BECA N D T H E N E W
WORLD OF COHERENT
MATTER WAVES

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After decades of effort to produce an atomic Bose condensate, this was
finally achieved in June, 1995 in a laser-cooled magnetically trapped gas
of $^{87}$Rb atoms by Eric Cornell and Carl Wieman at JILA (University of
Colorado and N.I.S.T.). As of July, 1999 (4 years later), there are now
over 20 experimental groups around the world who can routinely pro-
duce and study such atomic Bose condensates. In addition, over 1000 theo-
doretical papers have been published!

I will give three introductory lectures on this new Quantum Phase of
Matter, which I think will continue to be a growth point of fundamental
physics research for the next decade and will also be the source of new
technologies based on using this source of coherent matter waves. A brief
sketch of my three lectures is as follows:

1. A BRIEF HISTORY OF BEC STUDIES.
   • Before 1995, going back to pioneering work of Einstein (1925)
     and Fritz London (1938).
   • Introduce the concept of a Bose macroscopic wavefunction $\Phi(r, t)$
     describing a Bose condensate and the key Gross-Pitaevskii equa-
     tion of motion for $\Phi(r, t)$.
   • Review some of the results obtained over the last 4 years - and
     indicate why trapped Bose gases are so interesting, possibly even
     more than superfluid $^4$He and BCS superconductors, which also
     involve Bose condensation.

2. DYNAMICS OF A PURE CONDENSATE ($T \ll T_{BEC}$).
   • Time-dependent Gross-Pitaevskii (1961) equation of motion.
   • Crucial effect of weak interatomic interactions.
   • Collective oscillations of the condensate.
   • The Bose condensate as a classical quantum object!
3. DYNAMICS OF COUPLED CONDENSATE AND NON-CONDENSATE COMPONENTS (TWO-FLUID HYDRODYNAMICS):

- Derivation of a quantum Boltzmann equation for the non-condensate excited atoms and a generalized GP equation for the condensate.
- More complex behaviour than in superfluid $^4$He.
- Comparison with the well-known Landau two-fluid theory (1941).

Further references which are relevant for these lectures are:

1. BEC Homepage (maintained by a BEC theorist, Mark Edwards): http://amo.phy.gasou.edu/bec.html/. This widely-used homepage has played a crucial role in BEC research.

2. A recent article [1] by Dalfovo, Giorgini, Pitaevskii and Stringari is an authoritative review of recent theory on atomic Bose condensates and expands on the material I cover in Sections 1 and 2.

3. Articles in a book [2], *Bose-Einstein condensation in atomic gases*, *Proceedings of the International School of Physics “Enrico Fermi”*, ed by M. Inguscio, S. Stringari and C. Wieman. This contains many review articles on current BEC research. In particular, I call attention to (all these can be downloaded from the LANL website under cond-mat):

- W. Ketterle, D.S. Durfee and D.M. Stamper-Kurn, “Making, probing and understanding Bose-Einstein condensates” - a 100 page review of recent experiments [3].
- A. Griffin “A brief history of our understanding of BEC: From Bose to Beliaev” [4].
- A. Griffin, “Theory of excitations of the condensate and non-condensate at finite temperature” [5].
- A.L. Fetter, “Theory of a dilute low-temperature trapped Bose condensate.” A very detailed analysis at $T = 0$.

4. A long article by Zaremba, Nikuni and Griffin [6] on the non-equilibrium dynamics of trapped Bose gases at finite temperatures. This expands on the material covered in Section 3 of these lectures.

1 AN OVERVIEW OF PAST AND RECENT WORK
1.1 Some history before 1980

Einstein predicted that a non-interacting gas of atoms (Bosons) would undergo a phase transition at low temperatures, when a macroscopic \(0(N)\) number of atoms occupy the lowest energy level (in a uniform ideal Bose gas, this is the zero momentum single-particle state). His work was inspired by a novel derivation of the Planck distribution for photons by Bose in 1924. The basic physics of this phase transition is worked out in every text in statistical mechanics [6]. The simplest way of estimating \(T_{BEC}\) is to note that the transition occurs when

\[
\lambda_T \gtrsim d = \text{average distance between atoms} \sim \frac{1}{n^{1/3}},
\]

where \(\lambda_T\) is the thermal De Broglie wavelength of a gas of atoms at temperature \(T\),

\[
\lambda_T \equiv \left(\frac{2\pi \hbar^2}{mk_B T}\right)^{\frac{1}{2}}.
\]

(1.2)

The criterion in (1.1) is equivalent to \(n\lambda_T^3 \gtrsim 1\), while a more careful analysis [6] gives \(n\lambda_T^3 = \zeta(3/2) = 2.612\). One sees that \(\lambda_T \to \infty\) as \(T \to 0\). When \(\lambda_T \gtrsim d\), all the atoms become correlated and the gas exhibits new collective behaviour (even in absence of interactions). Using (1.3), one finds that

\[
k_B T_{BEC} \sim \frac{2\pi \hbar^2}{m} n^{2/3}.
\]

(1.3)

Below \(T_{BEC}\), the number of atoms in the \(p = 0\) single-particle state increases and is given by the well-known formula

\[
\frac{N_c(T)}{N} = \left[1 - \left(\frac{T}{T_{BEC}}\right)^{3/2}\right].
\]

(1.4)

At \(T = 0\), all the atoms in an ideal gas are in this \(p = 0\) state (this state is the Bose condensate in non-interacting 3D gas).

Nothing much happened until 1938. Then the neglected work of Einstein was re-discovered and developed by Fritz London [7], who suggested that it might be the basis of an explanation for the strange effects noticed in liquid \(^4\)He at \(T_c \sim 2.17K\). London’s suggestion was based on the fact that the \(^4\)He atom was a “composite” Boson \((S = 0)\) and the formula in (1.3) gives \(T_{BEC} \sim 3K\) if we used the density for liquid \(^4\)He. L. Tisza used London’s idea and suggested (somehow!) that the condensate atoms act in a coherent way - a new collective degree of freedom moving without friction. This “picture” led to a rudimentary two-fluid model that could explain experiments showing superfluidity (especially by Kapitza as well as by Allen and Meisner) as a counterflow of the superfluid and the normal
fluid. In a dilute, weakly interacting Bose gas, these two components are the condensate and non-condensate, respectively.

Until the 1960’s, the theory of BEC in interacting systems was dominated by efforts to use it to understand superfluid $^4$He. The London-Tisza scenario was essentially correct, but to formulate it properly needed field-theoretic many body techniques and the concept of broken-symmetry, which were only developed in the period 1957-1965. In this period, a large amount of work was done on a toy problem, a dilute weakly interacting Bose gas, since a Bose liquid like superfluid $^4$He was too difficult to deal with theoretically. These early studies are the foundation of our current understanding of trapped atomic gases.

A phenomenological theory of superfluid $^4$He was introduced by Landau in 1941, based on the idea of quasiparticles (phonon-roton spectrum) and a two-fluid superfluid hydrodynamics. This brilliant theory has been very successful and is the basis of modern descriptions of superfluid $^4$He. However, it made no explicit mention of BEC or even the fact that $^4$He atoms obeyed Bose statistics. Only in the 1960’s did it become clear that Landau formulation had it’s microscopic basis in the existence of a condensate macroscopic wavefunction,

$$\Phi(\mathbf{r},t) = \langle \hat{\psi}(\mathbf{r}) \rangle = \sqrt{N_c(\mathbf{r},t)} e^{i\theta(\mathbf{r},t)}, \quad (1.5)$$

where $\hat{\psi}(\mathbf{r})$ is the quantum field operator (see Section 2). This concept was first formally introduced by Beliaev in 1957, extending the pioneering work of Bogoliubov in 1947. The superfluid motion is associated with the gradient of the phase of this two-component order parameter,

$$e^{i\theta} = e^{i(\theta_0 + \mathbf{r} \cdot \nabla \theta)}; \quad \mathbf{k}_s \equiv \nabla \theta \equiv \frac{m v_s}{\hbar} \quad (1.6)$$

The essential relation between superfluidity and $\Phi(\mathbf{r},t)$ is simply and elegantly described in the classic monograph by Nozières and Pines. For further discussion of the development of our current understanding of Bose condensates, see the review article by Griffin in Ref. [2].

