Long-time asymptotic behavior of the fifth-order modified KdV equation in low regularity spaces

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Abstract
Based on the nonlinear steepest descent method of Deift and Zhou for oscillatory Riemann–Hilbert problems and the Dbar approach, the long-time asymptotic behavior of solutions to the fifth-order modified KdV (Korteweg–de Vries) equation on the line is studied in the case of initial conditions that belong to some weighted Sobolev spaces. Using techniques in Fourier analysis and the idea of the $I$-method, we give its global well-posedness in lower regularity Sobolev spaces and then obtain the asymptotic behavior in these spaces with weights.

KEYWORDS
fifth-order modified Korteweg–de Vries equation, Fourier analysis, long-time asymptotics, low regularity, nonlinear steepest descent method

1 | INTRODUCTION

The modified KdV (Korteweg–de Vries) equation is a fundamental completely integrable model in solitary waves theory and usually is written as

$$\partial_t u + 6u^2\partial_x u + \partial_x^3 u = 0,$$

where $u = u(x, t)$ is a real function with evolution variable $t$ and transverse variable $x$. This equation is well known as a canonical model for the description of nonlinear long waves in many
branches of physics when there is the polarity symmetry. For instance, applications in the context of electrodynamics are described in Ref. 1 in the context of stratified fluids in Ref. 2.

In the present paper, we investigate a fifth-order modified KdV equation taking the form

\[ \partial_t u + \partial_x^5 u + 30u^4 \partial_x u - 10u^2 \partial_x^3 u - 10(\partial_x u)^3 - 40u \partial_x u \partial_x^2 u = 0, \]  

(2)

which is the second equation from the modified KdV hierarchy, and the fixed coefficients in the nonlinearities could ensure the integrable structure. Many researchers have studied this equation (2) (see, e.g., Refs. 3–6). Similarly, Equation (2) also can be used to describe nonlinear wave propagation in physical systems with polarity symmetry. It is worth noting that Equation (2) is endowed with two important features. One is the complete integrability: in the sense that there are Lax pair formulations (see Ref. 5), thus (2) can be solved by means of the inverse scattering transformation formalism both in the case of vanishing as well as nonvanishing boundary conditions. Being integrable, Equation (2) admits an infinite number of conservation laws and its solution exists globally in time for any Schwartz initial data. The other one is the scaling symmetry: for any solution \( u(x, t) \) of (2) with initial data \( u_0(x) \), the scaling functions \( u_{\lambda}(x, t) := \lambda u(\lambda x, \lambda^5 t) \) also solve (2) with initial data \( u_{0,\lambda} := \lambda u_0(\lambda x) \). The Sobolev space \( H^{s_c} \) is called the critical space if it satisfies

\[ \|u_{0,\lambda}\|_{H^{s_c}} \sim \|u_0\|_{H^{s_c}}. \]

By simple calculation, one can know that \( s_c = -1/2 \). This scale invariance suggests ill-posedness for \( H^{s}, s < -1/2 \).

Using the Fourier analysis techniques and the theory of dispersive equations, one can study the low regularity theory for the fifth-order modified KdV equation. It seems natural that the regularity index of the well-posedness theory can be decreased to \(-1/2\). However, it is extremely difficult to achieve this goal by the known methods in Sobolev spaces. The best local well-posedness theory so far was given by Kwon.\(^7\) He expressed that Equation (2) is locally well-posed in Sobolev space \( H^{s}(\mathbb{R}) \) for \( s \geq 3/4 \) via the \( X^{s,b} \) method, which was posed by Bourgain, and showed that it is ill-posed when \( s < 3/4 \) in the sense that the data-to-solution map fails to be uniformly continuous.

In addition, the solutions to (2) enjoy mass conservation and energy conservation:

\[ M(u) = \int_{\mathbb{R}} u^2 dx; \quad E(u) = \int_{\mathbb{R}} (\partial_x u)^2 + u^4 dx. \]

(3)

Energy conservation and the local well-posedness immediately yield the global well-posedness of (2) from initial data in \( H^{1}(\mathbb{R}) \).

Many researchers are studying the local and global well-posedness theory of fifth-order KdV-type equations. For the fifth-order KdV equation, whose nonlinearities are quadratic, Chen and Guo\(^8\) gave the local and global well-posedness in \( H^{s} \) with \( s \geq -7/4 \) for the equation with the first-order derivative nonlinearities. The fifth-order KdV equation with the third-order derivative nonlinearities was considered by Kwon,\(^9\) and he showed that it is locally well-posed in \( H^{s}, s > 5/2 \). In Ref. 10, Guo et al. prove a priori bound of solutions to this equation in \( H^{s}, s \geq 5/4 \), and showed that the equation is globally well-posed in \( H^{2} \) due to the second energy conservation law. For the fifth-order modified KdV equation, whose nonlinearity has three \( u \) or even more \( u \), Linares\(^4\) gave the local well-posedness in \( H^{2} \) by using the dispersive smoothing effect, and then the global well-posedness is immediately obtained via the conservation law. Kwon\(^7\) improved the local well-posedness theory to the Sobolev spaces \( H^{s} \) with \( s \geq 3/4 \). Via the energy method and the
short-time Fourier restriction norm method, Kwak\textsuperscript{3} considered Equation (2) under the periodic boundary condition and proved the local well-posedness in $H^s(\mathbb{T})$, $s > 2$. In addition, by using the $I$-method, Gao\textsuperscript{11} prove the global well-posedness in $H^s$, $s > -3/2$ for the fifth-order modified KdV equation, but he only studied the first-order derivative nonlinearity. For more results, we lead the readers to Refs. 12–15 and the references therein.

Once one has established the well-posedness theory, another natural research topic for the dispersion equations is to study the large-time asymptotic behavior of the solutions. From the integrable point of view, in the context of inverse scattering, the first work to provide explicit formulas depending only on initial conditions for long-time asymptotics of solutions is due to Zakharov and Manakov,\textsuperscript{16} where the model considered was the cubic nonlinear Schrödinger (NLS) equation. In this setting, the inverse scattering map or the reconstruction of the potential is formulated through an oscillatory Riemann–Hilbert (RH) problem. Then the now well-known nonlinear steepest descent method for oscillatory RH problems introduced by Deift and Zhou\textsuperscript{17} provides a rigorous and transparent proof to analyze the long-time asymptotics of initial value problems for a large range of nonlinear integrable evolution equations with initial conditions lying in the Schwartz space $S(\mathbb{R})$. Numerous new significant results about asymptotics of solutions for the initial value or initial-boundary value problems to a lot of integrable systems were obtained under the assumptions that the initial or initial-boundary data belong to the Schwartz space.\textsuperscript{17–23}

To consider the asymptotic behavior of the solution in lower regularity spaces, we have to mention another meaningful result developed by Zhou\textsuperscript{24}, that is, the $L^2$-Sobolev space bijectivity of the direct and inverse scattering of the $2 \times 2$ AKNS (Ablowitz-Kaup-Newell-Segur) system for the initial data $u_0(x)$ belonging to the weighted Sobolev space $H^{i,j}(\mathbb{R})$, where

$$H^{i,j}(\mathbb{R}) = \{ f(x) \in L^2(\mathbb{R}) : x^j f(x), f^{(i)}(x) \in L^2(\mathbb{R}) \},$$

which is endowed with the following norm:

$$\| f(x) \|_{H^{i,j}(\mathbb{R})} = \left( \| f(x) \|_{L^2(\mathbb{R})}^2 + \| x^j f(x) \|_{L^2(\mathbb{R})}^2 + \| f^{(i)}(x) \|_{L^2(\mathbb{R})}^2 \right)^{1/2}.$$ 

And then Deift and Zhou\textsuperscript{25} obtained the long-time asymptotics for solutions of the defocusing NLS equation with initial data in a weighted Sobolev space $H^{1,1}$ based on this work, where the error term is $O(t^{-(1/2+\kappa)})$, $0 < \kappa < 1/4$. Moreover, inspired by the work\textsuperscript{26} and combined with Zhou’s result,\textsuperscript{24} the Dbar generalization of the nonlinear steepest descent method proposed by Dieng and McLaughlin\textsuperscript{27} was implemented in the analysis of the long-time asymptotic behavior of the solutions, where they derived the asymptotics for the defocusing NLS equation with the initial conditions $u_0(x)$ belonging to the weighted Sobolev space $H^{1.1}(\mathbb{R})$. This approach replaces the rational approximation of the reflection coefficient in the Deift–Zhou method\textsuperscript{17} by some nonanalytic extension of the jump matrices off the real axis, leading to a \( \tilde{\delta} \) problem to be solved in some sectors of the complex plane. The new \( \tilde{\delta} \) problem can be recast into an integral equation and solved by the Neumann series, which contributed to the error estimate. In addition, the error term can be improved to optimal $O(t^{-3/4})$. In the context of the NLS equation with soliton solutions, this method was successfully applied to prove asymptotic stability of $N$-soliton solutions to the defocusing NLS equation\textsuperscript{28} and address the soliton resolution problem for the focusing NLS equation in Ref. 29. Whereafter, the long-time asymptotic behavior of derivative NLS equation for generic initial data in $H^{2,2}(\mathbb{R})$ that do not support solitons and can support solitons (but exclude spectral singularities) were analyzed in Refs. 30 and 31, respectively. Recently, long-time behavior of
the defocusing modified KdV equation and the soliton resolution of the focusing modified KdV equation in weighted Sobolev spaces were reported in Refs. 32 and 33.

It is worth noting that some researchers also study the long-time asymptotic behavior of the dispersive equations by PDE (Partial Differential Equation) techniques, which do not need the complete integrability. For instance, one can see Refs. 34, 35, and 36 about the methods of factorization of operators, the space–time resonance, and wave packets analysis, respectively. However, they all need some smallness assumptions on the initial data. Specially, employing the method of testing by wave packets, Okamoto37 has proved the small-data global existence and derived the asymptotic behavior of solutions in the decaying, self-similar and oscillatory regions to the fifth-order modified KdV type equation in spaces $H^{2,1}(\mathbb{R})$, which is a very beautiful work, whereas the complex-valued function $W$ arising in the leading-order term cannot be written exactly in the oscillatory region.

In this paper, relying on complete integrability, we drop the smallness conditions and consider the long-time asymptotic behavior of the fifth-order modified KdV equation (2) by using the generalization of the nonlinear steepest descent method with the initial conditions $u_0(x)$ belonging to the weighted Sobolev space $H^{4,1}(\mathbb{R})$. Then taking advantage of Fourier analysis techniques and $I$-method, we extend the initial conditions to more general spaces $H^{s,1}(\mathbb{R})$, $s > 19/22$ (notice $H^s \subset H^s(\mathbb{R})$, $s \geq s_1$, $s \geq s_2$).

In short, the large-time behavior of the solution to Equation (2) is obtained through a sequence of transformations of a corresponding RH problem, which is finally transformed into solving the model RH problem near stationary phase points. More precisely, in the Region I: $k_0 > M$, $x \to \infty$ and Region II: $0 < k_0 \leq M$, $\tau = tk_0^5 \to \infty$, where $M$ is a positive constant, the phase function $\Phi(k)$ defined in (20) has two different real critical points located at $\pm k_0$, where $k_0 = \sqrt[5]{x/(80\tau)}$. To implement the approach, our first step is to introduce a scale function $\delta(k)$ conjugated to the solution $M(x, t; k)$ of the original RH problem 1.1. This operation is aimed to factorize the jump matrix (22) into a product of an upper/lower triangular and lower/upper triangular matrix, which is necessary for the following contour deformation. The second step is to deform the contour from $\mathbb{R}$ to a new contour depicted in Figure 3 so that the jump matrix involves the exponential factor $e^{-i\Phi}$ on the parts of the contour where Re$\Phi$ is positive and the factor $e^{i\Phi}$ on the parts where Re$\Phi$ is negative. To achieve this goal, the idea is to construct $\delta$ extensions $R_j(k)$ in $\Omega_j$ to have the prescribed boundary values and $\delta R_j(k)$ small in the sector (see Lemma 1). This will allow us to reformulate RH problem 2.1 as the mixed $\delta$-RH problem 2.2 for a new matrix-valued function $M^{(2)}(x, t; k)$. The next step is to extract from $M^{(2)}$ a contribution that is a pure RH problem. More exactly, we factorize $M^{(2)}(k) = M^{(3)}(k)M^{mod}(k)$, where $M^{mod}(k)$ is a solution of the RH problem 2.3 below with the jump matrix $J^{mod} = J^{(2)}$, and $M^{(3)}(k)$ has no jump across $\Gamma$ which will prove to be satisfied a pure $\delta$ problem 2.6. Since the reflection coefficient $r(k)$ in the jump matrix $J^{mod}$ is only fixed at $\pm k_0$ along the deformed contours, we then can solve this RH problem in terms of parabolic cylinder functions and the large-$k$ expansion can be exactly written with the estimate of the decay rate as $\tau \to \infty$ (see Subsection 2.3). The remaining $\delta$ problem may be written as an integral equation (refer to Equation (108)) whose integral operator has small norm at large times (see Equation (111)) and the large-time contribution to the asymptotics of $u(x, t)$ is negligible. The final step is to regroup all the transformations to find the behavior of the solution of the fifth-order modified KdV equation using the large-$k$ behavior of the RH problem solutions.

However, our main RH problem has only jump across the real axis $\mathbb{R}$, the main contributions to the asymptotic formula of the solution $u(x, t)$ come from the local RH problems near the two real critical points $\pm k_0$ even though there are two real and two pure imaginary critical points.
of the phase function $\Phi(k)$ located at the points $\pm k_0$ and $\pm ik_0$. Due to symmetries of reflection coefficient $r(k)$, the leading-order asymptotics of $u(x, t)$ exhibits decaying, of order $O(t^{-1/2})$, modulated oscillations in Regions I and II.

Another interesting region is Region IV: $\tau \leq M'$. Especially, as $t \to \infty$, the critical points $\pm k_0$ approach 0. The above steps are much easier to operate in this case. We can show that the asymptotics of the solution $u(x, t)$ in this region is expressed in terms of the solution of the fourth-order Painlevé II equation

$$u'''_p(y) - 40u^2_p(y)u''_p(y) - 40u'_p(y)u'^2_p(y) + 96u^5_p(y) - 4yu'_p(y) = 0. \tag{4}$$

It is a beautiful example that the asymptotic behavior of the solution of an integrable equation is expressed by the solution of the high order Painlevé II equation in the asymptotic analysis. Nevertheless, the asymptotic behaviors of solutions to the standard defocusing mKdV in sector $0 < x < M^{1/3}$ correspond to the solution of the standard Painlevé II equation $u''_p(y) - 2u^3_p(y) - yu'_p(y) = 0$, both on the line and half-line problem. We also note that for the Camassa–Holm and Sasa–Satsuma equations, Painlevé II-type asymptotics also exists in some certain regions. Interestingly, the Painlevé III hierarchy has appeared in a recent study of the fundamental rogue wave solutions of the focusing nonlinear Schrödinger equation in the limit of large order. However, for the fifth-order modified KdV equation (2), we find another interesting new asymptotic result which is related to the solution of the fourth-order Painlevé II equation (4). The study of asymptotic behaviors of the solution $u(x, t)$ in remaining regions is derived in Section 4, which follows the same strategy.

All in all, our asymptotic results can be divided into three categories: in Regions I and II (which is an oscillatory region), the leading-order asymptotics of $u(x, t)$ is described by the form of the cosine function; in the self-similar Regions III–V, the leading-order asymptotic behaviors can be expressed in terms of the solution of the fourth-order Painlevé II equation; in the Region VI, the solution $u(x, t)$ tends to 0 with fast decay.

To extend these asymptotic behavior results to lower regularity spaces, we have first to obtain the global well-posedness theory in some low regularity spaces by Fourier analysis techniques. As mentioned before, the global solution in space $H^1(\mathbb{R})$ is easily obtained owing to the energy conservation. The biggest obstacle in getting global solutions in $H^s$ with $0 < s < 1$ is the lack of any conservation law. In this paper, by utilizing the idea of the $I$-method, which was introduced by Colliander–Keel–Staffilani–Takaoka–Tao, we modify the $H^1$ energy to obtain an “almost conservation law,” whose increment is very small. Therefore, we can get the global well-posedness in low regularity Sobolev spaces and then extend the long-time asymptotic formulae to the initial data $u_0(x) \in H^{s,1}(\mathbb{R})$, $s > 19/22$.

Broadly speaking, the asymptotic analysis of (2) presents the following innovation points: (i) the phase function $\Phi(k)$ in the jump matrix raises to the fifth power of $k$, this will lead to the computations in the scaling transform and related estimates about the $\tilde{\mathcal{D}}$ problem more involved; (ii) we should establish a new suitable model RH problem which arises in the study of long-time asymptotics in the Region IV; (iii) because of the lack of any conservation law in $H^s$, $s < 1$, we have to get an “almost conservation law” by utilizing the $I$-method. As far as the authors can see, it is the first time that using the idea of the $I$-method to deal with the third-order derivative nonlinearities. Compared with the first-order derivative nonlinearities, the absence of the symmetries and the vanished property in the case of third-order derivatives makes the problem fairly tricky. Thanks
to an important observation, substantial technical difficulties caused by the third-order derivative nonlinearities could be solved.

1.1 Formulation of the Riemann–Hilbert problem

Equation (2) is the compatibility condition for the simultaneous linear equations of a Lax pair

\[ \Psi_x = Xa\Psi, \quad X = ik\sigma_3 + U, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}, \]  

(5)

and

\[ \Psi_t = T\Psi, \quad T = -16ik^5\sigma_3 + V, \]

\[ V = -8ik^3U^2\sigma_3 + 2ik(2UU_{xx} - U_x^2 - 3U^4)\sigma_3 - 16k^4U + 8ik^3\sigma_3U_x + k^2(4U_{xx} - 8U^3) \]

\[ + ik\sigma_3(12U^2U_x - 2U_{xxx}) - 6U^5 + 10U^2U_{xx} + 10U^2U_x - U_{xxxx}, \]

(6)

governing a 2 × 2 matrix-valued function \( \Psi(x, t; k) \) and the spectral parameter \( k \in \mathbb{C} \), where \( u \) is a real-valued function. It is noted that the linear system (5) and (6) belong to the classical ZS-AKNS system.\(^{44,45}\) For the given initial data \( u_0(x) \in L^1(\mathbb{R}) \), we define the Jost eigenfunctions \( \psi_\pm(x; k) = \psi_\pm(x; k)e^{ik\sigma_3} \) by

\[ \psi_\pm(x; k) = I + \int_{-\infty}^{x} e^{ik(x-x')\sigma_3}[U(x', 0)\psi_\pm(x'; k)]e^{-ik(x-x')\sigma_3} \, dx', \]

(7)

which satisfy

\[ \psi_\pm(x; k) - ik[\sigma_3, \psi_\pm(x; k)] = U(x, 0)\psi_\pm(x; k), \]

(8)

and the normalization conditions

\[ \lim_{x \to \pm \infty} \psi_\pm(x; k) = I. \]

If we denote \( \psi = (\psi^1, \psi^2) \) for a 2 × 2 matrix \( \psi \), it follows from (7) that for all \( (x, t) \):

(i) \( \det[\psi_\pm] = 1 \).
(ii) \( \psi^1_+ \) and \( \psi^2_+ \) are analytic and bounded in \( \{k \in \mathbb{C} | \text{Im} k > 0 \} \), and \( (\psi^1_+, \psi^2_+) \to I \) as \( k \to \infty \).
(iii) \( \psi^1_- \) and \( \psi^2_- \) are analytic and bounded in \( \{k \in \mathbb{C} | \text{Im} k < 0 \} \), and \( (\psi^1_-, \psi^2_-) \to I \) as \( k \to \infty \).
(iv) \( \psi_\pm \) are continuous up to the real axis.
(v) Symmetry:

\[ \psi_\pm(x; k) = \psi_\pm(x; -k) = \sigma_1\psi_\pm(x; k)\sigma_1, \]

(9)
where \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is the first Pauli matrix. The symmetry relation (9) can be proved easily due to the symmetries of the matrix \( X \):

\[
X(x, t; \overline{k}) = X(x, t; -k) = \sigma_1 X(x, t; k) \sigma_1.
\]

Both \( \psi_+ \) and \( \psi_- \) define a fundamental solution matrix for (8), and thus there exists a continuous matrix \( s(k) \) independent of \( x \), satisfying

\[
\Psi_+(x; k) = \Psi_-(x; k) s(k), \quad k \in \mathbb{R}.
\]  

Evaluation at \( x \to -\infty \) gives

\[
s(k) = \lim_{x \to -\infty} e^{-ikx \sigma_3} \psi_+(x; k) e^{ikx \sigma_3},
\]

\[
= I - \int_{-\infty}^{+\infty} e^{-ikx \sigma_3} [U(x, 0) \psi_+(x; k)] e^{ikx \sigma_3} dx.
\]  

(11)

It follows from \( \det[\psi_\pm(x; k)] = 1 \) and the symmetries (9) that the matrix-valued function \( s(k) \) can be expressed as

\[
s(k) = \begin{pmatrix} a(k) & b(k) \\ b(k) & a(k) \end{pmatrix}, \quad \det[s(k)] = 1, \quad k \in \mathbb{R}.
\]  

(12)

Moreover, by (11) and (12), we find that \( a(k) \) is analytic in \( \mathbb{C}_+ \), and \( b(k) \) is defined only for \( k \in \mathbb{R} \). The symmetry relation (9) implies

\[
a(k) = a(-k), \quad k \in \mathbb{C}_+, \quad b(k) = b(-k), \quad k \in \mathbb{R}.
\]  

(13)

However, it follows from \( \det[s(k)] = 1 \), that is,

\[
|a(k)|^2 - |b(k)|^2 = 1
\]  

(14)

that \( a(k) \) is zero-free.

Suppose \( u(x, t) \) is a solution of Equation (2). At each time \( t \), we define the spectral matrix \( s(k, t) \) according to (11), that is,

\[
s(k, t) = \lim_{x \to -\infty} e^{-ikx \sigma_3} \psi_+(x, t; k) e^{ikx \sigma_3}.
\]  

(15)

Then from the \( t \)-part Lax pair of (2), the evolution of the spectral matrix is given by

\[
s(k, t) = e^{-16ik^5 \sigma_3} s(k) e^{16ik^5 \sigma_3}.
\]  

(16)

More precisely, we have

\[
a(k, t) = a(k), \quad b(k, t) = b(k) e^{32ik^5 t}.
\]  

(17)
Now we define the Beals–Coifman solution

\[
M(x, t; k) = \begin{cases} 
\left( \frac{\psi_1^+(x, t; k)}{a(k)}, \frac{\psi_2^+(x, t; k)}{a(k)} \right), & k \in \mathbb{C}_+, \\
\left( \frac{\psi_1^-(x, t; k)}{a(k)}, \frac{\psi_2^-(x, t; k)}{a(k)} \right), & k \in \mathbb{C}_-. 
\end{cases}
\]  

(18)

Then, one can show that the boundary values \(M_\pm(x, t; k)\) of \(M\) as \(k\) approaches \(\mathbb{R}\) from the sides \(\pm \text{Im} k > 0\) are related as follows:

\[
M_+(x, t; k) = M_-(x, t; k) J(x, t; k), \quad k \in \mathbb{R},
\]  

(19)

with

\[
J(x, t; k) = \begin{pmatrix} 1 - |r(k)|^2 & - \overline{r(k)} e^{-i\Phi(k)} \\ r(k) e^{i\Phi(k)} & 1 \end{pmatrix},
\]

\[
r(k) = \frac{b(k)}{a(k)}, \quad |r(k)| < 1, \quad \Phi(k) = 32ik^5 - 2i \frac{x}{t}.
\]  

(20)

It follows from Zhou’s results\(^{24}\) that the following map is Lipschitz:

\[
D : H^{4,1}(\mathbb{R}) \ni \{u_0(x)\} \mapsto \{r(k)\} \in H^{1,4}(\mathbb{R}).
\]

Then, we arrive at our main RH problem which is formulated as follows.

