Theory of inner product vector and its application to multi-location damage detection

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Abstract. Structural damage detection methods using time domain vibration responses have shown appeal in recent years. In previous papers by the authors, the inner product vector (IPV) was proposed as a damage detection algorithm which uses cross correlation functions between vibration responses under white noise excitation or band pass white noise excitation. The proposed algorithm was verified by some simulative and experimental examples featuring a single damage location. However, damage at multiple locations was not considered. Therefore, this paper extends the application of IPV-based structural damage detection to the problem of multiple damage locations. Firstly, the theory of the IPV and its implementation in a damage detection context is reviewed. Then, two strategies for detecting multiple damages utilizing IPV are proposed. Finally, a damage detection experiment of a honeycomb sandwich composite beam is adopted to illustrate the feasibility and effectiveness of the IPV-based damage detection method.

1. Introduction
Vibration-based structural damage detection methods have received considerable attention in recent decades for a variety of reasons such as their potential to observe damage from sensors placed remote from an unknown damage site [1]. Non-model based structural damage detection methods are appealing in this research field as they utilize vibration responses directly and complex model updating procedures can be avoided [2]. Various dynamic properties, such as model shapes [3], frequency response functions [4], time domain vibration responses [5], cross correlation functions [2, 6], coherence functions [7] and transmissibility function [8] have been adopted in non-model based structural damage detection methods.

Utilizing the cross correlation functions between vibration responses under white noise excitation or band pass white noise excitation, the authors of this paper proposed a non-model based damage detection method, i.e. structural damage detection method based on inner product vector (IPV) [2, 6], and the proposed algorithm was verified by some simulative and experimental examples of single location damage detection. However, multiple damage locations, which might be present in a structure, were not considered. The purpose of this paper is to review the theory of IPV-based method and extend its application to the multi-location damage detection.

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2. Definition of inner product vector

Let us denote the displacement responses of measurement points 1, 2, ..., p as \( x_1, x_2, \ldots, x_p \). Then, considering the cross correlation function between all combinations of displacement responses, and setting the time lag \( T = 0 \), a \( p \times p \) dimensional matrix can be obtained,

\[
\begin{bmatrix}
    R_{x_1x_1}(0) & R_{x_1x_2}(0) & \cdots & R_{x_1x_p}(0) \\
    R_{x_2x_1}(0) & R_{x_2x_2}(0) & \cdots & R_{x_2x_p}(0) \\
    \vdots & \vdots & \ddots & \vdots \\
    R_{x_px_1}(0) & R_{x_px_2}(0) & \cdots & R_{x_px_p}(0)
\end{bmatrix}
\]

where \( R_{x_ix_j}(0) \) indicates the cross correlation function between displacement response \( x_i \) and \( x_j \). Then, the Inner Product Vector is defined by one row (or column) of the \( p \times p \) dimensional matrix, as follows [6]

\[
R^{\text{dis}}_{\text{IPV},j} = [R_{x_jx_1}(0), R_{x_jx_2}(0), \ldots, R_{x_jx_p}(0)]^T
\]

where the superscript \( \text{dis} \) indicates that the quantity is related to displacement.

Based on the auto spectral density of band pass white noise the IPV can be written as [6]

\[
R^{\text{dis}}_{\text{IPV},j} = \frac{1}{N_x} [\langle x_1, x_j \rangle, \langle x_2, x_j \rangle, \ldots, \langle x_p, x_j \rangle]^T
\]

where \( \phi_r = [\phi_{r1}, \phi_{r2}, \ldots, \phi_{rp}]^T \) is the \( r \)th mode shape, \( \kappa_{j,r}^{\text{dis}} \) is a coefficient dependent on the \( r \)th modal parameters and the response measurement point \( j \) and the excitation positions, \( \langle x, y \rangle \) is the inner product of the two vectors \( x \) and \( y \), \( N_x \) is the length of \( x \) or \( y \). Therefore, Eq.(3) shows that the IPV is a weighted summation of the mode shapes of the structure, and the weighting factor of each mode shape only depends on the modal parameters of the structure. Meanwhile, Eq.(4) shows that the IPV can be directly calculated by the time domain vibration responses. As we know, changes in local physical parameters can induce abrupt changes in some mode shapes. Accordingly, the IPV of the damaged structure may also have abrupt changes. Thus, the IPV may be adopted as a damage feature vector for structural damage detection.

