M 5-brane and superconformal (0,2) tensor multiplet in 6 dimensions

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ABSTRACT

We present a gauge-fixed M 5-brane action: a 6-dimensional field theory of a self-interacting (0,2) tensor multiplet with 32 world-volume supersymmetries. The quadratic part of this action is shown to be invariant under rigid $OSp(8|4)$ superconformal symmetry, with 16 supersymmetries and 16 special supersymmetries. We explore a deep relation between the superconformal symmetry on the worldvolume of the brane and symmetry of the near horizon anti-de Sitter infinite throat geometry of the M 5-brane in space-time.

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1 Introduction

M 5-brane \cite{1} is one of the most interesting supersymmetric extended objects with the rare property of being completely non-singular \cite{2}. The recently discovered kappa-symmetric action of the M 5-brane is the most mysterious one \cite{3, 4}. The superembedding approach to the M 5-brane theory was developed in \cite{5} and it gives a covariant set of equations of motion of the theory in a beautiful geometric setting. The gauge-fixed M 5-brane action describes a 6-dimensional action of a self-interacting (0,2) tensor multiplet with 32 worldvolume supersymmetries \cite{6}.

There is a growing interest to a 6-dimensional superconformal theory which should describe the small fluctuations (and possibly interactions) of the M 5-brane (M 5-branes) \cite{7, 8, 9}. The standard lore here goes as follows: “The (0, 2) theory of k 5-branes of M theory has a moduli space of vacua $\left( R^5 \right)^k / S_k$. At the origin of the moduli space, the theory is superconformal and has $U(k)$ gauge symmetry” \cite{10}. It seems also that not much of this is actually understood. The most difficult part of making this picture realistic is related to our inability at present to construct a non-Abelian interaction of k (0,2) supersymmetric tensor multiplets. This is different from the picture of interacting D-branes where the underlying Yang-Mills field theory of non-Abelian vectors fields is available. In fact, not much is known even about the superconformal theory of one tensor multiplet in d=6. Only the on shell superconformal tensor multiplet in d=6 as well as an on shell conformal supercurrent superfield are known \cite{11}.

The purpose of this paper is to study various aspects of the M 5-brane theory. We present a detailed exposition and derivation of the $\kappa$-symmetric M 5-brane action \cite{3, 4} in our notation which are given in Appendix A. We further present a detailed derivation of the gauge-fixed action \cite{6} describing the self-interacting (0,2) tensor multiplet and its 32 global supersymmetries, half of which are of the Volkov-Akulov type.

The quadratic part of this action, the free action of the tensor supermultiplet, will be explicitly shown to be invariant under superconformal symmetry with 16 supersymmetries and 16 special supersymmetries. This superconformal symmetry of the 6-dimensional theory has a bosonic part with $SO(6, 2) = SO(8^*)$-symmetry, which is exactly the symmetry of the spacetime M 5-brane near the horizon. The near horizon geometry is given by $adS_7 \times S^4$ \cite{12}. The superconformal symmetry forms an $OSp(8^*|4)$ algebra whose bosonic part is $SO(6, 2) \times USp(4)$. 

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In the process of proving the superconformal symmetry of the (0,2) free tensor multiplet action we have found that in general, except in the very first paper on Wess-Zumino model [13], the concept of rigid superconformal symmetry was not developed to the extent which is required to establish the presence or absence of superconformal symmetry of generic non-gravitational theories. In [14] rigid superconformal transformations for scalar multiplets in presence of a background metric are discussed. In most cases the local superconformal symmetry was developed in the past, in particular in 6-dimensional case the superconformal matter couplings to supergravity were presented in [15]. Since the actions of kappa-symmetric branes gauge-fixed in a flat target space are not gravitational actions but rather actions of matter multiplets, we have worked out the concept of rigid superconformal symmetry in general.

We observe\(^1\) the fascinating relation between the enhancement of supersymmetry near the M 5-brane horizon and emergence of superconformal supersymmetry for the small fluctuations of the 5-brane. The M 5-brane configuration in the bulk breaks 1/2 of the supersymmetry of 11-dimensional supergravity. The brane interpolates between the Minkowski $\mathbb{M}^{11}$ space at infinity, which is maximally supersymmetric, and $adS_7 \times S_4$ near the horizon, which is also maximally supersymmetric [12]. The symmetry of the supersymmetric $adS_7$ space-time is the symmetry of the superconformal field theory on the brane. At present we do not have a clear explanation of the relation between enhancement of supersymmetry in space-time and emergence of special conformal symmetry on the brane. However we have found the relation between the bosonic symmetries of the anti-de Sitter space and conformal symmetry on the brane.

In the past it was often conjectured that conformal supersingleton field theories will provide the action for small fluctuations of the superbranes. A discussion of this conjecture with respect to near horizon geometry of various branes with extensive list of references can be found in the paper of Gibbons and Townsend [12]. We recall the singletons are the most fundamental representations of the anti-de Sitter group $SO(p + 1, 2)$. In case of $SO(6, 2)$ they are called doubletons. The supersymmetric singleton (doubleton) field theories live on the boundary of $adS_{p+2}$ space, given by $S^p \times S^1$. The singleton field theories have some number of scalars and spinors and the action has a superconformal symmetry. The doubleton supermultiplet forms an ultra-short representation of the $OSp(8^*|4)$ superconformal algebra and

\(^1\)We are grateful to J. Maldacena who attracted our attention to this
has 5 scalars, a chiral spinor $(0,2)$ and an antisymmetric tensor with self-dual field strength \cite{16}. The equation of motions of the doubleton theory, which lives on $S^5 \times S^1$ boundary of the $adS_7$ space-time have a superconformal symmetry \cite{17}.

We will find that the small fluctuations of the M 5-brane are indeed given by the doubleton supermultiplet of $adS_7$ group. However, the quadratic part of the M 5-brane, gauge-fixed in a flat 11-dimensional space, defines fields on the 6-dimensional Minkowski space, it is superconformal invariant and different from the superconformal doubleton field theory since the equations of motion are different. After verifying the superconformal symmetry of the small excitations of the M 5-brane we will find that the self-interaction of the single M 5-brane in a flat 11-dimensional background violates the superconformal symmetry of the quadratic approximation.

The paper is organized as follows. In Sec. 2 we introduce the concept of rigid conformal symmetry of non-gravitational theories. For theories with supersymmetry the conformal symmetry leads to the appearance of the second special supersymmetry via the commutator of usual supersymmetry with special conformal symmetry. We discuss the superconformal algebra in general case and the one relevant to M 5-brane theory, an $OSp(8^*|4)$ algebra. Using a ‘triality’ of $SO(6,2)$ one can write down a manifestly symmetric superalgebra with graded (fermionic) bosonic (anti) commutators. In Sec. 3 we present the $OSp(8^*|4)$ superconformal algebra in more familiar form with only 6-dimensional symmetry manifest: we have $SO(5,1)$ Lorentz symmetry, translations, special conformal symmetry and dilatation, i.e. 28 generators of $SO(6,2)$ bosonic conformal symmetries. Next, there are 16 supersymmetries and 16 special supersymmetries and also the generators of internal symmetry, an $USp(4) \approx Spin(5)$ group. We further recall the properties of the doubleton representation of the $OSp(8^*|4)$ superconformal algebra \cite{16} and the doubleton field theory \cite{17}.

In Sec. 4 we present a free theory of the $(0,2)$ tensor multiplet in $d=6$ and prove that it has a superconformal symmetry. We contrast our superconformal action for the small excitations of the M 5-brane with the doubleton field theory \cite{17}.

Sec. 5 describes the M 5-brane action in flat 11-dimensional space and gauge-fixing of local symmetries on the brane, reparametrization and kappa, which leads to the theory of the self-interacting $(0,2)$ tensor multiplet. The quadratic part of this action is shown to coincide with the action of the free theory of the $(0,2)$ tensor multiplet. The interaction terms are shown to
break superconformal symmetry of the free action.

In Sec. 6 we explore the connection between superconformal symmetry on the worldvolume and M 5-brane as a classical BPS solution of 11d supergravity which near the horizon tends to $adS_7 \times S^4$ geometry with enhancement of supersymmetry near the horizon. We display the relation between the linearly realized $SO(6,2)$ part of superconformal symmetry of the $adS_7$ space (considered as a hypersurface in an 8-dimensional space) and non-linearly realized superconformal symmetry $SO(6,2)$ on the 6-dimensional worldvolume. In Conclusion we list our main new results. Appendix A contains the notations. They are rather extensive as we have to deal with spinor structures in $SO(10,1)$, $SO(5,1)$ and $SO(6,2)$ theories. Appendix B has some useful information on rigid superconformal symmetry for non-gravitational theories and finally, Appendix C has details on the derivation of $\kappa$-symmetry of the M 5-brane.

2 Superconformal symmetry in non-gravitational theories

The superconformal group in the past was used as a tool for obtaining supergravity actions with matter couplings invariant under local super-Poincaré transformations in various dimensions [15, 19, 20]. First the action was build with superconformal symmetry and then the conformal symmetry was gauge-fixed: the remaining action provided the action of supergravity coupled to all possible matter multiplets.

At present there is a growing interest to superconformal non-gravitational theories. In principle, for this purpose one can use the available information in the literature on local superconformal theories and decouple supergravity to be left with superconformal theories of matter multiplets. However, in practice, this is a rather complicated route. Therefore we will set up here a general procedure to define superconformal non-gravitational theories directly avoiding introducing and decoupling gravity. This we call rigid superconformal symmetry. First we consider the bosonic part of the superconformal symmetry and after that the enlarging with supersymmetry.
2.1 Conformal transformations in dimension \(d > 2\)

The bosonic part of conformal transformation in dimension \(d\) includes the Lorentz transformation \(M_{\mu\nu}\) of \(SO(d-1,1)\), the translation \(P_\mu\), the special conformal transformation \(K_\mu\) and the dilatation \(D\) \([21]\). Here \(\mu = 0,1,\ldots,d-1\). All these transformations can be nicely packaged in an algebra of the conformal group \(SO(d,2)\) with generators \(\hat{M}_{\hat{\mu}\hat{\nu}} = -\hat{M}_{\hat{\nu}\hat{\mu}}\), \(\hat{\mu} = 0,1,\ldots,d+1\).

Those include the Lorentz generators \(M_{\mu\nu}\) of the subgroup \(SO(d-1,1)\), whereas translations, special conformal transformations and dilatations form the rest of the \(SO(d,2)\) algebra as follows:

\[
P_\mu = 2(M_{\mu d} + M_{\mu(d+1)}), \quad K_\mu = 2(M_{\mu(d+1)} - M_{\mu d}), \quad D = 2M_{(d+1)d} . \quad (2.1)
\]

Thus the conformal group is \(SO(d,2)\) defined by the algebra

\[
[M_{\hat{\mu}\hat{\nu}}, M_{\hat{\rho}\hat{\sigma}}] = \eta_{\hat{\mu}\hat{\rho}}M_{\hat{\sigma}\hat{\nu}} - \eta_{\hat{\nu}\hat{\rho}}M_{\hat{\sigma}\hat{\mu}} , \quad (2.2)
\]

where \(\eta\) is the diagonal metric \((- + + \ldots + -)\). Corresponding to those conformal generators \(\{P_\mu, M_{\mu\nu}, D, K_\mu\}\), with the parameters \(\{a^\mu, \lambda_M^{\mu\nu}, \lambda_D, \Lambda_K^\mu\}\), the generator of the infinitesimal conformal transformation is:

\[
\delta_C = a^\mu P_\mu + \lambda_M^{\mu\nu} M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu . \quad (2.3)
\]

In general, fields \(\phi^i(x)\) of the \(d\)-dimensional theory have the following transformations under conformal group:

\[
\delta_C \phi^i(x) = \xi^\mu(x) \partial_\mu \phi^i(x) + \Lambda_M^{\mu\nu}(x) m_{\mu\nu}^i \phi^i(x)
+ w_i \Lambda_D(x) \phi^i(x) + \Lambda_K^\mu (k_\mu \phi)^i(x) . \quad (2.4)
\]

To specify for each field \(\phi^i\) its transformations under conformal group one has to specify:

i) transformations under the Lorentz group, encoded into the matrix \((m_{\mu\nu})^i_j\). The properties of the matrix \(m_{\mu\nu}\) which defines the Lorentz rotation can be found in appendix \([3]\).

ii) The Weyl weights, \(w_i\).

iii) Possible extra parts of the special conformal transformations \((k_\mu \phi)^i\) (apart from those connected to translations, rotations and dilatations as shown below in (2.5) and (2.6)). Examples of such transformations can be found e.g. in case of fields forming a representation of the real \(N = 1\) supermultiplet in \(d=4\) with Weyl weight \(w\): in particular, under special conformal
transformations the last component field $D$ transforms as
\[ \delta D = -2w \Lambda^K D \mu C \]
where $C$ is the first component of the real superfield \[22, 19\]. The Weyl weight
of $(k_\mu \phi^i)$ should be $w - 1$, and $k_\mu$ are mutually commuting operators.

We now can explain the various terms and why the transformations are
called rigid. The parameters \( \{a_\mu, \lambda^K_\mu, \lambda_D, \Lambda^K_\mu\} \) are all $x$-independent global,
but the transformation of the fields does depend on the coordinates $x$ of the
$d$-dimensional space via the $x$-dependent translation $\xi^\mu(x)$,
\[ \xi^\mu(x) = a^\mu + \lambda^K_\mu x_\nu + \lambda_D x^\mu + (x^2 \Lambda^K_\mu - 2x^\mu x \cdot \Lambda_K) \]  
\[ (2.5) \]
x-dependent rotation $\Lambda_{M \mu \nu}(x)$ and $x$-dependent dilatation $\Lambda_D(x)$:
\[ \Lambda_{M \mu \nu}(x) = \partial_{[\nu} \xi_{\mu]} = \lambda_{M \mu \nu} - 4x_{[\mu} \Lambda_K \nu] \]
\[ \Lambda_D(x) = \frac{1}{d} \partial_\rho \xi^\rho = \lambda_D - 2x \cdot \Lambda_K \]
\[ (2.6) \]
Here the $x$-dependent translation $\xi^\mu(x)$ is the Killing vector satisfying
\[ \partial_\rho (\partial_\xi_{\nu}) - \frac{1}{d} \eta_{\mu \nu} \partial_\rho \xi^\rho = 0 \]
\[ (2.7) \]
In $d=2$ the Killing equations are reduced to $\partial_2 \xi = \partial_2 \xi = 0$ and this leads to
an infinite dimensional conformal algebra. In dimensions $d > 2$ the conformal
algebra is finite-dimensional.

