DREAMING MACHINE LEARNING: LIPSCHITZ EXTENSIONS FOR REINFORCEMENT LEARNING ON FINANCIAL MARKETS

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ABSTRACT. We develop a new topological structure for the construction of a reinforcement learning model in the framework of financial markets. It is based on Lipschitz type extension of reward functions defined in metric spaces. Using some known states of a dynamical system that represents the evolution of a financial market, we use our technique to simulate new states, that we call "dreams". These new states are used to feed a learning algorithm designed to improve the investment strategy.

1. Introduction and basic definitions

The use of McShane-Whitney type extension of Lipschitz functions on metric spaces is a theoretical tool that have been often considered since the beginning of the development of the so called reinforcement learning methods in machine learning. Indeed, several theoretical aspects on Lipschitz extension of maps that are found in the fundamentals of reinforcement learning techniques were published in some early papers many years ago. The reader can find some information about in [2] and [11] for the mathematical results on the so called absolutely minimal extensions, and [3, 9] and the references therein for the concrete application for machine learning. Often, the metric space structure underlying the Lipschitz extension of maps is the usual finite dimensional space $\mathbb{R}^n$ with the Euclidean norm, or some classical modifications of this metric considering non-canonical scalar products acting in $\mathbb{R}^n$. Information about other related metric structures on which Lipschitz extensions of reward functions have been considered can be found for example in [8, 14], where metric graphs are studied.

Following the same general framework, the aim of this paper is to show a new theoretical environment for the development of mathematical tools for reinforcement learning. However, our ideas—which can be applied in much more general contexts—will focus on the rather specific issue of designing expert systems for the analysis of financial markets. In particular,
we will model the set of strategies to be applied in a financial market —a
dynamical system— as a metric space of finite sequences of $n$ items —states
of the system—, where $n$ is the number of times that a change of state
(purchase/sale event) could occur in the market. We will consider also a re-
ward function, that is supposed to be known for a certain subset of strategies
—initial “training set”—. Using well-known theoretical techniques of exten-
sion of Lipschitz functions on metric spaces, we will construct the necessary
tools for computing improved reward functions for bigger sets of strate-
gies by means of the search of “similarities” among different pieces of these
items. This will be used to feed the algorithm for creating new situations
—“dreams”— that will allow to increase the efficiency of the process by
increasing the size of the training set. The final result will be the definition
of a new reinforcement learning method.

Our arguments bring together ideas from abstract topology on quasi-
pseudo-metric spaces and Lipschitz maps and practical computational tools
for extending Lipschitz functions on metric vector spaces in which the dis-
tance is not given by a standard norm coming from an inner product. In
fact, our metric is not one of the classical distances used in machine learning
(see for example the comments in Section I and Section II in [6]). We use
the McShane and the Whitney extensions for Lipschitz maps in a special
way in order to extend some reward functions defined by a novel design.
The process of introduction of “dreams” to increase the size of the train-
ing set needs also some topological tools based on average values computed
on equivalence classes constructed by a specific metric similarity method.
Although our approach is new, the reader can find some related ideas in
[3, 4].

Concerning related work on mathematical economy and models for finan-
cial markets, we develop our method in a rather classical framework. The
definition of our reward function begins with a relationship of duality similar
to that of the commodity-prize duality that is at the core of market models
based on functional analytic tools (see for example [11 Ch.8]). Although our
method refers to some probabilistic tools, we do not consider our learning
method as based on stochastic arguments. However, philosophically we may
refer to some links with stochastic market modeling —concretely to the so
called continuous-time market model, see for example [7 Ch.2]—, since the
decision on the following step is given exactly in the previous one, based in
our case in a predictive reward function.

For clarity in the explanation of our technique, we will focus our presen-
tation on a particular problem related to the dynamics of a financial market.
In general terms, our technique is based on significantly extending the reward
function by creating new simulated situations to provide an improved tool for
decision making. As we said, this allows to mix known original situations
with new created states (dreams) to design a typical reinforcement learning
procedure. The calculations are simple, as the extension formulas are simple,
so the technique could be applied when dealing with a large amount of data.
The results will be presented in four sections. After this Introduction, we will explain the topological foundations on the metric representation spaces that will be used in the preliminary Section 2. In Section 3 we will describe the general facts for the definition of our procedure —mainly of mathematical nature—, and the model will be presented in a very concrete way in Section 4. The paper ends with some conclusions in Section 5.

We should note that the objective of the present paper is theoretical in nature, although a very explicit example is given. We do not intend to give an efficient algorithm for computing the mathematical elements that appear in the model in order to provide a concrete and effective tool: instead we are interested in explaining the fundamentals of our method.

