On the Mixed Connectivity Conjecture of Beineke and Harary

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September 2, 2019

Abstract

A pair of non-negative integers \((k, l)\) is called a connectivity pair for distinct vertices \(s\) and \(t\) of a graph \(G\) if they can be separated by removing \(k\) vertices and \(l\) edges from \(G\), but not with fewer elements. In this paper we consider the conjecture that if \((k, l)\) is a connectivity pair, then there are \(k + l\) edge-disjoint \(s\)-\(t\) paths of which \(k + 1\) are internally vertex-disjoint and prove it for \(l = 2\). Further we show that the conjecture holds true in general for graphs of treewidth at most 3.

1 Introduction

Connectivity is a well examined property of graphs. Many and more authors study various properties of edge or vertex connected graphs. A famous example is Menger’s Theorem where the vertex and edge connectivity is put in connection with the the number of vertex- and edge-disjoint paths. On the other hand properties of graphs that remain connected when removing vertices and edges at the same time have not been getting the same attention. In this paper we regard a form of mixed connectivity introduced by Beineke and Harary in [1], called connectivity pairs. A pair of non-negative integers \((k, l)\) is called a connectivity pair for distinct vertices \(s\) and \(t\) if they can be separated by removing \(k\) vertices and \(l\) edges, but not with fewer elements. In [1] Beineke and Harary claim to have proven a mixed version of Menger’s Theorem: If \((k, l)\) is a connectivity pair for \(s\) and \(t\) there exist \(k + l\) edge-disjoint \(s\)-\(t\) paths \(k\) of which are internally vertex-disjoint. Mader pointed out in [7], that the proof is erroneous. Sadeghi and Fan claim in [8] to have proven the following statement:

When \(V(G) \geq k + l + 1\), \(k \geq 0\) and \(l \geq 1\), a graph \(G\) has \(k + l\) edge-disjoint paths of which \(k + 1\) are internally vertex-disjoint between any two vertices, if and only if the graph cannot be disconnected by removing \(k\) vertices and \(l - 1\) edges.

Again the claimed proof has been found faulty as has been observed by Streicher. In particular the claimed statement is not correct, as the existence of

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the claimed paths are not sufficient for the connectivity statement to hold. We repeat the arguments by Streicher in Appendix 3 for the sake of completeness.

The most meaningful result on the conjecture to date is due to Enomoto and Kaneko, cf. [3]. They first extended the conjecture claiming that it is possible to find \( k + 1 \) internally vertex-disjoint paths instead of just \( k \) under the additional assumption that \( l \geq 1 \) and then proved their statement for certain \( k \) and \( l \). The exact result is restated as Theorem 3 in Section 2.

From our studies the following conjecture originally formulated by Beineke and Harary in [1] and extended by Enomoto and Kaneko in [6] may hold. In the remainder of this article we refer to the conjecture by the name Beineke-Harary-Conjecture.

**Conjecture** (Beineke-Harary-Conjecture). Let \( G \) be a graph, \( s, t \in V(G) \) distinct vertices and \( k, l \) non-negative integers with \( l \geq 1 \). If \((k, l)\) is a connectivity pair for \( s \) and \( t \) in \( G \), then there exist \( k + l \) edge-disjoint paths, of which \( k + 1 \) are internally vertex-disjoint.

Our main contribution is to prove the conjecture for \( l = 2 \) and any \( k \in \mathbb{N} \). It is worth noting that for \( l = 2 \) the conjecture has not been proven for any \( k > 1 \). In particular the result of Enomoto and Kaneko does not apply to these cases and their proof may not easily be altered to fit these cases. Further we prove that the Beineke-Harary-Conjecture also holds for all graphs that have treewidth at most 3 without any conditions on \( k \) or \( l \).

It can be observed that for a graph \( G \), two vertices \( s \) and \( t \) and fixed \( k \in \mathbb{N} \), there is a unique \( l \in \mathbb{N} \) such that \((k, l)\) is a connectivity pair for \( s \) and \( t \) in \( G \). In [2], Beineke mentions that it is an interesting open question to consider the complexity of computing that \( l \). In this paper we give a formal proof that the decision version of the raised question is \textsc{NP}-complete.

After we state some basic definitions and preliminary results in Section 2 we regard the complexity of computing the second coordinate in a connectivity pair in Section 3. The proof of our main result can be found in Section 4. We prove the conjecture for graphs of small treewidth in Section 5.

## 2 Preliminaries

Most of our notation is standard graph terminology as can be found in e.g. [5]. We recall some basic notations in the following. The considered graphs may contain parallels but no loops. For a graph \( G \), we refer to the vertex set of the graph \( G \) by \( V(G) \) and to the edge set by \( E(G) \). We denote an edge connecting vertices \( u, v \in V(G) \) by \( uv \). For \( U, V \subseteq V(G) \) and \( u \in U \) and \( v \in V \) we call \( uv \) a \( U-V \) edge. The set of all \( U-V \) edges in \( E(G) \) is denoted by \( E(U, V) \); instead of \( E([u], V) \) and \( E(U, \{v\}) \) we write \( E(u, V) \) and \( E(U, v) \). Note that the exact choice of the edge if parallel edges are present is not of relevance to any of our proofs. If \( H \) is another graph we denote by \( G \cup H \) the graph with vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \) where parallel edges are allowed. For a subset of vertices \( S \subseteq V(G) \) we denote by \( G[S] \) the graph induced by \( S \), that has vertex set \( S \) and all edges connecting vertices of \( S \). Further we denote by \( G - S \) the graph \( G[V(G) \setminus S] \). For a subset \( E' \subseteq E \) we write \( G - E' \) for the graph with vertex set \( V(G) \) and edge set \( E(G) \setminus E' \). For easier notation we write \( G - v \) and \( G - e \) instead of \( G - \{v\} \) and \( G - \{e\} \) for \( v \in V(G) \) and \( e \in E(G) \).
A path $P = v_0 \ldots v_k$ is a graph with vertex set $\{v_0, \ldots, v_k\}$ and edge set $\{v_iv_{i+1}: i = 0, \ldots, k-1\}$, where all vertices are distinct except possibly $v_0$ and $v_k$. We also write $v_0v_k$ path for the graph above. We denote by $v_iv_j$ with $i \leq j$ the subpath starting at $v_i$ and ending at $v_j$. If $v_i = v_0$ ($v_j = v_k$) for simplicity of notation we also write $Pv_i$ ($v_jP$). Two or more paths are edge-disjoint if no two paths use the same edge. Two or more $s$-$t$ paths are internally vertex-disjoint if they only share the vertices $s$ and $t$. If $P_1, \ldots, P_k$ are internally vertex-disjoint $s$-$t$ paths, we call the graph $\bigcup_{i=1}^k P_i$ an $s$-$t$ $k$-skein.

For distinct vertices $s$ and $t$ we say that the set $W \subseteq V(G) \setminus \{s, t\}$ ($F \subseteq E(G)$) separates $s$ and $t$ in $G$ if $s$ and $t$ are not connected in $G - W$ ($G - F$). In this case we call $W$ ($F$) a vertex-(edge-)separator. If $s$ and $t$ are non-adjacent, we denote by $\kappa_G(s, t)$ the size of a smallest vertex-separator for $s$ and $t$, where we omit the subscript $G$ if the graph is clear from context. For a graph $G$ a set $W$ is a vertex-separator if $G - W$ is not connected.

In the following we recall definitions and basic results concerning connectivity pairs and separators in general.

**Definition 1** (Disconnecting Pair). Let $G$ be a graph and $S, T \subseteq V(G)$. We call a pair $(W, F)$ with $W \subseteq V(G) \setminus (S \cup T)$ and $F \subseteq E(G)$ an $S$-$T$ disconnecting pair if in $G - W - F$ there is no path from a vertex in $S$ to a vertex in $T$.

We call the number of edges in a disconnecting pair its size, the number of vertices in a disconnecting pair its order and the number of elements $|W| + |F|$ to be its cardinality.

If $S = \{s\}$ or $T = \{t\}$ consist of only one element we omit the set brackets in the notation and also write $s$-$t$ disconnecting pair.

Beineke and Harary introduced *connectivity pairs* in their paper from 1967 [1]. We recall their definition in the following.

