NON-HARMONIC CONES ARE HEISENBERG UNIQUENESS PAIRS FOR THE FOURIER TRANSFORM ON $\mathbb{R}^n$

R. K. SRIVASTAVA

Abstract. In this article, we prove that a cone is a Heisenberg uniqueness pair corresponding to sphere as long as the cone does not completely lay on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$. We derive that $(S^2, \text{paraboloid})$ and $(S^2, \text{geodesic of } S_r(o))$ are Heisenberg uniqueness pairs for a class of certain symmetric finite Borel measures in $\mathbb{R}^3$. Further, we correlate the problem of Heisenberg uniqueness pair to the sets of injectivity for the spherical mean operator.

1. Introduction

A Heisenberg uniqueness pair is a pair $(\Gamma, \Lambda)$, where $\Gamma$ is a surface and $\Lambda$ is a subset of $\mathbb{R}^n$ such that any finite Borel measure $\mu$ which is supported on $\Gamma$ and absolutely continuous with respect to the surface measure, whose Fourier transform $\hat{\mu}$ vanishes on $\Lambda$, implies $\mu = 0$.

In general, Heisenberg uniqueness pair (HUP) is a question of asking about the determining properties of a finite Borel measures which is supported on some lower dimensional entities whose Fourier transform too vanish on lower dimensional entities. In fact, the main contrast in the HUP problem to the known results on determining sets for measures [11] is that the set $\Lambda$ has also been considered as a very thin set. In particular, if $\Gamma$ is compact, then $\hat{\mu}$ is real analytic having exponential growth and hence $\hat{\mu}$ can vanishes on a very delicate set. Thus, the HUP problem becomes little easier in this case. However, this problem becomes immensely difficult when the measure is supported on a non-compact entity. Eventually, the HUP problem is a natural invariant to the theme of the uncertainty principle for the Fourier transform.

In addition, the problem of getting the Heisenberg uniqueness pairs for a class of finite measures has also a few significant similarities with a well established result due to M. Benedicks [9]. That is, support of a function in $L^1(\mathbb{R}^n)$ and support of its Fourier transform both can not be of finite measure concurrently. Later, a series of analogous problems to the Beniticks theorem have been investigated in various set ups, including the Heisenberg group and the Euclidean motion groups etc (see [17][21][25]).
However, our main objective in this article is to discuss the concept of HUP, which was first introduced by Hedenmalm and Montes-Rodríguez in 2011. In the article [14] they have shown that (hyperbola, some discrete set) is a HUP in the plane. As a dual problem, they have constructed a weak∗ dense subspace of $L^\infty(\mathbb{R})$. Further, they have also given a complete characterization of the Heisenberg uniqueness pairs corresponding to the union of two parallel lines.

Thereafter, a considerable amount of work has been done on the HUP problem in the plane as well as in the Euclidean spaces.

Some of the Heisenberg uniqueness pairs corresponding to the circle have been independently investigated by N. Lev [16] and P. Sjolin [22] in 2011. In 2012, F. J. Gonzalez Vieli [32] has shown that any sphere whose radius does not lay in the zero sets of the Bessel functions $J_{(n+2k−2)/2}(r) \forall k \in \mathbb{Z}_+$, is a HUP corresponding to the unit sphere $S^{n−1}$.

In 2013, Per Sjolin [23] has investigated some of the HUP corresponding to the parabola. In 2013, D. Blasi Babot [10] has given a characterization of HUP corresponding to the certain system of three parallel lines in the plane. However, the analogous problem for the finitely many parallel lines is still unanswered. In 2014, P. Jaming and K. Kellay [15], have given a unifying proof for some of the Heisenberg uniqueness pairs corresponding to the hyperbola, polygon, ellipse and graph of the functions $\varphi(t) = |t|^\alpha$, whenever $\alpha > 0$.

In one another recent article [13], the authors have investigated some of the Heisenberg uniqueness pairs corresponding to the spiral, hyperbola, circle and the exponential curves. Further, they have given a complete characterization of the Heisenberg uniqueness pairs corresponding to the four parallel lines. In the latter case, a phenomenon of three totally disconnected interleaving sets that are given by zero sets of three trigonometric polynomials has been observed.

Let $\Gamma$ be a smooth surface or a finite disjoint union of smooth surfaces in $\mathbb{R}^n$. Suppose $\mu$ is a finite complex-valued Borel measure in $\mathbb{R}^n$ which is supported on $\Gamma$ and absolutely continuous with respect to the surface measure on $\Gamma$. Then for $\xi \in \mathbb{R}^n$, the Fourier transform of $\mu$ can be defined by

$$\hat{\mu}(\xi) = \int_{\Gamma} e^{-i\xi \cdot \eta} d\mu(\eta).$$

In the above context, the function $\hat{\mu}$ becomes a uniformly continuous bounded function on $\mathbb{R}^n$. Thus, we can analyse the point-wise vanishing nature of the function $\hat{\mu}$.

**Definition 1.1.** Let $\Lambda$ be a subset of $\mathbb{R}^n$. Then the pair $(\Gamma, \Lambda)$ is called a Heisenberg uniqueness pair if $\hat{\mu}|_{\Lambda} = 0$ implies $\mu = 0$.

In compliance with the fact that the Fourier transform is translation and rotation invariant, one can easily deduces the following invariance properties of a Heisenberg uniqueness pair.