With these rules the conformal algebra is satisfied. The question remains
when an action is conformal invariant. We consider local actions which can
be written as $S = \int d^d x \mathcal{L}(\phi^i(x), \partial_\mu \phi^i(x))$, i.e. with at most first order deriva-
tives on all the fields. For $P_\mu$ and $M_{\mu \nu}$ there are the usual requirements of a
covariant action. For the local dilatations we have the requirement that the
weights of all fields in each term should add up to $d$, where $\partial_\mu$ counts also for
1, as can be seen from \[2.3\]. Indeed, the explicit $\Lambda_D$ transformations finally
have to cancel with
\[ \xi^\mu(x) \partial_\mu \mathcal{L} \approx - (\partial_\mu \xi^\mu(x)) \mathcal{L} = -d \Lambda_D(x) \mathcal{L} \]
\[ (2.8) \]
For special conformal transformations one remains then with
\[ \delta_K S = 2 \Lambda^K_\mu \int d^d x \frac{\hat{\mathcal{L}}}{\partial \hat{\phi}^i} \left( -\eta_{\mu \nu} \phi^i + 2m_{\mu \nu}^i \phi^j \phi^j \right) + \Lambda^K_\mu \frac{S}{\partial \phi^i(x)} (k_\mu \phi^i)(x) \]
\[ (2.9) \]
where $\hat{\partial}$ indicates a right derivative. The first terms originate from the
$K$-transformations contained in \[2.3\] and \[2.6\]. In most cases these are
sufficient to find the invariance and no \((k_{\mu} \phi)\) are necessary. In fact, the latter are often excluded because of the requirement that they should have Weyl weight \(w_i - 1\), and in many cases there are no such fields available.

Although we will show that this condition is satisfied for many dilatational invariant theories, it is non-trivial. As a counterexample we give the action of the scalars \(\phi^1\) and \(\phi^2\) (with Weyl weights \((\frac{d}{2} - 1)\))

\[
\mathcal{L} = \left(1 + \frac{\phi^1}{\phi^2}\right) (\partial_{\mu} \phi^1)(\partial^\mu \phi^2).
\]

(2.10)

When proving special conformal symmetry of the (0,2) theory, we will use \((2.9)\) describing in general case the variation of the action under special conformal symmetries.

Thus the lesson from this section is that to establish a rigid conformal invariance of a non-gravitational action in general one has to use the transformations on fields as given above, see also Appendix B for additional details. The special conformal symmetry in general does not follow from Poincaré and dilatation symmetry and has to be established independently.

### 2.2 The supersymmetric algebra

Enlarging the conformal algebra with supersymmetry \(Q\) one automatically gets a second ‘special’ supersymmetry \(S\) as the commutator of \(K\) and \(Q\). Furthermore in the anticommutator of \(Q\) and \(S\) appears apart from the conformal transformations also an ‘internal’ symmetry group. The algebras were classified in \([23]\). To satisfy the theorem of Haag, Lopuszański and Sohnius, \([24]\) the conformal group should appear as a factored subgroup of the bosonic part of the superalgebra, and the fermionic generators should sit in a spinorial representation of that group. In 6 dimensions this is satisfied with \(OSp(8^*|2N)\).

At first sight, this would give spinor generators in the fundamental of \(SO(8^*)\). But \(SO(6,2)\) has ‘triality’. This means that the left-handed spinor, the right-handed spinor and the vector representations are all equivalent. We will therefore first rotate the generators \(M_{\mu\nu}\) to antisymmetric objects in chiral (right handed) spinor space. This is done using the invertibility of \((\hat{\Gamma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}:\)

\[
(\hat{\Gamma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} (\hat{\Gamma}^{\dot{\rho}\sigma})_{\dot{\alpha}\dot{\beta}} = 16\delta^{[\dot{\rho}}_{\mu}\delta^{\sigma]}_{\nu}.
\]

(2.11)
See appendix A.4 for the properties of gamma matrices in (6, 2) space. These properties imply that \( \hat{\Gamma}_{\hat{\mu}\hat{\nu}} \) are the only antisymmetric ones in chiral space (as they form already 28 independent matrices). We thus define

\[
M_{\hat{\alpha}\hat{\beta}} = \frac{1}{4} (\hat{\Gamma}_{\hat{\mu}\hat{\nu}})_{\hat{\alpha}\hat{\beta}} M_{\hat{\mu}\hat{\nu}} ; \quad M_{\hat{\mu}\hat{\nu}} = \frac{1}{4} (\hat{\Gamma}_{\hat{\mu}\hat{\nu}})^{\hat{\alpha}\hat{\beta}} M_{\hat{\alpha}\hat{\beta}} . \tag{2.12}
\]

In the new form, the \( SO(6, 2) \) algebra is

\[
\left[ M_{\hat{\alpha}\hat{\beta}}, M_{\hat{\gamma}\hat{\delta}} \right] = \hat{\mathcal{C}}_{\hat{\alpha}[\hat{\gamma}} M_{\hat{\delta]\hat{\beta}}} - \hat{\mathcal{C}}_{\hat{\beta}[\hat{\gamma}} M_{\hat{\delta]\hat{\alpha}}} , \tag{2.13}
\]

where the charge conjugation matrix (symmetric in (6, 2)) plays now the role of metric.

For (0,2) theories we want 16 supersymmetry generators + 16 special supersymmetries, which we find in \( OSp(8^*|4) \), which implies that the internal symmetry group is \( USp(4) \approx SO(5) \).

Orthogonal groups are defined by the transformations between vectors \( V^{\hat{\mu}} \), leaving invariant the inner product \( < V, V > = V^{\hat{\mu}} \eta_{\hat{\mu}\hat{\nu}} V^{\hat{\nu}} \) (symmetric metric \( \eta \)). For orthosymplectic groups we introduce superspace vectors \( V^\Lambda = (V^{\hat{\alpha}}, V^i) \), where the first ones are bosonic (8 in our case), and the latter ones fermionic (4 for us). The group is then defined by the transformations \( V^\Lambda \rightarrow M^\Lambda_{\Sigma} V^\Sigma \), leaving invariant the inner product

\[
<V, V > = V^\Lambda \eta_{\Lambda\Sigma} V^\Sigma , \tag{2.14}
\]

where we introduced the orthosymplectic metric

\[
\eta_{\Lambda\Sigma} = \begin{pmatrix} \hat{\mathcal{C}}_{\hat{\alpha}\hat{\beta}} & 0 \\ 0 & \Omega_{ij} \end{pmatrix} , \tag{2.15}
\]

graded symmetric, i.e. symmetric in the upper left part, and the lower right part contains the antisymmetric metric of \( USp(4) \).

Raising and lowering indices with this metric, we can write the generators with all indices down, and using the notation for signs where \( (-)^{\hat{\alpha}} = 1 \), and \( (-)^i = -1 \) (i.e. in these sign factors the bosonic indices \( \hat{\alpha} \) get the value 0, and \( i \) is 1), we then have

\[
M_{\Lambda\Sigma} = -M_{\Sigma\Lambda} (-)^{\Sigma\Lambda} , \tag{2.16}
\]

which are of statistics \( (-)^{\Sigma+\Lambda} \). Thus there are (introducing for future reference the notation \( U_{ij} = U_{ji} \) for what will become the \( USp(4) \) generators):

bosonic : \( M_{\hat{\alpha}\hat{\beta}} = -M_{\hat{\beta}\hat{\alpha}} \), \quad \( U_{ij} \equiv M_{ij} = M_{ji} \),

fermionic : \( M_{\hat{i}\hat{a}} = -M_{\hat{a}\hat{i}} \). \tag{2.17}
All the (anti)commutators can then be written by the rule
\[ [M_{\Lambda}, M_{\Gamma}] = \frac{1}{2} \eta_{\Sigma} M_{\Lambda \Pi} + \frac{1}{2} \eta_{\Pi} M_{\Gamma \Sigma} (-)^{\Pi (\Gamma + \Sigma)} - \frac{1}{4} \eta_{\Sigma} M_{\Gamma \Pi} (-)^{\Pi} - \frac{1}{4} \eta_{\Pi} M_{\Sigma \Gamma} (-)^{\Sigma}. \] (2.18)

The left hand side contains the graded commutator. This rule is thus the same as for the orthogonal groups, see e.g. (2.2), apart from signs corresponding to the interchanges of indices (and we used that the 2 indices on $\eta$ have the same statistics).

The bosonic subalgebra contains two parts. For all indices of the bosonic type, we recover (2.13). For the bosonic operators with two fermionic indices we find the unitary symplectic algebra:
\[ [U_{ij}, U_{k\ell}] = \Omega_{i(k} U_{\ell)j} + \Omega_{j(k} U_{\ell)i}. \] (2.19)

Due to the triality transformation, the commutators of the conformal algebra generators and the supersymmetries are now in the form
\[ [M_{\dot{\mu}}, M_{\dot{\alpha}i}] = -\frac{1}{4} (\hat{\Gamma}_{\dot{\mu}}) \hat{\beta} M_{\dot{\beta}i}, \] (2.20)
while the symplectic transformations of the fermionic generators are
\[ [U_{ij}, M_{k\dot{\alpha}}] = -\Omega_{k(i} M_{j)\dot{\alpha}}. \] (2.21)

The operator $D = 2 M_{76}$ in vector indices becomes now proportional to $\Gamma_{76}$. According to (A.33) its eigenvectors will thus distinguish the chiral and antichiral parts of a spinor in 6 dimensions. This implies that the $Q$ and $S$ supersymmetries, which have respectively weight $\frac{1}{2}$ and $-\frac{1}{2}$ under the dilatations, correspond with the parts in the splitting of the spinors $M_{i\dot{\alpha}}$ as in (A.34). We thus define
\[ M_{i\dot{\alpha}} = \frac{1}{4} \left( \frac{Q_{\alpha' \dot{i}}}{S_{\alpha \dot{\alpha}}} \right). \] (2.22)

The anticommutators of these fermionic generators are
\[ \{ M_{\dot{\alpha}i}, M_{\dot{\beta}j} \} = -\frac{1}{2} \left( \Omega_{ij} M_{\dot{\alpha} \dot{\beta}} + \hat{C}_{\dot{\alpha} \dot{\beta}} U_{ij} \right), \] (2.23)
which we will explicitize below.

Also here it is useful to introduce $x$-dependent supersymmetry parameters to describe rigid superconformal transformations [13]
\[ \epsilon(x) = \epsilon + \gamma_\mu x^\mu \eta, \] (2.24)
where $\epsilon$ and $\eta$ are the parameters of $Q$ and $S$ supersymmetry. The full generator of the infinitesimal transformations of the superconformal group is given by

$$
\delta_{SC} = \delta_C + \bar{\epsilon}Q + \bar{\eta}S + \delta_U, \quad \delta_U = \frac{1}{2} \alpha^{ij} U_{ij} = \alpha^{a'b'} U_{a'b'},
$$

(2.25)

where in addition to conformal transformations defined above we have supersymmetry, special supersymmetry and internal symmetry transformations $U$.

### 3 The superconformal algebra in six dimensions

Here we will rewrite the superalgebra $OSp(8^*|4)$ in (2.18), relevant for six-dimensional (0,2) non-gravitational theory, in more standard terms. First we consider the 28-parameter conformal group $SO(6, 2)$ which include the 21 parameter Poincaré group, dilatations and special conformal transformations. The algebra is given by (2.2), which, using the definitions (2.1), gets the form

$$
\begin{align*}
[M_{\mu\nu}, M^{\rho\sigma}] &= -2\delta_{[\mu}^{[\rho} M_{\nu]}^{\sigma]} , \\
[P_\mu, M_{\nu\rho}] &= \eta_{[\mu} P_{\nu\rho]} , \\
[K_\mu, M_{\nu\rho}] &= \eta_{[\mu} K_{\nu\rho]} , \\
[P_\mu, K_\nu] &= 2(\eta_{\mu\nu} D + 2M_{\mu\nu}) , \\
[D, P_\mu] &= P_\mu , \\
[D, K_\mu] &= -K_\mu .
\end{align*}
$$

(3.1)

On the fields this is realized as

$$
[\delta_C(\xi_1), \delta_C(\xi_2)] = \delta_C (\xi^\mu = \xi_2^\nu \partial_\nu \xi_1^\mu - \xi_1^\nu \partial_\nu \xi_2^\mu) .
$$

(3.2)

The anticommutator of supersymmetries $Q$ produces translation, that of the special supersymmetries $S$ produces a special conformal transfor-

\[1\text{In the 2-dimensional case the analogous set of generators }(M_{01}, D, P_0, P_1, K_0, K_1)\] corresponds to a finite subgroup of the infinite dimensional conformal group and is well known in terms of $L_{-1} = \frac{1}{2}(P_0 - P_1)$, $L_0 = \frac{1}{2}D + M_{10}$, $L_1 = \frac{1}{2}(K_0 + K_1)$, $L_{-1} = \frac{1}{2}(P_0 + P_1)$, $L_0 = \frac{1}{2}D - M_{10}$, $L_1 = \frac{1}{2}(K_0 - K_1)$. Higher order $L_n, |n| \geq 2$ have no analogs in $d > 2$. The $Q$ and $S$ supersymmetries are in $d = 2$ the Neveu-Schwarz generators $G_{-\frac{1}{2}}$ and $G_{\frac{1}{2}}$ respectively.

