THE CONVERGENCE OF BOUNDARY EXPANSIONS AND THE ANALYTICITY OF MINIMAL SURFACES IN THE HYPERBOLIC SPACE

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Abstract. We study expansions near the boundary of solutions to the Dirichlet problem for minimal graphs in the hyperbolic space and prove the local convergence of such expansions if the boundary is locally analytic. As a consequence, we prove a conjecture by F.-H. Lin that the minimal graph is analytic up to the boundary if the boundary is analytic and the minimal graph is smooth up to the boundary.

1. Introduction

Complete minimal hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$ demonstrate similar properties as those in the Euclidean space $\mathbb{R}^{n+1}$ in the aspect of the interior regularity and different properties in the aspect of the boundary regularity. Anderson [1], [2] studied complete area-minimizing submanifolds and proved that, for any given closed embedded $(n - 1)$-dimensional submanifold $N$ at the infinity of $\mathbb{H}^{n+1}$, there exists a complete area minimizing integral $n$-current which is asymptotic to $N$ at infinity. In the case $n \leq 6$, these currents are embedded smooth submanifolds; while in the case $n \geq 7$, as in the Euclidean case, there can be closed singular set of Hausdorff dimension at most $n - 7$. Hardt and Lin [18] discussed the $C^1$-boundary regularity of such hypersurfaces. Subsequently, Lin [28] studied the higher order boundary regularity. Recently, we studied the boundary expansions of the minimal graphs in the hyperbolic space and established optimal asymptotic expansions in the context of the finite regularity. In this paper, we will study the convergence of such expansions and the analyticity of minimal graphs up to the boundary.

Assume $\Omega$ is a bounded domain in $\mathbb{R}^n$. Lin [28] studied the Dirichlet problem of the form

$$\Delta f - \frac{f_i f_j}{1 + |Df|^2} f_{ij} + \frac{n}{f} = 0 \quad \text{in } \Omega,$$

with the condition

$$f > 0 \quad \text{in } \Omega,$$

$$f = 0 \quad \text{on } \partial \Omega.$$

We note that the equation (1.1) becomes singular on $\partial \Omega$ since $f = 0$ there. If $\Omega$ is a $C^2$-domain in $\mathbb{R}^n$ with a nonnegative boundary mean curvature $H_{\partial \Omega} \geq 0$ with respect to the
inward normal of \( \partial \Omega \), then (1.1) and (1.2) admit a unique solution \( f \in C(\bar{\Omega}) \cap C^\infty(\Omega) \). Moreover, the graph of \( f \) is a complete minimal hypersurface in the hyperbolic space \( \mathbb{H}^{n+1} \) with the asymptotic boundary \( \partial \Omega \). At each point of the boundary, the gradient of \( f \) blows up and hence the graph of \( f \) has a vertical tangent plane. Han, Shen and Wang [14] proved that \( f \in C^{1,n+1}(\bar{\Omega}) \).

Geometrically, it is more interesting to discuss the regularity of the graph of \( f \) instead of the regularity of \( f \) itself. Lin [28] and Tonegawa [37] proved the following result. If \( \partial \Omega \) is \( C^{n,\alpha} \) for some \( \alpha \in (0, 1) \), then the graph of \( f \) is \( C^{n,\alpha} \) up to the boundary. If \( \partial \Omega \) is smooth, then the graph of \( f \) is smooth up to the boundary if the dimension \( n \) is even or if the dimension \( n \) is odd and the principal curvatures of \( \partial \Omega \) satisfy a differential equation of order \( n + 1 \). See also [29].

Lin [28] conjectured that the graph of \( f \) is analytic up to the boundary if \( \partial \Omega \) is analytic. In this paper, we prove this conjecture under the necessary extra assumption that the graph is smooth up to the boundary. The first result is given by the following theorem.

**Theorem 1.1.** Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^n \) with \( H_{\partial \Omega} \geq 0 \) and \( f \in C(\bar{\Omega}) \cap C^\infty(\Omega) \) be a solution of (1.1)-(1.2). Assume \( \partial \Omega \) is analytic near \( x_0 \in \partial \Omega \). Then,

1. for \( n \) even, the graph of \( f \) is analytic up to \( \partial \Omega \) near \( x_0 \);
2. for \( n \) odd, the graph of \( f \) is analytic up to \( \partial \Omega \) near \( x_0 \) if it is smooth up to \( \partial \Omega \) near \( x_0 \).

Locally near each boundary point, the graph of \( f \) can be represented by a function over its vertical tangent plane. Specifically, we fix a boundary point of \( \Omega \), say the origin, and assume that the vector \( e_n = (0, \cdots, 0, 1) \) is the interior normal vector to \( \partial \Omega \) at the origin. Then, with \( x = (x', x_n) \), the \( x' \)-hyperplane is the tangent plane of \( \partial \Omega \) at the origin, and the boundary \( \partial \Omega \) can be expressed in a neighborhood of the origin as a graph of a smooth function over \( \mathbb{R}^{n-1} \times \{0\} \), say

\[
x_n = \varphi(x').
\]

We now denote points in \( \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \) by \((x', x_n, y_n)\). The vertical hyperplane given by \( x_n = 0 \) is the tangent plane to the graph of \( f \) at the origin in \( \mathbb{R}^{n+1} \), and we can represent the graph of \( f \) as a graph of a new function \( u \) defined in terms of \((x', 0, y_n)\) for small \( x' \) and \( y_n \), with \( y_n > 0 \). In other words, we treat \( \mathbb{R}^n = \mathbb{R}^{n-1} \times \{0\} \times \mathbb{R} \) as our new base space and write \( u = u(y) = u(y', y_n) \), with \( y' = x' \). Then, for some \( R > 0 \), \( u \) satisfies

\[
\Delta u - \frac{u_i u_j}{1 + |Du|^2} u_{ij} - \frac{nu_n}{y_n} = 0 \quad \text{in } B^+_R,
\]

and

\[
u = \varphi \quad \text{on } B'_R.
\]

Lin [28] and Tonegawa [37] proved their regularity result for the graph of \( f \) by proving the corresponding regularity of \( u \). Concerning the analyticity of \( u \) up to the boundary, we have the following result.
\textbf{Theorem 1.2.} Let \( \varphi \) be an analytic function in \( B_R^+ \) and \( u \in C(\bar{B}_R^+) \cap C^\infty(\bar{B}_R^+) \) be a solution of (1.3)-(1.4). Then, for any \( r \in (0, R) \),

1. for \( n \) even, \( u \) is analytic in \( \bar{B}_r^+ \);
2. for \( n \) odd, \( u \) is analytic in \( y', y_n \) and \( y_n \log y_n \) for \( (y', y_n) \in \bar{B}_r^+ \). In addition, if \( u \) is smooth in \( \bar{B}_R^+ \), then \( u \) is analytic in \( \bar{B}_r^+ \).

Obviously, Theorem 1.2 implies Theorem 1.1 and asserts that the minimal surfaces in the hyperbolic space are analytic up to their boundary at infinity if they are smooth up to boundary.

At the first glance, it seems strange that \( y_n \log y_n \) appears in the function of \( u \). In fact, the logarithmic factor shows up in the expansion of \( u \) near the boundary \( y_n = 0 \) and is the obstruction for \( u \) being smooth up to the boundary.

In [15], we studied the expansion of \( u \) near \( y_n = 0 \). Let \( k \geq n + 1 \) be an integer and set, for \( n \) even,

\begin{equation}
  u_k = \varphi + c_2 y_n^2 + c_4 y_n^4 + \cdots + c_n y_n^n + \sum_{i=n+1}^{k} c_i y_n^i,
\end{equation}

and, for \( n \) odd,

\begin{equation}
  u_k = \varphi + c_2 y_n^2 + c_4 y_n^4 + \cdots + c_{n-1} y_n^{n-1} + \sum_{i=n+1}^{k} \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} c_{i,j} y_n^i (\log y_n)^j,
\end{equation}

where \( c_i \) and \( c_{i,j} \) are functions of \( y' \in B_R^+ \). In the summation in \( u_k \), the lowest order is \( y_n^{n+1} \) if \( n \) is even and is \( y_n^{n+1} \log y_n \) if \( n \) is odd, and the highest order is given by \( y_n^k \).

According to the pattern in this expansion, if we intend to continue to expand \( u_k \), the next term has an order of \( y_n^{k+1} \) for \( n \) even and \( y_n^k \log y_n \) for \( n \) odd. In [15], we estimated the remainder \( u - u_k \) for appropriately determined coefficients \( c_i \) and \( c_{i,j} \). The coefficient \( c_{n+1} \) in (1.5) or \( c_{n+1,0} \) in (1.6) is the coefficient of the first nonlocal term. The coefficients of lower order terms, \( c_2, \ldots, c_n \) for \( n \) even and \( c_2, \ldots, c_{n-1} \) and \( c_{n+1} \) for \( n \) odd, can be expressed explicitly in terms of \( \varphi \) and are referred to as local terms.

If \( \varphi \) is smooth, then \( u_k \) is a smooth function in \( B_R^+ \) if \( n \) is even and is not necessarily smooth in \( y_n \) if \( n \) is odd due to the presence of \( \log y_n \). The first logarithmic term is given by \( y_n^{n+1} \log y_n \) with the coefficient \( c_{n+1,1} \). It is proved in [15] that there are no logarithmic terms in \( u_k \) if \( c_{n+1,1} = 0 \). For \( n = 3 \), \( c_{4,1} = 0 \) if and only if \( \partial \Omega \) is a Willmore surface.

We prove Theorem 1.2 or more general Theorem 1.1 in two steps. In the first step, we prove that all coefficients \( c_i, c_{i,j} \) in (1.5) and (1.6) are analytic in \( B_R^+ \). The crucial part is to prove the analyticity of the first nonlocal coefficient, \( c_{n+1} \) in (1.5) or \( c_{n+1,0} \) in (1.6).

In the second step, we prove \( u_k \to u \) uniformly in \( B_r^+ \) as \( k \to \infty \), for any \( r \in (0, R) \). In this step, we follow techniques in [22], [24], [25], and [36].

Logarithmic terms in the boundary expansions also appear in other problems, such as the singular Yamabe problem in [33], the complex Monge-Ampère equations in [6], [8] and [26], and the asymptotically hyperbolic Einstein metrics in [3], [5], [7] and
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[19], and many other problems. A result similar to Theorem 1.2 holds for the singular
Yamabe problem, which is also a consequence of Theorem 4.1.

Kichenassamy [22] constructed solutions in the form of convergent series to the local
embedding of an arbitrary real analytic manifold, of even dimension \( n \), into a Ricci-flat
manifold of dimension \( n + 2 \) admitting a homothety. See also [24], [25]. For the series
(1.6), his result can be reformulated as the following: given analytic functions \( \varphi \) and
\( c_{n+1,0} \) on \( B'_R \), the series in (1.6) converges to a solution of (1.3)-(1.4) in
\( B'_R \), for any \( r \in (0,R) \). One of the main contributions in this paper is the analyticity of
\( c_{n+1,0} \) for any solution \( u \) in \( B'_R \) with an analytic boundary value \( \varphi \) on \( B'_R \).

We finish the introduction with a brief outline of the paper. In Section 2, we prove
that \( u \) is tangentially analytic by the maximum principle and a scaling argument. In
Section 3, we prove that all coefficients in (1.5) and (1.6) are analytic by studying the
expressions of those coefficients. In Section 4, we prove that \( u \) is analytic in \( y_n \) and
\( y_n \log y_n \). The proof is based on an iteration of solutions of the corresponding ordinary
differential equations. We carried out the proof for a class of quasilinear elliptic equations
more general than the equation (1.3). In Section 5, we discuss the Loewner-Nirenberg
problem briefly.

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2. The Tangential Analyticity

In this section, we discuss the analyticity along the tangential directions.

We denote by \( x = (x', t) \) points in \( \mathbb{R}^n \), with \( x_n = t \), and, set, for any constant \( r > 0 \),
\( G_r = \{ (x', t) : |x'| < r, 0 < t < r \} \).

We assume \( u \in C(\overline{G}_1) \cap C^\infty(G_1) \) satisfies
\[
A_{ij} u_{ij} + P \frac{u_t}{t} + Q \frac{u}{t^2} + N = 0 \quad \text{in} \ G_1,
\]
where \( A_{ij}, P, Q \) and \( N \) are functions of the form
\[
A_{ij} = A_{ij} \left( x', t, Du, \frac{u}{t} \right), \quad P = P \left( x', t, Du, \frac{u}{t} \right), \quad Q = Q \left( x', t, Du, \frac{u}{t} \right),
\]
and
\[
N = N \left( x', t, Du, \frac{u}{t} \right).
\]

We assume (2.1) is uniformly elliptic; namely, there exists a positive constant \( \lambda \) such
that, for any \( (x', t, p, s) \in G_1 \times \mathbb{R}^n \times \mathbb{R} \) and any \( \xi \in \mathbb{R}^n \),
\[
\lambda^{-1} |\xi|^2 \leq A_{ij}(x', t, p, s)\xi_i \xi_j \leq \lambda |\xi|^2.
\]

Concerning the solution \( u \), we always assume, for some positive constant \( C_0 \),
\[
|u| \leq C_0 t^2,
\]
and
\[
|Du| \leq C_0 t.
\]
Throughout this section, we always assume that $A_{ij}, P, Q$ and $N$ are analytic in $G_1 \times \mathbb{R}^n$. For convenience, we denote by $(x, y) \in G_1 \times \mathbb{R}^{n+1}$ the arguments of $A_{ij}, P, Q$ and $N$. We assume, for any nonnegative integers $k$ and $l$,

$$\left| D_{(x, y)}^{k+l} A_{ij} \right| + \left| D_{(x, y)}^{k+l} P \right| + \left| D_{(x, y)}^{k+l} Q \right| \leq A_0 A^{k+l}(k - 2)! (l - 2)! \quad \text{in } G_1 \times \mathbb{R}^{n+1},$$

and

$$\left| D_{(x, y)}^{k+l} N \right| \leq A_0 A^{k+l}(k - 2)! (l - 2)! \quad \text{in } G_1 \times \mathbb{R}^{n+1},$$

for some positive constants $A_0$ and $A$. Here and hereafter, $m! = 1$ for any integer $m \leq 0$.

