LINEAR STABILITY OF THE COUETTE FLOW FOR THE NON-ISENTRPIC COMPRRESSIBLE FLUID

XIAOPING ZHAI

ABSTRACT. We are concerned with the linear stability of the Couette flow for the non-isentropic compressible Navier-Stokes equations with vanished shear viscosity in a domain $\mathbb{T} \times \mathbb{R}$. For a general initial data settled in Sobolev spaces, we obtain a Lyapunov type instability of the density, the temperature, the compressible part of the velocity field, and also obtain an inviscid damping for the incompressible part of the velocity field. Moreover, if the initial density, the initial temperature and the incompressible part of the initial velocity field satisfy some quality relation, we can prove the enhanced dissipation phenomenon for the velocity field.

1. Introduction and the main result

In this paper, we are interested in the long-time asymptotic behaviour of the linearized two dimensional non-isentropic compressible Navier-Stokes equations in a domain $\mathbb{T} \times \mathbb{R}$. The governing equations (in non-dimensional variables) are

\[
\begin{aligned}
\rho_t + (u \cdot \nabla)\rho + \rho \text{div } u &= 0, \\
\rho(u_t + u \cdot \nabla u) + \frac{1}{\gamma M^2} \nabla P &= \frac{1}{\text{Re}} \left( \mu \Delta u + (\nu + \mu) \nabla \text{div } u \right), \\
\rho(\vartheta_t + u \cdot \nabla \vartheta) + (\gamma - 1) P \text{div } u &= \frac{\gamma \mu}{\sigma \text{Re}} \Delta \vartheta + \frac{\gamma (\gamma - 1) M^2}{\text{Re}} \left( \frac{\mu}{2} |\nabla u + \nabla u^\top|^2 + \nu |\text{div } u|^2 \right).
\end{aligned}
\]

(1.1)

Here $t \geq 0$ is time, $(x, y) \in \mathbb{T} \times \mathbb{R}$ is the spatial coordinate and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The unknown $u$ is the velocity vector, $\rho$ is the density, $\vartheta$ is the temperature, $P = \rho \vartheta$ is the pressure. $\gamma > 1$ is the ratio of specific heats, $M > 0$ is the Mach number of the reference state, $\text{Re} > 0$ is the Reynolds number, and $\sigma > 0$ is the Prandtl number. The two constant viscosity coefficients $\mu$ and $\nu$ are the shear viscosity and the volume viscosity respectively. The equations (1.1) then express respectively the conservation of mass, the balance of momentum, and the balance of energy under internal pressure, viscosity forces, and the conduction of thermal energy.

A comprehensive understanding of the stability of compressible or incompressible shear flows is a fundamental problem in fluid mechanics and has been the subject of both theoretical and practical interest in astrophysics and engineering, see [1], [2], [6]–[9], [14], [25], [27], [29], [33], [38], [39] for the compressible fluid and [3]–[5], [10]–[13], [26], [28], [34]–[37] for incompressible fluid. The aim of the present paper is to study the long-time asymptotic behaviour of the linearized non-isentropic compressible Navier-Stokes equations around the Couette flow. That is we seek a stationary solution of (1.1) with a constant mean pressure which have the following form:

\[
\rho_{sh} = \rho_{sh}(y), \quad u_{sh} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \vartheta_{sh} = \vartheta_{sh}(y), \quad \text{with } \rho_{sh}(y)\vartheta_{sh}(y) = 1.
\]

(1.2)

Obviously, when $\mu \neq 0$, due to the strong nonlinear term $|\nabla u + \nabla u^\top|^2$ appeared in the third equation of (1.1), it is straightforward to verify that $\vartheta_{sh}(y)$ must satisfy the following restricted
\[ \frac{\gamma \mu}{\sigma \text{Re}} \partial_{yy} \vartheta_{sh}(y) = -\frac{\gamma \mu (\gamma - 1) M^2}{\text{Re}}. \] (1.3)

Solve (1.3), we can choose \( \vartheta_{sh}(y) \) as

\[ \vartheta_{sh}(y) = \vartheta_r \left[ r + (1 - r)y - (1 - \frac{1}{\vartheta_r})y^2 \right] \] (1.4)

where \( r > 0 \) is the temperature ratio and \( \vartheta_r \) is the recovery temperature defined as follows

\[ \vartheta_r \overset{\text{def}}{=} 1 + \frac{(\gamma - 1) \sigma M^2}{2}. \]

Due to the complicate form of \( \vartheta_{sh}(y) \), to study the long-time asymptotic behaviour of (1.1) around the stationary solution defined in (1.2) and (1.4) is a very difficult problem. To our best knowledge, there are few results in this direction, see [1], [2], [6], [7], [8], [15]–[25], [31], [33].

Due to mathematical challenges, to approach the problem, here, we are only consider a simple case of (1.1) with the shear viscosity coefficient \( \mu = 0 \) and the volume viscosity \( \nu \neq 0 \).

In this case, the system (1.1) can be rewritten as

\[
\begin{aligned}
\varrho_t + u \cdot \nabla \varrho + \varrho \text{div } u &= 0, \\
\varrho (u_t + u \cdot \nabla u) + \frac{1}{\gamma M^2} \nabla (\varrho \vartheta) &= \frac{\nu}{\text{Re}} \nabla \text{div } u, \\
\varrho (\vartheta_t + u \cdot \nabla \vartheta) + (\gamma - 1) \varrho \vartheta \text{div } u &= \frac{\nu \gamma (\gamma - 1) M^2}{\text{Re}} |\text{div } u|^2.
\end{aligned}
\] (1.5)

It’s straightforward to verify that the Couette flow,

\[ \varrho_{sh} = 1, \quad u_{sh} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad \vartheta_{sh} = 1, \] (1.6)

is a stationary solution of (1.5). Our goal is to understand the stability and large-time behavior of perturbations near this Couette flow.

Before presenting our main result, let us first give a short review of the extensive mathematical results on the compressible Navier-Stokes equations. Glatzel [17], [15] studied the linear inviscid and viscous stability properties of the compressible Couette flow via a normal mode analysis in simplified flow model with constant viscous coefficients and a constant density profile. Duck et al. [15] proved the linear stability of the plane Couette flow for the non-isentropic compressible Navier-Stokes equations. Chagelishvili et al. [8] considered the inviscid stability of the 2D Couette flow. By means of some formal approximation, they showed that the energy of acoustic perturbations grows linear in time due to the transfer of energy from the mean flow to perturbations. Taking advantage of a fourth-order finite-difference method and a spectral collocation method, Hu et al. [20] studied the viscous linear stability of supersonic Couette flow for a perfect gas governed by Sutherland viscosity law. Kagei [21] proved that the plane Couette flow in an infinite layer is asymptotically stable if the Reynolds and Mach numbers are sufficiently small. Li et al. [25] investigated the stability analysis of the plane Couette flow for the 3D compressible Navier-Stokes equations with Navier-slip boundary condition at the bottom boundary. They shown that the plane Couette flow is asymptotically stable for small perturbation provided that the slip length, Reynolds and Mach numbers satisfy some restricted relation. Recently, Antonelli et al. [11] studied the linear stability properties of the 2D isentropic compressible Euler equations linearized around a shear flow given by a monotone profile, close to the Couette flow, with constant density, in the domain \( \mathbb{T} \times \mathbb{R} \). Later then,
they in [2] also studied the linear stability properties of perturbations around the homogeneous Couette flow for a 2D isentropic inviscid or viscous compressible fluid. Moreover, in the inviscid case, they proved the inviscid damping for the solenoidal component of the velocity field and Lyapunov type instability for the density and the irrotational component of the velocity field. In the viscous case, they obtained the enhanced dissipation phenomenon. Zeng et al. [38] considered the linear stability of the three dimensional isentropic compressible Navier-Stokes equations on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$. They proved the enhanced dissipation phenomenon and the lift-up phenomenon around the Couette flow $(y, 0, 0)^T$. The motivation of the present paper is to generalize the results obtained by Antonelli et al. [1], [2] to the non-isentropic compressible Navier-Stokes equations with vanished shear viscosity.