(i) For $x_o, \xi_o \in \mathbb{R}^n$. Then the pair $(\Gamma, \Lambda)$ is a HUP if and only if the pair $(\Gamma + x_o, \Lambda + \xi_o)$ is a HUP.
(ii) Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation whose adjoint is $T^*$. Then $(\Gamma, \Lambda)$ is a HUP if and only if $(T^{-1} \Gamma, T^* \Lambda)$ is a HUP.

Now, we would like to state the first known result about the Heisenberg uniqueness pair due to Hedenmalm et al. [14].

Theorem 1.2. [14] Let $\Gamma$ be the hyperbola $x_1 x_2 = 1$ and let $\Lambda_{\alpha, \beta}$ be a lattice-cross defined by

$$\Lambda_{\alpha, \beta} = (\alpha \mathbb{Z} \times \{0\}) \cup \{\{0\} \times \beta \mathbb{Z}\},$$

where $\alpha, \beta$ are positive reals. Then $(\Gamma, \Lambda_{\alpha, \beta})$ is a Heisenberg uniqueness pair if and only if $\alpha \beta \leq 1$.

For $\zeta \in \Lambda$, define a function $e_\zeta$ on $\Gamma$ by $e_\zeta(X) = e^{i\pi X \cdot \zeta}$. Then as a dual problem to Theorem 1.2, Hedenmalm et al. [14] have proved the following density result which as an application solve the one dimensional Kein-Gordon equation.

Theorem 1.3. [14] The pair $(\Gamma, \Lambda)$ is a Heisenberg uniqueness pair if and only if the set $\{e_\zeta : \zeta \in \Lambda\}$ is a weak$^*$ dense subspace of $L^\infty(\Gamma)$.

Remark 1.4. In particular, the HUP problem has another formulation. That is, if $\Gamma$ is the zero set of a polynomial $P$ on $\mathbb{R}^2$, then $\hat{\mu}$ satisfies the PDE $P(-i\partial) \hat{\mu} = 0$ with initial condition $\hat{\mu}|_{\Lambda} = 0$. This may help in determining the geometrical structure of the set $Z(\hat{\mu})$, the zero set of the function $\hat{\mu}$. If we consider $\Lambda$ is contained in $Z(\hat{\mu})$, then $(\Gamma, \Lambda)$ is not a HUP. Hence, the question of the HUP arises when we consider $\Lambda$ to be located away from $Z(\hat{\mu})$.

A set $C$ in $\mathbb{R}^n$ ($n \geq 2$) which satisfy the scaling condition $\lambda C \subseteq C$, for all $\lambda \in \mathbb{R}$, is called a cone. In this article we prove the following result.

Let $\mu$ be a finite complex-valued Borel measure in $\mathbb{R}^n$ which is supported on the unit sphere $\Gamma = S^{n-1}$ and absolutely continuous with respect to the surface measure on $\Gamma$. For $\Lambda = C$, the pair $(\Gamma, \Lambda)$ is a Heisenberg uniqueness pair as long as $\Lambda$ does not lay on the level surface of any homogeneous harmonic polynomial on $\mathbb{R}^n$. We will call such cones as non-harmonic cones.

An example of such a cone had been produced by Armitage (see [1]). Let $0 < \alpha < 1$ and $G_l^\alpha$ denotes Gegenbauer polynomial of degree $l$ and order $\alpha$. Then

$$K_\alpha = \{x \in \mathbb{R}^n : |x_1|^2 = \alpha^2 |x|^2\}$$

is a non-harmonic cone if and only if $D^m G_l^{\alpha \frac{2}{2}}(\alpha) \neq 0$ for all $0 \leq m \leq l - 2$, where $D^m$ denotes the $m$th derivative.

2. Notation and Preliminaries

Next, we recall certain standard facts about spherical harmonics, for more details see [31], p. 12.

Let $K = SO(n)$ be the special orthonormal group and $M = SO(n - 1)$. Let $\tilde{K}_M$ denote the set of all the equivalence classes of irreducible unitary
representations of $K$ which have a nonzero $M$-fixed vector. It is well known that each representation in $\hat{K}_M$ has in fact a unique nonzero $M$-fixed vector, up to a scalar multiple.

For a $\sigma \in \hat{K}_M$, which is realized on $V_{\sigma}$, let $\{e_1, \ldots, e_{d(\sigma)}\}$ be an orthonormal basis of $V_{\sigma}$, with $e_1$ as the $M$-fixed vector. Let $t^{ij}_l(k) = \langle e_i, \sigma(k)e_j \rangle$, whenever $k \in K$. By Peter-Weyl theorem for the representations of a compact group, it follows that $\{\sqrt{d(\sigma)}t^{ij}_l : 1 \leq j \leq d(\sigma), \sigma \in \hat{K}_M\}$ is an orthonormal basis of $L^2(K/M)$.

We would further need a concrete realization of the representations in $\hat{K}_M$, which can be done in the following way.