**Definition 2** (Connectivity Pairs). Let $G$ be a graph and $s, t \in V(G)$ distinct vertices. We call a tuple of non-negative integers $(k, l)$ a connectivity pair for $s$ and $t$ in $G$ if

(i) there exists an $s$-$t$ disconnecting pair of order $k$ and size $l$ and

(ii) there is no $s$-$t$ disconnecting pair of cardinality less than $k + l$, order at most $k$ and size at most $l$.

It is easy to see, that property [ii] can be replaced by

[iii] there is no $s$-$t$ disconnecting pair of order $k$ and size $l - 1$ or order $k - 1$ and size $l$.

Further, if there are fewer than $l$ edges between $s$ and $t$, then we can replace property [iii] by

[iv] there is no $s$-$t$ disconnecting pair of order $k$ and size $l - 1$.

This is true since we may replace any edge in an $s$-$t$ disconnecting pair by a vertex incident to it unless the edge connects $s$ and $t$.

It can be readily observed, that for a fixed $k$ the second coordinate in a connectivity pair is unique in the following sense:
Observation 3. Let $G$ be a graph with distinct vertices $s, t \in V(G)$. Then for any $k$, with $0 \leq k \leq \kappa_{G-E(s,t)}(s,t)$ there exists a unique integer $l_k$ such that $(k,l_k)$ is a connectivity pair for $s$ and $t$ in $G$.

As a direct consequence of Observation 3 we get the following.

Observation 4. Let $G$ be a graph with distinct vertices $s, t \in V(G)$. If there exists an $s$-$t$ disconnecting pair of order $k$ and size $l$, then there exists an $l' \leq l$ such that $(k,l')$ is a connectivity pair.

The Beineke-Harary-Conjecture is, in some sense, a mixed form of Menger’s Theorem. As we will make use of it, we recall three versions of Menger’s theorem here.

Theorem 5 (Menger’s Theorem). Let $s$ and $t$ be two distinct vertices of a graph $G$.

(i) If $st \not\in E(G)$, then the minimum number of vertices separating $s$ and $t$ in $G$ is equal to the maximum number of internally vertex-disjoint $s$-$t$ paths.

(ii) The minimum number of edges separating $s$ and $t$ in $G$ is equal to the maximum number of edge-disjoint $s$-$t$ paths in $G$.

(iii) The maximum number of internally vertex-disjoint $s$-$t$ paths is equal to the minimum cardinality of an $s$-$t$ disconnecting pair.

With the help of Menger’s Theorem we can observe that for certain $k$ and $l$ the conjecture of Beineke and Harary is fulfilled.

Observation 6. Let $k \geq 0$ and $l \geq 1$ be integers and $s$ and $t$ be two distinct vertices of a graph $G$.

• If $(k,0)$ is a connectivity pair for $s$ and $t$, then $s$ and $t$ are $k$ vertex-connected and by Menger’s Theorem there are $k$ internally vertex-disjoint paths between $s$ and $t$.

• If $(k,1)$ is a connectivity pair for $s$ and $t$, then $s$ and $t$ are $k+1$ vertex-connected in $G$ and hence by Menger’s Theorem there are $k+1$ internally vertex-disjoint paths between $s$ and $t$.

• If $(0,l)$ is a connectivity pair for $s$ and $t$, then $s$ and $t$ are $l$ edge-connected in $G$ and hence by Menger’s Theorem there are $l$ edge-disjoint paths between $s$ and $t$.

It suffices to prove the Beineke-Harary-Conjecture for non-adjacent vertices $s$ and $t$, as we will see in the next two results.

Lemma 7. Let $G$ be a graph, $s, t \in V(G)$ be two distinct vertices and let $k, l$ be non-negative integers. Then, $(k,l)$ is a connectivity pair for $s$ and $t$ in $G$ if and only if $(k,l - |E(s,t)|)$ is a connectivity pair for $s$ and $t$ in $G - E(s,t)$.

Proof. Any $s$-$t$ disconnecting pair in $G$ has to contain all edges in $E(s,t)$. Thus, we get a one-to-one correspondence between the $s$-$t$ disconnecting pairs in $G$ and the ones in $G - E(s,t)$ by mapping a pair $(W,F)$ to the pair $(W,F \setminus E(s,t))$. The desired result follows immediately. \[\square\]
Lemma 8. If the Beineke-Harary-Conjecture holds true for all graphs $G$ and all vertices $s, t \in V(G)$ such that $s$ and $t$ are not adjacent, then the conjecture holds true for all graphs and all vertices.

Proof. Let $G$ be a graph, $s, t \in V(G)$ with $s \neq t$ and $|E(s, t)| \geq 1$ and let $(k, l)$ be a connectivity pair for $s$ and $t$ in $G$. By Lemma 7, $(k, l - |E(s, t)|)$ is a connectivity pair for $s$ and $t$ in $G - E(s, t)$. Thus, by assumption there exist $k + l - |E(s, t)|$ edge-disjoint $s$-$t$ paths of which at least $k$ are internally vertex-disjoint in $G - E(s, t)$. Note, that we cannot assume that $k + 1$ paths are internally vertex-disjoint, as $l - |E(s, t)| = 0$ is a possibility. Nevertheless, the $k + l - |E(s, t)|$ paths together with the edges in $E(s, t)$ yield $k + l$ edge-disjoint $s$-$t$ paths of which at least $k + 1$ are internally vertex-disjoint, as the edges in $E(s, t)$ are internally vertex-disjoint to all $s$-$t$ paths and by assumption $|E(s, t)| \geq 1$.

Their result directly yields the following corollary.

Corollary 10. Let $(1, l)$ be a connectivity pair for two distinct vertices $s$ and $t$ of a graph $G$, then there are $l + 1$ edge-disjoint $s$-$t$ paths of which two are internally vertex-disjoint.

Proof. For $l = 1$ the statement holds true due to Observation 6. For $l \geq 2$ and any $q, r \in \mathbb{N}$ with $1 + l = q \cdot 2 + r$ and $1 \leq r \leq l + 1$ we have $q + r > 1$ and by Theorem 9 we get the desired paths.

3 Complexity of Computing the Mixed Connectivity Function

In [2] Beineke and Wilson proposed the question how difficult it is to compute the second coordinate in a connectivity pair. In this section we deal with this question and show that it is NP-hard to compute the second coordinate in a connectivity pair.

To this end we consider the following decision problem.

Definition 11 (Second Coordinate in Connectivity Pair). Let $G = (V, E)$ be a graph with distinct vertices $s$ and $t$. Moreover let $k$ and $r$ be non-negative integers. The question of the Second Coordinate in Connectivity Pair (2-CP) problem is: Does there exist an $l \leq r$ such that $(k, l)$ is a connectivity pair for $s$ and $t$ in $G$.

We will prove the NP-completeness of the problem by a reduction from the Partial Vertex Cover problem. Let us recall the according definition first.
Let $G$ be a graph and $q, r$ non-negative integers. We call a set $C \subseteq V(G)$ a vertex cover if every edge of $G$ is incident with a vertex in $C$. Further we call $C' \subseteq V(G)$ a partial vertex cover with respect to $q$ and $r$ if $C'$ has cardinality at most $r$ and at least $q$ edges in $E(G)$ are incident to a vertex in $C'$.

**Definition 12** (Partial Vertex Cover problem). Let $G = (V, E)$ be a graph and $q$ and $r$ non-negative integers. The question of the Partial Vertex Cover (PVC) problem is if there is a set of vertices $C \subseteq V$ of cardinality at most $r$ that covers at least $q$ edges, i.e., for at least $q$ edges $uv \in E$ either $u \in C$ or $v \in C$.

In [4] Caskurlu and Subramani showed that the PVC problem is $NP$-complete on bipartite graphs. With this in mind we can prove the following result:

**Theorem 13.** The Second Coordinate in Connectivity Pair problem is $NP$-complete.

*Proof.* To see that 2-CP is contained in $NP$, we use as a certificate an $s$-$t$ disconnecting pair of order $k$ and size $r'$ for some $r' \leq r$. Clearly we may verify that the pair is disconnecting in polynomial time. Thus, by Observation 4 there exists an $l \leq r'$ such that $(k, l)$ is a connectivity pair.