Denote $$\rho = \varrho - \varrho_{sh}, \quad v = u - u_{sh}, \quad \theta = \vartheta - \vartheta_{sh}.$$ The linearized system of (1.5) around the Couette flow (1.6) read as follows

$$\begin{cases}
\partial_t \rho + y \partial_x \rho + \text{div} v = 0, \\
\partial_t v + y \partial_x v + \begin{pmatrix} v^y \\ 0 \end{pmatrix} + \frac{1}{\gamma M^2} (\nabla \rho + \nabla \theta) = \nu \nabla \text{div} v, \\
\partial_t \theta + y \partial_x \theta + \gamma \nabla \theta = 0.
\end{cases} \quad (1.7)$$

Before going into details of our theorem, we introduce several notations. Define $$\alpha = \text{div} v, \quad \omega = \nabla \perp \cdot v, \quad \text{with} \quad \nabla \perp = (-\partial_y, \partial_x)^T,$$ according to the Helmholtz projection operators, we have

$$v = (v^x, v^y)^T \overset{\text{def}}{=} P[v] + Q[v] \quad (1.8)$$

with $$P[v] \overset{\text{def}}{=} \nabla \perp \Delta^{-1} \omega, \quad Q[v] \overset{\text{def}}{=} \nabla \Delta^{-1} \alpha. \quad (1.9)$$

From the above definition, one can infer that

$$v^y = \partial_y (\Delta^{-1}) \alpha + \Delta_x (\Delta^{-1}) \omega, \quad (1.10)$$

hence, we can rewrite (1.7) in terms of $(\rho, \alpha, \omega, \theta)$ that

$$\begin{cases}
\partial_t \rho + y \partial_x \rho + \alpha = 0, \\
\partial_t \alpha + y \partial_x \alpha + 2 \partial_x (\partial_y (\Delta^{-1}) \alpha + \partial_x (\Delta^{-1}) \omega) + \frac{1}{\gamma M^2} (\Delta \rho + \Delta \theta) = \nu \Delta \alpha, \\
\partial_t \omega + y \partial_x \omega - \alpha = 0, \\
\partial_t \theta + y \partial_x \theta + (\gamma - 1) \alpha = 0.
\end{cases} \quad (1.11)$$

Obviously, the above system (1.11) is a closed system regarding of $(\rho, \alpha, \omega, \theta)$.

Let

$$\hat{f}(k, \eta) = \frac{1}{2\pi} \int_{\mathbb{T} \times \mathbb{R}} e^{-i(kx + \eta y)} f(x, y) \, dx \, dy,$$

and

$$f(x, y) = \frac{1}{2\pi} \sum_k \int_{\mathbb{R}} e^{i(kx + \eta y)} \hat{f}(k, \eta) \, d\eta,$$

then we define $f \in H^s(\mathbb{T} \times \mathbb{R})$ if

$$\|f\|^2_{H^s} = \sum_k \int (k, \eta)^{2s} |\hat{f}|^2(k, \eta) \, d\eta < +\infty.$$
Now, we can state the main result of the present paper.

**Theorem 1.1.** Let $\gamma > 1$, $0 < \nu < 1$ and $0 < M \leq \nu^{-1}$. Assume that $(\rho^{in}, \alpha^{in}, \omega^{in}, \theta^{in}) \in H^2(\mathbb{T} \times \mathbb{R})$ is the initial data of (1.11) with

$$
\int_{\mathbb{T}} \rho_{in} \, dx = \int_{\mathbb{T}} \alpha_{in} \, dx = \int_{\mathbb{T}} \omega_{in} \, dx = \int_{\mathbb{T}} \theta_{in} \, dx = 0. \tag{1.12}
$$

Then, there exists a positive constant $C$ independent of $\gamma, \nu, M$ such that

$$
\|P[v]^x(t)\|_{L^2} \leq C(t)^{-\frac{3}{2}} \gamma^{-1} \exp(CM(M + 1)) \times \left( \frac{1}{M} \|\rho^{in} + \theta^{in}\|_{H^4} + \|\alpha^{in}\|_{H^4} + \gamma \|\omega^{in}\|_{H^4} \right),
$$

$$
|\|P[v]^y(t)\|_{L^2} \leq C(t)^{-\frac{3}{2}} \gamma^{-1} \exp(CM(M + 1)) \times \left( \frac{1}{M} \|\rho^{in} + \theta^{in}\|_{H^4} + \|\alpha^{in}\|_{H^4} + \gamma \|\omega^{in}\|_{H^4} \right),
$$

and

$$
\|Q[v](t)\|_{L^2} + \frac{\gamma}{M} \|\rho(t)\|_{L^2} + \frac{\gamma}{M} \|\theta(t)\|_{L^2} \leq C(t)^{\frac{1}{2}} \left\{ \left( \frac{\gamma - 1}{1} \right) \|\rho^{in} - \theta^{in}\|_{L^2} \right\} (\gamma + 1) \exp(CM(M + 1)) \times \left( \frac{1}{M} \|\rho^{in} + \theta^{in}\|_{H^1} + \|\alpha^{in}\|_{H^1} + \gamma \|\omega^{in}\|_{H^1} \right).}
$$

Moreover, if $\rho^{in}, \theta^{in}, \omega^{in}$ additionally satisfy the following relation

$$
\rho^{in} + \gamma \omega^{in} + \theta^{in} = 0, \tag{1.13}
$$

we can obtain the enhanced dissipation for the velocity field

$$
\|P[v]^x(t)\|_{L^2} \leq C(t)^{-\frac{3}{2}} e^{-\frac{3}{16} \nu^{-1}} \exp(CM(M + 1)) \left( \|\alpha^{in}\|_{H^4} + \frac{1}{M} \|\omega^{in}\|_{H^4} \right),
$$

$$
\|P[v]^y(t)\|_{L^2} \leq C(t)^{-\frac{3}{2}} e^{-\frac{3}{16} \nu^{-1}} \exp(CM(M + 1)) \left( \|\alpha^{in}\|_{H^4} + \frac{1}{M} \|\omega^{in}\|_{H^4} \right),
$$

$$
\|Q[v](t)\|_{L^2} + \frac{1}{M} \|\rho(t) + \theta(t)\|_{L^2} \leq C(t)^{\frac{1}{2}} e^{-\frac{3}{16} \nu^{-1}} (1 + \gamma) \exp(CM(M + 1)) \left( \|\alpha^{in}\|_{H^1} + \frac{1}{M} \|\omega^{in}\|_{H^1} \right).}
$$

**Remark 1.2.** At first glance, the enhanced dissipation phenomenon of the velocity field is some surprising because of there is only dissipation for the compressible part of the velocity. This mainly benefits from the relation (1.13) which gives rise to $\omega = -\frac{1}{\gamma}(\rho + \theta)$. The special relation connects compressible and incompressible phenomena. Namely, an increase of the vorticity need to be compensated by a decrease for the density and the temperature.

