Towards the fundamentals of car following theory

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The problem of a car following a lead car driven with constant velocity is considered. To derive the governing equations for the following car dynamics a cost functional is constructed. This functional ranks the outcomes of different driving strategies, which applies to fairly general properties of the driver behavior. Assuming rational driver behavior, the existence of the Nash equilibrium is proved. Rational driving is defined by supposing that a driver corrects continuously the car motion to follow the optimal path minimizing the cost functional. The corresponding car-following dynamics is described quite generally by a boundary value problem based on the obtained extremal equations. Linearization of these equations around the stationary state results in a generalization of the widely used optimal velocity model. Under certain conditions (the “dense traffic” limit) the rational car dynamics comprises two stages, fast and slow. During the fast stage a driver eliminates the velocity difference between the cars, the subsequent slow stage optimizes the headway. In the “dense traffic” limit an effective Hamiltonian description is constructed. This allows a more detailed nonlinear analysis. Finally, the differences between rational and bounded rational driver behavior are discussed. The latter, in particular, justifies some basic assumptions used recently by the authors to construct a car-following model lying beyond the frameworks of rationality.

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I. CAR-FOLLOWING THEORIES AND BASIC PROPERTIES OF DRIVER BEHAVIOR

Recently, the theoretical and empirical foundations of the physics of traffic flow (for a review see Refs. 1, 2) has come into the focus of the physical community. The motion of individual cars has many peculiarities, since it is controlled by motivated driver behavior, together with some physical boundaries. Nevertheless, on macroscopic scales the vehicle ensembles display phenomena like phase formation and phase transitions widely met in physical systems (see, e.g., Refs. 1, 2, 3). So, the cooperative behavior of cars treated as active particles seems to be of a more general nature than the mechanical laws and constructing a consistent theory of traffic flow “from scratch” is up to now a challenging problem.

To describe individual car dynamics a great variety of microscopic models have been proposed. These models differ in the details of the interaction between cars and the time update rule, ranging from differential equations to cellular automata 1, 2. There has been a big deal of work on the macroscopic behavior emerging from the microscopic dynamics when exploring the behavior of systems of interacting cars. However, there is still a lot of controversy in both the macroscopic behavior when compared to reality 4, 5, and in the microscopic foundations of the individual car dynamics itself 6.

One currently adopted approach to specify the microscopic governing equations of the individual car motion is the so-called social force model. More details can be found in Refs. 1, 5, 6; here only the basic ideas are touched on. At each moment $t$ of time, a driver $\alpha$ changes the speed $v_\alpha$ of her car depending on the road conditions and the arrangement of the neighboring cars:

$$\frac{dv_\alpha}{dt} = g_\alpha(v_\alpha) + \sum_{\alpha' \neq \alpha} g_{\alpha\alpha'}(x_\alpha, v_\alpha | x_{\alpha'}, v_{\alpha'}).$$

(1.1)

The term $g_\alpha(v_\alpha)$ describes the motion of car $\alpha$ on the empty road, whereas the term $g_{\alpha\alpha'}(x_\alpha, v_\alpha | x_{\alpha'}, v_{\alpha'})$ allows for the interaction of car $\alpha$ with car $\alpha'$ ($\alpha' \neq \alpha$). The interaction is due to the necessity for driver $\alpha$ to keep a certain safe headway distance between the cars. All the models mentioned above use various Ansätze for the last term.

The most interesting special case, which covers the majority of all traffic flow situations, is that of single-lane traffic. Here, all cars can be ordered according to their position on the road in the car motion direction $x_\alpha < x_{\alpha+1}$, here $\alpha = 1, \ldots, N$. Most models take into account solely nearest neighboring cars $\alpha$ and $\alpha + 1$, i.e., $g_{\alpha\alpha'} = 0$ for $\alpha' = \alpha + 1$ and may be, $\alpha' = \alpha - 1$ only. However, more complicated models exist that can be described as models with anticipation 7, 10, 11, 12, 13, or the so-called intelligent driver model 14, 15.

The earliest “follow-the-leader” models 16, 17 relate the acceleration $a_\alpha$ of car $\alpha$ to the velocity difference $(v_\alpha - v_{\alpha+1})$ only, i.e.,

$$\frac{dv_\alpha}{dt} = -\frac{1}{\tau_v}(v_\alpha - v_{\alpha+1}),$$

(1.2)

where $\tau_v$ is the characteristic time scale of the velocity relaxation. In subsequent generalizations of this model $\tau_v$ became a function of the car motion state, in
particular, of the current velocity $v_{\alpha}$ and the headway $h_{\alpha} = x_{\alpha+1} - x_{\alpha} - \ell$ (for a review see Refs. [3, 15]). Here, $\ell$ is the car length. In Refs. [19, 20] another approach called the optimal velocity model is proposed that describes the individual car motion by

$$\frac{dv_{\alpha}}{dt} = -\frac{1}{\tau_v} [v_{\alpha} - \dot{v}_{\text{opt}}(h_{\alpha})], \quad (1.3)$$

where $\dot{v}_{\text{opt}}(h)$ is the steady-state velocity (the optimal velocity) chosen by drivers as function of the headway $h$ between the cars. It should be noted that this approach is related to much earlier safety distance models [21, 22, 23].

Concerning the fundamentals of approximations such as $a_\alpha = a(v_{\alpha}, v_{\alpha+1}, x_{\alpha}, x_{\alpha+1})$, it is noted that there are actually two stimuli affecting the driver behavior. One of them is the necessity to move at the mean speed of traffic flow, in the given case at the speed $v_{\alpha+1}$ of car $\alpha + 1$. So, first, driver $\alpha$ should control the velocity difference $(v_{\alpha} - v_{\alpha+1})$. The other is the necessity to maintain a safe headway distance $h_{\text{opt}}(v_{\alpha})$ depending on the current velocity $v_{\alpha}$. The “following-the-leader” models mainly take into account the former stimulus. The optimal velocity model, conversely, allows for the latter stimulus only. More sophisticated approximations, e.g., Refs. [14, 15, 24, 25, 26, 27, 28] to name but a few, allow for both stimuli. Note also a simple Ansatz called the combined model in Ref. [24] which is also related to the intelligent driver model [14, 15]:

$$\frac{dv_{\alpha}}{dt} = -\frac{(1 - \kappa)}{\tau_v} [v_{\alpha} - v_{\alpha+1}] - \frac{\kappa}{\tau_v} [v_{\alpha} - \dot{v}_{\text{opt}}(h_{\alpha})]. \quad (1.4a)$$

This equation takes into account both stimuli via a phenomenological coefficient $0 < \kappa < 1$. This is also the case for the Helly model [24] which can be written as (cf. Ref. [3])

$$\frac{dv_{\alpha}}{dt} = -\frac{1}{\tau_v} [v_{\alpha} - v_{\alpha+1}] + \frac{1}{\tau_v L_H} [(x_{\alpha+1} - x_{\alpha}) - h_{\text{opt}}(v_{\alpha})], \quad (1.4b)$$

where $L_H$ is a certain spatial scale and $h_{\text{opt}}(v)$ is the optimal headway distance chosen by drivers when moving at speed $v$. Ref. [24] used a linear Ansatz for $h_{\text{opt}}(v)$. Later [20], this model was generalized to also allow for the dependence of the kinetic coefficients on the motion state.

However, the question whether a collection of variables such as $\{v_{\alpha}, v_{\alpha+1}, x_{\alpha}, x_{\alpha+1}\}$ specifies the acceleration $a_{\alpha}$ completely is not trivial. Drivers are characterized by the motivated behavior rather than physical regularities. For example, memory effects may be essential and can destroy the direct relationship $a_{\alpha} = a(v_{\alpha}, v_{\alpha+1}, x_{\alpha}, x_{\alpha+1})$. Up to now, memory effects in car following modeling have been treated only in a simplified version. This has been done by relating the current acceleration $a_{\alpha}(t)$ to the velocities $v_{\alpha}(t - \tau_\alpha), v_{\alpha+1}(t - \tau_\alpha)$ and the headway distance $h_{\alpha}(t - \tau_\alpha)$ taken at a previous moment $(t - \tau_\alpha)$ of time (for a review of such an approach concerning with the “following-the-leader” models see Refs. [3, 16]).

Here $\tau_\alpha$ is the formal delay time in the driver response which is treated as a constant. Such an approach, however, is rather formal, because, first, it is not clear why the memory effects relate only two moments of time instead of a certain interval as a whole. Second, a simple physiological delay in the driver response as well as the mental driver estimation of the surrounding situation should contribute to the value of $\tau_\alpha$. If the latter contribution is essential the delay time $\tau_\alpha$ is to depend substantially on the state of car motion. Moreover, from our point of view the memory effects stem from the fact that the description of individual car motion is a boundary value problem rather than an initial value problem. Indeed, at the current moment of time the headway distance and the car velocity are quantities given for the driver beforehand. Correcting the car motion she should choose such a driving strategy that in a certain time interval the car velocity and headway distance attain their optimal stationary values, at least, approximately.

These memory effects are the topic of this paper. We propose that drivers plan their behavior for a certain time in advance [33] instead of simply reacting to the surrounding situation. Similar ideas related to the optimum design of a distance controlling driver assistance system are discussed in Ref. [34]. Mathematically, the driver’s planning of the further motion is just to find extremals of a certain priority functional that ranks outcomes of different driving strategies.

The derivation of microscopic governing equations for systems with motivated behavior based on a certain “optimal self-organization” principle has been discussed recently [2, 35, 36, 37, 38, 39]. The assumption adopted in these works is that individuals try to minimize the interaction strength or, equivalently, to optimize their own success and to minimize the efforts required for this. The approach discussed here applies to the concepts of mathematical economics, namely, to the notion of preferences and utility (see, e.g., Ref. [40]). In Ref. [33] a specific form of the priority functional has been proposed. From that, a certain Ansatz like the combined model [14, 15] or the Helly model [24] could be derived. Nevertheless, the question how to find the priority functional “from scratch” remains open.

In the present paper the priority functional is constructed by applying to general properties of the driver behavior for a simplified situation. A car following (no overtaking allowed) a lead car which is driven with constant speed $V$ is considered. The task is to derive governing equations for the following car motion specified by the time dependence of the velocity $v(t)$ and the headway distance $h(t)$. 