Let $\mathbb{Z}_+$ denote the set of all non-negative integers. For $l \in \mathbb{Z}_+$, let $P_l$ denote the space of all homogeneous polynomials $P$ in $n$ variables of degree $l$. Let $H_l = \{P \in P_l : \Delta P = 0\}$, where $\Delta$ is the standard Laplacian on $\mathbb{R}^n$. The elements of $H_l$ are called solid spherical harmonics of degree $l$. It is easy to see that the natural action of $K$ leaves the space $H_l$ invariant. In fact the corresponding unitary representation $\pi_l$ is in $\hat{K}_M$. Moreover, $\hat{K}_M$ can be identified, up to unitary equivalence, with the collection $\{\pi_l : l \in \mathbb{Z}_+\}$

Define the spherical harmonics on the sphere $S^{n-1}$ by $Y_{lj}(\omega) = \sqrt{\frac{d_l t^{lj}_l(k)}{d(\sigma)}}$, where $\omega = k.e_n \in S^{n-1}$, $k \in K$ and $d_l$ is the dimension of $H_l$. Then the set $\tilde{H}_l = \{Y_{lj} : 1 \leq j \leq d_l, l \in \mathbb{Z}_+, \sigma \in \hat{K}_M\}$ forms an orthonormal basis for $L^2(S^{n-1})$. Thus, we can expand a suitable function $g$ on $S^{n-1}$ as

$$g(\omega) = \sum_{l=0}^{\infty} \sum_{j=1}^{d_l} a_{lj} Y_{lj}(\omega) \quad (2.1)$$

For each fixed $\xi \in S^{n-1}$, define a linear functional on $\tilde{H}_l$ by $\xi \mapsto Y_l(\xi)$. Then there exists a unique spherical harmonic, say $Z^{(l)}(\xi) \in H_l$ such that

$$Y_l(\xi) = \int_{S^{n-1}} Z^{(l)}(\eta)Y_l(\eta)d\sigma(\eta) \quad (2.2)$$

The spherical harmonic $Z^{(l)}(\xi)$ is a $K$ bi-invariant real-valued function which is constant on the geodesics orthogonal to the line joining the origin and $\xi$. The spherical harmonic $Z^{(l)}(\xi)$ is called the zonal harmonic of the space $\tilde{H}_l$ around the point $\xi$ for the above and the various other peculiar reasons. For more details, see [30], p. 143.

Let $f$ be a function in $L^1(S^{n-1})$. For each $l \in \mathbb{Z}_+$, we define the $l$th spherical harmonic projection of the function $f$ by

$$\Pi_l f(\xi) = \int_{S^{n-1}} Z^{(l)}(\eta)f(\eta)d\sigma(\eta) \quad (2.3)$$

Then the function $\Pi_l f$ is a spherical harmonic of degree $l$. If for a $\delta > (n-2)/2$, we denote $A_l^m(\delta) = \left(\frac{(m-\delta)}{\delta}\right)^{-1}$, then the spherical harmonic expansion
$\sum_{l=0}^{\infty} \Pi_l f$ of the function $f \in L^1(\mathbb{R}^n)$ is $\delta$-Cesaro summable to $f$. That is,

$$\tag{2.4} f = \lim_{m \to \infty} \sum_{l=0}^{m} A_l^m(\delta) \Pi_l f,$$

where limit in the right hand side of (2.4) exists in $L^1(S^{n-1})$. For more details see [24].

We would like to mention that the proof of our main result is being carried out by concentrating the problem to the unit sphere $S^{n-1}$ in terms of averages of its geodesic spheres. This is possible because the cone $C$ is closed under scaling.

For $\omega \in S^{n-1}$ and $t \in (-1, 1)$, the set $S^t_\omega = \{\nu \in S^{n-1} : \omega \cdot \nu = t\}$ is a geodesic sphere on $S^{n-1}$ with pole at $\omega$. Let $f$ be an integrable function on $S^{n-1}$. Then by Fubini’s Theorem, we can define the geodesic spherical mean of the function $f$ by

$$\tilde{f}(\omega, t) = \int_{S^t_\omega} f d\nu_{n-2},$$

where $\nu_{n-2}$ is the normalized surface measure on the geodesic sphere $S^t_\omega$.

Since the zonal harmonic $Z^i_\xi(\eta)$ is $K$ bi-invariant, therefore, three exists a nice function $F$ in $(-1, 1)$ such that $Z^i_\xi(\eta) = F(\xi \cdot \eta)$. Hence the extension of the formula (2.2) becomes inevitable. An extension of formula (2.2) for the functions $F$ in $L^1(-1, 1)$ was obtained. This know is as Funk-Hecke Theorem. That is,

$$\tag{2.5} \int_{S^{n-1}} F(\xi \cdot \eta) Y_l(\eta) d\sigma(\eta) = C_l Y_l(\xi),$$

where the constant $C_l$ is given by

$$C_l = \alpha_l \int_{-1}^{1} F(t) G^\frac{n-2}{2}_l(t)(1 - t^2)^\frac{n-3}{2} dt$$

and $G^\beta_l$ stands for the Gegenbauer polynomial of degree $l$ and order $\beta$. As a consequence of the Funk-Hecke Theorem, it can be deduced that the geodesic mean of a spherical harmonic $Y_l$ can be expressed as

$$\tilde{Y}_l(\omega, t) = D_l(1 - t^2)^\frac{n-2}{2} G^\frac{n-2}{2}_l(t) Y_l(\omega),$$

where the constant $D_l = |S^{n-2}|/G^\frac{n-2}{2}_l(1)$. Here $|S^{n-2}|$ denotes the surface area of the unit sphere in $\mathbb{R}^{n-1}$. For more details see [2], p. 459. In order to prove our main result we need the following lemma which percolates the geodesic mean vanishing condition of $f \in L^1(S^{n-1})$ to each spherical harmonic component of $f$.

**Lemma 2.1.** Let $f \in L^1(S^{n-1})$. Then $\tilde{f}(\omega, t) = 0$ for all $t \in (-1, 1)$ if and only if $\Pi_l f(\omega) = 0$ for all $l \in \mathbb{Z}_+$. 
Notice that as a corollary to Lemma 2.1, it can be deduced that if \( \tilde{f}(\omega, t) = 0 \) for all \( t \in (-1, 1) \), then \( f = 0 \) a.e. on \( S^{n-1} \) if and only if \( \omega \) is not contained in the zero set of any homogeneous harmonic polynomial.

