ANTICHAIN GENERATING POLYNOMIALS OF POSETS

JIAN DING AND CHAO-PING DONG

Abstract. This paper gives a formula for the antichain generating polynomial \( N_{[k] \times Q} \) of the poset \([k] \times Q\), where \([k]\) is an arbitrary chain and \(Q\) is any finite graded poset. When \(Q\) specializes to be a connected minuscule poset, which was classified by Proctor in 1984, we find that the polynomial \( N_{[k] \times Q} \) bears nice properties. For instance, we will recover the \( B_n \)-Narayana polynomial and the \( D_{2n+2} \)-Narayana polynomial. We collect evidence for the conjecture that whenever \( N_{[k] \times P}(x) \) is palindromic, it must be \( \gamma \)-positive. Moreover, the family \( N_{[2] \times [n] \times [m]} \) should be real-rooted and \( N_{[2] \times [n] \times [n+1]} \) should be \( \gamma \)-positive. We also conjecture that \( N_{Q}(x) \) is log-concave (thus unimodal) for any connected Peck poset \(Q\).

1. Introduction

As on page 244 of Stanley [13], we call a finite poset \(Q\) graded if every maximal chain in \(Q\) has the same length. In this case, there is a unique rank function \(r : Q \to \mathbb{Z}_{\geq 0}\) such that all the minimal elements have rank 1, and \(r(x) = r(y) + 1\) if \(x\) covers \(y\). Let \(Q_i\) denote the set of all the elements in \(Q\) having rank \(i\). We call \(Q_i\) a rank level of \(Q\). Let \(d\) be the maximum of the rank function \(r\). Then we partition \(Q = \bigsqcup_{i=1}^{d} Q_i\), \(1 \leq i \leq d\), into \(d\) rank levels. If \(|Q_i| = |Q|_{d+1-i}\) for \(1 \leq i \leq \frac{d}{2}\), we say that \(Q\) is rank symmetric. If \(|Q_1| \leq |Q_2| \leq \cdots \leq |Q_k| \geq |Q_{k+1}| \geq \cdots \geq |Q_d|\) for some \(1 \leq k \leq d\), we say that \(Q\) is rank unimodal.

From now on, every poset \(Q\) is assumed to be finite, graded, and connected. A subset \(I\) of \(Q\) is called an ideal if \(x \leq y\) in \(Q\) and \(y \in I\) implies that \(x \in I\). Put
\[
\mathcal{M}_Q(x) := \sum_I x^{|I|},
\]
where \(I\) runs over the ideals of \(Q\). A subset \(A\) of \(Q\) is called an antichain if its elements are mutually incomparable. Put
\[
\mathcal{N}_Q(x) := \sum_A x^{|A|},
\]
where \(A\) runs over the antichains of \(Q\). Since ideals of \(Q\) are in bijection with antichains of \(Q\) via the map \(I \to \max(I)\), where \(\max(I)\) denotes the maximal elements of \(I\), we have that
\[
\mathcal{M}_Q(1) = \mathcal{N}_Q(1).
\]
The ideal generating polynomial $M_Q(x)$ has been addressed intensively in the literature, while the antichain generating polynomial $N_Q(x)$ seems to attract much fewer attention. One possible reason for this is that the computation of $N_Q(x)$ is much harder. The first result of the current paper is a formula for calculating the $N$-polynomial of $[k] \times Q$. Indeed, let us use $I, I_j, J, \ldots$ to denote ideals of $Q$. It is well-known that ideals of $[k] \times Q$ are in bijection with increasing $k$-sequences of ideals of $Q$: $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k$. Then one sees that the corresponding antichain has size

$$\sum_{j=1}^{k} \#(\max(I_j) \setminus I_{j-1}).$$

Here we make the convention that $I_0$ is the empty ideal. Let us define

$$N^k_I(x) := \sum_{I_1 \subseteq \cdots \subseteq I_{k-1} \subseteq I} x^{\sum_{j=1}^{k-1} \#(\max(I_j) \setminus I_{j-1})},$$

where $I_1 \subseteq \cdots \subseteq I_{k-1}$ runs over the increasing $(k-1)$-sequences of ideals of $Q$ which are contained in $I$. Then

$$N_{[k] \times Q}(x) = \sum_{I} N^k_I(x),$$

where $I$ runs over ideals of $Q$. Inspecting the formula (3) gives that

$$N^{k+1}_I(x) := \sum_{J \subseteq I} x^{\#(\max(I) \setminus J)} N^k_J(x),$$

where $J \subseteq I$ runs over ideals of $Q$. This leads us to the following.

