Matrix patterns with bounded saturation function

Benjamin Aram Berendsohn\textsuperscript{*}

January 1, 2021

A 0-1 matrix $M$ contains a 0-1 matrix pattern $P$ if we can obtain $P$ from $M$ by deleting rows and/or columns and turning arbitrary 1-entries into 0s. The saturation function $\text{sat}(P, n)$ for a 0-1 matrix pattern $P$ indicates the minimum number of 1s in an $n \times n$ 0-1 matrix that does not contain $P$, but changing any 0-entry into a 1-entry creates an occurrence of $P$. Fulek and Keszegh recently showed that the saturation function is either bounded or in $\Theta(n)$. Building on their results, we find a large class of patterns with bounded saturation function, including both infinitely many permutation matrices and infinitely many non-permutation matrices.

1. Introduction

In this paper, all matrices are 0-1 matrices. For a cleaner presentation, we write matrices with dots (\textbullet) instead of 1s and spaces instead of 0s, for example:

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} = (\textbullet \textbullet \textbullet)
\]

In line with this notation, we call a row or column empty if it only contains 0s. Furthermore, we refer to changing an entry from 0 to 1 as adding a 1-entry, and to the reverse as removing a 1-entry.

We index matrices as follows: The entry $(i, j)$ is in the $i$-th row (from top to bottom) and the $j$-th column (from left to right). For example, the above matrix has 1-entries $(1, 2)$, $(2, 3)$ and $(3, 1)$.

A pattern is a matrix that is not all-zero. A matrix $M$ contains a pattern $P$ if we can obtain $P$ from $M$ by deleting rows and/or columns, and turning arbitrary 1-entries into 0s. If $M$ does not contain $P$, we say $M$ avoids $P$. Matrix pattern avoidance can be seen as a generalization of two other areas in extremal combinatorics: Pattern avoidance in

\textsuperscript{*}Institut für Informatik, Freie Universität Berlin, beab@zedat.fu-berlin.de. Work supported by DFG grant KO 6140/1-1.
permutations (see, e.g., Vatter’s survey [Vat14]), which corresponds to the case where both $M$ and $P$ are permutation matrices; and forbidden subgraphs in bipartite graphs, which corresponds to avoiding a pattern $P$ and all other patterns obtained from $P$ by permutation of rows and/or columns.

A classical question in extremal graph theory is to determine the minimum number of edges in a $n$-vertex graph avoiding a fixed pattern graph $H$. The corresponding problem in forbidden submatrix theory is determining the maximum weight (number of 1s) of a $m \times n$ matrix avoiding the pattern $P$, denoted by $\text{ex}(P, m, n)$. We call $\text{ex}(P, n) = \text{ex}(P, n, n)$ the extremal function of the pattern $P$. The study of the extremal function originates in its applications to (computational) geometry [Mit87, Für90, BG91]. A systematic study initiated by Füredi and Hajnal [FH92] has produced numerous results [Kla00, Kla01, MT04, Tar05, Kes09, Ful09, Gen09, Pet11b, Pet11a] and further applications in the analysis of algorithms have been discovered [Pet10, CGK+15].

Clearly, for non-trivial patterns, $\text{ex}(P, n)$ is at least linear and at most quadratic. Large classes of patterns with linear and quasi-linear extremal functions have been identified [Kes09, Pet11b]. On the other hand, there are patterns with nearly quadratic extremal functions [ARSz99].

A natural counterpart to the extremal problem is the saturation problem. A matrix is saturated for a pattern $P$ if it avoids $P$ and is maximal in this respect, i.e., turning any 0-entry of $M$ into a 1 creates an occurrence of $P$. Clearly, $\text{ex}(P, m, n)$ can also be defined as the maximum weight of a $m \times n$ matrix that is saturated for $P$. The function $\text{sat}(P, m, n)$ indicates the minimum weight of a $m \times n$ matrix that is saturating for $P$. We focus on square matrices and the saturation function $\text{sat}(P, n) = \text{sat}(P, n, n)$.

The saturation problem for matrix patterns was first considered by Brualdi and Cao [BC20] as a counterpart of saturation problems in graph theory. Fulek and Keszegh [FK20] started a systematic study. They proved that, perhaps surprisingly, every pattern $P$ satisfies $\text{sat}(P, n) \in \Theta(1)$ or $\text{sat}(P, n) \in \Theta(n)$. This is in stark contrast to the extremal problem. Further, they present large classes of patterns with linear saturation functions, and a single non-trivial pattern with bounded saturation function.

