Towards analytic \((g - 2)_{\mu}\) at four loops

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In this contribution we present recent four-loop results for the muon anomalous magnetic moment based on analytic methods. In particular, fermionic corrections involving two or more closed electron loops or at least one tau lepton loop are discussed.

Keywords: anomalous magnetic moment, QED, multi-loop calculations

PACS numbers: 12.20.-m 12.38.Bx 14.60.Cd 14.60.Ef

1. Introduction

The anomalous magnetic moment of the muon constitutes both experimentally and theoretically a clean quantity which makes it an ideal object for precision studies. The current experimental value for \(a_{\mu} = (g - 2)_{\mu}/2\) (see Refs. 1, 2)

\[
a_{\mu}^{\exp} = 116 592 089(63) \times 10^{-11},
\]

matches the accuracy of the theoretical prediction which is given by 3 (see also Refs. 4, 5)

\[
a_{\mu}^{\text{th}} = 116 591 828(49) \times 10^{-11}.
\]

However, the comparison of Eqs. (1) and (2) shows that there is a discrepancy of about 3\(\sigma\) which persists for more than a decade.

The numerically most important contribution to \(a_{\mu}^{\text{th}}\) comes from QED followed by hadronic, light-by-light and electroweak corrections which are described in detail in the reviews 6, 7 and 8. In this contribution recent four-loop QED corrections 9, 10 are discussed which are based on analytic calculations with the purpose to provide an independent check of the purely numerical approach of Ref. 11. One of the motivations for such a cross-check is the fact that the four-loop QED contribution is of the same order of magnitude as the difference between Eqs. (1) and (2). Beyond one-loop order large QED corrections are obtained from Feynman diagrams containing closed electron loops. Since the electron mass cannot be set to zero such diagrams lead to sizeable logarithms \(\ln(M_{\mu}/M_e) \approx 5.3\) which occur up to third power at four loops. Among the various classes of Feynman diagrams those where the external photon couples to a closed electron loop, the so-called light-by-light-type diagrams,


Fig. 1. Two-loop Feynman diagrams contributing to \((g-2)\mu\). Thin and thick solid lines represent muon and tau leptons, respectively, and wavy lines denote photons.

give the most important contributions. At three loops, where analytic results are known, these diagrams come with an additional factor \(\pi^2\). In our approach the light-by-light diagrams are technically quite demanding and have not yet been considered. As a preparatory work we looked at other classes which are discussed in the following two sections: the contribution involving closed tau lepton loops (see Section 2) and the one with two or three closed electron loops (see Section 3). The corresponding results have been obtained in Refs. 10 and 9.

2. Closed tau loops

Starting from two loops there are Feynman diagrams contributing to \(a_\mu\) which contain a closed tau lepton loop. Actually there is only one such diagram at two-loop order (see Fig. 1) since the contributions where the external photon couples to the closed fermion loop are zero due to Furry’s theorem. At three loops one has to deal with 60 and at four loops with 1169 Feynman diagrams. The four-loop diagrams can be subdivided into twelve classes \(^{11}\) which are shown in Fig. 2.

The two-loop diagram can be computed exactly \(^{13}\) in terms of a function which depends on \(M_\mu/M_\tau\) (see also Refs. \(^{14}\) \(^{15}\) \(^{16}\)). We nevertheless want to use this simple example in order to demonstrate the method which we apply at four loops where an exact calculation is out of reach with the currently available technology. The basic idea is to obtain an expansion of \(a_\mu\) in the limit \(M_\tau \gg M_\mu\) by Taylor expanding the integrand in certain kinematical regions. The latter is visualized in Fig. 3 where the two contributions are shown which arise after applying the rules of asymptotic expansion \(^{17}\). The notation is as follows: left of the symbol \(\otimes\) one finds the so-called hard subgraphs which by definition contain all tau lepton propagators and which are one-particle irreducible with respect to the light lines. The subgraphs are expanded in all small quantities, in our case the external momenta and the muon mass, and afterwards the integrations over the hard loop momenta are performed. On the right of \(\otimes\) one finds the co-subgraphs. They are constructed from the original diagram by removing all lines which are part of the subgraph. The blob indicates the position where the result of the subgraph has to be inserted before integration over the loop momenta of the co-subgraph. At two-loop order the hard subgraphs lead to either one- or two-loop vacuum integrals with one mass scale, \(M_\tau\), whereas for the co-subgraphs tree-level contributions or one-loop on-shell integrals with \(q^2 = M_\mu^2\) have to be considered. Analogously, at four loops one has to deal with vacuum
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Fig. 2. Sample Feynman diagrams contributing to \((g-2)_\mu\) at four-loop order. The symbols label the individual diagram classes and are taken over from Refs. [12] [11].

Fig. 3. Graphical representation of the hard sub-graphs and co-subgraphs as obtained after applying the rules for asymptotic expansion to the two-loop diagram in Fig. 1.

integrals up to four loops and on-shell integrals up to three loops. Both classes of integrals are very well studied up to this loop-order (see Ref. [10] for references and more details).

