The $L^p$ Dirichlet Problem for the Stokes System on Lipschitz Domains

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Abstract
We study the $L^p$ Dirichlet problem for the Stokes system on Lipschitz domains. For any fixed $p > 2$, we show that a reverse Hölder condition with exponent $p$ is sufficient for the solvability of the Dirichlet problem with boundary data in $L^p_N(\partial \Omega, \mathbb{R}^d)$. Then we obtain a much simpler condition which implies the reverse Hölder condition. Finally, we establish the solvability of the $L^p$ Dirichlet problem for $d \geq 4$ and $2-\varepsilon < p < \frac{2(d-1)}{d-3} + \varepsilon$.

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1 Introduction

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 4$, with connected boundary, and let $N$ be the outward unit normal to $\partial \Omega$. Then, set

$$L^p_N(\partial \Omega, \mathbb{R}^d) = \left\{ \bar{g} \in L^p(\partial \Omega, \mathbb{R}^d) : \int_{\partial \Omega} \bar{g} \cdot N \, d\sigma = 0 \right\}.$$

In this paper we are interested in studying the $L^p$ Dirichlet problem for the Stokes system:

$$\begin{cases}
\Delta \bar{u} = \nabla p & \text{in } \Omega, \\
\text{div } \bar{u} = 0 & \text{in } \Omega, \\
\bar{u} = \bar{f} \in L^p(\partial \Omega, \mathbb{R}^d) & \text{on } \partial \Omega, \\
(\bar{u})^* \in L^p(\partial \Omega),
\end{cases}$$

(1)

where $(\bar{u})^*$ is the non-tangential maximal function of $\bar{u}$ and the boundary values are taken in the sense of non-tangential convergence. Note that using the divergence theorem we have the following necessary condition on the boundary data $\bar{f}$:

$$\int_{\partial \Omega} \bar{f} \cdot N \, d\sigma = \int_{\partial \Omega} \bar{u} \cdot N \, d\sigma = \int_{\Omega} \text{div } \bar{u} \, dx = 0,$$

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i.e. \( \vec{f} \in L^p_N(\partial\Omega, \mathbb{R}^d) \). We say that the \( L^p \) Dirichlet problem (1) on \( \Omega \) is uniquely solvable if given any \( \vec{f} \in L^p_N(\partial\Omega, \mathbb{R}^d) \), there exists a unique function \( \vec{u} \) and a unique function \( p \), up to constants, satisfying (1) where \( \vec{u} = \vec{f} \) in the sense of non-tangential convergence, i.e.

\[
\lim_{x \to Q} \vec{u}(x) = \vec{f}(Q) \quad \text{for a.e. } Q \in \partial\Omega.
\]

Here \( \Gamma(Q) = \{ x \in \Omega : |x - Q| < 2\text{dist}(x, \partial\Omega) \} \). Moreover, the solution \( \vec{u} \) satisfies \( \|(\vec{u})^*\|_{L^p(\partial\Omega, \mathbb{R}^d)} \leq C\|\vec{f}\|_{L^p(\partial\Omega, \mathbb{R}^d)} \), where \( C \) is independent of the boundary data \( \vec{f} \).

Since \( \Omega \) is a bounded Lipschitz domain there exists \( r_0 > 0 \) such that for each point \( P \in \partial\Omega \) there is a new coordinate system of \( \mathbb{R}^d \) obtained from the standard Euclidean coordinate system through translation and rotation so that \( P = (0, 0) \) and

\[
B(P, r_0) \cap \Omega = B(P, r_0) \cap \{(x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi(x')\},
\]

where \( \psi : \mathbb{R}^{d-1} \to \mathbb{R} \) is a Lipschitz function and \( \psi(0) = 0 \). Throughout the paper we let

\[
\Delta(Q, r) = B(Q, r) \cap \partial\Omega,
\]

\[
T(Q, r) = B(Q, r) \cap \Omega,
\]

\[
I_r = \{(x', \psi(x')) \in \mathbb{R}^{d-1} \times \mathbb{R} : |x_1| < r, \ldots, |x_{d-1}| < r\},
\]

\[
Z_r = \{(x', x_d) : |x_1| < r, \ldots, |x_{d-1}| < r, \psi(x') < x_d < C'r\},
\]

where \( Q \in \partial\Omega \), \( 0 < r < r_0 \), and \( C' = 1 + 10\sqrt{d}\|\nabla\psi\|_\infty > 0 \) is chosen so that \( Z_r \) is a star-shaped Lipschitz domain with Lipschitz constant independent of \( r \).

The main results of this paper are as follows:

**Theorem 1.1.** Let \( \Omega \) be a bounded Lipschitz domain with connected boundary in \( \mathbb{R}^d \), \( d \geq 4 \), and \( p > 2 \). If there exists \( C > 0 \) such that for any \( Q \in \partial\Omega \) and \( 0 < r < r_0 \), the reverse Hölder condition,

\[
\left( \frac{1}{|\Delta(Q, r)|} \int_{\Delta(Q, r)} |(\vec{u})^*|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{|\Delta(Q, 2r)|} \int_{\Delta(Q, 2r)} |(\vec{u})^*|^2 \, d\sigma \right)^{1/2},
\]

holds for any solution \( \vec{u} \) of the Stokes system (1) in \( \Omega \) with the properties that \( (\vec{u})^* \in L^2(\partial\Omega) \) and \( \vec{u} = 0 \) on \( \Delta(Q, 3r) \), then the \( L^p \) Dirichlet problem for the Stokes system (1) on \( \Omega \) is uniquely solvable.

