ON REGULARIZATION OF MELLIN PDO’S WITH SLOWLY OSCILLATING SYMBOLS OF LIMITED SMOOTHNESS

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Abstract
We study Mellin pseudodifferential operators (shortly, Mellin PDO’s) with symbols in the algebra $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ of slowly oscillating functions of limited smoothness introduced in [12]. We show that if $a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ does not degenerate on the “boundary” of $\mathbb{R}_+ \times \mathbb{R}$ in a certain sense, then the Mellin PDO $\text{Op}(a)$ is Fredholm on the space $L^p$ for $p \in (1, \infty)$ and each its regularizer is of the form $\text{Op}(b) + K$ where $K$ is a compact operator on $L^p$ and $b$ is a certain explicitly constructed function in the same algebra $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that $b = 1/a$ on the “boundary” of $\mathbb{R}_+ \times \mathbb{R}$. This result complements a known Fredholm criterion from [12] for Mellin PDO’s with symbols in the closure of $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$.

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1 Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space $X$, and let $\mathcal{K}(X)$ be the ideal of all compact operators in $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is called Fredholm if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number

$$\text{Ind} A := \dim \ker A - \dim \ker A^*$$

is referred to as the index of $A$ (see, e.g., [11 Sections 1.11–1.12], [3, Chap. 4]). For bounded linear operators $A$ and $B$, we will write $A \simeq B$ if $A - B \in \mathcal{K}(X)$.

Recall that an operator $B_r \in \mathcal{B}(X)$ (resp. $B_l \in \mathcal{B}(X)$) is said to be a right (resp. left) regularizer for $A$ if

$$AB_r \simeq I \quad \text{(resp.} B_l A \simeq I).$$

It is well known that the operator $A$ is Fredholm on $X$ if and only if it admits simultaneously a right and a left regularizers. Moreover, each right regularizer differs from each left regularizer by a compact operator (see, e.g., [3, Chap. 4, Section 7]). Therefore we may speak of a regularizer $B = B_r = B_l$ of $A$ and two different regularizers of $A$ differ from each other by a compact operator.

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$\mathcal{M} : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R}), \quad (\mathcal{M}f)(x) := \int_{\mathbb{R}_+} f(t)t^{-ix} \frac{dt}{t}.$$  

It is an invertible operator, with inverse given by

$$\mathcal{M}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+, d\mu), \quad (\mathcal{M}^{-1}g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} g(x)t^ix \, dx.$$

For $1 < p < \infty$, let $\mathcal{M}_p$ denote the Banach algebra of all Mellin multipliers, that is, the set of all functions $a \in L^{\infty}(\mathbb{R})$ such that $\mathcal{M}^{-1}aMf \in L^p(\mathbb{R}_+, d\mu)$ and

$$\|\mathcal{M}^{-1}aMf\|_{L^p(\mathbb{R}_+, d\mu)} \leq c_p\|f\|_{L^p(\mathbb{R}_+, d\mu)} \quad \text{for all} \quad f \in L^2(\mathbb{R}_+, d\mu) \cap L^p(\mathbb{R}_+, d\mu).$$

If $a \in \mathcal{M}_p$, then the operator $f \mapsto \mathcal{M}^{-1}aMf$ defined initially on $L^2(\mathbb{R}_+, d\mu) \cap L^p(\mathbb{R}_+, d\mu)$ extends to a bounded operator on $L^p(\mathbb{R}_+, d\mu)$. This operator is called the Mellin convolution operator with symbol $a$.

Mellin pseudodifferential operators are generalizations of Mellin convolution operators. Let $a$ be a sufficiently smooth function defined on $\mathbb{R}_+ \times \mathbb{R}$. The Mellin pseudodifferential operator (shortly, Mellin PDO) with symbol $a$ is initially defined for smooth functions $f$ of compact support by the iterated integral

$$[\text{Op}(a)f](t) = [\mathcal{M}^{-1}a(t, \cdot)Mf](t) = \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}_+} a(t, x) \left( \frac{1}{\tau} \right)^ix f(\tau) \frac{d\tau}{\tau} \quad \text{for} \quad t \in \mathbb{R}_+.$$  

In 1991 Rabinovich [14] proposed to use Mellin pseudodifferential operators techniques to study singular integral operators on slowly oscillating Carleson curves. This idea was exploited in a series of papers by Rabinovich and coauthors (see, e.g., [15], [16] and [17].
Sections 4.5–4.6] and the references therein). Rabinovich stated in [16, Theorem 2.6] a Fredholm criterion for Mellin PDO’s with $C^\infty$ slowly oscillating (or slowly varying) symbols on the spaces $L^p(\mathbb{R}_+,d\mu)$ for $1 < p < \infty$. Namely, he considered symbols $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ such that
\begin{equation}
\sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}} \left| (t\partial_t)^j \partial_x^k a(t,x) \right| (1 + x^2)^{j/2} < \infty \quad \text{for all } j, k \in \mathbb{Z}_+ \quad (1.1)
\end{equation}
and
\begin{equation}
\limsup_{t \to s, x \in \mathbb{R}} \left| (t\partial_t)^j \partial_x^k a(t,x) \right| (1 + x^2)^{j/2} = 0 \quad \text{for all } j \in \mathbb{N}, \ k \in \mathbb{Z}_+, \ s \in (0,\infty). \quad (1.2)
\end{equation}

Here and in what follows $\partial_t$ and $\partial_x$ denote the operators of partial differentiation with respect to $t$ and $x$. Notice that (1.1) defines nothing but the Mellin version of the Hörmander class $S^0_{1,0}(\mathbb{R})$ (see, e.g., [6], [13, Chap. 2, Section 1] for the definition of the Hörmander classes $S^m_{\rho,\sigma}(\mathbb{R}^n)$. If $a$ satisfies (1.1), then the Mellin PDO $\text{Op}(a)$ is bounded on the spaces $L^p(\mathbb{R}_+,d\mu)$ for $1 < p < \infty$ (see, e.g., [21] Chap. VI, Proposition 4) for the corresponding Fourier PDO’s). Condition (1.2) is the Mellin version of Grushin’s definition of slowly varying symbols in the first variable (see, e.g., [4], [13, Chap. 3, Definition 5.11]).

The above mentioned results have a disadvantage that the smoothness conditions imposed on slowly oscillating symbols are very strong. In this paper we will use a much weaker notion of slow oscillation, which goes back to Sarason [19]. A bounded continuous function $f$ on $\mathbb{R}_+ = (0,\infty)$ is called slowly oscillating at $0$ and $\infty$ if
\begin{equation}
\lim_{t \to s, \tau \in [t,2t]} \max |f(t) - f(\tau)| = 0 \quad \text{for } s \in [0,\infty].
\end{equation}
This definition can be extended to the case of bounded continuous functions on $\mathbb{R}_+$ with values in a Banach space $X$.

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a $C^*$-algebra. This algebra properly contains $C(\mathbb{R}_+)$, the $C^*$-algebra of all continuous functions on $\mathbb{R}_+ := [0,\infty]$. For a unital commutative Banach algebra $A$, let $M(A)$ denote its maximal ideal space. Identifying the points $t \in \mathbb{R}_+$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\mathbb{R}_+)$, we get $M(C(\mathbb{R}_+)) = \mathbb{R}_+$. Consider the fibers
\begin{equation}
M_s(SO(\mathbb{R}_+)) := \{ \xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\mathbb{R}_+)} = s \}
\end{equation}
of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in [0,\infty]$. By [12, Proposition 2.1], the set
\begin{equation}
\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))
\end{equation}
coincides with $(\text{clos}_{SO(\mathbb{R}_+)} \mathbb{R}_+) \setminus \mathbb{R}_+$ where $\text{clos}_{SO(\mathbb{R}_+)} \mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $SO(\mathbb{R}_+)$. Then $M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$.

The second author [10] developed a Fredholm theory for Fourier pseudodifferential operators with slowly oscillating $V(\mathbb{R})$-valued symbols where $V(\mathbb{R})$ is the Banach algebra of absolutely continuous functions of bounded total variation on $\mathbb{R}$. Those results were translated to the Mellin setting in [12]. In particular, the important algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ of slowly oscillating $V(\mathbb{R})$-valued functions was introduced and a Fredholm criterion for Mellin PDO’s with symbols in the closure of $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ in the norm of the Banach algebra
\( C_b(\mathbb{R}_+, C_p(\mathbb{R})) \) of bounded continuous \( C_p(\mathbb{R}) \)-valued functions was obtained on the space \( L^p(\mathbb{R}_+, d\mu) \) for all \( p \in (1, \infty) \) [12, Theorem 4.3]. Here \( C_p(\mathbb{R}) \) is the smallest closed subalgebra of the algebra \( M_p(\mathbb{R}) \) that contains the algebra \( V(\mathbb{R}) \). We refer, e.g., to [11 Sections 9.1–9.7], [2 Chap. 1], [5 Section 2.1], [18 Section 4.2], and [20] for properties of the algebras \( V(\mathbb{R}) \), \( C_p(\mathbb{R}) \), and \( M_p(\mathbb{R}) \).

