Pricing Derivatives by Path Integral and Neural Networks

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Abstract

Recent progress in the development of efficient computational algorithms to price financial derivatives is summarized. A first algorithm is based on a path integral approach to option pricing, while a second algorithm makes use of a neural network parameterization of option prices. The accuracy of the two methods is established from comparisons with the results of the standard procedures used in quantitative finance.

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1 Introduction

The development of efficient computational algorithms to price and hedge financial derivatives is a particularly lively topic of the research work in quantitative finance and econophysics.

In the classical theory of option pricing [1] by Black and Scholes, and Merton (BSM), the price of a financial derivative is given by a deterministic partial differential equation. In principle, by solving this equation with the appropriate boundary conditions, the value of the derivative of interest can be obtained. In
practice, the BSM equation can be analytically solved only in the most simple case of a so-called European option. Actually, if we consider other financial derivatives, which are commonly traded in real markets and allow anticipated exercise or depend on the history of the underlying asset, analytical solutions do not exist and numerical techniques are necessary. As discussed in the literature [2], the standard numerical procedures are binomial trees, the method of finite differences and the Monte Carlo simulation of random walks. Since binomial trees and finite difference methods are difficult to apply when a detailed control of the paths is required, the Monte Carlo simulation is the mostly used method to price path-dependent options. However, Monte Carlo is known to be time consuming if precise predictions are required and appropriate variance reduction techniques must be introduced to save CPU time [2].

This paper summarizes recent progress in the development of novel computational methods to price options, as alternatives to traditional numerical procedures used in finance. A first algorithm is based on a path integral approach to option pricing, as described in detail in [3]. A second algorithm relies upon neural networks and represents work in progress [4].

2 The path integral algorithm

The path integral method is an integral formulation of the dynamics of a stochastic process [5,6]. It allows to compute the transition probability associated to a given stochastic process. We consider, for definiteness, as stochastic model for the time evolution of the asset price $S$ the standard BSM Brownian motion, driven by the stochastic differential equation $d\ln(S) = \mu dt + \sigma dw$, where $\sigma$ is the volatility, $A = (\mu - \sigma^2/2)$, $\mu$ is the drift parameter and $w$ is a Wiener process. The probability associated to the above process for a transition from $z_i = \ln S_i$ to $z_f = \ln S_f$ over a time interval $\Delta t$ is given by

$$p(z_f|z_i) = \frac{1}{\sqrt{2\pi\sigma^2 \Delta t}} \exp \left\{ -\frac{(z_f - (z_i + A\Delta t))^2}{2\sigma^2 \Delta t} \right\}.$$  (1)

If we consider a finite-time interval $[t', t'']$ and we apply a time slicing in terms of $n + 1$ subintervals of length $\Delta t \hat{=} (t'' - t')/n + 1$, the finite-time transition probability can be expressed as a convolution of the form [3]

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{1}{\sqrt{(2\pi\sigma^2 \Delta t)^{n+1}}} \exp \left\{ -\frac{1}{2\sigma^2 \Delta t} \sum_{k=1}^{n+1} (z_k - (z_{k-1} + A\Delta t))^2 \right\}.$$  (2)
Once the transition probability is known, the price of an option can be computed as the conditional expectation value of a given functional of the stochastic process, involving integrals of the form [3,6]

$$E[O_i|S_{i-1}] = \int dz_i p(z_i|z_{i-1}) O_i(e^{z_i}).$$  \hspace{1cm} (3)

For example, for an European call option at the maturity $T$ the quantity of interest will be $\max\{S_T - X, 0\}$, $X$ being the strike price.

2.1 Transition probability

By means of appropriate manipulations of the integrand entering eq. (2), it is possible to obtain an expression for the transition probability suitable for an efficient numerical implementation [3]. If we define $y_k = z_k - kA\Delta t$, $k = 1, \ldots, n$, we can rewrite eq. (2) as

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dy_1 \cdots dy_n \frac{1}{\sqrt{(2\pi\sigma^2\Delta t)^{n+1}}} \cdot \exp \left\{ -\frac{1}{2\sigma^2\Delta t} \sum_{k=1}^{n+1} [y_k - y_{k-1}]^2 \right\}, \hspace{1cm} (4)$$

in order to get rid of the drift parameter. A convenient quadratic form can be extracted from the argument of the exponential function, to arrive, after some algebra, at the following formula for the probability distribution [3]

