Simplicial Waldhausen categories and topological K-theory

Yi-Sheng Wang

Abstract

Utilizing simplicial Waldhausen theory, we prove that the geometric realization of the topologized category of bounded chain complexes over $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{R}$) is an infinite loop space that represents connective complex (resp. real) topological $K$-theory. The key ingredient in our proof is a generalized Waldhausen comparison theorem.

Keywords: Simplicial Waldhausen categories; topological $K$-theory; topologized category of bounded chain complexes.

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Introduction

The standard category of finite-dimensional vector spaces over $\mathbb{F}$, denoted by $\mathcal{V}_F$, is an enriched category whose set of objects and space of morphisms are given by the set $\{\mathbb{F}^n\}_{n \in \mathbb{N} \cup \{0\}}$ and the space of $m$-by-$n$ matrices $\prod_{m,n \in \mathbb{N} \cup \{0\}} M_{m,n}(\mathbb{F})$, respectively, where $M_{m,n}(\mathbb{F})$ is topologized as $\mathbb{F}^{mn}$. Its associated category of bounded chain complexes, denoted by $\text{Ch}^b(\mathcal{V}_F)$, is an internal category in $\text{Top}$ (topologized category) consisting of bounded chain complexes in $\mathcal{V}_F$ and chain maps between them (Remark 2.5). It is not difficult to see that the
geometric realization of the nerve of the subcategory of isomorphisms in $\mathcal{V}_E$, denoted by $|w\mathcal{V}_E|$, classifies vector bundles over a paracompact Hausdorff space, and that the geometric realization of the nerve of the internal subcategory of quasi-isomorphisms in $\text{Ch}^b(\mathcal{V}_F)$, denoted by $|w\text{Ch}^b(\mathcal{V}_E)|$, is a group-like $H$-space with $\pi_0(|w\text{Ch}^b(\mathcal{V}_E)|) = \mathbb{Z}$. It is thus natural to ask if the canonical map

$$|w\mathcal{V}_E| \rightarrow |w\text{Ch}^b(\mathcal{V}_E)|,$$

is a topological group completion. In other words, we want to show that the space $|w\text{Ch}^b(\mathcal{V}_E)|$ represents topological $K$-theory. Indeed, this has been observed by Segal in [15, Section 2], where he suggests a geometric approach to this question using the notion of bundles of chain complexes developed in [14, Appendix]. Though we are not able to work out the approach proposed in [15], we discover a homotopy-theoretic way to solve this problem.

Our strategy can be summarized in the following commutative diagram

$$
\begin{array}{ccc}
|wS\mathcal{V}_E| & \xrightarrow{b} & |wS\text{Ch}^b(\mathcal{V}_E)| \\
\downarrow{a} & & \downarrow{c} \\
|wN\mathcal{V}_E| & \rightarrow & |wN\text{Ch}^b(\mathcal{V}_E)|
\end{array}
$$

where $\mathcal{V}_E$. (resp. $\text{Ch}^b(\mathcal{V}_E)$.) is the singularization of the internal category $\mathcal{V}_E$ (resp. $\text{Ch}^b(\mathcal{V}_E)$), and $N$. (resp. $S$.) stands for Segal’s $N$-construction (resp. Waldhausen’s $S$-construction). To make sense of the diagram, the simplicial category $\mathcal{V}_E$. (resp. $\text{Ch}^b(\mathcal{V}_E)$.) has to admit a simplicial Waldhausen structure. In fact, we shall prove a stronger statement:

**Theorem 0.1** (Theorem 2.1). The simplicial categories $\mathcal{V}_E$. and $\text{Ch}^b(\mathcal{V}_E)$. are simplicial exact categories.

By Theorem 0.1 we see that (2) is well-defined. The next step is then to prove that the solid arrows in (2) are homotopy equivalences. While the fact that the maps $a$ and $b$ are homotopy equivalences follows easily from Waldhausen’s comparison theorem [19, Section 1.8] and a straightforward generalization of the Gillet-Waldhausen theorem [17], to prove that the map $c$ is a homotopy equivalence is not an easy task; one reason is that, for a given $k$, cofibrations in $\text{Ch}^b(\mathcal{V}_E)_k$ are not always splittable. In fact, we need to make use of the simplicial structure of the simplicial category $\text{Ch}^b(\mathcal{V}_E)$ to connect a non-splittable cofibration to a splittable one. For arbitrary simplicial Waldhausen category, we prove a generalized Waldhausen comparison theorem.

**Theorem 0.2.** (Theorem 1.25) Let $C$. be a simplicial Waldhausen category with $C_k$ additive, for every $k$, and suppose that it is equipped with a sum functor, its cofibrations are weakly splittable, and $wC_k$ satisfies the extension axiom. Then the canonical simplicial exact functor

$$N.C. \rightarrow S.C.$$

induces a homotopy equivalence

$$|wN.C.| \rightarrow |wS.C.|.$$
A cofibration in $C_0$ is weakly splittable if and only if there exists a path—some 1-simplices—connecting the given cofibration to a splittable one. We shall see that the simplicial Waldhausen category $Ch^b(V_{\mathcal{F}})$ satisfies the conditions in Theorem 0.2. Now, observe that the group completion theorem for Segal $\Gamma$-spaces gives us the following topological group completion

$$|w'V_{\mathcal{F}}.| \to \Omega|wN.V_{\mathcal{F}}.|.$$ \hfill (3)

In addition, since the nerve of the enriched category $w'V_{\mathcal{F}}$ is proper and has the homotopy type of a CW-complex at each degree, we have the homotopy equivalence

$$|w'V_{\mathcal{F}}.| \to |w'V_{\mathcal{F}}|,$$ \hfill (4)

induced by the counit $|\text{Sing } (-)| \mapsto \text{Id}$, where $\text{Sing }$ is the singular functor. The homotopy equivalence (4) implies that the space $|w'V_{\mathcal{F}}.|$ also classifies vector bundles over a paracompact Hausdorff space, and hence, by (3) and (2), the loop space of any space in (2) represents topological $K$-theory. On the other hand, we shall also prove that the nerve of the internal category $w Ch^b(V_{\mathcal{F}})$ is proper and has the homotopy type of a CW-complex at each degree. Thus, there is a span of homotopy equivalences

$$|w Ch^b(V_{\mathcal{F}})| \leftrightarrow |w Ch^b(V_{\mathcal{F}})| \to \Omega|wN. Ch^b(V_{\mathcal{F}})|,$$

in view of the fact that $\pi_0(|w Ch^b(V_{\mathcal{F}})|) = \mathbb{Z}$. We summarize these results in our main theorem:

**Main Theorem.** There are natural homotopy equivalences of infinite loop spaces

$$|w Ch^b(V_{\mathcal{F}})| \leftrightarrow |w Ch^b(V_{\mathcal{F}})| \to \Omega|wN. Ch^b(V_{\mathcal{F}})|,$$

and any of the spaces above represents topological $K$-theory.

One important application of the Main Theorem is the topological group completion (1), and another is a new proof of the following theorem.

**Theorem 0.3.** Let $ku_{2,1}$ be the connected component of the topological $K$-theory space that contains the multiplicative identity. Then the $H$-structure induced by the tensor product of vector spaces gives an infinite loop space structure on $ku_{2,1}$.

The paper is organized as follows: In the first two subsections of Section 1, we recollect some basic notions and theorems in the theory of simplicial Waldhausen categories and simplicial $S$-categories, which are simplicial objects in the category of small categories equipped with categorical sums. In particular, the notion of simplicial sum functors (Definitions 1.4 and 1.18) and properties of simplicial bi-exact functors are discussed in details in these two subsections. The third subsection of Section 1 is devoted to generalized Waldhausen comparison theorems. Section 2 concerns the simplicial exact structure on $V_{\mathcal{F}}$. Section 3 contains the main applications of the paper. Especially, the proofs of the Main Theorem and Theorem 0.3 are completed in Section 3. In Appendix A, a version of the realization lemma needed in this paper is presented. Though this lemma appears to be well-known, we provide a detailed proof for the sake of completeness. The definition and properties of homotopy bi-linear maps, which seem to be less well documented, are recalled in Appendix B.

Throughout the paper, we work primarily in the Quillen model category of $k$-spaces, denoted by $\textbf{Top}$. However, on a few occasions, we shall make use of some properties of the Strøm model category of topological spaces as well; in those cases, the context shall make it clear the model structure we are referring to. We
use the category of prespectra, denoted by $\mathcal{P}$, to model the stable homotopy category; some basic properties of the category $\mathcal{P}$ can be found in [10] or [22, Appendix]. Given $C$ (resp. $C_\cdots$) an internal category in $\text{Top}$ (resp. multi-simplicial category), the space $|C|$ (resp. $|C_\cdots|$) stands for the geometric realization (resp. the iterated geometric realization (see Appendix [A]) of the nerve of $C$ (resp. $C_\cdots$).

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1. Simplicial Waldhausen theory

1.1. Simplicial Waldhausen categories

Simplicial Waldhausen categories appear in Waldhausen $K$-theory as part of the $S$-construction [19]. Though most theorems in [19, Section 1] are stated in terms of Waldhausen categories, almost all of them can be easily generalized to simplicial Waldhausen categories. In this subsection, we review some basic definitions in Waldhausen $K$-theory [19, Section 1] and their generalizations to simplicial Waldhausen categories.

A Waldhausen category is a small category $C$ with zero object $*$ and equipped with two special subcategories $\text{co} C$ and $\text{w} C$ such that they satisfy the following axioms:

1. The subcategory of isomorphisms $\text{iso} C$ is contained in the intersection of $\text{co} C$ and $\text{w} C$;
2. The morphism $* \to A$ is in $\text{co} C$, for every $A \in C$;
3. Pushouts along any morphism in $\text{co} C$ exist;
4. The subcategory $\text{w} C$ satisfies the gluing lemma.

The morphisms in $\text{w} C$ (resp. $\text{co} C$) are called weak equivalences (resp. cofibrations). An exact functor of Waldhausen categories is a functor that preserves cofibrations, weak equivalences and pushouts along a cofibration. A natural weak equivalence between two exact functors

$$\phi : F \Rightarrow G : C \to D$$

is a natural transformation that takes value in $\text{w} D$. $W$ denotes the 2-category comprises Waldhausen categories, exact functors and natural weak equivalences.

**Definition 1.1.** (i) A simplicial Waldhausen category is a simplicial object in $W$, meaning a (covariant) functor

$$C : \Delta^{op} \to W,$$

where $\Delta^{op}$ is the opposite of the simplex category.

(ii) A simplicial exact functor of simplicial Waldhausen categories

$$F : C \to D,$$

is a natural transformation that takes value in $\text{w} D$. or, equivalently, a simplicial functor with $F_k$ exact, for every $k$.

(iii) A simplicial natural weak equivalence between the simplicial exact functors

$$F : C \to D;$$

$$G : C \to D.$$
is a simplicial functor
\[ \phi : C \times [1] \to D. \]
such that \( \phi_k \) is a natural weak equivalence between \( F_k \) and \( G_k \), for every \( k \), where \([1] := \{0 < 1\} \) is considered as a simplicial category with trivial structure maps.

\( sW \) denotes the category consisting of simplicial Waldhausen categories, simplicial exact functors and simplicial natural weak equivalences.

**Remark 1.2.** In a similar manner, one can define multi-simplicial Waldhausen categories, multi-simplicial exact functors, multi-simplicial natural weak equivalences. The 2-category consisting of bi-simplicial Waldhausen categories, bi-simplicial exact functors and bi-simplicial weak equivalence is denoted by \( biW \), and \( mW \) denotes the category of multi-simplicial Waldhausen categories and the corresponding 1-(2-)morphisms.

Recall that Waldhausen’s \( S \)-construction induces a functor from the category \( mW \) to \( P \), the category of prespectra,
\[ K : mW \to P, \]
where the \( n \)-th component of \( K(C...) \), for any \( C... \in mW \), is given by the space \( |wS^{(n)}C...| \) when \( n \geq 1 \), and the space \( \Omega|wS^{(n)}C...| \) when \( n = 0 \).

As in Waldhausen \( K \)-theory, one can define the (2-) fiber product of a cospan of multi-simplicial Waldhausen categories \( C... \to E... \leftarrow D... \); the resulting multi-simplicial Waldhausen category is denoted by \( C... \times_{E...}^{(2)} D... \). For instance, given a multi-simplicial exact functor \( C... \to D... \), we can define the following cospan of multi-simplicial Waldhausen categories
\[ S.C... \to S.D... \leftarrow PS.D..., \]
where \( PS.D... \) is the path object in \( S \)-direction, and the corresponding fiber product is denoted by \( S.(C... \to D...) \). Another important example is the extension object \( E(A..., C..., B...) \), which denotes the fiber product of the cospan of inclusions
\[ \mathcal{A}... \leftarrow C... \leftarrow \mathcal{B}..., \]
where \( \mathcal{A}... \) and \( \mathcal{B}... \) are two multi-simplicial subcategories of \( C.... \)

The next two corollaries can be derived from the additivity theorem in [19, Proposition 1.3.2] and the realization lemma (Lemma [A.2]). For the sake of simplicity, they are stated in terms of simplicial Waldhausen categories even though both hold for multi-simplicial Waldhausen categories.

**Corollary 1.3 (The Additivity Theorem).** Let \( \mathcal{A}, \mathcal{B} \) and \( C \) be simplicial Waldhausen categories and \( \mathcal{A} \) and \( \mathcal{B} \) be simplicial subcategories of \( C \) such that the inclusions are simplicial exact functors. Then the simplicial exact functor
\[ E(\mathcal{A}..., C..., B...) \to \mathcal{A}... \times B... \]
\[ (A \to C \to B) \mapsto (A, B) \]
induces a homotopy equivalence
\[ |wS.E(\mathcal{A}..., C..., B...)| \to |wS.\mathcal{A}... \times wS.\mathcal{B}...|. \]
Proof. By Proposition 1.3.2 in [19], we know, for every $[k] \in \Delta^{op}$, the exact functor of Waldhausen categories

$$E(\mathcal{A}_k, \mathcal{B}_k) \to \mathcal{A}_k \times \mathcal{B}_k$$

$$A \to C \to B \mapsto (A, B)$$

induces a homotopy equivalence

$$|w.S.E(\mathcal{A}_k, \mathcal{B}_k)| \to |w.S.\mathcal{A}_k| \times |w.S.\mathcal{B}_k|.$$  

The claim then follows quickly from Lemma A.2 (the realization lemma). □

Now, in [19, Proposition 1.3.2], there are two more equivalent statements of the additivity theorem. The first one says the following two maps

$$t : |w.S.E(C)| \to |w.S.C|$$

$$s \lor q : |w.S.E(C)| \to |w.S.C|,$$

induced by the exact functors

$$E(C) \to C$$

$$(A \to C \to B) \mapsto C$$

$$\mapsto A \lor B,$$

are homotopic, whereas the second one asserts that, given an exact sequence of exact functors

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' : C \to \mathcal{D},$$

the exact functors $\mathcal{F}' \lor \mathcal{F}''$ and $\mathcal{F}$ induce two homotopic maps

$$\mathcal{F}' \lor \mathcal{F}'' : |w.S.C| \to |w.S.D|;$$

$$\mathcal{F} : |w.S.C| \to |w.S.D|.$$  

In the case of multi-simplicial Waldhausen categories, these two statements do not make sense, however. In fact, in order to define the simplicial exact functor $\mathcal{F}' \lor \mathcal{F}''$ and the assignment $(A \to C \to B) \mapsto A \lor B$, one needs a compatible way of choosing categorical sums in $C$ (or $\mathcal{D}$), which leads to the following definition.

