Order statistics applied to the most massive and most distant galaxy clusters

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ABSTRACT
In this work, we present an analytic framework for calculating the individual and joint distributions of the nth most massive or nth highest redshift galaxy cluster for a given survey characteristic allowing us to formulate Λ cold dark matter (ΛCDM) exclusion criteria. We show that the cumulative distribution functions steepen with increasing order, giving them a higher constraining power with respect to the extreme value statistics. Additionally, we find that the order statistics in mass (being dominated by clusters at lower redshifts) is sensitive to the matter density and the normalization of the matter fluctuations, whereas the order statistics in redshift is particularly sensitive to the geometric evolution of the Universe. For a fixed cosmology, both order statistics are efficient probes of the functional shape of the mass function at the high-mass end. To allow a quick assessment of both order statistics, we provide fits as a function of the survey area that allow percentile estimation with an accuracy better than 2 per cent. Furthermore, we discuss the joint distributions in the two-dimensional case and find that for the combination of the largest and the second largest observation, it is most likely to find them to be realized with similar values with a broadly peaked distribution. When combining the largest observation with higher orders, it is more likely to find a larger gap between the observations and when combining higher orders in general, the joint probability density function peaks more strongly. Having introduced the theory, we apply the order statistical analysis to the Southpole Telescope (SPT) massive cluster sample and metacatalogue of X-ray detected clusters of galaxies catalogue and find that the 10 most massive clusters in the sample are consistent with ΛCDM and the Tinker mass function. For the order statistics in redshift, we find a discrepancy between the data and the theoretical distributions, which could in principle indicate a deviation from the standard cosmology. However, we attribute this deviation to the uncertainty in the modelling of the SPT survey selection function. In turn, by assuming the ΛCDM reference cosmology, order statistics can also be utilized for consistency checks of the completeness of the observed sample and of the modelling of the survey selection function.

Key words: methods: statistical – galaxies: clusters: general – cosmology: miscellaneous.

1 INTRODUCTION
Clusters of galaxies represent the top of the hierarchy of gravitationally bound structures in the Universe and can be considered as tracers of the rarest peaks of the initial density field. This feature renders their abundance across the cosmic history a valuable probe of cosmology (for an overview of cluster cosmology see e.g. Voit 2005; Allen, Evrard & Mantz 2011, and references therein). The recent years brought significant advances to the field from an observational point of view. Past and present surveys, like e.g. the ROSAT All Sky Survey (RASS; Voges et al. 1999), the Massive Cluster Survey (MACS; Ebeling, Edge & Henry 2001) and the Southpole Telescope (SPT; Carlstrom et al. 2011), provided rich data for a multitude of massive clusters (\(>10^{15} \, M_\odot\)). In the near future, cluster data will be drastically extended in terms of...
completeness, coverage and depth by surveys like for instance Planck (Tauber, J. A. et al. 2010), eRosita (Cappelluti et al. 2011) and Euclid (Laureijs et al. 2011), allowing for statistical analyses of the samples with increasing quality.

A particular form of statistical analysis that recently entered focus is falsification experiments of the concordance Λ cold dark matter (ΛCDM) cosmology, based on the discovery of a single (or a number of) cluster(s) being so massive that it (they) could not have formed in the standard picture (Hotchkiss 2011; Hoyle, Jimenez & Verde 2011; Mortonson, Hu & Huterer 2011; Harrison & Coles 2012; Harrison & Hotchkiss 2012; Holz & Perlmuter 2012; Hoyle et al. 2012; Waizmann et al. 2012a,b). These studies were triggered by the discovery of massive clusters at high redshift (see e.g. Mullis et al. 2005; Jee et al. 2009; Rosati et al. 2009; Foley et al. 2011; Menanteau et al. 2012; Stalder et al. 2013).

However, the usage of a single observation for such falsification experiments requires statistical care since several subtleties have to be taken into account. From the theoretical point of view, it is necessary to include the Eddington bias (Eddington 1913) in mass, as discussed in Mortonson et al. (2011) and the bias that stems from the a posteriori choice of the redshift interval for the analysis (Hotchkiss 2011). From the observational point of view, it might, particularly for very high redshift systems, be difficult to define the survey area and selection function that are appropriate for the statistical analysis. Combining all of these effects, recent studies (Hotchkiss 2011; Harrison & Coles 2012; Harrison & Hotchkiss 2012; Waizmann et al. 2012a,b) converge to the finding that, when taken alone, none of the single most massive known clusters can be considered in tension with the concordance ΛCDM cosmology.

Conceptually, inference based on a single observation is not desirable, because by nature the extreme value might not be representative for the underlying distribution from which it is supposedly drawn. Thus, it is advised to incorporate statistical information from the sample of the most massive high-redshift clusters, which in turn are also particularly sensitive to the underlying cosmological model since they probe the exponentially suppressed tail of the mass function. Among previous works, Mortonson et al. (2011), Hotchkiss (2011) and Hoyle et al. (2012) dealt with the question of what is the mass and redshift of the nth most extreme cluster.

In this work, we introduce order statistics as a tool for analytically deriving distribution functions for all members of the mass and redshift hierarchy ordered by magnitude. By dividing our analysis in the observables mass and redshift, we avoid the bias due to an a posteriori definition of redshift intervals (Hotchkiss 2011) and avoid as well the arbitrariness of an a priori choice that had been necessary in our previous works based on the extreme value statistics. Furthermore, the formalism also allows for the formulation of joint probabilities of the order statistics. In the second part of this work, we compare our individual and joint analytic distributions to observed samples of massive galaxy clusters.

This paper is structured according to the following scheme. In Section 2, we introduce the statistical branch of order statistics by discussing the basic mathematical relations in Section 2.1 and by applying the formalism to the distribution of massive galaxy clusters in mass and redshift in Section 2.2. This is followed by a discussion of how the order statistics of haloes in mass and redshift depends on cosmological parameters in Section 3. In order to compare our analytic results to observations, we prepare observed cluster samples for the analysis in Section 4. Afterwards, we discuss the results of the comparison for the case of the individual order statistics in Section 5 and for the joint case in Section 6. Then, we summarize our findings in Section 7 and draw our conclusions in Section 8. In Appendix A, we give a more detailed overview of order statistics and in Appendix B, fitting formulae for the order statistics in mass and redshift are presented.

Throughout this work, unless stated otherwise, we adopt the Wilkinson Microwave Anisotropy Probe 7-year (WMAP7) parameters (Ω_m, Ω, h, σ_8) = (0.727, 0.273, 0.0455, 0.704, 0.811) (Komatsu et al. 2011).

## 2 ORDER STATISTICS

Order statistics (for an introduction, see e.g. Arnold, Balakrishnan & Nagaraja 1992; David & Nagaraja 2003) is the study of the statistics of ordered (sorted by magnitude) random variates. In this section, the basic mathematical relations and the connection to cosmology are introduced as they will be needed in remainder of this work.

