Abstract

We derive \( p+1 \)-dimensional \((p=1,2)\) maximally supersymmetric \( U(N) \) Yang-Mills theory from the wrapped supermembrane on \( R^{11-p} \times T^p \) in the light-cone gauge by using the matrix regularization. The elements of the matrices in the super Yang-Mills theory are given by the Fourier coefficients in the supermembrane theory. Although our approach never refers to both D-branes and superstring dualities, we obtain the relations which exactly represent T-duality.

1 Introduction

Supermembrane in eleven dimensions \cite{1} is believed to play an important role to understand the fundamental degrees of freedom in M-theory. Actually, the matrix-regularized light-cone supermembrane on \( R^{11} \) \cite{2,3}, which is 0+1-dimensional maximally supersymmetric \( U(N) \) Yang-Mills theory, is conjectured to describe the light-cone quantized M-theory in the large-\( N \) limit \cite{4}. Furthermore, Susskind suggested that even at finite \( N \), the super Yang-Mills theory describes the \( p^+ = N/R \) sector of discrete light-cone quantized (DLCQ) M-theory \cite{5}. \(^1\)

Hence, the super Yang-Mills theory is called Matrix theory, or M(atrix) theory, by identifying \( N \) of the gauge group \( U(N) \) with that of the light-cone momentum \( p^+ = N/R \). Susskind’s conjecture was explained in Refs.\cite{6,7}, where Matrix theory is interpreted as the low energy effective theory of \( N \) D0-branes and the 11th direction is chosen as the longitudinal direction. Thus, Matrix theory is looked on either as the matrix-regularized theory of the light-cone supermembrane on \( R^{11} \) or the low energy effective theory of D0-branes.

From the latter viewpoint, which we call the \( D\)-brane viewpoint, we can easily carry out the toroidal compactification of (the transverse directions of) Matrix theory. In fact, through the T-duality prescription à la Taylor \cite{8}, Matrix theory compactified on a circle is the low energy effective theory of \( N \) D1-branes, i.e., the 1+1-dimensional maximally supersymmetric \( U(N) \) Yang-Mills theory. The super Yang-Mills theory is called matrix string theory, which is conjectured to describe the light-cone quantized type-IIA superstring theory in the large-\( N \) limit \cite{9,10}. It is also conjectured to describe the \( p^+ = N/R \) sector of DLCQ type-IIA

\(^1\)In this paper we use a convention of the light-cone coordinates \( x^\pm \equiv (x^0 \pm x^{10})/\sqrt{2} \). Then, in DLCQ, \( x^- \) is compactified on \( S^1 \) with radius \( R \).
superstring theory even at finite $N$ \cite{5}. This proposal is explained by using the T- and S-dualities with the 9-11 flip of interchanging the role of the 11th and 9th directions \cite{9,10}. Furthermore, it is straightforward to compactify Matrix theory on $T^p$ ($p \leq 3$) and the resulting theory is the low energy effective theory of $N$ $D_p$-branes, i.e., the $p+1$-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory \cite{11,12,13}.

On the other hand, from the former point of view, which we call the \textit{membrane viewpoint}, it is not so straightforward to compactify Matrix theory on a torus. Actually, it is only recently that matrix string theory has deduced via matrix regularization of the light-cone wrapped supermembrane on $R^{10} \times S^1$ \cite{14,15,16}.\footnote{See Refs.\cite{17,18,19,20} for recent other approaches to the membrane theory.} Furthermore, Matrix theory compactified on higher dimensional torus from the membrane viewpoint is not known yet.

The purpose of this paper is to drive systematically Matrix theory compactified on $T^p$ ($p = 1, 2$) \textit{from the membrane viewpoint}. Namely, we derive $p+1$-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory through the matrix regularization of the wrapped supermembrane on $R^{11-p} \times T^p$. In the case of $p = 1$, it was already presented in Refs.\cite{14,15,16}. However, our explanation in this paper is improved and is successfully extended to the case of $p = 2$. Furthermore, we naturally obtain the relations which exactly agree with those inferred from T-duality. We also derive the dimensionless gauge coupling constant in the super Yang-Mills theory and find that it agrees with that obtained in Ref.\cite{13}, where they derived it from the D-brane viewpoint. Thus, this gives a consistency check of two kinds of approaches in the toroidal compactification of Matrix theory.

The plan of this paper is as follows. We start by giving the mode expansions of eleven-dimensional supermembrane in the light-cone gauge. In section 3, just as a warm-up, we present the derivation of 0+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory through the matrix regularization of the supermembrane on $R^{11}$. In section 4, we give a simpler derivation of 1+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory from the wrapped supermembrane on $R^{10} \times S^1$ \cite{14,15,16}. We also derive a certain relation which implies T-duality. In section 5, we go on to 2+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory with the wrapped supermembrane on $R^9 \times T^2$. Final section is devoted to summary and discussion.

## 2 Supermembrane in the light-cone gauge

Our starting point is the action of the eleven-dimensional supermembrane in the light-cone gauge,\footnote{Precisely speaking, when the membrane has a non-trivial space-sheet topology, we need to impose the global constraints to the action \cite{21,21}. For simplicity, such constraints are ignored in this paper.} (Here we just write it only with the bosonic degrees of freedom. Fermions are straightforwardly included.)

\begin{equation}
S = \frac{LT}{2} \int d\tau \int_0^{2\pi} d\sigma d\rho \left[ (D_\tau X^i)^2 - \frac{1}{2L^2} \{X^i, X^j\}^2 \right], \tag{2.1}
\end{equation}

\begin{equation}
D_\tau X^i = \partial_\tau X^i - \frac{1}{L} \{A, X^i\}, \tag{2.2}
\end{equation}

\begin{equation}
\{A, B\} \equiv \partial_\sigma A \partial_\rho B - \partial_\rho A \partial_\sigma B, \tag{2.3}
\end{equation}
where \( i,j = 1,2,\cdots,9, \) \( T \) is the tension of the supermembrane and \( L \) is an arbitrary length parameter.\(^4\) It is easy to see that the parameter \( L \) can be changed for \( L' \) just by a rescaling of \( \tau \rightarrow \tau L/L' \). Eq.\(^{23}\) represents a gauge theory of the area-preserving diffeomorphism. In this paper, we consider three kinds of the eleven-dimensional target space: \( R^{11}, R^{10} \times S^1 \) and \( R^9 \times T^2 \). On \( R^{11} \), the target-space coordinates \( X^i \) and gauge field \( A \) are expanded as\(^5\)

\[
X^i(\sigma, \rho) = \sum_{k_1,k_2=-\infty}^{\infty} X^i_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho},
\]

\[
A(\sigma, \rho) = \sum_{k_1,k_2=-\infty}^{\infty} A_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho}.
\]