Definition 1.4. A (simplicial) sum functor of a simplicial Waldhausen category $C$ is a map of simplicial sets

$$\theta : O(C) \times O(C) \to O(C)$$

such that

$$\theta_k(A, B) = (A \to A \lor B \leftarrow B)$$

is a diagram of categorical sum, for every $A, B \in C_k$ and $[k] \in \Delta^{op}$. In view of the universal property of categorical sums, the simplicial map $\theta$ can be extended to a simplicial exact functor

$$C. \times C. \to \text{Cosp}(C).$$
by the assignment:

\[
\begin{array}{ccc}
A & & B \\
A' & \\ \downarrow & & \downarrow \\
B' & & \downarrow \\
\end{array} & \mapsto & \\
\begin{array}{ccc}
A \to A' & \leftarrow & B \\
\downarrow & & \downarrow \\
A' & \leftarrow & A' \to B' \\
\end{array}
\]

This is why we call it (simplicial) sum functor. Precomposing the projection functor

\[
\text{Cospan}(C.) \to C. \times C.
\]

\[
(A \to C \leftarrow B) \mapsto (A, B)
\]

with the sum functor, one can further extend it to the simplicial exact functor below

\[
\text{Cospan}(C.) \to \text{Cospan}(C.)
\]

\[
(A \to C \leftarrow B) \mapsto (A \to A \obs B \leftarrow B).
\]

This extension comes with a simplicial natural transformation

\[
\phi : \theta \mapsto \text{id},
\]

that encodes the unique morphism from \(A \obs B\) to \(C\), meaning

\[
\phi : (A \to C \leftarrow B) \mapsto \begin{array}{ccc}
A \obs B & \leftarrow & B \\
\downarrow & & \downarrow \\
C & \leftarrow & \end{array}
\]  \hspace{1cm} (5)

The point here is that, once the choices of categorical sums are compatible, all these induced functors and natural transformations are automatically well-defined (compatible with the simplicial structure). For this reason, we use the same notation \(\theta\) for all these extensions of the (simplicial) sum functor.

With this definition, the following can be readily derived from Proposition 1.3.2 in [19].

**Corollary 1.5.**  \hspace{0.5cm} (i) Given \(C\) a simplicial Waldhausen category equipped with a sum functor \(\theta\), then the following simplicial exact functors

\[
E(C.) \to C.
\]

\[
(A \to C \to B) \mapsto C
\]

\[
\mapsto A \obs B.
\]

induce two homotopic maps

\[
t : \text{wS.E}(C.) \to \text{wS.C}.
\]

\[
s \obs q : \text{wS.E}(C.) \to \text{wS.C}.
\]

(ii) Let the sequence

\[
\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''
\]

be an exact sequence of simplicial exact functors from \(C\) to \(\mathcal{D}\). Namely, for every \(k\), the sequence \(\mathcal{F}'_k \to \mathcal{F}_k \to \mathcal{F}''_k\) is an exact sequence of exact functors. Suppose \(\mathcal{D}\) admits a sum functor \(\theta\). Then \(\mathcal{F}' \obs \mathcal{F}''\) is given by the composition of the following simplicial exact functors

\[
C \xrightarrow{(\mathcal{F}, \mathcal{F}')} \mathcal{D} \times \mathcal{D} \xrightarrow{\theta} \mathcal{D}.
\]
and the induced maps

\[ F' \vee \theta F'' : |wS.C.| \to |wS.D.|; \]
\[ F : |wS.C.| \to |wS.D.| \]

are homotopic.

**Remark 1.6.** A (simplicial) sum functor gives an alternative description of the \( H \)-structure on the loop space of \( K(C)_0 \) (Lemma B.8). More precisely, let \( C \) be a simplicial Waldhausen category equipped with a sum functor \( \theta \). Then the \( H \)-structure (addition) on \( K(C)_0 \) can be realized by

\[ K(\theta) : K(C)_0 \times K(C)_0 \to K(C)_0. \]

The following fibration theorem implies that \( K(C) \) is an \( \Omega \)-prespectrum, and hence \( K(C) \) is an infinite loop space.

**Theorem 1.7.** Given a simplicial exact functor

\[ A... \to B..., \]

then the following sequence

\[ |wS.B...| \to |wS.(A... \to B...)| \to |wS.A...| \]

is a homotopy \( q \)-fibration (\( := \) homotopy Serre fibration in \( \text{Top} \)).

**Proof.** From the proof of Proposition 1.5.5 in [19] and Lemma A.2, we know that all vertical maps in the commutative diagram below induce homotopy equivalences.

\[
\begin{array}{ccc}
|wS.B...| & \to & |wS.(A... \to B...)| \\
\downarrow & & \downarrow \\
|wS.A...| & \to & |wS.A...|
\end{array}
\]

Thus, for every \( n \), the upper sequence is a homotopy Kan fibration after applying the nerve construction (in \( w \)-direction) and the diagonal functor. Because \( |wS.A...| \) is connected, for every \( n \), the Bousfield-Friedlander theorem (see [6, IV.4]) applies to the sequence

\[ wS.B... \to wS.(A... \to B...), \]

and hence the diagonals of their nerves constitute a homotopy Kan fibration. The assertion then follows from the fact that the geometric realization of a homotopy Kan fibration is a homotopy \( q \)-fibration.

**Corollary 1.8.** Given a multi-simplicial Waldhausen category \( A... \), the prespectrum \( K(A...) \) is an \( \Omega \)-prespectrum.
Proof. Via Theorem 1.7 and the following identification
\[ S^{(i)} S.(\mathcal{A} \ldots \to \mathcal{B} \ldots) = S.S.(S^{(i-1)} \mathcal{A} \ldots \to S^{(i-1)} \mathcal{B} \ldots), \] (6)
we know the sequence
\[ |wS^{(i)} \mathcal{B} \ldots| \to |wS^{(i)} S.(\mathcal{A} \ldots \to \mathcal{B} \ldots)| \to |wS^{(i+1)} \mathcal{A} \ldots| \] (7)
is a homotopy $q$-fibration, for $i \geq 1$. When $\mathcal{F} = \text{id}$, it implies the sequence
\[ |wS^{(i)} \mathcal{A} \ldots| \to |wS^{(i)} PS.\mathcal{A} \ldots| \to |wS^{(i+1)} \mathcal{A} \ldots| \]
is a homotopy $q$-fibration.

Now recall that, given a simplicial object $X$, the inclusion $X_0 \hookrightarrow PX$, induced by $[n+1] \to [0] \in \Delta$, is a deformation retract with the retraction and the required homotopy given by
\[
[0] \to [n+1] \\
0 \mapsto 0
\]
and
\[
(a : [n] \to [1]) \mapsto (\phi_a : [n+1] \to [n+1])
\]
(8)
\[
\phi_a(i+1) := \begin{cases} 
  i+1 & \text{if } a(i) = 1 \\
  0 & \text{if } a(i) = 0,
\end{cases}
\]
respectively (See [19, p.341-342]). Applying this to our case, we obtain the homotopy,
\[ ws^{(i)} PS.\mathcal{A} \ldots \times \Delta^1 \to ws^{(i)} PS.\mathcal{A} \ldots, \] (9)
which induces the homotopy commutative diagram below
\[
\begin{array}{ccc}
\Omega|ws^{(i+1)} \mathcal{A} \ldots| & \to & * \\
|ws^{(i)} \mathcal{A} \ldots| & \to & |ws^{(i)} PS.\mathcal{A} \ldots| \\
|ws^{(i)} PS.\mathcal{A} \ldots| & \to & |ws^{(i+1)} \mathcal{A} \ldots|
\end{array}
\]
In particular, the vertical map on the right-hand side coincides with the adjoint of the inclusion
\[ |ws^{(i)} \mathcal{A} \ldots| \wedge S^1 \to |ws^{(i+1)} \mathcal{A} \ldots|. \]
Thus the map
\[ |ws^{(i)} \mathcal{A} \ldots| \to \Omega|ws^{(i+1)} \mathcal{A} \ldots| \]
is a homotopy equivalence by the five lemma, so $K(\mathcal{A} \ldots)$ is an $\Omega$-presepctrum. \qed
The following discusses a generalized Gillet-Waldhausen theorem. We consider only the exact categories equipped with an embedding into an abelian category such that the embedding is closed under extensions and kernels of epimorphisms—the latter means that, if \( \iota : C \to \mathcal{A} \) is an embedding of such, and \( \iota(f) \) is an epimorphism in \( \mathcal{A} \), then \( f \) is an admissible epimorphism in \( C \). The following criterion and construction come in handy when we want to check if a given exact category admits such an embedding. A morphism \( r : A \to B \) is a weakly split epimorphism in \( C \) if and only if there exists \( i : B \to A \) such that \( r \circ i = \text{id}_B \). If \( C \) is an exact category such that all weakly split epimorphisms are admissible, then the associated Gabriel-Quillen embedding \( \iota : C \to \mathcal{A}^C \) is closed under extensions and kernels of epimorphisms (see [17, Example 1.11.5, A.7.1 and A.7.6]). An exact functor of exact categories is an additive functor that preserves short exact sequences. We let \( \mathcal{E} \) denote the category of small exact categories with exact functors. The adjective “small” is often dropped as all exact categories considered in this article are assumed to be small.

Now, let \( C \) be an exact category satisfying the conditions described above. Then one can define the associated category of bounded chain complexes in \( C \), denoted by \( \text{Ch}^b(C) \). The category \( \text{Ch}^b(C) \) can be made into a Waldhausen category by declaring quasi-isomorphisms and degree-wise admissible monomorphisms to be weak equivalences and cofibrations, respectively. The Waldhausen structure admits a cylinder functor, and its subcategory of weak equivalences \( \text{wCh}^b(C) \) is extensional and saturated, and satisfies the cylinder axiom. The Gillet-Waldhausen theorem (see [17, Theorem 1.11.7]) asserts that the natural exact functor

\[
C \to \text{Ch}^b(C).
\]

induces a \( \pi_* \)-isomorphisms of prespectra

\[
\text{K}(C) \to \text{K(Ch}^b(C)).
\]

In particular, this gives us the following homotopy equivalence

\[
|\text{wS}^{(k)}C| \to |\text{wS}^{(k)} \text{Ch}^b(C)|,
\]

for every \( k \).

**Definition 1.9.**

(i) A simplicial exact category \( C_{\cdot} \), is a simplicial object in \( \mathcal{E} \) such that there exists an embedding \( C_k \hookrightarrow \mathcal{A}_k \) that is closed under extension and kernels of epimorphisms, for every \( k \).

(ii) Given a simplicial exact category \( C_{\cdot} \), its associated simplicial category of chain complexes \( \text{Ch}^b(C_{\cdot}) \) admits a simplicial Waldhausen category structure with the Waldhausen category structure on \( \text{Ch}^b(C_k) \) being the one defined earlier. In fact, \( \text{Ch}^b(C_{\cdot}) \) is a simplicial exact category (see [17, 1.11.8] and [13, Sec.9]).

**Remark 1.10.** Notice that the collection of ambient abelian categories \( \{\mathcal{A}_k\} \) does not need to constitute a simplicial object in the category of abelian categories. Note also that the closedness under kernel of epimorphisms ensures that it is independent of the choice of the ambient as one can define a quasi-isomorphism of chain complexes in such an exact category to be a chain map whose mapping cone is acyclic, where a chain complex \( C^\bullet \) is acyclic if and only if \( C^i \to C^{i+1} \) factors through \( Z^{i+1}C^\bullet \), namely

\[
C^i \to Z^{i+1}C^\bullet \to C^{i+1},
\]

for every \( i \), such that the sequence

\[
Z^iC^\bullet \to C^i \to Z^{i+1}C^\bullet
\]

is exact in the given exact category.
Theorem 1.11. Given a simplicial exact category $C$, the map of prespectra

$$K(C) \to K(Ch^b(C))$$

is a $\pi_*$-isomorphism in $P$.

Proof. By the Gillet-Waldhausen theorem, we have the following homotopy equivalence

$$|wS^{(n)}C_k| \to |wS^{(n)}Ch^b(C_k)|,$$
for every $n, k$. Applying the realization lemma (Lemma A.2), we find the map

$$|wS^{(n)}C| \to |wS^{(n)}Ch^b(C)|,$$
is also a homotopy equivalence, for every $n$. The theorem thus follows. \[\square\]

The notion of bi-exact functor can also be generalized to (multi-) simplicial Waldhausen categories. A bi-exact functor of Waldhausen categories is a bi-functor $\land : A \times B \to C$ such that the associated functors,

$$- \land B : A \to C;$$
$$A \land - : B \to C;$$

are exact, and given $A \to A' \in coA$ and $B \to B' \in coB$, the natural morphism from a pushout of the cospan $(A \land B' \leftarrow A \land B \to A' \land B)$ to $A' \land B'$ is in $coC$.

Definition 1.12. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be three simplicial Waldhausen categories. Then a simplicial bi-exact functor

$$\land : \mathcal{A} \times \mathcal{B} \to \mathcal{C};$$
is a simplicial bi-functor such that $\land_k : \mathcal{A}_k \times \mathcal{B}_k \to \mathcal{C}_k$ is bi-exact, for each $k$.

The lemma below is a simplicial version of [19, p.342]; we give a detailed proof using Waldhausen’s argument.

Lemma 1.13. A simplicial bi-exact functor

$$\land : \mathcal{A} \times \mathcal{B} \to \mathcal{C},$$
induces a map

$$\Lambda_{i,j} : |wS^{(i)}A_\land | \land |wS^{(j)}B| \to |wwS^{(i+j)}C|,$$
for every $i, j \in \mathbb{N} \cup \{0\}$ such that the diagram below commutes

$$\begin{array}{ccc}
|wS^{(i+1)}A_\land | \land |wS^{(j)}B| & \xrightarrow{\Lambda_{i+1,j}} & |wwS^{(i+j+1)}C| \\
\uparrow & & \uparrow \\
S^1 \land |wS^{(i)}A_\land | \land |wS^{(j)}B| & \xrightarrow{S^1 \land \Lambda_{i,j}} & S^1 \land |wwS^{(i+j)}C|
\end{array}$$
Proof. 1. Firstly, we observe that the assumption implies that the following simplicial bi-functors

\[ S_n \mathcal{A} \times \mathcal{B} \to S_n \mathcal{C} \]
\[ \langle \{A_{i,j}\}, B \rangle \mapsto \{A_{i,j} \land B\} =: \{C_{i,j}\}; \]
\[ \mathcal{A} \times S_n \mathcal{B} \to S_n \mathcal{C} \]
\[ (A, \{B_{i,j}\}) \mapsto \{A \land B_{i,j}\} =: \{C_{i,j}\} \]

are well-defined and bi-exact. Now, suppose the bi-functor

\[ S_{i}^j \mathcal{A} \times S_{i}^j \mathcal{B} \to S_{i}^j S_{i}^j \mathcal{C}. \]

is also defined and bi-exact, where \( I \in \mathbb{N}^l \) and \( J \in \mathbb{N}^j \). Then the observation (10) gives us the simplicial bi-exact functors

\[ S_{i+1}^j \mathcal{A} \times S_{i}^j \mathcal{B} \to S_{i+1}^j S_{i}^j \mathcal{C}. \]
\[ S_{i}^j \mathcal{A} \times S_{i+1}^j \mathcal{B} \to S_{i}^j S_{i+1}^j \mathcal{C}, \]

where \( I' \in \mathbb{N}^{l+1} \) extends \( I \in \mathbb{N}^l \), and \( J' \in \mathbb{N}^{j+1} \) extends \( J \in \mathbb{N}^j \). Therefore, by induction, the simplicial bi-functor

\[ S_{i}^j \mathcal{A} \times S_{i}^j \mathcal{B} \to S_{i}^j S_{i}^j \mathcal{C}. \]

is well-defined and bi-exact, for every \( i, j \in \mathbb{N} \cup \{0\}, I \in \mathbb{N}^l \) and \( J \in \mathbb{N}^j \). On the other hand, it is not difficult to see that (11) is compatible with the simplicial structures (\( S \)-direction). Thus, for every \( i, j \in \mathbb{N} \cup \{0\} \), we have the map

\[ \Lambda_{i,j} : |wS_{i}^j \mathcal{A}| \land |wS_{i}^j \mathcal{B}| \to |wwS_{i}^j \mathcal{C}|. \]

