The Gor’kov and Melik-Barkhudarov Correction to the Mean-Field Critical Field Transition to Fulde–Ferrell–Larkin–Ovchinnikov States

Heron Caldas* and Qijin Chen*

The Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) states, characterized by Cooper pairs condensed at finite-momentum are, at the same time, exotic and elusive. It is partially due to the fact that the FFLO states allow superconductivity to survive even in strong magnetic fields at the mean-field level. The effects of induced interactions at zero temperature are calculated in both clean and dirty cases, and it is found that the critical field at which the quantum phase transition to an FFLO state occurs at the mean-field level is strongly suppressed in imbalanced Fermi gases. This strongly shrinks the phase space region where the FFLO state is unstable and more exotic ground state is to be found. In the presence of high level impurities, this shrinkage may destroy the FFLO state completely.

1. Introduction

Fermionic particles with two different spins occupying states of momenta with equal size but in opposite directions close to their common Fermi surface form Cooper pairs, when subject to a pairing interaction. This is successfully explained by the Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity.[1] The presence of an imbalance between the two spin configurations prevents this mechanism, since there are now two Fermi surfaces that do not coincide so that pairing with zero total momentum for the BCS state is energetically unfavorable, as the formation of Cooper pairs implies equal densities of the two spin species.[2–5] The difficulty of BCS pairing caused by spin imbalance led to the proposal of possible energetically more favorable Fulde–Ferrell–Larkin–Ovchinnikov (FFLO) states,[6,7] which Bose condense into a finite momentum \( \vec{q} \) or a pair of momenta \( \pm \vec{q} \) (for FF and LO states, respectively). The FF state features a single-plane-wave superfluid order parameter, \( \Delta \equiv \Delta e^{i \vec{q} \cdot \vec{R}} \), which has a spatially uniform amplitude. And the LO phase has a standing-wave-like order parameter, \( \Delta = \Delta \cos(\vec{q} \cdot \vec{R}) \), which is a superposition of two counterpropagating plane waves, and is inhomogeneous in both amplitude and phase. The LO phase can be generalized to higher order crystalline states with multiple plane-wave components. While a stable FFLO state may exist in an anisotropic system[8,9] or in a lattice,[9–11] especially in a low dimensions,[12–14] however, it has been shown that the FFLO states are intrinsically unstable in clean homogeneous 3D and 2D continuum systems.[15] Instead, noncondensed pairing with the lowest pair energy at finite momenta is expected, which may lead to exotic ground states. Thus it is important to find the true solution where the unstable mean-field FFLO solutions exist. While this is a very difficult issue, in this paper, we aim to further constrain the phase space region where exotic pairing state may live. In particular, we find that particle-hole fluctuations help to significantly reduce the critical field transition window for the mean-field FFLO states. This also implies that the regions that otherwise have a mean-field FFLO solution should now exhibit more conventional solutions, such as polaronic normal phase[16,17] and phase separation.[18,19]

At the mean-field (MF) level, for small asymmetries between the two spin species, and at zero temperature \( T \), the system persists as a BCS superfluid of zero momentum. However, when the imbalance between the two Fermi surfaces is too large, superfluid pairing is broken apart so that the system undergoes a quantum phase transition to the normal state. Therefore, for a given imbalance, there exists a lower threshold of pairing strength for the BCS pairing solution to exist. On the other hand, for a given interaction strength, there exists an upper bound for the imbalance before pairing is broken. The existence of such a transition at a critical value of the polarization was first realized by Clogston[20] and Chandrasekhar,[21] who independently predicted the occurrence of a first-order phase transition from the superfluid to the normal state. This is known as the Clogston–Chandrasekhar (CC) limit of superfluidity, and was originally proposed in the context of conventional superconductivity.
Stability analysis based on energetic considerations reveals that the mean-field BCS solution at $T = 0$ is not stable in the presence of imbalance until the system enters the Bose–Einstein condensation (BEC) regime where the gap and hence the condensation energy become large.\cite{22–25} Indeed, the momentum of the minority fermions would have to be lifted up to match that of their majority partners, but the energy cost is larger than the condensation energy gain when the pairing gap is small. As a consequence, thermal smearing of the Fermi surfaces leads to possible intermediate temperature superfluidity, at both the mean-field level and with fluctuations included.\cite{22}

Theoretical investigations with ultracold imbalanced Fermi gases, where the numbers of atoms in the two spin states are different, have predicted that the first order transition between the superfluid at equal spin population and the imbalanced normal mixture brings about a phase separation between coexisting normal and superfluid phases.\cite{26,27} Recent experiments of Fermi gases in a trap using tomographic techniques have found a sharp separation between a superfluid core at the trap center and a partially polarized normal phase outside the core.\cite{18,19} So far, the exploration of a two-component Fermi gas with imbalanced populations remains a current and active area of research in the field of ultracold atoms in both theory and experiment.\cite{3,28–30} We find the GMB correction to the critical chemical potential imbalance $\mu_F$ necessary for the phase transitions in 3D systems. We find the GMB correction to the critical chemical potential imbalance $\mu_F$ responsible for the phase transitions from the partially polarized (PP) FFLO phase to a fully polarized (FP) normal state. In the presence of high level impurities, we show that short lifetimes necessarily further decrease $\mu_F$ and, consequently, reduce or even completely destroy the predicted FFLO region of existence. To our knowledge, this is the first time that the GMB correction is considered in the context of FFLO physics.

In this paper, we investigate at the mean-field level the effects of the GMB correction on the FFLO transition that may occur in Fermi gases with imbalanced spin populations, in the clean limit and in the presence of nonmagnetic impurities. We study the continuous phase transition that is triggered by an increase in the chemical potential imbalance $\mu_F$ in homogeneous 3D systems. We find the GMB correction to the critical chemical potential imbalance $\mu_F$ responsible for the phase transitions from the partially polarized (PP) FFLO phase to a fully polarized (FP) normal state. In the presence of high level impurities, we show that short lifetimes necessarily further decrease $\mu_F$ and, consequently, reduce or even completely destroy the predicted FFLO region of existence. To our knowledge, this is the first time that the GMB correction is considered in the context of FFLO physics.

The paper is organized as follows. In Section 2 we first calculate the generalized pair susceptibility in the clean case, associated with the onset of the instability of the PP normal phase. In Section 3 we obtain the induced interactions and find its effects on the critical chemical potential imbalance $\mu_F$ which sets the transition to the FFLO phase. In Section 4 we show how the GMB correction further reduces the FFLO window in the presence of nonmagnetic impurities. Finally, we conclude in Section 6.

