Higher-Rank Non-Abelian Tensor Field Theory:
Higher-Moment or Subdimensional Polynomial Global Symmetry,
Algebraic Variety, Noether’s Theorem, and Gauge

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Abstract

With a view toward a theory of fracton and embeddon in condensed matter, we introduce a higher-moment polynomial degree-(m-1) global symmetry, acting on complex scalar/vector/tensor fields. We relate this higher-moment global symmetry of \(n\)-dimensional space, to a lower degree (either ordinary or higher-moment, e.g., degree-(m-1-\(\ell\))) subdimensional or subsystem global symmetry on layers of \((n-\ell)\)-submanifolds. These submanifolds are algebraic affine varieties (i.e., solutions of polynomials). The structure of layers of submanifolds as subvarieties can be studied via mathematical tools of embedding, foliation and algebraic geometry. We also generalize Noether’s theorem for this higher-moment polynomial global symmetry. We can promote the higher-moment global symmetry to a local symmetry, and derive a new family of higher-rank-m symmetric tensor gauge theory by gauging. By further gauging a discrete charge conjugation symmetry, we derive a new more general class of non-abelian rank-m tensor gauge field theory: a hybrid class of (symmetric or non-symmetric) higher-rank-m tensor gauge theory and anti-symmetric tensor topological field theory, generalizing [arXiv:1909.13879]'s theory interplaying between gapless and gapped sectors.

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Dedicated to 90 years of Gauge Principle since Hermann Weyl [Elektron und Gravitation, Zeit. für Physik 56, 330-352 (1929)]
1 Introduction

Motivated by the fracton order in condensed matter [1], recently two of the present authors introduced a new hybrid family of tensor gauge field theories [2] mixing between the anti-symmetric tensor gauge fields and symmetric higher-rank tensor gauge fields in a delicate way. Their purpose was to formulate the first toy model of a gauge theory with a non-abelian continuous gauge structure for fracton order in condensed matter (see a review [1] on the fracton order and References therein). The toy model [2] suggested an interplay between the gapped anti-symmetric tensor gauge topological quantum field theory (TQFT) with topological order and the gapless symmetric higher-rank tensor gauge theory with gapless higher-spin photon modes.

The higher-rank tensor gauge theory in Ref. [2] combines the feature of:¹

1. Continuum TQFT formulation of group-cohomology topological gauge theory (known as Dijkgraaf-Witten theory or twisted gauge theory [4]) by anti-symmetric tensor differential form gauge fields (i.e., Kalb-Ramond fields [5]). There in [2] and here, we mainly use a particular continuum TQFT formalism set-up and notations presented in [6–9] that can capture all finite abelian unitary gauge group and some non-abelian unitary gauge group of Dijkgraaf-Witten theory with the group-cohomology cocycle twist. (See also other related general formulations for non-dynamical gauge background theories [10,11] and references therein.)

2. Symmetric tensor field theory: We will only apply a specific class of symmetric higher-rank tensor gauge theories or higher-spin theories studied in condensed matter literature, e.g. Ref. [12–17].

Ref. [2] finds that a known class of symmetric higher-rank tensor gauge field theory (from the Model 2) has a higher-moment abelian gauge structure $U(1)^x_{(n)}$² and also an ordinary 0-form global symmetry known as a $\mathbb{Z}_2^C$-charge conjugation symmetry. The higher-moment symmetry $U(1)^x_{(n)}$ and the $\mathbb{Z}_2^C$-charge conjugation symmetry do not commute, they have a semi-direct product (denoted $\ltimes$) structure. Ref. [2] dynamically gauge the $\mathbb{Z}_2^C$-charge conjugation symmetry to gain a non-abelian gauge structure:

$$\left[ \mathbb{Z}_2^C \ltimes \left( U(1)^x_{(n)} \right) \right]. \quad (1.1)$$

The gauge structure is the first example in the fracton order literature satisfying properties:³

- compact
- continuous⁴

¹We should focus on the limited references essential to the construction of our theories. We pardon for potentially leaving out some other important works from the References. For more References in condensed matter literature, the readers can find in the review Ref. [1]. For a short historical account of gauge theory and the earlier References, since the time of Maxwell electromagnetism and Weyl gauge principle [3], the readers can find in [2].

²Here we denote a vector global symmetry along $n$-dimensions as $U(1)^x_{(n)}$: the $U(1)^x_{(d+1)}$ means the vector global symmetry in $d+1$-dimensional spacetime, and the $U(1)^x_{(d)}$ means the vector global symmetry in $d$-dimensional spacetime. The $x$ means to be a set of Cartesian coordinates for the spacetime. Throughout this article, we should only focus on Cartesian coordinates. The Cartesian coordinates can be easily realized in a square, rectangular, cubic lattices in condensed matter systems. On the other hand, it is difficult to imagine how to rewrite the higher-moment global symmetry in curved coordinates such as spherical or cylindrical coordinates, and how to realize them in a lattice system with energy cutoffs in condensed matter.

The $d+1d$ means the $d+1$ spacetime dimensions, with $d$ spatial and 1 time dimensions. The $Dd$ means the $D$ spacetime dimensions. The $\bar{D}D$ means the $\bar{D}$ space dimensions. We denote $d+1D$ means the $d$ spatial and 1 time dimensions.

³We denote the global symmetry in the bracket [...] to imply that it is dynamically gauged.

⁴We should remind the readers that there are alternative pursuits to construct nonabelian fracton orders with discrete gauge structures on a lattice [18–22] (e.g. discrete gauge theories) instead of continuous gauge structures. It is possible to Higgs down our model with continuous gauge structures to obtain higher-rank nonabelian tensor field theories with nonabelian discrete gauge structures [23].
The $U(1)_{x(n)}$ actually means there are $n$-independent vector directions $x_i$ with $i = 1, \ldots, n$. It is easier to understand $U(1)_{x(n)}$ before gauging it. Thus, let us first recover the gauge structure Eq. (1.1) to the ungauged global symmetry:

$$Z_2^C \ltimes \left( U(1) \times U(1)_{x(n)} \right).$$

We can perform an ordinary 0-form global symmetry $U(1)$ and a vector global symmetry $U(1)_{x(n)}$ by transforming a complex matter field $\Phi$ (following the pioneer work of Pretko’s [17] and [2])

$$\Phi \rightarrow e^{iQ(x)} \Phi = e^{i(\Lambda_1 x_1 + \Lambda_0)} \Phi.$$ (1.3)

When the $\Lambda_i$ is nonzero, we have a degree-1 polynomial $Q(x) = (\Lambda_i x_i + \Lambda_0)$ of $x$ which $\Lambda_i x_i$ specifies the $U(1)_{x(i)}$ vector global symmetry. Since there are $n$ independent $\Lambda_i x_i$, we have indeed several copies of commuting $U(1)_{x_j}$-vector global symmetry:

$$U(1)_{x(n)} := U(1)_{x_1} \times U(1)_{x_2} \times \cdots \times U(1)_{x_n} = \prod_{j=1}^n U(1)_{x_j}.$$ (1.4)

We name this global symmetry as a degree-1 global symmetry thanks to the degree-1 polynomial $Q(x) = (\Lambda_i x_i + \Lambda_0)$.

When the $\Lambda_i$ is zero, we have a degree-0 polynomial $(\Lambda_0)$ independent of $x$. We name this global symmetry as a degree-0 global symmetry thanks to the degree-0 polynomial $Q(x) = \Lambda_0$.

