RANDOMIZATION-BASED CAUSAL INFERENCE FROM UNBALANCED $2^2$ SPLIT-Plot DESIGNS

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Given two 2-level factors of interest, a $2^2$ split-plot design (a) takes each of the $2^2 = 4$ possible factorial combinations as a treatment, (b) identifies one factor as ‘whole-plot,’ (c) divides the experimental units into blocks, and (d) assigns the treatments in such a way that all units within the same block receive the same level of the whole-factor. Assuming the potential outcomes framework, we propose in this paper a randomization-based estimation procedure for causal inference from $2^2$ designs that are not necessarily balanced. Sampling variances of the point estimates are derived in closed form as linear combinations of the between- and within-block covariances of the potential outcomes. Results are compared to those under complete randomization as measures of design efficiency. Interval estimates are constructed based on conservative estimates of the sampling variances, and the frequency coverage properties evaluated via simulation. Asymptotic connections of the proposed approach to the model-based super-population inference are also established. Superiority over existing model-based alternatives is reported under a variety of settings for both binary and continuous outcomes.

1. Introduction.

1.1. Split-plot designs for factorial experiments. Factorial experiments, originally developed in the context of agricultural experiments (Fisher, 1925, 1935; Yates, 1937) and later extensively used in industrial and engineering applications, are nowadays undergoing a third popularity surge among social, behavioral, and biomedical sciences, as a result of the massive trend in these areas to generalize the previous treatment-control experiments to include multiple factors. Among the plethora of possible multi-factor randomization schemes available, split-plot design, thanks to its flexibility and ease of application, has always remained a popular choice, especially when

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practical difficulties like economic constraints or hard-to-change factor preclude the use of simple, unrestricted randomizations (Jones and Nachtsheim, 2009). As a motivating example, consider a simplified version of the education experiment described in Dasgupta, Pillai and Rubin (2015). The goal is to evaluate the efficacies of two interventions — **A**: a mid-year quality review by a team of experts, and **B**: a bonus scheme to teachers — on 224 schools in the state of New York. Assume two possible actions for each intervention — application or non-application, a complete randomization of the four combinations likely scatters the schools to be reviewed throughout the state. Given the travel and time cost this may incur, a more practical alternative would be to divide the 224 schools by geographic proximity into sixteen ‘blocks,’ choose eight at random, and conduct expert quality review for all schools therein. The teacher bonus scheme can then be applied to half of the schools within each block. This exemplifies split-plot design. See Box, Hunter and Hunter (2005), Cochran and Cox (1957), and Wu and Hamada (2009) for formal definitions.

1.2. Randomization-based approach to analyzing split-plot designs. Most factorial experiments, like any experiment, receive regression-based methods as their default ‘treatment.’ For those under split-plot designs, this default is either the analysis of variance (ANOVA) or the linear mixed effects model (Wu and Hamada, 2009). Despite the good intention of both methods to adjust for the block structure that defines split-plot designs, the actual variance estimation often turns out inconsistent (Gelman, 2005; Hinkelmann and Kempthorne, 2008), likely due to the required model assumptions not being satisfied. A detailed examination of this argument can be found in Freedman (2006, 2008a), which recommended randomized-based inference as the proper solution.

Despite its long tradition in the context of treatment-control experiments (Ding and Dasgupta, 2015), randomization-based inference remains an almost uncharted field when it comes to factorial experiments. The recent works of Dasgupta, Pillai and Rubin (2015) and Espinosa, Dasgupta and Rubin (2015) are, to the best of our knowledge, the only literature along this line, each documenting improvements of randomization-based analysis over existing model-based methods in the context of multi-factor completely randomized designs. Generalizing their methods to split-plot designs could be a promising next step.

1.3. Contributions. The contribution of this paper is three-fold. First, we develop the first randomization-based estimation procedure for causal inference under $2^2$ split-plot designs, and demonstrate its superior frequency cov-
erage properties over existing alternatives. Second, motivated by split-plot designs' signature block structure, we propose a decomposition of the potential outcomes that links the relative efficiency between a split-plot design and a complete randomization of the same size to the level of heterogeneity among blocks. This allows any empirical knowledge about the latter, when available, to be admitted as possible aid for deciding between designs.

Third, in an attempt to reconcile the finite-population randomization-based perspective and a hypothetical super-population model-based perspective, we offer a heuristic argument that connects the two. This connection is established by using the asymptotics of the finite-population randomization-based residual covariances to justify the block-diagonal structure assumed by the linear mixed effects model for the covariances of its super-population sampling errors. This, to the best of our knowledge, is the very first attempt that aims at reconciling the difference between finite and super-population inferences.

1.4. Organization of the article. The article is organized as follows. We review in Section 2 the potential outcomes framework, and discuss possible extensions when the experimental units exhibit certain block structure. We define in Section 3 the causal questions in $2^2$ factorial experiments, and introduce in Section 4 the split-plot design as one possible randomization scheme. Sampling variances of the estimates are derived in Section 5, and their estimation addressed in Section 6. We discuss the connection and distinction between the model-based and randomization-based inferences in Section 7, and demonstrate the latter’s superior frequency coverage properties in Section 8. We conclude in Section 9. All proofs are deferred to the online supplementary material.

2. Potential outcomes and additivity assumptions. We review in this section the major concepts within the potential outcomes framework (Neyman, 1923; Rubin, 1974, 1978, 2005), and discuss some possible extensions when the experimental units are nested under blocks.

2.1. Potential outcomes framework for causal inference. Consider an experiment in which $K$ different treatments are to be tested on $N$ experimental units. The Stable Unit Treatment Value Assumption (Rubin, 1980) allows us to write the potential outcome of unit $i$ when exposed to treatment $k$ as $Y_i(k)$. Whereas causal effects are then defined as comparisons of such potential outcomes for a given set of units, any experiment, however well designed and implemented, allows us to observe at most one of $K$ potential outcomes per unit, according to the treatment it receives. This poses the fundamental
problem of causal inference (Holland, 1986). Various assumptions are introduced in this context as attempts to infer the unobserved from the observed, the most common being that of the strict additivity.

**Definition 1.** The potential outcomes \( Y_i(k) \) of \( N \) units under \( K \) treatments are ‘strictly additive’ if the differences between any two treatments are constant across all units, i.e., \( Y_i(l) = Y_i(k) + C(k,l) \) for some fixed real numbers \( C(k,l) \).

Let \( \bar{Y}(k) = N^{-1} \sum_{i=1}^{N} Y_i(k) \) be the population average under treatment \( k \),

\[
S^2(k,l) = (N - 1)^{-1} \sum_{i=1}^{N} \{Y_i(k) - \bar{Y}(k)\}\{Y_i(l) - \bar{Y}(l)\}
\]

be the finite-population covariance of \( Y_i(k) \) and \( Y_i(l) \), and

\[
S^2 = (S^2(k,l))_{K \times K}
\]

be the finite-population variance-covariance matrix. Lemma 1 gives an alternative characterization of strict additivity in terms of \( S^2(k,l) \).

**Lemma 1.** The potential outcomes \( Y_i(k) \) of \( N \) units under \( K \) treatments are strictly additive if and only if the finite-population covariances \( S^2(k,l) \) are the same for all \( k, l \in \{1, \ldots, K\} \), i.e., \( S^2 = S_0^2 J_K \), where \( J_K \) is a \( K \times K \) matrix of 1’s and \( S_0^2 \) is a fixed non-negative number.

For simplicity, we will omit the ‘finite-population’ before ‘covariance’ in the following text when no confusion would arise. All averages and covariances over a finite set of fixed numbers will be finite-population in nature, and defined the same way as \( \bar{Y}(k) \) and \( S^2(k,l) \) are defined for \( Y_i(k) \).

**2.2. Experimental units with block structure.** Whereas all definitions and discussion above apply universally to any \( K \)-treatment experiment with \( N \) experimental units, possible extensions arise when the experimental units in question exhibit certain block structure — due to either intrinsic characteristics like geographic proximity, or extrinsic arrangements as induced by the design.

Without essential loss of generality, assume the \( N \) experimental units are nested under \( W \) blocks, each of size \( M = N/W \). Generalization to unequal block sizes is straightforward. Index the blocks by \( w \), running from 1 to \( W \), and the units within block \( w \) by \( (wm) \), running from \( (w1) \) to \( (wM) \). Define the block average potential outcomes as

\[
Y_{(w)}(k) = M^{-1} \sum_{m=1}^{M} Y_{(wm)}(k) \quad (k = 1, \ldots, K).
\]
These aggregated potential outcomes enable the definitions of some weaker forms of additivity as compared to that in Definition 1.

**Definition 2.** The potential outcomes $Y_{(wm)}(k)$ of $N$ units in $W$ blocks under $K$ treatments are

- ‘between-block additive’ if the corresponding block average potential outcomes $Y_{(w)}(k)$ are strictly additive across all $w$, i.e., $Y_{(w)}(l) = Y_{(w)}(k) + C_{(k,l)}$ for some fixed real numbers $C_{(k,l)}$;
- ‘within-block additive’ if for each $w$, the potential outcomes $Y_{(wm)}(k)$ of the $M$ units within block $w$ are strictly additive, i.e., $Y_{(wm)}(l) = Y_{(wm)}(k) + C_{w(k,l)}$ for some fixed real numbers $C_{w(k,l)}$.

Strictly additive potential outcomes, if nested under blocks, must be strictly additive within each block and have strictly additive block averages. Lemma 2 asserts that the converse is also true.

**Lemma 2.** The potential outcomes of $N$ units in $W$ blocks are strictly additive if and only if they are both between- and within-block additive.

Define the between-block covariance of $Y_{(wm)}(k)$ and $Y_{(wm)}(l)$ by the covariance of $Y_{(w)}(k)$ and $Y_{(w)}(l)$:

$$S_{btw}^2(k,l) = (W - 1)^{-1} \sum_{w=1}^{W} \{Y_{(w)}(k) - \bar{Y}(k)\}\{Y_{(w)}(l) - \bar{Y}(l)\},$$

and their within-block covariance by

$$S_{in}^2(k,l) = W^{-1} \sum_{w=1}^{W} S_{(w)}^2(k,l),$$

where

$$S_{(w)}^2(k,l) = (M - 1)^{-1} \sum_{m=1}^{M} \{Y_{(wm)}(k) - Y_{(w)}(k)\}\{Y_{(wm)}(l) - Y_{(w)}(l)\}$$

is the covariance of $Y_{(wm)}(k)$ and $Y_{(wm)}(l)$ within block $w$. Let

$$S_{btw}^2 = \langle S_{btw}^2(k,l) \rangle_{K \times K}, \quad S_{in}^2 = \langle S_{in}^2(k,l) \rangle_{K \times K}.$$  

Applying Lemma 1 to Definition 2 allows us to characterize the between- and within-block additivities via their respective covariances as follows.

**Lemma 3.** The potential outcomes $Y_{(wm)}(k)$ of $N$ units in $W$ blocks under $K$ treatments are
• between-block additive if and only if \( S_{\text{btw}}^2 = S_{\text{btw}}^2 J_K \) for some non-negative number \( S_{\text{btw}}^2 \):

• within-block additive if and only if \( S_{\text{in}}^2 = S_{\text{in}}^2 J_K \) for some non-negative number \( S_{\text{in}}^2 \).

**Remark 1.** For potential outcomes that are between-block additive, the common value \( S_{\text{btw}}^2 \) provides a measure of the block variability.

### 2.3. Decomposition of covariances.

For any positive integer \( p \), let \( 1_p \) be the \( p \)-dimensional vector of 1’s, \( J_p = 1_p 1_p^T \) be the \( p \times p \) matrix of 1’s, \( I_p \) be the \( p \times p \) identity matrix, and \( P_p = I_p - p^{-1} J_p \) be the \( p \times p \) projection matrix with column space orthogonal to \( 1_p \). Let \( \otimes \) denote the Kronecker product.

Let \( Y(k) = (Y_1(k), \ldots, Y_N(k))^T = (Y_{(1)}(k), \ldots, Y_{(W_M)}(k))^T \) be the same potential outcomes vector indexed in two different ways — the running index \( i \) and the block-unit double-index \( (wm) \). Straightforward algebra determines

\[
 \begin{align*}
 P_{\text{in}} &= I_W \otimes P_M, \\
 P_{\text{btw}} &= P_W \otimes (M^{-1} J_M)
\end{align*}
\]
as two mutually orthogonal projection matrices, with

\[
\begin{align*}
 P_{\text{in}} Y(k) &= \left(Y_{(wm)}(k) - Y_{(w)}(k)\right)_{N \times 1}, \\
 P_{\text{btw}} Y(k) &= \left(Y_{(w)}(k) - \bar{Y}(k)\right)_{N \times 1}.
\end{align*}
\]

Let \( Y = (Y_i(k))_{N \times K} \) be the \( N \times K \) potential outcome matrix (POM) with \( Y(k) \) as its \( k \)th column. It follows from (2.3) that

\[
\begin{align*}
 P_N Y(k) &= (I_N - N^{-1} 1_N 1_N^T) Y(k) \\
 &= Y(k) - 1_N \bar{Y}(k) = \left(Y_{(wm)}(k) - \bar{Y}(k)\right) \\
 &= \left(Y_{(wm)}(k) - Y_{(w)}(k)\right) + \left(Y_{(w)}(k) - \bar{Y}(k)\right) \\
 &= P_{\text{in}} Y(k) + P_{\text{btw}} Y(k), \\
 P_N Y &= P_{\text{in}} Y + P_{\text{btw}} Y,
\end{align*}
\]

(2.4) \( Y^T P_N Y = Y P_{\text{in}} Y + Y P_{\text{btw}} Y \)

and that

\[
\begin{align*}
 S_{\text{in}}^2 (k, l) &= \frac{Y(k)^T P_{\text{in}} Y(l)}{W(M - 1)}, \\
 S_{\text{btw}}^2 (k, l) &= \frac{Y(k)^T P_{\text{btw}} Y(l)}{(W - 1)M}, \\
 S_{\text{in}}^2 &= \frac{Y^T P_{\text{in}} Y}{W(M - 1)}, \\
 S_{\text{btw}}^2 &= \frac{Y^T P_{\text{btw}} Y}{(W - 1)M}.
\end{align*}
\]

Combining (2.4) with (2.5) yields the first major result of this article.
Theorem 1. The variance-covariance matrix $\mathbf{S}^2$ is a linear combination of $\mathbf{S}_{btw}^2$ and $\mathbf{S}_{in}^2$:

$$\mathbf{S}^2 = \frac{(W - 1)M}{N - 1} \mathbf{S}_{btw}^2 + \frac{W(M - 1)}{N - 1} \mathbf{S}_{in}^2.$$ 

We have so far introduced, in the context of general $K$-treatment experiments, all concepts about the potential outcomes framework that we consider relevant to the current topic. Specific discussion of $2^2$ factorial experiments starts in the next section, in which we formally introduce this special type of 4-treatment experiment, together with its chief causal questions of interest.

3. Causal effects for $2^2$ factorial experiments. $2^2$ factorial experiments, as the name suggests, involve different $K = 4$ treatments as the $2^2$ possible combinations of two 2-level factors. Code the two factors as $A$ and $B$. Of chief causal interest are the main effect of factor $A$ (indexed by ‘$A$’), the main effect of factor $B$ (indexed by ‘$B$’), and the effect of interaction between $A$ and $B$ (indexed by ‘$AB$’, also refer to as factor $AB$). We set out in this section their formal definitions at unit, block, and population levels.

3.1. Causal effects at unit and population levels. Code the two levels of factor $A$ as $\{-1_A, +1_A\}$ and those of factor $B$ as $\{-1_B, +1_B\}$. We represent the four combinations as $(-1_A, -1_B), (1_A, +1_B), (+1_A, -1_B), (+1_A, +1_B)$, and name them in lexicographic order as treatments 1 to 4 (Table 1).

| Treatment | Factor $A$ | Factor $B$ | Interaction ($AB$) |
|-----------|------------|------------|-------------------|
| 1         | $-1_A$     | $-1_B$     | $+1_{AB}$         |
| 2         | $-1_A$     | $+1_B$     | $-1_{AB}$         |
| 3         | $+1_A$     | $-1_B$     | $-1_{AB}$         |
| 4         | $+1_A$     | $+1_B$     | $+1_{AB}$         |

Given a study population of $N$ units, denote by $\mathbf{Y}_i = (Y_i(1), Y_i(2), Y_i(3), Y_i(4))^T$ the potential outcomes vector of unit $i$. Let $\mathbf{g}_A = (-1, -1, +1, +1)^T$ summarize the levels of factor $A$ in treatments 1 to 4 — i.e., the ‘Factor $A$’ column in Table 1 — and $\mathbf{g}_B = (-1, +1, -1, +1)^T$, $\mathbf{g}_{AB} = (+1, -1, -1, +1)^T$ likewise. The factorial effect of factor $F \in \mathcal{F} = \{A, B, AB\}$ on unit $i$ is defined as

$$\tau_{i,F} = 2^{-1} \mathbf{g}^T_F \mathbf{Y}_i,$$
with population average

\[
\tau_F = N^{-1} \sum_{i=1}^{N} \tau_{i,F} = 2^{-1} g_F^T (\vec{Y}(1), \vec{Y}(2), \vec{Y}(3), \vec{Y}(4))^T.
\]

Let \(S^2_F = (N - 1)^{-1} \sum_{i=1}^{N} (\tau_{i,F} - \tau_F)^2\). Lemma 4 restates strict additivity in terms of the factorial effects and their variances.

**Lemma 4.** The \(4 \times N\) potential outcomes of \(N\) units in a \(2^2\) factorial experiment, \(Y_i(k)\ (i = 1, \ldots, N; k = 1, 2, 3, 4)\), are strictly additive if and only if all three unit-level factorial effects are constant across all units, i.e., \(\tau_{i,F} = \tau_F\) for all \(i \in \{1, \ldots, N\}\) and \(F \in \mathcal{F}\); or equivalently, \(S^2_F = 0\) for each \(F \in \mathcal{F}\).

**3.2. Causal effects at block level.** When the study population is nested under blocks, further define

\[
\tau_{(w)}-F = M^{-1} \sum_{m=1}^{M} \tau_{(wm)}-F
\]

\[
= 2^{-1} g_{F}^T (Y_{(w)}(1), Y_{(w)}(2), Y_{(w)}(3), Y_{(w)}(4))^T
\]

as the block average factorial effects. The \(\tau_{(wm)}-F\) in (3.2) are the same unit-level factorial effects as \(\tau_{i,F}\), only now under block-unit double-index \((wm)\).

