From GEM to Electromagnetism

A. Bakopoulos · P. Kanti

Received: date / Accepted: date

Abstract In the first part of the present work, we focus on the theory of gravitoelectromagnetism (GEM), and we derive the full set of equations and constraints that the GEM scalar and vector potentials ought to satisfy. We discuss important aspects of the theory, such as the presence of additional constraints resulting from the field equations and gauge condition, the requirement of the time-independence of the vector potential and the emergence of additional terms in the expression of the Lorentz force. We also propose an alternative ansatz for the metric perturbations that is found to be compatible only with a vacuum configuration but evades several of the aforementioned obstacles. In the second part of this work, we pose the question of whether a tensorial theory using the formalism of General Relativity could re-produce the theory of Electromagnetism. We demonstrate that the full set of Maxwell’s equations can be exactly re-produced for a large class of models, but the framework has several weak points common with those found in GEM.

Keywords General Relativity · GEM · Electromagnetism

1 Introduction

One of the most ambitious objectives of Theoretical Physics is the formulation of a theory that would unify all forces in nature. Maxwell’s theory of electro-
magnetism paved the way by unifying two, apparently distinct, sets of phenomena in nature associated with the electric and magnetic field, respectively. Decades later another unification was achieved, this time of electromagnetic and weak interactions in the context of Standard Model of particle physics. Although the true unification of forces met at a microscopic level, i.e. electromagnetic, weak and strong, will only take place – if it happens at all – at a much higher energy scale, the use of a common formalism, based on gauge theories, has already imposed a notion of ‘unification’ in people’s minds and certainly has deepened our understanding of particle interactions.

Gravity, on the other hand, is still resisting our attempts to implement it in a common framework with all other forces. Ironically, it was the profound similarity of the Newtonian and Coulomb potential, noticed centuries ago, that generated the idea of unification of forces at the first place. Even at the level of fields, the corresponding equations, namely\(^1\)

\[
\nabla \cdot \mathbf{E} = 4\pi \rho_E,
\]
\[
\nabla \cdot \mathbf{g} = -4\pi G \rho_M,
\]

where \((\mathbf{E}, \mathbf{g})\) and \((\rho_E, \rho_M)\) are the electric and gravitational fields and their corresponding charge and mass densities, clearly show the similarity in how electrostatics and gravity work. The counter-argument that the similarity breaks down as soon as we introduce the magnetic field was invalidated by the discovery of the Lense-Thirring effect [2] where the gravitational field of a rotating body may be expressed in terms of an additional ‘vector potential’ which at large distances creates a gravitational ‘magnetic’ dipole due to the mass current.

The first attempt to unify gravity with electromagnetism, the Kaluza-Klein theory, used the additional degrees of freedom of a higher-dimensional gravitational theory to accommodate the necessary gauge degrees of freedom. The theory was purely geometrical and revealed the relation between gauge invariance and coordinate transformation. The same idea was used much later in the formulation of string and M-theory [3][4]. Together with loop quantum gravity [5], these theories are the primary candidates for the unification of gravity with the other forces. However, a complete unification has not yet been achieved, and, as a result, several other attempts to derive a theory of electromagnetism, either classical or quantum, from a geometric theory have appeared in the literature. Among others, these include attempts to derive electromagnetism from General Relativity itself [6][7][8], from variants of General Relativity [9], from the Born-Infeld-Einstein theory [10] or from other geometric theories [11][12][13][14][15].

Motivated by the clear analogy between gravity and electromagnetism, supported further by the results of the work by Lense and Thirring, another idea, that of gravitoelectromagnetism (GEM) [16][17], has developed over the years. The idea amounts to studying gravity in the context of General Relativity by using the terminology of electromagnetism. For example, for a rotating gravitational source, the metric describing the spacetime around it may be

\(^1\) Throughout this work, we follow the conventions of [1].
expressed in terms of a scalar quantity $\Phi$ and a vector $\mathbf{A}$, that at large distances are associated to the Newtonian potential and angular momentum of the source. This idea has helped to investigate further the analogy between the two theories but also to understand gravity better by looking for gravitational analogues of electromagnetic phenomena in the context of General Relativity.

The first part of the present work focuses on the theory of gravitoelectromagnetism. Following the standard assumptions and conventions, we express the metric perturbations in terms of a scalar and vector potential, and we derive the full set of equations and constraints that these two quantities ought to satisfy. Apart from the set of equations that exhibit a close similarity to Maxwell’s equations, we derive a number of additional constraints from the remaining components of Einstein’s field equations. In addition, we demonstrate that the analogy between gravity and electromagnetism holds only under the assumption that the vector potential is time-independent; this feature is dictated also by the additional components of the transverse gauge condition, that is implemented in the standard form of the theory. We also re-derive the expression of the analogue of the Lorentz force as this follows from the geodesic equation: although we fully agree on the form of the terms appearing in previous works on GEM, we show that extra corrections emerge not all of which can be ignored in the classical limit.

In an effort to explore different directions in the context of gravitoelectromagnetism, we then deviate from the standard assumptions and we propose an alternative ansatz for the metric perturbations. The novel feature of this ansatz is that all components of the metric perturbations are assumed to be non-zero. In the context of GEM, where a specific form of the energy-momentum tensor is assumed, the spatial components of the metric perturbations are indeed significantly suppressed and thus ignored. Our ansatz, on the other hand, turns out to be compatible with a vacuum configuration where indeed all components can be of the same magnitude. We show that this ansatz is accompanied by a number of attractive features: any additional constraints from the field equations trivially vanish while the geodesics equation takes the exact form of the equation of the Lorentz force with no extra corrections. The use of alternative ansatze for the metric perturbations, such as the one studied here, could perhaps open new directions of thinking and lead to a novel class of phenomena in the context of gravitoelectromagnetism.

The similarity that the perturbed Einstein’s field equations have with those of electromagnetism naturally leads to the question of whether one could reproduce the true electromagnetism from a tensorial theory with a formalism similar to that of the General Theory of Relativity. In the second part of our paper, we attempt to answer this question. The starting point of our analysis are again Einstein’s field equations satisfied by the metric perturbations at a linear approximation – in the context of our analysis, however, the field equations will be modified to allow for the substitution of the gravitational constant by an, initially, undefined one; demanding that particular components of the field equations reduce to Maxwell’s equations, this constant will be determined solely in terms of the velocity of light. A scalar $\Phi$ and a vector potential $\mathbf{A}$ are
also introduced through the components of the metric perturbations. However, contrary to the case of gravitoelectromagnetism, these potentials will be assumed to be the true potentials of electromagnetism, and not ‘components’ of the gravitational field. We employ a general ansatz for the form of the metric perturbations, and we demonstrate that indeed the full set of exact Maxwell’s equations can be reproduced for a large class of choices. However, the ensuing model of electromagnetism suffers from a number of weak points which we discuss in detail.