**Remark 1.3.** In [1], Antonelli et al. studied the linear stability properties of the 2D isentropic compressible Euler equations linearized around a shear flow given by a monotone profile, close to the Couette flow, with constant density, in the domain $\mathbb{T} \times \mathbb{R}$. For the non-isentropic compressible fluid, how to obtain a similar result is an interesting problem. This is left in the future work.
2. The proof of the main theorem

2.1. Preliminary and the a priori estimates. First of all, we are concerned with the dynamics of the $x$-averages of the perturbations. In order to reveal the distinction between the zero mode case $k = 0$ and the nonzero modes $k \neq 0$. We define

$$f_0(y) \overset{\text{def}}{=} \frac{1}{2\pi} \int_T f(x, y) \, dx, \quad f_\neq(x, y) \overset{\text{def}}{=} f(x, y) - f_0(y),$$

which represents the projection onto 0 frequency and the projection onto non-zero frequencies.

Due to the structure of the Couette flow and the fact that the equations are linear, it is clear that the zero mode in $x$ has an independent dynamics with respect to other modes. Consequently, in our analysis we can decouple the evolution of the $k = 0$ mode from the rest of the perturbation. Integration in $x$ equations in (1.11), one infer that

$$\begin{cases}
\partial_t \rho_0 = -\alpha_0, \\
\partial_t \alpha_0 = -\frac{1}{\gamma M^2} \partial_{yy} \rho_0 - \frac{1}{\gamma M^2} \partial_{yy} \theta_0 + \nu \partial_{yy} \alpha_0, \\
\partial_t \omega_0 = \alpha_0, \\
\partial_t \theta_0 = - (\gamma - 1) \alpha_0.
\end{cases} \tag{2.1}$$

From the above equation (2.1), we can further get $\alpha_0, \rho_0 + \theta_0$ satisfy the following damped wave equations:

$$\begin{align*}
\partial_t \alpha_0 - \nu \partial_t \partial_{yy} \alpha_0 - \frac{1}{M^2} \partial_{yy} \alpha_0 &= 0, \quad \text{in } \mathbb{R}, \\
\partial_t (\rho_0 + \theta_0) - \frac{1}{M^2} \partial_{yy} (\rho_0 + \theta_0) &= 0, \quad \text{in } \mathbb{R}.
\end{align*} \tag{2.2} \tag{2.3}$$

Hence, given $\rho_0^{in} = \alpha_0^{in} = \theta_0^{in} = \omega_0^{in} = 0$, we can get for all $t \geq 0$

$$\rho_0(t) = \alpha_0(t) = \theta_0(t) = \omega_0(t) = 0.$$

Consequently, in our analysis we can decouple the evolution of the $k = 0$ mode from the rest of the perturbation. Let us consider the following coordinate transform

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x - yt \\ y \end{pmatrix}.$$

Under the new coordinate transform, the differential operators change as follows

$$\partial_x = \partial_X, \quad \partial_y = \partial_Y - t \partial_X, \quad \Delta = \Delta_L \overset{\text{def}}{=} \partial_{XX} + (\partial_Y - t \partial_X)^2.$$

Define

$$\begin{align*}
R(t, X, Y) &= \rho(t, X + tY, Y), \quad A(t, X, Y) = \alpha(t, X + tY, Y), \\
\Omega(t, X, Y) &= \omega(t, X + tY, Y), \quad \Theta(t, X, Y) = \theta(t, X + tY, Y).
\end{align*}$$

Then, the linear system (1.11) reduces to the following system in the new coordinates

$$\begin{cases}
\partial_t R = -A, \\
\partial_t A = \nu \Delta_L A - 2 \partial_X (\partial_Y - t \partial_X) (\Delta_L^{-1}) A - 2 \partial_X (\Delta_L^{-1}) \Omega - \frac{1}{\gamma M^2} (\Delta_L R + \Delta_L \Theta), \\
\partial_t \Omega = A, \\
\partial_t \Theta = - (\gamma - 1) A.
\end{cases} \tag{2.4}$$
We want to analyze the system (2.4) on the frequency space, in analogy with respect to the incompressible Couette flow. So we define the symbol associated to $-\Delta_L$ as
\[ p(t, k, \eta) = k^2 + (\eta - kt)^2, \]
and denote the symbol associated to the operator $2\partial_X(\partial_Y - t\partial_X)$ as
\[ (\partial_t p)(t, k, \eta) = -2k(\eta - kt). \]

In the moving frame, for the Laplacian operator, there holds the following inequalities.

**Lemma 2.1.** Let $p = -\hat{\Delta}_L = k^2 + (\eta - kt)^2$, then for any function $f \in H^{s+2\beta}(\mathbb{T} \times \mathbb{R})$, it holds that
\[
\|p^{-\beta}f\|_{H^{s}} \leq C\frac{1}{(t)^{2\beta}} \|f\|_{H^{s+2\beta}}, \quad \|p^{\beta}f\|_{H^{s}} \leq C\langle t \rangle^{2\beta} \|f\|_{H^{s+2\beta}},
\]
for any $\beta > 0$.

**Proof.** The bound (2.5) follows just by Plancherel Theorem and the basic inequalities for japanese brackets $\langle k, \eta \rangle \leq C\langle \eta - \xi \rangle\langle k, \xi \rangle$. \qed

Taking the Fourier transform of (2.4) gives rise to
\[
\begin{aligned}
\partial_t \hat{R} &= -\hat{A}, \\
\partial_t \hat{A} &= -\nu p\hat{A} + \frac{\partial_t p}{p} \hat{A} - \frac{2k^2}{p} \hat{\Omega} + \frac{p}{\gamma M^2} (\hat{R} + \hat{\Theta}), \\
\partial_t \hat{\Omega} &= \hat{A}, \\
\partial_t \hat{\Theta} &= -(\gamma - 1)\hat{A}.
\end{aligned}
\] (2.6)

To exploit the special structure of the system (2.6), we introduce the good unknowns $\Phi$ as
\[ \Phi = \frac{R + \Theta}{\gamma} \] (2.7)
from which we can rewrite (2.6) into
\[
\begin{aligned}
\partial_t \hat{\Phi} &= -\hat{A}, \\
\partial_t \hat{A} &= -\nu p\hat{A} + \frac{\partial_t p}{p} \hat{A} - \frac{2k^2}{p} \hat{\Omega} + \frac{p}{M^2} \hat{\Phi}.
\end{aligned}
\] (2.8)

In order to break through the barrier involve in the term $\Omega$ in (2.8), we deduce from
\[ \partial_t (R + \gamma\Omega + \Theta) = 0 \]
that there holds
\[ R + \gamma\Omega + \Theta = R^{in} + \gamma\Omega^{in} + \Theta^{in}. \]
Hence, combining with (2.7) leads to
\[ \Omega = \Phi^{in} + \Omega^{in} - \Phi. \] (2.9)

Substituting (2.9) into (2.8), we get a closed system only involved in $\hat{\Phi}, \hat{A}$ other than the initial data
\[
\begin{aligned}
\partial_t \hat{\Phi} &= -\hat{A}, \\
\partial_t \hat{A} &= -\nu p\hat{A} + \frac{\partial_t p}{p} \hat{A} + \left( \frac{p}{M^2} + \frac{2k^2}{p} \right) \hat{\Phi} - \frac{2k^2}{p} (\hat{\Phi}^{in} + \hat{\Omega}^{in}).
\end{aligned}
\] (2.10)
In the following, to obtain the enhanced dissipation, we introduce the “ghost multiplier” which has been used in [32, 33].

