Effects of anisotropic correlations in fermionic zero-energy bounds states of topological phases

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Topological phases of matter have been used as a fertile realm of intensive discussions about fermionic fractionalization. In this work, we study the effects of anisotropic superconducting correlations in the fermionic fractionalization on the topological phases. We consider a hybrid version of the SSH and Kitaev models with an anisotropic superconducting order parameter to investigate the unusual states with zero energy that emerges in a finite chain. To obtain these zero energy solutions, we built a chain with a well-defined domain wall at the middle of the chain. Our solutions indicate an interesting dynamic between the zero-energy state around the domain wall and the superconducting correlation parameters. Finally, we find that the presence of an isolated Majorana at the ends of the chain is strongly depending on the existence of the solitonic excitation at the middle of the chain.

I. INTRODUCTION

In condensed matter physics, non-local Majorana fermions are quasi-particle excitations that arise when a single electronic mode fractionalize into two halves [1]. The Kitaev chain has been used to study the properties of non-local Majorana zero-energy bound states that reside at edges of the chain [11,12]. The Kitaev model belongs to the BDI topological class [2]. In this class, the non-trivial topological phase depends on time-reversal symmetry, particle-hole symmetry and chiral symmetry [12,13]. The special particle-hole symmetry ensures that the Majorana zero-energy bound states behave like a superposition of particle and hole and have no well-defined charge [1,7]. However, besides the particle-hole symmetry, the number of zero-energy bound states at the edges is crucial to obtain one isolated Majorana zero-energy bound state. Any chain with an even number of zero-energy bound states per edge does not have an isolated Majorana at the edges. To obtain an isolated Majorana zero-energy bound state is necessary an odd number of zero energy at the ends of the chain [7].

In the last years, several models have been proposed to support isolated Majorana zero-energy bound states [4,5,8,10,11]. Recently, R. Wakatsuki et al [11] studied the dimerized Kitaev model. The dimerized Kitaev model exhibit a phase diagram composed by SSH-like ground state and Kitaev-like ground state. There are two zero-energy edges states in the SSH-like ground state and one zero-energy state in the Kitaev-like ground state. The SSH-like ground state displays an even number of zero-energy states, while the Kitaev-like ground states present an odd number. In the Kitaev-like ground states, the model can support an isolated Majorana zero-energy bound states at the ends of the chain [11,12].

Therefore, the dimerized Kitaev chain allows to study topological phase transitions from ground states with an even number of zero-energy states to the ground state with an odd number of zero-energy states. Considering a domain wall at the middle of the chain, the authors studied the evolution of the solitonic mode across the topological phase transition from SSH-like ground state to Kitaev-like ground state. The solitonic mode suffers a split across the quantum phase transition between the SSH-like and Kitaev-like phases. The splitting of the solitonic mode gives rise to one Majorana zero-energy bound state at each end of the chain. We point out that, the results from Ref [11] depend on a specific constraint between the hopping terms and superconducting correlation (i.e., a constraint between the hopping and superconducting parameters such that they are interdependent).

In this work, we explore the phase diagram of the Kitaev model with alternating hoppings and superconducting correlations. Our results do not depend on constraints between the hopping terms and superconducting parameters, as employed in Ref [11]. The effects of the anisotropic correlations have been clarified. We provide the correct phase transition between the hybrid model and the pure SSH model. We studied the effects of the anisotropic hopping terms and anisotropic superconducting correlations over the zero energy bound state around the domain wall at the middle of the hybrid chain. To this porpouse, we create a general kink (i.e., in hopping and superconducting correlations) at the middle of the chain.

The paper is organized as follows, in section [11] we defined the model and its important discrete symmetries. In section [11] we introduce the topological invariants related to each symmetry and the phase diagrams of the system. In section [11] we simulate a kink at the middle of the chain to obtain the zero energy states around the domain wall. We investigated the effects of the anisotropic superconducting correlation over the solitonic model. Finally, we summarize our results in the conclusion.
When $\Delta_1 = \Delta_2 = \mu = 0$ the system becomes the SSH model that belongs to the same topological class as the Kitaev model, as is well known \cite{7,11}.

The SSH model does not display Majorana zero-energy bound states at the ends of the chain \cite{7}.

The topological non-trivial ground state ($|t_2| > |t_1|$) of the SSH model supports one zero-energy edge states at the first and last sites of the chain. The zero-energy states of the SSH model are composed by superposition of particles and exhibit a well-defined charge. Therefore, a conventional fermionic excitation mode emerges \cite{7}.

