MEASURING QUASAR VARIABILITY WITH Pan-STARRS1 AND SDSS

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ABSTRACT

We measure quasar variability using the Panoramic Survey Telescope and Rapid Response System 1 Survey (Pan-STARRS1 or PS1) and the Sloan Digital Sky Survey (SDSS) and establish a method of selecting quasars via their variability in 10^5 deg^2 surveys. We use 10^3 spectroscopically confirmed quasars that have been well measured in both PS1 and SDSS and take advantage of the decadal timescales that separate SDSS measurements and PS1 measurements. A power law model fits the data well over the entire time range tested, 0.01–10 yr. Variability in the current PS1–SDSS data set can efficiently distinguish between quasars and nonvarying objects. It improves the purity of a griz quasar color cut from 4.1% to 48% while maintaining 67% completeness. Variability will be very effective at finding quasars in data sets with no u band and in redshift ranges where exclusively photometric selection is not efficient. We show that quasars’ rest-frame ensemble variability, measured as a root mean squared in z, L, and λ as V(z, L, λ, t) = 0.079(1 + z)^0.162(L/L_0)^-0.44(λ/1000 nm)^-0.44(t/1 yr)^0.246.

Key word: quasars; general

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1. INTRODUCTION

Quasars are supermassive black holes in the centers of galaxies that have large accretion rates and correspondingly large luminosities (Rees 1984; Antonucci 1993; Kembhavi & Narlikar 1999). They can be up to 100 times brighter than their host galaxies (e.g., Villata et al. 2006) and can thus be observed and analyzed in depth even when their host galaxies are not observable. Because of this, quasars up to z ≈ 7.1 have been detected and spectroscopically analyzed (Fan et al. 2001; Mortlock et al. 2011; Morganson et al. 2012), and moderately high redshift (z > 2) lensed quasars are among the most commonly detected galaxy scale gravitational lenses (Oguri et al. 2006; Inada et al. 2008; Oguri & Marshall 2010).

Despite the huge efforts undertaken to find quasars, there is a dearth of known quasars at z > 2.5 (Schneider et al. 2002, 2010) as shown in Figure 1. Survey depth is the major reason for this, but quasar selection incompleteness also plays a significant role. Nearly all quasars were initially detected in large photometric surveys like the Sloan Digital Sky Survey (SDSS; York et al. 2000). In these surveys, quasars are primarily detected as objects with excess u’ band flux (Richards et al. 2002, 2004). Quasars have relatively flat continuum spectra with no thermal drop-off and thus tend to be blue and particularly u’-bright relative to nearly any star or nonquasar galaxy. At z > 2, the quasars’ rest-frame Lyα absorption enters the observer-frame u’ band, and quasars cease to have exceptional u’ band fluxes. For z > 2.5 quasars, u’ band excess selection is no longer viable. This Lyα absorption becomes useful for higher redshift quasars, because the sudden drop-off in flux is also photometrically distinctive (Fan 1999). However, high redshift “dropout” searches only allow us to find quasars in relatively small redshift ranges where the dropout effect is distinct (Osmer 1982; Warren et al. 1991). In SDSS, z ≈ 3.5 quasars are g’ dropouts, z ≈ 4.5 quasars are r’ dropouts, and z ≈ 6 quasars are i’ dropouts. The result is that the population of z > 2.5 quasars is relatively small and nonuniform in redshift space.

Quasars vary nonperiodically in optical bands by several tenths of a magnitude over periods of months and years (Giveon et al. 1999; Vanden Berk et al. 2004). The main causes of this variability include accretion disk instabilities (Rees 1984; Kawaguchi et al. 1998; Pereyra et al. 2006) and inflow variation (Hopkins et al. 2006). Microlensing by intervening lensing galaxies (Wambsganss 2006; Morgan et al. 2010) also contributes in some cases. Regardless of which physical processes are responsible, optical quasar variability is distinctive and can be used to more efficiently select quasars (Kozlowski et al. 2010; Butler & Bloom 2011; Palanque-Delabrouille et al. 2011; Schmidt et al. 2010; MacLeod et al. 2011) as has already been done in relatively small (fewer than 1000 quasar) surveys (Geha et al. 2003; Kozlowski et al. 2011; Kim et al. 2011; Kozlowski et al. 2012, 2013).

Before we can use quasar variability to search for new quasars, we must understand, statistically, how quasars vary. Previous attempts to measure quasar variability have been limited by time range or sample size. Vanden Berk et al. (2004) measured the variability of 25,000 spectroscopically confirmed SDSS quasars by comparing photometry derived from follow-up spectroscopy to the initial detection photometry. This work was limited to a maximum time lag between photometry and spectroscopy of two years. Schmidt et al. (2010) and MacLeod et al. (2010) studied quasar variability in SDSS stripe 82 (Abazajian et al. 2009) on timescales up to five years but could only use the 9157 quasars in stripe 82. MacLeod et al. (2012) extends the SDSS
work even further by tracking variability of the 33,881 quasars that were either imaged in stripe 82 or were multiplied imaged because of overlaps in the larger SDSS surveys. Exploiting the full sample of the 105,783 spectroscopically confirmed quasars from Shen et al. (2011) and the more than 10 yr of time lag since SDSS began producing massive samples of quasars would increase our knowledge of quasar variability further.

The Panoramic Survey Telescope and Rapid Response System 1 (PS1; Kaiser et al. 2010) and its 3σ survey are powerful new tools for studying a new statistical regime of quasar variability. The PS1 3σ survey images the entire sky north of −30° declination, including the entire SDSS. It thus produces new photometry of the 10^5 spectroscopically confirmed SDSS quasars. The time difference between SDSS and PS1 measurements of an individual quasar are typically 5–10 yr.

For this paper, we cross-matched the PS1 and SDSS databases with the spectroscopically confirmed quasars from Shen et al. (2011) to precisely measure quasar variability and show that variability can be used to find new quasars in the future. We describe the PS1–SDSS cross-matched database that we produced and used for this work in the next section. In Sections 3 and 4, we describe how we measure and parameterize variability for every object in this database, including quasars. In Section 5, we briefly discuss the damped random walk (DRW) model of quasar variability. In Section 6, we analyze quasar variability in the observer-frame. We discuss the average PS1–SDSS magnitude offset for quasars in Section 7. In Section 8, we show that the variability measurements from the cross-matched PS1–SDSS database can be used to improve quasar selection efficiency significantly. In Section 9, we measure the rest-frame variability of the Shen quasars. Finally, we show how variability amplitude relates to luminosity, redshift, and wavelength in Section 10.

