COMBINATORICS OF ORBIT CONFIGURATION SPACES

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Abstract. From a group action on a space, define a variant of the configuration space by insisting that no two points inhabit the same orbit. When the action is almost free, this “orbit configuration space” is the complement of an arrangement of subvarieties inside the cartesian product, and we use this structure to study its topology.

We give an abstract combinatorial description of its poset of layers (connected components of intersections from the arrangement) which turns out to be of much independent interest as a generalization of partition and Dowling lattices. The close relationship to these classical posets is then exploited to give explicit cohomological calculations akin to those of (Totaro ‘96).

1. Introduction

1.1. Orbit configuration spaces. A fundamental topological object attached to a topological space $X$ is its ordered configuration space $\text{Conf}_n(X)$ of $n$ distinct points in $X$. Analogously, given a group $G$ acting freely on $X$ one defines the orbit configuration space by

$$\text{Conf}_n^G(X) = \{(x_1, \ldots, x_n) \in X^n \mid Gx_i \cap Gx_j = \emptyset \text{ for } i \neq j\}.$$ 

These spaces were first defined in [XM97] and come up in many natural topological contexts, including:

- Universal covers of $\text{Conf}_n(X)$ when $X$ is a manifold with $\dim(X) > 2$ [XM97].
- Classifying spaces of well studied groups, such as normal subgroups of surface braid groups with quotient $G^n$ [XM97].
- Arrangements associated with root systems [Bib17, Loo76, Moc08].
- Equivariant loop spaces of $X$ and $\text{Conf}_n(X)$ [Xic02].

A fundamental problem is thus to compute the cohomology $H^*(\text{Conf}_n^G(X))$. This has been previously studied e.g. by [Cas16, DS17, FZ02].

The current literature typically requires the action to be free, with main results relying on this assumption. For an action that is not free, one could simply throw out the set of singular points for the action and consider $\text{Conf}_n^G(X \setminus S)$, where

$$S := \text{Sing}_G(X) = \bigcup_{g \in G \setminus \{e\}} X^g,$$

the set of points fixed by a nontrivial group element. However, the excision can create more harm than good: e.g. when $X$ is a smooth projective variety, removing $S$ destroys the projective structure and causes mixing of Hodge weights in cohomology. In particular, having a projective structure makes a spectral sequence...
calculation more manageable (see Theorem D and §3.6). Furthermore, one is often interested in allowing orbit configurations to inhabit S, e.g. in arrangements arising from type B and D root systems (see §3.4).

We propose an alternative approach: observe that inside \( X^n \), the orbit configuration space \( \text{Conf}^G_n(X \setminus S) \) is the complement of an arrangement \( A_n(G,X) \) of subspaces. The cohomology \( H^*(\text{Conf}^G_n(X \setminus S)) \) can then be computed from the combinatorics of this arrangement and from \( H^*(X) \). Furthermore, the wreath product of \( G \) with the symmetric group \( S_n \), which we denote by \( S_n[G] \), acts naturally on the space \( X^n \), and this induces an \( S_n[G] \)–action on the set \( A_n(G,X) \) and its complement \( \text{Conf}^G_n(X \setminus S) \). The induced action on \( H^*(\text{Conf}^G_n(X \setminus S)) \) can also be traced through the combinatorial computation.

1.2. Running assumptions and notation. For our study, we from hereon assume that \( G \) and \( S \) are finite sets, so that the arrangement \( A_n(G,X) \) is finite. Moreover, by a “space” \( X \) we mean either CW-complex or an algebraic variety over an algebraically closed field.

When discussing cohomology below, we will always suppress the coefficients, as they never have a significant effect on the results. For \( X \) a CW-complex, one may take the cohomology \( H^*(X) \) to mean singular cohomology with coefficients in any ring \( R \); and for \( X \) an algebraic variety, \( H^*(X) \) may be taken to mean \( \ell \)-adic cohomology with coefficients in either \( \mathbb{Z}_\ell \) or \( \mathbb{Q}_\ell \).

1.3. Combinatorics. The combinatorics at play is the poset of layers: connected components of intersections from \( A_n(G,X) \), ordered by reverse inclusion. This poset admits an abstract combinatorial description, that does not in fact depend on \( X \) (only depending on the \( G \)–set \( S \)) and it is of much independent interest. For example,

- In the case of classical configuration spaces (\( G \) trivial), the poset is the lattice of set partitions of \( n = \{1, 2, \ldots, n\} \).
- In the case that \( G \) is a cyclic group acting on \( X = \mathbb{C} \) via multiplication by roots of unity, the poset is an instance of the Dowling lattice, described in [Dow73] as an analogue of the partition lattice which consists of partial \( G \)–partitions of \( n \).

In §2.1, we define the poset \( D_n(G,S) \) which specializes to these classical examples and discuss the natural action of the wreath product group \( S_n[G] \).

Even though \( D_n(G,S) \) is not in general a lattice, it supports a myriad of properties that have been fundamental in the modern study of posets, since it is essentially built out of partition and Dowling lattices as indicated in the following theorem (Theorem 2.4.2):

**Theorem A (Local structure of \( D_n(G,S) \)).** For any \( \alpha, \beta \in D_n(G,S) \) with \( \alpha < \beta \), the interval \([\alpha, \beta] \) is isomorphic to a product

\[ Q_{n_1} \times \ldots \times Q_{n_k} \times \mathcal{D}_{m_1}(G_1) \times \ldots \times \mathcal{D}_{m_k}(G_k) \]

where \( Q_{n_i} \) denotes a partition lattice and \( \mathcal{D}_{m_j}(G_j) \) denotes a Dowling lattice for some subgroup \( G_j \leq G \). In particular, every interval is a geometric lattice and has the homology of a wedge of spheres.

In the remainder of Section §2 we study the structure of these posets: In §2.3 we discuss their functoriality in the various inputs; in §2.4 we discuss local structure
and prove Theorem A. Of particular interest is §2.5, where we discuss the characteristic polynomial: a fundamental invariant of a ranked poset, which is a common generalization of the chromatic polynomial of a graph, and the Poincaré polynomial of the complement of a hyperplane arrangement. We give a factorization of the characteristic polynomial into linear factors, generalizing a long list of special cases stretching back to Arnol’d and Stanley’s work on the pure braid group and the partition lattice.

**Theorem B** (Characteristic polynomial). Let $S \neq \emptyset$ be a $G$-set. Then

$$
\chi(D_n(G,S);t) := \sum_{\beta \in D_n(G,S)} \mu(\hat{0}, \beta) t^{n-rk(\beta)} = \prod_{i=0}^{n-1} (t - |S| - |G|_i),
$$

where $\mu$ is the M"obius function of the poset and $\hat{0}$ is the minimum element. An analogous factorization for the case $S = \emptyset$ appears in Theorem 2.5.2 below.

Lastly, in §2.6, we consider the action of $\mathfrak{S}_n[G]$ on the poset $D_n(G,S)$, and describe its orbits. In §2.7, we study their Whitney homology as a representation of $\mathfrak{S}_n[G]$: this invariant has proved important both for topology and to the abstract theory of posets, and will be later used when discussing orbit configuration spaces in §3.

**1.4. Topology.** As mentioned above, the poset $D_n(G,S)$ arises naturally in the study of orbit configuration spaces, when we take $S$ to be the set of singular points for the action of $G$ on $X$. Section §3 is devoted to studying the topology of these spaces, and relating it with the combinatorics of Section §2.

In §3.1 we define an arrangement $A_n(G,X)$ in $X^n$, whose complement is the orbit configuration space $\text{Conf}_G^n(X \setminus S)$. Recall that the poset of layers of an arrangement $A_n(G,X)$ is the collection of connected components of intersections from $A_n(G,X)$, ordered by reverse inclusion. This poset encodes subtle aspects of the topology of $\text{Conf}_G^n(X \setminus S)$, as we shall see here (Theorem 3.2.5):

**Theorem C** (Poset of layers). The poset of layers of the arrangement $A_n(G,X)$ is naturally and $\mathfrak{S}_n[G]$-equivariantly isomorphic to the poset $D_n(G,S)$.

This description opens the door to cohomology calculations: considering a spectral sequence for complements of arrangements (see [Pet17] and also [Tot96, Bib16, Dup15]), one obtains a description of the $E_1$-page in terms of the poset’s Whitney homology. Furthermore, when $X$ is a smooth projective algebraic variety, a weight argument guarantees that there could be at most one nonzero differential. Thus, in this case one is closer to getting a hand on the cohomology.

We summarize the explicit description of the spectral sequence machinery, following the simplifications that arise from our combinatorial analysis, in the following (Theorem 3.6.1):

**Theorem D** (Simplified spectral sequence). There is a spectral sequence with

$$
E_1^{p,q} = \bigoplus_{\beta \in D_n(G,S)} H_c^p(X^{\beta}) \otimes \tilde{H}^{p-2}(\hat{0}, \beta) \implies H^{p+q}_c(\text{Conf}_n^G(X \setminus S)).
$$

Here, the summands are indexed by poset elements $\beta$ of rank $p$, and $X^{\beta}$ denotes the corresponding layer in $X^n$. The term $\tilde{H}^{p-2}(\hat{0}, \beta)$ denotes the reduced cohomology...
of the order complex for the interval $(0, \beta) \subset \mathcal{D}_n(G, S)$, and is therefore described explicitly by Theorem 2.4.2.

When $X$ is a smooth projective variety, the sequence degenerates at the $E_2$–page, i.e. all differentials vanish past the first page.

Recall that certain invariants of $\text{Conf}^G_n(X \setminus S)$ can be computed already from any page of a spectral sequence converging to $\mathrm{H}^*(\text{Conf}^G_n(X \setminus S))$. These are the generalized Euler characteristics, or cut-paste invariants, discussed briefly in §3.7. Universal among those is the motive, i.e. the class $[\text{Conf}^G_n(X \setminus S)]$ in the Grothendieck ring of varieties. Our combinatorial calculations then give:

**Theorem E (Motivic factorization).** Let $G$ act on an algebraic variety $X$ over and algebraically closed field $k$ as above, with singular set $S \neq \emptyset$. Then in the Grothendieck ring $K_0(k)$,

$$[\text{Conf}^G_n(X \setminus S)] = \prod_{i=0}^{n-1} ([X] - |S| - |G||i).$$

An analogous factorization for a free action is given in Theorem 3.7.1 below.

In particular, this gives a formula for the number of $\mathbb{F}_q$-points in $\text{Conf}^G_n(X \setminus S)$ for every $q$ divisible by $\text{char}(k)$. Alternatively, when $X \setminus S$ is smooth, one gets a formula for the classical Euler number of $\text{Conf}^G_n(X \setminus S)$.

In §3.5 we analyze the local structure of the arrangement $\mathcal{A}_n(G, X)$, i.e. its germ at every point in $X^n$. A surprising conclusion is that, to first order, the arrangement $\mathcal{A}_n(G, X)$ is isomorphic to a product of orbit configuration spaces for groups possibly different from $G$. As a byproduct of our analysis we get a new proof of the following result.

**Corollary F (Stabilizers on curves).** Suppose $G$ acts faithfully on an algebraic curve $C$ over some algebraically closed field $k$. Then the stabilizer in $G$ of any smooth point is cyclic.

Lastly, our handle on the combinatorics of these arrangements can be exploited to understand what happens when one removes from $X$ a set $T$ other than the set of singular points $S$. We consider this more general case in §3.4, but note now that all of our theorems hold true for these spaces as well.

For example, when $T$ is a $G$–invariant subset of $S$, the group $G$ now acts on $X \setminus T$ with nontrivial stabilizers. The resulting orbit configuration space $\text{Conf}^G_n(X \setminus T)$ is the complement in $X^n$ of a subarrangement of $\mathcal{A}_n(G, X)$. The new poset of layers is a subposet of $\mathcal{D}_n(G, S)$, which inherits many properties from $\mathcal{D}_n(G, S)$ to which our study applies. These types of arrangements arise naturally, e.g. from roots systems in $\mathbb{C}$, $\mathbb{C}^*$ and elliptic curves (see §3.4).

1.5. **Acknowledgements.** An extended abstract of this work will appear in the FPSAC (2018) proceedings volume of Séminaire Lotharingien Combinatoire. A follow-up paper by the same authors will continue this work in the realm of representation stability.

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2. A generalization of Dowling lattices

In [Dow73], Dowling defined a family of lattices \( \mathcal{D}_n(G) \) dependent on a positive integer \( n \) and a finite group \( G \). We recall his construction here.

Let \( n := \{1, 2, \ldots, n\} \). A partial \( G \)-partition of \( n \) is a set \( \tilde{\beta} = \{B_1, \ldots, B_\ell\} \) consisting of a partition \( \beta = \{B_1, \ldots, B_k\} \) of the subset \( \cup B_i \subseteq n \) along with projectivized \( G \)-colorings: functions \( b_i : B_i \to G \), defined up to the following equivalence: \( b : B \to G \) and \( b' : B \to G \) are equivalent if there is some \( g \in G \) for which \( b' = bg \).

The reader might benefit from thinking of such an equivalence class of colorings as a point in projective space
\[
[h_0 : \ldots : h_d] \sim [h_0 g : \ldots : h_d g] \quad \text{for} \quad g \in G.
\]
The zero block of a partial \( G \)-partition \( \tilde{\beta} \) of \( n \) is the set \( Z := n \setminus \cup_{B \in \tilde{\beta}} B \).

