Conformal Extension of Metrics of Negative Curvature

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In memory of Robert Brooks

Abstract

We consider the problem of extending a conformal metric of negative curvature, given outside of a neighbourhood of 0 in the unit disk $\mathbb{D}$, to a conformal metric of negative curvature in $\mathbb{D}$. We give conditions under which such an extension is possible, and also give obstructions to such an extension. The methods we use are based on a maximum principle and the Ahlfors–Schwarz Lemma. We also give an example in which no extension is possible, even when the conformality condition is dropped. We apply these considerations to compactification of Riemann surfaces.

1 Introduction

In [Bro99], Brooks considers the following problem: Let $S^O$ denote a complete hyperbolic Riemann surface of finite area, and let $S^C$ be its conformal compactification. In general, $S^C$ need not carry a hyperbolic metric, but it is shown in [Bro99] that if the cusps are large, in a sense to be defined below, then the hyperbolic metric on $S^C$ is close to the hyperbolic metric on $S^O$ outside cusp neighbourhoods. The result in [Bro99] is qualitative, and in this paper we quantify this result.

First, we would like to know how large a cusp should be in order to adjust the metric in its neighbourhood to a non-cusped metric with negative curvature. For example, if $S^O$ is the complete hyperbolic 3-punctured sphere, then $S^C$ has no hyperbolic structure at all. By contrast, we will show in Theorem 1 below that if a cusp has length $> 2\pi$, then we may modify the metric
conformally inside the cusp neighbourhood to obtain a metric of negative curvature. We will show that the constant $2\pi$ is sharp.

The above question leads us to the interesting problem of extension of metrics, with negative curvature and with a fixed conformal type, across a given boundary curve. In this paper, we consider this problem only for the complete hyperbolic metric on the punctured unit disk, since our motivation comes from hyperbolic Riemann surfaces of finite area. There is an obvious necessary condition for the extension which comes from the Gauss–Bonnet Theorem (cf. §7). We give several examples which show that this is not a sufficient condition for extension across general curves. Even under convexity restrictions, which seem natural, the Gauss–Bonnet condition is not enough.

A very similar problem was already addressed by M. Gromov in [Gro86], pp. 109–110. Gromov gives an obstruction for extension of metrics, not necessarily conformal, with curvature of a fixed sign, which goes beyond the Gauss–Bonnet obstruction. We present Gromov’s obstruction in section 7.3. It seems that in general, it is not easy to decide whether this obstruction is met. We give a class of curves which do satisfy the Gauss–Bonnet necessary condition for extension, but meet Gromov’s obstruction.

Next, we estimate how large a cusp should be in order to adjust the metric in its neighbourhood to a non-cusped metric with curvature close to $-1$. This gives a quantitative version of the theorem from [Bro99] which compares between the metrics on $S^O$ and $S^C$. We will show in Theorem 2 that if a cusp has length of the order $1/\sqrt{\varepsilon}$, then it is possible to modify the metric in a neighbourhood of the cusp to obtain a non-cusped metric with Gaussian curvature $\kappa$,

$$-(1 + \varepsilon) < \kappa < -\frac{1}{1 + \varepsilon}.$$  

We will show that this estimate is sharp.

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I dedicate this paper to the memory of Robert Brooks who inspired me greatly, and provided me with enormous support and encouragement. I will miss him dearly.
2 General Settings and the Main Theorems

Let \( \mathbb{D}^* \) be the complete hyperbolic punctured unit disk \( \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) with the unique complete hyperbolic metric \( ds^* \), conformally equivalent to the Euclidean metric. We will give this metric explicitly later. Let \( C_r \) be the Euclidean circle with center 0 and radius \( 0 < r < 1 \). \( C_r \) is a horocycle in \( \mathbb{D}^* \). Let \( B_r = \{ z : 0 < |z| < r \} \) be the interior of \( C_r \).

We are interested in adjusting the metric conformally in \( B_r \) in such a way that we get rid of the cusp and retain negative curvature. Thus, we will have a new metric of negative curvature on the unit disk \( \mathbb{D} \), which coincides with the metric on \( \mathbb{D}^* \) outside \( B_r \).