It might be useful to make a brief digression here on the BCS theory of superconductors based on formation of Cooper pairs (total spin $S = 0$), which was developed in 1957. In the BCS theory, superconductors exhibit the same kind of macroscopic quantum behaviour as superfluid $^4$He, as first argued in the late 1930’s by F.London. In the Gor’kov version of the BCS theory, the spin-singlet Cooper pair order parameter

$$\Phi(\mathbf{r},t) = \langle \hat{\psi}_\uparrow(\mathbf{r}) \hat{\psi}_\downarrow(\mathbf{r}) \rangle \quad (1.7)$$

is the equivalent of

$$\Phi(\mathbf{r},t) = \langle \hat{\psi}(\mathbf{r}) \rangle \quad (1.8)$$

in Bose superfluids. The essential equivalence of these two systems was obscured by the complexity of the original many-particle BCS-wavefunction,
due to the large spatial size of overlapping Cooper pairs. As a result, Cooper pairs form and condense into a coherent state at same temperature. It was only in the 1980’s that theorists (Leggett, Nozières and others [14]) realized that for small Cooper pairs (tightly bound and hence high $T_c$), the BCS theory smoothly goes over to theory of a weakly interacting Bose gas of non-overlapping Cooper pairs. Recently a gas of $^{40}$K atoms (a composite fermion) have been laser-cooled at JILA to slightly below the Fermi temperature [15]. There are ways to make the interaction attractive (see later) and then one might look for the formation of Cooper pairs [10] of ultra-cold Fermi atoms!

1.2 More recent developments (1980-1995)

Since the 1970’s, there has been increasing interest by experimentalists to find a “pure” form of BEC, namely, in a low temperature gas. The two early candidates for Bosons were excitons (electron-hole pairs) in semiconductors and spin-polarized hydrogen atoms (see articles by Greytak and by Wolfe et al. in the book mentioned in Ref. [14]). A gas of $H_\uparrow$ atoms was predicted to be stable as a gas even at $T = 0$. This is because the atoms cannot combine since there is no bound state of the interatomic potential between two spin-polarized H atoms. Thus, one cannot form liquid or solid phase. Many people got interested in BEC in $H_\uparrow$ gas, including theorists [17].

Several of the key ideas that led to success in alkali atoms in 1995 grew out of the pioneering work on $H_\uparrow$ gas in the 1980’s. However 3-body interactions become increasingly important at higher densities and these cause spin flips, allowing formation of H-molecules. High densities were needed since cooling was by cryogenic methods, which could reach $\sim 10^{-4}$K but no lower. BEC in $H_\uparrow$ gas was finally produced at MIT in June, 1998, after almost 20 years of work [18]. Unfortunately, the alkali atom condensates are much easier to create and study, and appear to be more interesting gases.

Since the early 1990’s, attention has focussed on the alkali atoms: Li, Na, K, Rb, Cs. The atoms are Bosons, with an even number of neutrons. The strategy was to use laser-cooling to get to very low temperatures, where the low density gas would Bose-condense. The essential idea behind laser cooling is that when an atom absorbs a photon, it slows down. In the summer of 1995, BEC was announced by three groups led by:

- C. Wieman and E. Cornell (JILA), using $^{87}$Rb atoms [19].
- W. Ketterle (MIT), using $^{23}$Na atoms [20].
- R. Hulet (Rice University), using $^7$Li atoms [21].

Parenthetically, it is now the general feeling that the original Rice data, interesting as it was, did not give an unambiguous signature of a (very small, since the interaction is attractive) condensate in $^7$Li gas [22].
Alkali atoms are perfect for BEC studies. They have a magnetic moment, and hence can be trapped by magnetic fields. They essentially have a “one-electron” structure. They are thus simple atoms, and have been well studied by atomic physicists. One can easily selectively flip the “spin” of higher energy trapped atoms. These “hot” atoms are then quickly ejected from the magnetic trap and the remaining atoms quickly thermalize to a lower temperature. This “evaporative cooling” is very efficient and quickly brings one into the temperature region required for BEC.

It is useful to mention a few experimental facts about the magnetic traps currently in use. As it turns out, these traps are well described as a harmonic potential

$$V_{ex}(r) = \frac{1}{2}m\omega_0^2r^2 \quad \text{(isotropic)} \quad (1.9)$$

$$= \frac{1}{2}m(\omega_{oz}^2x^2 + \omega_{oy}^2y^2 + \omega_{oz}^2z^2) \quad \text{(anisotropic)} \quad (1.10)$$

Most current traps are either:

- pancakes, $\omega_{oz} \gg \omega_{ox}(= \omega_{oy})$
- cigars, $\omega_{oz} \ll \omega_{ox}(= \omega_{oy}) \quad (1.11)$

and the trap frequencies are of the order $\omega_0 \sim 2\pi \times 100\text{Hz}$. In 1995, the first condensates were small $\sim 10^3$ atoms and $T_{BEC} \sim 100\text{nK}$. However in 1999, the condensates can be quite large $\sim 10^8$ atoms at $T_{BEC} \sim \mu\text{K}$. These have a size $\sim$ many microns, which can be easily seen optically. When the condensates are small, the trap is turned off and cloud allowed to expand, and then measured by optical methods. The results are simple to analyze if gas is non-interacting. However, more analysis is needed to include the effects of interactions during expansion.

Early reports on atomic condensates discussed the system as an ideal Bose gas. It was soon realized that even in these very dilute gases, the interactions played a crucial role. Indeed, since the 1960’s, it has been understood that an interacting Bose gas is quite different from an ideal Bose gas. In particular, interactions stabilize (or “lock”) the phase of the condensate and allow coherent properties to emerge. (We recall that a free Bose gas has a condensate but is not a superfluid). Of course, interactions are also crucial for cooling. After hot atoms are removed by rf-induced spin-flips, it is important that remaining atoms can quickly re-thermalize through collisions.

However, it is useful to first consider an ideal Bose gas in a trap, to illustrate some characteristic features. For atoms in an external potential, we have

$$N = \sum_i f_0(\epsilon_i), \quad (1.12)$$
1. BEC AND THE NEW WORLD OF COHERENT MATTER WAVES

where the Bose distribution is

\[ f_0(\epsilon_i) = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}. \]  

(1.13)

In a harmonic trap, the energy levels are:

\[ \epsilon_i = \epsilon_{n_x, n_y, n_z} = (n_x + n_y + n_z + \frac{3}{2})\hbar\omega_0, \]  

(1.14)

with \( n_x, n_y, n_z = 0, 1, 2, 3 \ldots \). The condensate is described by the ground state single-particle wavefunction

\[ \phi_0(r) \sim e^{-r^2/2a_{HO}^2}, \]  

(1.15)

where the S.H. oscillator length is \( a_{HO} \equiv \left( \frac{\hbar m \omega_0}{2} \right)^{\frac{1}{2}} \sim 1 \mu m \) in current traps. Clearly \( a_{HO} \) gives the “size” of condensate \( n_c(r) = |\phi_0(r)|^2 \) in a trapped gas. For the non-condensate density \( \tilde{n}_0(r) \) (often called the “thermal cloud”), we can use the semiclassical limit, since the thermal energy \( (k_B T) \) is much larger than the spacing between the S.H. energy levels \( (\hbar\omega_0) \).

Then we have

\[ \tilde{n}_0(r) \sim e^{-V_{ex}(r)/k_B T} = e^{-r^2/2R_T^2}, \]  

(1.16)

where

\[ R_T = \sqrt{\frac{k_B T}{m\omega_0}} = a_{HO} \left( \frac{kT}{\hbar\omega_0} \right)^{\frac{1}{2}} \gg a_{HO}. \]  

(1.17)

Thus we see that the size of the thermal cloud \( R_T \) is much larger than the condensate. The signature for condensate is this sharp high density peak at the centre of the trap, which suddenly starts to grow out of the broad thermal distribution at the predicted transition temperature \( T_{BEC} \). As \( T \to 0 \) (effectively \( T \lesssim 0.4 T_{BEC} \)), the thermal cloud steadily disappears as all atoms go into the ground state \( \phi_0(r) \) given by (1.13), which is the macroscopic wavefunction for a non-interacting trapped Bose gas. The temperature of the gas is measured from the temperature dependence of the tail of the thermal distribution given by (1.16).

It is easy to calculate the transition temperature for atoms in an harmonic trap. Separating out the condensate contribution in (1.13), we have

\[ N = N_c + \sum_{i \neq 0} f_0(\epsilon_i); \quad N_c = \int dr|\phi_0(r)|^2. \]  

(1.18)

Making a change of variable \( \beta\hbar \omega_0 n_x \equiv \bar{n}_x \), and using the continuum approximation, we have

\[ N - N_c \simeq \left( \frac{k_B T}{\hbar\omega_0} \right)^3 \int_0^\infty d\bar{n}_x \int_0^\infty d\bar{n}_y \int_0^\infty d\bar{n}_z \frac{1}{e^{(\bar{n}_x+\bar{n}_y+\bar{n}_z)} - 1} \]  

\[ \int_0^\infty d\bar{n}_x \int_0^\infty d\bar{n}_y \int_0^\infty d\bar{n}_z \frac{1}{e^{(\bar{n}_x+\bar{n}_y+\bar{n}_z)} - 1} \]
We note that the chemical potential \( \mu_0 = \frac{3}{2} \hbar \omega_0 \), but \( \frac{\hbar \omega_0}{k_B T} \ll 1 \) and hence the zero point energy has been neglected in (1.19). Since \( N_c = 0 \) at \( T_{BEC} \), we have

\[
k_B T_{BEC} \simeq 0.94(N_1^{\frac{1}{3}})\hbar \omega_0, \text{ with } \frac{N_c(T)}{N} = \left[ 1 - \left( \frac{T}{T_{BEC}} \right)^3 \right]. \tag{1.20}
\]

One can improve on these simple estimates for \( T_{BEC} \) and \( N_c(T) \), but the corrections are only a few percent at best \([1]\).

