Exact solutions of systems of nonlinear differential equations describing the evolution of interacting populations.

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Abstract

The generalization of the simplest equation method to look for exact solutions of systems of nonlinear differential equations is presented. The exact solutions of NDE systems describing the evolution of two interacting populations in two cases (both populations have the low critical density or low critical density is typical for only one of populations) are obtained.

Introduction.

Systems of nonlinear differential equations describe a lot of processes in different fields of science. Since XIX century it became clear that nonlinear phenomena are no less important than the linear ones. Therefore, the theory of nonlinear differential equations began its development.

Usually we use the approximate methods to solve the problems with the systems of nonlinear differential equations, but exact solutions are very useful too. Exact analytical solutions allow to determine the features of solutions behavior under some initial and boundary conditions and verify the numerical calculations.

In [1] the simplest equation method to look for exact solutions of nonlinear differential equations was presented (see also [2–5]). It summarizes a number of methods that were developed earlier, including the tanh-function method [6–9], the Jacobi elliptic function method [10,11] and the Weierstrass function method [12,13].

Here we generalize the simplest equation method for systems of ordinary differential equations. This method applies two main concepts: 1) the idea of the simplest nonlinear differential equation that has lesser order than the equations of the system studied and 2) accounting of possible singularities of the solutions of system studied.

The outline of this paper is as follows. The generalization of the simplest equation method to look for exact solutions of systems of nonlinear differential equations is given in section 1. Sections 2 and 3 are devoted to the exact solutions of NDE systems describing the evolution of two interacting populations in two cases 1) if both populations have the low critical density (section 2) and 2) if low critical density is typical for only one of populations (section 3) [14].
1 The simplest equation method to look for exact solutions of systems of nonlinear differential equations.

Consider the system of $m$ nonlinear differential equations in the polynomial form

$$M_i[y_1, y_2, \ldots y_m] = 0, \quad i = 1, 2, \ldots n$$

and the simplest equation

$$E[Y] = 0$$

with order lesser than the ones of equations $M_i$. The solution of simplest equation is assumed to be known.

For example, we can use as the simplest equation the Riccati equation, the equation for the Jacobi elliptic function, the equation for the Weierstrass elliptic function, and so on.

The unknown functions $y_j, j = 1, 2, \ldots m$ are supposed to be expressed via the solution of the simplest equation $Y$.

Let us describe the simplest equation method to look for exact solutions of systems of nonlinear differential equations. It contains three basic steps.

At the first step we determine the singularity orders of unknown functions $y_j, j = 1, 2, \ldots m$ by the analysis of the dominant terms of system (1) (this step coincides with the first step of the Painleve test).

At the second step the solutions of ODE system (1) are represented as selected simplest equation solution expansions, usually in the polynomial form

$$y_j = F_j(Y) = \sum_k A_{jk} f_k(Y)$$

The terms used in expansions are selected in consideration of singularity orders of simplest equation and singularity orders of the studied system solutions, that were determined at the first step.

At the third step we substitute expansion into the system studied. After consideration of used simplest equation properties, coefficients of various $Y$-function powers are set equal to zero. So we obtain the algebraic system, which solution determine coefficients $A_{jk}$ and, if it is necessary, the restrictions to the parameters of initial system.

Further we consider the applications of the simplest equation method to the systems of nonlinear differential equations that are used in ecology at the description of interacting species.
Exact solutions of the nonlinear differential equations system describing the evolution of two interacting species with low critical densities.

Evolution of two interaction populations with low critical densities can be described by the reaction-diffusion system in the form

\[
\begin{align*}
\frac{du}{dt} &= D_1 \frac{d^2 u}{dx^2} + \alpha_1 u(K_1 - u)(u - L_1) + E_1 - \varepsilon_1 uv \\
\frac{dv}{dt} &= D_2 \frac{d^2 v}{dx^2} + \alpha_2 v(K_2 - v)(v - L_2) + E_2 - \varepsilon_2 uv
\end{align*}
\]

Here \(u \equiv u(x, t)\) and \(v \equiv v(x, t)\) are the densities of populations, \(t\) means time, \(x\) is spatial variable.

