Double-scaling limits of random matrices and minimal \((2m, 1)\) models: the merging of two cuts in a degenerate case

O Marchal\textsuperscript{1,2} and M Cafasso\textsuperscript{3}

\textsuperscript{1} Département de Mathématiques et de Statistique, Université de Montréal, Canada
\textsuperscript{2} Institut de Physique Théorique, F-91191 Gif-sur-Yvette Cedex, France
\textsuperscript{3} Centre de Recherches Mathématiques, Concordia University, Montréal, Canada
E-mail: olivier.marchal@polytechnique.org and cafasso@crm.umontreal.ca

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Abstract. In this paper, we show that the double-scaling-limit correlation functions of a random matrix model when two cuts merge with degeneracy \(2m\) (i.e. when \(y \sim x^{2m}\) for arbitrary values of the integer \(m\)) are the same as the determinantal formulae defined by conformal \((2m, 1)\) models. Our approach follows the one developed by Bergère and Eynard in (2009 arXiv:0909.0854) and uses a Lax pair representation of the conformal \((2m, 1)\) models (giving a Painlevé II integrable hierarchy) as suggested by Bleher and Eynard in (2003 J. Phys. A: Math. Gen. 36 3085). In particular we define Baker–Akhiezer functions associated with the Lax pair in order to construct a kernel which is then used to compute determinantal formulae giving the correlation functions of the double-scaling limit of a matrix model near the merging of two cuts.

Keywords: correlation functions, Painlevé equations, matrix models, topological expansion

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1. Introduction

It has been known for a long time that the study of random matrix models in different scaling limits gives rise to a great number of well-known integrable equations: both PDEs of solitonic type (KdV and, more generally, Gelfand–Dikii equations) and ODEs arising from isomonodromic systems (like Painlevé equations). A key idea in these studies is the notion of spectral curves attached to algebraic equations $P(x, y) = 0$. The genus of the curve gives the number of intervals on which the eigenvalues of the matrices will accumulate when their size tends to infinity. It is well known that, in the generic case, the curve behaves like $y \sim \sqrt{x - a}$ near a branch point $a$ (an extremity of an interval); the appropriate double-scaling limit gives the celebrated Airy kernel in connection with the $(1,2)$ minimal model. But it may happen on taking a fine-tuned limit (see for instance [2,5]) that the behaviour near a branch point differs from the generic case and gives rise to the celebrated Painlevé I hierarchy.
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takes the form of \(y^q \sim (x - a)^p\). In such a case, it is expected that the double-scaling limit is related to the conformal \((p,q)\) minimal model. In their articles \([2]\) and \([3]\), the authors opened the way to rigorous mathematical proofs in order to establish that the correlation functions of the double-scaling limit of a matrix model are the same as the ones defined by determinantal formulae arising from \((p,q)\) models. In their articles, they apply this method to all \((2m+1, 2)\) models, i.e. suitable limits of matrix models where the spectral curve behaves like \(y^2 \sim x^{2m+1}\) near an endpoint. In this paper, we will use the same method for the \((2m, 1)\) case which corresponds to a point where two cuts are merging with a degeneracy \(2m\). For a generic merging, i.e. \(m = 1\), it has been proven in \([4]\) that the suitable double-scaling limit of the matrix model is connected to the Painlevé II equation. Some similar results have been established with the study of a suitable Riemann–Hilbert problem. For example the case of an even quartic polynomial has been studied in \([5]\). It would also be interesting to derive results, for these kernels, similar to the ones proved in \([9]\). Here, using the approach of \([2]\), we find, as expected, that the correlation functions of the double-scaling limit of the merging of two cuts with degeneracy \(2m\) are expressed through the Lax system of the Painlevé II hierarchy (see \([10,16]\)).

2. The double-scaling limit in random matrices: the merging of two cuts

2.1. Hermitian matrix models and equilibrium density

It is well known from the literature that the study of the Hermitian matrix model with the partition function

\[
Z_N = \int_{H_N} \exp(-N \text{Tr}(V(M))) \, dM
\]  

(2.1)

with an even polynomial potential

\[
V(x) = \sum_{i=1}^{2d} t_i x^i
\]  

(2.2)

can be reduced to an eigenvalue problem: \(\lambda = \{\lambda_j, j = 1, \ldots, N\}\) for the matrix \(M\) with distribution

\[
\tilde{Z}_N = \int_{\mathbb{R}^N} \exp \left( 2 \sum_{1 \leq j < k \leq N} \log |\lambda_j - \lambda_k| - N \sum_{i=1}^{N} V(\lambda_j) \right) \, d\lambda_1 \cdots d\lambda_N.
\]  

(2.3)

When \(N \to \infty\), the distribution of the eigenvalues on the line \(d\nu_N(x) = \rho_N(x) \, dx\) is defined (in the distribution theory sense) by the formula

\[
\int_{\mathbb{R}} \phi(x) \, d\nu_N(x) = \frac{1}{Z_N} \int_{\mathbb{R}^N} \left( \frac{1}{N} \sum_{j=1}^{n} \phi(\lambda_j) \right) \exp \left( 2 \sum_{1 \leq j < k \leq N} \log |\lambda_j - \lambda_k| - N \sum_{i=1}^{N} V(\lambda_j) \right) \, d\lambda_1 \cdots d\lambda_N.
\]  

(2.4)
For any test function $\phi(x)$ there is a weak limit $d\nu_\infty(x) := \lim_{N \to \infty} d\nu_N(x)$ which is the same as the equilibrium density $d\nu_{\text{eq}}(x)$ given by the limit of the empirical density:

$$d\nu_{\text{eq}}(x) = \frac{1}{N} \sum_{j=1}^{N} \delta(x - \lambda_j).$$  \hspace{1cm} (2.5)

For details about the existence of the distribution limits, the equality between the equilibrium density $d\nu_{\text{eq}}(x)$ and $d\nu_\infty(x)$, and the following characterizations we refer the reader to [8,15]. Nowadays, many properties of the equilibrium density are known. For example, we know [17] that the equilibrium density is supported by a finite number of intervals $[a_j,b_j], j = 1, \ldots, q$, and that it is absolutely continuous with respect to the Lebesgue measure:

$$d\nu_{\text{eq}}(x) = \rho(x)\, dx, \quad R(x)\, dx = \prod_{j=1}^{q} ((x - a_j)(x - b_j))\, dx \hspace{1cm} (2.6)$$

where $h(x)$ is a polynomial of degree $2d - q - 1$ and $R^{1/2}(x)$ is to be understood as the value on the upper cut of the principal sheet of the complex-valued function $R^{1/2}(z)$ with cuts on $J = \bigcup_{j=1}^{q} [a_j,b_j]$. In fact, the equilibrium density $d\nu_{\text{eq}}(x)$ is completely defined by the knowledge of the extremities $a_j$ and $b_j$ and the unknown coefficients of the polynomial $h(x)$. It has been proved [11] that such quantities are uniquely determined by the following set of equations.

1. Connection between $h(z)$ and the potential $V(z)$:

$$V'(z) = \text{Pol}_{z \to \infty} \left( h(z)R^{1/2}(z) \right).$$  \hspace{1cm} (2.7)

2. Residue constraint:

$$\text{Res}_{z \to \infty} (h(z)R^{1/2}(z)) = -2. \hspace{1cm} (2.8)$$

3. Integral constraints:

$$\int_{b_j}^{a_{j+1}} h(z)R^{1/2}(z)\, dz = 0, \quad \forall j \in \{1, \ldots, q - 1\}. \hspace{1cm} (2.9)$$

Note also that the relation between $h(z)$ and $V(z)$ (2.7) can be inverted as follows:

$$h(z) = \text{Pol}_{z \to \infty} \left( \frac{V'(z)}{R^{1/2}(z)} \right).$$ \hspace{1cm} (2.10)

In theory, the previous set of equations is sufficient for determining the whole solution $d\nu_{\text{eq}}(x)$ but, practically, since the equations are highly non-linear, it becomes very hard to compute the unknown coefficients for two or more intervals or for potentials of degree higher than 4. Moreover, in some exceptional situations, the previous set of equations has multiple solutions. In such situations, the good solution is determined using a positivity condition:

$$h(x) \geq 0, \quad \forall x \in J = \bigcup_{j=1}^{q} [a_j,b_j]. \hspace{1cm} (2.11)$$

\footnote{Here and below we denote as Pol$(A(z))$ the polynomial part of the asymptotic expansion of $A(z)$ at infinity.}
When $\forall x \in \bigcup_{j=1}^{q} [a_j, b_j]: h(x) > 0$, the potential $V(x)$ and the equilibrium measure $d\nu_{eq}(x)$ are called regular. Otherwise the equilibrium density is called singular and the corresponding potential is called critical, meaning that there is at least one point on $J$ where the equilibrium measure vanishes. For a regular potential, the situation can be summarized as in figure 1.

2.2. Singular densities for the $(2m, 1)$ case

In order to study what happens at a singular density, one embeds the potential $V(x)$ into a parametric family $V(x, t)$ such that for some $t = t_c$ the problem is at the critical potential: $V(x, t_c) = V_c(x)$. Then the interesting problem is determining the asymptotics of the eigenvalue correlation functions when $t \to t_c$. Indeed for $t \neq t_c$ the potential is regular and we are in the situation depicted in the figure above. Therefore one can define $a_j(t), b_j(t)$ and $h(x, t)$, determining completely the equilibrium density for $t \neq t_c$, and study their limits when $t \to t_c$. In matrix models, it is often interesting to study a modified version of the integral (2.1) by introducing a parameter $T$ often referred to as ‘the temperature’:

$$Z_N = \int_{\mathcal{H}_N} \exp \left( -\frac{N}{T} \text{Tr}(V(M)) \right) \, dM. \quad (2.12)$$

It turns out that $T$ can be used as a parameter for the study of singular densities. In order to fit this into our previous description, we need to introduce the following notation:

$$V(x, T) = \frac{V(x)}{T}. \quad (2.13)$$

To study the $(2m, 1)$ model, we consider a case such that at $T = T_c$ the potential $V(x, T_c) = V_c(x)$ becomes singular (see formula (2.29) below) and gives rise to a singular
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Figure 2. Example of a critical eigenvalue density; at the point \(b\epsilon\) the density is singular and it behaves like \((x - b\epsilon)^{2m}\).

density defined by the following \(2m\) singular density:

\[
\rho(x, T_c) = \rho_c(x) = \frac{1}{2i\pi}(x - b\epsilon)^{2m}\sqrt{x^2 - b^2} = \frac{1}{2i\pi}h_c(x)\sqrt{x^2 - b^2}
\]  
(2.14)

with \(\epsilon \in (-1, 1]\) representing the position of the singular point in the interval \([-b, b]\) supporting the distribution. For \(T \neq T_c\), we assume that the density is supported by two intervals \([a_1(T), b_1(T)]\) and \([a_2(T), b_2(T)]\) and define (note the normalization with \(1/T\))

\[
\rho(x, T) = \frac{1}{2i\pi T}h(x, T)\sqrt{(x - a_1(T))(x - b_1(T))(x - a_2(T))(x - b_2(T))}.
\]  
(2.15)

Note that in order to recover our singular density at \(T = T_c\) we must have:

1. \(a_1(T) \xrightarrow{T \to T_c} -b\).
2. \(b_1 \xrightarrow{T \to T_c} b\epsilon\).
3. \(a_2(T) \xrightarrow{T \to T_c} b\epsilon\).
4. \(b_2(T) \xrightarrow{T \to T_c} b\).
5. \((x - b\epsilon)h(x, T) \xrightarrow{T \to T_c} Th_c(x)\).