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mation and finally the anticommutator of supersymmetry with the special supersymmetry produces a dilatation, a Lorentz transformation and an $USp(4) \approx Spin(5)$ rotation $U^{ij}$:

$$\{Q^i_{\alpha'}, Q^j_{\beta'}\} = -2(\gamma_\mu)_{\alpha'\beta'} \Omega^{ij} P^\mu,$$
$$\{S^i_{\alpha'}, S^j_{\beta'}\} = -2(\gamma_\mu)_{\alpha'\beta'} \Omega^{ij} K^\mu,$$
$$\{Q^i_{\alpha'}, S^j_{\beta}\} = -2c_{\alpha'\beta}(\Omega^{ij} D + 4 U^{ij}) - 2(\gamma_\mu\nu)_{\alpha'\beta'} \Omega^{ij} M^{\mu\nu}.$$  \hspace{1cm} (3.3)

The $USp(4)$ part of the algebra is (2.13). There is also a set of commutators between generators of conformal group and fermionic generators which carry the information that the supersymmetry and special supersymmetry transform as spinors under the Lorentz group and that the Weyl weight of $Q(S)$ equals $1/2(1/2)$:

$$[M_{\mu\nu}, Q^i_{\alpha'}] = -\frac{1}{4}(\gamma_{\mu\nu}Q)^i_{\alpha'}, \hspace{1cm} [M_{\mu\nu}, S^i_{\alpha}] = -\frac{1}{4}(\gamma_{\mu\nu}S)^i_{\alpha},$$
$$[D, Q^i_{\alpha'}] = \frac{1}{2}Q^i_{\alpha'}, \hspace{1cm} [D, S^i_{\alpha}] = -\frac{1}{2}S^i_{\alpha}.$$  \hspace{1cm} (3.4)

There are two important commutators: i) between special conformal symmetry and supersymmetry, which produces special supersymmetry. This means that each time when the theory has special conformal symmetry and supersymmetry, the special supersymmetry is guaranteed; ii) Translation and special supersymmetry generate supersymmetry.

$$[K_\mu, Q^i_{\alpha'}] = (\gamma_\mu S)^i_{\alpha}, \hspace{1cm} [P_\mu, S^i_{\alpha}] = (\gamma_\mu Q)^i_{\alpha'}.$$  \hspace{1cm} (3.5)

Finally, there are commutators of supersymmetry and special supersymmetry with internal symmetry carry the information that the supersymmetry and special supersymmetry are in the fundamental representation of the internal group $USp(4)$ as given in (2.21).

This is a full set of generators of $OSp(8^*/4)$ supergroup which is a super extension of the anti-de Sitter group in 7 dimensions. All positive-energy unitary representations of this supergroup are constructed in [16] using super-oscillators. In particular, the ultrashort multiplet, called doubleton was found in [16] by using one pair of oscillators. The corresponding 6-dimensional field theory was expected to contain a two-index antisymmetric tensor field with a self-dual field strength, 5 scalars and a (0,2) chiral spinor. The superconformal doubleton field theory was constructed in [17].
In fact due to the self-duality of the tensor fields only the field equations were found. The theory was constructed for the fields which live on the boundary of the $\text{adS}_7$ which is an $S^p \times S^1$ space. The equations of motion were shown to transform into each other under the action of the superconformal transformations. In particular, the equation of motion for the 5 scalars of the doubleton field theory was found to be

$$ (\nabla^\mu \partial_\mu - 4) \phi^{a'} = 0 , \tag{3.6} $$

where the covariant derivative is due to the fact that the fields are coupled to the curved background of $S^p \times S^1$. We will show below that the action of the free tensor multiplet in Minkowski 6-dimensional space is superconformal invariant and the equation of motion for 5 scalars, being a free equation, is different from (3.6).

### 4 Small excitations of the M 5-brane

We consider the following action which will be later shown to be a quadratic approximation of the gauge-fixed M 5-brane action:

$$ S_{\text{lin}} = \int d^6 x \left[ -\frac{1}{2} H^{\ast \mu \nu} - \frac{1}{2} \partial_\mu X^{a'} \cdot \partial^\mu X^{a'} + 2 \lambda \not\partial \lambda \right] . \tag{4.1} $$

This is the action of the free superconformal (0,2) tensor multiplet in a 6-dimensional Minkowski space $\mathbb{R}^6$. There are 5 scalars $X^{a'}$, a right-handed 16-component spinor $\lambda$ and a tensor field $B_{\mu \nu}$ with on shell self-dual field strength. The auxiliary field $a(x)$ was introduced by Pasti-Sorokin-Tonin [26] in order to write down a 6-dimensional Lorentz covariant action for a self-dual tensor. The derivative of the auxiliary field $v_\mu = \frac{\partial a}{\sqrt{(\partial_\mu a)^2}}$ is used to convert a 3-index field $H_{\mu \nu \rho}$ into 2-index field $H_{\mu \nu}$:

$$ H_{\mu \nu \rho} = 3 \partial_\rho [B_{\mu \nu}] , \quad H_{\mu \nu} \equiv H_{\mu \nu \rho} v^\rho . \tag{4.2} $$

The action depends on

$$ H_{\mu \nu}^* = H_{\mu \nu}^* v^\rho , \quad H_{\mu \nu}^* \equiv \frac{1}{6} \epsilon_{\mu \nu \rho \sigma \tau \phi} H^{\sigma \tau \phi} , \tag{4.3} $$

3Recently an action of a free (0,1) tensor multiplet was presented in [25]. This action was shown to be supersymmetric. Our proof that the action of (0,2) tensor multiplet is not only supersymmetric but also superconformally symmetric can be easily applied to (0,1) theory since our theory consists of a (0,1) tensor multiplet and a hypermultiplet.
and
\[ H^-_{\mu\nu} \equiv \frac{1}{2}(H_{\mu\nu} - H^*_{\mu\nu}) . \] (4.4)

The action is invariant under the local symmetries I, II and III discovered in [25]:

I) \[ \delta_I B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]} , \quad \delta_I a = 0 , \]
II) \[ \delta_{II} B_{\mu\nu} = 2H^-_{\mu\nu} \frac{1}{\sqrt{u^2}} \varphi , \quad \delta_{IIa} = \varphi , \]
III) \[ \delta_{III} B_{\mu\nu} = \psi_{[\mu v_{\nu]} , \quad \delta_{IIIa} = 0 . \] (4.5)

Using these symmetries one can show that the field equation for the tensor field can be reduced to the self-duality condition:
\[ H^-_{\mu\nu\rho} = \frac{1}{2}(H_{\mu\nu\rho} - H^*_{\mu\nu\rho}) = 0 . \] (4.6)

The transformations I and III are reducible gauge invariances. First there is the usual one for gauge antisymmetric tensors. But there is more:

a) \[ \Lambda_\mu = \partial_\mu \Lambda , \]
b) \[ \psi_\mu = v_\mu \psi , \]
c) \[ \Lambda_\mu = u_\mu \Lambda' , \quad \psi_\mu = -\sqrt{u^2} \partial_\mu \Lambda' . \] (4.7)

The action is invariant under rigid conformal symmetry under condition that we assign the following weights to our fields
\[ w(X^a') = w(B_{\mu\nu}) = 2 , \quad w(a) = 0 , \quad w(\lambda) = \frac{5}{2} . \] (4.8)

It follows that
\[ w(v_\mu) = 0 , \quad w(H_{\mu\nu\rho}) = w(H_{\mu\nu\rho} v^\rho) = 3 . \] (4.9)

One can easily see the Poincaré symmetries. For the special conformal transformations we have to check (2.9). It is satisfied separately for \( \phi^i \) representing the fields \( a, X^a', \lambda \) and \( B_{\mu\nu} \). In fact the mechanisms at work are typical for the usual way in which dilatational invariant theories are often conformal invariant:

1. \( a \) is a scalar of Weyl weight 0, such that both terms in the bracket of (2.9) are vanishing.
2. The term with $\phi^i \to X^a'$ is a total derivative.

3. $\lambda^i$ falls in the category of spinors whose derivative appears in the action as $\partial \lambda$, and have Weyl weight $(d - 1)/2$. The various terms in (2.9) cancel.

4. Similarly $B_{\mu\nu}$ with Weyl weight $2$ belongs to the category of vectors or antisymmetric tensors whose derivatives appear as $\partial [\mu \nu \rho \ldots \rho]$ with Weyl weight $p - 1$. Again with this weight the terms cancel in (2.9). This value of the Weyl weight is what we need also in order that their gauge invariances and their zero modes commute with the dilatations.

The action is invariant under the superconformal supersymmetry transformations with left-handed parameter $\epsilon$ and right-handed $\eta$:

$$
\delta_{\epsilon} X^a' = -2\epsilon(x)\gamma^a' \lambda, \\
\delta_{\epsilon} \lambda = \frac{1}{2}(\partial X^a')\gamma_a' \epsilon(x) - \frac{1}{6}h^+_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon(x) + 2X^a' \gamma_a' \eta, \\
\delta_{\epsilon} B_{\mu\nu} = -2\epsilon(x)\gamma_{\mu\nu} \lambda .
$$

(4.10)

We remind that $\epsilon(x) = \epsilon + \gamma_\mu x^\mu \eta$ as explained in (2.24) with respect to rigid supersymmetries. Here the off shell (anti) self-dual tensor field strength is defined as

$$
h^\pm_{\mu\nu\rho} \equiv \frac{1}{4}H_{\mu\nu\rho} - \frac{3}{2}v_{[\mu} H^\pm_{\nu\rho]}, \\
= v^\sigma \left(v_{[\sigma} H_{\mu\nu\rho]} \pm \frac{1}{6}\epsilon_{\mu\nu\rho\tau\phi} u_{[\sigma} H_{\tau\phi\chi]}\right), \\
= \pm \frac{1}{4}H^*_{\mu\nu\rho} - 2\epsilon^\sigma v_{[\alpha} H^\pm_{\beta\gamma\delta]} .
$$

(4.11)

To prove the invariance one can use that under variations of $a$ and $H_{\mu\nu\rho}$ we have

$$
\delta S_{\text{lin}} = \int d^6x \left(\frac{1}{12}H^*_{\mu\nu\rho} - \frac{2}{3}h^+_{\mu\nu\rho}\right) \delta H^{\mu\nu\rho} + \frac{1}{2\sqrt{u^2}}\epsilon^{\mu\nu\rho\sigma\tau\phi} H^+_{\mu\nu\rho} H^-_{\sigma\tau\phi} v_\tau \partial_\phi \delta a .
$$

(4.12)

After taking the $\epsilon$-variations one adds partial derivatives such that the derivatives do not act on $\lambda$. Then one finds that there only remain derivatives on $\epsilon$ from the variation of $\lambda$. We thus get

$$
\delta_{\epsilon} S_{\text{lin}} = 2 \int d^6x \bar{\lambda} \gamma^\mu \left(\partial \gamma X^a' + \frac{1}{3}h^+_{\nu\rho\sigma} \gamma^{\nu\rho\sigma}\right) \partial_\mu \epsilon(x) .
$$

(4.13)
Using $\partial_\mu \epsilon(x) = \gamma_\mu \eta$, the second term drops due to the identity $\gamma^\mu \gamma^\nu \gamma_\mu \eta = 0$. The first term can then be cancelled by the explicit $\eta$ term in $\delta \lambda$. Thus we have established that the action has an off shell superconformal symmetry as well as a set of gauge symmetries defined in (4.3).

The equations of motion for the scalars of the free superconformal tensor multiplet in Minkowski space are

$$\partial^\mu \partial_\mu X^{a'} = 0, \quad (4.14)$$

which is different from the equations for the doubletons (3.6) which was presented in Sec. 3.

One can calculate the algebra of such transformations. The easiest part is the action of supersymmetries and special supersymmetries on the 5 scalars $X^{a'}$. It is given by

$$[\delta_1(\epsilon_1(x)), \delta_2(\epsilon_2(x))] X^{a'} = \delta C (\xi^\mu = 2\bar{\epsilon}_1(x) \gamma^\mu \epsilon_2(x)) X^{a'} + \delta U \left( \alpha^{a'b'} = 4\bar{\epsilon}_1 \gamma^{a'b'} \eta_2 - 4\bar{\epsilon}_2 \gamma^{a'b'} \eta_1 \right) X_{a'}. \quad (4.15)$$

This expression is a convenient way of representing all the anticommutators in (3.3). Note that the expression for $\xi^\mu$ implies

$$a^\mu = 2\bar{\epsilon}_1 \gamma^\mu \epsilon_2, \quad \lambda^{\mu\nu}_{\alpha'} = 2\bar{\epsilon}_1 \gamma^{\mu\nu} \eta_2 - 2\bar{\epsilon}_2 \gamma^{\mu\nu} \eta_1,$$

$$\lambda_D = 2\bar{\epsilon}_1 \eta_2 - 2\bar{\epsilon}_2 \eta_1, \quad \Lambda^K = 2\bar{\eta}_1 \gamma^\mu \eta_2. \quad (4.16)$$

The auxiliary field $a$ is a singlet on all the fermionic symmetries [27]. The algebra is

$$[\delta_1, \delta_2] a = 2\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu a + \phi(x) = 0, \quad (4.17)$$

which defines the gauge II transformation parameter $\phi(x) = -2\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu a(x)$ to be a function of a derivative of the $a$-field. On $B_{\mu\nu}$ field the algebra will be most complicated, as it will include all 3 type of gauge transformations I, II, III depending on fields and bilinear in susy and special susy parameters. On fermions the algebra will contain terms with equations of motion. Altogether this is an example of a so called ‘open’ and ‘soft gauge algebra’ [28], which often appear in supersymmetric gauge theories, supergravity and conformal theories. ‘Open algebra’ refers to the fact that equations of motion are necessary to close the algebra, the algebra is exact on physical states. ‘Soft algebra’ means that the commutators close with *field-dependent structure functions*:

$$[L_i, L_j] = c_{ij}^k L_k, \quad (4.18)$$
Thus the ‘structure constants’ of the group are not really constant but are field dependent and therefore space-time dependent. Starting from this type of a soft gauge algebra which follows from the symmetries of a given Lagrangian, one can define a so-called ‘hard gauge algebra’ where all fields are taken to be constant (consistent with the field equations). All structure constants of the algebra become then indeed constants $\partial_\mu c_{ij}^k = 0$. This can be understood as a bona fide algebra. It does not have to be associated with any particular Lagrangian and could have a potential of describing theories with symmetries of a given supergroup even in absence of a particular field theory Lagrangian. The corresponding abstract algebra for the 6-dimensional superconformal group was presented in the previous section on pure algebraic basis. After identification of the ‘soft algebra’ from the dynamical Lagrangian of the tensor multiplet, one can verify that the ‘hard algebra’ derived from this Lagrangian is exactly the one given in Sec. 3.

