Weingarten calculus for matrix ensembles associated with compact symmetric spaces

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Abstract

A method for computing integrals of polynomial functions on compact symmetric spaces is given. Those integrals are expressed as sums of functions on symmetric groups.

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1 Introduction

Let $G$ be the unitary group $U(n)$, orthogonal group $O(n)$, or (compact) symplectic group $Sp(2n)$, equipped with its Haar probability measure, and suppose that $G$ is realized as a matrix group. Consider a random matrix $X = (x_{ij})$ picked up from $G$. Moments of matrix elements

$$E[x_{i_1j_1} \cdots x_{i_kj_k} x_{i'_1j'_1} \cdots x_{i'_lk'_l}]$$

for $G = U(n)$,

$$E[x_{i_1j_1} \cdots x_{i_kj_k}]$$

for $G = O(n)$ or $Sp(2n)$ (1.1)

have been one of main interesting objects in random matrix theory. Weingarten [25] tried to understand the moments (1.1) for large $n$. Collins [3] established a method of computations of (1.1) for $U(n)$ with all finite $n$ and called his method the Weingarten calculus. The moments are expressed as sums in terms of class functions $W_g^{U}(\cdot; n)$ on the symmetric group $S_k$. The function $W_g^{U}(\cdot; n)$, called the unitary Weingarten function, has rich combinatorial structures involving Jucys-Murphy elements [22, 23]. The Weingarten calculus for $O(n)$ was constructed in [8, 5], and the corresponding Weingarten functions, called orthogonal Weingarten functions, are $H_k$-biinvariant functions on $S_{2k}$, where $H_k$ is the hyperoctahedral subgroup of $S_{2k}$. The orthogonal Weingarten function also bears nice combinatorics [19, 26]. The Weingarten calculus for $Sp(2n)$ was mentioned in [8, 9]. See also a recent preprint [10], in which Brownian motions on classical groups are studied. In the last decade, Weingarten calculus has been widely applied: Harish-Chandra–Itzykson–Zuber integrals [3, 16], quantum information theory [7, 4], designs [24], and statistics [6].
In his pioneering works for statistical mechanics, Dyson [12, 13, 14] introduced three important classes of random unitary matrices as modifications of Gaussian matrix ensembles. These unitary matrix ensembles are known as circular orthogonal/unitary/symplectic ensembles (COE/CUE/CSE). Whereas the CUE is nothing but the unitary group equipped with the Haar measure, the COE and CSE are not Lie groups but compact symmetric spaces.

Generally, a classical compact symmetric space is of the form $G/K$, where $G$ is a unitary, orthogonal, or symplectic group, and $K$ is a closed subgroup of $G$ consisting of fixed-points of an involution $\Omega$ on $G$. Let $\mathcal{S}$ be the image of the map $G \ni g \mapsto \Omega(g)^{-1}g$. We thus obtain a matrix realization of $G/K$:

$$G/K \simeq \mathcal{S}; \quad G \ni g \mapsto \Omega(g)^{-1}g \in \mathcal{S}, \quad K = \{h \in G \mid \Omega(h) = h\}.$$ 

Moreover, a $G$-invariant probability measure on $\mathcal{S}$ is induced from the Haar measure on $G$ via this map. Then the matrix space $\mathcal{S}$ equipped with the probability measure is the matrix ensemble associated with $G/K$. In the case of the COE, $G/K = U(n)/O(n)$, $\Omega(g) = \overline{g}$ (the complex-conjugation of $g$), and $\mathcal{S}$ is the set of $n \times n$ symmetric unitary matrices. Cartan [1] classified classical compact symmetric spaces into seven infinite series labeled by A I, A II, A III, BD I, C I, C II, and D III (see Figure I). Thus we obtain seven series of random matrix ensembles associated with these compact symmetric spaces. Eigenvalue distributions for them have been studied, see, e.g., [11], [2], and [15, §3.7.2 and 3.7.3]. In contrast, distributions for those matrix elements were not studied so much. Ones can see a small work for only classes A I and A II in [15, §2.3.2].

In this article, we will construct Weingarten calculus for random matrices associated with each class $\mathcal{C} = A I, A II, ...$ in Figure I. A development in a similar direction can be seen in [9], but only asymptotic behaviors of two-degree moments were observed in that article. As in the case of classical groups $U(n), O(n), Sp(2n)$, we will express the moments of random matrix elements as sums over symmetric groups. To do it, we need a distinguishing Weingarten function $W_{g^C}$ for each $\mathcal{C}$.

This article is organized as follows. In Section 2, we review the Weingarten calculus for $U(n), O(n),$ and $Sp(2n)$, developed in [3, 5, 8, 9]. Especially, the case for symplectic groups is discussed in detail. It is described in [9, 10], however, unlike their descriptions, we introduce the symplectic Weingarten function in terms of twisted zonal spherical functions for the twisted Gelfand pair $(S_{2k}, H_k, \epsilon|_{H_k})$. In Section 3, we discuss the Weingarten calculus for random matrices from class A I and A II, which are the COE and CSE, respectively. The COE case ($\mathcal{C} = A I$) is already given in [20, 21] in detail but we present it here again for readers’ convenience. In Section 4 and 5, we discuss the Weingarten calculus for chiral ensembles (A III, BD I, C II) and Bogoliubov-de Gennes (BdG) ensembles (D III, C I), respectively. In the final section, we give a short conclusion for new Weingarten functions.
Figure 1: List of classical compact symmetric spaces. Remark that we consider the full groups \(U(n)\) and \(O(n)\) rather than special ones \(SU(n)\) and \(SO(n)\), the latter groups of which are usually used in Lie theory.

## 2 Weingarten calculus for classical groups

### 2.1 Weingarten calculus for unitary groups

In this subsection, we review the Weingarten calculus for unitary groups. See [3, 8, 6] for details.

#### 2.1.1 Partitions and cycle-types

A partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)\) of a positive integer \(k\) is a weakly decreasing sequence of positive integers with \(|\lambda| = \sum_{i=1}^{l} \lambda_i = k\). We write \(\ell(\lambda)\) for the length \(l\) of \(\lambda\). When \(\lambda\) is a partition of \(k\), we write \(\lambda \vdash k\).

Let \(S_k\) be the symmetric group acting on \([k] = \{1, 2, \ldots, k\}\). A permutation \(\sigma \in S_k\) is usually expressed in the two-array notation \(\sigma = \begin{pmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{pmatrix}\), but transpositions are often written as \((i j)\) shortly. Denote by \(id\) the identity permutation in \(S_k\). If a permutation \(\sigma\) in \(S_k\) is decomposed into disjoint cycles with lengths \(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l\), then the partition \(\mu = (\mu_1, \mu_2, \ldots, \mu_l)\) of \(k\) is called the cycle-type of \(\sigma\). For example, the cycle-type of the permutation \((1 3 2) (4 5 6)\) in \(S_6\) is \((3, 2, 1)\).

#### 2.1.2 Unitary Weingarten functions

Let \(L(S_k)\) be the algebra of complex-valued functions on \(S_k\) with convolution

\[
(f_1 * f_2)(\sigma) = \sum_{\tau \in S_k} f_1(\tau) f_2(\tau^{-1}\sigma) \quad (f_1, f_2 \in L(S_k), \ \sigma \in S_k).
\]

The identity element in the algebra \(L(S_k)\) is the Dirac function \(\delta_{id_k}\).

Let \(Z(L(S_k))\) be the center of \(L(S_k)\): \(Z(L(S_k)) = \{ h \in L(S_k) \mid h * f = f * h \ (f \in L(S_k)) \}\). For a complex number \(z\), we define the element \(T^U(\cdot; z)\) in \(Z(L(S_k))\) by

\[
T^U(\sigma; z) = z^{\ell(\mu)} \quad (\sigma \in S_k),
\]
where $\mu$ is the cycle-type of $\sigma$. Note that the upper index $U$ stands for the unitary group. The class function $T^U(\cdot; z)$ can be expanded in terms of irreducible characters $\chi^{\lambda}$ of $S_k$ as follows:

$$T^U(\cdot; z) = \frac{1}{k!} \sum_{\lambda \vdash k} f^{\lambda} C_{\lambda}(z) \chi^{\lambda},$$

where $f^{\lambda} := \chi^{\lambda}(\text{id}_k)$ is the dimension of the irreducible representation associated with $\lambda$ and $C_{\lambda}(z)$ is the polynomial in $z$ given by

$$C_{\lambda}(z) = \prod_{(i,j) \in \lambda} (z + j - i).$$

Here $(i, j) \in \lambda$ stands for $1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i$. In other words, the $(i, j)$ are coordinates of the Young diagram of $\lambda$.

The unitary Weingarten function $W_g^U(\cdot; z)$ on $S_k$ is defined by

$$W_g^U(\cdot; z) = \frac{1}{k!} \sum_{\lambda \vdash k} \frac{f^{\lambda}}{C_{\lambda}(z)} \chi^{\lambda},$$

summed over all partitions $\lambda$ of $k$ with $C_{\lambda}(z) \neq 0$. It is the pseudo-inverse element of $T^U(\cdot; z)$, i.e., the unique element in $\mathcal{Z}(L(S_k))$ satisfying

$$T^U(\cdot; z) * W_g^U(\cdot; z) * T^U(\cdot; z) = T^U(\cdot; z).$$

In particular, unless $z \in \{0, \pm 1, \pm 2, \ldots, \pm (k-1)\}$, functions $T^U(\cdot; z)$ and $W_g^U(\cdot; z)$ are inverse of each other and satisfy $T^U(\cdot; z) * W_g^U(\cdot; z) = \delta_{\text{id}_k}$.

### 2.1.3 Integrals on unitary groups

The unitary group $U(n)$ is $U(n) = \{U \in \text{GL}(n, \mathbb{C}) \mid UU^* = I_n\}$, which has the Haar probability measure. Here $U^* := \overline{U^T}$ is the adjoint matrix of $U$ and $I_n = (\delta_{ij})_{1 \leq i,j \leq n}$ is the $n \times n$ identity matrix. We will also write $U$ for a random element of $U(n)$, with distribution given by Haar measure.

