The relative Howe–Moore property
for the universal group $U(F)^+$

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Abstract

The aim of this article is twofold: firstly, we give a characterization of the universal group $U(F)^+$, with $F$ being primitive, to have the Howe–Moore property. Secondly, we prove that $U(F)^+$ has the relative Howe–Moore property, when $F$ is primitive. These two results are a consequence of a strengthening of Mautner’s phenomenon for locally compact groups acting on $d$-regular trees and having Tits’ independence property.

1 Introduction

Known examples of locally compact groups that enjoy the Howe–Moore property, namely, the vanishing at infinity of all matrix coefficients of the group’s unitary representations that are without non-zero invariant vectors, are: connected, non-compact, simple real Lie groups and their totally disconnected analogs, isotropic simple algebraic groups over non Archimedean local fields and closed, topologically simple subgroups of $\text{Aut}(T)$ that act 2–transitively on the boundary $\partial T$, where $T$ is a bi-regular tree with valence $\geq 3$ at every vertex. We emphasize that both isotropic simple algebraic groups and 2–transitive subgroups of $\text{Aut}(T)$ admit a strongly transitive action on a locally finite thick Euclidean building. The strong transitivity of this action is essentially used to show the Howe–Moore property. Whether this condition is also necessary, is an open problem.

In this context, a particular locally compact group that is of special interest for us is the universal group $U(F)$ introduced by Burger–Mozes in [BM00b, Section 3]. This group is a closed subgroup of the full group of automorphisms $\text{Aut}(T)$ of a $d$-regular tree $T$, with $d \geq 3$. This group is defined as follows.

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Definition 1.1. Let $\iota : E(\mathcal{T}) \to \{1, \ldots, d\}$ be a function from the set $E(\mathcal{T})$ of unoriented edges of the tree $\mathcal{T}$ such that its restriction to the star $E(x)$ of every vertex $x \in \mathcal{T}$ is in bijection with $\{1, \ldots, d\}$. A function $\iota$ with those properties is called a legal coloring of the tree $\mathcal{T}$.

Definition 1.2. Let $F$ be a subgroup of permutations of the set $\{1, \ldots, d\}$ and let $\iota$ be a legal coloring of $\mathcal{T}$. The universal group, with respect to $F$ and $\iota$, is defined as

$$U(F) := \{ g \in \text{Aut}(\mathcal{T}) \mid \iota \circ g \circ (\iota|_{E(x)})^{-1} \in F, \text{ for every } x \in \mathcal{T} \}.$$ 

By $U(F)^+$ one denotes the subgroup generated by the edge-stabilizing elements of $U(F)$. Moreover, Amann [Ama03, Proposition 52] tells us that the group $U(F)$ is independent of the legal coloring $\iota$ of $\mathcal{T}$.

Immediately from the definition one deduces that $U(F)$ and $U(F)^+$ are closed subgroups of $\text{Aut}(\mathcal{T})$. Notice that, when $F$ is the full permutation group $\text{Sym}\{1, \ldots, d\}$, $U(F) = \text{Aut}(\mathcal{T})$ and $U(F)^+ = \text{Aut}(\mathcal{T})^+$, the latter group is an index 2, simple subgroup of $\text{Aut}(\mathcal{T})$ (for this see Tits [Tit70]). One of the important properties of these groups is that $U(F)$ and $U(F)^+$ act 2-transitively on the boundary $\partial \mathcal{T}$ if and only if $F$ is 2–transitive. Moreover, we have that $U(F)^+$ is either trivial or is a topologically simple group. This implies that when $F$ is 2–transitive, the group $U(F)^+$ is not trivial and has the Howe–Moore property.

One of the main theorems of this article provides a characterization of the group $U(F)^+$, with $F$ being primitive, but not cyclic of prime order, to have the Howe–Moore property.

**Theorem 1.3.** (See Theorem 5.1) Let $F \leq \text{Sym}\{1, \cdots, d\}$ be primitive, but not cyclic of prime order. Then $U(F)^+$ has the Howe–Moore property if and only if for every unitary representation $(\pi, H)$ of $U(F)^+$, without non-zero $U(F)^+$-invariant vectors, and for every $v \neq 0 \in H$ the closed isotropy subgroup $U(F)^+ v$ is compact.

As a second theorem we obtain:

**Theorem 1.4.** (See Theorem 5.1) Let $F \leq \text{Sym}\{1, \cdots, d\}$ be primitive, but not cyclic of prime order. Then for every $\xi \in \partial \mathcal{T}$ the group $U(F)^+$ has the relative Howe–Moore property with respect to its closed subgroup $(U(F)^+)_{\xi}^0 := \{ g \in U(F)^+ \mid g(\xi) = \xi \text{ and } g \text{ fixes a vertex of } \mathcal{T} \}$.

The relative Howe–Moore property was introduced and studied by Cluckers–de Cornulier–Louvet–Tessera–Valette [CdCL+11] and it is a weakening of the Howe–Moore property.

**Definition 1.5.** Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$. We say that the pair $(G, H)$ has the relative Howe–Moore property if every unitary representation $(\pi, H)$ of $G$ either has non-zero $H$–invariant vectors, or the restriction $\pi|_H$ is a unitary representation of $H$ that has all its matrix coefficients vanishing at infinity with respect to $H$. 

