Hairy black holes and holographic heat engine

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Abstract

By considering AdS charged black hole in extended thermodynamic context as the working substance we use it as a heat engine. We investigate the effect of hairy charge on the evolution of efficiency and Carnot’s efficiency along with Maxwell charge. Because of interesting thermodynamic behavior of hairy black holes it would be natural to know its effect when we use black hole as a heat engine. We show that this hairy charge could increase the efficiency, and maximum temperature would be happened for bigger Maxwell charge when this hairy charge grows. There is an inflection point for free Maxwell charge case at which efficiency accepts a minimum value. This point is not happened in the other cases or it would be un-physical and un-acceptable if it does. We also seek the behavior of the system in the large charge limit which is provided a model with low-temperature thermodynamic.

1 Introduction

Quantum field theory (QFT) begun to be considered in curved spacetime from 1970s and made a correspondence between some black hole characteristics like its area and entropy, or between surface gravity and temperature [1,2]. This implicated a basic connection among quantum theories, thermodynamic and gravity that leads to the birth of black hole thermodynamic. Hawking and Page in 1983 showed that there is a phase transition between a thermal AdS spacetime (a non-black hole solution) and Schwarzschild black hole [3]. Chamblin et al in 1999 tested it for a charged AdS background in a canonic ensemble and obtained a Van der Waals-like (VdW-like) phase transition between small and large black holes [4,5]. Despite such results,

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this similarity is not perfect due to the lack of a specific definition for the pressure and volume for black holes while VdW fluid presents a first order phase transition in $P-V$ plane.

In the other side we can see Smarr relation is incompatible with the first law of black hole thermodynamics. Kastor in 2009 tried to resolve the problem by varying the cosmological constant in the first law [6]. He reached to a generalized version of the first law of thermodynamics by considering cosmological constant and its conjugated quantity as the pressure and volume, respectively. By attention to this considerations one can see various behaviors and characteristics analogous to thermodynamic systems in black hole solutions like phase behaviors in gels and polymers or triple point in water. One of these aspects of a thermodynamic system is heat engine behavior which can be explored in black hole solution.

From AdS/CFT a gravity theory in the bulk in AdS spacetime corresponds to a conformal field theory on its boundary with one lower dimension. By attention to this duality varying the cosmological constant in the bulk perturb the dual side of our AdS gravity model and so initiate a renormalization group (RG) flow on the boundary. We can see from [7] that RG flow on the boundary could be understood by holographic heat engine when the black hole is considered as a working substance. It must be also interesting to study holographic heat engine at critical point at which its efficiency is very similar to Carnot engine at finite power [8]. This correspondence is studied for various aspects in [9-24] in details.

Pioneering works of Johnson on charged AdS black holes as the working substance [7,10] indicated that mechanical work can be extracted from heat energy via the "$PdV$" term in the first law of thermodynamics which is resulted by the relationship of the spacetime pressure and cosmological constant. This is in contrary with Penrose Process that leads to extracting of energy from rotating black holes in both asymptotically AdS and flat spacetime.

We know from no-hair theorem that hairy black hole solutions are forbidden in general relativity conformally coupled to the scalar field theory in asymptotically flat spacetime. In hairy black hole solutions in which the solving of backreaction problem in AdS spacetime would be possible analytically we can have the regular scalar fields configurations anywhere outside or on the horizon. This hairy black hole solution is defined in asymptotically flat or asymptotically (A)dS spaces in dimensions bigger than four [25]. In AdS spacetime the scalar field behaves asymptotically as the boundary conditions
related to AdS/CFT correspondence is considered. In [26] one can find hairy black holes more interesting case to study with respect to no-hair and thermal cases. So we choose this model for investigation in our work and study holographic heat engine for this kind of black hole solutions. We also interested in the studying of black holes in other gravity models such as modified gravity [27] at criticality which could be an interesting future work of this paper. (for recent reviews on modified gravity theories and the issue of dark energy to explain the late-time cosmic acceleration, see, for example [34-40]).

