An Improved Error Term for Turán Number of Expanded Non-degenerate 2-graphs

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Abstract

For a 2-graph $F$, let $H_F^{(r)}$ be the $r$-graph obtained from $F$ by enlarging each edge with a new set of $r - 2$ vertices. We show that if $\chi(F) = \ell > r \geq 2$, then $\text{ex}(n, H_F^{(r)}) = t_r(n, \ell - 1) + \Theta(\text{biex}(n, F)n^{r-2})$, where $t_r(n, \ell - 1)$ is the number of edges of an $n$-vertex complete balanced $\ell - 1$ partite $r$-graph and $\text{biex}(n, F)$ is the extremal number of the decomposition family of $F$. Since $\text{biex}(n, F) = O(n^{2-\gamma})$ for some $\gamma > 0$, this improves on the bound $\text{ex}(n, H_F^{(r)}) = t_r(n, \ell - 1) + o(n^r)$ by Mubayi (2016) [1]. Furthermore, our result implies that $\text{ex}(n, H_F^{(r)}) = t_r(n, \ell - 1)$ when $F$ is edge-critical, which is an extension of the result of Pikhurko (2013) [2].

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1. Introduction

An $r$-graph (or $r$-uniform hypergraph) $G$ consists of a vertex set and an edge set with exactly $r$ vertices in each edge. We sometimes identify an $r$-graph $H$ with its edge set, and denote by $V(H)$ its vertex set. An $r$-clique of order $j$, denoted by $K_j^{(r)}$, is an $r$-graph on $j \geq r$ vertices consisting of all $\binom{j}{r}$ different $r$-tuples. An $r$-graph $G$ is said to be $k$-partite if its vertex set can be partitioned into $k$ classes $V_1 \cup V_2 \cup \cdots \cup V_k$ such that every edge of $G$ contains at most one vertex in $V_i$, $i = 1, \ldots, k$. We say $G$ is a complete $k$-partite $r$-graph if $G$ consists of all $r$-tuples intersecting each vertex class in at most 1 vertex. Given two $r$-graphs $G$ and $F$, we say $G$ is $F$-free if $G$
does not have a (not necessarily induced) subgraph isomorphic to $F$. For a positive integer $\ell$, we denote by $[\ell]$ the set $\{1, \ldots, \ell\}$. For a set $V$ and an integer $r \geq 1$, let $[V]^r$ be the set of all $r$-element subsets of $V$. We write $[\ell]^r$ instead of $[[\ell]]^r$ for simple.

The Turán number $\text{ex}(n, F)$ is the maximum number of edges in an $n$-vertex $F$-free $r$-graph. A simple and important averaging argument of Katona, Nemetz and Simonovits shows that $(\binom{n}{r})^{-1} \text{ex}(n, F)$ form a decreasing sequence of real numbers in $[0, 1]$. It follows that the sequence has a limit, called the Turán density and denoted by $\pi(F)$.

The Turán density is an asymptotic result $\text{ex}(n, F) \sim \pi(F) \binom{n}{r}$. An important fundamental theorem proved by Erdős and Stone characterizes the Turán density of any 2-graph with its chromatic number.

Theorem 1.1 ([4]). Let $F$ be a 2-graph with $\chi(F) = \ell$, then $\pi(F) = \pi(K_\ell) = \frac{\ell - 2}{\ell - 1}$.

However, when $F$ is an $r$-graph, $\pi(F) \neq 0$, and $r > 2$, determining $\pi(F)$ is a hard problem, even for very simple $r$-graphs. In this paper, we focus on the non-degenerated expanded 2-graphs which are $r$-graphs defined as follows. Let $\ell > r \geq 2$ and let $F$ be a 2-graph, and $H_F^{(r)}$ be the $r$-graph obtained from $F$ by enlarging each edge with a new set of $r - 2$ vertices. Mubayi first determined the Turán density of expanded cliques and obtained a stability result.

Theorem 1.2 ([5]). Let $\ell > r \geq 2$. Then

$$\pi(H_F^{(r)}(K_\ell)) = \frac{r!}{(r - 1)^r} \binom{\ell - 1}{r}.$$

Later, using the stability method, Pikhurko obtained the exact number of $\text{ex}(n, H_F^{(r)}(K_\ell))$.