Type of interactions between species is determined by signs of parameters \(\varepsilon_i\) (competition at \(\varepsilon_1, \varepsilon_2 > 0\), symbiosis at \(\varepsilon_1, \varepsilon_2 < 0\), ”predator-prey” or ”parasite-host” at \(\varepsilon_1, \varepsilon_2 < 0\)).

Parameters \(D_i, \alpha_i, K_i, L_i\) and \(E_i\) are determined for each population separately and describe the evolution of population without species interactions \((i = 1, 2)\). Coefficient \(D_i\) characterizes the velocity of population diffusion by influence of random motion \([15]\), \(\alpha_i\) define the growth rate, \(K_i\) is the environment capacity, \(L_i\) is the low critical density (if population is under minimum viable level it become extinct), \(E_i\) describe the external influence on ecosystem \([14, 16, 17]\).

System of equations (4) can be reduced to the Burgers-Huxley equation \([18–20]\) by linear substitution if and only if the diffusion velocities of both populations coincide, i.e. \(D_1 = D_2\), and two additional conditions on parameters of system (4) hold. However, it is important to have exact solutions for different diffusion coefficients.

Substituting the variables

\[
\begin{align*}
u' &= \sqrt{\frac{2C_0^2}{\alpha_1 D_1}} u, & v' &= \sqrt{\frac{2C_0^2 D_2}{\alpha_2 D_1^2}} v, & z &= \frac{C_0}{D_1} (x - C_0 t)
\end{align*}
\]

and denoting

\[
\begin{align*}
p_1 &= \sqrt{\frac{2\alpha_1 D_1}{C_0^2}} (L_1 + K_1), & p_2 &= \frac{\alpha_1 D_1 K_1 L_1}{C_0^2}, & E_1' &= E_1 \sqrt{\frac{\alpha_1 D_1^3}{2C_0^5}}, \\
q_1 &= \sqrt{\frac{2\alpha_2 D_2}{C_0^2 D_2}} (L_2 + K_2), & q_2 &= \frac{\alpha_2 D_2 K_2 L_2}{C_0^2 D_2}, & E_2' &= E_2 \sqrt{\frac{\alpha_2 D_1^3}{2C_0^5 D_2^3}}, \\
d &= \frac{D_1}{D_2}, & \varepsilon_1' &= \varepsilon_1 \sqrt{\frac{2D_2}{\alpha_3 C_0^2}}, & \varepsilon_2' &= \varepsilon_2 \sqrt{\frac{2D_1}{\alpha_3 C_0 D_2^2}}
\end{align*}
\]

omitting strokes we obtain the dimensionless form of system (4) in traveling
wave variables

\[
\frac{d^2 u}{dz^2} + \frac{du}{dz} - 2u^3 + p_1 u^2 - p_2 u + E_1 - \varepsilon_1 uv = 0
\]

\[
\frac{d^2 v}{dz^2} + \frac{dv}{dz} - 2v^3 + q_1 v^2 - q_2 v + E_2 - \varepsilon_2 uv = 0
\]  \hspace{1cm} (5)

Use the simplest equation method.

The dominant terms of system (5) are the following

\[
\frac{d^2 u}{dz^2} - 2u^3 = 0
\]

\[
\frac{d^2 u}{dz^2} - 2v^3 = 0
\]  \hspace{1cm} (6)

This truncated system has solution

\[ u(z) = s_1 / z, \quad v(z) = s_2 / z, \]

where \(s_1\) and \(s_2\) can take on values ±1 independently. Therefore, solutions of system (5) have first order singularities.

As the simplest equation we use the Riccati equation in the form

\[
\frac{dY}{dz} = -Y^2 + P_1 Y + P_0
\]  \hspace{1cm} (7)

where \(Y \equiv Y(z)\) and \(P_0, P_1\) are constants.

