ON THE CHAOTIC BEHAVIOR OF THE DUNKL HEAT SEMIGROUP ON WEIGHTED $L^p$ SPACES

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Abstract. In this paper we study the chaotic behaviour of the heat semigroup generated by the Dunkl-Laplacian on weighted $L^p$ spaces. In the case of the heat semigroup associated to the standard Laplacian we obtain a complete picture on the spaces $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2 dx)$ where $\varphi_{i\rho}$ is the Euclidean spherical function. The behaviour is very similar to the case of the Laplace-Beltrami operator on non-compact Riemannian symmetric spaces studied by Pramanik and Sarkar.

1. Introduction

The study of chaotic dynamics of the heat semigroup on Riemannian symmetric spaces of noncompact type, which started with the work of Ji and Weber [10] has been completed recently by Pramanik and Sarkar [13] (see also Sarkar [17]). As they have remarked, the chaotic behavior of the heat semigroups on $L^p$ spaces seems to be a non-Euclidean phenomenon. In order to state the results of Pramanik and Sarkar and make a comparison with the Euclidean case, we need to recall several definitions from Ergodic theory. We closely follow the terminologies used in [13] referring to [4] and [10] for more details.

Let $T_t$, $t > 0$ be a strongly continuous semigroup on a Banach space $\mathcal{B}$.

1. We say that $T_t$ is hypercyclic if there exists a $v \in \mathcal{B}$ such that 
\[ \{ T_t v : t \geq 0 \} \] 
is dense in $\mathcal{B}$.
2. If there exist a $v \in \mathcal{B}$ such that $T_t v = v$ for some $t > 0$ then we say that $v$ is periodic for $T_t$.
3. We say that $T_t$ is chaotic if it is hypercyclic and if its periodic points are dense in $\mathcal{B}$.

Let $\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ be the standard Laplacian on the Euclidean space $\mathbb{R}^n$. The semigroup $T_t = e^{-t\Delta}$ generated by $\Delta$ fails to be chaotic on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. This can be easily checked by appealing to the following theorem proved in [11].

2010 Mathematics Subject Classification. Primary: 43A85; Secondary: 22E30.

Key words and phrases. Dunkl transform, heat semigroup, chaotic semigroup, hypercyclic, periodic points, spectrum of Dunkl Laplacian.
Theorem 1.1 (de Laubenfels-Emamirad [11]). If $T_t$ is a chaotic semigroup generated by $A$ in a Banach space $B$ then the cardinality of $\sigma_{pt}(A) \cap i\mathbb{R}$ is infinite, where $\sigma_{pt}(A)$ is the point spectrum of $A$.

Indeed, the spectrum of $\Delta$ on $L^p(\mathbb{R}^n)$ is independent of $p$ and equals $[0, \infty)$. Consequently, $e^{-t\Delta}$ cannot be chaotic on any of the $L^p$ spaces.

Compare this with the case of the heat semigroup generated by the Laplace-Beltrami operator $\Delta_X$ on a noncompact Riemannian symmetric space $X$. When $p > 2$, there are plenty of eigenfunctions, provided by the elementary spherical functions, in $L^p(X)$ with purely imaginary eigenvalues. This fact has been utilized in obtaining a complete picture of the chaotic behavior of $e^{-t\Delta_X}$ on $L^p(X)$ in the article [13] where the authors have established the following result.

Theorem 1.2 (Pramanik-Sarkar [13]). For any Riemannian symmetric space $X$ of non-compact type, let $T_t$ be the semigroup generated by $\Delta_X$, $T_c = e^{ct}T_t$ and let $c_p = \frac{4|\rho|^2}{pp'}$ where $\rho$ is the half-sum of the positive roots. Then the following conclusions hold: (a) For $2 < p < \infty$, $T_c$ is chaotic on $L^p(X)$ if and only if $c > c_p$. (b) For $p = \infty$, $T_c$ is non-chaotic on $L^\infty(X)$ for all $c \in \mathbb{R}$. (c) However, $T_c$ is subspace-chaotic on $L^\infty(X)$ if and only if $c > 0$.

The analogue of spherical functions in the Euclidean set up are the Bessel functions defined by

$$\varphi_\lambda(x) = \int_{S^{n-1}} e^{i\lambda x \cdot \omega} d\sigma(\omega)$$

where $d\sigma$ is the surface measure on $S^{n-1}$ and $\lambda \in \mathbb{C}$. These are all eigenfunctions of the Laplacian with eigenvalue $\lambda^2 : \Delta \varphi_\lambda = \lambda^2 \varphi_\lambda$ and when $\lambda \in \mathbb{R}$, $\varphi_\lambda \in L^p(\mathbb{R}^n)$, $p > \frac{2n}{n-1}$. But when $\lambda \in \mathbb{C}$, they have exponential growth. Indeed,

$$\varphi_\lambda(x) = c_n \frac{J_{\frac{n-1}{2}}(\lambda|x|)}{(\lambda|x|)^{\frac{n-1}{2}}}$$

where $J_\alpha(t)$ is the Bessel function of type $\alpha$. It follows that

$$|\varphi_\lambda(x)| \leq \varphi_{i\Im(\lambda)}(x) = c_n I_{\frac{n-1}{2}}(\Im(\lambda)|x|)(\Im(\lambda)|x|)^{-\frac{n-1}{2}+1}$$

where $I_\alpha(t) = J_\alpha(it)$ is the modified Bessel function. Using the asymptotic behavior of $I_\alpha(t)$ we see that

$$|\varphi_\lambda(x)| \leq C_\lambda |x|^{-\left(\frac{n-1}{2}\right)}e^{||\Im(\lambda)|||x|}.$$ 

It then follows that for any $\rho > 0$, and $p \neq 2$,

$$\int_{\mathbb{R}^n} |\varphi_\lambda(x)|^p (1 + |x|)^{(\frac{n-1}{2}p - \rho)p - 2} |x| dx < \infty$$
provided $|\Im(\lambda)| < \gamma_p \rho$ where $\gamma_p = \left| \frac{2}{p} - 1 \right|$. For $p > 2$, we can rewrite the above as

$$\int_{\mathbb{R}^n} |\varphi_\lambda(x) \varphi_{ip}(x)^{-1}|p (\varphi_{ip}(x))^2 dx < \infty.$$
hypercyclic nor has periodic points on $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2 dx)$.  

For $\nu > 0, z > 0$ let $K_\nu(z)$ be the Macdonald function given by the Sommerfeld integral

$$K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty e^{-(t + \frac{z^2}{4t})} t^{-\nu-1} dt.$$  

By making a change of variables, we observe that

$$z^n K_\nu(z) = 2^{-\nu-1} \int_0^\infty e^{-(tz^2 + \frac{1}{t})} t^{-\nu-1} dt.$$  

The asymptotic behavior of $K_\nu$ and $I_\nu(z)$ at infinity are given by

$$K_\nu(z) = \frac{\sqrt{\pi}}{\sqrt{2z}} e^{-z}(1 + O(1/z)), \quad I_\nu(z) = \frac{1}{\sqrt{2\pi z}} e^z(1 + O(1/z)),$$

see page 226 in [12]. Consequently, for $|\Im(\lambda)| < \gamma_\rho p$ we see that

$$\int_{\mathbb{R}^n} |\varphi_\lambda(x)|^p (\tilde{K}_{n/2}(\rho |x|))^{\gamma_\rho p} dx < \infty$$

where $\tilde{K}_\nu(z) = z^n K_\nu(z)$. If we take $\lambda = \beta(1 + i), |\beta| < \gamma_\rho p$, then $\Delta \varphi_\lambda = 2i\beta^2 \varphi_\lambda$ and hence we have plenty of eigenfunctions with purely imaginary eigenvalues which belong to the weighted $L^p$ spaces $L^p(\mathbb{R}^n, (\tilde{K}_{n/2}(\rho |x|))^{\gamma_\rho p} dx)$.

The behaviour of the modified semigroup $\tilde{T}_t$ on $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2 dx)$ is equivalent to the behaviour of $T_t$ on the spaces $L^p(\mathbb{R}^n, (\tilde{K}_{n/2}(\rho |x|))^{\gamma_\rho p} dx)$ for $p > 2$. Indeed, if we set $\tilde{I}_\nu(z) = \frac{I_\nu(z)}{z^n}$, then it follows from the asymptotic properties of $I_\nu$ and $K_\nu$ that

\begin{equation}
C_1 \leq \tilde{I}_\nu(z) \tilde{K}_{\nu+1}(z) \leq C_2, \quad z \geq 0.
\end{equation}

In view of this, $L^p(\mathbb{R}^n, (\tilde{K}_{n/2}(\rho |x|))^{p-2} dx)$ is the same as $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^{2-p} dx)$ whenever $p > 2$ as $\gamma_\rho p = p - 2$. It follows that $f \in L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2 dx)$ if and only if $f \varphi_{i\rho} \in L^p(\mathbb{R}^n, (\tilde{K}_{n/2}(\rho |x|))^{2-p} dx)$. It is therefore, natural to study the heat semigroup $e^{-t(\Delta + \rho^2)}$ on the weighted $L^p$ spaces $L^p(\mathbb{R}^n, (\tilde{K}_{n/2}(\rho |x|))^{\gamma_\rho p} dx)$. It turns out that the chaotic behavior of $e^{-t(\Delta + \rho^2)}$ on these spaces is very similar to the behavior of $e^{-t\Delta x}$ on $L^p(X), p > 2$. Indeed, we have the following theorem.

In what follows we write $L^p_\rho(\mathbb{R}^n)$ in place of $L^p(\mathbb{R}^n, (\tilde{K}_{n/2}(\rho |x|))^{\gamma_\rho p} dx)$ for the sake of notational convenience.

**Theorem 1.4.** For $1 \leq p \leq \infty$ let $c_p = \rho^2(1 - \gamma_\rho^2)$ and for $c \in \mathbb{R}$ define $T_t^c = e^{-t(\Delta + \rho^2 - c)}$ where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$. Then

1. For $1 \leq p < \infty, p \neq 2$, $T_t^c$ is chaotic on $L^p_\rho(\mathbb{R}^n)$ if $c > c_p$.
2. $T_t^c$ is not chaotic on $L^p_\rho(\mathbb{R}^n)$ for any $c \in \mathbb{R}$. 
(3) For $1 \leq p < \infty$ and $c \leq c_p$, $T_t^c$ is not chaotic on $L_p^p(\mathbb{R}^n)$ and for $c < c_p$ it is not even hypercyclic.

The proof of the above theorem depends on a sharp estimate for the heat semigroup $T_t$ on $L^p_{\rho}(\mathbb{R}^n)$ stated and proved in Proposition 3.4. And Theorem 1.3 is an immediate consequence of the above result.

In this paper we also work in a more general set up and study the chaotic dynamics of the heat semigroup generated by the Dunkl Laplacian $\Delta_\kappa$ on $\mathbb{R}^n$ associated to a finite reflection group. Let $G$ be such a group generated by the reflections associated to a root system on $\mathbb{R}^n$. Let $\kappa$ be a nonnegative multiplicity function and $h_\kappa^2(x)$ be the associated weight function. Let $T_j$, $j = 1, 2, \cdots, n$ be the Dunkl difference-differential operators and $\Delta_\kappa = -\sum_{j=1}^n T_j^2$ be the Dunkl Laplacian. For all the required definitions we refer to Section 3. We consider $T_t = e^{-t(\Delta_\kappa + \rho^2)}$ the heat semigroup generated by $A = \Delta_\kappa + \rho^2$ on the weighted $L^p$-space $L^p(\mathbb{R}^n, (\tilde{K}_\kappa/2+\gamma)(\rho|x|))^{\gamma p}h_\kappa^2(x)dx$, where $\gamma$ is defined in terms of the multiplicity function $\kappa$, see Section 3. We denote the above space by $L^p_{\rho,\kappa}(\mathbb{R}^n)$. Note that $L^p_{\rho,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n, (\tilde{K}_\kappa/2)(\rho|x|))^{\gamma p}dx$ which we have denoted by $L^p_{\rho}(\mathbb{R}^n)$. The chaotic behavior of the semigroup generated by $A$ is described in the following result. For $1 \leq p \leq \infty$ we denote $p'$ to be conjugate index of $p$ i.e. $1/p + 1/p' = 1$.

**Theorem 1.5.** Let $A = \Delta_\kappa + \rho^2$. For $1 \leq p \leq \infty$, let $c_p = \rho^2(1 - 1/p')$ and for $c \in \mathbb{R}$ define $T_t^c = e^{-t(A-c)}$. Then

1. For $1 \leq p < \infty$, $p \neq 2$, $T_t^c$ is chaotic on $L^p_{\rho,\kappa}(\mathbb{R}^n)$ if and only if $c > c_p$. (1)
2. $T_t^c$ is not chaotic on $L^\infty_{\rho,\kappa}(\mathbb{R}^n)$ for any $c \in \mathbb{R}$. (2)
3. $T_t^c$ is not chaotic on $L^2_{\rho,\kappa}(\mathbb{R}^n) = L^2(\mathbb{R}^n, h_\kappa(x)^2dx)$ for any $c \in \mathbb{R}$. (3)

For $c \leq c_p$, we would like to know which property of $T_t^c$ fails. The next theorem partially answers this question.

**Theorem 1.6.** With same notations as in the previous theorem we have the following results.

1. For $1 \leq p < 2$ and $c < 2\rho^2/p$, $T_t^c$ is not hypercyclic on $L^p_{\rho,\kappa}(\mathbb{R}^n)$. (1)
2. For $p > 2$ and $c < 2\rho^2/p$, $T_t^c$ is not hypercyclic on $L^p_{\rho,\kappa}(\mathbb{R}^n)$. (2)

**Remark 1.7.** When $1 \leq p < 2$ (when $p > 2$) and $2\rho^2/p \leq c \leq c_p$ (resp. when $2\rho^2/p \leq c \leq c_p$), we don’t know if $T_t^c$ fails to be hypercyclic or not. Also we are not able to say anything about the periodicity. This is due to the fact that we do not have sharp estimates on the operator norm of $T_t$ on $L^p_{\rho,\kappa}(\mathbb{R}^n)$. On the other hand when $\kappa = 0$ we do have better estimates for the operator norm of $T_t$ and hence we have a complete picture.
An examination of the proof of the sharp estimate for $T_t = e^{-t(\Delta + \rho^2)}$ in Theorem 2.1 reveals that we need to use the boundedness properties of translation operators on $\mathbb{R}^n$ on weighted $L^p$ spaces. If we want to prove an analogue of Theorem 2.1 for the Dunkl Laplacian, then we need to know the boundedness properties of Dunkl translation on the spaces $L^{p}_{\rho,\kappa}(\mathbb{R}^n)$. Unfortunately, the boundedness properties of these operators are not even known on $L^p$ spaces, see [21] for some results.

