Crystallographic T-duality

Kiyonori Gomi\textsuperscript{1} and Guo Chuan Thiang\textsuperscript{2}

\textsuperscript{1}Department of Mathematical Sciences, Shinshu University, Matsumoto, Nagano 390-8621, Japan
\textsuperscript{2}School of Mathematical Sciences, University of Adelaide, SA 5000, Australia

July 2, 2018

Abstract

We introduce the notion of crystallographic T-duality, inspired by the appearance of $K$-theory with graded equivariant twists in the study of topological crystalline materials. This gives a powerful tool for computing topological phase classification groups, and for understanding crystallographic bulk-boundary correspondences in physics through index theory.

Contents

1 Introduction

2 Generalities on crystallographic space groups 5
  2.1 Actions of the point group 5
    2.1.1 Linear and affine actions on position space torus 5
    2.1.2 Crystallography and group cohomology 6
    2.1.3 Dual action on Brillouin torus 7
    2.1.4 Dual cocycle on Brillouin torus 8

3 Generalities on twistings of $K$-theory 9
  3.1 Examples of special equivariant $H^3$ twists 9
  3.2 Examples of special equivariant $H^1$ twists 10
  3.3 Twisted composition rule for $H^3$ and $H^1$ twists 11
  3.4 Equivariant $K$-orientability of torus 12

4 Crystallographic T-duality 14
  4.1 T-duality of circle bundles 14
    4.1.1 Ordinary T-duality for a circle 14
    4.1.2 Topology change from $H^3$-twists 14
  4.2 Crystallographic T-duality and Baum–Connes assembly 15
1 Introduction

The role of \( K \)-theory in string theory [51] and in solid state physics [6] has been known for several decades. In the former, D-brane charges in various flavours of

| Section | Title | Page |
|---------|-------|------|
| 4.2.1  | Crystallographic T-duality and Poincaré bundle | 16 |
| 4.3    | T-dualities for circle bundles with involution | 17 |
| 4.3.1  | T-duality for ‘Real’ circle bundles and \( K \)-theory | 18 |
| 4.3.2  | \( \mathbb{Z}_2 \)-equivariant T-duality | 19 |
| 5      | ‘Real’ and \( \mathbb{Z}_2 \)-equivariant T-dualities over involutive circle base | 20 |
| 5.1    | Equivariant cohomology of \( \text{pt} \), \( S^1_{\text{triv}} \) and \( S^1_{\text{flip}} \) | 20 |
| 5.1.1  | \( \mathbb{Z}_2 \)-equivariant and Real T-duality over a point | 22 |
| 5.2    | ‘Real’ T-dualities over circle base | 22 |
| 5.2.1  | ‘Real’ T-duality over \( S^1_{\text{flip}} \) | 22 |
| 5.2.2  | ‘Real’ T-duality over \( S^1_{\text{triv}} \) | 22 |
| 5.3    | \( \mathbb{Z}_2 \)-equivariant T-duality over \( S^1_{\text{flip}} \) | 25 |
| 5.4    | \( \mathbb{Z}_2 \)-equivariant and ‘Real’ T-duality over \( S^1_{\text{free}} \) | 27 |
| 6      | 2D crystallographic T-dualities | 27 |
| 6.1    | Trivial point group: \( p1 \) | 28 |
| 6.2    | Order-2 point group | 28 |
| 6.2.1  | \( G = \mathbb{Z}_2 \): \( p2 \) | 28 |
| 6.2.2  | \( G = D_1 \cong \mathbb{Z}_2 \): \( \text{pm, pg, cm} \) | 28 |
| 6.2.3  | Exotic non-cocycle \( H^3 \)-twists from partial T-duality: \( \text{cm} \) | 30 |
| 6.3    | Point group \( D_3 \): Orbifold change under crystallographic T-duality | 31 |
| 6.4    | Remaining cases | 32 |
| 6.4.1  | Cyclic point groups \( p3, p4, p6 \) | 32 |
| 6.4.2  | Point group \( D_2, D_4, D_6 \) | 32 |
| 7      | 1D crystallographic T-dualities | 34 |
| 7.1    | 1D space groups, frieze groups, and graded twists | 34 |
| 7.1.1  | Point group \( 1: p1 \) | 34 |
| 7.1.2  | Point group \( \mathbb{Z}_2 \) acting by reflection: \( \text{pm1, p2} \) | 34 |
| 7.1.3  | Point group \( \mathbb{Z}_2 \) acting trivially: \( \text{p11m, p11g} \) | 35 |
| 7.1.4  | \( D_2 \) point group: \( \text{p2mm, p2mg} \) | 36 |
| 7.2    | T-duality with Möbius twists and \( G \)-equivariant T-duality | 37 |
| 8      | 3D dualities and applications | 38 |
| 8.1    | H-flux from partial T-duality: screw dislocations | 38 |
| 8.2    | Crystallographic bulk-boundary correspondence and super-indices for boundary zero modes | 39 |
| 8.3    | Spectral sequence extension problems and halving computations of topological phases | 39 |

A Appendix

41
(super)string theory live in appropriate $K$-(co)homology groups of spacetime, while in the latter, invariants of topological phases live in the $K$-theory of some (noncommutative) momentum space \([31, 18, 48]\). In both fields, the role of dualities is key. For instance, T-dualities relate complementary features of and account for different types of string theories \([12, 50]\), while a closely related position-momentum space duality was already observed in \([13]\) and features in the Fourier transform used heavily in solid state physics. Furthermore, index theory as formulated naturally in $K$-theoretic language appears in mathematical treatments of T-duality \([29]\), and in accounting for the quantum Hall effect \([7]\).

Mathematical interest in T-duality was stimulated by the discovery that the presence of H-flux generally requires the T-dual manifold to be topologically distinct from the original one \([11]\). The relevant $K$-theory is twisted by the degree-3 cohomology class of the H-flux, and the desire to understand the general mechanism behind T-duality led to a rekindling of interest in twisted $K$-theory. A very fruitful approach utilises $C^*$-algebras \([41, 36, 42, 15]\) and relates T-duality to the deep Baum–Connes isomorphisms \([5]\) and therefore to representation theory. Recently, twisted $K$-theory started to appear in solid-state physics from the work of Freed–Moore generalising the Bott-“Periodic Table” of topological insulators \([31]\) to the crystallographic setting. Here, the relevant twists are graded, equivariant and generally torsion classes.

These parallel developments serve as strong motivations for the notion of crystallographic T-duality introduced in this paper. Our main definition and result is in Theorem 4.1, which says that for each crystallographic space group $\mathcal{G}$ in $d$-dimensions (and there are many such groups), there is an isomorphism of twisted $K$-theories,

$$T_\mathcal{G} : K_G^{-•+\sigma_{\mathcal{G}}} (T^d_\mathcal{G}) \xrightarrow{\cong} K_G^{-•-d+\tau_{\mathcal{G}}}(\hat{T}^d).$$

On the LHS, $T^d_\mathcal{G}$ is a “position space” $d$-torus equipped with a naturally defined affine action of a finite quotient $G$ of $\mathcal{G}$, and $\sigma_{\mathcal{G}}$ is a graded $G$-equivariant twist ($\S3$) from the $K$-nonorientability of this $G$-action. On the RHS, $\hat{T}^d$ is the “momentum space” Brillouin torus equipped with the natural dual $G$-action, and $\tau_{\mathcal{G}}$ is a cocycle equivariant twist arising from group-theoretic properties of $\mathcal{G}$. Thus $T_\mathcal{G}$ has a physical interpretation as a “topological Fourier transform” adapted to $\mathcal{G}$, which passes between position and momentum space pictures. Strikingly, the data on one side appears at first glance to be of a different nature to the data on the other side, yet the total $K$-theoretic information is “conserved”. We also show, via a large number of explicit examples, that the crystallographic T-duality factorises through several circle bundle T-dualities — “partial Fourier transforms” — such as the ‘Real’ T-duality of \([19]\) involving $K_{\pm}$ groups. From this, we obtain intricate webs of T-dualities whose individual links are sometimes already known, but are now assembled together in coherent patterns ($\S6, \S7$).

Mathematically, our duality is shown by a chain of isomorphisms involving the Baum–Connes assembly map, and so implicitly passes through a (graded) $C^*$-algebraic formulation. The graded twists are subtle but essential, and we
provide many computable examples (§3.1-3.2). Because tori appear on both sides of the duality, the duality becomes a tool to compute certain (previously unknown) twisted equivariant $K$-theory groups “for free”, and to supplement spectral sequence methods by resolving extension problems (§8.3). We emphasise that the torsion part is of particular interest in physics, so rational methods are generally insufficient. We also define a crystallographic Fourier–Mukai transform $T^\text{FM}_G$ (§4.2.1), which is expected to also implement $T_G$.

**Solid state physics applications.** The RHS of the crystallographic T-duality, $K^{-\bullet-d+i\tau}_G(T^d)$, is supposed to be the group of bulk topological crystalline insulator phases (roughly: equivalence classes of $\mathcal{G}$-invariant Hamiltonians with a spectral gap at zero), assuming that one is working in the single-particle (i.e. non-interacting) framework [18, 48, 46, 47]. For $\bullet = 0$, these are Class A insulators, whereas $\bullet = 1$ is relevant for Class AIII ones which have an additional chiral symmetry (an odd “supersymmetry”).

Quite aside from tabulating the possible topological phases, it is critical that the actual experimental signatures of nontrivial topological (crystalline) insulators are expected to be topological zero modes localised at an appropriate boundary cut into the sample. Thus the somewhat “invisible” bulk topological invariant $K^{-\bullet-d+i\tau}_G(T^d)$ appears on the boundary as a phenomenon which is simultaneously topological and analytic in nature. This suggests that the so-called bulk-boundary correspondence is index-theoretic in nature, and indeed justifies the appropriateness of the $K$-theoretic classification in the first place.

In the non-crystalline case (i.e. $\mathcal{G} = \Pi \cong \mathbb{Z}^d$), such correspondences have been studied in mathematical physics for some time [27, 23, 40, 9], and with nonequivariant H-flux introduced in [24, 25]. An approach using coarse geometry and $C^*$-algebras appears in [34] and some crystalline symmetries were studied there. T-duality as a topological Fourier transform was introduced into this field in [37], and used to understand why certain Gysin (topological index) maps should implement the bulk-to-boundary homomorphisms [24, 25].

The main difficulty in studying the general crystallographic bulk-boundary correspondence rigorously is that the appropriate “index” for the topological boundary zero modes is not known. A major insight of [22] is that the symmetries of the boundary should be generalised from a lower-dimensional space group, to a subperiodic group, which is furthermore graded based on the data of how the boundary sits inside the bulk (§8.2). The linear space of zero modes should therefore allow for a graded representation of the boundary symmetries. The framework of graded groups and graded equivariant twistings of $K$-theory allows us to formulate appropriate “super-indices” for the exotic topological boundary modes arising in physics.

**Notation:** $\mathbb{Z}_2$ denotes the 2-element group \{±1\} written multiplicatively, while $\mathbb{Z}/2 = \{0, 1\}$ is the additive version. When necessary, objects (e.g. bundle, projection map, twist) on one side of a T-duality are denoted with a small hat ($\hat{\cdot}$) to distinguish them from similar objects on the other side. The Pontryagin dual of an abelian group $\mathcal{A}$ is denoted by $\hat{\mathcal{A}}$ (wide hat). Equations involving $K$-theory groups $K^{\bullet}(\cdot)$ hold for each $\bullet \in \mathbb{Z}/2$. 4
2 Generalities on crystallographic space groups

Let $\mathbb{R}^d$ be $d$-dimensional (affine) Euclidean space, which can be identified with its vector group $\mathbb{R}^d$ of translations upon choosing an origin. The Euclidean group $E(d)$ of isometries of $\mathbb{R}^d$ is then isomorphic to the semidirect product $E(d) \cong \mathbb{R}^d \rtimes O(d)$ where $O(d)$ is the orthogonal group fixing the origin.

**Definition 2.1** (e.g. [28, 44]). A $d$-dimensional crystallographic space group, or simply space group, is a discrete cocompact subgroup $G \subset E(d)$.

From various classical theorems of Bieberbach [8], the lattice $\Pi := G \cap \mathbb{R}^d$ of translations in $G$ is free abelian of rank $d$ (so isomorphic to $\mathbb{Z}^d$), with finite quotient $G = G/\Pi$. In fact, an abstract group $G$ is characterised as a $d$-dimensional space group, by virtue of it having a finite-index normal free abelian subgroup of rank $d$ which is maximal abelian [53].

To summarize, there is a commutative diagram of groups

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}^d & \longrightarrow & E(d) & \longrightarrow & O(d) & \longrightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Pi \cong \mathbb{Z}^d & \longrightarrow & G & \longrightarrow & G/\Pi & \longrightarrow & 1 \\
\end{array}
$$

in which the vertical maps are inclusions. In crystallography, the subgroup $\rho : G \hookrightarrow O(d)$, is called the point group. Note that $G$ need not be isomorphic to a semi-direct product $\mathbb{Z}^d \rtimes G$; if it is, we say that $G$ is symmorphic.

2.1 Actions of the point group

2.1.1 Linear and affine actions on position space torus

Since $\Pi$ is normal in $G$, the Euclidean action of $G$ on $\mathbb{R}^d$ descends to the $\Pi$-orbit space $T^d = \mathbb{R}^d/\Pi$, which is an affine torus with translation group $\mathbb{R}^d/\Pi \cong T^d$. Thus, there is a homomorphism $G = G/\Pi \rightarrow \text{Isom}(T^d)$.

More concretely, pick an origin for $\mathbb{R}^d$ and thus $T^d$. Then $\text{Isom}(T^d) \cong T^d \rtimes \text{Aut}(T^d)$ and we obtain a homomorphism

$$
\alpha \equiv (s, \alpha) : G \rightarrow T^d \rtimes \text{Aut}(T^d),
$$

with $s : G \rightarrow T^d$ the “translational” part and $\alpha : G \rightarrow \text{Aut}(T^d)$ the “linear” part fixing the origin. This terminology is based on the following. The linear action of $g \in G \subset O(d)$ on $\mathbb{R}^d$ taking $x \mapsto gx \in \mathbb{R}^d$ is, inside the Euclidean group, implemented by choosing a section

$$
g \mapsto \bar{g} \equiv (\bar{s}(g), g) \in G \subset \mathbb{R}^d \rtimes O(d) = E(d)
$$

and then conjugating $(x, 1) \in E(d)$ by $(\bar{s}(g), g)$ to get $(gx, 1)$. This conjugation preserves the subgroup of lattice translations $\Pi$, so there is an action $\alpha : G \rightarrow \text{Aut}(T^d)$ on the quotient group $T^d = \mathbb{R}^d/\Pi$. Note that the lifts $\bar{g}$, and thus the translational parts $\bar{s}(g) \in \mathbb{R}^d$, are specified up to a $\Pi$ ambiguity, so there is a
well-defined map $s : G \to T^d$. Points of $T^d$ are labelled, with respect to the origin, by equivalence classes $[x] \in T^d$ of translations $x \in \mathbb{R}^d$ modulo $\Pi$. Then the affine $G$-action $\alpha$ on $T^d$ can be written as

$$\alpha(g)[x] := [\bar{g} \cdot x] \equiv [\bar{s}(g) + gx] = [\bar{s}(g)] + [gx] = s(g) + \alpha(g)[x] = (s(g), \alpha(g))[x].$$

**Definition 2.2.** Let $\mathcal{G}$ be a $d$-dimensional space group with point group $G \subset \text{O}(d)$. We write $T^d_\mathcal{G}$ for the quotient torus $T^d = R^d / \Pi$ equipped with the induced affine $G$-action $\alpha$ described above.

We can identify $\Pi$ with $H_1(T^d, \mathbb{Z})$, and $\text{Aut}(T^d)$ with $\text{Aut}(\Pi)$ via the induced map on homology. Upon choosing a (not necessarily orthonormal) basis for $\Pi \cong \mathbb{Z}^d$, the injective homomorphism $\alpha : G \to \text{Aut}(T^d) = \text{Aut}((\Pi) \cong \text{GL}(d, \mathbb{Z})$ expresses $G$ as a finite subgroup of integral $d \times d$ matrices. The conjugacy class of $G$ in $\text{GL}(d, \mathbb{Z})$ is called the arithmetic crystal class. There may be several non-isomorphic space groups within the same arithmetic crystal class, due to possible translational parts in $\alpha \equiv (s, \alpha)$, and $s = 0$ gives the symmorphic $\mathbb{Z}^d \times_\alpha G$.

The fundamental domain $\mathcal{G} \backslash R^d = G \backslash (R^d / \Pi) = G \backslash T^d$ is an orbifold that is finitely covered by $T^d$. If $\alpha$ happens to be a free action of $G$ (thus $\mathcal{G}$ acts freely on $R^d$, is torsion-free, and is nonsymmorphic), then the fundamental domain is a flat manifold. An orbifold approach to crystallography can be found in [14].

**1D examples.** The 1D torus $T^1 = S^1 = \{u \in \mathbb{C} | |u| = 1\}$ admits three inequivalent $\mathbb{Z}_2$-actions: $S^1_{\text{triv}}, S^1_{\text{flip}}$ and $S^1_{\text{free}}$ have the trivial action $u \mapsto u$, the flip action $u \mapsto \bar{u}$ and the free action $u \mapsto -u$ respectively.

The two space groups in 1D are $\mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}_2$ with $\mathbb{Z}_2$ acting by reflection on $\mathbb{R} \supset \mathbb{Z}$, and are sometimes referred to as $\text{p1}, \text{p1m1}$ respectively. The $G$-tori $R^1 / \Pi = T^1$ are respectively $T^1_{\text{p1}} = S^1_{\text{triv}}$ (trivial $G$), and the involutive space $T^1_{\text{p1m1}} = S^1_{\text{flip}}$.

The two other involutive circles $S^1_{\text{triv}}, S^1_{\text{free}}$ do not come directly from 1D space groups, but they appear in a generalisation to frieze groups (Section 7.1).

**2D examples.** There are 13 arithmetic crystal classes in $d = 2$, whereas there are 17 wallpaper groups (2D space groups); the four extra ones are nonsymmorphic, see Table 1. For example, $\text{pm} \cong \mathbb{Z}^2 \times \mathbb{Z}_2$ has point group $\mathbb{Z}_2$ acting by reflection in one coordinate, while the nonsymmorphic version $\text{pg}$ has instead a glide reflection — a reflection followed by half a lattice translation along the orthogonal coordinate — which squares to a lattice translation, so it is of infinite order (§6.2.2). The quotient of the 2-torus $T^2_\text{pg}$ by its free involution $\alpha$ is the Klein bottle, whose torsion-free fundamental group recovers $\text{pg}$.

### 2.1.2 Crystallography and group cohomology

Unlike $\alpha$, the map $s$ is not generally a homomorphism but satisfies the condition

$$s(g_1g_2) = s(g_1) + \alpha(g_1)(s(g_2)).$$
Thus $s$ is a group 1-cocycle with values in $\mathbb{T}^d$ (regarded as a $G$-module via $\alpha$). A different choice of origin shifted by $t \in \mathbb{T}^d$ causes $(s(g), \alpha(g)) \in \mathbb{T}^d \times \text{Aut}(\mathbb{T}^d)$ to be conjugated by $(t, 1)$ into $(s'(g), \alpha(g)) = (t + s(g) - \alpha(g)(t), \alpha(g))$, thereby modifying $s$ by the 1-coboundary $g \mapsto t - \alpha(g)(t)$. Therefore, it is only the cohomology class $[s] \in H^1_{\text{group}}(G, \mathbb{T}^d)$ which matters.

