Neutron Star Structure in the Minimal Gravitational Standard-Model Extension and the Implication to Continuous Gravitational Waves

Rui Xu\textsuperscript{a,}, Junjie Zhao\textsuperscript{b}, Lijing Shao\textsuperscript{a,}\textsuperscript{*}

\textsuperscript{a}Kavli Institute for Astronomy and Astrophysics, Peking University, Beijing 100871, China

\textsuperscript{b}School of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China

Abstract

Tiny violation of Lorentz invariance has been the subject of theoretic study and experimental test for a long time. We use the Standard-Model Extension (SME) framework to investigate the effect of the minimal Lorentz violation on the structure of a neutron star. A set of hydrostatic equations with modifications from Lorentz violation are derived, and then the modifications are isolated and added to the Tolman-Oppenheimer-Volkoff (TOV) equation as the leading-order Lorentz-violation corrections in relativistic systems. A perturbation solution to the leading-order modified TOV equations is found. The quadrupole moments due to the anisotropy in the structure of neutron stars are calculated and used to estimate the quadrupole radiation of a spinning neutron star with the same deformation. The calculation puts forward a new test for Lorentz invariance in the strong-field regime when continuous gravitational waves are observed in the future.

Keywords: neutron star, Lorentz violation, Standard-Model Extension, continuous gravitational waves

1. Introduction

Neutron stars are important objects in astrophysics as the observation of them provides unique tests of gravitational theories and fundamental principles \cite{1, 2, 3}. The masses and orbital parameters of highly magnetized rotating neutron stars that emit radio waves in a binary system can be accurately determined by pulsar timing techniques \cite{1, 4, 5}. Another observation channel, the direct gravitational wave detection from binary neutron stars, though only had its practice in 2017 \cite{6}, is having more facilities in construction and developing promisingly \cite{7, 8, 9, 10, 11}. No matter through pulsar signals or gravitational waves, the information of neutron stars we receive not only supplies us knowledge about matter at supranuclear densities \cite{12, 13, 14}, but also constrains various alternatives to General Relativity (GR) \cite{15, 16, 17, 18}.

We are particularly interested in testing gravitational theories with tiny violation of Lorentz invariance, which is believed to be a possible quantum gravity effect \cite{19, 20, 21, 22, 23, 24, 25, 26}. At the low-energy level, any kind of such violation includes couplings between the Lorentz-violation fields and the conventional fields in effective field theory \cite{27}. The framework is called the Standard-Model Extension (SME) \cite{28, 29}, in which the complete Lagrangian density reads \cite{30}

\[ \mathcal{L}_{\text{SME}} = \mathcal{L}_{\text{GR}} + \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{LV}} + \mathcal{L}_k. \]  \hspace{1cm} (1)

In the expression, \( \mathcal{L}_{\text{GR}} \) represents the usual Einstein-Hilbert term for General Relativity, and \( \mathcal{L}_{\text{SM}} \) is the Lagrangian density of the Standard Model. One of the extra terms, \( \mathcal{L}_{\text{LV}} \), consists of Lorentz violation couplings in the form of \( (k^{(d)})^a J_a \), with \( (k^{(d)})^a \) being a Lorentz-violation field and \( J_a \) being an operator of mass-dimension \( d \) constructed from the conventional fields. The other term, \( \mathcal{L}_k \), describes the dynamics of the Lorentz-violation fields.

The Lorentz-violation couplings in \( \mathcal{L}_{\text{LV}} \) are naturally categorized by the mass-dimensions of the conventional field operators. For our purpose, we only consider the gravitational SME where the conventional field operators \( J_a \) are constructed using the Riemann tensor so that \( d \) starts with 4 \cite{30, 31}. The \( d = 4 \) coupling is simply \( \frac{1}{16\pi G} (k^{(4)})^{a\beta\gamma} R_{a\beta\gamma\delta} \), but intentionally introduced in terms of the trace-free components of \((k^{(4)})^{a\beta\gamma}\) and...
R_{\alpha\beta\gamma\delta} in Ref. [31] as
\[ L_{\text{NV}}^{(4)} = \frac{1}{16\pi G} \left( -u R + s^{\alpha\nu} R_{\mu\nu} + \rho^{\mu\nu} C_{\alpha\beta\gamma\delta} \right), \] (2)
where \( R_{\mu\nu} \) is the trace-free Ricci tensor and \( C_{\alpha\beta\gamma\delta} \) is the Weyl conformal tensor. By splitting the \( d = 4 \) Lorentz-violation model into three pieces, \( u, s^{\alpha\nu}, \) and \( \rho^{\mu\nu} \), Eq. (2) consists of all the Lorentz-violation couplings in the minimal gravitational SME. Any coupling with the mass dimension \( d \) of the conventional field operator greater than 4 belongs to the nonminimal sector of the SME framework [30, 32, 33, 34]. Nonminimal couplings involve more derivatives and are considered to be suppressed at least by factors of \( \frac{E}{M_P} \), where \( E \) is the energy below which effective field theory works and \( M_P \) is the Planck mass. Therefore, we will only consider the minimal couplings in our work but point out that extending the treatment to nonminimal Lorentz-violation couplings is still a relevant topic as in some specific Lorentz-violation models there is no minimal Lorentz-violation coupling and therefore the dominant effect comes from nonminimal terms [35, 36].