For symbols in the algebra \( \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \) the above mentioned Fredholm criterion has a simpler form [8, Theorem 3.6]. That result was already used in [7] (see also [8]) to prove that the simplest weighted singular integral operator with two shifts

\[
U_{\alpha}P_{\gamma}^+ + U_{\beta}P_{\gamma}^-
\]

is Fredholm of index zero on the space \( L^p(\mathbb{R}_+) \) with \( p \in (1, \infty) \), where \( \alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+ \) are orientation preserving diffeomorphisms with the only fixed points 0 and \( \infty \) such that \( \log \alpha', \log \beta' \) are bounded, \( \alpha', \beta' \in SO(\mathbb{R}_+) \),

\[
U_{\alpha}f = (\alpha')^{1/p}(f \circ \alpha), \quad U_{\beta}f = (\beta')^{1/p}(f \circ \beta), \quad P_{\gamma}^\pm := (I \pm S_{\gamma})/2,
\]

and \( S_{\gamma} \) is the weighted Cauchy singular integral operator given by

\[
(S_{\gamma}f)(t) := \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{f(\tau)}{\tau - t} d\tau
\]

with \( \gamma \in \mathbb{C} \) satisfying \( 0 < 1/p + \Re \gamma < 1 \) (for \( \gamma = 0 \) this result was obtained in [8]). To study more general operators than (1.3) in the forthcoming paper [9], we need not only a Fredholm criterion for \( \text{Op}(a) \) with \( a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \) given in [8, Theorem 3.6], but also an information on the regularizers of \( \text{Op}(a) \). Note that a full description of the regularizers of a Fredholm Mellin PDO \( \text{Op}(a) \) is available if \( a \in C^\infty(\mathbb{R}_+ \times \mathbb{R}) \) satisfies (1.1)–(1.2), see [16, Theorem 2.6]), however such a description is missing for the algebra \( \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \).

The aim of this paper is to fill in this gap and to complement the Fredholm criterion for Mellin PDO’s with symbols in \( \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \). Here we provide an explicit description of all regularizers of a Fredholm operator \( \text{Op}(a) \) with \( a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \). Namely, we prove that if \( a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \) does not degenerate on the “boundary” of \( \mathbb{R}_+ \times \mathbb{R} \) in a certain sense, then the Mellin PDO \( \text{Op}(a) \) is Fredholm on the space \( L^p(\mathbb{R}_+, d\mu) \) for \( p \in (1, \infty) \) and each its regularizer is of the form \( \text{Op}(b) + K \) where \( K \) is a compact operator on \( L^p(\mathbb{R}_+, d\mu) \) and \( b \) is a certain explicitly constructed function in the same algebra \( \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \) such that \( b = 1/a \) on the “boundary” of \( \mathbb{R}_+ \times \mathbb{R} \). By the “boundary” of \( \mathbb{R}_+ \times \mathbb{R} \) we mean the set

\[
(\mathbb{R}_+ \times \{\pm \infty\}) \cup (\Lambda \times \mathbb{R})
\]

(1.4)

The paper is organized as follows. In Section 2 we define the algebra \( C_b(\mathbb{R}_+, V(\mathbb{R})) \) of all bounded continuous \( V(\mathbb{R}) \)-valued functions and state that if \( a \in C_b(\mathbb{R}_+, V(\mathbb{R})) \), then \( \text{Op}(a) \) is bounded on \( L^p(\mathbb{R}_+, d\mu) \). In Section 3 we introduce the algebra \( SO(\mathbb{R}_+, V(\mathbb{R})) \) of slowly oscillating \( V(\mathbb{R}) \)-valued functions (a generalization of \( SO(\mathbb{R}_+) \)) and its subalgebra \( \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \). Further we explain how the values of a function \( a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \) on the boundary (1.4) are defined and recall that

\[
\text{Op}(a) \circ \text{Op}(b) \simeq \text{Op}(ab) \quad \text{whenever} \quad a, b \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})).
\]
In Section 4 we define our main algebra $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R})) \subset \mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$ and show that all algebras $C_0(\mathbb{R}^+, V(\mathbb{R}))$, $SO(\mathbb{R}^+, V(\mathbb{R}))$, $\mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$, and $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$ are inverse closed in $C_b(\mathbb{R}^+ \times \mathbb{R})$, the algebra of all bounded continuous functions on $\mathbb{R}^+ \times \mathbb{R}$. Combining the inverse closedness of the algebras $\mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$ (resp. $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$) with (1.5), we get a description of all regularizers for $\text{Op}(a)$ with $a \in \mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$ (resp. $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$) bounded away from zero on $\mathbb{R}^+ \times \mathbb{R}$. In Section 5 we show that the latter strong hypothesis can be essentially relaxed in the case of the algebra $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$. We show that if $a \in \widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$ does not degenerate on the “boundary” (1.4), then there exists $b \in \mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$ such that $b = 1/a$ on the “boundary” (1.4). This construction becomes possible for $a \in \widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$ because the limiting values of $a(t, \cdot)$ on $\Delta$ are attained uniformly in the norm of $V(\mathbb{R})$ (see Lemma 5.2). Finally we recall that if $c \in \widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$, then $\text{Op}(c)$ is compact if and only if its symbol $c$ degenerates on the “boundary” (1.4). Combining this result with our construction, we arrive at the main result of the paper.

2 Algebra $C_b(\mathbb{R}^+, V(\mathbb{R}))$ and Boundedness of Mellin PDO’s

2.1 Definition of the Algebra $C_b(\mathbb{R}^+, V(\mathbb{R}))$

Let $a$ be an absolutely continuous function of finite total variation

$$V(a) := \int_{\mathbb{R}} |a'(x)| dx$$

on $\mathbb{R}$. The set $V(\mathbb{R})$ of all absolutely continuous functions of finite total variation on $\mathbb{R}$ becomes a Banach algebra equipped with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a). \tag{2.1}$$

Following [10] [11], let $C_b(\mathbb{R}^+, V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions on $\mathbb{R}^+$ with the norm

$$\|a(\cdot, \cdot)\|_{C_b(\mathbb{R}^+, V(\mathbb{R}))} := \sup_{t \in \mathbb{R}^+} \|a(t, \cdot)\|_V.$$

2.2 Boundedness of Mellin PDO’s

As usual, let $C_0^\infty(\mathbb{R}^+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}^+$.

The following boundedness result for Mellin pseudodifferential operators can be extracted from [11] Theorem 6.1] (see also [10] Theorem 3.1]).

**Theorem 2.1.** If $a \in C_0^\infty(\mathbb{R}^+, V(\mathbb{R}))$, then the Mellin pseudodifferential operator $\text{Op}(a)$, defined for functions $f \in C_0^\infty(\mathbb{R}^+)$ by the iterated integral

$$\text{Op}(a)f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} dx \int_{\mathbb{R}^+} a(t, x) \left( \frac{1}{\tau} \right)^{ix} f(\tau) \frac{d\tau}{\tau} \quad \text{for} \quad t \in \mathbb{R}^+,$$

extends to a bounded linear operator on the space $L^p(\mathbb{R}^+, d\mu)$ and there is a positive constant $C_p$ depending only on $p$ such that

$$\|\text{Op}(a)\|_{B(L^p(\mathbb{R}^+, d\mu))} \leq C_p \|a\|_{C_0^\infty(\mathbb{R}^+, V(\mathbb{R}))}.$$
3 Algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and Compactness of Semi-Commutators of Mellin PDO’s

3.1 Definitions of the Algebras $SO(\mathbb{R}_+, V(\mathbb{R}))$ and $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$

Let $SO(\mathbb{R}_+, V(\mathbb{R}))$ denote the Banach subalgebra of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$-valued functions $a$ on $\mathbb{R}_+$ that slowly oscillate at 0 and $\infty$, that is,

$$\lim_{r \to 0} \sup_{\tau \in [r, 2r]} \|a(t, \cdot) - a(\tau, \cdot)\|_{L^\infty(\mathbb{R})} = 0,$$

where

$$\sup_{\tau \in [r, 2r]} \|a(t, \cdot) - a(\tau, \cdot)\|_{L^\infty(\mathbb{R})} = 0,$$

where $a^h(t, x) := a(t, x + h)$ for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.

