Moduli spaces and D-brane categories of tori using SCFT

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Abstract: We analyse the moduli spaces of superconformal field theories (SCFTs). For \(N = 2\) we find an enhanced moduli space which in geometrical terms corresponds to tori with two independent complex structures. To explain the precise relation with the moduli space of SCFTs on K3 surfaces as described by Aspinwall and Morrison, we discuss some subtleties with the precise interpretation of the \(N = 2\) and \(N = 4\) moduli spaces. We also explain why in some cases the SYZ-description of mirror symmetry as fibrewise T-duality seems to break down.

Using gluing matrices we give an algebraic description of D-branes and construct the corresponding boundary states. We study how isomorphisms of the SCFTs act on D-branes. Finally we give a geometrical interpretation of our algebraic constructions and make contact with the geometrical D-brane categories and Kontsevich’s homological mirror symmetry conjecture.

Keywords: Mirror symmetry, vertex algebra, conformal field theory, torus
1. Introduction

Tori are by far the simplest Calabi-Yau spaces and many aspects of string theory compactified on a torus are pretty well understood. Nevertheless some of the more subtle aspects still deserve careful investigation. In this paper we study the moduli space of conformal field theories with a torus as target space. We also investigate the description of D-branes. The conformal field theory point of view often leads to a rather algebraic description. We try to make the connection with the geometry of the tori precise. Let us give a quick overview and point out the most important results.

We start with some general remarks about the moduli space of $N = 1$ and $N = 2$ superconformal field theories (SCFTs). This allows us to introduce some notation and explain why from the SCFT point of view it makes sense to work with two independent complex structures to describe the $N = 2$ structures corresponding to a fixed $N = 1$ structure. We discuss how the $N = 2$ and $N = 4$ moduli spaces depend on the precise definition of $N = 2$ and $N = 4$ superconformal algebras. This leads to two slightly different moduli spaces for both $N = 2$ and $N = 4$. Using this description we can clarify the connection of our description for higher dimensional tori with the one found in the literature for K3 surfaces and 4-dimensional tori. It also leads to the notion of generalised morphisms of $N = 2$ SCFTs, which includes left and right mirror morphisms.
To obtain a concrete construction of the moduli space for tori we review the description by Kapustin and Orlov in [1]. For $N = 2$ SCFTs their description contains one not very natural condition. Dropping this condition we find a larger moduli space containing theirs. This enhanced moduli space turns out to be of the general form that we discussed before. To see this we discuss a geometrical interpretation of this larger moduli space involving a pair of complex structures on the torus. In these terms the extra condition used by Kapustin and Orlov has a natural interpretation: it is equivalent to the requirement that the two complex structures coincide.

The spaces that we discuss are not really moduli spaces because there are many isomorphisms of conformal field theories that identify points in these spaces. Kapustin and Orlov give a concrete description of isomorphisms and mirror morphisms of conformal field theories and their description generalises naturally to the enhanced moduli space we discuss. It turns out that in general the condition that the two complex structures coincide is not invariant under isomorphisms or mirror morphisms. This explains why for certain background field configurations mirror symmetry seemed to fail.

We continue by studying how D-branes fit into this picture. Using the conformal field theory description of D-branes in terms of gluing matrices, we define a large class of D-branes. We describe the corresponding boundary states and discuss how they transform under generalised morphisms. Following [2], we propose a geometrical interpretation of the D-branes as a module over the algebra of functions on the target space. This description is a slight modification of the standard description of D-branes as bundles on a submanifold.

According to the standard philosophy, D-branes can be interpreted as the objects of the so-called D-brane categories. After a topological twist these categories are expected to correspond to the Fukaya category for the A-twist and the derived category of coherent sheaves for the B-twist. In this geometrical interpretation a mirror morphism should induce an equivalence of the Fukaya category and the derived category of the mirror. This is Kontsevich’s famous homological mirror symmetry conjecture. For the torus our description is more detailed and at least in principle also works when mirror symmetry does not preserve the condition that both complex structures are equal. Using our description of the action of generalised morphisms on D-branes we also obtain a fairly good understanding of the equivalence functor on the level of objects. Making this more precise and extending it to morphisms would lead to a much better understanding of this conjecture.

2. Moduli space of SCFTs

2.1 General structure

Before specialising to SCFTs on tori we will discuss some general properties of SCFTs and their moduli spaces. Note that most of the time we do not discuss the moduli spaces themselves, but talk about the Teichmüller spaces instead. The actual moduli space can be obtained as the quotient of the Teichmüller space by a discrete group, the so-called duality group.

Our starting point is the Teichmüller space of $N = 1$ SCFTs, which we denote by $\mathcal{T}_{N=1}$. For any manifold $X$ with a Ricci-flat metric $G$ we can write down the action of a
nonlinear sigma model defining a SCFT with $N = 1$ supersymmetry. The most general form of this action includes a term depending on a closed 2-form $B$ on the target space $X$. This 2-form is often called the B-field. Different pairs $(G, B)$ and $(G', B')$ may correspond to isomorphic $N = 1$ SCFTs. Under favourable circumstances there exist extra conditions on the metric and the B-field that can always be fulfilled up to isomorphism. Then we can resolve part of the ambiguity by imposing these conditions. For the torus we can require the metric and the B-field to be constant. This defines the Teichmüller space $\mathcal{T}_{N=1}$.

However, these extra conditions will not get rid of all isomorphisms. At least in the case of the torus the identifications due to the remaining isomorphisms can be described by the action of a discrete duality group $\mathcal{G}$. Therefore the moduli space can be written as $\mathcal{M}_{N=1} = \mathcal{G} \backslash \mathcal{T}_{N=1}$.

To describe the Teichmüller space for SCFTs with more supersymmetry, we have to be more specific. First of all, note that we do not discuss space-time supersymmetry in this text. We also ignore the possibility of having different amounts of supersymmetry in the left- and right-moving sectors. So when we talk about $N = 1$ supersymmetry, this means $N = 1$ worldsheet supersymmetry in both the left- and the right-moving sector. The left-moving $N = 1$ superconformal algebra is generated by two fields $L(z)$ and $G(z)$ characterised by their operator product expansions (OPEs). These OPEs can be translated into commutation relations for the modes. In the geometrical situation where the SCFT corresponds to a nonlinear sigma model, these fields are defined in terms of the metric $G$

\[ L(z) = \frac{1}{2} :G(\partial X(z), \partial X(z)): - \frac{1}{2} :G(\psi(z), \partial \psi(z)):, \]

\[ G(z) = \frac{i}{2\sqrt{2}} :G(\psi(z), \partial X(z)):, \]

where $: :$ denotes operator normal ordering. In the sigma model language the bosonic field $X(z, \bar{z})$ corresponds to the embedding of the worldsheet into the target manifold. In itself it is neither left-moving (holomorphic) nor right-moving (antiholomorphic). However, the derivatives $\partial X(z)$ and $\bar{\partial} X(\bar{z})$ are left- respectively right-moving. The remaining field $\psi(z)$ is fermionic and has a right-moving counterpart $\bar{\psi}(\bar{z})$. The expressions for the generators $\hat{L}(\hat{z})$ and $\hat{G}(\hat{z})$ in the right-moving sector can be obtained from the ones above by replacing $\partial X(z)$ by $\bar{\partial} X(\bar{z})$ and $\psi(z)$ by $\bar{\psi}(\bar{z})$. Mathematically the precise interpretation of these expressions for general Calabi-Yau manifolds is delicate, but for tori, where the Ricci-flat metric $G$ can be represented by a constant matrix, everything can be made rigorous (see \[ ]).

For $N = 2$ superconformal algebras there are two slightly different definitions. The first definition says that an $N = 2$ superconformal algebra consists of an $N = 1$ superconformal algebra together with a choice of a $u(1)$ subalgebra with certain properties. The second definition defines an $N = 2$ superconformal algebra as an $N = 1$ superconformal algebra with a choice of fields $J(z)$ and $G^\pm(z)$ satisfying certain OPEs (also involving $L(z)$). The relation between these two definitions is that the field $J(z)$ is a generator of the $u(1)$ subalgebra. However, the $u(1)$ subalgebra only determines $J(z)$ up to a scalar. The required
OPEs restrict this scalar to ±1, but this remaining indeterminacy cannot be eliminated. Once we have fixed $J(z)$, we can define $G^\pm(z)$ using $J(z)$ and $G(z)$. The properties of the $u(1)$ subalgebra should ensure that the commutation relations are satisfied.

The Teichmüller space of $N = 2$ SCFTs depends on the definition we use. Let us write $\mathcal{T}_{N=2}$ in case of the first definition and $\mathcal{T}'_{N=2}$ when we use the second definition. Then $\mathcal{T}'_{N=2}$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-fibration over $\mathcal{T}_{N=2}$. The fibres correspond to the choice of the signs of $J(z)$ and the corresponding field $\bar{J}(\bar{z})$ in the right-moving sector. In this case the difference between the two Teichmüller spaces is rather small, but conceptually it is quite important. As we will see in the next section for $N = 4$ a similar phenomenon occurs which will be important to explain the relation between our description of the moduli space and the one in the literature.

Isomorphisms of $N = 2$ SCFTs can be defined as $N = 1$ isomorphisms that preserve the $u(1)$ algebras for the first definition or the generators for the second one. In both cases the duality group can be identified with the duality group $G$ for the $N = 1$ case. The reason is that any $N = 2$ isomorphism is also an $N = 1$ isomorphism. Conversely, every $N = 1$ isomorphism determines an $N = 2$ isomorphism. So the moduli spaces are given by $\mathcal{M}_{N=2} = \mathcal{G} \setminus \mathcal{T}_{N=2}$ and $\mathcal{M}'_{N=2} = \mathcal{G} \setminus \mathcal{T}'_{N=2}$ respectively.

In the sequel we will follow Kapustin and Orlov (see [1]) and use the second definition. It may be less elegant than the first one, but it has the significant advantage that the OPEs for $J(z)$ and $G^\pm(z)$ are well known, whereas the precise conditions for the $u(1)$ subalgebra are unclear. A consequence of using this definition is that mirror symmetry does not define an isomorphism of $N = 2$ SCFTs, as it would do if we used the first definition. Instead, we can define left mirror morphisms as $N = 1$ isomorphisms that in the left-moving sector change the sign of $J(z)$ and interchange $G^\pm(z)$, whereas in the right-moving sector they preserve $\bar{J}(\bar{z})$ and $\bar{G}^\pm(\bar{z})$. Similarly, right mirror morphisms change the sign of right-moving current $\bar{J}(\bar{z})$ and preserve $J(z)$. Together, left and right mirror morphisms generate a group $G' = G \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which we will call the extended duality group. Dividing out this group we obtain the moduli space corresponding to the first definition $\mathcal{M}_{N=2} = \mathcal{G} \setminus \mathcal{T}_{N=2}$.

Forgetting the $N = 2$ structure defines maps $\mathcal{T}_{N=2} \to \mathcal{T}_{N=1}$ and $\mathcal{T}'_{N=2} \to \mathcal{T}_{N=1}$. In both cases the fibres of these maps can be written as $S \times S$, where $S$ is the space of all $N = 2$ structures for a fixed $N = 1$ structure. Of course the space $S$ differs depending on which definition we use. The two factors correspond to the left- and right-moving sectors.

For the geometrical interpretation it is also advantageous to use the second definition. For a nonlinear sigma model to allow $N = 2$ supersymmetry the target space should be a Kähler manifold. Supposing that $X$ is a Kähler manifold, we can write down explicit formulae for the fields in terms of geometrical data

\[
G^\pm(z) = \frac{i}{4\sqrt{2}} G(\psi(z), \partial X(z)) \pm \frac{1}{4\sqrt{2}} \omega(\psi(z), \partial X(z));
\]

\[
J(z) = -\frac{i}{2} \omega(\psi(z), \psi(z));
\]  

(2.2)

Here $\omega$ is the Kähler form on the target space. Finding a Kähler form is equivalent to finding a complex structure $j$ compatible with the metric $G$ (i.e., satisfying $j^i G_{j} = G$).
The expressions for the right-moving sector can again be obtained by putting bars in the appropriate places. However, note that we could use a different complex structure (but the same metric) in the right-moving sector. So we can choose two independent complex structures $j_1$ and $j_2$ both compatible with the same metric $G$ and use $j_1$ to define $J(z)$ and $G^+(z)$ and $j_2$ to define $\bar{J}(\bar{z})$ and $\bar{G}^+(\bar{z})$. All we have to check is that these fields satisfy the required OPEs. For this it does not matter whether $j_1 = j_2$ or not: the OPEs within a sector are independent of the complex structure in the other sector and operators from different sectors should (anti)commute and that is independent of the complex structure.

This means that for the second definition we can interpret $S$ geometrically as the space of complex structures compatible with a given metric. Therefore the real dimension of $S$ is $2h^{2,0}$, where $h^{p,q} = h^{p,q}(X)$ are the Betti numbers of the target space $X$. In fact it seems that higher supersymmetries occur exactly when $h^{2,0} > 0$, i.e., when $S$ has a positive dimension. This situation can be summarised in the following diagram (with the real dimension above or below each space)

\[
\begin{array}{cccc}
D + 2h^{2,0} & D + 4h^{2,0} & 2 \times 2h^{2,0} \\
\mathcal{F}^\text{geom}_{N=2} & \mathcal{F}^\prime_{N=2} & S \times S
\end{array}
\]

\[\text{diag.}
\]

Here $n = \dim \mathbb{C} X$ and $D = 2(h^{1,1} + h^{n-1,1})$ is the real dimension of $\mathcal{F}_{N=1}$. This dimension can be computed as follows. Deformations of the complex structure contribute $2h^{n-1,1}$, the choice of the B-field $b_2 = h^{1,1} + 2h^{2,0}$ and the choice of the Kähler form $h^{1,1}$. Because for $N = 1$ the complex structure is irrelevant (only the metric and the B-field count), one should subtract $2h^{2,0}$ for the choice of a complex structure for one fixed metric.

The geometrical part $\mathcal{F}^\text{geom}_{N=2}$ is where the complex structures in both sectors are equal. So fibrewise it is the diagonal in $\mathcal{F}^\prime_{N=2}$. A slightly degenerate case of this picture is obtained when the target space is a strict Calabi-Yau manifold (i.e., a manifold with holonomy group equal to SU($n$) for $n > 2$). In that case there is just a $\mathbb{Z}_2$ of choices for the complex structure, so $S = \mathbb{Z}_2$. This also means that $\mathcal{F}_{N=2}$ is equal to $\mathcal{F}_{N=1}$. For strict Calabi-Yau manifolds $N = 2$ is the maximal amount of supersymmetry. It seems to be a general phenomenon that if one considers a SCFT on a certain target space with the maximal amount of supersymmetry $N = N_{\text{max}}$ possible for that target space, then $\mathcal{F}_{N=N_{\text{max}}} = \mathcal{F}_{N=1}$. Maximal supersymmetry should correspond to choosing the largest possible subalgebra from some series of subalgebras indexed by $N$. Then the conjecture is that for a given $N = 1$ SCFT there is just one possible choice for this maximal subalgebra. We will see another example of this in the next section.

In general mirror symmetry does not preserve $\mathcal{F}^\text{geom}_{N=2}$. A trivial example is the (left) mirror morphism that is the identity on the $N = 1$ level, but replaces $J(z)$ by $-J(z)$ and interchanges $G^+(z)$ and $G^-(z)$. Geometrically this corresponds to replacing $j_1$ by $-j_1$. Of course, this is a rather trivial mirror morphism. The geometrically interesting mirror morphisms do preserve $\mathcal{F}^\text{geom}_{N=2}$.
2.2 K3 surfaces and 4-tori

Apart from the degenerate case of strict Calabi-Yau manifolds the prime example in the literature of the structure discussed in the previous section is the moduli space of SCFTs on K3 surfaces and 4-tori. Their moduli spaces are very similar and have been studied in great detail (see e.g., [3, 4, 5, 6]). In those papers most attention was paid to the moduli space of \( N = 4 \) SCFTs. In the previous section we encountered some subtleties in the interpretation of \( N = 2 \) moduli spaces. For \( N = 4 \) superconformal algebras there are also two slightly different definitions. One possibility is to define an \( N = 4 \) superconformal algebra as an \( N = 1 \) superconformal algebra with an \( su(2) \) subalgebra. The other possibility is to choose generators and define an \( N = 4 \) algebra as an \( N = 1 \) algebra with extra fields \( J^{(i)}(z) (i = 1, 2, 3), G^\pm(z) \text{ and } G'^\pm(z) \) satisfying certain OPEs. The corresponding Teichmüller spaces are denoted by \( T_{N=4} \) and \( T'_{N=4} \) respectively.

Recall that K3 surfaces and 4-tori are hyperkähler manifolds. In particular for a fixed metric there exists a \( S^2 \) of compatible complex structures. More explicitly, we can choose three compatible complex structure \( j_i (i = 1, 2, 3) \) satisfying the quaternion relations \( j_1 j_2 = j_3 \) and cyclic permutations. An arbitrary compatible complex structure \( j \) can then be written as \( j = \sum_k a_k j_k \) with \( (a_1, a_2, a_3) \in S^2 \subset \mathbb{R}^3 \). If denote the corresponding Kähler forms by \( \omega_i \), then we can define the generators \( J^{(i)}(z) \) of the \( su(2) \) as

\[
J^{(i)}(z) = -i \sum_{\psi} \omega_i(\psi(z), \psi(z)) \; .
\]

These fields generate the \( su(2) \) subalgebra. Note that this subalgebra only depends on the metric. It follows that there is just one choice for the \( su(2) \) subalgebra and therefore \( T_{N=4} \cong T_{N=1} \).

