Detour global domination number of some graphs

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Abstract
In this paper, we introduce the concept detour global domination number of a graph. Also detour global domination number of certain classes of graphs are determined and some of its general properties are studied. A set $S$ of vertices in a connected graph $G = (V,E)$ is called a detour set if every vertex not in $S$ lies on a longest path between two vertices from $S$. A set $D$ of vertices in $G$ is called a dominating set of $G$ if every vertex not in $D$ has at least one neighbor in $D$. A set $H \subseteq V(G)$ is called a global dominating set of $G$ if it is a dominating set of both $G$ and $\overline{G}$. A set $S$ is called a detour global dominating set of $G$ if $S$ is both detour and global dominating set of $G$. The \textit{detour global domination number} is the minimum cardinality of a detour global dominating set in $G$.

Keywords
Detour set, Dominating set, Detour Domination, Global Domination, Detour Global Domination.

AMS Subject Classification
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1. Introduction

By a graph $G=(V,E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The order $|V|$ and size $|E|$ of $G$ are denoted by $p$ and $q$ respectively. For graph theoretic terminology we refer west\textsuperscript{[9]}. A vertex $v$ of $G$ is said to be an extreme vertex if the subgraph induced by its neighborhood is complete. The set of all extreme vertices is denoted by $\text{Ext}(G)$. For vertices $x$ and $y$ in a connected graph $G$, the detour distance $D(x,y)$ is the length of a longest $x-y$ path in $G$. An $x-y$ path of length $D(x,y)$ is called an $x-y$ detour. The closed interval $I_{D[x,y]}$ consists of all vertices lying on some $x-y$ detour of $G$. For $S \subseteq V, I_{D[S]} = \bigcup_{x,y \in S} I_{D[x,y]}$. A set $S$ of vertices is a detour set if $I_{D[S]} = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a minimum detour set \textsuperscript{[3]}.

A set $S \subseteq V(G)$ in a graph $G$ is a dominating set of $G$ if for every vertex $v$ in $V-S$, there exists a vertex $u \in S$ such that $v$ is adjacent to $u$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$\textsuperscript{[4]}. The complement $\overline{G}$ of a graph $G$ also has $V(G)$ as its point set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. A set $S \subseteq V(G)$ is called a global dominating set of $G$ if it is a dominating set of both $G$ and $\overline{G}$\textsuperscript{[8]}.

A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called an end vertex or a pendant vertex. A vertex that is adjacent to a pendant vertex is called a support vertex. A vertex of degree $p-1$ is called a full vertex. The set of all full vertices is denoted by $Fx(G)$. A cycle of length three is also a triangle and a graph $G$ containing no triangles is called triangle-free.

\textbf{Definition 1.1.} Let $G = (V,E)$ be a connected graph with at least two vertices. A set $S \subseteq V(G)$ is said to be a detour global dominating set of $G$ if $S$ is both detour and global dominating set of $G$. The detour global domination number, denoted by $\gamma_d(G)$ is the minimum cardinality of a detour global dominating set of $G$ and the detour global dominating set with cardinality $\gamma_d(G)$ is called the $\gamma_d$-set of $G$ or $\gamma_d(G)$-
set.

**Example 1.2.** For the graph $G$ in Figure 1.1, $S_1 = \{v_1, v_2\}$ is a detour dominating set of $G$ so that $\gamma_d(G) = 2$. But, $S_2 = \{v_1, v_2, v_3\}$ is a detour global dominating set of $G$ so that $\bar{\gamma}_d(G) = 3$.

![Figure 1.1](image)

**2. Some basic results**

In this section, we recall some definitions and basic results of detour number and detour domination number of a graph which will be used throughout the paper.

