Abstract—Diversity maximization is a fundamental problem with wide applications in data summarization, web search, and recommender systems. Given a set $X$ of $n$ elements, it asks to select a subset $S$ of $k \ll n$ elements with maximum diversity, as quantified by the dissimilarities among the elements in $S$. In this paper, we focus on the diversity maximization problem with fairness constraints in the streaming setting. Specifically, we consider the max-min diversity objective, which selects a subset $S$ that maximizes the minimum distance (dissimilarity) between any pair of distinct elements within it. Assuming that the set $X$ is partitioned into $m$ disjoint groups by some sensitive attribute, e.g., sex or race, ensuring fairness requires that the selected subset $S$ contains $k_i$ elements from each group $i \in [1,m]$. A streaming algorithm should process $X$ sequentially in one pass and return a subset with maximum diversity while guaranteeing the fairness constraint. Although diversity maximization has been extensively studied, the only known algorithms that can work with the max-min diversity objective and fairness constraints are very inefficient for data streams. Since diversity maximization is NP-hard in general, we propose two approximation algorithms for fair diversity maximization in data streams, the first of which is $\frac{1}{1+\varepsilon}$-approximate and specific for $m = 2$, where $\varepsilon \in (0,1)$, and the second of which achieves a $\frac{2}{m+1}$-approximation for an arbitrary $m$. Experimental results on real-world and synthetic datasets show that both algorithms provide solutions of comparable quality to the state-of-the-art algorithms while running several orders of magnitude faster in the streaming setting.

Index Terms—algorithmic fairness, diversity maximization, max-min dispersion, streaming algorithm

I. INTRODUCTION

Data summarization is a common approach to tackling the challenges of big data volume in data-intensive applications. That is because, rather than performing high-complexity analysis tasks on the entire dataset, it is often beneficial to perform them on a representative and significantly smaller summary of the dataset, thus reducing the processing costs in terms of both running time and space usage. Typical examples of data summarization techniques [3] include sampling, sketching, histograms, coresets, and submodular maximization.

In this paper, we focus on diversity-aware data summarization, which finds application in a wide range of real-world problems. For example, in database query processing [34], [41], web search [23], [35], and recommender systems [31], the output might be too large to be presented to the user entirely, even after filtering the results by relevance. One feasible solution is to present the user with a small but diverse subset that is easy for the user to process and representative of the complete results. As another example, when training machine learning models on massive data, feature and subset selection is a standard method to improve the efficiency. As indicated by several studies [11], [39], selecting diverse features or subsets can lead to better balance between efficiency and accuracy. A key technical problem in both applications is diversity maximization [1], [3], [7]–[10], [19], [20], [27], [32], [40].

In more detail, for a given set $X$ of elements in some metric space and a size constraint $k$, the diversity maximization problem asks for a subset of $k$ elements with maximum diversity. Formally, diversity is quantified by a function that captures how well the selected subset spans the range of elements in $X$, and is typically defined in terms of distances or dissimilarities among elements in the subset. Prior studies [23], [27], [31], [41] have suggested many different diversity objectives of this kind. Two of the most popular ones are max-sum dispersion, which aims to maximize the sum of distances between all pairs of elements in the selected subset $S$, and max-min dispersion, which aims to maximize the minimum distance between any pair of distinct elements in $S$. Fig. 1 illustrates how to select 10 most diverse points from a point set in 2D with each of the two diversity objectives. As shown in Fig. 1, the max-sum dispersion tends to select “marginal” elements and may include highly similar elements in the solution, which is not suitable for the applications requiring more uniform coverage. Therefore, we focus on diversity maximization based on the max-min dispersion problem in this paper.

![Fig. 1. Comparison of max-sum and max-min dispersion diversity objectives.](image-url)
In addition to diversity, fairness in data summarization is also attracting increasing attention. Several studies reveal that the biases w.r.t. sensitive attributes, e.g., sex, race, or age, in underlying datasets can be retained in the summaries and could lead to unfairness in data-driven socio-computational systems such as education, recruitment, and banking. One of the most common notions for fairness in data summarization is group fairness, which partitions the dataset into disjoint groups based on some sensitive attribute and introduces a fairness constraint that limits the number of elements from group $i$ in the summary to $k_i$ for every group $i \in [1, m]$. However, most existing methods for diversity maximization cannot be adapted directly to satisfy such fairness constraints. Moreover, a few methods that can deal with fairness constraints are specific for the max-sum dispersion problem \cite{1, 8, 9}. To the best of our knowledge, the methods in \cite{32} are the only ones for max-min diversity maximization with fairness constraints.

Furthermore, since the applications of diversity maximization are mostly in the realm of massive data analysis, it is important to design efficient algorithms for processing large-scale datasets. The streaming model is a well-recognized framework for big data processing. In the streaming model, an algorithm is only permitted to process each element in the dataset sequentially in one pass, is allowed to take time and space that are sublinear to or even independent of the dataset size, and is required to provide solutions of nearly equal quality to those returned by offline algorithms. However, the only known algorithms \cite{32} for fair max-min diversity maximization are designed for the offline setting and are thus very inefficient in data streams.

**Our Contributions:** In this paper we propose novel streaming algorithms for the fair diversity maximization (FDM) problem with the max-min objective. Our main contributions are summarized as follows.

- We formally define the fair max-min diversity maximization problem in metric spaces. Then, we describe the streaming algorithm for unconstrained max-min diversity maximization in \cite{7} and improve its approximation ratio from $\frac{1-\varepsilon}{\varepsilon}$ to $\frac{1-\frac{\varepsilon}{2}}{\varepsilon}$ for any parameter $\varepsilon \in (0, 1)$.

- We first propose a $\frac{1-\varepsilon}{\varepsilon}$-approximation streaming algorithm called SFDM1 for FDM when there are only two groups in the dataset. During stream processing, it maintains group-blind solutions and group-specific solutions for both groups using the streaming algorithm of \cite{7}. In the post-processing, each group-blind solution is balanced for the fairness constraint by swapping elements with group-specific solutions. It takes $O(\frac{k \log \Delta}{\varepsilon})$ time per element for streaming processing, where $\Delta$ is the ratio of the maximum and minimum distances between any pair of distinct elements in the dataset, spends $O(\frac{k^2 \log \Delta}{\varepsilon})$ time for post-processing, and stores $O(\frac{k m \log \Delta}{\varepsilon})$ elements in memory.

- We further propose a $\frac{1-\varepsilon}{\varepsilon}$-approximation streaming algorithm called SFDM2 for FDM with an arbitrary number $m$ of groups in the dataset. It uses a similar method for stream processing to SFDM1. In the post-processing, it first partitions the elements in all solutions into clusters based on their pairwise distances. Starting from a partial solution chosen from the group-blind solution, it utilizes an algorithm to find a maximum-cardinality intersection of two matroids, the first of which is defined by the fairness constraint and the second of which is defined on the clusters, to augment the partial solution to acquire the final fair solution. It takes $O(\frac{k^2 m \log \Delta}{\varepsilon})$ time per element for streaming processing, requires $O(\frac{k^2 m \log \Delta}{\varepsilon} (m^2 + k))$ time for post-processing, and stores $O(\frac{k^3 m \log \Delta}{\varepsilon})$ elements in memory.