As we study gravity at the classical level, in Eq. (1), \( L_{\text{SM}} \) is replaced by the Lagrangian density for macroscopic matter. Given an explicit expression for \( L_{\text{L}} \), we can get a set of modified Einstein field equations with Lorentz violation by taking the variation with respect to the metric \( g_{\mu\nu} \). Though the modified Einstein field equations depend on the specific dynamics of the Lorentz-violation fields in \( L_{\text{L}} \), it is shown in Ref. [31], that in the weak-field regime, the linearized modified Einstein field equations at the leading order of Lorentz violation can be expressed using the vacuum expectation values of the Lorentz-violation fields under several reasonable assumptions. Those vacuum expectation values of the Lorentz-violation fields, denoted as \( u, s^{\alpha\nu}, \) and \( \rho^{\mu\nu} \), and to be distinguished from the fields themselves, are called the Lorentz-violation coefficients. Introducing the Lorentz-violation coefficients makes the SME framework practically useful by allowing experiments to test the coefficients without worrying about the dynamics of the corresponding fields as long as gravity is weak [37, 18, 38]. A large amount of constraints have been put on the Lorentz-violation coefficients from various terrestrial experiments [39, 40, 41, 42] and astrophysical observations [43, 44, 32, 34, 45, 46, 47, 48, 49, 50] that assume the validity of weak gravity.

When applied to neutron stars, the nonlinearity in the Einstein field equations cannot be neglected. Directly using the linearized result in Ref. [31] leads to a set of Lorentz-violation hydrostatic equations only at the Newtonian level, causing inaccuracy in describing the structure of neutron stars. Inspired by the post-Tolman-Oppenheimer-Volkoff (post-TOV) approach [51, 52], we mend the inaccuracy by replacing the Lorentz-invariant Newtonian terms with the GR terms in the TOV equation. In this way, the Lorentz-violation terms, once isolated, can be used to calculate the leading-order modification to the structure of neutron stars.

We derive in Sec. 2 the Lorentz-violation hydrostatic equations by employing the linearized result in Ref. [31]. In Sec. 3, perturbative expansions of the fluid variables respect to the Lorentz-violation coefficient are performed so that the Lorentz-invariant terms can be identified and replaced by the GR terms in the TOV equation, leaving the Lorentz-violation terms forming the first-order equations. The first-order solution is provided and the calculation to determine the angular dependence is shown in Appendix A. Finally in Sec. 4, a spinning neutron star is assumed to estimate the amplitude of the quadrupole radiation from the Lorentz-violation induced deformation. The possibility of using continuous gravitational waves to constrain Lorentz violation is discussed. Throughout the work, we follow the notation and conventions of Ref. [29].

2. The Lorentz-violation hydrostatic equations

The TOV equation describes the distribution of the density and pressure inside a perfect fluid whose local energy-momentum tensor can be expressed as
\[ T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + pg^{\mu\nu}, \] (3)
where \( \epsilon \) is the proper energy density and \( p \) is the proper pressure of the fluid. The 4-velocity \( u^\mu \) can be taken as \( \left( 1, \frac{\sqrt{-g}}{\sqrt{-g_{00}}}, \frac{\partial g_{00}}{2g_{00}} \right) \) for a static configuration, and then the energy-momentum conservation equations \( D_\nu T^{\nu\mu} = 0 \) give
\[ \partial_\nu (\epsilon + p) \frac{\partial g_{00}}{2g_{00}} = 0. \] (4)

Equation (4) seems to indicate that static structures of fluids only depend on the metric component \( g_{00} \). However, other metric components come into the Einstein field equations in solving \( g_{00} \), and the presence of the fluid variables themselves in the field equations complicates the problem. In the case of a spherical fluid, the Einstein field equations imply [53]
\[ \frac{\partial_\nu g_{00}}{2g_{00}} = \left( \frac{Gm(r)}{r^2} + 4\pi G\rho r \right) \left( 1 - \frac{2Gm(r)}{r} \right)^{-1}, \] (5)
with $\tilde{\alpha}_{g00} = \tilde{\alpha}_{g00} = 0$ in the Schwarzschild coordinates. The mass function is defined as $m(r) = 4\pi \int_0^r \rho(r') r'^2 \, dr'$. Hence, Eq. (4) gives the standard TOV equation

$$\partial_r p = - (\epsilon + p) \frac{Gm(r) + 4\pi Gr^2 p}{r [r - 2Gm(r)]},$$

(6)

with the angular equations vanishing as expected.

