A Method to construct the Sparse-paving Matroids over a Finite Set

B. Mederos∗∗, M. Takane∗, G. Tapia-Sánchez∗∗ and B. Zavala∗∗∗

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Abstract

We give a method to construct the sparse-paving matroids over a finite set $S$. With it, we give an injective set-function $\Psi_r : \text{Matroid}_{n,r} \to \bigcup_{j=1}^{2(\lfloor r/2 \rfloor)} \text{Sparse}_{n,r}$ where $(\text{Sparse}_{n,r})_{\text{Matroid}}_{n,r}$ is the set of all (sparse-paving) matroids of rank $r$, over a set $S$ of cardinality $n$. Then, we give another proof of $\lim_{n \to \infty} \frac{\log_2 |\text{Matroid}_{n,r}|}{\log_2 |\text{Sparse}_{n,r}|} = 1$ and some new bounds of the cardinalities of these sets.

Keywords: matroid, paving matroid, sparse-paving matroid, combinatorial geometries, lattice of a matroid.

Introduction

We recall that a matroid $M = (S, \mathcal{I})$ consists of a finite set $S$ and a collection $\mathcal{I}$ of subsets of $S$ (called the independent sets of $M$) satisfying the following independence axioms:

(1) The empty set $\emptyset \in \mathcal{I}$.
(2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.
(3) Let $U, V \in \mathcal{I}$ with $|U| = |V| + 1$ then $\exists x \in U \setminus V$ such that $V \cup \{x\} \in \mathcal{I}$.

A subset of $S$ which does not belong to $\mathcal{I}$ is called a dependent set of $M$.

A basis [respectively, a circuit] of $M$ is a maximal independent [resp. minimal dependent] set of $M$. The rank of a subset $X \subseteq S$ is $\text{rk}X := \max\{|A|; A \subseteq X \text{ and } A \in \mathcal{I}\}$ and the rank of the matroid $M$ is $\text{rk}M := \text{rk}S$. A closed subset (or flat) of $M$ is a subset $X \subseteq S$ such that for all $x \in S \setminus X$, $\text{rk}(X \cup \{x\}) = \text{rk}X + 1$. Then can be defined the closure operator $\text{cl} : \mathcal{P}S \to \mathcal{P}S$ on the power set of $S$, as follows: $\text{cl}(X) := \min\{Y \subseteq S; X \subseteq Y \text{ and } Y \text{ is closed in } M\}$. The lattice of a matroid $M$, denoted by $\mathcal{L}_M$ is the lattice defined by the closed sets of $M$, ordered by inclusion where the meet is the intersection and the join the closure of the union of sets. For general references of Theory of Matroids, see [13], [11], [14] and [12]. For references of theory of lattices and theory of lattices of matroids, see [4], [5].
A matroid is **paving** if it has no circuits of cardinality less than \( \text{rk} M \). And a matroid \( M \) is **sparse-paving** if \( M \) and its dual \( M^* \) are paving matroids.

In [3], Blackburn, Crapo and Higgs wrote: "In the enumeration of (non-isomorphic) matroids on a set of 9 or less elements, (sparse-)paving matroids predominate. Does this hold in general?". There are several results which suggest that the answer should be positive, see for example [2], [1], [5]. We give a method to construct the sparse-paving matroids over a finite set \( S \) and an injective set-function \( \Psi_r : \text{Matroid}_{n,r} \to \bigcup_{j=1}^{2\binom{r}{r/2}} \text{Sparse}_{n,r} \) where \( (\text{Sparse}_{n,r}) \) is the set of all (sparse-paving) matroids of rank \( r \), over a set \( S \) of cardinality \( n \). Then, we give another proof of \( \lim_{n \to \infty} \frac{\log_2 |\text{Matroid}_{n,r}|}{\log_2 |\text{Sparse}_{n,r}|} = 1 \) and some new bounds of the cardinalities of these sets.

The material is organized as follows: In section 1, we give more definitions and known results that are useful along the paper. Also, we give an abstract construction of the sparse-paving matroids, \( M \), which leads us to a method to construct them explicitly, using matrices of \( r \)-subsets of \( S \), for any \( r \). In section 2, we prove the following inequalities: \(|\text{Sparse}_{n,r}| \leq |\text{Sparse}_{n+1,r}| \leq |\text{Sparse}_{n,r}| + |\text{Sparse}_{n,r-1}| \) where \( \text{Sparse}_{n,r} := \{ M = (S,I) : M \text{ is a sparse-paving matroid of rk} M = r \} \) and \(|S| = n\).

In section 3, we give a construction of a partition of the \( r \)-subsets of \( S \), \( \binom{S}{r} = \bigcup_{i=1}^{\gamma} \mathcal{U}_i \) such that each \( \mathcal{U}_i \) define a sparse-paving matroid of rank \( r \) and \( \gamma = 2\left(\frac{r}{r/2}\right) \). In section 4, we give an injective function that maps each matroid over \( S \) \( \text{Matroid}_{n,r} \), to a disjoint union of \( 2\left(\frac{r}{r/2}\right) \) sets of sparse-paving matroids on \( S \) of rank \( r \). And new bounds of these cardinalities are given.

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1 A description of the Sparse-paving Matroids through their set of circuits.

The study of sparse-paving and paving matroids helps to understand the behavior of the matroids in general and important examples of matroids are indeed sparse-paving matroids, as the combinatorial finite geometries. In 1959, Hartmanis [6] introduced the definition of paving matroid through the concept of \( d \)-partition in number theory. Then, following a Rota’s suggestion, Welsh [14] (1976) called these matroids paving. Later, Oxley [12] generalized the definition of paving matroid to include all possible ranks. And Jerrum [7] introduced
the notion of sparse-paving matroids. It is known that the sparse-paving matroids of rank $\geq 2$ have lattices which are atomic, semimodular and satisfies the Jordan-Hölder condition.

1.1 More definitions, notations and known results.

Let $S$ be a set of $n$ elements.

a. Any matroid $M = (S, \mathcal{I})$ is completely determined by its set of basis, $\mathcal{B}$. Namely, $\mathcal{I} = \{X \subseteq S; \exists B \in \mathcal{B} \text{ with } X \subseteq B\}$.

b. Let $M = (S, \mathcal{I})$ be a matroid of rank $\text{rk}M$. Any circuit $X$ of $M$, has cardinality $|X| \leq \text{rk}M + 1$.

c. Let $M = (S, \mathcal{I})$ be a matroid of rank $\text{rk}M$. Denote by $M^* = (S, \mathcal{I}^*)$ the dual matroid of $M$ whose set of basis is $\mathcal{B}^* := S\setminus \mathcal{B}$.