Interactions make a dilute, weakly-interacting Bose condensed gas into a full non-trivial many body problem, even though the system is very dilute and the interactions are weak. In a dilute gas, we need only consider binary collisions. The real interatomic potential \( v(r) \) has a hard core with a radius of a few Angstroms and a weak, long-range attractive tail. In a dilute, very cold gas, we can approximate \( v(r) \) using the \( s \)-wave scattering length approximation effectively replacing \( v(r) \) by a pseudopotential \([23]\)

\[
v(r) \Rightarrow \frac{4\pi \hbar^2}{m} a \delta(r) \equiv g \delta(r). \tag{1.21}
\]

We require \( a \ll \text{average distance between atoms}, \) or \( na^3 \ll 1 \), which is very well satisfied in these gases. For alkali atoms, \( v(r) \) almost has a bound state of two atoms. This quasi-bound state is very sensitive to the long range part of the potential, and thus the value of the \( s \)-wave scattering length \( a \) can be very large. Current values for atoms used in BEC experiments are:

\[
\begin{align*}
\text{\(^{87}\text{Rb} \)} & : \ a = 58 \ \text{\AA} \\
\text{\(^{23}\text{Na} \)} & : \ a = 28 \ \text{\AA} \\
\text{\(^{7}\text{Li} \)} & : \ a = -14 \ \text{\AA} \tag{1.22}
\end{align*}
\]

One can adjust the energy of the quasi-bound state and, as a result, change the value of \( s \)-wave scattering length \( a \) with a small magnetic field. Near a so-called Feshbach resonance, one can even change the interaction sign, going from repulsive \( (a > 0) \) to attractive \( (a < 0) \). There is a lot of current work \([24]\) trying to exploit this ability to change the interaction strength and sign by “turning a knob”, perhaps even making \( a \to 0! \)

The key equation for the macroscopic wavefunction for a \( T = 0 \) condensate was written down and discussed by Pitaevskii \([25]\) and Gross \([26]\) in 1961:

\[
i\hbar \frac{\partial \Phi(r,t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + gn_c(r,t) \right] \Phi(r,t), \tag{1.23}
\]
1. BEC AND THE NEW WORLD OF COHERENT MATTER WAVES

where $n_c = |\Phi|^2$. This equation describes the condensate atoms moving in
dynamic self-consistent Hartree field produced by the condensate,

$$V_H(r, t) = \int \, dr' \nu(r - r') n_c(r', t) = gn_c(r, t).$$

The non-linear GP equation (1.23) will be the subject of Section 2. Hundreds of papers have been written on it in the last four years. For $T < \sim 0.4 T_{BEC}$, it describes both the static properties and the dynamic fluctuations (linear and non-linear) very well, usually within a few percent [1, 2].

To complete this brief introduction, I mention several other research topics that make trapped Bose-condensed gases so exciting:

1. Alkali atoms have several different atomic hyperfine states. Apart from Cs, the alkali atoms have a nuclear spin $I = 3/2$ and an electron spin $S = 1/2$. Thus the total spin operator $F = I + S$ has values of 1 and 2, leading to 8 possible atomic states. One usually works with one of these atomic hyperfine states - trapped in a magnetic well. However, purely optical traps (MIT) using the dipole force of a laser beam can be used to trap low energy atoms in different hyperfine states. Thus one can now also deal with gases with several different states, ie, a multicomponent Bose gas. Moreover, one can induce transitions between different hyperfine states. One sees that spin is a new degree of freedom in such multicomponent Bose-condensed gases [1, 2].

2. Independent of the special “coherent” features of a Bose condensate, these trapped gases give us a source of high density, very cold atoms. Lene Hau [27] has used this high density to slow down the speed of light to that of slow car ($\sim$ 40 km/hour) using self-induced transparency. One can also switch on an optical lattice (produced by intersecting laser beams) on a trapped Bose gas [28]. Turning off the magnetic trap, the low energy atoms will occupy the potential minima of this periodic lattice. This could not be done with high energy atoms since the dipole-induced potentials of the optical lattice are very weak. With ultra-cold trapped fermions, one may also be able to produce a Hubbard model, of the kind extensively studied in connection with the cuprate-oxide high temperature superconductors.

We agree with the opinion of Pitaevskii [29] that the discovery of BEC in alkali gases “can be considered as one of the most beautiful results of experimental physics in our century”. The next two sections will flesh out this qualitative overview with some theoretical calculations on the collective oscillations at $T \ll T_{BEC}$ (pure condensate) and at finite temperatures $T \sim T_{BEC}$ (mixture of condensate and non-condensate) of these strange quantum “wisps of matter”.
2 DYNAMICS OF THE PURE CONDENSATE

The theory of interacting Bose-condensed fluids is most usefully discussed using quantum field operators. This procedure was formalized by Beliaev (1957) and developed by Bogoliubov [30], Gavoret and Nozières [31], Martin and Hohenberg [32], and others in the 1960’s [33]. We recall:

\[ \hat{\psi}^+(r) = \text{creates atom at } r \]
\[ \hat{\psi}(r) = \text{destroys atom at } r. \]  
(1.25)

These fields satisfy the usual Bose commutation relations, such as

\[ \left[ \hat{\psi}(r), \hat{\psi}^+(r') \right] = \delta(r - r'). \]  
(1.26)

All observables can be written in terms of these quantum field operators, such as the interaction energy

\[ \hat{V}_{\text{ext}} = \frac{1}{2} \int dr \int dr' \hat{\psi}^+(r')\hat{\psi}^+(r)\psi(r - r')\psi(r)\psi(r) \]
\[ = \frac{1}{2} g \int dr \hat{\psi}^+(r)\hat{\psi}^+(r)\psi(r)\psi(r). \]  
(1.27)

The crucial idea due to Bogoliubov (1947) and later generalized by Beliaev is to separate out the condensate part

\[ \hat{\psi}(r) = \langle \hat{\psi}(r) \rangle + \tilde{\psi}(r), \]  
(1.28)

where

\[ \langle \hat{\psi}(r) \rangle \equiv \Phi(r) = \text{Bose macroscopic wavefunction}. \]  
(1.29)

This quantity plays the role of the order parameter for the superfluid phase transition:

\[ \Phi(r) = 0 \quad T > T_c \]
\[ \neq 0 \quad T < T_c. \]  
(1.30)

We note that \( \Phi(r) \equiv \sqrt{N_c} e^{i\theta} \) is a 2-component order parameter. Clearly, \( \Phi(r) \) is not simply related to the many-particle wavefunction \( \Psi(r_1, r_2, \ldots, r_N) \).

The thermal average in \( \langle \hat{\psi}(r) \rangle \) involves a small symmetry-breaking perturbation to allow \( \Phi \) to be finite,

\[ \hat{H}_{\text{SB}} = \int dr \left[ \eta(r)\hat{\psi}^+(r) + \eta^*(r)\hat{\psi}(r) \right]. \]  
(1.31)

It is useful to make a few comments on the physics behind \( \Phi(r, t) \). \( \Phi(r, t) \) is a coherent state, with a “clamped” value of phase - rather than a Fock-state of fixed \( N \), with no well-defined phase. \( \Phi(r, t) \) acts like a classical
1. BEC AND THE NEW WORLD OF COHERENT MATTER WAVES

field, since quantum fluctuations are negligible when $N_c$ is large. Probably P.W. Anderson deserves the greatest credit for understanding (in the period 1958-1963) the new physics behind working with a broken-symmetry state $\Phi(r,t)$, both in BCS superconductors and in superfluid $^4$He [34]. It captures the physics of the new phase of matter (such as the occurrence of the Josephson effect) and the associated superfluidity. The symmetry-breaking perturbation allows $<\hat{\psi}>$ to be finite. More precisely, it allows the system to internally set up off-diagonal symmetry-breaking fields, which persist even when the external symmetry-breaking perturbation in (1.31) is set to zero at the end ($\eta \to 0$). The same sort of physics is behind the BCS theory of superconductors.