In the case of the minimal Lorentz violation, the static Newtonian solution for $g_{00}$ obtained in Ref. [31] reads

$$g_{00} = -1 + 2U + \tilde{s}^j k \tilde{U}^{jk} + O(1PN),$$

(7)

where $\tilde{s}^j k$ with $j,k = 1,2,3$ are the vacuum expectation values of the spatial components of the Lorentz-violation field $s^{0j}$, namely the spatial components of the Lorentz-violation coefficient $\tilde{s}^{0j}$. The Newtonian potentials $U$ and $\tilde{U}^{jk}$ are defined as

$$U = G \int \frac{1}{|\vec{x} - \vec{x}'|} \rho(\vec{x}') \, d^3 x',$$

$$\tilde{U}^{jk} = G \int \frac{(x^j - x'^j)(x^k - x'^k)}{|\vec{x} - \vec{x}'|^3} \rho(\vec{x}') \, d^3 x',$$

(8)

where $\rho$ is the baryonic rest mass density. The relative difference between $\rho$ and $\epsilon$, namely $\Pi = \frac{\epsilon - \rho}{\rho}$, is the internal energy per unit baryonic mass. Note that we have ignored the temporal component $\tilde{s}^{00}$ as it merely rescales the gravitational constant $G$ in a static system and hence does not produce any observable effect. For the same reason, the Lorentz-violation coefficient $\tilde{u}$ does not appear and a traceless condition

$$\eta_{jk} \tilde{s}^{jk} = 0,$$

(9)

can be imposed. As for the absence of the Lorentz-violation coefficient $\tilde{t}^{0j0k}$, it is proved in Ref. [31], that all the terms involving it automatically cancel out, though a plausible physical explanation for this remains missing at the moment (the so-called $t$ puzzle) [54].

Equation (7) represents the Lorentz-violation solution for $g_{00}$ generated by the couplings in Eq. (2) at the Newtonian level. When plugged into Eq. (4), we get a set of Newtonian hydrostatic equations with modifications from the minimal Lorentz violation. These equations are

$$\partial_r p = \rho \left( \partial_r U + \frac{1}{2} \tilde{s}^j k \partial_r \tilde{U}^{jk} + O(1PN) \right),$$

$$\partial_\theta p = \rho \left( \partial_\theta U + \frac{1}{2} \tilde{s}^j k \partial_\theta \tilde{U}^{jk} + O(1PN) \right),$$

$$\partial_\phi p = \rho \left( \partial_\phi U + \frac{1}{2} \tilde{s}^j k \partial_\phi \tilde{U}^{jk} + O(1PN) \right).$$

(10)

The terms in “$O(1PN)$” include both Lorentz-invariant and Lorentz-violation post-Newtonian corrections. The former are accounted once the TOV equation is used to replace the Lorentz-invariant terms in Eqs. (10), while the latter are neglected as higher-order corrections. Note that the higher-order Lorentz-violation corrections include nonlinear couplings between the Lorentz-violation coefficient and the gravitational field, and might become dominant in certain scenarios where the Lorentz-violation coefficient is fine-tuned so that the couplings blow up in the presence of strong gravitational fields. Our work is restricted to the case where the leading-order Lorentz-violation correction dominates. In the next section, we will identify the Lorentz-invariant terms in Eqs. (10) and extract the leading-order Lorentz-violation equations that describe the modification to the structure of the otherwise spherically static perfect fluid.

3. The leading-order modification to the TOV equation

The fact that any Lorentz-violation effect must be tiny to be consistent with the experimental support for Lorentz invariance naturally suggests us to treat the Lorentz-violation terms in Eqs. (10) as perturbations. To start, we write the fluid variables $\rho$, $p$, and the potentials $U$ and $\tilde{U}^{jk}$ as perturbation series

$$\rho = \rho^{(0)}(r) + \rho^{(1)}(\tilde{x}) + \ldots,$$

$$p = p^{(0)}(r) + p^{(1)}(\tilde{x}) + \ldots,$$

$$U = U^{(0)}(r) + U^{(1)}(\tilde{x}) + \ldots,$$

$$\tilde{U}^{jk} = U^{(0)}(r) + U^{(1)}(\tilde{x}) + \tilde{U}^{(k)(j)}(\tilde{x}) + \ldots.$$  

(11)

The zeroth-order fluid variables, $\rho^{(0)}(r)$ and $p^{(0)}(r)$, defined as the solution to Eqs. (10) in the absence of Lorentz violation given a proper equation of state (EOS), satisfy the usual Newtonian hydrostatic equation. The first-order corrections, $\rho^{(1)}(r)$ and $p^{(1)}(r)$, are then determined by

$$\partial_r p^{(1)} = \rho^{(0)} \left( \partial_r U^{(1)} + \frac{1}{2} \tilde{s}^j k \partial_r \tilde{U}^{jk}(0) + p^{(1)} \partial_r U^{(0)} \right),$$

$$\partial_\theta p^{(1)} = \rho^{(0)} \left( \partial_\theta U^{(1)} + \frac{1}{2} \tilde{s}^j k \partial_\theta \tilde{U}^{jk}(0) \right),$$

$$\partial_\phi p^{(1)} = \rho^{(0)} \left( \partial_\phi U^{(1)} + \frac{1}{2} \tilde{s}^j k \partial_\phi \tilde{U}^{jk}(0) \right).$$

(12)

Now that we have extracted the equations at the leading-order of Lorentz violation, as we advertised, to account for the relativistic corrections, the usual Newtonian hydrostatic equation at the zeroth order needs to
be replaced by the TOV equation (6). Namely, $\rho^{(0)}(r)$ and $p^{(0)}(r)$ are now regarded as the solution to the TOV equation, and Eqs. (12) describe the leading-order modification to the TOV equation due to Lorentz violation.

