Price’s law, mass inflation, and strong cosmic censorship

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Abstract

Two aspects of the widely accepted heuristic picture of the final state of gravitational collapse are the so-called Price law tails, describing the asymptotics of the exterior region of the black hole that forms, and Israel-Poisson’s mass inflation scenario, describing the internal structure of the black hole. (The latter scenario, if valid, would indicate in particular that the maximal development of initial data is extendible as a \( C^0 \) metric, putting into question the validity of Penrose’s strong cosmic censorship conjecture.) In this talk, I shall discuss a series of rigorous results proving both Price’s law and the mass inflation scenario in an appropriate spherically symmetric setting. The proof of Price’s law is joint work with I. Rodnianski.

1 The problem at hand

A central physical problem in general relativity is the study of the collapse of isolated self-gravitating systems. This process gives rise to spectacular predictions: the possible formation of black holes, naked singularities and Cauchy horizons. These predictions are given precise mathematical formulations in the celebrated weak and strong cosmic censorship conjectures of Penrose [31, 32]. Ultimately, resolution of these conjectures, without any additional symmetry assumptions, constitutes perhaps the central goal of mathematical relativity.

In reaching this goal, it is clear that there are several major obstacles that will have to be overcome. I believe that the first is the complete understanding of a “realistic” spherically symmetric formulation, i.e. a rigorous analysis of the global initial value problem for an appropriate Einstein-matter system, with spherically symmetric initial data, including, in particular, a proof (or disproof) of both cosmic censorship conjectures. With the word “appropriate”, it is required that the dynamical degrees of freedom of the gravitational field, together with the centrifugal force of angular momentum—both absent in spherical symmetry!
somehow, nevertheless, modeled. In the case where the effects of angular momentum are ignored, this problem was resolved by D. Christodoulou in a series of papers [17, 16, 14]. In this talk, I will describe my own contributions to this program [20, 22, 24] where the repulsive effects of angular momentum are modeled with charge. As we shall see, the implications for strong cosmic censorship are completely different. My results are motivated by previous heuristic and numerical work.

Part of the above program [24] involves proving the so-called Price’s law. This is joint work with Igor Rodnianski. Although our motivation was the role Price’s law plays in the mass inflation scenario, for which charge must be present, our proof of Price’s law applies in addition in the absence of charge, where it has independent interest. Moreover, our results also apply to the linearized problem of the wave equation on a Schwarzschild or Reissner-Nordström background. In this linear case, weaker results can also be deduced in the case where the scalar field is not spherically symmetric.

2 The Cosmic Censorship Conjectures

For appropriate coupled Einstein-matter systems, one can associate to initial data a unique maximal globally hyperbolic spacetime \((M, g)\), the so-called maximal development. The fundamental mathematical and physical questions about gravitational collapse refer then to the global geometry of this spacetime. In spherical symmetry, precise geometric properties can be read off a diagram which we can associate\(^1\) to \(M\), its so-called Penrose diagram.

The Penrose diagram of the classical conjectured picture of generic gravitational collapse is depicted below:

This “conjectured” diagram, in particular, encodes\(^2\) the statement that the following two conjectures are true:

**Weak cosmic censorship** For generic initial data, the maximal development possesses a complete future null infinity.

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\(^1\)The quotient manifold \(Q = M/\text{SO}(3)\) of the group action can be endowed with a 1+1 dimensional Lorentzian metric; a Penrose diagram of \(M\) is then just the image of a conformal representation of \(Q\) into a bounded domain of 1+1 dimensional Minkowski space. Thus the causal relation of two points can be read off the diagram, as if these points were in 1+1 dimensional Minkowski space.

\(^2\)We emphasize again that the diagrams have precise meanings and can be treated at the same level as formulas.
**Strong cosmic censorship** For generic initial data, the maximal development is inextendible as a $C^0$ metric.

A proper discussion of the terminology in the formulation of these conjectures\(^3\) is impossible here. Hopefully, even for the non-expert, the Penrose diagrams will be sufficiently suggestive. Informally, weak cosmic censorship says that there exist observers who observe a regular past forever, and strong cosmic censorship says that observers reaching the end of spacetime must be destroyed. Under the above formulations, the weak and the strong cosmic censorship conjectures are in fact independent.