The exact Heisenberg equation of motion for the field operator is

$$i\hbar \frac{\partial \hat{\psi}(r,t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + \delta V(r,t) \right] \hat{\psi}(r,t) + \eta(r) + g \hat{\psi}^+(r,t)\hat{\psi}(r,t)\hat{\psi}(r,t),$$

(1.32)

where $\delta V(r,t)$ is a small time-dependent driving potential. This gives an exact equation of motion for $\Phi(r,t) \equiv \langle \hat{\psi}(r,t) \rangle$,

$$i\hbar \frac{\partial \Phi(r,t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + \delta V(r,t) \right] \Phi(r,t) + \eta(r) + g \langle \hat{\psi}^+(r,t)\hat{\psi}(r,t)\hat{\psi}(r,t) \rangle,$$

(1.33)

with

$$\hat{\psi}^+\hat{\psi} = |\Phi|^2 \Phi + 2|\Phi|^2 \hat{\psi}\hat{\psi} + \Phi^*\hat{\psi}\hat{\psi} + 2\Phi\hat{\psi}^+\hat{\psi} + \hat{\psi}^+\hat{\psi} + \hat{\psi}^+\hat{\psi}.$$  

(1.34)

Taking the symmetry-breaking average, one finds

$$\langle \hat{\psi}^+\hat{\psi}\hat{\psi}\hat{\psi} \rangle = n_c\Phi + \vec{n}\Phi^* + 2\vec{n}\Phi + \langle \hat{\psi}^+\hat{\psi}\hat{\psi}\hat{\psi} \rangle,$$

(1.35)

where

$$n_c(r,t) \equiv |\Phi(r,t)|^2 = \text{condensate density}$$

$$\vec{n}(r,t) \equiv \langle \hat{\psi}^+ (r,t)\hat{\psi}(r,t) \rangle = \text{non-condensate density}$$

$$\vec{\vec{n}}(r,t) \equiv \langle \hat{\psi} (r,t)\hat{\psi}(r,t) \rangle = \text{off-diagonal (anomalous) density}.$$  

Here we have separated out the condensate and non-condensate parts

$$\hat{\psi} \Rightarrow \langle \hat{\psi} \rangle + \hat{\psi} = \Phi + \hat{\psi}.$$  

(1.36)

In general, the equation [1.33] for $\Phi(r,t)$ is not closed - it is coupled to the dynamics of the non-condensate. However, in this Section we limit
ourselves to $T \ll T_{BEC}$, where we can assume the non-condensate fraction is negligible, leaving
\[
\frac{i\hbar}{\partial t} \Phi(r, t) = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + g|\Phi(r, t)|^2 \right] \Phi(r, t).
\]  
This is the famous time-dependent Gross-Pitaevskii equation for the condensate macroscopic wavefunction. It gives a complete description of the dynamics of a coherent matter wave at $T = 0$.

2.1 Static condensate

We first consider the time-dependent stationary GP equation, which has the solution
\[
\langle \hat{\psi}(r, t) \rangle = \Phi(r, t) = \Phi_0(r) e^{-i\mu t/\hbar},
\]  
where $\mu$ is the chemical potential. The physics behind this can be seen from
\[
\langle N - 1|\hat{\psi}(r, t)|N \rangle = e^{iE_{N-1}t/\hbar} \langle N - 1|\hat{\psi}(r)|N \rangle e^{-iE_{Nt}/\hbar} = \langle N - 1|\sqrt{N}|N - 1 \rangle e^{-i(E_N - E_{N-1})t/\hbar} = \sqrt{N} e^{-i\mu t/\hbar}.
\]  
Using (1.38) in (1.37) gives
\[
\left( -\frac{\hbar^2 \nabla^2}{2m} - \mu + V_{ex}(r) + g|\Phi_0(r)|^2 \right) \Phi_0(r) = 0.
\]  
A simple approximation in solving (1.41) is to ignore the kinetic energy of the condensate, i.e., neglect the $-\frac{\hbar^2 \nabla^2}{2m}$ term. This is called the “Thomas-Fermi” approximation (TF) in the recent Bose gas literature [1]. In this TF approximation, the static GP (1.41) equation for $\Phi_0(r)$ reduces to
\[
V_{ex}(r) + g|\Phi_0(r)|^2 = \mu,
\]  
which is easily inverted to give the condensate density profile
\[
n_{c0}(r) = \frac{1}{g} \left[ \mu - V_{ex}(r) \right] = \frac{1}{g} \left[ \mu - \frac{1}{2} m\omega_0^2 r^2 \right] > 0.
\]
1. BEC AND THE NEW WORLD OF COHERENT MATTER WAVES

Clearly in the TF approximation, the “size” of the condensate is $R_{TF}$, where
\[ \mu = \frac{1}{2} m \omega_0^2 R_{TF}^2. \]  \hfill (1.44)

One finds $\mu$ from the condition $\int d\mathbf{r} n_c(\mathbf{r}) = N_c = N$, or
\[ N_c = 4\pi \int_0^{R_{TF}} dr r^2 \left( \mu - \frac{1}{2} m \omega_0^2 r^2 \right). \]  \hfill (1.45)

This gives
\[ \mu = \hbar \omega_0 \left( 15 \frac{N a}{a_{HO}} \right)^{2/5}; \quad a_{HO} \equiv (\hbar/m\omega_0)^{1/2}. \]  \hfill (1.46)

We note that SH oscillator length $a_{HO}$ is the size of the ground state wavefunction (1.13) of an atom in a parabolic potential. Combining (1.45) and (1.44) gives
\[ R_{TF} = a_{HO} \left( 15 \frac{N a}{a_{HO}} \right)^{1/5} \]
\[ \gg a_{HO}, \text{ if } \frac{N a}{a_{HO}} \gg 1. \]  \hfill (1.47)

We thus find the surprising result that interactions (while weak) spread out the ideal gas condensate ($R_{TF} \gg a_{HO}$) and decrease the density of the condensate at centre of trap. The TF approximation for $n_{c0}(\mathbf{r})$ is very good for large $N$, except for a small region near the edge of condensate ($\approx R_{TF}$). Experimental data confirms these GP predictions for the $n_{c0}(\mathbf{r})$ condensate profile, emphasizing that the condensate is not simply the ground state wavefunction of the harmonic trap potential.

2.2 Dynamics of the condensate (collective modes)

If we linearize around the static equilibrium value of the condensate
\[ \Phi(\mathbf{r}, t) = e^{-i\mu t/\hbar} [\Phi_0(\mathbf{r}) + \delta \Phi(\mathbf{r}, t)], \]  \hfill (1.48)

where $\delta \Phi \ll \Phi_0$, we see that
\[ i\hbar \frac{\partial \Phi}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(\mathbf{r}) + g \left[ |\Phi_0|^2 + \Phi_0^* \delta \Phi + \Phi_0 \delta \Phi^* \right] \right] \Phi_0 + \delta \Phi \right] e^{-i\mu t/\hbar}. \]  \hfill (1.49)

which gives
\[ i\hbar \frac{\partial \delta \Phi(\mathbf{r}, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ext}(\mathbf{r}) + 2g|\Phi_0|^2 - \mu \right] \delta \Phi(\mathbf{r}, t) + g \Phi_0^2 \delta \Phi^*(\mathbf{r}, t). \]  \hfill (1.50)
We also have a similar equation of motion for \( \delta \Phi^*(r, t) \). Solving these two coupled equations with the ansatz

\[
\delta \Phi(r, t) = u(r)e^{-i\omega t} + v(r)e^{i\omega t},
\]

we find two coupled “Bogoliubov equations” for the amplitudes \( u \) and \( v \):

\[
\begin{align*}
-\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) - \mu + 2gn_c(r) & \quad u(r) + gn_c(r)v(r) = E_iu(r) \\
-\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) - \mu + 2gn_c(r) & \quad v(r) + gn_c(r)u(r) = -E_iv(r)
\end{align*}
\]

Here \( E_i \equiv \hbar \omega \) are the excitation energies of the condensate.

The equations in (1.52) have been solved numerically by several groups and the observed oscillations are in good agreement with these predictions [1, 36]. As an illustration of the physics, it is useful to solve (1.52) for a uniform Bose gas. In this case we have

\[
\begin{align*}
u(r) &= u(r) = u e^{i k \cdot r} \\
v(r) &= v(r) = v e^{i k \cdot r},
\end{align*}
\]

which gives

\[
(\hbar \omega)^2 = \left[ \frac{\hbar^2 k^2}{2m} - \mu + 2gn_c \right]^2 - (gn_c)^2 = \epsilon_k^2 + 2gn_c \epsilon_k.
\]

This is the famous Bogoliubov spectrum at \( T = 0 \) [12, 35]. Here we have used \( \mu_0 = gn_c \) discussed earlier (see (1.42)). One finds a phonon region at long wavelengths

\[
\hbar \omega_k = \hbar v_B k; \quad v_B \equiv \left( \frac{gn_c}{m} \right)^{1/2}.
\]

The cross-over from particle-like to this collective phonon region occurs at \( k_c \), where

\[
\frac{\hbar^2 k_c^2}{2m} = 2gn_c \rightarrow k_c = \sqrt{4mn_c g/\hbar}.
\]

This shows how the interactions change the qualitative nature of low energy excitations in a Bose-condensed gas. This feature can be shown to stabilize superfluid motion against dissipation [13].