If we gauge only the vector global symmetry $U(1)_{x(n)}$ but not the ordinary symmetry $U(1)$, we gain the $[U(1)_{x(n)}]$ gauge structure, while the remained $U(1)$ is neither gauged nor global symmetry anymore. This particular way of gauging $[U(1)_{x(n)}]$ introduces the symmetric rank-2 tensor gauge field $A_{ij}$ which is compact along a 2-cycle integration

$$\oint A_{ij} \in 2\pi \mathbb{Z}$$
on any closed 2-surface. Ref. [2] dynamically gauge the discrete $Z_2^C$ which flips

$$A_{ij} \rightarrow -A_{ij}$$

to gain the non-abelian gauge structure. In particular the rank-2

In this work, we follow the setup in [2], and proceed to develop other families of new theories. We consider the following generalization:

(1). Higher-moment global symmetry for a complex scalar charge field $\Phi \in \mathbb{C}$: A general polynomial degree-$(m-1)$ global symmetry on $\Phi$ allows a symmetry transforation

$$\Phi \rightarrow e^{iQ(x)} \Phi := e^{i(\Lambda_{i_1,\ldots,i_{m-1}} x_{i_1} \cdots x_{i_{m-1}} + \cdots + \Lambda_{i,j} x_i x_j + \Lambda x_i + \Lambda_0)} \Phi.$$ (1.5)

When we gauge such a higher-moment polynomial degree-$(m-1)$ global symmetry, we will introduce a rank-m symmetric tensor gauge field $A_{i_1,\ldots,i_m}$ satisfying:

$$\oint A_{i_1,\ldots,i_m} \in 2\pi \mathbb{Z}$$
on any closed m-cycle or any closed m-surface. This new result is done in Sec. 2.
2. Higher-moment global symmetry for a complex vector charge field $\Phi_I \in \mathbb{C}$ (where the vector index is $I$, but each $\Phi_I$ still is a complex scalar field): A general polynomial degree-$(m - 1)$ global symmetry on $\Phi_I$ allows a symmetry transformation

$$\Phi_I \rightarrow e^{iQ_I(x)} \Phi_I := e^{i(\Lambda_{I; i_1, \ldots, i_{m-1}} x_{i_1} \cdots x_{i_{m-1}} + \cdots + \Lambda_{I; i, j} x_i x_j + \Lambda_{I; i} x_i + \Lambda_{I; 0})} \Phi_I.$$  

(1.7)

The degree-$(m - 1)$ polynomial $Q_I(x) = (\Lambda_{I; i_1, \ldots, i_{m-1}} x_{i_1} \cdots x_{i_{m-1}} + \cdots + \Lambda_{I; i, j} x_i x_j + \Lambda_{I; i} x_i + \Lambda_{I; 0})$ has an index $I$.

3. Higher-moment global symmetry for a complex rank-$M$ tensor charge field $\Phi_{I_1, \ldots, I_M} \in \mathbb{C}$ (where the tensor index is $I_1, \ldots, I_M$, but each $\Phi_{i_1, \ldots, i_M}$ still is a complex scalar field): A general polynomial degree-$(m - 1)$ global symmetry on $\Phi_{I_1, \ldots, I_M}$ allows a symmetry transformation

$$\Phi_{I_1, \ldots, I_M} \rightarrow e^{iQ_{I_1, \ldots, I_M}(x)} \Phi_{I_1, \ldots, I_M}.$$  

(1.8)

4. For all the above theories of higher-moment global symmetries, we construct their corresponding gauge theories by dynamically gauging the global symmetry. We introduce the abelian and non-abelian tensor gauge field, and their gauge invariant or covariant field strength tensor. We can use the field strength to construct the gauge invariant kinetic Lagrangian term of the non-abelian tensor gauge theory, shown in Sec. 2.1.2 and Sec. 2.1.3.

We also notice that the polynomial types or higher-moment types of global symmetries are also attempted to study systematically in [24, 25]. Although our motivations are somehow different from [24] and our framework is somehow different from [25]. We do not yet know a precise correspondence between our results [2, 23, 26] and theirs [24, 25].

Our theory can be formulated compatible with or without Euclidean, Poincaré, isotropic or anisotropic symmetry in $d + 1$ spacetime, at least in ultraviolet high or intermediate energy field theory, but not yet to a lattice cutoff scale, see more discussions in various versions of theories in [2]. Thus for Euclidean or Poincaré symmetry, we need to choose the $n$-dimensions in $U(1)_{x(n)}$ as $n = d + 1$ for dimensions. For an anisotropic symmetry, we can choose the $n$-dimensions in $U(1)_{x(n)}$ as $n \leq d$ for dimensions. Below we shall keep the general index $n$ in $U(1)_{x(n)}$, and leave the substitution of $n$ freely (to $n = d + 1$ or $n \leq d$) based on the specific needs of readers.

In a companion work, we explore the new types of sigma model that can interpolate between the disorder phases (as the present higher-rank tensor gauge theories) and the ordered phases. Similar to the famous quantum phase transition between insulator ($U(1)$ symmetry disorder described by a topological gauge theory or a disordered Sigma model) and superfluid/superconductivity ($U(1)$ global/gauge symmetry-breaking order described by a Sigma model with a $U(1)$ target space with Goldstone modes), we can explore phase structures of order–disorder phases by developing a new Sigma model [26].

## 2 Scalar Charge, Higher-Moment Polynomial Degree-$(m-1)$ Global Symmetry, and Rank-m Gauge Theory

First we consider how to gauge the following global symmetry for scalar field $\Phi$ on the $\mathbb{R}^n$ $n$-dimensional space or spacetime:

$$\Phi \rightarrow e^{iQ(x)} \Phi$$  

(2.1)

where $Q(x)$ is a polynomial with degree at most $(m - 1)$, say

$$Q(x) := (\Lambda_{i_1, \ldots, i_{m-1}} x_{i_1} x_{i_{m-1}} + \cdots + \Lambda_{i, j} x_i x_j + \Lambda_i x_i + \Lambda_0).$$  

(2.2)
Note that the gauge transformation can be written as:

$$\log \Phi \to \log \Phi + iQ(x)$$  \hspace{1cm} (2.3)

and the only invariant quantity under this transformation is

$$\partial^m \log \Phi,$$  \hspace{1cm} (2.4)

which is an order m symmetric tensor whose components are

$$\partial_{i_1} \cdots \partial_{i_m} \log \Phi,$$  \hspace{1cm} (2.5)

where the spacetime indices $i_k \in \{1, 2, \cdots, n\}$, with $k \in \{1, 2, \cdots, m\}$. In the next subsection, before gauging this higher-moment symmetry, we construct the covariant operator ($P_{i_1,\cdots,i_m}$ in Eq. (2.7)). After gauging this higher-moment symmetry, we also construct the gauge invariant operator (such as the Abelian gauge field strength in Sec. 2.1.2) or gauge covariant operator (such as the covariant derivative on the matter field $D_{i_1,\cdots,i_m}[[\Phi]]$ in Eq. (2.12) or the non-Abelian gauge field strength in Sec. 2.1.3).

### 2.1 Polynomial with arbitrary degree

By the law of differentiation, we know that

$$\partial_{i_1} \cdots \partial_{i_m} \log \Phi = \frac{P_{i_1,\cdots,i_m}(\Phi,\cdots,\partial^m \Phi)}{\Phi^m},$$  \hspace{1cm} (2.6)

so this order m tensor $P$ transforms as

$$P_{i_1,\cdots,i_m} \to e^{imQ(x)} P_{i_1,\cdots,i_m}.$$  \hspace{1cm} (2.7)

Under a more general gauge transformation

$$\Phi \to e^{i\eta(x)} \Phi,$$  \hspace{1cm} (2.8)

we find that

$$\log \Phi \to \log \Phi + i\eta(x).$$  \hspace{1cm} (2.9)

We shorthand $\partial_{i_1} \cdots \partial_{i_m} := \partial^m$ so

$$\partial^m \log \Phi \to \partial^m \log \Phi + i\partial^m \eta(x).$$  \hspace{1cm} (2.10)

This implies

$$P_{i_1,\cdots,i_m}(\Phi,\cdots,\partial^m \Phi) \to e^{im\eta(x)}(P_{i_1,\cdots,i_m}(\Phi,\cdots,\partial^m \Phi) + i\partial_{i_1} \cdots \partial_{i_m} \eta(x) \Phi^m).$$  \hspace{1cm} (2.11)

Therefore it is natural to introduce the connection-like symmetric rank-m tensor gauge field $A_{i_1,\cdots,i_m}$ and higher covariant derivative

$$D_{i_1,\cdots,i_m}[[\Phi]] := P_{i_1,\cdots,i_m}(\Phi,\cdots,\partial^m \Phi) - igA_{i_1,\cdots,i_m} \Phi^m,$$  \hspace{1cm} (2.12)

where we implicitly sum over all possible indices as

$$\sum_{\{i_1,\cdots,i_m\}} D_{i_1,\cdots,i_m}[[\Phi]] := \sum_{\{i_1,\cdots,i_m\}} \left( P_{i_1,\cdots,i_m}(\Phi,\cdots,\partial^m \Phi) - igA_{i_1,\cdots,i_m} \Phi^m \right),$$  \hspace{1cm} (2.13)

which transforms covariantly under a general gauge transformation

$$\Phi \to e^{i\eta(x)} \Phi.$$  \hspace{1cm} (2.14)
This implies that the tensor gauge field $A$ transforms as

$$A \rightarrow A + \frac{1}{g} \partial^m \eta.$$  \hspace{1cm} (2.15)

