Some Twisted Results

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March 29, 2022

Abstract

The Drinfeld twist for the opposite quasi-Hopf algebra, $H^{\text{op}}$ is determined and is shown to be related to the (second) Drinfeld twist on a quasi-Hopf algebra. The twisted form of the Drinfeld twist is investigated. In the quasi-triangular case it is shown that the Drinfeld $u$ operator arises from the equivalence of $H^{\text{op}}$ to the quasi-Hopf algebra induced by twisting $H$ with the $R$-matrix. The Altschuler-Coste $u$ operator arises in a similar way and is shown to be closely related to the Drinfeld $u$ operator. The quasi-cocycle condition is introduced, and is shown to play a central role in the uniqueness of twisted structures on quasi-Hopf algebras. A generalisation of the dynamical quantum Yang-Baxter equation, called the quasi-dynamical quantum Yang-Baxter equation is introduced.
1 Introduction

Quasi-Hopf algebras (QHA) were introduced by Drinfeld [6] as generalisations of Hopf algebras. QHA are the underlying algebraic structures of elliptic quantum groups [8, 9, 10, 14, 20] and hence have an important role in obtaining solutions to the dynamical Yang-Baxter equation. They arise in conformal field theory [3, 4], algebraic number theory [7] and in the theory of knots [1, 15, 16].

The antipode $S$ of a Hopf algebra $H$ is uniquely determined as the inverse of the identity map on $H$ under the convolution product. For a quasi-Hopf algebra, the triple $(S, \alpha, \beta) \in H$ consisting of the antipode $S$ and canonical elements $\alpha, \beta \in H$ is termed the quasi-antipode. The quasi-antipode of a QHA is not unique [2, 6, 17]. However, given two QHA which differ only in their quasi-antipodes, there exists a unique invertible element $v \in H$ relating them. Moreover, to each invertible element $v \in H$ there corresponds a quasi-antipode, so that the invertible elements $v \in H$ are in bijection with the quasi-antipodes. This allows us to work with a fixed choice for the quasi-antipode (more precisely, a fixed equivalence class for the quasi-antipode). We show that the operator $v \in H$ is universal i.e. invariant under an arbitrary twist $F \in H \otimes H$. In the quasi-triangular case, the equivalence of the quasi-antipode of the opposite QHA $H^{\text{cop}}$ and the quasi-antipode induced by twisting $H$ with the $R$-matrix, gives rise to a specific form of the $v$ operator, which we call the Drinfeld - Reshetikhin [5, 18] $u$ operator. The $u$-operator introduced by Altschuler and Coste [1], arises in a similar way and is shown to be simply related to the Drinfeld - Reshetikhin $u$ operator. In view of the invariance of the $v$ operators these $u$ operators are also invariant under twisting.

For a Hopf algebra $H$ the antipode $S$ is both an algebra and a co-algebra anti-homomorphism. In the QHA case Drinfeld has shown that the antipode $S$ is a co-algebra anti-homomorphism only upto conjugation by a twist, $F_3$ (the Drinfeld twist). Assuming the antipode $S$ is invertible with inverse $S^{-1}$, we show that $S^{-1}$ is a co-algebra anti-homomorphism upto conjugation by an invertible element $F_0$, which we call the second Drinfeld twist on $H$. The form of the Drinfeld twist for the opposite QHA $H^{\text{cop}}$ is determined and shown to be simply related to this second Drinfeld twist. The behaviour of the Drinfeld twist $F_3$ under an arbitrary twist $G \in H \otimes H$ is also investigated.

The set of twists on a QHA $H$ form a group. We study a sub-group of the group of twists on a QHA, namely those that leave the co-product $\Delta : H \to H \otimes H$ and co-associator $\Phi \in H \otimes H \otimes H$ unchanged. These twists are called compatible twists. Twists that leave the coassociator $\Phi$ unchanged are said to satisfy the quasi-cocycle condition. The quasi-cocycle condition is intimately related to the uniqueness of the structure obtained by twisting the quasi-bialgebra part of a QHA. In the quasi-triangular case we show that $R^T R$ and its powers are compatible twists.

Following on from our considerations of the quasi-cocycle condition we introduce the shifted quasi-cocycle condition on a twist $F(\lambda) \in H \otimes H$, where $\lambda \in H$ depends on one (or more) parameters. We conclude with the quasi-dynamical quantum Yang-Baxter equation (QQYBE), which is the quasi-Hopf analogue of the usual dynamical QYBE.

2 Preliminaries

We begin by recalling the definition [4] of a quasi-bialgebra.

Definition 1. A quasi-bialgebra $(H, \Delta, \epsilon, \Phi)$ is a unital associative algebra $H$ over a field $F$, equipped with algebra homomorphisms $\epsilon : H \to F$ (co-unit), $\Delta : H \to H \otimes H$ (co-product) and an invertible
element $\Phi \in H \otimes H \otimes H$ (co-associator) satisfying
\begin{align}
(1 \otimes \Delta)\Delta(a) &= \Phi^{-1}(\Delta \otimes 1)\Delta(a)\Phi, \quad \forall a \in H, \\
(\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi &= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)\Phi \cdot (1 \otimes \Phi), \\
(\epsilon \otimes 1)\Delta &= 1 = (1 \otimes \epsilon)\Delta, \\
(1 \otimes \epsilon \otimes 1)\Phi &= 1. 
\end{align}

It follows from equations (2.8), (2.9), and (2.10) that the co-associator $\Phi$ has the additional properties
\begin{align}
(\epsilon \otimes 1 \otimes 1)\Phi &= 1 = (1 \otimes 1 \otimes \epsilon)\Phi.
\end{align}

We now fix the notation to be used throughout the paper. For the co-associator we follow the notation of [12, 13] and write
\[ \Phi = \sum_{\nu} X_{\nu} \otimes Y_{\nu} \otimes Z_{\nu}, \quad \Phi^{-1} = \sum_{\nu} \tilde{X}_{\nu} \otimes \tilde{Y}_{\nu} \otimes \tilde{Z}_{\nu}. \]

We adopt Sweedler’s [19] notation for the co-product throughout. Since the co-product is quasi-coassociative we use the following extension of Sweedler’s notation
\begin{align}
(1 \otimes \Delta)\Delta(a) &= a_{(1)} \otimes \Delta(a_{(2)}) = a_{(1)} \otimes a_{(1)}^{(1)} \otimes a_{(2)}^{(2)}, \\
(\Delta \otimes 1)\Delta(a) &= \Delta(a_{(1)}) \otimes a_{(2)} = a_{(1)}^{(1)} \otimes a_{(1)}^{(2)} \otimes a_{(2)}. 
\end{align}

In general, the summation sign is omitted from expressions, with the convention that repeated indices are to be summed over.

**Definition 2.** A quasi-Hopf algebra $(H, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ is a quasi-bialgebra $(H, \Delta, \epsilon, \Phi)$ equipped with an algebra anti-homomorphism $S$ (antipode) and canonical elements $\alpha, \beta \in H$ such that
\begin{align}
S(X_{\nu})aY_{\nu} &= \beta S(Z_{\nu})a = \frac{\epsilon(a) \cdot \epsilon(\beta)}{\epsilon(S(a))} = \frac{\epsilon(S^{-1}(a))}{\epsilon(a)}, \quad \forall a \in H. 
\end{align}

Throughout we assume bijectivity of the antipode $S$ so that $S^{-1}$ exists. The antipode equations (2.10), (2.11) imply $\epsilon(a) \cdot \epsilon(\beta) = 1$ and $\epsilon(S(a)) = \epsilon(S^{-1}(a)) = \epsilon(a), \forall a \in H$. A triple $(S, \alpha, \beta)$ satisfying equations (2.10), (2.11) is called a quasi-antipode.

We shall need the following relations:
\begin{align}
X_{\nu}a \otimes Y_{\nu} &= a_{(1)}^{(1)}X_{\nu} \otimes a_{(2)}^{(2)}Y_{\nu}S(Z_{\nu})S(a_{(2)}), \quad \forall a \in H \\
\Phi \otimes 1 &= (\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi \cdot (1 \otimes \Phi^{-1}) \cdot (1 \otimes \Delta \otimes 1)\Phi^{-1} \\
&= X_{\nu}^{(1)}X_{\rho} \tilde{X}_{\rho} \otimes Y_{\nu}^{(2)}Y_{\rho} \tilde{Y}_{\rho}^{(1)} \otimes Z_{\nu} \tilde{Z}_{\rho}^{(2)} \otimes Z_{\nu} \tilde{Z}_{\rho}^{(2)} \otimes Z_{\nu} \tilde{Z}_{\rho}^{(2)} \\
&= \tilde{X}_{\nu} \tilde{X}_{\rho} X_{\nu}^{(1)} \tilde{X}_{\rho} \otimes \tilde{Y}_{\nu}^{(1)} \tilde{Y}_{\rho}^{(2)} \tilde{Z}_{\rho} \tilde{Z}_{\rho}^{(1)} \otimes \tilde{Z}_{\nu} \otimes Z_{\nu} Z_{\rho} Z_{\rho} \\
&= \tilde{X}_{\nu} \tilde{X}_{\rho} X_{\nu}^{(1)} X_{\rho}^{(2)} \otimes \tilde{Y}_{\nu}^{(1)} Y_{\rho}^{(2)} \otimes \tilde{Z}_{\rho} \tilde{Z}_{\rho}^{(1)} \otimes \tilde{Z}_{\nu} \otimes Z_{\nu} Z_{\rho} Z_{\rho} (2.10)
\end{align}

where we have adopted the notation of equation (2.8) in (2.9), and the obvious notation in (2.10) so that, for example
\begin{align}
\Delta(X_{\nu}) &= X_{\nu}^{(1)} \otimes X_{\nu}^{(2)}, \quad \text{etc.}
\end{align}

Equation (2.8) follows from applying $(1 \otimes m)(1 \otimes 1 \otimes \beta S)$ to equation (2.1), then using (2.4).
3 Uniqueness of the quasi-antipode.