2. To see the commutative diagram in the statement, we first recall the swallowing lemma in [19, Lemma 1.6.5], which says that, given \( \mathcal{A} \) a subcategory of \( \mathcal{B} \) that contains all objects of \( \mathcal{B} \), the inclusion \( \mathcal{B} \to \mathcal{A} \mathcal{B} \), considered as a functor of double categories, induces a homotopy equivalence after applying the geometric realization and the nerve functor. In view of the swallowing lemma, we have the following commutative diagram

\[
\begin{array}{ccc}
|wS_{i+1}^j \mathcal{A}| & \land & |wS_{i}^j \mathcal{B}| \\
\downarrow \quad \quad \quad \downarrow & \quad \quad \quad \downarrow \quad \quad \quad \downarrow & \quad \quad \quad \downarrow \\
S_1 \land |wS_{i}^j \mathcal{A}| & \land & |wS_{i}^j \mathcal{B}| \\
\downarrow \quad \quad \quad \downarrow & \quad \quad \quad \downarrow \quad \quad \quad \downarrow & \quad \quad \quad \downarrow \\
S_1 \land |wS_{i}^j \mathcal{A}| & \land & |wS_{i}^j \mathcal{B}| \\
\end{array}
\]

where the horizontal maps on the right-hand side are homotopy equivalences. \( \square \)

The lemma above can be stated in terms of prespectra as follows.
Theorem 1.14. Let $\land : \mathcal{A} \times \mathcal{B} \to C$ be a simplicial bi-exact functor and $\tilde{K}(C)$ be the $\Omega$-prespectrum given by

$$
\begin{align*}
\tilde{K}(C)_i & := \Omega|\omega S^{(i+1)}C|, \quad i > 0 \\
\tilde{K}(C)_0 & := \Omega^2|\omega S^{(2)}C|,
\end{align*}
$$

then $\tilde{K}(C) \cong K(C)$ in $\text{Ho}(\mathcal{P})$, and the simplicial bi-exact functor induces a map of prespectra:

$$
\Lambda : \mathbf{K}^{\mathcal{A}} \land K^{\mathcal{B}}_0 \to \tilde{K}(C).
$$

Proof. 1. The map of prespectra is defined as follows:

$$
\begin{align*}
\Lambda_k : |\omega S^{(k)}\mathcal{A}| \land \Omega|\omega S^{(k)}\mathcal{B}| & \to \Omega(|\omega S^{(k)}\mathcal{A}| \land |\omega S^{(k)}\mathcal{B}|) \to \Omega|\omega S^{(k+1)}C|, \quad k \geq 1; \\
\Lambda_0 : \Omega|\omega S^{(k)}\mathcal{A}| \land \Omega|\omega S^{(k)}\mathcal{B}| & \to \Omega^2(|\omega S^{(k)}\mathcal{A}| \land |\omega S^{(k)}\mathcal{B}|) \to \Omega^2|\omega S^{(k+1)}C|. \quad (12)
\end{align*}
$$

Now, let $A_k$, $B_k$ and $C_k$ denote $|\omega S^{(k)}\mathcal{A}|$, $|\omega S^{(k)}\mathcal{B}|$ and $|\omega S^{(k)}C|$, respectively. We have, for every $k \geq 1$, the following commutative diagram

$$
\begin{array}{ccc}
A_{k+1} \land \Omega B_1 & \to & \Omega(A_{k+1} \land B_1) \\
\uparrow & & \uparrow \\
S^1 \land A_k \land \Omega B_1 & \to & S^1 \land \Omega(A_k \land B_1)
\end{array}
$$

This implies that $\Lambda_i$ constitutes the map of prespectra $\Lambda$—notice that, at degree 0, the compatibility with the structure maps follows directly from Definition (12).

3. To see the equivalence $K(C) \cong \tilde{K}(C) \in \text{Ho}(\mathcal{P})$, we introduce another $\Omega$-prespectrum $\tilde{K}(C)$ which is given by

$$
\begin{align*}
\tilde{K}(C)_i & := |\omega \omega S^{(i)}C|, \quad i > 0 \\
\tilde{K}(C)_0 & := \Omega|\omega S^{(1)}C|.
\end{align*}
$$

There is a natural map $K(C) \to \tilde{K}(C)$ induced by the inclusion

$$
\omega S^{(i)}C \hookrightarrow \omega \omega S^{(i)}C.
$$

In view of the swallowing lemma [19, Lemma 1.6.3], it is a $\pi_\ast$-isomorphism. If we let $E$ be the prespectrum given by

$$
\begin{align*}
E_i & := |\omega \omega S^{(i+1)}C|, \quad i > 0 \\
E_0 & := \Omega|\omega \omega S^{(2)}C|,
\end{align*}
$$

then there is a $\pi_\ast$-isomorphism of prespectra

$$
\tilde{K}(C) \to \Omega^2 E,
$$

given by

$$
|\omega \omega S^{(i)}C| \to \Omega|\omega S^{(i)}C|;
$$

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on the other hand, by the definition of $\bar{K}(C)$, we know $\bar{K}(C) = \Omega^{alt}E$, where $\Omega^{alt}$ and $\Omega^{std}$ are the alternative and standard loop functors in $\mathcal{P}$. Then, the $\pi_*$-isomorphism in the statement

$$K(C) \cong \bar{K}(C) \in \text{Ho}(\mathcal{P})$$

ensues from the fact $\Omega^{std}E \cong \Omega^{alt}E \in \text{Ho}(\mathcal{P})$ [13, Lemma 0.81] or [22, Appendix A.1.3]. □

**Theorem 1.15.** Suppose the simplicial Waldhausen categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ admit the sum functors $\theta^{\mathcal{A}}$, $\theta^{\mathcal{B}}$ and $\theta^{\mathcal{C}}$, respectively. Then the following map

$$K(\mathcal{A})_0 \wedge K(\mathcal{B})_0 \rightarrow K(\mathcal{C})_0$$

is a homotopy bi-linear map (B.6), and it fits into the homotopy commutative diagram below.

$$\begin{array}{ccc}
K(\mathcal{A})_0 \wedge K(\mathcal{B})_0 & \xrightarrow{\text{wA}} & K(\mathcal{C})_0 \\
\text{|wA| \wedge |wB|} & \xrightarrow{\phi^C} & |wC| \\
\end{array}$$

**Proof.** 1. By symmetry, we only need to check the bi-linearity with respect to $K(\mathcal{A})_0$. To see this, we construct a natural transformation between the following two functors:

$$\mathcal{F}_1 : \mathcal{A} \times \mathcal{A} \times \mathcal{B} \xrightarrow{\theta^{\mathcal{A}} \times \text{id} \mathcal{B}} \mathcal{A} \times \mathcal{B} \xrightarrow{\wedge} \mathcal{C} ;
\quad A, A', B \mapsto (A \lor^{\mathcal{A}} A') \land B$$

and

$$\mathcal{F}_2 : \mathcal{A} \times \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \times \mathcal{B} \times \mathcal{A} \times \mathcal{B} \xrightarrow{\land \times \land} \mathcal{C} \times \mathcal{C} \xrightarrow{\theta^{\mathcal{C}}} \mathcal{C} ;
\quad (A, A', B) \mapsto (A, B, A', B) \mapsto (A \lor B) \lor^{\mathcal{C}} (A' \land B).$$

The required natural transformation is then given by the composition,

$$O(\mathcal{A} \times \mathcal{A} \times \mathcal{B}) \rightarrow O(\text{Cospan}(\mathcal{A}) \times \mathcal{B}) \xrightarrow{O(\theta^{\mathcal{A}} \times \text{id} \mathcal{B}) \ast \text{id}} O(\text{Cospan}(\mathcal{A}) \times \mathcal{B})$$

$$\rightarrow O(\text{Cospan}(\mathcal{C})) \xrightarrow{\phi^C} M(\text{Cospan}(\mathcal{C})) \xrightarrow{\pi} M(\mathcal{C}),$$

where the first map is given by the assignment that sends

$$(A, A', B) \in O(\mathcal{A}_k \times \mathcal{A}_k \times \mathcal{B}_k)$$

to

$$(A \rightarrow * \leftarrow A', B) \in O(\text{Cospan}(\mathcal{A}_k) \times \mathcal{B}_k),$$

the second, third and forth maps are given by the sum functor $\theta^{\mathcal{A}}$ times $\text{id}_{\mathcal{B}}$, the pairing $\land$, and the natural transformation $\phi^C$ from the sum functor to $\text{id}$ (Definition 1.4), and the last map is induced by the projection to
the universal morphism from \((A \land B) \lor^G (A' \land B)\) to \((A \lor^G A') \land B\). Notice that, because of the bi-exactness of \(\land\), the universal morphism
\[
(A \land B) \lor^G (A' \land B) \to (A \lor^G A') \land B
\]
is actually an isomorphism. Thus, we obtain a simplicial functor
\[
A \times A \times B \times [1] \to C.
\]
such that it restricts to \(F_1\) and \(F_2\) on \(A \times A \times B \times \{0\}\) and \(A \times A \times B \times \{1\}\), respectively. This induces a multi-simplicial functor
\[
wS(A \times A) \times wS(B \times [1]) \to wS(1).
\]
and, applying the geometric realization, we obtain a homotopy from \(|wS_{F_1}|\) to \(|wS_{F_2}|\):
\[
|wS| A \times |wS| \times |wS| B \times I \to |wS| C,
\]
which implies the homotopy bi-linear map
\[
|wS| A \times |wS| B \to |wS(1)|
\]
By Lemma B.7 we see the map
\[
K(A)_0 \land K(B)_0 \to K(C)_0
\]
is also homotopy bi-linear.

2. To see the diagram in the statement is commutative, we note that the commutative diagram
\[
|wS| A \times |wS| B \to |wS(1)|
\]
induces the commutative diagram
\[
\Omega^2(|wS| A \times |wS| B|) \to \Omega^2|wS(1)| \leftarrow \Omega^2|wS(1)|
\]
where the left vertical map is the composition
\[
|wS| A \times |wS| B \to \Omega|wS| A \times \Omega|wS| B \to \Omega^2(|wS| A \times |wS| B|).
\]
Thus, we have proved the theorem.
1.2. Simplicial S-categories

A category $C$ is a $S$-category if and only if any pair in $C$ admits a categorical sum, and $C$ contains a subcategory of weak equivalences $wC$ such that the morphisms in $wC$ are stable under categorical sums. An exact functor of $S$-categories

$$F : C \to D$$

is a functor that preserves categorical sums and sends $wC$ to $wD$. A natural weak equivalence of exact functors

$$F : \mathcal{G} : C \to D$$

is a natural transformation of $F$ and $\mathcal{G}$ that takes value in $wD$. $S$ denotes the 2-category comprising $S$-categories, exact functors and natural weak equivalences. $S$-categories have been studied in [19, Section 1.8] and [15, Section 2].

In this subsection, we discuss the simplicial analogue of $S$-categories.

**Definition 1.16.** (i) A simplicial $S$-category $C$ is a simplicial object in $S$, namely a functor

$$C : \Delta^{op} \to S.$$

(ii) A simplicial exact functor of simplicial categories is a simplicial functor

$$F : C \to D.$$

such that $F_k : C_k \to D_k$ is exact, for every $k$.

(iii) A simplicial natural weak equivalence of two simplicial exact functors

$$F : \mathcal{G} : C \to D.$$

is a simplicial functor

$$\phi : C \times [1] \to D.$$

such that $\phi_k$ is a natural weak equivalence between $F_k$ and $\mathcal{G}_k$, for every $k$.

**Remark 1.17.** The definition above can be easily generalized to multi-simplicial $S$-categories. $mS$ denotes the category of multi-simplicial $S$-categories, multi-simplicial exact functors and multi-simplicial natural weak equivalences, whereas $sS$ stands for its subcategory of simplicial $S$-categories.

Given a simplicial $S$-category $C$, one can always choose a categorical sum for every pairs in $C_k$, for each $k$, yet the choices might not be compatible with the simplicial structure of $C$. Nevertheless, in many cases, a compatible way of choosing categorical sums does exist. In fact, all the examples considered in this paper admit such compatible choices.

**Definition 1.18.** A (simplicial) sum functor of a simplicial $S$-category $C$ is a map of simplicial sets

$$\theta : O(C) \times O(C) \to O(\text{Cosp}(C)),$$

such that

$$\theta_k(A, B) := A \to A \vee^\theta B \leftarrow B$$
is a diagram of categorical sum. As sum functors of simplicial Waldhausen categories, the sum functor \( \theta \) can be extended to a simplicial exact functor

\[
\text{Cospan}(C) \to \text{Cospan}(C)
\]

\[(A \to C \leftarrow B) \mapsto (A \to A \uplus B \leftarrow B),\]

which comes with a simplicial natural transformation

\[\phi : \theta \mapsto \text{id}\]

that encodes the unique morphism from \( A \uplus B \) to \( C \).

Recall that, given a \( S \)-category \( C \), the \( N \)-construction gives a simplicial \( S \)-category \( N \cdot C \) \([19, \text{Sec.1.8}], [15, \text{Sec.2}] \), where \( N_k C \) consists of the functors

\[A : 2^k \to C \]

\[I \mapsto A_I\]

such that \( A_I \to A_{I \cup J} \leftarrow A_J \) is a diagram of categorical sum, where \( k = \{1, \ldots, k\} \) and the intersection \( I \cap J \) is empty. This construction can be extended to multi-simplicial \( S \)-categories and induces a functor

\[N : mS \to mS.\]

Iterating the \( N \)-construction, one obtains a functor \( K^S \) from \( mS \) to \( P \), where the presepctrum \( K^S(C\ldots) \) is given by

\[K^S(C\ldots)_i := |wN^i \cdot C\ldots| \quad i \geq 1\]

\[K^S(C\ldots)_0 := \Omega|wN \cdot C\ldots|.\]

Many constructions of multi-simplicial Waldhausen categories have their counterparts in the theory of multi-simplicial \( S \)-categories. For example, given a multi-simplicial exact functor

\[\mathcal{F} : C\ldots \to D\ldots,\]

one can define the 2-fiber product of the cospan

\[N \cdot C\ldots \to N \cdot D\ldots \leftarrow PN \cdot D\ldots,\]

denoted by \( N(C\ldots \to D\ldots) \), where \( PN \cdot D\ldots \) is the path object in the \( N \)-direction.

**Lemma 1.19.** Let

\[\mathcal{F} : C\ldots \to D\ldots\]

be a multi-simplicial exact functor, and assume \(|wC\ldots|\) is connected. Then the sequence

\[|wD\ldots| \to |wN(C\ldots \to D\ldots)| \to |N \cdot C\ldots|\]

is a homotopy \( q \)-fibration.

**Proof.** Observe first that the vertical simplicial exact functors in the commutative diagram below
induce weak equivalences after applying the nerve construction and diagonal functor, where $r$ is given by the forgetful functor

$$N_k(C \to D) \to wN_kC \times wD \to wN_kC$$

Hence, the sequence

$$wD \to wN_k(C \to D) \to N_kC$$

induces a homotopy Kan fibration. On the other hand, the forgetful functor

$$N_kC \to \text{forgetful functor}$$

gives a level-wise equivalence of multi-simplicial sets, for every $k$, and hence the space $|wN_kC|$ is connected. Applying the Bousfield-Friedlander theorem [6, IV.4], we obtain that the sequence

$$|wD| \to |wN(C \to D)| \to |wN.C|$$

is a homotopy $q$-fibration. □

**Corollary 1.20.** Given a multi-simplicial $S$-category $C$, the prespectra $K^S(C)$ is an $\Omega$-prespectrum.

**Proof.** It is proved in the same way as Corollary 1.8. □

Like in Waldhausen theory, one can define a bi-exact functor of $S$-categories to be a bi-functor

$$\land : \mathcal{A} \times \mathcal{B} \to C$$

such that, given any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the associated functors

$$A \land - : \mathcal{B} \to C;$$

$$- \land B : \mathcal{A} \to C$$

are exact.

