A DISCRETE COMPLEMENT OF LYAPUNOV’S INEQUALITY AND ITS INFORMATION THEORETIC CONSEQUENCES

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Abstract. We establish a reversal of Lyapunov’s inequality for monotone log-concave sequences, settling a conjecture of Havrilla-Tkocz and Melbourne-Tkocz. A strengthened version of the same conjecture is disproved through counter example. We also derive several information theoretic inequalities as consequences. In particular sharp bounds are derived for the varentropy, Rényi entropies, and the concentration of information of monotone log-concave random variables. Moreover, the majorization approach utilized in the proof of the main theorem, is applied to derive analogous information theoretic results in the symmetric setting, where the Lyapunov reversal is known to fail.

1. Introduction

In this paper we prove the following reversal of Lyapunov’s inequality\(^1\), conjectured in [33] and [18].

**Theorem 1.1.** For \(x\), a monotone, log-concave sequence in \(\ell_1\), the function

\[ t \mapsto \log \left( t \sum_i x_i^t \right) \]

is strictly concave for \(t \in (0, \infty)\).

This is anticipated by affirmative results in the continuous setting dating back to Cohn [10] on \(\mathbb{R}\), and Borell [9] in \(\mathbb{R}^d\). However in contrast to the continuous setting, the requirement that \(x\) is monotone cannot be dropped\(^2\), see [33] for examples. Moreover, we will also provide a counter example to a strengthening of Theorem 1.1 conjectured in [33, 18], further differentiating the continuous and discrete settings.

The main novelty in the proof is to establish a majorization between the distribution function of a monotone log-concave sequence and its geometric counterpart. Though we will not expound upon this outside of its application to this proof, it can be understood as a second order analog of the distributional majorization lemma utilized in [34, 32]. This alongside some further reductions, leaves one needing only the special case of a geometric sequence, which can be approached with direct computation, to complete the proof of Theorem 1.1.

This effort fits within a more general pursuit, developing discrete analogs for the continuous convexity theory, which in recent investigation has connected information theory and convex geometry (see [25] for background). One instantiation is the effort to understand the behavior of the entropy of discrete variables under independent summation, see [19, 27, 33, 24, 7]. Another is the pursuit of discrete Brunn-Minkowski type inequalities [15, 35, 30, 20, 16, 14, 40, 17]. In fact, in information theoretic language, the Brunn-Minkowski inequality can be understood as a “Rényi entropy power” inequality, see [5, 22, 8, 29, 26, 21, 38, 37], and in this sense, as an information theoretic inequality as well.

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\(^1\)By Lyapunov’s inequality we refer to the fact that \(p \mapsto \log \| f \|^p\) is convex in \(p\) for a general measurable function \(f\) and measure.

\(^2\)Note that for continuous variables on \(\mathbb{R}\), proving the result for monotone variables is equivalent to the general result since log-concavity is preserved under rearrangement, see for example [31].
We will see that Theorem 1.1 yields several information theoretic consequences for discrete monotone variables. We obtain sharp bounds on the varentropy to be compared to its continuous analog [13], and utilize this to derive concentration of information, analogous to [2, 13]. We give sharp reversals of the monotonicity of Rényi entropy general parameters, augmenting the recently obtained comparisons for the ∞-Rényi entropy given in [33], and as a consequence we obtain a sharp reverse entropy power type inequality for iid variables, tightening a result from [33]. We mention that this reverse entropy power is the discrete analog of an entropic Rogers-Shephard inequality pursued by Madiman and Kontoyannis in [23]. We also obtain a sharp comparison between the value of a log-concave sequence at its mean, and the value at its mode which we compare with the classical result of Darroch [12] for Bernoulli sums. We will also obtain as a Corollary of our arguments that for the monotone log-concave variables of a fixed p-Rényi entropy, the geometric distribution has maximal q-Rényi entropy for q ≥ p and minimal q-Rényi entropy for q ≤ p.

As mentioned, Theorem 1.1 can fail without the assumption of monotonicity. In particular, symmetric variables do not necessarily satisfy the conclusion of Theorem 1.1. However we will demonstrate that the majorization techniques used are robust enough to be applied in the symmetric case, and we use them to deliver sharp Rényi entropy comparisons, varentropy bounds, and concentration of information results in the symmetric setting. We also establish the “symmetric geometric” distribution as the maximal (resp. minimal) q-Rényi entropy distribution for fixed p-Rényi entropy among discrete symmetric log-concave variables for q ≥ p (resp. q ≤ p).

Let us outline the paper. In Section 2, we will define notation and derive applications of the main theorem. In Section 3 we give the proof of Theorem 1.1. In Section 4 we give a counter example to the strengthening of Theorem 1.1 conjectured in [18, 33], while in Section 5, we derive analogs of the consequences in Section 2 for symmetric log-concave variables. In the Appendix A we recall some elementary results from the theory of majorization for the convenience of the reader.

2. Applications

2.1. Definitions. For a real valued random variable Y, we let EY denote its expectation, and denote its variance Var(Y) := EY^2 − (EY)^2.

Definition 2.1. Let (E, µ) be a measure space and X an E valued random variable with density function f such that P(X ∈ A) = ∫_A f dµ. We define the information content I_X : E → R, as I_X(x) = − log f(x).

To avoid confusion, in sections where we discuss the information content random variable I_X(X), we will avoid the usual abuse of notation and write H(f) for the entropy of a variable X. Conversely, when there is no risk of confusion, and we are considering a single variable X, we will omit the subscript and write I for the information content. We write H(μ)(X) = H_μ(f) := E[I(X)] in the general case. For example, when E is discrete, and μ is the counting measure,

E[I(X)] = H(X)

is just the Shannon entropy of X. Observe that when μ is a probability measure given by a random variable Y, then the expectation of the information content is given by the relative entropy (or Kullback-Leibler divergence), H(μ)(f) = −D(X||Y), and the varentropy measures the deviation of −I(X) from D(X||Y).

In physical applications, it may be more natural to write the density of X in terms of a potential E, f(x) = e^{−E(x)}, in which case, E[I(X)] reflects the average energy of a system, and V(X) the average fluctuation.
Definition 2.2. For a random variable $X$ taking values on a measure space $(E, \mu)$ with density function $f$, define the varentropy functional,

\begin{equation}
V(X) = \mathbb{E}(\log f(X) - \log f(X))^2.
\end{equation}

Unless specified, we will consider $E = \mathbb{Z}$ and $\mu$ the standard counting measure, so that the density function $x_n := \mathbb{P}(X = n)$ of a variable $X$, can be expressed as a non-negative sequence.

Definition 2.3. A non-negative sequence $x_i$ is log-concave when

\[ x_i^2 \geq x_{i-1}x_{i+1} \]

and $i \leq j \leq k$ with $x_i x_k > 0$ implies $x_j > 0$.

We consider a sequence to be increasing when $x_i x_{i+1} > 0$ implies $x_{i+1} \geq x_i$, and decreasing when $x_i x_{i+1} > 0$ implies $x_{i+1} \leq x_i$. A sequence is monotone when it is either increasing or decreasing.

