CROSS CHARACTERISTIC REPRESENTATIONS OF $3D_4(q)$ ARE REDUCIBLE OVER PROPER SUBGROUPS

by

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Abstract. We prove that the restriction of any absolutely irreducible representation of Steinberg’s triality groups $3D_4(q)$ in characteristic coprime to $q$ to any proper subgroup is reducible.

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1. Introduction

Finite primitive permutation groups have been studied since the pioneering work of Galois and Jordan on group theory; they have had important applications in many different areas of mathematics.

If \( G \) is a primitive permutation group with a point stabilizer \( M \) then \( M < G \) is a maximal subgroup. Thanks to work of Aschbacher, O’Nan, Scott \([AS]\), and of Liebeck, Praeger, Saxl, and Seitz \([LPS], [LiS]\), most problems involving such a \( G \) can be reduced to the case where \( G \) is a finite classical group. In this case, Aschbacher's theorem \([A]\) describes all possible choices for the maximal subgroup \( M \) of \( G \). Work of Kleidman and Liebeck \([KL]\) and others essentially reduces the question of whether a subgroup \( M \) from this list is indeed maximal in \( G \) to the following problem.

**Problem 1.** Let \( \mathbb{F} \) be an algebraically closed field of characteristic \( \ell \). Classify all triples \((K,V,H)\) where \( K \) is a finite group with \( K/Z(K) \) almost simple, \( V \) is an \( \mathbb{F}K \)-module of dimension greater than one, and \( H \) is a proper subgroup of \( K \) such that the restriction \( V|_H \) is irreducible.

Our main focus is on the case where \( K \) is a finite group of Lie type in characteristic \( \neq \ell \). If, furthermore, \( K \) is of type \( A \), then Problem 1 has been solved recently in \([KT]\). On the other hand, the case where \( K \) is an exceptional group of type \( G_2, 2B_2, \) or \( 2G_2 \), was settled in \([N]\).

The main result of this paper is the following:

**Theorem 2.** Let \( G = \mathbb{3}D_4(q) \) and let \( \Phi \) be any irreducible representation of \( G \) in characteristic \( \ell \) coprime to \( q \). If \( H \) is any proper subgroup of \( G \) and \( \deg(\Phi) > 1 \), then \( \Phi|_H \) is reducible.

In the case of complex representations, Theorem 2 was proved by Saxl \([S]\).

2. Basic Reduction

Given a finite group \( X \), we denote by \( \mathfrak{d}_\ell(X) \) and \( \mathfrak{m}_\ell(X) \) the smallest and largest degrees of absolutely irreducible representations of degree greater than one of \( X \) in characteristic \( \ell \); furthermore, let \( \mathfrak{m}_\ell(X) = m_0(X) \). From now on, \( \mathbb{F} \) stands for an algebraically closed field of characteristic \( \ell \), and \( q \) is a power of a prime \( p \). If \( \chi \) is a complex character of \( X \), we denote by \( \hat{\chi} \) the restriction of \( \chi \) to \( \ell \)-regular elements of \( X \). By \( \text{IBr}_\ell(X) \) we denote the set of irreducible \( \ell \)-Brauer characters, or the set of absolutely irreducible \( \mathbb{F}X \)-representations, depending on the context.

First we record a few well-known statements.

**Lemma 3.** Let \( K \) be a finite group. Suppose \( V \) is an irreducible \( \mathbb{F}K \)-module of dimension greater than one, and \( H \) is a proper subgroup of \( K \) such that the restriction \( V|_H \) is irreducible. Then

\[
\sqrt{|H/Z(H)|} \geq \mathfrak{m}_\ell(H) \geq \mathfrak{d}_\ell(H) \geq \dim(V) \geq \mathfrak{d}_\ell(K).
\]
Lemma 4. [I, p. 190] Let $K$ be a finite group and $H$ be a normal subgroup of $K$. Let $\chi \in \operatorname{Irr}(K)$ and $\theta \in \operatorname{Irr}(H)$ be a constituent of $\chi_H$. Then $\chi(1)/\theta(1)$ divides $|K/H|$.

Lemma 5. Let $K$ be a simple group and $V$ an absolutely irreducible $\mathbb{F}K$-module of dimension greater than one. Suppose $H$ is a subgroup of $K$ such that $V|_H$ is irreducible. Then $Z(H) = C_K(H) = 1$.