It should be noted that nowhere in the proof of Theorem 1.1 is the condition \( d \geq 4 \) used. In the case \( d = 2, 3 \) the reverse Hölder Condition (1) can be established for \( 2 - \varepsilon < p < \infty \), providing another proof of the solvability of the \( L^p \) Dirichlet problem for \( d = 2, 3 \) and \( 2 - \varepsilon < p < \infty \).

Next we establish a simpler condition which implies the reverse Hölder condition given by estimate (2) using the square function estimates as well as the regularity estimate. This condition is given by the following theorem.
Theorem 1.2. Let $\Omega$ be a bounded Lipschitz domain with connected boundary in $\mathbb{R}^d$, $d \geq 4$. Suppose that there exists a constant $C_1 > 0$ and $\lambda \in (0, d]$ such that for $0 < r < R < r_0$ and $Q \in \partial \Omega$,

$$
\int_{T(Q,r)} |\tilde{u}|^2 \, dx \leq C_1 \left( \frac{r}{R} \right)^\lambda \int_{T(Q,R)} |\tilde{u}|^2 \, dx,
$$

whenever $\tilde{u}$ is a solution of the Stokes system (1) in $\Omega$ with the properties that $(\tilde{u})^* \in L^2(\partial \Omega)$ and $\tilde{u} = 0$ on $\Delta(Q,R)$. Then, if $2 < p < 2 + \frac{4}{d - \lambda}$, the $L^p$ Dirichlet problem (1) is uniquely solvable.

Finally, we establish condition (3) given in Theorem 1.2 to prove the following corollary.

Corollary 1.3. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 4$, with connected boundary. Then there exists $\varepsilon > 0$, depending only on the Lipschitz character of $\Omega$ such that, given $\tilde{f} \in L^p_N(\partial \Omega, \mathbb{R}^d)$ with $2 - \varepsilon < p < \frac{2(d-1)}{d-3} + \varepsilon$, the Dirichlet problem for the Stokes system (1) has a unique solution, and the solution satisfies the estimate $\|((\tilde{u})^*)_p \| \leq C \|\tilde{f}\|_p$.
Theorem 1.4. Let \( S = \{(x', \psi(x')) : x' \in \mathbb{R}^{d-1}\} \) be a Lipschitz graph in \( \mathbb{R}^d \). Let \( Q_0 \) be a surface cube in \( S \) and \( F \in L^2(2Q_0) \). Let \( p > 2 \) and \( g \in L^q(2Q_0) \) for some \( 2 < q < p \). Suppose that for each dyadic subcube \( Q \) of \( Q_0 \) with \( |Q| \leq \beta |Q_0| \), there exists two integrable functions \( F_Q \) and \( R_Q \) on \( 2Q \) such that

\[
\left( \frac{1}{|2Q|} \int_{2Q} |F_Q|^p \, d\sigma \right)^{1/p} \leq C_1 \left\{ \left( \frac{1}{|Q|} \int_{Q} |F|^2 \, d\sigma \right)^{1/2} + \sup_{Q' \supset Q} \left( \frac{1}{|Q'|} \int_{Q'} |g|^2 \, d\sigma \right)^{1/2} \right\},
\]

\[
\left( \frac{1}{|2Q|} \int_{2Q} |F|^{2} \, d\sigma \right)^{1/2} \leq C_2 \sup_{Q' \supset Q} \left( \frac{1}{|Q'|} \int_{Q'} |g|^2 \, d\sigma \right)^{1/2},
\]

where \( C_1, C_2 > 0 \) and \( 0 < \beta < 1 < \alpha \). Then,

\[
\left( \frac{1}{|Q_0|} \int_{Q_0} |F|^q \, d\sigma \right)^{1/q} \leq C_3 \left\{ \left( \frac{1}{|2Q_0|} \int_{2Q_0} |F|^2 \, d\sigma \right)^{1/2} + \left( \frac{1}{|2Q_0|} \int_{2Q_0} |g|^q \, d\sigma \right)^{1/q} \right\},
\]

where \( C_3 \) depends only on \( d, p, q, C_1, C_2, \alpha, \beta, \) and \( \|\nabla \psi\|_{\infty} \).

We will then show that the condition given by estimate (3) implies the reverse Hölder condition. Finally, condition (3) will be established for some \( \lambda > 3 \) to show that the Dirichlet problem is uniquely solvable for \( p \) in the range \( 2 - \varepsilon < p < \frac{2(d-1)}{d-3} + \varepsilon \). The proof will closely follow the argument used by Shen in [18].

The paper is organized as follows: Theorem 1.1 will be proved in section 2 and Theorem 1.2 will be proved in section 3. Finally, Corollary 1.3 will be proved in section 4.

In preparation of this paper, the author learned of the work of Matt Wright and Marius Mitrea [24] on the transmission problem for the Stokes system. As a corollary of their work they obtain a proof of Corollary 1.3. The proof provided in this paper is a more direct approach to the problem. Finally, the author would like to acknowledge Zhongwei Shen for many very helpful conversations.

2 A Sufficient Condition

The goal of this section is to prove Theorem 1.1 which establishes a sufficient condition for the solvability of the \( L^p \) Dirichlet problem when \( p > 2 \) and \( d \geq 4 \).

