SINGULAR SET OF A LEVI-FLAT HYPERSURFACE IS LEVI-FLAT

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Abstract. We study the singular set of a singular Levi-flat real-analytic hypersurface. We prove that the singular set of such a hypersurface is Levi-flat in the appropriate sense. We also show that if the singular set is small enough, then the Levi-foliation extends to a singular codimension one holomorphic foliation of a neighborhood of the hypersurface.

1. Introduction

A real smooth hypersurface $H$ in a complex manifold is said to be Levi-flat if the Levi-form vanishes identically, or in other words if it is pseudoconvex from both sides. Levi-flat hypersurfaces occur naturally, for example as invariant sets of holomorphic foliations. A real-analytic nonsingular Levi-flat hypersurface is locally biholomorphic to a hypersurface of the form $\{\text{Im } z_1 = 0\}$, and is therefore foliated by complex hypersurfaces (called the Levi-foliation). The definition of Levi-flat can be naturally extended to CR submanifolds of higher codimensions by requiring that the Levi-form vanishes identically. A real-analytic CR manifold is Levi-flat in this sense if in suitable local coordinates we can write its defining equations as $\text{Im } z_1 = \cdots = \text{Im } z_j = 0$ and $z_{j+1} = \cdots = z_k = 0$ for some $j$ and $k$ (where we interpret $j = 0$ and $j = k$ in the obvious sense). With this terminology we consider complex manifolds to be Levi-flat.

In this article, we consider singular Levi-flat real-analytic subvarieties. Local questions about singular Levi-flat hypersurfaces have been previously studied by Bedford [3], Burns and Gong [6], Fernández-Pérez [11], and the author [15, 16]. Real-algebraic singular Levi-flat hypersurfaces in complex projective space when written in homogeneous coordinates are real-algebraic Levi-flat complex cones and hence their classification is a local question as well, see [17]. A natural and well studied question is how to divide the projective space into pseudoconvex domains. A well known theorem of Lins Neto [18] says that a Levi-flat hypersurface in projective space is necessarily singular in dimension 3 and higher. We therefore need to understand the singular set of Levi-flat hypersurfaces. See the books [2, 9, 14] for the basic language and background.

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Theorem 1.1. Let \( U \subset \mathbb{C}^N \) be an open set and let \( H \subset U \) be a (closed) Levi-flat real-hypervariety. Then the singular set \( (\overline{H^*} \cap U)_s \) is Levi-flat near points where it is a CR real-analytic submanifold.

Furthermore, if \( (\overline{H^*} \cap U)_s \) is a generic submanifold, then \( (\overline{H^*} \cap U)_s \) is a generic Levi-flat submanifold of dimension \( 2N - 2 \).

A generic real submanifold \( M \subset \mathbb{C}^N \) is a submanifold with real defining equations \( r_1(z, \bar{z}) = \cdots = r_k(z, \bar{z}) = 0 \) such that \( \partial r_1, \ldots, \partial r_k \) are linearly independent. Here \( \partial r = \sum \frac{\partial r_k}{\partial z_k} \) refers to the part of the differential in the holomorphic variables. In particular, a generic submanifold is not contained in any proper complex variety.

The theorem is optimal in the sense that given simply the hypothesis that \( H^* \) is Levi-flat we cannot conclude that \( H_s \) is Levi-flat; there could be a lower dimensional component of \( H \) which need not be Levi-flat.

Since a semianalytic set is always contained in a real subvariety of the same dimension, the result also classifies singularities of semianalytic Levi-flat hypersurfaces.

Burns and Gong [6] construct many examples where the singularity is a complex variety. For example, \( \{ z \in \mathbb{C}^N : \text{Im}(z_1^2 + \cdots + z_k^2) = 0 \} \) is a Levi-flat real-hypervariety with \( \mathbb{C}^{N-k} \) as the singular set.

On the other hand, \( \{ z : (\text{Im} z_1)(\text{Im} z_2) = 0 \} \) is a Levi-flat real-hypervariety with a generic Levi-flat singular set \( \{ z : \text{Im} z_1 = \text{Im} z_2 = 0 \} \). It is possible to construct an irreducible Levi-flat real-hypervariety with a generic singular set. See Brunella [5] for an example.

We only study \( (\overline{H^*} \cap U)_s \) near points where it is a real-analytic CR submanifold (a real-analytic submanifold is CR on an open dense set). It is possible that \( (\overline{H^*} \cap U)_s \) is not a CR submanifold. For example the Levi-flat real-hypervariety \( \{ z : (\text{Re}(z_2 - z_1^2))(\text{Im} z_2) = 0 \} \) is a union of two nonsingular Levi-flat hypersurfaces whose intersection is not a CR submanifold at the origin.

As with all real-analytic varieties, the singular set \( H_s \) is not necessarily equal to \( H \setminus H^* \) even if \( H \) is irreducible. See [17] and [5] for examples of such Whitney-umbrella-type Levi-flat hypervarieties. The “umbrella handle” in those examples is also generic Levi-flat or complex analytic. The methods used in this present article only give information on \( \overline{H^*} \cap U \). It is not known if an “umbrella handle” of an irreducible \( H \) is necessarily Levi-flat. Note that while we know that \( (\overline{H^*} \cap U)_s \subset H_s \), the inclusion could be proper even if the singular set is contained in \( \overline{H^*} \) as points of \( H_s \) may in fact be intersections of nonsingular points of \( \overline{H^*} \cap U \) with \( H \setminus \overline{H^*} \).

Burns and Gong [6], and the author [15, 17] also studied Levi-flat real-hypervarieties defined by \( \text{Im} f = 0 \) for a holomorphic or a meromorphic function \( f \). Such Levi-flat real-hypervarieties have a complex analytic singular set, but it turns out that not every Levi-flat hypersurface can be obtained this way. If a meromorphic function defined on a neighborhood of \( H \) is constant on the leaves of \( H^* \), then the Levi-foliation extends to a possibly singular codimension one holomorphic foliation of a neighborhood of \( H \). That is, near each point of \( H \) there exists a nontrivial holomorphic one form \( \omega \) that is completely integrable \( (\omega \wedge d\omega = 0) \) and such that the leaves of the Levi-foliation of \( H^* \) are integral manifolds of \( \omega \) (tangent space of each leaf is annihilated by \( \omega \)). While near points of \( H^* \), the foliation always extends, it is not true that every Levi-flat real-hypervariety is such that the foliation extends near singular points, even if \( H \) is irreducible. Brunella [5] proved that the foliation does extend after lifting to the cotangent bundle.
In the proof of Theorem 1.1 we must find sufficient conditions for the foliation of $H^*$ to extend. Besides proving what is needed for Theorem 1.1 we have the following theorem that is of independent interest. Recently Cerveau and Lins Neto [8] proved a similar result. By $H$ being leaf-degenerate at $p \in H$, we mean that there are infinitely many distinct germs of complex hypervarieties $(X, p) \subset (H, p)$, see §6 for a more precise definition.

**Theorem 1.2.** Let $U \subset \mathbb{C}^N$ be an open set and let $H \subset U$ be a Levi-flat real-hypervariety that is irreducible as a germ at $p \in \overline{H} \cap U$. If either

(i) $\dim H_s = 2N - 4$ and $H$ is not leaf-degenerate at $p$, or

(ii) $\dim H_s < 2N - 4$,

then there exists a neighborhood $U'$ of $p$, and a nontrivial holomorphic one form $\omega$ defined in $U'$, such that $\omega \wedge d\omega = 0$ and such that the leaves of the Levi-foliation of $H^* \cap U'$ are integral submanifolds of $\omega$. In other words, near $p$ the Levi-foliation extends to a possibly singular codimension one holomorphic foliation.

A primary tool in the proofs is Lemma 5.4, which says that through every point $p \in \overline{H} \cap U$ for a Levi-flat real-hypervariety $H$, there exists a complex hypersurface $W$ such that $W \subset \overline{H} \cap U$. This $W$ is generally a branch of the Segre variety of $H$ at $p$, unless the Segre variety is degenerate at $p$. This lemma also implies that all sides of $\overline{H} \cap U$ are pseudoconvex. Hence a real-hypervariety $H$ is Levi-flat if and only if $\overline{H} \cap U$ is pseudoconvex from all sides.

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2. CR submanifolds

Background for CR geometry is taken from the books [2,9,14]. For background on complex varieties see the book [20].

Let $M \subset \mathbb{C}^N$ be a real-analytic submanifold (not necessarily closed) of dimension $n$. We consider the complexified tangent space $\mathbb{C} \otimes T_p M$. The tangent vectors of the form $\sum_{j=1}^{N} a_j \frac{\partial}{\partial z_j}$ tangent to $M$ are called the CR vectors. If the space of CR vectors at $p$, called $T^{0,1}_p M$, has constant dimension at all points of $M$, the submanifold is said to be a CR submanifold. The complex dimension of $T^{0,1}_p M$ is called the CR dimension of $M$. If a CR submanifold is not contained in a proper complex analytic subvariety, it is a generic submanifold. In fact a generic submanifold is automatically CR.

Let $\text{Orb}_p(M)$ denote the local CR orbit of $M$ at $p$, that is, the integral manifold of the distribution of CR vector fields and all of their commutators. For real-analytic $M$ the CR orbit is guaranteed to exist by a theorem of Nagano, and $\text{Orb}_p(M)$ is the germ of a CR submanifold of $M$ through $p$ of smallest dimension that has the same CR dimension as $M$ (see [2]).

If $\text{Orb}_p(M) = (M, p)$ as germs, then $M$ is said to be minimal at $p$. If a real-analytic submanifold is minimal at one point, then it is minimal outside a real-analytic subvariety (again see [2]).