The above picture of gravitational collapse was in fact rigorously obtained in 1939 by Oppenheimer and Snyder \(^2\) for the simple model of a homogeneous dust\(^4\); the significance of this work was not appreciated until the 1960’s. Christodoulou \(^1\), however, showed that the Oppenheimer-Snyder model is unstable to small inhomogeneities, as naked singularities form due to shell-crossing singularities. In the search for a more robust model that would not form such “spurious” singularities, Christodoulou was led to the Einstein-scalar field system. In a series of fundamental papers \(^17\, 13\, 16\, 14\), Christodoulou was able to show that generically, the above conjectured Penrose diagram is correct for this system under spherical symmetry.\(^5\) Christodoulou’s work initiated the rigorous theory of nonlinear p.d.e.’s as a tool for resolving the physical problems of gravitational collapse.

### 3 The challenge of angular momentum

In looking ahead to the non-spherically symmetric case, one can unfortunately not realistically hope to extrapolate from Christodoulou’s model. The spherically-symmetric Einstein-scalar field system does not account for effects of angular momentum. In the non-spherical case, the Newtonian theory indicates that these effects will dominate after collapse has progressed. It is precisely because the goal of the spherically symmetric study is to give insight into the non-spherical case that it is imperative to investigate a model that can encapsulate these effects. As we shall see below, angular momentum in general relativistic collapse in fact introduces phenomena that have no parallel in the Newtonian theory.

Fortunately for the prospect of studying this problem in the context of spherical symmetry, it turns out that there is a close connection between the repulsive mechanisms of charge and the centrifugal force of angular momentum; one manifestation of this is the analogy between the conformal structure of the rotating Kerr solution of the Einstein vacuum equations and the spherically symmetric Reissner-Nordström solution of the Einstein-Maxwell equations: The Penrose diagram of the latter is

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\(^3\)The above formulations are due to Christodoulou \(^15\).

\(^4\)The study of this model reduces to o.d.e.’s.

\(^5\)Moreover, he was also able to explicitly produce non-generic counterexamples to weak cosmic censorship.
depicted below:

![Diagram of Cauchy horizon and black hole]

Points beyond the Cauchy horizon are not part of the maximal development because the spacetime would fail to be globally hyperbolic. Yet there is no obstruction to extending beyond! It is clear, however, that such extensions are not uniquely determined by initial data. Thus, we see that the introduction of arbitrarily small angular momentum (or charge) indicates that Newtonian determinism fails in a spectacular way!

The purpose of strong cosmic censorship is precisely to exclude the above phenomena, at least generically. In fact, S.C.C. was originally conjectured on the basis of a geometric optics argument that indicated that linear fields blow up on the Reissner-Nordström Cauchy horizon.\(^6\) In a series of papers, which I will describe in this talk, I studied strong cosmic censorship in the context of the Einstein-Maxwell-neutral scalar field equations:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2 T_{\mu\nu},
\]

\[
g^{\mu\nu} F_{\lambda\mu,\nu} = 0, F_{[\lambda\mu,\nu]} = 0
\]

\[
g^{\mu\nu} \phi_{,\mu\nu} = 0,
\]

\[
T_{\mu\nu} = F_{\mu\lambda} F_{\nu\rho} g^{\lambda\sigma} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} g^{\lambda\sigma} + \phi_{,\mu\phi,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha\phi,\beta}.
\]

In spherical symmetry, the above system reduces to a hyperbolic system of p.d.e.'s in 2 independent variables. The Maxwell part decouples, and contributes an \(c^2 g_{\mu\nu}\) term to the energy-momentum tensor, where \(e\) is a constant to be called charge. The above system is in a sense quite peculiar, and in particular, cannot serve as a model for gravitational collapse from regular data with one asymptotically flat end.\(^7\) However, it is in some sense the simplest hyperbolic system which includes Reissner-Nordström as a special solution, and allows one to study the question of the stability or instability of the Cauchy horizon in a non-linear setting.

In \([20]\), I proved the following:

**Theorem 1** Consider a double characteristic initial value problem, where the outgoing characteristic is given the data of a Reissner-Nordström event horizon with \(e \neq 0\), and arbitrary sufficiently regular matching data are

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\(^6\)This insight is due to Penrose. There is a long history of numerical, heuristic, and analytic work on this linearized problem. See \([38\), \([11]\].