**Definition 1.21.** A simplicial bi-exact functor of simplicial $S$-categories

$$\land : \mathcal{A} \times \mathcal{B} \to C.$$

is a simplicial bi-functor such that, for every $k$, the functor

$$\land_k : \mathcal{A}_k \times \mathcal{B}_k \to C_k$$

is bi-exact.
We have the following counterparts of Lemma 1.13 and Theorem 1.14 for simplicial $S$-categories.

**Theorem 1.22.**  (i) Let $\tilde{K}^S(C)$ be the $\Omega$-prespectrum defined by

$$
\tilde{K}^S(C)_i := \begin{cases} 
\Omega|wN(i+1)C| & i \geq 1; \\
\Omega^2|wN(2)C| & i = 0,
\end{cases}
$$

and $\wedge : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a simplicial bi-exact functor of simplicial $S$-categories. Then there is a $\pi_*$-isomorphism $\tilde{K}^S(C) \simeq K^S(C)$ in $\text{Ho}(\mathcal{P})$, and the bi-simplicial exact functor induces a map of prespectra:

$$
\Lambda : K^S(\mathcal{A}) \wedge K^S(\mathcal{B})_0 \to \tilde{K}^S(C).
$$

(ii) Suppose further that the simplicial $S$-categories $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ admit sum functors. Then, when restricted to the zero component, the map

$$
K^S(\mathcal{A})_0 \wedge K^S(\mathcal{B})_0 \to K^S(C)_0
$$

is homotopy bi-linear [19, B.6], and it fits into the homotopy commutative diagram below.

\[
\begin{array}{ccc}
K^S(\mathcal{A})_0 \wedge K^S(\mathcal{B})_0 & \to & K^S(C)_0 \\
\uparrow & & \uparrow \\
|w\mathcal{A}| \wedge |w\mathcal{B}| & \to & |w\mathcal{C}|
\end{array}
\]

**Proof.** Replace the $S$-construction in the proofs of Lemma 1.13 and Theorems 1.14 and 1.15 by the $N$-construction. $\square$

1.3. Comparison theorems

Recall that, in order to prove the maps $a$ and $c$ in (2) are homotopy equivalences, we need some sort of comparison theorems. In this subsection, we introduce two comparison theorems. The first one, which implies the homotopy equivalence $a$ in (2), is a straightforward generalization of Theorem 1.8.1 in [19] and does not require the existence of a sum functor, whereas the second one, which implies the homotopy equivalence $c$ in (2), requires a weaker splittable condition on cofibrations, but makes use of the existence of a sum functor.

Recall from [19, Section 1.8], a cofibration $A \to B$ in a Waldhausen category $C$ is splittable up to weak equivalences if and only if there is a chain of weak equivalences, relative to $A$, relating $A \to B$ to $A \to B'$, where $B'$ is a categorical sum of $A$ and a pushout of the span $* \leftarrow A \to B'$. If all cofibrations in $C$ are splittable up to weak equivalences, we say $coC$ satisfies the splittable condition. A simplicial Waldhausen category $C$ satisfies the splittable condition if and only if $C_k$ satisfies the splittable condition, for every $k$.

**Theorem 1.23.** Let $C$ be a simplicial Waldhausen category with $C_k$ additive, for every $k$, and assume $wC*$ satisfies the extension axiom and $coC$ the splittable condition. Then the canonical map

$$
|wN.C| \to |wS.C|
$$

is a homotopy equivalence. In fact, this map can be extended to a $\pi_*$-isomorphism

$$
K^S(C) \to K(C).
$$
Proof. Firstly, we note that, given a multi-simplicial Waldhausen category $C$, the multi-simplicial category $N \cdot C$ admits a multi-simplicial Waldhausen category structure whose cofibrations and weak equivalences are point-wise cofibrations and point-wise weak equivalences, respectively. Secondly, we observe that the functor

$$\text{Ar}[n] \to 2^n$$

$$(i, j) \mapsto \{ k \mid i < k \leq j \}$$

induces a multi-simplicial exact functor (in $mW$)

$$N \cdot C \to S \cdot C.$$  \hspace{1cm} (13)

Now, assume we already have a well-defined multi-simplicial exact functor

$$N^{(i-1)} \cdot C \to S^{(i-1)} \cdot C.$$  \hspace{1cm} (14)

Then, we consider the composition

$$wN^{(i)} \cdot C \to wS N^{(i-1)} \cdot C \to wS^{(i)} \cdot C,$$

where the first multi-simplicial functor is induced by observation (13) and the second one by the induction hypothesis. Thus, for every $i$, we have a multi-simplicial exact functor

$$N^{(i)} \cdot C \to S^{(i)} \cdot C.$$  \hspace{1cm} (14)

In addition, it is not difficult to check (14) fits into the commutative diagram

$$S^1 \wedge |wN^{(i)} \cdot C.| \to |wN^{(i+1)} \cdot C.|$$

$$\downarrow \quad \downarrow$$

$$S^1 \wedge |wS^{(i)} \cdot C.| \to |wS^{(i+1)} \cdot C.|$$

and hence induces a map of $\Omega$-prespectra

$$K^S(C.) \to \textbf{K}(C.).$$

Now, Theorem 1.8.1 in [19] implies that the map

$$|wN \cdot C_k| \to |wS \cdot C_k|$$

is a homotopy equivalence, for every $k$, and thus, by Lemma [A.2], the map

$$|wN \cdot C.| \to |wS \cdot C.|$$

is also a homotopy equivalence. \hfill $\Box$
Remark 1.24. In the prove above, we have made use of the remark in [19, p.369] which says, if a Waldhausen category $C$ is additive, then there is a chain of weak equivalences connecting $X \to A \cup_X A$ to $X \to A \vee A/X$ in $C_X$, the category of cofibrant objects under $X$ [19, p.368], where $A \vee A/X$ is a categorical sum of $A$, and $A/X$ is a pushout of the span $* \leftarrow X \to A$. Following the hint given in [19], we provide a detailed proof for this claim.

1. Observe that, given an element $j : X \to A \in C_X$, there is a morphism

$$f : A \cup_X A \to A \vee A/X,$$

induced by the following commutative square

\[
\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow & & \downarrow i + q \\
A & \longrightarrow & A \vee A/X
\end{array}
\]

where $i$ and $q$ are the compositions

\[
\begin{align*}
i : A & \to A \to A \vee A/X; \\
q : A & \to A/X \to A \vee A/X,
\end{align*}
\]

respectively. In other words, $f$ fits in the triangular prism,

\[
\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow & & \downarrow i + q \\
A & \longrightarrow & A \vee A/X
\end{array}
\]

where the upper square is a pushout. Taking into account the quotients of the horizontal morphisms in (15) and the cofiber sequence $A \to A \vee A/X \to A/X$, we get a bigger triangular prism,

\[
\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow j & & \downarrow A/X \\
A & \longrightarrow & A \cup_X A \longrightarrow C_0
\end{array}
\]

\[
\begin{array}{ccc}
A & \longrightarrow & A \cup_X A \\
\downarrow i + q & & \downarrow A/X \\
A & \longrightarrow & A \vee A/X \longrightarrow C_1
\end{array}
\]

\[
\begin{array}{ccc}
* & \longrightarrow & A/X \\
\downarrow & & \downarrow A/X
\end{array}
\]
From this prism, we can see \( C_0 \to C_1 \) is an isomorphism, and by the extension axiom of \( wC \), we further obtain the morphism
\[
f : A \cup_X A \to A \vee A/X
\]
is a weak equivalence. Thus, the claim is proved.

The second comparison theorem requires a much weaker splittable condition, which we now describe. Given a simplicial Waldhausen category \( C \), we say its cofibrations are weakly splittable if and only if, for every \( X \to A \in \text{co}C_0 \), there exists \( s_0^i X \to A_i \in \text{co}C_1 \) such that
\[
\begin{align*}
d_0^i(s_0^i X \to A_i) &= X \to A; \\
d_1^i(s_0^i X \to A_i) &= X \to A_1
\end{align*}
\]
with \( X \to A_1 \) splittable up to weak equivalences.

**Theorem 1.25.** Let \( C \) be a simplicial Waldhausen category equipped with a sum functor \( \theta \) with \( C_k \) additive, for every \( k \). Assume also its cofibrations are weakly splittable and \( wC_k \) satisfies the extension axiom. Then the canonical simplicial exact functor
\[
N.C. \to S.C.
\]
duces a homotopy equivalence
\[
|wN.C.| \xrightarrow{\sim} |wS.C.|.
\]

**Proof.** 1. In view of Lemma 1.19 it suffices to show that the multi-simplicial exact functor
\[
wN.N.C. \to wN.S.C.
\]
duces a homotopy equivalence. Since the forgetful simplicial functors
\[
wN.N_n C. \to (wN.C.)^n
\]
\[
wN.S_n C. \to (wN.C.)^n,
\]
fit into the commutative diagram
\[
\begin{array}{ccc}
wN.N_n C. & \rightarrow & wN.S_n C. \\
\downarrow & & \downarrow \\
(wN.C.)^n & , &
\end{array}
\]
by Lemma A.2 and the induction, the question can be further reduced to showing the simplicial exact functor
\[
S_n C. \to S_{n-1} C. \times C.
\]
\[
\{A_{i,j}\}_{0 \leq i < j \leq n} \mapsto (\{A_{i,j}\}_{0 \leq i < j \leq n-1, A_{n-1,n}})
\]
duces a homotopy equivalence
\[
|wN.S_n C.| \xrightarrow{\sim} |wN.S_n C.| \times |wN.C.|.
\]

Now, consider the simplicial exact functor
\[
j : C. \to S_n C.
\]
which sends \( A \in C \) to \( \{A_{i,j}\} \) with \( A_{i,n} = A \) and \( A_{i,j} = * \) when \( j \neq n \). Observe also that \( j \) fits into the following commutative diagram of simplicial exact functors

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where $C \to S_{n-1}C \times C$ is the inclusion into the second component. Applying Lemma 1.19 to the diagram above, we get the horizontal sequences in the commutative diagram below are homotopy $q$-fibrations

$$
\begin{array}{ccc}
S_n C & \longrightarrow & S_{n-1} C \times C \\
\downarrow & & \downarrow \\
wN. S_n C & \longrightarrow & wN. N. (C \to S_n C)
\end{array}
\begin{array}{ccc}
\downarrow & & \downarrow \\
wN. (S_{n-1} C \times C) & \longrightarrow & wN. (C \to S_{n-1} C \times C)
\end{array}
\begin{array}{ccc}
\downarrow & & \downarrow \\
wN. N. (C \longrightarrow S_{n-1} C) & \longrightarrow & wN. N. C.
\end{array}
$$

Therefore, it is sufficient to show the bi-simplicial functor

$$wN. (C \to S_n C) \to wN. (C \to S_{n-1} C \times C) = wS_{n-1} C \times wN. (C \to C) \to wS_{n-1} C.$$

induces a homotopy equivalence after applying the geometric realization, where the last bi-simplicial functor is the projection. For the sake of convenience, we call the above composition $F$. The argument so far is a straightforward generalization of the one in [19, p.372].

2. To show that the bi-simplicial functor

$$F : wN. (C \to S_n C) \to wS_{n-1} C.$$

or equivalently, the simplicial functor

$$([k] \mapsto wN_k (C_k \to S_n C_k)) \xrightarrow{F} ([k] \mapsto wS_{n-1} C_k)$$

induces a homotopy equivalence, we employ Waldhausen’s variant of Quillen’s theorem A [18, Section 4]. Namely, given an object $B \in wS_{n-1} C_l$, we want to prove that the geometric realization of the simplicial left fiber of $B$,

$$[k] \mapsto \bigsqcup_{u : [k] \to [l]} \mathcal{F}_u / u^* B$$

(16)

is contractible. Recall that an object in

$$\bigsqcup_{u : [k] \to [l]} \mathcal{F}_u / u^* B$$

(17)

is a collection of objects $\{ A^I_{i,j} \in C_k \}_{(i,j) \in \mathcal{A}[n]}$ with the compatible morphisms induced by $I \subset J \in 2^k$ and $(i, j) \leq (i', j') \in \mathcal{A}[n]$ such that $A^I_{i,j} = *$ when $1 \notin I$ and $0 \leq i < j < n$, and there is an weak equivalence $A^{\{1\}}_{i,j} \to u^* B_{i,j}$ compatible with the morphisms inside the diagrams, for $0 \leq i < j < n$. Notice that the lower index $0 \leq i < j \leq n$ represents the $S$-direction, the upper index $I \in 2^k$ represents the $N$-direction, and $B = \{ B_{i,j} \}$.  

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On the other hand, utilizing the simplicial sum functor $\theta$, one can construct another simplicial category

$$[k] \mapsto \coprod_{u : [k] \to [l]} wN_k(C_k \to C_{k,u^*B_0^{u,n-1}}), \quad (18)$$

where the exact functor $C_k \to C_{k,u^*B_0^{u,n-1}}$ is given by the assignment $u^*B_0^{u,n-1} \to C \cup^0 u^*B_0^{u,n-1}$. This resembles the simplicial $S$-category in [19, Lemma 1.8.8]. In [19], one can directly construct a simplicial functor from (16) to (18), but in the present case, due to the lack of the compatible choices of pushouts, there is no direct passage from (16) to (18). Therefore, we consider an intermediate simplicial category

$$[k] \mapsto \coprod_{u : [k] \to [l]} D_u,$$

where an object in $D_k$ is an object in (17) plus a choice of pushout of the cospan

$$u^*B_{i,n-1} \leftarrow A^{l}_{i,n-1} \to A^{l}_{i,n},$$

for every $0 \leq i < n - 1$, and its morphisms are those level-wise weak equivalences. It is clear that there are forgetful simplicial functors

$$([k] \mapsto \coprod_{u : [k] \to [l]} D_u) \to ([k] \mapsto \coprod_{u : [k] \to [l]} wN_k(C_k \to C_{k,u^*B_0^{u,n-1}})) \quad (19)$$

$$([k] \mapsto \coprod_{u : [k] \to [l]} D_u) \to ([k] \mapsto \coprod_{u : [k] \to [l]} \mathcal{F}_k / u^*B) \quad (20)$$

which are level-wise equivalences of categories. In more details, for a fixed $k$, the inverse of the functor (19) is given by taking pushouts of $u^*B_{i,n-1} \leftarrow u^*B_0^{u,n-1} \to C_i$ and $* \leftarrow u^*B_0^{u,n-1} \to C_j$, where $1 \notin J$ and $1 \in I$; the inverse of the functor (20) is clear. In view of Theorem A’ in [18] (Waldhausen’s variant of Quillen’s theorem A), it is sufficient to show the geometric realization of (18) is contractible. For the sake of simplicity, we let $X = B_0^{u,n-1}$.

3. We claim the geometric realization of the simplicial category

$$[k] \mapsto \coprod_{u : [k] \to [l]} wN_k(C_k \to C_{k,u^*X})$$

is connected. Note that, at degree 0 ($k = 0$), we have the identity

$$\coprod_{u : [0] \to [l]} wN_0(C_0 \to C_{0,u^*X}) \cong \coprod_{u : [0] \to [l]} wN_1 C_{0,u^*X}.$$

Now, fix a morphism $u : [0] \to [l]$ and an object $u^*X \to Y$ in $wN_1 C_{0,u^*X}$. Then, since every morphism in $coC$ is weakly splittable, there is an object

$$s^0_0 u^*X \to Y_t \in coC_1 = N_0(C_1 \to C_{1,s^0_0 u^*X})$$

such that

$$d^0_t(s^0_0 u^*X \to Y_t) = u^*X \to Y;$$

$$d^0_t(s^0_0 u^*X \to Y_t) = u^*X \to Y_t,$$

with $u^*X \to Y_1$ splittable up to a chain of weak equivalences. Thus, the object
in \( N_1(\mathcal{C}_1 \to \mathcal{C}_{1,u^*X}) \), gives us a path connecting \( u^*X \to Y \) to \( u^*X \to Y_1 \). Via a chain of weak equivalences, one can further connect \( u^*X \to Y_1 \) to \( u^*X \to Y'_1 \), where \( Y'_1 \) is isomorphic to a categorical sum of \( u^*X \) and a pushout of \(* \leftarrow u^*X \to Y'_1 \). Applying \( s_0^* \) (in \( C \)-direction), we see the object

in \( N_1(\mathcal{C}_1 \to \mathcal{C}_{1,u^*X}) \) provides a path connecting \( u^*X \to u^*X \) and \( u^*X \to Y'_1 \). In this way, we find that, for any \( u : [0] \to [l] \), every object in

is connected to the base point \( u^*X \to u^*X \).