A matroid $M$ is called a 

**sparse-paving matroid** if $M$ and its dual $M^*$ are paving matroids.

d. Examples of sparse-paving matroids:

d.1. A matroid is called uniform of rank $r$ over a set of $n$ elements, denoted by $U_{r,n}$, if all the $r$-subsets are basis. The dual matroid $U^*_{r,n}$ of a uniform matroid is again uniform with rank $n - r$. Then any uniform matroid is sparse-paving. (This occurs, for example, when $r = 0$ or $r = n$).

d.2. Any matroid of rank 1 is paving (since the empty set is always independent).

d.3. By (d.1) and (d.2), for $n = 1, 2$ any matroid on $S$ is sparse-paving.

e. If a matroid $M$ has rank $\text{rk}M \geq 2$, the lattice of $M$, $\mathcal{L}_M$ is atomic (i.e., all the subsets of rank 1 are closed), semimodular and satisfies the Jordan-Hölder condition. An important class of these matroids are the combinatorial finite geometries, [12].

1.2. By (1.1.d), along this paper we can assume that $n \geq 3$ and $r \geq 2$.

Let $S$ be a set of cardinality $|S| = n$. Denote by $\binom{S}{t} := \{X \subseteq S; |X| = t\}$ for $0 \leq t \leq n$ the subsets of $S$ of cardinality $t$ (called $t$-subsets). Let $M = (S, \mathcal{I})$ be a paving matroid of rank $\text{rk}M$. Denote by $\mathcal{B}$ [resp. $\mathcal{C}_{\text{rk}M}$] the set of the basis [resp. $\text{rk}M$-circuits] of $M$.

**Lemma** [8][5]. For $n \geq 3$ and $\text{rk}M \geq 2$, there is an equivalent definition of being a sparse-paving matroid. Namely, A matroid $M = (S, \mathcal{I})$ with $|S| \geq 3$ and $\text{rk}M \geq 2$ is a sparse-paving matroid if and only if its set of $\text{rk}M$-circuits, $\mathcal{C}_{\text{rk}M}$ satisfies the following property:

\[
(*) \text{ For all } X, Y \in \mathcal{C}_{\text{rk}M} \text{ we have } |X \cap Y| \leq \text{rk}M - 2
\]

1.3. The next result is the counterpart of lemma in (1.2). That is, let $S$ be a set of cardinality $n \geq 3$ and $2 \leq r \leq n - 1$. Then any set $\mathcal{C} \subseteq \binom{S}{r}$ of $r$-subsets of $S$ satisfying property $(*)$ defines a sparse-paving matroid of rank $r$ with $\mathcal{C}$.
as its set of $r$-circuits. In other words, in this case, the ordered pair $(S, \mathcal{I})$ with $\mathcal{I} := \{X \subseteq S; \exists B \in (\binom{S}{r})\setminus \mathcal{C}\}$ is in fact a matroid (i.e., $\mathcal{I}$ satisfies the independent axioms of a matroid, see (Introduction)) and by (1.2), $(S, \mathcal{I})$ is sparse-paving.

**Proposition.** Let $S$ be a set of cardinality $|S| = n \geq 3$ and $2 \leq r \leq n - 1$. Let $C \subset (\binom{S}{r})$ be a set of $r$-subsets of $S$, satisfying the following property

$$(**) : \forall X, Y \in C \text{ with } X \neq Y \text{ then } |X \cap Y| \leq r - 2.$$  

Define $M := (S, I)$ where $I := (\binom{S}{r})\setminus C$ and $I := \{X \subseteq S; \exists B \in B \text{ with } X \subseteq B\}$. Then, $(A). M$ is a matroid of $rkM = r$ and $(B). M$ is sparse-paving.

**Proof.** Let $S$ be a set and take a subset $C \subset (\binom{S}{r})$ satisfying the property (**) . Take $M = (S, I)$ with set of basis $B = (\binom{S}{r})\setminus C$.

A. To prove $M$ is a matroid of rank $rkM = r$.

For this proof, we will use an equivalent definition of matroid, which says:

Let $M = (S, I)$ be a matroid if and only if $I$ satisfies $(I1),(I2)$ as in the introduction and $(I3)'$: let $B_1, B_2 \in B$ be two basis of $M$ and $x \in B_1 \setminus B_2$. To prove $\exists y \in B_2 \setminus B_1$ such that $(B_1 \{x\}) \cup \{y\} \in B$.

**case a.** If $|S| = 3$ and $rkM = 2$, the possibilities for $C$ to have property (**) are $C = \emptyset$ or $|C| = 1$. In both cases, $M$ is matroid and it is sparse-paving.

**case b.** $|S| \geq 4$.

(I1) To prove that $\emptyset$ is an independent set. It is enough to prove that $B$ is not empty.

Since $n \geq 4$, $2 \leq r \leq n - 1$ and $S = \{1, ..., r, r + 1, ..., n\}$. Take $A_1 = \{1, ..., r - 1, r\}$, $A_2 = \{1, ..., r - 1, r + 1\}$ which are subsets of $S$ with cardinality $r$ and $|A_1 \cap A_2| = r - 1$. Then by (**) , $\exists i \in \{1, 2\}$ such that $A_i \in B$. Then $B \neq \emptyset$.

(I2) Let $Y \subseteq X \subseteq S$ such that $\exists B \in B$ with $X \subseteq B$. Then $Y \subseteq B$, that is $Y$ is independent, by definition.

(I3)' Now, let $B_1, B_2 \in B$ be two basis of $M$ and $x \in B_1 \setminus B_2$. To prove $\exists y \in B_2 \setminus B_1$ such that $(B_1 \{x\}) \cup \{y\} \in B$.

(I3)'1. Assume $m := |B_2 \setminus B_1| = 1$. That is, $B_2 \cap B_1 = B_1 \{x\}$ and $B_2 = B_1 \{x\} \cup \{y\}$ for some $y \in S$. Then $(B_1 \{x\}) \cup \{y\} \in B$.

(I3)'2. Let define $m := |B_2 \setminus B_1| \geq 2$ and let $B_2 = (B_1 \setminus B_2) \cup \{y_1, y_2, y_3, ..., y_m\}$. Define $A_i := (B_1 \{x\}) \cup \{y_i\}$ for $i = 1, ..., m$. Since $\forall i \neq j, |A_i \cap A_j| = r - 1$ and $m \geq 2$, by (**), $\exists A_{i_0} \in B$. Therefore, $(B_1 \{x\}) \cup \{y_i\} = A_{i_0} \in B$, and $M$ is a matroid.