For Hopf algebras the antipode $S$ is uniquely determined as the inverse of the identity map on $H$ under the convolution product. The quasi-antipode $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ for a QHA, is not unique. Nevertheless it is almost unique as the following result due to Drinfeld [6] (whose proof is similar to the one given below) shows:

**Theorem 1.** Suppose $H$ is also a QHA, but with quasi-antipode $(\tilde{S}, \tilde{\alpha}, \tilde{\beta})$ satisfying (3.11), (3.12). Then there exists a unique invertible $v \in H$ such that

\[
v \alpha = \tilde{\alpha} , \tilde{\beta} v = \beta , \tilde{S}(a) = v S(a) v^{-1} , \forall a \in H.
\]

Explicitly

\[(i) \quad v = \tilde{S}(X_v) \tilde{\alpha} Y_v \beta S(Z_v) = \tilde{S}(S^{-1}(X_v)) \tilde{S}(S^{-1}(\beta)) \tilde{S}(Y_v) \tilde{\alpha} \tilde{Z}_v \]

\[(ii) \quad v^{-1} = S(X_v) \alpha Y_v \tilde{\beta} S(Z_v) = \tilde{X}_v \tilde{\beta} \tilde{S}(Y_v) \tilde{S}(S^{-1}(\alpha)) \tilde{S}(S^{-1}(\tilde{Z}_v)) \]

**Proof.** We proceed stepwise.

Applying $m \cdot (\tilde{S} \otimes 1)(1 \otimes \tilde{\alpha})$ to equation (2.6) gives

\[
\tilde{S}(X_v) a \tilde{\alpha} Y_v \beta S(Z_v) = \tilde{S}(a(1)_1 X_v) \tilde{\alpha} a(2)_1 Y_v \beta S(Z_v) S(a(2)_2) = v S(a), \quad \forall a \in H
\]

(3.13)

where $m : H \otimes H \rightarrow H$ is the multiplication map $m(a \otimes b) = ab, \forall a, b \in H$.

Next observe, from equation (2.7) that, in view of (2.7),

\[
v \otimes 1 = \tilde{S}(X_v) a \tilde{\alpha} Y_v \beta S(Z_v) = \tilde{S}(X_v) a \tilde{\alpha} Y_v \beta S(Z_v) = \tilde{S}(X_v) \tilde{\alpha} Y_v \beta S(Z_v) \tilde{\sigma} \tilde{Z}_v \]

Applying $m \cdot (1 \otimes \alpha)$ from the left gives

\[
v \alpha = \tilde{S}(X_v) \tilde{\alpha} Y_v \tilde{\beta} S(\tilde{Z}_v) \tilde{\alpha} = \alpha \tilde{\alpha} \beta S(\tilde{Y}_v) \tilde{\alpha} \tilde{Z}_v = \tilde{\alpha}.
\]

(3.14)

From this it follows that

\[
\tilde{S}(S^{-1}(X_v)) \cdot \tilde{S}(S^{-1}(\beta)) \cdot \tilde{S}(Y_v) \tilde{\alpha} \tilde{Z}_v
\]

\[
= \tilde{S}(S^{-1}(X_v)) \cdot \tilde{S}(S^{-1}(\beta)) \tilde{S}(Y_v) \cdot v \alpha \tilde{Z}_v
\]

\[
= v \cdot S(S^{-1}(X_v)) \cdot S(S^{-1}(\beta)) \cdot S(\tilde{Y}_v) \alpha \tilde{Z}_v
\]

(3.11)

\[
= v \cdot \tilde{X}_v \tilde{\beta} S(\tilde{Y}_v) \alpha \tilde{Z}_v = v
\]

(3.12)

which proves (3.12) (i). To see $v$ is invertible observe that

\[
v \cdot S(X_v) a \alpha Y_v \tilde{\beta} S(Z_v) = \tilde{S}(X_v) v \alpha Y_v \tilde{\beta} S(Z_v) = \tilde{S}(X_v) a \tilde{\alpha} Y_v \tilde{\beta} S(Z_v) \]

(3.12)

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so
\[ v^{-1} = S(X_v)\alpha Y_v \tilde{\beta} S(Z_v) \]
as stated.

Now using equation (2.10) we have
\[ 1 \otimes v^{-1} = \bar{X}_v \bar{X}_\mu X_\rho^{(1)} X_\sigma \otimes S(\bar{Y}_\mu X_\rho^{(2)} Y_\sigma) \alpha \bar{Y}_\mu X_\rho^{(2)} \bar{Z}_\mu Y_\rho \bar{Z}_\sigma^{(1)} \tilde{\beta} S(\bar{Z}_v Z_\mu Z_\sigma^{(2)}) \]
\[ \bar{X}_v X_\rho^{(1)} \otimes S(\bar{Y}_\mu X_\rho^{(2)}) \alpha \bar{Z}_\mu Y_\rho \tilde{\beta} S(Z_\rho). \]
Applying \( m \cdot (1 \otimes \beta) \) gives
\[ \beta v^{-1} = \bar{X}_v X_\rho^{(1)} \beta S(\bar{Y}_\mu X_\rho^{(2)}) \alpha \bar{Z}_\mu Y_\rho \tilde{\beta} S(Z_\rho) \]
\[ = \bar{X}_v \beta S(\bar{Y}_\mu) \alpha \bar{Z}_\mu \cdot \beta \overset{2.6}{=} \tilde{\beta} \]
which completes the proof of (3.11). As to (3.12)(ii) observe that
\[ \bar{X}_v \beta v^{-1} S(\bar{Y}_\mu) \tilde{S}(S^{-1}(\alpha)) S(S^{-1}(Z_\mu)) \]
\[ \overset{2.6}{=} \bar{X}_v \beta v^{-1} S(\bar{Y}_\mu) \tilde{S}(S^{-1}(\alpha)) S(S^{-1}(Z_\mu)) \]
\[ \overset{2.6}{=} \bar{X}_v \beta S(\bar{Y}_\mu) S(S^{-1}(\alpha)) S(S^{-1}(Z_\mu)) v^{-1} \]
\[ = \bar{X}_v \beta S(\bar{Y}_\mu) \alpha \bar{Z}_\mu \cdot v^{-1} \overset{2.6}{=} v^{-1} \]
as required. It finally remains to prove uniqueness. Hence suppose \( u \in H \) satisfies
\[ uS(a) = \tilde{S}(a)u, \quad \forall a \in H, \quad u\alpha = \tilde{\alpha}, \quad \tilde{\beta}u = \beta. \]

Then
\[ uv^{-1} = u \cdot S(X_v)\alpha Y_v \tilde{\beta} S(Z_v) \]
\[ = \tilde{S}(X_v)u\alpha Y_v \tilde{\beta} S(Z_v) \]
\[ = \tilde{S}(X_v)\tilde{\alpha} Y_v \tilde{\beta} S(Z_v) \overset{2.6}{=} 1 \]
which implies \( u = v \) as required. \( \square \)

In the special case \( \tilde{S} = S \) we obtain the following useful result.

**Corollary.** Suppose \( H \) is also a QHA with quasi-antipode \( (S, \tilde{\alpha}, \tilde{\beta}) \). Then there is a unique invertible central element \( v \in H \), given explicitly by equation (3.12)(i) (with \( \tilde{S} = S \)), such that
\[ \beta v = \beta. \]

It thus follows that the triple \( (S, \alpha, \beta) \) satisfying (2.6), (2.7) for a QHA is not unique. Indeed following Theorem 11 for arbitrary invertible \( v \in H \), the triple \( (\tilde{S}, \tilde{\alpha}, \tilde{\beta}) \) defined by
\[ \tilde{S}(a) = vS(a)v^{-1}, \quad \forall a \in H \quad ; \tilde{\alpha} = v\alpha, \quad \tilde{\beta} = \beta v^{-1} \]
is easily seen to satisfy (2.6), (2.7) and thus gives rise to a quasi-antipode \( (\tilde{S}, \tilde{\alpha}, \tilde{\beta}) \). Theorem 11 then shows that all such quasi-antipodes \( (\tilde{S}, \tilde{\alpha}, \tilde{\beta}) \) are obtainable this way: thus there is a 1–1 correspondence between the latter and invertible \( v \in H \). We say that these structures are equivalent since they clearly give rise to equivalent QHA structures. Throughout we work with a fixed choice for the quasi-antipode \( (S, \alpha, \beta) \).

We conclude this section with the following useful result, proved in [13], concerning the opposite QHA structure on \( H \):
Proposition 1. $H$ is also a QHA, with co-unit $\epsilon$, under the opposite co-product and co-associator $\Delta^T$, $\Phi^T \equiv \Phi_{\otimes 2}^T$ respectively, with quasi-antipode $(S^{-1}, \alpha^T = S^{-1}(\alpha), \beta^T = S^{-1}(\beta))$.

The QHA $H^{\text{cap}} \equiv (H, \Delta^T, \epsilon, \Phi^T, S^{-1}, \alpha^T, \beta^T)$ is called the opposite QHA structure. We remark that above we have adopted the notation of [12] and [13] so that $\Delta^T = T \cdot \Delta$, $T$ the usual twist map, and

$$\Phi_{\otimes 2}^{-1} = \tilde{Z}_\nu \otimes \bar{Y}_\nu \otimes \bar{X}_\nu.$$  

This latter notation extends in a natural way and will be employed throughout.

4 Twisting

Let $H$ be a quasi-bialgebra. Then $F \in H \otimes H$ is called a twist if it is invertible and satisfies the co-unit property

$$(\epsilon \otimes 1)F = (1 \otimes \epsilon)F = 1.$$  

We recall that $H$ is also a QBA with the same co-unit $\epsilon$ but with co-product and co-associator given by

$$\Delta_F(a) = F\Delta(a)F^{-1}, \quad \forall a \in H$$

$$\Phi_F = (F \otimes 1) \cdot (\Delta \otimes 1)F \cdot \Phi \cdot (1 \otimes \Delta)F^{-1} \cdot (1 \otimes F^{-1}),$$  

called the twisted structure induced by $F$. If moreover $H$ is a QHA with quasi-antipode $(S, \alpha, \beta)$ then $H$ is also a QHA under the above twisted structure with the same antipode $S$ but with canonical elements

$$\alpha_F = m \cdot (1 \otimes \alpha)(S \otimes 1)F^{-1} \quad \beta_F = m \cdot (1 \otimes \beta)(1 \otimes S)F$$  

respectively. A detailed proof of these well known results is given in [20]. We now investigate the behaviour of the operator $v$ of Theorem 1 under the twisted structure induced by $F$.