For further reference we will briefly sketch the known description of the moduli space for K3 surfaces and 4-tori. Using the uniform notation introduced in [3] we can discuss both cases simultaneously. Let \( X \) denote a K3 surface or a 4-torus, then we define \( \Lambda \) to be the lattice \( H^{ev}(X, \mathbb{Z}) = \oplus_k H^{2k}(X, \mathbb{Z}) \) of even integral cohomology classes. On this lattice there exists an even pairing \( q \) defined by \( q(\alpha, \beta) = -\int_X \alpha \wedge I(\beta) \), which is a slight modification of the intersection pairing. In this formula \( I : H^{ev}(X, \mathbb{Z}) \to H^{ev}(X, \mathbb{Z}) \) is defined by \( I|_{H^{2k}(X, \mathbb{Z})} = (-1)^k \text{ Id} \). The signature of this pairing is \( (4, 4 + \delta) \), where \( \delta = 0 \) for a 4-torus and \( \delta = 16 \) for a K3 surface.

A SCFT with target space a K3 surface or a 4-torus has \( N = 4 \) worldsheet supersymmetry. The \( N = 4 \) Teichmüller space is given by

\[
T_{N=4}(X) = O(\Lambda_{\mathbb{R}}, q)/O(4) \times O(4 + \delta) , \tag{2.4}
\]

where \( \Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R} = H^{ev}(X, \mathbb{R}) \). The duality group turns out to be \( O(\Lambda, q) \), the group of all lattice automorphisms of \( \Lambda \) preserving \( q \). For K3 surfaces this was originally a conjecture by Seiberg (see [6]). In [3] Aspinwall and Morrison analysed the duality group in great detail and argued that Seiberg’s guess is indeed correct. The remaining issues were settled by Nahm and Wendland in [5]. Using the duality group we find the following description of the \( N = 4 \) moduli space

\[
\mathcal{M}_{N=4}(X) = O(\Lambda, q)\backslash O(\Lambda_{\mathbb{R}}, q)/O(4) \times O(4 + \delta). \tag{2.5}
\]
Aspinwall and Morrison describe the $N = 2$ Teichmüller space for K3 surfaces as a fibration $T'_{N=2} \to T_{N=4}$ with fibres $S^2 \times S^2$. This coincides with the description that we found in the previous section when we use that $T_{N=4} = T_{N=1}$. Note that Aspinwall and Morrison do choose generators $J(z)$ and $\bar{J} (\bar{z})$ for the $N = 2$ SCFT, but they do not do this for the $N = 4$ SCFT. From our perspective, the existence of a map $T'_{N=2} \to T_{N=4}$ is a coincidence ultimately stemming from the fact that for K3 surfaces $T_{N=4}$ is equal to $T_{N=1}$ and there is a natural map $T'_{N=2} \to T_{N=1}$.

## 2.3 $N = 1$ SCFTs on tori

After these generalities we will restrict our attention to tori for the rest of this paper. To give a precise description of the moduli spaces, we will use the description of SCFTs for tori due to Kapustin and Orlov (see [1]). Note that they mostly discuss vertex algebras, which provide a mathematical formulation of most ingredients that go into a (S)CFT. Because the vertex algebras for tori that they describe can be extended to full (S)CFTs, we will be sloppy and use the names vertex algebra and SCFT interchangeably. Many of the formulae we discuss in this section are known in the physics literature (see e.g., [3]), but we want to collect them here to establish the notation and prepare the ground for the discussion of the $N = 2$ case which is less well known.

Let $X$ be a torus of real dimension $m$ equipped with a Ricci-flat metric $G$ and a closed 2-form $B$. Let $\Gamma$ be the lattice $H_1(X, \mathbb{Z})$. We will write $\Gamma$ for the corresponding real vector space $\Gamma \otimes \mathbb{R}$, so we can write $X = \Gamma / \Gamma$. The Ricci-flat metric and the B-field can then be represented by linear maps $\Gamma_{\mathbb{R}} \to (\Gamma^*_{\mathbb{R}})^{\mathbb{T}_R}$, which we will also denote by $G$ and $B$. Note that $G$ is symmetric, whereas $B$ is antisymmetric.

Using these data Kapustin and Orlov define an $N = 1$ SCFT for every torus. We will briefly discuss some essential points of their construction below, but first we want to present a different way of encoding the metric and the B-field. To do so, let us first introduce some notation. On the lattice $\Gamma \oplus \Gamma^*$ we have a symmetric bilinear form $q$ defined by $q(v \oplus f, w \oplus g) = f(w) + g(v)$. When we talk about a bilinear form $q$ in connection with a lattice of the form $\Gamma \oplus \Gamma^*$ we will always mean this particular bilinear form. A lattice isomorphism $g : (\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$ will be supposed to respect the bilinear forms in the sense that $g'q'g = q$.

In terms of the metric $G$ and the B-field $B$ we can define an endomorphism $K$ of $\Gamma_{\mathbb{R}} \oplus \Gamma_{\mathbb{R}}^*$ by

$$K = \begin{pmatrix} -G^{-1}B & G^{-1} \\ G - BG^{-1}B & BG^{-1} \end{pmatrix}.$$ (2.6)

One can easily check that $K^2 = \text{Id}$ and $K'qK = q$. To ensure that conversely any such $K$ can be written in the above form, we need one more requirement. This comes from the condition that $G$ be positive definite. To investigate this condition, let us define

$$\mathcal{R}(G,B) = \frac{1}{2} \begin{pmatrix} \text{Id}_m - G^{-1}B & G^{-1} \\ \text{Id}_m + G^{-1}B & -G^{-1} \end{pmatrix}.$$
This is a slight modification of the definition in [1]. A straightforward computation shows that
\[ qK = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} = \mathcal{R}(G, B)^t \begin{pmatrix} 2G & 0 \\ 0 & 2G \end{pmatrix} \mathcal{R}(G, B). \] (2.7)

So $G$ is positive definite if and only if $qK$ is. Putting all these ingredients together we are led to the following definition of the $N = 1$ Teichmüller space

**Definition 1.** The Teichmüller space of $N = 1$ SCFTs on a torus of real dimension $m$ is defined as

\[ \mathcal{T}_{N=1} := \left\{ (\Gamma, K) \mid \begin{array}{c} \Gamma \text{ lattice of rank } m, \, K \in \text{End}(\Gamma_{\mathbb{R}} \oplus \Gamma_{\mathbb{R}}^*) \\ K^2 = \text{Id}, \, K^t qK = q, \, qK \text{ pos. def.} \end{array} \right\}. \]

Here $q$ is the symmetric bilinear form on the lattice $\Gamma \oplus \Gamma$ discussed above. This space has several well known alternative descriptions which we summarise in the following proposition. For the sake of completeness we also sketch a proof.

**Proposition 1.** The $N = 1$ Teichmüller as defined in Definition 1 has the following alternative descriptions

\[ \mathcal{T}_{N=1} = \left\{ (\Gamma, G, B) \mid \begin{array}{c} \Gamma \text{ lattice of rank } m, \, B, G : \Gamma_{\mathbb{R}} \rightarrow \Gamma_{\mathbb{R}}^* \\ G \text{ symm., } B \text{ antisymm., } G \text{ pos. def.} \end{array} \right\} = \left\{ (\Gamma, V) \mid V \subset \Gamma_{\mathbb{R}} \oplus \Gamma_{\mathbb{R}}^*, \, V \text{ maximal pos. subspace w.r.t. } q \right\}. \]

**Proof.** For the first equality, note that the inclusion $\supset$ follows from (2.6). To prove the other inclusion we need to show that any $K \in \mathcal{T}_{N=1}(\Gamma)$ can be written in the form (2.6) with $G$ symmetric and positive definite and $B$ antisymmetric. To see this write $K = (\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix})$. Now note that $\beta : \Gamma_{\mathbb{R}}^* \rightarrow \Gamma_{\mathbb{R}}$ has to be invertible, because otherwise $\langle v, qKv \rangle = 0$ for $v \in 0 \oplus \Gamma_{\mathbb{R}}^*$ and that contradicts the requirement that $qK$ be positive definite. Knowing this, one can easily show that $K^2 = \text{Id}$ is equivalent to $\gamma = \beta^{-1} - \beta^{-1}\alpha^2$ and $\delta = -\beta^{-1}\alpha\beta$. Substituting that into $K^t qK = q$, one finds that $\beta^{-1}\alpha$ is antisymmetric and $\beta$ is symmetric. So if we identify $\beta$ with $G^{-1}$ and $\beta^{-1}\alpha$ with $-B$, then $K$ will have the form (2.6). The positive definiteness of $G$ follows from the positive definiteness of $qK$ using (2.7).

The second alternative is just a reformulation of the first one: $E$ can be defined as $B + G$ and $B$ and $G$ can be obtained from $E$ as the symmetric and antisymmetric part.

To show the third equality note that $K^2 = \text{Id}$ implies that $K$ is diagonalisable with eigenvalues $\pm 1$. Let $E_{\pm 1}$ be the eigenspaces of $K$ for the eigenvalues $\pm 1$. Then for all $v \in E_{\pm 1} \setminus \{0\}$ we have

\[ 0 < \langle v, qKv \rangle = \langle v, qv \rangle. \]

So $q$ restricted to $E_{\pm 1}$ is positive definite. Similarly, it follows that $q$ restricted to $E_{-1}$ is negative definite. This means that $E_{\pm 1}$ can each have at most dimension $m$. Because $E_{\pm 1} \oplus E_{\mp 1} = \Gamma_{\mathbb{R}} \oplus \Gamma_{\mathbb{R}}^*$, we can conclude that $E_{\pm 1}$ indeed have dimension $m$. So $E_{\pm 1}$ is a maximal positive subspace. Conversely, if $V$ is a maximal positive subspace for $q$,
then $W := V^\perp$ is a maximal negative subspace. So we can define an endomorphism $K$ of $\Gamma_R \oplus \Gamma_R^* = V \oplus W$ that is the identity on $V$ and minus the identity on $W$. One can easily check that $K$ defined in this way satisfies the conditions stated above in the definition of $T_{N=1}$.

Alternatively, note that if $E$ is a map $\Gamma_R \rightarrow \Gamma_R^*$ with positive definite symmetric part, then $\phi_E = (\frac{E}{E^t})$ defines a map $\Gamma_R \rightarrow \Gamma_R \oplus \Gamma_R^*$. One can easily check that the image $V = \text{im} \phi_E$ is a positive subspace and that $V^\perp = \text{im} \phi_{-E^t}$. □

Because any two lattices of the same rank are isomorphic, it often makes sense to make one fixed choice of the lattice. So we could define an alternative Teichmüller space as follows

$$T_{N=1}(\Gamma) := \{K \mid (\Gamma, K) \in T_{N=1}\}.$$ 

The statement of Proposition 1 continues to hold with the lattice $\Gamma$ fixed throughout. In fact we can even extend the list with following rather well known description of the Teichmüller space

$$T_{N=1}(\Gamma) = O(\Gamma_R \oplus \Gamma_R^*, q)/O(m) \times O(m).$$

(2.8)

Here the $O(\Gamma_R \oplus \Gamma_R^*, q)$ is group of automorphisms of the lattice $\Gamma \oplus \Gamma^*$ preserving the bilinear form $q$. This equality is based on the fact that the group $O(\Gamma_R \oplus \Gamma_R^*, q)$ acts transitively on the space of maximal positive subspaces of $(\Gamma_R \oplus \Gamma_R^*, q)$. Let $V$ be any such subspace, then $\Gamma_R \oplus \Gamma_R^* = V \oplus V^\perp$. The stabiliser of $V$ is $O(V) \times O(V^\perp)$. This proves the desired equality. For $m = 4$ this description coincides with the one from (2.5). In the sequel we will usually work with $T_{N=1}$ instead of $T_{N=1}(\Gamma)$, because this often leads to nicely coordinate invariant expressions.

For further reference we compute

$$R(G, B)^{-1} = \begin{pmatrix} \text{Id}_m & \text{Id}_m \\ B + G & B - G \end{pmatrix} = \begin{pmatrix} \text{Id}_m & \text{Id}_m \\ E & -E^t \end{pmatrix}$$

(2.9)

and note that this is precisely the linear transformation diagonalising $K$, so that

$$K = R(G, B)^{-1} \begin{pmatrix} \text{Id}_m & 0 \\ 0 & -\text{Id}_m \end{pmatrix} R(G, B).$$

(2.10)

In addition it also block diagonalises $q$ as a bilinear form, i.e.,

$$q = R(G, B)^t \begin{pmatrix} 2G & 0 \\ 0 & -2G \end{pmatrix} R(G, B).$$

(2.11)

This shows again that $qK$ can be written as in (2.7).

So far we have just claimed that there exists an $N = 1$ vertex algebra for any $(\Gamma, K) \in T_{N=1}$. Let us now be slightly more specific. We will denote the underlying (super) vector space by $V_{\Gamma,K}$. This space is also called the state space. Sometimes we will abuse this notation to denote the vertex algebra. In [1] Kapustin and Orlov describe in detail the construction of the state space together with its vertex algebra structure. Here we just review some of the essentials that we will need in the sequel.
Recall that a field is an $\text{End}(V)$-valued series in $z$ and $\bar{z}$. More explicitly we have the following \textit{mode expansions} for the basic fields

\begin{align}
\partial X(z) &= \sum_{s \in \mathbb{Z}} \alpha_s z^{-s-1}, & \psi(z) &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r z^{-r-\frac{1}{2}}, \\
\bar{\partial} X(\bar{z}) &= \sum_{s \in \mathbb{Z}} \bar{\alpha}_s \bar{z}^{-s-1}, & \bar{\psi}(\bar{z}) &= \sum_{r \in \mathbb{Z} + \frac{1}{2}} \bar{\psi}_r \bar{z}^{-r-\frac{1}{2}}.
\end{align}

(2.12)

Here $\alpha_s$, $\bar{\alpha}_s$, $\psi_r$, and $\bar{\psi}_r$ are vectors of endomorphisms of the state space $V$ satisfying the following (anti)commutations relations

$$[\alpha_s^\mu, \alpha_r^\nu] = s G^\mu\nu \delta_{s,-r}, \quad \{\psi_s^\mu, \psi_r^\nu\} = G^\mu\nu \delta_{s,-r},$$

and similar relations for $\bar{\alpha}_s$ and $\bar{\psi}_r$. These commutation relations allow us to construct the state space using the standard Fock space construction by declaring the negative modes to be creation operators and the positive modes to be annihilation operators. To deal with $\alpha_0$ and $\bar{\alpha}_0$ one has to be slightly more careful. In the end we obtain the following description of the state space

$$V = V_{\Gamma, K} = \bigoplus_{(w, m) \in \Gamma \oplus \Gamma^*} V_{w, m}.$$ 

Here each of the $V_{w, m}$ is a Fock space generated from a base state $|w, m\rangle$ by acting on it with the negative modes. The \textit{sectors} $V_{w, m}$ are joint eigenspaces of $\alpha_0$ and $\bar{\alpha}_0$ with eigenvalues $G^{-1}k_L$ and $G^{-1}k_R$, where

$$\begin{pmatrix} k_L \\ k_R \end{pmatrix} = \begin{pmatrix} Gw - Bw + m \\ -Gw - Bw + m \end{pmatrix} = 2 \begin{pmatrix} G & 0 \\ 0 & -G \end{pmatrix} \mathcal{R}(G, B) \begin{pmatrix} w \\ m \end{pmatrix}.$$ 

(2.13)

The lattice vectors $w$ and $m$ labelling the sectors are often referred to as \textit{winding number} and \textit{momentum}. As a side remark, note that using (2.11) and (2.7) it follows easily that

$$\frac{1}{2}(k_L^2 - k_R^2) = \frac{1}{2}(k_L^t k_R^t) \begin{pmatrix} G^{-1} & 0 \\ 0 & -G^{-1} \end{pmatrix} \begin{pmatrix} k_L \\ k_R \end{pmatrix} = (w^t m^t) q \begin{pmatrix} w \\ m \end{pmatrix},$$

$$\frac{1}{2}(k_L^2 + k_R^2) = \frac{1}{2}(k_L^t k_R^t) \begin{pmatrix} G^{-1} & 0 \\ 0 & -G^{-1} \end{pmatrix} \begin{pmatrix} k_L \\ k_R \end{pmatrix} = (w^t m^t) q K \begin{pmatrix} w \\ m \end{pmatrix}.$$ 

These formulae are well known in physics. Let us define the \textit{charge lattice} as the lattice of all $(k_L, k_R) \in \Gamma_R^* \oplus \Gamma_R^*$ that can be written in the form (2.13) for $(w, m) \in \Gamma \oplus \Gamma^*$ (see [4]). Then the first formula above means that the charge lattice with a bilinear form given by the symmetric matrix $\text{diag}(G^{-1}, -G^{-1})$ is isomorphic to the lattice $(\Gamma \oplus \Gamma^*, q)$.

An isomorphism of vertex algebras can be described as an isomorphism of the underlying vector spaces preserving the vertex algebra structure. Kapustin and Orlov give the following classification of isomorphisms of $N = 1$ vertex algebras.