**Rank:** By definition of $M$, $rkM = r$.

B. To prove $M$ is a sparse-paving matroid.

B.1. First we will prove that $M$ is a paving matroid. Equivalently, to prove $\forall Z \subseteq S \text{ of } |Z| = rkM - 1, Z \in I$. This proof is similar to the one of (I1). Namely:

Let $rkM \leq n - 1$. Since $n \geq 3$ and $|Z| = rkM - 1$, we have $S = Z \cup \{x_1, x_2, ..., x_m\}$ with $m \geq 2$. Let denote $A_i := Z \cup \{x_i\}$ for $i = 1, 2, ..., m$. By (**), $m \geq 2$, $\exists i_0 \in \{1, ..., m\}$ such that $(Z \subseteq)A_{i_0} \in B$. Then $Z \in I$. 

4
2 Recursive construction of bounds for the cardinality of the sparse-paving matroids on a finite set $S$ with $|S| = n$.

Recall, $\text{Sparse}_{n,r} := \{ M = (S, \mathcal{I}); M \text{ is a sparse-paving matroid with } rkM = r \}$ where $|S| = n$. By (1.3), to construct a sparse-paving matroid on $S$ of rank $r \geq 2$, it is enough to have a set $\mathcal{U} \subseteq \binom{S}{r}$ satisfying property (**): $\forall X, Y \in \mathcal{U}$ with $X \neq Y$, $|X \cap Y| \leq r - 2$, using this, we prove $|\text{Sparse}_{n,r}| \leq |\text{Sparse}_{n+1,r}| \leq |\text{Sparse}_{n,r}| + |\text{Sparse}_{n,r-1}|$.

In other hand, we prove that we can construct that $\mathcal{U} \subseteq \binom{S}{r}$ satisfying property (**)) and $\frac{1}{(n-r+1)!} \binom{|S|}{r} \leq |\mathcal{U}|$, this implies, $|\text{Sparse}_{n,r}| \geq 2^{\frac{|S|-|\text{Sparse}_{n,r}|}{n-r+1}}$.

Also we prove that any $\mathcal{U} \subseteq \binom{S}{r}$ with property (**), $|\mathcal{U}| \leq \frac{1}{r(n-r+1)!} \binom{|S|}{r}$.

2.1. Lemma. $\text{Sparse}_{n,r} \hookrightarrow \text{Sparse}_{n+1,r} \hookrightarrow \text{Sparse}_{n,r} \cup \text{Sparse}_{n,r-1}$.

Therefore, $|\text{Sparse}_{n,r}| \leq |\text{Sparse}_{n+1,r}| \leq |\text{Sparse}_{n,r}| + |\text{Sparse}_{n,r-1}|$.

Proof: Denote by $S := \{1, 2, ..., n\}$ and $\hat{S} := S \cup \{n + 1\}$.

2.1.1. The first inequality is given by the following canonical injective set-function $\iota_n : \text{Sparse}_{n,r} \hookrightarrow \text{Sparse}_{n+1,r}$: Let $M = (S, \mathcal{I})$ be a sparse-paving matroid of $rkM = r$ and $\mathcal{C}_r$ its set of $r$-circuits (recall that in the sparse-paving case $\binom{S}{r} = \mathcal{B} \sqcup \mathcal{C}_r$).

Define $\iota_n(M) = (\hat{S}, \hat{\mathcal{I}})$ where the $(\hat{S}, \hat{\mathcal{I}})$’s set of basis is $\hat{B} := \mathcal{B} \cup \{X \in \binom{\hat{S}}{r}; n + 1 \in X\}$ and $\hat{\mathcal{I}} = \{Y \subseteq \hat{S}; \exists \hat{B} \in \hat{\mathcal{B}} \text{ with } Y \subseteq \hat{B}\}$. Then $(\hat{S}, \hat{\mathcal{I}})$’s $r$-circuit set $\hat{\mathcal{C}}_r = \mathcal{C}_r$, the set of $r$-circuits of $M$. By (1.2), $\mathcal{C}_r = \mathcal{C}_r$ has property (**)) and by (1.3), $(\hat{S}, \hat{\mathcal{I}})$ is a sparse-paving matroid. And $\iota_n$ is injective, by definition of $\hat{B}$.

2.1.2. To prove the second inequality define $\zeta_{n+1} : \text{Sparse}_{n+1,r} \hookrightarrow \text{Sparse}_{n,r} \cup \text{Sparse}_{n,r-1}$ an injective set-function as follows: let $M = (\hat{S}, \hat{\mathcal{I}})$ be a sparse-paving matroid of $rk\hat{M} = r$ and $\hat{\mathcal{C}}_r$ its set of $r$-circuits (recall, $\binom{\hat{S}}{r} = \hat{\mathcal{B}} \sqcup \hat{\mathcal{C}}_r$). Define $\zeta_{n+1}(\hat{M}) := (M_1, r) \cup (M_2, r - 1)$ as follows: $\hat{\mathcal{C}}_r = \{X \in \hat{\mathcal{C}}_r; n + 1 \notin X\} \cup \{X \in \hat{\mathcal{C}}_r; n + 1 \in X\}$ a disjoint union of $\hat{\mathcal{C}}_r^{(1)} := \{X \in \hat{\mathcal{C}}_r; n + 1 \notin X\}$ and $\{X \in \hat{\mathcal{C}}_r; n + 1 \in X\}$.

Take $M_1 = (S, B_1)$ with $B_1 = \binom{\hat{S}}{r} \setminus \hat{\mathcal{C}}_r^{(1)}$. Now, since $\hat{\mathcal{C}}_r$ satisfies property (**)) for $r$, see (1.3), we get that $M_1 \in \text{Sparse}_{n,r}$.
In other hand, define $C_r^{(2)} := \{X \setminus \{n + 1\}; X \in \tilde{C}_r$ and $n + 1 \in X \} \subset (S_r^{-1})$ and observe again, since $\tilde{C}_r$ satisfies property $(**)$ for $r$. Now by property $(**)$ for $r$, if $X, Y \in \tilde{C}_r$ with $n + 1 \in X \cap Y$ then $|(X \setminus \{n + 1\}) \cap (Y \setminus \{n + 1\})| = |X \cap Y| - 1 \leq (r - 2) - 1 = (r - 1) - 2$. In other words by (1.3), $C_r^{(2)}$ defines a sparse-paving matroid on $S$ of rank $r - 1$, with $C_r^{(2)}_{r - 1}$ as $r - 1$-circuits of the matroid denoted as $M_2$.

Then $\zeta_{n + 1}$ is a well defined set-function. And it is injective by (1.1.a) and $\hat{\zeta} = C_r^{(1)} \cup \{Z \cup \{n + 1\}; Z \in C_r^{(2)}_{r - 1}\}$.