**Theorem 2.1.** [3] Every end vertex of $G$ belongs to every detour set of $G$.

**Theorem 2.2.** [3] For a non-trivial tree, $dn(G) = k$, where $k$ is the number of end vertices of $G$.

**Theorem 2.3.** [5] Every end vertex of $G$ belongs to every detour dominating set of $G$.

**Theorem 2.4.** [5] Every isolated vertex of $G$ belongs to every detour dominating set of $G$.

**Theorem 2.5.** [5] Every end vertex of $G$ belongs to every detour global dominating set of $G$.

**Theorem 2.6.** Let $S \subseteq V(G)$ be a detour dominating set of $G$. Then $S$ is a detour global dominating set of $G$ if and only if $S$ is a dominating set of $G$.

**Theorem 2.7.** Every full vertex of a connected graph $G$ of order $p$ belongs to every detour global dominating set of $G$.

**Remark 3.1.** There can be more than one $\bar{\gamma}_d$-set of $G$. For the graph $G$ given in Figure 1.1, $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_5\}$, $\{v_3, v_4, v_5\}$ and $\{v_1, v_4, v_5\}$ are $\bar{\gamma}_d$-sets of $G$.

**Theorem 3.1.** Every end vertex of $G$ belongs to every detour global dominating set of $G$.

**Proof.** Every detour global dominating set is a detour dominating set of $G$ and hence the result follows from Theorem 2.3.

**Theorem 3.2.** Every isolated vertex of $G$ belongs to every detour global dominating set of $G$.

**Proof.** By Theorem 2.4, every isolated vertex of $G$ belongs to every detour dominating set of $G$. Hence, every isolated vertex of $G$ belongs to every detour global dominating set of $G$.

**Theorem 3.3.** For any connected graph $G$, $2 \leq \gamma_d(G) \leq \bar{\gamma}_d(G) \leq p$.

**Proof.** Every detour dominating set contains at least two vertices and so $\gamma_d(G) \geq 2$. Since, every detour global dominating set is also a detour dominating set, it follows that $\gamma_d(G) \leq \bar{\gamma}_d(G)$. Also, the set of all vertices of $G$ is a detour global dominating set of $G$ and so $\bar{\gamma}_d(G) \leq p$. Thus, $2 \leq \gamma_d(G) \leq \bar{\gamma}_d(G) \leq p$.

**Remark 3.5.** All inequalities in Theorem 3.4 can be strict. For example, consider the graph $G$ given in Figure 3.1, where we have $2 < \gamma_d(G) = 3 < \bar{\gamma}_d(G) = 4 < p = 6$. Also, in $K_2$ all the inequalities in the above theorem become sharp.

![Figure 3.1](image)

**Theorem 4.1.** Let $S \subseteq V(G)$ be a detour dominating set of $G$. Then $S$ is a detour global dominating set of $G$ if and only if $S$ is a dominating set of $G$.

**Proof.** The proof is obvious from the Definition 1.1.

**Theorem 4.2.** Let $S \subseteq V(G)$ be a detour dominating set of $G$. Then $S$ is a detour global dominating set of $G$ if and only if $S$ is a dominating set of $G$.

**Proof.** The proof is obvious from the Definition 1.1.

**Theorem 4.3.** Every full vertex of a connected graph $G$ of order $p$ belongs to every detour global dominating set of $G$.

**Proof.** Let $S$ be a detour global dominating set of $G$ and $u$ be a full vertex of $G$. If $u \notin S$, then $u$ is an isolated vertex in $G$. Hence, $u$ is not dominated by any other vertices in $G$. Therefore, $S$ is not a detour global dominating set of $G$, contrary to our assumption. Hence, $u \in S$.

**Theorem 4.4.** For any complete graph $K_p, (p \geq 2), \bar{\gamma}_d(K_p) = p$.

**Proof.** All the vertices are isolated vertices in the complement graph of the complete graph $K_p$. Therefore, the detour global dominating set must contain all the vertices of $K_p$ and hence, $\bar{\gamma}_d(K_p) = p$.