More precisely, components by components, it transforms as

$$A_{i_1, \ldots, i_m} \rightarrow A_{i_1, \ldots, i_m} + \frac{1}{g} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-1}} \partial_{i_m} \eta := A_{i_1, \ldots, i_m} + \frac{1}{g} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{m-1}} \partial_{i_m} \eta.$$ \hspace{1cm} (2.16)

So a gauge invariant term in the Lagrangian, involving the interactions between the gauge field and the scalar field, is

$$|D_{i_1, \ldots, i_m}[^\Phi]|^2 := (P_{i_1, \ldots, i_m}(\Phi, \cdots, \partial^m \Phi) - ig A_{i_1, \ldots, i_m} \Phi^m) (P_{i_1, \ldots, i_m}(\Phi^\dagger, \cdots, \partial^m \Phi^\dagger) + ig A_{i_1, \ldots, i_m}(\Phi^\dagger)^m).$$ \hspace{1cm} (2.17)

2.1.1 Different meanings of “gauging”

We should mention that our gauging procedure is a generalization of [17], but our gauging procedure may be different from some others in the literature [27–31]. This implies that the meaning of “gauging” actually is not the most unique refined statement — there can be different ways of “gauging” although the initial global symmetry is the same. “Gauging” can implies many different things:

1. Promoting a global symmetry to a local symmetry, and the local symmetry fluctuation is absorbed by the gauge transformations of dynamical gauge fields. (This is our way of gauging the higher-moment global symmetry [2,26].)

2. Coupling the symmetry generator (the charge operator or the charge current) of the higher-moment symmetry to a background field. And then in the partition function, one makes the background field dynamical by summing over all the allowed background field configurations. This is also related to the orbifold procedure in field theory or string theory.

3. Gauging may also be interpreted as the condensation of the “charged object” $O$ of a global symmetry. The condensation means that the “charged object” becomes part of the property of the ground state wavefunction. Suppose the “charged object” $O$ can be a local point operator or an extended (line/surface/etc.) operator, then the new ground state $|\Psi_{\text{ground state}}\rangle$ (i.e., new vacuum) with the condensed “charged object” $O$ means that in the quantum mechanical sense, the ground state is in a coherent state

$$\hat{O} |\Psi_{\text{ground state}}\rangle \propto |\Psi_{\text{ground state}}\rangle.$$ \hspace{1cm} (2.18)

The $\hat{O}$ is correspondingly a local or an extended quantum mechanical operator of the “charged object” $O$ in field theory. Intuitively $\hat{O}$ can be created and annihilated from the vacuum for free – $\hat{O}$ can pop out or pop into the vacuum:

$$\langle \Psi_{\text{ground state}} |\hat{O} |\Psi_{\text{ground state}}\rangle \propto \langle \hat{O} \rangle \neq 0,$$ \hspace{1cm} (2.19)

which is known as the condensation in the vacuum.

All these above procedures are related to “gauging,” although we can gauge the same initial global symmetry, different gauging procedures may still give rise to different types of gauge theories. In our work, we study the gauging from the perspectives of continuum field theory. We do not yet attempt to make connections to other “gauging” procedures done on the lattice [27–29,31], but leave for future work.
2.1.2 Abelian gauge field strength and tensor gauge theory

Follow Sec. 2.1, to construct a rank-(m + 1) gauge invariant abelian gauge field strength, we simply define

\[ F_{\mu,\nu,i_2,\ldots,i_m} := \partial_\mu A_{\nu,i_2,\ldots,i_m} - \partial_\nu A_{\mu,i_2,\ldots,i_m}. \]  

(2.20)

Here \( F_{\mu,\nu,i_1,\ldots,i_m} \) is anti-symmetric respect to \( \mu \leftrightarrow \nu \). It is easy to check the gauge invariance of \( F_{\mu,\nu,i_1,\ldots,i_m} \) under the abelian gauge transformation Eq. (2.16):

\[ F_{\mu,\nu,i_2,\ldots,i_m} \rightarrow \partial_\mu (A_{\nu,i_2,\ldots,i_m} + \frac{1}{g} \partial_i A_{\mu,i_2,\ldots,i_m}) - \partial_\nu (A_{\mu,i_2,\ldots,i_m} + \frac{1}{g} \partial_i A_{\nu,i_2,\ldots,i_m}) = F_{\mu,\nu,i_2,\ldots,i_m}. \]  

(2.21)

It is easy to construct the gauge invariant kinetic Lagrangian term

\[ |\tilde{F}_{\mu,\nu,i_2,\ldots,i_m}|^2 := F_{\mu,\nu,i_2,\ldots,i_m} F^{\mu,\nu,i_2,\ldots,i_m}. \]

2.1.3 Non-abelian gauge field strength and tensor gauge theory

We can also promote the abelian gauge field strength Eq. (2.20) to a non-abelian gauge field strength, follow the trick of Ref. [2] by gauging the ordinary 0-form \( Z^C_2 \)-charge conjugation global symmetry. The \( Z^C_2 \) acts on the rank-m tensor gauge field via:

\[ A_{i_1,\ldots,i_m} \rightarrow -A_{i_1,\ldots,i_m}. \]  

(2.22)

By promoting the global \( Z^C_2 \) to a local symmetry, we introduce a new 1-form \( Z^C_2 \)-gauge field \( C \) coupling to the 0-form symmetry \( Z^C_2 \)-charged object \( A_{i_1,\ldots,i_m} \) with a new \( g_c \) coupling. The \( Z^C_2 \) local gauge transformation is:

\[ A_{i_1,\ldots,i_m} \rightarrow e^{i\gamma_c(x)} A_{i_1,\ldots,i_m}, \quad C_\nu \rightarrow C_\nu + \frac{1}{g_c} \partial_\nu \gamma_c(x). \]  

(2.23)

Note \( A_{i_1,\ldots,i_m} \) is real-valued, so a generic \( e^{i\gamma_c(x)} \) complexifies the \( A_{i_1,\ldots,i_m} \). However, what we can do is restricting gauge transformation so it is only \( Z^C_2 \)-gauged (not \( U(1)^C \)-gauged)

\[ e^{i\gamma_c(x)} := (-1)^{\gamma'_c(x)} \in \{ \pm 1 \}, \]  

(2.24)

so \( \gamma'_c(x) \) is an integer and \( A_{i_1,\ldots,i_m} \) stays in real. Thus \( \gamma'_c(x) \) jumps between even or odd integers in \( \mathbb{Z} \), while the \( Z^C_2 \)-gauge transformation can be suitably formulated on a lattice. We can directly rewrite the above Eq. (2.23) on a simplicial complex or a triangulable spacetime manifold. Follow Ref. [2], we also define a new covariant derivative with respect to \( Z^C_2 \):

\[ D^c_\mu := (\partial_\mu - ig_c C_\mu). \]  

(2.25)

We need to combine \( U(1)_{x(a)} \)-gauge transformation Eq. (2.16) and \( Z^C_2 \)-gauge transformation Eq. (2.23) to:

\[ A_{i_1,\ldots,i_m} \rightarrow e^{i\gamma_c(x)} A_{i_1,\ldots,i_m} + \frac{1}{(m!)} g (D^c_{i_1} D^c_{i_2} \cdots D^c_{i_m})(\eta_c(x)), \quad C_\nu \rightarrow C_\nu + \frac{1}{g_c} \partial_\nu \gamma_c(x). \]  

(2.26)

Here \( (D^c_{i_1} D^c_{i_2} \cdots D^c_{i_m}) := (D^c_{i_1} D^c_{i_2} \cdots D^c_{i_m} + D^c_{i_2} D^c_{i_3} \cdots D^c_{i_m} + \ldots) \) containing the permutation \((m!)\)-terms, which means to be a symmetrization over the subindices under the lower bracket \((i_1,\ldots,i_m)\).
We thus can promote the abelian gauge field strength Eq. (2.20)’s $F_{\mu,\nu,i_2,\ldots,i_m}$ into a new non-abelian
gauge field strength $\hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c$ after gauging $Z_2^C$:

$$
\hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c := D^c_\mu A_{\nu,i_2,\ldots,i_m} - D^c_\nu A_{\mu,i_2,\ldots,i_m} := (\partial_\mu - ig_c C_\mu)A_{\nu,i_2,\ldots,i_m} - (\partial_\nu - ig_c C_\nu)A_{\mu,i_2,\ldots,i_m}
$$

This $\hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c$ is covariant under the gauge transformation Eq. (2.26):

$$
\hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c \rightarrow e^{i\gamma_c(x)} \hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c.
$$