\textbf{Theorem 2 (Kapustin, Orlov).} \textit{Isomorphisms between the $N = 1$ SCFTs $V_{\Gamma, K}$ and $V_{\Gamma', K'}$ correspond 1-1 to lattice isomorphisms $g : (\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$ satisfying $K' = g K g^{-1}$.}
Let us write $f_g$ for the isomorphism of vertex algebras $V_{\Gamma,K} \to V_{\Gamma',K'}$ corresponding to a lattice isomorphism $g : (\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$. This map is defined by
\[
f_g|w,m) = |g(w,m)\rangle \tag{2.14}
\]
and
\[
\begin{align*}
(\alpha'_s f_g) &= \left( (a+bE)f_g\alpha_s \right) = M(g, E) (f_g\alpha_s), \\
(\bar{\psi}'_s f_g) &= \left( (a+bE)f_g\bar{\psi}_s \right) = M(g, E) (f_g\bar{\psi}_s). \\
(\psi'_s f_g) &= \left( (a+bE)f_g\psi_s \right) = M(g, E) (f_g\psi_s).
\end{align*} \tag{2.15}
\]
Here $a : \Gamma \to \Gamma'$ and $b : \Gamma^* \to \Gamma'$ are defined by writing the lattice isomorphism $g : (\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$ in block form as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The matrix $E = G + B$ is the background matrix introduced above and $M(g, E)$ is the block diagonal matrix $M(g, E) = \text{diag}(a+bE, a-bE^t)$. With (2.12) this leads to transformation formulae for the fields (see Table 2 on page 21, where we will collect the various transformation formulae). Note that compared with [1] we interchanged the transformation formulae for the left- and right-moving sectors. This is purely a matter of convention.

Using the precise definition of $f_g$ in [1] it is straightforward to check the following functionality property.

**Proposition 3.** Let $g_i : (\Gamma_i \oplus \Gamma_i^*, q_i) \to (\Gamma_{i+1} \oplus \Gamma_{i+1}^*, q_{i+1})$ be lattice isomorphisms, then we have the following functionality property
\[
f_{g_1} \circ f_{g_2} = f_{g_1 \circ g_2}.
\]

We can use Theorem 2 to associate a map $\mu_g$ between Teichmüller spaces to any lattice isomorphism $g$ preserving the bilinear form. This map is defined by the property that $g$ defines an isomorphism between the $N = 1$ SCFTs $V_{\Gamma, K}$ and $V_{\Gamma', \mu_g(K)}$.

**Definition 2.** Let $g : (\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$ be a lattice automorphism, then we define the map $\mu_g : \mathcal{T}_{N=1}(\Gamma) \to \mathcal{T}_{N=1}(\Gamma')$ by
\[
\mu_g(K) = gKg^{-1}.
\]

It is interesting to try and find a description of $\mu_g$ for the other descriptions of the moduli space from Proposition 3. Let us start with the description in terms of the background field $E = G + B$. As above we write the lattice isomorphism $g : \Gamma \oplus \Gamma^* \to \Gamma' \oplus \Gamma'^*$ as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a : \Gamma \to \Gamma'$, $b : \Gamma^* \to \Gamma'$ etc. Using (2.10) we find
\[
K' = gKg^{-1} = g\mathcal{R}(G, B)^{-1} \begin{pmatrix} \text{Id}_m & 0 \\ 0 & -\text{Id}_m \end{pmatrix} (g\mathcal{R}(G, B)^{-1})^{-1}. \tag{2.16}
\]
From (2.5) it follows
\[
g\mathcal{R}(G, B)^{-1} = \begin{pmatrix} a + bE & a - bE^t \\ c + dE & c - dE^t \end{pmatrix} = \begin{pmatrix} \text{Id}_m & \text{Id}_m \\ 0 & a - bE^t \end{pmatrix} \begin{pmatrix} a + bE & 0 \\ 0 & a - bE^t \end{pmatrix} \tag{2.17}
\]
and
\[
= \mathcal{R}(G', B')^{-1} M(g, E).
\]
For the second equality we defined $E' = (c + dE)(a + bE)^{-1}$. Recall that the lattice isomorphism takes $q$ to $q'$. Using the block form of $g$ the corresponding equation $g'q'g = q$ can be written as

\[
\begin{pmatrix}
  a' & c' \\
  b' & d'
\end{pmatrix}
\begin{pmatrix}
  0 & \text{Id} \\
  \text{Id} & 0
\end{pmatrix}
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} =
\begin{pmatrix}
  0 & \text{Id} \\
  \text{Id} & 0
\end{pmatrix}.
\]

This yields equations for $a$, $b$, $c$ and $d$ that can be used to show that $-E'^t = (c - dE^t)(a - bE^t)^{-1}$. This proves the second equality. The third equality is a matter of definition. If we define $G'$ and $B'$ as the symmetric and antisymmetric part of $E'$, then the first factor is equal to $R(G', B')^{-1}$. The second factor we defined as $M(g, E)$ above. Combining (2.16) and (2.17) one sees that $K'$ can again be written in the form (2.10) with $G'$ and $B'$ as defined above. This gives us an explicit transformation rule for $E$ and an indirect one for $G$ and $B$. For $G'$ one can obtain two different simplified expression starting from the expressions for $E'$ and $-E'^t$ (see Table 2 on page 21).

Because $K'$ is equal to $gKg^{-1}$, the eigenspaces of $K$ transform with $g$. So for the description of the Teichmüller space in terms of maximal positive subspaces of $\Gamma\mathbb{R} \oplus \Gamma\mathbb{R}^*$ we just use $g$ to transform the subspace.

To describe the moduli space we choose a fixed target space $X$ or equivalently a fixed lattice $\Gamma$. The duality group is given by $\mathcal{G} = O(\Gamma \oplus \Gamma^*, q)$. So using the description (2.3) of the moduli space $\mathcal{T}_{N=1}(\Gamma)$ we find the following description of the actual moduli space

\[
\mathcal{M}_{N=1}(\Gamma) = \mathcal{G}\backslash \mathcal{T}_{N=1}(\Gamma) = O(\Gamma \oplus \Gamma^*, q)\backslash O(\Gamma\mathbb{R} \oplus \Gamma\mathbb{R}^*, q)/O(m) \times O(m).
\]

For $\delta = 0$ and $m = 4$ this is exactly (2.5).

### 2.4 N = 2 SCFTs on tori

To discuss mirror symmetry we need the more refined structure of an $N = 2$ SCFT. Recall that we will use the definition of an $N = 2$ structure that involves a choice of the generators $J(z)$ and $\bar{J}(\bar{z})$. So we will study the Teichmüller space $\mathcal{T}'_{N=2}$. To define this space we slightly modify a definition due to Kapustin and Orlov.

**Definition 3.** The Teichmüller space of $N = 2$ SCFTs on a torus of real dimension $m$ is defined as

\[
\mathcal{T}'_{N=2} := \left\{ (\Gamma, I, J) \middle| \begin{array}{l}
\Gamma \text{ lattice of rank } m, I, J : \Gamma\mathbb{R} \oplus \Gamma^*_\mathbb{R} \to \Gamma\mathbb{R} \oplus \Gamma^*_\mathbb{R}, I^2 = J^2 = -\text{Id} \\
[I, J] = 0, I^t q I = q, J^t q J = q, q I J \text{ pos. def.}
\end{array} \right\}.
\]

As we will see below these data define two complex structures on the torus $X$, which implies that the real dimension $m$ of the torus has to be even for this space to be nonempty. Therefore we will often write $2n$ instead of $m$, where $n$ is the complex dimension of the torus. Comparing this definition to Definition 1 one can easily check that there exists a natural map $\pi : \mathcal{T}'_{N=2} \to \mathcal{T}_{N=1}$ defined by $K = IJ$. This is the concrete realisation of the map discussed in Section 2.1. The definition in 1 contains one extra condition, so the corresponding Teichmüller space is a subspace of the one defined here. Anticipating a
result below we will denote this subspace by \( \mathcal{T}_{\text{geom}}^{N=2} \). It can be defined in terms of \( \mathcal{T}_{\text{N=2}}' \) as follows
\[
\mathcal{T}_{\text{geom}}^{N=2} := \{(\Gamma, I, J) \in \mathcal{T}_{\text{N=2}}' \mid J(0 \oplus \Gamma^*_\mathbb{R}) \subset 0 \oplus \Gamma^*_\mathbb{R}\}. \tag{2.18}
\]
The extra condition allows us to find a nice geometrical description of this Teichmüller space.

**Proposition 4.** Geometrically the subspace \( \mathcal{T}_{\text{geom}}^{N=2} \) of \( \mathcal{T}_{\text{N=2}}' \) can be interpreted as follows
\[
\mathcal{T}_{\text{geom}}^{N=2} = \left\{(\Gamma, B, \omega, j) \mid \Gamma \text{ lattice of rank } m, B, \omega : \Gamma_\mathbb{R} \to \Gamma^*_\mathbb{R} \text{ antisymmetric,} \right. \\
\left. j : \Gamma_\mathbb{R} \to \Gamma_\mathbb{R}, j^2 = -\text{Id}, j^t \omega j = \omega, \omega j \text{ pos. def.} \right\}.
\]

**Proof.** To prove the inclusion \( \supset \) it suffices to check that \( I \) and \( J \) defined by
\[
I = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}, \\
J = \begin{pmatrix} \tilde{j} & 0 \\ Bj + j^tB & -j^t \end{pmatrix}.
\tag{2.19a}
\tag{2.19b}
\]
satisfy the conditions from Definition 3. For the other inclusion, note that the extra condition in \( (2.18) \) ensures that we can write \( J = \begin{pmatrix} \beta & 0 \\ \gamma & \alpha \end{pmatrix} \). From \( J^2 = -\text{Id} \) it follows that \( \alpha^2 = \gamma^2 = -\text{Id} \). Using \( J^t qJ = q \), we find that \( \gamma = -\alpha^t \). Because of the map \( \pi : \mathcal{T}_{\text{N=2}}' \to \mathcal{T}_{\text{N=1}} \) we know that \( K := IJ \) can be written in the form \( (2.6) \). So we can express \( I = -KJ \) in terms of \( \alpha, \beta, G \) and \( B \). From \( I^2 = -\text{Id} \) it follows that \( \beta = B\alpha + \alpha^tB \) (block in the upper right hand corner). If we now define \( \omega := -G\alpha \) and \( j := \alpha \), then using again \( I^2 = -\text{Id} \) (block in the lower right hand corner) it follows that \( j^t \omega j = \omega \). This shows that \( G = \omega j \), so \( \omega j \) is positive definite. Finally the antisymmetry of \( \omega \) follows from \( \tilde{t} qI = q \). \( \square \)

This description yields a geometrical interpretation because we can regard \( j \) as a complex structure on the torus compatible with the metric \( G \) in the sense that \( j^tGj = G \). The antisymmetric matrix \( \omega \) is the corresponding Kähler form. Combining Proposition 3 and 4 we see that the restriction of \( \pi \) to \( \mathcal{T}_{\text{geom}}^{N=2} \) is given by \( \pi = \omega j \). So the fibre of \( \pi|\mathcal{T}_{\text{geom}}^{N=2} \) over \((\Gamma, G, B)\) is just the space of all complex structures \( j \) satisfying \( j^tGj = G \). Note that \( \pi(\mathcal{T}_{\text{geom}}^{N=2}) = \mathcal{T}_{\text{N=1}} \), so the restriction is still surjective.

Let us analyse the fibres of the map \( \pi \) without imposing any extra restrictions. One can check that the fibres are given by
\[
\pi^{-1}(K) = \{(-JK, J) \in \text{End}(\Lambda_\mathbb{R})^2 \mid J^2 = -\text{Id}, [J, K] = 0, J^t qJ = q\}. \tag{2.20}
\]
Let \( E_{\pm 1} \) be the eigenspaces of \( K \) as discussed above. Because \( J \) and \( K \) commute, the endomorphism \( J \) should preserve these eigenspaces. So \( J \) can be defined by choosing complex structures on \( E_{\pm 1} \). Using the maps \( R(G, B) : \Lambda_\mathbb{R} \to \Gamma_\mathbb{R} \oplus \Gamma_\mathbb{R} \) to write \( J \) in block form we find
\[
J = R(G, B)^{-1} \begin{pmatrix} j_1 & 0 \\ 0 & j_2 \end{pmatrix} R(G, B), \tag{2.21}
\]
where $j_{1,2} \in \text{End}(\Gamma_{\mathbb{R}})$ are complex structures on $\Gamma_{\mathbb{R}}$. Using (2.11) we see that the last condition in (2.20) boils down to $j_{1,2}$ being compatible with the metric $G$. So if we define

$$S := \{ j \in \text{End}(\Gamma_{\mathbb{R}}) \mid j^2 = -\text{Id}_{\Gamma_{\mathbb{R}}}, j^* G j = G \},$$

then the fibres of $\pi$ can be described as $\pi^{-1}(K) = S \times S$. This leads to the following alternative description of the full $N = 2$ Teichmüller space.

**Proposition 5.** The full $N = 2$ Teichmüller space has the following geometrical description

$$\mathcal{T}'_{N=2} = \left\{ (\Gamma, G, B, j_1, j_2) \mid \text{\Gamma lattice of rank } m, B, G : \Gamma_{\mathbb{R}} \to \Gamma_{\mathbb{R}}^*, B = -B^t, G = G^t \right\}.$$

In this notation the subset $\mathcal{T}'_{N=2}^{\text{geom}}$ is given by the condition $j_1 = j_2$.

**Proof.** The alternative description of $\mathcal{T}'_{N=2}$ follows from the discussion above, so we only need to prove the last statement. If we write $J : \Gamma_{\mathbb{R}} \oplus \Gamma_{\mathbb{R}}^* \to \Gamma_{\mathbb{R}} \oplus \Gamma_{\mathbb{R}}^*$ in block form, the condition defining $\mathcal{T}'_{N=2}^{\text{geom}}$ corresponds to the vanishing of the block in the upper left which is a map from $\Gamma_{\mathbb{R}}^*$ to $\Gamma_{\mathbb{R}}$. Using (2.21) one can easily check that this map is given by $\frac{1}{2} (j_1 - j_2) G^{-1}$. So we see that the geometrical case is precisely when $j_1 = j_2 = j$. In hindsight this justifies the use of the notation $\mathcal{T}'_{N=2}^{\text{geom}}$ for the space defined in (2.18).

In other words fibrewise $\mathcal{T}'_{N=2}^{\text{geom}}$ is the diagonal in $\mathcal{T}'_{N=2}$. Using $j^* G j = G$ one finds that in that case $J$ is given by (2.19b). With $K$ fixed $I$ is determined by $J$. In the notation we established above we find

$$I = R(G, B)^{-1} \begin{pmatrix} -j_1 & 0 \\ 0 & j_2 \end{pmatrix} R(G, B).$$

(2.22)

Again one can check that in the geometrical case this reproduces (2.19a). Altogether these results mean that we reproduce exactly the diagram (2.3) that we found on general grounds in Section 2.1.

The underlying vector space of an $N = 2$ vertex algebra is the same as that of the corresponding $N = 1$ vertex algebra. The $N = 2$ structure is defined by a set of fields $J(z), G^\pm(z)$ and corresponding fields in the right-moving sector which generate the $N = 2$ superconformal algebra (see (2.2)). An $N = 2$ isomorphism is an $N = 1$ isomorphism that in addition preserves $J(z), G^\pm(z)$ and their right-moving counterparts. This means that *any* isomorphism $f : V \to V'$ of $N = 1$ vertex algebras that allow an $N = 2$ structure can be lifted as an $N = 2$ isomorphism by choosing an $N = 2$ structure on $V$ and then using $f$ to transport it to $V'$.

As discussed above we can generalise the notion of $N = 2$ isomorphisms by including mirror morphisms. The most general possibility is

$$f^{-1} J'(z) f = \epsilon_L J(z),$$

$$f^{-1} J'(\bar{z}) f = \epsilon_R J(\bar{z}),$$

(2.23)

where $(\epsilon_L, \epsilon_R) = (\pm 1, \pm 1)$. The map on $G^\pm(z)$ and $\bar{G}^\pm(\bar{z})$ has to satisfy the corresponding conditions. For $(\epsilon_L, \epsilon_R) = (1, 1)$ this reduces to the definition of $N = 2$ isomorphisms,
Let Proposition 7. \( \mu \) by \( \mu: G \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) a morphism of type \( (N,\Gamma) \) of the Teichmüller space and they only considered \( (\Gamma,\epsilon) \) case. Like we did in the \( (N,\Gamma) \) morphisms. However, one can check that their proof extends trivially to the more general \( (N,\Gamma) \) morphisms (see Table 1).

Theorem 6. Generalised \( N = 2 \) morphisms of type \( (\epsilon_L,\epsilon_R) = (\pm 1,\pm 1) \) between the \( N = 2 \) SCFTs \( V_{1,I,J} \) and \( V_{1,I',J'} \) correspond 1-1 to lattice isomorphisms \( g: (\Gamma \oplus \Gamma^*,q) \to (\Gamma' \oplus \Gamma'^*,q') \) such that \( I' \) and \( J' \) are given by Table 1.

| \( N = 2 \) isomorphism | \((\epsilon_L,\epsilon_R)\) | \( I'\) | \( J'\) | \( f^{-1}J'(z)f \) | \( f^{-1}J'(z)f \) |
|-------------------------|-------------------|-------|-------|----------------|----------------|
| left mirror morphism    | \((-1,1)\)        | \(gIg^{-1}\) | \(gJg^{-1}\) | \(J(z)\)       | \(\bar{J}(z)\) |
| right mirror morphism   | \((1,1)\)         | \(gJg^{-1}\) | \(gIg^{-1}\) | \(-J(z)\)      | \(-J(z)\)     |
| complex conjugation     | \((-1,-1)\)       | \(-gIg^{-1}\) | \(-gJg^{-1}\) | \(-J(z)\)      | \(-J(z)\)     |

whereas for \( (\epsilon_L,\epsilon_R) = (-1,1) \) and \( (\epsilon_L,\epsilon_R) = (1,-1) \) these conditions define left and right mirror morphisms (see Table 1).