Now, let \( s : [l+1] \to [1] \in \Delta \) sends 0 to 0 and \( i \neq 0 \) to 1, and observe that the object

provides a homotopy connecting all the base points. Thus, we have proved the claim, and hence, in view of Lemma 1.19 the question is reduced to showing the geometric realization of the bi-simplicial category

is contractible.

4. Observe that there is a commutative diagram of bi-simplicial sets

\[
\begin{array}{ccc}
([k], [j]) \mapsto \coprod_{u : [k] \to [l]} w_j \mathcal{N}_j \mathcal{C}_k & \longrightarrow & ([k], [j]) \mapsto \coprod_{u : [k] \to [l]} w_j \mathcal{N}_j \mathcal{C}_k \\
\downarrow & & \downarrow \\
([k], [j]) \mapsto \coprod_{u : [k] \to [l]} \ast & \longrightarrow & ([k], [j]) \mapsto \coprod_{u : [k] \to [l]} w_j \mathcal{N}_j \mathcal{C}_k
\end{array}
\]
and that, for every $k$, there is an equivalence of simplicial categories
\[
\coprod_{u:[k] \to [l]} \left( wN.N_k(C_k \to C_{k,u^*X}) \to \coprod_{u:[k] \to [l]} (wN.N_kC_k \times wN.C_{k,u^*X}) \right)
\]
which is induced by the simplicial exact functor of simplicial $S$-categories
\[
N_k(C_k \to C_{k,u^*X}) \to N_kC_k \times C_{k,u^*X}
\]
constructed in the proof of Lemma 1.19. Hence, fixing $k \in \mathbb{N} \cup \{0\}$, we have that (22) is weak equivalent to the commutative diagram,
\[
\begin{array}{ccc}
[j] \mapsto \coprod_{u:[k] \to [l]} w_j N_jC_k, u^*X & \cong & \coprod_{u:[k] \to [l]} \left( w_j N_jN_kC_k \times w_j N_jC_{k,u^*X} \right) \\
\downarrow & & \downarrow \\
[j] \mapsto \coprod_{u:[k] \to [l]} \ast & \cong & \coprod_{u:[k] \to [l]} w_j N_jN_kC_k
\end{array}
\]  
(23)

is clearly a homotopy cartesian square, and hence (22) is a level-wise homotopy cartesian square. The next step is to show that (22) induces a homotopy cartesian square after geometric realization. To see this, we employ the Bousfield-Friedlander theorem.

6. In order to utilize the Bousfield-Friedlander theorem [6, IV.4.9], [2, Appendix], we need to check if the bi-simplicial sets
\[
([k],[j]) \mapsto \coprod_{u:[k] \to [l]} \ast \\
([k],[j]) \mapsto \coprod_{u:[k] \to [l]} w_j N_jN_kC_k
\]
are $\pi_*$-Kan. The former is clear. To see the latter also satisfies the $\pi_*$-Kan condition, we consider the extension functor [6, p.188], and let
\[
X_{k,-} := \coprod_{u:[k] \to [l]} \text{Ex}^\infty([j] \mapsto w_j N_jN_kC_k).
\]
Observe that, for every $m \geq 1$, we have
\[
[k] \mapsto \coprod_{x \in X_{k,0}} \pi_m(X_{k,-}, x) = \coprod_{u:[k] \to [l]} \pi_m(\text{Ex}^\infty([j] \mapsto w_j N_jN_kC_k), \ast) = \pi_m(\text{Ex}^\infty([j] \mapsto w_j N_jN_kC_k), \ast) \times \Delta^l_k,
\]
and because $X_{k,0} = \coprod_{u:[k] \to [l]} \ast$, the simplicial map
\[
([k] \mapsto \coprod_{x \in X_{k,0}} \pi_m(X_{k,-}, x) = \pi_m(\text{Ex}^\infty([j] \mapsto w_j N_jN_kC_k), \ast) \times \Delta^l_k) \to ([k] \mapsto X_{k,0} = \Delta^l_k)
\]
is a Kan fibration as the simplicial set
\[
[k] \mapsto \pi_m(\text{Ex}^\infty([j] \mapsto w_j N_jN_kC_k), \ast)
\]
is a simplicial group, and is thus fibrant. For the last condition in the Bousfield-Friedlander theorem [6, Theorem 4.9], we note that the simplicial map of their vertical path components

\[ ([k] \mapsto \pi_0(\prod_{u[k] \to [l]} \ast) = \prod_{u[k] \to [l]} \ast \xrightarrow{=} ([k] \mapsto \pi_0(X_{k_*}) = \prod_{u[k] \to [l]} \ast) = (k \mapsto \triangle_k^l) \]

is clearly a Kan fibration. By the Bousfield-Friedlander theorem, the sequence

\[ |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j \mathcal{C}_{k,u' \cdot X} | \rightarrow |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j N_k (C_k \to C_{k,u' \cdot X}) | \]

\[ \rightarrow |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j N_k C_k | \]

is a homotopy \(q\)-fibration. Lastly, in view of the map of the map of homotopy \(q\)-fibrations below

\[ |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j \mathcal{C}_k | \rightarrow |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j N_k (C_k \to C_k) | \]

\[ \rightarrow |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j N_k C_k | \]

\[ \rightarrow |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j N_k C_k | \]

we can conclude that the geometric realization of (24) is contractible because the space

\[ |([k], [j]) \mapsto \prod_{u[k] \to [l]} w_j N_j N_k (C_k \to C_k) | \]

is contractible, and both the right and left vertical arrows in (24) are homotopy equivalences—Remark 1.24 implies that the map \(|w \mathcal{N} C_k| \to |w \mathcal{N} C_{k,u' \cdot X}|\) is a homotopy equivalence, for every \(k\) (see also [19, p.369]).

\[ \square \]

**Remark 1.26.** Let \(s' \mathcal{W}^a\) be the subcategory of \(s' \mathcal{W}\) consisting of those simplicial Waldhausen categories that satisfy either the conditions in Theorem 1.23 or in Theorem 1.25. Then the diagram below commutes

\[ \begin{array}{ccc}
    s' \mathcal{W}^a & \xrightarrow{K} & \text{Ho}(\mathcal{P}) \\
    \downarrow & & \\
    s \mathcal{S} & \xrightarrow{K^s} & \text{Ho}(\mathcal{P})
\end{array} \]
In the following, we discuss the relation between bi-exact functors of simplicial Waldhausen categories and those of simplicial $S$-categories.

**Lemma 1.27.** A bi-exact functor of simplicial Waldhausen categories

\[ \land : \mathcal{A} \times \mathcal{B} \to C. \]

induces a bi-exact functor of simplicial $S$-categories

\[ \land : \mathcal{A} \times \mathcal{B} \to C. \]

Furthermore, the following diagrams commute

\[
\begin{array}{ccc}
K^S(\mathcal{A}) \land K^S(\mathcal{B})_0 & \xrightarrow{\land} & K^S(C)

\\
\downarrow & & \downarrow

\\
K(\mathcal{A}) \land K(\mathcal{B})_0 & \xrightarrow{\land} & K(C)
\end{array}
\quad
\begin{array}{ccc}
K^S(\mathcal{A})_0 \land K^S(\mathcal{B})_0 & \to & K^S(C)_0

\\
\downarrow & & \downarrow

\\
K(\mathcal{A})_0 \land K(\mathcal{B})_0 & \to & K(C)_0
\end{array}
\]

**Proof.** Observe first that the following diagram is commutative

\[
\begin{array}{ccc}
wN^{(i)}\mathcal{A} \land wN^{(j)}\mathcal{B} & \to & wN^{(i+j)}C

\\
\downarrow & & \downarrow

\\
wS^{(i)}\mathcal{A} \land wS^{(j)}\mathcal{B} & \to & wS^{(i+j)}C
\end{array}
\]

(25)

because the canonical simplicial functor from $N^{(i)}C$ to $S^{(i)}C$ is induced by forgetting some data and reversing some morphisms via the universal property—that is, it respects simplicial structures. The assertion then follows from the fact that, given a commutative diagram of spaces

\[
\begin{array}{ccc}
\tilde{A} \land \tilde{B} & \to & \tilde{C}

\\
\downarrow & & \downarrow

\\
A \land B & \to & C
\end{array}
\]

the associated diagrams

\[
\begin{array}{ccc}
\tilde{A} \land \Omega\tilde{B} & \to & \Omega\tilde{A} \land \tilde{B}

\\
\downarrow & & \downarrow

\\
A \land \Omega B & \to & \Omega A \land B
\end{array}
\quad
\begin{array}{ccc}
\Omega\tilde{A} \land \Omega\tilde{B} & \to & \Omega^2\tilde{A} \land \tilde{B}

\\
\downarrow & & \downarrow

\\
\Omega A \land \Omega B & \to & \Omega^2 A \land B
\end{array}
\]

are also commutative. \qed
2. The category of vector spaces

Recall that the spaces of objects and morphisms of the standard category of vector spaces over \( \mathbb{F} \), denoted by \( \mathcal{V}_\mathbb{F} \), are given by

\[
\mathcal{V}_\mathbb{F} := \left\{ \begin{array}{l}
O(\mathcal{V}_\mathbb{F}) := \{ \mathbb{F}^n \}_{n \in \mathbb{N} \cup \{0\}} \\
M(\mathcal{V}_\mathbb{F}) := \prod_{m,n} \mathbb{M}_{m,n}(\mathbb{F}),
\end{array} \right.
\]

where \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \), \( \mathbb{M}_{m,n}(\mathbb{F}) \) is the space of \( m \times n \) matrices, and given an internal category in \( \text{Top}, O(C) \) and \( M(C) \) stands for the spaces of objects and morphisms of \( C \), respectively [21, Section 3]. In addition, we also denote the subspaces of full rank, injective, surjective and invertible matrices by \( FR_{m,n}(\mathbb{F}), FR_{m \geq n}(\mathbb{F}), FR_{m \leq n}(\mathbb{F}) \) and \( GL_n(\mathbb{F}) \), respectively, and let \( \mathcal{V}_\mathbb{F} : = \text{Sing}(\mathcal{V}_\mathbb{F}) \) be the singularization of \( \mathcal{V}_\mathbb{F} \).

2.1. The simplicial Waldhausen structure on \( \mathcal{V}_\mathbb{F} \).

**Theorem 2.1.** The simplicial category \( \mathcal{V}_\mathbb{F} \) is a simplicial exact category.

**Proof.** Note first that the constant map \( \Delta^k \to \mathbb{F}^0 = 0 \) is the zero object in \( \mathcal{V}_{\mathbb{F},k} \), for every \( k \in \mathbb{N} \cup \{0\} \), and the Ab-enrichment follows from the Lie group structure on \( \mathbb{M}_{m,n}(\mathbb{F}) \) induced by the addition of matrices \(+\). More precisely, given \( A, B \in \mathcal{V}_{\mathbb{F},k} \), we have the composition

\[
\Delta^k \xrightarrow{t \mapsto (A(t), B(t))} \mathbb{M}_{m,n}(\mathbb{F}) \times \mathbb{M}_{m,n}(\mathbb{F}) \xrightarrow{+} \mathbb{M}_{m,n}(\mathbb{F});
\]

the inverse element of \( A \) is \(-A\), and zero is the zero matrix. It is also clear that finite coproducts and hence finite products exist, so \( \mathcal{V}_{\mathbb{F},k} \) is an additive category, for every \( k \in \mathbb{N} \cup \{0\} \).

We define the set of short exact sequences in \( \mathcal{V}_{\mathbb{F},k} \) to be

\[
\{ \mathbb{F}^{n'} \xrightarrow{A} \mathbb{F}^n \xrightarrow{B} \mathbb{F}^{n''} \mid 0 \to \mathbb{F}^{n'} \xrightarrow{A(t)} \mathbb{F}^n \xrightarrow{B(t)} \mathbb{F}^{n''} \to 0 \text{ is exact, for every } t \in \Delta^k \}.
\]

(26)

To see the set (26) satisfies the axioms of an exact category. We first check that it is a kernel-cokernel pair. Given a short exact sequence

\[
\mathbb{F}^{n'} \xrightarrow{A} \mathbb{F}^n \xrightarrow{B} \mathbb{F}^{n''}
\]

and a morphism \( \mathbb{F}^m \xrightarrow{C} \mathbb{F}^m \) with \( CA = 0 \), where \( CA \) is the morphism given by \( t \mapsto CA(t) := C(t)A(t) \), then the morphism

\[
D := CB^T(BB^T)^{-1} : \mathbb{F}^{n''} \to \mathbb{F}^m
\]

satisfies \( DB = C \) because, for every \( t \in \Delta^k \), we have the identities

\[
C(t)B^T(t)(B(t)B(t)^T)^{-1}B(t)|_{\text{ker}(B(t))} = C(t)|_{\text{ker}(B(t))} = 0
\]

and

\[
C(t)B^T(t)(B(t)B(t)^T)^{-1}B(t)B(t)^Tv = C(t)B^T(t)v,
\]

for every \( v \in \mathbb{F}^{n''} \), meaning \( D(t)B(t)|_{\text{lim}(B^T(t))} = C(t)|_{\text{lim}(B^T(t))} \), where \( B^T \) is given by \( t \mapsto B^T(t) \). So, the diagram
is a pushout, and hence $B$ is a cokernel of $A$. Similarly, given a morphism $\mathbb{P}^m \xrightarrow{C} \mathbb{P}^n$ with $BC = 0$, we define

$$D := (A^T A)^{-1} A^T C : \mathbb{P}^m \to \mathbb{P}'$$

and observe that, for every $t$, $A^T(t)|_{\text{im}(A(t))}$ is an isomorphism, and there are

$$\text{im}(A(t)(A^T(t)A(t))^{-1} A^T(t)C(t)) \subset \text{im}(A(t))$$

$$\text{im}(C(t)) \subset \text{im}(A(t)).$$

With the observation, $A(t)D(t) = C(t)$ follows quickly from the identity

$$A^T(t)A(t)(A^T(t)A(t))^{-1} A^T(t)C(t) = A^T(t)C(t),$$

and hence, (27) is a pullback, and $A$ is a kernel of $B$.

Secondly, by the definition, it is clear that the short exact sequences are closed under isomorphisms, and the identities are admissible monomorphisms and epimorphisms.

Before proceeding to check the rest axioms of an exact category, we first prove a useful lemma:

**Lemma 2.2.** A morphism is an admissible monomorphism (resp. epimorphism) if and only if it is a point-wise injective (surjective) linear map. By point-wise injective (resp. surjective) linear map

$$A : \Delta^k \to \bigsqcup_{m,n} M_{m,n}(\mathbb{F})$$

in $\mathcal{V}_{k,\mathbb{F}}$, we understand $A(t)$ is injective (resp. surjective), for every $t \in \Delta^k$.

**Proof of the lemma.** Note that the direction “admissible monomorphism $\Rightarrow$ point-wise injection” (resp. “admissible epimorphism $\Rightarrow$ point-wise surjection”) is straightforward. Hence, we only need to check another direction. Let $A$ be a point-wise injection from $\mathbb{P}^m$ to $\mathbb{P}^n$, and thus $t \mapsto A(t)$ determines a moving frame in $\mathbb{P}^m$. Then, apply the Gram-Schmidt process to make it into an orthonormal frame—notice that the Gram-Schmidt process is continuous, meaning $A(t) = Q(t)R(t)$, where $Q(t)$ is the orthonormal frame in $\mathbb{P}^m$.