4.1 Universality of $v$

Recall that the operator $v$ is given by

$$v = \tilde{S}(X_\nu)\tilde{\alpha}Y_\nu \beta S(Z_\nu)$$

Let $F \in H \otimes H$ be an arbitrary twist. We use the following notation for the twist $F$ and its inverse $F^{-1}$,

$$F = f_i \otimes f^i, \quad F^{-1} = \bar{f}_i \otimes \bar{f}^i.$$  

The twisted form of the co-associator is given by

$$\Phi_F = X^F_\nu \otimes Y^F_\nu \otimes Z^F_\nu = f_i f_j f_k \otimes f^i f^j f^k Y_{(3)} \bar{f}_l \otimes f^i \bar{f}_l Z_{(2)} \bar{f}^k.$$  

For the twisted forms of the canonical elements we have from

$$\tilde{\alpha}_F = m \cdot (1 \otimes \tilde{\alpha})(\tilde{S} \otimes 1)F^{-1} = \tilde{S}(\bar{f}_p)\tilde{\alpha} \bar{f}^p$$

$$\beta_F = m \cdot (1 \otimes \beta)(1 \otimes S)F = f_q \beta S(f^q).$$  

We note that

$$\tilde{S}(f_j)\tilde{\alpha}_F f^j = \tilde{S}(\bar{f}_p f_j)\tilde{\alpha} \bar{f}^p f^j = m \cdot (1 \otimes \alpha)(\tilde{S} \otimes 1)(F^{-1}F) = \tilde{\alpha}.$$  

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and similarly,
\[ \mathcal{f}_j \beta_F S(\mathcal{f}^j) = \beta. \]  
(4.21)

The twisted form of \( v \) is given by
\[
v_F = \hat{S}(X^F) \hat{\alpha}_F Y^F \beta_F S(Z^F)
\]
\[
= \hat{S}(f_j X \hat{f}_k) \hat{\alpha}_F f^i \hat{f}^j Y_{j(1)} \hat{f}^k \beta_F S(f^i Z_{(2)} f^j)
\]
\[
= \hat{S}(f_j X \hat{f}_k) \hat{S}(f_i) \hat{\alpha}_F f^i \hat{f}^j Y_{j(1)} \hat{f}^k \beta_F S(f^i Z_{(2)} f^j)
\]
\[
= \hat{S}(f_j X \hat{f}_k) \hat{f}^{j(2)} Y_{j(1)} \hat{f}^k \beta_F S(f^j Z_{(2)})
\]
\[
= \hat{S}(X \hat{f}_k) \hat{\alpha} Y_{j(1)} \hat{f}^k \beta S(\hat{f}^j S(f^j Z_{(2)}))
\]
\[
= \hat{S}(X \hat{f}_k) \hat{\alpha} Y_{j(1)} \hat{f}^k \beta S(\hat{f}^j Z_{(2)} Z_{(2)}) = v,
\]
where, in the last two lines we have used the antipode properties of \( \alpha, \beta \) and the co-unit property of twists. We have thus proved

**Theorem 2.** The operator \( v \) is universal (i.e. invariant under twisting).

5 The Drinfeld twists

We turn our attention to the Drinfeld twist for the opposite structure of proposition 1. It is tempting to assume that \( F_T^0 \) qualifies as a Drinfeld twist for the opposite structure. However this is not true since the antipode for the latter is \( S^{-1} \) rather than \( S \). We shall show that the Drinfeld twist for the opposite structure is in fact related to the second Drinfeld twist which we define below. We begin with a review of the Drinfeld twist.

5.1 The Drinfeld twist

Observe that \( \Delta' \) defined by
\[
\Delta'(a) = (S \otimes S) \Delta^T (S^{-1} a), \quad \forall a \in H
\]
(5.22)
also determines a co-product on \( H \). Associated with this co-product we have a new QHA structure on \( H \), which was proved in [13] and which we restate here:

**Proposition 2.** \( H \) is also a QHA with the same co-unit \( \epsilon \) and antipode \( S \) but with co-product \( \Delta' \), co-associator \( \Phi' = (S \otimes S \otimes S) \Phi_{321} \), and canonical elements \( \alpha' = S(\beta), \ \beta' = S(\alpha) \) respectively.

Drinfeld has proved the remarkable result that this QHA structure is obtained by twisting with the Drinfeld twist, herein denoted \( F_\delta \), given explicitly by
\[
\text{(i)} \quad F_\delta = (S \otimes S) \Delta^T (X_{\nu}) \cdot \gamma \cdot \Delta (Y_{\nu} S(Z_{\nu}))
\]
\[
= \Delta'(X_{\nu}) \cdot \gamma \cdot \Delta(\hat{Z}_{\nu})
\]
where
\[ \gamma = S(B_i) \alpha C_i \otimes S(A_i) \alpha D_i \]

with
\[ A_i \otimes B_i \otimes C_i \otimes D_i = \begin{cases} (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1) \Phi \\ (1 \otimes \Phi) \cdot (1 \otimes 1 \otimes \Delta) \Phi^{-1}. \end{cases} \]

The inverse of \( F_\delta \) is given explicitly by
\[ F^{-1}_\delta = \Delta(\bar{X}_\nu) \cdot \bar{\gamma} \cdot \Delta'(S(Y_\nu)\alpha Z_\nu) \]

where
\[ \bar{\gamma} = \bar{A}_i \beta S(\bar{D}_i) \otimes \bar{B}_i \beta S(\bar{C}_i) \]

with
\[ \bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i = \begin{cases} (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \\ (1 \otimes 1 \otimes \Delta) \Phi \cdot (1 \otimes \Phi^{-1}). \end{cases} \]

The detailed proof that the QHA structure of proposition is obtained by twisting with \( F_\delta \), as given in (5.23), and in particular
\[ \Delta'(a) = F_\delta \Delta(a) F^{-1}_\delta, \quad \forall a \in H \]

is proved in [13]. We simply state here some properties of \( \gamma, \bar{\gamma} \) proved in [13] and which are crucial to the demonstration of Drinfeld's result:

**Proposition 3.**
\[ \begin{align*}
(i) & \quad (S \otimes S) \Delta^T(a_{1(1)}) \cdot \gamma \cdot \Delta(a_{1(2)}) = \epsilon(a) \gamma, \quad \forall a \in H \\
(ii) & \quad \Delta(a_{1(1)}) \cdot \bar{\gamma} \cdot (S \otimes S) \Delta^T(a_{1(2)}) = \epsilon(a) \bar{\gamma}, \quad \forall a \in H \\
(iii) & \quad F_\delta \Delta(a) = \gamma, \quad \Delta(\beta) F^{-1}_\delta = \bar{\gamma}.
\end{align*} \]

### 5.2 The second Drinfeld twist

Replacing \( S \) with \( S^{-1} \) we obtain yet another co-product \( \Delta_0 \) on \( H \):
\[ \Delta_0(a) = (S^{-1} \otimes S^{-1}) \Delta^T(S(a)), \quad \forall a \in H. \]

We have the following analogue of proposition the proof of which parallels that of [13] proposition 4, but with \( S \) and \( S^{-1} \) interchanged:

**Proposition 2** \( H \) is also a QHA with the same co-unit \( \epsilon \) and antipode \( S \) but with co-product \( \Delta_0 \), co-associator \( \Phi_0 = (S^{-1} \otimes S^{-1} \otimes S^{-1}) \Phi_{321} \) and canonical elements \( \alpha_0 = S^{-1}(\beta), \beta_0 = S^{-1}(\alpha) \) respectively.

By symmetry we would expect this structure to be obtainable twisting. Indeed we have
\textbf{Theorem 3.} \textit{The QHA structure of proposition \textbf{3}′ is obtained by twisting with}

\[ F_0 \equiv (S^{-1} \otimes S^{-1})F_3^T \]

\textit{herein referred to as the second Drinfeld twist, where }F_3\textit{ is the Drinfeld twist and }F_3^T = T \cdot F_3\text{.}

\textit{Proof.} It is clear that }F_0\text{ is invertible with inverse }F_0^{-1} = (S^{-1} \otimes S^{-1})(F_3^T)^{-1}\text{ and qualifies as a twist. For the co-product we observe,}

\[ F_0 \Delta(a)F_0^{-1} = (S^{-1} \otimes S^{-1})F_3^T \cdot \Delta(a) \cdot (S^{-1} \otimes S^{-1})(F_3^T)^{-1} = (S^{-1} \otimes S^{-1}) \cdot T \cdot [F_3^{-1} \cdot (S \otimes S) \Delta^T(a) \cdot F_3] = (S^{-1} \otimes S^{-1}) \cdot T \cdot [F_3^{-1} \Delta'(S(a))F_3] \]

\[ \overset{5.25}{=} (S^{-1} \otimes S^{-1}) \cdot T \cdot \Delta(S(a)) = (S^{-1} \otimes S^{-1})\Delta^T(S(a)) \]

\[ \overset{5.22}{=} \Delta_0(a), \quad \forall a \in H. \]

The co-associator is slightly more complicated, though also simple. We have from Drinfeld’s result

\[ \Phi' \equiv (S \otimes S \otimes S)\Phi_{321} = (F_3 \otimes 1) \cdot (\Delta \otimes 1)F_3 \cdot \Phi \cdot (1 \otimes \Delta)F_3^{-1} \cdot (1 \otimes F_3^{-1}) \]

which implies

\[ (S \otimes S \otimes S)\Phi = [(F_3 \otimes 1) \cdot (\Delta \otimes 1)F_3 \cdot \Phi \cdot (1 \otimes \Delta)F_3^{-1} \cdot (1 \otimes F_3^{-1})]_{321} = (1 \otimes F_3^T) \cdot (1 \otimes \Delta^T)F_3^T \cdot \Phi_{321} \cdot (\Delta^T \otimes 1)(F_3^T)^{-1} \cdot (F_3^{-1} \otimes 1). \]