**Riemann–Hilbert problem 1.1.** Given \(r(k) \in H^{1,4}(\mathbb{R})\). Find a \(2 \times 2\) matrix-valued function \(M(x, t; k)\) on \(\mathbb{C} \setminus \mathbb{R}\) with the following properties:

1. \(M(x, t; k)\) is analytic for \(k \in \mathbb{C} \setminus \mathbb{R}\) and is continuous for \(k \in \mathbb{R}\).
2. The boundary values \(M_\pm(x, t; k)\) satisfy the jump condition

\[
M_+(x, t; k) = M_-(x, t; k) J(x, t; k),
\]  

(21)

where

\[
J(x, t; k) = \begin{pmatrix} 1 - |r(k)|^2 & - \overline{r(k)} e^{-i\Phi(k)} \\ r(k) e^{i\Phi(k)} & 1 \end{pmatrix}, \quad r(-k) = \overline{r(k)}, \quad k \in \mathbb{R}.
\]  

(22)

3. \(M(x, t; k)\) has the asymptotics:

\[
M(x, t; k) = I + O \left( \frac{1}{k} \right), \quad k \to \infty.
\]

The jump matrix admits the following factorization:

\[
J(x, t; k) = J_-^{-1}(x, t; k) J_+(x, t; k) = (I - w_-(x, t; k))^{-1} (I + w_+(x, t; k)),
\]  

(23)
where
\[ w_- = \begin{pmatrix} 0 & -r(k)e^{-r(k)} \\ 0 & 0 \end{pmatrix}, \quad w_+ = \begin{pmatrix} 0 & 0 \\ r(k)e^{r(k)} & 0 \end{pmatrix}, \quad k \in \mathbb{R}. \]

By standard RH theory, the existence and uniqueness of the solution to RH problem 1.1 is determined by the existence and uniqueness of the following singular integral equation:
\[ \mu(x, t; k) = I + C_w \mu(x, t; k) = I + C_+(\mu w_-)(x, t; k) + C_-(\mu w_+)(x, t; k), \]
where
\[ \mu(x, t; k) = M_+(x, t; k)(I + w_+(x, t; k))^{-1} = M_-(x, t; k)(I - w_-(x, t; k))^{-1}, \]
and \( C_\pm \) denotes the Cauchy operators:
\[ (C_\pm f)(k) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{f(\zeta)}{\zeta - (k \pm i\varepsilon)} \frac{d\zeta}{2\pi i}. \]

It is shown in Propositions 4.1 and 4.2 of Ref. 46 that the operator \( 1 - C_w \) is Fredholm and has the Fredholm index zero. It also follows from Theorem 9.3 in Ref. 46 that the kernel of \( 1 - C_w \) is trivial because \( J + J^\dagger \) is a positive definite matrix. Thus, the operator \( 1 - C_w \) is invertible, as a result that RH problem 1.1 has a unique solution. And the solution \( M(x, t; k) \) is given by
\[ M(x, t; k) = I + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{[\mu(w_+ + w_-)](x, t; s)}{s - k} ds. \]

The solution \( u(x, t) \) of (2) in terms of \( M(x, t; k) \) is given by
\[ u(x, t) = -2i \lim_{k \to \infty} (kM(x, t; k))_{12} = \frac{1}{\pi} \left( \int_{\mathbb{R}} [\mu(w_+ + w_-)](x, t; s) ds \right)_{12}. \]

1.2 Main results

Our main results in this paper are summarized by the following theorems. The first one is about the asymptotic behavior of the Beals–Coifman solution to Equation (2).

**Theorem 1.** Let \( u_0(x) \in H^{4,1}(\mathbb{R}) \), \( M \), \( M' \), and \( \bar{M} \) be fixed constants larger than 1. Select a suitable constant \( \rho > 0 \). Then the solution \( u(x, t) \) of the fifth-order modified KdV equation (2) with the initial data \( u_0(x) \) admits the following uniform asymptotics in the six regions depicted in Figure 1. More precisely, set
\[ k_0 = \sqrt[4]{\frac{|x|}{80t}}, \quad \tau = tk_0^5 = \left( \frac{|x|}{80t^\frac{1}{5}} \right)^\frac{5}{4}, \]
FIGURE 1  The six regions

and let $u_0(y)$ be the solution of the fourth-order Painlevé II equation (4). Then, as $t \to \infty$, we have

In Region I: $k_0 > M$, $x \to \infty$,

$$u(x, t) = u_{as}(x, t) + O \left( x^{-1} + \left( k_0^{-1}x \right)^{-\frac{3}{4}} \right),$$

where

$$u_{as}(x, t) = \frac{\sqrt{\nu}}{2k_0 \sqrt{10k_0 t}} \cos \left( 128t k_0^5 + \nu \ln \left( 2560t k_0^5 \right) + \varphi(k_0) \right),$$

$$\varphi(k_0) = -\frac{3\pi}{4} - \arg{r(k_0)} + \arg{\Gamma(i\nu)} - \frac{1}{\pi} \int_{-k_0}^{k_0} \ln \left( \frac{1 - |r(s)|^2}{1 - |r(k_0)|^2} \right) \frac{ds}{s - k_0},$$

$$\nu = -\frac{1}{2\pi} \ln \left( 1 - |r(k_0)|^2 \right) > 0,$$

and $\Gamma(\cdot)$ denotes the standard Gamma function.

In Region II: $x > 0$, $k_0 \leq M$, $\tau \to \infty$,

$$u(x, t) = u_{as}(x, t) + O \left( \tau^{-1} + \left( k_0^2 t \right)^{-\frac{3}{4}} \right).$$

In Region III: $o(t^2) = \tau \geq M$,

$$u(x, t) = \left( \frac{8}{5t} \right)^{\frac{1}{5}} u_p \left( \frac{x}{(20t)^{\frac{1}{5}}} \right) + O \left( t^{-\frac{3}{10}} + \left( \frac{\tau}{t} \right)^{\frac{2}{5}} \right).$$

In Region IV: $\tau \leq M'$,

$$u(x, t) = \left( \frac{8}{5t} \right)^{\frac{1}{5}} u_p \left( \frac{x}{(20t)^{\frac{1}{5}}} \right) + O \left( t^{-\frac{3}{10}} \right).$$
In Region V: \( k_0 \leq M, \tau \geq \tilde{M} \),

\[
u(x,t) = \left(\frac{8}{5t}\right)^{\frac{1}{5}} u_p \left(\frac{x}{(20t)^{\frac{1}{5}}}\right) + O \left(t^{-\frac{3}{10}} + t^{-\frac{3}{10}} e^{-8(20\tau)^{\frac{4}{5}}}\right).
\]

In Region VI: \( k_0 \geq M, x \to -\infty \),

\[
u(x,t) = O \left(t^{-\frac{1}{5}} e^{-c\tau} + (-x)^{-\frac{3}{2}} + t^{-\frac{3}{10}} e^{-8(20\tau)^{\frac{4}{5}}}\right).
\]

Remark 1. The significance of the restriction \( \tau = o(t^{\frac{2}{7}}) \) in Region III is to ensure that the asymptotic formulas of the solution match up in the overlap section of Regions II and III. For more details, one can see Remark 4.

Remark 2. The condition \( u_0(x) \in H^{4,1}(\mathbb{R}) \) is used to ensure that the inverse scattering transform has nice mapping properties; however, our asymptotic results depend only on the \( H^1(\mathbb{R}) \) norm of \( r(k) \) in Theorem 1. In particular, the long-time asymptotic result presented in Theorem 1 is also valid for any \( u_0 \in L^2(\mathbb{R}, (1 + |x|)dx) \).

Remark 3. We note that the constants in the various error terms in Theorem 1 depend only on \( \|r(k)\|_{H^1(\mathbb{R})} \). This uniform estimate is a crucial point for the argument of asymptotic behavior in low regularity spaces, that is, the asymptotic formulae still hold under weaker regularity assumptions by applying a global approximation.

We next consider Equation (2) in low regularity spaces. From the PDE point of view, one always considers the solution given by an integral form. For convenience, we denote

\[
F(u) = 30u^4 \partial^4_x u - 10u^2 \partial^3_x u - 10(\partial^3_x u)^3 - 40u \partial^2_x u \partial^2_x u,
\]

then by Duhamel’s formula, (2) is equivalent to the following integral equation:

\[
u(x,t) = e^{-t\partial^5_x} u_0 - \int_0^t e^{-(t-t')\partial^5_x} F(u)(t') \, dt', \quad \text{where} \quad e^{-t\partial^5_x} = F^{-1} e^{-it\partial^5} F. \tag{29}
\]

Using the idea of the \( I \)-method,\(^{42,43}\) we obtain the global well-posedness theory.

**Theorem 2.** Let \( s > 19/22 \), the initial value problem of the fifth-order modified KdV equation (2) is globally well-posed from initial data \( u_0 \in H^s(\mathbb{R}) \).

To obtain an “almost conservation law,” we would modify the \( H^1 \) energy, so we just consider the case \( 19/22 < s \leq 1 \) in Section 5. For \( s > 1 \), one can use Sobolev embedding and standard arguments to get the global well-posedness results. In the end, cooperating the integrable structure with PDE techniques, we can give the long-time asymptotic behavior in lower regularity spaces.
**Theorem 3.** Let $u_0 \in H^{s,1}(\mathbb{R})$, $s > 19/22$, the solution given by the integral form (29) has the same asymptotic behavior as in Theorem 1.

Notations. $C$ denotes a universal positive constant which can always be different at different lines. $S(\mathbb{R})$ denotes the Schwartz space. $F$ ($F^{-1}$) denotes the (inverse) Fourier transform on $\mathbb{R}$. We also denote $\hat{\phi}$ the Fourier transform of a distribution $\phi$. We write $a \leq b$ if $a \leq Cb$, and analogous for $a \geq b$. We use the notation $a \sim b$ if $a \leq b \leq a$. We denote by $a+ = a + \varepsilon$ for some $0 < \varepsilon \ll 1$. We denote by $\bar{a}$ the complex conjugate of a complex number $a$. The three Pauli matrices are defined by

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The organization of this paper is as follows. In Section 2, we derive the long-time asymptotics of the solution for Equation (2) in Regions I and II. Section 3 aims to consider the asymptotics of $u(x, t)$ in the Region IV. The asymptotic behavior of the solution in the remaining regions is derived in Section 4. We then extend the long-time asymptotic behavior to the low regularity spaces in Section 5. A few facts related to the RH problem associated with the fourth-order Painlevé II equation are collected in the Appendix, for the detailed proof one can see Ref. 23.

## 2 | ASYMPTOTICS IN REGIONS I AND II

The jump matrix $J$ defined in (22) involves the exponentials $e^{\pm i\Phi}$; therefore, the sign structure of the quantity $\text{Re}\Phi(k)$ plays an important role as $t \to \infty$. In particular, in Regions I and II, it follows that there are two different real stationary points located at the points, where $\frac{\partial \Phi}{\partial k} = 0$, namely, at

$$
\pm k_0 = \pm i \sqrt{\frac{x}{80t}}.
$$

The signature table for $\text{Re}\Phi(k)$ is shown in Figure 2.
2.1 Conjugation

To apply the method of steepest descent, one goes from the original RH problem 1.1 for $M$ to the equivalent RH problem for the new unknown function $M^{(1)}$ defined by

$$M^{(1)}(x, t; k) = M(x, t; k)\delta^{(-1)}(k),$$

(31)

where the complex-valued function $\delta(k)$ is given by

$$\delta(k) = \exp \left\{ \frac{1}{2\pi i} \int_{-k_0}^{k_0} \ln(1 - |r(s)|^2) \frac{ds}{s - k} \right\}, \quad k \in \mathbb{C} \setminus [-k_0, k_0].$$

(32)

**Proposition 1.** The function $\delta(k)$ has the following properties:

(i) $\delta(k)$ is bounded and the analytic function of $k \in \mathbb{C} \setminus [-k_0, k_0]$ with continuous boundary values on $(-k_0, k_0)$, and satisfies the jump condition across the real axis oriented from $-\infty$ to $\infty$:

$$\delta_+(k) = \begin{cases} \delta_-(k)(1 - |r(k)|^2), & k \in (-k_0, k_0), \\ \delta_-(k), & k \in \mathbb{R} \setminus (-k_0, k_0). \end{cases}$$

(ii) As $k \to \infty$, $\delta(k)$ satisfies the asymptotic formula

$$\delta(k) = 1 + O(k^{-1}), \quad k \to \infty.$$  

(33)

(iii) $\delta(k)$ obeys the symmetry

$$\delta(k) = \delta(k)^{-1} = \delta(-k), \quad k \in \mathbb{C} \setminus [-k_0, k_0].$$

(iv)

$$\delta(k) = \left( \frac{k - k_0}{k + k_0} \right)^{iv} e^{\chi(k)},$$

(34)

where

$$\nu = -\frac{1}{2\pi} \ln \left( 1 - |r(k_0)|^2 \right) > 0,$$

(35)

$$\chi(k) = \frac{1}{2\pi i} \int_{-k_0}^{k_0} \ln \left( \frac{1 - |r(s)|^2}{1 - |r(k_0)|^2} \right) \frac{ds}{s - k}.$$  

(36)

Moreover, along any ray of the form $\pm k_0 + e^{i\phi} \mathbb{R}_+$ with $0 < \phi < \pi$ or $\pi < \phi < 2\pi$,

$$\left| \delta(k) - \left( \frac{k - k_0}{k + k_0} \right)^{iv} e^{\chi(\pm k_0)} \right| \leq C|k - k_0|^{1/2}.$$  

(37)
Proof. See Refs. 17, 25, and 32 and references therein.

Then $M^{(1)}(x, t; k)$ satisfies the following RH problem:

**Riemann–Hilbert problem 2.1.** Given $r(k) \in H^1(\mathbb{R})$. Find a $2 \times 2$ matrix-valued function $M^{(1)}(x, t; k)$ on $\mathbb{C} \setminus \mathbb{R}$ with the following properties:

1. $M^{(1)}(x, t; k)$ is analytic for $k \in \mathbb{C} \setminus \mathbb{R}$.
2. The boundary values $M^{(1)}(x, t; k)$ satisfy the jump condition

$$M^{(1)}_+(x, t; k) = M^{(1)}_-(x, t; k)J^{(1)}(x, t; k), \quad k \in \mathbb{R},$$

where

$$J^{(1)}(x, t; k) = \begin{pmatrix}
1 & -r(k)\delta^2(k)e^{-i\Phi(k)} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
r(k)\delta^{-2}(k)e^{i\Phi(k)} & 1
\end{pmatrix}, \quad |k| > k_0, \quad (38)$$

$$J^{(1)}(x, t; k) = \begin{pmatrix}
1 & 0 \\
r(k)\delta^{-2}(k)e^{i\Phi(k)} & 1
\end{pmatrix} \begin{pmatrix}
1 & \frac{r(k)}{1-|r(k)|^2}\delta^2_+(k)e^{-i\Phi(k)} \\
0 & 1
\end{pmatrix}, \quad |k| < k_0. \quad (39)$$

3. $M^{(1)}(x, t; k) = I + O\left(\frac{1}{k}\right)$, as $k \to \infty$.

2.2 | Contour deformation and introducing $\tilde{\delta}$ extensions

The next step is to introduce factorizations of the jump matrix whose factors admit continuous but not necessarily analytic extensions off the real axis by exploiting the methods in Refs. 27, 29–32. More precisely, define the contours

$$L := \left\{ k \in \mathbb{C} | k = k_0 + k_0\alpha e^{\frac{\pi i}{6}}, -\infty < \alpha \leq \frac{2\sqrt{3}}{3} \right\} \cup \left\{ k \in \mathbb{C} | k = -k_0 + k_0\alpha e^{\frac{\pi i}{6}}, -\infty < \alpha \leq \frac{2\sqrt{3}}{3} \right\}. \quad (40)$$

Then, $L$ and $\tilde{L}$ split the complex plane $\mathbb{C}$ into eight regions $\{\Omega_j\}_{j=1}^8$, for convenience, we also write $L \cup \tilde{L} = \cup_{j=1}^8 \Gamma_j$ (see Figure 3). Now, we define the $\tilde{\delta}$ extensions in the following lemma, which aims to separate the phases and performs the contour deformation.

**Lemma 1.** Let

$$T(k; k_0) = \left(\frac{k - k_0}{k + k_0}\right)^{\frac{i\gamma}{2}}. \quad (41)$$
It is possible to define functions $R_j : \Omega_j \mapsto \mathbb{C}$, $j = 1, 3, 4, 5, 6, 8$ with boundary values satisfying

$$R_1(k) = \begin{cases} -r(k)\delta^{-2}(k), & k \in (k_0, \infty), \\ -r(k_0)T^{-2}(k; k_0)e^{-2\chi(k_0)}, & k \in \Gamma_1, \end{cases}$$  \hspace{1cm} \text{(42)}

$$R_3^+(k) = \begin{cases} \frac{\overline{r(k)}}{1-|r(k)|^2} \delta^2_+(k), & k \in (0, k_0), \\ \frac{r(k_0)}{1-|r(k_0)|^2} T^2(k; k_0)e^{2\chi(k_0)}, & k \in \Gamma_2, \end{cases}$$  \hspace{1cm} \text{(43)}

$$R_3^-(k) = \begin{cases} \frac{\overline{r(k)}}{1-|r(k)|^2} \delta^2_+(k), & k \in (-k_0, 0), \\ \frac{r(-k_0)}{1-|r(-k_0)|^2} T^2(k; k_0)e^{2\chi(-k_0)}, & k \in \Gamma_3, \end{cases}$$  \hspace{1cm} \text{(44)}

$$R_4(k) = \begin{cases} -r(-k_0)T^{-2}(k; k_0)e^{-2\chi(-k_0)}, & k \in \Gamma_4, \end{cases}$$  \hspace{1cm} \text{(45)}

$$R_5(k) = \begin{cases} -\overline{r(k)}\delta^2(k), & k \in (-\infty, -k_0), \\ -\overline{r(-k_0)}T^2(k; k_0)e^{2\chi(-k_0)}, & k \in \Gamma_5, \end{cases}$$  \hspace{1cm} \text{(46)}

$$R_6^-(k) = \begin{cases} \frac{r(k)}{1-|r(k)|^2} \delta^{-2}(k), & k \in (-k_0, 0), \\ \frac{\overline{r(k)}}{1-|r(k)|^2} T^{-2}(k; k_0)e^{-2\chi(k_0)}, & k \in \Gamma_6, \end{cases}$$  \hspace{1cm} \text{(47)}

$$R_6^+(k) = \begin{cases} \frac{\overline{r(k)}}{1-|r(k)|^2} \delta^{-2}(k), & k \in (0, k_0), \\ \frac{r(k_0)}{1-|r(k_0)|^2} T^{-2}(k; k_0)e^{2\chi(k_0)}, & k \in \Gamma_7, \end{cases}$$  \hspace{1cm} \text{(48)}

$$R_8(k) = \begin{cases} -r(k)\delta^2(k), & k \in (k_0, \infty), \\ -r(-k_0)T^2(k; k_0)e^{2\chi(k_0)}, & k \in \Gamma_8, \end{cases}$$  \hspace{1cm} \text{(49)}

such that

$$|\delta R_j(k)| \leq c_1 |r'(Re k)| + c_2 |k \mp k_0|^{-\frac{1}{2}}, \hspace{1cm} \text{(50)}$$

for two positive constants $c_1, c_2$ depend on $\|r\|_{H^1(\mathbb{R})}$.
Proof. Define the functions 
\[ f_1(k) = -r(k_0)T^{-2}(k; k_0)e^{-2\chi(k_0)}\delta^2(k), \quad k \in \Omega_1, \]
\[ f_3^+(k) = \frac{r(k_0)}{1 - |r(k_0)|^2}T^2(k; k_0)e^{2\chi(k_0)}\delta^{-2}(k), \quad k \in \Omega_3^+. \]

Then, we can define the extensions for \( k \in \Omega_1 \),
\[ R_1(k) = [f_1(k) + (-r(Rek) - f_1(k))\chi(\phi)]\delta^{-2}(k), \quad (51) \]
where \( \phi = \arg(k - k_0) \) and \( \chi(\phi) \) is a smooth cut-off function with
\[
\chi(\phi) = \begin{cases} 
1, & \phi \in \left[0, \frac{\pi}{12}\right], \\
0, & \phi \in \left[\frac{\pi}{9}, \frac{\pi}{6}\right]. 
\end{cases} \quad (52)
\]

It is easy to check that \( R_1(k) \) as constructed has the boundary values \( (42) \). Let \( k - k_0 = se^{i\phi} \). It follows from
\[
\tilde{\delta} = \frac{1}{2} \left( \frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_2} \right) = \frac{1}{2}e^{i\phi} \left( \frac{\partial}{\partial s} + i\frac{\partial}{\partial \phi} \right),
\]
and Proposition 1 (iv) that
\[
|\tilde{\delta}R_1(k)| = -\frac{1}{2}r'(Rek)\chi(\phi)\delta^{-2}(k) + \frac{1}{2}ie^{i\phi} \frac{-r(Rek) - f_1(k)}{|k - k_0|} \chi'(\phi)\delta^{-2}(k) \leq c_1 |r'(Rek)| + c_2 |k - k_0|^{-\frac{1}{2}}. \quad (53)
\]

For \( k \in \Omega_3^+ \), let
\[ R_3^+(k) = \left[ f_3^+(k) + \left( \frac{r(Rek)}{1 - |r(Rek)|^2} - f_3^+(k) \right) \tilde{\chi}(\phi) \right]\delta^2(k), \quad (54) \]
where \( \tilde{\chi}(\phi) \) is a smooth cut-off function with
\[
\tilde{\chi}(\phi) = \begin{cases} 
1, & \phi \in \left[\frac{11\pi}{12}, \pi\right], \\
0, & \phi \in \left[\frac{3\pi}{4}, \frac{5\pi}{6}\right]. 
\end{cases} \quad (55)
\]

A direct computation implies that \( R_3^+(k) \) satisfies the boundary values \( (43) \) and the relation \( (50) \). In the remaining sectors, the proof is similar. \( \blacksquare \)

We now use the extensions in Lemma 1 to define a new unknown function \( M^{(2)} \) by
\[ M^{(2)}(x, t; k) = M^{(1)}(x, t; k)R^{(2)}(k), \quad (56) \]
where

\[
R^{(2)}(k) = \begin{cases} 
1 & , k \in \Omega_1, \\
R_1(k)e^{\Phi(k)} & , k \in \Omega_2, \\
1 & , k \in \Omega_3, \\
R_3(k)e^{\Phi(k)} & , k \in \Omega_4, \\
1 & , k \in \Omega_5, \\
R_4^+(k)e^{\Phi(k)} & , k \in \Omega_6, \\
1 & , k \in \Omega_7, \\
0 & , k \in \Omega_8.
\end{cases}
\] (57)

Let \( \Gamma = \{ \Gamma_j \} \cup i\kappa / \sqrt{3} (-1, 1) \). It is an immediate consequence of Lemma 1 and RH problem 2.1 that \( M^{(2)} \) satisfies a mixed \( \tilde{\delta} \)-RH problem.

