_2)$$
$$= V(R_1 R_2) V(R) V(B) = V(R_1 R_2 R) V(B) = V(R_1 R_2 RB) = V(\Lambda_1 \Lambda_2).$$ (2.19)

Thus $V(\cdot)$ is a continuous homomorphism from $\mathcal{L}_+^\uparrow$ into $\mathcal{J}$. It remains to extend $V(\cdot)$ to the component of $\mathcal{L}_+$ which is disconnected from unity. To this end we fix a reflection $\lambda_0 \in \mathcal{L}_+$ corresponding to some wedge $W_{e_0}$ and note that all elements in the disconnected part can be represented uniquely in the form $\lambda_0 \Lambda$, where $\Lambda \in \mathcal{L}_+^\uparrow$. We set

$$V(\lambda_0 \Lambda) \triangleq J(\lambda_0) V(\Lambda).$$ (2.20)

In view of the defining properties of reflection maps and the definition of $V(\cdot)$, we get

$$J(\lambda_0) V(\Lambda) J(\lambda_0) = V(\lambda_0 \Lambda \lambda_0).$$ (2.21)

(Note that the sets of rotations and boosts are mapped onto themselves by the adjoint action of $\lambda_0$, and the set of distinguished wedges $W_e$ is stable under the action of $\lambda_0$, as well.) Hence for any $\Lambda' \in \mathcal{L}_+^\uparrow$

$$V(\Lambda') V(\lambda_0 \Lambda) = J(\lambda_0)^2 V(\Lambda') J(\lambda_0) V(\Lambda) = J(\lambda_0) V(\lambda_0 \Lambda' \lambda_0) V(\Lambda)$$
$$= J(\lambda_0) V(\lambda_0 \Lambda' \lambda_0 \lambda_0) = V(\Lambda' \Lambda_0 \lambda_0) ,$$ (2.22)

and similarly $V(\lambda_0 \Lambda) V(\Lambda') = V(\lambda_0 \Lambda' \Lambda_0)$. Moreover,

$$V(\lambda_0 \Lambda) V(\lambda_0 \Lambda') = J(\lambda_0) V(\Lambda) J(\lambda_0) V(\Lambda')$$
$$= V(\lambda_0 \Lambda \lambda_0) V(\Lambda') = V(\lambda_0 \Lambda' \Lambda_0) .$$ (2.23)

Thus $V(\cdot)$ is a continuous homomorphism from $\mathcal{L}_+^\uparrow$ into $\mathcal{J}$.

As any reflection $\lambda$ can be represented in the form $\lambda = \Lambda \lambda_0 \Lambda^{-1}$ for some $\Lambda \in \mathcal{L}_+^\uparrow$, it follows that

$$J(\lambda) = J(\lambda_0)^2 V(\Lambda) J(\lambda_0) V(\Lambda)^{-1} = J(\lambda_0) V(\lambda_0 \Lambda \lambda_0 \Lambda^{-1}) = V(\lambda) .$$ (2.24)

Thus we finally see that $J(\cdot)$ is indeed the restriction of the continuous homomorphism $V(\cdot)$ to the set of reflections $\mathcal{R}$ and that $V(\Lambda)$ does not depend on the decomposition of $\Lambda$ into reflections for any $\Lambda \in \mathcal{L}_+^\uparrow$. Since $\mathcal{R}$ generates $\mathcal{L}_+^\uparrow$, $V$ is the only extension of the reflection map to a homomorphism from $\mathcal{L}_+^\uparrow$ into $\mathcal{T}$. We summarize these results in the following proposition.

**Proposition 2.8** Let $J$ be a continuous reflection map from the set of reflections $\mathcal{R} \subset \mathcal{L}_+^\uparrow$ into an arbitrary topological group $\mathcal{J}$. Then $J$ is the restriction to $\mathcal{R}$ of a unique continuous homomorphism mapping $\mathcal{L}_+^\uparrow$ into $\mathcal{J}$.
3 Continuity of modular reflection maps

In view of the preceding proposition it is of interest to clarify the continuity properties of the modular reflection maps appearing in quantum field theory. In order to reveal the pertinent structures, we discuss this problem in a setting which is slightly more general than that outlined in the introduction.

