Solution to Waves in Dissipative Media with Reciprocal Attenuation in

Time and Space Domains

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Abstract. Waves dissipate energy when they propagate through real medium. Theoretical study of waves is one of important way to understand the nature of waves in medium with dissipation. The study points out that the theoretical solution to the wave equation describing a disturbance propagating in a dissipative medium is not unique, which is determined by the dissipation mechanism of the medium. A new general solution is proposed by assuming that the attenuations of disturbance can occur in the time and space domains. The general solution is further used in case studies. The properties of viscoelastic waves propagating in the Kelvin–Voigt medium and electromagnetic waves propagating in conductive medium with the reciprocal attenuation in time and space domains are analyzed. The result shows that the attenuation mechanism has an obvious influence on the properties of waves in the dissipative medium when the wave equations are the same.
Introduction

Wave is an important form of substance movement, which exists widely in nature. According to physical quantities of disturbance, waves are divided into mechanical waves [1], electromagnetic waves [2] and gravitational waves [3], etc. When waves propagate in medium, they transfer the energy and information of disturbance source from one part of medium to another [4, 5]. Since the propagation properties of waves are determined by the properties of medium and the wave field is affected by the shape and distribution of media, the wave field includes the physical and geometric properties of media [6, 7]. Therefore by receiving the wave signal propagating in the media, the information of disturbance source and the properties of media can be obtained. As the carriers of energy and information, Waves have a wide range of applications in many fields including communication, remote controls, medical equipment, machinery manufacturing and earth sciences for information transmission [8, 9], damage detection [6, 7], energy harvesting [10, 11], geophysical exploration [12, 13] and so on.

The propagation of disturbance in a medium is closely related to the properties of the medium. When a wave propagates in a real medium, the medium absorbs the energy carried by the wave, and the amplitude of disturbance decreases with its propagation in the medium. In theory, the medium is regarded as an ideal medium when the energy dissipation of the wave in the medium is small or is not concerned like elastomer for mechanical disturbance [1, 14] and non-conductive medium for electromagnetic disturbance [2]. Otherwise, the medium is regarded as the dissipative medium like viscoelastic media [15] and conductive media [16]. The theoretical study of waves is
an effective way to describe and explain the properties of waves in media, and then apply waves to practice. To theoretically describe the propagation of disturbance in different media, different wave models are proposed based on experimental studies on the properties of media [1, 2, 4, 13, 15, 16]. For mechanical disturbances, mechanical waves are generated by the interaction between neighboring material elements in continuum. Energy and momentum are transferred from one material element to the next by this interaction. From the perspective of energy, the mechanical wave is the transformation between deformation energy and kinetic energy in the time domain and the propagation of mechanical energy in space domain. The wave equations derived from the mechanical disturbance models govern the displacement of material elements in different position at a certain time and the vibration of the material element at a certain point. To visualize the motion of the material element, the vibration of material elements in continuum is regarded as the mass spring damper system [15]. For electromagnetic disturbances, the electromagnetic wave is the transformation between electrical energy and magnetic energy in the time domain and the propagation of electromagnetic energy in the space domain. Although the mechanisms of mechanical waves and electromagnetic waves are different, their wave equations are the same in form. Therefore, it is concluded from wave equations that different kinds of waves have the same wave characteristics.

The analysis of the general solution of the wave equation is an important way to understand the property of wave propagation in a specific medium, especially for linear media, in which wave propagation satisfies the principle of superposition. For the wave
propagating in a linear medium, its properties can be obtained from the solution of harmonic wave. By viewing the disturbance source as the superposition of harmonic waves with different frequencies, the theoretical solution of the wave caused by arbitrary disturbance in a linear medium can be obtained. When the general solution of harmonic wave and the analytical solution of wave propagation in linear medium caused by arbitrary disturbances are derived in traditional way, the angular frequency of harmonic waves is always assumed to be a real number [15-19]. It is easily obtained from the solution that for any medium the energy of disturbance is only dissipated in the space domain. Since the wave equation describes energy transformation in the time domain and energy propagation in space domain, the interaction between adjacent substances should exist in both the time domain and space domain in the area where the wave passes. However, it is not clear from which process the dissipation comes. Therefore it is hard to say whether the traditional solution to the wave propagating in media with dissipation is correct.

