A flux reconstruction method for the Korteweg-de Vries equation

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Abstract. In this paper we aim to develop a flux reconstruction (FR) method for the KdV equation, which contains a third derivative term. By introducing two auxiliary variables, the third order derivative is written as a combination of first derivatives, such that the FR schemes for first derivative can be applied directly. To demonstrate the effectiveness of the schemes, numerical results of some benchmark examples are presented, including the examples of a single soliton, double solitons, multiple solitons and an extremely convection-dominated case.

1. Introduction

The flux reconstruction (FR) method was firstly proposed in [1] to solve hyperbolic conservation laws, providing a way to unify the discontinuous Galerkin (DG) method [2,3] and the spectral difference (SD) method [4]. The FR method was later extended to solve diffusion problems in [5]. At the same time, the way of implementing the method for the case with unstructured meshes was also proposed [6-8], where numerical results of the Euler equations and Navier-Stokes equations were presented. A class of single-parameter energy-stable FR (ESFR) method can be found in [9-11]. A so-called direct FR (DFR) method was also developed in [12] to simplify the algorithm. As the SD method, the computational domain is divided into small cells for the FR method, where the algorithm in each cell is identical, showing the advantage in terms of parallel efficiency. In addition, since the evaluation of integrals is avoided, the cost of the FR method is less than the traditional DG method. For more overview of the FR method, the reader is referred to the review paper [13].

As mentioned above, the FR method for problems with first and second derivatives has been well developed. However, to the best of our knowledge, the extension of the FR method to higher derivatives cannot be found in the literature. In mathematical physics, partial differential equations containing higher derivatives are often encountered. Thus, it is meaningful to study how to extend the aforementioned efficient FR method to deal with problems with higher derivatives. In this work, we confine the study to the case with third derivative. To be more specific, the well-known Korteweg-de Vries (KdV) equation is taken as an example to illustrate the algorithm. In one dimension, the KdV equation is expressed as

\[ u_t + c_1 u u_x + c_2 u_{xxx} = 0, \]  

(1)
where $c_1$ and $c_2$ are constant coefficients. Since the KdV equation was proposed in [14], it has attracted lots of attention in different disciplines. Since one can easily find many theoretical and numerical results related to the KdV equation in the literature, the review of the related field is beyond the purpose of this paper. Here we just intend to show how to discretize this equation with a third derivative term by using the efficient FR method. For this purpose, we mainly refer to [15], where the local DG method was developed to deal with the third derivative term.

The rest of this paper is organized as follows. In Sect. 2, the FR method for the discretization of the third derivative in Eq. (1) is described. Then some numerical examples are presented in Sect. 3 to demonstrate the validity and accuracy of the method. Finally, conclusions are drawn in Sect. 4.

2. Numerical method

For the KdV equation (1), the FR method can be applied directly to discretize the advection term $uu_x = \frac{1}{2}(u')_x$, where Roe’s method is applied to compute numerical flux as used in [1]. Thus, we only need to show how to deal with the dispersion term $u_{xxx}$. For simplicity of derivation, we consider the linear equation

$$u_t + u_{xxx} = 0$$  \hspace{1cm} (2)

to describe the algorithm.

Here, the computational domain $[x_L, x_R]$ is assumed to be divided into $J$ cells equally. The cells are denoted by $E_j = [x_{j-1/2}, x_{j+1/2}]$ with $x_{j+1/2} = x_j + jh$ for $1 \leq j \leq J$, where $h = (x_R - x_L)/J$ is the spatial step. By using the following mapping

$$\xi = \frac{2}{h}(x - x_{j+1/2}) - 1,$$

we can confine the study to the standard cell $\xi \in E = [-1, 1]$. In the computational domain, Eq. (2) is transformed into

$$u_t + \left(\frac{2}{h}\right)^3 u_{\xi\xi\xi} = 0.$$  \hspace{1cm} (4)

For self-contained purpose, we first give a brief review of the FR method [1]. To design a $K$-th order FR method, we need to place $K$ solution points in the cell $E$, denoted by $\xi_1, \xi_2, \ldots, \xi_K$. In this paper, the Gauss points [2] are chosen, which are presented in Table 1 for different values of $K$. The boundary points of the cell are named as flux points, i.e., $\xi = -1$ and $\xi = 1$.