With all blocks being of equal size, the three levels of factorial effects satisfy

\[
\tau_F = W^{-1} \sum_{w=1}^{W} \tau_{(w)}-F = N^{-1} \sum_{i=1}^{N} \tau_{i,F}.
\]

Define the between- and within-block variances of \(\tau_{i,F}\) the same way (2.1)–(2.2) defined \(S^2_{btw}(k,k)\) and \(S^2_{in}(k,k)\):

\[
S^2_{F, btw} = (W-1)^{-1} \sum_{w=1}^{W} (\tau_{(w)}-F - \tau_F)^2,
\]

\[
S^2_{F, in} = W^{-1} \sum_{w=1}^{W} \left\{ (M-1)^{-1} \sum_{m=1}^{M} (\tau_{(wm)}-F - \tau_{(w)}-F)^2 \right\}.
\]

These variances give an alternative characterization of the between- and within-block additivities as detailed in Lemma 5.

**Lemma 5.** Given \(N\) experimental units in a \(2^2\) factorial experiment that are nested under \(W\) blocks and indexed by \((wm)\), the corresponding \(4 \times N\) potential outcomes are
• between-block additive if and only if all three block average factorial effects \( \tau_{(w)} - \mathcal{F} \) are constant across all blocks, i.e., \( \tau_{(w)} - \mathcal{F} = \tau_\mathcal{F} \) for all \( w \in \{1, \ldots, W\} \) and \( \mathcal{F} \in \mathcal{F} \), or equivalently, \( S_{F \text{-in}}^2 = 0 \) for each \( \mathcal{F} \in \mathcal{F} \);

• within-block additive if and only if all three unit-level factorial effects \( \tau_{(wm)} - \mathcal{F} \) are constant within each block, i.e., \( \tau_{(wm)} - \mathcal{F} = \tau_{(w)} - \mathcal{F} \) for all \( w \in \{1, \ldots, W\} \) and \( \mathcal{F} \in \mathcal{F} \), or equivalently, \( S_{F \text{-btw}}^2 = 0 \) for each \( \mathcal{F} \in \mathcal{F} \).

4. \( 2^2 \) split-plot design. We introduce in this section the \( 2^2 \) split-plot design as a treatment assignment mechanism distinct from complete randomization.

4.1. Notation and definitions. Assume fixed treatment arm sizes \( N_k \) \( (k = 1, \ldots, K, \sum_{k=1}^K N_k = N) \). Let \( T_i \) be the assignment variable, taking the value \( k \) if unit \( i \) is assigned to treatment \( k \). Let \( Z(k) = (I_{\{T_1=k\}}, \ldots, I_{\{T_N=k\}})^T \), with \( \sum_{i=1}^N I_{\{T_i=k\}} = N_k \). Let \( Z^* = (N_1^{-1} Z(1)^T, \ldots, N_K^{-1} Z(K)^T) \), in which we normalize each \( Z(k) \) by the sum of its entries \( N_k \). Refer to \( Z^* \) as the assignment vector. It gives a full representation of the randomization result, in a form that promises easier algebra than \( \{T_i\}_{i=1}^N \).

4.2. A classical agricultural experiment. Consider a classical agricultural experiment in which two levels of irrigation (factor A) and two levels of fertilizers (factor B) are to be tested on \( N = 8 \) plots of land (experimental units) nested within four whole-plots (blocks) (Figure 1).

Assume each combination is to be replicated on \( N/K = 8/4 = 2 \) plots. The assignment process can be visualized as a distribution of the eight tags:

\[
\begin{align*}
(-1_A, -1_B) & \quad (-1_A, +1_B) & \quad (+1_A, -1_B) & \quad (+1_A, +1_B) \\
(-1_A, -1_B) & \quad (+1_A, -1_B) & \quad (+1_A, +1_B) & \quad (+1_A, +1_B)
\end{align*}
\]

to the eight plots. A completely randomized design, as the name suggests, distributes the tags at complete random. Any arrangement of the eight tags is equally likely, with Figures 2 and 3 being two examples.
A split-plot design, on the other hand, requires the two plots within each whole-plot to receive the same level of irrigation. This could be due to resource constraint, say, the technical difficulty in applying irrigation to areas smaller than a whole-plot, or on purpose, to minimize the bias from block heterogeneity when comparing the fertilizers. Whereas Figure 2 satisfies the requirement and remains a possible arrangement, the different irrigation levels in plots 1 and 2 disqualify Figure 3 from the candidate pool.

In general, with the experimental units in hand nested under several blocks, a split-plot design identifies one factor as the whole-plot factor, and restricts its level to be the same within each block. The possible assignments under a split-plot design thus constitute a proper subset of the possible assignments under a completely randomized one. This brings out the first and most salient distinction between the two designs.

Formal definitions of these two designs are given in the next section, along with the sampling moments of their respective assignment vectors $Z^*$. Not only do these sampling moments enable a first quantitative comparison between the two designs, but also provide the fundamental building blocks for computing the sampling variances of our major estimates to be introduced in Section 5.

**Fig 2. An assignment possible under both the completely randomized and split-plot designs.**

| Treatment $k$ | Combination  | Recipients $i$ | Indicators of recipients $Z(k)$ |
|---------------|--------------|----------------|-------------------------------|
| 1             | $(-1_A, -1_B)$ | 2, 8           | $(0, 1, 0, 0, 0, 0, 0, 1)^T$   |
| 2             | $(-1_A, +1_B)$ | 1, 7           | $(1, 0, 0, 0, 0, 1, 0)^T$      |
| 3             | $(+1_A, -1_B)$ | 4, 5           | $(0, 0, 0, 1, 1, 0, 0)^T$      |
| 4             | $(+1_A, +1_B)$ | 3, 6           | $(0, 0, 1, 0, 0, 1, 0)^T$      |

4.3. **Designs and their respective assignment vectors.** As far as $2^2$ factorial experiment is concerned, variations of complete randomization exist. The two factors of interest can be assigned either one at a time, each by a two-treatment complete randomization, or, jointly, via a single complete randomization of the treatment combinations 1 to 4. Being aware of such plurality, we qualify by Definitions 3 and 4 the particular $2^2$ completely
Fig 3. An assignment possible under the completely randomized design yet impossible under the split-plot design.

| Treatment $k$ | Combination Recipients $i$ | Indicators of recipients $Z(k)$ |
|---------------|-----------------------------|---------------------------------|
| 1             | $(-1_A, -1_B)$              | 1, 3                            |
| 2             | $(-1_A, +1_B)$              | 4, 8                            |
| 3             | $(+1_A, -1_B)$              | 2, 5                            |
| 4             | $(+1_A, +1_B)$              | 6, 7                            |

randomized (C-R) design’ and ‘$2^2$ split-plot (s-p) design’ on which we will base most of the quantitative derivations in this article.

**Definition 3.** Given treatments 1 to 4 in a $2^2$ factorial experiment and $N$ experimental units, a $2^2$ completely randomized design with planned treatment arm sizes $N_1$, $N_2$, $N_3$, and $N_4 = N - \sum_{k=1}^{3} N_k$ can be visualized as distributing a well shuffled deck of $N_1$ tags of treatment 1, $N_2$ tags of treatment 2, $N_3$ tags of treatment 3, and $N_4$ tags of treatment 4 to units 1 to $N$, such that all partitions of the $N$ units into the four treatment arms are equally likely.

**Lemma 6.** Under the $2^2$ completely randomized design qualified by Definition 3, the sampling expectation and variance-covariance matrix of the assignment vector $Z^*$ are

$$E_{C-R}(Z^*) = N^{-1}1_N, \quad \text{cov}_{C-R}(Z^*) = C \otimes P_N$$

where

$$C = \frac{1}{N(N-1)} \left( \text{diag} \left( \frac{N}{N_1}, \frac{N}{N_2}, \frac{N}{N_3}, \frac{N}{N_4} \right) - J_4 \right).$$

**Definition 4.** Given two 2-level factors of interest, whole-plot factor $A$ and sub-plot factor $B$, and $N$ experimental units nested within $W$ whole-plots (blocks), each of size $M = N/W$, a $2^2$ split-plot design with planned size parameters $W_{+1}$ and $M_{+1}$ consists of two separate randomizations:
• Whole-plot randomization that assigns $W_1$ of $W$ whole-plots chosen at complete random to $+1_A$ level of whole-plot factor $A$, and the remaining $W_1 = W - W_1$ ones to $-1_A$ level,

• Sub-plot randomization that assigns $M_1$ of $M$ sub-plots chosen at complete random within each whole-plot to $+1_B$ level of sub-plot factor $B$, and the remaining $M_1 = M - M_1$ ones to $-1_B$ level.

The final treatment for sub-plot (wm) will be the combination of the level of factor $A$ whole-plot $w$ receives in the whole-plot randomization and the level of factor $B$ itself receives in the sub-plot randomization.

We will use ‘whole-plot’ and ‘block,’ as well as ‘sub-plot’ and ‘experimental unit,’ interchangeably for the rest of the paper, so that the notations and definitions introduced in Section 2.2 apply directly. Let

$$r_A = W_1/W_1, \quad r_B = M_1/M_1$$

be the ratios of factor arm sizes for the whole-plot and sub-plot randomizations respectively.

**Theorem 2.** Under the $2^2$ split-plot design qualified by Definition 4, the sampling expectation and variance-covariance matrix of the assignment vector $Z^*$ are

$$E_{\text{s-p}}(Z^*) = N^{-1}1_N, \quad \text{cov}_{\text{s-p}}(Z^*) = C_{\text{btw}} \otimes P_{\text{btw}} + C_{\text{in}} \otimes P_{\text{in}}$$

where

$$C_{\text{btw}} = \frac{1}{N(W-1)} \begin{pmatrix} r_A & r_A & -1 & -1 \\ r_A & r_A & -1 & -1 \\ -1 & -1 & r_A^{-1} & r_A^{-1} \\ -1 & -1 & r_A^{-1} & r_A^{-1} \end{pmatrix},$$

$$C_{\text{in}} = \frac{1}{NW(M-1)} \begin{pmatrix} (1 + r_A)r_B & -(1 + r_A) & 0 & 0 \\ -(1 + r_A) & (1 + r_A)r_B^{-1} & 0 & 0 \\ 0 & 0 & (1 + r_A^{-1})r_B & -(1 + r_A^{-1}) \\ 0 & 0 & -(1 + r_A^{-1}) & (1 + r_A^{-1})r_B^{-1} \end{pmatrix}.$$
as its ‘(unrestricted) completely randomized counterpart.’ It follows from straightforward algebra that the respective coefficient matrices of the restricted and the unrestricted satisfy

\[ C = \frac{W - 1}{N - 1} C_{\text{btw}} + \frac{W (M - 1)}{N - 1} C_{\text{in}}. \]

This, together with Lemma 6 and Theorem 2, allows us to write the effect of ‘restriction’ on the variance-covariance matrix of \( Z^* \) as

\[ \text{cov}_{s-p}(Z^*) - \text{cov}_{c-r}(Z^*) = C_{\text{btw}} \otimes P_{\text{btw}} + C_{\text{in}} \otimes P_{\text{in}} - C \otimes P_N \]

5. Neymanian point estimates for \( 2^2 \) factorial effects. Neymanian causal inference focuses on the population-level effects, and takes the three population average factorial effects as its chief causal estimands of interest. We define in this section the Neymanian point estimates of these three estimands, and derive their respective sampling variances under \( 2^2 \) split-plot designs.

5.1. Point estimates and their sampling variances. Recall that \( T_i = k \) if unit \( i \) is assigned to treatment \( k \). Let

\[ \bar{Y}_{\text{obs}}(k) = N_k^{-1} \sum_{i:T_i = k} Y_{i\text{obs}} \]

be the average observed outcome of treatment arm \( k \). Estimating the unobservable \( \bar{Y}(k) \) by \( \bar{Y}_{\text{obs}}(k) \) in the definition of \( \tau_F \) in (3.1) yields the Neymanian point estimate of this population-level factorial effect:

\[ \hat{\tau}_F = 2^{-1} g_F(\bar{Y}_{\text{obs}}(1), \bar{Y}_{\text{obs}}(2), \bar{Y}_{\text{obs}}(3), \bar{Y}_{\text{obs}}(4))^T \quad (F \in \mathcal{F}). \]

Let \( \bar{Y} \) be the \( 4N \times 4 \) block-diagonal matrix with diagonal vectors \( Y(k) \):

\[ \bar{Y} = \begin{pmatrix} Y(1) & & & \\ & Y(2) & & \\ & & Y(3) & \\ & & & Y(4) \end{pmatrix}. \]

It follows from

\[ \bar{Y}_{\text{obs}}(k) = N_k^{-1} \sum_{i:T_i = k} Y_{i\text{obs}} = N_k^{-1} \sum_{i:T_i = k} Y_i(k) = Y(k)^T \{N_k^{-1} Z(k)\}, \]
that

\[
\begin{pmatrix}
\bar{Y}_{\text{obs}}^{(1)} \\
\bar{Y}_{\text{obs}}^{(2)} \\
\bar{Y}_{\text{obs}}^{(3)} \\
\bar{Y}_{\text{obs}}^{(4)}
\end{pmatrix}
= 
\begin{pmatrix}
Y^{(1)^T} \\
Y^{(2)^T} \\
Y^{(3)^T} \\
Y^{(4)^T}
\end{pmatrix}
\begin{pmatrix}
N_1^{-1} Z^{(1)} \\
N_2^{-1} Z^{(2)} \\
N_3^{-1} Z^{(3)} \\
N_4^{-1} Z^{(4)}
\end{pmatrix}
= \bar{Y}^* Z^*
\]

Substitute this into (5.1) to see

\begin{equation}
\hat{\tau}_F = 2^{-1} g_F^T \bar{Y}^* Z^* \quad (F \in \mathcal{F}),
\end{equation}

with assignment vector \(Z^*\) alone being stochastic on the right. The fact of (5.2) being true for any arbitrary \(Z^*\) allows us to take expectation and covariance of both sides with respect to any arbitrary \(2^2\) factorial assignment mechanism. This yields Lemma 7.

**Lemma 7.** The randomness in the Neymanian point estimate \(\hat{\tau}_F\), under any arbitrary \(2^2\) factorial assignment mechanism (A-M), originates solely from the randomness in the assignment vector \(Z^*\), with

\[
E_{\text{A-M}}(\hat{\tau}_F) = 2^{-1} g_F^T \bar{Y}^* E_{\text{A-M}}(Z^*), \quad \text{var}_{\text{A-M}}(\hat{\tau}_F) = 4^{-1} g_F^T \bar{Y}^* \text{cov}_{\text{A-M}}(Z^*) \bar{Y} g_F
\]

where \(E_{\text{A-M}}, \text{var}_{\text{A-M}}, \text{and} \text{cov}_{\text{A-M}}\) are the expectation, variance, and covariance with respect to the sampling distribution under A-M over all possible assignments.

Explicit formulae under completely randomized designs follow immediately from combining Lemma 7 with Lemma 6, and those under split-plot designs from combining Lemma 7 with Theorem 2:

**Theorem 3.** Under the \(2^2\) completely randomized design qualified by Definition 3, the Neymanian point estimate \(\hat{\tau}_F\) is unbiased for \(\tau_F\) with sampling variance

\begin{equation}
\text{var}_{\text{c-r}}(\hat{\tau}_F) = 4^{-1} (N - 1) g_F^T (C \circ S^2) g_F \quad (F \in \mathcal{F}).
\end{equation}

Here, ‘\(\circ\)’ denotes the entrywise product, and \(C\) is the coefficient matrix defined in Lemma 6.

**Theorem 4.** Under the \(2^2\) split-plot design qualified by Definition 4, the Neymanian point estimate \(\hat{\tau}_F\) is unbiased for \(\tau_F\) with sampling variance

\begin{equation}
\text{var}_{\text{s-p}}(\hat{\tau}_F) = 4^{-1} (W - 1) M g_F^T (C_{\text{btw}} \circ S^2_{\text{btw}}) g_F \\
+ 4^{-1} W (M - 1) g_F^T (C_{\text{in}} \circ S^2_{\text{in}}) g_F \quad (F \in \mathcal{F}).
\end{equation}
5.2. Comparison of precisions under strict additivity. Simplified forms of Theorems 3 and 4 are available when the potential outcomes are strictly additive, enabling intuitive comparisons of the estimation precision.

**Corollary 1.** For strictly additive potential outcomes, the sampling variances of $\hat{\tau}_A$, $\hat{\tau}_B$, and $\hat{\tau}_{AB}$ under the $2^2$ split-plot design in Theorem 4 reduce to

\[
\text{var}_{s.p.}(\hat{\tau}_A) = W^{-1}\gamma_A S_{btw}^2 + (4N)^{-1}\gamma_B(\gamma_B - 4)S_{in}^2,
\]

\[
\text{var}_{s.p.}(\hat{\tau}_B) = \text{var}_{s.p.}(\hat{\tau}_{AB}) = (4N)^{-1}\gamma_A\gamma_BS_{in}^2,
\]

where $\gamma_A = r_A + r_A^{-1} + 2$, and $\gamma_B = r_B + r_B^{-1} + 2$.

**Remark 2.** With $x+x^{-1}+2 = \{\sqrt{x}-(\sqrt{x})^{-1}\}^2+4$, we have $\min_{r_A} \gamma_A = \gamma_A|_{r_A=1} = 4$ and $\min_{r_B} \gamma_B = \gamma_B|_{r_B=1} = 4$. The increasing monotonicity of (5.5) in $\gamma_A$ and $\gamma_B$ suggests the three sampling variances be simultaneously minimized when $\gamma_A$ and $\gamma_B$ are at their respective minimums:

\[
\min_{\gamma_A,\gamma_B} \text{var}_{s.p.}(\hat{\tau}_A) = \text{var}_{s.p.}(\hat{\tau}_A)|_{\gamma_A=4,\gamma_B=4} = 4S_{btw}^2/W,
\]

\[
\min_{\gamma_A,\gamma_B} \text{var}_{s.p.}(\hat{\tau}_B) = \min_{\gamma_A,\gamma_B} \text{var}_{s.p.}(\hat{\tau}_{AB}) = \text{var}_{s.p.}(\hat{\tau}_B)|_{\gamma_A=4,\gamma_B=4} = 4S_{in}^2/N
\]

where $\gamma_A = 4, \gamma_B = 4$ imply $r_A = r_B = 1$ — i.e. the design being balanced. This establishes the optimality of balanced designs regarding strictly additive potential outcomes.

**Remark 3.** The sampling variances of $\hat{\tau}_A$ and $\hat{\tau}_B$ in (5.5) satisfy

\[
\text{var}_{s.p.}(\hat{\tau}_A) - \text{var}_{s.p.}(\hat{\tau}_B) = W^{-1}\gamma_A(S_{btw}^2 - S_{in}^2/M).
\]

This suggests more precise Neymanian estimation of the sub-plot factor $B$ than that of the whole-plot factor $A$ if $S_{btw}^2 - S_{in}^2/M > 0$, and vice versa if $S_{btw}^2 - S_{in}^2/M < 0$.