Applying \((S^{-1} \otimes S^{-1} \otimes S^{-1})\) gives

\[ \Phi = (F_0^{-1} \otimes 1) \cdot (\Delta_0 \otimes 1)F_0^{-1} \cdot \Phi_0 \cdot (1 \otimes \Delta_0)F_0 \cdot (1 \otimes F_0) = (\Delta \otimes 1)F_0^{-1} \cdot (F_0^{-1} \otimes 1) \cdot \Phi_0 \cdot (1 \otimes F_0) \cdot (1 \otimes \Delta)F_0 \]

with }F_0\text{ as in the Theorem. Thus

\[ \Phi_0 = (F_0 \otimes 1) \cdot (\Delta \otimes 1)F_0 \cdot \Phi \cdot (1 \otimes \Delta)F_0^{-1} \cdot (1 \otimes F_0^{-1}) \]

which shows that indeed }\Phi_0\text{ is obtained from }\Phi\text{ by twisting with }F_0\text{. The proof for the canonical elements is straightforward.}\]

\textbf{5.3 The Drinfeld twists for the opposite structure}

Recall that under the opposite structure of proposition\textbf{1} \(H\) is a QHA with antipode }S^{-1}\text{, co-product }\Delta^T\text{ and co-associator }\Phi^T = \Phi_{321}^{-1}\text{. It follows that if }F_3^0\text{ is the Drinfeld twist for this opposite structure then, }\forall a \in H\text{}

\[ F_3^0 \Delta^T(a)(F_3^0)^{-1} = (\Delta^T)'(a) = (S^{-1} \otimes S^{-1})\Delta(S(a)) = \Delta^T_0(a) \]

since }S^{-1}\text{ is the antipode for this structure. On the other hand if }F_0\text{ is the Drinfeld twist of equation \textbf{5.27} we have also

\[ F_0^T \Delta^T(a)(F_0^T)^{-1} = \Delta^T_0(a) \]
with $\Delta_\mu$ as in equation \eqref{5.22}. Here we show in fact that $F^0_\delta = F^T_0$. Before proceeding we note that the Drinfeld twist is given by the canonical expression of equation \eqref{5.23}(i) with $\gamma$ as in \eqref{5.23}(ii) constructed from the operator of \eqref{5.23}(iii); viz

$$A_i \otimes B_i \otimes C_i \otimes D_i = \begin{cases} (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1) \Phi & \text{or} \\ (1 \otimes \Phi) \cdot (1 \otimes 1 \otimes \Delta) \Phi^{-1}. \end{cases}$$

This gives rise to two equivalent expansions for $\gamma$. Using the first expression we have, in obvious notation,

$$A_i \otimes B_i \otimes C_i \otimes D_i = (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1) \Phi = X_\nu X^{(1)}_\mu \otimes Y_\nu Y^{(2)}_\mu \otimes Z_\mu \otimes Z_\mu$$

which gives, upon substitution into \eqref{5.23}(ii),

$$\gamma = S(Y_\nu X^{(2)}_\mu)\alpha \bar{Z}_\mu \otimes S(X_\nu X^{(1)}_\mu)\alpha Z_\mu$$

which is the expression obtained in \cite{13}. On the other hand using the second expression gives

$$A_i \otimes B_i \otimes C_i \otimes D_i = (1 \otimes \Phi) \cdot (1 \otimes 1 \otimes \Delta) \Phi^{-1} = X_\mu \otimes X_\nu \bar{Y}_\mu \otimes Y_\nu \bar{Z}^{(1)}_\mu \otimes Z_\nu \bar{Z}^{(2)}_\mu$$

and substituting into \eqref{5.23}(ii) gives the alternative expansion

$$\gamma = S(X_\nu \bar{Y}_\mu)\alpha Y_\nu \bar{Z}^{(1)}_\mu \otimes S(X_\nu \bar{Y}_\mu)\alpha Z_\nu \bar{Z}^{(2)}_\mu$$

\[5.28\] which is equivalent to the expression above \cite{13}.

Using \eqref{5.23}(i) for the opposite structure we have for the Drinfeld twist

$$F^0_\delta = (S^{-1} \otimes S^{-1}) \Delta(X^0_\mu) \cdot \gamma^0 \cdot \Delta^T(Y^0_\nu \beta^T S^{-1}(Z^0_\nu))$$

where we have used the fact that the co-product for the opposite structure is $\Delta^T$, the antipode is $S^{-1}$, with canonical elements $\alpha^T = S^{-1}(\alpha), \beta^T = S^{-1}(\beta)$ and where we have set

$$X^0_\nu \otimes Y^0_\nu \otimes Z^0_\nu = \Phi^T = \Phi^{-1}_{321},$$

which is the opposite co-associator, and where from \eqref{5.23}(ii)

$$\gamma^0 = S^{-1}(B^0_\nu)\alpha^T C^0_i \otimes S^{-1}(A^0_\nu)\alpha^T D^0_i$$

with

$$A^0_\nu \otimes B^0_i \otimes C^0_i \otimes D^0_i = [(\Phi^T)^{-1} \otimes 1] \cdot (\Delta^T \otimes 1 \otimes 1) \Phi^T$$

$$= (\Phi^{-1}_{321} \otimes 1) \cdot (\Delta^T \otimes 1 \otimes 1) \Phi^{-1}_{321}.$$ In obvious notation the latter is given by

$$(\Phi^{-1}_{321} \otimes 1) \cdot (\Delta^T \otimes 1 \otimes 1) \Phi^{-1}_{321} = Z_\nu \bar{Z}^{(2)}_\mu \otimes Y_\nu \bar{Z}^{(1)}_\mu \otimes X_\nu \bar{Y}_\mu \otimes X_\mu$$

so that, using $\alpha^T = S^{-1}(\alpha)$,

$$\gamma^0 = S^{-1}(Y_\nu \bar{Z}^{(1)}_\mu)S^{-1}(\alpha)X_\nu \bar{Y}_\mu \otimes S^{-1}(Z_\nu \bar{Z}^{(2)}_\mu)S^{-1}(\alpha)X_\mu$$

\[5.28\] (\ref{5.28}) \cite{13}. 

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Thus we may write, using $\beta^T = S^{-1}(\beta)$,

$$F^0_\delta = (S^{-1} \otimes S^{-1}) \Delta(X^0_\nu) \cdot (S^{-1} \otimes S^{-1})\gamma \cdot \Delta^T(Y^0_\nu S^{-1}(\beta) S^{-1}(Z^0_\nu))$$

so that, substituting

$$X^0_\nu \otimes Y^0_\nu \otimes Z^0_\nu = \Phi^T = \Phi^\varepsilon_{321} = \bar{Z}_\nu \otimes \bar{Y}_\nu \otimes \bar{X}_\nu,$$

gives

$$F^0_\delta = (S^{-1} \otimes S^{-1}) \Delta(\bar{Z}_\nu) \cdot (S^{-1} \otimes S^{-1})\gamma \cdot \Delta^T(\bar{Y}_\nu S^{-1}(\beta) S^{-1}(\bar{X}_\nu))$$

Thus we have proved

**Proposition 4.** The Drinfeld twist for the opposite QHA structure of proposition 1 is given explicitly by

$$F^0_\delta = (S^{-1} \otimes S^{-1}) F_\delta = F^T_0.$$

To see how $F^T_0$ fits into the picture we need to consider the second Drinfeld twist $F_0$ of Theorem 3 associated with the co-product of equation (5.22). We have immediately from proposition 4

**Corollary.** The second Drinfeld twist for the opposite structure is $F^T_0$.

**Proof.** Since the antipode for the opposite structure is $S^{-1}$, Theorem 3 implies that the second Drinfeld twist for this structure is $(S \otimes S)(F^0_\delta)^T$ where $F^0_\delta$ is the Drinfeld twist for the opposite structure, given explicitly in proposition 4. It follows that the second Drinfeld twist for the opposite structure is

$$(S \otimes S) \cdot [(S^{-1} \otimes S^{-1}) F^T_0] = F^T_\delta.$$

\[\square\]

### 5.4 Twisting the Drinfeld twist

It is first useful to determine the behaviour of $\tilde{\gamma}$ in equation (5.24)(ii) under an arbitrary twist $G \in H \otimes H$. Under the twisted structure induced by $G$ the operator $\tilde{\gamma}$ is twisted to $\tilde{\gamma}_G$, given by equation (5.24)(ii,iii) for the twisted structure, so that

$$(i) \quad \tilde{\gamma}_G = \tilde{A}_i^G \beta_G S(\tilde{D}_i^G) \otimes \tilde{B}_i^G \beta_G S(\tilde{C}_i^G),$$

where $(ii) \quad \tilde{A}_i^G \otimes \tilde{B}_i^G \otimes \tilde{C}_i^G \otimes \tilde{D}_i^G = (\Delta_G \otimes 1 \otimes 1) \Phi^{-1}_G \cdot (\Phi_G \otimes 1).$ (5.29)

We have

**Proposition 5.** Let $G = g_i \otimes g^i \in H \otimes H$ be a twist on a QHA $H$. Then

$$\tilde{\gamma}_G = G \cdot \Delta(g_i) \cdot \tilde{\gamma} \cdot (S \otimes S)(G^T \Delta^T(g^i)).$$
Proof. Throughout we write
\[ G^{-1} = \tilde{g}_i \otimes \tilde{g}^i. \]

For the RHS of equation (5.29) (ii) we have
\[ (\Delta_G \otimes 1 \otimes 1) \Phi_G^{-1} \cdot (\Phi_G \otimes 1) = \Delta_G \otimes 1 \otimes 1 \cdot \Phi^{-1} \cdot (\Delta \otimes 1) \Phi^{-1} \cdot (\Delta \otimes 1) \Phi^{-1} \cdot (G \otimes 1) \Phi^{-1} \cdot (G^{-1} \otimes 1) \]
\[ \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \·
Now using
\[ \tilde{g}_m \beta_G S(\tilde{g}^m) = (\beta_G)_{G^{-1}} = \beta_{G^{-1}G} = \beta \] (5.30)

and making repeated use of equation (2.7) gives
\[
\gamma_G = g_s g_j^{(1)} A_s \bar{g}_i^{(1)} \bar{g}_k \beta_G S(g_i g_j^{(2)} \bar{D}_i \bar{g}_j^{(1)}) \\
\otimes g^* j_j^{(2)} \bar{B}_i \bar{g}_k^{(2)} g_j^{(2)} \beta S(g_i g_j^{(2)} g_j^{(2)} \bar{C}_i) \\
= g_s g_j^{(1)} A_s \bar{g}_i \beta_G S(g_i g_j^{(2)} \bar{D}_i) \otimes g^* j_j^{(2)} \bar{B}_i \beta S(g_i g_j^{(2)} \bar{C}_i) \\
= g_s g_j^{(1)} A_s \beta_G S(\bar{D}_i) \otimes g^* j_j^{(2)} \bar{B}_i \beta S(\bar{C}_i) S(g_i g_j^{(2)}) \\
= G \cdot (g_j^{(1)} \otimes (\bar{g}_j^{(2)} \cdot \bar{\gamma}) \cdot (S \otimes S)(g_j^{(2)} \otimes g_i g_j^{(2)})) 
\]
which proves the result. \[\square\]

