Naked singularities, branes and Chern-Simons couplings: The dark side of the 2+1 black hole

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Abstract. Branes are naked singularities, analogous to linear or planar defects in crystals. Zero-branes in AdS spacetimes are “negative mass black holes”, which can be generalized to higher-dimensional branes. When these solutions are endowed with angular momentum, the extremal spinning branes correspond to BPS states. On the other hand, the \(2p\)-branes, spanning a \((2p+1)\)-dimensional worldsheet, provide a naturally coupling to CS field theories defined on a \(D\)-dimensional spacetime, with \(D > 2p + 1\). In this picture, the field that lives in the \(D\)-dimensional spacetime, as well as the sources that couple to it are made out of the same stuff—an \(SO(D-1, 2)\) connection. The fact that on the brane the AdS group is necessarily broken down to \(SO(2p, 2)\), brings in a number of tensor fields that play the role of charged matter living on the brane.

1. Introduction
Naked singularities are often viewed as unphysical, dreadful solutions that violate basic laws of physics. These structures would be some sort of Pandora’s box, from which anything can emerge, including “green slime, lost socks and TV sets” [1]. In view of this outrageous behavior, it has been suggested that nature abhors naked singularities, and a cosmic censorship mechanism, supposedly built-in in gravitation theory, would provide a forceful clothing for singularities. Clearly, an elegant way out of uncomfortable paradoxes.

Black hole solutions with mass and angular momentum parameters \(M\) and \(J\) exhibit a naked singularity at the center if \(1 \ M < |J|\). Hence, the existence of a cosmic censorship mechanism would imply that nature favors \(M \geq |J|\), and in particular, \(M \geq 0\). This agrees with the expectation that localized distributions of matter must have positive energy/mass. Not only elegant, but useful as well.

Numerous attempts to rigorously prove the cosmic conjecture have failed, and some numerical experiments of gravitational collapse under reasonable initial conditions, inevitably lead to the formation of naked singularities [2]. However, in particular cases it can be verified that nature doesn’t accept naked singularities quietly. It is a simple exercise to check that a charged...
extremal black hole repels a charged particle whose charge is larger than its mass [3]. Other numerical experiments also provide convincing evidence that extremal naked singularities are generically unstable under linearized perturbations [4]: a small localized perturbation around a naked singularity (e.g., a black hole with negative mass parameter), grows exponentially in finite time for generic initial data in the linearized approximation. Although the unbounded growth of the perturbation means that the validity of the linear approximation eventually breaks down, it is a strong indication that something smells rotten in the negative mass black hole.

The standard four-dimensional Schwarzschild black hole has a curvature that diverges for \( r \to 0 \), and if the mass parameter \( M < 0 \) there is no horizon around the singularity. This is a problem because a divergent Riemann tensor corresponds to an unbounded energy-momentum density. A naked singularity where the curvature tensor diverges implies an infinite accumulation of energy in a finite region and that would naturally lead to a breakdown in the laws of physics.

Possibly the simplest case of naked singularity is the static three-dimensional black hole with \( M < 0 \). The geometry of the 2+1 black hole, unlike its four dimensional counterpart, has constant negative curvature for any sign of \( M \) [5]. Thus, even in the absence of a horizon surrounding the singularity, an external observer would never have access to a region of infinite curvature. The naked singularity of a negative mass 2+1 black hole is similar to the singularity at the tip of a cone, a point where the manifold structure breaks down because differentiability fails. Nevertheless, this type of singularity is sufficiently mild to allow unambiguous interpretations. The curvature singularity at the tip of a cone the has the form of a Dirac delta, which means that its integral cannot represent an infinite amount of energy concentrated in a finite region. Thus, a conical singularity shouldn’t be expected to be the source of infinitely growing excitations like those observed in the instabilities mentioned above.

A naked singularity defined by the set of points where the manifold structure ends, such as a boundary, a domain wall, or a topological defect, is not necessarily a source of paradoxes, but determines the boundary conditions for the dynamical fields, including those that determine the spacetime geometry itself. In the presence of a boundary of this type it is necessary to specify the conditions on the fields, which must be in agreement with the topological obstruction produced by the singularity. In the action principle, this is afforded by the presence of an appropriate source term, that represents the coupling of the geometry to a point-like source or, more generically, to a 2p-brane.