### 2.1 Mathematical prerequisites

Let X_1, X_2, ..., X_n be a random sample of a continuous population with the probability density function (pdf), f(x), and the corresponding cumulative distribution function (cdf), F(x). Further, let X_{(1)} ≤ X_{(2)} ≤ ... ≤ X_{(n)} be the order statistics, the random variates ordered by magnitude, where X_{(1)} is the smallest (minimum) and X_{(n)} denotes the largest (maximum) variate. It can be shown (see Appendix A1) that the pdf of X_{(i)} (1 ≤ i ≤ n) is given by

\[ f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1 - F(x)]^{n-i} f(x). \]

(1)

The corresponding cdf of the ith order then reads

\[ F_{X_{(i)}}(x) = \sum_{k=i}^{n} \binom{n}{k} [F(x)]^k [1 - F(x)]^{n-k}, \]

(2)

and the distribution function of the smallest and the largest value are found to be

\[ F_{X_{(1)}}(x) = 1 - [1 - F(x)]^n, \]

(3)

and

\[ F_{X_{(n)}}(x) = [F(x)]^n. \]

(4)

In the limit of very large sample sizes both \( F_{X_{(i)}}(x) \) and \( F_{X_{(n)}}(x) \) can be described by a member of the general extreme value (GEV) distribution (Fisher & Tippett 1928; Gnedenko 1943)

\[ G(x) = \exp \left\{ - \left[ 1 + \gamma \left( \frac{x - \alpha}{\beta} \right) \right]^{-1/\gamma} \right\}, \]

(5)

where \( \alpha \) is the location, \( \beta \) the scale and \( \gamma \) the shape parameter. Usually these parameters are obtained directly from the data or from an underlying model (see for instance Coles 2001).

Apart from the distributions of the single order statistics, it is very interesting to derive joint distribution functions for several orders. The joint pdf of the two order statistics X_{(i)}, X_{(j)} (1 ≤ r < s ≤ n) is for \( x < y \) given by (see Appendix A for a more detailed discussion)

\[ f_{X_{(i)}, X_{(j)}}(x, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \times [F(x)]^{r-1} [F(y) - F(x)]^{i-r-1} [1 - F(y)]^{s-i} \times f(x)f(y). \]

(6)
The joint cdf can, e.g. be obtained by integrating the pdf above or by a direct argument and is found to be given by

$$F(x, y) = \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} \times \left[ F(x)^{j-1} \times \int F(y) - F(x) \, dx \right],$$

(7)

Analogously the above relations can be generalized to the joint pdf of $X_{z_1}$, $X_{z_2}$, ..., $X_{z_n}$ ($1 \leq z_1 < \cdots < z_n \leq z_0$) for $x_1 \leq \cdots \leq x_n$, which is given by

$$f_{(z_1), \ldots, (z_n)}(x_1, \ldots, x_n) = \frac{n!}{(n_1-1)! \cdots (n_k-1)!} \times \prod_{j=1}^{k} \int F(x_j)^{j-1} \times f(x_1) \times \prod_{j=2}^{k} \int F(x_j) - F(x_i) \, dx_j \times f(x_2) \cdots [1 - F(x_k)]^{n_k} \, f(x_k).$$

(8)

Further details and derivations concerning order statistics can be found in Appendix A. In the remainder of this work, we will repeatedly make use of percentiles. In statistics, a percentile is defined as the value of a variable below which a certain percentage, $p$, of observations fall. Percentiles can be directly obtained from the inverse of the cdf and will be hereafter denoted as $Q_p$.

### 2.2 Connection to cosmology

As outlined in the previous subsection, the only quantity that is needed for calculating the cdfs, $F_{(z)}(x)$, of the order statistics (see equation 2) is the cdf, $F(x)$, of the underlying distribution from which the sample is drawn. Assuming the random variates, $X_i$, to be the masses of galaxy clusters, then the cdf, $F(m)$, can be calculated (see e.g. Harrison & Coles 2012) by means of

$$F(m) = \frac{f_{\text{sky}}}{N_{\text{tot}}} \int_0^\infty \frac{dV}{dz} \, \frac{dV}{dM} \, n(M, z) \, dM,$$

(9)

where the total number of clusters, $N_{\text{tot}}$, is given by

$$N_{\text{tot}} = f_{\text{sky}} \int_0^\infty \frac{dV}{dz} \, \frac{dV}{dM} \, n(M, z) \, dM.$$

(10)

Here, $f_{\text{sky}}$ is the fraction of the full sky that is observed, $\frac{dV}{dz}$ is the volume element and $n(m, z)$ is the halo mass function. If needed, the corresponding pdf can always be obtained by $f(m) = \frac{dF(m)}{dm}$.

Analogously, the order statistics can be calculated as well for the redshift instead of the mass. In this case, the cdf reads

$$F(z) = \frac{f_{\text{sky}}}{N_{\text{tot}}} \int_0^\infty \frac{dV}{dz} \, \frac{dV}{dM} \, n(M, z) \, dM,$$

(11)

where

$$N_{\text{tot}} = f_{\text{sky}} \int_0^\infty \frac{dV}{dz} \, \frac{dV}{dM} \, n(M, z) \, dM.$$

(12)

For the latter, the order statistics does no longer depend only on the survey area via $f_{\text{sky}}$ but, in addition the selection function of the survey has to be included via a limiting survey mass, $m_{\text{lim}}(z)$. In this work, we do not attempt to model the possible redshift dependence of $m_{\text{lim}}$ and assume it to be constant throughout the remainder of this work.

With the distributions $F(m)$ and $F(z)$ at hand, we can now easily derive the cdfs of the corresponding order statistics. Since we will focus in this work on the few largest values, we will refer to the distribution of the largest, $F_{(n)}(x)$, as first order, to the second largest as second order and so on.

We calculated the distributions of the first 50 orders from $F_{(n)}(x)$ to $F_{(n-49)}(x)$, where $n = N_{\text{tot}}$, and present the results in Fig. 1 for the mass (left-hand panel) and redshift (right-hand panel). In both panels, the colour decodes the order of the distribution, ranging from the blue for $F_{(n)}(x)$ to the green for $F_{(n-49)}(x)$. For both cases, we assumed $f_{\text{sky}} = 1$, a redshift range of $0 \leq z < \infty$ and the Tinker et al. (2008) mass function. In the case of the order statistics in redshift, we assume a limiting survey mass of $m_{\text{lim}} = 10^{15} M_\odot$. It can be nicely seen how, with increasing order (from blue to green), the cdfs shift in both cases to smaller values of the mass or redshift.

A first important result is that, with the increasing order, the cdfs steepen, which results in an enhanced constraining power, since small shifts in the mass or redshift may yield large differences in the derived probabilities. In this sense, the higher orders will be more useful for falsification experiments than the extreme value distribution which, due to its shallow shape, requires extremely large values of the observable to statistically rule out the underlying assumptions. Since higher orders encode information from the $n$ most extreme objects, deviations from the expectation are

![Figure 1](https://academic.oup.com/mnras/article-abstract/432/2/914/1020807/4292140209827/4292140209827)

**Figure 1.** Cumulative distribution functions of the first 50 orders from $F_{(n-49)}$ to $F_{(1)}$ in mass (left-hand panel) and in redshift (right-hand panel). For comparison, the GEV distribution of the maxima is shown by the red, dashed line for both cases. All distributions were calculated for the full sky, assuming the Tinker mass function. For the order statistics in redshift, a limiting mass of $m_{\text{lim}} = 10^{15} M_\odot$ has been adopted.
statistically more significant for \( n \) values instead of a single extreme one.

In addition, we compare the distribution of the maxima \( F_{\alpha}(m) \) and \( F_{\alpha}(z) \) to those obtained from an extreme value approach (Davis et al. 2011; Waizmann et al. 2012a) based on the void probability (White 1979), using equation (5). For both cases presented in Fig. 1, the red curve, the GEV distribution, \( G(x) \), agrees very well with the directly calculated \( F_{\alpha}(x) \).