5 Classical action of the M 5-brane.

The classical kappa-symmetric action of the M 5-brane was discovered by Bandos, Lechner, Nurmagambetov, Pasti, Sorokin and Tonin in Lorentz covariant form [3] and by Aganagic, Park, Popescu and Schwarz [4] in Lorentz-non-covariant form. The covariance was achieved in [3] with the help of an auxiliary scalar field and additional gauge symmetries (type II and III). Upon gauge-fixing of these additional gauge symmetries a covariant M 5-brane action can be reduced to a non-covariant form [4].

The form of both actions is rather complicated, the Wess-Zumino term was not yet derived in an explicit form and finally there is a technical problem associated with different notations, sometimes not fully presented, in all available M 5-brane papers.

Therefore we will first present a detailed form of the Lorentz covariant classical action in a flat 11-dimensional background together with the complete set of notations in Appendix A and more details on $\kappa$-symmetry in Appendix C. We will also give an explicit form of the WZ terms.
5.1 The action and its symmetries

We start from the following fields on the 6-dimensional world-volume of the 5-brane.

\[ X^M(x) ; \quad \theta(x) ; \quad B_{\mu\nu}(x) ; \quad a(x) . \]  

Here 11 \( X^M(x) \) and 32 \( \theta(x) \) are coordinates of the 11-dimensional superspace which however are not considered as labels only but as dynamical variables on the brane. \( B_{\mu\nu}(x) \) is the tensor field on the brane and \( a(x) \) is an auxiliary scalar. The action in a flat 11-dimensional background is

\[ S_{M5} = \int d^6x \left( -\sqrt{-\det(g_{\mu\nu} + i\mathcal{H}^*_{\mu\nu} - \frac{\sqrt{g}}{4} \mathcal{H}^*_{\mu\nu} \mathcal{H}_{\mu\nu})} \right) + \int_{M_7} I_7 , \]  

The first term in the action is given as an integral over the 6-dimensional volume and the second term, Wess-Zumino term, is presented in the form of an integral over the 7-dimensional manifold \( M_7 \), which has as a boundary the 6-dimensional worldsheet, and \( I_7 \) is a total derivative (see section 5.2).

We used the following notations:

\[ g_{\mu\nu} = \Pi^M_{\mu} \eta_{MN} \Pi^N_{\nu} ; \quad g = -\det g_{\mu\nu} \]
\[ \Pi^M_{\mu} = \partial_\mu X^M - \bar{\theta} \Gamma^M \partial_\mu \theta \]
\[ H = dB , \quad \text{which is } H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} \]
\[ \mathcal{H} = H - c_3 , \quad \text{which is } \mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} - c_{\mu\nu\rho} \]
\[ u_\mu = \partial_\mu a ; \quad u^2 = u_\mu g^{\mu\nu} u_\nu \]
\[ v_\mu = \frac{u_\mu}{\sqrt{u^2}} ; \quad v^2 = 1 \]
\[ \mathcal{H}_{\mu\nu} = v^\rho \mathcal{H}_{\mu\nu\rho} \]
\[ \mathcal{H}^*_{\mu\nu} = v^\rho \mathcal{H}^*_{\mu\nu\rho} , \quad \text{where } \mathcal{H}^*_{\mu\nu\rho} = \frac{\sqrt{g}}{6} \epsilon^{\rho\sigma\tau\phi} \mathcal{H}_{\sigma\tau\phi} \]
\[ \mathcal{H}^\pm_{\mu\nu} = v^\rho \mathcal{H}^\pm_{\mu\nu\rho} \]
\[ I_7 = R_7 - \frac{1}{2} \mathcal{H} R_4 \]
\[ R_4 = \frac{1}{2} d\bar{\theta} \Gamma_{MN} d\theta \Pi^M \Pi^N = dc_3 \]
\[ R_7 = d\theta c_6 - \frac{1}{2} c_3 \wedge R_4 = \frac{1}{5!} \Pi_{M_1} \cdots \Pi_{M_5} d\bar{\theta} \Gamma_{M_1 \cdots M_5} d\theta . \]  

Here \( \mu, \nu \) indices are raised or lowered with \( g_{\mu\nu} \). The expression \( R_4 \) is \( d \)-closed, which shows the consistency of the definition of \( c_3 \), while the consistency of
the definition of $c_6$ follows from
\[
2dR_7 + R_4R_4 = 0 .
\] (5.4)

Explicit forms of $c_3$ and $c_6$ will be given below. It is further useful to introduce a notation
\[
\mathcal{G} = \sqrt{-\det(g_{\mu\nu} + i\mathcal{H}^*_{\mu\nu})} .
\] (5.5)

The gauge symmetries of the classical action in 6 dimensions are:

- local diffeomorphisms
- $\kappa$-symmetry (infinitely reducible)
- I, II, III (reducible)

\[ I \quad \delta_I B_{\mu\nu} = 2\partial_{[\mu}J_{\nu]} ; \quad \delta_I a = 0 \]
\[ II \quad \delta_{II} B_{\mu\nu} = \frac{1}{\sqrt{u^2}} \varphi \left( \mathcal{H}_{\mu\nu} + 2\frac{\partial\mathcal{G}}{\partial\mathcal{H}^*_{\mu\nu}} \right) ; \quad \delta_{II} a = \varphi \]
\[ III \quad \delta_{III} B_{\mu\nu} = \psi_{[\mu}J_{\nu]} ; \quad \delta_{III} a = 0 . \] (5.6)

For the zero modes of I, and III, see (4.7).

Furthermore there are rigid symmetries in 11 dimensions

- Poincaré (translations with parameter $a^M$ and rotations $\Lambda_{MN}$).
\[
\delta_{\text{P}} X^M = -a^M - \Lambda_{MN}X^N ; \quad \delta_{\text{P}} \theta = -\frac{1}{4}\Gamma_{MN}\Lambda^{MN}\theta
\] (5.7)

For later convenience we took a minus sign for the translations in 11 dimensions.

- supersymmetry

Under rigid space-time supersymmetry and 6-dimensional local $\kappa$-symmetry
\[
\delta_\epsilon \theta = \epsilon ; \quad \delta_\epsilon X^M = \epsilon\Gamma^M \theta
\]
\[
\delta_{\kappa} \theta = (1 + \Gamma)\kappa ; \quad \delta_{\kappa} X^M = \bar{\Gamma}^M \delta_{\kappa} \theta . \] (5.8)

The auxiliary field $a$ does not transform under $\epsilon$-supersymmetry and $\kappa$-transformations. The matrix $\Gamma$ and the $\kappa$-transformation of $B_{\mu\nu}$ will be given in section 5.3. For giving its $\epsilon$-transformation, we first write
\[
\delta_\epsilon R_4 = 0 ; \quad R_4 = dc_3 ; \quad \delta_\epsilon c_3 = dc_2(\epsilon) . \] (5.9)
This leads to explicit expressions \[29\]
\[
c_3 = \frac{1}{2} \bar{\theta} \Gamma_{MN} d\theta \left( \Pi^M \Pi^N + \bar{\theta} \Gamma^M d\theta \Pi^N + \frac{1}{3} \bar{\theta} \Gamma^M d\theta \bar{\theta} \Gamma^N d\theta \right)
\]
\[
c_2(\epsilon) = \frac{1}{2} \epsilon \Gamma_{MN} \theta \left( \Pi^M \Pi^N + \frac{2}{3} \bar{\theta} \Gamma^M d\theta \Pi^N + \frac{11}{15} \bar{\theta} \Gamma^M d\theta \bar{\theta} \Gamma^N d\theta \right)
\]
\[
+ \frac{1}{2} \epsilon \Gamma^M \theta \bar{\theta} \Gamma_{MN} d\theta \left( -\frac{1}{3} \Pi^N - \frac{4}{15} \bar{\theta} \Gamma^N d\theta \right). \tag{5.10}
\]
We can thus choose the transformation of $B_{\mu\nu}$ such that $\mathcal{H}$ is invariant under rigid supersymmetry:
\[
\delta_\epsilon \mathcal{H} = 0 \rightarrow \delta_\epsilon B = c_2(\epsilon). \tag{5.11}
\]

### 5.2 The Wess-Zumino term

Now we will define the Wess-Zumino term. Due to (5.4) we have that $dI_7 = 0$, and \[30\]
\[
I_7 = dc_6 - \frac{1}{2} H R_4 = d \left( c_6 - \frac{1}{2} B R_4 \right) = d \left( c_6 + \frac{1}{2} H c_3 \right). \tag{5.12}
\]
The Wess-Zumino term can thus also be defined as the 6-dimensional integral of one of the expression in brackets in the equation above. Its normalization can be fixed by the requirement that the invariance III should be valid\[7\]. We have that
\[
\delta_{III} H_{\mu\nu\rho} = 3 \sqrt{u^2 v^2} \partial_\nu \psi_\rho - \frac{1}{\sqrt{u^2}}; \quad \delta_{III} \mathcal{H}_{\mu\nu}^* = 0 \tag{5.13}
\]
\[
\delta_{III} \int d^6 x \sqrt{g} \mathcal{H}^{*\mu\nu} H_{\mu\nu} = -\frac{1}{3} \int d^6 x \sqrt{g} c^{*\mu\nu\rho} \delta_{III} H_{\mu\nu\rho} = 2 \int (\delta_{III} H) c_3,
\]
such that the appropriate normalization is
\[
I_{WZ} = \int_{M_7} I_7 = \int_{M_6} \left( c_6 + \frac{1}{2} H c_3 \right)
\]
\[
= \int d^6 x \frac{1}{6!} \epsilon_{\mu_1...\mu_6} \left( c_{\mu_1...\mu_6} - 10 H_{\mu_1\mu_2\mu_3} c_{\mu_4\mu_5\mu_6} \right). \tag{5.14}
\]

To get an explicit form for $c_6$, using the above result for $c_3$, we have to solve
\[
dc_6 = \frac{1}{3!} d\bar{\theta} \Gamma_{M_1...M_5} d\theta \Pi^{M_1} \cdots \Pi^{M_5}
\]
\[
+ \frac{1}{8} d\bar{\theta} \Gamma_{M_1M_2} d\theta d\bar{\theta} \Gamma_{M_3M_4} d\theta (\Pi^{M_1} \cdots \Pi^{M_4} + \bar{\theta} \Gamma^{M_1} d\theta \Pi^{M_2} \cdots \Pi^{M_4})
\]
\[
\quad + \frac{1}{3} d\bar{\theta} \Gamma^{M_1} d\theta \bar{\theta} \Gamma^{M_2} d\theta \Pi^{M_3} \Pi^{M_4}. \tag{5.15}
\]

\[4\]Alternatively, but more difficult, it is fixed by $\kappa$-symmetry.
We find the following solution (determined up to total derivatives)

\[
c_6 = \bar{\theta} \Gamma_{M_1 \ldots M_5} d\theta \left( \frac{1}{30} \Pi^{M_1} \ldots \Pi^{M_5} + \frac{1}{48} \bar{\theta} \Gamma^{M_1} d\theta \Pi^{M_2} \ldots \Pi^{M_5} \\
+ \frac{1}{30} \bar{\theta} \Gamma^{M_1} d\theta \bar{\theta} \Gamma^{M_2} d\theta \Pi^{M_3} \Pi^{M_4} \Pi^{M_5} \\
+ \frac{1}{48} \bar{\theta} \Gamma^{M_1} d\theta \bar{\theta} \Gamma^{M_2} d\theta \bar{\theta} \Gamma^{M_3} d\theta \Pi^{M_4} \Pi^{M_5} \\
+ \frac{1}{60} \bar{\theta} \Gamma^{M_1} d\theta \bar{\theta} \Gamma^{M_2} d\theta \bar{\theta} \Gamma^{M_3} d\theta \bar{\theta} \Gamma^{M_4} d\theta \Pi^{M_5} \\
- \bar{\theta} \Gamma_{M_1 M_2} d\theta \bar{\theta} \Gamma_{M_3 M_4} d\theta \left( \frac{1}{24} \bar{\theta} \Gamma^{M_1} d\theta \Pi^{M_2} \Pi^{M_3} \Pi^{M_4} \\
+ \frac{1}{48} \bar{\theta} \Gamma^{M_1} d\theta \bar{\theta} \Gamma^{M_2} d\theta \Pi^{M_3} \Pi^{M_4} \\
+ \frac{1}{60} \bar{\theta} \Gamma^{M_1} d\theta \bar{\theta} \Gamma^{M_2} d\theta \bar{\theta} \Gamma^{M_3} d\theta \Pi^{M_4} \right) \right) .
\] (5.16)

With indices we thus have

\[
c_{\mu \nu \rho} = -3 \bar{\theta} \gamma_{[\mu \nu} \partial_{\rho]} \theta + \ldots \\
c_{\mu_1 \ldots \mu_6} = 6 \bar{\theta} \gamma_{[\mu_1 \ldots \mu_5} \partial_{\mu_6]} \theta + \ldots .
\] (5.17)

Here we have introduced the notation

\[
\gamma_\mu = \Pi^{M} \Gamma_M ; \quad \gamma_{\mu \nu} = \gamma_{[\mu | \nu]} ; \ldots .
\] (5.18)

This accomplishes the derivation of the explicit form of the M 5-brane action.

We still have to show that the M 5-brane action upon gauge-fixing in a flat 11-dimensional background leads to a quadratic action for the tensor multiplet which in the quadratic approximation was shown to have a superconformal symmetry.