Definition 2.4. A random variable $X$ taking values in $\mathbb{Z}$ is log-concave when the sequence $x_i := \mathbb{P}(X = i)$ is log-concave. The variable $X$ is monotone when the sequence $x_i$ is monotone.

We say a non-negative sequence $x_i$ belongs to $\ell_p$ when $\sum_{i \in \mathbb{Z}} x_i^p < \infty$. Note that when $x_i$ is log-concave, $x_i$ belonging to $\ell_1$ implies that $x_i$ belongs to $\ell_p$ for all $p \in (0, \infty)$.

2.2. Varentropy bounds.

Theorem 2.5. For $X$ a monotone log-concave variable taking values in $\mathbb{Z}$, with the usual counting measure,

\[ V(X) < 1. \]

Proof. Define $\Psi(t) = \log(t \sum_n f^t(n))$, then

\[ \Psi'(t) = \frac{\sum_n \log f(n)f^t(n)}{\sum_n f^t(n)} + \frac{1}{t}, \]

and

\[ \Psi''(t) = \frac{(\sum_n f^t(n)) (\sum_n \log^2 f(n)f^t(n)) - (\sum_n \log f(n)f^t(n))^2}{(\sum_n f^t(n))^2} - \frac{1}{t^2}. \]

By concavity of $\Psi$, $\Psi''(1) = V(X) - 1 < 0$, and our result follows. \hfill \Box

The bound is sharp, the varentropy of a geometric distribution $Z_p$ with parameter $p$, can be explicitly computed as $V(Z_p) = \left( \frac{(1-p)\log(1-p)}{p} \right)^2$ which tends to 1 with $p \to 0$.

2.3. Rényi entropy comparisons.

Definition 2.6. For $X$ a random variable on $\mathbb{Z}$, and $p \in (0, 1) \cup (1, \infty)$ define

\[ H_p(X) := \frac{\log(\sum_i x_i^p)}{1-p}, \]

where $x_i := \mathbb{P}(X = i)$. Let $H_1(X) := H(X) = -\sum_i x_i \log x_i$ and $H_\infty(X) = -\log \|x\|_\infty$ where $\|x\|_\infty := \max_i x_i$ and $H_0(X) = \#\{i : x_i > 0\}$.

Theorem 2.7. When $X$ is a monotone and log-concave variable taking values in $\mathbb{Z}$ then $p > q > 0$ implies,

\[ H_p(X) > H_q(X) + \log \left( \frac{1}{q^{p-1}} \right) \]
Proof. Let \( x_i = \mathbb{P}(X = i) \). We prove the case \( p > q > 1 \), the other cases can be treated similarly. Letting \( \lambda = \frac{q-1}{p-1} \), \( q = \lambda p + (1 - \lambda)1 \), so that by strict concavity,
\[
\log \left( q \sum_i x_i^q \right) > \lambda \log \left( p \sum_i x_i^p \right) + \log \left( 1 \sum_i x_i^1 \right),
\]
and our result follows for \( p, q \not\in \{1, \infty\} \) from this inequality. Owing to log-concavity there in no difficulty obtaining the limiting cases through continuity.  

Note that when \( X_\lambda \) has geometric distribution \( \mathbb{P}(X = n) = (1 - \lambda)\lambda^n \) for parameter \( \lambda \in (0, 1) \), its Rényi entropy can be computed directly,
\[
H_p(X_\lambda) = \log \left( \frac{(1 - \lambda)^p}{1 - \lambda^p} \right)^\frac{1}{1-p}.
\]
Hence,
\[
H_p(X_\lambda) - H_q(X_\lambda) = \log \left( \frac{(1 - \lambda)^p}{1 - \lambda^p} \right)^\frac{1}{1-p} - \log \left( \frac{(1 - \lambda)^q}{1 - \lambda^q} \right)^\frac{1}{1-q}
\]
\[
= \log \left( \frac{1 - \lambda}{1 - \lambda^p} \right)^\frac{1}{1-p} - \log \left( \frac{1 - \lambda}{1 - \lambda^q} \right)^\frac{1}{1-q},
\]
which tends to \( \log \left( \frac{1}{q^{\lambda-1}} \right) \) with \( \lambda \to 1 \). Thus we see that Theorem 2.7 is sharp.

The following result is actually a consequence of the Rényi entropy comparison derived in [33]. It does not need the assumption of monotonicity. The result should be compared to the classical result of Darroch [12], that states that for independent sums of Bernoulli random variables the distance between the mean and mode is no greater than 1, see also [36, 41, 24] for background and recent developments on such variables. For the larger class of log-concave variables such a result is impossible. For example, a geometric distribution has mode at 0, but can have arbitrarily large expectation. However, the result below demonstrates that the value of any log-concave distribution at its mean approximates up to an absolute constant \( e \), the value of the distribution at its mode.

**Corollary 2.8.** For \( X \) with log-concave density function \( f \) with support \( A \subseteq \mathbb{Z} \),
\[
\max\{f([\mathbb{E}X]), f(\lceil\mathbb{E}X\rceil)\} \geq e^{-1} \|f\|_\infty,
\]
where \( [\cdot] \) and \( \lceil\cdot\rceil \) denote the usual floor and ceiling.

Note that inequality is sharp in the sense that the constant \( e^{-1} \) cannot be improved, as can be seen by choosing a geometric distribution with large, integer valued mean.

**Proof.** Define the log-affine interpolation of \( f \),
\[
\tilde{f}(x) = \begin{cases} 
  f^{1-(x-[x])}([x])f^{x-[x]}([x]) & \text{for } x \in \text{co}(A), \\
  0 & \text{otherwise.}
\end{cases}
\]
Then \( \log \tilde{f} \) is a concave function on \( \text{co}(A) \) the convex hull of \( A \), and by Jensen’s inequality and
\[
H(f) = -\mathbb{E} \log f(X) = -\mathbb{E} \log \tilde{f}(X) \geq -\log \tilde{f}(\mathbb{E}X).
\]
Using that by Theorem 2.7 and by Theorem 1.3 of [33] in the absence of monotonicity, $H(f) \leq H_\infty(f) + 1$, and inserting the inequality into exponentials we have

$$\exp(-\log \|f\|_\infty + 1) \geq \exp(-\log \tilde{f}(\mathbb{E}X))$$

$$\frac{e}{\|f\|_\infty} \geq \frac{1}{f(\mathbb{E}X)}$$

which yields

$$e \max\{f(\lceil \mathbb{E}X \rceil), f(\lfloor \mathbb{E}X \rfloor)\} \geq f^{1-(\mathbb{E}X-\lfloor \mathbb{E}X \rfloor)}(\lceil \mathbb{E}X \rceil)f^{\lfloor \mathbb{E}X \rfloor}(\lfloor \mathbb{E}X \rfloor) \geq \|f\|_\infty$$

\[ \square \]

### 2.4. Concentration of information content.

**Theorem 2.9.** For $X \sim f$ monotone log-concave variable on $\mathbb{Z}$, for $t > 0$

$$\mathbb{P}(I(X) \geq H(f) + t) \leq (1 + t)e^{-t},$$

and when $t \leq 1$,

$$\mathbb{P}(I(X) \leq H(f) - t) \leq (1 - t)e^t.$$

Note that when $t = 1$, we obtain $\mathbb{P}(I(X) \leq H(f) - 1) = 0$, implying that $-\log \|f\|_\infty = H_\infty(f) > H(f) - 1$ recovers the sharp comparison of min-entropy and Shannon entropy above. The inequality $H_\infty(X) \geq H(X) - 1$ holds without the monotonicity assumption, see [33].

