Provable non-convex projected gradient descent for a class of constrained matrix optimization problems

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Abstract

We propose a simple and scalable non-convex method for low-rank PSD matrix problems with a generic (strongly) convex objective \(f\), and additional matrix norm constraints. Such criteria appear in quantum state tomography and phase retrieval applications, among others. However, without careful design, existing methods quickly run into time and memory bottlenecks, as problem dimensions increase.

To remedy these shortcomings, we propose the \(\text{Projected Factored Gradient Descent (ProjFGD)}\) algorithm, that operates on the \(\text{low-rank factorization}\) of the variable space. Such factorization imputes non-convexity in the optimization; nevertheless, we show that our method favors local linear convergence rate in the non-convex factored space, for a class of convex norm-constrained problems. We build our theory on a novel \(\text{descent lemma}\), that extends recent results on the unconstrained version of the problem. Our findings are supported by empirical evidence on quantum state tomography and sparse phase retrieval applications.

1 Introduction

We consider matrix optimization problems that can be expressed as:

\[
\min_{X \in \mathbb{R}^{n \times n}} \quad f(X) \quad \text{subject to} \quad X \succeq 0, \ X \in C'.
\] (1)

Here, \(f\) is assumed to be strongly convex and Lipschitz gradient continuous. Moreover, let the optimum \(X^*\) of (1) satisfy \(\text{rank}(X^*) = r^*\), where \(r^* \leq n\), and let \(C' \subseteq \mathbb{R}^{n \times n}\) denote additional, convex constraints on \(X\). Observe that (1) is convex, under these assumptions. Nevertheless, in practice we usually force the estimate to be low-rank (even if \(r^* = n\))—e.g., due to statistical or computational reasons—in order to hope for a good approximation of \(X^*\) in reasonable time; in such cases, the problem becomes non-convex. We will focus on such low-rank cases where (i) either \(X^*\) is low-rank by nature, (ii) or we are interested in estimating a low-rank approximation of \(X^*\).

Such criteria appear naturally (or at least after simple transformations) in applications from diverse research fields; a non-exhaustive list includes density matrix estimation of quantum systems (where \(C'\) includes trace norm constraints on \(X\)) \([1, 23, 29]\), sparse phase retrieval applications such as X-ray crystallography and microscopy \([20, 12, 38]\), and sparse PCA \([31]\) (where \(C'\) contains \(\ell_1\)-norm constraints).
on $X$); see Section 4 for more details. In all cases, the number of variables grows fast, as the problem dimensions increase. Thus, due to their widespread occurrence, it is critical to devise easy-to-implement, efficient and provable algorithms.

For specific instances of (1), researchers have devised such algorithms that solve (1), even when non-convex rank constraints are present; even a brief overview of the techniques used would go beyond the page limits of this paper. As representative algorithmic realizations, we mention [27, 7, 4, 9, 44, 32, 30, 28, 43, 24, 47] and point to references therein: most of these schemes, though, focus on the matrix sensing / matrix completion problem, and, thus, are designed for specific instances of $f$. Moreover, the majority of them do not directly handle additional constraints.

To even a greater extent, there is ample discussion on how to solve rank-constrained instances of (1) in the convex setting, where the rank constraint is substituted by the nuclear norm of $X$; we refer the reader to [34, 6, 13, 5, 17, 46] and references therein for useful pointers. Nevertheless, these methods often involve top-$r$ singular value/vector computations –at least once per iteration– in order to hope for good approximate solution, due to rank/nuclear norm constraints (especially in under-determined problem cases). This constitutes their computational bottleneck in large-scale settings. Thus, it is desirable to find algorithms that scale well in practice.

Our approach. One way to achieve this is by low-rank matrix factorization techniques. In particular, we solve instances of (1) in the factored form as follows:

$$\min_{U \in \mathbb{R}^{n \times r}} f(UU^\top) \quad \text{subject to} \quad U \in C. \quad (2)$$

This formulation, popularized by Burer and Monteiro [11, 12], naturally encodes the PSD constraint in (2), removing the expensive eigen-decomposition projection step. Here, $r \leq r^*$ is often set to be much smaller than $r^*$, due to statistical or computational reasons, as mentioned above; thus, by construction, the $UU^\top$ matrix is low-rank and PSD.

$C \subseteq \mathbb{R}^{n \times r}$ is a compact convex set, that models well $C'$ in (1). While in practice we can assume any such constraint $C$ with tractable Euclidean projection operator$^1$ in our theory we mostly focus on norm constraint sets for $C$. As we describe next in more detail, in order to claim convergence to a useful estimate in the $X$ space through the factored $U$ space, we require $C$ and $C'$ be connected via a continuous map $U \mapsto X$, such that any point $U$ in $C$ has a representative point $X = UU^\top$ in $C'$. We defer this discussion to Section 3.

Our motivation for studying (2) origins from large-scale problem instances: when $r$ is much smaller than $n$, factor $U \in \mathbb{R}^{n \times r}$ contains much less variables to maintain and optimize than $X = UU^\top$. Thus, by construction, such parametrization also makes it easier to update and store the iterates $U$. Even more importantly, observe that $UU^\top$ reformulation automatically encodes any rank constraint. Standard approaches, that operate in the original variable space, either enforce such constraint at every iteration or involve a nuclear-norm projection. Doing so requires computing a truncated SVD per iteration, which can get cumbersome in large-scale settings.

Contributions. Our aim is to broaden the results on efficient, non-convex recovery for constrained low-rank matrix problems. In order to have a more user-friendly theory and algorithm, our developments maintain a connection with analogous results in convex optimization, where standard assumptions are made. To this end, we extend the results in [8] (see Section 1.1 for a more complete discussion on related work), and consider the common case where $f$ is smooth and (restricted) strongly convex $^2$; extension to just smooth $f$ cases is left for future work. From a practical point of view, we provide

$^1$In general, one could artificially introduce $C$ in (2), even when no $C'$ constraint is present in (1) (e.g., for better interpretation of results).

$^2$We consider here the strongly convex case since such cases are usually more interesting in practice, as we show in the experiments section.

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experimental results for two important tasks in physical sciences: quantum state density estimation and sparse phase retrieval.

Some highlights of our developments are the following:

• A key property for proving convergence in convex optimization, under common smoothness and strong convexity assumptions, is the notion of descent. I.e., given current, next and optimal points in the $X$ space, say $X_t, X_{t+1}, X^*$, respectively, and the recursion $X_{t+1} = X_t - \eta \nabla f(X_t)$, the condition $(X_t - X_{t+1}, X_t - X^*) \geq C$ — where $C > 0$ depends on the gradient norm — implies that $X_{t+1}$ “moves” towards the correct direction. Similar results have been proved in [8] on the non-convex factored-space, however without constraints. In this work, we present a novel descent lemma that non-trivially extends such conditions for (2), for a simple case of norm-based convex sets $C$. We hope that this result will trigger more attempts towards more generic convex sets.

• We propose ProjFGD, a non-convex projected gradient descent algorithm that solves instances of (2). ProjFGD has a novel constant step size selection procedure, following ideas from [8], that leads to favorable local convergence guarantees when $f$ is smooth and (restricted) strongly convex. We also present an initialization procedure with some guarantees in the supplementary material.

• Finally, we extensively study the performance of ProjFGD on two problem cases: (i) quantum state tomography and (ii) sparse phase retrieval. Our findings show significant acceleration when ProjFGD is used, as compared to state of the art.

1.1 Related work

Here, we elaborate a bit further on an evolving line of work that considers problem variants of (2), where factorizations of the form $UU^T$ are used.

The work of [18] proposes a first-order algorithm for (2), where the nature of the constraint set is more generic than the one we consider here, and depends on the problem at hand. The authors provided a set of conditions (local descent, local Lipschitz, and local smoothness) under which one can prove convergence to an $\varepsilon$-close solution with $O(1/\varepsilon)$ or $O(\log(1/\varepsilon))$ iterations. While the convergence proof is general, checking whether the three conditions hold is a non-trivial problem and requires different analysis depending on a particular estimation problem at hand.

[8] proposes the Factored Gradient Descent (FGD) algorithm for (2), where $C \equiv \mathbb{R}^{n \times r}$. FGD is also a first-order scheme. Key ingredient for convergence is a novel step size selection that can be used for any $f$, as long as it is Lipschitz gradient smooth (and strongly convex for faster convergence). However, [8] cannot accommodate any constraints on $U$. Notwithstanding this limitation, [8] is the first paper that provably solves the unconstrained re-parametrized problem in (2) for generic convex functions $f$, under common convex assumptions.

Concurrently, [48] presents a new analysis that handles non-square cases in (2). In that case, we look for a factorization $X = UV^T \in \mathbb{R}^{n \times p}$. The idea is based on the inexact first-order oracle, previously used in [3]. Similarly to [8], the proposed theory does not handle any constraints and restricts to the case of strongly convex and smooth convex $f$.

Roadmap. Section 2 contains some basic definitions and assumptions that are repeatedly used in the main text. Section 3 describes ProjFGD and its theoretical guarantees. In Section 4 we motivate the necessity of ProjFGD via some applications; due to space limitations, only one application is described in the main text (the second application is included in the supplementary material). This paper concludes with a discussion on future directions in Section 5. Supplementary material contains further experiments, all proofs of theorems in main text, and a proposed initialization procedure.

3To see this, observe that $X_t^* - X_t$ is the best possible direction to follow, while $X_t - X_{t+1}$ is the direction we actually follow. Then, such a condition implies that there is a non-trivial positive correlation between these two directions.
2 Preliminaries

Notation. For matrices $X, Y \in \mathbb{R}^{n \times n}$, $\langle X, Y \rangle = \text{Tr}(X^\top Y)$ represents their inner product. $X \succeq 0$ denotes $X$ is a positive semi-definite (PSD) matrix. We use $\|X\|_F$ and $\sigma_1(X)$ for the Frobenius and spectral norms of a matrix, respectively; we also use $\|X\|_2$ to denote the spectral norm. Moreover, we denote as $\sigma_i(X)$ the $i$-th singular value of $X$. $X$, denotes the best rank-$r$ approximation of $X$. For $X$ such that $X = UU^\top$, the gradient of $f$ with respect to $U$ is $(\nabla f(UU^\top) + \nabla f(UU^\top)^\top)U$. If $f$ is also symmetric, i.e., $f(X) = f(X^\top)$, then $\nabla f(X) = 2\nabla f(X) \cdot U$.

An important issue in optimizing $f$ over the factored space is the existence of non-unique possible factorizations. We use the following rotation invariant distance metric:

**Definition 2.1.** Let matrices $U, V \in \mathbb{R}^{n \times r}$. Define:

$$\text{DIST}(U, V) := \min_{R: R \in \mathcal{O}} \|U - VR\|_F,$$

where $\mathcal{O}$ is the set of $r \times r$ orthonormal matrices $R$.

**Assumptions.** We consider applications that can be described by strongly convex functions $f$ with gradient Lipschitz continuity. We state these standard definitions below for the square case.

**Definition 2.2.** Let $f : \mathbb{R}^{n \times n} \to \mathbb{R}$ be convex and differentiable. Then, $f$ is $\mu$-strongly convex if:

$$f(Y) \geq f(X) + \langle \nabla f(X), Y - X \rangle + \frac{\mu}{2} \|Y - X\|^2_F, \quad \forall X, Y \in \mathbb{R}^{n \times n}. \quad (3)$$

**Definition 2.3.** Let $f : \mathbb{R}^{n \times p} \to \mathbb{R}$ be a convex differentiable function. Then, $f$ is gradient Lipschitz continuous with parameter $L$ (or $L$-smooth) if:

$$\|\nabla f(X) - \nabla f(Y)\|_F \leq L \cdot \|X - Y\|_F, \quad \forall X, Y \in \mathbb{R}^{n \times n}. \quad (4)$$

For our proofs, we will also make the faithfulness assumption, as in [18]:

**Definition 2.4.** Let $\mathcal{E}$ denote the set of equivalent factorizations that lead to a rank-$r$ matrix $X^* \in \mathbb{R}^{n \times n}$; i.e., $\mathcal{E} := \{U^* \in \mathbb{R}^{n \times r} : X^* = U^* U^{*\top}\}$. Then, we assume $\mathcal{E} \subseteq \mathcal{C}$, i.e., the resulting convex set $\mathcal{C}$ in (2) (from $\mathcal{C}'$ in [1]) respects the structure of $\mathcal{E}$.

