Heat Transfer of Forced Fluid Flow in a Channel with Parallel Fillisters

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This study analyzes heat transfer and fluid dynamics in a forced convection laminar flow in a channel with parallel fillisters. The problem is solved by the point-matching method. The influence of the height and width of the fillisters on the thermal-fluid characteristics of a channel flow is discussed in the present research. The local dimensionless velocity, \( f \) Re values, local dimensionless temperature and mean Nusselt number of the fluid flow are all obtained for a channel flow under the influence of parallel fillisters.

1. Introduction

Internal flow plays an important role in the research and analysis of convective heat transfer and fluid dynamics. Research on internal flow is concerned with ducts, tubes or channels, and significant efforts have been made in related researches. Cheng [1] presented an analog solution of laminar heat transfer in noncircular ducts by point-matching, and the fully developed laminar flow characteristics in noncircular ducts as well as the steady temperature distribution in infinitely long prismatical bars were obtained. Hu and Chang [2] investigated the heat transfer of a fully developed laminar flow in internally finned tubes and discussed the heat transfer effect of the fins on the Nusselt number with or without heat generation. Shah [3] used a least-squares-matching technique to analyze a fully developed laminar fluid flow and heat transfer in ducts of arbitrary cross-section under an axially constant wall heat flux and peripherally arbitrary thermal boundary conditions. Sparrow et al. [4] presented an analysis of the laminar heat transfer characteristics of an array of longitudinal fins with an adiabatic shroud situated adjacent to the fin tips. Fin arrays with and without tip clearance were considered. Sparrow and Chukaev [5] investigated a fully developed laminar flow and heat

\[
\begin{align*}
\mu & = \text{viscosity (N-s/m}^2) \\
\theta_1 & = \text{dimensionless temperature (Eq. 29)} \\
\theta_2 & = \text{dimensionless temperature (Eq. 30)} \\
\theta_{wall} & = \text{dimensionless boundary condition of the temperature, } \theta_{wall} = 0
\end{align*}
\]

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transfer in a duct in which there were no cross-section uniformities in a spanwise-periodic array of rectangular protuberances that extended into the flow cross-section from an otherwise plane wall. Wang [6] presented a fully developed laminar flow and heat transfer between parallel plates with longitudinal fins. The friction factor and Nusselt number for both H1 and H2 problems were determined as a function of fin length and spacing. Tasnim and Mahmud [7] reported on the buoyancy-induced flow and heat transfer characteristics inside an inclined L-shaped enclosure and discussed the range of Rayleigh number and the inclined angle on heat transfer. Hu and Yeh [8] presented a study on laminar flow in a channel with longitudinal moving bars arrayed along the channel width. The effects of bars of motion characteristic on the friction of the fluid were discussed. Carlos et al. [9] investigated the effect of the dimple shape and orientation on the heat transfer coefficient of a vertical fin surface, with the results determined both numerically and experimentally. The research focused on the laminar channel flow between fins, with an Re=500 and 1000. Recently, Sundén [10] presented a convective heat transfer and separated flow in ribbed ducts. The impact of perforated ribs versus solid ribs was presented, some flow structures of the downstream ribs were highlighted and the importance of the rib arrangement was shown for V-shaped ribs. Although many researches on internal flow under different conditions have been discussed, the heat transfer of the fluid flow in a channel with arrayed fillisters also requires investigation. The present paper, thus, has analyzed the convective heat transfer of a laminar flow in a channel flow with a complex geometric pattern, i.e., the channel included an arrayed fillister. In addition, local dimensionless velocity, friction factor, temperature distributions and the mean Nusselt number were obtained in the present study.