In the following theorem, we use results and notation of [K].

Theorem 6 (Reduction Theorem). Let $G = {^3D_4}(q)$ and let $\varphi$ be an irreducible representation of $G$ in characteristic $\ell$ coprime to $q$. Suppose $\varphi(1) > 1$ and $M$ is a maximal subgroup of $G$ such that $\varphi|_M$ is irreducible. Then $M$ is $G$-conjugate to one of the following groups:

(i) $P$, a maximal parabolic subgroup of order $q^{12}(q^6 - 1)(q - 1)$,
(ii) $Q$, a maximal parabolic subgroup of order $q^{12}(q^3 - 1)(q^2 - 1)$,
(iii) $G_2(q)$,
(iv) $^3D_4(q_0)$ with $q = q_0^2$.

Proof. By [MMT, Theorem 4.1], $\mathfrak{d}_\ell(^3D_4(q)) \geq q^5 - q^3 + q - 1$ for every $\ell$ coprime to $q$. Next, according to [K], if $M$ is a maximal subgroup of $G$, but $M$ is not a maximal parabolic subgroup, then $M$ is $G$-conjugate to one of the following groups:

1) $G_2(q)$,
2) $PGL_3^\epsilon(q)$, where $4 \leq q \equiv 1(\text{mod } 3)$, $\epsilon = \pm$,
3) $^3D_4(q_0)$ with $q = q_0^2$, $\alpha$ prime, $\alpha \neq 3$,
4) $L_2(q^3) \times L_2(q)$, where $2 \nmid q$, a fundamental subgroup,
5) $C_2(q) = (SL_2(q^3) \circ SL_2(q)).2$, $q$ odd, involution centralizer,
6) $((\mathbb{Z}_{q^2+q+1}) \circ SL_2(q)).f_+2$, where $f_+ = (3, q^2 + q + 1)$,
7) $((\mathbb{Z}_{q^2-q+1}) \circ SU_3(q)).f_-2$, where $f_- = (3, q^2 - q + 1)$,
8) $((\mathbb{Z}_{q^2+q+1})^2 SL_2(3)$,
9) $((\mathbb{Z}_{q^2-q+1})^2 SL_2(3)$,
10) $\mathbb{Z}_{q^4-q^2+1}^4$.

We only need to consider the following cases.

• $M = PGL_3^\epsilon(q)$, where $4 \leq q \equiv 1(\text{mod } 3)$, $\epsilon = \pm$.

We have $|PGL_3^\epsilon(q)| = q^3(q^2 - 1)(q^3 \pm 1)$. So $m_C(M) \leq \sqrt{q^3(q^2 - 1)(q^3 + 1)} < q^5 - q^3 + q - 1$ for every $q \geq 4$. Therefore $m_C(M) < \mathfrak{d}_\ell(^3D_4(q))$, contradicting Lemma 3.

• $M = ^3D_4(q_0)$ with $q = q_0^2$, $\alpha$ prime, $\alpha \neq 2, 3$.

We have $|^3D_4(q_0)| = q_0^{12}(q_0^6 - 1)^2(q_0^4 - q_0^2 + 1) < q_0^{28}$. Hence, $m_C(M) \leq \sqrt{q_0^{28}} = q_0^{14}$. Since $\alpha$ is prime and $\alpha \neq 2, 3, 5 \geq 5$. It follows that $m_C(M) < q_0^{14/5} < q^5 - q^3 + q - 1 \leq \mathfrak{d}_\ell(^3D_4(q_0))$.

• $M = L_2(q^3) \times L_2(q)$, where $2 \nmid q$.

It is well known that $m_C(L_2(q)) = q + 1$ except that $m_C(L_2(2)) = 2$, $m_C(L_2(3)) = 3$ and $m_C(L_2(5)) = 5$. So we have $m_C(L_2(q^3)) = q^3 + 1$ for every $q$. Hence $m_C(M) =$
\((q + 1)(q^3 + 1)\) for \(q \geq 4\) and \(m_G(M) = 18\) for \(q = 2\). It is easy to see that \((q + 1)(q^3 + 1) < q^5 - q^3 + q - 1\) for every \(q \geq 4\). When \(q = 2\), we also have \(m_G(M) = 18 < 25 = 2^5 - 2^3 + 2 - 1\). Therefore, \(m_G(M) < \varphi(3D_4(q))\) for every \(q\).