It was mentioned in the survey of Mubayi that Alon and Pikhurko observed that the approach applied to prove Theorem 2 in [2] can be extended to any edge-critical graph $F$ with $\chi(F) > r$. More generally, the following results can be easily achieved through a result of Erdős, the supersaturation technique (see Erdős-Simonovits), and Theorem 1.2.

Theorem 1.3. Let $\ell > r \geq 2$. Let $F$ be any 2-graph with $\chi(F) = \ell$, then

$$\pi(H_F^{(r)}(K_\ell)) = \frac{r!}{(r - 1)^r} \binom{\ell - 1}{r}.$$

This asymptotically gave the Turán number of all non-degenerated expanded 2-graphs.

For self completeness, we will give a short proof of Theorem 1.3 in the next section. Then we prove a stability result of $H_F^{(r)}$. Denote by $T_\ell(n, \ell)$ the complete $\ell$-partite $r$-graph on $n$ vertices, where the size of each vertex class differs at most 1, and set $t_\ell(n, \ell) = |T_\ell(n, \ell)|$. We say two $r$-graphs $G_1$ and $G_2$ of order $n$ are $\varepsilon$-close if we
can add or remove at most \( \epsilon(n) \) edges from \( G_1 \) to make it isomorphic to \( G_2 \); in other words, for some bijection \( \sigma : V(G_1) \to V(G_2) \) the symmetric difference between \( \sigma(G_1) = \{ \sigma(D) : D \in G_1 \} \) and \( G_2 \) has at most \( \epsilon(n) \) edges.

**Theorem 1.4** (Stability of \( H_F^{(r)} \)). Fix \( \ell > r \geq 2 \) and 2-graph \( F \) with \( \chi(F) = \ell \). For every \( \epsilon > 0 \), there exist \( n_0 = n_0(r, \ell, \epsilon) > 0 \), \( \eta = \eta(r, \ell, \epsilon) > 0 \), such that if \( n > n_0 \) and \( G \) is an \( n \)-vertex \( H_F^{(r)} \)-free \( r \)-graph with \( |G| \geq |T_r(n, \ell - 1)| - \eta(n) \), then \( G \) is \( \epsilon \)-close to \( T_r(n, \ell - 1) \).

**Definition 1.1** (\( \text{biex}(n, F) \)). Given a 2-graph \( F \) with \( \chi(F) = \ell \), the decomposition family \( F_F \) of \( F \) is the set of bipartite graphs which are obtained from \( F \) by deleting \( \ell - 2 \) colour classes in some \( \ell \)-colouring of \( F \). Observe that \( F_F \) may contain graphs which are disconnected, or even have isolated vertices. Let \( F_F^* \) be a minimal subfamily of \( F_F \) such that for any \( H \in F_F \), there exists \( H' \in F_F^* \) with \( H' \subset H \). We define

\[
\text{biex}(n, F) := \text{ex}(n, F_F) = \text{ex}(n, F_F^*).
\]

Furthermore, we prove an improved bound for the Turán function \( \text{ex}(n, H_F^{(r)}) \).

**Theorem 1.5.** Given a 2-graph \( F \) with \( \chi(F) = \ell > r \), then

\[
\text{ex}(n, H_F^{(r)}) = t_r(n, \ell - 1) + \Theta(\text{biex}(n, F)n^{r-2}).
\]

We say a 2-graph \( F \) is edge-critical if there exists an edge \( e \in E(F) \) such that \( \chi(F) > \chi(F - e) \). The following theorem is a direct corollary of Theorem 1.5.

**Theorem 1.6.** Given a 2-graph \( F \) with \( \chi(F) = \ell > r \). If \( F \) is edge-critical, then

\[
\text{ex}(n, H_F^{(r)}) = t_r(n, \ell - 1).
\]

In the following section, we will provide a short proofs of Theorem 1.3 by using hypergraph Lagrange method. In section 3, we prove Theorem 1.4 based on the hypergraph removal lemma and a stability result of expanded cliques (see [2]). In the last section, we prove Theorem 1.5, the idea is first to identify a copy of \( F \) in the ‘2-shadow’ of an \( r \)-graph and then extend this copy to \( H_F^{(r)} \).

## 2. Proof of Theorem 1.3

The hypergraph Lagrange method was developed independently by Sidorenko [8] and Frankl and Füredi [9].