II. THE COST FUNCTION OF DRIVING

A. General properties of driver preference

Assuming a driver to be able of comparing any two states \( \{h_1, v_1\}, \{h_2, v_2\} \), the phase plane \( \{h > 0, v > 0\} \) can be ordered by a preference relation \( \preceq \). Therefore, the existence of a cost function \( F(h, v) \) may be assumed such that:

\[
\{h_1, v_1\} \preceq \{h_2, v_2\} \Leftrightarrow F(h_1, v_1) \geq F(h_2, v_2) .
\]

Typically, \(-F(h, v)\) is called the utility function and (for it) the relation \( \preceq \) matches the inequality \( \preceq \). Here, the use of condition (2.1) is preferred, because then the cost function \( F(h, v) \) is similar to the free energy of physical systems, with the minima as the stationary states.

Obviously, the cost function \( F(h, v) \) cannot be specified completely because a composite function \( \Psi[F(h, v)] \) also meets condition (2.1) for any increasing function \( \Psi[\cdot] \). Therefore, at the current stage of the theory development any approximation or Ansatz adopted for the cost function \( F(h, v) \) has no meaning.

To overcome this problem and to fix a certain class of the cost functions the preference relation defined on a set of car motion paths \( \{h(t), v(t)\} \) is considered. A car moving from one point to another can meet different states of traffic flow. It enables one to unite various points on the phase plane in an element treated as a whole with respect to the driver preference, reducing the numerical uncertainty in their evaluation. These parts are supposed, first, to connect the same origin-destination pair \( \{O, D\} \) and, second, to go through a traffic flow pattern whose structure is known to the driver beforehand. Such a set obviously can be ordered by the driver preference and the corresponding cost functional \( \mathcal{L}\{h(t), v(t)\} \) is assumed to exist. Here, only the fact that a car motion path can consist of various fragments of the phase plane is concerned with. Analysis of the real car dynamics is postponed to the following section.

Figure 1 shows two paths \( \alpha \) and \( \beta \) which are equivalent with respect to the driver preference. They differ from each other only in the order of passing through the traffic states, \( \{h_1, v_1\} \rightarrow \{h_2, v_1\} \rightarrow \{h_2, v_2\} \) for the path \( \alpha \) and \( \{h_1, v_1\} \rightarrow \{h_2, v_2\} \rightarrow \{h_1, v_1\} \) for the path \( \beta \). It demonstrates the fact that if a path goes through a certain small neighborhood of the point \( \{h, v\} \) one or many times then only the cumulative time interval \( \delta t \) during which the system was located in the given neighborhood contributes to the cost functional. The corresponding weight coefficient \( F(h, v) \) obviously plays the role of a certain cost function for the given traffic state. The resulting cost functional for the whole path is

\[
\mathcal{L}\{h(t), v(t)\} = \int_{\text{origin}}^{\text{destination}} F[h(t), v(t)] \, dt .
\]

If the driver has incomplete information about the traffic flow pattern ahead she plans the further motion only within some limits, spatial and temporal ones. This can be taken into account by introducing a certain cofactor in front of the function \( F(h, v) \) which depends on spatial coordinates or time.

The initial introduction of the cost function remains the freedom of its choice too “wide”, as was noted above. The constructed form of the cost functional reduces this freedom substantially. Namely, let \( F(h, v) \) and \( F(h, v) \) be two functions entering a cost functional similar to (2.2) with the prediction effect such that they preserve the preference relation with respect to the car motion paths on an on-one road. Then, as shown in Appendix A these cost functions are related by the expressions

\[
\begin{align*}
\tilde{F}(h, v) &= \bar{x}F(h, v) + C_a(h) v , \quad (2.3a) \\
\tilde{F}(h, v) &= \bar{x}F(h, v) + C_b(t) , \quad (2.3b)
\end{align*}
\]

depending on the spatial or temporal limitation in the driver prediction. Here \( \bar{x} > 0 \) is an arbitrary positive constant and \( C_a(h) \), \( C_b(t) \) are some functions of the headway distance \( h \) or the time \( t \). In other words, relations (2.3a), (2.3b) specify the family of the admissible type of cost functions preserving the driver preference relation and, in addition, correspond to linear transformation of the cost functional (Appendix A).

For example, \( [\mathcal{L}\{h(t)\}]^2 \) is also a cost functional. However, formula (2.2) enables us to fix its specific form being linear with respect to time integration along car motion paths. We note that this construction conforms with regarding the motion cost as the “physiological” cost of driving per time, i.e., having the meaning of a rate.

Both of these families possess the same freedom in choosing a specific form of the cost function especially dealing with the extremals of the cost functional. In fact, the type (2.3a) is chosen in the following.

B. Characteristic features of the car motion cost

In the following, the priority function of the car motion state with respect to the velocity \( v \) and the reciprocal
value $\rho = 1/h$ of the headway distance will be characterized. Note, that the value $\rho$ is not the real car density on a highway. The present analysis ignores the car length, so we have preferred to use the value $\rho$ defined as above. The most preferable state is the motion on an empty road ($\rho = 0$) at maximum speed $v_{\text{max}}$. This maximum speed is determined by external factors. So, at $\{\rho = 0, v = v_{\text{max}}\}$ the cost function $F(\rho, v) := F(h, v)|_{h=1/\rho}$ has its global minimum:

$$F(0, v_{\text{max}}) = 0, \quad (2.4)$$

set equal to zero keeping in mind the aforesaid about the freedom in specifying the cost function. Since there is no other minimum for the motion on an empty road, $\quad (2.5)$

$$F(0, 0) = 1$$

can be fixed.

Because of the car construction a driver can visually control the headway distance within some value $l \sim 1–2 \text{ m}$. When the headway distance $h$ attains such values a driver has to stop her car because of possible collision even at sufficiently slow velocities. So, $l$ coincides with the characteristic headway distance in dense jams where the car density attains the possible maximum. In the following, all headway distances are related to the scale $l$ and the dimensionless variable $\rho l$ is used.

When not moving at all ($v = 0$) and the density of cars surrounding the given car does not come close to the limit values, i.e., $\rho l \ll 1$, it does not matter how many cars are located in the vicinity. So the assumption that at $v = 0$ and for $h \gg l$ the cost function is independent of $\rho$ can be used:

$$F(\rho, 0) = 1 \quad \text{for} \quad \rho l \ll 1. \quad (2.6)$$

In all other cases, $F(\rho, v)$ depends on certain combinations of $\rho$ and $v$ rather than on $\rho$ individually, at least when $\rho l \ll 1$.

Considering the behavior of $F(\rho, v)$ for a fixed speed $v$ it can be stated that driving with small values of $h$ (large values of $\rho$) requires a lot of effort. Thus, $F(\rho, v)$ decreases with $\rho$. The opposite case of small values of $\rho$ (large values of $h$) deserves special attention. Without the possibility of overtaking no especially attractive headway distance can be marked. Therefore, we assume that the cost function $F(\rho, v)$ possesses the only one minimum attained at the boundary point $\rho = 0$ provided the velocity $v$ is fixed.

Keeping in mind the aforesaid it is reasonable to write

$$F(\rho, v) = F(0, v) + f(v_{\text{max}}/v) (\rho l)^m \quad (2.7)$$

for $\rho l \ll 1$. Here the exponent $m$ is a constant and the function $f(z) \to 0$ as $z \to 0$. Since the cost function attains its minimum at the boundary point we may set $m = 1$. Moreover, the latter term on the right-hand side of Exp. (2.7) can be interpreted as a certain “interaction” potential between the following and lead cars which is long-distance one for $m = 1$. A detailed analysis of effects caused by the value of the exponent $m$ requires an individual study. Here the value $m = 1$ is actually chosen as just a simple reasonable assumption. Since the effect of the surrounding cars is depressed for small values of the car velocity $v$ the function $f(z)$ is also to attain its minimum at $z = 0$. Therefore, $f(z) = z^2$ as $z \ll 1$ should hold. In other words, inside a certain neighborhood of the origin $\{\rho = 0, v = 0\}$ the following expansion

$$F(\rho, v) = F(0, v) + \frac{v^2}{\rho_{\text{max}}^2} \rho l \quad (2.8)$$

can be adopted.

Taking into account these speculations about the behavior of the cost function $F(\rho, v)$ caused by variations of both its arguments, the following simple Ansatz will be used subsequently:

$$F(\rho, v) = \left(1 - \frac{v}{v_{\text{max}}} \right)^2 + \frac{v^2}{\rho_{\text{max}}^2} \rho l. \quad (2.9)$$

Figure 2 displays this function. It has only one global minimum at $\rho = 0$ and $v = v_{\text{max}}$. For a fixed velocity $v$ it attains a local minimum at the boundary $\rho = 0$. Ansatz (2.9) generalizes the adopted assumptions about the cost function. The former term takes into account relations (2.4), (2.6), the latter one is based on approximation (2.8). Of course, the parabolic Ansatz (2.9) is approximate only in the limit $\rho l \ll 1$. Since this is the main region of interest, this does not constrain its usefulness.

To deal with the car dynamics we should construct the cost functional $\mathcal{L} \{h(t)\}$ for the car motion paths $\{h(t)\}$ treated now as continuous functions of time $t$. We note that the time dependence of headway distance $h(t)$ gives us the complete information about the car dynamics due to the relationship $dh/dt = V - v$. Leaping ahead, we say that facing this problem it is necessary to introduce additional notions. First, we should expand the phase space in describing the car motion state because transient
processes are now the subject of consideration. Second, drivers plan their behavior for a certain time in advance instead of simply reacting to the surrounding situation. So we should specify the region inside which a driver can monitor the traffic flow evolution and, thus, plan driving her car. A priority functional similar to Eq. (2.2) must span the time interval corresponding to this region. Beyond it the contribution of the path fragments to evaluating the car motion quality at the given moment of time has to be fairly minor.

III. RATIONAL DYNAMICS OF CAR MOTION

A. Cost functional and the extremal equation

Dealing with transient processes in the car motion we should consider once more the collection of phase variables characterizing the cost of car motion at the current moment of time. Keeping in mind conventional driver experience, we will expand the current state of car motion into a Taylor series with respect to aerodynamic development (3.1) with respect to acceleration $a$. This is essential because a driver cannot change the position and velocity of her car immediately, they vary continuously in time and contain no sharp jumps. Conversely, a driver controls the acceleration directly governing the car motion. Besides, she can change the acceleration practically without delay because in the present analysis it is quite reasonable to ignore time scales related to physiological properties of the driver or to the mechanical properties of the car. So, the cost function $F^d(h, v, a)$ for the motion state $\{h, v, a\}$ can be written:

$$F^d(h, v, a) = F(h, v) + \tau^2 a^2 \frac{\vartheta_{\text{max}}^2}{\vartheta_{\text{max}}}.$$  \hspace{1cm} (3.1)

where the time scale $\tau \gtrsim 1 \text{s}$ characterizes the acceleration capability of the car. In writing this expression we have assumed that driving without acceleration is preferable. Then, the cost function $F_t(h, v, a)$ has been expanded into a Taylor series with respect to $a$, keeping the leading term only. According to the result to be obtained the time scale $\tau$ entering expression (3.1) and the one used in equation (1.4) are practically the same. It should be noted that confining ourselves to expansion (3.1) with respect to $a$ the difference between acceleration and deceleration processes has been lost. In reality, they are different. Ignoring this difference leads to models where cars can crash. It is possible to take into account this effect using the approach under development, which, however, is worthy of individual investigation and will be done somewhere else.