In order to allow a quick estimation of the distributions of the order statistics, we also provide in Appendix B fitting formulae for \( F(x) \) as a function of the survey area for the cases of mass and redshift. The fitting formulae for the distribution in mass allow an estimation of the quantiles in the range from the 90.2 percentile, \( Q_{0.98} \), to the 98 percentiles, \( Q_{0.99} \), with an accuracy better than 1 per cent for \( \Lambda_{\text{CDM}} \). For \( m_{\text{lim}} = 10^{15} M_{\odot} \) an accuracy of better than 2 per cent can be achieved for \( \Lambda_{\text{CDM}} \). For \( m_{\text{lim}} = 5 \times 10^{14} M_{\odot} \) the same accuracy is obtained down to \( \Lambda_{\text{CDM}} = 100 \) deg. A more detailed discussion of the fitting functions and their performance can be found in Appendix B.

In the remaining part of this work, we will discuss how the underlying cosmological model affects the order statistics and confront the theoretically derived order statistics with observations, afterwards.

### 3 Dependence of the Order Statistics on the Underlying Cosmology

Eventually, the order statistics in mass and redshift is determined by the number of galaxy clusters in a given cosmic volume. The quantities that impact on this number can be categorized into two classes. The first one contains all effects that modify structure formation itself, like the choice of the mass function or the amplitude of the mass fluctuations, \( \sigma_{8} \), for instance. These effects manifest themselves most strongly in the exponentially suppressed tail of the mass function, hence at high masses. The second class contains all the effects that modify the geometric evolution of the Universe. By changing the evolution of the cosmic volume, the number of clusters in a given redshift range can be substantially different, even if both cosmologies yield the same number density of objects of a given mass (see e.g. Pace, Waizmann & Bartelmann 2010).

#### 3.1 Impact of cosmological parameters

In order to quantify the impact of different cosmological parameters on the order statistics in mass and redshift, we study the effect on the 98 percentile, \( Q_{0.98} \), which we use to define possible outliers from the underlying distribution. In Fig. 2, we present the relative difference in \( Q_{0.98} \) as a function of four different cosmological parameters comprising \( \sigma_{8}, \Omega_{m} \) (assuming the flatness constraint), the equation-of-state parameter, \( w_{0} \), and the derivative \( w_{a} \) from the relation \( w(a) = w_{0} + w_{a}(1 - a) \), where \( a \) denotes the scale-factor. A non-vanishing value of the latter indicates a time-varying equation of state. In each panel of Fig. 2, we show the relative differences for five different orders, of the order statistics in mass with \( z \in [0, \infty) \) (blue lines) and \( z \in [1, \infty] \) (green lines), as well as in redshift (red lines) assuming \( m_{\text{lim}} = 10^{15} M_{\odot} \). For all calculations, we assumed the full sky and the Tinker et al. (2008) mass function.

It can be seen that order statistics is very sensitive to \( \sigma_{8} \), such that the relative differences in \( Q_{0.98} \) would amount to \( \sim 7 \) per cent for the range allowed by WMAP7 of \( \sigma_{8} = 0.811 \pm 0.023 \). All three order statistics exhibit the same functional behaviour, with the mass-based ones being more sensitive than the redshift-based one. This can be understood by the fact that the mass-based order statistics probe the most massive clusters and hence the exponential tail of the mass function which is highly sensitive to \( \sigma_{8} \).

For modifications of the matter density, \( \Omega_{m} \), assuming the flatness constraint \( \Omega_{m} = 1 - \Omega_{\Lambda} \), the situation is substantially different from the previous case (see upper-right panel of Fig. 2). Overall, the order statistics are less sensitive and they do not exhibit the same functional behaviour. The order statistics in mass (blue lines) performs best for larger values of \( \Omega_{m} \) because the most massive clusters will reside at rather low redshifts. At high redshifts (green and red lines), the increase in \( \Omega_{m} \) and hence, the decrease in \( \Omega_{\Lambda} \), yields a smaller number of very massive clusters. Despite the increase in the matter density, the decrease in volume is dominating for the range of \( \Omega_{m} \) shown in the plot and, thus, the relative difference decreases. In this sense, the volume effects dominate at high redshifts. However, the order statistics in mass, whereas at low redshifts the increase in matter density dominates.

The lower-left panel of Fig. 2 shows the sensitivity of the order statistics to changes in the constant equation of state, \( w_{0} \). Evidently, the most massive clusters at low redshifts (blue line) have no sensitivity to \( w_{0} \), whereas at high redshifts (green and red line) the sensitivity is better. The volume effects are, compared to modifications in \( \Omega_{m} \), less important and the observed increase in the relative difference in \( Q_{0.98} \) with decreasing \( w_{0} \) is dominated by modifications of the exponential tail of the mass function (for a more thorough discussion, see e.g. Pace et al. 2010).

When assuming a time-dependent equation of state, modelled by \( w(a) = w_{0} + w_{a}(1 - a) \), as presented in the lower-right panel of Fig. 2, the observed functional behaviour can be explained by identical arguments as before. The results exhibit again the high sensitivity of the high-redshift order statistics on modifications of \( w_{a} \). It should be noted that we fixed \( w_{0} = -1.0 \) for all cases.

It can be summarized that for modifications that strongly affect the structure formation, like \( \sigma_{8} \) for instance, the order statistics in mass for \( z \in [0, \infty] \) is comparable in its sensitivity to the redshift-based order statistics. Modifications that strongly alter the geometric evolution of the Universe affect more strongly the order statistics in redshift. However, one should keep in mind that in the case of the order statistics in mass, the relative differences are on the same level as the inaccuracies in cluster mass estimates. This problem does not occur for redshifts, which can be measured to a very high accuracy. Of course, in this case the observational challenge is transferred to compiling a sample with a precise mass limit. Apart from the cosmological parameters also the choice of the mass function is expected to have a strong effect on the order statistics as will be discussed in the following subsection.

#### 3.2 Impact of the choice of the mass function

When performing a falsification experiment of \( \Lambda \)CDM using the \( n \) most massive or \( n \) highest redshift clusters, then one has to specify the reference model against which the observations have to be compared with. Apart from the cosmological parameters that are usually fixed to the obvious choice of the WMAP7 values, a halo mass function has to be chosen as well. As mentioned earlier, this
is particularly important for galaxy clusters since the exponentially suppressed tail of the mass function is naturally very sensitive to modifications.

In order to quantify the impact of different mass functions on the order statistics in mass and redshift, we computed the cdfs, $F_{(n)}$, for the Press & Schechter (1974) (PS), the Tinker et al. (2008) and the Sheth & Tormen (1999) (ST) mass functions for $f_{\text{sky}} = 1$ and present them from top to bottom in Fig. 3. Comparing the panels to each other reveals the tremendous sensitivity of the distributions to the choice of the mass function. Taking the Tinker mass function as a reference, the median, Q50, changes for both types of order statistics by $-20$ per cent for the PS case and by $+15$ per cent for the ST case. These differences can be explained by the fact that the ST mass function leads to a substantial increase in the number of haloes, particularly at the high-mass end, whereas the PS mass function results in much fewer haloes in the mass and redshift range of interest. For the remainder of this paper, we will use the Tinker mass function as reference because the halo masses are defined as spherical overdensities with respect to the mean background density, a definition that is closer to theory and actual observations than friend-of-friend masses. We also implicitly assume the universality of the mass function, which adds a further source of uncertainty to the analysis since Tinker et al. (2008) found evidence for non-universality.

However, considering that due to statistical limitations, current fits for the mass function are still not very accurate for the highest masses ($>3 \times 10^{15} M_\odot$) and that systematic uncertainties allow even smaller masses an accuracy of a few percent at most (Bhattacharya et al. 2011), one has to be very cautious with falsification experiments that are based on extreme objects. The uncertainty in the mass function alone will allow a rather wide range of distributions.

### 4 Suitable Samples of Galaxy Clusters for an Order Statistical Analysis

Having introduced the order statistics of the most massive or the highest redshift clusters, we intend now to compare observed clusters with the theoretical distributions. To do so, it is necessary to select suitable samples of galaxy clusters, which we will discuss in the following.