The previous assumptions correspond to the merging to two cuts with degeneracy \(2m\) (order of the singularity). The most general case would be a singular point \(a\) with \(\rho^a(x) \sim (x-a)^p\), \((p, q) \in \mathbb{N}^2\), which is expected to correspond to the \((p, q)\) minimal model (for \(q > 2\) we are speaking about multi-matrix models). In our case the situation can be summarized as in figure 2.

In [4], the authors studied the case \(m = 1\) in detail and conjectured some connections with the Painlevé II hierarchy for higher \(m\).

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2.3. Double-scaling limits in matrix models

In the study of matrix models, one is usually interested in the following functions, called resolvents:

\[ w_n(x_1, \ldots, x_n) := \langle \text{Tr} \left( \frac{1}{x_1 - M} \right) \cdots \text{Tr} \left( \frac{1}{x_n - M} \right) \rangle \]

\[ = \langle \sum_{i_1, \ldots, i_n} \left( \frac{1}{x_1 - \lambda_{i_1}} \right) \cdots \left( \frac{1}{x_n - \lambda_{i_n}} \right) \rangle \]  \hspace{1cm} (2.16)

and in their cumulants, also known as correlation functions:

\[ \hat{w}_n(x_1, \ldots, x_n) := \langle \text{Tr} \left( \frac{1}{x_1 - M} \right) \cdots \text{Tr} \left( \frac{1}{x_n - M} \right) \rangle_c \]

\[ = \langle \sum_{i_1, \ldots, i_n} \text{Tr} \left( \frac{1}{x_1 - \lambda_{i_1}} \right) \cdots \text{Tr} \left( \frac{1}{x_n - \lambda_{i_n}} \right) \rangle_c \]  \hspace{1cm} (2.17)

Here, the brackets stand for the integration relatively to the probability measure \( Z_N^{-1} \text{d}\nu_N(x) \), the \( \lambda_i \) are the eigenvalues of the matrices and the index \( c \) stands for the cumulant part defined as follows:

\[ \langle A_1 \rangle = \langle A_1 \rangle_c \]
\[ \langle A_1 A_2 \rangle = \langle A_1 A_2 \rangle_c + \langle A_1 \rangle_c \langle A_2 \rangle_c \]
\[ \langle A_1 A_2 A_3 \rangle = \langle A_1 A_2 A_3 \rangle_c + \langle A_1 A_2 \rangle_c \langle A_3 \rangle + \langle A_1 A_3 \rangle_c \langle A_2 \rangle + \langle A_2 A_3 \rangle_c \langle A_3 \rangle + \langle A_1 \rangle_c \langle A_2 \rangle_c \langle A_3 \rangle \]
\[ \langle A_1 \cdots A_n \rangle = \langle A_J \rangle = \sum_{k=1}^{n} \sum_{I_1 \sqcup I_2 \cdots \sqcup I_k = J} \prod_{i=1}^{k} \langle A_{I_i} \rangle_c. \]

The joint density correlation functions \( \rho_n(x_1, \ldots, x_n) \) can easily be deduced from the former correlation functions: densities are discontinuities of the resolvents and resolvents are Stieltjes transforms of densities. For example,

\[ \hat{w}_1(x) = \int \frac{\rho_1(x')}{x-x'} \text{d}x' \iff \rho_1(x) = \frac{1}{2i\pi} (\hat{w}_1(x-i0) - \hat{w}_1(x+i0)). \]  \hspace{1cm} (2.18)

Then we want to use a formal \( 1/T \) power series development which unfortunately is not necessarily well defined for all matrix models. Indeed, if one is interested in convergent matrix models, then one must be sure that such a series expansion commutes with integrations. In general, this does not happen and solutions of the convergent matrix model differ from the solutions of the formal matrix model (where by definition the development is assumed to exist and to commute with integrations). The explanation of this phenomenon is simple: when we use a series expansion, it automatically ignores the exponentially small factors (one can think, for example, of \( \exp(-x^2) \) which has at \( x = \infty \) the same asymptotic expansion as the zero function). To sum up, formal matrix models are easier to handle, because by definition the formal expansion exists and we can perform formal operations on it; but the price to pay is that we only get parts of the convergent solutions (we miss the exponentially decreasing terms). It could appear disappointing to consider just formal matrix models, since they do not carry the whole convergent solutions (and thus lead only to a significant but incomplete part of the convergent solutions),
but fortunately differences between formal and convergent matrix models have been well studied, and in [6,12], the authors show how to reconstruct with theta functions the convergent solutions from the formal ones. Moreover recently, in the article [7], the authors showed that once we assume the existence of the asymptotic expansion for the resolvent and the two-point function (formulae (2.20) below for $n = 1, 2$), then the same kind of expansion is guaranteed for all correlation functions and for the partition function as well. From now on, we will consider the case of formal matrix models, i.e. we assume that there automatically exists an expansion of the type

$$\ln Z_N \approx \sum_{g=0}^{\infty} \left( \frac{N}{T} \right)^{2-2g} \hat{f}_g$$

and

$$\hat{w}_n(x_1, \ldots, x_n) = \sum_{g=0}^{\infty} \left( \frac{N}{T} \right)^{2-2g-n} \hat{w}_n^{(g)}(x_1, \ldots, x_n).$$

The numbers $\hat{f}_g$ are called symplectic or spectral invariants of the model (invariant relatively to symplectic transformations of the spectral curve). We refer the reader to the recent article [7] concerning the existence of such an expansion. The previous expansion can be understood as a large $N$ expansion and therefore in the limit $N \to \infty$ one expects the leading value (for $g = 0$) to correspond to the ‘real’ large $N$ limit of the model. In fact this intuition is correct and it has been proved that

$$\hat{y}(x) = i\pi \rho_{eq}(x) = \frac{1}{2} V'(x) - \hat{w}_1^{(0)}(x).$$

This formula establishes a direct link between the equilibrium density and the leading order of the first correlation function. The function $\hat{y}(x)$ (which is up to a trivial rescaling the equilibrium density) is often named the spectral curve of the problem. In our case, it satisfies

$$\hat{y}^2(x) = \text{Polynomial}(x) = \frac{1}{2T} h^2(x, T)(x - a_1(T))(x - b_1(T))(x - a_2(T))(x - b_2(T)).$$

This identity defines the algebraic spectral curve $\hat{y}^2 = P(x)$ where $P$ is a polynomial. We remind the reader that Eynard and Orantin showed in [13] that for any algebraic curve $P(x, y) = 0$ we can associate some symplectic invariants $f_g$ and $w_n^{(g)}(x_1, \ldots, x_n)$. Moreover, when the algebraic curve comes from a matrix model, these invariants are the same as the ones that we defined earlier in (2.16) and (2.17).

In our case, the function $\hat{y}(x, T) = (1/2T)h^2(x, T)(x - a_1(T))(x - b_1(T))(x - a_2(T))(x - b_2(T))$ depends on the temperature $T$ and so do the corresponding invariants $\hat{w}_n(x_1, \ldots, x_n, T)$ and $f_g(T)$. For when $T \to T_c$ it is known that $\forall g > 1, f_g \to \infty$, and that the correlation functions diverges. This is so because the expansion (2.19) reaches its radius of convergence in $T$. In order to recover finite quantities, one has to rescale properly the variables at $T \sim T_c$. In our case we will prove that the good rescaling is given by

$$x_i = b \epsilon + (T - T_c)^{1/2m} \xi_i$$

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Figure 3. Example of a critical eigenvalue density near the critical temperature.

and so

$$
\hat{y}_{\text{rescaled}}(\xi) := \lim_{T \to T_c} \frac{\hat{y}(be + (T - T_c)^{1/2m}\xi, T)}{T - T_c}
$$

and

$$
\hat{w}^{(g)}_{\text{rescaled}, n}(\xi_1, \ldots, \xi_n) := \lim_{T \to T_c} \frac{\hat{w}^{(g)}(be + (T - T_c)^{1/2m}\xi_1, \ldots, be + (T - T_c)^{1/2m}\xi_n, T)}{(T - T_c)^n}
$$

and

$$
\hat{f}_{\text{rescaled}, g} := \lim_{T \to T_c} (T - T_c)^{-(2-2g)} \hat{f}_g
$$

are finite quantities, and the new $\hat{w}^{(g)}_{\text{rescaled}, n}(\xi_1, \ldots, \xi_n)$ and $\hat{f}_{\text{rescaled}, g}$ are the spectral invariants of the rescaled curve $\hat{y}_{\text{rescaled}}(\xi)$. In the general context of the matrix model, such a rescaling is called a double-scaling limit since we have performed a double limit $N \to \infty$ and $T \to T_c$, so $N(T - T_c)^{2m}$ remains finite:

$$
\ln Z_N = \sum_{g=0}^{\infty} \left( \frac{N}{T} \right)^{2-2g} \hat{f}_g \sim \sum_{g=0}^{\infty} \left( \frac{N}{T_c} \right)^{2-2g} (T - T_c)^{(2-2g)} \hat{f}_{\text{rescaled}, g}.
$$

From a geometric point of view, this double-scaling limit corresponds to a local zoom in the region of the degenerate point $be$. The rate of the zoom depends on both the temperature $T$ and the size of the matrices $N$, so $N(T - T_c)$ remains finite. It can be illustrated as in figure 3.

In the context of matrix models, double-scaling limits are often very important because they are expected to give universal (independent of the potential) rescaled spectral curve and correlation functions related to $(p, q)$ minimal models (and thus in our case the $(2m, 1)$ minimal model). On the other hand, $(p, q)$ minimal models are studied through string reductions of some well-known integrable systems. In the rest of the paper, we will prove that, in the case of the merging of two cuts, the rescaled spectral curve corresponds to the spectral curve of the $(2m, 1)$ minimal model. Then, using the method introduced by Bergère and Eynard in [2], we prove that the rescaled correlation functions and the spectral invariants correspond to some ‘correlation’ functions expressed with some determinantal formulae [3] for the $(2m, 1)$ minimal model.

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2.4. The rescaled spectral curve in our $2m$ degenerate matrix model case

In order to get the rescaled spectral curve, we need to perform a few consecutive steps. First we can express explicitly the corresponding critical potential corresponding to $\rho_c(x)$ in (2.14) using (2.7). The computation is straightforward and uses only the general Taylor expansion:

$$\sqrt{1 + x} = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n - 2)!}{n!(n-1)!2^{2n-1}} x^n. \quad (2.28)$$

It gives

$$V'_c(x) = \sum_{j=0}^{2m+1} \left( \begin{array}{c} 2m \\ j - 1 \end{array} \right) (-be)^{2m+1-j} E(2m+1-j/2) \sum_{n=1}^{\infty} \left( \begin{array}{c} 2m \\ 2n + j - 1 \end{array} \right) \frac{(-1)^j (2n - 2)! x^{2(m-n)+1-j}}{n!(n-1)!2^{2n-1}} x^j \quad (2.29)$$

where $E(2m+1-j/2)$ stands for the greatest integer lower than or equal to $2m+1-j/2$. The critical temperature is given by

$$T_c = \frac{b^{2m+2}}{2} \sum_{n=1}^{\infty} \frac{\epsilon^{2m-2n+2}(2m)!}{n!(2m - 2n + 2)! (n-1)!2^{2n-1}}. \quad (2.30)$$