\(^7\)This is due to the fact that, since there is no charged matter, the charge must be topological in origin. In particular, this implies that trapped or antitrapped surfaces must be present on initial Cauchy data. See the last section.
prescribed on the ingoing characteristic. Then the maximal development is as depicted:

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (2,2) node[midway,above] {Cauchy horizon};
\draw[->] (0,0) -- (-2,2) node[midway,below] {initial data segments};
\draw[->] (0,0) -- (0,-2) node[midway,right] {event horizon};
\end{tikzpicture}
\end{center}

In particular, the Cauchy horizon survives and the metric can be extended beyond it as a $C^0$ metric.

On the other hand, I also proved in [20]

**Theorem 2** Assuming that some appropriate quantity, computed from the data at the point of intersection of the initial characteristic segments, is non-zero, it follows then that the Hawking mass blows up identically along the Cauchy horizon. Thus, the spacetime cannot be extended as a $C^1$ metric.

These results were the first rigorous confirmation of a scenario that had been originally proposed by Israel and Poisson, and studied in a long series of heuristic and numerical work, e.g. [33, 7, 6, 8]. The picture of the above two theorems is known as mass inflation.

### 4 The geometry of the event horizon: Price’s law

While the above theorems indeed show that the stability of the Cauchy horizon (albeit only in the $C^0$ norm) can in principle occur in the non-linear theory, they do not show that it does occur in dynamical spacetimes arising from collapse. For it should be clear that the data considered above—with event horizon exactly Reissner-Nordström—are non-generic and unphysical. One could easily speculate that Theorems 1 and 2 are an artifice of the particular initial value problem posed, and that the conjectured classical picture would be restored were the “correct” problem considered.

What are then the “correct” characteristic data? At the time [20] was completed, this remained an open question. There was, however, a definite conjecture, formulated in 1972 by Price [34]. According to the so-called Price law, the scalar field should decay with respect to a naturally defined advanced time coordinate along the event horizon as $v^{-3.8}$.

In [21], I proved

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8Price’s heuristics, originally formulated with respect to Schwarzschild, were extended to the charged case in [5]. There has been a long series of heuristic and numerical work confirming Price’s law, see for example [3, 4, 8, 28].
**Theorem 3** Consider a spacetime with a black hole and regular event horizon, satisfying an appropriate upper bound formulation of Price’s law\(^9\) for the decay of the scalar field, and \(0 < c < \sqrt{2m_+r_+}\), where \(m_+\) and \(r_+\) denote the limiting values of the Hawking mass and area radius, respectively, along the event horizon. Then the black hole contains a Cauchy horizon over which the spacetime is extendible as a \(C^0\) metric.

In addition, I proved

**Theorem 4** Assuming now a lower bound formulation of Price’s law\(^10\), the Cauchy horizon is \(C^1\)-singular, in particular the curvature blows up.

The above two theorems showed thus that the results of [20] were not an artifice of the particular initial value problem posed, but rather, a consequence of Price’s law.

Complete elucidation of the Einstein-Maxwell-neutral scalar field picture thus reduced to proving Price’s law. In approaching this problem, however, there was a major obstacle. The physical intuition that motivated the conjecture of power-law decay was based on heuristic arguments for the linearized problem, i.e. the study of the wave equation on a fixed Schwarzschild or Reissner-Nordström background. These arguments rest for the most part on Fourier analytic and or spectral theoretical techniques, which do not appear sufficiently robust to carry over to the non-linear theory. A new way of looking at the problem was necessary. In collaboration with Igor Rodnianski, I developed a technique based on the interaction of the global conformal geometry, the celebrated red-shift effect, and local energy conservation, to understand the behaviour of the scalar field in the exterior. Some of the main points of this technique will be discussed in the next section. In [24], we proved

**Theorem 5** Consider spherically symmetric asymptotically flat initial data for the Einstein-Maxwell-scalar field equations, where the scalar field and its gradient are initially of compact support, and assume the data contain a trapped surface. Then the maximal development of initial data contains a domain of outer communications possessing a complete future null infinity, as depicted:

![Diagram](image)

*Defining a natural advanced time coordinate \(v\) on the event horizon, and a retarded time coordinate \(u\) on null infinity, then the decay rates as depicted*

\(^9\)The condition is \(|\partial_t \phi| \leq C v^{-1-\epsilon}\) for some \(\epsilon > 0\). See Theorem 5 for the definition of the advanced time coordinate \(v\).