\( \tilde{\delta} \)-Riemann–Hilbert problem 2.2. Find a function \( M^{(2)}(x, t; k) \) with the following properties:

1. \( M^{(2)}(x, t; k) \) is continuous with sectionally continuous first partial derivatives in \( \mathbb{C} \setminus \Gamma \).
2. Across \( \Gamma \), the boundary values satisfy the jump relation

\[
M^+_{(2)}(x, t; k) = M^-_{(2)}(x, t; k)J^{(2)}(x, t; k), \quad k \in \Gamma,
\] (58)

where the jump matrix \( J^{(2)}(x, t; k) \) is given by

\[
J^{(2)}(x, t; k) = \begin{cases} 
1 & , k \in \Gamma_1, \\
-\frac{(k_0)}{1-\rho(k_0)} e^{2i\rho(k_0)\Phi(k)} & , k \in \Gamma_2, \\
1 & , k \in \Gamma_3, \\
\frac{(k_0)}{1-\rho(k_0)} e^{2i\rho(k_0)\Phi(k)} & , k \in \Gamma_4, \\
0 & , k \in \Gamma_5, \\
\frac{(k_0)}{1-\rho(k_0)} e^{2i\rho(k_0)\Phi(k)} & , k \in \Gamma_6, \\
1 & , k \in \Gamma_7, \\
\frac{(k_0)}{1-\rho(k_0)} e^{2i\rho(k_0)\Phi(k)} & , k \in \Gamma_8, \\
1 & , k \in i\kappa (\tan \pi / 12, \sqrt{3}/ 3), \\
0 & , k \in i\kappa (\tan \pi / 12, \sqrt{3}/ 3), \\
\frac{(k_0)}{1-\rho(k_0)} e^{i\Phi(k)} & , k \in i\kappa \tan \pi / 12(-1, 1).
\end{cases}
\] (59)
3. For \( k \in \mathbb{C} \setminus \Gamma \), we have
\[
\bar{\partial}M^{(2)}(k) = M^{(2)}(k)\bar{\partial}R^{(2)}(k).
\] (60)

4. \( M^{(2)}(x, t; k) = I + O\left(\frac{1}{k}\right), \ k \to \infty. \)
   In the next subsection, we aim to extract from \( M^{(2)} \) a contribution that is a pure RH problem. More precisely, we write
\[
M^{(2)}(k) = M^{(3)}(k)M^{\text{mod}}(k)
\] (61)
and we request that \( M^{(3)}(k) \) has no jump across \( \Gamma \). Thus we look for \( M^{\text{mod}}(k) \) solution of the RH problem 2.3 below with the jump matrix \( J^{\text{mod}} = J^{(2)}. \)

## 2.3 Analysis of the Riemann–Hilbert problem 2.3

We now focus on \( M^{\text{mod}}(k) \), which satisfies the following RH problem.

**Riemann–Hilbert problem 2.3.** Find a \( 2 \times 2 \) matrix-valued function \( M^{\text{mod}}(x, t; k) \), analytic on \( \mathbb{C} \setminus \Gamma \) with the following properties:

1. \( M^{\text{mod}}(x, t; k) \) is analytic for \( k \in \mathbb{C} \setminus \Gamma \) and is continuous for \( k \in \Gamma \).
2. The boundary values \( M^{\text{mod}}_{\pm}(x, t; k) \) satisfy the jump condition
   \[
   M^{\text{mod}}_{+}(x, t; k) = M^{\text{mod}}_{-}(x, t; k)J^{\text{mod}}(x, t; k), \quad k \in \Gamma,
   \] (62)
   where \( J^{\text{mod}}(x, t; k) = J^{(2)}(x, t; k). \)
3. \( M^{\text{mod}}(x, t; k) = I + O\left(\frac{1}{k}\right), \text{ as } k \to \infty. \)

By using the method of Beals and Coifman, we set
\[
J^{\text{mod}} = \left(I - w^{\text{mod}}_{-}\right)^{-1}\left(I + w^{\text{mod}}_{+}\right), \quad w^{\text{mod}} = w^{\text{mod}}_{+} + w^{\text{mod}}_{-},
\]
and let
\[
(C_{\pm}f)(k) = \int_{\Gamma} \frac{f(s)}{s - k_{\pm}} \frac{ds}{2\pi i}, \quad k \in \Gamma, \ f \in L^2(\Gamma),
\]
denote the Cauchy operator on \( \Gamma \). Define the operator \( C_{w^{\text{mod}}} : L^2(\Gamma) + L^\infty(\Gamma) \to L^2(\Gamma) \) by
\[
C_{w^{\text{mod}}}f = C_{+}(f w^{\text{mod}}_{-}) + C_{-}(f w^{\text{mod}}_{+})
\] (63)
for any \( 2 \times 2 \) matrix-valued function \( f \). Let \( \mu^{\text{mod}}(x, t; k) \in L^2(\Gamma) + L^\infty(\Gamma) \) be the solution of the basic inverse equation
\[
\mu^{\text{mod}} = I + C_{w^{\text{mod}}} \mu^{\text{mod}}.
\] (64)
Then
\[ M^{\text{mod}}(x, t; k) = I + \int_{\Gamma'} \frac{(\mu^{\text{mod}} u^{\text{mod}})(x, t; s)}{s - k} \frac{ds}{2\pi i}, \quad k \in \mathbb{C} \setminus \Gamma \] (65)
is the unique solution of RH problem 2.3.

For a small enough constant $\varepsilon > 0$, we define
\[ L_\varepsilon : = \left\{ k \in \mathbb{C} \mid k = k_0 + k_0 e^{\frac{\pi i}{6}}, \varepsilon < \alpha \leq \frac{2\sqrt{3}}{3} \right\} \cup \left\{ k \in \mathbb{C} \mid k = -k_0 + k_0 e^{\frac{\pi i}{6}}, \varepsilon < \alpha \leq \frac{2\sqrt{3}}{3} \right\}. \] (66)

Denote $\Gamma_9 = \frac{i k_0}{\sqrt{3}} (-1, 1)$. Set $\Gamma' = \Gamma \setminus (\Gamma_9 \cup L_\varepsilon \cup \overline{L}_\varepsilon)$ with the orientation as in Figure 4. In the following, we reduce the RH problem 2.3 to a RH problem on the truncated contour $\Gamma'$ with controlled error terms. Let $u^{\text{mod}} = u^e + u'$, where $u' = u^{\text{mod}}|_{\Gamma'}$ and $u^e = u^{\text{mod}}|_{(\Gamma_9 \cup L_\varepsilon \cup \overline{L}_\varepsilon)}$.

**Lemma 2.** Let $\varepsilon > 0$ sufficiently small. For $0 < k_0 \leq M$, $\tau \to \infty$, we have
\[ \|u^e\|_{L^n(\Gamma_9 \cup L_\varepsilon \cup \overline{L}_\varepsilon)} \leq ce^{-32\varepsilon^2\tau}, \quad 1 \leq n \leq \infty, \] (67)
\[ \|u'\|_{L^\infty(\Gamma')} \leq ce^{-32\varepsilon^2\tau}. \] (68)

Furthermore,
\[ \|u'\|_{L^1(\Gamma')} \leq c\tau^{-\frac{1}{2}}, \quad \|u'\|_{L^2(\Gamma')} \leq c\tau^{-\frac{1}{4}}. \] (69)

For $k_0 > M$, $x \to \infty$,
\[ \|u^e\|_{L^n(\Gamma_9 \cup L_\varepsilon \cup \overline{L}_\varepsilon)} \leq ce^{-c^2k_0x}, \quad 1 \leq n \leq \infty, \] (70)
\[ \|u'\|_{L^\infty(\Gamma')} \leq ce^{-32\varepsilon^2k_0x}. \] (71)
\[ \|w'\|_{L^1(\Gamma')} \leq c(k_0x)^{-\frac{1}{2}}, \quad \|w'\|_{L^2(\Gamma')} \leq c(k_0x)^{-\frac{1}{4}}. \]  

(72)

Proof. For \( k \in L_\varepsilon, 0 < k_0 \leq M, \tau \to \infty \), we have

\[
\text{Re}\Phi(k) = 32\alpha^2k_0^5 \left( 5\sqrt{3} - 10\alpha + \frac{5\sqrt{3}}{2}\alpha^2 - \frac{1}{2}\alpha^3 \right)
\geq 32\alpha^2k_0^5 \left( 5\sqrt{3} + (5\sqrt{3} - 10)\alpha - \frac{5\sqrt{3}}{2}\alpha^2 - \frac{1}{2}\alpha^3 \right) \geq 32\alpha^2k_0^5. \quad (73)
\]

Thus, we find

\[
\left| R^\pm_3 e^{-t\Phi} \right| \leq ce^{-32\alpha^2k_0^5} \leq ce^{-32\varepsilon^2\tau}. \quad (74)
\]

This yields (67) and (68). On the other hand, a simple calculation implies

\[
\left\| e^{-32\alpha^2ik_0^5} \right\|_{L^1(\Gamma')} \leq c\tau^{-\frac{1}{2}}, \quad \left\| e^{-32\alpha^2ik_0^5} \right\|_{L^2(\Gamma')} \leq c\tau^{-\frac{1}{4}}. \quad (75)
\]

The estimate (69) immediately follows.

For \( k_0 > M, x \to \infty \), noting that \( x = 80k_0^4\tau \), thus, we get

\[
t\Phi(k) = x\Phi(k), \quad \tilde{\Phi}(k) = 2i \left( \frac{k^5}{5k_0^4} - k \right). \quad (76)
\]

Hence, for \( k \in L_\varepsilon \), we have

\[
\text{Re}\tilde{\Phi}(k) = \alpha^2k_0 \left( 2\sqrt{3} - 4\alpha + \sqrt{3}\alpha^2 - \frac{1}{5}\alpha^3 \right) \geq \alpha^2k_0. \quad (77)
\]

The estimates (70)–(72) follow. □

Lemma 3. In the case \( 0 < k_0 \leq M, \tau \to \infty \), or in the case \( k_0 > M, x \to \infty \), \((1 - C_{\omega'})^{-1} : L^2(\Gamma) \to L^2(\Gamma)\) exists and is uniformly bounded:

\[
\left\| (1 - C_{\omega'})^{-1} \right\|_{L^2(\Gamma)} \leq c.
\]

Moreover,

\[
\left\| (1 - C_{\omega\text{mod}})^{-1} \right\|_{L^2(\Gamma)} \leq c.
\]

Proof. See Ref. 17 and references therein. □
A simple computation implies that
\[
(1 - C_{w,\text{mod}}^{-1}I)w^{\text{mod}} = \left((1 - C_{w'})^{-1}I\right)w' + w^e + ((1 - C_{w'})^{-1}(C_{w'}I))w^{\text{mod}}
\]
\[
+ \left((1 - C_{w'})^{-1}(C_{w'}I)\right)w^e
\]
\[
+ \left((1 - C_{w'})^{-1}C_{w'}(1 - C_{w,\text{mod}})^{-1}\right)(C_{w,\text{mod}}I)w^{\text{mod}}.
\]

In the case \(0 < k_0 \leq M, \tau \to \infty\), it follows from Lemma 2 that
\[
\|w^e\|_{L^1(\Gamma)} \leq ce^{-32\varepsilon^2 \tau},
\]
\[
\|(1 - C_{w'})^{-1}(C_{w'}I)w^{\text{mod}}\|_{L^1(\Gamma)} \leq \|(1 - C_{w'})^{-1}\|_{L^2(\Gamma)} \|C_{w'}I\|_{L^2(\Sigma)} \|w^{\text{mod}}\|_{L^2(\Gamma)} 
\]
\[
\leq c \|w^e\|_{L^1(\Sigma)} \|w^{\text{mod}}\|_{L^2(\Gamma)} \leq ce^{-32\varepsilon^2 \tau - \frac{1}{2}},
\]
\[
\|(1 - C_{w'})^{-1}(C_{w'}I)w^e\|_{L^1(\Gamma)} \leq \|(1 - C_{w'})^{-1}\|_{L^2(\Sigma)} \|C_{w'}I\|_{L^2(\Gamma)} \|w^e\|_{L^2(\Gamma)}
\]
\[
\leq c \|w'\|_{L^2(\Gamma)} \|w^e\|_{L^2(\Gamma)} \leq ce^{-32\varepsilon^2 \tau - \frac{1}{2}},
\]
and
\[
\left\|\left((1 - C_{w'})^{-1}C_{w'}(1 - C_{w,\text{mod}})^{-1}\right)(C_{w,\text{mod}}I)w^{\text{mod}}\right\|_{L^1(\Gamma)} \leq c \|w^e\|_{L^\infty(\Gamma)} \|w^{\text{mod}}\|_{L^2(\Gamma)}^2 
\]
\[
\leq ce^{-32\varepsilon^2 \tau - \frac{1}{2}}.
\]

Thus, as \(\tau \to \infty\), we have the following result:
\[
\int_{\Gamma} \left(((1 - C_{w,\text{mod}}^{-1}I)w^{\text{mod}}\right)(x, t; s)ds = \int_{\Gamma'} \left(((1 - C_{w'})^{-1}I)w'\right)(x, t; s)ds + O(e^{-c\tau}).
\]
(78)

Accordingly, for \(k_0 > M\), as \(x \to \infty\),
\[
\int_{\Gamma} \left(((1 - C_{w,\text{mod}}^{-1}I)w^{\text{mod}}\right)(x, t; s)ds = \int_{\Gamma'} \left(((1 - C_{w'})^{-1}I)w'\right)(x, t; s)ds + O(e^{-c\varepsilon}).
\]
(79)

On \(\Gamma',\) set \(\mu' = (1 - C_{w'})^{-1}I\). Then it follows that
\[
M'(x, t; k) = I + \int_{\Gamma'} \frac{(\mu'w')(x, t; s)}{s - k} \frac{ds}{2\pi i}
\]
(80)
satisfies the following RH problem:

**Riemann–Hilbert problem 2.4.** Find a \(2 \times 2\) matrix-valued function \(M'(x, t; k)\) satisfying:

1. \(M'(x, t; k)\) is analytic for \(k \in \mathbb{C} \setminus \Gamma'\) and is continuous for \(k \in \Gamma'\).
2. The boundary values \( M'_\pm(x, t; k) \) satisfy the jump condition

\[
M'_+(x, t; k) = M'_-(x, t; k) J'(x, t; k), \quad k \in \Gamma',
\]

where \( J'(x, t; k) = J^\text{mod}(x, t; k)|_{\Gamma'} \).

3. \( M'(x, t; k) = I + O(\frac{1}{k}) \), as \( k \to \infty \).

Split \( \Gamma' \) into a union of two disjoint crosses, \( \Gamma' = \Gamma'_{-k_0} \cup \Gamma'_{k_0} \), where \( \Gamma'_{\pm k_0} \) denote the cross of \( \Gamma' \) centered at \( \pm k_0 \). Write \( w' = w'_{-k_0} + w'_{k_0} \), where \( w'_{-k_0} = 0 \) for \( k \in \Gamma'_{k_0} \) and \( w'_{k_0} = 0 \) for \( k \in \Gamma'_{-k_0} \). Define the operators \( C_{w'_{-k_0}} \) and \( C_{w'_{k_0}} \) as in definition (63). Then, analogous to Lemma 3.5 in Ref. 17, we obtain

\[
\left\| C_{w'_{-k_0}} C_{w'_{k_0}} \right\|_{L^2(\Gamma')} \left\| C_{w'_{k_0}} C_{w'_{-k_0}} \right\|_{L^2(\Gamma')} \leq c \tau^{-\frac{1}{2}},
\]

\[
\left\| C_{w'_{k_0}} C_{w'_{-k_0}} \right\|_{L^\infty(\Gamma') \to L^2(\Gamma')} \leq c \tau^{-\frac{3}{4}},
\]

\[
\left\| C_{w'_{-k_0}} C_{w'_{k_0}} \right\|_{L^\infty(\Gamma') \to L^2(\Gamma')} \leq c \tau^{-\frac{3}{4}}.
\] (82)

In the case \( k_0 > M \), we have \( \tau > cx \), thus,

\[
\left\| C_{w'_{-k_0}} C_{w'_{k_0}} \right\|_{L^2(\Gamma')} = \left\| C_{w'_{k_0}} C_{w'_{-k_0}} \right\|_{L^2(\Gamma')} \leq cx^{-\frac{1}{2}},
\]

\[
\left\| C_{w'_{k_0}} C_{w'_{-k_0}} \right\|_{L^\infty(\Gamma') \to L^2(\Gamma')} \leq cx^{-\frac{3}{4}},
\]

\[
\left\| C_{w'_{-k_0}} C_{w'_{k_0}} \right\|_{L^\infty(\Gamma') \to L^2(\Gamma')} \leq cx^{-\frac{3}{4}}.
\] (83)

From the identity

\[
\left(1 - C_{w'_{-k_0}} - C_{w'_{k_0}} \right) \left(1 + C_{w'_{-k_0}} \left(1 - C_{w'_{-k_0}} \right)^{-1} + C_{w'_{k_0}} \left(1 - C_{w'_{k_0}} \right)^{-1} \right) = 1 - C_{w'_{k_0}} C_{w'_{-k_0}} \left(1 - C_{w'_{-k_0}} \right)^{-1} - C_{w'_{-k_0}} C_{w'_{k_0}} \left(1 - C_{w'_{k_0}} \right)^{-1},
\]

it follows that

\[
(1 - C_{w'})^{-1} = 1 + C_{w'_{-k_0}} \left(1 - C_{w'_{-k_0}} \right)^{-1} + C_{w'_{k_0}} \left(1 - C_{w'_{k_0}} \right)^{-1}
\]

\[
+ \left[1 + C_{w'_{-k_0}} \left(1 - C_{w'_{-k_0}} \right)^{-1} + C_{w'_{k_0}} \left(1 - C_{w'_{k_0}} \right)^{-1} \right] \left[1 - C_{w'_{k_0}} C_{w'_{-k_0}} \left(1 - C_{w'_{-k_0}} \right)^{-1} \right]
\]

\[
- \left(C_{w'_{-k_0}} C_{w'_{k_0}} \left(1 - C_{w'_{k_0}} \right)^{-1} \right)^{-1} \left[C_{w'_{k_0}} C_{w'_{-k_0}} \left(1 - C_{w'_{-k_0}} \right)^{-1} + C_{w'_{-k_0}} C_{w'_{k_0}} \left(1 - C_{w'_{k_0}} \right)^{-1} \right].
\]
According to Lemmas 2 and 3 and estimate (82), proceeding the estimates as in Ref. 17, we obtain an important result stated as follows:

**Lemma 4.** For $0 < k_0 \leq M$, as $\tau \to \infty$,

$$
\int \left( \left( 1 - C_{w'}^{-1} I \right) w' \right)(x, t; s) ds = \int_{\Gamma_{-k_0}'} \left( \left( 1 - C_{w'_{-k_0}}^{-1} I \right) w'_{-k_0} \right)(x, t; s) ds
$$

$$
+ \int_{\Gamma_{k_0}'} \left( \left( 1 - C_{w'_{k_0}}^{-1} I \right) w'_{k_0} \right)(x, t; s) ds + O(\tau^{-1})(84)
$$

For $k_0 > M$, as $x \to \infty$,

$$
\int \left( \left( 1 - C_{w'}^{-1} I \right) w' \right)(x, t; s) ds = \int_{\Gamma_{-k_0}'} \left( \left( 1 - C_{w'_{-k_0}}^{-1} I \right) w'_{-k_0} \right)(x, t; s) ds
$$

$$
+ \int_{\Gamma_{k_0}'} \left( \left( 1 - C_{w'_{k_0}}^{-1} I \right) w'_{k_0} \right)(x, t; s) ds + O(x^{-1})(85)
$$

In the following, we aim to perform a scaling transform and then formulate the model RH problem. Extend the contours $\Gamma_{-k_0}'$ and $\Gamma_{k_0}'$ to the contours

$$
\hat{\Gamma}_{-k_0}' = \left\{ k = -k_0 + k_0 \alpha e^{\pm i \pi/6} : -\infty < \alpha < \infty \right\},
$$

$$
\hat{\Gamma}_{k_0}' = \left\{ k = k_0 + k_0 \alpha e^{\pm i 5\pi/6} : -\infty < \alpha < \infty \right\},
$$

respectively, and define $\hat{w}_{-k_0}'$, $\hat{w}_{k_0}'$ on $\hat{\Gamma}_{-k_0}'$, $\hat{\Gamma}_{k_0}'$, respectively, through

$$
\hat{w}_{-k_0}' = \begin{cases} w_{-k_0}', & k \in \Gamma_{-k_0}' \subset \hat{\Gamma}_{-k_0}' , \\
0, & k \in \hat{\Gamma}_{-k_0}' \setminus \Gamma_{-k_0}' , \end{cases}
$$

$$
\hat{w}_{k_0}' = \begin{cases} w_{k_0}', & k \in \Gamma_{k_0}' \subset \hat{\Gamma}_{k_0}' , \\
0, & k \in \hat{\Gamma}_{k_0}' \setminus \Gamma_{k_0}' . \end{cases}
$$

Let $\Gamma_{-k_0}$ and $\Gamma_{k_0}$ denote the contours $\{k = \alpha e^{\pm i \pi/6} : -\infty < \alpha < \infty \}$ centered at an original point and oriented as shown in Figure 5.
We next introduce the following scaling operators:

\[ S_{-k_0} : k \mapsto \frac{z}{8k_0 \sqrt{10k_0 t}} - k_0, \]  

(86)

\[ S_{k_0} : k \mapsto \frac{z}{8k_0 \sqrt{10k_0 t}} + k_0. \]  

(87)

Set \( w_{-k_0} = S_{-k_0} \hat{w}'_{-k_0}, \ w_{k_0} = S_{k_0} \hat{w}'_{k_0} \). A simple change of variables argument shows that

\[ C_{\hat{w}'_{-k_0}} = S_{-1} S_{-k_0} C_{\hat{w}'_{-k_0}}, \quad C_{\hat{w}'_{k_0}} = S_{-1}^{-1} C_{w_{k_0}} S_{k_0}, \]

where \( C_{w_{\pm k_0}} \) is a bounded map from \( \Gamma_{\pm k_0} \) to \( \Gamma_{\pm k_0} \). On the part

\[ L_{k_0} : \left \{ z = 8k_0^2 \sqrt{10k_0 t} e^{\frac{\im \pi}{6}} : -\varepsilon < \alpha < \infty \right \}, \]

we have

\[ w_{k_0} = w_{k_0}^+ = \begin{cases} 
0 - \frac{r(k_0)}{1 - |r(k_0)|^2} \left( \delta^0_{k_0} \right)^{-1} \left( \delta^1_{k_0} \right)^{-1} \\
0 \\
0 - r(k_0) \left( \delta^0_{k_0} \right)^{-1} \left( \delta^1_{k_0} \right)^{-1} \\
0 
\end{cases}, \]

and on \( \bar{L}_{k_0} \),

\[ w_{k_0} = w_{k_0}^- = \begin{cases} 
0 \\
\frac{r(k_0)}{1 - |r(k_0)|^2} \delta^0_{k_0} \delta^1_{k_0} \\
0 \\
-r(k_0) \delta^0_{k_0} \delta^1_{k_0} 
\end{cases}, \]

where

\[ \delta^0_{k_0} = e^{-2\chi(k_0)} (256k_0^5 t)^{\frac{\im}{2}}, \]  

(88)

\[ \delta^1_{k_0} = z^{-2\im \nu} \exp \left\{ \frac{\im z^2}{2} \left( 1 + \frac{z}{8k_0^2 \sqrt{10k_0 t}} + \frac{z^2}{1280k_0^3 t} + \frac{z^3}{51200k_0^4 t \sqrt{10k_0 t}} \right) \right\} \times \left( \frac{2k_0}{z/8k_0 \sqrt{10k_0 t} + 2k_0} \right)^{-2\im \nu}, \]  

(89)
otherwise, \( w_{k_0} = 0 \). Set \( J_{k_0}(z; r(k_0)) = (I - w_{k_0}^{-1}) (I + w_{k_0}^+), \) where

\[
\begin{align*}
    w_{k_0}^+ &= \left\{
        \begin{array}{ll}
            0 & z \in \Gamma_{k_0}^1, \\
            -\delta_{k_0}^{-1} z^{2i} e^{-\frac{iz^2}{2}} r(k_0) & z \in \Gamma_{k_0}^3,
        \end{array}
    \right.
\end{align*}
\]

\[
\begin{align*}
    w_{k_0}^- &= \left\{
        \begin{array}{ll}
            0 & z \in \Gamma_{k_0}^2, \\
            \delta_{k_0}^{-1} z^{2i} e^{\frac{iz^2}{2}} r(k_0) & z \in \Gamma_{k_0}^4,
        \end{array}
    \right.
\end{align*}
\]

We obtain the following estimate on the rate of convergence.