**Theorem 2.1** (Weingarten calculus for unitary groups [3, 8]). Let $U = (u_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ Haar unitary matrix. For four sequences $i = (i_1, \ldots, i_k), j = (j_1, \ldots, j_k), i' = (i'_1, \ldots, i'_k), j' = (j'_1, \ldots, j'_k)$ of positive integers in $[n]$, we have

$$\mathbb{E}[u_{i_1j_1} \cdots u_{i_kj_k} u_{i'_1j'_1} \cdots u_{i'_kj'_k}] = \sum_{\sigma, \tau \in S_k} \delta_\sigma(i, i') \delta_\tau(j, j') W_g^U(\sigma^{-1} \tau; n).$$

Here $\delta_\sigma(i, i')$ is defined by

$$\delta_\sigma(i, i') = \prod_{s=1}^k \delta_{i_{\sigma(s)}/i'_s}. \quad (2.2)$$
If $k \neq l$, then $E[\prod_{i,j} u_{i k j} u_{i' k' j'}] \equiv 0$ for any indices $i, \ldots, i_k$, $j, \ldots, j_k$, $i', \ldots, i'_k$, $j', \ldots, j'_l$.

Example 2.1. $W^U_g(id_1; n) = \frac{1}{n}$; $W^U_g(id_2; n) = \frac{1}{(n + 1)(n - 1)}$; $W^U_g((1 2); n) = -\frac{1}{n(n + 1)(n - 1)}$.

We can see more examples in [3].

Remark 2.1. In the sum of (2.1), we can drop the restriction for $\lambda$. Even if we replace the definition by $W^U_g(\cdot; z) = \sum_{\lambda \vdash k} f^\lambda C^\lambda_z \chi^\lambda$, which is a rational function in $z \in \mathbb{C}$ with finitely many poles, then Theorem 2.1 remains true after cancellations of poles. This enables us to ignore the restriction $C^\lambda_z \neq 0$ in latter sections. See [8, Proposition 2.5] for details.

2.2 Weingarten calculus for orthogonal groups

In this subsection, we review the Weingarten calculus for orthogonal groups, developed in [8, 5, 6]. The theory of zonal spherical functions for finite Gelfand pairs are seen in [18, Chapter VII].

2.2.1 Hyperoctahedral groups and coset-types

Let $H_k$ be the hyperoctahedral subgroup of $S_{2k}$ with order $2^k k!$, generated by adjacent transpositions $(2i - 1 2i)$ ($1 \leq i \leq k$) and double transpositions $(2i - 1 2i - 1)(2i 2j)$ ($1 \leq i < j \leq k$). Let $M_{2k}$ be the subset of permutations $\sigma \in S_{2k}$ satisfying

$$\sigma(2i - 1) < \sigma(2i) \quad (1 \leq i \leq k) \quad \text{and} \quad \sigma(1) < \sigma(3) < \cdots < \sigma(2k - 1).$$

The elements $\sigma$ in $M_{2k}$ form the complete set of the representatives of $S_{2k}/H_k$ and are sometimes identified with perfect matchings of the forms

$$\{\{\sigma(1), \sigma(2)\}, \{\sigma(3), \sigma(4)\}, \ldots, \{\sigma(2k - 1), \sigma(2k)\}\}$$

on $[2k]$.

For a permutation $\sigma \in S_{2k}$, we define the graph $\Gamma(\sigma)$ as follows. The vertex set is $\{1, 2, \ldots, 2k\}$ and the edge set consists of red edges $\{2r - 1, 2r\}$ and blue edges $\{\sigma(2r - 1), \sigma(2r)\}$, where $r$ runs over $1, 2, \ldots, k$. Then all connected components of the graph $\Gamma(\sigma)$ have even vertices with numbers $2\mu_1 \geq 2\mu_2 \geq \cdots \geq 2\mu_l$. We call the partition $\mu := (\mu_1, \mu_2, \ldots, \mu_l) \vdash k$ the coset-type of $\sigma$. For example, the coset-type of the permutation
Specifically, for two permutations \( \sigma \) in \( S_6 \) is (2, 1). The coset-type distinguishes with double cosets of \( H_k \) in \( S_{2k} \). Specifically, for two permutations \( \sigma, \tau \) in \( S_{2k} \), their coset-types coincides if and only if \( H_k \sigma H_k = H_k \tau H_k \) (\cite{Eilenberg} VII, (2.1)).

For a partition \( \mu = (\mu_1, \mu_2, \ldots, \mu_l) \) of \( k \), we define the permutation \( \sigma_\mu \) in \( S_{2k} \) as follows: For each \( r = 1, 2, \ldots, l \),

\[
\begin{align*}
\sigma_\mu (2 \sum_{i=1}^{r-1} \mu_i + 1) &= 2 \sum_{i=1}^{r-1} \mu_i + 1, \\
\sigma_\mu (2 \sum_{i=1}^{r-1} \mu_i + 2) &= 2 \sum_{i=1}^{r-1} \mu_i + 2 \mu_r, \\
\sigma_\mu (2 \sum_{i=1}^{r-1} \mu_i + p) &= 2 \sum_{i=1}^{r-1} \mu_i + p - 1 \quad \text{for } p = 3, 4, \ldots, 2 \mu_r.
\end{align*}
\]

For example, \( \sigma_{(3,1)} = (\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 3 & 6 & 4 \end{array}) \). It is easy to see that the permutation \( \sigma_\mu \) belongs to \( M_{2k} \) and has the coset-type \( \mu \) and signature +1. In particular, \( \sigma_{(1^k)} = id_{2k} \), the identity permutation in \( S_{2k} \). We often regard \( \sigma_\mu \) as a typical permutation of coset-type \( \mu \).

### 2.2.2 Orthogonal Weingarten functions and Gelfand pairs

Let \( L(S_{2k}, H_k) \) be the subspace of all \( H_k \)-biinvariant functions in \( L(S_{2k}) \):

\[
L(S_{2k}, H_k) = \{ f \in L(S_{2k}) \mid f(\zeta \sigma) = f(\sigma \zeta) = f(\sigma) \ (\sigma \in S_{2k}, \ \zeta \in H_k) \}.
\]

It is well known that \( (S_{2k}, H_k) \) is a Gelfand pair, i.e., \( L(S_{2k}, H_k) \) is a commutative algebra under the convolution \( (\ref{convolution}) \).

For two functions \( f_1, f_2 \) in \( L(S_{2k}) \), we define a new product \( f_1 \ast f_2 \) by

\[
(f_1 \ast f_2)(\sigma) = \sum_{\tau \in M_{2k}} f_1(\tau)f_2(\tau^{-1}\sigma) \quad (\sigma \in S_{2k}).
\]

**Example 2.2.** For \( f_1, f_2 \in L(S_4) \),

\[
\begin{align*}
(f_1 \ast f_2)(1, 2, 3, 4) &= f_1(1, 2, 3, 4) f_2(1, 2, 3, 4) + f_1(1, 2, 3, 4) f_2(1, 2, 3, 4) + f_1(1, 2, 3, 4) f_2(1, 2, 3, 4), \\
(f_1 \ast f_2)(1, 2, 3, 4) &= f_1(1, 2, 3, 4) f_2(1, 2, 3, 4) + f_1(1, 2, 3, 4) f_2(1, 2, 3, 4) + f_1(1, 2, 3, 4) f_2(1, 2, 3, 4), \\
(f_1 \ast f_2)(1, 2, 3, 4) &= f_1(1, 2, 3, 4) f_2(1, 2, 3, 4) + f_1(1, 2, 3, 4) f_2(1, 2, 3, 4) + f_1(1, 2, 3, 4) f_2(1, 2, 3, 4).
\end{align*}
\]

For \( f_1, f_2 \in L(S_{2k}, H_k) \), we have \( f_1 \ast f_2 = (2^k k!)^{-1} f_1 \ast f_2 \in L(S_{2k}, H_k) \). In fact, since \( M_{2k} \) is the complete set of representatives of cosets \( \sigma H_k \) in \( S_{2k} \) and since \( f_1, f_2 \) are \( H_k \)-biinvariant, we have

\[
(f_1 \ast f_2)(\sigma) = \sum_{\tau \in M_{2k}} \sum_{\zeta \in H_k} f_1(\tau \zeta)f_2((\tau \zeta)^{-1}\sigma) = \sum_{\tau \in M_{2k}} \sum_{\zeta \in H_k} f_1(\tau)f_2(\tau^{-1}\sigma) = |H_k| f_1 \ast f_2(\sigma).
\]

Hence \( L(S_{2k}, H_k) \) is a commutative algebra under the product \( \ast \) with the identity element

\[
1_k(\sigma) = \begin{cases} 
1 & \text{if } \sigma \in H_k, \\
0 & \text{if } \sigma \in S_{2k} \setminus H_k.
\end{cases}
\]
For each partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \vdash k \), we define the zonal spherical function \( \omega^\lambda \) of the Gelfand pair \((S_{2k}, H_k)\) by
\[
\omega^\lambda = (2^k k!)^{-1} \chi^{2\lambda} * 1_k
\]
with \( 2\lambda = (2\lambda_1, 2\lambda_2, \ldots) \). The \( \omega^\lambda, \lambda \vdash k \), form a linear basis of \( L(S_{2k}, H_k) \) and satisfy the orthogonality relation
\[
\omega^\lambda * \omega^\mu = \delta_{\lambda\mu} \frac{(2k)!}{2^k k!} \frac{1}{f^{2\lambda}} \omega^\lambda
\]  
([18, VII, (1.4)]).