2
Theorems 1.3 and 1.4 are a consequence of the following strengthening of Mautner’s phenomenon for locally compact groups acting on d-regular trees and having Tits’ independence property (see Definition 2.1). We state this result in the general framework of groups acting on locally finite trees.

**Proposition 1.6.** (See Proposition 4.7) Let $T$ be a locally finite tree. Let $G \leq \text{Aut}(T)$ be a closed, non-compact subgroup with Tits’ independence property and $(\pi, H)$ be a unitary representation of $G$. Assume there exists a sequence $\{g_n\}_{n>0} \subset G$ and a vector $v \in H$ such that $g_n \to \infty$ and $\{\pi(g_n)(v)\}_{n>0}$ weakly converges to a non-zero vector $v_0 \in H$.

Then for every $\xi \in \partial T$, with the property that there exists $x \in T$ and a subsequence $\{g_{n_k}\}_{n_k>0}$ such that $g_{n_k}(x) \to \xi$, we have that $\mathcal{N}_{\{g_{n_k}\}} \leq G^0_\xi \leq G_{v_0}$. Here $G^0_\xi := \{g \in G \mid g(\xi) = \xi \text{ and } g \text{ is not hyperbolic}\}$ and $G_{v_0} := \{g \in G \mid \pi(g)v_0 = v_0\}$.

Moreover, if $G_{v_0}$ is non-compact and does not contain hyperbolic elements then $G^0_\xi$, for a point $\xi \in \partial T$.

As a corollary we obtain:

**Corollary 1.7.** (See Corollary 4.8) Let $F$ be primitive but not 2-transitive. Let $(\pi, H)$ be a unitary representation of $U(F)^+$ without non-zero $U(F)^+$-invariant vectors and where $H$ is a separable complex Hilbert space. Consider $v \in H$ such that $(U(F)^+_v$ is not compact. Then $(U(F)^+_v = (U(F)^+_v)^0$ for a point $\xi \in \partial T$.

The structure of this article is the following. In the first place, Section 3 adds to the (long) list of Section 2 new equivalent conditions of the universal group $U(F)^+$ to act 2-transitively on the boundary $\partial T$. Section 4 gathers new algebraic and harmonic analytic properties of the group $U(F)^+$, when $F$ is primitive, but not cyclic of prime order. Those properties are used to prove Theorem 1.3 and Proposition 1.6. The second main result of this article, Theorems 1.3 and 1.4 is proved in Section 5.

### 2 Definitions and some known properties

First, let us fix some general notation. Denote by $T$ a $d$-regular tree, with $d \geq 3$, and by $\text{Aut}(T)$ its full group of automorphisms. Denote by $\text{dist}_T(\cdot, \cdot)$ the usual metric on $T$ where every edge has length one. Let $\text{Aut}(T)^+$ be the group of all type-preserving automorphisms of $T$. By a type-preserving automorphism of $T$ we mean one that preserves an orientation of $T$ that is fixed in advance; this is the same as saying that the automorphism acts without inversion. Denote by $\partial T$ the endpoints of $T$ (they are also called the ideal points of $T$) and we call $\partial T$ the boundary of $T$. For every two points $x, y \in T \cup \partial T$ we denote by $[x, y]$ the unique geodesic between $x$ and $y$ in $T \cup \partial T$.

For $G \leq \text{Aut}(T)$ and $x, y \in T \cup \partial T$ we define

$$G_{[x, y]} := \{g \in G \mid g \text{ fixes pointwise the geodesic } [x, y]\}.$$ 

In particular, $G_x = \{g \in G \mid g(x) = x\}$. For $\xi \in \partial T$ we define $G_\xi := \{g \in G \mid g(\xi) = \xi\}$ and $G^0_\xi := \{g \in G \mid \text{and } g \text{ fixes a vertex of } T\}$. Notice that $G_\xi$ can contain hyperbolic elements; if this is the case then $G^0_\xi \leq G_\xi$.
For a hyperbolic element \( \gamma \in \text{Aut}(T) \), we denote \( |\gamma| := \min_{x \in T} \{ \text{dist}_T(x, \gamma(x)) \} \), which is called the translation length of \( \gamma \), and set \( \text{Min}(\gamma) := \{ x \in T \mid \text{dist}_T(x, \gamma(x)) = |\gamma| \} \).

Before stating more important properties of the universal group that was defined in the Introduction, we record another definition which is given in general, for a locally finite tree, and which is used in the sequel as a key property.

**Definition 2.1** (See Tits [Tit70]). Let \( T \) be a locally finite tree and let \( G \leq \text{Aut}(T) \) be a closed subgroup. We say that \( G \) has **Tits’ independence property** if for every edge \( e \) of \( T \) we have the equality \( G_e = G_{T_1}G_{T_2} \), where \( T_i \) are the two infinite half sub-trees of \( T \) emanating from the edge \( e \) and \( G_{T_i} \) is the pointwise stabilizer of the half-tree \( T_i \).

