Mathematical modeling on conservation of depleted forestry resources

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Abstract
In this article, a nonlinear mathematical model is constructed to investigate the conservation of depleted forest resources due to the increase of population and associated pressures. Fundamental equations governing the dynamics of the system are defined by the set of highly nonlinear ordinary differential equations and solved numerically. The model is analyzed by using the nature of stability analysis theory of dynamical system. The numerical solutions and simulations of the system are carried out using ODE45 subroutine of MATLAB. Presentations of results are revealed using graphs and interpreted biologically. It is noted that the increase of population density and associated pressures causes the depletion of forestry resources. However, forest resources can be conserved by controlling man made fire, toxicant activities, applying economical incentives and technological efforts.

Recommendations for Resource managers
The forest resources are natural resources that can be used for ecosystem balancing mechanism in nature. However, forest resources are depleted as a result of augmented population and associated pressures. Therefore,
When population and associated pressures increase, the depletion of forestry resources increases. 
As conservation efforts applied, the density of forestry resources increases.

KEYWORDS
conservation, depletion, forest resources, mathematical modeling, numerical simulation

1 INTRODUCTION

Forests are one of the most valuable gifts that nature has given to mankind. The forest can be used for food, material, medicine, building materials, fertilization, and so on by the community. Forest resources support activities include things like sustainable forest support activities, timberland and timberland forest production, natural variety protection, endemic resource preservation, climate change rationalization, recreational and spiritual purification, and climate change rationalization. Forest resources serve as the lungs of the land, regulating environmental stability, carbon dioxide (CO₂), and oxygen exchange (O₂) (Teru & Koya, 2020).

Deforestation is a primary driver of land cover change, and it is one of the most pressing concerns facing the world's tropical moist forests, which span around 550 million hectares (Cantin & Verdière, 2020; Lata et al., 2018). Desertification, loss of diversity, humiliation of the earth, economic failure, disruption of horological cycles, environmental condition, flooding, global warming, soil erosion, landslides, and social conflict are all negative consequences of deforestation. The rate of deforestation can be reduced by increasing conservation and management of forest resources through the development of programs and organizations (Qureshi & Yusuf, 2019; Ramdhani, 2015; Teru & Koya, 2020).

To safeguard the remaining earthly resources, forest resources must be conserved to supply alternative energy sources, support plantations, and support the biological process of abandoned cultivated fields. Dense population, loss of livelihood, and increased ignorance are all societal consequences of deforestation. Poverty, a lack of other employment possibilities, and a lack of income phase strategy and morphological change are all economic reasons that contribute to deforestation (Dubey et al., 2009).

The major factors of forest resources depletion in developing country like Ethiopia are increased population, limestone quarries, wood and paper-based enterprises, population growth, improved agriculture, and village development (Yitebitu et al., 2010). When it comes to forest resources, the main subjects of discussion includes upgrading landed estate, developing policies and strategies for forest establishment, and boosting investment funds for challenges. Reforestation can be carried out by reducing population growth, controlling human fire and environmental pollution, taking care of soil erosion and land degradation, increasing timberland industry, decreasing forest humiliation and so on Yitebitu et al. (2010).

To the best of our knowledge, the combined effects of augmented population and associated pressures on conservation of depleted forestry resources have been not investigated. Due to this, we have constructed and analyzed a non linear mathematical model to study the
conservation of depleted forestry resources due to the increment of both population and associated pressures. The numerical solutions of the equations governing the system are obtained using ODE45 subroutine of MATLAB and analyzed and interpreted biologically.

2 | MATHEMATICAL FORMULATION

In this study, we have considered the biomass of forest resources as the study variable, and its depletion factors such as human population and associated pressures. Moreover, conservation mechanisms including plantation, economic incentives, and controlling man made fire and environmental pollution have been assumed. Furthermore, the model can be formulated with consideration of the following assumptions.

1. Human population and population pressures are considered.
2. The rate of population pressure is directly related to the density of population.
3. The density of forest resources grow logistically in the absence of population pressure and the human population with inherent development rank and susceptibility.
4. Plantation, controlling man made fire, distribution of toxicant activities and economic efforts are directly related to the low level of forest resource from its susceptibility.
5. Economic incentive is proportional to population pressures.

1. Equation of forest resource $B(t)$: Here, we assume that the density of forest resources denoted by $B(t)$ is logistically growth. Suppose that the depletion rate parametrized by $\beta_1$ can be caused by increase of population rate (Misra & Lata, 2015a; Teru & Koya, 2020). The depletion rate due to population pressure can be parametrized by $\lambda_1$. Rate of reforestation due to plantation can be parametrized by $\phi_1$. Rate of reforestation by controlling man made fire can be denoted by $\sigma_0$ and rate of forest conservation by controlling economic efforts can be parametrized by $\psi_1$. The natural depletion rate of the forest resources is denoted by $s_0$. Here, $L$ is carrying capacity of forest resources and $s$ is growth rate of forest resources. Under the above assumptions, the model for density of forest resources is given as follows:

$$\frac{dB}{dt} = s \left(1 - \frac{B}{L}\right)B - s_0B - \beta_1NB - \lambda_1BP + \sigma_0FB + \phi_1BT + \psi_1BE,$$

with initial condition, $B(0) = B_0 \geq 0$.

2. Equation of human population $N(t)$: We assume that the human population denoted by $N(t)$ logistically growth and density of forest resource is directly proportional to human population which is denoted by $\beta$. Moreover, the rate of natural depletion of forest resources due to human population is parameterized by $r_0$. Here, $K$ is carrying capacity of population and $R$ is the growth rate of population. Then, depending on the above assumptions, equation of human population can be given as follows Misra et al. (2013) and Misra and Lata (2015a):

$$\frac{dN}{dt} = R \left(1 - \frac{N}{K}\right)N - r_0N + \beta NB,$$
with initial condition, \( N(0) = N_0 \geq 0 \).