3.2 Limiting Values of Functions in the Algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$

Let $a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For every $t \in \mathbb{R}_+$, the function $a(t, \cdot)$ belongs to $V(\mathbb{R})$ and, therefore, has finite limits at $\pm \infty$, which will be denoted by $a(t, \pm \infty)$. Now we explain how to extend the function $a$ to $\Delta \times \mathbb{R}$. By analogy with [10, Lemma 2.7] one can prove the following.

**Lemma 3.1.** Let $s \in \{0, \infty\}$ and $\{a_k\}_{k=1}^{\infty}$ be a countable subset of the algebra $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. For each $\xi \in M_s(SO(\mathbb{R}_+))$ there is a sequence $\{t_j\}_{j=1}^{\infty} \subset \mathbb{R}_+$ and functions $a_k(\xi, \cdot) \in V(\mathbb{R})$ such that $t_j \to s$ as $j \to \infty$ and

$$a_k(\xi, x) = \lim_{j \to \infty} a_k(t_j, x)$$

for every $x \in \mathbb{R}$ and every $k \in \mathbb{N}$.

The following lemma will be of some importance in applications we have in mind [9] (although it will not be used in the current paper).

**Lemma 3.2.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that the series $\sum_{n=1}^{\infty} a_n$ converges in the norm of $C_b(\mathbb{R}_+, V(\mathbb{R}))$ to a function $a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$. Then

$$a(t, \pm \infty) = \sum_{n=1}^{\infty} a_n(t, \pm \infty) \text{ for all } t \in \mathbb{R}_+, \quad a(\xi, x) = \sum_{n=1}^{\infty} a_n(\xi, x) \text{ for all } (\xi, x) \in \Delta \times \mathbb{R}. \quad (3.2)$$

**Proof.** Fix $\varepsilon > 0$. For $N \in \mathbb{N}$, put

$$s_N := \sum_{n=1}^{N} a_n.$$

By the hypothesis, there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$,

$$\sup_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}} |a(t, x) - s_N(t, x)| \leq \|a - s_N\|_{C_b(\mathbb{R}_+, V(\mathbb{R}))} < \varepsilon/3. \quad (3.3)$$
Fix some \( t \in \mathbb{R}_+ \). For every \( N > N_0 \) there exists \( x(t,N) \in \mathbb{R}_+ \) such that for all \( x \in (x(t,N),+\infty) \),
\[
|a(t,\infty) - a(t,x)| < \varepsilon/3, \quad |\hat{s}_N(t,\infty) - \hat{s}_N(t,x)| < \varepsilon/3.
\] (3.4)

From (3.3) and (3.4) it follows that for every \( N > N_0 \) and \( x \in (x(t,N),+\infty) \),
\[
|a(t,\infty) - \hat{s}_N(t,\infty)| \leq |a(t,\infty) - a(t,x)| + |a(t,x) - \hat{s}_N(t,x)| + |\hat{s}_N(t,x) - \hat{s}_N(t,\infty)| < \varepsilon.
\]

This implies the first equality in (3.2) for the sign “+”. The proof for the sign “−” is analogous.

Fix \( s \in [0,\infty) \) and \( \xi \in M_s(SO(\mathbb{R}_+)) \). In view of Lemma 3.1 there exists a sequence \( \{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+ \) such that \( t_j \to s \) as \( j \to \infty \) and functions \( a(\xi,\cdot) \in V(\mathbb{R}_+) \) and \( \hat{s}_N(\xi,\cdot) \in V(\mathbb{R}_+) \), \( N \in \mathbb{N} \), such that
\[
a(\xi,x) = \lim_{j \to \infty} a(t_j,x), \quad \hat{s}_N(\xi,x) = \lim_{j \to \infty} \hat{s}_N(t_j,x)
\]
for all \( x \in \mathbb{R} \) and all \( N \in \mathbb{N} \).

Fix \( x \in \mathbb{R} \). For every \( N > N_0 \) there exists \( j_0(x,N) \in \mathbb{N} \) such that for \( j > j_0(x,N) \),
\[
|a(\xi,x) - a(t_j,x)| < \varepsilon/3, \quad |\hat{s}_N(\xi,x) - \hat{s}_N(t_j,x)| < \varepsilon/3.
\] (3.5)

From (3.3) and (3.5) we obtain that for \( N > N_0 \) and \( j > j_0(x,N) \),
\[
|a(\xi,x) - \hat{s}_N(\xi,x)| \leq |a(\xi,x) - a(t_j,x)| + |a(t_j,x) - \hat{s}_N(t_j,x)| + |\hat{s}_N(t_j,x) - \hat{s}_N(\xi,x)| < \varepsilon,
\]
which concludes the proof of the second equality in (3.2).

3.3 Compactness of Semi-Commutators of Mellin PDO’s

Let \( E \) be the isometric isomorphism
\[
E : L^p(\mathbb{R}_+,d\mu) \to L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}.
\] (3.6)

Applying the relation
\[
Op(a) = E^{-1}a(x,D)E
\] (3.7)

between the Mellin pseudodifferential operator \( Op(a) \) and the Fourier pseudodifferential operator \( a(x,D) \) considered in [10], where
\[
a(t,x) = a(t\ln x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},
\] (3.8)

we infer from [10] Theorem 8.3] the following compactness result.

**Theorem 3.3.** If \( a, b \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \), then \( Op(a)Op(b) = Op(ab) \).
4 Regularization of Mellin PDO’s with Symbols
Globally Bounded Away from Zero

4.1 Definition of the Algebra $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$
We denote by $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$ the Banach algebra consisting of all functions $a \in \mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$ that satisfy the condition
\[
\lim_{m \to \infty} \sup_{x \in \mathbb{R}} \int_{-m, m} |\partial_x a(t, x)| \, dx = 0. \tag{4.1}
\]
This algebra plays a crucial role in the paper.

4.2 Inverse Closedness of the Algebras $C_b(\mathbb{R}^+, V(\mathbb{R}))$, $SO(\mathbb{R}^+, V(\mathbb{R}))$, $\mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$, and $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$ in the Algebra $C_b(\mathbb{R}^+ \times \mathbb{R})$
Let $\mathfrak{B}$ be a unital Banach algebra and $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$, which contains the identity element of $\mathfrak{B}$. The algebra $\mathfrak{A}$ is said to be inverse closed in the algebra $\mathfrak{B}$ if every element $a \in \mathfrak{A}$, invertible in $\mathfrak{B}$, is invertible in $\mathfrak{A}$ as well.

**Lemma 4.1.** The algebras $C_b(\mathbb{R}^+, V(\mathbb{R}))$, $SO(\mathbb{R}^+, V(\mathbb{R}))$, $\mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))$, and $\widetilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))$ are inverse closed in the Banach algebra $C_b(\mathbb{R}^+ \times \mathbb{R})$ of all bounded continuous functions on the half-plane $\mathbb{R}^+ \times \mathbb{R}$.