2. Review of the 2+1 Black Hole

The geometry of the 2+1 black hole with mass \( M \) and angular momentum \( J \) is given by the metric

\[
ds^2 = -f(r)dt^2 + [f(r)]^{-1}dr^2 + r^2(Ndt + d\phi)^2,
\]

where \( f(r) = -M + \frac{r^2}{r_+^2} - \frac{\ell^2}{4r_+^2} \), \( N = -\frac{J}{2r_+} \), and \( \ell \) is a yardstick that defines the unit of length (AdS radius). The physical spectrum is the region of the \( M - J \) plane defined by \( M \geq |J|\ell^{-1} \geq 0 \), that corresponds to black hole states. Additionally, the point \( M = -1, J = 0 \) represents anti-de Sitter space. The states \( M = |J|\ell^{-1} > 0 \) are extremal black holes, and the state \( M = |J|\ell^{-1} = 0 \) is the lowest energy (ground state) configuration. As noted in [6], this locally AdS geometry can be obtained by a finite isometric identification in the AdS covering space. That is, the black hole manifold is a quotient of AdS3 by a Killing vector of AdS3.

\[
\Gamma = H^{2,2}/x(M, J),
\]

where the Killing vector \( x(M, J) \) is a boost. The AdS3 geometry is defined as a pseudosphere embedded in a four-dimensional flat spacetime \( \mathbb{R}^{2,2} \),

\[
-x_0^2 + x_1^2 + x_3^2 - x_4^2 = -\ell^2
\]
The explicit form of the identification in the coordinates of the embedded pseudosphere is

\[ \xi_{\text{Generic}} = r + J_{12} - r - J_{03}, \] (4)

where

\[ r^2 = \frac{M}{2} \left[ 1 \pm \sqrt{1 - \frac{J^2}{M^2 \ell^2}} \right] \ell^2, \] (5)

and \( J_{ab} \) is the generator of rotations/boosts in the \( a - b \) plane.

The ground state (\( M = 0 = J \)) and the extremal (\( M = J \)) black holes are obtained by means of the following identifications:

\[ \xi_{\text{Vac}} = \frac{1}{2} (J_{12} + J_{03} + J_{02} - J_{13}), \] (6)

\[ \xi_{\text{Ext}} = r + (J_{01} - J_{23}) + J_{12} - J_{20} + J_{03} - J_{13}, \] (7)

respectively. Notice that the Killing vector for the extremal and the vacuum solutions are not obtained by just taking the appropriate limits in the space of parameters of \( \xi_{\text{Generic}} \). For a good review of the geometry and physics of the 2+1 black hole, see, e.g., [7].

3. Conical defect in AdS

It was observed by Izquierdo and Townsend, that a static naked singularity solution (\( M < 0, J = 0 \)) corresponds to conical defect [8]. To illustrate the point, consider a two-dimensional cone, described by locally flat coordinates \( (x^1, x^2) = (r \cos \phi, r \sin \phi) \), where the radial coordinate takes values \( 0 \leq r < \infty \), and the azimuthal angle \( \phi \) has a deficit, \( 0 \leq \phi \leq 2\pi(1 - \alpha) \). The angular deficit produces a naked singularity at the center, \( r = 0 \), where the curvature has a Dirac \( \delta \). This cannot be seen just from reading the metric, which in these coordinates has the standard flat form, \( ds^2 = dr^2 + r^2 d\phi^2 \). It is the identification \( \phi_{12} \simeq \phi_{12} + 2\pi(1 - \alpha) \), with \( 0 \leq \alpha < 1 \), that is responsible for the singularity at \( r = 0 \). A standard azimuthal angle \( \phi \) of period \( 2\pi \) can be introduced by rescaling \( \phi_{12} \) \( (\phi_{12} = [1 - \alpha] \phi) \), making the topological defect manifest through a factor in the angular component of the metric. The resulting Riemann curvature two-form in two dimensions is \( R^1{}_{2} = d\omega^1{}_{2} \), where \( \omega^1{}_{2} = -d\phi_{12} \). Then, as pointed out in [9], the identity \( dd\phi_{12} = -2\pi\alpha \delta(T_{12}) d\Omega_{12} \) is valid in the sense of Stokes’ theorem upon integration, and the curvature exhibits the singularity at the origin,

\[ R^1{}_{2} = 2\pi\alpha \delta(T_{12}) d\Omega_{12}. \] (8)

The two-form Dirac delta \( \delta(T_{12})d\Omega_{12} \) has support at \( r = 0 \) on the two-dimensional plane \( T_{12} \). It can also be checked that the torsion tensor vanishes, thanks to the property of the Dirac distribution, \( r\delta(r) = 0 \).