2. Analysis

This study considered a steady-state, fully developed laminar fluid flow in a channel. The height of the channel is aH. The left and right sides of the channel are assumed to be infinite. The channel includes parallel fillisters arrayed along the X direction. The height of the fillister is bH and the width of the fillister is 2(1-c)H. The cross-section of the channel and the physical model are shown in Fig. 1. The direction of the flow is normal to this paper (Z-Dimensional). Since the velocity of the fluid is very slow, it was assumed that the inertia force was negligible. The momentum equation for the velocity of the fluid can be expressed as:

\begin{equation}
\nabla^2 \Delta w(x, y) = \frac{1}{\mu} \frac{dP}{dz} \tag{1}
\end{equation}

and the energy equation for the fluid flow can be expressed as:

\begin{equation}
\frac{d}{dX} \left( \frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right) = W_z(X, Y) \frac{\partial T}{\partial Z} \tag{2}
\end{equation}

By introducing the dimensionless parameter \( w = -W_z \left( \frac{H^3 dp}{\mu dz} \right) \) into Eq. (1), the governing equation can be rewritten in the dimensionless form with Poisson’s equation as:

\begin{equation}
\nabla^2 w(x, y) = -1 \tag{3}
\end{equation}

The study considered the domain with a specific period, 2H, and focused on the range of \( x = 0 \) to \( x = H \), \( y = 0 \) to \( y = aH/2 \) (shown in Fig. 1). The range was symmetric to the y-axis in \( x = 0 \), and symmetric to the x-axis in \( y = 0 \). One part of the channel, an L-shaped region with two combined rectangles, as shown in Fig. 2, was investigated. The L-shaped region in Fig. 1 when magnified is as Fig. 3, in a dimensionless form.

The governing dimensionless equation for the left rectangle of the L-shaped domain is:

\begin{equation}
\nabla^2 w_1(x, y) = -1 \tag{4}
\end{equation}

and the boundary conditions are:

\begin{equation}
w_1(x, a/2) = 0, \quad \frac{\partial w_1}{\partial y}(x, 0) = 0, \quad \frac{\partial w_1}{\partial x}(0, y) = 0 \tag{5}\end{equation}

Next, the governing equation of the right rectangle can be expressed as:

\begin{equation}
\nabla^2 w_2(x, y) = -1 \tag{6}
\end{equation}

subject to the boundary conditions below:

\begin{equation}
w_2(x, a/2 - b) = 0, \quad \frac{\partial w_2}{\partial y}(x, 0) = 0, \quad \frac{\partial w_2}{\partial x}(1, y) = 0 \tag{7}\end{equation}

With the boundary conditions in Eqs. (5) and (7), the governing equations, Eqs. (4) and (6), can be solved and obtained as follows:

\begin{equation}
w_1(x, y) = \frac{\sigma^2}{8} \frac{y^2}{2} + \sum_{n} A_n \sin(\alpha_n y) \left(e^{\alpha_n (x-c)} + e^{-\alpha_n (x+c)} \right) \tag{8}
\end{equation}
The other boundary condition for Eq. (4) is:

\[ w_1(c, y) = 0; \]

Substituting the boundary condition into Eq. (8), the following equation can be obtained:

\[ \sum A_m \sin(\alpha_m y) \left[ 1 + e^{-2a\alpha_m} \right] = \frac{\alpha^2}{2} \sum B_m \sin(\beta_m y) \left[ 1 + e^{-2(1-c)\beta_m} \right] \]

Integrating Eq. (19) gives:

\[ w_{mean} = \frac{1}{a/2 - b(1-c)} \left[ \frac{c/2}{0} \int_0^c w_1 dy dx + \frac{1/2 - b}{0} \int_c^w w_2 dy dx \right] \]

where \( y_i \) in Eqs. (12), (15) and (16) and \( N \) points along the boundary at \( y=c \) are chosen to obtain:

\[ y_i = (i-1)a/(2N), \quad i=1 \text{ to } N. \]

We truncated \( A_n \) to \( N \) terms and \( B_m \) to \( M \) terms. Note that \( M \) can be calculated from the equation below:

\[ M=\text{floor function} \left[ (N1-2b/a) + 1 \right] \]

where \( N=30 \).

The linear system of \( M+N \) equations with \( M+N \) unknowns are solved for the coefficients of \( A_n \) and \( B_m \).