On the other hand, instead of $L^{p}_{\rho,\kappa}(\mathbb{R}^n)$ for $1 \leq p < \infty$, if we consider the mixed norm spaces $L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$, then we can improve the estimates. These are defined as the space of all functions $f$ for which

\[
\int_0^{\infty} \left( \int_{S^{n-1}} |f(re^{i\omega})|^2 h_{n}^{\rho}(\omega) d\sigma(\omega) \right)^{\frac{p}{2}} (\tilde{K}_{n/2+\gamma}(\rho r))^{p/2} r^{n+2\gamma-1} dr < \infty.
\]

The $p$-th root of the above quantity will be denoted by $\|f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)}$. On this space we have better estimates for the Dunkl heat semigroup, see Theorem 3.5. Consequently, we can prove the following result.

**Theorem 1.8.** For $1 \leq p < \infty$ let $c_p = \rho^2 (1 - \gamma_p^2)$ and for $c \in \mathbb{R}$ define $T^c_t = e^{-t(A-c)}$. Then

1. For $1 \leq p < \infty$, $p \neq 2$, $T^c_t$ is chaotic on $L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$ if $c > c_p$.
2. For $1 \leq p < \infty$ and $c < c_p$, $T^c_t$ is not hypercyclic on $L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$ and hence not chaotic.

A comparison of these theorems with the results of [13] (see Theorems 1.2, 1.3 and 1.4) shows the similarity between the behavior of $e^{-t(A-c)}$ on $L^p(\mathbb{R}^n)$ and $e^{-t(\Delta x-c)}$ on $L^p(X)$. It is also interesting to compare our results to the unweighted case of the heat semigroup $e^{-t(-\Delta-c)}$ on $L^p(\mathbb{R}^n)$ stated and proved in Section 9 of [13].

2. Chaotic behavior of the heat semigroup on weighted $L^p$ spaces

2.1. The heat semigroup on $L^p(\mathbb{R}^n)$: In this subsection we will estimate the operator norm of $T_t$ acting on the weighted space $L^p(\mathbb{R}^n)$. In proving the following result we will make use of the fact that for $1 < p < \infty$, the dual of $L^p(\mathbb{R}^n)$ can be identified with $L^q(\mathbb{R}^n, (\varphi_{1p}(x))^{q'} dx)$ if the duality bracket is taken as

\[
(f, g) = \int_{\mathbb{R}^n} f(x)g(x)dx,
\]

for $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n, (\varphi_{1p}(x))^{q'} dx)$. This follows from the estimates (1.1). When $p = 1$ the dual of $L^1(\mathbb{R}^n)$ is taken as $L^\infty(\mathbb{R}^n)$ and we use the standard duality bracket

\[
(f, g) = \int_{\mathbb{R}^n} f(x)g(x)\tilde{K}_{n/2}(x)dx.
\]
Theorem 2.1. Let $T_t$ be the semigroup generated by $\Delta + \rho^2$. Then for any $1 \leq p < \infty$ it is strongly continuous on $L_p^p(\mathbb{R}^n)$. Moreover, we have the estimate

$$\|T_tf(x)\|_{L_p^p(\mathbb{R}^n)} \leq C(1 + t)^{-\frac{n+1}{2} + \gamma_p} e^{-\frac{\rho^2}{p\gamma_p} t} \|f(x)\|_{L_p^p(\mathbb{R}^n)}$$

for all $f \in L_p^p(\mathbb{R}^n)$ and $1 \leq p < \infty$.

Proof. In view of the asymptotic behavior of the Macdonald function, it is enough to consider the space defined using $(1 + |x|)^{(n-1)/2} e^{-\rho|x|}$ in place of $\tilde{K}_{n/2}(\rho|x|)$. For the sake of brevity, just for this section, we denote the weight function $(1 + |x|)^{(n-1)/2} e^{-\rho|x|}$ by $w_\rho(x)$. It is therefore enough to prove

$$\left( \int_{\mathbb{R}^n} |T_tf(x)|^p (w_\rho(x))^{p\gamma_p} dx \right)^{\frac{1}{p}} \leq C(1 + t)^{-\frac{n+1}{2} + \gamma_p} e^{-\frac{\rho^2}{p\gamma_p} t} \left( \int_{\mathbb{R}^n} |f(x)|^p (w_\rho(x))^{p\gamma_p} dx \right)^{\frac{1}{p}}.$$

The strong continuity of $T_t$ on $L_p^p(\mathbb{R}^n)$ follows from the norm estimates. Indeed, for $0 < t \leq 1$, the operators $T_t$ are uniformly bounded on $L_p^p(\mathbb{R}^n)$, $1 \leq p < \infty$. As $L_p^p(\mathbb{R}^n, dx)$ is dense in $L_p^p(\mathbb{R}^n)$, the strong continuity of $T_t$ on $L_p^p(\mathbb{R}^n)$ follows from the same on $L_p^p(\mathbb{R}^n, dx)$ in view of the continuous inclusion $L_p^p(\mathbb{R}^n, dx) \subset L_p^p(\mathbb{R}^n)$.

First assume that $1 < p < \infty$. For any $f \in L_p^p(\mathbb{R}^n)$ and $g \in (L_p^p(\mathbb{R}^n))^* \simeq L^p(\mathbb{R}^n, (\varphi_{\rho^2}(x))^{\gamma_p} dx)$ consider

$$\int_{\mathbb{R}^n} T_tf(x)g(x) dx = e^{-\frac{p\gamma_p}{2} t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)h_t(x-y)g(x) dx dy$$

where $h_t$ is the heat kernel which is explicitly given by

$$h_t(x) = (4\pi t)^{-n/2} e^{-\frac{1}{4t}|x|^2}.$$

By making a change of variables in the $x$-integral, the above reads as

$$e^{-\frac{p\gamma_p}{2} t} \int_{\mathbb{R}^n} h_t(x) \left( \int_{\mathbb{R}^n} f(y)g(x+y) dy \right) dx.$$

Now the inner integral can be estimated by Hölder's inequality after rewriting it as

$$\int_{\mathbb{R}^n} f(y)(w_\rho(y))^{\gamma_p} g(x+y)(w_\rho(y))^{-\gamma_p} dy.$$

Since $\gamma_p = \gamma_p'$ the result is the bound

$$\left( \int_{\mathbb{R}^n} |f(y)|^p (w_\rho(y))^{p\gamma_p} dy \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} |g(x+y)|^{p'} (w_\rho(y))^{-\gamma_p'} dy \right)^{\frac{1}{p'}}.$$

By making a change of variables, the second integral can be written as

$$\left( \int_{\mathbb{R}^n} |g(y)|^{p'} (w_\rho(y-x))^{-\gamma_p'} dy \right)^{\frac{1}{p'}}.$$
In view of the inequality $(1 + |y|) \leq (1 + |x - y|)(1 + |x|)$, we see that

$$(w_p(y - x))^{-\gamma_p \rho'} \leq (1 + |x|)^{\frac{n-1}{2} \gamma_p \rho e^{\rho|\rho| |x|}} (w_p(y))^{-\gamma_p \rho'}.$$ 

By making use of this, the above integral is bounded by

$$(1 + |x|)^{\frac{n-1}{2} \gamma_p \rho} e^{\gamma_p \rho |x|} \left( \int_{\mathbb{R}^n} |g(y)|^{\rho'} (\varphi_{i\rho}(y))^{\rho' \gamma_p \rho} dy \right)^{\frac{1}{\rho'}}.$$ 

Thus $|\int_{\mathbb{R}^n} T_t f(x) g(x) dx|$ is bounded by

$$e^{-t \rho^2} \|f\|_{L^p_w(\mathbb{R}^n)} \|g\|_{(L^p_w(\mathbb{R}^n))^*} \int_{\mathbb{R}^n} h_t(x)(1 + |x|)^{\frac{n-1}{2} \gamma_p \rho} e^{\gamma_p \rho |x|} dx.$$ 

A simple calculation shows that

$$e^{-t \rho^2} \int_{\mathbb{R}^n} h_t(x)(1 + |x|)^{\frac{n-1}{2} \gamma_p \rho} e^{\gamma_p \rho |x|} dx \leq C(1 + t)^{\frac{n-1}{2}(1 + \gamma_p)} e^{-t \rho^2(1 - \gamma_p^2)}$$

which completes the proof as $1 - \gamma_p^2 = \frac{4}{p^2}$. When $p = 1$ we can directly estimate

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h_t(x - y)|f(y)|(1 + |x|)^{(n-1)/2} e^{-\rho |x|} dx dy.$$ 

Just make a change of variables in the $x$-integral and proceed as before to get the required estimate. \(\square\)

### 2.2. Spectrum of the Laplacian on weighted $L^p$ spaces.

In case of noncompact Riemannian symmetric spaces $G/K$ the $L^p$ spectrum of the Laplace-Beltrami operator $\Delta$ is precisely known. It has been proved in Taylor [19] that the $L^p$ spectrum is equal to the parabolic neighborhood

$$\mathfrak{P}_p = \{ \lambda^2 + |\rho|^2 : |\Im(\lambda)| \leq \gamma_p |\rho| \}$$

of the half line $[|\rho|^2, \infty)$. This follows from a multiplier theorem proved in [19] for general Riemannian manifolds. It would be nice to see if we have precise information about the spectrum of the Dunkl-Laplacian $\Delta_{\kappa}$ acting on the spaces $L^p_{\rho,\kappa}(\mathbb{R}^n)$. In this generality, we are not able to determine precisely the spectrum of $\Delta_{\kappa}$. However, when $\kappa = 0$ i.e. for the standard Laplacian $\Delta$ on $\mathbb{R}^n$ we do have the following result. In view of the asymptotic behavior of the Macdonald function, it is enough to consider the space defined using $w_\rho(x) := (1 + |x|)^{(n-1)/2} e^{-\rho |x|}$ in place of $K_{n/2}(\rho |x|)$.

**Theorem 2.2.** For any $1 \leq p < \infty$, the spectrum of $\Delta + \rho^2$ on $L^p_w(\mathbb{R}^n)$ is precisely the set

$$\mathfrak{P}_p = \{ \lambda^2 + \rho^2 : |\Im(\lambda)| \leq \gamma_p |\rho| \}.$$ 

For $p > 2$, the spectrum of $\tilde{\Delta}$ on $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2 dx)$ is also $\mathfrak{P}_p$.

As in the case of symmetric spaces this result can be deduced from the following multiplier theorem for the Laplacian on the weighted $L^p$ spaces $L^p_{\rho}(\mathbb{R}^n)$. 





In order to state the result we recall some definitions from [19]. Let $\Omega_W$ be the set $\{ \lambda \in \mathbb{C} : |\Im(\lambda)| < W \}$ and set $\mathcal{E}_W^m$ to be the set of all even holomorphic functions $\varphi$ on $\Omega_W$ satisfying
\[
|\varphi^{(j)}(\lambda)| \leq C_j (1 + \lambda^2)^{\frac{m-j}{2}}
\]
on the closure $\overline{\Omega}_W$ for all $j = 0, 1, 2, \cdots$.

**Theorem 2.3.** For every $1 < p < \infty$ we have $\varphi(\sqrt{\Delta}) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ provided $\varphi \in \mathcal{E}_W^0$ with $W \geq \gamma_p \rho$.

In proving this theorem we closely follow [19] (see proof of Theorem A). We use the functional calculus to write
\[
\varphi(\sqrt{\Delta}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \varphi(t) \cos t \sqrt{\Delta} \, dt.
\]
Using a partition of unity we write $\hat{\varphi} = \hat{\varphi}_1 + \hat{\varphi}_2$ where $\hat{\varphi}_1$ is compactly supported and $\hat{\varphi}_2(t) = 0$ for $|t|$ small. As a consequence of this decomposition we have

**Lemma 2.4** (see Lemma 1.3 in [19]). Given $\varphi \in \mathcal{E}_W^m$ we can write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1$ has compact support, $\varphi_1 \in \mathcal{E}_W^m$, for all $W' < W$ and $\varphi_2 \in \mathcal{E}_W^m$.

In an earlier paper [1] it has been proved that $\varphi_1(\sqrt{\Delta})$ is a pseudo-differential operator whose distribution kernel is supported near the diagonal. Consequently, the boundedness of pseudo differential operators of order 0 on $L^p$ spaces gives us

**Lemma 2.5.** If $\varphi_1$ is as in the previous lemma with $m = 0$ then for any $1 < p < \infty$, $\varphi_1(\sqrt{\Delta})$ is bounded on $L^p(\mathbb{R}^n)$.