An intuitive link between the discriminant $S_{btw}^2 - S_{in}^2/M$ and the block heterogeneity can be established from a super-population perspective for potential outcomes generated from linear mixed effects models. Specifically, assume the study population in question to be a random sample from some super-population such that

\[
Y_{(wm)}(k) = \mu(k) + \eta_w + \xi_{(wm)} \quad (w = 1, \ldots, W; m = 1, \ldots, M)
\]

follow the linear mixed effects model with fixed treatment effects $\mu(k)$, random block effects $\eta_w \sim \mathcal{N}(0, \sigma_\eta^2)$, and individual sampling errors $\xi_{(wm)} \sim \mathcal{N}(0, \sigma_\xi^2)$ jointly independent of $\eta_w$. 
Assume, without loss of generality, $\mu(1) = 0$. The $W$ block average potential outcomes under treatment 1 constitute $W$ iid normals with mean 0 and variance $\sigma_\eta^2 + \sigma_\xi^2/M$:

$$Y_{(w)}(1) = M^{-1} \sum_{m=1}^{M} Y_{(wm)}(1) = \eta_w + M^{-1} \sum_{m=1}^{M} \xi_{(wm)} \sim \mathcal{N}(0, \sigma_\eta^2 + \sigma_\xi^2/M).$$

$S_{btw}^2(1, 1)$, as the finite-population variance of $Y_{(w)}(1)$, is thus unbiased for the super-population variance parameter $\sigma_\eta^2 + \sigma_\xi^2/M$:

$$(5.8) \quad E^*\{S_{btw}^2(1, 1)\} = \text{var}^*\{Y_{(w)}(1)\} = \sigma_\eta^2 + \sigma_\xi^2/M$$

where $E^*$ and var* are the expectation and variance with respect to the sampling distribution represented via model (5.7). Likewise, with

$$S_{(w)}^2(1, 1) = (M - 1)^{-1} \sum_{m=1}^{M} \{Y_{(wm)}(1) - Y_{(w)}(1)\}^2$$

$$= (M - 1)^{-1} \sum_{m=1}^{M} \left( \xi_{(wm)} - M^{-1} \sum_{m=1}^{M} \xi_{(wm)} \right)^2$$

simplifying to the finite-population variance of iid normals $\{\xi_{(wm)}\}_{m=1}^{M}$, we have $E^*\{S_{(w)}^2(1, 1)\} = E^*(\xi_{(wm)}) = \sigma_\xi^2$, and thus

$$(5.9) \quad E^*\{S_{in}^2(1, 1)\} = E^*\left\{ W^{-1} \sum_{w=1}^{W} S_{(w)}^2(1, 1) \right\} = \sigma_\xi^2.$$

Under strict additivity — as it is guaranteed by model (5.7), abbreviate $S_{btw}^2(1, 1)$ as $S_{btw}$ and $S_{in}^2(1, 1)$ as $S_{in}$ (by summoning Lemmas 2 and 3). Formulae (5.8) and (5.9) together yield

$$(5.10) \quad E^*(S_{btw}^2 - S_{in}^2/M) = E(S_{btw}^2) - E(S_{in}^2)/M = \sigma_\eta^2 \geq 0,$$

equating the super-population expectation of the discriminant to the super-population variance of the random block effects $\eta_w$. This, coupled with formula (5.6), suggests the average sampling variance of the sub-plot estimate $\hat{\tau}_B$ be strictly smaller than that of the whole-plot estimate $\hat{\tau}_A$ — unless $\sigma_\eta^2 = 0$ and (5.7) degenerates to a simple linear model that admits no random block effects.
Recall from Theorem 4 and Corollary 1 the decomposition of overall sampling variances under split-plot designs into the between- and within-whole-plot parts. Analogous results for completely randomized designs follow from substituting Theorem 1 into formula (5.3):

\[
\text{var}_{\text{c-r}}(\hat{\tau}_F) = 4^{-1}(W - 1)M g_F^T(C \circ S_{\text{btw}}^2)g_F + 4^{-1}W(M - 1)g_F^T(C \circ S_{\text{in}}^2)g_F \quad (F \in \mathcal{F}).
\]

Contrasting this with Theorem 4 yields Corollary 2.

**Corollary 2.** Assume common treatment arm sizes \((4.1)\). The sampling variance of \(\hat{\tau}_F\) under a \(2^2\) split-plot design \((s-p)\) differs from that under a \(2^2\) completely randomized design \((c-r)\) by

\[
\text{var}_{s-p}(\hat{\tau}_F) - \text{var}_{c-r}(\hat{\tau}_F) = C_0 g_F^T \left\{ (C_{\text{btw}} - C_{\text{in}}) \circ (S_{\text{btw}}^2 - S_{\text{in}}^2/M) \right\} g_F,
\]

where \(C_0\) is a positive constant.

The difference in Corollary 2 informs us of not only the relative efficiency of split-plot designs with regard to each \(F \in \mathcal{F}\), but also the discrepancy in variance estimation when a split-plot experiment is wrongfully analyzed as a completely randomized one.

**Corollary 3.** For strictly additive potential outcomes, the sampling variance under \(2^2\) completely randomized design in (5.11) reduces to

\[
\text{var}_{c-r}(\hat{\tau}_F) = \frac{\gamma_A \gamma_B}{4(N-1)} \left( \frac{W - 1}{W} S_{\text{btw}}^2 + \frac{M - 1}{M} S_{\text{in}}^2 \right) \quad (F \in \mathcal{F}).
\]

**Corollary 4.** For strictly additive potential outcomes, the differences in Corollary 2 reduce to

\[
\text{var}_{s-p}(\hat{\tau}_A) - \text{var}_{c-r}(\hat{\tau}_A) = C_1 (S_{\text{btw}}^2 - S_{\text{in}}^2/M),
\]

\[
\text{var}_{s-p}(\hat{\tau}_B) - \text{var}_{c-r}(\hat{\tau}_B) = \text{var}_{s-p}(\hat{\tau}_{AB}) - \text{var}_{c-r}(\hat{\tau}_{AB}) = -C_2 (S_{\text{btw}}^2 - S_{\text{in}}^2/M),
\]

where \(C_1\) and \(C_2\) are two positive constants.

With the same discriminant \(S_{\text{btw}}^2 - S_{\text{in}}^2/M\) as that in (5.6), the intuition from Remark 3 translates into Corollary 4 with almost no need for change: Assume super-population model (5.7), it follows from \(E^* (S_{\text{btw}}^2 - S_{\text{in}}^2/M) = \sigma_\eta^2 \geq 0\) in formula (5.10) that

\[
E^* \{\text{var}_{s-p}(\hat{\tau}_A)\} \geq E^* \{\text{var}_{c-r}(\hat{\tau}_A)\}, \quad E^* \{\text{var}_{s-p}(\hat{\tau}_B)\} \leq E^* \{\text{var}_{c-r}(\hat{\tau}_B)\}.
\]

The inequalities are strict unless \(\sigma_\eta^2 = 0\), in which case (5.7) degenerates to a simple linear model that admits no random block effects.
5.3. Simplified expressions under balanced designs. Recall $S^2_{F-btw}$ and $S^2_{F-in}$ from (3.3) as the between- and within-block variances of $\tau_{(wm)}-F$. Define analogously

$$S^2_{\mu-btw} = \frac{1}{W} \sum_{w=1}^{W} (\mu_{(w)} - \mu)^2,$$

$$S^2_{\mu-in} = \frac{1}{W} \sum_{w=1}^{W} \left\{ \frac{1}{M} \sum_{m=1}^{M} (\mu_{(wm)} - \mu_{(w)})^2 \right\}$$

for $\mu_{(wm)} = \frac{1}{W} \sum_{k=1}^{4} Y_{(wm)}(k)$, with $\mu_{(w)} = M^{-1} \sum_{m=1}^{M} \mu_{(wm)}$ and $\mu = N^{-1} \sum_{(wm)} \mu_{(wm)}$ being the block and population averages respectively.

**Corollary 5.** Under a balanced $2^2$ split-plot design with $W_1 = W_{+1}$ and $M_1 = M_{+1}$, the sampling variances in Theorem 4 reduce to

$$\text{var}_{s-p}(\hat{\tau}_A) = 4W^{-1}S^2_{\mu-btw} + N^{-1}(S^2_{B-in} + S^2_{AB-in}),$$

$$\text{var}_{s-p}(\hat{\tau}_B) = W^{-1}S^2_{AB-btw} + N^{-1}(4S^2_{\mu-in} + S^2_{A-in}),$$

$$\text{var}_{s-p}(\hat{\tau}_{AB}) = W^{-1}S^2_{B-btw} + N^{-1}(4S^2_{\mu-in} + S^2_{A-in}).$$

Analogous results for balanced complete randomizations follow from letting $N_1 = N_2 = N_3 = N_4$ in (5.3):

$$\text{var}_{c-r}(\hat{\tau}_F) = N^{-1} g_F^T (P_4 \circ S^2) g_F = N^{-1} \sum_{k=1}^{4} S^2(k,k) - N^{-1} S^2_F \quad (F \in \mathcal{F}).$$

This is the exact form of Theorem 2 in Dasgupta, Pillai and Rubin (2015) when the number of factors equals two.

**Corollary 6.** For within-block additive potential outcomes, the sampling variances of $\hat{\tau}_F$ under a balanced $2^2$ split-plot design reduce from Corollary 5 to

$$\text{var}_{s-p}(\hat{\tau}_A) = W^{-1}S^2_{\mu-btw},$$

$$\text{var}_{s-p}(\hat{\tau}_B) = W^{-1}S^2_{AB-btw} + 4N^{-1}S^2_{\mu-in},$$

$$\text{var}_{s-p}(\hat{\tau}_{AB}) = W^{-1}S^2_{B-btw} + 4N^{-1}S^2_{\mu-in}.$$

**Corollary 7.** For between-block additive potential outcomes, the sampling variances of $\hat{\tau}_B$ and $\hat{\tau}_{AB}$ under a balanced $2^2$ split-plot design reduce from Corollary 5 to

$$\text{var}_{s-p}(\hat{\tau}_B) = \text{var}_{s-p}(\hat{\tau}_{AB}) = N^{-1} (4S^2_{\mu-in} + S^2_{A-in}).$$
6. Estimating the sampling variances. The sampling variances by formula (5.4) are in practice unobservable. We address in this section their estimation, and use the results to construct Neymanian interval estimates.

Recall from Definition 4 that the whole-plot randomization assigns \( W_{-1} \) whole-plots to \(-1_A\) level of factor \( A \) and the rest \( W_{+1} \) to \(+1_A\) level. Let

\[
W_{-1} = \{ w : \text{whole-plot } w \text{ is assigned to } -1_A \text{ level} \},
\]

\[
W_{+1} = \{ w : \text{whole-plot } w \text{ is assigned to } +1_A \text{ level} \}.
\]

For each \( w \in W_{-1} \), whole-plot \( w \) ends up with — maybe ‘sees’?? \( M_{-1} \) of its \( M \) sub-plots in treatment arm 1 and the rest \( M_{+1} \) in treatment arm 2. Define for such whole-plots

\[
Y_{(w)}^{\text{obs}}(1) = M_{-1}^{-1} \sum_{m : \text{T}(wm) = 1} Y_{(wm)}^{\text{obs}}, \quad Y_{(w)}^{\text{obs}}(2) = M_{+1}^{-1} \sum_{m : \text{T}(wm) = 2} Y_{(wm)}^{\text{obs}}
\]

as the sample versions of \( Y_{(w)}(1) \) and \( Y_{(w)}(2) \) respectively. Assume \( |W_{-1}| = W_{-1} \geq 2 \),

\[
s_{\text{btw}}^2(k, l) = (W_{-1} - 1)^{-1} \sum_{w \in W_{-1}} \{ Y_{(w)}^{\text{obs}}(k) - \bar{Y}^{\text{obs}}(k) \} \{ Y_{(w)}^{\text{obs}}(l) - \bar{Y}^{\text{obs}}(l) \},
\]

as the covariance of \( Y_{(w)}^{\text{obs}}(k) \) and \( Y_{(w)}^{\text{obs}}(l) \) over \( w \in W_{-1} \), defines a sensible sample version of the between-whole-plot covariance

\[
S_{\text{btw}}^2(k, l) = (W - 1)^{-1} \sum_{w=1}^{W} \{ Y_{(w)}(k) - \bar{Y}(k) \} \{ Y_{(w)}(l) - \bar{Y}(l) \}
\]

for \( k, l = 1, 2 \). Likewise, define

\[
Y_{(w)}^{\text{obs}}(3) = M_{-1}^{-1} \sum_{m : \text{T}(wm) = 3} Y_{(wm)}^{\text{obs}}, \quad Y_{(w)}^{\text{obs}}(4) = M_{+1}^{-1} \sum_{m : \text{T}(wm) = 4} Y_{(wm)}^{\text{obs}}
\]

for each \( w \in W_{+1} \), now that whole-plots in this set end up with \( M_{-1} \) of its \( M \) sub-plots in treatment arm 3 and the rest \( M_{+1} \) in treatment arm 4. The corresponding

\[
s_{\text{btw}}^2(k, l) = (W_{+1} - 1)^{-1} \sum_{w \in W_{+1}} \{ Y_{(w)}^{\text{obs}}(k) - \bar{Y}^{\text{obs}}(k) \} \{ Y_{(w)}^{\text{obs}}(l) - \bar{Y}^{\text{obs}}(l) \}
\]

defines a sensible sample version of \( S_{\text{btw}}^2(k, l) \) for \( k, l \in \{3, 4\} \).
Lemma 8. Under the $2^2$ split-plot design qualified by Definition 4, the sampling expectations of $s_{\text{btw}}^2(k,l)$ satisfy

$$
E \begin{pmatrix}
S_{\text{btw}}^2(1,1) & S_{\text{btw}}^2(1,2) \\
S_{\text{btw}}^2(2,1) & S_{\text{btw}}^2(2,2)
\end{pmatrix}
+ M^{-1} \begin{pmatrix}
R_B & -1 \\
-1 & R_B^{-1}
\end{pmatrix}
\begin{pmatrix}
S_{\text{in}}^2(1,1) & S_{\text{in}}^2(1,2) \\
S_{\text{in}}^2(2,1) & S_{\text{in}}^2(2,2)
\end{pmatrix},
$$

$$
E \begin{pmatrix}
S_{\text{btw}}^2(3,3) & S_{\text{btw}}^2(3,4) \\
S_{\text{btw}}^2(4,3) & S_{\text{btw}}^2(4,4)
\end{pmatrix}
+ M^{-1} \begin{pmatrix}
R_B & -1 \\
-1 & R_B^{-1}
\end{pmatrix}
\begin{pmatrix}
S_{\text{in}}^2(3,3) & S_{\text{in}}^2(3,4) \\
S_{\text{in}}^2(4,3) & S_{\text{in}}^2(4,4)
\end{pmatrix},
$$

As illustrated by Lemma 8, the sampling expectations of $s_{\text{btw}}^2(k,l)$ contain not only their ‘potential outcomes prototypes’ $S_{\text{btw}}^2(k,l)$ but also the within-whole-plot covariances $S_{\text{in}}^2(k,l)$. This renders them ‘self-sufficient’ for estimating the $\text{var}_{s-p}(\hat{\tau}_F)$ in (5.4), requiring no extra help from the not-yet-defined ‘$s_{\text{in}}^2(k,l)$’.

Theorem 5. Under the $2^2$ split-plot design qualified by Definition 4, the sampling variance of $\hat{\tau}_F$ can be conservatively estimated by

$$
\hat{V}_F = 4^{-1} g_F^T W^{-1} \begin{pmatrix}
S_{\text{btw}}^2(1,1) & S_{\text{btw}}^2(1,2) \\
S_{\text{btw}}^2(2,1) & S_{\text{btw}}^2(2,2)
\end{pmatrix}
0
0
W_+^{-1} \begin{pmatrix}
S_{\text{btw}}^2(3,3) & S_{\text{btw}}^2(3,4) \\
S_{\text{btw}}^2(4,3) & S_{\text{btw}}^2(4,4)
\end{pmatrix} g_F
$$

in the sense that

$$
\text{var}_{s-p}(\hat{\tau}_F) - E_{s-p}(\hat{\tau}_F) = -(4W)^{-1} S_{F-\text{btw}}^2 \leq 0.
$$

The last inequality is strict unless the block average factorial effects $\tau_{(w).F}$ are constant across all $w = 1, \ldots, W$, i.e., $S_{F-\text{btw}}^2 = 0$.

Remark 4. Whereas we left $s_{\text{btw}}^2(k,l)$ undefined for treatment pairs that can never be observed together within the same whole-plot, the definition of ‘$s_{\text{in}}^2(k,l)$,’ as some sensible sample version of $S_{\text{in}}^2(k,l)$, could be even more ‘selective.’ In particular, any candidate of the form

$$
\sum \{Y_{(wm)}(k) - Y_{(w)}^{\text{obs}}(k)\} \{Y_{(wm)}(l) - Y_{(w)}^{\text{obs}}(l)\},
$$

would require $Y_{(wm)}(k)$ and $Y_{(wm)}(l)$ to be both observed for at least some sub-plots. This is possible if and only if $k = l \in \{1, 2, 3, 4\}$, leaving the rest $4 \times 4 - 4 = 12$ pairs of $(k,l)$ indefinite.
Let $q_{1-\alpha/2}$ be the 100$(1-\alpha/2)\%$ quantile of standard normal distribution. It follows from the finite population central limit theorem (Hajek, 1960) that interval

$$ (\hat{\tau}_F - q_{1-\alpha/2}\sqrt{\hat{V}_F}, \hat{\tau}_F + q_{1-\alpha/2}\sqrt{\hat{V}_F}}) $$

will cover $\tau_F$ with at least 100$(1-\alpha)\%$ long-run relative frequency as $W$ and $M$ approach infinity. We thus define (6.1) as the 100$(1-\alpha)\%$ Neymanian split-plot interval estimate of $\tau_F$, intending for approximate exact-coverage under between-block or stricter additivity and over-coverage if otherwise at reasonably large $W$ and $M$. This completes the estimation procedure.

7. Based on randomization vs. based on model. Before turning to performance evaluation of the proposed procedure, let us take a brief detour in this section, and discuss some of the key features that set this randomization-based approach apart from existing model-based alternatives.