Layout of this work is as follows.
In section 2 we set up hairy black hole solution and its thermodynamic aspects in $\text{AdS}_5$ space. We discuss relationship between the hairy and Maxwell charges that divides physical and non-physical critical points. In section 3 we calculate the efficiency of the heat engine by regarding the effects of hairy and Maxwell charge. We find various areas of efficiency which is restricted by attention to the signature of Hawking temperature and entropy. We also study Carnot’s efficiency and its evolution with respect to the efficiency of heat engine. At last we show that for large limit of hairy charge both these efficiency approach to each other. All these results would be satisfied by the universal trade off relation for which power vanishes in this limit. At last section we present conclusion of this work and its outlook.

2 Thermodynamics of hairy black holes in $AdS_5$ space revisited

To discuss thermodynamics of hairy black holes in $AdS_5$ in the extended phase space we begin with the action as follows[26,28]:

$$I = \frac{1}{\kappa} \int d^5 x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} F^2 + \kappa L(\phi, \nabla \phi) \right)$$

(2.1)

where $\kappa = 16\pi G$ and $F$ is the gauge invariant scalar of the Maxwell tensor. In this action the Lagrangian of a conformal coupled real scalar field is expressed by

$$L(\phi, \nabla \phi) = \phi^{15} \left( b_0 S^{(0)} + b_1 \phi^{-8} S^{(1)} + b_2 \phi^{-16} S^{(2)} \right)$$

(2.2)
where
\[
S^{(0)} = 1,
\]
\[
S^{(1)} = S \equiv g^{\mu\nu}S_{\mu\nu} = g^{\mu\nu}\delta^\rho_\sigma S_{\mu\nu}^\sigma,
\]
\[
S^{(2)} = S_{\mu\nu\alpha\beta}S^{\mu\nu\alpha\beta} - 4S_{\mu\nu}S_{\mu\nu} + S^2,
\]

and
\[
S_{\mu\nu}^\gamma^\delta = \phi^2 R_{\mu\nu}^\gamma^\delta - 12\delta^{[\gamma}_{[\mu} \delta^{\delta}_{\nu]} \nabla_\rho \phi \nabla^\rho \phi - 48\phi \delta^{[\gamma}_{[\mu} \nabla_{[\nu]} \phi \nabla^{\delta]} \phi + 18\delta^{[\gamma}_{[\mu} \nabla_{[\nu]} \phi \nabla^{\delta]} \phi.
\] (2.3)

\(b_0, b_1\) and \(b_2\) are given in the equation (2.2) are coupling constants which are conformally invariant under the following transformations.
\[
g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \quad \phi \to \Omega^{-1/3} \phi.
\] (2.4)

Applying (2.4) the tensor (2.3) can be changed as
\[
S_{\mu\nu}^\gamma^\delta \to \Omega^{-8/3} S_{\mu\nu}^\gamma^\delta.
\] (2.5)

Varying the action (2.1) with respect to the metric field \(g_{\mu\nu}\) we can obtain the Einstein equations
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}
\] (2.6)

in which the energy momentum tensor is given by
\[
T^\nu_{\mu} = \sum_{k=0}^{D-1} \left( \frac{k!b_k}{2k+1} \phi^{3D-8k} \delta^\nu_{[\mu} \delta^{[\lambda_1} \cdots \delta^{\lambda_k]}_{\rho_{2k+1}}} \right) \left( S_{\rho_1\rho_2\cdots\rho_{2k}\lambda_1\cdots\lambda_{2k}}^{\rho_1\rho_2\cdots\rho_{2k-1}\lambda_{2k-1}\lambda_{2k}} \right),
\] (2.7)

The above metric equation has a spherically symmetric static metric solution with a black hole topology as follows [26,28]
\[
ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2
\] (2.8)

with
\[
f(r) = 1 - \frac{m}{r^2} - \frac{h}{r^3} + \frac{q^2}{r^4} - \frac{\Lambda}{6} r^2,
\] (2.9)
in which \( d\Omega_3^2 \) is the line element of the unit 3-sphere which reads \( \omega_3 = \int d\Omega_3^2 = 2\pi^2 \). In this ansatz \( q \) is Maxwell charge and \( h \) represents hair charge for which the real scalar field or the hair configuration is a function of \( r \) that is regular except for \( r = 0 \). As we can see from (2.9) the hair counterpart in the metric potential (2.9) behaves as \( \sim 1/r^3 \) and defines as,

\[
\phi(r) = \frac{n}{r^{1/3}}. 
\]  
(2.10)