2. THE PS1–SDSS DATA SET

PS1 (Kaiser et al. 2002, 2010; Chambers 2011) is a 1.8 m optical telescope with a 7 deg² field of view that images the sky in the $g_{P1}$, $r_{P1}$, $i_{P1}$, and $z_{P1}$ filters that cover the 4000 Å < λ < 9200 Å spectral range similarly to the analogously named SDSS $g'$, $r'$, $i'$, and $z'$ filters. It also has a $y_{P1}$ filter that, including the spectral response of the camera, covers the 9200 Å < λ < 10500 Å range. These filters are described in detail in Tonry et al. (2012). The telescope is producing several surveys, including a solar system Near Earth Object survey, a Stellar Transit Survey, a Deep Survey of M31, a Medium Deep survey consisting of 10 PS1 footprints spaced around the sky, and a 3σ survey that covers three-fourths of the sky (30,000 deg²) in all five bands (Chambers 2011). This latter survey is the focus of our work here as it contains the entire SDSS survey and approximately 10^6 spectroscopically confirmed quasars.

The PS1 3σ survey takes four exposures per year with each of the $g_{P1}, r_{P1}, i_{P1}, z_{P1}, y_{P1}$ filters. The yearly fill factor is roughly 90% in each band. The missing area is due mostly to nondetection areas on the camera plane and weather restricting the survey to 2 or rarely 0 exposures per filter in some areas of the sky. Individual $g_{P1}, r_{P1}, i_{P1}, z_{P1}, y_{P1}$ exposures have median 5σ limiting AB magnitudes of 22.1, 21.9, 21.6, 20.9, and 19.9, as summarized in Table 1. These PS1 limits are median results within the SDSS area. Stacked images are not uniformly available, and the work presented here is based on single exposure detections. However, when stacks are made, we expect a single year’s stacked image to increase each limiting magnitude by approximately 0.7 (accounting for some survey incompleteness), and the stacks of the proposed three year duration of the survey to increase them by 1.2. In this work, we use the uncalibrated data from Schlafly et al. (2012). This database includes four-fifths of the PS1 data up through 2012 August and is calibrated to 0.01 mag or better. The remaining one-fifth of the data was excluded because it could not be calibrated because of weather or technical issues. This database also excludes detections flagged by PS1 as cosmic rays, edge effects, and other defects. The PS1 3σ survey is still collecting more data, and the techniques developed here will become even more powerful as new epochs are added to the database.

We use SDSS photometry from SDSS Data Release 8 (DR8; Aihara et al. 2011). SDSS DR8 covers 14,555 deg² to u' g' r' i' z' to 22.3, 23.3, 23.1, 22.3, and 20.8. To convert between SDSS and PS1 magnitudes, we use the conversions from D. P. Finkbeiner et al. (2014, in preparation), which use the equation

$$m_{P1} - m_{SDSS} = a_0 + a_1 g_{SDSS} + a_2 g_{SDSS}^2 + a_3 g_{SDSS}^3,$$

where $m = g_{riiz}$ and $a_{0-3}$ are in Table 2. Tonry et al. (2012) provides a similar conversion from SDSS to PS1 calculated from PS1 filter curves. However, we use the Finkbeiner conversion because it is optimized for a broad stellar population. It is also calculated within the Schlafly et al. (2012) uncalibrated system that we are using for our photometry. For the nonvarying
stars for which these coefficients were fit, these conversions are good to roughly 0.01 mag. We add this 0.01 mag in quadrature to our statistical error. In Section 9 we find that these conversions are not as accurate for quasars and that we must use an additional correction to match PS1 to SDSS. When using SDSS magnitudes, we convert them to standard logarithmic magnitudes, rather than the default arcsinh-based “Luptitudes” that SDSS reports.

We use the 105,783 spectroscopically confirmed quasars from Shen et al. (2011) as our known quasar population. These quasars all have redshifts and model bolometric luminosities. They are quite bright, as we see in the $z$ band distribution in Figure 2, and 98.7% have PS1-converted SDSS magnitudes brighter than the PS1 limiting magnitude in each of the four PS1 bands in Table 1. We considered analyzing the sample of $\approx 10^6$ SDSS photometric quasars (Richards et al. 2009). However, the sources in that sample are not all quasars, do not have precisely measured redshifts or luminosities, and many would be too faint for single epoch PS1 detections.

All database work and cross-matching of surveys is performed with the Large Survey Database software (Juric & Bonaca 2011). We cross-match and parameterize the variability of all 252,567,124 objects that exist in both PS1 and SDSS DR8. We make a separate catalog for the spectroscopic quasars from Shen et al. (2011). We compare PS1 and SDSS point spread function (PSF) magnitudes in all cases and take care to exclude extended objects (with techniques described in Section 6) from our analysis in this paper.

3. THE STRUCTURE FUNCTION

To parameterize quasar variability efficiently in large surveys, we modify the quasar variability statistic from Schmidt et al. (2010). This statistic assumes that for an individual quasar in an individual filter, the magnitude difference, $m$, between two measurements, $j$ and $k$, separated by a time difference $t = t_j - t_k$ is a Gaussian with the form

$$P(m) = \frac{1}{\sqrt{2\pi(V^2(t) + \sigma_j^2)}} e^{-\frac{m^2}{2V^2(t) + \sigma_j^2}},$$

$$V^2(t) = A^2 t^\gamma,$$

$$\sigma_j^2 = V_j^2 + \sigma_k^2.$$

(2)

Here $P(m)$ is the normalized probability distribution of $m$, and $\sigma_j^2$ and $\sigma_k^2$ are the statistical magnitude uncertainties for quasar measurements $j$ and $k$ so that $\sigma^2$ is the statistical uncertainty of the magnitude difference $m$. MacLeod et al. (2012) finds that ensemble quasar variability is consistent with a Gaussian model for individual quasar variability. The $V(t)$, known as the structure function, is an ensemble measurement and fit of quasar variability. Formally, $V^2(t)$ is the average variance of $m$ due to actual astrophysical variation, and $V(t)$ is just the root mean squared of the average physical variability in units of magnitudes. The power law $V^2(t) = A^2 t^\gamma$ is just an empirical fit and not a physical model. In this paper, the time difference $t$ is always in years (observer-frame in Section 6 and rest-frame in Section 9); $A$ is then just the amplitude of variation at one year. While $V(t) = A r^\gamma$ is the more intuitive function, for mathematical ease we use $V^2, A^2,$ and $\Gamma = 2\gamma$ in our derivations and convert into the conventional $A$ and $\gamma$ for plots and final results.