Recall $g_A = (-1, -1, +1, +1)$, $g_B = (-1, +1, -1, +1)$, and $g_{AB} = g_A \circ g_B$, such that the $k$th entry in $g_F$ equals the level of factor $F$ in treatment $k$. Let $D = 2^{-1}(1_4, g_A, g_B, g_{AB})$ be the design matrix, and let $g_F(k)$ be the $k$th entry in $g_F$. It follows from the orthogonality of $D$ that

$$ Y_{(wm)} = DD^T Y_{(wm)} = D \left\{ 2^{-1}(1_4, g_A, g_B, g_{AB})^T Y_{(wm)} \right\} $n

$$ = D \left\{ 2^{-1}1_4^T Y_{(wm)}, 2^{-1}g_A^T Y_{(wm)}, 2^{-1}g_B^T Y_{(wm)}, 2^{-1}g_{AB}^T Y_{(wm)} \right\}^T $n

$$ = D \left( 2\mu_{(wm)}, \tau_{(wm)-A}, \tau_{(wm)-B}, \tau_{(wm)-AB} \right)^T, $$n

with

$$ Y_{(wm)}(k) = 2^{-1}(1, g_A(k), g_B(k), g_{AB}(k)) \left( 2\mu_{(wm)}, \tau_{(wm)-A}, \tau_{(wm)-B}, \tau_{(wm)-AB} \right)^T $$n

$$ = \mu_{(wm)} + \sum_{F \in F} 2^{-1}g_F(k)\tau_{(wm)-F} $$n

in the $k$th row. Averaging (7.2) over all $(wm)$ yields

$$ \bar{Y}(k) = \mu + \sum_{F \in F} 2^{-1}g_F(k)\tau_F. $$n

Recall that $T_{(wm)} = k$ if sub-plot $(wm)$ is assigned to treatment $k$. We have $Y_{(wm)}^{obs} = Y_{(wm)}(T_{(wm)})$. The derived linear model (Hinkelmann and Kempthorne, 2008) treats the population average $\bar{Y}(T_{(wm)})$ as the part in
$Y_{(wm)}(T_{(wm)})$ explainable by the treatment, and decomposes the observed outcomes as

\begin{equation}
Y_{(wm)}^{\text{obs}} = Y_{(wm)}(T_{(wm)}) = \bar{Y}(T_{(wm)}) + \epsilon_{(wm)} = \mu + \sum_{F \in \mathcal{F}} 2^{-1} g_F(T_{(wm)}) \tau_F + \epsilon_{(wm)},
\end{equation}

where $\epsilon_{(wm)} = Y_{(wm)}^{\text{obs}} - \bar{Y}(T_{(wm)})$ are the unit-level random errors, and the last equality follows from letting $k = T_{(wm)}$ in (7.3). Let

$$\delta_{(wm)-\mu} = \mu_{(wm)} - \mu, \quad \delta_{(wm)-F} = \tau_{(wm)-F} - \tau_F \quad (F \in \mathcal{F})$$

be the deviations of unit-level parameters from the finite-population averages. Plug (7.2), with $k$ set at $T_{(wm)}$, into (7.4) to see

\begin{equation}
\epsilon_{(wm)} = \delta_{(wm)-\mu} + \sum_{F \in \mathcal{F}} 2^{-1} g_F(T_{(wm)}) \delta_{(wm)-F}.
\end{equation}

The $g_F(T_{(wm)})$ in (7.4), despite the compound definition as ‘the level of factor $F$ in the treatment $T_{(wm)}$ received by sub-plot $(wm)$,’ has the straightforward interpretation as the level of factor $F$ received by sub-plot $(wm)$. This, together with the functional form of (7.4), unsurprisingly reminds us of the family of additive regression models:

\begin{equation}
Y_{(wm)}^{\text{obs}} = \beta_0 + \sum_{F \in \mathcal{F}} g_F(T_{(wm)}) \beta_F + \epsilon_{\text{model}}^{(wm)}.
\end{equation}

Despite the apparent resemblance between (7.4) and (7.6), however, their difference is fundamental, with the source of randomness being the first and foremost.

The family of additive regression models (7.6), on the one hand, conditions on the treatment assignments $T_{(wm)}$ for all its inference, and attributes the randomness in $Y_{(wm)}^{\text{obs}}$ to the study population being a random sample of some hypothetical super-population, reflected via $\epsilon_{\text{model}}^{(wm)}$ as the individual sampling errors. The regression coefficients $\beta_F$ are treated as super-population causal parameters, and the linear combinations $\beta_0 + \sum_{F \in \mathcal{F}} g_F(T_{(wm)}) \beta_F$ as deterministic super-population means.

The derived linear model (7.4), on the other hand, conditions on the composition of study population for all its inference, and attributes the randomness in $Y_{(wm)}^{\text{obs}}$ solely to the random assignment of treatments, reflected via the joint distribution of treatment assignment variables $T_{(wm)}$. As a result, not only the residuals $\epsilon_{(wm)}$, but the linear combinations $\mu +$
\[ \sum_{F \in \mathcal{F}} 2^{-1} g_F(T_{(wm)}) \tau_F \] too, are now stochastic via their dependence on \( T_{(wm)} \) (Freedman, 2008a,b,c), with coefficients \( \tau_F \), by definition (3.1), describing the finite study population. See formula (7.5) for a full specification of \( \epsilon_{(wm)} \) in terms of \( g_F(T_{(wm)}) \).

More quantitative comparison follows from the difference in residual covariance structure. Whereas the covariances of the \( \epsilon_{(wm)} \) in (7.4) are in general specified as model assumptions, those of the \( \epsilon_{(wm)} \) in (7.5) follow naturally from identity (7.5) and the joint distribution of \( T_{(wm)} \) as determined by the treatment assignment mechanism.

To start with, viewing (7.5) in conjunction with Lemma 4 renders the computation of \( \text{cov}_{s-p}(\epsilon_{(wm)}, \epsilon_{(w'm')}) \) almost trivial under strict additivity: With \( \delta_{(wm)\cdot F} = 0 \) for all \((wm)\) and \( F \in \{A, B, AB\} \), the residuals in (7.5) reduce to constants \( \epsilon_{(wm)} = \delta_{(wm)\cdot \mu} \), and the covariance of constants is always zero, i.e., \( \text{cov}_{s-p}(\epsilon_{(wm)}, \epsilon_{(w'm')}) = 0 \) for all \((wm)\) and \((w'm')\) under strict additivity.

Without strict additivity, the algebra becomes tedious. To avoid unnecessary complexity, we defer the exact formulas for \( \text{cov}_{s-p}(\epsilon_{(wm)}, \epsilon_{(w'm')}) \) at each finite \((W,M)\) to the online supplementary material, and save Theorem 6 for but the ‘punch line’ in terms of finite-population asymptotics (Hajek, 1960)

\[ \lim_{W,M \to \infty} \text{cov}_{s-p}(W,M,r_A,r_B)(\epsilon_{(wm)}, \epsilon_{(w'm')}). \]

The asymptotic condition ‘\( W, M \to \infty \)’ can be visualized as keeping adding till infinity new whole-plots to the current study population, and new sub-plots to the current whole-plots. The covariance at each finite \((W,M)\) is computed under split-plot design ‘s-p\((W,M,r_A,r_B)\)’ as qualified by Definition 4 with \( W+1 = r_A(r_A+1)^{-1}W \) and \( M+1 = r_B(r_B+1)^{-1}M \).

**Theorem 6.** Fix \( r_A \) and \( r_B \). As \( W \) and \( M \) approach infinity, the residual covariance \( \text{cov}_{s-p}(W,M,r_A,r_B)(\epsilon_{(wm)}, \epsilon_{(w'm')}) \) for sub-plots \((wm)\) and \((w'm')\) in the current study population will converge to

\[ \frac{r_A}{(r_A+1)^2} \left\{ \delta_{(wm)\cdot A} + \left( \frac{r_B - 1}{r_B + 1} \right) \delta_{(wm)\cdot AB} \right\} \left\{ \delta_{(w'm')\cdot A} + \left( \frac{r_B - 1}{r_B + 1} \right) \delta_{(w'm')\cdot AB} \right\} \]

if the two are in the same whole-plot, and to zero if they are not.

**Corollary 8.** When the design series ‘s-p\((W,M,r_A,r_B)\)’ is balanced, i.e., \( r_A = r_B = 1 \), the asymptotic residual covariance in Theorem 6 reduces to \( 4^{-1} \delta_{(wm)\cdot A} \delta_{(w'm')\cdot A} \) for sub-plots \((wm)\) and \((w'm')\) in the same whole-plot.
Theorem 6 and Corollary 8 provide an explicit account of the non-vanishing within-whole-plot correlation of $\epsilon_{(wm)}$ under $2^2$ split-plot designs (Freedman, 2008a; Lin, 2013), and thereby justify heuristically the block-diagonal covariance structure that a linear mixed effects (LME) model assumes for its sampling errors. With

$$\epsilon_{(wm)}^{LME} = \eta_w + \xi_{(wm)}$$

where $\eta_w \sim \mathcal{N}(0, \sigma_{\eta}^2)$ and $\xi_{(wm)} \sim \mathcal{N}(0, \sigma_{\xi}^2)$ are jointly independent, the covariance of $\epsilon_{(wm)}^{LME}$ and $\epsilon_{(w'm')}^{LME}$ equals $\sigma_{\eta}^2$ if $w = w'$, and 0 if otherwise. Despite the ‘qualitative’ similarity in structure, two salient quantitative differences remain:

- First, whereas the linear mixed effects model assumes equal covariances for all pairs of residuals from the same whole-plot, those under the derived linear model, as is clear from Theorem 6 and Corollary 8, vary from pair to pair even in the asymptotics.

- Second, whereas the linear mixed effects model assumes independence between whole-plots at any finite $(W, M)$, formula (7.5) suggests otherwise for the derived model. Intuitively, $\epsilon_{(wm)}$ and $\epsilon_{(w'm')}^{LME}$ from two different whole-plots are correlated at any finite $W$ via their respective dependence on $T_{(wm)}$ and $T_{(w'm')}$ and the mutual dependence between $T_{(wm)}$ and $T_{(w'm')}$ — given that knowing whole-plot $w$ lowers the probability of whole-plot $w'$ to receive the same level, the two assignment variables $T_{(wm)}$ and $T_{(w'm')}$ are mutually correlated even if $w \neq w'$. See the online supplementary material for exact formulas for $\text{cov}_{s,p}(\epsilon_{(wm)}, \epsilon_{(w'm')})$ at each finite $(W, M)$.

8. Simulations. We evaluate in this section the frequency coverage property of the proposed Neymanian split-plot interval estimates via simulation.

8.1. Generative models for the POMs. Refer to the condition of all four $Y(k) = (Y_{(11)}(k), \ldots, Y_{(WM)}(k))$ being blockwise constant — i.e., $Y_{(wm)}(k) = Y_{(w)}(k)$ for all $w, m$, and $k$ — as ‘ultimate block effect.’ We consider here five types of potential outcomes:

(i) binary potential outcomes without block effect,
(ii) binary potential outcomes with ultimate block effect,
(iii) continuous potential outcomes without block effect,
(iv) continuous potential outcomes with block effect,
(v) continuous potential outcomes with ultimate block effect
in combination with three types of additivity assumption:

(i) strict additivity,
(ii) between-block additivity,
(iii) no assumption about additivity.

This gives a total of $5 \times 3 = 15$ types of POM, from which specific POMs are generated in two steps:

1. Generate $Y(1)$ according to the designated potential outcomes type. See Table 2 for details about the generative models.
2. Conditional on $Y(1)$, generate $Y(k) \ (k = 2, 3, 4)$ according to the designated additivity type. See Table 3 for details about the generative models.

Strict additivity for all five potential outcomes types is imposed by letting $Y(k) = Y(1) \ (k = 2, 3, 4)$, and between-block additivity by letting

\begin{equation}
Y_{(w)}(k) = Y_{(w)}(1) \ (k = 2, 3, 4; w = 1, \ldots, W),
\end{equation}

such that the resulting POMs satisfy Definitions 1 and 2 respectively with all differential constants being zero. No generality is lost so far as the coverage rate is concerned.

8.2. Interval estimates and their coverage rates. For each realized POM, coverage rates of the proposed Neymanian split-plot interval estimates are summarized over 1,000 independent split-plot randomizations and compared to those of the following three alternatives:

- **GLM interval estimates.**
  The $100(1 - \alpha)$% confidence intervals under the standard generalized linear model (GLM) with the levels of factors $A$ and $B$ and their interaction as explanatory variables.

- **GLME interval estimates.**
  The $100(1 - \alpha)$% confidence intervals under the standard generalized linear mixed effects model (GLME) that includes also whole-plot dummy, in addition to the levels of factors $A$ and $B$ and their interaction, as explanatory variable.

- **C-R interval estimates.**
  The $100(1 - \alpha)$% Neymanian interval estimates for $2^2$ completely randomized (C-R) design proposed by Dasgupta, Pillai and Rubin (2015).
All GLMs are fitted by the standard R function ‘glm,’ and all GLMEs by ‘glmer,’ both with ‘binomial’ link for binary potential outcomes types (i)–(ii) and ‘identity’ link for continuous potential outcomes types (iii)–(v). We abbreviate ‘GLM’ to ‘LM,’ and ‘GLME’ to ‘LME’ in the latter case, inasmuch as the identity link reduces the two generalized models to linear and linear mixed effects models, respectively.

8.3. Results. We realize each of the 15 POM types at two sizes: \((W, M) = (40, 40)\) and \((80, 80)\), and construct the intervals at confidence level \(\alpha = 0.05\). Results for the 15 POMs at \((W, M) = (40, 40)\) are shown in Figure 4; the overall superiority of s-p interval is evident. Results at \((W, M) = (80, 80)\) exhibit quite similar patterns, and are thus not included here to avoid redundancy.

The intended ‘approximate exact-coverage under between-block or stricter additivity and over-coverage if otherwise’ is fulfilled by the proposed s-p interval for all but potential outcomes types (ii) and (v) under strict additivity. Despite its undue conservativeness towards \(\tau_B\) and \(\tau_{AB}\) in these two cases, the proposed s-p interval remains to be the only interval that ‘does not under-cover’ — see Table 4 for the untruncated statistics regarding the severe under-coverage of \(\tau_A\) by LM and LME intervals. The fact that \(\hat{\tau}_B\) and \(\hat{\tau}_{AB}\) in these two cases are virtually constant at their respective true values \(\tau_B\) and \(\tau_{AB}\) over all possible assignments, as a result of the ultimate block effect, may render even s-p’s undue conservativeness excusable.

For potential outcomes type (iv) in particular, s-p markedly outperforms LM (C-R) in all three factorial effects, matches LME in the main effect of whole-plot factor A, and beats the latter in all other cases. The fact of potential outcomes type (iv) being actually generated from LME model accentuates s-p’s victory even further.

The general inadequacy of C-R, LM, and GLM intervals for potential outcomes types (ii), (iv), and (v), on the other hand, exemplifies the possible severe under-coverage when split-plot experiments are wrongfully analyzed as completely randomized ones, even when the preferred randomization-based perspective is adopted.

9. Discussion. Randomization-based causal inference, originally developed by Neyman (1923) and Neyman (1935) in the context of completely randomized, randomized block, and Latin square designs, (a) attributes the randomness in experimental data to the actual physical randomization of the experiments, (b) allows for the definition of causal effects over a finite population of interest, and (c) extends the super-population notions of ‘unbiased’ point estimates and ‘conservative’ interval estimates to the finite-
Fig 4. Coverage rates summarized over 1,000 independent split-plot randomizations with \( r_A = r_B = 1 \) at \( (W,M) = (40,40) \) (\( \alpha = 0.05 \)). All bars start from the nominal coverage rate 0.95 and grow upwards/downwards to the actual values, truncated at 0.85. Results of c-r and lm are combined for potential outcomes (PO) types (III)–(V), since the procedure by which Dasgupta, Pillai and Rubin (2015) constructed the c-r renders it numerically identical to the lm.

| PO Type | Strict Additivity | Between-Block Additivity | No Assumption about Additivity |
|---------|------------------|--------------------------|-------------------------------|
| (I)    |                  |                           |                               |
| (II)   |                  |                           |                               |
| (III)  |                  |                           |                               |
| (IV)   |                  |                           |                               |
| (V)    |                  |                           |                               |
Table 2

| PO Type | Generative Model for $Y(1)$ under potential outcomes (PO) types (I)–(V). |
|---------|-----------------------------------------------------------------------------------------------------|
| (I)     | $Y_{(wm)}(1) \overset{iid}{\sim} \text{Bern}(0.5)$.                                                   |
| (II)    | $Y(w)(1) \overset{iid}{\sim} \text{Bern}(0.5)$, and $Y_{(wm)}(1) = Y(w)(1)$.                      |
| (III)   | $Y_{(wm)}(1)$ are independent normals with means $\mu_{(wm)} = 2(-1)^{\lfloor m \leq M/2 \rfloor}$ and |
|         | variances $(\sigma^2_{(11)}, \ldots, \sigma^2_{(WM)})$ being a random permutation of $2(1_{N/2}^T, 0_{N/2}^T)$. |
| (IV)    | $Y_{(wm)}(1) = \eta_w + \epsilon_{(wm)}$, where $\eta_w$ and $\epsilon_{(wm)}$ are iid standard normals. |
| (V)     | $Y(w)(1) \overset{iid}{\sim} \mathcal{N}(0, 1)$, and $Y_{(wm)}(1) = Y(w)(1)$.                      |

population settings. Under this inferential framework, we proposed a new procedure for analyzing $2^2$ split-plot designs, and demonstrated its superior frequency coverage property over existing model-based alternatives.

Whereas the length limit restrains us from going any further, the interested reader may find the following three directions, among others, worthy of future exploration. First, Rubin (1978) and Dasgupta, Pillai and Rubin (2015) discussed Bayesian causal inference for completely randomized designs in the context of treatment-control and $2^K$ factorial experiments respectively. How to extend the same framework to split-plot designs in a way that also guarantees frequency properties is yet unclear. Second, Fisher (1935) proposed the use of randomization test for sharp null hypotheses regarding the treatment effects at unit level. Extension of such framework to split-plot designs should complement the current Neymanian framework’s focus on the population-level parameters. Third, extension of the current results to multi-level, multi-factor, or other more complex forms of split-plot designs, as documented in Federer and King (2007), would be of both theoretical and practical interest.