**Lemma 5.** Let \( 0 < \kappa < 1/2 \) be a fixed small number, then we have as \( \tau \to \infty \)

\[
\begin{align*}
    \left\| \delta_{k_0}^1 - z^{-2i} e^{\frac{iz^2}{2}} \right\|_{L^\infty(I_{k_0})} \leq c \left| e^{\frac{iz^2}{2}} \right| \tau^{-\frac{1}{2}}. \tag{92}
\end{align*}
\]

As a result, we have

\[
\begin{align*}
    \left\| \delta_{k_0}^1 - z^{-2i} e^{\frac{iz^2}{2}} \right\|_{(L^\infty \cap L^1 \cap L^2)(I_{k_0})} \leq c \tau^{-\frac{1}{2}}. \tag{93}
\end{align*}
\]

*Proof. Let*

\[
\Delta = \frac{z}{8k_0^2 \sqrt{10k_0 t(1 - 2\kappa)}} + \frac{z^2}{1280k_0^5 t(1 - 2\kappa)} + \frac{z^3}{51200k_0^7 t \sqrt{10k_0 t(1 - 2\kappa)}}.
\]

A direct calculation yields

\[
\begin{align*}
    \left| \delta_{k_0}^1 - z^{-2i} e^{\frac{iz^2}{2}} \right| &\leq c \left| e^{\frac{iz^2}{2}} \right| \left| \left( \frac{2k_0}{z/8k_0 \sqrt{10k_0 t + 2k_0}} \right)^{-2i} - 1 \right| \\
    &+ \left( \left| \exp \left( \frac{i(1 - 2\kappa)z^2}{2} (1 + \Delta) \right) - e^{\frac{iz^2}{2}} \right| \right) \leq c \left| e^{\frac{iz^2}{2}} \right| \tau^{-\frac{1}{2}}.
\end{align*}
\]

\[\square\]
Therefore, we have

\[
\int_{\Gamma_{k_0}} \left( \left( 1 - C_{w_{k_0}} \right)^{-1} I \right) w_{k_0}'(s) ds = \int_{\Gamma_{k_0}} \left( \left( 1 - C_{\tilde{w}_{k_0}} \right)^{-1} I \right) \tilde{w}_{k_0}'(s) ds
\]

\[
= \int_{\Gamma_{k_0}} \left( S_{k_0}^{-1} \left( 1 - C_{w_{k_0}} \right)^{-1} S_{k_0} I \right) (s) \tilde{w}_{k_0}'(s) ds
\]

\[
= \int_{\Gamma_{k_0}} \left( \left( 1 - C_{w_{k_0}} \right)^{-1} I \right) \left( (s - k_0) 8k_0 \sqrt{10k_0 t} \right) S_{k_0} \tilde{w}_{k_0}'(s) \left( (s - k_0) 8k_0 \sqrt{10k_0 t} \right) ds
\]

\[
= \frac{1}{8k_0 \sqrt{10k_0 t}} \int_{\Gamma_{k_0}} \left( \left( 1 - C_{w_{k_0}} \right)^{-1} I \right) w_{k_0}(s) ds
\]

\[
= \frac{1}{8k_0 \sqrt{10k_0 t}} \int_{\Gamma_{k_0}} \left( \left( 1 - C_{w_{k_0}} \right)^{-1} I \right) w_{k_0}(s) ds + O(\tau^{-1}).
\]

Together with a similar computation for $C_{w_{k_0}}$ and (78) and (84), we have obtained as $\tau \to \infty$

\[
\int_{\Gamma} \left( \left( 1 - C_{w_{\text{mod}}} \right)^{-1} \right) w_{\text{mod}}(s) ds = \frac{1}{8k_0 \sqrt{10k_0 t}} \left( \int_{\Gamma_{k_0}} \left( \left( 1 - C_{w_{k_0}} \right)^{-1} I \right) w_{k_0}(s) ds \right)
\]

\[
+ \left( \int_{\Gamma_{k_0}} \left( \left( 1 - C_{w_{k_0}} \right)^{-1} I \right) w_{k_0}(s) ds \right) + O(\tau^{-1}). \quad (94)
\]

For $z \in \mathbb{C} \setminus \Gamma_{k_0}$, let

\[
M^{k_0}(z) = I + \int_{\Gamma_{k_0}} \left( \left( 1 - C_{w_{k_0}} \right)^{-1} I \right) \frac{w_{k_0}}{s - z} ds \quad (2\pi i)^{-1},
\]

then $M^{k_0}(z)$ satisfies the following RH problem:

**Riemann–Hilbert problem 2.5.** Find a $2 \times 2$ matrix-valued function $M^{k_0}(z)$ with the following properties:

1. $M^{k_0}(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma_{k_0}$ and is continuous for $z \in \Gamma_{k_0}$.
2. The boundary values $M^{k_0}_\pm(z)$ satisfy the jump condition

\[
M^{k_0}_+(z) = M^{k_0}_-(z) J^{k_0}(z; r(k_0)), \quad z \in \Gamma_{k_0}. \quad (96)
\]

3. $M^{k_0}(z) = I + O\left( \frac{1}{z} \right)$, as $z \to \infty$. 

In particular, if we assume
\[ M_{k_0}(z) = I + \frac{M_{k_0}^1}{z} + O(z^{-2}), \quad z \to \infty, \tag{97} \]
then
\[ M_{k_0}^1 = -\int_{\Gamma_{k_0}} \left( \left( 1 - C_{w, k_0} \right)^{-1} I \right) w^{k_0}(s) \frac{ds}{2\pi i}. \tag{98} \]

There is an analogous RH problem on \( \Gamma_{-k_0} \), which satisfies
\[
\begin{cases}
M_{-k_0}(z) = M_{-k_0}^+(z) J_{-k_0}(z; r(-k_0)), & z \in \Gamma_{-k_0}, \\
M_{-k_0}(z) \to I, & z \to \infty.
\end{cases} \tag{99}
\]

According \((59)\), \( r(-k) = \overline{r(k)} \) and the analogously computation for \( w^{-k_0} \), one can find that
\[ J_{-k_0}(z; r(-k_0)) = J^{k_0}_{-k_0}(-\overline{z}; r(k_0)), \tag{100} \]
which in turn implies, by uniqueness, that
\[ M_{-k_0}(z) = M_{k_0}(\overline{z}), \quad M_{-k_0}^1 = -M_{1}^1. \tag{101} \]

On the other hand, it follows from Refs. 17, 20, and 48 that the solution \( M_{k_0}(z) \) of RH problem 2.5 can be explicitly solved in terms of parabolic cylinder functions
\[ M_{k_0}(z) = I + \frac{i}{z} \begin{pmatrix} 0 & -\beta(r(k_0))(\delta_{k_0}^1)^{-1} \\ \beta(r(k_0))\delta_{k_0}^1 & 0 \end{pmatrix} + O \left( \frac{1}{z^2} \right), \tag{102} \]
where the function \( \beta(r(k_0)) \) is defined by
\[ \beta(r(k_0)) = \sqrt{\nu} e^{i \left( \frac{\pi}{2} - \arg(r(k_0)) + \arg(\Gamma(\nu)) \right)}, \tag{103} \]
and \( \Gamma(\cdot) \) denotes the standard Gamma function. Thus, we have found that
\[ M_{k_0}^1 = i \begin{pmatrix} 0 & -\beta(r(k_0))(\delta_{k_0}^1)^{-1} \\ \beta(r(k_0))\delta_{k_0}^1 & 0 \end{pmatrix}. \tag{104} \]

Taking into account that \((65)\), \((94)\), \((97)\), \((101)\), and \((102)\), we get
\[ M^{mod}(x, t; k) = I + \frac{M_{1}^{mod}}{k} + O \left( \frac{1}{k^2} \right), \quad k \to \infty, \tag{105} \]
where
\[ M^\text{mod}_1 = \frac{1}{8k_0 \sqrt{10k_0 t}} \left( 2i \text{Im} M^k_1 \right) + O(\tau^{-1}), \quad \text{as } \tau \to \infty. \quad (106) \]

### 2.4 Analysis of the remaining $\bar{\partial}$ problem

Recalling the transform (61), it is easy to verify that $M^{(3)}(k)$ is a continuously differentiable function satisfying the following pure $\bar{\partial}$ problem.

**$\bar{\partial}$ problem 2.6.** Find a function $M^{(3)}(k)$ with the following properties:

1. $M^{(3)}(k)$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus \Gamma$.
2. For $k \in \mathbb{C} \setminus \Gamma$, we have
   \[
   \bar{\partial} M^{(3)}(k) = M^{(3)}(k) W^{(3)}(k),
   \]
   \[
   W^{(3)}(k) = M^{\text{mod}}(k) \bar{\partial} R^{(2)}(k) [M^{\text{mod}}(k)]^{-1}.
   \quad (107)
   \]
3. $M^{(3)}(k)$ admits asymptotics: $M^{(3)}(k) = I + O\left(\frac{1}{k}\right), k \to \infty$.

It follows from the Section 7.6: The DBAR problem of Ref. 49 that the $\bar{\partial}$ problem 2.6 is equivalent to the integral equation

\[ M^{(3)}(k) = I - \frac{1}{\pi} \int_{\mathbb{C}} \frac{M^{(3)}(s) W^{(3)}(s)}{s - k} dA(s), \quad (108) \]

where $dA(s)$ is the Lebesgue measure on the plane. Equation (108) can be rewritten as the following operator form:

\[ (1 - K_W) [M^{(3)}(k)] = I, \quad (109) \]

where the integral operator $K_W$ is defined by

\[ (K_W f)(k) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(s) W^{(3)}(s)}{s - k} dA(s). \quad (110) \]

**Proposition 2.** There exists a constant $C > 0$ such that for $0 < k_0 \leq M$ and enough large $t > 0$, the operator (110) obeys the estimate

\[ \|K_W\|_{L^\infty \to L^\infty} \leq C \left( k_0^2 t \right)^{-\frac{1}{2}}. \quad (111) \]

For $k_0 > M$, as $x \to \infty$, the operator (110) obeys the estimate

\[ \|K_W\|_{L^\infty \to L^\infty} \leq C \left( k_0^{-1} x \right)^{-\frac{1}{4}}. \quad (112) \]
Proof. We only discuss the case in $\Omega_1$. We write $s = k_0 + u + iv$, then $u \geq \sqrt{3}v \geq 0$. Note that
\[
\text{Re}\Phi = 32v \left( 10u^2v^2 + 20uv^2k_0 + 10k_0^2v^2 - 5u^4 - 20u^3k_0 - 20uk_0^3 - 30u^2k_0^2 - v^4 \right) \\
\leq -640k_0^3uv.
\]
Set $k = \alpha + i\beta$. Let $f \in L^\infty(\Omega_1)$, we have
\[
|\langle K_{W_1} f \rangle(k)| \leq \|f\|_{L^\infty(\Omega_1)} \int \int_{\Omega_1} \left| \frac{|W_1(s)|}{|s-k|} \right| dA(s) \leq C\|f\|_{L^\infty(\Omega_1)} \int \int_{\Omega_1} \frac{|\tilde{\Phi}_1(s)|e^{-640k_0^3uv}}{|s-k|} dA(s).
\]
Thus, we find
\[
\|K_{W_1}\|_{L^\infty \to L^\infty} \leq C(I_1 + I_2),
\]
where
\[
I_1 = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{3}v} \frac{1}{|s-k|} |r'(u + k_0)|e^{-640k_0^3uv} dudv,
\]
\[
I_2 = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{3}v} \frac{1}{|s-k|} e^{-640k_0^3uv} dudv.
\]
Recalling the estimates from Ref. 27, we can similarly get $|I_1|, |I_2| \leq C(k_0^3t)^{-\frac{1}{4}}$.

For $k_0 > M$, as $x \to \infty$, recall $t\Phi(k) = x\Phi(k)$, $\tilde{\Phi}(k) = 2i(\frac{k^5}{5k_0^5} - k)$. We have
\[
\text{Re}\Phi = \frac{1}{5k_0^5} \left( -40k_0^3uv - 60k_0^2u^2v - 40k_0u^3v - 10u^4v + 20k_0^2v^3 + 40k_0uv^3 + 20u^2v^3 - 2v^5 \right) \\
\leq -8k_0^{-1}uv.
\]
The estimate (112) follows. \[\square\]

To recover the long-time asymptotic behavior of $u(x, t)$ using (27), it is necessary to determine the asymptotic behavior of the coefficient of the $k^{-1}$ term in the Laurent expansion of $M^{(3)}$ at infinity. In fact, we have
\[
M^{(3)}(k) = I - \frac{1}{\pi} \int_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s-k} dA(s) = I + \frac{M^{(3)}_1}{k} + \frac{1}{\pi} \int_{\mathbb{C}} sM^{(3)}(s)W^{(3)}(s) \frac{dA(s)}{k(s-k)},
\]
where
\[
M^{(3)}_1 = \frac{1}{\pi} \int_{\mathbb{C}} M^{(3)}(s)W^{(3)}(s) dA(s).
\]
Proposition 3. For \(0 < k_0 \leq M\) and all sufficiently large \(t > 0\), there exists a constant \(C > 0\) such that
\[
\left| M_1^{(3)} \right| \leq C \left( k_0^3 t \right)^{-\frac{3}{4}}. \tag{117}
\]

In the case \(k_0 > M\), \(x \to \infty\),
\[
\left| M_1^{(3)} \right| \leq C \left( k_0^{-1} x \right)^{-\frac{3}{4}}. \tag{118}
\]

Proof. We estimate the integral (117) as follows:
\[
\left| M_1^{(3)} \right| \leq C \int \int_{\Omega_1} |\partial R_1(s)| e^{-640k_0^3tuv} dA(s) \leq C \left( \int_0^\infty \int_{\sqrt{3}v}^\infty |r'(u + k_0)| e^{-640k_0^3tuv} du dv \\
+ \int_0^\infty \int_{\sqrt{3}v}^\infty \frac{1}{|u + iv|^\frac{1}{2}} e^{-640k_0^3tuv} du dv \right) \leq C(I_3 + I_4).
\]

It follows from the Proposition D.2 in \(^29\) that \(I_3\) and \(I_4\) satisfy
\[
|I_3|, |I_4| \leq C \left( k_0^3 t \right)^{-\frac{3}{4}}. \tag{119}
\]

\[\blacksquare\]

2.5 Asymptotics for \(u(x, t)\)

We are now ready to find the asymptotic behavior of the solution \(u(x, t)\) to the fifth-order modified KdV equation (2) in Regions I and II. Working through the transformations, we have
\[
M(x, t; k) = M^{(3)}(k)M^{\text{mod}}(k)[R^{(2)}(k)]^{-1}\delta_{R_3}(k). \tag{120}
\]

Using the reconstruction formula (27) and (104)–(106), (115), and (117), taking \(k \to \infty\) vertically, eventually \(k \in \Omega_2\) so that \(R^{(2)}(k) = I\), we immediately find in the case \(0 < k_0 \leq M\), as \(\tau \to \infty\)
\[
u \lim_{k \to \infty} (kM(x, t; k))_{12} = -2i \left( M_1^{(3)} + M_1^{\text{mod}} \right)_{12}
= \frac{\sqrt{\nu}}{2k_0 \sqrt{10k_0 t}} \cos \left( 128tk_0^5 + \nu \ln \left( 2560tk_0^5 \right) + \varphi(k_0) \right) + O \left( \tau^{-1} + \left( k_0^3 t \right)^{-\frac{3}{4}} \right), \tag{121}
\]
and in the case \(k_0 > M\) (so that \(\tau > Mx\), as \(x \to \infty\)
\[
u \lim_{k \to \infty} (kM(x, t; k))_{12} = \frac{\sqrt{\nu}}{2k_0 \sqrt{10k_0 t}} \cos \left( 128tk_0^5 + \nu \ln \left( 2560tk_0^5 \right) + \varphi(k_0) \right) + O \left( x^{-1} + \left( k_0^{-1} x \right)^{-\frac{3}{4}} \right), \tag{122}
\]
where
\[
\varphi(k_0) = -\frac{3\pi}{4} - \arg r(k_0) + \arg \Gamma(i\nu) - \frac{1}{\pi} \int_{-k_0}^{k_0} \ln \left( \frac{1 - |r(s)|^2}{1 - |r(k_0)|^2} \right) \frac{ds}{s - k_0},
\]
\[
\nu = -\frac{1}{2\pi} \ln (1 - |r(k_0)|^2) > 0,
\]
and the constants in the error terms depend only on \( \|r\|_{H^1(\mathbb{R})} \).

3 \quad ASYMPTOTICS IN REGION IV

We now consider the asymptotics of \( u(x, t) \) in the Region IV: \( \tau \leq M' \), that is, \( |x| \leq 80M' t^{\frac{1}{5}} \). First, for \( x > 0 \), the function \( \Phi(k) \) has two real stationary phase points
\[
\pm k_0 = \pm \sqrt[4]{\frac{x}{80t}},
\]
however, as \( t \to \infty \), the critical points \( \pm k_0 \) approach 0 at least as fast as \( t^{-\frac{1}{5}} \), that is, \( k_0 \leq M' t^{-\frac{1}{5}} \).

Note that the jump matrix \( J(x, t; k) \) enjoys the factorization
\[
J(x, t; k) = \begin{pmatrix}
1 & -r(k)e^{-i\Phi(k)} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
r(k)e^{i\Phi(k)} & 1
\end{pmatrix}.
\]

Introducing the following scaling transform:
\[
k \to (20t)^{-\frac{1}{5}} z,
\]
and letting
\[
y = \frac{x}{(20t)^{\frac{1}{5}}}
\]
we then have
\[
J(x, t; z) = \begin{pmatrix}
1 & -r \left( (20t)^{-\frac{1}{5}} z \right) e^{-2i \left( \frac{4}{5} z^5 - yz \right)} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
r \left( (20t)^{-\frac{1}{5}} z \right) e^{2i \left( \frac{4}{5} z^5 - yz \right)} & 1
\end{pmatrix}.
\]

The first step also is to introduce the continuous but not necessarily analytic extensions. Let
\[
z_0 = (20t)^{\frac{1}{5}} k_0.
\]
Define the new contour $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, where the line segments

$$
\begin{align*}
\Sigma_1 &= \left\{ \left(z_0 + le^{\frac{ni}{6}} \right) \mid l \geq 0 \right\} \cup \left\{ \left(-z_0 + le^{\frac{ni}{6}} \right) \mid l \geq 0 \right\}, \\
\Sigma_2 &= \left\{ \left(z_0 + le^{\frac{ni}{6}} \right) \mid l \geq 0 \right\} \cup \left\{ \left(-z_0 + le^{-\frac{ni}{6}} \right) \mid l \geq 0 \right\}, \\
\Sigma_3 &= \left\{ l \mid |l - z_0| \leq l \leq z_0 \right\}
\end{align*}
$$

oriented with increasing real part and denote the four open sectors $\{D_j\}_{1}^{4}$ in $\mathbb{C}$ (see Figure 6).

**Lemma 6.** There exist functions $R_j(z)$ on $\bar{D}_j$ for $j = 1, 4$ with boundary values satisfying

$$
R_1(z) = \begin{cases} 
- r \left( 20t \right)^{\frac{1}{5}} z, & z \in (-\infty, -z_0) \cup (z_0, \infty), \\
- r(k_0), & z \in \Sigma_1,
\end{cases}
$$

$$
R_4(z) = \begin{cases} 
- r \left( 20t \right)^{\frac{1}{5}} z, & z \in (-\infty, -z_0) \cup (z_0, \infty), \\
- r(k_0), & z \in \Sigma_2,
\end{cases}
$$

such that

$$
|\tilde{\partial}R_j(z)| \leq c_1 t^{\frac{1}{5}} \left| r' \left( 20t \right)^{\frac{1}{5}} \text{Re} z \right| + c_2 t^{\frac{1}{5}} \left| (20t)^{\frac{1}{5}} z - k_0 \right|^{\frac{1}{2}}
$$

for two positive constants $c_1, c_2$ depend on $\|r\|_{H^1(\mathbb{R})}$.

**Proof.** We only consider $z \in \overline{\Omega}_1 \cap \{\text{Re} z > z_0\}$. Define the extension

$$
R_1(z) = -r(k_0) + \left( r(k_0) - r \left( 20t \right)^{\frac{1}{5}} \text{Re} z \right) \cos(3\phi).
$$
Let $z - z_0 = se^{i\phi}$. It follows from
\[
\bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial z_1} + i \frac{\partial}{\partial z_2} \right) = \frac{1}{2} e^{i\phi} \left( \frac{\partial}{\partial s} + \frac{i}{s} \frac{\partial}{\partial \phi} \right)
\]
that
\[
|\bar{\partial} R_1(z)| = \left| \frac{1}{2} (20t)^{-\frac{1}{5}} r' \left( (20t)^{-\frac{1}{5}} \text{Re} z \right) \cos(3\phi) + \frac{3i}{2} e^{i\phi} \frac{r \left( (20t)^{-\frac{1}{5}} \text{Re} z \right) - r(k_0)}{|z - z_0|} \sin(3\phi) \right| \leq c_1 t^{-\frac{1}{5}} |r' \left( (20t)^{-\frac{1}{5}} \text{Re} z \right)| + c_2 t^{-\frac{1}{5}} |(20t)^{-\frac{1}{5}} z - k_0|^{-\frac{1}{2}}. \tag{134}
\]

Next, we use the extensions in Lemma 6 to define
\[
M^{(1)}(y, t; z) = M(x, t; k)\mathcal{R}^{(1)}(z), \tag{135}
\]
where
\[
\mathcal{R}^{(1)}(z) = \begin{cases}
1 & z \in D_1, \\
R_1(z)e^{2i\left(\frac{4}{5} z^5 - yz \right)} & z \in D_2 \cup D_3, \\
1 & z \in D_4.
\end{cases}
\tag{136}
\]

Then it is an immediate consequence of Lemma 6 and RH problem 1.1 that $M^{(1)}(y, t; z)$ satisfies the following $\bar{\partial}$-RH problem.

**$\bar{\partial}$-Riemann–Hilbert problem 3.1.** Find a function $M^{(1)}(y, t; z)$ with the following properties:

1. $M^{(1)}(y, t; z)$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus \Sigma$.
2. Across $\Sigma$, the boundary values satisfy the jump relation
\[
M_+^{(1)}(y, t; z) = M_-^{(1)}(y, t; z)J^{(1)}(y, t; z), \quad z \in \Sigma, \tag{137}
\]
where the jump matrix $J^{(1)}(y, t; z)$ is given by
\[
J^{(1)}(y, t; z) = \begin{cases}
1 & z \in \Sigma_1, \\
\begin{pmatrix}
1 \\
r(k_0)e^{2i\left(\frac{4}{5} z^5 - yz \right)} \\
0 \end{pmatrix} & z \in \Sigma_2, \\
\begin{pmatrix}
1 \\
r(k_0)e^{2i\left(\frac{4}{5} z^5 - yz \right)} \\
0 \end{pmatrix} & z \in \Sigma_3.
\end{cases}
\tag{138}
\]
3. For $z \in \mathbb{C} \setminus \Sigma$, we have

$$\tilde{\delta} M^{(1)}(z) = M^{(1)}(z) \tilde{\delta} R^{(1)}(z),$$

$$\tilde{\delta} R^{(1)}(z) = \begin{cases} 
1 & z \in D_1, \\
\tilde{\delta} R_1(z) e^{2i \left( \frac{4}{5} z^5 - yz \right)} & z \in D_4, \\
0 & z \in D_2 \cup D_3.
\end{cases} \quad (139)$$

4. $M^{(1)}(y, t; z)$ enjoys asymptotics: $M^{(1)}(y, t; z) = I + O\left( \frac{1}{z} \right), z \to \infty$.

Let $M^{\text{RHP}}(y, t; z)$ be the solution of the RH problem resulting from setting $\tilde{\delta} R^{(1)} \equiv 0$ in $\tilde{\delta}$-RH problem 3.1. It is then turned out that $M^{\text{RHP}}(y, t; z)$ can be approximated by the solution of the model RH problem A.2 presented in Appendix A as $t \to \infty$. In fact, we have the following lemma.

