On Cosmological Perturbations on a Brane in an Anti–de Sitter Bulk

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(8 December 2000)

In this paper, we consider cosmological perturbations on a brane universe embedded in an Anti–de Sitter bulk. We use a novel gauge, in which the full five–dimensional problem is in principle solvable. In this gauge we derive the equations for scalar, vector and tensor perturbations. These equations are necessary in order to calculate microwave background anisotropies in this particular scenario. Throughout the paper, we draw attention to the influence of the bulk gravitons, which act as a source for the perturbations on the brane. In addition, we find that isocurvature modes are generated due to the influence of bulk gravitons.

I. INTRODUCTION

In a now famous paper, Randall and Sundrum have argued that compactification is not the only way one should think on how to hide extra spacetime dimensions [1]. In fact, rather than compactifying the extra dimension(s), they showed that not only matter could be confined on a brane with tension, but also gravity itself. In their setup they discussed the case for a brane embedded in an Anti–de Sitter spacetime with a $\mathbb{Z}_2$–symmetry. They found that, due to the curvature of the Anti–de Sitter spacetime, the Kaluza–Klein zero mode was indeed confined on the brane. Thus, for energy densities lower than the brane tension, gravity is effectively four–dimensional from the point of view of an observer confined on the brane.

The cosmological consequences for the background evolution in this type of brane world scenario were discussed intensively (for a rather incomplete list see, e.g. [2]). However, a definite test for this alternative to compactification is still missing. As such, the study of cosmological perturbations in brane worlds is very important. Indeed, perturbations can evolve differently in these models. Thus, there is the hope that the signs of extra dimensions may be found in the anisotropies of the cosmic microwave background or in the matter distribution of the universe (for a discussion on perturbations in brane worlds, see [3], [4], [5] and [6]).

In an interesting geometric approach, Einsteins equations were derived on the brane [7]. It was found that they can be written as follows,

$$G^{(4)}_{\mu\nu} = -\Lambda_{\text{eff}} g_{\mu\nu} + 8\pi G T_{\mu\nu} + \kappa_5^2 \pi_{\mu\nu} - E_{\mu\nu},$$

being $\kappa_5$ the five–dimensional gravitational coupling constant. The terms in this equation have the following physical meaning:

i) $g_{\mu\nu}$ is the induced metric on the brane,

ii) $\Lambda_{\text{eff}}$ is the effective cosmological constant on the brane, which has contributions from the bulk cosmological constant and brane tension, and could be tuned to be zero,

iii) $G$ is Newton’s gravitational constant, which is directly connected to the brane tension $U_b$ by $48\pi G = \kappa_5^2 U_b$,

iv) $\pi_{\mu\nu}$ is a tensor quadratic in the stress–energy tensor $T_{\mu\nu}$ for the brane matter,

v) and $E_{\mu\nu}$ is the projection of the five–dimensional Weyl tensor onto the brane. This is a non–local contribution which physically describes the influence of the bulk gravitons. For a homogeneous brane universe, $E_{\mu\nu}$ gives rise to a “dark radiation” term. That is, matter with the equation of state $p = \rho/3$. The amplitude of this radiation term is arbitrary, and actually it might set to be zero. Indeed, it was argued, that this amplitude measures the mass of a black hole inside the bulk [8].

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For an homogeneous and isotropic brane universe, equation (1) can be used to derive an effective Friedmann equation, which besides the usual four–dimensional Friedmann equation, it also contains a term quadratic in the energy density on the brane.

In [9], the equations for scalar cosmological perturbations in such a framework where derived. It was shown that the usual four–dimensional equations appear, together with a quadratic correction. Again, the influence of the bulk appeared in form of a fluid with an equation of state \( p = \rho/3 \). However, these equations where derived using gaussian normal coordinates on the brane. In this paper, we derive the perturbation equations in a gauge in which, at least in principle, the full five–dimensional dynamics is solvable. We find, as expected, that the form of Einstein equations is the same as found in [9]. In addition to that, we derive the equations for vector and tensor perturbations, on the believe that they could also be of relevance in calculating the anisotropies in the cosmic microwave background.