The mean value for \( w(x, y) \) is derived as:

\[ w_{mean} = \frac{1}{a/2 - b(1-c)} \left[ \frac{c/2}{0} \int_0^c w_1 dy dx + \frac{1/2 - b}{0} \int_c^w w_2 dy dx \right] \]

Integrating Eq. (19) gives:

\[ w_{mean} = \left( \frac{ca^3}{24} + \frac{1-c}{24} (a-2b)^2 + \sum A_n \frac{1}{a} (-1)^{n+1} (1-e^{-2ca_n}) + \sum B_m \frac{1}{a} (-1)^{n+1} (1-e^{-2b\beta_m}) \right) \]

Furthermore, \( \text{Re} \) can be expressed as the function of the volumetric flow rate as described in Wang [11]:

\[ f \text{ Re} = \frac{D_h^2}{2w_{mean}H^2} \]

or in a more detailed form,

\[ f \text{ Re} = \frac{2[a-2b(1-c)]^2}{(1+b)^2w_{mean}} \]

where \( D_h \) is the hydraulic diameter.

Substituting the velocity distribution (Eqs. 8 and 9) into the energy equation (Eq. 2), the dimensionless energy equations for the two regions can be expressed as:

\[ \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = w_1(x, y) \]

with the boundary conditions:

\[ \frac{\partial \theta}{\partial y} (x, 0) = 0, \quad \frac{\partial \theta}{\partial y} (0, y) = 0, \quad \theta(x, a) = \theta_{\text{wall}} \]

\[ \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = w_2(x, y) \]

Fig. 3 Enlarged view of the L-shaped regions with dimensionless parameters
where \( w_1(x,y) \) and \( w_2(x,y) \) in Eqs. (23) and (25) are expressed in Eqs. (8) and (9). Then, Eqs. (23) and (25) can be rewritten as follows:

\[
\frac{\partial^2 \theta_1(x,y)}{\partial x^2} + \frac{\partial^2 \theta_1(x,y)}{\partial y^2} = \frac{\partial^2}{8} - \frac{y^2}{2} + \sum_n A_n \sin(\alpha_n y)
\]

(26)

\[
\frac{\partial^2 \theta_2(x,y)}{\partial x^2} + \frac{\partial^2 \theta_2(x,y)}{\partial y^2} = \frac{1}{2} \left( \frac{a}{2} - b \right)^2 - \frac{y^2}{2} + \sum_m B_m \sin(\beta_m y) \left[ \theta_2^{m-2}(x-c) + e^{-\beta_m(x-c)} \right]
\]

(27)

with the boundary conditions:

\[
\frac{\partial \theta_1}{\partial y}(x,0) = 0, \quad \frac{\partial \theta_1}{\partial y}(0,y) = 0, \quad \theta_1(x, a/2) = \theta_{wall}
\]

\[
\frac{\partial \theta_2}{\partial y}(x,0) = 0, \quad \frac{\partial \theta_2}{\partial y}(1,y) = 0, \quad \theta_2(x, a/2) = \theta_{wall}
\]

(28)

where dimensionless temperature \( \theta \) is defined as:

\[
\theta_1 = \frac{T_1(X,Y) - T_{wall}}{\Delta T_{wall}} \quad \text{and} \quad \theta_2 = \frac{T_2(X,Y) - T_{wall}}{\Delta T_{wall}}
\]

(29)

The analytical solution for the dimensionless temperature can be expressed as:

\[
\theta_1(x,y) = \frac{y^2}{16} - \frac{y^4}{24} + \frac{C_n}{2 \alpha} \cos(\alpha_n y) \left[ e^{\alpha_n(x-c)} + e^{-\alpha_n(x+c)} \right] + \sum_n D_n \cos(\alpha_n y) \left[ e^{\alpha_n(x-c)} + e^{-\alpha_n(x+c)} \right]
\]

\[
\theta_2(x,y) = \frac{y^2}{4} \left( \frac{a}{2} - b \right)^2 - \frac{y^4}{24} + \frac{E_m}{2 \beta_m} \cos(\beta_m y) \left[ e^{\beta_m(x-2c)} + e^{-\beta_m(x+c)} \right] + \sum_m F_m \cos(\beta_m y) \left[ e^{\beta_m(x-2c)} + e^{-\beta_m(x+c)} \right]
\]

