On the relationships between Fourier-Stieltjes coefficients and spectra of measures

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Abstract

We construct examples of uncountable compact subsets of complex numbers with the property that any Borel measure on the circle group taking values of its Fourier coefficients from this set has natural spectrum. For measures with Fourier coefficients tending to 0 we construct the open set with this property.

1 Introduction

Let $M(\mathbb{T})$ denote the convolution algebra of Borel measures on the unit circle group. For details of notation and basic definitions see [K1]. The closure of values of the Fourier coefficients of $\mu \in M(\mathbb{T})$ are obviously a subset of a spectrum $\sigma(\mu)$ of $\mu$. However, as was observed by Wiener and Pitt (see [WP]) in general it is a proper subset. There are several different proofs of this phenomenon - cf. [S], [G]. Wiener - Pitt phenomenon is equivalent

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to the inversion problem which states that assumption $|\hat{\mu}(n)| > c > 0$ for all $n \in \mathbb{Z}$ and constant $c$ does not ensure invertibility of $\mu$ as an element in Banach algebra $M(\mathbb{T})$. Moreover, it is closely related to the remarkable asymmetry of algebra $M(\mathbb{T})$ which is presented in [R].

On the other hand there are classes of Borel measures for which spectrum equals to the closure of the set of Fourier coefficients. Such a measures are said to have a natural spectrum. It is known that absolutely continuous and purely discrete measures have natural spectrum (cf. [Z1]). The more natural question is how to recognize a measure with natural spectrum using only the information about its Fourier coefficients. Motivated by this problem we introduce a notion of Wiener - Pitt sets. We say that a compact set $A \subset \mathbb{C}$ is a Wiener - Pitt set whenever $\hat{\mu}(\mathbb{Z}) \subset A$ implies that $\mu$ has natural spectrum.

Finite sets are easy examples of Wiener - Pitt sets. Indeed, if $A = \{a_1, \ldots, a_k\}$ then by Gelfand theory the polynomial $P(z) = (z - a_1) \ldots (z - a_k)$ satisfies $\hat{P}(\mu)(n) = 0$ for every $n \in \mathbb{Z}$. Therefore $P(\mu) = 0$ which in turn yields that $P(\psi(\mu)) = 0$ for every linear multiplicative functional $\psi$. Hence $\psi(\mu)$ is a root of polynomial $P$ and therefore $\sigma(\mu) \subset A$. Finite sets are the only known class of Wiener - Pitt sets. The aim of this paper is to construct the infinite (even uncountable) Wiener - Pitt compacta. Our construction gives quite flexible family of zero dimensional examples and, moreover, the examples are stable under suitable small perturbation which proves their rather non-algebraic flavor. Even more we prove about the subalgebra of continuous measures. We construct the open subset $U \subset \mathbb{C}$ such that $0 \in U$ and any continuous measure with $\hat{\mu}(\mathbb{Z}) \subset U \cup \{0\}$ has natural spectrum.

**Theorem 1.** There exists an open set $U \subset \mathbb{C}$ with $0 \in U$ such that every continuous measure $\mu$ with $\hat{\mu}(\mathbb{Z}) \subset U \cup \{0\}$ has natural spectrum.

**Theorem 2.** There exists a set $K$ homeomorphic to the Cantor set such that $0 \in K$ and every measure $\mu$ with $\hat{\mu}(\mathbb{Z}) \subset K$ has natural spectrum.

To complete the above result we provide the example of a singular measure with spectrum contained in the set $U$ constructed in Theorem 1. The example, interesting itself, is given in Section 6 and combines the technics of the Riesz products with Rudin - Shapiro polynomials to get the singular measure with coefficients from the sequence converging to 0 arbitrarily fast.

The construction is quite involved - it uses four main ingredients. First is the Zafran characterization of measures with Fourier coefficients tending to zero with natural spectrum. Second is the Katznelson - DeLeeuw theorem
which is the main ingredient of their qualitative version of Grużewska - Rajchman theorem (nevertheless in the paper we use stronger and more involved result from [GM] - just to omit unnecessary complications) Third is the Bożejko - Pełczyński theorem on uniform invariant approximation property of $L^1(\mathbb{T})$. Fourth ingredient is the Littlewood Conjecture proved by McGehee Pigno Smith and independently by Konyagin.

The main construction constituting the proof of Theorem 1 under additional assumption that the regarding measure has Fourier coefficients tending to 0, is presented in Section 3. The aim of Section 4 is to prove that this additional assumption can be omitted. In Section 5 we complete the proof of Theorem 1 and we show how Theorem 2 could be derived from Theorem 1. Section 2 is devoted to prove of the auxiliary lemmas. Here we combine two main analytical ingredients, the Bożejko - Pełczyński uniform invariant approximation property and the Littlewood conjecture, to derive Lemma 8 - the main tool used in further inductions.

2 Preparatory lemmas

In this section we prove crucial lemmas which are also of independent interest. The following definition will be useful.

**Definition 3.** For $\mu \in M(\mathbb{T})$ and $\varepsilon > 0$ we write $\mu \in F(\varepsilon)$ iff $|\hat{\mu}(n)| < \varepsilon$ for all $n \in \mathbb{Z}$. Set of trigonometric polynomials will be denoted $\mathcal{P}$. The abbreviation $\#f$ will be used for number of elements in support of $\hat{f}$ (for $f \in \mathcal{P}$), i. e. $\#f = \#\text{supp} \hat{f} = \#\{n \in \mathbb{Z} : \hat{f}(n) \neq 0\}$. Also, for $f \in \mathcal{P}$ we say that $f \in G(|a|)$ for some complex number $a$ when inequality $|\hat{f}(n)| \geq |a|$ is true for all integers $n$ such that $\hat{f}(n) \neq 0$.

We will use in the sequel two powerful results. First is the Littlewood conjecture (for a proof consult [MPS] and [K1])

**Theorem 4.** (McGehee, Pigno, Smith; Konyagin) For every $f \in \mathcal{P}$ of the form

$$f(t) = \sum_{k=1}^{N} c_k e^{i n_k t},$$

where $n_k$ are sequence of increasing integers and $|c_k| \geq 1$, $1 \leq k \leq N$, we have

$$\|f\|_{L^1(\mathbb{T})} > L \ln N;$$

where the constant $L > 0$ does not depend on $N$. 

Second fact is the invariant uniform approximation property of $L^1(\mathbb{T})$ (proofs are contained in papers [BP] and [B] or in book [W2]).

**Theorem 5** (Bożejko, Pełczyński; Bourgain). Let $\Lambda \subset \mathbb{N}$ be a finite set with $\#\Lambda = k$. Then for every $\varepsilon > 0$ there exists $f \in \mathcal{P}$ such that

1. $\hat{f}(n) = 1$ for $n \in \Lambda$.
2. $\|f\|_{L^1(\mathbb{T})} \leq 1 + \varepsilon$
3. $\#\{n \in \mathbb{N} : \hat{f}(n) \neq 0\} \leq \left(\frac{c}{\varepsilon}\right)^{2k}$ for some $c > 0$.

We will write $f \in BPN_\varepsilon(\Lambda)$ for polynomials with described properties.

It is now time to formulate our first lemma.

**Lemma 6.** Let $\mu \in \mathcal{P}$. If $\#\mu > d$ for some integer $d$, then there exists two sided arithmetical progression $\Gamma \subset \mathbb{Z}$ such that

$$d < \#(\text{supp} \hat{\mu} \cap 1_\Gamma) < 2d.$$ 

**Proof.** If $d < \#\mu < 2d$, we take $\Gamma = \mathbb{Z}$. Otherwise $\#\mu > 2d$ and we taking $\Gamma_1 = 2\mathbb{Z}$ and $\Gamma_2 = 2\mathbb{Z} + 1$ we get that $d < \#(\text{supp} \hat{\mu} \cap 1_{\Gamma_i})$ for some $i = 1, 2$. If moreover $\#(\text{supp} \hat{\mu} \cap 1_{\Gamma_i}) < 2d$, we put $\Gamma = \Gamma_i$. Otherwise we repeat this procedure. 

Second lemma is much more interesting.

**Lemma 7.** There exists a function $\varepsilon = \varepsilon(K, |a|)$ and $c > 0$ such that whenever $||f + \nu||_{M(\mathbb{T})} < K$ for some $f \in G(|a|)$ and $\nu \in F(\varepsilon)$, then $\#f < \exp\left(\frac{cK}{|a|}\right)$.

