A symmetry theorem in two-phase heat conductors

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Abstract

We consider the Cauchy problem for the heat diffusion equation in the whole Euclidean space consisting of two media with different constant conductivities, where initially one medium has temperature 0 and the other has temperature 1. Under the assumptions that one medium is bounded and the interface is of class $C^{2,\alpha}$, we show that if the interface is stationary isothermic, then it must be a sphere. The method of moving planes due to Serrin is directly utilized to prove the result.

Key words. heat diffusion equation, two-phase heat conductors, Cauchy problem, stationary isothermic surface, method of moving planes, transmission conditions.

AMS subject classifications. Primary 35K05; Secondary 35K10, 35K15, 35J05, 35J25, 35B06.

1 Introduction

In the previous paper [KS], we considered the Cauchy problem for the heat diffusion equation in the whole Euclidean space consisting of two media with different constant conductivities, where initially one medium has temperature 0 and the other has temperature 1. There, the large time behavior, either stabilization to a constant or oscillation, of temperature was studied. The present paper deals with the case where one medium is bounded and the interface is of class $C^{2,\alpha}$, and introduces an overdetermined problem with the condition that the interface is stationary isothermic.

To be precise, let $\Omega$ consist of a finite number, say $m$, of bounded domains $\{\Omega_j\}$ in $\mathbb{R}^N$ with $N \geq 2$, where each $\partial \Omega_j$ is of class $C^{2,\alpha}$ for some $0 < \alpha < 1$ and $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$. 

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Denote by $\sigma = \sigma(x)$ ($x \in \mathbb{R}^N$) the conductivity distribution of the whole medium given by

$$
\sigma = \begin{cases} 
\sigma_+ & \text{in } \Omega = \bigcup_{j=1}^{m} \Omega_j, \\
\sigma_- & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

(1.1)

where $\sigma_-, \sigma_+$ are positive constants with $\sigma_- \neq \sigma_+$. The diffusion over such multiphase heat conductors has been dealt with also in [Sa1, Sa2, Sa3, CS U, CMS].

We consider the unique bounded solution $u = u(x,t)$ of the Cauchy problem for the heat diffusion equation:

$$
u_t = \text{div}(\sigma \nabla u) \quad \text{in } \mathbb{R}^N \times (0, +\infty) \quad \text{and} \quad u = \chi_{\Omega} \quad \text{on } \mathbb{R}^N \times \{0\},
$$

(1.2)

where $\chi_{\Omega}$ denotes the characteristic function of the set $\Omega$. The maximum principle gives

$$0 < u(x,t) < 1 \quad \text{for every } (x,t) \in \mathbb{R}^N \times (0, +\infty).
$$

(1.3)

Our symmetry theorem is stated as follows.

**Theorem 1.1** If there exists a function $a : (0, +\infty) \to (0, +\infty)$ satisfying

$$u(x,t) = a(t) \quad \text{for every } (x,t) \in \partial \Omega \times (0, +\infty),
$$

(1.4)

then $\Omega$ must be a ball.

If $\partial \Omega$ is of class $C^6$, then Theorem 1.1 can be proved by the method employed in [CMS] Theorem 1.5 with the proof, pp. 335–341], where concentric balls are characterized. The proof there consists of four steps summarized as follows: (i) reduction of (1.2) to elliptic problems by the Laplace-Stieltjes transform $\lambda \int_0^\infty e^{-\lambda t} u(x,t) dt$ for all sufficiently large $\lambda > 0$, (ii) construction of precise barriers based on the formal WKB approximation where the fourth derivatives of the distance function to $\partial \Omega$ together with the assumption (1.4) are used, (iii) showing that the mean curvature of $\partial \Omega$ is constant with the aid of the precise asymptotics as $\lambda \to \infty$ and the transmission conditions on the interface $\partial \Omega$, (iv) Alexandrov’s soap bubble theorem [Al] from which we conclude that $\partial \Omega$ must be a sphere.