These oscillations of the condensate can be understood as excitations involving the non-condensate. Using (1.36), the Hamiltonian is given by

\[
\hat{H} - \mu \hat{N} = \int d\mathbf{r} \hat{\psi}^+(\mathbf{r}) \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(\mathbf{r}) - \mu \right] \hat{\psi}(\mathbf{r})
\]
\[
H = \int dr \Phi_0^* \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{ex}(r) - \mu \right) \Phi_0(r) 
+ 2g \int dr |\Phi_0(r)|^2 \psi^+(r) \psi(r) 
+ \frac{1}{2} g \int dr \Phi_0^2(r) \psi^+(r) \psi^+(r) 
+ \frac{1}{2} g \int dr \Phi_0^2(r) \psi^+(r) \psi(r) \psi^+(r),
\]
(1.57)

We can diagonalize the quadratic part of the Hamiltonian, using
\[
\psi(r) = \sum_i \left[ u_i(r) \hat{\alpha}_i + v_i^*(r) \hat{\alpha}_i^+ \right],
\]
(1.58)
where \([\hat{\alpha}_i, \hat{\alpha}_j^+] = \delta_{ij}\) (Boson quasiparticles). Thus one finds
\[
\hat{H} - \mu \hat{N} = \text{const.} + \sum_i \hbar \omega_i \hat{\alpha}_i^+ \hat{\alpha}_i.
\]
(1.59)

This transformation shows how the non-condensate part of Hamiltonian can be reduced to a system of non-interacting quasiparticles with a spectrum identical to the condensate fluctuations. This equivalence is easy to understand. The condensate fluctuations
\[
\delta \Phi = \langle \hat{\psi}(r) \rangle - \Phi_0
\]
(1.60)
can be calculated to first order in the symmetry-breaking perturbation (1.31),
\[
H_{sb} = \int dr \left[ \eta \hat{\psi}^+ + \eta^* \hat{\psi} \right].
\]
(1.61)
Then standard linear response theory [4] gives (schematically)
\[
\delta \Phi \sim \int < [\hat{\psi}, H_{sb}] >= \int < [\hat{\psi}, \hat{\psi}^+] > \eta + < [\hat{\psi}, \hat{\psi}] > \eta^*.
\]
(1.62)

This shows that the single-particle Green's functions of the non-condensate fields have the same spectrum as \(\delta \Phi\). This identity of the spectrum of density fluctuations and single-particle excitations is a characteristic signature of all Bose-condensed systems which persists at finite temperatures [4, 8].

One interesting collective oscillation is the dipole mode corresponding to rigid oscillation of the centre of mass of the static condensate profile, and predicted to have the trap frequency \(\omega_0\). This mode is described by
\[
n_c(r, t) = n_{c0}(r - \eta(t)), \quad \dot{\eta}(t) = v_c,
\]
(1.63)
where the time-dependent centre of mass satisfies
\[
\frac{\partial^2 \eta(t)}{\partial t^2} = -\omega_0^2 \eta(t). \tag{1.64}
\]
This mode at frequency \(\omega_0\) is a special feature of a parabolic trap and is called the Kohn mode for the case of interacting fermions \([33]\). This “sloshing mode” is used in BEC experiments to measure the natural frequency \(\omega_0\) of the trap and it exists at finite temperatures as well (see Section 3).

2.3 Quantum hydrodynamic formulation

One often rewrites the time-dependent GP equation using the amplitude and phase variables \(\Phi(\mathbf{r}, t) = \sqrt{n_c} e^{i\theta}. \tag{1.65}\)
Enter this into the GP equation \(1.37\) and separating out the real and imaginary parts of the equation gives:
\[
\frac{\partial n_c(\mathbf{r}, t)}{\partial t} + \nabla \cdot n_c(\mathbf{r}, t) \mathbf{v}_c(\mathbf{r}, t) = 0, \ \text{continuity equation} \tag{1.66}
\]
\[
\hbar \frac{\partial \theta(\mathbf{r}, t)}{\partial t} = -\left[ \mu_c(\mathbf{r}, t) + \frac{1}{2} m \mathbf{v}_c^2(\mathbf{r}, t) \right], \ \text{Josephson equation} \tag{1.67}
\]
Here the gradient of the phase is related to the superfluid velocity by
\[
m \mathbf{v}_c(\mathbf{r}, t) \equiv \hbar \nabla \theta(\mathbf{r}, t) \tag{1.68}
\]
and the condensate chemical potential is
\[
\mu_c(\mathbf{r}, t) \equiv -\frac{\hbar^2 \nabla^2 \sqrt{n_c}}{2m \sqrt{n_c}} + V_{\text{ex}}(\mathbf{r}) + g n_c(\mathbf{r}, t). \tag{1.69}
\]
Taking the gradient of \(1.67\) gives
\[
m \left( \frac{\partial v_c}{\partial t} + \frac{1}{2} \nabla v_c^2 \right) = -\nabla \mu_c. \tag{1.70}
\]
The equations in \(1.66\) and \(1.70\) “look” like those in classical hydrodynamic theories. They show that the condensate can be described in terms of coherent motions involving two variables:
\[
n_c(\mathbf{r}, t), \ \mathbf{v}_c(\mathbf{r}, t) \tag{1.71}
\]
The Landau 2-fluid equations \([9, 10]\) reduce to these same equations at \(T = 0\) (where \(\rho_s = \rho, \ \rho_n = 0\)), namely
\[
\frac{\partial n}{\partial t} + \nabla \cdot n \mathbf{v}_c = 0
\]
\[
m \left( \frac{\partial v_c}{\partial t} + \frac{1}{2} \nabla v_c^2 \right) = -\nabla \mu. \tag{1.72}
\]
1. BEC AND THE NEW WORLD OF COHERENT MATTER WAVES

We will find these equations useful in Section 3, where they complement the hydrodynamic equations describing the non-condensate in the collision-dominated region.

This approach also allows a simple theory developed by Stringari [40] for linearized collective modes when we use the TF approximation. Taking the time-derivative of (1.66) gives

\[ \frac{\partial^2 \delta n_c}{\partial t^2} = -\nabla \cdot \left[ n_c \left( \frac{\partial \delta v_c}{\partial t} \right) \right]. \]  

(1.73)

Using (1.70), we have

\[ \frac{\partial \delta v_c}{\partial t} = -\frac{1}{m} \nabla [V_{ex}(r) + g n_c(r) + g \delta n_c(r, t)] \]
\[ = -\frac{g}{m} \nabla \delta n_c(r, t). \]  

(1.74)

Combining this last result with (1.73), we obtain the very useful Stringari equation of motion [40]

\[ \frac{\partial^2 \delta n_c}{\partial t^2} = \nabla \cdot \left\{ \left[ \mu - \frac{1}{2} m \omega_0^2 \right] \nabla \delta n_c \right\}. \]  

(1.75)

This describes the collective oscillations of the condensate in terms of a single differential equation. As one example, the breathing mode of the condensate has a frequency \( \bar{\hbar} \omega = \sqrt{5} \bar{\hbar} \omega_0 \). This example points out that in the TF limit (large \( N_c \)), the frequencies are independent of the interaction strength and the size of the condensate \( N_c \).

We also note that using (1.74), (1.75) can be equally well rewritten in terms of the superfluid velocity \( v_c(r, t) \) defined in (1.68). This emphasizes that the condensate fluctuations are directly related to the existence of phase fluctuations. Their existence may thus be viewed as “evidence” of superfluidity, the latter being always a consequence of the phase coherence of the macroscopic wavefunction given by (1.65) [13].

The great thing about the collective oscillations of a condensate in a trapped gas is you can “see” them. A beautiful example from MIT is shown in Fig. 2 of Ref. [1]. As Ketterle has remarked, these condensates are robust—one can kick them, shake them and these “wisps” of Bose-condensed matter keep their integrity.