The result can be re-interpreted as an identification by the Killing vector \( \xi = \xi^i \partial_i = -2\pi\alpha \partial_{\phi_{12}} \), where \( \partial_{\phi_{12}} = x_1 \partial_2 - x_2 \partial_1 \) is the generator of rotations around the origin in the 1-2 plane. The angular defect results from the identification \( x^i \simeq x^i + \xi^i \),

\[ \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \simeq \begin{pmatrix} \cos 2\pi\alpha & \sin 2\pi\alpha \\ -\sin 2\pi\alpha & \cos 2\pi\alpha \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}. \] (9)

Note that the strength of the curvature singularity in the identified geometry is the magnitude of the angular deficit, \( 2\pi\alpha \), times the two-form delta distribution, \( \delta(T_{12})d\Omega_{12} \), which can be identified as the source of the conical singularity. This happens whenever a curvature singularity is the result of an identification by a spacelike Killing vector that leaves fixed points [10].
Killing vector $\xi$ is everywhere spacelike, and hence it does not produce closed timelike curves. This geometry has constant negative curvature everywhere, except at the point $r = 0$. Therefore, this is a solution of the Einstein equations with a source, obtained from the action

$$I[g] = \int_{M^3} \left( \frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right) - \int_{\Gamma^1} \langle j \wedge A \rangle,$$

(10)

where $M^3$ is the three-dimensional spacetime manifold, $\Gamma^1$ is the worldline defined by the point $r = 0$, and $j$ is the two-form source defined by the right hand side of (8).

4. The “negative mass” black hole

Consider now a conical topological defect in the AdS spacetime. In polar coordinates (here we follow the notation and conventions of [11]),

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 (1 - \alpha)^2 d\phi^2,$$

(11)

where the azimuthal angle has the range $0 \leq \phi \leq 2\pi$, and $0 \leq \alpha \leq 1$ is the angular deficit. With a coordinate rescaling, it is possible to rewrite this metric as a black hole of mass $M$,

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2}\right) dt^2 + \left(-M + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\phi^2,$$

(12)

where the mass parameter is related to the angular defect by

$$M = -(1 - \alpha)^2.$$

(13)

Clearly this means that the conical defects belong to the negative part of the black hole energy spectrum. The configuration without angular deficit ($\alpha = 0$) corresponds to the AdS spacetime ($M = -1$), while the maximal deficit, $\alpha = 1$, represents the zero-mass black hole.

A natural question arises here: Are these solutions stable configurations? In order to answer this, one could analyze the perturbative spectrum. However, as is well known, 2+1 gravity has no local degrees of freedom, and hence, the corresponding spectrum is rather trivial: the excitations are all pure gauge transformations connected to the identity. The only nontrivial excitations are of topological character, resulting from identifications on AdS$_3$: black holes ($M \geq 0$) are obtained by identifying with a timelike Killing vector, corresponding to a boost; the topological defect states ($M < 0$) result from identifying with a spatial rotation. There is nothing to prevent these states from decaying – e. g., via Hawking radiation – into a zero-mass black hole or into global AdS$_3$, the only two stable configurations continuously connected with $-1 \leq M \leq 0$, $J = 0$.

5. BPS States

The only stable states are those for which a Bogomol’nyi bound is saturated (BPS states). In those cases, the existence of a globally defined Killing spinor (covariantly constant spinor), is sufficient to guarantee that a residual supersymmetry is preserved. The presence of some residual unbroken supersymmetries implies the existence of a lower bound for the sum of charges, the Bogomol’nyi bound. This leads to the positivity of energy in standard supergravity [12, 13, 14, 15], and ensures the stability of the configurations that saturate the energy bound (BPS states). These BPS configurations define good vacua for a perturbative expansion and, in order to establish that a given geometry is a stable ground states, one should investigate whether the geometry admits a globally defined covariantly constant spinor or not [16].
A globally defined Killing spinor is a solution of the equation \( D\psi = 0 \), in the spacetime region surrounding the topological defect, where \( D \) is the covariant derivative operator in \((2+1)\)-dimensional, locally AdS spacetime. In the present case, this means
\[
D\psi = \left( d + \frac{1}{4} \omega^{ab} \Gamma_{ab} + \frac{1}{2\ell} e^a \Gamma_a \right) \psi = 0,
\]
where the spacetime region is defined by \( r > 0, -\infty < t < +\infty \), and \( 0 \leq \phi \leq 2\pi \). The spin connection and vielbein for the locally AdS and torsion-free spacetime determined by the metric (12) are
\[
e^0 = \ell^{-1} \left( -M + \frac{r^2}{\ell^2} \right)^{1/2} dt, \quad \omega^{01} = \frac{r}{\ell^2} dt
\]
\[
e^1 = \left( -M + \frac{r^2}{\ell^2} \right)^{-1/2} \ell dr, \quad \omega^{12} = - \left( -M + \frac{r^2}{\ell^2} \right)^{1/2} d\phi
\]
\[
e^2 = r d\phi, \quad \omega^{02} = 0.
\]
(15)