- Finally, we evaluate the performance of our proposed algorithms against the state-of-the-art algorithms on several real-world and synthetic datasets. The results demonstrate that our proposed algorithms provide solutions of comparable quality to those returned by the state-of-the-art algorithms while running several orders of magnitude faster than them in the streaming setting.

The rest of this paper is organized as follows. The related work is discussed in Section \[II]. In Section \[III] we introduce the basic concepts and definitions and describe the streaming algorithm for unconstrained diversity maximization. In Section \[IV] we present our streaming algorithms for fair diversity maximization. Our experimental setup and results are reported in Section \[V]. Finally, we conclude this paper in Section \[VI].

## II. RELATED WORK

Diversity maximization has been extensively studied over the last two decades. Existing studies mostly focus on two popular diversity objectives — i.e., *max-sum dispersion* \cite{1, 2, 4, 7, 9, 10, 12, 13, 25, 27, 36} and *max-min dispersion* \cite{2, 7, 10, 20, 25, 27, 32, 36}, as well as their variants \cite{15, 27}.

An early study \cite{22} proved that the max-sum and max-min diversity maximization problems are NP-hard even in metric spaces. The classic approaches to both problems are the greedy algorithms \cite{25, 36}, which achieves the best possible approximation ratio of $\frac{1}{2}$ unless P=NP. Indyk et al. \cite{27}
proposed composable coreset-based approximation algorithms for diversity maximization. Aghamolaei et al. [2] improved the approximation ratios in [27]. Ceccarello et al. [10] proposed composable coreset-based approximation algorithms for diversity maximization in MapReduce and streaming settings where the metric space has a bounded doubling dimension. Borassi et al. [7] proposed sliding-window streaming algorithms for diversity maximization. Drosou and Pitoura [20] studied max-min diversity maximization on dynamic data. They proposed methods that cannot be directly used for our problem.

There have been several studies on diversity maximization under matroid constraints, of which the fairness constraints are special cases. Abbassi et al. [1] proposed a $\left(\frac{1}{2} - \varepsilon\right)$-approximation local search algorithm for max-sum diversification. Bhaskara et al. [5] proposed a $(\frac{1}{2} - \varepsilon)$-approximation algorithm for maximizing the sum of negative type. They also proposed a PTAS for the max-min dispersion objective. Zhang and Gionis [40] generalized the notion of distance to an element and a set $S$ as the distance between $x$ and its nearest neighbor in $S$—i.e., $d(x, S) = \min_{y \in S} d(x, y)$.

Our focus in this paper is to find a small subset of most diverse elements from $X$. Given a subset $S \subseteq X$, its diversity $\text{div}(S)$ is defined as the minimum of the pairwise distances between any two distinct elements in $S$—i.e., $\text{div}(S) = \min_{x,y \in S, x \neq y} d(x, y)$. The unconstrained version of diversity maximization (DM) asks for a subset $S \subseteq X$ of $k$ elements maximizing $\text{div}(S)$—i.e., $S^* = \arg\max_{S \subseteq X : |S| = k} \text{div}(S)$.

We use $\text{OPT} = \text{div}(S^*)$ to denote the diversity of the optimal solution $S^*$ for DM. This problem has been proven to be NP-complete [22], and no polynomial-time algorithm can achieve an approximation factor of better than $\frac{1}{2}$ unless P=NP. One approach to DM is the $\frac{1}{2}$-approximation algorithm [24], (known as GMM) in the offline setting.

We introduce fairness to diversity maximization in case that $X$ is comprised of demographic groups defined on some sensitive attribute, e.g., sex or race. Formally, suppose that $X$ is divided into $m$ disjoint groups $\{1, \ldots, m\}$ (in short) and a function $c : X \to [m]$ maps each element $x \in X$ to the group it belongs to. Let $X_i = \{x \in X : c(x) = i\}$ be the subset of elements from group $i$ in $X$. Obviously, we have $\bigcup_{i=1}^m X_i = X$ and $X_i \cap X_j = \emptyset$ for any $i \neq j$. The fairness constraint assigns a positive integer $k_i$ to each of the $m$ groups and restricts the number of elements from group $i$ in the solution to $k_i$. We assume that $\sum_{i=1}^m k_i = k$. The fair diversity maximization (FDM) problem is defined as follows.

**Definition 1 (Fair Diversity Maximization).** Given a set $X$ of $n$ elements with $m$ disjoint groups $X_1, \ldots, X_m$ and size constraints $k_1, \ldots, k_m \in \mathbb{Z}^+$, find a subset $S$ that contains $k_i$ elements from $X_i$ and maximizes $\text{div}(S)$—i.e., $S^*_f = \arg\max_{S \subseteq X : |S| = k \land \forall i \in [m]} \text{div}(S)$.

We use $\text{OPT}_f = \text{div}(S^*_f)$ to denote the diversity of the optimal solution $S^*_f$ for FDM. Since DM is a special case of FDM when $m = 1$, FDM is NP-hard up to a $\frac{1}{2}$-approximation as well. In addition, our FDM problem is closely related to the concept of matroid in combinatorics. Given a ground set $V$, a matroid is a pair $\mathcal{M} = (V, \mathcal{I})$ where $\mathcal{I}$ is a family of subsets of $V$ (called the independent sets) with the following properties: (1) $\emptyset \in \mathcal{I}$; (2) for each $A \subseteq B \subseteq V$, if $B \in \mathcal{I}$ then $A \in \mathcal{I}$ (hereditary property); and (3) if $A \in \mathcal{I}$, $B \in \mathcal{I}$, and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$ (augmentation property). An independent set is maximal if it is not a proper subset of any other independent set. A basic property of $\mathcal{M}$ is that its all maximal independent sets have the same size, which is denoted as the rank of the matroid. As is easy to verify, our fairness constraint is a case of rank-$k$ partition matroids, where the ground set is partitioned into disjoint groups and the independent sets are exactly the sets in which, for each group, the number of elements from this group is at most the group capacity. And our streaming algorithms in Section IV will be built on the properties of matroids. In this paper, we study the FDM problem in the streaming setting, where the elements in $X$ arrive one at a time and an algorithm must process each element sequentially in one pass using limited space (typically independent of $n$) and return a valid approximate solution $S$ for FDM.
Algorithm 1: Streaming Diversity Maximization

Input: A stream $X$, a distance metric $d$, a parameter $\varepsilon \in (0, 1)$, a size constraint $k \in \mathbb{Z}^+$

Output: A set $S \subseteq X$ with $|S| = k$

1. $U = \{ \frac{d_{\min}}{(1-\varepsilon)/2} : j \in \mathbb{Z}^+ \land \frac{d_{\min}}{(1-\varepsilon)/2} \in [d_{\min}, d_{\max}] \};$
2. Initialize $S_\mu = \emptyset$ for each $\mu \in U$;
3. foreach $x \in X$
4. foreach $\mu \in U$
5. if $|S\mu| < k$ and $d(x, S\mu) \geq \mu$ then
6. $S\mu \leftarrow S\mu \cup \{x\};$
7. return $S \leftarrow \arg\max_{\mu \in U : |S\mu| = k} \text{div}(S\mu);$

B. Streaming Algorithm for Diversity Maximization

Before presenting our proposed algorithms for FDM, we first recall the streaming algorithm for (unconstrained) DM proposed by Borassi et al. [7]. In Algorithm 1, let $d_{\min} = \min_{x, y \in X, x \neq y} d(x, y)$, $d_{\max} = \max_{x, y \in X} d(x, y)$, and $\Delta = \frac{d_{\max}}{d_{\min}}$. Obviously, it always holds that $OPT \in [d_{\min}, d_{\max}]$.