The following is a general and elementary technique for deriving concentration of the information content based on uniform bounds on the varentropy of the “canonical ensemble”. In [13], it is assumed that $X$ takes values in $\mathbb{R}^d$, and has a density with respect to the Lebesgue measure. We include the proof adapted from [13] below, for the convenience of the reader.

**Lemma 2.10** (Fradelizi-Madiman-Wang [13]). For a random variable $X$ on $E$ with density $f \in L^\alpha(\mu)$ for all $\alpha > 0$, and $X_\alpha \sim \int f'^\alpha \, d\mu$ satisfying $V(X_\alpha) \leq K$, then for $t > 0$

$$\mathbb{P}(I(X) - H(\mu)(f) \geq t) \leq e^{-Kr(t/K)}$$

and

$$\mathbb{P}(I(X) - H(\mu)(f) \leq -t) \leq e^{-Kr(-t/K)}$$

where $r(t) = t - \log(1 + t)$ for $t \geq -1$ and is infinite otherwise.

The proof is a combination of results from [13], Theorem 3.1 and Corollary 3.4 in particular.

**Proof.** Observe that the function $F(\alpha) = \log \int f^{\alpha}(x) \, d\mu(x)$ is infinitely differentiable\(^3\)

$$K = \sup_{\alpha > 0} V(X_\alpha) = \sup_{\alpha > 0} \alpha^2 F''(\alpha).$$

By applying $F''(t) \leq K/t^2$ to the Taylor expansion,

$$F(\alpha) = F(1) + (\alpha - 1)F'(1) + \int_1^\alpha (\alpha - t)F''(t) \, dt$$

\(^3\)Indeed, the $n$-th derivative of $\alpha \mapsto f^{\alpha}(x)$, $f^{\alpha}(\log f)^n$ is measurable as the composition of a measurable function $f$, with a continuous function $x^n(\log x)^n$, and that further, $|f^{\alpha}(\log f)^n| \leq \mathbb{1}_{\{f > 1\}} f^{\alpha+4}C(n, \varepsilon) + \mathbb{1}_{\{f < 1\}} f^{\alpha+4}c(n, \varepsilon)$ for $\alpha' \in (\alpha - \varepsilon/2, \alpha + \varepsilon/2)$, where $C$ and $c$ are uniform bounds on $(\log x)^n/x^{\varepsilon/2}$ for $x \geq 1$ and $x^{\varepsilon/2}|\log x|^n$ for $x \leq 1$ respectively, so that the requisite domination exists for Lebesgue dominated convergence to pass the derivative and integrals.
yields
\[ F(\alpha) = F(1) + (\alpha - 1)F'(1) + K(\alpha - 1 - \log \alpha). \]

With the substitution \( \alpha = 1 - \beta \), and the insertion of \( F(1) = 0 \), and \( F'(1) = -H(\mu)(X) \) we can rewrite (4) as
\[ \mathbb{E} \left( e^{\beta(I(X) - H(\mu)(f))} \right) \leq e^{Kr(-\beta)}. \]

For \( \beta, t > 0 \), taking exponentials and applying Markov’s inequality,
\[ \mathbb{P}(I(X) - H(\mu)(f) \leq -t) \leq \mathbb{E} \left[ e^{-\beta(I(X) - H(\mu)(f))} \right] e^{-\beta t} \leq e^{K(r(\beta) - \frac{\beta t}{K})} \]
Standard calculus allows minimization over \( \beta \) and yields, \( \inf \beta r(\beta) - \frac{\beta t}{K} = -r(-t/K) \) which gives (3). Applying the same ideas yields (2) as well. \( \Box \)

Proof of Theorem 2.9. If \( X \sim f \), is log-concave and monotone, then \( X_\alpha \sim f_\alpha := f^\alpha / \sum_{n \geq 0} f^n(n) \) is as well. Hence by Theorem 2.5, \( V(X_\alpha) \leq 1 \). Applying Lemma 2.10 with \( K = 1 \) yields the result. \( \Box \)

2.5. Renyi Entropy Power Reversals. The entropy power inequality, is a fundamental inequality in information theory that gives a sharp lower bound on the amount of entropy increase in summation of continuous independent variables, explicitly taking \( \mu \) to be the Lebesgue measure on \( \mathbb{R}^d \), and denoting for \( X \) with density \( f \) with respect to \( \mu \), \( N(X) = e^{\frac{2}{\beta}H(\mu)(f)} \), Shannon’s entropy power inequality states that \( N(X + Y) \geq N(X) + N(Y) \) for independent random vectors \( X \) and \( Y \). More generally, super-additivity properties of the Renyi entropy have been studied, extending the Shannon’s EPI, see [4, 37, 25, 21, 22, 29, 38]. We consider a Renyi Entropy Power reversal to be any non-trivial upper bound on the entropy of a sum of random variables, see [3, 6, 11, 43, 1, 42, 7].

Theorem 2.11. For \( X, Y \) iid, log-concave, and monotone on \( \mathbb{Z} \), and \( \alpha \in [2, \infty] \)
\[ H_\alpha(X - Y) \leq H_\alpha(X) + \log 2. \]

The inequality is a sharp improvement for monotone log-concave variables of Theorem 6.2 of [33], where it is proven that \( H_\alpha(X - Y) \leq H_\alpha(X) + \alpha \log 2 \) for \( X \) and \( Y \) iid and log-concave. To see that the constant 2 cannot be improved, take \( X \) to have density \( f(n) = (1 - p)p^n \) so that for \( n \geq 0 \), \( f_{X - Y}(n) = \frac{1 - p}{1 + p}p^n \). Taking the limit with \( p \to 1 \) shows the inequality to be sharp. An alternative motivation for the inequality is its relationship to an entropic generalization conjectured by Madiman and Kontoyannis [23] of the Rogers-Shephard inequality from convex geometry [39], see also [33] for further discussion.

The proof relies on an elementary trick, known to specialists, that \( H_2(X) = H_\infty(X - Y) \) holds for iid variables \( X \) and \( Y \). We include a proof for completeness and emphasize that this equality is independent of any property of the distribution\(^4\).

Lemma 2.12. For \( X \) and \( Y \) iid on \( \mathbb{Z} \),
\[ H_2(X) = H_\infty(X - Y). \]

\(^4\)The proof is given for iid log-concave \( X \) and \( Y \) in [33].
Proof. Let $f$ denote the shared distribution of $X$ and $Y$ and $f_{X−Y}$ the distribution of $X−Y$. We compute directly,

$$\sum_k f^2(X = k) = \sum_k \mathbb{P}(X = k)\mathbb{P}(Y = k)$$

$$= \mathbb{P}(X − Y = 0)$$

After taking logarithms, this shows that

$$(7) \quad H_2(X − Y) = −\log f_{X−Y}(0),$$

thus the result follows from demonstrating that $f_{X−Y}(0) = \|f_{X−Y}\|_{\infty}$. To this end, we recall the elementary rearrangement inequality (see for instance [28]) that for non-negative sequences $x, y \in \ell_2$,

$$\sum_i x_i y_i \leq \sum_i x_i^\downarrow y_i^\downarrow$$

where $x^\downarrow$ is the sequence $x$ rearranged in decreasing order. If we denote $\tau_n f(k) = f(n + k)$ then

$$f_{X−Y}(n) = \sum_k \tau_n f(k)f(k)$$

$$\leq \sum_k (\tau_n f)^\downarrow(k)f^\downarrow(k).$$

However since $\tau_n f$ is just a translation of $f$, $(\tau_n f)^\downarrow = f^\downarrow$ and since $\sum_k (f^\downarrow)^2(k) = \sum_k f^2(k) = f_{X−Y}(0)$ our result follows.