On \( R^{10} \times S^1 \), we can consider the wrapped supermembrane. We take \( X^9 \) as a coordinate of \( S^1 \) with the radius \( L_1 \). Then \( X^i \) \( (i = k,9 \ (k = 1,2,\cdots,8)) \) and \( A \) are expanded as

\[
X^9(\sigma, \rho) = w_1 L_1 \rho + \sum_{k_1,k_2=-\infty}^{\infty} Y^1_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho} \equiv w_1 L_1 \rho + Y^1(\sigma, \rho),
\]

\[
X^k(\sigma, \rho) = \sum_{k_1,k_2=-\infty}^{\infty} X^k_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho},
\]

\[
A(\sigma, \rho) = \sum_{k_1,k_2=-\infty}^{\infty} A_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho},
\]

where \( w_1 \) is an integer. If we regard \( X^9 \) as the 11th direction, this supermembrane corresponds to the light-cone type-IIA superstring on \( R^{10} \) via the double-dimensional reduction, which has a tension \( 1/(2\pi \alpha') = 2\pi L_1 T \)\(^6\).

Finally, we consider the wrapped supermembrane on \( R^9 \times T^2 \). We take \( X^8 \) and \( X^9 \) as the coordinates of two cycles of \( T^2 \). Then we can expand \( X^i \) \( (i = m,8,9 \ (m = 1,2,\cdots,7)) \) and \( A \) as

\[
X^9(\sigma, \rho) = w_1 L_1 \rho + \sum_{k_1,k_2=-\infty}^{\infty} Y^1_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho} = w_1 L_1 \rho + Y^1(\sigma, \rho),
\]

\[
X^8(\sigma, \rho) = w_2 L_2 \sigma + \sum_{k_1,k_2=-\infty}^{\infty} Y^2_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho} \equiv w_2 L_2 \sigma + Y^2(\sigma, \rho),
\]

\[
X^m(\sigma, \rho) = \sum_{k_1,k_2=-\infty}^{\infty} X^m_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho},
\]

\[
A(\sigma, \rho) = \sum_{k_1,k_2=-\infty}^{\infty} A_{(k_1,k_2)} e^{i k_1 \sigma + i k_2 \rho},
\]

where \( L_1 \) and \( L_2 \) are the radii of two cycles of \( T^2 \) and \( w_1 \) and \( w_2 \) are integers. The wrapping number on \( T^2 \) is given by \( w_1 w_2 \). If we also consider \( X^9 \) as the 11th direction, this supermembrane corresponds to the light-cone type-IIA superstring on \( R^9 \times S^1 \) (radius \( L_2 \)), which is T-dual to the light-cone type-IIB superstring on \( R^9 \times S^1 \) (radius \( \tilde{L}_2 = \alpha'/L_2 \)).

\(^4\)Note that the mass dimensions of the world-volume parameters, \( \tau, \sigma \) and \( \rho \), are 0.

\(^5\)For simplicity, we consider only toroidal supermembrane in this paper.

\(^6\)The double-dimensional reduction was discussed classically in Ref.\(^{23}\). In quantum mechanically, it is subtle whether such a reduction is realized or not\(^{23,13,24}\).
3 From supermembrane to $0+1$-D super Yang-Mills

In this section, we consider the light-cone supermembrane on $R^{11}$, where the mode expansions are given in eqs. (2.4) and (2.5). It is known that we shall obtain $0+1$-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory through the matrix regularization of the supermembrane [2, 3]. We pursue the procedure as a warm-up before the subsequent sections.

The matrix regularization is the following mathematical procedure: First, we introduce the noncommutativity for the space-sheet coordinates of the supermembrane, $[\sigma, \rho] = i\Theta$ ($\Theta$: constant). This noncommutativity is encoded in the star product for functions on the space sheet,

$$ f * g = f \exp \left( i \frac{1}{2} \Theta e^{\alpha\beta} \partial_\alpha \partial_\beta \right) g. \quad (\alpha, \beta = \sigma, \rho) \tag{3.1} $$

Then, the star-commutator for the Fourier modes in eqs. (2.4) and (2.5) is given by [25]

$$ [e^{ik_1\sigma+ik_2\rho}, e^{ik'_1\sigma+ik'_2\rho}]_* = -2i \sin \left( \frac{1}{2} \Theta k \times k' \right) e^{i(k_1+k'_1)\sigma+i(k_2+k'_2)\rho}. \tag{3.2} $$

In the $\Theta \to 0$ limit, the space-sheet Poisson bracket is obtained,

$$ \{ f, g \} = -i \lim_{\Theta \to 0} \Theta^{-1} [f, g]_* \tag{3.3} $$

If $\Theta = 2\pi q/N$ where $q$ and $N$ are mutually prime integers, the sine functions in the structure constants have zeros. Henceforth we set $\Theta = 2\pi/N$ with $N = 2M+1$ being odd number for simple presentation here. Then, the Fourier modes $e^{ipN\sigma}$, $e^{irN\rho}$ ($p, r \in \mathbb{Z}$) commute with any modes and hence they are central elements in the star-commutator algebra. Thus we can identify them with the identity operator consistently in the algebra and then we obtain the following equivalences:7

$$ e^{i(k_1+pN)\sigma+ik_2\rho} \approx (-1)^{pk_2} e^{i(k_1+k_2)\sigma+ik_2\rho}, \tag{3.4} $$

$$ e^{ik_1\sigma+i(k_2+rN)\rho} \approx (-1)^{rk_1} e^{i(k_1+k_2)\sigma+ik_2\rho}. \tag{3.5} $$

Under the equivalences, the infinite dimensional algebra is consistently truncated to the finite dimensional algebra $u(N)$. Then, the number of the modes $e^{ik_1\sigma+ik_2\rho}$ are restricted to $k_1, k_2 = 0, \pm 1, \pm 2, \cdots, \pm M$ and the truncated mode expansions are given by

$$ X^i(\sigma, \rho) = \sum_{k_1, k_2=-M}^{M} X^i_{(k_1, k_2)} e^{ik_1\sigma+ik_2\rho}, \tag{3.6} $$

$$ A(\sigma, \rho) = \sum_{k_1, k_2=-M}^{M} A_{(k_1, k_2)} e^{ik_1\sigma+ik_2\rho}. \tag{3.7} $$