Algorithm 1 maintains a sequence $U$ of values for guessing $OPT$ within relative errors of $1 - \varepsilon$ and initializes an empty solution $S\mu$ for each $\mu \in U$ before processing the stream. For each $x \in X$ and each $\mu \in U$, if $S\mu$ contains less than $k$ elements and the distance between $x$ and $S\mu$ is at least $\mu$, it will add $x$ to $S\mu$. After processing all elements in $X$, the candidate solution that contains $k$ elements and maximizes the diversity is returned as the solution $S$ for DM.

Algorithm 1 has been proven to be $\frac{1}{2-\varepsilon}$-approximate for many diversity objectives [2] including max-min dispersion. In Theorem 1, we improve its approximation ratio for max-min dispersion to $\frac{1}{2}$ by refining the analysis of [7].

Theorem 1. Algorithm 1 is a $\frac{1}{2}$-approximation algorithm for the max-min diversity maximization problem.

Proof. For each $\mu \in U$, there are two cases for $S\mu$ after processing all elements in $X$: (1) if $|S\mu| = k$, the condition of Line 5 guarantees that $\text{div}(S\mu) \geq \mu$; (2) if $|S\mu| < k$, it holds that $d(x, S\mu) < \mu$ for every $x \in X \setminus S\mu$ since the fact that $x$ is not added to $S\mu$ implies that $d(x, S\mu) < \mu$, as $|S\mu| < k$. Let us consider some $S\mu$ with $|S\mu| < k$. Suppose that $S^* = \{s^*_1, \ldots, s^*_k\}$ is the optimal solution for DM on $X$. We define a function $f : S^* \rightarrow S\mu$ that maps each element in $S^*$ to its nearest neighbor in $S\mu$. As shown above, $d(s^*, f(s^*)) < \mu$ for each $s^* \in S^*$. Because $|S\mu| < k$ and $|S^*| = k$, there must exist two distinct elements $s^*_a, s^*_b \in S^*$ with $f(s^*_a) = f(s^*_b)$. For such $s^*_a, s^*_b$, we have

$$d(s^*_a, s^*_b) \leq d(s^*_a, f(s^*_a)) + d(s^*_b, f(s^*_b)) < 2\mu$$

according to the triangle inequality. Thus, $OPT = \text{div}(S^*) \leq \text{div}(s^*_a, s^*_b) < 2\mu$ if $|S\mu| < k$. Let $\mu'$ be the smallest $\mu \in U$ with $|S\mu| < k$. We have got $\text{div}(S\mu) < 2\mu$ from the above results. Additionally, for $\mu'' = (1 - \varepsilon)\mu'$, we must have $|S\mu| = k$ and $\text{div}(S\mu) \geq \mu''$. Therefore, we have $\text{div}(S) \geq \mu'' = (1 - \varepsilon)\mu' \geq \frac{1-\varepsilon}{2} \text{div}(S^*)$ and conclude the proof.

Algorithm 2: SFDM1

Input: A stream $X = X_1 \cup X_2$, a distance metric $d$, a parameter $\varepsilon \in (0, 1)$, two size constraints $k_1, k_2 \in \mathbb{Z}^+$ (at $k = k_1 + k_2$)

Output: A set $S \subseteq X$ s.t. $|S \cap X_i| = k_i$ for $i \in \{1, 2\}$

/* Streaming processing */
1. $U = \{ \frac{d_{\min}}{(1-\varepsilon)/2} : j \in \mathbb{Z}^+ \land \frac{d_{\min}}{(1-\varepsilon)/2} \in [d_{\min}, d_{\max}] \};$
2. Initialize $S\mu, S\mu,i = \emptyset$ for every $\mu \in U$ and $i \in \{1, 2\};$
3. foreach $x \in X$
4. foreach $\mu \in U$ and $i \in \{1, 2\}$
5. if $|S\mu| < k$ and $d(x, S\mu) \geq \mu$ then
6. $S\mu \leftarrow S\mu \cup \{x\};$
7. if $c(x) = i \land |S\mu,i| < k_i \land d(x, S\mu,i) \geq \mu$ then
8. $S\mu,i \leftarrow S\mu,i \cup \{x\};$
9. return $S \leftarrow \arg\max_{\mu \in U} \text{div}(S\mu);$;

In terms of space and time complexity, Algorithm 1 stores $O\left(\frac{\log \Delta}{\varepsilon^2}\right)$ elements and takes $O\left(\frac{\log \Delta}{\varepsilon^2}\right)$ time per element since it makes $O\left(\frac{\log \Delta}{\varepsilon^2}\right)$ guesses for OPT, keeps at most $k$ elements in each candidate, and requires at most $k$ distance computations to determine whether to add an element to a candidate or not. In Section IV, we will show how Algorithm 1 serves a building block in our proposed algorithms for FDM.

IV. ALGORITHMS

As shown in Section III-A, the fair diversity maximization (FDM) problem is NP-hard in general. Therefore, we will focus on efficient approximation algorithms for this problem. We first propose a $\frac{1}{\varepsilon}$-approximation streaming algorithm for FDM in the special case that there are only $m = 2$ groups in the dataset. Then, we propose a $\frac{1}{\varepsilon}$-approximation streaming algorithm for an arbitrary number $m$ of groups.

A. Streaming Algorithm for $m = 2$

Now we present our streaming algorithm in case of $m = 2$ called SFDM1. The detailed procedure of SFDM1 is described in Algorithm 2. In general, it runs in two phases: streaming processing and post-processing. In stream processing, for each guess $\mu \in U$ of OPT, it utilizes Algorithm 1 to keep a group-blind candidate $S\mu$ with size constraint $k$ and two group-specific candidates $S\mu,1$ and $S\mu,2$ with size constraints
We first prove that \( \text{OPT} \) in one pass, it will post-process the group-blind candidates maintaining \( S \).

\[
\text{OPT} = \min \{ S \subseteq U \mid X \cap S = \emptyset \text{ and } \mu, i \in \{2\} \}
\]

Proof. First of all, it is obvious that \( \text{OPT} \leq \text{OPT}_f \), where \( \text{OPT}_f \) is the optimal diversity of unconstrained DM with \( k = k_1 + k_2 \) on \( X \), since any valid solution for FDM must also be a valid solution for DM. Moreover, it holds that \( \text{OPT}_f \leq \text{OPT}_{k_i} \), where \( \text{OPT}_{k_i} \) is the optimal diversity of unconstrained DM with size constraint \( k_i \) on \( X_i \) for each \( i \in \{1,2\} \), because the optimal solution must contain \( k_i \) elements from \( X_i \) and \( \text{div}(\cdot) \) is a monotonically non-increasing function – i.e., \( \text{div}(S \cup \{x\}) \leq \text{div}(S) \) for any \( S \subseteq X \) and \( x \in S \setminus X \). Therefore, we prove that \( \text{OPT}_f \leq \text{div}(S^* \cap X_i) \leq \text{OPT}_{k_i} \).