**Proof.** By the hypothesis, we have \( \tilde{f}(\omega, t) = 0 \) for all \( t \in (-1, 1) \). Now, by taking geodesic mean in (2.4) and then using (2.6), we arrive at

\[
\lim_{m \to \infty} \sum_{k=0}^{m} A_{l}^{m}(\delta)C_{l}G_{l}^{n-2}(t)\Pi_{l}f(\omega) = 0.
\]

Since the set \( \left\{ G_{l}^{n-2} : l \in \mathbb{Z}_{+} \right\} \) form an orthonormal set on \((-1, 1)\) with weight \((1 - t^2)^{-1/2}\). Therefore, from (2.7) it follows that

\[
\lim_{m \to \infty} A_{l}^{m}(\delta)C_{l}\left\| G_{l}^{n-2} \right\|_{2}^{2}\Pi_{l}f(\omega) = 0.
\]

By using the fact that for each fixed \( l \), we have \( \lim_{m \to \infty} A_{l}^{m}(\delta) = 1 \), we conclude that \( \Pi_{l}f(\omega) = 0 \) for all \( l \in \mathbb{Z}_{+} \). In particular, if \( \omega \) is not contained in \( Y_{l}^{-1}(o) \) for all \( l \in \mathbb{Z}_{+} \), then it follows that \( f(\omega) = 0 \) a.e. on \( S^{n-1} \). This completes the proof of Lemma 2.1.

\( \square \)

### 3. Proofs of the main result

In this section, we first prove our main result that a non-harmonic cone is a Heisenberg uniqueness pair corresponding to the unit sphere.

**Theorem 3.1.** Let \( \Lambda = C \) be a cone in \( \mathbb{R}^{n} \). Then \((S^{n-1}, \Lambda)\) is a Heisenberg uniqueness pair if and only if \( \Lambda \) is not contained in \( P^{-1}(o) \), whenever \( P \in H_{l} \) and \( l \in \mathbb{Z}_{+} \).

**Proof.** Since \( \mu \) is absolutely continuous with respect to the surface measure on \( S^{n-1} \), therefore, by Radon-Nikodym theorem, there exists a function \( f \) in \( L^{1}(S^{n-1}) \) such that \( d\mu = f(\eta)d\sigma(\eta) \), where \( d\sigma \) is the normalized surface measure on \( S^{n-1} \). Suppose \( \tilde{\mu}_{|\Lambda} = 0 \). Then

\[
\hat{\mu}(\xi) = \int_{S^{n-1}} e^{-i\xi \cdot \eta} f(\eta)d\sigma(\eta) = 0
\]

for all \( \xi \in S^{n-1} \). Let \( \xi = r\omega \), where \( r > 0 \) and \( \omega \in S^{n-1} \). By decomposing the integral in (3.1) into the geodesic spheres at pole \( \omega \), we get

\[
\int_{-1}^{1} \left( \int_{S_{\omega}^{l}} e^{-ir\omega \cdot \nu} f(\nu)d\sigma_{n-2}(\nu) \right) dt = 0,
\]

where \( S_{\omega}^{l} = \{ \nu \in S^{d-1} : \omega \cdot \nu = t \} \). That is,

\[
\int_{-1}^{1} e^{irt} \tilde{f}(\omega, t)dt = 0,
\]
for all \( r > 0 \). Since \( f \in L^1(S^{n-1}) \), therefore, the geodesic mean \( f(\omega, t) \) will be a continuous function on \((-1, 1)\). Thus for each fixed \( \omega \), the left hand side of (3.2) can be viewed as the Fourier transform of the compactly supported function \( \hat{f}(\omega, \cdot) \) on \( \mathbb{R} \). Hence, it can be extended holomorphically to \( \mathbb{C} \). Then, in this case, the Fourier transform of \( \hat{f}(\omega, \cdot) \) can vanish at most on a countable set. Thus, by the continuity of \( \hat{f}(\omega, \cdot) \) it follows that \( \hat{f}(\omega, t) = 0 \) for all \( t \in (-1, 1) \).

Hence, in view of Lemma 2.1, we conclude that \( f = 0 \) a.e. on \( S^{n-1} \) if and only if \( \omega \) is not contained in \( Y_{l-1}(o) \) for all \( l \in \mathbb{Z}_+ \). Since the cone \( \Lambda \) is closed under scaling, we infer that \( f = 0 \) a.e. if and only if \( \Lambda \) is not contained in \( P^{-1}(o) \) for any \( P \in H_I \) and for all \( l \in \mathbb{Z}_+ \). Thus \( \mu = 0 \).

Conversely, suppose the cone \( C \) is contained in the zero set of a homogeneous harmonic polynomial, say \( P_j \in H_I \). Then, we can construct a finite complex Borel measure \( \mu \) in \( \mathbb{R}^n \) such that \( d\mu = Y_j(\eta)d\sigma(\eta) \), where \( Y_j \in \mathcal{H}_I \).

Using Funk-Hecke Theorem, it had been shown that for spherical harmonic \( Y_j \in \mathcal{H}_I \), the following identity holds.

\[
(3.3) \quad \int_{S^{n-1}} e^{-ix\cdot\eta}Y_j(\eta)d\sigma(\eta) = i^n J_{j+(n-2)/2}(r) \rho^{(n-2)/2} Y_j(\xi),
\]

where \( x = r\xi \), for some \( r > 0 \). For a proof of identity (3.3), see [2], p. 464. This in turn implies that \( \hat{\mu}|_{C} = 0 \).