Similarly to Eq.(2), the IPVs \( R^{\text{vel}}_{\text{IPV},j} \) and \( R^{\text{acc}}_{\text{IPV},j} \), which are constructed by velocity and acceleration, respectively, are defined as

\[
R^{\text{vel}}_{\text{IPV},j} = [R_{x_jx_1}(0), R_{x_jx_2}(0), \ldots, R_{x_jx_p}(0)]^T
\]

\[
R^{\text{acc}}_{\text{IPV},j} = [R_{x_jx_1}(0), R_{x_jx_2}(0), \ldots, R_{x_jx_p}(0)]^T
\]

It is straightforward to verify that IPVs of velocity and acceleration can be written as

\[
R^{\text{vel}}_{\text{IPV},j} = \sum_{r=1}^{n} \kappa_{j,r}^{\text{vel}} [\phi_{r1}, \phi_{r2}, \ldots, \phi_{rp}]^T = \sum_{r=1}^{n} \kappa_{j,r}^{\text{vel}} \phi_r
\]

\[
R^{\text{acc}}_{\text{IPV},j} = \sum_{r=1}^{n} \kappa_{j,r}^{\text{acc}} [\phi_{r1}, \phi_{r2}, \ldots, \phi_{rp}]^T = \sum_{r=1}^{n} \kappa_{j,r}^{\text{acc}} \phi_r
\]

and computed by

\[
R^{\text{vel}}_{\text{IPV},j} = \frac{1}{N_x} [\langle \dot{x}_1, \dot{x}_j \rangle, \langle \dot{x}_2, \dot{x}_j \rangle, \ldots, \langle \dot{x}_p, \dot{x}_j \rangle]^T
\]

\[
R^{\text{acc}}_{\text{IPV},j} = \frac{1}{N_x} [\langle \ddot{x}_1, \ddot{x}_j \rangle, \langle \ddot{x}_2, \ddot{x}_j \rangle, \ldots, \langle \ddot{x}_p, \ddot{x}_j \rangle]^T
\]
where $\kappa_{j,r}^{vel}$ and $\kappa_{j,r}^{acc}$ are coefficients dependent on the $r$th modal parameters and the response measurement point $j$ and the excitation positions, the superscript $vel$ or $acc$ indicate that the quantity is related to velocity or acceleration, respectively. Similarly to the IPV defined by displacement, the IPV defined by velocity or acceleration may also be adopted as a damage feature vector for structural damage detection.

### 3. Damage index using inner product vector

It was verified previously [2, 6] that whereas the point inner product (i.e. the autocorrelation of a response) in the IPV is polluted by measurement noise the cross inner product (i.e. cross correlation between two different responses) is not. However, this result assumes that the measurement noise 1) has zero mean; 2) is independent of the vibration response without measurement noise; and 3) of different measurement points are independent of each other.

As the measurement noise does not satisfy the above three assumptions well in practice, both the cross inner product and point inner product will be polluted by measurement noise. Thus, a local damage index defined by the difference between the IPVs of the intact and damaged structures is utilized to decrease the effect of measurement noise [6], i.e.

$$D_{IPV,j}^d = R_{IPV,j}^d - R_{IPV,j}^{ia}$$

where $R_{IPV,j}^d$ and $R_{IPV,j}^{ia}$ indicate the $i$th element in the IPVs of the intact and damage structures, respectively. Then, the damage index is defined as $D_{IPV} = \{D_{IPV,1}^d, D_{IPV,2}^d, \ldots, D_{IPV,N_e}^d\}$.

In order to utilize the local maximum of the damage index to locate the damage, three different damage indices were proposed for three different cases [6]: 1) damage index $D_{IPV}$ is adopted when the abrupt changes in $D_{IPV}$ is “impulse change”; 2) damage index $D_{IPV}'$ (i.e. the first-order difference of $D_{IPV}$) is adopted when the abrupt changes in $D_{IPV}$ is “step change”; 3) damage index $D_{IPV}''$ (i.e. the second-order difference of $D_{IPV}$) is adopted when the abrupt changes in $D_{IPV}$ is “weak impulse change”.

