Where are ELKO Spinor Fields in Lounesto Spinor Field Classification?

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Abstract

This paper proves that from the algebraic point of view ELKO spinor fields belong together with Majorana spinor fields to a wider class, the so-called flagpole spinor fields, corresponding to the class 5, according to Lounesto spinor field classification. We show moreover that algebraic constraints imply that any class 5 spinor field is such that the 2-component spinor fields entering its structure have opposite helicities. The proof of our statement is based on Lounesto general classification of all spinor fields, according to the relations and values taken by their associated bilinear covariants, and can eventually shed some new light on the algebraic investigations concerning dark matter.

1 Introduction

In order to find an adequate mathematical formalism for representing dark matter, Ahluwalia-Khalilova and Grumiller have recently introduced the Eigenspinoren des Ladungskonjugationsoperators (ELKO) spinor fields\textsuperscript{1} \cite{Ahluwalia:2010ep}, which are shown to belong to a non-standard Wigner class, and to exhibit non-locality. They claim that an ELKO spinor field is a new fermion described by a spinor field that has not been identified, in the Physics literature, with any particle or more general physical entity yet. In the low-energy limit ELKO comports as a representation of the Lorentz group. However, mathematicians have already known since sometime ago that all spinors that are elements of the carrier spaces of the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representations of $Sl(2, \mathbb{C})$ belong to one of the six classes found by Lounesto in his theory of the classification of spinor fields.

\textsuperscript{1}Dual-helicity eigenspinors of the charge conjugation operator.
of spinor fields. Such an algebraic classification is based on the values assumed by their bilinear covariants, the Fierz identities, aggregates and boomerangs (see Eq. (7) below) [3, 4]. In this paper we prove that from the algebraic point of view ELKO spinor fields belong to Lounesto class 5 spinor fields, also called flagpole spinor fields due to the intrinsic flagpole structure they carry. It is a general property of class 5 spinor fields that they satisfy the Majorana condition, i.e., if ψ denotes an ELKO spinor field, λ a complex number of unitary modulus and C the charge conjugation operator then \( C \psi = \lambda \psi \). In Section 2, after presenting the bilinear covariants, that completely characterize a spinor field through Fierz identities and through Fierz aggregates (or boomerangs), Lounesto classification of spinor fields is reviewed. Section 3 recalls the definition of ELKO spinor fields [1] and shows that ELKO is indeed a flagpole spinor field. We show moreover in Section 4 that algebraic constraints imply that any class 5 spinor field is such that their 2-component spinor fields have opposite helicities. Now, it is well known that Majorana spinor fields on Minkowski spacetime are defined as eigenvalues of the charge operator, and it is easy to verify that any spinor field that is eigenspinor of the charge operator necessarily belongs to Lounesto class 5. So, what differentiates ELKO from Majorana spinor fields? In [1] authors quote that the difference is that according to Peskin and Schroeder [13] and Marshak and Sudarshan [12] it is imposed that the 2-component spinor fields of a Majorana spinor field have the same helicity. Is this reasonable? We briefly discuss such issue.

2 Bilinear Covariants

In this paper all spinor fields live in Minkowski spacetime \((M, \eta, D, \tau_\eta, \uparrow)\). Here, the manifold \(M \simeq \mathbb{R}^4\), \(\eta\) denotes a constant metric of signature \((1,3)\), \(D\) denotes the Levi-Civita connection of \(\eta\), \(M\) is oriented by the 4-volume element \(\tau_\eta \in \bigwedge T^*M\) and time-oriented by \(\uparrow\). As usual \(T^*M\) denotes the cotangent bundle and \(TM\) the tangent bundle over \(M\). By a constant metric we mean the following: let \(\{x^\mu\}\) be global coordinates in the Einstein-Lorentz gauge, naturally adapted to an inertial reference frame [15] \(e_0 = \partial/\partial x^0\). Let also \(e_i = \partial/\partial x^i, i = 1, 2, 3\). Then, \(\eta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). Also, \(\{e_\mu\}\) is a section of the frame bundle \(P_{SO(1,3)}(M)\) and \(\{e^\mu\}\) is its reciprocal frame satisfying \(\eta(e^\mu, e_\nu) := e^\mu \cdot e_\nu = \delta^\mu_\nu\). Let \(\Xi\) be a section of the principal spin structure bundle [10] \(P_{\text{Spin}^e_{1,3}}(M)\) such that \(s(\Xi) = \{e_\mu\}\). Classical spinor fields\(^2\) carrying a \(D^{(1/2,0)} \oplus D^{(0,1/2)}\), or \(D^{(1/2,0)}\), or \(D^{(0,1/2)}\) representation of \(Sl(2, \mathbb{C}) \simeq \text{Spin}^e_{1,3}\) are sections of the vector bundle
\[
P_{\text{Spin}^e_{1,3}}(M) \times_\mu \mathbb{C}^4,
\]

