Papillon graphs: perfect matchings, Hamiltonian cycles and edge-colourings in cubic graphs

Marién Abreu
Dipartimento di Matematica, Informatica ed Economia
Università degli Studi della Basilicata, Italy

John Baptist Gauci
Department of Mathematics, University of Malta, Malta

Domenico Labbate, Federico Romaniello
Dipartimento di Matematica, Informatica ed Economia
Università degli Studi della Basilicata, Italy

Jean Paul Zerafa
Department of Technology and Entrepreneurship Education
University of Malta, Malta

Abstract

A graph $G$ has the Perfect-Matching-Hamiltonian property (PMH-property) if for each one of its perfect matchings, there is another perfect matching of $G$ such that the union of the two perfect matchings yields a Hamiltonian cycle of $G$. The study of graphs that have the PMH-property, initiated in the 1970s by Las Vergnas and Häggkvist, combines three well-studied properties of graphs, namely matchings, Hamiltonicity and edge-colourings. In this work, we study these concepts for cubic graphs in an attempt to characterise those cubic graphs for which every perfect matching corresponds to one of the colours of a proper 3-edge-colouring of the graph. We discuss that this is equivalent to saying that such graphs are even-2-factorable (E2F), that is, all 2-factors of the graph contain only even cycles. The case for bipartite cubic graphs is trivial, since if $G$ is bipartite then it is E2F. Thus, we restrict our attention to non-bipartite cubic graphs. A sufficient, but not necessary, condition for a cubic graph to be E2F is that it has the PMH-property. The aim of this work is to introduce two infinite families of non-bipartite cubic graphs, which we term papillon graphs and unbalanced papillon graphs, and determine the values of their respective parameters for which these graphs have the PMH-property or are just E2F.

Keywords: Cubic graph, perfect matching, Hamiltonian cycle, 3-edge-colouring.

Math. Subj. Class.: 05C15, 05C45, 05C70
1 Introduction

Let $G$ be a connected graph of even order with vertex set $V(G)$ and edge set $E(G)$. A $k$-factor of $G$ is a $k$-regular spanning subgraph of $G$ (not necessarily connected). Two very well-studied concepts in graph theory are perfect matchings and Hamiltonian cycles, where the former is the edge set of a 1-factor and the latter is a connected 2-factor of a graph. For $t \geq 3$, a cycle of length $t$ (or a $t$-cycle), denoted by $C_t = (v_1, \ldots, v_t)$, is a sequence of mutually distinct vertices $v_1, v_2, \ldots, v_t$ with corresponding edge set $\{v_1v_2, \ldots, v_{t-1}v_t, v_tv_1\}$.

For definitions not explicitly stated here we refer the reader to [4]. A graph $G$ admitting a perfect matching is said to have the Perfect-Matching-Hamiltonian property (for short the PMH-property) if for every perfect matching $M$ of $G$ there exists another perfect matching $N$ of $G$ such that the edges of $M \cup N$ induce a Hamiltonian cycle of $G$. For simplicity, a graph admitting this property is said to be PMH. This property was first studied in the 1970s by Las Vergnas [13] and Häggkvist [9], and for more recent results about the PMH-property we suggest the reader to [1, 2, 3, 7, 8]. In [3], a property stronger than the PMH-property is studied: the Pairing-Hamiltonian property, for short the PH-property. Before proceeding to the definition of this property, we first define what a pairing is. For any graph $G$, $K_G$ denotes the complete graph on the same vertex set $V(G)$ of $G$. A perfect matching of $K_G$ is said to be a pairing of $G$, and a graph $G$ is said to have the Pairing-Hamiltonian property if every pairing $M$ of $G$ can be extended to a Hamiltonian cycle $H$ of $K_G$ such that $E(H) - M \subseteq E(G)$. Clearly, a graph having the PH-property is also PMH, although the converse is not necessarily true. Amongst other results, the authors of [3] show that the only cubic graphs admitting the PH-property are the complete graph $K_4$, the complete bipartite graph $K_{3,3}$, and the cube $Q_3$. However, this does not mean that these are the only three cubic graphs admitting the PMH-property. For instance, all cubic 2-factor Hamiltonian graphs (all 2-factors of such a graph form a Hamiltonian cycle) are PMH (see for example [5, 6, 10, 11, 12]).

If a cubic graph $G$ is PMH, then every perfect matching of $G$ corresponds to one of the colours of a (proper) 3-edge-colouring of the graph, and we say that every perfect matching can be extended to a 3-edge-colouring. This is achieved by alternately colouring the edges of the Hamiltonian cycle containing a predetermined perfect matching using two colours, and then colouring the edges not belonging to the Hamiltonian cycle using a third colour. However, there are cubic graphs which are not PMH but have every one of their perfect matchings that can be extended to a 3-edge-colouring (see for example Figure 1).

The following theorem characterises all cubic graphs for which every one of their perfect matchings can be extended to a 3-edge-colouring of the graph.

Figure 1: The bold edges can be extended to a proper 3-edge-colouring but not to a Hamiltonian cycle
Theorem 1.1. Let $G$ be a cubic graph admitting a perfect matching. Every perfect matching of $G$ can be extended to a 3-edge-colouring of $G$ if and only if all 2-factors of $G$ contain only even cycles.

Proof. Let $F$ be a 2-factor of $G$, and let $M$ be the perfect matching $E(G) - E(F)$. Since $M$ can be extended to a 3-edge-colouring of $G$, $F$ can be 2-edge-coloured, and hence $F$ does not contain any odd cycles. Conversely, let $M'$ be a perfect matching of $G$, and let $F'$ be its complementary 2-factor, that is, $E(F') = E(G) - M'$. Since $F'$ contains only even cycles, $M'$ can be extended to a 3-edge-colouring, by assigning a first colour to all of its edges and then alternately colour the edges of the 2-factor $F'$ using another two colours.

We shall call graphs in which all 2-factors consist only of even cycles as even-2-factorable graphs, denoted by E2F for short. In particular, from Theorem 1.1, if a cubic graph $G$ has the PMH-property, then it is also E2F. As in the proof of Theorem 1.1, in the sequel, given a perfect matching $M$ of a cubic graph $G$, the 2-factor obtained after deleting the edges of $M$ from $G$ is referred to as the complementary 2-factor of $M$.

If a cubic graph is bipartite, then trivially, each of its perfect matchings can be extended to a 3-edge-colouring, since it is E2F. But what about non-bipartite cubic graphs? In Table 1, we show that the number of non-bipartite cubic graphs $G$ (having girth at least 4) such that each one of their perfect matchings can be extended to a 3-edge-colouring is not insignificant, and the data suggests that this number increases considerably with the order of $G$. The numbers shown in this table were obtained thanks to a computer check done by Jan Goedgebeur, and the data is sorted according to the cyclic connectivity of the graphs considered. We remark that the reason why only graphs having girth at least 4 were considered, is because cubic graphs admitting the above properties and having girth 3 are somewhat different and shall be discussed further in an upcoming paper.

| Number of vertices | Cyclic connectivity | 3 | 4 | 5 | 6 | TOTAL |
|-------------------|-------------------|---|---|---|---|-------|
| 8                 |                   |   |   |   |   | 1     |
| 10                |                   |   |   |   |   | 0     |
| 12                | 2                 | 5 | 2 | / |   | 9     |
| 14                | 2                 | 2 | 2 | / |   | 6     |
| 16                | 35                | 56| 4 | / |   | 95    |
| 18                | 84                | 21| 9 | / |   | 114   |
| 20                | 926               | 655| 15| 2 |   | 1598  |
| 22                | 2978              | 331| 17| 6 |   | 3332  |

Table 1: The number of non-bipartite cubic 3-connected graphs with girth at least 4 which are E2F

A complete characterisation of which cubic graphs are PMH is still elusive, so considering the Class I non-bipartite cubic graphs having the property that each one of their perfect matchings can be extended to a 3-edge-colouring may look presumptuous. As far as we
know this property and the corresponding characterisation problem were never considered before and tackling the following problem seems a reasonable step to take.