**Proof.** Suppose $\#f > d$ for some integer $d > 0$. By Lemma 6 there exists $\Gamma \subset \mathbb{Z}$ such that

$$d < \#(\text{supp} \hat{\nu} \cap 1_\Gamma) < 2d.$$ 

We define $f_1 \in \mathcal{P}$ and $\nu_1 \in M(\mathbb{T})$ by taking

$$\hat{f}_1 = \hat{f} \cdot 1_\Gamma \text{ and } \hat{\nu}_1 = \hat{\nu} \cdot 1_\Gamma.$$ 

Since this is a convolution with a measure of norm one.

$$||f_1 + \nu_1||_{M(\mathbb{T})} < K$$

holds. By the definition of $\Gamma$,

$$\#f_1 < 2d.$$
It follows from Theorem 4 that
\[ ||f_1||_{M(T)} \geq a \cdot L \ln d \]
Let \( \Theta \in BPN_1(\text{supp} f_1) \). Then
1. \( ||\Theta||_{L^1(T)} < 2 \).
2. \( \hat{\Theta}(n) = 1 \) for \( n \in \text{supp} f_1 \).
3. \( \#\Theta < C^{2d} \) for some \( C > 0 \).
By the triangle inequality,
\[ 2K > ||(f_1 + \nu_1) * \Theta||_{M(T)} = ||f_1 + \nu_1 * \Theta||_{M(T)} \geq ||f||_{L^1(T)} - ||\Theta * \nu_1||_{M(T)} . \]
Estimating the \( L^1 \)-norm by the \( L^2 \)-norm we get
\[ ||\Theta * \nu_1||_{L^1(T)} \leq ||\Theta * \nu_1||_{L^2(T)} \leq 2\varepsilon C^d . \]
All together gives
\[ 2K > |a| \cdot L \ln d - 2\varepsilon C^d . \]
Hence the assumption \( d = \exp(\frac{4K}{|a|L}) \) leads to the contradiction if we put \( \varepsilon(K, |a|) = \frac{|a|L \ln d}{4C^d} . \) This ends the proof of the lemma with \( c = \frac{1}{L} \). \( \square \)

Next lemma gives more information.

**Lemma 8.** For every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon, |a|, K) \) such that if \( ||f + \nu||_{M(T)} < K \) for \( f \in G(|a|) \) and \( \nu \in F(\delta) \), then \( ||f||_{L^1(T)} < K(1 + \varepsilon) \).

**Proof.** Let \( \Theta \in BPB_{\frac{\varepsilon}{2}}(\text{supp} \hat{f}) \). Then
1. \( ||\Theta||_{L^1(T)} < 1 + \frac{\varepsilon}{2} \).
2. \( \hat{\Theta}(n) = 1 \) for \( n \in \text{supp} \hat{f} \).
3. \( \#\Theta < \left(\frac{\lambda}{\varepsilon}\right)^{2\# f} = \exp(2\# f \ln \frac{\lambda}{\varepsilon}) \) for some \( \lambda > 0 \).
By Lemma 7 for sufficiently small \( \delta \) we have
\[ \#\Theta < \exp(2 \ln \frac{\lambda}{\varepsilon} (\exp(cK|a|^{-1})) , \]
where \( c \) is as in Lemma 7. By the triangle inequality
\[ \left(1 + \frac{\varepsilon}{2}\right) K \geq ||\Theta * (f + \nu)||_{L^1(T)} = ||f + \Theta * \nu||_{L^1(T)} \geq ||f||_{L^1(T)} - ||\Theta * \nu||_{L^1(T)} . \]
Since $|\hat{\Theta}| < 1 + \frac{\xi}{2}$, we have $\Theta \ast \nu \in F((1 + \frac{\xi}{2})\delta)$. Obviously
\[
\#(\Theta \ast \nu) \leq \#\Theta < \exp(2\ln \frac{\lambda}{\varepsilon}(\exp(cK|a|^{-1}))).
\]
Estimating the $L^1$- norm by the $L^2$-norm we get
\[
||\Theta \ast \nu||_{L^1} \leq ||\Theta \ast \nu||_{L^2} \leq \left(1 + \frac{\varepsilon}{2}\right) \delta \exp(\ln \frac{\lambda}{\varepsilon}(\exp(cK|a|^{-1}))).
\]
Completing all together we get
\[
\left(1 + \frac{\varepsilon}{2}\right) K > ||f||_{L^1} - \left(1 + \frac{\varepsilon}{2}\right) \delta \exp(\ln \frac{\lambda}{\varepsilon}(\exp(cK|a|^{-1}))).
\]
If we put
\[
\delta = \frac{\varepsilon}{1 + \varepsilon} K \exp(-\ln \frac{\lambda}{\varepsilon}(\exp(cK|a|^{-1}))),
\]
then the assumption
\[
||f||_{L^1} > (1 + \varepsilon)K
\]
leads to a contradiction. \(\square\)

3 The case of $M_0(\mathbb{T})$

We will make use of the following proposition contained in [W1].

**Proposition 9.** Let $A$ be a commutative Banach algebra with unit and $x \in A$ has finite spectrum, $\sigma(x) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Put $\delta = \min_{i \neq j} |\lambda_i - \lambda_j|$. Then there exist orthogonal idempotents $x_1, x_2, \ldots, x_n \in A$ (i.e. $x_i^2 = x_i$ and $x_i x_j = 0$ for $i \neq j$, $i, j = 1, \ldots, n$) such that
\[
x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n.
\]
Moreover, the inequality
\[
||x_i|| \leq \delta^{-n+1}2^{n-1}||x||^{n-1}
\]
holds for $i = 1, 2, \ldots, n$.

One abbreviation is useful when it comes to manipulate with convolution powers. We will write $f^m = f \ast f \ast \ldots \ast f - m$ - times. Now, we prove lemma which is simple corollary of the last proposition.

**Lemma 10.** Let $f$ be a trigonometric polynomial such that
\[
\hat{f}(\mathbb{Z}) = \{0, \lambda_1, \lambda_2, \ldots, \lambda_k\}.
\]
Put $\lambda_0 = 0$ and define $\delta = \min_{i \neq j} |\lambda_i - \lambda_j|$, $\lambda_{max} = \max\{||\lambda| : \lambda \in \hat{f}(\mathbb{Z})\}$. Then, for every $m \in \mathbb{N}$
\[
||f^m|| \leq k\delta^{-k} 2^k ||f||^k \lambda_{max}^m.
\]
Proof. We easily see that, if \( f \) is a polynomial, its spectrum in the algebra \( M(\mathbb{T}) \) is equal to \( \hat{f}(\mathbb{Z}) \). Hence, by Proposition 9, we have

\[ f = \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_k f_k \]

for some orthogonal idempotent polynomials \( f_k \). Simple calculation shows that

\[ f^m = \left( \sum_{l=1}^{k} \lambda_l f_l \right)^m = \sum_{l=1}^{k} \lambda_l^m f_l. \]

Applying the estimate from Proposition 9, we complete the proof

\[ ||f^m|| \leq \sum_{l=1}^{k} |\lambda_l|^m ||f_l|| \leq k\delta^{-k} ||f||^m \lambda_{\max}^m. \]

Next theorem gives a possibility to relax strict spectral conditions of previous lemma.

**Theorem 11.** Let \( \Lambda \subset \mathbb{C} \) be finite set with \( \# \Lambda = m + 1 \) such that \( 0 \in \Lambda \) and let us denote \( \lambda_{\max} = \max\{|\lambda| : \lambda \in \Lambda\} \). Then there exists \( C = C(\Lambda) \) such that, for every \( K > 0 \) and every \( k > m \) there exists \( \varepsilon = \varepsilon(K, \Lambda) \) with property: if \( ||f|| \leq K \) satisfies \( \hat{f}(\mathbb{Z}) \subset \Lambda + B(0, \varepsilon) \), then the following inequality holds true

\[ ||f^k|| \leq C\lambda_{\max}^{k-m} ||f||^m, \]

where \( C = C(\Lambda) = C(t \cdot \Lambda) \) for any \( t \in \mathbb{C} \setminus \{0\} \).

