On twin edge mean colorings of graphs

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Abstract. Let \(k \geq 2\) be an integer and \(G\) be a connected graph of order at least 3. In this paper, we introduce a new neighbor-distinguishing coloring called twin edge mean coloring. A proper edge coloring of \(G\) that uses colors from \(\mathbb{N}_k = \{0, 1, \ldots, k - 1\}\) is called a twin \(k\)-edge mean coloring of \(G\) if it induces a proper vertex coloring of \(G\) such that the color of each vertex \(v\) of \(G\) is the average of the colors of the edges incident with \(v\), and is an integer. The minimum \(k\) for which \(G\) has a twin \(k\)-edge mean coloring is called the twin chromatic mean index of \(G\) and is denoted by \(\chi'_tm(G)\).

First, we establish lower and upper bounds for \(\chi'_tm(G)\) under general or more specific assumptions. Then we determine the twin chromatic mean indices of paths, cycles, and stars.

1. Introduction

Let \(G = (V, E)\) be a simple graph. A proper vertex coloring (resp. proper edge coloring) of \(G\) is a function from \(V\) (resp. \(E\)) to a given set of colors such that adjacent vertices (resp. edges) are colored differently. The minimum number of colors needed in a proper vertex coloring (resp. proper edge coloring) of \(G\) is called the chromatic number (resp. chromatic index) of \(G\) and is denoted by \(\chi(G)\) (resp. \(\chi'(G)\)). An edge coloring \(c\) of a graph \(G\) is called a neighbor-distinguishing edge coloring if it induces a proper vertex coloring \(c'\) of \(G\).

In recent decades, various neighbor-distinguishing edge colorings have been introduced and studied in the literature. Some of these studies are the works of Chartrand and Zhang [5], Karonski et al. [8], and Chartrand et al. [6]. In [5], Chartrand and Zhang introduced a neighbor-distinguishing edge coloring of a graph which was called the proper sum \(k\)-edge coloring. In particular, for a connected graph \(G\) of order at least 3, an edge coloring \(c : E(G) \to [k]\), where \(k \in \mathbb{N}\), is called a proper sum \(k\)-edge coloring of \(G\) if \(c'(x) \neq c'(y)\) for every pair \(x, y\) of adjacent vertices of \(G\) where \(c'(v)\) is the sum of the colors of edges incident for each \(v \in V(G)\). The minimum \(k\) for which a graph \(G\) has a proper sum \(k\)-edge coloring is called the sum distinguishing index of \(G\) and is denoted by \(sd(G)\). In [5], Chartrand and Zhang determined the sum distinguishing
indices of complete graphs $K_n$ and complete bipartite graphs $K_{s,t}$, where $n \geq 3$ and $s + t \geq 3$. In [8], Karonski et al. conjectured that for every connected graph $G$, $sd(G)$ is either 1, 2, or 3. This conjecture became known as the $1-2-3$ Conjecture. Recently, in [6], Chartrand et al. introduced a new neighbor-distinguishing edge coloring which is called the rainbow mean coloring. They defined a rainbow mean coloring of a graph as follows:

**Definition 1.** [6] An edge coloring $c$ of a connected graph $G$ of order at least 3 is called a mean coloring if the chromatic mean $cm(v)$ of each vertex $v$ of $G$, defined by

$$cm(v) = \frac{\sum_{e \in E_v} c(e)}{\deg(v)},$$

where $E_v$ is the set of all edges of $G$ incident with $v$, is an integer. If distinct vertices have distinct chromatic means, then the edge coloring $c$ is called a rainbow mean coloring of $G$.

For a rainbow mean coloring $c$ of a graph $G$, the maximum vertex color is the rainbow mean index of $c$ and is denoted by $rm(c)$. The rainbow mean index of the graph $G$ itself, denoted by $rm(G)$ is defined as

$$rm(G) = \min \{rm(c) \mid c \text{ is a rainbow mean coloring of } G\}.$$  

In their paper, they investigated the rainbow mean indices of paths, cycles, complete graphs, and stars. On the other hand, Hallas et al. [7] investigated the rainbow mean indices of bipartite graphs.

In 2014, Andrews et al. [1] initially studied a relatively new kind of edge coloring that uses colors from $\mathbb{Z}_k$ and induces a proper vertex coloring. This edge coloring is called the twin edge coloring and is defined as follows:

**Definition 2.** [1] For a connected graph $G$ of order at least 3, a proper $k$-edge coloring $c : E(G) \rightarrow \mathbb{Z}_k$ for some integer $k \geq 2$ is called a twin $k$-edge coloring of $G$ if the induced vertex coloring $c' : V(G) \rightarrow \mathbb{Z}_k$ defined by

$$c'(v) = \sum_{e \in E_v} c(e) \text{ in } \mathbb{Z}_k,$$

where $E_v$ is the set of edges of $G$ incident with $v$, is proper as well. The minimum $k$ for which $G$ has a twin $k$-edge coloring is the twin chromatic index of $G$, denoted by $\chi'_t(G)$.