For a complex number \( z \), we define the element \( T^O(\sigma; z) \) in \( L(S_{2k}, H_k) \) by
\[
T^O(\sigma; z) = z^{\ell(\mu)} \quad (\sigma \in S_{2k}),
\]
where \( \mu \) is the coset-type of \( \sigma \). We emphasize that \( T^O \) is different from \( T^U \). The function \( T^O(\cdot; z) \) is expanded in terms of \( \omega^\lambda \) as follows ([5, (4.5)]):
\[
T^O(\cdot; z) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f^{2\lambda} D_\lambda(z) \omega^\lambda,
\]
where \( D_\lambda(z) \) is the polynomial in \( z \) given by
\[
D_\lambda(z) = \prod_{(i, j) \in \lambda} (z + 2j - i - 1).
\]

The orthogonal Weingarten function \( Wg^O(\cdot; z) \) on \( S_{2k} \) is defined by
\[
Wg^O(\cdot; z) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k, D_\lambda(z) \neq 0} f^{2\lambda} D_\lambda(z) \omega^\lambda,
\]
which is the pseudo-inverse element of \( T^O(\cdot; z) \), i.e., the unique element in \( L(S_k, H_k) \) satisfying
\[
T^O(\cdot; z) * Wg^O(\cdot; z) * T^O(\cdot; z) = T^O(\cdot; z).
\]  
(2.5)

In particular, if \( D_\lambda(z) \neq 0 \) for all partitions \( \lambda \) of \( k \), functions \( T^O(\cdot; z) \) and \( Wg^O(\cdot; z) \) are their inverse of each other and satisfy \( T^O(\cdot; z) * Wg^O(\cdot; z) = 1_k \). Those relations follow from (2.4) and expansions of \( T^O(\cdot; z) \) and \( Wg^O(\cdot; z) \) in terms of \( \omega^\lambda \).

### 2.2.3 Integrals on orthogonal groups

The (real) orthogonal group is \( O(n) = \{ R \in GL(n, \mathbb{R}) \mid RR^T = I_n \} \) and has the Haar probability measure.
Theorem 2.2 (Weingarten calculus for orthogonal groups \[8, 5\]). Let \( R = (r_{ij})_{1 \leq i,j \leq n} \) be an \( n \times n \) Haar orthogonal matrix. For two sequences \( i = (i_1, \ldots, i_{2k}) \) and \( j = (j_1, \ldots, j_{2k}) \) of positive integers in \([n]\), we have

\[
\mathbb{E}[r_{i_1 j_1} \cdots r_{i_{2k} j_{2k}}] = \sum_{\sigma, \tau \in M_{2k}} \Delta_\sigma(i) \Delta_\tau(j) Wg^O(\sigma^{-1} \tau; n).
\]

Here \( \Delta_\sigma(i) \) is defined by

\[
\Delta_\sigma(i) = \prod_{s=1}^k \delta_{i_{(2s-1)i_{2s}}} \quad (2.6)
\]

Furthermore, \( \mathbb{E}[r_{i_1 j_1} \cdots r_{i_{2k+1} j_{2k+1}}] = 0 \) for any \( i_1, \ldots, i_{2k+1}, j_1, \ldots, j_{2k+1} \).

Example 2.3. \( Wg^O(id_2; n) = \frac{1}{n} \); \( Wg^O(id_4; n) = \frac{n+1}{n(n+2)(n-1)} \), \( Wg^O(\sigma(2); n) = \frac{-1}{n(n+2)(n-1)} \).

We can see more examples in \[8, 5\].

2.3 Weingarten calculus for symplectic groups

In this subsection, we give the Weingarten calculus for symplectic groups. It was given in \[9, 10\], however, unlike their descriptions, we employ the theory of twisted Gelfand pairs (\[18, VII, Examples 1-10, 1-11, 2-6, and 2-7\]).

2.3.1 Twisted Gelfand pairs

Let \( \epsilon \) be the signature function on \( S_{2k} \) and consider the linear space

\[
L^\epsilon(S_{2k}, H_k) = \{ f \in L(S_{2k}) \mid f(\zeta \sigma) = f(\sigma) \zeta = \epsilon(\zeta) f(\sigma) \quad (\sigma \in S_{2k}, \zeta \in H_k) \}.
\]

The space \( L^\epsilon(S_{2k}, H_k) \) is closed under the convolution \(*\), and it becomes a \( \mathbb{C}\)-algebra. Furthermore, it is known that \( L^\epsilon(S_{2k}, H_k) \) is a commutative algebra, which means that the triple \((S_{2k}, H_k, \epsilon|_{H_k})\) is, by definition, a twisted Gelfand pair (\[18, VII, Example 2-6\]). It is immediate to see that, if \( f_1, f_2 \in L^\epsilon(S_{2k}, H_k) \), then \( f_1 * f_2 = (2^k k!)^{-1} f_1 * f_2 \in L^\epsilon(S_{2k}, H_k) \).

Thus, \( L^\epsilon(S_{2k}, H_k) \) is a commutative algebra under the product \(*\) with the identity element

\[
1_k^\epsilon(\sigma) = \begin{cases} 
\epsilon(\sigma) & \text{if } \sigma \in H_k, \\
0 & \text{if } \sigma \in S_{2k} \setminus H_k.
\end{cases}
\]

For each partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \vdash k \), we define the twisted spherical function \( \pi^\lambda \) of the twisted Gelfand pair by

\[
\pi^\lambda = (2^k k!)^{-1} \chi^{\lambda \cup \lambda} * 1_k^\epsilon
\]
with $\lambda \cup \lambda = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)$. The map
\[
L(S_{2k}, H_k) \to L^\epsilon(S_{2k}, H_k) : f \mapsto f^\epsilon, \quad f^\epsilon(\sigma) = \epsilon(\sigma)f(\sigma) \quad (\sigma \in S_{2k})
\]
defines a $\mathbb{C}$-algebra isomorphism. The twisted spherical function $\pi^\lambda$ is the image of $\omega^{\lambda'}$ with $\pi^\lambda(\sigma_\mu) = \omega^{\lambda'}(\sigma_\mu)$. Here $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the conjugate partition of $\lambda$, which is characterized by $(i, j) \in \lambda \iff (j, i) \in \lambda'$. Hence, the $\{\pi^\lambda | \lambda \vdash k\}$ form a linear basis of $L^\epsilon(S_{2k}, H_k)$ and satisfy the orthogonality relation
\[
\pi^\lambda \star \pi^\mu = \delta_{\lambda\mu} \frac{(2k)!}{2^k k!} \frac{1}{f_{\lambda \cup \lambda}} \pi^\lambda.
\]

### 2.3.2 Symplectic Weingarten functions

Let $z$ be a complex number and consider the function $T^{Sp}(\cdot; z)$ in $L^\epsilon(S_{2k}, H_k)$ defined by
\[
T^{Sp}(\sigma; z) = (-1)^k\epsilon(\sigma)(-2z)^{\ell(\mu)} \quad (\sigma \in S_{2k}),
\]
where $\mu$ is the coset-type of $\sigma$. Let
\[
D_\lambda'(z) = \prod_{(i,j) \in \lambda} (2z - 2i + j + 1).
\]
The expansion of $T^{Sp}(\sigma; z)$ in terms of linear basis $\pi^\lambda$ is given as follows.

**Lemma 2.3.**
\[
T^{Sp}(\cdot; z) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f_{\lambda \cup \lambda} D_\lambda'(z) \pi^\lambda. \tag{2.7}
\]

**Proof.** We first suppose that $z = n$, a positive integer. We employ symmetric function theory. Recall power-sum symmetric functions $p_\mu$ and twisted zonal functions $Z'_\lambda$. We only use the following properties for them here (see [18, VII, Example 2-7]). For partitions $\lambda, \mu$ of $k$,
\[
p_\mu(1^n) = n^{\ell(\mu)}, \quad Z'_\lambda(1^n) = D_\lambda'(n),
p_\mu = \frac{2^k k!}{(2k)!} (-1)^{k-\ell(\mu)} 2^{-\ell(\mu)} \sum_{\lambda \vdash k} f_{\lambda \cup \lambda} \pi^\lambda(\sigma_\mu) Z'_\lambda.
\]
Here $(1^n) = (1, 1, \ldots, 1)$ with $n$ times. Hence, if $\mu$ is the coset-type of $\sigma \in S_{2k}$, then
\[
T^{Sp}(\sigma; n) = \epsilon(\sigma)(-1)^{k-\ell(\mu)} 2^{\ell(\mu)} p_\mu(1^n) = \epsilon(\sigma) \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f_{\lambda \cup \lambda} \pi^\lambda(\sigma_\mu) D_\lambda'(n),
\]
which implies the desired formula for $z = n$.

Since the both sides on (2.7) are polynomials in $z$, the equalities at all positive integers $z = n$ implies the ones at all complex numbers $z$. \qed
The symplectic Weingarten function with parameter $z$ is the function in $L^*(S_{2k}, H_k)$ defined by
\[
Wg^{Sp}(\cdot; z) = \frac{2^k k!}{(2k)!} \sum_{D'_{\lambda}(z) \neq 0} f_{\lambda}^{\mu} \pi_{\lambda}.
\] (2.8)

Note that $T^{Sp}(\sigma; z) = (-1)^k \epsilon(\sigma) T^{O}(\sigma; -2z)$ and $Wg^{Sp}(\sigma; z) = (-1)^k \epsilon(\sigma) Wg^{O}(\sigma; -2z)$. Equation (2.5) is equivalent to
\[
T^{Sp}(\cdot; z) \ast Wg^{Sp}(\cdot; z) \ast T^{Sp}(\cdot; z) = T^{Sp}(\cdot; z).
\] (2.9)

### 2.3.3 Integrals on symplectic groups

Consider the vector space $\mathbb{C}^{2n}$ of column vectors with standard basis $(e_1, e_2, \ldots, e_{2n})$. Define the skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{2n}$ by
\[
\langle v, w \rangle = v^T J w
\] (2.10)

with
\[
J = J_n := \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}.
\]

Let $(e_1^\vee, e_2^\vee, \ldots, e_{2n}^\vee)$ be the dual basis of $(e_1, \ldots, e_{2n})$ with respect to $\langle \cdot, \cdot \rangle$:
\[
\langle e_i^\vee, e_j \rangle = \delta_{ij} \quad (i, j \in [2n]).
\]

More specifically, $(e_1^\vee, \ldots, e_n^\vee, e_{n+1}^\vee, \ldots, e_{2n}^\vee) = (-e_{n+1}, \ldots, -e_{2n}, e_1, \ldots, e_n)$. For convenience, we use the following notation:
\[
\langle i, j \rangle := \langle e_i, e_j \rangle = \langle e_i^\vee, e_j^\vee \rangle \begin{cases} 
1 & \text{if } 1 \leq i \leq n \text{ and } j = i + n, \\
-1 & \text{if } 1 \leq j \leq n \text{ and } i = j + n, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that
\[
e_i = J^T e_i^\vee = \sum_{p=1}^{2n} \langle i, p \rangle e_p^\vee, \quad e_i^\vee = J e_i = \sum_{p=1}^{2n} \langle p, i \rangle e_p \quad (i = 1, 2, \ldots, 2n)
\] (2.11)

and $J = (\langle i, j \rangle)_{1 \leq i, j \leq 2n}$.