**Proof.** Let \( \lambda_0 = 0, \Lambda = \{0, \lambda_1, \lambda_2, \ldots, \lambda_m\} \) and put \( \delta = \min_{i \neq j} |\lambda_i - \lambda_j| \). We may assume that \( \varepsilon < \delta \) which guarantees \( B(\lambda_i, \varepsilon) \cap B(\lambda_j, \varepsilon) = \emptyset \) for \( i \neq j \). Let us take \( n \in \mathbb{Z} \), then there exists unique \( \lambda_{i_n} \) with property \( \min_{j=0,1,\ldots,m} |\lambda_j - \hat{f}(n)| = |\lambda_{i_n} - \hat{f}(n)|. \) Now, we define the polynomial \( f_0 \) by the condition \( \hat{f}_0(n) = \lambda_{i_n} \in \Lambda \) for every \( n \in \mathbb{Z} \). It is obvious that \( \hat{f}(\mathbb{Z}) \subset \Lambda \). Moreover, \( g = f - f_0 \) satisfies the property \( \hat{g}(\mathbb{Z}) \subset B(0, \varepsilon) \). We have to estimate \( ||g|| \) (an upper bound for \( ||f_0|| \) follows from Lemma 10). A simple observation gives that, if \( \hat{f}(l) = 0 \) for some \( l \in \mathbb{Z} \), then \( \hat{g}(l) = 0 \). For any polynomial \( h \), let us write \( \#h = \#\{n \in \mathbb{Z} : \hat{h}(n) \neq 0\} \). Using Parseval’s identity we obtain

\[ ||g||_{L^1(\mathbb{T})} \leq ||g||_{L^2(\mathbb{T})} \leq \varepsilon \sqrt{\#g} \leq \varepsilon \sqrt{\#f}. \]

Putting \( \gamma = \min_{n \in \mathbb{Z}} |\hat{f}(n)| \), we obtain from the McGehee-Pigno-Smith + Konyagin theorem

\[ \gamma ||f|| \geq L \ln(\#f). \]
which leads to
\[
\sqrt{\#f} \leq \exp\left(\frac{\gamma}{2L} \|f\|\right) \leq \exp\left(\frac{\gamma}{2L} K\right).
\]
Taking it all together we have
\[
\|g\| \leq \varepsilon \exp\left(\frac{\gamma}{2L} K\right).
\]
Finally
\[
\|f^k\| \leq \|f_0^k\| + \sum_{l=0}^{k-1} \binom{k}{l} \|f_0^l\| \|g\|^{k-l}.
\]
Using Lemma 10 we obtain
\[
\|f_0^l\| \leq m\delta^{-m^2} 2^m \lambda_{\max}^l \|f\|^m \text{ for } l = 1, 2, \ldots, k
\]
Moreover, we have
\[
\|g\|^{k-l} \leq \varepsilon^{k-l} \exp\left(\frac{\gamma(k-l)}{2l} K\right).
\]
Collecting these information we get
\[
\|f^k\| \leq m\delta^{-m^2} 2^m \|f\|^m (\lambda_{\max}^k + \varepsilon \sum_{l=1}^{k-1} \binom{k}{l} \lambda_{\max}^l \varepsilon^{k-l-1} \exp\left(\frac{\gamma(k-l)}{2l} K\right) + \frac{1}{\|f\|^m} \varepsilon^{k-1} \exp\left(\frac{m\gamma}{2l} K\right))
\]
Taking \(\varepsilon\) so small that the expression in parenthesis is smaller than \(2\lambda_{\max}^k\) we finally get
\[
\|f^k\| \leq m\delta^{-m^2} 2^m \|f\|^m \lambda_{\max}^k.
\]
Putting \(C = m2^{m+1} \lambda_{\max}^m \delta^m\) we have \(C = C(\Lambda) = C(t\cdot \Lambda)\) for every \(t \in \mathbb{C} \setminus \{0\}\) which gives the desired estimation. \(\square\)

It is useful now to introduce the following definition. Let \(C > 0\) and \(k \in \mathbb{N}, k \geq 2\). We say that a compact set \(\Lambda \subset \mathbb{C}\) with \(0 \notin \Lambda\) belongs to the class \(U(C,k)\) provided for every \(K > 0\) there exists an open neighborhood \(V_K\) of \(\Lambda\) such that for every \(\mu \in M_0(\mathbb{T})\) satisfying \(\|\mu\| \leq K\) and \(\sigma(\mu) \subset V_K \cup \{0\}\) we have
\[
\|\mu^k\|_{M(\mathbb{T})} \leq C \|\mu\|^{k-1}_{M(\mathbb{T})}.
\]
By Theorem 11 every finite set \(\Lambda \subset \mathbb{C}\) with \(0 \notin \Lambda\) and \(#\Lambda = k - 1\) belongs to the class \(U(C,k)\).

**Theorem 12.** Let \(C > 0\) and \(k \in \mathbb{N}\). Assume that the sets \(X,Y \in U(C,k)\) are such that \(X \subset B(0,r)\) and \(Y \subset \{z \in \mathbb{C} : |z| > R\}\) for some \(R,r > 0\). Then for every \(C' > C\), there exists \(\varepsilon = \varepsilon(r,R,C') > 0\) such that \(\varepsilon X \cup Y \in U(C',k)\).
Proof. Let us fix $K > 0$ and take $\mu \in M_0(\mathbb{T})$ such that

$$\sigma(\mu) \subset (\varepsilon X \cup Y + B(0, \delta)) \cup \{0\}$$

and $||\mu|| < K$. Since $\mu \in M_0(\mathbb{T})$, there are only finitely many $n \in \mathbb{Z}$ satisfying $\hat{\mu}(n) \in Y + B(0, \delta)$. Hence, we may define the polynomial $f$ by conditions $\hat{f}(n) = \hat{\mu}(n)$, if $\hat{\mu}(n) \in Y + B(0, \delta)$ and $\hat{f}(n) = 0$ otherwise. Then the measure $\nu = \mu - f$ satisfies $\hat{\nu}(\mathbb{N}) \subset (\varepsilon X \cup B(0, \delta)) \cup \{0\}$ and the equality $\mu = f + \nu$ holds. Now, we apply Lemma 8. For sufficiently small $\varepsilon = \varepsilon(c)$ we get $||f||_{L^1(\mathbb{T})} \leq c||\mu||$, where $c$ is any number greater than 1 and, consequently $||\nu|| \leq ||\mu|| + ||f|| \leq (1 + c)||\mu||$. The measure $\varepsilon^{-1}\nu$ has spectrum contained in $(X + B(0, \varepsilon^{-1}\delta)) \cup \{0\}$. By the assumption there exists $\delta > 0$ such that

$$||(\varepsilon^{-1}\nu)^k|| \leq C||\varepsilon^{-1}\nu||^{k-1}$$

which yields

$$||\nu^k|| \leq C\varepsilon||\nu||^{k-1}.$$ 

We also have $\hat{f}(\mathbb{N}) = \sigma(f) \subset (Y + B(0, \delta)) \cup \{0\}$ and $Y \in U(\varepsilon, k)$. Hence, taking smaller $\delta$ if necessary, we get

$$||f^k|| \leq C||f||^{k-1}.$$ 

Clearly, we may assume that sets $\varepsilon X + B(0, \delta)$ and $Y + B(0, \delta)$ are disjoint which leads to $\nu \ast f = 0$. Performing a simple calculation we obtain

$$||\mu^k|| = ||f^k + \nu^k|| \leq ||f^k|| + ||\nu^k|| \leq C||f||^{k-1} + C\varepsilon||\nu||^{k-1} \leq C(c^{k-1} + \varepsilon(1 + c)^{k-1}) \cdot ||\mu||^{k-1}.$$ 

Since $c$ can be chosen arbitrary close to 1 and $\varepsilon$ may be as small as we wish, the theorem follows. \hfill \Box

**Theorem 13.** Let $0 < a < b$. There exists continuous $\psi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that whenever decreasing sequence $s(n)$ tending to zero satisfies

$$b \cdot \frac{s(2^l \cdot n + 2^l-1 + 1)}{s(2^l \cdot n + 1)} < \psi \left( a \cdot \frac{s(2^l \cdot n + 2^l-1)}{s(2^l \cdot n + 1)} \right)$$

for every $l, n \in \mathbb{N}$ and $A_n \in U(\varepsilon, k), \ A_n \subset B(0, b) \cap \{z \in \mathbb{C} : |z| > a\}$ and $(r_n)$ tends to zero rapidly enough, then any measure $\mu \in M_0(\mathbb{T})$ with $||\mu||_{M(\mathbb{T})} \leq K$ and with property

$$\hat{\mu}(\mathbb{Z}) \subset \bigcup (s_n \cdot A_n + B(0, r_n)) \cup \{0\}$$

satisfies

$$\mu^k \in L^1(\mathbb{T}).$$
Proof. For simplicity assume \( s(1) = 1 \). We show first by induction, that for every \( m, n \geq 0 \),

\[
B_{m,n} = \bigcup_{j=n \cdot 2^m + 1}^{(n+1)2^m} \frac{s(j)}{s(n \cdot 2^{m+1})} \cdot A_j \in U(C_m, k),
\]

(1) Indeed, for \( m = 0 \) clearly \( B_{0,n} = A_{n+1} \). For \( m > 0 \) the interval \([n \cdot 2^m + a, (n + 1) \cdot 2^m]\) is a disjoint union of the intervals \([(2n) \cdot 2^{m-1} + 1, (2n + 1) \cdot 2^{m-1}]\) and \([(2n + 1) \cdot 2^{m-1} + 1, (2n + 2) \cdot 2^{m-1}]\), which implies

\[
B_{m,n} = B_{m-1,2n} \cup \frac{s((2n + 1)2^{m-1} + 1)}{s(n \cdot 2^{m+1})} \cdot B_{m-1,2n+1}.
\]

(2) It is easy to check that

\[
B_{m-1,2n} \subset \{ z \in \mathbb{C} : |z| > a \cdot \frac{s(n \cdot 2^{m} + 2^{m-1})}{s(n \cdot 2^{m} + 1)} \}
\]

and

\[
B_{m-1,2n+1} \subset B(0, b).
\]

By the previous theorem it follows that taking in (2) the coefficient

\[
\frac{s((2n + 1)2^{m-1} + 1)}{s(n \cdot 2^{m+1})}
\]

sufficiently small with respect to

\[
a \cdot \frac{s(n \cdot 2^{m} + 2^{m-1})}{s(n \cdot 2^{m} + 1)}
\]

the resulting union belongs to \( U(C_m, k) \). Moreover, by suitable adjusting of the inductive construction, we may assume that constants \( C_m \) are uniformly bounded by some fixed constant \( C \).

