On Lie algebras associated with representation directed algebras

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Abstract

Let $B$ be a representation-finite $C$-algebra. The $Z$-Lie algebra $L(B)$ associated with $B$ has been defined by Ch. Riedtmann in [17]. If $B$ is representation-directed there is another $Z$-Lie algebra associated with $B$ defined by C. M. Ringel in [20] and denoted by $K(B)$.

We prove that the Lie algebras $L(B)$ and $K(B)$ are isomorphic for any representation-directed $C$-algebra $B$.

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1. Introduction

Let $C$ be the field of complex numbers and let $B$ be a finite dimensional associative basic $C$-algebra with unit element. By a $B$-module we mean a finite dimensional right $B$-module. Assume that $B$ is a representation-directed algebra (see Section 2 for definitions). In particular, the algebra $B$ is representation-finite, that is, there is only finitely many isomorphism classes of indecomposable $B$-modules. Let $\mathcal{M}(B)$ (resp. $\text{ind}(B)$) be a set of representatives of all isomorphism classes of $B$-modules (resp. indecomposable $B$-modules). In [17], with any representation-finite algebra $B$, Ch. Riedtmann associated the $Z$-Lie algebra $L(B)$, which is the free $Z$-module with basis $\{v_X ; X \in \text{ind}(B)\}$. If $B$ is representation-directed, then the Lie algebra structure on $L(B)$ is defined in the following way. Let $X$, $Y$ be non-isomorphic indecomposable $B$-modules such that $\text{Ext}^1_B(X, Y) = 0$. Put

$$\mathcal{E}(X, Y ; Z) = \{ U \subseteq Z ; \ U \cong X, \ Z/U \cong Y \}.$$
This is locally a closed subset of a product of Grassmann varieties, see [17] and Section 3.1 for details. We set
\[
[v_X, v_Y] = \begin{cases} 
\chi(\mathcal{E}(X, Y; Z)) \cdot v_Z & \text{if there is an indecomposable } B\text{-module } Z \\
0 & \text{and a short exact sequence } 0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0,
\end{cases}
\]
otherwise,

where \(\chi(\mathcal{E}(X, Y; Z))\) denotes the Euler-Poincaré characteristic of \(\mathcal{E}(X, Y; Z)\). This formula defines the unique Lie algebra structure on \(L(B)\).

On the other hand, in [20], C. M. Ringel associated with a representation-directed algebra \(B\) the \(\mathbb{Z}\)-Lie algebra \(K(B)\), which is the free \(\mathbb{Z}\)-module with basis \(\{u_X : X \in \text{ind}(B)\}\). If \(X, Y\) are non-isomorphic indecomposable \(B\)-modules such that \(\text{Ext}^1_B(X, Y) = 0\), we set
\[
[u_Y, u_X] = \begin{cases} 
\varphi_{Y,X}^\mathbb{Z}(1) \cdot u_Z & \text{if there is an indecomposable } B\text{-module } Z \\
0 & \text{and a short exact sequence } 0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0,
\end{cases}
\]
otherwise,

where \(\varphi_{Y,X}^\mathbb{Z}\) are Hall polynomials (see [20] and Section 3.2 for details). This formula defines the unique Lie algebra structure on \(K(B)\).

If \(B = \mathbb{C}Q\) is the path algebra of a Dynkin quiver \(Q\), then \(\mathbb{C} \otimes_{\mathbb{Z}} L(B) \cong \mathfrak{n}_+(Q)\) and \(\mathbb{C} \otimes_{\mathbb{Z}} K(B) \cong \mathfrak{n}_+(Q)\), where \(\mathfrak{n}_+(Q)\) is the positive part of the complex semisimple Lie algebra of type \(Q\) (see [17] Proposition 5.3 and [21] Corollary 3). Therefore for path algebras \(B = \mathbb{C}Q\) of Dynkin quivers the Lie algebras \(L(B)\) and \(K(B)\) are isomorphic. Ch. Riedtmann conjectured in [17] that the Lie algebras \(L(B)\) and \(K(B)\) are isomorphic for any representation-directed \(\mathbb{C}\)-algebra \(B\). The main aim of this paper is to prove this conjecture, more precisely we prove the following theorem.

**Theorem 1.1.** Let \(B\) be a representation-directed \(\mathbb{C}\)-algebra. There is a Lie algebra isomorphism
\[
F : L(B) \rightarrow K(B)
\]
given by the formula
\[
F(u_X) = (-1)^{\text{dim} X - 1} v_X.
\]

Theorem 1.1 is an immediate consequence of the following fact.

**Theorem 1.2.** Let \(B\) be a representation-directed \(\mathbb{C}\)-algebra and let \(X, Y, Z\) be \(B\)-modules (not necessarily indecomposable). Then
\[
\chi(\mathcal{E}(X, Y; Z)) = \varphi_{Y,X}^\mathbb{Z}(1).
\]
The reason for the appearance of \((-1)^{(\dim C - 1)}\) in the formula in 1.1 is that the original definitions of the lie brackets in \(L(B)\) and \(K(B)\) are not "compatible" in the sense that \([u_X, u_Y]\) refers to the extensions of \(Y\) by \(X\), whereas \([u_X, u_Y]\) to the extensions of \(X\) by \(Y\).

The paper is organized as follows.

- In Section 2 we collect some facts of the representation theory of algebras. In particular, we recall basic properties of representation-directed algebras and their Auslander-Reiten quivers.
- In Section 3.1 we recall definitions and basic properties of the \(\mathbb{Z}\)-Lie algebra \(L(B)\).
- In Section 3.2 we recall definitions and basic properties of the \(\mathbb{Z}\)-Lie algebra \(K(B)\), Hall polynomials and Ringel-Hall algebras.
- In Section 4 we show that the modules over a representation-directed algebra can be "defined over \(\mathbb{Z}\)" independently on the base field.
- In Section 5 we investigate the subset \(\mathcal{E}(X, Y; Z)\) of a product of Grassmann varieties. Results of Section 4 and 5 allow to describe \(\chi(\mathcal{E}(X, Y; Z))\) in terms of Hall polynomials \(\varphi^Z_{Y,X}\).
- Section 6 contains proofs of Theorems 1.1 and 1.2 and consequences of the theorems.

The motivation for the study of Lie algebras \(L(B)\) and \(K(B)\) is their connection with Ringel-Hall algebras, generic extensions, and quantum groups (see [14], [19], [20], [22] and [21]). Hall polynomials and the Euler-Poincaré characteristic \(\chi\) are developed in [5], where the authors investigate connections between indecomposable representation of Dynkin quivers of type \(A, D, E\) and the cluster variables of cluster algebras of type \(A, D, E\).