(30)

The other boundary conditions are as follows:

\[
\theta_1(x) = \theta_{wall} \quad \text{and} \quad \frac{a}{2} - b \leq y \leq \frac{a}{2}
\]

(31)

Next, the velocity and shear stress of the two regions of the L-shaped domain can be matched along the common boundary:

\[
\theta_1(x,y) = \frac{b + c}{1 - c} \theta_2(x,y), \quad 0 \leq y < \frac{a}{2} - b
\]

\[
\frac{\partial \theta_1}{\partial x}(x,y) = \frac{b + c}{1 - c} \frac{\partial \theta_2}{\partial x}(x,y), \quad 0 \leq y < \frac{a}{2} - b
\]

(32)

Substituting the boundary conditions (Eqs. 33-35) into Eqs. (31) and (32) yields:

\[
\sum_n D_n \cos(\alpha_n y) \left[ e^{\alpha_n(x-c)} + e^{-\alpha_n(x+c)} \right] + \sum_m F_m \cos(\beta_m y) \left[ e^{\beta_m(x-2c)} + e^{-\beta_m(x+c)} \right] = 0
\]

\[
\sum_n C_n \cos(\alpha_n y) \left( 1 + e^{2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 + e^{2\alpha_n c} \right)
\]

\[
\sum_n D_n \cos(\alpha_n y) \left( 1 + e^{-2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 - e^{-2\alpha_n c} \right)
\]

\[
= \frac{y_1^4}{24} \left( \frac{a}{2} - b \right)^2 + \frac{y_1^4}{16}
\]

(33)

\[
\sum_n C_n \cos(\alpha_n y) \left( 1 + e^{2\alpha_n c} \right) + \sum_m D_n \cos(\alpha_n y) \left( 1 + e^{-2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 + e^{2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 - e^{-2\alpha_n c} \right)
\]

\[
\sum_m F_m \cos(\beta_m y) \left( 1 + e^{2\beta_m c} \right) + \sum_m F_m \cos(\beta_m y) \left( 1 - e^{-2\beta_m c} \right)
\]

\[
= \frac{5a^4}{24(1-c)} \left( \frac{a}{2} - b \right)^4
\]

(34)

where \( y_i \) in Eqs. (36), (37) and (38) and 2N points along the boundary at \( y=c \) are chosen to obtain:

\[
y_i = (i-1)a/(2N), \quad i=1 \text{ to } 2N.
\]

(35)

The linear system of 2M+2N equations with 2M+2N unknowns are solved for the coefficients of \( C_n, D_n, E_m \) and \( F_m \).

The analytical solution for the dimensionless temperature can be expressed as:

\[
\sum_n C_n \cos(\alpha_n y) \left( 1 + e^{2\alpha_n c} \right) + \sum_m D_n \cos(\alpha_n y) \left( 1 + e^{-2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 + e^{2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 - e^{-2\alpha_n c} \right)
\]

\[
= \frac{5a^4}{24(1-c)} \left( \frac{a}{2} - b \right)^4
\]

(36)

\[
\sum_n C_n \cos(\alpha_n y) \left( 1 + e^{2\alpha_n c} \right) + \sum_m D_n \cos(\alpha_n y) \left( 1 + e^{-2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 + e^{2\alpha_n c} \right) - \beta_m \theta_{wall} \left( 1 - e^{-2\alpha_n c} \right)
\]

\[
\sum_m F_m \cos(\beta_m y) \left( 1 + e^{2\beta_m c} \right) + \sum_m F_m \cos(\beta_m y) \left( 1 - e^{-2\beta_m c} \right)
\]

\[
= \frac{5a^4}{24(1-c)} \left( \frac{a}{2} - b \right)^4
\]

(37)

The linear system of 2M+2N equations with 2M+2N unknowns are solved for the coefficients of \( C_n, D_n, E_m \) and \( F_m \).