3. **Equation of population pressure denoted by \( P(t) \):** It is assumed that the increase of population pressure is proportional to the density of the population. The growth rate of population pressure is denoted by \( \lambda \). The natural depletion rate of forest resources due to population pressure is parameterized by \( \lambda_0 \). The rate of economic incentive which decreases population pressures is given by \( \psi_2 \). Using the above assumptions, the equation of population pressure is given as (Misra et al., 2013):

\[
dP \over dt = \lambda N - \lambda_0 P - \psi_2 P,
\]

with initial condition, \( P(0) = P_0 \geq 0 \).

4. **Equation of controlling man made fire and toxicant activity \( F(t) \):** It is assumed that the forest resources can be conserved by controlling man made fire and toxicant substances which can be denoted by \( \Delta \). Here, rate of natural depletion of the forest is parametrized by \( \sigma \) and growth rate of conservation of forest by controlling fire and toxicant activity is denoted by \( d \). Then, equation of controlling man made fire and toxicant activity is given as follows (Shuklaa et al., 2007):

\[
dF \over dt = \Delta \left( 1 - \frac{B}{L} \right) - \sigma F + dF,
\]

with initial condition, \( F(0) = F_0 \geq 0 \).

5. **Equation of plantation denoted by \( T(t) \):** It is one of the method of conservation of forest resource. Therefore, we assume that by applying plantation rate which is denoted by \( \phi \), the density of forest resources increases. In addition, the natural depletion rate is parameterized by \( \phi_0 \). Hence, the equation of plantation is given as follows (Shuklaa et al., 2007):

\[
dT \over dt = \phi (L - B) - \phi_0 T,
\]

with initial condition, \( T(0) = T_0 \geq 0 \).

6. **Equation of economical incentives denoted by \( E(t) \):** Conservation rate of forest resources due to economical incentives can be parametrized by \( \mu \). The natural depletion rate of economical incentives can be parameterized by \( \psi_0 \). The economical incentives growth rate is denoted by \( \psi \). Depending on these assumption, equation of economical efforts is given as follows (Misra et al., 2013):

\[
dE \over dt = \psi P - (\psi_0 - \mu)E,
\]

with initial condition, \( E(0) = E_0 \geq 0 \).

Hence, by combining above equations, the system of differential equations governing dynamics of the system can be written as:
\[ \frac{dB}{dt} = s \left(1 - \frac{B}{L}\right)B - s_0B - \beta_1NB - \lambda_1BP + \sigma_0FB + \phi_1BT + \psi_1BE, \]
\[ \frac{dN}{dt} = R \left(1 - \frac{N}{K}\right)N - \eta_0N + \beta NB, \]
\[ \frac{dP}{dt} = \lambda N - \lambda_0P - \psi_2P, \]
\[ \frac{dF}{dt} = \Delta \left(1 - \frac{B}{L}\right) - \sigma F + dF, \]
\[ \frac{dT}{dt} = \phi (L - B) - \phi_0 T, \]
\[ \frac{dE}{dt} = \psi P - (\psi_0 - \mu) E, \]

with initial condition, \( B(0) = B_0 \geq 0, N(0) = N_0 \geq 0, P(0) = P_0 \geq 0, F(0) = F_0 \geq 0, T(0) = T_0 \geq 0, E(0) = E_0 \geq 0. \)

3 | MODEL ANALYSIS

3.1 | Positivity solution of the model

**Theorem 1.** If \( B(0) > 0, N(0) > 0, P > 0, F(0) > 0, T(0) > 0, E(0) > 0 \) are positive in feasible set \( \omega \), then the solution set \{ \( B(t), N(t), P(t), F(t), T(t), E(t) \) \} of system (1) is positive for all \( t \geq 0 \).

**Proof.** Note that the growth rate of density of forest resources and population density are assumed to be logistically growth. If \( B \) is very small relative to carry capacity of forest resource \( L \), then \( \left(1 - \frac{B}{L}\right) \) approaches to 1. Now, \( \frac{dB}{dt} = Bs \), here \( s \) is intrinsic growth rate which is positive and solution of \( B \) is \( Be^{At} > 0 \), because solution of \( B \) is an exponential function with \( A > 0 \). This shows that \( s \) and \( B \) are positive. If \( N \) is very small relative to carry capacity of population density \( K \), then \( \left(1 - \frac{N}{K}\right) \) approaches to 1. Now, \( \frac{dN}{dt} = NR \), here, \( R \) is intrinsic growth rate and positive and solution of \( N \) is \( Ne^{Rt} > 0 \), because the solution of \( N \) is an exponential function with \( C > 0 \). This shows that \( R \) and \( N \) are positive.

From the first equation of system (1), we can write in inequality form \( \frac{dB}{dt} = s \left(1 - \frac{B}{L}\right)B - s_0B - \beta_1NB - \lambda_1BP + \sigma_0FB + \phi_1BT + \psi_1BE \geq -s_0B - \beta_1NB - \lambda_1BP \). After some simplifications, we get the analytical solution of \( B(t) > B(0)e^{-(\eta_0 + \beta_1N + \lambda_1P)t} \). Since \( B(0) \) is constant of integration and represents initial value of forest resources and a positive quantity. Since analytical solution leads to \( t \to \infty \) and \( B(t) > 0 \). From this argument, we can understand that \( B(t) \) is always positive.

