A slope conjecture for links

Roland van der Veen

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Abstract

The slope conjecture [7] gives a precise relation between the degree of the colored Jones polynomial of a knot and the boundary slopes of essential surfaces in the knot complement. In this note we propose a generalization of the slope conjecture to links and prove this conjecture for $B$-adequate links and torus links.

1 Introduction

As predicted by Witten’s path integral formulation [18], the Jones polynomial exhibits many intriguing relations to classical geometry and topology. These become even more apparent when considering the colored Jones polynomial [11]. A new instance of these relations is suggested by slope conjecture [7]. It describes a fascinating relationship between the degree of the colored Jones polynomial of a knot and the slopes of essential surfaces in its complement. This conjecture is closely related to the AJ-conjecture [5, 2] but may be more tractible. See Section 3 for more details on the connection between the two conjectures.

For a knot $K$ in $S^3$, let $N$ denote a tubular neighborhood of $K$ and let $M = S^3 - N$ denote the exterior of $K$. Let $(\mu, \lambda)$ be the canonical meridian-longitude basis of $H_1(\partial N, K)$. An element $p/q \in \mathbb{Q} \cup \{\frac{1}{0}\}$ is called a boundary slope of $K$ if there is a properly embedded essential surface $(\Sigma, \partial \Sigma) \subset (M, \partial N)$, such that every circle of $\partial \Sigma$ is homologous to $p\mu + q\lambda \in H_1(\partial N)$.

Let $K$ be a knot and consider the unnormalized framing independent colored Jones polynomial $J_N(K; v)$, see Section 2 for a definition. It is well known to be a Laurent polynomial in the variable $v = A^{-1}$ where $A$ is the Kauffman variable. One can therefore speak about its maximal degree $\text{maxdeg}$ and ponder its topological meaning.

Conjecture 1. (Slope conjecture [7])

Let $K$ be a knot. The accumulation points of the set

$$\left\{ \frac{\text{maxdeg} J_N(K; v)}{N^2} \right\}_{N \in \mathbb{N}}$$

is contained in the set of boundary slopes of essential surfaces in the knot complement.
The conjecture was proven for $B$-adequate knots by [3] and also for some non-adequate knots including the $(−2, 3, 7)$-Pretzel knot and torus knots [7].

There is a similar statement for the minimal degree of the knot but this follows from the present conjecture by replacing the knot by its mirror image.

It is tempting to ask whether there is an analogous conjecture for links and indeed it appears that such conjecture exists.

To formulate it properly we need the following definition. To motivate it, recall that in the above slope conjecture for knots we need to pick out those quadratic parts $sN^2$ that approximate part of the degree sequence well. Since we have multiple colors, say $N = (N_1, N_2)$ in the two component link case, we now need to find those quadratic forms $NQN^t = Q_{11}N_1^2 + 2Q_{12}N_1N_2 + Q_{22}N_2^2$ that approximate part of the degree multi-sequence well.

**Definition 1.** Let $f$ be a function defined on $\mathbb{N}^c$. Write $N = (N_1, \ldots, N_c) \in \mathbb{N}^c$. A $c \times c$ symmetric matrix $Q$ is called a slope matrix for $f$ if there exist infinite subsets $U_1, \ldots, U_c \subset \mathbb{N}$ and a constant $C$ such that for $N \in \prod_{j=1}^c U_j$ we have

$$|f(N_1, \ldots, N_c) - NQN^t| < C \max N_j$$

In case $c = 1$ and $f(N) = \maxdeg J_N(K; v)$ the slope matrices exactly correspond to the accumulation points and hence the suggested slopes in Conjecture [1].

We are now ready to formulate the slope conjecture for links.

**Conjecture 2.** (Slope conjecture for links)

Let $L$ be a $c$ component link. If $Q$ is a slope matrix for $\maxdeg J_N(L; v)$ then there exists an essential surface $\Sigma$ such that the slope of the $j$-th boundary component equals $\sum_i Q_{ij}$.

**Theorem 1.** The slope conjecture is true for all $B$-adequate links.

By taking mirror images there is a similar statement for the minimal degree of $A$-adequate links.