For a real driver, various thresholds in the driver recognition of hazards and obstacles exist. One of them is the distance $\lambda$ at which a driver can recognize the behavior of other objects. This distance is usually related to the threshold of the visual angle $\theta_c$ subtended, for example, by vehicles ahead (Fig. 3). The value of $\theta_c$ can be estimated as $\theta_c \sim 15–30 \text{min of arc}$. The critical angle $\theta_c$, the characteristics height $\vartheta$ of cars, and the corresponding mean distance $\lambda$ are related by

$$\theta_c \sim \frac{\vartheta}{\lambda}.$$  \hspace{1cm} (3.2)

This allows for the estimation of the recognition distance as $\lambda \sim 200–400 \text{m}$ setting $\vartheta \sim 2 \text{m}$ and $\theta_c \sim 30–15 \text{min of arc}$. As previously, we relate the driver anticipation with the region of size $\lambda$ in front that is clearly observable and where she can recognize the car behavior. A driver plans her motion based on the information received by monitoring traffic flow inside the observable region. Under normal conditions this region should enable her to govern the motion effectively, for example, to decelerate in advance avoiding a possible accident. Therefore its size $\lambda$ has to meet the inequality $\lambda \gtrsim \vartheta_{\text{max}} \tau$. Leaping ahead, we introduce the value

$$\sigma = \frac{\vartheta_{\text{max}} \tau}{\lambda} \ll 1 \hspace{1cm} (3.2)$$

treated as a small parameter in the theory to be developed. In particular, for $\tau \approx 1 \text{s}$, $\vartheta_{\text{max}} \approx 100 \text{km/h}$, and $\lambda \approx 300 \text{m}$ we have $\sigma \approx 0.1$.

The size $\lambda$ of the recognition distance can be estimated using another argumentation. If $\tau_d \sim 10 \text{s}$ is the typical deceleration time from the velocity $\vartheta_{\text{max}}$ attained on the given empty road to zero value then the estimate $\lambda \sim \vartheta_{\text{max}} \tau_d$ should hold. For $\vartheta_{\text{max}} \approx 100 \text{km/h}$ we have again $\lambda \approx 300 \text{m}$.

Now the cost functional of car motion can be written in an integral form as expression (2.2) containing the integrand. The car is assumed to be located at point $x$ along the road and to move with speed $v$ at the current moment of time. The possible paths of further motion form the set $\Theta(t, x, v)$ on which the cost functional is defined. Here and below Gothic letters will be used to label the path variables. The function $\{\kappa(t, t), t \geq t\}$ allows for a derivation of all the dynamical variables, the headway distance $h(t)$, the car velocity $u(t)$, and the acceleration $a(t)$ of the trial car path. The cost functional evaluating the quality of the path $\{\kappa(t, t)\}$ can be written as

$$L\{\kappa\} = \int_t^\infty \exp \left[ \frac{-\kappa(t) - x}{\lambda} \right] F^d(h, u, a) \, dt,$$  \hspace{1cm} (3.3a)

or

$$L\{h\} = \int_t^\infty \exp \left[ \frac{-V(t - t)}{\lambda} \right] F^d(h, u, a) \, dt.$$

\hspace{1cm} (3.3b)
Forms (3.3a) and (3.3b) correspond to the driver prediction with spatial or temporal limitations, respectively. The cost functional of form (3.3a) was used in (3.3). Subsequently, functional (3.3b) is used because its form simplifies the mathematical manipulations when the velocity $V$ of the lead car is fixed. In particular, dealing with functional (3.3b) we can specify directly the set of trial paths $\mathcal{S}(t, h, v)$ using solely the time dependence of $\{\mathfrak{h}(t, t)\}$ of the headway distance. The path variables $u(t, t), a(t, t)$ are completely determined by the dependence $\{\mathfrak{h}(t, t)\}$:

$$u(t, t) = V - \frac{\partial \mathfrak{h}(t, t)}{\partial t}, \quad a(t, t) = -\frac{\partial^2 \mathfrak{h}(t, t)}{\partial t^2}. \quad (3.4)$$

Furthermore, the cost functional (3.3b) matches the driver prediction with temporal limitation, the mechanism which also allows for the interpretation of the recognition region size in terms of $\lambda \sim \vartheta_{\text{max}} \tau_d$. However, both of the cost functionals lead practically to the same results for the analyzed situation.

Each one of trial paths originates at time $t$ and starts from $(h, v)$ on the phase plane determined by the headway distance and velocity of the car at the current time $t$. Since we investigate the car motion inside traffic flow but not processes of leaving it we assume that time variations $h(t), v(t)$ of the headway distance are bounded. Therefore the trial paths fulfill

$$h(t, t)|_{t=t} = h(t), \quad u(t, t)|_{t=t} = v(t) \quad (3.5)$$

and do exhibit bounded variations only as time goes to infinity. In other words, there is a constant $C > 0$ with

$$|\mathfrak{h}(t, t)| < C \quad \text{for} \quad t > t. \quad (3.6)$$

In what follows, the driver behavior will be described by the extremals of the cost functional (3.3b). Using the standard variational technique and taking into account conditions (3.5), (3.6) the governing equation for these extremals can be derived:

$$\frac{d^2}{dt^2} \partial_a F^d - \frac{V}{\lambda} \frac{d}{dt} \partial_a F^d + \frac{V^2}{\lambda^2} \partial_a F^d - \frac{d}{dt} \partial_h F^d + \frac{V}{\lambda} \partial_a F^d - \partial_h F^d = 0. \quad (3.7)$$

To study the spectral properties of Eq. (3.7) it is linearized around the stationary solution $(h_V, V)$. Its eigenfunctions can be found by the Ansatz

$$\mathfrak{h}_\zeta(t) \propto \exp \left( -\frac{\zeta}{\tau} \right). \quad (3.8)$$

The obtained eigenvalue equation is given by

$$(\zeta + \phi)^2 \zeta^2 - \Lambda (\zeta + \phi) \zeta + \frac{1}{4} \Omega = 0, \quad (3.8)$$

with the coefficients

$$\phi = \frac{V}{\vartheta_{\text{max}}} \sigma, \quad \Lambda = \frac{1}{2} \partial^2_{\text{max}} \partial^2_v F > 0, \quad (3.9)$$

$$\Omega = 2\vartheta^2 \partial^2_{\text{max}} \left( \partial_h F - \frac{V}{\lambda} \partial_h \partial_v F \right). \quad (3.10)$$

and the derivatives are taken at the stationary point $(h_V, V)$. The inequality $\partial^2_v F > 0$ is supposed to hold beforehand, in particular, it is the case for the cost function (2.6). Equation (3.11) possesses four roots, one pair $\{\zeta_+, \zeta_-\}$ of them have positive real parts, the other $\{\zeta'_+, \zeta'_-\}$ have negative ones. Since the eigenfunctions with the eigenvalues $\{\zeta'_+, \zeta'_-\}$ diverge as $t \to \infty$ we must omit them by virtue of condition (3.7). Roughly speaking, these divergent eigenfunctions describe the process of a driver leaving traffic flow, for example, to stop the car. Such processes are not under consideration. The former pair of eigenvalues are given by:

$$\zeta_{\pm} = -\frac{1}{2} \phi + \left[ \frac{1}{4} \phi^2 + \frac{1}{2} \Lambda \pm \frac{1}{2} \sqrt{\Lambda^2 - \Omega} \right]^{\frac{1}{2}}. \quad (3.11)$$

So, in a small neighborhood of the stationary point $(h_V, V)$ of equation (3.7) any extremal $\mathfrak{h}_{\text{opt}}(t, t)$ can be written as

$$\mathfrak{h}_{\text{opt}}(t, t) = h_V + h_+ \exp \left( -\zeta_{+} \frac{t - t}{\tau} \right) + h_- \exp \left( -\zeta_{-} \frac{t - t}{\tau} \right), \quad (3.12)$$

where $h_+$ and $h_-$ are some constants.

Equation (3.7) is of fourth order, so its general solution is specified by four conditions. Solutions that diverge as time goes to infinity have to be omitted (they are related to $\{\zeta'_+, \zeta'_-\}$). Their divergence is due to the exponential cofactor in integral (3.3x) so it is retained beyond a small neighborhood of the stationary point $(h_V, V)$. Therefore, only two conditions are needed to specify the desired solution of Eq. (3.7). In particular, the initial headway distance $h$ and the car velocity $v$ determine it completely. The latter statement can also be proved for the more general form (3.3x) of the cost functional (3.3).

### B. Rational driver behavior and Nash equilibrium

In the previous section the dynamical cost functional was stated. At the next step the driver strategy based on this evaluation of the motion quality should be described (Fig. 4). Following (3.3), it is supposed that the driver, first, in planning the further motion chooses the path $\mathfrak{h}_{\text{opt}}(t, t)$ minimizing the cost motion functional (3.3x):

$$\text{driver: } \Rightarrow \mathfrak{h}_{\text{opt}}(t, t) \Rightarrow \min_{\mathfrak{h}(t, t) \in \mathcal{S}(t, h, v)} \mathcal{L} \{\mathfrak{h}(t, t)\}. \quad (3.13)$$

To do this the driver “solves” equation (3.3x) subject to conditions (3.6), (3.7) and, thus, get the optimal path of further driving, $\mathfrak{h}_{\text{opt}}(t, t)$. As noted in the previous section the optimal path choice is completely determined by the current car velocity $v$ and the headway distance $h$. The terminal conditions, i.e., the goal of reaching the steady-state motion is implied. Since the driver controls the car motion through choosing the adequate value a
of acceleration she has to "find" the second derivative of \( h_{\text{opt}}(t,t) \) with respect to the former argument \( t \) (see Exp. (3.14)) and, then, to "calculate" the result at the current time. Therefore,

\[
a(t) = - \lim_{t \rightarrow t+0} \frac{\partial^2 h_{\text{opt}}(t,t)}{\partial t^2},
\]

relates the current car acceleration \( a(t) \) to the current headway distance \( h \) and the car velocity \( v \).