2. Preliminaries and topological tools

Let us present some relevant mathematical concepts. A quasi-pseudo-metric on a set $M$ is a function $d : M \times M \rightarrow \mathbb{R}^+$ —the set of non-negative real numbers— such that

1. $d(a, b) = 0$ if $a = b$, and
2. $d(a, b) \leq d(a, c) + d(c, b)$

for $a, b, c \in M$. A topology is defined by such a function $d$: the open balls define the basis of neighborhoods. For $\varepsilon > 0$, we define the ball of radius $\varepsilon$ and center in $a \in M$ as

$$B_\varepsilon(a) := \{ b \in M : d(a, b) < \varepsilon \}.$$ 

$(M, d)$ is called a quasi-pseudo-metric space. We will work in this paper mainly with pseudo-metrics, that is, $d(a, b) = d(b, a)$ for all $a, b \in M$, or metrics, that in addition satisfy that $d(a, b) = 0$ if and only if $a = b$. In this case, the topology defined by $d$ satisfies the Hausdorff separation axiom. However, we prefer to present some of our ideas in this more general context, since the basic elements of our technique can be easily extrapolated to the more general quasi-pseudo-metric case. This fact is relevant, since asymmetry in the definition of metric notions (quasi-metric case) could be crucial for the modeling of dynamical processes, in which the dependence on the time variable changes the concepts related to distance. As usual, we will use both the words metric and distance as synonyms. We will use also classical notation for distances from a point to a set: if $d$ is a (pseudo-)metric in a set $M$, $a \in M$ and $B \subset M$, we will write $d(a, B)$ for the distance from $a$ to $B$, that is $d(a, B) = \inf_{b \in B} d(a, b)$.

Let us recall now some definitions regarding functions. Let $(M, d)$ be a metric space. A function $f : M \rightarrow \mathbb{R}$ is a Lipschitz function if there is a positive constant $K$ such that

$$|f(a) - f(b)| \leq K d(a, b), \quad a, b \in M.$$ 

The infimum of such constants as $K$ is called the Lipschitz constant of $f$. 

Regarding extensions of Lipschitz maps, recall that the classical McShane-Whitney theorem states that if \((M_0, d)\) is a subspace of a metric space \((M, d)\) and \(T : M_0 \to \mathbb{R}\) is a Lipschitz function with Lipschitz constant \(K\), there always exists a Lipschitz function \(T^M : M \to \mathbb{R}\) extending \(T\) and with the same Lipschitz constant. There are also known extensions of this result to the general setting of real-valued semi-Lipschitz functions acting in quasi-pseudo-metric spaces; see for example \([2, 10, 12, 13, 15]\) and the references therein. The function
\[
T^M(a) := \sup_{b \in M_0} \{T(b) - K d(a, b)\}, \quad a \in M,
\]
provides such an extension; it is sometimes called the McShane extension. We will use it for giving a constructive tool for our approximation. The Whitney formula, given by
\[
T^W(a) := \inf_{b \in M_0} \{T(b) + K d(a, b)\}, \quad a \in M,
\]
provides also an extension. We will use the first one in this paper, although some results are also true when using the second, as will be explained.

Regarding references to some previous work on reinforcement learning, the reader can find some recent information directly related with our ideas in \([3, 9]\) and the references therein. Concretely, some applications of Lipschitz extensions of functions to machine learning can be found in \([5, 8, 9]\). General explanations about applications of mathematical analysis in Machine Learning can be found in \([17]\); in particular, basic definitions, examples and results on Lipschitz maps can be found in Section 5.10 of this book.

3. Metric spaces of states and Lipschitz maps: an algorithm for machine learning

3.1. Mathematical framework. Consider a subset \(M_0\) of vectors of the finite dimensional real linear space \(\mathbb{R}^n\) not containing the 0. Let us write \(M = \mathbb{R}^n \setminus \{0\}\). We start by defining an adequate metric on \(M\). As the reader will see, the difference of our technique with other methods of reinforcement learning begins at this point. The main reason is that our choice does not allow to define the distance by means of a norm in \(\mathbb{R}^n\). We mix the angular pseudo-distance—geodesic distance—and the Euclidean norm in this space. Thus, since the cosine of the angle among elements \(s_1\) and \(s_2\) in \(M\) is given by
\[
\cos(s_1, s_2) = \frac{s_1 \cdot s_2}{\|s_1\| \|s_2\|}, \quad s_1, s_2 \in M,
\]
we define a distance by mixing this angle
\[
\Theta(s_1, s_2) = \frac{1}{\pi} \arccos\left(\frac{s_1 \cdot s_2}{\|s_1\| \|s_2\|}\right),
\]
and an Euclidean component
\[ E(s_1, s_2) = \|s_1 - s_2\|_2 = \sqrt{\sum_{k=1}^{n} (s_{1k} - s_{2k})^2}, \]
where \( s_1 = (s_{11}, \ldots, s_{1n}) \) and \( s_2 = (s_{21}, \ldots, s_{2n}) \). This Euclidean term can be substituted by any other norm in \( \mathbb{R}^n \). For each \( \epsilon \geq 0 \), we define now the function
\[ d_\epsilon(s_1, s_2) = \Theta(s_1, s_2) + \epsilon E(s_1, s_2), \quad s_1, s_2 \in M, \]
that will become the general formula for the distance we want to use in our model. As usual, we use the same symbol \( d_\epsilon \) when it is restricted to any subset of \( M \).

\[ \text{Lemma 3.1.} \quad \text{Let } \epsilon > 0. \text{ With the definitions given above, the following statements hold.} \]

(i) The function \( d_\epsilon \) is a pseudo-metric on \( M \) for every \( \epsilon \geq 0 \). Moreover, it is a metric on \( M \) if and only if \( \epsilon > 0 \).