This assumption is necessary for arguments regarding the quality of solution obtained in the factored $U$ space, w.r.t. the original $X$ space.

3 The Projected Factored Gradient Descent (ProjFGD) algorithm

Let us first describe the ProjFGD algorithm, a projected, first-order scheme. The pseudocode is provided in Algorithm 1. Let $\Pi_{\mathcal{C}}(V)$ denote the projection of an input matrix $V \in \mathbb{R}^{n \times r}$ onto the convex set $\mathcal{C}$. The starting point is then computed as follows: we first compute $X_0 := \frac{1}{L} \cdot \Pi_+(-\nabla f(0))$, where $\Pi_+(\cdot)$ denotes the projection onto the set of PSD matrices and $\hat{L}$ represents an approximation of $L$; see also [8]. Then, ProjFGD requires a top-r SVD calculation, only once, to compute $\tilde{U}_0 \in \mathbb{R}^{n \times r}$, such that $X_0 = \tilde{U}_0 \tilde{U}_0^\top$; using $\tilde{U}_0$, the initial point $U_0$ satisfies $U_0 = \Pi_{\mathcal{C}}(\tilde{U}_0)$, in order to accommodate constraints $\mathcal{C}$.

The main iteration of ProjFGD applies the simple rule for any iteration $t$:

$$U_{t+1} = \Pi_{\mathcal{C}} \left(U_t - \eta \nabla f(U_t U_t^\top) \cdot U_t\right), \quad (5)$$

[4] Our ideas can be extended in a similar fashion to the case of restricted strong convexity [2].
Algorithm 1 ProjFGD method

input Function \( f \), target rank \( r \), \# iterations \( T \).
1. Compute \( X_0 := 1/L \cdot \Pi_+ (-\nabla f(0)) \).
2. Set \( \tilde{U}_0 \in \mathbb{R}^{n \times r} \) such that \( X_0 = \tilde{U}_0 \tilde{U}_0^\top \).
3. Compute \( U_0 = \Pi_C \left( \tilde{U}_0 \right) \).
4. Set step size \( \eta \) as in (4).
5. for \( t = 0 \) to \( T - 1 \) do
6. \( U_{t+1} = \Pi_C \left( U_t - \eta \nabla f(U_t U_t^\top) \cdot U_t \right) \).
7. end for
output \( X = U_T U_T^\top \).

with step size:
\[
\eta \leq \frac{1}{128(L\|X_0\|_2 + \|\nabla f(X_0)\|_2)}.
\] (5)

Here, one can use \( \tilde{L} \) to approximate \( L \).

Key ingredients to achieve provable convergence are the initialization step –so that initial point \( U_0 \) leads to \( \text{Dist}(U_0, U^*) \) sufficiently small– and the step size selection. For the initialization, apart from the procedure mentioned above, we could also use more specialized spectral methods –see [18, 49]– or even run algorithms, applied on the original variable space (1), for only a few iterations –this does not affect any theoretical guarantees of ProjFGD. However, this assumption does not affect any theoretical guarantees of ProjFGD.

3.1 When constrained non-convex problems can be scary?

In stark contrast to the convex projected gradient descent method, proving convergence guarantees for (2) is not a straightforward task. First, if we are interested in quantifying the quality of the solution in the factored space w.r.t. \( X^* \), \( C \) should be faithful, according to Definition 2.1. Furthermore, there should exist a continuous map \( U \rightarrow X \) that relates the constraint set \( C' \), in the original variable space (see (1)), to the factored one \( C \) (see (2)). In that case, claims about convergence to a point \( U^* \), in the factored space, can be “transformed” into claims about convergence to a point close to \( X^* \), in the original space. As an example, consider the case where, for any \( X = UU^\top \), \( \text{Tr}(X) \leq \lambda \Leftrightarrow \|U\|_F^2 \leq \lambda \), and, thus, satisfying \( \|U\|_F^2 \leq \lambda \), for any \( U \), guarantees that \( \text{Tr}(X) \leq \lambda \) for \( X = UU^\top \). Apart from the example above, other characteristic cases include Schatten norms.

Contrary to this example, consider the case \( C' := \{X \in \mathbb{R}^{n \times n} : \|X\|_1 \leq \lambda'\} \), where, \( \|X\|_1 = \sum_{ij} |X_{ij}| \). A natural choice for \( C \) would be \( C := \{U \in \mathbb{R}^{n \times r} : \|U\|_1 \leq \lambda \} \), for \( \lambda, \lambda' > 0 \); however, depending on the selection of \( \lambda \), w.r.t. \( X' \), points in \( U \in C \) might result into points in the original space \( X = UU^\top \) that \( \notin C' \). In this case, \( U^* \) of (2) could be \( U^* \notin E \) and, thus, convergence guarantees to \( U^* \) have no meaning in convergence in \( X \) space. However, as we show in Section 6.1, in this case \( C \) “simulates” fairly well \( C' \) in the original space: if \( U \) is sparse enough, then \( X = UU^\top \) could also be fairly sparse, so proper selection of \( \lambda \) plays a key role. Even in this case, ProjFGD still performs competitively compared to state-of-the-art approaches.

Second, the projection step itself complicates considerably the analysis due to non-convexity, as we show in the supplementary material. In our theory, we focus on convex sets \( C \) that satisfy (6)
where \( \Pi_C(V) \) can be equivalently seen as scaling the input. E.g., when \( C \equiv \{ U \in \mathbb{R}^{n \times r} : \|U\|_F \leq \lambda \} \), \( \Pi_C(V) = \xi(V) \cdot V \) where \( \xi(V) := \frac{\lambda}{\|V\|_F} \), for \( V \not\in C \). Our theory highlights that, even for this simple case, proving convergence is not a straightforward task.

3.2 Theoretical guarantees of \( \text{ProjFGD} \) for \( C := \{ U \in \mathbb{R}^{n \times r} : \|U\|_F \leq \lambda \} \)

We provide theoretical guarantees for \( \text{ProjFGD} \) in the case where the constraint satisfies

\[
\Pi_C(V) = \arg\min_{U \in C} \frac{1}{2}\|U - V\|_F^2 = \begin{cases} V & \text{if } V \in C, \\ \xi(V) \cdot V & \text{if } V \not\in C, \end{cases}
\]

i.e., the projection operation is an entry-wise scaling. Such settings include the Frobenius norm constraint \( C = \{ U \in \mathbb{R}^{n \times r} : \|U\|_F \leq \lambda \} \), which appears in quantum state tomography. Moreover, for this case, the constraint has one-to-one correspondence with the trace constraint in the original X space; thus any argument in the \( U \) space applies for the \( X \) space also.

We assume the optimum \( X^* \) satisfies \( \text{rank}(X^*) = r^* \). To solve \( \text{ProjFGD} \), we optimize over the re-parameterized problem \( \widetilde{f} \), with the faithful constraint set \( \tilde{C} \) satisfying the properties above.

For our analysis, we will use the following step sizes:

\[
\tilde{\eta} = \frac{1}{128(L\|X_t\|_2 + \|Q_{\tilde{\lambda}}Q_{\tilde{t}}\nabla f(X_t)\|_2)}, \quad \eta^* = \frac{1}{128(L\|X^*\|_2 + \|\nabla f(X^*)\|_2)},
\]

where \( Q_{\lambda} \) is a basis for column space of \( A \). By Lemma A.5 in [S], we know that \( \tilde{\eta} \geq \frac{2}{5}\eta \) and \( \frac{10}{7} \eta^* \leq \eta \leq \frac{11}{10} \eta^* \). Due to such relationships, in our proof we will work with step size \( \tilde{\eta} \); this is equivalent –up to constants– to the original step size \( \eta \), used in the algorithm. Thus, any results below will automatically imply similar results hold for \( \eta \), by using the bounds between step sizes.

**Theorem 3.1** (Local) convergence rate for restricted strongly convex and smooth \( f \). Let \( C \subseteq \mathbb{R}^{n \times r} \) be a convex, compact, and faithful set, with projection operator satisfying (6). Let \( U_t \in C \) be the current estimate and \( X_t = U_tU_t^\top \). Assume current point \( U_t \) satisfies \( \text{DIST}(U_t, U^*) \leq \rho' \sigma_r(U^*) \), for \( \rho' := c \cdot \frac{\sigma_r(X_t)}{\|X_t\|_2} \), \( c \leq \frac{1}{200} \), and given \( \xi_t(\cdot) \geq 0.78 \) per iteration, the new estimate of \( \text{ProjFGD} \), \( U_{t+1} = \Pi_C \left( U_t - \tilde{\eta} \nabla f(U_tU_t^\top) \cdot U_t \right) = \xi_t \left( (U_t - \tilde{\eta} \nabla f(U_tU_t^\top) \cdot U_t) \right) \) satisfies

\[
\text{DIST}(U_{t+1}, U^*)^2 \leq \alpha \cdot \text{DIST}(U_t, U^*)^2,
\]

where \( \alpha := 1 - \frac{\mu \sigma_r(X_t)}{500(L\|X_t\|_2 + \|\nabla f(X^*)\|_2)} < 1 \). Further, \( U_{t+1} \) satisfies \( \text{DIST}(U_{t+1}, U^*) \leq \rho' \sigma_r(U^*) \).

The complete proof of the theorem is provided in the supplementary material. The assumption \( \text{DIST}(U_t, U^*) \leq \rho' \sigma_r(U^*) \) only leads to a local convergence result. [IS] provide some initialization procedures for different applications, where we can find an initial point \( U_0 \) such that \( \text{DIST}(U_0, U^*) \leq \rho' \sigma_r(U^*) \) is satisfied. In the supplementary material, we present a similar generic initialization procedure that results in exact recovery of the optimum, under further assumptions. We borrow such procedure in Section 4 for our experiments.

**\( \xi_t(\cdot) \) requirement.** The assumption \( \xi_t(\cdot) \geq 0.78 \) implies the iterates of \( \text{ProjFGD} \) (before the projection step) are retained relatively close to the set \( C \). For some cases, this can be easily satisfied by setting the step size small enough, as indicated below; the proof can be found in Section 7.

**Corollary 3.2.** If \( C = \{ U \in \mathbb{R}^{n \times r} : \|U\|_F \leq \lambda \} \), then \( \text{ProjFGD} \) inherently satisfies \( 128 \leq \xi_t(\cdot) \leq 1 \), for every \( t \). I.e., it guarantees [7] without assumptions on \( \xi_t(\cdot) \).

Intuitively, we expect the estimates \( U_t \), before the projection, to be further from \( C \) during the first steps of \( \text{ProjFGD} \); as the number of iterations increases, the sequence of solutions gets closer to \( U^* \) and thus \( \xi_t(\cdot) \rightarrow 1 \).
We present two characteristic applications. For each application, we define the problem, enumerate the so-called rank-and,

\[ \text{Lemma 3.3 (Descent lemma). Let } \hat{U}_{t+1} = U_t - \hat{\eta} \nabla f(U_t^\top) \cdot U_t. \text{ For } f \text{ L-smooth and } \mu \text{-strongly convex, and under the same assumptions with Theorem 3.1, the following inequality holds true:} \]

\[
2\hat{\eta} \langle \nabla f(U_t U_t^\top) \cdot U_t, U_t - U^* R_t^* \rangle + \|U_{t+1} - \hat{U}_{t+1}\|^2_F \geq \hat{\eta}^2 \|\nabla f(U_t U_t^\top) U_t\|^2_F + \frac{3\mu \hat{\eta}}{2} \cdot \sigma_r(X^*) \cdot \text{Dist}(U_t, U^*)^2. \tag{8}
\]

In the unconstrained case, \cite{8} proposes a similar bound (Lemma 6.1 in \cite{8}):

\[
2\hat{\eta} \langle \nabla f(U_t U_t^\top) \cdot U_t, U_t - U^* R_t^* \rangle \geq \frac{3\mu \hat{\eta}}{2} \|\nabla f(U_t U_t^\top) U_t\|^2_F + \frac{3\mu \hat{\eta}}{2} \cdot \sigma_r(X^*) \cdot \text{Dist}(U_t, U^*)^2. \tag{8}
\]

However, their result and its analysis does not directly apply in our case: their techniques are oblivious of any convex constraints on the X-space (or the U-space).\footnote{To see this, in the unconstrained case, if }\hat{U}_t \equiv U^* \text{ (up to some rotation), then the following holds}

\[
0 := 2\hat{\eta} \langle \nabla f(X^*) \cdot U^*, U^* - U^* R_t^* \rangle \geq \frac{3\mu \hat{\eta}}{2} \|\nabla f(X^*) U^*\|^2_F + \frac{3\mu \hat{\eta}}{2} \cdot \sigma_r(X^*) \cdot \text{Dist}(U^*, U^*)^2 =: 0,
\]

since at the optimum we have \( \nabla f(X^*) U^* = 0 \). Though, given constraints, such condition does not hold in \cite{8}.