Next, we give a matrix representation for the $N^2$ generators of $u(N)$, $\{e^{ik_1\sigma+ik_2\rho} | k_1, k_2 = 0, \pm 1, \pm 2, \cdots, \pm M\}$. Actually, the generators are represented by $N \times N$ matrices [25]

$$ e^{ik_1\sigma+ik_2\rho} \to \lambda^{-k_1k_2/2} V^{k_2} U^{k_1}, \tag{3.8} $$

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7In eqs. (3.4) and (3.5), the sign factors depending on the Fourier modes are attached. These factors can be eliminated by choosing $\Theta = 4\pi/N$ (for odd $N$). Actually, in our previous paper [10], such a convention was adopted. In a mathematical point of view, this is nothing but a convention. In this paper, however, we adopt $\Theta = 2\pi/N$ and we will explain the physical meaning later.
where $\lambda \equiv e^{i2\pi/N}$, $U$ and $V$ are the clock and shift matrices, respectively,

\[
U = \begin{pmatrix}
1 & \lambda & 0 \\
\lambda & \lambda^2 & \cdots \\
0 & \cdots & \lambda^{N-1}
\end{pmatrix},
\]

\[
V = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 1 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]  
(3.9)

$U$ and $V$ have the following properties,

\[
U^N = V^N = 1,
\]

\[
VU = \lambda UV.
\]  
(3.11)

For $k < 0$, $U^k \equiv (U^\dagger)^{-k}$, $V^k \equiv (V^\dagger)^{-k}$. By using eq.(3.12), it is easy to show that the representation (3.8) satisfy the commutator 3.2 with $\Theta = 2\pi/N$. Furthermore, from eq.(3.11), equivalences (3.4) and (3.5) are satisfied. From eqs. (3.8) and (3.9), we see that $N \times N$ matrices of the zero-modes with respect to $\rho$ ($k_2 = 0$ in (3.8)) are diagonalized. However, this basis is a convention. We can transform to another basis where the zero-modes with respect to $\sigma$ ($k_1 = 0$ in (3.8)) are diagonalized by using the following unitary matrix,

\[
S = \frac{1}{\sqrt{N}} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & \lambda & \lambda^2 & \cdots & \lambda^{(N-1)} \\
1 & \lambda^2 & \lambda^4 & \cdots & \lambda^{2(N-1)} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & \lambda^{N-1} & \lambda^{2(N-1)} & \cdots & \lambda^{(N-1)^2}
\end{pmatrix}.
\]  
(3.13)

since $U$ and $V$ are transformed as $S^\dagger VS = U$, $S^\dagger US = V^{-1}$. According to (3.8), the truncated mode expansions (3.6) and (3.7) are represented by $N \times N$ matrices,

\[
X^i(\sigma, \rho) \rightarrow X^i = \sum_{k_1, k_2 = -M}^{M} X^i_{(k_1, k_2)} \lambda^{-k_1k_2/2} V^{k_2} U^{k_1},
\]

\[
A(\sigma, \rho) \rightarrow A = \sum_{k_1, k_2 = -M}^{M} A_{(k_1, k_2)} \lambda^{-k_1k_2/2} V^{k_2} U^{k_1}.
\]  
(3.14)

Finally, we show that the matrix-regularized action of the light-cone supermembrane on $R^{11}$ agrees with that of 0+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory. In the matrix regularization, the functions $X^i, A$ of $\sigma$ and $\rho$ are represented by
the $N \times N$ matrices (3.14) and (3.15) and the Poisson bracket and the double integral are represented as follows,

\[
\{ \cdot, \cdot \} \rightarrow -i \frac{N}{2\pi} [\cdot, \cdot], \tag{3.16}
\]

\[
\int_0^{2\pi} d\sigma d\rho \rightarrow \frac{(2\pi)^2}{N} \text{Tr}. \tag{3.17}
\]

From these results and a rescaling $\tau \rightarrow \tau/N$, the action (2.1) is mapped to

\[
S_{0+1} = (2\pi)^2 LT \int d\tau \text{Tr}\left[ (D_\tau X^i)^2 + \frac{1}{2(2\pi L)^2} [X^i, X^j]^2 \right], \tag{3.18}
\]

\[
D_\tau X^i = \partial_\tau X^i + i \frac{1}{2\pi L} [A, X^i], \tag{3.19}
\]

which is just a bosonic part of 0+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory, i.e., Matrix theory.

## 4 From supermembrane to 1+1-D super Yang-Mills

In this section, we consider the light-cone wrapped supermembrane on $R^{10} \times S^1$, which has the mode expansions (2.6)-(2.8). From this supermembrane, we can obtain 1+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory by introducing the noncommutativity $[\sigma, \rho] = i2\pi/N$ ($N = 2M + 1 : \text{odd number}$) and giving a matrix representation of the star-commutator algebra. We shall immediately notice that we need to handle the wrapping of the supermembrane this time. It was pointed out that we should add a linear term $\rho$ to the generators of the star-commutator algebra [15]. Then the star commutators are given by

\[
[e^{ik_1\sigma+ik_2\rho}, e^{ik'_1\sigma+ik'_2\rho}]_* = \frac{2\pi}{N} \delta_{k_2k'_2} e^{i(k_1+k'_1)\sigma+i(k_2+k'_2)\rho}, \tag{4.1}
\]

\[
[\rho, e^{ik_1\sigma+ik_2\rho}]_* = \frac{2\pi}{N} e^{ik_1\sigma+ik_2\rho}. \tag{4.2}
\]

In this algebra, we should notice that the Fourier modes $e^{ipN\sigma}$ cannot be the central elements and hence the equivalence (3.4) is no longer valid this time. On the other hand, $e^{irN\rho}$ are the central elements and the equivalence (3.5) is still valid. Then, we can consistently truncate only the Fourier modes with respect to $\rho$ and the truncated generators are given by $\{e^{ik_1\sigma+ik_2\rho}, \rho \mid k_1 = 0, \pm 1, \pm 2, \ldots, \pm \infty, k_2 = 0, \pm 1, \pm 2, \ldots, \pm M\}$ [15]. The truncated mode expansions are given by

\[
X^9(\sigma, \rho) = w_1L_1\rho + \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-M}^{M} Y^1_{(k_1,k_2)} e^{ik_1\sigma+ik_2\rho} \tag{4.3}
\]
\[ X^k(\sigma, \rho) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-M}^{M} X^k_{(k_1,k_2)} e^{ik_1 \sigma + ik_2 \rho} \]

\[ = \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} \sum_{k=-M}^{M} X^k_{(pN+q,k)} e^{i(pN+q) \sigma + ik \rho}, \tag{4.4} \]

\[ A(\sigma, \rho) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-M}^{M} A_{(k_1,k_2)} e^{ik_1 \sigma + ik_2 \rho} \]

\[ = \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} \sum_{k=-M}^{M} A_{(pN+q,k)} e^{i(pN+q) \sigma + ik \rho}. \tag{4.5} \]