\(^2\)Quantum spinor fields are operator valued distributions, as well known. It is not necessary to introduce quantum fields in order to know the algebraic classification of ELKO spinor fields.
where $\rho$ stands for the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ (or $D^{(1/2,0)}$ or $D^{(0,1/2)}$) representation of $\text{SL}(2, \mathbb{C}) \simeq \text{Spin}^+_4 \mathbb{C}$ in $\mathbb{C}^4$. Other important spinor fields, like Weyl spinor fields are obtained by imposing some constraints on the sections of $\mathbf{P}_{\text{Spin}^+_4 \mathbb{C}} \times \rho \mathbb{C}^4$. See, e.g., [3, 4] for details. Given a spinor field $\psi \in \text{sec} \, \mathbb{P}_{\text{Spin}^+_4 \mathbb{C}}(M) \times \rho \mathbb{C}^4$ the bilinear covariant fields are the following sections of $\mathbf{P}_{\text{Spin}^+_4 \mathbb{C}} \times \rho \mathbb{C}^4$, denoted by $\psi, J, K, S$:

\begin{align*}
\sigma &= \psi^\dagger \gamma_0 \psi, \\
\mathbf{J} &= J_\mu \mathbf{e}^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \mathbf{e}^\mu, \\
\mathbf{S} &= S_{\mu \nu} \mathbf{e}^{\mu \nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_\mu \gamma_\nu \psi \mathbf{e}^\mu \wedge \mathbf{e}^\nu, \\
\mathbf{K} &= \psi^\dagger \gamma_0 \gamma_{0123} \gamma_\mu \psi \mathbf{e}^\mu, \\
\omega &= -\psi^\dagger \gamma_0 \gamma_{0123} \psi,
\end{align*}

(1)

where $\mu, \nu, \rho = 0, 1, 2, 3$, and $\mu < \nu < \rho$ is a basis for $\text{Cl}(M, \eta)$, and

\begin{align*}
\{1, \mathbf{e}^\mu, \mathbf{e}^\mu \mathbf{e}^\nu, \mathbf{e}^\mu \mathbf{e}^\nu \mathbf{e}^\rho, \mathbf{e}^0 \mathbf{e}^1 \mathbf{e}^2 \mathbf{e}^3\},
\end{align*}

is a basis for $\mathbb{C}(4)$. In addition, these bases satisfy the respective Clifford algebra relations [3]

\begin{align*}
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu &= 2 \eta_{\mu \nu} \mathbf{1}_4, \\
\mathbf{e}^\mu \mathbf{e}^\nu + \mathbf{e}^\nu \mathbf{e}^\mu &= 2 \eta^{\mu \nu},
\end{align*}

(2)

where $\mathbf{1}_4 \in \mathbb{C}(4)$ is the identity matrix, and $\eta^{\mu \nu} = \text{diag}(1, -1, -1, -1)$. When there is no opportunity for confusion we shall omit the $\mathbf{1}_4$ identity matrix in our formulas. We observe that the Clifford product of two Clifford fields is denoted by juxtaposition of symbols. For the orthonormal vector fields $\mathbf{e}^\mu$ and $\mathbf{e}^\nu$, $\mu \neq \nu$, their Clifford product $\mathbf{e}^\mu \mathbf{e}^\nu$ is equal to the exterior product of those vectors, i.e., $\mathbf{e}^\mu \mathbf{e}^\nu = \mathbf{e}^\mu \wedge \mathbf{e}^\nu = \mathbf{e}^{\mu \nu}$. Also, for $\mu \neq \nu \neq \rho$, $\mathbf{e}^\mu \mathbf{e}^\nu = \mathbf{e}^\mu \mathbf{e}^\nu \mathbf{e}^\rho$, etc. More details on our notations, if needed can be found in [10].