When $\alpha \leq 2$ a constant depending on $\alpha$ can be found using only monotonicity of Rényi, see Theorem 6.2 in [33].

Proof of Theorem 2.11. We use the notation $c(\alpha) = \frac{\alpha−1}{\alpha−1}$, with $c(\infty) := 1$. We prove the case that $\alpha > 2$.

$$H_\alpha(X − Y) \leq H_\infty(X − Y) + \log \frac{c(\alpha)}{c(\infty)}$$

$$= H_2(X) + \log \frac{c(\alpha)}{c(\infty)}$$

$$\leq H_\alpha(X) + \log \frac{c(2)}{c(\alpha)} + \log \frac{c(\alpha)}{c(\infty)}$$

$$= H_\alpha(X) + \log 2.$$

□

3. Proof of Theorem 1.1

For $x$ a monotone, log-concave sequence $\ell_1$ sequence, we denote $\Phi_x : (0, \infty) \to \mathbb{R}$,

$$\Phi_x(t) := \log \left( t \sum_i x_i^\downarrow \right).$$

To prove that $\Phi_x$ is always strictly concave, we will first start with some reductions. For $x$ a log-concave sequence and $p > q$ we wish to prove,

$$(8) \quad \Phi_x((1−s)p + sq) − (1−s)\Phi_x(p) − s\Phi_x(q) \geq 0.$$
If we denote by \( x^q \), the monotone log-concave sequence \((x^q)_i = (x^q_i)\) and \( \tilde{p} = p/q \), then by algebraic manipulation the left hand side of (8) is exactly
\[
\Phi_{x^q}((1-s)\tilde{p} + s1) - (1-s)\Phi_{x^q}(\tilde{p}) - s\Phi_{x^q}(1) \geq 0.
\]

Additionally observe that for a constant \( c > 0 \), with \( cx \) denoting the sequence \((cx)_i = cx_i\) that \( \Phi_{cx}(t) = \Phi_x(t) + t \log c \). Thus we can and will without loss of generality assume that \( \sum_i x_i = 1 \) and need only prove that for \( p > 1 \), and \( s \in (0,1) \)
\[
\Phi_x((1-s)p + s) \geq (1-s)\Phi_x(p) + s.
\]

For the proof of this result we will derive the following lemma.

**Lemma 3.1.** For \( x \) a non-Dirac, monotone log-concave probability sequence, \( p > 1 \), and \( q \in (1,p) \), there exists a \( \lambda \in (0,1) \) such that the sequence \( z \) given by \( z_k = (1-\lambda)\lambda^k \) satisfies
\[
\sum_i x_i^q = \sum_i z_i^q
\]
and
\[
\sum_i x_i^q \geq \sum_i z_i^q
\]

As we will see Lemma 3.1 reduces our problem to proving (10) for the geometric distribution. To prove the Lemma, we establish a majorization between the distribution function of a monotone log-concave variable and its geometric counterpart.

**Proposition 3.2.** For a sequence \( x \), define \( F_x(t) := \#\{i : x_i > t\} \). Let \( x \) be a log-concave, non-increasing sequence, and \( z_k = C \lambda^k \) for \( C > 0 \) and \( p \in (0,1) \). Then there exist a finite interval \( I \) such that \( F_x(t) \leq F_z(t) \) if \( t \in I \) and \( F_x(t) \geq F_z(t) \) if \( t \notin I \).

**Proof.** Define \( a := \min\{k : x_k \geq z_k\} \), and \( b := \sup\{k : x_k \geq z_k\} \). It follows from the log-concavity of \( x \) and the log-affinity of \( z \) that \( \{k : x_k \geq z_k\} \) is a discrete interval. Thus, the interval\(^5\)
\[
[a,b] = \{k : x_k \geq z_k\}.
\]

Let \( I = [z_b, x_a] \), with \( z_b = 0 \) in the case \( b = +\infty \). Let \( t \in I \). Two cases will be considered: \( t < m \leq b \). First assume \( z_b \leq t < z_a \). Let \( m = \min\{i : z_i \leq t\} = F_z(t) \). See that \( a < m \leq b \); since \( z_b \leq t \), then \( m \leq b \) because \( m \) is the minimum index such that \( z \) satisfies such inequality. Also, if \( m \leq a \), then we have \( z_m \geq z_a \) because \( z \) is decreasing, which gives us both \( z_m \leq t \) by definition of \( m \) and \( x_m \geq t \) because \( z_a > t \). This is a contradiction, thus \( a < m \). Finally, since \( x_i \) is non-increasing and \( a < m \leq b \), we must have
\[
\sum z_m \leq t < \sum z_{m-1} \leq \sum x_{m-2} \leq \cdots \leq \sum x_0.
\]
From (13) we see that \( F_x(t) \geq m = F_z(t) \). Now, suppose \( z_a \leq t < x_a \). Since \( z_a \leq t \) then \( F_z(t) \leq a \). Now, since \( t < x_a \) and \( x_i \) is non-increasing, so \( x_0 \geq x_1 \geq \cdots \geq x_a \) and thus \( F_x(t) \geq a+1 \). Therefore \( F_x(t) \leq a < a+1 \leq F_z(t) \).

The following is a standard fact that holds for general measure spaces. It follows from the layer-cake representation of a non-negative function, a change of variables, and an application of Fubini-Tonelli.

**Proposition 3.3.** Let \( X \) be a random variable on the non-negative integers and the sequence \( x_i := \mathbb{P}(X = i) \), then for \( t \geq 1 \), \( F_x(\lambda) \) as defined in Proposition 3.2 satisfies
\[
\sum_i x_i^q = t \int_0^\infty \lambda^{a-1} F_x(\lambda)d\lambda.
\]

\(^5\)With the interpretation that \([a,b] = [a, \infty) \cap \mathbb{Z} \) when \( b = \infty \).
In particular, $F_x$ is a probability distribution function on $(0, \infty)$ when $x$ is a log-concave probability sequence.

**Lemma 3.4.** If $U, V$ are non-negative random variables with densities $f, g$ respectively, such that $E(U) = E(V)$, and $f \leq g$ on an interval $I$, and $f \geq g$ outside $I$, then

$$E(w(U)) \geq E(w(V))$$

for any convex function $w$. The inequality reverses if $w$ is concave.

The proof of Lemma 3.4 is classical, and given as Theorem A.2 in the Appendix for completeness.

