PASSIVE LINEAR SYSTEMS
CHARACTERIZATION THROUGH STRUCTURE

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Abstract. We here show that continuous-time passive linear systems are intimately linked to the structure of maximal, matrix-convex, cones, closed under inversion. Moreover, this observation unifies three setups: (i) differential inclusions, (ii) matrix-valued rational functions, (iii) realization arrays associated with rational functions.

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1. Introduction
Let \( \mathbb{C}_R \) (\( \mathbb{C}_C \)) denote the open (closed) right\(^1\) half of the complex plane. Let \( (P_n) \) be the set of \( n \times n \) positive (semi)-definite matrices.

Definition 1.1. A set of \( n \times n \) matrices is said to be a Convex Cone if it is closed under positive scaling and summation.
A set of \( n \times n \) matrices is said to be Invertible (=“closed under inversion”) if whenever a matrix \( M \) in it is non-singular, its inverse \( M^{-1} \) belongs to the same set.
A set of \( n \times n \) matrices\(^2\) combining both properties is called a Convex Invertible Cone, cic in short.

\(^1\)The subscript stands for right.
\(^2\)Convex Invertible Cones were originally defined over any real unital algebra, see [20], [23], [43]. For simplicity of exposition, we here start with matrices.
Example 1.2. 1. The set $H_n$ of $n \times n$ non-singular Hermitian matrices is a cone, closed under inversion, but not convex as $\pm H$ may belong to it, but not their sum\(^3\).

2. The set of $2 \times 2$ matrices with det = 1 is closed under inversion, but not convex. Its convex subset of matrices of the form $\left( \begin{array}{cc} 0 & -1 \\ 1 & c \end{array} \right)$, $c \in \mathbb{C}$, is not closed under inversion, as $\left( \begin{array}{cc} 0 & -1 \\ 1 & c \end{array} \right)^{-1} = \left( \begin{array}{cc} c & 1 \\ 1 & 0 \end{array} \right)$.

3. The set $P_n$ is a convex invertible cone, although it contains singular matrices. □

For future reference we recall the following.

Definition 1.3. For a square matrix $A$, with no eigenvalues on the imaginary axis, its sign matrix $E_A = \text{Sign}(A)$, may be defined as follows,

(i) $E_A^{-1} = E_A$ (ii) $E_A A = A E_A$ and (iii) the spectrum of $A E_A$ lies within $\mathbb{C}_R$. □

It is easy to check that if $A \in \mathbb{C}^{n \times n}$ has $\nu$ and $n - \nu$ (for some $\nu \in [0, n]$) eigenvalues in $\mathbb{C}_L$ and $\mathbb{C}_R$ respectively, then

\[
E_A = \text{Sign}(A) = U^* \left( \begin{array}{cc} -I_\nu & \nu \\ 0 & I_{n-\nu} \end{array} \right) U \\
U U^* = U^* U = I_n \\
T \in \mathbb{C}^{\nu \times (n-\nu)}.\]

We next recall in the following properties of convex cones of matrices, closed under inversion.

Proposition 1.4. Let $A \subset \mathbb{C}^{n \times n}$ be a convex set of matrices, closed under inversion.

(I) \([20]\) Proposition 2.6]. If all matrices in $A$ are non-singular, then for some $\nu \in [0, n]$, each matrix has exactly $\nu$ and $n - \nu$, eigenvalues in $\mathbb{C}_L$ and in $\mathbb{C}_R$, respectively.

(II) \([20]\) Proposition 2.5]. Whenever $A \in A$ has no eigenvalues on $i \mathbb{R}$, it implies that its sign, $\text{Sign}(A) \in A$ as well.

For more information on the sign matrix, see the proof of Corollary 5.3 below and as sample references, \([35]\) Chapter 5, \([38]\) Chapter 22, \([11]\) and \([13]\).

We next consider the set of matrices $A$ all satisfying a Lyapunov inclusion with the same Hermitian factor $H$. Formally, for $H \in H_n$ denote

\[
L_H := \{ A : HA + A^* H \in P_n \}
\]

\[
\overline{L}_H := \{ A : HA + A^* H \in \overline{P}_n \}.
\]

Adopting the convention that $\overline{P}_n$ is the closure (in $\overline{H}_n$) of the open set $P_n$, $\overline{L}_H$ is the closure of the open set $L_H$.

As a motivation to resorting to the set $L_H$, we recall in the following connection to what engineers colloquially refer to as “robust exponential stability” or “quadratic stability”.

For a set $A = \{ A_1, \ldots, A_m \}$ of $n \times n$ matrices let the differential inclusion,

\[
\frac{dx}{dt} \in A x \quad x \in \mathbb{R}^n,
\]

mean that there exists an unknown map $\phi(t, x)$ so that $A_{\phi(t,x)} \in A$ for all $x(t)$, $t \geq 0$ and the differential equation

\[
\frac{dx}{dt} = A_{\phi(t,x)} x,
\]

\(^3\)The set $\overline{H}_n$ of all $n \times n$ Hermitian matrices, will be addressed Observation 2.2.
has a unique solution $x(t)$, for all $t \geq 0$ and for all initial conditions $x(0)$.

The following is well known, see e.g. [14, Section 5.1] and for a special case [25].

**Observation 1.5.** If for some $-H \in P_n$ one has that $L_H$ from Eq. (1.2), is so that

\begin{equation}
A \subset L_H,
\end{equation}

then one can find $\alpha > 0$ and $\beta \geq 1$ so that the solution $x(t)$ of the equation in (1.3) uniformly satisfies,

\begin{equation}
\beta \|x(t_o)\| e^{\alpha(t_o-t)} \geq \|x(t)\| \quad \forall x(t_o) \quad \forall t \geq t_o \geq 0.
\end{equation}

The celebrated Linear Matrix Inequality (LMI) technique\(^4\), see e.g. [14], [32], is the prominent engineering tool to finding whether or not for $A = \{A_1, \ldots, A_m\}$ there exists $H$ satisfying (1.4).

Already here we need to recall that the converse of Observation 1.5 is in general not true. Namely Eq. (1.5) does not imply Eq. (1.4). For a special case where the two conditions are equivalent see, [25].

The first fundamental structural result is the following, see e.g. [20, Lemma 3.5, Proposition 3.7].

**Theorem 1.6.** The set $L_H$ in (1.2), where $H \in H_n$ is a maximal open convex cone of non-singular matrices, containing the matrix $H$.

Maximality is in the sense that whenever $HB + B^*H$ has (at least one) negative eigenvalue for some matrix $B$, then there is always $A \in L_H$ so that $A + B$ is singular, see proof of item (i) in Theorem 3.1.

Starting from [20], in a series of papers [1], [2], [21] - [25], [40] and [43] we explored the following:

Continuous-time passive linear systems imply the structure of maximal convex cones closed under inversion.

In [10] Section 3 T. Ando characterized the set $L_H$ for $H \in P_n$ and in [11] Theorem 3.5 he extended it to $H \in H_n$. In particular, he showed that the conditions in Theorem 1.6 fall short from characterizing the set $L_H$, namely the converse statement is (significantly) more involved.