We mention that Tits’ independence property guarantees the existence of ‘enough’ rotations in the group \( G \). It is used in the work of Tits as a sufficient condition to prove simplicity of ‘large’ subgroups of \( \text{Aut}(T) \) (see Tits [Tit70]). In his thesis, Amann employs it to give a complete classification of all super-cuspidal representations of a closed subgroup in \( \text{Aut}(T) \) acting transitively on the vertices and on the boundary of \( T \) and having Tits’ independence property. For a closed subgroup \( G \) of \( \text{Aut}(T) \) that acts transitively on the vertices and on the boundary of \( T \) but which does not have Tits’ independence property less is known about the complete classification of all its super-cuspidal representations. In contrast, for the above mentioned groups, with or without Tits’ independence property, the remaining two classes of irreducible unitary representations, namely the special and respectively, the spherical ones, are completely classified in Figà-Talamanca–Nebbia [FTN91, Chapter III, resp. Chapter II].

Following Burger–Mozes and Amann [BM00b, Ama03], we enumerate the following properties of the groups \( U(F) \) and \( U(F)^+ \):

1) \( U(F) \) and \( U(F)^+ \) have Tits’ independence property;

2) \( U(F)^+ \) is trivial or simple;

3) if \( F \) acts transitively on the set \( \{1, ..., d\} \) then \( U(F) \) acts transitively on the edges of \( T \) and \( U(F), U(F)^+ \) are unimodular groups;

4) \( U(F) \) and \( U(F)^+ \) act 2–transitively on the boundary \( \partial T \) if and only if \( F \) is 2–transitive;

5) When \( F \) is transitive and generated by its point stabilizers, by Amann [Ama03 Prop. 57], we have that \( U(F)^+ \) is edge-transitive, \( U(F)^+ = U(F) \cap \text{Aut}(T)^+ \) and \( U(F)^+ \) is of index 2 in \( U(F) \). For example, this is the case when \( F \) is primitive but not cyclic of prime order. When \( F \) is primitive and not generated by its point stabilizers, all point stabilizers of \( F \) are equal. This implies that all point stabilizers of \( F \) are just the identity, that \( F \) is cyclic of prime order and that the group \( U(F)^+ \) is trivial. Under these hypotheses, we obtain that \( U(F) \) is a nontrivial, discrete subgroup of \( \text{Aut}(T) \). In order to avoid heavy formulation, we simply use “\( F \) is primitive” to mean that “\( F \) is primitive but not cyclic of prime order”. 

4
Regarding the 2–transitive action on the boundary $\partial T$ of a group $G \leq \text{Aut}(T)$, we have the following general well-known equivalences.

**Proposition 2.2.** (See [FTN91], [BM00b, Lemma 3.1.1]) For a closed subgroup $G \leq \text{Aut}(T)$ the following are equivalent:

i) $G_x$ is transitive on $\partial T$, for every $x \in T$;

ii) $G$ is non-compact and there exists $x \in T$ such that $G_x$ acts transitively on $\partial T$;

iii) $G$ is non-compact and transitive on $\partial T$;

iv) $G$ is 2-transitive on $\partial T$.

**Remark 2.3.** (See [FTN91, Prop. 10.2]) One can prove that if a closed, non-compact subgroup $G \leq \text{Aut}(T)$ is transitive on the boundary $\partial T$, then either $G$ is transitive on the vertices of $T$ or it has exactly two $G$-vertex-orbits in $T$.

The following proposition gives us a slightly different, but equivalent, description of a 2–transitive action on $\partial T$.

**Proposition 2.4.** (See [CC15, Cio14]) Let $G$ be a closed, non-compact subgroup of $\text{Aut}(T)$ of type-preserving automorphisms. The following are equivalent:

i) $G$ is strongly transitive on $T$, where $T$ is viewed as a 1-dimensional Euclidean building;

ii) $G$ is 2–transitive on $\partial T$;

iii) $G$ has no fixed point in $\partial T$ and for some point $\xi \in \partial T$, the stabilizer $G_\xi$ acts co-compactly on $T$;

iv) $G$ acts co-compactly on $T$, without a fixed point in $\partial T$, and there exists a compact open subgroup in $G$ having finitely many orbits on $\partial T$;

v) $G$ acts co-compactly on $T$, without a fixed point in $\partial T$, and every compact open subgroup has finitely many orbits on $\partial T$;

vi) $G$ acts co-compactly on $T$ and $(G, G_x)$ is a Gelfand pair, where $x$ is a vertex of $T$.

**Convention 2.5.** Let $F$ be primitive. Consider fixed a coloring $\iota$ of $T$. Take a vertex $x \in T$ and an edge $e$ of the star of $x$. From now on and for simplicity, set $G := U(F)^+_x$ and $K := U(F)^+_x$ and let $T_{x,e}$ be the half-tree of $T$ that emanates from the vertex $x$ and that contains the edge $e$.

When $F$ is primitive but not 2–transitive, the universal group $G$ still has some of the properties of closed, non-compact subgroups of $\text{Aut}(T)$ that act 2–transitively on the boundary $\partial T$, see for example Ciobotaru [Cio, Lemma 2.6].