**Theorem 2.1.** Let $A_{ij}, P, Q$ and $N$ be smooth in its arguments and satisfy (2.1) and (2.5), and $u \in C^1(G_1) \cap C^\infty(G_1)$ be a solution of (2.1) and satisfy (2.2) and (2.3). Assume, for some $c_0 > 0$,

$$2A_{nn} + 2P + Q \leq -c_0 \quad \text{in } G_1 \times \mathbb{R}^{n+1}.$$  

Then, there exists a positive constant $R \in (0, 1)$, depending only on $n$, $A_0$, $A$ and $c_0$, such that, for any $(x', t) \in G_R$ and any nonnegative integer $l$,

$$\left| D_{x'}^l u(x', t) \right| \leq B_0 B^{l(l-1)^+} (l - 1)! t^2 (R - |x'|)^{-(l-1)^+},$$

$$\left| DD_{x'}^l u(x', t) \right| \leq B_0 B^{l(l-1)^+} (l - 1)! t (R - |x'|)^{-(l-1)^+},$$

$$\left| D^2 D_{x'}^l u(x', t) \right| \leq B_0 B^{l(l-1)^+} (l - 1)! (R - |x'|)^{-(l-1)^+},$$

where $B_0$ and $B$ are positive constants depending only on $n$, $A_0$, $A$ and $c_0$.

**Proof.** We note that the equation (2.1) is elliptic wherever $t$ is positive. Hence by the interior analyticity, we assume that, for any $R \in (0, 1)$, (2.7)-(2.9) hold in $B_R \cap \{ R/2 \leq t \leq R \}$, for some constants $B_0$ and $B$.

For each positive integer $l$, define

$$T_l = \left\{ (x', t) \in G_R : t < \frac{1}{l} (R - |x'|) \right\}.$$ 

Hence, $T_l$ is a circular cone and shrinks while $l$ increases. In this way, we decompose $G_R$ into two parts $T_l$ and $G_R \setminus T_l$.

In the following, we set

$$L = A_{ij} \partial_{ij} + \frac{P}{t} \partial_t + \frac{Q}{t^2}.$$ 

By applying $D_{x'}^l$ to (2.1), we obtain

$$L(D_{x'}^l u) + N_l = 0,$$

where $N_l$ is given by

$$N_l = \sum_{m=0}^{l-1} \left( \begin{array}{c} l-1 \nonumber \end{array} \right) A_{ij} \cdot D_{x'}^{l-m} u_{ij} + D_{x'}^{l-m} P \cdot \frac{D_{x'}^m u}{t} + D_{x'}^{l-m} Q \cdot \frac{D_{x'}^m u}{t^2} + D_{x'}^l N.$$ 

Derivatives of $A_{ij}, P, Q$ and $N$ also result in derivatives of $u$. 

We now prove (2.7)-(2.9) by induction. By (2.6), we can apply Theorem 4.2 in [15] and obtain (2.7)-(2.9) for \( l = 0, 1 \) and \( R = 1 \). Let \( p \geq 2 \) be an integer and assume (2.7)-(2.9) hold for all \( l < p \).

**Step 1.** We prove (2.7) for \( l = p \) in \( G_R \). We consider the cases \( T_p \) and \( G_R \setminus T_p \) separately.

We first take an \( x_0 = (x_0', t_0) \in T_p \). Set 
\[
\rho = \frac{1}{p}(R - |x_0'|),
\]
and 
\[
G_\rho(\tilde{x}_0) = \{(x', t) : |x' - x_0'| < \rho, t \in (0, \rho)\},
\]
where \( \tilde{x}_0 = (x_0', 0) \). The definition of \( T_p \) implies \( t_0 < \rho \). Next, we take any \( (x', t) \in G_\rho(\tilde{x}_0) \). Then,
\[
(R - |x_0'|) - (R - |x'|) = |x'| - |x_0'| \leq |x' - x_0'| < \rho = \frac{1}{p}(R - |x_0'|).
\]
Hence,
\[
(2.12) \quad R - |x_0'| < \frac{p}{p-1}(R - |x'|).
\]
A similar argument yields
\[
(2.13) \quad R - |x'| < \frac{p+1}{p}(R - |x_0'|).
\]
With \( t < \rho \) in \( G_\rho(\tilde{x}_0) \), we have 
\[
t < \rho = \frac{1}{p}(R - |x_0'|) < \frac{1}{p-1}(R - |x'|).
\]
This implies 
\[
(2.14) \quad G_\rho(\tilde{x}_0) \subset T_{p-1}.
\]

Consider, for some positive constant \( \varepsilon \) to be determined, 
\[
w(x', t) = M(\varepsilon|x - x_0'|^2 + t^2).
\]
Then,
\[
(2.15) \quad Lw = M \left( 2A_{nn} + 2P + Q + 2\varepsilon \sum_{\alpha=1}^{n-1} A_{\alpha\alpha} \right).
\]
By (2.6) and taking \( \varepsilon \) small, we have 
\[
Lw \leq -\frac{1}{2}Mc_0.
\]
For simplicity, we assume \( c_0 \in (0, 1] \). Next, the definition of \( w \) implies 
\[
w \geq M\varepsilon\rho^2 \quad \text{on } \partial B_\rho'(x_0') \times (0, \rho),
\]
\[
w \geq M\rho^2 \quad \text{on } B_\rho'(x_0') \times \{\rho\}.
\]
By the induction hypotheses (2.8) for \( l = p - 1 \), we have

\[
|D^p_{x'} u(x)| \leq B_0 B^{p-2}(p - 2)! t(R - |x'|)^{-p+2}.
\]

Note that (2.12) implies, for \((x', t) \in G_\rho(\tilde{x}_0)\),

\[
(R - |x'|)^{-p+2} < \left( \frac{p-1}{p} \right)^{-p+2} (R - |x'_0|)^{-p+2} \leq c_1 (R - |x'_0|)^{-p+2},
\]

where \( c_1 \) is a positive constant independent of \( p \). Hence by the definition of \( \rho \), we get, for any \((x', t) \in G_\rho(\tilde{x}_0)\),

\[
|D^p_{x'} u(x)| \leq c_1 B_0 B^{p-2}(p - 2)! \rho (R - |x'_0|)^{-p+2} = c_1 B_0 B^{p-2}(p - 1)! \rho^2 (R - |x'_0|)^{-p+1}.
\]

In order to have \( w \geq |D^p_{x'} u| \) on \( \partial G_\rho(\tilde{x}_0) \), we need to choose, by renaming \( c_1 \),

\[
(2.16) \quad M \geq c_1 B_0 B^{p-2}(p - 1)! (R - |x'_0|)^{-p+1}.
\]

Next, we consider (2.10) and estimate (2.11) for \( l = p \). We claim that, by taking \( B \) sufficiently large depending only on \( A_0, B_0 \) and \( A \), we have, for any \((x', t) \in G_\rho(\tilde{x}_0)\),

\[
(2.17) \quad |N_{\rho}(x', t)| \leq C_1 B_0 B^{p-2}(p - 1)! (R - |x'_0|)^{-p+1},
\]

where \( C_1 \) is a positive constant depending only on \( A_0, B_0 \) and \( A \). By renaming \( C_1 \), we may require \( C_1 \geq c_1 \), for \( c_1 \) in (2.10), and \( C_1 \geq 2c_0^{-1} \), for \( c_0 \) as in (2.6). Set

\[
(2.18) \quad M = C_1 B_0 B^{p-2}(p - 1)! (R - |x'_0|)^{-p+1}.
\]

Therefore, we obtain

\[
L(\pm D^p_{x'} u) \geq Lw \quad \text{in} \ G_\rho(\tilde{x}_0),
\]

\[
\pm D^p_{x'} u \leq w \quad \text{on} \ \partial G_\rho(\tilde{x}_0).
\]

By the maximum principle, we have

\[
\pm D^p_{x'} u \leq w \quad \text{in} \ G_\rho(\tilde{x}_0),
\]

or

\[
|D^p_{x'} u| \leq w \quad \text{in} \ G_\rho(\tilde{x}_0).
\]

By taking \( x' = x'_0 \), we obtain, for any \((x'_0, t) \in G_\rho(\tilde{x}_0)\),

\[
|D^p_{x'} u(x'_0, t)| \leq Mt^2.
\]

In conclusion, by (2.18), we obtain, for any \((x', t) \in T_p\),

\[
|D^p_{x'} u(x', t)| \leq C_1 B_0 B^{p-2}(p - 1)! t^2 (R - |x'|)^{-p+1}.
\]

We now prove (2.17). In view of (2.11) with \( l = p \), we first estimate \( D^p_{x'} N \). For any \( k = 1, \cdots, p \), by taking \( l = k - 1 < p \) in the induction hypothesis (2.8) and (2.9), we have

\[
|D^k_{x'} u|^t \leq \frac{1}{t} |D^k_{x'} D u| \leq B_0 B^{(k-2)^+} (k - 2)! (R - |x'|)^{-(k-2)^+},
\]

\[
|D^k_{x'} D u| \leq |D^{k-1}_{x'} D^2 u| \leq B_0 B^{(k-2)^+} (k - 2)! (R - |x'|)^{-(k-2)^+}.
\]
By Lemma A.1 and Remark A.2, we obtain
\begin{equation}
|D^p_{x'} N| \leq \tilde{B}_0 B^{p-2}(p-2)! (R - |x'|)^{-(p-2)}.
\end{equation}

Next, we estimate terms involving $A_{ij}$ in (2.11), i.e.,
\begin{equation}
I = \sum_{m=0}^{p-1} \binom{p}{m} D^{p-m}_{x'} A_{ij} \partial_{ij} D^m_{x'} u.
\end{equation}

Similar as (2.19), we have, for any $k = 0, 1, \ldots, p$,
\[ |D^k_x A_{ij}| \leq \tilde{B}_0 B^{(k-2)^+} (k - 2)! (R - |x'|)^{-(k-2)^+}. \]

In expanding the summation in $I$, we consider $m = 0, 1, p-1$ separately. By the induction hypotheses (2.10) for $l < p$, we have
\begin{align*}
|I| &\leq \tilde{B}_0 B^{p-2}(p-2)! (R - |x'|)^{-p+2} + \tilde{B}_0 B^{p-3}(p-3)! (R - |x'|)^{-p+3} \\
&+ \tilde{B}_0 B^{p-3}(p-1)! (R - |x'|)^{-p+3} \sum_{m=2}^{p-2} \frac{p}{m(p-m)(p-m-1)} \\
&+ \tilde{B}_0 B^{p-2}(p-2)! (R - |x'|)^{-p+2}.
\end{align*}

We note that the last term in the right-hand side above has the order $B^{p-2}(p-1)!$. A straightforward calculation yields
\[ |I| \leq B_1 B_0 B^{p-2}(p-1)! (R - |x'|)^{-p+1}. \]

We have similar results for other terms in $N_p$ by employing (2.7) and (2.8) for $l < p$. Therefore, we obtain (2.17).

Next, we take $(x', t) \in G_R \setminus T_p$. By the induction hypotheses (2.8) for $l = p-1$, we have
\[ |D^{p-1}_x u(x', t)| \leq B_0 B^{p-2}(p-2)! t (R - |x'|)^{-p+2}. \]

Note $R - |x'| \leq pt$ in $G_R \setminus T_p$. Then,
\[ |D^p_x u(x', t)| \leq \frac{p}{p-1} B_0 B^{p-2}(p-1)! t^2 (R - |x'|)^{-p+1}. \]

By combining the both cases for points in $T_p$ and $G_R \setminus T_p$, we obtain, for any $(x', t) \in G_R$,
\begin{equation}
|D^p_x u(x', t)| \leq C_1 B_0 B^{p-2}(p-1)! t^2 (R - |x'|)^{-p+1}.
\end{equation}

This implies (2.7) for $l = p$, if $B \geq C_1$. The extra factor $B^{-1}$ is for later purposes.

\textit{Step 2.} We prove (2.8) for $l = p$ in $G_R$. Again, we consider the cases $T_p$ and $G_R \setminus T_p$ separately.

Take any $x_0 = (x_0', t_0) \in T_p$ and set $\rho = t_0$. Then, $B_\rho(x_0) \subset G_R$. By a similar argument, (2.12) and (2.13) hold in $B_\rho(x_0)$. Similar to (2.17), we have, in $B_\rho(x_0)$,
\begin{equation}
|N_\rho| \leq C_1 B_0 B^{p-2}(p-1)! (R - |x_0'|)^{-p+1}.
\end{equation}
We now consider (2.10) in $B_{3\rho/4}(x_0)$ for $l = p$. Note
\[ |A_{ij}|L^{\infty}(B_{3\rho/4}(x_0)) + \rho |t^{-1}P|L^{\infty}(B_{3\rho/4}(x_0)) + \rho^2 |t^{-2}Q|L^{\infty}(B_{3\rho/4}(x_0)) \leq C. \]

We fix an arbitrary constant $\alpha \in (0, 1)$. The scaled $C^{1,\alpha}$-estimate implies
\[ \rho^\alpha |D_x^p u|C^\alpha(B_{\rho/2}(x_0)) + \rho |DD_x^p u|L^{\infty}(B_{\rho/2}(x_0)) + \rho^{1+\alpha} |DD_x^p u|C^\alpha(B_{\rho/2}(x_0)) \leq c_2 \left( |D_x^p u|L^{\infty}(B_{3\rho/4}(x_0)) + \rho^2 |N_p|L^{\infty}(B_{3\rho/4}(x_0)) \right). \]

By (2.21) and (2.22), we have
\[ \rho^\alpha |D_x^p u|C^\alpha(B_{\rho/2}(x_0)) + \rho |DD_x^p u|L^{\infty}(B_{\rho/2}(x_0)) + \rho^{1+\alpha} |DD_x^p u|C^\alpha(B_{\rho/2}(x_0)) \leq C_2 B_0 B^{p-2}(p-1)! \rho^2 (R - |x'|)^{-p+1}. \]

In particular, we get
\[ |DD_x^p u(x_0)| \leq C_2 B_0 B^{p-2}(p-1)! \rho(R - |x'|)^{-p+1}. \]

Next, we take $(x', t) \in G_R \setminus T_p$. By the induction hypotheses (2.9) for $l = p - 1$, we have
\[ |DD_x^p u(x', t)| \leq B_0 B^{p-2}(p-2)! (R - |x'|)^{-p+2}. \]

Note $R - |x'| \leq pt$ in $G_R \setminus T_p$. Then,
\[ |DD_x^p u(x', t)| \leq \frac{p}{p-1} B_0 B^{p-2}(p-1)! (R - |x'|)^{-p+1}. \]

By combining the both cases for points in $T_p$ and $G_R \setminus T_p$, we obtain, for any $(x', t) \in G_R$,
\[ (2.24) \quad |DD_x^p u(x', t)| \leq C_2 B_0 B^{p-2}(p-1)! (R - |x'|)^{-p+1}. \]

This implies (2.8) for $l = p$, if $B \geq C_2$.