(ii) For every \( \epsilon > 0 \), the metric space \( d_\epsilon \) is (topologically) equivalent to \( E \).

(iii) Let \( \epsilon > 0 \) and \( S_0 \subset \mathbb{R}^n \) a set that includes an open segment containing 0. Then, for any extension \( d_\epsilon^* \) of \( d_\epsilon \) to \( S_0 \), the metrics \( d_\epsilon^* \) and \( E \) are not equivalent on \( S_0 \).

\[ \text{Proof.} \quad \text{(i) Note first that } \Theta \text{ is well-defined on } M. \text{ The triangle inequality and the symmetry are satisfied by both the functions } \Theta \text{ and } E. \text{ Indeed, it is known that } \Theta \text{ is a metric on the Euclidean unit sphere, and so if } s_1, s_2, s_3 \in M, \]
\[ \Theta(s_1, s_2) = \Theta\left(\frac{s_1}{\|s_1\|}, \frac{s_2}{\|s_2\|}\right) \leq \Theta\left(\frac{s_1}{\|s_1\|}, \frac{s_3}{\|s_3\|}\right) + \Theta\left(\frac{s_3}{\|s_3\|}, \frac{s_2}{\|s_2\|}\right) = \Theta(s_1, s_3) + \Theta(s_3, s_2). \]

Moreover, any linear combination with non-negative coefficients of \( \Theta \) and \( E \) is a pseudo-metric. Also, if \( \epsilon > 0 \) then \( d(s_1, s_2) = \Theta(s_1, s_2) + \epsilon E(s_1, s_2) = 0 \) implies \( E(s_1, s_2) = 0 \), and so \( s_1 = s_2 \). The converse is obvious too.

(ii) Take an element \( s \in M \) and an open ball \( B_{d_\epsilon}(s) \) of radius \( r > 0 \) for the metric \( d_\epsilon \). Take the elements \( s' \in M \) in this set satisfying that \( \Theta(s, s') < r/2 \) and \( E(s, s') < r/(2\epsilon) \), and note that all of them are in \( B_{d_\epsilon}(s) \). Then, since \( s \neq 0 \), by the continuity of \( \Theta \) with respect to the Euclidean metric \( E \) we can find a ball of radius \( r' > 0 \) such that
\[ B_{E, r'}(s) \subset \{ s' \in M : \Theta(s, s') < r/2 \}. \]

Thus, taking \( r'' = \min\{r/(2\epsilon), r'\} \) we get that \( B_{E, r''}(s) \subseteq B_{d_\epsilon}(s) \). The obvious inequality
\[ E(s_1, s_2) = \|s_1 - s_2\|_2 \leq \frac{1}{\epsilon} d_\epsilon(s_1, s_2), \quad s_1, s_2 \in M, \]
gives the converse relation needed for the equivalence.
(iii) Consider without loss of generality the vectors \( b = (\alpha, 0, 0, 0, \ldots) \), \( -b = (-\alpha, 0, 0, 0, \ldots) \) ∈ \( M \), for some \( \alpha > 0 \). It is enough to notice that we can construct a sequence converging to 0 with respect to \( E \) and which does not converge for \( d^*_\epsilon \). Indeed,

\[
\lim_{0<\alpha\to0} \| b - (-b) \|_2 = \lim_{0<\alpha\to0} 2\alpha = 0,
\]

but

\[
\lim_{0<\alpha\to0} d_\epsilon(b, -b) = \lim_{0<\alpha\to0} \arccos\left( \frac{b \cdot (-b)}{\|b\| \| -b \|} \right) + \lim_{0<\alpha\to0} \epsilon \|b - (-b)\|_2 = 1.
\]

Thus, both metrics cannot be equivalent. □

Of course, Lemma 3.1 can be automatically stated if we change the Euclidean norm by any other norm on \( \mathbb{R}^n \), since all norms are equivalent on finite dimensional spaces. The metric \( d_\epsilon \) is defined to indicate the Euclidean distance among states \( s_1 \) and \( s_2 \) but also the trend that they represent: indeed, in terms of the financial model we are constructing, if two vectors have small size —in fact as small as we want—, but they represent opposite trends in the market, the distance among them is always bigger or equal than 1. The relative weight of \( \Theta \) and \( E \) in the definition of \( d_\epsilon \) is modulated by the parameter \( \epsilon \). We will fix \( \epsilon = 1/10 \) in the present paper, since we are mainly interested in analyzing the behavior of the market under small changes in the trends, trying to focus the model to be sensitive to these trends.

We will define a reward function acting in \( M_0 \) that will be given, as a primary formula, by a duality relation among the elements \( s \in M_0 \subset \mathbb{R}^n \) and partitions of the unity acting on these elements given by constant coefficients. We will call these elements actions, and they will be represented by vectors of the unit sphere of the space \( (\mathbb{R}^n, \| \cdot \|_1) \) having all the coordinates bigger or equal than 0, that will be called \( S^n_{1^+} \).