Let us clarify that our analysis in the second part of our paper does not propose a unification theory since gravity would in fact be absent. Nor does it aim to replace the current theory of electromagnetism - after all, its quantum version, the Quantum Electrodynamics, is one of the most successful and accurate theories in Physics. It rather aims at investigating how far the analogy between gravity and electromagnetism, that was noted in, and employed by, GEM, extends, and where and why it breaks down. We believe that such an analogy, if found to work satisfactorily, would link the two theories at a deeper level and assist, in a complimentary way, in the formulation of a unified theory that is still missing.

The outline of our paper is as follows: in section 2, we present the theoretical framework and basic tools for our analysis. In Section 3, we focus on the theory of gravitoelectromagnetism: we perform a comprehensive study of the full set of equations and constraints arising, and we re-derive the equivalent of the equation for the Lorentz force; at the end, we propose a novel ansatz for the metric perturbations and study its consequences. In Section 4, we turn to a tensorial theory that uses the formalism of General Relativity and we attempt to re-produce the full set of Maxwell’s equations; we discuss in detail the successes and weak points of such an approach. We present our conclusions in Section 5.

2 The Theoretical Framework

Our starting point will be Einstein’s field equations, and more specifically the equations obeyed by the metric perturbation $h_{\mu\nu}$ defined through the relation

$$g_{\mu\nu}(x^\mu) = \eta_{\mu\nu} + h_{\mu\nu}(x^\mu),$$

(2)

where $\eta_{\mu\nu}$ is the Minkowski metric of the flat spacetime and $x^\mu = (ct, \mathbf{x})$. In the context of General Relativity, the perturbations $h_{\mu\nu}$ are assumed to be sourced by gravitating bodies and to obey the inequality $|h_{\mu\nu}| \ll 1$ so that a linear-approximation analysis may be followed. Here, we will work along the same lines - to this end, we briefly review the corresponding formalism that leads to the equations in the linear-order approximation (for a more detailed analysis, see for example [1]). Employing Eq. (2) and keeping only terms linear

\footnote{Throughout this work, we will use the (+1, −1, −1, −1) signature for the Minkowski tensor $\eta_{\mu\nu}$.}
in the perturbation \( h_{\mu\nu} \), we easily find that the Christoffel symbols assume the form

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \eta^{\alpha\rho} (h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho}).
\]  

(3)

The above leads, in turn, to the following expression for the Ricci tensor

\[
R_{\mu\nu} = \frac{1}{2} (h^\rho_{\mu,\rho\nu} + h^\rho_{\nu,\rho\mu} - \partial^2 h_{\mu\nu} - h_{\mu\nu}) ,
\]  

(4)

where \( \partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu \), and indices are raised and lowered by the Minkowski metric \( \eta_{\mu\nu} \). In the above, we have also defined the trace of the perturbation \( h = \eta^{\mu\nu} h_{\mu\nu} \). Similarly, the Ricci scalar is found to be

\[
R = h^{\mu\nu}_{\phantom{\mu\nu},\mu\nu} - \partial^2 h .
\]  

(5)

Combining all the above, the Einstein tensor, \( G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \), takes the form

\[
G_{\mu\nu} = \frac{1}{2} (h^\alpha_{\mu,\alpha\nu} + h^\alpha_{\nu,\alpha\mu} - \partial^2 h_{\mu\nu} - h_{\mu\nu} - \eta_{\mu\nu} h^{\alpha\beta}_{\phantom{\alpha\beta},\alpha\beta} + \eta_{\mu\nu} \partial^2 h) .
\]  

(6)

For simplicity, as is usually the case in gravity, we will also define the new perturbations

\[
\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h,
\]  

(7)

in terms of which the Einstein tensor takes the simpler form

\[
G_{\mu\nu} = \frac{1}{2} (h^\alpha_{\mu,\alpha\nu} + h^\alpha_{\nu,\alpha\mu} - \partial^2 \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \tilde{h}^{\alpha\beta}_{\phantom{\alpha\beta},\alpha\beta}) .
\]  

(8)

We will also assume that the above tensor satisfies Einstein’s field equations, i.e. \( G_{\mu\nu} = k T_{\mu\nu} \), where \( T_{\mu\nu} \) is the energy-momentum tensor and \( k \) a constant whose value will differ in the first and second part of this work. Overall, our basic equations will be the following ones

\[
\hat{h}^\alpha_{\mu,\alpha\nu} + \hat{h}^\alpha_{\nu,\alpha\mu} - \partial^2 \hat{h}_{\mu\nu} - \eta_{\mu\nu} \hat{h}^{\alpha\beta}_{\phantom{\alpha\beta},\alpha\beta} = 2k T_{\mu\nu} .
\]  

(9)

In what follows, we will assume that the distribution of energy in the system is described by the expression \( T_{\mu\nu} = \rho u_\mu u_\nu \), where \( \rho \) the charge density and \( u^\mu = (u^0, u^i) = (c, \mathbf{u}) \) the velocity of the source.

3 GravitoElectroMagnetism

3.1 The traditional ansatz for the perturbations

In the context of the theory of gravitoelectromagnetism, Eqs. (9) are Einstein’s linearised gravitational field equations with \( k = 8\pi G/c^4 \). The components of the gravitational perturbations \( \tilde{h}_{\mu\nu} \) have the form \([17]\)

\[
\tilde{h}_{00} = \frac{4\Phi}{c^2}, \quad \tilde{h}_{0i} = -\frac{2A^i}{c^2}, \quad \tilde{h}_{ij} = 0,
\]  

(10)
and are expressed in terms of a scalar $\Phi(x^\mu)$ and a vector potential $\mathbf{A}(x^\mu)$, the so-called GEM potentials. The $h_{00}$ component yields the Newtonian potential $\Phi$, while the $h_{0i}$ component is associated to the ‘vector’ potential $\mathbf{A}$ generated by a rotating body; the $h_{ij}$ component is usually assumed to be negligible due to the suppression of the corresponding source by a $1/c^4$ factor. To see the above, we may contract Eq. (7) by $\eta^{\mu\nu}$ and find $h = -\tilde{\eta}$; this allows us to write the inverse relation between the original and the new perturbations as

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \tilde{h}. \quad (11)$$

Then, by use of the definition (2), the spacetime line-element assumes the form

$$ds^2 = c^2 \left(1 + 2\Phi \right) dt^2 - \frac{4}{c} \left(\mathbf{A} \cdot d\mathbf{x}\right) dt - \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j. \quad (12)$$

