Polygamy relations of multipartite systems

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We investigate the polygamy relations of multipartite quantum states. General polygamy inequalities are given in the \( \alpha \)th \(( \alpha \geq 2 \)) power of concurrence of assistance, \( \beta \)th \(( \beta \geq 1 \)) power of entanglement of assistance, and the squared convex-roof extended negativity of assistance (SCRENoA).

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INTRODUCTION

Quantum entanglement is an important kind of quantum correlation, plays essential roles in quantum information processing.1,2 One of the fundamental differences between classical and quantum correlations lies on the sharability among the subsystems. Different from the classical correlation, quantum correlation cannot be freely shared. Monogamy relation is important in the sense that it gives rise to the distribution of correlation in the multipartite quantum system and has a unique feature of keeping security in quantum key distribution.3

For the systems of three qubits, a kind of monogamy of bipartite quantum entanglement in concurrence can be described by Coffman-Kundu-Wootters CKW inequality.4,5 \( \mathcal{E}_{ABC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC} \), where \( \mathcal{E}_{ABC} \) denotes the entanglement between systems \( A \) and \( BC \). Whereas monogamy of entanglement shows the restricted sharability of multipartite entanglement, the distribution of entanglement, or entanglement of assistance6,7 between \( A \) and \( BC \), in multipartite quantum systems was shown to have a dually monogamous (polygamous) property. Note that the monogamy of entanglement inequalities provide an upper bound for bipartite entanglement of entanglement in a multipartite system, and the same quantity sets a lower bound for the distribution of bipartite entanglement.8,9

For pure states \( |\psi\rangle_{ABC} \), the concurrence of assistance is defined by \( \tau_{a}^{\alpha} \), where \( \tau_{a}^{\alpha} = \max_{\{p_{i},|\psi_{i}\rangle\}} \sum_{i} p_{i} C(|\psi_{i}\rangle) \), with \( p_{i} \geq 0 \), \( \sum_{i} p_{i} = 1 \), and \( |\psi_{i}\rangle \in \mathcal{H}_{A} \otimes \mathcal{H}_{B} \).

For a tripartite state \( |\psi\rangle_{ABC} \), the concurrence of assistance is defined by \( C_{a}(\psi_{ABC}) = \max_{\{p_{i},|\psi_{i}\rangle\}} \sum_{i} p_{i} C(|\psi_{i}\rangle) \), where the maximum is taken over all possible pure state decompositions of \( \rho_{AB} = \sum_{i} p_{i} |\psi_{i}\rangle\langle\psi_{i}| \), with \( p_{i} \geq 0 \), \( \sum_{i} p_{i} = 1 \), and \( |\psi_{i}\rangle \in \mathcal{H}_{A} \otimes \mathcal{H}_{B} \).

For pure states \( \rho_{AB} = |\psi\rangle\langle\psi|_{AB} \), one has \( C(|\psi\rangle_{AB}) = C_{a}(\rho_{AB}) \).

For an \( N \)-qubit state, \( \rho_{AB_{1}} \cdots B_{N-1} \in \mathcal{H}_{A} \otimes \mathcal{H}_{B_{1}} \otimes \cdots \otimes \mathcal{H}_{B_{N-1}} \), the concurrence \( C(\rho_{AB_{1}} \cdots B_{N-1}) \) of the state \( \rho_{AB_{1}} \cdots B_{N-1} \), viewed as a bipartite state under the partitioning \( A \) and \( B_{1} \cdots B_{N-1} \), is given by \( C(\rho_{AB_{1}} \cdots B_{N-1}) = C_{a}(\rho_{AB_{1}} \cdots B_{N-1}) \).
tition $A$ and $B_1, B_2, \cdots, B_{N-1}$, satisfies the Coffman-Kundu-Wooters inequality\cite{Coffman2000, Leung2000},

$$C^2(\rho_{AB}) = \sum_{i=1}^{N-1} C^2(\rho_{AB_i}) \geq \sum_{i=1}^{N-1} C^2(\rho_{AB_i}) \quad (1)$$

where $\rho_{AB_i} = \text{Tr}_{B_{i-1}B_{i+1}} B_i (\rho_{AB_1 \cdots B_{N-1}})$. Further improved monogamy relations are presented in\cite{Pan2006, Horodecki2007}. The dual inequality in terms of the concurrence of assistance for $N$-qubit states has the form\cite{Winter2005},

$$C^2(\rho_{AB}) = \sum_{i=1}^{N-1} C^2(\rho_{AB_i}) \leq \sum_{i=1}^{N-1} C^2(\rho_{AB_i}) \quad (2)$$