Then, according to the results of Theorem 1, we have \( \text{OPT} < 2\mu \) if \( S_{\mu} < k \) and \( \text{OPT}_{k_i} < 2\mu \) if \( S_{\mu,i} < k_i \) for each \( i \in \{1,2\} \). Note that \( \mu' \) is the largest \( \mu \in U \) such that \( |S| = k \), \( |S_{\mu,i}| = k_1 \), and \( |S_{\mu,2}| = k_2 \) after stream processing. For \( \mu'' = \frac{\mu}{\epsilon} \in U \), we have either \( |S_{\mu''|} < k \) or \( |S_{\mu''|,i} < k_i \) for some \( i \in \{1,2\} \). Therefore, it holds that \( \text{OPT}_f < 2\mu'' \leq \frac{\mu}{\epsilon} \cdot \mu' \) and we conclude the proof.

Lemma 2. For each \( \mu \in U' \), \( S_{\mu} \) must satisfy \( \text{div}(S_{\mu}) \geq \frac{\mu}{2} \) and \( |S_{\mu} \cap X_i| = k_i \) for both \( i \in \{1,2\} \) after post-processing.

Proof. The candidate \( S_{\mu} \) before post-processing has exactly \( k = k_1 + k_2 \) elements but may not contain \( k_i \) elements from \( X_i \) and \( k_2 \) elements from \( X_2 \). If \( S_{\mu} \) has exactly \( k_i \) elements from \( X_i \) and \( k_2 \) elements from \( X_2 \) and thus the post-processing is skipped, we have \( \text{div}(S_{\mu}) \geq \mu \) according to Theorem 1. Otherwise, assuming that \( |S_{\mu} \cap X_1| = k'_1 < k_i \), we will add \( k_i - k'_i \) elements from \( S_{\mu,1} \) to \( S_{\mu} \) and remove \( k_i - k'_i \) elements from \( S_{\mu} \cap X_2 \) for ensuring the fairness constraint. In Line 13 all the \( k_i \) elements in \( S_{\mu,1} \) can be selected for insertion. Since the minimum distance between any pair of elements in \( S_{\mu,1} \) is at least \( \mu \), we can find at most one element \( x \in S_{\mu,1} \) such that \( d(x,y) < \frac{\mu}{2} \) for each \( y \) in \( S_{\mu} \cap X_1 \). This means that there are at least \( k_i - k'_i \) elements from \( S_{\mu,1} \) whose distances to all the existing elements in \( S_{\mu} \cap X_1 \) are greater than \( \frac{\mu}{2} \). Accordingly, after adding \( k_i - k'_i \) elements from \( S_{\mu,1} \) to \( S_{\mu} \), greedily, it still holds that \( d(x,y) \geq \frac{\mu}{2} \) for any \( x, y \in S_{\mu} \cap X_1 \). In Line 16 for each element \( x \in S_{\mu} \cap X_2 \), there is at most one (newly added) element \( y \in S_{\mu} \cap X_1 \) such that \( d(x,y) < \frac{\mu}{2} \). Meanwhile, it is guaranteed that \( y \) is the nearest neighbor of \( x \) in \( S_{\mu} \) in this case. So, in Line 16 every \( x \in S_{\mu} \cap X_2 \) with \( d(x,S_{\mu} \cap X_2) < \frac{\mu}{2} \) is removed, since there are at most \( k_i - k'_i \) such elements and the one with the smallest \( d(x,S_{\mu} \cap X_2) \) is removed at each step. Therefore, \( S_{\mu} \) contains \( k_i \) elements from \( X_1 \) and \( k_2 \) elements from \( X_2 \) and satisfies that \( \text{div}(S_{\mu}) \geq \frac{\mu}{2} \) after post-processing.

Theorem 2. SFDM1 returns a \( \frac{1+\epsilon}{2} \)-approximate solution for the fair diversity maximization problem.

Proof. According to Lemmas 1 and 2, \( \text{div}(S) \geq \text{div}(S_{\mu}) \geq \frac{\mu}{2} \geq \frac{1+\epsilon}{2} \cdot \text{OPT}_f \), where \( \mu' \) is the largest \( \mu \in U' \).

Complexity Analysis: We analyze the time and space complexities of SFDM1 in Theorem 3.

Theorem 3. SFDM1 stores \( O(\frac{k \log \Delta}{\epsilon}) \) elements in memory, takes \( O(\frac{k \log \Delta}{\epsilon}) \) time per element for streaming processing, and spends \( O(\frac{k^2 \log \Delta}{\epsilon}) \) time for post-processing.

Proof. SFDM1 keeps 3 candidates for each \( \mu \in U \) and \( O(k) \) elements in each candidate. Hence, the total number of stored elements is \( O(\frac{k \log \Delta}{\epsilon}) \) since \( |U| = O(\frac{\log \Delta}{\epsilon}) \). The stream processing performs at most \( O(\frac{k \log \Delta}{\epsilon}) \) distance computations per element. Finally, for each \( \mu \in U' \) in the post-processing, at most \( k(k_i - k'_i) \) distance computations are performed to select the elements in \( S_{\mu,i} \) to be added to \( S_{\mu} \). To find the elements to be removed, at most \( k(k_i - k'_i) \) distance computations are performed to select the elements in \( S_{\mu,1} \) to be added to \( S_{\mu} \).
For each group
\[ C \leftarrow C \{ \text{Merge} \} \]
the offline setting, which keeps the whole dataset in memory
used for
\[ \text{SFDM1} \]

Comparison with Prior Art: The idea of finding an initial solution and balancing it for fairness in SFDM1 has also been used for FairSwap \([32]\). However, FairSwap only works in the offline setting, which keeps the whole dataset in memory and needs random accesses over it for solution computation, whereas SFDM1 works in the streaming setting, which scans the dataset in one pass and uses only the elements in the candidates for post-processing. Compared with FairSwap, SFDM1 reduces the space complexity from \(O(n)\) to \(O(\frac{k^2 \log \Delta}{\varepsilon})\) and the time complexity from \(O(nk)\) to \(O(\frac{k^2 \log \Delta}{\varepsilon^2})\) at the expense of lowering the approximation ratio by a factor of \(1 - \varepsilon\).

\[ \text{B. Streaming Algorithm for General} \ m \]

Now we introduce our streaming algorithm called SFDM2 that can work with an arbitrary \(m \geq 2\). The detailed procedure of SFDM2 is presented in Algorithm 3. Similar to SFDM1, it also has two phases: \textit{stream processing} and \textit{post-processing}. In stream processing, it utilizes Algorithm 1 to keep a group-blind candidate \(S_{\mu}\) and \(m\) group-specific candidates \(S_{\mu,1}, \ldots, S_{\mu,m}\) for all the \(m\) groups. The difference from SFDM1 is that the size constraint of each group-specific candidate is \(k\) instead of \(k_i\) for each group \(i\). Then, after processing all elements in \(X\), it requires a post-processing scheme for ensuring the fairness of candidates as well. Nevertheless, the post-processing procedures are totally different from SFDM1, since the swap-based balancing strategy cannot guarantee the validity of the solution with any theoretical bound. Like SFDM1, the post-processing is performed on a subset \(U'\) where \(S_h\) has \(k\) elements and \(S_{\mu,i}\) has at least \(k_i\) elements for each group \(i\). For each \(\mu \in U'\), it initializes with a subset \(S'_{\mu}\) of \(S_{\mu}\). For an over-filled group \(i - \text{i.e.,}\), \(|S_{\mu,i} \cap X_i| > k_i\), \(S'_{\mu}\) contains \(k_i\) arbitrary elements from \(S_{\mu}\). For an under-filled or exactly filled group \(i - \text{i.e.,}\), \(|S_{\mu,i} \cap X_i| \leq k_i\), \(S'_{\mu}\) contains all \(k_i' = |S_{\mu,i} \cap X_i|\) elements from \(S_{\mu}\). Next, new elements from under-filled candidates should be added to \(S'_{\mu}\) so that \(S'_{\mu}\) is a maximum cardinality set in \(I_1 \cap I_2\).