For the evaluation of the field equations (9), we will need the components of $\tilde{h}_{\mu\nu}$ in mixed form - these are:

$$\tilde{h}^0_0 = \frac{4\Phi}{c^2}, \quad \tilde{h}^0_i = -\frac{2A^i}{c^2}, \quad \tilde{h}^i_0 = \frac{2A^i}{c^2}, \quad \tilde{h}^i_j = 0. \quad (13)$$

Then, using the above, the field equations (9) reduce to the following system of equations

$$\frac{\delta^{ij}}{c^2} \partial_i \partial_j \Phi = \frac{k}{2} \rho u_0 u_0 \quad (14)$$

$$\frac{1}{c^2} \partial_i \left(\frac{1}{2} \partial_k A^k + \frac{1}{c} \partial_i \Phi\right) - \frac{1}{2c^2} \delta^{ij} \partial_k \partial_i A^i = \frac{k}{2} \rho u_0 u_i \quad (15)$$

$$-\frac{1}{2c^2} \partial_i \left(\partial_i A^j + \partial_j A^i\right) + \delta_{ij} \left[\frac{1}{c^3} \partial_i^2 \Phi + \frac{1}{c^3} \partial_i (\partial_i A^k)\right] = \frac{k}{2} \rho u_i u_j, \quad (16)$$

for $(\mu, \nu) = (0, 0), (0, i)$ and $(i, j)$, respectively.

Adopting a more familiar notation and using $k = 8\pi G/c^4$, Eq. (14) readily takes the analogue of Poisson’s law

$$\nabla^2 \Phi = 4\pi G \rho, \quad (17)$$

while Eq. (15) in turn can be rewritten as

$$\nabla \left[\nabla \cdot \left(\frac{\mathbf{A}}{2}\right) + \frac{1}{c} \partial_i \Phi\right] - \nabla^2 \left(\frac{\mathbf{A}}{2}\right) = \frac{4\pi G}{c} \rho \mathbf{u}. \quad (18)$$

Defining, in analogy with electromagnetism, the GEM fields $\mathbf{E}$ and $\mathbf{B}$ in terms of the GEM potentials [17]

$$\mathbf{E} \equiv -\frac{1}{c} \partial_i \left(\frac{\mathbf{A}}{2}\right) - \nabla \Phi, \quad \mathbf{B} \equiv \nabla \times \left(\frac{\mathbf{A}}{2}\right), \quad (19)$$
one may easily see that Eqs. (17) and (18) reduce to

\[ \nabla \cdot \mathbf{E} = 4\pi G\rho, \quad \nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E} + \frac{4\pi G}{c} \mathbf{j}, \quad (20) \]

respectively, where we have used the definition of the current vector \( \mathbf{j} = \rho \mathbf{u} \).

The aforementioned discussion on the equations satisfied by the GEM fields \( \mathbf{E} \) and \( \mathbf{B} \), as these follow from the linearised Einstein’s equations, first appeared in [17]. However, the explicit form of the equations (14-16) obeyed by the GEM potentials \( \Phi \) and \( \mathbf{A} \) was not given. A careful inspection of Eqs. (14-15), or equivalently (17-18), reveals that these reduce indeed to the form of (20) if and only if the vector potential is static. Therefore, the constraint \( \partial_t \mathbf{A} = 0 \), that in the context of the analysis of [17] is often presented as merely a simplifying assumption, is in fact a direct result of the analysis.

The same result follows from the application of the gauge condition – that aims to remove some of the arbitrariness in the theory caused by its invariance under coordinate transformations – to the system of equations. The most usual gauge condition, and the one used in the context of [17], is the so-called transverse condition \( \tilde{h}^{\mu\nu},_{\nu} = 0 \). The time-component of the gauge condition takes the Lorentz form

\[ \frac{1}{c} \partial_t \Phi + \nabla \cdot \left( \frac{\mathbf{A}}{2} \right) = 0, \quad (21) \]

in clear analogy to electromagnetism. Nevertheless, due to its tensorial structure, the gauge condition has three more components that may be collectively written as

\[ \frac{1}{c} \partial_t \mathbf{A} = 0. \quad (22) \]

The above constraint was not discussed in [17], however, it is an indispensable part following from the applied gauge condition that re-affirms the necessary time-independence of the vector potential \( \mathbf{A} \) (for different approaches regarding the role of this constraint in the context of gravito-electromagnetism, see [18][19][20][21][22]).

Let us now return to the linearised field equations (9): their spatial components lead to a third set of constraints, Eq. (16), that is largely ignored in the literature \(^3\). Its diagonal components (i.e. for \( i = j \)) reduce to the relation

\[ \partial_t^2 \Phi = -\frac{\pi}{3} \rho |\mathbf{u}|^2, \quad (23) \]

with \( (u^1)^2 = (u^2)^2 = (u^3)^2 \), while the off-diagonal ones (for \( i \neq j \)) give

\[ \partial_0 \left( \partial_i A^j + \partial_j A^i \right) = 8\pi G\rho \frac{u_i u_j}{c^2}. \quad (24) \]

Equation (23) therefore demands that the distribution of sources is isotropic (i.e. the current vector has the same absolute magnitude along all three spatial

\(^3\) While this manuscript was at the stage of production, we were notified by the authors of [21][22] that a study of these additional equations was previously performed in their works.
directions); it also restricts the magnitude of $\partial_t^2 \Phi$, a quantity that does not appear in the other two derived equations. The latter equation (24) dictates that the velocity of the source is strictly non-relativistic, $|\mathbf{u}| \ll c$: it is only then that the time variation of the vector potential is extremely small, due to the suppression factor $1/c^2$ on its right-hand-side, and the consistency of the complete set of derived equations is guaranteed.