Using the fact that the distribution kernel of $\varphi_1(\sqrt{\Delta})$ is supported in a neighborhood of the diagonal, say $|x - y| \leq \frac{1}{2}$, we can actually prove the boundedness of $\varphi_1(\sqrt{\Delta})$ on the weighted $L^p$ spaces $L^p_w(\mathbb{R}^n)$. To see this let $k(x, y)$ be the distribution kernel of $\varphi_1(\sqrt{\Delta})$ which is supported in $|x - y| \leq \frac{1}{2}$ so that
\[
\varphi_1(\sqrt{\Delta}) f(x) = \int_{|x - y| \leq \frac{1}{2}} k(x, y) f(y) \, dy.
\]
Consider now $\int_{\mathbb{R}^n} |\varphi_1(\sqrt{\Delta}) f(x)|^p (w_\rho(x))^{\rho_n} \, dx$ which is equal to
\[
\sum_{m=0}^{\infty} \int_{m - \frac{1}{2} \leq |x| < m + \frac{1}{2}} |\varphi_1(\sqrt{\Delta}) f(x)|^p (w_\rho(x))^{\rho_n} \, dx 
\]
\[
\leq \ C \sum_{m=0}^{\infty} (m + 2)^{(n-1)\rho_n} e^{-\rho_n \rho (m - \frac{1}{2})} \int_{m - \frac{1}{2} \leq |x| < m + \frac{1}{2}} |\varphi_1(\sqrt{\Delta}) f(x)|^p \, dx.
\]
Since the kernel $k(x, y)$ is supported in $|x - y| \leq \frac{1}{2}$ we observe that
\[
\chi_{m - \frac{1}{2} \leq |x| < m + \frac{1}{2}}(x) \varphi_1(\sqrt{\Delta}) f(x) = \int_{|x - y| \leq \frac{1}{2}} k(x, y) f(y) \chi_{m - 1 \leq |y| < m + 1} f(y) \, dy.
\]
Consequently, the boundedness of $\varphi_1(\sqrt{\Delta})$ gives the estimate
\[
\int_{m-\frac{1}{2} \leq |x| < m + \frac{1}{2}} |\varphi_1(\sqrt{\Delta}) f(x)|^p dx \leq C \int_{m-1 \leq |y| \leq m+1} |f(y)|^p dy
\]
which is easily seen to be bounded by
\[
C(1+m)^{-\frac{n-1}{2} p\gamma_p e^{p\gamma_p(m+1)}} \int_{m-1 \leq |y| < m+1} |f(y)|^p (w_\rho(y))^{p\gamma_p} dy.
\]
Summing over $m$ we obtain
\[
\int_{\mathbb{R}^n} |\varphi_1(\sqrt{\Delta}) f(x)|^p (w_\rho(x))^{p\gamma_p} dx \leq C \int_{\mathbb{R}^n} |f(y)|^p (w_\rho(y))^{p\gamma_p} dy
\]
which takes care of $\varphi_1(\sqrt{\Delta})$. The proof of Theorem 2.3 will be complete once we prove the following result.

**Theorem 2.6.** If $\varphi \in \mathcal{S}_{W^{-\infty}}$, $W \geq \gamma_p \rho$ then
\[
\varphi(\sqrt{\Delta}) : L^p_\rho(\mathbb{R}^n) \rightarrow L^p_\rho(\mathbb{R}^n)
\]
is bounded for $1 \leq p < \infty$.

**Proof.** Once again the proof is a modification of the proof of Proposition 1.4 in [19]. We only prove the theorem when $1 \leq p \leq 2$. The case $p > 2$ can be handled by duality. Since $\varphi(\sqrt{\Delta})$ is bounded on $L^2(\mathbb{R}^n)$, it is enough to prove the boundedness of $\varphi(\sqrt{\Delta})$ on $L^1_\rho(\mathbb{R}^n) = L^1(\mathbb{R}^n, \tilde{K}_{n/2}(\rho|x|)dx)$. For then, we can appeal to Stein-Weiss interpolation theorem with change of measures to get the desired result, see [19]. In fact our choice of the measure, namely $(\tilde{K}_{n/2}(\rho|x|))^{p\gamma_p}$ is motivated by this theorem.

If $k_\varphi(x,y)$ stands for the kernel of $\varphi(\sqrt{\Delta})$, we need to show that
\[
\sup_{y \in \mathbb{R}^n} (w_\rho(y))^{-1} \int_{\mathbb{R}^n} |k_\varphi(x,y)| w_\rho(x) dx \leq C.
\]
Let $A_y(m)$ be the annulus $\{x : m \leq |x - y| < m + 1\}$ and consider
\[
\int_{\mathbb{R}^n} |k_\varphi(x,y)| w_\rho(x) dx = \sum_{m=0}^{\infty} \int_{A_y(m)} |k_\varphi(x,y)| w_\rho(x) dx.
\]
By Cauchy-Schwarz we estimate the above by
\[
\sum_{m=0}^{\infty} \left( \int_{A_y(m)} (w_\rho(x))^2 dx \right)^{\frac{1}{2}} \left( \int_{A_y(m)} |k_\varphi(x,y)|^2 dx \right)^{\frac{1}{2}}.
\]
Now
\[
(w_\rho(y))^{-2} \int_{A_y(m)} (w_\rho(x))^2 \, dx = \int_{A_y(m)} \left( 1 + \frac{|x|}{1+|y|} \right)^{(n-1)} e^{2\rho(|y|-|x|)} \, dx
\]
\[
\leq (m+1)^{n-1} e^{2\rho(m+1)} \int_{A_y(m)} \, dx
\]
\[
\leq C(m+1)^{2(n-1)} e^{2m\rho}
\]
and consequently
\[
(w_\rho(y))^{-1} \int_{\mathbb{R}^n} |k_\varphi(x,y)| w_\rho(x) \, dx
\]
\[
\leq C \sum_{m=0}^{\infty} (m+1)^{n-1} e^{m\rho} \left( \int_{A_y(m)} |k_\varphi(x,y)|^2 \, dx \right)^{\frac{1}{2}}.
\]
Let \( L^2 \) norm of \( k_\varphi(x,y) \) over the annulus \( A_y(m) \) can be estimated as in \([19]\). For the convenience of the reader we give some details.

Let \( \delta_y \) stand for the Dirac delta distribution at \( y \). Then we can find functions \( g_\rho \) and \( h_\rho \) both in \( L^2(|x-y| \leq 1) \) such that \( \delta_y = \Delta^2 g_\rho + h_\rho \) where
\[
s = \left[ \frac{n}{2} \right] + 1.
\]
We can assume that \( \|g_\rho\|_2 \) and \( \|h_\rho\|_2 \) are bounded uniformly in \( y \). With this decomposition of \( \delta_y \) we obtain
\[
k_\varphi(x,y) = \varphi(\sqrt{\Delta}) \delta_y(x) = \varphi_s(\sqrt{\Delta}) g_\rho(x) + \varphi(\sqrt{\Delta}) h_\rho(x)
\]
where \( \varphi_s(\lambda) = \lambda^{2s} \varphi(\lambda) \). By the finite propagation speed, on the annulus \( A_y(m) \) we have
\[
\varphi_s(\sqrt{\Delta}) g_\rho(x) = \int_{|t| \geq m-1} \tilde{\varphi}(t)(\cos t\sqrt{\Delta}) g_\rho(x) \, dt.
\]
Since \( \varphi_s \in \mathcal{G}_{\mathcal{W}}^{-\infty} \), we have the estimate
\[
|\tilde{\varphi}(t)| \leq C_N (1 + t^2)^{-\frac{N}{2}} e^{-W|t|}, \quad N = 0, 1, 2, \ldots.
\]
Using the boundedness of \( \cos t\sqrt{\Delta} \) on \( L^2(\mathbb{R}^n) \) we have, for any \( N \),
\[
\left( \int_{A_y(m)} |\varphi_s(\sqrt{\Delta}) g_\rho(x)|^2 \, dx \right)^{\frac{1}{2}} \leq C_N \int_{|t| \geq m-1} (1 + t^2)^{-\frac{N}{2}} e^{-W|t|} \, dt
\]
\[
\leq C_N (1 + m^2)^{-\frac{N}{2}} e^{-Wm}.
\]
A similar estimate holds for \( \varphi(\sqrt{\Delta}) h_\rho \) on \( A_y(m) \). Putting these estimates together we have
\[
(w_\rho(y))^{-1} \int_{\mathbb{R}^n} |k_\varphi(x,y)| w_\rho(x) \, dx
\]
\[
\leq C \sum_{m=0}^{\infty} (m+1)^{n-1} e^{m\rho(1 + m^2)^{-\frac{N}{2}}} e^{-Wm}
\]
which is finite provided \( W \geq \rho \) if we take \( N > n + 1 \). This completes the proof of Theorem 2.6. \( \square \)
2.3. The chaotic behavior of the heat semigroup: In this subsection we prove one of the main theorems namely Theorem 1.4 regarding the chaotic behavior of the semigroup \( T_t^c = e^{ct}e^{-t(-\Delta + \rho^2)} \), \( c \in \mathbb{R} \) on the space \( L_p^p(\mathbb{R}^n) \). In proving the result we closely follow the proofs given in [13] for the case of symmetric spaces of non-compact type. As in [13] we let

\[
\Lambda_p = \{ \lambda \in \mathbb{C} : |\Im(\lambda)| \leq \gamma_p \rho \}
\]

and define \( c_p = (1 - \gamma_p^2)\rho^2 = \frac{4\rho^2}{pp'} \). For any \( c > c_p \) we also define a map

\[
\omega_c : \Lambda_p \to \mathbb{C}, \quad \omega_c(\lambda) = \lambda^2 + \rho^2 - c.
\]

Using this we define the following three subsets of \( \Lambda^0_p \), the interior of \( \Lambda_p \);

\[
A_1 = \{ \lambda \in \Lambda^0_p : \Re(\omega_c(\lambda)) > 0 \};
\]

\[
A_2 = \{ \lambda \in \Lambda^0_p : \Re(\omega_c(\lambda)) < 0 \};
\]

\[
A_3 = \{ \lambda \in \Lambda^0_p : \omega_c(\lambda) \in i\mathbb{Q} \};
\]

where \( \mathbb{Q} \) is the set of all rationals. In [13] the authors have proved that all these sets are non-empty and \( A_3 \) has infinitely many points (see Lemma 4.1 in [13]) for \( c > c_p \). Also note that \( A_1 \) and \( A_2 \) are both open subsets of \( \Lambda^0_p \).

To each of these \( A_j \)'s we associate certain subsets \( A_j \) as follows. The translation of the spherical functions \( \varphi_\lambda(y) \) are given by the equation

\[
\tau_x \varphi_\lambda(y) = \int_{\mathbb{S}^{n-1}} e^{i\lambda(x+y) \cdot \omega} d\sigma(\omega).
\]

It is therefore clear that \( \tau_x \varphi_\lambda \) are eigenfunctions of \( \Delta + \rho^2 \) with eigenvalues \( \lambda^2 + \rho^2 \) and that \( \tau_x \varphi_\lambda \in L_p^p(\mathbb{R}^n) \) whenever \( \lambda \in \Lambda^0_p \) and \( p \neq 2 \). This follows from the estimate \( |e^{i\lambda x \cdot \omega}| \leq C e^{\Im(\lambda)} |x| \). For each \( 1 \leq j \leq 3 \) we set

\[
A_j = \{ \tau_x \varphi_\lambda(x) : x \in \mathbb{R}^n, \lambda \in A_j \}.
\]

It is then clear that \( A_j \subset L_p^p(\mathbb{R}^n) \) and that for \( f \in A_j \), \( T_t f = e^{-t\omega_c(\lambda)} f \).

We now recall certain results from the general theory of chaotic semigroups. Given a strongly continuous semigroup \( T_t \) on a Banach space \( \mathcal{B} \) the following three subsets of \( \mathcal{B} \) are important in detecting the chaotic behavior of \( T_t \):

\[
\mathcal{B}_0 = \{ x \in \mathcal{B} : \lim_{t \to \infty} T_t x = 0 \};
\]

\[
\mathcal{B}_\infty = \{ x \in \mathcal{B} : \forall \epsilon > 0 \exists w \in \mathcal{B} \text{ and } t > 0 \text{ such that } \|w\| < \epsilon \text{ and } \|T_t w - x\| < \epsilon \};
\]

\[
\mathcal{B}_{per} = \text{ the set of all periodic points of } T_t.
\]

**Theorem 2.7** (see [4], Theorem 2.3). Let \( \mathcal{B} \) be a separable Banach space at let \( T_t \) be a strongly continuous semigroup on \( \mathcal{B} \). If both \( \mathcal{B}_0 \) and \( \mathcal{B}_\infty \) are dense in \( \mathcal{B} \) then \( T_t \) is hypercyclic.
Corollary 2.8. Let $\mathcal{B}$ and $T_t$ be as above. If all $\mathcal{B}_0$, $\mathcal{B}_\infty$ and $\mathcal{B}_{Per}$ are dense in $\mathcal{B}$ then $T_t$ is chaotic.

Once we have the above Theorem 2.7 and Corollary 2.8, the sufficiency part of the Theorem 1.4 follows immediately from the next proposition. Recall that we are considering the Banach spaces $L^p_\rho(\mathbb{R}^n)$ and the semigroup $T^\rho_t = e^{-t(\Delta + \rho^2 - c)}$.

Proposition 2.9. For each $1 \leq j \leq 3$, $A_j$ is dense in $L^p_\rho(\mathbb{R}^n)$, $1 \leq p < \infty$, $p \neq 2$ and span$(A_1) \subset \mathcal{B}_0$, span$(A_2) \subset \mathcal{B}_\infty$ and span$(A_3) \subset \mathcal{B}_{Per}$ provided $c > c_p$.