Now we give a matrix representation of generators \( \{ e^{i(pN+q) \sigma + ik \rho}, p = 0, \pm 1, \pm 2, \ldots, \pm \infty, q, k = 0, \pm 1, \pm 2, \ldots, \pm M \} \). Actually, the generators are represented as \( N \times N \) matrices with a continuous parameter \( \theta \) [16],

\[ e^{i(pN+q) \sigma + ik \rho} \rightarrow e^{i(pN+q) \theta_1/N \lambda^{kq}/2 V^k U^q}, \tag{4.6} \]

\[ \rho \rightarrow -2\pi i \partial_{\theta_1} I, \tag{4.7} \]

where \( I \) is the \( N \times N \) unit matrix. From eqs. (4.11) and (4.12), it is easy to see that the representation (4.6) and (4.7) satisfy the commutators (4.1) and (4.2) and also the equivalence (3.5). In the \( N \times N \) matrix representation (4.6) and (4.7), the truncated mode expansions (4.3)–(4.5) are given by

\[ X^0(\sigma, \rho) \rightarrow -2\pi i w_1 L_1 \partial_{\theta_1} I + Y^1(\theta_1) \]

\[ = -2\pi i w_1 L_1 \partial_{\theta_1} I + \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} \sum_{k=-M}^{M} Y^1_{(pN+q,k)} e^{i(pN+q) \theta_1/N \lambda^{kq}/2 V^k U^q}, \tag{4.8} \]

\[ X^k(\sigma, \rho) \rightarrow X^k(\theta_1) = \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} \sum_{k=-M}^{M} X^k_{(pN+q,k)} e^{i(pN+q) \theta_1/N \lambda^{kq}/2 V^k U^q}, \tag{4.9} \]

\[ A(\sigma, \rho) \rightarrow A(\theta_1) = \sum_{p=-\infty}^{\infty} \sum_{q=-M}^{M} \sum_{k=-M}^{M} A_{(pN+q,k)} e^{i(pN+q) \theta_1/N \lambda^{kq}/2 V^k U^q}. \tag{4.10} \]

Here we find that \( Y^1(\theta_1), X^k(\theta_1) \) and \( A(\theta_1) \) satisfy the boundary conditions,

\[ Y^1(\theta_1 + 2\pi) = V Y^1(\theta_1) V^\dagger, \tag{4.11} \]

\[ X^k(\theta_1 + 2\pi) = V X^k(\theta_1) V^\dagger, \tag{4.12} \]

\[ A(\theta_1 + 2\pi) = V A(\theta_1) V^\dagger, \tag{4.13} \]

because of \( V U V^\dagger = \lambda U \). This boundary conditions means that via the double-dimensional reduction, the wrapped supermembrane becomes to correspond to a long string in matrix string theory. In Ref. [14], the boundary conditions were assumed, while they are deducible in our case [16].

Next, we show that the matrix-regularized action of the light-cone supermembrane on \( R^{10} \times S^1 \) agrees with that of 1+1-dimensional maximally supersymmetric \( U(N) \) Yang-Mills theory. In such a truncation, the functions \( X^0, X^k, A \) of \( \sigma \) and \( \rho \) are represented by the
matrices (4.8)-(4.10) and the Poisson bracket and the double integral are represented as follows,

\[
\{ \cdot, \cdot \} \rightarrow -i \frac{N}{2\pi} [\cdot, \cdot], \tag{4.14}
\]

\[
\int_0^{2\pi} d\sigma d\rho \rightarrow \frac{2\pi}{N} \int_0^{2\pi} d\theta_1 \text{Tr}. \tag{4.15}
\]

From these results and a rescaling \( \tau \rightarrow \tau/N \), the action (2.1) in the \( w_1 = 1 \) case is mapped to [14, 15, 16]

\[
S_{1+1} = \frac{2\pi LT}{2} \int d\tau \int_0^{2\pi} d\theta_1 \text{Tr} \left[ (F_{\tau\theta_1})^2 + (D_\tau X^k)^2 - (D_{\theta_1} X^k)^2 + \frac{1}{2(2\pi L)^2} [X^k, X^l]^2 \right], \tag{4.16}
\]

\[
F_{\tau\theta_1} = \partial_\tau Y^1 - \frac{L_1}{L} \partial_{\theta_1} A + i \frac{1}{2\pi L} [A, Y^1], \tag{4.17}
\]

\[
D_\tau X^k = \partial_\tau X^k + i \frac{1}{2\pi L} [A, X^k], \tag{4.18}
\]

\[
D_{\theta_1} X^k = \frac{L_1}{L} \partial_{\theta_1} X^k + i \frac{1}{2\pi L} [Y^1, X^k]. \tag{4.19}
\]

Here we should note that the fields \( Y^1(\theta_1), X^k(\theta_1), A(\theta_1) \) have mass dimension \(-1\) and the parameters \( \tau, \theta_1 \) have mass dimension 0. We rewrite the action (4.16) to the standard form of Yang-Mills theory. The kinetic term of the gauge field in \( D \) dimensions is given by

\[
S_{YM} = -\frac{1}{4g_{YM}^2} \int d^D x \text{Tr} F_{\mu\nu} F^{\mu\nu}, \tag{4.20}
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu], \tag{4.21}
\]

where the mass dimensions of the gauge field \( A_\mu(x) \), the parameter \( x^\mu \) and the gauge coupling constant \( g_{YM} \) are 1, \(-1\) and \( 2 - D/2 \), respectively. In order to adjust their mass dimensions we shall introduce some dimensionful constants \( \alpha, \Sigma, \Sigma_1 \) for the time being and make a change of variables,

\[
Y^1(\theta_1) \rightarrow \alpha A_1(x^1), \tag{4.22}
\]

\[
X^k(\theta_1) \rightarrow \alpha \phi^k(x^1), \tag{4.23}
\]

\[
A(\theta_1) \rightarrow \alpha A_0(x^1), \tag{4.24}
\]

\[
\theta_1 \rightarrow \frac{x^1}{\Sigma_1}, \tag{4.25}
\]

\[
\tau \rightarrow \frac{x^0}{\Sigma}, \tag{4.26}
\]

where \( \alpha \) has mass dimension \(-2\) and \( \Sigma \) and \( \Sigma_1 \) have mass dimension \(-1\). Then, the action (4.16) is rewritten by

\[
\begin{align*}
S_{1+1} &= \frac{2\pi LT}{2} \frac{1}{\Sigma_1 \Sigma} \int dx^0 \int_0^{2\pi \Sigma_1} dx^1 \text{Tr} \left[ (F_{\tau\theta_1})^2 + (D_\tau X^k)^2 \right].
\end{align*}
\]