In the past years, several studies on twin edge coloring have been published (see [2], [3], [9], [10], for example). In [2], Andrews et al. verified the conjecture that the twin chromatic index of $G$ is at most $\Delta(G) + 2$ for several classes of cubic graphs, all permutation graph of $C_5$, prisms, and all trees of maximum degree at most 6. On the other hand, in [3], Andrews et al. also verified the said conjecture for several classes of trees such as brooms, double stars, and regular trees. In addition, Rajarajachozhan and Sampathkumar [9] investigated the twin chromatic indices of squares of paths and cycles, and the cartesian product of paths and cycles while Tolentino et al. [10] investigated the twin chromatic indices of some graphs with maximum degree 3.
In this paper, we introduce a new concept which combines some of the characteristics of “twin edge colorings” and “rainbow mean colorings”. Throughout the paper, all graphs to be considered are simple, finite, undirected, and connected. Basic notions and definitions will follow the book of Bondy and Murty [4], unless stated otherwise.

2. Twin Chromatic Mean Index

Definition 3. Let $G$ be a connected graph, $k \geq 2$ be an integer, and $c : E(G) \to \mathbb{N}_k$ ($\mathbb{N}_k = \{0, 1, \ldots, k - 1\}$) be a proper edge coloring of $G$. For $v \in V(G)$, define

$$c'(v) = \frac{1}{\deg(v)} \sum_{e \in E_v} c(e),$$

where $E_v$ is the set of edges incident with $v$. We call $c$ a twin $k$-edge mean coloring of $G$ if $c'(v) \in \mathbb{N}_k$ for all $v \in V(G)$ and $c' : V(G) \to \mathbb{N}_k$ is a proper vertex coloring of $G$. The minimum $k$ for which $G$ has a twin $k$-edge mean coloring is the twin chromatic mean index of $G$ and is denoted by $\chi_{tm}(G)$.

Since a twin edge mean coloring of $G$ is a proper edge coloring, $\chi_{tm}(G) \geq \Delta(G)$. In this paper, the twin chromatic mean indices of paths, cycles, and stars will be discussed.

The following theorem shows that the twin chromatic mean index of a connected graph of order at least 3 exists.

Theorem 4. Every connected graph $G$ of order at least 3 has a twin edge mean coloring.

Proof. Let $G$ be a connected graph with $|V(G)| \geq 3$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$ where $m \geq 2$. Let $q$ be the least common multiple of the degrees of the vertices of $G$, $t = 2\Delta$ and let $d = qt^m$. Define the edge coloring $c : E(G) \to \mathbb{N}_{d+1}$ by $c(e_i) = qt^i$ for $1 \leq i \leq m$.

We will show that $c$ is a twin edge mean coloring of $G$. It is straightforward to see that for any $e_i, e_j \in E(G)$, we have $c(e_i) \neq c(e_j)$; hence, $c$ is proper.

Next, we show that the induced vertex coloring $c'$ is proper; that is, $c'(u) \neq c'(v)$ for any two adjacent vertices $u$ and $v$ of $G$. Let $\deg(u) = r$ and $\deg(v) = s$ with $r \leq s$, and let $E_u = \{e_{i_1}, e_{i_2}, \ldots, e_{i_r}\}$ and $E_v = \{e_{j_1}, e_{j_2}, \ldots, e_{j_s}\}$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_s \leq m$. Accordingly,

$$c'(u) = \frac{q}{r} (t^{i_1} + t^{i_2} + \cdots + t^{i_r})$$

$$c'(v) = \frac{q}{s} (t^{j_1} + t^{j_2} + \cdots + t^{j_s})$$

where, by the choice of $q$, $c'(u)$ and $c'(v)$ are both positive integers.

Case 1: Suppose $r = s$. First, suppose that $i_r \neq j_r$. We may assume that $p = i_r > j_r$. 


Then,

\[ c'(u) = \frac{q}{r}(t^{i_1} + t^{i_2} + \cdots + t^{i_r}) \]
\[ \geq \frac{q}{r}(t^{i_r}) = \frac{q}{r}(t^p) \]
\[ > \frac{q}{r}(t + t^2 + \cdots + t^{p-1}) \]
\[ \geq \frac{q}{r}(t^{j_1} + t^{j_2} + \cdots + t^{j_r}) \]
\[ = c'(v). \]

Next, suppose that \( i_r = j_r \). Then,

\[ c'(u) = \frac{q}{r}(t^{i_1} + t^{i_2} + \cdots + t^{i_r}) \]
\[ = \frac{q}{r}(t^{i_1} + t^{i_2} + \cdots + t^{i_{r-1}}) + \frac{q}{r}(t^{i_r}). \]