### 5.3 \( \kappa \)-transformations and gauge fixing

The form of the \( \kappa \)-transformations acting on all fields is defined by the form of \( \kappa \)-transformations acting on \( \theta \):

\[
\delta_\kappa \theta(x) = (1 + \Gamma) \kappa(x) .
\] (5.19)

Here \( \Gamma \) is a function of fields of the theory:

\[
\Gamma = \frac{1}{\mathcal{G}} \left( \sqrt{\gamma} + \sqrt{\gamma} H^*_{\mu \nu} v^\rho \bar{\gamma}^{\mu \nu \rho} - \frac{1}{16} e^{\mu \nu \rho \sigma \tau \phi} \mathcal{H}^{*}_{\mu \nu} \mathcal{H}^{*}_{\rho \sigma} \bar{\gamma}^{\tau \phi} \right) \\
\bar{\gamma} = \frac{1}{6! \sqrt{\mathcal{G}}} \epsilon^{\mu_1 \ldots \mu_6} \bar{\gamma}_{\mu_1 \ldots \mu_6} .
\] (5.20)
satisfying $\Gamma^2 = 1$. On the remaining fields the $\kappa$-transformations are

$$
\delta_\kappa B = \frac{1}{2} \bar{\theta} \Gamma_{MN} \delta_\kappa \theta (\Pi^M \Pi^N + \bar{\theta} \Gamma^M \bar{\theta} \Pi^N - \frac{1}{2} \bar{\theta} \Gamma^M \bar{\theta} \Gamma^N \bar{\theta} \theta)
- \frac{1}{2} \bar{\theta} \Gamma^M \delta_\kappa \bar{\theta} \Gamma_{MN} \bar{\theta} \theta (\Pi^N + \frac{2}{3} \bar{\theta} \Gamma^N \bar{\theta} \theta)$$

$$
\delta_\kappa \Pi^M = \frac{1}{2} \bar{\theta} \Gamma^M \delta_\kappa \theta , \quad \delta_\kappa \alpha(x) = 0 .
$$

(5.21)

We will summarize a proof of these results in appendix C.

The $\kappa$-symmetry defined above is infinite reducible since it is given in terms of 32-component spinor $\kappa$, which however one can change by $(1 - \Gamma)\kappa_1(x)$, for any function $\kappa_1$ since $(1 + \Gamma)(1 - \Gamma)\kappa_1(x) = 0$.

We first define the irreducible $\kappa$-symmetries, following [6]. The matrix $\Gamma$ in (5.20) has square 1, such that the transformations where $\kappa$ is proportional to $1 - \Gamma$ are zero modes of (5.8). To identify the irreducible part, we consider first the classical values. We consider the classical solution of the equations of motion

$$
X^\mu = x^\mu ; \quad \theta_\alpha = \theta_\alpha' = 0 ; \quad X^{a'} = \text{constant} ; \quad B_{\mu \nu} = 0 .
$$

(5.22)

Then

$$
\Gamma|_{cl} = \bar{\gamma}|_{cl} = \Gamma_*. \quad (5.23)
$$

Therefore the irreducible $\kappa$ symmetries are $\kappa_\alpha$, and we put

$$
\frac{1}{2} (1 - \Gamma_*) \kappa = \begin{pmatrix} 0 \\ \kappa_{\alpha'} \end{pmatrix} = 0 .
$$

(5.24)

The tracelessness and unipotency of $\Gamma$ allow to write it in terms of $16 \times 16$ matrices as

$$
\Gamma = \begin{pmatrix} C & B \\ A & -C' \end{pmatrix} \text{ with } \text{Tr} C' = \text{Tr} C \text{ and } AC = C' A ,
$$

(5.25)

and if $A$ is invertible: $B = (1 - C^2) A^{-1} = A^{-1}(1 - C'^2)$. With the reduced $\kappa$-symmetries, we thus have under the fermionic symmetries (5.8)

$$
\delta_f \theta_\alpha = \epsilon_\alpha + (1 + C)_\alpha^\beta \kappa_\beta
$$

$$
\delta_f \theta_\alpha' = \epsilon_\alpha' + A_{\alpha'}^\beta \kappa_\beta .
$$

(5.26)

In the classical limit $A = 0$ and $C = 1$, therefore one can take the gauge

$$
\kappa - \text{gauge} : \frac{1}{2} (1 + \Gamma_*) \theta = \begin{pmatrix} \theta_\alpha \\ 0 \end{pmatrix} = 0 .
$$

(5.27)

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The corresponding decomposition law is
\[ \delta \theta_L = 0 \Rightarrow \kappa_L = (1 + C)^{-1} \left( -\epsilon_L + \frac{1}{2} \gamma_{a'} \gamma_a \Lambda^{a'} \Lambda_R \right), \quad (5.28) \]
where we have renamed the remaining fermion is \( \theta_{a'} \), as \( \lambda \), which is then right-handed.

We also gauge-fix the 6-dimensional diffeomorphisms by
\[ 6 \text{-dimensional diffeomorphisms - gauge : } X^a(x) = \delta^a_{\mu} x^\mu. \quad (5.29) \]
The resulting decomposition law is:
\[ \delta X^\mu = 0 \Rightarrow \delta^a_{\mu} = a^a + \Lambda^a \delta_{\mu}^{b'} + \Lambda^a_{a'} X^{a'} - \bar{\epsilon} \gamma^a \theta - \bar{\lambda} R \gamma^a A \kappa_L. \quad (5.30) \]
This implies that after the gauge fixing the fields \( X^{a'} \) and \( \lambda \) transform under the 32 rigid supersymmetries as follows
\[ \delta \epsilon_{a'} = -2\bar{\epsilon} \gamma^a \lambda_L + \left( \partial_{a'} X^{a'} \right) \delta^a_{\mu} \bar{\lambda} R \gamma^a \left( \epsilon_R + A(1 + C)^{-1} \epsilon_L \right) \]
\[ \delta \epsilon_{\lambda} = \epsilon_R - \left( A(1 + C)^{-1} \right) \epsilon_L + \left( \partial_{a} \lambda R \right) \delta^a_{\mu} \bar{\lambda} R \gamma^a \left( \epsilon_R + A(1 + C)^{-1} \epsilon_L \right). \quad (5.31) \]
We further have
\[ \Pi^a_{\mu} = \delta^a_{\mu} - \bar{\lambda} \gamma^a \partial_{\mu} \lambda; \quad \Pi^{a'}_{\mu} = \partial_{a'} X^{a'}. \quad (5.32) \]
Due to the chirality restriction of the gauge fixed \( \theta \), which becomes now \( \lambda \), there are further simplifications. E.g.
\[ c_{\mu\nu} = 3 \bar{\lambda} \gamma_a \gamma_{a'} \partial_{\mu} \lambda \left( 2 \delta^a_{\nu} - \bar{\lambda} \gamma^a \partial_{\nu} \lambda \right) \partial_{\nu} X^{a'}. \quad (5.33) \]
After gauge-fixing, we can use in the M 5-brane action (5.2) (action of the self-interacting tensor multiplet)
\[ g_{\mu\nu} = \eta_{\mu\nu} - \bar{\lambda} \gamma_{a'} \delta_{\mu} \partial_{\nu} + \delta^a_{\mu} \partial_{\nu} \lambda + \partial_{\mu} X^{a'} \partial_{\nu} X^{a'} + \left( \bar{\lambda} \gamma^a \partial_{\mu} \lambda \right) \left( \bar{\lambda} \gamma_{a'} \partial_{\nu} \lambda \right) \quad (5.34) \]
and
\[ H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} - 3 \bar{\lambda} \gamma_a \gamma_{a'} \partial_{\rho} \lambda \left( 2 \delta^a_{\nu} - \bar{\lambda} \gamma^a \partial_{\nu} \lambda \right) \partial_{\rho} X^{a'}. \quad (5.35) \]
\[ ^{5} \text{For convenience, we indicate here } L \text{ for left-handed and } R \text{ for right-handed spinors.} \]
The following explicit form of the Born-Infeld expression $G$ for what concerns the dependence on $H^*_{\mu\nu}$ can be given:

$$G^2 = - \det(g_{\mu\nu} + iH^*_{\mu\nu})$$

$$= g \left( 1 - \frac{1}{2}H^*_{\mu\nu}H^{*\mu\nu} + \frac{1}{8}(H^*_{\mu\nu}H^{*\tau\rho})^2 - \frac{1}{4}(H^*_{\mu\nu}H^{*\rho\sigma}H^*_{\tau\sigma}) \right).$$

For the Wess-Zumino term we can write

$$I_{WZ} = \int d^6 x \epsilon^{\mu\nu\rho\sigma\tau\phi} \left[ \frac{1}{6!}c_{\mu\nu\rho\sigma\tau\phi} + \frac{1}{8}(\partial_\mu B_{\nu\rho\tau}) \bar{\lambda}\gamma_{\alpha\gamma_\mu} \partial_\sigma \lambda \left( 2\delta^a_\alpha - \bar{\lambda}\gamma_{\alpha} \partial_\tau \lambda \right) \partial_\phi X^{a'} \right],$$

where in the expression for $c_6$ in $(5.16)$ one has to replace $\theta$ by the chiral spinor $\lambda$, and can use $(5.32)$.

The action is invariant under the 32 global supersymmetries on the worldvolume. The supersymmetry transformations for the scalars $X^{a'}$ and the spinors are given in $(5.31)$ where again $A$ and $C$ have to be taken at the values of $g_{\mu\nu}$ and $H_{\mu\nu\rho}$ presented in $(5.34)$ and $(5.35)$. The worldvolume supersymmetry transformations of $B_{\mu\nu}$ can be derived in analogous fashion from its $\kappa$-symmetry, space-time supersymmetry and 6-dimensional diffeomorphism transformation. Remark that also $a$ is not inert under these transformations, due to $\delta a = \xi^\mu \partial_\mu a$. The $SO(10,1)$ rotations of the original theory break up in linearly and non-linearly realised symmetries. The linear ones are the $SO(5,1) \times SO(5)$ transformations. The first factor gets from $\xi^\mu$ in $(5.31)$ the extra contributions such that they are the Lorentz rotations as in section 2.1. For the $SO(5)$ transformations we have $\alpha^{a'b'} = -\Lambda^{a'b'}$. The remaining transformations in the coset $SO(10,1)/SO(5,1) \times SO(5)$ are non-linear in the gauge-fixed theory.

### 5.4 Cutting to the linearized action

By linearized, we imply that we count powers of $X^{a'}$, $\theta$ and $B_{\mu\nu}$, leaving $a$ arbitrary. We keep in the action terms quadratic in these fields, and in transformation laws linear terms. We obtain

$$-G = -1 + \bar{\lambda}\phi\lambda - \frac{1}{2}\partial_\mu X^{a'}\partial_\mu X^{a'} + \frac{1}{4}H^*_{\mu\nu}H^{*\mu\nu}$$

$$c_3 = 0$$

$$I_{WZ} = \int c_6 = \int \frac{1}{6!}\epsilon^{\mu_1...\mu_6} \bar{\lambda}\gamma_{\mu_1...\mu_5} \partial_\mu_6 \lambda = \int d^6 x \bar{\lambda}\phi\lambda$$

(5.38)
Putting everything together we have

\[ S_{lin} = \int d^6x \left[ -\frac{1}{2} \mathcal{H}^{* \mu \nu} \mathcal{H}_{\mu \nu} - \frac{1}{2} \partial_\mu X^{a'} \partial^\mu X_{a'} + 2\lambda \bar{\phi} \lambda \right]. \]  

(5.39)

For the transformation laws, the remaining parts of the decomposition law are:

\[ \xi^\mu = a^\mu + \Lambda^\mu_\nu x^\nu \]

\[ (1 + C)\kappa = \frac{1}{2} \gamma_{a'} \gamma^a \Lambda^{a'd} a^d - \epsilon . \]  

(5.40)

The remaining part from the 6-dimensional general coordinate transformations are thus the rigid 6-dimensional Poincaré transformations (for the spinors they combine with the 6×6 part of the 11-dimensional Lorentz transformations). There remain also the translations in the extra 5 dimensions, as constant shifts of the \( X^{a'} \), and the Lorentz rotations in these 5 dimensions. These become the internal symmetry \( USp(4) \). The remaining off-diagonal parts of the Lorentz transformations remain only as

\[ \delta_{off-d} X^{a'} = -\Lambda^{a'd} x_\mu . \]  

(5.41)

Remains the supersymmetry part. We have

\[ \delta_f X^{a'} = -2\bar{\epsilon}_L \gamma^{a'} \lambda_R \]

\[ \delta_f \lambda_R = -A(1 + C)^{-1} \epsilon_L \]

\[ \delta_f B_{\mu \nu} = -2\bar{\epsilon}_L \gamma_{\mu \nu} \lambda_R . \]  

(5.42)

At the linear level we have

\[ \Gamma = \bar{\gamma} + \frac{1}{2} \gamma^{\mu \rho} \mathcal{H}^*_{\mu \nu} \nu_\rho \]

\[ \bar{\gamma} = \left( \begin{array}{c} 1 \\ -\frac{1}{5!} \epsilon_{\mu_1...\mu_6} \Pi_{\mu_1}^{a'} \delta^{a_2...a_6} \gamma \gamma_{a_2...a_6} \end{array} \right) \]

\[ C = 1 \]

\[ A = -\gamma_{a'} \partial X^{a'} + \frac{1}{3} h_{\mu \nu \rho} \gamma^{\mu \nu \rho} . \]  

(5.43)

For the last term, we use \((A.22)\), and

\[ (\mathcal{H}^*_{\mu \nu} \nu_\rho)^* = \frac{4}{3} v^\sigma v_{[\sigma} \mathcal{H}_{\mu \nu]} . \]  

(5.44)

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Then we use the second line of (4.11). This leads to the transformation
\[ \delta f \lambda_R = \frac{1}{2} \gamma^{a'} \partial X^a' \epsilon_L - \frac{1}{6} h_{\mu \nu \rho} \gamma^{\mu \nu \rho} \epsilon_L . \] 
(5.45)

Thus we have derived the action for the small excitations of the M 5-brane from the non-linear action. The supersymmetries of the quadratic action follow from the supersymmetries of the full non-linear action (however only 16 of the 32 are linear supersymmetries acting on the states of the multiplet). The special supersymmetries presented in Sec. 3. emerge only for quadratic approximation of the full M 5-brane theory.