\(^9\)Henceforth we set \( w_1 = 1 \) for simplicity. See Ref.[16] for the discussion in the case of an arbitrary integer \( w_1 \).
where $g_{YM}$ is the gauge coupling constant of mass dimension 1, which is given by $g_{YM}^2 = (2\pi)^{-3} \Sigma^{-2} l_1^{-3} T^{-1}$. We also define the dimensionless gauge coupling constant $\tilde{g}_{YM}$ by

$$\tilde{g}_{YM}^2 \equiv g_{YM}^2 (2\pi \Sigma_1)^2 = (2\pi)^{-1} l_1^{-3} T^{-1} = 2\pi \frac{l_1^3}{L_1^3},$$  \hspace{1cm} (4.40)$$

where eleven-dimensional Planck length $l_{11}$ is defined by $T = (2\pi)^{-2} l_{11}^{-3}$. The dimensionless gauge coupling constant in eq. (4.40) agrees with that in Ref. [13] including the numerical

\hspace{1cm} 

\footnote{From the D-brane viewpoint based on Refs. [5, 7], where the 11th direction is chosen as the longitudinal direction, the parameter $\alpha$ should be identified with the inverse of the string tension $2\pi RT$ where $R$ is a radius of the $x^-$ direction in DLCQ [12, 13]. However, in this paper we have taken a different approach and hence $\alpha$ is an arbitrary parameter having the mass dimension $-2$ in our case.}
constant.\footnote{Note that the parameters Σ and L in Ref.\cite{13} represent the circumferences but not the radii.} Note that in Ref.\cite{13} such a coupling constant was derived by regarding the super Yang-Mills theory as the low energy effective theory of D-branes (D-brane viewpoint), while we have obtained the coupling constant by the matrix regularization of the light-cone supermembrane in this paper (membrane viewpoint). Here we make a comment on the convention of the noncommutative parameter \( Θ \). It is nothing but a mathematical convention whether we adopt \( Θ = 2π/N \), \( Θ = 4π/N \) and so on in the matrix regularization. However, if we would not adopt \( Θ = 2π/N \), eq.\eqref{eq:4.40} has a different numerical constant which does not agree with that in Ref.\cite{13}. Thus the \( N \) in a choice of \( Θ = 2π/N \) has the physical meaning of the \( N \) in the light-cone momentum \( p^+ = N/R \) in DLCQ.

Finally, we consider \( X^9 \) as the 11th direction and relate to type-IIA superstring theory. The precise relation is that the string tension \((2πα'/L)^{-1}\) and string coupling constant \( g_s \) are given by \cite{22, 26}

\[
\frac{1}{2πα'} = 2πL_1T, \tag{4.41}
\]

\[
g_s = \frac{L_1}{\sqrt{α'}}. \tag{4.42}
\]

Evidently, the dimensionless gauge coupling constant \( \tilde{g}_{YM} \) and string coupling constant \( g_s \) are inversely related to one another \cite{13}

\[
\tilde{g}_{YM}^2 = \frac{2π}{g_s^2}. \tag{4.43}
\]

Thus the super Yang-Mills theory in the strong coupling limit is equivalent to free type-IIA superstring theory in the light-cone gauge.

## 5 From supermembrane to 2+1-D super Yang-Mills

In this section, we consider the light-cone wrapped supermembrane on \( R^9 \times T^2 \), which has the mode expansions \eqref{2.9}-\eqref{2.12}. Starting from this supermembrane, we can obtain 2+1-dimensional maximally supersymmetric \( U(N) \) Yang-Mills theory by introducing the noncommutativity \([σ, ρ] = i2π/N (N = 2M + 1 : \text{odd number})\) and giving a matrix representation of the star-commutator algebra. In this case, we need to add two linear terms, \( σ \) and \( ρ \), to the generators of the star-commutator algebra. Then the star commutators are given by

\[
\left[ e^{ik_1σ+ik_2ρ}, e^{ik_1'σ+ik_2'ρ} \right]_s = -2i \sin \left( \frac{π}{N} k \times k' \right) e^{i(k_1+k_1')σ+i(k_2+k_2')ρ}, \tag{5.1}
\]

\[
\left[ σ, e^{ik_1σ+ik_2ρ} \right]_s = -\frac{2πk_2}{N} e^{ik_1σ+ik_2ρ}, \tag{5.2}
\]

\[
\left[ ρ, e^{ik_1σ+ik_2ρ} \right]_s = \frac{2πk_1}{N} e^{ik_1σ+ik_2ρ}, \tag{5.3}
\]

\[
\left[ σ, ρ \right]_s = i\frac{2π}{N}. \tag{5.4}
\]

In this algebra, both \( e^{ipNσ} \) and \( e^{irNρ} \) cannot be the central elements and hence the equivalences \eqref{3.4} and \eqref{3.5} do not hold. Thus in this case, we cannot consistently truncate the
Fourier modes at all,

\[ X^9(\sigma, \rho) = w_1 L_1 \rho + \sum_{k_1, k_2 = -\infty}^{\infty} Y_{(k_1, k_2)}^1 e^{ik_1 \sigma + ik_2 \rho} \]
\[ = w_1 L_1 \rho + \sum_{p, r = -\infty}^{\infty} \sum_{q, s = -M}^{M} Y_{(pN + q, rN + s)}^1 e^{ip(N + q) + ir(N + s) \rho}, \]  
\( \tag{5.5} \)

\[ X^8(\sigma, \rho) = w_2 L_2 \sigma + \sum_{k_1, k_2 = -\infty}^{\infty} Y_{(k_1, k_2)}^2 e^{ik_1 \sigma + ik_2 \rho} \]
\[ = w_2 L_2 \sigma + \sum_{p, r = -\infty}^{\infty} \sum_{q, s = -M}^{M} Y_{(pN + q, rN + s)}^2 e^{ip(N + q) + ir(N + s) \rho}, \]  
\( \tag{5.6} \)