Results of this paper were presented by the second named author during the conference "Colloque d’algebre non-commutative" in University of Sherbrooke (June 2008) and on the "Seminar Darstellungstheorie" in Bielefeld during her stay in University of Bielefeld supported by SFB (September 2008).

2. Preliminaries

In this section we shortly recall basic definitions, notation and facts of the representation theory of representation-directed algebras. For the basic concepts of representation theory the reader is referred to [1] and [2].

We consider algebras of the form \(B = KQ/I\), where \(Q = (Q_0, Q_1)\) is a finite quiver and \(I\) is an admissible ideal of the path algebra \(KQ\) of \(Q\). Recall, that an ideal \(I\) is admissible if it is contained in the ideal generated by the paths of length at least 2 and there is a number \(N\) such that every path of length \(N\) belongs to \(I\). If \(Q\) is acyclic, that is, contains no oriented cycles, the latter condition is satisfied automatically.
If $B$ is as above then it is a finite dimensional associative basic $K$-algebra. All modules are assumed to be right finite dimensional. Denote by $\text{mod}(B)$ the category of all such $B$-modules. The category $\text{mod}(B)$ is equivalent to the category $\text{rep}_K(Q, I)$ of finite dimensional $K$-representations of $Q$ satisfying the relations from $I$, see [1, Definition 1.4]. We identify the two categories.

Given a field $K$ and a dimension vector $d \in \mathbb{N}^{Q_0}$ let $\text{rep}_K(Q, d)$ be the variety of $K$-representations of $Q$ with the dimension vector $d$. That is,

$$\text{rep}_K(Q, d) = \prod_{\alpha \in Q_1} \mathbb{M}_{d_{\tau(\alpha)}d_{s(\alpha)}}(K),$$

where $t(\alpha)$ and $s(\alpha)$ denote the terminus and the source of $\alpha$, respectively, and $\mathbb{M}_{a,b}(K)$ is the space of all $a \times b$-matrices. We identify a point of $\text{rep}_K(Q, d)$ with the corresponding representation of $Q$ in the usual way.

The group $\text{Gl}(d, K) = \prod_{x \in Q_0} \text{Gl}(d_x, K)$ acts "by conjugations" on $\text{rep}_K(Q, d)$ and the orbit of the point $M$ is the set of points corresponding to the representations isomorphic to $M$.

For any $B$-module $M$, identified with the representation $(M_x, M_\alpha)_{x \in Q_0, \alpha \in Q_1}$, denote by $\text{dim} \ M \in \mathbb{N}^{Q_0}$ the dimension vector of $M$ defined by $\text{dim} \ M_x = \dim_K M_x$, for any $x \in Q_0$.

Let $\mathcal{M}(B)$ (resp. $\text{ind}(B)$) be a set of representatives of all isomorphism classes of $B$-modules (resp. indecomposable $B$-modules).

Denote by $\Gamma_B = ((\Gamma_B)_0, (\Gamma_B)_1)$ the Auslander-Reiten quiver of $B$ and by $\tau_B$ the Auslander-Reiten translation in $\Gamma_B$. Recall that there is a bijection $(\Gamma_B)_0 \leftrightarrow \text{ind}(B)$.

Given two indecomposable $B$-modules $X$, $Y$ we write $X \leq Y$ if there is a sequence of nonzero maps $X = Y_0 \rightarrow Y_1 \rightarrow \ldots \rightarrow Y_i = Y$ with $Y_0, \ldots, Y_i$ indecomposable. Following [23], we say that a $B$-module $X$ is directing if there do not exist indecomposable direct summands $X_1$, $X_2$ of $X$ and an indecomposable nonprojective module $Y$ such that $X_1 \leq \tau_B Y$ and $Y \leq X_2$.

An algebra $B$ is said to be representation-finite (resp. representation-directed), if there is only finitely many isomorphism classes of indecomposable $B$-modules (resp. any indecomposable $B$-module is directing). It is well known that any representation-directed algebra is representation-finite [23 page 78].

If $B$ is a connected representation-directed algebra then the Auslander-Reiten quiver of $B$ coincides with its (unique) preprojective component. Moreover, if $B$ is representation-directed then we may enumerate the indecomposable $B$-modules $X_1, \ldots, X_m$ in such a way that $\text{Ext}_B^1(X_i, X_j) = 0$, if $i \leq j$, and $\text{Hom}_B(X_i, X_j) = 0$, if $j < i$.

If in addition $B$ is of the form $KQ/I$ and $I$ is an admissible ideal, then every arrow of $\Gamma_B$ has trivial valuation [2, Section VII].

Assuming this, let $P^B_x$ be an indecomposable projective $B$-module associated with the vertex $x \in Q_0$ and decompose the radical $\text{rad} P^B_x$ of $P^B_x$ into indecomposable direct summands:

$$\text{rad} P^B_x \cong M^1_x \oplus \ldots \oplus M^{r_x}_x.$$
We denote the set \( \{ \dim M_1^x, \ldots, \dim M_r^x \} \) of their dimension vectors by \( R^B_x \).

Observe that these dimension vectors are pairwise different.

**Lemma 2.1.** Let \( Q \) be a connected acyclic quiver, \( B = KQ/I, C = LQ/J \), for some fields \( K, L \) and admissible ideals \( I \) and \( J \) of \( KQ \) and \( LQ \) respectively. Assume that \( \dim P^B_x = \dim P^C_x, R^B_x = R^C_x \), for any \( x \in Q_0 \).

If \( B \) is representation-directed then \( C \) is also representation-directed and there is an isomorphism \( \sigma : \Gamma_B \rightarrow \Gamma_C \) of translation quivers such that the dimension vectors of the \( B \)-module corresponding to the vertex \( m \) of \( \Gamma_B \) and the \( C \)-module corresponding to the vertex \( \sigma(m) \) of \( \Gamma_C \) are equal, for any \( m \).

**Proof.** The input data of the usual algorithm for constructing preprojective components of algebra (“knitting procedure”) are the dimension vectors of

(i) the indecomposable projective modules and

(ii) the indecomposable direct summands of the radicals of the indecomposable projective modules

(see [6], [12]). We mean here an algorithm determining the “combinatorial data” of preprojective components: the dimension vectors of indecomposable preprojective modules and the numbers of arrows between the vertices.