4. PARAMETERIZING QUASAR VARIABILITY

To apply Equation (2) to a practical problem, we start with a quasar that has been measured a total of $N$ times in a PS1 or (PS1-converted) SDSS filter. A series of $N$ measurements produces $n = N(N - 1)/2$ discrete magnitude differences. We use $i$ to index over $n$. Following the normal $\chi^2$ type derivation, we find that the log probability of Equation (2) is

$$\log P(m_i, \sigma_i, t_i | A^2, \Gamma) = -\frac{1}{2} \sum_i^{N(N-1)/2} \log 2\pi + \log \left(V^2(t_i) + \sigma_i^2\right)$$

$$+ \frac{m_i^2}{V^2(t_i) + \sigma_i^2}. \quad (3)$$

Schmidt et al. (2010) maximizes Equation (3) using Monte Carlo methods. Since we are implementing this over $10^8$ sources, we search for faster, analytic approximations. We assume that $V^2(t) = A^2 t^\gamma$, and in our derivations, we fix $\Gamma$, taking it from the fitted ensemble value that we describe at the end of this section. Even with this assumption, it is difficult to analytically maximize Equation (3) for an arbitrary set of $t_i$ values. To simplify the problem further, we bin our $t_i$ values so that within a bin the time lags are roughly constant. We then analytically maximize the likelihood in Equation (3) of each bin with respect to $A^2$ and include the differences in time lags as a first order perturbation. We perform a lengthy derivation in the Appendix. The final result is that our estimator for $A^2$ for each

Table 2

| Filter | $a_0$ | $a_1$ | $a_2$ | $a_3$ |
|--------|-------|-------|-------|-------|
| g      | 0.00128 | -0.10699 | 0.00392 | 0.00152 |
| r      | -0.00518 | -0.03561 | 0.02359 | -0.00447 |
| i      | 0.00958 | -0.01287 | 0.00707 | -0.00178 |
| z      | 0.00144 | 0.07379 | -0.03566 | 0.00765 |

Notes. Ensemble error bars are insignificant, and for individual stars, these conversions are good to 0.01 mag.
The number of degrees of freedom, $N_{\text{Dof}}$, is not strictly correct, because our $N(N-1)/2$ time lags are taken from only $N$ independent measurements. In addition, our weights in Equations (5) and (7) assume a typical quasar amplitude variation. For sources that do not vary, we significantly underestimate the weight and produce tiny $\chi^2$. We can roughly correct for this by multiplying $\chi^2$ by $A_{\text{model}}^2/A^2$ as we do here. In general, we do not rely on this $\chi^2$ going to $N_{\text{Dof}}$ as a proper $\chi^2$ would.

When studying populations of quasars, it is easy enough to calculate the ensemble $A^2$ for a time bin, $t_i$, as

$$A^2(t_i) = \frac{\sum_j A^2_j(t_i) \text{weight}_j(t_i)}{\sum_j \text{weight}_j(t_i)},$$

where $j$ indexes our list of quasars. To calculate the uncertainty of a single bin in our ensemble average, we must account for the fact that each quasar has its own individual structure function, so the variance of $A^2$ is much larger than would be suggested by the error bars in Equation (8). The variance divided by the number of quasars produces much more reasonable ensemble error bars than the inverse sum of weights. For some time bins at high time lag, there may be very few measurements, and the observed population variance can be zero or negligibly small. In these cases, we use the variance of all $t_i > 1$ yr measurements of $A^2$. Our ensemble error bars for individual time bins are then

$$\sigma^2_{A^2} = \max \left( \frac{\text{Var}_{t_i}}{n_i}, \frac{\text{Var}_{t_i>1}}{n_i} \right),$$

$$\text{Var}_{t_i} = \frac{1}{n_i - 1} \left( \sum_j (A^2_j)^2 - \frac{1}{n_i} \left( \sum_j A^2_j \right)^2 \right),$$

and

$$\text{Var}_{t_i>1} = \frac{1}{n_{t_i>1} - 1} \left( \sum_{k>t_i,j} (A^2_j)^2 - \frac{1}{n_{t_i>1}} \left( \sum_{k>t_i} A^2_j \right)^2 \right).$$

Here $n_i$ is the number of quasars that had at least one measurement pair in time bin $t_i$, and $n_{t_i>1}$ is the sum of all $n_i$ values for $t_i$ greater than one year.

Finally, to produce an ensemble structure function, we multiply each $A^2$ and $\sigma^2_{A^2}$ by $t_i^1$ (using a model $\Gamma$). We refit a single $A^2$ and $\Gamma$ to the resulting curve using a $\chi^2$ minimization routine in log $A$–log $t$ space. We then iterate the process using the fit $A^2$ and $\Gamma$ as the model $A^2$ and $\Gamma$ until the two match. This also optimizes the fit. We then switch to the conventional $V(t) = A t^\gamma$, $\gamma = \Gamma/2$ for plotting and further analysis. We repeat this process across each of the $g_{P1}, r_{P1}, i_{P1}, z_{P1}$ filters, which are taken to be independent.

5. THE DAMPED RANDOM WALK MODEL

The power law model we use to parameterize the ensemble structure function is purely empirical and not physically motivated. Kelly et al. (2009), Kozłowski et al. (2010), MacLeod et al. (2010), and others have fit the variation of individual quasars as a DRW with a structure function parameterized as

$$V(t) = V_\infty (1 - e^{-t/\tau})^{0.5}.$$
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Figure 3. Observer-frame ensemble structure function, $V(t)$, of all quasars identified by Shen et al. (2011) in g_P1 (upper left), r_P1 (upper right), i_P1 (lower left), and z_P1 (lower right).

Table 3
Observer-frame Variability Parameters for the Complete Set of Shen et al. (2011) Quasars Fit with a Power Law Model

| Filter | $N_{\text{quasars}}$ | Mean($N_{\text{obs}}$) | Offset | $A$ | $\gamma$ | $\text{COV}_{\alpha\gamma}$ | $\chi^2_{\text{red}}$ |
|--------|----------------------|------------------------|--------|-----|---------|--------------------------|------------------|
| g_P1   | 87725                | 6.93                   | -0.0125| 0.1899 $\pm$ 0.0014 | 0.2395 $\pm$ 0.0037 | -1.4e-06 | 1.13 |
| r_P1   | 87401                | 7.05                   | 0.0142 | 0.1574 $\pm$ 0.0015 | 0.2531 $\pm$ 0.0047 | -2.1e-06 | 0.87 |
| i_P1   | 87860                | 7.03                   | 0.0088 | 0.1430 $\pm$ 0.0016 | 0.2489 $\pm$ 0.0057 | -3.5e-06 | 1.26 |
| z_P1   | 89917                | 7.31                   | 0.0308 | 0.1426 $\pm$ 0.0016 | 0.2202 $\pm$ 0.0061 | -4.3e-06 | 4.45 |

Note. The “Offset” column is the average PS1–SDSS value for each filter.

random walk (setting the exponent to 0.5) with an exponential damping term with timescale $\tau$ that prevents the quasar from randomly walking by arbitrarily large values. MacLeod et al. (2010) finds that $\tau$ is typically between 0.1 and 3 yr and $V_\infty$ is typically between 0.1 and 0.5 mag. We discuss the DRW here only to provide context for why we use a power law structure function model. MacLeod et al. (2012) and Zu et al. (2012) offer more thorough statistical discussions of the DRW structure function, while Mushotzky et al. (2011) and Lovegrove et al. (2011) offer detailed time series analyses of very small quasar samples.