4 Applications

We present two characteristic applications. For each application, we define the problem, enumerate state-of-the-art algorithms and provide numerical results. We refer the reader to Section 6 for additional experiments.

4.1 Quantum state tomography

Building on Aaronson’s work on quantum state tomography (QST) \cite{11}, we are interested in learning the (almost) pure\footnote{Purity is a structural property of the density matrix: A quantum systems is pure if its density matrix is rank one and, almost pure if it can be well-approximated by a low rank matrix.} q-bit state of a quantum system –known as the density matrix– via a limited set of measurements. In math terms, the problem can be cast as follows. Let us define the density matrix \( X^* \in \mathbb{C}^{n \times n} \) of a q-bit quantum system as an unknown Hermitian, positive semi-definite matrix that satisfies \( \text{rank}(X^*) = r \) and is normalized as \( \text{Tr}(X^*) = 1 \) \cite{23}; here, \( n = 2^q \). Our task is to recover \( X^* \) from a set of QST measurements \( y \in \mathbb{R}^m \), \( m \ll n^2 \), that satisfy \( y = \mathcal{A}(X^*) + \eta \). Here, \( (\mathcal{A}(X^*))_i = \text{Tr}(E_i X^*) \) and \( \eta_i \) can be modeled as independent, zero-mean normal variables. The operators \( E_i \in \mathbb{R}^{n \times n} \) are typically the tensor product of the 2 \( \times 2 \) Pauli matrices.\footnote{\cite{23}}

The above lead to the following non-convex problem formulation\footnote{\cite{12} showed that, for almost all such tensor constructions – of } m = O(rn \log^c n), c > 0, \text{ Pauli measurements – satisfy the so-called rank-r restricted isometry property (RIP) for all } X \in \{X : X \succeq 0, \text{rank}(X) \leq r, \|X\|_1 \leq \sqrt{r\|X\|_F}\}:

\[
(1 - \delta_r) \|X\|_2^2 \leq \|\mathcal{A}(X)\|_2^2 \leq (1 + \delta_r) \|X\|_2^2, \tag{9}
\]

where \( \| \cdot \| \) is the nuclear norm (i.e., the sum of singular values), which reduces to \( \text{Tr}(X) \) since \( X \succeq 0 \).

\footnote{As pointed out in \cite{23}, it is in fact advantageous in practice to choose \( \text{Tr}(X) \neq 1 \), as it improves the robustness to noise. Here, we force \( \text{Tr}(X) \leq 1 \).}
The number of measurements $r$ was through convexification \[40\]: this includes nuclear norm minimization approaches \[23\], as well as

\[
\min_{X \geq 0} \|A(X) - y\|_F^2 + \lambda \|X\|_*. \tag{11}
\]

Here, \(\|\cdot\|_*\) reduces to $\text{Tr}(X)$ since $X \succeq 0$. This approach is considered in the seminal work \[23\] and is both tractable and amenable to theoretical analysis. The approach does not include any constraint on $X$.[11] As one of the most recent algorithms, we mention the work of \[40\] where a universal primal-dual convex framework is presented, with the QST problem as application.

From a non-convex perspective, \[25\] presents $\text{SparseApproxSDP}$ algorithm that solves \[10\], when the objective is a generic gradient Lipschitz smooth function. $\text{SparseApproxSDP}$ solves \[10\] by updating a putative low-rank solution with rank-1 refinements, coming from the gradient. This way, $\text{SparseApproxSDP}$ avoids computationally expensive operations per iteration, such as full SVDs. In theory, at the $r$-th iteration, $\text{SparseApproxSDP}$ is guaranteed to compute a $\frac{1}{r}$-approximate solution, with rank at most $r$, i.e., achieves a sublinear $O\left(\frac{1}{r}\right)$ convergence rate. However, depending on $\varepsilon$, $\text{SparseApproxSDP}$ might not return a low rank solution. Finally, \[17\] propose Randomized Singular Value Projection ($\text{RSVP}$), a projected gradient descent algorithm for \[10\], which merges gradient calculations with truncated SVDs via randomized approximations for computational efficiency.

Since the size of these problems grows exponentially with the number of quantum bits, designing fast algorithms that minimize the computational effort required for \[10\] or \[11\] is mandatory.

**Numerical results.** In this case, the factorized version of \[10\] can be described as:

\[
\min_{U \in \mathbb{R}^{n \times r}} \|A(UU^\top) - y\|_F^2 \quad \text{subject to} \quad \|U\|_F^2 \leq 1. \tag{12}
\]

We compare $\text{ProjFGL}$ with the algorithms described above; as a convex representative implementation, we use the efficient scheme of \[40\]. We consider two settings: $X^* \in \mathbb{R}^{n \times n}$ is (i) a pure state (i.e., rank($X^*) = 1$) and, (ii) an almost pure state (i.e., rank($X^*) = r$, for some $r > 1$). For all cases, $\text{Tr} (X^*) = 1$ and $y = A(X^*)$ (noiseless setting). We use Pauli operators for $\mathcal{A}$, as described in \[35\]. The number of measurements $m$ satisfy $m = C_{\text{sam}} \cdot r \cdot n \log(n)$, for various values of $C_{\text{sam}}$.

For all algorithms, we used the correct rank input and trace constraint parameter. All methods that require an SVD routine use $\text{rank}(\cdot)$ from the PROPACK software package. Experiments and algorithms are implemented on MATLAB environment; we used non-specialized and non-mexified code parts for all

---

[11] E.g., in order to take the trace constraint $\text{Tr}(X) = 1$ into account, either $\lambda$ should be precisely tuned to satisfy this constraint or the final estimator is normalized heuristically to satisfy this constraint \[20\].
algorithms. For initialization, we use the same starting point for all algorithms, which is either specific (Section 8) or random. We set the tolerance parameter to $\text{tol} := 5 \times 10^{-6}$.

Convergence plots. Figure 1 (two-leftmost plots) illustrates the iteration and timing complexities of each algorithm under comparison, for a pure state density recovery setting ($r = 1$). Here, $q = 12$ which corresponds to a $\frac{n(n+1)}{2} = 8,390,656$ dimensional problem; moreover, we assume $C_{\text{sam}} = 3$ and thus the number of measurements are $m = 12,288$. For initialization, we use the proposed initialization in Section 8 for all algorithms: we compute $-A^*(y)$, extract factor $U_0$ as the best-$r$ PSD approximation of $-A^*(y)$, and project $U_0$ onto $C$.

It is apparent that ProjFGD converges faster to a vicinity of $X^*$, compared to the rest of the algorithms; observe also the sublinear rate of SparseApproxSDP in the inner plots, as reported in [25].

Figure 2 contains recovery error and execution time results for the case $q = 13$ ($n = 8096$); in this case, we solve a $\frac{n(n+1)}{2} = 33,558,528$ dimensional problem. For this case, RSVP and SparseApproxSDP algorithms were excluded from the comparison. Appendix provides extensive results, where similar performance is observed for other values of $q$, $C_{\text{sam}}$.

Figure 1 (rightmost plot) considers the more general case where $r = 20$ (almost pure state density) and $q = 12$. In this case, $m = 245,760$ for $C_{\text{sam}} = 3$. As $r$ increases, algorithms that utilize an SVD routine spend more CPU time on singular value/vector calculations. Certainly, the same applies for matrix-matrix multiplications; however, in the latter case, the complexity scale is milder than that of the SVD calculations. Further metadata are also provided in Figure 3.

For completeness, in the appendix we also provide results that illustrate the effect of random initialization: Similar to above, ProjFGD shows competitive behavior by finding a better solution faster, irrespective of initialization point.

Timing evaluation (total and per iteration). Figure 4 highlights the efficiency of our algorithm in terms of time complexity, for various problem configurations. Our algorithm has fairly low per iteration complexity (where the most expensive operation for this problem is matrix-matrix and matrix-vector multiplications). Since our algorithm shows also fast convergence in terms of # of iterations, this overall results into faster convergence towards a good approximation of $X^*$, even as the dimension increases. Figure 4(right) shows how the total execution time scales with parameter $r$.

Overall performance. ProjFGD shows a competitive performance, as compared to the state-of-the-art algorithms; we would like to emphasize also that projected gradient descent schemes, such as [7], are also efficient in small- to medium-sized problems, due to their fast convergence rate. Moreover, convex approaches might show better sampling complexity performance (i.e., as $C_{\text{sam}}$ decreases). For more experimental results, we defer the reader to Appendix, due to space restrictions.
Figure 4: Timing bar plot: y-axis shows total execution time (log-scale) and x-axis corresponds to different q values. Left panel corresponds to r = 1 and C_{sam} = 6; right panel corresponds to q = 10 and C_{sam} = 6.

4.2 Sparse phase retrieval

Consider the sparse phase retrieval (SPR) problem [16, 14, 33]: we are interested in recovering a (sparse) unknown vector $x^* \in \mathbb{C}^n$, via its lifted, rank-1 representation $X^* = x^*x^H \in \mathbb{C}^{n \times n}$, from a set of quadratic measurements:

$$y_i = \text{Tr}(a_i^HXa_i) + \eta_i, \quad i = 1, \ldots, m.$$ 

Here, $a_i \in \mathbb{C}^n$ are given measurement vectors (often Fourier vectors) and $\eta_i$ is an additive error term. The above description leads to the following non-convex optimization criterion:

$$\begin{align*}
\underset{X \succeq 0}{\text{minimize}} & \quad \|A(X) - y\|_F^2, \\
\text{subject to} & \quad \text{rank}(X) = 1, (\|X\|_1 \leq \lambda).
\end{align*} \tag{13}$$

Here, $A : \mathbb{C}^{n \times n} \to \mathbb{C}^m$ such that $(A(X))_i = \text{Tr}(\Phi_iX)$ where $\Phi_i = a_i a_i^H$. In the case where we know $x^*$ is sparse [33, 39], we can further constrain the lifted variable $X$ to satisfy $\|X\|_1 \leq \lambda, \lambda > 0$; this way we implicitly also restrict the number of non-zeros in its factors and can recover $X^*$ from a limited set of measurements.

**Transforming (13) into a factored formulation.** Given the rule $X = uu^H$, where $u \in \mathbb{C}^n$, one can consider the factored problem re-formulation:

$$\begin{align*}
\underset{u \in \mathbb{C}^n}{\text{minimize}} & \quad \|A(uu^H) - y\|_F^2, \\
\text{subject to} & \quad \|u\|_1 \leq \lambda'.
\end{align*} \tag{14}$$

for some $\lambda' > 0$.