Let multiplier $m$ solve the linear ODE for $k \neq 0$:

$$\frac{\partial m}{m} = -\frac{\nu^{1/3}}{\left[\nu^{1/3}t - \frac{t}{k}\right]^2 + 1}$$

$$m(0, k, \eta) = 1.$$ 

Notice that there is a constant $c$ (independent of $k$, $\eta$, $t$, and $\nu$) such that $c < m(t, k, \eta) \leq 1$. In particular, its presence does not change a norm:

$$\|m(t, \nabla)\langle\nabla\rangle^\sigma f\|_{L^2} \approx \|\langle\nabla\rangle^\sigma f\|_{L^2}. \quad (2.11)$$

The crucial property that $m$ satisfies is:

$$1 \lesssim \nu^{-1/6} \left(\sqrt{-\frac{\partial m}{m}(t, k, \eta) + \nu^{1/2}|k, \eta - kt|}\right) \quad \text{for } k \neq 0, \quad (2.12)$$

which implies that

$$\|f_\#\|^2_{L^2} \lesssim \nu^{-1/3} \left(\left\|\sqrt{-\frac{\partial m}{m}f_\#}\right\|^2_{L^2} + \nu\|\nabla_L f_\#\|^2_{L^2}\right)^{1/2}. \quad (2.13)$$

The following lemma plays a crucial role in our subsequent analysis.

**Lemma 2.2.** For any $(\rho^i, \alpha^i, \omega^i, \theta^i) \in H^s(\mathbb{T} \times \mathbb{R})$ with $s \geq 0$. Assume that $\gamma > 1$, $0 < \nu < 1$, and $0 < M \leq \nu^{-1}$. Then there exists a positive constant $C$ independent of $\gamma, \nu, M$ such that

$$\frac{1}{M} \left\|\rho^{-\frac{3}{4}}\hat{\Phi}(t)\right\|_{H^s} + \left\|\rho^{-\frac{3}{4}}\hat{A}(t)\right\|_{H^s} \leq C \exp(CM(M + 1)) \left(\frac{1}{M} \left\|\hat{\Phi}^i\right\|_{H^s} + \left\|\hat{A}^i\right\|_{H^s} + \left\|\hat{\Phi}^i + \hat{\Omega}^i\right\|_{H^s}\right). \quad (2.14)$$

**Proof.** For any $s \geq 0$, we define two weighted functions involved in $\hat{\Phi}$ and $\hat{A}$ as

$$Z_1(t) \overset{\text{def}}{=} \frac{1}{M} \langle k, \eta \rangle^s (m^{-1}p^{-\frac{3}{4}}\hat{\Phi})(t), \quad (2.15)$$

$$Z_2(t) \overset{\text{def}}{=} \langle k, \eta \rangle^s (m^{-1}p^{-\frac{3}{4}}\hat{A})(t). \quad (2.16)$$

To begin with, from equations in (2.10) and definitions of $Z_1$ and $Z_2$, a simple computations gives

$$\begin{cases}
\partial_t Z_1 = -\frac{\partial m}{m} Z_1 - \frac{1}{4} \frac{\partial p}{p} Z_1 - \frac{1}{M} p^{\frac{3}{2}} Z_2, \\
\partial_t Z_2 = -\left(\frac{\partial m}{m} + \nu p\right) Z_2 + \frac{1}{4} \frac{\partial p}{p} Z_2 \\
\quad + \left(\frac{1}{M} p^{\frac{3}{2}} + 2M \frac{k^2}{p^2}\right) Z_1 - \langle k, \eta \rangle^s \frac{2m^{-1}k^2}{p^2} (\hat{\Phi}^i + \hat{\Omega}^i).
\end{cases} \quad (2.17)$$

Now, we get by multiplying the first equation by $\bar{Z}_1$ and the second equation by $\bar{Z}_2$ in (2.17) respectively, that

$$\frac{1}{2} \frac{d}{dt} |Z_1|^2 = -\frac{\partial m}{m} |Z_1|^2 - \frac{1}{4} \frac{\partial p}{p} |Z_1|^2 - \frac{1}{M} p^{\frac{3}{2}} \text{Re}(Z_1Z_2), \quad (2.18)$$
\[
\frac{1}{2} \frac{d}{dt} |Z_2|^2 = - \left( \frac{\partial_t m}{m} + \nu \frac{p}{p} \right) |Z_2|^2 + \frac{1}{4} \frac{\partial_p}{p} |Z_2|^2 + \frac{1}{M} \frac{\partial^2}{p^2} \text{Re}(Z_1 Z_2)
\]
\[
+ 2M \frac{k^2}{p^2} \text{Re}(Z_1 Z_2) - \langle k, \eta \rangle^2 \frac{2m-1}{p^2} k^2 \text{Re}(\hat{\Phi}^m + \hat{\Omega}^m Z_2).
\]  

From \( p(t, k, \eta) = k^2 + (\eta - kt)^2 > 0 \), one has for any \( t > \eta/k \) there holds \( \partial_p/p > 0 \), the third term on the right-hand side of the first equation in (2.17) acts as a damping term for \( Z_1 \). Instead, \( \partial_p/p < 0 \) for \( t < \eta/k \), hence it induces a growth on \( Z_1 \). However, the situation is opposite for the second equation involved in \( Z_2 \). That is to say, for \( t > \eta/k \), the term \( (\partial_p/p) Z_2 \) induces a growth, for \( t < \eta/k \), the term \( (\partial_p/p) Z_2 \) acts as a damping term. Thus, there is a competition between \( Z_1 \) and \( Z_2 \). To balance this relation, we have to consider the time derivative of the mixed terms involved in \( Z_1, Z_2 \):

\[
\frac{d}{dt} \left( \frac{\partial_p}{p^2} Z_1 \right) = - \frac{\partial_t m}{m} \frac{\partial_p}{p^2} Z_1 + \frac{2k^2}{p^2} \left( \frac{\partial_p}{p^2} - \frac{7}{4} \frac{(\partial_p)^2}{p^4} \right) Z_1 - \frac{1}{M} \frac{\partial_p}{p} Z_2
\]  

(2.20)

from which and the second equation in (2.17), we can further get

\[
\frac{M}{4} \frac{d}{dt} \left( \frac{\partial_p}{p^2} \text{Re}(\bar{Z} Z_2) \right) = - \frac{1}{4} \frac{\partial_p}{p} (|Z_2|^2 - |Z_1|^2) + \frac{M}{4} \left( \frac{2k^2}{p^2} - \frac{3}{2} \frac{(\partial_p)^2}{p^4} \right) \text{Re}(\bar{Z} Z_2)
\]
\[
- \frac{M \partial_t m}{m} \frac{\partial_p}{p^{3/2}} \text{Re}(\bar{Z} Z_2) - \frac{M}{4} \frac{\partial_p}{p^2} \text{Re}(\bar{Z} Z_2) + M \frac{k^2}{2p^3} |Z_1|^2
\]
\[
- \langle k, \eta \rangle^2 \frac{m-1}{p^2} \text{Re}(\hat{\Phi}^m + \hat{\Omega}^m Z_2).
\]  

(2.21)

It’s obvious that the first term on the right hand side of (2.21) could cancel two bad terms \(- \frac{1}{4} \frac{\partial_p}{p} |Z_1|^2 \) appeared in (2.18) and \( \frac{1}{4} \frac{\partial_p}{p} |Z_1|^2 \) appeared in (2.19).