Formula (3.14) describes the driver's choice at the current moment \( t \) of time. To convert it into the governing equation of car motion we adopt the second assumption that the driver performs this choice continuously:

\[
a = R(h, v, V).
\]

All the car following models based on this equation, but with different particular forms of the cost function \( F^d(h, u, a) \) may be categorized as the rational car dynamics approximation.

Equation (3.14) holds even if the cost function \( F(h, u, t | t) \) depends explicitly on the time \( t \) at which the driver plans her further motion. Then, the driver evaluation of traffic flow state will change in time. Therefore, the resulting path \( \{h(t)\} \) of the real car motion envelops the family of the optimal paths \( \{h_{\text{opt}}(t,t)\} \) generated at different moments \( t \) of time (Fig. 4 left panel).

The present paper assumes driving to be perfect. The driver's choice is based on the precise knowledge of the cost function \( F^d(h, v, a, t) \) depending solely on the current headway distance \( h \), car velocity \( v \), acceleration \( a \), and, may be, the current time \( t \). It does not depend on the moment of time when the driver evaluates the possible paths of her further motion. This assumption leads to what is called the Nash equilibrium in game theory [33]. If, at time \( t_0 \) the driver has found the optimal path \( h_{\text{opt}}(t, t_0) \) of the car motion (which meets equation (3.17) and does not contain explicitly \( t_0 \)), then any recomputation a certain time \( t > t_0 \) later gives the same result, \( h_{\text{opt}}(t, t) = h_{\text{opt}}(t, t_0) \) for \( t \geq t_0 \). In other words, if at time \( t_0 \) the driver has chosen an optimal path \( h_{\text{opt}}(t, t_0) \) then the further motion will be described by it independently of either the driver follows it without correction or optimizes the car motion continuously (Fig. 4 right panel).

The Nash equilibrium in the driver strategy enables us to conclude that equation (3.17) describes the real car dynamics, not only the imaginary paths existing in the driver's mind during her planing of the further motion. In particular, it can be rewritten in a form containing the real acceleration \( a \), velocity \( v \), and headway distance \( h \):

\[
\left( \frac{d^2 a}{dt^2} - \frac{2 V}{\lambda} \frac{da}{dt} + \frac{V^2}{\lambda} a \right) \\
- \frac{\partial^2_{\text{max}}}{2\tau^2} \left( \frac{d}{dt} \partial_{a} F - \frac{V}{\lambda} \partial_{h} F + \partial_{h} F \right) = 0.
\]

In the vicinity of the stationary point \((h_V, V)\) its solution is actually given by Ansatz (3.12), which immediately leads us to the following expressions for the amplitudes

\[
h_+ = \frac{\tau (v - V) - \zeta_-(h - h_V)}{\zeta_+ - \zeta_-},
\]

\[
h_- = \frac{\zeta_+ (h - h_V) - \tau (v - V)}{\zeta_+ - \zeta_-}
\]

and relates the car acceleration \( a \) to the car velocity \( v \) and the headway distance \( h \):

\[
a = - \frac{(\zeta_+ + \zeta_-)}{\tau} \left[ (v - V) - \frac{\zeta_+ - \zeta_-}{(\zeta_+ + \zeta_-)} (h - h_V) \right].
\]

These results are analyzed individually.

1. Optimal driving condition

The stationary point \((h = h_V, v = V, a = 0)\) of equation (3.17) gives the headway \( h_V \) which the driver chooses in order to follow the lead car at speed \( V \). In particular, the expression

\[
\partial_{h} F - \frac{V}{\lambda} \partial_{a} F = 0
\]

specifies the relationship between the values of the headway \( h \) and the car velocity \( v \) for the stationary traffic flow imitated by the given car following problem. Solving Eq. (3.19) for the car velocity \( v \) the optimal velocity approximation is obtained, \( v = \partial_{a} F (h) \). In particular, for the specific form (2.9) of the cost function \( F(h, v) \) we
immediately get for \( l \ll h \ll \lambda \)

\[
\vartheta_{\text{opt}}(h) = \vartheta_{\text{max}} \frac{h^2}{h^2 + D^2},
\]
where the spatial scale \( D \) is given by expression

\[
D = \sqrt{\frac{\lambda}{2}}, \quad l \ll D \ll \lambda.
\]
Relation (3.20) or similar sigmoid functions are widely used in current literature. In particular, for \( l = 1 \) m and \( \lambda = 300 \) m Exp. (3.21) gives the estimate \( D \approx 12 \) m typically ascribed to the spatial scale \( D \).

2. Linear governing equation for the car following problem

By introducing the time scale \( \tau_v = \tau/(\zeta_+ + \zeta_-) \) and the coefficient

\[
g_h = \frac{\zeta_+ - \zeta_-}{(\zeta_+ + \zeta_-)^2},
\]
the car dynamics equation (3.18) can be rewritten as

\[
a = -\frac{1}{\tau_v} \left[ (v - V) - g_h \frac{(h - h_V)}{\tau_v} \right].
\]
Equation (3.22) plays a significant role in models that describe non-rational driver behavior [44], where expression (3.22) specifies an optimal acceleration that could be chosen by rational drivers and the parameter \( g_h \) was introduced phenomenologically. Here, \( g_h \) can be computed using the cost function (2.10). It is plotted as well as the ratio \( \tau_v/\tau \) as a function of the variable \( \Omega \) (Fig. 5).

This representation is due to the fact that the coefficient \( A \sim 1 \) (for the cost function (2.10) \( A \sim 1 \) for \( l \ll h \ll \lambda \)), the coefficient \( \phi \ll 1 \) by virtue of the adopted assumption (3.2), whereas the coefficient \( \Omega \) varies around unity. Below, the latter will be justified and the dependence of the coefficient \( \Omega \) on the car motion state will be analyzed. As seen in Fig. 5 the time scales \( \tau_v \) and \( \tau \) practically coincide with each other and the parameter \( g_h \) is a small value.

3. Relaxation curves vs system parameters

Using again the cost function Eq. (2.9), and taking into account \( l \ll h_V \ll \lambda \) and \( A \simeq 1 \), the optimal velocity dependence \( \vartheta_{\text{opt}}(h_V) \) is given by Eq. (3.20). The parameter \( \Omega \) depends as follows on the car motion state

\[
\Omega = 4\sigma^2 \frac{D \vartheta_{\text{opt}}(h_V)}{h_V} \bigg|_{h = h_V} = \frac{8\sigma^2}{D} \frac{\lambda D^2 h_V}{(h_V^2 + D^2)^2}.
\]
It attains the maximum

\[
\Omega_{\text{max}} = \frac{3\sqrt{3}(\tau \vartheta_{\text{max}})^2}{2D\lambda} = \frac{3\sqrt{3}\sigma^2 \lambda}{2D}
\]
at \( h_{\Omega} = D/\sqrt{3} \). Using \( \tau \sim 1 \) s, \( \vartheta_{\text{max}} \sim 100 \) km/h, \( D \sim 15 \) m, and \( \lambda \sim 300 \) m, respectively, \( \Omega_{\text{max}} \sim 0.5 \) is obtained. Of course, this is not a precise numerical value, but in general \( \Omega_{\text{max}} \lesssim 1 \) may be assumed.

The precise value of \( \Omega_{\text{max}} \) matters. According to Sec. III A the car dynamics is characterized by two eigenfunctions with eigenvalues given by Eq. (6.11). When \( \Omega < 1 \), the relaxation is a pure fading process characterized by time scales \( \tau/\zeta_+ \) and \( \tau/\zeta_- \). Otherwise, \( \Omega > 1 \), the relaxation process is characterized by the complex eigenvalues, \( \zeta_+, \zeta_- \). However in this region their dependence...
on $\Omega$ is sufficiently weak, $|\zeta_{\pm}| \sim \sqrt{\Omega}$ for $\Omega \gg 1$ as follows from Exp. (3.11). Therefore the car relaxation processes should be rather insensitive to the value of $\Omega$ as it varies from unity to about ten. Figure 5 demonstrates this fact showing the relaxation curves induced by deviations from the stationary values of the headway and the car velocity individually. All the corresponding curves lie near each other, only for $\Omega = 0.1$ a larger deviation is visible. This is because for such values of $\Omega$ the eigenvalues $\tau/\zeta_{\pm}$ differ essentially. By contrast, the cost of acceleration, i.e., the cost of correcting the car motion depends strongly on $\Omega$ because this component of the cost function (3.1) varies linearly with $\Omega$ to the first approximation. Since the maximum $\Omega_{\text{max}}$ of the parameter $\Omega$ is attained at $h \sim D$ this means that the greater the value of $\Omega$, the larger is the cost of the car governing with respect to the cost of motion in its own accord.

**Hypothesis about the value of $\vartheta_{\text{max}}$**

Keeping in mind the aforementioned speculations let us assume that the composed parameter $\Omega_{\text{max}}$ takes a certain fixed numerical value, $\Omega_{\text{max}} = \Omega_c \leq 1$. It leads immediately to a certain relationship between the initial quantities of the model, $\vartheta_{\text{max}}$, $\lambda$, $\tau$, $l$. In this collection only the optimal velocity $\vartheta_{\text{max}}$ on empty road is determined solely by the driver behavior. The other parameters, namely, the size $\lambda$ of recognition region, the characteristic time scale $\tau$ of the car ability to accelerate or decelerate, and the characteristic headway $l$ that drivers can control reliably and should be about the typical headway distance in dense jams are determined by other mechanisms, e.g., car properties, driver physiology, visual conditions and the like. At first glance such relations seem to be impossible, because a driver chooses the optimal velocity on empty road depending on the road conditions, speed regulations, weather, etc. Nevertheless, there is an additional factor affecting the car motion on empty roads, the driver experience. If the driver preference for the empty road velocity stems from here experience of driving in low density traffic flow then such relations can exist.

In the given model this assumptions lead to the expression

$$\vartheta_{\text{max}} = \left(\frac{\sqrt{2} \Omega_c}{3 \sqrt{3}}\right)^{\frac{1}{2}} \left(\frac{l + \lambda^2}{\tau}\right)$$  \hspace{1cm} (3.26)

Naturally, the specific form of relation (3.26) will change if, for example, another Ansatz of the cost function is used, however, its qualitative features should be retained. These speculations, definitely, are not enough to regard relations similar to formula (3.26) as an established theoretical result. We have presented it for discussion only.