The Riccati equation belongs to the class of exactly solvable ones, its solutions have first order singularities. Nonhomogeneous solutions of equation (7) can be written as

- At \(P_1^2 + 4P_0 > 0\)

  \[ Y(z) = \frac{1}{2} \left( P_1 + \sqrt{P_1^2 + 4P_0} \tanh \left( \sqrt{P_1^2 + 4P_0}(z - z_0) \right) \right) \]  \hspace{1cm} (8)

  \[ Y(z) = \frac{1}{2} \left( P_1 + \sqrt{P_1^2 + 4P_0} \coth \left( \sqrt{P_1^2 + 4P_0}(z - z_0) \right) \right) \]  \hspace{1cm} (9)

- At \(P_1^2 + 4P_0 = 0\)

  \[ Y(z) = \frac{P_1}{2} + \frac{1}{z - z_0} \]  \hspace{1cm} (10)

- At \(P_1^2 + 4P_0 < 0\)

  \[ Y(z) = \frac{1}{2} \left( P_1 - \sqrt{-(P_1^2 + 4P_0)} \tan \left( \sqrt{-(P_1^2 + 4P_0)}(z - z_0) \right) \right) \]  \hspace{1cm} (11)

  \[ Y(z) = \frac{1}{2} \left( P_1 + \sqrt{-(P_1^2 + 4P_0)} \cot \left( \sqrt{-(P_1^2 + 4P_0)}(z - z_0) \right) \right) \]  \hspace{1cm} (12)

Here \(z_0\) is the constant of integration. Note, that function (5) is the only one bounded on complete number scale, however other solutions can be used on bounded intervals.

As solutions of system (5) are to have first order singularities, we use the substitution

\[ u(z) = \tilde{A}_0 Y + \tilde{A}_1 + \tilde{A}_2 \frac{Y^2}{Y} \]

\[ v(z) = \tilde{B}_0 Y + \tilde{B}_1 + \tilde{B}_2 \frac{Y^2}{Y} \]
Taking into account the properties of function $Y$ (i.e. the Riccati equation (7)), the equivalent form of this substitution is

$$u(z) = A_0 Y + A_1 + \frac{A_2 P_0}{Y}$$

$$v(z) = B_0 Y + B_1 + \frac{B_2 P_0}{Y}$$

(13)

Nonhomogeneity condition for solution (13) is

$$A_0^2 + A_2^2 P_0^2 \neq 0, \quad B_0^2 + B_2^2 P_0^2 \neq 0$$

(14)

**Statement.**

We can assume $A_0 \neq 0$ without loss of generality.

**Proof.** Suppose that $A_0 = 0$. If $A_2 P_0 = 0$ then the nonhomogeneity condition (13) fails, therefore $P_0 \neq 0$, $A_2 \neq 0$. Denote $\hat{Y} = P_0/Y$.

$$\hat{Y}_z = (P_0/Y)_z = -\frac{P_0}{Y^2} Y_z = -\frac{P_0}{Y^2} (-Y^2 + P_1 Y + P_0) =$$

$$= -\frac{P_0^2}{Y^2} - P_1 \frac{P_0}{Y} + P_0 = -\hat{Y}^2 + \hat{P}_1 \hat{Y} + P_0$$

(15)

so function $\hat{Y}$ satisfy the Riccati equation in the form (7) with $\hat{P}_1 = -P_1$.

Therefore solution (13) at $A_0 = 0$ can be presented as

$$u(z) = \hat{A}_0 \hat{Y} + A_1$$

$$v(z) = \hat{B}_0 \hat{Y} + B_1 + \frac{\hat{B}_2}{Y}$$

where $\hat{A}_0 = A_2$, $\hat{B}_0 = B_2$ and $\hat{B}_2 = B_0$.

Substituting (13) in system (5), taking into account

$$Y_z = -Y^2 + P_1 Y + P_0$$

$$Y_{zz} = 2Y^3 - 3P_1 Y^2 + (P_1^2 - 2P_0) Y + P_0 P_1$$

(16)

and setting coefficients of various $Y$-function powers equal to zero, we get the system of fourteen algebraic equations. Consider its solutions.

Equations generated by the largest and the least powers of $Y$ are

$$A_0 (A_0^2 - 1) = 0, \quad A_2 (A_2^2 - 1) = 0, \quad B_0 (B_0^2 - 1) = 0, \quad B_2 (B_2^2 - 1) = 0$$

so we can sort possible solutions by values of parameters $A_0$, $A_2$, $B_0$ and $B_2$.

- Case $A_2 = B_2 = 0$, $A_0^2 = B_0^2 = 1$.