**Proof.** (of Theorem 1.1)

First, note that the uniqueness for \( p > 2 \) follows from the uniqueness for \( p = 2 \). Let \( \tilde{f} \in L^p_{\nu}(\partial \Omega, \mathbb{R}^d) \) and let \( \tilde{u} \) be the solution of the \( L^2 \) Dirichlet problem with data \( \tilde{f} \). We’ll show that if \( p > 2 \), then
\[
\left( \frac{1}{s^{d-1}} \int_{B(P,s) \cap \partial \Omega} |(\bar{u})^*|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{s^{d-1}} \int_{B(P,cs) \cap \partial \Omega} |(\bar{u})^*|^2 \, d\sigma \right)^{1/2} + \frac{C}{s^{d-1}} \int_{B(P,cs) \cap \partial \Omega} |\bar{f}|^p \, d\sigma \right)^{1/p},
\]

for any \( P \in \partial \Omega \) and \( 0 < s \leq cr_0 \). Then, by covering \( \partial \Omega \) with a finite number of balls of radius \( cr_0 \), estimate (5) implies that

\[
\|(\bar{u})^*\|_p \leq C_p \left( |\partial \Omega|^{\frac{1}{p} - \frac{1}{2}} \|(\bar{u})^*\|_2 + \|\bar{f}\|_p \right)
\leq C_p \left( |\partial \Omega|^{\frac{1}{p} - \frac{1}{2}} \|\bar{f}\|_2 + \|\bar{f}\|_p \right)
\leq C_p \|\bar{f}\|_p.
\]

Here we used the fact that \( \bar{u} \) is the solution of the \( L^2 \) Dirichlet problem with data \( \bar{f} \) and Hölder’s inequality. It remains to establish estimate (5); its proof relies on Theorem 1.4.

Fix \( Q \in \partial \Omega \) and \( 0 < r < r_0 \). By rotation and translation we may assume that \( Q = 0 \) and

\[
B(0,cr_0) \cap \Omega = B(0, cr_0) \cap \{(x', x_d) \in \mathbb{R}^d : x_d > \psi(x')\},
\]

\[
B(0, cr_0) \cap \partial \Omega = B(0, cr_0) \cap \{(x', \psi(x')) : x' \in \mathbb{R}^{d-1}\},
\]

where \( \psi \) is a Lipschitz function on \( \mathbb{R}^{d-1} \).

Consider the surface cube \( I_r \). Write \( \bar{u} = \bar{u}_1 + \bar{u}_2 \) where \( \bar{u}_1 \) is a solution to the \( L^2 \) Dirichlet problem with boundary data

\[
\bar{f}_1 = \begin{cases} 
\bar{f} & \text{on } I_{8r} \\
\beta \bar{\alpha} & \text{on } \partial \Omega \setminus I_{8r},
\end{cases}
\]

where \( \bar{\alpha} \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) \) is chosen so that \( |\bar{\alpha}| \leq C_0 \) and \( \bar{\alpha} \cdot N \geq C_1 > 0 \) (such a vector field has been shown to exist; for example, see the work of Verchota in [22]). Here \( \beta \) is a constant depending on \( \bar{f} \) chosen so that \( \bar{f}_1 \in L^2_N(\partial \Omega, \mathbb{R}^d) \), i.e.

\[
\beta = -\frac{\int_{I_{8r}} \bar{f} \cdot N \, d\sigma}{\int_{\partial \Omega \setminus I_{8r}} \bar{\alpha} \cdot N \, d\sigma}.
\]

Next, we verify the conditions of the real variable argument in Theorem 1.4. Let

\[
F = |(\bar{u})^*|,
\]

\[
g = |\bar{f}|,
\]

\[
F_Q = 2|(\bar{u}_1)^*|,
\]

\[
R_Q = 2|(\bar{u}_2)^*|.
\]
Now, using the $L^2$ estimates for the Dirichlet problem we obtain

\[
\frac{1}{|I_{2r}|} \int_{I_{2r}} |F_q|^2 \, d\sigma \quad \leq \quad \frac{C}{|I_{2r}|} \int_{\partial \Omega} |(\bar{u}_1)^*|^2 \, d\sigma
\]

\[
\leq \quad \frac{C}{|I_{2r}|} \int_{\partial \Omega} |\bar{f}|^2 \, d\sigma
\]

\[
\leq \quad \frac{C}{|I_{sr}|} \int_{I_{sr}} |\bar{f}|^2 \, d\sigma + \frac{C|\beta|^2}{|I_{sr}|} \int_{\partial \Omega \setminus I_{sr}} |\bar{\alpha}|^2 \, d\sigma
\]

\[
\leq \quad \frac{C}{|I_{sr}|} \int_{I_{sr}} |\bar{f}|^2 \, d\sigma + C \left( \frac{|\partial \Omega \setminus I_{sr}|}{|I_{sr}|} \right) |\beta|^2
\]

\[
\leq \quad \frac{C}{|I_{sr}|} \int_{I_{sr}} |\bar{f}|^2 \, d\sigma + C \left( \frac{|\partial \Omega \setminus I_{sr}|}{|I_{sr}|} \right) \left( \int_{\partial \Omega \setminus I_{sr}} \bar{\alpha} \cdot N \, d\sigma \right)^2
\]

\[
\leq \quad \frac{C}{|I_{sr}|} \int_{I_{sr}} |\bar{f}|^2 \, d\sigma + \frac{C}{|\partial \Omega \setminus I_{sr}|} \int_{I_{sr}} |\bar{f}|^2 \, d\sigma
\]

\[
\leq \quad \frac{C}{|I_{sr}|} \int_{I_{sr}} |\bar{f}|^2 \, d\sigma \leq C \sup_{Q' \supset I_{r}} \left( \frac{1}{|Q'|} \int_{Q'} |g|^2 \, d\sigma \right)^{1/2}.
\]