We will finally address the question of the equation of motion of a test particle propagating in the background (12). This has also been derived and discussed in the literature before [17] but only in a very simplified form. Here, we keep all corrections and comment on their importance at the final stage of the calculation. As demonstrated above, the consistency of the set of derived equations dictates that we work in the non-relativistic limit, in which case we may write

$$ds^2 = c^2 dt^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = c^2 dt^2 \left( 1 - \frac{|\mathbf{u}|^2}{c^2} \right) \simeq c^2 dt^2. \quad (25)$$

Then, the spatial components of the geodesics equation

$$\frac{d^2 x^\rho}{ds^2} + \Gamma^\rho_{\mu \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \quad (26)$$

take the explicit form

$$\frac{d^2 x^i}{dt^2} + c^2 \Gamma^i_{00} + 2c \Gamma^i_{0j} \frac{dx^j}{dt} + c \Gamma^i_{kj} \frac{dx^k}{dt} = 0. \quad (27)$$

In the linear approximation, the Christoffel symbols are given by Eq. (3). Reading the form of the initial perturbations $h_{\mu \nu}$ from the line-element (12), we find

$$\Gamma^i_{00} = \frac{1}{c^2} \partial_i \Phi + \frac{2}{c^3} \partial_i A^i, \quad (28)$$

$$\Gamma^i_{0j} = \frac{1}{c^2} F_{ij} - \frac{1}{c^3} \delta^i_j \partial_t \Phi, \quad (29)$$

$$\Gamma^i_{kj} = -\frac{1}{c^3} \left( \delta^i_j \partial_k \Phi + \delta^i_k \partial_j \Phi - \delta_{kj} \partial_i \Phi \right), \quad (30)$$

where, in the second of the above equations, we have used the definition $F_{ij} \equiv \partial_i A_j - \partial_j A_i$. Substituting the above into Eq. (27) and using vector notation, we find

$$m \mathbf{a} = \mathbf{F} = m \mathbf{E} \left( 1 + \frac{|\mathbf{u}|^2}{c^2} \right) + \frac{4m}{c} \mathbf{u} \times \mathbf{B} + 2m \left[ \frac{\mathbf{u}}{c} \frac{\partial \Phi}{c} - \frac{\mathbf{u}}{c} \left( \frac{\mathbf{u}}{c} \cdot \mathbf{E} \right) \right], \quad (31)$$

where, in accordance to the field equations and gauge condition, we have set $\partial_t \mathbf{A} = 0$ and thus $\mathbf{E} = -\nabla \Phi$. A simplified form of the above equation, namely the following

$$m \mathbf{a} = \mathbf{F} = m \mathbf{E} + \frac{2m}{c} \mathbf{u} \times \mathbf{B}, \quad (32)$$
is the one that has appeared in the literature before [17] expressed, as claimed, to the lowest order in \( u/c, \Phi \) and \( A \). As is evident from Eq. (31) we agree on the form of the two terms, proportional to \( mE \) and \( u \times B \), appearing in the simplified equation – the apparent disagreement in the numerical coefficient of the second term is only due to the different definition (19) of the GEM field \( B \) in terms of the vector potential. Also, the two additional terms proportional to \( E \) appearing on the right-hand-side of Eq. (31) may also be discarded in the non-relativistic limit in which we work. However, there is one more term, proportional to the combination \( u \partial_t \Phi/c^2 \), whose suppression of magnitude is not evident. If the time variation of the scalar potential could be roughly associated with the magnitude of the fluid velocity, then this term might also be discarded. Nevertheless, apart from the constraint equation (23) – which until now was ignored in the context of gravitoelectromagnetism – that may hint towards that direction, there is no reason why this term should not be present. In this aspect, our results agree with the ones presented in [21][22] where the interested reader may find a more extended analysis on the link between the time-independence of the GEM potentials and the analogy between gravity and electro-magnetism.

3.2 An Alternative ansatz for the metric perturbations

According to the usual assumptions of General Relativity, the scalar potential \( \Phi \) is only associated with the \( \hat{h}_{00} \) component of the metric perturbations. To preserve the analogy with electromagnetism, the vector potential should appear linearly in the expression of \( \hat{h}_{\mu \nu} \), and thus can only be accommodated by the \( \hat{h}_{0i} \) component. Then, that leaves \( \hat{h}_{ij} \) to be either zero, as was the case in the previous subsection, or to be also associated to the scalar potential \( \Phi \). Although this may at first seem peculiar, in the course of our analysis it will be justified and shown to exhibit interesting features.

To this end, we will now investigate the following alternative assumption for the perturbations \( \hat{h}_{\mu \nu} \)

\[
\hat{h}_{00} = \frac{\Phi}{c^2}, \quad \hat{h}_{0i} = -\frac{A^i}{c^2}, \quad \hat{h}_{ij} = \frac{\Phi}{c^2} \delta_{ij} .
\]  

(33)

Then, the spacetime line-element assumes the simplified form

\[
ds^2 = c^2 \left(1 + \frac{2\Phi}{c^2}\right) dt^2 - \frac{2}{c} (A \cdot dx) dt - \delta_{ij} dx^i dx^j ,
\]

(34)

with the potentials \( \Phi \) and \( A \) having again the same interpretation.

Working as before, we first derive the components of \( \hat{h}_{\mu \nu} \) in mixed form and then, from the field equations (9), we obtain the following system

\[
0 = 2k \rho u_0 u_0 ,
\]

(35)

\[
\frac{1}{c^2} \partial_i (\partial_k A^k) - \frac{1}{c^2} \delta^{kt} \partial_t A^t = 2k \rho u_0 u_i ,
\]

(36)
\[-\frac{1}{c^3} \partial_t \left( \partial_i A^j + \partial_j A^i \right) - \frac{2}{c^2} \partial_i \partial_t \Phi
\]
\[+ \delta_{ij} \left[ \frac{2}{c^2} \delta^{kl} \partial_k \partial_l \Phi + \frac{2}{c^3} \partial_k (\partial_t A^k) \right] = 2k \rho u_i u_j. \quad (37)\]

Setting again \( k \equiv 8\pi G/c^4 \), Eq. (36) takes a form similar to that of the fourth Maxwell’s equation for both a static scalar and vector potential, \( \partial_t \Phi = \partial_t A = 0 \),

\[\nabla (\nabla \cdot A) - \nabla^2 A = 16\pi G \rho \frac{u}{c}. \quad (38)\]

This equation differs from the exact Maxwell equation by a factor of 4 on the right-hand-side but, as will see, this will be irrelevant. Equation (35), that in the previous case gave us Poisson’s law, has now reduced to the trivial result \( \rho = 0 \), which when combined with the fact that \( \partial_t A = 0 \), leads to the demand that \( \Phi \) satisfies the equation

\[\nabla^2 \Phi = 0. \quad (39)\]

Then, we conclude that this choice for the gravitational perturbations leads to a static model of gravity in vacuum. For \( \rho = 0 \), the right-hand-side of Eq. (38) also vanishes making the numerical factor irrelevant. Also, in retrospect, our assumption of non-vanishing \( \tilde{h}_{ij} \) seems justified: although this component is indeed significantly suppressed in the presence of an energy-momentum tensor of the form \( T_{\mu \nu} = \rho u_\mu u_\nu \), in vacuum all components are of the same order.

