A Model of Blood Flow in a Circulation Network

Weihua Ruan†, M.E. Clark‡, Meide Zhao‡ and Anthony Curcio‡

†Department of Mathematics, Computer Science and Statistics, Purdue University Calumet
and
‡VasSol, Inc.

Abstract. We study a mathematical model of a blood circulation network which is a generalization of the coronary model proposed by Smith, Pullan and Hunter. We prove the existence and uniqueness of the solution to the initial-boundary value problem and discuss the continuity of dependence of the solution and its derivatives on initial, boundary and forcing functions and their derivatives.

1 Introduction

In a recent paper [13], Smith, Pullan and Hunter propose a mathematical model of blood circulation in the coronary network, and conduct a numerical analysis. In their model, major vessels (with cross-sectional areas larger than a certain value) are treated as a connected one-dimensional network, and small vessels, such as arterioles, capillaries and venules, are treated as lumped elements which are connected to the network of vessels. The flow on vessels are assumed to be incompressible, homogeneous, Newtonian, and has a small Reynolds number. Thus, the mass balance equation and Navier-Stokes equation can be written to describe the pressure and the flow rate on vessels. Equations on lumped elements are written in analogy with the current and voltage in an electric circuit. The result is an initial-boundary value problem of a system of hyperbolic type partial differential equations coupled at junctions of the network. Although the result of the numerical analysis conducted in [13] matches closely with measured data, the well-posedness problem of the system of partial differential equations, that is, the existence, uniqueness and the continuous dependence on initial and boundary data of the solution, has not been established before. The main objective of this paper is to establish the well-posedness. We prove that the system is well-posed under certain natural conditions. This work is an extension of our earlier work [12] on a model of blood circulation in the brain. The main differences between the two models are that the network configuration in [12] is more complicated owing to the presence of Willis loops, but the coupling junction conditions in the model of [13] are more complicated due to the different formulation and the inclusion of the capillaries and veinal system. We combine both features in a more general system with the hope that our result will be useful in the
modelling of circulation systems of higher complexity, including the whole body circulation system.

Before stating our system, let us briefly describe the model in [13]. Let $P_i$ and $R_i$ represent the pressure and radius on the $i$-th vessel, respectively, and let $V_i$ be the cross-sectional average of the axial component $v_{i,x}$ of the velocity on the $i$-th vessel. Assuming that the radial component $v_{i,r}$ of the velocity is small compared to the axial component $v_{i,x}$ of the velocity, one can write equations of mass balance

$$\frac{\partial R_i}{\partial t} + V_i \frac{\partial R_i}{\partial x} + \frac{R_i}{2} \frac{\partial V_i}{\partial x} = 0,$$

(1.1)

and momentum balance

$$\frac{\partial V_i}{\partial t} + 2(1 - \alpha_i) \frac{V_i}{R_i} \frac{\partial R_i}{\partial t} + \alpha_i \frac{V_i}{\rho} \frac{\partial R_i}{\partial x} + \frac{1}{\rho} \frac{\partial P_i}{\partial x} = \frac{2\nu}{R_i} \left[ \frac{\partial v_{i,x}}{\partial r} \right]_{r=R_i}.$$

(1.2)

Here $\nu$ is the viscosity constant and

$$\alpha_i = \frac{1}{R_i^2 V_i^2} \int_0^{R_i} 2r v_{i,x}^2 dr$$

is the energy quantity. Taking into consideration of no-slip boundary condition ($v_{i,x} = 0$ if $r = R_i$), the viscous axisymmetry ($\partial v_{i,x}/\partial r = 0$ if $r = 0$), and the fact that $V_i$ is the cross-sectional average of $v_{i,x}$, Smith, Pullan and Hunter propose the velocity profile

$$v_{i,x}(r, x) = \frac{\gamma_i + 2}{\gamma_i} V_i(x) \left[ 1 - \left( \frac{r}{R_i} \right)^{\gamma_i} \right]$$

where $\gamma_i$ is a positive number. Using this profile and the mass balance condition (1.1), Eq. (1.2) becomes

$$\frac{\partial V_i}{\partial t} + (2\alpha_i - 1) V_i \frac{\partial V_i}{\partial x} + 2(\alpha_i - 1) \frac{V_i^2}{R_i} \frac{\partial R_i}{\partial x} + \frac{1}{\rho} \frac{\partial P_i}{\partial x} = -\frac{2\nu}{\alpha_i - 1} \frac{V_i}{R_i^2}$$

(1.3)

with

$$\alpha_i = \frac{\gamma_i + 2}{\gamma_i} \in (1, \infty).$$

The pressure $P_i$ and the radius $R_i$ are related by a function

$$P_i = P_i(x, R_i).$$

In [13], it is assumed that

$$P_i(x, R_i) = C \left[ \left( \frac{R_i}{R_0} \right)^{\beta} - 1 \right]$$

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where \( C, R_0 \) and \( \beta \) are constants. We do not make such an assumption, only assume that it is a differentiable function and

\[
\frac{\partial P_i}{\partial R_i} > 0
\]

for all \( x \) and \( R_i \). Let

\[
A_i = \pi R_i^2, \quad Q_i = A_i V_i
\]

be the cross-section area and the flow rate, respectively. It can be shown that the system of equations (1.1) and (1.3) is equivalent to

\[
\begin{align*}
\frac{\partial A_i}{\partial t} + \frac{\partial Q_i}{\partial x} &= 0, \\
\frac{\partial Q_i}{\partial t} + \alpha_i \frac{\partial}{\partial x} \left( \frac{Q_i^2}{A_i} \right) + \frac{A_i \partial P_i}{\rho \partial x} &= -\frac{4\pi \nu \alpha_i Q_i}{\alpha_i - 1 A_i}.
\end{align*}
\] (1.4)

Also, by rescaling the spatial variable \( x \), we may assume that each vessel is parameterized to \( x \in (0, 1) \).

The system of differential equations are supplemented with the initial condition

\[ P_i(x, 0) = P_i^I(x), \quad Q_i(x, 0) = Q_i^I(x) \] (1.5)

and boundary conditions. Boundary conditions at each end of the vessel are given according to the type of the end. If it is an external end of the network, either the pressure

\[ P_i = P_i^B(t) \] (1.6)

or the flow rate

\[ Q_i = Q_i^B(t) \] (1.7)

is specified. If the end is a branching junction, a junction connecting several vessels, let \( j_1, \ldots, j_\nu \) and \( j_{\nu+1}, \ldots, j_\mu \) denote the incoming and outgoing vessels, respectively. One imposes the mass balance condition

\[ \sum_{l=1}^\nu Q_{jl}(1, t) = \sum_{l'=\nu+1}^\mu Q_{jl'}(0, t), \] (1.8)

and the momentum balance condition

\[ \rho_{jl} \frac{\partial Q_{jl}}{\partial t} = A_{jl} (P_{jl} - P_{junc}), \quad \rho_{jl'} \frac{\partial Q_{jl'}}{\partial t} = A_{jl'} (P_{junc} - P_{jl'}) \] (1.9)

for \( l = 1, \ldots, \nu, l' = \nu+1, \ldots, \mu \), where \( \rho_l \) are small positive constants and \( P_{junc} \) is the pressure at the junction. (In [3], every branching junction connects only three vessels, it is either a
bifurcation point of one artery into two smaller ones or a joining point of two veins into a bigger one. Our prescription allows more general configuration of the network, including the presence of Willis loops.) If the end is a transitional junction, which connects the vessel to a network of arterioles, capillaries and venules, we follow the so called microcirculation model proposed in [13, 14]. Generalizing from [13], arterioles or venules connected to the vessel \(j_l\) are represented by a lumped resistive element \(R_{jl}\). The capillary bed is also represented by a resistive element \(R_C\). \(R_{jl}\)'s are connected to \(R_C\) through capacitive elements \(C_1\) and \(C_2\) on the two ends.

![Microcirculation model](image)

Figure 1: Microcirculation model of the network of arterioles, capillaries and venules.