Note that $\bar{u}_2$ is a solution with $(\bar{u}_2)^* \in L^2(\partial \Omega)$ and $\bar{u}_2 = 0$ on $I_{sr}$. Then, using the reverse Hölder Inequality (2) and the same estimates on $\bar{u}_1$ as above, we obtain

\[
\left( \frac{1}{|I_{sr}|} \int_{I_{sr}} |R_q|^p \, d\sigma \right)^{1/p} \leq \left( \frac{C}{|I_{sr}|} \int_{I_{sr}} |(\bar{u}_2)^*|^p \, d\sigma \right)^{1/p}
\]

\[
\leq \quad C \left( \frac{1}{|I_{sr}|} \int_{I_{sr}} |(\bar{u}_2)^*|^2 \, d\sigma \right)^{1/2}
\]

\[
\leq \quad C \left( \frac{1}{|I_{sr}|} \int_{I_{sr}} |(\bar{u})|^2 \, d\sigma \right)^{1/2} + C \left( \frac{1}{|I_{sr}|} \int_{I_{sr}} |(\bar{u}_1)^*|^2 \, d\sigma \right)^{1/2}
\]

\[
\leq \quad C \left( \frac{1}{|I_{sr}|} \int_{I_{sr}} |F|^2 \, d\sigma \right)^{1/2} + C \sup_{Q' \supset I_{r}} \left( \frac{1}{|Q'|} \int_{Q'} |g|^2 \, d\sigma \right)^{1/2}.
\]

Thus, both conditions of Theorem 1.4 are satisfied and estimate (5) is proven. Therefore, the solvability of the $L^p$ Dirichlet problem has been established. 

**Remark.** In establishing estimate (7), the reverse Hölder condition was applied to a surface cube but was stated in Theorem 1.1 for surface balls. The reverse Hölder inequality for surface cubes can be obtained from that for surface balls by subdividing the surface cube into a finite number of smaller cubes that are contained in slightly larger surface balls. This estimate can be made in such a way that the constant still depends only on the Lipschitz character of $\Omega$. 

3 A Simpler and Stronger Sufficient Condition

We begin this section by recalling a few known results. The following Caccioppoli’s inequality is contained in Theorem 2.2 of [13].

Lemma 3.1 (Caccioppoli’s Inequality). Let \( x_0 \in \Omega \) and \( r > 0 \) be small. Assume that \((\bar{u}, p)\) is a solution of the Stokes system (1) in \( T(x_0, 3r) \) such that \( \bar{u} = 0 \) on \( \Delta(x_0, 3r) \). Then,
\[
\int_{T(x_0, r)} |\nabla \bar{u}|^2 dx \leq \frac{C}{r^2} \int_{T(x_0, 2r)} |\bar{u}|^2 dx. \tag{8}
\]

Then, Lemma 3.1 and standard arguments can be used to prove the following lemma.

Lemma 3.2 (Higher Integrability). Under the same assumptions as in Lemma 3.1 we have
\[
\left( \frac{1}{r^d} \int_{T(x_0, r)} |\nabla \bar{u}|^q dx \right)^{1/q} \leq C \left( \frac{1}{r^d} \int_{T(x_0, 2r)} |\nabla \bar{u}|^2 dx \right)^{1/2}, \tag{9}
\]
where \( q > 2 \) depends only on the Lipschitz character of \( \Omega \).

Finally, we recall the interior estimates for the Stokes system.

Lemma 3.3 (Interior Estimates). Let \( \bar{u} \) be a solution of the Stokes System (1) in \( \Omega \). Suppose that \( B(x, r) \subset \Omega \). Then,
\[
|D^\alpha \bar{u}(x)| \leq \frac{C_\alpha}{r^{d+|\alpha|}} \int_{B(x, r)} |\bar{u}(y)| dy,
\]
for any multi-index \( \alpha \), where \( C_\alpha \) depends only on \( |\alpha| \) and \( d \).

We will also need to use the square function estimates. Recall that the square function \( S(w) \) is defined by
\[
S(w)(Q) = \left( \int_{\Gamma(Q)} \frac{|\nabla w(x)|^2}{|x - Q|^{d-2}} dx \right)^{1/2}, \tag{10}
\]
for \( Q \in \partial \Omega \). We also define
\[
\tilde{S}(w)(Q) = \left( \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^2}{|x - Q|^{d-4}} dx \right)^{1/2}, \tag{11}
\]
for \( Q \in \partial \Omega \). The following square function estimates for solutions of the Stokes system (1) established in [1, 7] will be needed:

\[
\|S(\bar{u})\|_{L^p(\partial \Omega)} \leq C\|\bar{u}\|_{L^p(\partial \Omega)}, \tag{12}
\]
\[
\|(\bar{u})^*\|_{L^p(\partial \Omega)} \leq C \|\bar{u}\|_{L^p(\partial \Omega)} + C|\bar{u}(P_0)|\|\partial \Omega\|^{1/p}, \tag{13}
\]
where \( 0 < p < \infty, P_0 \in \Omega, \) and \( C \) depends on the Lipschitz character of \( \Omega \). Then, using Lemma 2 on page 216 of [21] along with the square function estimate (13) we get that
\[
\|(\bar{u})^*\|_{L^p(\partial \Omega)} \leq C \||\bar{u}\|_{L^p(\partial \Omega)} + C|\partial \Omega|^{\frac{1}{p}} \|(\bar{u})^*\|_{L^2(\partial \Omega)}. \tag{14}
\]