We may specify all the possible space groups within an arithmetic crystal class $\alpha : G \hookrightarrow \text{GL}(d, \mathbb{Z})$ by specifying a group cohomology class $[s] \in H^1_{\text{group}}(G, \mathbb{T}^d)$, see e.g. Theorem 5.2 in [28]. Via the connecting homomorphism $\delta$ coming from the exact sequence of $G$-modules $0 \rightarrow \Pi \rightarrow \mathbb{R}^d \rightarrow \mathbb{T}^d \rightarrow 0$, we have $H^1_{\text{group}}(G, \mathbb{T}^d) \cong H^2_{\text{group}}(G, \Pi)$, so that we are equivalently looking for inequivalent extensions of the point group $G$ by $\Pi$ (e.g. §3.4 of [44], remark after Theorem 5.2 of [28]). Explicitly, a lifting map $\tilde{s}$ as in Eq. (2) determines the 2-cocycle

$$\nu(g, h) := \delta s(g, h) \equiv \tilde{s}(g) + g \cdot \tilde{s}(h) - \tilde{s}(gh) \in \Pi, \tag{4}$$

which twists the product rule in $\Pi \times \alpha : G$ to give the space group $\mathcal{G}$ as an extension of $G$ by $\Pi$. The extension is symmorphic iff the cocycle class of $\nu$ is trivial. Starting from $\mathcal{G}$, we see that its $\Pi$-valued 2-cocycle is

$$\nu'(g, h) := \tilde{g} \tilde{h} \tilde{g}^{-1} \equiv (\tilde{s}(g), g)(\tilde{s}(h), h)(\tilde{s}(gh), gh)^{-1} = (\tilde{s}(g) + g \cdot \tilde{s}(h) - \tilde{s}(gh), 1)$$

recovering the formula Eq. (4). To emphasise that (the class of) $s$ is determined by $\mathcal{G}$, we shall sometimes write the affine torus action as $\alpha = (s_{\mathcal{G}}, \alpha)$.

### 2.1.3 Dual action on Brillouin torus

Under Pontryagin duality $\text{Hom}(\cdot, \mathbb{U}(1)) \equiv \hat{\cdot}$, conventionally denoted with a wide hat, the sequence of abelian groups

$$0 \rightarrow \Pi \cong \mathbb{Z}^d \rightarrow \mathbb{R}^d \rightarrow \mathbb{T}^d \rightarrow 0$$

| p1 | p2 | p3 | p4 | p6 | pm | pg | cm | pmm | pmg | pgg | cmm | p3m1 | p31m | p4m | p4g | p6m |
|----|----|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1  | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_4$ | $\mathbb{Z}_6$ | $D_1$ | $D_1$ | $D_2$ | $D_2$ | $D_3$ | $D_3$ | $D_4$ | $D_4$ |

Table 1: The thirteen arithmetic crystal classes, with symmorphic representatives listed first. $D_n$ is the dihedral group of order $2n$. Note that we have written $D_1$ here for the point group of pm, pg, cm to emphasise that it contains a reflection, whereas p2 with isomorphic point group $\mathbb{Z}_2$ contains a rotation.
are two Remark 2.3. Since we can think of $\hat{\chi}_{\alpha}$ of not be conjugate to its contragredient subgroup (inverse transpose). In group (which would give back the lattice $\Pi$).

Confused with taking the Pontryagin dual of the Brillouin torus as an abelian hat notation is usual in the string theory literature. This is not to be necessarily conjugate. This is due to the fact that a subgroup of $\hat{G}$ is a group 2-cocycle with values in the dual group for the $\mathbb{Z}$ torsional part, so it is itself associated to the twist $\tau$ with no translational part, so it is itself associated to the position space side in the affine action, where

Whether $\hat{\alpha} : G \to \text{GL}(d, \mathbb{Z}) \cong \text{Aut}(\Pi)$, there is a canonical (linear) dual action $\hat{\alpha}$ of $G$ on the Brillouin torus $T^d = \hat{\Pi}$, defined in the usual way: $\hat{\alpha}(g)(\chi) = \chi \cdot \alpha(g^{-1})$, $\chi \in T^d = \hat{\Pi}$. For convenience, we also write this as the equation

$$(g \cdot \chi)(n) = \chi(g^{-1} \cdot n), \quad g \in G, \ \chi \in T^d = \hat{\Pi}, \ n \in \Pi.$$  

**Remark 2.3.** Since we can think of $\hat{\alpha}$ as $\hat{\alpha} : G \to \text{Aut}(\Pi)$, there are two maps $\alpha, \hat{\alpha}$ into $\text{GL}(d, \mathbb{Z})$ (upon choosing bases for $\Pi, \Pi^\perp$), which are not necessarily conjugate. This is due to the fact that a subgroup of $\text{GL}(d, \mathbb{Z})$ need not be conjugate to its contragredient subgroup (inverse transpose). In $d = 2$, $\alpha$ and $\hat{\alpha}$ are conjugate for any space group, except for $p\overline{3}1m$ and $p3m1$ (Lemma 2.4 of [20]), see Section 6.3. In higher dimensions, the general relation between $\alpha$ and $\hat{\alpha}$ appears to be difficult to ascertain, but see [38] for some 3D examples.

### 2.1.4 Dual cocycle on Brillouin torus

Whether $\mathcal{G}$ is symmorphic or not, the Brillouin torus $T^d$ is a $G$-space under $\hat{\alpha}$ with no translational part, so it is itself associated to the symmorphic space group for the dual arithmetic crystal class $\hat{\alpha}$ of $\mathcal{G}$. To achieve a full duality, the nonsymmorphicity data $s$ should also appear on the Brillouin torus side.

Let us write $g \cdot \chi := \hat{\alpha}(g)(\chi)$ for $\chi \in \hat{\Pi} = T^d$ to simplify notation. The group 2-cocycle $\nu : G \times G \to \Pi$ for $\mathcal{G}$ has a Fourier transformed version as a $U(1)$-valued function $\tau_\mathcal{G} : G \times G \to C(\Pi, U(1)) \equiv U(C(T^d))$; explicitly,

$$\tau_\mathcal{G}(g_1, g_2)(\chi) = (g_1 g_2 \cdot \chi)(\nu(g_1, g_2)) \in U(1), \quad g_1, g_2 \in G. \quad (5)$$

The algebra $C(T^d)$ and also its unitary group $U(C(T^d))$ admits a natural left action of $G$ by taking $(g \cdot f)(\chi) := f(g^{-1} \cdot \chi)$, $f \in C(T^d)$. Then we see that $\tau_\mathcal{G}$ is a group 2-cocycle with values in the $G$-module $U(C(T^d))$. As explained in §3.1, the class of $\tau_\mathcal{G}$ in $H^2_{\text{group}}(G, U(C(T^d)))$ can be regarded as a $G$-equivariant twist $\tau_\mathcal{G} \in H^2_C(T^d, \mathbb{Z})$.

**Idea of crystallographic T-duality.** The nonsymmorphicity data of a space group appears on the position space side in the affine action, whereas
it is a $K$-theory twist on the momentum space side. The basic idea behind
crystallographic T-duality for a space group $G$, is that the position space data
$\alpha = (s_G, \alpha)$ determines dual data $(\hat{\alpha}, \tau_G)$ in momentum space, and that despite
this drastic-looking change, the $G$-equivariant $K$-(co)homology theories adapted
to $(T^d, (s_G, \alpha))$ and $(\hat{T}^d, (\hat{\alpha}, \tau_G))$ are isomorphic in a natural way. Its precise
statement requires a discussion of graded $K$-theory twists.

3 Generalities on twistings of $K$-theory

In what follows, we are concerned with compact spaces $E$ equipped with contin-
uous actions of a finite group $G$. The transformation groupoid $E//G$ is a special
case of a local quotient groupoid, and its complex (equivariant) $K$-theory has
a category $\text{Twist}(E//G)$ of twists, whose isomorphism classes $\pi_0(\text{Twist}(E//G))$
fit into a short exact sequence of groups $1 \rightarrow H^3_G(E, \mathbb{Z}) \rightarrow \pi_0(\text{Twist}(E//G)) \rightarrow H^1_G(E, \mathbb{Z}_2) \rightarrow 1$.

Under the bijection $\pi_0(\text{Twist}(E//G)) = H^3_G(E, \mathbb{Z}) \times H^1_G(E, \mathbb{Z}_2)$ of
sets, the group law is

$$(h_1, w_1)(h_2, w_2) = (h_1 + h_2 + \beta(w_1 \cup w_2), w_1 + w_2), \quad (6)$$

where $\beta : H^2_G(E, \mathbb{Z}_2) \rightarrow H^3_G(E, \mathbb{Z})$ is the Bockstein homomorphism associated
to the mod 2 reduction $\mathbb{Z} \rightarrow \mathbb{Z}_2$ of coefficients. The ungraded twists have $w = 0$.

In the non-equivariant case, $h \in H^3(E, \mathbb{Z})$ is the Dixmier–Douady invariant
when the $K$-theory of $E$ twisted by $h$ is modelled by gerbes [10] or by continuous-
trace algebras [41], and is also called a H-flux in string theory. In solid state
physics, it arises after a partial Fourier transform of a screw-dislocated lattice
[24], see Section 6.2.3. In the equivariant world, $H^3_G(E, \mathbb{Z})$ need not vanish
even if dim$(E) < 3$. In fact, there is a nice interpretation of the various “lower-
dimensional” terms that appear in the Leray–Serre spectral sequence computing
$H^3_G(E, \mathbb{Z})$ [20], intimately related to crystallographic groups when $E$ is a torus.
Some examples are given in Section 3.1.

Much less studied are the gradings $w \in H^1_G(E, \mathbb{Z}_2)$ of twists. This co-
homology group classifies $G$-equivariant real line bundles, or equivalently $G$-
equivariant principal O(1) bundles over $E$, which we can think of as an “orientation
field”. Graded twists are required in crystallographic T-duality because
of orientation reversing operations like reflections; they are needed to equivari-
antly implement push-forwards and Poincaré duality.

3.1 Examples of special equivariant $H^3$ twists

**Twist from $U(1)$ central extension of group.** For a finite group $G$ acting
on a point, $H^3_G(\text{pt}, \mathbb{Z}) \cong H^3(BG, \mathbb{Z}) \cong H^3_{\text{group}}(G, \mathbb{Z}) \cong H^2_{\text{group}}(G, U(1))$, so the
equivariant $H^3$-twists come from $U(1)$-valued group 2-cocycles of $G$, i.e. central
extensions of $G$ by $U(1)$. This cannot occur for $G = \mathbb{Z}_n$, but $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ has a non-trivial 2-cocycle $\omega$ specified by

$$\omega(((1)^{k_1}, (1)^{k_2}), ((1)^{l_1}, (1)^{l_2})) = (1)^{k_1 l_1},$$

which generates $H^3_{D_2}(pt, \mathbb{Z}) \cong \mathbb{Z}/2$. When a $D_2$-space $E$ has a fixed point, the pullback of $H^3_{D_2}$ from a point to $E$ is split injective and we continue to write the pullback as $\omega \in H^3_{D_2}(E, \mathbb{Z})$. Similarly, if $G \to D_2$ splits, we continue to write $\omega$ for its pullback in $H^3_{G}(pt, \mathbb{Z})$.

For the dihedral groups $D_n$ of order $2n$, which appear as point groups in crystallography, it is known that (e.g. Theorem 5.2 of [26])

$$H^3_{D_n}(pt, \mathbb{Z}) = H^3_{\text{group}}(G, \mathbb{Z}) = \begin{cases} 0 & n \text{ odd}, \\ \mathbb{Z}/2 & n \text{ even}. \end{cases}$$

For $n$ even, there are split surjections $D_n \to D_2$ so $\omega$ generates $H^3_{D_n}(pt, \mathbb{Z})$.

**Twist from group 2-cocycle in** $U(C(E))$. We have $H^3_{\mathbb{Z}_2}(S^1_{\text{triv}}, \mathbb{Z}) \cong H^3(S^1, \mathbb{Z}) = 0$, while $H^3_{\mathbb{Z}_2}(S^1_{\text{triv}}, \mathbb{Z}) \cong 0$ and $H^3_{\mathbb{Z}_2}(S^1_{\text{triv}}, \mathbb{Z}) \cong \mathbb{Z}/2$ are recalled in §5.1. The generating twist for the latter is represented by a group 2-cocycle $\tau_S$ for $\mathbb{Z}_2$ with coefficients in the (trivial) $\mathbb{Z}_2$-module $U(C(S^1_{\text{triv}}))$,

$$\tau_S(-1, -1) = \{k \mapsto e^{ik}\} \in U(C(S^1_{\text{triv}})).$$

(8)

In fact, $\tau_S$ is the Fourier transformed version, in the sense of §2.1.4, of the $\mathbb{Z}$-valued group 2-cocycle $\nu$ corresponding to the extension

$$0 \to \mathbb{Z} \xrightarrow{x^2} \mathbb{Z} \xrightarrow{(-1)^i} \mathbb{Z}_2 \to 1,$$

which has value $\nu(-1, -1) = 1$ and $0$ otherwise.

Generally, for a $\mathbb{Z}_2$-space $E$ with an equivariant map to $S^1_{\text{triv}}$, we will continue to write $\tau_S$ for its pullback to $E$, unless several such maps are possible and cause ambiguity.

**Non-cocycle twists.** For $T^2$, there may be general $G$-equivariant $H^3$-twists $h$ which are not represented by a 2-cocycle. Examples arising from crystallography will be discussed in Section 6.2.3. We mention that these non-cocycle twists can be represented by central extensions of the groupoid $T^2/G$, see [20] for a discussion. In higher dimensions, there may also be $H^3$ twists which are non-equivariantly nontrivial.

### 3.2 Examples of special equivariant $H^1$ twists

**$d = 0$: Twist from grading of group.** For a finite group $G$ acting on a point, $H^1_G(pt, \mathbb{Z}_2) \cong \text{Hom}(H_1(BG, \mathbb{Z}), \mathbb{Z}_2) \cong \text{Hom}(G, \mathbb{Z}_2)$, so a $H^1$-twist is essentially a homomorphism $c : G \to \mathbb{Z}_2$ making $G$ into a graded group. The equivariant line bundle over pt is just $pt \times \mathbb{R}$ with $g \in G$ acting by multiplication by $c(g)$.

The pullback of $c$ from pt to a $G$-space $E$ gives a twist in $H^1_G(E, \mathbb{Z}_2)$, corresponding to the $G$-equivariant product line bundle $E \times \mathbb{R}$ with $G$ action taking
the fiber over $x \in E$ to the fiber over $g \cdot x$ followed by multiplication by $c(g)$; we call such twists $c$-type. When there is only one possible surjective homomorphism, e.g. when $G$ is $\mathbb{Z}_2$ or $D_3$, we will just write $c$ for the unique nontrivial $H^1$-twist of $c$-type.

For a space group $\mathcal{G}$ with point group $\rho : G \to O(d)$, there is a distinguished twist from the orientability homomorphism

$$c_\rho : G \ni g \mapsto \det g \in \mathbb{Z}_2.$$  

Note that $\mathcal{G}, \mathcal{G}'$ can have the same $G$ as an abstract group but different $c_\rho$, e.g. $p2$ contains rotations so $c_{p2}$ is the trivial map, while $pm$ contains a reflection so $c_{pm}$ is the identity map on $G \cong \mathbb{Z}_2$.

$d = 1$: Möbius-type equivariant twists on circles. If $E$ has a $G$-fixed point, there is a splitting $H^1_G(E, \mathbb{Z}_2) = H^1_G(\text{pt}, \mathbb{Z}_2) \oplus \overline{H}^1_G(E, \mathbb{Z}_2)$, and the reduced part may be represented by non-trivial line bundles over $E$ made $G$-equivariant. This is a consequence of a calculation with the Leray–Serre spectral sequence. In such cases, equivariant $H^1$-twists come in $c$-type (from a point), in “$M$-type” (for Möbius, coming from the reduced part), or a sum of the two types.

For $S^1_{\text{triv}}$, we have

$$H^1_{\mathbb{Z}_2}(S^1_{\text{triv}}, \mathbb{Z}_2) \cong H^1_{\mathbb{Z}_2}(\text{pt}, \mathbb{Z}_2) \oplus \overline{H}^1_{\mathbb{Z}_2}(S^1_{\text{triv}}, \mathbb{Z}_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

where the first generator $c$ comes from $\mathbb{Z}_2 \xrightarrow{id} \mathbb{Z}_2$, and the second generator $M$ is the Möbius bundle over $S^1$ with $\mathbb{Z}_2$ acting trivially on the total space. The mixed twist $c + M$ is the Möbius bundle with $-1 \in \mathbb{Z}_2$ acting fiberwise by multiplication by $-1$.

For $S^1_{\text{flip}}$, we have

$$H^1_{\mathbb{Z}_2}(S^1_{\text{flip}}, \mathbb{Z}_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$

with one generator $c$ as above. The other generator $M$ is the Möbius line bundle with $\mathbb{Z}_2$-action given locally by $(e^{ik}, v) \mapsto (e^{-ik}, v), k \in (-\frac{2\pi}{3}, \frac{2\pi}{3})$ and $(e^{ik}, v) \mapsto (e^{-ik}, -v), k \in (\frac{2\pi}{3}, \frac{4\pi}{3})$. Thus on the fiber over the fixed point $k = 0$ (resp. $k = \pi$), the $\mathbb{Z}_2$-representation is trivial (resp. sign). Similarly, the mixed twist $c + M$ is the Möbius bundle with trivial (resp. sign) representation at $k = \pi$ (resp. $k = 0$).

For $S^1_{\text{tree}}$, we have

$$H^1_{\mathbb{Z}_2}(S^1_{\text{tree}}, \mathbb{Z}_2) \cong H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}/2.$$

The generator $c$ is pulled back from a point; explicitly, take the product bundle $S^1_{\text{tree}} \times \mathbb{R}$ with involution $(e^{ik}, v) \mapsto (-e^{ik}, -v), k \in [0, 2\pi]/0\sim 2\pi$.

### 3.3 Twisted composition rule for $H^3$ and $H^1$ twists

An important example where the composition rule for graded twists, Eq. (6), is modified nontrivially, is $\text{pt} // D_2$. From the usual cohomologies of $BD_2 =$
$BZ_2 \times BZ_2$, we obtain the $D_2$-equivariant cohomologies of pt:

| n = 0 | n = 1 | n = 2 | n = 3 |
|-------|-------|-------|-------|
| $H^0_{D_2}(pt, Z)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^3$ |
| basis | 1 | $c_1, c_2$ | $c_1^2, c_1 c_2, c_2^2$ | $c_1^2, c_1 c_2, c_1 c_2^2, c_2^3$ |
| $H^1_{D_2}(pt, Z)$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| basis | 1 | 0 | $t_1, t_2$ | $\omega$ |

Here, the generators $c_i \in H^1_{D_2}(pt, Z_2) \cong Z_2^2$ come from the $i$-th projections $p_i : D_2 = Z_2 \times Z_2 \rightarrow Z_2$, $i = 1, 2$. We had already seen that $H^1_{D_2}(pt, Z) \cong Z_2$, generated by the group 2-cocycle $\omega$ in Eq. (7).

**Proposition 3.1.** For the composition of graded twists of $pt//D_2$, we have $(0, c_1) + (0, c_1) = (0, 0), i = 1, 2$, but

$$(0, c_1) + (0, c_2) = (0, c_2) + (0, c_1) = (\omega, c_1 + c_2).$$

**Proof.** Putting the $D_2$-equivariant cohomology groups for pt tabulated above into the Bockstein sequence

$$\cdots \rightarrow H^0_{D_2}(pt, Z) \xrightarrow{\partial} H^0_{D_2}(pt, Z) \rightarrow H^3_{D_2}(pt, Z_2) \xrightarrow{\partial} H^2_{D_2}(pt, Z) \rightarrow \cdots,$$

we find that $\beta(c_i) = x, \beta(c_i^2) = 0, i = 1, 2$ and $\beta(c_1 c_2) = \omega$. \hfill \Box

Although $\pi_0(\text{Twist}(pt//D_2)) \cong Z_2^3$, it is not split as $H^3_{D_2}(pt, Z) \times H^3_{D_2}(pt, Z_2)$.