We are now in a position to determine the action of an arbitrary twist \( G \in H \otimes H \) on the inverse Drinfeld twist \( F^G_\delta^{-1} \), given in equation (5.24). Under the twisted structure induced by \( G \), \( F^G_\delta^{-1} \) is twisted to \( (F^G_\delta)^{-1} \equiv (F^G_\delta)^{-1}_G \), given as in equation (5.24), but in terms of the twisted structure, so that, with the notation of equation (5.24), we have from (5.24) (ii)
\[
(F^G_\delta)^{-1} = \Delta_G (S(X^G_\nu)^G \alpha_G Y^G_\nu) \cdot \bar{\gamma}_G \cdot (S \otimes S) \Delta^T_G (Z^G_\nu)
\]
with \( \bar{\gamma}_G \) as in proposition (5)

In obvious notation we may write
\[
X^G_\nu \otimes Y^G_\nu \otimes Z^G_\nu = \Phi_G = (G \otimes 1) \cdot (\Delta \otimes 1) G \cdot \Phi \cdot (1 \otimes \Delta) G^{-1} \cdot (1 \otimes G^{-1}) \\
= g_s g_j^{(1)} X_\nu \bar{g}_k \otimes g^i j_j^{(2)} Y_\nu \bar{g}_k \bar{g}_i \otimes g^j Z_\nu \bar{g}_k \bar{g}_j
\]
which implies
\[
(F^G_\delta)^{-1} = \Delta_G [S(g_s g_j^{(1)} X_\nu \bar{g}_k) \alpha_G g^i j_j^{(2)} Y_\nu \bar{g}_k \bar{g}_i] \cdot \bar{\gamma}_G \cdot (S \otimes S) \Delta^T_G (g^i Z_\nu \bar{g}_k \bar{g}_j) \\
= \Delta_G [S(X_\nu \bar{g}_k) S(g_j^{(1)}) S(g_i) \alpha_G g^i j_j^{(2)} Y_\nu \bar{g}_k \bar{g}_i] \cdot \bar{\gamma}_G \\
\cdot (S \otimes S) \Delta^T_G (g^i Z_\nu \bar{g}_k \bar{g}_j).
\]

Using
\[
S(g_i) \alpha_G g^i = (\alpha_G)_{G^{-1}} = \alpha_{G^{-1}G} = \alpha,
\]
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and equation (27), then gives

\[(F^G_\delta)^{-1} = G \cdot \Delta[S(X_{\nu}\tilde{g}_k)\alpha Y_{\nu}\tilde{g}^k_{(1)}\tilde{g}_l] \cdot \tilde{\gamma}_G \cdot (S \otimes S)\Delta^T(Z_{\nu}\tilde{g}^k_{(2)}\tilde{g}^l)\]

\[= G \cdot \Delta[S(X_{\nu}\tilde{g}_k)\alpha Y_{\nu}\tilde{g}^k_{(1)}\tilde{g}_l] \cdot G^{-1} \cdot \tilde{\gamma}_G \cdot (S \otimes S)(G^T)^{-1} \cdot (S \otimes S)\Delta^T(Z_{\nu}\tilde{g}^k_{(2)}\tilde{g}^l) \cdot (S \otimes S)G^T\]

\[\text{prop.} \] 

\[= G \cdot \Delta[S(X_{\nu}\tilde{g}_k)\alpha Y_{\nu}\tilde{g}^k_{(1)}\tilde{g}_l] \cdot \Delta(g_{i1}) \cdot \tilde{\gamma}_i \cdot (S \otimes S)\Delta^T(g^i) \cdot (S \otimes S)\Delta^T(Z_{\nu}\tilde{g}^k_{(2)}) \cdot (S \otimes S)\Delta^T(Z_{\nu}\tilde{g}^l) \cdot (S \otimes S)G^T\]

\[= G \cdot \Delta[S(X_{\nu}\tilde{g}_k)\alpha Y_{\nu}\tilde{g}^k_{(1)}\tilde{g}_l] \cdot \Delta(\tilde{g}_{i1}) \cdot \gamma \cdot (S \otimes S)\Delta^T(\tilde{g}^i) \cdot (S \otimes S)\Delta^T(Z_{\nu}\tilde{g}^k_{(2)}) \cdot (S \otimes S)G^T\]

where we have used the obvious result that

\[\tilde{g}_i g_i \otimes \tilde{g}^i g^i = G^{-1}G = 1 \otimes 1.\]

It then follows from proposition 3 that

\[(F^G_\delta)^{-1} = G \cdot \Delta[S(X_{\nu}\tilde{g}_k)\alpha Y_{\nu}] \cdot \tilde{\gamma}_i \cdot (S \otimes S)\Delta^T(Z_{\nu}) \cdot (S \otimes S)G^T\]

\[\text{prop.} \]

\[= G \cdot F^1_\delta \cdot (S \otimes S)G^T.\]

We have thus proved

**Theorem 4.** Let \(G \in H \otimes H\) be a twist on a QHA \(H\). Then under the twisted structure induced by \(G\), \(F^{-1}_\delta\) is twisted to

\[(F^G_\delta)^{-1} = G \cdot F^{-1}_\delta \cdot (S \otimes S)G^T.\]

Equivalently, the Drinfeld twist is twisted to

\[F^G_\delta = (F_\delta)G = (S \otimes S)(G^T)^{-1} \cdot F^{-1}_\delta \cdot G^{-1}.\]

**Corollary.** \(F_0\) as in equation (27) is twisted to

\[F^G_0 = (F_0)G = (S^{-1} \otimes S^{-1})(G^T)^{-1} \cdot F^{-1}_0 \cdot G^{-1}.\]

**Proof.** Follows from the definition of \(F_0 \equiv (S^{-1} \otimes S^{-1})F^T_\delta\) and the Theorem above. \(\square\)

When \(H\) is quasi-triangular the opposite structure of proposition 3 is obtainable, up to equivalence modulo \((S, \alpha, \beta)\), via twisting. In such a case the results of Section 3 have further useful consequences.
6 Quasi-triangular QHAs

A QHA $H$ is called quasi-triangular if there exists an invertible element

$$R = \sum_i e_i \otimes e^i \in H \otimes H$$

called the $R$-matrix, such that

(i) $\Delta^T(a)R = R\Delta(a), \quad \forall a \in H$

(ii) $(\Delta \otimes 1)R = \Phi^{-1}_{23}R_{13}\Phi_{132}R_{23}\Phi^{-1}_{123}$

(iii) $(1 \otimes \Delta)R = \Phi_{312}R_{13}\Phi^{-1}_{213}R_{12}\Phi_{123},$ \hspace{1cm} (6.31)

where

$$R_{12} = e_i \otimes e^i \otimes 1, \quad R_{13} = e_i \otimes 1 \otimes e^i,$$ etcetera.

We first summarise some well known results for quasi-triangular QHAs. It was shown in \[13\] that

**Proposition 1.** Under the opposite QHA structure of proposition 1, $H$ is also quasi-triangular with $R$-matrix $R^T = T \cdot R$, called the opposite $R$-matrix.

It follows from (6.31) (ii,iii) that

$$(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1$$

so that $R$ qualifies as a twist. Moreover if $F \in H \otimes H$ is any twist then, as shown in \[13\], $H$ is also quasi-triangular under the twisted structure of equations (4.16, 4.17) with $R$-matrix

$$R_F = F^T R F^{-1}. \hspace{1cm} (6.32)$$

It was shown in \[13\] that

**Proposition 6.** Under the QHA of proposition 2 $H$ is also quasi-triangular with $R$-matrix

$$R' = (S \otimes S)R.$$

We have seen that the QHA structure of proposition 2 is obtainable by twisting with the Drinfeld twist $F_\delta$. It was further shown in \[13\] that the full structure of proposition 5 is also obtained by twisting with $F_\delta$ which, in view of equation (6.32), is equivalent to

$$(S \otimes S)R = F_\delta^T R F_\delta^{-1}. \hspace{1cm} (6.33)$$

This result in fact follows from the following relation

$$(S \otimes S)R \cdot \gamma = \gamma^T R,$$

where $\gamma^T = T \cdot \gamma$, proved in \[13\]. In view of proposition 3 this last equation is equivalent to

$$R\tilde{\gamma} = \tilde{\gamma}^T \cdot (S \otimes S)R,$$

where $\tilde{\gamma}^T = T \cdot \tilde{\gamma}$, with $\gamma$ and $\tilde{\gamma}$ as in equations (5.23, 5.24).

In view of (6.31) (i) the opposite co-product is obtained from $\Delta$ by twisting with $R$. In fact we have the following result proved in \[13\]:

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Proposition 7. The opposite structure of propositions 14 is obtainable by twisting with the R-matrix \( R \) but with antipode \( S \) and canonical elements \( \alpha_R, \beta_R \) respectively.