The paper is organized as follows: after discussing very briefly the background equations in the next section, we derive the perturbed Einstein equations on the brane for scalar perturbations in section 3. As a result, we explicitly see that bulk gravitons act as a source for scalar perturbations. Following that, we derive the brane equations for vector and tensor perturbations in section 4 and section 5. We summarize our findings and discuss consequences for structure formation in section 6.

II. THE BACKGROUND

In this section, we briefly recapitulate the background equations which govern the expansion of the homogenous and isotropic brane universe (see [2] and [3]). We assume that the bulk is Anti–de Sitter, i.e. it contains only a negative cosmological constant. In particular, we will consider here the case of an isolated brane only, i.e. a cosmological version of the second Randall–Sundrum model.

We use the equations, which where derived in [4] and [6]. Restricting the 55–component of the background Einstein equations, using the proper time \( \tau \) of the brane (i.e. \( d\tau = abdt \)) and setting \( \kappa_5 = 1 \), one can easily derive,

\[
\frac{a_{\tau\tau}}{a} + \frac{a^2}{a^2} + \frac{k}{a^2} = -\frac{1}{3} \left[ \frac{1}{12} \rho_b (\rho_b + 3p_b) + q \right].
\]

In this equation, \( \rho_b \) and \( p_b \) are the total brane energy density and brane pressure respectively and \( q \) is the 55–component of the bulk energy–momentum tensor. We follow now the Randall–Sundrum construction, in which the total brane density is a sum of the matter density plus a contribution from the brane tension, i.e.

\[
\rho_b = \rho_M + U_b, \quad p_b = p_M - U_b,
\]

where \( \rho_M \) and \( p_M \) denote the energy density and pressure of the “ordinary” matter on the brane, respectively, and \( U_b \) is the brane tension. Then, equation (3) has the following solution (see [2] or [3] for details)

\[
\frac{a^2}{a^2} = \frac{\rho_M^2}{36} + \frac{8\pi G}{3} \rho_M + \Lambda_{\text{eff}} - \frac{k}{a^2} - \frac{C}{a^4},
\]

where we identify \( U_b = 48\pi G \), being \( G \) the four–dimensional Newton’s constant. Also, \( C \) is an arbitrary constant and \( \Lambda_{\text{eff}} \) is an effective cosmological term given by,

\[
\Lambda_{\text{eff}} = \frac{1}{6} \left( \frac{U_b^2}{6} - q \right).
\]

Following Randall and Sundrum, we assume that the brane tension cancels the contribution from the bulk cosmological constant, so that

\[
\rho = -p = -q = -\frac{U_b^2}{6},
\]

for which \( \Lambda_{\text{eff}} = 0 \). We additionally assume that the arbitrary constant \( C = 0 \), i.e. we neglect the contribution from the projected bulk Weyl tensor. This is justified, given the assumptions we make for the symmetries of the bulk [3]. Thus, the Friedmann equation on the brane that we will consider is

\[
\mathcal{H}^2 = \frac{\rho_M^2}{36} + \frac{8\pi G}{3} \rho_M - \frac{k}{a^2},
\]

where we use the notation \( \mathcal{H} = a_{\tau}/a \) for the Hubble parameter on the brane.
III. THE SCALAR PERTURBATIONS

In the following sections we would like to derive the effective Einstein equations for first order perturbations on the brane. In order to do that, we have to restrict the full Einstein equations onto the brane. In doing so, we will make use of the formalism presented in [4], where the full five–dimensional equations can be found. Since we are not going to repeat the equations for scalar perturbations here, we refer the interested reader to [4] for further details.

We start with the scalar perturbations. The perturbed metric we use takes the form

$$ds^2 = a^2 b^2 \left[ (1 + 2\phi)dt^2 - (1 - 2\Gamma)dy^2 - 2Wdydt - 2Bdydx^i \right] - a^2 \Omega_{ij}(1 - 2\psi)dx^i dx^j. \quad (8)$$

As explained in [4], the brane can be located at \( y = \text{const.} \) along the fifth dimension in this gauge, and the variable \( B \) contains only information about the anisotropic stress on the brane. In the case for vanishing anisotropic stress on the brane, the variable \( B \) can be set to be zero everywhere and the metric would coincide with the generalized longitudinal gauge.