Motivated by physical considerations, in this work we focus on the special case where in Eq. (1.2) $H = I_n$ and refine the main statement so that,

Continuous-time passive linear systems are closely associated with maximal open matrix-convex cones, closed under inversion.

Furthermore, this formulation suits well, three different frameworks:

(i) Differential inclusions,
(ii) Positive real rational functions,
(iii) Families of realization arrays of positive real rational functions.

This work is organized as follows. In Section 2 we lay the foundations to the structure used in the sequel. Differential inclusions, positive real rational functions and families of

\(^4\)where $m$ and $n$ are “modest”.
realization arrays of positive real rational functions are addressed in Sections 3, 4 and 5, respectively. Brief concluding remarks are given in Section 6.

2. MATRIX-CONVEX SETS AND CONES OF MATRICES

We next resort to the notion of a matrix-convex set, see e.g. [29] and more recently, [30], [31], [34], [45].

**Definition 2.1.**

a. A family \( \mathbf{A} \), of \( n \times n \) matrices is said to be a **matrix-convex set** if for all natural \( k \), for all \( A_1, \ldots, A_k \in \mathbf{A} \) and for all \( nk \times n \) isometries \( \Upsilon \), i.e

\[
\Upsilon^* \Upsilon = I_n,
\]

one has that also

\[
\Upsilon^* \begin{pmatrix}
A_1 & 0 & 0 & 0 \\
0 & A_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & A_k
\end{pmatrix} \Upsilon \in \mathbf{A}.
\]

b. A family of \( n \times n \) matrices \( \mathbf{A} \) is said to be a **matrix-convex cone** if Eq. (2.1) is relaxed to having the \( nk \times n \) matrices \( \Upsilon \), of a full rank, i.e \( \Upsilon^* \Upsilon \in \mathbb{P}_n \). □

The definition suggests that a notation like \( \Upsilon_k \) is more appropriate, however for simplicity we drop the subscript \( k \).

Note that sometimes matrix-convexity is equivalently written as for all natural \( k \) and for all \( v_j \in \mathbb{C}^{n \times n} \) so that

\[
\sum_{j=1}^k v_j^* v_j = I_n
\]

one has that having \( A_1, \ldots, A_k \) in \( \mathbf{A} \) implies that

\[
\sum_{j=1}^k v_j^* A_j v_j \in \mathbf{A}.
\]

For matrix-convex cone, condition (2.3) is relaxed to \( \sum_{j=1}^k v_j^* v_j \in \mathbb{P}_n \).

Matrix-convex cones are closely related to the classical notion of **Complete Positivity**, see e.g. [9], [19], and for a comprehensive account of the subject, see [13]. In recent years it has been applied to the study of **Quantum Channels**, see e.g. [39].

We next present some prime examples of matrix-convex sets and cones. To this end, recall that we denote by \( \mathbb{H}_n \) the set of (possibly singular) \( n \times n \) Hermitian matrices. Skew-Hermitian matrices are denoted by \( i\mathbb{H}_n \). It is common to consider \( \mathbb{H} \) and \( i\mathbb{H} \) as the matricial extensions of \( \mathbb{R} \) and \( i\mathbb{R} \), respectively.

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\footnote{We do not assume that \( \mathbf{A} \subset \mathbb{H} \).}
Observation 2.2. (I) \( \overline{A}_2(\alpha) \) (\( A_2(\alpha) \)), the closed (open) family of \( n \times n \) matrices whose spectral norm is uniformly bounded (with a prescribed \( \alpha > 0 \)),

\[
\overline{A}_2(\alpha) = \{ A : \alpha \geq \| A \|_2 \}, \\
A_2(\alpha) = \{ A : \alpha > \| A \|_2 \},
\]
is a matrix-convex set.

(II) Each of the following families of \( n \times n \) matrices,

\[
\overline{H}_n, \quad i\overline{H}_n, \quad \overline{P}_n, \quad P_n,
\]
is a matrix-convex cone.

Verification of (I) and (II) is self-evident and thus omitted.

Substituting in Eq. (2.4) \( \alpha = 1 \), one obtains the matrix-convex sets \( \overline{A}_2(1) \) and \( A_2(1) \), which are pivotal to the work in [42].

**Remark 2.3.**

(i) Substituting in Eq. (2.2) \( k = 1 \), reveals that matrix-convexity in particular implies that the set \( A \) is invariant under unitary similarity.

Thus in particular, in Eq. (2.4) the spectral norm \( \| \|_2 \), can not be substitute d by another unitarily-variant, induced matrix norm e.g. \( \| \|_1 \) or \( \| \|_\infty \), see e.g. [36, items 5.6.4, 5.6.5].

(ii) Similarly, taking

\[
\Upsilon = \begin{pmatrix} r_1 I_n \\ \vdots \\ r_k I_n \end{pmatrix}, \quad r_1, \ldots, r_k \in \mathbb{R}, \quad r_1^2 + \ldots + r_k^2 = 1
\]

reveals that matrix-convexity in particular implies classical convexity.

(iii) In the above two items, we have pointed that matrix-convexity implies both convexity and closure under unitary similarity. We here show that the converse need not be true: Consider the set of scaled \( n \times n \) identity matrices, i.e.

\[
\{ c I_n : c \in \mathbb{C} \}.
\]

Trivially, this set is convex and each matrix is invariant under unitary similarity. Yet this set is not *matrix-convex*. Indeed, already for \( k = 2 \) and arbitrary \( c_1 \neq c_2 \),

\[
(2.5) \quad \begin{pmatrix} 1 & 0_{1 \times n} & 0_{1 \times (n-1)} \\ 0 & 0_{(n-1) \times n} & I_{n-1} \end{pmatrix} \begin{pmatrix} c_1 I_n \\ 0 \\ c_2 I_n \end{pmatrix} \begin{pmatrix} 1 \\ 0_{n \times 1} \\ 0_{(n-1) \times 1} \end{pmatrix} = \begin{pmatrix} c_1 I_n \end{pmatrix}.
\]

Hence, the set of scaled identity matrices is not matrix-convex.

(iv) Note that in Eq. (2.4) the spectral norm \( \| \|_2 \) can be substituted neither by another induced matrix norm nor by another unitarily-invariant matrix norm. This conforms with the fact, see e.g. [36 Corollary 5.6.35 and Theorem 5.6.36], that the spectral norm, \( \| \|_2 \), is the *minimal* unitarily-invariant, vector-induced, matrix norm.

The following Example 2.4 illustrates the fact that even within the set \( \overline{H}_n \), matrix-convexity is rather stringent.
Example 2.4. (i) To identify matrix-convex sets within $\mathbb{H}_n$, the family of non-singular Hermitian matrices, this family is first partitioned into $n + 1$ subsets: $\mathbb{H}_n(\nu)$, with $\nu$ negative, and $n - \nu$ positive, eigenvalues, with $\nu \in [0, n]$. For $\nu \in [1, n - 1]$, the set $\mathbb{H}_n(\nu)$, is not even convex, as both matrices $(-I_\nu 0_{n-\nu})$ and $(I_{n-\nu} - I_\nu)$ belong to it, but not their sum. Convex subsets of $\mathbb{H}_n(\nu)$ were characterized in [37]. In contrast, each of the two families $\mathbb{H}_n(\nu)|\nu=0$ and $\mathbb{H}_n(\nu)|\nu=n$, i.e. the sets $P_n$ and $-P_n$, respectively, is matrix-convex and non-singular.