Let $(G, q, r)$ be an instance of the PVC problem, where $G = (U \cup V, E)$ is a bipartite graph whose partition has the parts $U$ and $V$. As we can decide in polynomial time if there exists a vertex cover with at most $r$ vertices on bipartite graphs, we may assume that any vertex cover of $G$ has cardinality of at least $r + 1$. We construct an instance $(G', s, t, k, l)$ of 2-CP as follows: Set $V(G') := V(G) \cup \{s, t\}$ and the set $E(G')$ consists of the set $E(G)$ and for every $u \in U$ we add $|E(G)|$ many times the edge $su$ and for every $v \in V$ we add $|E(G)|$ many times the edge $vt$. Moreover set $k := r$ and $l := |E(G)| - q$. We note that $s$ and $t$ are at least $(k + 1)$-vertex-connected since a vertex cover in $G$ has cardinality at least $r + 1 > k$. All these steps can be done in polynomial time.

First assume that $(G, q, r)$ is a YES-instance for the PVC problem and let $C$ be a partial vertex cover of size at most $r$. Then $C$ and $E(G) \setminus E(C, V(G) \setminus C)$ is an $s$-$t$ disconnecting pair in $G'$ with $|C| \leq r = k$ and $|E(G) \setminus E(C, V(G) \setminus C)| \leq |E(G)| - q = l$. Any $s$-$t$ disconnecting pair with $k$ vertices and a minimal number of edges in $G'$ only uses edges in $E(G)$ since each edge in $E(G') \setminus E(G)$ exists $|E(G)|$ times in $G'$. Since we already found an $s$-$t$ disconnecting pair with $k$ vertices and at most $l$ edges we know by Observation 4 that there is an $l' \leq l$ such that $(k, l')$ is a connectivity pair for $s$ and $t$ in $G'$.

Now assume that $(G', s, t, k, l)$ is a YES-instance for 2-CP and let $(k, l')$ be a connectivity pair for $s$ and $t$ in $G$ with $l' \leq l$. Then there is an $s$-$t$ disconnecting pair $(W, F)$ of size of at most $l$ and order at most $k$. Again we may assume that $F \subseteq E(G)$ with the same argument as before. But then the vertices in $W$ are incident to at least $|E(G)| - |F| \geq |E(G)| - l = q$ edges in $E(G)$ in $G$. This shows that $W$ is a partial vertex cover in $G$ of cardinality at most $r$ that covers at least $q$ edges.

As a direct consequence of Theorem 13 deciding for given integers $k$ and $l$ if $(k, l)$ is a connectivity pair is $NP$-hard. From Theorem 13 the desired result follows:
Corollary 14. Unless \( P=NP \), there is no polynomial time algorithm that, given any graph \( G \), distinct vertices \( s \) and \( t \) and an integer \( 0 \leq k \leq \kappa_{G-E(s,t)}(s,t) \), returns the integer \( l \) such that \((k,l)\) is a connectivity pair for \( s \) and \( t \) in \( G \).

Note that it is possible to formulate a version of 2-CP for the first coordinate of a connectivity pair. We would get \( NP \)-completeness in a similar manner as in Theorem 13. We do not explicitly define this problem here, as for a fixed \( l \) there does not necessarily exist a \( k \) such that \((k,l)\) forms a connectivity pair and the question of computing the first coordinate in a connectivity pair is thereby undefined. In particular, as for any fixed \( k \) the corresponding connectivity pair is unique by Observation 3, the problem of computing all connectivity pairs for a graph \( G \) and vertices \( s \) and \( t \) can be solved by computing the unique connectivity pair for all \( k \) with \( 0 \leq k \leq \kappa_{G-E(s,t)}(s,t) \).

4 The Beineke-Harary-Conjecture for the connectivity pair \((k,2)\)

In this section we show that the Beineke-Harary-Conjecture holds true for all non-negative integers \( k \) if \( l=2 \).

Theorem 15. Let \( G \) be a graph and \( s,t \in V(G) \). Further let \((k,2)\) be a connectivity pair for \( s \) and \( t \). Then, there exist \( k+2 \) edge-disjoint \( s-t \) paths of which \( k+1 \) are internally vertex-disjoint.

Before we begin with the proof, note that the result of Theorem 15 has only been proven for \( k=1 \). In particular, the result of Enomoto and Kaneko, cf. Theorem 9, basically tackles the conjecture from a different angle, as in their statement for \( k \geq 2 \), the sum \( q+r \) always equals 2, which leads to a big gap between \( k \) and \( q+r \) for large \( k \).

We will now prove a more general version of Theorem 15 and afterwards we conclude the correctness of Theorem 15.

Theorem 16. Let \( G \) be a graph, \( s_1, s_2, t \in V(G) \) with \( s_1 \neq t \). Further assume that

(i) there exists an \( s_2-t \) path in \( G \),

(ii) there exist \( k+1 \) internally vertex-disjoint \( s_1-t \) paths in \( G \) and

(iii) there is no \( \{s_1, s_2\} \)-t disconnecting pair of cardinality \( k+1 \) and order at most \( k \) in \( G \).

Then, there exist \( k+2 \) edge-disjoint \( s_1-t \) paths, \( k+1 \) of which are internally vertex-disjoint \( s_1-t \) paths and one of which is an \( s_2-t \) path.

Proof. Let \( G \) be a graph, \( s_1, s_2, t \in V(G) \) with \( s_1 \neq t \) fulfilling properties (i) to (iii). We prove the claim by induction on the number of edges \( |E(G)| \). If \( |E(G)| \leq k \) there cannot be \( k+1 \) internally vertex-disjoint \( s_1-t \) paths, as \( s_1 \neq t \). Thus from now on we may assume the induction hypothesis:

(*) Let \( G' \) be a graph with \( |E(G')| < |E(G)| \) and vertices \( s'_1, s'_2, t' \in V(G) \) with \( s'_1 \neq t' \). If properties (i) to (iii) are fulfilled, then there exist \( k+2 \) edge-disjoint \( s'_1-t' \) paths, \( k+1 \) of which are internally vertex-disjoint \( s'_1-t' \) paths and one of which is an \( s'_2-t' \) path.
We begin by proving the induction step for the case that \( s_2 \) is contained in an \( s_1 \)-\( t \) \( k + 1 \)-skein and afterwards use this result to prove the induction step for the case that \( s_2 \) is not contained in such a skein.

**Case 1:** \( s_2 \) is contained in an \( s_1 \)-\( t \) \( k + 1 \)-skein.

If \( s_2 = t \), then the \( k + 1 \) internally vertex-disjoint paths from Property \([\text{ii}]\) together with the \( s_2 \)-\( t \) path \( s_2 = t \) form the desired paths. Thus, we may assume that \( s_2 \neq t \). Denote by \( P_1, \ldots, P_{k+1} \) the \( s_1 \)-\( t \) paths of an \( s_1 \)-\( t \) \( k + 1 \)-skein containing \( s_2 \). Without loss of generality we may assume \( s_2 \in V(P_{k+1}) \). Denote by \( s_2' \) the vertex after \( s_2 \) on \( P_{k+1} \), i.e. \( P_{k+1} = s_1 \ldots s_2 s_2' \ldots t \). Note that \( s_1 = s_2 \) is not forbidden at this point. We now want to use the induction hypothesis for \( G - s_2 s_2' \) and the vertices \( s_1, s_2' \) and \( t \).

Property \([\text{ii}]\) is clearly fulfilled as \( s_2' P_{k+1} \) is an \( s_2' \)-\( t \) path in \( G - s_2 s_2' \). Suppose that there do not exist \( k + 1 \) internally vertex-disjoint \( s_1 \)-\( t \) paths in \( G - s_2 s_2' \). By Menger’s Theorem, cf. Theorem 5.(iii), there is an \( s_1 \)-\( t \) disconnecting pair \((W, F)\) of cardinality \( k \). Since in \( G - s_2 s_2' \) the internally vertex-disjoint paths \( P_1, \ldots, P_k \) still exist, all elements of \((W, F)\) are contained in the paths \( P_1, \ldots, P_k \). Thus, the path \( P_{k+1} s_2' \) still exists in \( G - s_2 s_2' - W - F \) and \((W, F)\) is an \( \{s_1, s_2'\} \)-\( t \) disconnecting pair in \( G - s_2 s_2' \). Therefore \((W, F \cup \{s_2 s_2'\})\) is an \( \{s_1, s_2\} \)-\( t \) disconnecting pair in \( G \), contradicting Property \([\text{iii}]\). Hence, Property \([\text{ii}]\) is fulfilled in \( G - s_2 s_2' \). Suppose now there exists an \( \{s_1, s_2'\} \)-\( t \) disconnecting pair \((W, F)\) of cardinality \( k + 1 \) and order at most \( k \) in \( G - s_2 s_2' \). As the paths \( P_1, \ldots, P_k, s_2' P_{k+1} \) are internally vertex-disjoint, each element of \((W, F)\) is contained in one of these paths. Thus, \( P_{k+1} s_2' \) still exists in \( G - s_2 s_2' - W - F \) and neither \( s_2 \) nor \( s_2' \) are contained in the same component as \( t \) in \( G - s_2 s_2' - W - F \). This implies that \((W, F)\) is an \( \{s_1, s_2\} \)-\( t \) disconnecting pair in \( G \). Again this is a contradiction to Property \([\text{iii}]\) in \( G \) and hence Property \([\text{iii}]\) is fulfilled for \( G - s_2 s_2', \{s_1, s_2'\} \) and \( t \).