**Remark 3.2. (a).** A set which is determining set for any real analytic function is called \( NA \)-set. For instance, the spiral is an \( NA \)-set in the plane (see [20]). The set

\[
\Lambda_{\varphi} = \{ (x_1, x_2, x_3) : x_3(x_1^2 + x_2^2) = x_1\varphi(x_3) \}
\]

where function \( \varphi \) is given by \( \varphi(x_3) = \exp \frac{1}{x_3^2 - 1} \), for \( |x_3| < 1 \) and 0 otherwise.

The set \( \Lambda_{\varphi} \) is an \( NA \)-set. For more details see [20]. Since, the Fourier transform of a finite measure \( \mu \) which is supported on the boundary \( \Gamma = \partial \Omega \) of a bounded domain \( \Omega \in \mathbb{R}^n \) can be extended holomorphically to \( \mathbb{C}^n \). Therefore, \( (\partial \Omega, NA \text{-} set) \) is a Heisenberg uniqueness pair. However, converse is not true. Hence, all together with the result of Gonzalez Vieli [32], it would be an interesting question to examine, whether the exceptional sets for the HUPs corresponding to \( \Gamma = S^{n-1} \), are eventually contained in the zero sets of all homogeneous harmonic polynomials and the countably many spheres whose radii are contained in the zero set of the certain class of Bessel functions. We leave it open for the time being.

(b). For \( \Gamma = S^{n-1} \), it is easy to verify that \( \hat{\mu} \) satisfies the Helmholtz’s equation

\[
(3.4) \quad \Delta \hat{\mu} + \hat{\mu} = 0
\]

with initial condition \( \hat{\mu}|_{\Lambda} = 0 \). For a continuous function \( f \) on \( \mathbb{R}^n \), \( n \geq 2 \), the spherical mean \( Rf \) of \( f \) over the sphere \( S_r(x) = \{ y \in \mathbb{R}^n : |x - y| = r \} \) is
defined by

\[ Rf(x, r) = \int_{S_r(x)} f(y) d\sigma_r(y), \]

where \( d\sigma_r \) is the normalized surface measure on the sphere \( S_r(x) \). Then \( \hat{\mu} \) will satisfy the functional equation

\[ R\hat{\mu}(x, r) = c_n \frac{J_n(n-2)/2}{r^{(n-2)/2}} \hat{\mu}(x). \]

Thus, we infer that \( \hat{\mu}(x) = 0 \) if and only if \( R\hat{\mu}(x, r) = 0 \) for all \( r > 0 \).

In an interesting article by Zalcman et al. [8], it has been shown that for \( f \) to be continuous function on \( \mathbb{R}^n \) if \( Rf(x, r) = 0 \) for all \( r > 0 \) and for all \( x \in C \). Then \( f \equiv 0 \) if and only if \( C \) is a non-harmonic cone in \( \mathbb{R}^n \). In integral geometry, such sets are called sets of injectivity for the spherical means. We skip here to write more histories of sets of injectivity for the spherical means in various set ups. Though, we would like to refer [3–8,18,19,26–28], however, this is an incomplete list of the articles on the sets of injectivity.

Thus in view of the above result, it follows that \( \hat{\mu} \equiv 0 \) if and only if \( C \) is a non-harmonic cone in \( \mathbb{R}^n \). As \( \mu \) is a signed measure, we again need to go through the proof of Theorem 3.1, in order to show that \( \mu = 0 \).

Now, consider \( \Lambda \) to be an arbitrary set in \( \mathbb{R}^n \). Then, it is clear that \( (S^{n-1}, \Lambda) \) is HUP if and only if \( \Lambda \) is a set of injectivity for spherical mean over a class of certain real analytic functions. However, the latter problem is yet not settled.

4. SOME OBSERVATIONS FOR A SPECIAL CLASS OF MEASURES IN \( \mathbb{R}^3 \)

In this section, we shall prove that paraboloid is a HUP corresponding to the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) for a class of finite Borel measure which are given by certain symmetric functions in \( L^1(S^2) \). Further, we prove that a geodesic on the sphere \( S_{\mathbb{R}}(o) \) is a HUP corresponding to \( S^2 \) for the above class of measures.

We need the following lemma for proofs of our results of this section.

**Lemma 4.1.** Let \( f \in L^1(S^{n-1}) \) be such that \( \int_{S^{n-1}} e^{-ix \cdot \eta} f(\eta)d\sigma(\eta) = 0 \). Then

\[ \lim_{m \to \infty} \sum_{k=0}^{m} i^k A_k \frac{J_{k+(n-2)/2}(r)}{r^{(n-2)/2}} \Pi_k f(\xi) = 0, \]

where \( x = r\xi \), for some \( r > 0 \) and \( \xi \in S^{n-1} \).
Proof. We have
\[
\left| \sum_{k=0}^{m} A_k^m \int_{S^{n-1}} e^{-i\xi \cdot \eta} \Pi_k f(\eta) d\sigma(\eta) \right| \\
= \sum_{k=0}^{m} \int_{S^{n-1}} e^{-i\xi \cdot \eta} (A_k^m \Pi_k f(\eta) - f(\eta)) d\sigma(\eta) \\
\leq \sum_{k=0}^{m} \int_{S^{n-1}} |(A_k^m \Pi_k f(\eta) - f(\eta))| d\sigma(\eta)
\]
In view of Equation (2.4), it follows that
\[
\lim_{m \to \infty} \sum_{k=0}^{m} A_k^m \int_{S^{n-1}} e^{-i\xi \cdot \eta} \Pi_k f(\eta) d\sigma(\eta) = 0.
\]
This in turn from (3.3) implies that (4.1) holds. \qed