The study tries to give a new wave equation solution by assuming that the energy of waves propagating in a dissipative medium dissipates in both the time and space domains. Then the general solution is further used to analyze the properties of viscoelastic waves propagating in the Kelvin–Voigt medium and electromagnetic waves propagating in conductive medium when the energy of waves dissipate in both the time and space domains.
Solution to wave equation of dissipative medium

When mechanical waves propagate in media, the vibration of material element is often treated as the block vibration of mass spring system [15]. For a mass spring damper system, the vibration of mass block is formulated as [15, 18]:

\[
m \frac{d^2x}{dt^2} + \eta \frac{dx}{dt} + kx = 0,
\]

where \( m \) is the mass of block, \( x \) is the displacement of block, \( k \) is the stiffness of spring, \( t \) is the time and \( \eta \) is the damping coefficient. For the mass spring damper system, its general form solution to harmonic vibration can be expressed as:

\[
x = e^{-\beta t} \left( A_1 e^{i\omega t} + A_2 e^{-i\omega t} \right),
\]

\[
\beta = \frac{\eta}{2m},
\]

\[
\alpha = i \sqrt{\omega^2 - \beta^2},
\]

where \( \omega = \sqrt{k/m} \) is the angular frequency, \( A_i \) is the constants determined by initial conditions. For the mass spring damper system, the amplitude of mass block vibration decreases with the time. When the damping coefficient \( \eta = 0 \), Equation (1) degenerates to the case without damping. In this case, the mass block vibrates freely without attenuation under the initial disturbance.

For objectively existing medium, it absorbs the energy of waves, so that the energy carried by waves decreases with its propagation in a real medium. For different media, the mechanisms or dominant mechanisms that cause the decay of wave energy are different, such as the viscous absorption, the heat conduction absorption, the relaxation absorption and so on [20, 21]. However, their wave equations are the same in form,
which are written as follows when the propagation of mechanical disturbances is along the coordinate axis [15]:

$$\rho \frac{\partial^2 u}{\partial t^2} - \hat{K} \frac{\partial^2 u}{\partial x^2} = 0, \quad (5)$$

where $\rho$ is the density, $u$ is the displacement, and $\hat{K}$ is the complex stiffness, a complex number related to angular frequency. Equation (5) formulates the vibration of a material element at a certain point when the spatial coordinate $x$ is regarded as a constant, and the propagation of mechanical disturbance in media when time $t$ is regarded as a constant.

The general form solution to Equation (5) for harmonic vibration propagating forward along coordinate axis $x$, is considered to be formulated as:

$$u(x,t) = u_0 e^{i(\kappa x - \omega t)}, \quad (6)$$

where $u_0$ is the amplitude of displacement at the origin of coordinates, $\kappa$ is the complex wave vector, whose imaginary part reflects the attenuation of a wave during propagation. Regarding the time $t$ as a constant, Equation (6) describes the displacement of material element in different position at a certain moment. Regarding the spatial coordinate $x$ as a constant, Equation (6) describes the vibration of the material element at a point with position $x$, which is equivalent to Equation (2). It is obtained from Equation (6) that the amplitude of displacement propagating in media with dissipation only decreases in space domain and does not change in the time domain. That is, for the wave propagating in the dissipative medium described with Equation (6), its energy only dissipates in the progress of disturbance propagation, and no energy dissipates in the progress of the vibration of material element. Reviewing the traditional solution
progress of Equation (5), we can found that the vibration attenuation is eliminated by assuming that the angular frequency is a real. In other words, Equation (6) describes the energy transfer at a certain point when the material element vibrates without dissipation. It can be obtained from Equation (5) that the partial derivatives of displacement with respect to time and space are symmetric. If the wave vector, describing the propagation of mechanical disturbance, is assumed to be a real, a harmonic solution similar in form to Equation (6) can be obtained. It's just that the angular frequency is a complex. In this case, the amplitude of displacement propagating in media with dissipation only decreases in the time domain. This means that the solution to the wave equation describing the disturbance propagation in dissipative medium is not unique, and different solutions can be derived from different dissipation mechanisms. At present, the process of dissipation has not been discussed in depth and which process the dissipation exists in is unclear. Therefore it is hard to say whether the traditional solution is correct and universal.