Then, we need to use some reformulations of conditions (2.8) and (2.10). Indeed, it has been known for a long time (a proof can be found in appendix A of [2] but the results were derived well before) and has been used intensively in [1] that the set of equations (2.8) and (2.10) leads to the following ordinary differential equations (sometimes called hodograph equations):

$$\frac{da_1(T)}{dT} = \frac{4(a_1(T) - x_0(T))}{h(a_1(T), T)(a_1(T) - b_1(T))(a_1(T) - a_2(T))(a_1(T) - b_2(T))} \quad (2.31)$$

$$\frac{da_2(T)}{dT} = \frac{4(a_2(T) - x_0(T))}{h(a_2(T), T)(a_2(T) - b_2(T))(a_2(T) - a_1(T))(a_2(T) - b_1(T))}$$

$$\frac{db_1(T)}{dT} = \frac{4(b_1(T) - x_0(T))}{h(b_1(T), T)(b_1(T) - a_1(T))(b_1(T) - a_2(T))(b_1(T) - b_2(T))}$$

$$\frac{db_2(T)}{dT} = \frac{4(b_2(T) - x_0(T))}{h(b_2(T), T)(b_2(T) - a_2(T))(b_2(T) - a_1(T))(b_2(T) - b_1(T))}$$

where the point $x_0(T)$ ($b_1(T) \leq x_0(T) \leq a_2(T)$) is determined by

$$\int_{b_1(T)}^{a_2(T)} \frac{z - x_0(T)}{\sqrt{(b_1(T) - z)(z - a_1(T))(b_2(T) - z)(z - a_2(T))}} dz = 0. \quad (2.32)$$

This set of equations taken at $T = T_c$ for $a_1$ and $b_2$ gives

$$\frac{da_1(T)}{dT} \bigg|_{T=T_c} = \frac{2}{(1 + \epsilon)^{2m}b^{2m+1}}$$

$$\frac{db_2(T)}{dT} \bigg|_{T=T_c} = \frac{2}{(1 - \epsilon)^{2m}b^{2m+1}} \quad (2.33)$$

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so in a neighbourhood of \( T_c \),

\[
a_1(T) \sim b - \frac{2}{(1 + \epsilon)^{2m}b^{2m+1}}(T - T_c) + o(T - T_c) \tag{2.34}
\]

\[
b_2(T) \sim b + \frac{2}{(1 - \epsilon)^{2m}b^{2m+1}}(T - T_c) + o(T - T_c). \tag{2.35}
\]

As mentioned earlier, we expect that the functions \( a_j(T) \) and \( b_j(T) \) will be analytic functions of \( \Delta = (T - T_c)^\nu \), where \( \nu \) is an exponent that we will determine later. Therefore we introduce the following notation:

\[
b_1(T) = b_\epsilon + \alpha \Delta + \sum_{n=1}^{\infty} b_{1,n} \Delta^n
\]

\[
a_2(T) = b_\epsilon + \gamma \Delta + \sum_{n=1}^{\infty} a_{2,n} \Delta^n
\]

\[
x_0(T) = b_\epsilon + X_0 \Delta + \sum_{n=1}^{\infty} x_n \Delta^n
\]

\[
h(z, T) = T(z - b_\epsilon)^{2m-1} + P(z) \Delta + \sum_{n=1}^{\infty} h_n(z) \Delta^n
\]

where \( P(z) \) and \( h_n(z) \) are polynomials of degree at most \( 2m - 2 \). In equations (2.31) for \( a_2(T) \) and \( b_1(T) \) we see that the lhs is of order \( (T - T_c)^{\nu-1} \) whereas the rhs is of order \( (T - T_c)^{-(2m-1)\nu} \). Hence, to have compatible equations we must have, as announced in section 2.3, that

\[
\nu = \frac{1}{2m} \tag{2.37}
\]

The next step is purely technical and consists in proving that \( \alpha = -\gamma \). Since it is only a technical point, we postpone this discussion to appendix B. With the help of this relation we can now determine the rescaled spectral curve.

First remember that for \( T = T_c \), we have (2.14):

\[
(z - b_\epsilon)^{2m} = h_c(z) = \text{Pol}_{z \to \infty} \left( \frac{V_c'(z)}{\sqrt{z^2 - b^2}} \right). \tag{2.38}
\]

For \( T \neq T_c \), recalling that \( V(z, T) = (V(z)/T) \) and that \( \rho(z, T) \) is defined with a factor \( 1/T \) in (2.15) (which will cancel that of \( V(x, T) \)), we have

\[
h(z, T) = \text{Pol}_{z \to \infty} \left( \frac{V'(z)}{\sqrt{(z - a_1)(z - b_2)(z - a_2)(z - b_1)}} \right). \tag{2.39}
\]

We now use the fact that up to order \( \Delta^{2m-1} \), \( a_1 \) and \( b_2 \) are respectively equal to \( -b \) and \( b \) (2.34). Therefore we get

\[
h(z, T) = \text{Pol}_{z \to \infty} \left( \frac{V'(z)}{\sqrt{(z^2 - b^2)(z - a_2)(z - b_1)}} + O(\Delta^{2m}) \right). \tag{2.40}
\]
Then, from the definition of $h_c(x)$ we have that
\[
(z - b\epsilon)^{2m} = \frac{1}{T_c} \text{Pol} \left( \frac{V'(z)}{\sqrt{z^2 - b^2}} \right)
\]
and so
\[
\frac{V'(z)}{\sqrt{z^2 - b^2}} = T_c(z - b\epsilon)^{2m} + O \left( \frac{1}{z} \right).
\] (2.41)

Putting this identity back into (2.40) and noticing that $1/\sqrt{(z^2 - b^2)(z - a_2)(z - b_1)}$ only gives negative powers of $z$ that will disappear when taking the polynomial part, we find that
\[
h(z, T) = T_c \text{Pol} \left( \frac{(z - b\epsilon)^{2m}}{\sqrt{(z - a_2)(z - b_1)}} + O(\Delta^{2m}) \right)
\]
\[
= T_c \text{Pol} \left( \frac{(z - b\epsilon)^{2m-1}}{1 + \frac{2b\epsilon - a_2 - b_1}{z - b\epsilon} + \frac{(b\epsilon - a_2)(b\epsilon - b_1)}{(z - b\epsilon)^2}} + O(\Delta^{2m}) \right).
\] (2.42)

We can now insert the Taylor series of the square root:
\[
(1 + x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n!)^2 2^n} x^n
\] (2.43)
to get
\[
\frac{h(z, T)}{T_c} = \text{Pol} \left( (z - b\epsilon)^{2m-1} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{(n!)^2 2^n} \left( \frac{2b\epsilon - a_2 - b_1}{z - b\epsilon} + \frac{(b\epsilon - a_2)(b\epsilon - b_1)}{(z - b\epsilon)^2} \right)^n \right) \right) + O(\Delta^{2m})
\]
\[
= (z - b\epsilon)^{2m-1} + \text{Pol} \left( \sum_{n=1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n (2n)!}{(n!)^2 2^n} \frac{(2b\epsilon - a_2 - b_1)^k ((b\epsilon - a_2)(b\epsilon - b_1))^{n-k}}{k^n} \right)
\]
\[
\times (z - b\epsilon)^{2m-1+k-2n} + O(\Delta^{2m}).
\] (2.44)

Let us now introduce the following ensemble:
\[
I_m = \{ (n, k) \in (\mathbb{N}^* \times \mathbb{N})/2n - k \leq 2m - 1 \text{ and } k \leq n \}.
\] (2.45)

Clearly $I_m$ is a finite set and we can rewrite the previous identity as
\[
\frac{h(z, T)}{T_c} = (z - b\epsilon)^{2m-1} + \sum_{(n,k) \in I_m} \frac{(-1)^n (2n)!}{(n!)^2 2^n} \frac{(2b\epsilon - a_2 - b_1)^k ((b\epsilon - a_2)(b\epsilon - b_1))^{n-k}}{k^n} (z - b\epsilon)^{2m-1+k-2n}
\]
\[
+ O(\Delta^{2m}).
\] (2.46)
We can now introduce the series expansion in $\Delta$:

$$2b\epsilon - a_2 - b_1 = -(\alpha + \gamma)\Delta + O(\Delta^2)$$

and

$$(b\epsilon - a_2)(b\epsilon - b_1) = \left(\alpha\Delta + \sum_{n=2}^{\infty} a_{2,n}\Delta^n\right)\left(\gamma\Delta + \sum_{n=2}^{\infty} b_{1,n}\Delta^n\right) = \Delta^2\alpha\gamma + O\left(\Delta^3\right).$$

Then we perform the rescaling

$$z = b\epsilon + \Delta\xi.$$  \hspace{1cm} (2.47)

We only need to take into account terms with degree strictly less than $\Delta^{2m}$, so only a few terms remain:

$$\frac{h(\xi, \Delta)}{T_c} = \left(\xi^{2m-1} + \sum_{(n,k)\in I_m} \frac{(-1)^n(2n)!}{(n!)^22^{2n}} \left(\sum_{\ell=1}^{\infty} \frac{1}{\ell!} \right) (-1)^{n} (\alpha + \gamma)^{k} (\alpha\gamma)^{n-k} \xi^{2m-1+k-2\ell} \right) \Delta^{2m-1}$$

and so

$$h_{\text{rescaled}}(\xi) = T_c \left(\xi^{2m-1} + \sum_{(n,k)\in I_m} \frac{(-1)^n(2n)!}{(n!)^22^{2n}} \left(\sum_{\ell=1}^{\infty} \frac{1}{\ell!} \right) (-1)^{n} (\alpha + \gamma)^{k} (\alpha\gamma)^{n-k} \xi^{2m-1+k-2\ell} \right).$$  \hspace{1cm} (2.49)

Eventually we get the rescaled spectral curve by taking into account the trivial term $R^{1/2}(z, T) = \sqrt{(z - a_1(T))(z - a_2(T))(z - b_1(T))(z - b_2(T))}$ with the rescaling (2.47):

$$R^{1/2}(b\epsilon + \xi\Delta, \Delta) = \sqrt{(b\epsilon + \xi\Delta - a_1(\Delta))(b\epsilon + \xi\Delta - a_2(\Delta))(b\epsilon + \xi\Delta - b_1(\Delta))(b\epsilon + \xi\Delta - b_2(\Delta))}$$

$$= b\sqrt{\epsilon^2 - 1} \sqrt{(b\epsilon + \xi\Delta - a_2(\Delta))(b\epsilon + \xi\Delta - b_1(\Delta))} + O\left(\Delta^{2m}\right)$$

and so

$$\rho(b\epsilon + \xi\Delta, \Delta) = \frac{b\sqrt{1 - \epsilon^2}}{2\pi} \sqrt{(\xi - \alpha)(\xi - \gamma)} \left(\xi^{2m-1} + \sum_{(n,k)\in I_m} \frac{(-1)^n(2n)!}{(n!)^22^{2n}} \left(\sum_{\ell=1}^{\infty} \frac{1}{\ell!} \right) (-1)^{n} (\alpha + \gamma)^{k} (\alpha\gamma)^{n-k} \xi^{2m-1+k-2\ell} \right) \Delta^{2m} + O(\Delta^{2m+1})$$

(2.51)

giving that

$$\hat{y}_{\text{rescaled}}(\xi) = b\pi \sqrt{1 - \epsilon^2} \sqrt{(\xi - \alpha)(\gamma - \xi)} \left(\xi^{2m-1} + \sum_{(n,k)\in I_m} \frac{(-1)^n(2n)!}{(n!)^22^{2n}} \left(\sum_{\ell=1}^{\infty} \frac{1}{\ell!} \right) (-1)^{n} (\alpha + \gamma)^{k} (\alpha\gamma)^{n-k} \xi^{2m-1+k-2\ell} \right).$$  \hspace{1cm} (2.52)
In appendix B, we prove that \( \alpha = -\gamma \), so this eventually leads to

\[
\alpha = -\gamma: \quad \hat{y}_{\text{rescaled}}(\xi) = b\pi \sqrt{1 - \epsilon^2} \sqrt{(\gamma^2 - \xi^2)} \left( \xi^{2m-1} + \sum_{n=1}^{m-1} \frac{(2n)!}{(n!)^2 2^{2n}} \xi^{2n} \xi^{2m-1-2n} \right). \tag{2.53}
\]