For a $2n \times 2n$ matrix $X$, we define the dual matrix of $X$ by
\[
X^D := JX^T J^T.
\]

Then $\langle v, Xw \rangle = \langle X^D v, w \rangle$ for all $v, w \in \mathbb{C}^{2n}$.
We realize the (unitary) symplectic group $\text{Sp}(2n)$ by

$$\text{Sp}(2n) = \{ S = (s_{ij})_{1 \leq i,j \leq 2n} \in U(2n) \mid SS^\text{T} = I_{2n} \}.$$ 

It is equipped with the Haar probability measure. The following theorem was first given in [9] without the explicit expression (2.8) for $W_{\text{Sp}}$.

**Theorem 2.4.** Let $S = (s_{ij})_{1 \leq i,j \leq 2n}$ be a Haar symplectic matrix. For two sequences $i = (i_1, \ldots, i_{2k})$ and $j = (j_1, \ldots, j_{2k})$ of positive integers in $[2n]$, we have

$$\mathbb{E}[s_{i_1 j_1} s_{i_2 j_2} \cdots s_{i_{2k} j_{2k}}] = \sum_{\sigma, \tau \in M_{2k}} \Delta'_\sigma(i) \Delta'_\tau(j) W_{\text{Sp}}^\sigma(\sigma^{-1} \tau; n).$$

Here the symbol $\Delta'_\sigma(i) \in \{0, 1, -1\}$ is defined by

$$\Delta'_\sigma(i) := \prod_{r=1}^k (i_{2(r-1)}, i_{2r}). \quad (2.12)$$

Furthermore, $\mathbb{E}[s_{i_1 j_1} \cdots s_{i_{2k+1} j_{2k+1}}] = 0$ for any $i_1, \ldots, i_{2k+1}, j_1, \ldots, j_{2k+1}$.

We postpone the proof of this theorem in the next subsubsection.

**Example 2.4.** $W_{\text{Sp}}^\sigma(\sigma; n) = \frac{\epsilon(\sigma)}{2n}$ for $\sigma \in S_2$;

$$W_{\text{Sp}}^\sigma(\sigma; n) = \epsilon(\sigma) \frac{2n - 1}{4n(n - 1)(2n + 1)} \quad \text{for } \sigma \in H_2;$$

$$W_{\text{Sp}}^\sigma(\sigma; n) = \epsilon(\sigma) \frac{1}{4n(n - 1)(2n + 1)} \quad \text{for } \sigma \in S_4 \setminus H_2.$$

### 2.3.4 Proof of Theorem 2.4

This proof was given by Collins and Stolz [9] (see also [5]), however, they did not give any explicit expression for the symplectic Weingarten function. We here reconstruct their proof and observe how $W_{\text{Sp}}$ arises.

We first introduce the following useful notation. For a permutation $\sigma \in S_{2k}$ and two matrices $X = (x_{ij})_{1 \leq i,j \leq n}$ and $Y = (y_{ij})_{1 \leq i,j \leq n}$, put

$$\mathcal{T}_\sigma(X, Y) = \sum_{p_1, \ldots, p_{2k} \in [n]} \prod_{r=1}^k x_{p_{\sigma(2r-1)} p_{\sigma(2r)}} y_{p_{2r-1} p_{2r}} = \sum_{q_1, \ldots, q_{2k} \in [n]} \prod_{r=1}^k x_{q_{2r-1} q_{2r}} y_{q_{\sigma^{-1}(2r-1)} q_{\sigma^{-1}(2r)}}.$$

Note that $\mathcal{T}_{\sigma^{-1}}(X, Y) = \mathcal{T}_\sigma(Y, X)$ and $\mathcal{T}_{\sigma_\mu}(X, Y) = \prod_{i=1}^{\ell(\mu)} \mathcal{T}_{\sigma(\mu)_{ij}}(X, Y)$, where $\sigma_\mu$ is defined in (2.3). Since

$$\mathcal{T}_{\sigma(m)}(X, Y) = \sum_{p_1, \ldots, p_{2m}} x_{p_1 p_{2m}} y_{p_1 p_2} x_{p_2 p_3} y_{p_3 p_4} \cdots x_{p_{2m-2} p_{2m-1}} y_{p_{2m-1} p_{2m}};$$
we have
\[
\mathcal{T}_{\sigma(m)}(X, Y) = \begin{cases} 
\text{Tr}[(XY)^m] & \text{if } X \text{ is a symmetric matrix}, \\
-\text{Tr}[(XY)^m] & \text{if } X \text{ is a skew-symmetric matrix}.
\end{cases}
\] (2.13)

As a particular case, we can see
\[
\mathcal{T}_\sigma(J, J) = \mathcal{T}^{\text{Sp}}(\sigma; n) \quad (\sigma \in S_{2k}).
\] (2.14)

Indeed, the skew-symmetry of $J$ implies that the function $\sigma \mapsto \mathcal{T}_\sigma(J, J)$ belongs to $L^\epsilon(S_{2k}, H_k)$, and hence it is enough to check $\mathcal{T}_{\sigma_{\mu}}(J, J) = \mathcal{T}^{\text{Sp}}(\sigma_{\mu}; n)$ for all partitions $\mu$. However, it follows from (2.13) that $\mathcal{T}_{\sigma_{\mu}}(J, J) = \prod_{\ell=1}^{\ell(\mu)} t_i^{\mu_i}(J, J) = \prod_{\ell=1}^{\ell(\mu)} [-\text{Tr}(-I_{2n})^{\mu_i}] = (-1)^{k-\ell(\mu)}(2n)^{\ell(\mu)} = \mathcal{T}^{\text{Sp}}(\sigma_{\mu}; n)$.

We next recall the invariant theory for symplectic groups. Consider the tensor product $(\mathbb{C}^{2n})^\otimes 2k$ and define a bilinear form on $(\mathbb{C}^{2n})^\otimes 2k$ by
\[
\langle \bigotimes_{j=1}^{2k} v_j, \bigotimes_{j=1}^{2k} w_j \rangle := \prod_{j=1}^{2k} \langle v_j, w_j \rangle, \quad (v_1, \ldots, v_{2k}, w_1, \ldots, w_{2k} \in \mathbb{C}^{2n}),
\]
where the skew-symmetric bilinear form $\langle v, w \rangle$ on the right hand side is defined in (2.10). Note that this bilinear form on $(\mathbb{C}^{2n})^\otimes 2k$ is symmetric.

Put
\[
\theta_k = \sum_{p_1, \ldots, p_k \in [2n]} e_{p_1}^\vee \otimes e_{p_1} \otimes \cdots \otimes e_{p_k}^\vee \otimes e_{p_k} \in (\mathbb{C}^{2n})^\otimes 2k.
\]
The symmetric group $S_{2k}$ acts on $(\mathbb{C}^{2n})^\otimes 2k$ by
\[
\rho_{2k}(\sigma)(v_1 \otimes \cdots \otimes v_{2k}) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(2k)},
\]
while the symplectic group $\text{Sp}(2n)$ acts by
\[
S(v_1 \otimes \cdots \otimes v_{2k}) = Sv_1 \otimes \cdots \otimes Sv_{2k}.
\]
Then the First Fundamental Theorem for the symplectic group states that $\{\rho_{2k}(\sigma)\theta_k \mid \sigma \in M_{2k}\}$ spans the vector subspace of $(\mathbb{C}^{2n})^\otimes 2k$ consisting of invariant elements under the action of $\text{Sp}(2n)$ (see, e.g., [17, Theorem 5.3.4]).