From the second equation of the system (1), it can be expressed as the form of inequality \( \frac{dN}{dt} = R \left(1 - \frac{N}{K}\right)N - \eta_0N + \beta NB \geq -\eta_0N \). Integrating this equation we get the solution \( N(t) > N(0)e^{-(\eta_0)t} \). Since \( N(0) \) is constant of integration and represents initial value of human population and a positive quantity. Now, as \( t \to \infty \) the solution leads to \( N(t) > 0 \). Hence, \( N(t) \) is always positive.

From the third equation of the system (1), we have inequality form \( \frac{dP}{dt} = \lambda N - \lambda_0P - \psi_2P \geq -\lambda_0P - \psi_2P \), we get the solution \( P(t) > P(0)e^{-(\lambda_0 + \psi_2)t} \). Now,
\( P(0) \) is constant of integration and represents initial value of human population pressure and a positive quantity as \( t \to \infty \), the solution leads to \( P(t) > 0 \). Hence \( P(t) \) is always positive.

In system (1) of fourth equation, we can express inequality form
\[
\frac{dF}{dt} = \delta \left( 1 - \frac{B}{L} \right) - \sigma F + dF \geq -\sigma F
\]
and integrating this equation, we get the analytical solution as \( F(t) > F(0)e^{-(\sigma)t} \). So, \( F(0) \) is an integration of constant and represents initial population of the susceptible compartment and hence it is a positive quantity. Now, as \( t \to \infty \) the analytically solution leads to \( F(t) > 0 \). Hence \( F(t) \) is always positive.

Fifth equation in system (1), \( \frac{dT}{dt} = \phi(L - B) - \psi_0 T \geq -\psi_0 T \) and integrating the inequality, we obtain the analytic solution as \( T(t) > T(0)e^{-(\psi_0)t} \). Since \( T(0) \) is constant of integration and represents the initial value of plantation and it is a positive quantity. Then, the analytical solution as \( t \to \infty \) leads to \( T(t) > 0 \). Hence \( T(t) \) is always positive.

Finally, sixth equation in system (1), \( \frac{dE}{dt} = \psi P - (\psi_0 - \mu)E \geq -\psi_0 E \). After some simplifications, we obtain analytic solution as, \( E(t) > E(0)e^{-(\psi_0)t} \). Here, \( E(0) \) is integration of constant and represents initial population of the susceptible compartment and hence it is a positive quantity. Now within the limit \( t \to \infty \), the analytical solution leads to \( E(t) > 0 \). Hence \( E(t) \) is always positive.

Therefore, the solution set \( \{ (B(t), N(t), P(t), F(t), T(t), E(t)) : 0 \leq B \leq M, 0 \leq N \leq \epsilon, 0 \leq P \leq \eta, 0 \leq F \leq \theta, 0 \leq T \leq T_m, 0 \leq E \leq E_n \} \) of system (1) is all nonnegative for all \( t \geq 0 \). Hence in biological validity of the model, if the population such as density of forest resource, density of population, population pressure, plantation, applying alternative resource, and controlling man made fire are negative. Then, they are not biologically feasible, so we have checked positivity of system. Hence the theorem is proved.

\[ \square \]

3.2 Boundedness of solution of the model

The region of attraction is the measurement of the given set distance from the mechanical phenomenon which can start and point of stability and convergence by using Lyapunov’s functions.

**Lemma 1.** The set \( A = \{ (B, N, P, F, T, E) : 0 \leq B \leq M, 0 \leq N \leq \epsilon, 0 \leq P \leq \eta, 0 \leq F \leq \theta, 0 \leq T \leq T_m, 0 \leq E \leq E_n \} \) is a region of attraction for all solution initiating in the interior of the positive octant.

Since, \( \epsilon = \frac{K}{R}(R + \beta L), \eta = \frac{\lambda e}{\lambda_0 + \psi_2}, \theta = \frac{\Delta(L-M)}{L(\sigma - d)} , T_m = \frac{\phi(L-M)}{\phi_0} , E_n = \frac{\psi_0}{\psi_0 - \mu} , \) and \( M = \frac{L}{s}(\sigma + \sigma_0 F + \phi_1 T + \psi_1 E) \).

**Proof:** From the first equation of system (1), \( \frac{dB}{dt} = s \left( 1 - \frac{B}{L} \right) B - s_0 B - \beta_1 N B - \lambda_1 BP + \sigma_0 FB + \phi_1 BT + \psi_1 BE \). Since \( B(t) \) is positive which implies that \( \frac{dB}{dt} \geq 0 \) or \( 0 \leq \frac{dB}{dt} \).

By comparison theorem, \( 0 \leq \frac{dB}{dt} \leq s \left( 1 - \frac{B}{L} \right) B + \sigma_0 FB + \phi_1 BT + \psi_1 BE \), which implies that \( 0 \leq s \left( 1 - \frac{B}{L} \right) B + \sigma_0 FB + \phi_1 BT + \psi_1 BE \).
After some simplifications we get, \( B \leq \frac{L}{\sigma}(s + \sigma_0 F + \phi_1 T + \psi_1 E) \). Depending on the definition \( 0 \leq B_0 \leq B(t) \) and \( 0 \leq B(t) \leq \frac{L}{\sigma}(s + \sigma_0 F + \phi_1 T + \psi_1 E) \). Let \( M = \frac{L}{\sigma}(s + \sigma_0 F + \phi_1 T + \psi_1 E) \), then \( 0 \leq B(t) \leq M \) implies \( B(t) \leq M \).