In the SSH model, the chiral symmetry results from an equivalence between the sublattices A and B. We notice that the chiral symmetry protects the non-trivial topological region and the zero-energy edge states \cite{7}.

Now, when $t_1 = t_2$ and $\Delta_1 = \Delta_2$, the Hamiltonian, Eq. 1 becomes the Kitaev model, which possesses particle-hole symmetry, given ($\Xi = \sigma_2 K$), time reversal symmetry ($\Theta = K$) and also chiral symmetry ($\Pi = \Xi \Theta$) \cite{11}.

Nevertheless, although the Kitaev model and the SSH model belongs to the same topological classification (i.e. BDI), it is important to notice that the origin of the particle-hole symmetry is physically distinct for each model \cite{7,11}.

As pointed out in Ref \cite{7}, for the SSH model, the particle-hole symmetry was induced by time-reversal and chiral symmetries, while for the Kitaev model, the particle-hole symmetry is essentially a characteristic of the superconducting phase.

III. PHASE DIAGRAM ON THE PARAMETER SPACE

A. One half limit, $\mu = 0$

Since the model in Eq. 1 has been classified into the BDI topological class, the $Z$ topological invariant that should be calculated to characterize the trivial and non-trivial phases is the well known Winding number \cite{14}

$$W_i = \frac{1}{4\pi i} \int_0^{2\pi} dk Tr(C_i \mathcal{H}_{k}^{-1} \partial_k \mathcal{H}_k) ,$$

where $C_{i=1,2}$ are the symmetry operators and $H_k$ is given by Eq. 3. The operators $C_{i=1,2}$ are two matrices that anti-commute with the Hamiltonian ($\{ C_i, H_k \} = 0$), and defines two distinct topological indexes $W_{i=1,2}$.

In the case of $\mu = 0$, the Hamiltonian, Eq 3 anti-commutes with two matrices, $C_1 = \sigma_0 \otimes \sigma_3$ and $C_2 = \sigma_0 \otimes \sigma_1$. Here, the matrix $C_1$ is the chiral operator or sublattice symmetry and $C_2$ is the particle-hole symmetry operator. Thus, $W_1$ (or the chiral index $N_1$ in the Ref \cite{11}) is related to the sublattice symmetry, and $W_2$ (or the chiral index of Majorana fermion $N_2$ in Ref \cite{11}) refers to the particle-hole symmetry.
Besides, as demonstrated by R. Wakatsuki et al Ref [11], $W_1$ is equal to the number of zero energy states per end of the chain and $|W_2|$ is equal to the number of Majorana zero-energy bound states. When $\Delta_1 = \Delta_2 = \mu = 0$, the chiral operator $C_1 \sigma_0 \otimes \sigma_3$ becomes $\sigma_3$. This reduction, $C_1 = \sigma_3$, is necessary to obtain a quantum phase transition from the hybrid chain to the SSH limit.

We calculated $W_1$ for $\mu = 0$ and the phase diagram results are shown in Fig. 2. All parameters are expressed in units of $t_1$ in the plot. The results for $W_2$, can be seen in Fig. 3. The analytical expressions for the winding numbers when $\mu = 0$ are given in Appendix A. As we can see, in Fig 2 and Fig. 3 these phase diagrams are presented as function of the parameter $\frac{\Delta_1}{t_1}$ and $\frac{\Delta_2}{t_1}$ for fixed values of $\frac{\Delta_1}{t_1}$.

Fig. 2 (a) shows the phase diagram of the pure SSH model without the superconducting correlation terms.

Fig. 2 (b) shows the case of $t_2/t_1=0.2$. For $\Delta_1/t_1=0, \Delta_2/t_1=2$ of the phase diagrams (b), (c) and (d). Here, the black solid lines are the energy of the bulk states while the blue and green lines are the energy of the edges states. The circle (yellow), square (orange) and triangle (orange) in the energy spectrum (a) indicates the value of the ratio $(t_2/t_1)$. (b) Topological phase diagram with $\Delta_1/t_1=0.2$, (c) Topological phase diagram with $\Delta_1/t_1=1$ and (d) Topological phase diagram with $\Delta_1/t_1=1.6$. A new topological phase with $W_1 = 2$ (white regions) arises in the superconducting hybrid system.