Now, let us consider a bipartite pure state of arbitrary dimension $d_1 \times d_2$, $|\phi\rangle_{AB} = \sum_{i=1}^{d_1} \sum_{k=1}^{d_2} a_{ik} |k\rangle_{AB}$ in $C^{d_1} \otimes C^{d_2}$. The squared concurrence of $|\phi\rangle_{AB}$ can be expressed as \cite{Plenio2007},

$$C^2(|\phi\rangle_{AB}) = 2(1 - \text{Tr}(\rho_A^2)) = 4 \sum_{i<j} \sum_{k<l} |a_{ik} a_{jl} - a_{il} a_{jk}|^2 \quad (3)$$

For a mixed state $\rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB} \langle \phi_i|$, its concurrence of assistance satisfies\cite{Fan2009},

$$C_A(\rho_{AB}) = \max_{\{p_i, |\phi_i|\}} \sum_i p_i C(|\phi_i|) \quad (4)$$

where

$$D_1 = d_1(d_1-1)/2, \quad D_2 = d_2(d_2-1)/2 \quad \text{(5)}$$

$$P_A^m = P^m_A(\cdot) |\langle j | A(\cdot) | A(i) \rangle| P^m_A \quad \text{(6)}$$

$$P_B^m = P^m_B(|l\rangle_B | l\rangle_B) P_B^m \quad \text{(7)}$$

with $P_A^m = |i\rangle_A(\cdot) + |j\rangle_A(\cdot)$ and $P_B^m = |k\rangle_B(\cdot) + |l\rangle_B(\cdot)$ being the projectors to the subspaces spanned by $\{|i\rangle_A, |j\rangle_A\}$ and $\{|k\rangle_B, |l\rangle_B\}$, respectively. A general polygamy inequality for any multipartite pure state $|\phi\rangle_{A_1 \cdots A_n} \in C^{d_1} \otimes \cdots \otimes C^{d_n}$ was established as\cite{Fan2009},

$$\tau^2_a(|\phi\rangle_{A_1A_2 \cdots A_n}) \leq \sum_{i=1}^{n} \tau^2_a(\rho_{A_1A_i}) \quad \text{(8)}$$

where $\rho_{A_1A_i}$ is the reduced density matrix $|\phi\rangle_{A_1A_2A_i} \cdots A_n$ with respect to subsystem $A_k A_k, k = 2, \cdots, n$.

**POLYGAMY RELATION FOR CONCURRENCE OF ASSISTANCE**

[Lemma 1]. For any real numbers $x$ and $t$, $t \geq 1$, $x \geq 1$, we have $(1 + t)^2 \leq (1 + 2x - 1)t^2$.\[Proof]. Let $f(x, y) = (1+y)^x - y^x$ with $x \geq 1$, $0 < y \leq 1$, $\frac{df}{dy} = x[(1+y)^{x-1} - y^{x-1}] \geq 0$. Therefore, $f(x, y)$ is an increasing function of $y$, i.e., $f(x, y) \leq f(x, 1) = 2x - 1$. Set $y = \frac{1}{t^2}$, $t \geq 1$. We obtain $(1 + t)^2 \leq 1 + (2x - 1)t^2$. Notice when $t = 1$, the inequality is true. Q.E.D.

The following theorem provides a class of polygamy inequalities satisfied by the $\alpha$-power of $\tau_a$. For convenience, we denote $\tau_a(\rho_{AB}) = \tau_a(\rho_{AB})$ the concurrence of assistance $\rho_{AB}$, and $\tau_a(\rho_{AB_1B_2B_3B_4}) = \tau_{A_1B_1A_2B_2}$. \[Theorem 1]. For any tripartite pure state $\rho_{ABC} \in H_A \otimes H_B \otimes H_C$:

1. If $\tau_{A|BC} \geq \tau_{A|AC}$, the concurrence of assistance satisfies

$$\tau_{A|BC}^a \leq \tau_{A|AC}^a + (2x - 1)\tau_{A|AB}^a \quad \text{(9)}$$

for $\alpha \geq 2$.