### 3.4 Equivariant $K$-orientability of torus

Non-equivariantly, the torus $T^d$ is Spin$^c$ thus $K$-oriented, and Poincaré duality has the simple form $K_*(T^d) \cong K^{d-*}(T^d)$. If $T^d$ is replaced by an oriented compact $d$-manifold $M$, a twist by the Spin$^c$ obstruction class $W_3(M) \in H^3(M, Z)$ is needed on the $K$-theory side, e.g. Prop. 9.2 of [1].

When $T^d$ has an action $\alpha$ of a finite group $G$, the obstruction to it being oriented in $G$-equivariant $K$-theory is

$$\sigma = (W^G_3(T^d), w^G_1(T^d)) \in H^3_G(T^d, Z) \times H^1_G(T^d, Z_2),$$

where $W^G_3(T^d)$ and $w^G_1(T^d)$ are respectively the $G$-equivariant third integral Stiefel–Whitney class and first Stiefel–Whitney class of the tangent bundle $TT^d$ of $T^d$. The Poincaré duality in this case is

$$K^G_*(T^d) \cong K^{d-*}_G(T^d),$$

a special case of general dualities, e.g. Theorem 2.1 of [49], Theorem 2.9 of [16].

We are interested in $T^d_G$ as in Definition 2.2, i.e. the affine $G$-actions $\alpha$ on $T^d$ arising from a space group $\mathcal{G}$, as explained in Section 2.1. Recall that $g \in G$ is an orthogonal transformation under $\rho : G \hookrightarrow O(d)$, and via a lift $\tilde{g} = (\tilde{s}(g), g) \in \mathcal{G} \subset \mathbb{R}^d \times O(d)$ specified by a splitting map $\tilde{s} : G \rightarrow \mathbb{R}^d$, there is the induced $G$-action $\alpha$ on $T^d_G$ by $\alpha(g)[x] = [\tilde{s}(g) + gx]$ as in Eq. (3).
Lemma 3.2. Let $\mathcal{G}$ be a space group and $\alpha$ be the associated affine action of the point group $G \subset O(d)$ on $T^d_{\mathcal{G}}$. The tangent bundle of $T^d_{\mathcal{G}}$ is a $G$-equivariant vector bundle isomorphic to $T^d_{\mathcal{G}} \times \mathbb{R}^d$ with the product $G$-action.

Proof. The tangent bundle $TR^d$ of $R^d$ is trivialised by the translation action of $\mathbb{R}^d$, and so an element $(r, O) \in \mathbb{R}^d \times O(d) = \delta'(d)$ acts on $TR^d = R^d \times \mathbb{R}^d$ by $(x, v) \mapsto (r + O x, O v)$ in this trivialisation. In particular, a lift $\tilde{g}$ of $g \in G$ in $\mathcal{G}$ acts this way, and passing to quotients we get

$$TT^d_{\mathcal{G}} = TR^d/\Pi = T^d_{\mathcal{G}} \times \mathbb{R}^d \ni ([x], v) \mapsto (\tilde{g}(g)x, gv) = (\alpha(g)[x], gv).$$

From this lemma, the obstruction classes $W_3^G(T^d_{\mathcal{G}}), w_1^G(T^d_{\mathcal{G}})$ are respectively the pullback of

$$W_3^G(\mathbb{R}^d_\rho) \in H^3_G(pt, \mathbb{Z}), \quad w_1^G(\mathbb{R}^d_\rho) \in H^1_G(pt, \mathbb{Z}_2), \quad (13)$$

where $\mathbb{R}^d_\rho \to pt$ is the $G$-representation given by $\rho : G \to O(d)$. Then $w_1^G(\mathbb{R}^d_\rho)$ vanishes iff $\rho$ is orientable, i.e. factors through $SO(d)$. Thus under the identification $H^1_G(pt, \mathbb{Z}_2) \cong \text{Hom}(G, \mathbb{Z}_2)$, we have $w_1^G(\mathbb{R}^d_\rho)$ being the orientability homomorphism $c_{\mathcal{G}}$ of Eq. (9).

Similarly, $W_3^G(\mathbb{R}^d_\rho)$ vanishes iff $\rho$ is $\text{Pin}^c$, factoring through the projection in

$$1 \to U(1) \to \text{Pin}^c(d) \to O(d) \to 1. \quad (14)$$

It is well-known that for finite $G$, central extensions of $G$ by $U(1)$ are classified by $H^2_{\text{group}}(G, U(1)) \cong H^3_G(pt, \mathbb{Z})$. Then $W_3^G(\mathbb{R}^d_\rho) \in H^3_G(pt, \mathbb{Z})$ corresponds to the pullback of Eq. (14) under $\rho : G \to O(d)$, which only depends on the point group. For the 2D space groups, only the point groups $G = D_2, D_4, D_6$ have a potential obstruction ($0 \neq H^2_G(pt, \mathbb{Z}) \cong \mathbb{Z}/2$ in these cases), and by Lemma A.2 in the Appendix, the obstruction in all these cases are nontrivial and therefore equal to the unique nontrivial $\omega$.

Remark 3.3. The pullbacks of $\omega$ from $pt$ to $T^2_{\mathcal{G}}$ for $\mathcal{G} = \text{pgg}$, $\text{pmg}$ (with point group $G = D_2$) can be shown to vanish. In these cases, there are maps of groupoids $T^2_{\text{pgg/pmg}} \parallel D_2 \to S^1_{\text{lip} \times \text{free}} \parallel D_2 \to pt \parallel D_2$ and one takes a quotient to see that $H^3_{D_2}(S^1_{\text{lip} \times \text{free}}, \mathbb{Z}) \cong H^3_{\mathbb{Z}_2}(S^1_{\text{lip}}, \mathbb{Z}) = 0$ (§5.1). On the $T$-dual side, on the other hand, the pullback of $\omega$ to the Brillouin torus $T^2_{\text{pmm}}$ is nonzero. Nevertheless, it was observed in [46] that there is a $D_2$-equivariant automorphism of $T^2_{\text{pmm}}$ which takes $\tau_{\mathcal{G}}$ to $\tau_{\mathcal{G}} + \omega$, so there is consistency with the vanishing of the pullback of $\omega$ in $H^3_{D_2}(T^2_{\mathcal{G}}, \mathbb{Z}).$

Definition 3.4. For the $G$-space $T^d_{\mathcal{G}}$ of Definition 2.2, we will write its $K$-orientability obstruction class $\sigma = (W_3^G(T^d_{\mathcal{G}}), w_1^G(T^d_{\mathcal{G}}))$ as $\sigma_{\mathcal{G}}.$
4 Crystallographic T-duality

4.1 T-duality of circle bundles

4.1.1 Ordinary T-duality for a circle

The basic T-duality in string theory exchanges a circle $S^1$ of radius $L$ with a circle $\hat{S}^1$ of radius $\frac{1}{L}$. There are degree-shifted isomorphisms

$$K^\bullet(S^1) \xleftrightarrow{T} K^{\bullet-1}(\hat{S}^1)$$

It is important to remember that $S^1$ and $\hat{S}^1$ are not canonically the same space, so that Eq. (15) is not the (non-trivial) observation that $K^0(S^1) \cong \mathbb{Z} \cong K^{-1}(S^1)$, but is rather a type of topological Fourier transform. This is apparent from its implementation as a Fourier–Mukai transform [29], Eq. (17).

For applications in solid-state and quantum physics, it is useful to view the two circles $S^1, \hat{S}^1$ as coming from the group $\mathbb{Z}$ in two different ways. On the one hand, the affine $S^1$ is the quotient of the Euclidean line $\mathbb{R}$ by a lattice $\mathbb{Z}$ of translations, i.e. the unit cell. Since $\mathbb{R}$ is contractible, $S^1 = \mathbb{BZ}$ is a classifying space for $\mathbb{Z}$. On the other hand, the Pontryagin dual of $\mathbb{Z}$ is topologically another circle $\hat{S}^1$, namely the 1D Brillouin torus. Notice that if $S^1$ has radius $L$ in the Euclidean metric, then $\hat{S}^1$ has radius $\frac{1}{L}$ in the dual metric on $\hat{\mathbb{R}}$, exactly as in the string theory story.

As explained in Section 4.2, this duality between $S^1 = \mathbb{BZ}$ and $\hat{S}^1 = \hat{\mathbb{Z}}$ can be regarded in noncommutative topology language as a Baum–Connes isomorphism, which is useful for formulating our crystallographic generalisation in which $\mathbb{Z}$ is replaced by a space group.

4.1.2 Topology change from $H^3$-twists

When trying to formulate the T-dual of a circle bundle $E \to X$ with some $h \in H^3(E, \mathbb{Z})$, one finds the remarkable fact that the dual circle bundle $\hat{E} \to X$ may be non-isomorphic to $E$ [11]. Furthermore, a dual twist $\hat{h} \in H^3(\hat{E}, \mathbb{Z})$ which completes the duality in the reverse direction always exists.

More precisely, for a pair $(E, h)$ on $X$ as above, there exists another pair $(\hat{E}, \hat{h})$ on $X$ such that

$$\pi_* h = c_1(\hat{E}), \quad \hat{\pi}_* \hat{h} = c_1(E), \quad p^* h = \hat{p}^* \hat{h}$$

where $p : \pi^* \hat{E} \to E$ and $\hat{p} : \hat{\pi}^* \hat{E} \to \hat{E}$ are projections from the fiber product $E \times_X \hat{E} = \pi^* \hat{E} = \hat{\pi}^* E$ as summarized in the diagram

$$\begin{align*}
\begin{array}{ccc}
E & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
\hat{E} & \xleftarrow{\hat{\pi}} & \hat{X}
\end{array}
\end{align*}$$

$$p \quad \hat{p}$$
Such dual pairs enjoy the property that there is a T-duality isomorphism
\[ T : K^{*+b}(E) \cong K^{*-1+b}(\hat{E}). \]
In this way, there is a “conservation of topological invariants” on each side of the duality, even though the two sides may look very different.

The special case where \( X \) is a point has \( E = S^1, \hat{E} = \hat{S}^1 \), and recovers the basic circle T-duality in Eq. (15). There are no possible \( H^3 \)-twists, and an explicit formula is the Fourier–Mukai transform
\[ T_{\text{FM}} : K^0(S^1) \ni [L] \mapsto \hat{p}_*(\mathcal{P} \otimes p^*L) \in K^{-1}(\hat{S}^1), \]
where \( \mathcal{P} \to S^1 \times \hat{S}^1 \) is the Poincaré line bundle, recalled in §4.2.1.

4.2 Crystallographic T-duality and Baum–Connes assembly

Recall that the Baum–Connes assembly map [5] for a discrete group \( \mathcal{G} \) is a map
\[ K_{\text{eq}}(E\mathcal{G}) \xrightarrow{\mu} K_{\star}(C^r_\star(\mathcal{G})) \]
where the left-hand-side is the equivariant \( K \)-homology (with compact supports) of the universal space \( E\mathcal{G} \) for proper \( \mathcal{G} \)-actions, and the right-hand-side is the \( K \)-theory of the reduced group \( C^r \)-algebra of \( \mathcal{G} \).

A crystallographic space group \( \mathcal{G} \) is a discrete subgroup of the Euclidean group \( \mathbb{R}^d \rtimes O(d) \), and it also acts properly on \( \mathbb{R}^d \). Then we see that \( \mathbb{R}^d \) is an \( E\mathcal{G} \) (cf. [5] Section 2), and it has only finite isotropy groups. Furthermore, the lattice subgroup \( \Pi \) acts freely, so after quotienting by \( \Pi \), we can rewrite
\[ K_{\text{eq}}(E\mathcal{G}) \cong K_{\text{eq}}/\Pi(R^d/\Pi) = K_{\star}(T^d_{\mathcal{G}}), \]
where the finite point group \( G \) acts on \( T^d_{\mathcal{G}} \) by \( \alpha \) given in Eq. (1).

\( C^r_\star(\mathcal{G}) \) is a noncommutative \( C^* \)-algebra, but since \( \mathcal{G} \) is virtually abelian with twisted product \( \mathcal{G} = \Pi \rtimes_{\alpha,\nu} G \), we can understand \( C^r_\star(\mathcal{G}) \) in a “virtually commutative” way by decomposing it as a twisted crossed product [39]
\[ C^r_\star(\mathcal{G}) \cong C^r_\star(\Pi) \rtimes_{\alpha,\nu} G \cong C(\hat{T}^d) \rtimes_{(\hat{\alpha},\tau_{\mathcal{G}})} G, \]
where \( \tau_{\mathcal{G}} \) is the Fourier transformed 2-cocycle of Eq. (5) valued in the \( G \)-module \( U(C(\hat{T}^d)) \). The K-theory of Eq. (19) turns into a \( \tau_{\mathcal{G}} \)-twisted G-equivariant K-theory of \( T^d \) via a twisted Green–Julg theorem (Theorem 4.10 in [33]),
\[ K_{\star}(C^r_\star(\mathcal{G})) \cong K_{\star}(C(\hat{T}^d) \rtimes_{(\hat{\alpha},\tau_{\mathcal{G}})} G) \cong K^G_{\star+\tau_{\mathcal{G}}}(C(\hat{T}^d)) \cong K^G_{\star+\tau_{\mathcal{G}}}(T^d), \]
where \( G \) acts on \( \hat{T}^d \) via the dual action \( \hat{\alpha} \).

The Baum–Connes conjecture is verified for \( \mathcal{G} \) so that \( \mu \) in (18) is an isomorphism, which we rewrite using Eq. (20) as
\[ \mu : K^G_{\star}(T^d_{\mathcal{G}}) \xrightarrow{\cong} K^G_{\star+\tau_{\mathcal{G}}}(T^d), \]
We can convert the LHS to K-theory using \( \sigma_{\mathcal{G}} \)-twisted Poincaré duality Eq. (12). Assembling these isomorphisms, we finally obtain:
Proof. We can readily check the commutativity of the diagram

\[
\begin{array}{c}
\mathbb{R}^d \times \hat{\Pi} \times C \\
\downarrow n \\
\mathbb{R}^d \times \hat{\Pi} \times C
\end{array}
\quad
\begin{array}{c}
\mathbb{R}^d \times \hat{\Pi} \times C \\
\downarrow n g \\
\mathbb{R}^d \times \hat{\Pi} \times C
\end{array}
\]

\[
\begin{array}{c}
\tilde{\gamma}_\alpha(g) \\
\gamma_\alpha(g)
\end{array}
\]

\[
\begin{array}{c}
\mathbb{R}^d \times \hat{\Pi} \times C \\
\mathbb{R}^d \times \hat{\Pi} \times C
\end{array}
\]

Theorem 4.1 (Crystallographic T-duality). Let \( T^d_\mathcal{G} \) be the d-torus with the affine action \( \alpha \equiv (s_\mathcal{G}, \alpha) \) associated to a space group \( \mathcal{G} \), as in Definition 2.2, and let the graded twist \( \sigma_\mathcal{G} \) be its G-equivariant Spin\(^c\) obstruction class as in Definition 3.4. Let \( T^d \) be the d-torus with the dual G-action \( \hat{\alpha} \) and 2-cocycle twist \( \tau_\mathcal{G} \) as defined in Section 2.1.4. Then \( (T^d_\mathcal{G}, \sigma_\mathcal{G}) \) and \( (T^d, \tau_\mathcal{G}) \) are crystallographic T-dual in the sense that there is an isomorphism

\[
T^d_\mathcal{G} : K^{-\bullet+\sigma_\mathcal{G}}(T^d_\mathcal{G}) \xrightarrow{\cong} K^{-\bullet+\tau_\mathcal{G}}(T^d).
\]  

(21)

Remark 4.2. Twists from a homomorphism \( c : \mathcal{G} \to G \to \mathbb{Z}_2 \) are relevant whenever \( (\mathcal{G}, c) \) arises as a graded group in physical applications. We anticipate that such c-twists can be added to both sides of Eq. (21), amounting to the statement that a “super Baum–Connes assembly map” for \( (\mathcal{G}, c) \) is an isomorphism. We leave the verification and the application of these conjectures for a future work.

4.2.1 Crystallographic T-duality and Poincaré bundle

We wish to define a variant of the Fourier–Mukai transform adapted to \( \mathcal{G} \),

\[
\mathcal{T}^{FM}_\mathcal{G} : K^{-\bullet+\sigma_\mathcal{G}}(T^d_\mathcal{G}) \to K^{-\bullet+\tau_\mathcal{G}}(T^d).
\]  

(22)

Recall that the Poincaré line bundle \( \mathcal{P} \to \mathbb{T}^d \times \hat{\Pi} \) is defined as the quotient of the product line bundle \( \mathbb{R}^d \times \hat{\Pi} \times C \to \mathbb{R}^d \times \hat{\Pi} \) under the \( \Pi \cong \mathbb{Z}^d \) action,

\[
n : \mathbb{R}^d \times \hat{\Pi} \times C \ni (x, \chi, z) \mapsto (x + n, \chi(n)z), \quad n \in \Pi.
\]  

(23)

To incorporate \( \mathcal{G} \), choose an origin to identify \( T^d_\mathcal{G} \) with \( T^d = \mathbb{R}^d/\Pi \), and recall that the G-action \( \alpha = (s_\mathcal{G}, \alpha) \) on \( T^d_\mathcal{G} \cong \mathbb{T}^d \) is obtained by first picking a map \( \tilde{s} : G \to \mathbb{R}^d \) which lifts \( G \ni g \mapsto \tilde{g} = (\tilde{s}(g), g) \in \mathcal{G} \subset \mathbb{R}^d \times O(d) \), and then taking \( \alpha(g)[x] = [\tilde{s}(g) + gx] \) for \( x \in \mathbb{T}^d \cong T^d_\mathcal{G} \) (Eq. (3)). On \( \mathbb{R}^d \times \hat{\Pi} \times C \), we can further define a G-action \( \tilde{\gamma}_\alpha \)

\[
\tilde{\gamma}_\alpha(g) : \mathbb{R}^d \times \hat{\Pi} \times C \ni (x, \chi, z) \mapsto (\tilde{g} \cdot x, g \cdot \chi, z) \equiv (\tilde{s}(g) + gx, g \cdot \chi, z),
\]  

(24)

where we write \( g \cdot \chi \equiv \hat{\alpha}(g)(\chi) \).

Theorem 4.3. \( \tilde{\gamma}_\alpha \) descends to a twisted G-action \( \gamma_\alpha : \mathcal{P} \to \mathcal{P} \) on the Poincaré line bundle, with cocycle the pullback of \( \tau_\mathcal{G}^{-1} \in \mathcal{Z}^2(\mathcal{U}(C(\Pi))) \) under \( \tilde{p} : \mathbb{T}^d \times \hat{\Pi} \to \hat{\Pi} \).

Proof. We can readily check the commutativity of the diagram
for all \( g \in G, n \in \Pi \), so that each \( \tilde{\gamma}_\alpha(g) \) gives a bundle map \( \gamma_\alpha(g) : P \to P \) covering the \( G \)-action \( \alpha \times \hat{\alpha} \) on \( T^d \times \hat{\Pi} \). Using the action formulae Eq. (24) and Eq. (23), along with Eq. (4) and Eq. (5), we compute for \( g, h \in G \),

\[
\begin{align*}
\tilde{\gamma}_\alpha(g)\tilde{\gamma}_\alpha(h)[x, \chi, z] &= \tilde{\gamma}_\alpha(g)[\tilde{s}(h) + hx, h \cdot \chi, z] \\
&= [\tilde{s}(g) + g(\tilde{s}(h) + hx), gh \cdot \chi, z] \\
&= [\nu(g, h) + \tilde{s}(g, h) + ghx, gh \cdot \chi, z] \\
&= [\tilde{s}(g, h) + ghx, gh \cdot \chi, gh(\chi(-\nu(g, h)))z] \\
&= \tilde{\gamma}_\alpha(gh)[x, \chi, z]gh \cdot \chi(-\nu(g, h)) \\
&= (\tau_{gh}^{-1}(g, h)(\chi))\tilde{\gamma}_\alpha(gh)[x, \chi, z],
\end{align*}
\]

verifying that the \( \tilde{\gamma}_\alpha \) action on \( P \) is twisted by the pullback of \( \tau_{gh}^{-1} \) in \( Z^2(G, U(C(\hat{\Pi}))) \).