One defined the heat transfer coefficient and the Nusselt number as:

\[
\overline{h} = \frac{\text{total heat in Area} \cdot \left( T_{wall} - T_m \right)}{\text{Area} \cdot \left( T_{wall} - T_m \right)} = \left( 1 + b \right) \phi \theta_{wall}^2 - \int_0^{\frac{a}{2}} \left( \frac{a/2}{c} \theta_1 w_1 dxdy + \frac{a/2 - b}{c} \theta_2 w_2 dxdy \right)
\]

(38)

(39)

(40)

(41)
\[ \bar{N}u = \frac{kD_b}{h} = \frac{4(1+b)kW_{mean}^2}{\left[ \int_{0}^{a/2} \int_{0}^{b/2} \theta_1 w_1 dydx + \int_{a/2-b}^{a/2} \int_{0}^{b/2} \theta_2 w_2 dydx \right]} \]

The method of numerical analysis was the Gauss-Seidel method. The coefficients for velocity field \( A_n \) and \( B_m \) in Eqs. (8) and (9) were evaluated with the aid of Eqs. (12), (15) and (16) to obtain \( w_1(x,y) \) and \( w_2(x,y) \). The coefficients for the temperature distributions \( C_n, D_n, E_n \) and \( F_n \) in Eqs. (31) and (32) were evaluated with the help of Eqs. (36), (37) and (38) to obtain \( \theta_1(x,y) \) and \( \theta_2(x,y) \). Then \( f \text{Re} \) and \( Nu \) can be obtained in Eqs. (22) and (42).

3. Results and Discussion

Figs. 4 and 5 plot profiles of the local fluid velocity in the channel with triple periods \((-2 \leq x \leq 4\) ). As illustrated, the local fluid velocity on the surface of the channel was zero under the no-slip boundary conditions and the velocity gradually increased away from the wall. At a fixed \( a/2 \), the local fluid velocity between the two neighboring fillisters increased as the height of the fillisters decreased, due to the fact that the cross-section area between the fillisters was smaller for a higher \( b \). As can be clearly seen in the figures, the maximum value of \( w \) fell on \( w_1 \) and occurred at \( x=0, y=0 \). The maximum \( w \) then gradually decreased to zero on the wall. When the width of the fillisters became larger, the shear stress decreased and led to an increase in the fluid velocity.

Fig. 6 plots the effects of different \( c \) on \( f \text{Re} \) for the channel under six different \( b \) values. Under smaller values of \( c \) \((c<0.4)\), it is shown that the height of the fillisters, \( b \) influenced \( f \text{Re} \) significantly, and the \( f \text{Re} \) value decreased as the \( b \) value decreased. Under larger values of \( c \) \((c>0.4)\), the trend of the fluid flow was like that of the flow between two flat plates without fillisters, and the friction factor of the flow gradually came close to 24. As the figure shows, when \( c=1.0 \), and the fillisters absent, the friction factor of the channel flow between parallel flat plates became 24.

Figs. 7 and 8 show the contour plot of the temperature distribution under the isothermal conditions for one period \((-1 \leq x \leq 1\) ). As the present study was conducted under the isothermal condition, the dimensionless temperature of the fluid on the wall of the channel was zero. The temperature gradually decreased as the fluid moved away from the wall. The minimum temperature of the fluid occurred at the geometric center of the channel, i.e., \( x=0, y=0 \). When the values of \( b \) and \( c \) were lower, the cross-section area of the fillisters became larger, and then the effect of the heat transfer and the fluid temperature in the geometric center were enhanced.
Fig. 8 Local dimensionless temperature distribution for \( a=1.0, b=0.25, c=0.8 \)

Fig. 8 Local dimensionless temperature distribution for \( a=1.0, b=0.29, c=0.5 \)

Fig. 9 shows the influence of \( c \) on the mean Nusselt number under six different values of \( b \). As the figure shows, under smaller values of \( f \), \( c < 0.4 \), an increased \( b \) decreased the distance between the top and bottom plates of the fillisters, and led to an increase in the Nusselt number. Besides, when the \( c \) value increased, the flow field became the flow between two flat plates, and then the Nusselt number gradually decreased to 7.54.

3. In this study, with the help of the point-matching method, the distributions of the local dimensionless velocity and the local dimensionless temperature in a channel with fillisters were accurately computed.

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