Applying (1) to \( n = 1 \) and \( n = 0 \), we get that for every \( m = 0, 1, 2, \ldots \)

\[
B_{m-1,1} = \bigcup_{j=2^{m-1}+1}^{2^m} \frac{s(j)}{s(2^{m-1} + 1)} \cdot A_j \in U(C, k)
\]

and

\[
B_{m,0} \setminus B_{m-1,0} = \bigcup_{j=2^{m-1}+1}^{2^m} s(j) \cdot A_j = s(2^{m-1} + 1) \cdot B_{m-1,1}
\]

(3) Then it follows from Theorem 12 that there exists \( r'_m \) such that for every measure \( \nu \in M(\mathbb{T}) \) with

\[
||\nu||_{M(\mathbb{T})} \leq \frac{4K}{s(2^{m-1} + 1)} \quad \text{and} \quad \hat{\nu}(\mathbb{Z}) \subset B_{m-1,1} + B(0, r'_m)
\]

(4)
there is

\[ ||\nu^k||_{M(T)} < C||\nu||^{k-1}. \]

Let \( \mu \in M_0(T) \) with \( ||\mu||_{M(T)} \leq K \) satisfy

\[ \hat{\mu}(Z) \subseteq \bigcup_{m=1}^{\infty} ((B_{m,0} \setminus B_{m-1,0}) + B(0, r_m)) \]

where \( r_m = r_m' \cdot s(2^{m-1} + 1) \). Since \( \mu \in M_0(T) \), for any fixed \( m \in \mathbb{N} \) there exist only finitely many \( p \in \mathbb{N} \) such that \( \hat{\mu}(p) \in (B_{m,0} \setminus B_{m-1,0}) + B(0, r_m) \) (we may assume that these sets are disjoint for different \( m' \)'s). Hence we are able to define polynomials \( f_m \) by the condition \( \hat{f}_m(p) = \hat{\mu}(p) \) for \( p \in Z \) such that \( \hat{\mu}(p) \in (B_{m,0} \setminus B_{m-1,0}) + B(0, r_m) \) and \( \hat{f}_m(p) = 0 \) for other \( p' \)'s. Then we have

\[ \hat{\mu} = \sum_{m=0}^{\infty} \hat{f}_m. \]

Polynomials

\[ f'_m = \frac{f_m}{s(2^{m-1} + 1)} \]

satisfies (4). Indeed, the second part of (4) follows directly from (3). For the first part it is enough to show that \( ||f_m|| \leq 4K \). This follows from the observation that

\[ f_m = \left( \sum_{j=0}^{m} f_j \right) - \left( \sum_{j=0}^{m-1} f_j \right), \]

the triangle inequality, Lemma 8 and properties

\[ \sum_{j=0}^{m} f_j \in G(a \cdot s(2^m)) \text{ and } \sum_{j=m+1}^{\infty} f_j \in F(b \cdot s(2^m + 1)) \]

we have, by (5)

\[ ||f_m^k||_{L^1(T)} < s(2^{m-1} + 1)C||f_m||^{k-1}_{L^1(T)} < s(2^{m-1} + 1) \cdot C \cdot (4K)^{k-1}. \]

It leads to conclusion that the series

\[ \sum_{m=0}^{\infty} f_m^k \]

is absolutely convergent in \( L^1(T) \) to \( \mu^k \) which finishes the proof. \( \square \)
4 Reduciton to the case of $M_0(\mathbb{T})$

First result which will be used in this section is the following theorem taken from the book [G], closely related to results from [KD].

**Theorem 14.** Let $r \in \mathbb{N}$, $r \geq 2$ and $\mu \in M_c(\mathbb{T})$. Let us define set $Q = Q(\mu)$

$$Q = \{n \in \mathbb{Z} : |\hat{\mu}(n)| \geq 1\}$$

and suppose that $|\hat{\mu}(n)| \leq e^{-r}$ for $n \notin Q$.

If $||\mu|| < \frac{r^2}{4}$, then $Q$ is a finite set.

If $||\mu|| < \frac{r^2}{4}$ and $N \in \mathbb{N}$ is such that $r \leq \left(\ln \frac{N}{4 \ln \ln N}\right)^{\frac{1}{2}}$ then $\#Q < N$.

Applying the above theorem we prove that for continuous measures belongs to $M_0(\mathbb{T})$ if the special assumptions is imposed on the range of their Fourier transforms. In the next lemma we denote $L(r, t) = \{z \in \mathbb{C} : w < |z| < t\}$ ($0 < w < t$).

**Lemma 15.** Let $w_k, t_k, w_k < t_k$ be sequences of positive real numbers such that

1. $$\lim_{k \to \infty} t_k = 0$$

2. Sequence $t_k \sqrt{\ln \frac{t_k}{w_k}}$ is increasing and divergent to $\infty$.

Let $L_k = L(w_k, t_k)$. If $\mu \in M_c(\mathbb{T})$ satisfies $\hat{\mu}(\mathbb{Z}) \cap L_k = \emptyset$ for all $k \in \mathbb{N}$, then $\mu \in M_0(\mathbb{T})$.

**Proof.** Let us define a sequence $x_k$ by formula $x_k = \ln \frac{t_k}{w_k}$. Then, the second condition on $w_k, t_k$ implies that the sequence $t_k \sqrt{x_k}$ is increasing and divergent to $\infty$. Hence, there exists $k_0 \in \mathbb{N}$ such that

$$||\mu|| < \frac{t_{k_0} \sqrt{x_{k_0}}}{4}.$$ 

If $\mu$ satisfies assumptions of our theorem, then $\hat{\mu}(\mathbb{Z}) \cap L_{k_0} = \emptyset$. That means that

$$\frac{\hat{\mu}(m)}{t_{k_0}} \notin L(e^{-x_{k_0}}, 1)$$

for all $m \in \mathbb{Z}$.

From theorem 14 we obtain that the set

$$A_{k_0} = \{n \in \mathbb{Z} : \left|\frac{\hat{\mu}(n)}{t_{k_0}}\right| > 1\} = \{n \in \mathbb{Z} : |\hat{\mu}(n)| > t_{k_0}\}$$
is finite. In order to finish the proof of the lemma it is enough to show that for all $k > k_0$ the set

$$A_k = \{ n \in \mathbb{Z} : |\hat{\mu}(n)| > t_k \}$$

is finite. However, this is true by previous argument, since $t_k \sqrt{x_k} > t_{k_0} \sqrt{x_{k_0}}$ and $\hat{\mu}(\mathbb{Z}) \cap L_k = \emptyset$ which gives us opportunity to apply theorem 14 for every $k > k_0$.

**Remark 16.** It is straightforward that the assumption $\hat{\mu}(\mathbb{Z}) \cap L_k$ from preceding lemma can be replaced by requirement $\hat{\mu}(\mathbb{Z}) \cap L_{k_n}$ for any subsequence $k_n$.

This will be used in the proof of Theorem 1 in the next section. Second reduction, needed in the proof of Theorem 2 is as follows: having arbitrary $\mu \in M(\mathbb{T})$ we split it in standard way $\mu = \mu_c + \mu_d$ where $\mu_c$ is continuous part of the measure and $\mu_d$ it’s discrete part. Then from the assumptions on set $\hat{\mu}(\mathbb{Z})$ we would like to extract information about set $\hat{\mu}_c(\mathbb{Z})$ which with aid of last lemma leads to the conclusion that $\mu_c \in M_0(\mathbb{T})$ and for measures from this class we shall apply Theorem 13. One thing remains to be proved - if $\mu \in M_0(\mathbb{T})$ has natural spectrum and $\nu$ is an arbitrary measure with natural spectrum, then $\mu + \nu$ also has natural spectrum. The key to obtain this fact is an important theorem of Zafran (see [Z1]). To formulate it we introduce the following definition.

