LOOP HOMOLOGY OF QUATERNIONIC PROJECTIVE SPACES

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Abstract. We determine the Batalin-Vilkovisky algebra structure of the integral loop homology of quaternionic projective spaces and octonionic projective plane.

1. Introduction

Let $M$ be a closed oriented manifold of dimension $d$ and let $LM = \text{Map}(S^1, M)$ denote its free loop space. By loop homology we understand the homology groups of $LM$ with the degree shifted by $-d$

$$H_*(LM) = H_{*+d}(LM).$$

In [2] it was shown that this graded group can be equipped with a product and an operator $\Delta$ giving $\mathbb{H}_*(LM)$ the structure of a Batalin-Vilkovisky algebra. The methods computing the product on concrete manifolds are based either on the modified Serre spectral sequence derived in [4] or on the isomorphism of the loop homology of $M$ with the Hochschild cohomology $HH^*(C^*(M); C^*(M))$ of the cochain complex as rings, [3]. There is also a way of defining a BV-structure on $HH^*(C^*(M); C^*(M))$, [11].

The BV-algebra structures on the loop homology and the Hochschild cohomology are isomorphic over the fields of characteristic zero ([5]) but not over other coefficients in general. Hence the computation of the BV operator is more subtle. So far the BV-algebra structure of the loop homology with integral coefficients has been determined for the Lie groups [7], for the spheres [9], for the complex Stiefel manifolds [10] and for the complex projective spaces [8]. Over rationals it has been described for the quaternionic projective spaces [13] and the surfaces [12].

The aim of this note is to describe the BV-algebra structure of the integral loop homology of the quaternionic projective spaces $\mathbb{H}P^n$ and the octonionic projective plane $\mathbb{O}P^2$.

Theorem 1.1. The string topology BV-algebra structure of $\mathbb{H}P^n$ is given by

$$\mathbb{H}_*(L\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[a, b, x] / \langle a^{n+1}, b^2, a^n \cdot b, (n+1)a^n \cdot x \rangle$$

with $a \in \mathbb{H}_{-4}(L\mathbb{H}P^n; \mathbb{Z})$, $b \in \mathbb{H}_{-1}(L\mathbb{H}P^n; \mathbb{Z})$ and $x \in \mathbb{H}_{4n+2}(L\mathbb{H}P^n)$, and

$$\Delta(a^px^q) = 0, \quad \Delta(a^p bx^q) = [(n-p) + q(n+1)]a^p x^q$$

Date: April 9, 2010.

2000 Mathematics Subject Classification. 55P35; 55R20.

Key words and phrases. Quaternionic projective space, octonionic projective plane, free loop space, integral loop homology, Batalin-Vilkovisky algebra.

This work was supported by the grant MSM0021622409 of the Czech Ministry of Education and the grant 0964/2009 of Masaryk University.

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for all $0 \leq p \leq n$, $0 \leq q$.

Let us note that for $n = 1$ the quaternionic projective space is $S^4$ and the statement agrees with the result obtain by L. Menichi in [9] for even dimensional spheres.

**Theorem 1.2.** There are elements $a \in \mathbb{H}_{-8}(L\Omega P^2; \mathbb{Z})$, $b \in \mathbb{H}_{-1}(L\Omega P^2; \mathbb{Z})$ and $x \in \mathbb{H}_{22}(L\Omega P^2)$ such that the string topology BV-algebra structure of $\Omega P^2$ is given by

$$
\mathbb{H}_*(L\Omega P^2; \mathbb{Z}) \cong \frac{\mathbb{Z}[a,b,x]}{(a^3, b^2, a^{2}\cdot b, 3a^n \cdot x)}
$$

and

$$
\Delta(a^p x^q) = 0, \quad \Delta(a^p b x^q) = (2 + 3q - p)a^p x^q
$$

for all $0 \leq p \leq 2$, $0 \leq q$.