**Lemma 7.** As $t \to \infty$, we have

$$M^{\text{RHP}}(y, t; z) = \left( I + \frac{O(t^{-\frac{1}{5}})}{z} \right) M^{Z}(y; z, z_0), \quad (140)$$

where the matrix-valued function $M^{Z}(y; z, z_0)$ is the solution of model RH problem A.2 with $s = r(0)$ and satisfies

$$M^{Z}(y; z, z_0) = I + \frac{M^{Z}_1(y)}{z} + O\left( \frac{1}{z^2} \right), \quad z \to \infty, \quad (M^{Z}_1(y))_{12} = -i u_p(y), \quad (141)$$

and $u_p(y)$ denotes the solution of the fourth-order Painlevé II equation (A5).

**Proof.** For $z = z_0 + le^{\frac{\pi i}{5}}$ and $z = -z_0 + le^{\frac{5\pi i}{6}}$ with $l \geq 0, z_0 \geq 0$ and $y = 4z_0^4$, we find

$$\text{Re} \left( 2i \left( \frac{4}{5} z^5 - yz \right) \right) = l^2 \left( -\frac{4}{5} l^3 - 4\sqrt{3} z_0 l^2 - 16z_0^3 l - 8\sqrt{3} z_0^3 \right) \leq -8\sqrt{3} z_0^3 l^2 - \frac{4}{5} l^5.$$

For $z \in \Sigma_3, |e^{\pm 2i \left( \frac{4}{5} z^5 - yz \right)}| = 1$. Then, a basic integral estimate shows that

$$\left\| r(k_0) e^{2i \left( \frac{4}{5} z^5 - yz \right)} - r(0) e^{2i \left( \frac{4}{5} z^5 - yz \right)} \right\|_{L^p(\Sigma)} \leq ct^{-\frac{1}{5}}, \quad p \in [1, \infty). \quad (142)$$
A simple calculation shows that the jumps $J^e$ of the quantity $e(z) = M^{\text{RHP}}(z)[M^Z(z)]^{-1}$, satisfy
\[
\|J^e - I\|_{L^p(\Sigma)} \leq ct^{-1/5}.
\] (143)

Using the theory of small-norm RH problems (see Ref. 17 for details), the conclusion follows. ■

Next, we define the ratio
\[
M^{(2)}(z) = M^{(1)}(z)[M^{\text{RHP}}(z)]^{-1},
\] (144)
then it is easy to verify that $M^{(2)}(z)$ is a continuously differentiable function satisfying the following pure $\bar{\partial}$ problem.

\textbf{Problem 3.2.} Find a function $M^{(2)}(z)$ with the following properties:

1. $M^{(2)}(z)$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus \Sigma$.
2. For $z \in \mathbb{C} \setminus \Sigma$, we have
   \[
   \bar{\partial} M^{(2)}(z) = M^{(2)}(z)W^{(2)}(z),
   \]
   \[
   W^{(2)}(z) = M^{\text{RHP}}(z)\bar{\partial} R^{(1)}(z)[M^{\text{RHP}}(z)]^{-1},
   \] (145)
   where $\bar{\partial} R^{(1)}(z)$ is defined by (139).
3. $M^{(2)}(z)$ admits asymptotics: $M^{(2)}(z) = I + O\left(\frac{1}{z}\right), z \to \infty$.

Proceeding as in the previous section, we find that the $\bar{\partial}$ problem 3.2 is equivalent to the integral equation
\[
M^{(2)}(z) = I - \frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{M^{(2)}(s)W^{(2)}(s)}{s - z} dA(s).
\] (146)

\textbf{Proposition 4.} For sufficiently large $t > 0$, the operator $K_W$ defined by
\[
(K_W f)(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{f(s)W^{(2)}(s)}{s - z} dA(s)
\] (147)

satisfies
\[
\|K_W\|_{L^\infty \to L^\infty} \leq Ct^{-1/10}.
\] (148)

\textbf{Proof.} In $D_1 = D_1 \cap \{\text{Re} z > z_0\}$, we write $s = z_0 + u + iv$, then $u \geq \sqrt{3}v \geq 0$. We first note that
\[
\text{Re} \left( 2i \left( \frac{4}{5} e^5 - y_5 \right) \right) = \frac{8}{5} v \left( 10u^2 v^2 + 20uv^2 z_0 + 10v^2 z_0^2 - 5u^4 - 20u^3 z_0 - 20u z_0^3 - 30u^2 z_0^2 - v^4 \right)
\]
\[
\leq \frac{8}{5}v \left(10u^2 \cdot \frac{u^2}{3} - 5u^4\right) \leq -2u^4v.
\]

Set \(z = \alpha + i\beta\). Let \(f \in L^\infty(D^1_1)\), we have

\[
|(K_W f)(z)| \leq \|f\|_{L^\infty(D^1_1)} \int_{D^1_1} \frac{|W^{(2)}(s)|}{|s - z|} dA(s) \leq C \|f\|_{L^\infty(D^1_1)} \int_{D^1_1} \frac{|\mathcal{S}R_1(s)| e^{-2u^4v}}{|s - z|} dA(s).
\]  

(149)

Thus, we find

\[
\|K_W\|_{L^\infty \rightarrow L^\infty} \leq C (I_1 + I_2),
\]  

(150)

where

\[
I_1 = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{3v}} \left| \frac{1}{t-\frac{1}{5}r'} \left( (20t)^{-\frac{1}{5}} \text{Res} \right) \right| e^{-2u^4v} du dv,
\]

\[
I_2 = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{3v}} \left| \frac{1}{t^\frac{1}{10} \sqrt{u+iv}} \right| e^{-2u^4v} du dv.
\]

We first note that

\[
\left( \int_{\mathbb{R}} \left| t^{-\frac{1}{5}} r' \left( (20t)^{-\frac{1}{5}} \text{Res} \right) \right|^2 \frac{1}{2} du \right)^\frac{1}{2} \leq Ct^{-\frac{1}{10}}.
\]

Recall the bound

\[
\left\| \frac{1}{|s - z|} \right\|_{L^2(\sqrt{3v}, \infty)} \leq \left( \frac{\pi}{|v - \beta|} \right)^\frac{1}{2}.
\]

Using these results and Schwarz’s inequality on the \(u\)-integration, we may bound \(I_1\) by

\[
|I_1| \leq Ct^{-\frac{1}{10}} \int_0^\infty \frac{e^{-18u^5}}{|v - \beta|^{\frac{1}{2}}} dv \leq Ct^{-\frac{1}{10}}.
\]  

(151)

For \(p > 2\), \(\frac{1}{p} + \frac{1}{q} = 1\), we recall the estimate from Appendix D of Ref. 29,

\[
\left\| \frac{1}{|u + iv|^{\frac{1}{2}}} \right\|_{L^p(\sqrt{3v}, \infty)} \leq C\nu^{\frac{1}{p} - \frac{1}{2}}, \quad \left\| \frac{1}{|s - z|} \right\|_{L^q(\sqrt{3v}, \infty)} \leq C|v - \beta|^{-\frac{1}{q} - 1}.
\]
Thus, we get
\[
|I_2| \leq C t^{-\frac{1}{10}} \int_0^\infty v_p^{1 - \frac{1}{2}} |v - \beta|^\frac{1}{4} e^{-18v^5} dv \leq C t^{-\frac{1}{10}}. \tag{152}
\]

Finally, we consider the Laurent expansion of $M^{(2)}$ as $z \to \infty$. In fact, we have
\[
M^{(2)}(z) = I + M_1^{(2)} \frac{1}{z} + \frac{1}{\pi} \int_C \frac{sM^{(2)}(s)W^{(2)}(s)}{z(s-z)} dA(s), \tag{153}
\]
where
\[
M_1^{(2)} = \frac{1}{\pi} \int_C \frac{M^{(2)}(s)W^{(2)}(s)}{dA(s)}. \tag{154}
\]

**Proposition 5.** For all large $t > 0$, the following estimate holds:
\[
|M_1^{(2)}| \leq C t^{-\frac{1}{10}}. \tag{155}
\]

**Proof.** We estimate the integral (154) as follows:
\[
|M_1^{(2)}| \leq C \int \int |\delta R_1(s)| e^{-2u^4v} dA(s) \leq C \left( \int_0^\infty \int_0^\infty \left| t^{-\frac{1}{5}} r' \left( 20t^{\frac{1}{2}} \text{Res} \right) e^{-2u^4v} \right| dv \right) du
\]
\[
+ \int_0^\infty \int_0^\infty \frac{1}{\sqrt{3v}} \frac{1}{t^{\frac{1}{10}} |u + iv|^\frac{1}{2}} e^{-2u^4v} dv du \leq C(I_3 + I_4).
\]
We bound $I_3$ by applying the Cauchy–Schwarz inequality:
\[
|I_3| \leq C t^{-\frac{1}{10}} \int_0^\infty e^{-9v^5} \left( \int_0^\infty e^{-2u^4v} du \right)^{\frac{1}{2}} dv
\]
\[
\leq C t^{-\frac{1}{10}} \sqrt{\Gamma(1/4)} \int_0^\infty e^{-9v^5} \frac{1}{\sqrt{2v}} dv \leq C t^{-\frac{1}{10}}. \tag{156}
\]
For $I_4$, applying Hölder’s inequality, we find
\[
|I_4| \leq C t^{-\frac{1}{10}} \int_0^\infty v_p^{1 - \frac{1}{2}} e^{-9v^5} \left( \int_0^\infty e^{-qu^4v} du \right)^{\frac{1}{q}} dv
\]
Recalling the transformations \((135)\) and \((144)\), we have

\[
M(x, t; k) = M^{(2)}(z) M^{RHP}(z) [R^{(1)}(z)]^{-1}.
\]  

Using the reconstruction formula \((27)\) and \((140)-(141), (153)-(155)\), taking \(z \to \infty\) vertically, eventually \(z \in D_2\) so that \(R^{(1)}(z) = I\), we immediately find the asymptotics of the solution \(u(x, t)\) in Region IV when \(x > 0\):

\[
\begin{aligned}
    u(x, t) &= -2i \lim_{k \to \infty} (kM(x, t; k))_{12} \\
    &= -2i(20t)^{-\frac{1}{5}} \left( M_1^{(2)} + M_1^{RHP} \right)_{12} \\
    &= \left( \frac{8}{5t} \right)^{\frac{1}{5}} u_p \left( \frac{x}{(20t)^{\frac{1}{5}}} \right) + O \left( t^{-\frac{3}{10}} \right).
\end{aligned}
\]  

For \(x < 0\), the two stationary points become

\[
\pm k_0 = \pm i \sqrt{\frac{-x}{80t}}.
\]

We again perform the scaling \(k \to (20t)^{-\frac{1}{5}} z\) and the contour deformation as shown in Figure 7.
Now the $\tilde{\partial}$ extension in $D_1$ turns into

$$R_1(z) = \begin{cases} -r \left( (20t)^{-\frac{1}{5}} z \right), & z \in (-\infty, \infty), \\ -r(0), & z \in \Sigma_1, \end{cases}$$

and the interpolation is given by

$$R_1(z) = -r(0) + \left( -r \left( (20t)^{-\frac{1}{5}} z \right) + r(0) \right) \cos(3\phi).$$

Then we have

$$|\tilde{\partial}R_1(z)| = \left| -\frac{1}{2} (20t)^{-\frac{1}{5}} r' \left( (20t)^{-\frac{1}{5}} \text{Re} z \right) \cos(3\phi) + \frac{3i}{2} v^\phi \frac{r \left( (20t)^{-\frac{1}{5}} \text{Re} z \right) - r(0)}{|z|} \sin(3\phi) \right|$$

$$\leq c_1 t^{-\frac{1}{5}} \left| r' \left( (20t)^{-\frac{1}{5}} \text{Re} z \right) \right| + c_2 t^{-\frac{1}{10}} |z|^{-\frac{1}{2}}.$$  

And if we write $s = u + iv$ in $D_1 = D_1 \cap \{\text{Re} z > 0\}$, then $u \geq \sqrt[3]{3}v \geq 0$. Note that $y < 0$, and then

$$\text{Re} \left( 2i \left( \frac{4}{5} s^5 - ys \right) \right) = \frac{8}{5} v (10u^2v^2 - 5u^4 - v^4) + 2yv \leq -2u^4v.$$  

Thus we can repeat the analysis as the case above for $x > 0$ and obtain the same long-time asymptotics as (159) for the solution $u(x, t)$.

## 4 \ ASYMPTOTICS IN REGIONS III, V, AND VI

We first derive the asymptotics for the fifth-order modified KdV equation (2) in Region V, then VI and finally III.

### 4.1 Region V: $k_0 \leq M, \tau \geq \bar{M}$

Again, we perform the scaling transformation

$$k \rightarrow (20t)^{-\frac{1}{5}} z.$$  

Letting $s = u + iv$, we find $\text{Re}(2i(\frac{4}{5}s^5 - ys)) = \frac{8}{5} v (10u^2v^2 - 5u^4 - v^4) + 2yv$. As $y = -4(20t)^{\frac{4}{5}} < 0$, we can select $\rho > 0$ which is sufficiently small and independent of $y$ and construct a new contour $\gamma = \gamma_1 \cup \gamma_2$ given in Figure 8, such that in sector $E_1^1$, 

The contour $\gamma = \gamma_1 \cup \gamma_2$ and the open sets $\{E_j\}_{j=1}^4$ in the complex $z$-plane.

$$\text{Re} \left( 2i \left( \frac{4}{5}s^5 - ys \right) \right) \leq \frac{8}{5}vu^2(10\rho^2 - 5u^2) - 8(20\tau)\frac{4}{5}v \leq -u^2v < 0. \quad (166)$$

We define the extension in $\bar{E}_1$ by $R_1(z) = -r((20t)^{-\frac{1}{5}}\text{Re}z)$, in $\bar{E}_3$ by $R_3(z) = -r((20t)^{-\frac{1}{5}}\text{Re}z)$. Then, we set

$$M^{(1)}(y, t; z) = M(x, t; k) \mathcal{R}^{(1)}(z), \quad (167)$$

where

$$\mathcal{R}^{(1)}(z) = \begin{cases} 
\begin{pmatrix} 
1 & 0 \\
R_1(z)e^{2i\left(\frac{4}{5}s^5 - zs\right)} & 1 
\end{pmatrix}, & z \in E_1, \\
\begin{pmatrix} 
1 & R_3(z)e^{-2i\left(\frac{4}{5}s^5 - zs\right)} \\
0 & 1 
\end{pmatrix}, & z \in E_3, \\
I, & \text{elsewhere}.
\end{cases} \quad (168)$$

Thus, $M^{(1)}(y, t; z)$ satisfies the following $\delta$-RH problem.

**$\delta$-Riemann–Hilbert problem 4.1.** Find a function $M^{(1)}(y, t; z)$ with the following properties:

1. $M^{(1)}(y, t; z)$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C} \setminus \{z|\text{Im}z = \pm \rho\}$.
2. Across $\text{Im}z = \pm \rho$, the boundary values satisfy the jump relation

$$M_+^{(1)}(y, t; z) = M_-^{(1)}(y, t; z)J^{(1)}(y, t; z), \quad (169)$$
where

\[
J^{(1)}(y, t; z) = \begin{cases} 
1 & \text{if } z \in \{z | \text{Im}z = \rho, \text{Re}z > 0\}, \\
\frac{r}{(20t)\frac{1}{5} \text{Re}z} e^{2i(\frac{4}{5}z^5 - yz)} & \text{if } z \in \{z | \text{Im}z = -\rho, \text{Re}z > 0\}, \\
1 & \text{elsewhere.}
\end{cases}
\] (170)

3. For \( z \in \mathbb{C} \setminus \{z | \text{Im}z = \pm\rho\} \), we have

\[
\delta M^{(1)}(z) = M^{(1)}(z) \delta R^{(1)}(z),
\]

\[
\delta R^{(1)}(z) = \begin{cases} 
1 & \text{if } z \in E^1_1, \\
\bar{\partial}R_1^1(z) e^{2i(\frac{4}{5}z^5 - yz)} & \text{if } z \in E^1_3, \\
0 & \text{elsewhere.}
\end{cases}
\] (171)

4. \( M^{(1)}(y, t; z) = I + O\left(\frac{1}{z}\right), z \to \infty. \)

Let \( M^{(2)}(y, t; z) \) be the solution of the RH problem resulting from setting \( \bar{\partial}R^{(1)} \equiv 0 \) in \( \bar{\partial} \)-RH problem 4.1. We now define the ratio

\[
M^{(3)}(z) = M^{(1)}(z)[M^{(2)}(z)]^{-1},
\] (172)

then it is easy to verify that \( M^{(3)}(z) \) satisfies the following pure \( \bar{\partial} \) problem.

\[\text{\( \bar{\partial} \) problem 4.2.} \text{ Find a function } M^{(3)}(z) \text{ satisfying the following properties:} \]

1. \( M^{(3)}(z) \) is continuous with sectionally continuous first partial derivatives in \( \mathbb{C} \setminus \{z | \text{Im}z = \pm\rho\}. \)
2. For \( z \in \mathbb{C} \setminus \{z | \text{Im}z = \pm\rho\}, \) we have

\[
\bar{\partial}M^{(3)}(z) = M^{(3)}(z)W^{(3)}(z),
\]

\[
W^{(3)}(z) = M^{(2)}(z)\bar{\partial}R^{(1)}(z)[M^{(2)}(z)]^{-1},
\] (173)

where \( \bar{\partial}R^{(1)}(z) \) is defined by (171).

3. \( M^{(3)}(z) \) admits asymptotics: \( M^{(3)}(z) = I + O\left(\frac{1}{z}\right), z \to \infty. \)
Consider the integral
\[
\int\int_{E_1^1} \frac{|W^{(3)}(s)|}{|s - z|} dA(s) \leq \int\int_{E_1^1} \frac{|\tilde{\Delta}R_1(s)| e^{-u^2v}}{|s - z|} dA(s)
\]
\[
\leq \int_0^\rho \int_{-\infty}^\infty \frac{1}{|s - z|} \left| t^{-\frac{1}{2}} \left( (20t)^{-\frac{1}{2}} \text{Res} \right) \right| e^{-u^2v} du dv
\]
\[
\leq C t^{-\frac{1}{10}} \int_0^\rho \frac{1}{|v - \beta|^{\frac{1}{2}}} dv \leq C t^{-\frac{1}{10}}. \tag{174}
\]

On the other hand, we have
\[
\int\int_{E_1^1} |W^{(3)}(s)| dA(s) \leq \int\int_{E_1^1} |\tilde{\Delta}R_1(s)| e^{-u^2v} dA(s)
\]
\[
\leq \int_0^\rho \int_{-\infty}^\infty \left| t^{-\frac{1}{2}} \left( (20t)^{-\frac{1}{2}} \text{Res} \right) \right| e^{-u^2v} du dv
\]
\[
\leq C t^{-\frac{1}{10}} \int_0^\rho \frac{1}{\sqrt{v}} dv \leq C t^{-\frac{1}{10}}. \tag{175}
\]

Thus, $M^{(3)}(z)$ exists and if we expand $M^{(3)}(z)$ as
\[
M^{(3)}(z) = I + \frac{M_1^{(3)}}{z} + O \left( \frac{1}{z^2} \right), \quad z \to \infty, \tag{176}
\]
we have
\[
\left| M_1^{(3)} \right| \leq C t^{-\frac{1}{10}}. \tag{177}
\]

Next, we analyze $M^{(2)}(y, t; z)$ which satisfies the conditions 1, 2, 4 of $\tilde{\Delta}$-RH problem 4.1. We aim to deform the contour $\{z | \text{Im} z = \pm \rho\}$ to $\gamma = \gamma_1 \cup \gamma_2$. For this purpose, we introduce a new unknown $M^{(4)}$ obtained from $M^{(2)}$ as
\[
M^{(4)}(z) = M^{(2)}(z) R^{(2)}(z). \tag{178}
\]

We choose $R^{(2)}(z)$ to remove the jump on the contour $\{z | \text{Im} z = \pm \rho\}$. More precisely, we define functions $R_1^2$ and $R_2^2$ satisfying
\[
R_1^2(z) = \begin{cases} 
-r \left( (20t)^{-\frac{1}{2}} z \right), & \text{Im} z = \rho, \\
-r(0), & z \in \gamma_1, 
\end{cases} \tag{179}
\]
\[ R^2_3(z) = \begin{cases} \frac{-r(20t)^{-\frac{1}{5}} z}{-r(0)}, & \text{Im} z = -\rho, \in \gamma_2, \\ -r(20t)^{-\frac{1}{5}} \left( \frac{4}{5} z^5 - yz \right), & \text{otherwise} \end{cases} \] (180)

and then we can select \( R^{(2)}(z) \) as follows:

\[ R^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ R^2_1(z) e^{2i\left(\frac{4}{5} z^5 - yz\right)} & 1 \end{pmatrix}, & z \in E^2_1 \cup E^3_1, \\ \begin{pmatrix} 1 & 0 \\ R^2_3(z) e^{-2i\left(\frac{4}{5} z^5 - yz\right)} & 1 \end{pmatrix}, & z \in E^2_3 \cup E^3_3, \\ I, & \text{elsewhere} \end{cases} \] (181)

**Lemma 8.** There exists functions \( R^2_1(z) \) in \( E^2_1 \cup E^3_1 \) and \( R^2_3(z) \) in \( E^2_3 \cup E^3_3 \) satisfying the boundary value conditions (179) and (180), such that

\[ |\partial^j \tilde{R}| \leq c_1 t^{-\frac{1}{3}} \left| r' \left( (20t)^{-\frac{1}{5}} \text{Re} z \right) \right| + c_2 t^{-\frac{1}{10}} |z|^{-\frac{1}{2}}, \quad j = 1, 3. \] (182)

Thus, \( M^{(4)}(z) \) satisfies the following \( \tilde{\delta} \)-RH problem.

**\( \tilde{\delta} \)-Riemann–Hilbert problem 4.3.** Find a function \( M^{(4)}(z) \) with the following properties:

1. \( M^{(4)}(z) \) is continuous with sectionally continuous first partial derivatives in \( \mathbb{C} \setminus \gamma \).
2. Across \( \gamma \), the boundary values \( M^{(4)}_\pm(z) \) satisfy the jump relation

\[ M^{(4)}_+(z) = M^{(4)}_-(z) J^{(4)}(y, t; z), \] (183)

where

\[ J^{(4)}(y, t; z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r(0) e^{2i\left(\frac{4}{5} z^5 - yz\right)} & 1 \end{pmatrix}, & z \in \gamma_1, \\ \begin{pmatrix} 1 & 0 \\ -r(0) e^{-2i\left(\frac{4}{5} z^5 - yz\right)} & 1 \end{pmatrix}, & z \in \gamma_2. \end{cases} \] (184)

3. For \( z \in \mathbb{C} \setminus \gamma \), we have

\[ \tilde{\delta} M^{(4)}(z) = M^{(4)}(z) \tilde{\delta} R^{(2)}(z), \]
\[ \tilde{\partial} R^{(2)}(z) = \begin{cases} 
1 & z \in E_1^2 \cup E_1^3, \\
\delta R_1^2(z)e^{2i\left(\frac{4}{5}z^5 - yz\right)} & z \in E_2^2 \cup E_3^2, \\
1 & \delta R_3^2(z)e^{-2i\left(\frac{4}{5}z^5 - yz\right)} & z \in E_2^3 \cup E_3^3, \\
0 & 1 & \text{elsewhere}. 
\end{cases} \] (185)

4. \( M^{(4)}(z) = I + O\left(\frac{1}{z}\right), \ z \to \infty. \)

Letting \( \tilde{\partial} R^{(2)} = 0 \) in \( \tilde{\partial} \)-RH problem 4.3, it easy to see that the remaining pure RH problem is equivalent to the fourth-order Painlevé II RH problem A.1 by setting \( s = r(0) \) up to a trivial contour deformation. Thus, the ratio

\[ M^{(5)}(z) = M^{(4)}(z)[M^p(z)]^{-1} \] (186)

is a continuously differentiable function satisfying the following \( \tilde{\partial} \) problem, where \( M^p(z) \) is the solution to the RH problem A.1.