**Problem 1.2.** Characterise the Class I non-bipartite cubic graphs for which each one of their perfect matchings can be extended to a 3-edge-colouring, that is, are E2F.

We remark that although the PMH-property is an appealing property in its own right, Problem 1.2 continues to justify its study in relation to cubic graphs. Observe that in the family of cubic graphs, whilst snarks are not 3-edge-colourable, even-2-factorable graphs are quite the opposite being “very much 3-edge-colourable”, since the latter can be 3-edge-coloured by assigning a colour to one of its perfect matchings, and then alternately colour the edges of the complementary 2-factor.

### 1.1 Cycle permutation graphs

Consider two disjoint cycles each of length \( t \), referred to as the first and second \( t \)-cycles and denoted by \((x_1, \ldots, x_t)\) and \((y_1, \ldots, y_t)\), respectively. Let \( \sigma \) be a permutation of the symmetric group \( S_t \) on the \( t \) symbols \( \{1, \ldots, t\} \). The *cycle permutation graph corresponding to* \( \sigma \) is the cubic graph obtained by considering the first and second \( t \)-cycles in which \( x_i \) is adjacent to \( y_{\sigma(i)} \), where \( \sigma(i) \) is the image of \( i \) under the permutation \( \sigma \).

![Figure 2: Two different drawings of the smallest non-bipartite E2F cubic graph](image)

The smallest non-bipartite cubic graph which is E2F is in fact a cycle permutation graph corresponding to \( \sigma = (1 \ 2) \in S_4 \), where \( \sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3, \) and \( \sigma(4) = 4 \) (see Table 1 and Figure 2). This shows that the edges between the vertices of the first and second 4-cycles of the cycle permutation graph are \( x_1y_2, x_2y_1, x_3y_3, x_4y_4 \). In what follows we shall denote permutations in cycle notation and, for simplicity, fixed points shall be suppressed. According to another computer check we conducted through Wolfram Mathematica [14], quite a large number of the non-bipartite E2F cubic graphs given in Table 1 are also cycle permutation graphs (see Table 2). We remark that, in the sequel, cycle permutation graphs with total number of vertices equal to twice an odd number are not considered because, in this case, the first and second cycles form a 2-factor consisting of two odd cycles, and so they are trivially not E2F.

Recall that PMH cubic graphs are also E2F, and so, PMH cycle permutation graphs should be searched for from amongst the cycle permutation graphs which are E2F. However, as Table 2 suggests, for a given number of vertices \( \nu \), the number of non-bipartite cycle permutation graphs on \( \nu \) vertices which are PMH is very small compared to the number of non-bipartite E2F cycle permutation graphs on the same number of vertices. This suggests
Table 2: The number of non-bipartite cycle permutation graphs with girth at least 4 which are E2F and PMH

| No. of vertices | E2F | PMH |
|-----------------|-----|-----|
| 8               | 1   | 0   |
| 12              | 5   | 1   |
| 16              | 28  | 2   |
| 20              | 175 | 0   |

that non-bipartite cycle permutation graphs which are PMH are hard to find. As already mentioned, by Theorem 1.1, bipartite cubic graphs are trivially E2F but are not necessarily PMH (see for example Figure 1). Although bipartite cubic graphs are not the main scope of this paper, we remark that the number of bipartite cycle permutation graphs which are PMH seems to be very small. By doing a computer check as the one conducted above, the only bipartite cycle permutation graphs having the PMH-property on at most 20 vertices are two: the cube and the bipartite cycle permutation graph on 16 vertices corresponding to the permutation $(3 \ 7)(4 \ 8) \in S_8$.

This work is a first structured attempt at tackling Problem 1.2. We give an infinite family of non-bipartite cycle permutation graphs which admit the PMH-property. In Section 2, we generalise the smallest cubic graph which is E2F into a family of non-bipartite cycle permutation graphs $\{P_n\}_{n \in \mathbb{N}}$ (which we term papillon graphs) whose first member is, in fact, the graph in Figure 2. We show that $P_n$ is E2F for every $n \in \mathbb{N}$ and PMH for every even $n \in \mathbb{N}$. In Section 3, we then generalise further the family of papillon graphs by defining the unbalanced papillon graphs on two parameters $r$ and $\ell$, for $1 \leq r < \ell$. We prove that unbalanced papillon graphs are also E2F for all values of $r$ and $\ell$ and PMH if and only if both $r$ and $\ell$ are even.

2 Papillon graphs

Definition 2.1. For $n \in \mathbb{N}$, the $n^{th}$ papillon graph $P_n$ is the graph on $8n$ vertices such that $V(P_n) = \{u_1, \ldots, u_{4n}, v_1, \ldots, v_{4n}\}$, where:

(i) $(u_1, u_2, \ldots, u_{4n})$ is a cycle of length $4n$;

(ii) $u_i$ is adjacent to $v_i$, for each $i \in [4n]$; and

(iii) if $n = 1$, then $(v_1, v_2, v_4, v_3)$ is a cycle of length 4, whereas if $n \geq 2$, then the adjacencies between the vertices $v_i$, for $i \in [4n]$, form a cycle of length $4n$ given by the edge set

$$\{v_{2i-1}v_{2i} : i \in [2n]\} \cup \{v_{2i-1}v_{2i+2} : i \in [2n-1] \setminus \{n\}\} \cup \{v_2v_{2n+2}, v_{2n-1}v_{4n-1}\}.$$

The papillon graph $P_n$ for $n \geq 2$ is depicted in Figure 3. The $4n$-cycle induced by the vertices $\{u_i : i \in [4n]\}$ is referred to as the outer-cycle, whilst the $4n$-cycle induced by the vertices $\{v_i : i \in [4n]\}$ is referred to as the inner-cycle. The edges on these two $4n$-cycles are said to be the outer-edges and inner-edges accordingly, whilst the edges $u_iv_i$...
are referred to as *spokes*. The edges $u_1 u_{4n}, v_2 v_{4n−1}, v_2 v_{4n+2}, u_2 u_{2n+1}$, are denoted by $a, b, c, d$, respectively, and we shall also denote the set $\{a, b, c, d\}$ by $X$. The set $X$ is referred to as the principal 4-edge-cut of $P_n$.