**Definition 1.** Let \( R \) be a region in \( \mathbb{D} \) which contains 0. We say that a metric \( ds \) on \( \mathbb{D} \) is a negatively curved completion of \( ds^* \) inside \( R \) if

(i) \( ds \) is conformally equivalent to \( ds^* \).

(ii) \( ds = ds^* \) outside \( R \).

(iii) The curvature \( \kappa(ds) \) of the metric \( ds \) satisfies

\[ \kappa(ds) < 0. \]

We may also say that \( ds \) is a completion of \( ds^* \) across \( \partial R \).

The main theorems we prove are

**Theorem 1.** \( ds^* \) has a negatively curved completion inside \( B_r \) if and only if

\[ \text{length}(C_r) > 2\pi. \]

In §7 we give several examples which show that the horocycle condition in the above theorem is sharp.

The next theorem is in the same spirit, but it gives a more accurate estimation of the curvature of the adjusted metric. This theorem leads us in §9 to a quantitative version of the comparison theorem from [Bro99].

**Theorem 2.** There exist \( C_1 > 0, C_2 > 0 \) such that

(i) For all \( 0 < \varepsilon < 1 \), if \( r > 1 - C_1\sqrt{\varepsilon} \), then \( ds^* \) has a conformal completion \( ds_\varepsilon \) in \( B_r \) with curvature

\[ -\left(1 + \varepsilon \right) < \kappa(ds_\varepsilon) < -\frac{1}{1 + \varepsilon}. \]
For all small $\varepsilon > 0$, if $ds^*$ has a conformal completion $ds_\varepsilon$ in $B_r$ with curvature

$$\kappa(ds_\varepsilon) < -\frac{1}{1+\varepsilon},$$

then $r > 1 - C_2\sqrt{\varepsilon}$.

3 Proof of Necessity in Theorem 1

In this section, we prove the necessity part of Theorem 1. The necessity of the horocycle condition is a direct consequence of the Gauss–Bonnet Theorem. We denote by $\kappa$ the Gaussian curvature, and by $\kappa_g$ the geodesic curvature of a curve.

Lemma 3. Let $R$ be a region in $\mathbb{D}$ which contains 0. If $ds$ is a negatively curved completion of $ds^*$ in $R$, then

$$\oint_{\partial R} \kappa_g \, d(\text{length}) > 2\pi.$$

Proof. By the Gauss–Bonnet Theorem in the region $R$ equipped with the metric $ds$ we have

$$\iint_R \kappa(ds) \, d(\text{area}) + \oint_{\partial R} \kappa_g \, d(\text{length}) = 2\pi,$$

and the first term is negative. \hfill \square

Proof of necessity in Theorem 1. Let $ds$ be a negatively curved completion of $ds^*$ in $B_r$. By the previous lemma

$$\oint_{C_r} \kappa_g \, d(\text{length}) > 2\pi. \quad (1)$$

Since the geodesic curvature of a horocycle is 1, we obtain $\text{length}(C_r) > 2\pi$. \hfill \square

4 The Complete Hyperbolic Punctured Disk

Before proving the sufficiency part in Theorem 1, we devote the next two sections to present some elementary formulas in the complete hyperbolic punctured unit disk.
Let \( \mathbb{D}^* \) denote the complete hyperbolic punctured unit disk, and let \( \mathbb{H}^2 \) denote the upper-half-plane \( \{ z : \Im(z) > 0 \} \) with the hyperbolic metric

\[
  ds^2 = \frac{dx^2 + dy^2}{y^2}.
\]

\( \mathbb{D}^* \) is isometric to \( \mathbb{H}^2 / \{ z \mapsto z + 1 \} \) via the map

\[
  z \in \mathbb{H}^2 \mapsto e^{2 \pi i z} \in \mathbb{D}^*.
\]  

(2)

This isometry lets us compute easily the metric on \( \mathbb{D}^* \):

\[
  ds^* = -\frac{1}{r \log r} |dz|, \tag{3}
\]

where \( r = |z| \).