4. **Following-the-leader model vs optimal velocity model**

As discussed already there are two stimuli for a driver to change the current motion state. One is the velocity difference $v - V$ between the leading car and her car. The second is the difference between the current speed $v$ and the optimal speed $\vartheta_{\text{opt}}(h)$ as function of headway. Models like (1.4) take into account both of them, leaving the determination of the weight coefficients to appropriate empirical or experimental data. The results obtained so far actually allow to predict these coefficients.

This can be demonstrated with the combined model (1.4a). Linearizing Eq. (1.4a) around the stationary point $(h_V, V)$ and comparing the result with Eq. (3.11) we get

$$\kappa = \left(\frac{\zeta_{\pm}}{\zeta_{\pm} + \zeta_{\pm}}\right) \left[\frac{\tau \vartheta_{\text{opt}}}{dh} \bigg|_{h=h_V}\right]^{-1}. \hspace{1cm} (3.27)$$

This expression gives $\kappa$ as function of the car motion state. Figure 7 plots this dependence for the cost function (2.8). As seen in Fig. 7 the weight coefficient $\kappa$ is small for all interesting values of the headway distance in the car following regime. Only for small headways $h_V \sim l$ (dense jams) or for large headways $h_V$ exceeding $D$ substantially (free flow) $\kappa$ approaches unity.

5. **Different types of the car dynamics**

As the headway distance $h_V$ varies in the interval $l \ll h_V \ll \lambda$ the coefficient $\Omega(h_V)$ changes essentially, leading to qualitatively different car dynamics. As mentioned before, when $\Omega(h_V) > 1$ the relaxation exhibits damped oscillations around the stationary point $(h_V, V)$. This is due to the eigenvalues $\{\zeta_{\pm}, \zeta_{\pm}\}$ (see Exp. (3.11)) having non-zero imaginary parts. Since for $\Omega(h_V) \gtrsim 1$ their real and imaginary parts are of the same order the car motion relaxation is characterized by one time scale about one (in units of $\tau$). This is true also for $\Omega(h_V) \lesssim 1$ because
\( \zeta_+ \approx \zeta_- \) there, although the dynamics is a pure fading process now. For still smaller values of \( \Omega(h_V) \approx 0.6 \) the ratio \( r = |\zeta_-|/|\zeta_+| \) of these eigenvalues becomes smaller than one half, which may be defined as the two-scale regime of the dynamics, see Fig. 8.

The change between the two-scale and the one-scale regime corresponds to a value of \( h_V \approx 1.5/D \). Below this value, the dynamical behavior is one-scale, above it is two-scale and will be called “fast-and-slow” in the following.

Concerning with the “fast-and-slow” dynamics the velocity relaxation and the headway relaxation can be analyzed individually. According to Exps (3.17) the initial difference \( h - h_V \) contributes mainly to the eigenfunction with the eigenvalue \( \zeta_- \). Thereby, the velocity difference \( v - V \) contributes mostly to the amplitude \( h_+ \). The amplitude \( h_+ \) also contains the term of the same magnitude, however, the time scale \( \tau/\zeta_- \) on which the corresponding eigenfunction varies is much bigger than the time scale \( \tau/\zeta_+ \) of the other eigenfunction. So, the velocity relaxation falls on the first eigenfunction. Thus, the velocity difference \( v - V \) disappears practically completely during the time \( \tau/\zeta_+ \), which forms the “fast” stage of the car relaxation. At the next “slow” stage of duration \( \tau/\zeta_- \) the headway deviation from the equilibrium value \( h_V \) disappears. To summarize, the “fast-and-slow” car dynamics is a two-stage process where the velocity difference between the cars is eliminated first. Later on, the headway is optimized. While this is being done, the resulting velocity difference is not essential and the driver can govern the car motion without the necessity of responding fast.

Another important characteristics, separating dense traffic and quasi-free flow can be derived as follows. The solution of the eigenvalue equation (3.11) depend actually on two values, \( \phi \) and \( \Omega \), since \( \Lambda \sim 1 \) for the cost functional (3.3a). The parameter \( \phi < \sigma \ll 1 \) by virtue of the adopted assumption (3.2). The maximum of \( \Omega \) attained at \( h_V = D/\sqrt{\lambda} \) is much larger than \( \sigma^2 \) as it results from Exp. (3.24). However, as the headway distance increases the value \( \Omega(h_V) \) decreases as \( h_V^{-3} \) (see Exp. (3.24)) whereas \( \phi \rightarrow \sigma \). So there is a value \( h_c \) of the headway distance at which both these terms contribute to the eigenvalues to the same extent. To find \( h_c \) consider \( h \gg D \) where \( \Omega(h_V) \ll 1 \) and \( \phi \approx \sigma \), thereby, \( \zeta_+ \approx 1 \) and

\[
\zeta_- \approx \frac{\Omega(h_V)}{2(\sigma + \sqrt{\sigma^2 + \Omega(h_V)})} \quad (3.28)
\]

by virtue of formula (3.11). So, \( h_c \) is specified by

\[
\Omega(h_c) = \sigma^2 \quad \Rightarrow \quad h_c = 2 \left( \frac{\lambda}{D} \right)^{\frac{1}{3}} D. \quad (3.29)
\]

The value \( h_c \) divides the headway distance region into two parts, see Fig. 9. When \( h_V \gtrsim h_c \) the velocity of the following car is close to \( \theta_{\max} \), so this type of car motion will be referred to as the quasi-free flow. In this case the eigenvalue equation (3.15) cannot be simplified, so to describe the quasi-free flow the cost functional (3.3b) only in its full form may be used. In the opposite case, \( h \ll h_c \), which will be called dense traffic mode, the term \( \phi \) is ignorable, reducing the eigenvalue equation (3.15) to

\[
\zeta^2 - \Lambda\zeta^2 + \frac{1}{4} \Omega = 0. \quad (3.30)
\]

Moreover, in the latter case the description of car dynamics can be reduced to the standard form of classical mechanics, enabling us to analyze the nonlinear stage of the car dynamics.

**IV. NONLINEAR CAR DYNAMICS**

**A. Effective cost functional for dense traffic flow**

In the dense traffic limit \( h \ll h_c \), the variational principle based on optimizing the cost functional (3.3a) can be simplified essentially. In the given limit the characteristic time scales of the car dynamics are \( \tau \) and \( \tau/\Omega(h_V) \), with both of them being small in comparison with \( \lambda/\theta_{\max} \). The latter follows from the adopted assumption (3.2) and the condition \( \Omega(h_V) \gg \sigma^2 \) for \( h \ll h_c \).
In this case, as shown in Appendix [4], the cost functional \( \mathcal{L} \) can be replaced by the following effective functional whose integrand does not contain a time dependent factor

\[
\mathcal{L} \{ h(t) \} = \int_t^\infty \mathcal{F}_{\text{eff}} (h, v, a \mid V) \, dt'.
\]  

(4.1)

where the integrand, which will be called the Lagrangian of the car dynamics is given by

\[
\mathcal{F}_{\text{eff}} (h, v, a \mid V) = \mathcal{F}^d (h, v, a) - \left( \frac{V}{\lambda} h + v \right) \partial_v \mathcal{F}_{V,hV} |_{V,hV}. \]

(4.2)

Interestingly, \( \mathcal{F}_{\text{eff}} (h, v, a \mid V) \) contains the lead car velocity \( V \) as a parameter and attains its extremal value with respect to \( h \) at the point \( h_V \) corresponding to the optimal driving with the velocity \( V \). The term \( v \partial_v \mathcal{F}_{V,hV} |_{V,hV} \) has been introduced for the sake of convenience only, it does not affect the extremal equation but enables the Lagrangian to attain a minimum with respect to the car velocity \( v \) at the stationary point \( (h_V, V) \).

Both functionals \( \mathcal{F}^d \) and \( \mathcal{F}_{\text{eff}} \) possess the same extremals to the first order in the small parameter \( \sigma / \sqrt{\Omega(h_V)} \). In particular, the extremals of functional \( \mathcal{F}_{\text{eff}} \) meet the equation

\[
\frac{2r^2}{\partial_{v_{\text{max}}}^2} \frac{d^2a}{dt^2} - \frac{d}{dt} \partial_v \mathcal{F} - \partial_h \left( \mathcal{F} - \frac{V}{\lambda} \partial_v \mathcal{F}_{V,hV} h \right) = 0.
\]

(4.3)

It corresponds directly to the initial full equation \( \mathcal{F} \) where the second and third terms in the former parentheses are ignored whereas the second term in the latter parentheses is replaced by its value taken at the stationary point. It is justified because these terms are due to variations of the time dependent cofactor.

Keeping in mind its following applications we rewrite Lagrangian \( \mathcal{F}_{\text{eff}} (h, v, a \mid V) \) also in the form

\[
\mathcal{F}_{\text{eff}} (h, v, a \mid V) = \frac{r^2 a^2}{\partial_{v_{\text{max}}}^2} + \mathcal{F}_0 (v) + \mathcal{F}_{\text{int}} (h, v \mid V).
\]

(4.4)

In particular, for the cost function \( \mathcal{F} \) we get

\[
\mathcal{F}_0 (v \mid V) = \frac{(v - V)^2}{\partial_{v_{\text{max}}}^2} + \left( 1 - \frac{V^2}{\partial_{v_{\text{max}}}^2} \right),
\]

(4.5)

\[
\mathcal{F}_{\text{int}} (h, v \mid V) = \frac{H_v^2}{4 \partial_{v_{\text{max}}}^2 v_{\text{max}}^2} \left( \frac{v^2 h_v v + h}{V^2 h} \right).
\]

(4.6)

In obtaining formula \( \mathcal{F}_0 \) we have taken into account the expression for \( \mathcal{F}_0 \) relating the velocity \( v \) to the headway distance \( h_v \), expression \( \mathcal{F}_{\text{int}} \), and omitted some insignificant terms. The latter term in expression \( \mathcal{F}_{\text{int}} \) can be also omitted because it has no effect on the extremal equation.

The most essential feature of the effective cost functional \( \mathcal{F}_{\text{eff}} \) is the absence of a time dependent cofactor. This enables us to reformulate the car dynamics in terms of autonomous Hamiltonian equations.