The graph in Figure 2 is actually the first papillon graph $P_1$, and in Figure 4 we depict the papillon graph $P_3$. We first note that these graphs are non-bipartite since the cycle $(u_1, u_2, \ldots, u_{2n}, u_{2n+1}, v_{2n+1}, v_{2n+2}, v_2, v_1)$ is a cycle of $P_n$ on $2n + 5$ vertices. Furthermore, since $\{u_i : i \in [4n]\}$ and $\{v_i : i \in [4n]\}$ induce two disjoint $4n$-cycles in $P_n$, and since every vertex belonging to the outer-cycle is adjacent to exactly one vertex on the inner-cycle, there exists an isomorphism $\pi$ between the papillon graph $P_n$ and a cycle permutation graph corresponding to some $\sigma \in S_{4n}$ satisfying $\pi(x_i) = u_i$ and $\pi(y_i) = v_{\sigma^{-1}(i)}$, for each $i \in [4n]$. In fact, the $n^{th}$ papillon graph $P_n$ is the cycle permutation graph corresponding to the permutation $\sigma_1 := (1 2)$ when $n = 1$, to $\sigma_2 := (1 2)(3 4)(5 7)(6 8)$ when $n = 2$, and to $\sigma_n := (1 2) \ldots (2n − 1 2n)(2n + 1 4n − 1)(2n + 2 4n)(2n + 3 4n − 3)(2n + 4 4n − 2) \ldots (\alpha \beta)$, otherwise, where $(\alpha \beta) = (3n 3n + 2)$ if $n$ is even, and $(\alpha \beta) = (3n − 1 3n + 3)$ if $n$ is odd. We remark that $\sigma_n$ has no fixed points when $n$ is even, but, when $n$ is odd, $3n$ and $3n + 1$ are fixed points of the permutation, and thus, in this case, $x_{3n}$ is adjacent to $y_{3n}$ and $x_{3n+1}$ is adjacent to $y_{3n+1}$ in $P_n$. Note that since $\sigma_n$ is an involution for all positive integers $n$, the isomorphism $\pi$ mentioned above can be rewritten as follows: $\pi(x_i) = u_i$ and $\pi(y_i) = v_{\sigma(i)}$, for each $i \in [4n]$. Moreover, the papillon graph $P_n$ admits a natural automorphism $\psi$ which exchanges the two cycles, given by $\psi(u_i) = v_{\sigma_n(i)}$ and $\psi(v_i) = u_{\sigma_n(i)}$, for each $i \in [4n]$. In fact, the function $\psi$ is clearly bijective. Moreover, it maps edges of the outer-cycle to edges of the inner-cycle (and vice-versa), and maps spokes to spokes, since the edges $u_i v_i$ are mapped to $u_{\sigma_n(i)} v_{\sigma_n(i)}$.

Before proceeding we introduce multipoles which generalise the notion of graphs. This will become useful when describing papillon graphs. A *multipole* $Z$ consists of a set of vertices $V(Z)$ and a set of generalised edges such that each generalised edge is either
an edge in the usual sense (that is, it has two endvertices) or a semiedge. A semiedge is a generalised edge having exactly one endvertex. The set of semiedges of $Z$ is denoted by $\partial Z$ whilst the set of edges of $Z$ having two endvertices is denoted by $E(Z)$. Two semiedges are joined if they are both deleted and their endvertices are made adjacent. A $k$-pole is a multipole with $k$ semiedges. A perfect matching $M$ of a $k$-pole $Z$ is a subset of generalised edges of $Z$ such that every vertex of $Z$ is incident with exactly one generalised edge of $M$. In what follows, we shall construct papillon graphs by joining together semiedges of a number of multipoles. In this sense, given a perfect matching $M$ of a graph $G$, and a multipole $Z$ used as a building block to construct $G$, we shall say that $M$ contains a semiedge $e$ of the multipole $Z$, if $M$ contains the edge in $G$ obtained by joining $e$ to another semiedge in the process of constructing $G$.

The $4$-pole $Z$ with vertex set $\{z_1, z_2, z_3, z_4\}$, such that $E(Z)$ induces the $4$-cycle $(z_1, z_2, z_3, z_4)$ and with exactly one semiedge incident to each of its vertices is referred to as a $C_4$-pole (see Figure 5). For each $i \in [4]$, let the semiedge incident to $z_i$ be denoted by $f_i$. The semiedges $f_1$ and $f_2$ are referred to as the upper left semiedge and the upper right semiedge of $Z$, respectively. On the other hand, the semiedges $f_3$ and $f_4$ are referred to as the lower left semiedge and the lower right semiedge of $Z$, respectively (see Figure 5).

For some integer $n \geq 1$, let $Z_1, \ldots, Z_n$ be $n$ copies of the above $C_4$-pole $Z$. For each $j \in [n]$, let $V(Z_j) = \{z_1^j, z_2^j, z_3^j, z_4^j\}$, and let $f_1^j, f_2^j, f_3^j, f_4^j$ be the semiedges of $Z_j$ respectively incident to $z_1^j, z_2^j, z_3^j, z_4^j$ such that $f_1^j$ and $f_2^j$ are the upper left and upper right semiedges of $Z_j$, whilst $f_3^j$ and $f_4^j$ are the lower left and lower right semiedges of $Z_j$. A chain of $C_4$-poles of length $n \geq 2$, is the $4$-pole obtained by respectively joining $f_2^1$ and $f_4^1$ (upper and lower right semiedges of $Z_j$) to $f_1^{j+1}$ and $f_3^{j+1}$ (upper and lower left semiedges of $Z_{j+1}$), for every $j \in [n-1]$. When $n = 1$, a chain of $C_4$-poles of length 1 is just a $C_4$-pole. For simplicity, we shall refer to a chain of $C_4$-poles of length $n$, as a $n$-chain of $C_4$-poles, or simply a $n$-chain. The semiedges $f_1^1$ and $f_3^1$ (similarly, $f_2^1$ and $f_4^1$) are referred...
to as the upper left and lower left (respectively, upper right and lower right) semiedges of the \( n \)-chain. A chain of \( C_4 \)-poles of any length has exactly four semiedges. For simplicity, when we say that \( e_1, e_2, e_3, e_4 \) are the four semiedges of a chain \( Z' \) of \( C_4 \)-poles (possibly of length 1), we mean that \( e_1 \) and \( e_2 \) are respectively the upper left and upper right semiedges of \( Z' \), whilst \( e_3 \) and \( e_4 \) are respectively the lower left and lower right semiedges of the same chain \( Z' \) (see Figure 6). The semiedges \( e_1 \) and \( e_2 \) (similarly, \( e_3 \) and \( e_4 \)) are referred to collectively as the upper semiedges (respectively, lower semiedges) of \( Z' \). In a similar way, the semiedges \( e_1 \) and \( e_3 \) (similarly, \( e_2 \) and \( e_4 \)) are referred to collectively as the left semiedges (respectively, right semiedges) of \( Z' \).

![Figure 5: A \( C_4 \)-pole \( Z \) and the 4-pole \( T_j \) in \( P_n \)](image)

![Figure 6: A chain of \( C_4 \)-poles of length 3 having semiedges \( e_1, e_2, e_3, e_4 \)](image)

In order to construct the papillon graph \( P_n \) using \( C_4 \)-poles as building blocks, for each \( j \in [2n] \), we consider the 4-pole \( T_j \) arising from the cycle \((u_{2j-1}, u_{2j}, v_{2j}, v_{2j-1})\) of \( P_n \), whose semiedges are \( e_1^j, e_2^j, e_3^j, e_4^j \) as in Figure 5. The two \( n \)-chains giving rise to \( P_n \) consist of \( T_1, \ldots, T_n \) (referred to as the right \( n \)-chain of \( P_n \)), and \( T_{n+1}, \ldots, T_{2n} \) (referred to as the left \( n \)-chain of \( P_n \)), which have semiedges \( e_1^1, e_2^1, e_3^1, e_4^1 \), and \( e_1^{n+1}, e_2^{n+1}, e_3^{n+1}, e_4^{2n} \), respectively. The papillon graph \( P_n \) is then obtained by joining the semiedges in pairs as follows: \( e_1^1 \) to \( e_2^{2n} \), \( e_2^1 \) to \( e_1^{n+1} \), \( e_3^1 \) to \( e_3^{n+1} \), and \( e_4^1 \) to \( e_4^{2n} \).