To solve $\rho^{(1)}$ and $p^{(1)}$ in Eqs. (12), let us keep in mind that Lorentz violation not only raises corrections to the fluid variables but also changes the shape of the fluid, hence the boundary conditions. Assuming the radius of the fluid sphere $S$ to be $R$ for a given GR solution, then taking the perturbative change due to Lorentz violation into consideration, the shape of the fluid can be written as

$$r = (1 + \alpha(\theta, \varphi)) R,$$

(13)

where $\alpha(\theta, \varphi)$ is to be determined up to the first order of $\bar{s}^R$. Therefore, the boundary conditions for $\rho$ and $p$ are

$$0 = \rho^\alpha = \rho^{(0)}(R + \alpha(\theta, \varphi) R) + \rho^{(1)}(\bar{R}),$$

$$0 = p^\alpha = p^{(0)}(R + \alpha(\theta, \varphi) R) + p^{(1)}(\bar{R}),$$

(14)

where $\Sigma$ is the surface described by Eq. (13) and $\bar{R}$ represents the position vectors for points on the sphere $S$.

One difficulty to solve Eqs. (12) comes from the fact that $U^{(1)}$ depends nontrivially on $\rho^{(1)}$. We can handle this by writing $U^{(1)}$ as

$$U^{(1)}(\bar{x}) = \int_{\Sigma-\Sigma} \frac{1}{|\bar{x} - \bar{x}'|} \rho^{(0)}(r) d^3 x' + \int_{\Sigma} \frac{1}{|\bar{x} - \bar{x}'|} \rho^{(1)}(\bar{x}') d^3 x'.$$

(15)

At the first order of $\bar{s}^R$, the first integral vanishes because it can be approximated at $r = R$ and $\rho^{(0)}(R) = 0$ is guaranteed in GR solutions. Another difficulty is that since $\rho^{(1)}$ and $p^{(1)}$ are Lorentz-violation induced anisotropic corrections, the usual isotropic EOS that relates $\rho^{(0)}$ and $p^{(0)}$ does not apply to them. This issue is compensated by the requirement that the second derivatives of $p^{(1)}$ exist, which implies

$$\partial_\theta p^{(1)}(\bar{x}) = \partial_\theta p^{(0)} \left( \partial_\theta U^{(1)} + \frac{1}{2} \bar{s}^R \partial_\theta U^{(0)} \right),$$

$$\partial_\varphi p^{(1)}(\bar{x}) = \partial_\varphi p^{(0)} \left( \partial_\varphi U^{(1)} + \frac{1}{2} \bar{s}^R \partial_\varphi U^{(0)} \right).$$

(16)

Skipping the tedious calculation using series expansion, we display the surprisingly tidy and simple perturbation solution

$$p^{(1)}(\bar{x}) = -\alpha(\theta, \varphi) r \partial_\theta \rho^{(0)}(r),$$

$$\rho^{(1)}(\bar{x}) = -\alpha(\theta, \varphi) r \partial_\varphi \rho^{(0)}(r).$$

(17)

It is straightforward to verify that the solution (17) satisfies Eqs. (12) and Eqs. (16), as long as the spherical harmonic expansion of $\alpha(\theta, \varphi)$ is (see Appendix A)

$$\alpha(\theta, \varphi) = \sum_{m = -2}^2 s_m^{(m)} \gamma_{2m}(\theta, \varphi),$$

(18)

where $s_m^{(m)}$ are the spherical components of $\bar{s}^R$. The explicit relations between $s_m^{(m)}$ and the cartesian components are

$$s_{2,2}^{(m)} = \sqrt{\frac{2\pi}{15}} (3s^{xx} - s^{yy} + 2is^{yz}),$$

$$s_{2,1}^{(m)} = \sqrt{\frac{2\pi}{15}} (-s^{xx} + s^{yy} + 2is^{yz}),$$

$$s_{2,0}^{(m)} = \sqrt{\frac{2\pi}{15}} (-s^{zz} + i2s^{yz}),$$

$$s_{2,-1}^{(m)} = \sqrt{\frac{2\pi}{15}} (-s^{zz} - i2s^{yz}).$$

(19)

In addition, it is clear that the boundary conditions (14) are also satisfied by the solution (17) given $\rho^{(0)}(R) = p^{(0)}(R) = 0$.