**Theorem 3.5.** If $U, V$ are non-negative random variables with densities $f$ and $g$ respectively, that satisfy $E(U^p) = E(V^p)$ for $p > 0$ and $f \leq g$ on an interval $I$, and $f \geq g$ outside of $I$, then

$$E(w(U^p)) \geq E(w(V^p))$$

for any convex function $w$. The inequality reverses if $w$ is concave.

**Proof.** Follows directly from Lemma 3.4. Indeed, $U^p$ has density $\tilde{f}(x) = f(x^p)x^{1/p}p^{-1}$ while $V^p$ has density $\tilde{g}(x) = g(x^p)x^{1/p}p^{-1}$ so that $U^p$ and $V^p$ satisfy the hypothesis of Lemma 3.4 for the interval $I^p := \{w : w = x^p, x \in I\}$. □

**Proof of Lemma 3.1.** For $p > 1$, and $x$ not a point mass, $0 < \sum x_i^p < \sum x_i = 1$. Then, observe that $\Psi(\lambda) := \sum_{k=0}^{\infty}((1 - \lambda)\lambda^k)^p = (1 - \lambda)^p$. By the intermediate value theorem, since $\Psi(0) = 1$ and $\lim_{\lambda \to 1} \Psi(\lambda) \to 0$ as $\lambda \to 1$ (L’Hospital), there exists $\lambda$ such that (11) holds.

Let $x$ be log-concave, non-increasing with $\sum_i x_i = 1$, let $z$ be geometric and let $p$ be such that $\sum x_i^p = \sum z_i^p$. Let $U$ be a random variable with density $F_x$, and $V$ be a random variable with density $F_z$. Since $\sum x_i^p = \sum z_i^p$, then $1/p \sum x_i^p = 1/p \sum z_i^p$, which implies $E(V^{p-1}) = E(U^{p-1})$ by Proposition 3.3. With $p > 1$ and $q \in (1, p)$, we have that $g(x) = x^{p-1}$ is concave, thus $E \left( g(V^{p-1}) \right) \geq E \left( g(U^{p-1}) \right)$ by Proposition 3.2 and Theorem 3.5. Thus $E(V^{q-1}) \geq E(U^{q-1})$ and, multiplying both sides by $q$ and using Proposition 3.3, we get $\sum x_i^q \geq \sum z_i^q$. □

The last ingredient of the proof of Theorem 1.1 is to prove it in the special case that the sequence is geometric.

**Proposition 3.6.** Let $z = (z_k)$ be a geometric distribution, i.e., $z_k = (1 - \lambda)^k \lambda^k$ for $\lambda \in (0, 1)$ and $k \in \{0, 1, \ldots\}$. Then

$$\Phi_z(t) = \log \left[ t \sum_i z_i^t \right]$$

is a concave function in $(0, +\infty)$.

**Proof.** See that

$$\Phi_z(t) = \log \left[ t(1 - \lambda)^t \right] + \log \left[ \sum_i (\lambda^i)^t \right]$$

$$= \log t + t \log(1 - \lambda) - \log(1 - \lambda^t),$$
It is clear that \( g \) only if \( y > H \) and some \( s \). Hence (10) holds, and we have concavity for any \( \Phi \).

Thus

\[
\Phi''_x(t) = -\frac{1}{t^2} + \frac{\lambda'}{(1-\lambda^2)} \log^2 \lambda
\]

\[
= \frac{\lambda' \log^2 \lambda - (1-\lambda^2)^2}{(1-\lambda^2)^2}
\]

so \( f''(t) \leq 0 \) if and only if \( \lambda' \log^2 \lambda - (1-\lambda^2)^2 \leq 0 \). To prove this, let us consider a variable \( y = \lambda^t \) and \( g(y) := y \log^2 y - (1-y)^2 \). To see \( g(y) \leq 0 \) we proceed in the following way. Clearly, \( g(1) = 0 \); we want to show this is the maximum value of \( g \). This will occur if and only if \( g'(1) = 0, g'(y) > 0 \) for \( y < 1 \) and \( g'(y) < 0 \) for \( y > 1 \), where

\[
g'(y) = 2 \log y + \log^2 y + 2(1-y).
\]

It is clear that \( g'(1) = 0 \) and that there exist some \( y < 1 \) for which \( g'(y) > 0 \) (e.g., \( y = 1/e \)) and some \( y > 1 \) for which \( g'(y) < 0 \) (e.g., \( y = e \)), so it suffices to prove \( g' \) is monotonically decreasing to conclude \( y = 1 \) is the only critical value of \( g \). To that end, see that

\[
g''(y) = \frac{2}{y} + \frac{2 \log y}{y} - 2
\]

is always non-negative. Indeed, \( g''(y) \leq 0 \iff \frac{1}{y} + \frac{\log y}{y} \leq 1 \), which is equivalent to say \( h(y) = \frac{1}{y} + \frac{\log y}{y} \) has a maximum value of 1. This is easy to see as \( h'(y) = -\frac{\log y}{y^2} \) is zero at \( y = 1 \), is positive on \((0, 1)\) and negative on \((1, +\infty)\), and \( h(1) = 1 \).

**Proof of Theorem 1.1.** By the aforementioned reductions, let \( x \) be a non-increasing log-concave probability sequence, and \( s \in (0, 1) \). Then there exists, by Lemma 3.1, a geometric distribution \( z \) such that \( \sum z^p = \sum x^p \) and moreover for all \( q \in (1, p) \),

\[
\sum z^q_i \geq \sum x^q_i.
\]

Taking \( q = s + (1-s)p \), we have by Lemma 3.1

\[
\Phi_x(s + (1-s)p) \geq \Phi_z(s + (1-s)p).
\]

By Proposition 3.6 \( \Phi_z \) is concave, and hence

\[
\Phi_z((1-s)p + s) \geq (1-s)\Phi_z(p) + s.
\]

Then by hypothesis, \( \Phi_x(p) = \Phi_z(p) \), and \( \Phi_x(1) = \Phi_z(1) = 0 \). Compiling these results gives the following sequence of equalities and inequalities,

\[
\Phi_x(s + (1-s)p) \geq \Phi_x(s + (1-s)p)
\]

\[
\geq (1-s)\Phi_x(p) + s
\]

\[
= (1-s)\Phi_x(p) + s.
\]

Hence (10) holds, and we have concavity for any \( \Phi_x \). 

**Corollary 3.7.** For \( X \) a monotone log-concave random variable, and \( Z \) a geometric random variable such that

\[
H_p(X) = H_p(Z)
\]

then for \( q \geq p \),

\[
H_q(X) \geq H_q(Z)
\]

while

\[
H_q(X) \leq H_q(Z)
\]

for \( q \leq p \).
The proof is omitted as it is the same as the symmetric case which is given in detail in Section 5.

4. Extensions

A natural generalization of Theorem 1 was first conjectured in an early version of [18], and reiterated in [33].

**Question 4.1.** For \( \gamma > 0 \) and a positive monotone concave sequence \((y_n)_{n=1}^N\) then the function

\[
\Phi_y(t) := \log \left( (t + \gamma) \sum_{n=1}^N y_n\right)
\]

is concave for \( t > -\gamma \).

However the following counterexample precludes an affirmative answer.