We will define the reward function \( R : M_0 \to \mathbb{R} \) as a mean of actions like

\[
R_0(s) = s \cdot a, \quad s \in M_0, \quad a \in B_s,
\]

where \( B_s \) is an \( s \)-dependent set defined using a mix among some experience on the system and a random procedure. The final function will be called \( R \), and will be the real function to be extended with the McShane formula for getting the reward function acting in all the space \( M \). In any case, as we will see in Section 4, it will be always possible to write \( R(s) \) as \( s \cdot a_s \) for a given action \( a_s \) of the selected set of actions \( A \) for the elements of \( M_0 \). We will define the set \( A \) by \( A := 100 \times S^n_{1^+} \), in order to work with bets given as %.

This representation formula \( R(s) = s \cdot a_s, s \in M_0 \), for a certain \( a_s \in A \) will not be always possible to get for all the extended values \( R^M(s^*) \), \( s^* \in M \). Let us show this fact with the following very simple example. However, due
to the particular formula that we have used for the definition of \( d_\epsilon \), it is possible to get a meaningful bound at least.

**Example 3.2.** Fix \( \epsilon > 0 \). Consider a market with two products and just two states \((n = 2)\). Consider the set \( M_0 = \{(1,0), (2,0)\} \). Both vectors represent increasing states of the market. Consider the reward function given for both states by the actions \( a_1 = (50,50) \) and \( a_2 = (0,100) \). That is, 
\[
R((1,0)) := (1,0) \cdot a_1 = 50 \quad \text{and} \quad R((2,0)) := (2,0) \cdot a_2 = 0.
\]

Note that \( d_\epsilon((1,0),(2,0)) = \epsilon \). The Lipschitz constant \( K \) is given by
\[
K = |0 - 50|/d_\epsilon((1,0),(2,0)) = 50/\epsilon.
\]

Therefore, the McShane extension of \( R \) is given by 
\[
R^M((x,y)) := \max\{ 50 - (50/\epsilon)d_\epsilon((x,y),(1,0)), 0 - (50/\epsilon)d_\epsilon((x,y),(2,0)) \}.
\]
for any possible state \((x,y) \in \mathbb{R}^2 \setminus \{0\}\). Take now \((x,y) = (-1,0)\), and note that 
\[
d_\epsilon((1,0),(-1,0)) = 1 + 2\epsilon \quad \text{and} \quad d_\epsilon((2,0),(-1,0)) = 1 + 3\epsilon.
\]
Then we have 
\[
R^M((-1,0)) = \max\{ 50 - (50/\epsilon)d_\epsilon((-1,0),(1,0)), 0 - (50/\epsilon)d_\epsilon((-1,0),(2,0)) \} = \max\{ 50 - \frac{50}{\epsilon} \cdot (1+2\epsilon), 0 - \frac{50}{\epsilon} \cdot (1+3\epsilon) \}.
\]
Take now \( \epsilon = 1/2 \). Then 
\[
R^M((-1,0)) = \max\{-150, -250\} = -150.
\]
Since all the actions in \( A \) belong to the ball of radius 100 of \( \ell^1 \), we cannot write 
\[
R^M((-1,0)) = (-1,0) \cdot a
\]
for any \( a \in A \).

To get the bound it is necessary to prove that the model is consistent, in the sense that the size of the extension \( R^M(s^*) \) is coherent with the size of \( s^* \), and respects the proportionality with \( R \) that appears in the seminal set \( M_0 \). We write \( \| \cdot \| _\infty \) for the \( \ell^\infty \)-norm as usual.

We also define the “dual set” of \( M_0 \), with respect to \( R \) as 
\[
\mathcal{A}_{M_0,R} = \{ a \in 100 \times S^{n,+}_\ell : a = a_s \text{ for some } s \in M_0 \text{ such that } R(a) = s \cdot a_s \}.
\]

**Proposition 3.3.** Let \( M_0 \subset M \) be a compact subset of \((\mathbb{R}^n \setminus \{0\}), \| \cdot \|_2\). Consider a function \( R : M_0 \to \mathbb{R} \) such that for each \( s \in M_0 \) there is a functional \( a_s \in \mathcal{A}_{M_0,R} \subset 100 \times S^{n,+}_\ell \) such that 
\[
R(s) := s \cdot a_s, \quad s \in M_0.
\]
Then for each \( s^* \in M \) there is a functional \( a_{s^*} \in \mathcal{A}_{M_0,R} \) such that 
\[
|R^M(s^*) - s^* \cdot a_{s^*}| \leq \min_{s \in M_0} \left( 100\|s - s^*\|_\infty + K\Theta(s, s^*) + \epsilon KE(s, s^*) \right).
\]
Proof. Fix $s^* \in M$. First note that, since $R^M$ is a Lipschitz function with the same Lipschitz constant $K$ than $R$, for each element $s \in M_0$ we have
\[ |R^M(s^*) - R(s)| \leq Kd_{\epsilon}(s^*, s'). \]
Fix now $s \in M_0$. Then by hypothesis there is a functional $a_s \in A_{M_0, R}$ such that
\[ |R^M(s^*) - a_s| = |R^M(s^*) - s \cdot a_s| \leq |R^M(s^*) - s \cdot a_s| + |(s - s^*) \cdot a_s| \leq Kd_{\epsilon}(s^*, s) + |(s - s^*) \cdot a_s|. \]
Therefore,
\[ |R^M(s^*) - s \cdot a_s| \leq K(\Theta(s^*, s) + \epsilon \|s - s^*\|_2) + 100 \|s - s^*\|_\infty. \]
Since this happens for all the elements $s \in M_0$, we have that the inequality holds for the infimum. Finally, note that the set $M_0$ is compact. Indeed, by Lemma 3.1 $d$ and $E$ are equivalent metrics on $M$. we have that the infimum is attained, and so we get the result by taking $a_{s^*} = a_{s_0}$ for the state $s_0$ that attains the minimum.