For the damage detection method in which damage is detected by the local maximum of the damage index, a threshold for classifying the damaged and intact structure should be selected. In this paper, the threshold utilized in our previous research is adopted [6], i.e.

$$t_h = \mu_D + \alpha_c \sigma_D$$

$$t_l = \mu_D - \alpha_c \sigma_D$$

where $\mu_D$ and $\sigma_D$ are the mean value and standard deviation of $D_{IPV}$ (or $D_{IPV}'$ or $D_{IPV}''$), respectively, and $\alpha_c$ is a coefficient corresponding to a confidence interval. When the elements of $D_{IPV}$ (or $D_{IPV}'$ or $D_{IPV}''$) fall in the region between $t_l$ and $t_h$, the structure is judged to be intact; otherwise, the structure is damaged.

### 4. Multi-location damage detection strategy using inner product vector

In our previous research involving the IPV-based damage detection method, only a single damage location was considered. However, damage might occur at multiple locations and so this section proposes two potential strategies for multi-location damage detection using the IPV.

Strategy 1: Compare the IPV of the structure with multiple damages to the intact structure as before and look for multiple outliers. The confidence interval factor $\alpha_c$ may need to be adjusted. However, similar to many damage detection methods in which damage is detected by the local maximum of the
damage index, it is difficult to select an efficient confidence interval factor $\alpha_c$, and the confidence interval factor $\alpha_c$ can only be selected based on experience.

Strategy 2: Compare the IPV of the structure with multiple damages to a previously measured IPV with one fewer damage locations (assuming that damages occur singly and responses are continually monitored). By a process of iteration damage locations can thereby be identified one by one.

5. Experimental validation

Figure 1 honeycomb sandwich composite beam and test setup [6]

In our previous research, a honeycomb sandwich composite beam with a single damage was adopted as the experimental example to validate the IPV-based method. In this research, a second debonding damage is inflicted on the same beam. The test setup is identical to the previous research [6], as shown in Figure 1. The damage sites are at about 100–150mm and 250–300mm from the clamped end.

Figure 2 shows indices $D_{IPV}$ and $D'_{IPV}$ for strategy 1 which feature abrupt changes at the damage locations. However, the choice of confidence interval factor $\alpha_c$ seriously affects whether the damages are detected. When the confidence interval factor $\alpha_c = 1.8$, neither of the two damages can be detected; when the confidence interval factor $\alpha_c = 1.5$, only one of the damages can be detected; and when the confidence interval factor is decreased to $\alpha_c = 1.2$, both the damages can be detected. Thus, in this case a lower confidence interval factor should be selected for multi-location damage when utilizing strategy 1. As shown in Figure 3, the second damage (i.e. debonding between 250–300mm) in the beam can be clearly detected using the confidence interval factor utilized by single-location damage (i.e. $\alpha_c = 1.5$ or $\alpha_c = 1.8$). Thus, the confidence interval factor for the multi-location damage detection might be the same as the confidence interval factor for the single-location damage detection.
when utilizing strategy 2. It should be notified that: as the first order difference of $D_{IPV}$ is utilized to locate damage in this paper, one needs at least four sensors around the damage location (i.e. two sensors on one side and two sensors on the other side).

Figure. 2 damage detection results using strategy 1

Figure. 3 damage detection results using strategy 2

6. Conclusions
The theory of inner product vector which uses cross correlation functions between vibration responses under band pass white noise excitation is reviewed, including the definition of IPV and the damage index using IPV. Two strategies for multi-location damage detection utilizing IPV are then proposed. The feasibility and effectiveness of using the IPV technique for multi-location damage detection is illustrated by a two-location damage detection experiment of a honeycomb sandwich composite beam. However, the influence of other factors (such as temperature, gradual global stiffness changes, etc) on the method is not investigated in the current research, and the filter technique or principle component analysis may be adopted in the future work to consider these factors.

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