As above we can argue that any \( N = 1 \) isomorphism \( f: V \to V' \) can be lifted to a generalised \( N = 2 \) morphism of type \( (\epsilon_L,\epsilon_R) \) by using (2.23) to define the \( N = 2 \) structure on \( V' \). In the case of tori we can classify the generalised \( N = 2 \) morphisms using lattice isomorphisms. The precise relation is given by the following generalisation of a theorem from [1].

Here we use \( V_{1,I,J} \) to denote an \( N = 2 \) SCFT. As a vector space it is equal to \( V_{1,K} \), but it comes with a choice of \( N = 2 \) generators. We will write \( f_{\epsilon_L,\epsilon_R}^{(g)} \) for the generalised \( N = 2 \) morphism of type \( (\epsilon_L,\epsilon_R) \) corresponding to the lattice isomorphism \( g: (\Gamma \oplus \Gamma^*,q) \to (\Gamma' \oplus \Gamma'^*,q') \). Kapustin and Orlov formulated this classification only for the geometrical part of the Teichmüller space and they only considered \( N = 2 \) isomorphisms and left mirror morphisms. However, one can check that their proof extends trivially to the more general case. Like we did in the \( N = 1 \) case we can define maps on the Teichmüller space for each lattice isomorphism \( g \) preserving the bilinear form.

Definition 4. Let \( g: (\Gamma \oplus \Gamma^*,q) \to (\Gamma' \oplus \Gamma'^*,q') \) be a lattice isomorphism and let \( (\epsilon_L,\epsilon_R) = (\pm 1,\pm 1) \), then we define the map \( \mu_g^{(\epsilon_L,\epsilon_R)}: \mathbb{T}_{N=2}(\Gamma) \to \mathbb{T}_{N=2}(\Gamma') \) by \( \mu_g^{(\epsilon_L,\epsilon_R)}(I,J) = (I',J') \), with \( I' \) and \( J' \) as defined in Table 1.

Again we can choose a fixed lattice \( \Gamma \) to study the duality group and the moduli space. The duality group \( G \) is still \( O(\Gamma \oplus \Gamma^*,q) \). The action of an element \( g \in G \) on \( \mathbb{T}_{N=2} \) is given by \( \mu_g^{(1,1)} \). An element \( (\epsilon_L,\epsilon_R,g) \) of the extended duality group \( G' = O(\Gamma \oplus \Gamma^*,q) \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts as \( \mu_g^{(\epsilon_L,\epsilon_R)} \). The simplest generalised \( N = 2 \) morphism are \( f_{\epsilon_L,\epsilon_R}^{(g)} \). Together with the \( N = 2 \) isomorphisms they generate the extended duality group.

The generalised \( N = 2 \) morphisms have the following functoriality property.

Proposition 7. Let \( g_i: (\Gamma_i \oplus \Gamma^*_i,q_i) \to (\Gamma_{i+1} \oplus \Gamma^*_{i+1},q_{i+1}) \) be lattice isomorphisms and let \( (\epsilon_L,i,\epsilon_R,i) = (\pm 1,\pm 1) \), then we have the following functoriality property

\[
 f_{g_2}^{(\epsilon_L,2,\epsilon_R,2)} \circ f_{g_1}^{(\epsilon_L,1,\epsilon_R,1)} = f_{g_2 \circ g_1}^{(\epsilon_L,2,\epsilon_R,2)}.
\]
This follows almost directly from Proposition 3, because a generalised $N=2$ isomorphism is just an $N=1$ isomorphism satisfying some extra conditions depending on the type $(\epsilon_L, \epsilon_R)$. Using (2.23) it is easy to check that $f_2 \circ g_1$ is of type $(\epsilon_L, 2\epsilon_R, 2\epsilon_R, 1\epsilon_R, 1\epsilon_R)$ if the $f_g$ are of type $(\epsilon_L, \epsilon_R)$. On this algebraic level this is a rather trivial statement. It becomes more significant when we consider the geometrical interpretation and the relation with D-branes.

To find out how the complex structures $j_1$ and $j_2$ transform, one can use the transformation rules for $I$ and $J$ from Table 1. Combining (2.22) and (2.21) with (2.17), one easily finds the formulae listed in Table 2 on page 21. Note that the signs for $j_1$ and $j_2$ coincide with the ones for $J(z)$ and $\bar{J}(\bar{z})$. This is natural because the fields $J(z)$ and $\bar{J}(\bar{z})$ are defined using the complex structures $j_1$ and $j_2$ respectively.

2.5 Geometrical interpretation

So far our description has been algebraic and based on a mathematical description of SCFTs. In this section we try to interpret the isomorphisms and generalised $N=2$ morphisms geometrically. As discussed above they all correspond to a lattice isomorphisms $g : (\Gamma \oplus \Gamma^*, q) \rightarrow (\Gamma' \oplus \Gamma'^*, q')$. A general geometrical interpretation is complicated, but we will discuss three classes of lattice isomorphisms that have a nice geometrical interpretation.

Coordinate transformations

The first class of lattice isomorphisms is $g = g_A = \left( \begin{smallmatrix} 1 & 0 \\ 0 & A^{-1} \end{smallmatrix} \right)$, where $A : \Gamma \rightarrow \Gamma'$. These correspond to isomorphisms of tori $\phi_A : X \rightarrow X' : x \mapsto Ax$. So they have clear geometrical interpretation. In terms of the metric and the B-field the map on Teichmüller space is given by $\mu_g(G, B) = (A^{-t}GA^{-1}, A^{-t}BA^{-1})$, which is exactly the expected transformation behaviour for a metric and a two-form.

Let us now consider the $N=2$ structure. Then for a given lattice isomorphism $g$ there are four generalised $N=2$ morphisms $f_g(\epsilon_L, \epsilon_R)$ for $(\epsilon_L, \epsilon_R) = (\pm 1, \pm 1)$. However, if we want a geometrical interpretation, the map on Teichmüller space should preserve the geometrical part of the Teichmüller space. So if we start with a point $(B, G, j_1, j_2) \in T'_{N=2}$ with $j_1 = j_2$, then we should check using the transformation rules for $j_1$ and $j_2$ from Table 3 that this condition is preserved. Because $\epsilon_L$ and $\epsilon_R$ determine the sign of $j_1$ and $j_2$ respectively, the only thing that matters is the relative sign. One can easily check that $T'_{N=2}$ is preserved when $\epsilon_L \epsilon_R = 1$, i.e., for $N=2$ isomorphisms and for complex conjugation (i.e., $(\epsilon_L, \epsilon_R) = (-1, -1))$.

Shifts in the B-field

For the second class of lattice isomorphisms we assume that $\Gamma' = \Gamma$. Then any antisymmetric map $C : \Gamma \rightarrow \Gamma^*$ defines a lattice isomorphism $g = g_C = \left( \begin{smallmatrix} \text{Id} & 0 \\ 0 & C \text{Id} \end{smallmatrix} \right)$. In this case $\mu_g(G, B) = (G, B + C)$, so this lattice isomorphism corresponds to a shift in the B-field. As above $f_g(\epsilon_L, \epsilon_R)$ preserves the geometrical part of the Teichmüller space if and only if $\epsilon_L \epsilon_R = 1$. 

\[ -16 - \]
T-duality

For the last class of lattice isomorphisms we consider lattices that are the direct sum of two lattice \( \Gamma = \Gamma_1 \oplus \Gamma_2 \). That allows us to define \( \Gamma' = \Gamma_1 \oplus \Gamma_2^* \) and the lattice isomorphism

\[
g_{\Gamma_1,\Gamma_2} = \begin{pmatrix} \text{Id}_{\Gamma_1^*} & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id}_{\Gamma_2^*} \\ 0 & 0 & \text{Id}_{\Gamma_1^*} & 0 \\ 0 & \text{Id}_{\Gamma_2} & 0 & 0 \end{pmatrix}, \tag{2.24} \]

Geometrically this means that we write the torus as a product \( X = X_1 \times X_2 \), where \( X_i = \Gamma_i \mathbb{R} / \Gamma_i \). The second torus is then given by \( X' = X_1 \times X_2^* \), where \( X_2^* = \Gamma_2 \mathbb{R} / \Gamma_2^* \). This is T-duality in the subtorus \( X_2 = \Gamma_2 \mathbb{R} / \Gamma_2^* \).

For this class of lattice isomorphisms it is more complicated to check if they preserve \( \mathcal{J}_{N=2}^{\text{geom}} \) as generalised \( N = 2 \) morphisms. Let us write \( x \) and \( y \) for coordinates on \( \Gamma_1 \) and \( \Gamma_2 \) and write \( \tilde{x} \) and \( \tilde{y} \) for the dual coordinates. To analyse what happens to \( I \) and \( J \) in this case note that \( g \) is a permutation matrix interchanging \( y \) and \( \tilde{y} \). So we can easily compute

\[
g I g^{-1} = \begin{pmatrix} I_{xx} & I_{x\tilde{x}} & I_{x\tilde{y}} & I_{xy} \\ I_{\tilde{y}x} & I_{\tilde{y}y} & I_{\tilde{y}\tilde{x}} & I_{\tilde{y}x} \\ I_{\tilde{x}x} & I_{\tilde{x}\tilde{y}} & I_{\tilde{x}\tilde{x}} & I_{\tilde{x}y} \\ I_{y\tilde{x}} & I_{y\tilde{y}} & I_{y\tilde{x}} & I_{yy} \end{pmatrix} \tag{2.25} \]

and similarly for \( g J g^{-1} \). Here we use the variable names to label the various components of linear maps. In general it is difficult to tell if a generalised \( N = 2 \) morphism corresponding to T-duality preserves \( \mathcal{J}_{N=1}^{\text{geom}} \). However, two important special cases can be analysed.

The first special case is when \( \dim X_1 = \dim X_2 \). In this case the projection \( \pi : X \to X_1 \) defines a torus fibration of the type needed for the Strominger-Yau-Zaslow conjecture describing mirror symmetry (see [10]). In their description mirror symmetry should be T-duality in \( X_2 \). We can describe exactly when \( \mu^{(\ell_1,\ell_2)}_{\Gamma_1,\Gamma_2} \) with \( \ell_1 \ell_2 = -1 \) preserves \( \mathcal{J}_{N=2}^{\text{geom}} \). Note that the condition \( \ell_1 \ell_2 = -1 \) singles out the left and right mirror morphisms.

**Proposition 8.** Let \( \Gamma = \Gamma_1 \oplus \Gamma_2 \) and \( \Gamma' = \Gamma_1 \oplus \Gamma_2^* \) with \( \text{rk} \Gamma_1 = \text{rk} \Gamma_2 \), then \( \mu^{(\ell_1,\ell_2)}_{\Gamma_1,\Gamma_2} \) with \( \ell_1 \ell_2 = -1 \) maps \( (B,\omega,j) \in \mathcal{J}_{N=2}^{\text{geom}}(\Gamma) \) to \( \mathcal{J}_{N=2}^{\text{geom}}(\Gamma') \) if and only if \( \omega | X_2 = 0 = B | X_2 \).

**Proof.** Let \( (I,J) \) correspond to \( (B,\omega,j) \in \mathcal{J}_{N=2}^{\text{geom}}(\Gamma) \) according to \( (2.19a) \) and \( (2.19b) \). Using \( \ell_1 \ell_2 = -1 \) and the expressions for \( I' \) and \( J' \) from Table [4], we find \( (I',J') = \mu^{(\ell_1,\ell_2)}_{\Gamma_1,\Gamma_2} (I,J) = \epsilon_R (gJg^{-1},gIg^{-1}) \). The condition that \( (I',J') \) be in \( \mathcal{J}_{N=2}^{\text{geom}}(\Gamma') \) is equivalent to \( J'(0 \oplus \Gamma'^* \subset 0 \oplus \Gamma^* \). Because \( \epsilon_R J' \) is given by \( (2.25) \), this condition is equivalent to

\[
(I_{xx} I_{xy} \\
I_{\tilde{y}x} I_{\tilde{y}y}) = 0.
\]

From \( I_{xx} = 0 \) it follows using \( (2.19a) \) that \( \omega_{\tilde{y}y} = 0 \). Here it is essential that \( \Gamma_1 \) and \( \Gamma_2 \) have equal rank. Combining this with \( I_{xy} = 0 \) we find \( B_{\tilde{y}y} = 0 \). It can easily be checked that these two conditions also guarantee that the remaining components vanish. \( \square \)
The condition $\omega|_{X_2} = 0$ can be interpreted geometrically as the requirement that the fibres of the SYZ-fibration be Lagrangian. The corresponding condition for the B-field does not have such a nice interpretation. This proposition also gives a nice explanation why the cases when these conditions are not fulfilled are so difficult to understand. Mirror symmetry then takes us to a non geometrical part of the Teichmüller space where many of our standard assumptions will cease to be valid. This is probably why so far all attempts to describe mirror symmetry on higher dimensional tori contain some assumptions that exclude those cases.

A remark is in order about what we call mirror symmetry. Above we associated left and right mirror morphisms to any lattice isomorphism preserving the bilinear form. This seems to be at odds with the SYZ-conjecture, which states that mirror symmetry corresponds to T-duality in the fibres of a very special torus fibration. A full reconciliation of these two points of view will probably require a careful analysis of the arguments in \cite{10} leading to the SYZ-conjecture. The key point seems to be that mirror morphisms generally do not preserve $\mathcal{T}_{N=2}^{geom}$. The above proposition shows that for the class of mirror morphisms corresponding to the SYZ-conjecture there is at least a big subspace of $\mathcal{T}_{N=2}^{geom}(\Gamma)$ that is mapped to $\mathcal{T}_{N=2}^{geom}(\Gamma')$. The argument in \cite{10} depends on the standard geometrical interpretation of D-branes which is only possible in the geometrical part of the Teichmüller space. That might restrict the class of mirror morphism to which the argument applies to fibrewise T-dualities as predicted by the SYZ-conjecture.

It is possible to define new Teichmüller spaces that are invariant under a fixed $g = g_{\Gamma_1, \Gamma_2}$.

$$\mathcal{T}_{N=2}^{SYZ} := \{(B, \omega, j) \in \mathcal{M}_{N=2}^{geom} \mid B_{\bar{g}y} = \omega_{\bar{g}y} = 0\}$$

$$\mathcal{T}_{N=1}^{SYZ} := \{(B, G) \in \mathcal{M}_{N=1}^{geom} \mid B_{\bar{g}y} = 0\}$$

Note that $\mathcal{T}_{N=2}^{SYZ}$ depends on the choice of the SYZ-fibration. All these spaces can be put together in the following diagram

This diagram is an extension of the one in \cite{2.3}. Recall that the Betti numbers of a torus $X$ of complex dimension $n$ are given by $h^{p,q}(X) = \binom{n}{p} \binom{n}{q}$. Then it is easy to check that the dimensions agree with the general formulae from \cite{2.3}. One can also check that this diagram commutes. It is tempting to restrict to $\mathcal{T}_{N=2}^{SYZ}$ and $\mathcal{T}_{N=1}^{SYZ}$, because the geometrical interpretation is clearer. However, to obtain a completely general description it is necessary to work with the most general Teichmüller spaces.

The second case that can be analysed is when $\Gamma_2 = \Gamma$. This corresponds to a Fourier-Mukai transform on the whole torus. In this case we want to see when $f_g^{(\epsilon_L, \epsilon_R)}$ with $\epsilon_L \epsilon_R = 1$
preserve $\mathcal{T}_{N=2}^{\text{geom}}$. So suppose that $j_1 = j_2 = j$, then the condition $j'_1 = j'_2$ is equivalent to

$$EjE^{-1} = E'jE'^{-1}.$$ 

Note that $G$ is Kähler so $Gj = -j^tG$. If we suppose that $B$ is a $(1,1)$-form, then we also have $Bj = -j^tB$, so $Ej = -j^tE$ and $E'j = -j^tE'$. This suffices to show that the above condition for $j'_1 = j'_2$ is fulfilled. It is not clear to me if this condition is also necessary.

On this level one can easily check that for a fixed decomposition $\Gamma = \Gamma_1 \oplus \Gamma_2$, doing a fibrewise T-duality in the $\Gamma_1$-directions and then in the $\Gamma_2$-directions (or the other way around) yields full Fourier-Mukai transform. If one starts with a SYZ-decomposition and lifts both fibrewise T-dualities as mirror morphisms, then this is exactly the situation of the conjecture discussed in Section 4.5.4 of [11].