Let \( W \mapsto R(W) \) be any net of von Neumann algebras indexed by wedge regions, which satisfies the condition of wedge duality, \( R(W)' = R(W') \), and let \( \Omega \in \mathcal{H} \) be any vector which is cyclic and separating for all algebras \( R(W) \). We denote the modular conjugation corresponding to the pair \( (R(W), \Omega) \) by \( J_W \). We shall show that the map \( W \mapsto J_W \) from the family of wedges \( W \) into the group of (anti)unitary operators on \( \mathcal{H} \) is continuous under quite general conditions. Making use of the fact that \( \mathcal{P}^+ \) acts transitively on \( W \), we identify \( W \), as a topological space, with the quotient space \( \mathcal{P}^+ / \mathcal{P}_0 \), where \( \mathcal{P}_0 \subset \mathcal{P}^+ \) is the invariance subgroup of any given wedge \( W_0 \in \mathcal{W} \); note that the topology does not depend on the choice of \( W_0 \). On the group of (anti)unitary operators we use the strong–*–topology.

As we shall see, the desired result follows from the assumption that the net \( W \mapsto R(W) \) is locally generated in the following specific sense: Let \( C \) be a family of closed regions \( C \subset \mathbb{R}^4 \) subject to the conditions:

(a) Each \( C \in C \) can be approximated from the outside by wedges \( W \in \mathcal{W} \), i.e. \( C = \bigcap_{W \supset C} W \). Here the inclusion relation \( W \supset C \) means that there is some open neighborhood of \( W \) in \( \mathcal{W} \) all of whose elements contain \( C \).

(b) Each wedge \( W \in \mathcal{W} \) can be approximated from the inside by regions \( C \in C \), i.e. \( W = \bigcup_{C \in W} C \), where \( C \in W \) if \( C \) is contained in all wedges in some neighborhood of \( W \).

(c) The family \( C \) is stable under the action of \( \mathcal{P}^+_+ \).

We say in this case that \( C \) is a generating family of regions. A familiar example of such a generating family is the set of closed double cones in Minkowski space; another one is the family of closed spacelike cones considered in [3] in the context of theories with topological charges.

Given a generating family \( C \), we define corresponding algebras \( \mathcal{R}(C), C \in C \), setting

\[
\mathcal{R}(C) = \bigcap_{W \supseteq C} \mathcal{R}(W).
\] (3.1)

Clearly, \( \mathcal{R}(C) \subset \mathcal{R}(W) \) whenever \( C \in W \).

**Definition:** The net \( W \mapsto \mathcal{R}(W) \) is said to be locally generated if there is a generating family \( C \) of regions such that \( \Omega \) is cyclic for \( \mathcal{R}(C), C \in C \), and

\[
\mathcal{R}(W) = \bigvee_{C \in W} \mathcal{R}(C), \quad W \in \mathcal{W}.
\] (3.2)
Note that the nets affiliated with quantum field theories satisfying the Wightman axioms are locally generated \[15\].

We shall establish the continuity properties of the modular conjugations by first showing that any locally generated net satisfying wedge duality complies with the net continuity condition introduced in \[7\], see below. Let \(\{W_\delta\}_{\delta > 0}\) be a family of wedges converging to some wedge \(W_0\) as \(\delta\) converges to 0. We define corresponding von Neumann algebras

\[
\mathcal{R}_\varepsilon = \bigwedge_{0 \leq \delta \leq \varepsilon} \mathcal{R}(W_\delta), \quad \mathcal{R}_\varepsilon' = \bigvee_{0 \leq \delta \leq \varepsilon} \mathcal{R}(W_\delta),
\]

and note that, by construction,

\[
\mathcal{R}_\varepsilon \subset \mathcal{R}(W_\delta) \subset \mathcal{R}_\varepsilon', \quad (3.3)
\]

for any \(0 \leq \delta \leq \varepsilon\). Moreover, for any \(\varepsilon_1 \geq \varepsilon_2 \geq 0\), one has \(\mathcal{R}_{\varepsilon_1} \subset \mathcal{R}_{\varepsilon_2}\) and \(\mathcal{R}_{\varepsilon_1} \supset \mathcal{R}_{\varepsilon_2}\).