The DRW model of quasar variability has gained popularity in recent years, and it may be a superior model of quasar variability for well-sampled light curves. That being said, we employ the power law model for a variety of reasons. First, for light curves with only a few data points, the difference between the two fits is negligible. Second, ensemble structure functions are generally found to be consistent with power laws even if individual quasars have DRW variability. It is difficult to relate an individual $V_\infty$ and $\tau$ to this power law, but assuming a constant $\gamma$ reduces individual quasar variability to a measurement of amplitude, $A$. This simplifying assumption facilitates the analysis in Section 10. Finally, this work aims to parameterize variability for every source in the PS1–SDSS overlap. This is computationally reasonable for a simple power law fit but would be prohibitively time consuming for a DRW fit, which generally require a Monte Carlo analysis for each light curve.

6. QUASAR VARIABILITY IN THE OBSERVER-FRAME

With the algorithm described in Section 4, we can perform the first of our scientific tasks, which is measuring the observer-frame variability of $10^5$ quasars. Ultimately, we are going to use our variability measurement of known quasars as a template for finding new quasars. So we must be able to distinguish a quasar from other sources by its variability without knowing its redshift or intrinsic luminosity. This includes using observer-frame times rather than correcting times by $(1+z)$. We show our ensemble results in Table 3 and Figure 3. Many bins in
Figure 3 have statistical large error bars because they represent time lags of roughly 6, 18, or 30 months, lags for which it is difficult to observe the same source. Reassuringly, we reproduce the general trend of quasar variability being the largest for the bluer bands.

We made several restrictions on these quasars to avoid biasing our PS1–SDSS comparisons. Some quasars have some extended source flux from the galaxy surrounding the quasars’ black holes. The different PSFs from PS1 and SDSS would likely lead to a biased luminosity difference, so we only use sources that SDSS categorizes as point sources (SDSS type = “star”). PS1 does not have a uniformly applied star–galaxy separation.

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Table 4

| Filter | OffsetSDSS | OffsetPS1 | Offsetboth |
|--------|------------|-----------|------------|
| g_p1  | 0.0425     | -0.0628   | -0.0010    |
| r_p1  | 0.0163     | -0.0739   | -0.0224    |
| i_p1  | 0.0275     | -0.0528   | -0.0084    |
| z_p1  | 0.0234     | -0.0498   | -0.0125    |

Notes. The Residual column is just the average of the two offset and estimates the residual bias after accounting for selection bias. The SDSS magnitudes here are SDSS magnitudes converted into the PS1 system.

7. THE AVERAGE MAGNITUDE OFFSET BETWEEN PS1 AND SDSS MEASUREMENTS OF QUASARS

When comparing PS1 and (converted) SDSS magnitudes of known quasars, we noticed an unexpected phenomenon: quasars, on average, apparently became dimmer by the amount listed in the “Offset” column in Table 3. This effect is strongest in the bluer filters, making it superficially consistent with physical variation.

We considered five possible origins for this effect. (1) A few outliers skewed the mean, (2) PS1–SDSS differences in measuring (barely) extended sources, (3) an astrophysical tendency of quasars to get dimmer on decadal timescales, (4) bias from the initial SDSS quasar selection and (5) our conversion in Equation (1) producing biased offset for quasars. To check against the “outlier” hypothesis (1), we used the mean, median, and outlier rejected mean PS1–SDSS offset. To check against the “extended source” hypothesis (2), we used quasars with \( m_{\text{g}_P} - m_{\text{g}_{SP}} < 0.3 \) in all filters. We also require that SDSS detect the object in the filter being measured as well as the \( g' \) and \( i' \) filters required for our PS1–SDSS conversion in Equation (1). The \( g' \) band requirement rejects many \( z > 3 \) quasars. When we study the redshift dependence of quasar variability, we do not extend any analysis into that redshift range. Finally, we require that there be at least 1 PS1 measurement of the quasar with the filter in question. The number of quasars being studied in each filter is listed as \( N_{\text{quasars}} \) in Table 3. The average number of PS1 measurements in each filter of quasars being studied in that filter is listed as mean \( (N_{\text{obs}}) \). A typical quasar has seven observations per filter from PS1. This small number of data points is another reason we do not try to fit \( \Gamma \) for each quasar in each filter. Combining PS1 with SDSS is useful to get more time information, and our variability characterizations of individual quasars tend to be weak.

Table 3 and Figure 3 have some important qualitative implications for selecting quasars by their variability. The amplitude of variability at one year, \( A \), is 0.1899 mag in \( g_{\text{P1}} \). This makes it very difficult to use variability to select quasars in surveys or redshift ranges where statistical or calibration uncertainty is \( \geq 0.1 \). Additionally, using the structure function in Equation (2), we can project our fit down to the 3 day scale where typical variability is 0.060 mag or to the 10 yr scale where it is 0.330 mag. Using a small number of measurements over a decadal timescale can thus be more effective than using many measurements over a small time period.

When studying the rest-frame variability of known quasars in Section 9, we add the “Offset” term from Table 3 to our SDSS magnitude to produce a more precise measurement of quasar variability. However, we do not use the “Offset” term when measuring observer-frame variability. This would add a great deal of false variability to otherwise static sources for which the conversion in Equation (1) works. Ultimately, the variability of quasars on the multi-year timescales that separate SDSS and PS1 measurements is much larger than this offset, so we are still able to produce sensible variability results, despite this offset.
time = 2 yr region where we transition from PS1–PS1 variability to PS1–SDSS variability.

8. VARIABILITY SELECTION OF QUASAR CANDIDATES

Several groups (Eyer 2002; Kożłowski et al. 2010; Schmidt et al. 2010; Butler & Bloom 2011; Kim et al. 2011; MacLeod et al. 2011; Palanque-Delabrouille et al. 2011) have produced criteria for selecting quasars using variability. However, these groups all used relatively small surveys with many observations and focused on the single filter variability. In Table 3, we see that we typically have only seven observations per object per filter. This means that we often cannot robustly fit anything beyond a variability amplitude in a single filter. The variability selection method we present here distinguishes quasars from other objects even if we only have a small number of observations.

A simple approach to selecting quasars with variability is to reduce all the information about source variability down to a two variable schema for identifying objects that vary in a quasar-like way. In principle, we could cross-match measurements from different filters at different times to get more time information. However, this would require us to implement a short-term color variability model that could get significant information from nonsimultaneous measurements from different filters. For simplicity, our method just combines the four single filter results into one weighted mean result. For each quasar candidate, we calculate two quantities: the one year amplitude of variability assuming a structure function model, $A$, and the difference between the quasar goodness of fit (with nonzero $\gamma$) and the RR Lyrae goodness of fit (with $\gamma = 0$), $\Delta \chi^2 = \chi^2_{qso} - \chi^2_{rrl}$.

A negative $\Delta \chi^2$ indicates that an object varies more on long timescales, while a positive $\Delta \chi^2$ is consistent with an object that varies in a fast, periodic way. Making cuts in this two-dimensional space allows us to distinguish between quasars, nonvarying objects, and quickly varying objects like RR Lyrae.