### 2.1 Main results

Let \( M \) be a perfect matching of \( P_n \). Since \( X = \{a, b, c, d\} \) is a 4-edge-cut of \( P_n \), \( |M \cap X| \equiv 0 \pmod{2} \), that is, \( |M \cap X| \) is 0, 2 or 4. The following is a useful lemma which shall be used frequently in the results that follow.

**Lemma 2.2.** Let \( M \) be a perfect matching of the papillon graph \( P_n \) and let \( X \) be its principal 4-edge-cut. If \( |M \cap X| = k \), then \( |M \cap \partial T_j| = k \), for each \( j \in [2n] \).

**Proof.** Let \( M \) be a perfect matching of \( P_n \). We first note that the left semiedges of a \( C_4 \)-pole are contained in a perfect matching if and only if the right semiedges of the \( C_4 \)-pole are contained in the same perfect matching. The lemma is proved by considering three
cases depending on the possible values of \( k \), that is, 0, 2 or 4. When \( n = 1 \), the result clearly follows since \( \mathcal{X} \) is made up by joining \( \partial \mathcal{T}_1 \) and \( \partial \mathcal{T}_2 \) accordingly. So assume \( n \geq 2 \).

**Case I.** \( k = 0 \).

Since \( a \) and \( c \) do not belong to \( M \), the left semiedges of \( \mathcal{T}_1 \) are not contained in \( M \), and so \( M \) cannot contain its right semiedges. Therefore, \(|M \cap \partial \mathcal{T}_1| = 0\). Consequently, the left semiedges of \( \mathcal{T}_2 \) are not contained in \( M \) implying again that \(|M \cap \partial \mathcal{T}_2| = 0\). By repeating the same argument up till the \( n^{th} \) \( C_4 \)-pole, we have that \(|M \cap \partial \mathcal{T}_j| = 0\), for every \( j \in [n] \).

By noting that \( c \) and \( d \) do not belong to \( M \) and repeating a similar argument to the \( 4 \)-poles in the left \( n \)-chain, we can deduce that \(|M \cap \partial \mathcal{T}_j| = 0\) for every \( j \in [2n] \).

**Case II.** \( k = 4 \).

Since \( a \) and \( c \) belong to \( M \), the left semiedges of \( \mathcal{T}_1 \) are contained in \( M \), and so \( M \) contains its right semiedges as well. Therefore, \(|M \cap \partial \mathcal{T}_1| = 4\). Consequently, the left semiedges of \( \mathcal{T}_2 \) are contained in \( M \) implying again that \(|M \cap \partial \mathcal{T}_2| = 4\). As in Case I, by noting that both \( c \) and \( d \) belong to \( M \) and repeating a similar argument to the \( 4 \)-poles in the left \( n \)-chain, we can deduce that \(|M \cap \partial \mathcal{T}_j| = 4\) for every \( j \in [2n] \).

**Case III.** \( k = 2 \).

We first claim that when \( k = 2 \), \( M \cap \mathcal{X} \) must be equal to \( \{a, d\} \) or \( \{b, c\} \). For, suppose that \( M \cap \mathcal{X} = \{a, c\} \), without loss of generality. This means that the right semiedges of \( \mathcal{T}_1 \) are also contained in \( M \), implying that \(|M \cap \partial \mathcal{T}_1| = 4\). This implies that the left semiedges of \( \mathcal{T}_2 \) are contained in \( M \), which forces \(|M \cap \partial \mathcal{T}_2| \) to be equal to 4, for every \( j \in [2n] \). In particular, \(|M \cap \partial \mathcal{T}_n| = 4\), implying that the edges \( b \) and \( d \) belong to \( M \), a contradiction since \( M \cap \mathcal{X} = \{a, c\} \). This proves our claim. Since the natural automorphism \( \psi \) of \( \mathcal{P}_n \), which exchanges the outer- and inner-cycles, exchanges also \( \{a, d\} \) with \( \{b, c\} \), without loss of generality, we may assume that \( M \cap \mathcal{X} = \{a, d\} \). Since \( c \notin M \), \( 1 \leq |M \cap \partial \mathcal{T}_1| < 4 \).

But, \( \partial \mathcal{T}_1 \) corresponds to a \( 4 \)-edge-cut in \( \mathcal{P}_n \), and so, by using a parity argument, \(|M \cap \partial \mathcal{T}_1| \) must be equal to 2, implying that exactly one of the right semiedges of \( \mathcal{T}_1 \) is contained in \( M \). This means that exactly one left semiedge of \( \mathcal{T}_2 \) is contained in \( M \), and consequently, by a similar argument now applied to \( \mathcal{T}_2 \), we obtain \(|M \cap \partial \mathcal{T}_2| = 2\). By repeating the same argument and noting that \( \mathcal{T}_{n+1} \) has exactly one left semiedge (corresponding to the edge \( d \)) contained in \( M \), one can deduce that \(|M \cap \partial \mathcal{T}_j| = 2\) for every \( j \in [2n] \).

The following two results are two consequences of the above lemma and they both follow directly from the proof of Case III. In a few words, if a perfect matching \( M \) of \( \mathcal{P}_n \) intersects its principal \( 4 \)-edge-cut in exactly two of its edges, then these two edges are either the pair \( \{a, d\} \) or the pair \( \{b, c\} \), and, for every \( j \in [2n] \), \( M \) contains only one pair of semiedges of \( \mathcal{T}_j \) which does not consist of the pair of left semiedges of \( \mathcal{T}_j \) nor the pair of right semiedges of \( \mathcal{T}_j \).

**Corollary 2.3.** Let \( M \) be a perfect matching of \( \mathcal{P}_n \) and let \( \mathcal{X} \) be its principal \( 4 \)-edge-cut. If \(|M \cap \mathcal{X}| = 2\), then \( M \cap \mathcal{X} \) is equal to \( \{a, d\} \) or \( \{b, c\} \).

**Corollary 2.4.** Let \( M \) be a perfect matching of \( \mathcal{P}_n \) and let \( \mathcal{X} \) be its principal \( 4 \)-edge-cut such that \(|M \cap \mathcal{X}| = 2\). For each \( j \in [2n] \), \( M \) contains exactly one of the following sets of semiedges: \( \{e^1_1, e^2_2\} \), \( \{e^3_3, e^4_4\} \), \( \{e^1_1, e^3_3\} \), \( \{e^2_2, e^4_4\} \), that is, of all possible pairs of semiedges of \( \mathcal{T}_j \), \( \{e^1_1, e^3_3\} \) and \( \{e^2_2, e^4_4\} \) cannot be contained in \( M \).
In the sequel, the process of traversing one path after another shall be called concatenation of paths. If two paths \( P \) and \( Q \) have endvertices \( x, y \) and \( y, z \), respectively, we write \( PQ \) to denote the path starting at \( x \) and ending at \( z \) obtained by traversing \( P \) and then \( Q \). Without loss of generality, if \( x \) is adjacent to \( y \), that is, \( P \) is a path on two vertices, we may write \( xyQ \) instead of \( PQ \).

**Lemma 2.5.** Let \( M_1 \) be a perfect matching of \( P_n \) such that \( |M_1 \cap \mathcal{X}| = 2 \).

(i) There exists a perfect matching \( M_2 \) of \( P_n \) such that \( |M_2 \cap \mathcal{X}| = 2 \) and \( M_1 \cap M_2 = \emptyset \).

