Expectation Value of $\sigma(1)\sigma(2)$-Wave Functions
Don’t Matter

Larry Zamick

Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08854

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Abstract

We consider the expectation value of the quantity $[3+ \sigma(1)\sigma(2)]/4$. This has a value +1 for 2 nucleons with spin $S=1$ and zero for $S=0$. We show that for the jj coupling 2 particle configuration $[j(1) j(2)]$ the expectation value has the structure $A+B J(J+1)$ where $A$ and $B$ are constants. We then show that for a 2proton-2neutron configuration with total angular momentum $I$ the expectation value per pair is independent of the details of the wave function and has a similar structure $A’ +B’ I(I+1)$ with $B’=B/6$.

1 Introduction

In this work we wish to study the spin contents of nuclear wave function in a single $j$ shell model space. For convenience we use the spin one projection operator $[3+ \sigma(1)\sigma(2)]/4$ which has a value of one for two particles coupled to $S=1$ and zero for 2 particles coupled to $S=0$. However it does not make any difference what linear combination of a constant and a term proportional to $\sigma(1)\sigma(2)$ one takes.

2 Two particles in jj coupling

We consider, in jj coupling, the two particle expectation values of the spin one projection operator: $P_1(J)= \langle jj| [3+ \sigma(1)\sigma(2)]/4 \rangle (jj) J$. Here $j$ is an abbreviation for $(l 1/2)j$. As mentioned above this interaction is non vanishing only for $S=1$. The total angular momentum is $J$. We can use the unitary $9j$ coefficients to evaluate this matrix element. The expression is

$$
P_1(J)= \sum L |\langle l, 1/2|l, 1/2\rangle| (l l 1/2, 1.2)^{(S=1)}> J |^2
$$

We here give the results for $j= f_{7/2}$. The values $P_1(J)$ from $J=0$ to $J=7$ are shown in Table I.
| J | Spin Expectation |
|---|------------------|
| 0 | 0.42857          |
| 1 | 0.44898          |
| 2 | 0.48980          |
| 3 | 0.55101          |
| 4 | 0.63263          |
| 5 | 0.73469          |
| 6 | 0.85714          |
| 7 | 1.00000          |

Note that for J=7 we have L=6 and S=1, i.e. it is a pure S=1 state so we are not surprised that P1(7)=1.

We can rewrite this as P1(J) = 0.42857 + 0.010204 J (J+1).

We can obtain these results in a simpler way. In a single j shell we can replace σ by g_j. For j=L+1/2 we have g_j=1/j , whilst for j=L-1/2 g_j = -1/(j+1).

Thus, for j= L+1/2, σ(1).σ(2) is replaced by j(1).j(2)/j^2.

Now j(1).j(2) is equal to [j(1)+j(2)]^2/2 - j(j+1) = J(J+1)/2 - j(j+1). Putting this all together we have for j=l+1/2

P1(J)= 3/4-(j+1)/4j +J(J+1)/(8j^2).

3 A system of 2 protons and 2 neutrons (^{44}_{Ti}).

We next consider the expectation value of P1(J) for a system of 2 protons and 2 neutrons i.e. ^{44}_{Ti}.

The J=0^+ wave functions for 2 protons and 2 neutrons in ^{44}_{Ti} with the MBZE interaction [1] are given in the appendix. For any four particle angular momentum I they are of the form

\[ \sum D^I(J_P J_N) \left[ \langle jj \rangle^J \langle jj \rangle^J \right]^I \]

Here D(J_P J_N) is the probability amplitude that the 2 protons couple to J_P and the 2 neutrons to J_N. The normalization condition is:

\[ \sum |D^I(J_P J_N)|^2 = 1 \]

The expectation value of \[3 + \sigma(i).\sigma(j)/4\] for J=0^+ states is given, per pair (there are 6 pairs) by

EXV = 1/6 (C1+C2)

C1 = \[\sum_{J_P J_N} |D^I(J_P J_N)|^2 (P1(J_P) + P1(J_N)) \]

C2 = 4.0*\[\sum_{J_A} F(J_A) \times P1(J_A) \]

where F(J_A) = \[\sum_{J_B} \sum_{J_N} \langle jj \rangle^{J_B} \langle jj \rangle^{J_N} |(jj)^{J_B} \langle jj \rangle^{J_N} \rangle^I \times D^I(J_P J_N) \]

We have what is an initially surprising result. The expectation per pair that S=1 is independent of the values of D^I(J_P J_N) . The value in fact is 0.643 for all 4 states. Thus the J=0^+ have more S=1 than S=0 but one cannot talk of correlations since the results do not depend on the details of the wave functions.

It should be noted that although EXV does not depend on the wave function components the quantities C1 and C2 do. This is shown in Table II.

Table II The spin expectation for J=0^+ per pair.
We should compare this with the value of \( S \) for a single nucleon. For \( j = L + 1/2 \) we have \( S = 1 \). Thus the results for \( I = 0^+ \) represent some spin suppression.

We have repeated the calculations for states with higher angular momentum. We find again that for any given the wave functions don’t matter. The expectation of \( P_1(I) \) does depend on \( I \).