**Theorem 2.** The slope conjecture holds for all torus links.

As additional evidence for the conjecture we note that if we take two knots for which the slope conjecture is known then the slope conjecture also holds for their distant union.

An amusing corollary is that if one colors the link with equal colors, the maximum degree of the colored Jones polynomial will detect the sum of the slopes of a surface bounding the link.

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2 Colored Jones polynomial for links

Since multiple conventions are in use in the literature we briefly give a
definition of the colored Jones polynomial of a link. The present definition
is perhaps not the most natural but has the advantage of being relatively
crude and easy to state. Assuming the reader is familiar with simple
skein theory it is easiest to construct the colored Jones polynomial
from the colored Kauffman bracket as follows.

Recall that the skein space of the annulus can be regarded as the poly-
nominal ring in the variable $x$ corresponding to its core. Given a framed
knot diagram $D$ we can insert an element $P(x)$ of the skein of the annulus
and take the Kauffman bracket in the plane. The resulting Laurent poly-
nomial in $A$ will be written $\langle D(P(x)) \rangle$. More generally for a $c$-component
link diagram $D$ and polynomial $P_j(x)$, on for each component we will
write $\langle D(P_1(x), \ldots, P_c(x)) \rangle(A)$.

Because of representation theory a special role is played by the Cheby-
shev polynomials $S_n(x)$ defined by

$$S_{n+1}(x) = xS_n - S_{n-1}$$

and $S_0 = 1$ and $S_1 = x$.

Definition 2. Let $L$ be a $c$-component link in $S^3$ with diagram $D$ where the writhe of the $j$-th component is $w_j$.

1. The $N = (N_1, \ldots, N_c)$-colored Kauffman bracket of $L$ is defined to be

$$\langle L \rangle_N(A) = \langle D(S_{N_1}, \ldots, S_{N_c}) \rangle(A)$$

2. Set $v = A^{-1}$. The $N = (N_1, \ldots, N_c)$-colored Jones polynomial of a $c$-component link $L$ is defined to be

$$J_N(L; v) = \prod_{j=1}^c (-1)^{N_j-1}((-1)^{N_j-1}q^{N_j^2-1})^{w_j} \langle L \rangle_{N_1-1, N_2-1, \ldots, N_c-1}(v)$$

Note the shift from $N$ to $N - 1$ and the change from $A$ to $v$. The
version of the colored Jones polynomial we defined is unnorma-
ized and framing independent because of the writhe term. The value of the $N$-
 colored unknot is $\overline{v}^{2N-2N}$. It is perhaps more standard to replace $v$ by
$q^{\frac{1}{4}}$ but the $v$-variable neatly absorbs the factor of 4 otherwise present in the slope conjecture.

3 Connection to the AJ conjecture

In this section we briefly motivate the slope conjecture in the knot case
by pointing out its connection to the AJ conjecture. This argument is due
Garoufalidis [8].

To introduce the AJ conjecture we first need to know that every knot
satisfies a $q$-recurrence [9]. This may be written as a non-commutative
polynomial $A_K(v, L, M)$ in the operators $M$ and $L$. These operators are
defined on functions $f$ on $\mathbb{N}$ as follows:

$$(Mf)(N) = v^{-2N} f(N) \quad \text{and} \quad (Lf)(N) = f(N + 1)$$
Viewing the colored Jones polynomial as a function on $N$ the recursion is then expressed as $\hat{A}_K(v, L, M)J = 0$.

The AJ conjecture states that at $v = 1$ the $\hat{A}$ is closely related to the famous A-polynomial defined in [1]. More precisely:

**Conjecture 3.** (AJ conjecture)

> For every knot $K$, the polynomial $\hat{A}_K(1, L, M)$ is equal to the A-polynomial of $K$, up to a polynomial factor depending on $M$ only.

It was shown in [1] that the slopes of the Newton polygon of the A-polynomial are all slopes of essential surfaces in the complement of the knot. So to motivate the slope conjecture we should explain how the certain slopes of the Newton polygon of the A-polynomial are detected by the growth of the degree of the colored Jones polynomial.