**Lemma 4.** Let \( \gamma \) be a simple closed \( C^1 \)-curve around 0 in \( \mathbb{D}^* \). Denote by \( \text{int}(\gamma) \) the finite-area component of \( \mathbb{D}^* \setminus \gamma \). The total geodesic curvature of \( \gamma \) is given by

\[
  \oint_{\gamma} \kappa_g \, d(\text{length}) = \text{area}(\text{int}(\gamma)).
\]

**Proof.** The Euler characteristic of \( \text{int}(\gamma) \), \( \chi(\text{int}(\gamma)) \), is 0. So, by the Gauss–Bonnet Theorem we have:

\[
  \oint_{\gamma} \kappa_g \, d(\text{length}) + \iint_{\text{int}(\gamma)} \kappa \, d(\text{area}) = 2\pi \chi(\text{int}(\gamma)) = 0.
\]

The lemma follows at once since \( \kappa = -1 \). \( \square \)

**Lemma 5.** \( \text{area}(B_r) = \text{length}(C_r) = -2\pi / \log r \).

**Proof.** By (3),

\[
  \text{area}(B_r) = \int_0^{2\pi} \int_0^r \frac{t}{(t \log t)^2} \, dt \, d\theta = -2\pi \left. \frac{1}{\log t} \right|_0^r = -\frac{2\pi}{\log r}.
\]

Next, we can compute the length of \( C_r \) directly, or we can apply the Gauss–Bonnet Theorem to show that it is equal to the area (recall that \( \kappa_g = 1 \) for \( C_r \)). \( \square \)
We would like now to see some simple properties of the metric on \( D^* \) given in (3). Set
\[
\lambda^*(r) = -\frac{1}{r \log r}, \tag{4}
\]
\[
u^*(r) = \log(\lambda^*(r)) = \log \left( \frac{1}{r \log(\frac{1}{2})} \right). \tag{5}
\]
The function \( u^* \) is shown in Figure 1. For comparison, we show in Figure 2 the corresponding function \( u_D \) for the complete hyperbolic metric \( ds_D \) on the unit disk:
\[
u_D = \log \left( \frac{2}{1 - r^2} \right).
\]

**Lemma 6.** \( u^* \) satisfies

(i) \( \lim_{r \to 0} u^*(r) = \infty \).

(ii) \( \lim_{r \to 1} u^*(r) = \infty \).
Figure 2: The function $u_D$, where $ds_D = e^{u_D} |dz|

(iii) $(u^*)(r)' < 0$ for $r < \frac{1}{e}$.
(iv) $(u^*)(r)' > 0$ for $r > \frac{1}{e}$.
(v) For all $0 < r < 1$, $(u^*)''(r) > 0$.

Proof. The proof is a simple computation. □

5 Curvature for Conformally Euclidean Metrics

We recall that if

$$ds^2 = \lambda(x, y)^2(dx^2 + dy^2)$$

is a conformally Euclidean Riemannian metric in a domain $U \subseteq \mathbb{R}^2$, then its curvature is given by the well known formula

$$\kappa(ds) = -\frac{\Delta \log \lambda}{\lambda^2},$$

(6)
where $\Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$.

In particular, for a radially symmetric Riemannian metric

$$ds = e^{u(r)}|dz|$$

on the unit disk $\mathbb{D}$, (6) reduces to

$$\kappa(ds) = -\frac{u'' + \frac{1}{r}u'}{e^{2u}}.$$  (7)

6 Proof of Sufficiency in Theorem 1

We are ready to finish the proof of Theorem 1.

Proof of sufficiency in Theorem 1. Suppose $\text{length}(C_{r_0}) > 2\pi$. By Lemma 5, $r_0 > \frac{1}{e}$. Then, Lemma 6 tells us that $(u^*)'(r_0) > 0$ and $(u^*)''(r_0) > 0$. Therefore, there exists a convex monotonically increasing $C^2$-function $u(r)$ on $[0,1)$ such that

$$u(r) = u^*(r) \text{ for } r > r_0,$$

$$u'(0) = 0.$$  

Indeed, we can construct $u(r)$ as follows: First, define a positive continuous function $w(r)$ such that,

$$\int_{0}^{r_0} w(r) \, dr = (u^*)'(r_0),$$

$$w(r) = (u^*)''(r) \text{ for } r \geq r_0.$$  

Next, define the function $v(r)$ by

$$v(r) = \int_{0}^{r} w(s) \, ds.$$  

Observe that $v(r)$ is a $C^1$-function which coincides with $(u^*)'(r)$ for $r \geq r_0$. Finally, define $u(r)$ by