For a connected real-analytic CR submanifold $M$, we find that $\text{Orb}_p(M)$ attains a maximal dimension for $p$ in a dense open subset of $M$. Near a point where the dimension of $\text{Orb}_p(M)$ is maximal we have the following well known theorem.
Theorem 2.1 (see Baouendi-Ebenfelt-Rothschild [1]). Let $M \subset \mathbb{C}^N$ be a real-analytic CR submanifold. Let $p \in M$ be such that $\text{Orb}_p(M)$ is of maximal dimension. Then there are coordinates $(z, w, w', w'') \in \mathbb{C}^n \times \mathbb{C}^{d-q} \times \mathbb{C}^q \times \mathbb{C}^k = \mathbb{C}^N$, vanishing at $p$, where $k$ denotes the complex dimension of the intrinsic complexification of $M$ of such that near $p$ $\phi$ where $H := \{\phi(\bar{z}, w, w') = 0\}$ with respect to $\phi$. Moreover, the local CR orbit of the point $(z, w, w', w'') = (0, 0, s', 0)$, for $s' \in \mathbb{R}^q$, is given by

$$\begin{align*}
\text{Im } w &= \varphi(z, \bar{z}, \text{Re } w, \text{Re } w'), \\
\text{Im } w' &= 0, \\
w'' &= 0,
\end{align*}$$

where $\varphi$ is a real valued real-analytic function with $\varphi(z, 0, s, s') \equiv 0$. Moreover, the local CR orbit of the point $(z, w, w', w'') = (0, 0, s', 0)$, for $s' \in \mathbb{R}^q$, is given by

$$\begin{align*}
\text{Im } w &= \varphi(z, \bar{z}, \text{Re } w, s'), \\
w' &= s', \\
w'' &= 0.
\end{align*}$$

A CR submanifold $M$ where $\text{Orb}_p(M)$ is of maximal dimension is Levi-flat if and only if $\text{Orb}_p(M)$ is a complex manifold, that is when $q = d$. This definition is the same as in the introduction and also includes complex manifolds.

Let $M$ be a CR submanifold. A function $f : M \to \mathbb{C}$ such that $Lf = 0$ for every $L \in T^01M$ is called a CR function. For example, a restriction to $M$ of a holomorphic function defined in a neighborhood of $M$ is a CR function. On the other hand, if $M$ and $f$ are both real-analytic, then $f$ extends to holomorphic function defined on a neighborhood of $M$.

3. Segre varieties

Let $U \subset \mathbb{C}^N$ be a connected open set, and write $^*U = \{z : \bar{z} \in U\}$. Let $H \subset U$ be defined by $r(z, \bar{z}) = 0$ and suppose that $r$ can be complexified (by complexifying its power series) as a function $r(z, \bar{w})$ on $U \times ^*U$. Let $p \in H$.

Definition 3.1. We write

$$\Sigma_p(U, r) := \{z \in U : r(z, \bar{p}) = 0\}. \quad (3)$$

We call $\Sigma_p(U, r)$ the Segre variety of $H$ at $p$ with respect to $r$.

We need a short lemma that is proved in [6] that says that a germ of a real-analytic function is irreducible if and only if its complexification is irreducible.

Lemma 3.2. If $\rho$ is an irreducible germ of a real-analytic function near 0 in $\mathbb{C}^N$, and $H := \{z : \rho(z, \bar{z}) = 0\}$ has dimension $2N - 1$, then for any neighborhood $U$ of 0, there is a smaller neighborhood $U' \subset U$ of 0, such that if $\hat{\rho}$ is any real-analytic function on $U$ that vanishes on an open set of $H^* \cap U'$, then $\rho$ divides $\hat{\rho}$ on $U'$. Further, $\rho$ is irreducible as a germ of a holomorphic function near origin in $\mathbb{C}^{2N}$.

The variety $\Sigma_p(U, r)$ depends on both $U$ and $r$. However, it is possible to talk uniquely about a germ $\Sigma_p(H)$ not depending on $U$ and $r$. First, we note that the ideal $I_p(H)$ of germs at $p$ of real-analytic functions vanishing on $H$ is generated by some real-analytic germ $r$. Let us take a small enough connected neighborhood $U$ of $p$ such that $r$ complexifies to $U \times ^*U$ and such that $\{r(z, \bar{p}) = 0\}$ contains only components passing through $p$. If $\varphi$ is another
real-analytic germ defining the ideal \( I_p(H) \), then \( \varphi = \alpha r \) where \( \alpha(p, \bar{p}) \neq 0 \). It is then easy to see that as germs at \( p \) we have \( \{ r(z, \bar{p}) = 0 \} = \{ \varphi(z, \bar{p}) = 0 \} \). Therefore there is a well defined germ of a complex variety at \( p \). Denote by \((\Sigma_p(U, r), p)\) the germ of \( \Sigma_p(U, r) \) at \( p \).

**Definition 3.3.** Define the germ \( \Sigma_p(H) \) as the germ \((\Sigma_p(U, r), p)\) for \( U \) small enough and \( r \) as given above.

That is, for each point, we can pick a small enough neighborhood and a defining function \( r \) such that \( \Sigma_p(H) \) is well defined. We have proved above that for any \( U \) and \( r \) we have, as germs at \( p \),

\[
\Sigma_p(H) \subset (\Sigma_p(U, r), p). \tag{4}
\]

The following proposition is classical and not hard to prove by complexification.

**Proposition 3.4.** Let \( U \subset \mathbb{C}^N \) be an open set, \( S \subset U \) be a real-analytic subvariety. Suppose that \( r: U \to \mathbb{R} \) is real-analytic, complexifies to \( U \times \ast U \), vanishes on \( S \), and suppose \( W \subset U \) is a complex subvariety such that \( W \subset S \). Then for \( p \in W \) we have \( W \subset \Sigma_p(U, r) \).

### 4. Degenerate singularities

In the sequel, we say \( H \subset \mathbb{C}^N \) is a local real-hypervariety to mean that \( H \) is a closed subvariety of some open set \( U \subset \mathbb{C}^N \). Also instead of writing \( \overline{H^r} \cap U \) we simply use \( H^r_{\text{rel}} \) to mean the relative closure of \( H^r \) in \( U \) (or equivalently the closure in the subspace topology on \( H^r \)).

**Definition 4.1.** Let \( H \subset \mathbb{C}^N \) be a Levi-flat local real-hypervariety. A point \( p \in H \) is said to be a Segre-degenerate singularity if \( \Sigma_p(H) \) is of dimension \( N \), that is, \( \Sigma_p(H) = (\mathbb{C}^N, p) \).

In other words, \( p \) is a degenerate singularity of \( H \) if \( z \mapsto r(z, \bar{p}) \) is identically zero for every local defining function of \( H \) at \( p \).

Suppose that \((V, p) \subset (H, p)\) is a germ of a complex subvariety. By Proposition 3.4 \((V, p) \subset \Sigma_p(H)\). As a nonsingular Levi-flat hypersurface contains a unique nonsingular complex analytic hypersurface through every point, we obtain the following well-known result.

**Proposition 4.2.** Let \( H \subset \mathbb{C}^N \) be a Levi-flat real analytic manifold of dimension \( 2N - 1 \). Then \( \Sigma_p(H) \subset (H, p) \) for every \( p \) and \( \Sigma_p(H) \) is nonsingular.

The proposition implies that only singular points can be Segre-degenerate. In fact the set of Segre-degenerate singularities must be small.

**Proposition 4.3.** Let \( H \subset \mathbb{C}^N \) be a Levi-flat local real-hypervariety. The set \( S \subset H \) of Segre-degenerate singularities is contained in a complex subvariety of (complex) dimension \( N - 2 \) or less.

**Proof.** Fix a point \( p \in S \) and take a defining function \( r \) for \( H \) in some neighborhood \( U \) of \( p \). Let us suppose that \( r \) complexifies to \( U \times \ast U \). We can assume that \( U \) is a polydisc. As \( r \) is real then \( x \in \Sigma_q(U, r) \) implies \( y \in \Sigma_x(U, r) \). Hence, \( p \in \Sigma_q(U, r) \) for every \( q \in U \). Take the set

\[
S_r = \bigcap_{q \in U} \Sigma_q(U, r). \tag{5}
\]

As \( \Sigma_q(U, r) \) must be proper subvariety for most \( q \) (otherwise \( r \) would be identically zero), \( S_r \) is a proper complex subvariety of \( U \). In fact we obtain that \( S_r \subset H \) because if \( r(z, \bar{q}) \) is zero

for all \( q \), then in particular \( r(z, \bar{z}) = 0 \). Obviously we also have \( S \subset S_r \). We simply need to show that \( S_r \) must not be of dimension \( N - 1 \).

Let us suppose that \( S_r \) contains a branch \( X \) of dimension \( N - 1 \). We assume that \( p \in X \). As \( U \) is a polydisc we can choose a a defining function \( f \) for \( X \) in \( U \), such that \( f \) generates the ideal \( I_U(X) \) of functions holomorphic on \( U \) that vanish on \( X \). As we have that \( f \mid r(\cdot, \bar{q}) \) and \( \bar{f} \mid r(q, \bar{\cdot}) \) for all \( q \in U \), (6)

we get that \( |f(z)|^2 \) divides \( r(z, \bar{z}) \).

We can assume that \( r \) is not divisible by any \( |f(z)|^2 \). If \( H \) had a complex component we could replace the factor \( |f(z)|^2 \) by for example \( (\text{Re} \ f(z))^2 + (\text{Im} \ 2f(z))^2 \). Therefore, \( S_r \) must be of dimension \( N - 2 \) or lower.

The set of Segre-degenerate singularities is also closed. In fact, we have proved that for every given defining function the set of Segre-degenerate singularities with respect to that defining function must be a complex subvariety.

**Proposition 4.4.** Let \( H \subset \mathbb{C}^N \) be a Levi-flat local real-hypervariety. Then the set \( S \) of Segre-degenerate singularities is closed.

In fact, when \( r \) is a defining function for \( H \) near \( p \) that complexifies to \( U \times U \) for some neighborhood \( U \) of \( p \), then the set

\[
S_r := \{ q \in U : \dim \Sigma_q(U, r) = N \}
\]

(7)

is a complex subvariety, and \( S_r \subset H \).