\[ \text{return} \ S \leftarrow \arg \max_{\mu \in \mathcal{U}'} |S_{\mu}| = k \operatorname{div}(S'_{\mu}); \]

\[ \text{Algorithm 3: SFDM2} \]

\textbf{Input}: A stream \(X = \bigcup_{i=1}^{m} X_i\), a distance metric \(d\), a parameter \(\varepsilon \in (0, 1)\), \(m\) size constraints \(k_1, \ldots, k_m \in \mathbb{Z}^+\) (\(k = \sum_{i=1}^{m} k_i\))

\textbf{Output}: A set \(S \subseteq X\) s.t. \(|S \cap X_i| = k_i, \forall i \in [m]\)

\[ \mathcal{U} = \{ \frac{d_{\min}}{\mathcal{V}} : j \in \mathbb{Z}_0^+ \wedge \frac{d_{\min}}{\mathcal{V}} \in [d_{\min}, d_{\max}] \}; \]

\[ \mathcal{V} = \{ \mathcal{V} : \mathcal{V} \cap \mathcal{V} \neq \emptyset \}; \]

\[ \mathcal{V} = \{ \mathcal{V} : \mathcal{V} \cap \mathcal{V} \neq \emptyset \}; \]

\[ \text{// Stream processing} \]

\[ \mathcal{S}_{\mu,i} = \emptyset \text{ for every } \mu \in \mathcal{U} \text{ and } i \in [m]; \]

\[ \text{// Post-processing} \]

\[ \mathcal{U}' = \{ \mu \in \mathcal{U} : |S_{\mu,i}| = k_i, \forall i \in [m] \}; \]

\[ \mathcal{U}' = \{ \mu \in \mathcal{U} : |S_{\mu,i}| = k_i, \forall i \in [m] \}; \]

\[ \text{// Rebuild} \]

\[ \text{Algorithm 4: Matroid Intersection} \]

\textbf{Input}: Two matroids \(\mathcal{M}_1 = (\mathcal{V}, \mathcal{I}_1)\), \(\mathcal{M}_2 = (\mathcal{V}, \mathcal{I}_2)\), a distance metric \(d\), an initial set \(S_0 \subseteq \mathcal{V}\)

\[ \text{Output}: \text{A maximum cardinality set } S \subseteq \mathcal{V} \text{ in } \mathcal{I}_1 \cap \mathcal{I}_2 \]

\[ \mathcal{I}_i = \{ \mathcal{I} : \mathcal{I} \cap \mathcal{I} \neq \emptyset \}; \]

\[ \mathcal{S}_{\mu,i} = \emptyset \text{ for every } \mu \in \mathcal{U} \text{ and } i \in [m]; \]

\[ \text{// Post-processing} \]

\[ \mathcal{U}' = \{ \mu \in \mathcal{U} : |S_{\mu,i}| = k_i, \forall i \in [m] \}; \]

\[ \text{// Rebuild} \]

\[ \text{return} \ S \leftarrow \arg \max_{\mu \in \mathcal{U}'} |S_{\mu}| = k \operatorname{div}(S'_{\mu}); \]

needed. Therefore, the time complexity for post-processing is \(O\left(\frac{k^2 \log \Delta}{\varepsilon}\right)\) since \(|\mathcal{U}'| = O(\frac{\log \Delta}{\varepsilon})\).
Matroid Intersection: Next, we describe how to use matroid intersection for solution augmentation in SFDM2. We define the first rank-\(k\) matroid \(M_1 = (V, I_1)\) based on the fairness constraint, where the ground set \(V\) is \(S_{all}\) and \(S \in I_1\) iff \(|S \cap X_i| \leq k_i, \forall i \in [m]|.\) Intuitively, a set \(S\) is fair if it is a maximal independent set in \(I_1\). Moreover, we define the second rank-\(l\) (\(l = |C|\)) matroid \(M_2 = (V = S_{all}, I_2)\) on the set \(C\) of clusters, where \(S \in I_2\) iff \(|S \cap C| \leq 1, \forall C \in C\). Accordingly, the problem of adding new elements to \(S_\mu\) for ensuring fairness can be seen as a matroid intersection problem, which aims to find a maximum cardinality set \(S \in I_1 \cap I_2\) for \(M_1 = (S_{all}, I_1)\) and \(M_2 = (S_{all}, I_2)\). A common solution for matroid intersection is the Cunningham’s algorithm \([18]\) based on the augmentation graph in Definition 2.

Definition 2 (Augmentation Graph \([18]\)). Given two matroids \(M_1 = (V, I_1)\) and \(M_2 = (V, I_2)\), a set \(S \in V\) such that \(S \in I_1 \cap I_2\), and two sets \(V_1 = \{x \in V : S \cup \{x\} \in I_1\}\) and \(V_2 = \{x \in V : S \cup \{x\} \in I_2\}\), an augmentation graph is a digraph \(G = (V \cup \{a, b\}, E)\) where \(a, b \notin V\). There is an edge \((a, x) \in E\) for each \(x \in V_1\). There is an edge \((x, b) \in E\) for each \(x \in V_2\). There is an edge \((y, x) \in E\) for each \(x \in V \setminus S\), \(y \in S\), such that \(S \cup \{x\} \notin I_1\) and \(S \cup \{x\} \notin I_2\). There is an edge \((x, y) \in E\) for each \(x \in V \setminus S, y \in S\), such that \(S \cup \{x\} \notin I_2\) and \(S \cup \{x\} \notin I_2\).

The Cunningham’s algorithm \([18]\) is initialized with \(S = \emptyset\) (or any \(S \in I_1 \cap I_2\)). At each step, it builds an augmentation graph \(G\) for \(M_1, M_2,\) and \(S\). If there is no directed path from \(a\) to \(b\) in \(G\), then \(S\) is returned as a maximum cardinality set. Otherwise, it finds the shortest path \(P^*\) from \(a\) to \(b\) in \(G\), and augments \(S\) according to \(P^*\): For each \(x \in V \setminus S\) except \(a\) and \(b\), add \(x\) to \(S\); For each \(x \in S\), remove \(x\) from \(S\).

We adapt the Cunningham’s algorithm \([18]\) for our problem as shown in Algorithm 4. Our algorithm is initialized with \(S_\mu\) instead of \(\emptyset\). In addition, to reduce the cost of building \(G\) and maximize the diversification, it first adds the elements in \(V_1 \cap V_2\) greedily to \(S_\mu\) until \(V_1 \cap V_2 = \emptyset\). This is because there exists a shortest path \(P^* = \{a, x, b\}\) in \(G\) for any \(x \in V_1 \cap V_2\), which is easy to verify from Definition 2. Finally, if \(|S| < k\) after the above procedures, the standard Cunningham’s algorithm will be used to augment \(S\) for ensuring its maximality.