In this case, a topological non-trivial phase arises when $|t_2| > |t_1|$ since $W_1 = 1$ (red region), and a trivial topological phase arises when $|t_1| > |t_2|$ ($W_1 = 0$) (blue region).

We then proceed our analysis introducing the superconducting correlations $\Delta_1$ and $\Delta_2$ to the SSH model. Starting from the three different ground states: $\frac{\Delta_1}{t_1} < 1$ (topological trivial), $\frac{\Delta_1}{t_1} = 1$ (phase transition line) and $\frac{\Delta_1}{t_1} > 1$ (topological non-trivial phase) in the absence of superconducting correlations($\Delta_1 = \Delta_2 = 0$), we varie the superconducting correlations for each case and the results are presented in Figs. 2 (b), (c) and (d). The ratio $t_2/t_1$ has been held fixed in each panel.

Fig. 2 (b) shows the case $\frac{\Delta_1}{t_1} < 1$ and we observe that
the superconducting correlations induce three different topological regions: blue \((W = 0)\), red \((W = 1)\), and an extra white region \((W = 2)\). Moreover, Fig. 2 (c) shows the results along the transition line, \(\frac{t_1}{t_2} = 1\), in the presence of the parameters \(\frac{\Delta_1}{t_1}, \frac{\Delta_2}{t_1} \neq 0\), and we get the same three different topological phases as previously obtained, indicated by the same color scheme as in Fig. 2 (b), however, notice that now, there is a black point located at the origin of the phase diagram, this point represent a gapless point of the SSH model, \(\frac{\Delta_1}{t_1} = \frac{\Delta_2}{t_1} = 0\), which is consistent with the phase diagram in Fig. 2 (a) at point \(t_1 = t_2 = 1\). Finally, the phase diagram shown in Fig. 2 (d) emerges from a topological phase of the SSH model, when \(\frac{\Delta_1}{t_1} = 1.6\).

The results are analogous to those previously obtained, except that, differently from the result obtained by R. Wakatsuki et al. 11, we have correctly determined the topological phase transition at \(\Delta_1 = \Delta_2 = 0\), see the red dot at the origin of the phase diagram.

Indeed, the model should fall into the topological phase of the SSH model, \(W_1 = 1\), when \(\frac{t_1}{t_2} > 1\) in the absence of superconducting correlations (see the red dot at the origin of the diagram in Fig. 2 (d)).

Fig. 3 (a) shows the energy spectrum in real space for \(N = 50\) as a function of \(\frac{\alpha}{t_1}\) for fixed values \(\frac{\Delta_1}{t_1} = 0\) and \(\frac{\Delta_2}{t_1} = 2\).

The zero-energy states located at the ends of the chain are highlighted by green and blue solid lines, and correspond to the orange square and triangle, indicated by the point \(\Delta_1 = 0\) and \(\Delta_2 = 2\) of the phase diagram in Fig. 3 (c) and (d). On the other hand, the bulk energy states have been highlighted by black solid lines, corresponding to the yellow circle in the energy spectrum of Fig. 3 (a), are related to the point \(\Delta_1 = 0\) and \(\Delta_2 = 2\) of the phase diagrams (b). and (c) of Fig. 3.

The zero-energy edge states solutions are shown in Fig. 3 (a) and indicated by the red dotted and green solid lines.

The circle indicates the topological non-trivial ground state (topological indexes \(W_1 = 2\) and \(W_2 = 0\)) with four zero-energy states at the ends of the chain, see Fig. 3 (a). The orange square and triangle shows the topological non-trivial region (topological indexes \(W_1 = 1\) and \(W_2 = -1\)) with two zero-energy states at the ends of the chain, Fig. 3 (a).

We see that, in the interval \(0 < \frac{\Delta_1}{t_1} < 1.0\) of the Fig. 3 (a) \(W_1 = 2\). Therefore, using the bulk boundary correspondence theorem 9, one realizes that \(W_1 = (4)/(2) = 2\) is the number of zero-energy states per end of the chain. Indeed, the white regions in Fig. 3 (b), (c) and (d) indicate two zero energy states per ends of the chain.

Moreover, we see that for \(1.0 < \frac{\Delta_1}{t_1} \leq 1.8\), there are only two zero-energy states in Fig. 3 (a) (see the two collapsed green lines in this interval). Again, using the bulk boundary correspondence theorem, we conclude that \(W_1 = 1\) is the number of zero-energy states per end of the chain (i.e. \(W_1 = (2)/(2) = 1\) states/end).