**Proof.** The proposition follows at once from the proof of Proposition 4.3 once we notice that the two definitions of the set \( S_r \) agree. \( \square \)

A useful corollary of this result is that if \( p \) is not a Segre-degenerate singularity then we can fix a neighborhood \( U \) of \( p \) and a defining function \( r \) such that \( H \) is not a Segre-degenerate singularity with respect to \( r \) at any point of \( U \).

5. LEAVES AT SINGULAR POINTS

We need the following well known result. See Diederich and Fornæss [10] (the claim in section 6).

**Lemma 5.1** (Diederich-Fornæss). Let \( S \subset \mathbb{C}^N \) be a local real-analytic subvariety. For every \( p \in S \), there exists a neighborhood \( U \) of \( p \) such that for every \( q \in U \) and every germ of a complex variety \( (V, q) \subset (S, q) \), there exists a (closed) complex subvariety \( W \subset U \) such that \( (V, q) \subset (W, q) \) and such that \( W \subset S \cap U \).

This lemma has an interesting and useful corollary that was pointed out to the author by Xianghong Gong.

**Corollary 5.2.** If \( X \subset \mathbb{C}^N \) is a local real-analytic subvariety such that for every \( p \in X_{\text{reg}} \) there exists a neighborhood \( U \) of \( p \) such that \( X \cap U \) is a complex manifold. Then \( X \) is a local complex analytic subvariety.

By \( X_{\text{reg}} \) we mean the set of points near which \( X \) is a real-analytic manifold (of any dimension).
Proof. Take $q \in X$ be a singular point. By considering the local complex subvariety $X_{\text{reg}}$ and appealing to Lemma 5.1, there exists a small neighborhood $U$ of $q$ and a complex subvariety $X' \subset U$ such that $X_{\text{reg}} \cap U \subset X' \subset X \cap U$. As $X_{\text{reg}}^{\text{rel}} = X$ we are done. \[
abla\]

We also need the following lemma of Fornæss. The proof is given in [13], Theorem 6.23. The statement we need is stronger, though more technical, and follows from minor modification of the proof in [13]. We reproduce the proof here with the necessary modifications.

Lemma 5.3 (Fornæss). Let $S \subset \mathbb{C}^N$ be a local real-analytic subvariety. Suppose that $W_k \subset S$ is a sequence of local complex subvarieties with $\dim W_k \geq m$. If $p \in S$ is a cluster point of this sequence, then there exists a neighborhood $U$ of $p$, a subsequence $\{W_{k_j}\}$, with $p$ still as a cluster point, and a complex subvariety $W \subset S \cap U$ with $\dim W \geq m$, such that $W$ contains the set $C$ of the cluster points (in $U$) of $\{W_{k_j} \cap U\}$. Furthermore, no such subvariety $W$ of dimension less than $m$ exists.

Proof. Let $U$ be a neighborhood of $p$ such that all $W_k \cap U$ extend to a closed complex subvariety of $U$ of dimension at least $m$, so assume that $W_k$ are closed subvarieties of $U$. We can also assume that the defining equation $r(z, \bar{z})$ complexifies to $U \times \mathbb{C}$. Let $p^{(1)}$ be a cluster point of $\{W_k\}$. We pass to a subsequence to find $p_k^{(1)} \in W_k$ such that $\lim p_k^{(1)} = p^{(1)}$. We proceed inductively. Let $C_n$ be the set of cluster points (in $U$) of the sequence $\{W_k\}$ at the $n$th step. Let $d$ be the supremum of the distance of a point $q \in C_n$ to the set $P_n = \{p_1^{(1)}, \ldots, p_{n-1}^{(n-1)}\}$. We choose $p_n^{(n)}$ to be the point of $C_n$ that is of distance at least $\frac{n}{n+1} d$ from $P_n$. We again pass to a subsequence of $\{W_k\}$ and choose $p_k^{(n)} \in W_k$ such that $\lim p_k^{(n)} = p^{(n)}$. Using diagonalization we obtain a subsequence $\{W_k\}$ such for each $j$ we have $p_k^{(j)} \in W_k$ and $\lim p_k^{(j)} = p^{(j)}$. The set $\{p^{(j)}\}$ is dense in the set $C$ of limit points of $\{W_k\}$. As $p_k^{(n)}, p_k^{(j)} \in W_k$ we have that $r(p_k^{(n)}, p_k^{(j)}) = 0$ by Proposition 3.4. Taking limits and using the density of $\{p^{(j)}\}$ in $C$, we have that $r(z, \bar{w}) = 0$ for all $z, w \in C$. Define closed complex subvarieties $W', W \subset U$ by

\begin{equation}
W' := \bigcap_{q \in C} \Sigma_q(U, r) \quad \text{and} \quad W := \bigcap_{q \in W'} \Sigma_q(U, r). \tag{8}
\end{equation}

If $q \in W'$ and $c \in C$, then $r(q, c) = 0$ and hence by reality of $r$, $r(c, q) = 0$. Therefore $C \subset W \subset W'$. Furthermore $r(z, \bar{w}) = 0$ for all $z \in W$ and so $W \subset S$.

Let us show that $W$ must be of (complex) dimension at least $m$. Suppose that $W$ is of dimension $d$. Pick a point $q \in C \cap W_{\text{reg}}$. If such a point does not exist, we then we have $C \subset W_{\text{reg}}$ and we could have taken $W_{\text{reg}}$ instead of $W$. We can assume that in a small neighborhood of $q$ we have local holomorphic coordinates such that $q$ is the origin and $W$ is given by $z_{d+1} = \cdots z_N = 0$. We can assume that $W$ and $W_k$ are closed in a neighborhood of the closure of the unit polydisc $\Delta$. We have that for large $k$ we have that if $z \in W_k \cap \Delta$, then $|z| < 1/2$ for $j = d + 1, \ldots, N$. Therefore, the projection of $W_k \cap \Delta$ onto $W \cap \Delta$ must be proper. Hence $d \geq m$. \[
\]

We require the following result. A somewhat weaker version of this fact was proved in [6], in the case the point is not a Segre-degenerate singularity, and not concluding that $W \subset \overline{\Pi}^{\text{rel}}$. The conclusion that $W \subset \overline{\Pi}^{\text{rel}}$ was also proved for a special case in [17].
Lemma 5.4. Let $H \subset \mathbb{C}^N$ be a Levi-flat local real-hypervariety. Suppose that $p \in \overline{H^{rel}}$, then there exists a neighborhood $U$ of $p$ and an irreducible complex subvariety $W \subset U$ of dimension $N - 1$ such that $W \subset \overline{H^{rel}}$ and $p \in W$.

Proof. Let $U$ be a neighborhood of $p$ as in Lemma 5.1.

We take a sequence $q_k \to p$, $q_k \in H^*$. For each $q_k$ we apply Proposition 4.2 and Lemma 5.1 to find a complex subvariety $W_k \subset U$ of dimension $N - 1$ such that $q_k \in W_k$ and $W_k \subset H$. We can also assume that $W_k \subset \overline{H^{rel}}$. That is because there can be at most finitely many $W_k$ such that $W_k \not\subset \overline{H^{rel}}$. By Lemma 5.3 we find a subsequence (calling it again $\{W_k\}$) and a complex subvariety $W \subset H$ of dimension $N - 1$ that contains all the cluster points of $\{W_k\}$ in $U$.

If $H$ has no branch of dimension $2N - 2$, which is a complex variety at some point, then we are done.

The set $C$ of cluster points of $\{W_k\}$ is a subset of $\overline{H^{rel}}$. By Lemma 5.3 the set $C$ of cluster points of $\{W_k\}$ cannot be contained in the set $S$ of Segre-degenerate singularities of $H$, because $S$ would be contained in a complex subvariety of dimension $N - 2$ or less.

Let us move to a point $q \in C \setminus S$. By Proposition 3.3 all germs of complex subvarieties of dimension $N - 1$ through $q$ contained in $H$ must be subsets (and hence branches) of the Segre variety $\Sigma_q(U', r)$ for some neighborhood $U'$ of $q$, which is a proper subvariety. After perhaps a linear change of variables we can assume that $U'$ is small enough such that we can apply Weierstrass preparation theorem on $r$ with respect the $z_N$ variable to obtain a new defining function $\tilde{r}$

$$\tilde{r}(z, \bar{z}) = z_N^d + \sum_{j=0}^{d-1} p_j(z', \bar{z'}, \bar{z}_N)z_N^j,$$  \hspace{1cm} (9)

where we use the we use the notation $z = (z_1, \ldots, z_N) = (z', z_N)$. We know that $W_k \cap U'$ are contained in $H$ and therefore for a sequence $q^{(k)} \in W_k$

$$W_k \cap U' \subset \left\{ z \in U' : z_N^d + \sum_{j=0}^{d-1} p_j(z', q^{(k)}((z', q^{(k)})z_N^j) \right\} = \Sigma_q(U', \tilde{r}).$$ \hspace{1cm} (10)

That means that $W_k \cap U'$ is multigraph of the holomorphic function $f_k : V' \to \mathbb{C}^{d}_{sym}$ for some neighborhood $V' \subset \mathbb{C}^{N-1}$. Here $\mathbb{C}^{d}_{sym}$ is the $d$th symmetric power and the multigraph is the set $\{(z, w) : w \in f_k(z)\}$. For more information on complex varieties as multigraphs of holomorphic mappings into the symmetric spaces see [20].

The functions $f_k$ are bounded and hence there exists a convergent subsequence, these converge to some $f : V' \to \mathbb{C}^{d}_{sym}$. Let us call $W'$ the multigraph of $f$. As $W_k \cap U' \subset \overline{H^{rel}} \cap U'$ then $W'$ is contained in $\overline{H^{rel}} \cap U'$. In fact the set of cluster points of $W_k \cap U'$ is in fact $W'$ so $W' \supset C \cap U'$. If $C \setminus W'$ is nonempty, we could repeat the procedure to get another branch. We only need to repeat the procedure finitely many times as $W'$ is of dimension $N - 1$ and therefore $C$ cannot be contained a complex subvariety of larger dimension. Therefore we can assume that $W' = C \cap U'$.