Above \( \alpha_R, \beta_R \) are given by equation (6.34), so that
\[
(i) \quad \alpha_R = m \cdot (1 \otimes \alpha)(S \otimes 1)R^{-1}, \quad \beta_R = m \cdot (1 \otimes \beta)(1 \otimes S)R.
\]

Below we set
\[
(ii) \quad R = e_i \otimes e^i, \quad R^{-1} = \bar{e}_i \otimes \bar{e}^i
\]
in terms of which we may write
\[
(iii) \quad \alpha_R = S(\bar{e}_i)\bar{e}^i, \quad \beta_R = e_i\beta S(e^i). \tag{6.34}
\]

Thus with the co-product \( \Delta^T \) and co-associator \( \Phi^T = \Phi_{321}^{-1} \) of proposition 11, we have two QHA structures with differing quasi-antipodes \((S, \alpha_R, \beta_R)\) and \((S^{-1}, \alpha^T, \beta^T)\) where, from proposition 11, \( \alpha^T = S^{-1}(\alpha), \beta^T = S^{-1}(\beta) \). It follows from Theorem 1 that

Theorem 5. There exists a unique invertible \( u \in H \) such that
\[
S(a) = uS^{-1}(a)u^{-1}, \text{ or } S^2(a) = uau^{-1}, \quad \forall a \in H
\]
and
\[
uS^{-1}(\alpha) = \alpha_R, \quad \beta_R u = S^{-1}(\beta). \tag{6.35}
\]

Explicitly,
\[
\begin{align*}
    u &= S(Y_\nu \beta S(Z_\nu))\alpha_R X_\nu = S(Z_\nu)\alpha_R \bar{Y}_\nu S^{-1}(\beta)S^{-1}(X_\nu) \\
    u^{-1} &= Z_\nu \beta_R S(S(X_\nu)\alpha Y_\nu) = S^{-1}(\bar{Z}_\nu)S^{-1}(X_\nu)\bar{Y}_\nu \beta_R S(\bar{X}_\nu). \tag{6.36}
\end{align*}
\]

Above we have used the fact that the opposite QHA structure has co-associator \( \Phi^T = \Phi_{321}^{-1} \) and quasi-antipode \((S^{-1}, \alpha^T, \beta^T)\). We have then applied Theorem 1 with \((\bar{S}, \bar{\alpha}, \bar{\beta}) = (S, \alpha_R, \beta_R)\) to give the result.

The above gives the \( u \)-operator of Drinfeld-Reshetikhin 5 [18]. It differs from, but is related to, the \( u \)-operator of Altschuler and Coste 11. To see how the latter arises, it is easily seen that \( \bar{R} \equiv (RT)^{-1} \) also satisfies equation (6.31) and thus constitutes an \( R \)-matrix. Thus proposition 7 and Theorem 5 also hold with \( R \) replaced by \( \bar{R} \). This implies the existence of a unique invertible \( \bar{u} \in H \) such that
\[
S^2(a) = \bar{u}au^{-1}, \quad \forall a \in H
\]
and
\[
\bar{u}S^{-1}(\alpha) = \alpha_R, \quad \beta_R \bar{u} = S^{-1}(\beta)
\]
with \( \alpha_R, \beta_R \) as in equation (6.34) but with \( R \) replaced by \( \bar{R} \). Explicitly we have, in this case,
\[
\begin{align*}
    \bar{u} &= S(Y_\nu \beta S(Z_\nu))\alpha_R X_\nu = S(Z_\nu)\alpha_R \bar{Y}_\nu S^{-1}(\beta)S^{-1}(X_\nu) \\
    \bar{u}^{-1} &= Z_\nu \beta_R S(S(X_\nu)\alpha Y_\nu) = S^{-1}(\bar{Z}_\nu)S^{-1}(X_\nu)\bar{Y}_\nu \beta_R S(\bar{X}_\nu). \tag{6.37}
\end{align*}
\]

Then, as can be seen from 12 \( \bar{u} \) is precisely the \( u \)-operator of Altschuler and Coste.

To see the relation between \( u \) and \( \bar{u} \) we first note that \( uS(u) = S(u)u \) is central. This follows by applying \( S \) to \( S(a) = uS^{-1}(a)u^{-1} \), giving
\[
S^2(a) = S(u^{-1})aS(u), \quad \forall a \in H.
\]

Before proceeding it is worth noting the following
Lemma 1.

\[(i) \quad \beta_R = S(u)S(\beta), \quad \alpha_R = S(\alpha)S(u^{-1})
\]
\[(ii) \quad \beta_R = S(\bar{u})S(\beta), \quad \alpha_R = S(\alpha)S(\bar{u}^{-1}). \tag{6.38}\]

**Proof.** By symmetry it suffices to prove (i). Now

\[
\begin{align*}
\beta_R &= m \cdot (1 \otimes \beta)(1 \otimes S)(R^T)^{-1} = \bar{e}^i \beta S(\bar{e}_i) \\
&= \bar{e}^i S(\beta_R u)S(\bar{e}_i) = \bar{e}^i S(u)S(\beta_R S(\bar{e}_i)) \\
&= \bar{e}^i S(u)S(\beta_R S(\bar{e}_j)S(\bar{e}_i)) \\
&= S(u)S(\bar{e}^i)S(\beta)S(\bar{e}_j)S(\bar{e}_i) \\
&= S(u)S(\bar{e}^i)S(\beta)S(\bar{e}_j)e_i = S(u)S(\beta).
\end{align*}
\]

where we have used the obvious result

\[
\bar{e}_i e_j \otimes \bar{e}^i e^j = R^{-1}R = 1 \otimes 1.
\]

Similarly

\[
\begin{align*}
\alpha_R &= m \cdot (1 \otimes \alpha)(S \otimes 1)R^T = S(\alpha)\alpha e_i \\
&= S(\alpha)S(u^{-1}\alpha_R e_i) = S(\alpha)S(\alpha_R S(u^{-1})e_i) \\
&= S(\alpha)S(\bar{e}_j\alpha e^j)S(u^{-1})e_i \\
&= S(\alpha)S(\bar{e}^i)S(\alpha)S(\bar{e}_j)S(u^{-1})e_i \\
&= S(\alpha)S(\bar{e}_j\alpha e^j)S(u^{-1})e_i.
\end{align*}
\]

We are now in a position to prove

**Lemma 2.**

\[
\tilde{u} = S(u^{-1})
\]

**Proof.** From equation (6.38) we have

\[
\begin{align*}
\tilde{u} &= S(Y_{\nu}/\beta S(Z_{\nu}))\alpha_R X_{\nu} \\
&\overset{(i)}{=} S(Y_{\nu}/\beta S(Z_{\nu}))S(\alpha)S(u^{-1})X_{\nu} \\
&= S(Y_{\nu}/\beta S(Z_{\nu}))S(\alpha)S^2(X_{\nu})S(u^{-1}) \\
&= S[S(X_{\nu})\alpha Y_{\nu}/\beta S(Z_{\nu})]S(u^{-1}) \\
&\overset{2.56}{=} S(u^{-1}).
\end{align*}
\]

The above result clearly shows the connection between the \(u\)-operator of Theorem 3 and that due to Altschuler and Coste. Obviously the existence of the \(u\)-operator in the quasi-triangular case is a direct consequence of Theorem 1 and proposition 7, the latter showing the equivalence
of the opposite structure of proposition 1 with that due to twisting with $R$. In the case $H$ is not quasi-triangular, this opposite structure is not in general obtainable by a twist.

The operators $u$ and $\tilde{u}$ are special cases of the $v$ operator of Theorem 1, it follows then from Theorem 2 that

**Theorem 6.** The operators $u$ and $\tilde{u}$ are invariant under twisting.

In section 3 we discussed the uniqueness of the quasi-antipode $(S, \alpha, \beta)$, but nothing has been said about the uniqueness of the twisted structures or the $R$-matrix in the quasi-triangular case. This is intimately connected with the quasi-cocycle condition to which we now turn.

## 7 The quasi-cocycle condition

The set of twists on a QHA $H$ forms a group, moreover, the twisted structure of equations (4.16, 4.17) induced on a QHA $H$ preserves this group structure in the following sense.

**Lemma 3.** Let $F, G \in H \otimes H$ be twists on a QHA $H$. Then in the notation of equations (4.16, 4.17)

1. $\Delta_{FG} = (\Delta_G)_F, \Phi_{FG} = (\Phi_G)_F$
2. $\alpha_{FG} = (\alpha_G)_F, \beta_{FG} = (\beta_G)_F$.

Moreover, if $H$ is quasi-triangular then

3. $R_{FG} = (R_G)_F$. (7.39)

In other words the structure obtained from twisting with $G$ and then with $F$ is the same as twisting with the twist $FG$. It is important that the right hand side of equation (7.39) is interpreted correctly, e.g. $(\Phi_G)_F$ is given as in equation (4.16) but with $\Phi$ replaced by $\Phi_G$ and $\Delta$ by $\Delta_G$ etc.

Given any QBA $H$ we may impose on a twist $F \in H \otimes H$ the following condition

$$ (F \otimes 1) \cdot (\Delta \otimes 1) F \cdot \Phi = \Phi \cdot (1 \otimes F) \cdot (1 \otimes \Delta) F $$

(7.40)

which we call the **quasi-cocycle condition**.

When $\Phi = 1 \otimes 1 \otimes 1$ this reduces to the usual cocycle condition on Hopf algebras. In the notation of equation (4.16), the quasi-cocycle condition is equivalent to

$$ \Phi_F = \Phi. $$

(7.40')

Thus twisting on a QBA by a twist $F$ satisfying the quasi-cocycle condition results in a QBA structure with the same co-associator.

It is thus not surprising that the quasi-cocycle condition (7.40) is intimately related to the uniqueness of twisted structures on a QHA $H$. Indeed, if $F, G \in H \otimes H$ are twists giving rise to the same QBA structure, so that

$$ \Delta_F = \Delta_G, \Phi_F = \Phi_G $$

(7.41)

then $C \equiv F^{-1}G$ must commute with the co-product $\Delta$ and satisfy the quasi-cocycle condition. Indeed in view of lemma 3 we have

$$ \Delta_C = \Delta_{F^{-1}G} = (\Delta_G)_{F^{-1}} = (\Delta_F)_{F^{-1}} = \Delta $$

$$ \Phi_C = \Phi_{F^{-1}G} = (\Phi_G)_{F^{-1}} = (\Phi_F)_{F^{-1}} = \Phi $$

This leads to the following
Definition 3. A twist $C \in H \otimes H$ on any QBA $H$ is called compatible if

(i) $C$ commutes with the co-product $\Delta$
(ii) $C$ satisfies the quasi-cocycle condition.

In other words twisting a QBA $H$ with a compatible twist $C$ gives exactly the same QBA structure. The set of compatible twists on $H$ thus forms a subgroup of the group of twists on $H$.

Proposition 8. Let $F, G \in H \otimes H$ be twists on a QBA $H$. Then the twisted structures induced by $F$ and $G$ coincide if and only if there exists a compatible twist $C \in H \otimes H$ such that $G = F C$.