Proof. We first prove the set inclusions span$(A_1) \subset \mathcal{B}_0$, span$(A_2) \subset \mathcal{B}_\infty$ and span$(A_3) \subset \mathcal{B}_{Per}$. When $\lambda \in A_1$, $\Re(\omega_c(\lambda)) > 0$ and hence for any $f \in$ span$(A_1)$, $T^\rho_t f = e^{-t\omega_c(\lambda)}f$ from which it is clear that $\lim_{t \to \infty} T^\rho_t f = 0$. If $g \in$ span$(A_2)$, then $g = \sum_{k=1}^{m} c_k \varphi_{\lambda_k}$ such that the map $\lambda \mapsto c_k \varphi_{\lambda_k}$ is holomorphic. Since $\Re(\omega_c(\lambda)) > 0$, for any $\epsilon > 0$ we can choose $t$ large enough so that $\|T^\rho_t f - g\|_{L^p_\rho(\mathbb{R}^n)} < \epsilon$. On the other hand $\|T^\rho_t f - g\|_{L^p_\rho(\mathbb{R}^n)} = 0$ and hence $g \in \mathcal{B}_\infty$. The proof of third inclusion is also easy. As it is similar to the case of symmetric spaces we leave the proof and refer to [13]. Now we need to prove the density of $\mathcal{B}_0$, $\mathcal{B}_\infty$ and $\mathcal{B}_{Per}$ which will follow once we prove that of $A_j$ in $L^p_\rho(\mathbb{R}^n)$. Suppose the span of $A_1$ is not dense in $L^p_\rho(\mathbb{R}^n)$, $1 < p < \infty$. As the dual of $L^p_\rho(\mathbb{R}^n)$ can be identified with $L^{p'}(\mathbb{R}^n, (\varphi_{\rho_p}(x))^{p'\gamma'} dx)$ (where $1/p + 1/p' = 1$), there exists $g \in L^{p'}(\mathbb{R}^n, (\varphi_{\rho_p}(x))^{p'\gamma'} dx)$ such that

$$\int_{\mathbb{R}^n} g(y)\tau_{\lambda}^\rho(\varphi)(y)dy = 0$$

for all $\lambda \in A_1$ and $x \in \mathbb{R}^n$. Note that $L^{p'}(\mathbb{R}^n, (\varphi_{\rho_p}(x))^{p'\gamma'} dx) \subset L^1(\mathbb{R}^n)$ for $1 < p' < \infty$ and hence the map

$$\lambda \mapsto \int_{\mathbb{R}^n} g(y)\tau_{\lambda}^\rho(\varphi)(y)dy$$

is a continuous function on $A_1^0$. Moreover, by Morera and Fubini, the map is holomorphic. Since $A_1$ is a nonempty open subset of $A_1^0$ it follows that

$$\int_{\mathbb{R}^n} g(y)\tau_{\lambda}^\rho(\varphi)(y)dy = 0$$

for all $\lambda \in A_1$; in particular, for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, we have $g * \varphi_{\lambda}(x) = \int_{\mathbb{R}^n} g(y)\tau_{-\lambda}(\varphi)(y)dy = 0$. In view of Fourier inversion formula, we have

$$g(x) = \int_{0}^{\infty} g * \varphi_{\lambda}(x)\lambda^{-n-1}d\lambda = 0.$$  

When $p = 1$ we have a bounded function $g_1$ such that

$$\int_{\mathbb{R}^n} g_1(y)\tau_{\lambda}^\rho(\varphi)(y)\tilde{K}_{n/2}(y)dy = 0.$$
for all $\lambda \in \Lambda_p$. Since the function $g(y) = g_1(y)k_{n/2}(y)$ belongs to $L^1(\mathbb{R}^n)$ we can conclude that $g_1 = 0$ as before. This proves the density of span of $\mathcal{A}_1$. The density of span of $\mathcal{A}_2$ and $\mathcal{A}_3$ are similarly proved. \hfill \Box

**Proof of the Theorem 1.4** For $1 \leq p < \infty$ and $c \in \mathbb{R}$ the semigroup $T^c_t = e^{-t(\Delta + \rho^2 - c)}$ is strongly continuous on $L^p_\rho(\mathbb{R}^n)$. Therefore, in view of the Corollary 2.8 and Proposition 2.9, $T^c_t$ is chaotic on $L^p_\rho(\mathbb{R}^n)$ for $c > c_p$ and $1 \leq p < \infty$. This proves the part (1) of Theorem 1.4. Now we proceed to prove part (2). We note that for any $c \in \mathbb{R}$, the semigroup $T^c_t = e^{-t(\Delta + \rho^2 - c)}$ is not hypercyclic on $L^\infty_\rho(\mathbb{R}^n)$, since for any $f \in L^\infty_\rho(\mathbb{R}^n)$, $T^c_tf$ is a continuous bounded function and hence the closure of the orbit $\{T^c_tf : t > 0\}$ in $L^\infty_\rho(\mathbb{R}^n)$ is a subset of the subspace of all continuous bounded functions which is strictly contained in $L^\infty_\rho(\mathbb{R}^n)$. This proves part (2) of Theorem 1.4. For part (3), we make use of Theorem 2.10 according to which the spectrum $\sigma_p(\Delta + \rho^2 - c)$ of the operator $(\Delta + \rho^2 - c)$ on $L^p_\rho(\mathbb{R}^n)$ is given by

$$\mathcal{P}_p - c = \{\lambda^2 + \rho^2 - c : |\Im(\lambda)| \leq \gamma_p\rho\}$$

for $1 \leq p < \infty$. By the geometric form of the above set, it can be easily seen that the set $\sigma_p(\Delta + \rho^2 - c) \cap i\mathbb{R}$ has at most one point for $c \leq c_p$ and hence in view of Theorem 1.1, $T^c_t$ is not chaotic. If $c < c_p$, the operators $T^c_t$ are uniformly bounded in $t$, as they satisfy the estimates

$$\|T^c_tf(x)\|_{L^p_\rho(\mathbb{R}^n)} \leq C(1 + t)^{n-1}(1+\gamma_p)e^{-t(\rho^2-c)}\|f(x)\|_{L^p_\rho(\mathbb{R}^n)}$$

for all $f \in L^p_\rho(\mathbb{R}^n)$ and $1 \leq p < \infty$. Consequently, for each $f \in L^p_\rho(\mathbb{R}^n)$ and $c < c_p$ the orbit $\{T^c_tf : t > 0\}$ is a bounded subset of $L^p_\rho(\mathbb{R}^n)$ and hence it cannot be dense in $L^p_\rho(\mathbb{R}^n)$. This proves that $T^c_t$ is not hypercyclic on $L^p_\rho(\mathbb{R}^n)$ for $c < c_p$ and $1 \leq p < \infty$ which completes the Theorem 1.4.

**Proof of the Theorem 1.3** Parts (a) and (b) of Theorem 1.3 are restatements of Theorem 1.1. Indeed, as we have already noted the chaotic behaviour of $T^c_t$ on $L^p_\rho(\mathbb{R}^n)$ is the same as that of $\tilde{T}^c_t$ on $L^p(\mathbb{R}^n, (\varphi_{\rho})(x))^2dx$ as long as $2 < p < \infty$. Thus Theorem 2.14 leads to the estimate

$$\|\tilde{T}^c_tf(x)\|_{L^p(\mathbb{R}^n, (\varphi_{\rho})(x))^2dx} \leq C(1 + t)^{n-1}(1+\gamma_p)e^{-\frac{4\lambda^2}{mp^2}}\|f(x)\|_{L^p(\mathbb{R}^n, (\varphi_{\rho})(x))^2dx}.$$ 

Similarly, from Theorem 2.10 it follows that the spectrum of $\Delta + \rho^2$ on $L^p_\rho(\mathbb{R}^n)$ is the same as the spectrum of $\tilde{\Delta} + \rho^2$ on $L^p(\mathbb{R}^n, (\varphi_{\rho})(x))^2dx$ for all $p > 2$. Thus the proof of parts (a) and (b) of Theorem 1.3 is almost the same as that of Theorem 1.4.

In order to treat the case $1 \leq p \leq 2$ we make use of the following theorem proved in [4] (see Theorem 2.5 in [13]).

**Theorem 2.10.** Let $T_t$ be a hypercyclic semigroup generated by $A$ in a Banach space $B$. Then the adjoint $A^*$ of $A$ and the dual semigroup $\tilde{T}^*_t$ on the dual space $B^*$ have the following properties: (a) The point spectrum $\sigma_{p\rho}(A^*)$
of $A^*$ is empty. (b) For any nonzero $\phi \in B^*$, the orbit $\{T^*_t \phi : t > 0\}$ is unbounded.

Recalling the definition of $\tilde{\Delta}$ we see that

$$\int_{\mathbb{R}^n} \tilde{\Delta} f(x)g(x)(\varphi_{i\rho}(x))^2dx = \int_{\mathbb{R}^n} f(x)\Delta g(x)(\varphi_{i\rho}(x))^2dx$$

which means that $\tilde{\Delta} + \rho^2$ is selfadjoint. If $T^*_t$ was chaotic on $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2dx)$ for $1 \leq p < 2$, then the point spectrum of $\tilde{\Delta} + \rho^2$ on $L^p$ should be empty. But this is not the case as $\varphi_{i\rho}$ with $|\Im(\lambda)| < p'\gamma_{\nu'}$ belongs to $L^{p'}(\mathbb{R}^n, (\varphi_{i\rho}(x))^2dx)$ is an eigenfunction of $\tilde{\Delta} + \rho^2$. For $p = 2$, the behaviour of $e^{-t(\Delta + \rho^2-c)}$ on $L^2(\mathbb{R}^n, (\varphi_{i\rho}(x))^2dx)$ is equivalent to the behaviour of $e^{-t(\Delta + \rho^2-c)}$ on $L^2(\mathbb{R}^n, dx)$. In [13] the authors have studied the latter semigroup and hence our results follow from theirs.

In order to show that $\tilde{T^*_t}$ has no periodic points in $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2dx)$ for $1 \leq p \leq 2$, assume, on the contrary that there is a nontrivial $f \in L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2dx)$ such that $\tilde{T^*_t}f = f$ for some $t = t_0 > 0$. This means that $g = f\varphi_{i\rho}$ which belongs to $L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2dx)$ is periodic for $e^{-t(\Delta + \rho^2-c)}$: that is, $e^{-t_0(\Delta + \rho^2-c)}g = g$. Since we are in the case $1 \leq p \leq 2, L^p(\mathbb{R}^n, (\varphi_{i\rho}(x))^2dx) \subset L^1(\mathbb{R}^n, dx)$ and hence by taking Fourier transform we obtain $(1 - e^{-t_0\omega(\lambda)})\hat{g}(\lambda \omega) = 0$ for all $\lambda > 0$ and $\omega \in S^{n-1}$. But then $\hat{g}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}^n$ which is a contradiction. This completes the proof of Theorem [13]

3. Chaotic behavior of the Dunkl heat semigroup

3.1. Coxeter groups and Dunkl operators: In this subsection we recall some definitions given in Introduction and we give some more preliminaries about Dunkl theory. Let $G$ be a Coxeter group (finite reflection group) associated to a fixed root system $R$ in $\mathbb{R}^n, n \geq 2$. We use the notation $\langle ., . \rangle$ for the standard inner product on $\mathbb{R}^n$ and $|x|^2 = \langle x, x \rangle$. We assume that the reader is familiar with the notion of finite reflection groups associated to root systems. Given a root system $R$ we define the reflections $\sigma_\nu, \nu \in R$ by

$$\sigma_\nu x = x - 2 \frac{\langle \nu, x \rangle}{|\nu|^2} \nu.$$  

Then $G$ is a subgroup of the orthogonal group generated by the reflections $\sigma_\nu, \nu \in R$. A function $\kappa$ defined on $R$ is called a multiplicity function if it satisfies $\kappa(g\nu) = \kappa(\nu)$ for every $g \in G$. We assume that our multiplicity function $\kappa$ is non negative.

In [5] Dunkl defined a family of first order differential-difference operators $T_j$ (which we call Dunkl operators) that play the role of partial differentiation
for the reflection group structure. Dunkl operators $T_j$ are defined by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\nu \in R_+} \kappa(\nu) \nu_j \frac{f(x) - f(\sigma_\nu x)}{(\nu, x)}$$

for $j = 1, 2, \cdots, n$, where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)$ and $R_+$ is the set of all positive roots in $R$. These operators map $\mathcal{P}_m^n$ to $\mathcal{P}_{m-1}^n$, where $\mathcal{P}_m^n$ is the space of homogeneous polynomials of degree $m$ in $n$ variables. More importantly, these operators mutually commute; that is $T_i T_j = T_j T_i$.

Recall that the Dunkl-Laplacian $\Delta_\kappa$ is defined to be the operator

$$\Delta_\kappa = \sum_{j=1}^d T_j^2$$

which can be explicitly calculated, see Theorem 4.4.9 in Dunkl-Xu [6]. The Dunkl Laplacian reduces to the standard Laplacian $\Delta$ when $\kappa = 0$. For all these facts we refer to Dunkl [5] and Dunkl-Xu [6]. The weight function $h_\kappa^2(x)$ associated to the group $G$ and the multiplicity function $\kappa$ is defined by

$$h_\kappa^2(x) = \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{2\kappa(\nu)}, \; x \in \mathbb{R}^n.$$  

Note that $h_\kappa^2(x)$ is a positive homogeneous function of degree $2\gamma$ where $\gamma = \sum_{\nu \in R_+} \kappa(\nu)$. We consider $L^p$ spaces defined with respect to the measure $h_\kappa^2(x) dx$. There exists a kernel $E_\kappa(x, \xi)$ which is a joint eigenfunction for all $T_j$:

$$T_j E_\kappa(x, \xi) = \xi_j E_\kappa(x, \xi).$$

This is the analogue of the exponential $e^{\langle x, \xi \rangle}$ and Dunkl transform is defined in terms of $E_\kappa(ix, \xi)$.

For $f \in L^1(\mathbb{R}^n, h_\kappa(x)^2 dx)$ we define the Dunkl transform of $f$ by

$$\mathcal{F}_\kappa f(\xi) = \int_{\mathbb{R}^n} f(x) E_\kappa(-ix, \xi) h_\kappa(x)^2 dx.$$ 

The Dunkl transform shares many important properties with the Fourier transform. For example, we have the Plancherel theorem

$$\int_{\mathbb{R}^n} |\mathcal{F}_\kappa f(\xi)|^2 h_\kappa(\xi)^2 d\xi = c_n \int_{\mathbb{R}^n} |f(x)|^2 h_\kappa(x)^2 dx$$

for all $f \in L^1 \cap L^2(\mathbb{R}^n, h_\kappa(x)^2 dx)$ and the inversion formula

$$f(x) = c_n \int_{\mathbb{R}^n} \mathcal{F}_\kappa f(\xi) E_\kappa(ix, \xi) h_\kappa(\xi)^2 d\xi$$

for all $f \in L^1(\mathbb{R}^n, h_\kappa(x)^2 dx)$ provided $\mathcal{F}_\kappa f$ is also in $L^1(\mathbb{R}^n, h_\kappa(x)^2 dx)$. In this paper we also make use of some properties of the Dunkl kernel $E_\kappa(x, \xi)$. For example we require $E_\kappa(\lambda x, \xi) = E_\kappa(x, \lambda \xi)$ for any $\lambda \in \mathbb{C}$ and also the
estimate $|E_\kappa(x,\xi)| \leq e^{\|x||\xi||}$ for all $x, \xi \in \mathbb{R}^n$. We refer to [6] for all these and more on Dunkl transform.

### 3.2. Dunkl heat semigroup on weighted $L^p$ spaces.

In [14] and [15], Rösler has studied the heat equation associated to the Dunkl Laplacian, viz.

$$\frac{\partial}{\partial t} u(x,t) = \Delta_\kappa u(x,t), \quad u(x,0) = f(x), \quad t > 0, x \in \mathbb{R}^n.$$  

The solution of this equation, for $f \in L^p(\mathbb{R}^n, h^2_\kappa dx)$ is given by

$$u(x,t) = \int_{\mathbb{R}^n} \Gamma_\kappa(t, x, y) f(y) h^2_\kappa(y) dy$$  

where $\Gamma_\kappa$ is the heat kernel associated to $\Delta_\kappa$. The kernel $\Gamma_\kappa$ is explicitly known and is given by

$$\Gamma_\kappa(t, x, y) = \frac{M_\kappa}{t^n(2\pi)^{\gamma}} e^{-\frac{1}{4t}(|x|^2 + |y|^2)} E_\kappa \left( \frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right).$$

We collect some important properties of this kernel in the following lemma.