**Proof.** The proof is developed by analogy with [10, pp. 755–756]. Let $a \in C_b(\mathbb{R}^+, V(\mathbb{R}))$ be invertible in $C_b(\mathbb{R}^+ \times \mathbb{R})$. Then
\[
\|a^{-1}\|_{C_b(\mathbb{R}^+ \times \mathbb{R})} = \sup_{(t, x) \in \mathbb{R}^+ \times \mathbb{R}} |a^{-1}(t, x)| = \left(\inf_{(t, x) \in \mathbb{R}^+ \times \mathbb{R}} |a(t, x)|\right)^{-1} < \infty.
\]
Therefore, for every $t \in \mathbb{R}^+$,
\[
\|a^{-1}(t, \cdot)\|_V = \|a^{-1}(t, \cdot)\|_{L^\infty(\mathbb{R}^+)} + V(a^{-1}(t, \cdot)) = \sup_{x \in \mathbb{R}} \left|\frac{a(t, x)}{a^2(t, x)}\right| + \int_{\mathbb{R}} \left|\frac{\partial_x a(t, x)}{a^2(t, x)}\right| \, dx \leq \|a^{-1}\|_{C_b(\mathbb{R}^+ \times \mathbb{R})}^2 (\|a(t, \cdot)\|_{L^\infty(\mathbb{R})} + V(a(t, \cdot))) = \|a^{-1}\|_{C_b(\mathbb{R}^+ \times \mathbb{R})}^2 \|a(t, \cdot)\|_V. \tag{4.2}
\]
Hence
\[
\|a^{-1}(\cdot, \cdot)\|_{C_b(\mathbb{R}^+, V(\mathbb{R}))} \leq \|a^{-1}\|_{C_b(\mathbb{R}^+ \times \mathbb{R})}^2 \|a(\cdot, \cdot)\|_{C_b(\mathbb{R}^+, V(\mathbb{R}))} \tag{4.3}
\]
and for every $t, \tau \in \mathbb{R}^+$,
\[
\|a^{-1}(t, \cdot) - a^{-1}(\tau, \cdot)\|_V \leq \|a^{-1}(t, \cdot)\|_V \|a^{-1}(\tau, \cdot)\|_V \|a(t, \cdot) - a(\tau, \cdot)\|_V \leq \|a^{-1}\|_{C_b(\mathbb{R}^+ \times \mathbb{R})}^2 \|a(\cdot, \cdot)\|_{C_b(\mathbb{R}^+, V(\mathbb{R}))} \|a(t, \cdot) - a(\tau, \cdot)\|_V. \tag{4.4}
\]
From inequalities (4.3)–(4.4) it follows that the function $a^{-1}$ is a bounded and continuous $V(\mathbb{R})$-valued function. Thus, $C_b(\mathbb{R}^+, V(\mathbb{R}))$ is inverse closed in $C_b(\mathbb{R}^+ \times \mathbb{R})$.

Suppose $a \in SO(\mathbb{R}^+, V(\mathbb{R}))$ is invertible in $C_b(\mathbb{R}^+ \times \mathbb{R})$. If $t, \tau \in \mathbb{R}^+$, then
\[
\|a^{-1}(t, \cdot) - a^{-1}(\tau, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|a^{-1}\|_{C_b(\mathbb{R}^+ \times \mathbb{R})}^2 \|a(t, \cdot) - a(\tau, \cdot)\|_{L^\infty(\mathbb{R})}. \tag{4.5}
\]
Therefore
\[ \text{cm}_r^C(a^{-1}) \leq \|a^{-1}\|^2_{C_b(\mathbb{R}_+ \times \mathbb{R})} \text{cm}_r^C(a), \quad r \in \mathbb{R}_+. \]

From the above inequality we conclude that \( a^{-1} \in SO(\mathbb{R}_+, V(\mathbb{R})) \). Thus, \( SO(\mathbb{R}_+, V(\mathbb{R})) \) is inverse closed in \( C_b(\mathbb{R}_+ \times \mathbb{R}) \).

Let \( a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) be invertible in \( C_b(\mathbb{R}_+ \times \mathbb{R}) \). Taking into account inequality (4.2) and that the norm in \( V(\mathbb{R}) \) is translation-invariant, we get for \( h \in \mathbb{R} \) and \( t \in \mathbb{R}_+ \),

\[ \|a^{-1}(t, \cdot) - (a^{-1})^h(t, \cdot)\|_V \leq \|a^{-1}(t, \cdot)\|_V \|(a^{-1})^h(t, \cdot)\|_V \|a(t, \cdot) - a^h(t, \cdot)\|_V \]

\[ \leq \|a^{-1}\|^4_{C_b(\mathbb{R}_+ \times \mathbb{R})} \|a(\cdot, \cdot)\|^2_{C_b(\mathbb{R}_+ \times \mathbb{R})} \|a(t, \cdot) - a^h(t, \cdot)\|_V. \quad (4.6) \]

From the above inequality and \( a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) it follows that

\[ \lim_{|h| \to 0} \sup_{t \in \mathbb{R}_+} \|a^{-1}(t, \cdot) - (a^{-1})^h(t, \cdot)\|_V = 0. \]

This means that \( a^{-1} \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \), whence the proof of the inverse closedness of the algebra \( \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) in the algebra \( C_b(\mathbb{R}_+ \times \mathbb{R}) \) is completed.

Finally, if \( a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \) is invertible in \( C_b(\mathbb{R}_+ \times \mathbb{R}) \), then

\[ \lim_{m \to \infty} \sup_{t \in \mathbb{R}_+} \int_{\mathbb{R} \setminus [-m, m]} |\partial_x a^{-1}(t, x)| \, dx \leq \|a^{-1}\|^2_{C_b(\mathbb{R}_+ \times \mathbb{R})} \lim_{m \to \infty} \sup_{t \in \mathbb{R}_+} \int_{\mathbb{R} \setminus [-m, m]} |\partial_x a(t, x)| \, dx = 0. \]

Therefore, \( a^{-1} \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \) and thus the algebra \( \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \) is inverse closed in the algebra \( C_b(\mathbb{R}_+, V(\mathbb{R})) \). \( \square \)

### 4.3 First Result on the Regularization of Mellin PDO’s

**Lemma 4.2.** If \( a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) (resp. \( a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \)) is such that

\[ \inf_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}} |a(t, x)| > 0, \quad (4.7) \]

then the Mellin pseudodifferential operator \( \text{Op}(a) \) is Fredholm on the space \( L^p(\mathbb{R}_+, d\mu) \) and each its regularizer is of the form \( \text{Op}(1/a) + K \) where \( K \) is a compact operator on the space \( L^p(\mathbb{R}_+, d\mu) \) and \( 1/a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) (resp. \( 1/a \in \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \)).

**Proof.** If \( a \) satisfies (4.7) and belongs to \( \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) (resp. to \( \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \)), then \( 1/a \) belongs to \( \mathcal{E}(\mathbb{R}_+, V(\mathbb{R})) \) (resp. to \( \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \)) in view of Lemma 4.1. Then in both cases from Theorem 3.3, we obtain \( \text{Op}(a) \text{Op}(1/a) \simeq \text{Op}(1) \simeq I \) and \( \text{Op}(1/a) \text{Op}(a) \simeq \text{Op}(1) \simeq I \), which completes the proof. \( \square \)

As it happens, the very strong hypothesis (4.7) can be essentially relaxed for Mellin PDO’s with symbols in the algebra \( \widetilde{\mathcal{E}}(\mathbb{R}_+, V(\mathbb{R})) \). This issue will be discussed in the next section.
5 Algebra \( \widetilde{\mathcal{C}}(\mathbb{R}_+, V(\mathbb{R})) \) and Fredholmness of Mellin PDO’s

5.1 Elementary Properties of Two Important Functions in \( V(\mathbb{R}) \)

We prelude our main construction with properties of two important functions in \( V(\mathbb{R}) \).

**Lemma 5.1.** (a) For \( x \in \mathbb{R} \), put

\[
p_-(x) := (1 - \tanh(\pi x))/2, \quad p_+(x) := (1 + \tanh(\pi x))/2.
\]

Then \( \|p_\|_V = \|p_+\|_V = 2 \).

(b) For every \( h \in \mathbb{R} \), put \( p_\pm^h(x) := p_\pm(x + h) \). Then

\[
\|p_\pm - p_\pm^h\|_V \leq 5\pi|h|/2.
\]

(c) For every \( m > 0 \),

\[
\int_{\mathbb{R} \cup [-m,m]} (p_\pm)'(x)|dx < e^{-2\pi m}.
\]

**Proof.** (a) Since the function \( p_\) (resp. \( p_- \)) is monotonically increasing (resp. decreasing), \( p_\)\((+\infty) = 0 \) and \( p_\)\((-\infty) = 1 \), we have \( \|p_\|_{L^\infty(\mathbb{R})} = 1 \) and \( V(p_\)\) = \( |p_\)\((+\infty) - p_\)\((-\infty) = 1 \). Thus \( \|p_\|_V = \|p_\|_{L^\infty(\mathbb{R})} + V(p_\)\) = 2. Part (a) is proved.