Hence $C \setminus S$ is a closed complex subvariety of $U \setminus S$. As $S$ is a subset of a complex variety of dimension $N - 2$, we can use the Remmert-Stein theorem to extend $C$ to a closed convex subvariety $W'' = C \setminus S^{rel} \subset \overline{H^{rel}}$ of dimension $N - 1$. It is not hard to see that $C = W''$ because $C$ is closed and all $W_k$ were subsets of $\overline{H^{rel}}$.

\[\square\]
As we said in the introduction, the lemma gives an alternative characterization of singular Levi-flat real-hypervarieties. That is a real-hypervariety \( H \subset U \) is Levi-flat if and only if all the components of \( U \setminus \overline{H}^{rel} \) are pseudoconvex.

The following corollary of Lemma 5.4 was already proved in [6] in the case that \( H \) is not Segre-degenerate.

**Corollary 5.5.** Suppose that \( H \subset \mathbb{C}^N \) is a Levi-flat local real-hypervariety that is reducible as a germ at \( p \in \overline{H}^{rel} \) into two distinct germs of real-hypervarieties \((H_1, p)\) and \((H_2, p)\). Then \( H_s \) is of dimension at least \( 2N - 4 \) and there exists a local complex subvariety \( X \) of (complex) dimension \( N - 2 \) such that \( X \subset (\overline{H}^{rel})_s \subset H_s \).

**Proof.** We can assume that we are working in a neighborhood \( U \) of \( p \) such that \( H_1, H_2, \) and \( H \) are closed subvarieties of \( U \). Let us assume that \( H_1 \) is irreducible as a germ at \( p \) and assume that \( H_2 \) does not have \( H_1 \) as one of its branches. We can thus assume that \( H_1^* \) and \( H_2^* \) do not meet on a set of dimension \( 2N - 1 \). There must exist two irreducible complex hypervarieties \( W_1 \subset \overline{H}^{rel}_1 \) and \( W_2 \subset \overline{H}^{rel}_2 \) through \( p \) by Lemma 5.4. As both \( W_1 \) and \( W_2 \) are in the closure of \( H_1^* \) and \( H_2^* \) and since \( H_1^* \cap H_2^* \) is not of dimension \( 2N - 1 \) it must be that \( W_1 \cap W_2 \) lies in the singularity \((\overline{H}^{rel})_s \). As \( W_1 \cap W_2 \) must be of (complex) dimension at least \( N - 2 \), the corollary follows.

\[ \square \]

6. **Leaf-degenerate points**

**Definition 6.1.** Let \( H \subset \mathbb{C}^N \) be a Levi-flat local real-hypervariety. For \( p \in H \), Lemma 5.1 implies that there exists a neighborhood \( U \) of \( p \) such that each germ of a complex subvariety \((V, p) \subset (H, p) \) extends to a (closed) subvariety of \( U \). Hence define \( \Sigma_p^l(H) \) as the germ at \( p \) of the union of complex subvarieties \( V \) of \( U \) of (complex) dimension \( N - 1 \) such that \( V \subset H \cap U \).

If \( p \in H \) is such that \( \Sigma_p^l(H) \) is not a complex variety of dimension \( N - 1 \) then we say that \( p \) is a leaf-degenerate point.

We show that the above definition of leaf-degenerate points is equivalent to the definition from the introduction.

**Lemma 6.2.** Let \( H \subset \mathbb{C}^N \) be a Levi-flat local real-hypervariety. If \( p \in \overline{H}^{rel} \) then \( \Sigma_p^l(H) \) is nonempty and in fact contains a local complex hypervariety \( W \) such that \( p \in W \subset \overline{H}^{rel} \).

Furthermore, if \( p \in \overline{H}^{rel} \) is a leaf-degenerate point then \( p \) is a Segre-degenerate singularity, and \( \dim(\overline{H}^{rel})_s \geq 2N - 4 \).

**Proof.** Lemma 5.4 says that \( \Sigma_p^l(H) \) is nonempty and contains a complex hypervariety \( W \subset \overline{H}^{rel} \). By Proposition 3.4 we have \( \Sigma_p^l(H) \subset \Sigma_p(H) \). Thus if \( p \) is leaf-degenerate, \( \Sigma_p^l(H) \) must contain infinitely many distinct complex subvarieties of dimension \( N - 1 \), and therefore \( \Sigma_p^l(H) \) must be open.

As \( \Sigma_p^l(H) \) is a union of infinitely many germs of complex subvarieties of dimension \( N - 1 \), suppose that \( V_1 \) and \( V_2 \) are two such subvarieties with no component in common. As there are infinitely many such subvarieties in \( \Sigma_p^l(H) \), and only finitely many complex subvarieties can contain points of \( H \setminus \overline{H}^{rel} \) (\( H \) can have at most finitely many components through \( p \) of dimension less than \( 2N - 1 \)), we can assume that \( V_1 \) and \( V_2 \) are subsets of \( \overline{H}^{rel} \). Then \( V_1 \cap V_2 \) is a complex variety of dimension \( N - 2 \), and we know that \( V_1 \cap V_2 \subset (\overline{H}^{rel})_s \) since at nonsingular points of \( \overline{H}^{rel} \) we have a unique leaf.

\[ \square \]
We can now classify those singular sets which are completely degenerate. That is, those singular sets where we cannot move to a generic point and expect a leaf-nondegenerate points in a neighborhood.

Lemma 6.3. Let $H \subset \mathbb{C}^N$ be a Levi-flat local real-hypervariety, such that $E = (\overline{H^{\ast \text{rel}}})_s$ is a connected real-analytic submanifold. Suppose that the set $S$ of leaf-degenerate points is dense in $E$, then $E$ must be a complex submanifold of dimension $N - 2$.

Proof. Let $p \in S \subset E$. The set $S$ is a subset of the Segre-degenerate singularities (see Lemma 6.2), and the Segre-degenerate singularities must be contained in a complex subvariety of (complex) dimension $N - 2$ or less (see Proposition 4.3). As $S$ is dense in $E$, then $E$ must be of (real) dimension $2N - 4$ or less.

As in the proof of Lemma 6.2, we have two complex subvarieties $V_1$ and $V_2$ of dimension $N - 1$ contained in $\overline{H^{\ast \text{rel}}}$ with no branch in common. As $V_1 \cap V_2 \subset E$ is a complex subvariety of dimension $N - 2$ and $E$ is a connected real-analytic submanifold of dimension at most $2N - 4$, we have $V_1 \cap V_2 = E$. □

7. Generic singular set

In [15] the author proved the following theorem.

Theorem 7.1. Let $H \subset \mathbb{C}^N$ be a Levi-flat local real-hypervariety. Let $M \subset \overline{H^{\ast \text{rel}}}$ be a real-analytic generic submanifold, then $M$ is not a minimal CR submanifold.

In the present paper we extend the proof of this result to prove the following lemma.

Lemma 7.2. Let $H \subset \mathbb{C}^N$ be a Levi-flat local real-hypervariety.

Suppose that $E = (\overline{H^{\ast \text{rel}}})_s$ is a connected generic real-analytic submanifold. Then $E$ is a generic Levi-flat submanifold of dimension $2N - 2$.

Large parts of the following proof already appeared in [15] in the proof of Theorem 7.1. As we need to modify the proof in many places, we simply reproduce the entire proof here with modifications as needed. Some of the techniques used are similar to those of Burns and Gong [6]. First we need the following short lemma, which also appears in [15]. We need a somewhat stronger conclusion than what is stated in [15] and hence we reprove it here.

Lemma 7.3. Let $H_1, H_2 \subset \mathbb{C}^N$, $N \geq 2$, be two connected nonsingular real-analytic Levi-flat hypersurfaces. If $p \in H_1 \cap H_2$, then there exists a neighborhood $U$ of $p$ and a complex subvariety $A \subset U$ such that $(U \cap H_1 \cap H_2) \setminus A$ is a generic Levi-flat submanifold of dimension $2N - 2$.

In fact, if $M = H_1 \cap H_2$ is a connected real-analytic CR submanifold, then $M$ is either a complex hypersurface or a generic Levi-flat submanifold of dimension $2N - 2$.

Proof. Take $U$ to be a small enough neighborhood of $p$ such that $H_1$ and $H_2$ are closed subsets. Change coordinates such that $p = 0$, and in $U$, $H_1$ is given by $\text{Im } z_1 = 0$, and $H_2$ is given by $\text{Im } f = 0$ for a holomorphic function with nonvanishing differential. Define $A$ to be the complex subvariety of $U$ where the differentials $dz_1$ and $df$ linearly dependent. Outside of $A$ we can change coordinates once again and assume that $H_2$ is given by $\text{Im } z_2 = 0$ hence the intersection is generic Levi-flat of dimension $2N - 2$. If the differentials are everywhere dependent, then $f$ depends only on $z_1$ and in this case the intersection is a complex hypersurface. The first part of the lemma is proved.
Thus assume that \( M = H_1 \cap H_2 \) is a connected real-analytic CR submanifold. At \( p \in M \) there exist the complex hypersurfaces \( W_1 \subset H_1 \) and \( W_2 \subset H_2 \) (closed in \( U \)). We note that \( W_1 \cap W_2 \subset H_1 \cap H_2 = M \). If \( W_1 = W_2 \) then \( M \) is a complex hypersurface and we are done. Otherwise \( W_1 \cap W_2 \) is of (complex) dimension \( N - 2 \). As above assume that \( H_1 \) is \( \{ \text{Im } z_1 = 0 \} \) and \( H_2 \) is \( \{ \text{Im } f = 0 \} \), where \( f(p) = 0 \). Unless \( M = H_1 = H_2 \) we can assume that \( \text{Im } f \) is positive somewhere on \( \{ \text{Im } z_1 = 0 \} \), and without loss of generality it can be on \( \{ z_1 = 0 \} \). Unless \( M \) is of dimension \( 2N - 2 \), this would mean that \( \text{Im } f \geq 0 \) on \( \{ \text{Im } z_1 = 0 \} \) and in fact \( \text{Im } f(z) > 0 \) for some \( z \) on \( \{ z_1 = 0 \} \). By the maximum principle this is impossible as \( f(p) = 0 \).