But \( i_{r-1} \neq j_{r-1} \), so we may assume that \( p = i_{r-1} > j_{r-1} \). Applying the same argument from \( i_r \neq j_r \), we get that

\[ \frac{q}{r}(t^{i_1} + t^{i_2} + \cdots + t^{i_{r-1}}) > \frac{q}{r}(t^{j_1} + t^{j_2} + \cdots + t^{j_{r-1}}). \]

This implies that

\[ c'(u) = \frac{q}{r}(t^{i_1} + t^{i_2} + \cdots + t^{i_{r-1}}) + \frac{q}{r}(t^{i_r}) \]
\[ > \frac{q}{r}(t^{j_1} + t^{j_2} + \cdots + t^{j_{r-1}}) + \frac{q}{r}(t^{j_r}) \]
\[ = \frac{q}{r}(t^{j_1} + t^{j_2} + \cdots + t^{j_{r-1}}) + \frac{q}{r}(t^{j_r}) \]
\[ = c'(v) . \]

In either case, we see that \( c'(u) > c'(v) \), hence \( c' \) is proper.

**Case 2:** Suppose \( r < s \). First, suppose that \( i_r \geq j_s \). Let \( p = i_r \geq 2 \). We see that

\[ t^{i_1} + t^{i_2} + \cdots + t^{i_r} \geq t^{i_r} = t^p > t + t^2 + \cdots + t^{p-1} = t^{j_1} + t^{j_2} + \cdots + t^{j_s}. \]

Since \( r < s \), this implies that \( \frac{1}{r} > \frac{1}{s} \). And so,

\[ c'(u) = \frac{q}{r}(t^{i_1} + t^{i_2} + \cdots + t^{i_r}) \]
\[ > \frac{q}{s}(t^{j_1} + t^{j_2} + \cdots + t^{j_s}) \]
\[ = c'(v). \]

Next, suppose that \( i_r < j_s \). Let \( p = j_s \geq 2 \). Since \( 1 > \frac{1}{t^p} + \frac{1}{t^{p-1}} + \cdots + \frac{1}{t} + 1 \), it follows that

\[ 2 > \frac{1}{t^p} + \frac{1}{t^{p-1}} + \cdots + \frac{1}{t} + 1 > \frac{1}{t^{p-2}} + \frac{1}{t^{p-3}} + \cdots + \frac{1}{t} + 1. \]
Let \( \Delta(G) \) be the maximum degree of a graph \( G \). Then, \( \Delta(G) \geq \frac{2}{r} \), and it follows that
\[
\Delta(G) = \Delta(\frac{1}{t^p-2} + \frac{1}{t^p-3} + \cdots + \frac{1}{t} + 1), \quad \Delta \geq \frac{2}{r}
\]
where \( r \) is a proper \( k \)-edge coloring of \( G \).

Observation 6. Let \( G \) be a connected graph of order at least 3. If \( \Delta(G) \) is even, then \( \chi_{tm}(G) \geq \Delta(G) + 1 \).

Proof. Let \( \Delta(G) = k \) be even and \( v \in V(G) \) with \( \deg(v) = k \). If \( c : E(G) \rightarrow \mathbb{N}_k \) is a proper \( k \)-edge coloring of \( G \), then \( c'(v) = (\sum_{i=0}^{k-1} i)/k = (k-1)/2 \) is not an integer.

That is, \( \Delta(G) = \Delta(\frac{1}{t^p-2} + \frac{1}{t^p-3} + \cdots + \frac{1}{t} + 1), \quad \Delta \geq \frac{2}{r}
\]

In either case, we see that \( c' \) is proper.

Observation 5. Let \( G \) be a connected graph of order at least 3. If \( \Delta(G) \) is even, then \( \chi_{tm}(G) \geq \Delta(G) + 1 \).

Proof. Let \( \Delta(G) = k \) be even and \( v \in V(G) \) with \( \deg(v) = k \). If \( c : E(G) \rightarrow \mathbb{N}_k \) is a proper \( k \)-edge coloring of \( G \), then \( c'(v) = (\sum_{i=0}^{k-1} i)/k = (k-1)/2 \) is not an integer.

Thus, \( G \) has no twin \( k \)-edge mean coloring.

Observation 6. Let \( G \) be a connected graph of order at least 3. If there exist two adjacent vertices \( u \) and \( v \) of \( G \) such that \( \deg(u) = \deg(v) = \Delta(G) \), then \( \chi_{tm}(G) \geq \Delta(G) + 1 \).

Proof. Let \( \Delta(G) = k \). Suppose on the contrary that \( \chi_{tm}(G) = k \); that is, \( G \) has a twin \( k \)-edge mean coloring, say \( c : E(G) \rightarrow \mathbb{N}_k \). Then \( c'(v) = (k-1)/2 = c'(v) \), a contradiction.