3. D-branes

3.1 Gluing matrices

In [2] the authors analyse in detail the geometrical interpretation of D-branes. In the terminology developed above their analysis is restricted to $\mathcal{T}_{N=2}^{\text{geom}}$. Let us try to extend their description of D-branes to one that is valid over the entire Teichmüller space. Of course, it is likely to be difficult to find a completely geometrical interpretation outside $\mathcal{T}_{N=2}^{\text{geom}}$. The authors of [2] used a gluing field $R$ to describe boundary conditions (see also [12, 13, 14]). We will use the same description. In general $R$ is a field, but for a torus we can use a constant matrix, which we will call the gluing matrix. As we will discuss below, this amounts to restricting to affine D-brane. On the boundary of the worldsheet one imposes the following conditions

$$\partial X(z)|_{\sigma=0,\pi} = R\overline{\partial X}(\bar{z})|_{\sigma=0,\pi}, \quad (3.1a)$$

$$\psi(z)|_{\sigma=0,\pi} = \eta R\overline{\psi}(\bar{z})|_{\sigma=0,\pi}. \quad (3.1b)$$

Here we introduced a parameter $\eta$, which can have the values $\pm 1$. Because GSO-invariant states are linear combinations of states for both values of $\eta$ (see pages 15–16 in [15]), this parameter is necessary in the general theory. However, in this paper it does not play an important role and we will just carry it along. In the expression above we used the usual complex coordinate $z = \exp(\tau + i\sigma)$ of $z$ in terms of $\tau$ and $\sigma$. This means that the strip $\{(\tau,\sigma) \mid 0 < \sigma < \pi\}$ is identified with the upper half plane. In [16, 17] more general boundary conditions are discussed containing extra terms in the equations (3.1a) for the bosonic fields. However, in our case their boundary conditions reduce to (3.1), because $R$ is constant and we use flat coordinates on the torus.

Using the formulae from Table 2 one can easily check that after an isomorphism of $N = 1$ SCFTs corresponding to a lattice isomorphism $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the boundary conditions for the new fields are given by (3.1) with $R$ replaced by

$$R' = (a + bE)R(a - bE^t)^{-1}.$$
In [2] the authors give a different formula for the special case where \( g \) corresponds to a T-duality transformation. They identify the lattices \( \Gamma, \Gamma^*, \Gamma', \) and \( \Gamma'' \) and assume implicitly that the background field \( E \) is simply \( \text{Id} \). They claim that a T-duality transformation is then given by a symmetric matrix \( T \) satisfying \( T^2 = \text{Id} \). If \( H \subset \Gamma_\mathbb{R} \) is the subspace that is dualised, then we can define \( T \) by the requirements \( T|_{H} = - \text{Id}_H \) and \( T|_{H^\perp} = \text{Id}_{H^\perp} \). The claim of [2] is that in terms of \( T \) the transformation of the gluing matrix is given by \( R' = RT \). To compare to our description, we have to find the lattice isomorphism \( g \) corresponding to this T-duality transformation. Generalising (2.24), we find \( g = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \), where \( a = \frac{1}{2}(\text{Id} + T) \) and \( b = \frac{1}{2}(\text{Id} - T) \). With \( E = \text{Id} \) it then follows easily that both formulae agree.

The matrix \( R \) should satisfy several conditions in order to define a boundary condition for an \( N = 1 \) SCFT. In [17], three conditions are listed, which we will discuss in turn. The mathematical formulation of the first condition is rather natural.

**Condition 1 (orthogonality).** The matrix \( R \) should be orthogonal with respect to the metric \( G \), i.e., \( R^t G R = G \).

Physically this condition corresponds to the requirement \( L(z) = L(\bar{z}) \) on the boundary. As was argued in [2] the \(-1\) eigenspace of \( R \) should correspond to directions orthogonal to the D-brane. So let \( V \subset \Gamma_\mathbb{R} \) be the \(-1\) eigenspace of \( R \) and let \( W \) be its orthogonal complement with respect to the metric \( G \). Using \( V \) and \( W \) we can define \( V^* := G(V) \) and \( W^* := G(W) \). Note that \( \Gamma^*_\mathbb{R} = V^* \oplus W^* \). Equivalently, we can use the canonical pairing \( \langle \cdot, \cdot \rangle : \Gamma_\mathbb{R} \otimes \Gamma^*_\mathbb{R} \rightarrow \mathbb{R} \) and define \( V^* = W^\perp := \{ u \in \Gamma^* \mid \forall w \in W : \langle w, u \rangle = 0 \} \) and \( W^* = V^\perp \).

With respect to the direct sum decomposition \( \Gamma_\mathbb{R} = V \oplus W \), the matrix \( R \) has the block diagonal form

\[
R = \begin{pmatrix} -\text{Id} & 0 \\ 0 & \tilde{R} \end{pmatrix}.
\]

By definition the map \( \tilde{R} : W \rightarrow W \) does not have \(-1\) as an eigenvalue. It follows that there exists a unique map \( \mathcal{F} : V \rightarrow W^* \) such that \( \tilde{R} = (\tilde{G} - \mathcal{F})^{-1}(\tilde{G} + \mathcal{F}) \), where \( \tilde{G} : W \rightarrow W^* \) is the restriction of \( G \) to \( W \). In fact one can easily compute that \( \mathcal{F} \) is given by \( \mathcal{F} = \tilde{G}(\tilde{R} - \text{Id})(\tilde{R} + \text{Id})^{-1} \). We also know that \( \tilde{R} \) is orthogonal with respect to \( \tilde{G} \), which can be written as \( \tilde{G}^{-1} = \tilde{R}^{-1} \tilde{G}^{-1} \tilde{R}^{-t} \). Using this one can easily show that \( \mathcal{F} \) is antisymmetric. Here we use the decompositions \( \Gamma_\mathbb{R} = V \oplus W \) and \( \Gamma^*_\mathbb{R} = V^* \oplus W^* \) to identify the transpose of a map \( W \rightarrow W^* \) with a map \( W \rightarrow W^* \) again.

Geometrically \( W \) can be regarded as a subspace of the tangent space to \( X = \Gamma_\mathbb{R}/\Gamma \). The submanifold on which the D-brane lives should have \( W \) as its tangent space. In general we need an integrability condition to ensure that there exists at least locally a submanifold \( S \subset X \) such that its tangent space is \( W \). This is the second condition from [17]. In the present case \( W \) is constant and the integrability condition is automatically fulfilled. However, this condition only guarantees the local existence of the submanifold \( S \). For the torus we can do better by imposing the following condition which ensures the global existence of \( S \) as a submanifold of \( X \).
Table 2: Transformation rules

| Background data | Fields |
|-----------------|--------|
| $N = 1$         | $E' = (c + dE)(a + bE)^{-1}$ | $f_g^{-1} \partial X' f_g = (a + bE) \partial X$ |
|                 | $-E' = (c - dE')(a - bE')^{-1}$ | $f_g^{-1} \partial X' f_g = (a - bE') \partial X$ |
|                 | $G' = (a + bE)^{-1}G(a + bE)^{-1}$ | $f_g^{-1} \psi' f_g = (a + bE) \psi$ |
|                 | $= (a - bE')^{-1}G(a - bE')^{-1}$ | $f_g^{-1} \psi' f_g = (a - bE') \bar{\psi}$ |
| $N = 2$         | $j_1' = \epsilon_L(a + bE)j_1(a + bE)^{-1}$ | |
|                 | $j_2' = \epsilon_R(a - bE')j_2(a - bE')^{-1}$ | |
|                 | $\omega_1' = \epsilon_L(a + bE)^{-1}\omega_1(a + bE)^{-1}$ | |
|                 | $\omega_2' = \epsilon_R(a - bE')^{-1}\omega_2(a - bE')^{-1}$ | |
| D-branes        | $R' = (a + bE)R(a - bE')^{-1}$ | |

**Condition 2 (rationality).** Let $W$ be the orthogonal complement with respect to $G$ of the $-1$ eigenspace of $R$, then $W \cap \Gamma$ should have rank equal to the real dimension of $W$.

If this condition is not fulfilled, the integral manifold will not be compact. More precisely, it will be dense in a submanifold of $X$ with dimension higher than the dimension of $W$. For the 2-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$ this is familiar and corresponds to lines with irrational slope which are dense in $\mathbb{R}^2/\mathbb{Z}^2$. Note that because $V^* = W^\perp$, this condition is equivalent to the condition that $V^* \cap \Gamma^*$ have rank equal to the real dimension of $V^*$.

The third condition allows us to interpret $\bar{R}$ as defined above in more detail. This requires a slight modification of the condition as formulated in [17] to allow for D-branes with a nontrivial field strength. The condition from [17] can be written as $\bar{R} = \bar{E}^{-1} \bar{E} = (\bar{G} + \bar{B})^{-1}(\bar{G} - \bar{B})$. Here $\bar{B} : W \rightarrow W^*$ is obtained from $B$ by restricting $B$ to $W$ and then projecting to $W^*$. It can be checked using $V^* = W^\perp$ that $\langle w, \bar{B}w' \rangle = \langle w, Bw' \rangle$ for all $w, w' \in W$. In this sense $\bar{B}$ does define the restriction of $B$ to $W$ as a bilinear form.

When the field strength $F$ of the connection on the D-brane is nonzero, the condition discussed above has to be modified. To do so we define $\mathcal{F} := F - \bar{B}$ and replace $\bar{B}$ by $-\mathcal{F}$. For line bundles on a torus the field strength has to be integral as a bilinear form on the lattice defining the torus. As will be discussed below this condition can be relaxed to the requirement that $F$ be rational. This leads to the second rationality condition.

**Condition 3 (rationality).** If we write $R$ as in (2.2), then there exists an antisymmetric matrix $\mathcal{F}$ such that $\bar{R} = (\bar{G} - \mathcal{F})^{-1}(\bar{G} + \mathcal{F})$. As an antisymmetric bilinear form $F := \mathcal{F} + \bar{B}$ should map $W \cap \Gamma \otimes W \cap \Gamma$ to $\mathbb{Q}$.

The condition in [17] amounts to requirement that the field strength $F$ vanish, so our condition certainly is a generalisation of that condition.

The logical next step would be to check that these conditions are well behaved under $N = 1$ isomorphisms. The orthogonality condition is easy to check. From the transformation rules in Table 2 it is immediate that $R'^t G'R' = G'$. The rationality conditions are more
complicated, because the decomposition $\Gamma_R = V \oplus W$ changes under an isomorphism of $N = 1$ SCFTs. It can be shown in examples that the two rationality conditions mix in the sense that one of the rationality conditions after an isomorphism may depend on both rationality conditions before the isomorphism. Therefore we will describe in the next section an alternative description of D-branes which replaces the two separate rationality conditions with a single rationality condition. This will enable us to show that isomorphisms also preserve the rationality conditions.

### 3.2 Boundary states

An important way to describe D-branes is using boundary states. These are states in an extension of the closed CFT Hilbert space such that expectation values of bulk operators $\phi_1, \ldots, \phi_k$ are given by

$$\langle \phi_1 \ldots \phi_k \rangle_\alpha = \langle \phi_1 \ldots \phi_k \| \alpha \rangle,$$

where we use $\alpha$ to label the D-brane. These states are linear combinations of the so-called Ishibashi states. For the torus we can denote the Ishibashi states as $|R, w, m\rangle$ using the gluing matrix $R$ and $(w, m) \in \Gamma \oplus \Gamma^*$ to label them. The defining condition for the Ishibashi states is

$$\left(\alpha_\ell + R\bar{\alpha}_{-\ell}\right)|R, w, m\rangle = 0.$$  \hspace{1cm} (3.3)

This equation corresponds to (1.1b) after translation from the upper half plane to the complement of the unit disk (see [12]). For a torus we can write down explicit solutions of these equations

$$|R, w, m\rangle = \exp\left(-\sum_{\ell=1}^{\infty} \frac{1}{\ell} G(\alpha_\ell, R\bar{\alpha}_{-\ell})\right)|w, m\rangle.$$ \hspace{1cm} (3.4)

These solutions are straightforward generalisations of the solutions for the 1-dimensional case (see e.g., [13, 12]). Using the commutation relations for the $\alpha_\ell^\mu$ and the $\bar{\alpha}_{-\ell}^\mu$ we can check that these states satisfy (3.3) for $\ell \neq 0$. To check that condition for $\ell = 0$, recall that $V_{w,m}$ are eigenspaces of $\alpha_0$ and $\bar{\alpha}_0$. The eigenvalues of $\alpha_0$ and $\bar{\alpha}_0$ are $G^{-1}k_L$ and $G^{-1}k_R$ respectively, where $k_L$ and $k_R$ have been defined in terms of $(w, m) \in \Gamma \oplus \Gamma^*$ in (2.13). Substituting this into the requirement (3.3) for $\ell = 0$ we find

$$G^{-1}(Gw - Bw + m) = -RG^{-1}(-Gw - Bw + m).$$ \hspace{1cm} (3.5)

This condition defines a lattice $L_{R,K} \subset \Gamma \oplus \Gamma^*$ depending on the gluing matrix $R$ and the background fields $G$ and $B$. For a fixed background this lattice turns out to be equivalent to the gluing matrix $R$ as we see in the following theorem.

**Theorem 9.** Let $G$ and $B$ be fixed, then a gluing matrix $R$ satisfying the conditions [3] and [4] is equivalent to a sublattice $L \subset \Gamma \oplus \Gamma^*$ satisfying the conditions

$$q|_L = 0,$$

$$\text{rk } L = \text{rk } \Gamma.$$
Proof. Given a gluing matrix $R$ satisfying the conditions \[ \[1\] \text{ and } \[2\] \text{ we can take } L \text{ to be the sublattice } L_{R,K} \subset \Gamma \oplus \Gamma^* \text{ defined by } \{1,2\}. \text{ Using } \Gamma_R = V \oplus W \text{ we can rewrite this equation as two equations, one for the } V\text{-component and one for the } W\text{-component. These two equations can be simplified further using } \{2\} \text{ and } \tilde{R} = (\tilde{G} - \mathcal{F})^{-1}(\tilde{G} + \mathcal{F}). \text{ After some rewriting we find }

\begin{align*}
  w_V &= 0, \\
  (m - Bw)_W &= \mathcal{F}w_W. 
\end{align*}

(3.6)

Here the subscripts $V, W, V^*$, and $W^*$ label the components and we also used $\Gamma_R^* = V^* \oplus W^*$. The first equation implies that $w \in W$, so we can drop that equation and use the second equation to describe $L$ as a sublattice of $(W \cap \Gamma) \oplus \Gamma^*$. The fact that $w \in W$ allows us to omit the subscript $W$ and to replace $(Bw)_W$ by $\tilde{B}w$. If we also use $\mathcal{F} = F - \tilde{B}$, we are left with the following equation defining $L$ as a sublattice of $(W \cap \Gamma) \oplus \Gamma^*$

\begin{align*}
  m_{W^*} &= Fw. 
\end{align*}

(3.7)

This vector equation stands for $k$ independent equations. More explicitly, we can choose an integral basis $\{e_i\}$ for $W \cap \Gamma$. Then for every $i = 1, \ldots, k$ we obtain an equation

\begin{align*}
  \langle e_i, m_{W^*} \rangle &= \langle e_i, m \rangle = \langle e_i, Fw \rangle.
\end{align*}

Because of the second rationality condition \[3\], these equations are linear with rational coefficients. So the lattice $L$ is a codimension $k$ sublattice in the $(k + \text{rk } \Gamma)$-dimensional lattice $(W \cap \Gamma) \oplus \Gamma^*$. Therefore the rank of the lattice $L_{R,K}$ is indeed equal to the rank of $\Gamma$.

To see that the restriction of $q$ to $L$ vanishes we compute

\begin{align*}
  q\left(\begin{pmatrix} w \\ m \end{pmatrix}, \begin{pmatrix} w' \\ m' \end{pmatrix}\right) &= \langle w, m' \rangle + \langle w', m \rangle = \langle w_V, m_{W^*} \rangle + \langle w_V', m_{W^*} \rangle + \langle w'_V, m_{V^*} \rangle + \langle w'_V, m_{W^*} \rangle \\
  &= \langle w_W, Fw_W \rangle + \langle w'_W, Fw_W \rangle = 0.
\end{align*}

Here we used the antisymmetry of $F$ in the final step. This shows that we can associate a sublattice $L$ satisfying the conditions from the statement of the theorem to any gluing matrix $R$ satisfying the conditions \[1\] \[2\] and \[3\].

Conversely, if we start with such a sublattice $L$, then we find a sublattice of $\Gamma$ by projecting $L \subset \Gamma \oplus \Gamma^*$ to $\Gamma$. Tensoring this sublattice with $\mathbb{R}$ yields a subspace $W$ satisfying $\text{rk}(W \cap \Gamma) = \dim_{\mathbb{R}} W$. Using $G$ we can define $V$ as the orthogonal complement of $W$, and $V^*$ and $W^*$ as $G(V)$ and $G(W)$ respectively. As above, we can express $q$ in terms of the components. Let $(w, m)$ and $(w', m')$ be elements of the lattice $L$. Because $w_V$ and $w'_V$ vanish, we find

\begin{align*}
  q\left(\begin{pmatrix} w \\ m \end{pmatrix}, \begin{pmatrix} w' \\ m' \end{pmatrix}\right) &= \langle w_W, m_{W^*} \rangle + \langle w'_W, m_{W^*} \rangle.
\end{align*}

As $q|_L$ vanishes, this implies $\langle w'_W, m_{W^*} \rangle = -\langle w_W, m_{W^*} \rangle$. We can choose a basis $e_i \in W \cap \Gamma$ of $W$, such that there exist $f_i \in \Gamma^*$ satisfying $\langle e_i, f_i \rangle \in L$. Substituting $(e_i, f_i)$ for $(w', m')$, we find $\langle e_i, m_{W^*} \rangle = -\langle w_W, f_i, W^* \rangle$. It follows that $m_{W^*}$ is determined by $w_W$. Because $L$ is a sublattice, we must have $m_{W^*} = Fw_W$ for some linear map $F : W \to W^*$. The
map $F$ has to be antisymmetric for $q|_{L}$ to vanish. The fact that $L$ is a lattice of rank $\text{rk} \Gamma$ ensures that $F$ is rational. Using $F$ we can define $\mathcal{F}$ as $F - \hat{B}$ and $\hat{R}$ as $(\hat{G} - \mathcal{F})^{-1}(\hat{G} + \mathcal{F})$. Finally, the gluing matrix $R$ can be defined as the block matrix \((3.2)\) with respect to $\Gamma_R = V \oplus W$.