Imposing the transverse gauge condition \( \tilde{h}_{\mu \nu, \nu} = 0 \) results into two constraints, namely

\[\frac{1}{c} \partial_t \Phi + \nabla \cdot A = 0, \quad \frac{1}{c} \partial_t A + \nabla \Phi = 0, \quad (40)\]

from the temporal and spatial components, respectively. The first is the well-known Lorentz condition, while the latter demands the vanishing of the GEM field \( E \). Note that the different numerical coefficients that appear in the ansatz (33), compared to the traditional one (10) used in GEM, allows us to define the GEM fields \( E \) and \( B \) in an exact analogy with the electromagnetism

\[E \equiv -\frac{1}{c} \partial_t A - \nabla \Phi, \quad B \equiv \nabla \times A. \quad (41)\]

Although this case seems to be not particularly rich in content compared to the one studied in the previous subsection, it is in contrast free of additional constraints. The set of equations (37) – both the diagonal and off-diagonal components – are trivially satisfied if one uses the aforementioned constraints (40) following from the gauge conditions.

Turning finally to the geodesics equation, by using the expressions for the initial perturbations \( h_{\mu \nu} \) as these are read in the line-element (34), we arrive at particularly simple forms for the Christoffel symbols

\[\Gamma^i_{00} = \frac{1}{c^2} \partial_0 \Phi + \frac{1}{c^2} \partial_i A^i, \quad \Gamma^i_{0j} = \frac{1}{2c^2} F_{ij}, \quad \Gamma^i_{kj} = 0. \quad (42)\]
Substituting these into the geodesics equation (27), we obtain the exact functional analogue of the Lorentz force

\[ m \mathbf{a} = \mathbf{F} = m \mathbf{E} + \frac{m}{c} \mathbf{u} \times \mathbf{B}. \] (43)

Note that the same coefficient appears in front of the ‘electric’ and ‘magnetic’ terms, in exact analogy to electromagnetism. We also observe that no additional terms, not even sub-dominant ones in the non-relativistic limit, emerge in this case. We therefore conclude that this particular ansatz for the gravitational perturbations may lead to another class of phenomena observed in vacuum in the context of gravitoelectromagnetism: the field equations in conjunction to the gauge condition lead to a self-consistent set of fundamental equations with no additional constraints and a remarkable similarity to the corresponding formulae of electromagnetism.

### 4 True Electromagnetism?

In the context of gravitoelectromagnetism, the scalar \( \Phi \) and vector potential \( \mathbf{A} \), as well as the corresponding GEM fields \( \mathbf{E} \) and \( \mathbf{B} \), satisfy equations that are remarkably similar to the ones of electromagnetism. The question then follows of how much an accurate theory of true electromagnetism one could obtain starting from a tensorial theory similar to that of General Relativity and introducing the scalar \( \Phi(x^\mu) \) and vector \( \mathbf{A}(x^\mu) \) electro-magnetic potentials again via the metric perturbations \( h_{\mu\nu} \). As already mentioned in the Introduction, the objective of such an analysis would of course be not to replace Maxwell’s theory of electromagnetism by a tensorial theory but rather to investigate how far the analogy between the two dynamics can go.

From the analysis of the two subsections of Section 3, it is evident that the form of the metric perturbations \( h_{\mu\nu} \) does affect the equations for the scalar and vector potentials that are derived from the linearised Einstein’s equations. In order to collectively study a large class of choices, we will use the following general form of \( \tilde{h}_{\mu\nu} \)

\[
\begin{align*}
\tilde{h}_{00} &= \frac{\alpha \Phi}{c^2}, \\
\tilde{h}_{0i} &= -\frac{\beta \dot{A}^i}{c^2}, \\
\tilde{h}_{ij} &= \frac{\gamma \Phi}{c^2} \delta_{ij},
\end{align*}
\] (44)

where \((\alpha, \beta, \gamma)\) are arbitrary numerical coefficients \(^4\). The two ansatze used in subsections 3.1 and 3.2 correspond to the choices \((\alpha = \beta = 1, \gamma = 0)\) and \((\alpha = \beta = \gamma = 1)\), respectively. If one hopes to reduce the gravitational field equations to Maxwell’s equations, the chosen \( \tilde{h}_{\mu\nu} \) must be linear in the electromagnetic potentials and should not contain any derivatives. The above leads to the conclusion that the scalar \( \Phi \) and the vector \( \mathbf{A} \) potential can be accommodated in the scalar \( \tilde{h}_{00} \), vector-like \( \tilde{h}_{0i} \) and tensor-like \( \tilde{h}_{ij} \) components.

\(^4\) Contrary to what happens in the context of GEM, where the numerical coefficient in the expression of \( \tilde{h}_{00} \) is fixed so that \( \tilde{h}_{00} = 2\Phi/c^2 \) in accordance with the Newtonian limit, here, the coefficient \( \alpha \) can be arbitrary.
in a limited number of distinct ways – in fact, all the allowed choices are included in our ansatz (44). The above general form will allow us to carry out a generalised analysis and investigate the contribution that each component of \( \tilde{h}_{\mu\nu} \) would have to the field equations.

Starting again from the linearised form of Einstein’s field equations (9), but allowing now for a general coefficient \( k \) different from the usual one \( 8\pi G/c^4 \), and employing the general ansatz (44), we arrive at the following set of equations

\[
\frac{(\alpha - \gamma)}{c^2} \delta^{ij} \partial_i \partial_j \tilde{\Phi} = 2k \rho u_0 u_0, \tag{45}
\]

\[
\frac{(\alpha - \gamma)}{c^3} \partial_i \partial_t \tilde{\Phi} + \frac{\beta}{c^2} \partial_i \left( \partial_k \tilde{A}^k \right) - \frac{\beta}{c^2} \delta^{kl} \partial_k \partial_l \tilde{A}^i = 2k \rho u_0 u_i, \tag{46}
\]

\[
- \frac{\beta}{c^3} \partial_i \left( \partial_i \tilde{A}^i + \partial_j \tilde{A}^j \right) - \frac{2\gamma}{c^2} \partial_i \partial_j \tilde{\Phi} + \delta_{ij} \left[ \frac{2\gamma}{c^2} \delta^{kl} \partial_k \partial_l \tilde{\Phi} + \frac{2\beta}{c^3} \partial_t \left( \partial_k \tilde{A}^k \right) + \frac{(\alpha - \gamma)}{c^4} \partial_i^2 \tilde{\Phi} \right] = 2k \rho u_i u_j. \tag{47}
\]