Now let \(C \in \mathcal{C}\) be such that \(C \in W_0\). Bearing in mind the meaning of this inclusion relation, it is apparent that \(C \in W_\delta\) for all sufficiently small \(\delta > 0\) and consequently \(\mathcal{R}(C) \subset \mathcal{R}_\varepsilon\) for sufficiently small \(\varepsilon > 0\). Hence, \(\mathcal{R}(C) \subset \bigvee_{\varepsilon > 0} \mathcal{R}_\varepsilon\). As \(\bigvee_{\varepsilon > 0} \mathcal{R}_\varepsilon\) is a von Neumann algebra and the net is locally generated, it follows that \(\mathcal{R}(W_0) = \bigvee_{C \in W_0} \mathcal{R}(C) \subset \bigvee_{\varepsilon > 0} \mathcal{R}_\varepsilon\). On the other hand, the inclusion (3.4) implies \(\bigvee_{\varepsilon > 0} \mathcal{R}_\varepsilon \subset \mathcal{R}(W_0)\), proving the equality

\[
\bigvee_{\varepsilon > 0} \mathcal{R}_\varepsilon = \mathcal{R}(W_0). \quad (3.5)
\]

In a similar manner one shows that also

\[
\bigwedge_{\varepsilon > 0} \mathcal{R}_\varepsilon = \mathcal{R}(W_0), \quad (3.6)
\]

since, by wedge duality,

\[
\mathcal{R}_\varepsilon' = \bigwedge_{0 \leq \delta \leq \varepsilon} \mathcal{R}(W_\delta)' = \bigwedge_{0 \leq \delta \leq \varepsilon} \mathcal{R}(W_\delta'), \quad (3.7)
\]

and it is easy to see that the family of wedges \(\{W_\delta\}_{\delta > 0}\) converges to \(W_0'\). Applying the arguments in the preceding step, one obtains \(\bigvee_{\varepsilon > 0} \mathcal{R}_\varepsilon' = \mathcal{R}(W_0') = \mathcal{R}(W_0)'\). From this equality the assertion follows by taking commutants.

Relations (3.5) and (3.6) comprise the net continuity condition used in \[7\] as a crucial technical assumption. We have therefore shown that this condition is met by any net which is locally generated and satisfies wedge duality. This fact puts us into the position to apply Proposition 4.6 in \[7\], giving the following unexpected result.\(^1\)

\(^1\)Although the CGMA is mentioned in the statement of \[7\, Prop. 4.6\], only wedge duality and the cyclicity properties of \(\Omega\) given above enter into its proof.
**Proposition 3.1** Let \( W \mapsto \mathcal{R}(W) \) be a locally generated net which satisfies the condition of wedge duality. Then the map \( W \mapsto J_W \) from the wedges \( W \in \mathcal{W} \) to the modular conjugations \( J_W \) corresponding to \((\mathcal{R}(W), \Omega)\) is continuous.

The conditions underlying this result are natural from a physical point of view and obtain in many models. We therefore believe that the continuity of the modular conjugations is a generic feature in local quantum physics. As a matter of fact, such continuity properties can be established for locally generated nets on many space–times, provided they satisfy an analogue of the wedge duality condition.