\(^10\)The condition is as follows: There exist positive constants \(V, C_1, C_2,\) and a constant \(s > \frac{1}{2}\), such that \(v > V\) implies \(C_1 v^{-s} \geq |\partial_v \phi| \geq C_2 v^{-3s}\) along the event horizon. See Theorem 5 for the definition of the advanced time coordinate \(v\).
above hold.\textsuperscript{11}

The upper bound formulation of Price's law thus indeed holds. Moreover, the proof clarifies the physical mechanism generating decay, in particular, the origin of the power \(-3\). Our techniques also apply to the linearized problem of the wave equation on a fixed Schwarzschild or Reissner-Nordström background\textsuperscript{12}. Since, as we noted earlier, complete spherically symmetric initial data of the Einstein-Maxwell-real scalar field system necessarily possess a trapped surface\textsuperscript{13}, we obtain from Theorems 3 and 5:

**Theorem 6** Strong Cosmic Censorship is false for the Einstein-Maxwell-real scalar field system under spherical symmetry.

On the other hand, in view of Theorem 5, the generic C\textsuperscript{1}-inextendibility of the maximal development would be given by a positive resolution to the following:

**Conjecture 1** For generic initial data for the problem considered in Theorem 5, there exist positive constants \(\epsilon, V, \text{and} C\), such that along the event horizon \(|\partial_v \phi| \geq Cv^{-9+\epsilon}\) for \(v \geq V\).

5 Remarks on the proof of Price’s law

The proof of Price’s law is rather involved, even in skeleton form. The main ingredients, however, when viewed separately, are quite transparent. They can be thought of as relatively simple analytical manifestations or consequences of the following robust\textsuperscript{14} geometrical features of black hole spacetimes

(A) a well-defined notion of infinity,

(B) the global conformal geometry; in particular, the existence of an affine complete null hypersurface (the event horizon) which does not terminate at null infinity and whose past is the entire domain of outer communications,

(C) the celebrated red-shift effect.

In this section, we will be content to make some very general remarks on how these features can be connected to one another in the analysis of the system. For more details, the reader can consult [24].

We first discuss (A). This feature is of course the least exotic of the above, as it is familiar from Minkowski space and small data dispersing solutions. For us, (A), together with the special form of energy conservation (see below) that is valid for our system, will allow us to obtain a

\textsuperscript{11}The decay rate is actually \(|\phi| + |\partial_t \phi| \leq C_v^{-3+\epsilon}\) for all \(\epsilon > 0\), but this is sufficient for applying Theorem 3. In addition, it is necessary to assume that \(\epsilon < \sqrt{\frac{m_+}{r^+}}\), i.e. that the black hole is not extremal in the limit. The coordinate \(v\) is advanced time normalized so that \(v = r\) on some fixed outgoing null ray intersecting null infinity, while \(u\) is retarded time normalized so that \(\partial_u r = -1\) on null infinity.

\textsuperscript{12}Machedon and Stalker had announced a weaker version of Price’s law for the linear problem, prior to our non-linear proof.

\textsuperscript{13}as long as the Maxwell field does not vanish identically

\textsuperscript{14}i.e. features well-known from the Schwarzschild solution, but not special to it
priori uniform decay in \( r \) for the scalar field, and certain of its derivatives, especially \( \partial_r (r \phi) \), which, as we shall see, decays better. It is this \( r \)-decay that we hope to translate into \( v \)-decay along the event horizon, and \( u \)-decay along null infinity.\(^{15}\)

The analytic significance of (B) is that the event horizon carries a positive finite flux, arising from integrating

\[
d \left( m + \frac{e^2}{2r} \right) = \frac{1}{2} \left( r \partial_r \phi \right)^2 (1 - \frac{2m}{r})du + \frac{1}{2} \left( r \partial_r \phi \right)^2 (1 - \frac{2m}{r})dv
\]

in \( u \) and \( v \) throughout the domain of outer communications, and applying Stokes’ theorem.\(^{16}\) Since the event horizon is affine complete, every so often there must be large regions where the flux is quantitatively small:

Specifically, by dyadically decomposing the event horizon, and then further subdividing each dyadic interval, we will be able to choose subintervals of size increasing as a positive power of \( v \) such that the flux in each subinterval decays as some power of \( v \), where \( v \) measures the position of the interval. This smallness will act in our argument as a catalyst for turning decay in \( r \) into decay in \( v \).