2. If $\tau_{A|BC} \leq \tau_{A|AC}$, the concurrence of assistance satisfies

$$\tau_{A|BC}^a \leq \tau_{A|AC}^a + (2x - 1)\tau_{A|AB}^a \quad \text{(10)}$$

for $\alpha \geq 2$.

[Proof]. For arbitrary tripartite pure state $\rho_{ABC}$, one has\cite{Fan2009}, $\tau_{A|BC}^2 \leq \tau_{A|AC}^2 + \tau_{A|AB}^2$. If $\tau_{A|BC} = 0$, the inequality (9) or (10) are true obviously. Therefore, assuming $\tau_{A|BC} \geq \tau_{A|AC} > 0$, we have

$$\tau_{A|ABC}^2 \leq (\tau_{A|AC}^2 + \tau_{A|AB}^2)^x = \tau_{A|AC}^2 \left(1 + \frac{\tau_{A|AB}^2}{\tau_{A|AC}^2}\right)^x \leq \tau_{A|AC}^2 \left(1 + (2x - 1)\left(\frac{\tau_{A|AB}^2}{\tau_{A|AC}^2}\right)^x\right) = \tau_{A|AC}^2 + (2x - 1)\tau_{A|AB}^2 \quad \text{(11)}$$

where the second inequality is true due to the inequality $(1 + t)^2 \leq 1 + (2 - 1)t^2$ for $x \geq 1$ and $t = \frac{\tau_{A|AB}}{\tau_{A|AC}} \geq 1$. Denote $2x = \alpha$. We obtain $\alpha \geq 2$ as $x \geq 1$. Then we have the inequality (9). If $\tau_{A|BC} \leq \tau_{A|AC}$, similarly we get (10).

**Example 1.** Let us consider the three-qubit state $|\psi\rangle$ in the generalized Schmidt decomposition form,

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\tau}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \quad \text{(12)}$$

where $\lambda_i \geq 0$, $i = 0, 1, 2, 3, 4$. And $\sum_{i=0}^4 \lambda_i^2 = 1$. We have

$$\tau_{A|BC} = 2\lambda_0 \sqrt{\lambda_0^2 + \lambda_2^2 + \lambda_4^2}, \quad \tau_{A|AB} = 2\lambda_0 \sqrt{\lambda_0^2 + \lambda_3^2}, \quad \text{and} \quad \tau_{A|AC} = 2\lambda_0 \sqrt{\lambda_0^2 + \lambda_4^2}.$$ 

Without loss of generality, we set $\lambda_0 = \cos \theta_0$, $\lambda_1 = \sin \theta_0 \cos \theta_2$, $\lambda_2 = \sin \theta_0 \sin \theta_2$, $\lambda_3 = \sin \theta_0 \cos \theta_2$, $\lambda_4 = \sin \theta_0 \sin \theta_2 \sin \theta_3$, $\theta_i \in [0, \frac{\pi}{2}]$.

For $\lambda_3 \geq \lambda_2$, i.e. $\tau_{A|AC} \geq \tau_{A|AB}$:
(a) if \( \theta_2 = \frac{\pi}{2} \),
\[
\tau^\alpha_{A|BC} - \tau^\alpha_{AB} - (2^\frac{\pi}{2} - 1)\tau^\alpha_{AC} = (2\lambda_0)^\alpha \left[ (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)^\frac{\pi}{2} - (\lambda_2^2 + \lambda_3^2)^\frac{\pi}{2} \right. \\
\left. - (2^\frac{\pi}{2} - 1)(\lambda_3^2 + \lambda_4^2)^\frac{\pi}{2} \right] = 2^\alpha \cos^\alpha \theta_0 \sin^\alpha \theta_0 \sin^\alpha \theta_1 (2 - \sin^\alpha \theta_3 - 2^\frac{\pi}{2}) \leq 0,
\]
where \( \alpha \geq 2 \) and the inequality is due to \( \sin \theta_3 \geq 0 \).