Here substitution (13) is invariant under linear transformations, so we can consider $P_1 = 0$ without loss of generality. Therefore we define the parameters of solution

$$A_1 = \frac{A_0 q_1 - B_0 \epsilon_1 - 1}{6 A_0}$$

$$B_1 = \frac{B_0 q_1 - A_0 \epsilon_2 - \epsilon_1}{6 B_0}$$

$$P_0 = -1/12 \epsilon_1 (-\epsilon_2 + p_1) B_0 A_0 + 1/12 \epsilon_1 (d - 1) B_0 +$$

$$+ 1/12 p_1^2 - 1/12 \epsilon_1 q_1 - 1/12 - 1/2 p_2$$

(17)
and coefficients of initial system, that allow this solution

\[
q_2 = \frac{1}{6} q_1^2 - \frac{1}{6} p_1^2 + \frac{1}{6} \varepsilon_1 q_1 + p_2 + \frac{1}{6} - \frac{1}{6} \varepsilon_2 p_1 - \frac{1}{6} d^2 - \frac{1}{6} \varepsilon_2 (d-1) A_0 - \frac{1}{6} \varepsilon_1 (d-1) B_0 + (\frac{1}{6} \varepsilon_1 p_1 - \frac{1}{6} \varepsilon_2) B_0 A_0
\]

\[
E_1 = A_1 \left( 2 A_1^2 - p_1 A_1 + B_1 \varepsilon_1 + p_2 \right) - A_0 P_0
\]

\[
E_2 = B_1 \left( 2 B_1^2 - q_1 B_1 + \varepsilon_2 A_1 + q_2 \right) - d B_0 P_0
\]

(18)

- Case \( A_2 = B_0 = 0, \ A_0^2 = B_2^2 = 1 \).
  Here system (5) has solution in the form (13) with

\[\begin{align*}
A_1 &= B_1 = 0 \\
P_1 &= (p_1 A_0 - 1)/3 \\
P_0 &= (p_1 A_0 - 2 + p_1^2 - 9 p_2)/18
\end{align*}\]

(19)

at condition that

\[\begin{align*}
q_1 &= B_2 (1 + d - p_1 A_0) \\
q_2 &= (1 + d + 3 p_2 - p_1 (d + 1) A_0)/3 \\
E_1 &= P_0 A_0 (\varepsilon_1 B_2 - P_1 - 1) \\
E_2 &= P_0 B_2 (\varepsilon_2 A_0 + P_1 - d)
\end{align*}\]

(20)

- Case \( A_2 = 0, \ A_0^2 = B_0^2 = B_2^2 = 1 \).
  Here

\[\begin{align*}
A_1 &= 0 \\
P_1 &= (p_1 A_0 - 1 - \varepsilon_1 B_0)/3 \\
B_1 &= (q_1 + B_2 (3 P_1 - d))/6 \\
P_0 &= (P_1 + P_1^2 - p_2 - \varepsilon_1 B_1)/2 \\
\varepsilon_2 &= A_0 (d (B_0 B_2 - 1) - 3 P_1 (1 + B_0 B_2)) \\
E_1 &= A_0 P_0 (\varepsilon_1 B_2 - P_1 - 1) \\
E_2 &= P_0 \left( (-2 q_1 + \varepsilon_2 A_0 B_0 + 12 B_1) B_0 B_2 - d (B_0 + B_2) - P_1 (B_0 - B_2) \right) + 2 B_1^3 + q_2 B_1 - q_1 B_1^2 \\
q_2 &= P_1^2 - 6 B_1^2 + 2 B_1 q_1 - 2 P_1 - 2 P_0 (1 + 3 B_0 B_2) \\
q_1 &= B_2 (d - 3 P_1) + 12 P_1 d A_0 B_0 / \varepsilon_2
\end{align*}\]

(21)

- Case \( B_2 = 0, \ A_0^2 = A_2^2 = B_0^2 = 1 \).
Here

\[ B_1 = 0 \]
\[ P_1 = (q_1 B_0 - d - \varepsilon_2 A_0)/3 \]
\[ A_1 = (p_1 + A_2(3P_1 - 1))/6 \]
\[ P_0 = (dP_1 + P_1^2 - q_2 - \varepsilon_2 A_1)/2 \]
\[ \varepsilon_1 = B_0(A_0A_2 - 1 - 3P_1(1 + A_0A_2)) \]
\[ E_1 = P_0((-2p_1 + \varepsilon_2 A_0B_0 + 12A_1)A_0A_2 - (A_0 + A_2) - P_1 (A_0 - A_2)) + 
+ 2A_1^3 + p_2A_1 - p_1A_1^2 \]
\[ E_2 = B_0P_1(\varepsilon_2 A_2 - P_1 - d) \]
\[ p_2 = P_1^2 - 6A_1^2 + 2A_1p_1 - P_1 - 2P_0(1 + 3A_0A_2) \]
\[ p_1 = A_2(1 - 3P_1) + 12 P_1dA_0B_0/\varepsilon_1 \]

(22)

Other solutions of algebraic system either don’t satisfy the conditions of positiveness of \( p_i, q_i \) and \( d \) or reduce to solutions sited above.