It is necessary to prove a version of Proposition 3.1 formulated in more group theoretical terms. Because of wedge duality, the modular conjugations \( J_W, J_{W'} \) corresponding to the pairs \((\mathcal{R}(W), \Omega)\) and \((\mathcal{R}(W'), \Omega)\), respectively, coincide. We may therefore return to the notation used in the introduction, i.e. put \( J(\lambda) = J_W = J_{W'} \), where \( \lambda \in \mathcal{P}_+ \) is the reflection about the common edge \( E \) of the wedges \( W \) and \( W' \). This notation is consistent, since every two–dimensional spacelike plane \( E \) determines uniquely a corresponding pair of wedges \( W, W' \) having \( E \) as their edge.

As we shall see, Proposition 3.1 implies that the map \( \lambda \mapsto J(\lambda) \) from the family of reflections \( \lambda \in \mathcal{P}_+ \) into the group of (anti)unitary operators is continuous. In order to prove this fact, we have to show that for any given family of reflections \( \{\lambda_\delta\}_{\delta > 0} \) converging to some reflection \( \lambda_0 \) there exists a corresponding convergent family of wedges whose elements \( W_\delta \) have edges \( E_\delta \) which are pointwise fixed under the action of \( \lambda_\delta, \delta > 0 \).

The edges \( E_\delta \) can be specified easily: Let \( E_0 \) be the two–dimensional spacelike plane which is pointwise fixed under the action of \( \lambda_0 \). Introducing on \( \mathbb{R}^4 \) the maps \( x \mapsto \frac{1}{2}(1 + \lambda_\delta \lambda_0)x \) one finds that the planes \( E_\delta = \frac{1}{2}(1 + \lambda_\delta \lambda_0)E_0 \) are pointwise fixed under the action of the reflections \( \lambda_\delta, \delta > 0 \). Since \( \lambda_\delta \) converges to \( \lambda_0 \) it follows that \( \lambda_\delta \lambda_0 \) converges to the unit element of \( \mathcal{P}_+ \), so the above maps converge to the identity, uniformly on compact subsets of \( \mathbb{R}^4 \). Thus, for sufficiently small \( \delta > 0 \), each \( E_\delta \) is a two–dimensional spacelike plane and therefore constitutes the edge of a pair of wedges. Moreover, since the points on the edges \( E_\delta \) converge to points on \( E_0 \) in the limit of small \( \delta \), it is straightforward to exhibit Poincaré transformations \( v_\delta \in \mathcal{P}_+^\uparrow \) such that \( v_\delta E_0 = E_\delta, \delta > 0 \), and \( \{v_\delta\}_{\delta > 0} \) converges to the identity. Picking one of the two wedges with edge \( E_0 \), say \( W_0 \), we take as our family of wedges \( \{W_\delta \equiv v_\delta W_0\}_{\delta > 0} \) and note that it has all of the desired properties. Since \( J(\lambda_\delta) = J_{W_\delta} \), we are now in a position to apply Proposition 3.1 entail the following result.

**Corollary 3.2** Let \( W \mapsto \mathcal{R}(W) \) be a locally generated net which satisfies the condition of wedge duality. Then the map \( \lambda \mapsto J(\lambda) \) from the reflections \( \lambda \in \mathcal{P}_+ \) to the modular conjugations \( J(\lambda) \) corresponding to \( (\mathcal{R}(W), \Omega) \), hence also to \((\mathcal{R}(W'), \Omega)\), is continuous. Here the wedges \( W, W' \) are fixed by the condition \( \lambda W = W' \).

We conclude this section with a technical result pertaining to nets which satisfy the condition of geometric modular action, CGMA, and the modular stability
condition, CMS; see the next section for their definitions. The latter condition says that the elements of the modular groups $\Delta_{it}^W$, $t \in \mathbb{R}$, corresponding to pairs $(\mathcal{R}(W), \Omega)$ are contained in the group generated by all finite products of the modular conjugations $J_W$, $W \in \mathcal{W}$. Under these circumstances one can relax the condition that the nets are locally generated without changing the conclusions of the preceding discussion. In fact, one has the following result.