5.5 Breaking of superconformal symmetry by self interaction of the tensor multiplet

Consider the full non-linear action of the gauge-fixed M 5-brane (5.2). We can expand it around the quadratic approximation
\[ S = \text{const} + \int d^6 x \left( -\frac{1}{2} H^{\mu \nu} H_{\mu \nu} - \frac{1}{2} (\partial X^a')^2 + 2 \tilde{\lambda} \not{\partial} \lambda \right) + S_{\text{interaction}} . \] 
(5.46)

Interaction terms are cubic, quartic etc. in scalars \( X^a' \), spinors and tensors. For example we have terms of the form
\[ \partial_\mu X^a' \partial_\nu X^a' \mathcal{H}^{\mu \nu} , \quad \partial_\mu X^a' \partial_\nu X^a' \tilde{\lambda} \gamma^\mu \partial^\nu \lambda , \quad \mathcal{H}^{\mu \nu} \mathcal{H}_{\mu \nu} (\partial X^a')^2 \] 
(5.47)

etc. All of these terms are not invariant under dilatation and therefore the superconformal symmetry of the free action of the tensor multiplet is lost. Another way to see this is to introduce the tension \( l_P \) and rescale every field in the tensor multiplet by \( l_P \). The quadratic part of the action will be independent on \( l_P \) however, the interaction terms will all depend on \( l_P \): cubic terms will have \( l_P \) in front, quartic will have \( l_P^2 \) etc. An interesting question which remains is to find out what will happen if the M 5-brane action would be quantized in the curved background of 11-dimensional supergravity. This would correspond to taking into account terms proportional to \( l_P \) in the background. Is it possible to restore the superconformal invariance of the free tensor multiplet in presence of both self-interaction as well as interaction to supergravity? This we leave for further investigations.
6  $M5$ as a classical solution of 11-dimensional supergravity and conformal symmetry

One of the most puzzling issues of the current status of our understanding of the fundamental theory is the relation between the space-time and the worldvolume pictures. Here we suggest a step to uncover this relation: we will study the symmetry of the space-time near the horizon and symmetry of the quantum field theory on the worldvolume.

The space-time configuration corresponding to M 5-brane is

$$ ds^2 = \left[1 + \left(\frac{\mu}{\rho}\right)^3\right]^{-\frac{1}{3}} (dx^2_{\mu}) + \left[1 + \left(\frac{\mu}{\rho}\right)^3\right]^\frac{2}{3} dx^2_{a'}, $$

where $\mu$ is a parameter and $dx^2_{a'} = d\rho^2 + \rho^2 d^2\Omega$. Near the horizon, $\rho \to 0$, we have

$$ ds^2 \to \frac{\rho}{\mu} (dx^2_{\mu}) + \left(\frac{\mu}{\rho}\right)^2 dx^2_{a'} $$

$$ = \frac{\rho}{\mu} (dx^2_{\mu}) + \left(\frac{\mu}{\rho}\right)^2 d\rho^2 + \mu^2 d^2\Omega. $$

This is the $adS_7 \times S^4$ geometry. We introduce the variable $w$ by

$$ w = \left[1 + \left(\frac{\mu}{\rho}\right)^3\right]^{-\frac{1}{6}}, $$

or solving for $\rho$:

$$ \rho = \mu w^2 \left[1 - w^6\right]^{-\frac{1}{3}} = f(w)w^2, $$

where $f(w)$ is analytic at $w \to 0$. Thus

$$ H \equiv 1 + \left(\frac{\mu}{\rho}\right)^3 = \frac{1}{w^6}. $$

For $w \to 0$ the metric gets the $adS_7 \times S^4$ form

$$ ds^2 \to \left[w^2 dx^2_{\mu} + (2\mu)^2 \left(\frac{dw}{w}\right)^2\right] + \mu^2 d^2\Omega. $$
The $adS_7$ transformations form the group $SO(6,2)$. This can be made explicit by defining it as a submanifold of an 8-dimensional space

$$\{X^\mu, X^-, X^+\}$$

with metric (signature $(-+++-+-+-)$)

$$ds^2 = dX^\mu \eta_{\mu\nu} dX^\nu - dX^+ dX^- .$$

(6.7)

The submanifold is determined by the equation

$$X^\mu \eta_{\mu\nu} X^\nu - X^+ X^- + (2\mu)^2 = 0 .$$

(6.8)

On the hypersurface we will define the coordinates $\{x^\mu, w\}$ by

$$X^- = w$$

$$X^\mu = w x^\mu$$

$$X^+ = \frac{(2\mu)^2 + w^2 x^2_\mu}{w} .$$

(6.9)

The induced metric on the hypersurface is

$$ds^2 = w^2 dx^2_\mu + \frac{(2\mu)^2}{w^2} dw^2 .$$

(6.10)

The $SO(6,2)$ is linearly realized in the embedding 8-dimensional space, and these transformations, ($\hat{\mu} = \mu, +, -$ and $\Lambda^{\hat{\mu}\hat{\nu}} = -\Lambda^{\hat{\nu}\hat{\mu}}$)

$$\delta X^{\hat{\mu}} = \Lambda^{\hat{\nu}\hat{\rho}} M_{\hat{\rho}\hat{\nu}} X^{\hat{\mu}} = -\Lambda^{\hat{\mu}\hat{\nu}} X^{\hat{\nu}} ,$$

(6.11)

satisfy the algebra (2.2). Some of these transformations are then non-linearly realized on the $5 + 1$ space of the brane. Indeed we get

$$\delta x^\mu = -\xi^\mu ,$$

(6.12)

with $\xi^\mu$ as in (2.3), when we identify

$$a^\mu = \Lambda^\mu_- + \frac{(2\mu)^2}{w^2} \Lambda^\mu_+ ; \quad \lambda^\mu_{\nu\rho} = \Lambda^\mu\nu$$

$$\lambda_D = -\Lambda^-_- = \Lambda^+^+ ; \quad \Lambda^\mu_{\nu} = \Lambda^\mu_+ = \frac{1}{2} \Lambda^-^\mu .$$

(6.13)

Thus we have found that the fact that the geometry of the infinite throat is an $adS_7$ space leads to the conformal symmetry of the 6-dimensional world-volume.
7 Discussion

We have performed here a study of the M 5-brane theory whose small excitations are associated with $OSp(8^*|2N)$ superconformal theory in $d=6$. Our main new results are the following.

We explained the generic procedure which allows to establish the presence of rigid superconformal symmetry in non-gravitational theories in dimensions higher than 2.

We have derived an action of the $(0,2)$ free tensor multiplet in 6 dimensions. The action has a superconformal symmetry and it is Lorentz covariant due to the use of the Pasti-Sorokin-Tonin \cite{26} auxiliary field $a(x)$. The precise form of the superconformal invariant action is

$$L = -\frac{1}{2} H^*_\mu\nu H^{\mu\nu} - \frac{1}{2} (\partial X^{a'})^2 + 2 \lambda \phi \lambda .$$

(7.1)

where the two-index field $H_{\mu\nu}$ is a usual 3-index field strength of the tensor field contracted with the derivative of the auxiliary field and $*$ (−) on $H$ mean dual (anti-self-dual) combinations. The Weyl weights and properties of fields under special conformal transformations, under supersymmetry and special supersymmetry and internal symmetry group can be found in Sec. 4.

This action was derived as a truncation of the full gauge-fixed M 5-brane action. The meaning of scalar fields $X^{a'}(x)$ is that they are excitation of the five transverse directions of the brane in space-time, the meaning of the chiral worldvolume spinors $\lambda(x)$ is that they are excitations of (half) of the 11-dimensional superspace coordinates $\theta(x)$ and finally, $H(x)$ represent a tensor field on the brane which does not have a simple interpretation in space time but is necessary to complete the supersymmetric tensor multiplet on the worldvolume. The supersymmetry transformations of the free tensor multiplet are derived by the truncation of the non-linear supersymmetry, however the special supersymmetry emerges only for the quadratic action describing the small excitations of the M 5-brane.

The interaction terms of the M 5-brane action violate superconformal symmetry of the free action since they have wrong scaling behavior, e. g. we have terms $\partial_\mu X^{d'} \partial_\nu X^{d'} \lambda^{\gamma\mu} \partial^{\nu}\lambda$ etc.

We have given a detailed derivation of the gauge-fixing procedure for the full non-linear M 5-theory with 32 linear-non-linear global supersymmetries. This leads to a better understanding of the single M 5-brane dynamics. One may try to generalize the gauge-fixing procedure for the case of the non-
trivial backgrounds. So far we have only worked in a flat 11-dimensional superspace background.

Finally we focused on the deep relation between $adS_7$ geometry of the M5-brane throat and the supergroup generalization of it, which is precisely the superconformal algebra of the small excitation of the brane. We explain this relation in Sec. 6.

We have displayed the full superconformal algebra which corresponds to the symmetry of the $(0,2)$ tensor multiplet in six dimensions. We have presented the $OSp(8^*|2N)$ algebra in two forms: first, using a ‘triality’ of $SO(6, 2)$ we gave a manifestly symmetric superalgebra with graded (fermionic) bosonic (anti) commutators, see (2.18). Secondly, we presented the $OSp(8^*|4)$ superconformal algebra in more familiar form which includes 28 generators of $SO(6, 2)$ bosonic conformal symmetries and 16 supersymmetries and 16 special supersymmetries and also the generators of internal symmetry, an $USp(4) \approx Spin(5)$ group, see Sec. 3. We have only studied here the classical superconformal algebra since in $d=6$ even this is much less understood than in $d=2$ and in $d=4$. The next step will be to study the OPE’s and extract information about correlators of various operators in 6-dimensional superconformal field theories.

Acknowledgments.

We had stimulating discussions of various parts of the paper with M. Aganagic, E. Bergshoeff, J. Distler, S. Ferrara, M. Günaydin, I. Klebanov, J. Kumar, J. Maldacena, M. Peskin, J. Rahmfeld, A. Rajaraman, E. Silverstein, J. H. Schwarz, S. Shenker, K. Stelle. The work of R. K. is supported by the NSF grant PHY-9219345. A.V.P. thanks the Physics Department of the Stanford University for the hospitality during a fruitful visit in which this work was performed. He also thanks the FWO, Belgium, for the travel grant. Work supported by the European Commission TMR programme ERBFMRX-CT96-0045.

\[6\] Jacques Distler has informed us that he also studied this superconformal algebra and its representations (J. Distler, to appear).
A  Notations

We start from 11-dimensional superspace with coordinates

\[ Z^\Lambda = \{ X^M, \theta^A \}. \tag{A.1} \]

Let us recapitulate a list of indices and their range:

- \( M = 0, 1, \ldots, 10 \) 11-dim. space-time
- \( A = 1, \ldots, 32 \) 11 − dim. spinor indices, equivalent with \{\((\alpha i), (\alpha' i)\)\}
- \( \alpha = 1, \ldots, 4 \) chiral 6-dim. spinor indices
- \( \alpha' = 1, \ldots, 4 \) antichiral 6-dim. spinor indices
- \( i = 1, \ldots, 4 \) \(USp(4)\) indices
- \( \mu = 0, 1, \ldots, 5 \) curved on the 6-dimensional worldsheet
- \( a = 0, 1, \ldots, 5 \) flat on the 6-dimensional worldsheet
- \( a' = 6, \ldots, 10 \) \(SO(5)\) indices
- \( \hat{\mu} = 0, 1, \ldots, 7 \) (6,2) vector indices
- \( \hat{\alpha} = 1, \ldots, 8 \) (6,2) chiral (right) spinor indices. \tag{A.2}

The \( X^M \) thus reduce to \( \{ X^\mu, X^{a'} \} \), the former being gauge fixed to \( x^\mu \), worldsheet coordinates on the 5-brane.

The space-time metric \( \eta_{MN} \) is \((- + \ldots +\)).

Let us also repeat that all (anti)symmetrizations are with ‘weight 1’, e.g.
\[ A_{[\mu} B_{\nu]} = \frac{1}{2} (A_{\mu} B_{\nu} - A_{\nu} B_{\mu}). \]

A.1  Decomposing 11-dimensional gamma matrices and spinors

Many aspects about the Clifford algebras in various dimensions can be found in \cite{31}. In 11 dimensions the charge conjugation matrix is \( C \), which is antisymmetric, and \( C \Gamma_M \) is symmetric. Majorana spinors satisfy \( \bar{\lambda} \equiv -i \lambda^T \Gamma_0 = \lambda^T C \). With indices, the matrix \( C \) is \( C^{AB} \), and \( C^{AB} C_{CB} = \delta_A^A \), and gamma matrices are written as \( (\Gamma_M)^A_B \). And let us also repeat the equations which are heavily used

\[
\begin{align*}
\Gamma^{MN}_{(AB} \Gamma_{CD)} N = 0 \\
\Gamma_M (AB) \Gamma^{M_1 M_2 M_3 M_4}_{CD} = 3 \Gamma^{[M_1 M_2} (AB) \Gamma^{M_3 M_4]}_{CD} \tag{A.3}.
\end{align*}
\]
We now decompose the 32-component spinor index $A$ in $\{(\alpha i), (\alpha'i)\}$, and simultaneously $M$ in $\{a, a'\}$. The $\alpha$ and $\alpha'$ will be chiral and antichiral indices in 6 dimensions as we will take a representation where

$$\Gamma_5 = -\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 ,$$

where in the notation with $\otimes$ the first factor refers to the $8 \times 8$ matrix in $(\alpha, \alpha')$, while the second factor refers to the $4 \times 4$ matrix in indices $i$. In this sense we take a representation where

$$\Gamma_a = \begin{pmatrix} 0 & \gamma_a \\ \tilde{\gamma}_a & 0 \end{pmatrix} \otimes 1 ; \quad \Gamma_{a'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \gamma_{a'} ,$$

where $\Omega$ is the (real antisymmetric) symplectic metric, and there are further relations

$$c c^\dagger = 1$$
$$\tilde{\gamma}_a \gamma_b + \tilde{\gamma}_b \gamma_a = 2 \eta_{ab} ; \quad \gamma_a \tilde{\gamma}_b + \gamma_b \tilde{\gamma}_a = 2 \eta_{ab}$$
$$\tilde{\gamma}_a = \gamma_0 \gamma_a \tilde{\gamma}_0 ; \quad \text{or} \quad \begin{cases} \tilde{\gamma}_a = \gamma_a & \text{for } a \neq 0 \\ \tilde{\gamma}_0 = -\gamma_0 \end{cases}$$
$$c^T \gamma_a = -\gamma_a^T c \text{ implying also } c \tilde{\gamma}_a = -\tilde{\gamma}_a^T c^T$$
$$\gamma_a^T \gamma_b^T + \gamma_b \gamma_a^T = 2 \delta_{ab}$$
$$\gamma_a^T = \gamma_{a'} ; \quad \text{Tr} \gamma_a = 0 ; \quad \Omega \gamma_{a'} = - (\Omega \gamma_{a'})^T .$$

We then have indeed symmetry of $\mathcal{C} \Gamma_M$:

$$\mathcal{C} \Gamma_a = \begin{pmatrix} c \tilde{\gamma}_a & 0 \\ 0 & c^T \gamma_a \end{pmatrix} \otimes \Omega ; \quad \mathcal{C} \Gamma_{a'} = \begin{pmatrix} 0 & -c \\ c^T & 0 \end{pmatrix} \otimes \Omega \gamma_{a'} .$$

Finally (A.4) imposes on our representation that

$$\gamma_0 \tilde{\gamma}_1 \gamma_2 \tilde{\gamma}_3 \gamma_4 \tilde{\gamma}_5 = -1 ,$$

which then also implies $\tilde{\gamma}_0 \gamma_1 \tilde{\gamma}_2 \gamma_3 \tilde{\gamma}_4 \gamma_5 = 1$.