Most interesting for our purposes is a class of patterns we call once-separable: A pattern is once-separable if it has the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ or } \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

for two patterns $A$ and $B$, where 0 denotes an all-0 matrix of arbitrary dimensions.

**Theorem 1.1** ([FK20, Theorem 1.7]). If $P$ is once-separable, then $\text{sat}(P, n) \in \Theta(n)$.

In this paper, for the sake of simplicity, we only consider patterns with no empty rows or columns. However, we note that the saturation function, unlike the extremal function, may change considerably by the addition of an empty row or column. In particular, Fulek and Keszegh proved that if the first or last row or column of a pattern $P$ is empty, then $\text{sat}(P, n) \in \Theta(n)$.

---

1For this, we interpret the $M$ and $P$ as adjacency matrices of bipartite graphs.
Figure 1: $5 \times 5$ permutation matrices with bounded saturation function, up to rotation and reflection.

Note that if $P'$ can be obtained from $P$ by rotation, inversion$^2$, or reflection$^3$, then $\text{sat}(P, n) = \text{sat}(P', n)$.

**Permutation matrix patterns.** In this paper, we give special attention to permutation matrix patterns. A permutation matrix is a square matrix where every row and every column contains exactly one 1-entry. Theorem 1.1 already covers the once-separable permutation matrices.

We call the 1-entries in the first or last row or column the *outer* 1-entries. It is easy to see that a not-once-separable permutation matrix cannot have a 1-entry in one of its corners. As such, up to reflection, the outer four 1-entries form one of the patterns $Q_0$ and $Q_1$, where

$$Q_0 = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}, \quad Q_1 = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}.$$

In particular, all $3 \times 3$ permutation matrices are once-separable, and $Q_1$ is the only $4 \times 4$ pattern that is not once-separable. The $5 \times 5$ not-once-separable matrices are shown in Figure 1. Fulek and Keszegh already proved that $Q_2$ has bounded saturation function, and ask whether the same is true for $Q_1$.

Call a permutation matrix $Q_0$-like ($Q_1$-like) if the outer 1-entries form $Q_0$ (respectively, $Q_1$). We prove that all $Q_1$-like permutation matrices have bounded saturation function.

**Theorem 1.2.** Let $P$ be a $Q_1$-like $k \times k$ permutation matrix. Then $\text{sat}(P, n) \in \mathcal{O}(1)$.

This covers the pattern $Q_1$ (thus answering the question of Fulek and Keszegh) and the patterns $Q_2$, $Q_3$, and $Q_4$ in Figure 1. For permutation matrices of size at most 6, we obtain a full characterization of the saturation functions with the following theorem.

**Theorem 1.3.** Let $P$ be a not-once-separable $k \times k$ permutation matrix with $k \leq 6$. Then $\text{sat}(P, n) \in \mathcal{O}(1)$.

---

$^2$Swapping the role of rows and columns.

$^3$Reversing all rows or all columns.
Other patterns. We call a pattern non-trivial if it has two rows that only have a 1 in the leftmost (respectively rightmost) position, and two columns which only have a 1 in the topmost (respectively bottommost) position. Otherwise, we call the pattern trivial. Fulek and Keszegh show that each trivial pattern has linear saturation function [FK20, Theorem 1.11]. Note that every permutation matrix is non-trivial.

\[
\begin{pmatrix} 
1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
\end{pmatrix}, \quad \begin{pmatrix} 
1 & 0 & 0 & \ldots \\
0 & 0 & 1 & \ldots \\
0 & 1 & 0 & \ldots \\
\end{pmatrix}.
\]

Figure 2: A non-trivial pattern (left), and a trivial pattern (right).

Our techniques easily generalize to a more general class of non-trivial patterns (in fact, we only prove them in the general form). We restrict ourselves to the patterns without empty rows or columns where the first and last row and column each contain only a single 1-entry. Since the case of once-separable patterns is already solved, this again leaves us with patterns where the outer 1-entries form either $Q_0$ or $Q_1$ (up to reflection). We extend our previous definitions as follows: An arbitrary pattern is called $Q_0$-like ($Q_1$-like) if it has no empty rows and columns, and exactly four outer entries that form an occurrence of $Q_0$ (respectively, $Q_1$). We prove a generalization of Theorem 1.2.