Using this result with some restrictions on the geometry of the set $M_0$ and the relation with the particular elements $s^*$, we obtain useful bounds for the formulas that approximate $R^M$. We write one of them in the next corollary. Essentially, it reflects what happens with the extension of the reward function $R$ for a state $s^*$ that represents the same market trend as another state belonging to $M_0$, but with different norm.

**Corollary 3.4.** Let $M_0 \subset M$ be a compact subset of $(\mathbb{R}^n \setminus \{0\}, \| \cdot \|_2)$. Consider a function $R : M_0 \to \mathbb{R}$ satisfying the requirements in Proposition 3.3.

Suppose that an element $s^* \in M$ belongs to $\{\lambda > 0 : \lambda M_0\}$. Then there is a functional $a_{s^*} \in A_{M_0, R}$ such that
\[ |R^M(s^*) - s^* \cdot a_{s^*}| \leq \min_{0<\lambda} \left\{ \frac{|\lambda - 1|}{\lambda} \left( 100 \|s^*\|_\infty + \epsilon K \|s^*\|_2 \right) : \frac{s^*}{\lambda} \in M_0 \right\}. \]

Proof. By assumption, $s^* = \lambda s$ for a given $0 < \lambda$ and $s \in M_0$. For such an $s$ we have that $\Theta(s, s^*) = 0$. The rest of the right hand term in the inequality in Proposition 3.3 can be rewritten as
\[ 100 \|s - \lambda s\|_\infty + \epsilon KE(s, \lambda s) = |1 - \lambda| \|s\|_\infty + |1 - \lambda| \epsilon K \|s\|_2, \]
for $s^* = \lambda s$. This can be rewritten as
\[ \frac{|\lambda - 1|}{\lambda} \left( 100 \|s^*\|_\infty + \epsilon K \|s^*\|_2 \right). \]
This gives the result.

Depending on the geometry of the set $M_0$ and its relation with the chosen state $s^* \notin M_0$, we can also obtain a lower bound for the approximation formula for $R^M$ using actions $a \in A$. 


Proposition 3.5. Let $M_0 \subset M$ be a compact subset of $(\mathbb{R}^n \setminus \{0\}, \| \cdot \|_2)$, and $\epsilon > 0$. Consider a function $R : M_0 \rightarrow \mathbb{R}$ satisfying the requirements in Proposition 3.3. Let $s^* \in M \setminus M_0$ and $a \in A$ such that

$$s^* \cdot a \geq R(s) \quad \text{for all} \quad s \in M_0.$$  

Then for $\Theta(s^*, M_0) = \inf_{s \in M_0} \Theta(s^*, s)$ and $E(s^*, M_0) = \inf_{s \in M_0} \| s^* - s \|_2$, we have that

$$|s^* \cdot a - R^M(s^*)| \geq K \left( \Theta(s^*, M_0) + \epsilon E(s^*, M_0) \right).$$

Proof. Take $s^*$ and $a \in A$ as in the statement of the result. Then, using again compactness of $M_0$ we get an element $s_0 \in M_0$ such that $R^M(s^*) = R(s_0) - K d_\epsilon(s_0, s^*)$. We know that by hypothesis there is an element $a_{s_0} \in A$ such that $R(s_0) = s_0 \cdot a_{s_0}$, and so we have that

$$|s^* \cdot a - R^M(s^*)| = |s^* \cdot a - R(s_0) + K d_\epsilon(s_0, s^*)|$$

$$= (s^* \cdot a - R(s_0)) + K d_\epsilon(s_0, s^*) \geq K \left( \Theta(s^*, M_0) + \epsilon E(s^*, M_0) \right),$$

and the lower bound is proved.  

In particular cases, this bound can be used for getting clear negative results on the possibility of approximating the extended reward function $R^M$ by means of actions. We show one of them in the following corollary, which proof is obvious.

Corollary 3.6. Let $M_0 \subset M$ be a compact subset of $(\mathbb{R}^n \setminus \{0\}, \| \cdot \|_2)$, and $\epsilon > 0$. Consider a function $R : M_0 \rightarrow \mathbb{R}$ satisfying the requirements in Proposition 3.3. Let $s^* \in M \setminus M_0$ and $a \in A$.