It is obvious that we can construct the gauge invariant kinetic Lagrangian term

$$
|\hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c|^2 := \hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c \hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c.
$$

We can propose a schematic path integral form:

$$
Z_{\text{rk-m-sym}}^\text{asym-BF} := \int [DA_1,\ldots,i_m] [DB][DC] \exp(i \int_{M^{d+1}} \left( |\hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c|^2 d^{d+1}x + \frac{2}{2\pi} \sum_{I=1}^N B_I dC_I \right)) \cdot \omega_{d+1}\{C_I\}.
$$

More generally, we introduce the index $I$ for specifying the different copies/layers of tensor gauge theories,

$$
Z_{\text{rk-m-sym}}^\text{asym-BF} := \int \left( \prod_{I=1}^N [DA_{I,1},\ldots,i_m] [DB_I][DC_I] \right) \exp(i \int_{M^{d+1}} d^{d+1}x \left( \sum_{I=1}^N |\hat{F}_{\mu,\nu,i_2,\ldots,i_m}^c|^2 + \frac{2}{2\pi} \sum_{I=1}^N B_I dC_I \right)) \cdot \omega_{d+1}\{C_I\}.
$$

where the level-2 BF theory is used to constrain the flat $C$ gauge field to be a $Z_2$-valued 1-form gauge field via

a $Z_2$-valued $(d-1)$-form gauge field $B$, based on the trick of [2]. The cocycle $\omega_{d+1} \in H^{d+1}(Z_2^C, \mathbb{R}/\mathbb{Z})$ is a group cohomology data [4] that we apply its continuum field theory formulation [6–9] (see the overview [2]).

The cocycle $\omega_{d+1}$ couples different copies/layers of tensor gauge theories together, which can be viewed as interlayer interaction effects.

Above we formulate a general degree polynomial as a higher moment global symmetry and construct the

field strength $F$ for the gauge theory. Our theory presented above in Sec. 2.1 is general. Let us take two

special examples in the next subsections, for polynomial of degree 1 in Sec. 2.2 and degree 2 in Sec. 2.3.

### 2.2 Polynomial with degree 1: Vector symmetry

For a vector global symmetry of a polynomial with degree 1, the symmetry transformation on the scalar

field and the invariant quantity under this transformation are as follows:

$$
\Phi \rightarrow e^{iQ(x)} \Phi = e^{i(\Lambda_1 x_i + \Lambda_0)} \Phi,
$$

$$
\log \Phi \rightarrow \log \Phi + iQ(x) = \log \Phi + i(\Lambda_1 x_i + \Lambda_0),
$$

$$
\partial_i \partial_j \log \Phi = \frac{P_{x_i x_j}(\Phi, \partial \Phi, \partial^2 \Phi)}{\Phi^2} = \frac{\Phi \partial_i \partial_j \Phi - (\partial_i \Phi)(\partial_j \Phi)}{\Phi^2} \rightarrow \partial_i \partial_j \log \Phi,
$$

$$
\partial_i \partial_j \log \Phi \rightarrow \partial_i \partial_j \log \Phi + i\partial_i \partial_j \eta(x).
$$

In the last line, we simply shorthand $x_i, x_j$ as $i, j$. To gauge, we rewrite $Q(x)$ as a local gauge parameter

$\eta(x)$,
This implies that we can write the gauge covariant operator $D_{i,j}[^{\{\Phi}\}}$ via:
\[
P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) := (\Phi \partial_i \partial_j \Phi - (\partial_i \Phi)(\partial_j \Phi)) \to e^{i2\eta(x)}(P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) + i \partial_i \partial_j \eta(x)).
\]  
\[
A_{i,j} \to A_{i,j} + \frac{1}{g} \partial_i \partial_j \eta.
\]  
\[
D_{i,j}[\{\Phi\}] := P_{i,j}(\Phi, \partial\Phi, \partial^2\Phi) - igA_{i,j}\Phi^2 = (\Phi \partial_i \partial_j \Phi - (\partial_i \Phi)(\partial_j \Phi) - igA_{i,j}\Phi^2).
\]  
Physically we may define the symmetry transformation
\[
e^{i\lambda \cdot x} = e^{i2\pi(\lambda^{-1}) \cdot x}.
\]  
So the $\lambda$ is a vector of an effective wavelength while $\bar{\lambda} = |\Lambda| \lambda = \frac{2\pi}{|\Lambda|} \hat{\lambda}$, with the unit vector $\hat{\lambda} = \hat{\lambda}$. So a gauge invariant term can be $|D_{i,j}[\{\Phi\}]|^2$.

See another route of pursuit for the vector global symmetry recently by Seiberg [30].

2.3 Polynomial with degree 2: Higher-moment symmetry

For a vector global symmetry of a polynomial with degree 2, the symmetry transformation on the scalar field and the invariant quantity under this transformation are as follows:
\[
\Phi \to e^{iQ(x)}\Phi = e^{i(\Lambda_{i,j}x_i x_j + \Lambda_i x_i + \Lambda_0)}\Phi,
\]
\[
\log \Phi \to \log \Phi + iQ(x) = \log \Phi + i(\Lambda_{i,j}x_i x_j + \Lambda_i x_i + \Lambda_0),
\]
\[
\partial_i \partial_j \partial_k \log \Phi = \frac{P_{i,j,k}(\Phi, \cdots, \partial^3\Phi)}{\Phi^3}
\]
\[
:= \frac{\Phi^2(\partial_i \partial_j \partial_k \Phi) - \Phi((\partial_k \Phi)(\partial_i \partial_j \Phi) + (\partial_i \Phi)(\partial_j \partial_k \Phi) + (\partial_j \Phi)(\partial_k \partial_i \Phi)) + 2((\partial_i \Phi)(\partial_j \partial_k \Phi)(\partial_k \Phi))}{\Phi^3}.
\]
This implies that we can write the gauge covariant operator $D_{i,j,k}[\{\Phi\}]$ via:
\[
P_{i,j,k}(\Phi, \cdots, \partial^3\Phi) := \Phi^2(\partial_i \partial_j \partial_k \Phi) - \Phi((\partial_k \Phi)(\partial_i \partial_j \Phi) + (\partial_i \Phi)(\partial_j \partial_k \Phi) + (\partial_j \Phi)(\partial_k \partial_i \Phi)) + 2(\partial_i \Phi)(\partial_j \Phi)(\partial_k \Phi)
\]
\[
= \Phi^2(\partial_i \partial_j \partial_k \Phi) - 3\Phi(\partial_k \Phi \partial_i \partial_j \Phi) + 2(\partial_i \Phi)(\partial_j \Phi)(\partial_k \Phi)
\]
\[
\to e^{i\lambda \cdot x}(P_{i,j,k}(\Phi, \cdots, \partial^3\Phi) + i \partial_i \partial_j \partial_k \eta(x)).
\]
\[
A_{i,j,k} \to A_{i,j,k} + \frac{1}{g} \partial_i \partial_j \partial_k \eta.
\]
\[
D_{i,j,k}[\{\Phi\}] := P_{i,j,k}(\Phi, \cdots, \partial^3\Phi) - igA_{i,j,k}\Phi^3.
\]
The above we use the symmetrized tensor notation: $T_{(i_1i_2\cdots i_k)} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{i_{\sigma(1)i_{\sigma(2)}\cdots i_{\sigma(k)}}}$, with parentheses $(ijk)$ around the indices being symmetrized. The $S_k$ is the symmetric group of $k$ symbols. So a gauge invariant term can be $|D_{i,j,k}[\{\Phi\}]|^2$.

3 Vector Charge, Tensor Charge and General Higher-Moment Symmetry

3.1 Vector charge

We may also consider more general higher moment conservation laws with a set of number $r$ fields $\Phi_1, \cdots \Phi_r$, and gauge transformations
\[
\Phi_I \to e^{iQ_I(x)}\Phi_I
\]
where
\[ Q_I \in V_I \subseteq \bigoplus_{I=1}^r V_I = V \subseteq \mathbb{R}^r \otimes \mathbb{R}^{x_1, \ldots, x_n} = \bigoplus_{I=1}^r \mathbb{R}^{x_1, \ldots, x_n}. \] (3.2)

The \( V_I \) denotes the vector space where the polynomial \( Q_I = Q_I(x) \) lives in. A special case is \( V_I = \mathbb{R}^{x_1, \ldots, x_n} \). For this vector space \( \mathbb{R}^{x_1, \ldots, x_n} \), we have the vector addition in terms of the polynomial addition, while we have the scalar multiplication in terms of the scalar in the real number \( \mathbb{R} \) multiplying by the polynomial.