The statements of Theorem 1.1 concerning the ring structure are consequences of the computation of $HH^*(\mathbb{Z}[y]/y^{n+1}; \mathbb{Z}[y]/y^{n+1})$ in [13] and the ring isomorphism between the loop homology and the Hochschild cohomology. Nevertheless, we provide an alternative proof using the Serre spectral sequence for the fibrations $\Omega M \to LM \to M$ converging to the ring $\mathbb{H}_*(LM; \mathbb{Z})$. These computations will be carried out in the next section.

In the last section we will show what the BV operator $\Delta$ looks like. We use the knowledge of $\Delta$ on $S^4$ and $S^8$ and the inclusions $S^4 = \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^n$ and $S^8 \hookrightarrow \Omega P^2$. The computation will be completed by comparing $\Delta$ in integral homology with BV-operator $\Delta$ in rational cohomology computed by Yang in [13]. The results show that for the quaternionic projective spaces and the octonionic projective plane the BV-algebra structures on the loop homology and the Hochschild homology over integers are isomorphic (in contrast to the complex projective spaces, see [8]).

2. **The ring structure of loop homology**

According to [4] the spectral sequence for the fibration $\Omega M \to LM \to M$ with $E_2^{p,q} = H^{-p}(M; H_q(\Omega M; \mathbb{Z}))$ and the product coming from the Pontryagin product in $H_*(\Omega M; \mathbb{Z})$ and the cup product in $H^*(M; H_*(\Omega M; \mathbb{Z})$ converges to $\mathbb{H}_{p+q}(LM; \mathbb{Z})$ as an algebra. To apply this spectral sequence to $M = \mathbb{H}P^n$ we have to determine the Pontryagin ring $H_*(\Omega \mathbb{H}P^n; \mathbb{Z})$. We will consider $n \geq 2$ since for $\mathbb{H}P^1 = S^4$ the statement of Theorem 1.1 has been proved in [9].

**Lemma 2.1.** For $n \geq 2$ the Pontryagin ring structure of $H_*(\Omega \mathbb{H}P^n; \mathbb{Z})$ is given by

$$
H_*(\Omega \mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[x] \otimes \Lambda[t]
$$

where the degrees of $x$ and $t$ are $4n + 2$ and 3, respectively.

**Proof.** The Hopf fibration $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ gives us the fibration

$$
\Omega S^{4n+3} \xrightarrow{i} \Omega \mathbb{H}P^n \xrightarrow{p} S^3
$$

Since $p_* : \pi_k(\Omega \mathbb{H}P^n) \to \pi_k(S^3)$ is an isomorphism for $0 \leq k \leq 6$, there is up to homotopy a unique map $i : S^3 \to \Omega \mathbb{H}P^n$ such that $p \circ i$ is homotopic to the identity.
on $S^3$. Therefore the long exact sequence of homotopy groups for this fibration passes to short exact sequences which split:

$$0 \longrightarrow \pi_* (\Omega S^{4n+3}) \xrightarrow{j_*} \pi_* (\Omega \mathbb{H} P^n) \xrightarrow{p_*} \pi_* (S^3) \longrightarrow 0$$

Denote by $\mu$ the Pontryagin product on $\Omega \mathbb{H} P^n$. The map $h = \mu \circ (j, i) : \Omega S^{4n+2} \times S^3 \rightarrow \Omega \mathbb{H} P^n$ is a homotopy equivalence since it induces an isomorphism of homotopy groups. So we obtain an isomorphism of homology groups

$$H_*(\Omega \mathbb{H} P^n; \mathbb{Z}) \cong H_*(\Omega S^{4n+3}; \mathbb{Z}) \otimes H_*(S^3; \mathbb{Z}) \cong \mathbb{Z}[x] \otimes \Lambda[t].$$