The approach of the present paper is different from that in [CMS] and only requires $\partial \Omega$ to be of class $C^{2,\alpha}$ for some $\alpha > 0$. Here the proof consists of two ingredients: (i) reduction to elliptic problems by the Laplace-Stieltjes transform $\lambda \int_0^\infty e^{-\lambda t} u(x,t) dt$ for some $\lambda$, for instance $\lambda = 1$, (ii) the method of moving planes due to Serrin [Se, GNN, R, Si] with the aid of the transmission conditions on $\partial \Omega$. To apply the method of moving planes, the solutions need to be of class $C^2$ up to the interface $\partial \Omega$ from each side, which is guaranteed if $\partial \Omega$ is of class $C^{2,\alpha}$. 

2
2 Introducing a Laplace-Stieltjes transform

Let $u = u(x,t)$ be the unique bounded solution of (1.2) satisfying (1.4). We use the Gaussian bounds for the fundamental solutions of diffusion equations due to Aronson [Ar, Theorem 1, p. 891](see also [FS, p. 328]). Let $g = g(x,\xi,t)$ be the fundamental solution of $u_t = \text{div}(\sigma \nabla u)$. Then there exist two positive constants $\lambda < \Lambda$ such that

$$\lambda t^{-\frac{N}{2}} e^{-\frac{|x-\xi|^2}{\lambda t}} \leq g(x,\xi,t) \leq \Lambda t^{-\frac{N}{2}} e^{-\frac{|x-\xi|^2}{\Lambda t}}$$

(2.1)

for all $(x,t), (\xi,t) \in \mathbb{R}^N \times (0, +\infty)$. Note that $u$ is represented as $u(x,t) = \int_{\Omega} g(x,\xi,t) d\xi$ for $(x,t) \in \mathbb{R}^N \times (0, +\infty)$.

(2.2)

Define the function $v = v(x)$ by

$$v(x) = \int_0^\infty e^{-t} u(x,t) dt \text{ for } x \in \mathbb{R}^N.$$  

(2.3)

With the function $a$ in (1.4), we set $a^* = \int_0^\infty e^{-t} a(t) dt$. Then, (1.3) yields that $0 < a^* < 1$.

Set

$$v^+ = v \text{ for } x \in \overline{\Omega} \text{ and } v^- = v \text{ for } x \in \mathbb{R}^N \setminus \Omega.$$  

(2.4)

Then we observe that

$$a^* < v^+ < 1 \text{ and } -\sigma_+ \Delta v^+ + v^+ = 1 \text{ in } \Omega,$$

(2.5)

$$0 < v^- < a^* \text{ and } -\sigma_- \Delta v^- + v^- = 0 \text{ in } \mathbb{R}^N \setminus \overline{\Omega},$$

(2.6)

$$v^+ = v^- = a^* \text{ and } \sigma_+ \frac{\partial v^+}{\partial \nu} = \sigma_- \frac{\partial v^-}{\partial \nu} \text{ on } \partial \Omega,$$

(2.7)

$$\lim_{|x| \to \infty} v^-(x) = 0.$$  

(2.8)

Here, $\nu$ denotes the outward unit normal vector to $\partial \Omega$, the inequalities in (2.5) and (2.6) follow from the maximum principle, (2.7) expresses the transmission conditions on the interface $\partial \Omega$, and (2.8) follows from (2.1) and (2.2).

3 Proof of Theorem 1.1

Let us apply directly the method of moving planes due to Serrin [Sc, GNN, R, Si] to our problem in order to show that $\Omega$ must be a ball. The point is to apply the method to both the interior $\Omega$ and the exterior $\mathbb{R}^N \setminus \overline{\Omega}$ at the same time. For the method of moving planes
for $\mathbb{R}^N \setminus \overline{\Omega}$, we refer to $[R, Si]$. In this procedure, the supposition that $\Omega$ is not symmetric will lead us to the contradiction that the transmission conditions (2.7) do not hold.

Let $\gamma$ be a unit vector in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, and let $\pi_\lambda$ be the hyperplane $x \cdot \gamma = \lambda$. For large $\lambda$, $\pi_\lambda$ is disjoint from $\overline{\Omega}$; as $\lambda$ decreases, $\pi_\lambda$ intersects $\overline{\Omega}$ and cuts off from $\Omega$ an open cap $\Omega_\lambda = \Omega \cap \{x \in \mathbb{R}^N : x \cdot \gamma > \lambda\}$.