2. The Intermediate Normal-Mixed Phase

We consider a generic system of fermions characterized by an effective, short range pairing interaction $-g_s$, (where $g > 0$), with
where the bare dispersion \( \epsilon_{k\sigma} = \epsilon_k - \mu_\sigma = \frac{k^2}{2m} - \mu_\sigma \), and \( \sigma = \uparrow, \downarrow \) is the spin index. Here \( c^\dagger (\cdot) \) is the fermion creation (anihilation) operator, and we have set the system volume to unity. We shall also take the natural units, \( \hbar = k_B = 1 \). The average chemical potential \( \mu = (\mu_\uparrow + \mu_\downarrow)/2 \). The population imbalance is defined as the relative spin density difference, \( p = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow) \). At the mean-field level, the reduced Hamiltonian for one-plane-wave FFLO state with pairing between \( \vec{k} \) and \( -\vec{k} + \vec{q} \) states is given by[82]

\[
\hat{H}^{\text{FFLO}} = \sum_{\vec{k}} \left\{ \epsilon_{\vec{k}\uparrow} c_{\vec{k}\uparrow}^\dagger c_{\vec{k}\uparrow} + \epsilon_{\vec{k}\downarrow} c_{\vec{k}\downarrow}^\dagger c_{\vec{k}\downarrow} \right. \\
+ \Delta_0 c_{-\vec{k}+\vec{q}\downarrow}^\dagger c_{\vec{k}\uparrow} + \Delta_0 c_{-\vec{k}+\vec{q}\uparrow}^\dagger c_{\vec{k}\downarrow} \left. \right\}
\]

(2)

Here the order parameter carries momentum \( \vec{q} \), with the self-consistency condition \( \Delta_0 = \frac{\hbar^2}{\nu_c m^2} q^2 \). As usual, the constant term related to the condensation energy has been dropped from the reduced Hamiltonian Equation (2). Setting \( \vec{q} = 0 \) will reduce to the polarized BCS case.[83]

As mentioned in ref. [84], a very important, and still open issue, is the precise nature of the ground state in the regime \( h_c < h < h_s \), where \( h_c = \Delta_0/\sqrt{2} \) sets the CC transition. Let us now investigate the possible FFLO phase that may arise in the intermediate region. Suppose we are in the normal FP phase at some \( h > h_c \), and the “field” \( h \) is decreased until it enters the PP phase. In order to have a qualitative and quantitative description of this picture, for small \( |\Delta_0| \) one may expand the action in fluctuations \( |\Delta_0| \) a la Landau, since the transition from the FP to the normal-mixed phase is continuous.[85,86] We then expand the action up to the second order in the order parameter \( |\Delta_0| \)[85,86] and obtain

\[
S_{\text{eff}} = \sum_{\vec{q},\Omega} a(\vec{q}) \Omega |\Delta_\Omega|^2 + \mathcal{O}(|\Delta_0|^4)
\]

(3)

where \( a(\vec{q}) \Omega = 1/\hbar - \chi(\vec{q}, \Omega) \), with \( \chi(\vec{q}, \Omega) \) being the four-momentum of pairs, and \( \chi(\vec{q}, \Omega) \) is the bare pair susceptibility without feedback effect,

\[
\chi(\vec{q}, \Omega) = \sum_{\vec{k}} \left\{ 1 - f(\xi_{\vec{k}-\vec{q}+\vec{l}/2,1}) - f(\xi_{\vec{k}+\vec{q}-\vec{l}/2,1}) \right. \\
- f(\xi_{\vec{k}-\vec{q}+\vec{l}/2,1}) - f(\epsilon_{\vec{k}+\vec{q}-\vec{l}/2,1} - \Omega) \left. \right\}
\]

(4)

where \( f(x) = 1/(e^{x\beta} + 1) \) is the Fermi distribution function with \( \beta \equiv 1/k_B T \). Here \( \chi(\vec{q}, \Omega) \) can be obtained from analytical continuation of the thermal pair susceptibility, \( \chi(\Omega) \equiv \chi(\vec{q}, i\Omega) \),

\[
\chi(\Omega) = \frac{1}{\beta} \sum_{\vec{k},\Omega_{m}} G_0^\prime(\vec{k}) G_0^\prime(\Omega - K)
\]

(5)

where \( G_0^\prime(K) = (i\omega_0 - \xi_{\vec{k}})^{-1} \) is the bare thermal Green’s function. Here, the four-vector \( K \equiv (\vec{k}, \omega_0) \) and \( \Omega \equiv (\vec{q}, i\Omega) \), where \( \omega_0 = (2l + 1)/\beta + \Omega_1 = 2l\pi/\beta \) are the fermionic and bosonic Matsubara frequencies, respectively.

Apparently, for an isotropic system, \( \chi(\vec{q}, \Omega) \) does not depend on the direction of \( \vec{q} \). Evaluation of the equation above is straightforward. At zero temperature we find that \( \chi(\vec{q}, \Omega) \) at zero frequency is given by

\[
\chi(\vec{q}, 0) = N(0) \left\{ 1 + \ln \left( \frac{2m_{\Omega_1}}{\hbar^2} \right) - \frac{1}{2} \left[ \ln \left| 1 - q^2 \right| + \frac{1}{2} \ln \left| \frac{1 + \vec{q}^2}{1 - \vec{q}^2} \right| \right] \right\}
\]

(6)

where \( m_{\Omega_1} \) is an energy cutoff, \( N(0) = \frac{mk_B}{2\pi^2} \) is the density of states at the Fermi level for a single spin component, \( q \equiv v_F q \) is the dimensionless “measure” of the pair momentum, with \( q \equiv |\vec{q}| \) and \( v_F \) is the Fermi velocity.