**Lemma 3.1** (Rösler).

1. $\Gamma_\kappa(t, x, y) = c_\kappa^{-2} \int_{\mathbb{R}^n} e^{-t|x|^2} E_\kappa(ix, \xi) E_\kappa(-iy, \xi) h^2_\kappa(\xi) d\xi$

2. $\int_{\mathbb{R}^n} \Gamma_\kappa(t, x, y) h^2_\kappa(y) dy = 1$

3. $\Gamma_\kappa(t + s, x, y) = \int_{\mathbb{R}^n} \Gamma_\kappa(t, x, z) \Gamma_\kappa(s, y, z) h^2_\kappa(z) dz$. 

In view of these properties, it is not difficult to show that the family of operators $H_t$, $t > 0$ defined on $L^p(\mathbb{R}^n, h^2_\kappa dx)$ by

$$H_t f(x) = \int_{\mathbb{R}^n} \Gamma_\kappa(t, x, y) f(y) h^2_\kappa(y) dy$$

forms a strongly continuous semigroup on $L^p(\mathbb{R}^n, h^2_\kappa dx)$, $1 \leq p < \infty$. Indeed, this has been proved in [20]. Thus for $f \in L^p(\mathbb{R}^n, h^2_\kappa dx)$, $1 \leq p < \infty$, $H_t f$ converges to $f$ in the norm as $t \to 0$. In this article we are interested in the semigroup $T_t$ generated by $\Delta_\kappa + \rho^2$ on the spaces $L^p_{\rho,\kappa}(\mathbb{R}^n)$. Note that $T_t f(x) = e^{-t\rho^2} H_t f(x)$ is well defined even when $f \in L^p_{\rho,\kappa}(\mathbb{R}^n, dx)$ as the integral defining $H_t f$ converges.

We define spherical functions in the Dunkl set up by the equation

$$\varphi_{\lambda,\kappa}(x) = \int_{S_{n-1}} E_\kappa(ix, \lambda \omega) h^2_\kappa(\omega) d\sigma(\omega)$$

for $\lambda \in \mathbb{C}$. These are all eigenfunctions of the Dunkl-Laplacian with eigenvalue $\lambda^2$; $\Delta_\kappa \varphi_{\lambda,\kappa} = \lambda^2 \varphi_{\lambda,\kappa}$. For $\lambda \in \mathbb{C}$, $\varphi_{\lambda,\kappa}$ has exponential growth. Indeed, 

$$\varphi_{\lambda,\kappa}(x) = c_n J_{\frac{\gamma - 1}{2}}(\lambda \rho \sqrt{t}) \left( \frac{\lambda \rho}{\sqrt{t}} \right)^{\frac{\gamma - 1}{2}}$$
where $J_\alpha(t)$ is the Bessel function of type $\alpha$. It can be easily proved that $\varphi_{\lambda,\kappa} \in L^p_{\rho,\kappa}(\mathbb{R}^n)$ for $|\Im(\lambda)| < \gamma p \rho$ and $p \neq 2$.

**Theorem 3.2.** For each $1 \leq p < \infty$, $T_t$, $t > 0$ defines a strongly continuous semigroup on $L^p_{\rho,\kappa}(\mathbb{R}^n)$. Moreover, for any $f \in L^p_{\rho,\kappa}(\mathbb{R}^n)$, $1 \leq p \leq 2$,

$$
\|T_t f\|_{L^p_{\rho,\kappa}(\mathbb{R}^n)} \leq C t^{(n+2\gamma-1)/2} e^{-\frac{2p^2}{p^2}t} \|f\|_{L^p_{\rho,\kappa}(\mathbb{R}^n)}
$$

whereas for $p > 2$

$$
\|T_t f\|_{L^p_{\rho,\kappa}(\mathbb{R}^n)} \leq C t^{(n+2\gamma-1)/2} e^{-\frac{2p^2}{p^2}t} \|f\|_{L^p_{\rho,\kappa}(\mathbb{R}^n)}.
$$

**Proof.** First note that the dual space of $L^p_{\rho,\kappa}(\mathbb{R}^n)$ can be identified with $L^{p'}(\mathbb{R}^n, (\varphi_{i\rho,\kappa}(x))^{\rho e^{ip\rho}} h^2_{\kappa}(x)dx)$ and with this identification the operator $T_t$ will be self adjoint. In view of the asymptotic behaviour of the Macdonald function (and Bessel function), it is enough to consider the space defined using $(1 + |x|)^{(n+2\gamma-1)/2} e^{-\rho |x|}$ (respectively $(1 + |x|)^{-(n+2\gamma-1)/2} e^{\rho |x|}$) in place of $\tilde{K}_{n/2+\gamma}(\rho |x|)$ (respectively $(\varphi_{i\rho,\kappa}(x))^{\rho e^{ip\rho}}$). For the sake of brevity, just for this section, we denote the weight function $(1 + |x|)^{(n+2\gamma-1)/2} e^{-\rho |x|}$ by $w_{\rho,\kappa}(x)$.

We first consider the case $1 \leq p \leq 2$ for which we prove the above estimates for the semigroup $T_t$ on both weighted $L^p$-spaces $L^p_{\rho,\kappa}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n, (\varphi_{i\rho,\kappa}(x))^{\rho e^{ip\rho}} h^2_{\kappa}(x)dx)$. We make use of these estimates and duality to prove the required estimates for $p > 2$. Since

$$
T_t f(x) = e^{-tp^2} \int_{\mathbb{R}^n} F_{\rho,\kappa} f(x)e^{-t|x|^2} E_{\rho,\kappa}(ix, \xi) h^2_{\kappa}(\xi)d\xi
$$

it is clear that

$$
\|T_t f\|_2 \leq C e^{-tp^2} \|f\|_2, \ \ f \in L^2(\mathbb{R}^n, h^2_{\kappa}(x)dx).
$$

Since $L^2_{\rho,\kappa}(\mathbb{R}^n) = L^2(\mathbb{R}^n, h^2_{\kappa}(x)dx) = (L^2_{\rho,\kappa}(\mathbb{R}^n))^*$ the required estimate is true for $p = 2$. To prove the result for $p = 1$ we recall that

$$
T_t f(x) = e^{-tp^2} \int_{\mathbb{R}^n} \Gamma_{\kappa}(t, x, y)f(y) h^2_{\kappa}(y)dy
$$

where $\Gamma_{\kappa}(t, x, y)$ is the heat kernel defined in (3.1). As $\gamma_1 = 1$ we need to show that

$$
\sup_{y \in \mathbb{R}^n} (w_{\rho,\kappa}(y))^{\pm 1} \int_{\mathbb{R}^n} \Gamma_{\kappa}(t, x, y)(w_{\rho,\kappa}(x))^{\pm 1} h^2_{\kappa}(x)dx \leq C t^{(n+2\gamma-1)} e^{tp^2}.
$$

We consider the case of $L^1_{\rho,\kappa}(\mathbb{R}^n)$; the treatment of $L^1(\mathbb{R}^n, \varphi_{i\rho,\kappa}(x)h^2_{\kappa}(x)dx)$ is similar.
Since the heat kernel $\Gamma_\kappa(t, x, y)$ satisfies $\int_{\mathbb{R}^n} \Gamma_\kappa(t, x, y)h^2(x)dx = 1$ we immediately see that

$$(1+|y|)^{-(n+2\gamma-1)/2}e^{\rho|y|}\int_{|x|\geq|y|} \Gamma_\kappa(t, x, y)(1+|x|)^{(n+2\gamma-1)/2}e^{-\rho|x|}h^2(x)dx \leq C.$$  

In order to treat the remaining part of the integral, we make use of the explicit expression for $\Gamma_\kappa(t, x, y)$, viz.

$$\Gamma_\kappa(t, x, y) = M_\kappa t^{-N/2}e^{-\frac{x}{\sqrt{2t}}}(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}})$$

where we have written $N = n + 2\gamma$. Integrating in polar coordinates and making use of the formula (see Proposition 2.3 in [20])

(3.2) $$\int_{S^{n-1}} E_\kappa(rx', sy')h^2_n(x')d\sigma(x') = c_N \frac{J_{\frac{N}{2}-1}(irs)}{(irs)^{\frac{N}{2}-1}}$$

we need to estimate

$$t^{-\frac{N}{2}}(1+s)^{-\frac{N+1}{2}}e^{\rho s} \int_0^s e^{-\frac{1}{4}(r^2+s^2)} \frac{J_{\frac{N}{2}-1}(irs)}{(irs)^{\frac{N}{2}-1}}(1+r)^{\frac{N-1}{2}} e^{-\rho r} r^{N-1} dr.$$  

In view of the Poisson integral representation of Bessel functions, viz.

$$J_\alpha(t) = \left(\frac{t}{2}\right)^\alpha \int_{-1}^1 e^{\lambda t}(1-u^2)^{-\frac{1}{2}}du$$

we need to estimate the integral

$$t^{-\frac{N}{2}}(1+s)^{-\frac{N+1}{2}}e^{\rho s} \int_0^s e^{-\frac{1}{4}(r^2+s^2-2rsu)}(1+r)^{\frac{N-1}{2}} e^{-\rho r} r^{N-1} dr.$$  

Note that $(s-r)^2 = r^2 + s^2 - 2rs \leq r^2 + s^2 - 2rsu$ for any $-1 \leq u \leq 1$. Consequently, as $s \geq r$, $(s-r) \leq (r^2 + s^2 - 2rsu)^{\frac{1}{2}}$ and $(1+s) \geq (1+r)$, the above integral is bounded by

$$t^{-\frac{N}{2}} \int_0^s e^{-\frac{1}{4}(r^2+s^2-2rsu)} e^{\rho(r^2+s^2-2rsu)^{\frac{1}{2}}} r^{N-1} dr$$

$$\leq t^{-\frac{N}{2}} \int_0^\infty e^{-\frac{1}{4}(r^2+s^2-2rsu)} e^{\rho(r^2+s^2-2rsu)^{\frac{1}{2}}} r^{N-1} dr.$$  

Thus, the required integral is bounded by

$$t^{-\frac{N}{2}} \int_0^\infty \int_{-1}^1 (1-u^2)^{-\frac{N+1}{2}} e^{-\frac{1}{4}(r^2+s^2-2rsu)} e^{\rho(r^2+s^2-2rsu)^{\frac{1}{2}}} r^{N-1} du dr.$$  

The inner integral is the generalised Euclidean translation of $e^{-\frac{1}{4}r^2} e^{\rho r}$. As the $L^1$-norm is preserved by such a translation, the above is bounded by

$$t^{-\frac{N}{2}} \int_0^\infty e^{-\frac{1}{4}r^2} e^{\rho r} r^{N-1} dr$$

which can be easily seen to be bounded by $Ce^{\rho^2/4} t^{N-1/2}$.
We can now appeal to Stein-Weiss interpolation theorem (see in [18]) to prove the result for \(1 \leq p \leq 2\). Indeed, we have
\[
\int_{\mathbb{R}^n} |T_t f(x)|(w_{p,\kappa}(x))^{\frac{p}{2}}h^2_{\kappa}(x)dx \leq C t^{\frac{N}{2}} \int_{\mathbb{R}^n} |f(x)|(w_{p,\kappa}(x))^{\frac{p}{2}}h^2_{\kappa}(x)dx
\]
and also
\[
\left( \int_{\mathbb{R}^n} |T_t f(x)|^2 h^2_{\kappa}(x)dx \right)^{\frac{1}{2}} \leq C e^{-\nu^2} \left( \int_{\mathbb{R}^n} |T_t f(x)|^2 h^2_{\kappa}(x)dx \right)^{\frac{1}{2}}.
\]
Interpolation of these two estimates give us
\[
\left( \int_{\mathbb{R}^n} |T_t f(x)|^p (w_{p,\kappa}(x))^{\frac{p}{2}} h^2_{\kappa}(x)dx \right)^{\frac{1}{p}} \leq C t^{\frac{N-1}{2}} e^{-\frac{2p^2}{\nu^2}} \left( \int_{\mathbb{R}^n} |f(x)|^p (w_{p,\kappa}(x))^{\frac{p}{2}} h^2_{\kappa}(x)dx \right)^{\frac{1}{p}}
\]
which is the required inequality for \(1 \leq p \leq 2\). In order to prove Theorem 3.2 when \(p > 2\) we use duality. Observe that
\[
\int_{\mathbb{R}^n} T_t f(x)g(x)h^2_{\kappa}(x)dx = \int_{\mathbb{R}^n} f(x)T_t g(x)h^2_{\kappa}(x)dx.
\]
Writing the right hand side as
\[
\int_{\mathbb{R}^n} f(x)(w_{p,\kappa}(x))^{\gamma_p} T_t g(x)(w_{p,\kappa}(x))^{-\gamma_p} h^2_{\kappa}(x)dx
\]
and applying Hölder’s inequality we get
\[
\left| \int_{\mathbb{R}^n} T_t f(x)g(x)h^2_{\kappa}(x)dx \right| \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p (w_{p,\kappa}(x))^{\gamma_p} h^2_{\kappa}(x)dx \right)^{\frac{1}{p}} \times \left( \int_{\mathbb{R}^n} |T_t g(x)|^{p'} (w_{p,\kappa}(x))^{-\gamma_p} h^2_{\kappa}(x)dx \right)^{\frac{1}{p'}}.
\]
For \(p > 2\), \(p' < 2\) and hence by what we have already proved and the fact that \(\gamma_p = \gamma_{p'}\), we get
\[
\left| \int_{\mathbb{R}^n} T_t f(x)g(x)h^2_{\kappa}(x)dx \right| \leq C t^{\frac{N-1}{2}} e^{-\frac{2p^2}{\nu^2}} \|f\|_{L^p_{\rho,\kappa}(\mathbb{R}^n)} \|g\|_{(L^p_{\rho,\kappa}(\mathbb{R}^n))^*}.
\]
Taking supremum over all \(g \in (L^p_{\rho,\kappa}(\mathbb{R}^n))^*\) we obtain the required estimate.