**Remark 2.6.** As $F$ is primitive, given an edge $e' \in V(T)$ at odd distance from $e$, one can construct, using the definition of the group, a hyperbolic element in $G$ translating $e$ to $e'$. Moreover, every hyperbolic element in $G$ has even translation length.
3 New characterizations of 2–transitivity

In addition to property [1] from Section [2] the next propositions provide new characterizations of the 2–transitivity of the $G$–action on the boundary $\partial T$. These characterizations are expressed in terms of the subgroups $G_\xi$ and the existence of hyperbolic elements in $G_\xi$, where $\xi \in \partial T$.

**Proposition 3.1.** $G$ is transitive on the boundary $\partial T$ if and only if for every $\xi \in \partial T$, $G_\xi$ contains a hyperbolic element.

**Proof.** ‘⇒’ This implication is easy and follows from Propositions [2.2] and [2.4].

‘⇐’ Suppose that for every $\xi \in \partial T$ the group $G_\xi$ contains a hyperbolic element. We want to show that $F$ is 2–transitive, which by property [4] from Section [2] is equivalent to the transitivity of $G$ on the boundary $\partial T$.

Suppose that $F$ is not 2–transitive. Therefore, there exists a color, say 1, such that $F_1$ is not transitive on $\{2, ..., d\}$ (otherwise $F$ would be 2–transitive). Moreover, without loss of generality, we can suppose that $F_1$ does not contain any element mapping 2 to 3.

Consider a vertex $v \in T$ and starting from this vertex we want to construct a particular geodesic ray $[v, \eta)$, with $\eta \in \partial T$. We enumerate its vertices by $\{v_i\}_{i \geq 0}$, such that $v_0 = v$. We construct the geodesic ray $[v, \eta)$ by choosing the vertices $\{v_i\}_{i \geq 0}$ such that the successive colors of the edges along the path $[v, \eta)$ are:

$$(12)(13)(12)^2(13)(12)^3(13) \cdots (12)^m(13)(12)^{m+1}(13) \cdots,$$

where $(12)^m$ denotes the geodesic segment of length $2m$ colored successively with the pair of ordered colors $(1, 2)$. Notice that the choice of the vertices $\{v_i\}_{i \geq 0}$ with the above prescribed coloration is unique. Moreover, it determines a unique endpoint $\eta \in \partial T$ and thus, a unique geodesic ray $[v, \eta)$.

By hypothesis, $G_\eta$ contains at least one hyperbolic element. Take one of those and denote it by $b$; consider that its translation length is $2N$ and that its attracting endpoint is $\eta$. Notice there exists a vertex $v_b$ on the geodesic ray $[v, \eta)$ such that for every $v \in [v_b, \eta)$ we have that $b(v) \in [v_b, \eta)$. Moreover, if we travel deep enough on the geodesic ray $[v_b, \eta)$ we find a geodesic segment whose coloring is the following string $(12)^N(13)(12)^N$. Let $x$ be the midpoint of this geodesic segment. The two edges in $[v_b, \eta)$ emanating from $x$ have colors 1 and 3, where the two edges in $[v_b, \eta)$ emanating from $b(x)$ have colors 1 and 2. We obtained that $b$ sends the pair of ordered colors $(1, 2)$ into the pair of ordered colors $(1, 3)$. This contradicts that $F_1$ does not contain any element mapping 2 to 3. The proposition stands proven.

**Proposition 3.2.** Let $F$ be transitive and generated by its points stabilizers. Then $G$ acts transitively on the boundary $\partial T$ if and only if every hyperbolic element of $G$ is a power of a hyperbolic element of translation length two.

**Proof.** The implication ‘⇒’ follows by applying Propositions [2.2] and [2.4].

Consider the reverse implication ‘⇐’. Suppose moreover that $G$ is not transitive on the boundary $\partial T$. This is equivalent to $F$ not being 2–transitive. Therefore, there
exists a color, say 1, such that $F_1$ is not transitive on $\{2, \cdots, d\}$. Moreover, without loss of generality, we can assume that $F$ does not contain any element sending the pair of ordered colors $(1, 2)$ to the pair of ordered colors $(1, 3)$.

Construct in $T$ a bi-infinite geodesic line colored using only the array of ordered colors $(1, 2, 1, 3)$ and denote it by $\ell$. As $F$ is transitive and generated by its points stabilizers, the group $G$ is edge-transitive on $T$; therefore, $G$ acts co-compactly on $T$. Apply then [CC15, Proposition 2.9] to the geodesic line $\ell$. There exists thus a hyperbolic element $h \in G$ such that its translation axis intersects $\ell$ into a geodesic segment of a big length. By hypothesis $h$ is a power of a hyperbolic element of translation length 2. This implies that there exists an element in $G$ such that the pair of ordered colors $(1, 2)$ is sent into the pair of ordered colors $(1, 3)$. This gives an element of $F$ sending $(1, 2)$ to $(1, 3)$ which is a contradiction. The conclusion follows.

4 Towards the Howe–Moore property

Recall the following definition.

**Definition 4.1.** Let $G$ be a locally compact group and let $\alpha = \{g_n\}_{n \geq 0}$ be a sequence in $G$. Corresponding to $\alpha$ we define the set:

$$U_\alpha^+ := \{g \in G \mid \lim_{n \to \infty} g_n^{-1}gg_n = e\}.$$  

Notice that $U_\alpha^+$ is a subgroup of $G$. It is called the contraction group corresponding to $\alpha$. Because it does not need to be closed in general, we denote by $N_\alpha^+$ the closed subgroup $\overline{U_\alpha}$. In the same way, but using $g_n g_n^{-1}$ we define $U_\alpha^-$ and $N_\alpha^-$. When $g_n = a^n$, for every $n > 0$ and for $a \in G$, we simply denote $U_\alpha^+$ by $U_a^+$.