**Notation.** We take the convention of using an uppercase letter \( B \) for a set, the corresponding lowercase letter for the function \( b : B \to G \), and \( \tilde{B} \) for the equivalence class of \( b \) with \( B \to G \).

The Dowling lattice \( \mathcal{D}_n(G) \) is the set of partial \( G \)-partitions of \( n \). We will consider the elements of \( \mathcal{D}_n(G) \) as ordered pairs \( (\tilde{\beta}, Z) \) where \( \tilde{\beta} \) is a partial \( G \)-partition and \( Z \) is its zero block. This set is a lattice with partial order determined by the following covering relations:

1. \( (\tilde{\beta} \cup \{A, B\}, Z) \preceq (\tilde{\beta} \cup \{C\}, Z) \) where \( C = A \cup B \) with \( c = a \cup bg \) for some \( g \in G \), and
2. \( (\tilde{\beta} \cup \{B\}, Z) \preceq (\tilde{\beta}, Z \cup B) \).

The lattice \( \mathcal{D}_n(G) \) has rank function given by \( \text{rk}( (\tilde{\beta}, Z) ) = n - \ell(\tilde{\beta}) \), where \( \ell(\tilde{\beta}) \) is the number of blocks in the partition \( \tilde{\beta} \).

**Remark 2.0.1.** While it is not necessary to record the zero block in an element of \( \mathcal{D}_n(G) \), we do so because it is useful in understanding our generalization which involves adding a coloring to the zero block by some finite \( G \)-set.

2.1. Introducing the posets. Let \( G \) be a finite group acting on a finite set \( S \).

**Definition 2.1.1 (The \( S \)-Dowling poset).** Let \( \mathcal{D}_n(G, S) \) be the set of ordered pairs \( (\tilde{\beta}, z) \) where \( \tilde{\beta} \) is a partial \( G \)-partition of \( n \) and \( z \) is an \( S \)-coloring of its zero block, i.e. a function \( z : Z \to S \).

**Notation.** To denote an element \( (\tilde{\beta}, z) \) we will extend the standard notation of set partitions, as illustrated by the following example:
\[
[1 g_1, 3 g_3, 2 g_2 4 g_4, 6 g_6, 5 z_5, 7 z_7]
\]
denotes the partial set partition \([1324657] \) with projectivized colorings \( g_1 : g_3 \) and \( g_2 : g_4 : g_6 \) respectively, and zero block \([57] \) colored by the function \( z \).

The set \( \mathcal{D}_n(G, S) \) is partially ordered with similar covering relations, given by either merging two blocks or coloring one by \( S \).

- (merge): \( (\tilde{\beta} \cup \{A, B\}, z) \preceq (\tilde{\beta} \cup \{C\}, z) \) where \( C = A \cup B \) with \( c = a \cup bg \) for some \( g \in G \), and
- (color): \( (\tilde{\beta} \cup \{B\}, z) \preceq (\tilde{\beta}, z') \) where \( z' \) is the extension of \( z \) to \( Z' = B \cup Z \) given on \( B \) by a composition
\[
B \xrightarrow{b} G \xrightarrow{f} S.
\]
for some $G$–equivariant function $f$.

Just as with the Dowling lattice, the poset $\mathcal{D}_n(G, S)$ is ranked with the rank of $(\tilde{\beta}, z)$ given by $\text{rk}(\tilde{\beta}, z) = n - \ell(\beta)$.

**Remark 2.1.2.** When coloring a block of $\tilde{\beta}$, the $G$–equivariant function $f : G \to S$ is determined by a choice of $f(e) = s \in S$, where $e$ is the identity in $G$. Then one can extend $z$ to $B$ by setting $z'(i) = b(i)s$ for $i \in B$.

Recall the wreath product $\mathfrak{S}_n[G]$, sometimes denoted by $G \wr \mathfrak{S}_n$, is the semidirect product of $G^n$ with the symmetric group $\mathfrak{S}_n$. It acts on the $S$–Dowling poset $\mathcal{D}_n(G, S)$ as follows. Let $g = (g_1, \ldots, g_n, \sigma) \in \mathfrak{S}_n[G]$ and $(\tilde{\beta}, z) \in \mathcal{D}_n(G, S)$. Then we have $g.(\tilde{\beta}, z) = (\tilde{\beta}', z')$ where

- $\tilde{\beta}' = \{\sigma.B \mid B \in \beta\}$ with zero block $\sigma.Z$,
- $b'_j : (\sigma.B) \to G$ is given by $b'(\sigma(j)) = g_j b(j)$, and
- $z' : (\sigma.Z) \to S$ is given by $z'(\sigma(j)) = g_j z(j)$.

We leave it as an exercise to the reader to verify that the action preserves the order.

**Remark 2.1.3.** While it is convenient to consider (partial) partitions of the set $n = \{1, 2, \ldots, n\}$, it will sometimes prove to be more convenient to consider partitions of any finite set $\tau$, for example in Theorem 2.4.2. That is, one could define $\mathcal{D}_\tau(G, S)$ as the set of partial $G$–partitions of $\tau$ whose zero block is colored by $S$. In the case that $\tau = n$, we have $\mathcal{D}_n(G, S) = \mathcal{D}_n(G, S)$, and in general when $|\tau| = n$ we have $\mathcal{D}_\tau(G, S) \cong \mathcal{D}_n(G, S)$. Note that the latter isomorphism depends on the choice of bijection $\tau \simeq n$.

### 2.2. Examples

Here we introduce the primary examples, which will be carried throughout this paper. The first describes the partition and Dowling lattices as specializations of $S$–Dowling posets.

**Example 2.2.1.** The Dowling lattice $\mathcal{D}_n(G)$ is equal to $\mathcal{D}_n(G, S)$ whenever $S$ consists of a single point. The lattice $\mathfrak{Q}_{n}$ of set partitions of $n$ can be realized with the trivial group $G = \{1\}$ and no zero block, $\mathfrak{Q}_n \cong \mathcal{D}_n(\{1\}, \emptyset)$. As in [Dow73, Thm. 1(c)], we also have $\mathfrak{Q}_n \cong \mathfrak{Q}_{n-1}(\{1\}) = \mathcal{D}_{n-1}(\{1\}, \emptyset)$.

**Example 2.2.2 (Type C Dowling poset).** Let $G = \mathbb{Z}_2$ act trivially on a finite set $S$. In the case that $|S|$ is 2 or 4, the poset $\mathcal{D}_n(G, S)$ was studied in [Bib17]. Here, the poset describes the combinatorial structure of an arrangement arising naturally from the type C root system, which we will revisit in Example 3.2.3.

The Hasse diagram for $\mathcal{D}_2(G, S)$ when $S = \{\pm 1\}$ is depicted in Figure 1.

**Example 2.2.3.** In contrast to the last example, with $\mathbb{Z}_2$ acting trivially on $\{\pm 1\}$, consider the nontrivial action of $\mathbb{Z}_2$ on $\{\pm 1\}$. These two $S$–Dowling posets have the same underlying set but a different partial order. The Hasse diagrams for the trivial and nontrivial actions are depicted in Figures 1 and 2.

**Example 2.2.4 (Hexagonal Dowling poset).** Let $G = \mathbb{Z}_6$, which we identify with the group of 6th roots of unity $\{\pm 1, \pm \zeta_3, \pm \zeta_3^2\}$. Let us consider $G$ acting on the set $S = \{e, z_1, z_2, z_3, w_1, w_2\}$ so that the action of the generator $-\zeta_3$ is by the permutation given by cycle notation $(e)(z_1, z_2, z_3)(w_1, w_2)$. This poset also arises from an arrangement, which we will revisit in Example 3.3.1 below.

Even when $n = 2$ this poset is large: There are $6^2 = 36$ maximal elements, all with rank two, corresponding to the possible $S$–colorings of $\{1, 2\}$. We include in
Figure 1. Type C Dowling poset $\mathcal{D}_2(\mathbb{Z}_2, \{\pm 1\})$. See Examples 2.2.2 and 3.2.3, and note the isomorphism with the poset depicted in Figure 6. The quotient of this poset by the action of $\mathcal{G}_2[\mathbb{Z}_2]$ is depicted in Figure 4.

Figure 2. The Dowling poset $\mathcal{D}_2(\mathbb{Z}_2, \{\pm 1\})$ where $\mathbb{Z}_2$ acts non-trivially on $\{\pm 1\}$ (see Example 2.2.3). Compare to Figure 1, where there is an obvious set bijection that does not preserve the order.

Figure 5 the Hasse diagram for the orbits of $\mathcal{D}_2(G, S)$ under the $\mathcal{G}_2[G]$-action. We will revisit this orbit space in Example 2.6.4 below.

Example 2.2.5 (Square Dowling poset). Let $G = \mathbb{Z}_4 = \{\pm 1, \pm i\}$ act on the set $S = \{e, z_1, z_2, t\}$, where the action of $i$ on $S$ is by the permutation $(e)(z_1, z_2)(t)$. As with the previous examples, this is associated to an arrangement which we revisit in Example 3.3.2.
We depict a subposet of $D_2(\mathbb{Z}_4, S)$ in Figure 3, the intervals under $[0]|1_2, 2_1, 3_2]$ and $[0]|1_2, 2_1]$. It is interesting to note that these intervals are isomorphic to $D_2(\mathbb{Z}_2)$ and $D_2(\mathbb{Z}_4)$, respectively, which will turn out to be a general phenomenon (see Theorem 2.4.2 below).

2.3. **Functoriality.** The first property we state for the posets $D_n(G, S)$ is that it behaves naturally with respect to changes in the inputs of $n$, $G$, and $S$. This proposition is straightforward to verify.

**Proposition 2.3.1 (Functoriality).** The $S$–Dowling posets are functorial in the following ways:

1. The two inclusions $n, k \rightarrow n \sqcup k$ induce a $(\mathfrak{S}_n[\mathbb{Z}] \times \mathfrak{S}_k[\mathbb{Z}])$–equivariant injective map of ranked posets $D_{n}(G, S) \times D_{k}(G, S) \rightarrow D_{n+k}(G, S)$ defined by

   $$((\bar{\beta}, z), (\bar{\beta}', z')) \mapsto (\bar{\beta} \cup \bar{\beta}', z \cup z').$$

   Here the sets $n$ and $k$ could be replaced by any two finite sets.

   In particular, multiplying by $0 \in D_{k}(G, S)$ gives a canonical equivariant injection $D_{n}(G, S) \rightarrow D_{n+k}(G, S)$ and exhibits $D_{\bullet}(G, S)$ as a functor, as stated next.

2. If $i : n \rightarrow m$ is an injective map of sets, then there is an injective map of ranked posets $\iota_* : D_n(G, S) \rightarrow D_m(G, S)$ defined by

   $$\iota_*(\bar{\beta}, z) = (\bar{\beta}' \cup \{j\} | j \notin \iota(\bar{\beta}), z)$$

   where $\bar{\beta}'$ is defined via $b \circ \iota^{-1} : \iota(B) \rightarrow G$ for each $\bar{B} \in \bar{\beta}$.

3. If $\mu : S \rightarrow T$ is a $G$–equivariant map of sets, then there is a $\mathfrak{S}_n[\mathbb{Z}]$–equivariant map of ranked posets $\mu_* : D_n(G, S) \rightarrow D_n(G, T)$ defined by

   $$\mu_*(\bar{\beta}, z) = (\bar{\beta}, \mu \circ z).$$

   Furthermore, if $\mu$ is surjective (resp. injective) then $\mu_*$ is also surjective (resp. injective).

4. If $\nu : G \rightarrow H$ is a group homomorphism and $H$ acts on a set $S$ (hence also inducing an action of $G$ on $S$), then there is a $\mathfrak{S}_n[\mathbb{Z}]$–equivariant map of ranked posets $\nu_* : D_n(G, S) \rightarrow D_n(H, S)$ defined by

   $$\nu_*(\bar{\beta}, z) = (\bar{\beta}', z')$$

   where $\bar{\beta}'$ is defined via $\nu \circ b : B \rightarrow H$ for each $\bar{B} \in \bar{\beta}$. Furthermore, if $\nu$ is surjective (resp. injective) then $\nu_*$ is also surjective (resp. injective).

**Remark 2.3.2.**

1. The maps defined in Proposition 2.3.1(1) and (2) will be most useful in studying representation stability in our forthcoming paper, when we consider the sequence of posets $D_n(G, S)$ as $n$ grows. One can view this map $i$ as a stabilization operation, padding our partial partitions with singleton blocks.

2. Rephrasing Proposition 2.3.1(1) in view of Remark 2.1.3 and applying it to general finite sets, it follows that $D_{\bullet}(G, S)$ is a *monoidal* functor from $(\text{FI}, \|\|)$ of finite sets and injections, to the category $\text{Pos}$ of posets.

3. A consequence of Proposition 2.3.1(3) is that if $|S| > 1$, the Dowling lattice $D_n(G)$ is a quotient of the poset $D_n(G, S)$, where the fiber above an element $\bar{\beta} \in D_n(G)$ consists of all the possible $S$–colorings of its zero block.
2.4. Local structure of the poset. In this section, we give an explicit description of the intervals inside of the $S$–Dowling posets. The beauty of the local structure (the intervals) is that we can view our posets as being built out of partition and Dowling lattices, which we denote by $\Omega_n$ and $\mathcal{D}_n(G)$.