Step 3. We prove (2.9) in $T_p$ for $l = p$.

As in Step 2, we take any $x_0 = (x'_0, t_0) \in T_p$ and set $\rho = t_0$. A simple calculation yields
\[ \rho^\alpha |A_{ij}|C^\alpha(B_{\rho/2}(x_0)) + \rho^{1+\alpha} |t^{-1}P|C^\alpha(B_{\rho/2}(x_0)) + \rho^{2+\alpha} |t^{-2}Q|C^\alpha(B_{\rho/2}(x_0)) \leq c_3. \]

We now consider (2.10) in $B_{\rho/2}(x_0)$ for $l = p$. The scaled $C^{2,\alpha}$-estimate implies
\[ \rho^2 |D_x^p u|L^{\infty}(B_{\rho/2}(x_0)) + \rho^2 |N_p|L^{\infty}(B_{\rho/2}(x_0)) + \rho^{2+\alpha} |N_p|C^\alpha(B_{\rho/2}(x_0)) \leq c_3. \]

By (2.21) and (2.22), we have
\[ |D_x^p u(x_0)| \leq C_3 B_0 B^{p-2}(p-1)! (R - |x'_0|)^{-p+1} + c_3 \rho^\alpha |N_p|C^\alpha(B_{\rho/2}(x_0)). \]

We claim
\[ (2.25) \quad \rho^\alpha |N_p|C^\alpha(B_{\rho/2}(x_0)) \leq C_3 B_0 B^{p-2}(p-1)! (R - |x'_0|)^{-p+1}. \]

Hence,
\[ |D_x^p u(x_0)| \leq C_3 B_0 B^{p-2}(p-1)! (R - |x'_0|)^{-p+1}. \]
By taking $B \geq C_3$, we obtain, for any $(x', t) \in T_p$,

$$|D^2D_{x'}^p u(x', t)| \leq B_0B^{p-1}(p - 1)!(R - |x'_0|)^{-p+1}.$$ 

This is (2.31) for $l = p$ in $T_p$.

We now prove (2.25) by examining $N_p$ given by (2.11) for $l = p$. We note that $N_p$ consists of two parts. The first part is given by a summation and the second part by $D_{x'}^p N$. For $D_{x'}^p N$, we have

$$\rho^a[D_{x'}^p N]_{C^\alpha(B_{p/2}(x_0))} \leq \widetilde{B}_0B^{p-2}(p - 2)!(R - |x'_0|)^{-(p-2)}.$$ 

The proof is similar to that of (2.19). We point out that Lemma A.1 still holds if the $L^\infty$-norms are replaced by $C^\alpha$-norms and the needed estimates of the $C^\alpha$ semi-norms of $DD_{x'}^l u$ and $D_{x'}^l u/t$ are provided by (2.23), for $l \leq p$. Next, we examine the summation part in $N_p$ and discuss $I$ in (2.20) for an illustration. Similar to (2.26), we have, for any $l \leq p$,

$$\rho^a[D_{x'}^l A_{ij}]_{C^\alpha(B_{p/2}(x_0))} \leq \widetilde{B}_0B^{l(l-2)^+}(l - 2)!|R - |x'_0||^{-(l-2)^+}.$$ 

We note that $I$ is a linear combination of $DD_{x'}^m u$, for $m \leq p-1$, which can be written as $DD_{x'}^m u$ for $m \leq p$ and $\partial^2 D_{x'}^m u$ for $m \leq p - 1$. We estimate these two groups separately. To do this, we first have, for any $l \leq p$,

$$\rho^a[D_{x'}^l u]_{L^\infty(B_{p/2}(x_0))} + \rho^a[D_{x'}^l u]_{C^\alpha(B_{p/2}(x_0))} + \rho^a[D_{x'}^l u]_{C^\alpha(B_{p/2}(x_0))} \leq C_3B_0B^{(l-1)^+}!(p - 1)!|R - |x'_0||^{-(l-1)^+}.$$ 

We note that (2.27) implies by (2.21) and (2.23) for $l = p$. The proof in Step 2 actually shows that (2.27) holds for all $l \leq p$. Next, we prove, for $l \leq p - 1$,

$$\rho^a[\partial^2 D_{x'}^l u]_{L^\infty(B_{p/2}(x_0))} + \rho^a[\partial^2 D_{x'}^l u]_{C^\alpha(B_{p/2}(x_0))} \leq C_3B_0B^{(l-1)^+}!(p - 1)!|R - |x'_0||^{-(l-1)^+}.$$ 

To prove (2.28), we first have, by (2.1),

$$\partial^2_t u = -\frac{N}{A_{nm}} - \sum_{0 \leq i+j \leq 2n-1} \frac{A_{ij}}{A_{nm}} \partial_{ij} u - \frac{P}{t A_{nn}} \partial_t u - \frac{1}{t^2} \frac{Q}{A_{nn}} u.$$ 

Then, for $l \leq p - 1$,

$$\partial^2_t D_{x'}^l u = -D_{x'}^l \left(\frac{N}{A_{nm}} + \sum_{0 \leq i+j \leq 2n-1} \frac{A_{ij}}{A_{nn}} \partial_{ij} u + \frac{P}{t A_{nn}} \partial_t u + \frac{1}{t^2} \frac{Q}{A_{nn}} u\right).$$ 

We analyze the summation involving $A_{ij}$. For each pair $i$ and $j$ with $i + j < 2n$, $\partial_{ij} u$ is a part of $DD_{x'}^l u$. Hence, for $l \leq p - 1$, $D_{x'}^l(A_{nn}A_{ij} \partial_{ij} u)$ is a linear combination of $DD_{x'}^m u$, for $m = 1, \ldots, p$. The $C^\alpha$-norms of these derivatives of $u$ are already estimated by (2.27). We can analyze other terms similarly. Hence, we have (2.28). As a consequence, we get

$$\rho^a[I]_{C^\alpha(B_{p/2}(x_0))} \leq C_3B_0B^{p-2}(p - 1)!(R - |x'_0|)^{-p+1}.$$
We can analyze other terms in \( N_p \) similarly. Therefore, we obtain (2.25) and finish the proof of the claim.

**Step 4.** We prove (2.9) in \( G_R \setminus T_p \) for \( l = p \). We will fix \( R \) in this step.

Take any \( x_0 = (x'_0, t_0) \in G_R \setminus T_p \), with \( t_0 \leq R/2 \). Then, \( t_0 \geq (R - |x'_0|)/p \). Set

\[
\rho = \frac{1}{2p}(R - |x'_0|).
\]

Then, \( t_0 \geq 2\rho \). Hence, for any \((x', t) \in B_\rho(x_0)\), \( t \geq t_0 - \rho \geq \rho \). We now consider (2.10) in \( B_\rho(x_0) \) for \( l = p + 1 \).

Note

\[
|a^{ij}|_{L^\infty(B_\rho(x_0))} + \rho |t^{-1}b^i|_{L^\infty(B_\rho(x_0))} + \rho^2 |t^{-2}c|_{L^\infty(B_\rho(x_0))} \leq c_4.
\]

We fix an arbitrary constant \( \alpha \in (0, 1) \). The scaled \( C^{1,\alpha} \)-estimate implies

\[
\rho |D^2 D^{p+1}_x u(x_0)| \leq c_4 \left( |D^{p+1}_x u|_{L^\infty(B_\rho(x_0))} + \rho^2 |N_{p+1}|_{L^\infty(B_\rho(x_0))} \right).
\]

By the induction hypotheses (2.9) for \( l = p + 1 \), we have

\[
|D^{p+1}_x u(x)| \leq B_0 B^{p-2} (p - 2)! (R - |x'|)^{-p+2}.
\]

By a similar argument, (2.12) and (2.13) hold in \( B_\rho(x_0) \). Hence, for any \( x = (x', t) \in B_\rho(x_0) \),

\[
|D^{p+1}_x u(x)| \leq c_4 B_0 B^{p-2} (p - 2)! (R - |x'_0|)^{-p+2}
\leq c_4 B_0 B^{p-1} (p - 1)! (R - |x'_0|)^{-p+1}.
\]

Next, we consider (2.11) for \( l = p + 1 \). In the expression of \( N_{p+1} \), we single out the term \( D^2 D^p_x u \). We note that \( D^l_x u, DD^l_x u, D^2 D^l_x u \) can be estimated by the induction hypothesis, for \( l < p \), and that \( D^p_x u, DD^p_x u \) can be estimated by Step 1 and Step 2, respectively. Hence, a similar argument as in Step 1 yields

\[
|N_{p+1}|_{L^\infty(B_\rho(x_0))} \leq (p + 1) A_0 A |D^2 D^p_x u|_{L^\infty(B_\rho(x_0))} + C_1 B_0 B^{p-2} (p - 1)! (R - |x'_0|)^{-p+1}.
\]

By a simple substitution, we have

\[
|DD^{p+1}_x u(x_0)| \leq (p + 1) A_0 A |D^2 D^p_x u|_{L^\infty(B_\rho(x_0))} + C_1 B_0 B^{p-2} (p - 1)! (R - |x'_0|)^{-p+1}.
\]

Combining with (2.29) for \( l = p \), we get

\[
|D^2 D^p_x u(x_0)| \leq (p + 1) A_0 A |D^2 D^p_x u|_{L^\infty(B_\rho(x_0))} + C_4 B_0 B^{p-2} (p - 1)! (R - |x'_0|)^{-p+1}.
\]

We now fix a constant \( \varepsilon \in (0, 1) \). By the definition of \( \rho \), we can choose \( R \) sufficiently small such that

\[
|D^2 D^p_x u(x_0)| \leq \varepsilon |D^2 D^p_x u|_{L^\infty(B_\rho(x_0))} + C_4 B_0 B^{p-2} (p - 1)! (R - |x'_0|)^{-p+1}.
\]

Next, for any \( r \in (0, R) \), we define

\[
h(r) = \sup \{|D^2 D^p_x u| : x \in G_R \setminus T_p, |x'| \leq r\}.
\]

At points in \( B_\rho(x_0) \cap T_p \), \( D^2 D^p_x u \) is already bounded in Step 3. Hence, we have, for any \( r \in (0, R) \),

\[
h(r) \leq \varepsilon h(r + p^{-1}(R - r)) + C_4 B_0 B^{p-2} (p - 1)! (R - r)^{-p+1}.
\]
By applying Lemma 2.2 below to the function $h$, we obtain, for any $r \in (0, R)$,
\[ h(r) \leq CC_4 B_0 B^{p-2}(p-1)! (R-r)^{-p+1}. \]
We now choose $B \geq CC_4$. For each $(x', t) \in G_R \setminus T_p$, we take $r = |x'|$ and then obtain
\[ |D^2 D_x^p u(x', t)| \leq B_0 B^{p-1}(p-1)! (R - |x'|)^{-p+1}. \]
This ends the proof of (2.9) in $G_R \setminus T_p$ for $l = p$.
In summary, we take $B \geq \max\{C_1, C_2, C_3, CC_4\}$. \hfill $\Box$

We need the following lemma to finish the proof of Theorem 2.1. See Lemma 2.11.

**Lemma 2.2.** Let $p$ be a positive integer, $\varepsilon \in (0, 1)$ and $M > 0$ be constants, and $h(t)$ be a positive monotone increasing function defined in the interval $[0, R]$. Assume, for any $r \in (0, R)$,
\[ h(r) \leq \varepsilon h(r + p^{-1}(R-r)) + M(R-r)^{-p}. \]
Then, for any $r \in (0, R)$,
\[ h(r) \leq CM(R-r)^{-p}, \]
where $C$ is a positive constant depending only on $\varepsilon$, independent of $p$.

The proof is by a simple iteration and hence is omitted.

For convenience, we introduce the notion of the tangential analyticity. Let $v$ be a smooth function in $G_r$ for some $r > 0$. Then, $v$ is *tangentially analytic* in $G_r$ if, for any nonnegative integer $l$ and any $(x', t) \in G_r$,
\[ |D^l_x v(x', t)| \leq B_0 B^l l!, \]
for some positive constants $B_0$ and $B$. We denote by $v \in A(G_r)$. We note that the constants $B_0$ and $B$ are allowed to depend on $v$. It is easy to verify that, if $v \in A(G_r)$, then $D^l_x v \in A(G_r)$, for any $l \geq 0$.

**Corollary 2.3.** Under the assumptions of Theorem 2.1, there holds, for any $r \in (0, 1)$,
\[ \frac{u}{l^2} \cdot \frac{\partial u}{l} \cdot \partial^2_w u \in A(\bar{G}_r). \]

**Proof.** Fix an $r_0 \in (1/2, 1)$. Under the assumptions of Theorem 2.1 we have (2.7)-(2.9) in $G_R$, for some positive constants $R \in (0, 1/2)$, $B_0$ and $B$, independent of $l$. By taking $r = R/2$, we have, for any $(x', t) \in G_r$ and any nonnegative integer $l$,
\[ |t^{-2} D^l_x u(x', t)| \leq B_0 (r^{-1} B)^l l!, \]
\[ |t^{-1} \partial_t D^l_x u(x', t)| \leq B_0 (r^{-1} B)^l l!, \]
\[ |\partial^2_w D^l_x u(x', t)| \leq B_0 (r^{-1} B)^l l!. \]
Similar estimates also hold for any $(x', t) \in B^r_r(x_0') \times (0, r)$, with $x_0'$ an arbitrary point in $B^r_r$. These estimates hold in $B^r_r \times (r, r_0)$ by the interior analyticity. Therefore, we have the desired result. \hfill $\Box$
3. The Analyticity of Coefficients

In this section, we prove the analyticity of coefficients in the expansions near the boundary. The crucial step is to prove that the coefficient of the first nonlocal term is analytic.