From the second equation of system (1), \( \frac{dN}{dt} = R\left(1 - \frac{N}{K}\right)N - r_0 N + \beta NB \). Since \( N(t) \) is positive which implies that \( \frac{dN}{dt} \geq 0 \) or \( 0 \leq \frac{dN}{dt} \), by comparison theorem \( 0 \leq \frac{dN}{dt} \leq R\left(1 - \frac{N}{K}\right)N + \beta NB \). Now, \( 0 \leq R\left(1 - \frac{N}{K}\right)N + \beta NB \). After some simplifications we get \( N \leq \frac{K}{R}(R + \beta L) \). Let \( \varepsilon = \frac{K}{R}(R + \beta L) \), we have \( 0 \leq N(t) \leq \varepsilon \) implies \( N(t) \leq \varepsilon \).

From the third equation of system (1), \( \frac{dP}{dt} = \lambda N - \lambda_0 P - \psi_2 P \), now \( P(t) \) is positive and implies that \( \frac{dP}{dt} \geq 0 \) or \( 0 \leq \frac{dP}{dt} \) and \( \frac{dP}{dt} \leq \lambda \varepsilon - \lambda_0 P - \psi_2 P \). Now \( 0 \leq \lambda \varepsilon - \lambda_0 P - \psi_2 P \), after some simplification, \( 0 \leq P(t) \leq \eta \), since \( \eta = \frac{\lambda \varepsilon}{\lambda_0 + \psi_2} \).

From the fourth equation of system (1), \( \frac{dT}{dt} = \Delta \left(1 - \frac{B}{L}\right) - \sigma F + dF \), since \( F(t) \) is positive which implies that \( \frac{dT}{dt} \geq 0 \) or \( 0 \leq \frac{dT}{dt} \) and \( 0 \leq \frac{dT}{dt} \leq \Delta \left(1 - \frac{M}{L}\right) - \sigma F + dF \), after some simplification \( 0 \leq F(t) \leq \theta \), since \( \theta = \frac{\Delta (L - M)}{\sigma - d} \).

From the fifth equation of system (1), \( \frac{dE}{dt} = \phi (L - B) - \phi_0 T \). Since \( T(t) \) is positive and implies that \( \frac{dT}{dt} \geq 0 \) or \( 0 \leq \frac{dT}{dt} \) and \( 0 \leq \frac{dT}{dt} \leq \phi (L - M) - \phi_0 T \) and this shows that \( 0 \leq T(t) \leq T_m \), where \( T_m = \frac{\phi (L - M)}{\phi_0} \) From the sixth equation of system (1), \( \frac{dE}{dt} = \psi P - (\psi_0 - \mu) E \), since \( E(t) \) is positive and implies that \( \frac{dE}{dt} \geq 0 \) or \( 0 \leq \frac{dE}{dt} \) and \( 0 \leq \frac{dE}{dt} \leq \psi \eta - (\psi_0 - \mu) \) and \( 0 \leq \frac{dE}{dt} \leq \psi \eta - (\psi_0 - \mu) \) which means \( 0 \leq E(t) \leq E_n \), where, \( E_n = \frac{\psi_0}{\psi_0 - \mu} \). This indicates that, a region of attraction for all solution initiation in the interior of positive octant. Hence, it is bounded. The set \( A \) shows that all solutions of the model are nonnegative and bounded, and hence the model is biologically well behaved.

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### 3.3 Equilibrium point of the model

Consider the nonlinear differential equation of system (1) to determine equilibrium point with \( s - s_0 > 0 \), \( R - r_0 > 0 \) and \( B \leq L \) and substituting the values of \( B = 0 \), \( N = 0 \), and \( P = 0 \), into all equations of system (1), we have \( F = \frac{\lambda}{\sigma - d}, T = \frac{\phi L}{\phi_0}, E = 0 \), Then \( E_0 \left(0, 0, 0, 0, \frac{\Delta}{\sigma - d}, \frac{\phi L}{\phi_0}, 0\right) \) is a trivial case, always the equilibrium point exist. In biological meaning, if human population and population pressure exist, then forest resources disappear. If the value of \( N = 0 \), then the equilibrium point \( E_1(B_1, 0, 0, F_1, T_1, 0) \) does not exist. Where, \( B_1 = s - s_0 + \frac{c_0 \Delta}{\sigma - d} \left(1 - \frac{B}{L}\right) + \frac{\phi \psi}{\phi_0} (L - B), T_1 = \frac{\phi}{\phi_0} (L - B), F_1 = \frac{\Delta}{\sigma - d} \left(1 - \frac{B}{L}\right) \). If \( B = 0 \), then the equilibrium point \( E_2(0, N_2, P_2, T_2, F_2, E_2) \) is nontrivial. When forestry biomass vanishes, human population and population pressure exist. Where, \( N_2 = \frac{k}{R}(R - r_0), P_2 = \frac{\lambda k}{\lambda_0 + \psi_2} (R - r_0), F_2 = \frac{\Delta}{\sigma - d} T_2 = \frac{\phi_1}{\phi_0} E_2 = \frac{\psi_2}{\psi_0 - \mu} \). Assume that \( B \neq 0, N \neq 0 \).
0, P \neq 0, F \neq 0, T \neq 0 and E \neq 0, then \( E_3(B_3, N_3, P_3, F_3, T_3, E_{3a}) \) is interior equilibrium when all variables exist. Since, \( B_3 = s - \frac{k_2}{L} - s_0 - \beta_1 N_2 - \lambda_1 P_2 + \sigma_0 F_2 + \phi_1 T_2 + \psi_1 E_2, N_2 = \frac{k}{R}(R - r_0 + \beta B_2), P_3 = \frac{\lambda k}{\gamma (\lambda_0 + \psi_2)}(R - r_0 + \beta B_2), F_3 = \frac{\Delta}{\sigma - d}(1 - \frac{B_1}{L}), T_3 = \frac{\psi}{\phi_0}(L - B) \) and \( E_3 = \frac{\psi_p}{\psi_0 - \mu} \).