Theorem 1.4. Let $P$ be a non-trivial $Q_1$-like $k \times k$ pattern. Then $\text{sat}(P, n) \in O(1)$.

We prove Theorem 1.4 (which implies Theorem 1.2) and Theorem 1.3 in Section 3. All our results are based on the construction of a witness, a concept introduced by Fulek and Keszegh. In Section 2, we formalize and develop this notion, based on the proof by Fulek and Keszegh that $Q_2$ has bounded saturation function.

2. Witnesses

Let $P$ be a matrix pattern without empty rows or columns. An explicit witness (called simply witness by Fulek and Keszegh [FK20]) for $P$ is a matrix $M$ that is saturated for $P$ and contains at least one empty row and at least one empty column. Clearly, if $\text{sat}(P, n) \in O(1)$, then $P$ has an explicit witness. Fulek and Keszegh note that the reverse is also true: We can replace an empty row (column) by an arbitrary number of empty rows (columns), and the resulting arbitrarily large matrix will still be saturating for $P$.

As such, an $m_0 \times n_0$ explicit witness for $P$ of weight $w$ implies that $\text{sat}(P, m, n) \leq w$ for each $m \geq m_0$ and $n \geq n_0$.

We call a row (column) of a matrix $M$ expandable w.r.t. $P$ if the row (column) is empty and adding a single 1-entry anywhere in that row (column) creates a new occurrence of $P$ in $M$. A explicit witness for $P$ is thus a saturated matrix with at least one expandable row and an expandable column w.r.t. $P$. We define a witness for $P$ (used implicitly by

\[\text{Note that it is critical here that } P \text{ has no empty rows or columns. Otherwise, increasing the number of empty rows or columns in } M \text{ might create an occurrence of } P.\]
Fulek and Keszegh also considered the asymptotic behavior of the functions $\text{sat}(P, m_0, n)$ and $\text{sat}(P, m, n_0)$, where $m_0$ and $n_0$ are fixed. The dichotomy of $\text{sat}(P, n)$ also holds in this setting:

**Theorem 2.2** ([FK20, Parts of Theorem 1.3]). For every pattern $P$, and constants $m_0, n_0$,

(i) either $\text{sat}(P, m_0, n) \in O(1)$ or $\text{sat}(P, m_0, n) \in \Theta(n)$;

(ii) either $\text{sat}(P, m, n_0) \in O(1)$ or $\text{sat}(P, m, n_0) \in \Theta(m)$.

We can adapt the notion of witnesses in order to classify $\text{sat}(P, m_0, n)$ and $\text{sat}(P, m, n_0)$. Let $P$ be a matrix pattern without empty rows or columns. A horizontal (vertical) witness for $P$ is a matrix $M$ that avoids $P$ and contains an expandable column (row).\(^5\) Clearly, $P$ has a horizontal witness with $m_0$ rows if and only if $\text{sat}(P, m_0, n)$ is bounded; and $P$ has a vertical witness with $n_0$ columns if and only if $\text{sat}(P, m, n_0)$ is bounded. Further note that $M$ is a witness for $P$ if and only if $M$ is horizontal witness and a vertical witness.

Observe that rotation and inversion of $P$ may affect the functions $\text{sat}(P, m_0, n)$ or $\text{sat}(P, m, n_0)$, but reflection does not.

**Lemma 2.3.** Let $P$ be a matrix pattern without empty rows or columns, and only one entry in the last row (column). Let $W$ be a horizontal (vertical) witness for $P$. Then, appending an empty row (column) to $W$ again yields a horizontal (vertical) witness.

**Proof.** We prove the lemma for horizontal witnesses, and appending a row. The other case follows by symmetry. Let $W$ be a $m_0 \times n_0$ horizontal witness for $P$, where the $j$-th column of $W$ is expandable. Let $W'$ be a matrix obtained by appending a row. Clearly, $W'$ still does not contain $P$. Moreover, adding an entry in $W'$ at $(i, j)$ for any $i \neq n_0 + 1$

\(^5\)A horizontal witness can be expanded horizontally, a vertical witness can be expanded vertically.
creates a new occurrence of $P$. It remains to show that adding an entry at $(n_0 + 1, j)$ creates an occurrence of $P$.