Approximation Ratio: Next, we prove that SFDM2 achieves an approximation ratio of \(\frac{1 - \frac{1}{3m+2}}{}\) for FDM. For the proof, we first show that the set \(C\) of clusters has several important properties (Lemma 3). Then, we prove that Algorithm 4 can return a fair solution for a specific \(\mu\) based on the properties of \(C\) (Lemma 4).

Lemma 3. The set \(C\) of clusters has the following properties: (i) for any element \(x \in C_a\) and \(y \in C_b\) (\(a \neq b\), \(d(x, y) \geq \frac{\mu}{m+1}\)); (ii) each cluster \(C\) contains at most one element from \(S_\mu\) and \(S_{\mu,i}\) for any \(i \in [m]\); (iii) for any two elements \(x, y \in C\), \(d(x, y) < \frac{m}{m+1} \cdot \mu\).

Proof. First of all, Property (i) holds from Lines \([13][16]\) of Algorithm 3. Then, we prove Property (ii) by contradiction. Let us construct an undirected graph \(G = (V, E)\) for a cluster \(C \in C\), where \(V\) is the set of elements in \(C\) and there exists an edge \((x, y) \in E\) iff \(d(x, y) < \frac{\mu}{m}\). Based on Algorithm 3 for any \(x \in C\), there must exist some \(y \in C\) (\(x \neq y\)) such that \(d(x, y) < \frac{\mu}{m}\). Therefore, \(G\) is a connected graph. Suppose that \(C\) can contain more than one element from \(S_\mu\) or \(S_{\mu,i}\) for some \(i \in [m]\). Let \(P_{x,y} = (x, \ldots, y)\) be the shortest path of \(G\) between \(x\) and \(y\) where \(x\) and \(y\) are both from \(S_\mu\) or \(S_{\mu,i}\). Next, we show that the length of \(P_{x,y}\) is at most \(m + 1\). If the length of \(P_{x,y}\) is longer than \(m + 1\), there will be a sub-path \(P_{x',y'}\) of \(P_{x,y}\) where \(x'\) and \(y'\) are both from \(S_\mu\) or \(S_{\mu,i}\), and this violates the fact that \(P_{x,y}\) is the shortest. Since the length of \(P_{x,y}\) is at most \(m + 1\), we have \(d(x, y) < (m + 1) \cdot \frac{\mu}{m+1} = \frac{\mu}{m}\), which contradicts with the fact that \(d(x, y) \geq \mu\), as they are both from \(S_\mu\) or \(S_{\mu,i}\). Finally, Property (iii) is a natural extension of Property (ii): Since each cluster \(C\) contains at most one element from \(S_\mu\) and \(S_{\mu,i}\) for any \(i \in [m]\), \(C\) has at most \(m + 1\) elements. So, for any two elements \(x, y \in C\), the length of the path between them is at most \(m \in G\) and \(d(x, y) < m \cdot \frac{\mu}{m+1} = \frac{m}{m+1} \cdot \mu\).

Lemma 4. If \(OPT_f \geq \frac{3m+2}{m+1} \cdot \mu\), then Algorithm 4 returns a size-\(k\) subset \(S'_\mu\) such that \(S'_\mu \in I_1 \cap I_2\) and \(div(S'_\mu) \geq \frac{\mu}{m+1}\).

Proof. First of all, the initial \(S'_\mu\) is a subset of \(S_\mu\). According to Property (ii) of Lemma 3, all elements of \(S'_\mu\) are in different clusters of \(C\) and thus \(S'_\mu \in I_1 \cap I_2\). The analysis of \([18]\) guarantees that Algorithm 4 can find a size-\(k\) set in \(I_1 \cap I_2\) as long as it exists. Next, we will show such a set exists when \(OPT_f \geq \frac{3m+2}{m+1} \cdot \mu\). To verify this, we need to identify \(k_i\) clusters of \(C\) that contain at least one element from \(X_i\) for each \(i \in [m]\) and show that all \(k = \sum_{i=1}^{m} k_i\) clusters are distinct. Here, we consider two cases for each group \(i \in [m]\):

- Case 1: For each \(i \in [m]\) such that \(k_i \leq |S_{\mu,i}| < k\), we have \(d(x, S_{\mu,i}) < \mu\) for each \(x \in X_i\). Given the optimal solution \(S^*_\mu\), we define a function \(f\) that maps each \(x^* \in S^*_\mu\) to its nearest neighbor in \(S_{\mu,i}\). For two elements \(x^*_1, x^*_2 \in S^*_\mu\) in these groups, we have \(d(x^*_1, f(x^*_1)) < \mu, d(x^*_2, f(x^*_2)) < \mu,\) and \(d(x^*_1, x^*_2) \geq OPT_f = div(S^*_\mu)\). Therefore, \(d(f(x^*_1), f(x^*_2)) > OPT_f - 2\mu\). Since \(OPT_f \geq \)
\[
\frac{3m+2}{m+1} \cdot \mu, \ d(f(x^*_n), f(x^*_m)) > \frac{3m+2}{m+1} \cdot \mu - 2\mu = \frac{m}{m+1} \cdot \mu.
\]

According to Property (iii) of Lemma 3, it is guaranteed that \( f(x^*_n) \) and \( f(x^*_m) \) are in different clusters. By identifying all the clusters that contains \( f(x^*) \) for all \( x^* \in S^*_G \), we have found \( k_i \) clusters for each group \( i \in [m] \) such that \( k_i \leq |S_{\mu, i}| < k \). And all the clusters found are guaranteed to be distinct.

- **Case 2**: For all \( i \in [m] \) such that \( |S_{\mu, i}| = k \), we are able to find \( k \) clusters that contain one element from \( S_{\mu, i} \) based on Property (ii) of Lemma 3. For such a group \( i \), even though \( k-k_i \) clusters have been identified for all other groups, there are still at least \( k_i \) clusters available for selection. Therefore, we can always find \( k_i \) clusters that are distinct from all the clusters identified by any other group for such a group \( X_i \).

Considering both cases, we have proven the existence of a size-\( k \) set in \( I_1 \cap I_2 \). Finally, for any set \( S \in I_2 \), we have \( \text{div}(S) \geq \frac{n}{m+1} \) according to Property (i) of Lemma 3.

**Theorem 4.** SFDM2 achieves a \( \frac{1}{3m+2} \)-approximation for the fair diversity maximization problem.

**Proof.** Let \( \mu^- \) be the smallest \( \mu \) not in \( U' \). It holds that \( \mu^- \geq \frac{\text{OPT}_{U'}}{2} \) (see Lemma 1). Thus, there is some \( \mu < \mu^- \) in \( U' \) such that \( \mu \in \left( \frac{m+1}{3m+2}, \frac{\text{OPT}_U}{m+2}, \frac{\text{OPT}_U}{m+2} \right] \), as \( \frac{m+1}{3m+2} < \frac{1}{2} \) for any \( m \in \mathbb{Z}^+ \). Therefore, SFDM2 provides a fair solution \( S \) such that \( \text{div}(S) \geq \text{div}(S_{\mu}) \geq \frac{n}{m+1} \geq \frac{1}{3m+2} \cdot \text{OPT}_U \).

**Complexity Analysis:** We analyze the time and space complexities of SFDM2 in Theorem 5.

**Theorem 5.** SFDM2 stores \( O\left(\frac{km \log \Delta}{\epsilon}\right) \) elements, takes \( O\left(\frac{k \log \Delta}{\epsilon}\right) \) time per element for streaming processing, and spends \( O\left(\frac{k^2 m \log \Delta}{\epsilon} \cdot (m + \log^2 k)\right) \) time for post-processing.