Due to lack of a diffusive term in the equation of \( \hat{\Phi} \), we have to exploit the special structural characteristics (wave structure) of (2.17) to find hidden dissipation for \( Z_1 \). So, we also need to consider the time derivative of the mixed terms involved in \( Z_1, Z_2 \) with different weight as

\[
\frac{d}{dt} \left( p^{-\frac{1}{2}} Z_1 \right) = - \frac{\partial_t m}{m} p^{-\frac{1}{2}} Z_1 - \frac{3}{4} \frac{\partial_p}{p^2} Z_1 - \frac{1}{M} Z_2
\]  

(2.22)

which combines with the second equation in (2.17) give rise to

\[
- \frac{d}{dt} \left( p^{-\frac{1}{2}} \text{Re}(\bar{Z} Z_2) \right) = - \frac{1}{M} \left( 1 + 2M \frac{k^2}{p^2} \right) |Z_1|^2 + \frac{1}{2} \frac{\partial_p}{p^{3/2}} \text{Re}(\bar{Z} Z_2)
\]
\[
+ 2 \frac{\partial_t m}{m p^{3/2}} \text{Re}(\bar{Z} Z_2) + \frac{1}{M} |Z_2|^2 + \nu \frac{1}{p^{3/2}} \text{Re}(\bar{Z} Z_2)
\]
\[
+ \langle k, \eta \rangle^2 \frac{2m-1}{p^2} \text{Re}(\hat{\Phi}^m + \hat{\Omega}^m Z_1).
\]  

(2.23)

Finally, in order to define a coercive energy functional, we need to consider the time derivative of the term \( p^{-\frac{1}{2}} \partial_p Z_1 \) or \( p^{-\frac{1}{2}} \partial_p Z_2 \). Here, we choose the former.

\[
\frac{M^2}{2} \frac{d}{dt} \left| \frac{\partial_p}{p^2} Z_1 \right|^2 = M^2 \left( \frac{2k^2}{p^2} \frac{\partial_p}{p^3} - \frac{7}{4} \frac{(\partial_p)^3}{p^4} \right) |Z_1|^2
\]
\[
- M^2 \frac{\partial_t m}{m} \frac{(\partial_p)^2}{p^3} |Z_1|^2 - M \frac{(\partial_p)^2}{p} \text{Re}(\bar{Z} Z_2).
\]
Now, we define the following energy functional

\[
\mathcal{E}(t) = \frac{1}{2} \left( 1 + M^2 \frac{\partial p}{p^3} \right) |Z_1|^2(t) + \frac{1}{2} |Z_2|^2(t) \\
+ \left( \frac{M}{4} \frac{\partial p}{p^2} \text{Re}(\bar{Z}_1 Z_2) \right)(t) - \left( \frac{M \nu^\frac{1}{2}}{4} \frac{p^{-\frac{1}{2}}}{p^2} \text{Re}(\bar{Z}_1 Z_2) \right)(t).
\]  

(2.24)

Multiplying by \( \frac{M \nu^\frac{1}{2}}{4} \) on both hand side of (2.23) then summing up (2.18), (2.19), (2.21), and (2.24) gives

\[
\frac{d}{dt} \mathcal{E}(t) + \left( \frac{\partial m}{m} + \frac{\nu^\frac{1}{2}}{4} \left( 1 + 2M^2 \frac{k^2}{p^2} + M^2 \frac{\partial m}{m} \frac{(\partial p)^2}{p^3} \right) \right) |Z_1|^2 + \left( \frac{\partial m}{m} + \nu p \right) |Z_2|^2 \\
= \frac{\nu^\frac{1}{2}}{4} |Z_2|^2 + \frac{M \nu^\frac{1}{2}}{4} p^{\frac{1}{2}} \text{Re}(\bar{Z}_1 Z_2) - \frac{M \partial p}{4} \frac{p^{\frac{1}{2}}}{p^2} \text{Re}(\bar{Z}_1 Z_2) + M^2 \left( \frac{5k^2 \partial p}{2p^3} - \frac{7}{4} \frac{(\partial p)^3}{p^4} \right) |Z_1|^2 \\
+ M \left( \frac{\nu^\frac{1}{2}}{8} \frac{\partial p}{p^2} + \frac{5k^2}{2 p^3} - \frac{11}{8} \frac{(\partial p)^2}{p^5} + \frac{\nu^\frac{1}{2}}{2} \frac{\partial m}{m p^\frac{1}{2}} \right) \text{Re}(\bar{Z}_1 Z_2) \\
+ \langle k, \eta \rangle^s \frac{M \nu^\frac{1}{2} m^{-1} k^2}{2p^\frac{1}{2}} \text{Re}(\tilde{\Phi}_m^{\text{in}} + \tilde{\Omega}_m^{\text{in}}) - \langle k, \eta \rangle^s \frac{M m^{-1} k^2 \partial p}{2p^\frac{1}{2}} \text{Re}(\tilde{\Phi}_m^{\text{in}} + \tilde{\Omega}_m^{\text{in}}) Z_2) \\
\overset{\text{def}}{=} \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + \mathcal{D}_4 + \mathcal{D}_5 + \mathcal{D}_6 + \mathcal{D}_7.
\]  

(2.25)

Now we go to bound the right hand side of (2.25). First, from (2.12), there holds

\[
\frac{\partial m}{m} + \nu p \geq \nu^\frac{1}{2}.
\]  

(2.26)

Hence, the first term \( \mathcal{D}_1 \) can be absorbed directly by the left. We next consider \( \mathcal{D}_2 \). With the aid of the Cauchy-Schwarz inequality, one has

\[
|\mathcal{D}_2| \leq \frac{M \nu^\frac{1}{2}}{8} (\nu |Z_1|^2 + \nu p |Z_2|^2) \\
\leq \frac{M \nu^\frac{1}{2}}{8} (\nu |Z_1|^2) + \frac{M \nu^\frac{1}{2}}{8} (\nu p |Z_2|^2).
\]  

(2.27)

As a result, in order to absorb \( \mathcal{D}_2 \) by the left, we need the assumption \( M \nu \leq 1 \). Since \( |\partial p| \leq 2|k|p^\frac{1}{2} \), we can bound the terms \( \mathcal{D}_3 \) as follows

\[
|\mathcal{D}_3| \leq \frac{\nu}{4} \left( \frac{2M|k|p^\frac{1}{2}}{p^3} \text{Re}(\bar{Z}_1 Z_2) \right) \\
\leq \frac{\nu}{4} \left( \frac{4M^2k^2}{p} |Z_1|^2 + \frac{1}{4} (p|Z_2|^2) \right) \\
\leq \frac{\nu M^2 k^2}{p} |Z_1|^2 + \frac{1}{16} \nu p |Z_2|^2.
\]  

(2.28)

In the same manner, from \( |\partial p| \leq 2|k|p^\frac{1}{2} \) and the fact that \( |k|p^{-\frac{3}{4}} \leq 1 \), we have

\[
|\mathcal{D}_4| \leq 19M^2 \frac{|k|^3}{p^\frac{3}{4}} |Z_1|^2 \leq CM^2 \frac{k^2}{p} |Z_1|^2.
\]  

(2.29)
In the following, we bound the terms in $\mathcal{D}_5$. Thanks to $|\partial_t p| \leq 2|k|p^{\frac{1}{2}}$ again, we have

$$|\mathcal{D}_5| \leq \frac{M\nu^\frac{1}{2}}{4} |k| \text{Re}(\bar{Z}_1 Z_2) + \frac{M |k|^2}{4p^{\frac{3}{2}}} \text{Re}(\bar{Z}_1 Z_2) + \frac{M \nu^\frac{1}{2}}{2mp^{\frac{3}{2}}} \text{Re}(\bar{Z}_1 Z_2)$$

$$\text{def} = \mathcal{D}_{5,1} + \mathcal{D}_{5,2} + \mathcal{D}_{5,3}. \quad (2.30)$$

Due to $\nu \leq 1$ and $p^{-\frac{3}{2}} \leq 1$, we can bound $\mathcal{D}_{5,1}$ as

$$|\mathcal{D}_{5,1}| \leq CM \frac{k^2}{p} (|Z_1|^2 + |Z_2|^2). \quad (2.31)$$

The term $\mathcal{D}_{5,2}$ can be controlled similarly if noticing the fact that $|k|p^{-1} \leq 1$.