(ii) The complementary 2-factors of \( M_1 \) and \( M_2 \) are both Hamiltonian cycles.

**Proof.** (i) Since \( |M_1 \cap \mathcal{X}| = 2 \), by Lemma 2.2 we get that \( |M_1 \cap \partial T_j| = 2 \) for every \( j \in [2n] \). For each \( j \), let \( P^{(j)} \) be the subgraph of \( P_n \) which is induced by \( E(T_j) - M_1 \). Note that \( \cup_{j=1}^{2n} V(P^{(j)}) = V(P_n) \). By Corollary 2.4, each \( P^{(j)} \) is a path of length 3. Letting \( N \) be the unique perfect matching of \( P_n \) which intersects each \( E(P^{(j)}) \) in exactly two edges, we note that \( M_1 \cap N = \emptyset \). Let \( M_2 = E(P_n) - (M_1 \cup N) \). Since \( M_1 \) and \( N \) are two disjoint perfect matchings, \( M_2 \) is also a perfect matching of \( P_n \) and, in particular, \( M_2 \) contains \( \mathcal{X} - (M_1 \cap \mathcal{X}) \). Thus, \( |M_2 \cap \mathcal{X}| = 2 \) and \( M_1 \cap M_2 = \emptyset \), proving part (i).

(ii) Let \( M_2 \) be as in part (i), that is, \( |M_2 \cap \mathcal{X}| = 2 \) and \( M_1 \cap M_2 = \emptyset \). When \( n = 1 \), the result clearly follows. So assume \( n \geq 2 \). For distinct \( i \) and \( j \) in \([2n] \), let \( Q^{(i,j)} \) be the subgraph of \( P_n \) which is induced by \( M_2 \cap \{xy \in E(P_n) : x \in V(T_i), y \in V(T_j)\} \), that is, \( E(Q^{(i,j)}) \) is either empty or consists of exactly one edge, that is, \( Q^{(i,j)} \) is a path of length 1. When \( M_1 \cap \mathcal{X} = \{a, d\} \), we can form a Hamiltonian cycle of \( P_n \) (not containing \( M_1 \)) by considering the following concatenation of paths:

\[
P^{(1)}Q^{(1,2)} \ldots Q^{(n-1,n)}P^{(n)}Q^{(n,2n)}P^{(2n)}Q^{(2n,2n-1)} \ldots P^{(n+1)}Q^{(n+1,1)},
\]

where \( Q^{(1,2)} \) and \( Q^{(2n,2n-1)} \) are respectively followed by \( P^{(2)} \) and \( P^{(2n-1)} \), and, \( Q^{(n,2n)} \) and \( Q^{(n+1,1)} \) consist of the edges \( b \) and \( c \), respectively. On the other hand, when \( M_1 \cap \mathcal{X} = \{b, c\} \), we can form a Hamiltonian cycle of \( P_n \) (not containing \( M_1 \)) by considering the following concatenation of paths:

\[
P^{(1)}Q^{(1,2)} \ldots Q^{(n-1,n)}P^{(n)}Q^{(n,n+1)}P^{(n+1)}Q^{(n+1,n+2)} \ldots P^{(2n)}Q^{(2n,1)},
\]
where \( Q^{(1,2)} \) and \( Q^{(n+1,n+2)} \) are respectively followed by \( P^{(2)} \) and \( P^{(n+2)} \), and, \( Q^{(n,n+1)} \) and \( Q^{(2n,1)} \) consist of the edges \( d \) and \( a \), respectively. Thus, the complementary 2-factor of \( M_1 \) is a Hamiltonian cycle. This is depicted in Figure 7. The proof that the complementary 2-factor of \( M_2 \) is a Hamiltonian cycle follows analogously.

**Theorem 2.6.** The papillon graphs \( \{P_n\}_{n \in \mathbb{N}} \) are E2F.

**Proof.** Let \( M_1 \) be a perfect matching of \( P_n \). In order to show that \( P_n \) is E2F it suffices to show that the complementary 2-factor of \( M_1 \) consists only of even cycles. Equivalently, by Theorem 1.1, we can show that \( P_n \) admits a perfect matching \( N \) such that \( M_1 \cap N = \emptyset \). We consider three cases, depending on \( |M_1 \cap X| \).

**Case I.** \( |M_1 \cap X| = 0 \)

By Lemma 2.2, for each \( j \in [2n] \), \( |M_1 \cap \partial T_j| = 0 \), and consequently, \( |M_1 \cap E(T_j)| = 2 \). Since \( M_1 \) is a perfect matching, \( E(T_j) - M_1 \) is a matching consisting of two edges, for each \( j \in [2n] \). Letting \( M_2 = (\bigcup_{j=1}^{2n} E(T_j)) - M_1 \), we obtain a perfect matching of \( P_n \), since \( |M_2| = \frac{|V(P_n)|}{2} \). Moreover, \( M_1 \cap M_2 = \emptyset \) by definition of \( M_2 \), proving Case I.

**Case II.** \( |M_1 \cap X| = 2 \)

By Lemma 2.2, the complementary 2-factor of \( M_1 \) is a Hamiltonian cycle, and so, since \( V(P_n) \) is even, the result follows.

**Case III.** \( |M_1 \cap X| = 4 \)

The complementary 2-factor of \( M_1 \) consists of \( 2n \) 4-cycles, implying that the complementary 2-factor consists of even cycles only.

**Proposition 2.7.** Let \( n \) be a positive odd integer. Then, the papillon graph \( P_n \) is not PMH.

**Proof.** Consider the following perfect matching of the papillon graph \( P_n \):

\[
M = \bigcup_{i=1}^{2n} \{u_{2i-1}u_{2i}, v_{2i-1}v_{2i}\}.
\]

It is clear that when \( n = 1 \), the perfect matching \( M \) cannot be extended to a Hamiltonian cycle of the papillon graph \( P_1 \). So assume that \( n \geq 3 \). We claim that \( M \) cannot be extended to a Hamiltonian cycle of \( P_n \). For, let \( F \) be a 2-factor of \( P_n \) containing \( M \). Since \( u_1u_2 \in M \) and \( P_n \) is cubic, \( F \) contains exactly one of the following two edges: \( u_1u_{4n} \) or \( u_1v_1 \). In the former case, if \( u_1u_{4n} \in E(F) \), then, \( u_{2n}u_{2n+1} \) and all the edges of the outer- and inner-cycle will belong to \( F \) (at the same time, the choice of \( u_1u_{4n} \) forbids all the spokes of \( P_n \) to belong to \( F \)), yielding two disjoint cycles each of length \( 4n \). In the latter case, if \( u_1v_1 \in E(F) \), then \( F \) must also contain all spokes \( u_iv_i \), for \( 1 < i \leq 4n \). In fact, the subgraph induced by the set of spokes is exactly the complement of the 2-factor obtained in the former case. Consequently, \( F \) will consist of \( 2n \) disjoint 4-cycles.

Consider \( P_n \), with \( n \geq 2 \), and let \( M \) be a perfect matching of \( P_n \) with \( M \cap X = 0 \), which by Lemma 2.2 implies that \( |M_2 \cap \partial T_j| = 0 \) for all \( j \in [2n] \). Now consider \( j \in [2n] \setminus \{n, 2n\} \) and let \( T_{j,j+1} \) denote a 2-chain composed of \( T_j \) and \( T_{j+1} \). We say that \( T_{j,j+1} \) is symmetric with respect to \( M \) if exactly one of the following occurs:

(i) \( \{u_{2j-1}v_{2j-1}, u_{2j}v_{2j}, u_{2j+1}v_{2j+1}, u_{2j+2}v_{2j+2}\} \subset M \); or
(ii) \( \{u_{2j-1}u_{2j}, v_{2j-1}v_{2j}, u_{2j+1}u_{2j+2}, v_{2j+1}v_{2j+2}\} \subset M. \)

If neither (i) nor (ii) occur, \( T_{(j,j+1)} \) is said to be asymmetric with respect to \( M \). This is shown in Figure 8.