Let \( G \) be an \( r \)-graph on \( [n] = \{1, \ldots, n\} \) with edge set \( E \), and \( x = \{x_1, \ldots, x_n\} \) with \( x_i \geq 0 \) for all \( 1 \leq i \leq n \) and \( \sum_{i=1}^{n} x_i = 1 \). Define

\[
p_G(x) = \sum_{\{i_1, i_2, \ldots, i_r\} \in E} x_{i_1}x_{i_2} \cdots x_{i_r},
\]
The Lagrange of $G$ is defined as $\lambda(G) = \max_x p_G(x)$. $G$ is said to be dense if the inequality $\lambda(G') < \lambda(G)$ holds for all its proper subgraphs $G' \neq G$. We say that $G$ covers pairs if for every pair of vertices $i, j$ in $G$, there is an edge containing both $i$ and $j$. We need the following results.

**Lemma 2.1** (8, 9). Every dense graph covers pairs.

Given $r$-graphs $F$ and $G$, we say $f : V(F) \to V(G)$ is a homomorphism if $f(e) \in E(G)$ for all $e \in E(F)$. And we call $G$ is $F$-hom-free if there is no homomorphism from $F$ to $G$. The following theorem shows how to compute the Turán density of any $r$-graph.

**Lemma 2.2** (8). Let $F$ be an $r$-graph, then

$$\pi(F) = \sup \{ r!\lambda(G) : G \text{ is } F\text{-hom-free} \}.$$

**Proof** First, for the $(\ell - 1)$-clique $K_{\ell-1}^{(r)}$, we have that $K_{\ell-1}^{(r)}$ is $H_F^{(r)}$-hom-free since $\chi(F) = \ell$. Otherwise, there is a homomorphism $f : F \to K_{\ell-1}^{(r)}$, then $f^{-1}(v_i)$ form a vertex partition of $F$ and every edge of $H_F^{(r)}$ contains at most one vertex in $f^{-1}(v_i)$. Thus $f^{-1}(v_i)$ is an independent set in $F$, which is a contradiction.

On the other hand, for any dense graph $G$ on at least $\ell$ vertices, we can construct a homomorphism $g$ from $H_F^{(r)}$ to $G$. Since $\chi(F) = \ell$, so there is a partition of $V(F)$ into $\ell$ independent set. We map each independent set to $\ell$ distinct vertices of $G$. For the rest vertices in $H_F^{(r)}$, we denote the vertices in the edge containing $i, j$ by $v^k_{ij}$, $k = 1, \ldots, r - 2$. By Lemma 2.1 $G$ covers pairs. So there is an edge containing both $g(i)$ and $g(j)$. We map $v^k_{ij}$ to the rest $r - 2$ distinct vertices in that edge. Thus $g$ is a homomorphism and by Lemma 2.2, we have

$$\pi(F) = \sup \{ r!\lambda(G) : G \text{ is } F\text{-hom-free} \} = r!\lambda(K_{\ell-1}^{(r)}) = \frac{r!}{(\ell - 1)^r} \binom{\ell - 1}{r},$$

which complete the proof. □

### 3. Stability of $H_F^{(r)}$

To proof Theorem 1.3 we need the following stability result of $H_F^{(r)}$ and the hypergraph removal lemma.

**Lemma 3.1** (2). Fix $\ell > r \geq 2$. For every $\varepsilon_1 > 0$, there are $\eta_1 = \eta_1(r, \ell, \varepsilon_1) > 0$ and $n_0 = n_0(r, \ell, \varepsilon_1)$ such that any $H_F^{(r)}$-free $r$-graph $G$ of order $n \geq n_0$ and size at least $|T_r(n, \ell - 1)| - \eta_1 n^r$ is $\varepsilon_1$-close to $T_r(n, \ell - 1)$.

Hypergraph removal lemma is yield among a series extensions of the Szemerédi’s regularity lemma to $r$-graphs (see 10,11,12,13). Tao 14 also obtained such a generalization. In this paper, we will use two versions of the hypergraph removal lemma as follows.
Lemma 3.2 (Hypergraph Removal Lemma, [13]). Fix an $r$-graph $F$. For every $\varepsilon > 0$, there exist $\eta > 0$ and $n_0 > 0$ such that for every $n$-vertex $r$-graph $G$ with $n > n_0$, if $G$ contains at most $m|V(F)|$ copies of $F$, then one can delete at most $\varepsilon(n^r)$ edges to make it $F$-free.

The second version is as follows. The proof is also based on hypergraph regularity lemma and general dense counting lemma and similar to that of Lemma 3.2.