\( \tilde{\partial} \) problem 4.4. Find a function \( M^{(5)}(z) \) with the following properties:

1. \( M^{(5)}(z) \) is continuous with sectionally continuous first partial derivatives in \( \mathbb{C} \setminus \gamma \).
2. For \( z \in \mathbb{C} \setminus \gamma \), we have

\[ \tilde{\partial} M^{(5)}(z) = M^{(5)}(z)W^{(5)}(z), \]

\[ W^{(5)}(z) = M^p(z)\tilde{\partial} R^{(2)}(z)[M^p(z)]^{-1}, \] (187)

where \( \tilde{\partial} R^{(2)}(z) \) is defined by (185).

3. \( M^{(5)}(z) \) admits asymptotics: \( M^{(5)}(z) = I + O\left(\frac{1}{z}\right), z \to \infty. \)

To show the existence of \( M^{(5)}(z) \), as the discussion in Section 2, we need to check the boundedness of the operator \( K_W \) defined by

\[ (K_W f)(z) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{f(s)W^{(5)}(s)}{s - z} dA(s). \] (188)

Indeed, we have the following proposition.

**Proposition 6.** For large time, the integral operator \( K_W \) given by (188) obeys the estimate

\[ \|K_W\|_{L^\infty \to L^\infty} \leq Ct^{-\frac{1}{10}}e^{-8(20\tau)^{\frac{4}{5}}}. \] (189)
Proof. In $E_1^2$, we write $s = u + i(v + \rho)$, where $u \geq \sqrt{3}v \geq 0$. Then we have

$$\text{Re} \left( 2i \left( \frac{4}{5}s^5 - ys \right) \right) = \frac{8}{5}(v + \rho) \left( 10u^2(v + \rho)^2 - 5u^4 - (v + \rho)^4 \right) + 2y(v + \rho)$$

$$\leq -u^4v - 8(20\tau)^{\frac{4}{5}}\rho. \tag{190}$$

Thus, it follows from (182) that

$$\|K_W\|_{L^\infty \rightarrow L^\infty} \leq C(I_1 + I_2), \tag{191}$$

where

$$I_1 = \int_0^\infty \int_{\sqrt{3}(v+\rho)}^\infty \frac{1}{|s-z|} \left| t^{-\frac{1}{5}}r^{\frac{1}{2}} \left( (20t)^{\frac{1}{5}} \text{Res} \right) \right| e^{-u^4v - 8(20\tau)^{\frac{4}{5}}\rho} \, du \, dv,$$

$$I_2 = \int_0^\infty \int_{\sqrt{3}(v+\rho)}^\infty \frac{1}{|s-z|} \frac{1}{t^{\frac{1}{10}}|u + i(v + \rho)|} e^{-u^4v - 8(20\tau)^{\frac{4}{5}}\rho} \, du \, dv.$$

Proceeding the same procedure, we can get the following estimates:

$$|I_1|, |I_2| \leq Ct\frac{1}{10} e^{-8(20\tau)^{\frac{4}{5}}\rho}. \tag{192}$$

Moreover, letting

$$M^{(5)}(z) = I + \frac{M^{(5)}_1}{z} + O \left( \frac{1}{z^2} \right), \quad z \to \infty, \tag{193}$$

we then have

$$\left| M^{(5)}_1 \right| \leq Ct\frac{1}{10} e^{-8(20\tau)^{\frac{4}{5}}\rho}. \tag{194}$$

Combining all the transforms (167), (172), (178), and (186), we find

$$M(x, t; k) = M^{(3)}(z)M^{(5)}(z)M^P(z)[R^{(2)}(z)]^{-1}[R^{(1)}(z)]^{-1}. \tag{195}$$

We now can use (165), (176) and (177), (193) and (194), and (A3) and (A4) to obtain the long-time asymptotics of $u(x, t)$ in Region V,

$$u(x, t) = \left( \frac{8}{5t} \right)^{\frac{1}{5}} u_p \left( \frac{x}{(20t)^{\frac{1}{5}}} \right) + O \left( t^{-\frac{3}{10}} e^{-8(20\tau)^{\frac{4}{5}}\rho} + t^{-\frac{3}{10}} \right). \tag{196}$$
4.2 Region VI: $k_0 \geq M, x \to -\infty$

Observe that on $\gamma_1$,
\[ \left\| r(0)e^{2i\left(\frac{4}{5}\right)\left(5z-5z_0\right)} \right\|_{L^1 \cap L^2 \cap L^\infty} \leq Ce^{-8(20\tau)^{\frac{4}{5}} \rho}. \quad (197) \]

However, we may choose $\rho \geq (20\tau)^{\frac{1}{5}}$. It then follows that in Region VI
\[ u(x, t) = O \left( t^{-\frac{1}{5}} e^{-c\tau} + (-x)^{-\frac{3}{5}} + t^{-\frac{3}{10}} e^{-8(20\tau)^{\frac{4}{5}} \rho} \right). \quad (198) \]

4.3 Region III: $o(t^\frac{2}{7}) = \tau \geq \tilde{M}$

We now scale:
\[ k \to k_0 z, \quad (199) \]

and then have
\[ J(x, t; z) = \begin{pmatrix} 1 & -r(k_0 z)e^{-32i\tau(z^5 - s_0)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(k_0 z)e^{32i\tau(z^5 - s_0)} & 1 \end{pmatrix}. \quad (200) \]

Construct the contour $Y$ as shown in Figure 9 and define the function $R_1(z)$ in $F_1$ with the boundary values
\[ R_1(z) = \begin{cases} -r(k_0 z), & z \in (-\infty, -1) \cup (1, \infty), \\ -r(k_0), & z \in Y_1. \end{cases} \quad (201) \]

In fact, we can choose
\[ R_1(z) = -r(k_0) + (r(k_0) - r(k_0 \text{Re } z)) \cos(3\phi). \quad (202) \]
A simple computation shows that

\[
|\tilde{\delta} R_1(z)| \leq c_1 k_0 r'(k_0 \text{Re} z) + c_2 k_0^{\frac{1}{2}} |z - 1|^{-\frac{1}{2}}.
\]

(203)

However, if we write \( z = u + 1 + iv \) in \( F_1 \cap \{ \text{Re} z > 1 \} \), then one can get

\[
\text{Re}(32\tau(z^5 - 5z)) = 32\tau v(10u^2v^2 + 20uv^2 + 10v^2 - 5u^4 - 20u^3 - 30u^2 - 20u - v^4)
\]

\[
\leq -2\tau u^4 v.
\]

(204)

On the other hand, on \( Y_1 \), for \( \tau \geq \tilde{M} \), we have

\[
\left\| r(k_0)e^{32\tau(z^5 - 5z)} - r(0)e^{32\tau(z^5 - 5z)} \right\|_{L^p} \leq ck_0, \quad p \in [1, \infty].
\]

(205)

Following the notation and analysis of Section 3, we then only need to give the following estimates.

**Proposition 7.** The integral operator \( K_W \) given by (147)

\[
(K_W f)(z) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{f(s)W^{(2)}(s)}{s - z} dA(s)
\]

obeys the estimate

\[
\|K_W\|_{L^\infty \to L^\infty} \leq C_{k_0^2} \tau^{-\frac{1}{10}}.
\]

(206)

**Proof.** In \( F_1 := F_1 \cap \{ \text{Re} z > 1 \} \), it follows from (203) and (204) that

\[
\|K_W\|_{L^\infty \to L^\infty} \leq C(I_1 + I_2),
\]

(207)

where

\[
I_1 = \int_0^\infty \int_0^{\infty} \frac{1}{|s - z|} |k_0 r'(k_0 \text{Re} s)| e^{-2\tau u^4 v} dv du,
\]

\[
I_2 = \int_0^\infty \int_0^{\infty} \frac{k_0^{\frac{1}{2}}}{|s - z|} e^{-2\tau u^4 v} dv du.
\]

For \( I_1 \), we can get the following estimates:

\[
|I_1| \leq C_{k_0^2} \int_0^\infty \frac{e^{-18\tau u^5}}{|u - \beta|^{\frac{1}{2}}} dv \leq C_{k_0^2} \tau^{-\frac{1}{10}} \int_0^\infty \frac{e^{-18(\omega + \tau^5 \beta)}}{|\omega|^{\frac{1}{2}}} dw \leq C_{k_0^2} \tau^{-\frac{1}{10}}.
\]

(208)
For $I_2$, we have
\begin{equation}
|I_2| \leq k_0^2 \int_0^\infty e^{-18\tau v^5} v^{\frac{1}{p} - \frac{1}{2}} |v - \beta|^{\frac{1}{q} - 1} \, dv. \tag{209}
\end{equation}

Observe that $e^{-m} \leq m^{-\frac{1}{10}}$, thus, one can get
\begin{align*}
\int_0^\beta e^{-18\tau v^5} v^{\frac{1}{p} - \frac{1}{2}} |v - \beta|^{\frac{1}{q} - 1} \, dv &= \int_0^1 \beta^2 e^{-18\tau \beta^5 \omega^5} \omega^{\frac{1}{p} - \frac{1}{2}} (1 - \omega)^{\frac{1}{q} - 1} \, d\omega \\
&\leq \tau^{-\frac{1}{10}} \int_0^1 \omega^{\frac{1}{p} - \frac{1}{2}} (1 - \omega)^{\frac{1}{q} - 1} \, d\omega \leq \tau^{-\frac{1}{10}}.
\end{align*}

Finally, we have
\begin{equation}
\int_\beta^\infty e^{-18\tau v^5} v^{\frac{1}{p} - \frac{1}{2}} (v - \beta)^{\frac{1}{q} - 1} \, dv \leq \int_0^\infty e^{-18\tau v^5} w^{\frac{1}{2}} \, dw \leq \tau^{-\frac{1}{10}}.
\end{equation}

Next, we estimate the integral defined in (154).

**Proposition 8.** For all large $t > 0$, we find
\begin{equation}
\left| M_1^{(2)} \right| \leq C k_0^2 \tau^{-\frac{3}{10}}. \tag{210}
\end{equation}

**Proof.** We estimate the integral (154) as follows:
\begin{align*}
\left| M_1^{(2)} \right| &\leq C \int \int |\tilde{R}_1(s)| e^{-2\tau u^4 v} \, dA(s) \leq C \left( \int_0^\infty \int_{\sqrt{3} v}^\infty |k_0 r'(k_0 \text{Res})| e^{-2\tau u^4 v} \, du \, dv \\
&\quad + \int_0^\infty \int_{\sqrt{3} v}^\infty \frac{k_0^2}{|u + iv|^{\frac{1}{2}}} e^{-2\tau u^4 v} \, du \, dv \right) \leq C (I_3 + I_4).
\end{align*}

We bound $I_3$ as follows:
\begin{align*}
|I_3| &\leq C k_0 \int_0^\infty e^{-9\tau u^5} \left( \int_{\sqrt{3} v}^\infty e^{-2\tau u^4 v} \, du \right)^{\frac{1}{2}} \, dv \\
&\leq C k_0^2 \sqrt{\Gamma(1/4)} \int_0^\infty \frac{e^{-9\tau v^5}}{\sqrt{2\tau v}} \, dv \leq C k_0^2 \tau^{-\frac{3}{10}}. \tag{211}
\end{align*}
For $I_4$, applying Hölder's inequality, we find

$$|I_4| \leq C k_0^2 \int_0^\infty v^{\frac{1}{p} - \frac{1}{2}} e^{-\frac{9}{2} \tau \nu^2} \left( \int_0^\infty e^{-q\tau u^2} du \right) \frac{1}{q} dv$$

$$\leq C k_0^2 \tau^{-\frac{1}{10}} \int_0^\infty v^{\frac{5}{10} - \frac{3}{2}} e^{-\frac{9}{2} \tau \nu^2} dv \leq C k_0^2 \tau^{-\frac{3}{10}}.$$  \hspace{1cm} (212)

Using these estimates and the scaling transform (199), we immediately find the asymptotics of the solution $u(x, t)$ in Region III:

$$u(x, t) = \left( \frac{8}{5t} \right)^{\frac{1}{5}} u_p \left( \frac{x}{(20t)^{\frac{1}{5}}} \right) + O \left( k_0^{\frac{3}{2}} \tau^{-\frac{1}{10}} + k_0^2 \right)$$

$$= \left( \frac{8}{5t} \right)^{\frac{1}{5}} u_p \left( \frac{x}{(20t)^{\frac{1}{5}}} \right) + O \left( t^{-\frac{3}{10}} + \left( \frac{\tau}{t} \right)^{\frac{2}{5}} \right).$$  \hspace{1cm} (213)

**Remark 4.** We show how to match the asymptotic formulas of solution $u(x, t)$ in the overlaps of Regions II and III. It is easy to see that the final model RH problem on $Y$ corresponding to a special case in Region II in which one replaces

$$t \rightarrow \tau, \quad k_0 \rightarrow 1, \quad r(z) \rightarrow r(0).$$

Then, we find as $\tau \rightarrow \infty$,

$$\left( \frac{8}{5t} \right)^{\frac{1}{5}} u_p \left( \frac{x}{(20t)^{\frac{1}{5}}} \right) = k_0 \frac{\sqrt{\nu(r(0))}}{2 \sqrt{10 \tau}} \cos \left( \frac{128 \tau + \nu(r(0)) \ln(2560 \tau)}{2} \right)$$

$$- \left( \frac{3\pi}{4} - \arg r(0) + \arg \Gamma(i \nu(r(0))) \right) + k_0 O \left( \tau^{-\frac{3}{4}} \right).$$  \hspace{1cm} (214)

and thus in Region III

$$u(x, t)) = \frac{\sqrt{\nu(r(0))}}{2k_0 \sqrt{10\tau k_0}} \cos (128 \tau + \nu(r(0)) \ln(2560 \tau))$$

$$- \frac{3\pi}{4} - \arg r(0) + \arg \Gamma(i \nu(r(0))) \right) + k_0 O \left( \tau^{-\frac{3}{4}} \right) + O \left( \left( \frac{\tau}{t} \right)^{\frac{2}{5}} \right).$$  \hspace{1cm} (215)
On the other hand, in Region II, it follows from the similar analysis in Ref. 17 that

\[ u(x, t) = \frac{\sqrt{v(r(0))}}{2k_0 \sqrt{10k_0 t}} \cos \left( 128\tau + v(r(0)) \ln(2560\tau) - \frac{3\pi}{4} - \arg r(0) + \arg \Gamma(i\nu(r(0))) + (tk_0^3)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} + (k_0 t)^{-\frac{1}{4}} + k_0 \ln \tau \right). \]  

(216)

However, for \( \tau = o(t^{\frac{2}{7}}) \), we find

\[ k_0 \tau^{-\frac{3}{2}} = (tk_0^3)^{-\frac{1}{2}} \tau^{-\frac{1}{4}}, \quad k_0 \ln \tau = o \left( \ln \frac{t}{t^{\frac{1}{7}}} \right), \quad \left( \frac{\tau}{t} \right)^{\frac{3}{5}} = (tk_0^3)^{-\frac{1}{2}} \left( \frac{\tau^2}{t} \right)^{\frac{1}{5}}. \]  

(217)

Therefore, we have

\[ |u_{II}(x, t) - u_{III}(x, t)| = (tk_0^3)^{-\frac{1}{2}} O \left( \tau^{-\frac{1}{4}} + \left( \frac{\tau^2}{t} \right)^{\frac{1}{5}} + \ln t \frac{1}{t^{\frac{1}{7}}} + \tau^{-1} (tk_0^3)^{\frac{3}{2}} + (k_0^3 t)^{-\frac{1}{4}} \right), \]  

(218)

which is of order \( o(\sup |u_{as}(x, t)|) \) as \( \tau \to \infty \) with \( \tau = o(t^{\frac{2}{7}}) \).

## 5 ASYMPTOTIC BEHAVIOR IN LOW REGULARITY SPACES

In this section, we first obtain the global well-posedness for the initial value problem of Equation (2) and then extend the long-time asymptotic behavior of the solution to the low regularity spaces \( H^{s,1} \), \( s > 19/22 \).

The \( X^{s,b} \) method is extremely useful for studying the Cauchy problem of low regularity initial data. The motivation behind the \( X^{s,b} \) method is based on the dispersion relation. We consider the corresponding linear equation of the fifth-order KdV equation:

\[ \partial_\tau u + \partial_\xi^5 u = 0. \]

Taking the Fourier transform with respect to both space and time variable, we can get

\[ (\tau - \xi^5) \hat{u}(\xi, \tau) = 0. \]

It is easy to see that \( \hat{u}(\xi, \tau) \) is supported on the surface \{\( (\tau, \xi) : \tau = \xi^5 \)\}. \( \tau = \xi^5 \) and \( |\tau - \xi^5| \) are called the dispersion relation and dispersion modulation, respectively. Let \( s, b \in \mathbb{R} \), the Bourgain space \( X^{s,b}_{\tau = \xi^5}(\mathbb{R} \times \mathbb{R}) \), or simply denoted as \( X^{s,b} \), is defined to be the closure of the Schwartz functions \( S(\mathbb{R} \times \mathbb{R}) \) under the norm

\[ \|u\|_{X^{s,b}_{\tau = \xi^5}} = \left\| \langle \xi \rangle^s \langle \tau - \xi^5 \rangle^b \hat{u}(\xi, \tau) \right\|_{L^2_{\tau,\xi}}. \]
5.1 Estimates in Bourgain spaces

Let $\psi : \mathbb{R} \to [0, 1]$ denote an even smooth function supported in $[-2, 2]$ and equal to 1 in $[-1, 1]$. Denote $\psi_\mu(\cdot) = \psi(\cdot / \mu)$ for any $\mu > 0$. The $X^{s, b}$ method reduces to the following crucial estimates.

Lemma 9. For $s \in \mathbb{R}$ and $1/2 < b < b' < 1/2 + \epsilon$, then

\begin{enumerate}
  \item (Embedding) For any $u \in X^{s, b}$, we have $\|u\|_{L^\infty(\mathbb{R}; H^s(\mathbb{R}))} \lesssim \|u\|_{X^{s, b}}$.
  \item (Homogeneous estimate) $\|\psi(t) e^{-it\frac{\partial}{\partial x}} u_0\|_{X^{s, b}} \lesssim \|u_0\|_{H^s}$.
  \item (Inhomogeneous estimate) $\|\psi(t) \int_0^t e^{-i(t-t')\frac{\partial}{\partial x}} F(u)(t') \, dt'\|_{X^{s, b}} \lesssim \|F(u)\|_{X^{s, b-1}}$.
  \item (Cut-off estimate) Let $0 < \mu < 1$, then $\|\psi_\mu(t) F\|_{X^{s, b-1}} \lesssim \mu^{b'-b} \|F\|_{X^{s, b}}$.
  \item (Nonlinear estimates) Let $s \geq 3/4$, we have
    \begin{align*}
    \|u_1 u_2 \partial_x^3 u_3\|_{X^{s, b'-1}} + \|\partial_x u_1 \partial_x u_2 \partial_x u_3\|_{X^{s, b'-1}} + \|u_1 \partial_x u_2 \partial_x^2 u_3\|_{X^{s, b'-1}} &\lesssim \prod_{j=1}^3 \|u_j\|_{X^{s, b}}; \\
    \|\partial_x(u_1 u_2 u_3 u_4 u_5)\|_{X^{s, b'-1}} &\lesssim \prod_{j=1}^5 \|u_j\|_{X^{s, b}}.
    \end{align*}
\end{enumerate}

Proof. The proofs of (1–4) can be found in many papers, we refer to Refs. 7, 50, and 51. The nonlinear estimate (5) needs condition $s \geq 3/4$, and its proof was followed by Ref. 7.

We recall some important estimates for the fifth-order KdV equations. These estimates are due to semigroup estimates and the Extension Lemma of Ref. 51; their proofs can be obtained by the methods in Refs. 52–55.

Lemma 10. Let $u \in S(\mathbb{R} \times \mathbb{R})$, then we have

\begin{enumerate}
  \item (Strichartz estimates) Let $2 \leq q, r \leq \infty$, and $2/q + 1/r = 1/2$, then
    \begin{align*}
    \|D_x^{3/4} u\|_{L^q_t L^r_x} &\lesssim \|u\|_{X^{0,1/2}}. \\
    \end{align*}
  \item (Local smoothing effect estimates)
    \begin{align*}
    \|D_x^2 u\|_{L^\infty_t L^2_x} &\lesssim \|u\|_{X^{0,1/2}}. \\
    \end{align*}
  \item (Maximal function estimates)
    \begin{align*}
    \|u\|_{L^4_t L^\infty_x} &\lesssim \|u\|_{X^{1/2,1/2}}. \\
    \end{align*}
\end{enumerate}
Lemma 11 (Bilinear estimates). Suppose $u, v \in X^{0, \frac{1}{2}^+}$ be supported on spatial frequencies on $|\xi| \sim N_1, N_2$, respectively. Let $N_1 \gg N_2$, then we have

$$
\|uv\|_{L^2_{x,t}} \lesssim N_1^{-2}\|u\|_{X^{0, \frac{1}{2}^+}}\|v\|_{X^{0, \frac{1}{2}^+}}.
$$

(223)

5.2 Local well-posedness in $H^s$, $s \geq 3/4$

In this subsection, we use the $X^{s,b}$ method and the contraction principle to obtain the local existence of the solution in the Sobolev space $H^s$, $s \geq 3/4$. This result was mainly stated by Kwon. For the sake of completeness, we give the outline of its proof.

Theorem 4. Let $s \geq 3/4$ and $u_0 \in H^s(\mathbb{R})$. Then there exists a time $T = T(\|u_0\|_{H^s(\mathbb{R})}) > 0$ such that (2) has a unique solution in $C([0,T]; H^s(\mathbb{R}))$. Moreover, the solution map from data to the solutions is real-analytic.