- \(M = C_G(s) = (SL_2(q^3) \circ SL_2(q)).2\), \(s\) an involution.
  Here \(C_G(M) \ni s \neq 1\), contradicting Lemma 5.

- \(M = ((\mathbb{Z}_{q^2 + q + 1}) \circ SL_3(q)).f_+2\), where \(f_+ = (3, q^2 + q + 1)\).
  By Lemma 4, \(m_G(M) \leq 2.f_+.m_G((\mathbb{Z}_{q^2 + q + 1}) \circ SL_3(q)) \leq 2.f_+.m_G(SL_3(q))\). From [SF], we have

\[
m_G(SL_3(q)) = \begin{cases} 8, & q = 2, \\ 39, & q = 3, \\ 84, & q = 4, \\ (q + 1)(q^2 + q + 1), & q \geq 5. \end{cases}
\]

It is easy to check that \(2.f_+.m_G(SL_3(q)) < q^5 - q^3 + q - 1\) for every \(q \geq 2\). Therefore \(m_G(M) < q^5 - q^3 + q - 1\).

- \(M = ((\mathbb{Z}_{q^2 + q + 1}) \circ SU_3(q)).f_-2\), where \(f_- = (3, q^2 - q + 1)\).
  By Lemma 4, \(m_G(M) \leq 2.f_-m_G((\mathbb{Z}_{q^2 + q + 1}) \circ SU_3(q)) \leq 2.f_-m_G(SU_3(q))\). From [SF], we have

\[
m_G(SU_3(q)) = \begin{cases} 8, & q = 2, \\ (q + 1)^2(q - 1), & q \geq 3. \end{cases}
\]

It is easy to check that \(2.f_-m_G(SU_3(q)) < q^5 - q^3 + q - 1\) for every \(q \geq 3\). Therefore \(m_G(M) < q^5 - q^3 + q - 1\) for every \(q \geq 3\).

When \(q = 2\), we have \(m_G(M) \leq \sqrt{|M|} = 36\). Therefore if \(\varphi|_M\) is irreducible then \(\deg(\varphi) \leq 36\). Inspecting the character tables of \(3D_4(2)\) in [Atlas1] and [Atlas2], we see that \(\deg(\varphi) = 25\) for \(\ell = 3\) or \(\deg(\varphi) = 26\) for \(\ell \neq 3\). Moreover, when \(\ell \neq 3\), \(\varphi\) is the reduction modulo \(\ell\) of the unique irreducible complex representation \(\rho\) of degree 26. Since \(26 \nmid |M| = 1296\), \(M\) does not have any irreducible complex representation of degree 26, whence \(\rho|_M\) and \(\varphi|_M\) must be reducible. When \(\ell = 3\), \(M = ((\mathbb{Z}_3) \circ SU_3(2)).3.2 \simeq 3^{1+2}.2S_4\). So \(m_3(M) = m_3(3^{1+2}.2S_4) = m_3(2S_4) \leq m_G(2S_4) \leq \sqrt{|2S_4|} \leq 7\). Therefore if \(\deg(\varphi) = 25\) then \(\varphi|_M\) is reducible.

- \(M = (\mathbb{Z}_{q^2 + q + 1})^2.SL_2(3)\).
  We have \(m_G(M) \leq |SL_2(3)| = 24\). Since \(q^5 - q^3 + q - 1 > 24\) for every \(q \geq 2\), \(m_G(M) < \varphi_4(3D_4(q))\).

- \(M = \mathbb{Z}_{q^4 - q^2 + 1}.4\).
  We have \(m_G(M) \leq 4 < 4_4(G)\). The Theorem is proved. \(\square\)

3. Restrictions to \(G_2(q)\) and to \(3D_4(\sqrt{q})\)

In this section we handle two of the maximal subgroups singled out in Theorem 6.
**Theorem 7.** Let $M \simeq G_2(q)$ be a subgroup of $G = 3D_4(q)$ and $\varphi \in \mathrm{IBr}_q(G)$ be of degree $> 1$. Then $\varphi|_M$ is reducible.