Lemma 3.3. Fix an $r$-graph $F$. For every $\eta_2 > 0$, there exist $n_0 > 0$ such that for every $n$-vertex $r$-graph $G$ with $n > n_0$, if $G$ is $F$-free, then one can delete at most $\eta_2(n^r)$ edges to make it $F$-hom-free.

Proof of Theorem 4.1. We choose constant $\varepsilon_1 + \eta_2 < \varepsilon$ and $\eta + \eta_2 < \eta_1$.

According to Lemma 3.3, we can delete at most $\eta_2(n^r)$ edges, denoting the remain $r$-graph by $G'$, to make $G'$ $H_F^{(r)}$-hom-free, which implies $G'$ is $H_{K_\varepsilon}^{(r)}$-free, and

\[|G'| \geq |G| - \eta_2(n^r) \geq |T_r(n, \ell - 1)| - \eta(n^r) - \eta_2(n^r) \geq |T_r(n, \ell - 1)| - \eta_1(n^r).\]

Apply Lemma 3.1 to $G'$ for $\varepsilon_1$, we have $G'$ is $\varepsilon_1$-close to $T_r(n, \ell - 1)$. Thus $G$ is $(\varepsilon_1 + \eta_2)$-close to $T_r(n, \ell - 1)$, which complete the proof. \qed

4. Proof of Theorem 1.5

For real constants $\alpha, \beta$, and a non-negative constant $\xi$, we write

$$\alpha = \beta \pm \xi, \quad \text{if } \beta - \xi \leq \alpha \leq \beta + \xi.$$ 

For $U \subseteq V(H^{(r)})$, we denote by $H^{(r)}[U]$ the sub-hypergraph of $H^{(r)}$ induced on $U$ (i.e. $H^{(r)}[U] = H^{(r)} \cap [U]^r$).

Given vertex sets $V_1, \ldots, V_{\ell}$, let $K_\ell^{(j)}(V_1, \ldots, V_{\ell})$ be the complete $\ell$-partite, $j$-graph. If $|V_i| = m$ for all $i \in [\ell]$, then an $(m, \ell, j)$-graph $H^{(j)}$ on $V_1 \cup \cdots \cup V_{\ell}$ is any subset of $K_\ell^{(j)}(V_1, \ldots, V_{\ell})$. Also, we regard the vertex partition $V_1 \cup \cdots \cup V_{\ell}$ as an $(m, \ell, 1)$-graph $H^{(1)}$. For $j \leq i \leq \ell$ and set $A_i \in [\ell]^i$, we denote the $\cup_{\lambda \in A_i} V_\lambda$ induced sub-hypergraph of the $(m, \ell, j)$-graph $H^{(j)}$ by $H^{(j)}[A_i] = H^{(j)}[\cup_{\lambda \in A_i} V_\lambda]$.

To prove Theorem 1.5 it is sufficient to prove the following theorem.

Theorem 4.1. Given a 2-graph $F$ with $\chi(F) = \ell > r \geq 2$, there exist $c_1, c_2, n_0 > 0$ such that if $n \geq n_0$, we have

$$t_r(n, \ell - 1) + c_1 \text{biex}(n, F)n^{r-2} \leq \text{ex}(n, H^{(r)}_F) \leq t_r(n, \ell - 1) + c_2 \text{biex}(n, F)n^{r-2}.$$ 

Proof of Theorem 4.1. Firstly, the left hand-side inequality is obtained as follows. Let $H$ be an $n$-vertex $F$-free 2-graph with $\text{biex}(n, F)$ edges, and let $c = (\ell - 2)^{-2}, c_1 = \left(\frac{\ell - 2}{(r-2)(r-1)^{r-2}}\right)^{\varepsilon}$. Obviously, there exists an $n/(\ell - 1)$-vertex subgraph $H'$ of $H$ with at least $c|H|$ edges.
Next, we construct $G$ from $T_r(n, \ell - 1)$ as follows. Without loss of generality, let $V_1$ be the vertex class of $T_r(n, \ell - 1)$ with largest size, we insert $H'$ into $V_1$. Then for each edge $(u, v)$ in $H'$, add all the $r$-tuples that contains $u, v$ and $(r - 2)$ vertices chosen from different vertex classes except $V_1$ to $G$, i.e.,

$$G = T_r(n, \ell - 1) \cup (\cup_{(u,v)\in H'} E(u, v)), $$

where $E(u,v) = \{ \{ u, v \} \cup f : |f| = r - 2, |f \cap V_i| \leq 1, f \cap V_1 = \emptyset, i = 2, \ldots, \ell - 1 \}.$