Since the interaction between adjacent substances exists in both the time domain and space domain in the area where the wave passes, it is reasonable to assume that the attenuation of wave occurs in both the time domain and the space domain. In this case the solution to the wave equation should contain attenuations in vibration propagation and material element vibration. Correspondingly the wave vector representing the wave propagation and the angular frequency representing the vibration of material element should be complex. This further indicates that there are no ideal harmonic wave in dissipative medium and the amplitude of element vibration decreases with time when
the material elements of a dissipative medium vibrate harmonically. Hence the solution to harmonic wave propagating forward along coordinate axis \( x \) should be formulated as:

\[
    u(x, t) = u_0 e^{-(\lambda - i\nu)x - (\zeta + i\omega)t},
\]

where \( s \) is the real wave vector, \( \lambda \) and \( \zeta \) are the attenuation coefficients related to propagation and vibration, respectively, which are determined by the attenuation of the medium to the wave in the space domain and the time domain. We further assume that the attenuations in the time domain and the space domain are reciprocal. That is, the vibration attenuation in unit time is equal to the propagation attenuation in the distance that the wave propagates in unit time. As a result, the following relationship is true:

\[
    \frac{\zeta}{\lambda} = \frac{\omega}{s} = \nu,
\]  

where \( \nu \) is the velocity of wave.

**Case study**

**Kelvin–Voigt viscoelastic wave**

For a Kelvin–Voigt viscoelastic wave propagating along coordinate axis \( x \), the displacement equation of motion is formulated as [22]:

\[
    \rho \frac{\partial^2 u}{\partial t^2} - K \frac{\partial^3 u}{\partial x^3} - \eta \frac{\partial^3 u}{\partial x \partial t} = 0,
\]

where \( K \) is the stiffness. Substituting Equation (7) into Equation (9), and replacing \( \lambda \) and \( s \) by \( \zeta \) and \( \omega \) through Equation (8), the dispersion relation is obtained:

\[
    (\zeta + i\omega)^2 - \frac{1}{\nu^3} \left( \frac{K}{\rho} (\zeta - i\omega)^2 - \frac{\mu}{\rho} (\zeta - i\omega)^2 (\zeta + i\omega) \right) = 0.
\]

In general, there is an internal relationship between \( \zeta \) and \( \omega \), which is determined by
the property of medium, and the relationship can be assumed as follows:

\[ \zeta = f \omega, \quad (11) \]

where \( f \) is a positive real value. Submitting Equation (11) into Equation (10), the following equations are obtained:

\[ \omega = \frac{K}{\eta} \left( \frac{f(f^2 - 1)}{(f^2 + 1)(3f^2 - 1)} \right), \quad (12) \]

\[ v = \sqrt{\frac{K}{\rho}} \sqrt{1 - \frac{4f^2}{3f^2 - 1}}, \quad (13) \]

It is obtained from Equations (12) and (13) that the velocity and attenuation coefficient of a viscoelastic wave are affected by angular frequency. Since the velocity and angular frequency are positive, the range of coefficient \( f \) is figured out in:

\[ 0 < f < \frac{1}{\sqrt{3}}. \quad (14) \]