We are going to assume for simplicity that the bulk cosmological constant and the brane tension are real constants, so that \( \delta \rho_0 = \delta \rho_M \) and \( \delta \rho_b = \delta \rho_B \). Also, we will use \( \tau \) as the proper time on the brane, that is

$$a(\tau)b(\tau) = 1. \quad (9)$$

Then, the restriction of the perturbed Einstein equations on the brane is straightforward but rather tedious. Such a restriction proceeds as follows:

1. We first use the background Einstein equations to express the second \( y \)-derivatives of the scale factors \( a \) and \( b \) in terms of their first \( y \)-derivatives and first and second \( t \)-derivatives (see [4] for the background Einstein equations).
2. Then we substitute the jump conditions for the first derivatives of \( a, b, \phi, \psi, \Gamma \) and the jump conditions for the functions \( W \) and \( B \) making sure to take the lower sign in any \( \pm \) combination (which refers to the brane on the right hand side, i.e. \( y = R \), where we assume the brane to be located).
3. Finally, we note that the second \( y \)-derivatives of \( \phi, \psi \) and \( \Gamma \) together with the first \( y \)-derivatives of \( W \) and \( B \) are not locally determined on the brane. Thus, they measure a non–local interaction of the brane with the bulk. Following [8], it is rather convenient to write these terms according to,

$$\phi_5 = \frac{(\phi')^2}{a^2 b^2}, \quad \psi_5 = \frac{(\psi')^2}{a^2}, \quad \Gamma_5 = \frac{(\Gamma a^2 b^2)'}{a^2 b^2}, \quad W_5 = \frac{(Wa^2 b^2)'}{a^2 b^2}, \quad \Sigma_5 = \frac{(Ba^3)}{a^3}. \quad (10)$$

Because we are assuming that there are no matter perturbations in the bulk, we simply have \( \delta G_{\alpha \beta} = 0 \). Using the junction conditions for the scale factors \( a \) and \( b \) and for the metric perturbations, we find that the scalar part of Einstein's equations can be written as

$$\delta^{(4D)} G_0^0 = 8\pi G \delta \rho_M + \frac{\rho_M}{6} \delta \rho_M + \delta \rho_b, \quad (13)$$

$$\delta^{(4D)} G_j^0 = - \left\{ 8\pi Ga^2 (\rho_M + p_M) v_j + \frac{\rho_M a^2 (\rho_M + p_M) v_j + a^2 (\rho_M + p_M) v_j} \right\}. \quad (14)$$

$$\delta^{(4D)} G_j^i = - \left\{ \delta p_5 + \frac{\delta \rho_M}{6} (\rho_M + p_M) + 8\pi G \delta \rho_M + \frac{\delta \rho_M}{6} \rho_M \right\} \epsilon_j^i + 8\pi G \delta \rho_M + \frac{\rho_M}{6} \sigma_j^i + \sigma_j^i, \quad (15)$$

where \( \delta^{(4D)} G_j^i \) is the four–dimensional Einstein tensor, obtained with the four dimensional part of the metric \( G \) in which all perturbations are taken to be functions of the intrinsic brane coordinates \( t \) and \( x^i \). Also, we have conveniently defined

$$\delta p_5 = \frac{\delta \rho_b}{12} (2 \rho_b + 3 \rho_b) - 3 \psi_5 - \Sigma_5 |_{|k} + \frac{\rho_b}{6} B \Gamma |_{|k} + 3 \delta W_5 + \frac{\rho_b}{2} \Gamma_5 + 3 \delta \Gamma_\tau - \frac{1}{a^2} \Gamma |_{|k}, \quad (16)$$

