Structured Regularization of Functional Map Computations

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Shape Matching

Point-based methods
- [Bronstein et al. 2006],
- [Huang et. al 2008]...

Parameterization-based methods
- [Lipman and Funkhouser 2009]
- [Aigerman et al. 2017]...

Optimal transport
- [Solomon et al. 2016]
- [Mandad et al. 2017]...

Functional maps
- [Ovsjanikov et al. 2012]
- [Ezuz and Ben-Chen 2017]...
- ...

Source

Target
Functional map pipeline

Eigenfunctions of Laplace–Beltrami Operator

Helmholtz equation
\[ \Delta_S f = \lambda f \]

Shape \( S \)

\[ 0 = \lambda_1^S \leq \lambda_2^S \leq \lambda_3^S \leq \cdots \leq \lambda_i^S \leq \cdots \leq \lambda_k^S \]
Functional map pipeline

Function space basis

Shape $S$

$\phi_1^S$, $\phi_2^S$, $\phi_3^S$, ... $\phi_i^S$, $\phi_k^S$

Function $f$

$f \approx a_1 \phi_1^S + a_2 \phi_2^S + ... + a_i \phi_i^S + ... + a_k \phi_k^S = \Phi^S a$
Functional map pipeline

Functional map definition

$S_1 \quad \Phi^{S_1} \quad f \quad \Phi^{S_1} a \approx f \
S_2 \quad \Phi^{S_2} \quad g \quad \Phi^{S_2} b \approx g$

$Ca = b$

functional map: the matrix $C$ that transports the coefficients from $\Phi^{S_1}$ to $\Phi^{S_2}$
Functional map pipeline

\[ a = (\Phi^{S_1})^\dagger f \]

\[ \hat{g} = \Phi^{S_2} b \]
Functional map pipeline

\[ a = (\Phi^{S_1})^+ f \]

\[ \hat{g} = \Phi^{S_2} b \]
Functional map pipeline

\[ C_{12}^* = \arg\min_C \|CA - B\|_F^2 \]

\[ + w_1 \|C\Delta_1 - \Delta_2 C\|_F^2 \]

\[ + w_2 \|C\Omega_1^{\text{multi}} - \Omega_2^{\text{multi}} C\|_F^2 \]

\[ + w_3 \|C\Omega_1^{\text{orient}} - \Omega_2^{\text{orient}} C\|_F^2 \]

\[ + \ldots \]

Descriptor preservation
[OBCS*12]

Laplacian commutativity
[OBCS*12]

Multiplicative operators
[NO17]

Orientation preservation
[RPWO18]
Outline

- Laplacian commutativity – widely used
- **Drawbacks** of the standard Laplacian commutativity
  - Unbounded in the smooth setting
  - Not aligned with the ground-truth functional map
- Propose the **resolvent** Laplacian commutativity
  - Bounded operator
  - Better aligned
- Quantitative results
  - Better **stability**
  - Better **accuracy**
Reformulate the Laplacian–Commutativity term

\[ E(C) = \|C \Delta_1 - \Delta_2 C\|_F^2 \]

\[ = \|C \text{diag}(\Lambda_1) - \text{diag}(\Lambda_2) C\|_F^2 \]

\[ = \sum_{(i,j)} M_{ij} C_{ij}^2 \]

where \( M_{ij} = \left( \lambda_j^{S_1} - \lambda_i^{S_2} \right)^2 \)
Applications of the Laplacian commutativity

“Image Co-Segmentation via Consistent Functional Maps”
Fan Wang, Qixing Huang, Leonidas J. Guibas

(a) (b) (c) (d)

Figure 1: Overview of the proposed framework. (a) the original image and its super-pixel representation; (b) the first few Laplacian eigenvectors; (c) functional maps between pairs of images; (d) the resulting segmentation.
Applications of the Laplacian commutativity

“Partial Functional Correspondence”
E. Rodolà, L. Cosmo, M.M. Bronstein, A. Torsello, D. Cremers

$$\rho_{\text{corr}}(C) = \sum_{ij} W_{ij} C_{ij}^2 + \cdots$$
Drawbacks of the Laplacian commutativity

- Unboundedness
  - in the full LB basis (of smooth manifolds)
    \[ \| C_{12} \Delta_1 - \Delta_2 C_{12} \|^2 \rightarrow \infty \]
- Structure misalignment
Unboundedness Example