The Pontryagin ring structure of $H_*(\Omega \mathbb{H} P^n; \mathbb{Z})$ can be recovered using the duality between the Hopf algebras $H_*(\Omega \mathbb{H} P^n; \mathbb{Z})$ and $H^*(\Omega \mathbb{H} P^n; \mathbb{Z})$. The map $h$ induces an algebra isomorphism $h^* : H^*(\Omega \mathbb{H} P^n; \mathbb{Z}) \rightarrow H^*(\Omega S^{4n+3}; \mathbb{Z}) \otimes H^*(S^3; \mathbb{Z})$. We know that $H^*(\Omega \mathbb{H} P^n; \mathbb{Z})$ is a commutative associative Hopf algebra with $\mu^*$ as a coproduct. As an algebra $H^*(\Omega \mathbb{H} P^n; \mathbb{Z}) \cong \Gamma_\infty[\alpha_1, \alpha_2, \ldots] \otimes \Lambda[\beta]$, where $\Gamma_\infty[\alpha_1, \alpha_2, \ldots]$ is a divided polynomial algebra with generators $\alpha_i$ and relations $\alpha_i \alpha_j = \binom{i+j}{i} \alpha_{i+j}$. Since $j^* : H^*(\Omega \mathbb{H} P^n; \mathbb{Z}) \rightarrow H^*(\Omega S^{4n+3}; \mathbb{Z})$ is a homomorphism of Hopf algebras and the Hopf algebra structure of $H^*(\Omega S^{4n+3}; \mathbb{Z})$ is well known, the coproduct on $H_*(\Omega \mathbb{H} P^n; \mathbb{Z})$ is given by

$$\mu^*(\beta) = \beta \otimes 1 + 1 \otimes \beta, \quad \mu^*(\alpha_k) = \sum_{k=i+j} \alpha_i \otimes \alpha_j,$$

$$\mu^*(\alpha_k \beta) = \sum_{k=i+j} \alpha_i \beta \otimes \alpha_j + \sum_{k=i+j} \alpha_i \otimes \beta \alpha_j.$$

By duality this coproduct completely determines the Pontryagin product in $H^*(\Omega \mathbb{H} P^n; \mathbb{Z})$. Let $t \in H_*(\Omega \mathbb{H} P^n)$ be a dual element to $\beta$, $x_k$ be a dual to $\alpha_k$ and $z_k$ be a dual to $\alpha_k \beta$. Then

$$x_{i+j} = x_i x_j, \quad z_{i+j} = z_i x_j.$$

If we put $x = x_1$, we obtain $x_i = x^i$ and $z_i = x^i t$ for all $i \geq 0$. This completes the proof. □

Now we return to the spectral sequence converging to the algebra $\mathbb{H}_*(L \mathbb{H} P^n; \mathbb{Z})$. Its $E^2$ term is

$$E^2_{p,q} = H^p(\mathbb{H} P^n, H_q(\Omega \mathbb{H} P^n; \mathbb{Z})) \cong H^p(\mathbb{H} P^n; \mathbb{Z}) \otimes H_q(\Omega \mathbb{H} P^n; \mathbb{Z}) \cong \frac{\mathbb{Z}[a] \otimes \mathbb{Z}[x, t]}{(a^n+1, t^2)}.$$
where \( a \in H^4(M; \mathbb{Z}) \) and \( x, t \) as in Lemma 2.1. The stages \( E^4 \) and \( E^{4n} \) of the spectral sequence with possible nonzero differentials are shown in the following diagram:

\[
E^4_{p,q}, E^{4n}_{p,q} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
\]

Since \( E^\infty_p \Rightarrow \mathbb{H}^p + q_\mathbb{H}(\mathbb{H}P^n; \mathbb{Z}) = H_p + 4q_\mathbb{H}(\mathbb{H}P^n; \mathbb{Z}) \) we can determine the differentials from the knowledge of the additive structure of \( H_*(\mathbb{H}P^n; \mathbb{Z}) \).