We know that, given the particle-hole symmetry, one isolated Majorana zero-energy bound state emerges at the ends of the chain when each end of the chain has only one zero-energy state. For instance, in the interval \(1.0 < \frac{\Delta_1}{t_1} < 1.8\), and for \(\frac{\Delta_1}{t_1} = 0\) and \(\Delta_2 = 2\) there is only one Majorana zero-energy bound state per end of the chain, see solid green lines in Fig. 3 (a). Indeed, the point \(\left(\frac{\Delta_1}{t_1} = 0, \frac{\Delta_2}{t_1} = 2\right)\) from Fig. 3 (c) and (d) is located within the green region with \(W_2 = -1\), see yellow square and triangle in Fig. 3.

Actually, the green and red regions in Fig. 3 support one isolated Majorana zero-energy bound state at the edges, while blue regions do not exhibit an isolated Majorana zero-energy bound state. The boundaries of the phase diagrams Fig. 2 and Fig. 3 indicate the phase transitions between topological phases with different number of zero-energy edge states, such that green and red regions exhibit ground states with Majorana zero-energy bound states at the edges of the chain.

In order to investigate the topological phase transition from the hybrid phase to a pure SSH ground state, we introduced the following parameterizations; \(t_1 = -t(1 + \eta_1), t_2 = -t(1 - \eta_1), \Delta_1 = \Delta(1 + \eta_2)\) and \(\Delta_2 = \Delta(1 - \eta_2)\), where \(\eta_2 = \alpha \eta_1\) and \(\alpha\) is a real parameter. \(\eta_1\) was defined in Ref 11 as the dimerization parameter.

We notice that in Ref 11 the authors studied only the case \(\alpha = 1\), i.e \(\eta_2 = \eta_1\). We report that the choice \(\eta_1 = \eta_2\) creates a specific constraint between the hopping and superconducting parameters.

Replacing the above parametrization in the Hamiltonian 11, we obtain a new phase diagram, as seen in Fig. 4. Moreover, Fig. 4 provides a clear visualization of the topological quantum phase transition from the superconducting hybrid model to SSH model. The dotted line at \(\Delta = 0\) represents the topological insulating phase of the SSH model with \(W^{SSH} = 1\), see Fig. 4 (a) and (b).

In order to compare with previous results, in Fig. 4 (a) we fixed \(\eta_1 = \eta_2\). The phase diagram when \(\eta_1 \neq \eta_2\) can be visualized in Fig. 4 (b). In this case, we fixed \(\alpha = 1.5, \eta_2 = \alpha \eta_1\) and \(0 < \eta_1 < 1\). We can see three different topological phases, red \(W_1 = 1\), blue \(W_1 = 0\), white \(W_1 = 2\) and a red traced line with \(W_1^{SSH} = 1\) for \(\Delta/t = 0\) and \(\eta_1 < 0\). Here \(W_1^{SSH}\) is the winding number for the insulating phase of the SSH model. This result has not been obtained in Fig. 2-(a) of Ref 11, where for \(\Delta/t = 0\) and \(\eta < 0\), they found \(N_1 = 2\) in their whole SSH-like area.

**B. \(\mu \neq 0\)**

For \(|\mu| > 0\) the sublattice symmetry is explicitly broken and only the particle-hole symmetry (\(\hat{Z}_2 = \sigma_1 \otimes \sigma_0 \mathcal{K}\)) induce the topological index. We computed the number of zero-energy states and \(W_1\) as a function of \(\Delta_1, \Delta_2\) and \(\frac{\mu}{t_1}\). These results are summarized in figures 5 and 6.

Fig. 5 shows the effects of the chemical potential \(\mu\) over the number of zero-energy solutions per end of the
Figure 4. (Color online) Topological phase diagrams with respect to $W_1$ in-plane $\Delta - \eta_1$ with $\mu = 0$. Parametrization $t_1 = -t(1 + \eta_1)$, $t_2 = -t(1 - \eta_1)$, $\Delta_1 = \Delta(1 + \eta_1)$ and $\Delta_2 = \Delta(1 - \eta_1)$. The numbers in the figures denotes $W_1$. Red regions exhibit phase with $W_1 = 0$ and white regions possess $W_1 = 2$. (a) Case $\eta_2 = \eta_1$. (b) Case $\eta_2 = 1.5\eta_1$. All points that belong to the traced line at $\Delta = 0$ and $\eta_1 < 0$ are into a topologically non-trivial phase of the pure SSH model, since at these points, the winding number $W^{SSH} = 1$. The topological traced line for $\eta_1 < 0$ separates two topological regions with the same topological invariant $W_1 = 2$ (white region).