**Theorem A.** Let $Q$ be a finite, connected, and graded poset. Let $I_1, I_2, \ldots, I_{N(Q)}$ be a enumeration of all the ideals of $Q$. Let $A_Q$ be the $N(Q) \times N(Q)$ matrix whose $(i,j)$-entry $a_{ij}$ equals $x^{\#(\max(I_i) \setminus I_j)}$ if $I_j \subseteq I_i$; and $a_{ij}$ equals zero otherwise. Let $V^{(k)}_Q$ be the column vector whose $i$-th entry is $N^k_{I_i}(x)$ for $1 \leq i \leq N(Q)$. Then

$$V^{(k)}_Q = A_{k-1}^{(1)} V^{(1)}_Q.$$

The formula (6) turns the calculation of $V^{(k)}_Q$—hence the calculation of $N_{[k] \times Q}(x)$ in view of (4)—into matrix multiplication. See Example 4.1 for an illustration. Computationally, this is very efficient.

For the rest of the paper, let us specialize $Q$ to be a minuscule poset, and demonstrate that similar to the polynomial $M_{[k] \times Q}$, the $N$-polynomial could also bear very nice properties. However, the techniques for unveiling them should lie much deeper.

Minuscule posets arise from minuscule representations of simple Lie algebras over $\mathbb{C}$. According to Section 11 of Proctor [10], minuscule posets are ubiquitous in mathematics. Our limited understanding of this philosophy comes from the recent work [5], where certain root posets arising from $\mathbb{Z}$-gradings of simple Lie algebras turn out to be mainly decoded by $[k] \times P$ for minuscule posets $P$.

As been classified by Proctor [10], a connected minuscule poset $P$ is one of the following:
\textbullet \, [m] \times [n];
\textbullet \, H_n := ([n] \times [n])/S_2;
\textbullet \, K_n = [n] \oplus ([1] \cup [1]) \oplus [n] \text{ (the ordinal sum, see page 246 of Stanley [13]);}
\textbullet \, J^2([2] \times [3]) \text{ (see Fig. 11);}
\textbullet \, J^3([2] \times [3]).

Here \( J(P) \) is the poset consisting of the ideals of \( P \), partially ordered by inclusion; \( J^2(P) \) stands for \( J(J(P)) \) and so on. The \( M \)-polynomials of minuscule posets enjoy many nice properties. For instance, by Theorem 6 of Proctor [10],

\begin{equation}
M_{[k] \times P}(x) = \prod_{\alpha \in P} \frac{1 - x^{r(\alpha) + k}}{1 - x^{r(\alpha)}}.
\end{equation}

Due to the symmetry of \([k] \times P\), the polynomial \( M_{[k] \times P}(x) \) is palindromic.

Let \( f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}_{\geq 0}[x] \) be a polynomial of degree \( n \). We say that \( f(x) \) is palindromic if \( a_i = a_{n-i} \) for \( 0 \leq i \leq n \); that \( f(x) \) is monic if \( a_n = 1 \); and that \( f(x) \) is \( \gamma \)-positive if there exist some positive reals \( \gamma_0, \gamma_1, \ldots, \gamma_{\lfloor \frac{n}{2} \rfloor} \) such that

\begin{equation}
f(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i x^i (1 + x)^{n-2i}.
\end{equation}

In this case, we call \( \gamma_0, \ldots, \gamma_{\lfloor \frac{n}{2} \rfloor} \) the \( \gamma \)-coefficients of \( f \). We say that \( f(x) \) is real-rooted if every root of \( f(x) \) is real; that \( f(x) \) is log-concave if \( \gamma_i^2 \geq \gamma_{i-1} \gamma_{i+1} \) for \( 1 \leq i \leq n - 1 \). See Athanasiadis [11] and Stanley [12] for excellent surveys on \( \gamma \)-positivity, log-concavity and unimodality in algebra, combinatorics and geometry.

We shall collect evidence for the following two conjectures. Theorem A is very helpful in this process.

**Conjecture B.** Let \( P \) be a connected minuscule poset. Let \( k \) be any positive integer. If \( N_{[k] \times P}(x) \) is palindromic, then it must be \( \gamma \)-positive.

We also suspect that the family \( N_{[2] \times [n] \times [m]} \) should be real-rooted and \( N_{[2] \times [n] \times [n+1]} \) should be \( \gamma \)-positive, see Conjectures 4.3 and 4.4. What underlies the hypothetical real-rootedness should be abundance of interlacing relations, see Conjecture 4.2.

One may view \( \gamma \)-positivity as a delicate property which happens only under very special circumstances. On the other hand, log-concavity lives in a much broader domain. For the latter, we propose the following.