For the last term $\mathcal{D}_{5,3}$, we can use $M\nu^\frac{1}{2} \leq 1$ and $p^{-\frac{3}{2}} \leq 1$ to get

$$|\mathcal{D}_{5,3}| \leq C \frac{\partial_t m}{m} (|Z_1|^2 + |Z_2|^2). \quad (2.32)$$

Substituting the above estimates involved in $\mathcal{D}_{5,1}, \mathcal{D}_{5,2}, \mathcal{D}_{5,3}$ into $(2.30)$, we can get

$$|\mathcal{D}_5| \leq CM \frac{k^2}{p} (|Z_1|^2 + |Z_2|^2) + C \frac{\partial_t m}{m} (|Z_1|^2 + |Z_2|^2). \quad (2.33)$$

From $\nu \leq 1$, $p^{-1} \leq 1$, $|\partial_t p| \leq 2|k|p^{\frac{1}{2}}$ and the multiplier $m^{-1}$ is a bound Fourier multiplier, we can get

$$\frac{Mm^{-1}k^2 \partial_t p}{2p^{\frac{3}{2}}} + \frac{M \nu^\frac{1}{2} m^{-1}k^2}{2p^{\frac{3}{2}}} \leq CM \frac{|k|^2}{p},$$

from which and the Young inequality give rise to

$$|\mathcal{D}_5| + |\mathcal{D}_7| \leq CM \frac{k^2}{p} \left( \langle k, \eta \rangle^{2s} |\hat{\Phi}^{in} + \hat{\Omega}^{in}|^2 + (|Z_1|^2 + |Z_2|^2) \right). \quad (2.34)$$

Noticing the fact that

$$M^2 \frac{\partial_t m}{m} \frac{(\partial_t p)^2}{p^3} > 0,$$

and then inserting $(2.27), (2.28), (2.29), (2.30), (2.33), (2.34)$ into $(2.25)$, we can get

$$\frac{d}{dt} \mathcal{E}(t) + \frac{\nu^\frac{1}{2}}{16} \left( 1 + 4M^2 \frac{k^2}{p^2} \right) |Z_1|^2 + |Z_2|^2 \leq CM \frac{k^2}{p} \langle k, \eta \rangle^{2s} |\hat{\Phi}^{in} + \hat{\Omega}^{in}|^2 + C \left( M(M+1) \frac{k^2}{p} + \frac{\partial_t m}{m} \right) \mathcal{E}(t). \quad (2.35)$$

As

$$4M^2 \frac{k^2}{p^2} \geq M^2 \frac{(\partial_t p)^2}{p^3},$$

we can further get

$$\frac{d}{dt} \mathcal{E}(t) + \frac{\nu^\frac{1}{2}}{16} \mathcal{E}(t) \leq CM \frac{k^2}{p} \langle k, \eta \rangle^{2s} |\hat{\Phi}^{in} + \hat{\Omega}^{in}|^2 + C \left( M(M+1) \frac{k^2}{p} + 2\frac{\partial_t m}{m} \right) \mathcal{E}(t). \quad (2.36)$$

It’s easy to check that there holds

$$\int_0^t \frac{k^2}{p(\tau)} d\tau = \int_0^t \frac{d\tau}{(\frac{\eta}{k} - \tau)^2 + 1} = \left( \arctan(\frac{\eta}{k} - t) - \arctan(\frac{\eta}{k}) \right).$$
As a result, applying Gronwall’s inequality to (2.36) we have
\[ \mathcal{E}(t) \leq C \left( \mathcal{E}(0) + \langle k, \eta \rangle^{2s} |\Phi^m + \tilde{\Omega}^m|^2 \right) \exp(CM(M + 1)). \] (2.37)

From $|\partial_t p| < p$, it’s not hard to check that
\[ \mathcal{E}(t) \approx \frac{1}{4} \left( (1 + M^2 \frac{(\partial_t p)^2}{p^3})|Z_1|^2 + |Z_2|^2 \right)(t) \]
which combines the fact that $m$ is a bounded Fourier multiplier and the definitions of $Z_1, Z_2$, we can further get
\[ \sum_k \int \mathcal{E}(t) d\eta \approx \frac{1}{M^2} \left| p^{-\frac{1}{2}} \Phi(t) \right|^2_{H^s} + \left| p^{-\frac{1}{2}} \tilde{A}(t) \right|^2_{H^s} \] (2.38)
which implies that
\[ \frac{1}{M} \left| \left( p^{-\frac{1}{2}} \Phi(t) \right) \right|_{H^s} + \left| \left( p^{-\frac{1}{2}} \tilde{A}(t) \right) \right|_{H^s} \leq C \exp(CM(M + 1) \left( \frac{1}{M} \left| \Phi^m \right|_{H^s} + \left| \tilde{A}^m \right|_{H^s} + \left| \Phi^m + \tilde{\Omega}^m \right|_{H^s} \right)). \] (2.39)
This proves the lemma. □

2.2. The proof of Theorem 1.1 for general $\rho^m, \theta^m, \omega^m$. Thanks to the previous Lemma, we are now to conclude the proof of Theorem 1.1. First, from (2.9) and Lemma 2.2 we have
\[ \| \Omega(t) \|_{H^s} = \| -\Phi(t) \|_{H^s} \]
\[ = M \left| p^{-\frac{1}{2}} (M^{-1} p^{-\frac{1}{2}} \Phi(t)) \right|_{H^s} \]
\[ \leq CM \langle t \rangle^{\frac{1}{2}} \left| M^{-1} p^{-\frac{1}{2}} \Phi(t) \right|_{H^{s+\frac{1}{2}}} \]
\[ \leq C \gamma^{-1} \exp(CM(M + 1) \langle t \rangle^{\frac{1}{2}} C_{in,s+\frac{1}{2}} \] (2.40)
with
\[ C_{in,s+\frac{1}{2}} \overset{\text{def}}{=} \left( \frac{1}{M} \left| \rho^m + \theta^m \right| \right|_{H^{s+\frac{1}{2}}} + \left| \alpha^m \right| \right|_{H^{s+\frac{1}{2}}} + \gamma \left| \omega^m \right| \right|_{H^{s+\frac{1}{2}}} \)
Recall the definition of $\mathbb{P}[v]$ in (1.9), there holds
\[ \| \mathbb{P}[v]^\alpha(t) \|_{L^2} = \| (\partial_y \Delta^{-1} \omega(t) \|_{L^2} \]
\[ = \| (\partial_y t \partial_x) (\Delta^{-1} \Omega(t)) \|_{L^2} \]
\[ \leq C \left| \left| (-\Delta) \Omega(t) \right| \right|_{L^2} \]
Therefore, we get from
\[ p^{-\frac{1}{2}} \langle k t \rangle \geq C \langle \eta - k t \rangle \langle k t \rangle \geq C \langle \eta \rangle \]
and the estimate (2.40) that
\[ \| \mathbb{P}[v]^\alpha(t) \|_{L^2} \leq C \frac{1}{\langle t \rangle} \| \Omega(t) \|_{H^s} \]
\[ \leq C(t)^{-\frac{1}{2}} \gamma^{-1} \exp(CM(M + 1) C_{in,s^{\frac{1}{2}}} \] (2.41)
In the same manner, we can deal with the second component of $\mathbb{P}[v]^y$
\[
\|\mathbb{P}[v]^y(t)\|_{L^2} = \|\partial_x \Delta^{-\frac{1}{2}} \omega\|_{L^2} = \|\partial_x (\Delta_L^{-1} \Omega(t))\|_{L^2}
\]
\[
= \left\| \frac{k}{p} \Omega(t) \right\|_{L^2} \leq C\langle t \rangle^{-\frac{3}{2}} \|\Omega(t)\|_{H^1}
\]
\[
\leq C\langle t \rangle^{-\frac{3}{2}} \gamma^{-1} \exp(CM(M + 1)C_{in,\frac{1}{2}}). \tag{2.42}
\]