For example, the next easy lemma describes the contraction group of a sequence $\alpha = \{g_n\}_{n \geq 0} \subset \text{Aut}(T)$ that enjoys particular properties.

**Lemma 4.2.** Let $T$ be a locally finite tree and $\alpha = \{g_n\}_{n \geq 0} \subset \text{Aut}(T)$. Assume there exist $x \in T$ and $\xi \in \partial T$ with the property that $g_n(x) \to \xi$ when $n \to \infty$, the limit being considered with respect to the cone topology on $T \cup \partial T$. Then every element $g \in \text{Aut}(T)$ that fixes pointwise the half-tree of $T$, which emanates from the vertex $x$ and which contains $\xi$ in its boundary, is an element of $U_\alpha^+$.

**Proof.** Let $g \in \text{Aut}(T)$ that fixes pointwise the half-tree $T_1 \subset T$, which emanates from the vertex $x$ and which contains $\xi$ in its boundary $\partial T_1$. We have to prove that

$$\lim_{n \to \infty} g_n^{-1}gg_n = e.$$  

This is equivalent to showing that for every $r > 0$, there exists $N_r > 0$ such that $g_n^{-1}gg_n(B(x, r)) = B(x, r)$ pointwise, for every $n > N_r$, where $B(x, r) \subset T$ is the ball centered in $x$ and of radius $r$. The latter condition is translated as $g(B(g_n(x), r)) = B(g_n(x), r)$ pointwise, for every $n > N_r$. This equality is true by hypothesis, as $g_n(x) \to \xi$ and $g$ fixes pointwise the half-tree $T_1$. The conclusion follows.
Notice that the sequence \(\{g_n = a^n\}_{n>0}\), where \(a \in \text{Aut}(T)\) is a hyperbolic element, satisfies the hypotheses of Lemma 4.2: the point \(\xi \in \partial T\) is the attracting endpoint of \(a\) and \(x\) is a vertex of the translation axis of \(a\). In this particular case, one can prove more when considering the group \(G\). The next lemma emphasizes the importance of contractions groups.

**Lemma 4.3.** Let \(F\) be primitive. Let \(a\) be a hyperbolic element in \(G\). Then \(G = \langle U_a^+, U_a^- \rangle\).

In particular, for every unitary representation \((\pi, \mathcal{H})\) of \(G\), without non-zero \(G\)-invariant vectors and where \(\mathcal{H}\) a separable complex Hilbert space, all matrix coefficients of \((\pi, \mathcal{H})\) corresponding to the sequence \(\{a^n\}_{k \geq 0}\) are \(C_0\).

**Proof.** Denote the translation axis of the hyperbolic element \(a\) by \(\ell\) and its repelling and attracting unique endpoints by \(\xi_-\) and respectively, by \(\xi_+\).

Let \(g\) be an element in \(U_a^+\), which by definition has the property \(\lim_{n \to \infty} a^{-n}ga^n = e\). One deduces that \(g(\xi_+) = \xi_+\) and that \(g\) fixes at least one vertex of \(T\). Therefore \(g \in G_{\xi_+}^0 := \{h \in G \mid h(\xi_+) = \xi_+\}\) and \(h\) is not hyperbolic. We conclude that \(U_a^+ \leq G_{\xi_+}^0\).

In the same way we have that \(U_a^- \leq G_{\xi_-}^0\).

Moreover, we claim that \(G_{\xi_+}^0 \leq \langle U_a^+, U_a^- \rangle\). Indeed, let \(g \in G_{\xi_+}^0\) and consider a vertex \(x \in \ell\) fixed by \(g\). The geodesic ray \([x, \xi_+] \subset \ell\) is thus pointwise fixed by \(g\). Let \(e\) be an edge in the interior of \([x, \xi_+]\). By Tits’ independence property we obtain that \(g = g_1h_1\), where \(g_1\) fixes pointwise the half-tree emanating from the edge \(e\) and containing the endpoint \(\xi_+\) and \(h_1\) fixes pointwise the other half-tree emanating from \(e\), which moreover contains the endpoint \(\xi_-\). By Lemma 4.2 we have that \(h_1 \in U_a^-\) and \(g_1 \in U_a^+\). In particular \(G_e \leq \langle U_a^+, U_a^- \rangle\) and our claim follows.

To conclude, we obtain that \(G_{\xi_+}^0, G_{\xi_-}^0 \leq \langle U_a^+, U_a^- \rangle\) and \(\langle U_a^+, U_a^- \rangle\) is an open subgroup of \(G\), because it contains the open subgroup \(G_e\). But \(\langle U_a^+, U_a^- \rangle\) is not compact as otherwise it would be contained in a vertex stabilizer by [CDM11] Lemma 2.6], which is impossible (see also Lemma 4.5). Thus \(\langle U_a^+, U_a^- \rangle\) is an open, non-compact subgroup of \(G\). By Caprace–De Medts [CDM11 Prop. 4.1] we have that the subgroup \(F\) is primitive if and only if every proper open subgroup of \(G\) is compact, thus we must have that \(\langle U_a^+, U_a^- \rangle = G\).