To indicate our use of indices, we rewrite some matrices from above with indices:

$$\mathcal{C}^{AB} = \begin{pmatrix} 0 & c^\alpha \beta \Omega^{ij} \\ c^\alpha \beta \Omega^{ij} & 0 \end{pmatrix} ; \quad \mathcal{C}_{AB} = \begin{pmatrix} 0 & c_{\alpha'} \beta \Omega_{ij} \\ c_{\alpha'} \beta \Omega_{ij} & 0 \end{pmatrix}$$
\[(\Gamma_a)^B_A = \begin{pmatrix} 0 & (\gamma_a)^\beta \delta^j_i \\ (\gamma_a)^\alpha \delta^i_j & 0 \end{pmatrix} \]

\[(\Gamma_a)^{AB} \equiv \mathcal{C}^{AC}(\Gamma_a)^C_B = \begin{pmatrix} (\gamma_a)^{\alpha \beta} \Omega^{ij} & 0 \\ 0 & (\gamma_a)^{\alpha' \beta'} \Omega^{ij} \end{pmatrix} \]

\[(\Gamma_{a'})^B_A = \begin{pmatrix} \delta^\alpha_{\alpha'} (\gamma_{a'})^j_i \\ 0 & -\delta^\alpha_{\alpha'} (\gamma_{a'})^j_i \end{pmatrix} \]

(A.9)

where

\[c^{\alpha \alpha'} = c^{\alpha' \alpha} = (c_{\alpha \alpha'})^* ; \quad c^{\alpha \alpha'} c^{\alpha' \beta} = \delta^\alpha_\beta : \quad \Omega_{ij} = -(\Omega^{ij})^{-1} . \quad \text{(A.10)}\]

The matrix \(\tilde{\gamma}_a\) is now the matrix \(\gamma_a\) where the first (lower) index is the antichiral \(\alpha'\), and the second is \(\alpha\). We then have (for \(a \neq 0\))

\[((\gamma_a)^\alpha_{\alpha'})^* = (\gamma_a)^\alpha'_{\alpha} , \quad \text{(A.11)}\]

and the indices of this matrix are raised or lowered with \(c^{\alpha \alpha'}\) using the NW-SE convention:

\[\gamma^\alpha_{\alpha'} = c^{\alpha \alpha'} (\gamma_a)^\alpha'_{\alpha} ; \quad (\gamma_a)^\alpha_{\alpha'} = (\gamma_a)^\alpha'_{\alpha} c^{\alpha' \beta} . \quad \text{(A.12)}\]

Remark that the Majorana condition is

\[\bar{\lambda} = \lambda^T \mathcal{C} \quad \text{or} \quad \bar{\lambda}^A = \lambda_B \mathcal{C}^{BA} = -\mathcal{C}^{AB} \lambda_B = -\lambda^A , \quad \text{(A.13)}\]

using again the NW-SE index contraction.

### A.2 6-dimensional conventions

Writing the latter equation on the spinor split in 6-dimensional chiral parts, \(\lambda = \begin{pmatrix} \lambda_{\alpha i}^0 \\ \lambda_{\alpha' i}^0 \end{pmatrix}\), we obtain the 6-dimensional Majorana condition which we will adopt

\[\bar{\lambda}^{\alpha'} = -i (\lambda_{i\beta})^\dagger (\gamma_0)^{\alpha'}_{\beta} = \lambda_{j\beta} \Omega^{ji} c^{\beta \alpha'} = -\lambda^{\alpha'} , \quad \text{(A.14)}\]

which also applies for indices \(\alpha\) and \(\alpha'\) interchanged, and where the last equation uses raising of indices with NW-SE convention:

\[\lambda_i = \lambda^j \Omega_{ji} ; \quad \lambda^i = \Omega^{ij} \lambda_j . \quad \text{(A.15)}\]

We call a spinor of the form

\[\lambda = \begin{pmatrix} \lambda_{\alpha i}^0 \\ 0 \end{pmatrix} \quad \text{"left handed" or "positively chiral" .} \quad \text{(A.16)}\]
They thus satisfy $\lambda = \Gamma \lambda$. Note that the Majorana conjugate has the $\alpha'$ index (up).

The charge conjugation matrix in 6 dimensions is $e^{\alpha \alpha'}$, and the gamma matrices are the first factor of $\Gamma_a$ in (A.3). Writing $L$ and $R$ for left- and right-handed spinors we have

$$
\bar{\lambda}_L \Gamma^a \theta_L = \bar{\lambda}_L \gamma^a \theta_L ; \quad \bar{\lambda}_R \Gamma^a \theta_R = \bar{\lambda}_R \gamma^a \theta_R ; \quad \bar{\lambda}_L \Gamma^{a'} \theta_L = -\bar{\lambda}_L \gamma^{a'} \theta_R , \quad (A.17)
$$

such that although $\Gamma^a$ and $\Gamma^{a'}$ anticommute, $\gamma^a$ and $\gamma^{a'}$ commute.

We define

$$
\epsilon_{012345} = 1 = -\epsilon^{012345} \quad , \quad (A.18)
$$

and we thus have

$$
\epsilon^{a_1...a_m b_1...b_{6-m}} \epsilon_{a_1...a_m c_1...c_{6-m}} = -(6-m)! m! \delta^{b_1}_{[c_1} \ldots \delta^{b_{6-m}}_{c_{6-m}]} . \quad (A.19)
$$

It is useful to know the relation

$$
\gamma^{a_1...a_k} = \frac{S_k}{(6-k)!} \epsilon^{a_1...a_6} \gamma_{a_{k+1}...a_6} \Gamma^* , \quad S_k = \begin{cases} +1 : & k = 0, 1, 4, 5 \\ -1 : & k = 2, 3, 6 \end{cases} \quad (A.20)
$$

where $\Gamma^*$ is +1 on left-handed and −1 on right-handed spinors. We denote the dual, self-dual and anti-self-dual of a three index tensor as

$$
F^*_{abc} = \frac{1}{6} \epsilon_{abcdef} F^{def} ; \quad F^\pm_{abc} = \frac{1}{2} (F_{abc} \pm F^*_{abc}) . \quad (A.21)
$$

Remark that omitting the $\Omega_{ij}$ in $C$ this becomes the symmetric charge conjugation matrix from 6 dimension.

As we did already when using explicit indices in the second part of appendix [A.1], we will omit tildes on the $\gamma$ matrices when chiral spinors are used, as the place where these matrices occur shows automatically which one is meant.

For a left-handed spinor $\epsilon$ we have for any antisymmetric tensor $F_{abc}$:

$$
F_{abc} \gamma^{abc} \epsilon = F^+_{abc} \gamma^{abc} \epsilon ; \quad \gamma_d F^+_{abc} \gamma^{abc} \epsilon = 6 F^+_{abc} \gamma^{ab} \epsilon . \quad (A.22)
$$

Under complex conjugation we have

$$
\left( \bar{\lambda} \chi \right)^* = \left( \bar{\lambda}^i \chi_i \right)^* = \chi_i \left( \bar{\lambda}^i \right)^\dagger = \bar{\chi}^i \lambda_i = \bar{\lambda} \chi = \bar{\lambda} \chi . \quad (A.23)
$$
We can replace $USp(4)$ notation with $SO(5)$. We identify $\Omega_{ij}$ with the charge conjugation matrix in 5 dimensions, which is thus real and antisymmetric. An $\Omega$-traceless antisymmetric tensor $X^{ij}$ can then be translated to a 5-dimensional vector, and a symmetric tensor $U_{ij}$ to an antisymmetric tensor $U_{a'b'}$:

\[
X^{a'} = \frac{1}{2} \gamma^{a'}_{ij} X^{ij}; \quad X^{ij} = \frac{1}{2} \gamma^{a'ij} X^{a'}; \\
U_{a'b'} = \frac{1}{4} \gamma^{ij}_{a'b'} U_{ij}; \quad U_{ij} = \frac{1}{4} \gamma^{a'b'}_{ij} U_{a'b'}. \tag{A.24}
\]

Note that this implies

\[
\delta_U \equiv \frac{1}{2} \alpha^{ij} U_{ij} = \frac{1}{4} \alpha^{ij} \gamma^{a'b'}_{ij} U_{a'b'} = \alpha^{a'b'} U_{a'b'}. \tag{A.25}
\]

For vectors and spinors of $Spin(5)$ we have

\[
\delta_U X^{a'} = \alpha^{a'b'} X^{b'}; \quad \delta_U \lambda = \frac{1}{4} \alpha^{c'd'} \gamma^{a'b'} \lambda. \tag{A.26}
\]

We will often therefore omit the $USp(4)$ indices, implying a NW-SE convention for summation indices:

\[
\bar{\lambda} \xi = \bar{\lambda}^i \xi_i; \quad \bar{\lambda} \gamma_{a'} \xi = \bar{\lambda}^i (\gamma_{a'})^i \xi_j, \tag{A.27}
\]

so $\gamma_{a'}$ matrices are supposed to have their indices in the position $(\gamma_{a'})^j$.

When taking Majorana conjugates, $\gamma_{a}$, $\gamma_{ab}$ and $\gamma_{a'b'}$ are the matrices which lead to minus signs. E.g. in a supersymmetry commutator we can only have the following structures:

\[
\bar{\epsilon}_1 \gamma_{a} \epsilon_2; \quad \bar{\epsilon}_1 \gamma_{a} \gamma_{a'} \epsilon_2; \quad \bar{\epsilon}_1 \gamma_{abc} \gamma_{a'b'} \epsilon_2. \tag{A.28}
\]

This gives thus the antisymmetric structures between spinors of equal chirality. For spinors of opposite chirality the symmetric combinations are

\[
\bar{\lambda} \xi = \bar{\xi} \lambda; \quad \bar{\lambda} \gamma_{a'} \xi = \bar{\xi} \gamma_{a'} \lambda; \quad \bar{\lambda} \gamma_{a'b'} \gamma_{ab} \xi = \bar{\xi} \gamma_{a'b'} \gamma_{ab} \lambda. \tag{A.29}
\]

\section*{A.4 $(6, 2)$ Clifford algebra}

The general properties of this algebra can once more be extracted from \[31\]. We start from the 6-dimensional matrices in \[A.3\] (omitting the $USp(4)$ part). The $SO(6, 2)$ indices will be indicated by $\mu = 0, 1, \ldots, 7$, and we use
as signature \((-+++++++--\)). The gamma-matrices and charge conjugations are 16 \times 16 matrices\(^7\)

\[
\hat{\Gamma}_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \tilde{\gamma}_\mu & 0 \end{pmatrix} \otimes \sigma_3 \ ; \quad \hat{\Gamma}_6 = \mathbf{1} \otimes \sigma_1 \ ; \quad \hat{\Gamma}_7 = \mathbf{1} \otimes (-i)\sigma_2
\]

\[
\hat{\mathcal{C}} = \begin{pmatrix} 0 & c \\ c^T & 0 \end{pmatrix} \otimes \sigma_1 = \hat{\mathcal{C}}^T \ ; \quad (\hat{\mathcal{C}}\hat{\Gamma}_\mu)^T = \hat{\mathcal{C}}\hat{\Gamma}_\mu . \quad (A.30)
\]

The second factor here is thus not in the internal \(USp(4)\) space as in \(A.5\), but are 2 \times 2 matrices, with \(\sigma_1\sigma_2 = i\sigma_3\). To define chirality we first obtain

\[
\hat{\Gamma}_\ast = -\hat{\Gamma}_0\hat{\Gamma}_1 \ldots \hat{\Gamma}_7 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \sigma_3 . \quad (A.31)
\]

We will indicate with \(\hat{\alpha}\) the index of right-handed spinors in 8 dimensions. They are composed of a right-handed and a left-handed one in 6-dimensions as follows

\[
\hat{\lambda}_\hat{\alpha} = \left( \begin{pmatrix} \hat{0} \\ \lambda_{\hat{\alpha}}' \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + \left( \begin{pmatrix} \lambda_{\hat{\alpha}} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) . \quad (A.32)
\]

Its two parts are separated as eigenvectors of

\[
\hat{\Gamma}_{67} = \mathbf{1} \otimes \sigma_3 . \quad (A.33)
\]

Further, we have in the chiral subspace, where

\[
\hat{\lambda}_{\hat{\alpha}} = \begin{pmatrix} \lambda_{\hat{\alpha}}' \\ \lambda_{\hat{\alpha}} \end{pmatrix} , \quad (A.34)
\]

that

\[
\hat{\mathcal{C}} = \begin{pmatrix} 0 & c^T \\ c & 0 \end{pmatrix} ; \quad \hat{\Gamma}_{\mu\nu} = \begin{pmatrix} \tilde{\gamma}_{\mu\nu} & 0 \\ 0 & \gamma_{[\mu} \tilde{\gamma}_{\nu]} \end{pmatrix} ; \quad \hat{\Gamma}_{\mu7} + \hat{\Gamma}_{\mu6} = \begin{pmatrix} 0 & 0 \\ -2\gamma_\mu & 0 \end{pmatrix} ; \quad \hat{\Gamma}_{\mu7} - \hat{\Gamma}_{\mu6} = \begin{pmatrix} 0 & -2\tilde{\gamma}_\mu \\ 0 & 0 \end{pmatrix} . \quad (A.35)
\]

Let us repeat that we drop the tildes in the main text where these matrices act on chiral or antichiral 6\(d\) spinors.