2.1.3. Therefore, $|\text{Sparse}_{n,r}| \leq |\text{Sparse}_{n+1,r}| \leq |\text{Sparse}_{n,r}| + |\text{Sparse}_{n,r-1}|.$

2.2. A bound for $|C_{rkM}|$  the cardinality of the set of $rkM$-circuits, $C_{rkM}$, of a sparse-paving matroid $M$.

Since we want to find relations between the cardinality of all matroids on a finite $n$-set $S$ of rank $r$, $[\text{Matroid}_{n,r}]$ and its subset of sparse-paving matroids, $[\text{Sparse}_{n,r}]$, in this section we give easy-finding bounds of the set of $r$-circuits of the sparse-paving matroids (see, property $(**)$ in (1.2)).

Recall, the following property on $U$ and $r$, $(**)$: For all $X, Y \in U$ we have $|X \cap Y| \leq r - 2$.

**Technical steps to construct sets with property $(**)$:** Let $S$ with $|S| = n$ and $2 \leq r \leq n - 1$. Let $X_1 \in \binom{S}{r}$ be fix.

**A.** Want to find $\{X \in \binom{S}{r}; |X \cap X_1| \leq r - 2\}$. Equivalent, find $\{A \in \binom{S}{r}; |A \cap X_1| = r - 1\}$.

A.1. Take all the $r + 1$-subsets of $S$ containing $X_1$. Namely, $\{Y_{1,1}, Y_{1,2}, ..., Y_{1,n-r}\} = \{Y \in \binom{S}{r+1}; X_1 \subset Y\}$. That is, $Y_{1,i} = X_1 \cup \{v_i\}$ where $S \backslash X_1 = \{v_1, ..., v_{n-r}\}$.

A.2. By construction, $\{A \in \binom{S}{r}; |X_1 \cap A| = r - 1\} = \{A \in \binom{S}{r}; \exists i = 1, ..., n - r \text{ such that } A \subset Y_{1,i} \}\{X_1\}$. And then $\{A \in \binom{S}{r}; |X_1 \cap A| = r - 1\} \cup \{X_1\} = (r(n - r) + 1$.

A.3. And $\{A \in \binom{S}{r}; |X_1 \cap A| \leq r - 2\} = \binom{S}{r} \setminus \{A \in \binom{S}{r}; |X_1 \cap A| = r - 1\}$ has cardinality $\binom{n}{r} - (r(n - r) - 1$.

**B.** Next choose and fix $X_2 \in \{A \in \binom{S}{r}; |X_1 \cap A| \leq r - 2\}$. To get a set of $r$-subsets of $S$ with property $(**)$, we have to make stepA for $X_2$:

B.1. By (A.1) for $X_2$, let $\{Y_{2,1}, Y_{2,2}, ..., Y_{2,n-r}\} = \{Y \in \binom{S}{r+1}; X_2 \subset Y\}$.

Thus, $\binom{S}{r} \setminus \{A \in \binom{S}{r}; \exists h = 1, 2, \exists i = 1, ..., n - r \text{ such that } A \subset Y_{h,i}\} = \{A \in \binom{S}{r}; |X_1 \cap A| \leq r - 2 \text{ and } |X_2 \cap A| \leq r - 2\}$. Also by construction, $|X_1 \cap X_2| \leq r - 2$.

**C.** By construction, $\{Y_{1,1}, Y_{1,2}, ..., Y_{1,r-n}\} \cap \{Y_{2,1}, Y_{2,2}, ..., Y_{2,r-n}\} = \emptyset$, since $X_1 \not\subseteq Y_{2,j}$ and $X_2 \not\subseteq Y_{1,j}$ for all $j = 1, ..., r - n$.
D. In this way, by (C), we can continue the procedure at most \( \left\lfloor \frac{1}{n-r} \binom{n}{r+1} \right\rfloor \) steps. That is, we can construct \( U \subseteq \binom{S}{r} \) satisfying property (**) with \( |U| \leq \left\lfloor \frac{1}{n-r} \binom{n}{r+1} \right\rfloor \).

E. And all the sets \( C \subseteq \binom{S}{r} \) satisfying property (**) is a subset of a \( U \) that can be built with this procedure.

We proved the following

**Lemma.** Let \( S \) be a set of cardinality \( n \) and let \( 2 \leq r \leq n-1 \).

a) Assume that \( U \subseteq \binom{S}{r} \) satisfies property (**). Then \( |U| \leq \frac{1}{n-r} \binom{n}{r+1} \).

b) There exists \( U \subseteq \binom{S}{r} \) satisfying property (**) with cardinality at least \( \frac{1}{r(n-r)+1} \binom{n}{r} \leq |U| \).

2.3. For the next result, see also [10, (4.8)].

**Corollary.** Let \( S \) be a set of cardinality \( n \) and let \( 2 \leq r \leq n-1 \). Let \( M = (S, I) \) be a sparse-paving matroid of rank \( r \) and \( C_r \) its set of \( r \)-circuits. Then \( |C_r| \leq \frac{1}{r(n-r)+1} \binom{n}{r} \).

2.4. **Corollary.** Let \( S \) be a set of cardinality \( n \) and let \( 2 \leq r \leq n-1 \). Then \( |\text{Sparse}_{n,r}| \geq 2^{|\text{Sparse}_{n,r}|} \).

**Proof.** By lemma(b) in (2.2), there exists \( U \subseteq \binom{S}{r} \) with property (**) and \( \frac{1}{r(n-r)+1} \binom{n}{r} \leq |U| \). Let \( \mathcal{P}(U) = \{X \subseteq U\} \) be the power set of \( U \). Then \( \forall C \in \mathcal{P}(U), C \) satisfies property (**), then \( C \) defines a sparse-paving matroid. Moreover, if \( C \neq C' \) in \( \mathcal{P}(U) \), their respective sparse-paving matroids are different. And since \( \frac{1}{r(n-r)+1} \binom{n}{r} \leq |U| \), we have \( 2^{|\mathcal{P}(U)|} \leq |\mathcal{P}(U)| = 2^{|U|} \leq |\text{Sparse}_{n,r}| \).

3 A method to construct sets \( U \) with property (**) : \( \forall X, Y \in U \) with \( X \neq Y \), \( |X \cap Y| \leq r - 2 \).

In this section, we will construct matrices of \( r \)-subsets of \( S \) from a fixed \( r \)-subset \( X \) having the following properties:

a) Any \( r \)-subset of \( S \) is an entry of exactly one of these matrices.

b) In each matrix, any two entries which are in different rows and different columns have intersection less or equal to \( r - 2 \).

c) Any two entries in different matrices, \( S_h \) and \( S_{h'} \), with \( |h - h'| \geq 2 \), have intersection less or equal to \( r - 2 \).
3.1. Let $S$ be a set of cardinality $n$, $2 \leq r \leq n - 1$. Fix $X \in \binom{S}{r}$. For each $0 \leq h \leq r$, let $X_h = \{A_h^{(1)}, ..., A_h^{(r)}\}$ be the $h$-subsets of $X$ and $X_{r-h} = \{Z_1^{(h)}, ..., Z_{r-h}^{(h)}\}$ be the $(r-h)$-subsets of $S \setminus X$.

