Exact solutions of an SO(5)-invariant spin-(3/2) Fermi gas model

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Abstract – An exactly solvable model describing the dilute spin-(3/2) fermion gas in one-dimensional optical trap is proposed. The diagonalization of the model Hamiltonian is derived by means of the Bethe ansatz method. Exotic spin excitations such as the heavy spinon with fractional spin 3/2, the neutral spinon with spin zero and the dressed spinon with spin 1/2 are found basing on the exact solution.

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Introduction. – The study on the optically and magnetically trapped ultracold atoms has attracted a lot of attentions in recent years. By using the magnetic fields or laser beams, the atoms can be trapped and cooled down to very low temperatures [1–11]. A fascinating fact is that not only the components but also the physical parameters of such systems can be manipulated in optical lattice, which allows us to mimic the ordinary correlated electron systems and even to explore new states of matter. For example, by means of the Feshbach resonance techniques, one can tune the scattering lengths or the interactions among the atoms; putting an optical lattice on, one can study the behavior of the atoms in a tunable periodic potential and with strong anisotropic traps, the one- and two-dimensional quantum systems can be realized. One of the hot research topics in this field is the study on the high-spin cold-atom systems. The physics of bosonic ultracold atoms with hyperfine spin \( F = 1 \) have been extensively studied [12–17]. The degenerate Fermi gas with hyperfine spin \( F = 3/2 \) could be obtained by cooling alkali atoms \(^{132}\text{Cs}\), as well as alkaline-earth atoms \(^{9}\text{Be}, \, ^{135}\text{Ba}, \, ^{137}\text{Ba}\) and \(^{53}\text{Cr}\) in experiments. The ground states of these high-spin fermionic atoms in optical traps are very rich. Many peculiar quantum orders and exotic collective excitations appear in these systems, which are rare in the ordinary interacting electron systems[18]. Several methods have been developed to approach these ultracold-atom systems. For example, Wu et al. showed that the one-dimensional (1D) spin-(3/2) Fermi gas with s-wave scattering possesses the SO(5)-invariance and obtained the phase diagram of the system by means of bosonization [19,20]; the functional integral approach was applied to interacting spin-(3/2) fermionic ultracold atoms in 2D square optical lattice to study a variety of Mott insulating phases [21,22]; two different superfluid phases in \( F = N - 1/2 \) fermionic cold-atom system were found, i.e., an unconfined BCS pairing phase and a confined molecular-superfluid made of \( 2N \) fermions, depending on whether a discrete symmetry is spontaneously broken or not [23].

Despite the fast progress in this field, the understanding of the high-spin correlated systems is still far from satisfied. Reference models with exact solutions are undoubtedly needed to get deep insight into these interesting quantum many-body systems. An important progress in this aspect is made by Controzzi and Tsvelik, who constructed an exactly solvable model for isospin \( F = 3/2 \) fermionic system with linear dispersion relations [24]. In this letter, we propose a new exactly solvable model which describes properly a 1D spin-(3/2) cold-atom system. With the Bethe ansatz solutions, we derive exactly the ground state. Exotic spin excitations such as the heavy spinons with fractional spin 3/2, the neutral spinons with zero spin and the dressed spinons with spin 1/2 are found.

The model and the Bethe ansatz equation. – The fermions with hyperfine spin \( F = 3/2 \) in a 1D trap is appropriately described by the following model Hamiltonian

\[
H = \sum_{j=1}^{N} \left[ -\frac{\partial^2}{\partial x_j^2} + V_e(x_j) \right] + \sum_{l<j,m} g_m \delta(x_j - x_l) P^m_{jl}, \tag{1}
\]

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where the kinetic energy $g_m$ is the two
bunching energy, and $P_{m}^{2}$ is the potential.