Let \( N = 2, y = \{\lambda, 1 + \lambda\} \) and consider the points, \( \{0, \gamma, 2\gamma\} \subseteq (-\gamma, \infty) \). Concavity of \( \Phi_y \) would imply,

\[
\exp \Phi_y(\gamma) \geq \exp (\Phi_y(0)\Phi(2\gamma)),
\]

which is,

\[
4\gamma^2 (2\lambda + 1) \geq 6\gamma^2 (\lambda^2 + (1 + \lambda)^2).
\]

Taking the limit with \( \lambda \to 0 \) would imply \( 4 \geq 6 \).

5. Symmetric Variables

A random variable on \( \mathbb{Z} \) can be symmetric about a point \( m \in \mathbb{Z} \) \((f(m + n) = f(m - n))\) or it could be symmetric about \( n + \frac{1}{2} \) for \( n \in \mathbb{Z} \). For example \( P(X = 0) = P(X = 1) = \frac{1}{2} \) is symmetric about \( 0 + \frac{1}{2} \). In this case, when a log-concave sequence \((x_i)_{i \in \mathbb{Z}}\) is symmetric about a point \( n + \frac{1}{2} \),

\[
\log \left( \sum_{i} x_i^t \right) = \log \left( \sum_{i>n} x_i^t \right) + \log 2
\]

is concave by Theorem 1.1 as \((x_i)_{i>n}\) is monotone and log-concave. Thus, we have the following corollary.

**Corollary 5.1.** For \((x_i)_{i \in \mathbb{Z}}\) an \( \ell_1 \) log-concave sequence, symmetric about a point \( n + \frac{1}{2} \),

\[
t \mapsto \log \left( \sum_{i} x_i^t \right)
\]

is concave in \( t \). Moreover, if \( X \) is a random variable satisfying \( P(X = i) = x_i \) then

\[
V(X) < 1,
\]

and

\[
H_p(X) > H_q(X) + \log \left( \frac{p^{p-1}}{q^{q-1}} \right).
\]

If \( f \) denotes the density of \( X \) then,

\[
P(I(X) \geq H(f) + t) \leq (1 + t)e^{-t}
\]

and when \( t \leq 1 \),

\[
P(I(X) \leq H(f) - t) \leq (1 - t)e^t.
\]
Remark 5.2. See that this implies Theorem 2.9 is valid for sequences symmetric about a point $n + \frac{1}{2}$.

However, in the case that $(x_i)$ is symmetric about a point $n \in \mathbb{Z}$, the concavity of (18) is known to fail. In spite of this, we show in the sequel that arguments from the proof of Theorem 1.1 are able to recover sharp bounds on the varentropy and the Rényi entropy in this setting.

**Definition 5.3.** A sequence $z$ is symmetric geometric when there exists $t \in (0, 1)$ and $C > 0$ such that

$$z_n = C\lambda^{[n]}$$

for $n \in \mathbb{Z}$. When $C = \frac{1 - \lambda}{1 + \lambda}$ the sequence defines a probability distribution. A random variable $Z$ is symmetric geometric when

$$\mathbb{P}(Z = n) = \frac{1 - \lambda}{1 + \lambda}\lambda^{[n]}.$$

Given $p \in (0, \infty)$, and $X$ symmetric and log-concave, there exists $Z$ symmetric geometric, such that $\sum_n f_X^p(n) = \sum_n f_Z^p(n)$

**Proposition 5.4.** Let $x_i$ be a probability distribution over $\mathbb{Z}$. For $p \neq 1$, there exists a geometric sequence $z_i = \frac{1 - \lambda}{1 + \lambda}\lambda^{[i]}$ such that $\sum x_i^p = \sum z_i^p$.

**Proof.** Suppose $p > 1$. Then $0 \leq x_i^p \leq x_i$, so $0 \leq \sum_i x_i^p \leq \sum_i x_i = 1$. Now consider a geometric symmetric sequence $z_i$ with parameter $q$, and see that

$$\sum_i z_i^p = \left(\frac{1 - \lambda}{1 + \lambda}\right)^p \sum_i (\lambda^p)^[i] = \frac{(1 - \lambda)^p}{(1 + \lambda)^p} \sum_i (\lambda^p)^[i].$$

Let $S(\lambda) = \frac{(1 - \lambda)^p}{(1 + \lambda)^p}$. Clearly $S(0) = 1$. While $\lim_{\lambda \to 1} \frac{(1 - \lambda)^p}{1 + \lambda} = 0$. Also $\lim_{\lambda \to 1} \frac{1 + \lambda^p}{1 + \lambda} = \frac{1}{2^{p-1}}$. Therefore $\lim_{\lambda \to 1} S(\lambda) = 0$. By the intermediate value theorem, since $S$ is continuous for $\lambda \in (0, 1)$, there must be a $\lambda \in (0, 1)$ such that $S(\lambda) = \sum z_i^p = \sum_i x_i^p$.

A similar approach will handle the case that $p \in (0, 1)$.

**Proposition 5.5.** If $x_i$ is non-increasing for $i \geq 0$ and $z$ is symmetric geometric, then there exists a finite interval $I$ such that

$$(19) \quad F_x(t) \geq F_z(t) \quad t \in I$$

$$(20) \quad F_x(t) \leq F_z(t) \quad t \notin I$$

**Proof.** We know the result to be true for $x^* = (x_i)_{i \geq 0}$ and $z^* = (z_i)_{i \geq 0}$ by Proposition 3.2. Now, see that

$$2F_{x^*} - 1 = F_x,$$

and

$$2F_{z^*} - 1 = F_z.$$

Furthermore, $F_x \geq F_z$ if and only if $2F_{x^*} - 1 \geq 2F_{z^*} - 1$ if and only if $F_{x^*} \geq F_{z^*}$. Similarly for $F_x \leq F_z$. Therefore the same interval $I$ given by Proposition 3.2 satisfies our desired inequalities.
Lemma 5.6. Let $X$ be log-concave, symmetric about a point $n \in \mathbb{Z}$. Then there exists a symmetric geometric distribution $Z$, such that $H_p(X) = H_p(Z)$ and

$$H_q(X) \geq H_q(Z)$$

for $q \geq p > 0$, and

$$H_q(X) \leq H_q(Z),$$

for $0 < q \leq p$.

Proof. First, see that $H_p(X) = H_p(Z)$ if and only if $\sum_i x_i^p = \sum_i z_i^p$ so there must exist such geometric distribution for $p \neq 1$ by Proposition 5.4. Suppose $H_p(X) = H_p(Z)$. Let $U$ be a random variable with density $F_U$, and $V$ be a random variable with density $F_V$. Since $\sum x_i^p = \sum z_i^p$ then $\frac{1}{p} \sum x_i^p = \frac{1}{p} \sum z_i^p$, which implies $E(V^{p-1}) = E(U^{p-1})$ by Proposition 3.3.