Remark 2.4.1. While the local structure of the poset is familiar, it is important to note that the global structure is more complicated. When $|S| > 1$, the poset $\mathcal{D}_n(G, S)$ is not even a (semi)lattice: least upper bounds and greatest lower bounds need not exist. While $\mathcal{D}_n(G, S)$ has a (unique) minimum element $\hat{0}$ given by the partition of $n$ into $n$ singleton blocks, it may have several maximal elements corresponding to different $S$–colorings of $n$.

The following theorem describes the intervals in $\mathcal{D}_n(G, S)$. It is a generalization of Dowling’s [Dow73, Theorem 2 and Corollary 2.1], which are the case with $|S| = 1$. The surprising consequence of Theorem 2.4.2 below, is that when $|S| \geq 2$ the local picture is sensitive to the orbits and stabilizers of $S$. As mentioned in Remark 2.1.3, this theorem is most naturally stated by considering $\mathcal{D}_n(G, S)$ for a general finite set $\tau$ (not just $n$); we discuss this further in Remark 2.4.3 below.

Theorem 2.4.2 (Local structure). Let $S$ be a finite set with an action of a finite group $G$, and let $O(S)$ denote its set of $G$–orbits. For each orbit $\alpha \in O(S)$, pick a representative $s_\alpha \in \alpha$ and let $G_\alpha$ be the stabilizer of $s_\alpha$ in $G$.

For $(\beta, z_{\beta}) \in \mathcal{D}_n(G, S)$, we have

(1) \[
\mathcal{D}_n(G, S)_{\leq (\beta, z_{\beta})} \cong \prod_{B \in \beta} \Omega_B \times \prod_{\sigma \in O(S)} \mathcal{D}_{z_{\beta}^{-1}(\sigma)}(G_\sigma)
\]

and

(2) \[
\mathcal{D}_n(G, S)_{\geq (\beta, z_{\beta})} \cong \mathcal{D}_{\beta}(G, S).
\]

Furthermore, if $(\tilde{\alpha}, z_{\tilde{\alpha}}) \leq (\beta, z_{\beta})$, then

(3) \[
[(\tilde{\alpha}, z_{\tilde{\alpha}}), (\beta, z_{\beta})] \cong \prod_{B \in \beta} \Omega_{p_B} \times \prod_{\sigma \in O(S)} \mathcal{D}_{r_\sigma}(G_\sigma),
\]

where $p_B$ is the set of blocks $\tilde{A} \in \tilde{\alpha}$ for which $A \subseteq B$ and $r_\sigma$ is the set of blocks $\tilde{A} \in \tilde{\alpha}$ for which $A \subseteq z_{\tilde{\beta}}^{-1}(\sigma)$.

In particular, every closed interval is a geometric lattice.

Remark 2.4.3. (1) One can write the product decompositions above more explicitly by denoting $\beta = \{B_1, \ldots, B_t\}$, $O(S) = \{\alpha_1, \ldots, \alpha_k\}$, $n_i = |B_i|$, and $m_j = |z_{\tilde{\beta}}^{-1}(\sigma_j)|$. Then Theorem 2.4.2(1) says:

\[
[0, (\tilde{\beta}, z_{\beta})] \cong \Omega_{n_1} \times \cdots \times \Omega_{n_t} \times \mathcal{D}_{m_1}(G_{\alpha_1}) \times \cdots \times \mathcal{D}_{m_k}(G_{\alpha_k}).
\]

(2) Note that in Theorem 2.4.2(2) and (3), the base sets of the lattices are sets of blocks rather than subsets of $n$.

We will prove parts (1) and (2) of Theorem 2.4.2 below; part (3) follows by combining these two. To qualitatively explain part (1), recall that the covering relations state that an element $(\tilde{\alpha}, z_{\tilde{\alpha}})$ lies under $(\tilde{\beta}, z_{\beta})$ if the partition $\alpha$ is a refinement of the partition $\beta$, possibly excising blocks away from the zero block $Z_{\tilde{\beta}}$. In particular, $\alpha$ defines a partition of each block in $\beta$, and furthermore includes a partial partition of the zero block. To qualitatively explain part (2), recall that the
covering relation allows one to merge existing blocks, or throw entire blocks into the zero block. Thus an element above \((\tilde{\beta}, z_{\tilde{\beta}})\) is determined by specifying blocks to be merged and the \(G\)-ratios between their \(G\)-colorings, and by coloring the remaining blocks by \(S\).

**Proof of Theorem 2.4.2(1).** The isomorphism is not canonical; we make the following choices. For each \(\sigma \in \mathcal{O}(S)\), fix a representative \(\sigma_{\sigma} \in \sigma\) with stabilizer subgroup \(G_{\sigma}\). Then for each element \(t \in \sigma\) pick a ‘transporter’ \(g_t \in G\) so that \(g_t.\sigma_{\sigma} = t\).

We define a map from \([\emptyset, (\tilde{\beta}, z_{\tilde{\beta}})]\) to the product by an assignment
\[
(\tilde{\alpha}, z_{\tilde{\alpha}}) \mapsto ((\alpha_B)_{B \in \beta}, (\tilde{\alpha}_\sigma)_{\sigma \in \mathcal{O}(S)}).
\]
The first tuple is simple: for a block \(B \in \beta\), define a partition of \(B\) by
\[
\alpha_B := \{A \in \alpha \mid A \subseteq B\}.
\]
For the second tuple start with defining for every \(\sigma \in \mathcal{O}(S)\) a partial partition of \(z_{\beta}^{-1}(\sigma)\) by
\[
\alpha_{\sigma} := \{A \in \alpha \mid A \subseteq z_{\beta}^{-1}(\sigma)\}.
\]
For each \(A \in \alpha_{\sigma}\), the coloring relation in the definition of \(<\) shows that we may pick a representative \(a : A \rightarrow G\) such that \(z_{\beta}(i) = a(i).\sigma_{\sigma}\) for each \(i \in A\). Since \(z_{\beta}(i) = g_{z_{\beta}(i)}.\sigma_{\sigma}\), we can define a \(G_{\sigma}\)-coloring \(a' : A \rightarrow G_{\sigma}\) with \(a'(i) = g_{z_{\beta}(i)}^{-1}a(i)\).

To describe the inverse map, consider a pair of tuples \(((\alpha_B)_{B \in \beta}, (\tilde{\alpha}_\sigma)_{\sigma \in \mathcal{O}(S)})\). We recover a partial partition of \(n\) by unioning all of these (partial) partitions,
\[
\alpha := \{A \in \alpha_B \mid B \in \beta\} \cup \{A \in \alpha_{\sigma} \mid \sigma \in \mathcal{O}(S)\},
\]
and obtain \(G\)-colorings \(a : A \rightarrow G\) as follows. If \(A \in \alpha_B\), then inherit \(a = b|_A : A \rightarrow G\) from \(\tilde{B} \in \tilde{\beta}\). Otherwise \(A \in \alpha_{\sigma}\) is a block of a partial partition. Represent its \(G_{\sigma}\)-coloring by \(a' : A \rightarrow G_{\sigma}\) and recolor by \(a(i) = g_{z_{\beta}(i)}a'(i)\). Lastly, inherit the \(S\)-coloring of the zero block \(Z_{\tilde{\sigma}}\) from that of \(Z_{\tilde{\beta}}\). These constructions are clearly inverses. \(\square\)

**Proof of Theorem 2.4.2(2).** This isomorphism is also not canonical: Let us write \(\tilde{\beta} = \{\tilde{B}_1, \ldots, \tilde{B}_t\}\) and pick a representative \(b_i : \tilde{B}_i \rightarrow G\) for each \(\tilde{B}_i \in \tilde{\beta}\).

Then for \((\tilde{\alpha}, z_{\tilde{\alpha}}) \geq (\tilde{\beta}, z_{\tilde{\beta}})\) we will construct an element of \(\mathcal{D}_t(G, S)\). For \(\tilde{A} \in \tilde{\alpha}\), let
\[
C_A := \{i \in t \mid B_i \subseteq A\}.
\]

Now, pick a representative \(a : A \rightarrow G\) for \(\tilde{A} \in \tilde{\alpha}\). By the covering relations, we have that for each \(i \in C_A\) there is \(g_i \in G\) such that \(a = b_i g_i\). This defines a function \(c_A : C_A \rightarrow G\); a different representative map for \(\tilde{A}\) would define a function in the same equivalence class of \(c_A\). The collection \(\{\tilde{C}_A \mid \tilde{A} \in \tilde{\alpha}\}\) is a partial \(G\)-partition of the set \(t\).

We also have that \(Z_\alpha = Z_{\beta} \cup Z\) where \(Z\) is a union of blocks from \(\beta\) and \(Z_\alpha\). In fact, the zero block of \(\{\tilde{C}_A\} = Z' = \{i \in t \mid B_i \subseteq Z\}\). Since \(z_{\beta}|_{B_\beta} = f_{\beta} \circ b_i\) for some (unique) \(G\)-equivariant \(f_{\beta} : G \rightarrow S\), let us color \(Z'\) so that \(z'(i) = f_{\beta}(\epsilon)\). Then \(\{(\tilde{C}_A \mid \tilde{A} \in \tilde{\alpha}), z'\} \in \mathcal{D}_t(G, S)\), and one can recover \((\tilde{\alpha}, z_{\tilde{\alpha}})\) from this data. \(\square\)

**Example 2.4.4 (Hexagonal Dowling Poset).** Recall from Example 2.2.4 the Hexagonal Dowling poset \(\mathcal{D}_6(Z_6, S)\), where \(S = \{e, z_1, z_2, z_3, w_1, w_2\}\). The orbits
of $S$ are $\{e\}, \{z_1, z_2, z_3\}, \{w_1, w_2\}$, so that we have $\mathcal{O}(S) = \{\sigma(e), \sigma(z), \sigma(w)\}$ with stabilizers $G_e = \mathbb{Z}_4$, $G_z = \{\pm 1\} \cong \mathbb{Z}_2$, and $G_{w_i} = \{1, \zeta, \zeta^2\} \cong \mathbb{Z}_3$.

One already finds intervals that factor as Dowling lattices of different ranks and of different groups when $n = 5$: consider $(\tilde{\beta}, z) \in \mathcal{D}_5(\mathbb{Z}_6, S)$ given by

$$(\tilde{\beta}, z) = [\emptyset||1_z, 2_z, 3_z, 4_w, 5_w]$$

then

$$[\emptyset, (\tilde{\beta}, z)] \cong \mathcal{D}_{\{1, 2, 3\}}(\mathbb{Z}_2) \times \mathcal{D}_{\{4, 5\}}(\mathbb{Z}_3)$$

$$\cong \mathcal{D}_3(\mathbb{Z}_2) \times \mathcal{D}_2(\mathbb{Z}_3).$$

Next, the element $(\tilde{\alpha}, z_\alpha) \in [\emptyset, (\tilde{\beta}, z)]$ given by $(\tilde{\alpha}, z_\alpha) = [1_1 3_z||2_z, 4_w, 5_w]$ is mapped under the isomorphism to

$$([1_1 3_z||2_z, \emptyset||4, 5]) \in \mathcal{D}_3(\mathbb{Z}_2) \times \mathcal{D}_2(\mathbb{Z}_3)$$

Moreover, $\mathcal{D}_5(\mathbb{Z}_6, S) \cong_{(\alpha, z_\alpha)} \mathcal{D}_1(G, S)$ and the element $(\tilde{\beta}, z)$ is mapped under this isomorphism to $[\emptyset||1_z]$.

**Example 2.4.5 (Square Dowling poset).** Recall from Example 2.2.5 the square Dowling poset $\mathcal{D}_n(\mathbb{Z}_4, S)$ where $S = \{e, z_1, z_2, t\}$. The orbits of $S$ are $\{e\}, \{z_1, z_2\}$, and $\{t\}$, so that we have $\mathcal{O}(S) = \{\sigma(e), \sigma(z), \sigma(t)\}$ with stabilizers $G_e = G_t = \mathbb{Z}_4$ and $G_{z_1} = \{\pm 1\} \cong \mathbb{Z}_2$.

Figure 3 depicts two intervals inside $\mathcal{D}_2(\mathbb{Z}_4, S)$, where

$$[\emptyset||1_z, 2_z] \cong \mathcal{D}_2(\mathbb{Z}_2) \quad \text{and} \quad [\emptyset||1_t, 2_t] \cong \mathcal{D}_2(\mathbb{Z}_4)$$

![Figure 3. Two overlapping intervals inside the square Dowling poset $\mathcal{D}_2(G, S)$ with $G = \mathbb{Z}_4$ acting on $S = \{e, z_1, z_2, t\}$ as in Example 2.2.5. See Example 2.4.5.](image)

2.5. **Characteristic polynomial.** A fundamental invariant attached to a ranked poset is its characteristic polynomial. Recall (and see [OS80]) that when specialized to intersection lattices of hyperplane arrangements this polynomial gives a close relative of the Poincaré polynomial of the complement, and that in the further special case of graphical arrangements it computes the chromatic polynomial of the graph. The roots of this polynomial carry subtle information, e.g. for reflection arrangements it encodes the exponents of the Coxeter group (see [Bri73]). In Theorem 2.5.2 below we factorize the polynomial associated with $\mathcal{D}_n(G, S)$.