**Remark 1.** In contrast to the QST problem, where there is a continuous map between the constraints in the original $X$ space and in the factored $U$ space (i.e., $\text{Tr}(X) \leq \lambda \iff \|U\|_F^2 \leq \lambda$), this is not true for the SPR problem: As we state in the main text, points $U \in \mathcal{C}$ can result into $X \notin \mathcal{C}'$, depending on the selection of $\lambda, \lambda'$ values (i.e., $\mathcal{C}$ is unfaithful). In this case, the convergence theorem [33, 39] in the $U$ factor space only proves convergence to a point $U^*$ in the factored space, which is not necessarily related to the optimal point $X^*$ in the original space. However, as we show next, in practice, even in this case ProjFGD returns a competitive (if not better) solution, compared to state-of-the-art approaches.
State-of-the-art approaches. One of the most widely used methods for the phase retrieval problem comes from the seminal work of Gerchberg-Saxton [22] and Fienup [19]: they propose a greedy scheme that alternates projections on the range of \( \{a_i\}_{i=1}^m \) and on the non-convex set of vectors \( b \) such that \( b = |Az| \). Main disadvantage of such greedy methods is that often they get stuck to locally minimum points.

An popularized alternative to these greedy methods is via semidefinite relaxations. [14] proposes PhaseLift, where the rank constraint is replaced by the nuclear norm surrogate. However, it is well-known that such SDP relaxations can be computationally prohibitive, when solved using off-the-self software packages, even for small problem instances; some specialized and more efficient convex relaxation algorithms are given in [21].

In [15], the authors present Wirtinger Flow algorithm, a non-convex scheme for solving phase retrieval problems. Similar to our approach, Wirtinger Flow consists of three components: (i) a careful initialization step using a spectral method, (ii) a specialized step size selection and, (iii) a recursion where gradient steps on the factored variable space are performed. Other approaches include Approximate Message Passing algorithms [11] and ADMM approaches [39].

Numerical results. We test our algorithm on image recovery, according to the description given in [15] Section 4.2. Here, we consider grayscale images that are by nature also sparse (Figure 5 - left panel). This way, we can also consider \( \ell_1 \)-norm constraints, as in the criterion (14). We generate \( L = 21 \) random octanary patterns and, using these 21 samples, we obtain the coded diffraction patterns using the grayscale image as input. As dictated by [15] Section 4.2, we perform 50 power method iterations for initialization.

For this experiment, we highlight (i) how our algorithm ProjFGD performs in practice, and (ii) how the additional sparsity constraint could lead to better performance. Figure 6 (right panel) depicts the relative error \( \frac{\|\hat{X} - X^*\|_F}{\|X^*\|_F} \) w.r.t. the iteration count for two algorithms: (i) Wirtinger flow [15], and (ii) ProjFGD. We observe that ProjFGD shows a slightly better performance, compared to Wirtinger flow, both in terms of iterations –i.e., we reach to a better solution within the same number of iterations– and in terms of execution time –i.e., given a time wall, ProjFGD returns an estimate of better quality within the same amount of time. We note that both algorithms used step sizes that were slightly different in values, while ProjFGD further performs also a projection step.\(^{12}\) Figure 6 shows some

\(^{12}\) The step size in Wirtinger flow satisfies \( \eta := \frac{\mu_t}{\|\nabla \ell \|_F} \), for \( \mu_t = \min \left\{ 1 - e^{1/t_0}, 0.4 \right\} \) and \( t_0 \approx 330 \).
reconstructed images returned by the algorithms under comparison, during their execution. In all cases, both algorithms perform appealingly, finding a good approximation of the original image in less than 5 minutes; comparing the two algorithms, we note that ProjFGD returns a solution, within the same number of iterations, with at least 5 dB higher Peak Signal to Noise Ratio (PSNR), in less time.

5 Discussion

We consider a class of low-rank matrix problems where the solution is assumed PSD and the constraint set is simple enough, according to definition in [6]. This paper proposes ProjFGD, a non-convex projected gradient descent algorithm that operates on the factors of the PSD putative solution. When the objective function is smooth and strongly convex in the original variable space, ProjFGD has (local) linear rate convergence guarantees (which can become global, given a proper initialization).

The main shortcoming of our current analysis lies in the assumption that the constraint set is simple enough; extending the proof for more complex constraints sets is one possible research direction for future work, where an analogous of gradient mapping [72] might be required. Furthermore, considering barrier functions in the objective function, in order to accommodate the constraints, could be a possible extension. We hope this work will trigger future attempts along these directions.

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Figure 6: First two figures: Reconstructed image by using Wirtiger flow algorithm after 80 iterations (left panel) and 100 iterations (right panel). Last two figures: Reconstructed image by using ProjFGD algorithm after 80 iterations (left panel) and 100 iterations (right panel).
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6 Additional experiments

6.1 Quantum state tomography – more results

Figures 7-8 show further results regarding the QST problem, where \( r = 1 \) and \( q = 10, 12 \), respectively. For each case, we present both the performance in terms of number of iterations needed, as well as what is the cumulative time required. For all algorithms, we use as initial point \( U_0 = \Pi_C(\tilde{U}_0) \) such that \( X_0 = \tilde{U}_0\tilde{U}_0^\top \) where \( X_0 = \Pi_+(-A^\top(y)) \) and \( \Pi_+(-) \) is the projection onto the PSD cone. Configurations
Figure 7: Quantum state tomography: Convergence performance of algorithms under comparison w.r.t. $\| \hat{X} - X^* \|_F$ vs. (i) the total number of iterations (top) and (ii) the total execution time (bottom). First, second and third column corresponds to $C_{s\text{am}} = 3, 6$ and $10$, respectively. For all cases, $r = 1$ (pure state setting) and $q = 10$. Initial point is $U_0 = \Pi_C(U_0)$ such that $X_0 = \tilde{U}_0 \tilde{U}_0^\top$ where $X_0 = \Pi_{+}(-A^*(y))$. are described in the caption of each figure. Table 1 contains information regarding total time required for convergence and quality of solution for all these cases. Results on almost pure density states, i.e., $r > 1$, are provided in Figure 8.

For completeness, we also provide results that illustrate the effect of initialization. In this case, we consider a random initialization and the same initial point is used for all algorithms. Some results are illustrated in Figure 10, table 2 contains metadata of these experiments. Similar to above, ProjFID shows competitive behavior by finding a better solution faster, irrespective of initialization point.
Figure 8: **Quantum state tomography:** Convergence performance of algorithms under comparison w.r.t. $\|\hat{X} - X^*\|_F$ vs. (i) the total number of iterations (top) and (ii) the total execution time (bottom). First, second and third column corresponds to $C_{\text{sam}} = 3, 6$ and $10$, respectively. For all cases, $r = 1$ (pure state setting) and $q = 12$. Initial point is $U_0 = \Pi_{C}(U_0)$ such that $X_0 = \hat{U}_0 \hat{U}_0^\dagger$ where $X_0 = \Pi_+(-A^*(y))$.

Figure 9: **Quantum state tomography:** Convergence performance of algorithms under comparison w.r.t. $\|\hat{X} - X^*\|_F$ vs. (i) the total number of iterations (left) and (ii) the total execution time (right). The two left plots correspond to the case $r = 5$ and the two right plots to the case $r = 20$. In all cases $C_{\text{sam}} = 3$ and $q = 10$. Initial point is $U_0 = \Pi_{C}(U_0)$ such that $X_0 = \hat{U}_0 \hat{U}_0^\dagger$ where $X_0 = \Pi_+(-A^*(y))$. 

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Algorithm & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time \\
RSVP & 5.1496e-05 & 0.7848 & 1.8550e-05 & 0.3791 & 6.6328e-06 & 0.1203 \\
SparseApproxSDP & 4.6323e-03 & 3.7404 & 2.2469e-03 & 4.3775 & 1.4776e-03 & 3.8536 \\
AccUniPDGrad & 4.0388e-05 & 0.3634 & 2.064e-05 & 0.3311 & 1.9032e-05 & 0.4911 \\
ProjFGD & 2.4116e-05 & 0.0599 & 1.6052e-05 & 0.0441 & 1.1419e-05 & 0.0446 \\

Algorithm & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time \\
RSVP & 1.5774e-04 & 5.7347 & 5.2470e-05 & 3.8649 & 2.958e-04 & 4.6548 \\
SparseApproxSDP & 4.6323e-03 & 16.1074 & 2.2469e-03 & 33.7608 & 1.7631e-03 & 85.0633 \\
AccUniPDGrad & 3.5122e-05 & 1.1006 & 2.4634e-05 & 1.8428 & 1.7719e-05 & 3.9440 \\
ProjFGD & 2.4388e-05 & 0.6918 & 1.5431e-05 & 0.8989 & 1.0561e-05 & 1.8804 \\

Algorithm & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time & \|\hat{X} - X^\star\|_F / \|X^\star\|_F & Total time \\
RSVP & 4.5667e-04 & 545.5525 & 1.8500e-05 & 0.3791 & 1.5774e-04 & 5.7347 \\
SparseApproxSDP & 3.7592e-03 & 646.3486 & 2.2469e-03 & 4.3775 & 4.1639e-03 & 16.1074 \\
AccUniPDGrad & 3.6455e-05 & 24.8531 & 2.064e-05 & 0.3311 & 3.5122e-05 & 1.1006 \\
ProjFGD & 7.0096e-06 & 19.5502 & 1.6052e-05 & 0.0441 & 2.4388e-05 & 0.6918 \\

Table 1: **Quantum state tomography:** Summary of comparison results for reconstruction and efficiency. As a stopping criterion, we used \( \|X_i - X_i\|_2 / \|X_{i+1}\|_2 \leq 5 \cdot 10^{-6} \), where \( X_i \) is the estimate at the \( i \)-th iteration. Time reported is in seconds. Initial point is \( U_0 = \Pi \mathcal{C}(\tilde{U}_0) \) such that \( X_0 = \tilde{U}_0 \tilde{U}_0^\top \) where \( \tilde{X}_0 = \Pi_+ (-A^*(y)) \).

Table 2: **Quantum state tomography:** Summary of comparison results for reconstruction and efficiency for random initialization. As a stopping criterion, we used \( \|X_i - X_i\|_2 / \|X_{i+1}\|_2 \leq 5 \cdot 10^{-6} \), where \( X_i \) is the estimate at the \( i \)-th iteration. Time reported is in seconds.
Figure 10: **Quantum state tomography:** Convergence performance of algorithms under comparison w.r.t. $\|X - X^*\|_F$ vs. (i) the total number of iterations (left) and (ii) the total execution time (right). All results correspond to executions starting from a **random initialization** (but common to all algorithms). In all cases $r = 1$ and $q = 10$. 
7 Proofs of local convergence of the ProjFGD

Here, we present the full proof of Theorem 3.1. For clarity, we re-state the problem settings: We consider problem cases such as

$$\min_{X \in \mathbb{R}^{n \times n}} f(X) \quad \text{subject to} \quad X \succeq 0, \ X \in C'.$$  \hspace{1cm} (15)

We assume the optimum $X^*$ satisfies $\text{rank}(X^*) = r^*$. For our analysis, we assume we know $r^*$ and set $r^* = r$. We solve (15) in the factored space, by considering the criterion:

$$\min_{U \in \mathbb{R}^{n \times r}} f(UU^\top) \quad \text{subject to} \quad U \in C.$$  \hspace{1cm} (16)

By faithfulness of $C$ (Definition 2.4), we assume that $E \subseteq C$. This means that the feasible set $C$ in (16) contains all matrices $U^*$ that lead to $X^* = U^*U^*\top$ in (15). Moreover, we assume both $C, C'$ are convex sets and there exists a "mapping" of $C'$ onto $C$, such that the two constraints are "equivalent": for any $U \in C$, we are guaranteed that $X = UU^\top \in C'$. We restrict our discussion on norm-based sets for $C$ such that (16) is satisfied. As a representative example, in our analysis consider the case where, for any $X = UU^\top$, $\text{Tr}(X) \leq 1 \Leftrightarrow \|U\|^2 \leq 1$.

For our analysis, we will use the following step sizes:

$$\hat{\eta} = \frac{1}{128(L\|X\|_2 + \|Q\|_2 \|\nabla f(X)\|_2)}, \quad \eta^* = \frac{1}{128(L\|X^\top\|_2 + \|\nabla f(X^\top)\|_2)}.$$  

By Lemma A.5 in [8], we know that $\hat{\eta} \geq \frac{5}{6} \eta$ and $\frac{10}{11} \eta^* \leq \eta \leq \frac{11}{10} \eta^*$. In our proof, we will work with step size $\hat{\eta}$, which is equivalent up to constants to the original step size $\eta$ in the algorithm.