$$a^2 (\rho_M + p_M) v_j = \Gamma_\tau - \delta \Gamma + \frac{1}{2} W_5 + \frac{a^2}{2} \frac{\partial \Sigma_5}{\partial \tau} + \frac{a^2}{6} (2 \rho_b + 3 \rho_b) B_\tau - \frac{\rho_b}{12} W + a^2 \frac{3}{4} R (\rho_M + p_M) B \quad (17)$$

\[3\]
\[ \delta \rho_5 = \frac{\delta p_0}{12}(2\rho_b + 3p_b) + 2\psi_5 - \phi_5 + \frac{2}{3} \nabla^k [k - \frac{\rho_b}{9} \nabla^k B^k] - \frac{\partial W_5}{\partial \tau} - 2\mathcal{H} W_5 + \frac{p_b}{2} \Gamma_5 - \Gamma_{\tau\tau} - 2\mathcal{H}\Gamma_\tau \]

\[ - \left[ 6\mathcal{H}^2 + 2\mathcal{H}_\tau + \frac{4k^2}{a^2} - \frac{p_b^2}{4} \right] \Gamma + \frac{2}{3a^2} \nabla^k [k - \frac{\rho_b}{6} i | i j] \]

\[ \delta_i^j = \frac{\delta_i^j}{5} \nabla^k \left[ k + \frac{1}{a^2} \nabla^k \right] - \frac{\delta_i^j}{3a^2} \nabla^k [k - \frac{\rho_b}{6} i | i j] \]

The four–dimensional Einstein tensor \( \delta^{(4D)} G_{\mu \nu} \) reads

\[ \delta^{(4D)} G^0_0 = -6\mathcal{H}^2 \phi - 6\mathcal{H}_\psi + \frac{6k}{a^2} \psi + 2 \frac{\nabla^k \psi}{| k \nabla^k |} \]

\[ \delta^{(4D)} G^0_i = \left( 2\mathcal{H} \phi + 2\mathcal{H}_\psi \right) \psi \]

\[ \delta^{(4D)} G^i_j = \left[ -6(\mathcal{H}^2 + 4\mathcal{H}_\tau) \phi - 2\mathcal{H}_\phi - \frac{1}{a^2} \nabla^k \phi \left[ k - 2\psi_{\tau\tau} - 6\mathcal{H}\psi + \frac{1}{a^2} \nabla^k \psi \right] \right] \psi \]

In defining the quantities \( \delta \rho_5, \delta p_5, \psi_5 \) and \( \sigma_5 \), we have made explicit use of the junction conditions given in [4]. Note that these terms contain the information about the bulk perturbations. The quantities \( \mathcal{H}^2 \) and \( \mathcal{H}_\tau \) also contain information about the brane and bulk matter fields. Indeed, according to (19) and (20), they can be written as

\[ \mathcal{H}^2 = \frac{1}{3} \left[ \frac{\rho_M^2}{12} + \frac{U_B}{6} \right] \rho_M - \frac{k}{a^2} \]

\[ \mathcal{H}_\tau = - \left[ \frac{\rho_M}{12} + \frac{U_B}{12} \right] \left( \rho_M + p_M \right) + \frac{k}{a^2} \]

On the other hand the restriction of the 5\( \mu \)-components of the perturbed Einstein equations on the brane give the usual energy–momentum conservation for brane perturbations. Finally, rather than the 55–component of the perturbed Einstein equations, we conveniently use the 55–component of the perturbed Ricci tensor, i.e. \( \delta R^5_5 = 0 \). This is possible because the bulk contains only a pure cosmological constant. Thus,

\[ \delta R^5_5 = -3\psi_5 + \phi_5 - \nabla^k \left[ k + \frac{\partial W_5}{\partial \tau} + 3\mathcal{H} W_5 + \frac{1}{6} (\rho_b - 3p_b) \Gamma_5 + \left[ \frac{\delta \rho_5}{36} - \frac{\delta p_5}{12} \right] (2\rho_b + 3p_b) + \Gamma_{\tau\tau} + 3\mathcal{H}\Gamma_\tau - \frac{1}{a^2} \nabla^k [k - \frac{\rho_b}{6} i | i j] \]

\[ + \frac{\rho_b}{6} B^k + \left[ \mathcal{H}_\tau + 5\mathcal{H}^2 + \frac{4k}{a^2} - \frac{p_b^2}{4} + \frac{13}{36} U_B^2 \right] \Gamma = 0. \]