Let \(j_1, \ldots, j_\nu\) be the arteries and let \(j_{\nu+1}, \ldots, j_\mu\) be the veins that are connected to a arteriole-capillary-venule network. The boundary conditions for \(P_{jl}, Q_{jl}\) are

\[
\begin{align*}
R_{jl} Q_{jl} (1, t) &= P_{jl} (1, t) - P_{C1} (t), & \text{for } l = 1, \ldots, \nu, \\
R_{j_{l'}} Q_{j_{l'}} (0, t) &= P_{C2} (0, t) - P_{j_{l'}} (t), & \text{for } l' = \nu + 1, \ldots, \mu
\end{align*}
\tag{1.10}
\]

and

\[
\begin{align*}
C_1 \frac{dP_{C1}}{dt} &= \sum_{l=1}^\nu Q_{jl} (1, t) - Q_C, & \quad C_2 \frac{dP_{C2}}{dt} &= Q_C (t) - \sum_{l' = \nu + 1}^\mu Q_{j_{l'}} (0, t)
\end{align*}
\tag{1.11}
\]

where \(P_{Ci}, i = 1, 2\) represent the pressure in the capacitive elements \(C_1, C_2\), and

\[
Q_C = \frac{P_{C1} - P_{C2}}{R_C}
\tag{1.12}
\]

represents the flow rate in the resistive element \(R_C\). (In [13], there is only one artery and one vein connected to the system of arteriole-capillary-venule at the two ends. We do not rule out the possibility of multiple arteries and veins join together to such a system.)
The system we study in this paper consists of the equations
\[
\begin{align*}
\frac{\partial P_i}{\partial t} + a_i \frac{\partial Q_i}{\partial x} &= f_i, \\
\frac{\partial Q_i}{\partial t} + b_i \frac{\partial P_i}{\partial x} + 2c_i \frac{\partial Q_i}{\partial x} &= g_i,
\end{align*}
\tag{1.13}
\]
and the initial and boundary conditions given by (1.15)–(1.12). For convenience, we also use the vector form
\[
(U_i)_t + B_i(U_i)_x = F_i
\tag{1.14}
\]
where \(U_i = (P_i, Q_i)\), \(F_i = (f_i, g_i)\) and
\[
B_i = \begin{pmatrix}
0 & a_i \\
b_i & 2c_i
\end{pmatrix}.
\]

Eq. (1.14) is a special case of this system where
\[
a_i = \frac{\partial P_i}{\partial A_i}, \quad b_i = \frac{A_i}{\rho} - \frac{\alpha Q_i^2}{A_i^2} \left( \frac{\partial P_i}{\partial A_i} \right)^{-1}, \quad c_i = \frac{\alpha Q_i}{A_i}, \quad f_i = 0, \quad g_i = \frac{\alpha Q_i^2}{A_i^2} \frac{\partial A_i}{\partial x} - \frac{4\pi \nu \alpha Q_i}{\alpha - 1 A_i}.
\]

We do not assume any particular form of these functions though, they are general differentiable functions of \((x, t, P_i, Q_i)\). Our basic assumptions are \(a_i > 0\) and \(A_i > \varepsilon_0\) for some positive constant \(\varepsilon_0\). Other assumptions will follow. Apart from the junction conditions, this system is the same as the one we study in [12]. Also, the junction conditions in [12] is the special case of (1.18)–(1.19) above with \(\rho_i = 0\). As in [12], we use a fixed point principle to prove the solvability of the problem. Substituting a pair of functions \((p_i, q_i)\) for \((P_i, Q_i)\) in the coefficients \(a_i, b_i, c_i, A_i\) and forcing functions \(f_i, g_i\), the system becomes linear. That is, all the functions \(a_i\), etc. are independent of unknowns. If the linear system has a unique solution, then, one can establish a mapping from \((p_i, q_i)\) to the linear problem solution \((P_i, Q_i)\). If one also shows that this mapping has a unique fixed point, then the fixed point is necessarily the unique solution of the quasilinear system. Hence, we shall first give a condition for the linear system to have a unique solution, then examine under what conditions the mapping has a unique fixed point. The first aspect of the problem is investigated in Section 2 and the second in Section 3. We also prove a result on the continuity of dependence of solutions on the initial, boundary and forcing functions for linear and quasilinear systems, thus, completing the analysis of the well-posedness of the problem. In spite of similarity in parts of the analysis to the one used in [12], the more general branching junction condition and the new transitional junction conditions require more careful treatments. Hence, there are substantial variations in the analysis. For completeness and to benefit the reader, we include all the major arguments in this paper.
2 The linear system

In this section, we analyze (1.13) as a linear system with $a_i$, $b_i$, $c_i$, $f_i$, $g_i$, $A_i$ independent of $P_i$ and $Q_i$. The initial and boundary conditions are given by (1.5)–(1.12) except that the junction condition (1.9) is substituted by the more general condition

$$
\rho_j \frac{\partial Q_{jl}}{\partial t} = A_{ji} (P_{ji} - P_{junct}) + C_{ji}, \quad \rho_{ji'} \frac{\partial Q_{ji'}}{\partial t} = A_{ji'} (P_{junct} - P_{ji'}) + C_{ji'}
$$

(2.1)

where $C_i$ are differentiable functions of $(x, t)$. The inclusion of $C_i$ is needed in the next section in order that the result of this section can be extended to the quasilinear system. We give conditions for the linear system to have a unique global solution. The conditions are most naturally given in terms of the eigenvalues of the matrix $B_i$, which have the form

$$
\lambda^R_i = c_i + u_i, \quad \lambda^L_i = c_i - u_i,
$$

where

$$
u_i = \sqrt{c_i^2 + a_i b_i}.
$$

These eigenvalues are real if

$$c_i^2 + a_i b_i > 0, \quad x \in (0, 1), \quad t > 0, \quad i = 1, \ldots, n. \quad (2.2)
$$

In this case,

$$\lambda_i^R (x, t) > 0, \quad \lambda_i^L (x, t) < \lambda_i^R (x, t) \quad (2.3)
$$

and the system is hyperbolic. Under the condition (2.2), we show that the linear system has a unique solution if

$$\lambda_i^L (0, t) < 0, \quad \lambda_i^L (1, t) < 0, \quad i = 1, \ldots, n
$$

which is equivalent to

$$a_i b_i > 0, \quad t \geq 0, \quad i = 1, \ldots, n. \quad (2.4)
$$

at $x = 0, 1$ only. It needs not hold for $x \in (0, 1)$.

**Theorem 2.1** Assume that the functions $a_i$, $b_i$, $c_i$, $f_i$, $g_i$, $A_i$ and $C_i$ are independent of $(P_i, Q_i)$. Suppose that these functions and the initial and boundary functions $P_i^I$, $Q_i^I$, $P_i^B$, $Q_i^B$ all have bounded first-order derivatives. Suppose also that $a_i > 0$, $A_i > 0$ and that the conditions (2.2) and (2.4) hold. Then, for any $T > 0$ there is a unique solution in a bounded subset of the space $C([0, 1] \times [0, T], \mathbb{R}^{2n})$ to the linear system (1.13) with the initial and boundary conditions given by (1.13)–(1.3), (1.11)–(1.12), and (2.1).
**Proof.** We first show that the system has a unique solution for $0 < t < \delta$ for some $\delta > 0$. The proof is based on the method of characteristics and a fixed point principle. For systems defined on only one branch with boundary conditions of the forms of (1.6) or (1.7), this is a standard approach. In our case, special care is needed to handle the junction conditions.

Consider the $i$-th branch. From any point $(\xi, \tau)$ on the left, right, and lower boundary of the rectangle $D = [0,1] \times [0,T]$, we construct the left-going and right-going characteristic curves $x = x_i^L(t; \xi, \tau)$ and $x = x_i^R(t; \xi, \tau)$ by

\[
\frac{dx_i^L}{dt} = \lambda_i^L(x_i^L, t), \quad x_i^L(\tau) = \xi,
\]

\[
\frac{dx_i^R}{dt} = \lambda_i^R(x_i^R, t), \quad x_i^R(\tau) = \xi,
\]

respectively, where $\lambda_i^L$ and $\lambda_i^R$ are the two eigenvalues of the matrix $B_i$. By the uniqueness of solutions to these differential equations, a left-going (resp. right-going) characteristic curve cannot intersect with another left-going (resp. right-going) characteristic curve. Let $X_i^L$ and $X_i^R$ be the right-most left-going and left-most right-going characteristic curves,

\[
x = x_i^L(t; 1, 0) \quad \text{and} \quad x = x_i^R(t; 0, 0)
\]

starting from the lower boundary of $D$, respectively. It can be shown from (2.3) that the two curves can have at most one intersection. Let $t_i$ be the value of $t$ at the intersection. If the two curves do not intersect in $D$, we simply define $t_i = T$. By condition (2.4), $X_i^L$ cannot reach the right vertical line $x = 1$ at any $t > 0$, and by $\lambda_i^R > 0$, $X_i^R$ cannot reach the vertical line $x = 0$ at any $t > 0$. Thus, the rectangle $D_i = [0,1] \times [0,t_i]$ can be divided into three parts

\[
D_i = D_i^L \cup D_i^C \cup D_i^R,
\]

where $D_i^L$ is between the vertical line $x = 0$ and the characteristic curve $X_i^R$, $D_i^C$ is between the two characteristic curves, and $D_i^R$ is between $X_i^L$ and $x = 1$.

![Figure 2: Three parts of $D_i$](image)
We show that there is a $\delta_i \leq t_i$ such that the solution $(P_i, Q_i)$ for the $i$-th branch exists in the restriction of $D_i$ to the strip $\{0 \leq t \leq \delta_i\}$.