Let $g(x) = x^{\frac{q-1}{p-1}}$. If $p > 1$ and $q \in (1,p)$, we have that $g(x)$ is concave, thus $E(g(U^{p-1})) \geq E(g(V^{p-1}))$ by Proposition 3.2 and Theorem 3.5. Thus $E(V^{q-1}) \geq E(U^{q-1})$ and, multiplying both sides by $q$ and using Proposition 3.3, we get $\sum x_i^q \geq \sum z_i^q$. The same can be argued if $p < 1$ and $q \in (p,1)$. Now, when $p > 1$ and $q \in (0,1) \cup [p,\infty)$, and when $p < 1$ and $q \in (0,p] \cup (1,\infty)$, $g(x)$ is convex, so by Theorem 3.5 the inequality is reversed and $\sum x_i^q \leq \sum z_i^q$. Now, to pass from the sum to Rényi’s entropy, we must multiply by $\frac{1}{1-q}$, which reverses the inequality when $q > 1$. So we get

$$1 < q < p \Rightarrow H_q(X) \leq H_q(Z);$$

$$q < p < 1 \Rightarrow H_q(X) \leq H_q(Z);$$

$$q < 1 \Rightarrow H_q(X) \leq H_q(Z);$$

$$1 < p < q \Rightarrow H_q(X) \geq H_q(Z);$$

$$p < 1 < q \Rightarrow H_q(X) \geq H_q(Z);$$

$$p < q < 1 \Rightarrow H_q(X) \geq H_q(Z).$$

The limiting cases with $q$ or $p \in \{1,\infty\}$ can be easily handled using continuity and monotonicity of the Rényi entropy as a function of $\alpha \mapsto H_\alpha(X)$ and as a function of the parameter $\lambda$ of a symmetric geometric distribution $Z_\lambda$, $\lambda \mapsto H_\lambda(Z_\lambda)$.

Theorem 5.7. For $X$ log-concave and symmetric about a point $n \in \mathbb{Z}$, $p \geq q$, then

$$H_q(X) - H_p(X) \leq C(q,p) := \sup_Z H_q(Z) - H_p(Z)$$

where the supremum is taken over all $Z$ symmetric-geometric.

Proof. For any $p \neq 1$ and $q \leq p$ we have, by Lemma 5.6, a symmetric geometric $Z$ with $H_p(X) = H_p(Z)$ and

$$H_q(X) \leq H_q(Z),$$

which implies

$$H_q(X) - H_p(X) \leq H_q(Z) - H_p(Z) \leq \sup_Z H_q(Z) - H_p(Z).$$

Theorem 5.8. For $X$ log-concave and symmetric,

$$V(X) \leq V_S := \sup_Z V(Z)$$

where the supremum is taken over all $Z$ symmetric-geometric.
Proof. Let $\Psi_X(t) = \log \sum_i x_i^{t+1}$ where $x_i = \mathbb{P}(X = i)$. Observe that

$$\Psi_X(0) = 1$$

$$\Psi'_X(0) = -H(X)$$

$$\Psi''_X(0) = V(X)$$

Choose $Z$ to be a symmetric geometric distribution satisfying $H(Z) = H(X)$. By Lemma 5.6, $H_{1+t}(Z) \leq H_{1+t}(X)$ for $t > 0$, which corresponds to $\Psi_Z(t) \geq \Psi_X(t)$ for $t > 0$. By Taylor expansion,

$$\Psi_X(t) = \Psi_X(0) + \Psi'_X(0) t + \frac{t^2}{2} \Psi''_X(0) + o(t^2)$$

$$\leq \Psi_Z(0) + \Psi'_Z(0) t + \frac{t^2}{2} \Psi''_Z(0) + o(t^2)$$

Since the Taylor expansions are identical up to linear terms, it follows that $\Psi''_Z(0) = V(Z) \geq V(X) = \Psi''_X(0)$. □

Note that if one expresses the distribution of a symmetric-geometric variable $Z_\lambda$ as $\frac{1-\lambda}{1+\lambda} \lambda^{|k|}$, its varentropy has the closed form expression,

$$V(Z_\lambda) = \log^2(\lambda) \left( \frac{2\lambda}{1-\lambda} - \left( \frac{2\lambda}{(1-\lambda)(1+\lambda)} \right)^2 \right).$$

Numerically, we have $V_S \approx 1.16923$. This is used for the following corollary.

Corollary 5.9. For $X$ with distribution $f$ log-concave and symmetric on $\mathbb{Z}$, and $t \geq 0$,

$$\mathbb{P}(I(X) - H(f) \geq t) \leq \left( 1 + \frac{t}{V} \right)^V e^{-t}$$

and

$$\mathbb{P}(I(X) - H(f) \leq -t) \leq \left( 1 - \frac{t}{V} \right)^V e^t$$

where $V := V_S \approx 1.16923$ is defined in Theorem 5.8

Proof. The result follows from combining Lemma 2.10 and Theorem 5.8. □

Appendix A. Majorization

The following theorem is a well known characterization of the convex order, see [28] for proof and further background.

Theorem A.1. For $X$ and $Y$ are random variables on $[0, \infty)$ such that $\mathbb{E}X = \mathbb{E}Y < \infty$, then

$$\mathbb{E} \varphi(Y) \geq \mathbb{E} \varphi(X)$$

holds for all convex functions $\varphi$, if it holds for all $\varphi$ of the form $\varphi(x) = [x-t]_+$ for $t \in (0, \infty)$.

When $X$ and $Y$ satisfy (21) we say that $Y$ majorizes $X$ in the convex order, or just that $Y$ majorizes $X$ for short, and write $Y \succ X$.

Theorem A.2. For non-negative random variables $X \sim f$ and $Y \sim g$ with densities taking values on $[0, \infty)$ such that $\mathbb{E}X = \mathbb{E}Y < \infty$, if there exists an interval $I \subseteq [0, \infty)$ such that $g \leq f$ on $I$, and $g \geq f$ on $[0, \infty) - I$, then $Y \succ X$. 
Proof. For $t \in [0, \infty)$ define $\Psi(t) = \mathbb{E}[Y - t]_+ - \mathbb{E}[X - t]_+$. By assumption $\mathbb{E}X = \mathbb{E}Y$, and hence $\Psi(0) = 0$. By monotone convergence, $\lim_{t \to \infty} \Psi(t) = 0$. Computing the derivatives of $\Psi$, one obtains $\Psi'(t) = \mathbb{P}(X > t) - \mathbb{P}(Y > t)$, and $\Psi''(t) = g(t) - f(t)$. Observe that $\Psi'(0) = 0$, $\lim_{t \to \infty} \Psi'(t) = 0$, and $0 = \mathbb{E}Y - \mathbb{E}X = \int_0^\infty \Psi'(t) dt$. Thus, $\Phi'$ must be both positive and negative or it is exactly 0 and the problem is trivial. As such $\Phi''$ is positive, negative, and then positive. It follows that $\Phi'$ is positive and then negative, and hence $\Phi \geq 0$ and our result follows.

References

[1] K. Ball, P. Nayar, and T. Tkocz. A reverse entropy power inequality for log-concave random vectors. Studia Mathematica, 235:17–30, 2016.

[2] S. Bobkov and M. Madiman. Concentration of the information in data with log-concave distributions. Ann. Probab., 39(4):1528–1543, 2011.

[3] S. Bobkov and M. Madiman. Reverse Brunn-Minkowski and reverse entropy power inequalities for convex measures. J. Funct. Anal., 262:3309–3339, 2012.