Proof. We have already seen that if $F, G$ give rise to the same QBA structure then $C = F^{-1}G$ is a compatible twist and $G = F C$. Conversely, suppose $C$ is a compatible twist and set $G = F C$. Then

\[ \Delta G = \Delta F C = (\Delta C)_F = \Delta F \]

\[ \Phi G = \Phi F C = (\Phi C)_F = \Phi F \]

so that $G$ gives precisely the same twisted structure as $F$.

Corollary. Let $F \in H \otimes H$ be a twist on a QBA $H$. Then the twisted structure induced by $F$ coincides with the structure on $H$ if and only if $F$ is a compatible twist.

In view of the group properties of twists the above corollary is equivalent to proposition 8.

Let $H$ be a quasi-triangular QHA with $R$-matrix $R$ satisfying equation (6.31). From proposition 8 the opposite co-associator $\Phi^T = \Phi_{R^{-1}}$ and co-product $\Delta^T$ are obtained by twisting with $R$, so that $\Phi^T = \Phi_R$. The proof of this result utilises only the properties (6.31). Hence, since

\[ \Phi = \Phi_{R^{-1}R} = (\Phi_R)_{R^{-1}} = (\Phi^T)_{R^{-1}} \]

it follows that if $Q$ is another $R$-matrix for $H$ i.e. satisfies equation (6.31), then we must have also

\[ (\Phi^T)_{Q^{-1}} = \Phi. \]

Then $Q^{-1}R$ must qualify as a compatible twist. Indeed it obviously commutes with $\Delta$, while as to the quasi-cocycle condition, we have

\[ \Phi_{Q^{-1}R} = (\Phi_R)_{Q^{-1}} = (\Phi^T)_{Q^{-1}} = \Phi. \]

Note that $(Q^T)^{-1}, (R^T)^{-1}$ also determine $R$-matrices so the following must all determine compatible twists: $Q^{-1}R$, $Q^T R$, $R^{-1}Q$, $R^T Q$. In particular $R^T R$ must determine a compatible twist, as may be verified directly.

With the notation of section 4 it is easily seen that the operator

\[ A = \Delta(u^{-1})F_s^{-1}(u \otimes u)F_0 = F_s^{-1}(u \otimes u)F_0 \Delta(u^{-1}) \quad (7.42) \]

commutes with $\Delta$. This operator appears in the work of Altschuler and Coste in connection with ribbon QHAs. The operator $A$ satisfies the quasi-cocycle condition and thus determines a compatible twist.

For general QBAs $H$, to see that there are sufficiently many compatible twists, we have
**Lemma 4.** Let $z \in H$ be an invertible central element. Then

$$C = (z \otimes z) \Delta(z^{-1})$$

is a compatible twist.

**Proof.** Obviously $C$ commutes with the co-product $\Delta$ so it remains to prove that it satisfies the quasi-cocycle condition. To this end note that

$$(C \otimes 1)(\Delta \otimes 1)C = (z \otimes z \otimes 1)(\Delta(z^{-1}) \otimes 1)(\Delta(z) \otimes z)(\Delta \otimes 1)\Delta(z^{-1})$$

and similarly

$$(1 \otimes C)(1 \otimes \Delta)C = (1 \otimes z \otimes z)(1 \otimes \Delta(z^{-1}))(z \otimes \Delta(z))(1 \otimes \Delta)\Delta(z^{-1})$$

thus

$$(C \otimes 1)(\Delta \otimes 1)C \Phi = (z \otimes z \otimes z)(\Delta \otimes 1)\Delta(z^{-1})\Phi$$

$$= (z \otimes z \otimes z)(\Delta \otimes z)(\Delta(z^{-1}))$$

$$= (z \otimes z \otimes z)(1 \otimes \Delta)\Delta(z^{-1})$$

with $C$ as in the lemma, we see that

$$\epsilon \otimes 1)C = (1 \otimes \epsilon)C = \epsilon(z).$$

Thus, strictly speaking, $\epsilon(z^{-1})C$ qualifies as a compatible twist.

Following Altschuler and Coste [1], a quasi-triangular QHA is called a ribbon QHA if the operator $A$ of equation (7.42) is given by

$$A = (v \otimes v)\Delta(v^{-1})$$

for a certain invertible central element $v$, related to the $u$-operator $u$. This is consistent with the lemma above and the fact that $A$ determines a compatible twist.

In the case of ribbon Hopf algebras, we have $R^T R = (v \otimes v)\Delta(v^{-1})$, so that the compatible twist $R^T R$ is also of the form of lemma 4. This may not be the case for quasi-triangular QHAs in general.

It is worth noting that if $H$ is a QHA and $C \in H \otimes H$ a compatible twist then $H$ is also a QHA under the twisted structure induced by $C$ with exactly the same co-product $\Delta$, co-unit $\epsilon$, co-associator $\Phi$, antipode $S$, but with canonical elements given by equation (7.44); viz

$$\alpha_C = m \cdot (S \otimes 1)(1 \otimes \alpha)C^{-1}, \quad \beta_C = m \cdot (1 \otimes S)(1 \otimes \beta)C.$$

In view of Theorem 4 and its corollary, we have immediately
**Proposition 9.** Suppose $C \in H \otimes H$ is a compatible twist on a QHA $H$. Then there exists a unique invertible central element $z \in H$ such that

$$z\alpha = \alpha C, \quad \beta Cz = \beta.$$ 

Explicitly

$$z = S(X_{\nu})\alpha C Y_{\nu} \beta S(Z_{\nu}) = \bar{X}_{\nu} \beta S(\bar{Y}_{\nu}) \alpha C \bar{Z}_{\nu}$$

$$z^{-1} = S(X_{\nu})\alpha Y_{\nu} \beta C S(Z_{\nu}) = \bar{X}_{\nu} \beta C S(\bar{Y}_{\nu}) \alpha \bar{Z}_{\nu}.$$ 

In the case $H$ is quasi-triangular we have seen that $C = R^T R$ is a compatible twist. Since the latter form a group we have the infinite family of compatible twists $C = (R^T R)^m, m \in \mathbb{Z}$, in which case the central elements $z^{\pm 1}$ of proposition 9 give the quadratic invariants of $[12]$.

We conclude this section by noting, in the quasi-triangular case, that twisting the Drinfeld twist with the $R$-matrix $R$ gives, from Theorem 4, the twisted Drinfeld twist

$$F^R_\delta = (F_\delta)R = (S \otimes S)(R^T)^{-1} \cdot F_\delta \cdot R^{-1}.$$ 

On the other hand, since $(R^T)^{-1}$ is an $R$-matrix we have, from eq. (6.33),

$$(S \otimes S)(R^T)^{-1} = F^T_\delta (R^T)^{-1} F^{-1}_\delta$$

which implies

$$F^R_\delta = F^T_\delta (R^T)^{-1} \cdot R^{-1} = F^T_\delta (R^R)^{-1}$$

where $R^R$ and its inverse are compatible twists under the opposite structure. This shows that $F^T_\delta$ will give rise to a Drinfeld twist under the opposite structure of proposition 9 induced by twisting with $R$ (which has antipode $S$ rather than $S^{-1}$). Applying $T$ to the equation above gives

$$(F^R_\delta)^T = F_\delta (R^T R)^{-1}$$

which shows that, since $R^T R$ and its inverse are compatible twists, $(F^R_\delta)^T$ also gives rise to a Drinfeld twist on $H$.

### 8 Quasi-dynamical QYBE

Throughout we assume $H$ is a quasi-triangular QHA with $R$-matrix $R$ satisfying (6.31) which we reproduce here:

$$(i) \quad \Delta^T (a) R = R \Delta (a), \quad \forall a \in H$$

$$(ii) \quad (\Delta \otimes 1) R = \Phi_{231}^{-1} R_{13} \Phi_{312} R_{23} \Phi_{123}^{1}$$

$$(iii) \quad (1 \otimes \Delta) R = \Phi_{312} R_{13} \Phi_{213}^{-1} R_{12} \Phi_{123}.$$ 

Applying $T \otimes 1$ to (ii) and $1 \otimes T$ to (iii) then gives

$$(i i') \quad (\Delta^T \otimes 1) R = \Phi_{321}^{-1} R_{23} \Phi_{312} R_{13} \Phi_{213}$$

$$(i i') \quad (1 \otimes \Delta^T) R = \Phi_{321} R_{12} \Phi_{213}^{-1} R_{13} \Phi_{132}.$$ 

It follows that

$$R_{12} (\Delta \otimes 1) R = (\Delta^T \otimes 1) R \cdot R_{12}$$

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from which we deduce that $\mathcal{R}$ must satisfy the quasi-QYBE:

$$\mathcal{R}_{12}\Phi_{231}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1} = \Phi_{231}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1}\mathcal{R}_{12}. \tag{8.45}$$

If we twist $H$ with a twist $F \in H \otimes H$ then $H$ is also a quasi-triangular QHA under the twisted structure $\Phi_{123} \otimes \Phi_{231}$ induced by $F$ with universal $R$-matrix

$$\mathcal{R}_F = F^T \mathcal{R} F^{-1}.$$

Following equation (7.40) we say a twist $F(\lambda) \in H \otimes H$ satisfies the shifted quasi-cocycle condition if

$$[F(\lambda) \otimes 1] \cdot (\Delta \otimes 1) F(\lambda) \cdot \Phi = \Phi \cdot [1 \otimes F(\lambda + h^{(1)})] \cdot (1 \otimes \Delta) F(\lambda) \tag{8.46}$$

where $\lambda \in H$ depends on one (or possibly several) parameters and $h \in H$ is fixed. Alternatively, we may write in obvious notation

$$F_{12}(\lambda) \cdot (\Delta \otimes 1) F(\lambda) \cdot \Phi = \Phi \cdot F_{23}(\lambda + h^{(1)}) \cdot (1 \otimes \Delta) F(\lambda). \tag{8.46}$$

When $h = 0$, this reduces to the quasi-cocycle condition (7.40) satisfied by $F = F(\lambda)$. When $\Phi = 1 \otimes 1 \otimes 1$ (i.e. the normal Hopf-algebra case) equation (8.46) reduces to the usual shifted cocycle condition.