Remark 2.8. Let \( n \geq 2 \). Consider a perfect matching \( M_1 \) of \( P_n \) such that \( M_1 \) does not intersect the principal 4-edge-cut \( X \) of \( P_n \), that is, \( M_1 \cap X = \emptyset \), and consider a 2-chain of \( P_n \), say \( T_{(j,j+1)} \) with \( j \in [2n] \setminus \{n, 2n\} \), having semiedges \( e_1, e_2, e_3, e_4 \), where \( e_1 = e_j^1, e_2 = e_{j+1}^2, e_3 = e_3^j \) and \( e_4 = e_{j+1}^3 \). Assume there exists a perfect matching \( M_2 \) of \( P_n \) such that \( |M_2 \cap X| = 2 \) and \( M_1 \cap M_2 = \emptyset \) (see Figure 9). If \( T_{(j,j+1)} \) is symmetric with respect to \( M_1 \), then we have exactly one of the following instances:

\[
M_2 \cap \partial T_{(j,j+1)} = \{e_1, e_2\} \text{ (upper); or } M_2 \cap \partial T_{(j,j+1)} = \{e_3, e_4\} \text{ (lower)}. 
\]

Otherwise, if \( T_{(j,j+1)} \) is asymmetric with respect to \( M_1 \), then exactly one of the following must occur:

\[
M_2 \cap \partial T_{(j,j+1)} = \{e_1, e_4\} \text{ (upper left, lower right); or } M_2 \cap \partial T_{(j,j+1)} = \{e_2, e_3\} \text{ (upper right, lower left)}. 
\]

Notwithstanding whether \( T_{(j,j+1)} \) is symmetric or asymmetric with respect to \( M_1 \), \( (M_1 \cup M_2) \cap E(T_{(j,j+1)}) \) induces a path (see Figure 9) which contains all the vertices of \( V(T_{(j,j+1)}) \), and whose endvertices are the endvertices of the semiedges in \( M_2 \cap \partial T_{(j,j+1)} \).

Remark 2.9. Let \( n \geq 2 \). Consider a perfect matching \( M_1 \) of \( P_n \) such that \( M_1 \) does not intersect the principal 4-edge-cut \( X \) of \( P_n \), that is, \( M_1 \cap X = \emptyset \), and consider a 2-chain of \( P_n \), say \( T_{(j,j+1)} \) with \( j \in [2n] \setminus \{n, 2n\} \). Let \( M_2 \) be the perfect matching of \( P_n \) such that \( |M_2 \cap X| = 4 \). Clearly \( M_1 \cap M_2 = \emptyset \). Notwithstanding whether \( T_{(j,j+1)} \) is symmetric or asymmetric with respect to \( M_1 \), we have that \( (M_1 \cup M_2) \cap E(T_{(j,j+1)}) \) induces two disjoint paths of equal length (see Figure 10) whose union contains all the vertices of \( T_j \) and \( T_{j+1} \). Let \( Q \) be one of these paths. We first note that \( Q \) contains exactly one vertex from \( \{u_j, v_{j+1}\} \) and exactly one vertex from \( \{u_{j+3}, v_{j+2}\} \). If \( T_{(j,j+1)} \) is symmetric with respect to \( M_1 \), then \( Q \) contains \( u_j \) if and only if \( Q \) contains \( u_{j+3} \). Otherwise, if \( T_{(j,j+1)} \) is asymmetric with respect to \( M_1 \), then \( Q \) contains \( u_j \) if and only if \( Q \) contains \( v_{j+2} \).
Theorem 2.10. Let $n$ be a positive even integer. Then, the papillon graph $\mathcal{P}_n$ is PMH.

Proof. Let $M_1$ be a perfect matching of $\mathcal{P}_n$. We need to show that there exists a perfect matching $M_2$ of $\mathcal{P}_n$ such that $M_1 \cup M_2$ induces a Hamiltonian cycle of $\mathcal{P}_n$. Three cases, depending on the intersection of $M_1$ with the principal 4-edge-cut $\mathcal{X}$ of $\mathcal{P}_n$, are considered. If $|M_1 \cap \mathcal{X}| = 2$, then, by Lemma 2.5, there exists a perfect matching $N$ of $\mathcal{P}_n$ such that $|N \cap \mathcal{X}| = 2$ and $M_1 \cap N = \emptyset$. Moreover, the complementary 2-factor of $N$ is a Hamiltonian cycle. Since $M_1$ is contained in the mentioned 2-factor, the result follows. When $|M_1 \cap \mathcal{X}| = 4$, we can define $M_2$ to be the following perfect matching:

$$M_2 = \{u_1v_1, u_2v_2\} \cup \bigcup_{j=2}^{2n} \{u_{2j-1}u_{2j}, v_{2j-1}v_{2j}\}.$$

In fact, $M_1 \cup M_2$ induces the following Hamiltonian cycle: $(u_1, v_1, v_4, \ldots, v_{2n}, v_{2n-1}, v_{4n-1}, v_{4n}, v_{4n-3}, \ldots, v_{2n+1}, v_{2n+2}, v_2, u_2, u_3, u_4, \ldots, u_4n)$, where $v_4$ and $v_{4n-3}$ are respectively followed by $v_3$ and $v_{4n-2}$. 

Figure 9: 2-chains when $M_1 \cap \mathcal{X} = \emptyset$ and $|M_2 \cap \mathcal{X}| = 2$ (bold edges belong to $M_1$ and highlighted edges to $M_2$)

Figure 10: 2-chains when $M_1 \cap \mathcal{X} = \emptyset$ and $|M_2 \cap \mathcal{X}| = 4$ (bold edges belong to $M_1$ and highlighted edges to $M_2$)
What remains to be considered is the case when $|M_1 \cap \mathcal{X}| = 0$. Clearly, $|M_2 \cap \mathcal{X}|$ cannot be zero, because, if so, choosing $M_2$ to be disjoint from $M_1$, $M_1 \cup M_2$ induces 2n disjoint 4-cycles. Therefore, $|M_2 \cap \mathcal{X}|$ must be equal to 2 or 4. Let $\mathcal{R} = \{\mathcal{T}_{(1,2)}, \ldots, \mathcal{T}_{(n-1,n)}\}$ and $\mathcal{L} = \{\mathcal{T}_{(n+1,n+2)}, \ldots, \mathcal{T}_{(2n-1,2n)}\}$ be the sets of 2-chains within the left and right $n$-chains of $\mathcal{P}_n$—namely the right and left $n$-chains each split into $\frac{n}{2}$ 2-chains. We consider two cases depending on the parity of the number of 2-chains in $\mathcal{L}$ and $\mathcal{R}$ which are asymmetric with respect to $M_1$. Let the function $\Phi : \mathcal{R} \cup \mathcal{L} \to \{-1, +1\}$ be defined on the 2-chains $\mathcal{T} \in \mathcal{R} \cup \mathcal{L}$ such that:

$$
\Phi(\mathcal{T}) = \begin{cases} 
+1 & \text{if } \mathcal{T} \text{ is symmetric with respect to } M_1, \\
-1 & \text{otherwise.}
\end{cases}
$$

**Case 1.** $\mathcal{L}$ and $\mathcal{R}$ each have an even number (possibly zero) of asymmetric 2-chains with respect to $M_1$.

