SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS
ASSOCIATED TO A RICCATI EQUATION OF CONSTANT COEFFICIENTS

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Abstract
We present several second-order linear differential equations that are associated to a particular
Riccati equation with only one constant parameter in its coefficients through the technique of
supersymmetric factorizations and through a Dirac-like procedure. The latter approach is a
minimal extension of the results obtained with the first technique in the sense that it includes
up to two more constant parameters.

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1. Riccati solutions and factorizations

We start with one of the simplest Riccati equation, which nevertheless dictates for example the
behaviour of the Hubble constant of barotropic Friedmann-Robertson-Walker cosmologies [1]
\[ u' + cu^2 + \kappa c = 0, \] (1)
where \( \kappa = \pm 1 \) and \( c \) is a real constant. Employing \( u = \frac{1}{c} \frac{w'}{w} \) one gets the well-known second
order differential equation
\[ w'' + \kappa c^2 w = 0 . \] (2)
For \( \kappa = 1 \) the solution of the latter is \( w_1 = W_1 \cos(c \eta + \varphi) \), where \( \varphi \) is an arbitrary phase,
whereas for \( \kappa = -1 \) one gets \( w_{-1} = W_{-1} \sinh(c \eta) \) where \( W_{\pm 1} \) are amplitude parameters.

The point now is that the Riccati solution \( u_p = \frac{1}{c} \frac{w'}{w} \) mentioned above is only the particular
solution, i.e., \( u_p = -\tan(c \eta) \) and \( u_p = \coth(c \eta) \) for \( \kappa = \pm 1 \), respectively. The particular
Riccati solutions enter as nonoperatorial part in the common factorizations of the second-order
linear differential equations that are directly related to the well-known Darboux isospectral
transformations [2]. Indeed, Eq. (2) can be written
\[ w'' - c(-\kappa c)w = 0 \] (3)
and also in factorized form using Eq. (1) one gets (\( D_\eta = \frac{d}{d\eta} \))
\[ (D_\eta + cu_p) (D_\eta - cu_p) w = w'' - c(u'_p + cu_p^2)w = 0 . \] (4)
To fix the ideas, we shall use the terminology of Witten’s supersymmetric quantum mechanics
and call Eq. (4) the bosonic linear equation. On the other hand, the supersymmetric partner
(or fermionic) equation of Eq. (4) will be
\[ (D_\eta - cu_p) (D_\eta + cu_p) w_f = w''_f - c(-u'_p + cu_p^2)w_f = w'' - c \cdot c_{\kappa,f}(\eta)w_f = 0 , \] (5)
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which is related to the fermionic Riccati equation
\[ -u' + cu^2 - c_{\kappa,f}(\eta) = 0 . \]  
(6)

Notice that for this fermionic Riccati equation the free term is already not constant. Explicitly, one gets
\[ c_{\kappa,f}(\eta) = -u' + cu^2 = \begin{cases} c(1 + 2\tan^2\eta) & \text{if } \kappa = 1 \\ c(-1 + 2\coth^2\eta) & \text{if } \kappa = -1 \end{cases} \]

for the supersymmetric partner Riccati free term. The solutions \( w_f \) are \( w_f = \frac{c}{\cos(\eta + d)} \) and \( w_f = \frac{c}{\sinh(c\eta)} \) for \( \kappa = 1 \) and \( \kappa = -1 \), respectively.

Introducing the operator \( P_\eta = -iD_\eta \), we can write the fermionic equations as follows
\[ (-P_\eta - icu_p) (P_\eta - icu_p) w_f = -P^2_\eta w_f - c(-icu_p + cu^2_p)w_f , \]
(7)

whereas the bosonic case is
\[ (P_\eta - icu_p) (P_\eta - icu_p) w_b = -P^2_\eta w_b - c(icu_p + cu^2_p)w_b , \]
(8)

There is a more general bosonic factorization that has been introduced for the case of the quantum harmonic oscillator by Mielnik [5] that in our case reads
\[ \left(D_\eta + cu_{g,f}\right) \left(D_\eta - cu_{g,f}\right) w_g = w_g'' - c(u_{g,f} + cu^2_{g,f})w_g = w_g'' + \kappa c c(\eta;\lambda) w_g = 0 . \]
(9)

It is given in terms of the general Riccati solution \( u_{g,f}(\eta) \) of the fermionic Riccati equation (6)
\[ u_{g,f}(\eta;\lambda) = u_p(\eta) - \frac{1}{c} D_\eta \left[ \ln(I_\kappa(\eta) + \lambda) \right] = D_\eta \left[ \ln \left( \frac{w_k(\eta)}{I_k(\eta) + \lambda} \right) \right] \]
(10)

and yields the one-parameter family of Riccati free terms \( c_\kappa(\eta;\lambda) \)
\[ -\kappa c_\kappa(\eta;\lambda) = cu^2_{g}(\eta;\lambda) + \frac{d u_g(\eta;\lambda)}{d\eta} = -\kappa c - \frac{2}{c} D_\eta \left[ \ln(I_\kappa(\eta) + \lambda) \right] \]
(11)

where \( I_\kappa(\eta) = \int_0^\eta w^2_k(y) dy \), if we think of a half line problem for which \( \lambda \) is a positive integration constant that occurs as a free parameter of the method.