Let us go back to the proof of Theorem 2.4. Let $G$ be the symmetric matrix
\[
G = (\langle \rho_{2k}(\sigma)\theta_k, \rho_{2k}(\tau)\theta_k \rangle)_{\sigma, \tau \in M_{2k}}
\] (2.15)
and $W = (w(\sigma, \tau))_{\sigma, \tau \in M_{2k}}$ the pseudo-inverse matrix of $G$, i.e., $W$ is the unique symmetric matrix satisfying $GWG = G$. 
Let \( S = (s_{ij})_{1 \leq i,j \leq 2n} \) be a Haar symplectic matrix. Since each matrix element \( s_{ij} \) is expressed as \( s_{ij} = \langle e_i^\vee, S e_j \rangle \), we have
\[
E[s_{i_1j_1} \cdots s_{i_2j_2}] = E \left[ \left( e_{i_1}^\vee \otimes \cdots \otimes e_{i_2}^\vee, S(e_{j_1} \otimes \cdots \otimes e_{j_2}) \right) \right] = \left( e_{i_1}^\vee \otimes \cdots \otimes e_{i_2}^\vee, E[S(e_{j_1} \otimes \cdots \otimes e_{j_2})] \right).
\]
As mentioned above, for any \( \mathbf{v} \in V^{\otimes 2k} \), the \( \text{Sp}(2n) \)-invariant vector \( E[\mathbf{v}] \) can be expanded in terms of \( \rho_{2k}(\sigma) \theta_k \). From a discussion parallel to the proof of [5, Theorem 2.1] (see also [10]), the expansion is given by
\[
E[\mathbf{v}] = \sum_{\sigma \in M_{2k}} \left( \sum_{\tau \in M_{2k}} E(\sigma, \tau, v, \rho_{2k}(\tau)\theta_k) \right) \rho_{2k}(\sigma)\theta_k.
\]
Applying this to the previous equation, we obtain
\[
E[s_{i_1j_1} \cdots s_{i_2j_2}] = \sum_{\sigma, \tau \in M_{2k}} E(\sigma, \tau) \langle e_{i_1}^\vee \otimes \cdots \otimes e_{i_2}^\vee, \rho_{2k}(\sigma)\theta_k \rangle \cdot \langle e_{j_1} \otimes \cdots \otimes e_{j_2}, \rho_{2k}(\tau)\theta_k \rangle.
\]
Here the values of bilinear forms are computed as follows:
\[
\langle e_{i_1}^\vee \otimes \cdots \otimes e_{i_2}^\vee, \rho_{2k}(\sigma)\theta_k \rangle = \langle \rho_{2k}(\sigma^{-1})(e_{i_1}^\vee \otimes \cdots \otimes e_{i_2}^\vee), \theta_k \rangle
\]
\[
= \langle e_{i_{\sigma(1)}}^\vee \otimes \cdots \otimes e_{i_{\sigma(2k)}}^\vee, \theta_k \rangle = \prod_{r=1}^{2n} \left( \sum_{p=1}^{2n} \langle e_{i_{\sigma(2r-1)}}^\vee, e_{p}^\vee \rangle \langle e_{i_{\sigma(2r)}}, e_{p} \rangle \right)
\]
and, similarly, \( \langle e_{j_1} \otimes \cdots \otimes e_{j_2}, \rho_{2k}(\tau)\theta_k \rangle = \Delta_{\tau}(\theta) \). In conclusion, Theorem 2.4 follows from the next lemma.

**Lemma 2.5.** For \( \sigma, \tau \in M_{2k} \), \( w(\sigma, \tau) = Wg^\text{Sp}(\sigma^{-1}\tau; n) \).

**Proof.** If we put
\[
\mathcal{T}(\sigma) = \theta_k, \rho_{2k}(\sigma)\theta_k \quad (\sigma \in S_{2k}),
\]
then \( G = (\mathcal{T}(\sigma^{-1}\tau))_{\sigma, \tau \in M_{2k}} \). Using (2.11) we have
\[
\theta_k = \sum_{p_1, \ldots, p_{2k}} \langle p_1, p_2 \rangle \langle p_3, p_4 \rangle \cdots \langle p_{2k-1}, p_{2k} \rangle e_{p_1}^\vee \otimes e_{p_2}^\vee \otimes \cdots \otimes e_{p_{2k}}^\vee
\]
\[
= \sum_{q_1, \ldots, q_{2k}} \langle q_1, q_2 \rangle \langle q_3, q_4 \rangle \cdots \langle q_{2k-1}, q_{2k} \rangle e_{q_1} \otimes e_{q_2} \otimes \cdots \otimes e_{q_{2k}};
\]
and hence
\[
\mathcal{T}(\sigma) = \sum_{q_{\sigma(1)}, q_{\sigma(2)}} \cdots \langle q_{\sigma(2k-1)}, q_{\sigma(2k)} \rangle \langle q_1, q_2 \rangle \cdots \langle q_{2k-1}, q_{2k} \rangle = \mathcal{T}_{\tau}(J, J),
\]
which implies \( \mathcal{T}(\sigma) = T_{\text{Sp}}(\sigma; n) \) by (2.14). The matrix \( (Wg^\text{Sp}(\sigma^{-1}\tau; n))_{\sigma, \tau \in M_{2k}} \) is therefore the pseudo-inverse matrix of \( G = (T_{\text{Sp}}(\sigma^{-1}\tau; n))_{\sigma, \tau \in M_{2k}} \) by (2.9). \( \square \)
3 Circular ensembles

From now we consider random matrix ensembles $S$ associated with classical symmetric spaces $G/K$. As we mentioned in Introduction, such ensembles are realized in the following way.

$$G/K \simeq S; \quad G \ni g \mapsto \Omega(g)^{-1}g \in S.$$  

Here $\Omega$ is an involution on $G$ and $K$ is the fixed-point set of $\Omega$. If $X$ is a Haar random matrix picked up from $G$, then the matrix $V := \Omega(X)^{-1}X$ is a random matrix associated with $G/K$. We consider the seven series of random matrices associated with compact symmetric spaces given in Figure 1.

In this section, we deal with the most important classes: circular orthogonal ensembles (COE) and circular symplectic ensembles (CSE).

3.1 Class A I

The setting for A I: $G = U(n)$, $K = O(n)$, $\Omega(g) = g$. $S$ consists of $n \times n$ symmetric unitary matrices.

When $U$ is an $n \times n$ Haar unitary matrix, a random matrix corresponding to $U(n)/O(n)$ is defined by

$$V = V^A = \Omega(U)^{-1}U = U^T U$$

and is said to be a COE matrix. The Weingarten calculus for a COE matrix is constructed in [20]. For completeness of this paper, we review main results in [20]. Applying the Weingarten calculus for $U(n)$, we can obtain the following theorem.

**Theorem 3.1** (Theorem 1.1 and Proposition 3.1 in [20]). Let $V = V^A = (v_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ COE matrix. For two sequences $i = (i_1, \ldots, i_{2k})$ and $j = (j_1, \ldots, j_{2k})$, we have

$$E[v_{i_1i_2}v_{i_3i_4} \cdots v_{i_{2k-1}i_{2k}}v_{j_1j_2}v_{j_3j_4} \cdots v_{j_{2l-1}j_{2l}}] = \sum_{\sigma \in S_{2k}} \delta_{\sigma(i,j)} Wg^A(\sigma; n)$$

with the convolution $Wg^A(\cdot; n) := T^O(\cdot; n) \ast Wg^U(\cdot; n)$ in $L(S_{2k})$. If $k \neq l$ then

$$E[v_{i_1i_2}v_{i_3i_4} \cdots v_{i_{2k-1}i_{2k}}v_{j_1j_2}v_{j_3j_4} \cdots v_{j_{2l-1}j_{2l}}]$$

always vanishes. Moreover, the function $Wg^A(\cdot; n)$ belongs to $L(S_{2k}, H_k)$ and coincides with the orthogonal Weingarten function with parameter $n + 1$, i.e.,

$$Wg^A(\sigma; n) = Wg^O(\cdot; n + 1).$$
As a corollary of this theorem, the following moments for a single entry are computed (see [20 Theorems 4.1 and 4.2]):
\[
\mathbb{E}[|v_{ii}|^{2k}] = \frac{2^k k!}{(n+1)(n+3) \cdots (n+2k-1)},
\]
\[
\mathbb{E}[|v_{ij}|^{2k}] = \frac{k!}{n(n+1)(n+2) \cdots (n+k-2)(n+2k-1)} \quad (i \neq j).
\]

The first equation can be obtained easily from the previous theorem, but the derivation of the second one is somewhat complicated.

**Example 3.1.** From Example 2.3 we have
\[
Wg^{A_1}(\sigma; n) = \frac{1}{n+1} \quad \text{for } \sigma \in S_2;
\]
\[
Wg^{A_1}(\sigma; n) = \frac{-1}{n(n+1)(n+3)} \quad \text{for } \sigma \in S_4 \setminus H_2.
\]

### 3.2 Class A II

*The setting for A II: G = U(2n), K = Sp(2n), Ω(g) = (gD)^{-1}. \mathcal{S} consists of 2n × 2n unitary matrices g satisfying gD = g.*

When \( U \) is a 2n×2n Haar unitary matrix, a random matrix corresponding to \( U(2n)/Sp(2n) \) is defined by
\[
V = V^{A_II} = UD_U
\]
and is said to be a CSE matrix.

We would like to compute mixed moments for \( v_{ij} \) and \( \overline{v_{ij}} \). In order to simplify the notation, we deal with
\[
\tilde{v}_{ij} := \langle e_i, V e_j \rangle \quad (1 \leq i, j \leq 2n)
\]
instead of \( v_{ij} = \langle e_i^\vee, V e_j \rangle \). More specifically,
\[
v_{ij} = \begin{cases} 
-\tilde{v}_{i+n,j} & \text{if } 1 \leq i \leq n, \\
\tilde{v}_{i-n,j} & \text{if } n+1 \leq i \leq 2n.
\end{cases}
\]