#### 4.1 General considerations

The selection of a suitable sample of galaxy clusters for an order statistical analysis is by no means a trivial task. The necessary ordering of the quantities’ mass and redshift by magnitude requires
that they have been derived in an identical way across the sample. Otherwise, systematics and biases, like the differences between lensing and X-ray mass estimates for instance (see e.g. Mahdavi et al. 2008; Meneghetti et al. 2010; Zhang et al. 2010; Rasia et al. 2012; Planck Collaboration et al. 2013), will render the ordering meaningless. Despite an increasing amount of data from different surveys, a lack of large homogeneous samples persists. Thus, we decided to base our comparison on clusters that stem from catalogues like the SPT massive cluster sample (Williamson et al. 2011) and the metacatalogue of X-ray detected clusters of galaxies (MCXC) cluster catalogue (Piffaretti et al. 2011), which will be discussed in further detail below.

### 4.2 The SPT massive cluster sample

The SPT survey (Carlstrom et al. 2011) is ideally suited for the intended purpose of an order statistical analysis. Being based on the Sunyaev-Zeldovich (SZ) effect (Sunyaev & Zeldovich 1972, 1980) the SPT survey is able to detect massive galaxy clusters up to high redshifts. The fact that the limiting mass of SZ surveys varies weakly with redshift (Carlstrom, Holder & Reese 2002) allows in principle to construct mass limited cluster catalogues. However, it should be emphasized that the assumption of a $m_{\text{lim}}$ independent of redshift depends critically on the sensitivity and the beam width of an actual survey.
Table 1. Compilation of the 10 most massive galaxy clusters from the SPT massive cluster sample (Williamson et al. 2011) and the MCXC catalogue (Piffaretti et al. 2011), respectively. The masses $M_{200m}$ and $M_{500c}^{\text{Edd}}$ are with respect to the mean background density before and after the correction for the Eddington bias based on the estimated mass uncertainty $\sigma_{\ln M}$. The last column lists the references for the values of the observed mass, on which the analysis is based on.

| Rank | Cluster | $z$ | $M_{200m}$ in units of $M_\odot$ | $\sigma_{\ln M}$ | $M_{500c}^{\text{Edd}}$ in units of $M_\odot$ | Reference |
|------|---------|----|------------------------------|----------------|---------------------------------|-----------|
| 1st  | SPT-CL J0658--5556 | 0.296 | (3.12 ± 1.15) × 10^{15} | 0.15 | 3.33 ± 0.50 × 10^{15} | Williamson et al. (2011) |
| 2nd  | SPT-CL J2248--4431 | 0.348 | (2.90 ± 1.03) × 10^{15} | 0.15 | 2.82 ± 0.42 × 10^{15} | Williamson et al. (2011) |
| 3rd  | SPT-CL J0102--4915 | 0.870 | (2.16 ± 0.32) × 10^{15} | 0.15 | 2.67 ± 0.38 × 10^{15} | Williamson et al. (2011) |
| 4th  | SPT-CL J0549--6204 | 0.320 | (1.99 ± 0.67) × 10^{15} | 0.15 | 2.62 ± 0.34 × 10^{15} | Williamson et al. (2011) |
| 5th  | SPT-CL J0638--5358 | 0.222 | (1.91 ± 0.62) × 10^{15} | 0.15 | 2.15 ± 0.28 × 10^{15} | Williamson et al. (2011) |
| 6th  | SPT-CL J0232--4421 | 0.284 | (1.88 ± 0.59) × 10^{15} | 0.15 | 2.50 ± 0.51 × 10^{15} | Williamson et al. (2011) |
| 7th  | SPT-CL J0645--5413 | 0.167 | (1.81 ± 0.60) × 10^{15} | 0.15 | 2.15 ± 0.28 × 10^{15} | Williamson et al. (2011) |
| 8th  | SPT-CL J0245--5302 | 0.098 | (1.70 ± 0.46) × 10^{15} | 0.15 | 2.15 ± 0.28 × 10^{15} | Williamson et al. (2011) |
| 9th  | SPT-CL J2201--5956 | 0.300 | (1.70 ± 0.42) × 10^{15} | 0.15 | 2.15 ± 0.28 × 10^{15} | Williamson et al. (2011) |
| 10th | SPT-CL J2344--4243 | 0.450 | (1.65 ± 0.38) × 10^{15} | 0.15 | 2.15 ± 0.28 × 10^{15} | Williamson et al. (2011) |

For this work, we take the compilation of Williamson et al. (2011) which comprises the 26 most significant detections in the full survey area of $A_s^\text{SPT} = 2500$ deg$^2$. Ensuring a constant mass limit of $M_{200}$ ≈ 10^{15} $M_\odot$, clusters were selected on the basis of a signal-to-noise ratio threshold in the filtered SPT maps. For all 26 catalogue members, either photometric or spectroscopic redshifts were determined as well. The cluster masses given in the catalogue are defined with respect to the mean cosmic background density and need no further conversion to match the mass definition of the Tinker et al. (2008) mass function. To each cluster of the sample we assign the error bars that we obtained by adding the reported statistical and systematic errors in quadrature.

4.3 The MCXC cluster catalogue

The MCXC catalogue (Piffaretti et al. 2011) is based on the publicly available compilation of clusters’ detections from RASS (NORAS, REFLEX, BCS, GSP, NEP, MACS and CIZA) and other serendipitous surveys (160SD, 400SD, SHARC, WARPS and EMSS), and provides the physical properties of 1743 galaxy clusters systematically homogenized to an overdensity of 500 (with respect to the cosmic critical density). This metacatalogue is not complete in any sense, but it is constituted by X-ray flux-limited samples that ensure that the X-ray brightest objects in the nearby ($z \lesssim 0.3$) Universe, and therefore the most massive X-ray detected clusters, are all included.

We have then simply ranked the objects accordingly to their estimated $M_{200m}$, that is obtained from the tabulated $M_{500c}$ as

$$M_{200m} = M_{500c} \left( \frac{R_{200m}}{R_{500c}} \right)^3,$$

(13)

where $\Omega = \Omega_s(1 + z)^3 / E^2$, $E = (\Omega_s(1 + z)^3 + \Omega_L)^{1/2}$, and the ratio between the radii at different overdensities has been obtained by assuming an NFW profile (Navarro, Frenk & White 1996) with $c_{200} = 4$.

4.4 Preparations of the ordered samples

We order the SPT and MCXC catalogues by magnitude of the observed mass and present the 10 most massive systems in Table 1. For statistical comparisons the observed masses have to be corrected for the Eddington bias (Eddington 1913) in mass. As a result of the exponentially suppressed tail of the mass function and the substantial uncertainties in the mass determination of galaxy clusters, it is more likely that lower mass systems scatter up while higher mass systems scatter down, resulting in a systematic shift. Thus, before an observed mass can be compared to a theoretical distribution, this shift has to be corrected for. In the case of the SPT cluster sample, the reported masses in Williamson et al. (2011) have already been corrected for the Eddington bias. To apply the correction to the MCXC sample, we follow Mortonson et al. (2011) and shift the observed masses, $M_{\text{obs}}$, to the corrected masses, $M_{\text{corr}}$, according to

$$\ln M_{\text{corr}} = \ln M_{\text{obs}} + \sigma_{\ln M}^2,$$

(14)

where $\epsilon$ is the local slope of the mass function $(dn/d\ln M \propto M^\epsilon)$ and $\sigma_{\ln M}$ is the uncertainty in the mass measurement. It should be noted that we applied the full boost, $\sigma_{\ln M}$, to the MCXC catalogue to be consistent with the correction applied to the SPT sample. For the corrections, we used the values of $\sigma_{\ln M}$ listed in the fifth column of Table 1 which we deduced from the reported uncertainties in the nominal masses. The larger the observational errors are, the larger is the correction towards lower masses.
Table 2. Compilation of the 10 highest redshift clusters from the SPT massive cluster sample (Williamson et al. 2011). Here, (s) and (p) denote the spectroscopic and photometric redshifts, respectively.