Proof. We construct the mapping:

$$
\mathcal{M} : (u(x, t) \mapsto \psi(t)e^{-t\xi}u_0 - \psi(t) \int_0^t e^{-(t-t')\xi}F(u)(t') dt',
$$

(224)

and show that it is a contraction if $T < 1$ is sufficiently small. Assume $u_0 \in H^s(s \geq 3/4)$, we define the metric space:

$$
\mathfrak{D} = \{u : \|u\|_{X^{s,b}} \leq 2C\|u_0\|_{H^s}\}; \quad d(u, v) = \|u - v\|_{X^{s,b}}.
$$

Therefore, we have from the estimates in Lemma 9 that

$$
\|\mathcal{M}u\|_{X^{s,b}} \leq C\|u_0\|_{H^s} + C\|\psi_T(t)F(u)\|_{X^{s,b-1}}
$$

$$
\leq C\|u_0\|_{H^s} + CT^{b'-b}\|u\|_{X^{s,b}}^3 + CT^{b'-b}\|u\|_{X^{s,b}}^5
$$

$$
\leq C\|u_0\|_{H^s} + CT^{b'-b} \left( (2C)^3 \|u_0\|_{H^s}^2 + (2C)^5 \|u_0\|_{H^s}^4 \right) \|u_0\|_{H^s}.
$$

Thus we choose $T$ small enough such that

$$
T^{b'-b} \left( (2C)^3 \|u_0\|_{H^s}^2 + (2C)^5 \|u_0\|_{H^s}^4 \right) < 1/2,
$$

(225)

then we know that $\mathcal{M}u \in \mathfrak{D}$. Similarly, assume $(u_1, u_2) \in \mathfrak{D}$, we have

$$
\|\mathcal{M}u_1 - \mathcal{M}u_2\|_{X^{s,b}} \leq C\|\psi_T(t)(F(u_1) - F(u_2))\|_{X^{s,b-1}}
$$

$$
\leq CT^{b'-b} \left( \sum_{i=0}^2 \|u_1\|_{X^{s,b}}^i \|u_2\|_{X^{s,b}}^{2-i} + \sum_{j=0}^4 \|u_1\|_{X^{s,b}}^j \|u_2\|_{X^{s,b}}^{4-j} \right) \|u_1 - u_2\|_{X^{s,b}}
$$

$$
\leq 1/2\|u_1 - u_2\|_{X^{s,b}}.
$$
This implies the local well-posedness for large data. Furthermore, for the solutions $u$ and $v$ with the initial data $u_0$ and $v_0$ in the $H^s$ space, we have from (224) that

$$
||u - v||_{X^{s,b}} \leq C||u_0 - v_0||_{H^s} + C\|\psi_T(t)(F(u) - F(v))\|_{X^{s,b-1}}
\leq C||u_0 - v_0||_{H^s} + C T^{b-\frac{s}{2} - \frac{1}{2}} \left( \sum_{i=0}^2 ||u||_{X^{s,b}} ||v||_{X^{s,b}}^{2-i} + \sum_{j=0}^2 ||u||_{X^{s,b}}^j ||v||_{X^{s,b}}^{4-j} \right)||u - v||_{X^{s,b}}.
$$

(226)

Denote $M = \max\{||u_0||_{H^s}, ||v_0||_{H^s}\}$, we can choose $T$ small enough such that

$$
5CT^{b'-\frac{s}{2}} ((2C)^2 M^2 + (2C)^4 M^4) < 1/2,
$$

(227)

then (226) becomes

$$
||u - v||_{X^{s,b}} \leq C||u_0 - v_0||_{H^s} + 1/2||u - v||_{X^{s,b}},
$$

it follows that

$$
||u - v||_{X^{s,b}} \leq 2C||u_0 - v_0||_{H^s}.
$$

(228)

Therefore, the solution map is Lipschitz continuous. From (225) and (227), we know that the existence time $T$ of the solution depends only on the $H^s$ norm of the initial data. By a similar argument, we can get the solution map which is even real-analytic.

\[ \blacksquare \]

5.3 Global well-posedness in $H^s$, $19/22 < s \leq 1$

As mentioned in the Introduction, the global well-posedness of (2) in $H^1(\mathbb{R})$ can be immediately obtained by energy conservation. One can conjecture that (2) is in fact globally well-posed in time from all initial data contained in the local theory. The biggest obstacle in getting global solutions in $H^s$ with $0 < s < 1$ is the lack of any conservation law. However, the $I$-method is an effective method to overcome this difficulty. It was introduced by Colliander–Keel–Staffilani–Takaoka–Tao\textsuperscript{42,43} considering the global well-posedness theory of the Schrödinger equation and KdV equation. Therefore, in this subsection, we introduce an “almost conservation law” by utilizing the $I$-method, then we can get the global well-posedness.

First of all, we introduce the definition of the $I$-operator (see Ref. 42). Given $s < 1$, assume $\varphi(\xi) : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth, even, real-valued function, which is equal to 1 in $[-1, 1]$, and defined by $|\xi|^{-1}$ in the set $\{\xi : |\xi| \geq 2\}$. For a parameter $N \gg 1$, define

$$
m_N(\xi) := \varphi(\xi/N) = \begin{cases} 
1 & |\xi| \leq N; \\
\left( \frac{N}{|\xi|} \right)^{1-s} & |\xi| \geq 2N,
\end{cases}
$$

(229)

and the $I$-operator as follows:

$$
I_N u = F^{-1}_\xi m_N(\xi) Fu.
$$

(230)
For convenience, we will omit the parameter $N$, writing $m(\xi)$ and $Iu$ in (229) and (230).

Recall the energy $E(u)$ in (3), the quantity $E(Iu)(t)$ can be compared with $\|u(\cdot,t)\|_{H^s}^2$, indeed, we have

$$E(Iu)(t) \lesssim N^{2(1-s)} \|u(\cdot,t)\|_{H^s}^2 + \|u(\cdot,t)\|_{L^4}^4; \tag{231}$$

$$\|u(\cdot,t)\|_{H^s}^2 \lesssim E(Iu)(t) + \|u_0\|_{L^2}^2. \tag{232}$$

Moreover, we will show that $E(Iu)(t)$ is an “almost conservation law” in the $H^s$ level, and its increment can be described as the following proposition, whose proof shall be given later.

**Proposition 9.** Let $19/22 < s < 1$, $N \gg 1$. Given initial data $u_0 \in C_0^\infty(\mathbb{R})$ with $E(Iu_0) \leq 1$, then there exists a $\delta = \delta(\|u_0\|_{L^2}) > 0$ so that the solution

$$u(x,t) \in C([0, \delta], H^s(\mathbb{R})) \tag{233}$$

of (2) satisfies

$$E(Iu)(t) = E(Iu)(0) + O(N^{-1/2}) \text{ for all } t \in [0, \delta]. \tag{234}$$

**Proof of Theorem 2.** We only need to consider $19/22 < s < 1$. We may assume $u_0 \in C_0^\infty(\mathbb{R})$, and then show that the solution in the $H^s$ norm grows at most polynomially, that is to say there exist $C_1$, $C_2$, and $M$, which depend only on $\|u_0\|_{H^s}$ such that

$$\|u(\cdot,t)\|_{H^s} \leq C_1 t^M + C_2. \tag{235}$$

Then the global result is immediately obtained by (235), the local well-posedness and a standard density argument. From (232), we only need to show that

$$E(Iu)(t) \lesssim (1 + t)^M. \tag{236}$$

Recall the scaling symmetry, let $0 < \lambda < 1$, then (231) and the Sobolev embedding inequality imply that

$$E(Iu_{0,\lambda}) \lesssim N^{2(1-s)} \lambda^{2s+1} \|u_0\|_{H^s}^2 + \lambda^3 \|u_0\|_{L^4}^4 \lesssim N^{2(1-s)} \lambda^{2s+1} (1 + \|u_0\|_{H^s})^4. \tag{237}$$

Assuming $N \gg 1$ is given, we choose $\lambda = \lambda(N, \|u_0\|_{H^s})$

$$\lambda = N^{-\frac{2(1-s)}{2s+1}} (2C_0)^{-1} \left( 1 + \|u_0\|_{H^s} \right)^{-\frac{4}{2s+1}} \tag{238}$$

such that $E(Iu_{0,\lambda}) \leq 1/2$, where $C_0$ is the implicit constant in (237). We apply Proposition 9 at least $CN^{\frac{1}{2}}$ times to get

$$E(Iu_{\lambda})(CN^{\frac{1}{2}} \delta) \sim 1. \tag{239}$$
For any \( T_0 \gg 1 \), we choose \( N \gg 1 \) so that

\[
T_0 \sim \lambda^5 N^{\frac{1}{7}} \delta \sim N^{\frac{2s-19}{3(2s+1)}},
\]

(240)

where we used (238). To keep the exponent of \( N \) is positive, we need the condition \( s > \frac{19}{22} \).

From the scaling symmetry, we have that

\[
E(Iu)(t) \leq \lambda^{-3} E(Iu_\lambda)(\lambda^{-5} t).
\]

(241)

Therefore, from (238), (240), and (241), for any \( T_0 \gg 1 \), we have

\[
E(Iu)(T_0) \leq C T_0^{\frac{12(s-1)}{22s-19}},
\]

(242)

where \( N \) is determined by (240) and \( C = C(\|u_0\|_{H^s}, \delta) \). We now get the conclusion (236), and complete the proof. \( \square \)

To prove Proposition 9, we need the following variant local well-posedness result:

**Proposition 10.** Let \( \frac{3}{4} \leq s < 1 \). Assume \( u_0 \) satisfies \( E(Iu_0) \leq 1 \), then there exists a constant \( \delta = \delta(\|u_0\|_{L^2}) \) and a unique solution \( u \) to (2) such that

\[
\|Iu\|_{X^{\frac{1}{2}+}_s} \lesssim 1,
\]

(243)

where the implicit constant is independent of \( N \).

**Proof.** We have the following estimates, which are analogous to Lemma 9 (2-3):

\[
\left\| e^{-t \delta^5_x} u_0 \right\|_{X^{\frac{1}{2}+}_s} \lesssim \|u_0\|_{H^1};
\]

(244)

\[
\left\| \int_0^t e^{-(t-t') \delta^5_x} F(u)(t') \ dt' \right\|_{X^{\frac{1}{2}+}_s} \lesssim \|F(u)\|_{X^{\frac{1}{2}+}_s}.
\]

(245)

\( Iu \) satisfies the integral equation on \( t \in [0, \delta] \):

\[
Iu(x, t) = e^{-t \delta^5_x} Iu_0 - \int_0^t e^{-(t-t') \delta^5_x} \psi_\delta(t') IF(u)(t') \ dt'.
\]

By the definition of the restricted norm (219), we can choose \( \tilde{u} \in X^{\frac{1}{2}+}_s, \tilde{u}|_{[0,\delta]} = u \) such that

\[
\|I\tilde{u}\|_{X^{\frac{1}{2}+}_s} \sim \|Iu\|_{X^{\frac{1}{2}+}_s}.
\]

(246)
Duhamel’s principle, (244), (245), and (4) in Lemma 9 give us

$$\|Iu\|_{X^1_{\frac{1}{2}^+}} \lesssim \|Iu_0\|_{H^1} + \|\psi_\delta(t)IF(u)(t)\|_{X^1_{\frac{1}{2}^+}}$$

$$\lesssim \|Iu_0\|_{H^1} + \delta \|IF(u)(t)\|_{X^1_{\frac{1}{2}^+}}.$$  \hfill (247)

We divide each $\tilde{u}$ into a part supported on frequencies on $|\xi| \lesssim N$ and a part supported on frequencies $|\xi| \gg N$, then from nonlinear estimates in Lemma 9 we immediately obtain that

$$\|IF(u)(t)\|_{X^1_{\frac{1}{2}^+}} \lesssim \|\tilde{Iu}\|_{3} \|X^1_{\frac{1}{2}^+}} + ||\tilde{Iu}\|_{5} \|X^1_{\frac{1}{2}^+}}.$$  \hfill (248)

Thus (246) and (247) yield that

$$\|Iu\|_{X^1_{\frac{1}{2}^+}} \lesssim \|Iu_0\|_{H^1} + \delta \left(\|\tilde{Iu}\|_{3} \|X^1_{\frac{1}{2}^+}} + ||\tilde{Iu}\|_{5} \|X^1_{\frac{1}{2}^+}}\right).$$  \hfill (249)

It is easy to see that

$$\|Iu_0\|_{H^1} \lesssim \left(E(Iu_0)\right)^{1/2} + \|u_0\|_{L^2} \leq 1 + \|u_0\|_{L^2}.$$  \hfill (250)

Since $Q(\delta) := \|Iu\|_{X^1_{\frac{1}{2}^+}}$ is continuous in the variable $\delta$, a continuity argument implies the conclusion from (249) and (250).

\[\square\]

In the end, we turn to prove Proposition 9.

**Proof of Proposition 9.** By simple calculation, we can change all nonlinear terms in (2) into the divergence form, that is to say, (2) becomes the following form:

$$\partial_t u + \partial_x^5 u + 6\partial_x(u^5) - 5\partial_x \left(u\partial_x^2(u^2)\right) = 0.$$  \hfill (251)

The energy in (3) is known to be conserved by differentiating in time, using Equation (251) and integrating by parts, we have that

$$\frac{d}{dt} E(u) = \int \left(-2\partial_x^2 u + 4u^3\right) \cdot \partial_t u \, dx$$

$$= \int \left(-2\partial_x^2 u + 4u^3\right) \left(-\partial_x^5 u - 6\partial_x(u^5) + 5\partial_x \left(u\partial_x^2(u^2)\right)\right) \, dx$$

$$= 2 \int \partial_x^2 u \partial_x^5 u \, dx - 24 \int u^3 \partial_x(u^5) \, dx$$

$$- 2 \int 2u^3 \partial_x^5 u + 5\partial_x^2 u \partial_x \left(u\partial_x^2(u^2)\right) \, dx + 4 \int 3\partial_x^2 u \partial_x(u^5) + 5u^3 \partial_x \left(u\partial_x^2(u^2)\right) \, dx$$

$$:= T_1 + T_2 + T_3 + T_4 = 0,$$  \hfill (252)
In fact, we can get $T_1 = T_2 = T_3 = T_4 = 0$ only by integrating by parts, which will be used in the following discussion. We apply $I$ to (251), then

$$\partial_t I u = -\partial_x^5 Iu - 6\partial_x I (u^5) + 5\partial_x (u \partial_x^2 (u^2)). \quad (253)$$

Following the same strategy, we estimate the growth of $E(Iu)(t)$. By Equation (253), we know that

$$\frac{d}{dt} E(Iu)(t) = \int \left( -2\partial_x^5 Iu + 4(Iu)^3 \right) \partial_t Iu \, dx$$

$$= \int \left( -2\partial_x^5 Iu + 4(Iu)^3 \right) \left( -\partial_x^5 Iu - 6\partial_x I (u^5) + 5\partial_x (u \partial_x^2 (u^2)) \right) \, dx$$

$$= 2 \int \partial_x^3 Iu \cdot \partial_x^5 Iu \, dx + 72 \int \partial_x Iu \cdot (Iu)^2 \cdot I (u^5) \, dx$$

$$+ 4 \int 3\partial_x^2 Iu \cdot \partial_x I (u^5) + 5(Iu)^3 \cdot \partial_x (u \partial_x^2 (u^2)) \, dx$$

$$- 2 \int 2(Iu)^3 \cdot \partial_x^3 Iu + 5\partial_x^2 Iu \cdot \partial_x I (u \partial_x^2 (u^2)) \, dx$$

$$\vdots = T_{I1} + T_{I2} + T_{I3} + T_{I4}. \quad (254)$$

It is easy to see that

$$T_{I1} = -2 \int \partial_x^3 Iu \cdot \partial_x^4 Iu \, dx = - \int \partial_x \left( (\partial_x^3 Iu)^2 \right) \, dx = 0.$$

We recall the $k$-Parseval formula:

$$\int_{\mathbb{R}} f_1(x) f_2(x) \cdots f_k(x) \, dx = \int_{\xi_1 + \xi_2 + \cdots + \xi_k = 0} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \cdots \hat{f}_k(\xi_k), \quad (255)$$

For simplicity, denote $\Gamma_k = \{(\xi_1, \xi_2, \ldots, \xi_k) \in \mathbb{R}^k, \xi_1 + \xi_2 + \cdots + \xi_k = 0\}$. Therefore, the $k$-Parseval formula combined with integrating by parts yields that

$$T_{I2} = 72i \int_{\Gamma_8} \xi_1 \cdot \frac{m(\xi_4 + \xi_5 + \xi_6 + \xi_7 + \xi_8)}{m(\xi_4)m(\xi_5)m(\xi_6)m(\xi_7)m(\xi_8)} \hat{f}(\xi_1) \hat{f}(\xi_2) \cdots \hat{f}(\xi_8); \quad (256)$$

$$T_{I3} = 12i \int_{\Gamma_6} \left( \frac{\xi_3^3 m(\xi_2 + \xi_3 + \cdots + \xi_6)}{m(\xi_2)m(\xi_3) \cdots m(\xi_6)} + 5\xi_1^2 (\xi_2 + \xi_3)^2 \cdot \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \hat{f}(\xi_1) \hat{f}(\xi_2) \cdots \hat{f}(\xi_6); \quad (257)$$

$$T_{I4} = -2i \int_{\Gamma_4} \left( 2\xi_1^5 - 5\xi_1^3 (\xi_2 + \xi_3)^2 \cdot \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \right) \hat{f}(\xi_1) \hat{f}(\xi_2) \cdots \hat{f}(\xi_4). \quad (258)$$
Integrating in time for (254), it suffices to control

$$E(Iu(\delta)) - E(Iu(0)) = \int_0^\delta T_{12} + T_{13} + T_{14}. \tag{259}$$

The term $T_{12}$ is the easiest one to estimate, because there are fewer derivatives and more $u$s. On the contrary, the term $T_{14}$ is the worst one. We will consider every term in the following three propositions. Having these propositions in hand, we can complete the proof by utilizing Proposition 10.

It remains to control every term in (259). Before that, we give some observations. The same argument as $T_2 = T_3 = T_4 = 0$, using the $k$-Parseval formula and integrating by parts, show that

$$\int_{\Gamma_8} \xi_1 \hat{u}(\xi_1) \hat{u}(\xi_2) \cdots \hat{u}(\xi_8) = 0; \tag{260}$$

$$\int_{\Gamma_6} \left( \frac{\xi_1^3 + 5\xi_1(\xi_2 + \xi_3)^2}{2^{\frac{3}{2}} + 5 \xi_1^3(\xi_2 + \xi_3)^2} \right) \hat{u}(\xi_1) \hat{u}(\xi_2) \cdots \hat{u}(\xi_6) = 0; \tag{261}$$

$$\int_{\Gamma_4} \left( \frac{2\xi_1^3 - 5\xi_1^3(\xi_2 + \xi_3)^2}{2^{\frac{3}{2}} + 5 \xi_1^3(\xi_2 + \xi_3)^2} \right) \hat{u}(\xi_1) \hat{u}(\xi_2) \cdots \hat{u}(\xi_4) = 0. \tag{262}$$

**Proposition 11.** Let $3/4 \leq s < 1$ and $T_{12}$ be as in (256), then

$$|\int_0^\delta T_{12}| \lesssim N^{-5} \|Iu\|_{X^s_{\frac{1}{2} + \epsilon}}^8. \tag{263}$$

**Proof.** We divide $Iu$ into a sum of dyadic parts $P_k Iu$, whose frequency is supported on $\{\xi \mid |\xi| \sim 2^k\}, k = 0, 1, 2, \ldots$. It suffices to show that

$$\sum_{N_1, \ldots, N_8} |\int_0^\delta \int_{\Gamma_8} \xi_1 \cdot \frac{m(\xi_4 + \cdots + \xi_8)}{m(\xi_4) \cdots m(\xi_8)} \hat{u}_1(\xi_1) \cdots \hat{u}_8(\xi_8)| \lesssim N^{-5} \prod_{j=1}^8 \|u_j\|_{X^s_{\frac{1}{2} + \epsilon}}, \tag{264}$$

for any function $u_j, j = 1, 2, \ldots, 8$ with the frequency supported on $|\xi_j| \sim 2^k_j \equiv N_j, k_j \in \{0, 1, \ldots\}$. We may assume all $\hat{u}_j$s are nonnegative. In the following discussion, we will pull the symbol out of the integral, then reverse the $k$-Parseval formula and use Hölder’s inequality to estimate the remaining integrals.

Note that the derivative is located in $u_1$, and by the symmetry of the above multiplier in $\xi_4, \xi_5, \ldots, \xi_8$, we may assume that

$$N_1 \geq N_2 \geq N_3, \quad N_4 \geq N_5 \geq \cdots \geq N_8. \tag{265}$$

Let $N_{\text{max}}$ and $N_{\text{sec}}$ denote the maximum and the second maximum of the numbers $N_1, N_2, \ldots, N_8$. From the constraint condition $\Gamma_8 = \{\sum_{j=1}^8 \xi_j = 0\}$ in the above integration, we know that

$$N_{\text{max}} \sim N_{\text{sec}}.$$
Since \( m(\xi) = 1 \) for \( \xi \leq N \), (256) and (260) shows that \( T_{I2} \) vanishes when \( |\xi_4| \leq N/5 \). Thus, we may assume that

\[ N_4 \gtrsim N. \]

According to \( N_{\max} \) and \( N_{\sec} \), we split the proof into three cases: \( \{N_{\max}, N_{\sec}\} = \{N_1, N_4\} \), \( \{N_{\max}, N_{\sec}\} = \{N_4, N_5\} \), \( \{N_{\max}, N_{\sec}\} = \{N_1, N_2\} \). For the sake of brevity, we only consider the first case because the other two cases can be obtained by the same techniques.

If \( \{N_{\max}, N_{\sec}\} = \{N_1, N_4\} \), we have here \( N_1 \sim N_4 \gtrsim N \). Using local smoothing effect estimates (221) to \( u_1, u_4 \), maximal function estimates (222) to \( u_5-u_8 \), it follows that

\[
\text{LHS of (264)} \leq \sum_{N_1 \sim N_4 \gtrsim N} \sum_{N_1, \ldots, N_8} \frac{N_1 N_4^{1-s}}{N_1^{1-s} m(N_5) \cdots m(N_8)} \left\| u_1 \right\|_{L^\infty_x L^1_t} \left\| u_4 \right\|_{L^\infty_x L^2_t} \prod_{i=2}^{3} \left\| u_i \right\|_{L^\infty_x L^\infty_t} \prod_{j=5}^{8} \left\| u_j \right\|_{L^2_x L^\infty_t} \lesssim N^{3-1} \sum_{N_1 \sim N_4 \gtrsim N} \frac{N_1^{-2} N_4^{-2-s} (N_2 N_3)^{1/2} (N_5 N_6 N_7 N_8)^{-1/2}} \prod_{j=1}^{8} \left\| u_j \right\|_{X_0^{1/2} \dot{H}_x^{1/2}} \lesssim N^{-5} \prod_{j=1}^{8} \left\| u_j \right\|_{X_0^{1/2} \dot{H}_x^{1/2}},
\]

where we used Sobolev embedding and Strichart estimate to get \( \left\| u_i \right\|_{L^\infty_x L^2_t} \lesssim \left\| \langle \nabla \rangle^{1/2} u_i \right\|_{L^\infty_x L^2_t} \lesssim \left\| u_i \right\|_{X_0^{1/2} \dot{H}_x^{1/2}} \). Since \( 3/4 \leq s < 1 \), we know that \( m(N_j) N_j^{1/4} \geq 1 \) for \( j = 5, \ldots, 8 \).

**Proposition 12.** Let \( 3/4 \leq s < 1 \) and \( T_{I3} \) be as in (257), then

\[
\left| \int_0^\delta T_{I3} \right| \lesssim N^{-3} \left\| u \right\|_{X_0^{1/2} \dot{H}_x^{1/2}}^6.
\]  

**Proof.** At first, to get some vanishing properties, we insert the zero term (261) into \( T_{I3} \). Then as we discussed in the above proposition, it suffices to control

\[
I_1 := \sum_{N_1, \ldots, N_6} \left| \int_0^\delta \int_{\Gamma_6} \xi_3 \left( \frac{m(\xi_2 + \cdots + \xi_6)}{m(\xi_2) \cdots m(\xi_6)} - 1 \right) \hat{u}_1(\xi_1) \cdots \hat{u}_6(\xi_6) \right|,
\]

and

\[
I_2 := \sum_{N_1, \ldots, N_6} \left| \int_0^\delta \int_{\Gamma_6} 5\xi_1(\xi_2 + \xi_3)^2 \cdot \left( \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} - 1 \right) \hat{u}_1(\xi_1) \cdots \hat{u}_6(\xi_6) \right|,
\]

for any function \( u_j, j = 1, 2, \ldots, 6 \) with the frequency supported on \( |\xi_j| \sim N_j \).
For term $I_1$, by the symmetry of the multiplier in $\xi_2, \xi_3, \ldots, \xi_6$, we can assume that

$$N_2 \geq N_3 \geq \cdots \geq N_6, \quad N_2 \geq N,$$

(267)

where we use that if $N_2 \leq N/10$, $I_1$ vanishes. Because of $N_{\text{max}} \sim N_{\text{sec}}$, we divide the proof into two cases: $N_1 \sim N_2 \geq N$ and $N_2 \sim N_3 \geq N$, $N_3 \geq N_1$.