Clearly, we have

$$|G| \geq t_r(n, \ell - 1) + c \left( \frac{\ell - 2}{r - 2} \right) \frac{1}{(\ell - 1)^{r - 2}} \text{biex}(n, F)n^{r - 2},$$

and by definition of $F_F$, the graph $G$ is $H_F^{(r)}$-free, and therefore

$$t_r(n, \ell - 1) + c b_{\text{ex}}(n, F)n^{r - 2} \leq \text{ex}(n, H_F^{(r)}).$$

Secondly, the main idea to prove the right hand-side inequality is to find a copy of $F$ in the $2$-shadow of $G$, and then extend $F$ to $H_F^{(r)}$ in $G$. Here by saying $2$-shadow of $G$, denoted by $\Delta_2(G)$, we mean the set of all $2$-tuples $\{u, v\} \in [V(G)]^2$ that are contained in some edge of $G$.

Set $|V(F)| = m$ and choose $\varepsilon > 0$ small enough. Suppose $G$ is an $n$-vertex $H_F^{(r)}$-free $r$-graph with $|G| > t_r(n, \ell - 1) + c_2 b_{\text{ex}}(n, F)n^{r - 2}$, $n \geq n_0$, then by Theorem 3.3, $G$ is $\varepsilon$-close to $T_r(n, \ell - 1)$. Thus $V(G)$ can be partitioned into balanced $V_1 \cup \cdots \cup V_{\ell - 1}$ corresponding to $T_r(n, \ell - 1)$.

Since $\frac{m}{2} \leq b_{\text{ex}}(n, F) = o(n^2)$ or $b_{\text{ex}}(n, F) = 0$, so we have

**Fact 1.** $\frac{1}{2}n^{r - 1} < b_{\text{ex}}(n, F)n^{r - 2} = o(n^r)$ or $b_{\text{ex}}(n, F)n^{r - 2} = 0$.

We call a pair of vertices *bad* if it is covered by at most

$$\kappa(n, F) = |V(H_F^{(r)})| \left( \begin{array}{c} n \\ r - 3 \end{array} \right)$$

edges of $G$.

Let $G'$ be obtained from $G$ by deleting all edges containing bad pairs, at most $\left( \frac{m}{2} \right) \kappa(n, F) < \varepsilon \left( \frac{n}{r} \right)$. So $G'$ is $2\varepsilon$-close to $T_r(n, \ell - 1)$.

For any vertex $v$, we denote by $d(v)$ the vertex degree of $v$ in $G'$, and denote by $N(v)$ the neighbours of $v$ in $G'$, i.e., for each vertex $u$ in $N(v)$, there is an edge containing both $u$ and $v$. Let $N_{V_i}(v) = N(v) \cap V_i$ and $d_i(v) = |N_{V_i}(v)|$. An edge $e$ is *crossing* if $|e \cap V_i| \leq 1$ for $i \in [\ell - 1]$. Let $C(v)$ be the set of crossing edges containing $v$ and $\bar{C}(v)$ be the set of non-crossing edges containing $v$, and we call $d^c(v) = |C(v)|$ the *crossing degree* of $v$ and $d^\bar{e}(v) = |\bar{C}(v)|$ the non-crossing degree.
Observe first that we may assume without loss of generality that
\[ \delta(G') \geq \delta(T_r(n, \ell - 1)). \] (1)
where \( \delta(G') = \min\{d(v) : v \in V(G')\} \). Indeed, if this is not the case, we can repeatedly delete vertices of minimum degree of \( G' \) and delete all edges containing bad pairs until we arrive at a graph \( G'' \) on \( n^* \) vertices with \( \delta(G''') \geq \delta(T_r(n^*, \ell - 1)) \). Denote the sequence of graphs obtained in this way by \( G'_n := G', G'_{n-1}, \ldots, G'_0 \). We need to verify that \( n^* \geq n_0 \). Indeed, we have
\[ |G'_{n-1}| \geq |G'_n| - \delta(G'_n) - \left( \frac{n}{2} \right) \kappa(n, F) \]
\[ > |T_r(n, \ell - 1)| - \delta(T_r(n, \ell - 1)) + \left( c_2 \biex(n, F) n^{r-2} - \left( \frac{n}{2} \right) \kappa(n, F) \right) \]
\[ \geq |T_r(n - 1, \ell - 1)| + \frac{c_2}{2} \biex(n - 1, F) (n - 1)^{r-2} + 1. \]
Similarly, we have
\[ |G'_{n-1}| \geq |T_r(n - i, \ell - 1)| + \frac{c_2}{2} \biex(n - i, F) (n - i)^{r-2} + i. \]
Let \( i^* := n - n^* \). If \( n^* < n_0 \), then \( i^* > n - n \frac{\ell}{r} \), which implies \( i^* \geq \left( \frac{n - i^*}{r} \right) \), a contradiction. Hence we may assume \( i^* \geq n_0 \).