By using the coupling constants in (2.2) we can fix the parameters \( n \) and \( h \) as follows:

\[
h = \frac{64\pi G}{5} b_1 n^9, \quad n = \varepsilon \left( -\frac{18 b_1}{5 b_0} \right)^{1/6},
\]  
(2.11)

with \( \varepsilon = -1, 0, +1 \). By applying these fixed parameters on the field equations a constrain condition will be achieved between \( b_0, b_1 \) and \( b_2 \) such that

\[
10b_0 b_2 = 9b_1^2.
\]  
(2.12)

Thermodynamic quantities of the black hole solution are expressed by the following equations in presence of the hairy charges.

\[
Q = -\frac{\sqrt{3}\pi}{8} q
\]  
(2.13)

\[
M = \frac{3\pi}{8} m = \frac{3\pi}{8} \left( r_+^2 - \frac{h}{r_+} + \frac{q^2}{r_+^2} + \frac{r_+^4}{\ell^2} \right),
\]  
(2.14)

\[
S = \int_0^{r_+} \frac{1}{T} \frac{\partial M}{\partial r_+} dr_+ = 2\pi^2 \left( r_+^3 - \frac{5}{8} h \right),
\]  
(2.15)

\[
T(r_+) = \frac{1}{\pi \ell^2 r_+^4} \left( -\frac{q^2 \ell^2}{2r_+} + \frac{h \ell^2}{4} + \frac{\ell^2 r_+^3}{2} + r_+^5 \right)
\]  
(2.16)

where the black hole event horizon \( r_+ \) is determined by solving the horizon equation \( f(r_+) = 0 \). Since in extended thermodynamics the black hole mass is treated as the enthalpy, so we can reach to the first law of the black hole thermodynamics regarding (2.13), (2.14), (2.15) and (2.16) such that

\[
dM = TdS + VdP + Kdh + \Phi dQ,
\]  
(2.17)

where \( V \) is the thermodynamic volume. It is the conjugated potential for the pressure of spacetime which is related to the cosmological constant through
\[ P = -\Lambda/8\pi. \]  
\[ K \] and \( \Phi \) are hairy and Maxwell charges respectively. They can be calculated by

\[ V = \left( \frac{\partial M}{\partial P} \right)_{r^+} = \frac{\omega_3}{4} \ell^4, \]  
(2.18)

\[ K = \left( \frac{\partial M}{\partial h} \right)_{r^+} = \frac{\omega_3}{32\ell^2 r^5_+} \left( 20r^6_+ + 8r^4_+ \ell^2 + 3\ell^2 hr^+_+ \right), \]  
(2.19)

\[ \Phi = \left( \frac{\partial M}{\partial Q} \right)_{r^+} = -\frac{2\sqrt{3}}{r^2} q. \]  
(2.20)

The generalized Smarr relation regarding the scaling argument reads

\[ 2M = 3TS - 2VP + 3hK + 2\Phi Q. \]  
(2.22)

Now we can obtain the equation of state of our hairy black hole by using Hawking temperature (2.16) and definition of specific volume \( v = \frac{4}{3}r^+_+ \) [27],

\[ P = \frac{T}{v} - \frac{2}{3\pi v^2} - \frac{64}{81\pi v^5} h + \frac{512}{243\pi v^6} q^2. \]  
(2.23)

The critical points that will be computed by the following condition

\[ \left. \frac{\partial P}{\partial v} \right|_{T=T_{cr}} = \left. \frac{\partial^2 P}{\partial v^2} \right|_{T=T_{cr}} = 0, \]  
(2.24)

simply achieved by assuming \( q = 0 \) and leads to the following critical quantities that are the functions of the hairy charge:

\[ v_{cr} = \frac{4}{3}(-5h)^{\frac{3}{5}}, \]  
(2.25)

\[ T_{cr} = \frac{3}{20} \left( \frac{-5h}{h} \right)^{\frac{2}{5}}, \]  
(2.26)

\[ P_{cr} = \frac{9}{200\pi} \left( \frac{-\sqrt{5}}{h} \right)^{\frac{2}{5}}. \]  
(2.27)