The factorization formula in Theorem 2.5.2 specializes to that computed by Dowling [Dow73, Thm. 5] when $|S| = 1$ and by Ardila, Castillo, and Henley [ACH15,
Thm. 1.18] when $G = \mathbb{Z}_2$ and $|S| = 2$. The same formula for the partition lattice
is well-known and goes back to Arnol’d [Arn69] and Stanley [Sta72]. This is the
special case when $G$ is trivial and $S$ is empty.

Recall that the characteristic polynomial of a ranked poset $P$ with minimum
element $\hat{0}$ is defined by

$$
\chi(P; t) = \sum_{x \in P} \mu(\hat{0}, x)t^{\text{rk}(P) - \text{rk}(x)},
$$

where $\mu$ is the Möbius function. In [Sta72] Stanley defined the notion of a super-
solvable lattice, encompassing the cases of partition and Dowling lattices. There he
showed that for such lattices, a partition of the atoms gives a factorization of their
characteristic polynomial. We thus proceed by describing the atoms, generalizing
Corollaries 1.1 and 1.2 of [Dow73].

**Lemma 2.5.1.** The rank-one elements (or atoms) of $\mathcal{D}_n(G, S)$ are

1. $\alpha_{ij}(g) := [i_1 j_1 | i_2 | \ldots | i_j | \ldots | n | 0]$ where $1 \leq i < j \leq n$ and $g \in G$,
corresponding to the partial $G$-partition whose only non-singleton block is
$A = \{i, j\}$ with $G$-coloring $[1 : g]$; and

2. $\alpha_i^s := [1 | \ldots | i | \ldots | n | i_s]$, where $1 \leq i \leq n$ and $s \in S$, corresponding to the
partial $G$-partition with zero block $\{i\}$ colored by $s$ and the rest are singleton
blocks.

Moreover, if $(\tilde{\beta}, z) \in \mathcal{D}_n(G, S)$ has rank $n - \ell$, then it is covered by

$$
\ell|S| + \binom{\ell}{2}|G|
$$

elements of rank $n - \ell + 1$.

**Proof.** The description of atoms follows directly from the covering relations. Thus,
the poset has $n|S| + \binom{\ell}{2}|G|$ atoms. To count the number of elements covering
some $(\tilde{\beta}, z)$ of rank $n - \ell$, we recognize them as atoms of $\mathcal{D}_\ell(G, S)$ via Theorem
2.4.2(2).

**Theorem 2.5.2 (Characteristic polynomial).** If $S$ is a nonempty finite set, then

$$
\chi(\mathcal{D}_n(G, S); t) = \prod_{i=0}^{n-1} (t - |S| - |G|i).
$$

If $S = \emptyset$, then

$$
\chi(\mathcal{D}_n(G, \emptyset); t) = \prod_{i=1}^{n-1} (t - |G|i).
$$

**Proof.** The authors thank Emanuele Delucchi for suggesting this method of proof.

First assume that $S$ is nonempty. For each $k \in \mathbf{n}$, define

$$
A_k := \{\alpha_k^s \mid s \in S\} \cup \{\alpha_{ik}(g) \mid g \in G, i < k\}.
$$

The sets $A_1, \ldots, A_n$ give a partition of the atoms of $\mathcal{D}_n(G, S)$. Further define
$A_k := A_k \cup \{\emptyset\}$, a subposet of $\mathcal{D}_n(G, S)$.

By choice of the partition, least upper bounds of elements from distinct blocks
exist, and so there is a well-defined map $f : A_1 \times \cdots \times A_n \rightarrow \mathcal{D}_n(G, S)$ defined by
$f(x_1, \ldots, x_n) = x_1 \lor \cdots \lor x_n$. It is easy to verify that this is a complete transversal
function in the sense of [Hal14, Def. 4.2] and that the following properties hold:
Definition 2.6.1. Made precise by the following definition.

Remark 2.5.3. It is interesting to note that the characteristic polynomial does not depend on the action of \( G \) on \( S \), but rather only on \( n, |G|, \) and \(|S|\). For example, the posets \( \mathcal{D}_2(\mathbb{Z}_2, \{\pm 1\}) \) where \( \mathbb{Z}_2 \) acts either trivially or nontrivially on \( \{\pm 1\} \), depicted in Figures 1 and 2, have the same characteristic polynomial.

Example 2.5.4 (Hexagonal Dowling poset). If \( G = \mathbb{Z}_6 \) and \(|S| = 6\) (eg. in Example 2.2.4), the first few characteristic polynomials are

\[
\chi(\mathcal{D}_2(G, S); t) = (t - 6)(t - 12)
\]
\[
= t^2 - 18t + 72
\]
\[
\chi(\mathcal{D}_3(G, S); t) = (t - 6)(t - 12)(t - 18)
\]
\[
= t^3 - 36t^2 + 396t - 1296
\]
\[
\chi(\mathcal{D}_4(G, S); t) = (t - 6)(t - 12)(t - 18)(t - 24)
\]
\[
= t^4 - 60t^3 + 1260t^2 - 10800t + 31104
\]

Example 2.5.5 (Square Dowling poset). If \( G = \mathbb{Z}_4 \) and \(|S| = 4\) (eg. in Example 2.2.5), the first few characteristic polynomials are

\[
\chi(\mathcal{D}_2(G, S); t) = (t - 4)(t - 8)
\]
\[
= t^2 - 12t + 32
\]
\[
\chi(\mathcal{D}_3(G, S); t) = (t - 4)(t - 8)(t - 12)
\]
\[
= t^3 - 24t^2 + 176t - 384
\]
\[
\chi(\mathcal{D}_4(G, S); t) = (t - 4)(t - 8)(t - 12)(t - 16)
\]
\[
= t^4 - 40t^3 + 560t^2 - 3200t + 6144
\]

2.6. Orbits and labeled partitions. A set partition of \( n \) determines a partition of the number \( n \) by taking the size of its blocks. The action of \( S_n \) on the partition lattice \( \mathcal{Q}_n \) preserves this data, and in fact the \( S_n \)-orbits of \( \mathcal{Q}_n \) are in bijection with partitions of \( n \). The action of \( \mathfrak{S}_n[G] \) on the Dowling lattice \( \mathcal{D}_n(G) \) behaves similarly, except that the zero block cannot be permuted among the other blocks. The orbit can then be recorded by a partition of \( n \) with a distinguished part denoting the size of the zero block.

Let \((\tilde{\beta}, z) \in \mathcal{D}_n(G, S)\), and let \( \emptyset := \emptyset(S) \). The action of \( \mathfrak{S}_n[G] \) preserves the block sizes in the partial \( G \)-partition \( \tilde{\beta} \) and cannot permute the sets \( z^{-1}(\sigma) \) amongst each other or with blocks in \( \tilde{\beta} \). In particular, the sizes of the sets \( z^{-1}(\sigma) \) are also preserved. This data can be recorded by a partition of \( n \), where some parts are labeled by elements of \( \emptyset \), recording the sizes of \( z^{-1}(\sigma) \) for \( \sigma \in \emptyset \). This notion is made precise by the following definition.

Definition 2.6.1 (Labeled partitions). An \( \emptyset \)-labeled partition of \( n \) is an integer partition of \( n \), i.e. a collection of positive integers summing to \( n \), with some
parts colored by $\emptyset$ so that each color is used at most once. For $O = \{o_1, \ldots, o_s\}$ and an $O$-labeled partition $\lambda$, we will use the notation

$$\lambda = (\lambda_1, \ldots, \lambda_\ell | | \lambda_o_1, \ldots, \lambda_o_s)$$

where $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ are the uncolored (or unlabeled) parts, $\lambda_\emptyset = (\lambda_o_1, \ldots, \lambda_o_s)$ is the colored (or labeled) portion and $\sum \lambda_i + \sum \lambda_o = n$. We typically omit the labeled parts which are zero.

Let $Q_n(O)$ denote the set of all $O$-labeled partitions of $n$.

The set $Q_n(O)$ is partially ordered and graded with $\text{rk}(\lambda) = n - \ell$. Moreover, the following theorem exhibits $Q_n(O)$ as the quotient of $D_n(G, S)$ by the action of $\mathfrak{S}_n[G]$.

**Theorem 2.6.2 (Orbits).** Let $S$ be a finite $G$–set with orbit set $O := O(S)$. Then the $\mathfrak{S}_n[G]$–orbits of $D_n(G, S)$ are in bijection with the $O$–labeled partitions of $n$. More explicitly, the fibers of the surjective map $\pi_n : D_n(G, S) \to Q_n(O)$, defined below, are $\mathfrak{S}_n[G]$–orbits.

**Proof.** Write $O(S) = \{o_1, \ldots, o_k\}$. For $(\beta, z)$ with $\beta = \{B_1, \ldots, B_\ell\}$ written so that $|B_1| \geq \cdots \geq |B_\ell| > 0$, define

$$\pi_n((\beta, z)) := (|B_1|, \ldots, |B_\ell| | \ | z^{-1}(o_1)|o_1, \ldots, |z^{-1}(o_k)|o_k).$$

As discussed in the beginning of this subsection, the group action preserves the list of cardinalities, thus $\pi_n((\beta, z)) = \pi_n(g(\beta, z))$ for all $g \in \mathfrak{S}_n[G]$ and $\pi_n$ descends to a well-defined map on orbits. It remains to show that $\pi_n$ is injective.

Indeed, given a labeled partition $\lambda = (\lambda_1, \ldots, \lambda_\ell | | \lambda_o_1, \ldots, \lambda_o_s)$ there exists a partition of $n$ with blocks $(B_1, \ldots, B_\ell, Z_1, \ldots, Z_k)$ of respective sizes $|B_i| = \lambda_i$ and $|Z_j| = \lambda_o_j$. This will give a partial $G$-partition with zero block $Z = \cup Z_j$ once we assign colorings: for every $1 \leq i \leq \ell$ color $B_i$ with the constant function $b_i \equiv 1 \in G$; and for the zero block, pick orbit representatives $s_j \in o_j$ and define $z : Z \to S$ by mapping $Z_j$ to $s_j$ for every $1 \leq j \leq k$. The resulting partial $G$-partition $(\beta_\lambda, z_\lambda)$ maps to $\lambda$ under $\pi_n$.

To see that $\pi_n$ is injective, suppose $(\alpha, z_\alpha)$ also maps to $\lambda$. We construct an element $(g, \sigma) \in \mathfrak{S}_n[G]$ transporting $(\beta_\lambda, z_\lambda) \to (\alpha, z_\alpha)$. There exists some permutation $\sigma \in \mathfrak{S}_n$ such that $\sigma.B_i = A_i$ for all $i$ and $\sigma.z^{-1}_\lambda(o_i) = z^{-1}_\alpha(o_i)$ for all $j$, as these sets have equal size respectively. Next, fix representative colorings $a_i : A_i \to G$ for $A_i$. Define $g = (g_1, \ldots, g_n) \in G^n$ as follows: if $r \in B_i$ take $g_r = a_i(\sigma(r))$; otherwise $r \in Z_j$ so $z_\lambda(r), z_\alpha(\sigma(r))$ belong to the same orbit $o_j$ and we take $g_r$ to be such that $g_r.z_\lambda(r) = z_\alpha(\sigma(r))$. $\square$

**Example 2.6.3 (Type C Dowling poset).** Recall from Example 2.2.2 the poset $D_n(G, S)$ where $G = \mathbb{Z}_2$ acts on $S$ trivially. The set of orbits, viewed as a quotient of $D_2(G, S)$ by the action of $\mathfrak{S}_2[G]$, are depicted in Figure 4 for $S = \{\pm 1\}$.

**Example 2.6.4 (Hexagonal Dowling poset).** Recall the poset $D_n(G, S)$ from Example 2.2.4 where $G = \mathbb{Z}_4$ acts on $S = \{e, z_1, z_2, z_3, w_1, w_2\}$. Since $O(S) = \{o(e), o(z), o(w)\}$, the orbits are in bijection with $O$–labeled partitions of $n$ where $O$ is a 3-element set. The set of orbits when $n = 2$, viewed as a quotient of $D_2(G, S)$ by the action of $\mathfrak{S}_2[G]$, are depicted in Figure 5.

**Example 2.6.5 (Square Dowling poset).** Recall the poset $D_n(G, S)$ from Example 2.2.5 where $G = \mathbb{Z}_4$ acts on $S = \{e, z_1, z_2, t\}$. Here, $O(S) = \{o(e), o(z), o(t)\}$,
Figure 4. The $G_2[\mathbb{Z}_2]$–orbits of the type C toric Dowling poset $D_2(\mathbb{Z}_2, \{\pm 1\})$ from Figures 1 and 6. See also Examples 2.2.2, 2.6.3, and 3.2.3.

Figure 5. Hasse diagram for the quotient of the hexagonal Dowling poset $D_2(G, S)$ by $G_2[G]$, where $G = \mathbb{Z}_6$ and $O(S) = \{e, z, w\}$. If we replace all of the $w$’s with $t$’s, this is also the Hasse diagram for the quotient of the square Dowling poset $D_2(G, S)$ by $G_2[G]$, where $G = \mathbb{Z}_4$ and $O(S) = \{e, z, t\}$. See Examples 2.2.4, 2.2.5, 2.6.4, 2.6.5, 3.3.1, and 3.3.2.

and hence the orbits are in bijection with $O$–labeled partitions of $n$ for a 3-element set $O$. This means that the orbits of the square Dowling poset are in bijection with the orbits of the hexagonal Dowling poset, and are thus also depicted in Figure 5 for $n = 2$. 