(b) From (5.1) it follows that

\[
(p_\)'(x) = \pm \frac{\pi}{2 \cosh^2(\pi x)}, \quad (p_\)'(x) = \mp \frac{\pi^2 \tanh(\pi x)}{\cosh^2(\pi x)}, \quad x \in \mathbb{R}.
\]

Hence \( |(p_\)'(x)| \leq \pi/2 \) for all \( x \in \mathbb{R} \). From here, by the mean value theorem, we obtain

\[
|p_\(\pi x) - p_\(\pi(x + h)| \leq \pi|h|/2, \quad x, h \in \mathbb{R},
\]

whence

\[
\|p_\pm - p_\pm^h\|_{L^\infty(\mathbb{R})} \leq \pi|h|/2.
\]

Taking into account identities (5.4), we obtain

\[
|p_\)'(x)| \leq 2\pi p_\)'(x), \quad x \in \mathbb{R}.
\]

Then for \( h \in \mathbb{R} \),

\[
V(p_\pm - p_\pm^h) = \int_{\mathbb{R}} |p_\)'(x) - p_\)'(x + h)|dx = \int_{\mathbb{R}} \int_{x}^{x+h} p_\)'(y)dydx \\
\leq \int_{\mathbb{R}} dx \int_{x}^{x+h} |p_\)'(y)|dy \leq 2\pi \int_{\mathbb{R}} dx \int_{x}^{x+h} p_\)'(y)dy \\
= 2\pi \int_{\mathbb{R}} p_\)'(y)dy \int_{y|\pm h}^y dx = 2\pi|h|(p_\)\((+\infty) - p_\)\((-\infty) = 2\pi|h|.
\]

Combining (5.5) and (5.6), we arrive at (5.2).

(c) From (5.1) it follows that for \( m > 0 \),

\[
\int_{\mathbb{R} \cup [-m,m]} |p_\)'(x)|dx = \pi \int_{m}^{m+\infty} \frac{dx}{\cosh^2(\pi x)} = 1 - \tanh(\pi m) = \frac{1}{e^{2\pi m} + 1} < e^{-2\pi m},
\]

which completes the proof. \( \Box \)
5.2 Limiting Values of Elements of \( \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \)

For functions in the algebra \( a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \), we have a stronger result than Lemma [31] which follows from [10] Lemma 2.9 with the aid of the diagonal process.

**Lemma 5.2.** Let \( s \in (0, \infty) \) and \( \{a_k\}_{k=1}^{\infty} \) be a countable subset of the algebra \( \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \). For each \( \xi \in M_s(SO(\mathbb{R}_+)) \) there is a sequence \( \{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+ \) and functions \( a_k(\xi, \cdot) \in V(\mathbb{R}) \) such that \( t_j \to s \) as \( j \to \infty \) and

\[
\lim_{j \to \infty} \|a_k(t_j, \cdot) - a_k(\xi, \cdot)\|_V = 0 \quad \text{for all} \quad k \in \mathbb{N}. \tag{5.7}
\]

Conversely, every sequence \( \{\tau_j\}_{j \in \mathbb{N}} \subset \mathbb{R}_+ \) such that \( \tau_j \to s \) as \( j \to \infty \) contains a subsequence \( \{t_j\}_{j \in \mathbb{N}} \) such that (5.7) holds for some \( \xi \in M_s(SO(\mathbb{R}_+)) \).

As usual, the maximal ideal space \( M(SO(\mathbb{R}_+)) \) is equipped with the Gelfand topology. Then, in view of [1 Section 1.24], the set \( \Delta \) is a compact Hausdorff subspace of \( M(SO(\mathbb{R}_+)) \). It is equipped with the induced topology. Finally, the compact Hausdorff space \( \Delta \times \mathbb{R} \) is equipped with the product topology generated by the topologies of \( \Delta \) and \( \mathbb{R} \).

**Lemma 5.3.** For every \( a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R})) \), the function \( (\xi, x) \mapsto a(\xi, x) \) is continuous on the compact Hausdorff space \( \Delta \times \mathbb{R} \).

**Proof.** Fix \( \varepsilon > 0 \). It follows from (3.1) that there exists a \( \delta > 0 \) such that for all \( h \in (-\delta, \delta) \),

\[
\sup_{t \in \mathbb{R}_+} \sup_{x \in \mathbb{R}} |a(t, x) - a(t, x + h)| \leq \sup_{t \in \mathbb{R}_+} \|a(t, \cdot) - a(t, \cdot + h)\|_V < \varepsilon/6.
\]

Hence there is an \( h \in (0, \infty) \) such that, for all \( t \in \mathbb{R}_+ \) and all \( x, y \in \mathbb{R} \) with \( |x - y| < h \),

\[
|a(t, x) - a(t, y)| < \varepsilon/6. \tag{5.8}
\]

By Lemma 5.2 for every \( s \in (0, \infty) \) and \( \xi \in M_s(SO(\mathbb{R}_+)) \), there is a sequence \( \{t_j\}_{j \in \mathbb{N}} \) and a function \( a(\xi, \cdot) \in V(\mathbb{R}) \subset C(\mathbb{R}) \) such that \( t_j \to s \) as \( j \to \infty \) and

\[
\lim_{j \to \infty} \sup_{x \in \mathbb{R}} |a(t_j, x) - a(\xi, x)| \leq \lim_{j \to \infty} \|a(t_j, \cdot) - a(\xi, \cdot)\|_V = 0. \tag{5.9}
\]

From the above inequality it follows that there is a \( J \in \mathbb{N} \) such that for all \( j \geq J \),

\[
|a(t_j, x) - a(\xi, x)| < \varepsilon/6, \quad |a(t_j, y) - a(\xi, x)| < \varepsilon/6.
\]

Combining these inequalities with (5.8), we deduce for all \( x, y \in \mathbb{R} \) satisfying \( |x - y| < h \), all \( j \geq J \), all \( s \in (0, \infty) \), and all \( \xi \in M_s(SO(\mathbb{R}_+)) \) that

\[
|a(\xi, x) - a(\xi, y)| \leq |a(t_j, x) - a(\xi, x)| + |a(t_j, y) - a(\xi, y)| + |a(t_j, x) - a(t_j, y)| < \varepsilon/2.
\]

Therefore, for all \( x, y \in \mathbb{R} \) satisfying \( |x - y| < h \) we have

\[
\sup_{\xi \in \Delta} |a(\xi, x) - a(\xi, y)| \leq \varepsilon/2. \tag{5.10}
\]
Fix $\xi \in \Delta$. Since the function $a(\cdot, x)$ belongs to the algebra $SO(\mathbb{R}_+)$, there exists an open neighborhood $U_\eta(\xi) \subset \Delta$ of $\xi$ such that

$$|a(\eta, x) - a(\xi, x)| < \varepsilon/2 \quad \text{for all } \eta \in U_\delta(\xi). \quad (5.11)$$

Consequently, we infer from (5.10) and (5.11) that

$$|a(\eta, y) - a(\xi, x)| \leq |a(\eta, y) - a(\eta, x)| + |a(\eta, x) - a(\xi, x)| < \varepsilon$$

for all $(\eta, y) \in U_\delta(\xi) \times (x - h, x + h)$, which means that the function $(\xi, x) \mapsto a(\xi, x)$ is continuous on $\Delta \times \mathbb{R}$.

It remains to show that actually the function $(\xi, x) \mapsto a(\xi, x)$ is continuous on $\Delta \times \mathbb{R}$. By (4.11), for every $\varepsilon > 0$ there is an $M > 0$ such that

$$\sup_{t \in \mathbb{R}_+} |a(\xi, x) - a(t, +\infty)| \leq \sup_{t \in \mathbb{R}_+} \int_M^\infty |\partial_x a(t, x)| dx < \varepsilon/6 \quad \text{for all } y > M. \quad (5.12)$$

By Lemma 5.2, for every $s \in [0, \infty]$ and every $\xi \in M_s(SO(\mathbb{R}_+))$ there exist a sequence $\{t_j\} \in \mathbb{N}$ and a function $a(\xi, \cdot) \in V(\mathbb{R}) \subset C(\mathbb{R})$ such that $t_j \to s$ as $j \to \infty$ and (5.9) is fulfilled. From (5.9) it follows that there is a $J \in \mathbb{N}$ such that for all $j \geq J$, all $s \in [0, \infty)$, and all $\xi \in M_s(SO(\mathbb{R}_+))$,

$$|a(\xi, y) - a(\xi, +\infty)| \leq |a(t_j, y) - a(\xi, y)| + |a(t_j, +\infty) - a(\xi, +\infty)| + |a(t_j, y) - a(t_j, +\infty)| < \varepsilon/2.$$

Therefore, for all $y > M$ we have

$$\sup_{\xi \in \Delta} |a(\xi, y) - a(\xi, +\infty)| \leq \varepsilon/2. \quad (5.13)$$

Fix $\xi \in \Delta$. Since the function $a(\cdot, +\infty)$ belongs to $SO(\mathbb{R}_+)$, there is an open neighborhood $U_{+\infty}(\xi) \subset \Delta$ of $\xi$ such that

$$|a(\eta, +\infty) - a(\xi, +\infty)| < \varepsilon/2 \quad \text{for all } \eta \in U_{+\infty}(\xi). \quad (5.14)$$

Then similarly to (5.11) we deduce from (5.13) and (5.14) that

$$|a(\eta, y) - a(\xi, +\infty)| \leq |a(\eta, y) - a(\eta, +\infty)| + |a(\eta, +\infty) - a(\xi, +\infty)| < \varepsilon \quad (5.15)$$

for all $(\eta, y) \in U_{+\infty}(\xi) \times (M, +\infty]$.