**Proof.** Assume the contrary: $\varphi|_M$ is irreducible. By Lemma 3, $\varphi(1) < \sqrt{|M|} < q^7$. We will identify the dual group $G^*$ with $G$. By the fundamental result of Broué and Michel [BM], $\varphi$ belongs to a union $\mathcal{E}_\ell(G,s)$ of $\ell$-blocks, labeled by a semisimple $\ell'$-element $s \in G$. Moreover, by [HM], $\varphi(1)$ is divisible by $(G : C_G(s))_{\ell'}$. Assume $s \neq 1$. Then it is easy to check, using [DM] for instance, that $(G : C_G(s))_{\ell'} \geq q^8 + q^4 + 1$. Since $\varphi(1) < q^7$, it follows that $s = 1$, i.e. $\varphi$ belongs to a unipotent block.

According to [K], $M = C_G(\tau)$ for some (outer) automorphism $\tau$ of order 3 of $G$. Furthermore, the degrees of all complex irreducible characters of $G$ are listed in [DM]. An easy inspection reveals that $G$ has a unique irreducible character of degree $\varphi(1)$ for every unipotent character $\psi \in \mathrm{Irr}(G)$. It follows that every unipotent (complex) character of $G$ is $\tau$-invariant.

Next we show that $\varphi$ is also $\tau$-invariant. First consider the case where $q$ is odd. Then Corollary 6.9 of [G1] states that the $\ell$-modular decomposition matrix of $G$ has a lower unitriangular shape. In particular, this implies that $\varphi$ is an integral linear combination of $\hat{\psi}$, with $\psi \in \mathcal{E}(G,1)$, the set of unipotent characters of $G$. But each such $\hat{\psi}$ is $\tau$-invariant, whence $\varphi$ is $\tau$-invariant. Now assume that $q$ is even. Then $\ell \neq 2$, and so it is a good prime for $G$, and $\ell$ does not divide $|Z(G)|$, where $G$ is the simple, simply connected algebraic group of type $D_4$. Hence, by the main result of [G2], $\{\hat{\psi} \mid \psi \in \mathcal{E}(G,1)\}$ is a basic set of Brauer characters of $\mathcal{E}_\ell(G,1)$. It follows that $\varphi$ is an integral linear combination of $\hat{\psi}$, with $\psi \in \mathcal{E}(G,1)$, and so it is $\tau$-invariant as above.

Consider the semidirect product $\hat{G} = G \rtimes \langle \phi \rangle$. Then $G \triangleleft \hat{G}$, and $\hat{G}/G$ is cyclic. Since $\varphi$ is $\hat{G}$-invariant, it extends to $\hat{G}$ by [F, Theorem III.2.14]. But $C_{\hat{G}}(M) \ni \tau \neq 1$, hence $\varphi|_M$ cannot be irreducible by Lemma 5. \hfill $\square$

**Theorem 8.** Let $H \simeq 3D_4(q^2)$ be a maximal subgroup of $G = 3D_4(q^2)$ and $V \in \mathrm{IBr}_q(G)$ be of dimension $> 1$. Then $V|_H$ is reducible.

**Proof.** Again assume the contrary. We consider a long-root parabolic subgroup $P = q^{2+16} \cdot SL_2(q^6) \cdot \mathbb{Z}_{q^2-1}$ of $G$, which also contains a long-root parabolic subgroup $P_H = q^{1+8} \cdot SL_2(q^3) \cdot \mathbb{Z}_{q-1}$ of $H$.

It is well known that $V|_Z$ affords all the nontrivial linear characters $\lambda$ of the long-root subgroup $Z := Z(P')$ (which is elementary abelian of order $q^2$), and the corresponding eigenspaces $V_\lambda$ are permuted regularly by the torus $\mathbb{Z}_{q^2-1}$. Let $U = q^{2+16}$ denote the unipotent radical of $P$ and consider any such $\lambda$. Then $\mathrm{IBr}_q(U)$ contains a unique representation (of degree $q^8$), on which $Z$ acts via the character $\lambda$. Moreover, since $P'/U \simeq SL_2(q^6)$ has trivial Schur multiplier and is perfect, this representation of $U$ extends to a unique representation of $P'$, which we denote by $E_\lambda$. By Clifford
theory, the $P'$-module $V_\lambda$ is isomorphic to $E_\lambda \otimes A$ for some $A \in \text{IBr}_f(P'/U)$. Suppose that $A$ contains a nontrivial composition factor, as a $SL_2(q^6)$-module. Then $\dim(A) \geq (q^6 - 1)/2$. It follows that

$$\dim(V) \geq (q^2 - 1)q^8(q^6 - 1)/2.$$  

On the other hand, the irreducibility of $V|_H$ implies that

$$\dim(V) < \sqrt{|H|} < q^{14},$$

contradicting (1). Thus all composition factors of $A$ are trivial. In particular, the $P'$-module $V_\lambda$ contains a simple submodule which is isomorphic to $E_\lambda$.