We start with the equation (2.1) and assume we can write it in the form

$$u_{tt} + p u_t + q \frac{u}{t^2} = F,$$

where $p$ and $q$ are constants and $F$ is a function in $x', t$ and

$$\frac{u}{t}, u_t, D_{x'}u, D_{x'}u_t, D_{x'}^2u, \frac{u^2}{t^3}, \frac{uu_t}{t^2}, \frac{u^2}{t^2}.$$

In the applications later on, $F$ is smooth in all of its arguments. In the following, we denote by $'$ the derivative with respect to $t$. This should not be confused with $x'$, the first $n-1$ coordinates of the point.

Throughout this section, we assume that $t^m$ and $t^m$ are solutions of the linear homogeneous equation corresponding to (3.1); namely,

$$p = 1 - (m + m), \quad q = m \cdot m.$$

We always assume that $m$ and $m$ are integers and satisfy

$$m \leq 0, \quad mm \geq 3.$$

We have the following simple result concerning the solution of (3.1).

Lemma 3.1. Let $u$ be a solution of (3.1) in $G_r$ satisfying

$$t^{-m}u \to 0 \quad as \quad t \to 0.$$

Then,

$$u(x', t) = \left[ u(x', r)r^{-m} + \frac{r^{-m}}{m - m} \int_0^r s^{1-m}F(x', s)ds \right] t^{m}$$

$$(3.6)$$

$$- \frac{1}{m - m} \int_0^t s^{1-m}F(x', s)ds$$

$$- \frac{1}{m - m} \int_t^r s^{1-m}F(x', s)ds.$$

Let $u \in C^1(\tilde{G}_R) \cap C^\infty(G_R)$ be a solution of (2.1) and satisfy (2.2) and (2.3). Then, $u$ admits a formal expansion of the form

$$u(x', t) = \sum_{i=2}^{m-1} c_i(x') t^i + \sum_{i=\overline{m}}^{\infty} \sum_{j=0}^{N_i} c_{i,j}(x') t^i (\log t)^j.$$

In [15], we discussed the regularity of coefficients and the estimates of remainders. In particular, if $A_{ij}, P, Q$ and $N$ are smooth in its arguments, then all coefficients $c_i$ and $c_{i,j}$ are smooth functions of $x'$. In the next result, we prove that all coefficients are analytic if $A_{ij}, P, Q$ and $N$ are analytic in its arguments,
Theorem 3.2. Let $F$ in $[3.1]$ be analytic in all of its arguments given by $[3.2]$ and $m$ and $\overline{m}$ be integers satisfying $[3.3]$ and $[3.4]$. Suppose that $u \in C^1(\overline{G}_1) \cap C^\infty(G_1)$ is a solution of $[3.1]$ satisfying $[2.30]$. Then, all $c_1$ and $c_{i,j}$ are analytic in $B'_1$.

Proof. The coefficients in $[3.7]$ can be divided into two groups. The first group consists of $c_2, \cdots, c_{m-1}$ and $c_{m,1}$, and the second group consists of the rest. Any coefficients in the first group can be expressed in terms of $c_2$ and its derivatives as well as $A_{ij}, P, Q$, and $N$, and any coefficients in the second group can be expressed in terms of $c_2, c_{m,0}$ and their derivatives as well as $A_{ij}, P, Q$, and $N$. See [15] for details. We only need to prove $c_2$ and $c_{m,0}$ are analytic in $x'$.

We first note
\[ c_2(x') = -N \frac{2A_{nn} + 2P + Q}{(x', 0)}. \]
Hence, the analyticity of $c_2$ follows from that of $A_{nn}, P, Q$ and $N$. Alternatively, we note
\[ c_2(x') = \lim_{t \to 0} \frac{u(x', t)}{t^2}. \]
Then, $c_2$ is analytic in $x'$ since $u/t^2 \in A(\overline{G}_r)$ by $[2.30]$. It remains to prove that $c_{m,0}$ is analytic in $x'$. To this end, we recall an integral expression of $c_{m,0}$.

We now write (3.1) as
\[ u'' + p' u' + q u = F \quad \text{in } G_1, \tag{3.8} \]
and set $p_0 = p, q_0 = q, u_0 = u, F_0 = F, \overline{m}_0 = \overline{m}$ and $\overline{m}_0 = \overline{m}$. Set, for $l \geq 1$ inductively,
\[ u_l = u_{l-1} - \frac{2u_{l-1}}{t}. \tag{3.9} \]
Then,
\[ u_l'' + p_l u_l' + q_l u_l = F_l \quad \text{in } G_1, \tag{3.10} \]
where, inductively,
\[ p_l = p_{l-1} + 2, \quad q_l = p_{l-1} + q_{l-1}, \]
and
\[ F_l = \partial_t F_{l-1}. \]
A simple calculation yields
\[ p_l = 2l + p, \quad q_l = l^2 + (p-1)l + q, \]
and
\[ F_l = \partial_t^l F. \]
We note that
\[ p_l = 1 + 2l - (\overline{m} + \overline{m}), \quad q_l = (\overline{m} - l)(\overline{m} - l), \tag{3.11} \]
and the general solutions of the homogeneous linear equation corresponding to (3.10) are spanned by $t^{\overline{m}-l}$ and $t^{\overline{m}-l}$. 
The solution $u$ can be expressed in terms of $u_l$. To see this, we first rewrite (3.9) as

$$\frac{u_l}{t^2} = \partial_t \left( \frac{u_l-1}{t^2} \right).$$

(3.12)

Then, inductively,

$$\frac{u_l}{t^2} = \partial_t \left( \frac{u_l}{t^2} \right).$$

To proceed, we set

$$v_l = \frac{u_l}{t^2} = \partial_t \left( \frac{u_l}{t^2} \right).$$

(3.13)

Then, $v_l = v_{l-1}$. If $v_0, v_1, \ldots, v_{l-1}$ are continuous in $G_1$ and $v_l$ is integrable in $G_1$, a successive integration yields

$$u(x', t) = v_0(x', 0)t^2 + v_1(x', 0)t^3 + \cdots + \frac{1}{(l-1)!} v_{l-1}(x', 0)t^{l+1}$$

$$+ t^2 \int_0^t \int_0^{s_1} \cdots \int_0^{s_{l-1}} v_l(x', s_l) ds_l \cdots ds_1.$$

(3.14)

For any $l \geq 0$, we now apply Lemma 3.1 to (3.10). Specifically, we replace $m$ and $m$ by $m - l$ and $m - l$ in (3.6) and obtain

$$u_l(x', t) = \left[ u_l(x', r)r^{l-m} + \frac{r^{m-m}}{m-m} \int_0^r s^{l+1-m} F_l(x', s) ds \right] t^{m-l}$$

$$- \frac{1}{m-m} t^{m-l} \int_0^t s^{l+1-m} F_l(x', s) ds$$

$$- \frac{1}{m-m} t^{m-l} \int_0^t s^{l+1-m} F_l(x', s) ds.$$

(3.15)

Now, we take $l = m - 2$. Then,

$$u_{m-2}(x', t) = \left[ u_l(x', r)r^{l-m} + \frac{r^{m-m}}{m-m} \int_0^r s^{m-m-1} F_{m-2}(x', s) ds \right] t^2$$

$$- \frac{1}{m-m} t^{m-m+2} \int_0^t s^{m-m-1} F_{m-2}(x', s) ds$$

$$- \frac{1}{m-m} t^2 \int_0^t s^{l} F_{m-2}(x', s) ds.$$
where
\[ c_{2,1}(x') = \frac{1}{m - m} F_{m-2}(x', 0), \]
\[ c_{2,0}(x') = u_{m-2}(x', r) r^{-2} + \frac{r^m - m}{m - m} \int_0^r s^{m-1} F_{m-2}(x', s) ds \]
(3.16)
\[ \frac{1}{(m - m)^2} F_{m-2}(x', 0) - \log \frac{r}{m - m} F_{m-2}(x', 0) \]
\[ - \frac{1}{m - m} \int_0^r s^{-1} F_{m-2}(x', s) ds, \]
and \( \tilde{w}_{m-2} \) is the higher order term, which does not play any role in the present proof.

Then, \( c_{m,0} \) in (3.7) is a linear combination of \( c_{2,0}, c_{2,1} \) in (3.16). See [15], Lemma 5.2 in particular, for details. In the following, we will prove that \( c_{2,0}, c_{2,1} \) in (3.16) are analytic in \( \bar{B}'_r \), for any \( r \in (0, 1) \).

We first prove by induction, for \( k = 1, \cdots, m - 2 \),
\[ \partial^k \left( \frac{u}{t} \right), \partial^{k+1} u \in \mathcal{A}(G_r), \]
and
\[ \partial^k \left( \frac{u^2}{t^3} \right), \partial^k \left( \frac{uu_t}{t^2} \right), \partial^k \left( \frac{u^2}{t} \right) \in \mathcal{A}(\bar{G}_r). \]
(3.17)

First, Corollary 2.3 implies (3.17) and (3.18) for \( k = 1 \). We assume (3.17) and (3.18) hold for \( k = 1, \cdots, l \), for some \( l \leq m - 3 \). Since \( F \) is analytic in \( x', t \) and quantities in (3.2), then \( F_l = \partial^l_F \in \mathcal{A}(G_r) \). By (3.15), we get
\[ \frac{u_l}{t^l}, \frac{u_l'}{t^l}, u_l'' \in \mathcal{A}(\bar{G}_r). \]

It is essential here to assume \( l \leq m - 3 \), because of the last integral in (3.15). With \( v_l \) given by (3.13), we have
\[ v_l, tv_l', t^2 v_l'' \in \mathcal{A}(\bar{G}_r). \]

By (3.14), we have
\[ u(x', t) = v_0(x', 0) t^2 + v_1(x', 0) t^3 + \cdots + \frac{1}{(l - 1)!} v_{l-1}(x', 0) t^l + t^2 R_l(x', t), \]
where
\[ R_l(x', t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{l-1}} v_l(x', s_1) ds_1 ds_{l-1} \cdots ds_1. \]

Note
\[ \partial^l+2 R = t^2 \partial^l+2 R_l + 2(l + 2) t \partial^l+1 R_l + (l + 2)(l + 1) \partial^l R_l \]
\[ = t^2 v''_l + 2(l + 2) t v'_l + (l + 2)(l + 1) v_l. \]

Then, \( \partial^l+2 u \in \mathcal{A}(\bar{G}_r) \). Similarly, we have \( \partial^{l+1} (u/t) \in \mathcal{A}(\bar{G}_r) \). This is (3.17) for \( k = l + 1 \).

We can also conclude (3.18) for \( k = l + 1 \).

By taking \( k = m - 2 \) in (3.17) and (3.18), we conclude \( F_{m-2} \in \mathcal{A}(\bar{G}_r) \). In particular, \( F_{m-2}(\cdot, 0) \) is analytic in \( \bar{B}'_r \), and hence \( c_{2,1} \) in (3.16) is analytic in \( \bar{B}'_r \).
We now proceed to prove that \( c_{2,0} \) in (3.16) is analytic in \( \mathcal{B}' \). By the expression of \( c_{2,0} \) in (3.16), it suffices to prove that

\[
\int_{0}^{r} s^{-1} \left[ F_{m-2}(x', s) - F_{m-2}(x', 0) \right] ds
\]

is an analytic function of \( x' \) in \( \mathcal{B}' \). By (3.13) and (3.15) for \( l = m - 2 \), we have

\[
v_{m-2}(x', t) = \left[ u_{m-2}(x', r) r^{-2} + \frac{m-m}{m-m} \int_{0}^{r} s^{m-m-1} F_{m-2}(x', s) ds \right]
\]

\[
- \frac{1}{m-m} \int_{0}^{r} s^{m-m-1} F_{m-2}(x', s) ds
\]

\[
- \frac{1}{m-m} \int_{t}^{r} s^{-1} F_{m-2}(x', s) ds.
\]

We note that the dominant term is the last integral. By the simple integral \( \int_{0}^{r} s^{-1} ds = \log r - \log t, \) we obtain, for any nonnegative integer \( l \) and any \((x', t) \in \mathcal{G}_{r},\)

\[
|D_{x}^{l} v_{m-2}(x', t)| \leq B_{0} B_{l} l! \log t^{-1},
\]

for some positive constants \( B_{0} \) and \( B. \) Moreover, a straightforward calculation yields

\[
|t^{2} D_{x}^{l} v_{m-2}(x', t)| + |t D_{x}^{l} v_{m-2}(x', t)| + \left| \int_{0}^{t} D_{x}^{l} v_{m-2}(x', s) ds \right| \leq B_{0} B_{l} l! t \log t^{-1}.
\]

By (3.14) with \( l = m - 2 \), we have

\[
u(x', t) = P_{m-1}(x', t) + t^{2} R_{m}(x', t),
\]

where

\[
P_{m-1}(x', t) = v_{0}(x', 0) t^{2} + v_{1}(x', 0) t^{3} + \cdots + \frac{1}{(m-3)!} v_{m-3}(x', 0) t^{m-1},
\]

\[
R_{m}(x', t) = \int_{0}^{t} \int_{0}^{s_{1}} \cdots \int_{0}^{s_{m-3}} v_{m-2}(x', s_{m-2}) ds_{m-2} \cdots ds_{1}.
\]

Then,

\[
\partial_{t}^{m-1} u = \partial_{t}^{m-1} P_{m-1} + t^{2} v_{m-2} + 2(m-1) t v_{m-2} + (m-1)(m-2) \int_{0}^{t} v_{m-2}(x', s) ds.
\]

Note that \( \partial_{t}^{m-1} P_{m-1} \) is a function of \( x' \). Then, for any nonnegative integer \( l \) and any \((x', t) \in \mathcal{G}_{r},\)

\[
|D_{x}^{l} \left[ \partial_{t}^{m-1} u(x', t) - \partial_{t}^{m-1} u(x', 0) \right]| \leq B_{0} B_{l} l! t \log t^{-1}.
\]

Similar estimates hold for \( \partial_{t}^{l} v_{m-2}(u/t) \), \( \partial_{t}^{l-2} (u^{2}/t^{3}) \), \( \partial_{t}^{l-2} (u u_{t}/t^{2}) \) and \( \partial_{t}^{l-2} (u^{2}/t) \). Then, the expression of \( F_{m-2} \) implies

\[
|D_{x}^{l} \left[ F_{m-2}(x', t) - F_{m-2}(x', 0) \right]| \leq B_{0} B_{l} l! t \log t^{-1}.
\]
Therefore,
\[
\left| D_{x'}^k \int_0^r s^{-1} \left[ F_{m-2}(x', s) - F_{m-2}(x', 0) \right] ds \right|
\leq \int_0^r s^{-1} |D_{x'}^k \left[ F_{m-2}(x', s) - F_{m-2}(x', 0) \right]| ds
\leq \int_0^r s^{-1} B_0 B^l! s \log s^{-1} ds = \tilde{B}_0 B^l!.
\]
This implies that the function in (3.19) is analytic in $B'_p$ for some $\rho < 1/2$. \hfill \Box

4. The Analyticity along the Normal direction

In this section, we prove the convergence of our boundary expansion. We adapt the methods by Nirenberg [30], by Kichenassamy and Littman [24], [25], and by Kichenassamy [22]. We adopt the norm used by Nirenberg.