**Conjecture C.** Let \( Q \) be a connected finite Peck poset. The polynomial \( N_Q(x) \) is log-concave (hence unimodal).

Recall that \( Q \) is said to be Sperner if no antichain has more elements than the largest rank level of \( Q \) does. We say that \( Q \) is strongly Sperner if for every \( k \geq 1 \) no union of \( k \) antichains contains more elements than the union of the \( k \) largest rank levels of \( Q \) does. We call \( Q \) Peck if it is strongly Sperner, rank symmetric and rank unimodal.

The paper is organized as follows. We collect necessary preliminaries in Section 2. Section 3 aims to collect evidence for Conjectures B and C. Sections 4 focuses on the family \( N_{[2] \times [n] \times [n+1]}(x) \).
2. Preliminaries

This section aims to give some preliminaries.

Lemma 2.1. Let $f(x)$ be a polynomial in $\mathbb{R}_{>0}[x]$. If $f(x)$ is real-rooted and palindromic, then it is $\gamma$-positive.

Proof. See Lemma 4.1 of Brändén [3], Remark 3.1.1 of Gal [6], and Sun-Wang-Zhang [14]. □

Lemma 2.2. Let $Q = \bigsqcup_{i=1}^{d} Q_i$ be the decomposition of a connected finite graded poset into rank levels. Assume that $Q$ is rank unimodal. Then $[k] \times Q$ has a unique rank level of the largest size if and only if $k \leq d$ and that there exists a unique $i_0 \in [k,d]$ such that the statistic $|Q_i| + |Q_{i-1}| + \cdots + |Q_{i+k-1}|$ attains the maximum at $i_0$.

Proof. When $k \geq d+1$, one sees easily that $[k] \times Q$ has $k-d+1$ rank levels of size $|Q|$, which must be the largest. Thus it remains to consider $k \leq d$, then observe that the rank levels of $[k] \times Q$ have sizes $|Q_i| + |Q_{i-1}| + \cdots + |Q_{i+k-1}|$ for $i \in [1,d+k-1]$. Here we interpret $Q_j$ as the empty set if $j$ does not fall in $[1,d]$. Since $Q$ is assumed to be rank unimodal, we see that a necessary condition for the size to be largest is that $i \leq d$ and that $i+1-k \geq 1$, i.e., $i \in [k,d]$. Now the desired conclusion is obvious. □

Lemma 2.3. Let $P$ be a connected minuscule poset. The polynomial $N_{[k] \times P}$ is monic precisely in the following cases:

(a) $P = [m] \times [n]$, $m \leq n$, and $k = n - m + 1, n - m + 3, n - m + 5, \ldots, n + m - 1$;
(b) $P = H_n$, $n$ is odd, and $k = 1, 5, 9, \ldots, 2n - 1$;
(c) $P = H_n$, $n$ is even, and $k = 3, 7, 11, \ldots, 2n - 1$;
(d) $P = K_n$, and $k = 1, 2n + 1$;
(e) $P = J^2([2] \times [3])$, and $k = 5, 11$;
(f) $P = J^3([2] \times [3])$, and $k = 1, 9, 17$.

Proof. Since each connected minuscule poset is Peck and the product of Peck posets is Peck (see Theorem 2 of Proctor [9]), we have that $[k] \times P$ is Peck. Therefore, $[k] \times P$ is Sperner. Thus the polynomial $N_{[k] \times P}$ is monic if and only if $[k] \times P$ has a unique rank level of the largest size. Checking the latter condition via Lemma 2.2 leads us to the desired conclusion. □

Proposition 2.4. The polynomials $N_{[n] \times [n]}(x) = \sum_{i \geq 0} \binom{n}{i} x^i$ are $\gamma$-positive.

Proof. By (62) of [1], the polynomial $N_{[n] \times [n]}(x)$ coincides with $\text{Cat}(B_n, x)$—the $B_n$-Narayana polynomial. Now by Proposition 11.15 of [11] Postnikov, Reiner and Williams (see also Theorem 2.32 of Athanasiadis [1]), these polynomials are $\gamma$-positive. Indeed, we have that

$$\text{Cat}(B_n, x) = \sum_{i=0}^{|\mathcal{D}|} \binom{n}{i, i, n-2i} x^i(1 + x)^{n-2i}.$$ □