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### Table 3

Generative models for $Y(k)$ ($k = 2, 3, 4$) under the 15 pom types as combinations of the five potential outcomes (PO) types (I)–(V) in Table 2, and the three additivity (ADT) types: (i) strict, (ii) between-block, and (iii) no assumption about additivity.

| ADT Type | PO Type | Generative Model for $Y(k)$ ($k = 2, 3, 4$) |
|----------|---------|------------------------------------------|
| (i)      | (I)–(V)| $Y(k) = Y(1)$. |
|          | (I)     | $Y(k)$ are independent blockwise permutations of $Y(1)$, such that the numbers of 1's within each block are the same for $Y(k)$ and $Y(1)$. This ensures (8.1). |
| (ii)     | (II), (V) | $Y(k) = Y(1)$. Under ultimate block effect, we have $Y_{(wm)}(1) = Y_{(w)}(1)$ and $Y_{(wm)}(k) = Y_{(w)}(k)$; (8.1) holds if and only if $Y_{(wm)}(k) = Y_{(w)}(1)$. |
|          | (III), (IV) | $Y_{(wm)}(k) = Y_{(wm)}'(k) - \{Y_{(w)}'(k) - Y_{(w)}(1)\}$, where $Y'(k)$ are iid as $Y(1)$. Subtracting $Y_{(w)}'(k) - Y_{(w)}(1)$ ensures (8.1). |
| (iii)    | (I)–(V) | $Y(k)$ are iid as $Y(1)$. |

### Table 4

Coverage rates (%) averaged over 1,000 independent split-plot randomizations with $r_A = r_B = 1$ at $(W,M) = (40,40)$ for potential outcomes (PO) types (II) and (V).

| PO Type (II) | S-P | C-R | GLM | GLME  |
|--------------|-----|-----|-----|-------|
| $\tau_A$    | 95.0| 0.0 | 0.0 | 32.5  |
| $\tau_B$    | 100.0| 100.0| 100.0| 100.0 |
| $\tau_{AB}$ | 100.0| 100.0| 100.0| 100.0 |

| PO Type (V)  | S-P | LM (C-R) | LME  |
|--------------|-----|-----------|------|
| $\tau_A$    | 99.3| 33.9 | 34.4 | 84.3  |
| $\tau_B$    | 99.7| 42.3 | 44.0 | 33.2  |
| $\tau_{AB}$ | 98.7| 36.8 | 47.3 | 26.2  |
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SUPPLEMENTARY MATERIAL

APPENDIX A: MATRIX ALGEBRA

For any $p$-dimensional vector $a = (a_1, \ldots, a_p)^T$, let $\text{diag}\{a\} = \text{diag}\{a_1, \ldots, a_p\}$ denote the $p \times p$ diagonal matrix with $a_i$ as the $i$th diagonal entry. For any arbitrary matrices (including vectors and scalars) $A_1, \ldots, A_q$, let

$$B\text{diag}\{A_1, A_2, \ldots, A_q\} = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_q \end{pmatrix}$$

denote the block-diagonal matrix with $A_i$ as the $i$th diagonal block.

Before proceeding to the formal proofs, let us first establish some properties of the Kronecker product ($\otimes$) and the entrywise product ($\circ$) that will be invoked repeatedly throughout this appendix.

**Lemma A.1.**

1. For any vectors $a, b \in \mathbb{R}^L$, and matrix $Q \in \mathbb{R}^{L \times L}$,

$$\text{(A.1)} \quad (ab^T) \circ Q = \text{diag}\{a\}Q\text{diag}\{b\}.$$  

2. For any random vectors $X_1$ and $X_2$, and constant vectors $a$ and $b$,

$$\text{(A.2)} \quad \text{cov}(X_1 \otimes a, X_2 \otimes b) = \text{cov}(X_1, X_2) \otimes (ab^T).$$

3. Let $\prod$ denote the usual matrix product. For any matrices $A_i$ and $B_i$ ($i = 1, \ldots, n$) such that $\prod_{i=1}^n (A_i \otimes B_i), \prod_{i=1}^n A_i$, and $\prod_{i=1}^n B_i$ are all well-defined,

$$\text{(A.3)} \quad \prod_{i=1}^n (A_i \otimes B_i) = \left( \prod_{i=1}^n A_i \right) \otimes \left( \prod_{i=1}^n B_i \right).$$

4. Given $X_1, \ldots, X_K \in \mathbb{R}^N$, let $X = (X_1, \ldots, X_K)$ be the $N \times K$ matrix with $X_i$ as the $i$th column, and $\tilde{X} = B\text{diag}\{X_1, \ldots, X_K\}$ be the $(NK) \times K$ block-diagonal matrix with $X_i$ as the $i$th diagonal block. For any $K \times K$ matrix $A$ and $N \times N$ matrix $B$,

$$\text{(A.4)} \quad \tilde{X}^T (A \otimes B) \tilde{X} = A \circ \{X^T BX\}.$$
APPENDIX B: ALGEBRAIC PROPERTIES OF THE SCIENCE

Proof of Lemma 1. On the one hand, strict additivity implies the existence of $c_k$ such that $Y(k) = Y(1) + c_k 1_N (k = 2, \ldots, K)$. We have $Y_i(k) = Y_i(1) + c_k$, $\bar{Y}(k) = \bar{Y}(1) + c_k$, and $Y_i(k) - \bar{Y}(k) = Y_i(1) - \bar{Y}(1)$. This, coupled with the definition of $S^2(k, l)$, proves $S^2(k, l) = S^2(1, 1)$ for all $(k, l)$. On the other hand, given $S^2 = S^2_{btw} J_4$, we have

$$||P_N \{Y(k) - Y(l)\}||^2$$

$$= \{Y(k) - Y(l)\}^T P_N \{Y(k) - Y(l)\}$$

$$= (N-1)(S_{k,k}^2 + S_{l,l}^2 - 2S_{k,l}^2) = (N-1)(S_0^2 + S_0^2 - 2S_0^2) = 0$$

for any $(k, l) \in \{1, \ldots, K\}^2$. Therefore, $P_N \{Y(k) - Y(l)\} = 0_N$, and the difference

$Y(k) - Y(l) = P_N \{Y(k) - Y(l)\} + 1_N \{\bar{Y}(k) - \bar{Y}(l)\} = 1_N \{\bar{Y}(k) - \bar{Y}(l)\}$

equals $\bar{Y}(k) - \bar{Y}(l)$ in all dimensions. This completes the proof. \hfill \Box

Proof of Lemma 3. The equivalence follows immediately from the definitions of $S^2_{btw}$ and $S^2_{in}$. \hfill \Box

Proof of Lemma 4. On the one hand, under strict additivity, there exists some constants $c_2, c_3, c_4$ such that $Y_i(2) = Y_i(1) + c_2, Y_i(3) = Y_i(1) + c_3, Y_i(4) = Y_i(1) + c_4$ for all $i \in \{1, \ldots, N\}$, and we can write $Y_i$ as

$Y_i = (Y_i(1), Y_i(2), Y_i(3), Y_i(4))^T = Y_i(1) 1_4 + (0, c_2, c_3, c_4)^T$.

Thus, for any $F \in \mathcal{F}$,

$\tau_{i-F} = 2^{-1} g_F^T Y_i = 2^{-1} g_F^T \{Y_i(1) 1_4 + (0, c_2, c_3, c_4)^T\}$

$$= 2^{-1} Y_i(1) g_F^T 1_4 + 2^{-1} g_F^T (0, c_2, c_3, c_4)^T = 2^{-1} g_F^T (0, c_2, c_3, c_4)^T$$

are constant for all $i \in \{1, \ldots, N\}$. This proves the necessity of the condition.

On the other hand, given $\tau_{i-F}$ being constant across all units ($F \in \mathcal{F}$), it follows from (7.1) in the main text that

\begin{align}
(B.1) \quad Y_i &= D(2\mu_i, \tau_{i-A}, \tau_{i-B}, \tau_{i-AB})^T = D(2\mu_i, \tau_A, \tau_B, \tau_{AB})^T, \\
Y &= (2\mu, \tau_A 1_N, \tau_B 1_N, \tau_{AB} 1_N) D^T.
\end{align}
 Neymanian Causal Inference for \(2^2\) Split-Plot Designs

We have

\[
P_N Y = P_N(2\mu, \tau_A 1_N, \tau_B 1_N, \tau_{AB} 1_N) D^T
= (2P_N\mu, \tau_A P_N 1_N, \tau_B P_N 1_N, \tau_{AB} P_N 1_N) D^T
= (2P_N\mu, 0_N, 0_N, 0_N) D^T = (2P_N\mu, 0_N, 0_N, 0_N)(1_4, g_A, g_B, g_{AB})^T
= 2P_N\mu 1_4^T,
\]

\[
S^2 = (N - 1)^{-1} Y^T P_N Y = (N - 1)^{-1}(P_N Y)^T (P_N Y)
= (N - 1)^{-1}(2P_N\mu 1_4^T)^T(2P_N\mu 1_4^T) = 4(N - 1)^{-1} 1_4(\mu^T P_N\mu)1_4^T
= \frac{4(\mu^T P_N\mu)}{N - 1} 1_4 1_4^T = \frac{4(\mu^T P_N\mu)}{N - 1} J_4.
\]

By Lemma 1, this implies strict additivity with \(S_0^2 = 4(N - 1)^{-1} \mu^T P_N\mu\).

**Proof of Lemma 5.** Treating block \(w\) as unit \(i\) in Lemma 4 proves the between-block part, whereas the within-block part follows straightforwardly from

\[
Y_{(wm)} = D(2\mu_{(wm)}, \tau_{(wm) - A}, \tau_{(wm) - B}, \tau_{(wm) - AB})^T
= D(2\mu_{(wm)}, \tau_{(w) - A}, \tau_{(w) - B}, \tau_{(w) - AB})^T
\]
as a modification of (B.1).

**APPENDIX C: SAMPLING MOMENTS OF THE ASSIGNMENT VECTORS**

**Lemma C.2.** For a completely randomized design with \(N\) experimental units, \(K\) different treatments, and planned treatment arm sizes \(N_k\) with \(\sum_{k=1}^{K} N_k = N\), the treatment assignment vectors \(Z(k) = (I_{\{T_1 = k\}}, \ldots, I_{\{T_N = k\}})^T\) satisfy

\[
E_{C-R}\{Z(k)\} = \frac{N_k}{N} 1_N, \quad \text{cov}_{C-R}\{Z(k)\} = \frac{N_k(N - N_k)}{N(N - 1)} P_N.
\]

**Proof.** For any given unit \(i\), the probability of it receiving treatment \(k\) equals \(pr_{C-R}(T_i = k) = N_k/N\). The indicator \(I_{\{T_i = k\}}\) thus follows a Bernoulli\((N_k/N)\) distribution with \(E_{C-R}(I_{\{T_i = k\}}) = N_k/N\) — from which follows immediately the first equality in (C.1) — and

\[
\text{var}_{C-R}(I_{\{T_i = k\}}) = \frac{N_k}{N} \left(1 - \frac{N_k}{N}\right).
\]
The covariances of any two dimensions \( i \) and \( j \) \((i \neq j)\) in \( Z(k) \) satisfy

\[
\text{cov}_{c-r}(I_{\{T_i = k\}}, I_{\{T_j = k\}}) = E_{c-r}(I_{\{T_i = k\}} I_{\{T_j = k\}}) - E_{c-r}(I_{\{T_i = k\}}) E_{c-r}(I_{\{T_j = k\}}) = \text{pr}_{c-r}(T_i = T_j = k) - \frac{N_k^2}{N^2} = \frac{N_k(N_k - 1)}{N(N - 1)} - \frac{N_k^2}{N^2}.
\]

Given the right-hand side of (C.2) satisfies

\[
\frac{N_k}{N} \left( 1 - \frac{N_k}{N} \right) = \frac{N_k(N - N_k)}{N^2} = \frac{N_k(N - N_k)}{N(N - 1)} - \frac{N_k(N - N_k)}{N^2(N - 1)},
\]

organizing (C.2) and (C.3) into variance-covariance matrix form yields

\[
\text{cov}_{c-r} \{Z(k)\} = \frac{N_k(N - N_k)}{N(N - 1)} I_N - \frac{N_k(N - N_k)}{N^2(N - 1)} J_N = \frac{N_k(N - N_k)}{N(N - 1)} P_N.
\]

This completes the proof. \( \square \)

**Proof of Lemma 6.** Given any two treatments \( k \) and \( l \) \((k \neq l)\), the covariance of the entries of \( Z(k) \) and \( Z(l) \) can be computed as

\[
\text{cov}_{c-r}(I_{\{T_i = k\}}, I_{\{T_j = l\}}) = E_{c-r}(I_{\{T_i = k\}} I_{\{T_j = l\}}) - E_{c-r}(I_{\{T_i = k\}}) E_{c-r}(I_{\{T_j = l\}}) = \begin{cases} -N_k N_l / N^2, & \text{if } i = j, \\ N_k N_l / \{N^2(N - 1)\}, & \text{if } i \neq j. \end{cases}
\]

The variance-covariance matrix of \( Z(k) \) and \( Z(l) \) is thus

\[
\text{cov}_{c-r}\{Z(k), Z(l)\} = -\frac{N_k N_l}{N(N - 1)} P_N \quad (k \neq l).
\]

This, together with (C.1), yields

\[
\text{cov}_{c-r}\{N_k^{-1} Z(k), N_k^{-1} Z(k)\} = \frac{1}{N(N - 1)} \left( \frac{N}{N_k} - 1 \right) P_N,
\]

\[
\text{cov}_{c-r}\{N_k^{-1} Z(k), N_l^{-1} Z(l)\} = -\frac{1}{N(N - 1)} P_N \quad (k \neq l)
\]
and
\[
\text{cov}_{C-R} (Z^*) = \text{cov}_{C-R} \left\{ (N_1^{-1} Z(1)^T, N_2^{-1} Z(2)^T, N_3^{-1} Z(3)^T, N_4^{-1} Z(4)) \right\}
\]
\[
= \frac{1}{N(N-1)} \begin{pmatrix}
\frac{N}{N_1} - 1 & -1 & -1 & -1 \\
-1 & \frac{N}{N_2} - 1 & -1 & -1 \\
-1 & -1 & \frac{N}{N_3} - 1 & -1 \\
-1 & -1 & -1 & \frac{N}{N_4} - 1 \\
\end{pmatrix} \otimes P_N
\]
\[
= \frac{1}{N(N-1)} \left( \text{diag} \left\{ \frac{N}{N_1}, \frac{N}{N_2}, \frac{N}{N_3}, \frac{N}{N_4} \right\} - J_4 \right) \otimes P_N.
\]

This completes the proof. \( \square \)

Under the \( 2^2 \) split-plot design qualified by Definition 4, let \( A_w \) be the level of factor \( A \) for whole-plot \( w \) in the whole-plot randomization, and let \( B_{(wm)} \) be the level of factor \( B \) for sub-plot \( (wm) \) in the sub-plot randomization. Recall from the main text that \( T_{(wm)} = k \) if sub-plot \( (w,m) \) receives treatment \( k \), and that \( g_A(T_{(wm)}) \) and \( g_B(T_{(wm)}) \) indicate the levels of factors \( A \) and \( B \) in treatment \( T_{(wm)} \) respectively. We have

(C.4) \[
A_w = g_A(T_{(wm)}), \quad B_{(wm)} = g_B(T_{(wm)}).
\]

For \( z \in \{-1,+1\} \), define \( Z_A(z) = (I_{(A_1=z)}, \ldots, I_{(A_W=z)})^T \in \{0,1\}^W \) and \( Z_B(z) = (I_{(B_{11} = z)}, \ldots, I_{(B_{W,M} = z)})^T \in \{0,1\}^N \) as the factorial analogues of \( Z(k) \) for the whole-plot and sub-plot randomizations respectively.

For treatment \( k \) with \( g_A(k) \in \{-1,+1\} \) level of factor \( A \) and \( g_B(k) \in \{-1,+1\} \) level of factor \( B \), introduce shorthand notations \( Z_A(k) = Z_A \{ g_A(k) \} \) to indicate the whole-plots that receive \( g_A(k) \) level of factor \( A \), and \( Z_B(k) = Z_B \{ g_B(k) \} \) to indicate the sub-plots that receive \( g_B(k) \) level of factor \( B \). Further define

- \( W(1) = W(2) = W(-1), W(3) = W(4) = W(+1) \) such that \( W(k) \) indicates the total number of whole-plots in which treatment \( k \) will be observed, and
- \( M(1) = M(3) = M(-1), M(2) = M(4) = M(+1) \) such that, for each whole-plot that receives \( g_A(k) \) level of factor \( A \) in the whole-plot randomization, \( M(k) \) of its \( M \) sub-plots end up in treatment arm \( k \).

The following lemma gives the covariance structures of \( Z_A(k) \) and \( Z_B(k) \) as a central building block for the proof of Theorem 2.

**Lemma C.3.** \( Z(k) \) can be expressed as

(C.5) \[
Z(k) = [Z_A(k) \otimes I_M] \circ Z_B(k) = [\text{diag} \{ Z_A(k) \} \otimes I_M] Z_B(k),
\]
where \( \{ Z_A(k) \}_{k=1}^4 \) and \( \{ Z_B(k) \}_{k=1}^4 \) are mutually independent with expectations and covariances

(C.6) \[
\text{cov}_{s-p} \{ Z_A(k), Z_A(l) \} = g_A(k)g_A(l) \frac{W_{+1}W_{-1}}{W(W-1)} P_W,
\]

(C.7) \[
E_{s-p} \{ Z_B(k) \} = \frac{M(k)}{M} 1_N,
\]

(C.8) \[
\text{cov}_{s-p} \{ Z_B(k), Z_B(l) \} = g_B(k)g_B(l) \frac{M_{+1}M_{-1}}{M(M-1)} P_{in}.
\]

**Proof of Lemma C.3.** It follows from identities

\[
I_{T_{(w_m)=k}} = I_{\{A_w=g_A(k)\}} I_{\{B_{(w_m)}=g_B(k)\}} \quad (w = 1, \ldots, W; m = 1, \ldots, M)
\]

that

\[
Z(k) = (I_{\{T_{(1)}=k\}}, \ldots, I_{\{T_{(W)}=k\}})^T
\]

\[
= (I_{\{A_1=g_A(k)\}} I_{M_1}, \ldots, I_{\{A_W=g_A(k)\}} I_{M_1})^T \circ (I_{\{B_{(1)}=g_B(k)\}}, \ldots, I_{\{B_{(W)}=g_B(k)\}})^T
\]

\[
= \{ Z_A(g_A(k)) \otimes 1_M \} \circ Z_B(g_B(k)) = \{ Z_A(k) \otimes 1_M \} \circ Z_B(k)
\]

\[
= [\text{diag} \{ Z_A(k) \} \otimes 1_M] Z_B(k)
\]

This proves (C.5).

Applying Lemma C.2 to the whole-plot randomization yields

\[
E_{s-p} \{ Z_A(k) \} = \frac{W(k)}{W} 1_W, \quad \text{cov}_{s-p} \{ Z_A(k) \} = \frac{W_{+1}W_{-1}}{W(W-1)} P_W,
\]

and it follows immediately from \( Z_A(-1) = 1_W - Z_A(+1) \) that

\[
\text{cov}_{s-p} \{ Z_A(-1), Z_A(+1) \} = -\text{cov}_{s-p} \{ Z_A(+1) \} = - \frac{W_{+1}W_{-1}}{W(W-1)} P_W.
\]

As a result, we have

\[
\text{cov}_{s-p} \{ Z_A(k), Z_A(l) \} = \text{cov}_{s-p} \{ Z_A(g_A(k)), Z_A(g_A(l)) \}
\]

\[
= (-1)^I_{\{g_A(k) \neq g_A(l)\}} \frac{W_{+1}W_{-1}}{W(W-1)} P_W = g_A(k)g_A(l) \frac{W_{+1}W_{-1}}{W(W-1)},
\]

where the last equality holds because \( g_A(k) \neq g_A(l) \) implies \( \{ g_A(k), g_A(l) \} = \{-1, +1\} \). This proves (C.6).