For each $0 \leq h \leq r$ and $n - r \geq r - h$, we build the following $\binom{|S|}{r-h} \times \binom{|X|}{h}$-matrix, $s_h$:

$$s_h := \begin{bmatrix}
A_1^{(h)} \cup Z_1^{(h)} & \cdots & A_r^{(h)} \cup Z_1^{(h)} \\
A_1^{(h)} \cup Z_2^{(h)} & \cdots & A_r^{(h)} \cup Z_2^{(h)} \\
\vdots & \cdots & \vdots \\
A_1^{(h)} \cup Z_{r-h}^{(h)} & \cdots & A_r^{(h)} \cup Z_{r-h}^{(h)}
\end{bmatrix}_{\binom{|S|}{r-h} \times \binom{|X|}{h}}$$

3.1.1. Properties of $s_h$:

a) By construction of the matrices $s_h$, $\forall Y \in \binom{S}{r}$, there exists a unique $0 \leq h \leq r$ such that $Y$ is an entry of $s_h$.

b) Let $0 \leq h \leq r$ and $s_h$. Now take $1 \leq i \neq j \leq \binom{n-r}{r-h}$, $1 \leq t \neq k \leq \binom{r}{h}$ and the entries $A_i^{(h)} \cup Z_t^{(h)}$, $A_k^{(h)} \cup Z_j^{(h)}$. Then $\left| \left( A_i^{(h)} \cup Z_t^{(h)} \right) \cap \left( A_k^{(h)} \cup Z_j^{(h)} \right) \right| \leq r - 2$. That is, a pair of entries in different columns and different rows have intersection $\leq r - 2$.

c) For $0 \leq h, h' \leq r$ such that $|h - h'| \geq 2$ and for all $i, j, t$ and $k$,

$$\left| \left( A_i^{(h)} \cup Z_t^{(h)} \right) \cap \left( A_k^{(h')} \cup Z_j^{(h')} \right) \right| \leq r - 2.$$

Proof. (b). $\left| \left( A_i^{(h)} \cup Z_t^{(h)} \right) \cap \left( A_k^{(h')} \cup Z_j^{(h')} \right) \right| = \left| A_i^{(h)} \cap A_k^{(h')} \right| + \left| Z_t^{(h)} \cap Z_j^{(h')} \right| \leq (h - 1) + (r - h - 1) = r - 2$, since $A_i^{(h)} \neq A_k^{(h')} \subseteq X$ and $Z_t^{(h)} \neq Z_j^{(h')} \subseteq S \setminus X$.

c) Let $0 \leq h, h' \leq r$ and $|h - h'| \geq 2$. Take any $1 \leq i, j \leq \binom{n-r}{r-h}$ and $1 \leq t, k \leq \binom{r}{h}$.

To prove that $\left( A_i^{(h)} \cup Z_t^{(h)} \right) \cap \left( A_k^{(h')} \cup Z_j^{(h')} \right) \leq r - 2$.

Since $|h - h'| \geq 2$, we can assume that $h = h' + m$ with $2 \leq m \leq r - h'$. Then $\left( A_i^{(h)} \cup Z_t^{(h)} \right) \cap \left( A_k^{(h')} \cup Z_j^{(h')} \right) = \left| A_i^{(h)} \cap A_k^{(h')} \right| + \left| Z_t^{(h)} \cap Z_j^{(h')} \right| \leq h + (r - h') = r - m \leq r - 2$.

3.2. Example. Let $S = \{1, 2, 3, 4, 5, 6\}$, $r = 3$ and fix $X = \{1, 2, 3\}$.

$s_0 := \begin{bmatrix}
\{4, 5, 6\} \\
\{1, 2, 3\}
\end{bmatrix}_{\binom{3}{2} \times \binom{6}{3}}$

$s_1 := \begin{bmatrix}
\{1\} \cup \{4, 5\} & \{2\} \cup \{4, 5\} & \{3\} \cup \{4, 5\} \\
\{1\} \cup \{5, 6\} & \{2\} \cup \{5, 6\} & \{3\} \cup \{5, 6\}
\end{bmatrix}_{\binom{3}{2} \times \binom{6}{3}}$

$s_2 := \begin{bmatrix}
\{1, 2\} \cup \{4\} & \{1, 3\} \cup \{4\} & \{2, 3\} \cup \{4\} \\
\{1, 2\} \cup \{5\} & \{1, 3\} \cup \{5\} & \{2, 3\} \cup \{5\} \\
\{1, 2\} \cup \{6\} & \{1, 3\} \cup \{6\} & \{2, 3\} \cup \{6\}
\end{bmatrix}_{\binom{3}{2} \times \binom{6}{3}}$
3.3. A partition of \((\binom{S}{r})\) by subsets satisfying property (**).

By (3.1.1), we will construct \(U_s\)'s having property (**), which form a partition of \((\binom{S}{r})\). Let \(S = \{1, 2, ..., n\}, 2 \leq r \leq n - 1\) and let \(X \in \binom{S}{r}\) be fixed.

3.3.1. Let \(0 \leq h \leq r\) and take the \((\binom{S \setminus X}{r-h}) \times (\binom{X}{h})\)-matrix \(s_h\). By (3.1.1.b), we can make \(\max\{\binom{n-r}{r-h}, \binom{h}{r}\}\) different sets consisting of the entries of \(S_h\) satisfying property (**). Namely, take each set constructing with the entries of each major diagonal of \(S_h\). In this way, we get:

3.3.1.a. \(\max\{\binom{n-r}{r-h}, \binom{h}{r}\}\) different sets of cardinality \(\min\{\binom{n-r}{r-h}, \binom{h}{r}\}\).

Graphically speaking, for
\[
\begin{bmatrix}
\bullet_1 & \triangleright_2 & \circ_3 \\
\#_1 & \#_2 & \#_3 \\
\triangleright_1 & \triangleright_2 & \#_3 \\
\circ_1 & \circ_2 & \#_3
\end{bmatrix}_{4 \times 3} \leftrightarrow \begin{bmatrix}
\bullet_1 & \#_1 & \#_2 \#_3 \\
\#_1 & \#_2 \#_3 \\
\triangleright_1 & \triangleright_2 \#_3 \\
\circ_1 & \circ_2 \#_3
\end{bmatrix}_{4 \times 3},
\]
we obtain \(\{\bullet_1, \#_2, \#_3\}, \{\#_1, *, \#_3\}, \{\#_1, \circ_2, \circ_3\}\) and \(\{\triangleright_1, \triangleright_2, \circ_3\}\).