Proposition 2.5. The polynomials $N_{H_n}(x) = \sum_{i \geq 0} \binom{n+1}{2i} x^i$ are $\gamma$-positive for $n$ odd.
Proof. Let $E_2$ be the linear operator on the space $\mathbb{R}[x]$ of polynomials with real coefficients defined by setting $E_2(x^m) = x^{m/2}$ if $m$ is even, and $E_2(x^m) = 0$ otherwise. Then one sees easily that

$$E_2(1 + x)^{n+1} = \sum_{i \geq 0} \binom{n+1}{2i} x^i.$$ 

Since $(1 + x)^{n+1}$ is real-rooted, we conclude from Lemma 7.4 of Athanasiadis and Savvidou [2] that $E_2(1 + x)^{n+1}$ is real-rooted. Now Lemma 2.1 finishes the proof. \qed

Remark 2.6. As suggested by Athanasiadis, it would be interesting to find combinatorial interpretations of the $\gamma$-coefficients of $N_{H_n}(x)$ when $n$ is odd.

Given two real-rooted polynomials $f(x) = \prod_{i=1}^{\deg f} (x - x_i)$ and $g(x) = \prod_{j=1}^{\deg g} (x - \xi_j)$, we say that $f(x)$ interlaces $g(x)$—denoted by $f(x) \preceq g(x)$—if their roots alternate in the following way:

$$\cdots \leq x_2 \leq \xi_2 \leq x_1 \leq \xi_1.$$ 

Note that a necessary condition for $f(x) \preceq g(x)$ is that $\deg f \leq \deg g \leq \deg f + 1$.

Theorem 2.7. (Obreschkoff [3, Satz 5.2]) Let $f(x), g(x) \in \mathbb{R}[x]$ be two polynomials such that $\deg f \leq \deg g \leq \deg f + 1$. Then $f(x)$ interlaces $g(x)$ if and only if $c_1 f(x) + c_2 g(x)$ is real-rooted for any $c_1, c_2 \in \mathbb{R}$.

Given three real-rooted polynomials $f(x), g(x)$ and $h(x)$, after Chudnovsky and Seymour [1], we say that $h(x)$ is a common interleaver for $f(x)$ and $g(x)$ if $f(x) \preceq h(x)$ and $g(x) \preceq h(x)$. Note that if $f(x) \preceq g(x)$, then $f(x)$ and $g(x)$ have a common interleaver $g(x)$.

3. Evidence for Conjectures B and C

This section aims to collect evidence for Conjectures B and C.

Firstly, let us verify Conjecture B for $k = 1$.

Proposition 3.1. Let $P$ be a connected minuscule poset. The following are equivalent:

(a) $N_P(x)$ is monic;
(b) $N_P(x)$ is palindromic;
(c) $N_P(x)$ is $\gamma$-positive.

Proof. It suffices to show that (a) implies (c). By Lemma 2.3, $N_P(x)$ is monic precisely when $P$ is $[n] \times [n]$, or $H_n$ for $n$ odd, or $K_n$, or $J^3([2] \times [3])$. Moreover, we have that

- $N_{[n] \times [n]}(x) = \sum_{i \geq 0} \binom{n}{i} \binom{n}{i} x^i$.
- $N_{H_n}(x) = \sum_{i \geq 0} \binom{n+1}{2i} x^i$ for $n$ odd.
- $N_{K_n}(x) = 1 + (2n + 2)x + x^2 = (1 + x)^2 + 2nx$.
- $N_{J^3([2] \times [3])} = 1 + 27x^2 + 27x^3 + x^4 = (1 + x)^3 + 24x(1 + x)$.

Now it follow from Propositions 2.1 and 2.5 that these polynomials are all $\gamma$-positive. \qed

Secondly, let us verify Conjecture B for $P = K_n$.

Proposition 3.2. The following are equivalent:

(a) $N_{[k] \times K_n}(x)$ is monic;
Therefore, it follows from (62) of [1] that (b) \(N_{[k] \times K_n}(x)\) is palindromic; (c) \(N_{[k] \times K_n}(x)\) is \(\gamma\)-positive.

**Proof.** It suffices to show that (a) implies (c). By Lemma 2.3, \(N_{[k] \times K_n}(x)\) is monic precisely when \(k = 1\) or \(2n + 1\). It remains to consider \(k = 2n + 1\). Indeed, by Theorem 7.2 of [5],

\[
N_{[2n+1] \times K_n}(x) = (1 + x^{2n+2}) + (4n^2 + 6n + 2)(x + x^{2n+1}) + \sum_{i=2}^{2n} \left( \binom{2n}{i-2} \binom{2n+1}{i-1} + \binom{2n}{i} \binom{2n+1}{i} + \binom{2n+1}{i} \right) x^i.
\]