Twisting $H$ with a twist $F$ satisfying the (unshifted) quasi-cocycle condition results in a QHA with the same co-associator $\Phi$, co-unit $\epsilon$ and antipode $S$ but with the twisted co-product $\Delta_F$, $R$-matrix $\mathcal{R}_F$ (and canonical elements $\alpha_F, \beta_F$). We now consider twisting $H$ with a twist $F = F(\lambda)$ satisfying the shifted condition (8.46). Then under this twisted structure $H$ is also a quasi-triangular QHA with the same co-unit $\epsilon$ and antipode $S$ but with the co-associator $\Phi(\lambda) = \Phi_{F(\lambda)}$, and co-product and $R$-matrix given by

$$\Delta_\lambda(a) = F(\lambda)\Delta(a)F(\lambda)^{-1}, \quad \forall a \in H, \quad \mathcal{R}(\lambda) = F^T(\lambda)\mathcal{R}F(\lambda)^{-1} \tag{8.47}$$

with canonical elements $\alpha_\lambda = \alpha_{F(\lambda)}, \beta_\lambda = \beta_{F(\lambda)}$.

In view of equation (8.46) we have for the co-associator

$$\Phi(\lambda) = F_{12}(\lambda) \cdot (\Delta \otimes 1) F(\lambda) \cdot \Phi \cdot (1 \otimes \Delta) F(\lambda)^{-1} \cdot F_{23}(\lambda)^{-1}
= \Phi \cdot F_{23}(\lambda + h^{(1)}) \cdot (1 \otimes \Delta) F(\lambda) \cdot (1 \otimes \Delta) F(\lambda)^{-1} \cdot F_{23}(\lambda)^{-1}
= \Phi \cdot F_{23}(\lambda + h^{(1)}) \cdot F_{23}(\lambda)^{-1} \tag{8.48}$$

which implies

$$\Phi(\lambda)^{-1} = F_{23}(\lambda) \cdot F_{23}(\lambda + h^{(1)})^{-1} \cdot \Phi^{-1}.$$

In the Hopf-algebra case equation (8.48) reduces to the expression for $\Phi(\lambda)$ obtained in (13) ($\Phi = 1 \otimes 1 \otimes 1$).

Under the above twisted structure equation (8.31)(ii) becomes

$$(\Delta_\lambda \otimes 1)\mathcal{R}(\lambda) = \Phi_{231}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132}(\lambda) \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{123}^{-1}(\lambda).$$

Now

$$\Phi_{132}(\lambda) = (1 \otimes T)\Phi_{123}(\lambda) \tag{8.48}$$

$$\Phi_{132} \cdot F_{23}^T(\lambda + h^{(1)}) \cdot F_{23}^T(\lambda)^{-1} \tag{8.49}$$
which implies
\[
(\Delta_\lambda \otimes 1)\mathcal{R}(\lambda) = \Phi_{231}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132} \cdot F_{23}^T(\lambda + h^{(1)}) \cdot F_{23}^T(\lambda)^{-1} \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{123}(\lambda)
\]

Similarly equation (6.31)(iii) becomes
\[
(1 \otimes \Delta_\lambda)\mathcal{R}(\lambda) = \Phi_{312}(\lambda) \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{213}(\lambda)^{-1} \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{123}(\lambda).
\]

Now
\[
\Phi_{312}(\lambda) = (T \otimes 1)(1 \otimes T)\Phi_{123}(\lambda)
\]

while
\[
\Phi_{213}(\lambda)^{-1} = (T \otimes 1)[\Phi_{132} \cdot F_{23}^T(\lambda + h^{(1)}) \cdot F_{23}^T(\lambda)^{-1}]
\]

Therefore
\[
(1 \otimes \Delta_\lambda)\mathcal{R}(\lambda) = \Phi_{312} \cdot F_{13}(\lambda + h^{(2)}) \cdot F_{13}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot F_{13}(\lambda) \cdot F_{13}(\lambda + h^{(2)})^{-1} \cdot \Phi_{213} \cdot \mathcal{R}_{12}(\lambda) \cdot \Phi_{123}(\lambda)
\]

We thus arrive at

**Lemma 5.** \(\mathcal{R}(\lambda)\) satisfies the co-product properties

\[
(i) \quad (\Delta_\lambda \otimes 1)\mathcal{R}(\lambda) = \Phi_{231}(\lambda)^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132} \cdot \mathcal{R}_{23}(\lambda + h^{(1)}) \cdot \Phi_{123}^{-1}
\]

\[
(ii) \quad (1 \otimes \Delta_\lambda)\mathcal{R}(\lambda) = \Phi_{312} \cdot R_{13}(\lambda + h^{(2)}) \cdot \Phi_{213}^{-1} \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{123}(\lambda)
\]

\[
(iii) \quad (\Delta_\lambda^T \otimes 1)\mathcal{R}(\lambda) = \Phi_{321}(\lambda) \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{312} \cdot \mathcal{R}_{13}(\lambda + h^{(2)}) \cdot \Phi_{213}^{-1}
\]

\[
(iv) \quad (1 \otimes \Delta_\lambda^T)\mathcal{R}(\lambda) = \Phi_{321} \cdot R_{12}(\lambda + h^{(2)}) \cdot \Phi_{231}^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132}(\lambda)
\]

(8.50)

**Proof.** We have already proved (i) and (ii) while (iii) follows by applying \((T \otimes 1)\) to (i) and (iv) by applying \((1 \otimes T)\) to (ii).
We are now in a position to determine the QQYBE satisfied by $\mathcal{R} = \mathcal{R}(\lambda)$ for this twisted structure. We have

\[
\mathcal{R}_{23}(\lambda) \cdot \Phi_{312} \cdot \mathcal{R}_{13}(\lambda + h^{(2)}) \cdot \Phi_{213}^{-1} \cdot \mathcal{R}_{12}(\lambda)
\]

where for the last three terms we have

\[
\Phi_{132}(\lambda) \cdot \mathcal{R}_{23}(\lambda) \cdot \Phi_{123}^{-1}(\lambda) = \Phi_{132} \cdot F_{23}^{T}(\lambda + h^{(1)}) \cdot F_{23}^{T}(\lambda)^{-1}
\]

\[
\cdot \mathcal{R}_{23}(\lambda) \cdot F_{23}(\lambda) \cdot F_{23}(\lambda + h^{(1)})^{-1} \cdot \Phi_{123}^{-1}
\]

\[
\Phi_{132} \cdot \mathcal{R}_{23}(\lambda + h^{(1)}) \cdot \Phi_{123}^{-1}.
\]

Hence

\[
\mathcal{R}_{23}(\lambda) \cdot \Phi_{312} \cdot \mathcal{R}_{13}(\lambda + h^{(2)}) \cdot \Phi_{213}^{-1} \cdot \mathcal{R}_{12}(\lambda)
\]

\[
= \Phi_{321} \cdot \mathcal{R}_{12}(\lambda + h^{(3)}) \cdot \Phi_{231}^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132} \cdot \mathcal{R}_{23}(\lambda + h^{(1)}) \cdot \Phi_{123}^{-1}.
\]

We thus arrive at

**Proposition 10.** $\mathcal{R}(\lambda)$ satisfies the quasi-dynamical QYBE

\[
\mathcal{R}_{12}(\lambda + h^{(3)}) \cdot \Phi_{231}^{-1} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{132} \cdot \mathcal{R}_{23}(\lambda + h^{(1)}) \cdot \Phi_{123}^{-1}
\]

\[
= \Phi_{321} \cdot \mathcal{R}_{12}(\lambda + h^{(3)}) \cdot \Phi_{231}^{-1} \cdot \mathcal{R}_{13}(\lambda + h^{(2)}) \cdot \Phi_{213}^{-1} \cdot \mathcal{R}_{12}(\lambda).
\]

(8.51)

In the Hopf algebra case ($\Phi = 1 \otimes 1 \otimes 1$) equation (8.51) reduces to the usual dynamical QYBE. If we set $h = 0$ then equation (8.51) reduces to the quasi-QYBE satisfied by $\mathcal{R} = \mathcal{R}(\lambda)$. Hence the term quasi-dynamical QYBE for (8.51) we could, alternatively, refer to (8.51) as the dynamical quasi-QYBE (dynamical QQYBE), since it is obviously the quasi-Hopf algebra analogue of the usual dynamical QYBE.

With respect to the QHA structure of propositions 2 we have the $R$-matrices

\[
\mathcal{R}'(\lambda) = (S \otimes S)\mathcal{R}(\lambda), \quad \mathcal{R}_0(\lambda) = (S^{-1} \otimes S^{-1})\mathcal{R}(\lambda)
\]

respectively. Then applying $(S \otimes S \otimes S)$, $(S^{-1} \otimes S^{-1} \otimes S^{-1})$ respectively to equation (8.51) it follows that both of these $R$-matrices satisfy the opposite quasi-dynamical QYBE

\[
\mathcal{R}_{12}(\lambda) \cdot \mathcal{R}_{13}(\lambda + h^{(2)}) \cdot \mathcal{R}_{23}(\lambda + h^{(1)}) \cdot \mathcal{R}_{12}(\lambda)
\]

\[
= \Phi_{321}^{-1} \cdot \mathcal{R}_{23}(\lambda + h^{(3)}) \cdot \Phi_{312} \cdot \mathcal{R}_{13}(\lambda) \cdot \Phi_{213}^{-1} \cdot \mathcal{R}_{23}(\lambda + h^{(1)}) \cdot \Phi_{123}^{-1},
\]

where $\Phi$ is the co-associator of propositions 2 and $\mathcal{R}(\lambda)$ denotes $\mathcal{R}'(\lambda)$, $\mathcal{R}_0(\lambda)$ respectively. Moreover applying $(T \otimes 1)(1 \otimes T)(T \otimes 1)$ to equation (8.51) it is easily seen that $\mathcal{R}^T(\lambda)$ also satisfies the above opposite quasi-dynamical QYBE but with respect to the opposite co-associator $\Phi^T$ of proposition 11.

We anticipate that the quasi-dynamical QYBE will play an important role in obtaining elliptic solutions to the QQYBE from trigonometric ones via twisted QUEs. Of particular interest is the quasi-dynamical QYBE for elliptic quantum groups.
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