3.3.1.b. Now, the Partition of \((\binom{S}{r})\):

Case \(\binom{n-r}{r-h} \geq \binom{h}{r}\). Then \(\max\{\binom{n-r}{r-h}, \binom{h}{r}\}\).

As we mentioned before, the set of the entries of \(S_h:\) \(\left\{A_i^{(h)} \cup Z_{\sigma_j^{(i)}}^{(h)}\right\}_{1 \leq i \leq \binom{n-r}{r-h}, 1 \leq j \leq \binom{h}{r}} = \bigcup_{j=1}^{\binom{n-r}{r-h}} \bigcup_{i=1}^{\binom{h}{r}} A_i^{(h)} \cup Z_{\sigma_j^{(i)}}^{(h)}(t)\), where for all \(j = 1, ..., \binom{n-r}{r-h}\), \(\sigma_j^{(h)}\) is the set-function \(\sigma_j^{(h)} : \{1, 2, ..., \binom{h}{r}\} \rightarrow \{1, 2, ..., \binom{n-r}{r-h}\}\) given by \(\sigma_j^{(h)}(t) = [j + t - 1]_{\mod \{\binom{n-r}{r-h}\}}\).

And by (3.1.1.b), for each \(j \in \{1, ..., \binom{n-r}{r-h}\}\), \(s_h(j) := \bigcup_{i=1}^{\binom{n-r}{r-h}} \left\{A_i^{(h)} \cup Z_{\sigma_j^{(i)}}^{(h)}(t)\right\}\) satisfies property (**).

Case \(\binom{n-r}{r-h} < \binom{h}{r}\) the proof is similar.

3.3.1.c. In conclusion, for all \(0 \leq h \leq r\) and for all \(j = 1, ..., \max\{\binom{n-r}{r-h}, \binom{h}{r}\}\),

\[s_h(j) := \min_{1 \leq l \leq \binom{n-r}{r-h}} \left\{A_i^{(h)} \cup Z_{\sigma_j^{(i)}}^{(h)}(t)\right\}\]

satisfies property (**)

where \(\sigma_j^{(h)}\) is the set-function
\(\sigma_j^{(h)} : \{1, 2, ..., \min\{\binom{n-r}{r-h}, \binom{h}{r}\}\} \rightarrow \{1, 2, ..., \max\{\binom{n-r}{r-h}, \binom{h}{r}\}\}\) given by
\(\sigma_j^{(h)}(t) = [j + t - 1]_{\mod \{\binom{n-r}{r-h}\}}\).

And the set of entries of \(S_h\) is \(\bigcup_{0 \leq h \leq r} \bigcup_{1 \leq j \leq \max\{\binom{n-r}{r-h}, \binom{h}{r}\}} s_h(j)\).

3.3.2. Now, using (3.3.1), we will give bigger subsets of \((\binom{S}{r})\) which still have property (**).
Define for \( j = 1, \ldots, \max_{0 \leq k \leq r, \ k \ odd} \{ \max \left\{ \binom{n-r}{j-k}, \binom{k}{j} \right\} \} \), \( U_j^{(odd)} := \bigcup_{0 \leq h \leq r, \ h \ odd} S_h(j) \)
and for \( j = 1, \ldots, \max_{0 \leq k \leq r, \ k \ even} \{ \max \left\{ \binom{n-r}{j-k}, \binom{k}{j} \right\} \} \), \( U_j^{(even)} := \bigcup_{0 \leq h \leq r, \ h \ even} S_h(j) \)
where in both cases, for all \( h \), \( S_h(j) := \emptyset \) if \( j > \max \left\{ \binom{n-r}{j-k}, \binom{k}{j} \right\} \).

3.3.3. Therefore by (3.1.1.c), for each \( h \), \( U_j^{(odd)} \) and \( U_j^{(even)} \) satisfy property (**), since \( \forall h \neq h' \) odd (resp. even) numbers \( |h - h'| \geq 2 \).
And \( \left\{ U_j^{(odd)} \right\} \) and \( \bigcup_{j=1}^{n} \max_{0 \leq k \leq r, \ k \ odd} \binom{n-r}{j-k} \) is a partition of \( (\frac{S}{r}) \).

It will be useful later, to note that this partition has exactly
\[
\gamma := \max_{0 \leq h \leq r, \ h \ odd} \{ \max \{ \binom{n-r}{j-k}, \binom{k}{j} \}, \max_{0 \leq h \leq r, \ h \ even} \} \max \{ \binom{n-r}{j-k}, \binom{k}{j} \} = \gamma \]
even \( r \geq n - r \).

3.3.4. Known \( (\frac{r}{h}) = (\frac{r}{r-h}) \). Thus, \( \max \{ \binom{n-r}{j-k}, \binom{k}{j} \} = \gamma \) \( \iff \ r \geq n - r \).

3.3.5. Example. Continue with the example (3.2), we have: \( S = \{1, \ldots, 6\} \) and \( r = 3 \).
\[
s_1 := \begin{pmatrix}
(a)\{1, 4, 5\} & (c)\{2, 4, 5\} & (b)\{3, 4, 5\} \\
(b)\{1, 4, 6\} & (a)\{2, 4, 6\} & (c)\{3, 4, 6\} \\
(c)\{1, 5, 6\} & (b)\{2, 5, 6\} & (a)\{3, 5, 6\}
\end{pmatrix}.
\]

By (3.1.1.b), each of \( \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\} \), \( \{1, 4, 6\}, \{2, 5, 6\}, \{3, 4, 5\} \) and \( \{1, 5, 6\}, \{2, 4, 5\}, \{3, 4, 6\} \) satisfies property (**). And by (3.1.1.c),
\[
U_1^{(even)} := \{4, 5, 6\}, \{1, 2, 4\}, \{1, 3, 5\}, \{2, 3, 6\},\]
\[
U_2^{(even)} := \{1, 2, 5\}, \{1, 3, 6\}, \{2, 3, 4\},\]
\[
U_3^{(even)} := \{1, 2, 6\}, \{1, 3, 4\}, \{2, 3, 5\},\]
\[
U_4^{(odd)} := \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\},\]
\[
U_2^{(odd)} := \{1, 4, 6\}, \{2, 5, 6\}, \{3, 4, 5\} \text{ and }\]
\[
U_3^{(odd)} := \{1, 5, 6\}, \{2, 4, 5\}, \{3, 4, 6\}.
\]

That is, each \( U_i \) satisfies property (**) and \( \left(\frac{S}{3}\right) = \bigcup_{i=1}^{3} U_i^{(odd)} \).\bigcup_{i=1}^{3} U_i^{(even)} \).

4 \( |\text{Matroid}_{n,r}| \leq 2^{r/(r/2)} |\text{Sparse}_{n,r}| \) if \( r \geq \frac{n}{2} \).