\(^7\)We work here in flat space where \(\gamma_\mu\) are the constant matrices \(\gamma_a\) from above.
A.5 Forms and integration

We define the tangent space components of a generic $p$–forms, $\phi_p$, according to

$$\phi_p = \frac{1}{p!} e^{\mu_1} \ldots e^{\mu_p} \phi_{\mu_\nu \ldots \mu_1},$$  \hspace{1cm} (A.36)

where the wedge product between forms will always be understood. The following relations are easily derived,

$$dx^{\mu_0} \ldots dx^{\mu_5} = -\varepsilon_{\mu_0 \ldots \mu_5} dx^0 \ldots dx^5 = -\varepsilon_{\mu_0 \ldots \mu_5} d^6 x.$$ \hspace{1cm} (A.37)

It follows that

$$\int e^\mu e^\nu e^\rho e^\sigma e^\tau e^\phi = \int d^6 x \varepsilon^{\rho \sigma \mu \nu \tau \phi} = -\int d^6 x \varepsilon^{\mu \rho \sigma \tau \phi},$$ \hspace{1cm} (A.38)

and for the 3-forms $H$ and $G$

$$\int G H = \int e^\mu e^\nu e^\rho e^\sigma e^\tau e^\phi \frac{1}{6!} G_{\mu \nu \rho} H_{\sigma \tau \phi} = -\frac{1}{6} \int d^6 x \sqrt{g} G_{\mu \nu \rho} H^*_{\mu \nu \rho}.$$ \hspace{1cm} (A.39)

Note also that e.g. in contrast to [4] we consider the differentials as space-time derivatives, commuting with the spinors.

B More on the conformal algebra

In the general rule for the transformation of a field, (2.4), appears the Lorentz transformation matrix $m_{\mu \nu}$, which should satisfy

$$m_{\mu \nu} k m_{\rho \sigma} k - m_{\rho \sigma} k m_{\mu \nu} k = -\eta_{\mu [\rho} m_{\sigma] \nu j} + \eta_{\nu [\rho} m_{\sigma] \mu j}.$$ \hspace{1cm} (B.1)

Note the sign difference between the commutator of these matrices and the commutator of the generators, which is due to the difference between ‘active’ and passive’ transformations. See e.g. also for transformations on a field of zero Weyl weight: (transformations act only on fields, not on explicit $x^\mu$)

$$\lambda_D a^\mu [D, P_\mu] \phi(x) = (\delta_D (\lambda_D) \delta_F (a^\mu) - \delta_F (a^\mu) \delta_D (\lambda_D)) \phi(x)$$

$$= \delta_D (\lambda_D) a^\mu \partial_\mu \phi(x) - \delta_F (a^\mu) \lambda_D x^\mu \partial_\mu \phi(x)$$

$$= a^\mu \partial_\mu (\lambda_D x^\nu) \partial_\nu \phi(x)$$

$$= a^\mu \lambda_D \partial_\mu \phi(x) = \lambda_D a^\mu P_\mu \phi(x).$$ \hspace{1cm} (B.2)
The explicit form for Lorentz transformation matrices is for vectors (the indices $i$ and $j$ are of the same kind as $\mu$ and $\nu$)

$$m_{\mu\nu}^\rho \sigma = -\delta^\rho_\mu \eta^\sigma_\nu ,$$  \hspace{1cm} (B.3)

while for spinors, (where $i$ and $j$ are (unwritten) spinor indices)

$$m_{\mu\nu} = -\frac{1}{4} \gamma_{\mu\nu} .$$  \hspace{1cm} (B.4)

It is interesting to see how the transformations of derivatives of fields get similar to those of covariant derivatives in gauged conformal gravity. E.g. for a scalar of weight $w$ (and without extra special conformal transformations) we get

$$\delta_C \partial_\mu \phi(x) = \xi^\nu(x) \partial_\nu \partial_\mu \phi(x) + w \Lambda_D(x) \partial_\mu \phi(x)$$

$$- \Lambda_{M\mu'}(x) \partial_{\nu'} \phi(x) + \Lambda_D(x) \partial_\mu \phi(x) - 2w \Lambda_{K\mu} \phi(x) .$$  \hspace{1cm} (B.5)

If one would consider the covariant derivatives in a local approach then the appropriate covariant derivative is

$$D_a \phi(x) = e_\mu^a (\partial_\mu - w b_\mu) \phi(x) .$$  \hspace{1cm} (B.6)

Due to a ‘theorem on covariant derivatives’ (see section 2.5 of [19]) one only has to consider some transformations of the gauge fields to obtain the full result of its transformation. One discovers that the last line of (B.5) exactly contains these extra terms.

C Details on M 5-brane and $\kappa$-symmetry

To determine the $\kappa$ transformations, we first do not fix $\delta_\kappa \theta$. The transformation of $X^M$ in (5.8) is sufficient to determine that $c_3$ transforms as

$$\delta_\kappa c_3 = (\delta_\kappa \bar{\theta}) \Gamma_{MN} d\theta \Pi^M \Pi^N + d(\text{something}) .$$  \hspace{1cm} (C.1)

We then fix the $\kappa$-transformation of $B$ such that it cancels with the second term above in $\delta_\kappa \mathcal{H}$. We thus obtain $\delta_\kappa B$ as in (5.21) and

$$\delta_\kappa c_3 = (\delta_\kappa \bar{\theta}) \Gamma_{MN} d\theta \Pi^M \Pi^N + \frac{1}{2} d\delta_\kappa B$$

$$\delta_\kappa \mathcal{H} = \Pi^{M_1} \Pi^{M_2} d\bar{\theta} \Gamma_{M_1M_2} \delta_\kappa \theta \quad \text{or} \quad \delta_\kappa \mathcal{H}_{\mu\nu\rho} = 6 (\delta_\kappa \bar{\theta}) \gamma_{[\mu\nu} \partial_{\rho]} \theta .$$  \hspace{1cm} (C.2)
This fixes all the transformations in terms of $\delta, \theta$.

Now calculate progressively the $\kappa$-transformations of the objects which appear in the action:

$$
\delta_{\kappa} g_{\mu\nu} = 4 \left( \partial_{(\mu} \theta \right) \gamma_{\nu)} \delta_{\kappa} \theta ; \quad \delta_{\kappa} \sqrt{g} = 2 \sqrt{g} \left( \partial_{(\mu} \theta \right) \gamma^{\mu} \delta_{\kappa} \theta
$$

$$
\delta_{\kappa} R_7 = \frac{2}{4!} d^7 \Gamma^{M_1} \delta_{\kappa} \theta d\theta \Gamma_{M_1 \ldots M_5} d\theta \Pi^{M_1} \ldots \Pi^{M_4}
+ \frac{2}{5!} d\theta \Gamma_{M_1 \ldots M_5} d\delta_{\kappa} \theta \Pi^{M_1} \ldots \Pi^{M_5}
$$

$$
\delta_{\kappa} R_4 = 2 d\theta \Gamma_{M_1 M_2} d\theta \Pi^{M_2} + d\theta \Gamma_{M_1 M_2} d\delta_{\kappa} \theta \Pi^{M_1} \Pi^{M_2}
$$

$$
\delta_{\kappa} I_7 = d \left( - \frac{2}{5!} d\theta \Gamma_{M_1 \ldots M_5} \delta_{\kappa} \theta \Pi^{M_1} \ldots \Pi^{M_5} - \frac{1}{2} H d\theta \Gamma_{M_1 M_2} \delta_{\kappa} \theta \Pi^{M_1} \Pi^{M_2} \right)
$$

$$
\delta_{\kappa} \sqrt{u^2} = - \frac{1}{\sqrt{g}} \gamma^{\mu \nu \rho \sigma \lambda \tau} v_\mu \delta_{\theta} \gamma_{\rho \sigma} \delta_{\kappa} \theta
$$

$$
\delta_{\kappa} = \frac{1}{4} \epsilon^{\mu \nu \rho \sigma \lambda \tau} \gamma_{\mu \nu \rho} + \frac{1}{\sqrt{g}} \gamma_{\mu \nu \rho} H_{\mu \nu \rho} \delta_{\kappa} \theta
$$

We define (inspired by [4], but with some other factors)

$$
\delta_{\kappa} \left( - \int d^6 x \frac{\sqrt{g}}{4} H^{* \mu \nu} H_{\mu \nu} + I_{WZ} \right) = \frac{\sqrt{g}}{2} \partial_{(\mu} \theta T^{\nu)} \delta_{\kappa} \theta
$$

$$
\delta_{\kappa} G = \frac{g}{2G} \partial_{(\mu} \theta U^{\nu)} \delta_{\kappa} \theta .
$$

We make use of the expression $\bar{\gamma}$, defined in (5.20), and satisfying

$$
\bar{\gamma}^2 = 1 ; \quad \gamma^{\mu_1 \cdots \mu_6} = (-1)^{(k+1)(k+2)/2} \frac{1}{k! \sqrt{g}} \epsilon^{\mu_1 \cdots \mu_6} \gamma_{\mu_1 \cdots \mu_k} . \quad (C.5)
$$

This leads to

$$
T^\mu = 4 \gamma^{\mu} \bar{\gamma} + 6 \gamma_{\nu \rho} H^{* [\mu \nu \rho]} + 2 H_{[\nu \rho} \gamma^{\nu \rho]} - 2 H_{[\nu \rho} \gamma^{\nu \rho]} - \frac{1}{\sqrt{g}} \epsilon^{\rho \sigma \lambda \tau \phi} H_{[\nu \rho} \gamma^{\nu \rho]} H^{* \lambda \tau \phi} \gamma_{\sigma]}
$$

using

$$
H_{\mu \nu \rho} = 3 \epsilon_{[\mu \nu \rho} H^{* \lambda \tau \phi} \gamma_{\sigma]}
$$

using

$$
H_{\mu \nu \rho} = 3 \epsilon_{[\mu \nu \rho} H^{* \lambda \tau \phi} \gamma_{\sigma]}
$$

$$
H_{\mu \nu \rho} = 3 \epsilon_{[\mu \nu \rho} H^{* \lambda \tau \phi} \gamma_{\sigma]}
$$
To calculate $U_{\mu}$, we make use of (5.36):

$$\delta_\kappa \mathcal{G} = \frac{1}{2 G} \delta_\kappa G^2$$

$$\frac{1}{g} \delta_\kappa G^2 = g^{\mu\nu} \delta_\kappa g_{\mu\nu} \left( \frac{1}{2} \mathcal{H}_{\rho\sigma}^{*} \mathcal{H}^{*\rho\sigma} - \frac{3}{8} \mathcal{H}^{*\lambda\tau} \mathcal{H}^{*}_{\tau\rho} \mathcal{H}^{*\lambda\sigma} \mathcal{H}^{*\rho\sigma} \right)$$

$$- \left( \mathcal{H}_{\rho\sigma}^{*} + \frac{3}{2} \mathcal{H}^{*\lambda\tau} \mathcal{H}^{*}_{\tau\rho} \mathcal{H}^{*\lambda\sigma} \right) \delta_\kappa \mathcal{H}^{*\rho\sigma}$$

$$= g^{\mu\nu} \delta_\kappa g_{\mu\nu} \left( \frac{1}{2} \mathcal{H}_{\rho\sigma}^{*} \mathcal{H}^{*\rho\sigma} - \frac{3}{8} \mathcal{H}^{*\lambda\tau} \mathcal{H}^{*}_{\tau\rho} \mathcal{H}^{*\lambda\sigma} \mathcal{H}^{*\rho\sigma} \right)$$

$$- \left( \mathcal{H}_{\rho\sigma}^{*} + \frac{3}{2} \mathcal{H}^{*\lambda\tau} \mathcal{H}^{*}_{\tau\rho} \mathcal{H}^{*\lambda\sigma} \right) \delta_\kappa \mathcal{H}^{*\rho\sigma}$$

This leads to

$$U^\mu = 4 \tilde{\gamma}^{\mu} - \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\rho\lambda\tau\sigma} \mathcal{H}_{\rho\sigma}^{*} \mathcal{H}^{*\nu} \gamma_{\tau\sigma} - 4 \mathcal{H}_{\nu\rho}^{*} \mathcal{H}^{*\mu\nu} \gamma_{\nu} - 2 \mathcal{H}_{\rho\sigma}^{*} \mathcal{H}^{*\rho\sigma} v^{\mu} v^{\nu} \gamma_{\nu}$$

$$- \frac{3}{2} \mathcal{H}^{*\rho\sigma} \mathcal{H}_{\rho|\nu} \mathcal{H}^{*}_{\nu|\rho} \left[ \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\rho\lambda\phi\xi} \mathcal{H}^{*\rho\sigma} \gamma_{\tau\sigma} - \mathcal{H}^{*\rho\sigma} \gamma_{\nu} + 4 \mathcal{H}^{*\rho\sigma} \gamma_{\nu} \right]$$

$$+ 2 \mathcal{H}^{*\rho\sigma} v^{\mu} v^{\nu} \gamma_{\nu}$$

(C.9)

The quantities $T^\mu$ and $U^\mu$ are related by

$$U^\mu = T^\mu \rho \text{ with } \rho = \tilde{\gamma} + \frac{1}{2} \mathcal{H}_{\mu\nu}^{*} v_{\nu} \gamma^{\mu\nu} - \frac{1}{16 \sqrt{g}} \epsilon^{\mu\nu\rho\sigma\tau\phi} \mathcal{H}_{\mu\nu}^{*} \mathcal{H}^{*\rho\sigma} \gamma_{\tau\phi}$$

(C.10)

The quantity $\rho$ squares to

$$\rho^2 = \frac{G^2}{g}$$

(C.11)

The kappa invariance thus requires

$$\frac{\sqrt{g}}{2} \partial_\mu \bar{\theta} T^\mu \left( 1 - \frac{\sqrt{g}}{G} \rho \right) \delta_\kappa \theta = 0$$

(C.12)

This leads us to define

$$\Gamma = \frac{\sqrt{g}}{G} \rho$$

(C.13)

consistent with $\Gamma^2 = 1$. Indeed, then invariance is obtained by the rule (5.19), and $\Gamma$ is given by (5.20).
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