Thus \( P \) is a \( \tau_{gh}^{-1} \)-twisted \( G \)-equivariant line bundle, and the Theorem implies

**Corollary 4.4.** The dual Poincaré line bundle \( P^* \) trivialises \( \hat{p}^*\tau_{gh} \) as a twist in \( H^3_G(T^d \times \hat{\Pi}, \mathbb{Z}) \).

Corollary 4.4 allows us to construct a “Fourier–Mukai” transform adapted to \( \mathcal{G} \), based on the diagram (cf. Eq. (16)),

\[
\begin{array}{ccc}
P & \xrightarrow{\hat{p}} & \mathbb{T}^d \times \hat{\Pi} \\
\downarrow & & \downarrow \hat{p} \\
\mathbb{T}^d & \xrightarrow{\pi} & \hat{\Pi}
\end{array}
\]

as the composition

\[
K_G^{•+\sigma_\mathcal{G}}(\mathbb{T}^d) \xrightarrow{\mathbb{L}^\mathcal{G}} K_G^{•+p^*\sigma_\mathcal{G}}(\mathbb{T}^d \times \hat{\Pi}) \xrightarrow{\hat{p}^*} K_G^{•+p^*\sigma_\mathcal{G} + \hat{p}^*\tau_\mathcal{G}}(\mathbb{T}^d \times \hat{\Pi}) \xrightarrow{\hat{\Pi}^*} K_G^{•-d + \tau_\mathcal{G}}(\hat{\Pi}),
\]

where \( \sigma_\mathcal{G} \) is the \( K_G \)-orientability obstruction for \( \mathbb{T}^d \) needed for the push-forward \( \hat{p}_* \) along \( \mathbb{T}^d \). Recalling that \( T^d_\mathcal{G} \simeq \mathbb{T}^d \) and \( \hat{T}^d = \hat{\Pi} \), we obtain the desired map \( T^d_\mathcal{G} \) in Eq. (22).

It is anticipated that \( T^d_\mathcal{G} \) is an isomorphism, implementing crystallographic T-duality, Eq.(21), and that twists that are pulled back from the common base (a point in the above case) can be added to both sides, cf. Remark 4.2. The latter is a common feature of well known dualities like ordinary circle bundle T-duality \( T \), as well as \( T_{\mathbb{Z}_2}, T_R \) recalled below.

### 4.3 T-dualities for circle bundles with involution

There are several variants of ordinary \( K \)-theory groups that we can apply to ‘Real’ or involutive spaces, i.e., spaces \( X \) equipped with a continuous \( \mathbb{Z}_2 \) action.
x ↦→ \bar{x}. We shall assume that X is a finite \( \mathbb{Z}_2 \)-CW complex for simplicity. There is equivariant \( K_{\mathbb{Z}_2} \), and also a variant \( K_{\pm} \) introduced by Witten in his study of orientifold string theory \cite{51} and studied by Atiyah–Hopkins \cite{3} in connection with Dirac operators. A definition of \( K_{\pm} \) is

\[
K_{\pm}^*(X) = K_{\mathbb{Z}_2}^{*+1}(X \times \tilde{I}, X \times \partial \tilde{I}),
\]

where \( \tilde{I} \) is the interval \([-1, 1]\) and \( X \times \tilde{I} \) has the involution \( (x, t) \mapsto (\bar{x}, -t) \). By a Thom isomorphism, one gets the relation \cite{19}

\[
K_{\pm}^{*+h} \cong K_{\mathbb{Z}_2}^{*+(h,c)}(X), \quad K_{\mathbb{Z}_2}^{*+h} \cong K_{\pm}^{*+(h,c)}(X)
\]

(25)

where \( h \in H_{\mathbb{Z}_2}^2(X, \mathbb{Z}) \) and \( c \in H_{\mathbb{Z}_2}^1(X, \mathbb{Z}_2) \) is the graded twist coming from the unique nontrivial homomorphism \( \mathbb{Z}_2 \overset{\text{id}}{\to} \mathbb{Z}_2 \).

There is also Atiyah’s \( KR \)-theory \cite{2}, constructed out of complex vector bundles equipped with antilinear involutions lifting the involution on the base. It turns out that \( S^1_{\text{triv}} \) and \( S^1_{\text{flip}} \) are T-dual in this context, in that there is a naturally defined isomorphism between \( KR^*(S^1_{\text{triv}}) \equiv KO^*(S^1_{\text{triv}}) \) and \( KR^{*-1}(S^1_{\text{flip}}) \). Such dualities were studied in the context of orientifold string theories in \cite{29, 15} and in the context of topological insulators in \cite{37}, and are related to the Baum–Connes conjecture over the reals \cite{42}. In the latter setting, \( S^1_{\text{flip}} \) can be thought of as a 1D Brillouin torus \( \tilde{\mathbb{Z}} \) with the flip involution induced by complex conjugating characters. It is possible to understand \( KR \)-theory as equivariantly twisted (complex) \( K \)-theory, provided we expand the notion of twists to “\( \phi \)-twists” \cite{18, 21}. This roughly means that \( \mathbb{Z}_2 \) (or more generally \( G \)) is allowed to act complex antilinearly on fibres, and is motivated to a large extent by quantum physics where time-reversal is a basic example of such an antilinear symmetry operator.

In this paper, we study T-dualities in the purely complex (twisted) equivariant setting, i.e. \( K_{\mathbb{Z}_2}, K_{\pm}, K_G \) and their twisted versions (with no further \( \phi \)-twisting), their relationship with crystallographic T-duality, and therefore their remarkable appearance in solid state physics.

4.3.1 T-duality for ‘Real’ circle bundles and \( K_{\pm} \)-theory

**Notation:** For a space \( X \) with \( \mathbb{Z}_2 \)-action, \( H^n_{\mathbb{Z}_2}(X; \mathbb{Z}) \) denotes its Borel equivariant cohomology with integer coefficients. We will sometimes write \( H^n_{\mathbb{Z}_2}(X) \equiv H^n_{\mathbb{Z}_2}(X; \mathbb{Z}) \) for simplicity. The variant \( H^n_{\pm}(X) := H^{n+1}_{\mathbb{Z}_2}(X \times \tilde{I}, X \times \partial \tilde{I}, \mathbb{Z}) \) is, by a Thom isomorphism \cite{19}, isomorphic to \( H^n_{\mathbb{Z}_2}(X; \mathbb{Z}(1)) \), which is the equivariant cohomology with coefficients in the local system \( \mathbb{Z}(1) \) in which \( \mathbb{Z}_2 \) acts by \( n \mapsto -n \in \mathbb{Z} \).

A ‘Real’ circle bundle \( E \) over a space \( X \) with involution is defined to be a principal \( S^1 \) bundle \( E \to X \) with an involution \( \varrho \) lifting that on \( X \), such that \( \varrho(\xi u) = \varrho(\xi) \bar{u} \) for all \( \xi \in E, u \in S^1 \cong U(1) \). A basic example is \( S^1_{\text{flip}} \to \text{pt} \) with trivial involution on pt. Such bundles are classified \cite{30} by their first ‘Real’ Chern class (Euler class) \( c_1^R(E) \in H^2_{\pm}(X) \). An important tool for computing
$H^n_{\mathbb{Z}_2}, H_\pm^n$ for ‘Real’ circle bundles $E$ is the ‘Real’ Gysin sequence (Corollary 2.11 of [19]),

$$\cdots \to H^{n-2}_{\mathbb{Z}_2}(X) \xrightarrow{c^R(\hat{E})} H^n_{\mathbb{Z}_2}(X) \xrightarrow{\pi} H^n_{\mathbb{Z}_2}(E) \xrightarrow{\pi} H^n_{\mathbb{Z}_2}(X) \to \cdots$$

$$\cdots \to H^{n-2}_{\pm}(X) \xrightarrow{c^R(\hat{E})} H^n_{\pm}(X) \xrightarrow{\pi} H^n_{\pm}(E) \xrightarrow{\pi} H^n_{\pm}(X) \to \cdots$$

A (‘Real’) pair $(E, h)$ on $X$ comprises a ‘Real’ circle bundle $E \to X$ and a class $h \in H^3_{\mathbb{Z}_2}(E, \mathbb{Z})$.

**Definition 4.5** (Theorem 1.1 of [19]). $(E, h)$ and $(\hat{E}, \hat{h})$ are called (‘Real’) T-dual pairs if

$$c^R(\hat{E}) = \pi_*(h), \quad c^R(E) = \hat{\pi}_*(\hat{h}), \quad p^*h = \hat{p}^*\hat{h},$$

where $p, \hat{p}, \pi, \hat{\pi}$ are the maps in the correspondence diagram Eq. (16) regarded in the ‘Real’ circle bundle sense.

Existence and uniqueness of T-dual pairs was established in [19].

**Theorem 4.6.** Let $(E, h)$ and $(\hat{E}, \hat{h})$ be T-dual pairs over an involutive space $X$. Then there are $K^*_X(E)$-module isomorphisms

$$T_R : K^{\bullet +h}_X(E) \to K^{\bullet +1+h}(\hat{E}), \quad T_R : K^{\bullet +h}_X(E) \to K^{\bullet -1+h}_X(\hat{E}). \quad (26)$$

### 4.3.2 $\mathbb{Z}_2$-equivariant T-duality

A $\mathbb{Z}_2$-equivariant circle bundle $E \to X$ over an involutive space $X$ is a principal $S^1$ bundle $E \to X$ with involution $\varrho$ lifting that on $X$, such that $\varrho(\xi u) = \varrho(\xi)u$ for all $\xi \in E, u \in S^1 \cong \text{U}(1)$. A basic nontrivial example is $S^1_{\text{free}} \to \text{pt}$ with trivial involution on pt. Such bundles are classified by their first equivariant Chern class $c^2_1(E) \in H^2_{\mathbb{Z}_2}(X, \mathbb{Z})$. The Gysin sequence for such an $E$ is

$$\cdots \to H^{n-2}_{\mathbb{Z}_2}(X) \xrightarrow{c^2_1(E)} H^n_{\mathbb{Z}_2}(X) \xrightarrow{\pi} H^n_{\mathbb{Z}_2}(E) \xrightarrow{\pi} H^n_{\mathbb{Z}_2}(X) \to \cdots$$

A ($\mathbb{Z}_2$-equivariant) pair $(E, h)$ on $X$ comprises a $\mathbb{Z}_2$-equivariant circle bundle $E \to X$ and a class $h \in H^3_{\mathbb{Z}_2}(E, \mathbb{Z})$. Generalising Definition 4.5 and Theorem 4.6 along the lines of the arguments for ‘Real’ T-duality in [19], we have

**Definition 4.7.** $(E, h)$ and $(\hat{E}, \hat{h})$ are called ($\mathbb{Z}_2$-equivariant) T-dual pairs if

$$c^2_1(\hat{E}) = \pi_*(h), \quad c^2_1(E) = \hat{\pi}_*(\hat{h}), \quad p^*h = \hat{p}^*\hat{h},$$

where $p, \hat{p}, \pi, \hat{\pi}$ are as in Eq. (16) regarded in the $\mathbb{Z}_2$-equivariant sense.

**Theorem 4.8.** Let $(E, H)$ and $(\hat{E}, \hat{h})$ be ($\mathbb{Z}_2$-equivariant) T-dual pairs over an involutive space $X$. Then there are $K^*_X(E)$-module isomorphisms

$$T_{\mathbb{Z}_2} : K^{\bullet +h}_{\mathbb{Z}_2}(E) \to K^{\bullet -1+h}_{\mathbb{Z}_2}(\hat{E}), \quad T_{\mathbb{Z}_2} : K^{\bullet +h}_X(E) \to K^{\bullet -1+h}_X(\hat{E}). \quad (27)$$

Note that $T_{\mathbb{Z}_2}$ is in particular a $R(\mathbb{Z}_2) = K^0(\mathbb{Z}_2)$-(pt)-module map.
5 ‘Real’ and $\mathbb{Z}_2$-equivariant T-dualities over involutive circle base

In this section, we will provide all the ‘Real’ or $\mathbb{Z}_2$-equivariant circle bundle T-dualities with base space one of the three involutive circles $S^1_{\text{triv}}, S^1_{\text{flip}}, S^1_{\text{free}}$. The total space of the circle bundle is necessarily a 2-torus, which admits six inequivalent $\mathbb{Z}_2$-actions, five of which are products of circles with $\mathbb{Z}_2$-action, $S^1_{\text{triv}} \times S^1_{\text{triv}}, S^1_{\text{triv}} \times S^1_{\text{flip}}, S^1_{\text{flip}} \times S^1_{\text{flip}}, S^1_{\text{flip}} \times S^1_{\text{free}}, S^1_{\text{triv}} \times S^1_{\text{free}}$.

The following crystallographic interpretations are available for three of these:

$T^2_\text{pm} \cong S^1_{\text{triv}} \times S^1_{\text{flip}}, T^2_\text{p2} \cong S^1_{\text{flip}} \times S^1_{\text{flip}}, T^2_\text{pg} \cong S^1_{\text{flip}} \times S^1_{\text{free}}.$

$S^1_{\text{triv}} \times S^1_{\text{triv}}, S^1_{\text{triv}} \times S^1_{\text{free}}$ do not arise directly from 2D wallpaper groups, but from a slight generalisation called layer groups [32] associated to 2D “layers” in 3D. The sixth and final $\mathbb{Z}_2$ action is induced from the wallpaper group $cm$,

$T^2_\text{cm} = S^1 \times S^1, \quad (u_1, u_2) \mapsto (u_2, u_1).$

The fiberings of $T^2_\text{p2}, T^2_\text{pm}, T^2_\text{pg}, T^2_\text{cm}$ as ‘Real’ and/or $\mathbb{Z}_2$-equivariant circle bundles are illustrated in Fig. 1, and explained in further detail in the subsequent Subsections.

5.1 Equivariant cohomology of pt, $S^1_{\text{triv}}$ and $S^1_{\text{flip}}$

Let pt be the point with the trivial $\mathbb{Z}_2$-action. The equivariant cohomology rings of pt, $S^1_{\text{triv}}$, and $S^1_{\text{flip}}$ were computed as ([19], Proposition 2.4, Lemma 2.15, Lemma 2.12 respectively)

$$H^*_{\mathbb{Z}_2}(\text{pt}) \oplus H^*_{\mathbb{Z}}(\text{pt}) \cong \mathbb{Z}[t^{1/2}]/(2t^{1/2}),$$

$$H^*_{\mathbb{Z}_2}(S^1_{\text{triv}}) \oplus H^*_{\mathbb{Z}}(S^1_{\text{triv}}) \cong \mathbb{Z}[t^{1/2}, e]/(2t^{1/2}, e^2),$$

$$H^*_{\mathbb{Z}_2}(S^1_{\text{flip}}) \oplus H^*_{\mathbb{Z}}(S^1_{\text{flip}}) \cong \mathbb{Z}[t^{1/2}, \chi]/(2t^{1/2}, \chi^2 - t^{1/2}\chi),$$

where

$t^{1/2} \in H^1_{\mathbb{Z}_2}(\text{pt}) \cong \mathbb{Z}_2, \quad e \in H^1_{\mathbb{Z}_2}(S^1_{\text{triv}}) \cong \mathbb{Z}, \quad \chi \in \tilde{H}^1_{\mathbb{Z}_2}(S^1_{\text{flip}}) \cong \mathbb{Z}$

are generators such that the equivariant push-forward along $\tilde{\pi} : S^1_{\text{triv}} \to \text{pt}$ provides $\tilde{\pi}_* e = 1$ and the ‘Real’ push-forward along $\pi : S^1_{\text{flip}} \to \text{pt}$ provides

$\footnote{Explicitly, $S^1_{\text{triv}} \times S^1_{\text{triv}}$ would correspond to p11m (reflection plane symmetries) while $S^1_{\text{triv}} \times S^1_{\text{free}}$ corresponds to p11a (glide plane symmetries), although the point group needs to be regarded as a graded group, as in the frieze group p11g.}$
Figure 1: For $\mathcal{G} = p2, pm, pg, cm$, the lattice $\Pi$ is taken to be square, generated by a horizontal and a vertical translation. Their unit cells $T^2_\mathcal{G} = \mathbb{R}^2/\Pi$ are drawn, with arrows indicating how pairs of points are exchanged under the induced involution $\alpha$ on $T^2_\mathcal{G}$. There is a product fibering of $T^2_{p2}, T^2_{pm}, T^2_{pg}$ as $\mathbb{Z}_2$-equivariant circle bundles (horizontal fibers) or as ‘Real’ circle bundles (vertical fibers). For $cm$ we redraw $T^2_{cm}$ as a rhombus in two different ways to illustrate its non-product (diagonal) fibering as a $\mathbb{Z}_2$-equivariant circle bundle and as a ‘Real’ circle bundle. Thick lines indicate reflection axes, dashed lines indicate glide axes, and circles indicate $\pi$-rotation centers.

$\pi_*\chi = 1$. In low degrees, these groups are:

| $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ |
|---------|---------|---------|---------|---------|
| $H^n_{p2}(pt)$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ |
| basis | 1 | $t$ | $t^2$ |
| $H^n_{pm}(pt)$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | 0 |
| basis | $t^{1/2}$ | $t^{3/2}$ |
| $H^n_{pg}(S^1_{triv})$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| basis | 1 | $e$ | $t$ | $te$ | $t^2$ |
| $H^n_{cm}(S^1_{triv})$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| basis | $t^{1/2}$ | $t^{1/2}e$ | $t^{3/2}$ | $t^{3/2}e$ |
| $H^n_{cm}(S^1_{flip})$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ |
| basis | 1 | $t, t^{1/2}\chi$ | 0 | $t^2, t^{3/2}\chi$ |
| $H^n_{cm}(S^1_{flip})$ | 0 | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ | 0 |
| basis | $t^{1/2}, \chi$ | $t^{3/2}, t^{3/2}\chi$ |

Note that $te \in H^3_{cm}(S^1_{triv}) \cong \mathbb{Z}_2$ can be represented by the group 2-cocycle $\tau_{S^1}$ of Eq. (8). Also, the generator $\chi$ depends on the choice of the base points on $S^1_{flip}$. The multiplication by $-1$ is an automorphism of $S^1_{flip}$ which exchanges the two fixed points. This automorphism acts on the basis by $(-1)^*\chi = \chi + t^{1/2}$ and
\((-1)^*t^{1/2} = t^{1/2}\).

### 5.1.1 $\mathbb{Z}_2$-equivariant and Real T-duality over a point

Note also that the generator $t \in H^2_{\mathbb{Z}_2}(pt)$ corresponds to the sign representation of $\mathbb{Z}_2$, and the representation ring of $\mathbb{Z}_2$ is

$$R(\mathbb{Z}_2) = \mathbb{Z}[t]/(t^2 - 1).$$

The unit circle bundle for this sign representation is just $S^1_{\text{free}}$, so that $c^Z_1(S^1_{\text{free}}) = t$, and there are no twists on $S^1_{\text{free}}$ since $H^3_{\mathbb{Z}_2}(S^1_{\text{free}}) \cong H^3(S^1) = 0$. Also, under $\tilde{\pi} : S^1_{\text{triv}} \to pt$, we have $\tilde{\pi}_*(\tau_{S^1}) \equiv \tilde{\pi}_*(te) = t$. Obviously $c^Z_1(S^1_{\text{triv}}) = 0$, so by Definition 4.7, we have

**Proposition 5.1.** The $\mathbb{Z}_2$-equivariant T-dual pairs over the point are

$$(S^1_{\text{free}}, 0) \leftrightarrow (S^1_{\text{triv}}, \tau_{S^1}) \quad \text{and} \quad (S^1_{\text{triv}}, 0) \leftrightarrow (S^1_{\text{triv}}, 0).$$

Since $H^2_{\mathbb{Z}_2}(pt) = 0$ the only possibility for ‘Real’ T-duality over a point is between $S^1_{\text{flip}}$ and itself, and there are no twists since $H^3_{\mathbb{Z}_2}(S^1_{\text{flip}}) = 0$.