**Theorem 3.2.** Let \( V \) be a 2n × 2n CSE matrix. For two sequences \( i = (i_1, \ldots, i_{2k}) \) and \( j = (j_1, \ldots, j_{2k}) \) in \( [2n]^{2k} \), we have
\[
\mathbb{E}[\tilde{v}_{i_1,i_2} \tilde{v}_{i_3,i_4} \cdots \tilde{v}_{i_{2k-1},i_{2k}} \overline{\tilde{v}_{j_1,j_2} \tilde{v}_{j_3,j_4} \cdots \tilde{v}_{j_{2k-1},j_{2k}}}] = \sum_{\sigma \in S_{2k}} \delta_{\sigma}(i,j)Wg^{A_II}(\sigma; n)
\]
with \( Wg^{A_II}(\cdot; n) := T^{Sp}(\cdot; n) * Wg^{U}(\cdot; 2n) \). If \( k \neq l \) then
\[
\mathbb{E}[\tilde{v}_{i_1,i_2} \tilde{v}_{i_3,i_4} \cdots \tilde{v}_{i_{2k-1},i_{2k}} \overline{\tilde{v}_{j_1,j_2} \tilde{v}_{j_3,j_4} \cdots \tilde{v}_{j_{2l-1},j_{2l}}}] = 0.
\]
Proof. Each element of a $2n \times 2n$ CSE matrix $V = U^D U$ is given as
\[
\bar{v}_{i,i'} = \langle e_i, U^D U e_{i'} \rangle = \langle U e_i, U e_{i'} \rangle = \sum_{p,q \in [2n]} u_{p,i} u_{q,i'} \langle p, q \rangle \quad (i, i' \in [2n]).
\]
Hence
\[
\mathbb{E}[\bar{v}_{i_1,i_2} \bar{v}_{i_3,i_4} \cdots \bar{v}_{i_{2k-1},i_{2k}} \bar{v}_{j_1,j_2} \bar{v}_{j_3,j_4} \cdots \bar{v}_{j_{2k-1},j_{2k}}] = \sum_{p=(p_1,\ldots,p_{2k})} \sum_{q=(q_1,\ldots,q_{2k})} \mathbb{E}[u_{p_1,i_1} \cdots u_{p_k,i_k} u_{q_1,j_1} \cdots u_{q_k,j_k}] \prod_{r=1}^{k} \langle p_{2r-1}, p_{2r} \rangle \langle q_{2r-1}, q_{2r} \rangle.
\]
(When $k \neq l$, $\mathbb{E}[u_{p_1,i_1} \cdots u_{p_k,i_k} u_{q_1,j_1} \cdots u_{q_l,j_l}] \equiv 0$.) Applying the Weingarten calculus for a Haar unitary matrix $U$, we have
\[
\mathbb{E}[\bar{v}_{i_1,i_2} \bar{v}_{i_3,i_4} \cdots \bar{v}_{i_{2k-1},i_{2k}} \bar{v}_{j_1,j_2} \bar{v}_{j_3,j_4} \cdots \bar{v}_{j_{2k-1},j_{2k}}] = \sum_{\sigma \in S_{2k}} \delta_{\sigma}(i, j) \sum_{\tau \in S_{2k}} Wg^U(\tau^{-1} \sigma; 2n) \tilde{T}(\tau),
\]
where
\[
\tilde{T}(\tau) = \sum_{p=(p_1,\ldots,p_{2k}) \in [2n]^{\times 2k}} \sum_{q=(q_1,\ldots,q_{2k}) \in [2n]^{\times 2k}} \delta_{\tau}(p, q) \prod_{r=1}^{k} \langle p_{2r-1}, p_{2r} \rangle \langle q_{2r-1}, q_{2r} \rangle =~ T(\lambda, \lambda) = T^{\text{Sp}}(\tau; n).
\]
The last equality above follows from (2.14). \hfill \square

**Proposition 3.3.** The function $Wg^{\text{All}}(\cdot; n) = T^{\text{Sp}}(\cdot; n) * Wg^{U}(\cdot; 2n)$ coincides with the symplectic Weingarten function with parameter $n - \frac{1}{2}$. Specifically, for each $\sigma \in S_{2k}$, we have
\[
Wg^{\text{All}}(\sigma; n) = Wg^{\text{Sp}}(\sigma; n - \frac{1}{2}).
\]

**Proof.** It follows from (2.7) and (2.1) that
\[
T^{\text{Sp}}(\cdot; n) * Wg^{U}(\cdot; 2n) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f^{\lambda \vdash \lambda} D_{\lambda}(n) \frac{1}{(2k)!} \sum_{\mu \vdash 2k} f^{\mu} C_\mu(2n) \pi^{\lambda} \chi^{\mu}.
\]
Since $\pi^{\lambda} \chi^{\mu} = \delta_{\lambda \vdash \lambda} \chi^{(2k)!} \pi^{\lambda}$ and since
\[
\frac{D_{\lambda}(z)}{C_{\lambda \vdash \lambda}(2z)} = \frac{\prod_{(i,j) \in \lambda}(2z - 2i + j + 1)}{\prod_{(i,j) \in \lambda}(2z + j - (2i - 1))(2z + j - 2i)} = \frac{1}{\prod_{(i,j) \in \lambda}(2z + j - 2i)} = \frac{1}{D_{\lambda}(z - \frac{1}{2})},
\]
we have $T^{\text{Sp}}(\cdot; n) * Wg^{U}(\cdot; 2n) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f^{\lambda \vdash \lambda} D_{\lambda}(n - \frac{1}{2}) \pi^{\lambda} = Wg^{\text{Sp}}(\cdot; n - \frac{1}{2}).$ \hfill \square

**Example 3.2.** From Example 2.4 we have $Wg^{\text{All}}(\sigma; n) = Wg^{\text{Sp}}(\sigma; n - \frac{1}{2}) = \frac{\epsilon(\sigma)}{2n-1}$ for $\sigma \in S_2$;
\[
Wg^{\text{All}}(\sigma; n) = Wg^{\text{Sp}}(\sigma; n - \frac{1}{2}) = \begin{cases} \frac{\epsilon(\sigma)}{n(2n-1)(2n-3)} & \text{for } \sigma \in H_2, \\ \frac{\epsilon(\sigma)}{(2n)(2n-1)(2n-3)} & \text{for } \sigma \in S_4 \setminus H_2. \end{cases}
\]
4 Chiral ensembles

In this section, we deal with random matrix ensembles associated with classes A III, BD I, C II. They are known as chiral ensembles.

4.1 Class A III

The setting for A III: $G = U(n)$, $K = U(a) \times U(b)$, $\Omega(g) = I'_ab g I'_ab$, where $n = a + b$ with $a \geq b \geq 1$ and

$$I'_ab = \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix}.$$ 

Let $U$ be an $n \times n$ Haar unitary matrix. A random matrix corresponding to $U(n)/U(a) \times U(b) = SU(n)/S(U(a) \times U(b))$ is defined by $V = V^{AIII} = I'_ab U^* I'_ab U$. For the sake of ease, we consider a Hermitian and unitary random matrix $W = W^{AIII} = U^* I'_ab U$, instead of $V = I'_ab W$.

We define the function $T^{AIII}_{ab}$ in $Z(L(S_k))$ by

$$T^{AIII}_{ab}(\sigma) = \text{Tr}(I'_ab U^* I'_ab U \text{Tr}(\sigma)).$$

where $\text{Tr}(A) = \prod_{j=1}^{\ell(\mu)} \text{Tr}(A^{\mu_j})$ if $\mu$ is the cycle-type of $\sigma$. More specifically, we have

$$T^{AIII}_{ab}(\sigma) = (a + b)^{\ell^e(\mu)}(a - b)^{\ell^o(\mu)}.$$

Here $\ell^e(\mu)$ (resp. $\ell^o(\mu)$) is the number of parts $\mu_j$ with even lengths (resp. odd lengths).

Theorem 4.1. Let $W = W^{AIII}$ be the random matrix defined as above. For two sequences $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_k)$, we have

$$\mathbb{E}[w_{i_1j_1} w_{i_2j_2} \cdots w_{i_kj_k}] = \sum_{\sigma \in S_k} \delta_\sigma(i, j) W^{AIII}_g(\cdot; a, b).$$

Here the function $W^{AIII}_g(\cdot; a, b)$ in $Z(L(S_k))$ is defined by

$$W^{AIII}_g(\cdot; a, b) = T^{AIII}_{ab} \ast W^U_g(\cdot; n).$$

Proof. The random matrix $W$ is Hermitian and the distribution of $W$ is invariant under the unitary transform $W \mapsto gwg^*$, where $g$ is a fixed unitary matrix. Hence we can apply Theorem 3.1 in [6] and obtain

$$\mathbb{E}[w_{i_1j_1} w_{i_2j_2} \cdots w_{i_kj_k}] = \sum_{\sigma, \tau \in S_k} \delta_\sigma(i, j) W^U(\tau^{-1}\sigma; n) \mathbb{E}[\text{Tr}_\tau(W)].$$

Here, we see that $\mathbb{E}[\text{Tr}_\tau(W)] = \mathbb{E}[\text{Tr}_\tau(I'_ab)] = \text{Tr}_\tau(I'_ab) = T^{AIII}_{ab}(\tau)$, and we obtain the desired identity. \square
Example 4.1. \( n = a + b \). \( W_{\text{III}}^A(id_1; a, b) = \frac{a-b}{n} \),

\[
W_{\text{III}}^A(id_2; a, b) = \frac{(a-b+1)(a-b-1)}{(n+1)(n-1)}, \quad W_{\text{III}}^A((1\ 2); a, b) = \frac{4ab}{n(n-1)(n+1)}.
\]

Recall power symmetric functions \( p_\mu \) and Schur functions \( s_\lambda \). They have the relation

\[
p_\mu = \sum_{\lambda \vdash k} \chi^\lambda(\sigma)s_\lambda
\]

if \( \mu \vdash k \) is the cycle-type of \( \sigma \) ([15], (7.8)). Furthermore, it is easy to see that

\[
p_\mu(1^a, (-1)^b) = p_\mu(1, 1, \ldots, 1, -1, -1, \ldots, -1) = T_{ab}^{\text{III}}(\sigma).
\]

Hence we obtain the expansion of \( T_{ab}^{\text{III}} \) in terms of irreducible characters \( \chi^\lambda \):

\[
T_{ab}^{\text{III}} = \sum_{\lambda \vdash k} s_\lambda(1^a, (-1)^b)\chi^\lambda.
\]

On the other hand, from the well-known identity \( s_\lambda(1^n) = \frac{f_\lambda C_\lambda(n)}{k!} \), the unitary Weingarten function is expressed as

\[
W_{\text{U}}^U(\cdot; n) = \frac{1}{(k!)^2} \sum_{\lambda \vdash k} (f^\lambda)^2 s_\lambda(1^n)\chi^\lambda.
\]

Consequently, using the relation \( \chi^\lambda \ast \chi^\mu = \frac{k!}{f^\lambda} \delta_{\lambda\mu} \chi^\lambda \), we obtain the following expansion of \( W_{\text{III}}^A(\cdot; a, b) \) in terms of \( \chi^\lambda \):

\[
W_{\text{III}}^A(\cdot; a, b) = T_{ab}^{\text{III}} \ast W_{\text{U}}^U(\cdot; n) = \frac{1}{k!} \sum_{\lambda \vdash k} f^\lambda s_\lambda(1^a, (-1)^b) s_\lambda(1^{a+b})\chi^\lambda.
\]

4.2 Class BD I

The setting for BD I: \( G = O(n) \), \( K = O(a) \times O(b) \), \( \Omega(g) = I_{ab}^gI_{ab}^g \), where \( n = a + b \) with \( a \geq b \geq 1 \).