**Proof.** SFDM2 keeps \( m+1 \) candidates for each \( \mu \in U \) and \( O(k) \) elements in each candidate. So, the total number of elements stored by SFDM2 is \( O\left(\frac{km \log \Delta}{\epsilon}\right) \). Only 2 candidates are checked in streaming processing for each element and thus \( O\left(\frac{k \log \Delta}{\epsilon}\right) \) distance computations are needed. In the post-processing of each \( \mu \), we need \( O(k) \) time to get the initial solution, \( O(k^2 m^2) \) time to cluster \( S_{\mu} \), and \( O(k^2 m) \) time to augment the candidate using Lines 3-7 of Algorithm 4. The time complexity of the Cunningham’s algorithm is \( O(k^2 m \log^2 k) \) according to [13], [33]. To sum up, the overall time complexity of post-processing is \( O\left(\frac{k^2 m \log \Delta}{\epsilon} \cdot (m + \log^2 k)\right) \).

**Comparison with Prior Art:** Finding a fair solution based on matroid intersection has been used by existing methods for fair k-center [16], [17], [28] and fair diversity maximization [32]. SFDM2 adopts a similar method to FairFlow [32] to construct the clusters and matroids. But FairFlow solves matroid intersection as a max-flow problem on a digraph. Its solution is of poor quality in practice, particularly so when \( m \) is large. Thus, SFDM2 uses a different method for matroid intersection based on the Cunningham’s algorithm, which initializes with a partial solution instead of \( \emptyset \) for higher efficiency and adds elements greedily like GMM [24] for higher diversity. Hence, SFDM2 has significantly higher solution quality than FairFlow though its approximation ratio is lower.

**V. Experiments**

In this section, we evaluate the performance of our proposed algorithms on several real-world and synthetic datasets. We first introduce our experimental setup in Section V-A. Then, the experimental results are presented in Section V-B.

**A. Experimental Setup**

**Datasets:** We perform our experiments on four publicly available real-world datasets as follows:

- **Adult** is a collection of 48,842 records extracted from the 1994 US Census database. We select 6 numeric attributes as features and normalize each of them to have zero mean and unit standard deviation. The Euclidean distance is used as the distance metric. The groups are generated from two demographic attributes: sex and race. By using them individually and in combination, there are 2 (sex), 5 (race), and 10 (sex+race) groups, respectively.

- **CelebA** is a set of 202,599 images of human faces. We use 41 pre-trained class labels as features and the Manhattan distance as the distance metric. We generate 2 groups from sex (‘female’, ‘male’), 2 groups from age (‘young’, ‘not young’), and 4 groups from both of them, respectively.

- **Census** is a set of 2,426,116 records obtained from the 1990 US Census data. We take 25 (normalized) numeric attributes as features and use the Manhattan distance as the distance metric. We generate 2, 7, and 14 groups from sex, age, and both of them, respectively.

- **Lyrics** is a collection of 122,448 documents, each of which is the lyrics of a song. We train a topic model with 50 topics using LDA [6] implemented in Gensim [5]. Each document is represented as a 50-dimensional feature vector and the angular distance is used as the distance metric. We generate 15 groups based on the primary genres of songs.

We generate different synthetic datasets with varying \( n \) and \( m \) for scalability tests. In each synthetic dataset, we generate ten 2-dimensional Gaussian isotropic blobs with random centers in \([-10, 10]^2 \) and identity covariance matrices. We assign points to groups uniformly at random. The Euclidean distance is used as the distance metric. The total number \( n \) of points varies

| dataset | \( n \) | \( m \) | # features | distance metric |
|---------|------|------|------------|----------------|
| Adult   | 48,842 | 2/5/10 | 6          | Euclidean      |
| CelebA  | 202,599 | 2/4   | 41         | Manhattan      |
| Census  | 2,426,116 | 2/7/14 | 25         | Manhattan      |
| Lyrics  | 122,448 | 15    | 50         | Angular        |
| Synthetic | 10\(^{d-10}\) | 2-20  | 2          | Euclidean      |

[1] https://archive.ics.uci.edu/ml/datasets/adult
[2] https://mmlab.ie.cuhk.edu.hk/projects/CelebA.html
[3] https://archive.ics.uci.edu/ml/datasets/US+Census+Data+(1990)
[4] http://millionsongdataset.com/musixmatch
[5] https://radimrehurek.com/gensim
[6] https://mmlab.ie.cuhk.edu.hk/projects/CelebA.html
from $10^3$ to $10^7$ with fixed $m = 2$ or 10. And the number $m$ of groups varies from 2 to 20 with fixed $n = 10^5$. The statistics of all datasets are summarized in Table I.

### Algorithms

We compare our streaming algorithms — i.e., SFDM1 and SFDM2, with three offline algorithms for FDM in [32]: the $\frac{1}{5m}$-approximation FairSwap algorithm for an arbitrary $m$, the $\frac{1}{5}$-approximation FairGMM algorithm for small $k$ and $m$, and the $\frac{1}{5}$-approximation FairSwap algorithm for $m = 2$. Since no implementation for the algorithms in [32] is available, they are implemented by ourselves following the description of the original paper. We implement all the algorithms in Python 3.8. Our code is published on GitHub.

All the experiments are run on a server with an Intel Broadwell 2.40GHz CPU and 296GB memory running Ubuntu 16.04.

For each experiment, SFDM1 and SFDM2 are invoked with parameter $\varepsilon = 0.1$ ($\varepsilon = 0.05$ for Lyrics) by default. For a given size constraint $k$, the group-specific size constraint $k_i$ for each group $i \in [m]$ is set based on equal representation, which has been widely used in the literature [17], [28], [29], [38]: If $k$ is divisible by $m$, $k_i = \frac{k}{m}$ for each $i \in [m]$; If $k$ is not divisible by $m$, $k_i = \lfloor \frac{k}{m} \rfloor$ for some groups or $k_i = \lceil \frac{k}{m} \rceil$ for the others with $\sum_{i=1}^{m} k_i = k$. We also compare the performance of different algorithms for proportional representation [11], [21], [38], another popular notion of fairness that requires the proportion of elements from each group in the solution generally preserves that in the original dataset.

### Performance Measures

The performance of each algorithm is evaluated in terms of efficiency, quality, and space usage. The efficiency is measured as average update time — i.e., the average wall-clock time used to compute a solution for each arrival element in the stream. The quality is measured by the value of the diversity function of the solution returned by an algorithm. Since computing the optimal diversity $OPT_f$ of FDM is infeasible, we run the GMM algorithm [24] for unconstrained diversity maximization to estimate an upper bound of $OPT_f$ for comparison. The space usage is measured by the number of distinct elements stored by each algorithm. Only the space usages of SFDM1 and SFDM2 are reported because the offline algorithms keep all elements in memory for random access and thus their space usages are always equal to the dataset size. We run each experiment 10 times with different permutations of the same dataset and report the average of each measure over 10 runs for evaluation.