To be able to assume that \( H \) is irreducible we need the following proposition.

**Proposition 7.4.** Let \( H \subset \mathbb{C}^N \) be a Levi-flat local real-hypervariety. Also suppose that \((\overline{H}^s)^\text{rel}_s\) is a connected real-analytic CR submanifold that is not a generic Levi-flat submanifold of dimension \( 2N - 2 \) nor a complex submanifold of (complex) dimension \( N - 1 \).

Then there exists a neighborhood \( U \) of \( p \), and real-hypervariety \( \bar{H} \subset H \cap U \), irreducible as germ at \( p \), such that \((\overline{H}^s)^\text{rel}_s \cap U = (\overline{H}^s)^\text{rel}_s\).

**Proof.** Pick a point \( p \in (\overline{H}^s)^\text{rel}_s \). Take the irreducible components \( H_1, \ldots, H_k \) of \( H \) at \( p \). These are irreducible real-analytic subvarieties of some neighborhood \( U \) of \( p \). We can also assume that \( U \) is such that \((\overline{H}^s)^\text{rel}_s \cap U \) is connected. As there are only finitely many components \( H_j \), and \((\overline{H}^s)^\text{rel}_s \subset (\overline{H}^s)^\text{rel}_s \cap U \), then the manifold \((\overline{H}^s)^\text{rel}_s \cap U \) is either the singularity of some \((\overline{H}^s_j)^\text{rel}_s\) or there must exist a point \( q \in (\overline{H}^s)^\text{rel}_s \cap U \) where \( \overline{H}^s \) is a union of at least two real-analytic submanifolds of dimension \( 2N - 1 \). Applying Lemma 7.3 would violate the hypothesis.

Now we have the tools to prove Lemma 7.2.

**Proof of Lemma 7.2.** We can move to a generic point on \( E = (\overline{H}^s)^\text{rel}_s \). Therefore, we can avoid arbitrary proper complex local subvarieties, as \( E \) is not contained in any such subvariety (\( E \) is a generic submanifold). Therefore, we can assume that \( H \) does not have a Segre-degenerate singularity at \( p \in E \) by applying Proposition 4.3.

By Proposition 7.4 we can assume that \( H \) is irreducible as a germ at \( p \).

We fix a connected neighborhood \( U \) of \( p \), and a defining equation \( r(z, \bar{z}) = 0 \) for \( H \) such that \( r \) complexifies to \( U \times \mathbb{C}U \). We define all Segre varieties using this \( U \) and \( r \) from now on. We also assume that both \( H \) and \( E \) are closed subsets of \( U \). We can assume that \( H \) is irreducible in \( U \), and \( r \) is also irreducible as a holomorphic function of \( z \) and \( \bar{z} \), see Lemma 3.2.

We can assume that \( U \) is small enough to be able to apply Lemma 5.1. Thus we write \( \Sigma'_q(H; U) \) when we are talking about the smallest (closed) complex subvariety of \( U \) contained in \( H \) and containing \( \Sigma'_q(H) \).

By Proposition 4.3 we know we could have picked \( U \) small enough such that \( \dim \Sigma'_q(H) = N - 1 \) for all \( q \in U \).

By Lemma 5.3 \( \Sigma'_p(H; U) \) is nonempty. As \( E \) is generic, no branch of \( \Sigma'_p(H; U) \) contains \( E \). Furthermore because \( E \) is generic, no branch of \( \Sigma'_p(H; U) \) lies in \( E \). Thus there must exist a \( q \) on \( E \) such that \( \Sigma'_q(H; U) \) intersects \( H^* \). We set \( p = q \) and again apply Proposition 7.4 to assume that \( H \) is irreducible at \( p \).
We find a point \( \zeta \in H^* \cap \Sigma'_p(H;U) \). As there is a unique complex hypersurface in \( H \) through \( \zeta \), we know that \( \Sigma'_{\zeta}(H;U) \) contains a branch of \( \Sigma'_p(H;U) \).

We pick \( \zeta \) to lie in a topological component of \( (\Sigma'_p(H;U))_{reg} \cap H^* \) (where \( (\Sigma'_p(H;U))_{reg} \) is the nonsingular part of \( \Sigma'_p(H;U) \)), such that \( p \) is in the closure of this component. Now pick a nonsingular real-analytic curve \( \gamma: (-\epsilon, \epsilon) \to H \) such that \( \gamma(0) = \zeta \), \( \{\gamma\} \subset H^* \), and such that \( \gamma \) is transverse to the Levi-foliation of \( H^* \). The function \( t \mapsto r(p, \gamma(t)) \) is not identically zero. If it were identically zero, then \( \Sigma_p(U, r) \) would contain an open set (the union of representatives of \( \Sigma'_{\gamma(t)}(H) \)) and we assumed that \( H \) was Segre-nondegenerate at \( p \) with respect to \( r \).

We complexify \( t \) in \( r(z, \gamma(t)) \), and apply the Weierstrass preparation theorem to \( r(z, \gamma(t)) \) in some neighborhood \( U' \times D \) where \( p \in U' \subset U \) and \( D \subset \mathbb{C} \). We obtain

\[
F(z, t) = t^m + \sum_{j=0}^{m-1} a_j(z)t^j,
\]

with the same zero set in \( U' \times D \) as \( r(z, \gamma(t)) \). Let \( \Delta \subset U' \) be the discriminant set of \( F \). Then near each point of \( U' \setminus \Delta \) we (locally) have \( m \) holomorphic functions \( \{e_j\}_1^m \) that are solutions of \( F(z, e_j(z)) = 0 \). We wish to study the set where at least one of the \( e_j \) is real-valued, that is \( e_j - \overline{e_j} = 0 \). We define

\[
\varphi(z, \overline{z}) = t^m \prod_{j,k=1}^{m} (e_j(z) - \overline{e_k(z)}) \tag{12}
\]

The expression on the right is real-valued and symmetric both in the \( e_j(z) \) and the \( \overline{e_k(z)} \). Therefore, after complexification we have a well defined function on \( (U' \times U') \setminus (\Delta \times \Delta) \), which extends to be continuous in all of \( U' \times U' \) and thus holomorphic in \( U' \times U' \), see [20]. Thus we have a real-analytic function \( \varphi: U' \to \mathbb{R} \) that is locally outside of \( \Delta \) given by (12).

Let \( K = \{z \in U' : \varphi(z, \overline{z}) = 0\} \). We have that \( \Sigma'_{\gamma(0)}(H;U) \cap U' \) is a subset of \( K \). We cannot immediately conclude that \( \Sigma'_{\gamma(t)}(H;U) \cap U' \) is a subset of \( K \) for \( t \) other than zero as \( \{\gamma\} \) might not lie in \( U' \).

We pick a point

\[
\zeta' \in (\Sigma'_{\gamma(0)}(H;U))_{reg} \cap H^* \cap U'
\]

As \( \zeta \) was in the topological component of \( (\Sigma'_{\gamma(0)}(H;U))_{reg} \cap H^* \) containing \( p \) in its closure, we pick a path from \( \zeta' \) to \( \zeta \) in \( \Sigma'_{\gamma(0)}(H;U) \cap H^* \) and a finite sequence of overlapping neighborhoods \( \{V_j\} \) whose union contains the path and such that inside each \( V_j \), \( H \) is given by \( \text{Im} f_j(z) = 0 \) (for some \( f_j \) holomorphic in \( V_j \)). We assume that \( \zeta' \in V_0 \subset U' \). The Levi-foliation is given by \( f_j(z) = c \) for real \( c \), and these sets must agree on \( V_j \cap V_k \). That is, we have a nonsingular holomorphic codimension one foliation of a neighborhood of the path from \( \zeta' \) to \( \zeta \). Therefore for some small interval of \( t \), we have that the sets \( \Sigma'_{\gamma(t)}(H;U) \cap V_0 \) are nonempty and are in fact equal to sets \( \{z : f_0(z) = c(t)\} \) for some real \( c(t) \).

Thus \( \Sigma'_{\gamma(t)}(H;U) \cap U' \) are subsets of \( K \). Therefore an open set of \( H \) is a subset of \( K \). As \( H \) is irreducible, then \( H \subset K \).

As \( E \) is generic, \( (E \cap U') \setminus \Delta \) is an open dense subset of \( E \cap U' \). Hence at a point \( q \in (E \cap U') \setminus \Delta \) there is a small neighborhood \( U'' \) of \( q \) such that in \( U'' \) \( \varphi \) is given by \( im \prod_{j,k=1}^{m} (e_j(z) - \overline{e_k(z)}) \). As \( e_j(z) - \overline{e_k(z)} \) is pluriharmonic its real and imaginary parts are pluriharmonic, meaning that we can represent them as the imaginary part of a holomorphic
function, that is $e_j(z) - e_k(z) = \text{Im} f_{jk}(z) + i \text{Im} g_{jk}(z)$. Therefore $H \cap U''$ is contained in the zero set of

$$im \prod_{j,k=1}^m (\text{Im} f_{jk}(z) + i \text{Im} g_{jk}(z)). \quad (14)$$

The zero set of each $\text{Im} f_{jk}(z) + i \text{Im} g_{jk}(z)$ is a real-analytic subvariety of real dimension $2N - 1$ or $2N - 2$. Hence, there is some finite set of holomorphic functions $\{h_k\}$ defined in $U''$ such that

$$\text{H}_{\text{red}} \cap U'' \subset \{ z : \prod_k \text{Im} h_k(z) = 0 \}. \quad (15)$$

The set where the differentials of $h_k$ vanish is a complex subvariety of $U''$. As $E$ is generic, there must be a point $q' \in E$ and a neighborhood $U'''$ of $q'$ such that $\text{H}_{\text{red}} \cap U'''$ is contained in the union of finitely many nonsingular real-analytic Levi-flat hypersurfaces. Therefore $\text{H}_{\text{red}} \cap U'''$ itself must be the union of finitely many nonsingular real-analytic Levi-flat hypersurfaces. We apply Lemma 7.3. Outside of a complex analytic subvariety $A$ of $U'''$ we have that $E$ is a dimension $2N - 2$ Levi-flat submanifold. Again as $E$ is generic, $(E \cap U''') \setminus A$ is nonempty. Therefore there exists a point on $E$ where $E$ is Levi-flat dimension $2N - 2$ generic submanifold. As $E$ is a connected generic real-analytic submanifold, then $E$ is a Levi-flat dimension $2N - 2$ generic submanifold at every point.