Applying the algorithm we construct the Auslander-Reiten quiver of \( B \) (equal to its unique preprojective component). Since the input data of the algorithm are the same for both algebras, then there exist unique preprojective component \( P \) of \( C \) and an isomorphism \( \sigma \Gamma_B \rightarrow P \). Moreover \( \sigma \) preserves dimension vectors. The component \( P \) is finite thus coincides with \( \Gamma_C \) by [2, Theorem 2.1], and the lemma follows. \( \square \)

In order to investigate modules over representation directed algebras independently on the base field we need the concept of lattices over orders.

By a \( \mathbb{Z} \)-**order** we mean a unital ring \( A \) which is free and finitely generated as a \( \mathbb{Z} \)-module. An \( A \)-**lattice** is a right \( A \)-module, free and finitely generated as a \( \mathbb{Z} \)-module. We denote by \( \text{latt}(A) \) the category of \( A \)-lattices. Given a unital commutative ring \( R \) let \( A^{(R)} \) denote the \( R \)-algebra \( A \otimes \mathbb{Z} R \). We call it **specialization** of \( A \) to \( R \). Similarly, given an \( A \)-lattice \( X \) we denote by \( X^{(R)} \) the \( A^{(R)} \)-module \( A \otimes \mathbb{Z} R \).

We are interested in \( \mathbb{Z} \)-orders of the form \( \mathbb{Z}Q/I \), where \( \mathbb{Z}Q \) is the path \( \mathbb{Z} \)-algebra of some quiver \( Q \) and \( I \) is an admissible ideal of \( \mathbb{Z}Q \). The ring \( \mathbb{Z}Q/I \) is a \( \mathbb{Z} \)-order if an only if \( I \) is a pure \( \mathbb{Z} \)-submodule of \( \mathbb{Z}Q \).

If \( A = \mathbb{Z}Q/I \) is a \( \mathbb{Z} \)-order and \( L \) is a field, then the \( L \)-algebra \( A^{(L)} \) is isomorphic to \( LQ/I^{(L)} \), where \( I \) is the ideal of \( LQ \) generated by the image of \( I \) under the canonical homomorphism \( \mathbb{Z}Q \rightarrow LQ \). Observe that if \( I \) is admissible, then \( I^{(L)} \) is also.

Given a quiver \( Q \) let \( \text{rep}_\mathbb{Z}(Q) \) be the category of all finitely generated \( \mathbb{Z} \)-representations of \( Q \), that is the systems \( (N_x, N_\alpha)_{x \in Q_0, \alpha \in Q_1} \), where \( N_x \) are
finitely generated \( \mathbb{Z} \)-modules and \( N_\alpha : N_{s(\alpha)} \rightarrow N_{t(\alpha)} \) are \( \mathbb{Z} \)-homomorphisms, for \( \alpha \in Q_1 \). The morphisms are defined in the usual way.

We denote by \( \text{rep}_Z(Q) \) the full subcategory of \( \text{rep}_Z(Q) \) consisting of the representations \( (N_x, N_\alpha)_{x \in Q_0, \alpha \in Q_1} \) such that \( N_x \) is a free \( \mathbb{Z} \)-module, for any \( x \in Q_0 \).

Given an ideal \( I \) of \( \mathbb{Z}Q \) we denote by \( \text{rep}_Z(Q, I) \) and \( \text{rep}_Z^f(Q, I) \) the full subcategories of \( \text{rep}_Z(Q) \) and \( \text{rep}_Z^f(Q) \), respectively, consisting of the representations satisfying all relations in \( I \).

It is clear that if \( A = \mathbb{Z}Q/I \) is a \( \mathbb{Z} \)-order then there is an equivalence of categories
\[
\text{latt}(A) \cong \text{rep}_Z^f(Q, I).
\]

From now on we identify the two categories.

Given an \( A \)-lattice \( N \) corresponding to the representation \( (N_x, N_\alpha)_{x \in Q_0, \alpha \in Q_1} \) we denote by \( \text{dim} \ N \in \mathbb{N}^{Q_0} \) the dimension vector of \( N \), that is \( (\text{dim} \ N)_x \) equals the rank of \( N_x, x \in Q_0 \).

3. Lie algebras associated with representation directed algebras

Given two natural numbers \( e \leq d \) and a field \( K \) we denote by \( \text{Gr}_e(d, K) \) the Grassmann variety of \( e \)-dimensional subspaces of \( K^d \), embedded into the projective space \( \mathbb{P}^{(d-1)} \) via Plücker embedding.

Let \( Q = (Q_0, Q_1) \) be a finite quiver and let \( K \) be a field. Given two vectors \( e, d \in \mathbb{N}^{Q_0} \) such that \( e_x \leq d_x \), for any \( x \in Q_0 \), we set
\[
\text{Gr}_e(d, K) = \prod_{x \in Q_0} \text{Gr}_{e_x}(d_x, K).
\]

Let \( M = (M_x, M_\alpha)_{x \in Q_0, \alpha \in Q_1} \) be a \( K \)-representation of \( Q \) with the dimension vector \( d \). We identify \( M_x \) with \( K^{d_x} \), for \( x \in Q_0 \). Let
\[
\text{Sub}_e(M) \subseteq \text{Gr}_e(d, K)
\]
be the set of the tuples \( N = (N_x)_{x \in Q_0} \) of subspaces which form a subrepresentation of \( M \), that is, such that \( M_\alpha(N_{s(\alpha)}) \subseteq N_{t(\alpha)} \), for any \( \alpha \in Q_1 \). Given \( N \in \text{Sub}_e(M) \) we treat \( N \) as a representation of \( Q \) in the natural way.

Assume that \( i_x \) indicates a homogeneous coordinate in \( \mathbb{P}^{(d_x-1)} \), for \( x \in Q_0 \), and put \( i = (i_x)_{x \in Q_0} \). We denote by \( U_i \) the open subset of \( \text{Gr}_e(d, K) \) consisting of the tuples \( N = (N_x)_{x \in Q_0} \) of subspaces such that the \( i_x \)th coordinate of \( N_x \) is nonzero, for \( x \in Q_0 \).

The following assertion follows by standard arguments.

**Lemma 3.1.** We keep the notation introduced above.

(1) The set \( \text{Sub}_e(M) \) is a Zariski-closed subset of \( \text{Gr}_e(d, K) \).
(2) For any possible \( i \), there exist regular maps
\[
\text{rep}_K(Q, e) \xrightarrow{\eta_i} \text{Sub}_e(M) \cap \mathcal{U}_i \xrightarrow{\rho_i} \text{rep}_K(Q, d - e)
\]
such that a subrepresentation \( N \) of \( M \) corresponds to \( \eta_i(N) \) and the factor-representation \( M/N \) corresponds to \( \rho_i(N) \), for any \( N \in \text{Sub}_e(M) \cap \mathcal{U}_i \).