### 3.4 Local and global stability analysis

A nonlinear ordinary differential equations in system \((1)\), depend on the sign of the eigenvalues corresponding to Jacobi matrices to determine the local stability. In other words, if the eigenvalues are a negative, then they are stable, otherwise unstable. The characteristic equation of the determinant of the Jacobean matrix of model \((1)\) at \( E_0(0, 0, 0, \frac{\Delta}{\sigma - d}, \frac{\phi}{\phi_0}, 0) \) is 
\[
(c - s - s_0 + \frac{\Delta_0}{\sigma - d} + \frac{\phi_1 \phi}{\phi_0})(c - (R - r_0))(c + \lambda_0 + \psi_2)(c + \sigma - d)(c + \phi_0)(c + \psi_0 - \mu) = 0,
\]
where \( c = (s - s_0 + \frac{\Delta_0}{\sigma - d} + \frac{\phi_1 \phi}{\phi_0}) \geq 0, -\lambda_0 - \psi_2, R - r_0 > 0, -\sigma + d, -\phi_0, -\psi_0 + \mu \) is the eigenvalue of system of Equation \((1)\) at \( E_0 \). If \( d > \sigma, \mu > \psi_0, R > r_0 \), then we get two or more positive eigenvalues. For this case \( E_0(0, 0, 0, \frac{\Delta}{\sigma - d}, \frac{\phi}{\phi_0}, 0) \) is unstable.

The characteristic equation of the determinant of the Jacobean matrix of system \((1)\) at \( E_1(B_1, 0, 0, F_1, T_1, 0) \) is 
\[
(c - a)(c - b)(c + \lambda_0 + \psi_2)(c + \sigma - d)(c + \phi_0)(c + \psi_0 - \mu) = 0,
\]
where, \( b = R - r_0 + \beta B_1 \) and \( a = s - s_0 - \frac{2k_1}{L} + \frac{\Delta_0 (1 - \frac{B_1}{L})}{\sigma - d} + \frac{\phi_1 (L - R)}{\phi_0} \) and \( c = (a, b > 0, -\lambda_0 - \psi_2, -\sigma + d, -\phi_0, -\psi_0 + \mu) \) is the eigenvalues of system \((1)\) at \( E_1 \). Depending on the some parameters, if we get two or more positive eigenvalues, then \( E_1(B_1, 0, 0, F_1, T_1, 0) \) is unstable.

The characteristic equation of the determinant of the Jacobean matrix of system \((1)\) at \( E_2(0, N_2, P_2, T_2, F_2, E_{2a}) \) is equal to 
\[
(c - a_2)(c - b_2)(c + \lambda_0 + \psi_2)(c + \sigma - d)(c + \phi_0)(c + \psi_0 - \mu) = 0,
\]
where, \( b_2 = R - r_0 + \frac{2N_2}{k} \) and \( a_2 = s - s_0 - \beta_1 N_2 - \lambda_1 P_2 + \frac{\Delta_0}{\sigma - d} + \frac{\phi_1}{\phi_0} \) and \( c = (a_2, b_2 > 0, -\lambda_0 - \psi_2, -\sigma + d, -\phi_0, -\psi_0 + \mu) \) is the eigenvalues of system \((1)\) at \( E_2 \). Depending on some parameters, if we get two or more than two positive eigenvalues, then \( E_2(0, N_2, P_2, T_2, F_2, E_{2a}) \) becomes unstable.

The equilibrium point of \( E_3(B_3, N_3, P_3, F_3, T_3, E_{3a}) \) is difficult to analyze using the Jacobean matrix. To analysis the local and global asymptotically stability, we use the following theorem, lemma and different assumptions.

**Theorem 2.** The interior equilibrium \( E_3(B_3, N_3, P_3, F_3, T_3, E_{3a}) \) if exists, is locally asymptotically stable in region \( A \).

**Proof.** Nontrivial equilibrium \( E_3 \) is difficult to determine the nature of eigen-values of Jacobin by using Routh-Herwitz criterion, so we use Lyapunov’s stability theory. By using the Taylor’s series expansion, the linearized system of model \((1)\) at the equilibrium point \( E_3(B_3, N_3, P_3, F_3, T_3, E_{3a}) \) can be transformed as \( B = B_3 + b_1, N = N_3 + n_1, P = P_3 + p_1, F = F_3 + f_1, T = T_3 + t_1, E = E_3 + e_1 \), where \( b_1, n_1, p_1, f_1, t_1, \) and \( e_1 \) are perturbations. By applying the positive definite function, we have
\[
M = \frac{1}{2} \left( \frac{n_1 b_1^2}{B_3} + \frac{n_2 n_1^2}{N_3} + n_3 p_1^2 + n_4 f_1^2 + n_5 t_1^2 + n_6 e_1^2 \right), \tag{2}
\]
\[
\frac{dM}{dt} = \frac{n_1 b_1 \dot{B}}{B_3} + \frac{n_2 n_1 \dot{N}}{N_3} + n_3 p_1 \dot{P} + n_4 f_1 \dot{F} + n_5 t_1 \dot{T} + n_6 e_1 \dot{E}, \tag{3}
\]

