The role of the Polyakov loop in the Dirac operator of QCD at finite temperature

Vicente Azcoiti
Departamento de Física Teórica, Universidad de Zaragoza,
Plaza San Francisco, 50009 Zaragoza, Spain

Abstract: We show how all the contributions to the determinant of the Dirac-Kogut-Susskind operator of QCD at finite temperature containing a net number of Polyakov loops become irrelevant in the infinite volume limit. We discuss also on two of the most interesting physical implications of this result: i) the restoration of the Polyakov symmetry in the full theory with dynamical fermions and ii) the total suppression of baryonic thermal fluctuations in QCD at finite temperature.

1 Introduction

The fermion determinant in QCD and in general in any gauge theory is a gauge invariant non local operator, the main contributions of which being complex combinations of closed Wilson loops including the Wilson line or Polyakov loop [1] which is closed through the boundary of the finite time direction.

In the pure gauge model the gauge action is invariant under Polyakov transformations and, because of that, the mean value of the Polyakov loop has been extensively used as order parameter in the investigations of the finite temperature phase transition. The situation however change in the full theory with dynamical fermions where the Polyakov symmetry is explicitly broken by the contribution to the integration measure of the determinant of the Dirac operator. The general wisdom in this case is that the Polyakov loop is no more an order parameter for the finite temperature phase transition.

Notwithstanding that I will show how in lattice QCD with staggered fermions all the contributions to the fermion determinant containing a net number of Polyakov loops become irrelevant in the infinite spatial volume limit. The Polyakov symmetry is recovered in this limit and we can therefore kill all these contributions from the beginning and work with a theory with dynamical fermions which preserves the Polyakov symmetry.
But this is, maybe, neither the only nor the most interesting physical implication of this result. In fact the Hilbert space of physical states can be decomposed as a direct sum of Hilbert spaces, each one of them corresponding to a fixed value of the quark number operator. The conservation of baryonic charge in QCD implies that all these spaces are invariant under the hamiltonian and we can therefore write the partition function as a sum of canonical partition functions, each one of them corresponding to a fixed value of the quark number operator. This decomposition of the partition function coincides with the integrated partition function obtained from the Polyakov loop expansion of the fermion determinant, the quark content being the net number of Polyakov loops. Since, as we will show, all the contributions with a net number of Polyakov loops are irrelevant, we conclude that thermal fluctuations of physical states with non vanishing baryonic charge are fully suppressed in QCD at finite temperature.

A method to implement in standard simulations the Polyakov symmetry and to kill baryonic thermal fluctuations in full QCD will also be developed here. The rest of the paper is organized as follows: section 2 describes some general features of the partition function of QCD at finite temperature. In section 3 we show the main result of this work with the help of an extra abelian degree of freedom. Section 4 is devoted to discuss some interesting physical consequences which follow from the results of section 3. In section 5 we discuss on the generalization of these results for Wilson fermions.

## 2 The Partition Function

The partition function of QCD at finite temperature $T$

$$Z = Tr(e^{-\frac{H}{T}})$$

is the trace over the Hilbert space of physical states of minus the inverse temperature times the hamiltonian. Taking into account the conservation of baryonic charge we can write the partition function as a sum of canonical partition functions at fixed baryon number as follows:

$$Z = \sum_k Tr_k(e^{-\frac{H}{T}}),$$

where $Tr_k$ in (2) indicates the trace over the subspace of fixed quark number $k$ and $k = 0, +3, -3, +6, -6, ...$

The decomposition of the partition function given in (2) is the standard representation used in the investigations of QCD at finite baryon density. In this last case the partition function (2) is slightly modified by the introduction of a chemical potential $\mu$ in the following way [2, 3]

$$Z = \sum_k e^{\mu k} Tr_k(e^{-\frac{H}{T}}).$$

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The previous decomposition (2) of the partition function as a sum of canonical partition functions at fixed baryon number corresponds in lattice regularized QCD to the Polyakov loop expansion of the integrated partition function which, for staggered fermions, can be written as follows

\[ Z = \bar{a}_{3V_x} + \bar{a}_{3(V_x - 1)} + \ldots + \bar{a}_0 + \bar{a}_{-3} + \ldots + \bar{a}_{-3V_x}, \]  

where \( \bar{a}_i \) \((\bar{a}_{-i})\) is the integral over the gauge group of all the contributions to the fermion determinant containing \( i \) forward (backward) Polyakov loops respectively, the integral being weighted with the pure gauge Boltzmann factor. The maximum number of Polyakov loops which can appear in a given coefficient of (4) for SU(3) and Kogut-Susskind fermions is three times the spatial lattice volume. \( \bar{a}_0 \) is the contribution with no net number of Polyakov loops and of course we have also all the symmetric contributions corresponding to backward Polyakov loops.