**Lemma 3.3** Let $W \mapsto \mathcal{R}(W)$ be a net satisfying the CGMA and the CMS. If there is a generating family of regions $\mathcal{C}$ such that $\Omega$ is cyclic for $\mathcal{R}(C)$, given any $C \in \mathcal{C}$, then the net is locally generated.

**Proof.** As was shown in [7], the CGMA entails relation (1.1), hence the CMS implies that, for any given $W \in \mathcal{W}$,

$$\Delta_{it}^W \mathcal{R}(W_0) \Delta_{it}^{-W} = \mathcal{R}(\nu_W(t)W_0), \quad W_0 \in \mathcal{W},$$

where $\nu_W(t) \in \mathcal{P}_+$ for $t \in \mathbb{R}$; as a matter of fact, since $\Delta_{it}^W = (\Delta_{it}^{W/2})^2$, $\nu_W(t)$ is the square of an element of $\mathcal{P}_+$ and thus lies in the identity component of this group. Moreover, since the modular unitaries $\Delta_{it}^W$ induce automorphisms of $\mathcal{R}(W)$ and since, by the CGMA, the map $W \mapsto \mathcal{R}(W)$ is a bijection, one finds that $\nu_W(t)$ must be an element of the stability group of $W$, $t \in \mathbb{R}$.

Now let $\mathcal{C}$ be a generating family of regions as hypothesized, and let $C \in \mathcal{C}$ be such that $C \in W$ for a given $W \in \mathcal{W}$. Bearing in mind the definition of the algebras $\mathcal{R}(C)$ and the stability of the family $\mathcal{C}$ under Poincaré transformations, one obtains $\Delta_{it}^W \mathcal{R}(C) \Delta_{it}^{-W} = \mathcal{R}(\nu_W(t)C)$ and $\nu_W(t)C \in \mathcal{C}$. Since $\nu_W(t)$ is an element of the stability group of $W$, one also finds, after a moment’s reflection, that $\nu_W(t)C \in W$. It follows that $\Delta_{it}^W \left( \bigvee_{C \in W} \mathcal{R}(C) \right) \Delta_{it}^{-W} = \bigvee_{C \in W} \mathcal{R}(C) \subset \mathcal{R}(W)$, $t \in \mathbb{R}$. But $\Omega$ is cyclic for the algebras $\mathcal{R}(C)$, hence $\bigvee_{C \in W} \mathcal{R}(C) = \mathcal{R}(W)$, by a well known result of Takesaki.

Analogous results can be established for other space–times, a prominent example being de–Sitter space.

**4 Modular action and Poincaré covariance**

Making use of the results obtained in the preceding two sections, we are now able to improve considerably on the analysis of theories complying with the CGMA carried out in [5,7]. For the convenience of the reader, we recall this condition in a form appropriate for the present discussion as well as some of the important consequences established in the cited articles.

A state vector $\Omega$ and a net $W \mapsto \mathcal{R}(W)$ from the family of wedge regions $W$ in Minkowski space to von Neumann algebras are said to comply with the CGMA if the following conditions are satisfied.
(a) $W \mapsto \mathcal{R}(W)$ is an order-preserving bijection.

(b) If $W_1 \cap W_2 \neq \emptyset$, then $\Omega$ is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$. Conversely, if $\Omega$ is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$, then $\overline{W_1} \cap \overline{W_2} \neq \emptyset$, where the bar denotes closure.

(c) For each $W \in \mathcal{W}$, the adjoint action of the modular conjugation $J_W$ corresponding to the pair $(\mathcal{R}(W), \Omega)$ leaves the set $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ invariant.

(d) The group of (anti)automorphisms generated by the adjoint action of the modular conjugations $J_W$, $W \in \mathcal{W}$, acts transitively on $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$.