Notice that we can embed $P_H$ in $P$ in such a way that $Z$ contains $Z_H := Z(P'_H)$ (a long-root subgroup in $H$, which is elementary abelian of order $q$), and $U$ contains the unipotent radical $U_H = q^{1+8}$ of $P_H$. Now choose $\lambda$ such that $Z_H \leq \text{Ker}(\lambda)$. Then it is easy to see that $E_\lambda|_{U_H}$ is just the regular representation, whence the subspace $L$ of $U_H$-fixed points in it is one-dimensional, and, since $U_H \not\subseteq P'_H$, this subspace is acted on by $P'_H$. But $P'_H/U_H \cong SL_2(q^3)$ is perfect, hence $P'_H$ acts trivially on $L$.

We have shown that, for the given choice of $\lambda$, $P'_H$ has nonzero fixed points in $V_\lambda$. Let $W$ be the subspace consisting of all $P'_H$-fixed points in $V$. Then $P_H/P'_H \cong \mathbb{Z}_{q-1}$ acts on $W$ and so $W$ contains a one-dimensional $P_H$-submodule $T$. By the Frobenius reciprocity, $0 \neq \dim \text{Hom}_{P_H}(T,V|_{P_H}) = \dim \text{Hom}_H(\text{Ind}_{P_H}^{H}(T),V|_{H})$. But $V|_H$ is irreducible, hence it is a quotient of $\text{Ind}_{P_H}^{H}(T)$. In particular,

$$\dim(V) \leq (H : P_H) \cdot \dim(T) = (q + 1)(q^8 + q^4 + 1).$$

On the other hand, Theorem 4.1 of [MMT] implies that

$$\dim(V) \geq q^2(q^8 - q^4 + 1) - 1,$$

contradicting (2). \hfill \Box

4. Restriction to maximal parabolic subgroups

**Lemma 9.** Let $Q$ denote the maximal parabolic subgroup of order $q^{12}(q^3 - 1)(q^2 - 1)$ of $G = 3D_4(q)$, and let $U := O_p(Q)$. Then

(i) For any prime $r \neq p$, $O_r(Q) = 1$.

(ii) Let $\varphi \in \text{IBr}_f(Q)$ be an irreducible Brauer character of $Q$ whose kernel does not contain $U$. If $q$ is odd, assume in addition that $\varphi$ is faithful. Then $\varphi$ lifts to a complex character $\chi$ of $Q$. Moreover, $\chi$ is also faithful if $\varphi$ is faithful.

**Proof.** (i) Since $O_r(Q), U < Q$ and $O_r(Q) \cap U = 1$, any element $g \in O_r(Q)$ is centralized by $U$, which has order $q^{11}$. Thus $q^{11}$ divides $|C_Q(g)|$. Assuming $g \neq 1$, we see by [H1] and [H3] that $g$ is $Q$-conjugate to the long-root element $u = x_{3\alpha+2\beta}(1)$. But then $g$ is a $p$-element, a contradiction. Hence $O_r(Q) = 1$.

(ii) Let $\lambda$ be an irreducible constituent of $\varphi|_U$, and let $I$ denote the inertia group of $\lambda$ in $Q$. By Clifford theory, $\varphi = \text{Ind}_I^Q(\psi)$ for some $\psi \in \text{IBr}_f(I)$ whose restriction
to $U$ contains $\lambda$. Since $p \neq \ell$, we may view $\lambda$ as an ordinary character of $U$. By our assumption, $\lambda \neq 1_U$. The structure of $I/U$ is described in [H1], [H3]. In particular, if $2|q$, then $I/U$ is always solvable. On the other hand, if $q$ is odd, then $I/U$ is solvable, except for one orbit, the kernel of any character in which however contains a long-root element $x_{3a+2\beta}(1)$ (in the notation of [H1]). Recall we are assuming that $\varphi$ is faithful if $q$ is odd. It follows that in either case $I/U$ is solvable, and so $I$ is solvable. By the Fong-Swan Theorem, $\psi$ lifts to a complex character $\rho$ of $I$. Hence $\varphi$ lifts to the complex character $\chi:=\operatorname{Ind}_{I}^{G}(\rho)$.