\[ X^m(\sigma, \rho) = \sum_{k_1, k_2 = -\infty}^{\infty} X_{(k_1, k_2)}^m e^{ik_1 \sigma + ik_2 \rho} \]
\[ = \sum_{p, r = -\infty}^{\infty} \sum_{q, s = -M}^{M} X_{(pN + q, rN + s)}^m e^{ip(N + q) + ir(N + s) \rho}, \]  
\( \tag{5.7} \)

\[ A(\sigma, \rho) = \sum_{k_1, k_2 = -\infty}^{\infty} A_{(k_1, k_2)} e^{ik_1 \sigma + ik_2 \rho} \]
\[ = \sum_{p, r = -\infty}^{\infty} \sum_{q, s = -M}^{M} A_{(pN + q, rN + s)} e^{ip(N + q) + ir(N + s) \rho}. \]  
\( \tag{5.8} \)

Although the consistent truncation does not exist, we can give a matrix representation of the generators \( \{e^{ip(N+q)\sigma+i(rN+s)\rho}, \sigma, \rho \mid p, r = 0, \pm 1, \pm 2, \ldots, \pm \infty, \ q, s = 0, \pm 1, \pm 2, \ldots, \pm M\} \). Actually, the generators are represented as \( N \times N \) matrices with two continuous parameters \( \theta_1, \theta_2 \),

\[ e^{ip(N+q)\sigma+i(rN+s)\rho} \rightarrow e^{ip(N+q)\theta_1/N} e^{-ir(N+s)\theta_2/N} \lambda^{-sq/2} V^s U^q, \]  
\( \tag{5.9} \)

\[ \rho \rightarrow -2\pi i\partial_{\theta_1} I, \]  
\( \tag{5.10} \)

\[ \sigma \rightarrow -2\pi i\partial_{\theta_2} I + \frac{\theta_1}{N} I. \]  
\( \tag{5.11} \)

Due to the properties of \( U \) and \( V \) in eqs. (3.11) and (3.12), it is easy to show that representation (5.9)-(5.11) satisfy the commutators (5.1)-(5.4). Then in the \( N \times N \) matrix representation (5.9)-(5.11), mode expansions (5.5)-(5.8) are given by

\[ X^9(\sigma, \rho) \rightarrow -2\pi i w_1 L_1 \partial_{\theta_1} I + Y^1(\theta_1, \theta_2) \]
\[ = -2\pi i w_1 L_1 \partial_{\theta_1} I \]
\[ + \sum_{p, r = -\infty}^{\infty} \sum_{q, s = -M}^{M} Y_{(pN + q, rN + s)}^1 e^{ip(N + q)\theta_1/N} e^{-ir(N + s)\theta_2/N} \lambda^{-sq/2} V^s U^q, \]  
\( \tag{5.12} \)

\[ X^8(\sigma, \rho) \rightarrow -2\pi i w_2 L_2 \partial_{\theta_2} I + \frac{w_2 L_2}{N} \theta_1 I + Y^2(\theta_1, \theta_2) \]
\[ = -2\pi i w_2 L_2 \partial_{\theta_2} I + \frac{w_2 L_2}{N} \theta_1 I. \]  

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$$X^m(\sigma, \rho) \rightarrow X^m(\theta_1, \theta_2)$$

$$= \sum_{p,r=-\infty}^{\infty} \sum_{q,s=-M}^{M} Y^2_{(pN+q,rN+s)} e^{i(pN+q)\theta_1/N} e^{-i(rN+s)\theta_2/N} \lambda^{-sq/2V^s U^q}, \quad (5.13)$$

$$V^s U^q$$

$$Y^1(\theta_1 + 2\pi, \theta_2) = VY^1(\theta_1, \theta_2)V^\dagger, \quad (5.16)$$

$$Y^2(\theta_1 + 2\pi, \theta_2) = VY^2(\theta_1, \theta_2)V^\dagger, \quad (5.17)$$

$$X^m(\theta_1 + 2\pi, \theta_2) = VX^m(\theta_1, \theta_2)V^\dagger, \quad (5.18)$$

$$A(\theta_1 + 2\pi, \theta_2) = VA(\theta_1, \theta_2)V^\dagger, \quad (5.19)$$

$$Y^1(\theta_1, \theta_2 + 2\pi) = UX^1(\theta_1, \theta_2)U^\dagger, \quad (5.20)$$

$$Y^2(\theta_1, \theta_2 + 2\pi) = UX^2(\theta_1, \theta_2)U^\dagger, \quad (5.21)$$

$$X^m(\theta_1, \theta_2 + 2\pi) = UX^m(\theta_1, \theta_2)U^\dagger, \quad (5.22)$$

$$A(\theta_1, \theta_2 + 2\pi) = UA(\theta_1, \theta_2)U^\dagger, \quad (5.23)$$

because of $VUV^\dagger = \lambda U$ and $UVU^\dagger = \lambda^{-1}V$.

Next, we show that after the introduction of the noncommutativity $[\sigma, \rho] = i2\pi/N$, the action of the light-cone wrapped supermembrane on $R^9 \times T^2$ agrees with that of 2+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory with constant magnetic flux. This is an example of the mapping from noncommutative gauge theory to ordinary (commutative) gauge theory with constant magnetic flux [27]. Actually, the functions $X^9, X^8, X^m, A$ of $\sigma$ and $\rho$ are represented by the matrices $[5.12]-[5.15]$ and the Poisson bracket and the double integral are represented as follows,

$$\{ \cdot, \cdot \} \rightarrow -\frac{iN}{2\pi} [\cdot, \cdot], \quad (5.24)$$

$$\int_0^{2\pi} d\sigma d\rho \rightarrow \frac{1}{N} \int_0^{2\pi} d\theta_1 d\theta_2 \text{Tr.} \quad (5.25)$$