## 4 Simpler Considerations

In the above we addressed the problem of the expectation value of the spin operator. The essence of the problem, however, can be dealt with more simply if we just take the expectation of \( J(J+1) \) where \( J \) is the 2 particle angular momentum e.g. 0 for \( J = 0,1^*2 \) for \( J = 1,2^*3 \) for \( J = 2 \) e.t.c. It turns out that for any \( J = 0^+ \) state in \(^{44}\text{Ti}\) the non-normalized expectation value is 126, which is the same as \( 8j(j+1) \) (with \( j = 7/2 \) in this case). The normalized value is 21. For a 4 particle state of angular momentum \( I \) the expectation value is simply 126 + \( I(I+1) \).

We can see better what is happening if we treat \( J(J+1) \) as 2-body interaction and perform a matrix diagonalization. The we find that all states of a given \( I \) are degenerate and the spectrum in the 4 particle sytem is the same as for the 2 particle system, namely \( E(I) = I(I+1) + \text{constant} \). The multi-degeneracy can easily be explained by the fact that the 2-body \( JJ \) interaction between basis states \((J_P, J_N)\) and \((J_P', J_N')\) vanishes unless \( J_P = J_P' \) and \( J_N = J_N' \). In more detail:

The interaction summed over all pairs is \( 1/2 \sum J(t) \cdot \sum J(q) - 1/2 \sum J(t)^2 \). The second term is a constant and can be ignored as far as mixing is concerned. The first term, acting on a 4 particle state of total angular momentum \( I \) yields an eigenvalue \( I(I+1) \). Clearly one will not get off diagonal matrix elements from this term either. This leads to the multi-degenerate states of a given angular momentum. The eigenvalues have the structure \( I(I+1) + \text{constant} \).

If we diagonalize rather \( P_1(J) \) and divide by the number of pairs (in this case 6) we get the spin probability per pair. The maximum possible value of this quantity is one. We show the results in Table III.

### Table III

|       | State 1 | State 2 | State 3 (T=2) | State 4 |
|-------|---------|---------|---------------|---------|
| C1/6  | 0.488   | 0.452   | 0.428         | 0.400   |
| C2/6  | 0.155   | 0.191   | 0.215         | 0.243   |
| Total | 0.643   | 0.643   | 0.643         | 0.643   |
| Pair Spin Prob. |
|----------------|
| 0  | 0.6428 |
| 1  | 0.6462 |
| 2  | 0.6530 |
| 3  | 0.6632 |
| 4  | 0.6768 |
| 5  | 0.6938 |
| 6  | 0.7141 |
| 7  | 0.7380 |
| 8  | 0.7652 |
| 9  | 0.7959 |
| 10 | 0.8446 |
| 11 | 0.8680 |
| 12 | 0.9081 |

Note that whereas with 2 nucleons the pair spin expectation value is $A + B \sqrt{J(J+1)}$, whilst with 2 protons and 2 neutrons it is $A' + B' \sqrt{I(I+1)}$ with $B' = B/6$. The values of $A, A', B$ and $B'$ are respectively 0.4286, 0.6428, 0.0102 and 0.0034.

5 Conclusions

In talking about the amount of $S=0$ and $S=1$ content in nuclear wave functions one must take care to separate results that do depend on correlations and those that do not. One often hears the phrases “$S=0 \ J=0$ pairing” and “$S=1 \ T=0$ pairing” which imply correlation dependent results. In the context in which these phrases are used they may well be justified. But one should make sure that this is the case. In this work we offer a counterpoint in which the results do not depend on the detailed wave functions. Our result 0.643, as the pair probability of $S=1$ in any $J=0^+$ in the $f_{7/2}$ shell model space, is the same for all 4 $J=0^+$ states, 3 with isospin zero and one with isospin 2. That the pair spin probability is independent of the wave functions in our single $j$ shell model space is true for all $I$. Not surprisingly this quantity increases with increasing $I$ as $0.6428 + 0.0102 I(I+1)$. For $I=12^+$ we come close to the maximum value of one.

The value of this work is that it sets up a base for comparison. If one wants to talk about $S=1$ correlations in a nucleus one should compare the results with those here shown in which the results do not depend on the wave functions. The fact that the expectation of the spin projection operator is here greater than 0.5 is simply due to the fact that we are in a $j=L+1/2$ model space and is not due to any subtle correlations.
6 Appendix

In Table III we give the J=0⁺ wave functions in $^{44}$Ti with the MBZE interaction [1].

| Energy (MeV) | $J^P_N$ | $J^P_N$ | $J^P_N$ | $J^P_N$ |
|-------------|---------|---------|---------|---------|
| 0.00000     | 0.78776 | -0.35240 | -0.50000 | 0.07248 |
| 5.58610     | 0.56165 | 0.73700  | 0.37268  | 0.04988 |
| 8.28402**   | 0.72080 | -0.37028 | 0.50000  | -0.75109|
| 8.78750     | 0.12340 | -0.44219 | 0.60092  | 0.65431 |

The 3rd state has isospin $T=2$, all the others $T=0$.

References

[1] A. Escuderos, L. Zamick and B.F. Bayman, [arXiv:nucl-th/0506050](http://arxiv.org/abs/nucl-th/0506050).