The strong continuity of \(T_t\) on \(L^p_{\rho,\kappa}(\mathbb{R}^n)\) follows from the norm estimates. Indeed, for \(0 < t \leq 1\), the operators \(T_t\) are uniformly bounded on \(L^p_{\rho,\kappa}(\mathbb{R}^n)\), \(1 \leq p < \infty\). As \(L^p(\mathbb{R}^n, h^2_{\kappa}(x)dx)\) is dense in \(L^p_{\rho,\kappa}(\mathbb{R}^n)\), the strong continuity of \(T_t\) on \(L^p_{\rho,\kappa}(\mathbb{R}^n)\) follows from the same on \(L^p(\mathbb{R}^n, h^2_{\kappa}(x)dx)\) in view of the continuous inclusion \(L^p(\mathbb{R}^n, h^2_{\kappa}(x)dx) \subset L^p_{\rho,\kappa}(\mathbb{R}^n)\). The proof of strong continuity of \(T_t\) on \(L^p(\mathbb{R}^n, h^2_{\kappa}(x)dx)\) was given in Theorem 5.3 of [20].
We now turn our attention to the Dunkl heat semigroup on the weighted mixed norm space \( L_{p, \kappa}^{0, 2}(\mathbb{R}^n) \). The semigroup \( T_t \) can be extended to the space \( L_{p, \kappa}^{0, 2}(\mathbb{R}^n) \). In fact we will show below that the weighted mixed norm (\( L_{p, \kappa}^{0, 2}(\mathbb{R}^n) \)-norm) estimate of \( T_t f \) can be reduced to a vector valued inequality for a sequence of Bessel semigroups of different types.

The Bessel semigroup \( B_t^\alpha \) of type \( \alpha \) is initially defined on \( L^2(\mathbb{R}^+, r^{2\alpha+1} dr) \) by
\[
B_t^\alpha f(r) = \int_0^\infty f(s) b_t^\alpha(r, s) s^{2\alpha+1} ds
\]
where the kernel \( b_t^\alpha(r, s) \) is given by
\[
b_t^\alpha(r, s) = (2t)^{-1} e^{-\frac{1}{4t}(r^2+s^2)} (rs)^{-\alpha} J_\alpha \left( \frac{ir}{2t} \right)
\]
where \( J_\alpha \) is the standard Bessel function of type \( \alpha \) of first kind.

We can identify \( L_{p, \kappa}^{0, 2}(\mathbb{R}^n) \) with \( L^p(\mathbb{R}^+, \mathcal{H}) \), \((\tilde{K}_{n/2+\gamma}(pr))^{p\gamma r^{n+2\gamma-1} dr}\) where \( \mathcal{H} = L^2(S^{n-1}, h_\kappa^2(\omega) d\sigma(\omega)) \) and \( \tilde{K}_{n/2+\gamma}(pr) \) is the orthogonal direct sum of the finite dimensional spaces \( \mathcal{H}_{m}^h \) consisting of h-harmonics. These are analogues of spherical harmonics and defined using \( \Delta_\kappa \) in place \( \Delta \). A homogeneous polynomial \( P(x) \) is said to be a solid h-harmonic if \( \Delta_\kappa P(x) = 0 \). Restrictions of such solid harmonics to \( S^{n-1} \) are called spherical h-harmonics. The space \( L^2(S^{n-1}, h_\kappa^2(\omega) d\sigma(\omega)) \) is the orthogonal direct sum of the finite dimensional spaces \( \mathcal{H}_{m}^h \) consisting of h-harmonics of degree \( m \). We can choose an orthonormal basis \( \{Y_{m,j}^h: j = 1, 2, \ldots, d(m), \ d(m) = dim(\mathcal{H}_m^h)\} \) for \( \mathcal{H}_m^h \) so that the collection \( \{Y_{m,j}^h: j = 1, 2, \ldots, d(m), \ m = 0, 1, 2, \ldots\} \) is an orthonormal basis for \( L^2(S^{n-1}, h_\kappa^2(\omega) d\sigma(\omega)) \).

If \( f \in L_{p, \kappa}^{0, 2}(\mathbb{R}^n) \), then \( f(r \cdot) \in L^2(S^{n-1}, h_\kappa^2(\omega) d\sigma(\omega)) \) for almost every \( r \). Hence we have the following h-harmonic expansion: for a.e. \( r > 0 \),
\[
f(r \omega) = \sum_{m=0}^\infty \sum_{j=1}^{d(m)} f_{m,j}(r) Y_{m,j}^h(\omega)
\]
where \( f_{m,j}(r) = \int_{S^{n-1}} f(r \omega) Y_{m,j}^h(\omega) h_\kappa^2(\omega) d\sigma(\omega) \) are the spherical harmonic coefficients of \( f \). In view of Plancherel formula, we also have the following expression for \( \|f\|_{L_{p, \kappa}^{0, 2}(\mathbb{R}^n)} \):
\[
\left( \int_0^\infty \left( \sum_{m=0}^\infty \sum_{j=1}^{d(m)} |f_{m,j}(r)|^2 \right)^{\frac{p}{2}} \tilde{K}_{n/2+\gamma}(pr)^{p\gamma r^{n+2\gamma-1} dr} \right)^{\frac{1}{p}}.
\]
With the above notations, the following proposition gives the relation between the Dunkl heat semigroup and the Bessel semigroups.

**Proposition 3.3.** For $1 \leq p < \infty$, let $T_t$ be the semigroup generated by $A = \Delta_\kappa + \rho^2$ and $f \in L_{p,\kappa}^2(\mathbb{R}^n)$. Then we have
\[
\int_{S^{n-1}} T_t f(r\omega) Y_{m,j}^h(\omega) h^2_\kappa(\omega) d\sigma(\omega) = c_{n,\kappa} e^{-t\rho^2} r^m B_t^{\frac{N}{2}+m-1}(\tilde{f}_{m,j})(r)
\]
for $m = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, d(m)$ where $\tilde{f}_{m,j}(r) = r^{-m} f_{m,j}(r)$.

**Proof.** To prove the proposition, we make use of the following formula
\[
\int_{S^{n-1}} E_{\kappa}(x,y) Y_{m,j}^h(x') h^2_\kappa(x') d\sigma(x') = c_{n,\kappa} (|x| |y|)^{-\left(\frac{N}{2}-1\right)} J_{\frac{N}{2}+m-1}(i|x| |y|) Y_{m,j}^h(y).
\]

In view of (3.1) and (3.6), we have
\[
\int_{S^{n-1}} \Gamma_\kappa(t, r\omega, s\eta) Y_{m,j}^h(\omega) h^2_\kappa(\omega) d\sigma(\omega)
= M_\kappa t^{\frac{N}{2}} e^{-t\rho^2} e^{-\frac{1}{\pi}(r^2+s^2)} \int_{S^{n-1}} E_{\kappa}(\frac{r\omega}{\sqrt{2t}}, \frac{s\eta}{\sqrt{2t}}) Y_{m,j}^h(\omega) h^2_\kappa(\omega) d\sigma(\omega)
= c_{n,\kappa} e^{-t\rho^2} (2t)^{-1} e^{-\frac{1}{\pi}(r^2+s^2)} (rs)^{-\left(\frac{N}{2}-1\right)} J_{\frac{N}{2}+m-1}(i/rs t^\frac{1}{2}) Y_{m,j}^h(\eta).
\]

Recalling the definition of $T_t f$ and making use of the above formula,
\[
\int_{S^{n-1}} T_t f(r\omega) Y_{m,j}^h(\omega) h^2_\kappa(\omega) d\sigma(\omega)
= c_{n,\kappa} e^{-t\rho^2} (2t)^{-1} \int_0^\infty e^{-\frac{1}{\pi}(r^2+s^2)} (rs)^{-\left(\frac{N}{2}-1\right)} J_{\frac{N}{2}+m-1}(i/rs t^\frac{1}{2}) f_{m,j}(s) s^{N-1} ds
= c_{n,\kappa} e^{-t\rho^2} r^m B_t^{\frac{N}{2}+m-1}(\tilde{f}_{m,j})(r).
\]

This proves the proposition. \qed

**Proposition 3.4.** For $1 \leq p < \infty$, let $T_t$ be the semigroup generated by $A = \Delta_\kappa + \rho^2$ and $f \in L_{p,\kappa}^2(\mathbb{R}^n)$. Then
\[
\| T_t f \|_{L_{p,\kappa}^2(\mathbb{R}^n)} \leq A(p, t) \| f \|_{L_{p,\kappa}^2(\mathbb{R}^n)}
\]
if and only if the vector-valued inequality holds:
\[
\left( \int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |r^m B_t^{\frac{N}{2}+m-1}(\tilde{f}_{m,j})(r)|^\frac{p}{\gamma} (\tilde{K}_{n/2+\gamma}(pr))^{p \gamma p r^{-N-1}} dr \right)^\frac{1}{p} \right)^\frac{1}{p}
\]
\[
\leq A(p, t) \left( \int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |f_{m,j}(r)|^\frac{p}{\gamma} (\tilde{K}_{n/2+\gamma}(pr))^{p \gamma p r^{-N-1}} dr \right)^\frac{1}{p} \right)^\frac{1}{p}.
\]
Here \( N = n + 2\gamma \), \( f_{m,j}(s) = s^{-m}f_{m,j}(s) \) and \( A(p,t) \) is a constant depending on \( p \) and \( t \).

**Proof.** With \( F = T_t f \) we use the h-harmonic expansion to get

\[
\|T_t f\|_{L^{p,2}(\mathbb{R}^n)} = \left( \int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |F_{m,j}(r)|^2 \right)^{\frac{2}{p}} (\tilde{K}_{N/2}(pr))^{p} pr^{-\gamma} dr \right)^{\frac{1}{p}}.
\]

Since \( F_{m,j}(r) = \int_{\mathbb{S}^{n-1}} T_t f(r\omega) Y^m_{m,j}(\omega) h_2^\gamma(\omega) d\sigma(\omega) \), in view of the previous proposition and the above, the norm \( \|T_t f\|_{L^{p,2}(\mathbb{R}^n)} \) is equal to

\[
c_n r^{-p} \left( \int_0^\infty \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} |r^m B_t^N \tilde{f}_{m,j}(r)|^2 \right)^{\frac{2}{p}} (\tilde{K}_{N/2}(pr))^{p} pr^{-\gamma} dr \right)^{\frac{1}{p}}
\]

which proves the proposition.

\[\square\]

In Theorem 3.2, we have obtained a bound for the operator norm of \( T_t \) on \( L^{p,2}_{\rho,\kappa}(\mathbb{R}^n) \) which is given by \( C t^{\frac{n+2\gamma-1}{2}\gamma} e^{-\frac{2\rho^2}{p}t} \) or \( C t^{\frac{n+2\gamma-1}{2}\gamma} e^{-\frac{2\rho^2}{p}t} \) depending on whether \( 1 \leq p \leq 2 \) or \( p > 2 \). This bound can be improved if we consider the heat semigroup on \( L^{p,2}_{\rho,\kappa}(\mathbb{R}^n) \) under the added assumption that \( 2\gamma \) is an integer.

**Theorem 3.5.** Let \( 1 \leq p < \infty \) and let \( 2\gamma \) be an integer. Then \( T_t \), \( t > 0 \) defines a strongly continuous semigroup on \( L^{p,2}_{\rho,\kappa}(\mathbb{R}^n) \). Moreover, for any \( f \in L^{p,2}_{\rho,\kappa}(\mathbb{R}^n) \) we have

\[
(3.7) \quad \|T_t f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)} \leq C(1 + t)^{\frac{n+2\gamma-1}{2}\gamma} e^{-\frac{4\rho^2}{p}t} \|f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)}.
\]

**Proof.** Let \( f(x) = f_0(r) Y(\omega) \) where \( x = r\omega, \omega \in \mathbb{S}^{n-1}, r = |x| \) and let \( Y(\omega) \) be a spherical h-harmonic of degree \( m \). In view of Proposition 3.3, we have

\[
T_t f(r\omega) = c_n, r^{-p} \left( \int_0^\infty B_t^N \tilde{f}_0(r) Y(\omega) \right)
\]

where \( \tilde{f}_0(r) = r^{-m} f_0(r) \). Note that \( \|T_t f - f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)} \) which is equal to the product of

\[
\left( \int_0^\infty \left| c_n, r^{-p} \left( \int_0^\infty B_t^N \tilde{f}_0(r) \right) - f_0(r)(\tilde{K}_{N/2}(pr))^{p} pr^{-\gamma} dr \right|^p \right)^{\frac{1}{p}}
\]

with \( \left( \int_{\mathbb{S}^{n-1}} |Y(\omega)|^p h_2^\gamma(\omega) d\sigma(\omega) \right)^{\frac{1}{p}} \), tends to 0 as \( t \to 0 \) by the strong continuity of \( T_t \) on \( L^{p,2}_{\rho,\kappa}(\mathbb{R}^n) \) for \( 1 \leq p < \infty \). This implies \( \|T_t f - f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)} \to 0 \).
as $t \to 0$, as $Y \neq 0$. Similarly, if $f \in L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$ is of the form

$$(3.8) \quad f(r\omega) = \sum_{m=0}^{M} \sum_{j=1}^{d(m)} f_{m,j}(r)Y_{m,j}(\omega)$$

where $M$ is a positive integer, then it follows that $\|T_tf - f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)} \to 0$ as $t \to 0$. Since the space of all such functions $f$ having the form (3.8) is dense in $L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$, once we prove that $T_t$ are uniformly bounded on $L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$ for $0 < t \leq 1$ it is immediate that $\|T_tf - f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)} \to 0$ as $t \to 0$ for every $f \in L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$ and hence $T_t$ is strongly continuous on $L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)$.