If we consider solutions bounded on complete number scale only, the solutions obtained describe two types of populations behavior: 1) simultaneous monotone change of both populations densities (fig. 1) and 2) change of one of populations state by act of the interactions with the solitary wave of the other population (fig. 2)

Note, that behavior of the second type is generated by solutions in case 3 at \( B_2 = -B_0 \) and in case 4 at \( A_2 = -A_0 \). Thus solution given on fig. 2 is obtained at \( p_1 = 4.8, p_2 = 2.8, d = 0.8, \varepsilon_1 = 3.1, q_1 = 7.7, q_2 = 2.45, \varepsilon_2 = -1.6, E_1 = 0.234, E_2 = 0 \) and has the form
\[ u(z) = \frac{23}{20} + \frac{\sqrt{61}}{20} \tanh \left( \frac{\sqrt{61}}{20} (z - z_0) \right), \]
\[ v(z) = \frac{23}{20} - \frac{\sqrt{61}}{20} \tanh \left( \frac{\sqrt{61}}{20} (z - z_0) \right) - \frac{117}{5 \left( 23 + \sqrt{61} \tanh \left( \frac{\sqrt{61}}{20} (z - z_0) \right) \right)} \]

where \( z_0 \) is the arbitrary constant (at fig. 2 it assumed to be zero). It corresponds to the situation when the predators rise results in the prey extinction, but predators can subsist in the bounded domain only.

3 Exact solutions of the nonlinear differential equations system describing the evolution of two interacting species, one of which is characterized by low critical density.

Consider the system of nonlinear differential equations in the form
\[
\frac{du}{dt} = D_1 \frac{d^2u}{dx^2} + 2\alpha_1 u(K_1 - u)(u - L_1) + E_1 - \varepsilon_1 uv
\]
\[
\frac{dv}{dt} = D_2 \frac{d^2v}{dx^2} + \alpha_2 v(K_2 - v) + E_2 - \varepsilon_2 uv
\]

(23)

In contrast to system (4), it describes the situation, when only one of populations is characterized by low critical density, and the other one in the absence of interacting species and external influence complies with the Ferhulst-Pearl equation [14, 16, 17, 21]. The meanings of parameters of system coincide with the ones declared in section 2.
Substituting the variables

\[ u' = 2(K_1 + L_1)u, \quad v' = 2\frac{\alpha_1 D_2}{\alpha_2 D_1} (K_1 + L_1)^2 v, \quad z = \sqrt{\frac{D_1}{2\alpha_1}} (x - C_0 t) \]

and denoting

\[ p = \frac{K_1 L_1}{2(K_1 + L_1)^2}, \quad E_1' = \frac{E_1}{4\alpha_1(L_1 + K_1)^3}, \quad \varepsilon_1' = \frac{\varepsilon_1 D_2}{D_1 \alpha_2}, \]

\[ q = \frac{\alpha_2 D_1}{\alpha_1 D_2(L_1 + K_1)^2} K_2, \quad E_2' = \frac{D_1^2 \alpha_2 E_2}{4\alpha_1^2 D_1^2 (K_1 + L_1)^4}, \quad d = \frac{D_1}{D_2}, \]

\[ C_1 = \frac{C_0}{\sqrt{2\alpha_1 D_1(K_1 + L_1)}}, \quad \varepsilon_2' = \frac{\varepsilon_2}{\alpha_1 D_2(K_1 + L_1)} \]

omitting strokes we obtain the dimensionless form of system (23) in traveling waves variables

\[ d^2 u dz^2 + C_1 \frac{du}{dz} - 2u^3 + u^2 - pu + E_1 - \varepsilon_1 uv = 0 \]

\[ d^2 v dz^2 + dC_1 \frac{dv}{dz} - v^2 + qv + E_2 - \varepsilon_2 uv = 0 \]

(24)

The dominant terms of system (24) are the following

\[ \frac{d^2 u}{dz^2} - 2u^3 - \varepsilon_1 uv = 0 \]

\[ \frac{d^2 v}{dz^2} - 2v^2 = 0 \]

(25)

The solution of this truncated system is \( u(z) = s_1 \sqrt{1 - 3\varepsilon_1 / z}, \quad v(z) = 6 / z^2 \), where \( s_1 \) can take on value \( \pm 1 \).