From these results and a rescaling $\tau \rightarrow \tau/N$, the action (2.31) in the case of $w_1 = w_2 = 1$ is mapped to

$$S_{2+1} = \frac{LT}{2} \int d\tau \int_0^{2\pi} d\theta_1 d\theta_2 \text{Tr} \left[ (F_{\tau \theta_1})^2 + (F_{\tau \theta_2})^2 - (F_{\theta_1 \theta_2})^2 + (D_{\tau} X^m)^2 - (D_{\theta_1} X^m)^2 - (D_{\theta_2} X^m)^2 + \frac{1}{2(2\pi L)^2} [X^m, X^n]^2 \right], \quad (5.26)$$

\[12\text{Henceforth, we set } w_1 = w_2 = 1 \text{ for simplicity.}\]
\[ F_{\tau \theta_1} = \partial_\tau Y^1 - \frac{L_1}{L} \partial_{\theta_1} A - i \frac{1}{2\pi L} [A, Y^1], \]  
\[ F_{\tau \theta_2} = \partial_\tau Y^2 - \frac{L_2}{L} \partial_{\theta_2} A + i \frac{1}{2\pi L} [A, Y^2], \]  
\[ F_{\theta_1 \theta_2} = \frac{1}{NL} L_1 L_2 I + \frac{L_1}{L} \partial_{\theta_1} Y^2 - \frac{L_2}{L} \partial_{\theta_2} Y^1 + i \frac{1}{2\pi L} [Y^1, Y^2], \]  
\[ D_\tau X^m = \partial_\tau X^m + i \frac{1}{2\pi L} [A, X^m], \]  
\[ D_{\theta_1} X^m = \frac{L_1}{L} \partial_{\theta_1} X^m + i \frac{1}{2\pi L} [Y^1, X^m], \]  
\[ D_{\theta_2} X^m = \frac{L_2}{L} \partial_{\theta_2} X^m + i \frac{1}{2\pi L} [Y^2, X^m]. \]

Here we should note that the fields \( Y^1(\theta_1, \theta_2), Y^2(\theta_1, \theta_2), X^m(\theta_1, \theta_2), A(\theta_1, \theta_2) \) have mass dimension \(-1\) and the parameters \( \tau, \theta_1, \theta_2 \) have mass dimension \(0\). We also rewrite the action \( S_{1+1} \) to the standard form of Yang-Mills theory. In order to adjust the mass dimensions of the fields and the parameters, we rewrite them by introducing some dimensionful constants,

\[
Y^1(\theta_1, \theta_2) \rightarrow \alpha A_1(x^1, x^2), \tag{5.33}
\]

\[
Y^2(\theta_1, \theta_2) \rightarrow \alpha A_2(x^1, x^2), \tag{5.34}
\]

\[
X^m(\theta_1, \theta_2) \rightarrow \alpha \phi^m(x^1, x^2), \tag{5.35}
\]

\[
A(\theta_1, \theta_2) \rightarrow \alpha A_0(x^1, x^2), \tag{5.36}
\]

\[
\begin{align*}
\theta_1 & \rightarrow \frac{x^1}{\Sigma_1}, \\
\theta_2 & \rightarrow \frac{x^2}{\Sigma_2}, \\
\tau & \rightarrow \frac{x^0}{\Sigma},
\end{align*} \tag{5.37-5.39}
\]

where \( \alpha \) has mass dimension \(-2\) and \( \Sigma_1, \Sigma_2 \) and \( \Sigma \) have mass dimension \(-1\). Then, the action \( S_{1+1} \) is rewritten by

\[
S_{1+1} = \frac{LT}{2} \frac{1}{\Sigma_1 \Sigma_2 \Sigma} \int dx^0 \int_0^{2\pi \Sigma_1} dx^1 d^2 \Sigma \int_0^{2\pi \Sigma_2} dx^2 \text{Tr} \left[ (F_{\tau \theta_1})^2 + (F_{\tau \theta_2})^2 - (F_{\theta_1 \theta_2})^2 + (D_\tau X^m)^2 - (D_{\theta_1} X^m)^2 - (D_{\theta_2} X^m)^2 + \frac{\alpha^4}{2(2\pi L)^2} [\phi^m, \phi^n]^2 \right], \tag{5.40}
\]

\[
\begin{align*}
F_{\tau \theta_1} & = \Sigma \alpha \partial_\theta A_1 - \frac{L_1}{L} \Sigma_1 \alpha \partial_\tau A_0 + i \frac{\alpha^2}{2\pi L} [A_0, A_1], \\
F_{\tau \theta_2} & = \Sigma \alpha \partial_\theta A_2 - \frac{L_2}{L} \Sigma_2 \alpha \partial_\tau A_0 + i \frac{\alpha^2}{2\pi L} [A_0, A_2], \\
F_{\theta_1 \theta_2} & = \frac{1}{NL} L_1 L_2 I + \frac{L_1}{L} \Sigma_1 \alpha \partial_\tau A_2 - \frac{L_2}{L} \Sigma_2 \alpha \partial_\tau A_1 + i \frac{\alpha^2}{2\pi L} [A_1, A_2], \\
D_\tau X^m & = \Sigma \alpha \partial_\theta \phi^m + i \frac{\alpha^2}{2\pi L} [A_0, \phi^m], \\
D_{\theta_1} X^m & = \frac{L_1}{L} \Sigma_1 \alpha \partial_\tau \phi^m + i \frac{\alpha^2}{2\pi L} [A_1, \phi^m], \\
D_{\theta_2} X^m & = \frac{L_2}{L} \Sigma_2 \alpha \partial_\tau \phi^m + i \frac{\alpha^2}{2\pi L} [A_2, \phi^m].
\end{align*} \tag{5.41-5.46}
\]
In order to bring the field strength (5.41)-(5.43) into the standard form (4.21), we obtain the following relations,

\[
\begin{align*}
\Sigma &= \frac{\alpha}{2\pi L}, \\
\Sigma_1 &= \frac{\alpha}{2\pi L_1}, \\
\Sigma_2 &= \frac{\alpha}{2\pi L_2}.
\end{align*}
\]  

Eqs. (5.48) and (5.49) represent the T-duality which relates the lengths \(\Sigma_1\) and \(\Sigma_2\) in the super Yang-Mills theory and the lengths \(L_1\) and \(L_2\) in M-theory. From the D-brane viewpoint [6, 7], this super Yang-Mills theory is regarded as the low energy effective theory of \(N\) D2-branes and hence this is just a T-duality between D0-branes and D2-branes [8]. We should stress, however, that we have obtained the same relations from the membrane viewpoint. Then, we have obtained the standard form of a bosonic part of 2+1-dimensional maximally supersymmetric \(U(N)\) Yang-Mills theory with constant magnetic flux,

\[
S_{1+1} = \frac{1}{2g_{YM}^2} \int dx^0 \int_0^{2\pi \Sigma_1} dx^1 \int_0^{2\pi \Sigma_2} dx^2 \text{Tr} \left[ (F_{01})^2 + (F_{02})^2 - (F_{12})^2 + (D_0 \phi^m)^2 - (D_1 \phi^m)^2 + \frac{1}{2} [\phi^m, \phi^n]^2 \right],
\]

\[(5.50)\]  