In the low-density case, the external

is no longer conserved because of the broken

case, the particle number of an individual spin component

Sutherland model [25, 26]. An obvious fact is that in our

physical properties are quite different to those of the

three independent conserved quantities hold:


\[ \sum_{I} c_{I} = \sum_{I} b_{I} = \sum_{I} a_{I} = 0 \]

Nevertheless, the system (2) is still

another integrable line


\[ \sum_{j} c_{j} \delta(x_j - x_i) = 0 \]

where $c_0 = g_0/3 + 2g_2/3$, $c_2 = g_2/3 - g_0/3$ and $S$ is the

spin-(3/2) operator. If $c_2 = 0$, the system (2) is $SU(4)$-

invariant and was solved exactly by Sutherland [25]. In

the present letter, we show that the system (2) possesses

another integrable line $c_0 = c/2$, $c_2 = -2c/3$ at which the

physical properties are quite different to those of the

Sutherland model [25, 26]. An obvious fact is that in our

case, the particle number of an individual spin component

is no longer conserved because of the broken $SU(4)$-
symmetry. Nevertheless, the system (2) is still $SO(5)$-
invariant at this new integrable line, and the following

three independent conserved quantities hold:

\[ I_1 = N_{3/2} + N_{1/2} + N_{-1/2} + N_{-3/2}, \]
\[ I_2 = N_{3/2} - N_{-3/2}, \]
\[ I_3 = N_{1/2} - N_{-1/2}, \]

(3)

where $N_s$ indicates the particle number with the spin

component $s = \pm 1/2, \pm 3/2$.

Assume the wave function takes the following form:

\[ \Psi(x_1 s_1, \ldots, x_N s_N) = \sum_{Q, \alpha} \theta(x_{Q_1}, \ldots, x_{Q_N}) \times A_{s_1, \ldots, s_N}(P) e^{i \sum_{j<k} \epsilon_{j} x_{Q_{j}}}, \]

(4)

where $Q = (Q_1, \ldots, Q_N)$ and $P = (P_1, \ldots, P_N)$ are the permutations of the integers $1, \ldots, N$; $k_j$ are the quasi-

momenta carried by the particles; $\theta(x_{Q_1}, \ldots, x_{Q_N}) = \theta(x_{Q_1} - x_{Q_{2}} - x_{Q_N}) \ldots \theta(x_{Q_2} - x_{Q_{N}})$ and $\theta(x - y)$ is the step

function. With the standard coordinate Bethe ansatz method,

we obtain the two-body scattering matrix as

\[ S_{jl} = k_j - k_l - i \frac{\epsilon_{j}}{2} \frac{P_{j}^{0} + P_{l}^{0}}{k_j - k_l + i \frac{\epsilon_{j}}{2}} \frac{P_{j}^{2} + P_{l}^{2}}{k_j - k_l + i \frac{\epsilon_{j}}{2}}, \]

(5)

which satisfies the Yang-Baxter equation

\[ S_{12}(k_1 - k_2) S_{13}(k_1 - k_3) S_{23}(k_2 - k_3) = S_{23}(k_2 - k_3) S_{13}(k_1 - k_3) S_{12}(k_1 - k_2), \]

(6)

Applying further the nested algebraic Bethe ansatz with

periodic boundary conditions [27], we obtain the following

Bethe ansatz equations (BAE):

\[ e^{ik_j L} = \prod_{\alpha=1}^{M_k} \frac{k_j - \lambda_{\alpha} + i \frac{\epsilon_j}{2}}{k_j - \lambda_{\alpha} - i \frac{\epsilon_j}{2}} j = 1, \ldots, N, \]
\[ \prod_{l=1}^{N} \lambda_j - k_l + i \frac{\epsilon_j}{2} \prod_{\gamma=1}^{M_j} \lambda_j - \mu_{\gamma} + i \frac{\epsilon_j}{2} = - \prod_{\alpha=1}^{M_k} \lambda_j - \lambda_{\alpha} + i \frac{\epsilon_j}{2} \]
\[ \prod_{\alpha=1}^{M_j} \lambda_j - \lambda_{\alpha} - i \frac{\epsilon_j}{2} = \prod_{\gamma=1}^{M_j} \lambda_j - \mu_{\gamma} - i \frac{\epsilon_j}{2} \]

(7)

where $N = N_{3/2} + N_{1/2} + N_{-1/2} + N_{-3/2}$, $M_k = N_{1/2} + 2N_{3/2} + N_{-1/2}$. $M_j = N_{-1/2} + N_{-3/2}$ and $L$ is the length

of the system. The corresponding eigenenergy of the

Hamiltonian is $E = \sum_{j=1}^{N} \lambda_j$.