Substitute the values of \(\frac{dB}{dt}, \frac{dN}{dt}, \frac{dP}{dt}, \frac{dF}{dt}, \frac{dT}{dt}, \frac{dE}{dt}\) into this equation,
\[
\frac{dM}{dt} = n_1 b_1 \left[s - \frac{s B}{L} - s_0 - \lambda_1 P - \sigma_0 F + \phi_1 T + \psi_1 E\right] + n_2 n_1 \left[R - \frac{RN}{K} - r_0 + \beta B\right] + n_3 p_1 (\lambda N - \lambda_0 P - \psi_2 P) + n_4 f_1 \left( \Delta \left(1 - \frac{B}{L}\right) - \sigma F + \Delta F \right) + n_5 t_1 (\phi L - \phi B - \phi_0 T) + n_6 e_1 (\psi P - (\psi_0 - \mu) E),
\]
\[
\Rightarrow -n_1 \frac{s}{L} b_1^2 - n_2 \frac{R}{K} n_1^2 - n_3 (\lambda_0 + \psi_0) p_1^2 - n_4 (\sigma - d) f_1^2 - n_5 \phi t_1^2 - n_6 (\psi_0 - \mu) e_1^2
\]
\[
- (n_1 \beta_1 - n_2 \beta) b_1 n_1 - (n_5 \phi - n_1 \phi_1) b_1 t_1 - n_1 (\lambda_1 p_1 b_1 + \sigma_0 b_1 f_1 - \psi_1 e_1 b_1)
\]
\[
+ n_3 \lambda p_1 n_1 - n_4 \frac{\Delta}{L} f_1 b_1 + n_6 \psi p_1 e_1, \tag{4}
\]

now choosing \(n_1 = 1, n_2 = \frac{\beta_1}{\beta}, n_3 = 1, n_4 = 1, n_5 = \frac{\phi_1}{\phi}, n_6 = 1,\)
\[
\Rightarrow -\frac{s}{L} b_1^2 - \frac{\beta_1}{\beta} \frac{R}{K} n_1^2 - (\lambda_0 + \psi_0) p_1^2 - (\sigma - d) f_1^2 - \phi_1 t_1^2 - (\psi_0 - \mu) e_1^2
\]
\[
- \left( \lambda_1 p_1 b_1 + \left( \sigma_0 + \frac{\Delta}{L} \right) b_1 f_1 - \psi_1 e_1 b_1 \right) + \lambda p_1 n_1 + \psi p_1 e_1. \tag{5}
\]

Since, \(\frac{dM}{dt} < 0\) will be negative definite inside the region of attraction \(A\). This implies that \(E_3(B_3, N_3, P_3, F_3, T_3, E_{3a})\) is local asymptotically stable. \(\square\)

**Theorem 3.** The interior equilibrium \(E_3(B_3, N_3, P_3, F_3, T_3, E_{3a})\) if exists, is globally asymptotically stable in region \(A\).

**Proof.** Let us apply lyapunov function in system (1), thus
\[
M = \left( B - B_3 - B_3 \ln \frac{B}{B_3} \right) + a \left( N - N_3 - N_3 \ln \frac{N}{N_3} \right) + \frac{b(P - P_3)^2}{2} + \frac{c(F - F_3)^2}{2} \tag{6}
\]
\[
+ \frac{d(T - T_3)^2}{2} + \frac{e(E - E_3)^2}{2},
\]
where \( a, b, c, d, \) and \( e \) are positive constants with an approximate values. This equation is a positive definite function if

\[
M = \left( B - B_3 - B_3 \ln \frac{B}{B_3} \right) + a \left( N - N_3 - N_3 \ln \frac{N}{N_3} \right) + \frac{b(P - P_3)^2}{2} + \frac{c(F - F_3)^2}{2} + \frac{d(T - T_3)^2}{2} + \frac{e(E - E_3)^2}{2} > 0.
\]

Differentiate Equation (7) with respect to time \( t \), we get

\[
\frac{dM}{dt} = (B - B_3) \frac{\dot{B}}{B} + c_1(N - N_3) \frac{\dot{N}}{N} + c_2(P - P_3) \dot{P} + c_3(F - F_3) \dot{F} + c_4(T - T_3) \dot{T} + c_5(E - E_3) \dot{E},
\]

Now using the system of differential equations and applying Jacobean matrix, we get,

\[
\frac{dM}{dt} = -\frac{s}{L} (B - B_3)^2 - \alpha \frac{R}{K} (N - N_3)^2 - b(\lambda_0 + \psi_2)(P - P_3)^2 - c(\sigma - d)(F - F_3)^2
\]

\[
- e(\psi_0 - \mu)(E - E_3)^2 - (\beta_1 - a\beta)(B - B_3)(N - N_3) - \lambda_1(B - B_3)(P - P_3)
\]

\[
+ \sigma_0((B - B_3)(F - F_3) + b\lambda(N - N_3)(P - P_3) + (\phi_1 - d\phi)(B - B_3)(T - T_3)
\]

\[
+ \psi_1(B - B_3)(E - E_3) - \frac{\Delta}{L}(B - B_3)(F - F_3) + e\psi(P - P_3)(E - E_3).
\]

Then, let us consider, \( a = \frac{\beta_1}{\beta} \), \( b = c = e = 1 \) and \( d = \frac{\phi_1}{\phi} \) and substituting into Equation (8), we get

\[
\frac{dM}{dt} = -\frac{s}{L} (B - B_3)^2 - \frac{R\beta_1}{\beta K} (N - N_3)^2 - (\lambda_0 + \psi_2)(P - P_3)^2 -
\]

\[
(\sigma - d)(F - F_3)^2 - \frac{\phi_1}{\phi} \phi_0(T - T_3)^2 - (\psi_0 - \mu)(E - E_3)^2
\]

\[
+ \psi(P - P_3)(E - E_3) - \lambda_1(B - B_3)(P - P_3) - \left( \frac{\Delta}{L} - \sigma_0 \right)(B - B_3)(F - F_3)
\]

\[
+ \lambda(N - N_3)(P - P_3) + \psi_1(B - B_3)(E - E_3) < 0.
\]