We consider the equation (3.1), with $F$ a function of $x'$, $t$ and those in (3.2). We will prove that under appropriate assumptions, solutions are analytic in $x'$, $t$ and $t \log t$. The main result in this paper is the following theorem.

**Theorem 4.1.** Let $F$ in (3.1) be analytic in all of its arguments given by $x'$, $t$ and those in (3.2) and let $m$ and $\overline{m}$ be integers satisfying (3.3) and (3.4). Suppose that $u \in C^1(\overline{G}_1) \cap C^\infty(G_1)$ is a solution of (3.1) satisfying (2.30). Then, $u$ is analytic in $x'$, $t$ and $t \log t$, and the series in (3.7) converges uniformly to $u$ in $G_{1/2}$.

We first present a brief description of the proof. For $k \geq \overline{m}$, set
\[
u_k(x', t) = \sum_{i=2}^{\overline{m}-1} c_i(x') t^i + \sum_{i=\overline{m}}^k \sum_{j=0}^{N_i} c_{i,j}(x') t^i (\log t)^j.
\]
Then, $\nu_k$ can be considered as a partial sum of the series (3.7). Our goal is to prove that $u$ is analytic in $x'$, $t$ and $t \log t$ and that $u_k$ converges to $u$ uniformly. We can write $u_k$ as
\[
u_k(x', t) = \sum_{i=2}^{\overline{m}-1} c_i(x') t^i + \sum_{i=\overline{m}}^k \sum_{j=0}^{N_i} c_{i,j}(x') t^i (\log t)^j.
\]
In other words, $\nu_k$ is the expansion of $u$ with respect to $t$ and $t \log t$ up to order $k$. In the proof below, we will construct another sequence $\{v_k\}$, analytic in $x'$, $t$ and $t \log t$ and converging uniformly to $v$, and we will prove that $u = v$. The first element $v_{\overline{m}}$ is given by $u_{\overline{m}+1}$ without the $t^{\overline{m}+1}$-term.

**Proof.** The proof is quite long and is divided into two steps after the initial setup.

For any function $v = v(x', t)$, we set
\[
V = \left( \frac{v}{t}, v', \frac{D_{x'} v}{t}, D_{x'} v', D_{x'}^2 v, \frac{v^2}{t^3}, \frac{vv'}{t^2}, \frac{v'^2}{t} \right),
\]
and write
\[
F = F(x', t, V).
\]
We assume that there exist constants $M > 0$ and $R \in (0, 1)$, such that, for any $(x', t, V)$ with $|x'| + t + |V| < R$,

$$|F(x', t, V)| \leq M \left[ 1 - \frac{1}{R}(|x'| + t + |V|) \right]^{-1}.$$  

We set

$$L_0v = v'' + \frac{v'}{t} + g \frac{v}{t^2}.$$

**Step 1.** We prove that (3.1) admits a unique solution satisfying (2.30). Let $u$ be given as in the statement of Theorem 4.1. Then, $u$ is analytic in $x'$ by Corollary 2.3. We extend arguments of all functions to the complex field. Set

$$v_m(x', t) = \sum_{i=2}^{\overline{m}-1} c_i(x')t^i + c_{\overline{m}, 1}(x')t^{\overline{m}} \log t + c_{\overline{m}, 1}(x')t^{\overline{m}}$$

$$(4.2)$$

where $c_i$ and $c_{i,j}$ are as in (3.7). Note that $v_m$ is holomorphic in $x'$ and is the expansion of $u$ before the term $t^{\overline{m}+1}$, i.e.,

$$|u - v_m| \leq C t^{\overline{m}+1}.$$  

Inductively, for $k \geq \overline{m} + 1$, we define $w_k$ and $v_k$ by

$$w_k = \frac{1}{\overline{m} - m} \int_0^t \rho^{1-m}F_k d\rho - \frac{1}{\overline{m} - m} \int_0^t \rho^{1-\overline{m}}F_k d\rho,$$

$$v_k = v_{k-1} + w_k = v_m + \sum_{i=m+1}^{k} w_i,$$

where

$$F_{\overline{m}+1} = F(x', t, V_{\overline{m}}) - L_0v_{\overline{m}},$$

and, for $k \geq \overline{m} + 2$,

$$F_k = F(x', t, V_{k-1}) - F(x', t, V_{k-2}).$$

Here,

$$V_k = \left( \frac{v_k}{t}, \frac{v_k'}{t}, \frac{D_x v_k}{t}, \frac{D_x v_k'}{t}, \frac{D_x^2 v_k}{t}, \frac{v_k^2}{t^2}, \frac{v_k' v_k}{t^2}, \frac{v_k'^2}{t} \right).$$

It is easy to see that $w_k$ is a solution of the equation $L_0w_k = F_k$. Hence,

$$L_0v_k = F(x', t, V_{k-1}).$$  

In the following, we will prove that $v_k$ is holomorphic in $x'$ and converges uniformly in a fixed region $|x'| + t < r$. To this end, we need to introduce appropriate domains and norms.
Let \( w = w(x') \) be a holomorphic function in \( B'_1 \subset \mathbb{C}^{n-1} \). Define, for any \( r \in (0, 1) \),
\[
\|w\|_r = \sup_{|x'| < r} |w(x')|.
\]
By the usual estimate for derivatives of holomorphic functions, we have, for any \( 0 < r' < r \) and any \( \alpha = 1, \ldots, n-1 \),
\[
(4.5) \quad \|\partial^\alpha w\|_{r'} \leq \frac{1}{r - r'} \|w\|_r.
\]
This estimate will be used repeatedly in the following.

With \( a_0 \) a positive constant to be determined later, define inductively, for any \( k \geq 0 \),
\[
(4.6) \quad a_{k+1} = a_k (1 - (k+1)^{-2}).
\]
Then,
\[
a \equiv \lim_{k \to \infty} a_k = a_0 \prod_{k=0}^{\infty} (1 - (k+1)^{-2}) > 0.
\]
For some fixed \( s_0 > 0 \), set
\[
\Omega_k = \left\{ (x', t) : |x'| + \frac{t}{a_k} < s_0, t > 0 \right\}.
\]
For any function \( w(x', t) \) defined in \( \Omega_k \), holomorphic in \( x' \) and continuous in \( t \), we write \( w(t) = w(\cdot, t) \) and define
\[
M_k[w] = \sup_{0 < s < s_0} \left[ \frac{\|w(t)\|_s}{t^{m-1}} \left( \frac{a_k(s_0 - s)}{t} - 1 \right) \right].
\]
We also define
\[
\Omega_\infty = \left\{ (x', t) : |x'| + \frac{t}{a} < s_0, t > 0 \right\},
\]
and
\[
M_\infty[w] = \sup_{0 < s < s_0} \left[ \frac{\|w(t)\|_s}{t^{m-1}} \left( \frac{a(s_0 - s)}{t} - 1 \right) \right].
\]
Inductively, we will prove that there exist positive constants \( A \) and \( s_0 \) such that, for any \( k \geq m+1 \), any \( 0 < s < s_0 \) and \( 0 < t < a_k(s_0 - s) \),
\[
(4.7) \quad \|V_{k-1}(t)\|_s \leq \frac{R}{4},
\]
\[
(4.8) \quad w_k \text{ is holomorphic in } x' \text{ in } \Omega_k,
\]
\[
(4.9) \quad M_k\left[ \frac{w_k}{t} \right], M_k[w'_k], M_k[D_{x'} w_k] \leq \frac{A}{2^k}.
\]
For \( k = m+1 \), we can set \( A \) large and \( s_0 \) small such that (4.7), (4.8) and (4.9) hold. We can set \( s_0^{m-2} A \) small for later purposes. For (4.7) with \( k = m+1 \), we require a stronger estimate
\[
\|V_m(t)\|_s \leq \frac{R}{64}.
\]
We assume that (4.7), (4.8) and (4.9) hold for \( k - 1 \), for some \( k \geq m + 2 \), and proceed to consider for \( k \).

First, we prove (4.7). There are eight components in \( V_{k-1} \). We consider \( D_{x'}^2 v_{k-1} \) for an illustration. For each \( i = m + 1, \ldots, k - 1 \), \( w_i \) is holomorphic in \( x' \) for any \( (x', t) \in \Omega_i \).

Hence, we can apply (4.5) to each \( w_i \). For any \( 0 < s < s_0 \) and \( 0 < t < a_k(s_0 - s) \), we set

\[
\tau_i = \frac{1}{2}(s_0 + s - \frac{t}{a_i}).
\]

Then,

\[
\|D_{x'}^2 w_i(t)\|_s \leq \frac{\|D_{x'}^2 w_i(t)\|_{\tau_i - s}}{\tau_i - s} \leq \frac{t^{m-1} M_i[D_{x'} w_i]}{(\tau_i - s)(a_i(s_0 - \tau_i)/t - 1)}
\]

\[
\leq \frac{4a_i t^{m-1}}{t(a_i(s_0 - s)/t - 1)^2} M_i[D_{x'} w_i]
\]

\[
\leq 4a_0 t^{m-2} A \left( \frac{a_i}{a_{i+1}} - 1 \right)^{-2} 2^{-i},
\]

and hence

\[
\|D_{x'}^2 v_{k-1}(t)\|_s \leq \|D_{x'}^2 v(m)\|_s + \sum_{i=m+1}^{k-1} \|D_{x'}^2 w_i(t)\|_s
\]

\[
\leq \frac{R}{64} + C a_0 s_0^{-2} A \sum_{i=m+1}^{k} \left( \frac{a_i}{a_{i+1}} - 1 \right)^{-2} 2^{-i} \leq \frac{R}{32},
\]

since \( s_0^{-2} A \) is small. We can discuss other components in \( V_{k-1} \) similarly.

Second, we prove (4.8). Recall that \( w_k \) is given by (4.3). Note that

\[
F_k = \int_0^1 D_V F(x', t, \theta V_{k-1} + (1 - \theta)V_{k-2})d\theta \cdot (V_{k-1} - V_{k-2}),
\]

where, for \( s_0 < R/4 \),

\[
\|D_V F\| \leq M \left[ 1 - \frac{1}{R}(s_0 + a_0 s_0 + \frac{R}{4}) \right]^{-1} \leq C,
\]

and

\[
V_{k-1} - V_{k-2} = \left( \frac{w_{k-1}}{t}, \frac{D_{x'} w_{k-1}}{t}, \frac{D_{x'}^2 w_{k-1}}{t}, \frac{D_{x'}^2 w_{k-1}}{t}, \frac{v_{k-1}}{t^2}, \frac{v_{k-1}'}{t}, \frac{v_{k-1}'}{t^2}, \frac{v_{k-1}'}{t}, \frac{v_{k-2}'}{t} \right).
\]

For the last three components, we have

\[
\frac{v_{k-1}'}{t^2} - \frac{v_{k-2}'}{t^2} = \frac{v_{k-1}'}{t^2} w_{k-1} + \frac{v_{k-2}'}{t} w_{k-1},
\]
and similar identities for the other two components. Note that \(v_{k-1}/t^2, v'_{k-2}/t\) are bounded. So, we need to prove \(t^{-m}w_{k-1}\) and \(t^{1-m}w'_{k-1}\) are holomorphic in \(x'\) in \(\Omega_k\). For any \(0 < s < s_0\) and \(0 < t < a_k(s_0 - s)\), we take

\[
\tau = \frac{1}{2}(s_0 + s - \frac{t}{a_{k-1}}),
\]

and get

\[
t^{-m}\|D_{x'}w_{k-1}(t)\|_s \leq Ct^{-m}\|w_{k-1}(t)\|_\tau \leq C\frac{4a_{k-1}M_{k-1}[\frac{w_{k-1}}{t}]}{t(a_{k-1}(s_0 - s)/t - 1)^2} \leq C\frac{4a_02^{-k+1}At}{(a_{k-1} - a_k)^2(s_0 - s)^2}.
\]

We can discuss other terms similarly.