To compute it we use the result of [1] on the existence of a stable decomposition

\[
(L\mathbb{H}P^n)_+ \simeq \mathbb{H}P^n_+ \vee \bigvee_{l \geq 1} S(\eta)^{l \xi \oplus (l-1)\zeta}
\]

where \( \eta \) is tangent bundle of the quaternionic projective space \( \mathbb{H}P^n \), \( \xi \) is the 3-dimensional Lie algebra bundle over \( \mathbb{H}P^n \) and \( \zeta \) is the fibrewise tangent bundle of \( S(\eta) \) and \( S(\eta)^\omega \) stands for the Thom space of the vector bundle \( \omega \) over \( S(\eta) \). Note that \( \dim S(\eta) = 8n - 1 \) and \( \dim \zeta = 4n - 1 \). Using the Gysin long exact sequence for the fibration \( S^{4n-1} \to S(\eta) \to \mathbb{H}P^n \) and the fact that the Euler characteristic class of \( \eta \) is an \((n+1)\)-multiple of the generator \( a^n \in H_{4n}(\mathbb{H}P^n; \mathbb{Z}) \) we get

\[
H_i S(\eta) = \begin{cases} 
\mathbb{Z} & i = 0, 4, \ldots, 4n - 4, 4n + 3, 4n + 7, \ldots, 8n - 1, \\
\mathbb{Z}_{n+1} & i = 4n - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

The dimension of the vector bundle \( l \xi \oplus (l-1)\zeta \) is \( 4n(l-1) + 2l + 1 \), so due to the Thom isomorphism

\[
H_*(L\mathbb{H}P^n; \mathbb{Z}) \cong H_*(\mathbb{H}P^n; \mathbb{Z}) \oplus \bigoplus_{l \geq 1} H_{*+4n(l-1)+2l+1}(S(\eta); \mathbb{Z}).
\]
Since $\mathbb{H}_s(L\mathbb{H}P^n; \mathbb{Z}) \cong E_{s,s}^\infty$, the $E^\infty$ stage of the spectral sequence is the following:

\[
\begin{array}{cccccccccccc}
 & & & & & & & & & & & & \downarrow q \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\downarrow p & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
- & - & - & - & Z & - & - & - & Z & - & - & 0 && 8n + 7 \\
- & Z_{n+1} & - & - & - & Z & - & - & Z & - & - & Z && 8n + 4 \\
- & - & Z & - & - & - & Z & - & - & Z & - & - & Z && 4n + 5 \\
- & Z_{n+1} & - & - & - & Z & - & - & Z & - & - & Z && 4n + 2 \\
- & - & Z & - & - & - & Z & - & - & Z & - & - & Z && 3 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
- & - & - & - & Z & - & - & - & Z & - & - & 0 & 3 \\
- & Z_{n+1} & - & - & - & Z & - & - & Z & - & - & Z & 0 \\
\end{array}
\]

It forces the differentials $d^4$ in $E^4$ to be zero and the differentials $d^{4n} : E_{0,(4n+2)i+3}^4 \to E_{4n,(4n+2)(i+1)}^4$ to be the multiplication by $n + 1$.

So $E_{s,s}^\infty$ as a ring is generated by the group generators $a \in E_{-4,0}^\infty \cong H^4(\mathbb{H}P^n; \mathbb{Z})$, $x \in E_{0,4n+2}^\infty \cong H_{4n+2}(\Omega\mathbb{H}P^n; \mathbb{Z})$ and $b \in E_{-4,3}^\infty \cong H^4(\mathbb{H}P^n; \mathbb{Z}) \otimes H_3(\Omega\mathbb{H}P^n; \mathbb{Z})$ which satisfy relations $a^{n+1} = 0$, $(n+1)x \otimes a^n = 0$, $b \otimes a^n = 0$, $b^2 = 0$. We conclude that as rings

\[
\mathbb{H}_s(L\mathbb{H}P^n; \mathbb{Z}) \cong E_{s,s}^\infty \cong \mathbb{Z}[a, b, x]/\langle a^{n+1}, b^2, a^n b, (n+1)a^n x \rangle.
\]

In the case of the octonionic projective plane the derivation of the ring structure of the loop homology follows the same lines.

**Lemma 2.2.** The Pontryagin ring structure of $H_*(\Omega\mathbb{O}P^2; \mathbb{Z})$ is given by

\[
H_*(\Omega\mathbb{O}P^2; \mathbb{Z}) \cong \mathbb{Z}[x] \otimes \Lambda[t]
\]

where $|x| = 22$ and $|t| = 7$.