Note that the full vector space \( V \) is fully characterized by another vector space \( D \) of differential operators, which annihilate the space \( V \). This space \( D_I \) is not finite dimensional, but we may take a finite dimensional subspace \( \tilde{D} \) generating the vector space \( D \). Namely, we can take differential operators \( \mathcal{D}_J^I \) such that
\[ \sum_I \mathcal{D}_J^I Q_I = 0 \] (3.3)
for any \( Q_I \in V \). If
\[ \sum_I \mathcal{D}_J^I Q_I = 0 \] (3.4)
for any \( Q \) we have
\[ \mathcal{D}_J^I = \sum a_J^I \mathcal{D}_J^I \] (3.5)
for some \( a_J^I \). In the previous example, we took homogeneous polynomials of degree \( m \).

Following the same logic, we may show that each elements in \( \tilde{D} \) gives an invariant field strength. To gauge this symmetry, we need to introduce gauge fields which one to one corresponds to elements in \( \tilde{D} \). The gauge transformation law is transparent: they are just differential operators \( \mathcal{D}_J^I \) in \( \tilde{D} \) acting on the gauge variational parameter say \( \eta_I \).

Let us be more concrete: under the gauge transformation \( \Phi_I \rightarrow e^{iQ_I(x)}\Phi_I \), we have \( \log \Phi_I \rightarrow \log \Phi_I + iQ_I(x) \) and
\[ \sum_I \mathcal{D}_J^I \log \Phi_I \rightarrow \mathcal{D}_J \log \Phi_I \] (3.6)
as \( \sum_I \mathcal{D}_J^I Q_I = 0 \). We can compute
\[ \sum_I \mathcal{D}_J^I \log \Phi_I = \frac{P_J}{Q_J} \] (3.7)
where \( P_J \) is a polynomial in fields \( \Phi_I \) and their derivatives, and \( Q_J \) is a monomial in \( \Phi_I \). Namely, \( Q_J = \prod_I (\Phi_I)^{n_J^I} \) is a product of \( \Phi_I \), where \( n_J^I \) is an integer power of some \( \Phi_I \).

Therefore we can see that because the denominator \( Q_J \) as a monomial transforms in the following way, same for the numerator \( P_J \):
\[ Q_J \rightarrow e^{i\zeta_I(x)}Q_J, \] (3.8)
\[ P_J \rightarrow e^{i\zeta_I(x)}P_J, \] (3.9)
for some polynomial \( \zeta_I = \zeta_I(x) \) depending on our data. Each of these \( P_J \) corresponds to a covariant derivative under a general gauge transformation
\[ \Phi_I \rightarrow e^{i\eta(x)\Phi_I}. \] (3.10)
We can compute
\[ \sum_I \mathcal{D}_I^J \log \Phi_I = P_J \frac{Q_J}{\mathcal{Q}_J} + \sum_I \mathcal{D}_I^J \eta_I, \] (3.11)
hence \( P_J \) transform as
\[ P_J \rightarrow e^{i\zeta(x)} P_J + \sum_I \mathcal{D}_I^J \eta_I Q_J. \] (3.12)

Therefore we introduce a new covariant derivative
\[ D_J[\{\Phi_I\}] := R_J \equiv P_J - igA_J \mathcal{Q}_J \] (3.13)
where under the gauge transformation of \( \Phi_I \rightarrow e^{i\eta_I(x)} \Phi_I \), we have the gauge transformation of:
\[ A_J \rightarrow A_J + \frac{1}{g} \sum_I \mathcal{D}_I^J \eta_I. \] (3.14)

Then \( R_J \) transforms as \( R_J \rightarrow e^{i\zeta(x)} R_J \). We construct the gauge invariant matter-gauge field interaction term in the Lagrangian
\[ |D_J[\{\Phi_I\}]|^2 := |R_J|^2 := (R_J)(\bar{R}_J) = (P_J - igA_J \mathcal{Q}_J)(P_J^\dagger + igA_J \mathcal{Q}_J^\dagger). \] (3.15)

### 3.2 Tensor charge

It is embarrassingly trivial to generalize to this tensor-index complex scalar field for the global symmetry transformation:
\[ \Phi_{I_1,\ldots,I_M} \rightarrow e^{iQ_{I_1,\ldots,I_M}(x)} \Phi_{I_1,\ldots,I_M}, \] (3.16)
and the gauge symmetry transformation with a local dependent gauge parameter \( \eta_{I_1,\ldots,I_M}(x) \):
\[ \Phi_{I_1,\ldots,I_M} \rightarrow e^{i\eta_{I_1,\ldots,I_M}(x)} \Phi_{I_1,\ldots,I_M}. \] (3.17)

Nonetheless, we just need to give a one-to-one map \((I_1,\ldots,I_M) \rightarrow I\), so the tensor indices can be mapped to a vector index, which transforms the above two equations to:
\[ \Phi_I \rightarrow e^{i\alpha_I(x)} \Phi_I \]
and
\[ \Phi_I \rightarrow e^{i\eta_I(x)} \Phi_I \]
respectively. Thus the tensor charge higher-moment global symmetry can be treated as the same way as the vector charge higher-moment global symmetry in Sec. 3.2 under the one-to-one map \((I_1,\ldots,I_M) \rightarrow I\).

### 3.3 Example 1: Vector charge with an exclusive degree-1 polynomial

As a special case, we can recover the vector charged tensor gauge theory by taking the vector space of degree-1 polynomial
\[ V_I = \{1, x_{I-1}, x_{I+1}\} \] (3.18)
which means that it is spanned by the vectors of $1$, $x_{I-1}$, and $x_{I+1}$. The global symmetry acts as
\[
\Phi_I(x) \to e^{iQ_I(x)}\Phi_I(x) = e^{iQ_I(x_{I-1}, x_{I+1})}\Phi_I(x) = e^{i(\Lambda_{I+1}x_{I-1} - \Lambda_{I-1}x_{I+1} + \lambda_0)}\Phi_I(x).
\] (3.19)

Here we may define
\[
x_{I+l'} := x_{I+l'} \mod n, \text{ with the subindex where } I + l' := I + l' \mod n.
\]

In fact, our specific example here is a generalization of one example in Pretko’s [17]. We may call this type of $Q_I(x) = Q_I(x_{I-1}, x_{I+1})$ as an exclusive polynomial which the $Q_I$ excludes the $x_I$ dependency, thus it is $x_I$ independent.

The vector space $V$ is fully characterized by another vector space $D$ of differential operators, which annihilate $V$ by differential. This space $D_I$ is not finite dimensional, but we may take a finite dimensional subspace $\bar{D}$ generating the vector space $D$, here
\[
\bar{D} = \{\partial_{(i=I)} \Phi_I, \partial_{(j=J)} \Phi_K + \partial_{(k=K)} \Phi_J\},
\] (3.21)
where $j = J$ and $k = K$ are related by
\[j = k \pm 1 \mod n.
\]

Here the spacetime index $(i, j, k, \ldots)$ and the internal vector index $(I, J, K, \ldots)$ of $\Phi$ fields are locked.

Furthermore, we can effectively construct the gauge theory explicitly, given by the rule of gauge principle. For this special case, we can recover the vector charged tensor gauge theory and covariant derivatives:
\[
\{\partial_{(i=I)} \Phi_I - igA_I \Phi_I, \quad \Phi_J \partial_{(j=J)} \Phi_K + \Phi_K \partial_{(k=K)} \Phi_J - igA_{(j=J)(k=K)} \Phi_J \Phi_K\}.
\] (3.22)

In short, by locking $i = I$, $j = J$ and $k = K$, we simply write
\[
\{\partial_i \Phi_i - igA_i \Phi_i, \quad \Phi_J \partial_J \Phi_k + \Phi_k \partial_J \Phi_j - igA_{jk} \Phi_J \Phi_K\}.
\] (3.23)

We can effectively construct everything explicitly, given the rule of gauge transformations:
\[
\Phi_I \to e^{i\eta_I(x)}\Phi_I, \quad (3.24)
\]
\[
A_{jk} \to A_{jk} + \frac{1}{g}(\partial_j \eta_k + \partial_k \eta_j). \quad (3.25)
\]

### 3.4 Example 2: Vector charge with an inclusive degree-1 polynomial

Let us consider another simple example: Given fields $\Phi_1, \Phi_2, \ldots$, and the higher-moment global symmetry:
\[
\Phi_1 \to e^{i\Lambda_1 x_1} \Phi_1.
\] (3.26)

We may call this type of $Q_I(x) = Q_I(x_I)$ as an inclusive polynomial which the $Q_I$ include only the $x_I$ dependency. We have an invariant Lagrangian term $|\Phi_2 \partial_1 \Phi_1 - \Phi_1 \partial_2 \Phi_2|^2$. We can introduce a tensor connection field $A_{12}$, then the covariant derivative type of Lagrangian term $|\Phi_2 \partial_1 \Phi_1 - \Phi_1 \partial_2 \Phi_2 - igA_{12} \Phi_1 \Phi_2|^2$. More generally, we have
\[
|\Phi_i \partial_j \Phi_j - \Phi_j \partial_i \Phi_i - igA_{ij} \Phi_i \Phi_j|^2
\] (3.27)

---

5Our result in Eq. (3.19) generalizes Pretko’s
\[
\Phi_I(x) \to e^{i\sum_{I,J,K} \varepsilon_{IJK} \Lambda_{IJK} \Phi_I(x)}.
\] (3.20)

where $\varepsilon_{IJK} = \varepsilon^{IJK}$ is just a Levi-Civita symbol, or a so-called alternating tensor.