3. Paths, Cycles, and Stars

To illustrate the concept of twin edge mean coloring, we determine the twin chromatic mean indices of paths, cycles, and stars. We begin with paths.

Theorem 7. If \( P_n \) is a path of order \( n \geq 3 \), then
\[
\chi_{tm}(P_n) = \begin{cases} 
3 & \text{if } n = 3, \\
5 & \text{if } n \geq 4.
\end{cases}
\]

Proof. Let \( G = P_n \) be a path with \( V(G) = \{v_0, v_1, \ldots, v_{n-1}\} \) and \( E(G) = \{e_i = v_i v_{i+1} \mid 0 \leq i \leq n-2\} \). Observe that, if \( c : E(G) \rightarrow \mathbb{N}_k \) is a twin \( k \)-edge mean coloring of \( G \), then \( c(e_i) \) and \( c(e_j) \) must have the same parity for each \( 0 \leq \ell < j \leq n - 2 \). Moreover, for \( n \geq 4 \), \( c(e_i) \neq c(e_{i+2}) \) for each \( 0 \leq i \leq n - 4 \).

Case 1: Suppose \( n = 3 \). Since \( c(e_0) \) and \( c(e_1) \) must have the same parity for any twin edge mean coloring of \( G \), we can say that \( \chi_{tm}(G) \neq 2 \); so \( \chi_{tm}(G) \geq 3 \). We need to
show that $\chi'_tm(G) \leq 3$. Now, define an edge coloring $c : E(G) \rightarrow N_3$ by $c(e_0) = 0$ and $c(e_1) = 2$. One can check that $c$ is a twin 3-edge mean coloring of $G$. Hence, $\chi'_tm(G) \leq 3$.

Case 2: Suppose $n \geq 4$. Let $c : E(G) \rightarrow N_k$ be a twin $k$-edge mean coloring of $G$. By our observations, we conclude that $c(e_0), c(e_1),$ and $c(e_2)$ are three distinct numbers. Without loss of generality, we consider the parity of $c(e_0)$. If $c(e_0)$ is even, then $\max\{c(e_i) \mid 0 \leq i \leq n - 2\} \geq 4$. On the other hand, if $c(e_0)$ is odd, then $\max\{c(e_i) \mid 0 \leq i \leq n - 2\} \geq 5$. In either case, $\max\{c(e_i) \mid 0 \leq i \leq n - 2\} \geq 4$. This implies that $k$ must be at least 5. Therefore, $\chi'_tm(G) \geq 5$.

We now show that $G$ has a twin 5-edge mean coloring. Define $c : E(G) \rightarrow N_5$ by

$$c(e_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{3}, \\
2 & \text{if } i \equiv 1 \pmod{3}, \\
4 & \text{if } i \equiv 2 \pmod{3}.
\end{cases}$$

By definition, we can easily see that $c$ is proper. Now, for $n \equiv 0 \pmod{3}$, we have

$$c'(v_i) = \begin{cases} 
0 & \text{if } i = 0, \\
1 & \text{if } i \equiv 1 \pmod{3}, \\
2 & \text{if } (i \equiv 0 \pmod{3} \text{ and } i \neq 0) \text{ or } (i = n - 1), \\
3 & \text{if } i \equiv 2 \pmod{3} \text{ and } i \neq n - 1.
\end{cases}$$

If $n \equiv 1 \pmod{3}$, we have

$$c'(v_i) = \begin{cases} 
0 & \text{if } i = 0, \\
1 & \text{if } i \equiv 1 \pmod{3}, \\
2 & \text{if } i \equiv 0 \pmod{3} \text{ and } i \notin \{0, n - 1\}, \\
3 & \text{if } i \equiv 2 \pmod{3}, \\
4 & \text{if } i = n - 1.
\end{cases}$$

Lastly, if $n \equiv 2 \pmod{3}$, we have

$$c'(v_i) = \begin{cases} 
0 & \text{if } i \in \{0, n - 1\}, \\
1 & \text{if } i \equiv 1 \pmod{3} \text{ and } i \neq n - 1, \\
2 & \text{if } i \equiv 0 \pmod{3} \text{ and } i \neq 0, \\
3 & \text{if } i \equiv 2 \pmod{3}.
\end{cases}$$

In any case, the induced vertex coloring $c'$ is proper. Hence, $c$ is a twin 5-edge mean coloring of $G$. □
Theorem 8. If $C_n$ is a cycle of order $n \geq 3$, then

$$\chi'_{tm}(C_n) = \begin{cases} 5 & \text{if } n \equiv 0 \pmod{3}, \\ 7 & \text{if } n \not\equiv 0 \pmod{3} \text{ and } n \neq 5, \\ 9 & \text{if } n = 5. \end{cases}$$