It must be noted that by regarding [29], there is a constraint between hairy and Maxwell charges which divides the critical points to physical and unphysical points. For both of them the critical points are positive but their
entropy could be positive or negative for physical and un-physical ones, respectively. For negative hairy charge, \( h < 0 \), we have a single critical point for any value of Maxwell charge and entropy would always be positive. So in this case critical points are physical. But when \( h > 0 \) the situation is different. For all values of positive hairy charges critical points are positive but when \( h > 1.3375q^{2/3} \) the entropy will be negative. So physical critical points happens for \( h \leq 1.3375q^{2/3} \) where equality satisfies entropy with zero value.

3 Holography heat engines

In this section we study hairy black holes in \( AdS_5 \) spacetime as the working substance of the heat engine. For this purpose we consider a rectangular cycle in \( P - V \) diagram as shown in figure 1. The cycle is made up of four processes: two constant volume process (isocore) and two constant pressure process (isobar). During this rectangular cycle, the system receives heat during \( b \rightarrow c \) and \( c \rightarrow d \), so the net amount of heat transferred to the system is \( Q_H = Q_{b \rightarrow c} + Q_{c \rightarrow d} \). Then expels heat during \( d \rightarrow a \) and \( a \rightarrow b \). Because the process of \( b \rightarrow c \) takes place in constant volume, the amount of heat entered to the system would be equal to the change in the system’s internal energy \( \Delta U = Q_{bc} = C_v \Delta T_{b \rightarrow c} \). But since the volume and entropy of a black hole are both depend only on the radius of the horizon, we can conclude that the specific heat at constant volume vanishes so the system does not receive any heat during the \( b \rightarrow c \) process. In the other side, the heat of the system absorbed during the isobaric process \( c \rightarrow d \) can also be obtained from \( Q_{cd} = C_P \Delta T_{b \rightarrow c} \). So the total heat entered into the system in this cycle is given by

\[
Q_H = \int_{T_b}^{T_d} C_P dT, \tag{3.1}
\]

where \( C_P \) is the specific heat at constant pressure defined by the following equation.

\[
C_P = T \left( \frac{\partial S}{\partial T} \right)_P, \tag{3.2}
\]
The above heat capacity could be computed for the black hole metric solution (2.8) as

\[ C_P = \frac{3(2PA^2\pi^2 + 12h\pi^4B^\frac{2}{3} + 3(A\pi)^\frac{2}{3} - 48q^2\pi^5)}{16(240q^2\pi^5 + 2PA^2\pi^2 - 48h\pi^4B^\frac{2}{3} - 3\pi(A\pi)^\frac{2}{3})}, \]  
(3.3)

where \( A = 20\pi^2h + 16S \) and \( B = 20\pi^3h + 16S \). The inflow heat relation will be simplified by using (3.1) and (3.2) in the expansion process of \( c \to d \) for fixed all other quantities as

\[ Q_H = \int_{S_c}^{S_d} C_P \left( \frac{\partial T}{\partial S} \right)_P dS = \int_{S_c}^{S_d} TdS = \int_{M_c}^{M_d} dM = M_d - M_c. \]  
(3.4)

From (2.14) for hairy \( AdS_5 \) black holes we have

\[ M_i = \frac{3\pi}{8}(r_{+i}^2 - \frac{h}{r_{+i}} + \frac{q^2}{r_{+i}^2} + \frac{4\pi}{3}Pr_{+i}^4), \]  
(3.5)

where the event horizon could be defined from (2.15) like \( r_{+i} = \frac{1}{2\pi^2\pi}A_i^\frac{1}{3} \) with \( i = c, d \) and \( A_i = 20\pi^2h + 16S_i \). So the net amount of heat which the system
receives is
\[ Q_H = M_d - M_c = \frac{3\pi}{8} \left( \frac{1}{4\pi^{4/3}} (A_d^2 - A_c^2) - 2\pi^{2/3} h (A_d^{3/2} - A_c^{3/2}) \right. \\
\left. + 4\pi^{4/3} q^2 (A_d^{2/3} - A_c^{2/3}) + \frac{P}{12\pi^{5/3}} (A_d^{4/3} - A_c^{4/3}) \right). \quad (3.6) \]