The following lemma found in [18] is stated and proved here for the sake of completeness.
**Lemma 3.4.** Let $p > 2$. Then for any $\gamma \in (0, 1)$ and $w \in C^2(\Omega)$ we have

$$
\int_{\partial \Omega} |\tilde{S}(w)|^p \, d\sigma \leq C \{diam(\Omega)\}^\gamma \int_{\Omega} |\nabla^2 w(x)|^p |\delta(x)|^{2p-1-\gamma} \, dx,
$$

where $\delta(x) = \text{dist}(x, \partial \Omega)$.

**Proof.** We begin by re-writing $\tilde{S}(w)$ in the following manner

$$
\tilde{S}(w)(Q) = \left( \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^2}{|x - Q|^{(d+\gamma-2)p/2}} \cdot \frac{dx}{|x - Q|^{(d-2p)/2}} \right)^{1/2}.
$$

Then, using Hölder’s inequality we obtain

$$
\begin{align*}
\tilde{S}(w)(Q) & \leq C \left( \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^p}{|x - Q|^{d+\gamma-2p}} \, dx \right)^{1/p} \left( \int_{\Gamma(Q)} \frac{dx}{|x - Q|^{d-2p/2}} \right)^{p-2/2p} \\
& \leq C \left( \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^p}{|x - Q|^{d+\gamma-2p}} \, dx \right)^{1/p} \left( \int_0^{\text{diam}(\Omega)} t^{2p/2-2} \, dt \right)^{p-2/2p} \\
& \leq C \{\text{diam}(\Omega)\}^{\gamma/p} \left( \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^p}{|x - Q|^{d+\gamma-2p}} \, dx \right)^{1/p}.
\end{align*}
$$

Finally, integrating $|\tilde{S}(w)(Q)|^p$ over $\partial \Omega$ we obtain

$$
\begin{align*}
\int_{\partial \Omega} |\tilde{S}(w)(Q)|^p \, d\sigma & \leq C \{\text{diam}(\Omega)\}^{\gamma} \int_{\partial \Omega} \left( \int_{\Gamma(Q)} \frac{|\nabla^2 w(x)|^p}{|x - Q|^{d+\gamma-2p}} \, dx \right) \, d\sigma \\
& \leq C \{\text{diam}(\Omega)\}^{\gamma} \int_{\Omega} |\nabla^2 w(x)|^p |\delta(x)|^{2p-1-\gamma} \, dx.
\end{align*}
$$

\[\square\]

**Lemma 3.5.** Let $p > 2$. Suppose that $\Delta \tilde{u} = \nabla p$ and $\text{div}(\tilde{u}) = 0$ in $\Omega$. Then, for any $\gamma \in (0, 1)$

$$
\int_{\partial \Omega} |(\tilde{u})^*|^p \, d\sigma \leq C |\partial \Omega|^{1-\gamma/2} \left( \int_{\partial \Omega} |(\tilde{u})^*|^2 \, d\sigma \right)^{p/2} \\
+ C \{\text{diam}(\Omega)\}^{\gamma} \sup_{x \in \Omega} |\nabla^2 \tilde{u}|^{2p-1-\gamma} \int_{\partial \Omega} |(\nabla \tilde{u})^*|^2 \, d\sigma.
$$

**Proof.** Using Lemma 3.4 as well as the square function estimate (12) we obtain
\[
\int_{\partial \Omega} |\tilde{S}(\tilde{u})|^p \, d\sigma \leq C_\gamma \{\text{diam}(\Omega)\}^\gamma \int_\Omega |\nabla^2 \tilde{u}|^p |\delta(x)|^{2p - 1 - \gamma} \, dx \\
\leq C_\gamma \{\text{diam}(\Omega)\}^\gamma \sup_{x \in \Omega} |\nabla^2 \tilde{u}|^p |\delta(x)|^{2p - 2 - \gamma} \int_\Omega |\nabla^2 \tilde{u}|^2 \delta(x) \, dx \\
\leq C_\gamma \{\text{diam}(\Omega)\}^\gamma \sup_{x \in \Omega} |\nabla^2 \tilde{u}|^p |\delta(x)|^{2p - 2 - \gamma} \int_{\partial \Omega} |(\nabla \tilde{u})|^2 \, d\sigma.
\]

Combining this with estimate (14) we obtain
\[
\int_{\partial \Omega} |(\tilde{u})^*|^p \, d\sigma \leq C \int_{\partial \Omega} |\tilde{S}(\tilde{u})|^p \, d\sigma + C |\partial \Omega|^{1 - \frac{\gamma}{2}} \left( \int_{\partial \Omega} |(\tilde{u})^*|^2 \, d\sigma \right)^{p/2} \\
\leq C |\partial \Omega|^{1 - \frac{\gamma}{2}} \left( \int_{\partial \Omega} |(\tilde{u})^*|^2 \, d\sigma \right)^{p/2} \\
+ C_\gamma \{\text{diam}(\Omega)\}^\gamma \sup_{x \in \Omega} |\nabla^2 \tilde{u}|^{p - 2} |\delta(x)|^{2p - 2 - \gamma} \int_{\partial \Omega} |(\nabla \tilde{u})|^2 \, d\sigma.
\]