In Ref. [10] three expansion terms in \(M_\mu^2/M_\tau^2 \approx 0.0035\) have been computed for all twelve classes of diagrams shown in Fig. 2. After adding all contributions one obtains

\[ A_{2,\mu}^{(8)}(M_\mu/M_\tau) \approx 0.0421670 + 0.0003257 + 0.0000015 \approx 0.0424941(2)(53), \]  

where \((\alpha/\pi)^4 A_{2,\mu}^{(8)}(M_\mu/M_\tau)\) represents the four-loop contribution to \(a_\mu\) induced by virtual tau lepton loops. The first and second uncertainty in Eq. (3) indicates the truncation error and the error in the input quantity \(M_\mu/M_\tau\), respectively. Due to the smallness of the expansion parameter we observe a rapid convergence. Actually, as can be seen after the first approximation sign in Eq. (3) each subsequent term is about a factor 100 smaller than the previous one. Thus it is safe to assign 10% of the
last computed term as uncertainty of the truncation after \((M^2_\mu/M^2_\mu)^3\). Comparing the result of Eq. (3) with the one from Ref. 11 based on numerical integration, 0.04234(12), shows good agreement. However, the analytic result is significantly more precise.

3. Closed electron loops

In Ref. 9 a first step towards a systematic study of four-loop on-shell integrals has been undertaken. More precisely, all classes of Feynman integrals have been studied which are needed to compute QED or QCD corrections to a massive fermion propagator with on-shell external momenta and two or three closed massless loops. Thus, contributions to \(a_\mu\) from diagrams as shown in Fig. 4 can be evaluated.

The four-loop integrals considered in Ref. 9 only contain either massless or massive lines. For this reason, as a start, the electrons have to be chosen as massless which leads to finite results as long as the fine structure coupling is renormalized in a mass independent renormalization scheme. Thus, initially, we renormalize the muon mass on-shell but \(\alpha\) in the \(\overline{\text{MS}}\) scheme. The (finite) result contains both constant terms and logarithms \(\ln(\mu^2/M^2_\mu)\) where \(\mu\) is the renormalization scale of the fine structure constant. Afterwards we transform \(\alpha\) to the on-shell scheme which introduces \(\ln(\mu^2/M^2_\mu)\) terms. In this way the \(\mu\) dependence cancels and \(\ln(M^2_\mu/M^2_\mu)\) terms remain.

By construction the described approach can only be applied to those Feynman diagrams where the closed electron loops are related to the renormalization of \(\alpha\). This excludes the light-by-light-type Feynman diagrams where the external photon couples to an electron.

As an example we want to discuss the four-loop contribution to \(a_\mu\) which contains two closed electron loops but no additional closed muon loop. In Ref. 9 this contribution has been denoted by \(a^{(42)a}_\mu\) and the analytic expression is given by

\[
a^{(42)a}_\mu = L^2_{\mu e} \left[ \pi^2 \left( \frac{5}{36} - \frac{a_1}{6} \right) + \frac{\zeta_3}{4} - \frac{13}{24} \right] + L_{\mu e} \left[ \frac{a_1^4}{9} + \pi^2 \left( -\frac{2a_1^2}{9} + \frac{5a_1}{3} - \frac{79}{54} \right) \right. \\
- \frac{8a_1}{3} - 3\zeta_3 + \frac{11\pi^4}{216} + \frac{23}{6} \left] - \frac{2a_1^5}{45} + \frac{5a_1^4}{9} + \pi^2 \left( -\frac{4a_1^3}{27} + \frac{10a_1^2}{9} \right) \right.
\]

\(^a\text{For similar corrections of this type see also Refs. 19, 20.}\)
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\[
- \frac{235a_1}{54} - \zeta_3 - \frac{595}{162} + \pi^4 \left( - \frac{31a_1}{540} - \frac{403}{3240} \right) + 40a_4 + \frac{16a_5}{3} - \frac{37\zeta_5}{6} \\
+ \frac{11167\zeta_3}{1152} - \frac{6833}{864} + \pi \left( - \frac{31}{540} - \frac{403}{3240} \right)
\]

with \(a_1 = \ln 2\), \(a_n = \text{Li}_n(1/2)\) \((n \geq 1)\), \(\zeta_n\) is Riemann’s zeta function and \(L_{\mu e} = \ln(M_{\mu}^2/\mu^2)\). It is interesting to note that quantities up to transcendentality level five appear in Eq. (4). The numerical evaluation leads to \(a^{(42)}_{\mu e} = -3.62427\) which should be compared to \(-3.64204(112)\). The difference is of order \(10^{-2}\) (i.e. 0.5%) and can be explained by missing \(M_e/M_{\mu}\) terms in the analytic expression.

Acknowledgments

I would like to thank Alexander Kurz, Roman Lee, Tao Liu, Peter Marquard, Alexander Smirnov and Vladimir Smirnov for a fruitful collaboration on the topics discussed in this contribution. This work was supported by the DFG through the SFB/TR 9 “Computational Particle Physics”.

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