So the solution of system (24) we are looking in the form

\[ u(z) = A_0 Y + A_1 + \frac{A_2 P_0}{Y} \]

\[ v(z) = B_0 Y^2 + B_1 Y + B_2 + \frac{B_3 P_0}{Y} + \frac{B_4 P_0^2}{Y^2} \]

(26)

where \( Y \equiv Y(z) \) is the solution of the Riccati equation (7) (here we take into account that function \( u \) has first order singularities and function \( v \) has the second order singularities).

The nonhomogeneity condition for solution in the form (26) is the following

\[ A_0^2 + A_2 P_0^2 \neq 0, \quad B_0^2 + B_2 P_0^4 \neq 0 \]

(27)

Substituting transformation (26) in system (24), taking into account the Riccati equation (7) and setting coefficients of various \( Y \)-function powers equal to zero, we obtain sixteen algebraic equations. Equations generated by the largest and the least powers of \( Y \) are

\[ A_0(2 - 2A_0^2 - \varepsilon_1 B_0) = 0 \]

\[ A_2 P_0^3 (2 - 2A_2^2 - \varepsilon_1 B_4) = 0 \]

\[ B_0(B_0 - 6) = 0 \]

\[ B_4 P_0^4 (B_4 - 6) = 0 \]
We can sort three cases by values of parameters $A_0$, $A_2$, $B_0$ and $B_4$, which specify all possible solutions of system (24) in the form (26) under natural conditions of parameters of initial system. Consider them sequentially.

- **Case $A_0^2 = 1 - 3\varepsilon_1$, $B_0 = 6$, $A_2 = B_4 = 0$.**

As a result we have that $B_3 = 0$. At $A_2 = B_4 = 0$ transformation (29) is invariant under linear transformations, so we assume $P_1 = 0$ without loss of generality. Then we can find sequentially determine the solution parameters

$$B_1 = -3(\varepsilon_2 A_0 + 2C_1 d)/5$$

$$A_1 = \frac{(1 - 3\varepsilon_1)(5 + 3\varepsilon_1\varepsilon_2 - A_0 C_1 (5 - 6\varepsilon_1 d)}{30(1 - 2\varepsilon_1)}$$

$$P_0 = \frac{2\varepsilon_2 A_0 \varepsilon_2^2 - 3\varepsilon_1\varepsilon_2^2 + 125\varepsilon_2 A_1 - 125 q + 17C_1^2 d^2}{200(12C_1 d + \varepsilon_2 A_0)}$$

$$+ 3dC_1 \frac{8C_1^2 d^2 + 27\varepsilon_1\varepsilon_2^2 - 9\varepsilon_2^2}{200(12C_1 d + \varepsilon_2 A_0)}$$

$$B_2 = (6q - B_1^2 - C_1 d B_1 - 48 P_0 - 6\varepsilon_2 A_1 - \varepsilon_2 A_0 B_1)/12$$

and coefficients of the initial system, at which this solution exists

$$p = 2A_1 - 6A_1^2 - 2P_0 - \varepsilon_1 B_2 - \frac{\varepsilon_1 A_1 B_1}{A_0}$$

$$E_1 = pA_1 - A_1^2 - C_1 A_0 P_0 + \varepsilon_1 A_1 B_2 + 2A_1^3$$

$$E_2 = \varepsilon_2 A_1 B_2 - 12P_0^2 - C_1 d B_1 P_0 + B_2^2 - qB_2$$

- **Case $A_0^2 = 1 - 3\varepsilon_1$, $A_2^2 = 1$, $B_0 = 6$, $B_4 = 0$.**

Here the solution parameters and coefficients of system (24) are to satisfy the relations