As \( G - s_2 s_2' \) contains \(|E(G)| - 1\) edges, by (\(\star\)), there exist \( k + 2 \) edge-disjoint paths, \( k + 1 \) of which are internally vertex-disjoint \( s_1 \)-\( t \) paths, say \( P'_1, \ldots, P'_{k+1} \), and one of which is an \( s_2' \)-\( t \) path, say \( P'_k \). If \( s_2' \in V(P'_{k+1}) \) the paths \( P'_1, \ldots, P'_{k+1}, s_2' P'_{k+1} \) are the desired paths in \( G \). Otherwise the paths \( P'_1, \ldots, P'_{k+1}, s_2 s'_2 \cup P'_{k+1} \) form the desired paths.

Thus from now on, in addition to (\(\star\)), we may assume:

(\(\star\star\)) Let \( G' \) be a graph with \(|E(G')| = |E(G)|\) and vertices \( s'_1, s'_2, t' \in V(G') \) such that \( s'_1 \neq t \) and \( s'_2 \) is contained in an \( s_1 \)-\( t \) \( k + 1 \)-skein. If Properties \([\text{i}]\) through \([\text{iv}]\) are fulfilled, then there exist \( k + 2 \) edge-disjoint paths, \( k + 1 \) of which are internally vertex-disjoint \( s'_1 \)-\( t' \) paths and one of which is an \( s'_2 \)-\( t' \) path.

**Case 2:** \( s_2 \) is not contained in any \( s_1 \)-\( t \) \( k + 1 \)-skein.

Denote by \( s'_2 \) a vertex on an \( s_1 \)-\( t \) \( k + 1 \)-skein that is closest (with respect to the number of edges) to \( s_2 \) among all vertices on \( s_1 \)-\( t \) \( k + 1 \)-skineins. Note that this is well defined as there exists an \( s_2 \)-\( t \) path. Now we show that the assumptions still hold true if we replace \( s_2 \) by \( s'_2 \).

Note that Properties \([\text{i}]\) and \([\text{ii}]\) are fulfilled when replacing \( s_2 \) by \( s'_2 \). To see that Property \([\text{iii}]\) still holds, suppose that there exists an \( \{s_1, s'_2\} \)-\( t \) disconnecting pair \((W, F)\) of cardinality \( k + 1 \) and order at most \( k \). As there cannot
be any \( s_1 \)-\( t \) path left in \( G - W - F \), all elements of the disconnecting pair are contained in \( s_1 \)-\( t \) \( k + 1 \)-skeins. As \( W \) may also not contain \( s'_2 \) by definition, the vertices \( s_2 \) and \( s'_2 \) are contained in the same component of \( G - W - F \). Thus, \((W, F)\) is an \( \{s_1, s_2\}\)-\( t \) disconnecting pair in contradiction to Property (iii) for \( \{s_1, s_3\} \) and \( t \).

Thus, by (**) there exist \( k + 2 \) edge-disjoint paths, say \( P_1, \ldots, P_{k+2} \) such that \( P_1, \ldots, P_{k+1} \) are internally vertex-disjoint \( s_1 \)-\( t \) paths and \( P_{k+2} \) is an \( s'_2 \)-\( t \) path. Denote by \( P' \) a shortest \( s_2 \)-\( s'_2 \) path. Note that no element of \( P' \), except possibly \( s'_2 \), is contained in \( P_1, \ldots, P_{k+1} \) as no vertex or edge on an \( s_1 \)-\( t \) \( k + 1 \)-skein is closer to \( s_2 \) than \( s'_2 \). Further denote by \( s' \) the vertex on \( P' \) closest to \( s_2 \) that is also contained in \( P_{k+2} \). Then \( P's' \cup s'P_{k+2} \) is an \( s_2 \)-\( t \) path that is edge-disjoint to all \( P_1, \ldots, P_{k+1} \) and we obtain the desired paths.

As claimed we are now ready to prove Theorem 15.

**Proof of Theorem 15** By Lemma 8 it suffices to prove the claim for non-adjacent vertices \( s, t \in V(G) \). We show that Properties (i) through (iii) of Theorem 16 hold for \( G, s_1 = s_2 = s \) and \( t \).

By the definition of a connectivity pair, there is no \( s \)-\( t \) disconnecting pair of cardinality less than \( k + 2 \), order at most \( k \) and size at most 2. Thus, by Menger’s Theorem, cf. Theorem 16(iii), there exist \( k + 1 \) internally vertex-disjoint \( s \)-\( t \) paths and Properties (ii) and (iii) are fulfilled. Now assume Property (iii) is not fulfilled and let \((W, F)\) be an \( s \)-\( t \) disconnecting pair of cardinality \( k + 1 \) and order at most \( k \). As \( s \) and \( t \) are not adjacent any edge in \( F \) has an endpoint which is not \( s \) or \( t \). Thus, replacing all but one edge in \((W, F)\) with an endpoint that is not \( s \) or \( t \) we get an \( s \)-\( t \) disconnecting pair of cardinality \( k + 1 \), order \( k \) and size 1. Such a pair may not exist, as \((k, 2)\) is a connectivity pair for \( s \) and \( t \). This yields a contradiction. Thus, the assumptions of Theorem 15 are fulfilled and there exist \( k + 2 \) edge-disjoint \( s \)-\( t \) paths of which \( k + 1 \) are internally vertex-disjoint.

5 The Beineke-Harary-Conjecture for Graphs with Small Treewidth

In this section we prove the Beineke-Harary-Conjecture for graphs with treewidth at most 3. To this end we recall the definition of treewidth and some basic results on tree decompositions. For more details, see i.e. [3].

For a graph \( G \) a **tree decomposition** \((\mathcal{B}, \mathcal{T})\) of \( G \) consists of a tree \( \mathcal{T} \) and a set \( \mathcal{B} = \{B_i : i \in V(\mathcal{T})\} \) of bags \( B_i \subseteq V(G) \) such that \( V(G) = \bigcup_{i \in V(\mathcal{T})} B_i \). Further for each edge \( vw \in E(G) \) there exists a node \( i \in V(\mathcal{T}) \) such that \( v, w \in B_i \), and if \( v \in B_{j_1} \cap B_{j_2} \), then \( v \in B_i \) for each node \( i \) on the simple path connecting \( j_1 \) and \( j_2 \) in \( \mathcal{T} \). A tree decomposition \((\mathcal{B}, \mathcal{T})\) has **width** \( k \) if each bag is of cardinality at most \( k + 1 \) and there exists some bag of size \( k + 1 \). The **treewidth** of \( G \) is the smallest integer \( k \) for which there is a width \( k \) tree decomposition of \( G \). We write \( \text{tw}(G) = k \).

The following result is well known and can be found in [3]. We formulate it here as an observation:
Observation 17. Let $G$ be a graph and $(\mathcal{B}, \mathcal{T})$ a tree decomposition of $G$. Let $ij \in E(T)$ and denote by $T_i$ and $T_j$ the two subtrees of $T - ij$ with $i \in V(T_i)$ and $j \in V(T_j)$. If $u \in B_i \setminus (B_i \cap B_j)$ for some $i' \in V(T_i)$ and $v \in B_j \setminus (B_i \cap B_j)$ for some $j' \in V(T_j)$, then $B_i \cap B_j$ is a separator for $u$ and $v$ in $G$.