Now we want to extend this orthonormal moving frame to a moving basis in $\mathbb{P}^m$, and it amounts to finding the following lifting—the dashed arrow,
where $O_m(\mathbb{F})$ is the space of orthogonal matrices, $V_n(\mathbb{F}^m)$ is the Stiefel manifold of $n$ orthonormal vectors of an orthogonal matrix, and the map

$$O_m(\mathbb{F}) \to V_n(\mathbb{F}^m)$$

is given by taking the first $n$ columns to $V_n(\mathbb{F}^m)$, which is a fiber bundle with fiber $O_{m-n}(\mathbb{F})$. The lifting always exists by obstruction theory because of the contractibility of the triangle $\Delta^k$. We pick up one lifting and denote it by $\bar{Q} : \Delta^k \to O_m(\mathbb{F})$.

Then the cokernel of $A(t)$ can be constructed by the assignment

$$\mathbb{F}^m \to \mathbb{F}^{m-n}$$

$$\bar{Q}(t)e_i \mapsto 0, \quad i \leq n$$

$$\bar{Q}(t)e_i \mapsto e_{i-n}, \quad i > n$$

The assignment can be written down in terms of the standard basis:

$$B(t) := [0, \text{id}]\bar{Q}^{-1}(t), \quad (28)$$

where $[0, \text{id}]$ is the projection onto the last $m - n$ vectors. Then the morphisms $B$ and $A$ constitute a short exact sequence

$$\mathbb{F}^m \xrightarrow{A} \mathbb{F}^m \xrightarrow{B} \mathbb{F}^{m-n}, \quad (29)$$

and hence, the morphism $\mathbb{F}^m \xrightarrow{A} \mathbb{F}^m$ is an admissible monomorphism.

In a similar manner, we can prove "point-wise surjection $\Rightarrow$ admissible epimorphism". Assume $B : \mathbb{F}^m \to \mathbb{F}^m$ is a point-wise surjection, and consider the transpose

$$B^T : \mathbb{F}^m \to \mathbb{F}^m$$

which determines a basis on $\ker(B(t))^\perp = \text{im}(B^T(t))$, for every $t$. Then, applying the Gram-Schmidt process, we obtain its QR-decomposition

$$B^T(t) = Q(t)R(t),$$

where $Q(t)$ represents an orthonormal moving frame in $\mathbb{F}^m$. As before, $t \mapsto Q(t)$ can be extended to a moving basis $t \mapsto \bar{Q}(t)$ of $\mathbb{F}^m$ by obstruction theory. More precisely, we can always find the dashed arrow in the diagram

Therefore, for every $t$, we can define a linear map

$$\mathbb{F}^{m-n} \to \mathbb{F}^m$$

$$e_i \mapsto \bar{Q}(t)e_{m+i}, \quad 1 \leq i \leq m - n.$$
which, with respect to the standard basis, is represented by the matrix

\[
\tilde{Q}(t) \begin{bmatrix} 0 \\ \text{id} \end{bmatrix} = A(t),
\]

where \([0 \ \text{id}]\) is the inclusion onto the last \(m - n\) vectors. In this way, we see that the sequence

\[
\mathbb{F}^{m-n} \xrightarrow{A} \mathbb{F}^m \xrightarrow{B} \mathbb{F}^n
\]
is point-wise exact and \(B\) is admissible. \(\square\)

In view of the lemma above, we see immediately that admissible monomorphisms and admissible epimorphisms are closed under composition. Now, we want to verify that a pushout (resp. pullback) along an admissible monomorphism (resp. an admissible epimorphism) exists and admissible monomorphisms (resp. epimorphisms) are stable under pushout (resp. pullback). We begin with admissible monomorphisms. Given a short exact sequence \(\mathbb{F}^{n'} \xrightarrow{A} \mathbb{F}^m \xrightarrow{B} \mathbb{F}^{n''}\) and a morphism \(\mathbb{F}^{m'} \xrightarrow{C} \mathbb{F}^{n'}\), we consider the morphism

\[
D : \mathbb{F}^m \rightarrow \mathbb{F}^{m+n''}
\]

\[
t \mapsto D(t) := \begin{bmatrix} 0 \\ \text{id} \end{bmatrix} B(t) + \begin{bmatrix} \text{id} \\ 0 \end{bmatrix} C(t)(A^T(t)A(t))^{-1}A^T(t),
\]

\(t \in \Delta^k\)

and check that it satisfies \(DA = \begin{bmatrix} 0 \\ \text{id} \end{bmatrix} C\), namely, the following commutative diagram,

\[
\begin{array}{ccc}
\mathbb{F}^{n'} & \xrightarrow{A} & \mathbb{F}^m \\
\downarrow C & & \downarrow D \\
\mathbb{F}^m & \xrightarrow{\begin{bmatrix} 0 \\ \text{id} \end{bmatrix}} & \mathbb{F}^{m+n''}.
\end{array}
\]

We observe that (30) is a pushout because, given two morphisms \(E : \mathbb{F}^n \rightarrow \mathbb{F}^l\) and \(F : \mathbb{F}^m \rightarrow \mathbb{F}^l\) such that \(EA = FC\), one may consider the morphism

\[
G : \mathbb{F}^{m+n''} \rightarrow \mathbb{F}^l
\]

\[
t \mapsto G(t) := E(t)B^T(t)(B(t)B(t)^T)^{-1}[0, \text{id}] + F(t)[\text{id}, 0],
\]

\(t \in \Delta^k\)

which satisfies the identities \(G(t)D(t) = E(t)\) and \(G(t)[\begin{bmatrix} 0 \\ \text{id} \end{bmatrix}] = F(t)\). Hence, the commutative square above is indeed a pushout, and the morphism \(\begin{bmatrix} 0 \\ \text{id} \end{bmatrix} : \mathbb{F}^m \rightarrow \mathbb{F}^{m+n''}\) is clearly an admissible monomorphism, in view of the Lemma above. Similarly, given a short exact sequence \(\mathbb{F}^{n'} \xrightarrow{A} \mathbb{F}^m \xrightarrow{B} \mathbb{F}^{n''}\) and a morphism \(\mathbb{F}^{m} \xrightarrow{C} \mathbb{F}^{n''}\), we have the morphism

\[
D : \mathbb{F}^{n'} \rightarrow \mathbb{F}^n
\]

\[
t \mapsto D(t) := B(t)(B(t)B^T(t))^{-1}C(t)[0, \text{id}] + A(t)[\text{id}, 0],
\]

\(t \in \Delta^k\)

and \(D\) fits in the following commutative diagram
We observe that (31) is a pullback because, given two morphisms \( E : \mathbb{F}^l \to \mathbb{F}^m \) and \( F : \mathbb{F}^l \to \mathbb{F}^m \) such that \( CF = BE \), one can define the morphism

\[
G : \mathbb{F}^l \to \mathbb{F}^{m' + m}
\]

\[
t \mapsto G(t) = \begin{bmatrix} \text{id} & 0 \\ 0 & 0 \end{bmatrix} (A^T(t)A(t))^{-1}A^T(t)E(t) + \begin{bmatrix} 0 & \text{id} \end{bmatrix} F(t),
\]

\( t \in \Delta^k \).

Again, it is straightforward to check the identities \( \begin{bmatrix} 0 & \text{id} \end{bmatrix} G(t) = F(t) \) and \( D(t)G(t) = E(t) \), and hence, (31) is a pullback with the morphism \( \mathbb{F}^{m' + m} \xrightarrow{\begin{bmatrix} 0 & \text{id} \end{bmatrix}} \mathbb{F}^m \) being point-wisely surjective obviously.

Lastly, by Lemma 2.2, given any morphism \( [l] \to [k] \in \Delta \), the induced functor \( \mathcal{V}_{\mathbb{F},k} \to \mathcal{V}_{\mathbb{F},l} \), preserves short exact sequences and finite products, and thus, \( \mathcal{V}_{\mathbb{F},k} \) is a simplicial exact category. □

Remark 2.3. (i) The proof given above implies that a commutative square in \( \mathcal{V}_{\mathbb{F},k} \) is a point-wise pushout along a point-wise monomorphism if and only if it is a pushout along an admissible monomorphism.

(ii) Notice that the category \( \mathcal{V}_{\mathbb{F},k} \) does not have an abelian category structure as not every morphism has a kernel and a cokernel—the dimension of the kernel might change when \( t \in \Delta^k \) varies.

It is clear that all weakly split epimorphism in \( \mathcal{V}_{\mathbb{F},k} \) is an admissible epimorphism, for every \( k \in \mathbb{N} \cup \{0\} \), by Lemma 2.2. Therefore, we have the following corollary of Theorem 1.11:

Corollary 2.4. The canonical map

\[
K(\mathcal{V}_{\mathbb{F},_}) \to K(\text{Ch}^b(\mathcal{V}_{\mathbb{F},_}))
\]

is a \( \pi_* \)-isomorphism in \( \mathcal{P} \).

Remark 2.5. Consider the internal category of bounded chain complexes \( \text{Ch}^b(\mathcal{V}_{\mathbb{F}}) \) whose space of objects is the space of bounded chain complexes topologized as the subspace of the space

\[
\coprod_{m \geq n_i \geq 0, n_i \leq m} M_{n_{i+1}, n_i}(\mathcal{F}) \coprod_{m \geq n_i \geq 0, n_i \leq m} \mathbb{F}^m,
\]

and whose space of morphisms is the space of the chain maps topologized as the subspace of the disjoint union

\[
\coprod_{k < k', m \geq m' \geq 0, i = m} M_{n_{i+1}, n_i}(\mathcal{F}) \coprod_{i = \max(m, k)} M_{i+1, i}(\mathcal{F}) \coprod_{i = k} M_{i+1, i}(\mathcal{F}) \coprod_{n_i \geq 0} M_{n_i, i}(\mathcal{F}).
\]

By the identification \( \text{Sing}(\text{Ch}^b(\mathcal{V}_{\mathbb{F}})) =: \text{Ch}^b(\mathcal{V}_{\mathbb{F}}) = \text{Ch}^b(\mathcal{V}_{\mathbb{F},_}) \), we know that the simplicial category \( \text{Ch}^b(\mathcal{V}_{\mathbb{F}}) \) is also a simplicial exact category.
2.2. The $H$-semiring structure on $|w^\mathcal{V}_E|$.

Recall, from [21, Lemma 3.1], that the enriched category $\mathcal{V}_E$ admits two functors: one is given by the direct sum of vector spaces ($\oplus$) and another by the tensor product of vector spaces ($\otimes$). These two functors give the space $|w^\mathcal{V}_E|$ a $H$-semiring space structure (see Definition B.9). Applying the singularization functor $\text{Sing}$, we obtain the corollary:

**Corollary 2.6.** $|w^\mathcal{V}_E|$ is a $H$-semiring space with addition induce by $\oplus$ and multiplication $\otimes$ and the canonical map $|w^\mathcal{V}_E| \to |w^\mathcal{V}_E|$ is a homotopy equivalence of $H$-semiring spaces.

**Proof.** It follows from the fact that $\text{Sing}$ preserves finite products, and thus, $\otimes$ and $\oplus$ induce the additive and multiplicative structures on $|w^\mathcal{V}_E|$. The homotopy equivalence results from the fact that the nerve of $w^\mathcal{V}_E$ is proper, and has the homotopy type of a CW-complex at each level. \[ \square \]

**Geometric Meaning:**
Recall from [20, Section 5], we have the following corollary, which implies that the space $|w^\mathcal{V}_E|$ classifies vector bundles over a paracompact Hausdorff space.

**Corollary 2.7.** There is a chain of homotopy equivalences of $H$-spaces

$$|w^\mathcal{V}_E| \xrightarrow{\sim} |w^\mathcal{V}_E| \xrightarrow{\sim} |\mathcal{G}| \xrightarrow{\sim} |\mathcal{G}|,$$

where $\mathcal{G}$ is the topological groupoid given by

$$O(\mathcal{G}) = M(\mathcal{G}) = \bigsqcup_k \text{Gr}_k(\mathbb{R}^\infty),$$

and trivial structure maps, and $\mathcal{V}_E^I$ is the fattened internal category of vector spaces [20, Section 3.3]. Note that, given $\mathcal{C}$ an internal category in $\text{Top}$, $C_\cdot$ stands for its singularization.

**Tensor product:** Applying the singular functor $\text{Sing}$ to the (internal) bi-functor

$$\otimes : \mathcal{V}_E \times \mathcal{V}_E \to \mathcal{V}_E,$$

we obtain a simplicial bi-functor

$$\otimes : \mathcal{V}_E^I \times \mathcal{V}_E^I \to \mathcal{V}_E^I,$$

which is bi-exact because of Lemma 2.2 and Remark 2.3. Hence, we have the following corollary, in view of Corollary [1.15]

**Corollary 2.8.** The map $K(\mathcal{V}_E^I)_0 \wedge K(\mathcal{V}_E^I)_0 \to K(\mathcal{V}_E^I)_0$ induced by $\otimes$ is a homotopy bi-linear map, and it fits into the homotopy commutative diagram

$$
\begin{array}{ccc}
K(\mathcal{V}_E^I)_0 \wedge K(\mathcal{V}_E^I)_0 & \to & K(\mathcal{V}_E^I)_0 \\
\uparrow & & \uparrow \\
|w^\mathcal{V}_E| \wedge |w^\mathcal{V}_E| & \to & |w^\mathcal{V}_E|.
\end{array}
$$
3. Applications

In this section, we present some applications of the theory of simplicial Waldhausen and S-categories. In particular, we shall see that the prespectra $K(V, F)$ and $K(\text{Ch}^b(V, F))$ represent connective topological $K$-theory and the space $|w \text{Ch}^b(V, F)|$ is homotopy equivalent to their infinite loop spaces—the latter implies the claim in [15, the last example in Section 2].

3.1. Connective topological $K$-theory

To explain what we mean by connective topological $K$-theory, we first recall some facts about vector bundles over a compact Hausdorff space $X$. Given $Y$ and $Z$, two pointed spaces, $[Y, Z]$ denotes the set of the homotopy classes of pointed maps from $Y$ to $Z$.

**Lemma 3.1.** (i) Let $\coprod_k \text{Vect}_k(X)$ be the abelian monoid of isomorphic classes of vector bundles over $X$. Then we have the algebraic group completion

$$\bigcoprod_k \text{Vect}_k(X) \to K(\bigcoprod_k \text{Vect}_k(X)) = K(X),$$

where $K(X)$ is the abelian group generated by elements in

$$\bigcoprod_k \text{Vect}_k(X)$$

subject to the relation $V \oplus W - V - W \sim 0$, for any $V, W \in \bigcoprod_k \text{Vect}_k(X)$. In particular, every element in $K(X)$ can be written as the difference $V - W$, for some $V, W \in \coprod_k \text{Vect}_k(X)$, and the homomorphism (32) can be realized by the assignment:

$$V \mapsto [V - 0].$$

Also, we consider the homomorphism given by the dimension of a vector space

$$K(X) \xrightarrow{\dim} \mathbb{Z}$$

and denote its kernel by $\tilde{K}(X)$.