If $N_1 \sim N_2 \geq N$, we take local smoothing effect estimates (221) to $u_1, u_2$, maximal function estimates (222) to $u_3 - u_6$, and then get that

$$I_1 \lesssim \sum_{N_1 \sim N_2 \geq N, N_1 \cdots N_6} \frac{N_3^3 m(N_1)}{m(N_2) \cdots m(N_6)} \|u_1\|_{L^6_x L^2} \|u_2\|_{L^6_x L^2} \|u_3\|_{L^6_x L^2} \prod_{j=3}^6 \|u_j\|_{L^4_x L^{\infty}_t} \tag{268}$$

$$\lesssim \sum_{N_1 \sim N_2 \geq N, N_1 \cdots N_6} \frac{N_3^3}{m(N_3) \cdots m(N_6)} \|u_1\|_{X^{-\frac{1}{2}+}_N} \|u_2\|_{X^{-\frac{1}{2}+}_N} \|u_3\|_{X^{-\frac{1}{2}+}_N} \prod_{j=3}^6 \|u_j\|_{X^{\frac{1}{2}+}_N} \tag{269}$$

If $N_2 \sim N_3 \geq N, N_3 \geq N_1$, we can get similarly that

$$I_1 \lesssim \sum_{N_2 \sim N_3 \geq N, N_1 \cdots N_6} \frac{N_3^3 m(N_1)}{m(N_2) \cdots m(N_6)} \|u_2\|_{L^6_x L^2} \|u_3\|_{L^6_x L^2} \|u_1\|_{L^6_x L^2} \prod_{j=4}^6 \|u_j\|_{L^4_x L^{\infty}_t} \tag{270}$$

$$\lesssim N^{1-s} \sum_{N_1 \sim N_2 \sim N_3 \geq N, N_1 \cdots N_6} \frac{N_3^{3+2}}{m(N_2) \cdots m(N_6)} \|u_1\|_{X^{-\frac{1}{2}+}_N} \|u_2\|_{X^{-\frac{1}{2}+}_N} \|u_3\|_{X^{-\frac{1}{2}+}_N} \prod_{j=4}^6 \|u_j\|_{X^{\frac{1}{2}+}_N} \tag{271}$$

$$\lesssim N^{1-s} \sum_{N_1 \sim N_2 \sim N_3 \geq N, N_1 \cdots N_6} N_1^{3+\frac{5}{4}} N_2^{-\frac{11}{4}} N_3^{-\frac{11}{4}} (N_4 N_5 N_6)^{-\frac{1}{2}} \prod_{j=1}^6 \|u_j\|_{X^{\frac{1}{2}+}_N} \tag{272}$$

For term $I_2$, the same argument as before, we may assume that

$$N_1 \geq N_5 \geq N_6, \quad N_2 \geq N_3 \geq N_4, \quad N_2 \geq N.$$

(270)

If $N_1 \sim N_2 \geq N$, the symbol is controlled by $N_3^3 m(N_1)/m(N_2) m(N_3) m(N_4)$, which is smaller than $N_1^3 m(N_1)/m(N_2) \cdots m(N_6)$; therefore, we can get the same bound as (268). If $N_2 \sim N_3 \geq N$, using the same techniques as (269) yields the estimates. If $N_1 \sim N_5 \geq N$, this case is better, thus we can obtain the conclusion as before.
Proposition 13. Let $3/4 \leq s < 1$ and $T_{14}$ be as in (272), then

$$\left| \int_{0}^{\delta} T_{14} \right| \lesssim N^{-\frac{s}{4}} \left\| Iu \right\|^4_{X^{1,\frac{1}{2}}_{s}}.$$  \hspace{1cm} (271)

Proof. As the above proposition, inserting the zero term (262) into $T_{14}$, it suffices to control

$$\sum_{N_1,\ldots,N_4} \left| \int_{0}^{\delta} \int_{\Gamma_{4}} \xi_1^3 (\xi_2 + \xi_3)^2 \cdot \left( \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} - 1 \right) \hat{u}_1(\xi) \cdots \hat{u}_4(\xi) \right|,$$  \hspace{1cm} (272)

for any function $u_j, j = 1, 2, 3, 4$ with the frequency supported on $|\xi_j| \sim N_j$. A glance at the symbol, we may assume that

$$N_2 \geq N_3 \geq N_4, \quad N_2 \gtrsim N,$$  \hspace{1cm} (273)

since that if $N_2 \leq N/8$, the symbol vanishes. The other orders of $N_2, N_3,$ and $N_4$ are similar and much easier. Then we control the above summation by considering the following three cases:

1. the nonresonant case, where $N_2 \gg N_3 \geq N_4$;
2. the semiresonant case, where $N_2 \sim N_3 \gg N_4$;
3. the resonant case, where $N_2 \sim N_3 \sim N_4$.

1. In the nonresonant case, from the constraint condition $\Gamma_4 = \{ \sum_{j=1}^{4} \xi_j = 0 \}$, it follows immediately that $N_1 \sim N_2 \gg N_3 \geq N_4$. Utilizing bilinear estimates (223) can efficiently control (272) by

$$\sum_{N_1,N_2,N_3,N_4} \frac{N_1^3 N_2^2 m(N_1)}{m(N_2)m(N_3)m(N_4)} \left\| u_1 u_3 \right\|_{L^2_{x,t}} \left\| u_2 u_4 \right\|_{L^2_{x,t}} \leq \sum_{N_1,N_2,N_3,N_4} \frac{N_1}{m(N_3)m(N_4)} \prod_{j=1}^{4} \left\| u_j \right\|_{X^{1/2}_{s}} \lesssim N^{-1} \prod_{j=1}^{4} \left\| u_j \right\|_{X^{1/2}_{s}}.$$  \hspace{1cm} (274)

2. In the semiresonant case, it is easy to see that $N_2 \sim N_3 \gg N_4 \sim N_1$. Bilinear estimates (223) imply that (272) is controlled by

$$\sum_{N_2,N_3,N_4} \frac{N_1^3 N_2^2 m(N_1)}{m(N_2)m(N_3)m(N_4)} \left\| u_1 u_3 \right\|_{L^2_{x,t}} \left\| u_2 u_4 \right\|_{L^2_{x,t}}$$
\[
\lesssim \sum_{N_2 \sim N_3 \gg N_4} \frac{N_2^2 N_3^{-2}}{m(N_2)m(N_3)} \prod_{j=1}^{4} \| u_j \|_{X_{\frac{3}{5}, \frac{3}{2}+}}^4.
\]

3. In the resonant case, we have \( N_1 \sim N_2 \sim N_3 \sim N_4 \). Strichartz estimates \((220)\) yield that
\[
\lesssim \delta \sum_{N_1 \sim N_2 \sim N_3 \sim N_4} \frac{N_1^2 N_2^4 N_3^{-11}}{m(N_2)m(N_3)m(N_4)} \prod_{j=1}^{4} \| u_j \|_{X_{\frac{3}{5}, \frac{3}{2}+}}^4 \lesssim \delta N^{-\frac{5}{2}} \prod_{j=1}^{4} \| u_j \|_{X_{\frac{3}{5}, \frac{3}{2}+}}^4.
\]

We now complete the proof. \(\blacksquare\)

### 5.4 | Global approximation in low regularity spaces

In this subsection, we extend the long-time asymptotic behavior to the low regularity data. At the beginning, we show that the Beals–Coifman solution is equal to the solution given by \((29)\).

**Lemma 12.** Let \( u_0 \in H^{4,1}(\mathbb{R}) \), the Beals–Coifman solution and the solution given by \((29)\) are the same (up to a measure zero set)

\[
 u = \frac{1}{\pi} \left( \int \mu(w_+ + w_-) \right)_{12} = e^{-\delta \tau} u_0 - \int_0^t e^{-(t-t')} \delta \tau F(u(t')) \, dt'
\]

in \([-T, T]\) where \( T \) is given as in Theorem 4.

**Proof.** For \( u_0 \in H^{4,1}(\mathbb{R}) \), there exists a Cauchy sequence \( u_{0,k} \in S(\mathbb{R}) \), such that

\[
\lim_{k \to \infty} \| u_{0,k} - u_0 \|_{H^{4,1}} = 0, \quad \text{and} \quad \| u_{0,k} \|_{H^{4,1}} \leq \| u_{0,k} \|_{H^{4,1}} \leq C, \quad \forall k \in \mathbb{N}^+.
\]

On the one hand, from Theorem 4 and the Lipschitz continuity \((228)\), there exists a solution \( u_k \) with initial data \( u_{0,k} \) such that for \( b > 1/2 \),

\[
\| u_k - u_l \|_{X^{4,b}} \leq \| u_{0,k} - u_{0,l} \|_{H^{4,1}} \to 0 \quad \text{as} \quad k, l \to \infty,
\]
Therefore, there exists \( u_\infty \) such that
\[
\sup_{t \in [-T, T]} \| u_k - u_\infty \|_{L^\infty} \lesssim \sup_{t \in [-T, T]} \| u_k - u_\infty \|_{H^{1,4}} \lesssim \| u_k - u_\infty \|_{X^{4, b}} \to 0, \quad \text{as} \ k \to \infty. \tag{277}
\]

On the other hand, from the inverse scattering transform, we have the Beals–Coifman solutions \( \tilde{u}_k \) with initial data \( u_{0,k} \). By utilizing the bijectivity of the transformation and bi-Lipschitz continuity, we know reflection coefficients satisfy
\[
\| r_k - r_l \|_{H^1} \lesssim \| u_{0,k} - u_{0,l} \|_{H^{4,1}} \to 0 \quad \text{as} \ k, l \to \infty.
\]

By the resolvent estimates, that is, \( \|(1 - C_w)^{-1}\|_{L^2} \leq 1 / (1 - \| r \|_{L^\infty}) \) (see Refs. 24 and 25),
\[
\| \tilde{u}_k - \tilde{u}_l \|_{L^\infty} \lesssim \| r_k - r_l \|_{H^1} \to 0 \quad \text{as} \ k, l \to \infty.
\]

Therefore, there exists \( \tilde{u}_\infty \) such that
\[
\tilde{u}_\infty = \lim_{k \to \infty} \tilde{u}_k \tag{278}
\]
in the \( L^\infty \) sense. Since \( u_{0,k} \in S(\mathbb{R}) \), \( u_k \) and \( \tilde{u}_k \) are also Schwartz functions, then \( u_k = \tilde{u}_k \). Therefore, we can get the conclusion from (277) and (278).

In the end, we divide the proof of Theorem 3 into the following two theorems. We first consider the long-time behavior in the Sobolev space \( H^{1,1}(\mathbb{R}) \). It is well known that \( u(\cdot, t) \) is uniformly bounded in \( H^1(\mathbb{R}) \). Indeed, from the mass conservation and energy conservation, we know that
\[
\| u \|_{H^1(\mathbb{R})} = \| u \|_{L^2} + \| u \|_{H^1} \leq \| u_0 \|_{L^2} + \| \partial_x u_0 \|_{L^2} + \| u_0 \|_{L^4}^2.
\]

From the Gagliardo–Nirenberg’s inequality and Young’s inequality,
\[
\| u_0 \|_{L^4}^2 \lesssim \| \partial_x u_0 \|_{L^2}^{1/2} \| u_0 \|_{L^2}^{3/2} \lesssim \frac{1}{\varepsilon} \| \partial_x u_0 \|_{L^2} + \varepsilon \| u_0 \|_{L^2}^3.
\]

We may choose \( \varepsilon \) satisfying \( \varepsilon \| u_0 \|_{L^2}^2 \leq 1 \) and get that
\[
\| u(\cdot, t) \|_{H^1} \lesssim \| u_0 \|_{H^1}. \tag{279}
\]

**Theorem 5.** Let \( u_0 \in H^{1,1}(\mathbb{R}) \), the solution given by the integral form (29) has the same asymptotic behavior as in Theorem 1.

**Proof.** For \( u_0 \in H^{1,1}(\mathbb{R}) \), there exists a sequence \( u_{0,k} \in H^{4,1}(\mathbb{R}) \), such that
\[
\lim_{k \to \infty} \| u_{0,k} - u_0 \|_{H^{1,1}} = 0, \quad \text{and} \quad \| u_{0,k} \|_{H^1} \leq \| u_{0,k} \|_{H^{1,1}} \leq C, \quad \forall k \in \mathbb{N}^+.
\]
Following the same strategy as in the proof of Lemma 12, on the one hand, there exists a solution \( u_k \) with initial data \( u_{0,k} \) from Theorem 4 and the Lipschitz continuity (228) such that for \( b > 1/2 \),
\[
\|u_k - u_l\|_{X^1,b} \lesssim \|u_{0,k} - u_{0,l}\|_{H^1} \to 0 \quad \text{as} \quad k, l \to \infty,
\]
Therefore, there exists \( u_{\infty} \) such that
\[
\sup_{t \in [-T,T]} \|u_k - u_{\infty}\|_{L^\infty} \lesssim \sup_{t \in [-T,T]} \|u_k - u_{\infty}\|_{H^1} \lesssim \|u_k - u_{\infty}\|_{X^1,b} \to 0, \quad \text{as} \quad k \to \infty.
\]
On the other hand, from Lemma 12, the Beals–Coifman solutions \( \tilde{u}_k \) with initial data \( u_{0,k} \) are equal to \( u_k \). By utilizing the bijectivity of the transformation, bi-Lipschitz continuity, and the resolvent estimates, we know that
\[
\|\tilde{u}_k - \tilde{u}_l\|_{L^\infty} \lesssim \|r_k - r_l\|_{H^1} \lesssim \|u_{0,k} - u_{0,l}\|_{H^{1,1}} \to 0 \quad \text{as} \quad k, l \to \infty.
\]
Therefore, there exists \( \tilde{u}_{\infty} \) such that
\[
\tilde{u}_{\infty} = \lim_{k \to \infty} \tilde{u}_k,
\]
in the \( L^\infty \) sense. Therefore, one has \( u_{\infty} = \tilde{u}_{\infty} \) pointwise for \( t \in [-T,T] \). Since \( u(\cdot, t) \) is uniformly bounded in \( H^1(\mathbb{R}) \) as (279), one can repeat the above argument many times to extend the time interval to whole line \( \mathbb{R} \). Since every \( \tilde{u}_k \) satisfies the asymptotic formulas in Theorem 1 obtained from the nonlinear steepest descent with uniform error terms estimates, which are independent of \( k \), then \( \tilde{u}_{\infty} \) has asymptotic behavior as in Theorem 1, \( u_{\infty} \) also does.

Theorem 6. Let \( u_0 \in H^{s,1}(\mathbb{R}) \), \( 19/22 < s < 1 \), the solution given by the integral form (29) has the same asymptotic behavior as in Theorem 1.

Proof. For \( u_0 \in H^{s,1}(\mathbb{R}) \), \( 19/22 < s < 1 \), there exists a sequence \( u_{0,k} \in H^{4,1}(\mathbb{R}) \), such that
\[
\lim_{k \to \infty} \|u_{0,k} - u_0\|_{H^{s,1}} = 0, \quad \text{and} \quad \|u_{0,k}\|_{H^s} \leq \|u_{0,k}\|_{H^{s,1}} \leq C, \quad \forall k \in \mathbb{N}^+.
\]
On the one hand, similar to Lemma 12 and Theorem 5, there exists a solution \( u_k \) with initial data \( u_{0,k} \) such that for \( b > 1/2 \),
\[
\|u_k - u_l\|_{X^1,b} \lesssim \|u_{0,k} - u_{0,l}\|_{H^1} \to 0 \quad \text{as} \quad k, l \to \infty,
\]
Therefore, there exists \( u_{\infty} \) satisfying the integral form (29) such that
\[
\sup_{t \in [-T,T]} \|u_k - u_{\infty}\|_{L^\infty} \lesssim \sup_{t \in [-T,T]} \|u_k - u_{\infty}\|_{H^1} \lesssim \|u_k - u_{\infty}\|_{X^1,b} \to 0, \quad \text{as} \quad k \to \infty.
\]
On the other hand, by utilizing the bijectivity of the transformation and bi-Lipschitz continuity results of Theorem 3.2 in Deift–Zhou,\(^{25}\) we know that
\[
\|r_k - r_l\|_{H^1} \lesssim \|u_{0,k} - u_{0,l}\|_{H^{s,1}} \to 0 \quad \text{as} \quad k, l \to \infty.
\]
As before, by the resolvent estimates,
\[
\|\tilde{u}_k - \tilde{u}_l\|_{L^\infty} \lesssim \|r_k - r_l\|_{H^1} \to 0 \quad \text{as} \quad k, l \to \infty.
\]
Therefore, there exists \(\tilde{u}_\infty\) such that
\[
\tilde{u}_\infty = \lim_{k \to \infty} \tilde{u}_k,
\]
in the \(L^\infty\) sense. Lemma 12 shows that \(\tilde{u}_k = u_k\), so we obtain that \(u_\infty = \tilde{u}_\infty\) pointwise for \(t \in [-T, T]\).

From the global well-posedness theory Theorem 2, \(u_\infty\) exists in \(H^s\) globally. One can also construct \(\tilde{u}_\infty\) globally. Suppose that \(u_\infty = \tilde{u}_\infty\) does not hold for all \(t\). We may assume
\[
T_* = \inf \{t \geq 0; u_\infty(t) \neq \tilde{u}_\infty(t)\}.
\]
(280)

From (235), we know that
\[
\|u(\cdot, T_*)\|_{H^s} \leq C_1 T_*^M + C_2,
\]
(281)
then as in the proof of Theorem 4, there exists \(t_0 > 0\) such that
\[
\sup_{t \in [0, T_* + t_0]} \|u_k - u_\infty\|_{L^\infty} \lesssim \|u_k - u_\infty\|_{X^{s,b}} \to 0, \quad \text{as} \quad k \to \infty.
\]
Combining
\[
\sup_{t \in [0, T_* + t_0]} \|\tilde{u}_k - \tilde{u}_\infty\|_{L^\infty} \to 0, \quad \text{as} \quad k \to \infty, \quad \text{and} \quad u_k = \tilde{u}_k,
\]
we can conclude that \(u_\infty(t) = \tilde{u}_\infty(t)\) pointwise for \(t \in [0, T_* + t_0]\), this is a contradiction with (280). Hence we can obtain that \(u_\infty(t) = \tilde{u}_\infty(t)\) pointwise for all \(t \geq 0\). Since \(\tilde{u}_\infty\) is the pointwise limit of \(\tilde{u}_k\) which has the asymptotic behavior as in Theorem 1 with uniform error terms estimates that are independent of \(k\), \(\tilde{u}_\infty\) also has the same asymptotic behavior as in Theorem 1. Therefore, \(u_\infty\) also has the same asymptotic behavior as desired. ■

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APPENDIX A
In the appendix, we present the RH problem related to the fourth-order Painlevé II equation and give a model RH problem in Region IV. The proof can be derived similarly as in Ref. 23 by Liu and Guo of the present authors and other coauthors. One also can refer to Refs. 40 and 56.

A.1 | Fourth-order Painlevé II RH problem
Let $P$ denote the contour $P = P_1 \cup P_2$ oriented as in Figure A1, where

$$P_1 = \{le^{\pi i} | l \geq 0\} \cup \{le^{5\pi i} | l \geq 0\}, \quad P_2 = \{le^{-\pi i} | l \geq 0\} \cup \{le^{-5\pi i} | l \geq 0\}.$$ 

Fourth-order Painlevé II RH problem A.1.Let $s \in \mathbb{R}$ be a real number with $|s| < 1$. Find an analytic function $M^P(y; z)$ in $\mathbb{C} \setminus P$ parameterized by $y \in \mathbb{R}$, $s \in \mathbb{R}$ such that

1. For $z \in P$, the continuous boundary values $M^P_{\pm}(y; z)$ satisfy

$$M^P_{+}(y; z) = M^P_{-}(y; z)v^P(s, y; z), \quad z \in P,$$

where the jump matrix $v^P(s, y; z)$ is defined by

$$v^P(s, y; z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
se^{2i\left(\frac{4}{5}z^{5/3} - yz\right)} & 1 \end{pmatrix}, & z \in P_1, \\
\begin{pmatrix} 1 & 0 \\
-se^{-2i\left(\frac{4}{5}z^{5/3} - yz\right)} & 1 \end{pmatrix}, & z \in P_2.
\end{cases}$$

FIGURE A1 The oriented contour $P$
Lemma A1. The RH problem A.1 has a unique solution $M^p(y; z)$ for each $y \in \mathbb{R}$. Moreover, there exist smooth functions $\{M_j^p(y)\}_{j=1}^4$ of $y \in \mathbb{R}$ with decay as $y \to -\infty$ such that

$$M^p(y; z) = I + \sum_{j=1}^4 \frac{M_j^p(y)}{z^j} + O(z^{-5}), \quad z \to \infty, \quad \text{(A3)}$$

uniformly for $y$ in compact subsets of $\mathbb{R}$ and for $\arg z \in [0, 2\pi]$. The leading coefficient $M_1^p$ is given by

$$M_1^p(y) = i \begin{pmatrix} -2 \int_{-\infty}^y u_p^2(y') dy' & u_p(y) \\ -u_p(y) & 2 \int_{-\infty}^y u_p^2(y') dy' \end{pmatrix}, \quad \text{(A4)}$$

where the real-valued function $u_p(y)$ satisfies the following fourth-order Painlevé II equation (see Ref. 57):

$$u'''_p(y) - 40u_p^2(y)u''_p(y) - 40u_p(y)u'^2_p(y) + 96u_p^5(y) - 4yu_p(y) = 0. \quad \text{(A5)}$$

A.2 | Model RH problem in Region IV

Given a number $z_0 \geq 0$, let $Z$ denote the contour $Z = Z_1 \cup Z_2 \cup Z_3$, where the line segments

$$Z_1 = \left\{ z_0 + le^{\frac{\pi i}{6}} | l \geq 0 \right\} \cup \left\{ -z_0 + le^{\frac{5\pi i}{6}} | l \geq 0 \right\},$$

$$Z_2 = \left\{ z_0 + le^{-\frac{\pi i}{6}} | l \geq 0 \right\} \cup \left\{ -z_0 + le^{-\frac{5\pi i}{6}} | l \geq 0 \right\},$$

$$Z_3 = \left\{ |l| - z_0 \leq l \leq z_0 \right\} \quad \text{(A6)}$$

are oriented as in Figure A2.

Model RH problem A.2. Find an analytic function $M^Z(y; z, z_0)$ in $\mathbb{C} \setminus Z$ parameterized by $y > 0, s \in \mathbb{R}, |s| < 1, z_0 \geq 0$ such that:
1. The continuous boundary values $M_{\pm}^Z(y; z, z_0)$ satisfy the jump condition

$$M_{+}^Z(y; z, z_0) = M_{-}^Z(y; z, z_0)v^Z(s, y; z, z_0), \quad z \in Z,$$

where the jump matrix $v^Z(s, y; z, z_0)$ is defined by

$$v^Z(s, y; z, z_0) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\ se^{2i\left(\frac{4}{5}z^5 - yz\right)} & 1 \end{pmatrix}, & z \in Z_1, \\
\begin{pmatrix} 1 & -se^{-2i\left(\frac{4}{5}z^5 - yz\right)} \\ 0 & 1 \end{pmatrix}, & z \in Z_2, \\
\begin{pmatrix} 1 & 0 \\ se^{2i\left(\frac{4}{5}z^5 - yz\right)} & 1 \end{pmatrix}, & z \in Z_3.
\end{cases}$$

(A7)

2. As $z \to \infty$, $M^Z(y; z, z_0) = I + O\left(\frac{1}{z}\right)$.

Lemma A2. Define the parameter subset

$$\mathbb{P} = \{(y, t, z_0) \in \mathbb{R}^3 | 0 < y < C_1, t \geq 1, \sqrt[3]{y}/\sqrt{2} \leq z_0 \leq C_2\},$$

(A9)

where $C_1, C_2 > 0$ are constants. Then for $(y, t, z_0) \in \mathbb{P}$, the RH problem A.2 has a unique solution $M^Z(s, y; z, z_0)$ which satisfies

$$M^Z(y; z, z_0) = I + \frac{i}{z} \begin{pmatrix} \int_{-\infty}^{y} u_p'(y')dy' & -u_p(y) \\ 2 \int_{-\infty}^{y} u_p^2(y')dy' & 1 \end{pmatrix} + O\left(\frac{1}{z^2}\right), \quad z \to \infty,$$

(A10)

where $u_p(y)$ denotes the solution of the fourth-order Painlevé II equation (A5) and $M^Z(y, z, z_0)$ is uniformly bounded for $z \in \mathbb{C} \setminus Z$. Furthermore, $M^Z$ obeys the symmetries

$$M^Z(y; -z, z_0) = \overline{M^Z(y; z, z_0)} = \sigma_1 M^Z(y; z, z_0)\sigma_1.$$

(A11)