$$u(r) = u^*(r_0) + \int_{r_0}^{r} v(s) \, ds.$$  

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$u$ is a $C^2$-function which satisfies:

\[ \forall r \geq r_0, \quad u(r) = u^*(r), \]
\[ u'(0) = v(0) = 0, \]
\[ \forall r > 0, \quad u'(r) > 0 \Rightarrow \text{$u$ is monotonically increasing}, \]
\[ u'' = w > 0 \Rightarrow \text{$u$ is convex}. \]

To conclude, set $\lambda(r) = e^{u(r)}$, and $ds = \lambda(r)|dz|$. Then, we have:

a) $\lambda'(0) = 0 \Rightarrow ds$ is smooth at 0.

b) $ds$ coincides with $ds^*$ for $r > r_0$.

c) $\kappa(ds) < 0$ since $\log \lambda = u$ is convex, monotonically increasing, and since the curvature is given by $\kappa$. 

\[ \square \]

7 A Natural Question

Lemma 3 raises the possibility that there might be a negatively curved completion of $ds^*$ in any region $R$ in $\mathbb{D}$ which contains 0 and satisfies the Gauss–Bonnet restriction

\[ \oint_{\partial R} \kappa_g \, d(\text{length}) \geq 2\pi, \quad (8) \]

which would generalize Theorem 1. In this section we show, that the supposedly more general theorem is false. We build counterexamples of several different types.

7.1 A First Example

We give an example which shows that the Gauss–Bonnet restriction (8) is not enough for having a negatively curved completion of $ds^*$ in $R$. We use the following general criterion for not having a completion:

**Theorem 7.** Let $\gamma$ be a curve around 0 in $\mathbb{D}$ such that the function $\lambda^*(z)$ attains its maximum on $\gamma$ at a point $z_0$ for which $|z_0| < 1/e$. Then, $ds^*$ has no negatively curved completion across $\gamma$. 

Proof. Suppose, on the contrary, that $ds = \lambda(z)|dz|$ is a negatively curved completion of $ds^*$ across $\gamma$. On the one hand, since $\kappa(ds) < 0$, formula (9) tells us that $u = \log \lambda$ is a subharmonic function. Hence, by the maximum principle, $u(z_0)$ is the maximal value of $u$ in $\text{int}(\gamma)$. So, if $\hat{n}$ is the outer unit normal to $\gamma$, then

$$\frac{\partial u}{\partial \hat{n}}(z_0) \geq 0.$$  \hfill (9)

On the other hand by Lemma 6,

$$\frac{\partial u}{\partial \hat{r}}(z_0) < 0,$$  \hfill (10)

where $\hat{r}$ is a unit vector from 0 to $z_0$. Since $z_0$ is a maximal point, we have

$$\frac{\partial u}{\partial \hat{r}}(z_0) = \frac{\partial u}{\partial \hat{n}}(z_0) \langle \hat{r}, \hat{n} \rangle,$$  \hfill (11)

Now, notice that by Lemma 6, the ray from 0 to $z_0$ is contained in $\text{int}(\gamma)$. Therefore, $\langle \hat{r}, \hat{n} \rangle > 0$. From here we get that (9) and (11) are in contradiction with (10).

Theorem 8. There exists a simple closed smooth curve $\gamma$, homologous to $C_r$ in $\mathbb{D}^*$, such that

(i) $\oint_{\gamma} \kappa_g \, d(\text{length}) > 2\pi$.

(ii) $ds^*$ has no negatively curved completion across $\gamma$.  \hfill $\square$
Proof. Let \( \gamma \) be a horocycle with a slit (see Figure 3), i.e. smooth the curve which is the boundary of the region

\[
V = \{ |z| < R_2 \} \setminus \{ |z| > R_1 \text{ and } |\arg z| < \theta \},
\]

with \( R_1, R_2, \theta \) chosen as follows: Let \( R_2 \) satisfy \( \frac{1}{e} < R_2 < 1 \). By Lemma 6 we can find \( 0 < R_1 < \frac{1}{e} \) such that

\[
u^\ast(R_1) > \nu^\ast(R_2).
\]

(12)

By Lemma 5 \( \text{area}(B_{R_2}) > 2\pi \). Therefore, we can find \( \theta \) small enough, such that the area of \( V > 2\pi \).