**Proposition 5.2.** $(S^1_{\text{flip}}, 0)$ is a ‘Real’ self-dual pair over the point.

### 5.2 ‘Real’ T-dualities over circle base

#### 5.2.1 ‘Real’ T-duality over $S^1_{\text{flip}}$

By $H^2_\pm(S^1_{\text{flip}}) = 0$, there is only the trivial ‘Real’ circle bundle $S^1_{\text{flip}} \times S^1_{\text{flip}} \to S^1_{\text{flip}}$, which can be identified with $T^{2\mathbb{Z}}_{\mathbb{Z}_2}$. By the splitting of the Gysin sequence, we find $H^3_{\mathbb{Z}_2}(S^1_{\text{flip}} \times S^1_{\text{flip}}) = 0$.

**Proposition 5.3.** There is only one ‘Real’ self-dual pair over $S^1_{\text{flip}}$

$$(S^1_{\text{flip}} \times S^1_{\text{flip}}, 0) \leftrightarrow (S^1_{\text{flip}} \times S^1_{\text{flip}}, 0).$$

The $K$-theories verifying Prop. 5.3 was computed in [19] (Proposition 4.1):

- $K^0_{\mathbb{Z}_2}(S^1_{\text{flip}} \times S^1_{\text{flip}}) \cong (R(\mathbb{Z}_2) \oplus (1 - t)) \oplus 2$,
- $K^1_{\mathbb{Z}_2}(S^1_{\text{flip}} \times S^1_{\text{flip}}) \cong 0$,
- $K^0_{\pm}(S^1_{\text{flip}} \times S^1_{\text{flip}}) \cong 0$,
- $K^1_{\pm}(S^1_{\text{flip}} \times S^1_{\text{flip}}) \cong (R(\mathbb{Z}_2) \oplus (1 - t)) \oplus 2$.

#### 5.2.2 ‘Real’ T-duality over $S^1_{\text{triv}}$

The T-dualities over $S^1_{\text{triv}}$ were given in [19] as an example (§5.5), but they are somewhat intricate and we recall some of the details here.

From $H^2_\pm(S^1_{\text{triv}}) \cong \mathbb{Z}_2$, there are two inequivalent ‘Real’ line bundles on $S^1_{\text{triv}}$ whose ‘Real’ Chern classes are 0 and $t^{1/2}e$.

- $(c^R_1 = 0)$ The trivial ‘Real’ line bundle with unit circle bundle $S^1_{\text{triv}} \times S^1_{\text{flip}}$, is identifiable with $T^{2\mathbb{Z}}_{\mathbb{Z}_2}$.
\( c_1^R = t^{1/2}e \) The non-trivial ‘Real’ line bundle \( R \to S^1_{\text{flip}} \) is \( S^1_{\text{triv}} \times \mathbb{C} \) with the ‘Real’ structure \((u, z) \mapsto (u, u \bar{z})\), and ‘Real’ Chern class \( c_1^R(R) = t^{1/2}e \).

We write \( \pi_R : S(R) \to S^1_{\text{triv}} \) for its unit circle bundle. As a \( \mathbb{Z}_2 \)-space, \( S(R) \) is identified with \( T^2 \).

\textbf{(Case} \( c_1^R = 0 \)). The relevant low degree cohomology groups for the (split) Gysin sequence are the middle two columns of:

| \( n \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
|---|---|---|
| \( n = 4 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 3 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 2 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 1 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 0 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |

With some extra computations, the generators for the desired shaded column are (suppressing the pullback notation)

\[ H^0_{\mathbb{Z}_2}(S^1_{\text{triv}}) \xrightarrow{\pi_*} H^0_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{flip}}) \xrightarrow{\pi_*} H^1_{\mathbb{Z}_2}(S^1_{\text{triv}}) \]

Under the projection \( \pi : S^1_{\text{triv}} \times S^1_{\text{flip}} \to S^1_{\text{triv}} \), we have \( \pi_* \chi = 1 \). In \( H^3_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{flip}}) \), the basis element \( te \) is represented by (the pull-back from \( S^1_{\text{triv}} \) of) the group 2-cocycle \( \tau_{S^1} \), whereas \( h_{pm} := t^{1/2}e\chi \) cannot be represented by any group 2-cocycle, according to the classification of twists [20]. In the Gysin sequence,

\[ H^3_{\mathbb{Z}_2}(S^1_{\text{triv}}) \xrightarrow{\pi_*} H^3_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{flip}}) \xrightarrow{\pi_*} H^1_{\mathbb{Z}_2}(S^1_{\text{triv}}) \]

we have

\[ \pi_*(\tau_{S^1}) \cong \pi_*(te) = 0, \quad \pi_*(h_{pm}) \cong \pi_*(t^{1/2}e\chi) = t^{1/2}e = c_1^R(R). \]

\textbf{(Case} \( c_1^R = t^{1/2}e \)). For the Gysin sequence, we need

| \( n \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
|---|---|---|
| \( n = 4 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 3 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 2 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 1 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
| \( n = 0 \) | \( \mathbb{Z}_2t^{1/2}e \) | \( \mathbb{Z}_2t^2 \) |
It turns out that $H^3_{Z_2}(S(R)) = H^3_{Z_2}(T^2_{cm}) \cong \mathbb{Z}_2$, and that its generator $h_{cm}$ is not representable by a group 2-cocycle \cite{20}. A part of the Gysin sequence is:

$$H^3_{Z_2}(S^1_{triv}) \xrightarrow{\pi_R^*} H^3_{Z_2}(S(R)) \xrightarrow{(\pi_R)^*} H^2_{\pm}(S^1_{triv}),$$

\[
\begin{array}{ccc}
\mathbb{Z}_2 \tau_{S^1} & \mathbb{Z}_2 h_{cm} & \mathbb{Z}_2 c^0_1(R) \\
\pi_R \tau_{S^1} = 0, & (\pi_R), h_{cm} = c^0_1(R).
\end{array}
\]

From the computations above, we get from Definition 4.5 that:

**Proposition 5.4.** The following pairs are ‘Real’ T-dual over $S^1_{triv}$:

- $(S^1_{triv} \times S^1_{flip}, 0) \leftrightarrow (S^1_{triv} \times S^1_{flip}, 0)$,
- $(S^1_{triv} \times S^1_{flip}, \tau_{S^1}) \leftrightarrow (S^1_{triv} \times S^1_{flip}, \tau_{S^1})$,
- $(S^1_{triv} \times S^1_{flip}, h_{pm}) \leftrightarrow (S(R), 0)$,
- $(S^1_{triv} \times S^1_{flip}, h_{pm} + \tau_{S^1}) \leftrightarrow (S(R), 0)$,
- $(S(R), h_{cm}) \leftrightarrow (S(R), h_{cm})$.

**Remark 5.5.** Notice that $S^1_{triv} \times S^1_{flip}$ has the $\mathbb{Z}_2$-equivariant automorphism given by multiplying $-1$ with $S^1_{flip}$. From $(-1)^* \chi = \chi + t^{1/2}$, we have

$$(-1)^* h_{pm} = (-1)^*(t^{1/2} e\chi) = t^{1/2} e(\chi + t^{1/2}) = t^{1/2} e\chi + tc = h_{pm} + \tau_{S^1}.$$

We summarize the $K$-theories of the T-dual pairs over $S^1_{triv}$, which verify Proposition 5.4. (In the following table, two errors/typo in \cite{19} are replaced by the correct results, which are shaded.)

| $h$ | $w$ | $K^{(h,w)}_{Z_2}(S^1_{triv} \times S^1_{flip})$ | $K^{(h,w)}_{Z_2}(S^1_{triv} \times S^1_{flip})$ |
|-----|-----|--------------------------------|--------------------------------|
| 0   | 0   | $R(\mathbb{Z}_2) \oplus (1 - t)$ | $R(\mathbb{Z}_2) \oplus (1 - t)$ |
| 0   | id | $R(\mathbb{Z}_2) \oplus (1 - t)$ | $R(\mathbb{Z}_2) \oplus (1 - t)$ |
| $\tau_{S^1}$ | 0 | $t^{1/2} \tau_{S^1}$ | $t^{1/2} \tau_{S^1}$ |
| $\tau_{S^1}$ | id | $(1 + t) \oplus \mathbb{Z}/2$ | $(1 + t) \oplus \mathbb{Z}/2$ |
| $h_{pm}, h_{pm} + \tau_{S^1}$ | 0 | $(1 + t) \oplus (1 - t)$ | $R(\mathbb{Z}_2)$ |
| $h_{pm}, h_{pm} + \tau_{S^1}$ | id | $(1 + t) \oplus (1 - t)$ | $R(\mathbb{Z}_2)$ |

| $h$ | $w$ | $K^{(h,w)}_{Z_2}(S(R))$ | $K^{(h,w)}_{Z_2}(S(R))$ |
|-----|-----|----------------|----------------|
| 0   | 0   | $R(\mathbb{Z}_2) \oplus (1 - t)$ | $R(\mathbb{Z}_2) \oplus (1 - t)$ |
| 0   | id | $R(\mathbb{Z}_2) \oplus (1 - t)$ | $R(\mathbb{Z}_2) \oplus (1 - t)$ |
| $h_{cm}$ | 0 | $(1 + t)$ | $(1 + t)$ |
| $h_{cm}$ | id | $(1 + t)$ | $(1 + t)$ |
5.3 \( \mathbb{Z}_2\)-equivariant T-duality over \( S^1_{\text{flip}} \)

By \( H^2_{\mathbb{Z}_2}(S^1_{\text{flip}}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), there are four inequivalent \( \mathbb{Z}_2\)-equivariant line bundles on \( S^1_{\text{flip}} \) whose \( \mathbb{Z}_2\)-equivariant Chern classes (Euler classes) \( c_1^{\mathbb{Z}_2} \) are the four elements \( 0, t, t\chi, t + t\chi \). (Here the basis \( \chi \in H^2_\pm(S^1) \) is chosen with respect to the base point \( 1 \in S^1_{\text{flip}} \subset \mathbb{C} \).)

- \((c_1^{\mathbb{Z}_2} = 0)\) The \( \mathbb{Z}_2\)-equivariant line bundle associated to the trivial representation. Its unit circle bundle is \( S^1_{\text{triv}} \times S^1_{\text{flip}} \), which is identified with the torus \( T^2_{\text{cm}} \).

- \((c_1^{\mathbb{Z}_2} = t)\) The \( \mathbb{Z}_2\)-equivariant line bundle associated to the sign representation. Its unit circle bundle is \( S^1_{\text{flip}} \times S^1_{\text{free}} \), which is identified with the torus \( T^2_{\text{cm}} \) with free involution and quotient manifold the Klein bottle \( K \).

- \((c_1^{\mathbb{Z}_2} = t^{1/2}\chi)\) The product bundle \( L = S^1_{\text{flip}} \times \mathbb{C} \) with the \( \mathbb{Z}_2\)-equivariant structure \((u, z) \mapsto (\bar{u}, uz)\). We write \( \pi_L : S(L) \rightarrow S^1_{\text{flip}} \) for its unit circle bundle. The total space \( S(L) \) can be identified with the torus \( T^2_{\text{cm}} \).

- \((c_1^{\mathbb{Z}_2} = t + t^{1/2}\chi)\) The product bundle \( L' = S^1_{\text{flip}} \times \mathbb{C} \) with the \( \mathbb{Z}_2\)-equivariant structure \((u, z) \mapsto (\bar{u}, -uz)\). We write \( \pi_{L'} : S(L') \rightarrow S^1_{\text{flip}} \) for its unit circle bundle. The total space \( S(L') \) can also be identified with \( T^2_{\text{cm}} \).

Remark 5.6. Notice that \( L \) and \( L' \) are non-isomorphic equivariant line bundles on \( S^1_{\text{flip}} \), despite both being \( T^2_{\text{cm}} \) as \( \mathbb{Z}_2\)-spaces. Nevertheless, we can take the bundle map \( L \rightarrow L' \), \((u, z) \mapsto (-u, z))\) covering the base automorphism \(-1 : S^1_{\text{flip}} \rightarrow S^1_{\text{flip}} \), then \((-1)^*t^{1/2}\chi = t + t^{1/2}\chi\) exchanges the equivariant Chern classes.

In order to find T-dual partners for these \( \mathbb{Z}_2\)-equivariant circle bundles, we need to compute the push-forward of each of their possible twists to the base \( S^1_{\text{flip}} \) under their respective bundle projection maps \( \hat{\pi} \).

**Case** \( c_1^{\mathbb{Z}_2} = t \). Since the quotient Klein bottle is two-dimensional,

\[
H^3_{\mathbb{Z}_2}(S^1_{\text{flip}} \times S^1_{\text{free}}) \cong H^3(K) = 0.
\]

**Case** \( c_1^{\mathbb{Z}_2} = 0 \). Recall from §5.2.2 the basis for \( H^n_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{flip}}) \) in low degrees:

| \( n \quad n = 0 \quad n = 1 \quad n = 2 \quad n = 3 \) | \( H^n_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{flip}}) \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) |
|---|---|---|---|---|
| basis | 1 | \( e \) | \( t, t^{1/2} \chi \) | \( te, t^{1/2} \chi e \) |

In the Gysin sequence for \( \hat{\pi} : S^1_{\text{triv}} \times S^1_{\text{flip}} \rightarrow S^1_{\text{flip}} \),

\[
\begin{align*}
H^3_{\mathbb{Z}_2}(S^1_{\text{flip}}) & \xrightarrow{\hat{\pi}^*} H^3_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{flip}}) \\
& \xrightarrow{\hat{\pi}^*} H^3_{\mathbb{Z}_2}(S^1_{\text{flip}}),
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
0 & \mathbb{Z}_2 e \oplus \mathbb{Z}_2 e^{1/2} \chi & \mathbb{Z}_2 e \oplus \mathbb{Z}_2 e^{1/2} \chi
\end{array}
\]
the push-forward $\tilde{\pi}_*$ is bijective, and we have

$$
\tilde{\pi}_*(\tau_{S^1}) \equiv \tilde{\pi}_*(t e) = t = c_1^{Z_2}(S_{\text{triv}}^1 \times S_{\text{free}}^1),
$$

$$
\tilde{\pi}_*(h_{\text{pm}}) \equiv \tilde{\pi}_*(t^{1/2}e\chi) = t^{1/2}\chi = c_1^{Z_2}(L),
$$

$$
\tilde{\pi}_*(\tau_{S^1} + h_{\text{pm}}) \equiv \tilde{\pi}_*(t e + t^{1/2}e\chi) = t + t^{1/2}\chi = c_1^{Z_2}(L').
$$

\textbf{(Cases $c_1^{Z_2} = t^{1/2}\chi$ and $t + t^{1/2}\chi$.)} We have $\mathbb{Z}_2$-equivariant homeomorphism $S(L) \cong S(R) \cong T_{\text{cm}}^2$, and $H_{Z_2}^n(S(R))$ was already computed in §5.2.2. The relevant groups in the Gysin sequence for $\tilde{\pi}_L : S(L) \to S_{\text{lip}}^1$ are:

| $n = 4$ | $\mathbb{Z}_2 t \oplus \mathbb{Z}_2 t^{1/2}\chi$ | $\mathbb{Z}_2 t^2 \oplus \mathbb{Z}_2 t^{3/2}\chi$ |
| $n = 3$ | 0 | 0 |
| $n = 2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 t \oplus \mathbb{Z}_2 t^{1/2}\chi$ | $\mathbb{Z}_2$ |
| $n = 1$ | 0 | 0 | $\mathbb{Z}_2$ |
| $n = 0$ | 0 | $\mathbb{Z}_2$ |

Putting $c_1^{Z_2}(L) = t^{1/2}\chi$ into the Gysin sequence and Eq. (28), we have for the generator $h_{\text{cm}} \in H_{Z_2}^3(S(L))$:

$$
(\tilde{\pi}_L)_*(h_{\text{cm}}) = t + t^{1/2}\chi = c_1^{Z_2}(L'),
$$

The computation for $S(L') \cong S(L)$ is similar, except that in the Gysin sequence for $\tilde{\pi}_{L'} : S(L') \to S_{\text{lip}}^1$, we have $c_1^{Z_2}(L') = t + t^{1/2}\chi$. Therefore we find

$$
(\tilde{\pi}_{L'})_*(h_{\text{cm}}) = t^{1/2}\chi = c_1^{Z_2}(L).
$$

Summarizing the calculations above, we have:

\textbf{Proposition 5.7.} The following pairs are $\mathbb{Z}_2$-equivariant T-dual over $S_{\text{lip}}^1$:

$$
(S_{\text{triv}}^1 \times S_{\text{lip}}^1, 0) \leftrightarrow (S_{\text{triv}}^1 \times S_{\text{lip}}^1, 0),
$$

$$
(S_{\text{triv}}^1 \times S_{\text{lip}}^1, \tau_{S^1}) \leftrightarrow (S_{\text{lip}}^1 \times S_{\text{free}}^1, 0),
$$

$$
(S_{\text{triv}}^1 \times S_{\text{lip}}^1, h_{\text{pm}}) \leftrightarrow (S(L), 0),
$$

$$
(S_{\text{triv}}^1 \times S_{\text{lip}}^1, h_{\text{pm}} + \tau_{S^1}) \leftrightarrow (S(L), 0),
$$

$$
(S(L), h_{\text{cm}}) \leftrightarrow (S(L'), h_{\text{em}}).
$$

In Proposition 5.7, all the pairs except the second one are also T-dual in the ‘Real’ sense by Proposition 5.3, and all their $K$-theories had been presented at the end of §5.2.2 except for $S_{\text{lip}}^1 \times S_{\text{free}}^1$ which is given at the end of §5.4. These $K$-theory computations verify the dualities in Proposition 5.7.
5.4 \( \mathbb{Z}_2 \)-equivariant and ‘Real’ T-duality over \( S^1_{\text{free}} \)

From \( H^2_{\mathbb{Z}_2}(S^1_{\text{free}}) \cong H^2(S^1, \mathbb{Z}) = 0 \), \( S^1_{\text{free}} \) admits only the trivial \( \mathbb{Z}_2 \)-equivariant line bundle, whose unit circle bundle is \( S^1_{\text{triv}} \times S^1_{\text{free}} \) and has no nontrivial twists. Also, \( H^2(\mathbb{Z}_2) = 0 \) so \( S^1_{\text{free}} \) admits only the trivial ‘Real’ line bundle, with unit circle bundle \( S^1_{\text{flip}} \times S^1_{\text{free}} \) (also \( T^2_{\text{rg}} \) encountered earlier in §5.3) having no \( H^3 \)-twists. From these computations, we get:

**Proposition 5.8.** The following is the \( \mathbb{Z}_2 \)-equivariant T-dual pair over \( S^1_{\text{free}} \):

\[
(S^1_{\text{triv}} \times S^1_{\text{free}}, 0) \leftrightarrow (S^1_{\text{triv}} \times S^1_{\text{free}}, 0).
\]

The following is the ‘Real’ T-dual pair over \( S^1_{\text{free}} \):

\[
(S^1_{\text{flip}} \times S^1_{\text{free}}, 0) \leftrightarrow (S^1_{\text{flip}} \times S^1_{\text{free}}, 0).
\]

To verify Proposition 5.8, we compute the relevant \( K \)-theories. First of all, we regard \( S^1_{\text{free}} \) as the unit circle bundle of the \( \mathbb{Z}_2 \)-equivariant line bundle associated to the sign representation. Thus, its Euler class in \( K \)-theory is \( 1 - t \in K_{\mathbb{Z}_2}(\text{pt}) \). Using the Gysin exact sequence in \( K \)-theory, we get:

\[
K^0_{\mathbb{Z}_2}(S^1_{\text{free}}) \cong (1 + t), \quad K^1_{\mathbb{Z}_2}(S^1_{\text{free}}) \cong (1 + t),
\]

\[
K^0_{\pm}(S^1_{\text{free}}) \cong 0, \quad K^1_{\pm}(S^1_{\text{free}}) \cong \mathbb{Z}/2.
\]

Then, regarding \( S^1_{\text{flip}} \times S^1_{\text{free}} \) as the trivial ‘Real’ circle bundle on \( S^1_{\text{free}} \), we apply the splitting of the Gysin sequence to have

\[
K^0_{\mathbb{Z}_2}(S^1_{\text{flip}} \times S^1_{\text{free}}) \cong (1 + t) \oplus \mathbb{Z}/2, \quad K^1_{\mathbb{Z}_2}(S^1_{\text{flip}} \times S^1_{\text{free}}) \cong (1 + t),
\]

\[
K^0_{\pm}(S^1_{\text{flip}} \times S^1_{\text{free}}) \cong (1 + t), \quad K^1_{\pm}(S^1_{\text{flip}} \times S^1_{\text{free}}) \cong (1 + t) \oplus \mathbb{Z}/2.
\]

Similarly, by the splitting of the Gysin sequence for \( S^1_{\text{triv}} \times S^1_{\text{free}} \to S^1_{\text{free}} \), we get

\[
K^0_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{free}}) \cong (1 + t) \oplus (1 + t), \quad K^1_{\mathbb{Z}_2}(S^1_{\text{triv}} \times S^1_{\text{free}}) \cong (1 + t),
\]

\[
K^0_{\pm}(S^1_{\text{triv}} \times S^1_{\text{free}}) \cong \mathbb{Z}/2, \quad K^1_{\pm}(S^1_{\text{triv}} \times S^1_{\text{free}}) \cong \mathbb{Z}/2.
\]

**Remark 5.9.** There are also \( \mathbb{Z}_2 \)-equivariant T-dualities for circle bundles over \( S^1_{\text{triv}} \), but we omit these as we do not use them in our examples.