We know that adding an entry at $(n_0, j)$ in $W'$ creates an occurrence of $P$. Let $I$ the set of positions of 1-entries in $W(P)$ that form the occurrence of $P$. Since $P$ has only one entry in the last row, all positions $(i', j') \in I \setminus \{(n_0, j)\}$ satisfy $i' < n_0 + 1$. Thus, adding a 1-entry at $(n_0 + 1, j)$ instead of $(n_0, j)$ creates an occurrence of $P$ at positions $I \setminus \{(n_0, j)\} \cup \{(n_0 + 1, j)\}$, which implies that $W'$ is a horizontal witness.

We now prove the following handy lemma, that allows us restrict our attention to the classification of sat($P, m_0, n$) and sat($P, m, n_0$). It essentially is a generalization of the technique used by Fulek and Keszegh to prove that sat($Q_2, n$) ∈ $O(1)$.

**Lemma 2.4.** Let $P$ be a not-once-separable pattern without empty rows or columns, and with only one 1-entry in the last row and one 1-entry in the last column. Then sat($P, n$) ∈ $O(1)$ if and only if there exist constants $m_0, n_0$ such that sat($P, m_0, n$) ∈ $O(1)$ and sat($P, m, n_0$) ∈ $O(1)$.

**Proof.** Suppose that sat($P, n$) ∈ $O(1)$. Then $P$ has a $m_0 \times n_0$ witness $M$, and thus sat($P, m_0, n$) is at most the weight of $M$, for every $n \geq n_0$. Similarly, sat($P, m, n_0$) ∈ $O(1)$.

Now suppose that sat($P, m_0, n$) ∈ $O(1)$ and sat($P, m, n_0$) ∈ $O(1)$. Then, for some $m_1, n_1$, there exists a $m_0 \times n_1$ horizontal witness $W_H$ and a $m_1 \times n_0$ vertical witness $W_V$. Consider the following $(m_0 + m_1) \times (n_0 + n_1)$ matrix, where $0_{m \times n}$ denotes the all-0 $m \times n$ matrix:

$$W = \begin{pmatrix} 0_{m_0 \times n_0} & W_H \\ W_V & 0_{m_1 \times n_1} \end{pmatrix}$$

We first show that $W$ does not contain $P$. Suppose it does. Since $P$ is contained neither in $W_H$ nor in $W_V$, an occurrence of $P$ in $W$ must contain 1-entries in both the bottom left and top right quadrant. But then $P$ must be once-separable, a contradiction.

By Lemma 2.3, $W_V = (W_V, 0_{m_1 \times n_1})$ is a vertical witness, and $W_H = (0_{m_0 \times n_1}, W_H)$ is a horizontal witness. The expandable row in $W_V$ and the expandable column in $W_H$ are both also present in $W$. This implies that $W$ is a witness for $P$, so sat($P, n$) ∈ $O(1)$. □

Figure 3 shows an example of a witness for $Q_1$, constructed with Lemma 2.4, using vertical/horizontal witnesses presented later in Section 3, and an explicit witness constructed using Lemma 2.1.

For certain classes of patterns closed under rotation or inversion, we can further restrict our attention to only vertical witnesses.

**Lemma 2.5.** Let $\mathcal{P}$ be a class of not-once-separable patterns without empty rows or columns, and with only one 1-entry in the last row and one 1-entry in the last column. If $\mathcal{P}$ is closed under rotation or inversion and each pattern in $\mathcal{P}$ has a vertical witness, then sat($P, n$) ∈ $O(1)$ for each $P \in \mathcal{P}$.
Figure 3: A witness (left) and an explicit witness (right) for the pattern $Q_1$. The small dots indicate the expandable row/column.

\[
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{pmatrix}
\]

Figure 4: Construction of $W(Q_1)$ from $Q_1$. The small dots indicate the expandable row.

Proof. By Lemma 2.4, it suffices to show that each pattern in $\mathcal{P}$ has a horizontal witness. Let $P \in \mathcal{P}$ and let $P' \in \mathcal{P}$ be obtained by rotating $P$ by 90 degrees clockwise (respectively, by inverting $P$). Let $W'$ be a vertical witness for $P'$, and let $W$ be obtained by rotating $W'$ by 90 degrees counterclockwise (respectively, by inverting $W'$). Clearly, $W$ is a horizontal witness for $P$. Lemma 2.4 concludes the proof. \qed

3. A simple witness construction

We present a construction that yields vertical witnesses for certain non-trivial matrices with only one 1-entry in the first and last column. Figure 4 shows an example of the construction. The idea is simple: Make two copies $P_1$ and $P_2$ of $P$, and arrange them in a way that the rightmost 1-entry of $P_1$ coincides with the leftmost 1-entry of $P_2$ (increase the matrix size as necessary, without creating empty rows or columns). Then, delete the column where $P_1$ and $P_2$ intersect. Note that this creates an empty row, which formerly contained the intersection of $P_1$ and $P_2$. Adding a 1-entry in that row creates an occurrence of $P$ by either completing $P_1$ or $P_2$, so that row is expandable. If now the constructed matrix also avoids $P$ (which is not necessarily the case), then it is a vertical witness for $P$. We now proceed with the formal definition and proof.