2.4 Interference of coherent matter waves

In the pioneering matter wave interference experiments done at MIT using a de-tuned cigar-shaped trap [41], one first destroys the condensate at the centre using laser beam. Then the confining trap is turned off and the two condensates are allowed to expand and interfere. One observes nice
interference fringes at the mid-point, as expected. Using
\[ \Phi_{\text{system}} = \Phi_A(r, t) + \Phi_B(r, t) \]
\[ = \sqrt{N_A} e^{i\theta_A} + \sqrt{N_B} e^{i\theta_B}, \]
(1.76)
the density is given by
\[ n(r, t) = |\Phi_{\text{system}}|^2 \]
\[ = N_A + N_B + 2 \sqrt{N_A N_B} \cos \Delta \theta, \]
(1.77)
where \( \Delta \theta = \theta_A - \theta_B \). In the region of interference, the density is low and hence interaction effects are small (i.e., the \( gn_c \) term is small in the GP equation). Asymptotically, the solution of GP equation gives \( \theta(r, t) = \frac{mr^2}{2\hbar t} \).
This implies \[ \Delta \theta = \frac{m(z + \frac{d}{2})^2}{2\hbar t} - \frac{m(z - \frac{d}{2})^2}{2\hbar t} = \frac{mzd}{\hbar t} \equiv \frac{2\pi z}{\lambda(t)}, \]
(1.78)
where \( \lambda(t) = \frac{2\pi \hbar t}{md} \). This wavelength is in good agreement with experimental observations (See Fig. 2 of Ref. [41]).

A condensate described by \( \Phi(r, t) \) may be viewed as a “classical” matter wave, as recently emphasized by Pitaevskii and Stringari [42]. This is quite different from ordinary quantum deBroglie waves, since one can ignore quantum fluctuations (large \( N_c \)). It is also quite different than ordinary (classical) macroscopic objects and electromagnetic waves, since \( \Phi(r, t) \) is described by the GP equation (1.37) which involves Planck’s constant \( \hbar \). Thus \( \Phi(r, t) \) is a classical object which is described by a quantum equation!! This promises to be a challenge for the quantum theory of measurement.

One can describe a two-component Bose gas (see Section [3]) using coupled equations:
\[ i\hbar \frac{\partial \Phi_1}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_1(r) + g_1|\Phi_1|^2 + g_{12}|\Phi_2|^2 \right] \Phi_1 \]
\[ i\hbar \frac{\partial \Phi_2}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_2(r) + g_2|\Phi_2|^2 + g_{12}|\Phi_1|^2 \right] \Phi_2. \]
(1.79)
There are two coupled GP equations for two macroscopic wave functions. Extensive studies [43] have been made at JILA using the two atomic hyperfine states of \(^{87}\text{Rb} \):
\[ |F = 1, m_F = -1 >, \quad |F = 2, m_F = 1 >. \]
(1.80)
In particular, one can study interesting interference effects between these coupled wave functions. Recent work at JILA has used such two-component Bose fluids to produce the long sought-for vortex state in one of the components [44].
In this Section, we switch our attention from $T = 0$ (i.e., $T \lesssim 0.4 \ T_{\text{BEC}}$) to finite temperatures, where $N_c$ and $\tilde{N}$ are comparable in size. We first consider how the GP equation of motion for $\Phi(r, t)$ is modified. As a first step, we could use [4, 45],

$$i\hbar \frac{\partial \Phi(r, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ex}}(r) + gn_c(r, t) + 2g\tilde{n}(r, t) \right] \Phi(r, t). \quad (1.81)$$

The last term takes into account that the condensate moves in the dynamic Hartree-Fock (HF) field produced by non-condensate atoms. Immediately, one sees this generalized GP equation is no longer closed. It requires a theory of the non-condensate fluctuations, as described by $\tilde{n}(r, t) = \tilde{n}_0(r) + \delta\tilde{n}(r, t)$.

A simpler version of (1.81) is to treat the effect of the non-condensate as a static HF field [38, 46]:

$$2g\tilde{n}(r, t) \simeq 2g\tilde{n}_0(r). \quad (1.82)$$

1. This corresponds to treating the condensate moving in a static HF field of the non-condensate.

2. $\tilde{n}_0(r)$ can be calculated (self-consistently) using the fluctuations of $\Phi$, as discussed in Section 4. One finds that the depletion of the condensate at $T = 0$ is only a few percent.

3. This procedure gives reasonable results for the thermodynamic properties at finite temperatures, as discussed in the recent literature [1]. Within the Thomas-Fermi approximation (good for $N \gtrsim 10^4$ atoms), the linearized version of (1.81) using (1.82) leads to the same Stringari equation at finite $T$ as the $T = 0$ result in (1.75). Since the solutions of (1.75) do not depend on the magnitude of the condensate, one concludes that the collective modes of the condensate will show no temperature dependence, even though the condensate is being thermally depleted. This prediction does not appear to agree with experimental results when $T \gtrsim 0.6 \ T_{\text{BEC}}$. This suggests that the dynamics of the non-condensate has to be included.

We now go on to determining $\tilde{n}(r, t)$ directly by deriving a quantum Boltzmann equation for the single-particle distribution function of excited atoms $f(p, r, t)$ and then use:

$$\tilde{n}(r, t) \equiv \int \frac{dp}{(2\pi)^3} f(p, r, t). \quad (1.83)$$
This procedure generalizes the approach of Boltzmann (1880’s) for a classical gas, including the effect of binary collisions. Clearly one must make some approximations! One wants, initially, to find a useful kinetic equation that builds in just enough physics. Here I will discuss such a quantum Boltzmann equation for a trapped Bose-condensed gas at finite temperatures, which has been extensively discussed by Zaremba, Nikuni and the author [5, 47]. It is only valid in the so-called semi-classical limit, where it is sufficient to work with \( f(p, r, t) \). The conditions are

\[
k_B T \gg g n, \quad k_B T \gg \hbar \omega_0.
\]  

(1.84)

In this domain, one also can assume that the important thermal excitations can be approximated by simple Hartree-Fock particle-like spectrum:

\[
\tilde{\varepsilon}(r, t) = \frac{p^2}{2m} + 2g [n_c(r, t) + \bar{n}(r, t)] + V_{ex}(r) \equiv \frac{p^2}{2m} + U(r, t).
\]  

(1.85)

Clearly the resulting kinetic equation is not valid at very low temperatures, where the thermal excitations are described by a Bogoliubov-type spectrum.

We simply write down our quantum kinetic equation [5],

\[
\frac{\partial f(p, r, t)}{\partial t} + \frac{p \cdot \nabla_r f(p, r, t) - \nabla_r U(r, t) \cdot \nabla_p f(p, r, t)}{m} = C_{22}[f] + C_{12}[f].
\]

(1.86)

The right hand side describes how binary collisions effect the value of the single-particle distribution function \( f(p, r, t) \). The effect of collisions between excited atoms in the non-condensate is described by:

\[
C_{22}[f] = \frac{2g^2}{(2\pi)^5 \hbar^7} \int dp_2 \int dp_3 \int dp_4 \delta(p + p_2 - p_3 - p_4) \times \\
\times \delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_p - \tilde{\varepsilon}_p - \tilde{\varepsilon}_p) \times \left[(1 + f)(1 + f_2)f_3f_4 - f f_2(1 + f_2)(1 + f_4)\right].
\]

(1.87)

This collision integral was discussed in detail in 1933 by Uehling and Uhlenbeck for \( T > T_{BEC} \) [48]. We recall that creating a Boson gives a factor \( (1 + f) \) and destroying a Boson gives \( f \). In the classical high temperature limit, \( f \ll 1 \) and the collision integral \( C_{22} \) considerably simplifies.

Where does \( f(p, r, t) \) come from in a microscopic derivation of (1.86)? Basically we calculate the non-equilibrium real-time single-particle Green’s functions of the non-condensate field operators (using the Kadanoff-Baym formalism [49]). This gives (schematically)

\[
g_1(1, 1') \sim \left< \hat{\psi}^+(1)\hat{\psi}(1') \right>.
\]

(1.88)
where \( 1 \equiv r_1, t_1; \quad 1' = r'_1, t'_1 \). We then express this as \( g_1(r, t; R, T) \), where the relative and centre of mass coordinates are

\[
\begin{align*}
  r &= r_1 - r'_1; \quad R = \frac{1}{2} (r_1 + r'_1) \\
  t &= t_1 - t'_1; \quad T = \frac{1}{2} (t_1 + t'_1).
\end{align*}
\] (1.89)

Finally we Fourier transform \( g_1(r, t; R, T) \) to find \( g_1(p, \omega; R, T) \), which gives the number of atoms at \( R, T \) with momentum \( p \) and energy \( \hbar \omega \). The single-particle Wigner distribution function is given by

\[
 f(p, R, T) \equiv \int_{-\infty}^{\infty} d\omega g_1(p, \omega; R, T). \tag{1.90}
\]

The Wigner distribution function \( f(p, R, T) \) is the quantum generalization the classical single-particle distribution function \([49]\). These remarks should indicate how we can go from an equation of motion for the single-particle Green’s function (within a given self-energy approximation) to a kinetic equation for \( f(p, R, T) \). We refer to the classic account given (for non-Bose-condensed gases) in the book by Kadanoff and Baym \([49]\) for further details. This powerful approach was generalized to uniform Bose-condensed gases by Kane and Kadanoff \([50]\), and has been extended to trapped gases in recent work \([51]\).