We know that for \( n = 3 \), a typical spherical harmonic of degree \( k \) can be expressed as \( Y^k_l(\theta, \varphi) = e^{il\varphi} P^l_k(\cos \theta) \), where \( P^l_k \)'s are the associated Legendre functions. In fact, the set \( \{ Y^k_l : -k \leq l \leq k \} \) form an orthonormal basis for \( \tilde{H}_k \), (see [29], p. 91). Hence, the \( k^{th} \) spherical harmonic projection \( \Pi_k f \) can be expressed as
\[
\Pi_k f(\theta, \varphi) = \sum_{l=-k}^{k} C^l_k(f) e^{il\varphi} P^l_k(\cos \theta),
\]
where \( 0 \leq \theta < \pi \) and \( 0 \leq \varphi < 2\pi \). Thus, an integrable function \( f \) on \( S^2 \) has the spherical harmonic expansion as
\[
f(\theta, \varphi) = \sum_{k=0}^{\infty} \sum_{l=-k}^{k} C^l_k(f) e^{il\varphi} P^l_k(\cos \theta)
\]
Let \( L^1_{\text{sym}}(S^2) \) denotes the space of all those functions \( f \) in \( L^1(S^2) \) that satisfy a set of symmetric-coefficient conditions \( C^l_k(f) = C^l_{k'}(f) \), for \( |l| \leq \min\{k, k'\}\).

**Theorem 4.2.** Let \( \Lambda = \{(x_1, x_2, x_3) : x_3 = x_1^2 + x_2^2\} \). Then \( (S^2, \Lambda) \) is a Heisenberg uniqueness pair with respect to \( L^1_{\text{sym}}(S^2) \).

**Proof.** Since \( \mu \) is absolutely continuous with respect to the surface measure on \( S^2 \), therefore, there exists a function \( f \in L^1_{\text{sym}}(S^2) \) such that \( d\mu = f(\eta) d\sigma(\eta) \), where \( d\sigma \) is the normalized surface measure on \( S^2 \). Suppose \( \mu|_{\Lambda} = 0 \). Then
\[
\int_{S^2} e^{-i\xi \cdot \eta} f(\eta) d\sigma(\eta) = 0
\]
for all \( \xi \in S^2 \). Now, consider the spherical polar co-ordinates \( x_1 = r \sin \theta \cos \varphi \), \( x_2 = r \sin \theta \sin \varphi \) and \( x_3 = r \cos \theta \), where \( 0 \leq \theta < \pi \) and \( 0 \leq \varphi < 2\pi \). Then, in
view of Lemma 4.1, Equation (4.4) becomes
\[
\lim_{m \to \infty} \sum_{k=0}^{m} i^k A_k^m J_{\frac{l+k}{2}}(r) \Pi_k f(\theta, \varphi) = 0
\]
for all \( \varphi \in [0, 2\pi) \). Notice that the rotation \( \varphi \) is independent of choice of \( r \), because, the paraboloid is completely determined by \( \cos \theta = r \sin^2 \theta \). Since the set \( \{e^{i\varphi} : l \in \mathbb{Z}_+\} \) form an orthonormal set in \( L^2[0, 2\pi) \) and \( f \in L^1_{\text{sym}}(S^2) \), then a simple calculation gives
\[
\lim_{m \to \infty} \sum_{k=0}^{m} i^k A_k^m J_{\frac{l+k}{2}}(r) \Pi_k f(\theta, \varphi) = 0
\]
(4.6)
\[
\int \Pi_k f(\theta, \varphi) \Pi_d f(\theta, \varphi) d\varphi = \begin{cases} \| \Pi_k f(\theta, .) \|^2_2, & \text{if } k < d \\ \| \Pi_d f(\theta, .) \|^2_2, & \text{if } k \geq d. \end{cases}
\]
After multiplying (4.5) by \( \Pi_d f(\theta, \varphi) \) and then using (4.6), we conclude that
\[
\lim_{m \to \infty} \left[ \sum_{k=0}^{m} A_k^m \left| J_{\frac{l+k}{2}}(r) \right|^2 \| \Pi_k f(\theta, .) \|^2_2 + \sum_{k=d}^{m} A_k^m \left| J_{\frac{l+k}{2}}(r) \right|^2 \| \Pi_d f(\theta, .) \|^2_2 \right] = 0.
\]
Thus, using the fact that \( \lim_{m \to \infty} A_k^m = 1 \) and the second sum goes to zero as \( d \to \infty \), we obtain that
\[
\sum_{l=0}^{\infty} \left| J_{\frac{l+k}{2}}(r) \right|^2 \| \Pi_l f(\theta, .) \|^2_2 = 0.
\]
That is, \( \| \Pi_l f(\theta, .) \|^2_2 = 0 \) for all \( r > 0 \). Since the Bessel functions can have at most countably many zeros, it follows that
\[
\Pi_l f(\theta, \varphi) = \sum_{d=-l}^{l} C_d^l(f) e^{id\varphi} P^d_1(\cos \theta) = 0.
\]
This in turn, due to orthogonality of the set \( \{e^{i\varphi} : l \in \mathbb{Z}_+\} \), implies that \( C_d^l(f) P^d_1(\cos \theta) = 0 \). However, on the paraboloid we have \( \cos \theta = r \sin^2 \theta \), which gives \( \cos \theta = \frac{-1 + \sqrt{1 + 4r^2}}{2r} \). Since, the Legendre functions can vanish only at countably many points, therefore, it follows that \( C_d^l(f) = 0 \) for all \( d \) such that \( -l \leq l \leq l \). That is, \( \Pi_l f = 0 \) for all \( l \in \mathbb{Z}_+ \). Thus \( f = 0 \) a.e. This complete the proof. \( \Box \)

**Remark 4.3.** (a). We observe that Theorem 1.2 could be worked out for the higher dimensions in a similar way. However, to avoid the complexities of notations and calculation, we prove the result for \( n = 3 \).