6In this notation below $\Phi_I \to e^{iQ_I(x)}\Phi_I = e^{iQ_I(x_{I-1}, x_{I+1})}\Phi_I$, we just focus on the dependence of $x$ only on $Q_I(x)$, not $\Phi_I$. 

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invariant under a general gauge transformation
\[ \Phi_j \rightarrow e^{i\eta_j(x)}\Phi_j, \]  
\[ A_{ij} \rightarrow A_{ij} + \frac{1}{g} (\partial_i \eta_i - \partial_j \eta_j), \]  
where \( i,j \) can be any coordinate since we have this specific global symmetry: \( \Phi_i \rightarrow e^{iA_i x_i} \Phi_i \) for any \( x_i \). Importantly, the generic gauge field \( A_{ij} \) is not symmetric under \( i \leftrightarrow j \).\(^7\) This example reveals that the generic higher-moment global symmetry for a vector-index charge field, after gauging, does not yield a symmetric tensor gauge field.

3.5 Example 3: Vector charge with a mixed degree-1 polynomial

Consider the vector-index charge fields: \( \Phi_j \). Consider the higher-moment global symmetry:
\[ \Phi_j \rightarrow e^{iA_j x_j} \Phi_j, \]  
where \( j = 1, 2, \ldots \) We have an invariant Lagrangian term \( |\Phi_2 \partial_1 \Phi_1 - \Phi_1 \partial_1 \Phi_2|^2 \), and other invariant Lagrangian terms \( |\Phi_j \partial_1 \Phi_1 - \Phi_1 \partial_1 \Phi_j|^2 \). We can introduce a tensor gauge connection field \( A_{ij} \), then the covariant derivative
\[ \Phi_j \partial_1 \Phi_1 - \Phi_1 \partial_1 \Phi_j - iA_{ij} \Phi_i \Phi_j \]  
is invariant under a general gauge transformation
\[ \Phi_j \rightarrow e^{i\eta_j(x)}\Phi_j, \]  
\[ A_{ij} \rightarrow A_{ij} + (\partial_1 \eta_i - \partial_1 \eta_j). \]  
Again \( i,j \) can be any coordinate since we have this specific global symmetry: \( \Phi_i \rightarrow e^{iA_i x_i} \Phi_i \) for any \( x_i \). Importantly, similar to Sec. 3.5, the generic gauge field \( A_{ij} \) is not symmetric under \( i \leftrightarrow j \).

4 Generalizing Noether’s Theorem for Higher-Moment Global Symmetry

Suppose we have a set of \( r \) fields \( \Phi_I \) (\( 1 \leq I \leq r \)), a Lagrangian term \( L \) which is invariant under a global transformations \( \Phi_I \rightarrow e^{iQ_I(x)}\Phi_I \) where \((Q_1, \cdots, Q_r) \in V \subset \oplus_{I=1}^r \mathbb{R}[x_1, \cdots, x_d]\) is a specified vector space of allowed polynomials.

Noether’s theorem guarantees that we have a conserved current corresponding to each global symmetry. Suppose the constant U(1) transformation for each field is a global symmetry, Noether’s theorem says that we have a one form current
\[ j_I = j_{I\mu} dx^\mu, \]  
\(7\)\( A_{ij} \) can be made symmetric if we revise the transformation law, for a specific pair of \( (i,j) \), such that \( \Phi_i \rightarrow e^{iA_i x_i} \Phi_i \) and \( \Phi_j \rightarrow e^{-iA_j x_j} \Phi_j \), so that the Lagrangian term
\[ |\Phi_i \partial_j \Phi_j + \Phi_j \partial_i \Phi_i - igA_{ij} \Phi_i \Phi_j|^2 \]  
invariant under a general gauge transformation
\[ \Phi_j \rightarrow e^{i\eta_j(x)}\Phi_j, \quad A_{ij} \rightarrow A_{ij} + \frac{1}{g}(\partial_i \eta_i + \partial_j \eta_j). \]
such that under the general infinitesimal variation
\[ \Phi_I \to e^{i\epsilon I(x)} \Phi_I, \] (4.2)
the Lagrangian density transforms as
\[ \delta L = \epsilon I \wedge \ast j_I. \] (4.3)
The \( \epsilon \) is an infinitesimal variational parameter. Here \( \rho_I = j_{I0} \) is the spatial density of the conserved charge.

Now let us take \( \alpha_I = Q_I(x) \) as the higher-moment global symmetry polynomial for
\[ \Phi_I \to e^{iQ_I(x)} \Phi_I \]
we said earlier, we see that \( \int \delta L = 0 \), and therefore
\[ \int_{\text{space}} \sum_I \rho_I Q_I = \int_{\text{space}} \sum_I j_{I0} Q_I \] (4.4)
is a conserved charge. That is, we have a conserved charge for each of the global symmetry we have, their number is precisely the dimension of the vector space \( V \) we started with.

By doing the above calculation, we need to be careful about the boundary conditions of the space manifold, or the infinite faraway field configurations of the space manifold. In most cases, we can assume that the density of field configurations decays sharply at the infinite faraway.

Let us take \( Q_I(x) \) is a polynomial over the spatial coordinates. Here are some examples:

1. For a single field \( \Phi \), when \( Q(x) = \Lambda \) is a constant, we have the usual Noether’s theorem for the ordinary \( U(1) \) global symmetry, with a conserved charge:
\[ \int_{\text{space}} \rho = \int_{\text{space}} j_0 \] (4.5)

2. For a single field \( \Phi \), when \( Q(x) = \Lambda_i x_i + \Lambda_0 \) is a linear degree-1 polynomial, we have a conservation theorem for the vector \( U(1) \) global symmetry. This coincides an example of Pretko’s [14].
\[ \Lambda_0 \text{'s}: \int_{\text{space}} \rho = \int_{\text{space}} j_0 \] (4.6)
\[ \Lambda_i \text{'s}: \int_{\text{space}} \rho x_i = \int_{\text{space}} j_0 x_i. \] (4.7)

There are the same number of conserved charges as the dimensions of the vector space (the independent parameters \( \Lambda_0 \) and \( \Lambda_i \) of the degree-1 polynomials).

3. For a single field \( \Phi \), when we follow Eq. (2.2) with
\[ Q(x) := (\Lambda_{i_1 \ldots i_{m-1}} x_{i_1} \ldots x_{i_{m-1}} + \cdots + \Lambda_{i,j} x_i x_j + \Lambda_i x_i + \Lambda_0) \]
of a degree-(\( m - 1 \)) polynomial, we have a conservation theorem for all independent \( \Lambda_{i_1 \ldots i_k} \)
\[ \Lambda_{i_1 \ldots i_k} \text{'s}: \int_{\text{space}} (\rho) \cdot (x_1 \ldots x_k) = \int_{\text{space}} (j_0) \cdot (x_1 \ldots x_k). \] (4.8)
for the higher-moment \( U(1) \) global symmetry. There are the same number of conserved charges as the dimensions of the vector space (the independent parameters \( \Lambda_{i_1 \ldots i_k} \)).
For a vector-index field $\Phi_I$, with 

$$Q_I(x) := (\Lambda_{i_1,\ldots,i_m} x_{i_1} \ldots x_{i_{m-1}} + \Lambda_{i,j} x_i x_j + \Lambda_{i,0}),$$

There are the same number of conserved charges as the dimensions of the vector space (the independent parameters $\Lambda_{i_1,\ldots,i_k}$).