String structure and free energy. – We consider $c > 0$, i.e., the repulsive interaction case. In the thermodynamic limit, the complex solutions of the rapidities symmetrically distribute in the complex plane with the real axis in the string hypothesis. The structure of the strings can be obtained by carefully checking the BAE (7). It can be found from the first equation of (7) that if there is a complex $k$ with a positive (negative) imaginary part, the modulus of the left hand of this equation is always less (greater) than 1, while modulus of the right side is greater (less) than 1 because $\lambda$’s appear always in conjugate pairs. Thus all the charge rapidities $k_j$ take real values, indicating the absence of charge bound state. However, the spin rapidities $\lambda_{\beta}$ and $\mu_{\nu}$ may form strings with the following form in the thermodynamic limit:

\[ \lambda_{\beta}^{(n)} = \frac{i c}{4} (n + 1 - 2j), \quad j = 1, \ldots, n, \]
\[ \mu_{\nu}^{(m)} = \frac{i c}{2} (n + 1 - 2j), \quad j = 1, \ldots, m, \]

(8)

(9)

where $\lambda_{\beta}^{(n)}$ and $\mu_{\nu}^{(m)}$ are the real parts of the $n$-string of $\lambda$ and the $m$-string of $\mu$, respectively. Denote $\rho$, $\eta_n$ and $\sigma_m$ as the densities of $k$, $\lambda$ n-arcs and $\mu$ m-arcs in the thermodynamic limit $N \to \infty$, $L \to \infty$ and $N/L \to$ finite, respectively, and $\rho^{h}$, $\eta^{h}$ and $\sigma^{h}$ the corresponding densities of holes. At temperature $T$, the Gibbs free energy of the system with an external magnetic field $h$ and chemical potential $A$ is

\[ F = E - AN - h \left( \frac{3}{2} N - M_1 - M_2 \right) - TS, \]

(10)

where $E = L \int k^2 \rho(k) dk$ is the energy; $N = L \int \rho(k) dk$ is the particle number; $M_1 = L \sum_{n} \eta_n \int \sigma_m(\mu) d\mu$; $M_2 = L \sum_{m} \sigma_m \int \sigma_m(\mu) d\mu$; $S = L \int \{(\rho + \rho^h) \ln(\rho + \rho^h) - \rho \ln \rho - \rho^h \ln \rho^h + \sum \{(\eta_n + \eta^h) \ln(\eta_n + \eta^h) - \eta_n \ln \eta_n - \eta^h \ln \eta^h \} + \sum \{(\sigma_m + \sigma^h) \ln(\sigma_m + \sigma^h) - \sigma_m \ln \sigma_m - \sigma^h \ln \sigma^h \}) dk$
denotes the entropy. Minimizing the Gibbs free energy at the thermal equilibrium, we obtain the following thermodynamic Bethe ansatz equations (TBAE):

\[
\ln \tilde{\rho} = \frac{k^2 - A}{T} - \sum_n \alpha_n \ln (1 + \tilde{\eta}_n^{-1}),
\]

\[
\ln \tilde{\eta}_1 = G_1 \ln ((1 + \tilde{\eta}_2)(1 + \tilde{\rho}^{-1})),
\]

\[
\ln \tilde{\sigma}_1 = G_2 \ln \left( \frac{1 + \tilde{\sigma}_2}{(1 + \tilde{\eta}_1^{-1})(1 + \tilde{\eta}_3^{-1})} \right)
\]

\[
- \frac{G_2}{G_1} \ln (1 + \tilde{\eta}_2^{-1}),
\]

\[
\ln \tilde{\eta}_{n \in \text{even}} = G_1 \ln \left( \frac{1 + \tilde{\eta}_{n-1}^{-1}(1 + \tilde{\eta}_{n+1})}{1 + \tilde{\eta}_{n/2}^{-1}} \right),
\]

\[
\ln \tilde{\eta}_{n \in \text{odd}} = G_1 \ln \left( \frac{(1 + \tilde{\eta}_{n-1}^{-1})}{(1 + \tilde{\eta}_{n+1}^{-1})} \right),
\]

\[
\ln \tilde{\sigma}_m = G_2 \ln \left( \frac{1 + \tilde{\sigma}_{m-1}}{(1 + \tilde{\eta}_{m-1}^{-1})(1 + \tilde{\eta}_{m+1}^{-1})} \right)
\]

\[
- \frac{G_2}{G_1} \ln (1 + \tilde{\eta}_{2m-1}^{-1}),
\]

\[
\lim_{n \to \infty} \frac{\ln \tilde{\eta}_n}{n} = h, \quad \lim_{m \to \infty} \frac{\ln \tilde{\sigma}_m}{m} = \frac{h}{T},
\]

where \( \tilde{\rho} = \rho_n / \rho, \tilde{\eta}_n = \eta_n / \eta_1, \tilde{\sigma}_m = \sigma_m / \sigma_n, G_n = a_n / (a_0 + a_{2n}), a_0 \equiv \delta(x), a_n(x) = 4nc / [\pi(16x^2 + n^2c^2)] \) and \( a_n \star f(x) = \int a_n(x-y)f(y)dy \).