2.7. Whitney homology. The homology of partition lattices has been studied as a representation of the symmetric group by Stanley [Sta82], Lehrer and Solomon [LS86], Barcelo and Bergeron [BB90], and Wachs [Wac98]. For Dowling lattices, this has been carried out by Hanlon [Han84] and Gottlieb and Wachs [GW00]. Since the partition and Dowling lattices are geometric, it follows from Folkman [Fol66] that they are Cohen Macaulay and hence $\tilde{H}_i(D_n(G)) = 0$ unless $i = n - 2$. Dowling [Dow73] computed the Whitney numbers for the lattices $D_n(G)$ and shows that

$$\dim \tilde{H}_{n-2}(D_n(G)) = \prod_{j=1}^{n-1}(1 + j|G|).$$

This number can be computed from the characteristic polynomial in Theorem 2.5.2 via its characterization as $(-1)^n \chi(D_n(G); 0)$.

Since the poset $D_n(G, S)$ does not have a unique maximum element when $|S| > 1$, it is interesting to investigate the homology of the poset obtained by adding a maximum $\hat{1}$ of rank $n + 1$ to obtain a bounded poset $\hat{D}_n(G, S)$. Delucchi, Girard, and Paolini [DGP17] prove that $\hat{D}_n(Z_2, S)$ is EL-shellable, hence Cohen-Macaulay, when $Z_2$ acts trivially on $S$. Based on this, we suggest the following.

**Conjecture 2.7.1 (Shellability).** All $\hat{D}_n(G, S)$ are shellable. In particular, using Theorem 2.5.2, the order complex of $\hat{D}_n(G, S)$ is homotopy equivalent to a wedge of

$$\prod_{i=0}^{n-1}(1 + |S| + |G|i)$$

spheres of dimension $n - 1$.

In this paper, though, we are interested in the Whitney homology, defined by

$$\text{WH}_r(D_n(G, S)) = \bigoplus_{(\hat{\beta}, z) \in D_n(G, S)} \tilde{H}_{r-2}(\hat{1}, (\hat{\beta}, z)).$$

Since the intervals $[\hat{1}, (\hat{\beta}, z)]$ are products of partition and Dowling lattices (Theorem 2.4.2) hence geometric lattices, we get:

**Corollary 2.7.2.** The group $\tilde{H}_{r-2}(\hat{1}, (\hat{\beta}, z))$ is trivial unless $\text{rk}((\hat{\beta}, z)) = r$. Thus, the summation in $\text{WH}_r(D_n(G, S))$ includes only the rank-$r$ elements of $D_n(G, S)$.

The explicit decomposition of intervals $[\hat{1}, (\hat{\beta}, z)]$ from Theorem 2.4.2(1) also gives a formula for the dimension of the Whitney homology, but this can alternatively be derived from our characteristic polynomial calculation:

**Theorem 2.7.3.** If $S$ is nonempty, the Hilbert series for Whitney homology is

$$P_{WH}(t) := \sum_{r \geq 0} \dim \text{WH}_r(D_n(G, S)) t^r = \prod_{i=0}^{n-1}(1 + (|S| + |G|i) t).$$

**Proof.** This can be obtained through the following specialization of the characteristic polynomial in Theorem 2.5.2:

$$P_{WH}(t) = (-t)^n \chi(D_n(G, S); -\frac{1}{t}) = (-t)^n \prod_{i=0}^{n-1} \left(-\frac{1}{t} - |S| - |G|i \right).$$
This is because the Whitney numbers (of the first kind) are simultaneously the coefficients in the characteristic polynomial and (up to sign and reordering) the rank of the Whitney homology groups.

We aim to describe the Whitney homology as an $\mathfrak{S}_n[G]$–module. We do so by using familiar representations and hence we fix the following notation:

- $\iota_j$ is the trivial representation of $\mathfrak{S}_j$.
- $\epsilon_j$ is the sign representation of $\mathfrak{S}_j$.
- $\pi_m$ is the $\mathfrak{S}_m$–module structure on the nontrivial homology group of the partition lattice, $\tilde{H}_{m-3}(Q_m)$. By Stanley’s work [Sta82, Thm. 7.3], this representation, up to a twist by a sign representation, is an induced representation $\text{Ind}_{(c_m)}^{S_m} \chi_m$ where $c_m$ is an $m$-cycle and $\chi_m$ is a faithful character.

- $\rho_m(H)$ is the $\mathfrak{S}_m[H]$–module structure on the nontrivial homology group of a Dowling lattice, $\tilde{H}_{m-2}(D_m(H))$, whose character was described by Hanlon [Han84, Thm. 3.4].
- If $U$ and $V$ are $\mathfrak{S}_n$– and $\mathfrak{S}_m$–modules, respectively, then $U[V] := U \otimes (V \otimes \cdots \otimes \rho)$ is naturally a representation of the wreath product group $\mathfrak{S}_n[\mathfrak{S}_m]$.

Let $S$ be a finite $G$-set with orbits $\mathcal{O}(S) = \{g_1, \ldots, g_k\}$. Given an $\mathcal{O}(S)$–labeled partition $(\lambda)[m_1, \ldots, m_k)$ of $n$, with $\lambda = 1^{g_1}2^{g_2}\cdots n^{g_n}$, construct a subgroup of $\mathfrak{S}_n[G]$ by

$$\mathfrak{S}(\lambda)[m_1, \ldots, m_k) := \mathfrak{S}_{a_1}[\mathfrak{S}_1] \times \cdots \times \mathfrak{S}_{a_n}[\mathfrak{S}_n] \times \mathfrak{S}_{m_1}[G_1] \times \cdots \times \mathfrak{S}_{m_k}[G_k]$$

and construct a representation of it $V(\lambda)[m_1, \ldots, m_k)$ by

$$\iota_{a_1}[\pi_1] \otimes \iota_{a_2}[\pi_2] \otimes \cdots \otimes \iota_{a_n}[\pi_n] \otimes \epsilon_1[\pi_1] \otimes \epsilon_2[\pi_2] \otimes \cdots \otimes \epsilon_{a_n}[\pi_n] \otimes \rho_{m_1}(G_1) \otimes \cdots \otimes \rho_{m_k}(G_k)$$

Inducing such modules to $\mathfrak{S}_n[G]$ then allows us to describe the $\mathfrak{S}_n[G]$–module structure on the Whitney homology of $\mathcal{D}_n(G, S)$. This is done in the following theorem, generalizing a result of Lehrer and Solomon [LS86, Thm. 4.5] for partition lattices.

**Theorem 2.7.4.** As an $\mathfrak{S}_n[G]$–module,

$$\text{WH}_r(\mathcal{D}_n(G, S)) \cong \bigoplus \text{Ind}^{\mathfrak{S}_n[G]}_{\mathfrak{S}(\lambda)[m_1, \ldots, m_k)} V(\lambda)[m_1, \ldots, m_k),$$

where the sum is over all $\mathcal{O}(S)$–labeled partitions of $n$ such that $\ell(\lambda) = n - r$.

**Proof.** By our description of the orbits in Theorem 2.6.2, it is clear that the representation decomposes over the $\mathcal{O}(S)$–labeled partitions, and by Corollary 2.7.2 all such partitions must have $\ell(\lambda) = n - r$. Let us consider a single orbit corresponding to $(\lambda)[m_1, \ldots, m_k)$, which by Theorem 2.4.2 contributes summands of the form $(\tilde{H}_{-2}(Q_1))^{a_1} \otimes \cdots \otimes (\tilde{H}_{n-3}(Q_n))^{a_n} \otimes \tilde{H}_{m_1-2}(D_{m_1}(G_1)) \otimes \cdots \otimes \tilde{H}_{m_k-2}(D_{m_k}(G_k)).$

The stabilizer of this summand is $\mathfrak{S}(\lambda)[m_1, \ldots, m_k)$, and as a representation of the stabilizer it is $V(\lambda)[m_1, \ldots, m_k)$. 

3. Symmetric arrangements

Throughout this section, one may take the word ‘space’ to mean either a CW-complex or an algebraic variety over some algebraically closed field. Fix a finite group $G$, acting almost freely on a connected space $X$. By this we mean that the
set of singular points of the $G$–actions, i.e. the set points with non-trivial stabilizer, is finite. Denote this singular set by

$$S := \text{Sing}_G(X) = \bigcup_{g \in G \setminus \{e\}} X^g.$$ 

Note that $G$ acts freely on $X \setminus S$, and also the action of $G$ on $X$ restricts to an action of $G$ on the set $S$. Denote the set of $G$–orbits of $S$ by

$$\emptyset := \{G.x \mid x \in S\}.$$ 

When discussing cohomology below, we will always suppress the coefficients, as they never have a significant effect on the results. For $X$ a CW-complex, one may take the cohomology $H^*(X)$ to mean singular cohomology with coefficients in any ring $R$; and for $X$ an algebraic variety, $H^*(X)$ may be taken to mean $\ell$-adic cohomology with coefficients in either $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$.

3.1. Introducing the arrangements. We define an arrangement $A_n = A_n(G, X)$ as the collection of the following (closed) subspaces in $X^n$:

1. $H_{ij}(g) = \{x_1, \ldots, x_n\} \in X^n \mid g.x_i = x_j\}$ for $1 \leq i < j \leq n$ and $g \in G$, and
2. $H^i_s = \{(x_1, \ldots, x_n) \in X^n \mid x_i = s\}$ for $1 \leq i \leq n$ and $s \in S$.

Note that for the first type of subspace, we may extend our notation to allow $j < i$ by observing $H_{ij}(g) = H_{ji}(g^{-1})$.

The wreath product group $\mathfrak{S}_n[G]$ acts naturally on $X^n$: first, the group $G^n$ acts on $X^n$ coordinatewise, and then $\mathfrak{S}_n$ permutes the coordinates. Explicitly, $(g_1, \ldots, g_n, \sigma) \in \mathfrak{S}_n[G]$ acts on $(x_1, \ldots, x_n) \in X^n$ by sending it to the tuple whose $\sigma(i)$-th entry is $g_i.x_i$. This action induces an action on $A_n(G, X)$ as described here:

1. $g.H_{ij}(h) = H_{\sigma(i)\sigma(j)}(g_j h g_i^{-1})$, and
2. $g.H^i_s = H^i_{\sigma(i)}$.

Because the arrangement $A_n(G, X)$ is $\mathfrak{S}_n[G]$–invariant, the action of $\mathfrak{S}_n[G]$ on $X^n$ also restricts to an action of $\mathfrak{S}_n[G]$ on the complement, which we denote by

$$M_n(G, X) := X^n \setminus \bigcup_{H \in A_n} H.$$ 

In fact, the action of $\mathfrak{S}_n[G]$ on $M_n(G, X)$ is free.

Remark 3.1.1. The space $M_n(G, X)$ is what we denote by $\text{Conf}_n^G(X \setminus S)$ in the introduction. We use this notation to emphasize our perspective of studying $M_n(G, X)$ as a subspace of $X^n$ rather than $(X \setminus S)^n$.

3.2. The poset of layers. A fundamental combinatorial object attached to the arrangement $A_n$ is its poset of layers.

Definition 3.2.1. A layer of $A_n$ is defined to be a connected component of an intersection $\cap_{H \in B} H$ for some subset $B \subseteq A_n$.

The set of all layers is partially ordered under reverse inclusion. We call the resulting poset the poset of layers and denote it by $P_n(G, X)$.

The poset $P_n(G, X)$ has a unique minimum $\hat{0}$ corresponding to the empty intersection $X^n$. Moreover, the action of $\mathfrak{S}_n[G]$ on $X^n$ and $A_n(G, X)$ induces an action of $\mathfrak{S}_n[G]$ on $P_n(G, X)$.

Note that intersections of layers need not be connected; a simple example of this phenomenon appears in Example 3.2.3 below. However, it will prove useful to
observe that the intersection of enough subspaces gives a connected space. This fact will follow from the proof of Theorem 3.2.5 and is a special property of the type of arrangements we consider here.

**Example 3.2.2.** It can be helpful to keep in mind the classical example of the trivial group $G = \{1\}$ acting on any space $X$. Here, the arrangement $A_n(\{1\}, X)$ is called the braid arrangement in $X^n$. The name comes from its complement $\mathcal{M}_n(\{1\}, X)$ which is an ordered configuration space whose fundamental group is a generalized pure braid group.

Since $S = \emptyset$ in this example, the arrangement $A_n$ consists only of subspaces $H_{ij}$ for $1 \leq i < j \leq n$. All intersections will be connected, and the poset of layers is the partition lattice $\mathcal{Q}_n$. One can see this by viewing an intersection as putting an equivalence relation on the coordinates of $X^n$, since $(x_1, \ldots, x_n) \in H_{ij}$ whenever $x_i = x_j$.

**Example 3.2.3 (Type C Dowling poset).** Let $X$ be one of $\mathbb{C}$, $\mathbb{C}^\times$, or a complex elliptic curve, and let $G = \mathbb{Z}_2$ act on $X$ by using the group inversion. The arrangements $A_n(G, X)$ arise naturally from the type C root system, viewed as characters on the complex torus, and were studied in [Bib17] where a specialization of Theorem 3.2.5 appeared as Theorem 1.