Now assume that $\varphi$ is faithful but $K:=\operatorname{Ker}(\chi)$ is non-trivial; in particular, $\ell \neq 0$. If $K$ is not an $\ell$-group, then $K$ contains a non-trivial $\ell'$-element $g$. Since $\varphi(g) = \chi(g) = \chi(1) = \varphi(1)$, we see that $\varphi$ is not faithful, a contradiction. Hence $K$ is an $\ell$-group, and so $O_\ell(Q) \neq 1$, contradicting (i). □

**Theorem 10.** Let $M$ be a maximal parabolic subgroup of $G = 3D_4(q)$ and $\varphi \in \operatorname{IBr}_\ell(G)$ be of degree $>1$. Then $\varphi|_M$ is reducible.

**Proof.** First suppose that $M = P$, the long-root parabolic subgroup of $G$. Then the statement follows from Theorem 1.6 of [11]. So we may assume that $M = Q$, the other maximal parabolic subgroup of $G$. Also assume the contrary: $\varphi|_Q$ is irreducible.

We will consider two particular long-root elements $u = x_{3a+2\beta}(1)$ and $v = x_\beta(1)$ of $Q$, in the notation of [H1], [H2], [H3]. Clearly, they are conjugate in $G$, so $\varphi(u) = \varphi(v)$. By Lemma[9] $\varphi|_Q$ lifts to a complex irreducible character $\chi$ of $Q$ which is also faithful. Since $u$ and $v$ are $\ell'$-elements, we have $\varphi(u) = \chi(u)$ and $\varphi(v) = \chi(v)$. It follows that

$$\chi(u) = \chi(v).$$

Note that $Z := Z(O_\ell(Q)) = X_{3a+2\beta}X_{3a+3\beta}$ has order $q^3$, and consists of the $q^3 - 1$ $Q$-conjugates of $u$ and $1$. Thus $Q$ acts transitively on $Z \setminus \{1\}$ and on $\operatorname{Irr}(Z) \setminus \{1_Z\}$. Since $\operatorname{Ker}(\chi) = 1$, we conclude that $\chi(u) = -\chi(1)/(q^3 - 1)$.

First consider the case $q$ is odd. Then $u$, resp. $v$, belongs to the $Q$-conjugacy class $c_{1,1}$, resp. $c_{1,2}$, in the notation of [H1]. According to [H1], the faithful character $\chi$ must be one of $\chi_j(k)$, $16 \leq j \leq 20$. If $j = 16$ or 17, then $\chi(v)$ is explicitly computed in [H1], and one sees that (3) is violated. Now suppose that $j = 18$ or 19. Then $\chi(u) = -q^3(q^3 - 1)/2$. On the other hand, according to Proposition 2.1 of the Appendix, $\chi(v) = mq(q^3 - 1)$ with $m \geq -(q^2 - 1)/2$. It follows that $\chi(v) > \chi(u)$, violating (3). Finally, suppose that $j = 20$. Then $\chi(u) = -q^3(q^3 - 1)$. Meanwhile, by Proposition 2.1 of the Appendix, $\chi(v) = mq(q^3 - 1)$ with $m \geq -(q^2 - 1)$. It follows that $\chi(v) > \chi(u)$, again violating (3).

Next we consider the case $q$ is even. Then $u$, resp. $v$, belongs to the $Q$-conjugacy class $c_{1,1}$, resp. $c_{1,7}$, in the notation of [H3]. According to [H3], the faithful character $\chi$ must be one of $\chi_j(k)$, $14 \leq j \leq 16$. If $j = 14$ or 15, then $\chi(u)$ is explicitly computed in [H3], and one sees that (3) is violated. Finally, suppose that $j = 16$. Then $\chi(u) = -q^3(q^3 - 1)/2$. On the other hand, according to Proposition 2.1 of the Appendix, $\chi(v) = mq(q^3 - 1)$ with $m \geq -(q^2 - 1)/2$. It follows that $\chi(v) > \chi(u)$, again violating (3).
Then $\chi(u) = -q^3(q^3 - 1)$. On the other hand, by Proposition 1.1 of the Appendix, $\chi(v) = mq(q^3 - 1)$ with $m \geq -(q^2 - 1)$. It follows that $\chi(v) > \chi(u)$, again violating (3).