Lemma 4 assures us that the Gauss–Bonnet restriction (i) is satisfied, and from Theorem 7, we immediately get that \( ds^\ast \) has no negatively curved completion across \( \gamma \).

7.2 A Convex Example

Now, we impose convexity on our curve. A convex curve is a curve whose geodesic curvature is non-negative.

**Theorem 9.** There exists a convex simple closed smooth curve \( \gamma \), homologous to \( C_r \) in \( \mathbb{D}^\ast \), such that

(i) \( \oint_{\gamma} \kappa_g \, d(\text{length}) > 2\pi \).

(ii) \( ds^\ast \) has no negatively curved completion across \( \gamma \).

Moreover, such a curve can be taken arbitrarily close to the horocycle of length \( 2\pi \).

**Proof.** Take \( \gamma \) to be composed of a horocyclic segment and a geodesic segment as in Figure 4: \( \gamma \) is the boundary of the domain

\[
V = \left\{ z \in \mathbb{H}^2 : |z - \frac{1}{2}|^2 > R^2, \ 0 \leq \Re z < 1, \ \Im z > R \cos \theta \right\}
\]

The total geodesic curvature of \( \gamma \) is given by the area of \( V \) (see Lemma 4):

\[
\oint_{\gamma} \kappa_g \, d(\text{length}) = \frac{1 - 2R \sin \theta}{R \cos \theta} + 2\theta,
\]

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Figure 4: A convex curve

which we would like to be $> 2\pi$ (for convenience, we write this in terms of $y = R \cos \theta$):

$$\frac{1}{y} - 2 \tan \theta + 2\theta > 2\pi.$$  \hspace{1cm} (13)

In order to apply the non-extendibility criterion of Theorem 7, we should also like to have (recall the map in Equation (2))

$$u^*(e^{-2\pi R}) > u^*(e^{-2\pi y}) \iff e^{2\pi R} > e^{2\pi y}.$$  \hspace{1cm} (14)

We may first find a pair $y_0, \theta_0 > 0$ which satisfies equality in (13) and satisfies inequality (14), and then decrease $\theta_0$ a little, keeping $y_0$ fixed. So, extracting $y$ from (13), we obtain that we should find $0 < \theta < \pi/2$ such that

$$\frac{\pi(1 - \cos \theta)}{\pi + \tan \theta - \theta} > -\cos \theta \log \cos \theta.$$  \hspace{1cm} (15)

The existence of such a $\theta$ can be shown by elementary calculus. In fact, the first three derivatives of the difference are 0 at $\theta = 0$, and the fourth derivative of the difference at $\theta = 0$ is $> 0$. So, we can find $\theta$ arbitrarily close to 0, which satisfies the last inequality. This shows that we can take our curve arbitrarily close to the horocycle of length $2\pi$, as stated. $\square$
7.3 The Condition of Gromov

In [Gro86], pp. 109–110, Gromov gives a necessary condition for extension of metrics, which is based on the Gauss–Bonnet Theorem. The condition as stated in Gromov’s book is a little obscure, and we rephrase it here for the sake of clarity:

**Theorem 10 ([Gro86], pp. 109–110).** Let $\gamma$ be a simple closed smooth curve bounding a disk $V$. Suppose that outside of $V$, we are given a Riemannian metric $d_s$ of negative curvature. Let $p, q \in \gamma$. $p, q$ dissect $\gamma$ into two segments $\alpha$ and $\beta$. Let $\text{length}(\beta) \leq \text{length}(\alpha)$. Suppose that every $\gamma^+ \subseteq \alpha$, possibly disconnected, with $\text{length}(\gamma^+) \leq \text{length}(\beta)$, satisfies

$$\int_{\alpha \setminus \gamma^+} \kappa_g d(\text{length}) \leq 0.$$ 

Then, $d_s$ cannot be extended to a negatively curved metric in $V$.