The Thouless criterion for pairing instability, which corresponds to the divergence of the T matrix \( t(\vec{q}, \Omega) \),

\[
t^{-1}(\vec{q}, 0) = -\frac{1}{\beta} + \chi(\vec{q}, 0) = 0
\]

(7)

yields,

\[
\frac{\hbar}{\Delta_0} = \frac{e}{2|1 + \vec{q}| \left( 1 + \frac{\vec{q}^2}{12} + \mathcal{O}(\vec{q}^4) \right)}
\]

(8)

where \( \Delta_0 = 2\omega_0 \exp(-1/N(0)q) \) is the zero temperature BCS gap. For a contact potential in 3D, which is relevant for Fermi gases, one needs to replace the interaction strength \( g \) with the dimensionless parameter \( 1/k_a a \) via the Lippmann–Schwinger equation, \( m/4\pi a = -1/g + \sum_\Omega \hbar \omega_1 \), where \( a = \) the two-body scattering length. Then the zero temperature gap is given by \( \Delta_0 = \frac{\hbar^2}{\nu_c m^2} e^{2\pi^2 a} \). Note that the exact expression for \( a \) is not crucial here, although these specific expressions are appropriate only for the weak coupling BCS regime. At the same time, it has been known that a possible FFLO phase mainly exists on the BCS side of unitarity. As shown in the inset of Figure 1, at high imbalances, it extends slightly into the BEC side.[85]

We determine the critical reduced momentum \( \vec{q}_c \) by imposing an extremal condition on the pair susceptibility. Thus, extremizing \( \chi(\vec{q}, 0) \) with respect to \( \vec{q} \) yields

\[
2\vec{q}_c = \ln \left| 1 + \frac{\vec{q}_c}{1 - \vec{q}_c} \right|
\]

(9)

A numerical solution of the equation above gives \( \vec{q}_c = 0 \), and \( \vec{q}_c \approx 1.2 \). However, the locus of continuous transitions may be determined from the value of \( \vec{q}_c \) at which \( \chi(\vec{q}_c) \) is both minimized and passes through zero, and this happens only for \( \vec{q}_c \approx 1.2 \) or, equivalently, at a wave-vector \( \vec{q}_c \approx 2.4h/v_F \).

The \( \vec{q} = \vec{q}_c \) limit of \( \chi(\vec{q}) \) gives \( N(0)[1 - 0.59 + \ln(\frac{\pi}{\pi})] \), so that \( a(\vec{q} = \vec{q}_c) = 0 \) is \( -1/g + \chi(\vec{q} = \vec{q}_c) = -1/g + N(0)[0.41 + \ln(\frac{\pi}{\pi})] = 0 \), which leads to \( h_c \approx 0.75\Delta_0 \), for the location of the FFLO transition, agreeing with the findings of Shimahara,[88] Burkhhardt and Rainer,[89] and Combescot and Mora.[90]

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The critical $\tilde{h}_c$ in turn yields the magnitude of the wave-vector $q_c \approx 1.8\Delta_0/\sqrt{\gamma}$. These results also agree with the ones obtained in Ref. [88] by a variational approach for a 3D FF superconductor with a spherical symmetric Fermi surface. The FFLO window is then $h_c < h < h_c$, where the phase transition at $h = h_{C} \approx 0.71\Delta_0$ is of first order, and that at $h_c \approx 0.75\Delta_0$ is of second order. The same results and conclusions are obtained when the calculations are performed with the interaction $g$ replaced by $1/k_c a$ which is appropriate for a short range interaction,[91] as they should.

It should be mentioned that the original Clogston derivation equates the free energy of the superfluid state at zero-field (i.e., $h = 0$) with that of a polarized normal state at the threshold $h_c$, both at $T = 0$. This approach is expected to be valid for the small $\Delta_0$ case in the perturbative sense. However, we argue that the balanced and the imbalanced cases are really distinct and cannot connect to each other continuously. This can be told from the fact that in the BCS regime, an arbitrarily small but nonzero population imbalance is sufficient to destroy superfluidity at precisely $T = 0$ in the 3D homogeneous case (when stability is taken into account).[12] For a finite $\Delta_0$, the “magnetic field” $h$ would have to jump from 0 of the balanced case to a value comparable to $\Delta_0$, implying that $h$ should not be treated perturbatively. Furthermore, there is no guarantee that the normal state in Clogston’s approach is a solution of the BCS gap equation in the zero gap limit. To check the CC limit, we calculate for a 3D homogeneous Fermi gas the gap $\Delta_0$ in the balanced case at zero $T$ and the field $h$ in the imbalanced case when the mean-field $T_c$ (also referred to as $T^*$) approaches 0, both as a function of pairing strength. The result is shown in Figure 1, where we plot $\Delta_0$ for $h = 0 (\text{red})$ and $h$ along the $T^* = 0$ curve (black dashed), as well as their ratio $h(T^* = 0)/\Delta_0$ (blue dashed), as a function of $1/k_c a$. This $h$ should be taken as $h_c$ since it is the boundary between a normal phase and polarized superfluid (also referred to as Sarma phase[81]) with $q = 0$ at $T = 0$. The $T^* = 0$ curve for the Sarma phase in the $p - 1/k_c a$ plane can be easily obtained from the BCS-like mean-field $T_c$ equation with $\tilde{g} = 0$ and $T_c = 0$, along with the fermion number constraints.[82] The figure indicates that the exact mean-field solution yields $h_c/\Delta_0 \approx 0.5$ in the BCS regime, substantially different from $1/\sqrt{2}$ given by CC, and this ratio increases to about 0.733 at unitarity. This result suggests that exact calculation is needed in order to obtain quantitatively accurate value for $h_c$. It corresponds to $q = 0$ limit of Equation (8) and is not stable. The difference between this result and that of CC can likely be attributed to the possibility that the CC normal state does not satisfy the Thouless criterion while the present case does.

In the inset of Figure 1, we show the stable FFLO phase at the mean-field level, as the yellow shaded region. The upper boundary (green curve) is given by the zero gap solution, $T^*_{\text{FFLO}} = 0$, with a finite $q$ vector, which separates the FFLO phase from the normal Fermi gases. The lower phase boundary (magenta curve) is given by the instability condition of the FFLO phase against phase separation. Both boundary lines were taken from ref. [82]. Next to but on the lower right side of this boundary are phase separated states. It is clear that the Sarma mean-field $T_c$ curve line (black dashed) lies completely within the stable FFLO phase, in agreement with the fact that the Sarma states along this curve are unstable against FFLO. We plot $h$ along these two boundaries (green and magenta solid curves) in the main figure. Interestingly, it turns out that, in the BCS limit, the ratio $h/\Delta_0$ along the lower boundary is close to $1/\sqrt{2}$, in agreement with ref. [92]. Meanwhile, the ratio along the upper boundary is close to 0.75. This leaves us with roughly the same FFLO window of 0.71 < $h/\Delta_0 < 0.75$ in the absence of the induced interactions.