**Proof.** Using the fact that

\[
H^*(\Omega\mathbb{O}P^2) \cong \Gamma[\alpha_1, \alpha_2, \ldots] \otimes \Lambda[\beta]
\]

where $|\alpha_i| = 22i$ and $|\beta| = 7$, proved in [6], we can proceed in the same way as in the proof of Lemma 2.1. \[\square\]
The additive structure of \( H_*(L\mathcal{O}P^2; \mathbb{Z}) \) was found in [1] using a stable decomposition of \( L\mathcal{O}P^2 \) derived there:

\[
H_i(L\mathcal{O}P^2) = \begin{cases} 
\mathbb{Z} & i = 0, 8, 16, 22m - 15, 22m - 7, 22m + 8, 22m + 16, \\
\mathbb{Z}_3 & i = 22m, \\
0 & \text{otherwise}.
\end{cases}
\]

It yields that in the spectral sequence starting with

\[
E^2_{p,q} = H^p(\mathcal{O}P^2; H_q(\Omega\mathcal{O}P^2; \mathbb{Z})) \cong H^p(\mathcal{O}P^2; \mathbb{Z}) \otimes H_q(\Omega\mathcal{O}P^2; \mathbb{Z}) \cong \mathbb{Z}[a] \otimes \mathbb{Z}[x, b]
\]

all the differentials are zero with the exception of the differentials \( d^{16} : E^{16}_{0,22m-15} \rightarrow E^{16}_{1,22m} \) which act as the multiplication by \( 3 \). The group generators \( a \in E^{\infty}_{-8,0} \cong H^*(\mathcal{O}P^2; \mathbb{Z}), \ x \in E^{\infty}_{0,22} \cong H_{22}(\Omega\mathcal{O}P^2; \mathbb{Z}) \) and \( b \in E^{\infty}_{8,7} \cong H^8(\mathbb{H}\mathcal{P}^n; \mathbb{Z}) \otimes H_7(\Omega\mathcal{O}P^2; \mathbb{Z}) \), generate \( E^{\infty}_{*,*} \cong \mathbb{H}_*(L\mathcal{O}P^2; \mathbb{Z}) \) as a ring satisfying relations \( a^3 = 0, \ b^2 = 0, \ 3ax = 0 \) and \( a^2b = 0 \).

### 3. The BV Operator

The BV operator \( \Delta : \mathbb{H}_*(LM) \rightarrow \mathbb{H}_{*+1}(LM) \) and its unshifted version \( \Delta' : H_*(LM) \rightarrow H_{*+1}(LM) \) come from the canonical action of \( S^1 \) on \( LM \). So any map \( f : N \rightarrow M \) between manifolds induces a homomorphism \( H_*(LN) \rightarrow H_*(LM) \) which commutes with \( \Delta' \). To determine the BV operator on \( \mathbb{H}_*(L\mathbb{H}\mathcal{P}^n; \mathbb{Z}) \) and \( \mathbb{H}_*(L\mathcal{O}P^2; \mathbb{Z}) \) we use this fact for the inclusions \( S^1 \hookrightarrow \mathbb{H}\mathcal{P}^n \) and \( S^8 \hookrightarrow \mathcal{O}P^2 \) together with the knowledge of the BV operator on \( \mathbb{H}_*(S^n; \mathbb{Z}) \), see [9].

We start with the quaternionic projective space. First, \( \Delta(a^p x^q) = 0 \) because \( \mathbb{H}_{[a^p x^q]+1}(L\mathbb{H}\mathcal{P}^n; \mathbb{Z}) = 0 \). Since \( \mathbb{H}_{[a^p b]+1}(L\mathbb{H}\mathcal{P}^n; \mathbb{Z}) \cong \mathbb{Z} \) is generated by \( a^p b \), there is an integer \( \nu_p \) such that \( \Delta(a^p b) = \nu_p a^p b \). Due to the relation

\[
\Delta(xyz) = \Delta(x)yz + (-1)^{|x|}\Delta(y)z + (-1)^{|y|}y\Delta(z) - \Delta(x)y - (-1)^{|y|}y\Delta(x)z - (-1)^{|x|+|y|}xy\Delta(z)
\]

we obtain

\[
\Delta(a^pb) = \Delta(a^{p-1}ab) = a^{p-1}\Delta(ab) + a\Delta(a^{p-1}b) - a^p\Delta(b).
\]