Next we move the vertices to get a max \((\ell - 1)\)-cut of \( G' \), i.e., maximise the number of crossing edges. For \( v \in V_i \) and \( i \neq j \in [\ell - 1] \), let
\[ E^i_{i,j}(v) = \{ e \in C(v) \cup \bar{C}(v) : |e \cap V_i| = s, |e \cap V_j| = t, |e \cap V_k| \leq 1, k \neq i, j \} \]
Then, the max \((\ell - 1)\)-cut implies a vertex partition such that for each vertex \( v \in V_i \), we have
\[ |E^i_{i,j}(v)| \leq |E^i_{i,j}(v)| \quad j \neq i, j \in [\ell - 1] \] (2)
Since the number of crossing edges is at least \( \ell r(n, \ell - 1) - 2 \varepsilon \binom{n}{r} \), so a simple computation would indicate that
\[ |V_i| = (1 \pm \sqrt{\varepsilon}) \frac{n}{\ell - 1}, \quad i \in [\ell - 1] \] (3)
Note that \( d^r(v) + d^r(v) = d(v) \). Let \( X = \{ v \in V(G') : d^r(v) > \varepsilon^{1/(4(r-1))} n^{r-1} \} \) and \( X_i = X \cap V_i \). Set \( V'_i = V_i \backslash X \). Since \( G' \) is \( 2\varepsilon \)-close to \( T_r(n, \ell - 1) \), so
\[ \frac{1}{r} |X| \varepsilon^{1/(4(r-1))} n^{r-1} \leq 2 \varepsilon \binom{n}{r} \]
which implies \( |X| \leq \varepsilon^{2/3} n \).

Then for every \( v \in V'_i, i \in [\ell - 1], \) we have
\[ d^r(v) = d(v) - d^r(v) \geq \delta(T_r(n, \ell - 1)) - \varepsilon^{1/(4(r-1))} n^{r-1} \] (4)
and this implies that for every $j \neq i, j \in [\ell - 1]$,
\[
|N_{V_i}(v) \setminus X| \geq |V_i| - |X| - (\ell - 1)^{r-2} \varepsilon \frac{1}{(r-1)} \cdot n
\]
\[
\geq (1 - 2(\ell - 1)^{r-2} \varepsilon \frac{1}{(r-1)} ) \frac{n}{\ell - 1}
\]
\[
\geq (1 - \varepsilon \frac{1}{(r-1)} ) \frac{n}{\ell - 1}
\]
(5)

Let $q$ be a positive constant depending only on $|V(F)|$ and $\varepsilon$. Its value will be given later.

**Case 1.** If $|X| < q(m - 1)$ and $\text{biex}(n, F)n^{r-2} > 0$. Since $|G| > t_r(n, \ell - 1) + c_2 \text{biex}(n, F)n^{r-2}$, so the number of non-crossing edges in $G'$ is at least
\[
\frac{c_2}{2} \text{biex}(n, F)n^{r-2}.
\]
(6)

For every $i \in [\ell - 1]$, we denote by $E(V_i')$ the set of non-crossing edges in $G'$ that contains at least 2 vertices in $V_i'$. We have
\[
\sum_{i=1}^{\ell-1} |E(V_i')| \geq \frac{c_2}{2} \text{biex}(n, F)n^{r-2} - |X|n^{r-1} \geq \frac{c_2}{3} \text{biex}(n, F)n^{r-2},
\]
where the last inequality is due to Fact 1 and $n$ sufficiently large. Then, there exists some $i^*$ such that
\[
|E(V_{i^*}')| \geq \text{biex}(n, F)n^{r-2}.
\]
(7)