(i) If $s^* \cdot a \geq 100 \| s \|_2$ for all $s \in M_0$, sup$_{s \in M_0} \| s \|_2 = B$ and $s^* \in \cup_{\lambda > 0} \lambda M_0$, then

$$|s^* \cdot a - R^M(s^*)| \geq K \epsilon (\| s^* \|_2 - B).$$

(ii) If $M_0 \in C$, where $C$ is a closed convex cone (with vertex in 0) that do not contain $s^*$, then

$$|s^* \cdot a - R^M(s^*)| \geq K \Theta(s^*, C).$$

Remark 3.7. As we have demonstrated, The mathematical model imposes the restriction that valid states are always different than 0. That is, there are no states that represent that the system has not changed, or that there is no trend. Therefore, these states must be eliminated if they appear in the experience.
3.2. The procedure. We will work with the following metric space structure as a model for the dynamical system defined by a financial market with \( n \) products. We will assume that there are \( m \) times in which there are share purchase/sale events.

(A) Take a subset \( M_0 \) of vectors of \( M = \mathbb{R}^n \) representing the states of the market. Each of the vectors in \( M_0 \) describes a state of the market in the following way: each coordinate gives the value of the increment of the corresponding product at this moment. In fact, we will write at each coordinate \( i \) the difference of the value at the moment \( i \in \{1, ..., m\} \) and the value at \( i - 1 \). This means, in particular, that the original values of the products is not relevant for defining the states, just the variations.

We will fix the value \( \epsilon = 1/10 \) for the definition of the metric in the next sections. That is, we will use

\[
d(s_i, s_j) = d_{1/10}(s_i, s_j) = \Theta(s_i, s_j) + \frac{1}{10} E(s_j, s_i), \quad s_i, s_j \in M_0.
\]

(B) We are interested in measuring the success of a concrete action in the market, that is, the success of a share purchase/sale event that a decision-maker has executed on the market. So we have to define what an action is in the model. Formally, we have already defined them as elements of the dual of \( \mathbb{R}^n \). As we said, at each step the state of the system is defined by an \( n \)-coordinate vector; each coordinate represents the increase/decrease of the value of each product with respect to the previous step. An action is a suitable share purchase/sale event that the decision maker could execute, represented as follows: it is supposed that he has 100 monetary units to invest at every step, so an action is a vector of \( n \)-coordinates (\( n + 1 \) if we want to consider leaving some of the money out of the buying process). In Section 4 we will call “bets” to the actions to reinforce their meaning in the model. Mathematically, they are positive elements of the algebraic dual of \( \mathbb{R}^n \) having \( \ell^1 \)-norm equal to 1. Let us write \( \mathcal{A} \) for the set of all the actions.

The natural reward function to be defined in the model must be related to the evaluation of the success of an action when it is applied to a certain state of the system. Therefore, it must be defined as a functional acting in \( \mathcal{A} \) once a given state of the system has been fixed, and so it is a two-vector-variables function \( R_0 \) acting in \( M_0 \times \mathcal{A} \).

However, the reward function must evaluate states of the market—an element of \( M_0 \)—, taking into account how the decision maker acts in it and the success of his actions. Therefore, we will finally consider a reward function \( R \) acting in \( M_0 \), but we will use all the information we have about the system to estimate it. That is, we will use the function \( R_0 \) for defining the function \( R \). We will see
that, finally, for each state $s \in M_0$ there is an action $a \in A$ such that $R(s) = s \cdot a$, or a mean of such values.

(C) After this, we are interested in extending the reward function $R$ to the whole linear space $M$ preserving the Lipschitz constant. In order to assure that this constant is a (positive) real number, it is enough to take into account that the set $M_0$ is defined by a finite set of vectors. In the model, $R$ is supposed to measure “how successful” is a given state. We will use the McShane formula for the extension. The extension $R^M$ is supposed to extrapolate the same concept — success of a given state—, preserving the metric relations of $M_0$ and $M$. Since it appears explicitly in the formula, we have to compute the Lipschitz constant $K$ for the reward function $R$ in order to get the extension $R^M$, for which the same $K$ works. The way we have defined the metric in the space allows to obtain a theoretical bound for this extension, as stated in Proposition 3.3. However, note that in general we cannot expect that $R^M(s)$ can be represented as an action belonging to the positive part of $100 \times B_{\ell_1}(M, \| \cdot \|_{\ell_1})$. This was shown in the previous section in Example 3.2; the general behavior of such kind of representation formula was discussed also there, as a consequence of Proposition 3.5 and its corollaries.

(D) Finally, we will use $R^M$ for simulating the reward of new time sequences of states in order to perform our reinforcement learning algorithm. In order to do this, we generate randomly new states for increasing the set $M_0$. We create in this way a new seminal set $M_1$ bigger than $M_0$, in which we are mixing “known situations” ($s \in M_0$) and new ones, that we call “dreams” ($s^* \in M_1 \setminus M_0$). The rate of elections of known cases and dreams that we have chosen is $\beta = 50\%$.

4. Training and dreaming: a Lipschitz approximation to a real market reward function for reinforcement learning

Let us continue with the explanation of the procedure by further specifying the example explained on the financial market that we started in the previous section. Suppose that we are analyzing a market with four similar products. In fact, there is a clear correlation among their values, as the reader can see in the figures that will be shown below. We have the complete behavior of the values of all of them each minute of a sequence of 800 minutes. As we said in Section 3 and for the aim of simplicity, we assume that at $t = 0$ the values of all the products equal 0.