(ii) A classical example of a matrix convex set (see e.g. the discussion preceding [29, Lemma 3.1]) is where for real scalars $r \geq r_0$ (the product $rr$ need not be positive),
\[
\{ A \in \mathbb{H}_n : (rI_n - A) \in P_n, (A - rI_n) \in P_n \}.
\]
□

3. Maximal matrix-convex cones, closed under inversion

As already mentioned, we here focus on the special case of the set $L_H$ (or $\overline{L_H}$) in Eq. (1.2), where one substitutes $H = I_n$, i.e.

\[
L_{I_n} := \{ A : A + A^* \in P_n \}
\]

(3.1)

\[
\overline{L}_{I_n} = \{ A : A + A^* \in \overline{P}_n \}.
\]

Note now that the sets in Eq. (3.1) may be viewed as a matricial extensions of $\mathbb{C}_R$, $\overline{\mathbb{C}}_R$, respectively. Indeed, one can equivalently write these sets as,

\[
L_{I_n} = \{ P + iH : P \in P_n, H \in \mathbb{H}_n \}
\]

(3.2)

\[
\overline{L}_{I_n} = \{ P + iH : P \in \overline{P}_n, H \in \overline{\mathbb{H}}_n \}.
\]

Under the assumption $H = I_n$, Observation 1.5 takes the form that Eq. (1.5) holds with $\beta = 1$ and the norm used is the spectral norm (i.e. $\|x\|_2 = \sqrt{x^*x}$).

Here is the first motivation to resorting to the notion of matrix-convexity.

**Theorem 3.1.** The following statements are true.

(i) The set $L_{I_n}$ in (3.1) is a maximal open matrix-convex cone of non-singular matrices, closed under inversion.

(ii) Conversely, a maximal, open, matrix-convex, cone of non-singular matrices, closed under inversion, is the set $L_{I_n}$ in Eq. (3.1) (or $-L_{I_n} = L_{-I_n}$).

(iii) The set $\overline{L}_{I_n}$ in (3.1) is a closed matrix-convex invertible cone containing the matrix $I_n$, and on its boundary the matrix $iI_n$.

(iv) $L_{I_n} \cap L_{-I_n} = i\overline{\mathbb{H}}_n$. The set $i\overline{\mathbb{H}}_n$ is a matrix-convex cone, closed under inversion; in fact, a maximal convex subset of $\mathbb{C}^{n\times n}$, which does not contain an involution.

**Proof :** (i) The fact that this is a convex cone closed under inversion, follows from Theorem 1.6 upon substituting $H = I_n$.

We next show that the set $L_{I_n}$ is matrix-convex.

From Eq. (3.2) it follows that for $j = 1, \ldots, k$ (where $k$ is a parameter), $A_j \in L_{I_n}$ can
be written as $A_j = P_j + iH_j$ with $P_j \in \mathbf{P}_n$ and $H_j \in \mathbf{H}_n$. Now, following Definition 2.1 and item (II) of Observation 2.2, one has that,

$$P_1, \ldots, P_k \in \mathbf{P}_n, \quad H_1, \ldots, H_k \in \mathbf{H}_n \quad \Rightarrow \quad \Upsilon^* \begin{pmatrix} P_{i}+iH_{i} & 0 & 0 & 0 \\ 0 & P_{i}+iH_{i} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & P_{k}+iH_{k} \end{pmatrix} = P_{o} + iH_{o} \quad P_{o} \in \mathbf{P}_n, H_{o} \in \mathbf{H}_n.$$  

To show invertibility of this set, multiply Eq. (3.2), by $A^{-1}$ and $(A^{-1})^*$ from the left and from the right, respectively, so that $A^{-1} + (A^{-1})^* = A^{-1}Q(A^{-1})^* \in \mathbf{P}_n$.

For maximality, we show that whenever $B \notin \mathbf{T}_{I_n}$ one can find $A \in \mathbf{L}_{I_n}$ so that $A + B$ is singular. Indeed, $B \notin \mathbf{T}_{I_n}$ is equivalent to,

$$0 > \min_{j=1, \ldots, n} \lambda_j(B + B^*) \quad \alpha > 0,$$

where $\lambda_j(M)$ denotes the $j$-th eigenvalue of a matrix $M$. Take now $A = \frac{1}{\alpha}(\alpha I_n + B^* - B)$ so by construction $A \in \mathbf{L}_{I_n}$ and $A + B$ is indeed singular, so this part of the claim is established.

(ii) For the converse, we start by showing that a non-singular matrix-convex set, closed under inversion, is comprised of matrices whose spectrum is confined to $\mathbb{C}_R$ (or to $\mathbb{C}_L$).

Using Observation 1.4 it follows that a non-singular convex set closed under inversion contains involutions in the form of Eq. (1.1). Technically, it is enough to focus on a $2 \times 2$ sub-block of $E_A$ in Eq. (1.1) (up to unitary similarity, take the $(1, 1)$, $(1, n)$, $(n, 1)$ and $(n, n)$ elements).

Indeed, if $A$ have eigenvalues in both half-planes, $E_A$ has a $2 \times 2$ sub-block of the form $\begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}$ for some $t \in \mathbb{C}$. Then consider the following convex combination of this $2 \times 2$ block along with its unitary similar version,

$$M := \theta \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix} + (1+\theta)U^* \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix} U = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & t=0 \\ \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} & t \neq 0 \end{cases} \quad UU^* = U^*U = I_2 \quad \theta \in (0, 1).$$  

Then, $M = \begin{pmatrix} \frac{2\theta-1}{\theta-1} & \frac{\theta t}{\theta-1} \\ \frac{2\theta-1}{\theta-1} & \frac{\theta t}{\theta-1} \end{pmatrix}$ and thus for $\theta = \frac{1}{2} \pm \frac{|t|}{2\sqrt{|t|^2+4}}$ one has that $\det(M) = 0$, so this matrix-convex set contains singular matrices. Hence one can conclude that indeed the spectrum of all matrices in this set must be confined to either $\mathbb{C}_R$ or to $\mathbb{C}_L$.

To guarantee that the set is at least maximal non-singular convex cone, closed under inversion, we employ Theorem 1.6 and assume that the sought set is of the form $\mathbf{L}_{H}$ for some $H \in \mathbf{P}_n$ (or $-H \in \mathbf{P}_n$).

Recall (e.g. [20] Lemma 3.4) that for a unitary matrix $U$ and $H \in \mathbf{H}_n$, both arbitrary, one has that $U^* \mathbf{L}_H U = \mathbf{L}_{U^*HU}$. Now the fact that a matrix-convex set is in particular closed under unitary similarity, implies that here $U^* \mathbf{L}_H U = \mathbf{L}_H$, thus indeed $H = I_n$ (or $H = -I_n$) and this item is established.

(iii) This follows from the previous items along with the already mentioned fact that $\overline{\mathbf{L}}_H$ is the closure of the open set $\mathbf{L}_H$ for all $H \in \mathbf{H}_n$, and in particular for $H = I_n$.