In Fig. 5 (a), we calculated the energy spectrum as a function of $t_2/t_1$ for $t_1 = 0.9$, $\Delta_1/t_1 = 0$ and $\Delta_2/t_1 = 0.4$.

The zero-energy states located at the ends of the chain are highlighted by green solid lines, and correspond to the orange triangle, indicated by the point $\Delta_1 = 0$ $\Delta_2 = 2$ of the phase diagram in Fig. 5 (d). On the other hand, the bulk energy states have been highlighted by black solid lines, corresponding to the yellow circle and square in the energy spectrum of Fig. 5 (a), are related to the point $\Delta_1 = 0$ and $\Delta_2 = 2$ of the phase diagrams (b) and (c) of Fig. 5.

Fig. 5 (b) shows the phase diagram in plane $\Delta_1 - \Delta_2$ for $\mu/t = 0.9$ and $t_2/t_1 = 0.2$. Again, the regions with $W_2 = 0$ are colored with blue color, while the regions with $W_2 = -1$ and $W_2 = 1$ are colored with green and red colors respectively. The yellow circle indicates the topological trivial ground states ($W_2 = 0$) for $\mu = 0.9$, $t_2/t_1 = 0.2$, $\Delta_1 = 0$ and $\Delta_2 = 2$.

Fig. 5 (c) shows the phase diagram when $t_2/t_1 = 1$, where the yellow square indicates the topological trivial ground state ($W_2 = 0$) for $\mu = 0.9$, $t_2/t_1 = 0.2$, $\Delta_1 = 0$ and $\Delta_2 = 2$.

The Fig. 5 (d) shows the phase diagram for $t_2/t_1 = 1.6$. The orange triangle indicates the topological non-trivial ground state ($W_2 = 0$) for $\mu = 0.9$, $t_2/t_1 = 1.6$, $\Delta_1 = 0$ and $\Delta_2 = 2$. We can see clearly that the topological regions with $W_2 \neq 0$ increasing when the ratio $t_2/t_1$ increases (i.e. the size of the red and green regions from the Fig. 5 (b) ($t_2/t_1 = 0.2$), (c)($t_2/t_1 = 1.0$) (d) ($t_2/t_1 = 1.6$) increases as the ratio ($t_2/t_1$) increase.).

In order to study the effects of the chemical potential and the superconducting correlation function over the topological phases, we defined the parameter $r$ as $\Delta_2 = r \Delta_1$. Fig. 5 (a) shows the energy spectrum in real space as a function of the ratio $\mu/t_1$ for $t_2/t_1 = 1.6$ (here we fixed $r = -1$ and $N = 60$ unit cell.). We depicted the zero-energy states by green solid line and bulk edge states by blue color.

In Fig. 5 (b), (c) and (d), we show the phase diagram for $W_2$ in plane $\mu/t_1$ for $t_2/t_1 = 0.2$ (b), $t_2/t_1 = 1.0$ and (d) $t_2/t_1 = 1.6$. In this case, we observe non-trivial topological phases with $W_2 = 1$ (red regions), non-trivial topological phases with $W_2 = -1$ (green regions) and trivial topological phases with $W_2 = 0$ (blue regions). Note that, the topological non-trivial phases are surrounded by circular
Figure 6. (Color online) Energy spectrum of the finite chain and topological phase diagrams with respect to $W_2$ in-plane $\mu/t_1 - r$. Here $r = \frac{2\Delta}{t_2}$. The numbers in the figures (b), (c) and (d) denotes $W_2$. (a) Energy spectrum as a function of $\frac{\mu}{t_1}$ with $\frac{\mu}{t_1} = 1.6$ and $r = -1$. (b) Topological phase diagram for $\frac{\mu}{t_1} = 0.2$ (c) Topological phase diagram for $\frac{\mu}{t_1} = 1.0$ and (d) Topological phase diagram for $\frac{\mu}{t_1} = 1.6$

In this section, we consider a kink at the middle of the hybrid chain, according the Fig. 7. In Fig. 7(a), we considered a kink only in the hopping terms ($t_2$), see green dotted circle. In Fig. 7(b), we considered a kink only in the superconducting correlations $\Delta_i$, while in Fig. 7(c), we considered a kink in both, hopping and superconductor correlation parameters.