Proof. Let $G = C_n$ be a cycle with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(G) = \{e_i = v_iv_{i+1} \mid 0 \leq i \leq n - 1\}$ (subscripts are reduced modulo $n$). Observe that $c : E(G) \to \mathbb{N}_k$ is a twin $k$-edge mean coloring of $G$ if and only if $c(e_i)$ and $c(e_{i+1})$ have the same parity, and $|\{c(e_i), c(e_{i+1}), c(e_{i+2})\}| = 3$ for any $0 \leq i \leq n - 1$. Without loss of generality, we consider the parity of $c(e_0)$. If $c(e_0)$ is even, then $\max\{c(e_i) \mid 0 \leq i \leq n - 1\} \geq 4$. On the other hand, if $c(e_0)$ is odd, then $\max\{c(e_i) \mid 0 \leq i \leq n - 1\} \geq 5$. This implies that $\chi'_{tm}(G) \geq 5$.

Case 1: Suppose that $n \equiv 0 \pmod{3}$. We will show that $\chi'_{tm}(G) \leq 5$, that is, $G$ has a twin 5-edge coloring. Define an edge coloring $c : E(G) \to \mathbb{N}_5$ by

$$c(e_i) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod{3}, \\ 2 & \text{if } i \equiv 1 \pmod{3}, \\ 4 & \text{if } i \equiv 2 \pmod{3}. \end{cases}$$

By definition of $c$, it is easy to see that $c(e_i)$ and $c(e_{i+1})$ have the same parity, and $|\{c(e_i), c(e_{i+1}), c(e_{i+2})\}| = 3$ for any $0 \leq i \leq n - 1$. Therefore, $c$ is a twin 5-edge mean coloring of $G$. Hence, $\chi'_{tm}(G) \leq 5$.

Case 2: Suppose that $n \not\equiv 0 \pmod{3}$ and $n \neq 5$. Suppose $c : E(G) \to \mathbb{N}_k$ is a twin $k$-edge coloring of $G$. First, we will show that $k \geq 7$. Suppose on the contrary that $k < 7$. Then $k \in \{5, 6\}$. If $k = 5$, then $c(E(G)) = \{0, 2, 4\}$. On the other hand, if $k = 6$, $c(E(G)) = \{1, 3, 5\}$. Let $c(E(G)) = \{x, y, z\}$, where $\{x, y, z\} = \{0, 2, 4\}$ or $\{x, y, z\} = \{1, 3, 5\}$. Without loss of generality, let $c(e_0) = x$, $c(e_1) = y$, and $c(e_2) = z$. Since $|\{c(e_i), c(e_{i+1}), c(e_{i+2})\}| = 3$ for any $0 \leq i \leq n - 1$, we have

$$c(e_i) = \begin{cases} x & \text{if } i \equiv 0 \pmod{3}, \\ y & \text{if } i \equiv 1 \pmod{3}, \\ z & \text{if } i \equiv 2 \pmod{3}. \end{cases}$$

Since $n \not\equiv 0 \pmod{3}$, $c(e_{n-1}) \in \{x, y\}$. This is a contradiction since $c(e_0) = x$ and $c(e_1) = y$. Therefore, $k \geq 7$; so $\chi'_{tm}(G) \geq 7$. We now show that $G$ has a twin 7-edge mean coloring.
**Subcase 2.1:** Suppose $n \equiv 1 \pmod{3}$. Define $c : E(G) \to \mathbb{N}_7$ by

$$
c(e_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{3} \text{ and } i \neq n - 1, \\
2 & \text{if } i \equiv 1 \pmod{3}, \\
4 & \text{if } i \equiv 2 \pmod{3}, \\
6 & \text{if } i = n - 1.
\end{cases}
$$

By definition of $c$, we can easily observe that $c(e_i)$ and $c(e_{i+1})$ have the same parity, and $|\{c(e_i), c(e_{i+1}), c(e_{i+2})\}| = 3$ for any $0 \leq i \leq n - 1$. Therefore, $c$ is a twin 7-edge mean coloring of $G$.