Before applying it to neutron stars, we would like to clarify that the solution (17) is physical though its particular form suggests that it can be generated from the Lorentz-invariant quantities $\rho^{(0)}(r)$ and $p^{(0)}(r)$ by a coordinate transformation

$$r \rightarrow r' = (1 - \alpha(\theta, \varphi)) r.$$

(20)

The inverse transformation seems to eliminate the Lorentz-violation corrections, but actually just hides the effects into the spatial part of the metric. Taking the Newtonian limit as an example, the boundary of the fluid becomes the sphere $r' = R$ in the $(r', \theta, \varphi)$ coordinates. However, the spatial part of the metric in these coordinates are

$$g'_{jk} = \begin{pmatrix}
1 + 2\alpha & r' \partial_\theta \alpha & r' \partial_\varphi \alpha \\
 r' \partial_\theta \alpha & r'^2 (1 + 2\alpha) & 0 \\
 r' \partial_\varphi \alpha & 0 & r'^2 (1 + 2\alpha) \sin^2 \theta
\end{pmatrix},$$

(21)

with $\alpha(\theta, \varphi)$ describing the same Lorentz-violation effects as one would experience in the coordinates $(r, \theta, \varphi)$ where $g_{jk} = \eta_{jk}$.

4. Newtonian quadrupole of a neutron star

Deformed neutron stars emit continuous gravitational waves when rotating [55]. The quadrupole radiation
Figure 1: Different views of a neutron star with minimal Lorentz violation. For illustrative purpose, we choose $\bar{s}^{xy} = 0.5$ and all the other components of the Lorentz-violation coefficient $\bar{s}^{ik}$ vanish.

Figure 2: The distribution of the relative density correction, $\rho^{(1)}(\mathbf{r})/\rho^{(0)}(\mathbf{r})$, in the equatorial section (the $X$-$Y$ plane) of a neutron star. We have assumed that all the other independent components of the Lorentz-violation coefficient vanish except for $\bar{s}^{xy} = 10^{-10}$. The zeroth-order solution is obtained numerically with the EOS AP4 for a neutron star with mass 1.44 $M_\odot$.

is the leading term in the post-Newtonian expansion. As the quadrupole moments themselves are defined at the Newtonian level using the baryonic rest mass density $\rho$, the solution (17) is just accurate to give us the quadrupole moments caused by the minimal Lorentz violation.

To illustrate the effect of Lorentz violation on the deformation of a neutron star, we plot in Fig. 1 the shape viewed from three different angles with an unrealistically large component of the Lorentz-violation coefficient $\bar{s}^{xy}$. In addition, in Fig. 2 we plot the fractional correction to the density of a neutron star, whose mass is fixed to 1.44 $M_\odot$ with the EOS AP4 [12]. The more relevant value chosen for the component of $\bar{s}^{ik}$ in Fig. 2 is based on its current bounds in Ref. [56].

Figures 1 and 2 show that in general both the shape of the star and the density of the star become anisotropic under the influence of Lorentz violation. This indicates anisotropic quadrupole moments. Using the solution (17), the quadrupole moments are found to be

$$I^{ik} = \int \Sigma x^i x^k (\rho^{(0)}(r) + \rho^{(1)}(\vec{s})) \, d^3 x = \frac{1}{3} (\delta^{ik} + \bar{s}^{ik}) I,$$  

where $\delta^{ik}$ is the Kronecker delta and $I = 4\pi \int_0^R r^4 \rho^{(0)} \, dr$ is the rotationally invariant trace of the quadrupole moments. The anisotropy related to the quadrupole radiation is measured by the ellipticity

$$e = \frac{I^{YY} - I^{XX}}{I^{XX} + I^{YY}},$$

where $\{I^{XX}, I^{YY}, I^{ZZ}\}$ are the eigenvalues of $I^{ik}$, and $I^{ik}$ is diagonalized in the $(X, Y, Z)$ coordinates. The ellipticity (23) applies to rotations along the $Z$-axis, and the case for a general spin direction can be obtained with 3-dimensional rotations.

As an example, we consider the quadrupole radiation of a deformed neutron star due to $\bar{s}^{xy}$ alone spinning in the $z$-direction. The coordinates that diagonalize $I^{ik}$ have the $Z$-axis along the $z$-axis, while the $X$ and $Y$ coordinates are related to $(x, y)$ by the coordinate transformation

$$X = \frac{1}{\sqrt{2}} (x + y), \quad Y = \frac{1}{\sqrt{2}} (x - y).$$
The eigenvalues of \( I^R \) are

\[
I^{XX} = \frac{1}{3} (1 + 3\bar{y}^y), \\
I^{YY} = \frac{1}{3} (1 - 3\bar{y}^y), \\
I^{ZZ} = I^z = \frac{2}{3}I.
\] (25)

Then, the ellipticity associated with rotations along the Z-axis is simply \( \bar{y}^y \) at the leading order of Lorentz violation, and the amplitude of the quadrupole radiation can be estimated as [55]

\[
h_0 = \frac{4G\Omega^2}{d} \frac{2}{3} I^{yy} \\
= 7 \times 10^{-28} \left( \frac{1\text{ms}}{P} \right)^2 \left( \frac{1\text{kpc}}{d} \right) \left( \frac{\bar{y}^y}{10^{-10}} \right),
\] (26)

where \( \Omega = \frac{2\pi}{P} \) is the angular velocity of the neutron star with \( P \) being the spin period, and \( d \) is the distance of the neutron star. To obtain the approximate numerical value, we used \( M = 1.4M_\odot \) for the mass of the neutron star and \( R = 12 \text{ km} \) for its radius. A uniform density is assumed to estimate the trace of the quadrupole moments, namely \( I = \frac{1}{2}MR^2 = 2.4 \times 10^{38} \text{ kg m}^2 \). Note that the moment of inertia along any diameter of the uniform sphere is \( \frac{4}{5}I \).