**B. Hamiltonian description of car dynamics**

Finding the extremals of functional \( \mathcal{F}_{\text{eff}} \) can be done as follows. The phase space \( \{ h, v, a \} \) is expanded to \( \{ h, h_v, v, \dot{v} \} \) by adding the relationship between the headway distance \( h \) and the car velocity \( v \) as an additional constraint:

\[
\text{minimize} \int_0^\infty \mathcal{F}_{\text{eff}} (h, v, \dot{v}) \, dt \quad \text{subject to the dynamical equation}
\]

\[
\dot{h} = V - v
\]

(4.7)

and the initial and final conditions,

\[
h(0) = h_0, \quad v(0) = v_0, \quad h(\infty) = h_V, \quad v(\infty) = V.
\]

(4.8)

For the sake of simplicity the parameter \( V \) has been omitted in the list of variables of \( \mathcal{F}_{\text{eff}} \), the current time is set to zero. By using a Lagrange multiplier, the problem is rewritten in the standard form. Namely, the extremals of

\[
\int_0^\infty \left[ \mathcal{F}_{\text{eff}} (h, v, \dot{v}) + p (h + v - V) \right] \, dt
\]

(4.11)

are sought. They are defined on the extended set of variables \( \{ h(t), v(t), p(t) \} \) and subject to conditions \( \mathcal{F}_{\text{int}} \). The extremals of functional \( \mathcal{F}_{\text{eff}} \) obey the classical Lagrange equations.

In constructing the Hamiltonian \( H(h, v, p, q) \) that produces the same extremal equations the Pontryagin technique is used \[4\]. Introducing the new variable \( q \)

\[
q = \frac{\partial \mathcal{F}_{\text{eff}} (h, v, \dot{v})}{\partial \dot{v}} \quad \text{(4.12)}
\]

and solving for \( \dot{v} \), i.e., finding \( \dot{v} = \dot{v}(h, v, q) \), the desired Hamiltonian is written as

\[
H(h, v, p, q) = q \dot{v} - \mathcal{F}_{\text{eff}} (h, v, \dot{v}) - p (v - V). \quad \text{(4.13)}
\]

It can be demonstrated directly that the desired extremals meet the standard Hamiltonian system of equations

\[
\dot{h} = \partial_p H, \quad \dot{v} = \partial_q H, \quad \dot{p} = -\partial_h H, \quad \dot{q} = -\partial_v H. \quad \text{(4.14)}
\]

Again, the optimization of functional \( \mathcal{F}_{\text{eff}} \) actually leads to a boundary value problem because the extremal is specified by both the initial and final conditions \( \mathcal{F}_{\text{int}} \). The Hamiltonian approach \( \mathcal{F}_{\text{eff}} \) also shares this property. Namely, the initial values of the variables \( h \) and \( v \) are given by conditions \( \mathcal{F}_{\text{int}} \), whereas the initial values \( p_0 \) and \( \eta_0 \) of the quasi-momenta \( p \) and...
$q$ should be chosen such that the system tend to the sta-

However, the Hamiltonian description has a certain ad-
vantage. First, the Hamiltonian itself is conserved,

$$\frac{d\mathcal{H}(h, v, p, q)}{dt} = 0.$$

Second, an additional autonomous first integral of the
system \((4.14), (4.15)\), can be found, at least, for a certain
stage of the car dynamics. Fortunately, this is the case
for the "fast-and-slow" car dynamics analyzed below.

Note, that this Hamiltonian description of the car mo-
tion relaxation towards the stationary state does not con-
tradict the conservation of phase volume being the gen-
eral property of Hamiltonian systems. The matter is that
the relaxation process is described by only one path lead-
ing to the stationary point of the saddle type. Other
possible paths are not considered.

**Physical meaning of the Hamiltonian variables**

The two Eqns \((4.14)\) can be understood readily. The
equation for $\dot{h}$ is just $h = V - v$, while the equation for
$\dot{q}$, or, what is the same, expression \((4.12)\) shows that the
quasi-momentum $q$ is proportional to the car acceleration
$a = \dot{v}$:

$$q = \frac{2\tau^2}{\nu^2_{\text{eff}}} v.$$

The physical meaning of the quasi-momentum $p$ is
more complex. To clarify it, rewrite the equation line \((4.15)\) in terms of partial derivatives of the La-
grangian $\mathcal{F}_{\text{eff}}(h, v, \dot{v})$. Namely, the definitions of the
quasi-momentum $q$ \((4.12)\) and the Hamiltonian \((4.13)\) en-
ables us to represent these equations as

$$\dot{q} = \partial_v \mathcal{F}_{\text{eff}} + p, \quad \dot{p} = \partial_h \mathcal{F}_{\text{eff}}.$$

In particular, expressions \((4.17)-(4.19)\) immediately lead,
as it must, to equation \((4.18)\).

Expression \((4.18)\) demonstrates that the quasi-
momentum $p$ includes the rate $\dot{a}$ of acceleration varia-
tions. In this way the variable $\dot{a}$ implicitly enters the
Hamiltonian $\mathcal{H}(h, v, p, q)$ specifying a hyper-surface in
the four-dimensional phase space $\{h, v, a, \dot{a}\}$. Even more,
by ignoring the dependence of $\mathcal{F}_{\text{eff}}(h, v, \dot{v})$ on the head-
way distance $h$, the velocity relaxation towards the sta-
tionary value $V$ is governed by the equation

$$\dot{\dot{v}} = \partial_v \mathcal{F}_{\text{eff}}.$$

It results from the effective cost functional \((4.2)\) where the
car velocity $v$ is treated as a primary argument.

As will be seen below, there is a stage of the car dy-
namics where this assumption is justified and the quasi-
momentum $p = 0$ is conserved. Only the dependence of
the Lagrangian $\mathcal{F}_{\text{eff}}(h, v, \dot{v})$ on the headway distance $h$, see Eq. \((4.19)\), causes variations in the quasi-momentum $p$.
Since the velocity control is of primary importance in
the driving strategy, the variable $p$ can be regarded as a
certain measure of the necessity to control also the head-
way distance when eliminating the velocity difference.

**C. “Fast-and-slow” dynamics**

For $\Omega \ll 1$ the last term on the right-hand side of
Eq. \((4.1)\) is small. Since the headway $h$ enters the
Hamiltonian \((4.13)\) exactly via this term time variations
in the quasi-momentum $p$ are retarded by virtue of the
first equation at line \((4.15)\). The same follows from
Eq. \((4.19)\). The other variables $h, v, q$ can vary substan-
tially on scales independent of the value $\Omega \ll 1$. Then,
the car dynamics can exhibit multi-scale relaxation. In
particular, Sec. \((4.3.5)\) demonstrated this fact, namely,
it has been shown that in the given limit the system
approximation can be used to analyze the system evolu-
tion.

The results below are exemplified with the cost func-
tion \((2.2)\). They can be easily generalized to other cost
functions.

**1. Fast stage**

At the zeroth approximation in the small parameter
$\Omega$ the quasi-momentum $p$ is a constant. During the fast
stage mainly the velocity difference $v - V$ is eliminated,
so Eq. \((4.13)\) formally describes the relaxation process
of the car velocity $v$ to the stationary value $V$. There-
fore, by virtue of Exp. \((4.1)\), the quasi-momentum $p$ takes
zero value, $p = 0$. In this limit the Lagrangian component
$\mathcal{F}_{\text{int}}(h, v | V)$ as well as the additive constant of the
component $\mathcal{F}_0(v | V)$ can be omitted. So, the Hamiltoni-
nian \((4.13)\) becomes

$$\mathcal{H}(v, a) = \mathcal{H}_a(a) - \mathcal{H}_v(v),$$

where

$$\mathcal{H}_a(a) = \frac{\tau^2 a^2}{\nu^2_{\text{max}}}, \quad \mathcal{H}_v(v) = \frac{(v - V)^2}{\nu^2_{\text{max}}},$$

and the non-canonical variables $\{h, v, a\}$ are used.

Since the Hamiltonian $\mathcal{H}(v, a)$ is conserved and the
fast stage describes the velocity relaxation to $V$ the car
dynamics obeys the equation
\[ H_a(a) = H_v(v) . \] (4.21)
This immediately gives the relationship between the acceleration \( a \) and the velocity \( v \)
\[ a = -\frac{1}{\tau} (v - V) . \] (4.22)
The sign in the latter equation has been chosen so to allow for the system relaxation.

Equation (4.21) is, in fact, of the general form and the linear form of Eq. (4.22) is due to the adopted quadratic Ansatz of the cost function but not a consequence of linearization. Besides, equality (4.21) can be read as follows. The comparison of the Hamiltonian parts \( H_a(a) \) and \( H_v(v) \) with the corresponding components of Lagrangian (4.13) demonstrates that the function \( H_a(a) \) actually measures the cost of the car acceleration and the function \( H_v(v) \) does the same with respect to the car motion relative to the optimal driving conditions. Thereby, during the fast stage the driver corrects the car dynamics so that the cost of acceleration be equal to the cost of current motion measured relative to the stationary conditions.

The headway distance \( h \) does not enter explicitly the governing equation of the fast stage. However, it varies during the fast stage and finally attains a value \( h_0 \) differing from the initial value \( 0 \). In the adopted simple approximation of the cost function Eq. (4.22) enables to estimate easily the value \( h_0 \). Namely, the direct integration of Eq. (4.22) yields the relationship
\[ h_0 = h_0 - \tau (v_0 - V) . \] (4.23)
Applying to formula (4.23), it can be seen that expression (4.24) holds until the headway distance \( h \) becomes too small and it is impossible to ignore the effect of the term \( F_{int}(h, v) \), i.e., when
\[ h \lesssim \sqrt{\Omega} h V \ll h V . \] (4.24)
The car dynamics in the region of small values of the headway distance when the probability of collision is high is worth of an individual consideration. Here we only touch on this problem by assuming the collision to happen when the value \( h_e \) given by Exp. (4.24) becomes equal to zero. This assumption is justified as a rough approximation due to estimate (4.24), see Fig. 10. The boundary of the collisionless region can be shifted to the right substantially by dropping the assumed quadratic dependence of the cost function on acceleration. This will be done elsewhere.