This equation allows to find the following relationship between \( \delta \rho_5 \) and \( \delta p_5 \):

\[ \delta \rho_5 = \frac{1}{3} \delta p_5. \]

Recall that the origin of the terms involving \( \delta \rho_5 \) and \( \delta p_5 \) comes from a non-local interaction between the brane and the bulk gravitons. Therefore, in view of (27), these gravitons can be seen as a source for the fluctuations on the brane [9]. In particular, the bulk quantities \( \Gamma_5 \) and \( B \) give rise to an induced anisotropic stress on the brane.

We remark that, regardless of having a vanishing anisotropic stress on the brane, the quantities \( \phi \) and \( \psi \) no longer commute. This is true due to the appearance of an effective anisotropic stress and it is in clear contrast to the usual four–dimensional case. Indeed, the traceless part of the \( ij \)-component reads

\[ \phi_{ij} = \psi^i_j - \sigma^i_j = \frac{\rho_b}{6} \sigma^i_j \]

Assuming that the anisotropic stress on the branes vanishes, we can replace \( \psi \) in favor of \( \phi \) and \( \sigma_5 \) (observe that \( \sigma_5^k = 0 \)). Then, the 00–component and the trace of the \( ij \)-component can be written as

\[ \phi_{\tau\tau} + 4\mathcal{H}_\phi + \left[ 3\mathcal{H}^2 + 2\mathcal{H}_\tau - \frac{k}{a^2} \right] \phi = 4\pi G \delta \rho_M + \frac{\rho_M}{12} \delta \rho_M + \frac{1}{2} \delta \rho_5 - 3 \left[ \mathcal{H} \sigma_{\tau\tau} - \frac{k}{a^2} \sigma_5 \right] \]

\[ \phi_{\tau\tau} + 4\mathcal{H}_\phi + \left[ 3\mathcal{H}^2 + 2\mathcal{H}_\tau - \frac{k}{a^2} \right] \phi = 4\pi G \delta \rho_M + \frac{\rho_M}{12} \delta \rho_M + \frac{\rho_M + p_M}{12} \delta \rho_M + \frac{1}{2} \delta \rho_5 + \sigma_{5,\tau\tau} + 3\mathcal{H} \sigma_5, \tau - \frac{k}{a^2} \sigma_5 \]

Now, decomposing the pressure perturbations as
\[ \delta \rho = c_s^2 \delta \rho + T \delta S, \]  

where \( \delta S \) is the entropy production and \( c_s \) the sound velocity, we can combine both equations to get

\[ \phi_{\tau\tau} + \left[ 4 + 3c_s^2 \right] H \phi_{\tau} + \left[ 2H + 3H^2 \left( 1 + c_s^2 \right) - \frac{k}{a^2} \left( 1 + 3c_s^2 \right) \right] \phi - c_s^2 \frac{\phi_{\tau}^k}{a^2} = 4\pi G T \delta S + \frac{\rho M}{12} \delta S + \frac{\rho M + \rho M}{12} \delta \rho + \frac{1}{2} \left( \delta p_5 - c_s^2 \delta \rho_5 \right) + \sigma_{5,\tau\tau} + 3H \left( 1 + c_s^2 \right) \sigma_{5,\tau} + \frac{k}{a^2} \left( 1 + 3c_s^2 \right) \sigma_5. \]  

The right hand side corresponds to the usual 4D case, without any further corrections. Also, the first term on the left hand side corresponds to the term usually found in four dimensions, which describes entropy production. Note, that the combination \( \delta \rho_5 - c_s^2 \delta \rho_5 \) describes an effective entropy production, induced by bulk gravitons. In addition, we find the usual corrections quadratic in the energy density and terms involving the effective anisotropic stress induced from bulk gravitons. Since we cannot a priori neglect these extra terms, which contain information from the bulk, a study of the full set of five-dimensional equations is required.