(ii) The abelian monoid $\coprod_k \text{Vect}_k(X)$ and the abelian group $K(X)$ are representable, and there is a topological group completion

$$U \to K$$

such that

$$\coprod_k \text{Vect}_k(X) \cong [X_+, U],$$

$$K(X) \cong [X_+, K]$$

and the induced homomorphism $[X_+, U] \to [X_+, K]$ realizes the homomorphism (32).
(iii) Given a bilinear map of abelian monoids

\[ b : A \times B \to C \]

there exists a unique bi-linear map of abelian groups

\[ K(b) : K(A) \times K(B) \to K(C) \]

such that the following diagram commutes

\[
\begin{array}{ccc}
A \times B & \longrightarrow & C \\
\downarrow & & \downarrow \\
K(A) \times K(B) & \longrightarrow & K(C)
\end{array}
\]

Proof. The proof of the first and third assertion can be found in [16, Chapter 11], whereas in [20, Section 2], we give a detailed construction of the topological group completion

\[
\coprod_k Gr_k(\mathbb{F}^\infty) \to \mathbb{Z} \times G(\mathbb{F}^\infty),
\]

which realizes the homomorphism (32), where \(Gr_k(\mathbb{F}^\infty)\) is the Grassmannian of \(k\)-planes in \(\mathbb{F}^\infty\) and \(G(\mathbb{F}^\infty)\) is an enlargement of the colimit of \(\{Gr_k(\mathbb{F}^\infty)\}_{k \in \mathbb{N}}\) [20, Section 2.2]. □

Definition 3.2. Connective topological \(K\)-theory is a generalized cohomology theory represented by a \((-1)\)-connective \(\Omega\)-prespectrum \(E\) equipped with a map of prespectra

\[ B : E \to \Omega^d E \]

such that, for any pointed compact Hausdorff space \(X\), there is an isomorphism

\[ [X, E_0] \cong \tilde{K}(X) \]

such that the diagram

\[
\begin{array}{ccc}
[X, E_0] & \xrightarrow{B_0} & [X, \Omega^d E_0] \\
\downarrow & & \downarrow \\
\tilde{K}(X) & \xrightarrow{B} & \tilde{K}(X \wedge S^d)
\end{array}
\]

commutes, where \(B\) is the Bott homomorphism (see [1] and [2]). When \(d = 2\) (resp. \(d = 8\)), it represents connective complex (resp. real) topological \(K\)-theory.
3.2. Main theorems

**Theorem 3.3.** The $\Omega$-prespectra $K^S(V, \cdot)$ and $K(V, \cdot)$ represent connective topological $K$-theory.

**Proof.**

1. We first compare the following two topological group completions (see [21])

$$|w V| \to K^S(V, \cdot)_0;$$

$$\coprod_k \text{Gr}_k(\mathbb{P}^\infty) \to \mathbb{Z} \times G(\mathbb{P}^\infty).$$

Observe that the chain of homotopy equivalences of $H$-spaces in Corollary [27] gives us a homotopy equivalence of $H$-spaces from $|w V|$ to $\coprod_k \text{Gr}_k(\mathbb{P}^\infty)$, and since the maps (33) and (34) are topological group completions, the universal property of the topological group completion (Theorem B.5) implies that there is a homotopy equivalence of weak $H$-spaces

$$K^S(V, \cdot)_0 \to \mathbb{Z} \times G(\mathbb{P}^\infty),$$

unique up to homotopy. Hence, the homomorphism

$$[X, |w V|] \to [X, K^S(V, \cdot)_0]$$

is an algebraic group completion, for every pointed compact Hausdorff space $X$.

2. Now, we want to construct a map of prespectra

$$K^S(V, \cdot) \to \Omega^d K^S(V, \cdot)$$

that realizes the Bott isomorphism. Notice first, by Proposition [122], there is a map of homotopy bi-linear map

$$|w V| \wedge |w V| \to |w V|$$

$$K^S(V, \cdot)_0 \wedge K^S(V, \cdot)_0 \to K^S(V, \cdot)_0$$

Secondly, with the uniqueness clause in the third statement of Lemma [31], and the algebraic group completion (3.2), we can deduce the following isomorphism of bi-linear maps

$$K(X) \otimes K(Y) \to K(X \times Y)$$

$$[X_+, K^S(V, \cdot)_0] \otimes [Y_+, K^S(V, \cdot)_0] \to [X_+ \wedge Y_+, K^S(V, \cdot)_0]$$

where $X$ and $Y$ are compact Hausdorff spaces. If $X$ and $Y$ are pointed, we have the following reduced version:
Now, let  \( b : S^d \to \mathbb{Z} \times G^\infty \) be the Bott element, and define  \( b^S \) to be the composition—using (35):  
\[
b^S : S^d \to \mathbb{Z} \times G^\infty \to K^S(V_{F_2},) ;
\]
it induces a map of prespectra:  
\[
K^S(V_{F_2}) \wedge S^d \to K^S(V_{F_2}) \wedge K^S(V_{F_2},) \to K^S(V_{F_2}).
\]
which, when restricted to the zero component, yields a map  
\[
K^S(V_{F_2},) \wedge S^d \to K^S(V_{F_2},) \wedge K^S(V_{F_2},) \to K^S(V_{F_2},) .
\]
Combining the map above with (37), we obtain the diagram of isomorphisms below  
\[
K(X) \otimes K(Y) \to K(X \wedge Y)
\]
\[
[X, K^S(V_{F_2},)] \otimes [Y, K^S(V_{F_2},)] \to [X \wedge Y, K^S(V_{F_2},)].
\]
In this way, we see that the prespectrum  \( K^S(V_{F_2},) \) along with the adjoint of the map (38)  
\[
B^S : K^S(V_{F_2},) \to \Omega^d K^S(V_{F_2},)
\]
represents connective topological  \( K \)-theory.

4. Coming on to the case  \( K(V_{F_2},) \), we let  \( b^W \) be the composition  
\[
S^d \xrightarrow{b^S} K^S(V_{F_2},) \to K(V_{F_2}),
\]
and observe that Lemma 1.27 implies the following commutative diagram  
\[
K^S(V_{F_2}) \wedge S^d \to K^S(V_{F_2}) \wedge K^S(V_{F_2},) \to K^S(V_{F_2})
\]
\[
K(V_{F_2}) \wedge S^d \to K(V_{F_2}) \wedge K(V_{F_2},) \to K(V_{F_2}).
\]
Therefore, in view of Theorem 1.23, it suffices to verify \( V \) satisfies the extension and splittable axioms to show that the vertical maps are \( \pi, \cdot \)-isomorphisms. It is clear that \( wV \) is extensional as a morphism is a point-wise weak equivalence if and only if it is a weak equivalence in \( V_{F,k} \). To see \( coV \) is splittable, we recall that it is proved in Lemma 2.2 that, given a cofibration \( F \to F' \to A \) in \( coV \), there exists a morphism \( t \mapsto B(t) : \Delta^k \to FR_{n-n' \leq n} \) such that there is a short exact sequence

\[
F \to F' \to F_n \to F_n' \to 0
\]

(see (28) and (29)). Now, let \( D(t) := B(t)^T (B(t)B(t)^T)^{-1} \). Then there is an isomorphism \( FR' \oplus FR'' \to FR \) in \( V_{F,k} \) such that the diagram below commute

\[
\begin{array}{ccc}
FR' & \oplus & FR'' \\
\downarrow & & \downarrow \\
FR_n & & FR_{n-n'}
\end{array}
\]

where \( \oplus \) is the operation on \( V \) induced by the direct sum of vector spaces. In this way, we find that \( coV \) satisfies the splittable condition. Hence, by Theorem 1.25, all vertical maps in (39) are homotopy equivalences, and the prespectrum \( K(V,F,\cdot) \) along with the map \( Bw \) given by the adjoint of the composition

\[
K(V,F,\cdot) \wedge S^d \to K(V,F,\cdot) \wedge K(V,F,\cdot)_0 \to K(V,F,\cdot)
\]

also represents connective topological \( K \)-theory. \( \square \)

Now, consider the category of bounded chain complexes over \( F \), and note that the tensor product of chain complexes over \( F \) induces a simplicial bi-exact functor

\[
Ch^b(V,F,\cdot) \times Ch^b(V,F,\cdot) \to Ch^b(V,F,\cdot)
\]

which makes the following commute

\[
\begin{array}{ccc}
V_{F,k} \times V_{F,k} & \to & V_{F,k} \\
\downarrow & & \downarrow \\
Ch^b(V,F,\cdot) \times Ch^b(V,F,\cdot) & \to & Ch^b(V,F,\cdot)
\end{array}
\]

Define \( B_{ch}^w \) to be the adjoint of the composition

\[
K(Ch^b(V,F,\cdot)) \wedge S^d \xrightarrow{id \wedge B_{ch}^w} K(Ch^b(V,F,\cdot)) \wedge K(V,F,\cdot)_0 \to K(Ch^b(V,F,\cdot)) \wedge K(Ch^b(V,F,\cdot))_0 \to K(Ch^b(V,F,\cdot)).
\]

Then Theorem 3.3 and Corollary 2.4 imply the following theorem concerning Waldhausen \( K \)-theory of the simplicial Waldhausen category \( wCh^b(V,F,\cdot) \).
\textbf{Theorem 3.4.} The prespectrum $K(Ch^b(V_{\mathbb{E}}))$ along with the map $B^W_{\text{ch}}: K(Ch^b(V_{\mathbb{E}})) \to \Omega^d K(Ch^b(V_{\mathbb{E}}))$ represents connective topological $K$-theory.

Applying Theorem 1.25 one can obtain an analogous theorem concerning the $N$-construction of $Ch^b(V_{\mathbb{E}})$:

\textbf{Theorem 3.5.} The canonical map $|wN. Ch^b(V_{\mathbb{E}})| \to |wS. Ch^b(V_{\mathbb{E}})|$ is a homotopy equivalence, and the prespectrum $K^S(Ch^b(V_{\mathbb{E}}))$ along with the map $B^S_{\text{ch}}$ given by the adjoint of the composition

\[
K^S(Ch^b(V_{\mathbb{E}})) \wedge S^d \xrightarrow{id \wedge b^S} K^S(Ch^b(V_{\mathbb{E}})) \wedge K^S(Ch^b(V_{\mathbb{E}}))_0 \to K^S(Ch^b(V_{\mathbb{E}}))_0 \to K^S(Ch^b(V_{\mathbb{E}})).
\]

represents connective topological $K$-theory.

\textbf{Proof.} It is sufficient to check $Ch^b(V_{\mathbb{E}})$ satisfies the conditions in Theorem 1.25. Theorem 2.1 implies that $Ch^b(V_{\mathbb{E}})$ is a simplicial Waldhausen category and $Ch^b(V_{\mathbb{E},k})$ is additive, for every $k$. Using the long exact sequence associated with a short exact sequence of chain complexes

\[
0 \to C^* \to D^* \to E^* \to 0
\]

and the five lemma, we find that $wCh^b(V_{\mathbb{E}})$ is extensional. To see every morphism in $co Ch^b(V_{\mathbb{E}})$ is weakly splittable, we consider a point-wise monomorphism

\[
C^* \to D^*
\]

and observe that it admits a decomposition

\[
D^k = C^k \oplus E^k; \quad d^k_D = \begin{bmatrix} d^k_C & f^k \\ 0 & d^k_E \end{bmatrix},
\]

where $d^k_C$ and $d^k_E$ are the differential of $C^*$ and $D^*$ at degree $k$, respectively. Then define a family of chain complexes $D^*_t$ by letting

\[
D^*_t := C^k \oplus E^k; \quad d^*_t := \begin{bmatrix} d^k_C & tf^k \\ 0 & d^k_E \end{bmatrix},
\]

where $0 \leq t \leq 1$, and observe that it satisfies the following properties

\[
d^*_t \circ d^*_t = 0,
\]

\[
C^* \to D^*_t = D^*,
\]

\[
C^* \to D^*_0 \text{ is splittable},
\]

\[
C^* \to D^*_t \in co Ch^b(V_{\mathbb{E},k}).
\]

Thus, the simplicial Waldhausen category $Ch^b(V_{\mathbb{E}})$ satisfies the conditions in Theorem 1.25 \hfill \Box

Now, we shall explain how to see the claim in [15, the last example in Section 2]. Let $O$ and $M$ be the spaces of objects and morphisms in $w Ch^b(V_{\mathbb{E}})$, respectively, and observe that $O$ is the disjoint unions of the collection \{ $O((n_i)^m_m)$, $n_i > 0, m \leq m'$ \}, where $O((n_i)^m_m)$ contains those chain complexes whose vector spaces at degree $i$ are $\mathbb{P}^m$, for every $m \leq i \leq m'$ and 0 otherwise. Let $M((n_i)^m_m) := (s, t)^{-1}O((n_i)^m_m) \times O((n_i)^m_m)$, where $s, t$ are the source and target maps from $M$ to $O$. Then we have the spaces

\[
O((n_i)^m_m) \hookrightarrow M((n_i)^m_m)
\]
are algebraic sets in $\mathbb{R}^N$, for some large $N$, and hence they have a relative CW-complex structure (see [8, Section 1.2]). In particular, the map $O \hookrightarrow M$ is a closed cofibration. Similarly, one can show that the map

$$M \times_O M \times_O \ldots \times_O M \xrightarrow{\times_i} M \times_O M \times_O \ldots \times_O M$$

is a closed cofibration, where $s_i : [k] \to [k - 1] \in \Delta$ is the standard degeneracy map sending $i + 1$ and $i$ to $i$. Therefore, the nerve of $w \text{Ch}^b(V_b)$ is proper and has the homotopy type of a CW-complex at each degree. Combing this observation with the group completion theorem in [21] and Theorem 3.5, we obtain the assertion in [15, p.300]:

**Theorem 3.6.** The geometric realization $|w \text{Ch}^b(V_b)|$ is an infinite loop space representing connective topological $K$-theory.

**Proof.** From the discussion above, we see that the canonical map

$$|w \text{Ch}^b(V_b)| \to |w \text{Ch}^b(V_b)|$$

is a homotopy equivalence. On the other hand, by the group completion theorem [21], the map

$$|w \text{Ch}^b(V_b)| \to K^S(\text{Ch}^b(V_b),)_0$$

is a group completion. Notice that we have made use of the fact $\text{Ch}^b(V_b) = \text{Ch}^b(V_b)$ (Remark 2.5). Now, since $\pi_0(|w \text{Ch}^b(V_b)|) = \mathbb{Z}$, the map (41) is actually a homotopy equivalence. \hfill \Box

**Corollary 3.7.** Let $\text{Ch}^b_0(V_b)$ be the internal subcategory of $\text{Ch}^b(V_b)$ consisting of chain complexes with zero Euler characteristic. Then the geometric realization of the internal subcategory of quasi-isomorphisms $|w \text{Ch}^b_0(V_b)|$ is an infinite loop space representing the $0$-connective cover of topological $K$-theory.

**Proof.** Note first that the simplicial category $\text{Ch}^b_0(V_b)$ is a simplicial $S$-category with its simplicial subcategory of weak equivalences given by the intersection $\text{Ch}^b_0(V_b) \cap w \text{Ch}^b(V_b)$. Notice also that the inclusion

$$\text{Ch}^b_0(V_b) \hookrightarrow \text{Ch}^b(V_b)$$

is a simplicial exact functor. Now, observe that there is a chain of homotopy equivalences

$$|w \text{Ch}^b_0(V_b)| \xrightarrow{\sim} |w \text{Ch}^b_0(V_b)| \xrightarrow{\sim} K^S(\text{Ch}^b_0(V_b),)_0$$

because the nerve of $w \text{Ch}^b_0(V_b)$ is proper and has the homotopy type of a CW-complex at each level, and $\pi_0(|w \text{Ch}^b_0(V_b)|) = 0$. Since the homomorphism

$$\pi_i(|w \text{Ch}^b_0(V_b)|) \to \pi_i(|w \text{Ch}^b(V_b)|)$$

is an isomorphism, for every $i \geq 1$, we find that the map of prespectra

$$K^S(\text{Ch}^b_0(V_b),) \to K^S(\text{Ch}^b(V_b),)$$

is indeed a $0$-connective cover. \hfill \Box

Our proof of Segal’s example ([15, the last example in Section 2]) is completed in following corollary.
Corollary 3.8. [12, VI.4; VII] The tensor product of vector spaces induces an infinite loop space structure on the component of the topological $K$-theory space that contains the multiplicative identity.