An efficient use of the event horizon flux in the above role is only possible when one is able to transport decay from the event horizon without much loss to other regions of spacetime, in particular, to regions where \( r \) is on the order of \( v \) to some positive power. In other words, one needs to be able to derive good estimates in the direction indicated in the figure below.

One might expect that the smallness of the flux on the “good” segments of the event horizon, which we hope to preserve, is completely dwarfed by initial data on the conjugate null segment depicted above, on which we know nothing. The remarkable fact is that this is not the case. Herein lies the significance of the red-shift effect (C).

Feature (C) is tied to the very name of black holes, as it is the red-shift effect that makes the black holes of gravitational collapse look “black”,

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\(^{15}\)See the footnote to Theorem \( \text{in} \) for the definitions of the \( u \) and \( v \) coordinate.

\(^{16}\)This encorporates energy conservation in our system. It is interesting to note that in the linearized case, the 1-form on the right hand side of the above equation is again closed; it is precisely the form arising from contracting the energy momentum tensor of the scalar field with the \( \partial_t \) Killing vector field.
rather than like frozen stars. To measure quantitatively this effect, one considers the ratio of the proper time between the reception of two signals, as measured by an observer travelling to timelike infinity, to the proper time between the emission of the two signals by an observer entering the black hole. As the emitter approaches the black hole, and the proper time between his emissions tends to zero, this ratio goes to $\infty$. In the context of the scalar field, this shifts the frequency of the radiation on the incoming null ray to the red. This will allow us to essentially “forget” about this part of the initial data when estimating as in the previous diagram.

Finally, it is worth commenting on how the power $-3$ appears. As noted in the above discussion, we derive decay in $v$ ultimately from decay in $r$, specifically, from the decay in $r$ of the quantity $\partial_t r \phi$, which satisfies:

$$\partial_u (\partial_t r \phi) = \frac{\phi}{r^2} \frac{\partial_t r \partial_u r}{1 - \frac{2m}{r}} \left( m - \frac{e^2}{2r} \right)$$

Recall that in Minkowski space, $\partial_t r \phi$ vanishes (for large $v$) for spherically symmetric solutions of the wave equation with compactly supported data. In our case, however, due to backscattering off the curved background, $\partial_t r \phi$ will immediately acquire an $r^{-3}$ tail on an outgoing null cone. It is for this reason that we cannot improve beyond $-3$.

6 A complete “relatistic” spherically symmetric picture

The Einstein-Maxwell-neutral scalar field system may indeed capture some of the phenomenology of vacuum collapse in the black hole interior. As remarked earlier, however, it cannot represent a truly complete model in spherical symmetry, as, if $e \neq 0$, complete initial data must possess two asymptotically flat ends. Black holes are thus “built in” to the problem from the start. A more interesting model arises when the matter is endowed with charge, for instance a complex-valued scalar field $\phi$, interacting with an electromagnetic potential $A_\mu$, defined up to a phase (see [25]). Large time existence results for small data solutions of such systems have been proven by Chae [10]. In view of the results described here, a working conjecture for the evolution of generic spacelike initial data is as follows:

Conjecture 2 For generic, asymptotically flat, spacelike spherically symmetric data with one end, the maximal development has Penrose diagram...
In particular, according to the above, the trapped surfaces conjecture (and thus W.C.C.) is true, but S.C.C., as formulated, is false.\footnote{A $C^1$-formulation, however, of S.C.C. may still be true.}

It should be emphasized once again that the primary motivation for the matter described here is as a spherically symmetric model problem for the non-spherically symmetric vacuum. Another source for interesting matter models in gravitational collapse is kinetic theory. The reader should consult [1, 36, 35]. In this context, certain results have been recently extended to the charged case in [30, 29], motivated by similar considerations to those described here.

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