[4] S. G. Bobkov and G. P. Chistyakov. Entropy power inequality for the Rényi entropy. IEEE Trans. Inform. Theory, 61(2):708–714, February 2015.

[5] S. G. Bobkov and G. P. Chistyakov. On concentration functions of random variables. J. Theoret. Probab., 28:976–988, 2015.

[6] S. G. Bobkov and M. M. Madiman. On the problem of reversibility of the entropy power inequality. In Limit theorems in probability, statistics and number theory, volume 42 of Springer Proc. Math. Stat., pages 61–74. Springer, Heidelberg, 2013. Available online at arXiv:1111.6807.

[7] S. G. Bobkov, A. Marsiglietti, and J. Melbourne. Concentration functions and entropy bounds for discrete log-concave distributions. Combinatorics, Probability and Computing, pages 1–19, 2020.

[8] S.G. Bobkov and A. Marsiglietti. Variants of the entropy power inequality. IEEE Transactions on Information Theory, 63(12):7747–7752, 2017.

[9] C. Borell. Complements of Lyapunov’s inequality. Math. Ann., 205:323–331, 1973.

[10] J. H. E. Cohn. Some integral inequalities. Quart. J. Math. Oxford Ser. (2), 20:347–349, 1969.

[11] T. M. Cover and Z. Zhang. On the maximum entropy of the sum of two dependent random variables. IEEE Trans. Inform. Theory, 40(4):1244–1246, 1994.

[12] J. N. Darroch. On the distribution of the number of successes in independent trials. The Annals of Mathematical Statistics, 35(3):1317–1321, 1964.

[13] M. Fradelizi, M. Madiman, and L. Wang. Optimal concentration of information content for log-concave densities. In C. Houdré, D. Mason, P. Reynaud-Bouret, and J. Rosinski, editors, High Dimensional Probability VII: The Cargèse Volume, Progress in Probability. Birkhäuser, Basel, 2016. Available online at arXiv:1508.04093.

[14] R. J. Gardner and P. Gronchi. A Brunn-Minkowski inequality for the integer lattice. Trans. Amer. Math. Soc., 353(10):3995–4024 (electronic), 2001.

[15] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali. Displacement convexity of entropy and related inequalities on graphs. Probab. Theory Related Fields, 160(1-2):47–94, 2014.

[16] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali. Transport proofs of some discrete variants of the Prékopa-Leindler inequality. arXiv preprint arXiv:1905.04038, 2019.

[17] D. Halikias, B. Klartag, and B. A. Slomka. Discrete variants of Brunn-Minkowski type inequalities. arXiv preprint arXiv:1911.04392, 2019.

[18] A. Havrilla and T. Tkocz. Sharp Khinchin-type inequalities for symmetric discrete uniform random variables. arXiv preprint arXiv:1912.13345, 2019.

[19] O. Johnson and Y. Yu. Monotonicity, thinning, and discrete versions of the entropy power inequality. IEEE Trans. Inform. Theory, 56(11):5387–5395, 2010.

[20] B. Klartag and J. Lehec. Poisson processes and a log-concave Bernstein theorem. Studia Mathematica, 247:1, 2019.

[21] J. Li. Rényi entropy power inequality and a reverse. Studia Mathematica, 242:303–319, 2018.

[22] J. Li, A. Marsiglietti, and J. Melbourne. Further investigations of Rényi entropy power inequalities and an entropic characterization of s-concave densities. In Geometric Aspects of Functional Analysis, pages 95–123. Springer, 2020.

[23] M. Madiman and I. Kontoyiannis. Entropy bounds on abelian groups and the Ruzsa divergence. IEEE Transactions on Information Theory, 64(1):77–92, 2016.
[24] M. Madiman, J. Melbourne, and C. Roberto. Bernoulli sums and Rényi entropy inequalities. *arXiv preprint arXiv:2103.00896*, 2021.

[25] M. Madiman, J. Melbourne, and P. Xu. Forward and reverse entropy power inequalities in convex geometry. *Convexity and Concentration*, pages 427–485, 2017.

[26] M. Madiman, J. Melbourne, and P. Xu. Rogozin's convolution inequality for locally compact groups. *arXiv preprint arXiv:1705.00642*, 2017.

[27] M. Madiman, L. Wang, and J. O. Woo. Majorization and Rényi entropy inequalities via Sperner theory. *Discrete Mathematics*, 342(10):2911–2923, 2019.

[28] A. W. Marshall, I. Olkin, and B. C. Arnold. *Inequalities: Theory of Majorization and Its Applications*. Springer Science & Business Media, 2010.

[29] A. Marsiglietti and J. Melbourne. On the entropy power inequality for the Rényi entropy of order [0, 1]. *IEEE Transactions on Information Theory*, 65(3):1387–1396, 2018.

[30] A. Marsiglietti and J. Melbourne. Geometric and functional inequalities for log-concave probability sequences. *arXiv preprint arXiv:2004.12005*, 2020.

[31] J. Melbourne. Rearrangement and Prékopa–Leindler type inequalities. In *High Dimensional Probability VIII*, pages 71–97. Springer, 2019.

[32] J. Melbourne and C. Roberto. Transport-majorization to analytic and geometric inequalities. *arXiv preprint arXiv:2110.03641*, 2021.

[33] J. Melbourne and T. Tkocz. Reversals of Rényi entropy inequalities under log-concavity. *IEEE Transactions on Information Theory*, 67(1):45–51, 2020.

[34] F. L. Nazarov and A. N. Podkorytov. Ball, Haagerup, and distribution functions. In *Complex analysis, operators, and related topics*, volume 113 of *Oper. Theory Adv. Appl.* Birkhäuser, Basel, 2000.

[35] Y. Ollivier and C. Villani. A curved Brunn-Minkowski inequality on the discrete hypercube, or: what is the Ricci curvature of the discrete hypercube? *SIAM J. Discrete Math.*, 26(3):983–996, 2012.

[36] J. Pitman. Probabilistic bounds on the coefficients of polynomials with only real zeros. *Journal of Combinatorial Theory, Series A*, 77(2):279–303, 1997.

[37] E. Ram and I. Sason. On Rényi entropy power inequalities. *IEEE Transactions on Information Theory*, 62(12):6800–6815, 2016.

[38] O. Rioul. Rényi entropy power inequalities via normal transport and rotation. *Entropy*, 20(9):641, 2018.

[39] C. A. Rogers and G. C. Shephard. Convex bodies associated with a given convex body. *J. London Math. Soc.*, 33:270–281, 1958.

[40] B. A. Slomka. A remark on discrete Brunn-Minkowski type inequalities via transportation of measure. *arXiv preprint arXiv:2008.00738*, 2020.

[41] W. Tang and F. Tang. The Poisson binomial distribution–old & new. *arXiv preprint arXiv:1908.10024*, 2019.

[42] P. Xu, J. Melbourne, and M. Madiman. Reverse entropy power inequalities for s-concave densities. In *Proc. IEEE Intl. Symp. Inform. Theory.*, pages 2284–2288, Barcelona, Spain, July 2016.

[43] Y. Yu. Letter to the editor: On an inequality of Karlin and Rinott concerning weighted sums of i.i.d. random variables. *Adv. in Appl. Probab.*, 40(4):1223–1226, 2008.

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