**Proof of Theorem** Assume the contrary: $\Phi|_H$ is irreducible. Without loss we may assume that $\Phi$ is absolutely irreducible and that $H$ is a maximal subgroup of $G$. Now we can apply Theorem 6 to $H$ to get four possibilities (i) – (iv) for $H$. None of them cannot however occur by Theorems 7, 8 and 10.

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Appendix

Faithful characters of a parabolic subgroup of $^3D_4(q)$

by Frank Himstedt

Let $q$ be a prime power, $^3D_4(q)$ Steinberg’s triality group, and $Q$ a maximal parabolic subgroup of order $q^{12}(q^3-1)(q^2-1)$ of $^3D_4(q)$. A classification and construction of all irreducible characters of $Q$ is described in [H1, H3], and the values of almost all of these characters are given by Tables A.13 and A.14 in [H1, H3]. This appendix deals with some values of the faithful irreducible characters of $Q$ which are not contained in [H1, H3]. More specifically, we are interested in the values of the faithful irreducible characters of $Q$ on the “long root element” $x_\beta(1)$ (for a definition of $x_\beta(1)$, see [G1], for example). In particular, we prove bounds on these character values by considering scalar products with restrictions of unipotent irreducible characters of $^3D_4(q)$.

Suppose that $q$ is even. We use the same notation as in [H3]. The faithful irreducible characters of $Q$ are the characters $Q\chi_{14}(k)$, $Q\chi_{15}(k)$, $Q\chi_{16}(k)$ with $k$ as in [H3, Table A.13]. The values of $Q\chi_{14}(k)$ and $Q\chi_{15}(k)$ are given in [H3, Table A.14], in particular those on $x_\beta(1)$. So we only deal with the values of $Q\chi_{16}(k)$ on $x_\beta(1)$.

**Proposition 11.** The values of $Q\chi_{16}(k)$ on $x_\beta(1)$ satisfy:

$$Q\chi_{16}(k)(x_\beta(1)) \in \{ q(q^3-1) \cdot m \mid m \in \mathbb{Z} \text{ with } -(q^2-1) \leq m \leq q^2(q-1) \}.$$  

**Proof.** We recall the definition of $Q\chi_{16}(k)$ (see [H3, p. 258 and p. 256]): number the elements of the field $F_q$ in some way, say $F_q = \{ x_1, x_2, \ldots, x_q \}$ with $x_1 = 0$. Then, $Q\chi_{16}(k)$ is the character of $Q$ which is induced from the following linear character of the subgroup $X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$:

$$x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6) \mapsto \phi'(x_k \cdot d_2 + d_4 + d_5)$$

where $\phi'$ is a linear character of the additive group of $F_q^\times$ restricting nontrivially on $F_q$. Using the definition of induced characters and the relations in Tables 2.2-2.4 in [G1] we see that the value of $Q\chi_{16}(k)$ on $x_\beta(1)$ is:

$$y_k := \sum_{i=1}^{q^2-1} \sum_{j=1}^{q-1} \sum_{s \in F_{q^3}} \phi'(\tau(-q^2+q+1)i s^{q+1} + x_k \pi^{2j-i} + \pi^{j-i} s^{q^2+q+1})$$

where $\tau$ is a generator of the multiplicative group $F_{q^3}^\times$ and $\pi := \tau q^2+q+1$ a generator of $F_{q^3}^\times$. Since $x_\beta(1)$ is an involution we know $y_k \in \mathbb{Z}$ and so $y_k \leq q^3(q^3-1)(q-1)$. Let $[\varepsilon_2]_Q$ be the restriction of the unipotent irreducible character $[\varepsilon_2]$ of degree $q^2(q^4-q^2+1)$ of $^3D_4(q)$ (see [Sp]). Because we know the values of $Q\chi_{16}(k)$ on all conjugacy classes of $Q$ where $[\varepsilon_2]_Q$ is nonzero, we can compute the scalar product.
(q \chi_{16}(k), [\varepsilon_2]Q) = \frac{q^6 - q^4 - q^3 + q^2 + q + 1}{q(q - 1)}$ using CHEVIE. Since this scalar product is a nonnegative rational integer the statement about $Q\chi_{16}(k)(x_\beta(1))$ follows.

\[ \square \]

**Remark:** The value of $Q\chi_{16}(1)$ on $x_\beta(1)$ can be evaluated explicitly: using \([\text{4}]\) and \[ \sum_{i=1}^{q^3-1} \phi^i(\tau^i) = \sum_{j=1}^{q^3-1} \phi^j(\pi^j) = -1 \] one gets $Q\chi_{16}(1)(x_\beta(1)) = q(q^3 - 1).