### 3. Effects of the Induced Interaction on the FFLO Window

The induced interaction was obtained originally by GMB in the BCS limit by the second-order perturbation.[64] For a scattering process with $p_1 + p_2 \rightarrow p_1 + p_3$, the induced interaction for the diagram in Figure 2 is expressed as

$$U_{\text{ind}}(p_1, p_3) = -g^2 \chi_{\text{ph}}(p_1 - p_3)$$  \hspace{1cm} (10)$$

where $p_i = (\mathbf{k}_i, \omega_n)$ is a four vector. Including the induced interaction, the effective pairing interaction between atoms with

![Figure 2. The lowest-order diagram representing the induced interaction $U_{\text{ind}}(p_1, p_3)$. The solid and dashed lines represent fermion propagators and the interaction $g$ between fermions, respectively.](image-url)
different spins is given by

\[ U_{\text{eff}}(p_1, p_s) \equiv U_{\text{eff}} = -g + U_{\text{int}}(p_1, p_s) \]

\[ = -g - g^2 \chi_{\text{ph}}(p_1 - p_s) \]  

(11)

The polarization function \( \chi_{\text{ph}}(p') \) is given by

\[ \chi_{\text{ph}}(p') = \sum_p G^0(p) G^0(p + p') \]

\[ = \int \frac{d^3k}{(2\pi)^3} \frac{f^0_k - f^0_{k+q}}{i\Omega + \xi_{k,a} - \xi_{k+q,b}} \]  

(12)

where \( f^0_k \equiv f(\xi_{k,a}) \), and \( p' = (q, \Omega) \). This means that \( U_{\text{eff}} \) is a function of momentum and frequency. The static polarization function is then,

\[ \chi_{\text{ph}}(q, h) = \frac{2m}{(2\pi)^2} \int \frac{dk}{2q} \left[ f^1_k \ln \left( \frac{q^2 - 4mh + 2qk}{q^2 - 4mh - 2qk} \right) + f^1_k \ln \left( \frac{q^2 + 4mh + 2qk}{q^2 + 4mh - 2qk} \right) \right] \]

(13)

where \( q \equiv |\vec{q}| \). The above expression is usually computed in the zero temperature limit, with \( f^1_k \to \Theta(k^* - k) \), where \( \Theta(x) \) is the step function, such that the induced correction to the coupling \( g \) is a (temperature independent) constant.

\[ \chi_{\text{ph}}(q, h) \equiv \chi^1(q, h) + \chi^1(q, h) \]

\[ = -\frac{m}{(2\pi)^2} \int_0^{k^*_\parallel} \frac{dk}{2q} \ln \left( \frac{q^2 - 4mh + 2qk}{q^2 - 4mh - 2qk} \right) \]

\[ = -\frac{m}{(2\pi)^2} \int_0^{k^*_\parallel} \frac{dk}{2q} \ln \left( \frac{q^2 + 4mh + 2qk}{q^2 + 4mh - 2qk} \right) \]  

(14)

Equation (14) shows that the static polarization function in the case of a spin imbalanced Fermi gas separates into contributions from the spin-down and the spin-up like susceptibilities. The integration in \( k \) gives

\[ \chi^1_{\text{ph}}(q, h) = -\frac{m}{8\pi^2} \left\{ \left[ k^*_\parallel^2 - \left( \frac{q^2 - 4mh + 2qk^*_\parallel}{2q} \right)^2 \right] \ln \left[ \frac{q^2 - 4mh + 2qk^*_\parallel}{q^2 - 4mh - 2qk^*_\parallel} \right] \right\} \]

\[ + k^*_\parallel^2 \left( \frac{q^2 - 4mh}{q} \right) \]  

\[ \chi^1_{\text{ph}}(q, h) = -\frac{m}{8\pi^2} \left\{ \left[ k^*_\parallel^2 - \left( \frac{q^2 + 4mh + 2qk^*_\parallel}{2q} \right)^2 \right] \ln \left[ \frac{q^2 + 4mh + 2qk^*_\parallel}{q^2 + 4mh - 2qk^*_\parallel} \right] \right\} \]

\[ + k^*_\parallel^2 \left( \frac{q^2 + 4mh}{q} \right) \]  

(15)

The equations above can be put in a more convenient form, \( \chi^\alpha_{\text{ph}}(x, y) \equiv \chi^\alpha_{\text{ph}}(q, h) \), where

\[ \chi^1_{\text{ph}}(x, y) = -\frac{N(0)}{4} \left\{ \left[ \frac{1}{\sqrt{1+y^2}} \left( 1 - \frac{y}{2x^2} \right) \right] \right\} \]

\[ - \frac{1}{2x} \left[ 1 - \frac{y - x^2}{1 - \frac{y}{2x^2}} \right] \ln \left[ \frac{\sqrt{1+y+x^2} - x}{\sqrt{1+y-x^2} + x} \right] \]  

(17)

and

\[ \chi^1_{\text{ph}}(x, y) = -\frac{N(0)}{4} \left\{ \left[ \frac{1}{\sqrt{1+y^2}} \left( 1 + \frac{y}{2x^2} \right) \right] \right\} \]

\[ - \frac{1}{2x} \left[ 1 + y - x^2 \right] \ln \left[ \frac{\sqrt{1+y+x^2} - x}{\sqrt{1+y-x^2} + x} \right] \]  

(18)

where \( x \equiv \frac{\mu}{\sqrt{2} \nu} \), and \( y \equiv \frac{\nu}{\mu} \). This allows us to write the polarization function of an imbalanced Fermi gas as

\[ \chi_{\text{ph}}(x, y) \equiv -N(0)L(x, y) \]  

(19)

where \( L(x, y) \equiv L^1(x, y) + L^1(x, y) \) is the generalized Lindhard function.