Let \( B = KQ/I \) for some admissible ideal \( I \) of \( KQ \), \( e \in \mathbb{N}_Q^0 \), and let \( N_1, N_2, M \) be \( B \)-modules identified with representations of \( Q \). We set
\[
(E(N_1, N_2; M)) = \{ U \subseteq M ; U \in \text{mod}(B), U \cong N_1, X/U \cong N_2 \}.
\]
Note that \( E(N_1, N_2; M) \) is empty unless \( \dim M = \dim N_1 + \dim N_2 \). The set \( E(N_1, N_2; M) \) is a subset of
\[
\text{Gr}_{\dim N_1}((\dim M, K)).
\]

**Lemma 3.3.** Let \( K \) be an algebraically closed field. Then \( E(N_1, N_2; M) \) is a locally closed subset of \( \text{Gr}_{\dim N_1}((\dim M, K)) \).

**Proof.** The assertion follows from the well-known fact that an orbit of an algebraic group action is locally closed by Lemma 3.3. \( \square \)

### 3.1. Riedtmann’s Lie algebras

Let \( B = \mathbb{C}K/I \) be a representation-finite \( \mathbb{C} \)-algebra. In [17], Ch. Riedtmann associated with \( B \) the Lie algebra \( L(B) \) as follows. Let \( \chi(E(N_1, N_2; M)) \) denote the Euler-Poincaré characteristic of a locally closed subset \( E(N_1, N_2; M) \) defined in 3.2. Let \( R(B) \) be the free \( \mathbb{Z} \)-module with basis \( \{ v_M ; M \in \mathcal{M}(B) \} \). The formula
\[
v_M \cdot v_N = \sum_{X \in \mathcal{M}(B)} \chi(E(M, N; X)) v_X
\]
defines an associative \( \mathbb{Z} \)-algebra structure on \( R(B) \) (see [17, 2.3]). Note that the sum in (3.4) is finite, since there is only finitely many isomorphism classes of modules \( X \) with \( \dim X = \dim N + \dim M \) and \( \chi(E(M, N; X)) = 0 \) if \( \dim X \neq \dim N + \dim M \). The \( \mathbb{Z} \)-submodule
\[
L(B) = \bigoplus_{X \in \text{ind}(B)} \mathbb{Z} v_X
\]
of \( R(B) \) is a Lie subalgebra of finite rank \( |\text{ind}(B)| \) of the Lie algebra \( L(B) \) with respect to the Lie bracket \( [x, y] = xy - yx \), where \( |S| \) denotes the cardinality of a finite set \( S \) (see [17, 2.3]).
3.2. Ringel’s Lie algebras

Following ideas of C. M. Ringel from [20], we define Hall and Lie algebras of representation-directed algebras. Assume that \( \Gamma = \Gamma_B \) is the Auslander-Reiten quiver of a representation-directed \( K \)-algebra \( B = KQ/I \). For any \( x \in \Gamma_0 \), we denote by \( M(B, x) \in \text{ind}(B) \) the indecomposable \( B \)-module corresponding to \( x \). Denote by \( B(\Gamma) \) the set of all functions \( a : \Gamma_0 \rightarrow \mathbb{N} \). With any function \( a : \Gamma_0 \rightarrow \mathbb{N} \), we associate the \( B \)-module \( M(B, a) = \bigoplus_{x \in \Gamma_0} M(B, x)^{a(x)} \). This establishes a bijection between the set \( B(\Gamma) \) and the set \( \mathcal{M}(B) \) of all isomorphism classes of \( B \)-modules.

Let \( K \) be a finite field, \( C \) be a \( K \)-algebra and \( N_1, N_2, M \in \text{mod}(C) \). We set

\[
F_{M}^{N_2, N_1} = |\mathcal{E}(N_1, N_2; M)|,
\]

where \(|X|\) denotes the cardinality of a finite set \( X \).

The following theorem is due to Ringel ([20, Theorem 1]).

**Theorem 3.7.** Let \( \Gamma \) be a directed Auslander-Reiten quiver (i.e. there exists a representation directed \( C \)-algebra \( B \) with \( \Gamma_B = \Gamma \)), and \( a, b, c \in B(\Gamma) \). There exists a polynomial \( \varphi_{ba}^{c} \in \mathbb{Z}[T] \) (called Hall polynomial) with the following property: if \( K \) is a finite field, and \( C \) is a \( K \)-algebra with \( \Gamma = \Gamma_C \), then

\[
F_{M(C,b), M(C,c)}^{M(C,a)} = \varphi_{ba}^{c}(|K|).
\]

Let \( \Gamma \) be a directed Auslander-Reiten quiver. Following [20], we define the degenerate Ringel-Hall algebra \( \mathcal{H}(\Gamma)_1 \) to be the free \( \mathbb{Z} \)-module with basis \( \{u_a\}_{a \in B(\Gamma)} \) and multiplication given by the formula

\[
u_c u_a = \sum_{b \in B(\Gamma)} \varphi_{ba}^{c}(1) u_b.
\]

By [20, Proposition 5], \( \mathcal{H}(\Gamma)_1 \) is an associative \( \mathbb{Z} \)-algebra with unit element and the \( \mathbb{Z} \)-submodule

\[
\mathcal{K}(\Gamma) = \bigoplus_{x \in \Gamma_0} \mathbb{Z} u_x
\]

of \( \mathcal{H}(\Gamma)_1 \) is a Lie subalgebra of finite rank \(|\Gamma_0| = |\text{ind}(B)|\) of the Lie algebra \( \mathcal{H}(\Gamma)_1 \) (with the Lie bracket \( [x, y] = xy - yx \)).

Assume that \( B \) be a representation-directed algebra over the field \( \mathbb{C} \) of complex numbers. Let \( \Gamma_B \) be the Auslander-Reiten quiver of \( B \). We set \( \mathcal{H}(B)_1 := \mathcal{H}(\Gamma_B)_1 \) and \( \mathcal{K}(B) := \mathcal{K}(\Gamma_B) \).

4. Lattices

In this section we prove that for any representation-directed algebra \( B = KQ/I \) there is a \( \mathbb{Z} \)-order \( A \) such that the \( K \)-algebras \( A(K) \) and \( B \) are isomorphic and the
specialization $A^{(L)}$ of $A$ to any other field $L$ is representation-directed. Moreover, the $A^{(L)}$-modules are induced from some $A$-lattices chosen independently on $L$.