Let $P = (p_{i,j})_{i,j}$ be a $k \times k$ pattern with exactly one entry in the first column and exactly one entry in the last column. Let $s$ and $t$ be the rows of the leftmost and rightmost 1-entry in $P$, i.e., $p_{s,1} = 1$ and $p_{t,k} = 1$. Without loss of generality, assume that $s < t$. We define the $(k + t - s) \times (2k - 2)$ matrix $W(P) = (w_{i,j})_{i,j}$ as follows:

\[
w_{i,j} = \begin{cases} 
p_{i,j}, & \text{if } j < k, i \leq k \\
p_{i-(t-s),j-(k-2)}, & \text{if } j \geq k, i \geq (t-s) + 1 \\
0, & \text{otherwise.}
\end{cases}
\]
Lemma 3.1. Let \( P \) be a non-trivial pattern without empty rows and columns and with exactly one entry in the first and last column. If \( W(P) \) avoids \( P \), then \( W(P) \) is a vertical witness for \( P \).

Proof. Since \( P \) is non-trivial, the \( s \)-th and \( t \)-th rows of \( P \) each only contain one entry, so the \( t \)-th row of \( W(P) \) is empty. It remains to show that adding a 1-entry in the \( t \)-th row of \( W(P) \) creates a new occurrence of \( P \).

Let \( M \) be the matrix obtained by adding a 1-entry \((t, u)\) in \( W(P) \). If \( u \leq k - 1 \), we remove the first \( t - s \) rows and all columns other than the \( u \)-th and the last \( k - 1 \). The result is \( P \) with an additional 1-entry in the first column (which was the \( u \)-th column in \( M \)). If \( u > k - 1 \), we remove the last \( t - s \) rows and all columns except the \( u \)-th and the first \( k - 1 \). The result is \( P \) with an additional 1-entry in the last column.

3.1. Non-trivial \( Q_1 \)-like patterns

Lemma 3.2. Let \( P \) be a non-trivial \( Q_1 \)-like pattern. Then \( W(P) \) avoids \( P \).

Proof. Suppose \( W(P) \) contains an occurrence of \( P \), say at positions \( I \). Consider the bottommost and topmost positions \((i_B, j_B), (i_T, j_T) \in I \). Since \( P \) is \( Q_1 \)-like, we have \( j_B < j_T \). Moreover, \( i_T - i_B \geq k - 1 \) (since \( P \) has \( k - 2 \) rows between the bottommost and topmost 1-entry).

Consider first the case that \( j_B \leq k - 1 \). Then, by construction, \( i_B \leq k \), which implies that \( I \) is completely contained in the first \( k \) rows, including the empty \( t \)-th row. However, an occurrence of \( P \) must have 1-entries in \( k \) distinct rows, a contradiction.

Second, consider the case that \( j_B > k - 1 \). Then \( j_T > k \). By construction, this implies that \( i_T \geq t - s + 1 \). Since \( W(P) \) has \( t - s + k \) rows in total, \( I \) is contained in the last \( k \) rows, including the empty \( t \)-th row. This is again a contradiction.

The class of non-trivial \( Q_1 \)-like patterns is closed under rotation, so Lemma 2.5 and Lemma 3.1 imply Theorem 1.4.

Theorem 1.4. Let \( P \) be a non-trivial \( Q_1 \)-like \( k \times k \) pattern. Then \( \text{sat}(P, n) \in \mathcal{O}(1) \).