Continuous gravitational waves are important signals for the LIGO/Virgo detectors. Various algorithms are being developed for possible events [57, 58, 59, 60, 61, 62, 63, 64, 65, 66]. Although there is no detection yet, meaningful constraints are already set from the advanced detectors for different systems at levels of \( h_0 \lesssim 10^{-25} \) [60, 67, 68, 65, 66] and \( \epsilon \lesssim 10^{-9} \) to \( 10^{-8} \) [69, 64, 70]. The estimated quadrupole radiation from Lorentz violation is too weak to be detected currently. The idea of Lorentz violation, anyway, provides a possible cause of continuous gravitational waves for future observations. Also, it is important to keep in mind that current constraints on \( \bar{y}^y \) [56] all assume experiments and observations involving only weak gravitational fields. We expect Lorentz violation to be comparatively larger in a strong gravitational scenario. It is possible that the quadrupole radiation due to Lorentz violation might be comparable to or even greater than that of conventional deformations, like a mountain on the star or the tidal interactions in a close binary [55]. Despite the fact that the amplitude of the quadrupole radiation does not distinguish a Lorentz-violation deformation from conventional deformations, upper bounds on Lorentz violation in strong-field systems will be told once any continuous gravitational waves are detected in the future.

Finally, there remains a natural and essential question on whether there are any signatures in the continuous gravitational waves that distinguish Lorentz violation from conventional deformations when more stringent constraints on Lorentz violation are to be extracted. The answer requires a full Lorentz-violation quadrupole radiation formula which in principle follows the 1PN metric solution with Lorentz violation and a nonstatic fluid configuration. Therefore, any Lorentz-violation signature in the quadrupole radiation is a second-order effect in the sense that the Lorentz-violation coefficient couples with 1PN terms instead of the Newtonian term as we study here. Such effects most likely show up in the phases and polarizations of the waves as the amplitude is dominated by the first-order effect that we have calculated. While a detailed derivation is beyond the scope of the present work, the question is certainly worth an investigation.

5. Summary

We calculated the leading-order modification to a static perfect fluid due to the minimal Lorentz violation and applied it to neutron stars to find an estimate for the quadrupole radiation. The corrections were first derived at the Newtonian level as shown in Eqs. (10). Then, the Lorentz-violation corrections and the Lorentz-invariant terms are separated using the perturbation method, and the TOV equation replaces the zeroth-order Lorentz-invariant hydrostatic equation to account for relativistic effects in strong gravitational systems like neutron stars. The perturbation solution (17) to the first-order equations (12) was found and used to estimate the quadrupole gravitational radiation for continuous gravitational waves. Our calculation shows that the amplitude is too weak to be detectable at the moment, but we expect that future observations of continuous gravitational waves can make use of our result and set constraints on Lorentz violation in the strong-field regime.

We point out that it is possible, if not straightforward, to generalize our results to the nonminimal gravitational SME [30]. We conjecture that the solution still takes the form of (17) with higher spherical harmonics involved in the angular function \( \alpha \). The specific dependence of the factors in front of the spherical harmonics on the nonminimal Lorentz-violation coefficients as an analog of Eq. (18) requires further calculation.

Acknowledgements

We are grateful to Quentin G. Bailey and M. Alessandra Papa for comments. This work was sup-
ported by the National Natural Science Foundation of China (11975027), the Young Elite Scientists Sponsorship Program by the China Association for Science and Technology (2018QNRC001), and the High-performance Computing Platform of Peking University. It was partially supported by the National Natural Science Foundation of China (11721303), and the Strategic Priority Research Program of the Chinese Academy of Sciences through the Grant No. XDB23010200. R.X. is supported by the Boya Postdoctoral Fellowship at Peking University.