First, observe that the initial conditions alone determine the solution completely in the central region $D^C_i$. This follows from the theory of first-order linear hyperbolic systems and the fact that from any point $(x, t) \in D^C_i$, the two characteristic curves, followed backwards, must land on the horizontal line $t = 0$. (The latter is a consequence of (2.3).) To extend the solution to other parts of $D_i$, we make a change of unknowns and derive a set of integral equations. Note that $l^R_i = (\lambda^L_i, a_i)$ and $l^L_i = (\lambda^R_i, a_i)$ are the left eigenvectors of $B_i$ corresponding to $\lambda^R_i$ and $\lambda^L_i$, respectively. Introduce new unknowns

$$r_i = l^R_i U_i \equiv -\lambda^L_i P_i + a_i Q_i, \quad s_i = l^L_i U_i \equiv -\lambda^R_i P_i + a_i Q_i.$$  \hspace{1cm} (2.5)

The system (1.13) can be written in terms of $r_i$ and $s_i$ by multiplying the left eigenvectors to (1.14) and substituting in

$$P_i = \frac{1}{2u_i} (r_i - s_i), \quad Q_i = \frac{1}{2u_i a_i} (\lambda^R_i r_i - \lambda^L_i s_i).$$  \hspace{1cm} (2.6)

This results in the equations

$$\partial^R_i r_i = F^R_i (x, t, r_i, s_i), \quad \partial^L_i s_i = F^L_i (x, t, r_i, s_i),$$  \hspace{1cm} (2.7)

where

$$\partial^R_i = \frac{\partial}{\partial t} + \lambda^R_i \frac{\partial}{\partial x}, \quad \partial^L_i = \frac{\partial}{\partial t} + \lambda^L_i \frac{\partial}{\partial x},$$  \hspace{1cm} (2.8)

and

$$F^R_i (x, t, r_i, s_i) = l^R_i F_i + (\partial^R_i l^R_i) U_i, \quad F^L_i (x, t, r_i, s_i) = l^L_i F_i + (\partial^L_i l^L_i) U_i.$$  \hspace{1cm} (2.9)

(A differential operator acting on a vector means that it acts on each component of the vector.) Let $(x, t) \in D_i$. We integrate the first equation of (2.7) along the right-going characteristic curve $x^R (t; \xi, \tau)$ which passes through $(x, t)$ and reaches the left or lower boundary of $D_i$ at $(\xi, \tau)$. It can be shown that for $(x, t) \in D^C_i \cup D^R_i$, $\tau = 0$, and for $(x, t) \in D^L_i$, $\xi = 0$. In the former case, we obtain

$$r_i (x, t) = r_i^R (\xi) + \int_0^t F^R_i (x^R_i (t'; \xi, 0), t', r_i, s_i) dt'$$  \hspace{1cm} (2.10)

In the latter case, we have

$$r_i (x, t) = r_i (0, \tau) + \int_\tau^t F^R_i (x^R_i (t'; 0, \tau), t', r_i, s_i) dt'.$$  \hspace{1cm} (2.11)
Similarly, by integrating the second equation of (2.7) along the left-going characteristic curve $x_i^L(t; \xi, \tau)$ that passes through both $(x, t)$ and $(\xi, \tau)$ (which is on either the right or lower boundary of $D_i$), the equations are

$$s_i(x, t) = s_i^L(\xi) + \int_0^t F_i^L \left( x_i^L(t'; \xi, 0), t', r_i, s_i \right) dt'$$  \hspace{1cm} (2.12)

if $(x, t) \in D_i^L \cup D_i^C$ and

$$s_i(x, t) = s_i(1, \tau) + \int_\tau^t F_i^L \left( x_i^L(t'; 1, \tau), t', r_i, s_i \right) dt'$$  \hspace{1cm} (2.13)

if $(x, t) \in D_i^R$. These are the integral equations we need.

For any $\delta_i \leq t_i$ we use $D_{i,\delta_i}^L$, $D_{i,\delta_i}^C$ and $D_{i,\delta_i}^R$ to denote the restrictions of $D_i^L$, $D_i^C$ and $D_i^R$ to the strip $\{0 \leq t \leq \delta_i\}$, respectively. First, consider the case where the end of the branch is an external end. We discuss the case of a left end only, the case of a right end can be treated similarly. If the boundary condition is given by (1.6), we define $\hat{s}_i = s_i/\varepsilon$ where $\varepsilon < 1$ is any constant. Using the first equation of (2.6) in the integral equations (2.11) and (2.12),

$$\begin{pmatrix} r_i(x, t) \\ \hat{s}_i(x, t) \end{pmatrix} = \begin{pmatrix} 2u_i(0, \tau) P_i^B(\tau) + \varepsilon \hat{s}_i(0, \tau) + \int_\tau^t F_i^R \left( x_i^R(t'; 0, \tau), t', r_i, \varepsilon \hat{s}_i \right) dt' \\ \frac{1}{\varepsilon} s_i^L(\xi) + \frac{1}{\varepsilon} \int_0^t F_i^L \left( x_i^L(t'; \xi, 0), t', r_i, \varepsilon \hat{s}_i \right) dt' \end{pmatrix}.$$  \hspace{1cm} (2.14)

This is a fixed point equation for $(r_i, \hat{s}_i)$ if we define the right hand side as a mapping of an operator $K$ on $(r_i, \hat{s}_i)$ in a bounded subset of $C(D_{i,\delta_i}^L \cup D_{i,\delta_i}^C, \mathbb{R}^2)$. In a standard approach, it can be shown that $K$ is a contraction mapping if $\delta_i$ is sufficiently small. Hence, the fixed point exists and is unique, and the solution $(r_i, s_i)$ is uniquely extended to $D_{i,\delta_i}^L \cup D_{i,\delta_i}^C$. If the boundary condition is given by (1.7), we define $\hat{s}_i = s_i/\varepsilon$, where $\varepsilon > 0$ is so small such that

$$\varepsilon \left| \frac{\lambda_i^L(0, \tau)}{\lambda_i^R(0, \tau)} \right| < 1, \quad \tau \in (0, t_i).$$

The fixed point equation is then

$$\begin{pmatrix} r_i(x, t) \\ \hat{s}_i(x, t) \end{pmatrix} = \begin{pmatrix} \frac{2u_i(0, \tau)}{\lambda_i^R(0, \tau)} Q_i^B(\tau) + \frac{\lambda_i^L(0, \tau)}{\lambda_i^R(0, \tau)} \varepsilon \hat{s}_i(0, \tau) + \int_\tau^t F_i^R \left( x_i^R(t'; 0, \tau), t', r_i, \varepsilon \hat{s}_i \right) dt' \\ \frac{1}{\varepsilon} s_i^L(\xi) + \frac{1}{\varepsilon} \int_0^t F_i^L \left( x_i^L(t'; \xi, 0), t', r_i, \varepsilon \hat{s}_i \right) dt' \end{pmatrix}.$$  \hspace{1cm} (2.15)

By a similar argument, the solution can again be uniquely extended.

We next extend the solution to either $D_{i,\delta_i}^L$ or $D_{i,\delta_i}^R$ if the end is a branching junction. In this case, we shall extend the solution on all the branches that are connected to the
same junction simultaneously. Let \( j_1, \ldots, j_\nu \) be the incoming and \( j_{\nu+1}, \ldots, j_\mu \) the outgoing branches to the junction. Equations (1.8), (2.1) and (2.6) give rise to a \( 2\mu \times \mu \) homogenous system of linear (ordinary) differential equations for \( r_i (1, t), s_i (1, t), i = j_1, \ldots, j_\nu \) and \( r_i (0, t), s_i (0, t), i = j_{\nu+1}, \ldots, j_\mu \):

\[
\frac{\rho_{j_1}}{A_{j_1}} \frac{d}{dt} Q_{j_1} (1, t) - \frac{C_{j_1}}{A_{j_1}} - P_{j_1} (1, t) = \frac{\rho_i}{A_i} \frac{d}{dt} Q_i (1, t) - \frac{C_i}{A_i} - P_i (1, t), \quad i = j_2, \ldots, j_\nu,
\]

\[
\frac{\rho_{j_1}}{A_{j_1}} \frac{d}{dt} Q_{j_1} (1, t) - \frac{C_{j_1}}{A_{j_1}} - P_{j_1} (1, t) = \frac{\rho_i}{A_i} \frac{d}{dt} Q_i (0, t) - \frac{C_i}{A_i} - P_i (0, t), \quad i = j_{\nu+1}, \ldots, j_\mu,
\]

\[
\sum_{\nu=1}^\nu Q_{j_\nu} (1, t) - \sum_{\nu=\nu+1}^\mu Q_{j_\nu} (0, t) = 0.
\]