For this action of $\mathbb{Z}_2$ on $X$, the set $S$ is the set of two-torsion points $X[2]$, and the Hasse diagram of $\mathcal{P}_n(\mathbb{Z}_2, \mathbb{C}^\times)$ is given in Figure 6. To foreshadow Theorem 3.2.5 below, one can see an obvious isomorphism between this poset and $\mathcal{D}_n(\mathbb{Z}_2, \{\pm 1\})$, whose Hasse diagram is shown in Figure 1.

![Figure 6. Poset of layers for the type C toric arrangement, $\mathcal{P}_2(\mathbb{Z}_2, \mathbb{C}^\times)$, which is isomorphic to $\mathcal{D}_2(\mathbb{Z}_2, \{\pm 1\})$ from Figure 1. See also Figure 4 and Examples 2.2.2 and 3.2.3.](image)

To demonstrate the possibility of disconnected intersections, observe that when $X = \mathbb{C}^\times$ and $n = 2$, the intersection of the two diagonals is a set of two points:
$H_{12}(1) \cap H_{12}(-1) = \{(1,1),(-1,-1)\}$. This can be seen in the Hasse diagram since there is not a unique least upper bound of $H_{12}(1)$ and $H_{12}(-1)$.

**Remark 3.2.4.**  
(1) When $X = \mathbb{C}$, the arrangement $A_n(G, X)$ is a (complex) hyperplane arrangement. The intersection of hyperplanes are all connected, and the poset of layers is a geometric lattice usually referred to as the intersection lattice.

(2) One may wonder why we consider the set of connected components of intersections, rather than the set of intersections themselves. A case for this choice is made by having codimension be a strictly increasing function on $P_n(G, X)$, which is in fact proportional to the intrinsic rank.

Meanwhile, the proof of Theorem 3.2.5 below shows that every layer is in fact an intersection. That is, the poset of layers is a subposet of the poset of all intersections, and one that appears easier to work with and study. It is important to note that this relationship does not exist for general arrangements.

The main theorem of this subsection is that the poset of layers is in fact familiar from our above work.

**Theorem 3.2.5 (Combinatorial description of the poset of layers).** Let $X$ be an almost free $G$-space with

$$S = \text{Sing}_G(X) = \bigcup_{g \in G - \{e\}} X^g.$$ 

There is a $S_n[G]$-equivariant isomorphism of posets

$$P_n(G, X) \cong D_n(G, S).$$

Considering just the atoms in these two posets for a moment, isomorphism (5) is given by the obvious bijection between $\alpha_{ij}(g)$, $\alpha^*_{ij}$ (from Lemma 2.5.1) and the subspaces $H_{ij}(g)$, $H^s_{ij}$ in the arrangement $A_n(G, X)$. We make note of some key observations about how these subspaces intersect: e.g. imposing both $g.x_i = x_j$ and $h.x_i = x_j$ has the consequence that $g^{-1}h.x_i = x_i$, so $x_i \in X^{g^{-1}h}$ and $x_j \in X^{h^{-1}g}$ are singular points. More generally, one can check the following.

**Lemma 3.2.6.**  
(1) For $i, j \in \mathfrak{n}$ and $g, h \in G$ distinct, the intersection $H_{ij}(g) \cap H_{ij}(h)$ has connected components $H^s_{ij} \cap H^z_{ij}$ where $s \in X^{g^{-1}h}$.

(2) For $i \in \mathfrak{n}$ and $s, t \in S$ distinct, we have $H^s_i \cap H^t_i = \emptyset$.

(3) For $i, j, k \in \mathfrak{n}$ distinct and $g, h \in G$, $H_{ij}(g) \cap H_{jk}(h) \subseteq H_{ik}(hg)$.

**Proof of Theorem 3.2.5.** Start by defining a map $\phi : D_n(G, S) \to P_n(G, X)$. For $(\bar{\beta}, z) \in D_n(G, S)$ with partition $\beta = (B_1, \ldots, B_t)$ we get a product decomposition

$$X^n = X^{B_1 \cup \ldots \cup B_t \cup \bar{Z}} \cong X^{B_1} \times \ldots \times X^{B_t} \times X^{\bar{Z}}$$

where $X^{B_i}$ is the space of functions $B_i \to X$. Now, for every projectively colored block $\bar{B}_i \in \bar{\beta}$ we define a connected subspace $X^{\bar{B}_i} \subseteq X^{B_i}$, and one $X^z \subseteq X^{\bar{Z}}$ so that their product gives a connected subspace

$$X^{(\bar{\beta}, z)} := X^{\bar{B}_1} \times \ldots \times X^{\bar{B}_t} \times X^{\bar{Z}} \subseteq X^{B_1} \times \ldots \times X^{B_t} \times X^{\bar{Z}} \cong X^n.$$

Then $\phi$ will be defined to be

$$\phi : (\bar{\beta}, z) \mapsto X^{(\bar{\beta}, z)}.$$
To illustrate the construction of $X\tilde{B}$, consider first a projectivized $G$-coloring on \{1, \ldots, d\} denoted by the suggestive notation $[g_1 : \ldots : g_d]$. This defines a subspace $X^{[g_1: \ldots : g_d]} \subseteq X^d$ by imposing the equations $(g_i^{-1}.x_i = g_j^{-1}.x_j)$ for all $i$ and $j$. Then the map $t_g : X \mapsto X^d$ given by

$$t_g : x \mapsto (g_1.x, g_2.x, g_3.x, \ldots, g_d.x)$$

gives an isomorphism $X \cong X^{[g_1: \ldots : g_d]}$, and thus the space we defined is connected. Note that this is well defined, as a different representative $(g_1h, \ldots, gdh)$ gives equations $h^{-1}g_i^{-1}.x_i = h^{-1}g_j^{-1}.x_j$ which are clearly equivalent constraints. Also note that the set of defining equations has many redundancies, and in fact it suffices to only consider pairs $(i, j)$ with $i = 1$. However, we avoid making such choices for the purpose of having a canonical construction.

For a general block $\tilde{B} \in \tilde{\beta}$ we follow the same procedure: pick a representative coloring $b : B \rightarrow G$ and consider the subspace $X^B \subseteq X^d$ consisting of functions $x_B : B \rightarrow X$ satisfying

$$b(i)^{-1}.x_B(i) = b(j)^{-1}.x_B(j) \quad \forall i, j \in B.$$ 

As in the previous paragraph, this definition gives a connected space isomorphic to $X$, and independent of the choice of representative coloring $b$.

As for the zero block, the function $z : Z \rightarrow S \subseteq X$ gives an element in $X^Z$, which we take to be the (connected) space $X^\tilde{Z}$. One should consider the special case $Z = \{1, \ldots, d\}$, in which this point will be

$$(z(1), \ldots, z(d)) \in X^d.$$ 

With the definition of $X^{(\tilde{\beta}, z)}$ in hand, the map $\phi$ is defined. We must now show that $\phi$ is order-preserving, $\mathfrak{S}_n[G]$-equivariant, and bijective.

Equivariance will follow from the next observation: $X^{(\tilde{\beta}, z)}$ is cut out of $X^n$ by equations of the form $g_i.x_i = x_j$ and $x_i = s$. This gives an alternative description of $X^{(\tilde{\beta}, z)}$ as the intersection of atomic layers $H_{ij}(g)$ and $H_{ij}^s$. Explicitly, for every $\tilde{B} \in \tilde{\beta}$ define

$$A(\tilde{B}) := \{H_{ij}(g) \mid i, j \in B, gb(i) = b(j)\}$$

and let

$$A(z) := \{H_{ij}^s \mid i \in Z, z(i) = s\}.$$ 

Denote the union of these subspaces by $A(\tilde{\beta}, z)$, and observe that $X^{(\tilde{\beta}, z)}$ is precisely their intersection.

Now equivariance is quick: For $g \in \mathfrak{S}_n[G]$ and $(\tilde{\beta}, z) \in \mathcal{D}_n(G, S)$, we have $A(g(\tilde{\beta}, z)) = gA(\tilde{\beta}, z)$ and hence $X_{g.\tilde{(\beta), z}} = g.X_{\tilde{(\beta), z}}$.

To see that $\phi$ is order-preserving, we consider our two covering relations from Definition 2.1.1. First, consider a merge $(\tilde{\beta} \cup \{A, \tilde{B}\}, z) \prec (\tilde{\beta} \cup \{C\}, z)$ where $C = A \cup B$ and $c = a \cup bg$ for $g \in G$. Then $X^C \cong X^A \times X^B$ and its subspace $X^{\tilde{C}} \subseteq X^C$ is defined by equations of the form $(c(i)^{-1}.x(i) = c(j)^{-1}.x(j))$. When $i, j \in A$ these are the defining equations of $X^{\tilde{A}} \subseteq X^A$, and when $i, j \in B$ the analogous statement holds for $X^{\tilde{B}}$. Therefore, every point $x_C \in X^{\tilde{C}}$ satisfies the conditions of being in $X^{\tilde{A}} \times X^{\tilde{B}}$, and the inclusions $X^{\tilde{C}} \subseteq X^{\tilde{A}} \times X^{\tilde{B}}$ and $X^{(\tilde{\beta} \cup \{C\}, z)} \subseteq X^{(\tilde{\beta} \cup \{A, \tilde{B}\}, z)}$ follow.

Next, consider a coloring $(\tilde{\beta} \cup \{\tilde{B}\}, z) \prec (\tilde{\beta}', z')$, where $Z' = Z \cup B$ and $f : G \rightarrow S$ is an equivariant function such that $z' = z \circ f \circ b$. Then $X^{Z'} \cong X^B \times X^Z$ and the
subspace $X^{z'}$ is the point $z'$. Since $z'$ restricts to the function $z$ on $Z$, it maps into the subspace $X^z$ under the projection $X^{z'} \to X^z$. As for its projection to $X^B$, on every $i, j \in B$ we need to check that $b(i)^{-1}z'(i) = b(j)^{-1}z'(j)$. Indeed, by the equivariance of $f$,

$$b(i)^{-1}z'(i) = b(i)^{-1}f(b(i)) = f(1) = b(j)^{-1}z'(j)$$

and the inclusion $X^{z'} \subseteq X^B \times X^z$ follows.

Finally, to show that $\phi$ is bijective we will construct an inverse map. Given $Y \in \mathcal{P}_n(G, X)$, let $A(Y)$ be the subset of $A_n(G, X)$ consisting of subspaces which contain $Y$. Define $Z_Y = \{ i \in n \mid H_i^s \in A(Y), \text{ some } s \in S \}$. Then by Lemma 3.2.6(2), we have for each $i \in Z_Y$ a unique $s \in S$ for which $H_i^s \in A(Y)$; this defines a map $\beta_Y : Z_Y \to S$. Next, by Lemma 3.2.6(1), for $i, j \notin Z_Y$, if there is some $g \in G$ for which $H_{ij}(g) \in A(Y)$ then this $g$ is unique and we denote it by $g_{ij}$. Define a partition $\beta_Y$ of $n \setminus Z_Y$ from the equivalence relation with $i \sim j$ if there is such a $g_{ij}$. Moreover, for $B \in \beta_Y$, one can always construct a coloring $b : B \to G$ so that $g_{ij}b(i) = b(j)$, giving a partial $G$-partition $\beta_Y$ of $n$. It is now easy to check that the assignment $Y \mapsto (\beta_Y, Z_Y)$ is indeed an inverse to $\phi$. \hfill \Box

**Remark 3.2.7.** As we saw in Remark 3.1.1, the two complements $M_n(G, X)$ and $M_n(G, X \setminus S)$ coincide. However, the former is viewed as the complement of an arrangement in $X^n$ while the latter sits inside $(X \setminus S)^n$. The two arrangements have rather different combinatorics, even though their complements are equal: the poset of layers of $M_n(G, X)$ is most apparent when $X$ is compact, e.g. a smooth projective variety, while $X \setminus S$ is not. In particular, when the cohomology of $X$ has pure Hodge structure, many spectral sequence calculations simplify greatly.

### 3.3. More examples.

Our first motivating example is the Dowling lattice. Dowling [Dow73] showed that when $G$ acts on $A_k^1$ linearly, i.e. via a character $G \to k^*$, the lattice $\mathcal{D}_n(G)$ is the intersection lattice of $A_n(G, A_k^1)$. In the case that $X = C$, one considers $G = \mu_d$ the group of $d$th roots of unity, and the hyperplanes in the arrangement are the reflecting hyperplanes for reflections in $\mathfrak{S}_n[G]$. Hence these are complex reflection arrangements, and their complements are $K(\pi, 1)$ spaces (see [Nak88]) and their combinatorics is particularly interesting.

In this subsection we consider examples of interest with varying $X$ and $G$, and relate them to the examples of posets $\mathcal{D}_n(G, S)$ explored in Section 2.2.

Recall one of our motivating examples: the type $C$ arrangements, described in Example 3.2.3 above. Generalizing on this, one can consider any algebraic group $X$ and let $G = Z_2$ act by the group inversion. In this case, the set $S$, where inversion fails to be free, is the set $X[2]$ of two-torsion points. Even more generally, one can take any finite subgroup $G \subseteq \text{Aut}(X)$ of algebraic group automorphisms, for which the set $S$ is finite, and consider the resulting arrangements $A_n(X, G)$.