| Rank | Cluster        | $\bar{z}$ | $M_{200m}$ in units of $M_\odot$ |
|------|----------------|----------|----------------------------------|
| 1st  | SPT-CL J2106−5844 | 1.132 (s) | (1.27 ± 0.21) $\times 10^{15}$ |
| 2nd  | SPT-CL J0615−5746 | 0.972 (s) | (1.32 ± 0.40) $\times 10^{15}$  |
| 3rd  | SPT-CL J0102−4915 | 0.870 (s) | (2.16 ± 0.32) $\times 10^{15}$  |
| 4th  | SPT-CL J2337−5942 | 0.775 (s) | (1.99 ± 0.20) $\times 10^{15}$  |
| 5th  | SPT-CL J2344−4243 | 0.620 (p) | (1.91 ± 0.50) $\times 10^{15}$  |
| 6th  | SPT-CL J0417−4748 | 0.620 (p) | (1.88 ± 0.20) $\times 10^{15}$  |
| 7th  | SPT-CL J0243−4833 | 0.530 (p) | (1.81 ± 0.23) $\times 10^{15}$  |
| 8th  | SPT-CL J0304−4401 | 0.520 (p) | (1.70 ± 0.33) $\times 10^{15}$  |
| 9th  | SPT-CL J0438−5419 | 0.450 (p) | (1.70 ± 0.38) $\times 10^{15}$  |
| 10th | SPT-CL J0254−5856 | 0.438 (s) | (1.65 ± 0.25) $\times 10^{15}$  |

As an exemplary exception from the SPT catalogue, we used for the mass of SPT-CL J0102−4915 the value reported by Menanteau et al. (2012), which is based on a combined SZ+X-rays+optical+infrared analysis. The multwavelength study shifts $M_{\text{obs}} = (1.89 ± 0.45) \times 10^{15} M_\odot$ (Williamson et al. 2011) to a larger value of $M_{\text{obs}} = (2.16 ± 0.32) \times 10^{15} M_\odot$, changing the rank from the fifth to the third most massive. This shows that with the expected increase in the quality of cluster mass estimates, the ordering of the most massive cluster will undergo significant changes. We expect that the reshuffling will affect more strongly the most massive clusters due to the fact that the large error bars will cause lower ranked clusters to scatter up. We will discuss the impact of the reshuffling in more detail in Section 5.1.

In addition, we sorted the SPT catalogue by redshift and list the 10 highest redshift clusters above $m_{\text{lim}} \approx 10^{15} M_\odot$ in Table 2.

5 Comparison of the Individual Order Statistics with Observations

In this section, we will compare the individual ranked systems listed in Table 1 for the mass and in Table 2 for the redshift with the individual distributions for each rank, as e.g. shown in Fig. 3

5.1 Order statistics in cluster mass

In order to demonstrate the impact of the survey area on the distributions of the order statistics in mass, we show in Fig. 4 the dependence of different quantiles ($Q_2$, $Q_25$, $Q_50$, $Q_75$ and $Q_98$) on the survey area for the nine most massive clusters. In addition, the green error bars show the clusters from the SPT and MCXC catalogues listed in Table 1 for the respective survey areas of $A_{\text{SPT}} = 2500$ deg$^2$ and $A_{\text{MCXC}} = 27490$ deg$^2$.

From the individual panels in Fig. 4 it can be inferred that, as expected, a larger survey area yields a larger expected mass for the individual rank. Furthermore, with increasing rank towards higher orders, the interquartile range, like ($Q_2$–$Q_98$), narrows. A behaviour that can also be seen in Fig. 1 as steepening of the cdf with increasing rank. Therefore, the largest mass (first order) is expected to be realized in a much wider mass range than the higher orders.

We will now compare the observations in more detail with the theoretical expectations in the form of box and whisker diagrams as shown in Fig. 5. Here, the blue-bordered, grey filled box denotes the interquartile range (IQR) which is bounded by the 25 and 75 percentiles ($Q_{25}$, $Q_{75}$) and the median ($Q_{50}$) is depicted as a red line. The black whiskers denote the 2 and 98 percentiles ($Q_2$, $Q_{98}$) and we follow the convention that observations that fall outside are considered as outliers. As before, the nominal observed cluster masses are denoted as green error bars where for the left-hand column the SPT catalogue and for the right-hand column the MCXC catalogue was used. For the SPT sample the nominal masses have already been corrected for the Eddington bias. In addition, we plot for the MCXC case the Eddington bias corrected masses, $M_{\text{Edd}}$, from the sixth column of Table 1 as orange triangles with dashed error bars. We performed the analysis for three different mass functions, comprising from the top to the bottom panel, the PS, the Tinker and the ST mass functions. In addition to the Eddington bias in mass, we expect a shift to larger masses caused by the reshuffling of orders due to the uncertainties in mass. In order to quantify this effect, we use Monte Carlo (MC) simulated 10 000 realizations of the 26 SPT and 123 MCXC (with $M > 10^{15} M_\odot$) cluster masses after their correction for the Eddington bias and order them by mass. The masses were randomly drawn from the individual error interval, assuming Gaussian distributions. For the SPT case, we used only the statistical errors since they can be considered independent for each cluster. The systematic errors, often larger than the statistical ones, might move all clusters masses in the same direction. We present the results of the reshuffling as the violet, empty circles with dash-dotted 1σ error bars in Fig. 5. It can be seen that the highest ranks are more strongly affected by the reshuffling than the lower ones and that they are on average shifted to larger values. Of course, the amount of this effect will depend on the size of the error bars. Further, the reshuffling yields mass values that fall between the nominal (green error bars) and the Eddington bias corrected ones (orange error bars).

For the SPT catalogue, it can be seen from the top-left panel of Fig. 5 that the outdated PS mass function seems to be disfavoured by the resuffled and the nominal masses of the 10 largest objects. However, the error bars are large and do not allow an exclusion of the PS mass function. For the Tinker and the ST mass function, the boxes indicating the theoretical distributions move to larger mass values and therefore they match the observed masses better than the outdated PS mass function. In particular, the third ranked system SPT-CL J0102 with its smaller errors and, hence, giving the tightest constraints, is consistent with $\Lambda$CDM for both mass functions. All other ranks are consistent as well due to their large error bars, consolidating the conclusion that the most massive clusters of the SPT sample are in agreement with the statistical expectations. The conclusions for the MCXC catalogue are identical; however, the jump between the fourth and the fifth largest order yield to an inconsistency of the observed higher orders with the expectations based on the ST mass function. This jump is clearly caused by the incompleteness of the MCXC catalogue and, thus, the inclusion of the missing clusters would most certainly move the observed sample to higher masses in the direction of the results we obtained from the SPT sample. In this sense, we do not see any indication of a substantial difference between the small- and wide-field survey.