**Ground state.** The ground-state configuration of the system can be obtained by taking the limit of \( T \to 0 \) and \( \hbar \to 0 \). In this case, most of the string densities are zero and the TBAE (11) are reduced to

\[
\rho(k) = \frac{1}{2\pi} + a_1 \ast \eta_1(k) + a_2 \ast \eta_2(k),
\]

\[
\eta_1(\lambda) = \int_{-Q}^{Q} a_1(\lambda - k)\rho(k)dk - a_2 \ast \eta_1(\lambda)
\]

\[
- (a_1 + a_3) \ast \eta_2(\lambda) + a_1 \ast \sigma(\lambda),
\]

\[
\eta_2(\xi) = \int_{-Q}^{Q} a_2(\xi - k)\rho(k)dk - (a_1 + a_3) \ast \eta_1(\xi)
\]

\[
- (a_2 + a_4) \ast \eta_2(\xi) + (a_1 + a_3) \ast \sigma(\xi),
\]

\[
\sigma(\mu) = a_1 \ast \eta_1(\mu) + (a_1 + a_3) \ast \eta_2(\mu) - a_4 \ast \sigma(\mu).
\]

The Fermi point \( Q \) is determined by the density of particles \( N/L = \int_{-Q}^{Q} \rho(k)dk \). \( M_1/L = \int [\eta_1(\lambda)]d\lambda \) and \( M_2/L = \int \sigma(\mu)d\mu \). Such a ground-state configuration is quite different from that of the SU(4) Sutherland model, where there is no string or spin bound state in the ground state. In the present SO(5) case, part of the spin rapidities form 2-strings which heavily affect the spin excitations as we shall show below. From the solutions of eq. (12), we can easily derive \( M_1 = N \) and \( M_2 = N/2 \), which give the total spin of the ground state \( S = 3N/2 - M_1 - M_2 = 0 \), indicating a spin singlet ground state. When \( c \) tends to zero, the function \( a_n(x) \to \delta(x) \). In this case we recover the free Fermi gas solutions

\[
\rho(k) = \frac{2}{\pi}, \quad \frac{E}{L} = \frac{4}{3\pi} Q^3, \quad \frac{N}{L} = \frac{4}{\pi} Q.
\]

When \( c \to \infty \), \( a_n(x) \to 0 \), we recover the Tonks-Girardeau solutions [26,28]

\[
\rho(k) = \frac{1}{2\pi}, \quad \frac{E}{L} = \frac{1}{3\pi} Q^3, \quad \frac{N}{L} = \frac{1}{\pi} Q.
\]

**Elementary excitations.** Basing on the ground-state configuration, the elementary excitations of the system can be studied exactly. In the integrable models, the excitation energies are uniquely defined by the so-called dressed energies \( \epsilon = T \ln \tilde{\rho}, \xi_1 = T \ln \tilde{\eta}_1 \) and \( \xi_m = T \ln \tilde{\sigma}_m \). In our case, when \( T \to 0 \), only \( \epsilon, \xi_1, \xi_2 \) and \( \xi_3 \) are left. The dressed energies of the ground state for \( Q = 1 \) and \( c = 1 \) are shown in fig. 1. The low-lying excitations of the system can be studied systematically by adding particles, holes or strings into the ground state configuration of the rapidities. The excitation energy reads

\[
\Delta E = \sum_{j=1}^{n_h} \epsilon(k^h_{j}) + \sum_{j=1}^{n_p} \epsilon(k^p_{j}) + \sum_{j=1}^{m_1} \xi_1(\lambda^h_{j,1})
\]

\[
+ \sum_{j=1}^{m_2} \xi_2(\lambda^h_{j,2}) + \sum_{j=1}^{m_3} \xi_3(\mu^h_{j}),
\]