$$E_1 = A_1 (2A_1^2 - A_1 + p) - P_0 (P_1 + C_1) A_0 -$$

$$- 3(\varepsilon_2 A_0 + 2C_1 d)/5$$

$$A_1 = \frac{(1 - 3\varepsilon_1)(5 + 3\varepsilon_1\varepsilon_2 - A_0 C_1 (5 - 6\varepsilon_1 d)}{30(1 - 2\varepsilon_1)}$$

$$P_0 = \frac{2\varepsilon_2 A_0 \varepsilon_2^2 - 3\varepsilon_1\varepsilon_2^2 + 125\varepsilon_2 A_1 - 125 q + 17C_1^2 d^2}{200(12C_1 d + \varepsilon_2 A_0)}$$

$$+ 3dC_1 \frac{8C_1^2 d^2 + 27\varepsilon_1\varepsilon_2^2 - 9\varepsilon_2^2}{200(12C_1 d + \varepsilon_2 A_0)}$$

$$B_2 = (6q - B_1^2 - C_1 d B_1 - 48 P_0 - 6\varepsilon_2 A_1 - \varepsilon_2 A_0 B_1)/12$$

$$p = -6A_0 A_2 P_0 - 2P_0 - C_1 P_1 - 6A_1^2 + P_1^2 + 2A_1$$

$$q = -4P_1^2 + 1/2B_1 P_1 + \varepsilon_2 A_1 + 1/6B_1^2 + 1/6C_1 d B_1 +$$

$$+ 1/6\varepsilon_2 A_0 B_1 - 2C_1 d P_1 + 8P_0$$

$$P_0 = -B_1 (\varepsilon_2 A_0 d P_0 - 18 P_1^2 + 3B_1 P_1 + B_1^2 + C_1 d B_1 + \varepsilon_2 A_0 B_1)$$

$$36(B_1 + 6P_1 - \varepsilon_2 A_2 + 2C_1 d)$$

$$B_1 = -3/5\varepsilon_2 A_0 - 6/5C_1 d - 6P_1$$

$$A_1 = 1/6 + (1/2P_1 - 1/6C_1) A_2$$

$$B_2 = B_3 = 0$$

$$P_1 = \frac{C_1 (6\varepsilon_1 d - 5) A_0 - 5C_1 (-1 + 2\varepsilon_1) A_2 - \varepsilon_1 (9\varepsilon_1\varepsilon_2 + 5 - 3\varepsilon_2)}{15(1 - 2\varepsilon_1) (A_0 + A_2)}$$

(30)
and condition
\[(C_1 P_1 - 6 A_1^2 - p + 2 A_1 + P_1^2 - 2 P_0) A_0 - 6 (1 - 3 \varepsilon_1) A_2 P_0 - \varepsilon_1 (A_1 B_1 + 6 A_2 P_0) = 0 \quad (31)\]

(which can be reduced to the cubic equation in $C_1$) must hold.

- Case $A_0^2 = 1 - 3 \varepsilon_1$, $A_2 = -A_0$, $B_0 = B_4 = 6$.

Solution with parameters
\[
A_1 = \frac{2d - 5}{6(d - 5)}, \quad B_1 = -\frac{2d}{d - 5} A_0 - \frac{6d}{5} C_1, \\
B_2 = \frac{9d^2}{200} C_1^2 + \frac{5d(2d - 5)}{16(d - 5)^2}, \quad B_3 = \frac{2d}{d - 5} A_0 - \frac{6d}{5} C_1, \\
P_1 = -\frac{d}{3(d - 5)} A_0, \quad P_0 = \frac{d^2}{400} C_1^2 - \frac{19d(2d - 5)}{288(d - 5)^2}
\]

is allowed at
\[
p = \frac{d(4d - 25)}{400} C_1^2 - \frac{5(4d + 3)(2d - 5)}{96(d - 5)^2}, \\
q = \frac{3d^2}{20} C_1^2 + \frac{3d(2d - 5)}{8(d - 5)^2}, \\
\varepsilon_1 = \frac{5}{6d^2}, \quad \varepsilon_2 = \frac{20d}{3(d - 5)}, \quad E_1 = 0, \\
E_2 = \frac{351d^4}{40000} C_1^4 - \frac{21d^4(2d - 5)}{64(d - 5)^2} C_1^2 - \frac{9d^2(2d - 5)^2}{256(d - 5)^4}
\]

Solutions, given in this section, can demonstrate more complicated behavior in comparison with ones obtained in section 2. Some examples are presented.
at figures 3, 4 and 5. Thus at fig. 4 density of one of populations under the influence of the other population diffusion falls before the equilibrium state mounts.