We can even compute the precise value of \( \gamma \). Indeed, using (2.45) to compute the leading term of the \( \Delta \)-expansion of \( h(a_2, T) \) and putting it back into (2.31) (and using the fact that with the definition of \( x_0 \), (2.32), we have \( X_0 = 0 \) when \( \alpha + \gamma = 0 \)), we have

\[
\alpha = -\gamma \quad \text{with} \quad \gamma^{2m} = \alpha^{2m} = -\frac{4m}{b^2 (1 - \epsilon^2) \left( \sum_{n=0}^{m-1} \frac{(2n)!}{(n!)^2 2^{2n}} \right)} = -\frac{(m!)^2 2^{2m+1}}{b^2 (1 - \epsilon^2) (2m)!}. \tag{2.54}
\]

In this case, introducing the new variable \( s \) by \( \xi = \gamma s \) or equivalently

\[
z = b\epsilon + \gamma \Delta s \tag{2.55}
\]

we get

\[
\alpha = -\gamma: \quad \hat{y}_{\text{rescaled}}(s) = b\pi \gamma^{2m} \sqrt{1 - \epsilon^2} \sqrt{(1 - s^2)} \left( s^{2m-1} + \sum_{n=1}^{m-1} \frac{(2n)!}{(n!)^2 2^{2n}} s^{2n-1-2n} \right). \tag{2.56}
\]

Eventually (2.56) shows as expected that when performing a double-scaling limit \( z = b\epsilon + \gamma \Delta s \) (with \( \gamma \) a complex number given by (2.54) whose argument gives oscillations in the \((\text{Re}(z), \text{Im}(z))\) plane), we recover a universal curve. In section 3, we will see that this rescaled spectral curve (2.53) is exactly (up to the trivial normalization factor \( b\sqrt{1 - \epsilon^2} \)) the spectral curve arising in the Lax pair representation of the Painlevé II hierarchy with \( t_m = 1 \), with all other \( t_j \) (see section 3 for a definition) taken to zero and the identification \( u_0(t) = \gamma \) made (consistently with (3.14)). Before proceeding with the study of the Lax pair representation, we remind the reader that from general results of Eynard and Orantin [13], the rescaled invariants and correlation functions \( \hat{w}_{\text{rescaled}, n}^{(g)} \) and \( \hat{f}_{\text{rescaled}, g} \) are automatically the symplectic invariants and correlation functions of the new rescaled spectral curve \( \hat{y}_{\text{rescaled}}(\xi) \) and thus do automatically satisfy the famous loop equations [13].

### 3. Correlation functions and invariants arising in the Lax pair representation of the (2m, 1) minimal model

In section 2.4, we have found the rescaled spectral curve for a double-scaling limit of a 2m degenerate merging of two cuts in matrix models. As conjectured in [4], we expect this universal double-scaling limit to be connected to the Painlevé II hierarchy. In order to prove this result, we will follow the approach that [2] developed and successfully applied for the \((2m + 1, 2)\) models. It consists in finding a natural spectral curve \( y_{\text{Lax}}(x) \) from a Lax pair representation of the hierarchy and checking that it is equal to our rescaled
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curve defined in section 2.4. Then from another work of Bergère and Eynard [3], we can define from the Lax pair representation some new correlation functions \(W_n^{(g)}(x_1, \ldots, x_n)\) and invariants \(F_g\) using some determinantal formulae and a suitable kernel. In particular, they proved that these new functions satisfy the same loop equations as our correlation functions. Eventually, with the study of the pole structure and \(W_2^{(0)}\) we will end up by proving that our new correlation functions \(W^{(g)}(x_1, \ldots, x_n)\) and invariants \(F_g\) are identical to the rescaled ones defined in section 2.4.

3.1. A Lax pair representation for the \((2m,1)\) minimal model

In their paper [4], the authors claimed that a good Lax pair representation for the \((2m,1)\) minimal model should be given by a set of two \(2 \times 2\) matrices \(\mathcal{R}(x, t)\) and \(\mathcal{D}(x, t)\) satisfying the following Lax pair representation:

\[
\frac{1}{N} \frac{\partial}{\partial x} \Psi(x, t) = \mathcal{D}(x, t) \Psi(x, t) \\
\frac{1}{N} \frac{\partial}{\partial t} \Psi(x, t) = \mathcal{R}(x, t) \Psi(x, t)
\]

(3.1)

where \(\Psi(x, t)\) is a two by two matrix whose entries will be written as

\[
\Psi(x, t) = \begin{pmatrix}
\psi(x, t) & \phi(x, t) \\
\tilde{\psi}(x, t) & \tilde{\phi}(x, t)
\end{pmatrix}
\]

(3.2)

and satisfies the normalization \(\det \Psi(x, t) = 1\).

The compatibility condition of the Lax pair is then

\[
\left[ \frac{1}{N} \frac{\partial}{\partial x} - \mathcal{D}(x, t), \mathcal{R}(x, t) - \frac{1}{N} \frac{\partial}{\partial t} \right] = 0.
\]

(3.3)

In order to specify completely the Lax pair, we need to impose some conditions concerning the shape of the matrices \(\mathcal{R}(x, t)\) and \(\mathcal{D}(x, t)\). In our case we will assume that

\[
\mathcal{R}(x, t) = \begin{pmatrix}
0 & x + u(t) \\
-x + u(t) & 0
\end{pmatrix}
\]

(3.4)

and

\[
\mathcal{D}(x, t) = \sum_{k=0}^{m} t_k \mathcal{D}_k(x, t)
\]

(3.5)

with

\[
\mathcal{D}_k(x, t) = \begin{pmatrix}
-A_k(x, t) & xB_k(x, t) + C_k(x, t) \\
xB_k(x, t) - C_k(x, t) & A_k(x, t)
\end{pmatrix}
\]

(3.6)

and \(A_k, B_k, C_k\) are polynomials of \(x\) of degree respectively \(2k - 2, 2k - 2, 2k\). Note that in the literature one can find several different Lax pairs corresponding to the same problem. Indeed any conjugation (change of basis) gives equivalent matrices that describe the same problem but in different coordinates (see section 4). In fact any equivalent Lax pair can be used since the quantities that we will define later will be invariant regardless of which choice is made. In order to have more compact notation, we will use the following...
convention: a dot will indicate a derivative with respect to $t$ normalized by a coefficient $1/N$, namely
\[
\dot{f}(x, t) \overset{\text{def}}{=} \frac{1}{N} \frac{\partial f(x, t)}{\partial t}.
\]
(3.7)

Putting back this specific shape of matrices into the compatibility equation gives the following recursion:
\[
\begin{align*}
A_0 &= 0, \quad B_0 = 0, \quad C_0 = 1 \\
C_{k+1} &= x^2 C_k + \hat{R}_k(u) \\
B_{k+1} &= x^2 B_k + \hat{R}_k(u) \\
A_{k+1} &= x^2 A_k + \frac{1}{2} \hat{\dot{R}}(u)
\end{align*}
\]
(3.8)

where $\hat{R}_k$ and $\hat{\dot{R}}_k$ are the modified Gelfand–Dikii polynomials given by the following recursion:
\[
\begin{align*}
\hat{R}_0(u) &= u \hat{R}_0(u) = \frac{u^2}{2} \\
\hat{R}_{k+1}(u) &= u \hat{R}_k(u) - \frac{1}{4} \frac{d^2}{dt^2} \hat{R}_k(u) \\
\frac{d}{dt} \hat{R}_k(u) &= u \frac{d}{dt} \hat{R}_k(u).
\end{align*}
\]
(3.9)

It is then easy to see that the matrices $R(x, t)$ and $D(x, t)$ satisfy (3.1) if and only if $u(t)$ satisfies the string equation (see the details in [4])
\[
\sum_{k=0}^{m} t_k \hat{R}_k(u(t)) = -tu(t)
\]
(3.10)

which gives an explicit differential equation of order $m$ satisfied by $u(t)$ (since the polynomials $\hat{R}_k$ can be explicitly computed from the recursion (3.9)). In particular the case $m = 1$ gives the Painlevé II equation:
\[
\frac{d^2u}{dt^2}(t) = 2u^3(t) + 4(t + t_0)u(t)
\]
(3.11)

where $t_0$ is a free parameter that can be set to 0 by a time translation $\tilde{t} = t + t_0$.

**Remark.** Secular equations.

As is always the case for a linear differential equation, we can get a secular equation for $\psi(x, t)$ by combining the two components of the differential equation in $t$ given by (3.1). In our case, we find that both $\psi(x, t)$ and $\phi(x, t)$ are solutions of the secular equation
\[
\dddot{\psi}(x, t) - \frac{\ddot{u}(t)\dot{\psi}(x, t)}{x + u(t)} = (u^2(t) - x^2) \psi(x, t)
\]
(3.12)

which by a simple standard change of variable can be transformed into a Schrödinger-like equation.
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3.2. Large \(N\) development

From the fact that a dot derivative contributes a factor \(1/N\), it is easy to see from the string equation (3.10) that \(u(t)\) admits a series development at large \(N\):

\[
u(t) = \sum_{j=0}^{\infty} \frac{u_j(t)}{N^{2j}} = u_0(t) + \frac{u_1(t)}{N^2} + \cdots \tag{3.13}
\]

**Note.** The fact that \(u(t)\) admits such a development in \(1/N^2\) and not \(1/N\) comes from the fact that the modified Gelfand–Dikii polynomials \(\hat{R}_k\) are sums of terms involving only even numbers of dot derivatives (i.e. even powers of \(1/N\)).

Putting this expansion back into the string equation (3.10) and looking at the power of \(N^0\) of the series gives us that \(u_0(t)\) must satisfy the following algebraic relation:

\[
-t = \sum_{j=1}^{m} t_j \frac{(2j)!}{2^{2j}(j!)^2} u_0(t)^{2j}. \tag{3.14}
\]

From that result, it is then easy to see that the matrices \(R(x, t)\) and \(D(x, t)\) also admit a large \(N\) expansion:

\[
R(x, t) = \begin{pmatrix} 0 & x + u_0(t) \\ -x + u_0(t) & 0 \end{pmatrix} + \frac{1}{N^2} \begin{pmatrix} 0 & u_1(t) \\ u_1(t) & 0 \end{pmatrix} + \cdots = \sum_{j=0}^{\infty} \frac{R_j(x, t)}{N^{2j}} \tag{3.15}
\]

and

\[
D(x, t) = \sum_{j=0}^{\infty} \frac{D_j(x, t)}{N^j} \tag{3.16}
\]

where the first matrix can be explicitly computed:

\[
D_0(x, t) = \begin{pmatrix} 0 & t + B_0 + C_0 \\ -t + B_0 - C_0 & 0 \end{pmatrix} \tag{3.17}
\]

with

\[
B_0 = \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} x^{2(j-k)-1} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k+1}
\]

\[
C_0 = \sum_{j=1}^{m} t_j \left( x^{2j} + \sum_{k=1}^{j} x^{2(j-k)} \frac{(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right). \tag{3.18}
\]

It should also be possible to find equations defining recursively the next matrices \(R_j(x, t)\) and \(D_j(x, t)\) by looking at the next orders in the series expansion. But since we will have no use for such results we do not mention them here.
3.3. The spectral curve attached to the Lax pair

By definition, the spectral curve of a differential system like (3.1) is given by det\((y \text{Id} - D_0(x, t)) = 0\), that is to say by the large \(N\) limit of the eigenvalues of the spectral problem (which we expect to give the large \(N\) limit of our matrix model). Note in particular that this definition is independent of a change of basis (conjugation by a matrix). From all the previous results, we can compute this two by two determinant and get

\[
y^2 = (B_0 + C_0 + t)(B_0 - C_0 - t) = B_0^2 - (C_0 + t)^2. \tag{3.19}
\]

Then, since we have \(xB_0 = u_0(t)(C_0 + t)\) from (3.14) and (3.18) a straightforward computation gives the product as

\[
y^2_{\text{Lax}} = P(x, t) = (u_0(t)^2 - x^2)^{\left(\sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^{2(j-k)-1}(2k)!}{2k(k!)^2} u_0(t)^{2k}\right)^2}. \tag{3.20}
\]

In particular in the specific case where \(\forall j < m : t_j = 0\), and \(t_m = 1\), we find that the spectral curve reduces to

\[
\forall j < m : t_j = 0, t_m = 1 \Rightarrow y_{\text{Lax}}(x) = \sqrt{u_0(t)^2 - x^2}^{\sum_{k=0}^{m-1} \frac{x^{2(m-k)-1}(2k)!}{2k(k!)^2} u_0(t)^{2k}}. \tag{3.21}
\]

As expected, with the identification \(u_0(t) = \gamma\) we recover exactly the rescaled spectral curve of our matrix model (2.53).