Equation (45) reveals that, for \( \alpha = \gamma \), we inevitably obtain a model of electro-magnetism in vacuum, i.e. with \( \rho = 0 \), as it was also found in subsection 3.2 in the context of GEM. The reason for that is that, for \( \alpha = \gamma \), the \( \tilde{h}_{00} \) and \( \tilde{h}_{ij} \) components, that are both associated with \( \tilde{\Phi} \), have contributions to the (00)-component of the perturbed field equations that are of equal magnitude but of opposite sign. For \( \alpha \neq \gamma \), on the other hand, we obtain Poisson’s law

\[
\nabla^2 \tilde{\Phi} = -4\pi \rho, \tag{48}
\]

under the identification

\[
k \equiv -\frac{2\pi}{c^4} (\alpha - \gamma). \tag{49}
\]

Adopting the above value for \( k \), and using vector notation, Eq. (46) in turn can be written in the form

\[
\nabla \left( \nabla \cdot \tilde{\mathbf{A}} + \frac{1}{c} \partial_t \tilde{\Phi} \right) - \nabla^2 \tilde{\mathbf{A}} = 4\pi \rho \frac{\mathbf{u}}{c}, \tag{50}
\]

under the assumption that \( \beta = \alpha - \gamma \). Therefore, there are apparently an infinite number of choices one could make concerning the numerical coefficients appearing in the perturbations \( \tilde{h}_{\mu\nu} \) and still be able to derive the electromagnetic equations from the field equations (9). Recalling the usual definitions of the electric \( \tilde{\mathbf{E}} \) and magnetic field \( \tilde{\mathbf{B}} \) in terms of the potentials, i.e.

\[
\tilde{\mathbf{E}} \equiv -\frac{1}{c} \partial_t \tilde{\mathbf{A}} - \nabla \tilde{\Phi}, \quad \tilde{\mathbf{B}} \equiv \nabla \times \tilde{\mathbf{A}}, \tag{51}
\]

one may easily see that Eqs. (48) and (50) are the first and fourth, respectively, Maxwell’s equations, namely

\[
\nabla \cdot \tilde{\mathbf{E}} = 4\pi \rho, \quad \nabla \times \tilde{\mathbf{B}} = \frac{1}{c} \partial_t \tilde{\mathbf{E}} + \frac{4\pi}{c} \mathbf{j}, \tag{52}
\]
under the constraint that the vector potential is again static, i.e. $\partial_t \hat{A} = 0$, and for $\mathbf{j} = \rho \mathbf{u}$. Note, that, by using the definitions (51) for the electric and magnetic fields, the remaining two Maxwell’s equations follow automatically - in this case, these have the form:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0. \quad (53)$$

We would also like to note that due to the numerical factor $(\alpha - \gamma)$ appearing in Eqs. (45) and (46), Maxwell’s equations remain unchanged under the simultaneous changes $(\alpha \leftrightarrow \gamma)$ and $\hat{\Phi} \rightarrow -\hat{\Phi}$. Therefore, contrary to what happens in GEM where $\tilde{h}_{00}$ is necessarily tied to the Newtonian potential and thus should always be non-vanishing, here, the whole set of Maxwell’s equations could also be recovered e.g. in the simple case where $\tilde{h}_{00}$ is zero and the potential $\hat{\Phi}$ is introduced solely through the $\tilde{h}_{ij}$ component. However, as in GEM, Maxwell’s equations are accompanied by an additional set of equations, Eqs. (47), where this symmetry is broken: for the chosen values of $k$ and $\beta$, these take the form

$$\frac{1}{c} \partial_t \left( \partial_i \hat{A}^j + \partial_j \hat{A}^i \right) + \frac{2 \gamma}{\alpha - \gamma} \partial_i \partial_j \hat{\Phi} - \delta_{ij} \left[ \frac{2 \gamma}{\alpha - \gamma} \delta^{kl} \partial_k \partial_l \hat{\Phi} + \frac{2}{c} \partial_l (\partial_k \hat{A}^k) + \frac{1}{c^2} \partial_t^2 \hat{\Phi} \right] = 4 \pi \rho u_i u_j / c^2. \quad (54)$$

For $i = j$, the above reduces to the relation

$$\partial_t^2 \hat{\Phi} = -\frac{4\pi}{3} \rho |\mathbf{u}|^2, \quad (55)$$

with $(u^1)^2 = (u^2)^2 = (u^3)^2$, for all values of $\alpha$ and $\gamma$. For $i \neq j$, the term proportional to $\delta_{ij}$ in Eq. (54) vanishes - the remaining terms on the left-hand-side of the equation can be combined to form the components of the electric field under the assumption that $\alpha = 2\gamma$. In that case, the constraint reads

$$\partial_i \hat{E}_j + \partial_j \hat{E}_i = -4 \pi \rho u_i u_j / c^2. \quad (56)$$

In the opposite case, $\alpha \neq 2\gamma$, the constraint is imposed independently on the time and space-derivatives of the electromagnetic potentials $\hat{A}$ and $\hat{\Phi}$. In both cases, the right-hand-side is suppressed by a $u_i u_j / c^2$ factor, that, in the no-relativistic limit, is always small.

Therefore, although the four Maxwell’s equations are correctly recovered, these are supplemented by additional constraints resulting from the remaining components of the linearised field equations - the situation is thus similar to the one encountered in section 3.1. The additional constraints dictate that the distribution of charges should be again isotropic. Also, here, the form of the derived equations match the ones of Maxwell’s equations under the assumption that the vector potential $\hat{A}$ be static.

Imposing a gauge condition yields additional constraints to the model; some of them act complimentary completing the theory, however, others lead
to unnecessary restrictions to the electromagnetic potentials. If we consider again the transverse gauge condition \( \tilde{h}^{\mu\nu} = 0 \), then its time-component takes the explicit form

\[
\frac{\alpha}{c} \partial_t \hat{\Phi} + \beta \partial_i \hat{A}^i = 0.
\] (57)

For all cases with \( \alpha = \beta \), the above reduces exactly to the Lorentz condition

\[
\frac{1}{c} \partial_t \hat{\Phi} + \nabla \cdot \hat{A} = 0.
\] (58)

Its spatial components, though, lead to the unusual constraint

\[
\frac{\beta}{c} \partial_i \hat{A} + (\alpha - \beta) \nabla \hat{\Phi} = 0,
\] (59)

where the relation \( \beta = \alpha - \gamma \) has again been used. Here, the choice \( \alpha = \beta \) leads to the time-independence of the vector potential \( \hat{A} \), a demand that was already evident from the field equations. For the choice \( \alpha = 2\beta \), however, this condition demands that the electric field itself \( \hat{E} = -\partial_i \hat{A}/c - \nabla \hat{\Phi} \) should vanish. For all other cases, the above equation imposes an additional constraint between the scalar and vector potentials which is not present in the traditional electromagnetism. The choice of the gauge condition is of course not unique, therefore the above analysis is by no means exhaustive; it merely acts in an indicative way regarding the type of the constraints that one would end up with. Another usual gauge condition, the transverse-traceless one is further supplemented by the demand that the trace of \( \tilde{h} \) should be zero which eliminates altogether the scalar potential \( \hat{\Phi} \) from the theory.