Proof. (of Theorem 1.2)

Fix \( \Delta(Q_0, r) \) with \( Q_0 \in \partial \Omega \) and \( 0 < r < r_0 \). Let \( \tilde{u} \) be a solution of the Stokes system (11) in \( \Omega \) with the properties \((\tilde{u})^* \in L^2(\partial \Omega)\) and \( \tilde{u} = 0 \) on \( \Delta(Q_0, 3r) \). Now, using the assumption given by estimate (3), Caccioppoli’s inequality, and the interior estimates, for any \( x \in T(Q_0, r) \) we obtain
\[
[\delta(x)]^2 |\nabla^2 \tilde{u}(x)| \leq C[\delta(x)]^2 \left( \frac{1}{[\delta(x)]^d} \int_{B(x, c\delta(x))} |\nabla^2 \tilde{u}(y)|^2 \, dy \right)^{1/2} \\
\leq \frac{C}{[\delta(x)]^{d/2}} \left( \int_{T(Q_0, c\delta(x))} |\tilde{u}(y)|^2 \, dy \right)^{1/2} \\
\leq \frac{C}{[\delta(x)]^{d/2}} \left\{ \left( \frac{\delta(x)}{r} \right)^\lambda \int_{T(Q_0, 2r)} |\tilde{u}(y)|^2 \, dy \right\}^{1/2} \\
= C \left( \frac{\delta(x)}{r} \right)^\frac{\lambda d}{2} \left( \frac{1}{r^d} \int_{T(Q_0, 2r)} |\tilde{u}(y)|^2 \, dy \right)^{1/2}.
\]

Thus, for \( x \in T(Q_0, r) \) we have
\[
|\nabla^2 \tilde{u}(x)| \leq \frac{C}{[\delta(x)]^2} \left( \frac{\delta(x)}{r} \right)^\frac{\lambda d}{2} \left( \frac{1}{r^d} \int_{T(Q_0, 2r)} |\tilde{u}(y)|^2 \, dy \right)^{1/2}.
\]  

By rotation and translation we may assume that \( Q_0 = 0 \) and \( r_0 = r_0(\Omega, d) > 0 \) is small enough so that
\[ B(0, C_0 r_0) \cap \Omega = B(0, C_0 r_0) \cap \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : x_d > \psi(x')\}, \]
\[ B(0, C_0 r_0) \cap \partial \Omega = B(0, C_0 r_0) \cap \{(x', \psi(x')) : x' \in \mathbb{R}^{d-1}\}, \]

where \( \psi \) is a Lipschitz function on \( \mathbb{R}^{d-1} \) and \( \psi(0) = 0 \). For \( \rho \in (1, 4) \) we also define

\[
I_{pr} = \{(x', \psi(x')) : |x'| < \rho C_2 r\},
\]
\[
Z_{pr} = \{(x', x_d) : |x'| < \rho C_2 r, \psi(x') < x_d < \psi(x') + \rho C_2 r\},
\]

where \( C_2 = C_2(d, \|\nabla \psi\|_{\infty}) > 0 \) is small enough that \( I_{3r} \subset \Delta(0, r) \) and \( Z_{3r} \subset B(0, r) \cap \Omega \). Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be defined as follows:

\[
\mathcal{M}_1(\bar{u})(Q) = \sup\{||\bar{u}(x)| : x \in \Gamma(Q), |x - Q| < C_0 r\},
\]
\[
\mathcal{M}_2(\bar{u})(Q) = \sup\{||\bar{u}(x)| : x \in \Gamma(Q), |x - Q| \geq C_0 r\}.
\]

We begin by estimating \( \mathcal{M}_2(\bar{u}) \). Using the interior estimates we obtain

\[
|\bar{u}(x)| \leq \frac{C}{|B(x, cr)|} \int_{B(x, cr)} |\bar{u}| \, dy \\
\leq C \left( \frac{1}{r^{d-1}} \int_{\Delta(0,2r)} |(\bar{u})^*|^2 \, d\sigma \right)^{1/2},
\]

for \( x \in \Gamma(Q) \) such that \( |x - Q| \geq cr \). This implies that

\[
\left( \frac{1}{r^{d-1}} \int_{I_{pr}} |\mathcal{M}_2(\bar{u})|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{d-1}} \int_{\Delta(0,2r)} |(\bar{u})^*|^2 \, d\sigma \right)^{1/2}.
\]

(17)

Next we estimate \( \mathcal{M}_1(\bar{u}) \) on \( I_r \) which is much more involved. Applying Lemma 3.5 to \( \bar{u} \) on the Lipschitz domain \( Z_{pr} \) for \( \rho \in (\frac{3}{2}, 2) \) we obtain

\[
\frac{1}{r^{d-1}} \int_{I_{r}} |\mathcal{M}_1(\bar{u})|^p \, d\sigma \leq \frac{1}{r^{d-1}} \int_{\partial Z_{pr}} |(\bar{u})^*|^p \, d\sigma \\
\leq C \left( \frac{1}{r^{d-1}} \int_{\partial Z_{pr}} |(\bar{u})^*|^2 \, d\sigma \right)^{p/2} \\
+ C \gamma_{r^{\gamma}} \sup_{x \in Z_{pr}} |\nabla^{2}\bar{u}(x)|^{p-2} [\delta_{\rho}(x)]^{2p-2-\gamma} \frac{1}{r^{d-1}} \int_{\partial Z_{pr}} |(\nabla^{2}\bar{u})_{\rho}|^{2} \, d\sigma,
\]

