A permutation pattern that illustrates the strong law of small numbers

DAVID CALLAN
Dept. of Statistics, University of Wisconsin-Madison, Madison, WI 53706
callan@stat.wisc.edu

Abstract

We obtain an explicit formula for the number of permutations of $[n]$ that avoid the barred pattern 1432. A curious feature of its counting sequence, 1, 1, 2, 5, 14, 43, 145, 538, 2194, ..., is that the displayed terms agree with A122993 in the On-Line Encyclopedia of Integer Sequences, but the two sequences diverge thereafter.

1 Introduction

A permutation $\pi$ avoids the barred pattern 1432 if each instance of a not-necessarily-consecutive 432 pattern in $\pi$ is part of a 1432 pattern in $\pi$, and similarly for other barred patterns. This paper is one of a series of notes counting permutations avoiding a 5 letter pattern with 2 bars that do not yield to Lara Pudwell’s method of Enumeration Schemes [3]. The question of whether there may be an automated method to fill in these and other gaps in Pudwell’s enumeration remains open. Here we treat the pattern 1432. A curious feature of the counting sequence is that it agrees through the $n = 8$ term with sequence A122993 in the On-Line Encyclopedia of Integer Sequences [4], an instance of the Strong Law of Small Numbers [5, 6].

Our method is to identify the structure of a 1432-avoider. This permits a direct count as a 5-summation formula according to five statistics of the permutation, four of which are the first entry $a$, the immediate predecessor of 1 denoted $b$, the position of 1 denoted $j$, and the number of left to right maxima that occur after 1 denoted $k$. One of these sums can be evaluated, leading to a faster formula.

2 1432-Avoiders

A “typical” 1432-avoider is illustrated in Figure 1 in matrix form. It has first entry $a = 5$, 1 is in position $j = 4$, the immediate predecessor of 1 is $b = 16$, and there are $k = 5$ left to right maxima that occur after 1. Here $j \geq 3$ so that 1, $a$, $b$ are all distinct. The special cases $j = 1$ or 2 are treated later. There is a vertical blue line through the bullet representing the entry 1, and yellow vertical lines through the left to right maxima that occur after 1. These $k$ yellow lines divide the the part of the matrix to the right of the blue line into $k + 1$ vertical strips (in white, one of which is vacuous in Figure 1).
Furthermore, horizontal lines through 1, a and b determine three horizontal strips indexed by $A = [2, a - 1]$, $B = [a + 1, b - 1]$, $C = [b + 1, n]$. There are $j - 3$ bullets to the left of the blue line in strip $B$ and none in $A$ or $C$. Hence, to the right of the blue line there are $A := a - 2$ bullets in strip $A$, $B := |B| - (j - 3) = b - a - j + 2$ bullets in strip $B$, and $C := n - b$ bullets in strip $C$. The following properties of a 14352-avoider are evident in the illustration and easily proved from the definition.

- The entries to the left of 1 are increasing, else together with 1, there is a 432 pattern with no available 1. Equivalently, the bullets in the gray vertical strip on the left are rising.
- The entries 2, 3, ..., $a - 1$ occur in that order, else together with $b$, there is a 432 pattern with no available 1. Equivalently, the bullets in horizontal strip $A$ are rising.
- Entries in the interval $(a, b)$ lie either to the left of 1 or to the right of $a - 1$, else together with $b$, there is a 432 pattern with no available 1. Equivalently, all bullets in horizontal strip $B$ to the right of 1 are also to the right of $a - 1$.
- Every descent initiator after 1 is a left to right maximum, else together with a left to right maximum to its left (there is one), we have a 432 pattern with no available 5. Equivalently, the bullets in each vertical white strip $A$ are rising.

Conversely, when the position $j$ of 1 is $\geq 3$, one can check that a permutation with these properties is 14352-avoiding.
To count permutations with these four properties, let \( i \in [1, k + 1] \) denote the left to right position of the first white strip containing an entry in \((a, b)\), that is, containing a bullet in horizontal strip \( B \) (when there is one).

The subpermutation of entries in \( C = [b + 1, n]\), when split at its left to right maxima, forms a partition in a canonical form: in each block, the largest entry occurs first and the rest of the block is increasing, and the blocks are ordered by increasing first entry. This yields \( \binom{C}{k} \) choices to determine the relative positions of entries in \( C \).

Next, choose \( j - 3 \) elements from \([a + 1, b - 1]\) to precede \( 1 - \binom{b-a-1}{j-3} \) choices. The entries in \( B \) following 1 must be distributed into boxes (white strips) labeled \( i, i+1, \ldots, k + 1 \) in such a way that box \( i \) is nonempty—\((k - i + 2)^B - (k - i + 1)^B\) choices when \( B > 0 \). The bullets for entries in \( A \) must be distributed into boxes \( 1, 2, \ldots, i \)—\( \binom{A+i-1}{i-1} \) choices when \( B > 0 \). In case \( B = 0 \), we merely distribute the bullets for entries in \( A \) into \( k + 1 \) boxes—\( \binom{A+k}{k} \) choices.

Recalling that \( A = a - 2 \), \( B = b - a - j + 2 \), \( C = n - b \), the contribution of the case \( j \geq 3 \) to the desired count is now seen to be

\[
\sum_{a=2}^{n-1} \sum_{b=a+1}^{n} \sum_{j=3}^{b-a+1} \sum_{k=1}^{n-b} \sum_{i=1}^{n-b-k+1} \binom{n-b}{k} \binom{b-a-1}{j-3} ((k - i + 2)^b - (k - i + 1)^b)^{a-i-3} \times \binom{a+i-3}{i-1} + \sum_{a=2}^{n-1} \sum_{b=a+1}^{n} \binom{n-b}{k} \binom{a+i-3}{i-1} \quad (1)
\]

When \( j = 1 \), the map “delete first entry” is a bijection to 4352-avoiding permutations of size \( n - 1 \), counted by the Bell number \( B_{n-1} \) [7]. When \( j = 2 \), we have \( a = b \) in Figure 1, and the count reduces to \( \sum_{a=2}^{n} \sum_{k=0}^{a-2} \binom{k+a-2}{a-2} \binom{n-a}{k} \) where \( \binom{0}{1} := 1 \).

The sum over \( j \) in (1) can be evaluated using the binomial theorem, and putting it all together we have, after minor simplifications, the following result.

**Theorem.** For \( n \geq 2 \), the number of permutations of \([n]\) avoiding the barred pattern 14352 is

\[
B_{n-1} + 1 + 2^{n-2} - n + \\
\sum_{a=0}^{n-3} \sum_{b=0}^{a-1} \sum_{k=0}^{a-b} \left( \sum_{i=0}^{k} \binom{n-4-a+k-i}{k-i} (i+2)^b - \binom{n-3-a+k}{k} \right) \binom{a-b}{k} + \\
\sum_{a=0}^{n-2} \sum_{k=0}^{n-2-a} \binom{k+a+1}{k+1} \binom{n-2-a}{k}.
\]

The first few terms of the counting sequence, starting at \( n = 1 \), are 1, 2, 5, 14, 43, 145, 538, 2194, 9790, 47491, 248706.
References

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