Which implies that

\[
\frac{dM}{dt} < 0. \quad (9)
\]
Now, $\frac{dM}{dt} < 0$ is negative definite inside the region of attraction $A$. This show that the equilibrium point $E_3(B_3, N_3, P_3, F_3, T_3, E_{3*})$ is globally asymptotically stable. This theorem implies that under certain condition, the system (1) highly resistance to change in species composition and all species coexist in same habitat. This indicates that the density of forest resource decrease due to increase of the density of population and associated pressures. Generally, the analytical analysis shows that, the density of forest resource are extinction in the absence of conservation. The presence of conservation, the density of forest resource can be maintained to its original level.

4 | NUMERICAL SIMULATION, RESULT AND DISCUSSION

Numerical simulations of the system (1) are obtained by using ODE45 subroutine of MATLAB. The stability analysis of the given system of nonlinear ordinary differential equation (1) is carried out by substituting the following parameter values in Table 1, then the given model is systematically computed.

The interior equilibrium points $E_3(B_3, N_3, P_3, F_3, T_3, E_{3*})$ from the data of Table 1, are $B_3 = 49.6205143$, $N_3 = 27.595446$, $P_3 = 3.41632162$, $F_3 = 0.03794855$, $T_3 = 0.12649523$, $E_3 = 0.569387$. Substituting these values in system (1), we get negative eigenvalues such as $-0.5002$, $-0.3635$, $-0.0927$, $-0.0517$, $-0.0704$, and $-0.0300$. Hence, interior equilibrium point $E_3(B_3, N_3, P_3, F_3, T_3, E_{3*})$ is locally asymptotically stable. This indicates that forest resources are stable with small disturbance.

Figures 1–4 show the mechanical phenomenon in the interior of the domain of the system (1) which are approaching towards the equilibrium points $E_3(B_3, N_3, P_3, F_3, T_3, E_{3*})$. Here, we can see that the trajectories starting with different initial points A, B, C, and D converge to an interior equilibrium point $E_3$. Hence, we can say that the equilibrium point $E_3$ is globally asymptotically stable.

Figure 5 shows the effect of depletion rate of forest due to population on the density of forest resources at time $t$. Here, it can be seen that as the depletion rate of forestry resources which is denoted by $\beta_1$ increases, the density of forest resources decreases. From the biological fact we can understand that forest resources can be conserved by reducing the depletion rates due to population.

Figure 6 shows the effect of depletion rate due to population pressure on the density of forestry resources. Here, we can see that as depletion rate due to population pressure increases, the density of forest resources reduces. Biologically, it is a fact that forestry resources can be conserved by reducing the depleting factors.

Figure 7 shows the effect of conservation rate of forests due to controlling man made fire and toxicant activities on the density of forestry resources. Here, we can see that as conservation rate increases, the density of forestry resources increases. From the biological point of view, forests can be conserved by applying controlling mechanisms of man made fire and toxicant activities.
| Parameters | Description | Value (per year) | Source |
|------------|-------------|-----------------|--------|
| $s$        | Intrinsic growth rate of forest resource | 0.6 | Misra and Lata (2015a) |
| $\beta$    | growth rate of population due to forest | 0.00004 | Misra and Lata (2015a) |
| $L$        | Carrying capacity of forest resource | 50 | Pathak (2018), Misra and Lata (2015b), Misra et al. (2013) |
| $\lambda$  | The growth rate of population pressure | 0.1 | Misra and Lata (2015a) |
| $s_0$      | Natural depletion rate of forest resource | 0.02 | Assumed |
| $\lambda_0$ | Natural depletion rate of population pressure | 0.8 | Misra and Lata (2015a) |
| $\beta_1$  | Depletion rate of forest due to population | 0.0004 | Assumed |
| $\Delta$  | Rate of effort to conserve the forest | 3.14, 10, 18 | Assumed |
| $\lambda_1$ | Depletion rate of forest due to population pressure | 0.0011 | Misra and Lata (2015a) |
| $\sigma$  | Depletion rate of controlling of fire | 0.04, 0.5 | Dubey and Hussain (2015), Agarwal and Pathak (2015) |
| $\sigma_0$ | Conservation rate of forests due to controlling fire | 0.3 | Agarwal and Pathak (2015) |
| $\eta_0$  | Natural depletion rate of population | 0.01 | Assumed |
| $\phi_1$  | Conservation rate of forest due to reforestation | 0.02 | Misra and Lata (2015a) |
| $\phi$    | The growth rate of plantation | 0.01 | Misra and Lata (2015a), Agarwal and Pathak (2015) |
| $\psi_1$  | Conservation rate of forest due to economical incentive | 2, 0.05 | Assumed |
| $\phi_0$  | Natural depletion rate of plantation | 0.03, 0.02 | Misra and Lata (2015a), Agarwal and Pathak (2015) |
| $R$       | Intrinsic growth rate of human population | 0.1 | Misra and Lata (2015a), Qureshi and Yusuf (2019) |
| $\psi$    | The growth rate of economic incentives | 0.05 | Misra and Lata (2014) |
| $K$       | Carrying capacity of human population | 100 | Teru and Koya (2020), Misra and Lata (2015a), Misra and Lata (2014) |
| $\psi_0$  | Natural depletion rate of economic incentives | 0.1 | Misra and Lata (2015a) |
Figure 8 shows the impact of conservation rate of forestry resources due to reforestation on the density of forestry resources. From this figure, we can see that as conservation rate of forests increases, the density of forestry resources increases. Biologically, conservation rate of forests due to reforestation causes the conservation of depleted forestry resources.