Finally, we estimate the compressible part of the velocity. On the one hand, from the Helmholtz decomposition and the change of coordinates, we get
\[
\|Q[v](t)\|_{L^2} + \frac{1}{M} \|\rho(t) + \theta(t)\|_{L^2}
\]
\[
= \left\| (-\Delta)^{-\frac{1}{2}} \alpha(t) \right\|_{L^2} + \frac{1}{M} \|\rho(t) + \theta(t)\|_{L^2}
\]
\[
= \left\| (-\Delta_L)^{-\frac{1}{2}} A(t) \right\|_{L^2} + \frac{1}{M} \|R(t) + \Theta(t)\|_{L^2}
\]
\[
= \left\| (-\Delta_L)^{-\frac{1}{2}} A(t) \right\|_{L^2} + \frac{\gamma}{M} \|\Phi(t)\|_{L^2}. \tag{2.43}
\]

As a result, we can further deduce from (2.43), Lemma 2.2 and the fact that $p \leq \langle t \rangle^2 \langle k, \eta \rangle^2$ that
\[
\|Q[v](t)\|_{L^2} + \frac{1}{M} \|\rho(t) + \theta(t)\|_{L^2}
\]
\[
= \left\| \gamma \left(p^{-\frac{1}{2}} A(t)\right) \right\|_{L^2} + \frac{\gamma}{M} \left\|\gamma \left(p^{-\frac{1}{2}} \Phi\right)(t)\right\|_{L^2}
\]
\[
\leq C\langle t \rangle^\frac{1}{2} \left( \left\| \gamma \left(p^{-\frac{1}{2}} A(t)\right) \right\|_{H^1} + \frac{\gamma}{M} \left\|\gamma \left(p^{-\frac{1}{2}} \Phi\right)(t)\right\|_{H^1} \right)
\]
\[
\leq C\langle t \rangle^\frac{1}{2} (1 + \gamma) \gamma^{-1} \exp(CM(M + 1)C_{in,1}) . \tag{2.44}
\]

On the other hand, by (1.11), there holds
\[
(\partial_t + y \partial_x)((\gamma - 1)\rho - \theta) = 0
\]
which implies that
\[
(\gamma - 1)\rho - \theta = (\gamma - 1)\rho^{in} - \theta^{in}.
\]

Moreover, we have
\[
\left\| \frac{(\gamma - 1)\rho(t) - \theta(t)}{M} \right\|_{L^2}^2 = \left\| \frac{(\gamma - 1)\rho^{in} - \theta^{in}}{M} \right\|_{L^2}^2.
\]

A simple computation gives
\[
\frac{\gamma}{M} \rho = \frac{(\gamma - 1)\rho - \theta}{M} + \frac{\rho + \theta}{M},
\]
\[
\frac{\gamma}{M} \theta = -\frac{(\gamma - 1)\rho - \theta}{M} + (\gamma - 1)\frac{\rho + \theta}{M}.
\]

Hence
\[
\frac{\gamma}{M} \|\rho(t)\|_{L^2} = \left( \left\| \frac{(\gamma - 1)\rho - \theta}{M} \right\|_{L^2} + \left\| \frac{\rho + \theta}{M} \right\|_{L^2} \right)
\]
\[
\leq C\langle t \rangle^\frac{1}{2} \left\{ \left\| \frac{(\gamma - 1)\rho^{in} - \theta^{in}}{M} \right\|_{L^2} + (1 + \gamma) \gamma^{-1} \exp(CM(M + 1)C_{in,1}) \right\},
\]
and
\[ \frac{\gamma}{M} \|\theta(t)\|_{L^2} = \left\| \frac{(\gamma - 1)\rho - \theta}{M} \right\|_{L^2} + (\gamma - 1) \left\| \frac{\rho + \theta}{M} \right\|_{L^2} \leq C(\gamma - 1) \left\| \frac{(\gamma - 1)\rho^{in} - \theta^{in}}{M} \right\|_{L^2} + C(t)^{\frac{1}{2}}(\gamma - \gamma^{-1}) \exp(CM(M + 1)C_{in,1}). \]

This proves the first case for general \( \rho^{in}, \theta^{in}, \omega^{in} \).

2.3. The proof of Theorem 1.1 with special \( \rho^{in}, \theta^{in}, \omega^{in} \) satisfying (1.13). If \( \rho^{in} + \gamma \omega^{in} + \theta^{in} = 0 \), from \( \partial_t(R + \gamma \Omega + \Theta) = 0 \) we can infer that
\[ \Omega = -\frac{R + \Theta}{\gamma} = -\Phi. \]

Thus, we get a closed system only involved in \( \hat{\Phi} \) and \( \hat{A} \)
\[ \begin{cases} 
\partial_t\hat{\Phi} = -\hat{A}, \\
\partial_t\hat{A} = -\nu p\hat{A} + \frac{\partial_p\hat{A}}{p} + \frac{2k^2}{p}\hat{\Phi}.
\end{cases} \tag{2.45} \]

For the above system (2.45), we can make a similar argument as the proof of Lemma 2.2 to get another version of (2.36) which don’t involve in \( \hat{\Phi}^{in}, \hat{\Omega}^{in} \) that
\[ \frac{d}{dt}\mathcal{E}(t) + \frac{\nu_1^2}{16} \mathcal{E}(t) \leq C \left( M(M + 1) \frac{k^2}{p} + \frac{\partial_m}{m} \right) \mathcal{E}(t), \tag{2.46} \]

Consequently, applying Gronwall’s inequality to (2.46) we have
\[ \mathcal{E}(t) \leq C \exp(CM(M + 1)) e^{-\frac{1}{16}t} \mathcal{E}(0) \]
which combines with the equivalent relation (2.38) give
\[ \frac{1}{M} \left\| (\rho^{-\frac{1}{2}}\hat{\Phi})(t) \right\|_{H^s} + \left\| (\rho^{-\frac{1}{2}}\hat{A})(t) \right\|_{H^s} \leq C \exp(CM(M + 1)) e^{-\frac{1}{16}t} \left( \frac{1}{\gamma M} \left\| \Phi^{in} \right\|_{H^s} + \left\| \alpha^{in} \right\|_{H^s} \right). \tag{2.47} \]

With (2.47) in hand, we can follow the same argument as the derivation of (2.41), (2.42), (2.44) to get
\[ \|\mathbb{P}[v](t)^{\bar{r}}\|_{L^2} \leq C \exp(CM(M + 1)) \langle t \rangle^{\frac{1}{2}} e^{-\frac{1}{16}t} \left( \frac{1}{\gamma M} \left\| \rho^{in} + \theta^{in} \right\|_{H^{\frac{s}{2}}} + \left\| \alpha^{in} \right\|_{H^{\frac{s}{2}}} \right), \]
\[ \|\mathbb{P}[v](t)^{\bar{y}}\|_{L^2} \leq C \exp(CM(M + 1)) \langle t \rangle^{\frac{3}{2}} e^{-\frac{1}{16}t} \left( \frac{1}{\gamma M} \left\| \rho^{in} + \theta^{in} \right\|_{H^{\frac{s}{2}}} + \left\| \alpha^{in} \right\|_{H^{\frac{s}{2}}} \right), \]
and
\[ \|\mathbb{Q}[v](t)^{\bar{r}}\|_{L^2} + \frac{1}{M} \|\rho(t) + \theta(t)\|_{L^2} \leq C(1 + \gamma) \exp(CM(M + 1)) \langle t \rangle^{\frac{1}{2}} e^{-\frac{1}{16}t} \left( \frac{1}{\gamma M} \left\| \rho^{in} + \theta^{in} \right\|_{H^1} + \left\| \alpha^{in} \right\|_{H^1} \right). \]
The proof of Theorem 1.1 is completed.
Acknowledgement. This work is supported by NSFC under grant number 11601533.