For ease of exposition, we re-define the sequence of updates: $U_t$ is the current estimate in the factored space, $\tilde{U}_{t+1} = U_t - \hat{\eta} \nabla f(X_t)U_t$ is the putative solution after the gradient step (observe that $\tilde{U}_{t+1}$ might belong in $C$), and $U_{t+1} = \Pi_C(\tilde{U}_{t+1})$ is the projection step onto $C$. Observe that for the constraint cases we consider in this paper, $U_{t+1} = \Pi_C(\tilde{U}_{t+1}) = \xi_t(\tilde{U}_{t+1}) \cdot \tilde{U}_{t+1}$, where $\xi_t(\cdot) \in (0, 1)$; in the case $\xi_t(\cdot) = 1$, the algorithm boils down to the FGD algorithm. For simplicity, we drop the subscript and the parenthesis of the $\xi$ parameter; these values are apparent from the context.

We assume that ProjFGD is initialized with a “good” starting point $X_0 = U_0U_0^\top$, such that:

(A1) $U_0 \in C$ and $\text{Dist}(U_0, U^*) \leq \rho(\sigma_r(U^*))$ for $\rho(\cdot) = c' \cdot \frac{\sigma_r(U^*)}{\sigma_1(U^*)}$, where $c' \leq \frac{1}{200}$.

By the assumptions above, $X_0 = U_0U_0^\top \in C'$. Next, we show that the above lead to a local convergence result. A practical initialization procedure is given in Section 8 and follows from [3]; this also is used in the experimental section 4.

7.1 Proof of Theorem 3.1

For our analysis, we make use of the following lemma [10, Chapter 3], which characterizes the effect of projections onto convex sets w.r.t. to inner products, as well as provides a type-of triangle inequality for such projections; see also Figure 11 for a simple illustration.

Lemma 7.1. Let $U \in C \subseteq \mathbb{R}^{n \times r}$ and $V \in \mathbb{R}^{n \times r}$ where $V \notin C$. Then,

$$\langle \Pi_C(V) - U, V - \Pi_C(V) \rangle \geq 0.$$  \hspace{1cm} (17)
Proof of Theorem 3.1. We start with the following series of (in)equalities:

\[
\text{DIST} \left( U_{t+1}, \ U^* \right)^2 = \min_{R \in \mathcal{D}} \| U_{t+1} - U^* R \|_F^2
\]

\[
\overset{(i)}{\leq} \| U_{t+1} - U^* R_{U_{t+1}}^* \|_F^2
\]

\[
\overset{(ii)}{=} \| U_{t+1} - \tilde{U}_{t+1} + \tilde{U}_{t+1} - U^* R_{U_{t+1}}^* \|_F^2
\]

\[
= \| U_{t+1} - \tilde{U}_{t+1} \|_F^2 + \| \tilde{U}_{t+1} - U^* R_{U_{t+1}}^* \|_F^2
\]

\[
+ 2 \left\langle U_{t+1} - \tilde{U}_{t+1}, \ \tilde{U}_{t+1} - U^* R_{U_{t+1}}^* \right\rangle
\]

where (i) is due to the fact $R_{U_{t+1}}^* := \arg\min_{R \in \mathcal{D}} \| U_{t} - U^* R \|_F^2$, (ii) is obtained by adding and subtracting $\tilde{U}_{t+1}$.

Focusing on the second term of the right hand side, we substitute $\tilde{U}_{t+1}$ to obtain:

\[
\| \tilde{U}_{t+1} - U^* R_{U_{t+1}}^* \|_F^2 = \| U_t - \hat{\eta} \nabla f \left( U_t U_t^T \right) U_t - U^* R_{U_{t+1}}^* \|_F^2
\]

\[
= \| U_t - U^* R_{U_{t+1}}^* \|_F^2 + \hat{\eta}^2 \| \nabla f \left( U_t U_t^T \right) U_t \|_F^2
\]

\[
- 2 \hat{\eta} \left\langle \nabla f \left( U_t U_t^T \right) U_t, U_t - U^* R_{U_{t+1}}^* \right\rangle
\]

Then, our initial equation transforms into:

\[
\text{DIST} \left( U_{t+1}, \ U^* \right)^2 \leq \| U_{t+1} - \tilde{U}_{t+1} \|_F^2 + \text{DIST} \left( U_t, \ U^* \right)^2 + \hat{\eta}^2 \| \nabla f \left( U_t U_t^T \right) U_t \|_F^2
\]

\[
- 2 \hat{\eta} \left\langle \nabla f \left( U_t U_t^T \right) U_t, U_t - U^* R_{U_{t+1}}^* \right\rangle
\]

\[
+ 2 \left\langle U_{t+1} - \tilde{U}_{t+1}, \ \tilde{U}_{t+1} - U^* R_{U_{t+1}}^* \right\rangle
\]

Focusing further on the last term of the expression above, we obtain:

\[
\left\langle U_{t+1} - \tilde{U}_{t+1}, \ \tilde{U}_{t+1} - U^* R_{U_{t+1}}^* \right\rangle = \left\langle U_{t+1} - \tilde{U}_{t+1}, \ \tilde{U}_{t+1} - U_{t+1} + U_{t+1} - U^* R_{U_{t+1}}^* \right\rangle
\]

\[
= \left\langle U_{t+1} - \tilde{U}_{t+1}, \ \tilde{U}_{t+1} - U_{t+1} \right\rangle
\]

\[
+ \left\langle U_{t+1} - \tilde{U}_{t+1}, \ U_{t+1} - U^* R_{U_{t+1}}^* \right\rangle
\]

Observe that, in the special case where $\tilde{U}_{t+1} \equiv U_{t+1}$ for all $t$, i.e., the iterates are always within $\mathcal{C}$ before the projection step, the above equation equals to zero and the recursion is identical to that of [8] [Proof of Theorem 4.2]. Here, we are more interested in the case where $\tilde{U}_{t+1} \neq U_{t+1}$ for some $t$—thus $\tilde{U}_{t+1} \not\in \mathcal{C}$. By faithfulness (Definition 2.1), observe that $U^* R_{U_{t+1}}^* \in \mathcal{C}$ and $X^* = U^* R_{U_{t+1}}^* \left( U^* R_{U_{t+1}}^* \right)^\top = U^* U^\top$. Moreover, $U_{t+1} = \Pi_{\mathcal{C}}(\tilde{U}_{t+1})$: Then, according to Lemma 7.1 and focusing on eq. (17), for $U := U^* R_{U_{t+1}}^*$ and $V := \tilde{U}_{t+1}$, the last term in the above equation satisfies:

\[
\left\langle U_{t+1} - \tilde{U}_{t+1}, \ U_{t+1} - U^* R_{U_{t+1}}^* \right\rangle \leq 0,
\]

and, thus, the expression above becomes:

\[
\left\langle U_{t+1} - \tilde{U}_{t+1}, \ \tilde{U}_{t+1} - U^* R_{U_{t+1}}^* \right\rangle \leq -\| U_{t+1} - \tilde{U}_{t+1} \|_F^2.
\]
Therefore, going back to the original recursive expression, we obtain:

\[
\text{DIST} (U_{t+1}, U^*)^2 \leq -\|U_{t+1} - \tilde{U}_{t+1}\|_F^2 + \text{DIST} (U_t, U^*)^2 + \tilde{\eta}^2 \|\nabla f \left( U_t U_t^\top \right) U_t \|_F^2 \\
- 2\tilde{\eta} \left( \nabla f \left( U_t U_t^\top \right) U_t, U_t - U^* R_{U_t^\top}^* \right)
\]

For the last term, we use the descent lemma [3.3] in the main text; the proof is provided in Section 7.2. Thus, we can conclude that:

\[
\text{DIST} (U_{t+1}, U^*)^2 \leq \left( 1 - \frac{3\tilde{\eta} \mu}{10} \cdot \sigma_r(X^*) \right) \cdot \text{DIST}(U_t, U^*)^2.
\]

The expression for \( \alpha \) is obtained by observing \( \tilde{\eta} \geq \frac{5}{6} \eta \) and \( \frac{10}{11} \eta^* \leq \eta \leq \frac{11}{10} \eta^* \), from Lemma 20 in [8]. Then, for \( \eta^* \leq \frac{1}{L} \|X^*\|_2 + \|\nabla f(X^*)\|_2 \) and \( C = 1/128 \), we have:

\[
1 - \frac{3\tilde{\eta} \mu}{10} \cdot \sigma_r(X^*) \leq 1 - \frac{3 \cdot \frac{10}{11} \cdot \frac{5}{6} \eta^* \mu}{10} \cdot \sigma_r(X^*) = 1 - \frac{15 \eta^* \mu}{66} \cdot \sigma_r(X^*)
\]

\[
= 1 - \frac{15 \mu \sigma_r(X^*)}{66 \cdot 128 (L \|X^*\|_2 + \|\nabla f(X^*)\|_2)} \leq 1 - \frac{1}{550 (L \|X^*\|_2 + \|\nabla f(X^*)\|_2)} =: \alpha
\]

where \( \alpha < 1 \).

Concluding the proof, the condition \( \text{DIST}(U_{t+1}, U^*)^2 \leq \rho' \sigma_r(U^*) \) is naturally satisfied, since \( \alpha < 1 \).

\[\Box\]

### 7.2 Proof of Lemma 3.3

First we recall the definition of restricted strong convexity:

**Definition 7.2.** Let \( f : \mathbb{R}^{n \times n} \to \mathbb{R} \) be convex and differentiable. Then, \( f \) is \((\mu, r)\)-restricted strongly convex if:

\[
f(Y) \geq f(X) + \langle \nabla f(X), Y - X \rangle + \frac{\mu}{2} \|Y - X\|_F^2, \quad \forall X, Y \in \mathbb{R}^{n \times n}, \text{rank-}r \text{ matrices.}
\] (18)

The statements below apply also for standard \( \mu \)-strong convex functions, as defined in Definition 2.2.

Recall \( \tilde{U}_{t+1} = U_t - \tilde{\eta} \nabla f(X_t)U_t \) and define \( \Delta := U_t - U^* R_{U_t^\top}^* \). Before presenting the proof, we need the following lemma that bounds one of the error terms arising in the proof of Lemma 3.3. This is a variation of Lemma 6.3 in [8]. The proof is presented in Section 7.3.

**Lemma 7.3.** Let \( f \) be \( L \)-smooth and \((\mu, r)\)-restricted strongly convex. Then, under the assumptions of Theorem 3.1 and assuming step size \( \tilde{\eta} = \frac{1}{128 (L \|X^*\|_2 + \|\nabla f(X^*)\|_2)} \), the following bound holds true:

\[
\langle \nabla f(X_t), \Delta \Delta^\top \rangle \geq -\frac{\tilde{\eta}}{2} \|\nabla f(X_t)U_t\|_F^2 \leq -\frac{\mu \sigma_r(X^*)}{10} \cdot \text{DIST}(U_t, U^*)^2.
\] (19)

Now we are ready to present the proof of Lemma 3.3.
Proof of Lemma 3.3 First we rewrite the inner product as shown below.

\[
\langle \nabla f(X_t)U_t, U_t - U^*R^*_tU_t^\top \rangle = \frac{1}{2} \langle \nabla f(X_t), X_t - U^*R^*_tU_t^\top \rangle
\]

\[
= \frac{1}{2} \langle \nabla f(X_t), X_t - X^* \rangle + \frac{1}{2} \langle X_t + X^* - U^*R^*_tU_t^\top, U_t \rangle
\]

\[
= \frac{1}{2} \langle \nabla f(X_t), X_t - X^* \rangle + \frac{1}{2} \langle \nabla f(X_t), \Delta \Delta^\top \rangle,
\]  

(20)

which follows by adding and subtracting \( \frac{1}{2} X^* \).