Last, but not least, to better understand the covariance structure of \( Z_B(k) \), let us introduce \( Z_{B,(w)}(k) = (I_{\{B_{(w_1)}=g_B(k)\}}, \ldots, I_{\{B_{(w_3)}=g_B(k)\}})^T \) as the \( M \)-dimensional sub-vector of \( Z_B(k) \) that corresponds to whole-plot \( w \).
For any fixed \( k \in \{1, 2, 3, 4\} \), the sub-plot randomization mechanism renders \( Z_{B,(w)}(k) \ (w = 1, \ldots, W) \) iid with
\[
E_{s-p}\{Z_{B,(w)}(k)\} = \frac{M(k)}{M} 1_M, \quad \text{cov}_{s-p}\{Z_{B,(w)}(k)\} = \frac{M+1M-1}{M(M-1)} P_M
\]
as follows from Lemma C.2. The expectation and covariance of \( Z_B(k) \) can thus be computed block by block as
\[
E_{s-p}\{Z_B(k)\} = (E_{s-p}\{Z_{B,(1)}(k)\}^T, \ldots, E_{s-p}\{Z_{B,(W)}(k)\}^T)^T = \frac{M(k)}{M} 1_N,
\]
which proves (C.7), and
\[
\text{cov}_{s-p}\{Z_B(k)\} = \text{Bdiag}[\text{cov}_{s-p}\{Z_{B,(1)}(k)\}, \ldots, \text{cov}_{s-p}\{Z_{B,(W)}(k)\}]
= I_W \otimes \text{cov}_{s-p}\{Z_{B,(1)}(k)\} = \frac{M+1M-1}{M(M-1)} I_W \otimes P_M = \frac{M+1M-1}{M(M-1)} P_{in}.
\]
Thus,
\[
\text{cov}_{s-p}\{Z_B(-1)\} = \text{cov}_{s-p}\{Z_B(+1)\} = \frac{M+1M-1}{M(M-1)} P_{in},
\]
and it follows immediately from identity \( Z_B(+1) = 1_N - Z_B(-1) \) that
\[
\text{cov}_{s-p}\{Z_B(-1), Z_B(+1)\} = - \frac{M+1M-1}{M(M-1)} P_{in}.
\]
Finally, the fact that \( g_B(k)g_B(l) \) equals 1 if \( g_B(k) = g_B(l) \) and equals \(-1\) if \( g_B(k) \neq g_B(l) \) allows us to unify (C.9) and (C.10) into one formula as
\[
\text{cov}_{s-p}\{Z_B(k), Z_B(l)\} = \text{cov}_{s-p}[Z_B\{g_B(k)\}, Z_B\{g_B(l)\}]
= g_B(k)g_B(l) \frac{M+1M-1}{M(M-1)} P_{in},
\]
which proves (C.8). This completes the proof of Lemma C.3. \( \square \)

**Proof of Theorem 2.** We again approach the mean and variance-covariance matrix of \( Z^* \) from those of the \( Z(k) \ (k = 1, 2, 3, 4) \).

In particular, let \( Z_A = \{Z_A(k)\}_{k=1}^4 \). The law of iterated expectations allows us to decompose the covariance of \( Z(k) \) and \( Z(l) \) into
\[
\text{cov}_{s-p}\{Z(k), Z(l)\} = \text{cov}_{s-p}\{E_{s-p}\{Z(k) \mid Z_A\}, E_{s-p}\{Z(l) \mid Z_A\}\}
+ E_{s-p}\{\text{cov}_{s-p}\{Z(k), Z(l) \mid Z_A\}\}.
\]
Refer to the two components on the right as the covariance of expectations and the expectation of covariance, respectively. Given

\[ E_{s,p} \{ Z(k) | Z_A \} = E_{s,p} \{ \{ Z_A(k) \otimes 1_M \} \circ Z_B(k) | Z_A \} \]

\[ = \{ Z_A(k) \otimes 1_M \} \circ E_{s,p} \{ Z_B(k) | Z_A \} \]

\[ = \{ Z_A(k) \otimes 1_M \} \circ E_{s,p} \{ Z_B(k) \} \]

\[ = \{ Z_A(k) \otimes 1_M \} \circ \left( \frac{M(k)}{M} 1_N \right) = \frac{M(k)}{M} \{ Z_A(k) \otimes 1_M \}, \]

we have

\[ \text{cov}_{s,p} \{ E_{s,p} \{ Z(k) | Z_A \}, E_{s,p} \{ Z(l) | Z_A \} \} \]

\[ = \frac{M(k) M(l)}{M^2} \text{cov}_{s,p} \{ Z_A(k) \otimes 1_M, Z_A(l) \otimes 1_M \} \]

\[ = \frac{M(k) M(l)}{M^2} \text{cov}_{s,p} \{ Z_A(k), Z_A(l) \} \otimes J_M \]

\[ = \frac{M(k) M(l)}{M^2} g_A(k) g_A(l) \frac{W+1 W-1}{W(W-1)} P \otimes J_M \]

\[ = g_A(k) g_A(l) \frac{W+1 W-1 M(k) M(l)}{N(W-1)} P_{btw}. \]

This gives the covariance of expectations component of (C.11). Likewise, given

\[ \text{cov}_{s,p} \{ Z(k), Z(l) | Z_A \} \]

\[ = \text{cov}_{s,p} \{ [\text{diag} \{ Z_A(k) \} \otimes I_M] Z_B(k), [\text{diag} \{ Z_A(l) \} \otimes I_M] Z_B(l) | Z_A \} \]

\[ = [\text{diag} \{ Z_A(k) \} \otimes I_M] \text{cov}_{s,p} \{ Z_B(k), Z_B(l) | Z_A \} [\text{diag} \{ Z_A(l) \} \otimes I_M] \]

\[ = [\text{diag} \{ Z_A(k) \} \otimes I_M] \text{cov}_{s,p} \{ Z_B(k), Z_B(l) \} [\text{diag} \{ Z_A(l) \} \otimes I_M] \]

\[ = [\text{diag} \{ Z_A(k) \} \otimes I_M] \left\{ g_B(k) g_B(l) \frac{M+1 M-1}{M(M-1)} I_W \otimes P_M \right\} [\text{diag} \{ Z_A(l) \} \otimes I_M] \]

\[ = g_B(k) g_B(l) \frac{M+1 M-1}{M(M-1)} [\text{diag} \{ Z_A(k) \} \otimes I_M] (I_W \otimes P_M) [\text{diag} \{ Z_A(l) \} \otimes I_M], \]
we have
\[ E_{s-p} \left[ \left\{ g_B(k)g_B(l) \frac{M_{k-1}M_{l-1}}{M(M-1)} \right\}^{-1} \text{cov}_{s-p} \{ Z(k), Z(l) \mid Z_A \} \right] = E_{s-p} \left[ (\text{diag} \{ Z_A(k) \} \otimes I_M) (I_W \otimes P_M) [\text{diag} \{ Z_A(l) \} \otimes I_M] \right] = E_{s-p} \left[ (\text{diag} \{ Z_A(k) \}) I_W \cdot (I_M P_M I_M) \right] = E_{s-p} \left[ \text{diag} \{ Z_A(k) \circ Z_A(l) \} \otimes P_M \right] = \{ E_{s-p} (I_{\{Z_1=g_A(k)\}} I_{\{Z_1=g_A(l)\}}) \cdot I_W \} \otimes P_M = E_{s-p} (I_{\{Z_1=g_A(k)\}=g_A(l)}) (I_W \otimes P_M) = \left( I_{\{g_A(k)=g_A(l)\}} \frac{W(k)}{W} \right) (I_W \otimes P_M) = I_{\{g_A(k)=g_A(l)\}} \frac{W(k)}{W(M-1)} P_{in}.
\]

Multiply both sides by \( g_B(k)g_B(l)M_{k+1}M_{l-1}/\{M(M-1)\} \) to have (C.13)
\[ E_{s-p} \left[ \text{cov}_{s-p} \{ Z(k), Z(l) \mid Z_A \} \right] = I_{\{g_A(k)=g_A(l)\}} g_B(k)g_B(l) \frac{W(k)M_{k+1}M_{l-1}}{N(M-1)} P_{in}.
\]

This gives the expectation of covariance component of (C.11). Substituting (C.12) and (C.13) into (C.11) yields
\[ \text{cov}_{s-p} \{ Z(k), Z(l) \} = g_A(k)g_A(l) \frac{W_{k+1}W_{l-1}M_{k}M_{l}}{N(W-1)} P_{btw} + I_{\{g_A(k)=g_A(l)\}} g_B(k)g_B(l) \frac{W(k)M_{k+1}M_{l-1}}{N(M-1)} P_{in}, \]

and
\[ \text{cov}_{s-p} \{ N_k^{-1}Z(k), N_l^{-1}Z(l) \} = \left( W_{k}M_{k}W_{l}M_{l} \right)^{-1} \text{cov}_{s-p} \{ Z(k), Z(l) \} = C_{k,l}^{\text{btw}} P_{btw} + C_{k,l}^{\text{in}} P_{in}, \]

where
\[ C_{k,l}^{\text{btw}} = g_A(k)g_A(l) \frac{W_{k+1}W_{l-1}}{W(k)W(l)} \frac{M_{k+1}M_{l-1}}{N(W-1)}, \quad C_{k,l}^{\text{in}} = \frac{I_{\{g_A(k)=g_A(l)\}} g_B(k)g_B(l)}{W_{k}M_{k}W_{l}M_{l}} \frac{M_{k+1}M_{l-1}}{N(M-1)}. \]

It is straightforward to verify that \( C_{k,l}^{\text{btw}} \) is the \((k,l)\)th entry of \( \text{C}_{\text{btw}} \) and \( C_{k,l}^{\text{in}} \) is the \((k,l)\)th entry of \( \text{C}_{\text{in}} \). This, coupled with (C.15), completes the proof. \( \square \)
An interesting observation is that the three coefficient matrices $C_{btw}$, $C_{in}$, and $C$ satisfy

$$(C.16) \quad (W - 1)C_{btw} + W(M - 1)C_{in} = (N - 1)C.$$  

**Proof of Identity (C.16).** The result follows from

$$N(W - 1)C_{btw} + NW(M - 1)C_{in}$$

$$= \begin{pmatrix} r_A & -1 \\ -1 & r_A \end{pmatrix} \otimes J_2 + \left\{ \begin{pmatrix} r_A & -1 \\ -1 & r_A \end{pmatrix} \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B \end{pmatrix} + J_2 \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} r_A & -1 \\ -1 & r_A \end{pmatrix} + J_2 \right\} \otimes \left\{ J_2 + \begin{pmatrix} r_B & -1 \\ -1 & r_B \end{pmatrix} \right\} - J_4$$

$$= \text{diag} \left\{ \frac{N}{N_1}, \frac{N}{N_2}, \frac{N}{N_3}, \frac{N}{N_4} \right\} - J_4 = N(N - 1)C.$$  

\[ \square \]

**APPENDIX D: SAMPLING VARIANCES OF THE ESTIMATORS**

**Proof of Lemma 7.** Straightforward.  

**Proof of Theorem 3.** By Lemmas 6 and 7 we have

$$E_{C-R}(\hat{\tau}_F) = 2^{-1} g_F^T \tilde{Y}^T E_{C-R}(Z^*) = 2^{-1} g_F^T \tilde{Y}^T (N^{-1} 1_{4N})$$

$$= 2^{-1} g_F^T (N^{-1} \tilde{Y}^T 1_{4N}) = 2^{-1} g_F^T \tilde{Y} = \tau_F$$

and

$$\text{var}_{C-R}(\hat{\tau}_F) = 4^{-1} g_F^T \tilde{Y}^T \text{cov}_{C-R}(Z^*) \tilde{Y} g_F = 4^{-1} g_F^T \tilde{Y}^T (C \otimes P_N) \tilde{Y} g_F$$

$$= 4^{-1} g_F^T \tilde{Y}^T (C \otimes P_N) \tilde{Y} g_F \overset{(A.4)}{=} 4^{-1} g_F^T \{ C \circ (Y^T P_N Y) \} g_F.$$  

This, coupled with $S^2 = (N - 1)^{-1} Y^T P_N Y$, proves (5.3).

When the design is balanced, the coefficient matrix $C$ in Lemma 6 reduces to $C = (4I_4 - J_4)/\{N(N - 1)\} = 4P_4/\{N(N - 1)\}$. Substituting this simplified version into (5.3) yields

$$N \cdot \text{var}_{C-R}(\hat{\tau}_F) = g_F^T (P_4 \circ S^2) g_F = g_F^T \{ (I_4 - J_4/4) \circ S^2 \} g_F$$

$$= g_F^T (I_4 \circ S^2) g_F - 4^{-1} g_F^T (J_4 \circ S^2) g_F$$

$$= g_F^T \text{diag} \{ S^2(1,1), S^2(2,2), S^2(3,3), S^2(4,4) \} g_F - 4^{-1} g_F^T S^2 g_F$$

$$= \sum_{k=1}^{4} S^2(k,k) - 4^{-1} g_F^T Y^T P_N Y g_F = \sum_{k=1}^{4} S^2(k,k) - S_F^2.$$
This completes the proof. 

**Proof of Theorem 4.** By Theorem 2 and Lemma 7, we have

\[
E_{s-p}(\tau_F) = 2^{-1} g_F^T \bar{Y}^T E_{s-p}(Z^*) = 2^{-1} g_F^T \bar{Y}^T (N^{-1} 1_{4N}) \\
v_{s-p}(\tau_F) = 4^{-1} g_F^T \bar{Y}^T \text{cov}_{s-p}(Z^*) \bar{Y} g_F \\
= 4^{-1} g_F^T \bar{Y}^T (C_{btw} \otimes P_{btw} + C_{in} \otimes P_{in}) \bar{Y} g_F \\
= 4^{-1} g_F^T (\bar{Y}^T (C_{btw} \otimes P_{btw}) \bar{Y} + \bar{Y} (C_{in} \otimes P_{in}) \bar{Y}) g_F \\
\overset{(A.4)}{=} 4^{-1} g_F^T \{C_{btw} \odot (Y^T P_{btw} Y) + C_{in} \odot (Y^T P_{in} Y)\} g_F.
\]

This, coupled with the definitions of \(S_{btw}^2\) and \(S_{in}^2\) in (2.5), completes the proof. 

**Proof of Corollary 5.** When the design is balanced, we have \(r_A = r_B = 1\), and the coefficient matrices \(C_{btw}\) and \(C_{in}\) in (5.4) reduce to

\[
C_{btw} = \frac{1}{N(W-1)} g_A g_A^T, \quad C_{in} = \frac{1}{NW(M-1)} (g_B g_B^T + g_{AB} g_{AB}^T).
\]

Substituting these simplified versions, together with the definitions of \(S_{btw}^2\) and \(S_{in}^2\) in (2.5), into (5.4) yields

\[
(D.1) \quad v_{s-p}(\tau_F) = \frac{g_F^T \{(g_A g_A^T) \odot (Y^T P_{btw} Y)\} g_F}{4N(W-1)} + \frac{g_F^T \{(g_B g_B^T + g_{AB} g_{AB}^T) \odot (Y^T P_{in} Y)\} g_F}{4NW(M-1)} \\
= \frac{g_F^T \{(g_A g_A^T) \odot (Y^T P_{btw} Y)\} g_F}{4N(W-1)} \\
+ \frac{g_F^T \{(g_B g_B^T + g_{AB} g_{AB}^T) \odot (Y^T P_{in} Y)\} g_F}{4NW(M-1)}.
\]

Introduce shorthand notations for the entrywise products in (D.1):

\[
H_{btw} = (g_A g_A^T) \odot (Y^T P_{btw} Y), \quad H_{in-F} = (g_F g_F^T) \odot (Y^T P_{in} Y) (F = B, AB).
\]

It follows from

\[
H_{btw} = (g_A g_A^T) \odot (Y^T P_{btw} Y) \overset{(A.1)}{=} \text{diag}(g_A)(Y^T P_{btw} Y)\text{diag}(g_A) \\
= \text{[Ydiag(g_A)]}^T P_{btw} [Y\text{diag}(g_A)].
\]
To prove Corollary 1 thus reduces to computing the quadratic forms 
\[ S \] and 
\[ g_F^T H_{btw} g_F = \begin{bmatrix} Y \text{diag}\{g_A\} g_F \end{bmatrix}^T P_{btw} \begin{bmatrix} Y \text{diag}\{g_A\} g_F \end{bmatrix}^T = \begin{bmatrix} Y(g_A \circ g_F) \end{bmatrix}^T P_{btw} \begin{bmatrix} Y(g_A \circ g_F) \end{bmatrix}. \]

This, coupled with 
\[ Y(g_A \circ g_A) = Y1_4 = 4\mu, \quad Y(g_A \circ g_B) = Yg_{AB} = 2\tau_{AB}, \]
and 
\[ Y(g_A \circ g_{AB}) = Yg_B = 2\tau_B, \] yields

\[ g_A^T H_{btw} g_A = 16\mu^T P_{btw} \mu = \{16(W - 1)M\} S_{\mu,btw}^2, \]
\[ g_B^T H_{btw} g_B = 4\tau_{AB}^T P_{btw} \tau_{AB} = \{4(W - 1)M\} S_{AB,btw}^2, \]
\[ g_{AB}^T H_{btw} g_{AB} = 4\tau_{B}^T P_{btw} \tau_{B} = \{4(W - 1)M\} S_{B,btw}^2. \]

Analogues of (D.2) for \( H_{in-B} \) and \( H_{in-AB} \) follow from similar algebra as

\[ g_F^T H_{in-B} g_F = \{Y(g_B \circ g_F)\}^T P_{in}\{Y(g_B \circ g_F)\}, \]
\[ g_F^T H_{in-AB} g_F = \{Y(g_{AB} \circ g_F)\}^T P_{in}\{Y(g_{AB} \circ g_F)\}. \]

This, coupled with

\[ Y(g_B \circ g_A) = 2\tau_{AB}, \quad Y(g_B \circ g_B) = 4\mu, \quad Y(g_B \circ g_{AB}) = 2\tau_A, \quad Y(g_B \circ g_{AB}) = 2\tau_B, \quad Y(g_{AB} \circ g_B) = 2\tau_A, \quad Y(g_{AB} \circ g_B) = 4\mu \]
yields

\[ g_A^T H_{in-B} g_A = 4W(M - 1)S_{AB,in}^2, \quad g_A^T H_{in-AB} g_A = 4W(M - 1)S_{B,in}^2, \]
\[ g_B^T H_{in-B} g_B = 16W(M - 1)S_{\mu,in}^2, \quad g_B^T H_{in-AB} g_B = 4W(M - 1)S_{A,in}^2, \]
\[ g_{AB}^T H_{in-B} g_{AB} = 4W(M - 1)S_{A,in}^2, \quad g_{AB}^T H_{in-AB} g_{AB} = 16W(M - 1)S_{\mu,in}^2. \]