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{n} dh_i \frac{1}{\sqrt{2\pi\sigma^2\Delta t\det(M)}} \exp \left\{ -\frac{1}{2\sigma^2\Delta t} \left[ y_0^2 + y_{n+1}^2 + \sum_{i=1}^{n} \frac{(y_0 O_{1i} + y_{n+1} O_{ni})^2}{m_i} \right] \right\}, \hspace{1cm} (5)$$

where we have introduced new variables $h_i$ obeying the relation

$$dh_i \doteq \sqrt{\frac{m_i}{2\pi\sigma^2\Delta t}} \exp \left\{ -\frac{m_i}{2\sigma^2\Delta t} \left[ w_i - \frac{(y_0 O_{1i} + y_{n+1} O_{ni})}{m_i} \right] \right\} dw_i. \hspace{1cm} (6)$$

The gaussian variables $h_i$ can be extracted from a normal distribution with mean $(y_0 O_{1i} + y_{n+1} O_{ni})^2/m_i$ and variance $\sigma^2\Delta t/m_i$, where $m_i$ are the eigenvalues of a real, symmetric and tridiagonal matrix $M$ and $O_{ij}$ are the matrix elements of the orthogonal matrix $O$ which diagonalizes $M$, with $w_i = O_{ij} y_j$. The probability distribution, as given by eq. (5), is an integral whose kernel is a constant function and can be very efficiently and precisely computed as shown in [3].
2.2 Option pricing

In the case of options with possibility of anticipated exercise before the expiration date, it is necessary to check at any time \(t_i\), and for any value of the stock price \(S_i\), whether early exercise is more convenient with respect to holding the option for a future time. Therefore, the value of the option at a time slice \(i\) is given by

\[
O_i(S_i) = \max \left\{ O_i^Y(S_i), e^{-r \Delta t} E[O_{i+1}|S_i] \right\},
\]

where \(E[O_{i+1}|S_i]\) is the expectation value of \(O\) at time \(i+1\) under the hypothesis of having the price \(S_i\) at the time \(t_i\), \(O_i^Y\) is the option value in the case of anticipated exercise and \(r\) is the risk-free interest. To keep under control the computational complexity, it is mandatory to limit the number of points for the integral evaluation of \(E[O_{i+1}|S_i]\). To this end, we can create a grid of possible prices, according to the dynamics of the stochastic process. Starting from \(z_0\), we evaluate the expectation value \(E[O_1|S_0]\) with \(p = 2m + 1, m \in \mathbb{N}\) values of \(z_1\) centered on the mean value \(E[z_1] = z_0 + A \Delta t\) and differing of a quantity of the order of \(\sigma \sqrt{\Delta t}\)

\[
z_1^j = z_0 + A \Delta t + j \sigma \sqrt{\Delta t}, \quad j = -m, \ldots, +m.
\]

We can then evaluate each expectation value \(E[O_2|z_1^j]\) obtained from each one of the \(z_1^j\)’s created above with \(p\) values for \(z_2\) centered around the mean value

\[
E[z_2|z_1^j] = z_1^j + A \Delta t = z_0 + 2A \Delta t + j \sigma \sqrt{\Delta t}.
\]

Iterating the procedure until the maturity, we create a deterministic grid of points such that, at a given time \(t_i\), there are \((p - 1)i + 1\) values of \(z_i\), in agreement with a request of linear growth.

In the particular case of an American option, the possibility of exercise at any time up to the expiration date allows to develop a specific semi-analytical procedure, which is precise and very fast [3]. In the limit \(\Delta t \rightarrow 0\) and \(\sigma \rightarrow 0\), the transition probability of eq. (1) approximates a delta function. This means that, apart from volatility effects, the price \(z_i\) at time \(t_i\) will have a value remarkably close to the expected value \(\check{z} \doteq z_{i-1} + A \Delta t\), given by the drift. This suggests to evaluate integrals as in eq. (3) by performing a Taylor expansion of the kernel function \(O_i(e^z)\) around the expected value \(\check{z}\), to arrive, after inserting the analytical expression (1), at the following semi-analytical approximation

\[
E[O_i|S_{i-1}] = O_i(\check{z}) + \frac{\sigma^2}{2} \Delta t O_i''(\check{z}) + \ldots,
\]
Table 1
Price of an American put option in the BSM model for the parameters $t = 0$ year, $T = 0.5$ year, $r = 0.1$, $\sigma = 0.4$, $X = 10$, as a function different stock prices $S_0$.
The path integral 1 is performed with 200 time slices and $p = 13$ integration points, while path integral 2 is obtained with 300 time slices and $p = 3$.