Spectrum of torus and sphere with unit area

$\lambda_k$

$\|M_{\text{standard}}\|_F^2$ v.s. increasing size of $M_{\text{standard}}$

$\|M_{\text{standard}}\|_F^2$

size of $M$

Torus
Sphere
Unboundedness Example

\[ S_2: \Delta_2 = c\Delta_1 \]
\[ c \neq 1 \]

\[ S_1: \Delta_1 \]

\[ \|C_{12}\Delta_1 - \Delta_2 C_{12}\|^2 = (c - 1)^2 \|\Delta_1\|^2_F \rightarrow \infty \]
Structure misalignment

Mask $M_{\text{standard}}$

where $M_{ij} = (\lambda_j^{S_1} - \lambda_i^{S_2})^2$

$(C_{\text{ground\_truth}})^2$

Funnel-shape
Our solution

- **Boundedness**: $\Delta \rightarrow \text{resolvent of } \Delta$
- **Structure alignment**: $\Delta \rightarrow \Delta^Y$
Resolvent operator

Definition

Let $A$ be a possibly unbounded linear operator (with some technical assumption), the resolvent of $A$ at $\mu$ is defined as

$$R_\mu(A) = (A - \mu I)^{-1}$$

- $\mu$ is a complex number
- $R_\mu(A)$ is defined for all $\mu$ NOT in the spectrum of $A$

$R_{a+ib}(\Delta)$ is well-defined for any $(a + ib)$ NOT in the non-negative real line (which contains the spectrum of $\Delta$)
Important tool in operator theory

- **Spectral theory**: used in the definition of spectrum
- **Unbounded self-adjoint operators**: norm–resolvent convergence $d(A, B) = \|R_{\mu}(A) - R_{\mu}(B)\|$
**Theorem 1 (Bounded Resolvent Commutativity)** Let $C_{12}$ be a bounded functional map. Then in the operator norm,

$$\|C_{12}R(\Delta_1^\nu) - R(\Delta_2^\nu)C_{12}\|_{op}^2 < \infty$$
Bounded resolvent Laplacian–Commutativity

The graph plots the size of $M$ against the squared Frobenius norm of $M_{\text{standard}}$ and $M_{\text{resolvent}}$. As the size of $M$ increases, the norm of $M_{\text{standard}}$ and $M_{\text{resolvent}}$ also increases, showing the relationship between the size of the matrix and its norm.
Bounded resolvent Laplacian–Commutativity

- \( \Delta \rightarrow \) standard Laplacian commutator
- \( R_{a+ib}(\Delta^\gamma) \): well-defined and bounded
  - Introduce \( \gamma \) to tune the structure of the mask
  - Our resolvent Laplacian commutator

\[
E(C_{12}) = \| C_{12} \Delta_1 - \Delta_2 C_{12} \|_F^2 = \| C_{12} R(\Delta_1^\gamma) - R(\Delta_2^\gamma) C_{12} \|_F^2
\]
**Resolvent mask**

- $\Delta$ has eigenvalues $\lambda_k$
- $R_i(\Delta^{1/2})$ has eigenvalues $\sqrt{\lambda_k}$

Mask $M_{\text{standard}}$

\[
 M_{ij} = \left(\lambda_j^{S_1} - \lambda_i^{S_2}\right)^2
\]

Real part

\[
 M_{ij}^{\text{Re}} = \left(\frac{\sqrt{\lambda_j^{S_1}}}{\lambda_j^{S_1} + 1} - \frac{\sqrt{\lambda_i^{S_2}}}{\lambda_i^{S_2} + 1}\right)^2
\]

Imaginary part

\[
 M_{ij}^{\text{Im}} = \left(\frac{1}{\lambda_j^{S_1} + 1} - \frac{1}{\lambda_i^{S_2} + 1}\right)^2
\]

* Def: $R_\mu(A) = (A - \mu I)^{-1}$
Resolvent mask

\[ \| C_{12} R(\Delta^2_1) - R(\Delta^2_2) C_{12} \|_F^2 = \sum_{i,j} M_{ij} C_{12}^2 \]

Mask \( M_{\text{resolvent}} \)

where \( M_{ij} = M_{ij}^{\text{Re}} + M_{ij}^{\text{Im}} \)

Funnel-shape

\((C_{\text{ground\_truth}})^2\)
Mask reformulation of the resolvent commutativity

\[ E(C_{12}) = \| C_{12} R(\Delta_1^\gamma) - R(\Delta_2^\gamma) C_{12} \|_F^2 = \sum_{i,j} M_{ij} C_{12}^2 \]

\( \gamma = 0.25 \)  \( \gamma = 0.5 \)  \( \gamma = 0.75 \)  \( \gamma = 1 \)
Penalty mask v.s. ground-truth functional map

Standard mask  Slanted mask  Resolvent mask  Mean squared ground-truth

“Partial Functional Correspondences”  Rodolà et al

\( \gamma = 0.5 \)
Results: **Stability** (example)