In the other side, the work performed by engine equals to the area enclosed by the cycle which is computed regarding figure 1 as follows:
\[ W = (V_d - V_c)(P_c - P_b), \quad (3.7) \]
where subscripts denote the point of evaluated thermodynamic quantities in the figure. So by attention to (2.18) it reads
\[ W = \frac{(A_d\pi)^{\frac{2}{3}} - (A_c\pi)^{\frac{2}{3}}}{32\pi^2} (P_c - P_b). \quad (3.8) \]

Finally we can demonstrate the performance of the heat engine by a thermal efficiency \( \eta \)
\[ \eta = \frac{W}{Q_H}. \quad (3.9) \]

To investigate the effect of hairy and Maxwell charges on the performance of heat engine it would be useful to set one charge with unit and let the other one varies in ranges with positive Hawking temperature. In figures 2.a and 2.b we show the maximum value of charge with a red spot before which the Hawking temperature stays positive. As we can see for this maximum charge the thermal efficiency has also its maximum value.

In figure 2.a we plotted the heat engine efficiency with respect to Maxwell charge by fixing parameters as \( P_c = 2, P_b = 1, S_c = 8, S_d = 10 \) and for various fixed hairy charges. The analyzing of the diagram shows that the efficiency grows by increasing the Maxwell charge for any fixed hairy black holes until ceased at a maximum value point \((q_{\text{max}}, \eta_{\text{max}})\) after which temperature would be negative. Of course by attention to the point is mentioned in the last paragraph of section 2 there is a relation between two charges which determines areas of physical and un-physical critical points. For example blue line in figure 2.a indicates \( h = 1 \), so only critical points for \( Q \geq 0.6464862821 \) will be physical and \( 0 < q < 0.6464862821 \) stands for un-physical points. The green line corresponds to free hairy charge case or simply AdS-RN solution
which is studied in [7]. As we can see the effect of hairy charge could be effective on the efficiency as by increasing $h$ the maximum point of $(q_{\text{max}}, \eta_{\text{max}})$ has a bigger amount and so the physical critical points which is restricted by negative entropy and negative temperature borders accept a wider range of Maxwell charges. Diagrams are different when we fix the Maxwell charge

![Figure 2](image-url)

**Figure 2:** Behavior of efficiency with respect to charges by fixing other charge. In (a) hairy charge fixes and diagrams end at the maximum point. Green, purple and blue lines indicate fixed $h = 0, 0.5$ and 1, respectively. Maximum point for green line is $q_{\text{max}} = 4.890$, $\eta_{\text{max}} = 0.824$, for purple line is $q_{\text{max}} = 8.572$, $\eta_{\text{max}} = 0.885$ and for blue line is $q_{\text{max}} = 12.234$, $\eta_{\text{max}} = 0.915$. Diagram (b) is plotted for fixed Maxwell charge and maximum points $(h_{\text{max}}, \eta_{\text{max}})$ are in the the left side. Here Green, purple and blue lines indicate fixed $q = 0, 0.25$ and 0.5, respectively.