6 2D crystallographic T-dualities

In this section, we apply crystallographic T-duality, \( T_{\varphi} \) of Eq. (21), to the 17 wallpaper groups. When the point group \( G \) is \( \mathbb{Z}_2 \), \( T_{\varphi} \) fibers as a ‘Real’ or \( \mathbb{Z}_2 \)-equivariant circle bundle over another circle (or even both), so that we can T-dualise only the fiber circle while keeping the base circle fixed. This “partial Fourier transform” produces an intermediate 2-torus which we denote by \( \hat{T}^2 \); if there is a second way to fiber and T-dualise, we denote the resulting space by \( \hat{T}^2 \). Subsequently, we may be able to re-fiber \( \hat{T}^2 \) (or \( \hat{T}^2 \)) such that what was considered the base is now a fiber. A second “partial Fourier transform” produces \( T^2 \) on the right-hand-side of Eq. (21), showing that \( T_{\varphi} \) factorises into partial T-dualities. In these cases, the factorisation is essentially a combination of the circle bundle T-dualities in Propositions 5.3, 5.4, 5.7.
6.1 Trivial point group: p1

Despite $T_p^1 : K^\bullet(T_{p1}^2) \to K^\bullet(\hat{T}_{p1}^2)$, this is not simply the identity map. First, the spaces $T_{p1}^2$ and $\hat{T}_{p1}^2$ are not naturally identified, and second, the rank and Hopf generators for their $K^0$ theory are actually exchanged under $T_{p1}$ [29, 37].

Pick a principal fibration $T_{p1}^2 = S^1_y \times S^1_y \to S^1_y$, then the circle bundle T-dual is $\hat{T}_{p1}^2 = \hat{S}^1_y \times S^1_y \to S^1_y$; fibering over $S^1_y$ instead will yield $\hat{T}_{p1}^2 = \hat{S}^1_y \times S^1_x \to S^1_x$. Refibering $\hat{T}_{p1}^2$ to exchange base and fiber allows a second circle bundle T-duality transformation to arrive at $\hat{\hat{T}}_{p1}^2$; similarly for $\hat{T}_{p1}^2$. In summary, we have a web of T-duality isomorphisms

6.2 Order-2 point group

6.2.1 $G = \mathbb{Z}_2$: p2

The wallpaper group $p2$ has point group $\mathbb{Z}_2$ comprising $\pi$ rotations about an origin in Euclidean space. $T_{p2}^2 = S_{\text{flip}}^1 \times S_{\text{flip}}^1$ is a ‘Real’ circle bundle over $S_{\text{flip}}^1$ (in two different ways, see Fig. 1) whose T-dual is $\hat{T}_{p2}^2 = \hat{S}_{\text{flip}}^1 \times S_{\text{flip}}^1 \to S_{\text{flip}}^1$, or $\hat{T}_{p2}^2 = S_{\text{flip}}^1 \times S_{\text{flip}}^1 \to S_{\text{flip}}^1$ by Proposition 5.3. Exchanging base and fiber for $\hat{T}_{p2}^2$ and T-dualising again gives $\hat{T}_{p2}^2 = \hat{S}_{\text{flip}}^1 \times \hat{S}_{\text{flip}}^1$; in summary,

6.2.2 $G = D_1 \cong \mathbb{Z}_2$: pm, pg, cm

The three wallpaper groups pm, pg, cm have point group $D_1 \cong \mathbb{Z}_2$ acting on $T^2$ by an orientation-reversing involution. For pm and pg, the linear actions $\alpha, \hat{\alpha}$ on $T^2, \hat{T}^2$ both correspond to the $\text{GL}(2, \mathbb{Z})$ subgroup generated by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus the Brillouin torus $\hat{T}^2$ in these two cases is identified with $\hat{T}^2_{p2}$. The difference is
that $T_{pg}^2$ has a further translational component, because $pg$ is nonsymmorphic. Explicitly, $pg \cong \mathbb{Z} \times \mathbb{Z}$ with the second copy of $\mathbb{Z}$ acting on the first by reflection, and it is a non-split extension

$$0 \to \Pi = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2 = G \to 1,$$

with 2-cocycle $\nu(-1, -1) = (0, 1)$. The Fourier transform of $\nu$ is the $U(C(T_{pg}^2))$-valued 2-cocycle $\tau_{pg}(-1, -1) = \{(k_1, k_2) \mapsto e^{ik_2}\}$, which is $\tau_{S^1}$ of Eq. (8) (pulled back under an inclusion $S^1_{triv} \hookrightarrow T_{pg}^2$).

For $cm$, both $\alpha, \hat{\alpha}$ correspond to the $GL(2, \mathbb{Z})$ subgroup generated by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In each of $pm, pg, cm$, the orientability obstruction homomorphism $c_{pg}$ is the unique nontrivial homomorphism $c = \text{id}$. Thus, their crystallographic T-dualities, Eq. (21), are

$$K_{S^2_2}^{\bullet +c}(T_{pm}^2) \cong K_{Z^2_2}(T_{pm}^2),$$

$$K_{S^2_2}^{\bullet +c}(T_{pg}^2) \cong K_{Z^2_2}^{\bullet +\tau_{S^1}}(T_{pm}^2),$$

$$K_{S^2_2}^{\bullet +c}(T_{cm}^2) \cong K_{Z^2_2}^{\bullet +\tau_{S^1}}(T_{cm}^2).$$

**Cystallographic T-duality diagram for pm.** Using Propositions 5.4, 5.7, we can apply $T_R$ to the trivial ‘Real’ circle bundle $T_{pm}^2 = S^1_{\text{flip}} \times S^1_{\text{triv}} \to S^1_{\text{triv}}$ to get another trivial ‘Real’ circle bundle $\hat{T}_{pm}^2 = \hat{S}^1_{\text{flip}} \times S^1_{\text{triv}} \to S^1_{\text{triv}}$. Now regard $\hat{T}_{pm}^2$ as a trivial $\mathbb{Z}_2$-equivariant circle bundle over $S^1_{\text{flip}}$, and apply $T_{Z^2_2}$ to get $\hat{T}_{pm}^2 = S^1_{\text{flip}} \times S^1_{\text{flip}}$. Similarly, we can start from $T_{pg}^2$ as a $\mathbb{Z}_2$-equivariant circle bundle over $S^1_{\text{flip}}$, and apply $T_{Z^2_2}$ to obtain $\hat{T}_{pm}^2 = \hat{S}^1_{\text{triv}} \times S^1_{\text{flip}}$. Now T-dualise $\hat{T}_{pm}^2$ as a ‘Real’ circle bundle over $S^1_{\text{triv}}$ to get $\hat{T}_{pm}^2$. To summarize, we have

![Cystallographic T-duality diagram for pm](image)

**Cystallographic T-duality diagram for pg.** Using Propositions 5.7, 5.8, we can apply $T_R$ to $T_{pg}^2 = S^1_{\text{flip}} \times S^1_{\text{free}} \to S^1_{\text{free}}$ as a trivial ‘Real’ circle bundle to get $\hat{T}_{pg}^2 = \hat{S}^1_{\text{flip}} \times S^1_{\text{free}} \to S^1_{\text{free}}$, then regard $\hat{T}_{pg}^2$ as a $\mathbb{Z}_2$-equivariant circle bundle over $S^1_{\text{flip}}$ and apply $T_{Z^2_2}$ to get $\hat{T}_{pm}^2$ twisted by $\tau_{S^1}$. As in the pm case, there is a second factorisation route, applying $T_{Z^2_2}$ then $T_R$. To summarize,

![Cystallographic T-duality diagram for pg](image)
Remark 6.1. The $H^1$-twist $c$ may be identified as the orientation class of the Klein bottle $K$ after passing to the quotient in $H^1_{\mathbb{Z}_2}(T^2_{pg}, \mathbb{Z}_2) = H^1(K, \mathbb{Z}_2)$. Then the map $T_R$ on the top left of Eq. (28) says that the $T$-dual of a Klein bottle is another Klein bottle with orientation twist. The same isomorphism was obtained in [4], §9.1, in the context of $T$-dualising general (non-principal) circle bundles such as $K$.

Remark 6.2. An easy way [22] to compute the groups in Eq. (28) is to calculate the $K$-homology groups

$$K_0(K) \cong H_{\text{even}}(K) = \mathbb{Z}, \quad K_1(K) \cong H_{\text{odd}}(K) = \mathbb{Z} \oplus \mathbb{Z}/2,$$

where passage to ordinary homology is justified by low-dimensionality of $K$. By the Baum–Connes isomorphism for $pg$, this computes the RHS of Eq. (28), and also the LHS by Poincaré duality. The top entry of Eq. (28) is just the ordinary $K$-theory of $K$, which is

$$K^1(K) \cong H^{\text{odd}}(K) = \mathbb{Z}, \quad K^0(K) \cong H^{\text{even}}(K) = \mathbb{Z} \oplus \mathbb{Z}/2,$$

thus independently verifying the ‘Real’ $T$-duality in the diagram.

6.2.3 Exotic non-cocycle $H^3$-twists from partial $T$-duality: cm

In §5.2.2, we saw that $T^2_{pm}$ and $T^2_{cm}$ admitted equivariant $H^3$-twists $h_{pm}, h_{cm}$ which are not representable by cocycles. These somewhat mysterious twists are not from the family of “special twistings” associated with groupoid central extensions [17] which had appeared naturally in solid state physics in [18].

A natural question is whether these extra $H^3$-twists have any realisation in solid state physics. We answer this in the affirmative, namely, they appear when we do a partial Fourier transform adapted to cm; that is, an exotic twist is required in a mixed position-momentum space description. A similar observation was made in [24], where partial Fourier transform for the nonabelian integer Heisenberg group (roughly: a screw-dislocated 3D lattice) was found to induce (non-equivariant) H-flux on a 3-torus (cf. §8.1).

Crystallographic $T$-duality diagram for cm. By Proposition 5.4, we can take $T_R$ for the ‘Real’ circle bundle $T^2_{cm}$ over $S^1_{\text{triv}}$ to get $T^2_{pm}$ twisted either by $h_{pm}$ or $h_{pm} + \tau_{S^1}$, the two options being related by the automorphism $f$ of $T^2_{pm}$ which multiplies its $S^1_{\text{flip}}$ circle by $-1$ (Remark 5.5). By Proposition 5.7, we can then take $T_{\mathbb{Z}_2}$ to arrive at $\hat{T}^2_{cm}$. A second (pair of) factorisations is possible by first performing $T_{\mathbb{Z}_2}$ on $T^2_{cm}$ regarded as either of the $\mathbb{Z}_2$-equivariant circle
bundles $S(L)$ or $S(L')$ over $S^1$, (cf. Remark 5.6), and then $T_R$. To summarize,

$$K^\bullet_{Z_2} \xrightarrow{f} K^\bullet_{Z_2} \xrightarrow{T_R} K^\bullet(T^2)$$

The exotic twist $h_{cm}$ appears in the last pair of Proposition 5.7 and of Proposition 5.4, and these lead to a modified web of dualities,

$$K^\bullet_{Z_2} \xrightarrow{T_R} K^\bullet_{Z_2} \xrightarrow{T_R} K^\bullet(T^2)$$

where the composition $T_{cm}^{h_{cm}}$ of the two partial T-dualities may be interpreted as crystallographic T-duality $T_{cm}$ enhanced by twisting both sides with $h_{cm}$.

6.3 Point group $D_3$: Orbifold change under crystallographic T-duality

The two symmorphic wallpaper groups $p3m1, p31m$ have the dihedral group $D_3$ as their point group, and have the interesting feature that the $\alpha$ for $p31m$ has dual action $\hat{\alpha}$ equivalent to the $\alpha$-action of $p3m1$, and vice versa (see Eq. 27 of [38], cf. Lemma 2.4 of [20]), as illustrated in Fig. 2. Thus the position space orbifold $T^2_{p3m1}$ dualises into $\hat{T}^2_{p31m}$ while $T^2_{p31m}$ dualises into $\hat{T}^2_{p3m1}$.

The orientability homomorphism $c : D_2 \to Z_2$ is the unique surjective one, and we have the dualities

$$K^\bullet_{D_2}(T^2_{p3m1}) \xrightarrow{T_{p3m1}} K^\bullet_{D_2}(\hat{T}^2_{p31m})$$

$$K^\bullet_{D_2}(T^2_{p31m}) \xrightarrow{T_{p31m}} K^\bullet_{D_2}(\hat{T}^2_{p3m1})$$

Without the $c$-twist, the equivariant $K$-theories of both $T^2_{p3m1}$ and $T^2_{p31m}$ were computed in [46], and also via the $K$-groups of the group $C^*$-algebra in [52, 35], to be $Z^3$ in degree 0 and $Z$ in degree 1. It is also possible, although not detailed in this paper, to compute directly that the $c$-twisted equivariant $K$-theories are also $Z^3$ or $Z$, verifying the crystallographic T-dualities for these two wallpaper groups.
Figure 2: Unit cells for \( p3m1 \) and \( p31m \). Solid lines indicate reflection axes, solid circles indicate points with full point group \( D_3 \) isotropy, hollow circles (only applicable to \( p31m \)) indicate points with only \( \frac{2\pi}{3} \) rotation symmetry. The lattice for \( p3m1 \) has reciprocal lattice identifiable with the lattice for \( p31m \).

6.4 Remaining cases

In each of the remaining cases, \( \hat{\alpha} \) on \( \hat{T}^2 \) is conjugate to \( \alpha \) on \( T^2 \), and there are no equivariant fiberings as circle bundles for factorisation of \( T_\mathcal{G} \).

6.4.1 Cyclic point groups \( p3, p4, p6 \)

For point groups \( \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \) comprising order-3, 4 or 6 rotations with respective wallpaper groups \( p3, p4, p6 \), there are only the crystallographic T-dualities:

\[
K^{+}_{\omega/\mathbb{Z}_n} (T^2_{\mathcal{G}}) \xrightarrow{T_{\mathcal{G}}} K^{+}_{\omega/\mathbb{Z}_n} (T^2_{\mathcal{G}}).
\]

6.4.2 Point group \( D_2, D_4, D_6 \)

For the remaining seven wallpaper groups with point group \( D_2, D_4 \) or \( D_6 \), the orientability homomorphism is \( c_\mathcal{G} : D_n = \mathbb{Z}_n \times \mathbb{Z}_2 \to \mathbb{Z}_2 \). Of these, the four symmorphic ones \( \text{pmm}, \text{cmm}, \text{p4m}, \text{p6m} \) and also \( \text{p4g} \) have the \( H^3 \) obstruction \( \omega \) (pulled back from a point), cf. Remark 3.3. The nonsymmorphic \( \text{pmg}, \text{pgg}, \text{p4g} \), have nontrivial group 2-cocycles which give the cocycle \( H^3 \)-twists \( \tau_{\text{pmg}}, \tau_{\text{pgg}}, \tau_{\text{p4g}} \) on the Brillouin torus \( \hat{T}^2 \).

Crystallographic T-dualities for \( \mathcal{G} \) with \( G = D_2, D_4, D_6 \),

\[
K^+_{\mathcal{G}} (T^2_{\mathcal{G}}) \xrightarrow{T_{\mathcal{G}}} K^+_{\mathcal{G}} (T^2_{\mathcal{G}}), \quad \mathcal{G} = \text{pmm, cmm, p4m, p6m},
\]

\[
K^{+}_{D_2} (T^2_{\text{pmm/p4g}}) \xrightarrow{T_{\text{pmm/p4g}}} K^{+}_{D_2} (T^2_{\text{pmm/p4g}}),
\]

\[
K^{+}_{D_2} (T^2_{\text{p4g}}) \xrightarrow{T_{\text{p4g}}} K^{+}_{D_2} (T^2_{\text{p4g}}),
\]

\[
K^{+}_{D_4} (T^2_{\text{p4m}}) \xrightarrow{T_{\text{p4m}}} K^{+}_{D_4} (T^2_{\text{p4m}}).
\]
| Space | $H^3$-twist | $H^1$-twist | $K^\bullet$ | $K^{\bullet-1}$ | $H^1$-twist | $H^3$-twist | Space |
|-------|------------|------------|-----------|------------|-----------|------------|-------|
| $T^3_{p1}$ | N/A | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}^4$ | 0 | N/A | Self T-dual | $T^3_{p1}$ |
| $T^3_{p2}$ | N/A | 0 | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | N/A | Self T-dual | $T^3_{p2}$ |
| $T^3_{pm}$ | $\tau S^1$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | 0 | $\tau S^1$ | $T^3_{pm}$ |
| $T^3_{cm}$ | $h_{cm}$ | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | 0 | $h_{pm}$ or $h_{pm} + \tau S^1$ | $T^3_{cm}$ |
| $T^3_{p3}$ | N/A | 0 | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | Self crystal T-dual | $T^3_{p3}$ |
| $T^3_{p4}$ | N/A | 0 | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | $\tau S^1$ | $T^3_{p4}$ |
| $T^3_{p6}$ | N/A | 0 | $\mathbb{Z}^{10}$ | $\mathbb{Z}^{10}$ | 0 | $\tau S^1$ | $T^3_{p6}$ |
| $T^4_{pmm}$ | $\omega$ | $c_{pmm}$ | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | $\tau S^1$ | $T^4_{pmm}$ |
| $T^4_{pmm}$ | 0 | $c_{pmm}$ | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | $\tau S^1$ | $T^4_{pmm}$ |
| $T^4_{pgg}$ | $\omega$ | $c_{pgg}$ | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | $\tau S^1$ | $T^4_{pgg}$ |
| $T^4_{pgg}$ | $\omega$ | $c_{pgg}$ | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | $\tau S^1$ | $T^4_{pgg}$ |
| $T^4_{pdm}$ | $\omega$ | $c_{pdm}$ | $\mathbb{Z}^0$ | $\mathbb{Z}^0$ | 0 | $\tau S^1$ | $T^4_{pdm}$ |

Table 2: List of $K$-theory groups appearing in 2D crystallographic T-dualities, note the use of Eq. (25). Unshaded entries were computed in [52] through $K_\bullet(C^*_r(G))$ (see also [35]), and directly as twisted $K$-theory groups in [46]. Shaded entries indicate $K$-theories with graded twists that are further implied by various T-dualities, which for point group $\mathbb{Z}_2$ (middle set of rows) were independently computed in §5 (and partially in §5.5 of [19]).
7 1D crystallographic T-dualities

7.1 1D space groups, frieze groups, and graded twists

The two 1D space groups \(\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}_2\) are special cases of frieze groups, and we shall use the international notation \(p1, p1m1\). Our convention is to regard the \(\mathbb{Z}\) symmetry to be along the horizontal direction. A frieze group is a generalisation of a 1D space group to include an extra “internal” direction. Such generalisations of space groups are called subperiodic groups \([32]\). Sometimes, the extra direction is taken to be a time direction which can be reversed by symmetry group operations, and 1D frieze groups are examples of magnetic space groups.