For the sake of exposition we assume that the maximal degree of the colored Jones polynomial is given by a quadratic polynomial $\delta(N) = sN^2 + tN + u$. This actually happens for B-adequate knots (as will be seen below). In general however the degree is a quadratic quasi-polynomial [6].

Since $s$ is the coefficient of the quadratic part of $\delta$ we see it is the unique cluster point of the set

$$\left\{ \frac{\maxdeg J_N(K; v)}{N^2} | N \in \mathbb{N} \right\}$$

mentioned in the slope conjecture. In our terminology the numbers $s$ is the unique $1 \times 1$ slope matrix of the function $\maxdeg J_N(K; v)$.

What we would like to show now is that there exists an edge of the Newton polygon of the A-polynomial with slope $s$. We will however settle for something slightly weaker. Let $P\hat{A}$ be the projection of the Newton polytope of $A$ onto the $L, M$ plane.

**Proposition 1.** [8] There exists an edge of the projection $P\hat{A}$ with slope $s$.

**Proof.** Let us write the quantum A-polynomial as

$$\hat{A}_K(v; M, L) = \sum_{(i, j, k) \in A} v^k M^j L^i$$

where $A$ is some finite set. The basic idea is that since $\hat{A}_K(v; M, L)J(N) = 0$ for all $N$ there must be a lot of cancellation going on. More concretely this means that for each $N$ there must be two terms of maximal degree in $v$. Write $\mu(i, j, k)$ to be the max degree in $v$ of $(v^k M^j L^i)(N)$. We have

$$\mu(i, j, k) = k + 2jN\delta(N + i) = sN^2 + (2si - 2j + t)N + u + k$$

For each $N$ there must be $(i_0, j_0, k_0) \neq (i_1, j_1, k_1) \in A$ such that $\mu(i_0, j_0, k_0) = \mu(i_1, j_1, k_1)$ so we conclude $2si_0 - 2j_0 = 2si_1 - 2j_1$. If $i_0 = i_1$ then $j_0 = j_1$ but then also $k_0 = k_1$ contrary to our assumption so $i_0 \neq i_1$. We can therefore divide and obtain

$$s = \frac{j_1 - j_0}{i_1 - i_0}$$

It remains to check that the line between $(i_0, j_0)$ and $(i_1, j_1)$ actually bounds the projected Newton polygon $P\hat{A}(L, M)$. This is true because
for $N$ sufficiently large there can not be any term $(i, j, k)$ with $2si - 2j > 2si_0 - 2j_0$. Such a term would necessarily have a greater degree than the assumed maximal degree.

Of course it is often the case that $\hat{P}A$ is actually equal to the Newton polygon of $f(M)A(L, M)$ for some polynomial factor $f(M)$. Since the slopes of $f(M)A(L, M)$ are equal to the slopes of $A(L, M)$ the slope conjecture is then valid. It would be interesting to see what happens in cases where $\hat{P}A$ is actually greater than the classical Newton polygon.

We end this section by giving a simple and explicit example for the right hand trefoil depicted in the figure.

Figure 1: The right hand trefoil knot with together with the projection of the Newton polygon $\hat{P}A_K$. The power of $t$ is indicated by the number next to the dot corresponding to the term.

If we define the quantum integer $[k] = \frac{2k}{v^2 - v^{-2}}$ then the colored Jones polynomial for the right hand trefoil is computed to be [13]:

$$J_N(3_1; v) = \sum_{k=0}^{N-1} (-1)^{N-k-1} [2k + 1]v^{6(N^2 - k^2 - k - 1)}$$

its recursion relation is given by the quantum A-polynomial $\hat{A}_{3_1}(v, L, M) =$

$$(v^{-4}M^{10} - M^6)L^2 - (v^{-2}M^{10} + v^{18} - v^{10}M^6 - v^{14}M^4)L + v^{16} - v^4M^4$$

So we have the recursion relation $\hat{A}_{3_1}(v, L, M)J_N(3_1; v) = 0$. The simplest instance of this relation is the $N = 0$ case. Here $M$ acts as the identity so since $J_0(3_1; v) = 0$ and $J_1(3_1; v) = 1$ we can read off the value for $J_2(3_1; v)$:

$$(v^{-4} - 1)J_2(3_1; v) - (v^{-2} + v^{18} - v^{10} - v^{14}) = 0$$

Next to check the AJ-conjecture we set $v = 1$ and factor to find

$\hat{A}_{3_1}(1, L, M) = (M^4 - 1)(L - 1)(1 + M^6L) = (M^4 - 1)A_{3_1}(L, M)$
as predicted by the AJ-conjecture.