Notice that in the \( y \to 0 \) limit, \( k^*_\parallel = k^*_0 = k \), such that

\[ \chi^1_{\text{ph}}(x, 0) = \chi^1_{\text{ph}}(x, 0) \equiv \chi^1_{\text{ph}}(x) \]  

and we obtain the well-known (balanced) result

\[ \chi^1_{\text{ph}}(x) = -\frac{m}{4\pi^2} \left[ k_\parallel^2 q - \left( \frac{k_\parallel^2}{4} - \frac{q^2}{4} \right) \ln \left[ \frac{q^2 - 4qk_\parallel}{q^2 + 4qk_\parallel} \right] \right] \]

\[ = -N(0) L(x) \]  

(20)

where \( L(x) \equiv L(x, 0) \) is the standard Lindhard function.

\[ L(x) = \frac{1}{2} \left[ \frac{1}{x^2} \left( 1 - x^2 \right) \right] \ln \left[ \frac{1 - x}{1 + x} \right] \]  

(21)

In the scattering process the conservation of total momentum implies that \( \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \), with \( \vec{k}_4 = -\vec{k}_2 \) and \( \vec{k}_3 = -\vec{k}_1 \). The momentum \( q \) is equal to the magnitude of \( \vec{k}_1 + \vec{k}_2 \), so that \( q = \sqrt{\vec{k}_1 + \vec{k}_2} \), \( \vec{k}_1 + \vec{k}_2 = \sqrt{k_1^2 + k_2^2 + 2\vec{k}_1 \cdot \vec{k}_2} \), \( k_1 \) is the angle between \( \vec{k}_1 \) and \( \vec{k}_2 \). Since both particles are at the Fermi surface, \( |\vec{k}_1| = |\vec{k}_2| = k_f \equiv \sqrt{2m\mu} \), thus, \( q = k_f \sqrt{2(1 + \cos \phi)} \), and consequently \( x = k_f \sqrt{2(1 + \cos \phi)} \), which sets \( 0 \leq x \leq 1 \).

The s-wave part of the effective interaction is approximated by averaging the polarization function \( \chi_{\text{ph}}(q) \) over the Fermi sphere,
which means an average of the angle \( \phi \).

\[
\langle \chi_{\text{ph}}(x, y) \rangle = \frac{1}{2} \int_{-1}^{1} \cos \phi \, \chi_{\text{ph}}(x, y) \, d\phi
\]

\[= -N(0) \left[ \frac{1}{2} \int_{-1}^{1} \cos \phi \, L(x, y) \right] \equiv -N(0) \bar{L}(y) \tag{22}
\]

where we have made use of Equation (19). The quantity \( L \) characterizes the magnitude of GMB corrections in the presence of population imbalance. Shown in the inset of Figure 4 is the behavior of \( \bar{L}(y) \) as a function of imbalance \( y \). In the \( y \to 0 \) limit, we have precisely \( \bar{L}(y) = (1 + 2n(2)) / 3 = 0.795431454 \), as given in ref. [63] and other papers [66] for the balanced case. As \( y \) increases from 0 to 1, \( \bar{L}(y) \) decreases to 0.69, indicating that the particle-hole fluctuation effect becomes weaker due to the Fermi surface mismatch caused by population imbalance. This result is identical to that of ref. [66].

Taking into account the GMB correction, the divergence of the \( T \) matrix in Equation (7) is now given by

\[
\frac{1}{g} \left( -\frac{1}{g} + \langle \chi_{\text{ph}}(x, y) \rangle \right) + \chi(\bar{q}, 0) = 0 \tag{23}
\]

which can be obtained by replacing \( g \) in Equation (7) with \( U_{\text{eff}} \), as given in Equation (11). This expression has been shown to be correct when the more complicated matrix inversion is included self-consistently. This yields a GMB corrected solution \( \Delta_{0}^{\text{GMB}} \) satisfying

\[
\frac{h_{c}}{\Delta_{0}^{\text{GMB}}} = \left( \frac{h_{c}}{\Delta_{0}^{\text{MF}}} \right)^{\text{MF}}
\]

and \( \Delta_{0}^{\text{GMB}}(y) = \Delta(y)e^{-\bar{L}(y)} \), with \( \Delta(y) \equiv \Delta(y = 0, T = 0) \) and \( \Delta_{0}^{\text{GMB}}(y = 0, T = 0) \). This amounts to

\[
\frac{h_{c}}{\Delta_{0}^{\text{GMB}}} = \left( \frac{2}{e} \right)^{7/3} \mu \approx 0.49 \mu \tag{24}
\]

where \( \gamma_{c} = h_{c} / \mu \), and \( \left( \frac{h_{c}}{\Delta_{0}^{\text{MF}}} \right)^{\text{MF}} = 0.75 \) is the MF result without the GMB corrections.

It is well known that the zero temperature BCS pairing gap (at \( y = 0 \) is modified due to the particle-hole channel effect (or GMB correction) as

\[
\Delta_{0}^{\text{GMB}} = \frac{\Delta_{0}}{4(e)^{7/3}} = \frac{8}{e^{2} (4e)^{7/3}} \mu e^{-x / 2k_{B} |a|}
\]

\[= \left( \frac{2}{e} \right)^{7/3} \mu e^{-x / 2k_{B} |a|} \tag{25}
\]

Note here that in the expression for \( \Delta_{0} \), the chemical potential \( \mu \) plays the role of \( E_{F} \). This can be readily obtained following the standard derivation in the BCS framework, but allowing the Fermi level to evolve continuously as one does for the BCS-BEC crossover. This automatically corrects the moving density of states as the Fermi level changes, even though the approximation becomes less accurate by replacing the full momentum space integral by an energy integral with the density of states fixed at the Fermi level. Shown in Figure 3 is a comparison between the calculated \( \Delta_{0} \) and different analytical approximations. The blue dot-dashed line is the expression in the weak coupling limit, with \( \mu \) pinned at \( E_{F} \). Our corrected expression is shown as the red solid line. Both are to be compared with \( \Delta_{0} \) (green solid curve), which is calculated self-consistently in the context of BCS-BEC crossover. It is evident that our corrected expression is quantitatively good all the way from the BCS through the unitary limit.

In order to find the appropriate \( \gamma_{c} \) for consistently evaluating Equation 24, we take the expression for the GMB gap at unitarity, \( 1 / k_{F}a = 0 \), and obtain

\[
\Delta_{0}^{\text{GMB}} = \left( \frac{2}{e} \right)^{7/3} \mu \approx 0.49 \mu \tag{26}
\]

for \( y = 0 \). Note that this is very close to the more complete solution of \( \Delta_{0}^{\text{GMB}} = 0.42 E_{F} = 0.50 \mu \), with \( \mu = 0.837 E_{F} \), calculated with the full particle-hole \( T \) matrix included at the \( G_{c}G_{a} \) level. According to Equation (25), when taking into account the GMB correction, the original \( h_{c} / \Delta_{0} = 1 / \sqrt{2} \) is transformed to \( h_{c}^{\text{GMB}} / \Delta_{0}^{\text{GMB}} = 1 / \sqrt{2} \), which, with Equation (26), yields \( h_{c}^{\text{GMB}} / \mu \equiv \gamma_{c} = \left( \frac{2}{e} \right)^{7/3} / \sqrt{2} = 0.3455 \) at unitarity.