The core of this condition is part (c), which says that the modular conjugations generate a subgroup of the symmetric (permutation) group on the set $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$. So, in view of the correspondence between algebras and wedge regions, it is appropriate to say that these conjugations act geometrically. No a priori assumptions are made about the specific form of this action and the nature of the resulting group. This general formulation of the condition is also appropriate if one thinks of applications to nets labelled by other families of regions appearing, for example, in theories on curved space–times.

As was shown in [7], cf. also [1], the CGMA implies that the net satisfies wedge duality. So one may consistently label these conjugations by the reflections $\lambda \in \mathcal{P}_+$ about the edges of the wedges $W$, respectively $W'$, setting $J(\lambda) \equiv J_W = J_{W'}$. Moreover, the modular conjugations act on the net covariantly in the sense of relation (1.1) and satisfy the fundamental relation (1.2), see [7].

We make use now of the additional assumption that the net is locally generated. Then, by Corollary 3.2, $\lambda \mapsto J(\lambda)$ is a continuous map from the set of reflections $\lambda \in \mathcal{P}_+$, equipped with the topology induced by $\mathcal{P}_+$, to the group of (anti)unitary operators, equipped with the strong–*–topology. Using this information, we want to construct a continuous representation of the semidirect product $\mathcal{L}_+ \rtimes \mathbb{R}^4 = \mathcal{P}_+$ whose elements are denoted, as usual, by $(\Lambda, x)$.

Restricting attention first to the subgroup $\mathcal{L}_+$, we can apply Proposition 2.8 and extend the reflection maps $\lambda \mapsto J(\lambda, 0), \lambda \in \mathcal{L}_+$, to a continuous (anti)unitary representation $U$ of the proper Lorentz group $\mathcal{L}_+, (\Lambda, 0) \mapsto U(\Lambda, 0)$. Turning to the subgroup $\mathbb{R}^4$ of translations, let $\lambda \in \mathcal{L}_+$ be any reflection and let $x \in \mathbb{R}^4$ be such that $\lambda x = -x$. The set of such $x$ constitutes a two–dimensional subgroup of $\mathbb{R}^4$ containing timelike translations. Moreover, $(\lambda, x) = (1, x/2)(\lambda, 0)(1, -x/2)$, hence $(\lambda, x)$ is again a reflection. We put

$$U_\lambda(x) \equiv J(\lambda, x)J(\lambda, 0), \quad \lambda x = -x, \quad (4.1)$$

and note that it has been shown in [7, Sec. 4.3] that $U_\lambda$ defines a continuous unitary representation of the two–dimensional subgroup of translations $x$ satisfying the stated condition. As a matter of fact, the continuity of this representation follows

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2This result was established in [7] only for reflections about edges of certain specific wedges; the general statement follows from the special case by an application of relation (1.2).
directly from the CGMA without any further assumptions. We want to show that these representations of subgroups of translations can be combined with the above representation of the Lorentz group to yield a true representation of the proper Poincaré group.

Let \( x \in \mathbb{R}^4 \) be any timelike vector; so its stability subgroup in \( \mathcal{L}_+ \) is conjugate to the entire group of rotations. Hence, applying the arguments in the proof of Lemma 2.4 to the present situation, one finds that for all elements \( S \in \mathcal{L}_+ \) of the stability group of \( x \) one has \( U_{S\Lambda S^{-1}}(x) = U_\lambda(x) \). We may therefore omit the dependence of these operators on the reflections \( \lambda \) and set, for given timelike \( x \),

\[
U(1, x) \doteq U_\lambda(x),
\]

for any reflection \( \lambda \) such that \( \lambda x = -x \). Now if \( x, y \) are positive timelike, their sum is positive timelike, too, and there is a reflection \( \lambda \in \mathcal{L}_+ \) such that \( \lambda x = -x \), \( \lambda y = -y \). Hence we can compute

\[
U(1, x)U(1, y) = U_\lambda(x)U_\lambda(y) = U_\lambda(x+y) = U(1, x+y),
\]