The analysis of the SPT sample illustrates the potential of utilizing the most massive galaxy clusters to test underlying assumptions, e.g. the mass function. For instance, a multwavelength study of the 26 SPT clusters would reduce the error bars to the level of SPT-CL J0102 (the nominal third ranked cluster in the left-hand column of Fig. 5), which would significantly tighten the constraints on the underlying assumptions, e.g. the halo mass function. In turn, by assuming the $\Lambda$CDM reference cosmology, the comparison of the observed masses with the individual order
Figure 4. Functional box plots for the first nine orders in the observable mass as indicated in the individual panels. Here, the red line denotes the median ($Q_{50}$), the blue-bordered, grey region denotes the IQR and the black lines denote the 2 and the 98 percentile ($Q_2$, $Q_{98}$). The green error bars show the corresponding observed masses, $M_{200m}$, from the SPT (green circles) and the MCXC (green triangles) catalogues (see Table 1) for their respective survey areas of $A_{SPT} = 2500 \text{ deg}^2$ and $A_{MCXC} = 27490 \text{ deg}^2$.

In the upper panel of Fig. 6, we present the dependence of different percentiles ($Q_2$, $Q_{25}$, $Q_{50}$, $Q_{75}$ and $Q_{98}$) on the order for a survey area of $A_s = 20000 \text{ deg}^2$. Choosing the $Q_{98}$ percentile as exclusion criterion, one would need roughly to find ten clusters with $m \gtrsim 2.5 \times 10^{15} \text{ M}_\odot$, three clusters with $m \gtrsim 3.2 \times 10^{15} \text{ M}_\odot$ or one cluster with $m \gtrsim 5 \times 10^{15} \text{ M}_\odot$ in order to report a significant deviation from the $\Lambda$CDM expectations. Of course, the observed masses might have to be corrected for the Eddington bias in mass and a possible reshuffling as previously demonstrated. In general, exclusion criteria based on order statistics extend previous works (Mortonson et al. 2011; Waizmann et al. 2012a) from statements about single objects to statements about object samples, which considerably enhances the quantity of information available to be analysed.

5.2 Order statistics in cluster redshift

We performed an identical analysis for the individual order statistics for the SPT massive cluster catalogue ranked by redshift listed in Table 2. For the theoretical distributions, we assume a limiting mass of $m_{\text{lim}} = 10^{15} \text{ M}_\odot$ and a survey area of $A_{SPT} = 2500 \text{ deg}^2$. As before, we present in Fig. 7 the dependence of the order statistical distributions on the survey area for the first nine orders. Again, an increase in the survey area yields a shift of the theoretical distributions to higher redshifts and, as shown in the right-hand panel of...
Figure 5. Box and whisker diagram of the 10 most massive clusters from the SPT survey (left-hand column) and the MCXC catalogue (right-hand column) for three different choices of the mass function as denoted in the title of each panel. For each order, the red lines denote the median ($Q_{50}$), the blue-bordered, grey boxes give the IQR and the black whiskers mark the range between the 2 and 98 percentile ($Q_2$, $Q_{98}$) of the theoretical distribution. The green, filled circles denote the nominal observed cluster masses (already corrected for the Eddington bias in the SPT case), $M_{200m}$, the orange, empty triangles are the ones that are corrected for the Eddington bias in mass in the MCXC case and the violet, empty circles are the results of the MC reshuffling of the ranks. All error bars denote the $1\sigma$ range.

In Fig. 8, we present the box and whisker diagram in redshift, again for the PS, the Tinker and the ST mass functions (from top to bottom). The definition of boxes and whiskers remains unchanged with respect to Fig. 5. Again, the data from Table 2 are denoted by the green error bars, which are negligibly small in the case of spectroscopic redshifts. Thus, we abstained from the MC simulation of the reshuffling in the case of redshift. While for the order statistics in mass the results depended only on the choice of the survey area, the situation is different for the order statistics in redshift. Here, a constant survey limiting mass is assumed, which will be subject to uncertainties for a real survey and, furthermore, will also exhibit some redshift dependence. Thus, the theoretical distributions are
extrinsically less accurate than the ones with respect to cluster mass. Indeed, the comparison with the data in Fig. 8 exhibits a different behaviour with respect to the one in Fig. 5. Here, first four orders seem to be fitted better by the Tinker mass function while the higher orders seem to favour the PS mass function. Taking the Tinker mass function as reference it seems that a few systems with $M > 10^{15} \, M_\odot$ are missing at redshifts $z \gtrsim 0.7$. The difference with respect to the findings for the order statistics in mass for the same sample could, along the lines of Section 3, be interpreted as a signature of a deviation from the reference ΛCDM model. However, considering the previously mentioned simplifying assumptions in the modelling of the theoretical distributions, we do not infer any cosmological conclusions and leave a better, more realistic, modelling of $m_{\text{lim}}(z)$ of the SPT survey to a future work.

In the lower panel of Fig. 6, we present the dependence of different percentiles (Q2, Q25, Q50, Q75, and Q98) on the order for a survey area of $A_\text{s} = 20\,000 \text{ deg}^2$ and a constant limiting mass of $m_{\text{lim}} = 10^{15} \, M_\odot$. Taking the Q98 percentile as exclusion criterion, one would need to find ten clusters with $z \gtrsim 1$, three clusters with $z \gtrsim 1.2$ or one cluster with $z \gtrsim 1.55$ in order to report a significant deviation from the ΛCDM expectations. Currently, SPT-CL J2106 is the only known cluster of such a high mass having a redshift $z > 1$. With an assigned survey area of $A_\text{s} = 2800 \text{ deg}^2$ (ACT+SPT), it might from a statistical point of view still be possible to find 10 objects that massive at $z > 1$ in the larger survey area. The method presented in this work allows us to construct similar exclusion criteria for any kind of survey design.

6 COMPARISON OF THE JOINT ORDER STATISTICS WITH OBSERVATIONS

Having studied the individual order statistics in mass and redshift in the previous section, we turn now to the study of the joint distributions of the order statistics as introduced in Section 2.1.

The simplest case of a joint order distribution is two dimensional. In this case, the pdf and cdf are given by equations (6) and (7), respectively. Starting with the joint pdf, we present in Fig. 9 the joint distributions in mass (left-hand panel) and redshift (right-hand panel) for several order combinations as denoted in the individual panels. All calculations assume the full sky and the Tinker mass function. In the case of the joint distributions in redshift, we assume a constant limiting survey mass of $m_{\text{lim}} = 10^{15} \, M_\odot$. Due to the condition that $x < y$, all distributions are limited to a triangular domain.

An inspection of the different pdfs in Fig. 9 reveals that, for the combination of the first and the second largest order (upper leftmost panel), the most likely combination of the observables is very close to the diagonal. This means that it is more likely to find the two largest values close to each other at absolute values that are smaller than the extreme value statistics, which would imply for the maximum alone. Then, when moving to combinations of the first with higher orders (first row), it can be seen that the peaks of the pdfs move away from the diagonal and that they extend to larger values for the larger observable. This indicates that it is more likely to find the two systems with a larger separation in the observable when the difference between the considered orders is larger. Accordingly, for higher order combinations (lower rows), the peaks of the joint pdf move to smaller values of the observables. It should also be noted that the peaks steepen for higher order combinations, confining the pdfs to smaller and smaller regions in the observable plane. As an example, the first and second largest observations (upper leftmost panel) can be realized in much larger area than the sixth and eighth largest one (lower rightmost panel).

Apart from the joint pdfs, it is also instructive to study the joint cdfs as presented in Fig. 10 for the observed mass (left-hand panel) and redshift (right-hand panel). In order to add observational data from the SPT catalogue, we assume a survey area of $A^{\text{SPT}} = 2500 \text{ deg}^2$ and a $m_{\text{lim}} = 10^{15} \, M_\odot$ for the joint distribution in redshift. Additionally, we added the two largest nominal observed, Eddington bias corrected masses (red error bars) from Table 2 to the left-hand panel and the two highest redshifts of the SPT massive cluster sample from Table 2 to the right-hand panel. In the case of the mass, we find $F_{(n-1)(0)} \approx 0.92$ for the nominal, Eddington bias corrected masses. Hence, taking the error bars into account, in $\sim (2-70)$ per cent of the cases a mass larger than the one of SPT-CL J0658 and a mass larger than the one of SPT-CL J2248 are observed. Thus, also the joint cdf confirms that the two largest masses do not exhibit any tension with the concordance cosmology. The same conclusion applies in the case of the joint distribution in redshift.