In Equation (24), we approximate \( \gamma_{c} = h_{c} / \mu \) by \( \gamma_{c} \) in \( \bar{L}(y) \), and obtain \( \bar{L} = 0.7857 \), and thus

\[
\frac{h_{c}^{\text{GMB}}}{\Delta_{0}^{\text{GMB}}} = 0.4558 \left( \frac{h_{c}}{\Delta_{0}^{\text{MF}}} \right) \approx 0.342 \tag{27}
\]

Alternatively, one can take \( h_{c}^{\text{GMB}} / \Delta_{0}^{\text{GMB}} = \left( \frac{h_{c}}{\Delta_{0}^{\text{MF}}} \right) \) and obtain immediately

\[
\frac{h_{c}^{\text{GMB}}}{\Delta_{0}^{\text{GMB}}} = \frac{1}{(4e)^{7/3}} 0.75 = 0.339 \tag{28}
\]
which agrees with Equation (27). On the other hand, with the shifted interaction strength, the CC limit is modified to

$$\frac{\Delta_{GMB}}{\Delta_0} = \frac{1}{(4e)^{1/2}} \frac{1}{\sqrt{2}} = 0.319$$

Combining Equations (27) and (29), we conclude that the screening of the medium (i.e., the induced interactions) has shrunk the FFLO window to

$$0.32 \leq \frac{\hbar}{\Delta_0} \leq 0.34$$

4. Effects of Induced Interactions in the Presence of Impurities

The above calculations in Section 3 have been done assuming the system is clean. However, this is not always true, especially for a superconductor, for which impurities and dislocations are easy to find. Impurities may cause a finite lifetime for the quasiparticles. In quasi-1D organic superconductors, for instance, the issue of lifetime effects arise from nonmagnetic impurities or defects.[99] These impurities may add to the complexity of the effect of particle-hole fluctuations, and thus deserve careful inspection.

In this section we show that, indeed, the FFLO window may be strongly affected by impurity effects.[73] We shall only consider nonmagnetic weak impurities in the Born limit[100] which mainly lead to a spectral broadening $\gamma$ for the fermions.[101] Then we rederive everything in the presence of the spectral broadening. For the nonmagnetic impurities which we consider, possible modification to the real part of quasiparticle dispersion may be absorbed into the chemical potential. Along with the simplification of the imaginary part by a constant parameter $\gamma$, these nonmagnetic impurities satisfy the Anderson’s theorem in the BCS regime.[79,102] As the gap becomes large, the small gap approximation assumed by Anderson’s theorem is no longer valid. However, a large $s$-wave gap itself is very robust against weak impurities.[100]

In Figure 4, we show the numerical solutions of $h/\Delta_0$ as a function of $q$ from Equations (8) and (32), for both the clean limit, that is, $\gamma = 0$ (red curve), and the dirty case with $\gamma \equiv \gamma/2\hbar = 1$ (blue dashed line), where $\gamma = \gamma^{-1}$ is the inverse of the lifetime of a quasiparticle in the normal phase. The clean case has a maximum value at $q = q_c \approx 1.2$, which gives $h/\Delta_0 = h_c/\Delta_0 \approx 0.75$, as obtained by the analytical calculations in Section 2. In the dirty case, the maximum has shifted toward lower $q$, and the maximum ratio $h/\Delta_0$ has decreased significantly. This inevitably narrows the FFLO window. When this ratio drops below $1/\sqrt{2}$, the FFLO window will be gone and thus the FFLO phase will disappear at the mean-field level.

Impurities in principle have an effect on the GMB correction, mainly via changing the chemical potential $\mu$. However, we point out this is only a minor secondary effect, since the change in $\mu$ due to impurities is often very small, especially for Born impurities, which cannot induce an impurity band outside the Fermi sphere.[101] The GMB correction used in the present calculation has been done at the lowest level approximation,[63] without considering the gap effect at the Fermi level. Therefore, we believe that at this level, one can safely neglect the impurity effect on the GMB correction to the pairing strength. Therefore, we conclude that the effect of particle-hole fluctuations may be largely taken care of by assuming that it is encoded in an effective pairing strength, a la Equation (23). One only needs to roughly rescale $h_c$ obtained in the presence of impurities by same factors $(4e)^{-1/3}$ as in the clean case. Hence, it does not have to appear explicitly in our impurity derivations below. It should be noted, however, that $h_c$ is unaffected by impurities, for two reasons. As given by the Anderson’s theorem, $\Delta_0$ is unaffected by the weak nonmagnetic impurities. The thermodynamics calculation of $h_c$, which involves the free energy density of magnetic field, $H^2/8\pi$, as given by Clogston,[20] is insensitive to impurities. Therefore, $h_c$ is only subject to the GMB correction.

Considering the finite lifetime of the quasi-particle states in the momentum representation, the pair susceptibility $\gamma$ is found, via Equations (5) and (23), by the standard method of including a finite imaginary part to the Green’s function,[104,105] $\overline{\chi}(K) = 1/(i\omega_0 - i\gamma_{\omega_0} + i\gamma \text{sgn}(\xi_{\omega_0}))$. After somewhat lengthy but straightforward derivations (as shown in Appendix A), the real part of the particle-particle dynamic pair susceptibility can be written as,

$$\text{Re}(\overline{\chi}(q)) = N(0) \left[ 1 + \ln \left( \frac{2\omega_0}{2\hbar} \right) + \frac{1}{4\bar{q}} (1 - \bar{q}) \ln \left( (-1 - \bar{q})^2 + \bar{q}^2 \right) - \frac{1}{4\bar{q}} (1 + \bar{q}) \ln \left( (1 + \bar{q})^2 + \bar{q}^2 \right) \right]$$

which is the counterpart of Equation (6). This approach is formally close to that used to investigate the effect of nonmagnetic impurities in 1D imbalanced Fermi gases,[106] and in 2D[107] and 3D[108] FFLO superconductors.

With $\text{Re}(\overline{\chi}(q))$ from Equation (31) the divergence of the $T$ matrix now yields

$$\frac{\hbar}{\Delta_0} = \frac{e}{2} \left[ (1 - \bar{q})^2 + \bar{q}^2 \right] \frac{\gamma_{\omega_0}}{\gamma} \left[ (1 + \bar{q})^2 + \bar{q}^2 \right]^{-1/4}$$
Notice that in the limit \( \tilde{\gamma} \to 0 \) in Equations (31) and (32), the standard results in Equations (6) and (8) are recovered.