### IV. THE VECTOR PERTURBATIONS

In this section we draw our attention to the vector perturbations. It is commonly assumed that since these type of perturbations decay due to the cosmological expansion, they are not relevant for the study of cosmological perturbations. However, this may not be true in the context of brane world models. The reason is that, analogously to the case of scalar perturbations, vector and tensor perturbations are also sourced by bulk gravitons. Thus, the study of both vector and tensor perturbations may be also necessary.

The most general metric for vector perturbations in brane worlds with a brane located at a constant \( y \) position along the bulk is

\[ ds^2 = a^2 \left[ \gamma_{ab} dy^a dy^b - \left[ \Omega_{ij} + 2F_{(ij)} \right] dx^i dx^j - 2S_{ai} dy^a dx^i \right], \]  

where the three–vectors \( S_{ai} \) and \( F_i \) have vanishing divergences, i.e.

\[ S_{ai}^i = 0, \quad F_i^i = 0. \]  

In five dimensions, there exist two gauge–invariant vector perturbations (see e.g. \[ \Box \]). In the generalized longitudinal gauge, these two gauge–invariant vector perturbations can be identified to \( S_{ai} \), while the remaining vector perturbation \( F_i \) can be conveniently cancelled. Thus, working in this gauge, we have

\[ ds^2 = a^2 \left[ b^2 dt^2 - b^2 dy^2 - \Omega_{ij} dx^i dx^j - 2S_{ai} dt dx^i - 2S_{ai} dy dx^i \right], \]  

On the other hand, the perturbed stress-energy tensor for vector perturbations on the brane is given by

\[ \delta T^{a\beta} = \begin{pmatrix} 0 & -(\rho_b + p_b) b \cdot V_j & 0 \\ -(\rho_b + p_b) S_{0i}^i + (\rho_b + p_b) V_i & 2F_{(ij)} & U^i \\ 0 & p_b b^{-2} S_{5j}^j - b^{-2} U_j & 0 \end{pmatrix}, \]  

where \( V^i \) and \( U^i \) describe two “velocity” fields for the matter on the branes. In the bulk, the first order Einstein equations for the vector perturbations are

\[ a^2 b^2 \delta G_{ai} = - \frac{1}{2b^2} \left[ \frac{\partial^2}{\partial y^a \partial t} + \left( 3 \frac{a'}{a} - 2 \frac{b'}{b} \right) \frac{\partial}{\partial y} + 4b^2 k \right] S_{0i} - \frac{1}{2} S_{0i}^i + \frac{1}{2b^2} \left[ \frac{\partial^2}{\partial y \partial t} + \left( 3 \frac{a'}{a} - 2 \frac{b'}{b} \right) \frac{\partial}{\partial t} \right] S_{5i} = 0, \]  

\[ a^2 b^2 \delta G_{aj} = \left[ \frac{\partial}{\partial t} + 3 \frac{a'}{a} \right] S_{0i}^i - 3 \frac{a'}{a} \frac{\partial}{\partial y} S_{5i}^i = 0 \]  

\[ a^2 b^2 \delta G_{ai} = \frac{1}{2b^2} \left[ \frac{\partial^2}{\partial y \partial t} + \left( 3 \frac{a'}{a} - 2 \frac{b'}{b} \right) \frac{\partial}{\partial y} \right] S_{0i} - \frac{1}{2b^2} \left[ \frac{\partial^2}{\partial t^2} + \left( 3 \frac{a'}{a} - 2 \frac{b'}{b} \right) \frac{\partial}{\partial t} - 4b^2 k \right] S_{5i} + \frac{1}{2} S_{5i}^i = 0. \]  

The junction conditions can easily found to be

\[ 5 \]
\[ S_{5i} = \mathcal{F}_i, \]  
\[ S'_{0i} = \dot{S}_{5i} + (\rho_b + p_b)V_i, \]  
\[ \mathcal{F}_i = \frac{U_i}{p_b}. \]  