We now turn to the equation of motion that a test particle would satisfy in the context of the same formalism. We will conjecture that a modified geodesics equation, of the form

\[
m \frac{d^2 x^\rho}{ds^2} + q \Gamma^\rho_\mu\nu \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0,
\] (60)

can indeed describe the motion of a massive, charged particle inside the resulting electro-magnetic field. Working again in the linear approximation, the relevant components of the Christoffel symbols that appear in Eq. (60) take the form

\[
\Gamma^i_00 = \left( \frac{\alpha + 3\gamma}{4c^2} \right) \partial_i \hat{\Phi} + \frac{\beta}{c^3} \partial_i \hat{A}^i,
\] (61)

\[
\Gamma^i_0j = \frac{\beta}{2c^2} \tilde{F}_{ij} - \left( \frac{\alpha - \gamma}{4c^3} \right) \delta^i_j \partial_i \hat{\Phi},
\] (62)

\[
\Gamma^i_kj = -\left( \frac{\alpha - \gamma}{4c^2} \right) \left( \delta^i_k \partial_j \hat{\Phi} + \delta^i_j \partial_k \hat{\Phi} - \delta_{kj} \partial_i \hat{\Phi} \right),
\] (63)
where, we have defined the electro-magnetic field-strength tensor $\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i$. Substituting the above into Eq. (60), we find

$$m a^i + q \left[ \frac{\beta}{c} \partial_t \hat{A}^i + \frac{(\alpha + 3\gamma)}{4} \partial_i \hat{\Phi} \right] - \frac{\beta q}{c} \epsilon_{ijk} u^j \hat{B}_k$$

$$- \frac{(\alpha - \gamma)q}{2c^2} (\partial_t \hat{\Phi}) u^i + \frac{(\alpha - \gamma)q}{4c^2} \left[ \partial_t \hat{\Phi} (u^k u^k) - 2(\partial_j \hat{\Phi} u^j) u^i \right] = 0. \quad (64)$$

Under the assumption that a robust theory of electromagnetism may follow by applying the formalism of General Relativity, the above formula stands for the generalised equation of motion for a massive, charged test particle, that at the same time defines the Lorentz force of the system. We note that in the special case where $\alpha = \gamma$, where we recover a static electromagnetic theory in vacuum, all terms in the second line of the previous equation trivially vanish; then, we readily obtain the minimal form

$$m a = F = q \alpha \hat{E} + \frac{q \beta}{c} \hat{u} \times \hat{B}. \quad (65)$$

In this case the constraint $\beta = \alpha - \gamma$ does not hold, thus we are free to choose a perturbation configuration with $\alpha = \beta$ in order to restore a common numerical factor in front of the electric and magnetic terms. For the particular choice $\alpha = \beta = 1$, this factor disappears leaving behind the exact expression of the Lorentz force – alternatively, it could be eliminated by appropriately modifying the original form of the geodesics equation (60).

For $\alpha \neq \gamma$, on the other hand, it is evident that in trying to build a more realistic electromagnetic theory, i.e. a theory not in vacuum, the expression of the Lorentz force is bound to obtain corrections to its traditional form. Applying the constraint $\beta = \alpha - \gamma$, that, as found above, is necessary to restore the form of Maxwell’s equations, the expression of the Lorentz force is written as

$$m a = F = \frac{q}{4} \hat{E} \left[ (4\alpha - 3\beta) + \frac{\beta |\hat{u}|^2}{c^2} \right] + \frac{\beta q}{c} \hat{u} \times \hat{B} + \frac{\beta q}{2} \left[ \frac{\hat{u}}{c} \partial_t \hat{\Phi} - \frac{\hat{u}}{c} \left( \frac{\hat{u} \cdot \hat{E}}{c} \right) \right]. \quad (66)$$

The form of the additional terms appearing in the expression of the Lorentz force is similar to that arising in the context of GEM. The second and fifth term can be safely ignored in the non-relativistic limit since their magnitude is suppressed by a factor of $u^2/c^2$, while the role and magnitude of the fourth term should be more carefully looked at. The exact numerical coefficients appearing in front of the various terms differ depending on the particular choice for the $(\alpha, \beta)$ coefficients. Particular configurations leading to a common numerical factor in front of the dominant electric and magnetic terms can always be found (e.g. $\alpha = 7\beta/4$) while a remaining overall factor could again be eliminated by the modification of the original geodesics equation.

---

5 For $\alpha = \gamma$, all Maxwell’s equations in vacuum are correctly reproduced from Eqs. (45) and (46) under the identification $k \equiv -2\pi/\beta c^4$ and for arbitrary $\alpha$ and $\gamma$. 
5 Discussion and Conclusions

There is undoubtedly a striking similarity between the gravitational and electromagnetic forces at classical level. Even in the context of the General Theory of Relativity, the dynamics of the gravitational field resembles the one of the electromagnetic, a result that led to the development of the theory of gravitoelectromagnetism. The first part of the present analysis focused on the latter theory aiming at shedding light to particular aspects of it that have not been adequately discussed in the literature.

Starting from the perturbed Einstein’s field equations at linear approximation and employing the standard GEM ansatz for the form of the metric perturbations, we derived the equations that govern the dynamics of the scalar and vector potentials as well as that of the corresponding GEM fields. We confirmed that these equations have the form appearing in the literature but only under the assumption that the vector potential is time-independent – otherwise, important terms are unjustifiably missing and the analogy between gravity and electromagnetism, that is the cornerstone of GEM, breaks down. In fact, we showed that the time-independence of the vector potential is dictated by the transverse gauge condition that is already incorporated in the standard form of the theory. The equations for the GEM potentials and fields that resemble the ones of electromagnetism are, however, supplemented by additional constraints on their form; these follow from the remaining components of Einstein’s field equations that are usually ignored. Finally, the derivation of the equation for the Lorentz force revealed the presence of additional terms not all of which can be ignored in the non-relativistic limit.