Figure 1 shows the properties of viscoelastic waves whose attenuations are only in the space domain and reciprocal in the time and space domains, respectively. It is seen from Figure 1 that although the two attenuation mechanisms are different, their coefficient \( f \) and velocity have similar trends with frequency. The coefficient \( f \) and velocity increase monotonically with the increase of frequency. The coefficient \( f \) increases slowly in the low frequency range \((\omega < K/\eta)\) and high frequency range \((\omega > K/\eta)\), and increases sharply in the transition frequency range (Figure 1a). In high frequency limit, the coefficient \( f \) tends to \(1/\sqrt{3}\) when the attenuation is reciprocal in the time and space domains \((\zeta/\lambda = \nu)\) and to 1 when the attenuation is only in the space domain \((\zeta/\lambda = 0)\). The velocities of the two waves are almost the same and increase slowly in the low frequency range (Figure 1b). When the frequency is higher than the critical frequency \((\omega_c = K/\eta)\), the velocity difference of the two waves increases, but
the difference is not significant. This indicates that for Kelvin–Voigt viscoelastic medium, these two attenuation mechanisms can only be distinguished by attenuation measurement under the high frequency limit.

Using the same method, the properties of the viscoelastic wave with attenuation in time domain can be obtained. For, the Kelvin–Voigt viscoelastic medium with attenuation in time domain, The relationship between frequency and coefficient $f$ is $\omega/\omega_0 = 2f/(1 + f^2)$. The relationship between velocity and coefficient $f$ is $v/v_0 = 1/\sqrt{1 + f^2}$.

Figure 2 shows the properties of the viscoelastic wave in the Kelvin–Voigt viscoelastic medium with attenuation in time domain. It is seen that the properties of the viscoelastic wave with attenuation in time domain are obviously different from that of viscoelastic waves with the attenuation in time domain and the reciprocal attenuation in time and space domains. There are two propagation modes in the Kelvin–Voigt viscoelastic medium with attenuation in time domain: underdamped mode and overdamped mode. For underdamped mode, the velocity decreases with frequency increase and the coefficient $f$ decreases with frequency. For overdamped mode, the velocity increases with frequency and the coefficient $f$ decreases with frequency increase. When the frequency tends to the critical frequency, the velocity and coefficient $f$ for the two model tends to $v_0/\sqrt{2}$ and 1, respectively. No wave whose frequency higher than the critical frequency exists in this kind of media.

**Electromagnetic waves with reciprocal attenuation**

Similar to viscoelastic wave propagation, the energy carried by electromagnetic
waves decays in conductive medium. Assuming that an electromagnetic wave propagates along coordinate axis $x$ in conductive medium and no charge accumulates in the medium, the wave equation is expressed by the following formula [16]:

$$\mu \varepsilon \frac{\partial^2 E}{\partial t^2} + \mu \sigma \frac{\partial E}{\partial t} - \frac{\partial^2 E}{\partial x^2} = 0,$$

(15)

where $E$ is the electric field intensity, $\sigma$ is the electric field intensity, $\mu$ and $\varepsilon$ are the magnetic permeability and permittivity of medium, respectively. Here we assume that electromagnetic waves in conductive media are attenuated in both time and space domains, and the attenuation in time and space domain are reciprocal. Under the assumption, the dispersion relation of electromagnetic waves in media with dissipation is obtained:

$$\varepsilon \mu (\zeta + i\omega)^2 - \mu \sigma (\zeta + i\omega) - \frac{(\zeta - i\omega)^2}{v^2} = 0.$$

(16)

With Equation (16), the following equations are obtained:

$$\omega = \frac{\sigma}{\varepsilon} \frac{3f^2 - 1}{f (f^2 - 1)},$$

(17)

$$v = \frac{1}{\sqrt{\varepsilon \mu}} \sqrt{1 - \frac{f^2}{3f^2 - 1}}.$$

(18)

From Equations (17) and (18), the range of coefficient $f$ is acquired in:

$$0 < f < \frac{1}{\sqrt{3}} \quad and \quad f > 1.$$

(19)

It is seen from Equations (17) and (18) that like viscoelastic waves with attenuation in time domain, the attenuation coefficient and velocity of an electromagnetic wave are affected by frequency and two propagation modes of electromagnetic disturbance in the dissipative medium.