**Definition 17.** Let $\mathcal{C}$ denote the set of measures with natural spectrum with Fourier-Stieltjes coefficients from $c_0$, i.e.

$$\mathcal{C} = \{ \mu \in M_0(\mathbb{T}) : \sigma(\mu) = \sigma(\hat{\mu}(\mathbb{Z})) = \hat{\mu}(\mathbb{Z}) \cup \{0\} \}.$$  

For any Banach algebra $A$ we denote by $\mathfrak{M}(A)$ the space of maximal ideals of $A$ (cf. [Z]).

**Theorem 18** (Zafran). The following holds true:

1. if $h \in \mathfrak{M}(M_0(\mathbb{T})) \setminus \mathbb{Z}$, then $h(\mu) = 0$ for $\mu \in \mathcal{C}$.

2. $\mathcal{C}$ is closed ideal $M_0(\mathbb{T})$.

3. $\mathfrak{M}(\mathcal{C}) = \mathbb{Z}$.

On the contrary, sum of two measures with natural spectrum not necessarily has natural spectrum. Proof of this fact is based on the construction of measure supported on the independent Cantor set as in [R] (see also [Z1] for details). However, as we stated before, assuming more on one summand provides desired property.
Theorem 19. The sum of two measures with natural spectrum has natural spectrum, if one of them has Fourier coefficients tending to zero.

Proof. The spectrum of a measure is the image of its Gelfand transform. Hence using result of Zafran, we obtain

\[
\sigma(\mu + \nu) = \{\varphi(\mu + \nu) : \varphi \in \mathcal{M}(\mathcal{M}(\mathbb{T}))\} \\
= (\mu + \nu)(\mathbb{Z}) \cup \{\varphi(\nu) \in \mathcal{M}(\mathcal{M}(\mathbb{T})) \setminus \mathbb{Z}\}.
\]

Since \(\nu\) has natural spectrum then for every \(\varphi \in \mathcal{M}(\mathcal{M}(\mathbb{T}))\) we have \(\varphi(\nu) \in \hat{\nu}(\mathbb{Z})\). We consider two cases

1. \(\varphi(\nu) \in \hat{\nu}(\mathbb{Z}) \setminus \hat{\nu}(\mathbb{Z})\).
2. \(\varphi(\nu) \in \hat{\nu}(\mathbb{Z})\).

In the first case there exists increasing sequence \((n_k)_{k \in \mathbb{N}}\) of integers such that

\[
\lim_{k \to \infty} \hat{\nu}(n_k) = \varphi(\nu).
\]

Since \(\mu \in M_0(\mathbb{T})\), we get

\[
\varphi(\nu) = \lim_{k \to \infty} \hat{\nu}(n_k) = \lim_{k \to \infty} (\mu + \nu)(n_k).
\]

Hence \(\varphi(\nu) \in \hat{\mu + \nu}(\mathbb{Z})\), which completes the proof in this case.

In the second case \(\varphi(\nu) = \hat{\nu}(n)\) for some \(n \in \mathbb{Z}\). If \(\hat{\mu}(n) = 0\) then \(\varphi(\nu) = \hat{\nu}(n) + \hat{\mu}(n) \in \hat{\mu + \nu}(\mathbb{Z})\). Otherwise \(\hat{\mu}(n) \neq 0\). If \(\hat{\nu}(n)\) is an accumulation point of \(\sigma(\nu) = \hat{\nu}(\mathbb{Z})\), then we proceed as in the first case. It remains to consider the case when \(\hat{\nu}(n)\) is an isolated point of \(\hat{\nu}(\mathbb{Z})\). Because \(\mu \in M_0(\mathbb{T})\) we get that also \(\hat{\nu}(n)\) is an isolated point of \(\hat{\mu + \nu}(\mathbb{Z})\). We will prove stronger statement, that \(\hat{\nu}(n)\) is an isolated point of \(\sigma(\mu + \nu)\). Indeed, suppose for the contrary that there exists a sequence of complex numbers \((\lambda_k)_{k \in \mathbb{N}} \subset \sigma(\mu + \nu)\) tending to \(\hat{\nu}(n)\). Since the spectrum of a measure is the image of its Gelfand transform, we can choose a sequence \(\varphi \neq (h_k)_{k \in \mathbb{N}} \in \mathcal{M}(\mathcal{M}(\mathbb{T}))\) such that \(h_k(\mu + \nu) = \lambda_k\). Without losing of generality, we may assume that for sufficiently large \(k\), the functionals \(h_k\) are not the Fourier coefficients (otherwise \(\hat{\nu}(n) \in \hat{\mu + \nu}(\mathbb{Z})\) and the proof is finished). Using again the theorem of Zafran we get

\[
\lim_{k \to \infty} h_k(\mu + \nu) = \lim_{k \to \infty} h_k(\nu) = \hat{\nu}(n).
\]

But \(h_k(\nu) \in \sigma(\nu) = \hat{\nu}(\mathbb{Z})\). Hence \(\hat{\nu}(n)\) is not an isolated point of \(\hat{\nu}(\mathbb{Z})\), which contradicts the assumption. Since \(\sigma(\mu + \nu)\) is an isolated point, we can find
two open sets $A, B \subset \mathbb{C}$ such that $A \cap B = \emptyset$, $\sigma(\mu + \nu) \subset A \cup B$, $\hat{\nu}(n) \in B$ and $\sigma(\mu + \nu) \setminus \hat{\nu}(n) \subset A$. Let $f$ be a holomorphic function defined on $A \cup B$ by taking $f(z) = z$ for $z \in A$ and $f \equiv \hat{\nu}(n) + 1$ on $B$. By the spectral mapping theorem there exists a measure $f(\mu + \nu) \in M(\mathbb{T})$ satisfying

$$f(\mu + \nu)(m) = f((\mu + \nu)(m)) \text{ for all } m \in \mathbb{Z}.$$ 

By definition of $f$ we have $f((\mu + \nu)(m)) = (\mu + \nu)(m)$ for $m \neq n$. Moreover, $\hat{\mu}(n) + \hat{\nu}(n) \neq \hat{\nu}(n)$ and

$$f((\mu + \nu)(n)) = (\mu + \nu)(n).$$

Therefore the measures $f(\mu + \nu)$ and $\mu + \nu$ have the same Fourier coefficients, which implies $f(\mu + \nu) = \mu + \nu$. By properties of functional calculus, remembering that $\varphi(\mu) = 0$,

$$\varphi(f(\mu + \nu)) = f(\varphi(\mu + \nu)) = f(\varphi(\nu)) = f(\hat{\nu}(n)) = \hat{\nu}(n) + 1.$$

This contradiction completes the proof. \hfill \qed

5 Proofs of Main Theorems

We begin with proof of Theorem 1. Let sequences $t_k, w_k$ and $L_k$ be as in Lemma 15. We construct inductively sequence $\varepsilon_n$ as follows: $\varepsilon_1 = 1$ and if $\varepsilon_1, \ldots, \varepsilon_n$ are chosen let us take the smallest $k$ such that $t_k < \varepsilon_1 \cdot \ldots \cdot \varepsilon_n$ and rename it as $k_n$. Now we put $\varepsilon_{n+1}$ any number which satisfies $\varepsilon_{n+1} < \frac{1}{2} w_{k_n}$. Every $n \in \mathbb{N}$ has the unique binary expansion

$$n = \sum a_i 2^i, \quad a_i \in \{0, 1\}.$$

We put $s(n) = \prod \varepsilon_i^{a_i}$, where $a_i$ are coefficient of the binary expansion given above. We also put $A_n = A = \{-1, 1\}$ for $n \in \mathbb{N}$. We choose $r(n)$ from Theorem 13 and modify them (if necessary) to guarantee conditions: $r(2^{n-1}) < t_{k_n}$ and $r(2^n) < \frac{1}{2} w_{k_n}$. Eventually, we put

$$U = \bigcup_{n \in \mathbb{N}} (s(n) \cdot A_n + B(0, r(n))).$$

Let $\mu \in M_c(\mathbb{T})$ be a measure such that $\hat{\mu}(\mathbb{Z}) \subset U \cup \{0\}$. Then, by the construction $(U \cup \{0\}) \cap I_{k_n} = \emptyset$. By Lemma 15 and the following it remark, $\mu \in M_0(\mathbb{T})$. Finally Theorem 13 yields that $\mu^2 \in L^1(\mathbb{T})$. Hence $\mu \in \mathcal{C}$, i.e. $\mu$ has natural spectrum.

We are passing now to the proof of Theorem 2. We start from the following simple lemma whose proof is left to the reader.
Lemma 20. Let $S = \bigcup_{k=1}^{\infty} B_k \subset \mathbb{C}$ be the union of balls such that $0 \in \overline{S}$. Then there exists a (topological) Cantor set $K$ such that $K - K \subset S \cup -S$.