The discussion is parallel to that in the previous subsection. We deal with the symmetric and orthogonal random matrix

\[
W = W_{\text{BD I}} = R^T I_{ab}^g R,
\]

where \( R \) is an \( n \times n \) Haar orthogonal matrix. This random matrix is associated with the symmetric space \( O(n)/O(a) \times O(b) = \text{SO}(n)/\text{S(O(a)} \times \text{O(b))}. \)

We define the function \( T_{ab}^{\text{BD I}} \) in \( L(S_{2k}, H_k) \) by

\[
T_{\text{BD I}}(\sigma) = T_{\sigma}(I_{ab}^g, I_n).
\]
More specifically, if $\sigma \in S_{2k}$ has the coset-type $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, then

$$T_{ab}^{BDI}(\sigma) = \prod_{i=1}^{\ell(\mu)} \text{Tr}(I_{ab}'^{\mu_i}) = (a + b)^{\mu_1}(a - b)^{\mu_l}.$$ 

**Theorem 4.2.** Let $W = W^{BDI}$ be the random matrix defined as above. For a sequence $i = (i_1, \ldots, i_{2k})$, we have

$$\mathbb{E}[w_{i_1} \cdots w_{i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta_\sigma(i) Wg^{BDI}(\sigma; a, b).$$

Here the function $Wg^{BDI}(\cdot; a, b)$ in $L(S_{2k}, H_k)$ is defined by

$$Wg^{BDI}(\cdot; a, b) = T_{ab}^{BDI} \ast Wg^O(\cdot; n).$$

**Proof.** The proof is the same with that of Theorem 4.1. Apply Theorem 3.3 in [6]. \qed

**Example 4.2.** $n = a + b$. $Wg^{BDI}(\text{id}_2; a, b) = \frac{a - b}{n}$;

$$Wg^{BDI}(((1 \ 2 \ 3 \ 4) ; a, b) = \frac{(a - b)(n + 1) - 2n}{n(n + 2)(n - 1)},$$

$$Wg^{BDI}(((1 \ 2 \ 4 \ 3) ; a, b) = \frac{4ab}{n(n + 2)(n - 1)}.$$ 

Using zonal polynomials $Z_\lambda$ (see [18, VII-2]), we can obtain the expansion of $Wg^{BDI}(\cdot; a, b)$ in terms of zonal spherical functions $\omega^\lambda$:

$$Wg^{BDI}(\cdot; a, b) = \frac{2^{k} k!}{(2k)!} \sum_{\lambda \vdash k} f^{2\lambda} Z_\lambda(1^a, (-1)^b) \omega^\lambda.$$ 

**4.3 Class C II**

The setting for C II: $G = \text{Sp}(2n)$, $K = \text{Sp}(2a) \times \text{Sp}(2b)$, $\Omega(g) = I_{ab}''gI_{ab}''$, where $n = a + b$ with $a \geq b \geq 1$ and

$$I_{ab}'' = \begin{pmatrix} I_{ab}' & O \\ O & I_{ab}' \end{pmatrix}.$$ 

The discussion is parallel to the previous subsection again. We deal with the random matrix

$$W = W^{CII} = S^D I_{ab}'' S.$$ 

where $S$ is a $2n \times 2n$ Haar symplectic matrix. It is associated with the symmetric space $\text{Sp}(2n)/\text{Sp}(2a) \times \text{Sp}(2b)$. 
We define the function $T_{ab}^{\text{CH}}$ in $L^4(S_{2k}, H_k)$ by

$$T_{ab}^{\text{CH}}(\sigma) = T_\sigma(J, JI_{ab}''').$$

If $\sigma \in S_{2k}$ has the coset-type $\mu$, then

$$T_{ab}^{\text{CH}}(\sigma) = \epsilon(\sigma) \prod_{i=1}^{\ell(\mu)} T_{ab}^{\text{CH}}(\sigma_{(\mu_i)}) = \epsilon(\sigma) \prod_{i=1}^{\ell(\mu)} [-\operatorname{Tr}(JI_{ab}'')^{\mu_i}]
= (-1)^{k-\ell(\mu)} \epsilon(\sigma) \prod_{i=1}^{\ell(\mu)} \operatorname{Tr}(I_{ab}'')^{\mu_i} = (-1)^{k-\ell(\mu)} \epsilon(\sigma) 2^{\ell(\mu)} (a + b)\,\tau(\mu)(a - b)^{\tau(\mu)}.
$$

Here the second equality above follows by (2.13).

As in the case of class A II, we consider $\bar{w}_{ij} = \langle e_i, W e_j \rangle$ instead of matrix elements $w_{ij}$.

**Theorem 4.3.** Let $W = W^{\text{CH}}$ be the random matrix defined as above. For a sequence $i = (i_1, \ldots, i_{2k})$, we have

$$\mathbb{E}[\bar{w}_{i_1, i_2} \bar{w}_{i_3, i_4} \cdots \bar{w}_{i_{2k-1}, i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta'_\sigma(i) W^\text{CH}(\sigma; a, b).$$

Here the function $W^\text{CH}(\cdot; a, b)$ in $L^4(S_{2k}, H_k)$ is defined by

$$W^\text{CH}(\cdot; a, b) = T_{ab}^{\text{CH}} * W^\text{Sp}(\cdot; n).$$

**Proof.** Since

$$\bar{w}_{i,j} = \langle e_i, S^D I''_{ab} S e_j \rangle = \langle S e_i, I''_{ab} S e_j \rangle = \sum_{p,q=1}^{2n} s_{pi} s_{qj} \langle e_p, I''_{ab} e_q \rangle,$$

we have

$$\mathbb{E}[\bar{w}_{i_1, i_2} \bar{w}_{i_3, i_4} \cdots \bar{w}_{i_{2k-1}, i_{2k}}] = \sum_{p_1, \ldots, p_{2k}} \mathbb{E}[s_{p_1 i_1} s_{p_2 i_2} \cdots s_{p_{2k} i_{2k}}] \prod_{r=1}^{k} \langle p_{2r-1}, p_{2r} \rangle''',$n

with $\langle p, q \rangle'' := \langle e_p, I''_{ab} e_q \rangle$. The Weingarten calculus for symplectic groups gives

$$\mathbb{E}[\bar{w}_{i_1, i_2} \bar{w}_{i_3, i_4} \cdots \bar{w}_{i_{2k-1}, i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta'_\sigma(i) \sum_{\tau \in M_{2k}} W^\text{Sp}(\tau^{-1} \sigma; n) \tilde{T}(\tau),$$

where

$$\tilde{T}(\tau) := \sum_{p=(p_1, \ldots, p_{2k})} \Delta'_\tau(p) \prod_{r=1}^{k} \langle p_{2r-1}, p_{2r} \rangle''' = \sum_{p_1, \ldots, p_{2k}} \prod_{r=1}^{k} \langle p_{r(2r-1)}, p_{r(2r)} \rangle \langle p_{2r-1}, p_{2r} \rangle'''
= T_\tau(J, JI''_{ab}) = T_{ab}^{\text{CH}}(\tau).$$

\qed
**Example 4.3.** $n = a + b$. $Wg_{\mathbb{C}H}^C(id_2; a, b) = \frac{a - b}{n}$;

$$Wg_{\mathbb{C}H}^C \left( \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{array} \right); a, b \right) = a - b;$$

$$(-1)Wg_{\mathbb{C}H}^C \left( \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{array} \right); a, b \right) = \frac{4ab}{n(n-1)(2n+1)}.$$

Using twisted zonal polynomials $Z'_\lambda$ (see [18, VII, Example 2-7]), we obtain the expansion of $Wg_{\mathbb{C}H}^C(\cdot; a, b)$ in twisted spherical functions $\pi^\lambda$:

$$Wg_{\mathbb{C}H}^C(\cdot; a, b) = \frac{2^k k!}{(2k)!} \sum_{\lambda \vdash k} f_{\lambda^\mu} Z'_\lambda(1^a, (-1)^b) Z'_\lambda(1^{a+b}) \pi^\lambda.$$

## 5 BdG ensembles

In this section, we deal with matrix ensembles of types D III and C I. They are called Bogoliubov-de Gennes (BdG) ensembles.

### 5.1 Class D III

*The setting for D III: $G = O(2n)$, $K = O(2n) \cap \text{Sp}(2n) \cong U(n)$, $\Omega(g) = (g^D)^{-1}$.*

Let $R$ be a $2n \times 2n$ Haar orthogonal matrix. We consider the random matrix $V = V^{D\text{III}} = R^D R$,

associated to the symmetric space $O(2n)/U(n)$.

We define the function $T_{n}^{D\text{III}}$ on $S_{2k}$ by

$$T_{n}^{D\text{III}}(\sigma) = T_{\sigma}(I_{2n}, J_{n}).$$

It satisfies $T_{n}^{D\text{III}}(\zeta \sigma \zeta') = \epsilon(\zeta) T_{n}^{D\text{III}}(\sigma)$ for any $\sigma \in S_{2k}$ and $\zeta, \zeta' \in H_k$. Note that

$$T_{n}^{D\text{III}}(\sigma_\mu) = \prod_{i=1}^{\ell(\mu)} T_{n}^{D\text{III}}(\sigma(\mu_i)) = \prod_{i=1}^{\ell(\mu)} \text{Tr}(J^{\mu_i}) = \begin{cases} (-2n)^{\ell(\mu)} & \text{if } \mu \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Here a partition $\mu$ is said to be even if $\mu = 2\nu$ for some partition $\nu$. In particular, $T_{n}^{D\text{III}}(\sigma) = 0 (\sigma \in S_{2k})$ if $k$ is odd.