3.2. Some \( Q_0 \)-like permutation matrix patterns

One can manually check that \( W(P) \) avoids \( P \) even for many \( Q_0 \)-like patterns, such as \( Q_5 \). We refine Lemma 3.2 to cover more patterns, including \( Q_5 \) and all but four of the not-once-separable \( 6 \times 6 \) permutation matrices, up to reflection. For three of the remaining patterns, we individually show that \( W(P) \) yields a witness. For the last pattern, we construct a witness by modifying \( W(P) \) slightly. This shows that every \( Q_0 \)-like permutation matrix of size at most 6 has a vertical witness. Since these patterns are closed under inversion, they all have bounded saturation function. Together with Theorem 1.4, we obtain:

Theorem 1.3. Let \( P \) be a not-once-separable \( k \times k \) permutation matrix with \( k \leq 6 \). Then \( \text{sat}(P, n) \in \mathcal{O}(1) \).
Let $I = \{(i_B, j_B), (i_T, j_T)\}$ be an occurrence of $(\bullet \bullet)$ in some matrix, i.e., two 1-entries with $i_B > i_T$ and $j_B < j_T$. We define the **height** if $I$ as $i_T - i_B + 1$, the number of rows containing an entry in $I$ or between the two entries of $I$.

We first consider $Q_0$-like patterns that contain an occurrence of $(\bullet \bullet)$ with height $k - 1$, which, among others, covers all but four permutation matrices of size at most 6. Observe that permutation matrices of this type can be thought of as *almost* $Q_1$-like: Removing the top (or bottom) row and then the new empty column creates a $Q_1$-like permutation matrix. We first prove some facts about occurrences of $(\bullet \bullet)$ in $W(P)$.

**Lemma 3.3.** Let $P$ be a non-trivial $Q_0$-like $k \times k$ pattern. Then each occurrence $I$ of $(\bullet \bullet)$ in $W(P)$ has height at most $k - 1$.

**Proof.** Suppose there is an occurrence $I = \{(i_B, j_B), (i_T, j_T)\}$ of $(\bullet \bullet)$ in $W(P)$ of height at least $k$, i.e., $j_B < j_T$ and $i_B - i_T \geq k - 1$. We claim that $I$ is completely contained in one of the two partial copies of $P$ in $W(P)$, i.e., either $j_B < j_T \leq k - 1$ or $k - 1 < j_B < j_T$. This implies that there is also a height-$k$ occurrence of $(\bullet \bullet)$ in $P$, which contradicts the assumption that $P$ is $Q_0$-like. It remains to show our claim.

Let $s$ and $t$ be the rows of the leftmost and rightmost 1-entry in $P$. Towards our claim, suppose on the contrary that $j_B \leq k - 1 < j_T$. Then $i_T \geq (t - s) + 1$ by construction, and thus $i_B \geq k + (t - s) > k$. But then $(i_B, j_B)$ cannot be a 1-entry, a contradiction. \hfill \Box

**Lemma 3.4.** Let $P$ be a non-trivial $Q_0$-like $k \times k$ pattern. Then each occurrence $I$ of $(\bullet \bullet)$ in $W(P)$ with height $k - 1$ has the empty row between its two entries.

**Proof.** Let $s$ and $t$ be the rows of the leftmost and rightmost 1-entry in $P$, so $W(P)$ is a $(k + t - s) \times (2k - 2)$ matrix where the $t$-th row is empty. Since $P$ is $Q_0$-like, we have $s \geq 2$ and $t \leq k - 1$. Consider an occurrence $I = \{(i_B, j_B), (i_T, j_T)\}$ of $(\bullet \bullet)$ in $W(P)$ where $i_B - i_T = k - 2$. We have

\[
i_T = i_B - k + 2 \leq k + t - s - k + 2 = t - s + 2 \leq t, \quad \text{and} \quad i_B = k - 2 + i_T \geq k - 1 \geq t.
\]

Since the $t$-th row is empty, we also have $i_T \neq i_B$, and thus $i_T < t < i_B$. \hfill \Box

**Proposition 3.5.** Let $P$ be a non-trivial $Q_0$-like $k \times k$ pattern that contains an occurrence $(\bullet \bullet)$ of height $k - 1$. Then $W(P)$ avoids $P$.

**Proof.** Suppose that $P$ is contained in $W(P)$. Then $W(P)$ must contain an occurrence $I$ of $(\bullet \bullet)$, such that there are $k - 3$ non-empty rows between the two entries in $I$. This means that either $I$ has height at least $k_1$, or $I$ has height $k - 1$ and there are no empty rows between its two entries. The former is impossible by Lemma 3.3, the latter is impossible by Lemma 3.4. \hfill \Box

There are four remaining not-once-separable $Q_0$-like permutation matrices of size at most 6. Figure 5 shows them along with vertical witnesses.