where \( n_h, n_p \) and \( m_{1,2,3} \) are the numbers of the charge holes, excited charges, holes in real-\( \lambda \) sea, in 2-string \( \lambda \) sea and in real-\( \mu \) sea, respectively; \( k^h_{j} \), \( k^p_{j} \), \( \lambda^h_{j,1,2} \), and \( \mu^h_{j} \) are the positions of the corresponding charges and holes. Formally, the extra strings contribute nothing to the energy because the contribution of such strings is exactly canceled by the rearrangement of the Fermi sea. Some of the low-lying excitations are shown in fig. 2.
Fig. 2: (Color online) The low-lying excitations ($c = 1$, $Q = 1$). $\Delta E$ and $P$ are the energy and the momentum carried by the excitation. (a) The charge-hole excitations. (b) The excitation of two holes of real $\lambda$. (c) The excitation of two holes of real $\mu$. (d) The excitation of four holes of $\lambda$ 2-string.

Table 1: Possible hole configurations.

| $m_1$ | $m_2$ | $m_3$ |
|------|------|------|
| 2    | 0    | 0    |
| 0    | 2    | 4    |
| 0    | 1    | 2    |

The real-$k$ charge-hole excitation in our case is similar to that of $SU(4)$-invariant Sutherland model as shown in fig. 2(a). However, the excitations in the spin sector are quite different from those of the Sutherland model. The spin quanta carried by the spin excitations read

$$ S = \frac{3}{2} m_2 + 2m_3 + \sum_{l \geq 3} (2 - l)m_{\lambda(l)} + \sum_{t \geq 2} (1 - t)m_{\mu(t)}, $$

where $m_{\lambda(l)}$ and $m_{\mu(t)}$ are the numbers of $\lambda$ $l$-strings and the $\mu$ $t$-strings formed in the excitations, respectively. We note the numbers of holes and strings added are not independent but satisfy some constraints determined by the BAE:

$$ \Delta M_1^1 = -m_1 + \frac{1}{2}m_2, $$

$$ \Delta M_1^2 = \frac{1}{2}m_1 - \frac{3}{4}m_2 - \frac{1}{2}m_3 - \sum_{l \geq 3} m_{\lambda(l)}, $$

$$ \Delta M_2^1 = -\frac{1}{2}m_2 - m_3 - \sum_{t \geq 2} m_{\mu(t)}, $$

where $\Delta M_{1,2}$ (integers) indicate the number changes of $\lambda, \mu$ $l$-strings. Several possible hole configurations are listed in table 1. Nevertheless, the energies of the holes are additive as shown in eq. (15) and the thermodynamic behavior of the system is mainly determined by the dispersion relations of the individual holes, which are shown in fig. 3. Interestingly, the single $\lambda$ 2-string hole carries the lowest energy with spin $3/2$ (named as heavy spinon) and therefore dominate the low-temperature thermodynamics of the system.

The simplest spin excitation is a real-$\lambda$ hole pair (the first column in table 1), corresponding to the two domain walls of a single excited domain. However, unlike the usual spinons, such holes carry zero spin (named as neutral spinons). The $\lambda$ 2-string hole pair cannot exist independently but must be associated with a neutral spinon (the second column in table 1) or a real-$\mu$ hole (the third column in table 1). If we add further a $\lambda$ 4-string into a 2-string hole pair and a real-$\lambda$ hole configuration, the total spin of this excitation is $1$. In this case each of the $\lambda$ 2-string holes carries a spin $1/2$. Such a dressed hole is quite similar to the ordinary spinon (named as dressed spinon). Four $\lambda$ 2-string holes may exist independently (the fourth column in table 1). If we put further one $\lambda$ 6-string and one $\mu$ 3-string into this four-hole configuration, we get the $SO(5)$ spin singlet excitation. The simplest excitation in the $\mu$ sector is a pair of real-$\mu$ holes (the fifth column in table 1). This excitation is quite similar to a real-$\lambda$ hole pair but each of the real-$\mu$ hole carries a spin 2. Joint pair of a real-$\lambda$ hole and a real-$\mu$ hole may also happen as shown in the last column in table 1. Other kinds of spin excitations can be obtained by analyzing the BAE but most of the complex excitations are generally composed of the excitations listed in table 1.

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