(b) If \( \theta_2 \neq \frac{\pi}{2} \), we denote \( t_1 = \frac{\sin \theta_0 \cos \theta_3}{\cos \theta_2} \geq 1 \), then we have
\[
\tau^\alpha_{A|BC} - \tau^\alpha_{AB} - (2^\frac{\pi}{2} - 1)\tau^\alpha_{AC} = (2\lambda_0)^\alpha \left[ (\lambda_2^2 + \lambda_3^2 + \lambda_4^2)^\frac{\pi}{2} - (\lambda_2^2 + \lambda_3^2)^\frac{\pi}{2} \right. \\
\left. - (2^\frac{\pi}{2} - 1)(\lambda_3^2 + \lambda_4^2)^\frac{\pi}{2} \right] = 2^\alpha \cos^\alpha \theta_0 \sin^\alpha \theta_0 \sin^\alpha \theta_1 \left[ 1 - (\cos^2 \theta_2 + \sin^2 \theta_2 \sin^2 \theta_3)^\frac{\pi}{2} \\
- (2^\frac{\pi}{2} - 1) \sin^\alpha \theta_2 \right] \leq 2^\alpha \cos^\alpha \theta_0 \sin^\alpha \theta_0 \sin^\alpha \theta_1 \left[ 1 - \cos \theta_2 - (2^\frac{\pi}{2} - 1) \sin^\alpha \theta_2 \right] = 2^\alpha \cos^\alpha \theta_0 \sin^\alpha \theta_0 \sin^\alpha \theta_1 \left[ 1 - \cos \theta_2 (1 + t_1)^\frac{\pi}{2} \right] \leq 2^\alpha \cos^\alpha \theta_0 \sin^\alpha \theta_0 \sin^\alpha \theta_1 \left[ 1 - \cos \theta_2 (1 + t_1)^\frac{\pi}{2} \right] = 2^\alpha \cos^\alpha \theta_0 \sin^\alpha \theta_0 \sin^\alpha \theta_1 \left[ 1 - \cos \theta_2 \left( 1 + \frac{\sin^2 \theta_2}{\cos^2 \theta_2} \right)^\frac{\pi}{2} \right] = 0,
\]
where \( \alpha \geq 2 \) and the second inequality is due to Lemma 1.

Therefore, we have \( \tau^\alpha_{A|BC} \leq \tau^\alpha_{AB} + (2^\frac{\pi}{2} - 1)\tau^\alpha_{AC} \) for \( \alpha \geq 2 \).

When \( \lambda_3 \leq \lambda_2 \), i.e. \( \tau_{AC} \leq \tau_{AB} \), from similar analysis, we can obtain \( \tau^\alpha_{A|BC} \leq \tau^\alpha_{AC} + (2^\frac{\pi}{2} - 1)\tau^\alpha_{AB} \) for \( \alpha \geq 2 \).

Specially, when \( \theta_2 = \frac{\pi}{2}, \alpha = 2, \theta_3 = 0 \), i.e. \( |\psi\rangle = \cos \theta_0 |000\rangle + \sin \theta_0 \cos \theta_1 e^{i\varphi} |100\rangle + \sin \theta_0 \sin \theta_1 |110\rangle \), the inequality in (8) is saturated. Generalizing the conclusion in Theorem 1 to \( N \) partite case, we have the following result.

**Theorem 2.** For any multipartite pure state \( \rho_{AB_0\cdots B_{N-1}} \), if \( \tau^\alpha_{AB_i} \leq \sum_{j=i+1}^{N-1} \tau^\alpha_{AB_k} \) for \( i = 0, 1, \cdots, m \), and \( \tau^\alpha_{AB_i} \geq \sum_{j=i+1}^{N-1} \tau^\alpha_{AB_k} \) for \( j = m+1, \cdots, N-2 \), \( \forall 1 \leq m \leq N-3, N \geq 4 \), we have
\[
\tau^\alpha_{A|B_0B_1\cdots B_{N-1}} \leq \tau^\alpha_{AB_0} + (2^\frac{\pi}{2} - 1)\tau^\alpha_{AB_1} + \cdots + (2^\frac{\pi}{2} - 1)^m \tau^\alpha_{AB_m} + (2^\frac{\pi}{2} - 1)^{m+1} \tau^\alpha_{AB_{N-1}}, \tag{13}
\]
for \( \alpha \geq 2 \).