So instead of using a gluing matrix $R$ to describe a D-brane, we might as well use the corresponding lattice $L = L_{R,K}$. The rationality conditions on $R$ correspond to $\text{rk} \ L = \text{rk} \ \Gamma$. Note that this is the maximal rank for a lattice $L$ satisfying $q|_{L} = 0$. This is a much more manageable condition when checking the behaviour under $N = 1$ isomorphisms. However, before we can do that, we have to determine how the lattice $L$ transforms under $N = 1$ isomorphisms.

**Proposition 10.** Let $L_{R,K}$ be the sublattice from Theorem 9 corresponding to a gluing matrix $R$ and let $g : (\Gamma \oplus \Gamma^*, q) \rightarrow (\Gamma' \oplus \Gamma'^*, q')$ be a lattice isomorphism, then the sublattice $L_{R',K'}$ corresponding to transformed gluing matrix $R'$ is given by

$$L_{R',K'} = g(L_{R,K}).$$

**Proof.** The condition \((3.5)\) defining $L_{R,K}$ can be written as

$$\begin{pmatrix} G^{-1} & 0 \\ 0 & -RG^{-1} \end{pmatrix} \begin{pmatrix} E' & \text{Id} \\ -E & \text{Id} \end{pmatrix} \begin{pmatrix} w \\ m \end{pmatrix} = \begin{pmatrix} \text{Id} & 0 \\ 0 & R \end{pmatrix} \mathcal{R}(G,B) \begin{pmatrix} w \\ m \end{pmatrix} \in \Delta,$$

where $\Delta$ is the diagonal in $\Gamma_R \oplus \Gamma^*_R$. In other words the lattice $L_{R,K}$ can be defined as

$$L_{R,K} = \mathcal{R}(G,B)^{-1} \begin{pmatrix} \text{Id} & 0 \\ 0 & R^{-1} \end{pmatrix} \Delta.$$

Applying the lattice isomorphism $g$ to both sides, we obtain

$$g(L_{R,K}) = g\mathcal{R}(G,B)^{-1} \begin{pmatrix} \text{Id} & 0 \\ 0 & R^{-1} \end{pmatrix} \Delta = \mathcal{R}(G',B')^{-1} \begin{pmatrix} (a + bE) & 0 \\ 0 & (a - bE')R^{-1} \end{pmatrix} \Delta$$

$$= \mathcal{R}(G',B')^{-1} \begin{pmatrix} \text{Id} & 0 \\ 0 & R'^{-1} \end{pmatrix} \Delta = L_{R',K'}.$$

The second equality is based on \((2.17)\). For the next step we used the definition of $R'$ and the fact that $\text{diag}(a + bE, a + bE)$ leaves $\Delta$ invariant.

It is clear that this transformation of $L$ preserves the conditions from Theorem 9 defining the subspace $L$. It follows that $N = 1$ isomorphisms must also preserve the conditions defining a gluing matrix $R$. Note that although we motivated the introduction of the sublattice $L$ by studying Ishibashi states, this theorem is independent of that and is mathematically rigorous.

The Ishibashi state defined above is for a purely bosonic theory. In a supersymmetric theory there are additional requirements related to the fermions

$$(\psi_r + i\eta R\tilde{\psi}_{-r})|R, w, m, \eta\rangle\rangle.$$

\[(3.8)\]
Here we use again the parameter \( \eta = \pm 1 \) introduced above. As in the bosonic case, this equation can be obtained from the corresponding boundary condition (3.8) on the upper half plane. Also in this case we can write down explicit solutions

\[
|R, \eta, w, m\rangle = \exp\left(-\sum_{\ell=1}^{\infty} \frac{1}{\ell} G(\alpha_{-\ell}, R\alpha_{-\ell}) - i\eta \sum_{\ell=0}^{\infty} G(\psi_{-\ell}, R\tilde{\psi}_{-\ell})\right) |w, m\rangle.
\]

However, in the sequel we will continue to use the bosonic expressions, because they already contain all the information we need. To do the supersymmetric case properly we would have to go into things like the GSO-projection, which would lead us too far away (see [18, 17]).

Every boundary state has to satisfy the gluing conditions (3.3) (and (3.8) in the fermionic case) and can therefore be written as an infinite linear combination of Ishibashi states. There are additional conditions that boundary states have to satisfy, namely the Cardy condition and the sewing conditions. However, it is unclear if all these conditions together are sufficient. Therefore, we will refrain from studying these extra conditions in detail and instead write down explicit expressions for boundary states as linear combinations of Ishibashi states. It seems quite plausible that these expressions actually define boundary states, but the evidence we have is quite scarce. Firstly, the expressions are straightforward generalisations of well known expressions from the literature. Secondly, they lead to a description of D-branes that agrees quite well with the expectations based on the geometrical description of D-branes.

Let us now, as promised, return to the bosonic theory and write down an explicit expression for a boundary state as a linear combination of Ishibashi states

\[
\|R, \xi\rangle = \sum_{(w,m) \in L} e^{-2\pi i q(w,m,\xi)} |R, w, m\rangle.
\]

(3.9)

Here we introduce some extra data in the form of a vector \( \xi \in \Gamma_R \oplus \Gamma_R^* \). First note that because \( q|L \) vanishes, we can regard \( \xi \) as an element of \( \Gamma_R \oplus \Gamma_R^*/L_R \). A second remark is that shifts of \( \xi \) by elements of \( \Gamma \oplus \Gamma_R^* \) do not affect the boundary state \( \|R, \xi\rangle \). So we should consider \( \xi = (\xi_X, \xi_X^*) \) as an element of the torus \( X \times X^*/(L_R/L) \). Let us now investigate the geometrical significance of \( \xi \). The simplest case is a 0-brane \( (R = -1) \). In that case \( L = \Gamma^* \), so we can take \( \xi = (\xi_X, 0) \). We can interpret \( \xi_X \in X \) as the position of the 0-brane. To justify this interpretation note that for any \( k \in \Gamma^* \)

\[
:e^{2\pi i (k, X)}: \|R, \xi\rangle = \sum_{m \in \Gamma^*} e^{-2\pi i (m, \xi)} :e^{2\pi i (k, X)}: |R, 0, m + k\rangle = \sum_{m \in \Gamma^*} e^{-2\pi i (m, \xi)} |R, 0, m + k\rangle
\]

\[
= e^{2\pi i (k, \xi_X)} \|R, \xi\rangle.
\]

So \( \|R, \xi\rangle \) is an eigenvector of \( :e^{2\pi i (k, X)}: \) with eigenvalue \( e^{2\pi i (k, \xi_X)} \). For arbitrary gluing matrices we can obtain a similar expression, but the interpretation is not completely clear. Let us write the bosonic field \( X(z, \bar{z}) = X_L(z) + X_R(\bar{z}) \). The vertex operator corresponding to \( |w, m\rangle \) can then be written in a slightly imprecise notation as \( :e^{2\pi i (k_L, X_L(z)) + (k_R, X_R(\bar{z}))}: \); where \( k_L \) and \( k_R \) are defined in terms of \( (w, m) \in L \) (see (2.13)). For \( (w, m) \in L \) a computation analogous to the one above shows

\[
:e^{2\pi i ((k_L, X_L(z)) + (k_R, X_R(\bar{z})))}: \|R, \xi\rangle = e^{2\pi i q((w, m), \xi)} \|R, \xi\rangle.
\]
Defining \( \tilde{X}(z, \bar{z}) = X_L(z) - X_R(\bar{z}) \), we can rewrite the operator in the exponent as follows

\[
\langle k_L, X_L(z) \rangle + \langle k_R, X_R(\bar{z}) \rangle = q \left( \begin{pmatrix} w \\ m \end{pmatrix}, \begin{pmatrix} X(z, \bar{z}) \\ G\tilde{X}(z, \bar{z}) + BX(z, \bar{z}) \end{pmatrix} \right).
\]

So we can interpret the equation above as the statement that the boundary state \( \| R, \xi \| \) is a joint eigenvector of the vector of operators \( (X(z, \bar{z}), G\tilde{X}(z, \bar{z}) + BX(z, \bar{z})) \) with eigenvalues \( \xi \) up to elements of \( L_\mathbb{R} \) and up to elements of the lattice \( \Gamma \oplus \Gamma^* \). This seems to fit the geometrical interpretation of next section quite well, but we do not have such a clear interpretation as for the case \( R = -\text{Id} \).

As noted by Gaberdiel in \([15]\) we could generalise these boundary states by taking \( \xi \) in \( \Gamma_\mathbb{C} \oplus \Gamma_\mathbb{C}^* \). This may cause problems with unitarity. A different generalisation is to introduce Chan-Paton factors. In modern terminology Chan-Paton factors correspond to multiple D-branes stacked on top of each other. For \( r \) stacked D-branes Chan-Paton factors can be implemented by replacing the state space \( V \) by \( V \otimes M_r(\mathbb{C}) \). The fields also take values in \( M_r(\mathbb{C}) \) and the expectation values are changed to include a trace

\[
\langle \phi_1 \ldots \phi_k \rangle := \text{Tr}(\langle \phi_1 \ldots \phi_k \rangle) = \sum_{i_1, \ldots, i_k=1}^{r} \langle \phi_{i_1, i_2} \ldots \phi_{k, i_k} \rangle.
\]

Because the state space is tensored with \( M_r(\mathbb{C}) \), the same should be true for the boundary states. Such boundary states can easily be constructed by replacing \( \xi \) in \((3.9)\) by a matrix valued vector \( \Xi \in (\Gamma_\mathbb{R} \oplus \Gamma_\mathbb{R}^*) \otimes M_r(\mathbb{C}) \). The leads to the following more general definition of boundary states

\[
\| R, \Xi \| = \sum_{(w,m) \in L} e^{-2\pi i q((w,m),\Xi)} |R, w, m\rangle \langle R, w, m|.
\]

We will impose two restrictions on \( \Xi \in (\Gamma \oplus \Gamma^*) \otimes M_r(\mathbb{C}) \). The components \( \Xi_i \in M_k(\mathbb{C}) \) of \( \Xi \) \((i = 1, \ldots, 2m)\) should commute and all their eigenvalues should be real. In addition we have to identify \( \Xi \)'s that lead to the same boundary state. This leads to the following definition of the space of Chan-Paton matrices.

**Definition 5.** Let \( K \) be in \( \mathcal{J}_{N=1}(\Gamma) \) and let \( R \) be a gluing matrix, then the space of rank \( r \) Chan-Paton matrices is defined as

\[
\text{CP}(K, R, r) := \left\{ \Xi \in (\Gamma_\mathbb{R} \oplus \Gamma_\mathbb{R}^*) \otimes M_r(\mathbb{C}) \mid \text{all } \Xi_i \text{ (}i = 1, \ldots, 2\text{rk } \Gamma\text{) commute and have real eigenvalues} \right\} / \sim,
\]

where \( \sim \) is the equivalence relation defined by

\[
\Xi \sim \Xi' \iff \exists \xi \in \Gamma \oplus \Gamma^* \forall (w, m) \in L_{R,K} : q((w,m),\Xi'-\Xi) = q((w,m),\xi) \text{ Id}_r.
\]

Using the definition of a boundary state it is easy to check that \( \| R, \Xi \| = \| R, \Xi' \| \) when \( \Xi \sim \Xi' \). This equivalence relation also has the following more explicit description

\[
\Xi \sim \Xi' \iff \exists \xi \in \Gamma \oplus \Gamma^* \exists Y \in L_{R,K} \otimes M_r(\mathbb{C}) : \Xi' = \Xi + Y + \xi \text{ Id}_r.
\]

For \( r = 1 \) this space coincides with the space \( X \times X^*/((L_{R,K} \otimes \mathbb{R})/L_{R,K}) \) we found before.
When the D-brane is a vector bundle $E$ (i.e., the underlying submanifold is all of $X$), then the introduction of Chan-Paton matrices should correspond to replacing $E$ by $E \otimes F$, where $F$ is a flat vector bundle of rank $r$ with monodromies given by $\Xi_i$ ($i = m + 1, \ldots, 2m$). We will discuss this in more detail in the next section.

Combining the gluing matrix with the Chan-Paton matrices leads to the following definition of the set of D-branes.

**Definition 6.** The set of rank $r$ affine $N = 1$ D-branes on a torus $X = \Gamma_\mathbb{R}/\Gamma$ corresponding to a point $K \in \mathcal{T}_{N=1}$ in the Teichmüller space is defined as follows

$$C_{r}^{0}(X) := \{(R, \Xi) \mid R : \Gamma_\mathbb{R} \to \Gamma_\mathbb{R} \text{ satisfies conditions 1, 2, and 3}, \Xi \in \mathbb{C} \mathbb{P}(K, R, r)\}.$$

The set $C_{r}^{0}(X)$ of all D-branes is the union $\bigcup_{r > 0} C_{r}^{0}(X)$.

The transformation behaviour of these boundary states under $N = 1$ isomorphisms is easy to determine.

**Theorem 11.** An isomorphism $f_g$ of $N = 1$ SCFTs corresponding to a lattice isomorphism $g : (\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$ induces a bijective map $\phi_g : C_{K}(X) \to C_{K'}(X')$ defined by

$$\phi_g(R, \Xi) = (R', g\Xi),$$

with $R'$ as in Table 3. This map is compatible with $f_g$ in the sense that

$$\|R', g\Xi\| = f_g\|R, \Xi\|.$$

**Proof.** As already observed above, Theorem 3 and Proposition 11 together imply that $R'$ satisfies the conditions 1, 2, and 3 in terms of the metric and B-field corresponding to $K'$. To see that $\Xi' = g\Xi \in \mathbb{C} \mathbb{P}(K', R', r)$, note that the lattice isomorphism $g$ only acts on the $\Gamma_\mathbb{R} \oplus \Gamma_\mathbb{R}^*$ part of the tensor product $(\Gamma_\mathbb{R} \oplus \Gamma_\mathbb{R}^*) \otimes M_r(\mathbb{C})$. Therefore the components $\Xi'_i \in M_r(\mathbb{C})$ of $\Xi'$ are linear combinations with real (even integral) coefficients of the components of $\Xi$. This shows that they again commute and have real eigenvalues. Because $g$ is a lattice isomorphism $(\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$ and $L_{R', K'}^{g} = g(L_{R, K})$, mapping $\Xi$ to $g\Xi$ is compatible with the equivalence relation $\sim$ and $\Xi'$ is a well defined element of $\mathbb{C} \mathbb{P}(K', R', r)$. Combining all this it follows that $\|R', \Xi\| \in C_{K'}^{r}(X')$. Bijectivity follows from the functoriality property $\phi_{g_1} \circ \phi_{g_2} = \phi_{g_1 \circ g_2}$, which is easily verified.

To see that $\phi_g$ is compatible with the map $f_g$ on the boundary states, we first compute $f_g$ on Ishibashi states

$$f_g[R, w, m] = \exp\left(-\sum_{\ell = 1}^{\infty} \frac{1}{\ell} G(f_g\alpha_{-\ell}f_g^{-1}, Rf_g\bar{\alpha}_{-\ell}f_g^{-1})\right)f_g[w, m]$$
$$= \exp\left(-\sum_{\ell = 1}^{\infty} \frac{1}{\ell} G((a + bE)^{-1} \alpha_{-\ell}', R(a - bE')^{-1} \bar{\alpha}_{-\ell}')\right)|g(w, m)|$$
$$= |R', g(w, m)|$$

Here we started with the definition of the Ishibashi states in (3.4) and then used the transformation of the $\alpha_{-\ell}'$’s and $\bar{\alpha}_{-\ell}'$’s from (2.13) and finally used the expressions for $G'$ and $R'$ from Table 2 on page 21. Combining this with the definition (3.10) of boundary states, we can easily verify the required compatibility.
3.3 Geometrical interpretation

In this section we will try to translate the algebraic description of D-branes in terms of conformal field theory into more geometrical terms. We have alluded to the geometrical description of D-branes several times as a motivation for parts of our algebraic description. A first guess would be to represent a D-brane in a target space $X$ as a triple $(S, E, \nabla)$ of a submanifold $S \subset X$ and a vector bundle $E$ on $S$ with connection $\nabla$. However, for D-branes of positive codimension and rank $r > 1$ this description ignores part of the information contained in the Chan-Paton matrices $\Xi \in \text{CP}(K, R, r)$ introduced in the previous section. The description proposed here solves this problem and seems to fit everything known about the geometrical description of D-branes.