Denote by $\Omega^\lambda$ the reflection of $\Omega_\lambda$ with respect to the plane $\pi_\lambda$. Then, $\Omega^\lambda$ is contained in $\Omega$ at the beginning, and remains in $\Omega$ until one of the following events occurs:

(i) $\Omega^\lambda$ becomes internally tangent to $\partial \Omega$ at some point $p \in \partial \Omega \setminus \pi_\lambda$;

(ii) $\pi_\lambda$ reaches a position where it is orthogonal to $\partial \Omega$ at some point $q \in \partial \Omega \cap \pi_\lambda$ and the direction $\gamma$ is not tangential to $\partial \Omega$ at every point on $\partial \Omega \cap \{x \in \mathbb{R}^N : x \cdot \gamma > \lambda\}$.

Let $\lambda_\ast$ denote the value of $\lambda$ at which either (i) or (ii) occurs. We claim that $\Omega$ is symmetric with respect to $\pi_{\lambda_\ast}$. Suppose that $\Omega$ is not symmetric with respect to $\pi_{\lambda_\ast}$. Denote by $D$ the reflection of $(\mathbb{R}^N \setminus \overline{\Omega}) \cap \{x \in \mathbb{R}^N : x \cdot \gamma > \lambda_\ast\}$ with respect to $\pi_{\lambda_\ast}$. Let $\Sigma$ be the connected component of $(\mathbb{R}^N \setminus \overline{\Omega}) \cap \{x \in \mathbb{R}^N : x \cdot \gamma < \lambda_\ast\}$ whose boundary contains the points $p$ and $q$ in the respective cases (i) and (ii). Since $\Omega^{\lambda_\ast} \subset \Omega$, we notice that

$$\Sigma \subset (\mathbb{R}^N \setminus \overline{\Omega}) \cap \{x \in \mathbb{R}^N : x \cdot \gamma < \lambda_\ast\} \subset D.$$  

Let $x^{\lambda_\ast}$ denote the reflection of a point $x \in \mathbb{R}^N$ with respect to $\pi_{\lambda_\ast}$, namely,

$$x^{\lambda_\ast} = x + 2[\lambda_\ast - (x \cdot \gamma)]\gamma. \quad (3.1)$$

Using the functions $v^\pm$ defined in (2.4), we introduce the functions $w^\pm = w^\pm(x)$ by

$$w^+(x) := v^+(x) - v^+(x^{\lambda_\ast}) \quad \text{for} \quad x \in \overline{\Omega^{\lambda_\ast}},$$

$$w^-(x) := v^-(x) - v^-(x^{\lambda_\ast}) \quad \text{for} \quad x \in \Sigma. \quad (3.2)$$

It then follows from (2.5)–(2.8) that

$$-\sigma_+ \Delta w^+ + w^+ = 0 \quad \text{in} \quad \Omega^{\lambda_\ast} \quad \text{and} \quad w^+ \geq 0 \quad \text{on} \quad \partial \Omega^{\lambda_\ast}, \quad (3.3)$$

$$-\sigma_- \Delta w^- + w^- = 0 \quad \text{in} \quad \Sigma \quad \text{and} \quad w^- \geq 0 \quad \text{on} \quad \partial \Sigma, \quad (3.4)$$

and hence by the maximum principle

$$w^+ \geq 0 \quad \text{in} \quad \Omega^{\lambda_\ast} \quad \text{and} \quad w^- > 0 \quad \text{in} \quad \Sigma. \quad (3.5)$$
Note that $w^+$ can be zero in $\Omega^\lambda$ since some connected component $\Omega_j$ of $\Omega$ can be symmetric with respect to $\pi_{\lambda_*}$ and, in such a case, $w^+ \equiv 0$ in $\Omega_j$. But $w^-$ is strictly positive in $\Sigma$ since $\Omega$ is not symmetric with respect to $\pi_{\lambda_*}$.