[Proof]. From the inequality (8) and Theorem 1, we have
\[
\tau^\alpha_{A|B_0B_1\cdots B_{N-1}} \leq \tau^\alpha_{AB_0} + (2^\frac{\pi}{2} - 1)\left( \sum_{i=1}^{N-1} \tau^\alpha_{AB_i} \right)^\frac{\pi}{2} \leq \tau^\alpha_{AB_0} + (2^\frac{\pi}{2} - 1)\tau^\alpha_{AB_1} + (2^\frac{\pi}{2} - 1)^2 \left( \sum_{i=2}^{N-1} \tau^\alpha_{AB_i} \right)^\frac{\pi}{2} \leq \cdots \leq \tau^\alpha_{AB_0} + (2^\frac{\pi}{2} - 1)^m \tau^\alpha_{AB_m} + (2^\frac{\pi}{2} - 1)^{m+1} \left( \sum_{i=m+1}^{N-1} \tau^\alpha_{AB_i} \right)^\frac{\pi}{2}. \tag{14}
\]
Similarly, as \( \tau^\alpha_{AB_i} \geq \sum_{k=j+1}^{N-1} \tau^\alpha_{AB_k} \) for \( j = m + 1, \cdots, N-2 \), we get
\[
\left( \sum_{i=m+1}^{N-1} \tau^\alpha_{AB_i} \right)^\frac{\pi}{2} \leq (2^\frac{\pi}{2} - 1)\tau^\alpha_{AB_{m+1}} + \left( \sum_{i=m+2}^{N-1} \tau^\alpha_{AB_i} \right)^\frac{\pi}{2} \leq (2^\frac{\pi}{2} - 1)(\tau^\alpha_{AB_{m+1}} + \cdots + \tau^\alpha_{AB_{N-2}}) + \tau^\alpha_{AB_{N-1}}. \tag{15}
\]
Combining (14) and (15), we have Theorem 2. ☐

In Theorem 2, if \( \tau^\alpha_{AB_i} \leq \sum_{j=i+1}^{N-1} \tau^\alpha_{AB_j} \) for all \( i = 0, 1, \cdots, N-2 \), then we have the following conclusion:

**Theorem 3.** For any multipartite pure state \( \rho_{AB_0\cdots B_{N-1}} \), if \( \tau^\alpha_{AB_i} \leq \sum_{j=i+1}^{N-1} \tau^\alpha_{AB_j} \) for all \( i = 0, 1, \cdots, N-2 \), we have
\[
\tau^\alpha_{A|B_0B_1\cdots B_{N-1}} \leq \sum_{j=0}^{N-1} (2^\frac{\pi}{2} - 1)^j \tau^\alpha_{AB_j}, \tag{16}
\]
for \( \alpha \geq 2 \).

**Example 2.** We consider again the pure state (12). Setting \( \lambda_0 = \lambda_1 = \frac{1}{2}, \lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{3}}{6} \), one has \( \tau_{A|B} = \frac{\sqrt{2}}{2}, \tau_{AB} = \tau_{AC} = \frac{\sqrt{3}}{2} \). Let \( y = \tau^\alpha_{AB} + (2^\frac{\pi}{2} - 1)\tau^\alpha_{AC} - \tau^\alpha_{A|B} = 2^\frac{\pi}{2} (\frac{\sqrt{3}}{6})^\alpha, \alpha \geq 2 \), be the residual concurrence of assistance. From our results, one can see that \( y > 0 \) for \( \alpha \geq 2 \), which is the case that does not given in [28], see Fig. 1.

**POLYGAMY RELATIONS FOR ENTANGLEMENT OF ASSISTANCE**

For polygamy inequality beyond qubits, it was shown that the von Neumann entropy can be used to establish a polygamy inequality of tripartite quantum systems [16]. For any arbitrary dimensional tripartite
pure state $|\psi\rangle_{ABC}$, one has $E(|\psi\rangle_{A|BC}) \leq E_a(\rho_{AB}) + E_a(\rho_{AC})$, where $E(|\psi\rangle_{A|BC}) = S(\rho_A)$ is the entropy of entanglement between $A$ and $BC$ in terms of the von Neumann entropy $S(\rho) = -\text{Tr}\rho \ln \rho$, and $E_a(\rho_{AB}) = \max \sum_i p_i E(|\psi_i\rangle_{AB})$, with the maximization taking over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$. Later, a general polygamy inequality for any multipartite state $\rho_{A_1A_2...A_n}$ was established, $E_a(\rho_{A_1A_2...A_n}) \leq \sum_{i=2}^{n-1} E_a(\rho_{A_1i})$ [17].