We have drawn the projection \( N \hat{A}_3 \) of the Newton polytope onto the \((L, M)\) plane in the figure. For each term we have indicated the power of \( v \) corresponding to it. Notice that in this case the projection equals the Newton polygon of \((M^4 - 1)A_3(L, M)\).

It is not hard to read the maximal degree from the formula.

\[
\text{maxdeg } J_N(31; v) = 6N^2
\]

The number \( 6 = 2c^+ \) is the slope of the lower right edge of the hexagon and also the slope of one of the checkerboard surfaces.

The minimal degree (or equivalently the maximal degree of the mirror image) grows linearly, indicating a slope of 0. This can also be observed in the top or bottom edge of the hexagon. It is the slope of the other checkerboard surface.

4 Proof of the slope conjecture for \( B \)-adequate links

For the terminology of adequate links we refer to \[3\]. Let \( D \) be a \( B \)-adequate link diagram whose all \( B \)-resolution gives rise to \( v_B \) state circles. \( c^+_{ij} \) and \( c^-_{ij} \) are the number of positive and negative crossings between \( D_i \) and \( D_j \). The total number of such crossings is \( c_{ij} \) and the writhe is \( w_j = c^+_{jj} - c^-_{jj} \).

If we denote by \( D^N \) the \( N \)-parallel of the diagram (component \( j \) gets replaced by \( j \) parallel copies) then

\[
\text{maxdeg } J_N(L; v) = \sum_j w_j(N_j^2 - 1) + \text{maxdeg } \langle D^{N-1} \rangle(A^{-1})
\]

Next \( \text{maxdeg } \langle D^{N-1} \rangle(A^{-1}) = -\text{mindeg } \langle D^{N-1} \rangle(A) \) so using the well known estimate for adequate diagrams \[15\] we get:

\[
|\sum_{i \leq j} (N_i - 1)(N_j - 1)c_{ij} + \text{mindeg } \langle D^{N-1} \rangle(A)| \leq 2v_B \max N_j
\]

We can now show that the matrix \( Q \) defined below is a slope matrix in the sense of Definition 1.

\[
Q_{ij} = \begin{cases} 
\frac{1}{2}c_{ij} & \text{if } i \neq j \\
2c^+_{ii} & \text{if } i = j 
\end{cases}
\]

For this we take the infinite subsets from the definition to be \( \mathbb{N} \) and choose a constant \( C > 2v_B \). We will need to prove that

\[
|\text{maxdeg } J_N(L) - NQN^+| < C \max N_j
\]

But this follows directly from the above computations.

In order to prove Theorem 4 we now need to find an essential surface whose slope at the \( j \)-th component equals \( \sum_i Q_{ij} = \frac{1}{2} \sum_{i \neq j} c_{ij} + 2c^+_{jj} \).
Following [3] we consider the surface corresponding to the all B-state. This surface is essential according to Ozawa [16], see also [4] Theorem 3.19. The slope is found by calculating the linking number between the $j$-th component $D_j$ and a curve following it along the surface. For every positive crossing between $D_j$ and another component $D_i$ we find a contribution of 2. For negative such crossings we get a contribution of 0. For any type of crossing between $D_j$ and another component $D_i$ we obtain a contribution $\frac{1}{2}$. In total one thus gets a slope of $2c_{ij}^+ + \frac{1}{2}\sum_{i \neq j} c_{ij}$ as required.

5 The case for torus links

We consider the $(r, s)$-torus link $T^r_s$ with $r, s \in \mathbb{Z}$ and $s \geq 1$ to be the closure of the braid $(\sigma_1 \sigma_2 \cdots \sigma_{s-1})^r$ in $S^3$. In case $r < 0$ the standard braid diagram is B-adequate. If follows from the formula of the Jones polynomial below that the case $r > 0$ is not B-adequate and hence provides additional evidence to our version of the Slope conjecture for links.