Let us first consider the case (i). The first equality in (2.7) yields that $w^+(p) = w^-(p) = 0$. Then, it follows from (3.5) and Hopf’s boundary point lemma that

$$\frac{\partial w^+}{\partial \nu}(p) \leq 0 < \frac{\partial w^-}{\partial \nu}(p), \quad (3.6)$$

where we used the fact that $\nu$ is the outward unit normal vector to $\partial \Omega$ as well as the inward unit normal vector to $\partial \Sigma$. It thus follows from the definition (3.2) of $w^\pm$ that

$$\frac{\partial v^+(x)}{\partial \nu} \bigg|_{x=p} \leq \frac{\partial (v^+(x^{\lambda^*}))}{\partial \nu} \bigg|_{x=p} \quad \text{and} \quad \frac{\partial v^-(x)}{\partial \nu} \bigg|_{x=p} > \frac{\partial (v^-(x^{\lambda^*}))}{\partial \nu} \bigg|_{x=p}. \quad (3.7)$$

Reflection symmetry with respect to the plane $\pi_{\lambda_*}$ yields that

$$\frac{\partial (v^\pm(x^{\lambda^*}))}{\partial \nu} \bigg|_{x=p} = \frac{\partial v^\pm}{\partial \nu}(p^{\lambda^*}). \quad (3.7)$$

Indeed, we observe that

$$\nu(p) \cdot \gamma = -\nu(p^{\lambda^*}) \cdot \gamma \quad \text{and} \quad \nu(p) - (\nu(p) \cdot \gamma)\gamma = \nu(p^{\lambda^*}) - (\nu(p^{\lambda^*}) \cdot \gamma)\gamma,$$

and by using (3.1), we see that

$$\nabla (v^\pm(x^{\lambda^*})) = (\nabla v^\pm)(x^{\lambda^*}) - 2 \left( (\nabla v^\pm)(x^{\lambda^*}) \cdot \gamma \right) \gamma.$$

Then, combing these equalities yields (3.7). It thus follows that

$$\frac{\partial v^+}{\partial \nu}(p) \leq \frac{\partial v^+}{\partial \nu}(p^{\lambda^*}) \quad \text{and} \quad \frac{\partial v^-}{\partial \nu}(p) \geq \frac{\partial v^-}{\partial \nu}(p^{\lambda^*}). \quad (3.8)$$

On the other hand, the second equality in (2.7) shows that

$$\sigma_+ \frac{\partial v^+}{\partial \nu}(p) = \sigma_- \frac{\partial v^-}{\partial \nu}(p) \quad \text{and} \quad \sigma_+ \frac{\partial v^+}{\partial \nu}(p^{\lambda^*}) = \sigma_- \frac{\partial v^-}{\partial \nu}(p^{\lambda^*}),$$

which contradict (3.8).

Let us proceed to the case (ii). As in [Se], by a translation and a rotation of coordinates, we may assume:

$$\gamma = (1, 0, \ldots, 0), \quad q = 0, \quad \lambda_* = 0 \quad \text{and} \quad \nu(q) = (0, \ldots, 0, 1).$$
Since \( \partial \Omega \) is of class \( C^2 \), there exists a \( C^2 \) function \( \varphi : \mathbb{R}^{N-1} \rightarrow \mathbb{R} \) such that in a neighborhood of \( q = 0 \), \( \partial \Omega \) is represented as a graph \( x_N = \varphi(\hat{x}) \) where \( \hat{x} = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \), where

\[
\varphi(0) = 0, \quad \nabla \varphi(0) = 0, \quad \text{and} \quad \nu = \frac{1}{\sqrt{1 + |\nabla \varphi|^2}}(-\nabla \varphi, 1).
\]

Since the event (ii) occurs at \( \lambda = 0 \), we observe that the function \( \frac{\partial \varphi}{\partial x_j}(0, x_2, \ldots, x_{N-1}) \) achieves its local maximum 0 at \( (x_2, \ldots, x_{N-1}) = 0 \in \mathbb{R}^{N-2} \), and hence

\[
\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(0) = 0 \quad \text{for} \quad j = 2, \ldots, N - 1.
\] (3.9)

Notice that

\[
w^\pm(x) = v^\pm(x_1, x_2, \ldots, x_N) - v^\pm(-x_1, x_2, \ldots, x_N),
\] (3.10)

since \( x^{\lambda^*} = (-x_1, x_2, \ldots, x_N) \).