(iv) See Observation 2.2(I) along with [23] Proposition 3.2.5(i)].
We conclude this section by putting Theorem 3.1 into perspective:

1. A quantitative refinement of item (i) of Theorem 3.1 is introduced in [6, Theorem 3.4], where the Lyapunov inclusion in Eq. (1.2) is substituted by a Riccati inclusion.

2. As already mentioned, a complete characterization of the set $L_H$ for an arbitrary $H \in H_n$, appeared in [11]. The restriction in Theorem 3.1 to $H = I_n$ enables us, through resorting to the notion of matrix-convexity, to obtain a much simpler characterization of $L_{I_n}$, which in turn is exploited in describing Positive Real functions, see Definition 4.2.

4. **Maximal matrix-convex invertible cones of Rational Functions**

To simplify the exposition, we relegate the matrix-valued case to the next subsection.

4.1. **Scalar positive real rational functions.** A real scalar rational function $f(s)$ of a complex variable $s$, is called Positive Real, denoted by $\mathcal{P}$, if it analytically maps $\mathbb{C}_R$ to $\mathbb{C}_R$ and leaves $\mathbb{R}^+$ invariant.

For example, a scalar rational function of McMillan degree one, is positive real if and only if it is of the form of

\begin{equation}
(4.1) \quad \text{either } f(s) = a + bs \quad \text{or } h(s) = d + \frac{b}{s+a} \quad b > 0, \ a, d \geq 0.
\end{equation}

$(f(s), h(s)$ are $\mathcal{P}$ functions of degree zero, when $b = 0$).

Recall that in [15] Otto Brune showed the following:

*The driving point immittance of a lumped $R - L - C$ electrical network belongs to $\mathcal{P}$. Conversely, an arbitrary positive real rational function can be realized as the driving point immittance of a lumped $R - L - C$ electrical network.* See e.g. [12], [27].

For example, the rational function $h(s)$ in Eq. (4.1) can be realized as the driving point impedance of the simple circuit in Figure 1.

\[ Z_{in} = d + \frac{b}{s+a} \quad R_1 = d \quad \frac{1}{C} = b \quad \frac{1}{R_2 C} = a \]

\[ R_1 \quad \rightarrow \quad R_2 \quad C \]

**Figure 1.** $Z_{in} = d + \frac{b}{s+a}$, $R_1 = d$, $\frac{1}{C} = b$, $\frac{1}{R_2 C} = a$

Duality between rational positive real functions and the driving point immittance of $R - L - C$ electrical circuits, has already been recognized for about ninety years, e.g. [15], [16], [17]. This has lead to rich and well-established theory, see e.g. [8], [12], [27], [50]. For a recent comprehensive account of circuits describing $\mathcal{P}$ functions of degree two, see [44].
Recall also that the following analogy between $R - L - C$ electrical circuits and simple mechanical systems, is classical, see e.g. [48].

| electrical          | mechanical         |
|---------------------|--------------------|
| current             | force              |
| voltage             | velocity           |
| transformer         | gear transmission  |
| resistor (admittance)| damper             |
| inductor (admittance)| spring             |
| capacitor (admittance)| inerter.          |

Thus, one can conclude that the set $\mathcal{P}$ may serve as a prototype model to continuous-time, linear, passive systems see e.g. [8, Theorem 2.6.1], [12, Section 3.18], [28, page 314], [44], [47, Theorem 2]. [48].

As a first connection with the structure we focus on, we cite the following adapted version of [23, Proposition 5.3.2].

**Proposition 4.1.** Let $f(s) = \frac{1}{s}$ and $g(s) \equiv 1$ be a pair of scalar rational positive real functions, of degree 1 and 0, respectively.

A scalar positive real rational function can always be generated by iteratively taking positive scaling, summation and inversion of $f(s)$ and $g(s)$.

Thus, one can conclude that scalar rational $\mathcal{P}$ functions may be viewed as $\text{cic}(f, g)$ a convex invertible cone generated by the above $f(s)$ and $g(s)$. A comparable observation for state-space realization of the above $f$ and $g$, will be given in Example 5.8 below.

The fact that in the scalar case, matrix-convexity degenerates to classical convexity, simplified the discussion in this subsection. In the next subsection we address matrix-valued rational functions.

### 4.2. Matrix-valued positive real rational functions.

Recall that $\mathbf{L}_I$ is the matricial generalizations of $\mathbb{C}_R$. Thus, we find it convenient to employ the terminology of Eq. 3.1 to describe matrix-valued Positive Real functions.

**Definition 4.2.** Let $F(s)$ be an $m \times m$-valued rational function so that $F(s)_{|s \in \mathbb{R}} \in \mathbb{R}^{m \times m}$. $F(s)$ is said to be Positive Real, denoted by $\mathcal{P}$, if it analytically maps $\mathbb{C}_R$ to $\mathbf{L}_I \cup \infty$.

As it is well known, physically this set of $m \times m$-valued positive real rational functions corresponds to the driving point immittance of a lumped $R - L - C$ electrical networks with $m$ inputs and $m$ outputs.

Here is a fundamental structural property of this set.
**Theorem 4.3.** The family $\mathcal{P}$, of $m \times m$-valued positive real rational functions, is a maximal, matrix-convex, cone, closed under inversion, of functions analytic in $\mathbb{C}_R$.

Conversely, a maximal matrix-convex cone of $m \times m$-valued rational functions, containing the zero degree function $F_0(s) \equiv I_m$, is the set $\mathcal{P}$.

**Proof:** Using the fact that all functions in $\mathcal{P}$ analytically map $\mathbb{C}_R$ to $\mathbb{L}_{I_m}$, together with item (i) of Theorem 3.1 establishes the sought structure.

For maximality take $G(s)$ a rational function which does not belong to $\mathcal{P}$. To avoid triviality, assume that it is analytic in $\mathbb{C}_R$, but there exists $s_o \in \mathbb{C}_R$ so that

$$ (G(s_o)) v = (-a + ib)v \quad a > 0, \quad b \in \mathbb{R}, \quad 0 \neq v \in \mathbb{C}^m. $$

Note now that (with the $v$),

$$ (G(s) + aI_m + b^2 (G(s) + aI_m)^{-1}) |_{s_o} v = 0. $$

This means that the rational function

$$ (G(s) + aF_0(s) + b^2 (G(s) + aF_0(s))^{-1})^{-1}, $$

is within $\text{cic}(G, F_0)$, but as it has a right half plane pole at $s_o$, it is no longer analytic in $\mathbb{C}_R$. Hence, the claim is established.

A scalar (not necessarily rational) version of Theorem 4.3 appeared in [23, Proposition 4.1.1]

We conclude this section by illustrating an application of Theorem 4.3. Here are the details.