where, in the second equality, we made use of the fact established above that \( U_\lambda \) is a representation of the respective two-dimensional subgroup. We also note that, for timelike \( x \) and admissible \( \lambda \),

\[
U(1, x)^{-1} = J(\lambda, 0) J(\lambda, x) = J(\lambda, 0) J(\lambda, x) J(\lambda, 0)^2
\]

\[
= J(\lambda, -x) J(\lambda, 0) = U(1, -x),
\]

where we made use of relation (1.2). The preceding two relations allow us to extend the unitary operators \( U(1, z) \) to arbitrary translations \( z \in \mathbb{R}^4 \). Indeed, any \( z \) can be decomposed into \( z = x - y \), where \( x, y \) are positive timelike, and, if \( z = x' - y' \) is another such decomposition, one has \( x + y' = y + x' \). Hence \( U(1, x)U(1, y) = U(1, x + y') = U(1, y + x') = U(1, y)U(1, x') \). All operators appearing in this equality commute with each other according to relation (4.3). Thus, making use of relation (4.4), one gets \( U(1, x)U(1, -y) = U(1, x')U(1, -y') \). One can therefore consistently define for \( z \in \mathbb{R}^4 \)

\[
U(1, z) \doteq U(1, x)U(1, -y), \quad x, y \in V_+, x - y = z.
\]

Based on the equalities established thus far, one can show that \( U \) defines a continuous unitary representation of the subgroup \( \mathbb{R}^4 \) of translations. We omit the straightforward proof of this fact.

Since timelike vectors \( x \) are mapped by the elements \( \Lambda \in \mathcal{L}_+ \) to timelike vectors, one can also compute the adjoint action of the unitary operators \( U(\Lambda, 0) \) on \( U(1, x) \). Fixing \( x \) and making use of relations (1.1) and (1.2), one gets for any admissible reflection \( \lambda \) the equation

\[
U(\Lambda, 0)U(1, x)U(\Lambda, 0)^{-1} = U(\Lambda, 0) J(\lambda, x) J(\lambda, 0) U(\Lambda, 0)^{-1}
\]

\[
= J(\Lambda\Lambda\lambda^{-1}, \lambda x) J(\Lambda\lambda\lambda^{-1}, 0) = U(1, \lambda x).
\]

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This equation can be extended to arbitrary \( x \in \mathbb{R}^4 \) by means of relation (4.5). Hence, setting
\[
U(\Lambda, x) \doteq U(1, x) U(\Lambda, 0), \quad (\Lambda, x) \in \mathcal{P}_+,
\] we arrive at a continuous unitary representation of \( \mathcal{P}_+ \). Since for any reflection \( \lambda \in \mathcal{L}_+ \) and \( x \in \mathbb{R}^4 \) such that \( \lambda x = -x \) we have
\[
U(\lambda, x) = U(1, x)U(\lambda, 0) = U(1, x/2)U(\lambda, 0)U(1, x/2)^{-1} = U(1, x/2)J(\lambda, 0)U(1, x/2)^{-1} = J(\lambda, x),
\] we also see that the representation \( U \) extends the reflection map \( J \).

As a consequence of relation (1.1) and the fact that the operators \( U(\Lambda, x) \) are certain specific products of modular conjugations, these operators act covariantly on the net. Moreover, because of the invariance of \( \Omega \) under the action of the modular conjugations, \( \Omega \) is invariant under the action of \( U(\Lambda, x), (\Lambda, x) \in \mathcal{P}_+ \). We summarize these results in the following theorem.

**Theorem 4.1** Let \( W \mapsto \mathcal{R}(W) \) be a locally generated net and \( \Omega \) a state vector complying with the CGMA, i.e. conditions (a) to (d). Then the net satisfies wedge duality and there is a continuous (anti)unitary representation \( U \) of \( \mathcal{P}_+ \) which leaves \( \Omega \) invariant and acts covariantly on the net. Moreover, for any given wedge \( W \) and reflection \( \lambda \) about its edge, \( U(\lambda) \) is the modular conjugation corresponding to the pair \( (\mathcal{R}(W), \Omega) \).