Notice that

\[
\binom{2n+1}{i-1} \left( \binom{2n+1}{i} + \binom{2n}{i-2} \right) + \binom{2n}{i} \left( \binom{2n+1}{i} + \binom{2n}{i-1} \right) = \binom{2n+1}{i-1} \left( \binom{2n+1}{i} + \binom{2n}{i-2} \right) + \binom{2n}{i} \left( \binom{2n+1}{i} + \binom{2n}{i-1} \right) = \binom{2n+1}{i-1} \left( \binom{2n+1}{i} + \binom{2n}{i-2} \right) + \binom{2n}{i} \left( \binom{2n+1}{i} + \binom{2n}{i-1} \right) = \binom{2n+1}{i-1} \left( \binom{2n+1}{i} + \binom{2n}{i-2} \right) + \binom{2n}{i} \left( \binom{2n+1}{i} + \binom{2n}{i-1} \right) = \binom{2n+1}{i-1} \left( \binom{2n+1}{i} + \binom{2n}{i-2} \right) + \binom{2n}{i} \left( \binom{2n+1}{i} + \binom{2n}{i-1} \right).
\]

Therefore, it follows from (62) of [1] that \(N_{[2n+1] \times K_n}(x)\) coincides with \(\text{Cat}(D_{2n+2}, x)\)—the \(D_{2n+2}\)-Narayana polynomial. The latter polynomial is \(\gamma\)-positive for any \(n \in \mathbb{Z}_{>0}\) by Gorsky [7] (see also Theorem 2.32 of [1]). Indeed, we have that

\[
\text{Cat}(D_{2n+2}, x) = \sum_{i=0}^{n+1} \frac{2n+1-i}{2n+1} \binom{2n+2}{i, i, 2n+2-2i} x^i (1 + x)^{2n+2-2i}.
\]

\[
\square
\]

**Remark 3.3.** It would be interesting to find direct explanations for

\[
N_{[n] \times [n]}(x) = \text{Cat}(B_n, x), \quad N_{[2n+1] \times K_n}(x) = \text{Cat}(D_{2n+2}, x).
\]

Now let us verify Conjecture B for the two exceptional minuscule posets.

**Example 3.4.** We can parameterize the ideals of \(Q := J^2([2] \times [3])\) by those 4-tuples \((a, b, c, d)\) such that \(0 \leq a \leq 4, b = 0\) or \(3 \leq b \leq 6, c = 0\) or \(3 \leq c \leq 6, d = 0\) or \(5 \leq d \leq 8\) and that

\[
\text{(9)} \quad a \geq \min\{b, 4\}, \quad b \geq c, \quad c \geq \min\{6, d\}, \quad d \geq 0.
\]

Let \(I\) be the ideal of \(Q\) which is parameterized by one 4-tuple \((a, b, c, d)\) as above. One sees that \# max\((I)\) equals the number of inequalities in (9) which are strict. This allows us to obtain \(V_Q^{(1)}\). Let \(J\) be another ideal of \(Q\) which is parameterized by the 4-tuple \((a_1, b_1, c_1, d_1)\).
Then $J \subseteq I$ if and only if $a_1 \leq a$, $b_1 \leq b$, $c_1 \leq c$ and $d_1 \leq d$. Moreover, in such a case, 
\[\#(\max(I) \setminus J)\] equals the number of non-zero entries in the following sequence
\[(a - a_1)(a - \min\{b, 4\}), \quad (b - b_1)(b - c), \quad (c - c_1)(c - \min\{6, d\}), \quad (d - d_1)d.\]
This allows us to obtain $A_Q$. Then aided by Theorem A, we find that $N_{[k] \times J^2([2] \times [3])}(x)$ are $\gamma$-positive for $k = 5, 11$. Indeed, $N_{[5] \times J^2([2] \times [3])}(x)$ has degree 10 and $\gamma$-coefficients 
\[\gamma_0 = 1, 170, 745, 1850, 1025, 62; \quad \gamma_0 = 1, 160, 4900, 49280, 194810, 193760, 35840, 860.\]

Therefore, in view of Lemma 2.3(e), Conjecture B holds for $J^2([2] \times [3])$. Moreover, we have checked that $N_{[k] \times J^2([2] \times [3])}(x)$ is log-concave for $1 \leq k \leq 11$. □

**Remark 3.5.** Similarly, we have computed $N_{[k] \times P^2([2] \times [3])}(x)$ for $k$ up to 17. This polynomial is not palindromic when $k = 9$ or 17. For instance,
\[N_{[17] \times P^2([2] \times [3])}(x) = 1 + 459x + 51867x^2 + \cdots + 20564929719672x^{12} + 267230238920x^{13} + 26743721449352x^{14} + 20605116728504x^{15} + \cdots + 55461x^{25} + 483x^{26} + x^{27}.\]

Thus Conjecture B holds for $J^3([2] \times [3])$ in view of Lemma 2.3(f). Moreover, we have checked that this polynomial is log-concave for $1 \leq k \leq 17$.