Table 1. Locations of Gauss points (denoted by $\xi_k$, $k = 1, 2, \ldots, K$) for different values of order $K$.

| $K$ | $\xi_1$ | $\xi_2$ | $\xi_3$ | $\xi_4$ | $\xi_5$ |
|-----|---------|---------|---------|---------|---------|
| 2   | -0.577350 | 0.577350 |        |        |        |
| 3   | -0.774597 | 0       | 0.774597 |        |        |
| 4   | -0.861136 | -0.339981 | 0.339981 | 0.861136 |        |
| 5   | -0.906180 | -0.538469 | 0       | 0.538469 | 0.906180 |

To discretize Eq. (4) properly, we introduce two auxiliary variables (denoted by $p$ and $q$) and write the equation as a linear first-order system,
as done in [15] for the local DG method. Then at each time level, the algorithm of the FR method can be summarized as the following five steps [1]:

(i) In each cell, obtain the values of \( u \) at the flux points \( x_{j+1/2} \) and \( x_{j+1/2} \) (denoted by \( u_{j+1/2}^{+} \) and \( u_{j+1/2}^{-} \)) by using the following interpolation schemes,

\[
\begin{align*}
    u_{j+1/2}^{+} &= \sum_{i=1}^{K} u_{j,k} l_{k}(1), \\
    u_{j+1/2}^{-} &= \sum_{i=1}^{K} u_{j,k} l_{k}(-1),
\end{align*}
\]

where \( u_{j,k} \) represent the values of \( u \) at \( \xi_{k} \) in the cell \( E_{j} \) and

\[
l_{k}(\xi) = \prod_{i \neq k} \frac{\xi - \xi_{i}}{\xi_{k} - \xi_{i}}
\]

are the Lagrangian interpolation basis functions such that \( l_{k}(\xi_{k}) = 1 \) and \( l_{k}(\xi_{i}) = 0 \) for \( i \neq k \);

(ii) Get the flux values \( \hat{u}_{j+1/2} \) at flux points. According to [15], we set

\[
\hat{u}_{j+1/2} = u_{j+1/2}
\]
to take into account the upwind feature of the system (5).

(iii) Approximate the value of \( q = u_{\xi} \) at solution points by

\[
q_{j,k} = u_{j,k}(\xi_{k}) + \left( \hat{u}_{j+1/2} - u_{j}(-1) \right) g_{L}^{j}(\xi_{k}) + \left( \hat{u}_{j+1/2} - u_{j}(1) \right) g_{R}^{j}(\xi_{k}),
\]

where \( u_{j}(\xi) = \sum_{i=1}^{K} u_{j,i} l_{i}(\xi) \), \( g_{L}^{j}(\xi) \) and \( g_{R}^{j}(\xi) \) are the so-called left and right correction functions. For simplicity, we usually choose the correction functions such that \( g_{R}^{j}(\xi) = g_{L}(-\xi) \) in practice. Thus, we only need to present the form of \( g_{L}^{j}(\xi) \). Here we just choose one of the correction functions developed in [1], denoted by \( g_{DG}^{j}(\xi) \), where the subscript ‘DG’ indicates that the choice has a close relation with the DG method. For convenience of the reader, the expressions of the correction function for different orders are tabulated in Table 2;

**Table 2.** The expressions of the correction function \( g_{DG}^{j}(\xi) \) used for Eq. (8) for different values of order \( K \).