(b). It is interesting to mention that Theorem 1.2 need not be true in \( \mathbb{R}^2 \). Let \( \Lambda = \{(t, t^2) : t \in \mathbb{R}\} \) be a parabola. Then \( (S^1, \Lambda) \) is not a HUP. For this, consider the measure \( d\mu = f(\theta) d\theta \), where \( f \in C(S^1) \). Suppose
\[
\hat{\mu}(t) = \int_{-\pi}^{\pi} e^{i(t \cos \theta + t^2 \sin \theta)} f(\theta) d\theta = 0,
\]
for all $t \in \mathbb{R}$. By expanding the right hand side of (1.7) with the help of dominated convergence theorem, we conclude that if $f$ integrates to zero over $[-\pi, \pi)$, then $\hat{\mu}|_{\Lambda} = 0$. In fact, for $\Lambda$ to be a algebraic curve of the form $\{(p_1(t), p_2(t)) : t \in I\}$, where $p_j$, $j = 1, 2$ are polynomial on $\mathbb{R}$, the pair $(S^1, \Lambda)$ is not a HUP. Hence, it produces an evidence that the exceptional set for HUP corresponding to circle is very delicate.

Next, we prove that a geodesic sphere which is parallel to the equator of the sphere $S_R(o)$ is a HUP corresponding to the unit sphere $S^2$ with respect to $L^1_{\text{sym}}(S^2)$.

**Theorem 4.4.** Let $\Lambda_{\alpha,r} = \{(\alpha, \varphi) : R \cos \alpha = r \text{ and } 0 \leq \varphi < 2\pi\}$. Then $(S^2, \Lambda_{\alpha,r})$ is a HUP if and only if $J_{\frac{l+1}{2}}(R) \neq 0$ for all $l \in \mathbb{Z}_+$ and the ratio $r/R$ is not contained in the zero set of any Legendre function.

**Proof.** Suppose $\hat{\mu}|_{\Lambda_{\alpha,r}} = 0$. Then as similar to the proof of Theorem 4.2 we can reach to the conclusion that $|J_{\frac{l+1}{2}}(R)| \|\Pi_l f(\alpha, .)\|_2 = 0$. Then $\|\Pi_l f(\alpha, .)\|_2 = 0$ for all $l \in \mathbb{Z}_+$ if $|J_{\frac{l+1}{2}}(R)| \neq 0$ for all $l \in \mathbb{Z}_+$. That is,

$$\Pi_l f(\alpha, \varphi) = \sum_{d=-l}^{l} C_{\alpha}^d(r)e^{id\varphi}P_l^d(\cos \alpha) = 0.$$

By the uniqueness of the Fourier series, it follows that $C_{\alpha}^d(r)P_l^d(\frac{r}{R}) = 0$. Then $C_{\alpha}^d(r) = 0$ if $P_l^d(\frac{r}{R}) \neq 0$. Under the assumptions of the hypothesis, we conclude that $\Pi_l f = 0$ for all $l \in \mathbb{Z}_+$. Thus $f = 0$.

Conversely, if either of the conditions of Theorem 4.4 fails, then for the measure $d\mu = e^{id\varphi}P_l^d(\cos \theta)d\sigma(\theta, \varphi)$, it follows from Funk-Hecke identity (3.3) that $\hat{\mu}|_{\Lambda_{\alpha,r}} = 0$. This complete the proof. $\square$

**Remark 4.5.** It is reasonable to mention that if Theorem 4.4 can be worked out for a general class of finite Borel measures, then this result would have a sharp contrast, in terms of topological dimension of the pairing set, with the known results for HUP corresponding to sphere.

**Concluding remark:**

In this article we have shown that $(S^{n-1}, C)$ is a HUP as long as the cone $C$ is not contained in the zero set of any homogeneous harmonic polynomial. Now, it is a naturalistic observation to consider a compact subgroup $K$ of $SO(n)$ with $K_o$ the orbit of $K$ around the origin. Let $\Gamma_K = K/K_o$. We know that a unitary irreducible representation of $SO(n)$ can be decomposed into finitely many irreducible representations of $K$. Thus, the action of the group $K$ to a spherical harmonic $Y_l$ on the unit sphere $S^{n-1}$ will recompose $Y_l$ uniquely into a finite sum of spherical harmonics. Therefore, it would be an interesting question to find out the possibility that $(\Gamma_K, C)$ is a HUP as long as the cone $C$ does not lay on the level surface of any $K^*$-invariant homogeneous polynomial. We leave this question open for the time being.
Acknowledgements:
The author wishes to thank E. K. Narayanan and Rama Rawat for several fruitful discussions. The author would also like to gratefully acknowledge the support provided by IIT Guwahati, Government of India.