**Note.** In (3.20) we can see that the only simple zeros of \(P(x, t)\) are at \(x = \pm u_0(t)\). Moreover since the polynomial \(P(x, t)\) is obviously even and there is no constant term in \(x\) in the sum, we get that \(P(x, t)\) has a double zero at \(x = 0\) and has double roots at some points \(\pm \lambda_i, i = 1, \ldots, m - 1\).

3.4. Asymptotics of the matrix \(\Psi(x, t)\)

The next step in the method of [2] is to determine asymptotic forms of the functions \(\psi(x, t)\) and \(\phi(x, t)\). From the Schrödinger-like equation (3.12), we have a BKW expansion:

\[
\psi(x, t) = g(x, t)e^{Nh(x, t)} \left(1 + \frac{\psi_1(x, t)}{N} + \frac{\psi_2(x, t)}{N^2} + \cdots\right).
\]

Putting this back into the secular equation gives the following result:

\[
\begin{align*}
\psi(x, t) &= \frac{1}{\sqrt{2}} \left(\frac{u_0(t) + x}{u_0(t) - x}\right)^{1/4} e^{N \int x' \sqrt{u_0^2(t') - x'^2} \, dt'} \left(1 + \frac{\psi_1(x, t)}{N} + \cdots\right) \\
\phi(x, t) &= -\frac{1}{\sqrt{2}} \left(\frac{u_0(t) + x}{u_0(t) - x}\right)^{1/4} e^{-N \int x' \sqrt{u_0^2(t') - x'^2} \, dt'} \left(1 + \frac{\phi_1(x, t)}{N} + \cdots\right) \\
\tilde{\psi}(x, t) &= \frac{1}{\sqrt{2}} \left(\frac{u_0(t) - x}{u_0(t) + x}\right)^{1/4} e^{N \int x' \sqrt{u_0^2(t') - x'^2} \, dt'} \left(1 + \frac{\tilde{\psi}_1(x, t)}{N} + \cdots\right) \\
\tilde{\phi}(x, t) &= \frac{1}{\sqrt{2}} \left(\frac{u_0(t) - x}{u_0(t) + x}\right)^{1/4} e^{-N \int x' \sqrt{u_0^2(t') - x'^2} \, dt'} \left(1 + \frac{\tilde{\phi}_1(x, t)}{N} + \cdots\right). \tag{3.22}
\end{align*}
\]

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Then, using standard chain rule differentiation, one can compute
\[
\frac{\partial y}{\partial x} \frac{\partial x}{\partial z} - \frac{\partial y}{\partial t} \frac{\partial x}{\partial z} = -\frac{\partial \tilde{y}}{\partial t} \frac{\partial x}{\partial z}.
\] (3.24)

From the expression for the spectral curve (3.20) (which gives explicitly \( \tilde{y}(x,t) \)) one can compute \( \frac{\partial \tilde{y}}{\partial t} \):
\[
\frac{\partial \tilde{y}}{\partial t} = \frac{ xu_0(\partial_t u_0) }{ \sqrt{u_0^2 - x^2} } \left( \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^{2(j-1-k)}(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right)
+ x(\partial_t u_0) \sqrt{u_0^2 - x^2} \left( \sum_{j=1}^{m} t_j \sum_{k=1}^{j-1} \frac{x^{2(j-1-k)}(2k)!2k}{2^{2k}(k!)^2} u_0(t)^{2k-1} \right)
= \frac{x(\partial_t u_0)}{ \sqrt{u_0^2 - x^2} } \sum_{j=1}^{m} t_j \frac{(2j)!(2j)}{2^j(j!)^2} u_0(t)^{2j-1}
= -\frac{ x }{ \sqrt{u_0^2 - x^2} }.
\] (3.25)

To get the last identity, we have used the string equation (3.14) for \( u_0(t) \). Therefore by introducing the parametrization
\[
z^2 = u_0(t)^2 - x^2 \iff x^2 = u_0(t)^2 - z^2
\] (3.26)
on one finds that
\[
x'(z,t) = \frac{\partial x}{\partial z} = \frac{\sqrt{u_0^2 - x^2}}{x}, \quad \frac{\partial x}{\partial t} = \frac{-u_0(\partial_t u_0)}{x}
\] (3.27)
and so eventually
\[
\frac{\partial y}{\partial z} \frac{\partial x}{\partial t} - \frac{\partial y}{\partial t} \frac{\partial x}{\partial z} = -\frac{\partial \tilde{y}}{\partial t} \frac{\partial x}{\partial z} = -\frac{ x }{ \sqrt{u_0^2 - x^2} } \frac{ \sqrt{u_0^2 - x^2} }{ x } = -1.
\] (3.28)

The last identity can be rewritten as
\[
\frac{\partial y}{\partial t} \frac{\partial x}{\partial z} - \frac{\partial y}{\partial z} \frac{\partial x}{\partial t} = 1
\] (3.29)
and interpreted as the remaining of a non-commutative structure of \([P,Q] = 1/N\) in the limit \( N \to \infty \) which in such situations often transforms into a Poisson structure for
\[\text{doi:10.1088/1742-5468/2011/04/P04013}\]
$$y(z, t) \leftrightarrow P \text{ and } x(z, t) \leftrightarrow Q \text{ on simply replacing the commutator with a Lie bracket:}$$

$$\{y(z, t), x(z, t)\} = 1.$$

(3.30)

With the help of this structure, we can get a reformulation of the integral:

$$\frac{\partial \tilde{y}}{\partial t} = \frac{1}{x^*(z)}$$

(3.31)

and hence

$$\frac{\partial}{\partial t} \int^x \tilde{y} \, dx = z$$

(3.32)

and

$$\int^t \sqrt{u_0^2(t') - x^2} \, dt' = \int^t z \, dt = \int^x \tilde{y} \, dx.$$

(3.33)

Eventually we have the following large \(N\) developments:

$$\psi(x, t) = \frac{1}{\sqrt{2}} \left( \frac{u_0(t) + x}{u_0(t) - x} \right)^{1/4} e^{N f^* \tilde{y} \, dx} \left( 1 + \frac{\psi_1(x, t)}{N} + \cdots \right)$$

$$\phi(x, t) = -\frac{1}{\sqrt{2}} \left( \frac{u_0(t) + x}{u_0(t) - x} \right)^{1/4} e^{-N f^* \tilde{y} \, dx} \left( 1 + \frac{\phi_1(x, t)}{N} + \cdots \right)$$

(3.34)

$$\tilde{\psi}(x, t) = \frac{1}{\sqrt{2}} \left( \frac{u_0(t) - x}{u_0(t) + x} \right)^{1/4} e^{N f^* \tilde{y} \, dx} \left( 1 + \frac{\tilde{\psi}_1(x, t)}{N} + \cdots \right)$$

$$\tilde{\phi}(x, t) = \frac{1}{\sqrt{2}} \left( \frac{u_0(t) - x}{u_0(t) + x} \right)^{1/4} e^{-N f^* \tilde{y} \, dx} \left( 1 + \frac{\tilde{\phi}_1(x, t)}{N} + \cdots \right).$$

3.5. Kernels and correlation functions in the Lax pair formalism

It was established in [3] that one can define a kernel \(K(x_1, x_2)\) and define from it (through determinantal formulae) some functions \(W_n(x_1, \ldots, x_n)\) that have nice properties. In particular the authors showed in [3] that these functions do satisfy some loop equations and thus are likely to correspond to our matrix model correlation functions. Following [3] we define the kernel by

$$K(x_1, x_2) = \frac{\psi(x_1) \tilde{\phi}(x_2) - \tilde{\psi}(x_1) \phi(x_2)}{x_1 - x_2}.$$

(3.35)

Then we define the (connected) correlation functions by

$$W_1(x) = \psi'(x) \tilde{\phi}(x) - \tilde{\psi}'(x) \phi(x)$$

(3.36)

$$W_n(x_1, \ldots, x_n) = -\frac{\delta_{n,2}}{(x_1 - x_2)^2} - (-1)^n \sum_{\sigma \text{-cycles} \ i=1}^n K(x_{\sigma(i)}, x_{\sigma(i+1)})$$

(3.37)

and eventually we define non-connected functions \(W_{n,n-c}(x_1, \ldots, x_n)\) by determinantal formulae:

$$W_{n,n-c}(x_1, \ldots, x_n) = \det(K(x_i, x_j)).$$

(3.38)
where the notation $\det'$ means that the determinant is computed in the usual way as a sum over permutations $\sigma$ of products $(-1)^\sigma \prod_{i=1}^n K(x_i, x_n)$, except for terms where $i = \sigma(i)$ and where $i = \sigma(j), j = \sigma(i)$. In such cases, one must replace $K(x_i, x_i)$ with $W_1(x_i)$ and $K(x_i, x_j)K(x_j, x_i)$ with $-W_2(x_i, x_j)$. For additional details, we invite the reader to look at [3].