**Subcase 2.2:** Suppose $n \equiv 2 \pmod{3}$ and $(n > 5)$. Define $c : E(G) \to \mathbb{N}_7$ as follows:

If $n \equiv 0 \pmod{4}$, let

$$
c(e_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{4}, \\
2 & \text{if } i \equiv 1 \pmod{4}, \\
4 & \text{if } i \equiv 2 \pmod{4}, \\
6 & \text{if } i \equiv 3 \pmod{4};
\end{cases}
$$

if $n \equiv 1 \pmod{4}$, let

$$
c(e_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{4} \text{ and } i < n - 5, \\
2 & \text{(if } i \equiv 1 \pmod{4} \text{ and } i < n - 5) \text{ or } (i \in \{n - 5, n - 2\}), \\
4 & \text{(if } i \equiv 2 \pmod{4} \text{ and } i < n - 5) \text{ or } (i \in \{n - 4, n - 1\}), \\
6 & \text{(if } i \equiv 3 \pmod{4} \text{ and } i < n - 5) \text{ or } (i = n - 3);
\end{cases}
$$

if $n \equiv 2 \pmod{4}$, let

$$
c(e_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{4} \text{ and } i < n - 2, \\
2 & \text{(if } i \equiv 1 \pmod{4} \text{ and } i < n - 2) \text{ or } (i = n - 2), \\
4 & \text{(if } i \equiv 2 \pmod{4} \text{ and } i < n - 2) \text{ or } (i = n - 1), \\
6 & \text{(if } i \equiv 3 \pmod{4} \text{ and } i < n - 2);
\end{cases}
$$

and if $n \equiv 3 \pmod{4}$, let

$$
c(e_i) = \begin{cases} 
0 & \text{if } i \equiv 0 \pmod{4} \text{ and } i \neq n - 3, \\
2 & \text{(if } i \equiv 1 \pmod{4} \text{ and } i < n - 3) \text{ or } (i = n - 3), \\
4 & \text{(if } i \equiv 2 \pmod{4} \text{ and } i < n - 3) \text{ or } (i = n - 2), \\
6 & \text{(if } i \equiv 3 \pmod{4} \text{ and } i < n - 3) \text{ or } (i = n - 1).
\end{cases}
$$
In any case, we see that \( c(e_i) \) and \( c(e_{i+1}) \) have the same parity, and \( |c(e_i), c(e_{i+1}), c(e_{i+2})| = 3 \) for any \( 0 \leq i \leq n - 1 \). Therefore, \( c \) is a twin 7-edge mean coloring of \( G \).

**Case 3:** Suppose \( n = 5 \). Let \( c : E(G) \to \mathbb{N}_k \) be a twin \( k \)-edge mean coloring of \( G \). Then \( c(E(G)) = \{a, b, x, y, z\} \), where \( a, b, x, y, \) and \( z \) are five distinct elements of \( \mathbb{N}_k \).

Consider the parity of \( c(e_0) \). If \( c(e_0) \) is even, then \( \max\{c(e_i) \mid 0 \leq i \leq 4\} \geq 8 \). On the other hand, if \( c(e_0) \) is odd, then \( \max\{c(e_i) \mid 0 \leq i \leq 4\} \geq 9 \). Therefore, \( k \geq 9 \); so \( \chi_{tm}'(G) \geq 9 \).

We now show that \( \chi_{tm}'(G) \leq 9 \). Define \( c : E(G) \to \mathbb{N}_0 \) by \( c(e_i) = 2i \) for each \( 0 \leq i \leq 4 \). Then \( c(e_i) \) and \( c(e_{i+1}) \) have the same parity, and \( |c(e_i), c(e_{i+1}), c(e_{i+2})| = 3 \) for any \( 0 \leq i \leq 4 \). Thus, \( c \) is a twin 9-edge coloring of \( G \). Hence, \( \chi_{tm}'(G) \leq 9 \)

**Theorem 9.** If \( K_{1,s} \) is a star where \( s \geq 2 \), then

\[
\chi_{tm}'(K_{1,s}) = \begin{cases} 
s + 1 & \text{if } s \text{ is even}, \\
3 + 1 & \text{if } s \text{ is odd}. 
\end{cases}
\]

**Proof.** Let \( G = K_{1,s} \) be a star, where \( s \geq 2 \). Let \( V(G) = \{v_0, v_1, \ldots, v_s\} \) and \( E(G) = \{e_i = v_0v_i \mid i \in \{1, 2, \ldots, s\}\} \). Since \( \Delta(G) = s \), we have \( \chi_{tm}'(G) \geq s \). First, we show that \( \chi_{tm}'(G) \geq s + 1 \). To do that, we need to show that \( \chi_{tm}'(G) \neq s \), that is, \( G \) has no twin \( s \)-edge mean coloring. By Observation 5, we can assume that \( s \) is odd. Suppose on the contrary that \( \chi_{tm}'(G) = s \); that is, \( G \) has a twin \( s \)-edge mean coloring \( c : E(G) \to \mathbb{N}_s \). Then \( c(E(G)) = \mathbb{N}_s \), \( c'(v_i) = c(e_i) \) for \( 1 \leq i \leq s \), and

\[
c'(v_0) = \frac{1}{s} \sum_{j=0}^{s-1} j = \frac{1}{s} \left(\frac{s(s-1)}{2}\right) = \frac{s-1}{2}.
\]

But \( \frac{s-1}{2} = c'(v_i) \) for some \( i \in \{1, 2, \ldots, s\} \), a contradiction. Thus, \( \chi_{tm}'(G) \geq s + 1 \).