In addition, the spinor might be coupled to a connection field (electromagnetism or some other), but let us limit the discussion to the simplest possibility.

Direct computation shows that the solution takes the form,
\[
\psi = \exp \left[ f(r) \Gamma_1 \right] \exp \left[ \frac{i}{2} \alpha \left( \phi + \frac{t}{\ell} \right) \right] \psi_0,
\]
where \( \psi_0 \) is a constant spinor such that \( \Gamma_0 \psi_0 = -\psi_0 \). It is clear that the only acceptable conditions for a globally defined \( \psi \) are: \( \psi(\phi + 2\pi) = \psi(\phi) \) (single-valued), or \( \psi(\phi + 2\pi) = -\psi(\phi) \) (double-valued). The first case implies \( \alpha = 0 \), while the second requires \( \alpha = 1 \). These two cases are precisely the conditions for the globally AdS spacetime and the massless black hole, respectively. This means that these naked singularities are not BPS states and therefore are not protected from disintegration into more stable configurations (either the \( M = 0 \) or the \( M = -1 \) states).

This situation could have been anticipated: the \( M > 0, J = 0 \) black holes are not BPS either [17]. It is possible, however, to turn a positive mass black hole into a BPS state by boosting it with the right amount of angular momentum. The question then comes to mind: would it be possible to stabilize a naked singularity as well? The answer is yes, and the situation reflects in every detail the positive mass case: one can boost a black hole into a spinning state, and the same can be done with a topological defect. The result is a geometry that resembles the spinning BTZ solution,
\[
ds^2 = -f^2(r) dt^2 + \frac{dr^2}{f^2(r)} + r^2 (N dt + d\phi)^2,
\]
(17)
where, as in the standard case,
\[
f^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \quad \text{and} \quad N = -\frac{J}{2r^2},
\]
(18)
but now \( M\ell^2 < -|J| \) implies that both horizons are absent
\[
r^2_\pm = \frac{M\ell^2}{2} \left( 1 \pm \sqrt{1 - \frac{J^2}{M^2\ell^2}} \right), \quad \text{imaginary.}
\]
(19)

\(^2\) See, e.g., Refs. [11, 18] for details.
For the extreme case \( M \ell^2 = -|J| \) there is one globally defined Killing spinor given by [17]

\[
\psi = h(r) \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},
\]

where \( h(r) = \sqrt{\frac{\ell}{r} - \frac{M}{r}} \).

6. Generalizations

A similar result is obtained if the spinor is coupled to an electromagnetic potential [18]. In that case, the spinor must be complex and the Killing equation (14) has an additional term \( ieA\psi \).

An interesting electromagnetic source in this spacetime is the static magnetic pole at \( r = 0 \). Its electromagnetic potential is \( A = gd\phi \), which looks like an exact form, but it is not: The zero-form \( \phi \) is not a well defined (single-valued) function in the \( r - \phi \) plane, as can be easily checked by computing the integral of \( A \) around a closed loop around the origin (\( \int A = 2\pi g \neq 0 \)).\(^3\) In the presence of this magnetic pole, the Killing spinor gets an extra contribution to the phase from the electromagnetic flux through the center of the constant time slices. Now, if the flux exactly matches the angular deficit of the conical defect, a Killing spinor can be globally defined everywhere in the 3D spacetime for \( r \neq 0 \). The matching condition is [18],

\[
(1 - \alpha)(1 + 2eg) \in \mathbb{Z}.
\]

Note that there are multiple options to satisfy this condition for \( 2eg \in \mathbb{Z} \) and \( 0 < \alpha < 1 \). For instance, for \( eg = 1 \), \( \alpha \) can take the values 1/3 and 2/3, for \( eg = 8 \), \( \alpha = 1/17, 2/17, \ldots, 16/17 \), etc.