and let hairy charge varies. There is a inflection point at which diagram has a minimum efficiency, so one can see both increasing and decreasing behavior of $\eta$ by varying hairy charge. But what is physical range in which the above conditions are depended to sign of entropy and Hawking temperature. In figure 2.b we show the behavior of heat engine efficiency for various fixed electrical charge and fixing parameters as $P_c = 2$, $P_b = 1$, $S_c = 8$, $S_d = 10$ under the change of the hairy charge. The green line indicates case of free
Maxwell charge for which hairy charge must be negative, so all positive parts of green line does not indicate any critical points. The absolute value of maximum hairy charge ($h_{\text{max}} = -0.6066$ with $\eta_{\text{max}} = 0.553$) and inflection point (with $h_{\text{inf}} = -0.418$ and $\eta_{\text{inf}} = 0.476$) both are in negative side of $h$ and so are acceptable. However when $q \neq 0$ then hairy charge must have a positive value, so both maximum hairy charge and inflection point are not acceptable. For instance when $q = 0.5$ only for $0 \leq h \leq 0.8425722021$ we have physical critical points and for all bigger values they are un-physical. Since $h_{\text{max}}$ and inflection points are in negative side, so they does not imply on critical points. We also compare the heat engine efficiency $\eta$ with the Carnot’s efficiency $\eta_c = 1 - \frac{T_{\text{low}}}{T_{\text{high}}}$ where $T_{\text{low}}$ and $T_{\text{high}}$ are the lowest and the highest temperature of the heat engine in cycle under consideration. In figures 3.a and 3.b we can see that by fixing one charge, $\eta_c \rightarrow 1$ at maximum value of another charge. From figure 3.a it can be seen that for bigger fixed values of $q$ we have smaller absolute value of $h_{\text{max}}$ at which $\eta_c \rightarrow 1$, in contrary with figure 3.b that by increasing fixed value of hairy charge we have bigger $q_{\text{max}}$. There is same analyse for the relation of $q$ and $h$ and physical critical points as we discussed already. The ratio of $\eta/\eta_c$ is depicted for fixed hairy and Maxwell charge in figures 4.a and 4.b, respectively. In figure 4.a with $h = 1$ one can see decrease of this ratio by rasing the Maxwell charge, while with $q = 1$ in figure 4.b we see inverse behavior for it. One can see that $\eta/\eta_c$ increases by raising the hairy charge. It approaches to 1 for large enough $h$ that means $\eta \rightarrow \eta_c$. As it studied in [30, 31] when we put one of the corner of rectangular cycle in critical points of the system or near to them, the efficiency of heat engine could be approached to the Carnot’s efficiency having the finite power. In [8] we can see good results for charged-AdS black hole in the large charge limit. Following this work we put the critical point ($P_{\text{cr}}, V_{\text{cr}}$) in corner $a$ and choose the boundary conditions as

$$
\begin{align*}
P_a &= P_b = P_{\text{cr}}, & P_c &= P_d = \frac{3}{2}P_{\text{cr}}, \\
V_d &= V_a = V_{\text{cr}}, & V_b &= V_c = V_{\text{cr}} \left(1 - \frac{L}{h^{2/3}}\right),
\end{align*}
$$

(3.10)

in which $L$ is a constant considered for dimensional analyse. In the large charge limit we can put $\alpha \equiv \frac{L}{h^{2/3}} \rightarrow 0$. The work done by the engine could be simply calculated by the area of $\Delta V \Delta P$ in the cycle:

$$
W = \frac{1}{2}P_{\text{cr}} V_{\text{cr}} \alpha = \frac{9 \times 5^{2/3}}{160} \pi L.
$$

(3.11)
Figure 3: Behavior of Carnot’s efficiency for fixed $q$ in (a) and fixed $h$ in (b). Solid, dash and dotted lines represent respectively $q = 0.5, 0.25, 0$ in (a), and $h = 0, 0.25, 0.5$ in (b).

Note that the work done in cycle under these considerations is finite and independent of hairy charge. The heat that absorbed by the system is given by

$$Q_H = M_2 - M_1 = \frac{3\pi}{8} (5H)^{2/3} \left( (1-\alpha)^{1/2} + \frac{1}{20} (9\alpha + 24 - 4(1-\alpha)^{-1/4}) \right),$$  \hspace{1cm} (3.12)

where by attention to the negativity of hairy charge for free Maxwell charge case we put $H = -h$ as the absolute value of $h$. At large limit of hairy charge when $\alpha \to 0$, the heat which is absorbed by the system takes the following form

$$Q_H = \frac{27\pi}{80} (5H)^{2/3} \alpha + \frac{9\pi}{256} (5H)^{2/3} \alpha^2 + \frac{15\pi}{1024} (5H)^{2/3} \alpha^3 + \mathcal{O}(\alpha^4).$$  \hspace{1cm} (3.13)