For a tensor-index field $\Phi_{I_1,\ldots,I_M}$, we can map to a vector-index field $\Phi_I$ by a one-to-one map $(I_1,\ldots,I_M) \to I$, thus the result follows from the previous remark.

In all cases, if we have additional constraints (such as from the constraint of field strength, say the electric tensor in [14] to be traceless, say $\tilde{E}_j^j = 0$ for $\tilde{E}_{ij} = -\partial_0 A_{ij} + \partial_i \partial_j A_0 = -\partial_t A_{ij} + \partial_i \partial_j A_0$ for [2]'s notation), then we have additional new conservation laws, not accounted by the previously counted number of conserved laws as the dimensions of the vector space of $Q_I(x)$.

5 Conclusion, and Relations to Algebraic Geometry

In this section, we bridge the relations between our theories (both the matter or the gauge theories) by physics construction and the algebraic geometry in mathematics. We conclude with some final comments.

5.1 Algebraic (affine) Variety and Subvariety

In mathematics, the polynomials are related to geometric objects called the algebraic variety. More precisely, (affine) varieties are defined as the solutions of polynomial equations. The morphisms between them are maps defined by polynomials. Here we review their basic definitions for both physicists and mathematicians:

**Definition 1.** An affine algebraic variety over real numbers $\mathbb{R}$ is the zero-locus in the affine space $\mathbb{R}^n$ of some finite family of polynomials of $n$ variables with coefficients in $\mathbb{R}$.

**Definition 2.** A morphism, or a regular map, of affine varieties is a function between affine varieties which is polynomial in each coordinate: more precisely, for affine varieties $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$, a morphism from $V$ to $W$ is a map $\phi : V \to W$ of the form $\phi(a_1,\ldots,a_n) = (f_1(a_1,\ldots,a_n),\ldots,f_m(a_1,\ldots,a_n))$, where $f_i \in \mathbb{R}[X_1,\ldots,X_n]$ for each $i = 1,\ldots,m$. Here $(a_1,\ldots,a_n) \in V \subseteq \mathbb{R}^n$ and $\phi(a_1,\ldots,a_n) \in W \subseteq \mathbb{R}^m$.

**Definition 3.** Given two affine varieties $V,W \subseteq \mathbb{R}^n$, $V$ is called a subvariety of $W$, if $V \subseteq W$ as subsets of $\mathbb{R}^n$.

**Definition 4.** Two affine varieties $V$ and $W$ are isomorphic if there exist morphisms $\phi : V \to W$ and $\psi : W \to V$ such that $\psi \circ \phi = \text{id}_V$ and $\phi \circ \psi = \text{id}_W$, where $\text{id}$ is the identity map.

For example, $x^2 + y^2 = 1$ defines the unit circle and $t^2 + w^2 = 1$ defines an ellipse, both on the plane $\mathbb{R}^2$. There is a polynomial map $t \to 2x, w \to y$ which identifies circle and ellipse, with inverse given by a rescaling $x \to t/2, y \to w$ therefore in algebraic geometry they are isomorphic.

In our setting, the higher-moment global symmetry transformations are given by polynomials on the space (here we focus on the Cartesian $\mathbb{R}^n$ or $\mathbb{R}^d$ stated since Sec. 1), and the contours (or constant hypersurfaces) are given by solutions of polynomials: they are subvarieties of our space (here on the Cartesian $\mathbb{R}^n$ or $\mathbb{R}^d$).
We should mention that Ref. [27, 32] has a different look on the algebraic variety: the topological degeneracy of the gapped fractonic topologically ordered state in [27] are encoded also in an algebraic variety, which is defined by the common zeros of a set of polynomials over a finite field.

In contrast, the algebraic variety in our case is a way to organize the data of generalized higher-moment or subdimensional polynomial global symmetry or its gauge theory. The use of algebraic variety for our wide classes of theories do not require to be a gapped (fractonic) topological order. Our theories include gapless or gapped theories.

5.2 From Higher-Moment to Subdimensional or Subsystem Polynomial Global Symmetry

Let us relate the algebraic (affine) variety and subvariety in Sec. 5.1 to the patterns of polynomial in the higher-moment or subdimensional or subsystem polynomial global symmetry, or their gauge theories. The studies of subdimensional or subsystem global symmetries can be traced back to as early as Ref. [33, 34] in condensed matter literature. Here we generalize the concept to study the subdimensional or subsystem polynomial global symmetry. For instance, subsystem global symmetry can act on lines [35, 36] or planes [27, 37, 38], for the bulk of 2+1D systems [35, 36] or 3+1D systems [27, 35, 38].

1. From a degree-1 higher-moment symmetry to a degree-0 ordinary global symmetry in subdimensions:
   Recall the degree-1 polynomial global symmetry of Eq. (2.30), \( \Phi \rightarrow e^{iQ(x)} \Phi = e^{i(\Lambda_i x_i + \Lambda_0)} \Phi \) acting on the matter field \( \Phi \) on \( \mathbb{R}^n \). We can relate this degree-1 polynomial global symmetry to a degree-0 ordinary global symmetry by taking the constant surface solution of
   \[
   (\Lambda_i x_i + \Lambda_0) = \Lambda_{\text{constant}}
   \]
   for a certain \((n - 1)D\) subdimensional space (e.g. plane) of \( x_i \in \mathbb{R}^n \).

2. From a degree-2 higher-moment symmetry to a degree-1 higher-moment or degree-0 ordinary global symmetry:
   Recall the degree-2 polynomial global symmetry of Eq. (2.39), \( \Phi \rightarrow e^{iQ(x)} \Phi = e^{i(\Lambda_{i,j} x_i x_j + \Lambda_i x_i + \Lambda_0)} \Phi \) acting on the matter field \( \Phi \) on \( \mathbb{R}^n \). We can relate this degree-2 polynomial global symmetry to a degree-1 symmetry by restricting to an appropriate the constant \( x_i = c_i \) space for some specific \( x_i \).
   For example, we have
   \[
   (\Lambda_{i,j} x_i x_j + \Lambda_i x_i + \Lambda_0) \big|_{x_i = c_i} = (\Lambda_{i,j} c_i c_j + \Lambda_i c_i + \Lambda_0)
   \]
   for a certain \((n - 1)D\) subdimensional space of \( x_i \in \mathbb{R}^n \). Moreover, we can reduce to a degree-0 ordinary global symmetry, if there is an intersecting subspace between the constant spaces of \( x_i = c_i \) and \( x_j = c_j \), e.g.
   \[
   (\Lambda_{i,j} x_i x_j + \Lambda_i x_i + \Lambda_0) \big|_{x_i = c_i, x_j = c_j} = (\Lambda_{i,j} c_i c_j + \Lambda_i c_i + \Lambda_0).
   \]
   Depend on the \( \Lambda_{i,j} \) and \( \Lambda_i \), there could be a different constant surface by solving the polynomial with different set of constraints.

For example, given a two-variable quadratic equation
   \[
   Q(x) := Q(x_1, x_2) = \Lambda_{1,1} (x_1)^2 + \Lambda_{1,2} x_1 x_2 + \Lambda_{2,2} (x_2)^2 + \Lambda_1 x_1 + \Lambda_2 x_2 + \Lambda_0,
   \]
   we can solve the constant space to be an ellipse, a parabola, a hyperbola, also possibly a circle, a line, or two crossing lines, etc. The solution is a quadratic algebraic curves through the well-known conic section. In other words, if we apply the degree-2 polynomial global symmetry of \( e^{iQ(x)} \) under
Eq. (5.1), we can find the degree-2 polynomial global symmetry on the $\mathbb{R}^n$ reduced to the ordinary degree-0 global symmetry on the algebraic curves (an ellipse, a parabola or a hyperbola, etc) through the well-known conic section.

3. From a degree-$(m-1)$ higher-moment symmetry to a subdimensional lower-degree (higher-moment or ordinary) global symmetry: Recall the general degree-$(m-1)$ polynomial global symmetry of Eq. (2.1),

$$\Phi \rightarrow e^{iQ(x)}\Phi$$

where $Q(x)$ is a polynomial with degree at most $(m-1)$, say Eq. (2.2)

$$Q(x) := (\Lambda_{i_1,...,i_{m-1}}x_{i_1} \cdots x_{i_{m-1}} + \cdots + \Lambda_{i,j}x_i x_j + \Lambda_i x_i + \Lambda_0).$$

We can reduce the degree-$(m-1)$ higher-moment symmetry in $\mathbb{R}^n$ to a lower degree-$(m-2)$ higher-moment symmetry in $\mathbb{R}^{n-1}$ by restricting to a specific subspace $x_i = c_i$.