The set $M_0$ of known states for which the reward function can be calculated is defined as the first 50% of the states that have been registered in the experience of a day.
1. A state of the system is given by a four-coordinate vector $s$: as we explained in Section 3, each minute the vector gives the cumulative increase or decrease of the values of each product. Since we want to define the reward function using the scalar product with a vector representing an action we will need to enlarge the vector $s$ by adding a new coordinate, with the value 0. We preserve the same symbol $s$ for the extended vector.

We consider series of “bets” applied at each minute. They correspond to series of what we called “actions” in Section 3, that in this particular case are described as the % of the money that the decision maker wants to apply in each market this minute (including not investing a certain part). As we said, it is supposed the decision maker is investing 100 monetary units at each step. A bet is then given by a five positive coordinates vector such that they sum 100; recall we have five coordinates because the decision maker could decide not to invest a part of the money.

2. Fix now a (five-coordinate) state of the system $s$. The reward function $R : M_0 \times A \rightarrow \mathbb{R}$ is then defined as a two-(vector)-variable function given by the scalar product of the state and the action $a$, $R_0(a, s) = a \cdot s$ where $s$ is the five-coordinate extension of the original 4-coordinate vector representing the state $s$.

At this point we introduce our first arguments regarding reinforcement learning. The main idea is to use the information that is known for similar situations in order to compute a reward function $R : M_0 \rightarrow \mathbb{R}$, depending only on the state. This is relevant, since we are going to evaluate the state of the system using this reward function. In order to define it, we use the following procedure. For a state of the system $s$, we define $R(s) := \text{mean} \{ R(a, s) : a \in A \cup B \}$, where the mean is computed over two sets $A$ and $B$ constructed as explained below, whose sizes are in a relation of 90% and 10%, respectively.

a) The first set $A$—90%— is defined by using actions/bets $a$ that have been already checked and have obtained good enough values of the reward functions when acting in states $s'$ that are similar to $s$. This is done by choosing the bets that give the highest values of the reward function when they act on these states $s'$. The similarity relation is given by proximity with respect to the distance $d$, that is $d(s, s') < \varepsilon$ for a given $\varepsilon > 0$ (for example, $\varepsilon = 0.5$).

b) The second one $B$—10%— is randomly obtained.
Note that this way of defining the reward function is not mathematically optimal since, given a state, the definition of the reward function allows to compute the better bet for it using elementary calculus. However, this is an example for which the function can be directly computed—an explicit formula is considered—and in the general case this would not be the case. Our method aims to introduce some “empirical information” from the system. Moreover, the given procedure allows some random elements to be introduced into the process, which is necessary to avoid, for example, overlearning.

This method is used for computing the reward function \( R \) for the elements of \( M_0 \). For states which do not belong to \( M_0 \), we will use the McShane formula for obtaining the extended function \( R^M \) as explained in Section 3.

3. We design in this way a procedure for obtaining a reward function on the whole set \( M \) of possible states of the system. We use the first 50% positions of the market (Figure 1) as the set \( M_0 \); it can be considered as the training set, and the function \( R \) is defined by the procedure explained in Point 2 above. Note that, although the original value is considered to be 0 for all the products, we represent the cumulative value from the starting point in Figure 1, that is, the sum of each coordinate of the vectors (states) representing the consecutive steps. The same is done in Figure 2, in which the testing set of states is given.

Using it, and after computing the corresponding Lipschitz constant \( K \), we use the McShane formula to obtain the extension \( R^M : M \to \mathbb{R} \), that is

\[
R^M(s^*) := \sup_{s \in M_0} \{ R(s) - K d(s^*, s) \}, \quad s^* \in M.
\]

The additional 50% (Figure 2) is used to check the performance of the model and the quality of the results by comparison.
4. Taking into account the procedure for obtaining the reward function $R$, given a state $s \in M_0$ we can find an action/bet $a_s \in A$ such that $R(s) = R_0(a_s, s)$ and is as good (high) as possible. Of course, $a_s$ is not unique. However, a random function can be defined from $M_0$ to $A$ in such a way that for a state $s$ the assignment $s \mapsto a_s$ provides a successful bet for $s$. One of these randomly defined functions $s \mapsto a_s$ is shown in Figure 3. Note that, certainly, all of them are successful actions/bets, since they provide the maximum possible value of the reward function for each element of $M_0$.

A similar definition can be done for suitable states that do not belong to $M_0$. We call dreams to such states. In this case, the reward function that should be considered is $R^M$, since this function plays the role of $R$ for states that have not been found in the experience in the market. However, note that we cannot say that, if $s^* \in M \setminus M_0$, there is a positive functional —an action— $a_{s^*}$ in the unit ball of $\ell^1$ such that $R^M(s^*) = s^* \cdot a_{s^*}$, as happens for $s \in M_0$ and $R$. This problem is solved just by taking a suitable “norm 100” functional $a_{s^*}$ such that $R^M(s^*) - s^* \cdot a_{s^*}$ attains its minimum value. We have already proved that in general, $R^M(s^*)$ cannot be attained by a value as $a_{s^*} \cdot s^*$. However, Proposition 3.3 gives precise bounds for this difference.