The free terms \( c_\kappa(\eta;\lambda) \) have the same supersymmetric partner free term \( c_{\kappa,f}(\eta) \). They may be considered as intermediates between the initial constant Riccati free term \( \kappa c \) and its supersymmetric partner free term \( c_{\kappa,f}(\eta) \). From Eq. (9) one can infer the new parametric ‘zero mode’ solutions of the linear equations entailing the functions \( c_\kappa(\eta;\lambda) \) as follows
\[ w_g(\eta;\lambda) = \frac{w_k(\eta)}{I_k(\eta) + \lambda} . \]
(12)

For \( \lambda \to \infty \) the solutions \( w_g \) vanish. For an application to the damped classical oscillator see [3]. In quantum mechanics, where we have different types of solutions and physical interpretation, one can obtain \( w_g \to w_k \) in the same asymptotic limit of the parameter if one introduces the appropriate normalization constant [4].
2. Dirac-like approach

The Dirac equation in the susy nonrelativistic formalism has been discussed by Cooper et al [6] already in 1988. They showed that the Dirac equation with a Lorentz scalar potential is associated with a susy pair of Schroedinger Hamiltonians. This result has been used later by many authors in the particle physics context. In mathematical terms, it is only a simple approach for matrix differential equations. Here we present several applications that may be considered the simplest extension of the results of the previous section in the sense that only up to two more constant parameters are introduced.

2.1 - Let us now consider the following two Pauli matrices

\[ \alpha = -i \sigma_y = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \]

and \[ \beta = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

and write a Dirac-like equation for zero mass and at fixed zero-energy of the form

\[ D1 \equiv [i \sigma_y \eta + \sigma_x (icu_p)]W = 0 \] (13)

where \[ W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \]

is a two component ‘zero-mass’ spinor. This is equivalent to the following decoupled equations

\[ -P \eta w_1 + icu_p w_1 = 0 \] (14)

\[ +P \eta w_2 + icu_p w_2 = 0 \] (15)

Solving these equations one gets \[ w_1 \propto 1/\cos(c \eta) \]

\[ w_2 \propto \cos(c \eta) \]

for the \[ \kappa = 1 \] case and \[ w_1 \propto 1/\sinh(c \eta) \]

\[ w_2 \propto \sinh(c \eta) \]

for the \[ \kappa = -1 \] case. Thus, we obtain

\[ W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_f \\ w_b \end{pmatrix} \] (16)

This shows that the matrix ‘zero-mass’ Dirac equation is equivalent to the two second-order linear differential equations of bosonic and fermionic type, Eq. (2) and Eq. (5), respectively [6].

2.2 - Consider now the following Dirac equation

\[ D2 \equiv [i \sigma_y \eta + \sigma_x (icu_p + K)]W = KW \] (17)

where \[ K \] is a positive real constant. In the left hand side, \[ K \] stands as a mass parameter of the Dirac spinor, whereas on the right hand side it corresponds to the energy parameter, i.e., \[ E = K \]. Thus, we have a Dirac equation for a spinor of mass \[ K \] at the fixed energy \[ E = K \]. This equation can be written as the following system of coupled equations

\[ -P \eta w_1 + (icu_p + K)w_1 = Kw_2 \] (18)

\[ +P \eta w_2 + (icu_p + K)w_2 = Kw_1 \] (19)

This system is equivalent to the following second order equations for the two spinor components, respectively

\[ -P^2 \eta w_i - c \left[ i(\mp P \eta - 2K)u_p + cu_p^2 \right] w_i = 0 \] (20)

where the subindex \[ i = 1, 2 \].