2. Slow stage

When the system attains a quasi-equilibrium with respect to the car velocity \( v \) its further evolution is due to the direct dependence of the Hamiltonian \( H(h, v, q, p) \) on the headway distance \( h \). It enters the Hamiltonian via \( F_{eff}(h, v | V) \). As a result, the velocity difference \( v - V \) and the acceleration \( a \) should be small. Keeping this in mind, Eq. (4.18) can be solved for \( p \). Substitution of the obtained expression into Eq. (4.19), and the further linearization with respect to the variable \( v - V \) and the component \( F_{int} \) results in
\[ \dot{q} - a \frac{∂ q}{F_{int}(v = V) = ∂_h F_{int} . \] (4.25)
If \( r_s \) is the characteristic time scale of the slow stage then
\[ (v - V) \sim \frac{h - h V}{τ_s}, \quad a \sim \frac{h - h V}{τ_s^2}, \quad q \sim \dot{h} \sim \frac{h - h V}{τ_s^3} . \]
These estimates together with Eq. (4.25) lead to
\[ τ_s \sim Ω^{-1/2} τ_s . \] (4.26)
In particular, the rate \( \dot{q} \) of time variations in the quasi-momentum \( q \) scales as \( \dot{q} \sim Ω^2 \) for \( Ω \to 0 \). Since the Lagrangian \( F_{eff} \) has a minimum at \( h = h V \) the quasi-momentum \( p \) can be estimated as
\[ p \sim τ_s ∂_h F_{int} \sim Ω^{1/2}(h - h V) \]
by virtue of Eq. (4.19). In the limit \( Ω \ll 1 \) the term \( \dot{q} \) can be ignored in comparison with the quasi-momentum \( p \). Then, Eq. (4.18) immediately leads to the relation
\[ p = -∂_h F_{eff} . \] (4.27)
which is no more than the standard expression for the momentum of a system described by the Lagrangian \( F_{eff}(h, v, 0) \) with \( v = V - h \).

By the same reasons all the terms in Hamiltonian (4.13) containing the quasi-momentum \( q \) may be omitted. Then the substitution of (4.24) into (4.19) yields the Hamiltonian of the slow stage
\[ H(h, v) = H_v(v) - H_h(h) , \] (4.28)
where
\[ H_h(h) = F_{int}(h, V | V) - F_{int}(h V, V | V) \]
\[ = \frac{Ω h V^2}{4 τ^2} \frac{(h - h V)^2}{h V h} . \] (4.29)
In the Lagrangian component \( F_{\text{int}}(h, v \mid V) \) the car velocity \( v \) has been replaced by its stationary value \( V \), and some negligible terms have been omitted. Then the conservation of the Hamiltonian \( \mathcal{H}(h, v) \) during the system dynamics gets the form

\[
\mathcal{H}_v(v) = \mathcal{H}_h(h) \tag{4.30}
\]

which leads to the following governing equation

\[
v - V = \frac{\sqrt{\Omega}}{2\tau} (h - h_V) \sqrt{\frac{h_v}{h}}. \tag{4.31}
\]

The conservation law (4.30) can be understood in analogy to that for the fast stage. However, during the slow stage of the car dynamics the car velocity plays the role of the control parameter. The driver changes the speed to correct the headway distance. Again the cost of deviation of the car velocity from the stationary value is chosen to be equal to the cost of the car motion measured relative to the optimal conditions.

It should be pointed out that the slow stage of the car dynamics is governed by the conservation law, Exp. (4.30), which contains only the headway distance \( h \) and the velocity \( v \). At the first approximation, the acceleration \( a \) does not enter at all. In this meaning, the slow stage is similar to other physical systems. However, the stationary point of the car dynamics is a saddle point rather than a minimum.

Up to now, the two different stages of car-following discussed already in Sec. III B have been identified with two different conservation laws. Both of them can be unified into just another effective conservation law interpolating Exps. (4.21) and (4.30):

\[
\mathcal{H}_a(v) = \mathcal{H}_a(a) + \mathcal{H}_h(h). \tag{4.32}
\]

Within the same accuracy it is possible to interpolate directly Eqs. (4.22) and (4.31), leading to

\[
a = -\frac{1}{\tau} \left[ (v - V) - \frac{\sqrt{\Omega}}{2\tau} \left( \frac{h_v}{h} \right)^{1/2} (h - h_V) \right]. \tag{4.33}
\]

Besides, the proposed interpretations of the conservation laws (4.21) and (4.30) enable us to formulate a generalized principle of adequate control. It declares that the effective cost of correcting the car motion via changing the car acceleration (fast stage) or the car velocity (slow stage) is equal to the cost of the current state of car motion measured with respect to the optimal driving conditions.

V. CONCLUSION AND DISCUSSION

A variational approach to the description of car dynamics has been developed in case of a car following a lead car moving at a constant speed. To derive governing equations for the following car motion the driver preference has been used to construct a cost functional. Its extremals specify the optimal paths of the further motion for the following car. Applying to the general properties of the driver behavior we analyzed the basic properties of the cost function and proposed a simple parabolic Ansatz, which, nevertheless, catches typical features of traffic properties.

The concept of a rational driver is formulated. It comprises the assumptions that the driver follows the optimal paths and corrects the car motion continuously. In this case the optimal path is a Nash equilibrium of the system, which is defined as follows. If the driver has chosen an optimal path at a certain moment of time then no further correction of the car motion is necessary because it leads to the same result. As the consequence of the Nash equilibrium the extremals of the cost functional specify the real dynamics of cars with rational drivers although originally they determine only the imaginary paths in the driver’s mind when planning the further motion. In this way we obtained several results. The optimal velocity approximation has been derived. The weight coefficient entering a car following model combining the “following-the-leader” model and the optimal velocity model has been found depending on the headway distance. As an important result it has been shown that the car dynamics can be categorized under different types according to its properties. First, we have shown that there can be two types of the car relaxation towards the stationary motion, the mono-scale dynamics and the “fast-and-slow” dynamics. Second, we singled out the quasi-free motion and the dense traffic mode.

The variational technique for the latter mode can be simplified essentially. Namely, it is possible to reformulate the cost functional so that it does not contain a time dependent cofactor. As a result, the autonomous Hamiltonian description for the car dynamics has been constructed. For the dense traffic mode the “fast-and-slow” dynamics has been analyzed for the nonlinear stage. In particular, different conservation laws for this stages have been found. A generalized principle of adequate car motion control has been proposed.

We assumed that the lead car moves at a fixed speed. If it is not the case two different situations should be singled out. For perfect drivers who can predict rigorously the motion of the car ahead the main results still hold. Otherwise the cost function will depend not only on the current time but also, what is crucial, on the time when the driver has started to evaluate the further car dynamics. This disturbs the Nash equilibrium and the car correction should be carried out continuously. Briefly this problem was studied in [55], but it actually requires a more detailed investigation.
Beyond the rationality

Real drivers have certain limitations. They are not capable of finding the optimal path precisely and they cannot correct the car motion continuously. So, the concept of rational driver behavior is just the first approximation of the real situation and deviations from this perfect behavior should be analyzed consistently. A first step towards this problem can be found in Ref. [44]. Here we justify some assumptions adopted there and substantiated them in a different way.

Let us again concern the driver evaluation of the car motion quality. For a given path of the car’s further motion \( \{ h(t, t) \} \) the cost functional (3.36) is written as

\[
\mathcal{L}\{ h \} = \int_t^\infty dt \, e^{-\frac{V}{\lambda}(t-t)} \times \left[ \frac{\tau^2 a^2}{\vartheta_\text{max}^2} + \left( 1 - \frac{u}{\vartheta_\text{max}} \right)^2 + \frac{u^2}{\vartheta_\text{max}^2} \right].
\]

Expression (5.1) can be represented also in the form

\[
\mathcal{L}\{ h \} = \left( \frac{\vartheta_\text{max}^2}{\tau^2 a^2} + \left( 1 - \frac{u}{\vartheta_\text{max}} \right)^2 + \frac{u^2}{\vartheta_\text{max}^2} \right) \left( 1 + \frac{\delta^2}{\vartheta_\text{max}^2} \right) \frac{2V}{\vartheta_\text{max}^2} \left( h - h_V \right) + \mathcal{L}_0,
\]

where \( \delta(t, t) \) describes the deviation of the given path from the stationary car motion trajectory,

\[
\mathcal{L}_0 = \frac{\lambda}{V} \left( \frac{V}{\vartheta_\text{max}} - 1 \right)^2.
\]

In addition, when deriving expression (5.1) we have used

\[
\frac{l}{h} = \frac{l}{h_V} + \frac{l}{h_V} \left( h - h_V \right),
\]

integrated various fragments of Eq. (5.1) by parts, assuming \( l \ll h \ll \lambda \), and dropped terms like \( l/h \) where ever possible.

When the driver plan her further motion the current values of the headway \( h \) and the velocity \( v \) are regarded as the initial conditions. So, when choosing the optimal path she can minimize only the first term \( L_c \{ h \} \) of expression (5.2). This optimization is implemented through the adequate control over the car acceleration \( a \), so, exactly the acceleration plays the role of the control parameter available for the driver actions.

A real driver can only approximately evaluate the quality of motion. Let us describe the threshold in the driver perception of the motion quality by

\[
\mathcal{L}_\text{thr} = \frac{\lambda}{V} \epsilon_c^2
\]

where \( \epsilon_c \) is a small constant, \( \epsilon_c \ll 1 \), because definitely a driver can recognize the difference in the state of staying and freely motion on an empty road.

When the controllable part \( L_c \{ h \} \) of the cost motion is much smaller than the threshold (5.3) the driver actually has no information of how to govern the car motion. In this case it is natural to assume that he will not do anything with respect to the car driving and, so, will fix the car motion at the current state, including the current acceleration \( a \). Thereby the inequality

\[
\mathcal{L}_c \{ h \} \lesssim \mathcal{L}_\text{thr}
\]

determine the region in the phase space \( \{ h, v, a \} \) inside which the driver cannot control the car motion. In evaluating such a driver behavior with expression (5.2) we can formally treat \( \delta \), \( u \), and \( a \) as constants independent of one another. So setting \( \delta = h - h_c \), \( u = v \), and \( a = a \) and taking into account the relation between \( h_V \) and \( V \) we get from condition (5.3) the approximate boundary of this region in the following form

\[
\tau^2 a^2 + (v - V)^2 + \frac{\Omega(h_V)}{4V^2} \frac{h}{h} (h - h_V)^2 \lesssim \epsilon_c^2 \vartheta_\text{max}^2.
\]

We recall that the parameter \( \vartheta_0 \) entering Eq. (3.28) coincides with \( \sqrt{\Omega/(2\tau)} \). So we reproduced here the expression for the rational driving boundary as has been introduced in paper [44] by a different line of reasonings.