We now consider the cases based on the parity of \( s \).

**Case 1:** Suppose \( s \geq 2 \) is even. We will show that \( \chi_{tm}'(G) = s + 1 \). Since \( \chi_{tm}'(G) \geq s + 1 \), we only need to show that \( \chi_{tm}'(G) \leq s + 1 \); that is, that \( G \) has a twin \( (s + 1) \)-edge mean coloring. If \( s = 2 \), then \( G \cong P_3 \); so, by Theorem 7, \( \chi_{tm}'(G) = 3 = 2 + 1 \) when \( s = 2 \). We now assume that \( s \geq 4 \). We define an \( (s + 1) \)-edge coloring \( c : E(G) \to \mathbb{N}_{s+1} \) by \( c(E(G)) = \mathbb{N}_{s+1} \setminus \{\frac{s}{2}\} \). It is straightforward to see that \( c \) is a proper edge coloring of \( G \). To show that \( c \) is a twin \( (s + 1) \)-edge mean coloring of \( G \), we need to show that \( c'(v_0) \) is an integer and \( c'(v_i) \neq c'(v_i) \) for any \( i \in \{1, 2, \ldots, s\} \). By definition of \( c \), we have

\[
c'(v_0) = \frac{1}{s} \left[ s \sum_{j=0}^{s-1} j - s \right] = \frac{1}{s} \left[ \frac{s(s+1)}{2} - s \right] = \frac{s}{2}.
\]

Since \( s \) is even, \( c'(v_0) = \frac{s}{2} \) is an integer. Since \( \frac{s}{2} \notin c(E(G)) \), \( c'(v_0) \neq c'(v_i) \) for any \( i \in \{1, 2, \ldots, s\} \). Hence, \( c \) is a twin \( (s + 1) \)-edge mean coloring.
Case 2: Suppose \( s \geq 3 \) is odd. We will show that \( \chi'_{tm}(G) = s + 3 \). First, we show that \( \chi'_{tm}(G) \geq s + 3 \). We do this by showing that \( G \) has no twin \( r \)-edge mean coloring where \( r \in \{ s+1, s+2 \} \). Suppose on the contrary that \( G \) has a twin \( r \)-edge mean coloring \( c : E(G) \to \mathbb{N}_r \), with \( r \in \{ s+1, s+2 \} \).

**Subcase 2.1:** Suppose \( r = s+1 \). Then \( c(E(G)) = \mathbb{N}_{s+1} \setminus \{x\} \) for some \( x \in \mathbb{N}_{s+1} \). It follows that

\[
c'(v_0) = \frac{1}{s} \left[ \sum_{j=0}^{s+1} j - x \right] = \frac{1}{s} \left[ \frac{s(s+1)}{2} - x \right].
\]

Since \( c \) is a twin \( r \)-edge mean coloring of \( G \), \( c'(v_0) \) is an integer. This implies that \( s \) divides \( \frac{s(s+1)}{2} - x \). Since \( s \) is odd, \( \frac{s+1}{2} \) is an integer. Then, \( s \) divides \( \frac{s(s+1)}{2} \). Since \( s \) divides \( \frac{s(s+1)}{2} - x \) and \( \frac{s(s+1)}{2} \), \( s \) also divides \( x \). Therefore, \( x \in \{ 0, s \} \). If \( x = 0 \), then \( c(E(G)) = \mathbb{N}_{s+1} \setminus \{0\} \) and \( c'(v_0) = \frac{s+1}{2} \in \mathbb{N}_{s+1} \setminus \{0\} \). Then, \( c'(v_0) = c'(v_i) \) for some \( i \in \{1, 2, \ldots, s\} \), a contradiction. If \( x = s \), then \( c(E(G)) = \mathbb{N}_s \) which implies that \( c \) is a twin \( s \)-edge mean coloring of \( G \), a contradiction.

**Subcase 2.2:** Suppose \( r = s+2 \). Then \( c(E(G)) = \mathbb{N}_{s+2} \setminus \{x, y\} \) for some \( x, y \in \mathbb{N}_{s+2} \) where \( x \neq y \) (say \( x < y \)). This implies that

\[
c'(v_0) = \frac{1}{s} \left[ \sum_{j=0}^{s+1} j - (x+y) \right] = \frac{1}{s} \left[ \frac{s(s+1)}{2} + (s+1) - (x+y) \right].
\]

Using similar arguments in subcase 2.1, we say that

\[
\frac{s(s+1)}{2} + (s+1) - (x+y) \equiv 0 \pmod{s}.
\]