Since we are interested only in the algebraic classification of spinor fields it is helpful, in order to consistently perform calculations with algebraic methods known to the majority of physicists, to introduce operator fields associated with the bilinear covariant fields. In a fixed spin frame these operator fields are, for each $x \in M$, mappings $\mathbb{C}^4 \rightarrow \mathbb{C}^4$. They will be represented by the same symbols, since from this usage (hopefully) no confusion will result. So, in what follows
the bilinear covariants are considered as being the following \textit{operator} fields:

\[
\sigma = \psi \dagger \gamma_0 \psi, \quad J = J_\mu \gamma^\mu = \psi \dagger \gamma_0 \gamma_\mu \psi \gamma^\mu, \quad S = S_{\mu\nu} \gamma^{\mu\nu} = \frac{1}{2} \psi \dagger \gamma_0 \gamma_{\mu\nu} \psi \gamma^{\mu\nu},
\]

\[
K = \psi \dagger \gamma_0 \gamma_{0123} \gamma_\mu \psi \gamma^\mu, \quad \omega = -\psi \dagger \gamma_0 \gamma_{0123} \psi.
\] (3)

In the case of the electron, described by Dirac spinor fields (classes 1, 2 and 3 below), \( J \) is a future-oriented timelike current vector which gives the current of probability. This means that the Clifford product of \( J \in \sec \bigwedge TM \hookrightarrow \mathcal{C}(M, \eta) \) with itself, i.e., \( J^2 \) is such that

\[
J^2 = J_\mu e^\mu J_\nu e^{\nu} = J_\mu J_\nu \frac{1}{2} (e^\mu e^{\nu} + e^{\mu} e^{\nu}) = \eta^{\mu\nu} J_\mu J_\nu = J_\mu J^\mu > 0. \quad (4)
\]

Of course, if \( J : M \ni x \mapsto \mathbb{C}(4) \) is interpreted as a vector (field) operator we have \( J^2 = J_\mu J^\mu 1_4 \). In this case writing \( J^2 > 0 \) means \( J_\mu J^\mu > 0 \).

Moreover, the bivector \( S \) is associated with the distribution of intrinsic angular momentum, and the spacelike vector \( K \) is associated with the direction of the electron spin. For a detailed discussion concerning such entities, their relationships and physical interpretation, and generalizations, see, e.g., [2, 3, 4, 6, 7].

The bilinear covariants satisfy the Fierz identities \( \mathbb{2} \mathbb{3} \mathbb{4} \mathbb{6} \mathbb{7} \).

\[ J^2 = \omega^2 + \sigma^2, \quad K^2 = -J^2, \quad J \cdot K = 0, \quad J \wedge K = -(\omega + \sigma \gamma_{0123})S. \] (5)

and also satisfy\(^3\):

\[
S \cdot J = \omega K, \quad S \cdot K = \omega J, \quad (\gamma_{0123}S) \cdot J = \sigma K,
\]

\[
(\gamma_{0123}S) \cdot K = \sigma J, \quad S \cdot S = -\omega^2 + \sigma^2, \quad (\gamma_{0123}S) \cdot S = -2\omega \sigma,
\]

\[
JS = -(\omega + \sigma \gamma_{0123})K, \quad KS = -(\omega + \sigma \gamma_{0123})J, \quad SJ = (\omega - \sigma \gamma_{0123})K
\]

\[
SK = (\omega - \sigma \gamma_{0123})J, \quad S^2 = (\omega - \sigma \gamma_{0123})^2 = \omega^2 - \sigma^2 - 2\omega \sigma \gamma_{0123},
\]

\[
S^{-1} = -\frac{S(\sigma - \omega \gamma_{0123})^2}{(\omega^2 + \sigma^2)^2} = \frac{KSK}{(\sigma^2 + \omega^2)^2}. \quad (6)
\]

In the formulas above \( \cdot \) denotes the scalar product and \( \wedge \) refers to the \textit{right} contraction product of Clifford fields (or Clifford operators). For details, please consult, e.g., \( \mathbb{2} \mathbb{3} \mathbb{4} \mathbb{15} \). Introduce the complex multivector field \( Z \in \sec \mathcal{C}(M, \eta) \) (where \( \mathcal{C}(M, \eta) \) denotes the complexified spacetime Clifford bundle, in which the typical fiber is \( \mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \mathbb{10} \)) and the corresponding complex multivector \textit{operator} (represented by the same letter):

\[
Z = \sigma + J + iS + iK \gamma_{0123} + \omega \gamma_{0123}. \quad (7)
\]

When the multivector operators \( \sigma, \omega, J, S, K \) satisfy the Fierz identities, then the complex multivector operator \( Z \) is denominated a \textit{Fierz aggregate}, and, when

\(^3\)Note that \( S^{-1} \) exists of course only if \( \omega \) and \( \sigma \) are not simultaneously null.
\(\gamma_0 Z^\dagger \gamma_0 = Z\), which means that \(Z\) is a Dirac self-adjoint aggregate\(^4\), \(Z\) is called a \textit{boomerang}.