Recently, another class of multipartite polygamy inequalities in terms of the $\beta$th power of entanglement of assistance (EOA) has been introduced [28]. For any multipartite state $\rho_{A_0B_1...B_{N-1}}$ and $0 \leq \beta \leq 1$, if $E_{a_{AB_i}} \geq \sum_{j=i+1}^{N-1} E_{a_{AB_j}}$ for $i = 0, 1, \ldots, N-2$, then $E_{a_{AB_0B_1...B_{N-1}}}^\beta \leq \sum_{j=0}^{N-1} \beta^j E_{a_{AB_j}}^\beta$, where $E_a(\rho_{AB_i}) = E_{a_{AB_i}}$, is the entanglement of assistance $\rho_{AB_i}$ and $E_a(\rho_{A_0B_1...B_{N-1}}) = E_{a_{A_0B_1...B_{N-1}}}$. But, for $\beta \geq 1$ the polygamy relations for the $\beta$th power of the entanglement of assistance is still not clear.

[Theorem 4]. For any multipartite state $\rho_{A_0B_1...B_{N-1}}$, if $E_{a_{AB_i}} \leq \sum_{j=i+1}^{N-1} E_{a_{AB_j}}$ for $i = 0, 1, \ldots, m$, and $E_{a_{AB_j}} \geq \sum_{k=j+1}^{N-1} E_{a_{AB_k}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, we have

\begin{equation}
E_{a_{AB_0B_1...B_{N-1}}}^\beta \leq E_{a_{AB_0}}^\beta + (2^\beta - 1)E_{a_{AB_1}}^\beta + \cdots + (2^\beta - 1)^m E_{a_{AB_m}}^\beta \\
+ (2^\beta - 1)^{m+2}(E_{a_{AB_{m+1}}}^\beta \cdots + E_{a_{AB_{N-2}}}^\beta) \\
+ (2^\beta - 1)^{m+1}E_{a_{AB_{N-1}}}^\beta,
\end{equation}

for $\beta \geq 1$.

[Proof]. From Lemma 1, we have

\begin{align*}
E_{a_{A|B_0B_1...B_{N-1}}}^\beta & \leq E_{a_{AB_0}}^\beta + (2^\beta - 1)
\left(\sum_{i=1}^{N-1} E_{a_{AB_i}}\right)^\beta \\
& \leq E_{a_{AB_0}}^\beta + (2^\beta - 1)E_{a_{AB_1}}^\beta + (2^\beta - 1)^2
\left(\sum_{i=2}^{N-1} E_{a_{AB_i}}\right)^\beta \\
& \leq \cdots \\
& \leq E_{a_{AB_0}}^\beta + (2^\beta - 1)E_{a_{AB_1}}^\beta + \cdots + (2^\beta - 1)^m E_{a_{AB_m}}^\beta \\
& \quad + (2^\beta - 1)^{m+1}
\left(\sum_{i=m+1}^{N-1} E_{a_{AB_i}}\right)^\beta. \quad (18)
\end{align*}

Similarly, as $E_{a_{AB_j}} \geq \sum_{k=j+1}^{N-1} E_{a_{AB_k}}$ for $j = m + 1, \ldots, N - 2$, we get

\begin{align*}
\left(\sum_{i=m+1}^{N-1} E_{a_{AB_i}}\right)^\beta & \leq (2^\beta - 1)E_{a_{AB_{m+1}}}^\beta + \left(\sum_{i=m+2}^{N-1} E_{a_{AB_i}}\right)^\beta \\
& \leq (2^\beta - 1)(E_{a_{AB_{m+1}}}^\beta + \cdots + E_{a_{AB_{N-2}}}^\beta) \\
& \quad + E_{a_{AB_{N-1}}}^\beta. \quad (19)
\end{align*}