These results can be generalized to other gauge groups, to higher dimensions, or both. The example of a zero brane in three spacetime dimensions analyzed above can be easily generalized to a 0-brane in higher dimensions, or to branes whose worldvolumes are codimension 2 manifolds. More generically, one can consider 2\( p \)-branes embedded in a \( (2n+1) \)-dimensional spacetime, with \( n > p \). These possibilities have been discussed in [10] for the case of gravitational Chern-Simons theories, as well as generic couplings between CS forms and 2\( p \)-branes.

The general result can be summarized as follows:

- A 2\( p \)-brane can be seen as a charged object that couples to a connection through the Chern-Simons form. The world history of the brane, \( \Gamma^{2p+1} \), is the manifold on which the CS form is defined. This world history is a submanifold that can be isometrically embedded in a \((2n+1)\)-dimensional spacetime \((n \geq p)\), where the brane is a topological defect.

- Zero-branes in spacetimes of \( D > 3 \), obtained by elimination of and angular sector are not constant curvature manifolds. This is because a solid angle deficit cannot be produced by an identification with a Killing vector, as is the case of the \( 2 + 1 \) black hole—with either positive or negative mass. The result is a naked curvature singularity which is likely to be unstable.

- Branes whose world history is a manifold of codimension 2, can be obtained by identification with a single Killing vector. The resulting spacetime has the same local curvature as the space before the identification, but the topology and the causal structure depend on the nature of the Killing vector used. These are topological defects in the same class with the conical singularities discussed earlier. Several identifications can be made simultaneously, provided the corresponding Killing vectors commute, as for example in a two-dimensional torus.

- Identifications can also be performed even if the resulting manifold is not a quotient space. For example, one could make a polyhedron by removing several angular sectors centered around different points. The resulting solution is a multi-centered collection of conical defects. The

\(^3\) This is consistent with the notation \( F = dA = 2\pi g d\phi(r) \), although this notation might be confusing since it suggests that \( r = 0 \) is a regular point in the manifold, while it is actually an excluded point in the \( r - \phi \) plane.
only constraint that seems to exist on such structure is that the sum of defects cannot exceed
the value for the zero-mass black hole,
\[ \sum_i \alpha_i \leq 1. \] (22)

7. Discussion
Action: The branes discussed here are geometrical structures obtained from the application of
certain boundary conditions that introduce peculiar topological features, but they are essentially
made from the same stuff as the rest of the spacetime geometry. These structures do not require
the introduction of additional matter: the same connection 1-form that defines the dynamics of
the bulk geometry can be used to couple to the brane.

For instance, the spacetime geometry in 2+1 dimensions is determined by a Chern-Simons
action for the \( SO(2, 2) \) algebra [19]. This has been generalized to higher dimensions and even to
the supersymmetric case [20]. What is remarkable in the theory that emerges when one allows
for conical defects or their higher dimensional generalizations, is that these new structures don’t
require additional fields to implement them: The dynamics of the intrinsic brane geometry is
governed by the CS action living on the brane history, as in (10) [21, 22]. For an observer living
in the bulk, however, the brane looks as some source that could even be charged, the charge
being related to the magnitude of the angular deficit at the brane.

This seems like a consistent picture –at least classically–, in which the distinction between
bulk degrees of freedom (connection fields \( A|_M \)) and brane degrees of freedom (connection
fields restricted to the brane world \( A|_\Gamma \)) is metaphysical. Naturally, the components of the
connection 1-form, when restricted to \( \Gamma \), split into the components of a connection for a smaller
symmetry group, and some 0-forms fields transforming in some tensor representations of the
reduced symmetry group. To an observer in the bulk, these tensor fields would look like “matter”
fields, charged under the reduced gauge group, in a standard field-theoretical interpretation [23].

Solutions: The “negative energy black hole states” identified as topological defects, are
clearly not black holes but naked singularities. In 2+1 dimensions these singularities are
topological defects, and hence quite harmless. These solutions are similar to ordinary (classical)
elementary particles: They are point-like objects, devoid of a horizon and therefore temperature
and entropy cannot be defined for them. An interesting question is whether these states belong
to the microscopic states that contribute to the black hole entropy count. It seems that in
principle they should, but it is not obvious how to count these states.

The interpretation of these states as classical particles seems more natural, and is also sup-
ported by the fact that some of these states, such as the extreme case \( M = -|J| \), or in the
electrically charged solution with (21), are BPS. These states are supersymmetric invariant
ground states that exhibit a positive energy spectrum of excitations, and therefore, perturba-
tively stable. This corresponds exactly to the idea of an elementary particle: a classically stable
state.

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