![Figure 2](image.jpg)

**Figure 2.** $Z_{\text{in}}(s) = \left( ((sC_a)^{-1} + sL_b)^{-1} + (sL_c)^{-1} + sC_d \right)^{-1}$. 

The driving point impedance of the circuit in Figure 2 is a standard positive real (odd a.k.a. lossless or Foster) rational function of degree four. Yet, employing the notation,

$$ \phi(X, Y) := (X^1 + Y)^{-1}, $$

this driving point impedance can also be written as,

$$ Z_{\text{in}}(s) = \phi(\phi(sC_a, sL_b), \phi(sL_c, sC_d)). $$

We now leave this circuit for a short while and address a $2m \times 2m$-valued feedback-loop network $H(s)$ in Figure 3.
In 1

Out 1

Out 2

Out 2

F_d(s)

F_c(s)

F_a(s)

F_b(s)

In 1

F_a(s)

F_c(s)

F_d(s)

\( F_b(s) \)

\( F_a(s) \)

\( F_c(s) \)

\( F_d(s) \)

**Figure 3.** \( 2m \times 2m \)-valued network

\[
\begin{pmatrix}
\text{Out}_1 \\
\text{Out}_2
\end{pmatrix}
= \begin{pmatrix}
(\hat{F}_c + \hat{F}_a)^{-1} - (\hat{F}_c + \hat{F}_a)^{-1} \hat{F}_a^{-1} \\
\hat{F}_a^{-1} (\hat{F}_c + \hat{F}_a)^{-1} - (\hat{F}_c + \hat{F}_a)^{-1} \hat{F}_a^{-1}
\end{pmatrix}
\begin{pmatrix}
\text{In}_1 \\
\text{In}_2
\end{pmatrix}
\text{Out}_{1|\text{In}_2=0} = ( (F_a(s))^{-1} + F_b(s) )^{-1} + (F_c(s))^{-1} + F_d(s) )^{-1} \text{In}_1 .
\]

Employing again the map \( \phi \) from Eq. (4.2), the relation in Eq. (4.5) can be compactly written as,

\[
\text{Out}_{1|\text{In}_2=0} = \phi ( \phi(F_c, F_d), \phi(F_a, F_b) ) \text{In}_1 .
\]

Now, in comparison to Eq. (4.3), one can formally identify the elements \( sC_a, sL_b, sL_c, sC_a \), in Figure 2 with the blocks \( F_a(s), F_b(s), F_c(s), F_d(s) \) in Eq. (4.5), respectively.

This calls for adapting one of the classical construction schemes of \( R-L-C \) circuits, e.g. Brune, Botte-Duffin, Darlington, Foster, Cauer, etc. see e.g. [8], [12], [27], [44], [50], to introducing a design tool for networks of feedback-loops, more elaborate than that in Figure 3 (and as mentioned, the building blocks need not be positive real).

A word of caution: The passage from one-port circuit design to that of feedback-loops networks can not be straightforward: Typically blocks like \( F_a(s), F_b(s), F_c(s), F_d(s) \) are non-commutative. Hence, one needs to formally introduce positive real rational functions of say \( k \) non-commuting variables, mapping \( \prod_{I_n} \times \cdots \times \prod_{I_n} \) to \( \prod_{I_n} \), where \( n \) is a parameter.

The potential significance of having a design tool for feedback-loops networks, justifies addressing the challenge involved. Networks as in Figure 3 are of interest in a wide variety of applications. Further pursuing this direction is beyond the scope of this work.
5. Matrix-convex invertible cones of Realization Arrays

The renowned Kalman-Yakubovich-Popov Lemma ties up two representations of positive real functions: Rational functions and corresponding state-space realizations.

**Theorem 5.1.** Let \( F(s) \) be an \( m \times m \)-valued rational function \( F(s) \) with no pole at infinity and let \( R_F \) be a corresponding \((n + m) \times (n + m)\) state-space realization array, i.e.

\[
F(s) = C(sI_n - A)^{-1}B + D, \quad R_F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

If for some \( H \in \mathbb{P}_n \) one has that

\[
\begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix} R + R^* \begin{pmatrix} -H & 0 \\ 0 & I_m \end{pmatrix} \in \mathbb{P}_{n+m},
\]

then \( F(s) \) is positive real.

If the realization \( R_F \) in Eq. (5.1) is minimal, i.e. the McMillan degree of \( F(s) \) is \( n \), then the converse is true as well.

This result first appeared in [7]. The formulation used here is due to [26], [49]. For further details, see e.g. [1], [14, Subsection 2.7.2].

Theorem 5.1 employs an elegant idea: To treat the above \((n + m) \times (n + m)\) \( R \) as having \( i \) of an array and \( ii \) of a matrix. This will be further adopted in Theorem 5.7 below.

The relevance of following result, goes beyond the scope of this work.

**Observation 5.2.** Let \( F(s) \) be \( p \times m \)-valued rational function analytic in \( \mathbb{C}_R \), with no pole at infinity. Then, if

\[
R_F = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

is a corresponding \((n + m) \times (n + p)\) balanced realization, its upper-left \( n \times n \) block satisfies

\[-A \in \mathbb{L}_{I_n}.\]

**Proof:** First recall that a realization \( R_F \) in Eq. (5.3) is said to be balanced, see e.g. [21, Section 4], [23, Section 4.8], if there exists a matrix \( H \), where \( -H \in \mathbb{P}_n \), so that simultaneously

\[
HA^* + AH = BB^* \quad \text{and} \quad H^{-1}A^* + AH^{-1} = (CH^{-1})^*(CH^{-1}).
\]

Consider next the following iterative procedure, of at most \( n \) steps, for obtaining, out of \( H \), the involution \( \text{Sign}(H) = -I_n \), see Definition 1.3.

Denote \( H_0 = H \) and for \( j = 0, 1, 2, \ldots \), let \( \alpha_j := \frac{1}{1 + \max(\|H_j\|_2, \|H_j^{-1}\|_2)} \), so by construction \( \frac{1}{2} \geq \alpha_j \). Now, as long as, \( \frac{1}{2} > \alpha_j \) let,

\[
H_{j+1} = \alpha_j H_j + (1 - \alpha_j) H_j^{-1}, \quad j = 0, 1, 2, \ldots.
\]

\footnote{Like Janus in the Roman mythology}
and thus
\[ \alpha_j > \alpha_{j+1} \geq \frac{1}{2}. \]

Denoting, \( B_oB_o^* = BB^* \), \( C_o^*C_o = C^*C \), in addition \( H_{j+1} \) satisfies,
\[
H_{j+1}^{-1}A + AH_{j+1}^{-1} = \alpha_j B_j B_j^* + (1-\alpha_j) \left( C_j H_j^{-1} \right)^* \left( C_j H_j^{-1} \right)^* \]
\[
H_{j+1}^{-1}A + A^* H_{j+1}^{-1} = \alpha_j C_j^* C_j + (1-\alpha_j) \left( H_j^{-1} B_j \right)^* \left( H_j^{-1} B_j \right)^* \quad j = 0, 1, 2, \ldots
\]

Once, \( \frac{1}{2} = \alpha_j := \alpha_{\hat{j}} \), (for some \( n \geq \hat{j} \)) one can conclude that \( H_{\hat{j}} = \text{Sign}(H) = -I_n \) and stop. In this case,
\[
-(A + A^*) = B_j B_j^* = C_j^* C_j^*,
\]

so the construction is complete. \( \square \)

The idea of the above proof is similar to that of [21 Observation 4.1].