Let us focus on bounding the first term on the right hand side of (20). Consider points \( X_t = U_t U_t^\top \) and \( X_{t+1} = U_{t+1} U_{t+1}^\top \); by assumption, both \( X_t \) and \( X_{t+1} \) are feasible points in (16). By smoothness of \( f \), we get:

\[
f(X_t) \geq f(X_{t+1}) - \langle \nabla f(X_t), X_{t+1} - X_t \rangle - \frac{\eta}{2} \|X_{t+1} - X_t\|_F^2.
\]

(21)

where (i) follows from optimality of \( X^* \) and since \( X_{t+1} \) is a feasible point \( (X_{t+1} \geq 0, \Pi_C(X_{t+1}) = X_{t+1}) \) for problem (15).

Moreover, by the \((\mu, r)\)-restricted strong convexity of \( f \), we get,

\[
f(X^*) \geq f(X_t) + \langle \nabla f(X_t), X^* - X_t \rangle + \frac{\eta}{2} \|X^* - X_t\|_F^2.
\]

(22)

Combining equations (21), and (22), we obtain:

\[
\langle \nabla f(X_t), X_t - X^* \rangle \geq \langle \nabla f(X_t), X_t - X_{t+1} \rangle - \frac{\eta}{2} \|X_{t+1} - X_t\|_F^2 + \frac{\eta}{2} \|X^* - X_t\|_F^2
\]

(23)

By the nature of the projection \( \Pi_C(\cdot) \) step, it is easy to verify that

\[
X_{t+1} = \xi^2 \cdot \left( X_t - \hat{\eta} \nabla f(X_t)X_t\Lambda_t - \hat{\eta} \Lambda_t^\top X_t^\top \nabla f(X_t)^\top \right),
\]

where \( \Lambda_t = I - \hat{\eta} Q_{U_t} Q_{U_t}^\top \nabla f(X_t) \in \mathbb{R}^{n \times n} \) and \( Q_{U_t} Q_{U_t}^\top \) denoting the projection onto the column space of \( U_t \). Notice that, for step size \( \hat{\eta} \), we have

\[
\Lambda_t > 0, \quad \sigma_1(\Lambda_t) \leq 1 + 1/256 \quad \text{and} \quad \sigma_n(\Lambda_t) \geq 1 - 1/256.
\]

Using the above \( X_{t+1} \) characterization in (23), we obtain:

\[
\langle \nabla f(X_t), X_t - X^* \rangle - \frac{\eta}{2} \|X^* - X_t\|_F^2 + \frac{\eta}{2} \|X_t - X_{t+1}\|_F^2
\]

\[
\geq \langle \nabla f(X_t), (1 - \xi^2) X_t \rangle + 2\hat{\eta} \cdot \xi^2 \cdot \langle \nabla f(X_t), \nabla f(X_t)X_t\Lambda_t \rangle
\]

\[
\geq (1 - \xi^2) \cdot \langle \nabla f(X_t)U_t, U_t \rangle + 2\hat{\eta} \cdot \xi^2 \cdot \text{Tr}(\nabla f(X_t)\nabla f(X_t)X_t) \cdot \sigma_n(\Lambda_t)
\]

\[
\geq (1 - \xi^2) \cdot \langle \nabla f(X_t)U_t, U_t \rangle + \frac{255\cdot\hat{\eta} \cdot \xi^2}{128} \|\nabla f(X_t)U_t\|_F^2,
\]

(24)

where: (i) follows from symmetry of \( \nabla f(X_t) \) and \( X_t \) and, (ii) follows from the sequence equalities an inequalities:

\[
\text{Tr}(\nabla f(X_t)\nabla f(X_t)X_t\Lambda_t) = \text{Tr}(\nabla f(X_t)\nabla f(X_t)U_tU_t^\top) - \frac{\eta}{2} \text{Tr}(\nabla f(X_t)\nabla f(X_t)U_tU_t^\top \nabla f(X_t))
\]

\[
\geq \left( 1 - \frac{\eta}{2}\|Q_{U_t} Q_{U_t}^\top \nabla f(X_t)\|_2 \right) \|\nabla f(X_t)U_t\|_F^2
\]

\[
\geq (1 - 1/256) \|\nabla f(X_t)U_t\|_F^2.
\]

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Combining the above in the expression we want to lower bound: $2\tilde{\eta}\langle \nabla f(X_t) \cdot U_t, U_t - U^* R_{U_t}^* \rangle + \|U_{t+1} - \tilde{U}_{t+1}\|_F^2$, we obtain:

$$
2\tilde{\eta}\langle \nabla f(X_t) \cdot U_t, U_t - U^* R_{U_t}^* \rangle + \|U_{t+1} - \tilde{U}_{t+1}\|_F^2 \\
= \tilde{\eta}\langle \nabla f(X_t), X_t - X^* \rangle + \tilde{\eta}\langle \nabla f(X_t), \Delta \Delta^\top \rangle + \|U_{t+1} - \tilde{U}_{t+1}\|_F^2 \\
\geq (1 - \xi^2) \cdot \tilde{\eta}\langle \nabla f(X_t)U_t, U_t \rangle + \frac{255\xi^2 \xi^2}{128}\|\nabla f(X_t)U_t\|_F^2 \\
+ \frac{\bar{\eta}u}{2}\|X^* - X_t\|_F^2 - \frac{\bar{\eta}L}{2}\|X_t - X_{t+1}\|_F^2 \\
- \frac{\xi^2}{5}\|\nabla f(X_t)U_t\|_F^2 - \frac{\bar{\eta}u\sigma(X^*)}{10}\cdot \text{DIST}(U_t, U^*)^2 \\
+ \|U_{t+1} - \tilde{U}_{t+1}\|_F^2 \\
(25)
$$

For the last term in the above expression and given \(U_{t+1} = \Pi_C (\tilde{U}_{t+1}) = \xi \cdot \tilde{U}_{t+1}\) for some \(\xi \in (0, 1)\), we further observe:

$$
\|U_{t+1} - \tilde{U}_{t+1}\|_F^2 = \|\xi \cdot \tilde{U}_{t+1} - \tilde{U}_{t+1}\|_F^2 \\
= (1 - \xi)^2 \cdot \|U_t\|_F^2 + (1 - \xi)^2 \tilde{\eta}^2 \cdot \|\nabla f(X_t)U_t\|_F^2 \\
- 2 (1 - \xi)^2 \cdot \tilde{\eta} \cdot \langle \nabla f(X_t)U_t, U_t \rangle
$$

Combining the above equality with the first term on the right hand side in (25), we obtain:

$$
(1 - \xi^2) \cdot \tilde{\eta}\langle \nabla f(X_t)U_t, U_t \rangle + (1 - \xi^2) \cdot \|U_t\|_F^2 + (1 - \xi^2) \tilde{\eta}^2 \cdot \|\nabla f(X_t)U_t\|_F^2 \\
- 2 (1 - \xi^2) \cdot \tilde{\eta} \cdot \langle \nabla f(X_t)U_t, U_t \rangle \\
= \left[(1 - \xi^2) - 2(1 - \xi)\right] \cdot \tilde{\eta}\langle \nabla f(X_t)U_t, U_t \rangle + (1 - \xi^2) \cdot \|U_t\|_F^2 + (1 - \xi^2) \tilde{\eta}^2 \cdot \|\nabla f(X_t)U_t\|_F^2 \\
= (3\xi - 1)(1 - \xi) \cdot \tilde{\eta}\langle \nabla f(X_t)U_t, U_t \rangle + (1 - \xi^2) \cdot \|U_t\|_F^2 + (1 - \xi^2) \tilde{\eta}^2 \cdot \|\nabla f(X_t)U_t\|_F^2 \\
= \left\|\frac{3\xi - 1}{2} U_t + (1 - \xi) \cdot \tilde{\eta}\nabla f(X_t) \cdot U_t \right\|_F^2 + (1 - \xi^2)^2 \cdot \|U_t\|_F^2.
$$

Focusing on the first term, let \(\Theta_t := I + \frac{2(1 - \xi)}{3\xi} \cdot \tilde{\eta} \cdot \nabla f(X_t)Q_{U_t}Q_{U_t}^\top\); then, \(\sigma_n(\Theta_t) \geq 1 - \frac{2(1 - \xi)}{3\xi} \cdot \frac{1}{128}\), by the definition of \(\tilde{\eta}\) and the fact that \(\tilde{\eta} \leq \frac{1}{128\|\nabla f(X_t)Q_{U_t}Q_{U_t}^\top\|_2^2}\). Then:

$$
\left\|\frac{3\xi - 1}{2} U_t + (1 - \xi) \cdot \tilde{\eta}\nabla f(X_t) \cdot U_t \right\|_F^2 = \left\|\frac{3\xi - 1}{2} \Theta_t \cdot U_t \right\|_F^2 \\
\geq \left\|\frac{3\xi - 1}{4} \cdot \Theta_t \right\|_F^2 \cdot \sigma_n(\Theta_t)^2 \\
\geq \left\|\frac{3\xi - 1}{4} \cdot \left(1 - \frac{2(1 - \xi)}{3\xi} \cdot \frac{1}{128}\right)^2 \cdot \|U_t\|_F^2
$$

Combining the above, we obtain the following bound:

$$
(1 - \xi^2) \cdot \tilde{\eta}\langle \nabla f(X_t)U_t, U_t \rangle + \|U_{t+1} - \tilde{U}_{t+1}\|_F^2 \\
\geq \left(1 - (\xi^2) - \frac{(3\xi - 1)^2}{4} \cdot \left(1 - \frac{2(1 - \xi)}{3\xi} \cdot \frac{1}{128}\right)^2 \right) \cdot \|U_t\|_F^2
$$

The above transform (25) as follows:

$$
2\tilde{\eta}\langle \nabla f(X_t) \cdot U_t, U_t - U^* R_{U_t}^* \rangle + \|U_{t+1} - \tilde{U}_{t+1}\|_F^2 \\
\geq \left(\frac{255\xi^2}{128} - \frac{1}{5}\right) \cdot \tilde{\eta}^2 \|\nabla f(X_t)U_t\|_F^2 + \frac{\bar{\eta}u}{2}\|X^* - X_t\|_F^2 - \frac{\bar{\eta}u\sigma(X^*)}{10}\cdot \text{DIST}(U_t, U^*)^2 \\
+ \left(1 - (\xi^2) - \frac{(3\xi - 1)^2}{4} \cdot \left(1 - \frac{2(1 - \xi)}{3\xi} \cdot \frac{1}{128}\right)^2 \right) \cdot \|U_t\|_F^2 - \frac{\bar{\eta}L}{2}\|X_t - X_{t+1}\|_F^2 \\
(26)
$$
Let us focus on the term $\frac{\tilde{\eta}}{2} \|X_t - X_{t+1}\|_F^2$; this can be bounded as follows:

$$\frac{\tilde{\eta}}{2} \|X_t - X_{t+1}\|_F^2 = \frac{\tilde{\eta}}{2} \|U_t U_t^\top - U_{t+1} U_{t+1}^\top\|_F^2 = \frac{\tilde{\eta}}{2} \|U_t U_t^\top - U_t U_{t+1}^\top + U_t U_{t+1}^\top - U_{t+1} U_{t+1}^\top\|_F^2$$

$$\leq \frac{\tilde{\eta}}{2} \|U_t (U_t - U_{t+1})^\top + (U_t - U_{t+1}) U_{t+1}^\top\|_F^2$$

$$\leq \eta L \left( \|U_t (U_t - U_{t+1})^\top\|_F^2 + \| (U_t - U_{t+1}) U_{t+1}^\top\|_F^2 \right)$$

$$\leq \eta L \left( \|U_t + 1\|_F^2 + \|U_t - 1\|_F^2 \right)$$

where $(i)$ is due to the identity $\|A + B\|_F^2 \leq 2\|A\|_F^2 + 2\|B\|_F^2$ and $(ii)$ is due to the Cauchy-Schwarz inequality. By definition of $U_{t+1}$, we observe that:

$$\|U_{t+1}\|_F^2 = \|\xi \cdot (U_t - \tilde{\eta} \nabla f(X_t) U_t)\|_F^2 \leq \eta^2 \cdot \|U_t\|_F^2 \cdot \|I - \tilde{\eta} \nabla f(X_t) Q_{U_t} Q_{U_t}^\top\|_F^2 \leq (1 + \frac{1}{128})^2 \cdot \|U_t\|_F^2.$$