Last, we prove (4.9). For any \(0 < s < s_0\) and \(0 < t < a_k(s_0 - s)\), we have

\[
\frac{\|w_{k}(t)\|_s}{\rho} \leq \frac{1}{m - m} t^{m-m} \int_0^t \rho^{1-m}\|F_k\|_s \rho + \frac{1}{m - m} \int_0^t \rho^{1-m}\|F_k\|_s \rho.
\]

By (4.10),

\[
|F_k| \leq C|V_{k-1} - V_{k-2}|.
\]

The integrals split to several parts. We first consider \(w_{k-1}/t\). By

\[
\int_0^t \rho^{1-m}\|w_{k-1}(\rho)\|_s \rho \leq \frac{1}{a_{k-1}(s_0 - s)/\rho - 1} M_{k-1}[\frac{w_{k-1}}{t}],
\]

we have

\[
\int_0^t \rho^{1-m}\|w_{k-1}(\rho)\|_s \rho \leq \frac{a_{k}s_0}{a_k(s_0 - s)/t - 1} M_{k-1}[\frac{w_{k-1}}{t}].
\]

We can discuss \(w'_{k-1}\) similarly. Next, we consider \(D_{x'}^2w_{k-1}\). For each \(\rho \in (0, t)\), we take \(s(\rho) < s_0 - \frac{\rho}{a_{k-1}}\) to be fixed. Then,

\[
\rho^{1-m}\|D_{x'}^2w_{k-1}(\rho)\|_s \leq \rho^{1-m}\|D_{x'}w_{k-1}(\rho)\|_s(\rho) - s
\]

(4.12)

By taking

\[
s(\rho) = \frac{1}{2}(s_0 + s - \frac{\rho}{a_{k-1}}),
\]

we have

\[
\int_0^t \rho^{1-m}\|D_{x'}^2w_{k-1}(\rho)\|_s \rho \leq \frac{Ca_k}{a_{k-1}(s_0 - s)/t - 1} M_{k-1}[D_{x'}w_{k-1}].
\]
Similar estimates hold for other integrals. By requiring $Ca_0$ to be small, we have

$$M_k \left[ \frac{w_k}{t} \right] \leq \frac{1}{6} (M_{k-1}[w_{k-1}] + M_{k-1}[w'_{k-1}] + M_{k-1}[D_{x'}w_{k-1}]).$$

Similar estimates hold for $w'_k$. Hence,

$$M_k \left[ \frac{w_k}{t} \right], M_k |w'_k| \leq \frac{A}{2^k}.$$  

Next, we note

$$D_{x'}w_k(t) = \int_0^t D_{x'}w'_k(\rho)d\rho.$$  

For $t < a_k(1-s)$, we have, similarly as in (4.12),

$$M_k[D_{x'}w] \leq \frac{1}{2} M_k[w'_k] \leq \frac{A}{2^k}.$$  

We hence have (4.9).

In conclusion, by (4.9), we have, for any $k \geq m + 1$,

$$M_\infty \left[ \frac{w_k}{t} \right], M_\infty [w'_k], M_\infty [D_{x'}w_k] \leq \frac{A}{2^k}.$$  

There exists a function $v$, holomorphic in $x'$ for any $(x', t) \in \Omega_\infty$, such that $v_k \to v$, $v'_k \to v'$, $D_{x'}v_k \to D_{x'}v$ in $\Omega_\infty$ as $k \to \infty$.

We also have, for $|x'| < s$ and $t < a(s_0 - s)$,

$$\left| t^{1 - \overline{m}}(\frac{a(s_0 - s)}{t} - 1) \frac{v - v_m}{t} \right| \leq \sum_{k=m+1}^{\infty} M_k \left[ \frac{w_k}{t} \right] \leq A,$$

and

$$\left| t^{1 - \overline{m}}(\frac{a(s_0 - s)}{t} - 1) D_{x'}(v - v_m) \right| \leq \sum_{k=m+1}^{\infty} M_k[D_{x'}w_k] \leq A.$$  

Hence, $v$ satisfies

(4.13) 

$$L_0 v = F(x', t, V),$$

and

(4.14) 

$$|v - v_m| \leq Ct^{\overline{m}+1}.$$  

Moreover, $t^{-\overline{m}}(v - v_m)$ is holomorphic in $x'$ in $\Omega_\infty$.

Last, we prove $u = v$. Note that $u$ also satisfies (4.13) and (4.14), with $v$ and $V$ replaced by $u$ and $U$, respectively. Set $w = u - v$. Then,

$$L_0 w = F(x', t, U) - F(x', t, V) \quad \text{in } \Omega_\infty,$$

and

$$|w| \leq Ct^{\overline{m}+1}.$$  

We can repeat the above iteration and have $M_k[u - v] = M_k[w_k] \to 0$, as $k \to \infty$. This implies $u = v$.  

Step 2. We prove that \( u \) is analytic in \( t, t \log t \). We treat \( t \) and \( t \log t \) as two independent variables and set
\[
T = t, \quad S = t \log t.
\]
For a function \( u = u(x', t, t \log t) = u(x', T, S) \), we have
\[
t \partial_t u = t \partial_T u + t(\log t + 1) \partial_S u = T \partial_T u + (T + S) \partial_S u.
\]
Set
\[
\Lambda = T \partial_T + (T + S) \partial_S.
\]
Then,
\[
t \partial_t u = \Lambda u.
\]
Next, we extend arguments \( x', T, S \) into the complex field. Since the complexified \( t \) or \( T \) cannot be the upper bounds in the integral, we need to make a change of variables so that \( t \) or \( T \) appears in the integrands. A simple substitution yields
\[
\int_0^t u(x', s, s \log s) ds = \int_1^0 T u(x', \rho T, \rho(\log \rho) T + \rho S) d\rho.
\]
We now take the same sequences \( v_k \) and \( w_k \) as in Step 1 and treat them as functions of \( x', T \) and \( S \). We start by writing \( v_m \) in the form
\[
v_m = c_2(x') T^2 + \cdots + c_{m-1}(x') T^{m-1} + c_m(x') T^{m-1} S
\]
(4.15)
\[+ \sum_{j=1}^{N_m+1} c_{m+j} \cdot j(x') T^{m+1-j} S^j.
\]
For \( k \geq m + 1 \), define \( w_k(x', T, S) \) and \( v_k(x', T, S) \) inductively by
\[
w_k = \frac{1}{m - m} T^2 \int_0^1 \rho^{1-m} F_k(x', \rho T, \rho(\log \rho) T + \rho S) d\rho
\]
(4.16)
\[\quad - \frac{1}{m - m} T^2 \int_0^1 \rho^{1-m} F_k(x', \rho T, \rho(\log \rho) T + \rho S) d\rho,
\]
and
\[
v_k = v_m + \sum_{i=m+1}^{k} w_i,
\]
where
\[
F_{m+1}(x', T, S) = F(x', T, V_m(x', T, S)) - \frac{1}{T^2} (\Lambda^2 V_m + (p - 1) \Lambda V_m + q V_m)(x', T, S),
\]
and, for \( k \geq m + 2 \),
\[
F_k = F(x', T, V_{k-1}(x', T, S)) - F(x', T, V_{k-2}(x', T, S)).
\]
Here,
\[
V_k = \left( \frac{v_k}{T}, \frac{\Lambda v_k}{T}, \frac{D_{x'} v_k}{T}, \frac{D_{x'} \Lambda v_k}{T}, \frac{D_{x'}^2 v_k}{T^3}, \frac{D_{x'}^2 v_k}{T^3}, \frac{(\Lambda v_k)^2}{T^3} \right).
\]
Then, \( w_k \) is a solution of
\[
\Lambda^2 w_k + (p - 1) \Lambda w_k + q w_k = T^2 F_k.
\]

In the following, we will prove that \( v_k \) is holomorphic in \( x', T, S \) and converges uniformly in a fixed region \( |x'| + |T| + |S| < r \).

We fix an arbitrary \( \theta \in (0, 1) \) and let \( \{a_k\} \) be introduced as in (4.6). For convenience, we write \( u(T, S) = u(\cdot, T, S) \) and set
\[
\delta = |T| + \theta |S|.
\]

We define
\[
\|u(T, S)\|_r = \sup_{|x'| < r} |u(x', T, S)|,
\]
and
\[
M_k[u] = \sup_{0 < s < s_0, T \neq 0 \atop \delta < a_k(s_0 - s)} \left[ \frac{\|u(T, S)\|_s}{|T|^{m-1}} \left( \frac{a_k(s_0 - s)}{\delta} - 1 \right) \right].
\]

Set
\[
\Omega_k = \{ (x', T, S) : |x'| + \delta/a_k < s_0 \}.
\]
We also define
\[
\Omega_\infty = \{ (x', T, S) : |x'| + \delta/a < s_0 \}.
\]
and
\[
M_\infty[w] = \sup_{0 < s < s_0, T \neq 0 \atop \delta < a(s_0 - s)} \left[ \frac{\|u(T, S)\|_s}{|T|^{m-1}} \left( \frac{a(s_0 - s)}{\delta} - 1 \right) \right].
\]

Inductively, we will prove that there exist positive constants \( A \) and \( s_0 \) such that, for any \( k \geq m + 1 \), any \( 0 < s < s_0 \) and \( \delta < a_k(s_0 - s) \),
\begin{align*}
(4.17) & \quad \|V_{k-1}(T, S)\|_s \leq \frac{R}{4}, \\
(4.18) & \quad \frac{w_k}{T^m}, \frac{\Lambda w_k}{T^m} \text{ are holomorphic in } (x', T, S) \in \Omega_k, \\
(4.19) & \quad M_k[\frac{w_k}{T}], M_k[\frac{\Lambda w_k}{T}], M_k[D_{x'} w_k] \leq \frac{A}{2^k}.
\end{align*}

For \( k = m + 1 \), we can set \( A \) large and \( s_0 \) small such that (4.17), (4.18) and (4.19) hold. We can set \( s_0^{-2} A \) small for later purposes. For (4.17), we require a stronger estimate
\[
\|V_m(T, S)\|_s \leq \frac{R}{64}.
\]

We assume that (4.17), (4.18) and (4.19) hold for \( k - 1 \), for some \( k \geq m + 2 \), and proceed to consider for \( k \).
First, we prove (4.17). There are eight components in $V_{k-1}$. We consider $D^2_{x'}v_{k-1}$ for an illustration. For each $i = \underline{m}+1, \ldots, k-1$, $w_i$ is holomorphic in $\Omega_i$. For any $0 < s < s_0$ and $\delta < a_k(s_0 - s)$, we set
$$\tau_i = \frac{1}{2}(s_0 + s - \frac{\delta}{a_i}).$$
Then,
$$\|D^2_{x'}w_i(T, S)\|_s \leq \frac{|D^2_{x'}w_i(T, S)\|_{\tau_i}}{\tau_i - s} \leq \frac{|T|^{m-1}M_i[D_{x'}w_i]}{(\tau_i - s)(a_i(s_0 - \tau_i)/\delta - 1)}$$
$$\leq \frac{4a_i|T|^{m-1}M_i[D_{x'}w_i]}{\delta(a_i(s_0 - s)/\delta - 1)^2}$$
$$\leq 4a_0|T|^{m-2}A(\frac{a_i}{a_{i+1}} - 1)^{-2}2^{-i},$$
and hence
$$\|D^2_{x'}v_{k-1}(T, S)\|_s \leq \|D^2_{x'}v_{m}(T, S)\|_s + \sum_{i=\underline{m}+1}^{k-1} \|D^2_{x'}w_i(T, S)\|_s$$
$$\leq \frac{R}{64} + C a_0 s_0^{m-2} A \sum_{i=\underline{m}+1}^{k-1} (\frac{a_i}{a_{i+1}} - 1)^{-2}2^{-i} \leq \frac{R}{32},$$
since $s_0^{m-2} A$ is small. We can discuss other components in $V_{k-1}$ similarly.

Second, we prove (4.18). Recall that $w_k$ is given by (4.16). Note that
$$F_k(x', T, S) = \int^1_0 \varepsilon F(x', T, 1a_{k-1} + (1 - \theta)(V_{k-1} - V_{k-2}),$$
where, for $s_0 < R/4$,
$$|D\varepsilon F| \leq M \left[ 1 - \frac{1}{R}(s_0 + a_0 s_0 + \frac{R}{4}) \right]^{-1} \leq C,$$}
and
$$v_{k-1} - v_{k-2} = \left( \frac{w_{k-1}}{T}, \frac{\Lambda w_{k-1}}{T}, \frac{D_{x'}w_{k-1}}{T}, \frac{D_{x'}\Lambda w_{k-1}}{T}, \frac{D^2_{x'}w_{k-1}}{T}, \frac{v_{k-1}}{T^3} - \frac{v_{k-2}}{T^3}, \frac{u_{k-1}v_{k-1}}{T^3} - \frac{u_{k-2}v_{k-2}}{T^3}, \frac{(\Lambda v_{k-1})^2}{T^3} - \frac{(\Lambda v_{k-2})^2}{T^3} \right).$$
In view of (4.16) and the induction that $\frac{\Lambda w_{k-1}}{T}$ and $\frac{\Lambda \Lambda w_{k-1}}{T}$ are holomorphic in $(x', T, S) \in \Omega_{k-1}$, we need to analyze the impact of the factor $\rho^{1-m}$ in the second term of (4.16).

Divided by $T^{m}$, the expression
$$T^{2-m} \int^1_0 \rho^{1-m} F_k(x', \rho T, \rho(\log \rho)T + \rho S) d\rho$$
is holomorphic in $x', T$ and $S$, since $V_{k-1} - V_{k-2}$ in (4.20) and their derivatives can absorb the $\rho^{1-m}$ factor. For example, corresponding to the first component in $V_{k-1} - V_{k-2}$, we write

$$T^{2-m} \int_0^1 \rho^{1-m} \frac{w_{k-1}}{\rho T^m} (x', \rho T, \rho (\log \rho) T + \rho S)d\rho = T \int_0^1 \frac{w_{k-1}}{(\rho T)^m} (x', \rho T, \rho (\log \rho) T + \rho S)d\rho,$$

which is holomorphic in $x', T$ and $S$, since the integrand is holomorphic in its arguments. Corresponding to the second component in $V_{k-1} - V_{k-2}$, we write

$$T^{2-m} \int_0^1 \rho^{1-m} \frac{\Lambda w_{k-1}}{\rho T^m} (x', \rho T, \rho (\log \rho) T + \rho S)d\rho$$

$$= \int_0^1 \frac{T D_T^{m-1} w_{k-1}}{(\rho T)^m} (x', \rho T, \rho (\log \rho) T + \rho S)$$

$$+ (m-1) \frac{w_{k-1}}{(\rho T)^m} (x', \rho T, \rho (\log \rho) T + \rho S)$$

$$+ (\log \rho) T D_S^{m-1} w_{k-1} (x', \rho T, \rho (\log \rho) T + \rho S)$$

$$+ S D_S^{m-1} w_{k-1} (x', \rho T, \rho (\log \rho) T + \rho S)d\rho,$$

which is holomorphic in $x', T$ and $S$. We can discuss other terms similarly.