In view of Proposition 3.3, in order to prove the weighted mixed norm estimate (3.7), it is enough to prove the following vector-valued inequality

$$
\left( \int_{0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} r^{m} B_{j}^{N/2} + m - 1 \left( \tilde{f}_{m,j}(r) \right)^{2} \right)^{p/2} \left( \tilde{K}_{N/2}(r) \right)^{p} r^{N-1} dr \right)^{1/p}
\leq A(p,t) \left( \int_{0}^{\infty} \left( \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \left| f_{m,j}(r) \right|^{2} \right)^{p/2} \left( \tilde{K}_{N/2}(r) \right)^{p} r^{N-1} dr \right)^{1/p}
$$

where $A(p,t) = C(1 + t)^{\frac{n+2\gamma-1}{2} (1+\gamma p) - \frac{4\gamma^{2}}{pp} t}$. In view of the same Proposition 3.4, the above vector valued inequality will follow once we prove

$$
\|T_tf\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)} \leq C(1 + t)^{\frac{n+2\gamma-1}{2} (1+\gamma p) - \frac{4\gamma^{2}}{pp} t} \|f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)}
$$

for the standard heat semigroup $T_t = e^{-t(\Delta + \rho^2)}$ on $\mathbb{R}^N, N = n + 2\gamma$. This is the content of the next theorem.

**Theorem 3.6.** Let $1 \leq p < \infty$, $\Delta$ be the standard Laplacian on $\mathbb{R}^n$ and $T_t$ be the semigroup generated by $(\Delta + \rho^2)$. Then

$$
\|T_tf\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)} \leq C(1 + t)^{\frac{n+2\gamma-1}{2} (1+\gamma p) - \frac{4\gamma^{2}}{pp} t} \|f\|_{L^{p,2}_{\rho,\kappa}(\mathbb{R}^n)}.
$$

We obtain the above result as a consequence of the weighted norm estimate proved in Theorem 2.1. To this end, we make use of a transference result due to Rubio de Francia, see [16]. For given $k \in SO(N)$, the special orthogonal group, we define the rotation operator $g(k)$ by $g(k)f(x) = f(kx)$. For a given radial weight function $w$ consider weighted mixed norm space $L_{\rho}^{p,2}(\mathbb{R}^N)$ consisting of all functions $f$ for which the norms

$$
\|f\|_{L_{\rho}^{p,2}(\mathbb{R}^N)} = \left( \int_{0}^{\infty} \left( \int_{S^{N-1}} |f(r\omega)|^{2} h_{\rho}^{2}(\omega) d\sigma(\omega) \right)^{p/2} w(r)r^{N-1} dr \right)^{1/p}
$$

are finite. We claim that for any bounded linear operator $T$ acting on $L^p(\mathbb{R}^N, w(|x|)dx)$ which commutes with rotations, i.e. $T g(k) = g(k) T$ for
every $k \in SO(N)$, there exists a bounded linear operator $\tilde{T}$ on $L^p_w(\mathbb{R}^N)$ such that

$$\tilde{T} f = T f$$

for $f \in L^p_w(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, w(|x|)dx)$ and $\|\tilde{T}\|_{\text{op}} \leq \|T\|_{\text{op}}$. In order to prove this claim, we make use of an idea due to Rubio de Francia [16]. This method described briefly in [16] is based on an extension of a theorem of Marcinkiewicz and Zygmund as expounded in Herz and Riviere [16].

**Lemma 3.7.** Let $(G, \mu)$ and $(H, \nu)$ be arbitrary measure spaces and $S : L^p(G) \to L^p(G)$ a bounded linear operator. Then if $p \leq q \leq 2$ or $p \geq q \geq 2$, there exists a bounded linear operator $\tilde{S} : L^p(G; L^q(H)) \to L^p(G; L^q(H))$ with $\|\tilde{S}\| \leq \|S\|$ such that for $g \in L^p(G; L^q(H))$ of the form $g(x, \xi) = f(\xi)u(x)$ where $f \in L^p(G)$ and $u \in L^q(H)$ we have

$$(\tilde{S}g)(\xi, x) = (Sf)(\xi)u(x).$$

The idea of Rubio de Francia is as follows. Since $T : L^p(\mathbb{R}^N, w(|x|)dx) \to L^p(\mathbb{R}^N, w(|x|)dx)$ is a bounded linear operator, by the lemma of Herz and Riviere, there exists a bounded linear operator $\tilde{T}$ on $L^p(\mathbb{R}^N, H, w(|x|)dx)$, the space of all $H$ valued functions $F$ on $\mathbb{R}^N$ for which

$$\int_{\mathbb{R}^N} \left( \int_{K} |F(x)(k)|^2 dk \right)^{\frac{p}{2}} w(|x|)dx$$

are finite. Here $H$ is the Hilbert space $L^2(K)$, $K = SO(N)$ and $dk$ is the Haar measure on $K$. Moreover, the operator $T$ satisfies $(Tf)(x, k) = Tg(x)h(k)$ if $f(x, k) = g(x)h(k)$, $x \in \mathbb{R}^N$, $k \in SO(N)$. Given a function $f \in L^p(\mathbb{R}^N, w(|x|)dx)$ consider $\tilde{f}(x, k) = g(k)f(x) = f(kx)$. Then

$$\int_{\mathbb{R}^N} \left( \int_{K} |\tilde{f}(x, k)|^2 dk \right)^{\frac{p}{2}} w(|x|)dx$$

can be calculated as follows. If $x = r\omega, \omega \in S^{N-1}$, $\tilde{f}(x, k) = f(rk\omega)$ and hence

$$\int_{K} |\tilde{f}(x, k)|^2 dk = \int_{K/\omega} \left( \int_{K/\omega} |f(rk\omega)|^2 d\nu \right) d\nu$$

where $K_{\omega} = \{k \in K : k\omega = \omega\}$ is the isotropy subgroup of $K$, $d\nu$ is the Haar measure on $K_{\omega}$ and $d\mu$ is the $K_{\omega}$ invariant measure on $K/K_{\omega}$ which can be identified with $S^{N-1}$. Hence

$$\int_{K} |\tilde{f}(x, k)|^2 dk = c_N \int_{S^{N-1}} |f(r\omega)|^2 d\sigma(\omega).$$

Therefore,

$$\int_{\mathbb{R}^N} \left( \int_{K} |\tilde{f}(x, k)|^2 dk \right)^{\frac{p}{2}} w(|x|)dx = c'_N \int_{0}^{\infty} \left( \int_{S^{N-1}} |f(r\omega)|^2 d\sigma(\omega) \right)^{\frac{p}{2}} w(r)r^{N-1}dr.$$
and hence \(L^p_w(\mathbb{R}^N)\) can be considered as a subspace of \(L^p(\mathbb{R}^N, \mathcal{H}, w(|x|)dx)\) with the identification \(f \mapsto \hat{f}\) and it is invariant under the operator \(\hat{T}\). Since \(T\) commutes with rotations, i.e. \(Tg(k) = \sigma(k)T\) we see that
\[
\hat{T} \hat{f}(x, k) = T(\sigma(k)f)(x) = \sigma(k)(Tf)(x) = (Tf)(kx).
\]
The boundedness of \(\hat{T}\) on \(L^p(\mathbb{R}^N, \mathcal{H}, w(|x|)dx)\) gives
\[
\int_{\mathbb{R}^N} \left( \int_K |Tf(kx)|^2 dk \right)^{\frac{p}{2}} w(|x|)dx \leq C \int_{\mathbb{R}^N} \left( \int_K |f(kx)|^2 dk \right)^{\frac{p}{2}} w(|x|)dx
\]
which translates into the boundedness of the restriction of \(\hat{T}\) to the weighted mixed norm space \(L^p_w(\mathbb{R}^N)\). This proves our claim.

### 3.3. The point spectrum of \(\Delta_\kappa\) on \(L^p_{p,\kappa}(\mathbb{R}^n)\)

In this subsection we precisely determine the point spectrum of the Dunkl-Laplacian on the weighted spaces \(L^p_{p,\kappa}(\mathbb{R}^n)\). In the unweighted case, the spectrum of \(\Delta_\kappa\) turns out to be the half line \([0, \infty)\) for all \(p\). This follows from a multiplier theorem for the Dunkl transform proved in [2]. On the other hand we do not have a multiplier theorem on the weighted spaces \(L^p_{p,\kappa}(\mathbb{R}^n)\) for \(p \neq 2\). However, it is not difficult to determine the point-spectrum \(\sigma_{pt}(\Delta_\kappa + \rho^2)\) on these spaces.

**Theorem 3.8.** For any \(1 \leq p < \infty\) we have \(\sigma_{pt}(\Delta_\kappa + \rho^2) = \mathcal{P}_p^0\), the interior of \(\mathcal{P}_p\).

**Proof.** It is enough to prove that if \(f\) is an eigenfunction of \(\Delta_\kappa\) in \(L^p_{p,\kappa}(\mathbb{R}^n)\), \(1 \leq p < \infty\) with eigenvalue \(\lambda^2\), then \(|\lambda(\lambda)| < \gamma_p \rho\). If \(f\) is such an eigenfunction, then
\[
\int_{S^{n-1}} \Delta_\kappa f(r\omega)Y^h_{m,j}(\omega)h^2_\kappa(\omega)d\sigma(\omega) = \lambda^2 \int_{S^{n-1}} f(r\omega)Y^h_{m,j}(\omega)h^2_\kappa(\omega)d\sigma(\omega)
\]
for \(m = 0, 1, 2, \ldots, j = 1, 2, \ldots, d(m)\). Here \(Y^h_{m,j}\) is a spherical h-harmonic of degree \(m\) taken from the orthonormal basis \(\{Y^h_{m,j} : j = 1, 2, \ldots, d(m), m = 0, 1, 2, \ldots\}\) for \(L^2(S^{n-1}, h^2_\kappa(\omega)d\sigma(\omega))\). The Dunkl-Laplacian has the explicit form (see page 159 in [3])
\[
\Delta_\kappa = - \left( \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} + \frac{1}{r^2} \Delta_{h,0} \right)
\]
where \(\Delta_{h,0}\) is the spherical part of \(\Delta_\kappa\) and \(N = n + 2\gamma\). In view of this, we have
\[
\left( \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} + \lambda^2 \right) f_{m,j}(r) = - \frac{1}{r^2} \int_{S^{n-1}} \Delta_{h,0} f(r\omega)Y^h_{m,j}(\omega)h^2_\kappa(\omega)d\sigma(\omega)
\]
where
\[
f_{m,j}(r) = \int_{S^{n-1}} f(r\omega)Y^h_{m,j}(\omega)h^2_\kappa(\omega)d\sigma(\omega)
\]
are the spherical harmonic coefficients of \(f\).
Making use of the facts that $\Delta_{h,0}$ is selfadjoint on $L^2(S^{n-1}, h^2(\omega) d\sigma(\omega))$ and $Y^h_{m,j}$ are eigenfunctions of $\Delta_{h,0}$ with eigenvalues $-m(m+N-2)$, we see that the functions $f_{m,j}$ satisfy the differential equation
\[
\left( r^2 \frac{d^2}{dr^2} + (N-1)r \frac{d}{dr} + \lambda^2 r^2 - m(m+N-2) \right) f_{m,j}(r) = 0
\]
for $r > 0$. If we define $g_{m,j}(r) = r^{N/2-1} f_{m,j}(r)$, then these functions satisfy the Bessel differential equation of type $(m + \frac{N}{2} - 1)$, i.e.
\[
\left( r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} + \lambda^2 r^2 - (m + \frac{N}{2} - 1)^2 \right) g_{m,j}(r) = 0
\]
for $r > 0$. We know that the linearly independent solutions of Bessel differential equation of type $\nu$ are given by the Bessel functions of first kind $J_\nu(\lambda r)$ and second kind $Y_\nu(\lambda r)$, see e.g. [12], page 219. Therefore,
\[
g_{m,j}(r) = C_1(\lambda)J_{m+N/2-1}(\lambda r) + C_2(\lambda)Y_{m+N/2-1}(\lambda r).
\]
In the above $C_2(\lambda)$ has to be 0 since $g_{m,j}(r)$ and $J_{m+N/2-1}(\lambda r)$ are locally integrable functions whereas $Y_{m+N/2-1}(\lambda r)$ is not locally integrable near the origin. This follows from the fact that
\[
Y_{m+N/2-1}(x) \approx -\frac{2^{m+N/2-1}\Gamma(m + N/2 - 1)}{\pi x^{m+N/2-1}}
\]
for $x \to 0$. Thus $g_{m,j}(r) = r^{N/2-1} f_{m,j}(r) = C_1(\lambda)J_{m+N/2-1}(\lambda r)$ and hence
\[
f_{m,j}(r) = C_1 \frac{J_{m+N/2-1}(\lambda r)}{(\lambda r)^{N/2-1}}
\]
for $r > 0$. Since $f \in L^p_{\mu,\nu}(\mathbb{R}^n)$, it can be easily seen that
\[
\int_0^\infty |f_{m,j}(r)|^p (K_{n/2+\gamma}(\rho r))^{\rho \gamma} r^{N-1} dr < \infty
\]
which implies that
\[
\int_0^\infty \left| \frac{J_{m+N/2-1}(\lambda r)}{(\lambda r)^{N/2-1}} \right|^p (1 + r)^{(m+2\gamma-1)/2} e^{-\rho \gamma r} r^{N-1} dr < \infty.
\]
This is possible only if $|\Im(\lambda)| < \gamma_r\rho$ which proves Theorem 3.8. \hfill \Box

3.4. The chaotic behavior of the Dunkl heat semigroup: In this subsection we prove the remaining main theorems, namely Theorem 1.2 and Theorem 1.3 regarding the chaotic behavior of the semigroup $T^c_t = e^{ct} e^{-t(\Delta_{\alpha,\beta^2})}$, $c \in \mathbb{R}$ on the spaces $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}_x^2)$, $c \in \mathbb{R}$ on that space. We prove these results by imitating the proofs presented in the Subsection 2.3 where we have discussed the chaotic behavior of heat semigroup in the Euclidean set up. So we give a very sketchy outline of these proofs. We use all the notations introduced in the Subsection 2.3 with some appropriate
As in Subsection 2.3. Note that other notations $\Lambda_p$, $\omega_c$, $c_p$, $A_j$, $A_j'$, $B_0$, $B_\infty$ and $B_{\text{Per}}$ are the same as in Subsection 2.3. Note that $A_j$'s are defined by using Dunkl translations of Dunkl spherical functions $\tau_x \varphi_{\lambda, \kappa}$ in place of Euclidean translations of spherical functions $\tau_x \varphi_{\lambda}$ and $\mathcal{B}_0, \mathcal{B}_\infty$ and $\mathcal{B}_{\text{Per}}$ are defined for the space $\mathcal{B} = L^p_{\rho, \kappa}(\mathbb{R}^n)$ and the semigroup $T_t^\rho = e^{ct}e^{-(\Delta+\rho^2-t)}$. Also note that $\tau_x \varphi_{\lambda, \kappa} \in L^p_{\rho, \kappa}(\mathbb{R}^n)$ whenever $\lambda \in \Lambda^0_p$ and $p \neq 2$. This follows from the estimate $|E_\kappa(i x, \lambda \omega)| \leq C e^{3|\lambda|}|x|$. See [9], where estimates of partial derivatives of $E_\kappa(x, z)$, $x \in \mathbb{R}^n$, $z \in \mathbb{C}^n$ are given. It is then clear that $A_j \subset L^p_{\rho, \kappa}(\mathbb{R}^n)$ and also for $f \in A_j$, $T_t f = e^{-t\omega_c(\lambda)} f$ for each $1 \leq j \leq 3$.