**Theorem 4.5.** Let $G$ be a connected graph of order $p \geq 3$. If $\gamma_d(G) = p - 1$, then $\bar{\gamma}_d(G) = p$.
Proof. Let $G$ be a connected graph of order $p \geq 3$ and let $\gamma_d(G) = p - 1$. Then by Theorem 2.5, it is clear that $G = K_1, p - 1$. Also, $G$ contains a full vertex. Therefore, by Theorem 3.7, $\bar{\gamma}_d(G) = p - 1 + 1 = p$. \qed

Remark 3.10. The converse of the Theorem 3.9 need not be true. For example consider any complete graph $K_p$ of order $p \geq 4$. It is clear that $\bar{\gamma}_d(K_p) = p$. But $\gamma_d(K_p) = 2 \neq p - 1$.

Theorem 3.11. For any star graph $K_{1,p-1}$, $(p \geq 2)$, $\bar{\gamma}_d(K_{1,p-1}) = p$.

Proof. For $p = 2$, $K_{1,1-1} = K_{1,1,0}$. By Theorem 3.8, $\bar{\gamma}_d(K_{1,1,0}) = 2$. Now, assume $p > 2$. Let $V(K_{1,p-1}) = \{v_1, v_2, \ldots , v_{p-1}\}$, where $v$ is the only vertex of degree $p - 1$ and each $v_i$ is an end vertex adjacent to $v$. By Theorem 2.1, $S = \{v_1, v_2, \ldots , v_{p-1}\}$ is a detour set in $K_{1,p-1}$ because $v$ lies on every $v_i - v_j$ detour path of $S$. Also, clearly $S$ is a dominating set of $K_{1,p-1}$. But the complement of $K_{1,p-1}$ contains two components $K_1$ and $K_{p-1}$ where $V(K_1) = \{u\}$ and $V(K_{p-1}) = \{v_1, v_2, \ldots , v_{p-1}\}$. Hence, $S_1 = V(G)$ is the detour global dominating set of $K_{1,p-1}$ and so $\bar{\gamma}_d(K_{1,p-1}) = p$. \qed

Theorem 3.12. For the path graph $P_p$, $(p \geq 4)$, $\bar{\gamma}_d(P_p) = \gamma_d(P_p) = \lceil \frac{p-2}{2} \rceil$.

Proof. Let $v_1v_2\ldots v_p$ be the path, $p \geq 4$. By Theorem 3.2, for any detour dominating set $S$, both $v_1$ and $v_p$ is in $S$. If $p \equiv 0$ (mod 3), then $S = \{v_1, v_4, \ldots , v_{p-2}, v_p\}$ is a minimum detour dominating set of $P_p$. Similarly, if $p \equiv 1$ (mod 3) or $p \equiv 2$ (mod 3), then $S = \{v_1, v_4, \ldots , v_{p-3}, v_p\}$ or $S = \{v_1, v_4, \ldots , v_{p-1}, v_p\}$ is a minimum detour dominating set of $P_p$ respectively and so $\gamma_d(P_p) = \lceil \frac{p-2}{2} \rceil$. Now we show that $S$ is a detour global dominating set of $P_p$. In $P_p$, there are two types of vertices, end vertices are of degree 1 and $p - 2$ internal vertices are of degree 2. In $P_p$, $v_1$ is adjacent to every vertex while $v_p$ and $v_{p-1}$ is adjacent to every vertex rather than $v_1$ and $v_p$ is adjacent to every vertex rather than $v_p$. Since, $v_1, v_p \in S$, $S$ is a dominating set of $P_p$. By Theorem 3.6, $S$ is a detour global dominating set of $P_p$ and so $\bar{\gamma}_d(P_p) \leq \gamma_d(P_p)$. By Theorem 3.4, $\gamma_d(P_p) \leq \bar{\gamma}_d(P_p)$ and hence we conclude that $\bar{\gamma}_d(P_p) = \gamma_d(P_p) = \lceil \frac{p-2}{2} \rceil$. \qed

Theorem 3.13. For the cycle graph $C_p$, where $p \geq 6$, $\bar{\gamma}_d(C_p) = \gamma_d(C_p) = \lceil \frac{p}{2} \rceil$.