For concreteness, take $X$ to be a complex elliptic curve. Most elliptic curves have $\text{Aut}(X) = Z_2$, and so the type $C$ elliptic arrangements are the only ones arising as $A_n(X, G)$. But when the $j$-invariant is either 0 or 1728, extra automorphisms appear. We describe the arrangements arising from the action of the automorphism group in the following two examples.
Example 3.3.1 (Hexagonal elliptic curve). Let $X$ be a complex elliptic curve with $j(X) = 0$. Alternatively, this is the complex torus $\mathbb{C}/(\mathbb{Z} \oplus \zeta_3 \mathbb{Z})$, corresponding to a tiling of the plane by equilateral triangles. Then the group of automorphisms is $G = \mathbb{Z}_6$, which we may consider as generated by multiplication with a primitive 6-th root of unity, $-\zeta_3$. The set of points where the action fails to be free is

$$S = \{e, z_1, z_2, z_3, w_1, w_2\},$$

represented in $\mathbb{C}$ by $e = 0$, $z_1 = \frac{1}{3}$, $z_2 = \frac{1}{3}\zeta_3$, $z_3 = \frac{1}{3}(1 + \zeta_3)$, $w_1 = \frac{1}{3}(1 + 2\zeta_3)$, and $w_2 = \frac{1}{3}(2 + \zeta_3)$. The action of $G$ on $S$ agrees with that in Example 2.2.4, hence the poset of layers is the hexagonal Dowling poset given in that example above.

Example 3.3.2 (Square elliptic curve). Let $X$ be a complex elliptic curve with $j(X) = 1728$, or alternatively, the complex torus $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. Then the automorphism group $G$ is $\mathbb{Z}_4$, which we may consider to be generated by multiplication with the primitive fourth root of unity $i$. The points where the action fails to be free are the two-torsion points

$$X[2] = \{e, z_1, z_2, t\},$$

represented in $\mathbb{C}$ by $e = 0$, $z_1 = \frac{1}{2}$, $z_2 = \frac{1}{2}i$, and $t = \frac{1}{2}(1 + i)$. The group $G$ acts on these points just as it did in Example 2.2.5, hence the resulting poset of layers is the square Dowling poset discussed in that example above.

Example 3.3.3 (Translation by torsion points). Another interesting example for $X$ an algebraic group is when $d$-torsion points $G = X[d]$ act by translation. A specific example of this is when $X = \mathbb{C}^\times$ so that $G = \mathbb{Z}_d$ are the $d$th roots of unity; here we note that $M_n(\mathbb{Z}_d, \mathbb{C}^\times) = M_n(\mathbb{Z}_d, \mathbb{C})$ is Dowling’s motivating example, by Remark 3.2.7. The action of $X[d]$ on $X$ is free, and thus the poset of layers for $A_n(G, X)$ is always the lattice of $G$-partitions $D_n(G, \emptyset)$.

3.4. Invariant arrangements. In the above treatment, we construct an arrangement whose complement is the orbit configuration space in $X^{reg} := X \setminus S$. Next, we consider a variant on this idea, in which one chooses to remove a different collection of points from $X$.

Let $T$ be finite a $G$-invariant subset of $X$, i.e. a finite union of $G$ orbits. Analogously to the discussion above, define an arrangement $A_n(G, X; T)$ in $X^n$ consisting of the subspaces:

1. $H_{ij}^g$ for $1 \leq i < j \leq n$ and $g \in G$, and
2. $H_i^t$ for $1 \leq i \leq n$ and $t \in T$.

Its complement $M_n(G, X; T)$ is the orbit configuration space in $X \setminus T$. We will also be interested in the resulting poset of layers, denoted by $P_n(G, X; T)$.

Because $T$ is $G$-invariant, the arrangement $A_n(G, X; T)$ is $S_n[G]$-invariant, and hence $S_n[G]$ acts on $M_n(G, X; T)$ as well as on $P_n(G, X; T)$. However, note that $S_n[G]$ not longer acts freely on $M_n(G, X; T)$ when $T$ does not contain $S$.

Let us start by mentioning two motivational examples of these invariant arrangements.

Example 3.4.1 (Punctured surface). Suppose that $G$ is a group acting on a Riemann surface $X$. Then if $T$ is a finite $G$-invariant subset, the complement of the invariant arrangement $A_n(G, X; T)$ is the orbit configuration space of the punctured surface. As mentioned in Remark 3.2.7, this is a scenario in which one can benefit from studying the (orbit) configuration space of a punctured surface inside of the compact $X^n$ rather than the usual (non-compact) $(X \setminus T)^n$. 
Example 3.4.2 (Types B and D). Consider the case that \( X = \mathbb{C}, \mathbb{C}^\times \), or a complex elliptic curve, and \( G = \mathbb{Z}_2 \) acts by using the group inversion, as in Examples 2.2.2 and 3.2.3. The arrangement \( \mathcal{A}_n(X, G) \), discussed in Example 3.2.3, naturally arises from the type C root system, viewed as characters on a torus. The type B and D root systems also define arrangements, which are subarrangements of the type C arrangement. In fact, they are invariant subarrangements, where the type B arrangement uses \( T = \{ e \} \subseteq X[2] \) and in type D we have \( T = \emptyset \). The poset of layers and representation stability for these subarrangements were studied in [Bib17].

Remark 3.4.3. It is important to note that while \( \mathcal{D}_n(G, S) \) describes the poset of layers of \( \mathcal{A}_n(G, X) \), it is not true that the poset \( \mathcal{D}_n(G, T) \) describes the layers of \( \mathcal{A}_n(G, X; T) \).

When \( T \subset S \), one immediately sees that the poset \( \mathcal{P}_n(G, X; T) \) is a subposet of \( \mathcal{P}_n(G, X) \). However, it is larger than one might at first expect. In fact, even when \( T \cap S = \emptyset \), the singular set \( S \) appears in the description of layers in \( \mathcal{A}_n(G, X; T) \). This phenomenon is explained fully next, in Theorem 3.4.4.

Theorem 3.4.4 (Removing general \( T \)). Let \( T \) be a finite \( G \)-invariant subset of \( X \), and let \( S = \text{Sing}_G(X) \) as above. Denote the respective sets of orbits by \( \mathcal{O}(T) \) and \( \mathcal{O}(S) \). Then there is a natural equivariant embedding

\[
\mathcal{P}_n(G, X; T) \hookrightarrow \mathcal{D}_n(G, T \cup S)
\]

whose image consists of all pairs \((\tilde{\beta}, z)\) for which \(|z^{-1}(\sigma)| \neq 1\) whenever \( \sigma \in \mathcal{O}(S) \setminus \mathcal{O}(T) \).

Proof. Given \((\tilde{\beta}, z) \in \mathcal{D}_n(G, T \cup S)\), one can construct the layer \( X^{(\tilde{\beta}, z)} \) of the arrangement \( \mathcal{A}_n(G, X; T \cup S) \) as in the proof of Theorem 3.2.5: by intersecting the subspaces in the collections

\[
A(\tilde{B}) := \{ H_{ij}(g) \mid i, j \in B, gb(i) = b(j) \}
\]

\[
A(\sigma) := \{ H^z(i) \mid i \in Z, z(i) = \sigma \}.
\]

We therefore only need to show that if \(|z^{-1}(\sigma)| \neq 1\) for all \( \sigma \in \mathcal{O}(S) \setminus \mathcal{O}(T) \), then \( X^{(\tilde{\beta}, z)} \) is in fact a layer of the subarrangement \( \mathcal{A}_n(G, X; T) \).

Consider the intersection of subspaces in \( A(\tilde{B}) \) along with only \( A(\sigma) \) for orbits \( \sigma \in \mathcal{O}(T) \). Its connected components are layers of \( \mathcal{A}_n(G, X; T) \), by definition. Furthermore, one of those connected components is indeed \( X^{(\tilde{\beta}, z)} \). This last claim follows from Lemma 3.2.6(1).

Conversely, consider a layer \( Y \) of the arrangement \( \mathcal{A}_n(G, X; T) \) and the corresponding pair \((\tilde{\beta}_Y, z_Y) \in \mathcal{D}_n(G, T \cup S)\) from the proof of Theorem 3.2.5. We need to show that \(|z_Y^{-1}(\sigma)| = 1\) implies \( \sigma \in \mathcal{O}(T) \). Suppose \( i \in n \) is the single element for which \( z_Y(i) \in \sigma \). Then there cannot exist another \( j \in n \) and \( g \in G \) for which \( H_{ij}(g) \supseteq Y \), since otherwise Lemma 3.2.6(1) would imply that \( z_Y(j) \in \sigma \) as well. But then the only way \( Y \) can be a layer of \( \mathcal{A}_n(G, X; T) \) is if \( H^z_Y(i) \in \mathcal{A}_n(G, X; T) \), which implies \( \sigma \in \mathcal{O}(T) \). \( \square \)

Next we discuss the intervals of the arising posets of layers. Theorem 2.4.2 above shows that closed intervals in \( \mathcal{D}_n(G, S) \) are products of Dowling and partition lattices. An analogue of this statement is true for the subposets in question, for which closed intervals are again geometric lattices. We prove this characterization
of intervals in Theorem 3.4.4 below, but before we can do that we must first revisit
the case of the Dowling lattice.

**Definition 3.4.5 (The posets \( \mathcal{D}^v_n(G) \)).** Let \( \mathcal{D}^v_n(G) \) be the subposet of \( \mathcal{D}_n(G) \)
consisting of partial partitions whose zero block is not a singleton.

Note that \( \mathcal{D}^v_n(G) \) is precisely the subposet of \( \mathcal{D}_n(G, \{0\}) \) corresponding to taking
\( T = 0 \) in Theorem 3.4.4. Observe that \( \mathcal{D}^v_n(G) \) is still a geometric lattice, and
hence intervals built out of these lattices are just as well-behaved as those of our
\( S \)-Dowling posets. Furthermore, one can extend to this context the description of
atoms (Lemma 2.5.1), functoriality (Proposition 2.3.1), and \( S_n[G] \)-orbits (Theorem
2.6.2), but we omit such details. In particular, the \( S_n[G] \)-orbits of \( \mathcal{P}_n(G, X; T) \)
are indexed by \( O(T) \cup S \)-labeled partitions \( \lambda \) of \( n \) for which \( \lambda_o \neq 1 \) whenever
\( o \in O(S) \setminus O(T) \).

**Theorem 3.4.6 (Local structure for general \( T \)).** Let \( T \) be a finite \( G \)-invariant
subset of \( X \), and let \( \mathcal{D}_n(G, S; T) \) be the subposet of \( \mathcal{D}_n(G, T \cup S) \)
consisting of pairs \((\bar{\beta}, z)\) for which \( |z^{-1}(o)| \neq 1 \) whenever
\( o \in O(S) \setminus O(T) \).

Then for every \((\bar{\beta}, z) \in \mathcal{D}_n(G, S; T)\),

\[
\mathcal{D}_n(G, S; T)_{\leq (\bar{\beta}, z)} \cong \prod_{\beta \in \bar{\beta}} Q_B \times \prod_{o \in O(S)} \mathcal{D}_{\bar{z}^{-1}(o)}(G_o) \times \prod_{o \in O(S) \setminus O(T)} \mathcal{D}^v_{\bar{z}^{-1}(o)}(G_o).
\]

**Proof.** Recall from Theorem 2.4.2(1) that the decomposition of the interval under
\((\bar{\beta}, z) \) in \( \mathcal{D}_n(G, S) \) assigns to some \((\alpha, z_o)\) a pair of tuples \(((\alpha_B)_{B \in \bar{\beta}}, (\alpha_o)_{o \in O(S)})\). If
we require \( |z^{-1}(o)| \neq 1 \) whenever \( o \in O(S) \setminus O(T) \), then for \( o \in O(S) \setminus O(T) \) the zero
block of \( \alpha_o \) could not be a singleton. Thus, the image of \( \mathcal{D}_n(G, S; T) \) under
the isomorphism would be the product decomposition stated in the theorem. \( \square \)

### 3.5. Local arrangements

This subsection will focus on the local structure of the arrangement \( \mathcal{A}_n(G, X) \). For this purpose, take the space \( X \) to be a smooth
manifold or variety. One way to understand what we mean by the local structure
of \( \mathcal{A}_n(G, X) \) is to consider a ‘scanning’ procedure: studying the germ of \( \mathcal{A}_n(G, X) \)
at every point \( p \in X^n \). Note that the germs are what appears in the Leray spectral
sequence for the inclusion \( \mathcal{M}_n(G, X) \hookrightarrow X^n \).

Since each element \((\bar{\beta}, z) \in \mathcal{D}_n(G, S)\) corresponds to a subspace \( X^{(\bar{\beta}, z)} \subseteq X^n \),
the incidence relation attaches to every point \( p \in X^n \) a subposet

\[
\mathcal{D}_n(G, S)_p := \{ (\bar{\beta}, z) \in \mathcal{D}_n(G, S) \mid p \in X^{(\bar{\beta}, z)} \}.
\]

This is exactly the subposet of those layers that meet every neighborhood of \( p \).