We claim that there exists a perfect matching such that its union with $M_1$ gives a Hamiltonian cycle of $\mathcal{P}_n$. Since the number of asymmetric 2-chains in $\mathcal{R}$ is even, $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = +1$, and consequently, appropriately concatenating paths as in Remark 2.8, there exists a path $R$ with endvertices $u_1$ and $u_{2n}$ whose vertex set is $\cup_{i=1}^{2n} \{u_i, v_i\}$ such that it contains all the edges in $M_1 \cap (\cup_{i=1}^{2n} E(\mathcal{T}_i))$. We remark that this path intersects exactly one edge of $\{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_j), y \in V(\mathcal{T}_{j+1})\}$, for each $j \in [n-1]$. By a similar reasoning, since $\prod_{\mathcal{T} \in \mathcal{L}} \Phi(\mathcal{T}) = +1$, there exists a path $L$ with endvertices $u_{2n+1}$ and $u_{4n}$ whose vertex set is $\cup_{i=2n+1}^{4n} \{u_i, v_i\}$, such that it contains all the edges in $M_1 \cap (\cup_{i=n+1}^{4n} E(\mathcal{T}_i))$. Once again, this path intersects exactly one edge of $\{xy \in E(\mathcal{P}_n) : x \in V(\mathcal{T}_j), y \in V(\mathcal{T}_{j+1})\}$, for each $j \in [n+1, \ldots, 2n-1]$. These two paths, together with the edges $a$ and $d$ form the required Hamiltonian cycle of $\mathcal{P}_n$ containing $M_1$, proving our claim. We remark that this shows that there exists a perfect matching $M_2$ of $\mathcal{P}_n$ such that $M_2 \cap \mathcal{X} = \{a, d\}$, $M_1 \cap M_2 = \emptyset$ and with $M_1 \cup M_2$ inducing a Hamiltonian cycle of $\mathcal{P}_n$. One can similarly show that there exists a perfect matching $M_2'$ of $\mathcal{P}_n$ such that $M_2' \cap \mathcal{X} = \{b, c\}$, $M_1 \cap M_2' = \emptyset$ and with $M_1 \cup M_2'$ inducing a Hamiltonian cycle of $\mathcal{P}_n$.

**Case 2.** One of $\mathcal{L}$ and $\mathcal{R}$ has an odd number of asymmetric 2-chains with respect to $M_1$.

Without loss of generality, assume that $\mathcal{R}$ has an odd number of asymmetric 2-chains with respect to $M_1$, that is, $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = -1$. Let $M_2$ be the perfect matching of $\mathcal{P}_n$ such that $|M_2 \cap \mathcal{X}| = 4$. We claim that $M_1 \cup M_2$ induces a Hamiltonian cycle of $\mathcal{P}_n$. Since $\prod_{\mathcal{T} \in \mathcal{R}} \Phi(\mathcal{T}) = -1$, by appropriately concatenating paths as in Remark 2.9 we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths $R_1$ and $R_2$, such that:

(i) $|V(R_1)| = |V(R_2)| = 2n$;

(ii) $V(R_1) \cup V(R_2) = \cup_{i=1}^{2n} \{u_i, v_i\}$;

(iii) the endvertices of $R_1$ are $u_1$ and $v_{2n-1}$; and

(iv) the endvertices of $R_2$ are $v_2$ and $u_{2n}$.

Next, we consider two subcases depending on the value of $\prod_{\mathcal{T} \in \mathcal{L}} \Phi(\mathcal{T})$. We shall be using the fact that $\{u_1u_{4n}, v_{2n-1}v_{4n-1}, v_2v_{2n+2}, u_{2n}u_{2n+1}\} = \{a, b, c, d\} = \mathcal{X} \subset M_2$.

**Case 2a** $\prod_{\mathcal{T} \in \mathcal{L}} \Phi(\mathcal{T}) = -1$
As above, by Remark 2.9, we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths $L_1$ and $L_2$, such that:

(i) $|V(L_1)| = |V(L_2)| = 2n$;
(ii) $V(L_1) \cup V(L_2) = \cup_{i=2n+1}^{4n} \{u_i, v_i\}$;
(iii) the endvertices of $L_1$ are $u_{2n+1}$ and $v_{4n-1}$; and
(iv) the endvertices of $L_2$ are $v_{2n+2}$ and $u_{4n}$.

The concatenation of the following paths and edges gives a Hamiltonian cycle of $P_n$ containing $M_1$:

$$R_1 v_{2n-1} v_{n-1} L_1 u_{2n+1} u_{2n} R_2 v_{2n+2} L_2 u_{4n+1} u_1.$$  

**Case 2b)** $\prod_{T \in \mathcal{L}} \Phi(T) = +1$.

Once again, by Remark 2.9 we can deduce that $M_1 \cup M_2$ contains the edge set of two disjoint paths $L_1$ and $L_2$, such that:

(i) $|V(L_1)| = |V(L_2)| = 2n$;
(ii) $V(L_1) \cup V(L_2) = \cup_{i=2n+1}^{4n} \{u_i, v_i\}$;
(iii) the endvertices of $L_1$ are $u_{2n+1}$ and $u_{4n}$; and
(iv) the endvertices of $L_2$ are $v_{2n+2}$ and $v_{4n-1}$.

The concatenation of the following paths and edges gives a Hamiltonian cycle of $P_n$ containing $M_1$:

$$R_1 v_{2n-1} v_{n-1} L_2 v_{2n+2} v_2 R_2 u_{2n} u_{2n+1} L_1 u_{4n} u_1.$$  

This completes the proof. \[\square\]

### 3 Unbalanced papillon graphs

Papillon graphs can be further generalised by adding or removing $C_4$-poles from the left and right $n$-chains of our original construction, or equivalently, by joining accordingly the semiedges of two chains of different lengths, as follows.

Let $r$ and $\ell$ be two positive integers such that $r < \ell$. Consider an $r$-chain and an $\ell$-chain whose semiedges are $e_1, e_2, e_3, e_4$ and $e'_1, e'_2, e'_3, e'_4$, respectively. The **unbalanced papillon graph** $P_{r,\ell}$ is the graph obtained by joining: $e_1$ to $e'_2$, $e_2$ to $e'_1$, $e_3$ to $e'_3$, and $e_4$ to $e'_4$.

For completeness, we also give the definition of unbalanced papillon graphs as in Definition 2.1 and subsequently in terms of cycle permutation graphs.

**Definition 3.1.** The **unbalanced papillon graph** $P_{r,\ell}$ is the graph on $4r + 4\ell$ vertices such that $V(P_{r,\ell}) = \{u_i, v_i : i \in [2r + 2\ell]\}$, where:

(i) $(u_1, u_2, \ldots, u_{2r+2\ell})$ is a cycle of length $2r + 2\ell$;
(ii) $u_i$ is adjacent to $v_i$, for each $i \in [2r + 2\ell]$; and
extended to unbalanced papillon graphs in the following way.