Our formulae for $A^2$ from Equation (4) and $V^2$ from Equation (6) apply to single filter measurements. To produce a single $A^2$ and $V^2$, we average the $A^2$ and $V^2$ from each band, weighted by the average values of $A^2$ for each filter list in Table 3. We index these average values as $A^2_f$, where $f = gP1, rP1, iP1, zP1$. The mean $A^2$ and $V^2$ for a quasar candidate is then

$$ A^2 = \sigma_A^2 \sum_f \frac{A^2_f}{A^2_0f \sigma_A^2_f} \quad (13) $$

$$ \sigma_A^2 = \left( \sum_f \frac{1}{A^2_0f \sigma_A^2_f} \right)^{-1} $$

$$ V^2 = \sigma_V^2 \sum_f \frac{V^2_f}{A^2_0f \sigma_V^2_f} \quad (14) $$

$$ \sigma_V^2 = \left( \sum_f \frac{1}{A^2_0f \sigma_V^2_f} \right)^{-1} $$

$$ f = gP1, rP1, iP1, zP1. \quad (15) $$

Hereafter, when discussing quasar selection, we take “$A$” and “$V$” to be this weighted mean $A$ and $V$. Note that $A^2$ is normalized by the single band ensemble average values of $A^2_0f$. Quasars’ average $A^2$, $A^2_{model}$, is thus normalized to 1. The ability to average
variability amplitudes across filters is a major advantage of using a power law structure function with a fixed $\gamma$. Combining four DRW structure functions or four structure functions with a variable $\gamma$ would not be more complicated.

We can also define a combined, multiband goodness of fit, $\chi^2$ as

$$
\chi^2_{\text{qso}} = \frac{1}{A^2} \sum_{f,i} (A_f^2(t_i) - A_f^2 A_{0f}) \text{weight}_f(t_i),
$$

$$
\chi^2_{\text{rrl}} = \frac{1}{A^2} \sum_{f,i} (V_f^2(t_i) - V_f^2 A_{0f}) \text{weight}_f(t_i),
$$

$$
\Delta \chi^2 = \chi^2_{\text{qso}} - \chi^2_{\text{rrl}}.
$$

(16)

Here we are summing over all four bands with $f$ and all 30 time bins, denoted by $t_i$. $A_f^2$ and $V_f^2$ have slightly different weights defined in Equations (5) and (7), respectively. Note that we multiplied our $\chi^2$ values by the analogous $A_{\text{model}}^2/A^2$ factor from Equations (5) and (7), but $A_{\text{model}}^2 = 1$. Because of the relatively small number of observations we make of the average quasar, the $\chi^2$ itself has a large variance and is not useful as a selection variable, but the difference, $\Delta \chi^2$, is more robust.

To examine the usefulness of $A-\Delta \chi^2$ space for selecting quasars, we define five test data sets. The first is just our set of quasars from Shen et al. (2011).

The second sample is a set of PS1–SDSS point sources (sources marked as type “star” by SDSS) with a broad quasar color cut:

$$
-0.2 < g' - r' < 0.9
$$

$$
-0.2 < r' - i' < 0.6
$$

$$
-0.15 < i' - z' < 0.5
$$

$$
i' < 20.1
$$

(17)

which we take from Schmidt et al. (2010). This is a reasonable quasar color selection for a region where there is no $u$ band data. We limit our sources to $i' < 20.1$ so that they have a similar magnitude distribution as the Shen et al. (2011) quasars, and we only use the 203,892 sources with R.A. $< 6^h$, $-2^\circ < $ decl. $< 2^\circ$ to make the sample more manageable.

Our third sample is the 483 RR Lyrae from Sesar et al. (2010). RR Lyrae are an interesting test case because their variability is of similar amplitude to quasar variability but is regular and occurs on timescales of days.

It would be difficult to estimate the purity and completeness of a variability-selected quasar sample with a mix of spectroscopically selected quasars and photometrically selected candidates. If we restrict ourselves to quasar candidates that satisfy Equation (17) and are also detected in the Wide-field Infrared Survey Explorer (WISE; Wright et al. 2010) survey, we can produce fairly pure and complete quasar samples. Wu et al. (2012) shows that requiring $z' - W1 > 0.66(g' - z') + 2.01$, $i' < 20.5$ produces a quasar sample that is roughly 98.3% complete and 95.6% pure. We call the sources that pass these cuts, our fourth and fifth populations, “SDSS-WISE quasars” and those that fail “SDSS-WISE stars.”

For all five samples, we only use sources that have a pair of measurements (in one band) with $0.01 \text{ yr} < \Delta t < 0.8 \text{ yr}$ and another pair with $\Delta t > 2 \text{ yr}$. This requirement ensures that we study the objects on both relatively short and relatively long timescales. Currently, 85% of spectroscopically confirmed quasars and 77% of RR Lyrae satisfy this requirement. We show the distribution of each of the five populations in $A-\Delta \chi^2$ space in Figure 5.

To select for quasars in $A-\Delta \chi^2$ space, we adopt a set of cuts in $A-\Delta \chi^2$ space:

$$
A > 0.2, \Delta \chi^2 < 0, \Delta \chi^2 < 7.2 \log_{10} A.
$$

(18)

In Figure 5 we see that the horizontal cut from Equation (18) distinguishes quasars from RR Lyrae and the vertical and diagonal cuts distinguish quasars from quasar-colored stars.

In Table 5, we see that even with a small amount of data, variability alone is a moderately effective method for selecting quasars when no $u$-band is available. Our proposed cut recovers
71% of known quasars while rejecting 92% of quasar-colored objects and 68% of RR Lyrae. PROJECTING our cut results over the entire SDSS area, we would select approximately 800,000 candidates; 60,274 of these would be known spectroscopically confirmed quasars, and we would expect that several times this number would be new quasars. To estimate purity more precisely, we examine the SDSS-WISE candidates. IGNORING the slight incompleteness and impurity of these samples, we find that our selection list is 67% complete and 48% pure. The initial sample, before variability selection, was 4.1% pure.

Our variability selection method cannot compete with u-band or even SDSS-WISE quasar selection over the broad population of quasars. But variability will be an effective method of quasar selection across the roughly 15,000 deg$^2$ of PS1 area with no quasars. But variability will be an effective method of quasar or even SDSS- Wise selection. Precisely, we examine the SDSS-WISE catalog of ensemble quasar variability with great precision. To do this accurately, we divide all measurements times by (1 + z) to account for residual uncertainty after the “Offset” is added as discussed in Section 7. With these slight modifications, we parameterize variability in each filter over all rest-frame–luminosity space as

$$A = A_0 (1 + z)^{B_0} \left( \frac{L}{10^{46} \text{ erg s}^{-1}} \right)^{B_L},$$

$$\log A = \log A_0 + B_L \log (1 + z) + B_L (\log_{10} L - 46) \log 10,$$

$$\gamma = \gamma_0 + \beta_L z + \beta_L (\log_{10} L - 46).$$  \hspace{1cm} (19)

Because of the large covariance between A and \(\gamma\), we fit the two variables jointly. For ease, we use the central log_{10} L_{Bol} and z of each bin.