As in our problem we will need the large $N$ developments of these functions, we introduce the notation

$$K(x_1, x_2) = K_0(x_1, x_2)e^{\int_{x_2}^{x_1} \tilde{y} \, dx} \left( 1 + \sum_{g=1}^\infty N^{-g} K(g)(x_1, x_2) \right)$$

$$W_n(x_1, \ldots, x_n) = \sum_{g=0}^\infty N^{2-2g-n} W_n^{(g)}(x_1, \ldots, x_n) \quad (3.39)$$

$$W_{n,n-c}(x_1, \ldots, x_n) = \sum_{g=0}^\infty N^{n-2g} W_{n,n-c}^{(g)}(x_1, \ldots, x_n).$$

Then, we can insert all our previous results concerning the leading terms of the series expansion (3.34), (3.37) and (3.39). This gives

$$K_0(x_1, x_2) = \frac{1}{2(x_1 - x_2)} \left( \frac{u_0 + x_1}{u_0 - x_1} \right)^{1/4} \left( \frac{u_0 - x_2}{u_0 + x_2} \right)^{1/4} + \left( \frac{u_0 - x_1}{u_0 + x_1} \right)^{1/4} \left( \frac{u_0 + x_2}{u_0 - x_2} \right)^{1/4}$$

$$W_1^{(0)}(x) = \tilde{y}(x) \quad (3.41)$$

and

$$W_2^{(0)}(x_1, x_2) = \frac{1}{4(x_1 - x_2)^2} \left( -2 + \sqrt{\frac{(u_0 + x_1)(u_0 - x_2)}{(u_0 - x_1)(u_0 + x_2)}} + \sqrt{\frac{(u_0 - x_1)(u_0 + x_2)}{(u_0 - x_1)(u_0 + x_2)}} \right). \quad (3.42)$$

In order to get rid of the square roots in the expressions above, it is better to introduce a proper parametrization of our spectral curve (3.20). Let us define

$$x = \frac{u_0}{2} \left( z + \frac{1}{z} \right) = \frac{u_0(z^2 + 1)}{2z} \iff z = \frac{1 + \sqrt{x^2 - u_0^2}}{u_0}. \quad (3.43)$$

In particular, under such a change of variables we obtain several useful identities:

$$\sqrt{\frac{u_0 - x}{u_0 + x}} = \frac{z - 1}{z + 1}$$

$$u_0 - x = - \frac{u_0(z - 1)^2}{2z}$$

$$u_0 - x = \frac{u_0(z + 1)^2}{2z}$$

$$\sqrt{\frac{u_0^2 - x^2}{u_0^2}} = \frac{u_0(z + 1)(z - 1)}{2z}$$

$$\frac{dx(z)}{dz} = \frac{u_0(z^2 - 1)}{2z^2}. \quad (3.44)$$

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Eventually we can rewrite $W_2^{(0)}$ in terms of the new variable $z$:

$$W_2^{(0)}(z_1, z_2) = \frac{4z_1^2z_2^2}{u_0^2(z_1^2 - 1)(z_2^2 - 1)(z_1z_2 - 1)^2}. \quad (3.45)$$

Although these functions have some interesting features, they still depend on the choice of coordinates on the Riemann surface defined by the spectral curve. Therefore, we introduce similarly to [2] and [13] the corresponding differential forms:

$$W_n^{(g)}(z_1, \ldots, z_n) = W_n^{(g)}(x(z_1), \ldots, x(z_n))x'(z_1) \cdots x'(z_n) + \delta_n, 2\delta g, 0 x'(z_1)x'(z_2) \left( x(z_1) - x(z_2) \right)^2. \quad (3.46)$$

These differentials are symmetric rational functions of all their variables. Moreover as proved in the crucial theorem 3.2, these functions only have poles at $z_i = \pm 1$ (except again $W_2^{(0)}(z_1, z_2)$ which may have a pole at $x(z_1) = x(z_2)$). Eventually, a direct computation from (3.45) gives

$$W_2^{(0)}(z_1, z_2) = \frac{1}{(z_2 - z_1)^2}. \quad (3.47)$$

### 3.6. Loop equations, determinantal formulae, pole structure and unicity

The previous determinantal definitions may seem rather arbitrary, but as we mentioned before they have the interesting property (proved in [3]) of satisfying the following loop equations.

**Theorem 3.1. Loop equations satisfied by the determinantal functions:**

$$P_n(x; x_1, \ldots, x_n) = W_{n+2,n-c}(x, x, x_1, \ldots, x_n)$$

$$+ \sum_{j=1}^n \frac{\partial}{\partial x_j} W_n(x, x_1, \ldots, x_j-1, x_{j+1}, \ldots, x_n) - W_n(x_1, \ldots, x_n) x - x_j \quad (3.48)$$

is a polynomial in the variable $x$. The previous theorem is equivalently reformulated for the standard connected functions after projection on $N^{2-2g}$ using the sets of equations (valid for every $g \geq 0$):

$$P_n^{(g)}(x; x_1, \ldots, x_n) = \sum_{h=0}^g \sum_{I \subset J} W_{1+|I|}^{(h)}(x, I)W_{1+n-|I|}^{(g-h)}(x, J/I)$$

$$+ \sum_{j=1}^n \frac{\partial}{\partial x_j} W_n^{(g)}(x, \{x_j\}) - W_n^{(g)}(x_1, \ldots, x_n) x - x_j \quad (3.49)$$

is a polynomial of the variable $x$.

We emphasize again that loop equations are essential, because it is well known in the matrix model world [17] that the correlation functions introduced in our section 1 do satisfy these loop equations. Unfortunately, loop equations generally admit several solutions encoded essentially in the unknown coefficients of the polynomial $P_n$. Therefore we need some additional results to get unicity. The first one deals with the pole structure:
Theorem 3.2. Pole Structure:
The functions $z \to \psi_k(z,t)$ are rational functions with poles only at $z \in \{\pm i, 0, \infty\}$. The coefficients of these fractions depend on $u_0(t)$ and its derivatives. Hence the determinantal correlation functions $W_n^{(g)}$ are symmetric and rational functions in the variables $z_i$ with poles only at $z_i = \pm 1$.

Proof. The last part of the theorem is obvious from the definitions as soon as the results regarding the $\psi_k(z,t)$ are established. This proof is presented in appendix A and is highly non-trivial. It uses the whole structure of integrability (i.e. the two differential equations (3.1)) to eliminate other possible poles (at the other zeros of $y_{\text{Lax}}(x)$).

With the knowledge of the pole structure of the $W_n^{(g)}$, the fact that they satisfy the loop equations and the knowledge of $W_2^{(0)}$, we have a unicity theorem. In fact under these conditions we can identify our differentials $W_n^{(g)}$ with the ones defined by the standard recursion relation introduced by Eynard and Orantin in [13].

Theorem 3.3. The differentials $W_n^{(g)}$ satisfy the following recursion:

$$W_{n+1}(z_1, \ldots, z_n, z_{n+1}) = \text{Res}_{z \to z_1} \frac{dz}{2u_0y(z)(1 - (z_{n+1}/z))((1/z) - z_{n+1})} \times \left[ W_{n+2}^{(g-1)}(z, \bar{z}, z_1, \ldots, z_n) + \sum_{h=0}^{n} \sum_{I \in J} W_{1+|I|}^{(h)}(z, I) W_{n+1-|I|}^{(g-h)}(\bar{z}, J/I) \right]$$

(3.50)

where $J$ is a short form for $J = (z_1, \ldots, z_n)$ and $\sum_{h=0}^{n} \sum_{I \in J}'$ means that we exclude the terms $(h, I) = (0, \emptyset)$ and $(h, I) = (g, J)$ from the sum. The notation $\bar{z}$ stands for the conjugate point of $z$ near the poles where the residue is taken. In our case, $\bar{z} = 1/z$.

Note. It is worth noticing that in Eynard and Orantin’s notation we have in our case (we omit the dependence on the $t$ parameter)

$$\omega(z) = y(z) \frac{u_0(z^2 - 1)}{z^2} \quad y(\bar{z}) = y(1/z) = -y(z)$$

$$dE_z(p) = \frac{1}{2} \int_{z}^{1/z} \frac{ds}{(s - p)^2} = \frac{1 - z^2}{2(z - p)(pz - 1)}$$

(3.51)

and so

$$\frac{dE_z(z_{n+1})}{\omega(z)} = \frac{z^2}{2u_0y(z)(z - z_{n+1})(1 - z_{n+1}z)} = \frac{1}{2u_0y(z)(1 - (z_{n+1}/z))((1/z) - z_{n+1})}.$$  

(3.52)

Proof of theorem 3.3. The unicity proof has been given in various articles, but for completeness we rederive it here with our notation. First of all Cauchy’s theorem states that

$$W_{n+1}^{(g)}(z_1, \ldots, z_{n+1}) = \text{Res}_{z \to z_{n+1}} \frac{dz}{z - z_{n+1}} W_{n+1}^{(g)}(z_1, \ldots, z_n).$$

(3.53)
We can move the integration contour to enclose all other poles, i.e. only ±1 in our case:

\[
\mathcal{W}^{(g)}_{n+1}(z_1, \ldots, z_{n+1}) = \text{Res}_{z \to \pm 1} \frac{dz}{z_{n+1} - z} \mathcal{W}^{(g)}_{n+1}(z_1, \ldots, z_{n+1}) = \text{Res}_{z \to \pm 1} \frac{x'(z)}{z_{n+1} - z} W^{(g)}_{n+1}(x(z_1), \ldots, x(z_{n+1})).
\] (3.54)

We observe that the loop equations (3.49) can be rewritten in accordance with (3.49) in the following form by isolating the coefficients \(W^{(0)}_1\) in the sum:

\[
-2W^{(0)}_1(x)W^{(g)}_{n+1}(x_1, \ldots, x_n, x) = \sum_{h=0}^{g} \sum_{I \subset J} W^{(h)}_{1+|I|}(x, I)W^{(g-h)}_{1+n-|I|}(x, J/I)
+ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} W^{(g)}_{n}(x, J/\{x_j\}) - W^{(g)}_{n}(x_j, J/\{x_j\}) - P^{(g)}_{n}(x_1, \ldots, x_n).
\] (3.55)

Then putting this back into the residue computation, observing that the polynomial \(P^{(g)}_{n}(x_1, \ldots, x_n)\) does not contribute to the residue, and using the relation between \(x\) and \(z\), we are left with (3.50).

### 4. Lax pairs for the \((2m,1)\) minimal model and for the Painlevé II hierarchy

#### 4.1. The \((2m,1)\) minimal model and the Flashka–Newell Lax pair

As observed in [4] the string equation (3.10) is nothing but the \(m^{th}\) member of the so-called Painlevé II hierarchy. The Painlevé II (PII) hierarchy, a collection of ODEs of order \(2m\), arises as a self-similar reduction of the mKdV hierarchy. In the papers [10,16] this relationship has been used to construct a Lax pair for the PII hierarchy starting from the relevant Lax pair for the modified KdV hierarchy. We call this PII Lax pair the Flashka–Newell Lax pair since the first member of the hierarchy was found, for the first time, in [14]. In this subsection we prove that, up to a linear transformation of the wavefunction and a rescaling of the variables, the Flashka–Newell Lax pair is equivalent to the \((2m,1)\) minimal model Lax pair. In order to simplify the notation we forget, in this section, the rescaling given by \(1/N\) over the variables \(x\) and \(t\). We begin with the case \(t_1 = 0 = t_2 = \cdots = t_{m-1} = 0\).

**Proposition 4.1.** Define \(\tilde{\Psi}\) as a new wavefunction

\[
\tilde{\Psi} := J\Psi
\]

with

\[
J := \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}
\]

and set \(t_m \mapsto (4^{m+1}/2)\) (all other parameters \(t_j\) being equal to 0). Then \(\tilde{\Psi}\) satisfies the Flashka–Newell Lax pair as written in [10].

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Proof. Since \(J\) is constant we observe that \(\tilde{\Psi}\) solves the Lax system

\[
\frac{\partial}{\partial x} \tilde{\Psi}(x, t) = \tilde{\mathcal{D}}_m(x, t) \tilde{\Psi}(x, t) \quad \frac{\partial}{\partial t} \tilde{\Psi}(x, t) = \tilde{\mathcal{R}}(x, t) \tilde{\Psi}(x, t)
\]

with \(\tilde{\mathcal{D}}(x, t), \tilde{\mathcal{R}}(x, t)\) obtained through conjugation with \(J\); i.e.,

\[
\tilde{\mathcal{R}}(x, t) = J \mathcal{R}(x, t) J^{-1} = \begin{pmatrix} -ix & u \\ u & ix \end{pmatrix}
\]

and

\[
\tilde{\mathcal{D}}_m(x, t) = \frac{4^{m+1}}{2} J \mathcal{D}_m(x, t) J^{-1}
\]

\[
= \frac{4^{m+1}}{2} \begin{pmatrix} -iC_m(x, t) & iA_m(x, t) + xB_m(x, t) \\ -iA_m(x, t) + xB_m(x, t) & iC_m(x, t) \end{pmatrix}
\]

These two matrices are exactly the ones appearing in (16a) and (16b) in [10] (modulo the identification \(u \leftrightarrow w, x \leftrightarrow \lambda, t \leftrightarrow z\)). For the matrix \(\tilde{\mathcal{R}}\) this is self-evident. For \(\tilde{\mathcal{D}}_m\) we just have to observe that it has the same shape as the matrix written in the right-hand side of (16b) (see equations (14); in particular the polar part in (16b) is zero thanks to (14b)). On the other hand this condition, plus the compatibility condition, determines uniquely \(\tilde{\mathcal{D}}_m\).