**Corollary 5.3.** In Theorem 5.1, up to a change of coordinates
\[
R_F \longrightarrow \hat{R}_F := \begin{pmatrix} H_F^* & 0 \\ 0 & I_m \end{pmatrix} R_F \left( \begin{pmatrix} H_F^* & 0 \\ 0 & I_m \end{pmatrix} \right)^*,
\]

Eq. (5.2) may be substituted by,
\[
(\begin{pmatrix} -I_n & 0 \\ 0 & I_m \end{pmatrix} \hat{R}_F + \hat{R}_F^* \left( \begin{pmatrix} -I_n & 0 \\ 0 & I_m \end{pmatrix} \right)) = Q, \quad Q \in \mathbf{F}_{n+m}.
\]

In particular, this is the case when the realization \( R_F \) in q. (5.1) is balanced.

The last phrase follows from Theorem 5.1 along with Observation 5.2. Recall also that by definition, balanced realization implies minimality. However, as before, the passage from Eq. (5.2) to Eq. (5.4), does not require minimality of the realization.

In [17] system satisfying Eq. (5.4) is called “internally passive”. From the above it follows that a balanced positive real system is internally passive.

To study families of realization simultaneously satisfying Theorem 5.1 we need to introduce a relaxed version of matrix-convexity. To this end, from small dimensions isometries we construct a more elaborate isometry: Let \( \Upsilon_a \) and \( \Upsilon_b \) be the following \((n+m)k \times nk\) and \((n+m)k \times mk\) isometries respectively,
\[
\Upsilon_a = \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0_{m \times n} & 0 & \cdots & 0 \\ 0 & I_n \cdots & \cdots & 0 \\ 0 & 0 \cdots & I_n & 0_{m \times n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \cdots & I_n & 0_{m \times n} \end{pmatrix}, \quad \Upsilon_b = \begin{pmatrix} 0_{n \times m} & 0 & \cdots & 0 \\ I_m & 0 & \cdots & 0 \\ 0 & 0_{n \times m} & \cdots & 0 \\ 0 & 0 \cdots & I_m & 0 \end{pmatrix}.
\]

Let now,
\[
\Upsilon_c = \begin{pmatrix} \upsilon_{1,n} \\ \upsilon_{2,n} \\ \vdots \\ \upsilon_{k,n} \end{pmatrix}, \quad \Upsilon_d = \begin{pmatrix} \upsilon_{1,m} \\ \upsilon_{2,m} \\ \vdots \\ \upsilon_{k,m} \end{pmatrix}.
\]
be arbitrary \( nk \times n \) and \( mk \times m \) isometries respectively.

Whenever defined, a product of isometries is an isometry. Thus, the products \( \Upsilon_a \Upsilon_c \) and \( \Upsilon_b \Upsilon_d \) are isometries of dimensions \((m + n)k \times n\) and \((m + n)k \times m\), respectively.

One can now construct the following \((n + m)k \times (n + m)\) isometry

\[
(5.5) \quad \Upsilon = \left( \Upsilon_a \Upsilon_c : \Upsilon_b \Upsilon_d \right) = \begin{pmatrix}
    v_{1,n} & 0 & \cdots & 0 \\
    0 & v_{1,m} & \cdots & 0 \\
    v_{2,n} & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    v_{k,n} & 0 & \cdots & 0
\end{pmatrix}.
\]

i.e.

\[
\Upsilon^* \Upsilon = \sum_{j=1}^{k} \left( \begin{pmatrix}
    v_{j,n} \\
    0
\end{pmatrix} \right)^* \left( \begin{pmatrix}
    v_{j,n} \\
    0
\end{pmatrix} \right) = \begin{pmatrix}
    I_n & 0 \\
    0 & I_m
\end{pmatrix}.
\]

**Definition 5.4.**

(I) A set of \((n + m) \times (n + m)\) matrices \( R \) is said to be \( n,m \)-matrix-convex if for all natural \( k \), for all \( R_1, ..., R_k \) in the set \( R \), and for all \((n + m)k \times (n + m)\) structured isometries \( \Upsilon \) of the form of Eq. (5.5), one has that also

\[
\Upsilon^* \Upsilon \in P_{n+m},
\]

belongs to \( R \).

(II) If the above structured \( \Upsilon \) in Eq. (5.5) is assumed to be of a full-rank (and not necessarily an isometry) i.e \( \Upsilon^* \Upsilon \in P_{n+m} \), then \( R \) is \( n,m \)-matrix-convex cone. \( \square \)

**Remark 5.5.** As already pointed out \( n,m \)-matrix-convexity is weaker than matrix-convexity. Note that in a way similar to Remark 2.3 one can conclude that \( n,m \)-matrix-convexity still implies classical convexity. \( \square \)

Here is our first motivation to resorting to the notion of \( n,m \)-matrix-convexity,

**Lemma 5.6.** For all \( n,m = 1,2, \ldots \), the sets \( L\left( -I_n \ 0 \right) \) and \( L\left( -I_n \ 0 \right) \) in (1.2) are, respectively (closed and) open, \( n,m \)-matrix-convex cones, closed under inversion.

Indeed, without loss of generality, a matrix \( R \) within \( L\left( -I_n \ 0 \right) \) can always be written as

\[
(5.6) \quad R = \begin{pmatrix}
    -P_n+iH_n & -M+T \\
    R^*+T^* & P_m+iH_m+M^*P_n^{-1}M
\end{pmatrix} \in P_n \oplus P_m
\]

where \( P_n \in \mathbb{P}_n, P_m \in \mathbb{P}_m, H_n \in \mathbb{H}_n, H_m \in \mathbb{H}_m, M,T \in \mathbb{C}^{n \times m}. \)
Now, substituting $R_1, \ldots, R_k$ of the from of Eq. (5.6) in Definition 5.4 one obtains,

$$\hat{R} = \Upsilon^* \left( \begin{array}{cccc}
R_1 & 0 & 0 & 0 \\
0 & R_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & R_k
\end{array} \right) \Upsilon$$

$$= \sum_{j=1}^{k} \begin{pmatrix}
v_{j,n} & 0 \\
0 & v_{j,m}
\end{pmatrix} \left( \begin{array}{cccc}
-P_{j,n+iH_{j,n}} & -R_j+T_j & R_j & T_j \\
R_j^*+T_j^* & P_{j,m}+R_j^*P_{j,n}^{-1}R_j+iH_{j,m} & 0 & 0
\end{array} \right) \begin{pmatrix}
v_{j,n} & 0 \\
0 & v_{j,m}
\end{pmatrix}.$$  

Next note that,

$$\left( -I_n \ 0 \ \ 0 \ \ I_m \right) \hat{R} + \hat{R}^* \left( -I_n \ 0 \ \ 0 \ \ I_m \right) = 2 \sum_{j=1}^{k} \begin{pmatrix}
v_{j,n} & 0 \\
0 & v_{j,m}
\end{pmatrix} \left( \begin{array}{cccc}
P_{j,n} & R_j & R_j & T_j \\
R_j^* & P_{j,m}+R_j^*P_{j,n}^{-1}R_j & 0 & 0
\end{array} \right) \begin{pmatrix}
v_{j,n} & 0 \\
0 & v_{j,m}
\end{pmatrix},$$  

namely using $\Upsilon$ from Eq. (5.5)

$$Q_o = \Upsilon^* \left( \begin{array}{cccc}
P_{k,n} & R_k & R_k & T_k \\
P_k^* & P_{k,m}+R_k^*P_{k,n}^{-1}R_k & 0 & 0
\end{array} \right) \Upsilon \in P_{n+m}.$$  