Combining (18) and (19), we have Theorem 4. □

As a special case of Theorem 4, if $E_{a_{AB_i}} \leq \sum_{j=i+1}^{N-1} E_{a_{AB_j}}$ for all $i = 0, 1, \ldots, N - 2$, we have the following conclusion:

[Theorem 5]. For any multipartite state $\rho_{A_0B_1...B_{N-1}}$, if $E_{a_{AB_i}} \leq \sum_{j=i+1}^{N-1} E_{a_{AB_j}}$ for all $i = 0, 1, \ldots, N - 2$, we have

\begin{equation}
E_{a_{A|B_0B_1...B_{N-1}}}^\beta \leq \sum_{j=0}^{N-1} (2^\beta - 1)^j E_{a_{AB_j}}^\beta, \quad (20)
\end{equation}

for $\beta \geq 1$.

Example 3. Let consider the three-qubit $W$ state $|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. We have $E_a(|W\rangle_{A|BC}) = S(\rho_A) = \log_2 3 - \frac{A}{2}$ and $E_a(\rho_{AC}) = \frac{A}{2}$. Set $y = E_{a_3}(\rho_{AB}) + (2^\beta - 1)E_{a_2}(\rho_{AC}) - E_{a_2}^\beta(|W\rangle_{A|BC}) = 2^\beta (\frac{A}{2})^\beta - (\log_2 3 - \frac{A}{2})^\beta$ to be the residual entanglement of assistance. Fig. 2 shows our polygamy inequality for $\beta \geq 1$.

**Polygamy Relations for ScrenoA**

Given a bipartite state $\rho_{AB}$ in $H_A \otimes H_B$, the negativity is defined by [41], $N(\rho_{AB}) = (\|\rho_{AB}^T\| - 1)/2$, where $\rho_{AB}^T$ is the partially transposed $\rho_{AB}$ with respect to the subsystem $\mathcal{A}$, $\|X\|$ denotes the trace norm of $X$, i.e., $\|X\| =$
For the purpose of discussion, we use the following definition of negativity, $N(\rho_{AB}) = ||\rho_{AB}||_{1} - 1$. For any bipartite pure state $|\psi\rangle_{AB}$, the negativity $N(\rho_{AB})$ is given by $N(|\psi\rangle_{AB}) = 2 \sum_{i<j} \sqrt{\lambda_i \lambda_j} = (\text{Tr} \sqrt{\rho_A})^2 - 1$, where $\lambda_i$ are the eigenvalues for the reduced density matrix $\rho_A$ of $|\psi\rangle_{AB}$. For a mixed state $\rho_{AB}$, the square of convex-roof extended negativity (SCREEN) is defined by

$$N_{sc}(\rho_{AB}) = \min \sum_{i} p_i N(|\psi_i\rangle_{AB})^2,$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle_{AB}\}$ of $\rho_{AB}$. Similar to the duality between concurrence and concurrence of assistance, we also define a dual quantity to SCREEN as

$$N_{sc}^\ast(\rho_{AB}) = \max \sum_{i} p_i N(|\psi_i\rangle_{AB})^2,$$

which we refer to as the SCREEN of assistance (SCREN0A), where the maximum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle_{AB}\}$ of $\rho_{AB}$. For convenience, we denote $N_{sc}^{\ast A}{\rho_{AB}} = N_{sc}^\ast(\rho_{AB}),$ the SCREN0A of $\rho_{AB}$, and $N_{sc}^{\ast A}(\rho_{AB}) = N_{sc}^\ast(|\psi\rangle_{AB})_{B_0...B_{N-1}}.$

In [27] it has been shown that $N_{sc}^{\ast A}(\rho_{AB}) \leq \sum_{j=0}^{N-1} N_{sc}^{\ast A}(\rho_{B_j})$. It is further improved that for $0 \leq \beta \leq 1$, $(N_{sc}^{\ast A}(\rho_{B_{i}})_{B_i...B_{N-1}}) \leq \sum_{j=0}^{N-1} \beta^j (N_{sc}^{\ast A}(\rho_{B_j})$. But, it is still not clear whether the polygamy relation still holds for the $\beta$th ($\beta \geq 1$) power of SCREN0A. With a similar consideration to $\tau_{B_0...B_{N-1}}$, we have the following result of SCREN0A for $\beta \geq 1$.