Given one pair of descriptors, compute a $k \times k$ functional map with $k^2$ variables!
Results: Stability (summary)

FAUST

per-vertex measure

Average error

\( k \)

standard
slanted
ours
Results: **Accuracy** (example)

Given one pair of descriptors
Compute a $100 \times 100$ functional map

Standard  Slanted  Resolvent  Ground-truth
Results: **Accuracy** (summary)

![Graph showing accuracy results for TOSCA](image)

- **standard (1):** 213.0
- **slanted (2):** 190.6
- **ours (3):** 125.7
- **(1) + ICP:** 156.5
- **(2) + ICP:** 163.2
- **(3) + ICP:** 90.2
- **(1) + BCICP:** 81.8
- **(2) + BCICP:** 125.3
- **(3) + BCICP:** 61.5
Results: Correlation (fMap penalty v.s. pMap accuracy)

![Graph showing correlation between mask penalty and average geodesic error (direct measure)]
Results: **Stability** under remeshing and coarsening

\[ n_v = 6890 \quad n_v = 200 \quad n_v = 300 \quad n_v = 500 \quad n_v = 1000 \quad n_v = 3000 \quad n_v = 5000 \quad n_v = 6890 \]
Summary

- Shape matching – functional map pipeline
- Laplacian commutativity – widely used
- Drawbacks of the standard Laplacian commutativity
  - Unbounded in the smooth setting
  - Not aligned with the ground-truth functional map
- Propose the resolvent Laplacian commutativity
  - Bounded operator
  - Aligned with the funnel shape
- Results
  - Better accuracy
  - Better stability
Thanks for your attention

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Lemma 2. Let $\Delta_1$ and $\Delta_2$ be Laplacians on compact, connected, oriented surfaces $M_1$ and $M_2$, respectively. Let $C_{12}: L_2(M_1) \to L_2(M_2)$ be a bounded operator. If $\gamma > \frac{1}{2}$, then:

$$\|C_{12} R_\mu (\Delta_1^\gamma) - R_\mu (\Delta_2^\gamma) C_{12}\|_{HS}^2 < \infty$$

Where $\mu$ is any complex number not on the non-negative real line.
Reformulate the Laplacian–Commutativity term

\[ E(C_{12}) = \|C_{12}\Delta_1 - \Delta_2 C_{12}\|_F^2 \]

\[ = \|C_{12}\text{diag}(\Lambda_1) - \text{diag}(\Lambda_2)C_{12}\|_F^2 \]

\[ = \|C_{12} \otimes (1_{k_2} \Lambda_1^T) - (\Lambda_2 1_{k_1}^T) \otimes C_{12}\|_F^2 \]

\[ = \|(1_{k_2} \Lambda_1^T - \Lambda_2 1_{k_1}^T) \otimes C_{12}\|_F^2 \]

\[ = \sum_{(i,j)} M \otimes (C_{12})^2 \]

Note: \( \otimes \) is the entry–wise matrix multiplication
Results: **Stability** (summary)

FAUST

**per-vertex measure**

- Standard
- Slanted
- Ours

FAUST

**direct measure**

- Initialization
- With ICP
Given one pair of descriptors
Compute a 100×100 functional map
Corresponding point-wise map
Unboundedness Example

Unit Sphere  
Unit Torus

\[ \lambda_k \]

| Torus | Sphere | Weyl Estimate |
|-------|--------|---------------|

| 0 | 20 | 40 | 60 | 80 | 100 |
|---|----|----|----|----|-----|
| 0 | 500 | 1000 |

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REFERENCES

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Unbounded standard Laplacian–Commutativity

\[ \|M_{\text{standard}}\|_F^2 \text{ w.r.t. increasing size of } M_{\text{standard}} \]
Definition 1 (Resolvent) Let $A$ be a closed operator on some Hilbert space. Let $\rho(A)$ be the set of all complex numbers $\mu$ such that $R_\mu(A) = (A - \mu I)^{-1}$ is defined and bounded.

$\rho(A)$: the resolvent set of operator $A$

$R_\mu(A)$: the resolvent operator of $A$ at $\mu$

- Given Laplace–Beltrami operator $\Delta$
- Define $R_{a+ib}(\Delta^\gamma)$, the resolvent operator of $\Delta^\gamma$ at $(a + bi)$
  - Parameters $\gamma = \frac{1}{2}, a = 0, b = 1$
  - $R_{a+ib}(\Delta^\gamma)$ is well-defined and bounded for any $(a + ib)$ not in the non-negative real line (where the spectra of $\Delta^\gamma$ lies in)