A spinor field such that \textit{not both} \(\omega\) and \(\sigma\) are null is said to be regular. When \(\omega = 0 = \sigma\), a spinor field is said to be \textit{singular}. In this case the Fierz identities are in general replaced by the more general conditions \([2]\) (which obviously also holds for \(\omega, \sigma \neq 0\)). These conditions are:

\[
\begin{align*}
Z^2 &= 4\sigma Z, \\
Z \gamma_\mu Z &= 4J_\mu Z, \\
Z i \gamma_\mu Z &= 4S_{\mu\nu} Z, \\
Z i \gamma_{0123} Z &= 4K_\mu Z, \\
Z \gamma_{0123} Z &= -4\omega Z.
\end{align*}
\]

(8)

Now, any spinor field (regular or singular) can be reconstructed from its bilinear covariants as follows. Take an arbitrary spinor field \(\xi\) satisfying \(\xi^\dagger \gamma_0 \psi \neq 0\). Then the spinor field \(\psi\) and the multivector field \(Z\xi\), differ only by a phase. Indeed, it can be written as

\[
\psi = \frac{1}{4N} e^{-ia} Z\xi,
\]

(9)

where \(N = \frac{1}{2} \sqrt{\xi^\dagger \gamma_0 Z\xi}\) and \(e^{-ia} = \frac{1}{N} \xi^\dagger \gamma_0 \psi\). For more details see, e.g., \([2, 11]\).

Lounesto spinor field classification is given by the following spinor field classes \([3, 4]\), where in the first three classes it is implicit that \(J, K, S \neq 0\):

1. \(\sigma \neq 0, \ \omega \neq 0\).
2. \(\sigma \neq 0, \ \omega = 0\).
3. \(\sigma = 0, \ \omega \neq 0\).
4. \(\sigma = 0 = \omega, \ \ K \neq 0, \ S \neq 0\).
5. \(\sigma = 0 = \omega, \ \ K = 0, \ S \neq 0\).
6. \(\sigma = 0 = \omega, \ \ K \neq 0, \ S = 0\).

The current density \(J\) is always non-zero. Type 1, 2 and 3 spinor fields are denominated \textit{Dirac spinor fields} for spin-1/2 particles and type 4, 5, and 6 are respectively called \textit{flag-dipole}, \textit{flagpole} and \textit{Weyl spinor fields}. Majorana spinor fields are a particular case of a type 5 spinor field. It is worthwhile to point out a peculiar feature of types 4, 5 and 6 spinor fields: although \(J\) is always non-zero, we have \(J^2 = -K^2 = 0\). We shall see, below, that the bilinear covariants related to an ELKO spinor field, satisfy \(\sigma = 0 = \omega, \ K = 0, \ S \neq 0\) and \(J^2 = 0\).

Lounesto proved that there are \textit{no} other classes based on distinctions between bilinear covariants. So, ELKO spinor fields must belong to one of the six classes.

Before ending this section we remark that the sum of two spinor fields belonging to a given Lounesto class is not necessarily a spinor field of the same class, as it is easy to verify.

\(^4\)It is equivalent to say that \(\omega, \sigma, J, K, S\) are real multivectors.
3 ELKO Spinor Fields

In this section we explore in details the algebraic properties of ELKO spinor fields as defined in [1].

A ELKO spinor field $\Psi$ corresponding to a plane wave with momentum $p = (p^0, \mathbf{p})$ can be written, without loss of generality, as $\Psi = \psi e^{-ip \cdot x}$ (or $\Psi = \psi e^{ip \cdot x}$) with

$$\psi = \begin{pmatrix} i\Theta \phi_+(p) \\ \phi_+(p) \end{pmatrix}, \quad \phi_+(p) = \begin{pmatrix} \sigma_2 \rho(p) \\ \rho(p) \end{pmatrix}, \quad \Theta J \Theta^{-1} = -J^*.$$}

Here, as in [1], the Weyl representation of $\gamma^\mu$ is used, i.e.,

$$\gamma^0 = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad -\gamma^k = \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix},$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.