[Theorem 6]. For any multipartite state $\rho_{AB_0...B_{N-1}}$, if $N_{sc}^{\ast A}(\rho_{AB_i}) \leq \sum_{k=i+1}^{N-1} N_{sc}^{\ast A}(\rho_{B_k})$ for $i = 0, 1, \ldots, m$, and $N_{sc}^{\ast A}(\rho_{AB_j}) \geq \sum_{k=j+1}^{N-1} N_{sc}^{\ast A}(\rho_{B_k})$ for $j = m + 1, \ldots, N - 2, \forall 1 \leq m \leq N - 3, N \geq 4$, we have

$$N_{sc}^{\ast A}(\rho_{AB_0B_1...B_{N-1}})^\beta \leq (N_{sc}^{\ast A}(\rho_{B_0B_1...B_{N-1}})^\beta$$

$$+ (2^\beta - 1)(N_{sc}^{\ast A}(\rho_{AB_1})_B^\beta + \ldots + (2^\beta - 1)^m (N_{sc}^{\ast A}(\rho_{AB_m})_B^\beta$$

$$+ (2^\beta - 1)^{m+1} (N_{sc}^{\ast A}(\rho_{AB_{N-1}})_B^\beta,$$

for $\beta \geq 1$.

[Proof]. From Lemma 1, we have

$$N_{sc}^{\ast A}(\rho_{AB_0B_1...B_{N-1}})^\beta \leq (N_{sc}^{\ast A}(\rho_{B_0B_1...B_{N-1}})^\beta$$

$$+ (2^\beta - 1)(N_{sc}^{\ast A}(\rho_{AB_1})_B^\beta + \ldots + (2^\beta - 1)^m (N_{sc}^{\ast A}(\rho_{AB_m})_B^\beta$$

$$+ (2^\beta - 1)^{m+1} (N_{sc}^{\ast A}(\rho_{AB_{N-1}})_B^\beta.$$

Similarly, as $N_{sc}^{\ast A}(\rho_{AB_j}) \geq \sum_{k=j+1}^{N-1} N_{sc}^{\ast A}(\rho_{B_k})$ for $j = m + 1, \ldots, N - 2, \forall 1 \leq m \leq N - 3, N \geq 4$, we get

$$N_{sc}^{\ast A}(\rho_{B_0B_1...B_{N-1}})^\beta \leq (2^\beta - 1)(N_{sc}^{\ast A}(\rho_{B_m+1})_B^\beta + \ldots + (2^\beta - 1)^m (N_{sc}^{\ast A}(\rho_{B_{N-1}})_B^\beta$$

$$+ (2^\beta - 1)^{m+1} (N_{sc}^{\ast A}(\rho_{B_{N-1}})_B^\beta.$$

Combining (24) and (25), we have the Theorem 6. □

Particularly, the equality in (23) can be established from Theorem 6.

[Theorem 7]. For any multipartite state $\rho_{AB_0...B_{N-1}}$, if $E_{AB_i} \leq \sum_{j=i+1}^{N-1} E_{AB_j}$ for all $i = 0, 1, \ldots, N - 2$, we have

$$N_{sc}^{\ast A}(\rho_{AB_0B_1...B_{N-1}})^\beta \leq \sum_{j=0}^{N-1} (2^\beta - 1)^j (N_{sc}^{\ast A}(\rho_{B_j})_B^\beta$$

for $\beta \geq 1$.
Entanglement monogamy and polygamy are fundamental properties of multipartite entangled states. We have investigated in this work the polygamy relations related to the concurrence of assistance, entanglement of assistance, and SCREN generally for multipartite states. We have found a class of polygamy inequalities of multipartite entanglement in arbitrary-dimensional quantum systems in the $\alpha$th ($\alpha \geq 2$) power of concurrence of assistance, a case that has not been studied before. Moreover, the $\beta$th power of polygamy inequalities have been obtained for the entanglement of assistance and SCRENoA for $\beta \geq 1$. The approach developed in this work is applicable to the study of monogamy properties in other quantum entanglement measures and quantum correlations.

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