As we will see in the next section, some D-branes are expected to have a description as coherent sheaves. That means that they should be modules over the structure sheaf $\mathcal{O}_X$ (see the description of skyscraper sheaves in [19]). For general D-branes we are not in a holomorphic context, but we could try to work with modules over $C^\infty(X)$ instead. Note that because we are in a smooth context, it is not necessary to use sheaves. According to the Serre-Swan theorem smooth vector bundles on a compact smooth manifold $X$ can be identified with projective modules over $C^\infty(X)$. The projective module corresponding to a vector bundle $E$ is its space of smooth global sections $\Gamma(E)$. Using $\Omega^k(E) := \Omega^k(X) \otimes \Gamma(E)$, we can define a connection as a map $\nabla : \Omega^k(E) \rightarrow \Omega^{k+1}(E)$ satisfying the Leibniz property $\nabla(f s) = df \otimes s + f \nabla(s)$. To describe a D-brane we drop the condition that the module be projective.

**Definition 7 (Geometrical D-brane).** A geometrical D-brane on a manifold $X$ is a pair $(M, \nabla)$ consisting of a module $M$ over $C^\infty(X)$ with a connection $\nabla : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ satisfying the Leibniz property. Here $\Omega^k(M)$ is defined as $\Omega^k(X) \otimes_{C^\infty(X)} M$.

It is likely that ultimately some replacement for the projectivity will be necessary. However, we will only consider concrete examples corresponding to the affine D-branes discussed above and those D-branes should have whatever additional property will be required.

To construct explicitly the module associated to an affine D-brane, let us first recall the definition of a vector bundle on a torus in terms of multipliers.

**Definition 8.** Let $\Lambda$ be a lattice, $F$ an antisymmetric map $\Lambda \otimes \Lambda \rightarrow \mathbb{Z}$, and $\alpha_0$ and $\beta_0$ matrix valued vectors in $\Lambda_\mathbb{R} \otimes M_r(\mathbb{C})$. Suppose that all components of $\alpha_0$ and $\beta_0$ commute. Then we define a multiplier $e(x, \lambda) := e^{2\pi i(Fx + \beta_0, \lambda)}$ and a connection 1-form $\alpha := 2\pi i(F x + \alpha_0, dx)$. These define a vector bundle $E_{F, \alpha_0, \beta_0}$ on the torus $\Lambda_\mathbb{R}/\Lambda$ with

$$\Gamma(E_{F, \alpha_0, \beta_0}) := \{s : \Lambda_\mathbb{R} \rightarrow \mathbb{C}^r \mid s(x + \lambda) = e(x, \lambda) \text{ for all } \lambda \in \Lambda\}$$

as its space of sections. The connection $\nabla$ on this vector bundle is defined by $\nabla(s) := ds + \alpha s$.

Note that using both $\alpha_0$ and $\beta_0$ is redundant, because $E_{F, \alpha_0, \beta_0}$ is isomorphic to $E_{F, \alpha_0 + \beta_0, 0}$. These bundles can be written as a tensor product $L \otimes E'$, where $L$ is a line...
bundle with first Chern class given by $F$ and $E' = E_{0,\alpha_0,\beta_0}$ is a flat vector bundle with rank $r$.

Let $(R, \Xi)$ be an element of $C^r_K(X)$. As we saw above the gluing matrix $R$ defines subspaces $V$ and $W$ of $\Gamma \cap \mathbb{R}$, such that $\Gamma \cap W$ has maximal rank and an antisymmetric map $F : (\Gamma \cap W) \otimes (\Gamma \cap W) \to \mathbb{Q}$. If we suppose that $F$ is in fact integral, then we can apply this definition to $\Lambda = \Gamma \cap W$ and $F$ to obtain a vector bundle on the subtorus $W/(\Gamma \cap W)$. This is in fact the starting point of our definition. Two issues complicates things a bit: we have to account for $\Xi$ and we need a module over $C^\infty(X)$ instead of over $C^\infty(W/(\Gamma \cap W))$. The following definition solves both problems.

**Definition 9.** Let $(R, \Xi)$ be an element of $C^r_K(X)$, then we define the associated geometrical D-brane $(M, \nabla)$ as follows. Write $\Gamma_R = V \oplus W$ and $\Gamma_R^* = V^* \oplus W^*$ as above. Suppose that $F : (\Gamma \cap W) \otimes (\Gamma \cap W) \to \mathbb{Q}$ is integral. Then we can apply Definition 8 to the lattice $\Gamma \cap W$ to obtain a module $M := \Gamma(E_F,\alpha_0,\beta_0)$, where $\alpha_0 = \Xi_{W^*}$ and $\beta_0 = F\Xi_W$. The connection $\nabla$ is the one from Definition 8. To make $M$ into an $C^\infty(X)$ module we define

$$f \cdot s(x) = (e^{\left(\sum \Xi_{W^*}\right)} f)(x)s(x).$$

A point that deserves some further explanation is the definition of the module structure. To understand the notation write the components of $\Xi_X$ in Jordan normal form $\Xi_X = \xi_X + N$, where $\xi_X \in \Gamma_R$ and $N$ is a vector of nilpotent matrices. Here it is essential that we assumed the eigenvalues to be real (see Definition 8). For simplicity we also assumed that there is just one Jordan block. In this notation the action can be written as follows

$$f \cdot s(x) = (e^{\left(\sum \xi_X\right)} f)(x + \xi_X)s(x) = \left(\sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial}{\partial x}\right)^{\alpha} f(x + \xi_X)N^{\alpha}\right)s(x).$$

The sum over the multi indices $\alpha$ is finite because the component matrices of $N$ are nilpotent. Here it is important to note that the differentiation and translation are not restricted to directions parallel to $W$. The restriction to $x$ in $W$, which is necessary to obtain an element of $M$, takes place afterwards. From this expression it is clear that we can consider $\xi_X$ to define the position of the D-brane. This allows us to define the support of the D-brane $(R, \Xi)$ as

$$\text{supp}(R, \Xi) = \xi_X + W/(\Gamma \cap W) \subset X. \quad (3.12)$$

If there is more than one Jordan block, there are several D-branes at different positions corresponding to the eigenvalues.

Note that in the expression above we describe the function $f$ on $X$ by a $\Gamma$-periodic function on $\Gamma_R$. Together with the fact that all components of $\Xi$ commute this ensures that $f \cdot s(x + \lambda) = e(x, \lambda)f \cdot s(x)$. The final thing to check is the Leibniz rule. This is a straightforward computation.

Because of the assumption that $F$ be integral, instead of just rational as required by Condition 8, this definition is not completely general. The necessary generalisation is more or less clear, but the necessary modifications would make the discussion less transparent. Therefore, we will briefly sketch the solution, but then return to the integral case for rest of this section.
The key idea is that we can replace the lattice \( \Gamma_W := \Gamma \cap W \) by a coarser lattice \( \Gamma'_W \subset \Gamma_W \) such that \( F : \Gamma'_W \otimes \Gamma_W \to \mathbb{R} \) is integral. In this way we obtain a vector bundle \( E' \) on \( W/\Gamma'_W \). Using the isogeny \( i : W/\Gamma'_W \to W/\Gamma_W \) we can define a vector bundle \( E := i_*E' \) on \( W/\Gamma_W \). The rank of this bundle is equal to the degree of the isogeny times the rank of \( E' \). Note that the sublattice \( \Gamma'_W \) is not unique, but if it is chosen such that the degree of the isogeny is minimal, the vector bundle \( E \) on \( W/\Gamma_W \) should be unique up to isomorphism. The module \( M \) can then be defined as the space of sections of the vector bundle \( E \).

Let us now return to integral \( F \). So far we have ignored a potential problem with the definition above, namely the fact that \( \Xi \) is only determined up to the equivalence relation from Definition 5. The following lemma shows that different choices lead to isomorphic geometrical D-branes.

**Lemma 12.** Let \((M, \nabla)\) and \((M', \nabla')\) be geometrical D-branes corresponding to the same gluing matrix and different, but equivalent Chan-Paton matrices \( \Xi \) and \( \Xi' \), then they are isomorphic.

**Proof.** We use the alternative description \([3,11]\) of the equivalence relation, i.e., we write \( \Xi' = \Xi + Y + \xi \text{Id}_r \), where \( Y \in L_{R,K} \otimes M_r(\mathbb{C}) \) and \( \xi \in \Gamma \oplus \Gamma^* \). Let us define a map \( \phi : M \to M' \) by

\[
\phi(s(x)) = (e^{i\frac{\partial}{\partial x} \Xi_W - \xi_W})(e^{-(\frac{\partial}{\partial x} \Xi'_W)})s(x)
\]

We should check that \( \phi(s) \) is indeed in \( M' \) and that the map \( \phi \) is compatible with all the structures present. Because \( s \) is in \( M \), we know that

\[
s(x + \lambda) = e(x,\lambda)s(x) = e^{2\pi i(F(x+\Xi_W),\lambda)}s(x).
\]

For our computations the following identity will be very useful

\[
e^{i\frac{\partial}{\partial x}(A)}(g(x)h(x)) = (e^{i\frac{\partial}{\partial x}A}g(x))(e^{i\frac{\partial}{\partial x}A}h(x)). \tag{3.13}
\]

This holds whenever all components \( A_i \) of \( A \) commute with \( g(x) \). When applying \( \phi \) to the expression for \( s(x + \lambda) \), we use this twice to find

\[
\phi(s(x + \lambda)) = (e^{\frac{\partial}{\partial x} \Xi_W - \xi_W})(e^{2\pi i(Fx,\lambda)}(e^{-(\frac{\partial}{\partial x} \Xi'_W)}s(x))) = e^{2\pi i(F(x+\Xi_W),\lambda)}\phi(s(x)).
\]

In the first step we could apply \((3.13)\), because all components of \( \Xi \) commute. That turns the first factor into a scalar, so that also in the second step there are no problems with non-commuting matrices. We omitted \( \xi_W \) in the final expression, because \( \langle F\xi_W,\lambda \rangle \) is integral because of our assumption that \( F \) is integral. This shows that \( \phi(s) \) is in \( M' \). The computation for the connection is slightly more complex

\[
\phi(\nabla s(x)) = \phi(ds(x) + 2\pi i(Fx + \Xi_{W^*}, dx)s(x))
\]

\[
= (e^{\frac{\partial}{\partial x} \Xi_W - \xi_W})(d(e^{-(\frac{\partial}{\partial x} \Xi'_W)}s(x))
\]

\[
+ 2\pi i(Fx - F\Xi_W + \Xi_{W^*}, dx)(e^{-(\frac{\partial}{\partial x} \Xi'_W)}s(x)))
\]

\[
= (e^{\frac{\partial}{\partial x} \Xi_W - \xi_W})(d(e^{-(\frac{\partial}{\partial x} \Xi'_W)}s(x))
\]

\[
+ 2\pi i(Fx - F\Xi'_W + \Xi_{W^*} + F\xi_W - \xi_W, dx)(e^{-(\frac{\partial}{\partial x} \Xi'_W)}s(x)))
\]

\[
d(\phi(s(x))) + 2\pi i(Fx + \Xi'_{W^*} - \xi_W, dx)\phi(s(x)).
\]
Here we used again $\Xi' = \Xi + Y + \xi \text{Id}_r$. Because $Y$ is in $L_{R,K} \otimes M_r(\mathbb{C})$, we know that $FY_W = Y_{W^*}$ (see (3.7)). Now recall that $\xi$ is an element of $\Gamma \oplus \Gamma^*$. This means that $(\xi_{W^*}, dx)$ defines an integral 1-form on $W/(\Gamma \cap W)$, so $\phi$ takes the connection on $M$ to a connection on $M'$ equivalent to $\nabla'$. Finally we check the compatibility of $\phi$ with the module structure

$$
\phi(f \cdot s(x)) = (e^{(\frac{\partial}{\partial x} \Xi'_{W^*} - \xi_{W^*})}((e^{(\frac{\partial}{\partial x} \Xi_X)} f)(x)s(x))
= (e^{(\frac{\partial}{\partial x} \Xi'_{W^*} - \xi_{W^*})}((e^{(\frac{\partial}{\partial x} \Xi_V)} f)(x)(e^{-(\frac{\partial}{\partial x} \Xi_{W^*})})s(x))
= (e^{(\frac{\partial}{\partial x} \Xi'_{X^*} - \xi_X)} f)(x)\phi(s(x)) = f \cdot \phi(s(x)).
$$

Because $Y_V$ vanishes (see (3.6)), we have $\Xi_V = \Xi'_V - \xi_V$. In the last step we use that $f$ is periodic, so $f(x - \xi_X) = f(x)$.

The proof of this lemma heavily uses the properties of the Chan-Paton matrices we defined in the previous section. This lends support to our definition of geometrical D-branes. On the other hand this geometrical definition sheds some light on the interpretation of $\Xi \in \text{CP}_K(X)$.

Finally note that it is expected that the geometry becomes non commutative when $B$ does not vanish. However, at this level of detail that seems to have no visible effects. This may be related to our interpretation of $F = F + \widetilde{B}$ as the curvature matrix.

### 3.4 D-brane categories

This section is rather speculative. We will rephrase some of the results we obtained so far in the language of D-brane categories and use them to motivate a number of conjectures.

All D-branes of a given type together define a D-brane category. The objects of such a category are the D-branes and the morphisms correspond to strings stretching between a pair of D-branes. So let us see if we can construct the category $C_K(X)$ of affine D-branes on a torus $X$. The objects are the affine D-branes discussed above. Often one also requires direct sums of D-branes to define an object in the category. This corresponds to the fact that one can put several D-branes together. Of course, such a configuration will be unstable, but we will mostly ignore questions about stability. Starting with a certain class of D-branes, we can define the objects to be formal direct sums of these D-branes. In this way the class of objects will automatically be closed under direct sums. A subtlety is that direct sums of D-branes corresponding to the same gluing matrix are already included, because we can simply take the direct sum of the Chan-Paton matrices. A D-brane category depends on where we are in the moduli space of the SCFT. This can be seen in our example from the fact that the set of affine D-branes $C_K(X)$ depends on $K \in \mathcal{J}_{N=1}(\Gamma)$.

The really problematic part in the construction of a D-brane category is to define the morphisms and the composition of morphisms. In general a D-brane category is expected to be a graded linear category, i.e., the spaces of morphisms are graded vector spaces and the composition is expected to be linear and compatible with the grading. To construct the space of morphisms, one should analyse the conformal field theory on the upper half plane with one D-brane on the positive real axis and another one on the negative real axis. The
Table 3: Boundary conditions

|                | Algebraic | Geometric |
|----------------|-----------|-----------|
| **A-type**     | $J(z) = -\bar{J}(\bar{z})$ | $R^*\omega_1 = -\omega_2$ (A1) |
| $\epsilon = -1$ | $G_\pm(z) = \eta \bar{G}_\mp(\bar{z})$ | $e^{i\phi(z)} = \eta^n e^{i\alpha} e^{-i\bar{\phi}(\bar{z})}$ | $R^*\Omega_1 = e^{i\alpha} \bar{\Omega}_2$ (A2) |
| **B-type**     | $J(z) = \bar{J}(\bar{z})$ | $R^*\omega_1 = \omega_2$ (B1) |
| $\epsilon = 1$ | $G_\pm(z) = \eta \bar{G}_\mp(\bar{z})$ | $e^{i\phi(z)} = \eta^n e^{i\beta} e^{i\bar{\phi}(\bar{z})}$ | $R^*\Omega_1 = e^{i\beta} \Omega_2$ (B2) |

The Hilbert space of this boundary conformal field theory (BCFT) is the space of morphisms between the two D-branes and it is graded by fermion degree. Elements in this Hilbert space can be identified with vertex operators on the boundary. Unfortunately the product of two fields in a CFT is in general not defined. So it seems that our whole construction breaks down at this point. However, by restricting the class of D-branes we consider, we also restrict the class of vertex operators under consideration. In addition we can replace the Hilbert space of the boundary conformal field theory by some subspace. Together these restrictions on the vertex operators that we consider may be enough to guarantee that for those vertex operators a sensible product can be defined. Below we will discuss two cases where a construction along these lines seems to work.

An alternative way out is to drop the requirement that a D-brane category be a true category. Associating D-branes with objects and the Hilbert spaces in BCFT with Hom-spaces, seems to work quite well, but the composition should be replaced with some other structure. This is not as absurd as it may seem. After all an $A_\infty$-category is also not a category in the strict sense. For such D-brane categories it should still be possible to define notions like functor, equivalence etc. To find a replacement for the composition we can probably use some of the structure defined by a BCFT (expectation values, OPEs etc.). The real question is if we can obtain a useful structure that can be described mathematically. We will not try to make this precise here, but let us assume for the moment that we have defined an appropriate D-brane category $\mathcal{C}_K(X)$. Because $\mathcal{C}_K(X)$ is defined in terms of conformal field theory, the whole construction should be compatible with $N = 1$ isomorphisms. For tori that means that each lattice isomorphism $g : (\Gamma \oplus \Gamma^*, q) \to (\Gamma' \oplus \Gamma'^*, q')$ defines a functor $\Phi_g : \mathcal{C}_K(X) \to \mathcal{C}_K(X')$. On objects this functor coincides with the map $\phi_g$ from Theorem [1]. Because of the corresponding property for $N = 1$ isomorphisms, we also expect that $\Phi_{g_2} \circ \Phi_{g_1} = \Phi_{g_2 \circ g_1}$. However, as we will discuss below, this is probably too optimistic and we should be content with both sides being isomorphic as functors (of D-brane categories).