Further the following lemma will come in handy during our proofs.

Lemma 18. Let $G$ be a graph with treewidth at most $k$ for some integer $k \geq 1$ and let $s, t \in V(G)$ be distinct and non-adjacent. If every tree decomposition of width $k$ has a bag containing $s$ and $t$, then $s$ and $t$ are contained in a vertex-separator in $G$ with at most $k$ vertices.

Proof. Let $G$ be a graph with treewidth $k$ and $s, t \in V(G)$ distinct and non-adjacent. Moreover, assume that every tree decomposition of width $k$ has a bag that contains $s$ and $t$.

Suppose that $(\mathcal{B}, \mathcal{T})$ is a tree decomposition of $G$ with treewidth $k$, such that $s$ and $t$ share exactly one bag $B_i$. Let $j_1, \ldots, j_r$ be the neighbors of $i$ in $\mathcal{T}$ whose bags contain $s$. We construct a new tree decomposition by replacing the node $i$ with two adjacent nodes $i_1$ and $i_2$ with corresponding bags $B_{i_1} = B_i \setminus \{t\}$ and $B_{i_2} = B_i \setminus \{s\}$, making $j_1, \ldots, j_r$ adjacent to $i_1$ and making the remaining neighbors of $i$ adjacent to $i_2$. As $s$ and $t$ are not adjacent, the result is in fact a tree decomposition of width at most $k$, in which no bag contains both $s$ and $t$. This is a contradiction.

As for any edge $ij \in E(T)$ with $B_i \subseteq B_j$ we can create a new tree decomposition by removing $i$ and making all its neighbors (except $j$) adjacent to $j$ we can also assume that there exists a tree decomposition of width $k$ with an edge $ij$ such that $s, t \in B_i \cap B_j$, $B_i \not\subseteq B_j$ and $B_j \not\subseteq B_i$.

Thus, we can find a tree decomposition $(\mathcal{B}, \mathcal{T})$ of width at most $k$ containing an edge $ij \in E(T)$ such that the bags $B_i$ and $B_j$ both contain $s$ and $t$ and have size of at most $k + 1$. Moreover we may assume that there exists $u \in B_i \setminus B_j$ and $v \in B_j \setminus B_i$ and by Observation 17 the set $W = B_i \cap B_j$ separates $u$ and $v$. Thereby $G - W$ is not connected and $s$ and $t$ are contained in a vertex-separator with at most $k$ vertices.

We begin by observing that the Beineke-Harary-Conjecture holds true for graphs of treewidth 1. If for a graph $G$ the underlying simple graph is a tree, either $s$ and $t$ are adjacent, in which case every $s$-$t$ path consists of a single edge, or there exists a vertex $a \in V(G) \setminus \{s, t\}$ separating $s$ and $t$. In the first case the Beineke-Harary-Conjecture clearly holds, as the cardinality of a minimal disconnecting pair is exactly the number of edges between $s$ and $t$. In the second case the only possible connectivity pairs for $s$ and $t$ are of the form $(0, l)$ for $l \geq 1$ or $(1, 0)$. The conjecture follows from Observation 5.

Observation 19. Let $G$ be a graph of treewidth 1 and vertices $s, t \in V(G)$. Further let $(k, l)$ be a connectivity pair for $s$ and $t$ in $G$, with $k \geq 0$ and $l \geq 1$. Then there exist $k + l$ edge-disjoint $s$-$t$ paths, $k + 1$ of which are internally vertex-disjoint.

In the next step we will prove the conjecture for graphs with treewidth at most 2. Although Theorem 20 is implied by Theorem 21 and the proof could be included into the one of Theorem 21 for better readability we prove the
theorems separately. The structure of the two proofs is similar and therefore the proof of Theorem 20 can be regarded as a warmup for the one of Theorem 21.

**Theorem 20.** Let $G$ be a graph of treewidth at most 2 with distinct vertices $s, t \in V(G)$ and $k \geq 0$ and $l \geq 1$ integers. If $(k, l)$ is a connectivity pair for $s$ and $t$, then $G$ contains $k + l$ edge-disjoint $s$-$t$ paths of which $k + 1$ are internally vertex-disjoint.

**Proof.** Let $G$ be a graph, $s, t \in V(G)$ distinct vertices and let $(k, l)$ be a connectivity pair for $s$ and $t$ with $l \geq 1$. If $\text{tw}(G) = 1$ the result follows from Observation 19 and by Lemma 8 we may assume that $s$ and $t$ are not adjacent in $G$.

First assume there exists a tree decomposition $(\mathcal{B}, \mathcal{T})$ of $G$ of width 2 such that no bag contains both, $s$ and $t$. Denote by $T_s$ ($T_t$) the subtree of $\mathcal{T}$ induced by all nodes corresponding to bags containing $s$ ($t$). As $V(T_s) \cap V(T_t) = \emptyset$ there exists an edge $ij$ that separates $V(T_s)$ form $V(T_t)$. Thus, by Observation 17 $W = B_i \cap B_j$ is an $s$-$t$ vertex-separator in $G$. As without loss of generality $B_i \neq B_j$, we get $|W| \leq 2$. As $l \geq 1$, this implies that $k \leq 1$ and the result follows from Observation 6 and Corollary 10.

Thus, we may assume that in every tree decomposition there is at least one bag containing both vertices $s$ and $t$. By Lemma 18 we get that $G - \{s, t\}$ is not connected. We prove the remainder of the theorem by induction on the number of vertices $|V(G)|$. If $|V(G)| \leq 3$, $\{s, t\}$ cannot be a vertex separator and the claim holds. So assume the claim holds for all graphs with less than $|V(G)|$ vertices. Let $C$ be a component of $G - \{s, t\}$ and denote by $G_1$ the graph induced by $C \cup \{s, t\}$ and let $G_2 = G - C$. Note that $|V(G_i)| < |V(G)|$ for $i \in \{1, 2\}$. Regard some $s$-$t$ disconnecting pair $(W, F)$ of order $k$ and size $l$ in $G$. For $i \in \{1, 2\}$, the pair induces an $s$-$t$ disconnecting pair $(W_i, F_i)$ in $G_i$, with $W_i = W \cap V(G_i)$ and $F_i = F \cap E(G_i)$. Let $k_i = |W_i|$ and $l_i = |F_i|$. It is easy to see that $(k_i, l_i)$ is a connectivity pair for $s$ and $t$ in $G_i$. Further $k_1 + k_2 = k$, $l_1 + l_2 = l$ and without loss of generality we may assume $l_2 \geq 1$. Thus, in $G_1$ there exist $k_1 + l_1$ edge-disjoint paths of which $k_1$ are internally vertex-disjoint. Note that we cannot assume that there exist $k_1 + 1$ internally vertex-disjoint paths as $l_1$ may be 0. In $G_2$, by induction we get $k_2 + l_2$ edge-disjoint paths of which $k_2 + 1$ are internally vertex-disjoint. As for any two paths $P_1$ in $G_1$ and $P_2$ in $G_2$ it holds true that $P_1$ and $P_2$ are internally vertex-disjoint in $G$ there exist $k_1 + k_2 + l_1 + l_2 = k + l$ edge-disjoint $s$-$t$ paths of which $k_1 + k_2 + 1 = k + 1$ paths are internally vertex-disjoint.

Finally we are ready to prove the Beineke-Harary-Conjecture for graphs with treewidth at most 3. A part of the proof is done by induction, where we move the induction step to Lemma 22 as it is somewhat technical and lengthy.

**Theorem 21.** Let $G$ be a graph with treewidth at most 3. Let $s, t \in V(G)$ be two distinct vertices and $k \geq 0$, $l \geq 1$ integers. If $(k, l)$ is a connectivity pair for $s$ and $t$, then $G$ contains $k + l$ edge-disjoint $s$-$t$ paths, $k + 1$ of which are internally vertex-disjoint.

**Proof.** Let $G$ be a graph, $s, t \in V(G)$ distinct vertices and let $(k, l)$ be a connectivity pair for $s$ and $t$ with $l \geq 1$. If $\text{tw}(G) \leq 2$ the result follows from Theorem 20 and by Lemma 8 we may assume that $s$ and $t$ are not adjacent in $G$. 