Since each coefficient in (4) represents the partition function at a fixed baryon number, this expression is also consistent with the fact that \( V_x \) is the maximum number of baryons allowed by the Fermi-Dirac statistics in a finite and discrete space of \( V_x \) points and staggered fermions.

All the averaged coefficients \( \bar{a}_i \) in (4) are positive definite since they correspond to the decomposition of the partition function as a sum of canonical partition functions over the subspaces of fixed baryon number. Due to the Z(3) Polyakov symmetry of the pure gauge action, the only non vanishing coefficients are those containing a multiple of three times Wilson lines, which on the other hand reflects the fact that the only physical states in QCD are mesons, baryons and combinations of them.

3 The Extra-Abelian Degree of Freedom

Let us consider the determinant of the Dirac-Kogut-Susskind operator for the slightly modified gauge configuration which consists in multiplying all link variables at a fixed time-slice and pointing forward in the time direction by the global phase factor \( e^{i\eta} \) and all the hermitian conjugates by \( e^{-i\eta} \). Taking into account that the standard way to introduce a chemical potential in the lattice is to multiply all links pointing forward (backward) in the time direction by a factor \( e^{i\mu} \) \((e^{-i\mu})\) respectively, this is just what corresponds to consider an imaginary chemical potential \( \mu = i\eta/L_t \), \( L_t \) being the lattice temporal extent.

The fermion determinant for a given gauge configuration and with the previous extra-degree of freedom can be written as follows

\[ Det\Delta(i\eta) = a_{3V_x}e^{i3V_x\eta} + \ldots + a_1e^{i\eta} + a_0 + a_{-1}e^{-i\eta} + \ldots + a_{-3V_x}e^{-i3V_x\eta}. \]  

The coefficients \( a_i \) \((a_{-i})\) in (5), averaged over the gauge group with the corresponding pure gauge Boltzmann factor, are the coefficients \( \bar{a}_i \) \((\bar{a}_{-i})\) which appear in (4).

\[ \bar{a}_i = \int [DU]a_i(U) e^{-\beta S_G(U)}. \]  

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The first interesting property of expression (5) which follows from the hermiticity and chiral properties of the fermion matrix $\Delta$ is that

$$\text{Det} \Delta(i\eta) \geq 0$$  \hspace{1cm} (7)

for every $\eta$.

By inverse Fourier transformation we can write

$$a_j = \frac{1}{2\pi} \int e^{-ij\eta} \text{det} \Delta(i\eta) d\eta,$$  \hspace{1cm} (8)

$a_j = 0$ if $j > 3V_x$, $j < -3V_x$. These relations and the inequality (7) together tell us that $a_0 \geq 0$ and the absolute values $|a_j| \leq a_0$ for every $j$. Furthermore the same relations also hold for the integrated coefficients which appear in (4), i.e.

$$\bar{a}_k \leq \bar{a}_0,$$  \hspace{1cm} (9)

where the absolute value in (9) disappears because of the fact that the integrated coefficients are real and positive.

Taking into account the symmetry properties of the coefficients ($\bar{a}_k = \bar{a}_{-k}$) we can write the partition function (4) as follows

$$Z = \bar{a}_0(1 + \frac{2\bar{a}_3}{\bar{a}_0} + \ldots + \frac{2\bar{a}_{3V_x}}{\bar{a}_0}),$$  \hspace{1cm} (10)

which gives for the free energy density $f$ the following expression

$$f = \frac{T}{V_x} \log \bar{a}_0 + \frac{T}{V_x} \log(1 + \frac{2\bar{a}_3}{\bar{a}_0} + \ldots + \frac{2\bar{a}_{3V_x}}{\bar{a}_0}).$$  \hspace{1cm} (11)

This expression and the inequalities (9) imply that in the thermodynamical limit $V_x \to \infty$ the free energy density can be computed as:

$$f = \frac{T}{V_x} \log \bar{a}_0,$$  \hspace{1cm} (12)

i.e., the only relevant contribution in the determinant of the Dirac operator to the thermodynamics of QCD at finite temperature is that corresponding to a zero net number of Polyakov loops. We can therefore kill all the irrelevant contributions from the beginning and restore the Polyakov symmetry in full QCD.