It yields the equation \( \nu_p = \nu_1 - \nu_0 + \nu_{p-1} \), which can be rewritten as

\[
\nu_p = p(\nu_1 - \nu_0) + \nu_0.
\]

The relation \( a^nb = 0 \) implies that \( \nu_n = 0 \). Consequently, for \( p = n \) the equation above gives \( n\nu_1 = (n-1)\nu_0 \). Hence for \( n \geq 2 \) the only possible integer solutions of this equation are

\[
\nu_1 = (n-1)\lambda_n, \quad \nu_0 = n\lambda_n,
\]

where \( \lambda_n \) is an integer. Consequently, we obtain \( \nu_p = (n-p)\lambda_n \).

For \( n = 1 \) the quaternionic projective space is \( S^4 \). According to [9] the generators of \( \mathbb{H}_*(L\mathbb{H}\mathcal{P}^1; \mathbb{Z}) \) as an algebra are \( a_1, b_1 \) and \( v_1 \) in degrees \(-4, -1 \) and \( 6 \), respectively, and \( \Delta(b_1) = 1 \).
The standard inclusion $i: S^4 = \mathbb{H}P^1 \hookrightarrow \mathbb{H}P^n$ induces the commutative diagram of fibrations

\[
\begin{array}{ccc}
\Omega \mathbb{H}P^1 & \longrightarrow & \Omega \mathbb{H}P^n \\
\downarrow & & \downarrow \\
L \mathbb{H}P^1 & \longrightarrow & L \mathbb{H}P^n \\
\downarrow & & \downarrow \\
\mathbb{H}P^1 & \longrightarrow & \mathbb{H}P^n
\end{array}
\]

The inclusion $i$ induces an isomorphism $H_4(\mathbb{H}P^1; \mathbb{Z}) \cong H_4(\mathbb{H}P^n; \mathbb{Z})$ and the inclusion $\Omega \mathbb{H}P^1 \hookrightarrow \Omega \mathbb{H}P^n$ yields an isomorphism $H_3(\Omega \mathbb{H}P^1; \mathbb{Z}) \cong H_3(\Omega \mathbb{H}P^n; \mathbb{Z})$.

The commutative diagram above gives us a homomorphism between the Serre spectral sequences of the corresponding fibrations. (Here we consider the spectral sequence of the inclusion $\mathbb{H}P^1 \hookrightarrow \mathbb{H}P^n$ yielding an isomorphism $H_4(\mathbb{H}P^1; \mathbb{Z}) \cong H_4(\mathbb{H}P^n; \mathbb{Z})$.)

Choose $b \in \mathbb{H}_1(L \mathbb{H}P^n; \mathbb{Z})$ and $a \in \mathbb{H}_2(\mathbb{H}P^n; \mathbb{Z})$ so that $a^{n-1}$ is the image of 1 and $a^{n-1}b$ is the image of $b_1 \in \mathbb{H}_1(L \mathbb{H}P^n; \mathbb{Z})$ under the above isomorphism. Since these isomorphisms commute with $\Delta$, we obtain $\Delta(a^{n-1}b) = a^{n-1}$. Consequently, $\lambda_n = 1$.