**Theorem 5.1.** Let $V = V^{D\text{III}}$ be the random matrix defined as above. For a sequence $\vec{i} = (i_1, i_2, \ldots, i_{2k})$ with an even number $k$, we have

$$\mathbb{E}[\vec{v}_{i_1, i_2} \vec{v}_{i_3, i_4} \cdots \vec{v}_{i_{2k-1}, i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta_{\sigma}(\vec{i}) Wg_{\mathbb{C}H}^{D\text{III}}(\sigma; n).$$
Here \( Wg^{DIII}(\cdot; n) \) is the function on \( S_{2k} \) defined by
\[
Wg^{DIII}(\cdot; n) = T_n^{DIII} \ast Wg^O(\cdot; 2n).
\]

If \( k \) is odd, then, for any sequence \( i = (i_1, i_2, \ldots, i_{2k}) \),
\[
E[\tilde{v}_{i_1, i_2} \tilde{v}_{i_3, i_4} \cdots \tilde{v}_{i_{2k-1}, i_{2k}}] = 0.
\]

**Proof.** The proof is similar to that of Theorem 4.3. The Weingarten calculus for a Haar orthogonal matrix \( R \) gives
\[
E[\tilde{v}_{i_1, i_2} \tilde{v}_{i_3, i_4} \cdots \tilde{v}_{i_{2k-1}, i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta_\sigma(i) \sum_{\tau \in M_{2k}} Wg^O(\tau^{-1} \sigma; 2n) \tilde{T}(\tau)
\]
where
\[
\tilde{T}(\tau) = \sum_{p=(p_1, \ldots, p_{2k})} \Delta_\tau(p) \prod_{r=1}^k (p_{2r-1}, p_{2r}) = T_\tau(I, J) = T_n^{DIII}(\tau)
\]
for any \( \tau \in S_{2k} \).

Note that
\[
Wg^{DIII}(\zeta \zeta'; n) = e(\zeta) Wg^{DIII}(\sigma; n) \quad (\sigma \in S_{2k}, \ \zeta, \zeta' \in H_k),
\]
and hence \( Wg^{DIII}(\zeta; n) = 0 \) for \( \zeta \in H_k \). Furthermore, \( Wg^{DIII}(\sigma; n) = 0 \) \((\sigma \in S_{2k})\) if \( k \) is odd.

**Example 5.1.** Since \((\frac{1}{1} \frac{2}{3} \frac{3}{2} \frac{4}{4}) = (3 4) (\frac{1}{1} \frac{2}{3} \frac{3}{2} \frac{4}{4})\) and \((3 4) \in H_2\), we have
\[
(-1) Wg^{DIII}((\frac{1}{1} \frac{2}{3} \frac{3}{2} \frac{4}{4}); n) = Wg^{DIII}((\frac{1}{1} \frac{2}{3} \frac{3}{2} \frac{4}{4}); n) = \frac{-1}{2n-1}.
\]

### 5.2 Class C I

The setting for C I: \( G = \text{Sp}(2n) \), \( K = \text{U}(n) \), \( \Omega(g) = I'_{nn}gI'_{nn} \). Here the fix-point set of \( \Omega \) in \( G \) is
\[
\left\{ \begin{pmatrix} U & O \\ O & U \end{pmatrix} \mid U \in \text{U}(n) \right\} \simeq \text{U}(n).
\]

Let \( S \) be a \( 2n \times 2n \) Haar symplectic matrix. We consider the random matrix
\[
W^C = S^DI'_{nn}S,
\]
instead of \( V^C = \Omega(S)^{-1}S = I'_{nn}S^D'I'_{nn}S \), which is associated to the symmetric space \( \text{Sp}(2n)/\text{U}(n) \).
We define the function $T_n^{C1}$ on $S_{2k}$ by

$$T_n^{C1}(\sigma) = T_\sigma(J, JJ'_{nn}).$$

Since $JJ'_{nn} = - (J_n^T J_n)_{nn}$ is symmetric and $J$ is skew-symmetric, the function $T_n^{C1}$ satisfies $T_n^{C1}(\zeta, \zeta') = T_n^{C1}(\sigma) Wg^{C1}(\sigma, n)$ for $\sigma \in S_{2k}$ and $\zeta, \zeta' \in H_k$. For $\sigma = \sigma_\mu$, we have

$$T_n^{C1}(\sigma_\mu) = \prod_{i=1}^{\ell(\mu)} T_n^{C1}(\sigma_{(i)}) = - \prod_{i=1}^{\ell(\mu)} \text{Tr}(JJ'_{nn}) = (-1)^{\ell(\mu)} \prod_{i=1}^{\ell(\mu)} \text{Tr}(-I_{nn})^{\mu_i}
= \begin{cases} (-2n)^{\ell(\mu)} & \text{if } \mu \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $T_n^{C1}(\sigma) = T_n^{DIII}(\sigma^{-1}) = \pm T_n^{DIII}(\sigma)$ for all $\sigma \in S_{2k}$.

**Theorem 5.2.** Let $W = W^{C1}$ be the random matrix defined as above. For a sequence $i = (i_1, i_2, \ldots, i_{2k})$ with an even number $k$, we have

$$\mathbb{E}[\hat{w}_{i_1, i_2} \hat{w}_{i_3, i_4} \cdots \hat{w}_{i_{2k-1}, i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta_{\sigma}^i Wg^{C1}(\sigma; n).$$

Here $Wg^{C1}(\cdot; n)$ is the function on $S_{2k}$ defined by

$$Wg^{C1}(\cdot; n) = T_n^{C1} \ast Wg^{Sp}(\cdot; n).$$

If $k$ is odd, then, for any sequence $i = (i_1, i_2, \ldots, i_{2k})$,

$$\mathbb{E}[\hat{w}_{i_1, i_2} \hat{w}_{i_3, i_4} \cdots \hat{w}_{i_{2k-1}, i_{2k}}] = 0.$$

**Proof.** It is proved in a usual way. The Weingarten calculus for a Haar symplectic matrix $S$ gives

$$\mathbb{E}[\hat{w}_{i_1, i_2} \hat{w}_{i_3, i_4} \cdots \hat{w}_{i_{2k-1}, i_{2k}}] = \sum_{\sigma \in M_{2k}} \Delta_{\sigma}^i \sum_{\tau \in M_{2k}} Wg^{Sp}(\tau^{-1} \sigma; n) \bar{T}(\tau)$$

where

$$\bar{T}(\tau) := \sum_{p=(p_1, \ldots, p_{2k})} \Delta_{\tau}(p) \prod_{r=1}^{k} e_{p_{2r-1}} e_{p_{2r}} = T_\tau(J, JJ'_{nn}) = T_n^{C1}(\tau)$$

for any $\tau \in S_{2k}$. □

The function $Wg^{C1}(\cdot; n)$ on $S_{2k}$ satisfies

$$Wg^{C1}(\zeta, \zeta'; n) = \epsilon(\zeta') Wg^{C1}(\sigma, n) \quad (\sigma \in S_{2k}, \zeta, \zeta' \in H_k),$$

and hence $Wg^{C1}(\zeta; n) = 0$ for $\zeta \in H_k$. Furthermore, $Wg^{C1}(\sigma; n) = 0$ (for $\sigma \in S_{2k}$ if $k$ is odd.

**Example 5.2.**

$$Wg^{C1}((1, 2, 3, 4); n) = Wg^{C1}((1, 4, 3, 2); n) = \frac{-1}{2n + 1}.$$
6 Conclusion

We have made methods for computations of moments of matrix elements from classical compact Lie groups and classical compact symmetric spaces. Write \( C = A, B/D, C \) for unitary, orthogonal, symplectic groups, respectively. The moment for a classical group is given by the double sum

\[
\sum_{\sigma} \sum_{\tau} (\Delta\text{-function in } \sigma) \times (\Delta\text{-function in } \tau) \times Wg^C(\sigma^{-1} \tau; n),
\]

whereas that for a classical compact symmetric space is given by the single sum

\[
\sum_{\sigma} (\Delta\text{-function in } \sigma) \times Wg^C(\sigma; n).
\]

Here the \( \Delta\text{-function is} \)

- \( \delta_\sigma(\cdot, \cdot) \) defined in (2.2) if \( C = A, A\ I, A\ II, A\ III, \)
- \( \Delta_\sigma(\cdot) \) defined in (2.6) if \( C = B/D, BD\ I, D\ III, \)
- \( \Delta'_\sigma(\cdot) \) defined in (2.12) if \( C = C, C\ I, C\ II, \)

and the Weingarten function \( Wg^C \) belongs to

- \( Z(L(S_k)) \) if \( C = A, A\ III, \)
- \( L(S_{2k}, H_k) \) if \( C = B/D, A\ I, BD\ I, \)
- \( L'(S_{2k}, H_k) \) if \( C = C, A\ II, C\ II. \)

The Weingarten functions for \( C = D\ III \) or \( C\ I \) differ from others. In fact, if we let \( F(\sigma) \) to be \( Wg^{D\ III}(\sigma; n) \) or \( Wg^{C\ I}(\sigma^{-1}; n) \), then \( F \) is the function on \( S_{2k} \) satisfying the property

\[
F(\zeta \sigma \zeta') = e(\zeta) F(\sigma) \quad (\sigma \in S_{2k}, \zeta, \zeta' \in H_k).
\]

In particular, \( F \) identically vanishes on \( H_k \), and, moreover, on \( S_{2k} \) if \( k \) is odd. Weingarten functions except \( D\ III \) and \( C\ I \) have Fourier expansions in \( \chi^\lambda, \omega^\lambda, \) or \( \pi^\lambda \). However, we could not find such expansions for these two cases.

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