Recall that Matroid \( n,r \) (resp., Sparse \( n,r \)) is the set of all matroids (resp. sparse-paving matroids) on a set \( S \) of cardinality \( n \) and rank \( r \). For other bounds of the cardinalities of these sets, see [19], [8], [13], [??], [11], [?].
Let $M$ be a matroid on $S$ of rank $r$. Then the $r-$subsets of $S$, $\binom{S}{r} = B \cup D_r \cup \mathcal{C}_r$ is the disjoint union of its set of basis, $B$, its set of $r$-circuits, $\mathcal{C}_r$ and the dependent $r$-subsets which are no circuits, $D_r$.

Now following the notation of (3.3), all the sets $(\mathcal{C}_r \cup D_r) \cap U_j^{(\text{odd})}$ and $(\mathcal{C}_r \cup D_r) \cap U_j^{(\text{even})}$ satisfy property (**). That is, each of these sets define sparse-paving matroids, see (1.3).

4.1. Theorem: The follow set-function $\Psi_r$ is injective, where

$\alpha := \max_{0 \leq h \leq r, \ h \ odd} \{ \max \{(\frac{n-h}{2}), (\frac{h}{2})\} \}$ and

$\beta := \max_{0 \leq h \leq r, \ h \ even} \{ \max \{(\frac{n-h}{2}), (\frac{h}{2})\} \}$.

$\Psi_r : \text{Matroid}_{n,r} \rightarrow \cup_{j=1}^n \text{Sparse}_{n,r} \times \{j\} \cup \bigcup_{j=\alpha+1}^{\alpha+\beta} \text{Sparse}_{n,r} \times \{j\}$ such that

$\Psi_r(M) = \bigcup_{j=1}^{\alpha+\beta} \left( M^{(r)}_{j,d_j}, (d_j) \right)$

where $M^{(r)}_{j,c_j}$ is the sparse-paving matroid with set of $r-$circuits $(\mathcal{C}_r \cup D_r) \cap U_j^{(\text{odd})}$ if $1 \leq j \leq \alpha$, $M^{(r)}_{d_j}$ is the sparse-paving matroid with set of $r-$circuits $D_r \cap U_j^{(\text{odd})}$ if $\alpha+1 \leq j \leq \alpha+\beta$, $M^{(r)}_{(c_j)}$ is the sparse-paving matroid with set of $r-$circuits $\mathcal{C}_r \cap U_j^{(\text{even})}$ if $1 \leq j \leq \alpha$ and $M^{(r)}_{(d_j)}$ is the sparse-paving matroid with set of $r-$circuits $D_r \cap U_j^{(\text{even})}$ if $\alpha+1 \leq j \leq \alpha+\beta$.

Proof. By (1.3), $\Psi_r$ is a well defined set-function, and since any matroid is defined by its sets of basis, $\Psi_r$ is injective.

4.2. Another version of (4.1) is the following, whose idea is to recognize in any matroid its $r-$circuits from its dependent $r-$subsets which are not circuits.

Theorem: The follow set-function $\overline{\Psi}_r$ is injective, where

$\alpha := \max_{0 \leq h \leq r, \ h \ odd} \{ \max \{(\frac{n-h}{2}), (\frac{h}{2})\} \}$ and

$\beta := \max_{0 \leq h \leq r, \ h \ even} \{ \max \{(\frac{n-h}{2}), (\frac{h}{2})\} \}$.

$\overline{\Psi}_r : \text{Matroid}_{n,r} \rightarrow \bigcup_{j=1}^\alpha \text{Sparse}_{n,r} \times \{(c_j)\} \cup \bigcup_{j=1}^{\alpha+\beta} \text{Sparse}_{n,r} \times \{d_j\}$

such that $\overline{\Psi}_r(M) = \bigcup_{j=1}^{\alpha+\beta} \left( M^{(r)}_{j,c_j}, (c_j) \right) \cup \bigcup_{j=1}^{\alpha+\beta} \left( M^{(r)}_{j,d_j}, (d_j) \right)$

where $M^{(r)}_{(c_j)}$ is the sparse-paving matroid with set of $r-$circuits $\mathcal{C}_r \cap U_j^{(\text{odd})}$ if $1 \leq j \leq \alpha$, $M^{(r)}_{(d_j)}$ is the sparse-paving matroid with set of $r-$circuits $D_r \cap U_j^{(\text{odd})}$ if $\alpha+1 \leq j \leq \alpha+\beta$, $M^{(r)}_{(c_j)}$ is the sparse-paving matroid with set of $r-$circuits $\mathcal{C}_r \cap U_j^{(\text{even})}$ if $1 \leq j \leq \alpha$ and $M^{(r)}_{(d_j)}$ is the sparse-paving matroid with set of $r-$circuits $D_r \cap U_j^{(\text{even})}$ if $\alpha+1 \leq j \leq \alpha+\beta$.

4.3. With the same proof of (4.1), we have the next

Proposition. The set-function $\Gamma_r$ is injective. Let denote by $M_C$ the sparse-paving matroid with $C$ its $r-$circuits.
4.5. Observations for $\Gamma_C$ and $\Gamma_D$.

Let $\gamma := \max_{0 \leq h \leq r} \max\{(\binom{n-r}{h}, \binom{r}{h})\}$. Then

\begin{align*}
\text{a) } & 2^{\left\lceil \frac{n-1}{r} \right\rceil \binom{n}{r}} \leq |\text{Sparse}_{n,r}| \leq |\text{Matroid}_{n,r}| \leq \gamma |\text{Sparse}_{n,r}|. \\
\text{b) } & |\text{Matroid}_{n,r}| \leq 2^{\left\lceil \frac{r}{r+1} \frac{n}{r} \right\rceil}.
\end{align*}

Proof. (b) By (4.3) and (2.2), for $\mathcal{U}$ satisfying property (**), $|\mathcal{U}| \leq \frac{1}{n-r} \binom{n}{r+1}$. Then $|\{M \in \text{Sparse}_{n,r}; |C \subseteq \mathcal{U}|\} = 2^{\left\lfloor \frac{n}{r+1} \right\rfloor} \leq 2^{\frac{n}{r+1} \binom{n}{r+1}}$ and $|\text{Matroid}_{n,r}| \leq 2^{\frac{r}{r+1} \binom{n}{r+1}}$.