The set of all —randomly chosen but optimal— bets as $a_s$ and $a_{s^*}$ represents how the decision maker should act when he faces the
Figure 2. Real market experience: set of states for testing the model (minutes from 400 to 800). Again, the accumulated value for all the products of the market is represented. In this case, the starting value is given by the last value of the previous states, given in Figure 1.

Problem of investing in the market. Figure 3 shows a representation of a suitable set of optimal bets for the states represented in Figure 1. Figure 4 represents a sequence of optimal actions in mixed situation of $\beta = 50\%$ real states and 50% dreams. At each time, the sum of the values in the five graphics sum 100%.

As we have shown, the main tool of our technique is the computation of the McShane extension of the reward function. In order to clarify this computation based on the McShane-Whitney extension theorem, we provide an scheme of the algorithm (Algorithm 1).

5. Finally, we check the results of the model. We assume that we start betting on the market at the time $t = 0$ with 1000 of monetary units and we stop when we lose all of them. In order to check the success of the model, we produce a simulation when the reward function is purely obtained by the information of the market (Figure 3), and using 50% of dreamed states (Figure 4).

We use the second part of the experience that was shown in Figure 2 for checking our results. The system has been trained using all the information of the first 400 minutes in the first case (Figure 5), and with just 50% of these states + 50% of dreams in the second (Figure 6). Thus, in Figure 5 and Figure 6 we have presented the value of
Algorithm 1 Computation of the McShane extension

1: Fix $M_0 = \{s_k : s_k$ is a state from the time series experience$\} \neq \emptyset$.

Require: $|M_0| \geq 100$

2: while $s_k \in M_0$ do
3:   For $i \in \{1, \cdots, 1000\}$, sort $a^k_i \in A := 100 \times S_i^{5,+}$ to define $A^k$.
4:   $R^k_i \leftarrow R_0(a^k_i, s_k) := a \cdot s_k$, $a \in A^k$
5:   $A^k_* := \{90$ states in $A^k$ such $R^k_i$ are the biggest ones in $A^k.$
6:   Define randomly a subset of 10 elements $B^k_* \subset A^k$.
7:   $R(s_k) \leftarrow \text{mean} \{ R(a^k_i, s_k) : a^k_i \in A^k_* \cup B^k_* \}$.  
8: end while
9: $K \leftarrow \max_{s, s' \in M_0} \frac{|R(s) - R(s')|}{d(s, s')}$. 
10: if $s \in M \setminus M_0$ then
11:   return $R^M(s) \leftarrow \sup_{s' \in M_0} R(s') - Kd(s, s')$.
12: else
13:   $R^M(s) \leftarrow R(s)$
14: end if

Figure 3. Sequence of (randomly chosen) actions that optimize the bets when applied to the set of real states $M_0$ (first 300 minutes). Note that for each fixed time, the five values equals 100%.

the sum of the four products of the market at each state, where the investment that has been made in each of them has been the result of the application of the action/bet obtained in the previous steps.
Figure 4. Sequence of randomly chosen actions that optimize the bets when applied to a mixed set given by 50% of real states in $M_0$ and 50% of dreams. (First 300 minutes, 150 randomly changed from the original experience by “dreams.”)

The measure of the success of the models is given by the survival time.

For the first case (Figure 5) we have used the set of actions obtained for the set $M_0$, which was shown in Figure 4. It is supposed that the situations should be similar than in the training part of the experience. However, in case the state $s$ was not exactly appearing in the market situations that was recorded in the first part of the experience, we approximate its value by distance similarity applying the action $a_{s'}$, where $s'$ is the element of $M_0$ that satisfies that $d(s, s')$ attains its minimum.

The second figure (Figure 6) shows the same cumulative result: the total value obtained at each state by applying to the same sequence of states the optimal sequence of actions, that has been obtained in this case with a 50% of dreams. As the reader can see, the evolution and the surveillance time are similar, and so the success of both models is comparable. That is, the same result can be obtained by using the McShane extension of the 50% of known data instead of 100% of real data.
Figure 5. Simulation with real data obtained from the experience.

Figure 6. Simulation with 50% of real data +50% of dream.

5. Conclusions

We have shown a reinforcement learning method to provide an expert system for investing in a financial market. The first introduced tool, that involves approximation of a reward function by using metric similarity with other known states of the system, is based on a classic machine learning scheme on metric spaces. Regarding this point, the main novelty is the non-standard metric that is used, that combines a geodesic distance —directly related with the cosine similarity of vectors and that models the directions of the trends of
the market—, and the Euclidean distance, which cannot be defined as associated to a norm in the underlying finite dimensional linear space.

The second part of our technique consists on the development of a new reinforcement learning procedure that allows the use of a smaller set $M_0$ of experiences on the financial market to obtain a good investment tool to act in the market. Basically, we combine the use of approximation of the reward function on neighbors of $M_0$ with a Lipschitz-preserving extension of the reward function by using the McShane formula. Thus, the main contribution of the present paper is to show that an expert system for investment in financial markets can be done by substituting a great set of experiences on the particular markets by a reinforcement learning method based on the extension of Lipschitz maps. Since the results obtained are comparable, our technique opens up the possibility of building models of similar efficiency using much less data from experience.

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