Last, we prove (4.19). Set

$$\delta(\rho) = \rho |T| + \theta |\rho (\log \rho) T + \rho S|,$$

and

$$h(\rho) = \rho - \theta \rho \log \rho.$$

Then, $h$ is an increasing function in $(0, 1)$ and hence $h(\rho) \leq h(1) = 1$ for any $\rho \in (0, 1)$. It is easy to check, for any $\rho \in (0, 1)$,

$$\delta(\rho) \leq h(\rho) \delta \leq \delta.$$

For any $0 < s < s_0$ and $\delta < a_k(s_0 - s)$, we have

$$\frac{|w_k(T, S)|}{|T|^{2-\alpha}} \leq \frac{1}{|T|} \int_0^1 \rho^{1-m} |F_k(x', \rho T, \rho (\log \rho) T + \rho S)|d\rho$$

$$+ \frac{1}{|T|^{2-\alpha}} \int_0^1 \rho^{1-m} |F_k(x', \rho T, \rho (\log \rho) T + \rho S)|d\rho.$$

By (4.20), we have

$$|F_k| \leq C |V_{k-1} - V_{k-2}|.$$
The integrals split to several parts. First, we have

\[
\|T^{2-\overline{m}} \int_0^1 \rho^{1-\overline{m}} \frac{w_{k-1}}{\rho T} (x', \rho T, \rho (\log \rho) T + \rho S) d\rho \|_s \leq M_{k-1} \left[ \frac{w_{k-1}}{T} \right] |T| \int_0^1 \frac{d\rho}{a_{k-1}(s_0 - s)/\delta(\rho) - 1} \leq \frac{|T|}{a_{k-1}(s_0 - s)/\delta - 1} M_{k-1} \left[ \frac{w_{k-1}}{T} \right].
\]

We can discuss \( T^{2-\overline{m}} \int_0^1 \rho^{1-\overline{m}} D_{x'}^2 w_{k-1} (x', \rho T, \rho (\log \rho) T + \rho S) d\rho \|_s \leq |T|^{2-\overline{m}} \int_0^1 \rho^{1-\overline{m}} \frac{\|D_{x'}^2 w_{k-1}(x', \rho T, \rho (\log \rho) T + \rho S)\|_{s(\rho)}}{s(\rho) - s} d\rho \leq C |T| M_{k-1} [D_{x'}^2 w_{k-1}] \cdot I,

where

\[
I = \int_0^1 \frac{d\rho}{[s(\rho) - s][a_{k-1}(s_0 - s(\rho))/\delta(\rho) - 1]}.
\]

Set

\[
s(\rho) = \frac{1}{2} \left( s_0 + s - \frac{h(\rho)}{a_{k-1}} \right).
\]

Then,

\[
I \leq \int_0^1 \frac{d\rho}{[s(\rho) - s][a_{k-1}(s_0 - s(\rho))/\delta h(\rho) - 1]} \leq \int_0^1 \frac{4a_{k-1} d\rho}{\delta h(\rho)[a_{k-1}(s_0 - s(\rho))/\delta h(\rho) - 1]^2}.
\]

Introduce a new variable

\[
\tau = \frac{\delta h(\rho)}{a_{k-1}(s_0 - s)}.
\]
Then, with $\delta < a_{k-1}(s_0 - s)$,

$$I \leq \int_0^{\delta/a_{k-1}(s_0 - s)} \frac{4a_{k-1}d\tau}{\delta \tau (\tau^{-1} - 1)^2 (1 - \theta - \theta \log \rho)}$$

$$\leq \frac{4a_{k-1}}{\delta(1 - \theta)} \int_0^{\delta/a_{k-1}(s_0 - s)} \frac{\tau d\tau}{(1 - \tau)^2}$$

$$\leq \frac{4a_{k-1}}{\delta(1 - \theta)} \int_0^{\delta/a_{k-1}(s_0 - s)} d\tau$$

$$= \frac{4a_{k-1}}{\delta(1 - \theta) a_{k-1}(s_0 - s)/\delta - 1}.$$

Hence, with $|T| \leq \delta$, we get

$$\left\| T^{2-\rho} \int_0^1 \rho^{1-\rho} D_x^2 w_{k-1}(x', \rho T, \rho (\log \rho) T + \rho S) d\rho \right\|_s \leq \frac{4a_{k-1}}{\delta(1 - \theta) M_{k-1}[D_x^2 w_{k-1}] a_{k-1}(s_0 - s)/\delta - 1}.$$

Similar estimates hold for other terms. By taking $Ca_0$ small, we have

$$M_k[w_k/T] \leq \frac{1}{6} (M_{k-1}[w_{k-1}/T] + M_{k-1}[\Lambda w_{k-1}/T] + M_{k-1}[D_x^2 w_{k-1}]).$$

Next, we consider $\Lambda w_k$. Note

$$D_{\rho} F_k(x', \rho T, \rho (\log \rho) T + \rho S) = \rho^{-1} \Lambda F_k(x', \rho T, \rho (\log \rho) T + \rho S),$$

and hence

$$\int_0^1 \rho^{1-\rho} \Lambda F_k(x', \rho T, \rho (\log \rho) T + \rho S) d\rho$$

$$= \int_0^1 \rho^{2-\rho} D_{\rho} F_k(x', \rho T, \rho (\log \rho) T + \rho S) d\rho.$$

Integrating by parts, we can estimate all terms similarly. Hence, we conclude

$$M_k[w_k/T], M_k[\Lambda w_k/T] \leq \frac{A}{2^k}.$$

Next, note

$$D_{x'} w_k(x', T, S) = \int_0^1 D_{\rho} D_{x'} w_k(x', \rho T, (\log \rho) T + \rho S) d\rho$$

$$= \int_0^1 T D_{x'} \frac{\Lambda w_k}{T}(x', \rho T, \rho (\log \rho) T + \rho S) d\rho.$$

For $\delta < a_k(1 - s)$, we have similarly

$$M_k[D_{x'} w_k] \leq \frac{1}{2} M_k[\Lambda w_k/T] \leq \frac{A}{2^k}.$$

Therefore, we conclude $v_k \to v$ in the norm $M_\infty$ and hence $v$ is holomorphic in $(x', T, S) \in \Omega_\infty$. Moreover, the Taylor series of $v$ in terms of $T$ and $S$ converges to $v$
uniformly for $|x'| + |T| + |S| < r$. By $u = v$ for $T = t$ and $S = t \log t$ and a comparison of coefficients, we obtain that $u$ is analytic in $x', t, t \log t$ and that the series in (3.7) converges uniformly to $u$ for $|x'| + t < r/2$. □

Theorem 1.2 follows easily from Theorem 4.1.

5. The Loewner-Nirenberg Problem

In this section, we discuss briefly the Loewner-Nirenberg problem. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, for some $n \geq 3$. Consider

\begin{align}
\Delta u &= \frac{1}{4} n(n-2) u^{\frac{n+2}{n-2}} \quad \text{in } \Omega, \\
u &= \infty \quad \text{on } \partial \Omega.
\end{align}

We let $d(x) = \text{dist}(x, \partial \Omega)$ be the distance of $x$ to the boundary $\partial \Omega$.

Assume that $\Omega$ has a $C^{1,1}$-boundary. Loewner and Nirenberg [30] proved that (5.1) and (5.2) admit a unique positive solution $u \in C^\infty(\Omega)$ and that there exists a constant $\mu > 0$ such that, for any $x \in \Omega$ with $d(x) < \mu$,

\begin{equation}
|d^{\frac{n-2}{2}}(x)u(x) - 1| \leq Cd(x),
\end{equation}

where $C$ is a positive constant depending only on $n$ and the $C^{1,1}$-norm of $\partial \Omega$.

Solutions of (5.1)-(5.2) are known to have formal expansions. In the case that $\Omega$ is a bounded smooth domain, $d$ is a smooth function near $\partial \Omega$. For each $x \in \Omega$ close to $\partial \Omega$, there exists a unique $z \in \partial \Omega$ such that $d(x) = |x - z|$. Then, a formal expansion of $u$ is given by

\begin{equation}
d^{-\frac{n-2}{2}} \left( 1 + \sum_{i=1}^{n-1} c_i d^i + \sum_{i=n}^{\infty} \sum_{j=0}^{N_i} c_{i,j} d^i \right),
\end{equation}

where $c_i$ and $c_{i,j}$ are smooth functions of $z \in \partial \Omega$, and $N_i$ is a nonnegative constant depending on $i$, with $N_n = 1$. A formal calculation can only determine finitely many terms in the formal expansion of $u$ near $\partial \Omega$. In fact, the coefficients $c_1, \ldots, c_{n-1}$ and $c_{n,1}$ have explicit expressions in terms of principal curvatures of $\partial \Omega$ and their derivatives. For example,

\begin{equation}
c_1 = \frac{n-2}{4(n-1)} H,
\end{equation}

and, for $n = 3$,

\begin{equation}
c_{3,1} = -\frac{1}{16} \left\{ \Delta_{\partial \Omega} H + 2H(H^2 - K) \right\},
\end{equation}

where $H$ and $K$ are the mean curvature and the Gauss curvature of $\partial \Omega$, respectively. We note that $c_{3,1} = 0$ if and only if $\partial \Omega$ is a Willmore surface.

Mazzeo [33] and Andersson, Chruściel, and Friedrich [4] proved that solution $u$ of (5.1)-(5.2) is polyhomogeneous if $\Omega$ has a smooth boundary.

To analyze behaviors of solutions near the boundary, we introduce a new function with the zero boundary value. Let $u \in C^\infty(\Omega)$ be a solution of (5.1)-(5.2). Set

\begin{equation}
u = d^{-\frac{n-2}{2}}(1 + v).
\end{equation}
If $\partial \Omega \in C^{1,1}$, then $v$ satisfies
\begin{equation}
S(v) = 0 \quad \text{in } \Omega,
\end{equation}
and, by (5.3),
\begin{equation}
|v| \leq Cd \quad \text{in } \Omega,
\end{equation}
where
\[ S(v) = d^2 \Delta v - (n-2)d\nabla d \cdot \nabla v - \frac{1}{2} (n-2)d \Delta d(1 + v) 
- \frac{1}{4} n(n-2) \left[ (1+v)^{\frac{n+2}{n-2}} - (1+v) \right]. \]

In particular, $v$ is continuous up to the boundary and $v = 0$ on $\partial \Omega$. We note that $S$ is a semilinear elliptic operator, degenerate along $\partial \Omega$. We rewrite $S$ as
\begin{equation}
S(v) = d^2 \Delta v - (n-2)d\nabla d \cdot \nabla v - nv - \frac{1}{2} (n-2)d \Delta dv - \frac{1}{2} (n-2)d \Delta d 
- \frac{1}{4} n(n-2) \left[ (1+v)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}v \right]. \end{equation}
In the expression of $S$ in (5.7), the first four terms are linear in $v$, the fifth term is the nonhomogeneous term, and the final term is a nonlinear expression of $v$. Methods in [15] and in this paper can be adapted to treat (5.5).

Let $k \geq n$ be an integer and set, for $z \in \partial \Omega$ and $d > 0$,
\begin{equation}
S_k(z,d) = 1 + \sum_{i=1}^{n-1} c_i(z)d^i + \sum_{i=n}^{k} \sum_{j=0}^{N_i} c_{i,j}(z)d^i(\log d)^j.
\end{equation}

We point out that the highest order in the parenthesis is given by $d^k$. According to the pattern in this expansion, if we intend to continue to expand, the next term has an order of $d^{k+1}(\log d)^{N_{k+1}}$.

Similarly as in [15], we have the regularity and growth of the remainder $d^{\frac{n+2}{2}}u - S_k$ as well as the regularity of the coefficients $c_i$ and $c_{i,j}$.

**Theorem 5.1.** For some integer $k \geq n$ and some constant $\alpha \in (0,1)$, assume $\partial \Omega \cap B_R(z_0)$ is $C^{k+1,\alpha}$, for some $z_0 \in \partial \Omega$ and $R > 0$, and let $u \in C^\infty(\Omega \cap B_R(z_0))$ be a solution of (5.1)-(5.2). Then, there exist functions $c_i, c_{i,j} \in C^{k-i,\epsilon}(\partial \Omega \cap B_R(z_0))$, for $i = 1, \cdots , k$ and $j = 0,1, \cdots , N_i$, and any $\epsilon \in (0,\alpha)$, such that, for $S_k$ defined as in (5.8), for any $m = 0,1, \cdots , k$, any $\epsilon \in (0, \alpha)$, and any $r \in (0, R)$,
\begin{equation}
\partial_r^m \left( d^{\frac{n+2}{2}}u(x) - S_k(z,d) \right) \in C^{\epsilon}(\bar{\Omega} \cap B_r(z_0)),
\end{equation}
and, for any $x \in \Omega \cap B_{R/2}(z_0)$,
\begin{equation}
\partial_d^m \left( d^{\frac{n+2}{2}}u(x) - S_k(z,d) \right) \leq Cd^{k-m+\alpha},
\end{equation}
where $d = d(x)$, $z \in \partial \Omega$ is the unique point with $d(x) = |x-z|$ and $C$ is a positive constant depending only on $n$, $k$, $\alpha$, $R$, the $L^\infty$-norm of $d^{\frac{n+2}{2}}u$ in $\Omega \cap B_R(z_0)$ and the $C^{k+2,\alpha}$-norm of $\partial \Omega \cap B_R(z_0)$.
Here we have one loss of regularity, but we remark there is no regularity loss for a similar result of function \(d^2u(x)\). Concerning the analyticity, we have the following result.

**Theorem 5.2.** Assume \(\partial \Omega \cap B_R(z_0)\) is analytic, for some \(z_0 \in \partial \Omega\) and \(R > 0\). Let \(u \in C^\infty(\Omega \cap B_R(z_0))\) be a solution of \((5.1)-(5.2)\). Then, \(u\) is analytic in \(z, d\) and \(d \log d\) in \(\Omega \cap B_{R/2}(x_0)\). Moreover, let \(S_k\) be defined as in \((5.8)\) satisfying \((5.10)\). Then,
\[
S_k(z, d) \to d^{\frac{n-2}{2}}u(x) \quad \text{uniformly in } \Omega \cap B_{R/2}(x_0).
\]

**Appendix A. Analyticity Estimates**

In this section, we present an analyticity type estimate for compositions of functions, which is due to Friedman. The following result is essentially Lemma 1 in [11] with \(M_l = l!\).