Figure 3 shows the properties of electromagnetic waves. It is seen from Figure 3
that electromagnetic waves in conductive medium propagate in two modes: underdamped mode and overdamped mode. The attenuation and velocity of these two modes are affected by frequency. For the electromagnetic wave propagating in the underdamped mode, the attenuation coefficient $f$ and velocity increase with the decrease of frequency (Figure 3a). The coefficient $f$ tends to 0 at the high frequency limit ($\omega \gg \sigma / \varepsilon$), and $\sqrt{\varepsilon / \omega}$ at the low frequency limit. The coefficient $f$ increases sharply with frequency in the transition frequency range. The velocity of electromagnetic wave tends to $v_0$ ($v_0 = \sqrt{\varepsilon / \mu}$) at the high frequency limit, and its variation with frequency can be ignored when the frequency is much higher than the critical frequency. When the frequency is close to the critical frequency, the velocity of electromagnetic wave varies markedly with frequency. At the low frequency limit, the velocity of electromagnetic wave tends to infinity. For the electromagnetic wave propagating in the overdamped mode, the attenuation coefficient $f$ and velocity also increase with the decrease of frequency (Figure 3b). The coefficient $f$ tends to 1 at the high frequency limit and its variation with frequency can be ignored when the frequency is much higher than the critical frequency. When the frequency is close to the critical frequency, the coefficient $f$ varies markedly with frequency. At the low frequency limit, the coefficient $f$ tends to infinity. The velocity of electromagnetic wave tends to $\sqrt{2/3} v_0$ at the high frequency limit, and $\sqrt{2/3} v_0$ at the low frequency limit. The velocity of electromagnetic wave increases sharply with frequency in the transition frequency range.
Conclusions

The study points out that the solution to the wave equation describing a wave propagating in a dissipative medium is not unique due to the uncertainty of the dissipation process. The traditional solution to the wave propagating in a dissipative medium describes the wave when the disturbance energy dissipates only in the progress of propagation. A new general solution to wave propagating in media with dissipation is proposed by assuming that the attenuation of disturbance occurs in both propagation and vibration and the two attenuations are reciprocal. Then the general solution is used to analyze the properties of viscoelastic waves propagating in the Kelvin–Voigt medium and electromagnetic waves propagating in conductive medium. The result shows the attenuation mechanism has an obvious influence on the properties of waves in the dissipative medium though the wave equations are the same.

Acknowledgments

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Figures

Figure 1 properties of viscoelastic waves with different attenuation mechanism.

Influence of angular frequency on coefficient \( f \) (a) and velocity (b). \( v_0 = \sqrt{K/\rho} \) is the velocity at low frequency limit, \( \omega_0 = K/\eta \) is the critical frequency. \( \zeta/\lambda = 0 \) and \( \zeta/\lambda = v \) correspond to the attenuation only in the space and the reciprocal attenuation in the time and space domains, respectively.

Figure 2 Properties of viscoelastic waves with attenuation in the time domain. The relationship between frequency and coefficient \( f \) is \( \omega/\omega_0 = 2f/(1+f^2) \). The relationship between velocity and coefficient \( f \) is \( v/v_0 = 1/\sqrt{1+f^2} \).
Figure 3 Properties of the electromagnetic wave with reciprocal attenuations in the time domain and space domain. (a) Influence of angular frequency on coefficient $f$, (b) dispersion relation. $v_0 = \sqrt{\varepsilon/\mu}$ is the velocity at low frequency limit, $\omega_0=\sigma/\varepsilon$ is the critical frequency.