Over, by following the proofs in Section 2, the results obtained for papillon graphs can be

\[ u_{x+1} \text{ and } u_2 \] respectively. As in the original case, the unbalanced papillon graph

\[ P_{r,ℓ} \] graph, with

\[ u_{x+1}^{r+1}, \ldots, u_2 \text{ is odd}, \]

\[ v_{i}^{r+2}, \text{ otherwise.} \]

The \( (2r + 2ℓ) \)-cycles induced by the sets of vertices \( \{u_i : i ∈ [2r + 2ℓ]\} \) and \( \{v_i : i ∈ [2r + 2ℓ]\} \) are the outer-cycle and the inner-cycle of our unbalanced papillon graph, respectively. As in the original case, the unbalanced papillon graph \( P_{r,ℓ} \) is the cycle permutation graph, with \( (u_1, \ldots, u_{2r+2ℓ}) \) as the first cycle, corresponding to the permutation:

\[
\begin{align*}
& (3 \ 4) \ldots (2ℓ + 1 \ 2ℓ + 2), \text{ with fixed points 1 and 2, when } r = 1; \\
& (1 \ 2)(3 \ 4)(5 \ 9)(6 \ 10), \text{ with fixed points 7 and 8, when } r = 2 \text{ and } ℓ = 3; \text{ and} \\
& (1 \ 2) \ldots (2r - 1 \ 2r)(2r + 1 \ 2r + 2ℓ - 1)(2r + 2 \ 2r + 2ℓ)(2r + 3 \ 2r + 2 + 2ℓ - 3)(2r + 4 \ 2r + 2ℓ - 2) \ldots (α \ β), \text{ otherwise, where } (α \ β) = (2r + ℓ \ 2r + ℓ + 2) \text{ if } ℓ \text{ is even,} \\
& \text{ and } (α \ β) = (2r + ℓ - 1 \ 2r + ℓ + 3) \text{ if } ℓ \text{ is odd.}
\end{align*}
\]

We remark that when \( r > 1 \), the above permutation has no fixed points when \( ℓ \) is even, but, when \( ℓ \) is odd, \( 2r + ℓ \) and \( 2r + ℓ + 1 \) are fixed points, that is, \( x_{2r+ℓ} \) is adjacent to \( y_{2r+ℓ} \) and \( x_{2r+ℓ+1} \) is adjacent to \( y_{2r+ℓ+1} \) in \( P_{r,ℓ} \). In particular, we note that now the principal 4-edge-cut \( κ \) of \( P_{r,ℓ} \) consists of the following edges: 

\[ \{u_1u_{2r+2ℓ}, u_{2r+1}v_{2r+2ℓ+1}, v_2v_{2r+2}, u_2u_{2r+1}\}, \text{ which can be respectively denoted by } a, b, c, d \text{ as in Section 2.} \]

We remark that unbalanced papillon graphs are also non-bipartite, because the cycle

\[ (u_1, u_2, \ldots, u_{2r}, u_{2r+1}, v_{2r+1}, v_{2r+2}, v_2, v_1) \] is a cycle of \( P_{r,ℓ} \) on \( 2r + 5 \) vertices. Moreover, by following the proofs in Section 2, the results obtained for papillon graphs can be extended to unbalanced papillon graphs in the following way.

\[ \text{Figure 11: } P_{1,3} \text{ and } P_{3,4}; \text{ unbalanced papillon graphs are not always PMH. The above perfect matchings do not extend to a Hamiltonian cycle.} \]
Theorem 3.2. The unbalanced papillon graphs $P_{r,\ell}$ are E2F.

Proof. It suffices to observe that the statements of Lemma 2.2, Corollary 2.3, Corollary 2.4 and Lemma 2.5 would also hold if restated for unbalanced papillon graphs, so the proof that unbalanced papillon graphs are E2F is the same as the one proposed in Theorem 2.6. \qed

Theorem 3.3. The unbalanced papillon graph $P_{r,\ell}$ is PMH if and only if $r$ and $\ell$ are both even.

Proof. This is an immediate consequence of Proposition 2.7 and Theorem 2.10. In particular, when at least one of $r$ and $\ell$ are odd, $P_{r,\ell}$ is not PMH because the perfect matching $\bigcup_{i=1}^{r+\ell} \{u_{2i-1}u_{2i}, v_{2i-1}v_{2i}\}$ of $P_{r,\ell}$ (illustrated in Figure 11) cannot be extended to a Hamiltonian cycle. \qed

Since $P_n$ is PMH for every even $n \in \mathbb{N}$, papillon graphs provide us with examples of non-bipartite PMH cubic graphs which are cyclically 4-edge-connected and have girth 4 such that their order is a multiple of 16. However, by considering unbalanced papillon graphs, say $P_{2,\ell}$, for some even $\ell > 2$, we can obtain non-bipartite PMH cubic graphs having the above characteristics (that is, cyclically 4-edge-connected and having girth 4) such that their order is $8\nu$, for odd $\nu \geq 3$. Moreover, as can be seen in Table 2, there are no cycle permutation graphs on 8 and 20 vertices. It would be interesting to see whether these are just isolated cases or if there are infinitely many other integers $\nu$, such that there does not exist a non-bipartite cycle permutation graph on $\nu$ vertices which is PMH.

Finally, we remark that it would be very compelling to see whether there exist other 4-poles instead of the $C_4$-poles that can be used as building blocks when constructing papillon graphs and which yield non-bipartite PMH or just E2F cubic graphs.

References

[1] M. Abreu, J.B. Gauci, D. Labbate, G. Mazzuoccolo and J.P. Zerafa, Extending perfect matchings to Hamiltonian cycles in line graphs, Electron. J. Combin. 28(1) (2021), #P1.7.

[2] M. Abreu, J.B. Gauci and J.P. Zerafa, Saved by the rook: a case of matchings and Hamiltonian cycles, submitted, arXiv:2104.01578.

[3] A. Alahmadi, R.E.L. Aldred, A. Alkenani, R. Hijazi, P. Solé and C. Thomassen, Extending a perfect matching to a Hamiltonian cycle, Discrete Math. Theor. Comput. Sci., 17(1) (2015), 241–254.

[4] R. Diestel, Graph Theory, Graduate Texts in Mathematics 173, Springer-Verlag, New York, 2000.

[5] M. Funk, B. Jackson, D. Labbate and J. Sheehan, 2-Factor hamiltonian graphs, J. Combin. Theory Ser. B 87 (2003), 138–144.

[6] M. Funk, D. Labbate. On minimally one-factorable r-regular bipartite graphs, Discrete Math. 216 (2000), 121–137.

[7] J.B. Gauci and J.P. Zerafa, A note on perfect matchings and Hamiltonicity in the Cartesian product of cycles, submitted, arXiv:2005.02913.

[8] J.B. Gauci and J.P. Zerafa, On a family of quartic graphs: Hamiltonicity, matchings and isomorphism with circulants, submitted, arXiv:2011.04327.

[9] R. Häggkvist, On $F$-Hamiltonian graphs, in: J.A. Bondy, U.S.R. Murty (eds.), Graph Theory and Related Topics, Academic Press, New York, 1979, 219–231.
[10] D. Labbate, Characterizing minimally 1-factorable r-regular bipartite graphs, *Discrete Math.* **248** (2002), 109–123.

[11] D. Labbate, On 3-cut reductions of minimally 1-factorable cubic bigraphs, *Discrete Math.* **231** (2001), 303–310.

[12] D. Labbate, On determinants and permanents of minimally 1-factorable cubic bipartite graphs, *Note Math.* **20** (2000/01), 37–42.

[13] M. Las Vergnas, *Problèmes de couplages et problèmes hamiltoniens en théorie des graphes*, Thesis, University of Paris 6, Paris, 1972.

[14] Wolfram Research, Inc., Mathematica, Version 12.1, *Wolfram Research, Inc.*, Champaign, Illinois (2020).