This internal direction is crucial in the bulk-boundary correspondence, where a 1D boundary line should be thought of as sitting in 2D, whence it has a notion of “above” and “below” the line \([22]\). For example, even though reflection of the vertical coordinate in 2D restricts to the trivial action on the invariant horizontal axis, the internal label “above/below” is changed, and this is recorded by giving the reflection the odd grading.

The seven frieze groups, with their natural gradings, are summarized in the following table. The point groups are either trivial, \(\mathbb{Z}_2\), or \(D_2\). In a semidirect product \(\mathbb{Z} \times \mathbb{Z}_2\), the point group \(\mathbb{Z}_2\) acts on \(\mathbb{Z}\) by reflection, while in \(\mathbb{Z} \times D_2\), the second factor of \(D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2\) acts on \(\mathbb{Z}\) by reflection while the first factor acts trivially. The projection of \(D_2\) onto the \(i\)-th \(\mathbb{Z}_2\)-factor is denoted by \(p_i, i = 1, 2\).

| IUC Name | Graded point group | Abstract graded group |
|----------|--------------------|-----------------------|
| p1       | \(\mathbb{Z}\)        | \(\mathbb{Z}\)      |
| p1m1     | \(\mathbb{Z}_2 \to 1\) | \(\mathbb{Z} \times \mathbb{Z}_2\) |
| p2       | \(\mathbb{Z}_2 \to \mathbb{Z}_2\) | \(\mathbb{Z} \times \mathbb{Z}_2^{1,1}\) |
| p11m     | \(\mathbb{Z}_2^{1,1}\) | \(\mathbb{Z} \times \mathbb{Z}_2^{1,1}\) |
| p11g     | \(\mathbb{Z}_2^{(-1)^n}\) | \(\mathbb{Z} \times \mathbb{Z}_2^{(-1)^n}\) |
| p2mm     | \(D_2^{1, p_1}\) | \(\mathbb{Z} \times D_2^{1, p_1}\) |
| p2mg     | \(D_2^{(-1)^n, 1}\) | \(\mathbb{Z} \times D_2^{(-1)^n, 1}\) |

7.1.1 Point group 1: p1

p1 case. \(T^1_{p1} = S^1, \hat{T}^1_{p1} = \hat{S}^1\), and \(T_{p1}\) is the basic T-duality in Eq. (15).

7.1.2 Point group \(\mathbb{Z}_2\) acting by reflection: p1m1, p2

The frieze groups p1m1 and p2 are both \(\mathbb{Z} \times \mathbb{Z}_2\) and come with the nontrivial orientability homomorphism \(c_\phi = c : \mathbb{Z}_2 \to \mathbb{Z}_2\). In both cases, \(T^1_\phi = R^1/\mathbb{Z} = S^1_{\text{flip}}\) has the flip involution, and the momentum space is \(\hat{S}^1_{\text{flip}}\). However, the point group in p1m1 implements reflection of the horizontal coordinate, whereas in p2 it implements \(\pi\)-rotation; only the latter exchanges the internal “above/below” label and is non-trivially graded by \(c : \mathbb{Z}_2 \to \mathbb{Z}_2\).
Figure 3: For each frieze group, three lattice points \( \bullet \) (two unit cells) are drawn. There may be additional symmetry operations which preserve a pattern of symbols indicated \( b, p, \lor, \land, S, B, \) or \( H \). Thick horizontal lines are horizontal reflection axes, dashed lines are glide reflection axes, circles are \( \pi \) rotation centres, and vertical lines | are vertical reflection axes. The \( \pi \)-rotation symmetries for \( p2mm \) and \( p2mg \) are not independent group generators, and are omitted.

The crystallographic T-duality, Eq. (21), for \( p1m1 \) is

\[
\mathcal{T}_{p1m1} : K^{\bullet}_{2} (S^{1}_{\text{flip}}) \cong K^{\bullet+\epsilon}_{2} (S^{1}_{\text{flip}}) \xrightarrow{\cong} K^{\bullet-1}_{2} (\hat{S}^{1}_{\text{flip}}),
\]

which is also \( \mathcal{T}_{p} \) for the ‘Real’ T-dual circle bundles \( S^{1}_{\text{flip}}, \hat{S}^{1}_{\text{flip}} \) over a point (Proposition 5.2). Adding a \( c \)-twist to both sides, (or exchanging the roles of \( S^{1}_{\text{flip}} \) and \( \hat{S}^{1}_{\text{flip}} \)), we get the crystallographic T-duality for the graded group \( p2 \),

\[
\mathcal{T}_{p2} : K^{\bullet+c\omega+c}_{2} (S^{1}_{\text{flip}}) = K^{\bullet}_{2} (S^{1}_{\text{flip}}) \xrightarrow{\cong} K^{\bullet-1+c}_{2} (\hat{S}^{1}_{\text{flip}}) \cong K^{\bullet-1}_{2} (\hat{S}^{1}_{\text{flip}}).
\]

7.1.3 Point group \( \mathbb{Z}_{2} \) acting trivially: \( p11m, p11g \)

The frieze group \( p11m \) is \( \mathbb{Z} \times \mathbb{Z}_{2} \), with the point group \( \mathbb{Z}_{2} \) reflecting the vertical coordinate and thus nontrivially graded. It has \( T^{p11m}_{3} = BZ = R^{1}/Z = S^{1}_{\text{triv}} \) and Brillouin torus \( \hat{S}^{1}_{\text{triv}} \).

As \( \mathbb{Z}_{2} \)-equivariant circle bundles over a point, \( S^{1}_{\text{triv}}, \hat{S}^{1}_{\text{triv}} \) are T-dual,

\[
\mathcal{T}_{S_{2}} : K^{\bullet}_{2} (S^{1}_{\text{triv}}) \xrightarrow{\cong} K^{\bullet-1}_{2} (\hat{S}^{1}_{\text{triv}}),
\]

and adding a \( c \)-twist on both sides gives crystallographic T-duality for \( p11m \),

\[
\mathcal{T}_{p11m} : K^{\bullet+c}_{2} (S^{1}_{\text{triv}}) \xrightarrow{\cong} K^{\bullet-1+c}_{2} (\hat{S}^{1}_{\text{triv}}).
\]

Remark 7.1. In §3.2, we saw that \( H^{1}_{2} (S^{1}_{\text{triv}}, \mathbb{Z}_{2}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \) with generators \( c \) and \( M \) the Möbius bundle over \( S^{1}_{\text{triv}} \) made \( \mathbb{Z}_{2} \)-equivariant in a trivial way. We sketch a strategy to T-dualise \( (S^{1}_{\text{triv}}, M) \) and \( (S^{1}_{\text{triv}}, M+c) \) in §7.2.

In §3.1, we also saw that \( H^{3}_{2} (S^{1}_{\text{triv}}, \mathbb{Z}) \cong \mathbb{Z}/2 \) generated by \( \tau_{S_{1}} \), defined by Eq. (8). This \( \mathbb{Z} \)-twist appears in the crystallographic T-dual of \( p11g \), the graded group \( \mathbb{Z} \) generated by an odd glide reflection, i.e. reflection of vertical coordinate followed by half a lattice translation, see Fig. 4. The (even) lattice subgroup \( \Pi \)
is a proper subgroup of \( p11g \) of index 2,

\[
0 \to \Pi \cong \mathbb{Z} \xrightarrow{x^2} p11g \cong \mathbb{Z} \xrightarrow{(-1)^f} \mathbb{Z}_2 \to 1
\]  

(30)

On \( T^1 = R^1/\Pi \), the translational part of the \( \mathbb{Z}_2 \)-action is \( s_{p11g} : -1 \mapsto e^{i\pi} = -1 \), so \( T^1_{p11g} = S^1_{\text{free}} \). The Brillouin zone is \( \tilde{\Pi} = \tilde{S}^1_{\text{triv}} \) but has the cocycle twist \( \tau_{p11g} = \tau_{S1} \) due to the 2-cocycle \( \nu_{p11g}(-1, -1) = 1 \) for Eq. (30). By Proposition 5.1, \((S^1_{\text{free}}, 0)\) and \((\tilde{S}^1_{\text{triv}}, \tau_{S1})\) are T-dual pairs in the \( \mathbb{Z}_2 \)-equivariant sense, so

\[
T_{Z_2} : K^*_s(S^1_{\text{free}}) \xrightarrow{\cong} K^*_{Z_2}(S^1_{\text{triv}}). 
\]

Adding a \( c \)-twist to both sides gives crystallographic T-duality for \( p11g \),

\[
T_{p11g} : K^*_s(S^1_{\text{free}}) \xrightarrow{\cong} K^*_{Z_2}(S^1_{\text{triv}}). 
\]

(31)

**Remark 7.2.** In [45], the odd glide reflection generating \( p11g \) was called a nonsymmetric chirality symmetry, and the \( K \)-theory of the \( (\tau_{S1}, c) \)-twisted \( \tilde{S}^1_{\text{triv}} \) was computed to be \( \mathbb{Z}/2 \). This \( K \)-theory group is important for the crystallographic bulk-boundary correspondence for \( G = \text{pg} \), as studied in [22] and briefly discussed in §8.2.

The T-dualities associated to frieze groups with \( G \subset \mathbb{Z}_2 \) appear in Table 3.

### 7.1.4  \( D_2 \) point group: p2mm, p2mg

For \( p2mm \), the point group is \( D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) with the first (resp. second) generator reflecting the vertical (resp. horizontal) coordinate, so \( T^1_{p2mm} = S^1_{\text{triv} \times \text{flip}} \) and similarly for the Brillouin torus \( T^1_{p2mm} = \tilde{S}^1_{\text{triv} \times \text{flip}} \). Let the \( i \)-th projection homomorphism \( p_i : D^2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \) define the \( H^1 \)-twist \( c_i, i = 1, 2 \), then the grading \( p_1 \) on \( D_2 \) gives the twist \( c_1 \), while the orientability homomorphism \( p_2 \) gives \( c_{p2mm} = c_2 = c_{p2mg} \).

If \( p2mm \) is regarded as an ungraded group \( \mathbb{Z} \times D_2 \), the crystallographic T-duality Eq. 21 would give

\[
T_{\text{ungraded}}_{p2mm} : K^*_{D_2}(S^1_{\text{triv} \times \text{flip}}) \cong K^*_D(\tilde{S}^1_{\text{triv} \times \text{flip}}). 
\]

We anticipate that when the grading twist \( c_1 \) is added, we will obtain

\[
T_{p2mm} : K^*_{D_2}(\omega, c_1 + c_2)(S^1_{\text{triv} \times \text{flip}}) \cong K^*_{D_2}(\tilde{S}^1_{\text{triv} \times \text{flip}}), 
\]

where on the LHS, we recall that \( (0, c_1) + (0, c_2) = (\omega, c_1 + c_2) \) in the group of graded \( D_2 \)-equivariant twists (pulled back from pt) due to Eq. (10).
So far, the $H^1$-twists that we have considered are of $c$-type, coming from a homomorphism $G \to \mathbb{Z}_2$. Consider $(S^1, M)$ where $M \in H^1(S^1, \mathbb{Z}_2)$ is the Möbius twist. Passing to the double cover $S^1_{\text{free}}$, the generating twist $c \in H^1_{\text{free}}(S^1_{\text{free}}, \mathbb{Z}_2) \cong \mathbb{Z}/2$ corresponds to $M$, and it is possible to show that

$$K^{\bullet+M}(S^1) \cong K^{\bullet+c}(S^1_{\text{free}}) \cong \begin{cases} 0, & \bullet = 0, \\ \mathbb{Z}/2, & \bullet = 1. \end{cases}$$

We had already seen that $S^1_{\text{free}}$ can be identified with $T^1_{\text{p11g}}$ and found the $\mathbb{Z}_2$-equivariant T-dual of $(S^1_{\text{free}}, c)$ in Eq. (31). Thus $(S^1, M)$ has a T-dual pair via passage to an equivariant double cover $S^1_{\text{free}}$. Now consider $S^1_{\text{triv}}$ which has $H^1_{\text{triv}}(S^1_{\text{triv}}, \mathbb{Z}_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $c$ and by $M$ made $\mathbb{Z}_2$-equivariant in the trivial way. A similar strategy to T-dualise $(S^1_{\text{triv}}, M)$ is to pass to a double cover $S^1_{\text{triv}} \times \mathbb{Z}_2 \to S^1_{\text{triv}}$ which has a $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ action with the first factor acting by deck transformations.

Table 3: $S^1$ and the three involutive circles with all possible graded twists, except those of $M$-type, are T-dualised as above. To T-dualise $(S^1, M)$ we pass to $(S^1_{\text{free}}, c)$ instead. To T-dualise $(S^1_{\text{triv}}, M)$, we need to pass to a double cover and take a conjectured $D_2$-equivariant T-dual.

For $p2mg$, the vertical coordinate reflection is replaced by a glide reflection so that $T^1_{p2mg} = S^1_{\text{free} \times \text{flip}}$. The Brillouin torus is again $\hat{S}^1_{\text{triv} \times \text{flip}}$, with a 2-cocycle twist $\tau_{p2mg}$ from the nonsymmmorphicity. The crystallographic T-duality for the group $p2mg$ is

$$T^\text{ungraded}_{p2mg} : K_{D_2}^{\bullet+c_2}(S^1_{\text{free} \times \text{flip}}) \cong K_{D_2}^{\bullet-1+\tau_{p2mg}}(\hat{S}^1_{\text{triv} \times \text{flip}}),$$

$$T_{p2mg} : K_{D_2}^{\bullet+c_1+c_2}(S^1_{\text{free} \times \text{flip}}) \cong K_{D_2}^{\bullet-1+(\tau_{p2mg}, c_1)}(\hat{S}^1_{\text{triv} \times \text{flip}})$$

where we note that $H^3_{D_2}(S^1_{\text{free} \times \text{flip}}, \mathbb{Z}) \cong H^3_{\hat{S}^1_{\text{triv} \times \text{flip}}}(\mathbb{Z}, \mathbb{Z}) = 0$ so $c_1, c_2$ add as graded twists in the naïve way as in $H^1_{D_2}(S^1_{\text{free} \times \text{flip}}, \mathbb{Z}_2)$.

7.2 T-duality with Möbius twists and $G$-equivariant T-duality

So far, the $H^1$-twists that we have considered are of $c$-type, coming from a homomorphism $G \to \mathbb{Z}_2$. Consider $(S^1, M)$ where $M \in H^1(S^1, \mathbb{Z}_2)$ is the Möbius twist. Passing to the double cover $S^1_{\text{free}}$, the generating twist $c \in H^1_{\text{free}}(S^1_{\text{free}}, \mathbb{Z}_2) \cong \mathbb{Z}/2$ corresponds to $M$, and it is possible to show that

$$K^{\bullet+M}(S^1) \cong K^{\bullet+c}(S^1_{\text{free}}) \cong \begin{cases} 0, & \bullet = 0, \\ \mathbb{Z}/2, & \bullet = 1. \end{cases}$$

We had already seen that $S^1_{\text{free}}$ can be identified with $T^1_{\text{p11g}}$ and found the $\mathbb{Z}_2$-equivariant T-dual of $(S^1_{\text{free}}, c)$ in Eq. (31). Thus $(S^1, M)$ has a T-dual pair via passage to an equivariant double cover $S^1_{\text{free}}$. Now consider $S^1_{\text{triv}}$ which has $H^1_{\text{triv}}(S^1_{\text{triv}}, \mathbb{Z}_2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $c$ and by $M$ made $\mathbb{Z}_2$-equivariant in the trivial way. A similar strategy to T-dualise $(S^1_{\text{triv}}, M)$ is to pass to a double cover $S^1_{\text{triv}} = S^1_{\text{free} \times \text{triv}} \to S^1_{\text{triv}}$ which has a $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ action with the first factor acting by deck transformations.
Now regard $M \to S^1_{\text{triv}}$ as a $D_2$-equivariant real line bundle $\tilde{M} \to S^1_{\text{free} \times \text{triv}}$ — explicitly, $\tilde{M}$ is the product bundle with $D_2$ acting via its first $\mathbb{Z}_2$ factor by the deck transformation on the base and $-1$ on the fiber. Thus $\tilde{M}$ can be regarded as the twist $c_1 \in H^1_D(S^1_{\text{free} \times \text{triv}}, \mathbb{Z}_2)$ coming from the homomorphism $p_1 : D_2 \to \mathbb{Z}_2$.

Conjecturally, there is a notion of $D_2$-equivariant T-duality $T_{D_2}$ (and also for more general groups $G$), generalising $T_{\mathbb{Z}_2}$ and $T_R$ in a natural way. Then we can T-dualise $(S^1_{\text{triv}}, M)$ by first passing to $(S^1_{\text{free} \times \text{triv}}, c_1)$ and then taking $T_{D_2}$. Furthermore, the frieze group dualities in §7.1.4, as well as the wallpaper group crystallographic T-dualities in Eq. (29), would be expected to be implemented by $T_{D_2}$. The circle can be made a $D_2$ space in several other ways such as $S^1_{\text{free} \times \text{triv}}$. These $D_2$-actions can arise from more general subperiodic groups such as the rod groups [32] associated to symmetries of a line in 3D space. For example, a two fold screw axis which is also on a reflection plane is preserved by a point group $D_2$. The $D_2$ action on a unit cell (a circle) for the lattice translation along the axis gives $S^1_{\text{free} \times \text{triv}}$. Generalising the particular case of $p1_{11g}$ studied in [22], we expect that the $K$-theories associated to rod groups will be important for crystallographic bulk-boundary correspondences with screw axes.

8 3D dualities and applications

8.1 H-flux from partial T-duality: screw dislocations

In [24], it was observed that $H^3$-flux (in the nonequivariant sense) is “produced” when a screw-dislocated lattice is partially Fourier transformed. In string theory, one might start with $T^3$ with “one unit of H-flux”, meaning that $K^{**+h}(T^3)$ is needed, for $h$ a generator of $H^3(T^3, \mathbb{Z})$, to study D-brane charges. As a circle bundle over $T^2$, the T-dual of the pair $(T^3, h)$ is $(\text{Nil}, 0)$ where the nilmanifold $\text{Nil}$ is the circle bundle over $T^2$ with Chern class the generator of $H^2(T^2, \mathbb{Z})$ as required. The name “$\text{Nil}$” comes from the fact that $\pi_1(\text{Nil}) = \text{Heis}^Z$, the integer Heisenberg group

$$\text{Heis}^Z = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\},$$

and $\text{Nil} = \text{Heis}^Z / \text{Heis}^Z$ is a $B\text{Heis}^Z$. This example illustrated “topology change from H-flux” as in [11].

The story is run from a different angle in [24], where the nonabelian lattice $\text{Heis}^Z$ was considered to be a screw-dislocated version of the standard (Euclidean) lattice $\mathbb{Z}^3$. One sees this by noticing that the commutator of the $a$ and $b$ translations in $\text{Heis}^Z$ is not zero (corresponding to a closed loop) but rather a translation in the third direction (corresponding to helical motion along an axis in this third direction). The fundamental domain (“unit cell” in position space) is no longer the 3-torus, but $\text{Nil}$. Despite $\text{Heis}^Z$ being nonabelian, it is built up from three copies of $\mathbb{Z}$, and the noncommutative “momentum space”
C_{r}(\text{Heis}^Z) can be understood as a field of noncommutative tori parameterised by the circle dual to the central Z [36]. Note that Heis\( ^Z \) is a discrete cocompact subgroup of the continuous version Heis\( ^R \) (with real number entries), and then “crystallographic T-duality” for Heis\( ^Z \) can be defined as Poincaré duality for Nil = BHeis\( ^Z \) = Heis\( ^R \)/Heis\( ^Z \) composed with the Baum–Connes assembly map. In summary, we have

\[
K^{•+h}(T^3) \xrightarrow{T_{\text{circle}}} K_{•}(C_{r}(\text{Heis}^Z)) \xrightarrow{T_{\text{crystal}}} K^{•+1}(\text{Nil})
\]

Then we see that \( T_{\text{circle}} \), interpreted as a partial Fourier transform, means that the mixed position-momentum space \( T^3 \) comes with a \( H^3 \)-twist. This is an other instance of the observation in \( §6.2.3 \).