Analogously we get $\Delta(bx^q) = \rho_q x^q$ for an integer $\rho_q$ and derive that

$$\Delta(a^p bx^q) = \Delta(a^p b)x^q + a^p \Delta(bx^q) - a^p x^q \Delta(b) = (n-p) + q(p_1 - n) |a^p x^q|.$$ 

In [13] T. Yang computed the BV-algebra structure of the Hochshild cohomology of truncated polynomials. Using Theorem 1 from [5] on the existence of a BV-algebra isomorphism between the loop homology $\mathbb{H}_*(LM; F)$ of a manifold and the Hochschild cohomology $HH^*(C^*(M); C^*(M))$ of the singular cochain complex over fields of characteristic zero, he was able to calculate the BV-algebra structure of $\mathbb{H}_*(L \mathbb{H}P^n; \mathbb{Q})$. This is given by

$$\mathbb{H}_*(L \mathbb{H}P^n; \mathbb{Q}) = \mathbb{Q}[\alpha, \beta, \chi]/(\alpha^{n+1}, \beta^2, \alpha^n \beta, \alpha^n \chi),$$

where $|\alpha| = -4$, $|\beta| = -1$, $|\chi| = 4n + 2$, and by

$$\Delta(a^p \chi^q) = 0, \quad \Delta(a^p \beta \chi^q) = [(n-p) + q(n+1)]a^p \chi^q.$$ 

Consider the homomorphism $r_*: \mathbb{H}_*(L \mathbb{H}P^n; \mathbb{Z}) \rightarrow \mathbb{H}_*(L \mathbb{H}P^n; \mathbb{Q})$ induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Let

$$r_*(a) = k \alpha, \quad r_*(b) = l \beta, \quad r_*(x) = m \chi,$$

where, $k, l, m \in \mathbb{Q} - \{0\}$. Since $r_*$ is a homomorphism of BV-algebras, we obtain

$$[(n-p) + q(p_1 - n)]k^p m^q a^p \chi^q = r_*(\Delta(a^p bx^q)) = r_*(\Delta(r_*(a^p bx^q))) = l[(n-p) + q(n+1)]k^p m^q a^p \chi^q.$$
Putting \( q = 0 \) we get \( l = 1 \). Then the choice \( p = 0, q = 1 \) yields \( \rho_1 = 2n + 1 \) which concludes our computation.

To compute the BV operator in \( \mathbb{H}_*(L\mathbb{O} P^2; \mathbb{Z}) \) we can follow the same procedure step by step replacing the inclusion \( S^4 \hookrightarrow \mathbb{H}P^n \) by the inclusion \( S^8 \hookrightarrow \mathbb{O}P^2 \).

References

[1] M.B. Böckstedt, I.M. Ottosen, The suspended free loop space of a symmetric space, arXiv:math.AT/0511086 (2005).
[2] M. Chas, D. Sullivan, String topology, arXiv:math.GT/9911159v1.
[3] R.L. Cohen, J.D.S. Jones, A homotopy theoretic realization of string topology, Math. Ann. 324 (2002), 773-798.
[4] R.L. Cohen, J.D.S. Jones, J. Yan, The loop homology algebra of sphere and projective spaces, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), 77–92, Progr. Math. 215, Birkhäuser, Basel, 2004.
[5] Y. Félix, J. Thomas, Rational BV-algebra in string topology, Bull. Soc. Math. France 136 (2008), no. 2, 311–327.
[6] E. Halpern, The cohomology algebra of certain loop spaces, Proc. Amer. Math. Soc. 9 (1958), 808-817.
[7] R.A. Hepworth, String topology for Lie groups, arXiv:math.AT/0905.1199v1.
[8] R.A. Hepworth, String topology for complex projective spaces, arXiv:math.AT/0908.1013v1.
[9] L. Menichi, String topology for spheres, Comment. Math. Helv. 84 (2009), 135-157.
[10] H. Tamanoi, Batalin-Vilkovisky Lie algebra structure on the loop homology of complex Stiefel manifolds, Int. Math. Res. Not.23 (2006), Art. ID 97193, 23 pp.
[11] T. Tradler, The Batalin-Vilkovisky algebra on Hochschild cohomology induced by infinity inner products, Ann. Inst. Fourier (Grenoble) 58 (2008), 2351–2379.
[12] D. Vaintrob, The string topology BV algebra, Hochschild cohomology and the Goldman bracket on surfaces, arXiv:math.AT/0702.2850.
[13] T. Yang, A Batalin-Vilkovisky algebra structure on the Hochschild cohomology of truncated polynomials, arXiv:math.AT/0707.4213.