The link $T^r_s$ has $g = \gcd(r, s) > 0$ components. If we define $a = r/g$ and $b = s/g$ then the colored Jones polynomial colored by $N = (N_1, N_2, \cdots, N_g)$ is given by the formula [17]

$$J_N(T^r_s) = v^{ab(|N|^2 - g)} \sum_{k=-|N|-g}^{|N|-g-1} \binom{|N|-g}{k} \left(\frac{a}{N}\right) v^{-a(bk+2)[bk+1]}$$

where the summation variable $k$ takes steps of two and we used the notation $|N| = N_1 + N_2 + \cdots + N_g$ and $|N|^2 = N_1^2 + N_2^2 + \cdots N_g^2$ and finally $\binom{N}{N_i}$ is the coefficient of $v^{2k}$ in $\prod_{j=1}^g |N_j|^{-1}$.

For $r < 0$ we see that the maximal degree is given by the term $k = (|N|-g)$. This term has maxdegree $-ab(|N|^2 - g) + ab(|N|-g)^2 + 2a(|N|-g) + 2b(|N|-g) = ab \sum_{i, j} N_i N_j + O(|N|)$. Hence the unique slope matrix $Q$ is given by $Q_{ij} = ab(1 - \delta_{ij})$. This is in agreement with Theorem 1 applied to the standard braid diagram. Hence the state-surface provides the surface with the correct slopes $ab(g - 1)$.

The case $r > 0$ is more interesting. The formula for the maximal degree is not a quadratic polynomial in $N$ but is piecewise quadratic depending on the parity of $|N| - g$. This already shows that such links cannot be B-adequate since then the degree would be a quadratic polynomial in $N$.

More specifically if $|N| - g$ is even then maximal degree is given by the term $k = 0$ and equals $ab(|N|^2 - g)$. Care has to be taken with the case $b = 1$ since there the term $k = -2$ contributes with the same degree but luckily with a different coefficient: For $g > 1$ and $N_j > 1$ we have $\binom{0}{N} \neq -\binom{g}{2}N$.

Next we look at the case $|N| - g$ odd. The term $k = -1$ contributes the maximal degree which now equals $ab(|N|^2 - g) + ab - 2a + 2b - 4$, except in the cases $ab \leq 2$. By the symmetry between $a$ and $b$ we may assume that $b = 1$ in which case the $k = -1$ term vanishes. The $k = 1$ term then contributes $ab(|N|^2 - g) - ab - 2a + 2b$.

In either case the correct slope matrix is $Q = abI$. To complete the slope conjecture we need to find an incompressible surface bounding the
link whose boundary slopes are all $ab$. Consider the canonical annuli defined by taking the complement of the $(r, s)$ torus link viewed as sitting on the torus surface. This surface is certainly essential and each component is seen to have slope $ab$ as required. This proves Theorem 2.

With a bit more work one may be able to prove the slope conjecture for all zero-volume links as these are very similar to torus links. Their colored Jones polynomials can be readily computed by repeatedly cabling and taking connected sums, see [17].

6 Further directions

In the knot case we have seen that the word 'slope' refers to three distinct notions. Slopes of surfaces in the complement, the slope of an edge of the Newton polygon of the A-polynomial and the growth rate of the degree of the colored Jones polynomial.

Our slope conjecture for links suggests further research in each of these three directions.

Slopes of surfaces in link complements are not yet well understood although there has been work on the two-bridge knot case [12]. One wonders if there a way to extract these slopes from the tropical geometry of the A-ideal that replaces the A-polynomial in the link case. This is natural since one expects an AJ-type conjecture for links.

On the colored Jones side one wonders whether the degree is still a (multivariate) quadratic quasipolynomial as in the knot case [6]. At least for $B$-adequate links and the torus links we considered this is true. Such a question brings us back to a possible link version of the AJ conjecture. An important first step towards such a conjecture has been set for two-bridge links in [13].

It would also be interesting to go beyond the slope conjecture as in [10] and consider stabilization properties for links. At least for alternating links their (multivariate) heads and tails and beyond can readily be computed.

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