In addition to \( C_{22} \) collisions, we also have collisions which involve one condensate atom:

\[
 C_{12} = \frac{2g^2}{(2\pi\hbar)^2} \int dp_1 \int dp_2 \int dp_3 \delta(mv_c + p_1 - p_2 - p_3) \\
 \times \delta((\varepsilon_c + \varepsilon_{p_1} + \varepsilon_{p_2} - \varepsilon_{p_3}) - \delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3)) \\
 \times [n_c(1 + f_1)f_2f_3 - n_c f_1(1 + f_2)(1 + f_3)]. \tag{1.91}
\]

Here the condensate atom has

- energy: \( \varepsilon_c = \mu_c + \frac{1}{2}mv_c^2 \); \( \mu_c = V_{ex} + gn_c + 2g\tilde{n} \)
- momentum: \( p_c = mv_c \) \tag{1.92}

We note the key difference between \( C_{12} \) and \( C_{22} \) collisions:

- \( C_{22} \) and \( C_{12} \) conserve energy and momentum in collisions.
- \( C_{12} \) does not (but \( C_{22} \) does) conserve the number of condensate atoms. \( C_{12} \) describes how atoms are “kicked” in and out of condensate.

It turns out the generalized GP equation \([1.81]\) is also modified by a term related to \( C_{12}[f] \). This makes sense, since the \( C_{12} \) collisions modify
the condensate wavefunction $\Phi(r, t)$. One finds the new GP equation is given by (see also Ref. [52]):

$$i\hbar \frac{\partial \Phi(r, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m} + V_{ex}(r) + g n_c(r, t) + 2g \tilde{n}(r, t) - iR(r, t) \right] \Phi(r, t),$$

(1.93)

where

$$R(r, t) \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} C_{12}[f(\mathbf{p}, r, t)]$$

(1.94)

More precisely, the dissipative $iR$ term in (1.93) arises from a three field correlation function in the exact equation of motion [see (1.33) and (1.35)], and is given by [5]

$$\int \frac{d\mathbf{p}}{(2\pi)^3} C_{12}[f] = \frac{2g}{\hbar} \sqrt{n_c} \text{Im} \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle.$$  

(1.95)

We have to solve for $f(\mathbf{p}, r, t)$ and $\Phi(r, t)$, treating $C_{12}[f]$ very carefully. We see that there will be an exchange of atoms between the $\tilde{n}(r, t)$ and $n_c(r, t)$ components through the $C_{12}$ collisions. We can use these coupled equations for a variety of problems. In these lectures, we will consider the collective oscillations of the combined system composed of condensate and non-condensate. It is useful to introduce two regimes to describe collective modes in interacting systems [4, 53]:

I. Collisionless (produced by mean fields)

$$\omega \tau_R \gg 1 \text{ or } T \ll \tau_R \left( \omega = \frac{2\pi}{T} \right)$$

II. Hydrodynamic (produced by collisions)

$$\omega \tau_R \ll 1 \text{ or } T \gg \tau_R,$$

where $\tau_R$ is some appropriate relaxation time. What should we use for $\tau_R$? For a classical gas, this is the collision time [6]

$$\frac{1}{\tau_c} = \tilde{n} \sigma \bar{v}$$

(1.96)

where

$$\sigma = 8\pi a^2 \text{ (for Bose particles); } a = s\text{-wave scattering length.}$$

$$\bar{v} \simeq \text{average velocity of atoms } \sim \sqrt{\frac{k_B T}{m}}.$$  

$\tilde{n} = \text{ density of excited atoms.}$
Even for a Bose-condensed gas, taking $\tau_R \sim \tau_c$ is a reasonable first estimate \[47\]. To get into the interesting hydrodynamic region ($\omega \tau_R \ll 1$), we need small $\tau_R$, i.e., a large density $\tilde{n}$ or a large collision cross-section $\sigma$ (perhaps using a Feshbach resonance, as discussed in Section 3).

Let us look at the kinetic equation (1.86), writing it in the schematic form:

$$\hat{L} f = C_{22}[f] + C_{12}[f].$$  \tag{1.97}

In the collisionless region, we need only solve $\hat{L} f = 0$. In contrast, in the hydrodynamic region, the collisions are so strong they produce local equilibrium \[6\]. That is, they force $f$ to satisfy $C_{22}[f] = 0$. The unique solution $\tilde{f}$ of this equation is well-known to be given by

$$\tilde{f}(p, r, t) = \frac{1}{e^{\beta(\tilde{\mu} - \mu_c)} - 1},$$  \tag{1.98}

where $v_n$ is the average local velocity and $\tilde{\mu}$ is the local chemical potential of the thermal atoms. This local equilibrium Bose distribution involves the local variables $\beta, v_n, \tilde{\mu}$ and $U$, all of which depend on $(r,t)$.

Why must $\tilde{f}$ have the form in (1.98)? To satisfy $C_{22}[f_1] = 0$, we must have [see (1.87)]

$$(1 + f_1)(1 + f_2)f_3f_4 - f_1f_2(1 + f_3)(1 + f_4) = 0,$$  \tag{1.99}

and this requires that $f$ be given by the Bose distribution. We have used the fact that

$$f(x) = \frac{1}{e^x - 1} = -[f(-x) + 1]$$  \tag{1.100}

and that

$$\begin{align*}
P_1 + P_2 &= P_3 + P_4 \quad \text{energy and momentum conservation,} \\
\tilde{\epsilon}_{P_1} + \tilde{\epsilon}_{P_2} &= \tilde{\epsilon}_{P_3} + \tilde{\epsilon}_{P_4}
\end{align*}$$  \tag{1.101}

where $\tilde{\epsilon}_P = \frac{p^2}{2m} + U(r,t)$. As an aside, using a kinetic equation is the most physical way of deriving the equilibrium Bose distribution. The standard approach in statistical mechanics texts based on calculating a partition function does not bring out the reason why

$$f_{B,F} = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}.$$  \tag{1.102}

However, while the fact that $\tilde{f}$ is given by the local equilibrium Bose distribution in (1.98) ensures that $C_{22}[\tilde{f}] = 0$, one finds that $C_{12}[\tilde{f}] \neq 0$. More precisely, we find from (1.91)

$$\left[(1 + \tilde{f}_1)\tilde{f}_2\tilde{f}_3 - \tilde{f}_1(1 + \tilde{f}_2)(1 + \tilde{f}_3)\right]$$

$$\propto \left[\frac{1}{e^{\beta(\tilde{\mu} - \mu_c)} - 1}(1 + \tilde{f}_1)\tilde{f}_2\tilde{f}_3\right].$$  \tag{1.103}
The expression in the square bracket only vanishes if the condensate and non-condensate are in \textit{diffusive} equilibrium, which requires that

$$\tilde{\mu} = \mu_c + \frac{1}{2}m(v_n - v_c)^2. \quad (1.104)$$

When we \textit{perturb} the system, this may not be true, ie, the two components may be out of diffusive equilibrium.

We can now derive hydrodynamic equations for non-condensate by taking moments of Boltzmann equation, the standard procedure used in classical gases \footnote{[6]}. The first moment gives a continuity equation with a source term:

$$\int dp \left\{ C \tilde{f} = C_{12}[\tilde{f}] \right\} = \frac{\partial \tilde{n}}{\partial t} = -\nabla \cdot (\tilde{n} \mathbf{v}_n) + \Gamma_{12}[\tilde{f}], \quad (1.105)$$

where

$$\tilde{n} \equiv \int \frac{dp}{(2\pi)^3} \tilde{f}(p, \mathbf{r}, t)$$

$$\tilde{n} \mathbf{v}_n \equiv \int \frac{dp}{(2\pi)^3} \frac{p}{m} \tilde{f}(p, \mathbf{r}, t)$$

$$\Gamma_{12}[\tilde{f}] \equiv \int \frac{dp}{(2\pi)^3} C_{12}[\tilde{f}]. \quad (1.106)$$