Analogously, for every $\xi \in \Delta$ there exist an open neighborhood $U_{-\infty}(\xi) \subset \Delta$ of $\xi$ and a number $M < 0$ such that

$$|a(\eta, y) - a(\xi, -\infty)| < \varepsilon \quad (5.16)$$

for all $(\eta, y) \in U_{-\infty}(\xi) \times [-\infty, M]$.

Finally, we conclude from (5.15)–(5.16) and the continuity of $(\xi, x) \mapsto a(\xi, x)$ on the set $\Delta \times \mathbb{R}$ that this function is continuous on the compact Hausdorff space $\Delta \times \mathbb{R}$. $\Box$
5.3 Key Construction

In this subsection we show that if \( a \in \tilde{E}(\mathbb{R}^+, V(\mathbb{R})) \) does not degenerate on the “boundary” (1.4), then there exists \( b \in \tilde{E}(\mathbb{R}^+, V(\mathbb{R})) \) such that \( b = 1/a \) on the “boundary” (1.4).

Lemma 5.4. If \( a \in \tilde{E}(\mathbb{R}^+, V(\mathbb{R})) \) and

\[
a(t, \pm \infty) \neq 0 \text{ for all } t \in \mathbb{R}^+, \quad a(\xi, x) \neq 0 \text{ for all } (\xi, x) \in \Delta \times \mathbb{R}. \tag{5.17}
\]

then

\[
A_\pm := \sup_{t \in \mathbb{R}^+} \frac{1}{|a(t, \pm \infty)|} < \infty \tag{5.18}
\]

and there exists an \( r > 1 \) such that

\[
A(r) := \sup_{(t, x) \in T_r \times \mathbb{R}} \frac{1}{|a(t, x)|} < \infty \tag{5.19}
\]

where \( T_r := (0, r^{-1}] \cup [r, \infty). \)

Proof. By Lemma 5.3, the function \( (\xi, x) \mapsto a(\xi, x) \) is continuous on the compact Hausdorff space \( \Delta \times \mathbb{R} \). Therefore, we infer from (5.17) that

\[
C := \min\{|a(\xi, x)| : (\xi, x) \in \Delta \times \mathbb{R}\} > 0. \tag{5.20}
\]

For every point \( (\xi, x) \in \Delta \times \mathbb{R} \) we consider its open neighborhood \( U_{a,\xi,x} \subset M(SO(\mathbb{R}^+)) \times \mathbb{R} \) such that

\[
|a(\eta, y) - a(\xi, x)| < C/2 \text{ for every } (\eta, y) \in U_{a,\xi,x}. \tag{5.21}
\]

We claim that there exists a number \( r > 1 \) such that

\[
T_r \times \mathbb{R} \subset \bigcup_{(\xi, x) \in \Delta \times \mathbb{R}} U_{a,\xi,x}. \tag{5.22}
\]

Assume the contrary. Then for every \( n \in \mathbb{N} \setminus \{1\} \) there exists a point \( (\tau_n, x_n) \in T_n \times \mathbb{R} \) such that

\[
(\tau_n, x_n) \notin \left( \bigcup_{(\xi, x) \in \Delta \times \mathbb{R}} U_{a,\xi,x} \right) \cup \left( \bigcup_{(\xi, x) \in \Delta \times \mathbb{R}} U_{a,\xi,x} \right). \tag{5.23}
\]

Since \( \tau_n \in T_n = (0, 1/n] \cup [n, \infty) \) for all \( n \geq 2 \), we can extract a subsequence \( \{\tau_{n_k}\}_{k \in \mathbb{N}} \) of the sequence \( \{\tau_n\}_{n \in \mathbb{N} \setminus \{1\}} \) such that

\[
\lim_{k \to \infty} \tau_{n_k} = s \quad \text{for some } s \in [0, \infty]. \tag{5.24}
\]

Further, we can extract a subsequence \( \{x_{n_{k_j}}\}_{j \in \mathbb{N}} \) of the corresponding sequence \( \{x_{n_k}\}_{k \in \mathbb{N}} \) such that the limit

\[
x_0 := \lim_{j \to \infty} x_{n_{k_j}} \in \mathbb{R} \tag{5.25}
\]
exists. Then, by Lemma 5.2 there exists a subsequence \( \{t_j\}_{j \in \mathbb{N}} = \{\tau_{n_j}\}_{j \in \mathbb{N}} \) of the sequence \( \{\tau_{n_k}\}_{k \in \mathbb{N}} \) and a point \( \xi_0 \in M_a(SO(\mathbb{R}^+)) \) such that
\[
\lim_{j \to \infty} \|a(t_j, \cdot) - a(\xi_0, \cdot)\|_V = 0. \tag{5.26}
\]

Put \( \{y_j\}_{j \in \mathbb{N}} = \{x_{n_j}\}_{j \in \mathbb{N}} \). Taking into account (5.23), (5.26), we have shown that if (5.22) is violated for all \( r > 1 \), then there exist \( s \in [0, \infty) \), \( \xi_0 \in M_a(SO(\mathbb{R}^+)) \), and a sequence \( \{(t_j, y_j)\}_{j \in \mathbb{N}} \) such that (5.26) is fulfilled,
\[
\{(t_j, y_j) : j \in \mathbb{N}\} \cap \bigcup_{(\xi, x) \in M_a(SO(\mathbb{R}^+) \times \overline{\mathbb{R}}) \setminus \emptyset} U_{a, \xi, x} = \emptyset, \tag{5.27}
\]
and
\[
\lim_{j \to \infty} y_j = x_0 \in \overline{\mathbb{R}}, \quad \lim_{j \to \infty} t_j = s. \tag{5.28}
\]

Since \( (\xi_0, x_0) \in M_a(SO(\mathbb{R}^+) \times \overline{\mathbb{R}}) \subset \Delta \times \overline{\mathbb{R}} \), from Lemma 5.3 and the first equality in (5.28) we deduce that
\[
\lim_{j \to \infty} |a(\xi_0, y_j) - a(\xi_0, x_0)| = 0. \tag{5.29}
\]

For every \( j \in \mathbb{N} \), we have
\[
|a(t_j, y_j) - a(\xi_0, x_0)| \leq |a(t_j, y_j) - a(\xi_0, y_j)| + |a(\xi_0, y_j) - a(\xi_0, x_0)|
\leq \sup_{y \in \overline{\mathbb{R}}} |a(t_j, y) - a(\xi_0, y)| + |a(\xi_0, y_j) - a(\xi_0, x_0)|
\leq ||a(t_j, \cdot) - a(\xi_0, \cdot)||_V + |a(\xi_0, y_j) - a(\xi_0, x_0)|.
\]

From (5.26), (5.29), and the above inequality we deduce that
\[
\lim_{j \to \infty} a(t_j, y_j) = a(\xi_0, x_0).
\]

This means that for all sufficiently large \( j \) the points \( (t_j, y_j) \) belong to the neighborhood \( U_{a, \xi_0, x_0} \) of the point \( (\xi_0, x_0) \in M_a(SO(\mathbb{R}^+) \times \overline{\mathbb{R}}) \), which is impossible in view of (5.27). Hence, we arrive at the contradiction.