Let $D = \{-1, 1\}$ and $E = \{-2, -1, 1, 2\}$. By Theorem 11, $D \in U(C, 2)$ and $E \in U(C, 4)$ for some $C > 0$. Let $(s_n)_{n=1}^{\infty}$ and $(r_n)_{n=0}^{\infty}$ satisfy conditions of Theorem 13 for $A_n = D$, $n = 1, 2, \ldots$ and $(s_n)_{n=1}^{\infty}$ and $(r_n)_{n=0}^{\infty}$ satisfy the conditions of Theorem 13 for $A_n = E$, $n = 1, 2, \ldots$ and, moreover, $2|s_m| + r_n' + r_n' < r_n$ for $m > n$. Let $G_n$, $n = 1, 2, \ldots$, be a Cantor set satisfying, by Lemma 18, $G_n - G_n \subset \bigcup_{k>n} B(s_k, r_k')$. We put

$$X = \bigcup_{n=1}^{\infty} s_n A_n + G_n.$$

Suppose now that $\hat{\mu}(\mathbb{Z}) \subset X$. By results from [GW] we have $\hat{\mu}_d(\mathbb{Z}) \subset X$. Therefore

$$\hat{\mu}_c(\mathbb{Z}) \subset \hat{\mu}(\mathbb{Z}) - \hat{\mu}_d(\mathbb{Z})$$

$$\subset X - X$$

$$\subset \bigcup_{n=1}^{\infty} s_n A_n + G_n - \bigcup_{n=1}^{\infty} s_n A_n + G_n$$

$$\subset \bigcup_{n<k} (s_n A_n + G_n) - (s_k A_k + G_k)$$

$$\bigcup \bigcup_{n} (s_n A_n + G_n) - (s_n A_n + G_n)$$

$$\subset \bigcup_{n<k} s_n A_n + B(0, 2|s_k| + r_n' + r_k')$$

$$\bigcup \bigcup_{n} s_n B_n + G_n - G_n$$

$$\bigcup \bigcup_{n} G_n - G_n$$

$$\subset \bigcup_{n<k} s_n B_n + B(0, 2|s_k| + r_n' + r_k')$$

$$\bigcup \bigcup_{n} s_n B_n + B(0, r_n')$$

By Theorem 1, we derive that $\mu_c$ has natural spectrum. Finally, since additionally $\mu_c \in M_0$, Theorem 19 yields that $\mu = \mu_c + \mu_d$ has natural spectrum.
6 The Example

In this section we construct the example of singular measures satisfying assumptions of Theorem 1 which we call product of Riesz-Rudin-Shapiro. At the beginning let us remind few results concerning usual Riesz products. They are continuous, probabilistic measures on the circle group given as weak-star limit of trygonometric polynomials of the form

$$\prod_{k=1}^{N} (1 + a_k \cos(n_k t)),$$

where $-1 \leq a_k \leq 1$ and $n_k$ is sequence of natural numbers satisfying lacunary condition $\frac{n_{k+1}}{n_k} \geq 3$. We will write

$$R(a_k, n_k) = \prod_{k=1}^{\infty} (1 + a_k \cos(n_k t))$$

for the Riesz product built on sequences $a_k$ and $n_k$ satisfying above conditions. One of the oldest result (see [Z2]) about Riesz products is that $R(1, 3^k)$ is singular with respect to Lebesgue measure (we will write simply "singular" for this situation). However, much more general theorem was proved in [BM]. In this formulation $\mu \perp \nu$ means that measures $\mu, \nu \in M(\mathbb{T})$ are mutually singular and $\mu \sim \nu$ denotes equivalence of measures, ie. $\mu$ is absolutely continuous with respect to $\nu$ and vice versa.

**Theorem 21** (Brown,Moran). If $a_k, b_k$ satisfies $-1 \leq a_k, b_k \leq 1$ and sequence of natural numbers has property $\frac{n_{k+1}}{n_k} \geq 3$ then

$$R(a_k, n_k) \perp R(b_k, n_k) \iff \sum_{k=1}^{\infty} (a_k - b_k)^2 = \infty,$$

$$R(a_k, n_k) \sim R(b_k, n_k) \iff \sum_{k=1}^{\infty} (a_k - b_k)^2 < \infty.$$

As we stated in the introduction, Riesz products may be used for a simple proof of Wiener-Pitt phenomenon (see [G]). Moreover, Zafran in his paper gave necessary and sufficient condition for the Riesz product $R(a_k, n_k)$ to have natural spectrum under assumption that the sequence $a_k$ converges to zero.

**Theorem 22** (Zafran). Let $a_k$ be a sequence tending to zero such that $-1 \leq a_k \leq 1$ and $n_k$ be sequence of natural numbers such that $\frac{n_{k+1}}{n_k} \geq 3$. Then
the Riesz product $R(a_k, n_k)$ has natural spectrum if and only if, there exists $m \in \mathbb{N}$ such that
\[ \sum_{k=1}^{\infty} |a_k|^m < \infty. \]

It is also proven in [BBM] that in the case when the Riesz product has all powers mutually singular then its spectrum is the whole disc $\{ z \in \mathbb{C} : |z| \leq 1 \}$. However, usual Riesz products are not sufficient for our needs and we pass to the construction of more general class of measures.

Let us start with some preliminary lemmas and notation. The first one is a very simple arithmetic argument needed in calculating Fourier-Stieltjes coefficients of our measure.

**Lemma 23.** Let $\{m_j\}_{j=1}^{\infty}$, $\{r_j\}_{j=1}^{\infty}$ and $\{n_j\}_{j=1}^{\infty}$ be increasing sequences of positive integers with property
\[ r_k > 2 \sum_{j=1}^{k-1} ((2^{n_j} - 1)m_j + r_j) \text{ for } k \geq 2. \]

Moreover, let $\{c_j\}_{j=1}^{\infty}$ be a sequence of positive integers satisfying for $j \in \mathbb{N}$ condition
\[ c_j \in \{r_j, m_j + r_j, 2m_j + r_j, \ldots, (2^{n_j} - 1)m_j + r_j\}. \]

Assume that an integer $s$ is expressible in the form
\[ s = \sum_{j=1}^{N} b_j c_j \text{ where } N \in \mathbb{N}, \ b_j \in \{-1, 0, 1\} \text{ and } b_N \neq 0. \]

Then this expression is unique.

Proof is obvious and we omit it.

The fundamental ingredient in our construction are Rudin-Shapiro polynomials. We recall them in the next definition.

**Definition 24.** Let $P_0 \equiv 1$ and $Q_0 \equiv 1$. We define inductively two sequences of polynomials by the formula
\[ P_{n+1}(t) = P_n(t) + e^{it}Q_n(t), \]
\[ Q_{n+1}(t) = P_n(t) - e^{it}Q_n(t). \]

We will reserve the name 'Rudin-Shapiro polynomials' for sequence $\{P_n\}_{n=0}^{\infty}$.

Now, we collect well-known properties of this polynomials.
Proposition 25. For every $n \in \mathbb{N}$ we have

$$P_n(t) = \sum_{k=0}^{2^n-1} a_k e^{ikt} \text{ where } a_k \in \{-1, 1\} \text{ for } k \in \{0, \ldots, 2^n - 1\}.$$ 

Hence $\|P_n\|_{L^2(T)} = 2^n$. Also, $\|P_n\|_{C(T)} \leq 2^{n+1}$.

Using the sequence $\{P_n\}_{n=1}^{\infty}$ we define another sequence $\{w_k\}_{k=1}^{\infty}$ of polynomials.

Definition 26. Let $\{P_n\}_{n=1}^{\infty}$ be the sequence of Rudin-Shapiro polynomials and $\{r_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$ be increasing sequences of positive integers. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers vanishing at infinity. We define polynomials $\{w_k\}_{k=1}^{\infty}$ by the formula

$$w_k = \varepsilon_k P_{n_k}(m_k t) e^{i r_k t} + \varepsilon_k \overline{P_{n_k}(m_k t)} e^{-i r_k t}.$$ 

We summarize properties of polynomials $w_k$.

Proposition 27. Polynomials $w_k$ are real-valued and have the following form

$$w_k(t) = \varepsilon_k \sum_{l=-2^{n_k-1}}^{2^{n_k}-1} a_l |e^{it(l m_k + sgn(l) r_k)}| \text{ where } a_l \in \{-1, 1\}. $$ 

Hence, $\|w_k\|_{L^2(T)}^2 = 2 \varepsilon_k^2 (2^{n_k} - 1)$. Moreover, $\|w_k\|_{C(T)} \leq \varepsilon_k 2^{n_k + 3}$. 

Proof: Polynomials $w_k$ are real-valued by definition 26. Equation (6) is straightforward by proposition 25. The latter properties follows by (6) and Proposition 25. \qed

Equation (6) exposes important feature of polynomials $w_k$, namely sequence of their Fourier coefficients has gaps with length equal to $m_k$. We are ready now to construct the Riesz-Rudin-Shapiro products.