Next we write $D = \{\{u, v\} \in [V_i]' : \text{there exists some } e \in E(V_i'), \text{ such that } \{u, v\} \subseteq e\}$. Because each vertex pair in $D$ is contained in at most $n^{r-2}$ edges in $E(V_i')$, so, by (7), we have $|D| \geq \text{biex}(n, F)$. That is, we can find some $H \in F^*_r$ in $\Delta_2(G')[V_i']$. Let such a copy of $H$ be fixed and assume without loss of generality that $V(H) \subseteq V_{i-1}$. Then we show that $H$ can be extended to a copy of $F$ in the 2-shadow of $G'$ by finding a complete $(\ell - 2)$-partite 2-graph in $\Delta_2(G')$.

Note that by (5), we have for any vertex set $S \subseteq V(G)$ with $S \leq \ell m$ and every $i \in [\ell - 1]$, the number of common neighbours in $V_i'$ of every vertex in $S$ is at least
\[
(1 - \ell m \varepsilon \frac{1}{(r-1)}) \frac{n}{\ell - 1} - \ell m - |X| \geq (1 - \varepsilon \frac{1}{(r-1)}) \frac{n}{\ell - 1}.
\]
(8)
The inequality is due to $\varepsilon$ is small enough and $n$ is sufficiently large.

We inductively find sets $S_i \subseteq V_i'$ of size $m$ which form the parts of the complete $(\ell - 2)$-partite 2-graph. For each $1 \leq i \leq \ell - 2$ in turn, we note that $|V(H)| + (i - 1)m \leq \ell |V(H)|$, and therefore the set $V(H) \cup S_1 \cup \cdots \cup S_{i-1}$ has at least $(1 - \varepsilon \frac{1}{(r-1)}) \frac{n}{\ell - 1} \geq m$ common neighbours in $V_i'$. We let $S_i$ be any set of size $m$ of these common neighbours. Hence we can extend $H$ to a copy of $F$ in $\Delta_2(G')$. 

Finally, recalling that we have deleted the edges that contains bad pairs, each vertex pair (or edge) in this copy of \( F \) is contained in at least \( (\ell + r - 2)\binom{n}{2} \binom{n-3}{r-3} \) edges of \( G' \). Thus we can choose, for each pair, one of these edges that is vertex-disjoint to the chosen ones to form a copy of \( H_F^{(r)} \) in \( G' \).

**Case 2.** If \(|X| \geq q(m-1)\) and \( \text{biex}(n,F)n^{r-2} > 0 \).

Let \( C_i(x) \) be the subset of \( \bar{C}(x) \) that contains at 2 vertices in \( V_i \setminus \{x\} \).

If there is \( i, j \in [\ell - 1] \) (Notice that \( i = j \) is possible), and \( x \in X_i \) such that \( |C_j(x)| \geq \sqrt{\varepsilon}n^{r-1} \).

We write \( D = \{\{u,v\} \in [V_j]^2 : \text{there exists some } e \in C_j(x), \ s.t. \ \{u,v\} \subseteq e \} \) and we claim \( |D[V_j]| \geq \text{biex}(n,F) \). Because the number of 2-tuples in \( D \) that contains at least 1 vertex in \( X_i \) is at most \( \varepsilon \frac{3}{2}n^2 \). Thus the number of edges in \( \bar{C}_j(x) \) that contains at least 1 vertex in \( X_i \) is at most \( \varepsilon \frac{3}{2}n^2 n^{r-3} \). Note that each 2-tuple in \( D[V_j] \) is contained in at most \( n^{r-3} \) edges in \( \bar{C}_j(x) \), thus

\[
|D[V_j]| \geq \frac{1}{n^{r-3}}(\sqrt{\varepsilon} - \varepsilon)\varepsilon n^{r-1} \geq \text{biex}(n,F).
\]

This means that we can again find a copy of some \( H \in \mathcal{F}_F \). And the extending from \( H \) to \( F \), and then to \( H_F^{(r)} \) is the same as that in Case 1.