![Figure 1. The Hasse diagram of $J^2([2] \times [3])$.](image)

To end up this section, let us present some examples suggesting that it is not easy to find infinite family of $\gamma$-positive polynomials among $N_{[k] \times P}(x)$ for $P$ minuscule.

**Example 3.6.** (a) The polynomial
\[N_{[3] \times [3] \times [3]}(x) = 1 + 27x + 162x^2 + 350x^3 + 310x^4 + 114x^5 + 15x^6 + x^7.\]
It is not palindromic, and sits in the family $N_{[3] \times [n] \times [n]}(x)$. 

(b) The polynomial $N_{[3] \times [3] \times [5]}(x)$ equals
\[ 1 + 45x + 495x^2 + 2155x^3 + 4360x^4 + 4360x^5 + 2141x^6 + 505x^7 + 49x^8 + x^9. \]
It is not palindromic and sits in the family $N_{[3] \times [3] \times [n+2]}(x)$.

(c) The polynomial $N_{[4] \times [3] \times [4]}(x)$ equals
\[ 1 + 48x + 576x^2 + 2800x^3 + 6525x^4 + 7848x^5 + 4957x^6 + 1644x^7 + 274x^8 + 22x^9 + x^{10}. \]
It is not palindromic, and sits in the family $N_{[4] \times [3] \times [n+1]}(x)$.

(d) The polynomial $N_{[4] \times [3] \times [6]}(x)$ looks like
\[ 1 + 72x + 1368x^2 + \cdots + 103200x^5 + 134806x^6 + 102912x^7 + \cdots + 1510x^{10} + 86x^{11} + x^{12}. \]
It is not palindromic, and sits in the family $N_{[4] \times [3] \times [n+3]}(x)$.

(e) The polynomial $N_{[5] \times [3] \times [7]}(x)$ looks like
\[ 1 + 105x + 3045x^2 + \cdots + 4080285x^7 + 4078275x^8 + \cdots + 3692x^{13} + 137x^{14} + x^{15}. \]
It is not palindromic, and sits in the family $N_{[5] \times [3] \times [n+4]}(x)$.

(f) The polynomial $N_{[3] \times H_6}(x)$ equals
\[ 1 + 63x + 840x^2 + 4088x^3 + 8736x^4 + 8736x^5 + 4060x^6 + 862x^7 + 69x^8 + x^9. \]
It is not palindromic, and sits in the family $N_{[3] \times H_n}$ for $n$ even.

(g) The polynomial $N_{[4] \times H_6}(x)$ looks like
\[ 1 + 135x + 4455x^2 + \cdots + 7209048x^7 + 7206012x^8 + \cdots + 4745x^{13} + 145x^{14} + x^{15}. \]
It is not palindromic, and sits in the family $N_{[2n-1] \times H_n}$.
All the polynomials above are log-concave. \qed

4. The family $N_{[2] \times [n] \times [n+1]}(x)$

This section aims to study the polynomials $N_{[2] \times [n] \times [n+1]}(x)$. It follows from Lemma 2.3(a) that $N_{[2] \times [n] \times [n+1]}(x)$ is monic and has degree $2n$. By Theorem 6 of Proctor [10], we have that
\[ N_{[2] \times [n] \times [n+1]}(1) = \prod_{\alpha \in [n] \times [n+1]} \frac{r(\alpha) + 2}{r(\alpha)} = (2n + 1)C_nC_{n+1}, \]
where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$-th Catalan number.

Let us warm up with an example.

**Example 4.1.** We can enumerate all the ideals of $Q := [2] \times [3]$ by the following 2-tuples:
\[ (0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3). \]
If an ideal $I$ corresponds to the 2-tuple $(a_1, a_2)$, we have that $\# \text{max}(I)$ equals $\#\{a_i \mid a_i > 0\}$. Thus one calculates that
\[ V_Q^{(1)} = [x, x, x, x, x^2, x^2, x, x^2, x, x^2, x]^T. \]
Summing up these components leads us to that $N_Q(x) = 1 + 6x + 3x^2$. 
If another ideal \( J \) corresponds to the 2-tuple \((b_1, b_2)\), then \( J \subseteq I \) if and only if \( b_1 \leq a_1 \) and \( b_2 \leq a_2 \). Then