The last part of the lemma follows by applying [Cio15] Lemma 2.9 and Lemma 3.1.

Lemma 4.3 motivates the following remark.

**Remark 4.4.** Let \(F\) be primitive but not 2–transitive. Let \((\pi, \mathcal{H})\) be a unitary representation of \(G\) without non-zero \(G\)-invariant vectors and where \(\mathcal{H}\) is a separable complex Hilbert space. Consider \(v \neq 0 \in \mathcal{H}\) and let \(G_v := \{g \in G \mid \pi(g)v = v\}\). We record the following two possibilities for the closed subgroup \(G_v\) of \(G\):

i) \(G_v\) is compact;
ii) $G_v$ is not compact. In this case, as the unitary representation $(\pi, \mathcal{H})$ is without non-zero $G$-invariant vectors, by Lemma 4.3, we obtain that $G_v$ does not contain hyperbolic elements. Applying the result of Tits [Tit78, Prop. 3.4] we conclude that $G_v$ is a closed subgroup of a stabilizer of an endpoint $\xi \in \partial T$.

Remark 4.4 is completed by the following lemma and proposition.

**Lemma 4.5.** Let $F$ be primitive. Then for every $\xi \in \partial T$ the subgroup $G^0_\xi$ is not compact, where $G^0_\xi := \{ g \in G \mid g(\xi) = \xi \text{ and } g \text{ is not hyperbolic} \}$.

**Proof.** Suppose there is $\xi \in \partial T$ such that $G^0_\xi$ is compact. Using [CDM11] Lemma 2.6 there exists a vertex $x \in T$ such that $G^0_\xi \leq G_x$. Take the geodesic ray $[x, \xi]$ and let $e$ be an edge in $[x, \xi]$ with $\text{dist}(e, x) > 1$. Using Tits’ independence property of $G$ we claim that for every $g \in G_x$, $g(e') = e'$, where $e'$ is an edge contained in $[x, \xi]$ such that $d(e', x) \leq d(e, x) - 1$. Indeed, we have $G_x = G_{T_1} G_{T_2}$, where $T_1$ and $T_2$ are the two infinite half sub-trees of $T$ emanating from the endpoints of $e$. Therefore, if $g \in G_{T_1}$ then $g(e') = e'$, where $T_1$ is the half-tree not containing the end $\xi$. If $g \in G_{T_2}$ then $g \in G_{\xi} \leq G_v$ and thus $g(e') = e'$.

We conclude that $G_x \subset G_{e'}$, for any two edges $e, e'$ of $[x, \xi]$ with $\text{dist}(e, x) > 1$ and $d(e', x) \leq d(e, x) - 1$. In particular, if we take $e'$ such that $d(e', x) = d(e, x) - 1$ then we obtain $F_{1(e)} \leq F_{1(e')}$, where by $F_{1(j)}$, with $j \in \{1, \ldots, d\}$, we denote the stabilizer of the color $j$ in the permutation subgroup $F$.

Now, as the geodesic ray $[x, \xi]$ is $d$-colored, there are at least two different colors on this geodesic ray, say $k \neq j$, for which $F_j = F_k$.

Choose an edge $e_1$ on the axis $[x, \xi]$ which is colored $j$. Consider the unique bi-infinite geodesic line $\ell$ containing $e_1$ and colored only with $j$ and $k$, alternatively. Now $G_\ell := \{ g \in G \mid g \text{ stabilizes } \ell \}$ contains $G_{e_1}$. This is because if the color $j$ is fixed then also is the color $k$, as $F_j = F_k$, and vice versa. Therefore, $G_\ell$ is an open subgroup in $G$.

But $G_\ell$ is not compact because it contains a hyperbolic element of translation length 2, i.e., whose translation axis is $\ell$. Indeed, observe that the group $U(1d)$ is a subgroup of $U(F)$ and it preserves the fixed coloring $i$ of the tree. In particular, we can construct in $U(1d)$ a hyperbolic element $h$ of translation length 2 whose translation axis is $\ell$. Moreover, this hyperbolic element $h$ belongs to $\text{Aut}(T)^+$. As $F$ is primitive, apply [Ama03] Prop. 57. We obtain that $U(F)^+ = U(F) \cap \text{Aut}(T)^+$ and so $h \in G_\ell \leq U(F)^+$. We conclude that $G_\ell$ is a non-compact, open, proper subgroup of $G$ contradicting [CDM11] Prop. 4.1. The conclusion follows.

Remark 4.6. Recall that a locally finite tree $T$ is endowed with the cone topology which turns $T \cup \partial T$ into a compact topological space. Let $x \in T$ and take a sequence $\{g_n\}_{n>0} \subset \text{Aut}(T)$ such that $g_n \rightarrow \infty$. Therefore $\text{dist}_T(x, g_n(x)) \rightarrow \infty$. By compactness, we can extract a subsequence $\{g_{n_k}\}_{n_k}$ such that $\{g_{n_k}(x)\}_{n_k}$ converges with respect to the cone topology to an endpoint $\xi \in \partial T$.