| \( K \) | \( g_{DG}^{j}(\xi) \) |
|---|---|
| 2 | \(-\frac{1}{4} - \frac{1}{2} \xi + \frac{3}{4} \xi^{2}\) |
| 3 | \(-\frac{1}{4} + \frac{3}{4} \xi + \frac{3}{4} \xi^{2} - \frac{5}{4} \xi^{3}\) |
| 4 | \(\frac{3}{16} + \frac{3}{4} \xi - \frac{15}{8} \xi^{2} - \frac{5}{4} \xi^{3} + \frac{35}{16} \xi^{4}\) |
| 5 | \(\frac{3}{16} - \frac{15}{8} \xi + \frac{35}{8} \xi^{2} + \frac{35}{16} \xi^{3} - \frac{63}{16} \xi^{4}\) |
(iv) Approximate the values of \( p = q_x \) and \( p_z \) subsequently by using similar algorithms as steps (i)-(iii) for the approximation to \( q = u_x \). The only difference is that the values at flux points are evaluated as

\[
\hat{q}_{j+1/2} = q_{j+1/2}, \quad \hat{p}_{j+1/2} = p_{j+1/2}.
\]

(9)

It is worth pointing out that we can also use the choice \( \hat{p}_{j+1/2} = p_{j+1/2}, \quad \hat{q}_{j+1/2} = q_{j+1/2} \) and \( \hat{u}_{j+1/2} = u_{j+1/2} \) to compute the aforementioned fluxes, as stated in [15].

(v) After the above four steps, we obtained from Eq. (5) the following system of ordinary differential equations

\[
\frac{du_j}{dt} = R(u_j),
\]

(10)

where \( R(u_j) \) denotes the right-hand side term with \( u_j^n = (u_{j,1}^n, u_{j,2}^n, \ldots, u_{j,K}^n) \). Finally, some time-marching schemes can be applied directly to solve the obtained system. Here the following third-order Runge-Kutta scheme is employed

\[
\begin{align*}
S_1 &= R(u_j^n), \\
S_2 &= R(u_j^n + \frac{1}{2} \Delta t S_1), \\
S_3 &= R(u_j^n - \Delta t S_1 + 2 \Delta t S_2), \\
S_t &= \frac{1}{6} \Delta t S_1 + 4 S_2 + S_3, \\
u_{j+1}^n &= u_j^n + \frac{1}{6} \Delta t (S_t + 4 S_2 + S_3),
\end{align*}
\]

(11)

3. Numerical experiments

In this section, some benchmark examples are presented to demonstrate the effectiveness of the FR method derived for the KdV equation (1). To keep the errors arising from time discretization negligible, the time step is set to be \( \Delta t = 0.1(\Delta x_{\text{min}})^2 \) to implement the fourth-order time-marching scheme (11) for all examples, where \( \Delta x_{\text{min}} \) represents the minimum distance between two solution points in the physical space.

3.1. Propagation of a single soliton

In this case, the coefficients of the KdV equation (1) are set to be \( c_1 = 6 \) and \( c_2 = 1 \). It is well known that for the given initial value condition

\[
u(x,0) = \text{sech}^2 \frac{x}{\sqrt{2}}
\]

(12)

Eq. (1) admits the following exact solution

\[
u(x,t) = \text{sech}^2 \left[ \frac{1}{\sqrt{2}} \left( x - 2t \right) \right].
\]

(13)

For the simulation, the computational domain is chosen to be \([-20,20]\) with periodic boundary conditions. In Table 3, numerical results at time \( t = 0.1 \) are presented for different orders of accuracy, where the design convergence rates are confirmed. For larger time, the propagation of one soliton is depicted in Figure 1, showing that the numerical solutions are captured well by the proposed FR method.
Table 3. Numerical test of accuracy for Eq. (1) with the initial value condition (12). As we can see, the expected convergence rates are achieved for different values of order $K$.

| $J$ | $K = 2$ | $K = 3$ | $K = 4$ | $K = 5$ |
|-----|---------|---------|---------|---------|
|     | $L^2$ error | order | $L^2$ error | order | $L^2$ error | order | $L^2$ error | order |
| 40  | 8.53E-3   | -  | 1.03E-3   | -  | 9.67E-5   | -  | 1.68E-5   | -  |
| 80  | 2.11E-3   | 2.02 | 1.21E-4   | 3.08 | 7.65E-6   | 3.66 | 4.77E-7   | 5.14 |
| 160 | 5.03E-4   | 2.07 | 1.54E-5   | 2.97 | 4.89E-7   | 3.97 | 1.53E-8   | 4.96 |
| 320 | 1.24E-4   | 2.02 | 1.94E-6   | 2.99 | 3.08E-7   | 3.97 | 4.81E-10  | 4.99 |

Figure 1. Solutions to Eq. (1) with $c_1 = 6$, $c_2 = 1$ and the initial value condition (12). Here we produced the results by setting the number of cells to be $J=40$ and the value of order $K=5$. The solid lines represent the exact solution (13) and the points stand for the numerical results.