Omitting the subindex of the spinor $\phi_+(p)$, which is denoted heretofore by $\phi$, the left-handed spinor field $\phi_+(p)$ can be represented by

$$\phi = \begin{pmatrix} \alpha(p) \\ \beta(p) \end{pmatrix}, \quad \alpha(p), \beta(p) \in \mathbb{C}.$$}

Now using Eqs.(9) it is now possible to calculate explicitly the bilinear covariants for ELKO spinor fields:

$$\sigma = \psi^\dagger \gamma_0 \psi = 0,$$

$$\omega = -\psi^\dagger \gamma_0 \gamma_{0123} \psi = 0$$

$$J_\mu \gamma^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \gamma^\mu = 2(\alpha^* \beta + \alpha \beta^*) \gamma^1 + 2i(\alpha^* \beta - \alpha \beta^*) \gamma^2 + 2(\beta \beta^* - \alpha \alpha^*) \gamma^3 + 2(\alpha \alpha^* + \beta \beta^*) \gamma^0,$$

$$K_\mu \gamma^\mu = \psi^\dagger i \gamma_{123} \gamma_\mu \psi \gamma^\mu = 0.$$
\[ S = \frac{1}{2} S_{\mu\nu} \gamma^{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \gamma^{\mu\nu} \]
\[ = \frac{i}{2} ((\alpha^*)^2 + (\beta^*)^2 - \beta^2 - \alpha^2) \gamma^{02} + \frac{1}{2} ((\alpha^*)^2 + (\beta^*)^2 + \beta^2 + \alpha^2) \gamma^{31} \]
\[ + \frac{1}{2} ((\beta^*)^2 + \beta^2 - (\alpha^*)^2 - \alpha^2) \gamma^{01} + \frac{i}{2} (-\beta^2 - \alpha^2 + (\alpha^*)^2 + (\beta^*)^2) \gamma^{02} \]
\[ + (\alpha \beta + \alpha^* \beta^*) \gamma^{03} + \frac{i}{2} (\alpha \beta - \alpha^* \beta^*) \gamma^{12} + \frac{i}{2} (\beta^2 - \alpha^2 + (\alpha^*)^2 - (\beta^*)^2) \gamma^{23}. \]

(19)

From the formulas in Eqs. (17, 18) it is trivially seen that that
\[ J \cdot K = 0. \]

(20)

Also, from Eq. (17) it follows that
\[ J^2 = 0, \]

and it is immediate that all Fierz identities introduced by the formulas in Eqs. (15) are trivially satisfied. It also follows directly from Eq. (19) (or easier yet, using the formula for \( S^2 \) in Eq. (6) and Eq. (16)) that
\[ S^2 = 0. \]

Now, any flagpole spinor field is an eigenspinor of the charge conjugation operator \( \mathcal{C} \psi = -\gamma^2 \psi^* \). We must have:
\[ -\gamma^2 \psi^* = \lambda \psi, \quad |\lambda| = 1. \]

(21)

where \( \lambda \) is a complex number of unitary modulus. Using Eq. (11) it follows that
\[ -\gamma^2 \psi^* = \left( \begin{array}{cc} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{array} \right) \left( \begin{array}{c} (\sigma_2 \phi^*)^* \\ \phi^* \end{array} \right) \]
\[ = \left( \begin{array}{c} \sigma_2 \phi^* \\ -\sigma_2 \sigma_2^* \phi^* \end{array} \right) \]
\[ = \left( \begin{array}{c} \sigma_2 \phi^* \\ -\sigma_2 \sigma_2^* \phi^* \end{array} \right) \]
\[ = \psi. \]

(22)

Now, recall that a Majorana spinor field (in Minkowski spacetime) is defined as an eigenvector of the charge operator, with \( \lambda = \pm 1 \). It follows, as can be easily verified that its structure implies immediately that it must be a flagpole spinor field, i.e., of Lounesto class 5 spinor fields. \[ \text{ [3, 4] } \]

So ELKO spinor field belongs to the same class as Majorana spinor fields, i.e., class 5 spinor fields, by Lounesto spinor field classification. So, the question arises: what is the difference between ELKO and Majorana spinor fields?