Appendix A. Determining $\alpha(\theta, \varphi)$

Substituting the solution (17) into Eqs. (12) and Eqs. (16), we find that $\alpha(\theta, \varphi)$ only needs to satisfy

$$-ar\partial_r U^{(0)} = U^{(1)} + \frac{1}{2} \bar{s}^\beta \bar{U}^{(0)} j^\beta.$$

(A.1)

The spherical expansion of the left-hand side is

$$-ar\partial_r U^{(0)} = 4\pi G \sum_{lm} \alpha_{lm} Y_{lm}(\theta, \varphi) \frac{1}{r} \int_0^\rho \rho^2 \rho^{(0)} (\rho') \, d\rho' ,$$

(A.2)

and the spherical expansions of the terms on the right-hand side are

$$U^{(1)} = 4\pi G \sum_{lm} \frac{4\pi}{2l+1} \alpha_{lm} Y_{lm}(\theta, \varphi) \left( \frac{l+3}{r+1} \int_0^\rho r^{l+2} \rho^{(0)} (\rho') \, d\rho' - (l-2)r \int_0^\rho r^{l-1} \rho^{(0)} (\rho') \, d\rho' \right) ,$$

(A.3)

and

$$\bar{s}^\beta \bar{U}^{(0)} j^\beta = 4\pi G \sum_{lm} \bar{s}^{(j) 2m} Y_{2m}(\theta, \varphi) \left( \frac{1}{r} \int_0^\rho r^2 \rho^{(0)} (\rho') \, d\rho' - \frac{1}{r^2} \int_0^\rho \rho' \rho^{(0)} (\rho') \, d\rho' \right) .$$

(A.4)

Note that we have used the solution (17) to compute $U^{(1)}$ from Eq. (15) and the condition $\bar{\eta} \bar{s}^\beta = 0$ to simplify $\bar{s}^\beta \bar{U}^{(0)} j^\beta$. By comparing, we obtain

$$\alpha_{lm} = \begin{cases} \frac{1}{2} \bar{s}^{(j) 2m} & \text{for } l = 2, \\ 0 & \text{for } l \neq 2. \end{cases}$$

(A.5)

References

[1] D. R. Lorimer, M. Kramer, Handbook of Pulsar Astronomy, Cambridge University Press, Cambridge, England, 2005.

[2] N. Wex, Testing Relativistic Gravity with Radio Pulsars, in: S. M. Kopeikin (Ed.), Frontiers in Relativistic Celestial Mechanics: Applications and Experiments, Vol. 2, Walter de Gruyter GmbH, Berlin/Boston, 2014, p. 39. arXiv:1402.5594.

[3] L. Shao, N. Wex, Tests of gravitational symmetries with radio pulsars, Sci. China Phys. Mech. Astron. 59 (9) (2016) 699501. arXiv:1604.03662, doi:10.1007/s11433-016-0687-6.

[4] F. Özel, P. Freire, Masses, Radii, and the Equation of State of Neutron Stars, Ann. Rev. Astron. Astrophys. 54 (2016) 401–440. arXiv:1601.02698, doi:10.1146/annurev-astro-081915-023322.

[5] J. M. Lattimer, Neutron Star Mass and Radius Measurements, Universe 5 (7) (2019) 159. doi:10.3390/universe5070159.

[6] B. Abbott, et al., GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, Phys. Rev. Lett. 119 (16) (2017) 161101. arXiv:1710.05832, doi:10.1103/PhysRevLett.119.161101.

[7] B. P. Abbott, et al., Exploring the Sensitivity of Next Generation Gravitational Wave Detectors, Class. Quant. Grav. 34 (4) (2017) 044001. arXiv:1607.06697, doi:10.1088/1361-6382/aa51f4.

[8] D. Shoemaker, Gravitational wave astronomy with LIGO and similar detectors in the next decade, arXiv:1904.03187.

[9] Y. Michimura, et al., Prospects for improving the sensitivity of KAGRA gravitational wave detector, in: 15th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG15) Rome, Italy, July 1-7, 2018, 2019. arXiv:1906.02866.

[10] K. A. Kuns, H. Yu, Y. Chen, R. X. Adhikari, Astrophysics and cosmology with a deci-hertz gravitational-wave detector: TianGO, arXiv:1908.06094.

[11] M. A. Sedda, et al., THe Missing Link in Gravitational-Wave Astronomy: Discoveries waiting in the decihertz range, arXiv:1908.11375.

[12] J. M. Lattimer, M. Prakash, Neutron star structure and the equation of state, Astrophys. J. 550 (2001) 426. arXiv:astro-ph/0004001.

[13] A. W. Steiner, J. M. Lattimer, E. F. Brown, The Equation of State of Hot, Dense Matter and Neutron Stars, Phys. Rept. 621 (2016) 126–164. arXiv:1512.07820, doi:10.1016/j.physrep.2015.12.005.

[14] E. Berti, et al., Testing General Relativity with Present and Future Astrophysical Observations, Class. Quant. Grav. 32 (2015) 243001. arXiv:1501.07274, doi:10.1088/0264-9381/32/24/243001.

[15] C. M. Will, Bounding the mass of the graviton using gravitational wave observations of inspiralling compact binaries, Phys. Rev. D57 (1998) 2061–2068. arXiv:gr-qc/9709011, doi:10.1103/PhysRevD.57.2061.

[16] X. Miao, L. Shao, B.-Q. Ma, Bounding the mass of graviton in a dynamic regime with binary pulsars, Phys. Rev. D99 (12) (2019) 123015. arXiv:1905.123015, doi:10.1103/PhysRevD.99.123015.