Thus $\hat{R} \in R$ so the claim is established. \hfill \Box

We now introduce families of realization arrays associated with rational functions. Before that, a word of caution: For example, $R_1 = \begin{pmatrix} A & B \\
C & D \end{pmatrix}$ and $R_2 = \begin{pmatrix} A & -B \\
-C & D \end{pmatrix}$ are two realization of the same rational function. Furthermore, $R_1$ is minimal (balanced) if and only if $R_2$ is minimal (balanced). However, $R_3 = \frac{1}{2}(R_1 + R_2) = \begin{pmatrix} A & 0 \\
0 & D \end{pmatrix}$ is only a non-minimal realization of a zero degree rational function $F(s) \equiv D$. Yet, as $(n+m) \times (n+m)$ matrices, if $R_1$ belongs to $L\left( \begin{pmatrix} -I_n & 0 \\
0 & I_m \end{pmatrix} \right)$, then also $R_2$ and $R_3$ belong to the same set.

More generally, when considering families of realizations $R$ satisfying Eq. (5.4) as matrices, one obtains only a proper subset of $L\left( \begin{pmatrix} -I_n & 0 \\
0 & I_m \end{pmatrix} \right)$.

**Theorem 5.7.** Given a family of $m \times m$-valued positive real rational functions of McMillan degree, of at most, $n$, with no poles at infinity. Consider the corresponding $(n+m) \times (n+m)$ realizations $R$ in Theorem 5.2 and in Eq. (5.4). Then (as matrices), this family of realizations is an $n, m$-matrix-convex cone, closed under inversion.

The fact that this is a matrix-convex cone follows from Eqs. (5.1), (5.4) along with Lemma 5.6. For invertibility note that assuming $R$ is non-singular, multiplying Eq. (5.4) by $(R^*)^{-1}$ and $R^{-1}$ from the left and from the right respectively, the resulting right-hand side is $\hat{Q} := (R^*)^{-1}QR^{-1}$. Now if $Q$ is in $P_{n+m}$, then so is $\hat{Q}$.

Note that in Theorem 5.7 we have not assumed minimality of the realizations.
To illustrate an application of Theorem 5.7 we next show how a set of realization arrays, may be parametrized by a pair of representatives.

**Example 5.8.** Recall that in Proposition 4.1 we stated that scalar positive real rational functions can be equivalently described as $\text{cic}(f,g)$, with $f(s) = \frac{1}{s}$ and $g(s) \equiv 1$. Let now,

$$R_f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R_g = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

be their (balanced) realizations, respectively. Treating $R_f$, $R_g$ as matrices, taking positive scaling, summation and inversion, one obtains $R_h$,

$$R_h = dR_g + b \left( aR_g + \sqrt{b}R_f \right)^{-1} = \left( \begin{array}{c} -a \\ \sqrt{b} \\ d \end{array} \right)$$

$a, b, d \geq 0$,

which in turn is a (balanced) realization of the function $h(s)$,

$$h(s) = \frac{b}{s + a} + d \quad a, b, d \geq 0.$$  

Recall now that in Eq. (4.1) we pointed out that $h(s)$ is a parametrization of all positive real rational functions of degree of at most one, with no pole at infinity. □

6. Concluding remarks

It appears that this work opens a door for further research on various questions. Here is a sample list.

- Balanced truncation model order reduction preserving passivity of a given system, is well established, see e.g. [46]. Now, one can combine Theorem 5.7 along with [21] to obtain a scheme of simultaneous balanced truncation model order reduction of a convex hull of realization arrays associated with positive real rational functions.
- Enhance the connection between LMI’s and matrix-convex cones, see e.g. [33]. As a simple illustration, let $A_1, \ldots, A_m$ be a given set of $n \times n$ matrices. Having all matrices in the convex cone, closed under inversion, generated by these $m$ matrices non-singular, is a necessary condition to satisfying $\{A_1, \ldots, A_m\} \subset \mathbb{L}_H$ for some $H_n$. This can be taken into account before applying the Matlab LMI-toolbox to try to find $H$ a common quadratic Lyapunov factor to $A_1, \ldots, A_m$.
- Quantitatively Hyper-Positive real rational functions are associated with the absolute stability (a.k.a the Lurie problem). These functions map $\mathbb{C}_R$ to a bounded disk within $\mathbb{C}_R$. Furthermore, under inversion this disk is mapped onto itself. For details, see [6].
- Recall that in discrete-time linear systems framework, passivity is described by Bounded Real rational functions, see e.g. [8, Chapter 7]. These are functions analytically mapping $\mathbb{C}_R$ to $\mathbb{R}_2(1)$, see Eq. (2.4). Recall also that Positive-Real and Bounded-Real rational functions are inter-related through the Cayley transform. In analogy to the currents work, in [12] structural properties of discrete-time passive systems are explored.
Extend the above results to certain classes of non-linear systems. Note that here, invertibility is in the sense that composition yields the identity map, e.g. \( \tan(x) \) and \( \arctan(x) \) or \( x^3 \) and \( x^{\frac{1}{3}} \).