8. Intersections of Levi-flats with complex manifolds

We need to see what happens to a Levi-flat real-hypervariety when we intersect it with a complex manifold. The following lemma is useful in proving results about Levi-flat hypersurfaces by induction on dimension.

**Lemma 8.1.** Let $H \subset \mathbb{C}^N$ be a Levi-flat local real-hypervariety and let $V \subset \mathbb{C}^N$ be a connected complex submanifold of positive dimension $k$. Suppose that there exists a point $p \in \text{H}_{\text{red}} \cap V$. Then exactly one of the following statements is true.

(i) $H \cap V$ is a complex variety of dimension $k - 1$ and $p$ is a leaf-degenerate point of $H$.

(ii) $H \cap V$ is a real-hypervariety of $V$ (is of dimension $2k - 1$).

(iii) $V \subset H$.

**Proof.** By induction on codimension of $V$ it is enough to consider $V$ of dimension $N - 1$. So let us suppose that $H \cap V$ is a proper subset of $V$ and hence of dimension at most $2N - 3$. By Lemma 5.3, $\Sigma_p(H)$ is nonempty, and contains at least one irreducible complex subvariety $W$ of dimension $N - 1$. If $W$ is not equal to $V$ then $W \cap V$ must be of dimension $N - 2$. Hence $H \cap V$ must be of dimension at least $2N - 4$. We simply need to show that if $H \cap V$ is a complex variety of complex dimension $N - 2$ then $p$ is a leaf-degenerate point.

By restricting to the correct 2 complex dimensional subspace it is enough to consider $N = 2$ with coordinates $(z, w) \in \mathbb{C}^2$ and it is also enough to consider $V = \{ z = 0 \}$ and $p = 0$. Suppose for contradiction that $V \cap H = \{ 0 \}$. Take $V_\epsilon = \{ z = \epsilon \}$, for a small complex $\epsilon$. We note that $V_\epsilon \cap H$ must be compact for small $\epsilon$ as $H$ is a closed subvariety of a neighborhood of the origin. If $V_\epsilon \cap H$ was isolated points (dimension 0) for all small $\epsilon$, then dimension of $H$ would be 2 which would be a contradiction. Thus $V_\epsilon \cap H$ must be of dimension 1 for $\epsilon$ arbitrarily close to 0 ($V_\epsilon \cap H$ cannot be dimension 2 and still compact as then $V_\epsilon$ would be a subset of $H$). Furthermore for $\epsilon$ arbitrary close to zero we must have that $V_\epsilon \cap H^*$ is of dimension 1. Since $V_\epsilon \cap H^*$ is of dimension 1 and $V_\epsilon \cap H$ is compact, there must
be infinitely many distinct leaves of $H^*$ that intersect $V_\epsilon \cap H^*$. As $V_\epsilon \cap H^*$ approaches the origin as $\epsilon$ goes to 0, we see that infinitely many distinct leaves of $H^*$ must have the origin in their closure. By Lemma 5.1 all of those leaves extend to a subvariety of a neighborhood of the origin and the origin must be a leaf-degenerate point.

All three cases are possible. The last two are obvious. For the first case consider the Levi-flat hypersurface $H$ given by $|z|^2 - |w|^2 = 0$. Then the set $V = \{z = 0\}$ intersects $H$ at the origin only. The origin is a leaf-degenerate point where for each $\theta$ we obtain a leaf $\{z = e^{i\theta}w\}$.

9. CR ORBITS OF MANIFOLDS IN LEVI-FLATS

We know that a complex subvariety of $H$ must lie in $\Sigma_p'(H)$, however it is also true that a minimal CR submanifold that lies inside $H$ also lies inside $\Sigma_p'(H)$ as we can prove that its intrinsic complexification does. In particular we prove the following lemma.

**Lemma 9.1.** Let $H \subset \mathbb{C}^N$ be a Levi-flat local real-hypervariety without degenerate singularities. Suppose that $M \subset \overline{H}^*_{rel}$ is a connected real-analytic CR submanifold and $p \in M$ is a point such that $\text{Orb}_p(M)$ is of maximal dimension.

Then $\text{Orb}_p(M) \subset \Sigma_p'(H)$.

**Proof.** First suppose that $M$ is minimal, that is, $\text{Orb}_p(M) = (M, p)$ as germs. If $M \not\subset (\overline{H}^*_{rel})_s$, then we can find a point $q \in M$ near which $\overline{H}^*_{rel}$ is nonsingular. Thus suppose that $H$ is nonsingular. In particular we have $H = \{\text{Im } f = 0\}$ as germs at $q$ for a holomorphic function $f$ defined near $q$. As $M$ is minimal, then $f$ is constant on $M$ (the sets $\{f = c\} \cap M$ define CR submanifolds of same CR dimension as $M$). Thus $f$ is constant on the intrinsic complexification of $M$. Therefore $M$ near $q$ is contained in the leaf of the Levi-foliation of $H^*$. As $M$ is connected, the closure of the leaf must contain $p$. The closure of the leaf that contains $p$ must extend to a neighborhood of $p$ by Lemma 5.1 and therefore as germs at $p$, $(M, p) \subset \Sigma_p'(H)$.

Now suppose that $M \subset (\overline{H}^*_{rel})_s$. As $M$ is minimal, it cannot be generic by Theorem 7.1.

We write coordinates vanishing at $p$ as in Theorem 2.1 $(z, w, w'') \in \mathbb{C}^n \times \mathbb{C}^d \times \mathbb{C}^k$ and define $M$ by

$$\begin{align*}
\text{Im } w &= r(z, \bar{z}, \text{Re } w), \\
2N - 3 &= 0.
\end{align*}$$

(16)

Write $w'' = (w''_1, \ldots, w''_k)$. If $k > 1$, then there is some affine function $L: \mathbb{C}^k \to \mathbb{C}$ such that $H \cap \{Lw'' = 0\}$ is of real dimension strictly less than $2N - 2$ and therefore of dimension $2N - 3$ by Lemma 8.1. Thus the case $k > 1$ is finished by induction on the dimension $N$.

Therefore we are left with the case that $k = 1$ (the intrinsic complexification of $M$ is a complex hypersurface). If $H \cap \{w'' = 0\}$ is of dimension strictly less than $2N - 2$ we are done by induction as above. Therefore assume that $\{w'' = 0\} \subset H$. But then $\{w'' = 0\} \subset \Sigma_p'(H)$ (as germs at $p$) by definition of $\Sigma_p'(H)$ and we are finished.

It is left to deal with the nonminimal case. In this case we use Theorem 2.1 to write $M$ in the coordinates $(z, w, w', w'') \in \mathbb{C}^n \times \mathbb{C}^{d-q} \times \mathbb{C}^q \times \mathbb{C}^k$ such that $M$ is defined by

$$\begin{align*}
\text{Im } w &= r(z, \bar{z}, \text{Re } w, \text{Re } w'), \\
\text{Im } w' &= 0, \\
2N - 3 &= 0.
\end{align*}$$

(17)
Write \( w' = (w'_1, \ldots, w'_p) \). Suppose that \( H \cap \{ w'_1 = 0 \} \) is of dimension strictly less than \( 2N - 2 \). Then \( H \cap \{ w'_1 = 0 \} \) is of dimension \( 2N - 3 \) by Lemma 8.1 and we can finish by induction.

Thus assume that \( \{ w'_1 = 0 \} \subset H \). Then \( \{ w'_1 = 0 \} \subset \Sigma'_p(H) \) (as germs at \( p \)). By Theorem 2.7 we obtain that \( \text{Orb}_p(M) \subset \{ w'_1 = 0 \} \) and we are done. \( \square \)

10. Holomorphic foliations

A possibly singular \textit{holomorphic foliation} \( F \) of codimension one of a complex manifold \( M \) is given by an open covering \( \{ U_i \} \) and a one-form \( \omega_i \) defined in \( U_i \) such that if \( U_i \cap U_k \neq \emptyset \), then \( \omega_i \) and \( \omega_k \) must be proportional at every point of \( U_i \cap U_k \). Furthermore \( \omega_i \) is completely integrable, \( \omega_i \wedge d\omega_i = 0 \). A complex submanifold \( L \subset M \) is called a solution if it satisfies \( \omega_i|_{T_L} = 0 \) (restricted to the tangent space of \( L \)) in each \( U_i \). The points where \( \omega_i \) vanishes are called the singular set of \( F \) and denoted \( \text{sing}(F) \). The set \( M \setminus \text{sing}(F) \) is then a union of immersed complex hypersurfaces called leaves of the foliation. The codimension of the singularity of the foliation can safely be taken to be at least 2, by dividing out the coefficients of the form by any common divisors. See [7,18] for more information on foliations in general.

If \( H \) is a nonsingular real-analytic Levi-flat hypersurface, then the foliation of \( H \) by complex hypersurfaces, the \textit{Levi-foliation}, is a real-analytic foliation with leaves that are complex hypersurfaces. As locally a real-analytic Levi-flat hypersurface can be defined by \( \{ \text{Im } f = 0 \} \) where \( df \neq 0 \), we can see that the Levi-foliation extends as a holomorphic codimension one foliation to a neighborhood of \( H \). It is not hard to see that locally the extended foliation is uniquely determined: if a one-form also \( \omega \) defines an extension of the foliation, then on \( H \) we have \( df = g\omega \) for a nonvanishing real-analytic CR function \( g \). A real-analytic CR function on a real-analytic hypersurface uniquely extends to a holomorphic function on a neighborhood of the hypersurface. As \( df \) and \( \omega \) are proportional, they define the same unique foliation in a neighbourhood. We thus have the following proposition.