Figure 9 shows the variation of density of forestry resources for the different values of $\psi_1$. Here, we can see that as conservation rate of forests due to economical incentives increases, the density of forestry resources increases. Biologically, conservation of forestry resources can be carried out by applying economical incentives.

Figure 10 shows the effect of the growth rate of population pressure on the density of population pressure. Here, as the growth rate of population pressure $\lambda$ increases, the density of

| Parameters | Description                          | Value (per year) | Source   |
|------------|-------------------------------------|------------------|----------|
| $\psi_2$   | Rate of population pressure due to economic incentives | 0.3              | Assumed  |
| $d$        | Conservation rate of controlling man made fire and toxicant activities | 0.01             | Assumed  |
| $\mu$      | Conservation rate of forest due to economic incentives | 0.1              | Assumed  |

FIGURE 1 Global stability of $E_3$ in BNP space

Figure 8 shows the impact of conservation rate of forestry resources due to reforestation on the density of forestry resources. From this figure, we can see that as conservation rate of forests increases, the density of forestry resources increases. Biologically, conservation rate of forests due to reforestation causes the conservation of depleted forestry resources.

Figure 9 shows the variation of density of forestry resources for the different values of $\psi_1$. Here, we can see that as conservation rate of forests due to economical incentives increases, the density of forestry resources increases. Biologically, conservation of forestry resources can be carried out by applying economical incentives.

Figure 10 shows the effect of the growth rate of population pressure on the density of population pressure. Here, as the growth rate of population pressure $\lambda$ increases, the density of
population pressure $P(t)$ increases. However, the density of forestry resources reduces due to increase of the growth rate of population pressures. Biologically, the population pressure causes depletion of forestry resources.

Figure 11 shows the effect of growth rate of population due to forests on human population. Here, one can see that as the growth rate of population ($\beta$) increases,
FIGURE 4  Global stability of $E_3$ in BNT space

FIGURE 5  Variation of $B(t)$ versus time ($t$) for the values of $\beta_1$
FIGURE 6  Variation of $B(t)$ versus time $(t)$ for the values of $\lambda_1$

FIGURE 7  Variation of $B(t)$ versus time $(t)$ for the values of $\sigma_0$
the human population increases at its equilibrium point. However, as the human population growth logistically, the density of forestry resources depleted. Moreover, if there is less population density in the forest area, then the forest resources highly increase.

Figure 12 shows the effect of the growth rate of economic incentives on economic incentives. Here, as the coefficient of economic incentives increases, the economic incentives and the density of forest resources increase. Biologically, conservation of forestry resources can be maintained by applying economic incentives.

Figure 13 shows that $T(t)$ increases at its equilibrium point with increases of $\phi$. Here, the rate of plantation $T(t)$ increases, the coefficient of plantation $\phi$ and the density of forest resource $B(t)$ increase. We conclude that the rate of forest resource $B(t)$ is conserved or protected by applying technological efforts $T(t)$.

Generally, the equilibrium level of forest resources can be conserved at the proper level for high growth rate of conservation parameters such as plantation or reforestation, control human fire and pollution, and use of alternative resources which have an important role in conservation of forest resources. The density of population and population pressures due o use of capacity are extinction of forest resources depending on their parameters.
FIGURE 9  Variation of $B(t)$ versus time ($t$) for the values of $\psi_1$

FIGURE 10  Variation of $P(t)$ versus time ($t$) for the values of $\lambda$
FIGURE 11  Variation of $N(t)$ versus time $t$ for the values of $\beta$

FIGURE 12  Variation of $E(t)$ versus time $t$ for the values of $\psi$
In this paper, we proposed and analyzed a nonlinear mathematical model to study the conservation of depleted forest resources due to augmented population and associated pressures. The stability analysis is carried out depending on the equilibrium points, Jacobean matrix, and eigenvalues of the given conservation model. The negative influences of forest resources increase with increased population and associated pressures. The positive impacts of the resource of forests is controlling man made fire and environmental pollution and applying economic and technological efforts. The global and local stability analysis of the equilibrium point of the system of equation have been demonstrated by applying different theorems. The conservation model is stable at the equilibrium point $E_0(B_0, N_0, P_0, F_0, T_0, E_{3b})$ for all negative eigenvalues, and other equilibrium points are unstable because we get a positive eigenvalue. The numerical simulations show that the depletion rate of forest resources due to human population and human pressure continuously increases without any protection mechanism, the equilibrium point of forest resources decreases. The analysis of conservation model implies, when technological effort, economical incentives, controlling man made fire, and environmental pollution are applied, the forest resource can be expressed at a proper level and thus, forest density may remain conserved at any time. If no efforts is made to preserve the forest resources, then the equilibrium points of the forest resources remain to be decrease.

**AUTHOR CONTRIBUTIONS**

**Masitawal Demsie Goshu:** conceptualization (lead); data curation (lead); formal analysis (lead); software (lead); writing—original draft (equal); writing—review and editing (equal).

**Mehari Fentahun Endalew:** conceptualization (equal); formal analysis (equal); investigation (equal); methodology (equal); validation (equal); writing—review and editing (equal).
DATA AVAILABILITY STATEMENT
We declare that the materials described in the manuscript, including all relevant raw data, will be freely available to any scientist wishing to use them for noncommercial purposes without breaching participant confidentiality.

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