8.2 Crystallographic bulk-boundary correspondence and super-indices for boundary zero modes

In a crystalline version of the bulk-boundary correspondence, a \( d \)-dimensional crystalline topological insulator should be detectable on some codimension-1 layer fixed under some point group operation. Such a layer need not only have an ordinary \( d − 1 \)-dimensional space group symmetry, but the isotropy can also contribute by toggling the “above/below” degree of freedom.

For example, if \( \mathcal{G} = \text{pg} \), there is a 2-torsion Class AIII phase because of \( K_{222}^{-1+\tau_{S1}}(\hat{T}_{\text{pm}}^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2 \) (recall that \( \tau_{\text{pg}} = \tau_{S1} \)). In [22], is was shown that this phase is detected by zero modes localised on a cut along a glide axis, and such an axis has precisely the 1D frieze group p11g symmetry with the generator given the odd grading, see Fig. 4. The graded group p11g gives \( K_{222}^{0+c+\tau_{S1}}(\hat{S}_{\text{triv}}^1) \cong \mathbb{Z}/2 \) on the RHS of crystallographic T-duality, and may be understood as \( K_{•}^{\text{graded}}(C_{r}(\text{p11g})) \) of the graded group \( C^{•} \)-algebra for p11g \( \cong \mathbb{Z}^{(−1)^{3} Z_2} \). A natural ‘Real’ Gysin map takes \( \pi_* : K_{222}^{-1+\tau_{S1}}(\hat{T}_{\text{pm}}^2) \to K_{222}^{0+c+\tau_{S1}}(\hat{S}_{\text{triv}}^1) \) along the fiber projection \( \pi : \hat{T}_{\text{pm}}^2 = \hat{S}_{\text{flip}}^1 \times \hat{S}_{\text{triv}}^1 \to \hat{S}_{\text{triv}}^1 \), and it was shown in [22] that \( \pi_* \) realises an analytic index map for a \( \tau_{S1} \)-twisted family of Toeplitz-like operators parametrised by \( \hat{S}_{\text{triv}}^1 \). In this way, the target group

\[
K_{•}^{\text{graded}}(C_{r}(\text{p11g})) \cong K_{222}^{0+c+\tau_{S1}}(\hat{S}_{\text{triv}}^1)
\]

is the “super-higher index” group for the p11g-symmetric topological boundary zero modes of pg-symmetric insulators.

8.3 Spectral sequence extension problems and halving computations of topological phases

Consider the symmorphic 3D space groups \( \mathcal{G} = \text{P222}, \text{C222}, \text{F222}, \text{I222} \), which have point group \( 222 \cong D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \to \text{O}(3) \) whose three nontrivial elements...
are π-rotations about three mutually orthogonal axes (say, x, y, z). Note that $H_{322}^3(\text{pt}, \mathbb{Z}) \cong \mathbb{Z}/2$ with generator $\omega$ which pulls back to a nontrivial twist on $T^3_D$. We do not need the precise description of these space groups, but just the fact that the arithmetic crystal classes for P222, C222 are self-dual but those for F222 and I222 are dual to each other, i.e. α for one is $\bar{\alpha}$ for the other (see (45)-(47) of [38] for a list of dual pairs of arithmetic crystal classes).

In [47], the Atiyah–Hirzebruch spectral sequence (AHSS) was used to compute the $K$-theories of $T^3_{P222}, T^3_{C222}, T^3_{F222}, T^3_{I222}$, with the results\(^3\)

\[
\begin{align*}
K^0_{D_2}(T^3_{P222}) &\cong \mathbb{Z}^{13}, & K^2_{D_2}(T^3_{P222}) &\cong \mathbb{Z} \oplus \mathbb{Z}/2, \\
K^0_{D_2}(T^3_{C222}) &\cong \mathbb{Z}^8, & K^2_{D_2}(T^3_{C222}) &\cong \mathbb{Z}^2 \oplus \mathbb{Z}/2, \\
K^0_{D_2}(T^3_{F222}) &\cong \mathbb{Z}^7, & K^2_{D_2}(T^3_{F222}) &\cong \mathbb{Z}^2 \oplus \mathbb{Z}/2, \\
K^0_{D_2}(T^3_{I222}) &\cong \mathbb{Z}^7 \oplus \mathbb{Z}/2, & K^2_{D_2}(T^3_{I222}) &\cong \mathbb{Z} \oplus \mathbb{Z}/2.
\end{align*}
\]

It is possible to resolve the ambiguity for the untwisted $K^1_{D_2}$ by a direct Mayer–Vietoris computation, but let us instead show how crystallographic T-duality comes to the rescue. In anticipation of this, notice that groups in the left column also appear in the right column.

First, note that the $222 \cong D_2$ point group action is orientable but has the Spin\(^c\) obstruction $\omega$, by Lemma A.3, and this is pulled back faithfully to the $K_{D_2}$-orientability obstruction for $T^3_D$. Consequently, the crystallographic T-dualities are (dropping the hats for now)

\[
\begin{align*}
T^*_{P222} : K^*_{D_2}(T^3_{P222}) &\xrightarrow{\cong} K^*_{D_2}(T^3_{F222}), \\
T^*_{C222} : K^*_{D_2}(T^3_{C222}) &\xrightarrow{\cong} K^*_{D_2}(T^3_{C222}), \\
T^*_{F222} : K^*_{D_2}(T^3_{F222}) &\xrightarrow{\cong} K^*_{D_2}(T^3_{I222}), \\
T^*_{I222} : K^*_{D_2}(T^3_{I222}) &\xrightarrow{\cong} K^*_{D_2}(T^3_{F222}).
\end{align*}
\]

These dualities enable the resolution of the $K^1$ ambiguities by referring to the unambiguous $K^0$ groups on the T-dual side, i.e.,

\[
K^1_{D_2}(T^3_{P222}) \cong \mathbb{Z}, \quad K^1_{D_2}(T^3_{C222}) \cong \mathbb{Z}^2, \quad K^1_{D_2}(T^3_{I222}) \cong \mathbb{Z}, \\
K^1_{D_2}(T^3_{F222}) \cong \mathbb{Z} \oplus \mathbb{Z}/2, \quad K^1_{D_2}(T^3_{F222}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2.
\]

These examples demonstrate how our crystallographic T-duality supplements the powerful general machinery of the AHSS. In effect, the number of computations is halved, some twisted $K$-theories can be computed more easily on

---

\(^3\)In Table 4 of [47], the entries for a space group are the $E_2$ terms for the untwisted (+0) and $\omega$-twisted ($-1/2$) $K$-theory for the corresponding Brillouin torus with dual point group action. So, e.g. the F222 entries compute the $K$-theory for $T^3_{I222}$.\]
the T-dual side (cf. Remark 6.2), and extension problems may be resolved by inspecting the T-dual computations. In the physics context, these $K^1$ groups (with no $\omega$-twist) classify the so-called Class AIII topological insulators with respective space group symmetries, and in particular (restoring the hat) $K^1_{D_3}(T_{222}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ shows that there is a 2-torsion chiral symmetric and I222-symmetric phase.

**Acknowledgements**

G.C.T. is supported by ARC grant DE170100149, and K.G. by JSPS KAKENHI Grant Number JP15K04871. Both authors would like to thank Siye Wu for his hospitality at the National Center for Theoretical Sciences (Physics Division) of Taiwan, where the ideas for this paper crystallized.

**A Appendix**

In order to determine the $K_G$-orientability obstruction $\sigma_g$ for a torus with action induced from a space group $G$, Eq. (11), we need to first compute the obstruction class $W^G_3(R^d)$ of Eq. (13) associated to the point group $\rho : G \to O(d)$ as follows.

To shorten notation, we just write $g$ for $\rho(g) \in O(d)$. Choose lifts $\tilde{g} \in \text{Pin}^c(d)$ of $g \in G \subset O(d)$ in the central extension

$$1 \to U(1) \to \text{Pin}^c(d) \to O(d) \to 1,$$

(32)

giving a projective representation of $G$ whose cocycle $\zeta \in Z^2(G, U(1))$ is

$$\tilde{g} \tilde{h} = \zeta(g, h)\tilde{g}\tilde{h}, \quad g, h \in G.$$

If $[\zeta] = 0 \in H^2(G, U(1))$, then we can actually choose the lifts $\tilde{g}$ to give a genuine representation of $G$ factoring through $\text{Pin}^c(d)$, otherwise there is an obstruction and $(W^G_3(R^d), W^G_1(R^d)) \neq 0$.

**Lemma A.1.** Let $\zeta \in Z^2(G, U(1))$ be a cocycle for a finite group $G$, and let $\epsilon(g, h) := \zeta(g, h)\zeta(h, g)^{-1}$. If $\epsilon(g, h) \neq 1$ for some $g, h$ such that $gh = hg$, then $[\zeta] \neq 0 \in H^2(G, U(1))$.

**Proof.** If $\zeta$ is a coboundary, we can verify that $\epsilon(g, h) = 1$ whenever $gh = hg$. □

Let us analyse the 2D point groups $G = D_2, D_4, D_6 \subset O(2)$. They are generated in $O(2)$ by a reflection $\varsigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and a rotation $r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $\theta = \frac{2\pi}{n}$. Let $\mathbb{C}l(2)$ be the complex Clifford algebra generated by $e_1, e_2$ with $e_1^2 = e_2^2 = -1, e_1 e_2 = e_2 e_1 = 0$. The $\text{Pin}(2)$ group is a double cover of $O(2)$ and can be realised concretely inside $\mathbb{C}l(2)$ as

$$\text{Pin}(2) = \{\cos \theta + \sin \theta e_1 e_2\} \cup \{\cos \theta e_1 + \sin \theta e_2\}_{\theta \in [0, 2\pi]}$$
The double-cover projection \( \text{Pin}(2) \xrightarrow{\cong} \text{O}(2) \) is then given by
\[
\varpi(\cos \theta + \sin \theta e_1 e_2) = r(2\theta), \\
\varpi(\cos \theta e_1 + \sin \theta e_2) = r(2\theta)\sigma
\]
The \( \text{Pin}^c(2) \) group is defined to be \( \text{Pin}(2) \times \text{U}(1) / \{(1,1),(-1,-1)\} \), and the projection in the central extension Eq. (32) is \( \varpi^c[x,u] = \varpi(x) \).

**Lemma A.2.** For the point groups \( G = D_2, D_4, D_6 \xrightarrow{\rho} \text{O}(2) \), the class \( W^G_3(\mathbb{R}^2) \in H^3_G(\text{pt}, \mathbb{Z}) \cong \mathbb{Z}/2 \) is the unique generator \( \omega \).

**Proof.** Note that \( D_2, D_4, D_6 \) each contain the commuting elements \( \varsigma \) and \( r_\pi \). Choose the lifts
\[
\tilde{r}_\pi = [e_1 e_2, 1], \quad \tilde{\varsigma} = [e_1, 1] \quad \tilde{r}_\pi \tilde{\varsigma} = \tilde{\varsigma} \tilde{r}_\pi = \tilde{r}_\pi \tilde{\varsigma}.
\]
Then Lemma A.1 applied to the computation
\[
\zeta(\varsigma, r_\pi) = \tilde{\varsigma} \tilde{r}_\pi \tilde{\varsigma}^{-1} = [e_1, 1][e_1 e_2, 1][e_1 e_2, 1][e_1, 1]^{-1} = -1
\]
shows that \( \rho \) is not \( \text{Pin}^c(2) \), and the obstruction \( W^G_3(\mathbb{R}^2) \) is nontrivial.

Next, we analyse the 3D point group \( 222 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \subset \text{O}(d) \) which contains \( \pi \) rotations \( r_x, r_y, r_z = r_x r_y = r_y r_z \) about the \( x, y, z \) axes. Since \( 222 \subset \text{SO}(3) \), we only need to check whether it is also \( \text{Spin}^c \), and we recall the exact sequence
\[
1 \to \text{U}(1) \to \text{U}(2) = \text{Spin}^c(3) \to \text{PU}(2) = \text{SO}(3) \to 1.
\]

**Lemma A.3.** For the point group \( G = 222 \xrightarrow{\rho} \text{SO}(3) \subset \text{O}(3) \), the class \( W^G_3(\mathbb{R}^3) \in H^3_G(\text{pt}, \mathbb{Z}) \cong \mathbb{Z}/2 \) is the unique generator \( \omega \).

**Proof.** Choose lifts of \( r_i \in G, i = x, y, z \) to be \( \tilde{r}_i = e^{\frac{\pi}{2}} \sigma_i = i \sigma_i \in \text{U}(2) = \text{Spin}^c(3) \) where \( \sigma_i \) are the Pauli spin matrices. Then the cocycle \( \zeta \) has
\[
\zeta(r_x, r_y) = \tilde{r}_x \tilde{r}_y (\tilde{r}_x \tilde{r}_y)^{-1} = (i \sigma_x)(i \sigma_y)(i \sigma_z)^{-1} = -1,
\]
and is nontrivial by Lemma A.1. Thus \( \rho \) is not \( \text{Spin}^c(3) \), and its obstruction \( W^G_3(\mathbb{R}^3) \) is nontrivial.

**References**

[1] M. Ando, A.J. Blumberg, D. Gepner.: Twists of \( K \)-theory and \( \text{TMF} \). Superstring, geometry, topology, and \( C^* \)-algebras, Proc. Sympos. Pure Math., vol. 81, Amer. Math. Soc., Providence, RI, 2010, pp. 27–63

[2] M.F. Atiyah.: \( K \)-theory and reality. Quart. J. Math. 17(1) 367–386 (1966)

[3] M.F. Atiyah, M. Hopkins.: A variant of \( K \)-theory: \( K^\pm \). Topology, geometry and quantum field theory 308 5–17 (2004)
[4] D. Baraglia.: Topological T-Duality for General Circle Bundles. Pure and Applied Mathematics Quarterly 10(3) 367–438 (2014)

[5] P. Baum, A. Connes, N. Higson.: Classifying space for proper actions and K-theory of group C*-algebras. C*-algebras: 1943–1993 (San Antonio, TX, 1993), Contemp. Math. 167, Amer. Math. Soc. (1994), 240–291.

[6] J. Bellissard.: K-theory of C*-algebras in solid state physics. Statistical mechanics and field theory: mathematical aspects. Springer, Berlin, Heidelberg (1986) 99–156.

[7] J. Bellissard, A. van Elst, H. Schulz-Baldes. The noncommutative geometry of the quantum Hall effect. J. Math. Phys. 35(10) 5373–5451 (1994)

[8] L. Bieberbach.: Über die Bewegungsgruppen der Euklidischen Räume I, Math. Ann., 70 297–336 (1910)

[9] C. Bourne, J. Kellendonk, A. Rennie.: The K-Theoretic Bulk-Edge Correspondence for Topological Insulators. Ann. Henri Poincar. 18(5) 1253–1273 (2017)

[10] P. Bouwknegt, A.L. Carey, V. Mathai, M.K. Murray, D. Stevenson.: Twisted K-theory and K-theory of bundle gerbes. Commun. Math. Phys. 228(1) 17–49 (2002)

[11] P. Bouwknegt, J. Evslin, V. Mathai.: T-duality: Topology change from H-flux. Commun. Math. Phys. 249(2) 383–415 (2004)

[12] T.H. Buscher.: A symmetry of the string background field equations. Phys. Lett. B 194(1) 59–62 (1987)

[13] P. Cartier.: Quantum mechanical commutation relations and theta functions. Proc. Sympos. Pure Math., vol. 9, Amer. Math., Soc., Providence, R. I., 1966, pp. 361–383

[14] J.H. Conway, O.D. Friedrichs, D.H. Huson, W.P. Thurston.: On Three-dimensional Orbifolds and Space Groups. Beiträge Algebra Geom. 42(2) 475–507 (2001)

[15] C. Doran, S. Méndez-Diez, J. Rosenberg.: T-duality for orientifolds and twisted KR-theory. Lett. Math. Phys. 104(11) 1333–1364 (2014)

[16] S. Echterhoff, H. Emerson, H.J. Kim.: KK-theoretic duality for proper twisted actions.: Math. Ann. 340(4) 839–873 (2008)

[17] D.S. Freed, M.J. Hopkins, C. Teleman.: Loop groups and twisted K-theory I. J. Topology 4 737–798 (2011)

[18] D.S. Freed, G. Moore.: Twisted equivariant matter. Ann. Henri Poincaré 14(8) 1927–2023 (2013)
[19] K. Gomi.: A variant of K-theory and topological T-duality for real circle bundles. Commun. Math. Phys. 334(2) 923–975 (2015)

[20] K. Gomi.: Twists on the Torus Equivariant under the 2-Dimensional Crystallographic Point Groups. SIGMA Symmetry Integrability Geom. Methods Appl. 13 014 (2017)

[21] K. Gomi.: Freed–Moore K-theory. arXiv:1705.09134

[22] K. Gomi, G.C. Thiang.: Crystallographic bulk-edge correspondence: glide reflections and twisted mod 2 indices. arXiv:1804:03945

[23] G.M. Graf, M. Porta.: Bulk-edge correspondence for two-dimensional topological insulators. Commun. Math. Phys. 324(3) 851–895 (2013)

[24] K. Hannabuss, V. Mathai, G.C. Thiang.: T-duality simplifies bulk-boundary correspondence: the parametrised case. Adv. Theor. Math. Phys. 20(5) 1193–1226 (2016)

[25] K. Hannabuss, V. Mathai, G.C. Thiang.: T-duality simplifies bulk-boundary correspondence: the noncommutative case. Lett. Math. Phys. 108(5) 1163–1201 (2018)

[26] D. Handel.: On products in the cohomology of the dihedral groups. Tohoku Math. J., Second Series 45(1) 13–42 (1993)

[27] Y. Hatsugai.: Chern number and edge states in the integer quantum Hall effect. Phys. Rev. Lett. 71(22) 3697 (1993)

[28] H. Hiller.: Crystallography and cohomology of groups. Amer. Math. Monthly 93(10) 765–779 (1986)

[29] K. Hori.: D-branes, T-duality, and Index Theory. Adv. Theor. Math. Phys. 3 281–342 (1999)

[30] B. Kahn.: Construction de classes de Chern équivariantes pour un fibré vectoriel réel. Commun. Algebra 15(4) 695–711 (1987)

[31] A. Kitaev.: Periodic table for topological insulators and superconductors. AIP Conf. Proc. 1134 22–30

[32] V. Kopsky, D.B. Litvin.: eds. International Tables for Crystallography, Volume E: Subperiodic groups, E (5th ed.), Berlin, New York (2002)

[33] Y. Kubota.: Notes on twisted equivariant K-theory for C*-algebras. Int. J. Math. 27(6) 1650058 (2016)

[34] Y. Kubota.: Controlled Topological Phases and Bulk-edge Correspondence. Commun. Math. Phys. 349(2) 493–525 (2017)

[35] W. Lück, R. Stamm.: Computations of K- and L-Theory of Cocompact Planar Groups. K-theory 21 249–292 (2000)
[52] M. Yang.: Crossed Products by Finite Groups Acting on Low Dimensional Complexes and Applications. Ph.D. thesis, University of Saskatchewan, Saskatoon (1997)

[53] H. Zassenhaus.: Beweis eines Satzes über diskrete Gruppen. Abh. Math. Sem. Univ. Hamburg, 12 276–288 (1938)