References

[1] P. Antonelli, M. Dolce, and P. Marcati, Linear stability analysis for 2D shear flows near Couette in the isentropic compressible Euler equations, arXiv:2003.01694.

[2] P. Antonelli, M. Dolce, and P. Marcati, Linear stability analysis of the homogeneous Couette flow in a 2D isentropic compressible fluid, arXiv:2101.01696.

[3] J. Bedrossian, P. Germain, and N. Masmoudi, On the stability threshold for the 3D Couette flow in Sobolev regularity, Ann. of Math., 185 (2017), 541–608.

[4] J. Bedrossian, P. Germain, and N. Masmoudi, Stability of the Couette flow at high Reynolds numbers in two dimensions and three dimensions, Bull. Amer. Math. Soc., 56 (2019), 373–414.

[5] R. Bianchini, M. Coti Zelati, and M. Dolce, Linear inviscid damping for shear flows near Couette in the 2D stably stratified regime, arXiv:2005.09058.

[6] W. Blumen, P. Drazin, and D. Billings, Shear layer instability of an inviscid compressible fluid, part 2, J. Fluid Mech., 71 (1975), 305–316.

[7] W. Blumen, Shear layer instability of an inviscid compressible fluid, J. Fluid Mech., 40 (1970), 769–781.

[8] G. Chagelishvili, A. Rogava, and I. Segal, Hydrodynamic stability of compressible plane Couette flow, Physical Review E, 50 (1994), R4283.

[9] G. Chagelishvili, A. Tevzadze, G. Bodo, and S. Moiseev, Linear mechanism of wave emergence from vortices in smooth shear flows, Physical review letters, 79 (1997), 3178.

[10] Q. Chen, T. Li, D. Wei, and Z. Zhang, Transition threshold for the 2D Couette flow in a finite channel, Arch. Ration. Mech. Anal., 238 (2020), 125–183.

[11] Q. Chen, T. Li, D. Wei, and Z. Zhang, Transition threshold for the 3D Couette flow in a finite channel, arXiv:2006.00721.

[12] M. Coti Zelati and M. Dolce, Separation of time-scales in drift-diffusion equations on R^2, J. Math. Pures Appl., 142 (2020), 58–75.

[13] W. Deng, J. Wu, and P. Zhang, Stability of Couette flow for 2D Boussinesq system with vertical dissipation, arXiv:2004.09292.

[14] P. Drazin and A. Davey, Shear layer instability of an inviscid compressible fluid, part 3, J. Fluid Mech., 82 (1977), 255–260.

[15] P. W Duck, G. Erlebacher, and M Y. Hussaini, On the linear stability of compressible plane Couette flow, J. Fluid Mech., 258 (1994), 131–165.

[16] B. Farrell and P. Ioannou, Transient and asymptotic growth of two-dimensional perturbations in viscous compressible shear flow, Physics of Fluids, 12 (2000), 3021–3028.

[17] W. Glatzel, Sonic instability in supersonic shear flows, Mon. Not. R. Astron. Soc., 233 (1988), 795–821.

[18] W. Glatzel, The linear stability of viscous compressible plane Couette flow, J. Fluid Mech., 202 (1989), 515–541.

[19] A. Hanifi, P. J Schmid, and D. S Henningson, Transient growth in compressible boundary layer flow, Physics of Fluids, 8 (1996), 826–837.

[20] S. Hu and X. Zhong, Linear stability of viscous supersonic plane Couette flow, Phys. Fluids, 10 (1998), 709–729.

[21] Y. Kagei, Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow, J. Math. Fluid Mech., 13 (2011), 1–31.

[22] Y. Kagei, Global existence of solutions to the compressible Navier-Stokes equation around parallel flows, J. Differential Equations, 251 (2011), 3248–3295.

[23] Y. Kagei, Asymptotic behavior of solutions to the compressible Navier-Stokes equation around a parallel flow, Arch. Ration. Mech. Anal., 205 (2012), 585–650.

[24] L. Lees and C. C. Lin, Investigation of the stability of the laminar boundary layer in a compressible fluid, NACA Tech. Note, (1946), No. 1115.

[25] H. Li and X. Zhang, Stability of plane couette flow for the compressible Navier-Stokes equations with Navier-slip boundary, J. Differential Equations, 263 (2017), 1160–1187.

[26] Z. Lin and C. Zeng, Inviscid dynamical structures near Couette flow, Arch. Ration. Mech. Anal., 200 (2011), 1075–1097.

[27] M. Malik, J. Dey, and M. Alam, Linear stability, transient energy growth, and the role of viscosity stratification in compressible plane Couette flow, Physical Review E, 77 (2008), 036322.
[28] N. Masmoudi and W. Zhao, Enhanced dissipation for the 2D Couette flow in critical space, *Comm. Partial Differential Equations*, **45** (2020), 1682–1701.

[29] P. Marcus, and W. Press, On Green’s functions for small disturbances of plane Couette flow, *J. Fluid Mech.*, **79** (1977), 525–534.

[30] C. Mouhot and C. Villani, On Landau damping, *Acta Math.*, **207** (2011), 29–201.

[31] M. Padmini and M. Subbiah, Stability of non-homentropic, inviscid, compressible shear flows, *J. Math. Anal. Appl.*, **241** (2000), 56–72.

[32] V.A. Romanov, Stability of plane-parallel Couette flow, *Funkcional Anal. i Priložen*, **7** (1973), 62–73.

[33] M. Subbiah and R. Jain, Stability of compressible shear flows, *J. Math. Anal. Appl.*, **151** (1990), 34–41.

[34] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping for a class of monotone shear flow in sobolev spaces, *Comm. Pure. Appl. Math.*, **71** (2018), 617–687.

[35] D. Wei, Z. Zhang, and W. Zhao, Linear inviscid damping and vorticity depletion for shear flows, *Ann. PDE*, **5** (2019), Art. 3, 101.

[36] D. Wei and Z. Zhang, Transition threshold for the 3D Couette flow in Sobolev space, *Comm. Pure Appl. Math.*, (2020), 0001–0082 (Preprint).

[37] J. Yang and Z. Lin, Linear inviscid damping for Couette flow in stratified fluid, *J. Math. Fluid Mech.*, **20** (2018), 445–472.

[38] L. Zeng, Z. Zhang, and R. Zi, Linear stability of the Couette flow in the 3D isentropic compressible Navier-Stokes equations, *arXiv*:2105.10200.

[39] X. Zhai, Linear stability analysis of the Couette flow for the two dimensional non-isentropic compressible Euler equations, *arXiv*:2105.07395.

(X. Zhai) **School of Mathematics and Statistics, Shenzhen University, Shenzhen, 518060, China.**

*Email address: zhaixp@szu.edu.cn*