4 Helicities

Consider any class 5 spinor field \( \psi = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) \). We already stated that any such \( \psi \) satisfy the equation \( \mathcal{C} \psi = \lambda \psi \) with \( \lambda \overline{\lambda} = 1 \). Such condition implies that \( \lambda \phi_1 = \sigma_2 \phi_2^* \).
Let as usual $\sigma \cdot \hat{p}$ be the helicity operator acting on 2-component spinor fields. Suppose now that $\sigma \cdot \hat{p}\phi_2 = \phi_2$ (respectively, $\sigma \cdot \hat{p}\phi_1 = \phi_1$). Then a trivial calculation shows that $\sigma \cdot \hat{p}\phi_1 = -\phi_1$ (respectively $\sigma \cdot \hat{p}\phi_2 = -\phi_2$), i.e., the 2-component spinor fields presented in the structure of a class 5 spinor field have necessarily opposite helicities.

Of course, this is also the case of an ELKO spinor field, since we have just proved that they belong to class 5 spinor fields. We are now prepared to give the answer to the question formulated at the end of the previous section, according to Ahluwalia-Khalilova and Grumiller [1]. They asserted that the difference between ELKO and Majorana spinor fields resides in the fact that the 2-component spinor fields entering the structure of a Majorana spinor field have the same helicity. They attribute this statement, e.g., to Peskin and Schroeder [13] and Marshak and Sudarshan [12]. Of course, this assumption if used by [13, 12] or by any other author must be considered completely ad hoc from the algebraic point of view. There is no justification for it, except an eventual desire to give to Majorana particles a well-defined helicity, something that is not endorsed by the Mathematics of spinor fields. Reading carefully Peskin and Schroeder’s book [13] we found that those authors propose as an exercise the possibility of writing a field equation for a 2-component spinor field of definite helicity (with Grassmann algebra-valued entries) encoding the contents of a Majorana field. In part (e) of that exercise those authors call the 2-component spinor field (of definite helicity) a Majorana field. However, the true Majorana spinor field is a 4-component spinor field, eigenvector of the charge operator and thus, as already proved, it must be composed by two 2-component spinor fields of opposite helicities. At the heart of the issue it is a real confusion between the concepts of chirality and helicity for massive fermions. Indeed, we can find papers, e.g., one by Hannestad [5] where it is stated that the Majorana quantum spinor field is indeed without definite chirality (i.e., “it is a linear combination of left handed and right handed parts”, these parts understood as chiral parts of a spinor field) but that the quantum state of a Majorana particle may be of definite helicity. Hannestad endorses his statement quoting the theory of Majorana particles as derived in the book by Mohapatra and Pal [9] and also on the book by Kim and Pevsner [8]. Also, Plaga [14] stated that Majorana fermions may have states of definite helicity. However, adding power to the confusion he indeed exhibits a “Majorana field” with definite helicity, something that according to our view is equivocated. Plaga also stated that physical states cannot have definite chirality. A complete discussion of these issues will be postponed to another paper. Here we only quote that Ahluwalia-Khalilova and Grumiller [1] showed that adhering to the correct mathematical result leads to interesting physical consequences, as, e.g., the issue of non-locality (see also [17]).

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5Exercise 3.4, page 73 of [13].
5 Concluding Remarks

We showed that ELKO spinor fields belong to the class of flagpole, class 5, spinor fields, according to Lounesto classification. We showed moreover that algebraic constraints imply that any class 5 spinor field is such that their 2-component spinor fields have opposite helicities. The statements that [1] attributes to [13, 12] asserting that Majorana spinor fields, (a particular class 5 spinor field) are such that their 2-component spinor fields have the same helicity seems to be ad hoc. Moreover, it is easy to verify that while the anticommutator between the charge conjugation and parity operators acting on a Dirac spinor field is equal to zero, the commutator of those operators acting on an ELKO spinor field (which do not satisfy Dirac equation) is also zero. ELKO dynamics is to be analyzed in a forthcoming paper. Also, a relation between Lounesto’s classification and Wigner’s classification of spinor fields (which plays an important role in [1]) needs some further study and will be presented elsewhere.

Finally, we take the opportunity to call the reader’s attention to the fact that no use has been made until now (to the best of our knowledge) of class 4 spinor fields. Eventually they may be the important spinor fields to describe dark matter and/or dark energy. This possibility will be to explored elsewhere.

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6As first observed by Ahluwalia-Khalilova and Grumiller in [1].
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