**Proposition 3.6.** For $i \in \{6, 7, 8, 9\}$, the matrix $W_i$ is a vertical witness for $Q_i$.  

9
Proof. For $i \in \{6, 7, 8\}$, we simply have $W_i = W(P_i)$. Thus, it suffices to show that $W_i$ avoids $Q_i$. For $i \in \{6, 7\}$, note that $W_i$ has height $k + 1$, and therefore only $k$ non-empty rows. As such, an occurrence of $Q_i$ in $W_i$ must map the topmost 1-entry of $Q_i$ to the topmost 1-entry of $W_i$ (marked red in the figure). It is easy to see that then the 1-entry in the second column of $Q_i$ must be mapped to the second column of $W_i$ (marked blue). But then the 1-entries in the second and third column of $Q_i$ cannot be correctly mapped to 1-entries in $W_i$.

Considering $i = 8$, observe that $Q_8$ has two occurrences $I_1, I_2$ of (••) of height $k - 2 = 4$ (marked red/blue in the figure). All occurrences of (••) in $W_8$ have height at most 4, and out of the occurrences of height 4, only two, say $I'_1$ and $I'_2$ (marked red/blue), do not contain the empty row. Thus, an occurrence of $Q_8$ in $W_8$ must map $I_1, I_2$ to $I'_1, I'_2$. However, $I_1, I_2$ span overlapping rows, while $I'_1, I'_2$ do not, a contradiction.

Finally, consider $i = 9$. The matrix $W_9$ is almost equal to $W(Q_9)$; the only difference is that the entry in the 6-th column is moved one row up. Note that this entry is the highest 1-entry in the right partial copy of $Q_9$ in $W(Q_9)$. Since we only move the highest entry up, the right partial copy stays intact in some sense. In particular, adding a 1-entry in the left half of the $t$-th row will still complete an occurrence of $Q_9$. The same is true for the right half of the $t$-th row, since the left partial copy is not changed. Thus, the $t$-th row of $W_9$ is expandable.

We still have to argue that $W_9$ does not contain $Q_9$. Suppose otherwise. Observe that $Q_9$ contains exactly one occurrence $I$ of (••) of height 4 (marked in red in the figure). All occurrences of (••) in $W_9$ of height at least 4 have the empty row in between their two entries, so $I$ must be mapped to the some occurrence $I'$ of (••) in $W_9$ of height larger than 4. There are only two such occurrences (marked in red), both involving the
entry in the sixth column of $W_9$. However, the top entry in $I'$ is in the first row of $W_9$, but the top entry of $I$ is in the second row of $Q_9$, leaving no room for the top entry of $Q_9$. This means $Q_9$ is not contained in $W_9$. \hfill \Box

Propositions 3.5 and 3.6 show that each not-once-separable $Q_0$-like permutation matrix of size at most 6 has a vertical witness. As discussed at the start of this section, this implies Theorem 1.3. For convenience, we list all not-once-separable $Q_0$-like permutation matrices of size at most 6 in Appendix A.

4. Conclusion

Fulek and Keszegh [FK20] showed that the saturation function of once-separable patterns is linear. We extend their result by showing that all non-trivial $Q_1$-like patterns have bounded saturation function. In particular, this is another step towards the classification of permutation matrices, leaving only the $Q_0$-like permutation matrices. We find many more $Q_0$-like permutation matrices with bounded saturation function. This completes the classification of permutation matrices of size at most 6, showing that a permutation matrix of size at most 6 has linear saturation function if and only if it is once-separable. It seems possible that this is true for all permutation matrices.

Open Question. Is the saturation function bounded for each not-once-separable permutation matrix?

Our witness construction $W(P)$ undoubtedly works for a larger class of matrices than we identified (cf. Proposition 3.6). However, we also provide an example of a not-once-separable $Q_0$-like permutation matrix ($Q_9$) for which our construction does not yield a vertical witness. It would be interesting to precisely identify the patterns where the construction works.

Open Question. Is there a simple characterization of patterns $P$ where $W(P)$ avoids $P$?

Our results also extend to certain non-permutation matrices, but we did not consider matrices with empty rows or columns or with more than one 1-entry in either of the first or last row or column. We note, however, that Lemma 2.5 still may be useful for patterns that have only one 1-entry in the last row and only one 1-entry in the last column, but multiple 1-entries in the first row and column.
References

[ARSz99] Noga Alon, Lajos Rónyi, and Tibor Szabó. Norm-graphs: Variations and applications. *Journal of Combinatorial Theory, Series B*, 76(2):280–290, 1999.