The next proposition is a strengthening of Mautner’s phenomenon when Tits’ independence property is fulfilled and is stated in the general framework of groups acting on locally finite trees.
Assume there exists a sequence \( \{g_n\}_{n>0} \subset G \) and a vector \( v \in \mathcal{H} \) such that \( g_n \to \infty \) and \( \{\pi(g_n)(v)\}_{n>0} \) weakly converges to a non-zero vector \( (v_0) \in \mathcal{H} \).

Then for every \( \xi \in \partial T \), with the property that there exists \( x \in T \) and a subsequence \( \{g_{n_k}\}_{n_k>0} \) such that \( g_{n_k}(x) \to \xi \), we have that \( N_{\{g_{n_k}\}}^{+} \leq G_{\xi}^0 \leq G_{v_0} \).

Moreover, if \( G_{v_0} \) is non-compact and does not contain hyperbolic elements then \( G_{v_0} = G_{\xi}^0 \), for a point \( \xi \in \partial T \).

**Proof.** First of all, by Mautner’s phenomenon [BM00, Ch. III, Thm. 1.4] (or [Cio15]) the vector \( v_0 \) is \( N_{\{g_n\}}^{+} \)-invariant. So \( N_{\{g_n\}}^{+} < G_{v_0} \).

Consider now a vertex \( x \in T \). As \( g_n \to \infty \), by Remark 4.6 we extract a subsequence \( \{g_{n_k}\}_{n_k>0} \subset \{g_n\}_{n>0} \) such that \( g_{n_k}(x) \to \xi \) in the cone topology of \( T \cup \partial T \).

For such an \( \{g_{n_k}\}_{n_k>0} \) and \( \xi \), we claim that \( N_{\{g_{n_k}\}}^{+} < G_{\xi}^0 \). As \( G_{\xi}^0 \) is a closed subgroup, it is enough to show that \( U_{\{g_{n_k}\}}^{+} < G_{\xi}^0 \). Indeed, let \( g \in U_{\{g_{n_k}\}}^{+} \). By definition we have \( \lim_{n \to \infty} g_{n_k}^{-1}gg_{n_k} = e \). As \( g_{n_k}(x) \to \xi \) in the cone topology, we deduce that \( g \) fixes the vertices \( g_{n_k}(x) \) and so also the endpoint \( \xi \). We conclude that \( g \in G_{\xi}^0 \).

Moreover, we claim that \( G_{\xi}^0 \leq G_{v_0} \).

Indeed, let \( g \in G_{\xi}^0 \) and we want to prove that \( g \in G_{v_0} \). Let \( [x_0, \xi] \) be a geodesic ray fixed by \( g \) and we index its vertices by \( x_j \). Take an edge \([x_j, x_{j+1}]\), with \( j \geq 0 \) and we have that \( g \in G_{[x_j, x_{j+1}]} := \{h \in G \mid h[x_j, x_{j+1}] = [x_j, x_{j+1}]\} \). By Tits’ independence property we have that \( g = t_jh_j \), where \( t_j \in G_{T_{j+1}} \), \( h_j \in G_{T_j} \), \( T_{j+1} \) is the half-tree emanating from the vertex \( x_{j+1} \) and containing the end \( \xi \), and \( T_j \) is the half-tree emanating from the vertex \( x_j \) and not containing the end \( \xi \). As \( g \) and \( t_j \) are stabilizing the end \( \xi \) we deduce that \( h_j \in G_{\xi}^0 \).

In addition, as \( t_j \) fixes pointwise the half-tree \( T_{j+1} \) containing the endpoint \( \xi \) and \( g_{n_k}(x) \to \xi \), by Lemma 4.2 we have that \( \lim_{n \to \infty} g_{n_k}^{-1}t_jg_{n_k} = e \). Thus \( t_j \in N_{\{g_{n_k}\}}^{+}h_k < G_{v_0} \).

Remark that \( h_j \to e \) when \( j \to \infty \). Therefore \( gh_k^{-1} = t_k \to g \). Because \( G_{v_0} \) is a closed subgroup and \( t_k \in G_{v_0} \), we deduce that \( g \in G_{v_0} \). The claim follows.

Let us now prove the last assertion of the proposition by supposing that \( G_{v_0} \) is non-compact and does not contain hyperbolic elements. By the result of Tits [Tit70, Prop. 3.4] we have that \( G_{v_0} \) is contained in the stabilizer \( G_{\eta}^0 \) of an endpoint \( \eta \in \partial T \). As \( G_{v_0} \) is non-compact, choose a sequence \( \{h_m\}_{m>0} \subset G_{v_0} \) such that \( h_m \to \infty \). Remark that \( \{\pi(h_m)(v_0)\}_{m>0} \) weakly converge to \( v_0 \) and for any vertex \( x \in T \) we have that \( h_m(x) \to \eta \), with respect to the cone topology of \( T \cup \partial T \). Applying the first part of the proposition we have that \( G_{\eta}^0 \leq G_{v_0} \) and the conclusion follows.