| $S_0$ | finite difference | binomial tree | GFDNM | path integral 1 | path integral 2 |
|-------|-------------------|---------------|-------|-----------------|-----------------|
| 6.0   | 4.00              | 4.00          | 4.00  | 4.00            | 4.00            |
| 8.0   | 2.095             | 2.096         | 2.093 | 2.095           | 2.095           |
| 10.0  | 0.921             | 0.920         | 0.922 | 0.922           | 0.922           |
| 12.0  | 0.362             | 0.365         | 0.364 | 0.362           | 0.362           |
| 14.0  | 0.132             | 0.133         | 0.132 | 0.132           | 0.132           |

where the second derivative $O''_i$ can be estimated numerically. It is worth noticing that each expectation value $E[O_i|S_{i-1}]$ can be now computed once only $p = 3$ points at each slice are known, in order to evaluate the second derivative, provided $\Delta t$ is taken sufficiently small.

2.3 Numerical results and comparisons

To test the algorithm, we present results for the particular case of the price of an American option in the BSM model in Tab. 1, as obtained with the calculation of the transition probability and the grid technique described above (path integral 1) and with the semi-analytical approximation of eq. (8) (path integral 2). As can be seen from Tab. 1, there is generally a good agreement of our path integral results with those known in the literature [6] and obtained by means of binomial trees, the finite difference method and the Green Function Deterministic Numerical Method (GFDNM) [6]. It is worth noticing that our results in the path integral 1 require only a few seconds on a PentiumIII 500MhZ PC, while the CPU time is negligible for path integral 2. Further numerical results for option prices and greek letters can be found in [3].

3 Neural networks

Because the pricing formulae are given by non-linear functions and neural networks are known to be well suited to approximate non-linear relations, it is conceivable to use the fast path integral algorithm above summarized to generate grids of option prices over which to train learning networks. The goal is to construct neural networks able to price and hedge derivatives with a sufficient degree of accuracy to be of practical use.
Fig. 1. Comparison for an European call option between the analytical results (left) and the neural network predictions (right), as a function of spot price and maturity. The BSM parameters are $X = 10$, $\sigma = 0.4$, $r = 0.1$. $20 \times 20$ data points are used for training and $40 \times 40$ for generalization.

Using the Neural Networks Toolbox of MatLab, we developed Radial Basis Functions networks, since it is known that this kind of networks have the best approximation properties for real-valued continuous functions [7]. For such networks, the activation functions are gaussians of variable width (a parameter called spread). We trained the networks using as training data the prices for European and American options obtained with the path integral algorithm. We generated several training grids of different dimensionality, starting from grids depending on one and two parameters to arrive at multi-parametric grids. After the training phase, we tested the generalization performances of the networks, comparing the predictions of the networks for input parameters not present in the training sample with the results of the benchmark. In general, we observed that the performances of the networks strongly depend on the value of the spread parameter and the number of data used in the training samples, as expected.

An example of the results obtained is shown in Fig. 1, for a two-parameter case of an European call option in the BSM model, as a function of spot price and maturity. The exact analytical values (left) differ on average from the neural networks predictions (right) well below the 1% level. More details and further numerical results will be presented elsewhere [4].

4 Conclusions

Computational algorithms based on the path integral approach to stochastic processes and neural networks can be successfully applied to the problem of option pricing in financial analysis.
The path integral can provide fast and accurate predictions for a large class of financial derivatives with path-dependent and early exercise features, by means of a careful evaluation of the transition probability and a suitable choice of the integration points needed to evaluate the quantities of financial interest. The computational cost of the algorithm is competitive with the most efficient strategies used in finance. Also neural networks are a powerful and flexible tool for option pricing. With them it is possible to parameterize functions emerging from financial models in a very efficient and flexible way. After an appropriate training, neural networks can predict option prices with good accuracy, even in the case of multi-parametric dependence.

The natural development of the present work concerns the application of our computational methods of option pricing to more realistic models of financial dynamics, beyond the gaussian approximation [8].

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