Recall, that a representation $(N_x, N_\alpha)_{x \in Q_0, \alpha \in Q_1}$ of $Q$ over a field $L$ is called thin if $\dim_L N_x \leq 1$, for $x \in Q_0$. We say that a $\mathbb{Z}$-representation $(N_x, N_\alpha)_{x \in Q_0, \alpha \in Q_1}$ of $Q$ is thin if $N_x \cong \mathbb{Z}$ or $N_x = 0$, for $x \in Q_0$ and $N_\alpha$ is either 0 or invertible, for any arrow $\alpha$.

If $A = \mathbb{Z}Q/I$ is an order and $I$ is an admissible ideal then we say that an $A$-lattice $N$ is thin provided so is the corresponding representation of $Q$.

**Lemma 4.1.** Let $A = \mathbb{Z}Q/I$ be a $\mathbb{Z}$-order, where $I$ is an admissible ideal of $\mathbb{Z}Q$ and assume that $A^{(L)}$ is a representation-directed algebra, for any field $L$. Assume that $X$ is an $A$-lattice such that $X^{(L)}$ is an indecomposable thin $A^{(L)}$-module, for any field $L$, and $Y$ is a thin indecomposable lattice. If $\dim(X^{(K)}) = \dim(Y^{(K)})$ for some field $K$, then $X \cong Y$ as $A$-lattices.

**Proof.** Given a thin representation $Z = (Z_x, Z_\alpha)_{x \in Q_0, \alpha \in Q_1}$ of $Q$ over a field let $N(Z) = (N_0, N_1)$ be the subquiver of $Q$ with the set of vertices $N_0 = \{x \in Q_0 : Z_x \neq 0\}$ and arrows $N_1 = \{\alpha \in Q_1 : Z_\alpha \neq 0\}$. Note that $Z$ is indecomposable if and only if $N(Z)$ is connected. It follows that the $A^{(L)}$-module $Y^{(L)}$ is indecomposable, for any field $L$. Therefore $X^{(L)} \cong Y^{(L)}$, since the dimension vector determines uniquely the isomorphism class of an indecomposable module over a representation-directed algebra [11]. The quiver $N(X^{(L)})$ does not depend on the field $L$, hence $X$ is a thin lattice. Now it is easy to observe that the lattices $X$ and $Y$ are isomorphic (apply the isomorphism $X^{(L)} \cong Y^{(L)}$ for a field $L$ of characteristic 0).

The following theorem is a version of Lemma 4.3 from [11].

**Theorem 4.2.** Let $B$ be a representation-directed algebra of the form $KQ/J$ for an acyclic quiver $Q$ and an admissible ideal $J$ in $KQ$. There exist:

(a) a $\mathbb{Z}$-order $A = \mathbb{Z}Q/I$, where $I$ is a two-sided admissible ideal $I$ of $\mathbb{Z}Q$.
(b) $A$-lattices $X_1, \ldots, X_r$,
(c) a set $\mathcal{J} \subseteq \{1, \ldots, r\}^2$ and a family of $A$-homomorphisms $f_{ij} : X_i \rightarrow X_j$, for $(i, j) \in \mathcal{J}$, such that

1. $A^{(K)} \cong B$,

and, for any field $L$:

2. $A^{(L)}$ is a representation-directed $L$-algebra,
3. $X_1^{(L)}, \ldots, X_r^{(L)}$ are indecomposable pairwise nonisomorphic $A^{(L)}$-modules,
4. $f_{ij}^{(L)} : X_i^{(L)} \rightarrow X_j^{(L)}$ are irreducible maps, for $(i, j) \in \mathcal{J}$, and if there is an irreducible map $X_i^{(L)} \rightarrow X_j^{(L)}$ for some $i, j$ then $(i, j) \in \mathcal{J}$.
5. the Auslander-Reiten quiver $\Gamma_{A^{(L)}}$ of $A^{(L)}$ has vertices

$$x_1 = [X_1^{(L)}], \ldots, x_r = [X_r^{(L)}]$$
and there is an arrow \( x_i \rightarrow x_j \) if and only if \((i, j) \in \mathcal{J}\),

(6) the dimension vector \( \text{dim}(X_i^{(L)}) \) does not depend on \( L \), for any fixed \( i \).

**Proof.** The algebra \( B \) is schurian. Thanks to multiplicative basis theorem (it is enough to apply the easier ”triangular” case, see [3]) we may assume that \( J \) is generated by zero-relations and commutativity relations. Let \( \mathcal{F} \) be the set of all paths in \( Q \) belonging to \( J \) and let \( \mathcal{C} \) denote the set of all elements \( u - w \in \mathcal{J} \), where the paths \( u, w \) do not belong to \( \mathcal{F} \). Clearly, \( J \) is the \( K \)-subspace of \( KQ \) spanned by \( \mathcal{F} \cup \mathcal{C} \). For any pair \( x, y \in Q_0 \) such that \( e_x B e_y \neq 0 \) choose a path \( u_{x,y} \notin \mathcal{J} \) starting at \( x \) and ending at \( y \). Let \( \mathcal{B} \) be a set of all the paths \( u_{x,y} \).

Let \( I \) be the \( Z \)-submodule of \( ZQ \) spanned by \( \mathcal{F} \cup \mathcal{C} \). It is easy to check that \( I \) is a two-sided ideal of \( ZQ \). Moreover, \( A = ZQ/I \) is isomorphic to \[ \bigoplus_{u \in \mathcal{B}} Zu \] as a \( Z \)-module, therefore \( A \) is a \( Z \)-order.

It is clear that \( A^{(K)} \cong B \).

We say that a set \( \mathcal{S} = \{ Z_1, ..., Z_l \} \) of pairwise nonisomorphic \( B \)-modules is **properly defined over** \( Z \) if there exist \( A \)-lattices \( \widetilde{Z}_1, ..., \widetilde{Z}_l \), a set \( \mathcal{J}_\mathcal{S} \subseteq \{1, ..., l\}^2 \) and a family of \( A \)-homomorphisms \( g_{ij} : \widetilde{Z}_i \rightarrow \widetilde{Z}_j \), for \((i, j) \in \mathcal{J}_\mathcal{S} \), such that \( \widetilde{Z}_i \cong Z_i \), for \( i = 1, ..., l \), and the conditions (3), (4), (6) of the theorem are satisfied, for any field \( L \), with \( X_1, ..., X_r, f_{ij}, (i, j) \in \mathcal{J}_\mathcal{S} \) interchanged by \( \widetilde{Z}_1, ..., \widetilde{Z}_l, g_{ij}, (i, j) \in \mathcal{J}_\mathcal{S} \), respectively. In this case we call \( \{ \widetilde{Z}_1, ..., \widetilde{Z}_l \} \) a **set of \( Z \)-frames of** \( \mathcal{S} \).