Otherwise for every \( i, j \in [\ell - 1], x \in X_i \), we have \( |C_j(x)| \leq \sqrt{\varepsilon}n^{r-1} \). Denote by \( C_1^i(x) \) the subset of \( \bar{C}(x) \) that contains exactly 1 vertex in \( V_i \setminus x \). Then by \( \sum_{j=1}^{\ell - 1} |C_j(x)| + |C_1^i(x)| \geq d(x) \), we know

\[
|C_1^i(x)| \geq \varepsilon \frac{1}{n^{r-1}} n^{r-1} - (\ell - 1)\varepsilon n^{r-1} \geq \varepsilon \frac{1}{n^{r-1}} n^{r-1}.
\]  

(9)

**Claim 1.** \( |N(x) \cap V_j'| \geq \varepsilon^{\frac{1}{2\ell-1}} n \) for every \( j \in [\ell - 1] \)

**Proof** By (9), it is easy to know that

\[
|N(x) \cap V_j'| \geq \varepsilon^{\frac{1}{2\ell-1}} n.
\]  

(10)

Moreover, by (9) and the assumption of \( x \), we have for every \( j \neq i, j \in [\ell - 1] \)

\[
|E_1^{i,j}(x)| \geq |E_2^{i,j}(x)| \geq |C_1^i(x)| - (\ell - 1)\varepsilon n^{r-1} \geq \varepsilon^{\frac{1}{2\ell-1}} n^r.
\]

So \( |N(x) \cap V_j'| \geq \varepsilon^{\frac{1}{2\ell-1}} n \) for \( j \in [\ell - 1] \).

Now we start identifying a copy of \( H_F^{(r)} \) in \( G' \) by 2 steps.

Set \( X' := X \). The first step is to identify \( m \) vertices in \( X' \) which are completely joined to an \((r-1)\)-partite \( r \)-graph \( H_{K_{(r-1)}(m)} \) in \( V(G') \setminus X \), with \( m \) vertices, one vertex class of \( K_{(r-1)}(m) \), in one vertex class of \( V(G') \setminus X \). The second step is to extend the structure identified in this way to a copy of \( H_F^{(r)} \) in \( G' \), which is similar as that in Case 1.
By Claim 1, we can choose for each $i$ a set $S_i \in N(x) \cap V'_i$ of size $\varepsilon^{1/2} n$. Since $G'$ is $2\varepsilon$-close to $T_r(n, \ell - 1)$, so the graph $G'[\cup S_i]$ has density at least

$$\frac{\binom{\ell-1}{r-1}((\varepsilon^{1/2} n)^{r-1} - 2\varepsilon n^{r-1})}{(\ell-1)\varepsilon^{1/2} n} \geq \frac{\ell!}{(\ell-1)^r} \left( \frac{\ell-1}{r} \right) \left( 1 - \frac{\varepsilon}{2} \right)$$

$$> \frac{\ell!}{(\ell-2)^r} \left( \frac{\ell-2}{r} \right)$$

$$= \pi(H^{(r)}_{K_{\ell-1}}).$$

Then by Theorem 1.3 we can not remove any edge set of size $\varepsilon n^r$ to make $r$-graph $G'[\cup S_i]$ contain no $H^{(r)}_{K_{\ell-1}(m)}$. So by Lemma 3.2 there are at least $\eta n |V(H^{(r)}_{K_{\ell-1}(m)})|$ copies of $H^{(r)}_{K_{\ell-1}(m)}$, where $\eta$ depends only on $m$ and $\varepsilon$. Choosing $q := 1/\eta$, we can then use the pigeonhole principle and the fact that $|X'| > q(m - 1)$ to infer that there are $m$ vertices in $|X'|$ which are all adjacent to the vertices of one specific copy of $H^{(r)}_{K_{\ell-1}(m)}$ in $G'[\cup S_i]$ as desired.

**Case 3.** When $\text{biex}(n, F)n^{r-2} = 0$, i.e., the single edge graph $H \in F.$

The only difference is that the condition \([\square]\) in Case 1 is no longer hold. We can change the proof slightly by using $G$ instead of $G'$.

First, we change the assumption $|X| < q(m - 1)$ of Case 1 to $X = \emptyset$. For $v \in V(G)$, We call the vertex $u$ a **good neighbour** of $v$ if $(u, v)$ are covered by at least $\kappa(n, F)$ edges of $G$. Note that, similar to \([\square]\), we still have the number of good neighbours in $V_j$ of $v \in V_i$ is at least $(1 - \varepsilon^{1/2} n^{r-1}) \frac{\mu_F}{r}$. Except the only non-crossing edge we identify as a copy of $H$, the rest of proof in Case 1 is the same.

And in Case 2, only one vertex in $X$ is enough for us to find a copy of $H^{(r)}$ because there is a 1-vertex class in some coloring of $F$.

\[\Box\]

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