Proof. Let $C_p$ be the cycle $v_1v_2\ldots v_pv_1$ where $p \geq 6$. If $p \equiv 0$ (mod 3), then $S = \{v_1, v_4, \ldots , v_{p-2}\}$ is a minimum detour dominating set of $C_p$. Similarly, if $p \equiv 1$ (mod 3) or $p \equiv 2$ (mod 3), then $S = \{v_1, v_4, \ldots , v_{p-3}\}$ or $S = \{v_1, v_4, \ldots , v_{p-1}\}$ is a minimum detour dominating set of $C_p$ respectively and so $\gamma_d(C_p) = \lceil \frac{p}{2} \rceil$. Now we show that $S$ is a detour global dominating set of $C_p$. By Theorem 3.6, it’s enough to prove that $S$ is a dominating set of $C_p$. In $C_p$, every vertex is of degree 2 and $C_p$ is a triangle free implies any two vertices in $S$ dominates $C_p$. Thus, $S$ is a detour global dominating set of $C_p$ and so $\bar{\gamma}_d(C_p) \leq |S| = \gamma_d(C_p)$. By Theorem 3.4, we conclude that $\bar{\gamma}_d(C_p) = \gamma_d(C_p) = \lceil \frac{p}{2} \rceil$. \qed

Theorem 3.14. For the wheel graph $W_p = K_1 + C_{p-1}$ $(p \geq 6)$, $\bar{\gamma}_d(W_p) = 3$.

Proof. Let $v_1v_2\ldots v_{p-1}v_1$ be the outer cycle $C_{p-1}$ and $v$ be the central vertex of $W_p$. Then $deg(v) = p - 1$ and $deg(v_i) = 3$ for each $i \in \{1, 2, \ldots , p - 1\}$.

Consider, $S = \{v, v_i\}$, then $S$ is a detour dominating set of $W_p$. Clearly $\bar{\gamma}_d(W_p) = K_1 \cup C_{p-1}$ where $V(K_1) = \{v\}$. Choose any vertex $v_j$ from $C_{p-1}$ such that $v_j \notin S$. Clearly, $v_i$ and $v_j$ dominates every vertex from $C_{p-1}$. Therefore, $S_1 = S \cup \{v_j\} = \{v, v_i, v_j\}$ is a minimum detour global dominating set of $W_p$ and hence $\bar{\gamma}_d(W_p) = 3$. \qed

Theorem 3.15. For a complete graph $K_p$ and $e \in E(K_p)$, then $\bar{\gamma}_d(K_p - e) = p - 1$.

Proof. Consider an edge $e = xy \in E(K_p)$. Let $S = \{x, z\}$ be a subset of $V(K_p - e)$, where $z$ is distinct from both $x$ and $y$. Then for every vertex $w \in V(K_p - e) - S$, there exists a $x - z$ detour path of length $p - 1$ containing all $w$. Hence, $S$ is a minimum detour set of $K_p - e$. Also, $N[S] = V(K_p - e)$. Therefore, $S$ is a dominating set of $K_p - e$. Let $T$ be the set of all full vertices in $K_p - e$. Then, $T = V - \{x, y\}$ and hence $|T| = p - 2$ and $z \in T$. Clearly, $K_p - e = K_2 \cup (p - 2)K_1$, where $K_2$ is the edge $xy$. By Theorem 3.3, $S' = S \cup T$ is a minimum detour global dominating set of $K_p - e$. Hence, $\bar{\gamma}_d(K_p - e) = |S'| = |S \cup T| = |S| + |T| - |S \cap T| = 2 + p - 2 - 1 = p - 1$.