We need to prove that \( \text{ind}(B) \) is properly defined over \( Z \).

Let \( P_x = P^{(B)}_x \) be the indecomposable projective \( B \)-module associated with the vertex \( x \in Q \) and assume that \( \text{rad} P_x \cong M_x^1 \oplus ... \oplus M_x^r \) is a decomposition of \( \text{rad} P_x \) into indecomposable direct summands. It is easy to see that the set \( \{ P_x, M_x^i : x \in Q_0, i = 1, ..., r_x \} \) is properly defined over \( Z \) and there is a set \( \{ \tilde{P}_x, M_x^i : x \in Q_0, i = 1, ..., r_x \} \) of \( Z \)-frames consisting of thin lattices. We assume that the notation is the natural one, that is, \( \tilde{P}_x^{(K)} \cong P_x \), etc. Then, for any field \( L \), \( \tilde{P}_x^{(L)} \) is the indecomposable projective \( A^{(L)} \)-module associated with the vertex \( x \), \( (M_x^i)^{(L)} \) are indecomposable \( A^{(L)} \)-modules and \[ P_x^{(L)} \cong (M_x^1)^{(L)} \oplus ... \oplus (M_x^r)^{(L)} \].

Similarly, the set of representatives \( I_x, x \in Q_0 \), of the isomorphism classes of indecomposable injective \( B \)-modules together with indecomposable direct summands of \( I_x/\text{soc} I_x \) is properly defined over \( Z \).

It follows by Lemma [3] that the algebra \( A^{(L)} \) is representation-directed, for any field \( L \). Moreover, the combinatorial data of the Auslander-Reiten quiver of \( A^{(L)} \) do not depend on the field \( L \). It remains to prove that the vertices of those quivers can be realized by lattices over \( A \).
Inductively we construct sets $S_i \subseteq \text{ind}(B)$, such that

(i) $S_0 = \{ P \}$, where $P$ is a simple projective $B$-module,

(ii) $S_l = \text{ind}(B)$, for some natural number $l$,

and, for any $i = 0, \ldots, l - 1$:

(iii) $S_i \subseteq S_{i+1}$,

(iv) $S_i$ is properly defined over $Z$,

(v) $S_i$ is closed under predecessors in $\Gamma_B$.

Suppose that the set $S_n$ is already defined for some $n$ and $S_n \neq \text{ind}(B)$. Let $Z \in \text{ind}(B) \setminus S_n$ be such that every proper predecessor of $Z$ belongs to $S_n$.

If $Z$ is projective, say $Z = P_\mathcal{L}$, then $M_1^\mathcal{L}, \ldots, M_r^\mathcal{L} \in S_n$ up to isomorphism.

Thus there are lattices $N_1, \ldots, N_r$ in the set $\tilde{S}_n$ of $Z$-frames of $S_n$ such that $N_j^{(K)} \cong M_j^\mathcal{L}$, for $j = 1, \ldots, r$. By Lemma 4.1, we conclude that $N_j \cong M_j^\mathcal{L}$, for $j = 1, \ldots, r$. We set $S_{n+1} = S_n \cup \{ Z \}$ and we see that $\tilde{S}_n \cup \{ \tilde{P}_\mathcal{L} \}$ is a set of $Z$-frames of $S_{n+1}$.

If $Z$ is not projective, then let $0 \to X \xrightarrow{\eta} \bigoplus_{i=1}^{m_Z} E_i \xrightarrow{\nu} Z \to 0$

be the Auslander-Reiten sequence terminating at $Z$, where $E_i$ are indecomposable modules, for $i = 1, \ldots, m_Z$. We denote by $\eta_i$ and $\nu_i$ the component maps of $\eta$ and $\nu$, respectively, for $i = 1, \ldots, m_Z$.

Let $\tilde{X}$ and $\tilde{E}_i$ be $A$-lattices in $\tilde{S}_n$ such that $\tilde{X}^{(K)} \cong X$ and $\tilde{E}_i^{(K)} \cong E_i$, for all $i$. There are nonzero $A$-homomorphisms $\tilde{\eta}_i : \tilde{X} \to \tilde{E}_i$ such that the induced map $\tilde{\eta}_i^{(L)} : \tilde{X}^{(L)} \to \tilde{E}_i^{(L)}$ is irreducible for every $i$. Since every arrow in $\Gamma_B$ (and consequently in $\Gamma_{A(L)}$, for any field $L$) has trivial valuation, then

\[(*) \quad X^{(L)} \xrightarrow{\tilde{\eta}^{(L)}} \bigoplus_{i=1}^{m_Z} E_i^{(L)},\]

where $\tilde{\eta} = [\tilde{\eta}_i]_{i=1}^{m_Z}$, is a left minimal almost split map, for any field $L$.

Moreover, $X^{(L)}$ is not injective, hence the map $\tilde{\eta}^{(L)}$ is a monomorphism, for any field $L$. The $A$-homomorphism

\[X \xrightarrow{\tilde{\eta}} \bigoplus_{i=1}^{m_Z} E_i\]

is a pure monomorphism and we denote its cokernel by $\tilde{Z}$. It follows that $Z^{(L)}$ is the cokernel of the map $(*)$ and thus it is the terminus of the Auslander-Reiten sequence starting at $\tilde{X}^{(L)}$, for any field $L$.

Then $\tilde{S}_n \cup \{ \tilde{Z} \}$ is a set of $Z$-frames of $S_{n+1} = S_n \cup \{ Z \}$. □
5. \(\mathbb{Z}\)-schemes of submodules

Let \(A = \mathbb{Z}Q/I\) be an order and \(N_1, N_2, M\) be \(A\)-lattices. We prove that the locally closed sets \(E(N_1(K), N_2(K); M(K)) \subseteq \text{Gr}_{\dim N_1} (\dim M, K)\) are defined "almost" independently on the base field \(K\).

If \(F_1, ..., F_r\) are homogeneous polynomials in \(n + 1\) variables with coefficients in a field \(K\), then we denote by \(V_K(F_1, ..., F_r)\) the Zariski-closed subset of \(\mathbb{P}^n(K)\) defined by \(F_1 = ... = F_r = 0\).

We denote by \(\mathbb{K}\) the algebraic closure of the field of rational numbers \(\mathbb{Q}\).