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First assume that there exists a tree decomposition \((B, T)\) of \(G\) with \(\text{tw}(G) = 3\) such that no bag contains both, \(s\) and \(t\). With the same arguments as in the proof of Theorem 20, there exists a set \(W\) with \(|W| \leq 3\) such that \(s\) and \(t\) are not connected in \(G - W\). As \(l \geq 1\), this implies that \(k \leq 2\), as otherwise \((k, l)\) is not a connectivity pair. If \(l = 1\) the theorem follows from Observation 9. If \(l = 2\) the result follows from Theorem 10. Further, if \(k = 1\) the result follows from Corollary 11. Finally, if \(l > 2, k = 2\) and \(q, r\) are integers such that \(2 + l = q \cdot 3 + r\) with \(1 \leq r \leq 3\), we get that \(q + r > 2 = k\) and the desired result follows from Theorem 9.

We now may assume that every tree decomposition of \(G\) has a bag containing \(s\) and \(t\). Then, by Lemma 15 there is a vertex-separator containing \(s\) and \(t\) in \(G\) with at most 3 vertices. If \(\{s, t\}\) is a vertex separator we get the result with the same arguments as in Theorem 20. So assume there is a vertex \(a \in V(G) \setminus \{s, t\}\), such that \(G - \{s, t, a\}\) is not connected. In this case we proof the claim by induction on the number of vertices in \(G\). If \(|V(G)| \leq 4\), \(\{s, t, a\}\) cannot be a vertex-separator and the claim holds. Thus, we may assume, that the theorem holds true for all graphs with treewidth at most 3 and less than \(V(G)\) vertices, integers \(k' \geq 0\) and \(l' \geq 1\) and non-adjacent vertices \(s'\) and \(t'\). In this case Lemma 22 gives us \(k + l\) edge-disjoint \(s-t\) paths, of which \(k + 1\) are internally vertex-disjoint, which completes the proof.

Lemma 22. Let \(G\) be a graph of treewidth 3 containing vertices \(s, t\) and \(a\), such that \(G - \{s, t, a\}\) is not connected and \(s\) and \(t\) are not adjacent. Further let \(k \geq 0\) and \(l \geq 1\) be integers such that \((k, l)\) is a connectivity pair for \(s\) and \(t\) in \(G\). Assume the Beineke-Harary-Conjecture holds for all graphs with treewidth at most 3 and less than \(V(G)\) vertices, integers \(k' \geq 0\) and \(l' \geq 1\) and vertices \(s'\) and \(t'\) such that \(s'\) and \(t'\) are not adjacent. Then there exist \(k + l\) edge-disjoint \(s-t\) paths of which \(k + 1\) are internally vertex-disjoint.

Proof. Let \(C\) be a component of \(G - \{s, t, a\}\), denote by \(G_1\) the graph induced by \(V(C) \cup \{s, t, a\}\) and let \(G_2\) be the graph \(G - V(C)\). Note that \(|V(G_i)| < |V(G)|\).

We distinguish two cases.

Case 1: The vertex \(a\) is contained in an \(s-t\) disconnecting pair of order \(k\) and size \(l\) in \(G\).

In \(G - a\) the pair \((k - 1, l)\) is a connectivity pair for \(s\) and \(t\). For \(i = 1, 2\) any \(s-t\) disconnecting pair \((W, F)\) of order \(k\) and size \(l\), with \(a \in W\) induces an \(s-t\) disconnecting pair \((W_i, F_i)\) in \(G_i - a\) of order \(k_i\) and size \(l_i\) such that \(k_1 + k_2 = k - 1\) and \(l_1 + l_2 = l\). Without loss of generality we may assume \(l_1 \geq 1\).

Claim 1: \((k_1, l_1)\) is a connectivity pair for \(s\) and \(t\) in \(G_i - a\).

Proof. Suppose \((k_1, l_1)\) is not a connectivity pair. As \((W_i, F_i)\) is a disconnecting pair of order \(k_i\) and size \(l_i\) such a pair clearly exists. Thus, there has to be a disconnecting pair \((W', F')\) of order \(k' \leq k_i\), size \(l' \leq l_i\) and cardinality \(k' + l' < k_i + l_i\), but then \((W' \cup W_j \cup \{a\}, F' \cup F_j)\) with \(j \in \{1, 2\} \setminus \{i\}\) is an \(s-t\) disconnecting pair in \(G\) of order at most \(k\), size at most \(l\) and cardinality less than \(k + l\). A contradiction.
Next we show that if \((k_2, l_2)\) is not a connectivity pair for \(s\) and \(t\) in \(G_2\) the desired paths exist. If \((k_2, l_2)\) is in fact not a connectivity pair, then \((k_2, l_2+x)\) is a connectivity pair for some integer \(x \geq 1\). By induction we get \(k_2 + l_2 + x\) edge-disjoint \(s\)-\(t\) paths in \(G_2\), \(k_2+1\) of which are internally vertex-disjoint. As \((k_1, l_1)\) is a connectivity pair in \(G_1 - a\) again by induction we get \(k_1 + l_1 - 1\) edge-disjoint \(s\)-\(t\) paths in \(G_1 - a\). If \(k_2 + l_2 + x + k_1 + l_1 \geq k + l\) edge-disjoint \(s\)-\(t\) paths in \(G\), \(k_1 + 1 + k_2 + 1 = k + 1\) of which are internally vertex-disjoint. Thus we may assume that \((k_2, l_2)\) is a connectivity pair for \(s\) and \(t\) in \(G_2\).

It is clear that \(a\) cannot be contained in any \(s\)-\(t\) disconnecting pair in \(G_2\) of order \(k_2\) and size \(l_2\), as otherwise \((k_2, l_2)\) would not be a connectivity pair for \(s\) and \(t\) in \(G_2 - a\). We fix some \(s\)-\(t\) disconnecting pair \((W'_2, F'_2)\) in \(G_2\) that has order \(k_2\) and size \(l_2\). As in \(G_2 - W'_2 - F'_2\), the vertex \(a\) cannot be connected to \(s\) and \(t\), without loss of generality we may assume that \(a\) is not connected to \(t\).

Thus, for the remainder of case 1 we assume

\[(*) \quad (k_2, l_2) \text{ is a connectivity pair for } s \text{ and } t \text{ in } G_2, (W'_2, F'_2) \text{ is an } s\text{-}t \text{ disconnecting pair of order } k_2 \text{ and size } l_2 \text{ that does not contain } a \text{ and the vertices } a \text{ and } t \text{ are not connected in } G - W'_2 - F'_2.\]

We define the integer \(0 \leq q \leq l_1\) to be the unique integer such that \((k_1 + 1, l_1 - q)\) is a connectivity pair for \(s\) and \(t\) in \(G_1\). Note that this is well defined as \((W_1 \cup \{a\}, F_1)\) is an \(s\)-\(t\) disconnecting pair of order \(k_1 + 1\) and size \(l_1\). Denote by \(H_1\) the graph arising from \(G_1\) by adding \(q\) parallel edges \(e_1, \ldots, e_q\) between \(a\) and \(s\). Then the following holds:

**Claim 2:** \((k_1 + 1, l_1)\) is a connectivity pair for \(s\) and \(t\) in \(H_1\).

**Proof.** As argued before \((W_1 \cup \{a\}, F_1)\) is an \(s\)-\(t\) disconnecting pair of order \(k_1 + 1\) and size \(l_1\) in \(G_1\) and thereby also disconnecting in \(H_1\). Suppose that there exists an \(s\)-\(t\) disconnecting pair of order \(k_1 + 1\) and size at most \(l_1 - 1\). Let \((W', F')\) be one such pair of minimal size. Suppose that \(e_i \in F'\) for some \(i \in \{1, \ldots, q\}\). As the size of the disconnecting pair is minimal, this implies \(e_i \in F'\) for all \(i \in \{1, \ldots, q\}\). If this is the case \((W', F' \setminus \{e_1, \ldots, e_q\})\) is an \(s\)-\(t\) disconnecting pair in \(G_1\) of order at most \(k_1 + 1\) and size at most \(l_1 - q\) in contradiction to \((k_1 + 1, l_1 - q)\) being a connectivity pair in \(G_1\). Thus, either \(a \in W'\) or \(a\) and \(s\) are contained in the same component of \(H_1 - W' - F'\). In particular there is no \(a\)-\(t\) path in \(G_1 - W' - F'\). But then \((W' \cup W'_1, F' \cup F'_1)\) is an \(s\)-\(t\) disconnecting pair in \(G\) of order at most \(k\) and size at most \(t - 1\) by \((*)\). A contradiction to \((k, l)\) being a connectivity pair for \(s\) and \(t\) in \(G\).