[17] L. Shao, N. Sennett, A. Buonanno, M. Kramer, N. Wex, Constraining nonperturbative strong-field effects in scalar-tensor gravity by combining pulsar timing and laser-interferometer
applications of the post-Tolman-Oppenheimer-Volkoff formalism, Phys. Rev. D94 (4) (2016) 044030. arXiv:1606.05196, doi:10.1103/PhysRevD.94.044030.

[53] S. Weinberg, Gravitation and Cosmology, John Wiley and Sons, New York, 1972.

[54] Y. Bondar, Lorentz violation in the gravity sector: The t puzzle, Phys. Rev. D94 (4) (2016) 044030. arXiv:1606.05106, doi:10.1103/PhysRevD.94.044030.

[55] S. Weinberg, Gravitation and Cosmology, John Wiley and Sons, New York, 1972.

[56] Y. Bonder, Lorentz violation in the gravity sector: The t puzzle, Phys. Rev. D91 (12) (2015) 125002. arXiv:1504.03636, doi:10.1103/PhysRevD.91.125002.

[57] E. Poisson, C. M. Will, Gravity: Newtonian, Post-Newtonian, Relativistic, Cambridge University Press, Cambridge, England, 2014. doi:10.1103/ChO09781139597486.

[58] V. A. Kostelecký, N. Russell, Data Tables for Lorentz and CPT Violation, Rev. Mod. Phys. 83 (2011) 11–31. arXiv:0801.0287, doi:10.1103/RevModPhys.83.11.

[59] J. T. Whelan, S. Sundaresan, Y. Zhang, P. Peiris, Model-Based Cross-Correlation Search for Gravitational Waves from Scorpius X-1, Phys. Rev. D91 (2015) 102005. arXiv:1504.05890, doi:10.1103/PhysRevD.91.102005.

[60] J. Ming, B. Krishnan, M. A. Papa, C. Aulbert, H. Fehrmann, Optimal directed searches for continuous gravitational waves, Phys. Rev. D93 (6) (2016) 064011. arXiv:1510.03417, doi:10.1103/PhysRevD.93.064011.

[61] S. Walsh, et al., Comparison of methods for the detection of gravitational waves from unknown neutron stars, Phys. Rev. D94 (12) (2016) 124010. arXiv:1606.06660, doi:10.1103/PhysRevD.94.124010.

[62] B. P. Abbott, et al., Upper Limits on Gravitational Waves from Scorpius X-1 from a Model-Based Cross-Correlation Search in Advanced LIGO Data, Astrophys. J. 847 (1) (2017) 47. arXiv:1706.03119, doi:10.3847/1538-4357/aa86f0.

[63] B. P. Abbott, et al., Search for gravitational waves from Scorpius X-1 in the first Advanced LIGO observing run with a hidden Markov model, Phys. Rev. D95 (12) (2017) 122003. arXiv:1704.03719, doi:10.1103/PhysRevD.95.122003.

[64] C. Dreissigacker, R. Prix, K. Wette, Fast and Accurate Sensitivity Estimation for Continuous-Gravitational-Wave Searches, Phys. Rev. D98 (8) (2018) 084058. arXiv:1808.02459, doi:10.1103/PhysRevD.98.084058.

[65] L. Sun, A. Melatos, P. D. Lasky, Tracking continuous gravitational waves from a neutron star at once and twice the spin frequency with a hidden Markov model, Phys. Rev. D99 (12) (2019) 123010. arXiv:1903.03866, doi:10.1103/PhysRevD.99.123010.

[66] B. P. Abbott, et al., Searches for Gravitational Waves from Known Pulsars at Two Harmonics in 2015-2017 LIGO Data, Astrophys. J. 879 (1) (2019) 10. arXiv:1902.08507, doi:10.3847/1538-4357/ab20cb.

[67] V. Dergachev, M. A. Papa, Sensitivity improvements in the search for periodic gravitational waves using O1 LIGO data, Phys. Rev. Lett. 123 (10) (2019) 101101. arXiv:1902.05539, doi:10.1103/PhysRevLett.123.101101.

[68] V. Dergachev, M. A. Papa, Results from an Extended Falcon All-Sky Survey for Continuous Gravitational Waves, Phys. Rev. D101 (2) (2020) 022001. arXiv:1909.09619, doi:10.1103/PhysRevD.101.022001.

[69] B. P. Abbott, et al., All-sky search for continuous gravitational waves from isolated neutron stars using Advanced LIGO O2 data, Phys. Rev. D100 (2) (2019) 022004. arXiv:1903.09190, doi:10.1103/PhysRevD.100.022004.

[70] B. P. Abbott, et al., Search for gravitational waves from Scorpius X-1 in the second Advanced LIGO observing run with an improved hidden Markov model, Phys. Rev. D100 (12) (2019) 122002. arXiv:1906.12648, doi:10.1103/PhysRevD.100.122002.

B. P. Abbott, et al., Searches for Continuous Gravitational Waves from 15 Supernova Remnants and Fomalhaut b with Advanced LIGO. Astrophys. J. 875 (2) (2019) 122. arXiv:1812.11656, doi:10.3847/1538-4357/ab113b.