Let us now return to the first solution, restricting the class of D-branes and morphisms. This allows us to stick to ordinary categories or well known generalisations thereof such as $A_\infty$-categories. If we choose an $N = 2$ structure, we can define two classes of BPS D-branes,
the so called A- and B-branes. These D-branes can also be considered as D-branes in a
topological string theory obtained from the original theory by the A- or B-twist respectively.
Because topological string theory is much simpler than ordinary string theory, that gives a
physical explanation why the behaviour of A- and B-branes is so much nicer. The A- and
B-type boundary conditions can be found in Table 3. Algebraically these conditions can
be formulated in terms of the fields defining the $N = 2$ structure. Note that this means
that to define A- and B-branes it is necessary to use the second definition of an $N = 2$
algebra discussed in Section 2.1. Using the definition (2.2) of these fields on a torus and the
boundary conditions (3.1), these algebraic conditions can be translated into geometrical
terms.

Following [2] we also use the spectral flow operators

$$e^{i \phi} = \Omega_1(\psi, \ldots, \psi),$$

and similarly for the right-moving sector. Here $\Omega_i$ is the holomorphic volume form corre-
sponding to the complex structure $j_i$. In the conditions involving these operators there are
real parameters $\alpha$ and $\beta$. The boundary conditions are that there exists an $\alpha \in \mathbb{R}$ or a
$\beta \in \mathbb{R}$ such that the equation holds.

Actually for affine D-branes the conditions (A2) and (B2) are implied by (A1) and
(B1) respectively. This can be seen as follows. Recall that $R^t G R = G$ (see Condition 1)
and that $\omega_i = -G j_i$. Let us do B-type boundary conditions first. We can write (B1) as

$$R^t \omega_1 R = -\omega_2.$$  

Then it follows that $j_1 R = R j_2$, which implies that $R^* \Omega_1 = \lambda \Omega_2$ for some
$\lambda \in \mathbb{C}$ (a holomorphic 1-form $\alpha$ satisfies $\alpha \circ j_1 = i \alpha$, so $R^* \alpha \circ j_2 = \alpha \circ R \circ j_2 = \alpha \circ j_1 \circ R = i \alpha \circ R = i R^* \alpha$). Using the normalisation $\Omega_i \wedge \bar{\Omega}_i = \frac{1}{n!} \omega_n$ it follows that $|\lambda| = 1$, so we
can write $\lambda = e^{i \beta}$ for some $\beta \in \mathbb{R}$. Pointwise this argument is always valid (also on more
general manifolds and for more general D-branes). However, the essential point is that $\beta$
should be constant. For tori and affine D-branes that is automatic, but for more general
situations that is a highly nontrivial condition.

For A-type boundary conditions the argument is similar. Using (A1) we find $j_1 R =
- R j_2$. This implies $R^* \Omega_1 = \lambda \bar{\Omega}_2$ for some $\lambda \in \mathbb{C}$. Using the normalisation of $\omega_1$ and $\Omega_1$ it follows that $(-1)^n |\lambda|^2 \Omega_2 \wedge \bar{\Omega}_2 = (-1)^n \frac{1}{n!} \omega_n \bar{\omega}_n$. Comparing with the normalisation of $\omega_2$ and
$\Omega_2$ yields $|\lambda| = 1$. Below we will discuss some hints that the conditions (A2) and (B2) can
be thought of as stability conditions. Apparently affine D-branes are automatically stable,
which explains why these conditions are automatically satisfied in that case.

Using A- and B-type boundary conditions we can define two subsets of $C_{I,J}(X)$, namely
the set $C_{I,J}^A(X)$ of A-branes and the set $C_{I,J}^B(X)$ of B-branes. Starting with these sets of
D-branes we hope to construct the corresponding D-brane categories $C_{I,J}^A(X)$ and $C_{I,J}^B(X)$.
When $(I, J)$ is in the geometrical part of the Teichmüller space, these categories should
also have a geometrical description, as was discussed in [2]. This is easiest for the B-branes,
so let us discuss that case first.

**Conjecture 1.** For $(I, J) \in T_{N=2}^{\text{geom}}$ the category of B-branes $C_{I,J}^B(X)$ is equivalent to the
derived category of coherent sheaves $D\text{Coh}_{I,J}(X)$.
To make this conjecture plausible, let us argue that an object \((R, \Xi) \in C_{I,J}^\text{B}(X)\) corresponds to a coherent sheaf on \(X\) (following [2]). Because \((I, J) \in \mathcal{T}^{\text{geom}}_{N=2}(X)\), there is a single complex structure \(j = j_1 = j_2\) and a single Kähler form \(\omega = \omega_1 = \omega_2 = -G_j\) on \(X\). So the equation \(j_1 R = R j_2\) we found above simplifies to \(j R = R j\). This means that \(R : \Gamma_R \to \Gamma_R\) is a complex linear map if we regard \(\Gamma_R\) as a complex vector space with complex structure defined by \(j\). So the \(-1\) eigenspace \(V \subset \Gamma_R\) is a complex subspace of \((\Gamma_R, j)\). Because \(j^* G_j = G\) it follows that the orthogonal complement \(W\) is a complex subspace as well. Because the support \(\text{supp}(R, \Xi)\) as defined in (3.14) is a shift of \(W/ (\Gamma \cap W)\), it is a holomorphic submanifold of \(X\). Using the fact that \(j^{-1} \tilde{R} j = \tilde{R}\) and \(j^* \tilde{G} j = \tilde{G}\) it follows that \(\tilde{R} = (\tilde{G} - j^* F_j)^{-1}(\tilde{G} + j^* F_j)\), so because of the uniqueness of \(\mathcal{F}\) we see that \(j^* F_j = \mathcal{F}\). In other words \(\mathcal{F}\) is a \((1, 1)\)-form, so the vector bundle \(E\) on \(\text{supp}(R, \Xi)\) is holomorphic. This argument only works when \(B\) is a \((1, 1)\)-form. If it is not, we have to twist the derived category (see [1]).

As discussed above the condition (B2) is automatically fulfilled. A physical argument that affine D-branes are likely to be stable is that they have minimal volume (mass) in their cohomology class. This only says something about the support, so it cannot be a complete argument. Mathematically, one can check that the vector bundles we obtain are precisely the semi-homogeneous vector bundles discussed by Mukai in [21]. Mukai also shows that semi-homogeneous vector bundles are Gieseker semi-stable and Gieseker stable when they are simple. This seems to confirm our suspicion that the condition (B2) is related to stability.

Not every object of the derived category corresponds to an element of \(C_{I,J}^\text{B}(X)\). However, Fukaya in [21] mentions a conjecture by Mukai that any coherent sheaf on an abelian variety has a resolution in terms of semi-homogeneous bundles. That would mean that although we only have a direct description of semi-homogeneous vector bundles on affine submanifolds, we can still ‘approximate’ arbitrary objects of the derived category in terms of them. In other words, semi-homogeneous vector bundles generate the derived category.

Actually, this is not too different from our understanding on physical side. We know how to describe D-branes corresponding to semi-homogeneous vector bundles on affine submanifolds in conformal field theory, but it is unclear how to describe more general sheaves. Recent work in physics suggests that the resolutions from mathematics also have a physical significance (see e.g., [22, 23, 24]).

The corresponding conjecture for A-branes is the following.

**Conjecture 2.** For \((I, J) \in \mathcal{T}^{\text{geom}}_{N=2}\) the category of A-branes \(C_{I,J}^\text{A}(X)\) is equivalent to the derived Fukaya category \(\mathcal{D}\mathcal{F}_{I,J}(X)\).

Let \((R, \Xi) \in C_{I,J}^\text{A}(X)\) be an A-brane and let us suppose that \(\mathcal{F} = 0\). This means that \(W\) is the eigenspace for eigenvalue \(+1\). So if \(v, w \in W\), then \(\omega(v, w) = -\omega(Rv, Rw) = -\omega(v, w)\). Therefore the restriction of \(\omega\) to \(\text{supp}(R, \Xi)\) vanishes. Similarly, we see that \(\omega(v, w) = 0\) for \(v, w \in V\). Because \(V \oplus W = \Gamma_R\) it follows that \(W\) has dimension \(n\), so \(\text{supp}(R, \Xi)\) is a Lagrangian submanifold of \(X\). At least when \(B|_W = 0\), we also have \(F = 0\), so the vector bundle \(E\) on \(S\) is flat. If \(v_1, \ldots, v_n \in W\) then \(\Omega(v_1, \ldots, v_n) = R^* \Omega(v_1, \ldots, v_n) = e^{-i\alpha/2} \Omega(v_1, \ldots, v_n)\), so \(\text{Im}(e^{-i\alpha/2} \Omega)|_S = 0\). Therefore \(\text{supp}(R, \Xi)\) is even
special Lagrangian and we find back the original definition of objects in the Fukaya category. The notion of being special Lagrangian is closely related to $\Pi$-stability (see [25, 22]).

However, this argument depends on the assumption $\mathcal{F} = 0$. In [26] Kapustin and Orlov showed that without this assumption one obtains a more general class of D-branes, the so-called coisotropic D-branes. So the Fukaya category should be a kind of generalised Fukaya category including coisotropic D-branes. Another problem used to be the construction of a derived version of the Fukaya category, but that problem has now been solved thanks to the work of Fukaya and collaborators (see [27]).

It is natural to expect that isomorphisms of $N = 2$ SCFTs correspond to isomorphisms of these categories, but for generalised $N = 2$ morphisms it is not quite clear what to expect. It may be helpful to return to the putative general D-brane category $\mathcal{C}_{K}(X)$ for a moment. To keep track of the $N = 2$ structure we define $\mathcal{C}_{I,J}(X) = \mathcal{C}_{K}(X)$, where $K = IJ$. The functor $\Phi_{g}$ induces functors $\Phi_{g}^{(\epsilon_{L},\epsilon_{R})}: \mathcal{C}_{I,J}(X) \to \mathcal{C}_{g,\epsilon_{L},\epsilon_{R}}^{(\epsilon_{L},\epsilon_{R})}(I,J)$. The justification for this notation is that we want to be able to consider $\mathcal{C}_{A,I,J}(X)$ and $\mathcal{C}_{B,I,J}(X)$ as subcategories of $\mathcal{C}_{I,J}(X)$. The functor $\Phi_{g}^{(\epsilon_{L},\epsilon_{R})}$ need not preserve these subcategories, because their definition depends on $J(z)$ and $\bar{J}(\bar{z})$. However, generalised $N = 2$ morphisms are well behaved on these fields (see Table 1 on page 150). Comparing to the algebraic formulation of A- and B-type boundary conditions in Table 3, one can easily check that $f_{g}^{(\epsilon_{L},\epsilon_{R})}$ interchanges A- and B-type boundary conditions when $\epsilon_{L}R_{\epsilon_{R}} = -1$ and preserves them when $\epsilon_{L}R_{\epsilon_{R}} = 1$. So the corresponding functor $\Phi_{g}^{(\epsilon_{L},\epsilon_{R})}$ should interchange the A- and B-type subcategories for $\epsilon_{L}R_{\epsilon_{R}} = -1$ and preserve them for $\epsilon_{L}R_{\epsilon_{R}} = 1$.

To make this more precise we can restrict the functor $\Phi_{g}^{(\epsilon_{L},\epsilon_{R})}$ to either $\mathcal{C}_{A,I,J}(X)$ or $\mathcal{C}_{B,I,J}(X)$. The image is again one of these subcategories. As these subcategories are much better defined, one may hope to be able to make sense of these restrictions even if it is difficult to make sense of $\Phi_{g}^{(\epsilon_{L},\epsilon_{R})}$ itself. To formulate a precise conjecture, it is useful to label the boundary condition with $\epsilon = -1$ for type A and $\epsilon = 1$ for type B. The first geometrical condition in Table 3 can then be written as $R^{*}\omega_{2} = \epsilon \omega_{1}$. Using this notation we can state the following conjecture.

**Conjecture 3.** Let $g: (\Gamma \oplus \Gamma^{*}, q) \to (\Gamma' \oplus \Gamma^{*}, q')$ be a lattice isomorphism and let $(\epsilon_{L},\epsilon_{R}) = (\pm 1, \pm 1)$. Then for all $(I, J) \in \mathcal{T}_{N=2}(\Gamma)$ and $\epsilon = \pm 1$ there exists a functor $\Phi_{g,\epsilon}^{(\epsilon_{L},\epsilon_{R})}: \mathcal{C}_{I,J}^{\epsilon}(X) \to \mathcal{C}_{I',J'}^{\epsilon}(X')$, where $(I', J') = \mu_{g}^{(\epsilon_{L},\epsilon_{R})}(I, J)$. These functors satisfy

$$\Phi_{g,\epsilon}^{(\epsilon_{L},\epsilon_{R})} \circ \Phi_{g,\epsilon}^{(\epsilon_{L}',\epsilon_{R}')}(\phi_{g}) \cong \Phi_{g,\epsilon}^{(\epsilon_{L},\epsilon_{R})},$$

where $\cong$ denotes isomorphism of functors.

Here $\Phi_{g,\epsilon}^{(\epsilon_{L},\epsilon_{R})}$ can be thought of as the restriction of $\Phi_{g}^{(\epsilon_{L},\epsilon_{R})}$ to $\mathcal{C}_{A,I,J}(X)$ for $\epsilon = -1$ or to $\mathcal{C}_{B,I,J}(X)$ for $\epsilon = 1$. The statement about the composition of functors reflects Proposition 4. Note that we have been careful and do not require equality, but just an isomorphism of functors in the last property. On the level of objects these functors are given by $\phi_{g}$ and we do have equality. However, as we saw above, the sets of D-branes $\mathcal{C}_{A,I,J}$ and $\mathcal{C}_{B,I,J}$ and the morphisms between them, are really just a starting point. To actually construct the D-brane categories one has to add formal direct sums, construct the derived category
etc. After all these constructions it is rather likely that the best one can hope for is an isomorphism of functors. Taking \( g_2 = g_1^{-1} \) and \((\epsilon_{L,2}, \epsilon_{R,2}) = (\epsilon_{L,1}, \epsilon_{R,1})\) it follows from this property that the functors \( \Phi_{g_2}^{(\epsilon_{L,2}, \epsilon_{R,2})} \) are equivalences.

For \( \epsilon_L \epsilon_R = -1 \) these functors define equivalences between the category of A-branes on \( X \) and the category of B-branes on \( X' \). When we are in the geometrical part of the moduli space we can use Conjecture 1 and Conjecture 2 and interpret these categories as the derived Fukaya category and the derived category of coherent sheaves. So we see that this conjecture is in fact a generalisation of Kontsevich’s homological mirror symmetry conjecture applied to tori (see [28]). From the discussion of T-duality in Section 2.5 it follows that mirror symmetry in the geometrically interesting cases can be described as T-duality in the fibres of a torus fibration (the SYZ-description). On the other hand if we fix a torus fibration and consider the mirror morphism corresponding to T-duality in the fibres of this fibration, then for certain values of the background field this mirror morphism will take us to a non geometrical part of the moduli space. In that case Kontsevich’s conjecture breaks down, but our conjecture still predicts an equivalence of categories. However, we can no longer interpret both categories as the derived Fukaya category or the derived category of coherent sheaves.

This description also points to a way of breaking down the proof of Kontsevich’s conjecture into several steps. One part would be proving Conjecture 2 and Conjecture 1. That would reduce the proof to Conjecture 3. Here we at least have a solid understanding of the map on the moduli and we also know the map on the objects for a large class of objects. In fact, the class of D-brane we describe is a generalisation of the class used by Polishchuk and Zaslow in their proof of the homological mirror symmetry conjecture for 2-dimensional tori. One can check that our description leads to exactly the same map on objects. The advantage of our description is that we have a largely systematic description of the map on D-branes. The disadvantage is that so far a description of the morphisms and the composition is lacking.

4. Conclusions

In this paper we tried to clarify some issues concerning the moduli spaces of tori and their D-brane categories. We emphasised the need to clearly distinguish the various moduli spaces. Only then is it possible to understand the relations between the moduli spaces. For tori we described these moduli spaces in great detail. For \( N = 2 \) this led us to a geometrical description using two independent complex structures. In itself this structure is general, but for tori it is particularly interesting because there are so many complex structures compatible with a given metric. Together with a precise description of isomorphisms of \( N = 1 \) and \( N = 2 \) SCFTs this allowed us to explain some issues with interpretation of mirror symmetry for general values of the background fields.

Turning our attention to D-branes, we found that this description of the moduli space interacts nicely with the conformal field theory description of D-branes using gluing matrices. Using the boundary state formalism we studied the transformation of D-branes under isomorphisms of the SCFT. Explicit formulae for the boundary states motivated the
introduction of extra data complementing the gluing matrix and necessary to completely
determine a D-brane.

The final sections were more speculative. This also takes us to options for further re-
search. Our preliminary geometrical description of D-branes should be made more precise
and also the categorical interpretation of D-brane needs further study. The best oppor-
tunity for progress on these topics seems to be an even more detailed investigation of
the relation between conformal field theory and geometry, especially the relation between
boundary conformal field theory and D-brane categories. That should lead to a better
understanding of how the Fukaya category and the derived category of coherent sheaves
have to be modified to obtain a completely general description of the categories of A-
and B-branes. That in turn should improve our understanding of the homological mirror
symmetry conjecture.

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