Observe, thus, that the velocity field \( U_i \) and the matter perturbation \( \mathcal{F}_i \) are not independent quantities but they are directly related by (42). Defining

\[ P_{5i} = \left( S_{5i} \frac{a^2}{a^2} \right)', \]  
\[ P_{0i} = \frac{1}{2a^5} \left[ a^3 \left( S'_{0i} - \dot{S}_{5i} \right) \right]', \]

the first order Einstein equations for vector perturbations on the brane can be written as

\[
\delta^{(4D)} G_{0i} = -8\pi G a^2 (\rho_M + p_M)V_i - \frac{\rho_M a^2}{6} (\rho_M + p_M)V_i - P_{0i}
\]  
\[
\delta^{(4D)} G_{ij} = 8\pi G \mathcal{F}_{(i|j)} + \frac{\rho_M}{6} \mathcal{F}_{(i|j)} + P_{5_{(i|j)}},
\]

where again, \( \delta^{(4D)} G_{\mu\nu} \) denote the four–dimensional Einstein tensor, which reads

\[
\delta^{(4D)} G_{0i} = -\frac{1}{2} S_{0i}^{\mid k} - 2kS_{0i},
\]  
\[
\delta^{(4D)} G_{ij} = \left[ \frac{\partial}{\partial \tau} + 3H \right] S_{0i}^{(i|j)},
\]

Finally, the remaining non-zero component of Einstein equations, \( \delta G_{i}^{5} \), gives the following energy conservation equation for vector perturbations on the brane,

\[
(\rho_M + p_M)V_i + [2H(\rho_M + p_M) + \dot{p}_M]V_i + \frac{\mathcal{F}_{i|k}}{a^2} + \frac{4k}{a^2} \mathcal{F}_i = 0.
\]

\section{V. THE TENSOR PERTURBATIONS}

Tensor perturbations may be physically related to linearized gravitational waves. For the case of a single extra dimension the most general metric is

\[
ds^2 = a^2 \left\{ b^2 dt^2 - b^2 dy^2 - \left[ \Omega_{ij} + h_{ij} \right] dx^i dx^j \right\},
\]

The 3–dimensional tensor \( h_{ij} \), which is already gauge–invariant, is traceless and divergenceless, i.e.

\[
h^{k\mid k} = 0 \quad \text{and} \quad h_{ij\mid k} = 0.
\]

We use now the following perturbed stress-energy tensor for tensor perturbations on the brane,

\[
\delta T^\alpha_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Pi^i_j & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]  
\[
\delta G^i_j = -\frac{1}{2} \left[ \Box - \frac{2k}{a^2} \right] h^i_j = 0,
\]

where \( \Pi^i_j \) is a 3–dimensional tensor describing the tensorial part of the anisotropic stress of the matter fields on the brane.

The first order Einstein equations for the tensor perturbations in the bulk take the very simple form
where $\Box$ stands for the 5–dimensional Laplacian, given by

$$\Box = \nabla^\mu \nabla_\mu = \frac{1}{a^2 b^2} \left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + 3 \frac{\dot{a}}{a} \frac{\partial}{\partial t} - 3 \frac{a'}{a} \frac{\partial}{\partial y} - b^2 \partial^k \partial_k \right].$$

(54)

Then the junction conditions read

$$h_{ij}' = \Pi_{ij}.$$  

(55)

If we now define the quantity

$$I_{ij} = -\frac{(h_{ij}' a^5)'}{2a^5},$$

(56)

which describes the influence on the brane of the bulk tensor perturbations, we can write the effective four–dimensional Einstein equation as

$$\delta^{(4D)} G^0_i = 8\pi G \Pi^i_j + \frac{\rho_m}{6} \Pi^i_j + I_j^i.$$  

(57)

Once more, $\delta^{(4D)} G^\mu_\nu$, denote the four–dimensional Einstein tensor, which reads

$$\delta^{(4D)} G^i_j = -\frac{1}{2} \left[ \Box_4 - \frac{2k}{a^2} \right] h^i_j,$$