where $(i)$ is due to Cauchy-Schwarz and $(ii)$ is obtained by substituting $\tilde{\eta} \leq \frac{1}{128 \|\nabla f(X_t) Q_{U_t} Q_{U_t}^\top\|_2}$ and since $\xi \in (0, 1)$. Thus, $\frac{\tilde{\eta}}{2} \|X_t - X_{t+1}\|_F^2$ can be further bounded as follows:

$$\frac{\tilde{\eta}}{2} \|X_t - X_{t+1}\|_F^2 \leq \tilde{\eta} L \cdot \left( 1 + \frac{1}{128} \right)^2 \cdot \|U_t\|_F^2 \cdot \|U_{t+1} - U_t\|_F^2$$

$$\leq \frac{\left( 1 + \frac{1}{128} \right)^2 + 1}{128} \cdot \|U_{t+1} - U_t\|_F^2$$

$$= \frac{128}{1 + \frac{1}{128}} \cdot \|U_{t+1} - U_t\|_F^2$$

$$= \|\xi \cdot \tilde{U}_{t+1} - U_t\|_F^2$$

$$\leq (1 - \xi)^2 \cdot \left( 1 + \frac{1}{128} \right)^2 \cdot \|U_t\|_F^2 + \frac{1}{128} \cdot \|\nabla f(X_t) \cdot U_t\|_F^2$$

where in the last inequality we substitute $\tilde{\eta}$; observe that $\tilde{\eta} \leq \frac{1}{128 L \|X_t\|_2}$ Combining this result with [20], we obtain:

$$2\tilde{\eta} \langle \nabla f(X_t) \cdot U_t, U_t - U^* R_{U_t}^\top + \tilde{U}_{t+1} \|U_{t+1} - \tilde{U}_{t+1}\|_F^2$$

$$\geq \left( \frac{255 \xi^2 - \frac{1}{4}}{128} - \frac{1}{128} \right) \cdot \xi \cdot \|\nabla f(X_t) U_t\|_F^2 + \frac{\tilde{\eta}}{2} \|X_t - X_{t+1}\|_F^2 - \frac{\tilde{\eta} \sigma_{2}(X^*)}{10} \cdot \text{Dist}(U_t, U^*)^2$$

$$\geq \left( \frac{255 \xi^2 - \frac{1}{4}}{128} - \frac{1}{128} \right) \cdot \xi \cdot \|\nabla f(X_t) U_t\|_F^2 + \frac{\tilde{\eta}}{2} \|X_t - X_{t+1}\|_F^2$$

where $(i)$ is due to the assumption $\xi \geq 0.78$ and thus $\left( \frac{255 \xi^2 - \frac{1}{4}}{128} - \frac{1}{128} \right) \cdot \xi^2 \geq 1$; see also Figure 12 (left panel), and $(ii)$ is due to the non-negativity of the constant in front of $\|U_t\|_F^2$; see also Figure 12 (right panel).

Finally, we bound $\frac{\tilde{\eta}}{2} \|X^* - X_t\|_F^2$ using the following Lemma by [41]:

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Lemma 7.4. For any $U, V \in \mathbb{R}^{n \times r}$, we have:

$$\|UU^T - VV^T\|^2 \geq 2 \cdot \left(\sqrt{2} - 1\right) \cdot \sigma_r(U)^2 \cdot \text{DIST}(U, V)^2.$$ 

Thus,

$$\tilde{\eta}_\mu \|X^* - X_i\|^2_F \geq \tilde{\eta}_\mu \cdot \left(\sqrt{2} - 1\right) \cdot \sigma_r(X^*) \cdot \text{DIST}(U_i, U^*)^2,$$

and can thus conclude:

$$2\tilde{\eta}_\mu \langle \nabla f(X_i) \cdot U_t, U_t - U^* R_{U_t}^* \rangle \right. + \left. \|U_{t+1} - \tilde{U}_{t+1}\|^2_F \right)$$

$$\geq \tilde{\eta}_\mu^2 \|\nabla f(X_i) U_t\|^2_F + \tilde{\eta}_\mu \cdot \left(\sqrt{2} - 1\right) \cdot \sigma_r(X^*) \cdot \text{DIST}(U_t, U^*)^2 - \frac{\tilde{\eta}_\mu \sigma_r(X^*)}{10} \cdot \text{DIST}(U_t, U^*)^2$$

$$= \tilde{\eta}_\mu^2 \|\nabla f(X_i) U_t\|^2_F + \left(\sqrt{2} - 1 - \frac{1}{10}\right) \cdot \tilde{\eta}_\mu \cdot \sigma_r(X^*) \cdot \text{DIST}(U_t, U^*)^2$$

$$= \tilde{\eta}_\mu^2 \|\nabla f(X_i) U_t\|^2_F + \frac{3\tilde{\eta}_\mu}{10} \cdot \sigma_r(X^*) \cdot \text{DIST}(U_t, U^*)^2$$

This completes the proof.

7.3 Proof of Lemma 7.3

Proof. We can lower bound $\langle \nabla f(X_i), \Delta \Delta^\top \rangle$ as follows:

$$\left\langle \nabla f(X_i), \Delta \Delta^\top \right\rangle \overset{(i)}{=} \left\langle Q_\Delta Q_\Delta^\top \nabla f(X_i), \Delta \Delta^\top \right\rangle$$

$$\geq -\left| \text{Tr} \left( Q_\Delta Q_\Delta^\top \nabla f(X_i) \Delta \Delta^\top \right) \right|$$

$$\overset{(ii)}{\geq} -\|Q_\Delta Q_\Delta^\top \nabla f(X_i)\|_2 \text{Tr}(\Delta \Delta^\top)$$

$$\overset{(iii)}{\geq} -\left( \|Q_U Q_U^\top \nabla f(X_i)\|_2 + \|Q_U Q_U^\top \nabla f(X_i)\|_2 \right) \text{DIST}(U_t, U^*)^2.$$  \hfill (28)

Note that (i) follows from the fact $\Delta = Q_\Delta Q_\Delta^\top$ and (ii) follows from $|\text{Tr}(AB)| \leq \|A\|_2 \text{Tr}(B)$, for PSD matrix $B$ (Von Neumann’s trace inequality [30]). For the transformation in (iii), we use that fact that the column space of $\Delta$, $\text{SPAN}(\Delta)$, is a subset of $\text{SPAN}(U_t \cup U^*)$, as $\Delta$ is a linear combination of $U_t$ and $U^* R_{U_t}$. 

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For the second term in the parenthesis above, we first derive the following inequalities; their use is apparent later on:

\[
\|\nabla f(X_t)U^*\|_2 \leq \|\nabla f(X_t)U_t\|_2 + \|\nabla f(X_t)\Delta\|_2
\]

\[
\leq \|\nabla f(X_t)U_t\|_2 + \|\nabla f(X_t)Q\Delta Q^\top\|_2\|\Delta\|_2
\]

\[
\leq \|\nabla f(X_t)U_t\|_2 + \left(\|\nabla f(X_t)Q_UQ_U^\top\|_2 + \|\nabla f(X_t)Q_UQ_U^\top\|_2\right)\|\Delta\|_2
\]

\[
\leq \|\nabla f(X_t)U_t\|_2 + \left(\|\nabla f(X_t)Q_UQ_U^\top\|_2 + \|\nabla f(X_t)Q_UQ_U^\top\|_2\right)\frac{1}{200}\sigma_r(U^*)
\]

\[
\leq \frac{1}{200}\|\nabla f(X_t)U_t\|_2 + \frac{1}{200}\|\nabla f(X_t)U^*\|_2.
\]

where (i) is due to triangle inequality on $U^*R^*_U = U_t - \Delta$, (ii) is due to generalized Cauchy-Schwarz inequality; we denote as $Q\Delta Q^\top$ the projection matrix on the column span of $\Delta$ matrix, (iii) is due to triangle inequality and the fact that the column span of $\Delta$ can be decomposed into the column span of $U_t$ and $U^*$, by construction of $\Delta$, (iv) is due to the assumption \(\text{Dist}(U_t, U^*) \leq \rho' \cdot \sigma_r(U^*)\) and \(\|\Delta\|_2 \leq \text{Dist}(U_t, U^*) \leq \frac{1}{200} \cdot \sigma_r(U^*)\).

Finally, (v) is due to the facts:

\[
\|\nabla f(X_t)U^*\|_2 = \|\nabla f(X_t)Q_UQ_U^\top U^*\|_2 \geq \|\nabla f(X_t)Q_UQ_U^\top\|_2 \cdot \sigma_r(U^*),
\]

and

\[
\|\nabla f(X_t)U_t\|_2 = \|\nabla f(X_t)Q_UQ_U^\top U_t\|_2 \geq \|\nabla f(X_t)Q_UQ_U^\top\|_2 \cdot \sigma_r(U_t)
\]

\[
\geq \|\nabla f(X_t)Q_UQ_U^\top\|_2 \cdot \left(1 - \frac{1}{200}\right) \cdot \sigma_r(U^*),
\]

by the proof of (a variant of) Lemma A.3 in [3]. Thus, for the term $\|\nabla f(X_t)Q_UQ_U^\top\|_2$, we have

\[
\|\nabla f(X_t)Q_UQ_U^\top\|_2 \leq \frac{1}{\sigma_r(U^*)} \|\nabla f(X_t)U^*\|_2
\]

\[
\leq \frac{1}{\sigma_r(U^*)} \frac{201}{199} \|\nabla f(X_t)U_t\|_2
\]

\[
\leq \frac{201}{200\sigma_r(U^*)} \frac{201}{199} \|\nabla f(X_t)Q_UQ_U^\top\|_2.
\]

Using (29) in (28), we obtain:

\[
\left< \nabla f(X_t), \Delta \Delta^\top \right> \geq -\left(\|Q_UQ_U^\top \nabla f(X_t)\|_2 + \frac{201}{200\sigma_r(U^*)} \|Q_UQ_U^\top \nabla f(X_t)\|_2\right)\text{Dist}(U_t, U^*)^2
\]

\[
\geq -\frac{21\tau(U^*)}{10} \|Q_UQ_U^\top \nabla f(X_t)\|_2 \cdot \text{Dist}(U_t, U^*)^2
\]