Instead of being a solution of Equation (9), the critical reduced momentum \( q_c \) is now given by the solution of

\[
2q_c \left[ \frac{(1 + q_c)^2}{(1 + q_c)^2 + \tilde{\gamma}^2} + \frac{(1 - q_c)^2}{(1 - q_c)^2 + \tilde{\gamma}^2} \right] = \ln \left[ \frac{(1 + q_c)^2 + \tilde{\gamma}^2}{(1 - q_c)^2 + \tilde{\gamma}^2} \right]
\]

(33)

For \( \tilde{\gamma} \approx 1 \), for instance, the numerical solution of the equation above yields \( q_c \approx 1.01 \), besides the trivial solution \( q_c = 0 \). This leads to \( \tilde{h}_c \approx 0.61 \Delta_0 \) for the location of the FFLO transition, which transforms to \( h^\text{GMB}_c \approx 0.28 \Delta_0 \), in agreement with Figure 4. However, this value is beyond the critical value \( \tilde{h}_c \approx 0.3 \), which gives \( \tilde{h}^\text{critical}_c / \Delta_0 \approx 0.71 = h_c / \Delta_0 \), or \( h^\text{GMB, critical}_c \approx 0.32 \Delta_0 = h^\text{GMB}_c \), for closing the FFLO window. This means that at this critical value of \( \gamma \) the system undergoes a first-order quantum phase transition from the BCS to the polarized normal phase. Conversely, with infinite life time (\( \gamma = 0 \)) the FFLO window remains open with the “unperturbed” limits \( h^\text{GMB}_c < h < h^\text{GMB}_c \), as given by Equation (30). This nontrivial result comes from the fact that \( h_c \) and \( h \) respond to impurities differently.

5. Further Discussions

It should be noted that we have in fact considered only the FF case. This is justified in that there is no self-consistent way to calculate the LO and higher order crystalline LOFF phases when the pairing gap is large. In the original LO paper,[7] the LO state order parameter is treated as a small perturbation to the noninteracting fermion propagator so that in the evaluation of all the diagrams, the Green’s function is treated at the noninteracting level. Such a perturbative treatment necessarily breaks down in the unitary regime, where the gap is large, comparable to the Fermi energy. In addition, in the presence of two wavevectors \( \pm q \), a simple diagrammatic analysis shows that it will generate an infinite series of components of wavevector \( \pm nq \) in the order parameter.[78] Indeed, many later works on LO and higher order crystalline states treat the order parameters as an expansion parameter, in a Ginzburg–Landau type of formalism,[73,74,76,108] and thus they are appropriate only in the weak pairing regime. Therefore, unlike the FF case, there is no simple field-theoretical approach to the LO and higher order crystalline phases beyond the perturbative mean-field treatment with a truncation of the series of the wavevectors.[2] This makes it more difficult to include the GMB or particle-hole fluctuation effect using field-theoretical techniques.[63,66,96] It remains a challenging issue for the future to treat self-consistently the effect of particle-hole fluctuations on the LO and higher order crystalline FFLO states beyond the mean-field level.

In the presence of inhomogeneity, the Bogoliubov– de Gennes (BdG) treatment is often used.[73] It is particularly useful for treating Fermi gases in a trap. However, it should be emphasized that BdG is also a mean-field treatment, albeit in real space. In the presence of multiple wavevectors, the complexity of the generalized formalism increases rapidly (see, ref. [78]). There has been thus far no report in the literature of incorporating particle-hole fluctuations in the BdG formalism.

It should be pointed out that, for population imbalanced Fermi gases in a trap, the local population imbalance \( p \) varies as a function of the radius. While an inverted density distribution is possible,[109,110] in most cases, \( p \) increases from zero (or nearly zero depending on the temperature) at the trap center to unity at the trap edge. One may think of the radius as an equivalent of the imbalance \( p \) in the inset of Figure 1. At low \( T \), except for the BEC regime, one may find that the FFLO solution exists at certain radius, or inside a narrow shell near this radius, under the local density approximation (LDA). The thickness of the shell, relative to the coherence length, may have a strong influence as to whether a FFLO solution exists. In such a case, BdG may have an advantage over LDA.

Our impurity treatment has been restricted to nonmagnetic impurities in the Born limit, following the approach of Anderson[79] and Abrikosov and Gor’kov.[105,111] This assumes randomly distributed weak impurities, whose effect can mainly be simplified as a finite life time effect in the quasiparticles. Recent works[100,112] show that weak disorders do not significantly affect \( T_c \) of an s-wave BCS superfluid in accordance with Anderson’s theorem,[79] and the superfluid is more robust to the presence of disorder in the unitary regime. There has been treatment beyond the Born limit, in the context of d-wave high \( T_c \) superconductors[103,113,114] and s-wave atomic Fermi gases in the BCS-BEC crossover.[100,115] There are of course also treatments of pair breaking, magnetic disorders or impurities in superconductors.[116–120] It would be interesting to investigate how an FFLO phase responds to magnetic impurities. Apparently, the impurity averaging technique has been widely applied to the impurity treatment for the FFLO states as well.[68,69,71,74,76] Beside treating random impurities in an averaged fashion, some studies treat impurities locally,[73,75,77,78] especially in the case of single, few, or non-uniformly distributed impurities. Such impurities, if strong enough, may lead to localized states.[119] When averaged over a large number of uniform random distributions, it is expected that, for weak impurities, these two approaches yield compatible results. Indeed, our result is in agreement with the BdG based findings of ref. [73] for s-wave pairing, in that the FFLO state is much more sensitive to disorders than the BCS state, and that it can survive moderate disorder strength but may be fully suppressed by higher impurity levels. Both results indicate that a low impurity level, or equivalently a long mean free path, is needed for the FFLO state to survive the disorder effect.

It should be noted that, BdG calculations are usually done in a discretized lattice, which necessarily needs to be much larger than the coherence length \( \xi_0 \) of the superfluid, of the order of \( \hbar v_f / \Delta \). In the BCS regime, the gap is small so that \( \xi_0 \) is huge. For a d-wave superconductor, due to the nonlocal effect,[121] \( \xi_0 \) diverges in the nodal directions. Both these cases raise a concern about the quantitative reliability for BdG calculations, when the lattice size is not big enough.