The Hamiltonian of the hybrid chain, Eq 1, after the inclusion of the kink is given by

$$H = \psi^\dagger (h_0 + V) \psi$$

where,

$$\psi^\dagger = (c^\dagger_{A1}, c^\dagger_{A2}, c^\dagger_{B1}, c^\dagger_{B2}, \ldots, c^\dagger_{AN}, c^\dagger_{BN}, c^\dagger_{C1}, c^\dagger_{C2}, \ldots, c^\dagger_{CN})$$

$\quad h_0 = \mu I \otimes (\sigma_z \otimes I) + T_{11} \otimes (\sigma_z \otimes \sigma_z) + T_{21} \otimes (\sigma_0 \otimes A) + h.c.$

Figure 7. (Color online) The hybrid chain in the presence of a kink at the middle of the chain. In panel (a), we considered a kink only in hopping terms, see green dotted circle at the middle of the chain. In panel (b), we considered a kink only in the superconducting correlations, while in panel (c), we considered a kink in both, hopping and superconductor correlation parameters.

IV. SIMULATION OF THE ZERO ENERGY STATES AROUND THE DOMAIN WALL AND EDGES OF THE HYBRID CHAIN

For instance, for $r = -1$ in Fig. 7(d) the topological phase transitions between the regions with $W_2 = 0$ (blue) and $W_2 = 1$ (red) happen at $\mu/t_1 = -1.75,$ $-0.6, 0.6,$ and $1.75.$

The fermionic fractionalization at the ends of the chain depends on the anisotropic parameter $r$ and chemical potential $\mu,$ as we can see in all phase diagrams from this section. Again, we can see the topological region increasing when the ratio $t_2/t_1$ increases.
where,
\[ I = \sum_{i,j=1}^{N} \delta_{i,j}, \]
\[ T_{12} = t_{1} \sum_{i,j=1}^{N/2} \delta_{i,j} + t_{2} \sum_{i,j=N/2+1}^{N} \delta_{i,j}, \]
\[ T_{21} = t_{2} \sum_{i,j=1}^{N/2} \delta_{i-1,j} + t_{b} \sum_{i,j=N/2+1}^{N} \delta_{i-1,j}, \]
\[ A = \sum_{i,j=1,2} \delta_{i+1,j}, \]
\[ \Delta_{12} = \Delta_{1} \sum_{i,j=1}^{N/2} \delta_{i,j} + \Delta_{2} \sum_{i,j=N/2+1}^{N} \delta_{i,j}, \]
\[ \Delta_{21} = \Delta_{2} \sum_{i,j=1}^{N/2} \delta_{i-1,j} + \Delta_{b} \sum_{i,j=N/2+1}^{N} \delta_{i-1,j}, \]
\[ B = \sum_{i,j=1,2} \delta_{i-1,j}. \]

The kink will occur only in the hopping term when \( t_{a} = t_{2}, \ t_{b} = t_{1}, \ \Delta_{a} = \Delta_{1} \) and \( \Delta_{b} = \Delta_{2} \), see Fig. 7(a).

Contrarily, the kink will occur only in the superconducting correlations when \( t_{a} = t_{1}, \ t_{b} = t_{2}, \ \Delta_{a} = \Delta_{2} \) and \( \Delta_{b} = \Delta_{1} \), see Fig. 7(b). Finally, when \( t_{a} = t_{2}, \ t_{b} = t_{1}, \ \Delta_{a} = \Delta_{2} \) and \( \Delta_{b} = \Delta_{1} \), we have the kink in hopping and superconducting correlations, Fig. 7(c). We simulated these three situations, depicted in the Fig. 7(a), (b) and (c). The results of these simulations are shown in Fig. 8 and Fig. 9.

When the space symmetry is broken, for instance, at the kink, zero-energy bound states can emerge around it, as we can see in Fig. 8 and Fig. 9.