We remind that the step size we use here is: $
\hat{\eta} = \frac{1}{128(L\|X_t\|_2 + \|Q_UQ_U^\top \nabla f(X_t)\|_2)}$. Then, we have:

\[
\frac{21\tau(U^*)}{10} \cdot \|Q_UQ_U^\top \nabla f(X_t)\|_2 \cdot \text{Dist}(U_t, U^*)^2
\]

\[
\leq \frac{21\tau(U^*)}{10} \cdot \hat{\eta} \cdot 128L\|X_t\|_2 \|Q_UQ_U^\top \nabla f(X_t)\|_2 \cdot \text{Dist}(U_t, U^*)^2
\]

\[
+ \frac{21\tau(U^*)}{10} \cdot \hat{\eta} \cdot \|Q_UQ_U^\top \nabla f(X_t)\|_2 \cdot \text{Dist}(U_t, U^*)^2
\]

\[
(30)
\]
To bound the first term on the right hand side, we observe that $\|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2 \leq \frac{\mu \sigma_r(X_t)}{\text{Dist}(U_t, U^*)^{10}}$ or $\|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2 \geq \frac{\mu \sigma_r(X_t)}{\text{Dist}(U_t, U^*)^{10}}$. This results further into:

$$
\frac{21\tau(U^*)^{10}}{10} \cdot \hat{\eta} \cdot 128 \hat{L} \cdot \|X_t\|_2 \cdot \|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2 \cdot \text{Dist}(U_t, U^*)^2 \leq \max \left\{ \frac{21\tau(U^*)^{10}}{10} \cdot 128 \cdot \hat{L} \cdot \|X_t\|_2 \cdot \mu \sigma_r(X_t), \text{Dist}(U_t, U^*)^2 \right\},
$$

$$
\hat{\eta} \left( \frac{21\tau(U^*)^{10}}{10} \right)^2 \cdot 128 \cdot 10\kappa \tau(X_t) \cdot \|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2^2 \cdot \text{Dist}(U_t, U^*)^2 \leq \frac{128 \hat{L} \cdot \|X_t\|_2 \cdot \mu \sigma_r(X_t)}{10} \cdot \text{Dist}(U_t, U^*)^2 + \hat{\eta} \left( \frac{21\tau(U^*)^{10}}{10} \right)^2 \cdot 128 \cdot 10\kappa \tau(X_t) \cdot \|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2^2 \cdot \text{Dist}(U_t, U^*)^2,
$$

where $\kappa := \frac{L}{\mu}$ and $\tau(X) := \frac{\sigma_1(X)}{\sigma_r(X)}$ for a rank-$r$ matrix $X$. Combining the above with (30):

$$
\frac{21\tau(U^*)^{10}}{10} \cdot \|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2 \cdot \text{Dist}(U_t, U^*)^2 \leq \frac{\mu \sigma_r(X_t)}{10} \cdot \text{Dist}(U_t, U^*)^2 + \left( 10\kappa \tau(X_t) \cdot \frac{21\tau(U^*)^{10}}{10} + 1 \right) \cdot \frac{21\tau(U^*)^{10}}{10} \cdot 128 \cdot \hat{\eta} \cdot \|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2^2 \cdot \text{Dist}(U_t, U^*)^2,
$$

$$
\frac{\mu \sigma_r(X_t)}{10} \cdot \text{Dist}(U_t, U^*)^2 + \left( \frac{11\kappa \tau(X^*) \cdot \frac{21\tau(U^*)^{10}}{10} + 1}{10} \right) \cdot \frac{21\tau(U^*)^{10}}{10} \cdot 128 \cdot \hat{\eta} \cdot \|\nabla f(X_t)\|_2^2 \cdot \left( \frac{\mu \sigma_r(X^*)}{10} \cdot \text{Dist}(U_t, U^*)^2 + \frac{\hat{\eta}}{\kappa} \|\nabla f(X_t)\|_2^2 \right),
$$

where (i) follows from $\hat{\eta} \leq \frac{1}{\|\nabla f(X_t)\|_2}$, (ii) is due to Lemma A.3 in [8] and using the bound $\text{Dist}(U_t, U^*) \leq \rho \sigma_r(U^*)$ by the hypothesis of the lemma, (iii) is due to $\sigma_r(X^*) \leq 1.1 \sigma_r(X_t)$ by Lemma A.3 in [8], due to the facts $\sigma_r(X_t) \|Q_{U_t}Q_{U_t}^\top \nabla f(X_t)\|_2^2 \leq \|U_t^\top \nabla f(X_t)\|_2^2$ and $(11\kappa \tau(X^*) \cdot \frac{21\tau(U^*)^{10}}{10} + 1) \leq 12\kappa \tau(X^*) \cdot \frac{21\tau(U^*)^{10}}{10}$, and $\tau(U^*)^2 = \tau(X^*)$. Finally, (iv) follows from substituting $\rho' := \frac{1}{\kappa} \cdot \frac{1}{\tau(X^*)}$ for $c = \frac{1}{2\mu}$ and using Lemma A.3 in [8] (due to the factor $\frac{1}{2\mu}$), all constants above lead to bounding the term with the constant $\frac{1}{2}$.

Thus, we can conclude:

$$
\langle \nabla f(X_t), \Delta \Delta^\top \rangle \geq \left( \frac{\hat{\eta}}{\kappa} \|\nabla f(X_t)\|_2^2 + \frac{\mu \sigma_r(X^*)}{10} \cdot \text{Dist}(U_t, U^*)^2 \right).
$$

This completes the proof. \(\square\)

### 7.4 Proof of Corollary 3.2

We have

$$
\|\tilde{U}_{t+1}\|_F \leq \|U_t\|_F + \hat{\eta} \cdot \|\nabla f(X_t)\|_2 \leq \|U_t\|_F + \hat{\eta} \cdot \|\nabla f(X_t)Q_{U_t}Q_{U_t}^\top\|_2 \cdot \|U_t\|_F = (1 + \hat{\eta} \cdot \|\nabla f(X_t)Q_{U_t}Q_{U_t}^\top\|_2) \cdot \lambda \leq (1 + \frac{1}{128}) \cdot \lambda
$$

where the first inequality follows from the triangle inequality, the second holds by the property $\|AB\|_F \leq \|A\|_2 \cdot \|B\|_F$, and the third follows because the step size is bounded above by $\hat{\eta} \leq \frac{128}{\|\nabla f(X_t)Q_{U_t}Q_{U_t}^\top\|_2}$.

Hence, we get $\xi(\tilde{U}_{t+1}) = \frac{\lambda}{\|\tilde{U}_{t+1}\|_F} \geq \frac{128}{129}$. 

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8 Initialization

In this section, we present a specific initialization strategy for the ProjFDG. For completeness, we repeat the definition of the optimization problem at hand, both in the original space:

$$\min_{X \in \mathbb{R}^{n \times n}} f(X) \quad \text{subject to} \quad X \in C'.$$  \hfill (31)

and the factored space:

$$\min_{U \in \mathbb{R}^{n \times r}} f(U U^\top) \quad \text{subject to} \quad U \in C.$$  \hfill (32)

For our initialization, we restrict our attention to the full rank \((r = n)\) case. Observe that, in this case, \(C'\) is a convex set and includes the full-dimensional PSD cone, as well as other norm constraints, as described in the main text. Let us denote \(\Pi_C'(\cdot)\) the corresponding projection step, where all constraints are satisfied simultaneously. Then, the initialization we propose follows similar motions with that in \([8]\): We consider the projection of the weighted negative gradient at 0, \(i.e., -\frac{1}{L} \cdot \nabla f(0)\), onto \(C'\)\footnote{As in \([8]\), one can approximate easily \(L\), if it is unknown.} \(i.e.,

$$X_0 = U_0 U_0^\top = \Pi_{C'}\left(-\frac{1}{L} \cdot \nabla f(0)\right).$$  \hfill (33)

Assuming a first-oracle model, where we access \(f\) only though function evaluations and gradient calculations, (33) provides a cheap way to find an initial point with some approximation guarantees as follows:\footnote{As we show in the experiments section, a random initialization performs well in practice, without requiring the additional calculations involved in (33). However, a random initialization provides no guarantees whatsoever.}

**Lemma 8.1.** Let \(U_0 \in \mathbb{R}^{n \times n}\) be such that \(X_0 = U_0 U_0^\top = \Pi_{C'}\left(-\frac{1}{L} \cdot \nabla f(0)\right)\). Consider the problem in (32), where \(f\) is assumed to be \(L\)-smooth and \(\mu\)-strongly convex, with optimum point \(X^*\) such that \(\text{rank}(X^*) = n\). We apply ProjFDG algorithm with \(U_0\) as the initial point. Then, in this generic case, \(U_0\) satisfies:

$$\text{Dist}(U_0, U^*) \leq \rho' \cdot \sigma_r(U^*),$$

where \(\rho' = \sqrt{\frac{1 - \eta^2}{2(\sqrt{2} - 1)}} \cdot \tau^2(U^*) \cdot \sqrt{\text{rank}(X^*)} \) and \(\text{rank}(X) = \frac{\|X\|_F}{\|X\|_2}\).

**Proof.** To show this, we start with:

$$\|X_0 - X^*\|_F^2 = \|X^*\|_F^2 + \|X_0\|_F^2 - 2 \langle X_0, X^* \rangle.$$  \hfill (34)

Recall that \(X_0 = U_0 U_0^\top = \Pi_{C'}\left(-\frac{1}{L} \cdot \nabla f(0)\right)\) by assumption, where \(\Pi_C'(\cdot)\) is a convex projection. Then, by Lemma 7.1 we get

$$\langle X_0 - X^*, -\frac{1}{L} \cdot \nabla f(0) - X^* \rangle \Rightarrow \langle -\frac{1}{L} \nabla f(0), X^* - X^* \rangle \geq \langle X^*, X^* - X^* \rangle.$$  \hfill (35)

Observe that 0 is a feasible point, since it is PSD and satisfy any common symmetric norm constraints, as the ones considered in this paper. Hence, using strong convexity of \(f\) around 0, we get,

$$f(X^*) - \frac{\mu}{2} \|X^*\|_F^2 \geq f(0) + \langle \nabla f(0), X^* \rangle$$

\(\overset{(i)}{=} f(0) + \langle \nabla f(0), X_0 \rangle + \langle \nabla f(0), X^* - X_0 \rangle$$

\(\overset{(ii)}{=} f(0) + \langle \nabla f(0), X_0 \rangle + \langle L \cdot X_0, X_0 - X^* \rangle.$$  \hfill (36)
where (i) is by adding and subtracting \(\langle \nabla f(0), X_0 \rangle\), and (ii) is due to (35). Further, using the smoothness of \(f\) around 0, we get:

\[
\begin{align*}
 f(X_0) & \leq f(0) + \langle \nabla f(0), X_0 \rangle + \frac{L}{2} \| X_0 \|^2_F \\
 & \leq f(X^*) - \frac{\mu}{2} \| X^* \|^2_F + \langle L \cdot X_0, X^* \rangle - \frac{\mu}{2} \| X_0 \|^2_F \\
 & \leq f(X_0) - \frac{\mu}{2} \| X^* \|^2_F + (L \cdot X_0, X^*) - \frac{\mu}{2} \| X_0 \|^2_F.
\end{align*}
\]

where (i) follows from (36) by upper bounding the quantity \(f(0) + \langle \nabla f(0), X_0 \rangle\), (ii) follows from the assumption that \(f(X^*) \leq f(X_0)\). Hence, rearranging the above terms, we get:

\[
\langle X_0, X^* \rangle \geq \frac{\mu}{2} \| X_0 \|^2_F + \frac{\mu}{2} \| X^* \|^2_F.
\]

Combining the above inequality with (34), we obtain,

\[
\| X_0 - X^* \|_F \leq \sqrt{1 - \frac{\mu}{L} \cdot \| X^* \|_F}.
\]

Given, \(U_0\) such that \(X_0 = U_0 U_0^\top\) and \(U^*\) such that \(X^* = U^* U^{*\top}\), we use Lemma 7.4 from [44] to obtain:

\[
\| U_0 U_0^\top - U^* U^{*\top} \|_F \geq \sqrt{2(\sqrt{2} - 1)} \cdot \sigma_r(U^*) \cdot \text{DIST}(U_0, U^*).
\]

Thus:

\[
\text{DIST}(U_0, U^*) \leq \frac{\| X_0 - X^* \|_F}{\sqrt{2(\sqrt{2} - 1)} \cdot \sigma_r(U^*)} \cdot \| X^* \|_F \\
\leq \rho' \cdot \sigma_r(U^*)
\]

where \(\rho' = \sqrt{\frac{1 - \mu/L}{2(\sqrt{2} - 1)}} \cdot \tau^2(U^*) \cdot \sqrt{\text{rank}(X^*)}.

Such initialization, while being simple, introduces further restrictions on the condition number \(\tau(X^*)\), and the condition number of function \(f\). Finding such simple initializations with weaker restrictions remains an open problem; however, as shown in [8, 44, 18], one can devise specific deterministic initialization for a given application.

As a final comment, we state the following: In practice, the projection \(\Pi_C(\cdot)\) step might not be easy to compute, due to the joint involvement of convex sets. A practical solution would be to sequentially project \(-\frac{1}{L} \cdot \nabla f(0)\) onto the individual constraint sets. Let \(\Pi_+(\cdot)\) denote the projection onto the PSD cone. Then, we can consider the approximate point:

\[
\tilde{X}_0 = \tilde{U}_0 \tilde{U}_0^\top = \Pi_+ \left( \tilde{X}_0 \right);
\]

Given \(\tilde{U}_0\), we can perform an additional step:

\[
U_0 = \Pi_C \left( \tilde{U}_0 \right),
\]

to guarantee that \(U_0 \in C\).