Differentiate the last equation with respect to \( t \) and regard \( s_{j_1} (1, t), \ldots, s_{j_\nu} (1, t), r_{j_{\nu+1}} (0, t), \ldots, r_{j_\mu} (0, t) \) as unknowns. The derivatives of unknowns can be solved from (2.16) because the coefficient matrix of \( ds_i / dt \) and \( dr_i / dt \) in (2.16),

\[
\left( \begin{array}{cccc}
-\frac{\rho_{j_1} \lambda_{j_1}^L (1, t)}{2u_{j_1} a_{j_1} A_{j_1} (1, t)} & \frac{\rho_{j_2} \lambda_{j_2}^L (1, t)}{2u_{j_2} a_{j_2} A_{j_2} (1, t)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\rho_{j_1} \lambda_{j_1}^L (1, t)}{2u_{j_1} a_{j_1} A_{j_1} (1, t)} & \frac{\rho_{j_2} \lambda_{j_2}^L (1, t)}{2u_{j_2} a_{j_2} A_{j_2} (1, t)} & \cdots & 0 \\
\end{array} \right)
\]

has the determinant

\[
\left( -\frac{1}{2} \right)^\mu \prod_{i=1}^{\nu} \frac{\rho_{j_i} \lambda_{j_i}^L (1, t)}{u_{j_i} a_{j_i} A_{j_i} (1, t)} \prod_{i=\nu+1}^{\mu} \frac{\rho_{j_i} \lambda_{j_i}^R (1, t)}{u_{j_i} a_{j_i} A_{j_i} (0, t)} \sum_{i=1}^{\mu} A_{j_i}.
\]

Since \( \lambda_L < 0 < \lambda_R \) at the junction, the determinant is not zero. Thus, the derivatives of the unknowns, \( s_{j_1} (1, t) \) and \( r_{j_1} (0, t) \), are each a linear combination of the functions \( r_{j_1} (1, t), s_{j_1} (1, t), r_{j_\nu} (0, t), s_{j_\nu} (0, t) \) together with the derivatives of \( r_{j_1} (1, t) \) and \( s_{j_1} (0, t), l = 1, \ldots, \nu, l' = \nu + 1, \ldots, \mu \). Integrating and using the initial condition determined by (1.3) and (2.3), we can write

\[
s_i (1, \tau) = s_i (1, 0) + \sum_{l=1}^{\nu} m_{j_l}^i (\tau) r_{j_l} (1, \tau) + \sum_{l'=\nu+1}^{\mu} m_{j_{l'}}^i (\tau) s_{j_{l'}} (0, \tau) + \int_0^\tau H_i dt', \quad (2.17)
\]

for \( i = j_1, \ldots, j_\nu \) and

\[
r_i (0, \tau) = r_i (0, 0) + \sum_{l=1}^{\nu} n_{j_l}^i (\tau) r_{j_l} (1, \tau) + \sum_{l'=\nu+1}^{\mu} n_{j_{l'}}^i (\tau) s_{j_{l'}} (0, \tau) + \int_0^\tau H_i dt', \quad (2.18)
\]
for \( i = j_{\nu+1}, \ldots, j_\mu \), where \( m^i_j, n^i_j \) are continuous functions and \( H_i \) are linear combinations of \( r_j, (1, t), s_j, (1, t), r_{j'} (0, t), s_{j'} (0, t) \) and \( C_i (t) \) with coefficients depending only on \( t \). Choose an \( \varepsilon > 0 \) such that

\[
\varepsilon \max \left\{ \sum_{l=1}^{\mu} \left| m^i_j (\tau) \right|, \sum_{l=1}^{\mu} \left| n^i_j (\tau) \right| \right\} < 1, \quad i = j_1, \ldots, j_\mu, \quad \tau \in [0, t_i]
\]

and introduce

\[
\hat{r}_j = \frac{r_j}{\varepsilon}, \quad \hat{s}_j = \frac{s_j}{\varepsilon}, \quad l = 1, \ldots, \nu, \quad l' = \nu + 1, \ldots, \mu.
\]

Then, from (2.10)–(2.13), the integral equations for the 2\( \mu \) unknowns \( \hat{r}_j, s_j, r_{j'}, \hat{s}_{j'}, l = 1, \ldots, \nu, \ l' = \nu + 1, \ldots, \mu \) constitute a fixed point equation, \( w = K w \), where

\[
w = (\hat{r}_1, \ldots, \hat{r}_\nu, s_1, \ldots, s_\nu, r_{j_1+1}, \ldots, r_{j_\mu}, \hat{s}_{j_1+1}, \ldots, \hat{s}_{j_\mu})
\]

(2.19)

and

\[
K w = \left( \frac{1}{\varepsilon} \int_{j_1}^t F^R_{j_1} dt', \ldots, \frac{1}{\varepsilon} \int_{j_1}^t F^R_{j_\mu} dt' \right)
\]

\[
\frac{1}{\varepsilon} \hat{s}^i_j (\xi_j) + \frac{1}{\varepsilon} \int_0^t F^L_{j+1} dt', \ldots
\]

\[
\frac{1}{\varepsilon} \hat{s}^i_{j_{\nu+1}} (\xi_{j_{\nu+1}}) + \frac{1}{\varepsilon} \int_0^t F^L_{j_{\nu+1}} dt', \ldots
\]

(2.20)

in which

\[
F^R_{j_i} = F^R_{j_i} \left( x^R_{j_i}, t', \varepsilon \hat{r}_j, s_j \right), \quad F^L_{j_i} = F^L_{j_i} \left( x^L_{j_i}, t', \varepsilon \hat{r}_j, s_j \right)
\]

for \( l = 1, \ldots, \nu \), and

\[
F^R_{j_{\nu}} = F^R_{j_{\nu}} \left( x^R_{j_{\nu}}, t', r_{j'_{\nu}}, \varepsilon \hat{s}_{j'_{\nu}} \right), \quad F^L_{j_{\nu}} = F^L_{j_{\nu}} \left( x^L_{j_{\nu}}, t', r_{j'_{\nu}}, \varepsilon \hat{s}_{j'_{\nu}} \right)
\]

for \( l' = \nu + 1, \ldots, \mu \). It can be shown by a standard argument that \( K \) is a contraction mapping in the space

\[
X_j = \prod_{l=1}^\nu C \left( D^C_{j_i, \delta_j} \cup D^R_{j_i, \delta_j}, \mathbb{R}^2 \right) \times \prod_{l=\nu+1}^\mu C \left( D^L_{j_i, \delta_j} \cup D^L_{j_i, \delta_j}, \mathbb{R}^2 \right)
\]

if \( \delta_j \) is sufficiently small. Hence, it has a unique fixed point in \( X_j \). This extends the solution \( (r_1, s_i) \) for the neighboring branches of the junction.

It remains to extend the solution to a region adjacent to a transitional junction. Similar to the case of a branching junction, we simultaneously treat all the branches that are connected to the same transitional junction. Let \( j_1, \ldots, j_\nu \) be the arteries and \( j_{\nu+1}, \ldots, j_\mu \) be the veins.
The condition connecting the vessels are given by (1.10), (1.11) and (1.12). Differentiate the equations in (1.10) with respect to \( t \), the resulting equations together with (1.11) is a linear system of the derivatives of the functions \( r_{ji} (1, t), s_{ji} (1, t), r_{j'i'} (0, t), s_{j'i'} (0, t), l = 1, \ldots, \nu, l' = \nu + 1, \ldots, \mu, \) and \( P_{C_1} (t), P_{C_2} (t) \). The coefficient matrix of \( ds_{ji} (1, t) / dt, dr_{j'i'} (0, t) / dt \), for \( l = 1, \ldots, \nu, l' = \nu + 1, \ldots, \mu \) and \( P_{C_1}', P_{C_2}' \) is

\[
\begin{bmatrix}
D_1 & B_1 \\
0 & D_2 & B_2 \\
0 & 0 & I_2
\end{bmatrix}
\]

where

\[
D_1 = \text{diag} \left( \frac{R_j \lambda^l_{j}}{2u_j a_j}, \ldots, \frac{R_j \lambda^l_{j'}}{2u_j a_j} + \frac{1}{2u_j} \right),
\]

\[
D_2 = \text{diag} \left( \frac{R_{j'} \lambda^{l'}_{j+1}}{2u_{j+1} a_{j+1}}, \ldots, \frac{R_{j'} \lambda^{l'}_{j'}}{2u_{j'} a_{j'}} + \frac{1}{2u_{j'}} \right),
\]

\( I_2 \) is the \( 2 \times 2 \) identity matrix, and \( B_1, B_2 \) are some constant matrices. Since all the elements of the diagonal matrices \( D_1 \) and \( D_2 \) are positive, the system can be uniquely solved for these derivatives. Thus, each of \( ds_{ji} (1, t) / dt, dr_{j'i'} (0, t) / dt \) for \( l = 1, \ldots, \nu, l' = \nu + 1, \ldots, \mu \) is a linear combination of \( r_{ji} (1, t), s_{ji} (1, t), r_{j'i'} (0, t), s_{j'i'} (0, t) \), and \( dr_{j'i'} (1, t) / dt, ds_{ji} (0, t) / dt, l = 1, \ldots, \nu, l' = \nu + 1, \ldots, \mu \) as well as \( P_{C_1} \) and \( P_{C_2} \). We can also eliminate \( P_{C_1} \) and \( P_{C_2} \) by (1.10). Integrating the resulting equations, we obtain equations (2.17)–(2.18) for some functions \( m^i_j, n^i_j \), and \( H_i \). The remaining of the previous paragraph can then be used here to give the extension of the solution to the left or right regions for the branches.