Of course the result above is extended to the case in which all \(t_j\) enter in \(\mathcal{D}\) just taking linear combinations of the matrices studied in the previous proposition. This has been done, for the Flashka–Newell pair, in [16] (note, nevertheless, that there the spectral parameter is rotated; \(\lambda \rightarrow -i\lambda\)). Hence we have proposition 4.2.

Proposition 4.2. Under a rescaling of all time variables \(t_j \rightarrow (4^{j+1}/2)t_j\), the \((2m, 1)\)-minimal model Lax pair is equivalent to the Flashka–Newell Lax pair for the PII hierarchy.

5. Conclusion and outlook

In section 1, we have established that the double-scaling limit of a matrix model with a \(2m\)-degenerate point can define a universal rescaled spectral curve \(\hat{y}_{rescaled}(x)\). In section 1 we also recalled that the correlation functions and symplectic invariants \(\hat{w}^{(g)}_n(x_1, \ldots, x_n)\) and \(\hat{f}_g\) can also be rescaled in a suitable way in order to give some new functions \(\hat{w}_{rescaled,n}(x_1, \ldots, x_n)\) and new symplectic invariants \(\hat{f}_{rescaled,g}\) corresponding respectively to the correlation functions and symplectic invariants of the rescaled curve \(\hat{y}_{rescaled}(x)\). Then, starting from a Lax pair of the Painlevé II hierarchy and using the same method as [2], we have constructed a spectral curve \(y_{\text{Lax}}(x)\) which coincides with \(\hat{y}_{rescaled}(x)\) for a natural choice of the flow parameters \(t_j\). Finally, with the definition of a suitable kernel and determinantal formulae, we have defined in the same way as [2] some functions \(W_{\alpha}^{(g)}\) having interesting properties (loop equations). Studying in detail the pole structure and computing \(W_2^{(0)}(z_1, z_2)\), we eventually showed that the functions \(W_{\alpha}^{(g)}\) are in fact exactly the correlation functions of the curve \(y_{\text{Lax}}(x)\). Since the two spectral curves are the same, we have proved the statement.

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Theorem 5.1. The correlation functions (and spectral curve) of the double-scaling limit of a $2m$-degenerate merging of two cuts are the same as the functions $W^{(g)}_n$ (and spectral curve) defined by determinantal formulae of the integrable Painlevé II hierarchy’s kernel.

This result reinforces the links between double-scaling limit in matrix models and integrable $(p, q)$ minimal models. With this new result and that of [2], the two models are shown to be identical for $(p = 2m, q = 1)$ and $(p = 2m + 1, q = 2)$ ($m \in \mathbb{N}^\ast$). However even if this identity is expected to hold for every $(p, q)$, some complete proofs such as the one presented here are still lacking. Indeed, while our reasoning may seem easy to generalize to arbitrary values of $p$ and $q$, the crucial theorem (3.49) establishing that the functions $W^{(g)}_n$ coming from determinantal formulae do satisfy the loop equations (proved in [3]) is only known to be valid for $q \leq 2$ at the moment. Therefore a good approach to the generalization for arbitrary values of $(p, q)$ could be to first extend this theorem to every $(p, q)$ and then to use the method presented here to extend the result.

Another approach could be to use this approach to study other integrable systems whose Lax pairs are known. Indeed, it is possible to perform the same method as is presented here for any Lax pair. In particular, for every Lax pair, it would be interesting to analyse the associated spectral curve and the corresponding determinantal correlation functions.

Appendix A. Pole structure for $\psi_k(z, t)$

In order to use the unicity theorem (3.49) showing that the $W^{(g)}_n$ are the expected correlation functions, we need to make precise the pole structure of the functions $\psi_k(x, t)$ and $\phi_k(x, t)$ from which they are defined. In order to determine the functions $\psi_k(x, t)$, one can insert the series expansion (3.34) into the secular equations. Since the cases $\psi_k(x, t)$ and $\phi_k(x, t)$ are similar (they satisfy the same secular equation), we will focus just on the $\psi_k(x, t)$. The main issue of this appendix is that putting the large $N$ asymptotics of $\psi(x, t)$ (3.34) into the secular equation a priori gives unwanted poles at the zeros of $y(x)$ for $\psi_k(x, t)$ that we need to rule out. It is the purpose of this appendix to explain how this can be done.

A.1. Study of the differential equation in $t$

From the fact that $u(t)$ satisfies the string equation we remind the reader that we have (3.14):

$$t = -\sum_{j=1}^{m} t_j \frac{(2j)!}{2^{2j}(j)!^2} u_0(t)^{2j} = P_0'(u_0).$$  \hfill (A.1)

From this, it follows that $(du_0/dt)$ is

$$\frac{du_0}{dt} = \frac{1}{P_0'(u_0)}.$$  \hfill (A.2)
Performing more differentiations with respect to $t$ can give the derivatives of $u_0(t)$ to any order as a fraction whose denominator is always a power of $P'_0(u_0)$. For example,

\[
\frac{d^2 u_0}{dt^2} = - \frac{P''_0(u_0)}{(P'_0(u_0))^3} \quad \frac{d^3 u_0}{dt^3} = - \frac{P'''_0(u_0)}{(P'_0(u_0))^4} + 3 \left( \frac{P''_0(u_0)}{(P'_0(u_0))^3} \right)^2 \]

and so on.

As a consequence, any power of any derivative of $u_0$ with poles only at the roots of $P'_0(x)$. For example, expressions like $(du_0/dt)(d^3 u_0/dt^3) + (du_0/dt)^2(d^2 u_0/dt^2)$ will be rational functions of $u_0$ with poles only at the roots of $P'_0(x)$.

Now, putting the development of $u(t) = u_0(t) + (u_2(t)/N^2) + (u_3(t)/N^3) + \cdots$ back into the full string equation (3.10) gives that any subleading order $u_k$ can be expressed as a rational function of $u_0$ with poles only at the roots of $P'_0(x)$.

Eventually, inserting the shape of the function $\psi(x,t)$ into the secular equation and evaluating the order $N^{-k}$ gives the following equation $\forall k \geq 2$:

\[
\partial_t \psi_{k-1} = \frac{\partial^2 g}{2gh} \psi_{k-2} - \frac{\partial g}{gh} \partial_t \psi_{k-2} - \frac{\partial^2 \psi_{k-2}}{2h} + \frac{1}{2} \left( \frac{\partial u}{u + x} \right)_k + \frac{1}{2} \sum_{i=0}^{k-2} \left( \frac{\partial u}{u + x} \right)_{k-i} \psi_i \\
+ \frac{\partial g}{2gh} \left( \frac{\partial u}{u + x} \right)_{k-1} + \frac{1}{2} \sum_{i=0}^{k-2} \left( \frac{\partial u}{u + x} \right)_{k-1-i} \left( \psi_i \frac{g}{gh} + \frac{\partial \psi_i}{h} \right) \\
+ \frac{1}{2} \sum_{i=0}^{k-2} (u^2)_{k-i} \psi_i \\
\]

where we have written for short

\[
\psi_0(x,t) = 1 \quad h(x,t) = \sqrt{u_0(t)^2 - x^2} \\
g(x,t) = \left( \frac{u_0(t) + x}{u_0(t) - x} \right)^{1/4} \quad (A.5)
\]

and the notation $(\partial_k u/u + x)_k$ stands for the term in $N^{-k}$ in the expansion of $(\partial_k u/u + x)$. Note in particular that these terms can be expressed as a fraction with poles at $u_0(t) + x = 0$ and at $P'(u_0(t)) = 0$ (the latter are independent of $x$). For example the first one is

\[
\partial_t \psi_1(x,t) = \frac{(\partial u_0)^2 x^2 (u_0 - x)^{3/2}}{4(u_0 + x)^{3/2}} + \frac{(u_0 + x)^{1/2}}{(u_0 - x)^{1/2}} u_2(t) \quad (A.6)
\]

where we remember that $u_2(t)$ can be expressed as a rational function of $u_0(t)$ whose poles are known to occur only when $u_0(t)$ is at a root of $P'_0$ (and thus are independent of $x$). From this expression, it is clear that $\psi_1(x,t)$ may only have $x$-dependent singularities at $x = \pm u_0$ and at $x = \infty$.

### A.2. Study of the differential equation in $x$

The technique presented in section A.1 can be carried out for the differential equation in $x$. Starting with the second equation of the Lax pair (3.1):

\[
\frac{1}{N} \frac{\partial}{\partial x} \Psi(x,t) = \begin{pmatrix} -A(x,t) & xB(x,t) + C(x,t) \\ xB(x,t) - C(x,t) & A(x,t) \end{pmatrix} \Psi(x,t) \quad (A.7)
\]

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we can derive another secular equation for both $\psi(x, t)$ and $\phi(x, t)$:

\[
0 = \frac{1}{N^2} \frac{\partial^2}{\partial x^2} \psi(x, t) - \frac{1}{N^2} \left( \frac{\partial_x (xB + C)}{xB + C} \right) \frac{\partial}{\partial x} \psi(x, t) + \frac{1}{N} \left( \partial_x A - A \frac{\partial_x (xB + C)}{xB + C} \right) \psi(x, t) - y^2(x, t)\psi(x, t) \tag{A.8}
\]

where we have used that

\[
\det(\Psi) = 1 \Leftrightarrow y^2(x, t) = A(x, t)^2 + x^2B(x, t)^2 - C(x, t)^2. \tag{A.9}
\]

Note in particular in the last identity that the rhs should have a large $N$ development whereas the lhs $y(x)$ given by (3.20) does not. Therefore, the lhs must have vanishing subleading orders in $1/N^k, \forall k > 0$.

Moreover, reformulating (3.17) gives

\[
A_0 = 0 \quad (xB + C)_0 = y(x, t)\sqrt{\frac{u_0 + x}{u_0 - x}} \tag{A.10}
\]

where the subscript 0 stands for the first order in the large $N$ expansion. Indeed, this comes from the fact that

\[
y^2(x, t) = (xB + C)_0(xB - C)_0 = P(x, t) = P_1(x, t)P_2(x, t)
\]

\[
(xB + C)_0 = P_1(x, t) = (u_0 + x) \left( \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^2(j-1-k)(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right) \tag{A.11}
\]

\[
(xB - C)_0 = P_2(x, t) = (u_0 - x) \left( \sum_{j=1}^{m} t_j \sum_{k=0}^{j-1} \frac{x^2(j-1-k)(2k)!}{2^{2k}(k!)^2} u_0(t)^{2k} \right)
\]

and eventually

\[
P_2(x, t) = P_1(x, t)\frac{u_0 - x}{u_0 + x}. \tag{A.12}
\]

With (A.10) it is easy to see that

\[
\left( \frac{\partial_x (xB + C)}{xB + C} \right)_0 = \frac{\partial_x y}{y} + \frac{u_0}{u_0^2 - x^2} \tag{A.13}
\]

which will be crucial for the consistency of the computation. Indeed, putting the large $N$ expansion of $\psi(x, t)$:

\[
\psi(x, t) = g(x, t)e^{Nh(x, t)} \left( 1 + \frac{\psi_1(x, t)}{N} + \frac{\psi_2(x, t)}{N^2} + \cdots \right)
\]

into (A.8) and comparing the first orders in $1/N$ gives

\[
0 = g(x, t)y^2(x, t) - g(x, t)y^2(x, t)
\]

\[
0 = \partial_x (g(x, t)y(x, t)) + y(x, t)\partial_x g(x, t) + g(x, t)y(x, t) \left( \frac{\partial_x (xB + C)}{xB + C} \right)_0. \tag{A.14}
\]
The second equation, with the help of (A.13), determines \( g(x, t) \) consistently with (A.5), that is to say,
\[
g(x, t) = \left( \frac{u_0(t) + x}{u_0(t) - x} \right)^{1/4}.
\]