Theorem 3.16. For the graph $G = K_p - \{e_1, e_2\}$, obtained by removing the edges $e_1$ and $e_2$ from $K_p$, where $(p > 4)$, $\bar{\gamma}_d(G) = p - 2$.

Proof. The edges $e_1$ and $e_2$ are either adjacent or non-adjacent in $K_p$. We consider the following two cases

Case 1. $e_1$ and $e_2$ are non-adjacent edges in $K_p$

Let $e_1 = uv$ and $e_2 = xy$. Let $S = \{u, x\} \cup \{v, y\}$ be a subset of $V(G)$. Then for every vertex $t \in V(G) - S$, there exists a $u - x$ or $v - y$ detour path of length $p - 1$ containing all $t$. Hence, $S$ is a minimum detour set of $G$ and also a dominating set of $G$, since $N[S] = V(G)$. Let $T$ be the set of all full vertices in $G$. Then, $|T| = p - 4$. Clearly, $G = 2K_2 \cup (p - 4)K_1$, where the $2K_2$'s are the edges $e_1$ and $e_2$ and the $K_1$'s are the vertices in $T$. Hence, by Theorem 3.3, $S' = S \cup T$ is a minimum detour global dominating set of $G$ and so $\bar{\gamma}_d(G) = |S'| = |S \cup T| = |S| + |T| - |S \cap T| = 2 + p - 4 - 0 = p - 2$.

Case 2. $e_1$ and $e_2$ are adjacent edges in $K_p$

Let $y$ be the common vertex to both $e_1$ and $e_2$. Let $e_1 = xy$ and $e_2 = yz$. Let $S = \{u, y\}$ be a subset of $V(G)$, where $u$ is distinct from $x$, $y$, and $z$. Then for every vertex $w \in V(G) - S$, there exists a $u - y$ detour path of length $p - 1$ containing all $w$. Hence, $S$ is a minimum detour set of $G$ and also a dominating set of $G$, since $N[S] = V(G)$. Therefore, $S$ is a dominating set of $G$. Let $T$ be the set of all full vertices in $G$ then, $|T| = p - 3$ and $|S \cap T| = 2 + p - 4 - 0 = p - 2$.
\{x, y, z\} and \(N[T] = \phi\) in \(G\). Hence, \(\gamma_d(G) = |S'| = |S \cup T| = |S| + |T| - |S \cap T| = 2 + p - 3 - 1 = p - 2.\)

4. Conclusion

In this paper, we get a deep knowledge about detour global domination of some graphs. It has many applications in the field of social networking and modern technologies. For our future work we can extend it for large families of graphs.

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References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, (1990).
[2] G. Chartrand, H. Escuadro and B. Zang, Detour distance in graph, *J. Combin. Math. Combin. Comput.*, 53 (2005), 75–94.
[3] G. Chartrand, N. Johns and P. Zang, Detour Number of a graph, *Util. Math.*, 64(2003), 97–113.
[4] T.W. Haynes, S.T. Hedetniemi and P.J. Slater *Fundamentals of Domination in Graphs*, Marcel Dekker,Inc., New York, (1998).
[5] J. John and N. Arianayagam, The detour domination number of a graph, *Discrete Mathematics, Algorithms and Applications*, 9(1)(2017), 1–10.
[6] A. Nellai Murugan, A. Esakkimuthu and G. Mahadevan, Detour Domination Number of a Graph, *International Journal of Science, Engineering and Technology Research (IJSETR)*, 5(2)(2016), 12–20.
[7] S. Robinson Chellathurai and X. Lenin Xaviour, Geodetic Global Domination in Graphs, *International Journal of Mathematical Archive*, (2018), 29–36.
[8] E. Sampath Kumar, The Global Domination Number of a Graph, *Journal of Mathematical and Physical Sciences*, 23(5)(1989), 377-385.
[9] D.B. West, *Introduction to Graph Theory*, Second Ed., Prentice-Hall, Upper Saddle River, NJ, (2001).