4.5. Observations for $\gamma$.

\begin{align*}
\gamma := & \alpha + \beta = \max_{0 \leq h \leq r} \max\{(\binom{n-r}{h}, \binom{r}{h})\} + \max_{0 \leq h \leq r} \max\{\binom{n-r}{h}, \binom{h}{r}\}. \\
\text{if } & 2 \leq r \leq \frac{n-r}{r} \Leftrightarrow 2 \leq r \leq \frac{4}{3}, \\
\gamma = & \binom{r}{\lfloor r/2 \rfloor} + \binom{r}{\lfloor (r+1)/2 \rfloor} = 2^{\frac{r}{(r/2)}} \leftrightarrow n-1 \geq r > n-r \leftrightarrow n-1 \geq r > \frac{r}{3}.
\end{align*}

And it is known, \(2^{\frac{r}{r+1}} \leq \gamma = 2^{\frac{r}{(r/2)}} \to \infty \frac{2^r}{r} < 2^r\).

\begin{itemize}
  \item [i)] In case, $r \leq \frac{4}{3}$, we have two cases:
    \begin{itemize}
      \item [i.1)] If $2 \leq r \leq \frac{n-r}{r} \Leftrightarrow 2 \leq \frac{r}{3} \leq \frac{4}{3}$, $\gamma = \binom{n-r}{r-1} + \binom{r-1}{r} = \binom{n-r+1}{r-1} \leq \binom{\frac{n-r+1}{2}}{\frac{n-r+1}{2}} \to \infty \frac{2^{n-r+1}}{r} < 2^{n-r+1} < 2^{\frac{n-r+1}{2}}$
      \item [i.2)] If $\frac{n-r}{r} \leq \frac{r}{2} \Leftrightarrow \frac{n-r}{r} \leq \frac{4}{3}$, $\gamma = 2^{\frac{r}{2}}$. And $2^{\frac{n-r}{2}} \leq 2^{\frac{r}{2}} \leq 2^{\frac{n-r}{2}} \leq \gamma = 2^{\frac{r}{2}} \to \infty \frac{2^{n-r}}{2^{\frac{r}{2}}} < 2^{n-r} < 2^\frac{r}{2}$
    \end{itemize}
  \item [ii)] Therefore, $\gamma < \begin{cases}
2^r & \text{if } \frac{r}{4} \leq r \leq n-1 \\
2^{n-r+1} & \text{if } 2 \leq r \leq \frac{4}{3} \text{ and } \frac{n-r}{3} \leq r \leq \frac{n-r}{2} \\
2^{n-r} & \text{if } \frac{n-r}{4} \leq r \leq \frac{n-r}{2}
\end{cases}$
  \item [iii)] $\log_2 \gamma < \begin{cases}
\frac{r}{3} & \text{if } \frac{r}{4} \leq r \leq n-1 \\
\frac{n-r+1}{n} & \text{if } 2 \leq r \leq \frac{4}{3} \\
\frac{n-r}{n} & \text{if } \frac{n-r}{4} \leq r \leq \frac{n-r}{2}
\end{cases}$
  \item [iv)] $\frac{1}{r(n-r+1)} \binom{n}{r} \geq \begin{cases}
\frac{2}{n-r+2} \frac{n}{n/2} & \text{if } \frac{r}{4} \leq r \leq n-1 \\
\frac{3}{n-r+2} \frac{n}{n/3} & \text{if } 2 \leq r \leq \frac{n-r}{3} \\
\frac{3}{n-r+2} \frac{n}{n/3} & \text{if } \frac{n-r}{4} \leq r \leq \frac{n-r}{2}
\end{cases}$
\end{itemize}

The proof is straightforward.

Proof. (ii) Looking at the Pascal’s triangle and $r \leq \frac{4}{3}$, we have two cases:

\begin{itemize}
  \item [ii.1)] If $r \leq \frac{n-r}{2}$ then $\max_{0 \leq h \leq r} \max\{\binom{n-r}{h}, \binom{r}{h}\} = \binom{n-r}{r}$ in $\bigtriangleup$-region.
  \item [ii.2)] If $\frac{n-r}{2} \leq r \leq \frac{4}{3}$ then $\max_{0 \leq h \leq r} \max\{\binom{n-r}{h}, \binom{r}{h}\} = \binom{r}{\lfloor (n-r)/2 \rfloor}$ in $\blacklozenge$-region.
\end{itemize}

Pascal’s triangle:
4.6. Corollary. \( \lim_{n \to \infty} \frac{\log_2 |\text{Matroid}_{n,r}|}{\log_2 |\text{Sparse}_{n,r}|} = 1 \) and \( \lim_{n \to \infty} \frac{\log_2 |\text{Matroid}_{n}|}{\log_2 |\text{Sparse}_{n}|} = 1 \)
where \( \text{(Sparse}_n \) Matroid\( _n \) is the set of the (sparse-paving) matroids over \( S, |S| = n \).

Proof. By (4.5),
\[
1 \leq \lim_{n \to \infty} \frac{\log_2 |\text{Matroid}_{n,r}|}{\log_2 |\text{Sparse}_{n,r}|} \leq \lim_{n \to \infty} \left( \frac{\log_2 |\text{Sparse}_{n,r}|}{\log_2 |\text{Sparse}_{n,r}|} + \frac{\log_2 \gamma}{\log_2 |\text{Sparse}_{n,r}|} \right) = 1 + \lim_{n \to \infty} \frac{\log_2 \gamma}{2^{\frac{1}{2} \binom{n}{r} \binom{\binom{n}{r}}{2}}}.
\]

On the other hand, by the duality \( M \leftrightarrow M^* \), we have \( |\text{Sparse}_{n,r}| = |\text{Sparse}_{n,n-r}| \).
Then without loss of generality we can assume \( n - 1 \geq r \geq \frac{n}{2} \).

Then by (4.4),
\[
2^{\frac{1}{2} \binom{n}{r} \binom{\binom{n}{r}}{2}} \geq 2^{\frac{n^2 - n + 2}{n} \binom{n}{r}} > 2^{\frac{n^2 - n + 2}{3}}.
\]

Thus, \( 0 \leq \lim_{n \to \infty} \frac{\log_2 \gamma}{2^{\frac{1}{2} \binom{n}{r} \binom{\binom{n}{r}}{2}}} \leq \lim_{n \to \infty} \frac{\max\{r,n-r+1\}}{2^{\frac{n^2 - n + 2}{3}}} = \lim_{n \to \infty} \frac{n+2}{2^{\frac{n^2 - n + 2}{3}}} = 0.
\]

Therefore, \( \lim_{n \to \infty} \frac{\log_2 |\text{Matroid}_{n,r}|}{\log_2 |\text{Sparse}_{n,r}|} = 1. \)

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(*)
takane@matem.unam.mx (email contact)
Instituto de Matemáticas
Universidad Nacional Autónoma de México (UNAM)
www.matem.unam.mx

(**)
boris.mederos@uacj.mx
gtapia@uacj.mx
Instituto de Ingeniería y Tecnología
Universidad Autónoma de Ciudad Juárez
http://www.uacj.mx/IIT

(***)
bzs@unam.mx
Facultad de Ciencias
Universidad Autónoma del Estado de México
http://www.uaemex.mx/fciencias/