Thus, condition (5.22) is fulfilled for some \( r > 1 \). Therefore, in view of (5.20) and (5.21), we obtain
\[
\inf_{(t, x) \in T \times \overline{\mathbb{R}}} |a(t, x)| > C/2 > 0.
\]

This inequality immediately yields (5.19). Finally, (5.19) and the first condition in (5.17) imply (5.18). \( \square \)

**Lemma 5.5.** Suppose \( a \in \tilde{E}(\mathbb{R}^+, V(\mathbb{R})) \) satisfies (5.17) and \( r > 1 \) is a number such that (5.19) holds (the existence of this number is guaranteed by Lemma 5.4). Put
\[
\ell_\pm(t) := \frac{\ln r \pm \ln t}{2 \ln r}, \quad c_\pm(t) := \frac{1}{a(t, \pm \infty)} - \frac{\ell_- (t)}{a(r^{-1}, \pm \infty)} = \frac{\ell_+ (t)}{a(r, \pm \infty)}, \quad t \in [r^{-1}, r], \tag{5.30}
\]
and consider the functions $p_\pm$ given by (5.1). Then the function
\[
 b(t, x) := \begin{cases} 
 \frac{1}{a(t, x)}, & (t, x) \in (\mathbb{R}_+ \setminus [r^{-1}, r]) \times \overline{\mathbb{R}}, \\
 \frac{\ell_-(t)}{a(r^{-1}, x)} + \frac{\ell_+(t)}{a(r, x)} + c_-(t) p_+(x) + c_+(t) p_-(x), & (t, x) \in [r^{-1}, r] \times \overline{\mathbb{R}},
 \end{cases} 
\] (5.31)
is continuous on $\mathbb{R}_+ \times \overline{\mathbb{R}}$ and is equal to $1/a$ on the set $((\mathbb{R}_+ \setminus (r^{-1}, r)) \times \overline{\mathbb{R}}) \cup ((r^{-1}, r) \times \{\pm \infty\})$.

**Proof.** Since $\ell_\pm(r^{-1}) = 0$ and $\ell_\pm(r^1) = 1$, we have $c_\pm(r) = c_\pm(r^{-1}) = 0$. Therefore
\[
 b(r^{-1}, x) = 1/a(r^{-1}, x) \quad \text{for all} \quad x \in \mathbb{R}. 
\] (5.32)
Taking into account that $p_\pm(\pm \infty) = 0$ and $p_\pm(\pm \infty) = 1$, we get from (5.30)–(5.31)
\[
 b(t, \pm \infty) = \frac{\ell_-(t)}{a(r^{-1}, \pm \infty)} + \frac{\ell_+(t)}{a(r, \pm \infty)} + c_+(t) = \frac{1}{a(t, \pm \infty)} \quad \text{for all} \quad t \in [r^{-1}, r]. 
\] (5.33)
Thus, the assertion of the lemma follows from (5.32)–(5.33) and the equality $b(t, x) = 1/a(t, x)$ for all $(t, x) \in (\mathbb{R}_+ \setminus [r^{-1}, r]) \times \overline{\mathbb{R}}$ (see (5.31)). \qed

**Lemma 5.6.** Suppose $a \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ satisfies (5.17) and $b$ is the function defined by (5.30)–(5.31) with $r > 1$ such that (5.19) holds (the existence of this number is guaranteed by Lemma 5.4). Then $b \in \mathcal{E}(\mathbb{R}_+, V(\mathbb{R}))$ and
\[
b(t, \pm \infty) = 1/a(t, \pm \infty) \quad \text{for all} \quad t \in \mathbb{R}_+, \quad b(\xi, x) = 1/a(\xi, x) \quad \text{for all} \quad (\xi, x) \in \Delta \times \overline{\mathbb{R}}. 
\] (5.34)

**Proof.** We divide the proof into five steps:
(a) First we prove that the function $b$ belongs to the algebra $C_0(\mathbb{R}_+, V(\mathbb{R}))$. Let
\[
 T_r := (0, r^{-1}] \cup [r, +\infty).
\] By Lemma 5.5
\[
b(t, x) = 1/a(t, x), \quad (t, x) \in T_r \times \overline{\mathbb{R}}. 
\] (5.35)
Since $a(t, \cdot)$ belongs to $V(\mathbb{R})$ for all $t \in \mathbb{R}_+$, by analogy with (4.2), we infer from (5.19) that
\[
\|b(t, \cdot)\|_V \leq A^2(r) \sup_{t \in T_r} \|a(t, \cdot)\|_V, \quad t \in T_r. 
\] (5.36)
From (5.18) and (5.30) it follows that
\[
0 \leq \ell_\pm(t) \leq 1, \quad |c_\pm(t)| \leq 3A_\pm, \quad t \in [r^{-1}, r]. 
\] (5.37)
From (5.31), (5.35)–(5.37), and Lemma 5.1(a) it follows that for $t \in (r^{-1}, r),$
\[
\|b(t, \cdot)\|_V \leq \ell_-(t)\|b(r^{-1}, \cdot)\|_V + \ell_+(t)\|b(r, \cdot)\|_V + |c_-(t)|\|p_-(\cdot)\|_V + |c_+(t)|\|p_+(\cdot)\|_V 
\leq 2A^2(r) \sup_{t \in T_r} \|a(t, \cdot)\|_V + 6A_- + 6A_+.
\] (5.38)
Combining (5.36) and (5.38), we arrive at
\[ \|b(t, \cdot)\|_{C^b_b(\mathbb{R}, V(\mathbb{R}))} = \sup_{r \in \mathbb{T}} \|b(t, \cdot)\|_V \leq 2A^2(r) \sup_{t \in \mathbb{T}} \|a(t, \cdot)\|_V + 6A_+ + 6A_+ < +\infty. \] (5.39)

From (5.19) and (5.35)–(5.36), by analogy with (4.4), we obtain for \( t, r(t) \in \mathbb{T} \),
\[ \|b(t, \cdot) - b(\tau, \cdot)\|_V \leq \|b(t, \cdot)\|_V \|b(\tau, \cdot)\|_V \|a(t, \cdot) - a(\tau, \cdot)\|_V \]
\[ \leq A^4(r) \left( \sup_{t \in \mathbb{T}} \|a(t, \cdot)\|_V \right)^2 \|a(t, \cdot) - a(\tau, \cdot)\|_V. \]

Since \( a \) is a continuous \( V(\mathbb{R}) \)-valued function, from the above inequality we conclude that \( t \mapsto b(t, \cdot) \) is a continuous \( V(\mathbb{R}) \)-valued function for \( t \in \mathbb{T} \).

Obviously, \( \ell_\pm \) are continuous on \([r^{-1}, r]\). Since \( a \) is a continuous \( V(\mathbb{R}) \)-valued function, taking into account (5.18), we also have for \( t, \tau \in [r^{-1}, r] \),
\[ \left| \frac{1}{a(t, \pm \infty)} - \frac{1}{a(\tau, \pm \infty)} \right| = \left| \frac{a(t, \pm \infty) - a(\tau, \pm \infty)}{a(t, \pm \infty)a(\tau, \pm \infty)} \right| \leq A^2_\pm \|a(t, \cdot) - a(\tau, \cdot)\|_V. \]

From this inequality and the definitions of \( c_\pm \) in (5.30) we see that the functions \( c_\pm \) are continuous on \([r^{-1}, r]\). Therefore, from the definition (5.31) we conclude that \( t \mapsto b(t, \cdot) \) is a continuous \( V(\mathbb{R}) \)-valued function on \([r^{-1}, r]\). From the continuity of the \( V(\mathbb{R}) \)-valued function \( t \mapsto b(t, \cdot) \) on \( \mathbb{R}_+ \) and inequality (5.39) we conclude that \( b \in C^b(C_b(\mathbb{R}_+, V(\mathbb{R})))). \)

(b) Now we prove that \( b \in SO(C_b(\mathbb{R}_+, V(\mathbb{R})))). \) By analogy with (4.5), from (5.19) and (5.35) we obtain
\[ \|b(t, \cdot) - b(\tau, \cdot)\|_{L^{\infty}(\mathbb{R})} \leq A^2(r) \|a(t, \cdot) - a(\tau, \cdot)\|_{L^{\infty}(\mathbb{R})}, \quad t, \tau \in \mathbb{T}. \]

Since \( a \in SO(C_b(\mathbb{R}_+, V(\mathbb{R})))), \) from this estimate we obtain
\[ \lim_{y \to s} cm^C_y(b) \leq A^2(r) \lim_{y \to s} cm^C_y(a) = 0, \]
which means that \( b \in SO(C_b(\mathbb{R}_+, V(\mathbb{R})))). \)

(c) On this step we show that \( b \in \mathcal{E}(C_b(\mathbb{R}_+, V(\mathbb{R}))). \) By analogy with (4.6), taking into account that the norm of \( V(\mathbb{R}) \) is translation-invariant, from (5.19) and (5.35)–(5.36) we get for \( h \in \mathbb{R} \) and \( t \in \mathbb{T} \),
\[ \|b(t, \cdot) - b^h(t, \cdot)\|_V \leq \|b(t, \cdot)\|_V \|b^h(t, \cdot)\|_V \|a(t, \cdot) - a^h(t, \cdot)\|_V \]
\[ \leq C(a, H) \sup_{t \in \mathbb{T}} \|a(t, \cdot) - a^h(t, \cdot)\|_V, \] (5.40)
where
\[ C(a, H) := A^4(r) \left( \sup_{t \in \mathbb{T}} \|a(t, \cdot)\|_V \right)^2. \]