By means of equation (8) these results can be extended to the n-dimensional case, allowing the formulation of a likelihood function of the ordered sample of the n most massive or highest redshift clusters.

7 SUMMARY

In this work, we studied the application of order statistics to the mass and redshifts of galaxy clusters and compared the theoretically derived distributions with observed samples of galaxy clusters. Our work extends previous studies that hitherto considered only the extreme value distributions in mass or redshift.

On the theoretical side, our results can be summarized as follows.

(i) We introduce all relations necessary to calculate pdfs and cdfs of the individual and joint order statistics in mass and redshift. In particular, we find a steepening of the cdfs for higher order distributions with respect to the extreme value distribution of both mass and redshift. This steepening corresponds to a higher constraining power from distributions of the n largest observations.
presented method extends previous works to include exclusion criteria based on the $n$ most massive or $n$ highest redshift clusters for a given survey set-up.

(ii) Conceptually, we avoid the bias due to an a posteriori choice of the redshift interval in the case of the order statistics in mass by selecting the interval $0 \leq z \leq \infty$. Hence, we study the statistics of the hierarchy of the most massive haloes in the Universe, which mostly stem from redshifts $z \lesssim 0.5$. On the contrary, when choosing the order statistics in redshift, focus is laid on haloes that stem from the highest possible redshifts. However, the calculations will require a model of the survey characteristics in the form of a limiting survey mass as a function of redshift.

(iii) By putting the emphasis on either the most massive or on the highest redshift clusters above a given mass limit, the order statistics is, e.g. particularly sensitive to the choice of the mass function. While the order statistics in mass is very sensitive to $\sigma_8$ and $\Omega_m$ due to the domination of low-redshift objects, the order statistics in redshift proves to be very sensitive to $w_0$ and $w_a$. For a fixed cosmology, both order statistics are efficient probes of the functional shape of the mass function at the high-mass end.

(iv) In addition to the individual order statistics, we study as example case also the joint two order statistics. We find that for the combination of the largest and the second largest observation, it is most likely to find them to be realized with very similar values with a relatively broadly peaked distribution.

(v) In order to allow a quick estimation of the distributions of the order statistics, we provide in Appendix B fitting formulae for
0.7. One explanation is that we ranked by the magnitude of the observables mass and redshift. We decided to compile two catalogues, the main one is based on the SPT massive cluster sample (Williamson et al. 2011) and additionally we analysed the MCXC catalogue (Piffaretti et al. 2011) based on publicly available flux-limited all-sky survey and serendipitous cluster catalogues. This metacatalogue can be considered as complete for $z \lesssim 0.3$ and, hence, by no means as complete as the SPT one. The results of the comparison can be summarized as follows.

(i) In the case of the order statistics in mass, we compared the theoretical expectations for the 10 largest masses for the PS, the Tinker and the ST mass functions. Assuming WMAP7 parameters, we find that the nominal and the Eddington bias corrected values for the observed masses favour the Tinker and the ST mass functions. When considering the possible bias due to a reshuffling of the ranks caused by the large error bars (taking only the statistical errors into consideration), we find that the SPT sample is consistent with the ST and Tinker mass functions. Though the ST mass function seems to provide a better match, the results are still inconclusive in view of the presence of systematic errors that might result in a common shift in mass to all orders. The constraints are expected to tighten considerably once the error bars of all objects are scaled down by combining several cluster observables in multiwavelength studies.

(ii) In contrast to the ranking in mass, the order statistics of the SPT clusters in redshift is less well fit by the theoretical distributions based on the Tinker mass function. It appears that a few systems with $M > 10^{15} M_\odot$ are missing at redshifts $z \gtrsim 0.7$. One explanation could be found in a non-standard cosmological evolution to which the order statistics in redshift is more sensitive. However, it is more likely that a more precise modelling (including the redshift dependence) of the true limiting survey mass of SPT will account for the observed deviations.

(iii) Instead of utilizing order statistics to perform exclusion experiments, it can also be used for consistency checks of the completeness of the observed sample and of the modelling of the survey selection function as indicated by the analysis of the MCXC (mass) and the SPT (redshift) samples.

8 CONCLUSIONS

We introduced a powerful theoretical framework which allows us to calculate the expected individual and joint distribution functions of the $n$ largest masses or the $n$ highest redshifts of galaxy clusters in a given survey area. This approach is more powerful than the extreme value statistics that focuses on the statistics of the single largest observation alone.

As a proof of concept, we compared the theoretical distributions with observed samples of galaxy clusters. However, data of sufficient quantity, uniformity and completeness is still sparse such that constraints are not particularly tight. This situation will most certainly improve in the near and intermediate future. Since the emphasis of this work lies on the introduction of the theoretical framework of order statistics and its application to galaxy clusters, we contended ourselves with a study of cluster masses and redshifts. Unfortunately, the mass of a galaxy cluster is not a direct observable and subject to large scatter and observational biases. In a follow-up work, we intend to extend the formalism to direct observables, like for instance X-ray luminosities, and to include the scatter in the scaling relations into the theoretical distributions.
Order statistics of galaxy clusters

Figure 9. Joint pdf \( f_{\text{Joint}}(x, y) \) (see equation 6) for the observable mass (left-hand panel) and redshift (right-hand panel) for different combinations of rank as indicated in the upper-left of each small panel. The distributions are calculated for the full sky and a constant limiting survey mass of \( m_{\text{lim}} = 10^{15} \, M_\odot \) has been assumed for the joint distribution in redshift. The colour bar is set to range from 0 to the maximum of the joint pdf for each rank combination.

Figure 10. Joint cdf \( F_{\text{Joint}}(x, y) \) (see equation 7) for the observable mass (left-hand panel) and redshift (right-hand panel) for the combination of the largest and second largest observation, assuming a survey area of \( A_{\text{SPT}} = 2500 \, \text{deg}^2 \) and a constant limiting survey mass of \( m_{\text{lim}} = 10^{15} \, M_\odot \) for the joint distribution in redshift. The red error bars denote the nominal, Eddington bias corrected values, \( M_{200m} \), for the SPT catalogue as listed in Table 1. In the case of the redshifts from Table 2, the error bars are too small to be printed.

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APPENDIX A: ORDER STATISTICS

In this appendix, we outline the derivation of the most important relations of the order statistics and some subtleties considering their implementation. For more details, we refer to the excellent textbooks on the topic by Arnold et al. (1992) and by David & Nagaraja (2003) which we closely follow for the remainder of this appendix.