Although \( \Omega \) is invariant under the action of \( U \) and as such clearly is a distinguished state, the CGMA does not imply that it is necessarily a ground state. In fact, there exist examples conforming with the hypothesis of Theorem 4.1 for which the joint spectrum \( \text{sp} U \) of the generators of the subgroup of translations is all of \( \mathbb{R}^4 \), cf. [7, Sec. 5.3]. So, for the characterization of ground states on Minkowski space, where \( \text{sp} U \) is contained in a light cone, one has to supplement the CGMA by additional constraints. A conceptually simple and quite general requirement is the modular stability condition CMS, proposed in [7]. We recall this condition here as the last item in our list of constraints characterizing Poincaré invariant ground states describing the vacuum.

(e) For any \( W \in \mathcal{W} \), the elements \( \Delta^t_W, t \in \mathbb{R}, \) of the modular group corresponding to \( (\mathcal{R}(W), \Omega) \) are contained in the group generated by all finite products of the modular involutions \( \{J_W\}_{W \in \mathcal{W}} \).

We refer the interested reader to [5, 7] for a discussion of the background of this condition and a brief account of other interesting approaches towards an algebraic characterization of ground states. Within the present context, the implications of condition (e) are twofold. On the one hand, we can apply Lemma 3.3 and replace in Theorem 4.1 the assumption that the net is locally generated by the weaker requirement that there is some generating family \( \mathcal{C} \) of regions such that \( \Omega \) is cyclic
for the algebras $\mathcal{R}(C)$, $C \in \mathcal{C}$. On the other hand, we can employ the results in [5,7], which imply that the representation $U$ has the desired spectral properties.

Theorem 4.2 Let $W \mapsto \mathcal{R}(W)$ be a net and $\Omega$ a state vector satisfying the CGMA and CMS, i.e. conditions (a) to (e), and let $\mathcal{C}$ be some generating family of regions such that $\Omega$ is cyclic for the algebras $\mathcal{R}(C)$, $C \in \mathcal{C}$. Then the net satisfies wedge duality and there is a representation $U$ of $\mathcal{P}_+$ with properties described in the preceding theorem such that $\text{sp} U \subset \overrightarrow{V}_+$ or $\text{sp} U \subset -\overrightarrow{V}_+$, where $\overrightarrow{V}_+$ denotes the closed forward lightcone.

The fact that both the forward and the backward lightcone appear as possible supports of the spectrum of the generators of the translations can be understood easily: Neither the CGMA nor the CMS contains any input about the arrow of time. By choosing proper coordinates, one may therefore assume without loss of generality that $\text{sp} U \subset \overrightarrow{V}_+$. With this convention, $U$ is then the only continuous unitary representation of the spacetime translations which acts covariantly on the given net and leaves $\Omega$ invariant [7, Prop. 5.2], cf. also [4, Prop. 2.4].

We have thus attained our goals, the characterization of vacuum states in Minkowski space and the construction of continuous unitary representations of the isometry group of this space, using conditions which are expressed solely in terms of the algebraically determined modular objects. All technical assumptions about continuity properties of the net have been eliminated from this analysis. Instead, they follow in a natural manner from the net structure. And Proposition 2.8 has made clear that once this continuity has been assured, the modular reflection map determines the representation of the isometry group in a canonical and unique manner.

There are already strong indications from studies of nets on de Sitter [7,11] and anti-de Sitter space–times [6], and also more general Robertson–Walker space–times [8,9] that this strategy is applicable to a large class of space–times of physical interest. This is true in spite of the fact that there is no translation subgroup in the isometry group of these spaces, and thus the standard definition of vacuum state is inapplicable. We therefore believe that the analysis of the modular data in quantum field theories on curved space–time deserves further attention.

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