**Proof.** Let $d_s$ be a negatively-curved smooth Riemannian metric in $V$, which extends the given metric outside of $V$. Let $\delta$ be a path of minimal length in $V$ from $p$ to $q$ which is different from $\alpha$. Then,

1. $\delta$ is a $C^1$ curve,
2. $\delta \cap \partial V$ has non-positive geodesic curvature.

Define $\gamma^+ = \delta \cap \alpha$. Apply the Gauss–Bonnet Theorem to each of the complementary regions of $V \setminus \delta$ which touch $\alpha \setminus \gamma^+$. Taking negative curvature into consideration, together with [1] and [2], we get

$$\int_{\alpha \setminus \gamma^+} \kappa_g > 0,$$

contradicting our assumptions. 

Gromov also gives obstructions for extension of metrics of positive curvature, and for arbitrary Euler characteristic of $V$. These cases are treated similarly, and we will omit them here.

We give a class of examples which satisfy the Gauss–Bonnet necessary condition for extension, but meet Gromov’s obstruction.

The idea, shown to us by Mikhail Katz, is to find a situation in which two points are connected by two geodesics. This is in contradiction with the
fact that in simply connected non-positively curved spaces there is at most one geodesic connecting two points. Alternatively, one can see the following as a special case of Gromov’s obstruction, taking $\alpha$ to be the geodesic.

**Theorem 11.** There exists a convex simple closed smooth curve $\gamma$, homologous to $C_r$ in $D^*$, such that

(i) $\int_{\gamma} \kappa_g \, d(\text{length}) > 2\pi$.

(ii) $\text{ds}^*$ has no negatively curved completion across $\gamma$, even if we allow completions with metrics not conformal to the Euclidean metric.

**Proof.** We construct $\gamma$ as follows (see Figure 5): Take any geodesic $\tau$ in $H^2$, whose end points are in $\mathbb{R}$. Take a symmetric geodesic segment $\alpha$ about the “center” of $\tau$ of hyperbolic length $l_\alpha > 2\pi$. Now, extend the end points $p, q$ of $\alpha$ by horocyclic segments of total hyperbolic length $2\pi < l_\beta < l_\alpha$. Then, identify $D^*$ with $H^2/\{z \mapsto z+C\}$, where $C$ is the Euclidean distance between the end points of the horizontal segments. We get a convex curve $\gamma$ in $D^*$ whose total geodesic curvature $> 2\pi$ (Lemma 4).
Suppose $d_*$ is a negatively curved completion of $d_*$ across $\gamma$ (it might not be conformally Euclidean). Since the distance between $p$ and $q$

\[ \text{dist}(p, q) \leq l_\beta < l_\alpha, \quad (16) \]

there exists a geodesic $\delta \neq \alpha$ which connects these points. But this is in contradiction with the fact that in a Riemannian surface of negative curvature there is at most one geodesic in each homotopy class. \hfill \square

**Remark.** The above construction shows the existence of such curves. We could also work as in the previous example in the fixed model $\mathbb{H}^2/\{z \mapsto z+1\}$. In that case, the existence is less obvious: Fixing $\theta$ (see Figure 4) determines the length of the geodesic segment $\alpha$. Then, we can take concentric segments until we get the desired inequality (16). In terms of the last example, we should find $R, \theta$ which satisfy the inequalities (13) and

\[ \frac{1}{\cos^2(\theta)} > \cosh \left( \frac{1}{R \cos \theta} - 2 \tan \theta \right). \quad (17) \]

Here, the left-hand side is the hyperbolic cosine of the length of the geodesic segment $\alpha$, while the right-hand side is the hyperbolic cosine of the length of the horocyclic segment.

### 7.4 A Remark on the Last Two Examples

The examples presented in sections §7.2 and §7.3 seem similar, but in fact there are curves which satisfy the non-extendibility criterion of §7.2 but not that of §7.3, and vice versa. There are also curves which satisfy simultaneously the non-extendibility criteria of §7.2 and §7.3.

To see this, we will give values for $R$, $\theta$ which either satisfy or do not satisfy the inequalities (13), (14) and (17). The curve with parameters (see Figure 4) $R = 0.2$, $\theta = 0.8$ satisfies only the non-extendibility criterion of §7.2. The curve with parameters $R = 0.48$, $\theta = 1.55$ satisfies only the non-extendibility criterion of §7.3. Finally, the curve with parameters $R = 0.4$, $\theta = 1.45$ satisfies both of the non-extendibility criteria.