**Lemma A.1.** Let \(\Omega\) be a domain in \(\mathbb{R}^n\) and \(p\) be a positive integer. Assume that \(\Phi\) is a \(C^p\)-function in \(\Omega \times \mathbb{R}^N\) satisfying, for any \((x, y) \in \Omega \times \mathbb{R}^N\) and any nonnegative integers \(j\) and \(k\) with \(j + k \leq p\),
\[
|\partial_j^k \Phi(x, y)| \leq A_0 A_1^j A_2^k (j - 2)! (k - 2)!,
\]
for some positive constants \(A_0, A_1\) and \(A_2\). Then, there exist positive constants \(B_0, \tilde{B}_0\) and \(B_1\), depending only on \(n, N, A_0, A_1\) and \(A_1\), such that, for any \(\Psi : \Omega \to \mathbb{R}^N\), if for any \(x \in \Omega\) and any nonnegative integer \(k \leq p\),
\[
\sum_{i=1}^{N} |\partial_y^k y_i(x)| \leq B_0 B_1^{(k-2)^+} (k - 2)!,
\]
then, for any \(x \in \Omega\),
\[
|\partial_x^p [\Phi(x, y(x))]| \leq \tilde{B}_0 B_1^{(p-2)^+} (p - 2)!. \tag{A.2}
\]

**Proof.** Set \(t = x_1 + \cdots + x_n\). Fix an \(x \in \Omega\) and write \(y = y(x)\). We will construct scalar-valued \(C^p\)-functions \(z(t)\) and \(\Psi(t, z)\) such that, for \(k = 1, \cdots, p\),
\[
|\partial_{(x,y)}^k \Phi(x, y)| \leq \partial_{(t,z)}^k \Psi(0, 0), \tag{A.4}
\]
and
\[
\sum_{i=1}^{N} |\partial_y^k y_i(x)| \leq \frac{d^k}{dt^k} z(0). \tag{A.5}
\]
Also, we require \(z(0) = 0\).

First, we note
\[
\partial_x^p [\Phi(x, y(x))] = \sum_{\alpha_0 + \cdots + \alpha_k = p} \frac{\partial^{\alpha_0+k} \Phi}{\partial x^{\alpha_0} y_{i_1} \cdots y_{i_k}}(x, y) \frac{\partial^{\alpha_1} y_{i_1}}{\partial x^{\alpha_1}}(x) \cdots \frac{\partial^{\alpha_k} y_{i_k}}{\partial x^{\alpha_k}}(x),
\]
where the summation is for $\alpha_0, \alpha_1, \cdots, \alpha_k \in \mathbb{Z}_n^+$ with $|\alpha_0| + |\alpha_1| + \cdots + |\alpha_k| = p$ and also for $i_1, \cdots, i_k$ from 1 to $N$. By (A.4) and (A.5), we have

$$|\partial_x^p \Phi(x, y(x))| \leq \sum \frac{\partial^{\alpha_0 + k}_t \Psi_{\alpha_0}}{\partial z^k_{\alpha_0} \partial z^k_{\alpha_1}}(0, 0) \frac{d^{\alpha_1}_t z(0)}{d t^{\alpha_1}_t} \cdots \frac{d^{\alpha_k}_t z(0)}{d t^{\alpha_k}_t}.$$

Hence,

(A.6) $$|\partial_x^p \Phi(x, y(x))| \leq \frac{d^p}{d t^p} [\Psi(t, z(t))]_{t=0}.$$

In view of (A.1), we set

(A.7) $$\Psi(t, z) = \Psi_1(t) \Psi_2(z),$$

where

(A.8) $$\Psi_1(t) = \sum_{i=0}^{p} \frac{A_i (i-2)!}{i!} t^i,$$

and

(A.9) $$\Psi_2(z) = A_0 \sum_{i=0}^{p} \frac{A_i (i-2)!}{i!} z^i.$$

Then, (A.4) holds.

Next, we set

(A.10) $$z(t) = B_0 \left[ t + \sum_{k=2}^{p} \frac{1}{k(k-1)} B_1^{k-2} i^k \right].$$

By (A.2), (A.5) holds.

Now we start to estimate the right-hand side of (A.6). We claim, for any $i = 1, \cdots, p$,

(A.11) $$[z(t)]^i = B_0^i \left[ t^i + \sum_{k=2}^{p} \sum_{k+i+1} a_{i,k} B_1^{k-i-1} i^k \right] + O(t^{p+1}),$$

where $a_{i,k}$ is a nonnegative constant satisfying, for $1 \leq i < k \leq p$,

(A.12) $$a_{i,k} \leq \frac{3^{i-1}}{(k-i+1)(k-i)}.$$

To prove (A.11) and (A.12), we first note that (A.10) implies (A.11) with $i = 1$ and the equality holds in (A.12) with $i = 1$. We assume that (A.11) and (A.12) hold for some $i = 1, \cdots, p-1$. Next, we consider $i+1$. A simple multiplication of (A.10) and (A.11) yields

$$[z(t)]^{i+1} = B_0^{i+1} \left[ t^{i+1} + \sum_{k=2}^{p} B_1^{k-2} a_{1,k} t^{k+i} + \sum_{l=i+1}^{p} B_1^{l-i-1} a_{i,l} t^{l+1} \right. \left. + \sum_{k=2, l=i+1}^{p} B_1^{k+i-l-3} a_{1,k} a_{i,l} t^{k+l} \right] + O(t^{p+1}).$$
By a change of indices in summations, we have
\[
[z(t)]_{i+1} = B_0^{i+1} \left[ t^{i+1} + \sum_{m=i+2}^{p} B_1^{m-i-2} a_{i+1,m} t^m \right] + O(t^{p+1}),
\]
where, for \( m = i + 2, \ldots, p, \)
\[
a_{i+1,m} = a_{1,m-i} + a_{i,m-1} + \text{sgn}(m-i-2)B_1^{-1} \sum_{k+l=m \atop k \geq 2, l \geq i+1} a_{1,k} a_{i,l}.
\]

By (A.12), we have
\[
a_{i+1,m} \leq \frac{1 + 3^{i-1}}{(m-i)(m-i-1)} + \sum_{k+l=m \atop k \geq 2, l \geq i+1} \frac{1}{B_1^k(k-1)(l-i+1)(l-i)}.
\]

Note \( k \leq 2(k-1) \) for \( k \geq 2 \) and \( l-i+1 \leq 2(l-i) \) for \( l \geq i+1 \). Then,
\[
\frac{1}{k(k-1)(l-i+1)(l-i)} \leq \frac{4}{k^2(l-i+1)^2} = \frac{4}{(k+l-i+1)^2} \left[ \frac{1}{k} + \frac{1}{l-i+1} \right]^2
\]
\[
\leq \frac{8}{(k+l-i+1)^2} \left[ \frac{1}{k^2} + \frac{1}{l-i+1} \right].
\]

Hence,
\[
\sum_{k+l=m \atop k \geq 2, l \geq i+1} \frac{1}{k(k-1)(l-i+1)(l-i)} \leq \frac{8}{(m-i+1)^2} \left[ \sum_{k \geq 2} \frac{1}{k^2} + \sum_{l \geq i+1} \frac{1}{l-i+1} \right]
\]
\[
\leq \frac{16}{(m-i+1)^2} \left( \frac{\pi^2}{6} - 1 \right) \leq \frac{16}{(m-i+1)^2}.
\]

A simple substitution yields
\[
a_{i+1,m} \leq \frac{1 + 3^{i-1}}{(m-i)(m-i-1)} + \frac{3^{i-1} \cdot 16B_1^{-1}}{(m-i)(m-i-1)}.
\]

If \( B_1 \geq 16 \), then
\[
a_{i+1,m} \leq \frac{3 \cdot 3^{i-1}}{(m-i)(m-i-1)} = \frac{3^{i}}{(m-i)(m-i-1)}.
\]

This proves (A.11) and (A.12) for \( i+1 \).

By (A.9), we write
\[
\Psi_2(z) = A_0 \left[ 1 + A_2 z + \sum_{i=2}^{p} \frac{A_i^2}{i(i-1)} z^i \right],
\]
and hence
\[
\Psi_2(z(t)) = A_0 \left[ 1 + A_2 z(t) + \sum_{i=2}^{p} \frac{A_i^2}{i(i-1)} [z(t)]^i \right].
\]
We claim, for \( k = 1, \ldots, p, \)
\[
\frac{d^k}{dt^k} \Psi_2(z(t))|_{t=0} \leq \hat{B}_0 B_1^{(k-2)^*} (k-2)!,
\]
where
\[
\hat{B}_0 = A_0 B_0 (9 A_2 + A_2^2 B_0).
\]

First, we have
\[
\frac{d}{dt} \Psi_2(z(t))|_{t=0} = A_0 A_2 \frac{d}{dt} (0) = A_0 A_2 B_0.
\]
Then, (A.13) holds for \( k = 1 \) since \( \hat{B}_0 \geq A_0 A_2 B_0. \) Next, for \( k = 2, \ldots, p, \)
\[
\frac{d^k}{dt^k} \Psi_2(z(t))|_{t=0} = A_0 A_2 \frac{d^k z}{dt^k} (0) + A_0 \sum_{i=2}^{p} \frac{A_i^2}{i(i-1)} \frac{d^k}{dt^k} [z(t)]^i|_{t=0}
\]
\[
= A_0 A_2 B_0 B_1^{k-2} (k-2)! + A_0 \sum_{i=2}^{k-1} \frac{A_i^2}{i(i-1)} a_{i,k} B_0^i B_1^{k-i-1} k!
\]
\[
+ A_0 A_2 B_0^k (k-2)!
\]

Hence, by (A.12),
\[
\frac{d^k}{dt^k} \Psi_2(z(t))|_{t=0} \leq A_0 B_0 B_1^{k-2} (k-2)!
\]
\[
\cdot \left[ A_2 + \sum_{i=2}^{k-1} \frac{3^{i-1} A_i^2 B_0^{i-1}}{B_1^{i-1}} \frac{k(k-1)}{i(i-1)(k-i+1)(k-i)} + \frac{A_2 B_0^{k-1}}{B_1^{k-2}} \right].
\]
We note, for \( 2 \leq i \leq k-1, \)
\[
\frac{k(k-1)}{i(i-1)(k-i+1)(k-i)} \leq 8.
\]
To prove this, we consider \( 2 \leq i \leq k/2 \) first and have
\[
\frac{k(k-1)}{(k-i+1)(k-i)} \leq \frac{k(k-1)}{(k/2+1)k/2} \leq \frac{4(k-1)}{k+2} \leq 4.
\]
For \( k/2 < i \leq k-1, \) we have \( k \geq 3 \) and
\[
\frac{k(k-1)}{i(i-1)} < \frac{k(k-1)}{k/2(k/2-1)} \leq \frac{4(k-1)}{k-2} \leq 8.
\]
Therefore,
\[
\frac{d^k}{dt^k} \Psi_2(z(t))|_{t=0} \leq A_0 B_0 B_1^{k-2} (k-2)!
\]
\[
\cdot \left[ A_2 + 8 A_2 \sum_{i=2}^{k-1} \left( \frac{3 A_2 B_0}{B_1} \right)^{i-1} + A_2^2 B_0 \left( \frac{A_2 B_0}{B_1} \right)^{k-1} \right].
\]
If \( B_1 \geq 6 A_2 B_0, \) we obtain (A.13) for \( k = 2, \ldots, p. \)
Next, by (A.8), we have, for any $i = 0, 1, \cdots, p$,

\begin{equation}
\frac{d^i \Psi_1}{dt^i}(0) = A_1^i (i - 2)!
\end{equation}

With (A.7), (A.13) and (A.14), we have

\[
\frac{dp}{dt} \Psi(t, z(t))|_{t=0} = \sum_{i=0}^{p} \frac{p!}{i!(p-i)!} \frac{d^i}{dt^i} \Psi_2(z(t))|_{t=0} \cdot \frac{dp-i}{dt^{p-i}} \Psi_1|_{t=0}
\]

\[
\leq \sum_{i=0}^{p} \frac{p!}{i!(p-i)!} (p-i-2)! (i-2)! \tilde{B}_0 A_1^{p-i} B_1^{i-2} \]

\[
= \tilde{B}_0 B_1^{p-2} (p-2)! \sum_{i=0}^{p} \frac{p(p-1)}{i!(p-i)!} (p-i-2)! (i-2)! \frac{A_1^{p-i} B_1^{i-2}}{B_1^{p-2}}.
\]

We consider $p \geq 3$. By considering $i = 0$ and $p, i = 1$ and $p-1$, and $2 \leq i \leq p-2$, we have

\[
\frac{p(p-1)}{i!(p-i)!} (p-i-2)! (i-2)! \leq 8.
\]

Therefore,

\[
\frac{dp}{dt} \Psi(t, z(t))|_{t=0} \leq \sum_{i=0}^{p} \tilde{B}_0 B_1^{p-2} (p-2)! \left[ (A_1^2 + 3A_1) \left( \frac{A_1}{B_1} \right)^{p-2} + 8 \sum_{i=2}^{p} \left( \frac{A_1}{B_1} \right)^{p-i} \right].
\]

By taking $B_1 \geq 2A_1$, we have

\[
\frac{dp}{dt} \Psi(t, z(t))|_{t=0} \leq (A_1^2 + 3A_1 + 16) \tilde{B}_0 B_1^{p-2} (p-2)!.
\]

In summary, we take $B_1 \geq \max\{16, 6A_2 B_0, 2A_1\}$ and

\[
\tilde{B}_0 = (A_1^2 + 3A_1 + 16) \tilde{B}_0 = A_0 B_0 (9A_2 + A_2^2 B_0) (A_1^2 + 3A_1 + 16),
\]

and then have the desired result. \hfill \Box

**Remark A.2.** Write $x = (x', x_n)$. In Lemma [A.1] if we assume (A.1) and (A.2) hold only for $D_{x'}$ instead of $D_x$, then (A.3) holds for $D_{x'}$.  

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