Finally, if we let \( \delta \) be the minimum of all \( \delta_i \) occurring above, we see that \( \delta > 0 \) and the solution exists and is unique in \( (x, t) \in D_\delta = [0, 1] \times [0, \delta] \). Observe that \( \delta \) depends only on the bounds of the system functions \( a_i \), etc., the initial and boundary functions \( P_t \), etc., and their first-order derivatives in \( D = [0, 1] \times [0, T] \). Hence, it is independent of \( t \), and we can extend the solution successively in the time intervals \( [0, \delta], [\delta, 2\delta], \) etc. In this way, the solution is obtained in \( D \) in finitely many steps.

We next derive an estimate of the deviation of solution in term of the deviations of the initial, boundary and forcing functions. This estimate is needed in the next section. For any vector function \( v = (v_1, \ldots, v_k) \) defined in \( C (X; \mathbb{R}^k) \), we use \( |v|_X \) to denote the norm \( \max_i \left\{ |v_i|_{C(X)} \right\} \), where \( X \) represents a closed subset of either \( \mathbb{R} \) or \( \mathbb{R}^2 \).

**Lemma 2.1** Let \( U = (P, Q) \) and \( \tilde{U} = (\tilde{P}, \tilde{Q}) \) be two solutions of the linear problem (1.14) with different initial, boundary, and forcing functions. Suppose the conditions of Theorem 2.1 hold for both solutions. Suppose also that there is a positive lower bound for all \( A_i \). Then,
there exists a constant $M > 0$, independent of initial, boundary and forcing functions, such that

$$\left| U - \bar{U} \right|_{C(\mathcal{D}_I)} \leq M \left( \left| P^I - \bar{P}^I \right|_{C[0,1]} + \left| Q^I - \bar{Q}^I \right|_{C[0,1]} + \left| P^B - \bar{P}^B \right|_{C[0,\delta]} + \left| Q^B - \bar{Q}^B \right|_{C[0,\delta]} + \left| C - \bar{C} \right|_{C[0,\delta]} + \delta \left| f - \bar{f} \right|_{C(\mathcal{D}_I)} + \delta \left| g - \bar{g} \right|_{C(\mathcal{D}_I)} \right).$$

(2.21)

**Proof.** We need only prove (2.21) for a $\delta \leq \min \{\delta_i\}$, where $\delta_i$ represents the constants occurring in the proof of Theorem 2.1. This is because for larger $\delta$, we can divide the interval $[0, \delta]$ into subintervals, each has a length less than $\min \{\delta_i\}$, and apply (2.21) in each subinterval. We can then take the maximum on each side of the inequalities to derive the inequality of in $[0, \delta]$. In the sequel, $D^C_i$, $D^L_i$ and $D^R_i$ are the restrictions of $D^C$, $D^L$ and $D^R$ to the strip $\{0 \leq t \leq \delta\}$, respectively.

By linearity, $U - \bar{U}$ is the solution of the system with the initial, boundary and forcing functions $P^I - \bar{P}^I$, $Q^I - \bar{Q}^I$, $P^B - \bar{P}^B$, $Q^B - \bar{Q}^B$, $f_i - \bar{f}_i$, $g_i - \bar{g}_i$ and $C_i - \bar{C}_i$. Let $r_i$, $\hat{r}_i$, $s_i$, $\hat{s}_i$, be defined as in the proof of Theorem 2.1, corresponding to $U - \bar{U}$. We show that these quantities have upper bounds in the form of the right hand side of (2.21) in $D^C_i$, $D^L_i$ and $D^R_i$.

In $D^C_i$, (2.10) and (2.12) hold. Notice that the functions $F^R_i$ and $F^L_i$ are linear in $r_i$, and $s_i$. Hence, there exists a constant $M$ (we will use $M$ generically for any constant bounds that are independent of solutions) such that

$$R^C_i(t) + S^C_i(t) \leq \left| r^I_i \right|_{C[0,1]} + \left| s^I_i \right|_{C[0,1]} + M \int_0^t (R^C_i(t') + S^C_i(t') + T^C_i(t')) \, dt',$n

where

$$R^C_i(t) = \sup_{x:(x,t) \in D^C_i} \left| r_i(x,t) \right|, \quad S^C_i(t) = \sup_{x:(x,t) \in D^C_i} \left| s_i(x,t) \right|, \quad T^C_i(t) = \sup_{x:(x,t) \in D^C_i} \left( \left| f_i(x,t) - \hat{f}_i(x,t) \right| + \left| g_i(x,t) - \hat{g}_i(x,t) \right| \right).$$

(2.22)

(2.23)

Hence, by Gronwall’s inequality (see, e.g. [8, p.327]),

$$R^C_i(t) + S^C_i(t) \leq M \left( \left| r^I_i \right|_{C[0,1]} + \left| s^I_i \right|_{C[0,1]} + \delta \sup_{t \in (0,\delta)} T^C_i(t) \right)$$

for $t \in [0, \delta]$. This proves that $R^C_i$ and $S^C_i$ have upper bounds in the form of the right side of (2.21).
We next consider the left or right regions if it is adjacent to an external end. Since the both cases are similar, we will only treat the case where the left end is an external. The integral equations to be used are (2.14) or (2.15) according to the type of the boundary condition. The resulting inequality has the form

$$R_i^L(t) + \hat{S}_i^L(t) \leq \sigma \hat{S}_i^L(t) + M \left( |s_i^I|_{C[0,1]} + |\xi_i^B|_{C[0,\delta]} + \int_0^t \left( R_i^L(\tau) + \hat{S}_i^L(\tau) + T_i^L(\tau) \right) d\tau \right)$$

where $\xi_i^B$ is either $P_i^B$ or $Q_i^B$ depending on the boundary condition, and $R_i^L$, $\hat{S}_i^L$ and $T_i^L$ are defined in the same way as in (2.22), (2.23), with $D_\delta^C$ substituted by $D_\delta^L \cup D_\delta^R$, and $\sigma > 0$ is a positive constant such that $\sigma = \varepsilon$ if the boundary condition is (1.6) and

$$\sigma = \varepsilon \sup_{t \in (0,\delta)} \left| \frac{\lambda_i^L(0,t)}{\lambda_i^R(0,t)} \right| < 1$$

if the boundary condition is (1.7). Replacing $M$ by $(1 - \sigma) M$, we can write

$$R_i^L(t) + \hat{S}_i^L(t) \leq M \left( |s_i^I|_{C[0,1]} + |\xi_i^B|_{C[0,\delta]} + \int_0^t \left( R_i^L(\tau) + \hat{S}_i^L(\tau) + T_i^L(\tau) \right) d\tau \right).$$

Hence, by Gronwall’s inequality

$$R_i^L(t) + \hat{S}_i^L(t) \leq M \left( |s_i^I|_{C[0,1]} + |\xi_i^B|_{C[0,\delta]} + \delta \max_{t \in (0,\delta)} T_i^L(t) \right).$$

This proves that both $R_i^L(t)$ and $S_i^L(t)$ have upper bounds in the form of the right hand side of (2.21).

We next extend the estimate to $D_{i,\delta}^L$ or $D_{i,\delta}^R$ if the end is either a branching junction or a transitional junction. In either case, the solutions on the branches $j_1, \ldots, j_\mu$ connecting to the junction constitute a fixed point of the operator $K$, which is defined in (2.20). Let

$$W(t) = \sum_{l=1}^\nu \left( \hat{R}_{jl}^R(t) + S_{jl}^R(t) \right) + \sum_\mu_{l'=\nu+1}^{\mu} \left( \hat{R}_{jl'}^L(t) + S_{jl'}^L(t) \right)$$

where $\hat{R}_{jl}^R$ and $S_{jl}^R$ are defined as in (2.22) with $D_\delta^C$ substituted by $D_\delta^L \cup D_\delta^R$. Then, from $w = Kw$ and in view of the assumption that $A_i$ has a positive lower bound for all $i$ and $t > 0$, we can deduce

$$W(t) \leq \sigma \left( \sum_{l=1}^\nu \hat{R}_{jl}^R(t) + \sum_{l'=\nu+1}^{\mu} \hat{S}_{jl'}^L(t) \right) + M \left( \sum_{l=1}^\nu |r_{jl}^I|_{C[0,1]} + \sum_{l'=\nu}^\mu |s_{jl'}^I|_{C[0,1]} + \int_0^t (W(\tau) + T(\tau)) d\tau \right).$$
where
\[ T(\tau) = \sum_{l=1}^{\nu} T_{jl}^R(\tau) + \sum_{l'=\nu+1}^{\mu} T_{jl}^L(\tau) + \sum_{l=1}^{\mu} C_{jl}(\tau) - \tilde{C}_{jl}(\tau) \]
and \( T_i^R(t) \) is defined as in (2.23) with \( D_i^C \) substituted by \( D_i^C \cup D_i^R \). Replacing \( M \) by \((1 - \sigma) M\), we obtain
\[ W(t) \leq M \left( \sum_{l=1}^{\nu} |r_{jl}|_{C[0,1]} + \sum_{l'=\nu}^{\mu} |s_{jl'}|_{C[0,1]} + \int_0^t (W(\tau) + T(\tau)) d\tau \right). \]

Hence, by Gronwall’s inequality,
\[ W(t) \leq M \left( \sum_{l=1}^{\nu} |r_{jl}|_{C[0,1]} + \sum_{l'=\nu}^{\mu} |s_{jl'}|_{C[0,1]} + \delta \max_{t \in (0,\delta)} T(t) \right). \]

This leads to an upper bound in the form of the right hand side of (2.21) for \( R_i^R(t), S_i^R(t), i = j_1, \ldots, j_\nu \) and \( R_i^L(t), S_i^L(t), i = j_{\nu+1}, \ldots, j_\mu \).