More generally, we can solve the polynomial with certain constraints as a lower-degree polynomial. This is related to the concepts of variety and subvariety in Sec. 5.1, and the mathematical concepts of embedding and foliation of subspaces. Indeed, the foliation concepts are powerful and applied recently in fracton literature, e.g. [39, 40]. It is also pointed out that the concept of spacetime embedding may be treated as a quantized excitation, named the embeddon [2]. Therefore, it will be illuminating to revisit all the above new gauge theories of higher-moment or subdimensional polynomial global symmetry in a fully quantum mechanical set up in the future.

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References

[1] R. M. Nandkishore and M. Hermele, Fractons, Ann. Rev. Condensed Matter Phys. 10 295–313 (2019), [arXiv:1803.11196].
[2] J. Wang and K. Xu, Higher-Rank Tensor Field Theory of Non-Abelian Fracton and Embeddon, [arXiv:1909.13879].
[3] H. Weyl, Electron and Gravitation. 1. (In German), Z. Phys. 56 330–352 (1929).

After the completion of our work, we thank Meng Cheng for pointing out a potentially related Ref. [24] and references therein on the study of the polynomial shift symmetries. This is a generalization to allow for an extension of the constant shift symmetry to a polynomial shift symmetry in the spatial coordinates. Although the essences of our and their ideas are related, the outcomes and motivations are dramatically different. We do not yet know the precise correspondence between our results and theirs.
[4] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, *Commun.Math.Phys.* **129** 393 (1990).

[5] M. Kalb and P. Ramond, *Classical direct interstring action*, *Phys. Rev.* **D9** 2273–2284 (1974).

[6] J. Wang, X.-G. Wen and S.-T. Yau, *Quantum Statistics and Spacetime Surgery*, arXiv:1602.05951.

[7] P. Putrov, J. Wang and S.-T. Yau, *Braiding Statistics and Link Invariants of Bosonic/Fermionic Topological Quantum Matter in 2+1 and 3+1 dimensions*, *Annals Phys.* **384** 254–287 (2017), [arXiv:1612.09298].

[8] J. Wang, K. Ohmori, P. Putrov, Y. Zheng, Z. Wan, M. Guo et al., *Tunneling Topological Vacua via Extended Operators: (Spin-)TQFT Spectra and Boundary Deconfinement in Various Dimensions*, *PTEP* **2018** 053A01 (2018), [arXiv:1801.05416].

[9] J. Wang, X.-G. Wen and S.-T. Yau, *Quantum Statistics and Spacetime Topology: Quantum Surgery Formulas*, *Annals Phys.* **409** 167904 (2019), [arXiv:1901.11537].

[10] A. Kapustin and R. Thorngrén, *Anomalies of discrete symmetries in various dimensions and group cohomology*, arXiv:1404.3230.

[11] J. C. Wang, Z.-C. Gu and X.-G. Wen, *Field theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology and beyond*, *Phys. Rev. Lett.* **114** 031601 (2015), [arXiv:1405.7689].

[12] A. Rasmussen, Y.-Z. You and C. Xu, *Stable Gapless Bose Liquid Phases without any Symmetry*, arXiv e-prints arXiv:1601.08235 (2016 Jan), [arXiv:1601.08235].

[13] M. Pretko, *Subdimensional Particle Structure of Higher Rank U(1) Spin Liquids*, *Phys. Rev.* **B95** 115139 (2017), [arXiv:1604.05329].

[14] M. Pretko, *Generalized Electromagnetism of Subdimensional Particles: A Spin Liquid Story*, *Phys. Rev.* **B96** 035119 (2017), [arXiv:1606.08857].

[15] M. Pretko, *Higher-Spin Witten Effect and Two-Dimensional Fracton Phases*, *Phys. Rev.* **B96** 125151 (2017), [arXiv:1707.03838].

[16] K. Slagle, A. Prem and M. Pretko, *Symmetric Tensor Gauge Theories on Curved Spaces*, *Annals Phys.* **410** 167910 (2019), [arXiv:1807.00827].

[17] M. Pretko, *The Fracton Gauge Principle*, *Phys. Rev.* **B98** 115134 (2018), [arXiv:1807.11479].

[18] S. Vijay and L. Fu, *A Generalization of Non-Abelian Anyons in Three Dimensions*, arXiv:1706.07070.

[19] H. Song, A. Prem, S.-J. Huang and M. A. Martin-Delgado, *Twisted Fracton Models in Three Dimensions*, *Phys. Rev.* **B99** 155118 (2019), [arXiv:1805.06899].

[20] A. Prem, S.-J. Huang, H. Song and M. Hermele, *Cage-Net Fracton Models*, *Phys. Rev.* **X9** 021010 (2019), [arXiv:1806.04687].

[21] D. Bulmash and M. Barkeshli, *Gauging fractons: immobile non-Abelian quasiparticles, fractals, and position-dependent degeneracies*, arXiv:1905.05771.

[22] A. Prem and D. J. Williamson, *Gauging permutation symmetries as a route to non-Abelian fractons*, arXiv:1905.06309.
[23] J. Wang and et al, *Higher-Rank Non-Abelian Tensor Field Theory: Higgs mechanism and Beyond*, arXiv:191n.nnnnn.

[24] T. Griffin, K. T. Grosvenor, P. Horava and Z. Yan, *Scalar Field Theories with Polynomial Shift Symmetries*, Commun. Math. Phys. 340 985–1048 (2015), [arXiv:1412.1046].

[25] A. Gromov, *Towards classification of Fracton phases: the multipole algebra*, Phys. Rev. X9 031035 (2019), [arXiv:1812.05104].

[26] J. Wang, K. Xu and S.-T. Yau, *Higher-Rank Non-Abelian Tensor Field Theory: Fully Gauged Fractonic Matter versus New Sigma Model*, arXiv:1910.nnnnn.

[27] S. Vijay, J. Haah and L. Fu, *Fracton Topological Order, Generalized Lattice Gauge Theory and Duality*, Phys. Rev. B94 235157 (2016), [arXiv:1603.04442].

[28] D. J. Williamson, *Fractal symmetries: Ungauging the cubic code*, Phys. Rev. B94 155128 (2016), [arXiv:1603.05182].

[29] W. Shirley, K. Slagle and X. Chen, *Foliated fracton order from gauging subsystem symmetries*, SciPost Phys. 6 041 (2019), [arXiv:1806.08679].

[30] N. Seiberg, *Field Theories With a Vector Global Symmetry*, arXiv:1909.10544.

[31] D. Radicevic, *Systematic Constructions of Fracton Theories*, arXiv:1910.06336.

[32] S. Vijay, J. Haah and L. Fu, *A New Kind of Topological Quantum Order: A Dimensional Hierarchy of Quasiparticles Built from Stationary Excitations*, Phys. Rev. B92 235136 (2015), [arXiv:1505.02576].

[33] C. D. Batista and Z. Nussinov, *Generalized Elitzur’s theorem and dimensional reduction*, Phys. Rev. B72 045137 (2005), [arXiv:cond-mat/0410599].

[34] Z. Nussinov and G. Ortiz, *A symmetry principle for topological quantum order*, Annals Phys. 324 977–1057 (2009), [arXiv:cond-mat/0702377].

[35] Y. You, T. Devakul, F. J. Burnell and S. L. Sondhi, *Subsystem symmetry protected topological order*, Phys. Rev. B98 035112 (2018), [arXiv:1803.02369].

[36] T. Devakul, D. J. Williamson and Y. You, *Classification of subsystem symmetry-protected topological phases*, Phys. Rev. B98 235121 (2018), [arXiv:1808.05300].

[37] Y. You, T. Devakul, F. J. Burnell and S. L. Sondhi, *Symmetric Fracton Matter: Twisted and Enriched*, arXiv:1805.09800.

[38] T. Devakul, W. Shirley and J. Wang, *Strong planar subsystem symmetry-protected topological phases and their dual fracton orders*, arXiv:1910.01630.

[39] W. Shirley, K. Slagle, Z. Wang and X. Chen, *Fracton Models on General Three-Dimensional Manifolds*, Phys. Rev. X8 031051 (2018), [arXiv:1712.05892].

[40] W. Shirley, K. Slagle and X. Chen, *Twisted foliated fracton phases*, arXiv e-prints arXiv:1907.09048 (2019 Jul), [arXiv:1907.09048].