[BC20] Richard A. Brualdi and Lei Cao. Pattern-avoiding (0,1)-matrices. *arXiv e-prints*, 2020.

[BG91] Dan Bienstock and Ervin Győri. An extremal problem on sparse 0-1 matrices. *SIAM Journal on Discrete Mathematics*, 4(1):17–27, 1991.

[CGK+15] Parinya Chalermsook, M. Goswami, L. Kozma, K. Mehlhorn, and Thatchaphol Saranurak. Pattern-avoiding access in binary search trees. *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*, pages 410–423, 2015.

[FH92] Zoltán Füredi and Péter Hajnal. Davenport-schinzel theory of matrices. *Discrete Mathematics*, 103(3):233 – 251, 1992.

[FK20] Radoslav Fulek and Balázs Keszegh. Saturation problems about forbidden 0-1 submatrices. *arXiv e-prints*, 2020.

[Ful09] Radoslav Fulek. Linear bound on extremal functions of some forbidden patterns in 0–1 matrices. *Discrete Mathematics*, 309(6):1736–1739, 2009.

[Für90] Zoltán Füredi. The maximum number of unit distances in a convex n-gon. *J. Comb. Theory, Ser. A*, 55(2):316–320, 1990.

[Gen09] Jesse T. Geneson. Extremal functions of forbidden double permutation matrices. *Journal of Combinatorial Theory, Series A*, 116(7):1235 – 1244, 2009.

[Kes09] Balázs Keszegh. On linear forbidden submatrices. *Journal of Combinatorial Theory, Series A*, 116(1):232 – 241, 2009.

[Kla00] Martin Klazar. The füredi-hajnal conjecture implies the stanley-wilf conjecture. In *Formal power series and algebraic combinatorics*, pages 250–255. Springer, 2000.

[Kla01] M Klazar. *Enumerative and extremal combinatorics of a containment relation of partitions and hypergraphs*. PhD thesis, Habilitation thesis, 2001.

[Mit87] Joseph S. B. Mitchell. Shortest rectilinear paths among obstacles. Technical report, Cornell University Operations Research and Industrial Engineering, 1987.
[MT04] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the
stanley–wilf conjecture. *Journal of Combinatorial Theory, Series A*,
107(1):153–160, 2004. 2

[Pet10] Seth Pettie. Applications of forbidden 0–1 matrices to search tree and
path compression-based data structures. In *Proceedings of the 2010 Annual ACM-
SIAM Symposium on Discrete Algorithms*, pages 1457–1467, 2010. 2

[Pet11a] S. Pettie. Generalized davenport–schinzel sequences and their 0–1 matrix
counterparts. *Journal of Combinatorial Theory, Series A*, 118(6):1863 –
1895, 2011. 2

[Pet11b] Seth Pettie. Degrees of nonlinearity in forbidden 0–1 matrix problems. *Dis-
crete Mathematics*, 311(21):2396 – 2410, 2011. 2

[Tar05] Gábor Tardos. On 0–1 matrices and small excluded submatrices. *Journal of
Combinatorial Theory, Series A*, 111(2):266 – 288, 2005. 2

[Vat14] Vincent Vatter. Permutation classes. *arXiv e-prints*, 2014. 2
A. Small permutation matrices

The following table lists all not-once-separable and $Q_0$-like permutation matrices of size at most $6 \times 6$, up to reflection. For each matrix, we reference the proof that it has bounded saturation function. Whenever Proposition 3.5 is used, the relevant occurrence of $(\cdot, \cdot)$ is highlighted in red.

| Matrix | Proposition |
|--------|-------------|
| $(\cdot, \cdot)$ | 3.5 $(Q_3)$ |
| $(\cdot, \cdot)$ | 3.5 |
| $(\cdot, \cdot)$ | 3.6 $(Q_6)$ |
| $(\cdot, \cdot)$ | 3.6 $(Q_8)$ |
| $(\cdot, \cdot)$ | 3.6 $(Q_7)$ |
| $(\cdot, \cdot)$ | 3.5 |
| $(\cdot, \cdot)$ | 3.5 |
| $(\cdot, \cdot)$ | 3.5 |
| $(\cdot, \cdot)$ | 3.5 |
| $(\cdot, \cdot)$ | 3.5 |

14