We have thus obtained an upper bound in the form of the right hand side of (2.21) for the quantities \( |r_i - \tilde{r}_i|_{C(D_\delta)} \) and \( |s_i - s_i|_{C(D_\delta)} \). The conclusion of the lemma follows now from (2.6).

3 The quasilinear system

In this section, we study the quasilinear system where the coefficients \( a_i, b_i, c_i, f_i, g_i, A_i \) and \( C_i \) depend on both \((x, t)\) and \((P_i, Q_i)\). Under certain conditions, we show that the system has a unique local solution. We then present a theorem on the continuity of dependence of the solution on initial, boundary and forcing function.

The basic idea in the proof of the existence of solution is to construct an iterative sequence. Substituting any vector function \((p_i, q_i)\) for \((P_i, Q_i)\) in \( a_i \), etc., the system becomes linear. Thus, we can use Theorem 2.1 to get a solution \((P_i, Q_i)\). This defines a mapping \( S \) from \( u =: (p_i, q_i) \) to \( U =: (P_i, Q_i) \), and the solution for the quasilinear system is a fixed point of \( S \). If there is a subset of a Banach space that is invariant under \( S \), then, we can construct a sequence
\[ u_{k+1} = Su_k, \quad k = 0, 1, \ldots. \]
In the case where the limit exists and is unique, it gives rise to fixed point of \( S \). This is our approach in this section.

In this approach, conditions (2.2) and (2.4) are repeatedly used. One might want to impose them for all the values of the variables. This would give the existence and uniqueness for the global solution, as in the case of the linear system. However, such a requirement is
so restrictive that even the original system (1.4) cannot meet it. Therefore, we will impose them only for $t = 0$, and obtain the local solution for the quasilinear system.

**Theorem 3.1** Assume that the initial and boundary functions $P_i^I$, $Q_i^I$, $P_i^B$, $Q_i^B$ and the system functions $a_i$, $b_i$, $c_i$, $f_i$, $g_i$, $A_i$ and $C_i$ all have continuous first-order derivatives with respect to each variable. Suppose that $a_i$ is positive and $A_i$ has a positive lower bound for all the values of their arguments, and that conditions (2.2)–(2.4) hold at $t = 0$. Suppose also that the initial functions $P_i^I$, $Q_i^I$ satisfy any relevant boundary conditions at $t = 0$. Then, for some $\delta > 0$, there is a unique solution for $0 \leq t < \delta$ to the quasilinear system (1.13) with the initial and boundary conditions given by (1.3)–(1.8), (2.4), and (1.10)–(1.12).

**Proof.** We first consider the simpler case where $U^I =: (P^I, Q^I) = 0$. Let $v = \{v_i\}$, $v_i = (p_i, q_i)$ be a family of vector functions (not necessarily constitutes a solution) that satisfy the initial and boundary conditions. Substitute $v$ for $U$ in the functions $a_i$, $b_i$, $c_i$, $f_i$, $g_i$, $A_i$ and $C_i$. Then, the system becomes linear and we can invoke Theorem 2.1 to obtain a solution $U$ to the linear system. This defines a mapping $S : v \mapsto U$. A solution to the quasilinear system is then a fixed point of $S$. We will choose a subset $X_{\delta,M_0}$ of a Banach space such that (1) $SX_{\delta,M_0} \subset X_{\delta,M_0}$, and (2) $S$ is contracting in $X_{\delta,M_0}$. For any scalar or vector function $f \in C^k(D_\delta)$, let $|f|_{k,\delta}$ denote the maximum norm of all the $k$-th order derivatives of $f$ in $D_\delta$. (If $f$ is a vector function, $|f|_{k,\delta} = \max_i \{|f_i|_{k,\delta}\}$.) Let $C_B(D_\delta, \mathbb{R}^{2n})$ denote the subset of the vector-valued functions in $C(D_\delta, \mathbb{R}^{2n})$ that satisfy the initial and boundary conditions. We seek $X_{\delta,M_0}$ in the form

$$X_{\delta,M_0} = \left\{ v \in C_B(D_\delta, \mathbb{R}^{2n}) : |v|_{0,\delta} \leq M_0, |v|_{1,\delta} \leq M_1 \right\} \quad (3.1)$$

where $M_0$ is an arbitrary positive constant and $M_1$ is a constant to be determined. Note that by the vanishing initial condition, for any $M_1$, $|U|_{1,\delta} \leq M_1$ implies $|U|_{0,\delta} \leq M_1 \delta$. Hence, for any $M_0$, we can ensure $|U|_{0,\delta} \leq M_0$ by reducing $\delta$. It remains, therefore, only to show that for $M_1$ sufficiently large and $\delta$ sufficiently small, $|v|_{1,\delta} \leq M_1$ implies $|Sv|_{1,\delta} \leq M_1$. Throughout this proof, we use $\bar{M}$ to represent any positive constant that may depend on $M_1$ but is otherwise independent of $v$ and $\delta$, and use $M$ for any constant that is independent of $M_1$, $v$ and $\delta$. The values of $M$ or $\bar{M}$ in different occurrences need not be equal.

Let $U = Sv$ and let $r_i$ and $s_i$ be defined by (2.5). On each branch, we show that

$$\max \{|(r_i)_x|, |(s_i)_x| \} \leq M_1 \quad (3.2)$$

and

$$\max \{|(r_i)_t|, |(s_i)_t| \} \leq M_1 \quad (3.3)$$
in $D^C_\delta$, $D^L_\delta$ and $D^R_\delta$ if $M_1$ is large and $\delta$ is small. (Recall that $D^C_\delta$ etc. are the intersections $D^C_\delta \cap D_\delta$ etc., respectively.) In fact, only $\eqref{3.2}$ needs to be shown. To see this, first observe that the vanishing initial condition and the compatibility of the initial and boundary conditions gives

$$\max_i \left\{ |P_i|_{C[0,\delta]} , |Q_i|_{C[0,\delta]} \right\} \leq M\delta.$$  

Hence, we obtain from Lemma 2.1 with $\tilde{U} = 0$ that

$$|U|_{0,\delta} \leq M\delta. \tag{3.4}$$

From $\eqref{2.7}$ and $\eqref{2.9}$, there are constants $\tilde{M}$ and $M$ such that

$$|\partial_i^R r_i| \leq |l_i^R F_i| + |\partial_i^R l_i^R| |U_i| \leq \tilde{M} + M\delta, \quad \tag{3.5}$$

$$|\partial_i^L s_i| \leq |l_i^L F_i| + |\partial_i^L l_i^L| |U_i| \leq \tilde{M} + M\delta$$

for each $i = 1, \ldots, n$. Hence, $\eqref{3.3}$ follows from $\eqref{3.2}$, $\eqref{3.5}$ and the definition of $\partial_i^L$ and $\partial_i^R$ in $\eqref{2.8}$. We also note that $\eqref{2.6}$ and $\eqref{3.3}$ imply

$$|\partial_i^R U_i|_{0,\delta} \leq \tilde{M} + M\delta, \quad |\partial_i^R U_i|_{0,\delta} \leq \tilde{M} + M\delta \tag{3.6}$$

for all $i$. This will be used later.