The collection of layers in \( \mathcal{D}_n(G, S)_p \) is clearly closed under intersection, and
thus has a maximum \((\bar{\beta}_p, z_p)\). It is also downward-closed (an order ideal), and is
therefore the interval \([0, (\bar{\beta}_p, z_p)]\) described by Theorem 2.4.2. Geometrically, this
characterizes the germ of \( \mathcal{A}_n(G, X) \) at \( p \) in \( X^n \): it is well known that when \( X \)
is a smooth manifold, the restriction of \( \mathcal{A}_n(G, X) \) to a small ball centered at \( p \) is
isomorphic to a linear subspace arrangement \( \mathcal{A}_p \), whose intersection poset is the
interval \([0, (\bar{\beta}_p, z_p)]\). One can see this, e.g. by choosing a Riemannian metric on \( X \)
and using the exponential map to identify a neighborhood of \( p \) with the tangent
space \( T_p X \) and the linear arrangement therein.

Theorem 2.4.2 thus translates to the following,
Theorem 3.5.1 (Local arrangements). For every \( p \in X^n \) the complement of the local arrangement \( A_p \) is isomorphic to a product of (free) orbit configuration spaces of points in \( \mathbb{R}^d \).

Equivalently, the restriction of \( \text{Conf}_n^G(X \setminus S) \) to any sufficiently small ball is isomorphic to a product of such orbit configuration spaces.

Remark 3.5.2. This observation is striking for the following two reasons:

1. The local picture involves orbit configuration spaces for groups different from \( G \), and possibly different actions on \( \mathbb{R}^d \) at every point.
2. One could not have avoided the difficulty of this description by removing a set of ‘bad’ points, as is typically done with non-free actions. This is since neighborhoods of these points record their bad behavior. The entire description of our posets \( D_n(G,S) \) was necessary, and before this work, the local structure described in Theorem 3.5.1 was generally unknown. This is while a description like above is rather important and comes up in applications, e.g. using the Leray spectral sequence for \( M_n(G, X) \hookrightarrow X^n \) for complete \( X \).

Let \( M_p \) denote the complement of \( A_p \) inside a small open ball. Since \( A_p \) is a linear subspace arrangement, the work of Goresky–MacPherson [GM88] ties together the cohomology of \( M_p \) with the Whitney homology of the interval:

\[
H^{(d-1)*}(M_p) \cong WH_*(\hat{0}, (\hat{\beta}_p, z_p)).
\]

Remark 3.5.3 (Realizability). A central question in matroid theory is whether a geometric lattice is realizable by an arrangement of hyperplanes over a field. Dowling [Dow73] completely settled this for his lattices: he showed that \( D_n(G) \) is realizable over \( \mathbb{C} \) if and only if \( G \) is cyclic, and changing the field amounts to putting restrictions on which cyclic groups are allowed.

Making contact with our local arrangements, one observes that if \( A_n \) is an arrangement of smooth hypersurfaces in some smooth algebraic variety over a field \( k \), then the tangent spaces to the arrangement at various points are realizations of intervals over \( k \).

In particular, Dowling’s realizability result implies the following:

Corollary 3.5.4 (Restriction on possible stabilizers). If an arrangement of hypersurfaces in a variety \( M \) has poset of layer \( \cong D_n(G,S) \), then the stabilizer subgroup \( G_s \) for every \( s \in S \) must be cyclic.

By considering \( A_n(G, X) \) we get, in particular, that if \( X \) is any Riemann surface or an algebraic variety over a field \( k = \mathbb{K} \) with an almost free \( G \) action, then the stabilizer of any point \( x \in X \) must be cyclic.

We wonder about the converse of this:

Question 3.5.5. Suppose that \( S \) is a \( G \)-set for which the stabilizer subgroups are all cyclic. When can one find an arrangement of hypersurfaces in a complex manifold whose poset of layers is \( \cong D_n(G,S) \)? Furthermore, if such an arrangement exists, must it be of the form \( A_n(G,X) \)?
3.6. Cohomology of the complement. This subsection is devoted to computing $H^\ast(M_n(G, X))$ as explicitly as possible. In a forthcoming sequel to this paper, we will apply ideas from representation stability to analyze the sequence of $\mathfrak{S}_n[G]$–representations $H^\ast(M_n(G, X))$ as $n$ varies. Recall from the introduction our running assumption on coefficients: When $X$ is a CW complex $H^\ast(\ldots)$ denotes singular cohomology with coefficients in any ring, and when $X$ is an algebraic variety it denotes $\ell$-adic cohomology with coefficients in either $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$.

In the case of where $G$ acts linearly on $X = \mathbb{A}^d$, the cohomology of the complement $M_n(G, X)$ can be described in a purely combinatorial way, as the Whitney homology of the intersection poset as we saw in (6), by the work of Goresky–MacPherson [GM88]. In general, though, there is a spectral sequence converging to $H^\ast(M_n(G, X))$, which combines the cohomology of the ambient space $X^n$ with the Whitney homology of the poset of layers. This spectral sequence can be realized as the Leray spectral sequence for the inclusion $M_n(G, X) \hookrightarrow X^n$ in some cases. We refer the reader to a clever construction by Petersen, which has appeared in various special cases beforehand but applies in the general context – see [Pet17].

The poset of layers $\mathcal{P}_n(G, X)$ gives rise to a stratification of the space $X^n$, and since the poset is (close to being) Cohen-Macaulay, we can use Example 3.10 in [Pet17] to simplify the $E_1$ page of the spectral sequence. Moreover, in the case that $X$ is projective, one can use the pure Hodge structure to conclude that most differentials must vanish, similar to Totaro’s argument in [Tot96]. It is also worth noting that one could apply Verdier or Poincaré duality to the following theorem to obtain a sequence more akin to that of Totaro, avoiding compactly supported cohomology.

**Theorem 3.6.1 (Spectral sequence for $H^\ast(M_n(G, X))$).** There is a spectral sequence

$$E_1^{pq} = \bigoplus_{(\tilde{\beta}, z) \in D_n^p(G, S)} H^{p-2}(\tilde{0}, (\tilde{\beta}, z)) \otimes H_c^q(X(\tilde{\beta}, z)) \implies H_c^{p+q}(M_n(G, X))$$

where $D_n^p(G, S)$ is the set of elements of rank $p$.

When $X$ is a smooth projective variety, this sequence degenerates at the $E_2$ page.

The main contributions of this theorem to Petersen’s general construction are the following:

- Every layer $(\tilde{\beta}, z)$ contributes exactly one term to $E_1^{pq}$ with homological degrees matching $(p, q)$, as opposed to summing over all combinations $H^i \otimes H^j_c$ with $i + j + 2 = p + q$.
- One gets control over weights. For example, when $X$ is a smooth complex projective variety then $E_1^{pq}$ will have pure weight.
- Our Theorem 2.4.2 gives an exact description of $[\tilde{0}, (\tilde{\beta}, z)]$ and its cohomology.

Next, we use our combinatorial formula for the characteristic polynomial in Theorem 2.5.2 to compute the Hilbert series for the above $E_1$ page. Recall that a specialization of the Hilbert series ($t = u = -1$) computes the Euler characteristic for the complement $\mathcal{M}_n(G, X)$. Note that our formula only works for the arrangements $\mathcal{A}_n(G, X)$, and not the invariant subarrangements discussed in Section 3.4. Unfortunately, the characteristic polynomial for the subposets $\mathcal{D}_n(G, S; T)$ does not in general factor.
We state these formulas for the case that \( S \neq \emptyset \) for simplicity, but one can easily extend to \( S = \emptyset \) using the second formula in Theorem 2.5.2.

**Proposition 3.6.2 (Hilbert series and Euler number).** Assume that \( S \neq \emptyset \).
The Hilbert series of the \( E_1 \) term of Theorem 3.6.1 is
\[
\sum_{p,q} (\dim E_{pq}^1) t^p u^q = \prod_{i=0}^{n-1} (P(u) + (|S| + |G|i) t),
\]
where \( P(u) \) is the compactly supported Poincaré polynomial of \( X \).

**Example 3.6.3.** The complement of the type C toric arrangement \( \mathcal{A}_n(\mathbb{Z}/2, \mathbb{C}^\times) \) has Poincaré polynomial
\[
\prod_{i=0}^{n-1} (1 + t + (2 + 2i) t) = \prod_{i=1}^{n} (1 + (1 + 2i) t).
\]
This can be seen from using an analogue of Proposition 3.6.2 for the Leray spectral sequence, so that \( P(u) = 1 + u \), the (ordinary) Poincaré polynomial. Since this sequence has no nontrivial differentials, the \( E_1 \) term gives exactly the cohomology groups. This method can be used anytime that the Leray spectral sequence degenerates immediately; for toric arrangements this formula could also be derived by the work of Moci [Moc12, Moc08].

From Proposition 3.6.2, one can now compute the Euler characteristic of \( \mathcal{M}_n(G, X) \) by substituting \( u = t = -1 \). However, a more conceptual and flexible approach to this calculation is presented next.

### 3.7. Motive of the complement

Recall that the compactly supported Euler characteristic is additive with respect to decompositions \( X = Z \cup U \) where \( Z \) is closed and \( U \) is its open complement. Other invariants with this property are called cut-paste invariants, or generalized Euler characteristics. The Grothendieck ring of varieties \( K_0 \) provides the universal example of such an invariant: it is generated by isomorphism classes of varieties, subject to the relation
\[
[X] = [Z] + [X \setminus Z]
\]
whenever \( Z \) is closed in \( X \), and \([X : Y] = [X \times Y]\). Every generalized Euler characteristic with values in some ring \( R \) can be identified with a homomorphism \( K_0 \to R \). The class \([X]\) associated to a variety \( X \) is called the motive of \( X \). The following equality generalizes the second author’s proof of Proposition 4.2 in [FW16].

**Theorem 3.7.1 (Motive of orbit configuration space).** The motive of the complement \( \mathcal{M}_n(G, X) \) factors as
\[
[M_n(G, X)] = \prod_{i=0}^{n-1} ([X] - |S| - |G|i).
\]

The universality of the motive now implies a long list of numerical identities, for example:

- Let \( \chi_c \) be the compactly supported Euler characteristic. Then
\[
\chi_c(\mathcal{M}_n(G, X)) = \prod_{i=0}^{n-1} (\chi_c(X) - |S| - |G|i).
\]
In particular, when $X$ is smooth, this computes the classical Euler characteristic via Poincaré duality.

- The Hodge-Deligne polynomial $HD(Y)$ is a generalized Euler characteristic, which on closed complex manifolds records the Hodge numbers
  \[ HD(Y) = \sum_{p,q} \dim H^{p,q}(Y) t^p u^q. \]

Theorem 3.7.1 then gives a formula for $HD(M_n(G, X))$ in terms of $HD(X)$.

- Over finite fields, the number of $F_q$-points on a variety is a generalized Euler characteristic. Therefore, when the set $S$ is fixed by the Frobenius action, we get the following point-count
  \[ \#M_n(G, X)(F_q) = \prod_{i=0}^{n-1} (\#X(F_q) - |S| - |G|)^i \]

**Proof of Theorem 3.7.1.** Our goal is to exhibit the following connection between the motive of $M_n(G, X)$ and the characteristic polynomial of the poset $D_n(G, S)$ (see §2.5):

\[ [M_n(G, X)] = \sum_{b \in D_n(G, S)} \mu(\hat{0}, b) [X]^{n-rk(b)} = \chi(D_n(G, S), [X]) \]

Recall that the the Möbius function $\mu(x) := \mu(\hat{0}, x)$ on a poset $P$ is the unique function for which

\[ \mu(\hat{0}) = 1 \text{ and } \sum_{y \leq x} \mu(y) = 0 \]

for all $x > \hat{0}$. We apply this property to the formal difference of layers

\[ [M] := \sum_{b \in D_n(G, S)} \mu(b) [X^b]. \]

In this difference, every point $p \in X^n$ is counted precisely

\[ \sum_{b \leq b_p} \mu(b) \]

times, where $b_p$ is the maximal layer that contains $p$ (see §3.5). But then, by Equation 9, it follows that the only points that contribute to $[M]$ are the ones in $X^0 \setminus \bigcup_{b > 0} X^b$, and those are counted precisely once. We therefore get

\[ [M_n(G, X)] = [M] = \sum_{b \in D_n(G, S)} \mu(b) [X^b]. \]

Lastly, in §3.2 we produced isomorphisms $X^{(\hat{b}, z)} \cong X^{n-rk(\hat{b}, z)}$ for every layer. Applying the multiplicative relation in $K_0$, we arrive at Equation 8. The factorization into linear factors now follows from that of the characteristic polynomial, see Theorem 2.5.2.

**Example 3.7.2.** (1) The Euler characteristic for the complement of the reflection arrangement $A_n(Z/d, C)$ is $(-d)^{n-1}(n-1)!$.

(2) The Euler characteristic for the type C toric and elliptic arrangements, $A_n(Z/2, C^\times)$ and $A_n(Z/2, E)$, from Example 3.2.3, are $(-2)^{n!}$ and $(-2)^n(n+1)!$, respectively.
(3) The Euler characteristic for the elliptic arrangement with hexagonal Dowling poset from Example 3.3.1 is $(-6)^n n!$.

(4) The Euler characteristic for the elliptic arrangement with square Dowling poset from Example 3.3.2 is $(-4)^n n!$.

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