Substituting (D.3) and (D.4) into (D.1) completes the proof. \( \square \)

**Proof of Corollary 1.** Under strict additivity, we have \( S_{btw}^2 = S_{btw}^2 J_4 \) and \( S_{in}^2 = S_{in}^2 J_4 \). Formula (5.4) simplifies to

\[ \text{var}_F(\hat{\tau}_F) = 4^{-1} \{((W - 1)M S_{btw}^2) g_F^T C_{btw} g_F + 4^{-1} \{W(M - 1)S_{in}^2\} g_F^T C_{in} g_F \}. \]

To prove Corollary 1 thus reduces to computing the quadratic forms \( g_F^T C_{btw} g_F \) and \( g_F^T C_{in} g_F \). Starting with \( F = A \), direct application of (A.3) to

\[ C_{btw} = \frac{1}{N(W - 1)} \begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} \otimes J_2, \]
\[ C_{in} = \frac{1}{NW(M - 1)} \left\{ \begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} + J_2 \right\} \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix} \]
\[ = \frac{1}{NW(M - 1)} \begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix} + \frac{1}{NW(M - 1)} J_2 \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix}, \]
\[ g_A = (-1, 1)^T \otimes 1_2 \]
yields

\[ g_A^T \{ N(W - 1)C_{btw} \} g_A = \{(−1, 1) \otimes 1 \} \left\{ \begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} \right\} \{(-1, 1)^T \otimes 1 \} \]

\[ = \{(−1, 1) \left( \begin{matrix} r_A & -1 \\ -1 & r_A^{-1} \end{matrix} \right) \} \otimes (1^TJ_21) \]

\[ = \{γ_A \otimes 4 \} = 4γ_A. \]

\[ g_A^T \{ NW(M - 1)C_{in} \} g_A \]

\[ = g_A \left\{ \begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix} \right\} \{(-1, 1)^T \otimes 1 \} \]

\[ + \{(−1, 1) \otimes 1 \} \left\{ \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix} \right\} \{(-1, 1)^T \otimes 1 \} \]

\[ = \gamma_A \otimes (γ_B - 4) + 0 = γ_A(γ_B - 4). \]

Thus

\[ g_A^T C_{btw} g_A = \frac{4γ_A}{N(W - 1)}, \quad g_A^T C_{in} g_A = \frac{γ_A(γ_B - 4)}{NW(M - 1)}. \]

Substituting \( g_A \) with \( g_B = 1_2 \otimes (−1, 1)^T \) and \( g_{AB} = (−1, 1)^T \otimes (−1, 1)^T \) respectively in the above computation yields

\[ g_B^T C_{btw} g_B \]

\[ \propto \left\{ \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix} \right\} \otimes \left\{ \begin{pmatrix} 1 \end{pmatrix} \right\} = 0, \]

\[ g_{AB}^T C_{btw} g_{AB} \]

\[ \propto \left\{ \begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} \right\} \otimes \left\{ \begin{pmatrix} 1 \end{pmatrix} \right\} = 0. \]
and
\[g_B^T\{NW(M-1)C_{in}\}g_B\]
\[= \{1^T_2 \otimes (-1,1)\}\left\{\begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix}\right\}\{1 \otimes (-1,1)^T\}
+ \{1^T_2 \otimes (-1,1)\}\left\{J_2 \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix}\right\}\{1 \otimes (-1,1)^T\}
= (\gamma_A - 4) \otimes \gamma_B + 4 \otimes \gamma_B = \gamma_A \gamma_B,
\]
\[g_{AB}^T\{NW(M-1)C_{in}\}g_{AB}\]
\[= \{(-1,1) \otimes (-1,1)\}\left\{\begin{pmatrix} r_A & -1 \\ -1 & r_A^{-1} \end{pmatrix} \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix}\right\}\{(-1,1)^T \otimes (-1,1)^T\}
+ \{(-1,1) \otimes (-1,1)\}\left\{J_2 \otimes \begin{pmatrix} r_B & -1 \\ -1 & r_B^{-1} \end{pmatrix}\right\}\{(-1,1)^T \otimes (-1,1)^T\}
= \gamma_A \otimes \gamma_B + 0 = \gamma_A \gamma_B.
\]
Thus
\[(D.9) \quad g_{AB}^T C_{in} g_{AB} = g_{AB}^T C_{in} g_{AB} = \frac{\gamma_A \gamma_B}{NW(M-1)}.
\]
Substituting (D.6)–(D.9) into (D.5) completes the proof. \[\square\]

**Proof of Corollary 3.** Under strict additivity, we have \(S^2_{btw} = S^2_{btw} J_4\), \(S^2_{in} = S^2_{in} J_4\), and
\[S^2 = \frac{(W-1)M}{N-1} S^2_{btw} + \frac{W(M-1)}{N-1} S^2_{in} = \left\{(\frac{(W-1)M}{N-1} S^2_{btw} + \frac{W(M-1)}{N-1} S^2_{in}\right\} J_4.
\]
Substituting this simplified expression for \(S^2\) into (5.11) yields
\[(D.10) \quad \text{var}_{C-R}(\hat{\tau}_F) = 4^{-1} \left\{(W-1)MS^2_{btw} + W(M-1)S^2_{in}\right\} g_F^T C g_F,
\]
where, by identities (C.16) and (D.6)–(D.9),
\[g_F^T C g_F = \frac{W-1}{N-1} g_F^T C_{btw} g_F + \frac{W(M-1)}{N-1} g_F^T C_{in} g_F = \frac{\gamma_A \gamma_B}{N(N-1)} (F \in \mathcal{F}).
\]
This, coupled with (D.10), proves (5.12). It then follows from (5.12) and
Corollary 1 that, for $F = A$,

$$\text{var}_{s-p}(\hat{\tau}_A) - \text{var}_{c-r}(\hat{\tau}_A)$$

$$= \frac{\gamma_A}{W} S_{btw} + \frac{\gamma_A(\gamma_B - 4)}{4N} S_{in}^2 - \frac{\gamma_A\gamma_B}{4(N-1)} \left( \frac{W - 1}{W} S_{btw}^2 + \frac{M - 1}{M} S_{in}^2 \right)$$

$$= \frac{\gamma_A}{4(N-1)W} \{4(N-1) - (W-1)\gamma_B\} S_{btw}^2$$

$$+ \frac{\gamma_A}{4N(N-1)} \{(N-1)(\gamma_B - 4) - (N-W)\gamma_B\} S_{in}^2$$

$$= \frac{\gamma_A}{4(N-1)W} \{4(N-W) - (W-1)(\gamma_B - 4)\} S_{btw}^2$$

$$+ \frac{\gamma_A}{4N(N-1)} \{(W-1)(\gamma_B - 4) - 4(N-W)\} S_{in}^2$$

$$= \frac{\gamma_A}{4(N-1)W} \{4(N-W) - (W-1)(\gamma_B - 4)\} \left( S_{btw}^2 - \frac{S_{in}^2}{M} \right)$$

where

$$4(N-W) - (W-1)(\gamma_B - 4) = 4W(M-1) - (W-1)(r_B + r_B^{-1} - 2)$$

$$\geq 4W(M-1) - (W-1) \left( \frac{M - 1}{1} + \frac{1}{M - 1} - 2 \right)$$

$$\geq 4W(M-1) - (W-1)(M-1) = (3W+1)(M-1) > 0,$$

and, for $F = B$ and $AB$,

$$\text{var}_{s-p}(\hat{\tau}_F) - \text{var}_{c-r}(\hat{\tau}_F) = \frac{\gamma_A\gamma_B}{4N} S_{in}^2 - \frac{\gamma_A\gamma_B}{4(N-1)} \left( \frac{W - 1}{W} S_{btw}^2 + \frac{M - 1}{M} S_{in}^2 \right)$$

$$= -\frac{\gamma_A\gamma_B(W-1)}{4(N-1)W} S_{btw}^2 + \frac{\gamma_A\gamma_B(W-1)}{4N(N-1)} S_{in}^2$$

$$= -\frac{\gamma_A\gamma_B(W-1)}{4(N-1)W} \left( S_{btw}^2 - \frac{S_{in}^2}{M} \right).$$

This completes the proof. \(\square\)

**APPENDIX E: VARIANCE ESTIMATION**

**Lemma E.4.** For treatments $k$ and $l$ with the same $z = g_A(k) = g_A(l) \in \{-1, +1\}$ level of factor $A$, we have

$$E_{s-p}\{\mathbf{Z}(l)\mathbf{Z}(k)^T\} = \frac{W+1}{N(W-1)} M_k M^{-1}_k \mathbf{P}_{btw} + \frac{g_B(k)g_B(l)}{N} \frac{W+1}{N(M-1)} M_k M^{-1}_k \mathbf{P}_{in}$$

$$+ \frac{W+1}{N^2} \mathbf{J}_N.$$
**Proof of Lemma E.4.** With $z = g_A(k) = g_A(l) \in \{-1, 1\}$, we have $g_A(k)g_A(l) = z^2 = 1$. Substituting this into (C.14) yields

$$\text{cov}_{s-p}\{\mathbf{Z}(l), \mathbf{Z}(k)\} = \frac{W_l W_m M_l M_k}{N(N-1)} \mathbf{p}_{btw} + g_B(k)g_B(l) \frac{W_l M_l M_k}{N(N-1)} \mathbf{p}_{in}.$$  

This, coupled with

$$E_{s-p}\{\mathbf{Z}(l)\mathbf{Z}(k)^T\} = \text{cov}_{s-p}\{\mathbf{Z}(l), \mathbf{Z}(k)\} + E_{s-p}\{\mathbf{Z}(l)\} E_{s-p}\{\mathbf{Z}(k)^T\},$$

$$E_{s-p}\{\mathbf{Z}(l)\} E_{s-p}\{\mathbf{Z}(k)\}^T = \left(\frac{W_l M_l}{N}\right) \mathbf{1}_N \left(\frac{W_m M_k}{N}\right) \mathbf{1}_N^T = \frac{W_l^2 M_l M_k}{N^2} \mathbf{J}_N,$$

completes the proof. \(\square\)

**Proof of Lemma 8.** Define

(E.1) \[m_w(k) = M_k^{-1} \sum_{m=1}^{M} Y_{(wm)}(k) I_{T_{(wm)} = k}\]

\[= M_k^{-1} \sum_{m=1}^{M} Y_{(wm)}(k) I_{A_w = g_A(k)} I_{B_{(wm)} = g_B(k)}\]

such that $m_w(k)$ equals $Y_{(w)}^{obs}(k)$ if $A_w = g_A(k)$, and equals 0 if otherwise. Let $\mathbf{m}(k) = (m_1(k), \ldots, m_W(k))^T$. The sample between-whole-plot covariances $s_{btw}(k, l)$ satisfy

$$W_z - 1)s_{btw}^2(k, l) = \sum_{w: A_w = z}\{Y_{(w)}^{obs}(k) - \bar{Y}_{(w)}^{obs}(k)\}\{Y_{(w)}^{obs}(l) - \bar{Y}_{(w)}^{obs}(l)\}$$

$$= \sum_{w: A_w = z} Y_{(w)}^{obs}(k) Y_{(w)}^{obs}(l) - W_z \bar{Y}_{(w)}^{obs}(k) \bar{Y}_{(w)}^{obs}(l)$$

$$= \mathbf{m}(k)^T \mathbf{m}(l) - W_z \bar{Y}_{(w)}^{obs}(l) \bar{Y}_{(w)}^{obs}(k),$$

with sampling expectations

(E.2) \[E_{s-p}\{s_{btw}^2(k, l)\} = E_{s-p}\{\mathbf{m}(k)^T \mathbf{m}(l)\} - W_z E_{s-p}\{\bar{Y}_{(w)}^{obs}(l) \bar{Y}_{(w)}^{obs}(k)\}.\]

We take the divide-and-conquer strategy here, and compute the two terms on the right-hand side of (E.2) one at a time.

To start with, let $\mathbf{Y}_w(k) = (Y_{(w_1)}(k), \ldots, Y_{(w_M)}(k))^T$ be the potential outcomes vectors for whole-plot $w$, and let $\mathbf{Z}_w(k) = (I_{T_{(w_1)} = k}, \ldots, I_{T_{(w_M)} = k})^T$
indicate the recipients of treatment \( k \) therein. That \( m_w(k) = M_{-1}^k Y_w(k)^T Z_w(k) \) allows us to write \( m(k) \) as
\[
m(k) = (m_1(k), \ldots, m_w(k))^T
\]
(E.3) \[
= M_{-1}^{-1} (Y_1(k)^T Z_1(k), \ldots, Y_W(k)^T Z_W(k)) = M_{-1}^{-1} Y^*(k)^T Z(k),
\]
where \( Y^*(k) = \text{Bdiag} \{ Y_1(k), \ldots, Y_W(k) \} \) is the block-diagonal matrix with
\( Y_w(k) \) as its \( \text{wth diagonal block}. \) Thus,
\[
m(k)^T m(l) = \{ M_{-1}^{-1} Y^*(k)^T Z(k) \}^T \{ M_{-1}^{-1} Y^*(l)^T Z(l) \}
\]
\[
= M_{-1}^{-1} M_{-1}^{-1} Z(k)^T Y^*(k) Y^*(l)^T Z(l)
\]
\[
= M_{-1}^{-1} M_{-1}^{-1} \text{tr} \{ Y^*(l)^T Z(l) Z(k)^T Y^*(k) \},
\]
with expectation
\[
(E.4) \quad E_{S-P} \{ m(k)^T m(l) \} = M_{-1}^{-1}(E(l)) \{ \text{tr} \{ Y^*(l)^T Z(l) Z(k)^T Y^*(k) \} \}
\]
\[
= M_{-1}^{-1}(E(l)) \text{tr} \{ Y^*(l)^T E_{S-P} \{ Z(l) Z(k)^T \} Y^*(k) \}.
\]
Given Lemma E.4 and the linearity of trace function, we have
\[
(E.5) \quad \text{tr} \{ Y^*(l)^T E_{S-P} \{ Z(l) Z(k)^T \} Y^*(k) \}
\]
\[
= \frac{W_1 W_{-1} M(l) M(k)}{N(W - 1)} \text{tr} \{ Y^*(l)^T P_{\text{btw}} Y^*(k) \}
\]
\[
+ g_B(k) g_B(l) \frac{W_z M_{l+1} M_{-1}}{N(M - 1)} \text{tr} \{ Y^*(l)^T P_{\text{in}} Y^*(k) \}
\]
\[
+ \frac{W_z^2 M(l) M(k)}{N^2} \text{tr} \{ Y^*(l)^T J_N Y^*(k) \},
\]
where it follows from straightforward yet tedious matrix algebra that
\[
(E.6) \quad \text{tr} \{ Y^*(l)^T P_{\text{btw}} Y^*(k) \} = \{ W^{-1}(W - 1) M \} Y_{\text{block}}(l)^T Y_{\text{block}}(k),
\]
\[
(E.7) \quad \text{tr} \{ Y^*(l)^T P_{\text{in}} Y^*(k) \} = W(M - 1) S_{in}^2(k, l),
\]
\[
(E.8) \quad \text{tr} \{ Y^*(l)^T J_N Y^*(k) \} = M^2 Y_{\text{block}}(l)^T Y_{\text{block}}(k)
\]
with \( Y_{\text{block}}(k) = (Y_1(k), \ldots, Y_W(k))^T \). We defer the algebraic details for
(E.6) – (E.8) after the main proof. Equalities (E.6) – (E.8) simplify (E.5) to
\[
\text{tr} \{ Y^*(l)^T E_{S-P} \{ Z(l) Z(k)^T \} Y^*(k) \}
\]
\[
= \frac{W_1 W_{-1} M(l) M(k)}{W^2} Y_{\text{block}}(l)^T Y_{\text{block}}(k) + g_B(k) g_B(l) \frac{W_z M_{l+1} M_{-1}}{M} S_{in}^2(k, l)
\]
\[
+ \frac{W_z^2 M(l) M(k)}{W^2} Y_{\text{block}}(l)^T Y_{\text{block}}(k)
\]
\[
= \frac{W_z M(l) M(k)}{W} Y_{\text{block}}(l)^T Y_{\text{block}}(k) + g_B(k) g_B(l) \frac{W_z M_{l+1} M_{-1}}{M} S_{in}^2(k, l).
\]
Substituting this back into (E.4) yields
\[ E_{\text{s-p}} \{ m(k)^T m(l) \} = \frac{W_z}{W} y_{\text{block}}(l)^T y_{\text{block}}(k) + g_B(k)g_B(l) \frac{W_z M_{+1} M_{-1}}{M M(k) M(l)} S_{\text{in}}^2(k, l). \]

This gives the first term of (E.2). For the second term of (E.2), it follows from \( \bar{Y}^{\text{obs}}(k) = N_k^{-1} Y(k)^T Z(k) = W_{(k)}^{-1} M_{(k)}^{-1} Y(k)^T Z(k) \) that
\[ (E.10) \]
\[ W_z E_{\text{s-p}} \{ \bar{Y}^{\text{obs}}(k) \bar{Y}^{\text{obs}}(l) \} = W_z E_{\text{s-p}} \{ W_{z}^{-2} M_{(k)}^{-1} M_{(l)}^{-1} Y(l)^T Z(l) Z(k)^T Y(k) \} \]
\[ = W_{z}^{-1} M_{(k)}^{-1} M_{(l)}^{-1} Y(l)^T E_{\text{s-p}} \{ Z(l) Z(k)^T \} Y(k), \]

in which, by Lemma E.4,
\[ Y(l)^T E_{\text{s-p}} \{ Z(l) Z(k)^T \} Y(k) \]
\[ = \frac{W_{+1} W_{-1} M_{(k)} M_{(l)}}{N(W - 1)} Y(l)^T P_{\text{btw}} Y(k) + g_B(k)g_B(l) \frac{W_z M_{+1} M_{-1}}{N(M - 1)} Y(l)^T P_{\text{in}} Y(k) \]
\[ + \frac{W_z^2 M_{(k)} M_{(l)}}{N^2} Y(l)^T J_N Y(k) \]
\[ = \frac{W_z W_{-z} M_{(k)} M_{(l)}}{W} S^2_{\text{btw}}(k, l) + g_B(k)g_B(l) \frac{W_z M_{+1} M_{-1}}{M} S_{\text{in}}^2(k, l) \]
\[ + \frac{W_z^2 M(k) M(l)}{W} \bar{Y}(k) \bar{Y}(l). \]