**Proposition 5.1.** Assume that the set \(U \subseteq \mathbb{P}^n(\mathbb{R})\) is invariant under the action of every automorphism of the field \(\mathbb{R}\). Then the Zariski closure of \(U\) is defined over \(\mathbb{Z}\), that is, there are homogeneous polynomials \(F_1, ..., F_r \in \mathbb{Z}[T_0, ..., T_n]\) such that

\[
\overline{U} = V_{\mathbb{R}}(F_1, ..., F_r).
\]

**Proof.** Let

\[
\overline{U} = V_{\mathbb{R}}(H_1, ..., H_s),
\]

where \(H_1, ..., H_s \in \mathbb{R}[T_0, ..., T_n]\) are homogeneous polynomials. Denote by \(L_1\) the subfield of \(\mathbb{R}\) generated by all coefficients of \(H_1, ..., H_s\). Let \(L\) be the normal closure of \(L_1\) over \(\mathbb{Q}\) and we denote by \(G\) the Galois group of the extension \(\mathbb{Q} \subseteq L\). Assume that

\[
G = \{\sigma_1, ..., \sigma_p\},
\]

where \(p = [L : \mathbb{Q}]\).

Given a polynomial \(H\) with coefficients in \(\mathbb{R}\) and an automorphism \(\sigma\) of \(\mathbb{R}\) we denote by \(H^\sigma\) the polynomial obtained from \(H\) by the action of \(\sigma\) on the coefficients of \(H\). Since \(U\) is invariant under the automorphisms of \(\mathbb{R}\), every \(H^\sigma_1, i = 1, ..., s, j = 1, ..., p\) vanishes on \(U\).

Let \(s_1, ..., s_p\) be the standard symmetric polynomials in \(p\) variables. Observe that

\[
H_{i,j} := s_i(H_1^{s_1}, ..., H_s^{s_p})
\]

is invariant under each element of \(G\), thus its coefficients belong to \(L^G = \mathbb{Q}\). Clearly, each \(H_{i,j}\) vanishes on \(U\). Moreover, since the unique common zero of \(s_1, ..., s_p\) is \((0, ..., 0)\) it follows that

\[
\overline{U} = V_{\mathbb{R}}(H_{i,j} : i = 1, ..., s, j = 1, ..., p).
\]

Therefore \(\overline{U}\) is defined by vanishing of polynomials with coefficients in \(\mathbb{Q}\). Multiplying by denominators we can assume that

\[
\overline{U} = V_{\mathbb{R}}(H_1, ..., H_q)
\]

for some homogeneous polynomials \(H_1, ..., H_q\) with coefficients in \(\mathbb{Z}\). \(\square\)
Corollary 5.2. Let \( U \subseteq \mathbb{P}^n(\mathfrak{R}) \) be a locally closed set which is invariant under each automorphism of the field \( \mathfrak{R} \). There are homogeneous polynomials \( F_1, \ldots, F_r, G_1, \ldots, G_s \in \mathbb{Z}[T_0, \ldots, T_n] \) such that

\[
U = V_\mathfrak{R}(F_1, \ldots, F_r) \setminus V_\mathfrak{R}(G_1, \ldots, G_s).
\]

Proof. Corollary follows immediately from Proposition 5.1 applied to \( U \) and \( \overline{U} \setminus U \). □

Let \( B = \mathbb{C}Q/J \) be a representation directed \( \mathbb{C} \)-algebra and assume that \( A = \mathbb{Z}Q/I \) is the \( \mathbb{Z} \)-order corresponding to \( B \) as in Theorem 4.2.

Lemma 5.3. Assume that \( K \) is a subfield of a field \( L \) and \( A \) is a \( \mathbb{Z} \)-order.

1. Let \( B \) be a \( K \)-algebra. If \( X, Y \) are \( B \)-modules then \( X \sim Y \) if and only if the \( B \otimes_K L \)-modules \( X \otimes_K L \) and \( Y \otimes_K L \) are isomorphic.

2. If \( N_1, N_2, M \) are \( A \)-lattices then

\[
\mathcal{E}(N_1^{(K)}, N_2^{(K)}; M^{(K)}) = \text{Gr} \dim_{N_1}(\text{dim} M, K) \cap \mathcal{E}(N_1^{(L)}, N_2^{(L)}; M^{(L)}).
\]

Proof. The assertion (1) is proved in [9, Lemma 3.2], whereas (2) follows directly from (1). □

Theorem 5.4. Let \( N_1, N_2, M \) be \( A \)-lattices. There exist a finite set \( \Sigma \) of prime numbers and there exist homogeneous polynomials \( F_1, \ldots, F_r, G_1, \ldots, G_s \) with integral coefficients such that

\[
\mathcal{E}(N_1^{(K)}, N_2^{(K)}; M^{(K)}) = V_K(F_1, \ldots, F_r) \setminus V_K(G_1, \ldots, G_s),
\]

for any field \( K \) of characteristic not belonging to \( \Sigma \).

Proof. Since any automorphism of the field \( \mathfrak{R} \) is constant on \( \mathbb{Z} \) and, by Theorem 4.2, \( B \)-modules are defined over \( \mathbb{Z} \), it is easy to see that the set \( \mathcal{E}(N_1^{(R)}, N_2^{(R)}; M^{(R)}) \) is closed under each automorphism of the field \( \mathfrak{R} \). Therefore it follows from Lemma 3.3 and Corollary 5.2 that there are homogeneous polynomials \( F_1, \ldots, F_r, G_1, \ldots, G_s \) such that

\[
\mathcal{E}(N_1^{(R)}, N_2^{(R)}; M^{(R)}) = V_\mathfrak{R}(F_1, \ldots, F_r) \setminus V_\mathfrak{R}(G_1, \ldots, G_s).
\]

Note that the fact \( x \in \mathcal{E}(N_1^{(K)}, N_2^{(K)}; M^{(K)}) \) can be written as a first order formula in the language of rings [10, Chapter 10]. Applying the Characteristic Transfer Principle [10, Theorem 1.14] to the formula:

\[
x \in \mathcal{E}(N_1^{(K)}, N_2^{(K)}; M^{(K)}) \iff [(F_1(x) = \ldots = F_r(x) = 0) \land (G_1(x) \neq 0 \lor \ldots \lor G_s(x) \neq 0)].
\]
yields the existence of a finite set $\Sigma$ such that the equality
\[ E(N_1^{(K)}, N_2^{(K)}; M^{(K)}) = V_K(F_1, ..., F_r) \setminus V_K(G_1, ..., G_s) \]
holds whenever $K$ is an algebraically closed field of characteristic not belonging to $\Sigma$.