Therefore, the efficiency of the heat engine (3.9) leads to the following form by using (3.11) and by putting $L = 1$.

$$\eta = \frac{1}{6} \left( 1 - \frac{5}{48H^{2/3}} - \frac{25}{768H^{4/3}} + \frac{875}{110592H^2} + \ldots \right).$$  \hspace{1cm} (3.14)
In the other side, the Carnot’s efficiency as we discussed before depends on the highest temperature \( T_H \) and the lowest temperature \( T_C \) of the heat engine which happens at the corners \( d \) and \( b \), respectively. In the large hairy charge limit they are given respectively by

\[
T_H = \frac{9}{50\pi} \left( \frac{25}{H} \right)^{1/3},
\]

(3.15)

and

\[
T_C = \frac{3}{20\pi} \left( \frac{25}{H} \right)^{1/3} - \frac{1}{640\pi} \left( \frac{25}{H^7} \right)^{1/3} + \cdots.
\]

(3.16)

So the Carnot’s efficiency would be obtained as follows

\[
\eta_c = 1 - \frac{T_C}{T_H} = \frac{1}{6} \left( 1 + \frac{5}{96H^2} + \cdots \right).
\]

(3.17)

By attention to (3.14) and (3.17) we can conclude that against the initial behavior of \( \eta \) and \( \eta_c \) they approach to each other for large limit of hairy charge. It could be seen from figure 5 that for \( H \to \infty \) with finite work while
Figure 5: Behavior of efficiency for free Maxwell charge solution with large hairy charge limit when efficiency of heat engine (solid line) approaches to Carnot’s efficiency (dash line).

the power vanishes and so we have $\eta = \eta_c$.

By attention to critical value of pressure (2.27) we can infer $P \sim H^{-2/3}$ and so $\tau \sim H^{2/3}$ which is the time for to complete a cycle scales at a finite hairy charge [8] leading to a finite power $W/\tau$ in which $W$ is work given by (3.11) (see [32, 33]).

$$\frac{W}{\tau} \leq \Theta \eta (\eta_c - \eta) \frac{T_C}{T_C}$$

(3.18)

where $\Theta$ is a model dependent constant and the right hand side would be expanded for large $H(\equiv -h)$ as follows:

$$\frac{\eta (\eta_c - \eta)}{T_C} = \frac{5^{4/3} \pi}{1296} \left( H^{-1/3} + \frac{5}{24} H^{-1} + \frac{827}{2304} H^{-5/3} + \ldots \right).$$

(3.19)

When hairy charge goes to infinity and $\eta = \eta_c$ the power (3.18) vanishes as we expect. The universal trade off relation as a bound between power and efficiency would be satisfied for any values of $h$. 

14
4 Conclusions

We studied thermodynamic of black hole solution with Maxwell and hairy charge in $AdS$ space when the cosmological constant plays as a thermodynamic variable. We restricted us to study hairy black holes in $AdS_5$ space due to interesting features of phase transition. From [28] we know Maxwell and hairy charges are bounded together and the sign of entropy depends on this bound. Positive critical points divides on two physical and un-physical branches for positive and negative entropy, respectively. We found out when hairy charge get fixed, the efficiency of black hole solution which acts as a heat engine changes. There is a maximum value for efficiency that happens at a maximum charge after which we have unacceptable negative temperature. This maximum point will be increased for bigger fixed hairy charges. We found a minimum value of efficiency when Maxwell charge vanishes and hairy charge varies. This inflection point only exist for this case and for other fixed values of $q$ will be un-physical and un-acceptable. For non-zero fixed Maxwell charge the maximum point happened in forbidden area and we have only an increasing of efficiency by the growth of hairy charge. We also compared this efficiency with Carnot’s efficiency and conclude that their ratio decreased by changing of Maxwell charge when $h = constant$, but when we fix hairy charge it could be seen that $\eta \rightarrow \eta_c$ for large $h$. By putting one corner of rectangular cycle on or near critical points of the system the efficiency approaches to Carnot’s efficiency having the finite power. We studied it in large hairy charge limit when the power vanishes.

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