While we consider the ground state only, the treatment in principle can be extended to finite temperatures at the mean-field level. Without considering the FFLO state, the GMB effect acts essentially as a reduction of the pairing interaction strength (with a small temperature dependence).[63] This would thus lead to a reduction to both \( T_c \) and \( \Delta_0 \), with a slight difference between finite and zero \( T \). (This difference vanishes in the BCS limit). In
this case, there should be a GMB-reduced pairing temperature, \( T_{c, \text{GMB}} \), at which pairs form but do not Bose condense. Then, at a lower temperature, \( T_{c, \nu, \text{GMB}} \), phase coherence sets in and pairs start to Bose condense. The situation is different with a nonzero FFLO wavevector \( \vec{q} \), which is pertinent to a high population imbalance or a high magnetic field. While the FFLO mean-field solution usually exists at low \( T \), when pairing fluctuations, which usually lead to the formation of a pseudogap, are taken into account, the mean-field FFLO states become unstable, in the absence of extrinsic symmetry breaking factors such as spatial anisotropy and lattices, as found in ref. [15]. Similar results were found by others as well.\(^{[59-61]} \) Even at the mean-field level, there may exist an intermediate temperature pseudogap regime, between \( T_{c, \nu, \text{GMB}} \) and \( T_{c, \text{FFLO}} \). An example of such a pseudogap regime, calculated in the absence of the GMB corrections, can be found in ref. [63].

Finally, it is known that in the \( H-T \) phase diagram of a superconductor, the existence of the mean-field FFLO phase extends the \( H_c,T \) line at low \( T \) towards the high field side of its BCS counterpart, leading to a kink-like feature at the tricritical point which signals the onset of the FFLO state. Since the field strength \( H \) is proportionally related to the population imbalance \( p \), a counterpart \( T-p \) phase diagram can often be found in the atomic Fermi gas literature, for example in refs. [87, 122]. Now that the GMB effect leads mainly to a reduction of the pairing interaction strength, it is expected that the \( T-p \) phase diagram looks qualitatively similar to its clean counterpart at the reduced pairing strength.

### 6. Conclusion

In summary, we have investigated in homogeneous 3D systems the GMB correction to the chemical potential difference \( h/\Delta_0 \), which is responsible for the transition to the FFLO phase. We find at the mean-field level that the window for the FFLO phase to exist has been reduced by a factor of \( (4e)^{-1/3} \). Therefore, the region in the phase space that otherwise possesses an FFLO order will take alternative solutions, such as phase separation and polaronic normal state. This shall thus further confine the phase space where the true stable solution is yet to be determined.

We have also considered the GMB effect on the FFLO window in the presence of weak (nonmagnetic) impurities or defects, in terms of a finite lifetime \( \tau = 1/\gamma \) of the quasi-particle excitations. We find that a high impurity level leads to a reduction in the critical field \( h_c \) of the continuous phase transition between the FFLO and the normal phase. This will shrink or completely destroy the FFLO window.

### Appendix A: Calculation of the Pair Susceptibility \( \chi \) in the Presence of Impurities

The dynamic pair susceptibility is given by,

\[
\chi(\vec{q}, \Omega) = \sum_{k} \frac{1 - f(\xi_{k,q/2}) - f(\xi_{k,q/2})}{\xi_{k,q/2} + \xi_{k,q/2} - \Omega} \equiv \frac{1}{2} \sum_{k} \frac{\tanh(\beta \xi_{k,q/2}) + \tanh(\beta \xi_{k,q/2})}{\xi_{k,q/2} + \xi_{k,q/2} - \Omega}
\]  

(A1)

which can be rewritten as

\[
\chi(\vec{q}, \Omega) = \frac{1}{2} \sum_{k} \left[ \frac{\tanh(\beta \xi_{k,q/2}) + \tanh(\beta \xi_{k,q/2})}{\xi_{k,q/2} + \xi_{k,q/2} - \Omega} \right]
\]  

(A2)

The denominators can be approximated as

\[
\xi_{k,q/2} + \xi_{k,q/2} - \Omega \approx 2(\xi_{k,q} + h + a \cos \theta - \Omega/2)
\]  

(A2.1)

and

\[
\xi_{k,q/2} - \Omega \approx 2(\xi_{k,q} - h - a \cos \theta - \Omega/2)
\]  

(A2.2)

where \( a \equiv kq/2m \), and \( \theta \) is the angle between \( \vec{k} \) and \( \vec{q} \). Terms of order \( q^2 \) and higher have been neglected. Then we obtain

\[
\chi(\vec{q}, \Omega) = \frac{1}{4} \sum_{k} \left[ \frac{\tanh(\beta \xi_{k,q}) + \tanh(\beta \xi_{k,q})}{\xi_{k,q} + h + a \cos \theta - \Omega/2} \right] + \frac{\tanh(\beta \xi_{k,q})}{\xi_{k,q} - h - a \cos \theta - \Omega/2}
\]  

(A3)

Now we first integrate out \( \theta \) over a narrow momentum shell with an energy cutoff \( \omega_0 \), near the Fermi level, followed by analytical continuation, \( \Omega \rightarrow \Omega + i\gamma \). Then we arrive in the static limit at

\[
\text{Re} \chi(\vec{q}, \gamma) = \frac{m^2}{4\pi^2 q} \int_{0}^{\omega_0} d\omega \tanh \left( \frac{\beta \omega}{2} \right) \ln \left( \frac{(\omega + h + a)^2 + \gamma^2/4}{(\omega - h - a)^2 + \gamma^2/4} \right)
\]  

(A4)

where \( a \rightarrow v_\parallel q/2 \), and \( v_\parallel = k_f/m \) is the Fermi velocity. Taking now the zero temperature limit, and integrating over \( \omega \) we obtain Equation (31).

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### Conflict of Interest

The authors declare no conflict of interest.

### Keywords

atomic Fermi gases, critical magnetic field, Fulde–Ferrell–Larkin–Ovchinnikov states, Gor’kov and Melik-Barkhudarov correction, induced interactions, particle-hole fluctuations, population imbalance
For nonmagnetic impurities in the Born limit, one has $\gamma = n_i u^2$, where $n_i$ is the impurity density, and $u$ is the impurity scattering strength. See, for example, ref. [105] for details.

[102] Such simplification is appropriate only for nonmagnetic impurities in the Born limit. One may see, for example, ref. [103] for an example how Anderson’s theorem may be broken when going beyond this simplified approximation.

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