Conclusion.

In this paper the generalization of simplest equation method to look for exact solutions of nonlinear differential equations is described.

Using the simplest equation method, we obtain the exact solutions of systems describing the evolution of two interacting species in two cases 1) when both populations are characterized by low critical densities and 2) when low critical density is important for one of populations only and the other one follows the Ferhulst-Pearl law.

All the solutions obtained describe the spatial transitions between the steady states of the system studied.

References

1. N. A. Kudryashov. Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos, Solitons and Fractals*, 24(5):1217–1231, 2005.

2. N. A. Kudryashov and M. V. Demina. Polygons of differential equations for finding exact solutions. *Chaos, Solitons & Fractals*, 33(5):1480–1496, 2007.

3. N. A. Kudryashov. Solitary and periodic solutions of the generalized Kuramoto-Sivashinsky equation. *Regular and Chaotic Dynamics*, 13(3):234–238, 2008.
4. N. A. Kudryashov and N. B. Loguinova. Be careful with the Exp-function method. *Communications in Nonlinear Science and Numerical Simulation*, 2008. In press.

5. N. A. Kudryashov and N. B. Loguinova. Extended simplest equation method for nonlinear differential equations. *Applied Mathematics and Computation*, 2008. In press.

6. S. Lou, G. Huang, and H. Ruan. Exact solitary waves in a convecting fluid. *Journal of Physics A: Mathematical and General*, 24(11):L587–L590, 1991.

7. N. A. Kudryashov and E. D. Zargaryan. Solitary waves in active-dissipative dispersive media. *J. Phys. A: Math. Gen.*, 29:8067–8077, 1996.

8. E. Fan. Extended tanh-function method and its applications to nonlinear equations. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 277(4-5):212–218, 2000.

9. S.A. Elwakil, S.K. El-labany, M.A. Zahran, and R. Sabry. Modified extended tanh-function method for solving nonlinear partial differential equations. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 299(2-3):179–188, 2002.

10. S. Liu, Z. Fu, S. Liu, and Q. Zhao. Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 289(1-2):69–74, 2001.

11. Z. Yan. The extended jacobian elliptic function expansion method and its application in the generalized Hirota-Satsuma coupled KdV system. *Chaos, Solitons and Fractals*, 15(3):575–583, 2003.

12. N. A. Kudryashov. Exact solutions of the generalized Kuramoto-Sivashinsky equation. *Phys. Lett. A*, 147(5,6):287–291, 1990.
13. A. V. Porubov. Exact travelling wave solutions of nonlinear evolution equation of surface waves in a convecting fluid. *Journal of Physics A: Mathematical and General*, 26(17):797–800, 1993.

14. A. D. Bazykin. *Nonlinear dynamics of interacting populations*. Moscow-Izhevsk: ICI, 2003. In Russian.

15. J. G. Skellam. Random dispersal in theoretical populations. *Bulletin of Mathematical Biology*, 53(1-2):135–165, 1991.

16. G. Yu. Riznichenko. *Lectures on mathematical models in biology*. Moscow-Izhevsk: RCD, 2002. In Russian.

17. G. Yu. Riznichenko and A. D. Rubin. *Mathematical models of biological productional processes*. Moscow-Izhevsk: Institute of Computer Investigations, 2004. In Russian.

18. P. G. Estevez and P. R. Gordoa. Painleve analysis of the generalized Burgers-Huxley equation. *J. Phys. A*, 23:4831–4837, 1990.

19. O. Yu. Yefimova and N. A. Kudryashov. Exact solutions of the Burgers-Huxley equation. *Journal of Applied Mathematics and Mechanics*, 68(3):413–420, 2004.

20. N. A. Kudryashov. *Analytical theory of nonlinear differential equations*. Moscow-Izhevsk: Institute of Computer Investigations, 2004. In Russian.

21. R. Pearl. *The biology of population growth*. New York: A.A. Knopf, 1930.