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APPENDIX A: ADMISSIBLE FAMILY OF THE COST FUNCTIONS

A set \( \mathcal{S} \) of one-dimensional paths \( \{ x(t) \}_{t=0}^\infty \) of car motion is considered. All the paths are supposed to originate from one point \( x_0 \) at the initial time \( t = 0 \), i.e. \( x(0) = x_0 \), and to go to infinity, \( x(t) \to \infty \), as \( t \to \infty \) such that the car velocity \( v(t) = dx/dt \) is bounded during the motion, \( |v(t)| < D \) (here \( D \) is some constant).

It is assumed that the set \( \mathcal{S} \) is ordered by a certain preference relation \( \preceq \) and there is a functional \( L \{ x \} \) of the form

\[
L\{ x \} = \int_0^\infty dt \, w(x, t) F(x, v, t),
\]

measuring numerically the preference relation, i.e. meeting the condition

\[
x_1(t) \preceq x_2(t) \iff L\{ x_1(t) \} \geq L\{ x_2(t) \}.
\]
Here, first, the integrand $F(x, v, t)$ depends on the car position on the road $x(t)$, the velocity $v(t)$, and, may be, the current moment $t$ of time. It plays the role of cost function, i.e. evaluates the quality of the car motion state at current time. Second, the weight factor takes one of the forms

$$w(x, t) = w_a(x) > 0 \quad \text{or} \quad w(x, t) = w_b(t) > 0,$$

where $w_a(x) \to 0$ and $w_b(t) \to 0$ as $x \to \infty$ or $t \to \infty$. The functions $w_a(x)$ and $w_b(t)$ describe the spatial or temporal limitations in the driver prediction, respectively. The examples

$$w_a(x) = \exp \left( -\frac{x}{\lambda} \right) \quad \text{and} \quad w_b(t) = \exp \left( -\frac{Vt}{\lambda} \right),$$

where $\lambda$ and $V$ are some constants, enable us to relate the cost functional (A1) to ones used in the main text. The specific form of $w_a(x)$ and $w_b(t)$, however, does not affect the results to be obtained in this Appendix.

Let $\mathcal{L}\{x(t)\}$ be another cost functional of form (A1) with its own integrand $\tilde{F}(x, v, t)$ that also describes the same preference relation. The purpose of the present Appendix is to derive the relationship between the cost functions $F(x, v, t)$ and $\tilde{F}(x, v, t)$.

For this purpose the path set $\mathcal{S}$ is divided into classes $\{\mathcal{S}\}$ of equivalent paths, i.e. $x_1(t), x_2(t) \in \mathcal{S}$ if $x_1(t) \sim x_2(t)$ or, what is the same, $\mathcal{L}\{x_1(t)\} = \mathcal{L}\{x_2(t)\}$. The cost functional $\mathcal{L}(x(t))$ based on the integrand $\tilde{F}(x, v, t)$ has to preserve this class partition. So the condition $\delta\mathcal{L}/\delta x = 0$ must hold for any infinitesimal perturbation $\delta x(t)$ of a path $x(t)$ within the class $\mathcal{S}\{x(t)\}$. Using the Lagrange multiplier technique and taking into account the initial condition $x(0) = x_0$ (thus, $\delta x(0) = 0$) the latter requirement is reduced to the equation

$$(\Delta - v\partial_v\Delta)\partial_x w - (\partial_v\Delta)\partial_t w + \left(\partial_x\Delta - \frac{d}{dt}\partial_t\Delta\right)w = 0, \quad (A3)$$

where the function $\Delta(x, v, t) := \tilde{F}(x, v, t) - \kappa F(x, v, t)$ and $\kappa$ is a constant for all the points $\{x, v = dx/dt\}$ on the phase plane $\{x, v\}$ related by the given curve $x(t)$. Let us show that the value of $\kappa$ is constant for all the points of the phase plane. The cost functions $\tilde{F}(x, v, t)$, $F(x, v, t)$ are chosen, thereby, if the quasi-local equation $A3$ holds, the value of $\kappa$ must be determined by any fragment of the path $x(t)$ under consideration, for example, by a fragment $x^t(t) := x(t)|t_1^t \in x(t)|^\infty_0$.

For $t < t_1$ and $t > t_2$ any point $(x, v)$ can be joined to the fragment $x^t(t)$ by a certain path $\tilde{x}(t)$ which must be characterized by the same value of $\kappa$. The latter proves this statement because the time moments $t_1 < t_2$ have been taken arbitrary.

The cost function $F(x, v, t)$ (as well as $\tilde{F}(x, v, t)$) and the weight factor $w(x, t)$ describe different effects of the car driving, so they are not related to each other. Thus the terms of Eq. (A3) containing $w$, $\partial_t w$, or $\partial_x w$ should be equal to zero individually. Thereby

$$\partial_x\Delta - \frac{d}{dt}\partial_t\Delta = 0 \quad (A4)$$

and

$$\Delta - v\partial_v\Delta = 0 \quad \text{for} \quad w(x, t) = w_a(x), \quad (A5a)$$
$$\partial_v\Delta = 0 \quad \text{for} \quad w(x, t) = w_b(t). \quad (A5b)$$

The solutions of system (A4), (A5a) and system (A4), (A5b) have the forms

$$\Delta(x, v, t) = C_a(x) v \quad \text{and} \quad \Delta(x, v, t) = C_b(t),$$

respectively, where $C_a(x)$ and $C_b(t)$ are arbitrary functions of $x$ and $t$. Thereby the desired relationship of the cost functions $F(x, v, t)$, $\tilde{F}(x, v, t)$ is given for $w(x, t) = w_a(x)$ by the expression

$$\tilde{F}(x, v, t) = \kappa F(x, v, t) + C_a(x) v \quad (A6a)$$

and for $w(x, t) = w_b(t)$ by the expression

$$\tilde{F}(x, v, t) = \kappa F(x, v, t) + C_b(t). \quad (A6b)$$

where the constant $\kappa$ must be positive, $\kappa > 0$.

Formulas (A6a) and (A6b) specify the admissible family of cost functions entering cost functional (A1) that preserve the preference relation. As seen directly, integral (A1) with the integrands $C_a(x)$ and $C_b(t)$ is independent of a particular form of the path $x(t)$. So the transformations (A6a) and (A6b) correspond to the linear transformation of the cost functional.

**APPENDIX B: LAGRANGIAN REPRESENTATION OF THE COST FUNCTIONAL**

The problem of finding the extremals of the functional

$$L\{h(t)\} = \int_0^\infty dt w(t)F(h, v, a) \quad (B1)$$

defined on a set of paths $\{h(t)\}|^\infty_0$ is considered. Here, first, the integrand $F(h, v, a)$ is a given function of the headway distance $h$, the current velocity $v = V - dh/dt$, and the acceleration $a = -d^2 h/dt^2$. Second, the weight factor $w(t) > 0$ is a function of time $t$ confined within some time interval $(0, \tau_v)$, i.e., $w(0) \sim 1$, $w(t) \ll 1$ for $t \gg \tau_v$, and $w(t) \to 0$ sufficiently fast as $t \to \infty$. So, $w(0) = 1$ can be adopted without loss of generality. Third, the trial paths $\{h(t)\}$ meet the initial conditions

$$h(0) = h_0, \quad v(0) = v_0, \quad (B2)$$

and do not diverge as time goes on, i.e., there is a number $C$ so that

$$|h(t)| < C. \quad (B3)$$
The properties of the functional extremals have been studied already. Keeping in mind the results in Sec. III, a special case is analyzed in the following. Let $\{\tau_d\}$ be the characteristic time scale of the velocity relaxation to the stationary value $V$. In particular, the values of $\{\tau_d\}$ are estimated by the eigenvalues of the corresponding extremal equation linearized near the stationary point. The case $\tau_d \ll \tau_V$ is considered here to show how the functional $L$ can be rewritten to eliminate the time dependent factor $w(t)$. In this way the problem of finding the extremals gets the classical form met in theoretical mechanics.

The stationary point $\{h, V, a = 0\}$ is determined by the properties of both the function $F(h, v, a)$ and the factor $w(t)$ (see Exp. (3.19)). As a result, at the stationary point (see again Exp. (3.19))

$$\partial_v F_{st} = 0, \quad \partial_h F_{st} \propto \frac{1}{\tau_V}.$$ (B4)

The derivative $\partial_v F_{st}$ does not contain factors similar to $\tau_d/\tau_V$. So, $w(t)$ cannot be omitted directly. As a first step the function $F(h, v, a)$ is replaced by the difference $\Delta F(h, v, a) := F(h, v, a) - F(h_v, V, 0)$. This does not affect the extremals. The difference $\Delta F(h, v, a)$ as a function of $t$ is practically confined within a time interval $(0, T_d)$ whose upper boundary $T_d \gtrsim \max\{\tau_d\}$ and, so, $T_d \ll \tau_V$. Therefore, at the next step expression (B4) is rewritten as

$$L\{h(t)\} \cong \int_0^\infty dt \left\{ \Delta F(h, v, a) - \frac{1}{\tau_V} (v - V) \partial_v F(h_v, V, 0) \right\}.$$ (B5)

This is justified to the first order in the small parameter $\max\{\tau_d\}/\tau_V$ by virtue of expressions (B4) and the fact that the second term in Exp. (B5) plays a substantial role only when the path $h(t)$ tends to the stationary point. Here the symbol $\cong$ means that functional (B5) possesses the same collection of the extremals as the initial functional $L\{h(t)\}$ within the adopted accuracy and the time dependent factor $w(t)$ has been approximated by

$$w(t) \approx 1 - \frac{t}{\tau_V}.$$ (B6)

Formula (B6) can be regarded actually as the definition of the time scale $\tau_V := [-dw(0)/dt]^{-1}$. Taking into account the relation $(v - V) = -dh/dt$ and integrating the second term in Exp. (B5) by parts yields

$$L\{h(t)\} \cong \int_0^\infty dt \left\{ F(h, v, a) - \frac{h}{\tau_V} \partial_h F(h_v, V, 0) \right\}.$$ (B7)

Omitting the constant components of the integrand which does not affect the extremals the required result follows

$$L\{h(t)\} \cong \int_0^\infty dt \left\{ F(h, v, a) - \frac{h}{\tau_V} \partial_h F(h_v, V, 0) \right\}.$$ (B8)

Minimization of functional (B8) gives the desired extremals and integral (B8) does not contain a time dependent factor. The function

$$F_{eff}(h, v, a | V) := F(h, v, a) - \frac{h}{\tau_V} \partial_h F(h_v, V, 0)$$ (B9)

can be regarded as a Lagrangian of the car dynamics.

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