Proof. In view of Corollary 3.7, it suffices to show that $|w \text{Ch}_b^1(\mathcal{V}_b)|$ is an infinite loop space, where $\text{Ch}_b^1(\mathcal{V}_b)$ is the topologized category of bounded chain complexes with Euler characteristic 1.

Let $K^M(C, \boxtimes, 1)$ be the $\Omega$-prespectrum associated with a symmetric monoidal category $(C, \boxtimes, 1)$ given by May’s delooping machine [11], [12, VII.3], and observe that the triple $(\mathcal{V}_b, \otimes, \mathbb{F})$ is a permutative category [12, Example 5.4], and therefore, the triple $(w \text{Ch}_b^1(\mathcal{V}_b), \otimes, \mathbb{F})$ is also a permutative category. Applying May’s delooping machine, we obtain a group completion [12, VII, Theorem 3.1]

$$|w \text{Ch}_b^1(\mathcal{V}_b)| \rightarrow K^M(w \text{Ch}_b^1(\mathcal{V}_b), \otimes, \mathbb{F})_0,$$

which is in fact a homotopy equivalence as $\pi_0(|w \text{Ch}_b^1(\mathcal{V}_b)|) = 0$.

Appendix A  The realization lemma

In this appendix, we give a detailed proof of a well-known lemma, called the realization lemma in [19]. Let $n$-$\text{Sets}$ denote the categories of $n$-fold simplicial sets, for every $n \geq 2$, and $s$-$\text{Sets}$ the category of simplicial sets. Then observe that, by induction, one can endow $n$-$\text{Sets}$ a Reedy model structure [6, Chapter VII], and there is a realization functor $|−| : n$-$\text{Sets} \rightarrow s$-$\text{Sets}$ given by applying the Quillen left adjoint

$$|−| : n$-$\text{Sets} \rightarrow (n − 1)$-$\text{Sets}$$

iteratively [6, VII.3]. On the other hand, there is another functor, called the diagonal functor, denoted by $d$

$$d : n$-$\text{Sets} \rightarrow s$-$\text{Sets}$$

$$X \mapsto d(X)$$

where $d(X) = X_{∗, ..., ∗}$

We claim that there is a natural isomorphism $\phi^n : |−|^{(n−1)} \simeq d(−)$. We construct this natural isomorphism by induction. When $n = 2$, the realization functor $|−|^{(1)} : 2$-$\text{Sets} \rightarrow s$-$\text{Sets}$ is given by the coequalizer

$$\text{coeq} \left\{ \coprod_{[k] \rightarrow [l]} X_{k, \Delta^k} \rightarrow \coprod_k [X_{k, \Delta^k}] \right\}$$

and the natural transformation $\phi^2$ at $X_.$ is induced by the simplicial map

$$\coprod_k [X_{k, \Delta^k}] \rightarrow d(X_.)$$

$$(x \in X_{k, m}, [m] \rightarrow [k]) \mapsto f^* x \in X_{m, m}.$$
it is known that $\phi_X$ is a natural isomorphism [6, p.198]. Now, suppose that $\phi^k$ is well-defined and is a natural isomorphism, for $k \leq n - 1$, and $X_n$ is a $n$-fold simplicial set. Then, we have, for every $p_n$, the simplicial map

$$\phi^{n-1}_X : |X_{n-1}| p_n \to d(X_{n-1} p_n).$$

(42)

is an isomorphism in $sSets$. On the other hand, since $\phi^2$ is a natural isomorphism, the simplicial map below

$$([q] \mapsto \langle([p_n], [p']) \mapsto d(X_{n-1} p_n) \sigma_{q}^{(1)} \rangle) \mapsto ([q] \mapsto d(X_{n-1} q) \sigma_{q}^{(1)})$$

(43)

is also an isomorphism of simplicial sets. Combining (43) with (42), we obtain the simplicial map

$$\phi^n_X : ([q] \mapsto |X_n (n-1)| : ([q] \mapsto \langle([p_n], [p']) \mapsto d(X_{n-1} p_n) \sigma_{q}^{(1)} \rangle) \mapsto ([q] \mapsto d(X_{n-1} q) \sigma_{q}^{(1)})$$

is an isomorphism. Thus, the claim is proved.

Notice that when we say “$|-|^{(n-1)}$ can be constructed inductively”, a preferred ordering on indices has been chosen. However, since the argument above is independent of the ordering, we can conclude that, for any given ordering on $n$, the set of $n$-elements, there is a natural isomorphism

$$\phi^n : |-|^{(n-1)} \to d(-).$$

In particular, this implies the following lemma:

**Lemma A.1.** Given a $n$-fold multi-simplicial set $X_n$ and a partition $I_1 \cup \ldots \cup I_k = \mathbf{n} := \{1, 2, \ldots, n\}$, then the associated $k$-fold simplicial set $X_{i_1, \ldots, i_k}$ is given by

$$(\{j_1, \ldots, j_k\} \mapsto X_{j_1, \ldots, j_k} := X_{i_1, \ldots, i_n}$$

with $i_s = i_t$ if $s, t \in I_i$, for some $i$, and the induced map

$$|\phi^k| : |[m] \mapsto |X_{i_1, \ldots, i_k}^{(k-1)}| \mapsto |[m] \mapsto d(X_{n-k})_m$$

is a homeomorphism.

Therefore, we may disregard the given partition or ordering on $\mathbf{n}$ and simply denote any of such iterated geometric realizations by $|X_n|$.

**Lemma A.2.** Given a map of $n$-fold multi-simplicial sets

$$f_n : X_n \to Y_n$$

and a partition $(I = \{i_1, \ldots, i_k\}) \cup (J = \{j_1, \ldots, j_{n-k}\}) = \mathbf{n}$, if $f_i$ induces a homotopy equivalence

$$|([i_1, \ldots, [i_k]) \mapsto X_{i_1, \ldots, i_k} \to Y_{i_1, \ldots, i_k}$$

for each $(n-k)$-tuple $(j_1, \ldots, j_{n-k})$, then the realization of $f_n$, $|f_n| : |X_n| \to |Y_n|$, is a homotopy equivalence.
Proof. By Proposition 1.7 in [6, IV], we know it is true when \( f \) is a map of bi-simplicial sets, \( I = \{ i_1 \} \) and \( J = \{ j_1 \} \). Now, in view of Lemma A.1 and the assumption of the present lemma, the map

\[
|d| \mapsto X^{I_{d_1, \ldots, d_{n-k}}} = \left| X^{i_1, \ldots, i_{n-k}} \right| \quad \text{and} \quad |J| \mapsto Y^{J_{d_1, \ldots, d_{n-k}}} = \left| Y^{j_1, \ldots, j_{n-k}} \right|
\]

is a homotopy equivalence, for each \( (n-k) \)-tuple \((j_1, \ldots, j_{n-k})\), where \( J_l = \{ j_l \} \). Applying Lemma A.1 again, we get the homeomorphisms

\[
|d'| \mapsto |d| \mapsto X^{I_{d, d'}} = \left| (j_1, \ldots, j_{n-k}) \right| \quad \text{and} \quad |d'| \mapsto |d| \mapsto Y^{J_{d, d'}} = \left| (j_1, \ldots, j_{n-k}) \right|
\]

The question is then reduced to the case of bi-simplicial sets, and hence, the map

\[
|f_m| : |X| = |X^{I_{n}}| \xrightarrow{\sim} |Y^{J_{n}}| = |Y|
\]

is a homotopy equivalence. \( \square \)

Appendix B Homotopy bi-linear maps

In this appendix, we shall review some basic facts about \( H \)-spaces, homotopy bi-linear maps and \( H \)-semiring spaces [4]. Throughout this section, we assume all spaces have the homotopy type of CW-complexes and all homotopies preserve base points unless otherwise specified. In addition, we assume all pointed spaces are well-pointed.

Definition B.1. (i) A pointed space \((X, *)\) is a \( H \)-space if and only if it admits an operation

\[
m : X \times X \to X
\]

such that the diagrams below commute, up to homotopy,

\[
\begin{align*}
& X \xrightarrow{(\text{id}_X, c_\cdot) / (c_\cdot, \text{id}_X)} X \times X \\
& \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow m
\end{align*}
\]

\[
\begin{align*}
& X \times X \xrightarrow{(m, \text{id}_X)} X \times X \\
& \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow m
\end{align*}
\]

\[
\begin{align*}
& X \times X \xrightarrow{(\text{id}_X, m)} X \times X \\
& \downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow m
\end{align*}
\]

\[
\begin{align*}
& X \times X \xrightarrow{\tau} X \times X \\
& \downarrow m \quad \quad \quad \quad \quad \quad \quad \downarrow m
\end{align*}
\]

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where $c_*$ is the constant map onto $*$, the base point, and $\tau(x, y) = (y, x)$.

(ii) A $H$-map of $H$-spaces $f : (X, *, m_X) \to (Y, *, m_Y)$ is a continuous map such that the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
X \times X & \xrightarrow{(f, f)} & Y \times Y \\
\downarrow m_X & & \downarrow m_Y \\
X & \xrightarrow{f} & Y
\end{array}
$$

(iii) A weak $H$-map of $H$-spaces $f : (X, *, m_X) \to (Y, *, m_Y)$ satisfies the same condition as a $H$-map except it commutes only up to weak homotopy.

Remark B.2. (i) In the literature, there are at least three different definitions of $H$-spaces. They all revolve around the question: how the base point is respected by homotopies. The strictest one asks that the base point * has to be a strict unit, whereas the weakest one does not even require that the homotopy in (44) preserves the base point. However, these definitions turn out to be equivalent when the $H$-space in question has the homotopy type of a CW-complex [8, exercise (3C.1)], [7].

(ii) The well-pointedness assumption comes in handy when we need the homotopy inverse of a homotopy equivalence of $H$-spaces.

Definition B.3. A $H$-space $X$ is group-like if and only if $\pi_0(X)$ is an abelian group.

Definition B.4. A group completion of a $H$-space $X$ is a $H$-map $X \to Y$ such that $Y$ is group-like and the induced homomorphism

$$f_* : H_*(X, R)[\pi_0(X)^{-1}] \to H_*(Y, R)$$

is a ring isomorphism for any commutative coefficient ring $R$.

Theorem B.5. Given a group completion of $H$-spaces $f : X \to Y$, and a $H$-map $g : X \to Z$ with $Z$ group-like, then there exists a weak $H$-map $h : Y \to Z$, unique up to homotopy, such that $h \circ f$ and $g$ are homotopic.

Proof. By Proposition 1.2 in [4], there exists a weak $H$-map $h : Y \to Z$. To see such $h$ is unique up to homotopy, we consider the induced maps $\tilde{f} : \tilde{X} \to \tilde{Y} \simeq Y$ and $\tilde{g} : \tilde{X} \to \tilde{Y} \simeq Z$, where $\tilde{X}$ (resp. $\tilde{Y}$ and $\tilde{Z}$) is the telescopes of $X$ (resp. $Y$ and $Z$) induced by the $H$-structure [4]. Now, suppose $h' : Y \to Z$ is another map such that $h' \circ f$ and $g$ are homotopic. Then both the maps $h$ and $h'$ fit in the following homotopy commutative triangle

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \tilde{Y} \simeq Y \\
\downarrow \tilde{g} & & \downarrow \kappa/h \\
\tilde{Z} \simeq Z
\end{array}
$$

The uniqueness clause then follows from [4, Lemma 1.1]. □
**Definition B.6.** Given three $H$-spaces $X$, $Y$, and $Z$, a map $b : X \wedge Y \rightarrow Z$ is homotopy bilinear if and only if it satisfies the following homotopy commutative diagrams

where the maps $d_1$ and $d_2$ are defined as follows:

$$d_1 : (X \times X) \times Y \rightarrow (X \times Y) \times (X \times Y)$$

$$(x, x, y) \mapsto (x, y, x, y)$$

$$d_2 : X \times (Y \times Y) \rightarrow (X \times Y) \times (X \times Y)$$

$$(x, y, y) \mapsto (x, y, x, y).$$

**Lemma B.7.** A homotopy bi-linear map $b : X \wedge Y \rightarrow Z$ naturally induces a homotopy bi-linear map between the associated loop spaces

$$\Omega b : \Omega X \wedge \Omega Y \rightarrow \Omega^2 Z,$$

where $\Omega b$ is given by the following composition,

$$\Omega^2 X \wedge \Omega^2 Y \rightarrow \Omega^2 (X \wedge Y) \rightarrow \Omega^2 Z$$

$$(\gamma(t), \delta(s)) \mapsto (s, t \mapsto \gamma(t) \wedge \delta(s)) \mapsto (s, t \mapsto b(\gamma(t) \wedge \delta(s))).$$

**Proof.** 1. Observe that the homotopy $(X \times X) \wedge Y \wedge I_+ \rightarrow Z$ for the left distribution,

induces a homotopy

$$(\Omega X \times \Omega X) \wedge \Omega Y \wedge I_+ \rightarrow \Omega^2 ((X \times X) \wedge Y \wedge I_+) \rightarrow \Omega^2 Z.$$  \hfill (46)

Now, consider the commutative diagrams below
where the upper routes of the first and second diagrams represent the restrictions of the map (46) at $t = 0$ and $t = 1$, respectively, and the lower routes of them represent the paths going through the left and right corners of the diagram below, respectively.

In this way, the homotopy (46) gives us the homotopy required, and hence $\Omega b$ is left linear. The right linearity can be derived in a similar way. □

Notice that the $H$-structure on $\Omega X$ (resp. $\Omega Y$ and $\Omega Z$) used in the proof above is the one given by the $H$-structure on $X$ (resp. $Y$ and $Z$). However, there is another $H$-structure on $\Omega X$ that is given by concatenating two loops. These two $H$-structures turn out to be equivalent.

**Lemma B.8.** Given a $H$-space $(X, a, *)$, let $\Omega a$ denote the $H$-structure on $\Omega X$ induced by the $H$-structure $a$ on $X$ and $\Omega_X$ the $H$-structure induced by concatenating two loops. Then $\Omega a$ and $\Omega_X$ are homotopic.

**Proof.** Without loss of generality, we may assume the identity element $*$ is strict with respect to the operation $a$. Now, consider the maps below:
\[\pi_1 : I \times I \longrightarrow I\]
\[(t, s) \mapsto \frac{2t - s}{2 - s}\]
\[t \geq \frac{s}{2}\]
\[t \leq \frac{s}{2}\]

\[\pi_0 : I \times I \longrightarrow I\]
\[(t, s) \mapsto \frac{2t}{2 - s}\]
\[t \leq 1 - \frac{s}{2}\]
\[t \geq 1 - \frac{s}{2}\]

which induce two maps from \(S^1 \land I_+\) to \(S^1\), denoted by \(\pi_0\) and \(\pi_1\) still. Then the required homotopy between \(\Omega a\) and \(\Omega X\) is given by the composition,

\[H_s : (\Omega X \times \Omega X) \land I_+ \rightarrow (\Omega X \land I_+) \times (\Omega X \land I_+) \xrightarrow{(\pi_0^*, \pi_1^*)} \Omega X \times \Omega X \rightarrow \Omega X,\]

where \(H_0\) is \(\Omega a\), and \(H_1\) is \(\Omega X\). \(\square\)

**Definition B.9.**

(i) A \(H\)-semiring space is a 5-tuple \((X, *, 1, a, m)\) such that \((X, *, a)\) constitutes a \(H\)-space and \((X, 1, m)\) a \(H\)-space in a weaker sense—the homotopies of the commutative diagrams in Definition \(B.1\) do not necessarily preserve the point 1. In addition, we require that \(*\) be a strict zero, meaning

\[m(*, x) = * = m(x, *),\]

for all \(x \in X\), and the induced map, denoted still by \(m\), \(m : X \land X \rightarrow X\) is homotopy bi-linear with respect to \(a\).

(ii) A map of \(H\)-semiring spaces \((X, *, 1, a, m) \rightarrow (X', *, 1', a', m')\) is a map \(f : X \rightarrow Y\) that preserves the points \(*\) and 1, and respects the operations \(a\) and \(m\).

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