In Fig. 8, we show the probability density \(|\Psi|^{2}\) (z-axis) of the zero-energy bound states as a function of the position of the site and the ratio \( \Delta_{2}/\Delta_{1} \) for three values of parameter \( t_{2}/t_{1} \). In Fig. 8(a) we fixed \( t_{2}/t_{1} = 1 \), in Fig. 8(b) \( t_{2}/t_{1} = 1.5 \) and finally in Fig. 8(c) \( t_{2}/t_{1} = 2 \). Note that, for \( t_{2}/t_{1} = 1 \), case (a), the zero-energy bound states are only localized at the edges of the chain. This result is independent of the value of the ratio \( \Delta_{2}/\Delta_{1} \).
Now, increasing the ratio $t_2/t_1$, for instance $t_2/t_1 = 1.5$ (Fig. 8 (b)), we can see that a zero-energy bound state emerges at the middle of the chain for $1 < \Delta_2/\Delta_1 < 1.7$. In this case, we observe zero-energy bound states at the middle and at the ends of the chain (both edges, left and right). Fig. 8 (c) shows the zero-energy states at the middle and at the edges of the chain for $1 < \Delta_2/\Delta_1 < 2$. The intensity of the zero-energy bound states at the middle of the chain depends on the ratio $t_2/t_1$, as we can see in Fig. 8 (a), (b) and (c).

In Fig. 9 we show the behavior of the zero-energy bound states as a function of the ratio $\Delta_2/\Delta_1$ and the site positions for the case (c) of the Fig. 8. As we can see, the presence of the zero-energy bound state at the middle of the chain seems to prohibit the formation of the zero-energy bound states at the right end of the chain, see Fig. 9 (c). Therefore, there is a destructive competition between the zero-energy bound states at the middle of the chain and the zero-energy bound states around the site 200.

In Fig. 10 (a), we show the energy spectrum as a function of the ratio $\Delta_2/\Delta_1$ for $t_2/t_1 = 1.5$ and $\mu = 0$. In Fig. 10 (b), we show the probability density $|\Psi_i|^2$ of the zero-energy bound states as a function of the site position and the ratio $\Delta_2/\Delta_1$ for $t_2/t_1 = 1.5$ and $\mu = 0$. For $\Delta_2/\Delta_1 < 1$ and/or $\Delta_2/\Delta_1 > 1.6$ there are two zero energy states, see red traced line inside the ellipse in panel (a). In this case, there is only one zero-energy per end of the chain. On the contrary, when $1 < \Delta_2/\Delta_1 < 1.6$, we can see four zero-energy states in the energy spectrum, see red and green traced lines inside the ellipse in panel (a). Here, while the red states are pinned at the edges (each edge possesses one red state), the green zero-energy states are pinned at the middle of the chain.

In Fig. 10 (a), we show the energy spectrum for $\Delta_2/\Delta_1 < 1$ and/or $\Delta_2/\Delta_1 > 1.6$ there are two zero-energy states, see red traced line along the axis $\Delta_2/\Delta_1$ for $E_n = 0$. In this case, there is only one zero-energy state per end of the chain, therefore, one Majorana zero-energy bound state emerges at the ends of the chain. Moreover, note that for $\Delta_2/\Delta_1 < 1$ and/or $\Delta_2/\Delta_1 > 1.6$ there is no zero-energy...
states at the middle of the chain (site 100).

On the contrary, when \( 1 < \Delta_2/\Delta_1 < 1.6 \), we can see zero-energy solutions at the middle and at the edges of the chain, see panel (b). In this case, inside the region \( 1 < \Delta_2/\Delta_1 < 1.6 \), the energy spectrum possesses four zero-energy states highlighted by red and green traces, see the yellow region inside the ellipse in panel (a). The two red zero-energy states inside the yellow ellipse are located at the ends of the chain (one zero-energy state per end), while the two green zero-energy states are located around the middle of the chain. At the ends of the chain, we found Majorana zero-energy (non-local) bound states, that are combinations of a particle and a hole.

At the middle of the chain, on the other hand, there are no Majorana zero-energy bound states. Since there are two zero-energy states that (although being also combinations of a particle and a hole each) combined to form non-conventional fermionic modes \([16]\).

V. CONCLUSION

In this work, we studied the fermionic fractionalization that emerges in the anisotropic superconducting Su-Schieriffer-Heeger (SSH) model. We generalize the phase diagrams obtained in Ref.\([11]\).

The hybrid SSH model exhibits two distinct discrete symmetries for zero chemical potential and therefore, in this case, these two symmetries (chiral and particle-hole symmetry) allow to calculate two distinct topological invariants \(W_1\) and \(W_2\), where the first can be associated to the number of the zero-edge states per end of the chain and the last tell us if a Majorana zero-energy bound state reside or not at the end of the chain. These results have been confirmed by the calculation of the number of the zero-energy edge states at each case of the phase diagrams obtained through these two topological invariants.