\(F_{01} = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1],\)  
\(F_{02} = \partial_0 A_2 - \partial_2 A_0 + i[A_0, A_2],\)  
\(F_{12} = \frac{1}{2\pi N \Sigma_1 \Sigma_2} I + \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2],\)  
\(D_0 \phi^m = \partial_0 \phi^m + i[A_0, \phi^m],\)  
\(D_1 \phi^m = \partial_1 \phi^m + i[A_1, \phi^m],\)  
\(D_2 \phi^m = \partial_2 \phi^m + i[A_2, \phi^m],\)

with the boundary conditions,

\[
\begin{align*}
A_0(x^1 + 2\pi \Sigma_1, x^2) &= V A_0(x^1, x^2) V^\dagger, \\
A_1(x^1 + 2\pi \Sigma_1, x^2) &= V A_1(x^1, x^2) V^\dagger, \\
A_2(x^1 + 2\pi \Sigma_1, x^2) &= V A_2(x^1, x^2) V^\dagger, \\
\phi^m(x^1 + 2\pi \Sigma_1, x^2) &= V \phi^m(x^1, x^2) V^\dagger, \\
A_0(x^1, x^2 + 2\pi \Sigma_2) &= U A_0(x^1, x^2) U^\dagger, \\
A_1(x^1, x^2 + 2\pi \Sigma_2) &= U A_1(x^1, x^2) U^\dagger, \\
A_2(x^1, x^2 + 2\pi \Sigma_2) &= U A_2(x^1, x^2) U^\dagger, \\
\phi^m(x^1, x^2 + 2\pi \Sigma_2) &= U \phi^m(x^1, x^2) U^\dagger,
\end{align*}
\]  

(5.57) - (5.64)  

where \(g_{YM}\) is the gauge coupling constant of mass dimension one half, which is given by \(g_{YM}^2 = (2\pi)^{-2}(\Sigma_1 \Sigma_2)^{-1/2}(L_1 L_2)^{-3/2}T^{-1}\). We also define the dimensionless gauge coupling constant \(g_{YM}\) by

\[
g_{YM}^2 \equiv g_{YM}^2(2\pi \Sigma_1 2\pi \Sigma_2)^{1/2} = (2\pi)^{-1}(L_1 L_2)^{-3/2} T^{-1} = 2\pi \frac{f_{11}^3}{(L_1 L_2)^{3/2}}.
\]

\[(5.65)\]
This dimensionless gauge coupling constant exactly agrees with that obtained in Ref. [13] including the numerical constant.\footnote{Note that the parameters $\Sigma_1, \Sigma_2$ and $L_1, L_2$ in Ref. [13] represent the circumferences but not the radii.} Note that in Refs. [13] the super Yang-Mills theory was regarded as the low energy effective theory of D-branes in deriving such a relation, while we have taken a different approach of matrix regularization of supermembrane in this paper. Furthermore, the constant magnetic flux $(2\pi N \Sigma_1 \Sigma_2)^{-1} I$ in eq. (5.53) agrees with that obtained in Refs. [12, 13] including the numerical constant.

Finally, we comment on the relation to string. If we consider $X^9$ as the 11th direction, this theory would be the light-cone type-IIB superstring theory with the string coupling constant $g_{sIIB} = L_1/L_2$. The flip of $X^8$ and $X^9$ directions corresponds to the S-duality in type-IIB superstring theory \cite{28, 29}.

6 Summary and discussion

In this paper, we have studied systematically matrix regularization for the supermembrane in the light-cone gauge. The regularization procedure is applicable for both unwrapped and wrapped supermembranes and is summarized as the following mathematical steps: (i) Introduce the noncommutativity on the space sheet of the supermembrane, i.e., replace the product of functions on the space sheet to the star product. (ii) If possible, truncate the generators of the star-commutator algebra in an algebraically consistent way. (iii) Give a matrix representation of the (truncated) star-commutator algebra. Following this procedure, we have deduced the $p+1$-dimensional ($p = 0, 1, 2$) maximally supersymmetric $U(N)$ Yang-Mills theory from the eleven-dimensional supermembrane in the light-cone gauge. We have given the complete correspondence of the super Yang-Mills theory to the eleven-dimensional supermembrane theory. That is, in eqs. (3.14)-(3.15), (4.8)-(4.10) and (5.5)-(5.8), the matrix elements in the super Yang-Mills theory are determined by the Fourier coefficients in the supermembrane theory. We stress that we have never regarded the super Yang-Mills theory as the low energy effective theory of D-branes in this paper. Nevertheless, we have obtained the T-duality relations which relates the lengths in the super Yang-Mills theory and M-theory and the same dimensionless gauge coupling constant as that obtained in Ref. [13] in which the super Yang-Mills theory is regarded as the low energy effective theory of D-branes. Thus our results gives a consistency check of two kinds of profiles of the super Yang-Mills theory, i.e., one is the matrix regularized theory of the eleven-dimensional light-cone supermembrane and the other is the low energy effective theory of D-branes.

Finally, we discuss the derivation of 3+1-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory from our point of view (the membrane viewpoint). Naively, it seems that it is obtained by the matrix regularization of the wrapped supermembrane on $R^8 \times T^3$. However, we shall come up against a problem immediately. If we take $X^9, X^8$ and $X^7$ as coordinates of three cycles of $T^3$, the supermembrane can wrap on $T^3$ in three ways, i.e., wrapping around $X^9$-$X^8$ surface, $X^8$-$X^7$ surface and $X^7$-$X^9$ surface of $T^3$. In the language of the 3+1-dimensional super Yang-Mills theory, three ways of the wrapping correspond to three components of the magnetic flux $F_{12}, F_{23}, F_{31}$ \cite{11, 12, 13}. Thus the 3+1-dimensional super Yang-Mills theory includes three ways of the wrapping simultaneously. On the other hand, eq. (2.14) is the action of the first-quantized supermembrane theory in the light-cone gauge, i.e., the action of the one-body system.\footnote{It is widely appreciated that the first-quantized supermembrane theory through its continuous spec-}
three ways of the wrapping simultaneously in the present formulation. This deserves to be studied further.

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trum \cite{30} is capable of describing multiple supermembranes. This picture may solve the difficulty in the derivation of the 3+1-dimensional super Yang-Mills theory through the matrix regularization of the wrapped supermembrane on $R^8 \times T^3$. 

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