On the other hand, from (5.31), (5.35), (5.37), (5.40), and Lemma 5.1(b) it follows that for \( h \in \mathbb{R} \) and \( t \in (r^{-1}, r), \)
\[ \|b(t, \cdot) - b^h(t, \cdot)\|_V \leq \ell_-(t) \|b(r^{-1}, \cdot) - b^h(r^{-1}, \cdot)\|_V + \ell_+(t) \|b(r, \cdot) - b^h(r, \cdot)\|_V \]
\[ + |c_-(t)| \|p_- - p^h_-\|_V + |c_+(t)| \|p_+ - p^h_+\|_V \]
\[ \leq 2C(a) \sup_{t \in \mathbb{T}} \|a(t, \cdot) - a^h(t, \cdot)\|_V + \frac{15\pi}{2}(A_+ + A_-)h. \] (5.41)
Combining (5.40)–(5.41), we arrive at
\[
\sup_{t \in \mathbb{R}^+} |b(t, \cdot) - b^\lambda(t, \cdot)|_V \leq 2C(\alpha) \sup_{t \in \mathbb{R}^+} |a(t, \cdot) - a^\lambda(t, \cdot)|_V + \frac{15\pi}{2} (A_- + A_+) |h|.
\]
Since \(a \in \mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))\), the right-hand side of the above inequality tends to zero as \(|h| \to 0\).
Hence
\[
\lim_{|h| \to 0} \sup_{t \in \mathbb{R}^+} |b(t, \cdot) - b^\lambda(t, \cdot)|_V = 0.
\]
Thus, \(b \in \mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))\).

(d) Now we prove that \(b \in \tilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))\). From (5.35) we obtain
\[
\partial_\lambda b(t, x) = -a^{-2}(t, x) \partial_\lambda a(t, x), \quad (t, x) \in T_\tau \times \mathbb{R}.
\]
From this identity and (5.19) it follows that for all \(m > 0\) and \(t \in T_\tau,
\[
\int_{\mathbb{R} \setminus [-m, m]} |\partial_\lambda b(t, x)| dx \leq A^2(r) \sup_{r \in \mathbb{R}^+} \int_{\mathbb{R} \setminus [-m, m]} |\partial_\lambda a(t, x)| dx.
\]
On the other hand, from (5.35), (5.37), (5.42), and Lemma 5.1(c) it follows that for all \(t \in (r^{-1}, r)\) and \(m > 0,
\[
\int_{\mathbb{R} \setminus [-m, m]} |\partial_\lambda b(t, x)| dx \leq \ell_-(t) \int_{\mathbb{R} \setminus [-m, m]} |\partial_\lambda b(r^{-1}, x)| dx + \ell_+(t) \int_{\mathbb{R} \setminus [-m, m]} |\partial_\lambda b(r, x)| dx + |c_-(t)| \int_{\mathbb{R} \setminus [-m, m]} |p'_-(x)| dx + |c_+(t)| \int_{\mathbb{R} \setminus [-m, m]} |p'_+(x)| dx \leq 2A^2(r) \sup_{r \in \mathbb{R}^+} \int_{\mathbb{R} \setminus [-m, m]} |\partial_\lambda a(t, x)| dx + 3(A_- + A_+) e^{-2\pi m}.
\]
Combining (5.42)–(5.43), we obtain for \(m > 0,
\[
\sup_{t \in \mathbb{R}^+, \mathbb{R} \setminus [-m, m]} |\partial_\lambda b(t, x)| dx \leq 2A^2(r) \sup_{t \in \mathbb{R}^+, \mathbb{R} \setminus [-m, m]} |\partial_\lambda a(t, x)| dx + 3(A_- + A_+) e^{-2\pi m}.
\]
Since \(a \in \tilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))\), the right-hand side of the above inequality tends to zero as \(m \to \infty\).
This implies that
\[
\lim_{m \to \infty} \sup_{t \in \mathbb{R}^+, \mathbb{R} \setminus [-m, m]} |\partial_\lambda b(t, x)| dx = 0.
\]

Thus, \(b \in \tilde{\mathcal{E}}(\mathbb{R}^+, V(\mathbb{R}))\).

(e) Finally, we prove (5.34). The first equality in (5.34) was proved in Lemma 5.5.

Fix \(s \in (0, \infty)\). Since \(a, b \in \mathcal{E}(\mathbb{R}^+, V(\mathbb{R}))\), from Lemma 5.1 it follows that for each \(\xi \in M_{\alpha}(S O(\mathbb{R}^+)) \subset \Delta\) there exists a sequence \(\{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+\) and functions \(a(\xi, \cdot), b(\xi, \cdot) \in V(\mathbb{R})\) such that \(t_j \to s\) as \(j \to \infty\) and
\[
a(\xi, x) = \lim_{j \to \infty} a(t_j, x), \quad b(\xi, x) = \lim_{j \to \infty} b(t_j, x), \quad x \in \mathbb{R}.
\]
For all sufficiently large \(j\), one has \(t_j \in T_\tau\). Then from (5.35) we get \(b(t_j, x) = 1/a(t_j, x)\) for all sufficiently large \(j\) and all \(x \in \mathbb{R}\). From this equality and (5.44) we obtain the second equality in (5.34). \(\square\)
5.4 Regularization of Mellin PDO’s with Symbols in $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$

From [12, Theorem 4.1] we can extract the following.

**Lemma 5.7.** If $c \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$, then $\text{Op}(c) \in \mathcal{K}(L^p(\mathbb{R}_+, d\mu))$ if and only if

$$c(t, \pm \infty) = 0 \text{ for all } t \in \mathbb{R}_+, \quad c(\xi, x) = 0 \text{ for all } (\xi, x) \in \Delta \times \mathbb{R}. \quad (5.45)$$

Now we are in a position to prove the main result of the paper.

**Theorem 5.8.** Suppose $a \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$.

(a) If the Mellin pseudodifferential operator $\text{Op}(a)$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$, then

$$a(t, \pm \infty) \neq 0 \text{ for all } t \in \mathbb{R}_+, \quad a(\xi, x) \neq 0 \text{ for all } (\xi, x) \in \Delta \times \mathbb{R}. \quad (5.46)$$

(b) If (5.46) holds, then the Mellin pseudodifferential operator $\text{Op}(a)$ is Fredholm on the space $L^p(\mathbb{R}_+, d\mu)$ and each its regularizer has the form $\text{Op}(b) + K$, where $K$ is a compact operator on the space $L^p(\mathbb{R}_+, d\mu)$ and $b \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that

$$b(t, \pm \infty) = 1/a(t, \pm \infty) \text{ for all } t \in \mathbb{R}_+, \quad b(\xi, x) = 1/a(\xi, x) \text{ for all } (\xi, x) \in \Delta \times \mathbb{R}. \quad (5.47)$$

**Proof.** Part (a) follows from the necessity portion of [12, Theorem 4.3], which was obtained on the base of [10, Theorem 12.2] and (3.6), (3.7)–(3.8).

The proof of part (b) is analogous to the proof of the sufficiency portion of [10, Theorem 12.2]. If (5.46) holds, then by Lemma 5.6 there exists a function $b \in \tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ such that (5.47) is fulfilled. Therefore, the function $c := ab - 1$ belongs to $\tilde{E}(\mathbb{R}_+, V(\mathbb{R}))$ and (5.45) holds. By Lemma 5.7 the operator $\text{Op}(c) = \text{Op}(ab) - I$ is compact on $L^p(\mathbb{R}_+, d\mu)$. From this observation and Theorem 3.3 we obtain

$$\text{Op}(a)\text{Op}(b) \simeq \text{Op}(ab) \simeq I, \quad \text{Op}(b)\text{Op}(a) \simeq \text{Op}(ab) \simeq I.$$  

Thus, the operator $\text{Op}(a)$ is Fredholm and each its regularizer is of the form $\text{Op}(b) + K$, where $K \in \mathcal{K}(L^p(\mathbb{R}_+, d\mu))$. □

For a symbol $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ satisfying (1.1)–(1.2) the corresponding result was obtained in [16, Theorem 2.6].

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