Then, \( x+y \equiv 1 \pmod{s} \). Therefore, \( x+y \in \{1, s+1, 2s+1\} \). If \( x+y = 1 \), then \( x = 0 \) and \( y = 1 \). This gives us \( c(E(G)) = \mathbb{N}_{s+2} \setminus \{0, 1\} \) and \( c'(v_0) = \frac{s+3}{2} \in c(E(G)) \). This contradicts our assumption that \( c' \) is proper since \( c'(v_0) = c'(v_i) \) for some \( i \in \{1, 2, \ldots, s\} \). If \( x+y = s+1 \), then \( c'(v_0) = \frac{s+1}{2} \). Since \( x \neq y \), neither \( x \) nor \( y \) is equal to \( \frac{s+1}{2} \); so \( \frac{s+1}{2} \in c(E(G)) \). Thus \( c'(v_0) = c'(v_i) \) for some \( i \in \{1, 2, \ldots, s\} \), a contradiction. If \( x+y = 2s+1 \), then \( x = s \) and \( y = s+1 \). Then \( c(E(G)) = \mathbb{N}_s \) which implies that \( c \) is a twin \( s \)-edge mean coloring of \( G \), an impossibility.

Therefore, \( \chi'_tm(G) \geq s+3 \). We now define an \( (s+3) \)-edge coloring \( c : E(G) \to \mathbb{N}_{s+3} \) by \( c(E(G)) = \mathbb{N}_{s+3} \setminus \{ \frac{s+1}{2}, \frac{s+3}{2}, s+1 \} \). By definition of \( c \), we conclude that \( c \) is proper. Moreover, we have

\[
c'(v_0) = \frac{1}{s} \left[ \sum_{j=0}^{s+2} j - \left( \frac{s+1}{2} + \frac{s+3}{2} + s+1 \right) \right] = \frac{s+1}{2}.
\]

Since \( s \) is odd, \( c'(v_0) = \frac{s+1}{2} \) is an integer. Moreover, since \( \frac{s+1}{2} \notin c(E(G)) \), we have \( c'(v_0) \neq c'(v_i) \) for any \( i \in \{1, 2, \ldots, s\} \). Thus \( c' \) is proper. Since \( c'(v) \) is an integer for any vertex \( v \) of \( G \), \( c \) is a twin \( (s+3) \)-edge mean coloring of \( G \). Hence \( \chi'_{tm}(G) \leq s+3 \). \( \Box \)
We end this section by proving that only the graph $P_3$ has twin chromatic mean index equal to 3.

**Proposition 10.** Let $G$ be a connected graph of order at least 3. Then $\chi'_{tm}(G) = 3$ if and only if $G \cong P_3$.

**Proof.** If $G \cong P_3$, then by Theorem 7, $\chi'_{tm}(G) = \chi'_{tm}(P_3) = 3$. Now, suppose that $\chi'_{tm}(G) = 3$. We will show that $G \cong P_3$. Suppose on the contrary that $G \not\cong P_3$. By Theorem 7, $G$ cannot be a path. Since $G$ is not a path, $\Delta(G) \geq 3$. Moreover, since $\chi'_{tm}(G) = 3$, $\Delta(G) \leq 3$; so $\Delta(G) = 3$. Let $u \in V(G)$ with $\deg(u) = 3$ and let $c : E(G) \to N_3$ be a twin 3-edge mean coloring of $G$. Since $\deg(u) = 3$, we have $c(E_u) = N_3$ and $c'(u) = 1$. If each of the three vertices of $G$ adjacent with $u$ has degree 1, then $G \cong K_{1,3}$. But by Theorem 9, $\chi'_{tm}(K_{1,3}) = 6$. This implies that at least one vertex of $G$ adjacent with $u$ has degree at least 2. Let $v$ be a vertex of $G$ adjacent with $u$ with $\deg(v) \geq 2$. Note that $\deg(v) \leq 3$. If $\deg(v) = 3$, then by Observation 6, $\chi'_{tm}(G) \geq 4$, a contradiction. Therefore, $\deg(v) = 2$. Since $\deg(v) = 2$, the colors of the edges incident with $v$ must have the same parity. Then $c(E_v) = \{0, 2\}$ and $c'(v) = 1 = c'(u)$. This contradicts our assumption that $c$ is a twin 3 edge mean coloring of $G$. \hfill \Box

4. Conclusion

In this paper, we introduced the notion of twin edge mean colorings that share some characteristics of twin edge colorings and rainbow mean colorings and have determined the twin chromatic mean indices of paths, cycles, and stars. Moreover, we have shown that the twin chromatic index of $G$ is 3 if and only if $G$ is isomorphic to $P_3$. The future researchers may investigate the upper and lower bounds for the twin chromatic mean index of a connected graph. The twin chromatic mean indices of families of graphs such as complete graphs and complete bipartite graphs may be determined.

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