References

[1] D. Alpay and I. Lewkowicz, “The Positive Real Lemma and Construction of all Realizations of Generalized Positive Rational Functions”, *Systems and Control Letters*, Vol. 60, pp. 985-993, 2011.
[2] D. Alpay and I. Lewkowicz, “Convex Cones of Generalized Positive Rational Functions and the Nevanlinna-Pick Interpolation”, *Linear Algebra and its Applications*, Vol. 438, pp. 3949-3966, 2013.
[3] D. Alpay and I. Lewkowicz, “Wrong Side Interpolation by Positive Real Rational Functions”, *Linear Algebra and its Applications*, Vol. 539, pp. 175-197, 2018.
[4] D. Alpay and I. Lewkowicz, “Composition of Rational Functions: State-space Realization and Applications”, to appear in *Linear Algebra and its Applications*. See [arXiv:1807.01753](https://arxiv.org/abs/1807.01753).
[5] D. Alpay and I. Lewkowicz, “Realization of Tensor-Product and of Tensor-Factorization of Rational Functions”, *Quantum Studies: Mathematics and Foundations*, Vol. 6, pp. 269-278, 2019.
[6] D. Alpay and I. Lewkowicz, “Quantitatively Hyper-Positive Real rational functions”, a manuscript.
[7] B.D.O. Anderson and J.B. Moore, “Algebraic Structure of Generalized Positive Real Matrices”, *SIAM J. Contr.*, Vol. 6, pp. 615-624, 1968.
[8] B.D.O. Anderson and S. Vongpanitlerd, *Networks Analysis & Synthesis, A Modern Systems Theory Approach*, Prentice-Hall, New Jersey, 1973.
[9] T. Ando, *Completely Positive Matrices*, Lecture Notes, Sapporo, Japan, 1991
[10] T. Ando, “Set of Matrices with Common Lyapunov Solution, *Arch. Math.*, Vol. 77, pp. 76-84, 2001.
[11] T. Ando, “Sets of Matrices with Common Stein Solutions and Hcontractions, *Linear Algebra and its Applications*, Vol. 383, pp. 49-64, 2004.
[12] V. Belevich, *Classical Network Theory*, Holden Day, San-Francisco, 1968.
[13] A. Berman and N. Shaked-Monderer, *Completely Positive Matrices*, World Scientific Publishing Co., 2003
[14] S. Boyd, L. El-Ghaoui, E. Ferron and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, SIAM books, 1994.
[15] O. Brune, “Synthesis of a Finite Two Terminal Network whose Driving Point Impedance is a Prescribed Function of Frequency”, *J. Math. Phys.*, Vol. 10, pp. 191-236, 1931.
[16] W. Cauer, “The Realization of Impedances of Prescribed frequency Dependence” (in German), *Archiv für Elektrotechnik*, Vol. 17, pp. 355-388, 1926.
[17] W. Cauer, “Über Funktionen mit positivem Realteil” (in German), *Mathematische Annalen*, Vol. 106, pp. 369-394, 1932
[18] W.H. Chen, *Linear Networks Design and Synthesis*, McGraw-Hill, Electrical and Electronic Engineering Series, 1964
[19] M-D. Choi, “Completely Positive Linear Maps on Complex Matrices”, *Linear Algebra and its Applications*, Vol. 10, pp. 285-290, 1975.
[20] N. Cohen and I. Lewkowicz, “Convex Invertible Cones and the Lyapunov Equation”, *Linear Algebra and its Applications*, Vol. 250, pp. 265-286, 1997.
[21] N. Cohen and I. Lewkowicz, “Convex Invertible Cones of State Space Systems”, *Mathematics of Control Signals and Systems*, Vol. 10, pp. 265-285, 1997.
[22] N. Cohen and I. Lewkowicz, “A Pair of Matrices Sharing Common Lyapunov Solutions - a Closer Look”, *Linear Algebra and its Applications*, Vol. 360, pp. 83-104, 2003.
[23] N. Cohen and I. Lewkowicz, “Convex Invertible Cones and Positive Real Analytic Functions”, *Linear Algebra and its Applications*, Vol. 425, pp. 797-813, 2007.
[24] N. Cohen and I. Lewkowicz, “The Lyapunov Order for Real Matrices”, *Linear Algebra and its Applications*, Vol. 430, pp. 1489-1866, 2009.
[25] N. Cohen, I. Lewkowicz and L. Rodman, “Exponential Stability of Triangular Differential Inclusion Systems”, *System and Control Letters*, Vol. 30, pp. 159-164, 1997.
[26] B. Dickinson, Ph. Delsarte, Y. Genin and Y. Kump, “Minimal Realization of Pseudo Positive and Pseudo Bounded Real Rational Matrices”, *IEEE trans. Circuits and Systems*, Vol. 32, pp. 603-605, 1985.

[27] R.J. Duffin, “Elementary Operations which Generate Network Matrices”, *Proc. Amer. Math. Soc.*, Vol. 6, pp. 335-339, 1955.

[28] G.E. Dullerud and F. Paganini, *A Course in Robust Control Theory - a convex approach*, Springer 2000.

[29] E.G. Effros and S. Winkler, “Matrix Convexity: Operator Analogues of the Bipolar and Han-Banach Theorems”, *Journal of Functional Analysis*, Vol. 144, pp. 117-152, 1997.

[30] E. Evert, “Matrix Convex Sets Without Absolute Extreme Points”, *Linear Algebra and its Applications*, Vol. 537, pp. 287-301, 2018.

[31] E. Evert, J.W. Helton, I. Klep and S. McCullough, “Extreme Points of Matrix Convex Sets, Free Spectrahedra and Dilation Theory”, *Journal of Geometric Analysis*, Vol. 28, pp. 1373-1408, 2018.

[32] P. Gahinet, A. Nemirovsky, A.J. Laub and M. Chilali, *Matlab- LMI control toolbox - user’s guide*, Mathworks 1995.

[33] J.W. Helton, S. McCullough and V. Vinnikov, “Noncommutative Convexity Arises from Linear Matrix Inequalities”, *J. Func. Anal.*, Vol. 240, pp. 105-191, 2006.

[34] J.W. Helton, I. Klep and S. McCullough, “The Matricial Relaxation of a Linear Matrix Inequality” *Math. Program.*, Vol. 138, pp. 401-445, 2013.

[35] N. J. Higham, *Functions of Matrices: Theory and Computation*, SIAM Edition, 2008.

[36] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[37] C.R. Johnson and L. Rodman, “Convex Sets of Hermitian Matrices with Constant Inertia”, *SIAM Journal on Algebraic and Discrete Methods*, Vol. 3 pp. 351-359, 1985.

[38] P. Lancaster and L. Rodman, *Algebraic Riccati Equation*, Clarendon Press, Oxford, 1995.

[39] J. Levick, “Factorization of Quantum Channels”, *Linear Algebra and its Applications* Vol. 553, pp. 145-166, 2018.

[40] I. Lewkowicz, “Convex Invertible Cones of Matrices - a Unified Framework for the Equations of Sylvester, Lyapunov and Riccati”, *Linear Algebra and its Applications* Vol. 286, pp. 107-133, 1999.

[41] I. Lewkowicz, “Matrix Sign Function Iterations - Geometric Point of View”, a manuscript.

[42] I. Lewkowicz, “Discrete-time Passive Linear Systems - Multiplicative Matrix-Convex Sets point of view”, a manuscript.

[43] I. Lewkowicz, L. Rodman and E. Yarkoni, “Convex Invertible Sets and Matrix Sign Function”, *Linear Algebra and its Applications*, Vol. 396, pp. 329-352, 2005.

[44] A. Morelli and M.C. Smith, *Passive Network Synthesis: An Approach to Classification* no. DC33 in *Advances in Design and Control* series by SIAM, 2019.

[45] B. Passer, Orr Shalit and B. Solel, “Minimal and Maximal Matrix Convex Sets”, *Journal of Functional Analysis*, Vol. 274 pp. 3197-3253, 2018.

[46] J.R. Philips, L. Daniel and L.M. Silveira, “Guaranteed Passive Balancing Transformation for Model Order Reduction”, *IEEE Transaction on Computer-Aided Design of Integrated Circuits and Systems*, Vol. 22, pp. 1027-1043, 2003.

[47] T. Reis and J.C. Willems, “A balancing Approach of Systems with Internal Passivity and Reciprocity”, *Systems and Control Letters*, Vol. 60, pp. 152-173, 2011.

[48] M.C. Smith, “Synthesis of Mechanical Networks: The Inerter”, *IEEE Trans. Auto. Contr.* Vol. 47, pp. 1648-1662, 2002.

[49] J.C. Willems, “Realization of Systems with Internal Passivity and Symmetry Constraints”, Journal of the Franklin Institute, Vol. 301, pp. 605-621, 1976.

[50] M. R. Wohlers, *Lumped and Distributed Passive Networks*, Acad. Press 1969.