(18)

where \( \delta_{\rho}(x) = \text{dist}(x, \partial Z_{pr}) \) and \( (\nabla^{2}\bar{u})_{\rho}^* \) is the non-tangential maximal function of \( \nabla \bar{u} \) with respect to the domain \( Z_{pr} \). Now, using the \( L^2 \) regularity estimate established by Fabes, Kenig, and Verchota in [11] and the fact that \( \bar{u} = 0 \) on \( \Delta(0,3r) \) we obtain
\[ \int_{\partial Z_{\rho r}} |(\nabla \tilde{u})^*|^2 d\sigma \leq C \int_{\partial Z_{\rho r}} |\nabla_{t\tilde{u}}|^2 d\sigma \leq C \int_{\Omega \cap \partial Z_{\rho r}} |\nabla \tilde{u}|^2 d\sigma \leq C \int_{\Omega \cap \partial Z_{\rho r}} |\nabla \tilde{u}|^2 d\sigma. \tag{19} \]

Note that when \( x \in Z_{\rho r} \) we have \( \delta_{\rho}(x) \leq \delta(x) \leq Cr \). Also note that the condition \( p < 2 + \frac{4}{d} \) implies that \( \frac{\lambda - d}{2}(p - 2) + 2 > 0 \).

Thus, we may choose \( \gamma > 0 \) so small that \( \frac{\lambda - d}{2}(p - 2) + 2 - \gamma > 0 \).

Thus, using estimate \( (16) \) we have that

\[ r^{\gamma} \sup_{x \in Z_{\rho r}} |\nabla^2 \tilde{u}(x)|^{p-2} [\delta_{\rho}(x)]^{2p-2-\gamma} \]

\[ \leq C r^{\gamma} \sup_{x \in Z_{\rho r}} [\delta_{\rho}(x)]^{2p-2-\gamma} \left( \frac{\delta_{\rho}(x)}{r} \right)^{\frac{\lambda - d}{2}(p - 2)} \left( \frac{1}{r} \right)^{\frac{1}{r^{d-1}} \int_{\Delta(0,2r)} |(\tilde{u})^*|^2 d\sigma}^{\frac{p-2}{2}} \]

\[ \leq C r^{\gamma} \sup_{x \in Z_{\rho r}} [\delta_{\rho}(x)]^{2p-2-\gamma} \left( \frac{1}{r} \right)^{\frac{1}{r^{d-1}} \int_{\Delta(0,2r)} |(\tilde{u})^*|^2 d\sigma}^{\frac{p-2}{2}} \]

\[ \leq C r^{2} \left( \frac{1}{r^{d-1}} \int_{\Delta(0,2r)} |(\tilde{u})^*|^2 d\sigma \right)^{\frac{p-2}{2}}. \tag{20} \]

Combining estimates \( (18), (19), \) and \( (20) \) we obtain

\[ \left( \frac{1}{r^{d-1}} \int_{\Delta(0,2r)} |\mathcal{M}_1(\tilde{u})|^p d\sigma \right)^{2/p} \]

\[ \leq C \left( \frac{1}{r^{d-1}} \int_{\Omega \cap \partial Z_{\rho r}} |\tilde{u}|^2 d\sigma \right) + C \left( \frac{1}{r^{d-1}} \int_{\Delta(0,2r)} |(\tilde{u})^*|^2 d\sigma \right)^{\frac{p-2}{p}} \left( \frac{1}{r^{d-3}} \int_{\Omega \cap \partial Z_{\rho r}} |\nabla \tilde{u}|^2 d\sigma \right)^{2/p}. \]

Now, using Young’s inequality we get that
\[
\left( \frac{1}{r^{d-1}} \int_{I_r} |\mathcal{M}_1(\bar{u})|^p \, d\sigma \right)^{2/p} \leq C \int_{\Omega \cap \partial Z_{pr}} |\bar{u}|^2 \, d\sigma + \frac{C}{r} \int_{\Delta(0,2r)} |(\bar{u})^*|^2 \, d\sigma \\
+ \frac{C}{r^{d-3}} \int_{\Omega \cap \partial Z_{pr}} |\nabla \bar{u}|^2 \, d\sigma.
\]

Integrating the above inequality in \( \rho \in \left( \frac{3}{2}, 2 \right) \) and using the Caccioppoli inequality we obtain
\[
\left( \frac{1}{r^{d-1}} \int_{I_r} |\mathcal{M}_1(\bar{u})|^p \, d\sigma \right)^{2/p} \leq C \int_{\Delta(0,2r)} |(\bar{u})^*|^2 \, d\sigma + \frac{C}{r} \int_{Z_{2r}} |\bar{u}|^2 \, dx \\
+ \frac{C}{r^{d-2}} \int_{Z_{4r}} |\nabla \bar{u}|^2 \, dx
\]
\[
\leq \frac{C}{r^{d-1}} \int_{\Delta(0,2r)} |(\bar{u})^*|^2 \, d\sigma + \frac{C}{r} \int_{Z_{4r}} |\bar{u}|^2 \, dx
\]
\[
\leq \frac{C}{r^{d-1}} \int_{\Delta(0,4r)} |(\bar{u})^*|^2 \, d\sigma.
\]

Thus, a simple covering argument gives that
\[
\left( \frac{1}{r^{d-1}} \int_{\Delta(0,r)} |(\bar{u})^*|^p \, d\sigma \right)^{1/p} \leq C \left( \frac{1}{r^{d-1}} \int_{\Delta(0,4r)} |(\bar{u})^*|^2 \, d\sigma \right)^{1/2}.
\]

Thus, by Theorem 1.1 this implies the solvability of the \( L^p \) Dirichlet problem on \( \Omega \) for
\[
2 < p < 2 + \frac{4}{d - \lambda}.
\]

\[\square\]

4 Solvability of the \( L^p \) Dirichlet Problem

We conclude with the proof of Corollary 1.3. To prove the corollary we show that condition (i) is satisfied for some \( \lambda > 3 \).