The results of the four fits are in Table 8. Note that \(A_0\) and \(\gamma_0\) are the values of these variables at \(L = 10^{46}, z = 0\), not some mean estimate. In every case, we see that variability amplitude does indeed increase with redshift and decrease with luminosity. Both these trends make intuitive sense. Higher luminosity quasars also tend to be more massive and larger. So whatever physical process drives an individual quasar’s variability, we would expect brighter quasars to vary more slowly. Quasars are also generally more variable in bluer wavelengths. For a given filter, higher redshift quasars are being observed at blue rest-frame wavelengths. This may explain the redshift dependence.
Figure 6. Rest-frame ensemble structure function, $V(t)$, of all quasars identified by Shen et al. (2011) in $g_{P1}$ (upper left), $r_{P1}$ (upper right), $i_{P1}$ (lower left), and $z_{P1}$ (lower right). Here we fit the data with a power law structure function.

Figure 7. Distribution of spectroscopically confirmed quasars from Shen et al. (2011) in redshift–log $L_{\text{Bol}}$ space. The dotted lines mark the boundaries of the regions we use in Table 7.
Figure 8. Estimated value of $A$ as a function of redshift and bolometric luminosity for quasars in $g_{P1}$ (upper left), $r_{P1}$ (upper right), $i_{P1}$ (lower left), and $z_{P1}$ (lower right).

Figure 9. Estimated value of $\gamma$ as a function of redshift and bolometric luminosity for quasars in $g_{P1}$ (upper left), $r_{P1}$ (upper right), $i_{P1}$ (lower left), and $z_{P1}$ (lower right).
Table 7

| $z$  | $\log L$ | $A_{P1}$ | $\gamma_{P1}$ | $\gamma_{P1}$ | $A_{P1}$ | $\gamma_{P1}$ | $\gamma_{P1}$ |
|------|----------|----------|---------------|---------------|----------|---------------|---------------|
| 0.25 | 45.2     | 0.179 ± 0.012 | 0.191 ± 0.029 | −1.9 ± 0.04 | 0.227 ± 0.011 | 0.179 ± 0.028 | −1.2 ± 0.04 |
| 0.25 | 45.6     | 0.236 ± 0.010 | 0.239 ± 0.026 | −8.7 ± 0.05 | 0.199 ± 0.011 | 0.258 ± 0.035 | −1.4 ± 0.04 |
| 0.35 | 46.0     | 0.260 ± 0.028 | 0.150 ± 0.069 | 8.1 ± 0.03 | 0.210 ± 0.026 | 0.212 ± 0.084 | −1.4 ± 0.03 |
| 0.45 | 46.8     | 0.217 ± 0.005 | 0.221 ± 0.016 | −2.4 ± 0.05 | 0.199 ± 0.005 | 0.254 ± 0.016 | −1.3 ± 0.05 |
| 0.55 | 46.4     | 0.183 ± 0.009 | 0.269 ± 0.034 | −7.2 ± 0.05 | 0.187 ± 0.010 | 0.185 ± 0.030 | −9.1 ± 0.05 |
| 0.75 | 46.6     | 0.267 ± 0.005 | 0.223 ± 0.011 | −9.0 ± 0.07 | 0.210 ± 0.005 | 0.262 ± 0.015 | −7.1 ± 0.06 |
| 0.85 | 46.4     | 0.220 ± 0.004 | 0.252 ± 0.011 | −2.1 ± 0.06 | 0.174 ± 0.004 | 0.263 ± 0.013 | −1.0 ± 0.06 |
| 1.05 | 46.8     | 0.196 ± 0.008 | 0.206 ± 0.025 | −1.5 ± 0.05 | 0.165 ± 0.009 | 0.197 ± 0.037 | −6.8 ± 0.05 |
| 1.15 | 46.4     | 0.250 ± 0.004 | 0.240 ± 0.010 | 2.1 ± 0.06 | 0.203 ± 0.004 | 0.284 ± 0.011 | 3.6 ± 0.06 |
| 1.55 | 46.8     | 0.205 ± 0.004 | 0.248 ± 0.013 | 4.4 ± 0.06 | 0.172 ± 0.004 | 0.299 ± 0.016 | 5.7 ± 0.06 |
| 2.05 | 47.2     | 0.204 ± 0.015 | 0.092 ± 0.061 | −4.6 ± 0.04 | 0.147 ± 0.012 | 0.308 ± 0.049 | 4.9 ± 0.05 |
| 2.05 | 46.8     | 0.255 ± 0.008 | 0.216 ± 0.021 | 2.3 ± 0.05 | 0.226 ± 0.008 | 0.256 ± 0.021 | 3.2 ± 0.05 |
| 2.05 | 46.8     | 0.212 ± 0.005 | 0.249 ± 0.017 | 1.3 ± 0.05 | 0.190 ± 0.005 | 0.287 ± 0.019 | 1.6 ± 0.05 |
| 2.05 | 47.2     | 0.188 ± 0.009 | 0.243 ± 0.033 | 3.4 ± 0.05 | 0.162 ± 0.009 | 0.341 ± 0.037 | 5.1 ± 0.05 |

Notes. Each row contains the results for all quasars with $z$ within 0.25 of the “$z$” and $\log L$ within 0.2 of “$\log L$.” Note that the $g$ and $r$ results are in the top half of the table and the $i$ and $z$ results are in the bottom half. For ease, we only list the parameters used to produce the result in Table 8.

Table 8

| Filter | $A_0$ | $B_L$ | $B_L$ | $\gamma_0$ | $\beta_0$ | $\beta_L$ | $\chi^2_{red}$ |
|--------|-------|-------|-------|------------|-----------|-----------|---------------|
| $g_{P1}$ | 0.1766 ± 0.0058 | 0.455 ± 0.045 | −0.162 ± 0.011 | 0.236 ± 0.016 | −0.003 ± 0.013 | 0.016 ± 0.017 | 2.98 |
| $r_{P1}$ | 0.1597 ± 0.0062 | 0.445 ± 0.053 | −0.213 ± 0.013 | 0.210 ± 0.017 | 0.010 ± 0.015 | 0.090 ± 0.018 | 9.38 |
| $i_{P1}$ | 0.1577 ± 0.0068 | 0.248 ± 0.060 | −0.125 ± 0.015 | 0.258 ± 0.020 | −0.006 ± 0.017 | −0.002 ± 0.022 | 5.58 |
| $z_{P1}$ | 0.1570 ± 0.0062 | 0.234 ± 0.055 | −0.148 ± 0.014 | 0.243 ± 0.020 | −0.004 ± 0.018 | 0.060 ± 0.023 | 7.91 |

Notes. Because of the significant covariance between $A$ and $\gamma$, the two variables are fit simultaneously and there is only one $\chi^2_{red}$. Note that $A_0$ and $\gamma_0$ are not mean values, but the fitted values at $L = 10^{46}$, $z = 0$.