The equalities (2.7) at \((\hat{x}, \varphi(\hat{x}))\) in a neighborhood of \( q = 0 \) are read as

\[
v^\pm = a^*,
\] (3.11)

\[
\sigma_+ \left( - \sum_{k=1}^{N-1} \frac{\partial \varphi}{\partial x_k} \frac{\partial v^+}{\partial x_k} + \frac{\partial v^+}{\partial x_N} \right) = \sigma_- \left( - \sum_{k=1}^{N-1} \frac{\partial \varphi}{\partial x_k} \frac{\partial v^-}{\partial x_k} + \frac{\partial v^-}{\partial x_N} \right).
\] (3.12)

Differentiating (3.11) in \( x_i \) for \( i = 1, \ldots, N - 1 \) yields that at \((\hat{x}, \varphi(\hat{x}))\)

\[
\frac{\partial v^\pm}{\partial x_i} + \frac{\partial v^\pm}{\partial x_N} \frac{\partial \varphi}{\partial x_i} = 0.
\] (3.13)

Then, differentiating (3.13) in \( x_j \) for \( j = 1, \ldots, N - 1 \) yields that at \((\hat{x}, \varphi(\hat{x}))\)

\[
\frac{\partial^2 v^\pm}{\partial x_j \partial x_i} + \frac{\partial^2 v^\pm}{\partial x_N \partial x_i} \frac{\partial \varphi}{\partial x_j} + \frac{\partial^2 v^\pm}{\partial x_N \partial x_j} \frac{\partial \varphi}{\partial x_i} + \frac{\partial^2 v^\pm}{\partial x_N^2} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \frac{\partial v^\pm}{\partial x_N} \frac{\partial^2 \varphi}{\partial x_j \partial x_i} = 0.
\] (3.14)

By letting \( \hat{x} = 0 \) in these equalities, we obtain from (3.9) that

\[
\frac{\partial v^\pm}{\partial x_i}(0) = \frac{\partial^2 v^\pm}{\partial x_1 \partial x_j}(0) = 0 \quad \text{for} \quad i = 1, \ldots, N - 1 \quad \text{and} \quad j = 2, \ldots, N - 1.
\] (3.15)

Next, differentiating (3.12) in \( x_i \) for \( i = 1, \ldots, N - 1 \) and letting \( \hat{x} = 0 \) give

\[
\sigma_+ \frac{\partial^2 v^+}{\partial x_i \partial x_N}(0) = \sigma_- \frac{\partial^2 v^-}{\partial x_i \partial x_N}(0) \quad \text{for} \quad i = 1, \ldots, N - 1.
\] (3.16)

Since the functions \( w^\pm \) are expressed as (3.10), with the aid of (3.15) we have that

\[
w^\pm(0) = \frac{\partial w^\pm}{\partial x_j}(0) = \frac{\partial^2 w^\pm}{\partial x_1 \partial x_j}(0) = 0 \quad \text{for} \quad j = 1, \ldots, N - 1.
\] (3.17)
The relations (3.3)–(3.5) enable us to apply Serrin’s corner point lemma (see [GNN, Lemma S, p. 214] or [R, Serrin’s Corner Lemma, p. 393]) to show that
\[ \frac{\partial^2 w^+}{\partial s_+^2}(0) \geq 0 \quad \text{and} \quad \frac{\partial^2 w^-}{\partial s_-^2}(0) > 0 \quad \text{with} \quad s_\pm = -\gamma \mp \nu = (-1, 0, \ldots, 0, \mp 1), \] (3.18)
where \( \frac{\partial^2 w^\pm}{\partial s_\pm^2} \) denotes the second derivative of \( w^\pm \) in the direction of \( s_\pm \). Note that each of the directions \( s_\pm \) respectively enters \( \Omega^{\lambda_1}, \Sigma \), transversally to both of the hypersurfaces \( \partial \Omega \) and \( \pi_{\lambda_1} \). Thus, we have from (3.10) and (3.17) that
\[ \frac{\partial^2 w^\pm}{\partial s_\pm^2}(0) = \pm 2 \frac{\partial^2 w^\pm}{\partial x_1 \partial x_N}(0) = \pm 4 \frac{\partial^2 v^\pm}{\partial x_1 \partial x_N}(0). \] (3.19)
It then follows from (3.18) that
\[ \frac{\partial^2 v^-}{\partial x_1 \partial x_N}(0) < 0 \leq \frac{\partial^2 v^+}{\partial x_1 \partial x_N}(0), \] (3.20)
which contradicts (3.16) with \( i = 1 \). Thus \( \Omega \) is symmetric with respect to \( \pi_{\lambda_1} \). Since the unit vector \( \gamma \) is arbitrary, \( \Omega \) must be a ball and Theorem 1.1 is proved.

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