Now suppose that $q$ is odd. We use the same notation as in [\text{11}]. The faithful irreducible characters of $Q$ are $Q\chi_{16}(k)$, $Q\chi_{17}(k)$, $Q\chi_{18}(k)$, $Q\chi_{19}(k)$, $Q\chi_{20}(k)$ with $k$ as in [\text{11} Table A.13]. The values of $Q\chi_{16}(k)$, $Q\chi_{17}(k)$ are given in [\text{11} Table A.14], in particular the values on $x_\beta(1)$. So we only deal with the values of $Q\chi_{18}(k)$, $Q\chi_{19}(k)$ and $Q\chi_{20}(k)$ on $x_\beta(1)$.

**Proposition 12.** The values $Q\chi_{18}(k)(x_\beta(1))$ and $Q\chi_{19}(k)(x_\beta(1))$ are elements of the set \(\{q(q^3 - 1) \cdot m \mid m \in \mathbb{Z} \text{ with } -(q^2 - 1)/2 \leq m \leq q^2(q - 1)/2\}\) and

\[ Q\chi_{20}(k)(x_\beta(1)) \in \{q(q^3 - 1) \cdot m \mid m \in \mathbb{Z} \text{ with } -(q^2 - 1) \leq m \leq q^2(q - 1)\}. \]

**Proof.** The proof is similar to that of Proposition [\text{11}]. By construction (see p. 795 and 783 in [\text{11}]), the value of $Q\chi_{18}(k)$ and $Q\chi_{19}(k)$ on $x_\beta(1)$ is:

\[ \sum_{i=1}^{q^3-1} \frac{(q-1)/2}{(q-1)} \sum_{j=1}^{q^3-1} \sum_{s \in \mathbb{F}_{q^3}} \phi^i(-\tau(-q^2-q+1)i\pi^j s - \pi^{-i-j+k}s^2+q+1) =: y_k'. \]

Because $x_\beta(1)$ is an involution we have $y_k' \in \mathbb{Z}$ and $y_k' \leq q^3(q^3 - 1)(q - 1)/2$. Using CHEVIE, we compute the scalar products $(Q\chi_{18}(k), [\varepsilon_2]Q) = (Q\chi_{19}(k), [\varepsilon_2]Q) = \frac{q^6 - q^4 - q^3 + q^2 + q + 1}{2q(q - 1)}$. The fact that this is a nonnegative integer implies the statement about $Q\chi_{18}(k)(x_\beta(1))$, $Q\chi_{19}(k)(x_\beta(1))$. By construction (see [\text{11} p. 795 and 784]):

\[ Q\chi_{20}(k)(x_\beta(1)) = \sum_{i=1}^{q^3-1} \sum_{j=1}^{q^3-1} \phi^i \left( \frac{2j-i+k}{2} \right) s^j - \pi^{-i-j+k}s^2+q+1) =: y_k'' \]

for $1 \leq k \leq (q - 1)/2$ and

\[ Q\chi_{20}(k)(x_\beta(1)) = \sum_{i=1}^{q^3-1} \sum_{j=1}^{q^3-1} \phi^i \left( \frac{2j-i+k}{2} \right) s^j - \pi^{-i-j+1}s^2+q+1) =: y_k'' \]

for $(q + 1)/2 \leq k \leq q - 1$. So $y_k'' \in \mathbb{Z}$ and $y_k'' \leq q^3(q^3 - 1)(q - 1)$. The fact $(Q\chi_{20}(k), [\varepsilon_2]Q) = \frac{q^6 - q^4 - q^3 + q^2 + q + 1}{q(q - 1)} \in \mathbb{Z}_{\geq 0}$ completes the proof.

\[ \square \]

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