We also studied the effects of a finite chemical potential over phase diagrams and the existence of fermionic fractionalization, like Majorana zero-energy bound states. Differently from previous works, the phase diagrams of our hybrid model was correctly reduced to the limit of a pure SSH ground state.

We found that a topological phase transition from a topological non-trivial phase of the hybrid chain to a non-trivial SSH topological phase can be induced for the limit \( \Delta_1 < 0 \), \( \Delta_2 < 0 \), \( \mu < 0 \) and \( t_2/t_1 > 1 \).

In the final part of this work, we simulated the behavior of the zero-energy states around the edges and the domain wall. After creating a domain wall through a kink, we have diagonalized the Hamiltonian in the real space for 200 sites. We obtained the zero energy solutions around the edges and the kink. We observed that the superconducting correlations dictate the existence of these zero energy states around the domain wall, such that, for some specific values of these correlations the zero energy states disappears from the middle and becomes majority localized around the ends of the chain.

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Appendix A: Analytical derivation of the winding numbers

In this section we provide the analytical expressions for the winding numbers. By definition, they are calculated from Eq.\([A3]\) where \(H_k\) is given by Eq.\([2]\) and it is explicitly expressed here in its matricial form,

\[
\mathcal{H}(k) = \begin{pmatrix}
-\mu & z & 0 & w \\
z^* & -\mu & -w^* & 0 \\
0 & -w & \mu & -z \\
w^* & 0 & -z^* & \mu
\end{pmatrix},
\]

with the parameters \(z\) and \(w\) given in terms of the hopping strengths and the superconducting gaps,

\[
z(k) = t_1 + t_2 e^{-ika},
\]

\[
w(k) = -\Delta_1 + \Delta_2 e^{-ika}.
\]

Notice that presently we take \(a \neq 1\) for the sake of clarity.

For the particular case of \(\mu = 0\), the winding number can be expressed as

\[
W = W_1 + W_2
= \sum_{i=1,2} \text{Tr} \int_0^{2\pi a} \frac{dk}{4\pi i} C_i H_k^{-1} \partial_k H_k
= -\sum_{i=1,2} \int_0^{2\pi a} \frac{dk}{2\pi i} \partial_k \log \det z_i.
\]

where

\[
z_1 = A_1 + B_1 e^{-ika},
\]

\[
z_2 = A_2 + B_2 e^{-ika}.
\]

with \(A_1 = t_1 - \Delta_1\), \(B_1 = t_2 + \Delta_2\), \(A_2 = -t_1 - \Delta_1\) and \(B_1 = \Delta_2 - t_2\).

Now, the integrals in Eq.\([A4]\) can be easily calculated making use of

\[
\int \frac{dk}{2\pi} \frac{B_k e^{-ika}}{A_i + B_k e^{-ika}} = \frac{2\pi}{a}, \quad \text{if } |B_i/A_i| > 1.
\]

Plugging this result in Eq.\([A1]\) we get

\[
W = \Theta (|\Delta_2 + t_2| - |t_1 - \Delta_1|)
+ \Theta (|\Delta_2 - t_2| - |t_1 - \Delta_1|),
\]

where \(\Theta(x)\) denotes the Heaviside step function.

Moreover, making use of the mappings \(t_1 = -t(1 + \eta_1)\) and \(t_2 = -t(1 - \eta_1)\), where \(t\) denotes their mean value.
and $\eta_1$ is the absolute difference between $t_1$ and $t_2$ divided by $t$, and also that $\Delta_1 = -\Delta (1 + \eta_2)$ and $\Delta_2 = -\Delta (1 - \eta_2)$, in a similar fashion, one can show that the winding number reduces to

$$\mathcal{W} = \Theta (\Delta - t\eta_1) + \Theta (-\Delta - t\eta_1).$$

(A9)

In this particular case, notice that the results does not depend on the difference between $\Delta_1$ and $\Delta_2$. Moreover, for the even more strict case when $\eta_1 = \eta_2$, this result is identical to the one obtained by R. Wakatsuki et al. Ref [11] and the phase diagram analysis presented in this paper constitutes a generalization of their previous investigation.

For the case $\mu \neq 0$, the topological number is given by Ref [11],

$$\mathcal{W} = -\int_\pi^\pi \frac{dk}{2\pi i} \partial_k \log Z(k),$$

(A10)

where

$$Z(k) = \mu^2 + (z(k) - w(k))(z(k)^* + w(k)^*).$$

(A11)

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