Proposition 28. Let $\{r_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$ be increasing sequences of positive integers. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers vanishing at infinity and $\{w_k\}_{k=1}^{\infty}$ be the sequence of polynomials corresponding to them. Assume that the condition

$$r_k > 2 \sum_{j=1}^{k-1} ((2^{n_j} - 1)m_j + r_j) \text{ for } k \geq 2$$ 

Therefore, $\|w_k\|_{L^2(T)} \leq 2^{n_k}$. Moreover, $\|w_k\|_{C(T)} \leq 2^{n_k + 3}$.
is satisfied and moreover, for all $k \in \mathbb{N}$ we have $\varepsilon_k 2^{\frac{n_k+1}{2}} < 1$. Then the sequence of polynomials

$$f_N(t) = \prod_{k=1}^{N} (1 - w_k(t)).$$

converges in the weak-star topology of $M(\mathbb{T})$ as $N \to \infty$ to some positive measure $\mu \in M_0(\mathbb{T})$ with $||\mu||_{M(\mathbb{T})} = 1$ with additional property

$$\hat{\mu}(\mathbb{Z}) = \{ \pm \prod_{k=1}^{m} \varepsilon_k^{l_k} : l_k \in \{0, 1, m\} \cup \{0\} \}.$$

Proof. Since $\varepsilon_k 2^{\frac{n_k+1}{2}} < 1$ and by Proposition 27 we see that $f_N \geq 0$. Hence, $||f_N||_{L^1(\mathbb{T})} = \hat{f}_N(0) = 1$. We easily see that $\hat{f}_N(s) = 0$ if $s$ is not expressible in the way presented in Lemma 23. On the other hand, if $s$ has the expression of the form

$$s = \sum_{j=1}^{N} b_j c_j$$

where $b_j \in \{-1, 0, 1\}$ not necessarily $b_N \neq 0$,

then by assumption (7) and Lemma 23 this expression is unique which gives

$$\hat{f}_N(s) = \pm \prod_{j=1}^{N} \varepsilon_j^{\lfloor \frac{|b_j|}{|c_j|} \rfloor} \hat{w}_j(c_j) = \pm \prod_{j=1}^{N} \varepsilon_j^{\lfloor \frac{|b_j|}{|c_j|} \rfloor}.$$

Hence, for every fixed $s \in \mathbb{Z}$, either $\hat{f}_N(s) = 0$ for all $N \in \mathbb{N}$ (if $s$ is not expressible in the form presented in Lemma 23) or the sequence $\{\hat{f}_N(s)\}_{N=1}^{\infty}$ stabilizes on the number given in equation (8). This, together with $||f_N||_{L^1(\mathbb{T})} = 1$, is sufficient to guarantee the weak-star convergence of the sequence $\{f_N\}_{N=1}^{\infty}$ to a positive measure with norm 1 with prescribed Fourier-Stieltjes coefficients. \qed

We will write

$$\mu = \prod_{k=1}^{\infty} (1 - w_k)$$

to denote the measure $\mu$ obtained by the procedure described above. Adding more restrictions on our sequences we get the following

Proposition 29. Let $\{r_k\}_{k=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty}$, $\{n_k\}_{k=1}^{\infty}$ be increasing sequences of positive integers. Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers
vanishing at infinity and \( \{w_k\}_{k=1}^{\infty} \) be the sequence of polynomials corresponding to them. Assume that the conditions

\[
\begin{align*}
    r_k &> 2 \sum_{j=1}^{k-1} ((2^{n_j} - 1)m_j + r_j) \quad \text{for } k \geq 2, \\
m_k &> 2 \sum_{j=1}^{k-1} ((2^{n_j} - 1)m_j + r_j) \quad \text{for } k \geq 2
\end{align*}
\]

are satisfied and moreover, for all \( k \in \mathbb{N} \) we have \( \varepsilon_k 2^{\frac{n_k + 3}{2}} < 1 \). Also, assume that there exists a constant \( c > 0 \) such that

\[
2 \varepsilon_k^2 (2^{n_k} - 1) > c \quad \text{for all } k \in \mathbb{N}.
\]

Then the measure \( \mu = \prod_{k=1}^{\infty} (1 - w_k) \) does not belong to \( L^2(\mathbb{T}) \).

**Proof.** The above assumptions are sufficient to prove the existence of a measure \( \mu = \prod_{k=1}^{\infty} (1 - w_k) \) as it was done in proposition 28. Let us fix \( N \in \mathbb{N} \) and consider the polynomial

\[
f_N(t) = \prod_{k=1}^{N} (1 - w_k(t)).
\]

Easy application of Parseval’s identity gives

\[
\sum_{k=-\infty}^{\infty} |\hat{\mu}(k)|^2 \geq \sum_{k=-\infty}^{\infty} |\hat{f_N}(k)|^2 = ||f_N||^2_{L^2(\mathbb{T})}.
\]

Hence it is enough to show that \( ||f_N||^2_{L^2(\mathbb{T})} \to \infty \) as \( N \to \infty \). We proceed with the following calculation (we use normalized Lebesgue’a measure on interval \([0, 2\pi]\)).

\[
||f_N||^2_{L^2(\mathbb{T})} = \int \prod_{k=1}^{N} (1 - w_k(t))^2 dt = \int \prod_{k=1}^{N} (1 - 2w_k(t) + w_k^2(t)) dt.
\]

Expanding the last product we get the terms of the form

\[
\pm \int w_{i_1}^{l_1}(t) \cdot w_{i_2}^{l_2}(t) \cdot \ldots \cdot w_{i_m}^{l_m}(t) dt = \pm \int h(t) dt
\]

where \( 1 \leq m \leq N, i_1 < i_2 < \ldots < i_m \) and \( l_1, l_2, \ldots, l_m \in \{1, 2\} \). Simple arithmetic argument based on equation (9) shows that the integral equals 0
unless \( l_1 = l_2 = \ldots = l_m \). Indeed, it is equal to \( \hat{h}(0) \). Let \( i_s = 1 \) for some \( 1 \leq s \leq m \). If we have \( \hat{h}(0) \neq 0 \), then there exist integers \( j_1, j'_1, j_2, j'_2, \ldots, j_m, j'_m \) (without \( j'_s \)) satisfying \( j_d, j'_d \in \{-2^{nd} + 1, \ldots, 2^{nd} - 1\} \) for \( d = 1, 2, \ldots, m \) such that

\[
0 = \sum_{k=1}^{m} (m_k(j_k + j'_k) + r_k(\text{sgn}(j_k) + \text{sgn}(j'_k))).
\]

Equation (9) implies that this is possible if and only if \( j_k + j'_k = 0 \) for all \( k \). However, this situation is excluded by the assumption \( i_s = 1 \), which leading to the non-presence of number \( j'_s \). Putting all this information together we obtain

\[
||f_N||_{L^2(T)}^2 = \int \prod_{k=1}^{N} (1 - w_k(t))^2 dt = \int \prod_{k=1}^{N} (1 + w_k^2(t)) dt.
\]

Omitting terms with order higher than two we have

\[
\int \prod_{k=1}^{N} (1 + w_k^2(t)) dt \geq 1 + \sum_{k=1}^{N} \int w_k^2(t) dt = 1 + \sum_{k=1}^{N} 2 \varepsilon_k^2 (2^{n_k} - 1).
\]

Using the assumption (10) we finally get

\[
||f_N||_{L^2(T)}^2 \geq 1 + Nc \rightarrow \infty \text{ as } N \rightarrow \infty.
\]

\[\square\]

In the main theorem of this section we prove that under additional assumptions our measure is singular. We also show how to chose sequences sufficient for that.

**Theorem 30.** Let \( \{\varepsilon_k\}_{k=1}^{\infty} \) be a decreasing sequence of positive numbers vanishing at infinity satisfying \( \varepsilon_{k+1} < \frac{1}{2} \varepsilon_k \) for all \( k \in \mathbb{N} \). Then there exist sequences \( \{r_k\}_{k=1}^{\infty}, \{m_k\}_{k=1}^{\infty}, \{n_k\}_{k=1}^{\infty} \) of positive integers satisfying conditions

\[
\begin{align*}
\varepsilon_k & \cdot 2^{\frac{n_k+3}{2}} < 1, \\
2^n \varepsilon_k^2 & > \frac{1}{32}.
\end{align*}
\]

\[
\begin{align*}
r_k & > 2 \sum_{j=1}^{k-1} \left( (2^{n_j} - 1) m_j + r_j \right) \text{ for } k \geq 2, \\
m_k & > 2 \sum_{j=1}^{k-1} \left( (2^{n_j} - 1) m_j + r_j \right) \text{ for } k \geq 2.
\end{align*}
\]
such that the positive measure $\mu \in M_0(\mathbb{T})$ with norm 1

$$\mu = \prod_{k=1}^{\infty} (1 - w_k)$$

satisfying condition

$$\hat{\mu}(\mathbb{Z}) \subset \{ \pm \prod_{k=1}^{m} \varepsilon_k^{l_k} : l_k \in \{0, 1\}, m \in \mathbb{N} \} \cup \{0\}$$

is singular.