We also see that variation in $\gamma$ is essentially consistent with 0. We refit the data with a single $B_L$, $B_L$, and $\gamma_0$ to obtain

$$A = A_0 (1 + z)^{B_L} \left( \frac{L}{10^{46} \text{ erg s}^{-1}} \right)^{B_L},$$

where

$$A_{g_{P1}} = 0.1897 ± 0.0037,$$

$$A_{r_{P1}} = 0.1620 ± 0.0032,$$

$$A_{i_{P1}} = 0.1465 ± 0.0030,$$

$$A_{z_{P1}} = 0.1416 ± 0.0028,$$

$$\gamma = 0.2457 ± 0.0025,$$

$$B_L = 0.365 ± 0.026,$$

$$B_L = −0.159 ± 0.006.$$  

For completeness, we note that $\chi^2 = 6.29$. It is a better fit than the two worst single filter fits. These results are consistent with but more precise than those found in Vanden Berk et al. (2004).

The above results are useful for quasars in PS1 or similar surveys, but to generalize, we can also account for wavelength dependency. For each quasar and in each filter, we calculate a variability amplitude assuming a constant $\gamma = 0.2457$. We also calculate a rest-frame wavelength, $\lambda_0$, defined as the central wavelength of the $g_{P1}/p_{P1}/i_{P1}/z_{P1}$ filters (483 nm, 619 nm, 752 nm, and 866 nm, respectively) divided by $1 + z$. We then bin our quasars by $\lambda_0$ in five evenly spaced bins between 1500 nm and 6500 nm and use the same $L$ and $z$ bins from Table 7. In each bin, we take the median as our amplitude measurement and the difference between the 10th and 90th percentile divided by the
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Figure 10. Distribution of actual variability amplitudes over the average amplitudes defined in Equation (21).

square root of the number of sources as our error bars. We also ignore bins with fewer than 30 light curves. These last steps ensure robustness. Finally, we fit the three-dimensional binned data with Equation (20) with an additional $\lambda$ term to obtain:

$$A = A_0 (1 + z)^{B_z} \left( \frac{L}{10^{46} \text{ erg s}^{-1}} \right)^{B_L} \left( \frac{\lambda}{1000 \text{ nm}} \right)^{\nu},$$

$$A_0 = 0.0789 \pm 0.0017,$$

$$B_z = 0.153 \pm 0.028,$$

$$B_L = -0.200 \pm 0.006,$$

$$\nu = -0.441 \pm 0.018.$$  

(21)

For completeness, we note that $\chi^2 = 113$ with 78 degrees of freedom, but our error bars are noncanonical.

The change in variability amplitude versus luminosity is surprisingly small in both Equations (20) and (21). It suggests that a “bright” quasar, 100 times more luminous than a “dim” quasar, varies with roughly half the amplitude. If one assumes that observed quasar luminosity scale roughly with mass, this defies simple dimensional arguments based on the Schwarzschild radius scaling as $M$. The increase of variability with redshift is consistent with the idea that galaxies were generally more active and dynamic in the early universe. The inverse relationship between variability and $\lambda$ is also generally accepted (MacLeod et al. 2012).

We anticipate that these results will be useful for estimating quasar variability in theoretical models and simulations. With that in mind, we make a simple fit for the distribution of variability amplitudes of all quasars with respect to the average amplitude defined in Equation (21). For each light curve (subscripted “$i$”), we take the ratio of variability amplitude over the expected amplitude, $A_i/A$, and bin the results in Figure 10. We find that global, individual variability amplitudes are distributed as:

$$P(x_i) = 1.16 e^{-1.43 x_i^6},$$

$$x_i = \frac{A_i}{A}.$$  

(22)

Again, $A$ is defined for all $z$, $L$, and $\lambda$ in Equation (21), and we have assumed a constant exponent of $\gamma = 0.2457$. This model of quasar variability is obviously simplistic but may nonetheless be useful for simulations and models across a wide range of $z$, $L$, and $\lambda$.

11. CONCLUSIONS

The cross-matched PS1–SDSS catalog we produced and studied here is a powerful tool for probing quasar variability. The large sample size of $10^5$ quasars and the 10 yr time spanned by SDSS and PS1 allowed us to study ensemble quasar variability with unprecedented precision. When we examined the typical quasar root mean squared magnitude variability as parameterized by the structure function, $V(t) = A \, t^\gamma$, we found $\gamma \approx 0.25$ regardless of quasar luminosity, quasar redshift, or observer filter. We confirmed the well-known trend that quasars are more variable in bluer bands than in redder bands.

After measuring observer-frame quasar variability, we examined the effectiveness of using quasar variability as a method for quasar selection with our current sample. We produced
with variability. As in the observer-frame case, the structure function studying the more astrophysical problem of rest-frame quasar to our database.

is only 4.1% pure. PS1–SDSS quasar variability selection will objects that can be fairly cleanly separated into quasars and spectroscopic quasars. If we restrict our search to SDSS–WISE projects with quasar-like colors while recovering 71% of known The Astrophysical Journal, 784:92 (16pp), 2014 April 1

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APPENDIX

PARAMETERIZING QUASAR VARIABILITY FOR MEASUREMENT PAIRS WITH APPROXIMATELY CONSTANT TIME INTERVALS

We want to estimate the variability amplitude, $A$, of a quasar implied by a set of measurement pairs that produces a set of magnitude differences, $m_i$, time differences, $t_i$, and statistical uncertainties, $\sigma_i$. We assume the time intervals are roughly constant (that we are binning measurement pairs into fairly fine time bins). We start with the likelihood in Equation (3) and differentiate it with respect to $A^2$ to find the probability maximum

$$\frac{\partial \log(P)}{\partial (A^2)} = - \frac{1}{2} \sum_i \frac{t_i^F}{A^2 t_i^F + \sigma_i^2} - \frac{m_i^2 t_i^F}{(A^2 t_i^F + \sigma_i^2)^2} = 0. \quad (A1)$$

In the case that the variation of $\sigma_i^2$ and $t_i^F$ are totally negligible, this is trivially solvable as

$$A_0^2 = \frac{m_0^2}{t_0^F} \sigma_0^2, \quad m_0^2 = \frac{1}{n} \sum_i m_i^2, \quad \sigma_0^2 = \frac{1}{n} \sum_i \sigma_i^2, \quad t_0^F = \frac{1}{n} \sum_i t_i^F. \quad (A2)$$

Formally, $A_0^2$ can be less than 0 for sources that randomly vary less than their error bars suggest. We allow this to happen for individual time bins for individual quasars to avoid biasing our results. The mean $A^2$ is positive for every time bin in the in the every ensemble average we produce.