References

[1] D. H. Armitage, *Cones on which entire harmonic functions can vanish*, Proc. Roy. Irish Acad. Sect. A 92 (1992), no. 1, 107-110.
[2] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
[3] M. L. Agranovsky, C. Berenstein and P. Kuchment, *Approximation by spherical waves in Lp-spaces*, J. Geom. Anal. 6 (1996), no. 3, 365–383 (1997).
[4] G. Ambartsumian and P. Kuchment, *On the injectivity of the circular Radon transform*, Inverse Problems 21 (2005), no. 2, 473–485.
[5] M. L. Agranovsky and E.K. Narayanan, *Injectivity of the spherical mean operator on the conical manifolds of spheres*, Siberian Math. J. 45 (2004), no. 4, 597 - 605.
[6] M. L. Agranovsky and E. T. Quinto, *Injectivity sets for the Radon transform over circles and complete systems of radial functions*, J. Funct. Anal., 139 (1996), no. 2, 383–414.
[7] M. L. Agranovsky and R. Rawat, *Injectivity sets for spherical means on the Heisenberg group*, J. Fourier Anal. Appl., 5 (1999), no. 4, 363–372.
[8] M. L. Agranovsky, V. V. Volchkov and L. A. Zalman, *Conical uniqueness sets for the spherical Radon transform*, Bull. London Math. Soc. 31 (1999), no. 2, 231-236.
[9] M. Benedicks, *On Fourier transforms of functions supported on sets of finite Lebesgue measure*, J. Math. Anal. Appl. 106 (1985), no.1, 180-183.
[10] D. B. Babot, *Heisenberg uniqueness pairs in the plane*, Three parallel lines, Proc. Amer. Math. Soc. 141 (2013), no. 11, 3899-3904.
[11] S. C. Bagchi and A. Sitaram, *Determining sets for measures on Rn*, Illinois J. Math. 26 (1982), no. 3, 419-422.
[12] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 2, Interscience, New York, 1962.
[13] Deb Kumar Giri and R. K. Srivastava, *Heisenberg uniqueness pairs for some algebraic curves in the plane*,
[14] H. Hedenmalm, A. Montes-Rodriguez, *Heisenberg uniqueness pairs and the Klein-Gordon equation*, Ann. of Math. (2) 173 (2011), no. 3, 1507-1527.
[15] P. Jaming and K. Kellett, *A dynamical system approach to Heisenberg uniqueness pairs*, arXiv:1312.6236, July, 2014.
[16] N. Lev, *Uniqueness theorem for Fourier transform*. Bull. Sc. Math., 135(2011), 134-140.
[17] E. K. Narayanan and P. K. Ratnakumar, *Benedick’s theorem for the Heisenberg group*, Proc. Amer. Math. Soc. 138 (2010), no. 6, 2135-2140.
[18] E. K. Narayanan, R. Rawat and S. K. Ray, *Approximation by K-finite functions in Lp spaces*, Israel J. Math. 161 (2007), 187-207.
[19] E. K. Narayanan and S. Thangavelu, *Injectivity sets for spherical means on the Heisenberg group*, J. Math. Anal. Appl. 263 (2001), no. 2, 565-579.
[20] V. Pati and A. Sitaram, *Some questions on integral geometry on Riemannian manifolds*, Ergodic theory and harmonic analysis (Mumbai, 1999). Sankhya Ser. A 62 (2000), no. 3, 419-424.
[21] F. J. Price and A. Sitaram, *Functions and their Fourier transforms with supports of finite measure for certain locally compact groups*, J. Funct. Anal. 79 (1988), no. 1, 166-182.

[22] P. Sjolin, *Heisenberg uniqueness pairs and a theorem of Beurling and Malliavin*, Bull. Sc. Math., 135(2011), 125-133.

[23] P. Sjolin, *Heisenberg uniqueness pairs for the parabola*, Jour. Four. Anal. Appl., 19(2013), 410-416.

[24] C. D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. 53 (1986), 43-65.

[25] A. Sitaram, M. Sundari and S. Thangavelu, *Uncertainty principles on certain Lie groups*, Proc. Indian Acad. Sci. Math. Sci. 105 (1995), no. 2, 135-151.

[26] R. K. Srivastava, *Sets of injectivity for weighted twisted spherical means and support theorems*, J. Fourier Anal. Appl., 18 (2012), no. 3, 592-608.

[27] R. K. Srivastava, *Coxeter system of lines and planes are sets of injectivity for the twisted spherical means*, J. Funct. Anal. 267 (2014) 352383.

[28] R. K. Srivastava, *Real analytic expansion of spectral projection and extension of Hecke-Bochner identity*, Israel J. Math. 200 (2014), 1-22.

[29] M. Sugiura, *Unitary representations and harmonic analysis*, North-Holland Mathematical Library, 44. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1990.

[30] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, No. 32. Princeton University Press (1971).

[31] S. Thangavelu, *An introduction to the uncertainty principle*, Prog. Math. 217, Birkhauser, Boston (2004).

[32] F. J. Gonzalez Vieli, *A uniqueness result for the Fourier transform of measures on the sphere*, Bull. Aust. Math. Soc. 86(2012), 78-82.

[33] F. J. Gonzalez Vieli, *A uniqueness result for the Fourier transform of measures on the paraboloid*, Matematicki Vesnik, Vol. 67(2015), No. 1, 52-55.

Department of Mathematics, Indian Institute of Technology, Guwahati, India 781039.

E-mail address: rksri@iitg.ernet.in