Thanks to Lemma 5.3 the equality holds for every field $K$ such that $\text{char } K \notin C$. □

6. Proof of Theorems [1.1] and [1.2]

In order to prove Theorem [1.2] we have to relate the Euler-Poincaré characteristic $\chi$ of $E(N_1, N_2; M)$ with Hall polynomials. For the properties of the Euler-Poincaré characteristic the reader is referred to [7] and [8]. We will need the following lemma, which can be deduced from [8] (see also [5], [15, Proposition 6.1] and [16, Lemma 8.1]).

**Lemma 6.1** Let $F_1, ..., F_r, G_1, ..., G_s$ be homogeneous polynomials with integral coefficients in $p$ variables. Given a field $K$ we denote $V_K = V_K(F_1, ..., F_r) \setminus V_K(G_1, ..., G_s)$.

Assume that there exists a polynomial $\phi \in \mathbb{Z}[t]$ and a finite set $C$ of prime numbers such that
\[ |V_K| = \phi(|K|) \]
for any finite field $K$ such that $\text{char } K \notin C$.

Then
\[ \chi(V_C) = \phi(1). \]

□

Let $B = \mathbb{C}Q/J$ be a representation-directed $\mathbb{C}$-algebra and $A = \mathbb{Z}Q/I$ the $\mathbb{Z}$-order corresponding to $B$ as in Theorem [4.2]. The $K$-algebras $A^{(K)}$ are representation-directed for all fields $K$ and their Auslander-Reiten quivers coincide. Denote this common Auslander-Reiten quiver by $\Gamma$.

**Proof of Theorem [1.2]** Let $B = \mathbb{C}Q/J$ be a representation directed $\mathbb{C}$-algebra and let $A = \mathbb{Z}Q/I$ be the corresponding $\mathbb{Z}$-order, such that $A^{(\mathbb{C})} \cong B$. Consider functions $a, b, c : \Gamma_0 \to \mathbb{N}$. By Theorem [3.7]

\[ |E(M(A^{(K)}, a), M(A^{(K)}, c); M(A^{(K)}, b))| = F_{M(A^{(K)}, c), M(A^{(K)}, a)}^{M(A^{(K)}, b)} = \varphi_{ca}^{b}(|K|), \]

for any finite field $K$, where $\varphi_{ca}^{b}$ are Hall polynomials associated with $a, b, c \in \Gamma$.

Thanks to Theorem [5.4] and Lemma [6.1] we obtain
\[ \chi(E(M(B, a), M(B, c); M(B, b))) = \varphi_{ca}^{b}(1), \]
and we are done. □

**Proof of Theorem 1.1.** Let $B$ be a representation-directed $\mathbb{C}$-algebra, $\text{ind}(B) = \{X_1, \ldots, X_m\}$ and let $\Gamma$ be the Auslander-Reiten quiver of $B$. Moreover let $x_1, \ldots, x_m \in \Gamma_0$ be such that $X_i \cong M(B, x_i)$, for all $i = 1, \ldots, m$. Note that the $\mathbb{Z}$-Lie algebras $L(B)$ and $\mathcal{K}(B)$ are free $\mathbb{Z}$-modules with basis $v_{X_1}, \ldots, v_{X_m}$ and $u_{x_1}, \ldots, u_{x_m}$, respectively. Let

$$F : \mathcal{K}(B) \rightarrow L(B)$$

be the homomorphism of $\mathbb{Z}$-modules given by

$$F(u_{x_i}) = \begin{cases} v_{X_i} & \text{if } \dim_c X_i \text{ is odd} \\ -v_{X_i} & \text{otherwise} \end{cases}.$$  

We show that $F$ is a homomorphism of Lie algebras. Let $x_i, x_j \in \Gamma_0$. Consider the case that there exist $x_t \in \Gamma_0$ and a short exact sequence

$$0 \rightarrow X_j \rightarrow X_t \rightarrow X_i \rightarrow 0,$$

then, by (3.9) and (3.8),

$$F([u_{x_i}, u_{x_j}]) = F(\varphi^{x_i}_{x_j}(1) \cdot u_{x_j}) = (-1)^{\dim_c X_i - 1} \varphi^{x_j}_{x_i}(1) \cdot v_{X_t}.$$  

On the other hand, by (3.5, 3.4 and Theorem 1.2),

$$[F(u_{x_i}), F(u_{x_j})] = (-1)^{\dim_c X_i + \dim_c X_j - 2} [v_{X_i}, v_{X_j}] = (-1)^{\dim_c X_i + \dim_c X_j - 1} [v_{X_j}, v_{X_i}] = (-1)^{\dim_c X_i - 1} \chi(\varphi^{x_j}_{x_i}(1)) \cdot v_{X_t} = (-1)^{\dim_c X_i - 1} \varphi^{x_j}_{x_i}(1) \cdot v_{X_t}.$$  

It follows that $F([u_{x_i}, u_{x_j}]) = [F(u_{x_i}), F(u_{x_j})]$. In the remaining cases the proof is analogous. Then $F$ is a homomorphism of Lie algebras. Obviously $F$ is an isomorphism of Lie algebras, because it is isomorphism of free abelian groups $\mathcal{K}(B)$ and $L(B)$. This finishes the proof. □

**Corollary 6.2.** Let $B$ be a representation-directed $\mathbb{C}$-algebra.

(a) The complex Lie algebras $L(B) \otimes_{\mathbb{C}} \mathbb{C}$ and $\mathcal{K}(B) \otimes_{\mathbb{C}} \mathbb{C}$ are isomorphic.

(b) The algebra $\mathcal{H}(B)_{1} \otimes_{\mathbb{C}} \mathbb{C}$ is the universal enveloping algebra of $L(B) \otimes_{\mathbb{C}} \mathbb{C}$.

**Proof.** The assertion (a) is obvious. By [20] Proposition 5], $\mathcal{H}(B)_{1} \otimes_{\mathbb{C}} \mathbb{C}$ is the universal enveloping algebra of $\mathcal{K}(B) \otimes_{\mathbb{C}} \mathbb{C}$, then the statement (b) follows from Theorem 1.1. □

**Remark 6.3.** Combinatorial properties of the Lie algebra $\mathcal{K}(B) \otimes_{\mathbb{C}} \mathbb{C}$ are investigated in [13]. In particular, the Lie algebra $\mathcal{K}(B) \otimes_{\mathbb{C}} \mathbb{C}$ is described by generators and relations, for any representation directed $\mathbb{C}$-algebra $B$. This together with Theorem 1.1 give us a description of the Lie algebra $L(B) \otimes_{\mathbb{C}} \mathbb{C}$ by generators and relations.
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