### 8 Curvature Estimations

In this section we prove Theorem 4. Until now, we were interested in adjusting the metric in $B_r$ to a metric of negative curvature. Now, we want
to have more control on the curvature of the new metric. Namely, we want it to be close to $-1$. We begin by an application of the Ahlfors–Schwarz Lemma (Ahl38).

### 8.1 An Ahlfors–Schwarz Bound

In this section we find a lower bound for $r$ in order to have a completion of $d s^*$ in $B_r$ with curvature as in Theorem 2. The proof is a direct application of the Ahlfors–Schwarz Lemma.

**Proof of Theorem 2, part(ii).** Suppose $d s_\varepsilon$ is a completion of $d s^*$ in $B_{r_\varepsilon}$ with curvature

$$\kappa(d s_\varepsilon^2) < -\frac{1}{1 + \varepsilon}. \tag{18}$$

The last inequality may also be written as

$$\kappa(d s_\varepsilon^2) < \kappa((1 + \varepsilon) d s_D^2) < 0. \tag{19}$$

Hence, from the Ahlfors–Schwarz Lemma, we obtain

$$\frac{d s_\varepsilon^2}{d s_D^2} < 1 + \varepsilon. \tag{20}$$

For $r > r_\varepsilon$ it means that

$$\frac{(1 - r_\varepsilon^2)^2}{4r^2(\log r)^2} < 1 + \varepsilon. \tag{21}$$

Expanding the left-hand side near $r = 1$ gives

$$1 + \frac{1}{3}(1 - r)^2 + o((1 - r)^2) < 1 + \varepsilon. \tag{22}$$

For an inequality of this sort to be valid, we must have

$$r_\varepsilon > 1 - C\sqrt{\varepsilon}$$

for all small $\varepsilon > 0$, with $C > 0$ which does not depend on $\varepsilon$. $\Box$
8.2 Proof of Theorem 2

Here we construct a completion of $d s^*$ in $B_r$, and we estimate its curvature. The existence of the constant $C_1$ as stated in Theorem 2 will follow.

We consider metrics with radial symmetry. Let
\[ d s = \lambda(r)|dz| \]
be such a metric on $\mathbb{D}$. We know that its curvature is given by
\[ \kappa(d s) = -\frac{u'' + \frac{1}{r}u'}{e^{2u}}, \quad (23) \]
where $u = \log \lambda$.

We denote by $u_D$, $u_D^*$ the $u$'s which correspond to the complete hyperbolic metrics on $\mathbb{D}$ and on $\mathbb{D}^*$, respectively. Explicitly (see Figure 1),
\[ u_D = \log \left( \frac{2}{1 - r^2} \right), \quad (24) \]
\[ u_D^* = \log \left( \frac{1}{r \log \left( \frac{1}{r} \right)} \right). \quad (25) \]

The idea is to find an intermediate function $u_\varepsilon$ which coincides with $u_{D^*}$ near $r = 1$, finite near $r = 0$, and is close to $u_D$ or $u_{D^*}$ for any $r$.

We begin with a technical lemma:

**Lemma 12.** $\exists A > 0$ such that for $r$ sufficiently close to 1 we have:
1. $0 < u_{D^*}(r) - u_D(r) < A(1 - r)^2$,
2. $1 - A(1 - r)^2 < \frac{u_{D^*}'(r)}{u_D'(r)} < 1$,
3. $1 < \frac{u_{D^*}''(r)}{u_D''(r)} < 1 + A(1 - r)^2$.