We first consider the middle region $D^C_\delta$, where the solution $(r_i, s_i)$ satisfies the integral equations $\eqref{2.10}$ and $\eqref{2.12}$ with $r_i^I = s_i^I = 0$. Differentiating the equations with respect to $x$, we have

$$(r_i)_x = (l_i^R)_x U_i (x, t) + \int_0^t \left[ \left( (l_i^R F_i)_x + (\partial_i^R l_i^R) (U_i)_x - (l_i^R)_x (\partial_i^R U_i) \right) (x, t) \right] \, dt,$$

$$(s_i)_x = (l_i^L)_x U_i (x, t) + \int_0^t \left[ \left( (l_i^L F_i)_x + (\partial_i^L l_i^L) (U_i)_x - (l_i^L)_x (\partial_i^L U_i) \right) (x, t) \right] \, dt. \tag{3.7}$$

Here, we used an identity from [3, p.469]:

$$\frac{d}{dx} \int_a^b f (x (t), t) \, dt = f (x (b), b) g_x (x (b), b) x (b) - f (x (a), a) g_x (x (a), a) x (a) \tag{3.8}$$

where $x (t)$ is a function such that $x (b) = \xi$ and $D = \frac{\partial}{\partial \xi} + x' (t) \frac{\partial}{\partial \xi}$. (Notice that $x (b) = 1$.) Let

$$R^C_i (t) = \sup_{x \in [x (t), t] \in D^C_\delta} \{ |(r_i)_x (x, t)| \}, \quad S^C_i (t) = \sup_{x \in [x (t), t] \in D^C_\delta} \{ |(s_i)_x (x, t)| \}. \tag{3.9}$$
From (3.4), (3.6) and (3.7), we derive

\[ R^C_i(t) + S^C_i(t) \leq M\delta + M \int_0^t (1 + R^C_i(t') + S^C_i(t')) \, dt' \]

for \( t \in [0, \delta] \). Hence, Gronwall’s inequality gives

\[ |(r_i)_x| \leq M\delta e^{M\delta}, \quad |(s_i)_x| \leq M\delta e^{M\delta} \]

in \( D^C_\delta \). This proves (3.2) in \( D^C_\delta \) if \( M_1 \) is sufficiently large and \( \delta \) is sufficiently small.

We next consider the left and right regions \( D^L_\delta, D^R_\delta \) which are next to an external end. Since the two cases are similar, we will consider the left region only. Let \( s_i = s_i/\varepsilon \) for any \( \varepsilon > 0 \). Then, the pair \((r_i, s_i)\) satisfies the fixed point equations of either (2.14) or (2.15), depending on the type of the boundary condition. Differentiating the equations with respect to \( x \) and using a slightly modified version of (3.8) where the lower limit \( a \) of the integral also depends on \( \xi \):

\[
\frac{d}{d\xi} \int_a^b f(x(t), t) \, Dg(x(t), t) \, dt = f(x(b), b) g_x(x(b), b) x_\xi(b) - f(x(a), a) g_x(x(a), a) x_\xi(a) \\
- f(x(a), a) Dg(x(a), a) a_\xi \\
+ \int_a^b [f_x(x(t), t) \, Dg(x(t), t) - Df(x(t), t) \, g_x(x(t), t)] x_\xi(t) \, dt,
\]

we have

\[
(r_i)_x = (\zeta_i - l_i^R F_i - (\partial^R_i l_i^R) U_i(0, \tau) \tau_x + (l_i^R)_x U_i(x, t) - (l_i^R)_x U_i(x^R, x)(0, \tau) \\
+ \int_0^t [(l_i^R F_i)_x + (\partial^R_i l_i^R)(U_i)_x - (l_i^R)_x (\partial^R_i U_i)] (x_i^R)_x \, dt,
\]

(3.10)

\[
(s_i)_x = \frac{1}{\varepsilon} (l_i^L)_x U_i(t, x) + \frac{1}{\varepsilon} \int_0^t [(l_i^L F_i)_x + (\partial^L_i l_i^L)(U_i)_x - (l_i^L)_x (\partial^L_i U_i)] (x_i^L)_x \, dt,
\]

where

\[
\zeta_i = 2 \left( u_i P^B_i \right)_t + \varepsilon (s_i)_t
\]

if the boundary condition is given by (2.9), and

\[
\zeta_i = 2 \left( \frac{a_i u_i}{\lambda_i^R} Q^B_i \right)_t + \varepsilon \left( \frac{\lambda_i^L}{\lambda_i^R} \right) (s_i)_t
\]

if the boundary condition is given by (2.7). This equation is valid for any \( \varepsilon \). So, we may choose \( \varepsilon \) so small such that

\[
\sigma := \varepsilon \left| \lambda_i^L \tau_x(0, t) \right| \max \left\{ 1, \left| \left( \frac{\lambda_i^L}{\lambda_i^R} \right)(0, t) \right| \right\} < 1, \quad t \in [0, \delta].
\]
To proceed further, we need an estimate of $|\tau_x(0, t)|$. Observe that $\tau(x)$ satisfies the equation

$$x_i^R(\tau; x, t) = 0$$

where $x_i^R(\tau; x, t)$ is the solution of the initial value problem

$$\frac{dx_i^R}{ds} = \lambda_i^R(x_i^R(s), x_i^R(t; x, t)) = x.$$

By differentiation,

$$\lambda_i^R(0, \tau(x)) \tau_x + \frac{\partial x_i^R}{\partial x}(\tau(x); x, t) = 0. \quad (3.11)$$

Let $w_i = \partial x_i^R / \partial x$. Then, $w_i$ is the solution to the linear equation

$$\frac{dw_i}{ds} = (\lambda_i^R)_{x_i^R(s; x, t), s} w_i, \quad w_i(t) = 1.$$

Solving the equation,

$$w_i(s) = \exp \left( \int_t^s (\lambda_i^R)_{x_i^R(s'; x, t), s'} ds' \right).$$

Returning to (3.11), we find

$$\tau_x = -\frac{1}{\lambda_i^R(0, \tau(x))} \exp \left( \int_t^{\tau(x)} (\lambda_i^R)_{x_i^R(s'; x, t), s'} ds' \right).$$

Observe that $0 < \tau(x) < t \leq \delta$ and the integrand is bounded. Hence,

$$|\tau_x| \leq M e^{M\delta}. \quad (3.12)$$

This is the estimate we need. By this estimate, for any $M_1$, we can choose $\delta$ small enough such that the constants $\sigma$ and $\varepsilon$ are independent of $M_1$. Let $R_i^L(t)$ and $\hat{S}_i^L(t)$ be defined as in (3.9) except that $s_i$ is substituted by $\hat{s}_i$ and $D_i^L$ is substituted by $D_i^L \cup D_i^C$. We derive from (3.10) and the identity

$$(\hat{s}_i)_t = \partial_i^L \hat{s}_i - \lambda_i^L(\hat{s}_i)_x$$

that

$$R_i^L(t) + \hat{S}_i^L(t) \leq \sigma \hat{S}_i^L(t) + \hat{M} + M\delta + M \int_0^t \left( 1 + R_i^L(t') + \hat{S}_i^L(t') \right) dt'.$$

Replacing $M$ and $\hat{M}$ by $M(1 - \sigma)$ and $\hat{M}(1 - \sigma)$, respectively, and applying Gronwall’s inequality, we obtain

$$R_i^L(t) + \hat{S}_i^L(t) \leq \left( \hat{M} + M\delta \right) e^{M\delta}.$$
Since $|s_i| \leq |\hat{s}_i|$, it follows that

$$\max \{(r_i)_x, (s_i)_x\} \leq \left( \tilde{M} + M\delta \right) e^{M\delta}$$

in $D^L_{\delta} \cup D^C_{\delta}$. This proves (3.2) in $D^L_{\delta} \cup D^C_{\delta}$ if $M_1$ is large and $\delta$ is small.

We next consider the case where the end of the branch is a branching or transitional junction. As before, all the branches that are connected to the same junction are considered simultaneously. Differentiating the fixed point equation $w = Kw$ where $w$ and $Kw$ are defined in (2.13) and (2.20), respectively, we obtain (3.10) in $D^L_{\delta} \cup D^C_{\delta}$ for $i = j_{\nu+1}, \ldots, j_\mu$ and

$$\left( \hat{r}_i \right)_x = \frac{1}{\varepsilon} \left( l^{Ri}_i \right)_x U_i (x, t) + \frac{1}{\varepsilon} \int_0^t \left[ \left( \hat{l}^{P}_i \right)_x + \left( \partial^P_i \hat{l}^{P}_i \right)_x \right] U_i - \left( l^{R}_i \right)_x \left( \partial^R_i U_i \right)_x \right) \left( x^{R}_i \right)_x dt,$$

$$\left( s_i \right)_x = \left( \theta_i - \hat{l}^{P}_i \right)_x U_i (x, t) + \left( l^{R}_i \right)_x U_i (x, t) - \left( l^{L}_i \right)_x \hat{U}_i (x, t) \right) \left( x^{L}_i \right)_x dt,$$

in $D^C_{\delta} \cup D^R_{\delta}$ for $i = j_1, \ldots, j_\nu$, where

$$\zeta_i = \varepsilon \sum_{l=1}^{\nu} \left( n^i_{jl} \right)_t \left( 1, \tau \right) + \varepsilon \sum_{l'=\nu+1}^{\mu} \left( n^i_{jl'} \hat{s}_{jl'} \right)_t \left( 0, \tau \right) + H_i,$$

$$\theta_i = \varepsilon \sum_{l=1}^{\nu} \left( m^i_{jl} \right)_t \left( 1, \tau \right) + \varepsilon \sum_{l'=\nu+1}^{\mu} \left( m^i_{jl'} \hat{s}_{jl'} \right)_t \left( 0, \tau \right) + H_i,$$