With these notations, we have the following proposition which is the analogue of Proposition [2.9].

**Proposition 3.9.** For each $1 \leq j \leq 3$, $A_j$ is dense in $L^p_{\rho, \kappa}(\mathbb{R}^n)$, $1 \leq p < \infty$, $p \neq 2$ and $\text{span}(A_1) \subset \mathcal{B}_0$, $\text{span}(A_2) \subset \mathcal{B}_\infty$ and $\text{span}(A_3) \subset \mathcal{B}_{\text{Per}}$ provided $c > c_p$.

**Proof.** The set inclusions $\text{span}(A_1) \subset \mathcal{B}_0$, $\text{span}(A_2) \subset \mathcal{B}_\infty$ and $\text{span}(A_3) \subset \mathcal{B}_{\text{Per}}$ can be proved by the same arguments given in the proof of Proposition [2.9]. Now we need to prove the densities of $\mathcal{B}_0$, $\mathcal{B}_\infty$ and $\mathcal{B}_{\text{Per}}$ which will follow once we prove that of $A_j$'s are dense in $L^p_{\rho, \kappa}(\mathbb{R}^n)$. Suppose the span of $A_1$ is not dense in $L^p_{\rho, \kappa}(\mathbb{R}^n)$, $1 < p < \infty$. As the dual of $L^p_{\rho, \kappa}(\mathbb{R}^n)$ can be identified with $L^{p'}(\mathbb{R}^n, (\varphi_{\rho, \kappa}(x))^{p'-1}h_\kappa(x)^2 dx)$ (where $1/p + 1/p' = 1$ and $1 < p' < \infty$), there exists $g \in L^{p'}(\mathbb{R}^n, (\varphi_{\rho, \kappa}(x))^{p'-1}h_\kappa(x)^2 dx)$ such that

$$\int_{\mathbb{R}^n} g(y) \tau_x \varphi_{\lambda}(y) h_\kappa^2(y) dy = 0$$

for all $\lambda \in A_1$ and $x \in \mathbb{R}^n$. Note that $L^{p'}(\mathbb{R}^n, (\varphi_{\rho, \kappa}(x))^{p'-1}h_\kappa(x)^2 dx)$ is a subspace of $L^1(\mathbb{R}^n, h_\kappa^2(x) dx)$ for $1 < p' < \infty$ and hence the map

$$\lambda \mapsto \int_{\mathbb{R}^n} g(y) \tau_x \varphi_{\lambda}(y) h_\kappa^2(y) dy$$

is a continuous function on $\Lambda^0_p$. Moreover, by Morera and Fubini, the map is holomorphic. Since $A_1$ is a nonempty open subset of $\Lambda^0_p$ it follows that

$$\int_{\mathbb{R}^n} g(y) \tau_x \varphi_{\lambda}(y) h_\kappa^2(y) dy = 0$$
for all \( \lambda \in \Lambda_p \); in particular, for all \( \lambda \in \mathbb{R} \). In view of the definition of \( \tau_x \varphi_\lambda \) and Lemma 3.1 (1), we infer that for any \( t > 0 \)
\[
e^{-t\Delta_\kappa} g(x) = c_N \int_0^\infty e^{-\lambda t^2} \left( \int_{\mathbb{R}^n} \tau_{-x} \varphi_\lambda(y) g(y) h_\kappa^2(y) dy \right) \lambda^{n + 2\gamma - 1} d\lambda = 0.
\]
Since the heat semigroup \( e^{-t\Delta_\kappa} \) is strongly continuous on \( L^1(\mathbb{R}^n, h_\kappa^2 dx) \), \( g \in L^1(\mathbb{R}^n, h_\kappa^2 dx) \), \( e^{-t\Delta_\kappa} g = 0 \) for every \( t > 0 \) implies \( g = 0 \). When \( p = 1 \) we have a bounded function \( g_1 \) such that
\[
\int_{\mathbb{R}^n} g_1(y) \tau_x \varphi_{\lambda, \kappa}(y) \tilde{K}_{n/2 + \gamma}(y) dy = 0
\]
for all \( \lambda \in \Lambda_1 \). Since \( g(y) = g_1(y) \tilde{K}_{n/2 + \gamma}(y) \) belongs to \( L^1(\mathbb{R}^n, h_\kappa^2(x) dx) \) we can conclude that \( g_1 = 0 \) as before. This proves the density of span of \( A_1 \). The density of the spans of \( A_2 \) and \( A_3 \) are similarly proved. \( \square 

Remark 3.10. By keen observation of the above proof, Proposition 3.9 still holds if we replace the space \( L^p_{\rho, \kappa}(\mathbb{R}^n) \) by the weighted mixed norm space \( L^{p, 2}_{\rho, \kappa}(\mathbb{R}^n) \) for \( 1 < p < \infty, p \neq 2 \). To see this we only have to check that \( A_j \) are subsets of \( L^{p, 2}_{\rho, \kappa}(\mathbb{R}^n) \) for \( 1 \leq j \leq 3 \) and the strong continuity of heat semigroup \( e^{-t\Delta_\kappa} \) on \( L^{p, 2}_{\rho, \kappa}(\mathbb{R}^n) \). The latter fact has been already proved in Theorem 3.5. To see the inclusions, we have \( \tau_x \varphi_\lambda \in L^{p, 2}_{\rho, \kappa}(\mathbb{R}^n) \) whenever \( \lambda \in \Lambda_0^p \). This also follows from the estimate \( |E_\kappa(ix, \lambda\omega)| \leq Ce^{3|\lambda|}|\omega| \). It is then clear that \( A_j \subset L^{p, 2}_{\rho, \kappa}(\mathbb{R}^n) \).

Now we are ready to prove the remaining main theorems (Theorem 1.5 and Theorem 1.8).

Proof of the Theorem 1.5: In Theorem 3.2 we already proved that \( T_t = e^{-tA} \) defines a strongly continuous semigroup on \( L^p_{\rho, \kappa}(\mathbb{R}^n) \) for \( 1 \leq p < \infty \).

In view of Corollary 2.8 and Proposition 3.9, \( T^c_\rho \) is chaotic on \( L^p_{\rho, \kappa}(\mathbb{R}^n) \) for \( c > c_p \) and \( 1 \leq p < \infty, p \neq 2 \). This proves the sufficient part of part (1) of Theorem 1.5. For necessary part we make use of Theorem 3.8 according to which the point spectrum \( \sigma_{pt}(\Delta_\kappa + \rho^2 - c) \) of the operator \( (\Delta_\kappa + \rho^2 - c) \) on \( L^p_{\rho, \kappa}(\mathbb{R}^n) \) is given by
\[
\mathcal{Q}^0_p\rho - c = \{ \lambda^2 + \rho^2 - c : |\Im(\lambda)| < \gamma_p \rho \}
\]
for \( 1 \leq p < \infty \). By the geometric form of the above set, it can be easily seen that the set \( \sigma_{pt}(\Delta + \rho^2 - c) \cap i\mathbb{R} \) is empty for \( c \leq c_p \) and hence in view of Theorem 1.1 \( T^c_\rho \) is not chaotic. This proves part (1) of Theorem 1.5. Now we proceed to part (2). We note that for any \( c \in \mathbb{R} \), the semigroup \( T_t^c = e^{-t(A-c)} \) is not hypercyclic on \( L^\infty_{\rho, \kappa}(\mathbb{R}^n) \), since for any \( f \in L^\infty_{\rho, \kappa}(\mathbb{R}^n) \), \( T_t^c f \) is a continuous bounded function and hence the closure of the orbit \( \{ T_t^c f : t > 0 \} \) in \( L^\infty_{\rho, \kappa}(\mathbb{R}^n) \) is a subset of the subspace of all continuous bounded functions which is a strictly smaller than \( L^\infty_{\rho, \kappa}(\mathbb{R}^n) \). This proves part (2). For part (3), we know that the \( L^2 \)-spectrum \( \sigma_2(\Delta_\kappa) \) of \( \Delta_\kappa \) on \( L^2_{\rho, \kappa}(\mathbb{R}^n) = L^2(\mathbb{R}^n, h_\kappa(x)^2 dx) \) is given by \( [0, \infty) \) and hence
σ_2(Δ_κ + ρ^2 - c) ∩ iR has at most one point. In view of Theorem 1.4, T^C is not chaotic on L^2_{ρ,κ}(R^n). This proves part (3) which completes the proof of Theorem 1.5.

Proof of the Theorem 1.6: Let us define a_p := \frac{2ρ^2}{p'} for 1 ≤ p < 2 and a_p := \frac{2ρ^2}{p} for p > 2. In view of Theorem 3.2, for any f ∈ L^p_{ρ,κ}(R^n), we have

\|T^C t f\|_{L^p_{ρ,κ}(R^n)} = e^{ct}\|T_t f\|_{L^p_{ρ,κ}(R^n)} ≤ Ct^{\frac{n+2γ-1}{2}}e^{-t(a_p-c)}\|f\|_{L^p_{ρ,κ}(R^n)}

where T_t = e^{-tA}. From the above, the operators T^C_t are bounded uniformly in t on L^p_{ρ,κ}(R^n) for c < a_p. Consequently, for each f ∈ L^p_{ρ,κ}(R^n) the orbit \{T^C_t f : t > 0\} is a bounded set and hence it cannot be dense in L^p_{ρ,κ}(R^n). Thus T^C_t fails to be hypercyclic. This proves part (1) and part (2) which completes the proof of Theorem 1.6.

Proof of the Theorem 1.8: In view of Theorem 3.5, the semigroup T^C_t is strongly continuous on L^{p,2}_{ρ,κ}(R^n) for 1 < p < ∞, p ≠ 2. In view of Remark 3.10 and Corollary 2.9, T^C_t is chaotic on the space L^{p,2}_{ρ,κ}(R^n) for c > c_p and 1 ≤ p < ∞, p ≠ 2. This proves the part (1). For part (2), in view of Theorem 3.5 for any f ∈ L^{p,2}_{ρ,κ}(R^n) we have

\|T^C_t f\|_{L^{p,2}_{ρ,κ}(R^n)} ≤ Ct^{\frac{n+2γ-1}{2}}e^{-t(c_p-c)}\|f\|_{L^{p,2}_{ρ,κ}(R^n)}.

For c < c_p, by the same arguments given in the proof part (3) of Theorem 1.5, T^C_t fails to be hypercyclic. This proves part (2) which completes the proof of Theorem 1.8.

Acknowledgments

The first author is thankful to CSIR, India, for the financial support. The work of the second author is supported by J. C. Bose Fellowship from the Department of Science and Technology (DST) and also by a grant from UGC via DSA-SAP. Both authors wish to thank Rudra Sarkar for some useful conversations regarding the subject matter of this article.

References

[1] J. Cheeger, M. Gromov and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differential Geom., 17 (1982), 15-53.
[2] F. Dai and H. Wang, A transference theorem for the Dunkl transform and its applications, J. Funct. Anal., 258 (2010), no. 12, 4052-4074.
[3] F. Dai and Y. Xu, Approximation theory and harmonic analysis on spheres and balls, Springer Monographs in Mathematics, Springer, 2013.
[4] W. Desch, W. Schappacher and G. F. Webb, Hypercyclic and chaotic semigroups of linear operators, Ergodic Theory Dynam. Systems, 17(1997), no. 4, 793-819.
[5] C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc., 311 (1989), 167-183.
[6] C. F. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*. Encyclopedia of Mathematics and its Applications, 81, Cambridge University Press, Cambridge, 2001.

[7] S. Helgason, *Topics in harmonic analysis on homogeneous spaces*, Prog. in Math. Vol. 13, Birkhäuser, (Boston) (1981).

[8] C. Herz and N. Riviere, *Estimates for translation invariant operators on spaces with mixed norms*, Studia Math., 44 (1972), 511-515.

[9] M. F. E. de Jeu, *The Dunkl transform*, Invent. Math., 113 (1993), 147-162.

[10] L. Ji and A. Weber, *Dynamics of the heat semigroup on symmetric spaces*, Ergodic Theory Dynam. Systems, 30 (2010), no. 2, 457-468.

[11] R. de Laubenfels and H. Emamirad, *Chaos for functions of discrete and continuous weighted shift operators*, Ergodic Theory Dynam. Systems, 21 (2001), no. 5, 1411-1427.

[12] A. V. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics: A Unified Introduction with Applications*, Translated from the Russian by Ralph P. Boas, Birkhäuser, Basel-Boston, 1988.

[13] M. Pramanik and R. P. Sarkar, *Chaotic dynamics of the heat semigroup on Riemannian symmetric spaces*, J. Funct. Anal., 266 (2014), no. 5, 2867-2909.

[14] M. Rösler, *Dunkl operators: theory and applications*, Orthogonal polynomials and special functions (Leuven, 2002), 93-135, Lecture Notes in Math., 1817, Springer, Berlin, 2003.

[15] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys., 192 (1998), no. 3, 519-542.

[16] J. L. Rubio de Francia, *Transference principles for radial multipliers*, Duke Math. J., 58 (1989), 1-19.

[17] R. P. Sarkar *Chaotic dynamics of the heat semigroup on the Damek-Ricci spaces*, Israel J. Math., 198 (2013), no. 1, 487-508.

[18] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc., 87 (1958), 159-172.

[19] M. E. Taylor, *Lp-estimates on functions of the Laplace operator*, Duke Math. J., 58 (1989), no. 3, 773-793.

[20] S. Thangavelu and Y. Xu, *Convolution operator and maximal function for Dunkl transform*, J. Anal. Math., 97 (2005) 25-55.

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