**Proposition 10.1.** Let \( H \subset \mathbb{C}^N \) be a real-analytic Levi-flat submanifold of dimension \( 2N - 1 \). Then there exists a nonsingular codimension one holomorphic foliation defined on a neighborhood \( U \) of \( H \) that extends the Levi-foliation of \( H \).

A singular Levi-flat local real-hypervariety \( H \) may have several components of \( H^* \) even if \( H \) is irreducible. We do, however, have the following lemma.

**Lemma 10.2.** Let \( H \subset \mathbb{C}^N \) be an irreducible Levi-flat local real-hypervariety and \( F \) a possibly singular codimension one holomorphic foliation defined on a neighborhood of \( H \). Suppose that there is an open subset \( G \subset H^* \) such that \( F \) extends the Levi-foliation of \( G \). Then \( F \) extends the Levi-foliation of \( H^* \).

**Proof.** By analytic continuation we see that \( F \) extends the foliation of the whole topological component of \( H^* \) that contains \( G \).

Therefore, it is enough to show that if \( H \) is irreducible as a germ at some point \( p \in H \) and \( F \) extends the Levi-foliation of some topological component \( H' \) of \( H^* \) such that \( p \in \overline{H'}^{\text{rel}} \), then \( F \) extends the Levi-foliation of \( H^* \) near \( p \). The global result then follows.

In some small neighborhood \( U \) of \( p \), \( F \) is defined by a 1-form \( \omega \). Suppose that \( r(z, \bar{z}) \) is the defining function for \( H \) in \( U \) and suppose that \( H \cap U \) is irreducible. That \( F \) extends the Levi-foliation of \( H' \cap U \) is the same as saying that \( \partial r \wedge \omega \) vanishes on \( H' \cap U \). As \( H \cap U \) is irreducible, then \( \partial r \wedge \omega \) must vanish on all of \( H \cap U \) and hence the result follows. \( \square \)
The following lemma is proved in [18], although it is not stated as a separate theorem. We state the theorem in a more general setting and so we reprove it here for completeness. A Riemann domain over \( \mathbb{C}^N \) is a path-connected Hausdorff space \( U \) together with a local homeomorphism \( \pi: U \to \mathbb{C}^N \). An envelope of meromorphy of \( U \) is a Riemann domain \( \hat{U} \) such that any meromorphic function on \( U \) extends to a meromorphic function on \( \hat{U} \).

**Lemma 10.3.** Let \( U \) be a connected Riemann domain over \( \mathbb{C}^N \), \( N \geq 2 \), and let \( \hat{U} \) be the envelope of meromorphy of \( U \). Let \( \mathcal{F} \) be a possibly singular codimension one holomorphic foliation on \( U \). Then \( \mathcal{F} \) extends to a possibly singular codimension one holomorphic foliation on \( \hat{U} \).

**Proof.** The foliation \( \mathcal{F} \) is defined locally by completely integrable 1-forms; there exists a covering of \( U \) by open sets \( \{U_i\} \) and 1-forms \( \{\omega_i\} \) such that \( \omega_i = 0 \) define the leaves of \( \mathcal{F} \). When \( U_{\iota\kappa} = U_i \cap U_\kappa \neq \emptyset \), there also exist functions \( \{h_{\iota\kappa}\} \) in \( \mathcal{O}^* (U_{\iota\kappa}) \) such that \( \omega_i = h_{\iota\kappa}\omega_\kappa \) on \( U_{\iota\kappa} \). We can assume that the codimension of the singularity of \( \mathcal{F} \) is 2 or greater.

The covering of \( U \) can be such that \( \pi \) is a homeomorphism of \( U_i \) onto \( \pi(U_i) \) and so we can think of each \( U_i \) as an open subset of \( \mathbb{C}^N \). We write

\[
\omega_i = \sum_{j=1}^{N} g_j^i dz_j. \tag{18}
\]

We note that when \( U_{\iota\kappa} \) is not empty then for all \( j \) we have

\[
g_j^i = h_{\iota\kappa} g_j^\kappa. \tag{19}
\]

As \( \hat{U} \) is connected, it follows that there exists a \( j \) such that for all \( \iota \) we have \( g_j^\iota \neq 0 \). We can suppose that \( j = N \).

For every \( j = 1, \ldots, N-1 \) we have meromorphic functions \( f_j^i = g_j^i / g_j^N \) defined on \( U_i \). By (19) on \( U_{\iota\kappa} \) we have \( f_j^i = f_j^\kappa \) for all \( j = 1, \ldots, N-1 \). As \( U \) is connected, for each \( j = 1, \ldots, N-1 \), there exists a well-defined meromorphic function \( f_j \) on \( U \).

Every meromorphic function on \( \hat{U} \) extends to a meromorphic function on \( \hat{U} \). Thus we have a meromorphic function \( f_j \) on \( \hat{U} \) such that \( f_j = f_j^\iota \) on \( U_i \).

Now we consider the meromorphic 1-form

\[
\eta = dz_N + \sum_{j=1}^{N-1} f_j^i dz_j. \tag{20}
\]

We can cover \( \hat{U} \) by polydiscs \( \{\hat{U}_\kappa\} \). In each \( \hat{U}_\kappa \) we find a nonzero holomorphic function \( \varphi_\kappa \) such that \( \varphi_\kappa \eta \) has only removable singularities. Thus we obtain a 1-form \( \hat{\omega}_\kappa \) on \( \hat{U}_\kappa \) that equals to \( \varphi_\kappa \eta \) where that makes sense, thus \( \hat{\omega}_\kappa \) is proportional to \( \eta \) outside the poles of the \( f_j \), and if \( \hat{U}_\kappa \) intersects \( U_\iota \), then \( \eta \) is proportional to \( \omega_\iota \) on \( U_\iota \) outside of the poles of the \( f_j \). Therefore, \( \{\hat{\omega}_\kappa\} \) extend the foliation on \( \hat{U} \). \( \square \)

We have the following result about extending a foliation of \( H^* \). Let \( U \subset \mathbb{C}^N \) be a (euclidean) Hartogs figure, that is

\[
U = (V' \times \Delta(r)) \cup (V \times (\Delta(r) \setminus \Delta(r'))), \tag{21}
\]

where \( V' \subset V \subset \mathbb{C}^{N-1} \) are two polydiscs and \( \Delta(r) \subset \mathbb{C} \) is a disc of radius \( r \), and \( 0 < r' < r \). By a theorem of Levi (see [12,19]) the envelope of meromorphy of \( U \) is \( \hat{U} = V \times \Delta(r) \). A
Lemma 10.4. Suppose that $H \subset \mathbb{C}^N$ is a Levi-flat local real-hypervariety that is irreducible as germ at $p \in \overline{H^{rel}}^*$. Suppose that there exists a nonsingular complex submanifold $W \subset \overline{H^{rel}}$ of (complex) dimension at least 2, such that there exists a generalized Hartogs figure $K \subset W \setminus (\overline{H^{rel}})^*$ and such that $p \in \hat{K}$.

Then there exists a possibly singular codimension one holomorphic foliation $\mathcal{F}$ extending the foliation of $H$ near $p$.

Proof. Let us take a connected component $H'$ of the nonsingular points $(\overline{H^{rel}})^*_{reg}$ such that $p$ lies in the closure of $H'$, and $H'$ contains the component of $W \setminus (\overline{H^{rel}})^*$ that contains $K$. We define a possibly singular codimension one holomorphic foliation in a neighborhood $V$ of $H'$, see Proposition 10.1.

We can now “fatten” the Hartogs figure $K$ to find a Hartogs figure $K' \subset V$ of dimension $N$. As the envelope of meromorphy of $K'$ is $\hat{K}'$, we extend the foliation $\mathcal{F}$ past $p$ by Lemma 10.3.

We have a possibly singular holomorphic foliation $\mathcal{F}$ of a neighborhood of $p$ that extends the foliation of $H'$. By Lemma 10.2 the foliation in fact agrees with the foliation on all of $H^*$ (near $p$) as $H$ is irreducible at $p$. □

We can now prove Theorem 1.2. That is, if the germ $(H, p)$ is irreducible and $\dim H_s < 2N - 4$ or $p$ is not a leaf-degenerate point and $\dim H_s = 2N - 4$, then the Levi-foliation extends to possibly singular codimension one holomorphic foliation in a neighborhood of $p$.

Proof of Theorem 1.2. We note that for a Riemann domain over $\mathbb{C}^N$, the domain of meromorphy is a Stein manifold (See e.g. Theorem 3.6.6 [12]). Therefore, it must be holomorphically convex.

Suppose that $\dim H_s \leq 2N - 4$. Take a neighborhood $U$ of $p$ in which we can apply Lemma 5.1, and such that $H \cap U$ is irreducible. We can assume that $H$ is closed in $U$.

Let us first suppose that $N = 2$, $H_s = \{p\}$, and $H$ is not leaf-degenerate at $p$. Take an irreducible complex subvariety $W' \subset \overline{H^{rel}}^*$ by Lemma 5.4. Pick the connected topological component $H'$ of $H^*$ such that $W' \setminus \{p\} \subset H'$. Define a holomorphic foliation $\mathcal{F}$ on a neighborhood $\Omega$ of $H'$ by Proposition 10.1.

As $H$ is not leaf-degenerate at $p$, then there exists a sequence of nonsingular leaves $L_j \subset H'$ such that $p$ is a cluster point of this sequence, but such that $p \notin L_j$ for all $j$. By Lemma 5.1, the $L_j$ must be closed in $U$ and they must be nonsingular (as a singular point of $L_j$ would mean a singular point of $H'$). Now consider $L_j$ intersected with a small ball $B$ centered at $p$. Let $K = \bigcup_j (L_j \cap \partial \Omega)$. Note that $K \subset \Omega$, in particular $K$ is a positive distance away from $p$. However, the holomorphic hull $\hat{K}$ (with respect to $\mathcal{O}(\Omega)$) contains the sets $L_j \cap B$, and $p$ is in the cluster set of the $L_j \cap B$. If $p \in \Omega$ then we were already done. If $p \notin \Omega$ then we see that no Riemann domain $\Omega'$ containing $\Omega$ can be Stein unless $p \in \Omega'$. Thus the envelope of meromorphy of $\Omega$ contains $p$ and hence a whole neighborhood of $p$. We finish by applying Lemma 10.3.