Next denote by \(H_2\) the graph arising from \(G_2\) by adding \(q\) parallel edges \(f_1, \ldots, f_q\) between \(a\) and \(t\). We prove the following:

**Claim 3:** \((k_2, l_2 + q)\) is a connectivity pair for \(s\) and \(t\) in \(H_2\).

**Proof.** If \(q = 0\) the statement holds true by \((*)\), so assume that \(q \geq 1\). The pair \((W'_2, F'_2 \cup \{f_1, \ldots, f_q\})\) is clearly an \(s\)-\(t\) disconnecting pair in \(H_2\) and \(a\) and \(t\) are not in the same component in \(H_2 - W'_2 - F'_2 \cup \{f_1, \ldots, f_q\}\). So suppose there exists an \(s\)-\(t\) disconnecting pair of order \(k_2\) and size \(l_2 + q - 1\).
Let \((W', F')\) be one such pair of minimal size. With the same arguments as in the previous claim we get that \(f_i \notin F'\) for all \(i \in \{1, \ldots, q\}\). Thus, \(a \in W'\) or \(a\) and \(t\) are in the same component in \(H_2 - W' - F'\). In particular there does not exist an \(s\)-\(a\) path in \(G_2 - W' - F'\). As \((k_1 + 1, l_1 - q)\) is a connectivity pair for \(G_1\), there exists an \(s\)-\(t\) disconnecting pair \((W'_1, F'_1)\) in \(G_1\) of order \(k_1 + 1\) and size \(l_1 - q\). Suppose there exists an \(s\)-\(a\) path in \(G_1 - W'_1 - F'_1\). Then there does not exist an \(a\)-\(t\) path and \((W'_1 \cup W'_2, F'_1 \cup F'_2)\) is an \(s\)-\(t\) disconnecting pair in \(G\) of order \(k\) and size \(l_1 + l - q < l\) by (s).

A contradiction. On the other hand if there does not exist an \(s\)-\(a\) path in \(G_1 - W'_1 - F'_1\), then the pair \((W'_1 \cup W', F'_1 \cup F')\) is \(s\)-\(t\) disconnecting in \(G\) and of order \(k\) and size \(l - 1\), which again yields a contradiction. ■

Note that it is \(V(H_i) = V(G_i) < V(G)\) for \(i = 1, 2\) and \(tw(H_i) \leq 3\) as by the proof of Lemma 18 there exists a tree decomposition of \(G\) that has a bag containing \(s\), \(t\) and \(a\). By Claim 2 and the induction hypothesis there are \(k_1 + 1 + l_1\) edge-disjoint \(s\)-\(t\) paths in \(H_1\), say \(P_1, \ldots, P_{k_1+l_1+1}\), of which \(k_1 + 2\) are internally vertex-disjoint. Without loss of generality let \(P_1, \ldots, P_{\ell}\) be the paths using edges from \(\{e_1, \ldots, e_q\}\), where from these we denote by \(P_{\ell}\) the path that is among the \(k_1 + 2\) internally vertex-disjoint paths, if one such path exists.

If \(q = l_2 = 0\), then \((k_2, 0)\) is a connectivity pair for \(s\) and \(t\) in \(G_2 - a\) by Claim 1 and by Observation 3 there are \(k_2\) internally vertex-disjoint \(s\)-\(t\) paths in \(G_2 - a\). Together with \(P_1, \ldots, P_{k_1+l_1+1}\) we get the desired paths for \(G\). So assume that \(q + l_2 > 0\). By Claim 3 and induction there exist \(k_2 + l_2 + q\) edge-disjoint \(s\)-\(t\) paths in \(H_2\), say \(Q_1, \ldots, Q_{k_2+l_2+q}\), \(k_2 + 1\) of which are internally vertex-disjoint. Without loss of generality let \(Q_1, \ldots, Q_{r_2}\) for \(r_2 \leq q\) be the paths using edges from \(\{e_1, \ldots, e_q\}\), where again from these we denote by \(Q_1\) the path that is among the \(k_2 + 1\) internally vertex-disjoint paths, if one such path exists. We now claim that for \(r := \min\{r_1, r_2\}\) the paths

\[
Q_1 a \cup aP_1, \ldots, Q_r a \cup aP_r, P_{r+1}, \ldots, P_{k_1+l_1+1}, Q_{r+1}, \ldots, Q_{k_2+l_2+q}
\]

are at least \(k + l\) edge-disjoint \(s\)-\(t\) paths, of which at least \(k + 1\) are internally vertex-disjoint. First note, that the number of paths is exactly

\[
r + (k_1 + l_1 + 1) - r_1 + (k_2 + l_2 + q) - r_2 = k + l + q + r - r_1 - r_2.
\]

As \(r\) is equal to \(r_i\) for some \(i\) and \(q\) is greater or equal to \(r_1\) and \(r_2\) we get that the number of paths is at least \(k + l\). To see that among the paths above are at least \(k + 1\) internally vertex-disjoint paths, note that we started off with a set of \(k_1 + 2 + k_2 + 1 = k + 2\) internally vertex-disjoint paths \(P \subseteq \{P_1, \ldots, P_{k_1+l_1+1}, Q_1, \ldots, Q_{k_2+l_2+q}\}\).

The only vertex that, besides \(s\) and \(t\) may be contained in more than one path of \(P\) is \(a\). If \(Q_1, P_1 \in P\) they are glued together and \(k + 1\) internally vertex-disjoint paths still remain. If only one of \(P_1\) and \(Q_1\), say \(P_1\), is among the internally vertex-disjoint paths, then \(P \setminus \{P_1\}\) is a set of \(k + 1\) internally vertex-disjoint paths, as only one other path than \(P_1\) may contain \(a\). Finally if neither \(P_1\) nor \(Q_1\) are among the internally vertex-disjoint paths, then \(P\) contains a subset of internally vertex-disjoint paths of size \(k + 1\) as at most two paths in \(P\) may contain \(a\).

This concludes Case 1.
Case 2: The vertex $a$ is not contained in any $s$-$t$ disconnecting pair of order $k$ and size $l$.

Denote by $(W, F)$ an $s$-$t$ disconnecting pair of order $k$ and size $l$ and for $i \in \{1, 2\}$ let $W_i = V(G_i) \cap W$, $k_i = |W_i|$, $F_i = E(G_i) \cap E$ and by $l_i = |F_i|$. Then $k_1 + k_2 = k$ and $l_1 + l_2 = l$. Without loss of generality we may assume that there is no $s$-$a$ path in $G - W - F$ and thereby also no $s$-$a$ path in $G - W_i - F_i$ for $i \in \{1, 2\}$.

Further, for $i \in \{1, 2\}$ denote by $0 \leq q_i \leq l_i$ the unique integer such that $(k_i, l_i - q_i)$ is a connectivity pair for $s$ and $t$ in $G_i$. Note that this is well defined as as $(W_i, F_i)$ is an $s$-$t$ disconnecting pair in $G_i$. We define $q = \max\{q_1, q_2\}$ and assume without loss of generality that $q = q_1$. Let $(W'_1, F'_1)$ be an $s$-$t$ disconnecting pair in $G_1$ of order $k_1$ and size $l_1 - q$ and denote by $H_1$ the graph arising from $G_1$ by adding $q$ edges $e_1, \ldots, e_q$ between $a$ and $t$.

Claim 4: $(k_1, l_1)$ is a connectivity pair for $s$ and $t$ in $H_1$.

Proof. If $q = 0$ the claim holds true by definition of $q$. So assume $q \geq 1$. Clearly $(W'_1, F'_1 \cup \{e_1, \ldots, e_q\})$ is an $s$-$t$ disconnecting pair in $H_1$ of order $k_1$ and size $l_1$. So suppose there exists an $s$-$t$ disconnecting pair of order $k_1$ and size at most $l_1 - 1$. Let $(W', F')$ be such a pair of minimal size. If $e_1, \ldots, e_q \in F'$ the pair $(W', F' \setminus \{e_1, \ldots, e_q\})$ is $s$-$t$ disconnecting in $G_1$ and of order $k_1$ and size at most $l_1 - q - 1$, contradicting the fact that $(k_1, l_1 - q)$ is a connectivity pair for $s$ and $t$ in $G_1$. Thus, either $a \in W'$ or $a$ and $t$ are contained in the same component in $H_1 - W' - F'$. In particular there is no $s$-$a$ path in $G_1 - W' - F'$ and thereby the pair $(W' \cup W_2, F' \cup F_2)$ is $s$-$t$ disconnecting in $G$ and of order $k$ and size at most $l - 1$. A contradiction to $(k, l)$ being a connectivity pair for $s$ and $t$ in $G$.