4 Some Relevant Physical Implications

The main result of the analysis here developed is contained in equation (12). This section will be devoted to discuss some physical consequences which follow from it.

i) Since the only relevant contribution to the partition function of QCD at finite temperature is that with zero net number of Polyakov loops, the Polyakov symmetry is restored in the infinite volume limit of full QCD with dynamical fermions. The Polyakov loop is therefore a good order parameter for full QCD. A practical way to
implement this result in numerical simulations is to include an extra-abelian degree of freedom, as done in section 3 of this paper.

ii) Another interesting conclusion which follows from the fact that the coefficient $\bar{a}_0$ in (12) does not depend on the boundary conditions for the fermion field is that periodic and antiperiodic boundary conditions give rise to the same physics.

iii) The physical meaning of equation (12) for the free energy is that the partition function of QCD at finite temperature is dominated by the canonical partition function computed over the Hilbert subspace of physical states of vanishing baryon number. In other words, baryonic thermal fluctuations are fully suppressed in QCD at finite temperature.

iv) It has been pointed out recently that in numerical simulations of quenched QCD at finite temperature and in the broken deconfined phase, the chiral condensate seems to depend crucially on the $Z_3$ phase in which the gauge dynamics settles and the chiral symmetry restoration transition appears to occur at different temperatures depending of the phase of the Polyakov loop. After the analysis here developed it is clear that the correct way to solve this puzzle and to implement the quenched approximation in QCD is to take for the chiral condensate operator its Polyakov loop invariant part. This result also suggest that a investigation of the finite size effects in finite temperature full QCD induced by the irrelevant contributions to the partition function could be of great interest.

5 Wilson Fermions

We have shown in the previous sections of this paper how the thermodynamics of QCD, when regularized in a space-time lattice and using staggered fermions, is controlled by the contribution to the fermion determinant with no net number of Polyakov loops, i.e., by the thermal fluctuations of physical states with vanishing baryon number.

The two main ingredients to get this result are the conservation of baryonic charge in QCD and the inequalities (9) of section 3 which tell us that the partition function at fixed baryon number reaches its maximum value in the Hilbert subspace corresponding to zero baryon number. The first one of the two ingredients is independent of the lattice regularization for the Dirac operator. The second one however is based on the positivity of the determinant of the Dirac operator for any gauge configuration and any value of the extra-abelian degree of freedom $e^{i\eta}$ (equation (7) of section 3).

Since we have made use of the hermiticity and chiral properties of the Dirac-Kogut-Susskind operator in order to get equation (7), it is natural to ask whether our result applies to any fermion regularization or rather it is related to the presence-absence of the chiral anomaly.

Let us say from the beginning that even if we have not yet a definite answer to this question, there are strong indications suggesting that the chiral anomaly does not play any relevant role here. These indications come from the analysis of the properties of the fermion determinant for Wilson fermions. As well known, the Dirac-Wilson operator $\Delta$ can be written as
\[ \Delta = I - \kappa M, \]  \hspace{1cm} (13)

where \( \kappa \) is the hopping parameter and the matrix \( M \) verifies the following chiral relation

\[ \gamma_5 M \gamma_5 = M^+. \]  \hspace{1cm} (14)

Equation (14) implies that if \( \lambda \) is eigenvalue of \( M \), \( \lambda^* \) is also eigenvalue of \( M \), i.e., the fermion determinant is always real. However it could be negative and in fact this unpleasant situation has been found for some gauge configurations in numerical simulations of the Schwinger model done in the unphysical strong coupling region \[6\]. However the unitary character of the gauge group implies that all the eigenvalues are upper bounded by the relation

\[ |\lambda| \leq 8 \]  \hspace{1cm} (15)

which implies that for \( \kappa \leq 1/8 \), \( \text{det}\Delta \geq 0 \). It is easy to verify that under the previous condition \( \kappa \leq 1/8 \), the positivity of \( \text{det}\Delta \) also holds in the presence of the extra-abelian degree of freedom introduced in section 3.

In other words, all the results of this paper can be extended in a straightforward way to Wilson fermions if we impose the restriction \( \kappa \leq 1/8 \), i.e., the hopping parameter region associated to a positive bare fermion mass.

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References

[1] A.M. Polyakov, Phys. Lett. B72 (1977) 477

[2] J. Kogut, H Matsuoka, M. Stone, H.W. Wyld, S. Shenker, J. Shigemitsu, D.K. Sinclair, Nucl. Phys. B225 (1983) 93

[3] P. Hasenfratz, F. Karsch, Phys. Lett. B125 (1983) 308

[4] S. Chandrasekharan, N. Christ, Nucl. Phys. B (Proc. Suppl.) 47 (1996) 527

[5] M.A. Stephanov, Phys. Lett. B375 (1996) 249

[6] V. Azcoiti, G. Di Carlo, A. Galante, A.F. Grillo, V. Laliena, C.E. Piedrafita, Phys. Rev. D53 (1996) 5069