(58)

where

$$\Box_4 = \left[ \frac{\partial^2}{\partial t^2} + 3 \mathcal{H} \frac{\partial}{\partial t} - \frac{1}{a^2} \partial^k \partial_k \right].$$

(59)

VI. CONSEQUENCES FOR STRUCTURE FORMATION

In this paper we have derived the equations for scalar, vector and tensor perturbations on a brane universe embedded in an Anti–de Sitter space. We have assumed a single brane and a bulk without matter perturbations. However, the equations are written in such a way, that, at least in principle, the full set of 5D equations could be solved. As such, the results could be extended to include a second brane, for instance.

What was found is that all kind of perturbations are sourced by bulk gravitons. This is not surprising, since perturbations on the brane induce perturbations in the bulk and vice versa. However, for structure formation on the brane, the dynamics of these source terms should be understood. In the particular case of scalar perturbations, the non–local interaction between the brane and the bulk is described by an effective energy density, pressure, velocity field and anisotropic stress. The effective pressure and energy density are related by an *radiation like* equation of state. Furthermore, the combination $\delta p_5 - c_s^2 \delta \rho_5$, where $c_s$ is the *matter sound speed*, describes an effective entropy production. This entropy production can in principle generate isocurvature modes. The efficiency of the production depends strongly on the evolution of $\delta \rho_5$, $\delta \rho_5$ and the sound speed $c_s$. On the other hand, within the formalism presented so far, the evolution of $\sigma_5$ is not constrained. Although the contribution from bulk gravitons (i.e. the dark radiation term) may be neglected for the background, it reappears in the cosmological perturbation equations. There, we cannot arbitrarily set its amplitude to zero. One important consequence is that, contrary to the usual four–dimensional case, the metric variables $\phi$ and $\psi$ do not necessarily commute, even in the absence of anisotropic stress on the brane. Thus, given that the stress history is very important for structure formation (see [10]), the evolution of $\sigma_5$ has to be studied carefully.

Besides scalar perturbations, we have also found the equations for vector and tensor perturbations. As expected, these perturbations are sourced by bulk gravitons as well. This means that any constraints given to this particular model (from the CMB, for example) have to take into account the contribution of tensor and, once produced, vector modes also. Of course, in order to have the information about bulk perturbations, one has to solve the full Einstein equation in the bulk. Although the gauge used here makes the 5D equations in principle solvable, this is a task beyond the scope of the present work.
In [9] the connection to “seed”–models of structure formation was made. However, within the model we discuss, it is not clear, if the seeds obey a scaling solution. Non–scaling seeds are rather peculiar and need to be studied further (for the case of non–scaling cosmic strings, see [12]). In addition, the projection onto the brane is non–local, which is another new effect in this kind of brane world scenarios. Thus, at this stage, it is not at all clear if the model could be ruled out by current observations.

Finally we would like to comment what would change if a second brane (with negative tension) is somewhere else in the bulk. In that case, all the quantities describing the influence of bulk gravitons (i.e. \( \delta \rho_5, \delta p_5 \), etc.) would carry the information about the perturbations on the other brane. Indeed, through the full 5D equations, it would be possible to relate \( \delta \rho_5, \delta p_5 \), etc. to the matter perturbations on the other brane. However, due to the warp factor these might be strongly suppressed if we lived on the positive tension brane. If that is the case, the low–effective theory relating perturbations of the matter fields on both branes would be, when the physical distance of the branes is stabilized, the usual four–dimensional Einstein gravity, with contributions from both branes.

**Acknowledgments:** We are grateful to Richard Battye, Robert Brandenberger, Andrew Mennim and Toby Wiseman for useful discussions. C.v.d.B. was supported by the Deutsche Forschungsgemeinschaft (DFG). M. Dorca is supported by the Fundación Ramón Areces. The research was supported in part (at Brown) by the U.S. Department of Energy under Contract DE-FG02-91ER40688, TASK A.

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\[1\] That the model has similarities with defect models of structure formation was independently pointed out by Richard Battye to us.