3.2. Interaction of double solitons

As [15], we consider here the case with a double soliton solution by setting $c_2 = 1$ and $c_1 = 4.84 \times 10^{-4}$ in Eq. (1) and choosing the following initial condition

$$u(x,0) = 3a_1 \text{sech}^2(k_1(x-x_1)) + 3a_2 \text{sech}^2(k_2(x-x_2)),$$

where $a_i = 0.3$, $a_2 = 0.1$, $x_i = 0.4$, $x_2 = 0.8$ and $k_i = \frac{1}{2}\sqrt{a_i / c_i}$ for $i = 1, 2$. The computational domain is set to be $[0,2]$ with periodic boundary conditions. As an example, we depict the numerical results obtained from the case with $J=40$ and $K=5$ in Figure 2. As we can see, the results are consistent with the results of [15] (see Example 4.5 therein), where the interaction of double solitons is captured well, demonstrating the effectiveness of the proposed FR method.
Figure 2. Numerical solutions to Eq. (1) with $c_1 = 1, c_2 = 4.84 \times 10^{-4}$ and the initial value condition (14). Here we produced the results by setting the number of cells to be $J=40$ and the value of order $K=5$.

Figure 3. Numerical solutions to Eq. (1) with $c_1 = 1, c_2 = 4.84 \times 10^{-4}$ and the initial value condition (15). Here we produced the results by setting the number of cells to be $J=40$ and the value of order $K=5$. 
3.3. Interaction of multiple solitons

This example is from the well-known letter [7], where the coefficients $c_1$ and $c_2$ are set as exactly the same values of the previous example, and the initial condition is chosen to be

$$u(x,0) = \cos(\pi x). \quad (15)$$

For the computational domain $[0,2]$ with periodic boundary conditions, the problem admits multiple solitons that are widely known. Here we present the numerical results of the proposed FR method in Figure 3, where $J = 40$ and $K = 5$. As we can see, the interaction of the multiple solitons is captured well (see also Figure 1 in [16]), indicating the effectiveness of the proposed method.

3.4. Convection-dominated case

To test the proposed FR method further, we consider the convection-dominated case with $c_1 = 1$ and $c_2 = 10^{-4}$, $10^{-5}$, $10^{-6}$, $10^{-7}$, respectively. It is noted that the solution of the KdV equation (1) contains more oscillations when the coefficient $c_2$ becomes smaller, resulting in a challenge for numerical schemes [17]. Here we set the initial condition to be

$$u(x,0) = 2 + 0.5\sin(2\pi x), \quad (16)$$

and choose different number of cells to implement the proposed FR method. In Figure 4, the results with $K = 5$ are depicted. As we can see, the oscillation solutions are captured well, which are in agree with the results presented in [15] (see Figure 4.4 therein).

Figure 4. Numerical solutions to Eq. (1) with $c_1 = 1$ and the initial value condition (16). Here we produced the results by setting $K = 5$. As we can see, the solution is more oscillated for smaller value of $c_2$. 

4. Conclusions
We have developed an efficient FR method for the KdV equation in this paper. The key is to write the third derivative by introducing auxiliary variables as done in the local DG method, such that arbitrary orders of FR schemes derived for first derivative could be applied directly. To demonstrate the effectiveness of the proposed method, some numerical results were presented. The results showed that the design order of accuracy is achieved and the solitary interaction is captured well, indicating that the efficient FR method is suitable for use in problems with high derivatives. In the future, the method can be extended to solve other equations with higher derivatives.

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