More explicitly, we find

$$\Gamma_{12}[\tilde{f}] = \frac{2\gamma^2 n_c}{(2\pi)^3 \hbar^7} \left[ e^{-\beta[\tilde{\mu} - \tilde{\mu}_c - \frac{1}{2} m(v_n - v_c)^2]} - 1 \right]$$

$$\times \int dp_1 \int dp_2 \int dp_3 \delta(m \mathbf{v}_c + p_1 - p_2 - p_3)$$

$$\times \delta(\tilde{\epsilon}_c + \tilde{\epsilon}_1 - \tilde{\epsilon}_2 - \tilde{\epsilon}_3)(1 + \tilde{f}_1)\tilde{f}_2\tilde{f}_3$$

$$\equiv \left[ e^{-\beta[\tilde{\mu} - \tilde{\mu}_c - \frac{1}{2} m(v_n - v_c)^2 - 1]} - 1 \right] \frac{n_c}{\tau_{12}}. \quad (1.107)$$

We note that \(\tau_{12}\) is a collision time \footnote{[7]} which describes the \(C_{12}\) collisions between the C and N.C. atoms. Combining (1.105) with the continuity equation which results from (1.93),

$$\frac{\partial n_c}{\partial t} = -\nabla \cdot (n_c \mathbf{v}_c) - \Gamma_{12}[\tilde{f}], \quad (1.108)$$

we see that the source term \(\Gamma_{12}\) cancels out to give

$$\frac{\partial (n_c + \tilde{n})}{\partial t} = -\nabla \cdot (n_c \mathbf{v}_c + \tilde{n} \mathbf{v}_n). \quad (1.109)$$

Thus our theory gives the exact continuity equation for the total local density \(n = n_c + \tilde{n}\).
Similarly, one finds

\[ \int d\mathbf{p} \left\{ \mathcal{L} \tilde{f} = C_{12}[\tilde{f}] \right\} \rightarrow m\tilde{n} \left( \frac{\partial \mathbf{v}_n}{\partial t} + \frac{1}{2} \nabla v_n^2 \right) \]

\[ = -\nabla \tilde{P}(\mathbf{r}, t) - \tilde{n} \nabla U(\mathbf{r}, t) - m(\mathbf{v}_n - \mathbf{v}_c) \Gamma_{12}[\tilde{f}], \quad (1.110) \]

where the kinetic pressure is given by

\[ \tilde{P}(\mathbf{r}, t) = \frac{m}{3} \int \frac{d\mathbf{p}}{(2\pi)^3} (\mathbf{p} - m\mathbf{v}_n)^2 \tilde{f}(\mathbf{p}, \mathbf{r}, t). \quad (1.111) \]

The second moment gives

\[ \int d\mathbf{p} p^2 \left\{ \mathcal{L} \tilde{f} = C_{12}[\tilde{f}] \right\} \rightarrow \frac{\partial \tilde{P}}{\partial t} + \nabla \cdot (\tilde{P} \mathbf{v}_n) \]

\[ = -\frac{2}{3} \tilde{P} \nabla \cdot \mathbf{v}_n + \frac{2}{3} \left[ \mu_c + \frac{1}{2} m(\mathbf{v}_n - \mathbf{v}_c)^2 - U \right] \Gamma_{12}[\tilde{f}], \quad (1.112) \]

The detailed derivation of these results is not important here. The main thing is that the hydrodynamic equations (1.105), (1.110) and (1.112) can be shown to describe the non-condensate in terms of three new "coarse-grained" variables:

\[ \tilde{n}(\mathbf{r}, t), \mathbf{v}_n(\mathbf{r}, t) \text{ and } \tilde{P}(\mathbf{r}, t). \]

These are coupled to the two additional variables which describe the condensate:

\[ n_c(\mathbf{r}, t), \mathbf{v}_c(\mathbf{r}, t). \]

We note that the two condensate equations of motion given by (1.70) and (1.108) are always "hydrodynamic" in form. In contrast, it is only in the collision-dominated region that the non-condensate dynamics can be described in terms of a few collective variables. We thus have 5 variables and 5 equations, which form a closed system. Both components exhibit coupled, coherent collective motions. This is the essence of two-fluid superfluid behaviour, a new unexplored frontier in trapped Bose gases.

What is new about the two-fluid hydrodynamic equations derived above is the role of the source term \( \Gamma_{12}[\tilde{f}] \). In a linearized theory expanded around the static equilibrium Bose distribution \( \hat{f}_0 \) (where \( \Gamma_{12}[\hat{f}_0] \) vanishes), one finds \[ \Gamma_{12}[\tilde{f}] = \delta \Gamma_{12}[\tilde{f}] = -\frac{\beta_0 n_{0c}}{\tau_{12}} \delta \mu_{diff}, \quad (1.113) \]

where \( \mu_{diff} \equiv \tilde{\mu} - \mu_c \). We find an equation of motion of the kind

\[ \frac{\partial \delta \mu_{diff}}{\partial t} = -\frac{\delta \mu_{diff}}{\tau_{12}} + \ldots, \quad (1.114) \]

where (see Eq.(87) in Ref. [3]).
\[ \frac{1}{\tau_\mu} \equiv \left( \frac{g n_{c0}}{k_B T} \right) \frac{1}{\sigma} \frac{1}{\tau_{12}}. \]  

(1.115)

Here \( \sigma \) involves various static equilibrium thermodynamic functions. The new relaxation time \( \tau_\mu \) (which we can calculate!) determines how fast \( \tilde{\mu} \to \mu_c \), ie, how fast we reach diffusive equilibrium between the condensate and non-condensate. We can have

\[
\begin{align*}
\omega \tau_{22} &\ll 1 \\
\omega \tau_{12} &\ll 1
\end{align*}
\]

required for hydrodynamics  

(1.116)

but simultaneously

\[ \omega \tau_\mu \gg 1 \]  

(1.117)

near \( T_{BEC} \), where \( n_{c0} \to 0 \). Our hydrodynamic equations predict the existence of a new relaxational mode \[5, 47\]

\[ \omega \approx -i/\tau_\mu. \]  

(1.118)

This mode is not included in the standard Landau 2-fluid equations (where \( \rho_s \) and \( \rho_n \) are assumed to be always in local equilibrium with each other).

In a uniform gas, the two-fluid hydrodynamic equations give two normal mode solutions \[54, 5\]:

- First sound (oscillation of the non-condensate mainly)
  
  \[ \omega = u_1 k, \ u_1^2 \approx \frac{5}{3 m n_{c0}} \sim \frac{kT}{m}. \]  

(1.119)

- Second sound (oscillation of the condensate mainly)
  
  \[ \omega = u_2 k, \ u_2^2 \approx \frac{g n_{c0}}{m}. \]  

(1.120)

We note the second sound mode is the hydrodynamic version of famous \( T = 0 \) Bogoliubov phonon mode discussed in Section \[. It is the “soft mode” at \( T_{BEC} \). This second sound mode couples to the new relaxational mode given in \(1.118\) and is damped as a result, the maximum damping occurring when \( \omega \tau_\mu = 1 \).

In a trapped gas, we can work out the spectrum of hydrodynamic oscillations (\( \sim e^{-i\omega t} \)). Both the condensate and non-condensate components have the same frequency. The most interesting one is the dipole mode, described by

\[
\begin{align*}
\tilde{n}(r,t) &= \tilde{n}_0(r - \eta_n(t)), \quad \tilde{\eta}_n(t) = v_n \\
n_c(r,t) &= n_{c0}(r - \eta_c(t)), \quad \dot{\eta}_c(t) = v_c.
\end{align*}
\]

(1.121)

One finds there are two modes of this kind \[\[45, 5\]:
1. BEC AND THE NEW WORLD OF COHERENT MATTER WAVES

• In-phase (or Kohn) mode, where \( \eta_n = \eta_c \) and \( \omega = \omega_0 \) (trap frequency). It is the finite temperature version of the sloshing mode described by (1.63) and (1.64). We note that this mode is generic (occurring in both the hydrodynamic and collisionless limit) and is not damped [5].

• Out-of-phase dipole mode, with \( \eta_n \neq \eta_c \) and in opposite directions. The frequency of this mode is different from the trap frequency. This out-of-phase mode is of special interest since it is the analogue of the out-of-phase second sound mode in superfluid \( ^4\text{He} \).

We conclude this Section with some remarks:

1. The specific calculation sketched above is also of interest in the general field of non-equilibrium statistical physics. It describes the detailed dynamics of a system with a two-component order parameter self-consistently coupled to a gas of excitations based on a fully microscopic theory.

2. More work is needed to extend our analysis to low but finite temperatures and also into the critical region very close to \( T_{\text{BEC}} \). In both cases, our simple Hartree-Fock particle-like thermal excitation spectrum (1.85) is no longer valid.

3. The classical kinetic theory of gases has been a rich subject in mathematical physics in the twentieth century, with well-known contributions by people like Boltzmann, Hilbert, Enskog, Chapman, Uhlenbeck and Burnett. These new equations of motion for a Bose-condensed gas promise to yield a lot of new physics in the next century - and surprises, as our work in this Section has already shown.

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