When $N > 2$ and $p$ is not leaf-degenerate we proceed similarly.

If $\dim H_s < 2N - 4$, then $H$ is not leaf-degenerate at any point by Lemma 6.2. □
When the foliation extends, we can show that the singular set must be Levi-flat.

**Lemma 10.5.** Suppose that $H \subset \mathbb{C}^N$ is a Levi-flat local real-hypervariety, and $\mathcal{F}$ is a possibly singular codimension one holomorphic foliation extending the Levi-foliation of $H$, then $E = (\overline{H^*})_s$ is Levi-flat wherever $E$ is a CR submanifold.

**Proof.** We can assume that $E$ is a connected real-analytic CR submanifold. Let us suppose for contradiction that $E$ is not a Levi-flat submanifold (that of course also means $E$ is not a complex submanifold).

Take $p \in E$. As we are interested in $(\overline{H^*})_s$, we can without loss of generality assume that all irreducible components of the germ $(H, p)$ are of dimension $2N - 1$. If the foliation is nonsingular at $p$, then it is easy to show that any $2N - 1$ dimensional component of $H$ at $p$ must be locally biholomorphic to $C \times \mathbb{C}^{N-1}$ for a one dimensional real-analytic curve $C \subset \mathbb{C}$. We simply look in a coordinate patch of the foliation where the leaves are given by $\{z_N = c\}$ for a constant $c$, and we note that the leaves of the foliation must agree with leaves of $H^*$. Therefore $E$ must be either empty or a complex hypersurface.

Therefore suppose that the foliation $\mathcal{F}$ is singular at $p \in E$. The singular set $\text{sing}(\mathcal{F})$ of the foliation is a complex subvariety. As noted above we can assume that $\text{sing}(\mathcal{F})$ is of complex dimension $N - 2$ or less and that no point of $\text{sing}(\mathcal{F})$ is a removable singularity.

Since at the nonsingular points $(\overline{H^*})_{\text{reg}}$ the foliation must be nonsingular we see that $\text{sing}(\mathcal{F}) \cap \overline{H^*}$ must be a subset of $E$. As we are assuming that $E$ is not a complex hypersurface, then $E = \text{sing}(\mathcal{F}) \cap \overline{H^*}$, as where $\mathcal{F}$ is nonsingular $E$ would be a complex hypersurface or empty. Thus for any complex hypersurface $W \subset \Sigma_p(H)$ we obtain

$$E \cap W = \text{sing}(\mathcal{F}) \cap W. \quad (22)$$

$\text{sing}(\mathcal{F}) \cap W$ is a complex subvariety. We can assume that $\text{Orb}_p(E)$ is of maximal dimension. By Lemma 9.1 we have that $\text{Orb}_p(E) \subset W$ for some $W$. Thus $\text{Orb}_p(E) = \text{Orb}_p(E \cap W)$ and $\text{Orb}_p(E \cap W) = E \cap W$ as it is complex. Therefore $\text{Orb}_p(E)$ is a complex subvariety and $E$ must be Levi-flat (as $\text{Orb}_p(E)$ was of maximal dimension).

**□**

11. **Proof of the Theorem**

First a technical lemma.

**Lemma 11.1.** Suppose that $V$ is a complex manifold of (complex) dimension 2 or more. Suppose that $M \subset V$ is a real-analytic CR submanifold, which is not Levi-flat (therefore also not complex analytic).

Then there exists a point $p \in M$ and a generalized Hartogs figure $K \subset V \setminus M$ such that $p \in \hat{K}$.

**Proof.** This is a local theorem and hence we can assume that $0 \in M$ and that $V$ is a small neighborhood of the origin in $\mathbb{C}^k$, where we can apply Theorem 2.1 on $M$.

Let $W \subset V$ be a 2 (complex) dimensional complex submanifold through the origin, we look at $M \cap W$. If we can construct the required Hartogs figure in $W$, then we can “fatten” it up to be of dimension $k$.

If the intersection $M \cap W$ is of dimension 0 or 1, then it is not hard to construct the required Hartogs figure with $p$ being the origin. If $M$ is not a generic submanifold, then as
it is not a complex submanifold, then we can find such a $W$ (simply setting all variables except $w_i$ and $w_i'$ to zero).

If $M$ is generic, then if the real codimension of $M$ is 2 or more, we can simply set $z = 0$ and all but 2 of the $w$ or $w'$ to zero and we have that $M \cap W$ is a totally real submanifold of $W$. It is then again not hard to construct the Hartogs figure with $p$ being the origin.

Hence what is left is the case when $M \subset V$ is a hypersurface. As $M$ is not Levi-flat, then the Levi-form of $M$ is not identically zero, then there must exist a point $p$ and an affine complex subspace $W$ of (complex) dimension 2 such, $p \in W$ such that $M \cap W$ is strictly pseudoconvex in $W$ and hence on the strictly pseudoconcave side of $M \cap W$ we can construct the Hartogs figure. □

Let us restate Theorem 1.1 for reader convenience.

**Theorem.** Let $U \subset \mathbb{C}^N$ be an open set and let $H \subset U$ be a (closed) Levi-flat real-hypervariety. Then the singular set $(\overline{H}^s \cap U)_s$ is Levi-flat near points where it is a CR real-analytic submanifold.

Furthermore, if $(\overline{H}^s \cap U)_s$ is a generic submanifold, then $(\overline{H}^s \cap U)_s$ is a generic Levi-flat submanifold of dimension $2N - 2$.

**Proof.** Let $E = (\overline{H}^s_{\text{rel}})_s$.

If $N = 1$, then the theorem has no content and is trivially true. If $N = 2$, then $E$ can be of dimension 1 or 2. Such real subvarieties are automatically Levi-flat near CR points (either totally-real or complex). Hence the theorem is true automatically for $N = 2$. From now on suppose that $N \geq 3$.

We only need to prove that near points where $E$ is a real-analytic CR submanifold it is Levi-flat. Thus we can assume without loss of generality that $E$ is a connected real-analytic CR submanifold.

If $E$ is a generic submanifold of $\mathbb{C}^N$, then by Lemma 7.2 we have that $E$ is a generic Levi-flat submanifold of codimension $2N - 2$ and we are done. We thus suppose that $E$ is not a generic submanifold.

By Lemma 6.3 we have that if the set $S$ of leaf-degenerate points of $H$ is dense in $E$, then $E$ is a complex submanifold of dimension $N - 2$. If the set $S$ is not dense in $E$, we can move to a neighborhood of a generic point of $E$ and assume that no point of $H$ is leaf-degenerate.

Let us suppose for contradiction that $E$ is not a complex nor a Levi-flat submanifold. Furthermore, by moving to a generic point $p$ of $E$ we can assume that $\text{Orb}_p(E)$ is of maximal possible dimension. Near $p$ we can work in a small neighborhood $U$ of $p$ and assume that any germs of complex varieties extend to the whole neighborhood $U$ by Lemma 5.1. Thus as before, we write $\Sigma_q'(H; U)$ when we are talking about the smallest (closed) complex subvariety of $U$ contained in $H$ and containing $\Sigma_q'(H)$. We also simply assume that $H$ and $E$ are closed subsets of $U$.

If $\Sigma_q'(H; U)$ is singular, then the singular set $S$ of $\Sigma_q'(H; U)$ would be a subset of $E$, by Proposition 4.2. If $(S, p) = (E, p)$ as germs, then we are done. We pick another point $q \in \Sigma_q'(H; U) \cap E \setminus S$. We note that $\Sigma_q'(H)$ must contain (as a germ at $q$) a nonsingular complex hypersurface as one of its components. Therefore, either $\Sigma_q'(H)$ is nonsingular as a germ, or it is reducible and has a singularity of complex dimension $N - 2$ which must be contained in $E$. If $\Sigma_q'(H)$ is singular (reducible) then the CR dimension of $E$ is at least $N - 2$ (as it contains the singularity of $\Sigma_q'(H)$). As $\text{Orb}_q(E) \subset \Sigma_q'(H)$, the CR dimension of $E$ is
exactly $N - 2$ and $\text{Orb}_q(E)$ is a complex submanifold of dimension $N - 2$. Because $\text{Orb}_q(E)$ is of maximal dimension, $E$ must be Levi-flat.

Therefore let us assume that the germ $\Sigma'_p(H;U)$ is nonsingular, and in fact we can assume that the germ $\Sigma'_q(H)$ is nonsingular as germ at $q$ for all $q \in \Sigma'_p(H;U)$.

As we are assuming $\text{Orb}_p(E)$ is of maximal dimension, we can pick coordinates vanishing at $p$ as in Theorem 2.1. Furthermore if $E \subset \Sigma'_p(H;U)$, then we can pick coordinates such that $\Sigma'_p(H) = \{ w'_i = 0 \}$ as germs at $p$.

If $E \not\subset \Sigma'_p(H;U)$, then as $\text{Orb}_p(E) \subset \Sigma'_p(H)$ by Lemma 9.1 we can pick coordinates such that $\Sigma'_p(H) = \{ w'_i = 0 \}$ as germs at $p$.

In either case, if $E$ was not Levi-flat, then $E \cap \Sigma'_p(H;U)$ is not Levi-flat. We now appeal to Lemma 11.1 to obtain a Hartogs figure $K$ inside $\Sigma'_p(H;U)$. To do so, we may have needed to perhaps move to yet another point $p' \in E \cap \Sigma'_p(H;U)$. This move is allowed as we are assuming that $\Sigma'_{p'}(H)$ is nonsingular.

As $E$ is a connected real-analytic CR submanifold that is neither generic Levi-flat nor complex analytic, we can apply Proposition 7.4 and assume that $H$ is irreducible at $p$. We can now appeal to Lemma 10.1 to obtain a foliation $\mathcal{F}$ near $p$. Next we appeal to Lemma 10.5 to get a contradiction ($E$ is Levi-flat though we assumed it was not).

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