The fermionic spinor components can be found directly as solutions of

\[ D^2 \eta w_i^+ = \left[ c^2(1 + 2 \tan^2 c \eta) + 2icK \tan c \eta \right] w_i^+ = 0 \] for \[ \kappa = 1 \] (21)
and
\[ D^2_\eta w^-_1 - \left[ c^2(-1 + 2\coth^2 c\eta) - 2icK\coth c\eta \right] w^-_1 = 0 \quad \text{for } \kappa = -1, \tag{22} \]
whereas the bosonic components are solutions of
\[ D^2_\eta w^+_2 + \left[ c^2 - 2icK\tan c\eta \right] w^+_2 = 0 \quad \text{for } \kappa = 1 \tag{23} \]
and
\[ D^2_\eta w^-_2 + \left[ -c^2 + 2icK\coth c\eta \right] w^-_2 = 0 \quad \text{for } \kappa = -1. \tag{24} \]
The solutions of the bosonic equations are expressed in terms of the Gauss hypergeometric functions \( _2F_1 \) in the variables \( y = e^{ic\eta} \) and \( y = e^{c\eta} \), respectively
\[ w^+_2 = Ay^{-p} \begin{pmatrix} -\frac{1}{2}(p + iq); -\frac{1}{2}(p - iq), 1 - p; -y^2 \end{pmatrix} + \]
\[ By^p \begin{pmatrix} \frac{1}{2}(p - iq), \frac{1}{2}(p + iq), 1 + p; -y^2 \end{pmatrix} \tag{25} \]
and
\[ w^-_2 = C(-1)^{-\frac{r}{2}}y^{-ir} \begin{pmatrix} -\frac{i}{2}(r + is), -\frac{i}{2}(r - is), 1 - ir; y^2 \end{pmatrix} + \]
\[ D(-1)^{\frac{s}{2}}y^{ir} \begin{pmatrix} \frac{1}{2}(r - s), \frac{1}{2}(r + s), 1 + ir; y^2 \end{pmatrix}, \tag{26} \]
respectively. The parameters are the following: \( p = (-1 - \frac{2K}{c})^{\frac{1}{2}} \), \( q = (1 - \frac{2K}{c})^{\frac{1}{2}} \), \( r = (-1 - i\frac{2K}{c})^{\frac{1}{2}} \), \( s = (-1 + i\frac{2K}{c})^{\frac{1}{2}} \), whereas \( A, B, C, D \) are superposition constants.

It is not necessary to try to find the general fermionic solutions through the analysis of their differential equations (21) and (22) because they are related in a known way to the bosonic solutions [7]. The general fermionic solutions can be obtained easily if one argues that the particular fermionic zero mode is the inverse of a particular bosonic zero mode and constructing the other independent zero mode solution as in textbooks. Thus
\[ w^\pm_1 = \frac{1 + k \int [w^\pm_2]^2 dz}{w^\pm_2}, \tag{27} \]
where \( k \) is an arbitrary constant.

2.3 - The most general case in this scheme with only constant parameters is to consider the following matrix Dirac-like equation
\[ D^3 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} P_\eta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[ \begin{pmatrix} icu_p + K_1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ icu_g + K_2 \end{pmatrix} \right] \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right) = \]
\[ \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \left( \begin{array}{c} w_1 \\ w_2 \end{array} \right). \tag{28} \]
Proceeding as in 2.2 one finds the coupled system of first-order differential equations
\[ \begin{align*}
P_\eta + icu_g + K_2 \end{align*} w_2 &= K_1 w_1 \tag{29} \\
- P_\eta + icu_p + K_1 \end{align*} w_1 &= K_2 w_2 \tag{30} \]
and the equivalent second-order differential equations

\[-P_\eta^2 w_i + \left[ ic (u_p - u_g) + (K_1 - K_2) \right] P_\eta w_i + \left[ ic (\pm P_\eta u_i + K_1 u_g + K_2 u_p) - c^2 u_p u_g \right] w_i = 0, \quad (31)\]

where the subindex \(i = 1, 2\), and \(u_1\) and \(u_2\) correspond to \(u_p\) and \(u_g\), respectively. In the \(D_\eta\) notation this equation reads

\[D_\eta^2 w_i + \left[ c \Delta u_p - i \Delta K \right] D_\eta w_i + \left[ c (\pm D_\eta u_i + (i K_1 u_g + K_2 u_p)) - c^2 u_p u_g \right] w_i = 0. \quad (32)\]

Under the gauge transformation

\[w_i = z_i \exp \left( - \frac{1}{2} \int_\eta \left[ c \Delta u_p - i \Delta K \right] d\tau \right) = z_i(\eta) e^{\frac{i}{2} \int_\eta \Delta K} \quad (33)\]

one gets

\[-P_\eta^2 z_i + Q_i(\eta) z_i = 0, \quad \text{or} \quad D_\eta^2 z_i + Q_i(\eta) z_i = 0, \quad (34)\]

where

\[Q_i(\eta) = c (\pm D_\eta u_i + (i K_1 u_g + K_2 u_p)) - c^2 u_p u_g - \frac{1}{2} D_\eta \left[ c \Delta u_p \right] - \frac{1}{4} \left[ c \Delta u_p - i \Delta K \right]^2 \quad (35)\]

for \(i = 1, 2\), respectively. Since \(Q_i\) are complicated functions we were not able to find analytical solutions of Eq. (34).

The corresponding Dirac spinor is of the following form

\[W(\eta; K_1, K_2, \lambda) = \left( \begin{array}{c} \psi_1(\eta; K_1) \\ \psi_2(\eta; K_2, \lambda) \end{array} \right) = \left( \begin{array}{c} w_1(\eta; K_1) \\ w_2(\eta; K_2, \lambda) \end{array} \right), \quad (36)\]

where \(w_2(\eta; K_2, \lambda)\) is given by Eq. (12) when \(K_1 = K_2 = 0\). The discussion of the asymptotic limit \(\lambda \to \infty\) is similar to that at the end of the first section.

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