Let now $H_2$ be the graph arising from $G_2$ by adding $q$ edges $f_1, \ldots, f_q$ between $a$ and $s$.

Claim 5: $(k_2, l_2 + q)$ is a connectivity pair for $H_2$.

Proof. Again, if $q = 0$ the claim is immediate by definition of $q$. So let $q \geq 1$. It is easy to see that $(W_2, F_2 \cup \{f_1, \ldots, f_q\})$ is an $s$-$t$ disconnecting pair of order $k_2$ and size $l_2 + q$ in $H_2$. So suppose there exists an $s$-$t$ disconnecting pair of order $k_2$ and size at most $l_2 + q - 1$. Let $(W', F')$ be such a pair of minimal size. If $f_1, \ldots, f_q \in F'$, then there is no $s$-$a$ path in $H_2 - W' - F'$. This implies that $(W' \cup W_1, F' \setminus \{f_1, \ldots, f_q\} \cup F_1)$ is a disconnecting pair in $G$ of order at most $k$ and size at most $l - 1$. A contradiction. Thus, either $a \in W'$ or $a$ and $s$ are contained in the same component in $H_2 - W' - F'$. In particular there is no $a$-$t$ path in $G_2 - W' - F'$. If there is also no $a$-$t$ path in $G_1 - W'_1 - F'_1$, the pair $(W' \cup W_1, F' \cup F'_1)$ is disconnecting in $G$ and of order at most $k$ and size at most $l - 1$. A contradiction to $(k, l)$ being a connectivity pair for $s$ and $t$ in $G$. So suppose that there is an $a$-$t$ path in $G_1 - W'_1 - F'_1$. Then there is no $s$-$a$ path in $G_1 - W'_1 - F'_1$ and $(W'_1 \cup W_2, F'_1 \cup F_2)$ is an $s$-$t$ disconnecting pair in $G$, that has order at most $k$ and size at most $l_1 - q + l_2 < l$ as $q \geq 1$. Again this contradicts the fact that $(k, l)$ is a connectivity pair for $s$ and $t$ in $G$. □
As in the proof of Case 1 we use the induction hypothesis on $H_1$ and $H_2$ to get the desired paths. If neither $l_1 = 0$ nor $l_2 + q = 0$ we get the paths in $G$ in the same manner as in Case 1 and therefore do not repeat the arguments here.

For the other case, let $l'_1 = l_1$ and $l'_2 = l_2 + q$. If we can show for $i \in \{1, 2\}$, that if $l'_i = 0$, then in $G_i - a$ the pair $(k_i, 0)$ is a connectivity pair, we can again proceed as in Case 1 and get the desired paths. To see this we simply observe that $a$ is not contained in any $k_i$-vertex separator in $G_i$ as this would imply that $a$ is contained in an $s$-$t$ disconnecting pair of order $k$ and size $l$ in $G$.

\[ \square \]

6 Conclusion

In this article we considered a form of mixed connectivity in graphs introduced by Beineke and Harary, namely connectivity pairs. First we showed that the decision version of computing the second component of such a pair is NP-complete. Then we considered the Beineke-Harary-Conjecture and proved it for $l = 2$. Finally, we presented a proof that the conjecture holds true in general for graphs with tree width at most 3.

From our studies the Beineke-Harary-Conjecture may hold:

**Conjecture** (Beineke-Harary-Conjecture). Let $G$ be a graph, $s, t \in V(G)$ distinct vertices and $k, l$ non-negative integers with $l \geq 1$. If $(k, l)$ is a connectivity pair for $s$ and $t$ in $G$, then there exist $k + l$ edge-disjoint paths, of which $k + 1$ are internally vertex-disjoint.

Acknowledgements

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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[8] Sadeghi, E. and Fan, N. (2019). On the survivable network design problem with mixed connectivity requirements. *Annals of Operations Research*. 
A Counterexample for the Claim by Sadeghi and Fan

In the paper by Sadeghi and Fan [8] for integers \( k, l \geq 1 \) a graph \( G \) with at least \( k + l + 1 \) vertices is called \((k, l)\)-connected if it cannot be disconnected by removing \( k \) vertices and \( l - 1 \) edges. The following claim is then made as Theorem 2.

\[(\star) \text{ Let } k, l \geq 1 \text{ and } G \text{ be a graph with at least } k + l + 1 \text{ vertices. Then } G \text{ is } (k, l)\text{-connected if and only if } G \text{ is } k + 1 \text{ vertex-connected and } k + l \text{ edge-connected.}\]

If \( G \) is in fact \((k, l)\)-connected it can readily be observed that it is also \( k + 1 \) vertex-connected and \( k + l \) edge-connected. On the other hand \( G \) being \( k + 1 \) vertex-connected and \( k + l \) edge-connected does not imply \((k, l)\)-connectivity.

To see this, consider the two complete graphs \(?G_1\) and \(?G_2\) on the vertex sets \(\{x_1, x_2, x_3, x_4\}\) and \(\{x_1, x_5, x_6, x_7\}\). We construct a graph \( G \) by regarding the union of \(?G_1\) and \(?G_2\) and additionally adding an edge between vertices \(x_5\) and \(x_2\).

Figure 1 displays the constructed graph. The graph \( G \) is 2-vertex-connected and 3-edge-connected, but it is not \((1, 2)\)-connected as the removal of the vertex \(x_1\) and the edge \(x_2x_5\) disconnects the graph. Thus, \((\star)\) cannot be true.

As a corollary of \((\star)\), Sadeghi and Fan claim the following:

\[\text{(\star\star) Let } k \geq 0, \ l \geq 1 \text{ and } G \text{ be a graph with at least } k + l + 1 \text{ vertices. Then } G \text{ is } (k, l)\text{-connected if and only if } G \text{ has } k + l \text{ edge-disjoint paths between every pair of vertices of which } k + 1 \text{ paths are internally vertex-disjoint.}\]

In the original conjecture by Beineke and Harary [1] and in the extension due to [6] it is never claimed that the existence of the desired paths is also sufficient for \((k, l)\)-connectivity. And in fact it is not. As a corollary of \((\star)\) the claim cannot be considered proven. Further Sadeghi and Fan do not give sufficient arguments why \((\star\star)\) would hold, even provided that \((\star)\) holds. For the sake of completeness we argue why the existence of the paths in \((\star\star)\) is not sufficient.

Theorem 23. There exists a graph with 7 vertices containing a separator of one vertex and one edge such that between any pair of vertices there exist three edge-disjoint paths two of which are internally vertex-disjoint.

Proof. Consider the graph \( G \) above, that also provided a counterexample to \((\star)\), see Figure 1. Again the vertex \( x_1 \) together with the edge \( x_5x_2 \) disconnects the graph. Now let \( v_1, v_2 \in V(G) \). If \( v_1, v_2 \in V(G_i) \) for some \( i \in \{1, 2\} \) there are

Figure 1: A counterexample to Sadeghi’s and Fan’s claim.

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three internally vertex-disjoint $v_1$-$v_2$ paths. Otherwise, without loss of generality $v_1 \in \{x_2, x_3, x_4\}$ and $v_2 \in \{x_5, x_6, x_7\}$. Denote by $P_1$ a shortest path from $v_1$ to $x_2$ (This is either a single edge or the path without edges) and by $P_2$ a shortest path from $x_5$ to $v_2$. We define the path $P = P_1 \cup P_2$. Further let $Q = v_1x_1v_2$. Finally let $w_1 \in \{x_3, x_4\} \setminus \{v_1\}$ and $w_2 \in \{x_6, x_7\} \setminus \{v_2\}$ and define the path $R = v_1w_1x_1w_2v_2$. It is easily verified that $P$, $Q$, $R$ are three edge-disjoint $v_1$-$v_2$ paths, two of which are internally vertex-disjoint.