**Proof.** From the Taylor expansions near $r = 1$, we see that any $A > 1/3$ will do.

Let $r_\varepsilon = 1 - \sqrt{\varepsilon/A}$. Define the intermediate function $u_\varepsilon$ in two steps. First, define
\[ \bar{u}_\varepsilon(r) = \begin{cases} u_{D^*}(r) & : r_\varepsilon \leq r < 1 \\ u_D(r) + u_{D^*}(r_\varepsilon) - u_D(r_\varepsilon) & : 0 \leq r < r_\varepsilon. \end{cases} \quad (26) \]
Then, smooth $\bar{u}_\varepsilon$ near $r(\varepsilon)$ without changing much its first and second derivatives.
Lemma 13. The function $u_\varepsilon$ satisfies

1. $0 < u_\varepsilon - u_D < \varepsilon$,
2. $1 - \varepsilon < \frac{u'_\varepsilon}{u_D} < 1$,
3. $1 < \frac{u''_\varepsilon}{u''_D} < 1 + \varepsilon$

for all $\varepsilon$ small enough.

Proof. This follows immediately from the construction of $u_\varepsilon$ and from Lemma 12.

We are ready to prove the theorem.

Corollary 14. The curvature $\kappa$ of the metric

$$d s_\varepsilon = e^{u_\varepsilon/4(r)} |dz|$$

satisfies

$$-(1 + \varepsilon) < \kappa(\varepsilon) < -\frac{1}{1 + \varepsilon},$$

for all $\varepsilon$ small.

Proof. We have by formula (23)

$$\frac{\kappa(d s_\varepsilon)}{\kappa(d s_D)} = \frac{u''_\varepsilon/4 + u'_\varepsilon/4}{u''_D + u'_D} \cdot \frac{1}{e^{2(u_\varepsilon/4 - u_D)}}.$$ 

Then, by Lemma 13 the following inequalities are true for small $\varepsilon$:

$$\frac{\kappa(d s_\varepsilon)}{\kappa(d s_D)} > \frac{1 - \varepsilon/4}{e^{\varepsilon/2}} > \frac{1}{1 + \varepsilon},$$

and

$$\frac{\kappa(d s_\varepsilon)}{\kappa(d s_D)} < 1 + \varepsilon/4 < 1 + \varepsilon.$$

Since $d s_\varepsilon$ coincides with $d s^* \text{ for } r > 1 - C/\sqrt{\varepsilon}$, where $C = 1/(2\sqrt{A})$ and $A$ is from Lemma 12, the last two inequalities finish the proof of Theorem 2, part (i).
9 Compactification of Riemann Surfaces

Let \( S^O \) be a complete hyperbolic Riemann surface of finite area. We compactify \( S^O \) conformally, i.e. we simply fill in the cusps, and take the unique conformal structure on the filled in surface to get a compact Riemann surface \( S^C \). We would like to adjust the metric on \( S^O \) in disjoint neighbourhoods of the cusps to get a smooth metric across the cusps and retain negative curvature.

**Definition 2.** We say that the length of a cusp, \( P \), on \( S^O \), is \( \geq l \), if there is a neighbourhood of \( P \), which is isometric to a domain \( V \) in the punctured unit disk, which contains a closed horocycle of length \( \geq l \).

**Definition 3.** We say that the lengths of the cusps on \( S^O \) is \( \geq l \), if there exist disjoint closed horocycles around each cusp, each of length \( \geq l \).

The following theorem is an immediate consequence of Theorem 3.

**Theorem 15.** Let \( S^O \) be a complete hyperbolic finite-area Riemann surface. We can adjust the metric in horocyclic neighbourhoods of the cusps to get a smooth metric of negative curvature on \( S^C \) if and only if the cusps of \( S^O \) have lengths \( \geq 2\pi \).

In [Bro99], Brooks compares between the complete hyperbolic metrics \( ds^o \), \( ds^c \) on \( S^O \) and \( S^C \), respectively. His result is based on the Ahlfors–Schwarz Lemma. The curvature estimates in §8 give a quantitative version of this result:

**Theorem 16 (compare [Bro99]).** For every \( \varepsilon > 0 \), there exists \( l(\varepsilon) = O(1/\sqrt{\varepsilon}) \) as \( \varepsilon \to 0 \), such that for any complete hyperbolic finite-area Riemann surface \( S^O \) with cusps \( \geq l(\varepsilon) \), we have

\[
\frac{1}{1+\varepsilon} (ds^o)^2 \leq (ds^c)^2 \leq (1+\varepsilon)(ds^o)^2,
\]

outside horocyclic neighbourhoods of length \( l(\varepsilon) \).

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