In the context of GEM we proposed an alternative ansatz for the metric perturbations. The novel feature of this ansatz was the presence of a non-vanishing $\tilde{h}_{ij}$ component – in GEM, this component is significantly suppressed compared to the other ones, and thus ignored. However, this ansatz led to a set of equations similar to the ones for static electromagnetism in vacuum. Then, our ansatz was justified: in the absence of an energy-momentum tensor, all components of the metric perturbations turn out to be of the same magnitude. The ensuing analysis of this particular ansatz revealed a number of attractive features: the additional constraints from the field equations trivially vanish while the geodesics equation takes the exact form of the equation of the Lorentz force.

The similarity in the dynamics of the gravitational and electromagnetic forces created also great expectations that their unification in the context of a common theory would be straightforward. However, centuries later, such a theory is still missing. Motivated by GEM, we posed the question whether a tensorial theory based on the formalism of General Relativity could exactly re-produce the theory of electromagnetism, and in the second part of this work we tried to answer this question. As noted in the Introduction, our analysis does not aim at replacing the theory of electromagnetism but at investigating how far the analogy between gravity and electromagnetism extends.
We employed again Einstein’s field equations, with the gravitational constant replaced by an, initially, undetermined one; that constant was later defined in terms of the velocity of light. The scalar and vector electromagnetic potentials were introduced via the form of the metric perturbations. We used a general ansatz and demonstrated that the field equations indeed reduce to the exact Maxwell’s equations for a large class of choices. However, this formulation of electromagnetism has a number of weak points, similar to those appearing in the context of GEM, namely: (i) additional constraints emerge from the field equations, (ii) employing a modified geodesics equation, the expression of the Lorentz force may be derived but it contains extra terms that are not always negligible, (iii) the type of gauge condition that one may choose to impose on the metric perturbations also yields new constraints on the electromagnetic potentials, and (iv) the vector potential should always be time-independent.

Our present analysis will hopefully work towards opening new directions of thinking in the context of gravitoelectromagnetism. But we also envisage that it will provide the basis for the formulation of a more sophisticated mathematical formalism, inspired by gravity, capable of re-producing the theory of electromagnetism. Such a formalism would link the two theories at a deeper level and work towards the formulation of a unified theory. Further work is clearly necessary: more thought should be devoted on whether the additional constraints derived from the field equations carry a physical content or not; the role of the gauge condition and its effect on the ensuing form of equations should be looked at; finally, the absence of time-dependence in the vector potential should be cured - preliminary studies show that a partial breaking of the symmetry $h_{0i} = h_{i0}$ for the spatial-temporal component of metric perturbations may solve this problem. We hope to return to these questions in a future work.

Acknowledgements The authors would like to thank Luis Filipe Costa and Jose Natario for useful comments on our manuscript and for communicating to us important information on previous analyses in this topic. This research has been co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF) - Research Funding Program: “THALIS. Investing in the society of knowledge through the European Social Fund”. Part of this work was supported by the COST Actions MP0905 “Black Holes in a Violent Universe” and MP1210 “The String Theory Universe”.

References

1. L.D. Landau and E.M. Lifshitz, “The Classical Theory of Fields”, Butterworth-Heinemann Editions, 2003.
2. H. Thirring, Phys. Z. 19 (1918) 204; J. Lense and H. Thirring, Phys. Z. 19 (1918) 156; B. Mashhoon, F.W. Hehl and D.S. Theiss, Gen. Rel. Grav. 16 (1984) 711.
3. M.B. Green, J.H. Schwarz and E. Witten, “Superstring Theory”, Cambridge University Press, 1987.
4. M. Duff, Int. J. Mod. Phys. A 11 (1996) 5623 [hep-th/9608117].
5. A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244.
6. M.W. Evans, Found. Phys. Lett. 9 (1996) 397; Found. Phys. 26 (1996) 1243.
7. N. Dadhich, Gen. Rel. Grav. 32 (2000) 1009 [gr-qc/9909067].
8. R. M. iAgut, arXiv:1012.4730 [gr-qc].
9. P.K. Anastasovski et al., Found. Phys. Lett. 16 (2003) 275.
10. D.N. Vollick, Phys. Rev. D 69 (2004) 064030 [gr-qc/0309101].
11. D. Weingarten, J. Math. Phys. 18 (1977) 165.
12. V.V. Kassandrov, Grav. Cosmol. 1 (1995) 216 [gr-qc/0007027].
13. A. Unzicker, gr-qc/9612061.
14. G.R. Gonzalez-Martin, physics/0009042.
15. R. G. Torrome, arXiv:0905.2060 [math-ph]; arXiv:1207.3791 [math.DG].
16. A. Matte, Canadian J. Math. 5 (1953) 1; L. Bel, Compt. Rend. 247 (1958) 1094; P. Teyssandier, Phys. Rev. D 16 (1977) 946; Phys. Rev. D 18 (1978) 1037; V. Braginsky, C. Caves and K.S. Thorne, Phys. Rev. D 15 (1977) 2047; R. Jantzen, P. Carini and D. Bini, Ann. Phys. (NY) 215 (1992) 1; I. Ciufolini and J.A. Wheeler, “Gravitation and Inertia”, Princeton University Press, Princeton, 1995; M.A.G. Bonilla and J.M.M. Senovilla, Phys. Rev. Lett. 11 (1997) 783; R. Maartens and B.A. Bassett, Class. Quantum Grav. 15 (1998) 705; S.J. Clark and R.W. Tucker, Class. Quantum Grav. 17 (2000) 4125; J.M.M. Senovilla, Mod. Phys. Lett. A 15 (2000) 159; Class. Quantum Grav. 17 (2000) 2799; M.L. Ruggiero and A. Tartaglia, Nuovo Cimento B 117 (2002) 743; L. Iorio and D.M. Lucchesi, Class. Quantum Grav. 20 (2003) 2477; B. Mashhoon, Phys. Lett. A 173 (1993) 347; Class. Quant. Grav. 17 (2000) 2399; Int. J. Mod. Phys. D 14 (2005) 2025; gr-qc/0311030; S. Kopeikin and B. Mashhoon, Phys. Rev. D 65 (2002) 064025 [gr-qc/0110101].
17. E. G. Harris, Am. J. Phys. 59(5) (1991) 421.
18. V. B. Braginsky, C. M. Caves and K.S. Thorne, Phys. Rev. D 15 (1977) 2047.
19. J. F. Pascual-Sanchez, Nuovo Cim. B 115 (2000) 725 [gr-qc/0010075].
20. L. Filipe Costa and C. A. R. Herdeiro, Phys. Rev. D 78 (2008) 024021 [gr-qc/0612140]; arXiv:0912.2146 [gr-qc].
21. L. F. O. Costa and J. Natario, arXiv:1207.0465 [gr-qc].
22. L. F. O. Costa and J. Natario, arXiv:1207.0465 [gr-qc].