\[
\#(\max(I) \setminus J) = \#\{a_i \mid a_i > b_i\},
\]

and one calculates that

\[
A_Q = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & x & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & x & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
x^2 & x^2 & x & 0 & x & 1 & 0 & 0 & 0 & 0 \\
x^2 & x^2 & x^2 & x & x & 1 & 0 & 0 & 0 & 0 \\
x & x & x & 0 & x & 0 & 1 & 0 & 0 & 0 \\
x^2 & x^2 & x^2 & x & x^2 & x & x & 1 & 0 & 0 \\
x & x & x & x & x & x & x & x & x & 1
\end{bmatrix}.
\]

Then for instance, by \((6)\), we have that

\[
V_{Q}^{(4)} = A_Q^3 V_{Q}^{(1)}
\]

\[
= [1, 4x, 4x + 6x^2, 4x + 12x^2 + 4x^3, 4x + 6x^2, 16x^2 + 14x^3, \\
16x^2 + 38x^3 + 11x^4, 4x + 18x^2 + 22x^3 + 6x^4, 16x^2 + 52x^3 + 48x^4 + 9x^5, \\
4x + 30x^2 + 70x^3 + 55x^4 + 15x^5 + x^6]T.
\]

Summing up these components gives that

\[
N_{4 \times Q}(x) = 1 + 24x + 120x^2 + 200x^3 + 120x^4 + 24x^5 + x^6.
\]

There are many interlacing relations among \(N_4^1(x)\). To mention a few, we have

\[
N_{(0,0)}^1(x) + N_{(0,1)}^1(x) \leq N_{(0,2)}^1(x),
\]

\[
N_{(0,0)}^1(x) + N_{(0,1)}^1(x) + N_{(0,2)}^1(x) \leq N_{(0,3)}^1(x),
\]

\[
N_{(1,1)}^1(x) + N_{(1,2)}^1(x) \leq N_{(1,3)}^1(x),
\]

\[
\sum_{j=0}^{3} N_{(0,j)}^1(x) \leq \sum_{j=1}^{3} N_{(1,j)}^1(x),
\]

\[
\sum_{i=0}^{2} \sum_{j=i}^{3} N_{(i,j)}^1(x) \leq N_{(3,3)}^1(x).
\]

Many other calculations lead us to the following.

**Conjecture 4.2.** Fix \( Q = [2] \times [n] \). Parameterize the ideals of \( Q \) by the 2-tuples \((a, b)\) such that \( 0 \leq a \leq b \leq n \). Fix any \( k \geq 0 \). Then the polynomials \( \sum_{i=0}^{s} \sum_{j=i}^{n} N_{(i,j)}^k(x) \) and \( \sum_{l=s+1}^{n} N_{(s+1,l)}^k(x) \) have common interleavers for \( 0 \leq s \leq n - 1 \).
If Conjecture 4.2 holds, setting \( s = n - 1 \) there would give us that 
\[
\sum_{i=0}^{n-1} \sum_{l=i}^{n} N_{(i,l)}^k(x) + N_{(n,n)}^k(x)
\]
would be real-rooted.

**Conjecture 4.3.** The polynomial 
\( N_{[2] \times [n] \times [k]}(x) \) is real-rooted for any \( n, k \in \mathbb{Z}_{>0} \).

**Remark 4.4.** The polynomial 
\( N_{[3] \times [3] \times [3]}(x) \) (see Example 3.6(a)) is not real-rooted.

The polynomials 
\( N_{[2] \times [n] \times [n+1]}(x) \) for the first few values of \( n \) are computed as below. For convenience, here we only present the \( \gamma \)-coefficients \( \gamma_0 = 1, \gamma_1, \ldots, \gamma_n \) such that
\[
N_{[2] \times [n] \times [n+1]}(x) = \sum_{i=0}^{n} \gamma_ix^i(1 + x)^{2n-2i}. 
\]

\[
\begin{array}{c|c}
 n & \text{The } \gamma \text{-coefficients of } N_{[2] \times [n] \times [n+1]}(x) \\
\hline
1 & 1, 2 \\
2 & 1, 8, 2 \\
3 & 1, 18, 33, 6 \\
4 & 1, 32, 150, 144, 12 \\
5 & 1, 72, 1020, 4480, 6300, 2400, 100 \\
6 & 1, 98, 2037, 14350, 37730, 35700, 9625, 350 \\
7 & 1, 128, 3668, 37856, 160020, 282240, 191100, 39200, 980 \\
8 & 1, 162, 6120, 87024, 539532, 1528632, 1933344, 987840, 156996, 3528 \\
9 & 1, 200, 9630, 180480, 1542660, 6408864, 13028400, 12418560, 4948020, 635040, 10584 \\
\end{array}
\]

**Conjecture 4.5.** The polynomial 
\( N_{[2] \times [n] \times [n+1]}(x) \) is palindromic and real-rooted (thus \( \gamma \)-positive) for every \( n \).