Note now that \( \forall k > 0 \), the function \( (\partial_x (xB + C)/xB + C)_k \) only has singularities at the singularities of \( (1/xB_0 + C_0) \) according to the standard rules of Taylor series for a fraction. The next order, \( 1/N^2 \), gives us the function \( \psi_1(x, t) \) (with the convention that a subscript \( k \) defines the term in \( N^{-k} \) in the expansion at large \( N \)):
\[
\partial_x \psi_1(x, t) = -\frac{\partial_x g}{2gy} + \frac{\partial_x g}{2gy} \left( \frac{\partial_x (xB + C)}{xB + C} \right)_0 + \frac{1}{2} \left( \frac{\partial_x (xB + C)}{xB + C} \right)_1 + \frac{1}{2} \left( \frac{\partial_x A - \partial_x (xB + C)}{xB + C} \right)_1.
\]

From the definition of \( g(x, t) \), it is easy to compute
\[
\frac{\partial_x g}{g} = \frac{1}{2} \frac{u_0}{u_0^2 - x^2} \quad \frac{\partial_x^2 g}{g} = \frac{u_0 x}{u_0^2 - x^2} + \frac{1}{4} \frac{u_0^2}{(u_0^2 - x^2)^2}
\]
and thus to see that \( \partial_x \psi_1(x, t) \) is a function of \( x \) that may only have singularities at \( x = \pm u_0 \), at \( x = \infty \) and at the other zeros of \( y(x) = 0 \). (This is so because \( (\partial_x (xB + C)/xB + C)_1 \) has the same singularities as \( (1/xB_0 + C_0) \), which by (A.10) are only at \( x = \pm u_0 \), at \( x = \infty \) and at the zeros of \( y(x) \)).

It is then possible to extend this result to higher terms in the large \( N \) expansion. The power \((1/N^k)\) gives
\[
\partial_x \psi_{k-1} = -\frac{\partial_x g}{2gy} \psi_{k-2} - \frac{\partial_x g}{gy} \partial_x \psi_{k-2} - \frac{1}{y} \partial_x \psi_{k-2}
+ \frac{1}{2} \sum_{i=0}^{k-2} \left( \frac{\partial_x (xB + C)}{xB + C} \right)_{k-1-i} \psi_i + \frac{\partial_x g}{2gy} \sum_{i=0}^{k-2} \left( \frac{\partial_x (xB + C)}{xB + C} \right)_{k-2-i} \psi_i
+ \frac{1}{2y} \sum_{i=0}^{k-2} \left( \frac{\partial_x (xB + C)}{xB + C} \right)_{k-2-i} \partial_x \psi_i
- \frac{1}{2y} \sum_{i=0}^{k-1-i} \left( \frac{\partial_x A - \partial_x (xB + C)}{xB + C} \right)_{k-1-i} \psi_i
\]
where we have defined \( \psi_0 = 1 \). The precise form of the relation is mostly irrelevant, but the main fact is that if all the \( \psi_i(x, t) \) with \( i < k \) are assumed to have singularities only at \( x = \pm u_0 \), at \( x = \infty \) and at the other zeros of \( y(x) = 0 \), then the same is true for \( \partial_x \psi_k \) by a simple recursion.

A.3. Pole structure of \( \psi_k(x, t) \)

With the help of (A.4) and (A.17) we are now able to prove that the only singularities of \( x \mapsto \psi_k(x, t) \) are at \( x = \pm u_0 \) and at \( x = \infty \).

From (A.4) we have shown that \( \partial_x \psi_k(x, t) \) can only have singularities at \( x = \pm u_0(t) \), at \( x = \infty \) and when \( u_0(t) \) is at a root of \( P'_0 \). But from (A.17) we have shown that \( \partial_x \psi_k(x, t) \)
can only have singularities at \( x = \pm u_0(t) \), at \( x = \infty \) and at the other zeros of \( y(x) = 0 \) given by \( x = \lambda_i(t) \), the solution of \( \sum_{j=1}^m t_j \sum_{k=0}^{j-1} (x^{2(j-k)-1}(2k)!/2^{2k}(k!)^2)u_0(t)^{2k} = 0 \) in (3.20). But these poles are incompatible with the former result. Indeed if \( \psi_k(x,t) \) had a pole at \( x = \lambda_i(t) \), then \( \partial_t \psi_k(x,t) \) would also have a pole at \( x = \lambda_i(t) \), but we have shown that the only \( x \)-dependent singularities of \( \partial_t \psi_k(x,t) \) are at \( x = \pm u_0(t) \) or \( x = \infty \), giving rise to a contradiction. Therefore, \( x \mapsto \psi_k(x,t) \) has only singularities at \( x = \pm u_0 \) (square-root poles) and at \( x = \infty \) (poles) and in particular has no pole at the other zeros of \( y(x) = 0 \). This result is highly non-trivial because we need to combine the two differential equations (i.e. the whole integrable structure) to get it. Hence, the structure of integrability seems to play an important underlying role in the pole structure and we can hope that such a result could extend to every integrable system.

### A.4. Pole structure in the \( z \) variable

In order to have only poles (and not square-root singularities), we want to shift the former result to the \( z \) variable defined by

\[
z^2 = \frac{u_0 - x}{u_0 + x} \iff x = u_0 \frac{1 - z^2}{1 + z^2}. \tag{A.18}
\]

Note that we have the identities

\[
\begin{align*}
\frac{\partial x}{\partial t} &= (\partial_t u_0) \frac{1 - z^2}{1 + z^2} \quad \frac{\partial x}{\partial z} = -\frac{4u_0 z}{(1 + z^2)^2} \\
u_0 + x &= \frac{2u_0}{1 + z^2} \\
g(z,t) &= \frac{(-u_0)^{1/4}}{z^{1/2}} \\
y(z,t) &= \frac{4z^2 u_0^2}{(1 + z^2)^2} P_0 \left( \frac{1 - z^2}{1 + z^2} \right) u_0 \tag{A.19}
\end{align*}
\]

Note also that every polynomial in \( x \) will give a polynomial in \( (1 - z^2)/1 + z^2 \), that is to say a rational function in \( z \) with poles at \( z^2 + 1 = 0 \).

The rules for differentiation give that

\[
\begin{align*}
\partial_t \psi_k(z,t) &= \partial_t \psi_k(x,t) + \frac{\partial x}{\partial t} \frac{\partial \psi_k(x,t)}{\partial x} \tag{A.20} \\
\partial_z \psi_k(z,t) &= \frac{\partial x}{\partial z} \partial_x \psi_1(x,t) \tag{A.21}
\end{align*}
\]

where all of these terms are already known from the previous sections. If one uses (A.19) and the remark that a polynomial in \( x \) will give a rational function in \( z \) with poles at \( z^2 + 1 = 0 \) (and remembers that functions \( A, B, C \) are polynomials in \( x \)), one can see that the singularities of \( \psi_k(x,t) \) at \( x = \pm u_0 \) (square-root type) and at \( x = \infty \) (poles) will transform into poles at \( z = 0 \) (\( \Leftrightarrow x = -u_0 \)), \( z = \infty \) (\( \Leftrightarrow x = u_0 \)) and \( z = \pm i \) (\( \Leftrightarrow x = \infty \)).

Hence we have the final result: \( \forall k \geq 0 \), the functions \( z \mapsto \psi_k(z,t) \) are rational functions with poles only at \( z \in \{ \pm i, 0, \infty \} \). The coefficients of these fractions depend on \( u_0(t) \) and its derivatives.

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Appendix B. Discussion concerning $\alpha = -\gamma$

When computing the rescaled spectral curve in the matrix model double-scaling limit, we need to find a relationship between the $\alpha$ and $\gamma$ that are given by (2.36):

\[
\begin{align*}
    b_1(T) &= b \epsilon + \alpha \Delta + \sum_{n=1}^{\infty} b_{1,n} \Delta^n \\
    a_2(T) &= b \epsilon + \gamma \Delta + \sum_{n=1}^{\infty} a_{2,n} \Delta^n
\end{align*}
\]

where we recall that $\Delta = (T - T_c)^{1/2m}$. A first argument in favour of the fact that $\alpha = -\gamma$ is given by the case when $\epsilon = 0$. Indeed, in such a case, the situation is fully symmetric around the singular point 0. Therefore, one expects the two endpoints $b_1(T)$ and $a_2(T)$ to be symmetric around $x = 0$ for every value of $T$ around $T_c$. In such a case the identity $\forall T \simeq T_c : a_2(T) = -b_1(T)$ gives $\alpha = -\gamma$. When $\epsilon \neq 0$, we can use a similar reasoning at first order in $\Delta$. Indeed, if we centre the origin at $b \epsilon$, then as we observed several times, the endpoints $a_1$ and $b_2$ can be considered to be respectively $-b$ and $b \epsilon$ up to order $\Delta^6$. Therefore in the function $R^{1/2}(x)$ they only add a multiplicative trivial factor depending on $\epsilon \left( \sqrt{1 - \epsilon^2} \right)$ to be precise which will not change the symmetry around $b \epsilon$ of the endpoints $a_2$ and $b_1$ at first order in $\Delta$.

Finally, another more explicit approach is to put the developments (2.36) into all the equations (2.10), (2.8), (2.9) and (2.32) determining $h(z,T)$, $x_0(T)$ and the endpoints $a_1(T)$, $b_1(T)$, $a_2(T)$ and $b_2(T)$. Doing so leads to an algebraic equation of degree $2m$ connecting $\alpha$ and $\gamma$:

\[
Q(\alpha, \gamma) = 0
\]

with $Q$ a symmetric, homogeneous polynomial of degree $2m$. Unfortunately the system does not just admit a unique solution as soon as $m > 1$. Indeed, although the solution $\alpha = -\gamma$ is always there, when $m > 1$ there are also other possibilities such as $\alpha = \lambda \gamma, \lambda \in \mathbb{C}$ and $\gamma$ satisfying an equation of degree $2m$ with complex coefficients.

Though it might appear surprising that the set of equations may have several distinct solutions (thus giving several eigenvalue densities), one must remember that there are some additional constraints on the solution. Indeed, if one wants to have a density distribution, it means that all quantities involved must at least be real and positive. Therefore only the solution $\alpha = -\gamma$ is possible.

Note. In fact $\alpha$ and $\gamma$ are not necessarily well defined. Indeed, they are only defined up to a multiplicative $(2m)$th root of unity since the equation defining them is homogeneous of degree $2m$. This is because the notion of $\Delta = (T - T_c)^{1/2m}$ is also ambiguous, whereas $\Delta^{2m}$, $\alpha^{2m}$ and $\gamma^{2m}$ are well-defined quantities (which explains why the development in $a_1(T)$ and $b_2(T)$ is well defined). Indeed, if one makes the change

\[
\forall n \in \{1, \ldots, 2m - 1\} : \Delta \rightarrow \tilde{\Delta} = \Delta e^{(2in\pi/2m)}, \quad \alpha \rightarrow \tilde{\alpha} = \alpha e^{-2in\pi/2m} \quad \text{and} \quad \gamma \rightarrow \tilde{\gamma} = \gamma e^{-(2in\pi/2m)}
\]

then (2.36) remains unchanged. With the change $\xi \rightarrow \tilde{\xi} = \xi e^{-(2in\pi/2m)}$, the rescaled spectral curve remains unchanged.

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