By Remarks 4.4 and 4.6 we obtain:

**Corollary 4.8.** Let \( F \) be primitive but not 2–transitive. Let \( (\pi, \mathcal{H}) \) be a unitary representation of \( \mathbb{G} \) without non-zero \( \mathbb{G} \)-invariant vectors and where \( \mathcal{H} \) is a separable complex Hilbert space. Consider \( v \in \mathcal{H} \) such that \( \mathbb{G}_v \) is not compact. Then \( \mathbb{G}_v = \mathbb{G}_{\xi}^0 \) for a point \( \xi \in \partial T \).
Using the results obtained until now, the next theorem provides an equivalent characterization for the group $G$ to have the Howe–Moore property. We stress here that in general, for locally compact groups an ‘if and only if’ condition for the Howe–Moore property is difficult to be achieved.

**Theorem 4.9.** Let $F$ be primitive. Then $G$ has the Howe–Moore property if and only if for every unitary representation $(\pi, \mathcal{H})$ of $G$, without non-zero $G$-invariant vectors, and for every $v \neq 0 \in \mathcal{H}$ the closed subgroup $G_v$ is compact.

**Proof.** Remark that, by [CdCL11] Prop. 3.2 and [CDM11] Prop. 4.1, the primitivity of $F$ is a necessary condition for the Howe–Moore property. Also, by [Cio15] Remark 2.5 and Lemma 2.4, it is enough to consider only unitary representations over separable Hilbert spaces.

Moreover, the implication “$G$ has Howe–Moore property implies $G_v$ is compact” is easy and holds in general. Indeed, if $G_v$ is non-compact then there exists a sequence $\{g_n\}_n \subset G_v$ with $g_n \to \infty$. Therefore, the corresponding matrix coefficients $\langle \pi(g_n)(v), v \rangle = 1$ which is a contradiction.

Consider now the reverse implication.

Suppose there exists a unitary representation $(\pi, \mathcal{H})$ of $G$, without non-zero $G$-invariant vectors and where $\mathcal{H}$ is a separable Hilbert space, two non-zero vectors $v, w \in \mathcal{H}$ and a sequence $\{g_n\}_{n>0} \subset G$ such that $g_n \to \infty$ and $|\langle \pi(g_n v), w \rangle| \not\to 0$. Restricting to a subsequence and without loss of generality we can consider also that $\{\pi(g_n)(v)\}_{n>0}$ weakly converges to $v_0 \neq 0 \in \mathcal{H}$.

Take a vertex $x \in T$. As $g_n \to \infty$, by Remark 4.6 we extract a subsequence $\{g_{n_k}\}_{n_k>0} \subset \{g_n\}_{n>0}$ such that $g_{n_k}(x) \to \xi$ in the cone topology of $T \cup \partial T$. Applying Proposition 4.7 we obtain that $G^0_\xi \leq G_{v_0}$. By Lemma 4.5 we conclude that $G_{v_0}$ is not compact, which contradicts our hypotheses. The theorem stands proven.

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**5 The relative Howe–Moore property**

**Theorem 5.1.** Let $F$ be primitive. Then for every $\xi \in \partial T$ the pair $(G, G^0_\xi)$ has the relative Howe–Moore property.

**Proof.** Let $\xi \in \partial T$ and let $(\pi, \mathcal{H})$ be a unitary representation of $G$ that does not have non-zero $G^0_\xi$-invariant vectors. We need to prove that $\pi|_{G^0_\xi}$ is a $C_0$ unitary representation of $G^0_\xi$. Suppose that this is not the case, so that $\pi|_{G^0_\xi}$ is not $C_0$. As $G^0_\xi$ is not compact by Lemma 4.5, there exists a sequence $\{g_n\}_{n \geq 0} \subset G^0_\xi$, with $g_n \to \infty$, and $v, w \in \mathcal{H} \setminus \{0\}$ such that $|\langle \pi(g_n v), w \rangle| \not\to 0$ when $g_n \to \infty$. Because the set $\{\pi(g_n v)\}_{n \geq 0}$ is bounded in the norm of $\mathcal{H}$, there exists $v_0 \in \mathcal{H}$ and a subsequence $\{n_k\}_{k \geq 0}$ such that $\{\pi(g_{n_k} v)\}_{k \geq 0}$ weakly converges to $v_0$. In particular $\lim_{n_k \to \infty} \langle \pi(g_{n_k} v), w \rangle = \langle v_0, w \rangle$, thus by the assumption above we have that $v_0$ is a non-zero vector. Moreover, we have that $g_{n_k} \to \infty$ and for a fixed vertex $x \in T$ we also have that $g_{n_k}(x) \to \xi$ with respect to the cone topology on $T$, when $n_k \to \infty$. The latter assertion follows because $\{g_{n_k}\}_{k \geq 0} \subset G^0_\xi$ and because $g_{n_k} \to \infty$. 

11
As the hypotheses of Proposition 4.7 are satisfied for the sequence \( \{g_{n_k}\}_{k \geq 0} \subset G_\xi^0 \leq G \), the fixed vertex \( x \in T \) and the vector \( v, v_0 \), we conclude that \( G_\xi^0 \leq G_{v_0} \). Thus \( v_0 \) is a non-zero vector of \( \mathcal{H} \) that is invariant under \( G_\xi^0 \), contradicting the assumption that \( (\pi, \mathcal{H}) \) does not have non-zero \( G_\xi^0 \)-invariant vectors. The conclusion follows. □

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