A1 Individual distributions

Let $X_1, X_2, \ldots, X_n$ be a random sample of a continuous population with the cdf, $F(x)$. Further, let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the order statistics, the random variates ordered by magnitude, where $X_{(1)}$ is the smallest (minimum) and $X_{(n)}$ the largest (maximum) variate. The event $x < X_{(i)} \leq x + \delta x$ is the same as the one depicted in panel (a) of Fig. A1 and, thus, we have $X_i \leq x$ for $i = 1$ of the $X_i$,

![Figure A1. Schematic for the derivation of $f_{(i)}(x)$ and $f_{(2)}(x, y)$.](https://academic.oup.com/mnras/article-abstract/432/2/914/1020807)

exactly one $X_i$ in $x < X_i \leq x + \delta x$ and the remaining $n - i$ of the $X_i$ in $X_i > x + \delta x$. Now, the number of ways how $n$ observations can be arranged in the three regimes is given by

$$A(n, i) = \frac{n!}{(i - 1)! (n - i)!},$$  \hspace{1cm} (A1)

where each of them has a probability

$$[F(x)]^{i - 1} [F(x + \delta x) - F(x)] [1 - F(x)]^{n - i - 1}. \hspace{1cm} (A2)$$

Therefore, under the assumption that $\delta x$ is small, we find for the probability

$$P_r [x < X_{(i)} \leq x + \delta x] = A(n, i) [F(x)]^{i - 1} \times [1 - F(x)]^{n - i} f(x) \delta x, \hspace{1cm} (A3)$$

neglecting terms of $O(\delta x)^2$. Dividing by $\delta x$ and performing $\delta x \to 0$ yields the pdf as given in equation (1)

$$f_{(i)}(x) = \lim_{\delta x \to 0} \frac{P_r [x < X_{(i)} \leq x + \delta x]}{\delta x} = A(n, i) [F(x)]^{i - 1} [1 - F(x)]^{n - i} f(x). \hspace{1cm} (A4)$$

The corresponding cdf of the $i$th order, as given by equation (2) in Section 2, can now either be obtained by integrating the above equation or by the following argument:

$$F_{(i)}(x) = P_r [X_{(i)} \leq x] = P_r \{\text{at least } i \text{ of } X_{(i)}, \ldots, X_{(n)} \text{ are at most } x\} = \sum_{k=i}^{n} \binom{n}{k} [F(x)]^k [1 - F(x)]^{n-k}, \hspace{1cm} (A5)$$

for $-\infty < x < \infty$. Hence, the cdf of $X_{(i)}$ is equivalent to the tail probability (starting from $i$) of a binomial distribution with $n$ trials and a success probability of $F(x)$. By setting $i = n$ or $i = 1$ one obtains the cdfs for the smallest and the largest order statistics as given by equations (3) and (4).

A2 Joint distributions

The joint pdf of the two order statistics $X_{(r)}, X_{(s)}$ ($1 \leq r \leq s \leq n$) for $x < y$ can be derived by similar arguments as for the single order statistics. The derivation scheme is now extended according
to panel (b) of Fig. A1. Analogously to equation (A4) we obtain
\[
    f_{\nu\,(r)}(x, y) = \lim_{\delta x \to 0, \delta y \to 0} \left\{ P_r(x < X_{(r)} \leq x + \delta x, y < X_{(s)} \leq y + \delta y) \right\}
\]
\[
    = A(n, r, s) \times \sum_{i=0}^{n} \sum_{j=0}^{r} \frac{n!}{i!(j-i)!(n-i)!} \times [F(x)^i \{1 - F(y)^i\}^{r-i} f(x)f(y).
\]
where
\[
    A(n, r, s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.
\]

The joint cdf can be obtained by integrating the pdf from above or again by the following direct argument
\[
    F_{\nu\,(r)}(x, y) = P_r[X_{(r)} \leq x, X_{(s)} \leq y] = P_r[\text{at least } r X_{(r)} \leq x \wedge \text{at least } s X_{(s)} \leq y]
\]
\[
    = \sum_{j=0}^{n} \sum_{i=0}^{r} \frac{n!}{i!(j-i)!(n-i)!} \times [F(x)^i \{1 - F(y)^i\}^{r-i} f(x)f(y).
\]
This is exactly identical to the tail probability of a bivariate binomial distribution.

Following the same line of reasoning as for the joint two order statistics, the above relations can be generalized to the joint pdf of \(X_{n_1}, \ldots, X_{n_k}\) (1 \(\leq n_1 < \cdots < n_k \leq n\)) for \(x_1 \leq \cdots \leq x_k\), which is given by
\[
    f_{\nu_{(x_1),\ldots,x_k}}(x_1, \ldots, x_k) = \frac{n!}{(n_1 - 1)!(n_2 - n_1 - 1)! \cdots (n_k - n_{k-1})!}
\]
\[
    \times [F(x_1)^{n_1-1} f(x_1)] \times [F(x_2) - F(x_1)]^{n_2-n_1-1} \times [F(x_3) - F(x_2)]^{n_3-n_2-1} \cdots [1 - F(x_k)]^{n_k-n_{k-1}} f(x_k).
\]

The right-hand side of this relation can be written in a more compact form (David & Nagaraja 2003) as
\[
    n! \left\{ \prod_{j=1}^{k} f(x_j) \right\} \left\{ \prod_{j=0}^{k} \frac{[F(x_{j+1}) - F(x_j)]^{n_{j+1} - n_j - 1}}{(n_{j+1} - n_j - 1)!} \right\},
\]
where we defined \(n_0 = 0, n_{k+1} = n + 1\).\(^{n_1}\)\(x_0 = -\infty\) and \(x_{k+1} = +\infty\).

\[1\] For the numerical calculation of \(n\), we limit the mass range without loss of generality to the interval relevant for galaxy clusters of \(10^{13} \, M_\odot \leq m \leq 10^{16} \, M_\odot\).
Relative differences, $\Delta = (Q_{\text{fit}} - Q_{\text{dir}})/Q_{\text{dir}}$, between the fitted and directly calculated percentiles (different line styles) as a function of the survey area for the order statistics in mass. The differences are shown for three different ranks, the largest (black lines), the fifth largest (blue lines) and the tenth largest (green lines) one.

B2 Order statistics in redshift

For fitting the order statistics in redshift, we proceed in a similar way as for the mass, setting $x = z$ in equation (B1). For calculating the GEV parameters as a function of the survey area, we follow the approach presented in Metcalf & Waizmann (in preparation). However, since in contrast to the order statistics in mass, the distributions depend on the choice of the limiting survey mass, we fitted the distributions for two choices of $m_{\text{lim}}$. First, we set $m_{\text{lim}} = 10^{15} M_\odot$, identical to the setup we discussed in this paper for the SPT massive cluster sample. Secondly, we lower the threshold to $m_{\text{lim}} = 5 \times 10^{14} M_\odot$. In the first case, we obtain

$$\alpha(y) = 1.13729 \ln(0.567735y + 0.332933),$$

$$\beta(y) = \exp[- \exp(-1.76728y^{-1.84932} + 0.929307)],$$

$$\gamma(y) = -2.23597 \ln(y^{-2.90376} + 1.01017),$$

$$n(y) = 10^{0.981095y - 1.52015},$$

and for the second choice we find

$$\alpha(y) = 2.1084\ln(0.284062y + 1.09002),$$

$$\beta(y) = \exp[- \exp(-0.905364y^{-0.375228} + 1.30066)],$$

$$\gamma(y) = -0.260275 \ln(y^{-1.55487} + 1.0592),$$

$$n(y) = 10^{0.998552y - 0.451364},$$

where $y = \log_{10}(A_s)$ for both cases. We present the results in Fig. B1 again as relative differences. It can be seen that in the case of high limiting mass (upper panel), the fit performs poorly for survey areas smaller than $\sim 1000 \text{deg}^2$ due to the insufficient number of haloes that are expected to be found. However, above $\sim 2000 \text{deg}^2$ the percentiles of the first 10 orders can be fitted with an accuracy better than 2 per cent.

If the limiting mass is lowered, the quality of the fit improves drastically as shown in the lower panel of Fig. B1 for $m_{\text{lim}} = 5 \times 10^{14} M_\odot$. In this case, sub-per cent-level accuracy is reached for $A_s \geq 1000 \text{deg}^2$ and an accuracy better than 2 per cent down to $100 \text{deg}^2$. This paper has been typeset from a \TeX/\LaTeX\ file prepared by the author.