**Proof.** We show first how to choose the sequence \(\{n_k\}_{k=1}^{\infty}\). We put \(n_k\) the smallest integer satisfying

$$2 \log_2 \frac{1}{\varepsilon_k} - 5 < n_k < 2 \log_2 \frac{1}{\varepsilon_k} - 3.$$ 

To guarantee \(n_{k+1} > n_k\) it is enough to have \(2 \log_2 \frac{1}{\varepsilon_k} - 3 < 2 \log_2 \frac{1}{\varepsilon_{k+1}} - 5\) which is equivalent to \(\varepsilon_{k+1} < \frac{1}{2} \varepsilon_k\). It easy two define two sequences \(\{r_k\}_{k=1}^{\infty}\) and \(\{m_k\}_{k=1}^{\infty}\). We put \(m_1 = r_1 = 1\) and then we choose inductively sequences with properties

\[ r_k > 2 \sum_{j=1}^{k-1} ((2^{n_j} - 1)m_j + r_j) \text{ for } k \geq 2, \]

\[ m_k > 2 \sum_{j=1}^{k-1} ((2^{n_j} - 1)m_j + r_j) \text{ for } k \geq 2. \]

We are now in the position of Proposition 29 and we know that our measure $\mu$ exists and does not belong to $L^2(\mathbb{T})$ (in fact the assumption $2^{n_k} \varepsilon_k^2 > \frac{1}{32}$ leads to this result without referring to Proposition 29). Proof of the singularity is essentially the same as given in [BM] and [GM] for the Riesz products.

We obviously have

$$\sum_{k=1}^{\infty} 2^{n_k} \varepsilon_k = \infty.$$ 

Hence, we may choose a sequence \(\{c_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})\) of real numbers and an increasing sequence \(\{l_k\}_{k=1}^{\infty}\) of integers such that

\[ \sum_{l=l_k+1}^{l_{k+1}} c_l 2^{n_l} \varepsilon_l = 1 \text{ for all } k \in \mathbb{N} \]
with the additional property

\[(13) \quad \sum_{l=1}^{\infty} c_l^2 2^{n_l} < \infty.\]

We introduce sets \(A_l \subset \mathbb{N}\) for \(l \in \mathbb{N}\)

\[A_l = \{r_l, r_l + m_l, r_l + 2m_l, \ldots, r_l + (2^{n_l} - 1)m_l\}.\]

Clearly, we have \(A_l \cap A_k = \emptyset\) for \(l \neq k\) and \(|A_l| = 2^{n_l}\). Moreover, we get \(\hat{\mu}(n) = \text{sgn}\hat{\mu}(n)\varepsilon_l\) for \(n \in A_l\). Let us consider polynomials \(f_k\) for \(k \in \mathbb{N}\) defined by the formula

\[(14) \quad f_k(t) = \sum_{l=l_k+1}^{l_{k+1}} c_l \sum_{n \in A_l} \text{sgn}\hat{\mu}(n)e^{int}.\]

By \((13)\) we have

\[(15) \quad \|f_k\|^2_{L^2(\mathbb{T}, \mu)} = \sum_{l=l_k+1}^{l_{k+1}} 2^{n_l} c_l^2 \to 0 \quad \text{as} \quad k \to \infty.\]

We perform now the crucial calculation of \(\|f_k\|^2_{L^2(\mathbb{T}, \mu)}\).

\[\|f_k\|^2_{L^2(\mathbb{T}, \mu)} = \int f_k(t)\overline{f_k(t)} d\mu = \sum_l 2^{n_l} c_l^2 + \sum_{l, m \in A_l, n \neq m} \text{sgn}\hat{\mu}(n)\text{sgn}\hat{\mu}(m)\hat{\mu}(m - n) + \sum_{l, r \neq r} c_l 2^{n_l}\varepsilon_l c_r 2^{n_r}\varepsilon_r.\]

However, \(\hat{\mu}(m - n) = 0\) for \(m, n \in A_l, n \neq m\) (this is true by \((11)\) and the construction of \(\mu\) in Proposition \((28)\) and the second sum vanishes. Simple manipulations on remaining terms give

\[\sum_l 2^{n_l} c_l^2 + \sum_{l, r \neq r} c_l 2^{n_l}\varepsilon_l c_r 2^{n_r}\varepsilon_r = \sum_l 2^{n_l} c_l^2 + \left(\sum_l 2^{n_l}\varepsilon_l c_l\right)^2 - \sum_l c_l^2 2^{2n_l}\varepsilon_l^2.\]

By \((12)\), the second term equals 1 and

\[\|f_k\|^2_{L^2(\mathbb{T}, \mu)} = 1 + \sum_{l=l_k+1}^{l_{k+1}} c_l^2 2^{n_l}(1 - 2^{n_l}\varepsilon_l^2).\]
The assumption $2^n \varepsilon_l^2 > \frac{1}{32}$ leads to $1 - 2^n \varepsilon_l^2 < \frac{31}{32}$ which, with aid of (13), gives
\[ \sum_{l=I_k+1}^{l_{k+1}} c_l^2 2^n (1 - 2^n \varepsilon_l^2) \to 0 \text{ as } k \to \infty. \]
Hence
\[(16) \quad ||f_k||_{L^2(\mathbb{T}, \mu)} \to 1 \text{ as } k \to \infty.\]

We shall show now that
\[(17) \quad \lim_{k \to \infty} f_k = 1 \text{ in } L^1(\mathbb{T}, \mu).\]

Applying Schwarz inequality we get
\[
\left( \int |f_k - 1| d\mu \right)^2 \leq \int |f_k - 1|^2 d\mu = \int |f_k|^2 d\mu - 2Re \int f_k d\mu + 1 =
\int |f_k|^2 d\mu - 2Re \sum_{l=I_k+1}^{l_{k+1}} c_l 2^n \varepsilon_l + 1 \to 0 \text{ as } k \to \infty.
\]
The last assertion follows from (12) and (16). This proves (17). The final argument is easy - let us take a subsequence $(k_j)_{j=1}^{\infty}$ such that (here the symbol $|F|$ denotes the Haar measure of $F$).
\[
\lim_{j \to \infty} f_{k_j}(t) = 1 \text{ for } t \in F \text{ where } F \text{ is a Borel set with } |F| = 1,
\]
\[
\lim_{j \to \infty} f_{k_j} = 0 \text{ } \mu - \text{ almost everywhere.}
\]
This leads to $\mu(F) = 0$. Both Haar measure and $\mu$ are positive and have norm 1. Hence we obtain that $\mu$ is singular with respect to Lebesgue measure which finishes the proof.

7 Final remarks

1. The components of the open set $U$ constructed in Theorem 1 tends to 0 very fast. The closer look on the proof shows that the reciprocal of the distance of $n$-th component to 0 grows as fast as the Ackermann function $A(4, 2n)$ which is truly fast. It is not easy to see in which places of the proof this growth could be optimized.

2. Our proof gives that the continuous part of a measure satisfying the assumptions of Theorem 1 has absolutely continuous convolution square.
We do not know however whether this could be improved. In particular it would be interesting to show an example of an open set $U_1$ which yields the continuous measure to be absolutely continuous if it has all its Fourier coefficient in $U_1$.

3. It would be also interesting to construct an open set $U_k$ with the property that any function with Fourier coefficients from $U_k$ has only $k$th convolution power absolutely continuous, and such that there exists a measure with Fourier coefficient in $U_k$ with all smaller convolution powers singular.

4. The above property uses the fact that the sum of measures with Fourier coefficients tending to 0 whose convolution powers are absolutely continuous belongs to the Zafran class $\mathcal{C}$. We conjecture that also converse property holds true, i.e. any measure from $\mathcal{C}$ could be decomposed onto an (infinite) sum of measures whose some convolution powers are absolutely continuous.

5. The crucial in our proof was the use of the Littlewood conjecture to estimate the number of repetitions of any specific value taken by the Fourier coefficients. In the case of Cantor group the Littlewood conjecture does not hold. This fact encourages us to ask whether any infinite Wiener - Pitt set exist for the convolution measure algebra on the Cantor group.

6. Our Lemma 7 is a stronger version of the second part of Theorem 14 which is taken from [GM]. Our proof uses exact version on the Littlewood conjecture, which was not available when [GM] was written. But our proof differs in more aspects - it uses Bożejko - Pełczyński’s invariant local approximation property which seems to be a simpler method.

7. While Theorem 7 does not hold for torsion abelian groups, because the Littlewood conjecture is false there, it seems likely that Theorem 8 may be extended to this case but it would require completely different proof.

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