**Proof.** (of Corollary 1.3) Let \( \bar{u} \) be a solution of the Stokes system (i) with the properties \((\bar{u})^* \in L^2(\partial \Omega) \) and \( \bar{u} = 0 \) on \( \Delta(Q_0, R) \). By rotation and translation we may assume that \( Q_0 = 0 \) and use the notation in Theorem 1.2. Let \( 0 < r < R/8 \). Note that
\[
|\bar{u}(x)| = |\bar{u}(x) - \bar{u}(Q)| \\
\leq \int_{c\delta(x)} |\nabla \bar{u}| \, dx_d \\
\leq C\delta(x)|\nabla \bar{u}|_\rho \\
\leq Cr|\nabla \bar{u}|_\rho^*(Q),
\]

\[\]
where \( x = (Q, x_d) \in Z_r \) and \((\nabla \vec{u})^*_\rho\) is the non-tangential maximal function of \( \nabla \vec{u} \) with respect to the Lipschitz domain \( Z_{\rho R} \) for \( \rho \in (\frac{1}{8}, \frac{1}{4}) \). Thus,

\[
\begin{align*}
\int_{Z_r} |\vec{u}|^2 \, dx &\leq C \int_{I_r} \int_0^{cr} r^2 |(\nabla \vec{u})^*_\rho|^2 \, dx \, d\sigma \\
&\leq C r^3 \int_{I_r} |(\nabla \vec{u})^*_\rho|^2 \, d\sigma \\
&\leq C r^{3+(d-1)(1-\frac{2}{q})} \left( \int_{I_{\rho R}} |(\nabla \vec{u})^*_\rho|^q \, d\sigma \right)^{2/q},
\end{align*}
\]

where \( \rho \in (\frac{1}{8}, \frac{1}{4}) \) and \( q > 2 \). Now, we choose \( q > 2 \) so that the regularity estimate holds uniformly on the Lipschitz domain \( Z_{\rho R} \) for \( \rho \in (\frac{1}{8}, \frac{1}{4}) \). Then,

\[
\begin{align*}
\left( \int_{Z_r} |\vec{u}|^2 \, dx \right)^{q/2} &\leq C r^{\frac{3q}{2}+(d-1)(\frac{q}{2}-1)} \int_{\partial Z_{\rho R}} |(\nabla \vec{u})^*_\rho|^q \, d\sigma \\
&\leq C r^{\frac{3q}{2}+(d-1)(\frac{q}{2}-1)} \int_{\partial Z_{\rho R}} |\nabla \vec{u}|^q \, d\sigma \\
&\leq C r^{\frac{3q}{2}+(d-1)(\frac{q}{2}-1)} \int_{\Omega \cap \partial Z_{\rho R}} |\nabla \vec{u}|^q \, d\sigma.
\end{align*}
\]

Next, we integrate both sides of the above inequality in \( \rho \in (\frac{1}{8}, \frac{1}{4}) \) to obtain

\[
\begin{align*}
\left( \int_{Z_r} |\vec{u}|^2 \, dx \right)^{q/2} &\leq C r^{\frac{3q}{2}+(d-1)(\frac{q}{2}-1)} \frac{1}{R} \int_{Z_{\frac{R}{4}}} |\nabla \vec{u}|^q \, dx. \tag{21}
\end{align*}
\]

Then, using Lemma 3.2 on the higher integrability and Caccioppoli’s inequality we obtain

\[
\begin{align*}
\int_{Z_r} |\vec{u}|^2 \, dx &\leq C r^{3+(d-1)(1-\frac{2}{q})} \frac{R^{2d/q}}{R^{2/q}} \left( \frac{1}{R^{d}} \int_{Z_{\frac{R}{4}}} |\nabla \vec{u}|^q \, dx \right)^{2/q} \\
&\leq C r^{3+(d-1)(1-\frac{2}{q})} R^{(d-1)(\frac{2}{q})} \frac{1}{R^{d+2}} \int_{Z_R} |\vec{u}|^2 \, dx \\
&\leq C \left( \frac{r}{R} \right)^{3+(d-1)(1-\frac{2}{q})} \int_{Z_R} |\vec{u}|^2 \, dx.
\end{align*}
\]

Thus, condition (3) holds for \( \lambda = 3 + (d-1)(1-\frac{2}{q}) > 3 \). Note that

\[
2 + \frac{4}{d-\lambda} = 2 + \frac{4}{d-3-(d-1)(1-\frac{2}{q})} \\
\geq 2 + \frac{4}{d-3} = \frac{2(d-1)}{d-3}.
\]

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Thus, the $L^p$ Dirichlet problem is solvable for

$$2 - \varepsilon < p < \frac{2(d-1)}{d-3} + \varepsilon.$$

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