Substituting this last expression into the right-hand side of (E.10) equates
\[ W_z E_{\text{s-p}} \{ \bar{Y}^{\text{obs}}(k) \bar{Y}^{\text{obs}}(l) \} \]
\[ (E.11) \]
\[ = \frac{W_z}{W} S^2_{\text{btw}}(k, l) + \frac{g_B(k)g_B(l) M_{+1} M_{-1}}{M M(k) M(l)} S_{\text{in}}^2(k, l) + W_z \bar{Y}(k) \bar{Y}(l). \]
Substituting (E.9) and (E.11) into (E.2) yields

\[(W_z - 1)E_{v \cdot v}\{s^2_{\text{btw}}(k, l)\}
\]
\[
= \left\{ \frac{W_z}{W} Y_{\text{block}}(l)Y_{\text{block}}(k) + \frac{g_B(k)g_B(l)W_zM_{+1}M_{-1}}{MM(k)M(l)}S^2_{\text{in}}(k, l) \right\}
\]
\[\quad - \left\{ \frac{W_z}{W} s^2_{\text{btw}}(k, l) + \frac{g_B(k)g_B(l)M_{+1}M_{-1}}{MM(k)M(l)}S^2_{\text{in}}(k, l) + W_z\bar{Y}(k)\bar{Y}(l) \right\}
\]
\[
= \frac{W_z}{W} \{ Y_{\text{block}}(l)Y_{\text{block}}(k) - W\bar{Y}(k)\bar{Y}(l) \} - \frac{W_z}{W} s^2_{\text{btw}}(k, l)
\]
\[\quad + g_B(k)g_B(l) \frac{(W_z - 1)M_{+1}M_{-1}}{MM(k)M(l)}S^2_{\text{in}}(k, l)
\]
\[
= \frac{W_z}{W} \{ (W - 1)s^2_{\text{btw}}(k, l) \} - \frac{W - W_z}{W} s^2_{\text{btw}}(k, l) + g_B(k)g_B(l) \frac{(W_z - 1)M_{+1}M_{-1}}{MM(k)M(l)}S^2_{\text{in}}(k, l)
\]
\[
= (W_z - 1)s^2_{\text{btw}}(k, l) + (W_z - 1)g_B(k)g_B(l) \frac{M_{+1}M_{-1}}{MM(k)M(l)}S^2_{\text{in}}(k, l),
\]

with

\[g_B(k)g_B(l) \frac{M_{+1}M_{-1}}{MM(k)M(l)} = \begin{cases} 
M^{-1}_{+1} & \text{if } (k, l) = (1, 1), (3, 3), \\
M^{-1}_{+1}r_B^{-1} & \text{if } (k, l) = (2, 2), (4, 4), \\
-M^{-1}_B & \text{if } (k, l) = (1, 2), (2, 1), (3, 4), (4, 3).
\end{cases}\]

Dividing both sides by \((W_z - 1)\) completes the proof. \(\square\)

We give the algebraic details for (E.6) – (E.8) below.
Proof. Equality (E.6) follows from
\[
\text{tr}\{(Y^*(l)^T P_i \otimes Y^*(k))\} = \text{tr}\{Y^*(l)^T (P_W \otimes (M^{-1} J_M)) Y^*(k)\}
\]
\[
= \text{tr}\{Y^*(l)^T [(I_W - W^{-1} J_W) \otimes (M^{-1} J_M)] Y^*(k)\}
\]
\[
= M^{-1} \text{tr}\{Y^*(l)^T (I_W \otimes J_M) Y^*(k)\} - N^{-1} \text{tr}\{Y^*(l)^T (J_W \otimes J_M) Y^*(k)\}
\]
\[
= M^{-1} \sum_{u=1}^{W} Y_w(l)^T J_M Y_w(k) - N^{-1} \sum_{u=1}^{W} Y_w(l)^T J_M Y_w(k)
\]
\[
= \frac{W - 1}{N} \sum_{u=1}^{W} Y_w(l)^T J_M Y_w(k) = \frac{W - 1}{N} \sum_{u=1}^{W} (Y_w(l)^T 1_M) \{1_M^T Y_w(k)\}
\]
\[
= \frac{W - 1}{N} \sum_{u=1}^{W} \{Y_w(l)M\} \{Y_w(k)M\} = \frac{(W - 1)M}{W} Y_{\text{block}}(l)^T Y_{\text{block}}(k).
\]

Equality (E.7) follows from
\[
\text{tr}\{(Y^*(l)^T P_i \otimes Y^*(k))\} = \text{tr}\{Y^*(l)^T (I_W \otimes P_M) Y^*(k)\}
\]
\[
= \text{tr}\{Y^*(l)^T \begin{bmatrix} P_M & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & Y_W(l)^T \end{bmatrix} \begin{bmatrix} Y_1(k) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & Y_W(k) \end{bmatrix} \}
\]
\[
= \sum_{u=1}^{W} Y_w(l)^T P_M Y_w(k) = W(M - 1)S_{ii}^2(k, l).
\]

Equality (E.8) follows from
\[
\text{tr}\{Y^*(l)^T J_N Y^*(k)\}
\]
\[
= \text{tr}\{Y^*(l)^T \begin{bmatrix} J_M & \ldots & J_M \\ \vdots & \ddots & \vdots \\ 0 & \ldots & J_M \end{bmatrix} \begin{bmatrix} Y_1(k) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & Y_W(k) \end{bmatrix} \}
\]
\[
= M^2 Y_{\text{block}}(l)^T Y_{\text{block}}(k).
\]
\[\square\]
LEMMA E.5. Under the $2^2$ split-plot design qualified by Definition 4, the sampling expectation of $\hat{V}_F$ equals

$$E_{s-p}(\hat{V}_F) = 4^{-1} g_F \left\{ \frac{1}{NW(M-1)} \begin{pmatrix} (1 + r_A) r_B & - (1 + r_A) & 0 & 0 \\ -(1 + r_A) & (1 + r_A) r_B^{-1} & 0 & 0 \\ 0 & 0 & -(1 + r_A^{-1}) & (1 + r_A^{-1}) r_B^{-1} \\ 0 & 0 & -(1 + r_A^{-1}) & (1 + r_A^{-1}) r_B^{-1} \end{pmatrix} \circ S_{btw}^2 + W(M-1) C_{in} \circ S_{in}^2 \right\} g_F.$$

**Proof of Lemma E.5.** Rewrite $C_{in}$ as

$$C_{in} = \frac{1}{NW(M-1)} \begin{pmatrix} (1 + r_A) r_B & - (1 + r_A) & 0 & 0 \\ -(1 + r_A) & (1 + r_A) r_B^{-1} & 0 & 0 \\ 0 & 0 & -(1 + r_A^{-1}) & (1 + r_A^{-1}) r_B^{-1} \\ 0 & 0 & -(1 + r_A^{-1}) & (1 + r_A^{-1}) r_B^{-1} \end{pmatrix} \circ \begin{pmatrix} r_B & -1 & 0 & 0 \\ -1 & r_B^{-1} & 0 & 0 \\ 0 & 0 & r_B & -1 \\ 0 & 0 & -1 & r_B^{-1} \end{pmatrix} \circ \begin{pmatrix} r_B & -1 & 0 & 0 \\ -1 & r_B^{-1} & 0 & 0 \\ 0 & 0 & r_B & -1 \\ 0 & 0 & -1 & r_B^{-1} \end{pmatrix}.$$

The result follows from identity

$$\left( \frac{1}{NW(M-1)} \begin{pmatrix} (1 + r_A) r_B & - (1 + r_A) & 0 & 0 \\ -(1 + r_A) & (1 + r_A) r_B^{-1} & 0 & 0 \\ 0 & 0 & -(1 + r_A^{-1}) & (1 + r_A^{-1}) r_B^{-1} \\ 0 & 0 & -(1 + r_A^{-1}) & (1 + r_A^{-1}) r_B^{-1} \end{pmatrix} \circ E_{s-p}(s_{btw}^2) \right) \circ \left\{ \begin{pmatrix} r_B & -1 & 0 & 0 \\ -1 & r_B^{-1} & 0 & 0 \\ 0 & 0 & r_B & -1 \\ 0 & 0 & -1 & r_B^{-1} \end{pmatrix} \circ S_{in}^2 \right\}$$

$$= \left( \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \circ S_{btw}^2 + M^{-1} \begin{pmatrix} r_B & -1 & 0 & 0 \\ -1 & r_B^{-1} & 0 & 0 \\ 0 & 0 & r_B & -1 \\ 0 & 0 & -1 & r_B^{-1} \end{pmatrix} \circ S_{in}^2 \right) \circ \left\{ S_{btw}^2 + W(M-1) C_{in} \circ S_{in}^2 \right\}.$$
Proof of Theorem 5. It follows from Theorem 4 and Lemma E.5 that

\[ \text{var}_{s,p}(\tilde{\tau}_F) - E_{s,p}(\tilde{V}_F) \]

\[ = 4^{-1}(W - 1)M g_F^T(C_{btw} \circ S_{btw}^2)g_F + 4^{-1}W(M - 1)g_F^T(C_{in} \circ S_{in}^2)g_F \]

\[ - 4^{-1}g_F^T \left\{ \begin{pmatrix} W^{-1}J_2 & 0 \\ 0 & W^{-1}J_2 \end{pmatrix} \circ S_{btw}^2 \right\} g_F - 4^{-1}W(M - 1)g_F^T(C_{in} \circ S_{in}^2)g_F \]

\[ = 4^{-1}g_F^T \left\{ (W - 1)MC_{btw} - \begin{pmatrix} W^{-1}J_2 & 0 \\ 0 & W^{-1}J_2 \end{pmatrix} \right\} \circ S_{btw}^2 g_F \]

\[ = 4^{-1}g_F^T \left\{ (W - 1)MC_{btw} - \begin{pmatrix} r_A J_2 & -J_2 \\ -J_2 & r_A^{-1} J_2 \end{pmatrix} - W^{-1} \begin{pmatrix} (1 + r_A) J_2 & 0 \\ 0 & (1 + r_A^{-1}) J_2 \end{pmatrix} \right\} \circ S_{btw}^2 g_F \]

\[ = 4^{-1}g_F^T \left\{ (-W^{-1}J_4) \circ S_{btw}^2 g_F = -(4W)^{-1}g_F^T(J_4 \circ S_{btw}^2)g_F \right\} \]

\[ = -(4W)^{-1}g_F^T S_{btw}^2 g_F = -(4W)^{-1}S_{btw}^2 \cdot \]

This completes the proof.

\[ \square \]

Appendix F: Covariance Structure of Residuals from the Derived Linear Model

Recall from (C.4) that \( g_A(T_{wm}) = A_w \), \( g_B(T_{wm}) = B_{wm} \), and thus \( g_{AB}(T_{wm}) = A_w B_{wm} \) for all \((wm)\). This allows us to rewrite formula (7.5) of the main text as

\[ (F.1) \]

\[ \epsilon_{(wm)} = \delta_{(wm)-\mu} + 2^{-1} \delta_{(wm)-A} A_w + 2^{-1} \delta_{(wm)-B} B_{wm} + 2^{-1} \delta_{(wm)-AB} A_w B_{wm} \cdot \]

Proof of Theorem 6. Let \( \mathcal{A} = \{A_w\}_{w=1}^W \). The law of iterated expectations allows us to write the covariance of \( \epsilon_{(wm)} \) and \( \epsilon_{(wm')} \) as

\[ (F.2) \]

\[ \text{cov}_{s,p}(\epsilon_{(wm)}, \epsilon_{(wm')}) = \text{cov}_{s,p} \left\{ E_{s,p}(\epsilon_{(wm)} \mid \mathcal{A}), E_{s,p}(\epsilon_{(wm')} \mid \mathcal{A}) \right\} \]

\[ + E_{s,p} \left\{ \text{cov}_{s,p}(\epsilon_{(wm)}, \epsilon_{(wm')} \mid \mathcal{A}) \right\} . \]

Refer to the first term on the right-hand side of (F.2) as the covariance of expectations, and the second the expectation of covariance.

Let \( e_B = E_{s,p}(B_{wm}) = (r_B - 1)/(r_B + 1) \) be the common expectation of the identically distributed \( \{B_{wm}\} \). With \( \epsilon_{(wm)} \) given by (F.1), it follows from the joint independence of \( B_{wm} \) and \( \mathcal{A} \) that

\[ E_{s,p}(\epsilon_{(wm)} \mid \mathcal{A}) = \delta_{(wm)-\mu} + 2^{-1} e_B \delta_{(wm)-B} + 2^{-1} \left( \delta_{(wm)-A} + e_B \delta_{(wm)-AB} \right) A_w . \]

This expression for \( E_{s,p}(\epsilon_{(wm)} \mid \mathcal{A}) \) allows us to compute the covariance of
expectations term in (F.2) as

\[\text{(F.3)}\]

\[
\text{cov}_{s-p} \left\{ E_{s-p}(\epsilon_{(wm)}) \mid A \right\}, E_{s-p}(\epsilon_{(w'm') \mid A})
\]

\[
= \text{cov}_{s-p} \left\{ 2^{-1} \left( \delta_{(wm)} - A + e_B \delta_{(wm) - AB} \right) A_w, 2^{-1} \left( \delta_{(w'm') - A} + e_B \delta_{(w'm') - AB} \right) A_{w'} \right\}
\]

\[
= 4^{-1} \left( \delta_{(wm)} - A + e_B \delta_{(wm) - AB} \right) \left( \delta_{(w'm') - A} + e_B \delta_{(w'm') - AB} \right) \text{cov}_{s-p}(A_w, A_{w'})
\]

\[
= 4^{-1} (\delta_{(wm)} - A, \delta_{(wm) - AB}) \left( \begin{array}{c} 1 \\ e_B \end{array} \right) \left( \begin{array}{c} \delta_{(w'm') - A} \\ \delta_{(w'm') - AB} \end{array} \right) \text{cov}_{s-p}(A_w, A_{w'}),
\]

where

\[
\text{cov}_{s-p}(A_w, A_{w'}) = \mathrm{var}_{s-p}(A_w) = \frac{4W_{+1}W_{-1}}{W^2} = \frac{4r_A}{(1 + r_A)^2},
\]

if \( w = w' \), and

\[
\text{cov}_{s-p}(A_w, A_{w'}) = -\frac{4W_{+1}W_{-1}}{W^2(W - 1)} = -\frac{4r_A}{(1 + r_A)^2(W - 1)},
\]

if \( w \neq w' \) by Lemma C.2. Similarly, by (F.1) and the joint independence of \( B_{(wm)} \) and \( A \),

\[
\text{cov}_{s-p}(\epsilon_{(wm)}, \epsilon_{(w'm') \mid A})
\]

\[
= \text{cov}_{s-p} \left\{ 2^{-1} (\delta_{(wm)} - B + \delta_{(wm) - AB} A_w) B_{(wm)}, 2^{-1} (\delta_{(w'm') - B} + \delta_{(w'm') - AB} A_w') B_{(w'm')} \right\}
\]

\[
= 4^{-1} (\delta_{(wm)} - B + \delta_{(wm) - AB} A_w) (\delta_{(w'm') - B} + \delta_{(w'm') - AB} A_{w'}) \cdot \text{cov}_{s-p}(B_{(wm)}, B_{(w'm')})
\]

\[
= 4^{-1} (\delta_{(wm)} - B, \delta_{(wm) - AB}) \left( \begin{array}{c} 1 \\ A_w \end{array} \right) \left( \begin{array}{c} \delta_{(w'm') - B} \\ \delta_{(w'm') - AB} \end{array} \right) \text{cov}_{s-p}(B_{(wm)}, B_{(w'm')}).
\]

This expression for \( \text{cov}_{s-p}(\epsilon_{(wm)}, \epsilon_{(w'm') \mid A}) \) allows us to compute the expectation of covariance term in (F.2) as

\[\text{(F.4)}\]

\[
E_{s-p} \left\{ \text{cov}_{s-p}(\epsilon_{(wm)}, \epsilon_{(w'm') \mid A}) \right\}
\]

\[
= 4^{-1} (\delta_{(wm)} - B, \delta_{(wm) - AB}) E_{s-p} \left( \begin{array}{c} 1 \\ e_A \end{array} \right) \left( \begin{array}{c} A_w \\ A_{w'} \end{array} \right) \left( \begin{array}{c} \delta_{(w'm') - B} \\ \delta_{(w'm') - AB} \end{array} \right) \text{cov}_{s-p}(B_{(wm)}, B_{(w'm')}),
\]

where \( e_A = E_{s-p}(A_w) = (r_A - 1)/(r_A + 1) \) is the common expectation of the identically distributed \( \{A_w\} \), \( \text{cov}_{s-p}(B_{(wm)}, B_{(w'm')}) = 0 \) if \( w \neq w' \) by
Definition 4, and
\[
\text{cov}_{\text{s-p}}(B_{(wm)}, B_{(w'm')}) = \text{cov}_{\text{s-p}}(B_{(wm)}, B_{(w'm')}) \\
= -\frac{4M + 1}{M^2(M - 1)} - \frac{4r_B}{(1 + r_B)^2(M - 1)}
\]
if \(w = w', m \neq m'\) by Lemma C.2. Given (F.3), (F.4), and the covariances of the treatment indicators, the decomposition (F.2) simplifies to
(F.5)
\[
\text{cov}_{\text{s-p}}(\epsilon_{(wm)}, \epsilon_{(w'm')}) \\
= 4^{-1}(\delta_{(wm)\cdot A}, \delta_{(wm)\cdot AB})\left(\begin{array}{ll}
1 & e_B \\
E_{\text{s-p}}(A_{w}) & e_B^2
\end{array}\right)\left(\begin{array}{l}
\delta_{(w'm')\cdot A} \\
\delta_{(w'm')\cdot AB}
\end{array}\right)\text{cov}_{\text{s-p}}(A_w, A_{w'}) \\
+ 4^{-1}(\delta_{(wm)\cdot B}, \delta_{(wm)\cdot AB})\left(\begin{array}{ll}
1 & e_A \\
E_{\text{s-p}}(A_{w}) & e_A^2
\end{array}\right)\left(\begin{array}{l}
\delta_{(w'm')\cdot B} \\
\delta_{(w'm')\cdot AB}
\end{array}\right)\text{cov}_{\text{s-p}}(B_{(wm)}, B_{(w'm')}) \\
\frac{r_A}{(r_A + 1)^2}(\delta_{(wm)\cdot A}, \delta_{(wm)\cdot AB})\left(\begin{array}{ll}
1 & e_B \\
e_B^2 & 1
\end{array}\right)\delta_{(w'm')\cdot AB}
\]
if \(w = w', m \neq m'\), and
(F.6)
\[
\text{cov}_{\text{s-p}}(\epsilon_{(wm)}, \epsilon_{(w'm')}) \\
= -(W - 1)^{-1}\left(\frac{r_A}{(r_A + 1)^2}(\delta_{(wm)\cdot A}, \delta_{(wm)\cdot AB})\left(\begin{array}{ll}
1 & e_B \\
e_B^2 & 1
\end{array}\right)\delta_{(w'm')\cdot AB}
\right).
\]
if \(w \neq w'\). Letting \(W\) and \(M\) approach infinity in (F.5) and (F.6) proves the result.