We can improve the result in Equation (A2) by allowing for variations from $A_0^2$, $\sigma_0^2$, and $t_0^F$ and treating them as perturbations. This produces

$$\frac{\partial \log(P)}{\partial (A^2)} = - \frac{1}{2} \sum_i \left( \frac{m_i^2 + t_i^F \delta A^2 + A_0^2 \delta t_i^F + \delta \sigma_i^2}{m_i^2 + t_i^F \delta A^2 + A_0^2 \delta t_i^F + \sigma_i^2} \right) = 0,$n \sum_i \frac{1 + \tau_{1i}}{1 + a + \tau_{2i} + s_i} - \frac{\mu_i (1 + \tau_{1i})}{(1 + a + \tau_{2i} + s_i)^2} = 0, \quad (A5)$$

a = \frac{t_0^F}{m_0^2} \delta A^2, \quad \mu_i = \frac{m_i^2}{m_0^2}, \quad s_i = \frac{\delta t_i^F}{m_0^2}, \quad \delta t_i^F = t_i^F - t_0^F \quad (A6)$$

A first order perturbation yields

$$\sum_i (1 + \tau_{1i} - a - \tau_{2i} - s_i)$$

(a - \mu_i \tau_{1i} + 2\mu_i \tau_{2i} + 2\mu_i s_i = 0, \quad (A5)$$

$$a = \frac{1}{n} \sum_i \mu_i \tau_{1i} - 2\mu_i \tau_{2i} - 2\mu_i s_i, \quad (A7)$$

We use the facts that $\sum_i \mu_i = n = \sum_i 1$ and that $\sum_i s_i = \sum_i \tau_{1i} = \sum_i \tau_{2i} = 0$ in the simplifications above.
We substitute more physical variables to obtain
\[
a = \frac{1}{n} \sum_i m_i^2 \left( \frac{t_i^\Gamma - t_i^0}{t_i^0} \right) - 2 \left( \frac{A_0^2 (t_i^\Gamma - t_i^0)}{m_0^2} \right) - 2 \left( \frac{\sigma_i^2 - \sigma_0^2}{m_0^2} \right),
\]
which we add as a perturbation to obtain \(A^2\):
\[
A^2 = A_0^2 + \frac{m_0^2 - \sigma_0^2}{t_0^2} a
\]
\[
A^2 = \frac{2m_0^2 - \sigma_0^2}{t_0^2} - \frac{1}{n} \sum_i m_i^2 t_i^\Gamma t_i^0
+ \frac{2}{nm_0^2 t_0^2} \left( \sigma_0^2 \sum_i m_i^2 t_i^\Gamma t_i^0 - \sum_i m_i^2 \sigma_i^2 \right).
\]

While Equation (A12) is formally correct and perfectly usable for objects with many observations, objects which have been observed only a few times can have \(m_0^2\) go to zero. Having \(m_0^2\) in the denominator can produce arbitrarily large variability estimates. To avoid this problem, we replace the denominator \(m_0^2\) with a model
\[
m_0^2 = \frac{1}{n} \sum_i A_{\text{model}}^2 t_i^\Gamma + \sigma_i^2 = A_{\text{model}}^2 t_0^\Gamma + \sigma_0^2
\]
and obtain a more robust solution
\[
A^2 = \frac{2m_0^2 - \sigma_0^2}{t_0^2} - \frac{1}{n} \sum_i m_i^2 t_i^\Gamma t_i^0
+ \frac{2}{nt_0^2} \left( \sigma_0^2 \sum_i m_i^2 t_i^\Gamma t_i^0 - \sum_i m_i^2 \sigma_i^2 \right).
\]

In the case where the time intervals are exactly constant or the variability has no apparent time dependence (for instance, a variable star that varies on timescales shorter than those that are measured), \(V(t)\) is just a constant \(V, \Gamma = 0\), and Equation (A14) can be simplified to produce
\[
V^2 = m_0^2 - \sigma_0^2 + \frac{2}{V_{\text{model}}^2 + \sigma_0^2} \left( \sigma_0^2 m_0^2 - \frac{1}{n} \sum_i m_i^2 \sigma_i^2 \right).
\]

If error bars are also constant, we obtain
\[
V^2 = m_0^2 - \sigma_0^2.
\]

We can derive a weight (inverse variance) for \(A^2\) by assuming it is Gaussian distributed. The weight is then just
\[
\text{weight} = -\frac{\partial^2 \log(P)}{\partial (A^2)^2},
\]
\[
\text{weight} = \frac{-1}{2} \sum_i \frac{t_i^{2 \Gamma}}{(V(t_i) + \sigma_i^2)^3} - \frac{2m_i^2 t_i^{2 \Gamma}}{(V^2(t_i) + \sigma_i^2)^3},
\]
\[
\text{weight} = \frac{1}{2} \sum_i \frac{t_i^{2 \Gamma}}{(V(t_i) + \sigma_i^2)^3} (2m_i^2 - V^2(t_i) - \sigma_i^2).
\]

Here \(V\) is necessarily a model \(V\) and not derived from the current \(m_0^2\). If one uses the \(V_i = m_i^2 - \sigma_i^2\) as in Equation (A2), the weight will be inversely proportional to \(m_0^4\). This effectively downweights all large variability data points and severely biases the final result downward. In fact, even in Equation (A19), \(\sum_i m_i^2\) will randomly be very small and can lead to negative weights. It is best to replace \(m_i^2\) with the model value \(V^2(t_i) - \sigma_i^2\) and obtain
\[
\text{weight} = \frac{1}{2} \sum_i \frac{t_i^{2 \Gamma}}{(V(t_i) + \sigma_i^2)^2},
\]
\[
\text{weight} = \frac{1}{2} \sum_i \frac{1}{(A_{\text{model}}^2 + \sigma_i^2/t_i^{2 \Gamma})^2}.
\]

Since the weight can be solved for directly, no further approximations are necessary.

The simple weight in Equation (A21) is proportional to the number of time intervals being examined and makes the faulty assumption that all time intervals in the set \(t_i\) are independent. This is not generally true, and many time intervals will overlap and should be statistically downweighted. If we have \(j\) time intervals \(t_0\) long over the time interval \(\Delta t\) that runs from the first measurement to the last measurement, then on average \(e^{-j \Delta t}/\Delta t\) of the larger interval \(\Delta t\) is not “covered” by at least one of the \(j\) measurements. The \(j + 1\) measurement should be weighted by approximately that factor. Thus, instead of being weighted by a factor “\(n\),” our measurements should be weighted by a factor
\[
\hat{n} = \sum_i e^{-i \Delta t} = 1 - \frac{1 - e^{-b_i/\Delta t}}{1 - e^{-b_i/\Delta t}},
\]
and our actual weight is
\[
\text{weight} = \frac{\hat{n}}{2n} \sum_i \frac{1}{(A_{\text{model}}^2 + \sigma_i^2/t_i^{2 \Gamma})^2}.
\]

In the quick sinusoidally varying case, this reduces to
\[
\text{weight} = \frac{\hat{n}}{2n} \sum_i \frac{1}{(V_{\text{model}}^2 + \sigma_i^2)^2}.
\]

If we also impose constant error bars, we obtain
\[
\text{weight} = \frac{\hat{n}}{2} (V_{\text{model}}^2 + \sigma_0^2)^2.
\]
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