Finally, let us deduce a recursive formula for 
\( N_{[2] \times [n] \times [n+1]}(x) \) by Young tableau. Indeed, fix a Young tableau with two rows. Let the first row have length \( n \), and the second row have length \( m \), where \( m \leq n \). We fill the boxes in the first row with numbers \( 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq k \), and fill the boxes in the second row with numbers \( 0 \leq b_1 \leq \cdots \leq b_m \leq k \). Require that \( a_i \leq b_i \) for \( 1 \leq i \leq m \). For instance, when \( n = 2 \), \( m = 1 \) and \( k = 1 \), there are five such Young tableaux:

\[
\begin{array}{c|c|c|c|c|c}
 & 0 & 0 & 0 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Given such a Young tableau, we associate the monomial
\[
x^{\#\{a_i | a_i > 0\} + \#\{b_j | b_j > a_j\}}
\]
to it. For instance, the monomials associated to the five Young tableaux above are \( 1, x, x, x^2, x \) respectively. We denote by \( f_{n,m}^k \) the sum of all the monomials associated to the Young tableaux defined above with row lengths \( n, m \) and filling numbers no bigger than \( k \). For example, \( f_{2,1}^1 = 1 + 3x + x^2 \).
Proposition 4.6. We have the recursive formula

\[
 f_{n,n}^{k+1} = f_{n,n}^k + x \sum_{s=1}^n f_{s,n}^k + \sum_{l=1}^n (xf_{n-l,n-l}^{k} + x^2 \sum_{s=1}^{n-l} f_{n-l,n-l-s}^k).
\]

Proof. Assume that in the first row we fill precisely the last \(l\) boxes with \(k+1\), while in the second row we fill precisely the last \(s+l\) boxes with \(k+1\). We proceed according to the following two cases.

Case 1: \(l = 0\). Then \(0 \leq s \leq n\).

Subcase 1a: \(s = 0\). Then one sees easily that the monomials associated to these Young tableaux sum up to \(f_{n,n}^k\).

Subcase 1b: \(1 \leq s \leq n\). Then we have that

\[
 \# \{a_i \mid a_i > 0\} + \# \{b_j \mid b_j > a_j\} = \# \{a_i \mid a_i > 0 \text{ for } i \in [1,n]\} + \# \{b_j \mid b_j > a_j \text{ for } j \in [1,n-s]\} + 1.
\]

The monomials associated to these Young tableaux sum up to \(x \sum_{s=1}^n f_{s,n}^k\).

Case 2: \(1 \leq l \leq n\). Then \(0 \leq s \leq n - l\).

Subcase 2a: If \(s = 0\), we obtain that

\[
 \# \{a_i \mid a_i > 0\} + \# \{b_j \mid b_j > a_j\} = \# \{a_i \mid a_i > 0 \text{ for } i \in [1,n-l]\} + \# \{b_j \mid b_j > a_j \text{ for } j \in [1,n-l]\} + 1.
\]

The monomials associated to these Young tableaux sum up to \(x \sum_{l=1}^n f_{n-l,n-l}^k\).

Subcase 2b: \(1 \leq s \leq n - l\). Then we have that

\[
 \# \{a_i \mid a_i > 0\} + \# \{b_j \mid b_j > a_j\} = \# \{a_i \mid a_i > 0 \text{ for } i \in [1,n-l]\} + \# \{b_j \mid b_j > a_j \text{ for } j \in [1,n-l-s]\} + 2.
\]

The monomials associated to these Young tableaux sum up to \(x^2 \sum_{l=1}^n \sum_{s=1}^{n-l} f_{n-l,n-l-s}^k\).

Adding up the four terms above gives the desired formula. \(\square\)

Note that \(f_{n,n}^{n+1}\) is the \(N\)-polynomial for \([2] \times [n] \times [n+1]\). Thus Proposition 4.6 may be useful for investigating the conjectures of this section.

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(Ding) College of Mathematics and Econometrics, Hunan University, Changsha 410082, China

E-mail address: dingjain@hnu.edu.cn

(Dong) Mathematics and Science College, Shanghai Normal University, Shanghai 200234, P. R. China

E-mail address: chaopindong@163.com