and $m^i_{jl}, n^i_{jl}$ are defined in the proof of Theorem 2.1. Note that the estimate (3.12) holds for $\tau_x$ in both (3.10) and (3.13), although in the latter case, $\tau$ is the $t$-coordinate of the intersection of the left-going characteristic curve $x^{L}_i$ with the vertical line $x = 1$. The derivation is identical. Hence, there is a constant $\varepsilon$, independent of $M_1$, such that

$$\varepsilon \left| \tau_x \right| \left( \sum_{k=1}^{\nu} \left| m^i_{jk} \right| (t) + \sum_{k'=\nu+1}^{\mu} \left| m^i_{jk'} \right| (t) \right) < 1,$$

$$\varepsilon \left| \tau_x \right| \left( \sum_{k=1}^{\nu} \left| n^i_{jk} \right| (t) + \sum_{k'=\nu+1}^{\mu} \left| n^i_{jk'} \right| (t) \right) < 1$$

in $[0, \delta]$. Let $\sigma$ be the maximum of the quantities on the left hand side of the above inequalities. Define $\hat{R}^{R}_i, S^{R}_i, R^{L}_i$ and $\hat{S}^{L}_i$ as in (3.9) with obvious modifications. We see that the function

$$W(t) = \sum_{l=1}^{\nu} \left( \hat{R}^{R}_{jl} (t) + S^{R}_{jl} (t) \right) + \sum_{l'=\nu+1}^{\mu} \left( R^{L}_{jl'} (t) + \hat{S}^{L}_{jl'} (t) \right)$$
satisfies the inequality
\[(1 - \sigma)W(t) \leq \sum_{i=1}^{\nu} \left( (1 - \sigma) \hat{R}_{j_i}^R(t) + \hat{S}_{j_i}^R(t) \right) + \sum_{j'=\nu+1}^{\mu} \left( R_{j'_i}^L(t) + (1 - \sigma) \hat{S}_{j'_i}^L(t) \right) \]
\[\leq \bar{M} + M\delta + M \int_0^t (1 + W(t')) dt'.\]

Hence, by rescaling and using Gronwall’s inequality, we achieve
\[W(t) \leq \left( \bar{M} + M\delta \right) e^{M\delta}.\]

This proves that
\[\max \{ |(r_i)_x|, |(s_i)_x| \} \leq M_1\]
in \[D_{\delta}^{\bar{R}}\] for \(i = j_1, \ldots, j_{\nu}\) and in \[D_{\delta}^{\bar{L}}\] for \(i = j_{\nu+1}, \ldots, j_{\mu}\) if \(M_1\) is sufficiently large and \(\delta\) is sufficiently small. We have thus proved (3.2) in this case.

This completes the proof of (3.2) in all cases. By choosing appropriate values of \(M_1\) and \(\delta\), we thus obtain a set \(X_{\delta,M_0}\) in the form of (3.1) which is invariant under the mapping \(S\).

We now show that \(S\) is a contraction in \(X_{\delta,M_0}\). Let \(U = Sv, \bar{U} = S\bar{v}\) for some \(v, \bar{v} \in X_{\delta}\), and let \(W = U - \bar{U}\). \(W\) satisfies the vanishing initial and external boundary conditions and its differential equations takes the form of (1.13) with the coefficients
\[a_i = a_i(x,t,v), \ b_i = b_i(x,t,v), \ c_i = c_i(x,t,v), \ A_i = A_i(x,t,v)\]
the forcing functions \(f_i\) and \(g_i\) replaced by
\[\hat{f}_i := f_i(x,t,v) - f_i(x,t,\bar{v}) + (a_i(x,t,v) - a_i(x,t,\bar{v})) \frac{\partial \hat{Q}_i}{\partial x}, \quad (3.14)\]
and
\[\hat{g}_i := g_i(x,t,v) - g_i(x,t,\bar{v}) + (b_i(x,t,v) - b_i(x,t,\bar{v})) \frac{\partial \hat{P}_i}{\partial x} + 2 (c_i(x,t,v) - c_i(x,t,\bar{v})) \frac{\partial \hat{Q}_i}{\partial x}, \quad (3.15)\]
respectively, and the functions \(C_i\) in (2.1) replaced by
\[\hat{C}_{j_i} = C_{j_i}(x,t,v) - C_{j_i}(x,t,\bar{v}) + (A_{j_i}(x,t,v) - A_{j_i}(x,t,\bar{v})) \left( \hat{P}_{j_i} - \tilde{P}_{junc} \right),\]
\[\hat{C}_{j'_i} = C_{j'_i}(x,t,v) - C_{j'_i}(x,t,\bar{v}) + (A_{j'_i}(x,t,v) - A_{j'_i}(x,t,\bar{v})) \left( \tilde{P}_{junc} - \hat{P}_{j'_i} \right)\]
for \( l = 1, \ldots, \nu, \ l' = \nu + 1, \ldots, \mu \). By the Lipschitz property and the boundedness \( |\tilde{U}|_{1,\delta} \leq M_1 \), there is a constant \( M \) such that

\[
|\tilde{f}|_{0,\delta} \leq M |v - \tilde{v}|_{0,\delta}, \quad |\tilde{g}|_{0,\delta} \leq M |v - \tilde{v}|_{0,\delta}, \quad |\tilde{C}|_{0,\delta} \leq M |v - \tilde{v}|_{0,\delta}
\]

Hence, by Theorem 2.1,

\[
|Sv - S\tilde{v}|_{0,\delta} \leq M \delta |v - \tilde{v}|_{0,\delta}.
\]

Therefore, \( S \) is contracting in \( X_{\delta,M_0} \) if \( \delta \) is sufficiently small.

The rest is standard (cf. e.g., [6]). Starting with a \( v_0 \in X_{\delta,M_0} \), we generate an iterative sequence \( v_{k+1} = Sv_k \). Clearly, each \( v_k \) lies in \( X_{\delta,M_0} \) and the sequence converges uniformly. The limit then satisfies the integral equations in the proof of Theorem 2.1, and hence, is differentiable. Therefore, it is the solution of the quasilinear differential equations. This proves the existence and uniqueness of the solution when \( U^I = 0 \).

If \( U^I \neq 0 \), we regard \( U^I \) as a vector function of \( x \) and \( t \) and introduce \( \tilde{U} = U - U^I \). It follows that \( \tilde{U} \) is a solution of the quasilinear equations \((1.13)\) with the forcing functions \( \tilde{f}_i \) and \( \tilde{g}_i \) given by

\[
\tilde{f}_i = f_i - (Q_i^I) \frac{\partial}{\partial x} a_i, \quad \tilde{g}_i = g_i - (P_i^I) \frac{\partial}{\partial x} b_i - (Q_i^I) \frac{\partial}{\partial x} 2c_i
\]

and the boundary functions are given by

\[
\tilde{P}_i = P_i - P_i^I, \quad \tilde{Q}_i = Q_i - Q_i^I,
\]

and

\[
\tilde{C}_{ji} = C_{ji} + A_{ji} P_{ji}^I, \quad \tilde{C}_{j'j} = C_{j'j} - A_{j'j} P_{j'j}^I
\]

for \( l = 1, \ldots, \nu, \ l' = \nu + 1, \ldots, \mu \). Since \( \tilde{U} \) has the vanishing initial values, it can be uniquely solved for an interval of \( t \in [0, \delta] \). This gives rise to a solution \( U \). \( \blacksquare \)

**Remark:** Examples can be constructed to show that if the condition (2.4) fails at \( t = 0 \), then, the local solution need not exist or may be not unique. In particular, if (2.4) fails at a source end, then, the system is under-determined, and if it fails at a terminal end, the system is over-determined.

We give next a result for the continuity of dependence of the solution and its derivatives on the initial, boundary and forcing functions and their derivatives. This follows from an argument similar to the proofs of Lemma 2.1 and Theorem 3.1.

**Corollary 3.1** Let \( U = (P, Q) \) and \( \tilde{U} = \left( \tilde{P}, \tilde{Q} \right) \) be two solutions of the quasilinear problem of Theorem 3.1. Suppose the conditions of that theorem hold for the initial and boundary conditions.
functions of both solutions. Then, there exists a constant $M > 0$, independent of initial, boundary and forcing functions, such that

$$ |U - \tilde{U}|_{k,\delta} \leq M \left( \left| P^I - \tilde{P}^I \right|_{C^k[0,1]} + \left| Q^I - \tilde{Q}^I \right|_{C^k[0,1]} + \left| P^B - \tilde{P}^B \right|_{C^k[0,\delta]} + \left| Q^B - \tilde{Q}^B \right|_{C^k[0,\delta]} + \delta \left| f - \tilde{f} \right|_{C^k(D_x)} + \delta \left| g - \tilde{g} \right|_{C^k(D_x)} + \delta \left| C - \tilde{C} \right|_{C^k[0,\delta]} \right). $$

(3.16)

for $k = 0, 1$.

**Proof.** For $k = 0$, the result follows from substituting one of the solutions into the coefficients, modifying the forcing functions by (3.14)–(3.15), and using Lemma 2.1. For $k = 1$, we differentiate the equations and apply the lemma to the resulting equations for the derivatives of the solution. The process is standard and is omitted.

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