### B. Experimental Results

#### Overview

Table II shows the performance of different algorithms for FDM on four real-world datasets with different group settings when the solution size $k$ is fixed to 20. Note that FairGMM is not included in Table II because it needs to enumerate at most $(\begin{pmatrix} m \\ k \end{pmatrix}) = O(m^k)$ candidates for solution computation and cannot scale to $k > 10$ and $m > 5$. First of all, compared with the unconstrained solution returned by GMM, all the fair solutions are less diverse because of additional fairness constraints. Since GMM is a $\frac{1}{5}$-approximation algorithm and $OPT \geq OPT_f$, $2 \cdot div_{GMM}$ can be seen as an upper bound of $OPT_f$, from which we can find that all four algorithms return solutions of much better approximations than the lower bounds.

In case of $m = 2$, SFDM1 runs the fastest among all four algorithms, which achieves two to four orders of magnitude speedups over FairSwap and FairFlow. Meanwhile, its solution quality is close or equal to that of FairSwap in most cases. SFDM2 shows lower efficiency than SFDM1 due to higher cost of post-processing. But it is still much more efficient than offline algorithms by taking the advantage of stream processing. In addition, the solution quality of SFDM2 benefits from the greedy selection procedure in Algorithm 4, which is not only consistently better than that of SFDM1 but also better than that of FairSwap on Adult and Census.

In case of $m > 2$, SFDM1 and FairSwap are not applicable any more and thus ignored in Table II. SFDM2 shows significant advantages over FairFlow in terms of both solution quality and efficiency. It provides solutions of up to 6.3 times more diverse than FairFlow while running several orders of magnitude faster.

In terms of space usage, both SFDM1 and SFDM2 store very small portions of elements (less than 0.1% on Census) on all datasets. SFDM2 keeps slightly more elements than SFDM1 because the capacity of each group-specific candidate for group $i$ is $k$ instead of $k_i$. For SFDM2, the number of stored elements increases near linearly with $m$, since the total number of candidates is linear to $m$.

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### Table II

Overview of the performance of different algorithms ($k = 20$)

| Dataset     | Group | $m$ | SFDM1 | SFDM2 |
|-------------|-------|-----|-------|-------|
|             | GMM   |     |       |       |
| Adult       | Sex   | 2   | 5.0226| 4.1485| 3.1190|
|             | Race  | 5   | -     | 1.3702| 7.951 |
|             | Sex+Race | 10 | -     | 1.0049| 8.732 |
| CelebA      | Sex   | 2   | 13.0  | 11.4  | 34.392|
|             | Age   | 2   |       | 11.4  | 36.606|
|             | Sex+Age | 4  |       | 8.5   | 25.950|
| Census      | Sex   | 2   | 35.0  | 27.0  | 355.315|
|             | Age   | 14  |       | -     | 8.5   |
|             | Sex+Age | 14 |       | -     | 5.0   |
| Lyrics      | Genre | 15  | 1.5476| -     | 0.2228|

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[https://github.com/yhwang1990/code-FDM](https://github.com/yhwang1990/code-FDM)
Effect of Parameter $\varepsilon$: Fig. 5 illustrates the performance of SFDM1 and SFDM2 with different values of $\varepsilon$ when $k$ is fixed to 20. We range the value of $\varepsilon$ from 0.05 to 0.25 on Adult, CelebA, and Census and from 0.02 to 0.1 on Lyrics. Since the angular distances between two vectors in Lyrics are at most $\frac{\pi}{2}$, too large values of $\varepsilon$ will lead to great estimation errors for $\text{OPT}_f$. Generally, SFDM1 has higher efficiency and smaller space usage than SFDM2 for different values of $\varepsilon$, but SFDM2 exhibits better solution quality. Furthermore, the running time and numbers of stored elements of both algorithms significantly decrease when the value of $\varepsilon$ increases. This is consistent with our analyses in Section IV because the number of guesses for $\text{OPT}_f$ and thus the number of candidates maintained by both algorithms are $O\left(\frac{\log \Delta}{\varepsilon}\right)$. A slightly surprising result is that the diversity values of the solutions do not degrade obviously even when $\varepsilon = 0.25$. This can be explained by the fact that both algorithms return the best solutions after post-processing among all candidates, which means that they can provide good solutions as long as there is some $\mu \in \mathcal{U}$ close to $\text{OPT}_f$. We infer that such $\mu$ still exists when $\varepsilon = 0.25$. Nevertheless, we note that the chance of finding an appropriate value of $\mu$ will be smaller when the value of $\varepsilon$ is larger, which will lead to less stable solution quality. In the remaining experiments, we always use $\varepsilon = 0.1$ for both algorithms on all datasets except Lyrics, where the value of $\varepsilon$ is set to 0.05.

Effect of Solution Size $k$: The impact of $k$ on the performance of different algorithms is illustrated in Fig. 6. Here we vary $k$ in $[5, 50]$ when $m \leq 5$, or $[10, 50]$ when $5 < m \leq 10$, or $[15, 50]$ when $m > 10$, since we restrict that an algorithm must pick at least one element from each group. In general, for each algorithm, the diversity value drops with $k$ as the diversity function is monotonically non-increasing while the running time grows with $k$ as their time complexities are
The gaps in diversity values become more obvious when the time complexity of an algorithm becomes linear or quadratic w.r.t. \( k \). Compared with the solutions of GMM, all fair solutions are slightly less diverse when \( m = 2 \). The gaps in diversity values become more obvious when \( m \) is larger. Although FairGMM achieves slightly higher solution quality than other algorithms when \( k \leq 10 \) and \( m = 2 \), it is not scalable to larger \( k \) and \( m \) due to the huge cost of enumeration. The solution quality of FairSwap, SFDM1, and SFDM2 is close to each other when \( m = 2 \), which is better than that of FairFlow. But the efficiencies of SFDM1 and SFDM2 are orders of magnitude higher than those of FairSwap and FairFlow when \( m = 2 \). Furthermore, when \( m > 2 \), SFDM2 outperforms FairFlow in terms of both efficiency and effectiveness across all \( k \) values. However, since the time complexity of SFDM2 is quadratic w.r.t. both \( k \) and \( m \), its running time increases drastically with \( k \) when \( m \) is large. Finally, in terms of space usage, the numbers of elements maintained by SFDM1 and SFDM2 both increase linearly with \( k \). In addition, it is also linearly correlated with \( m \) for SFDM2. In all experiments, both algorithms only store small portions of elements in the dataset.

**Equal Representation (ER) vs. Proportional Representation (PR):** Fig. 9 compares the solution quality and running time of different algorithms for two popular notions of fairness – i.e., equal representation (ER) and proportional representation (PR), when \( k = 20 \) on Adult with highly skewed groups, where 67% of the records are for males and 87% of the records are for Whites. The diversity value of the solution of each algorithm is slightly higher for PR than ER, as the solution for PR is closer to the unconstrained one. The running time of SFDM1 and SFDM2 is slightly shorter for PR than ER since fewer swapping and augmentation steps are performed on each candidate in the post-processing.
Scalability: We evaluate the scalability of each algorithm on synthetic datasets with varying the number of groups $m$ from 2 to 20 and the dataset size $n$ from $10^3$ to $10^7$. The results on solution quality and running time for different values of $n$ and $m$ when $k = 20$ are presented in Fig. 10 and 11 respectively. We omit the results on space usages because they are similar to previous results. First of all, SFDM2 shows much better scalability than FairFlow w.r.t. $m$ in terms of solution quality. The diversity value of the solution SFDM2 only slightly deceases with $m$ and is up to 3 times higher than that of FairFlow when $m > 10$. Nevertheless, its running time increases more rapidly with $m$ due to the quadratic dependence on $m$. Furthermore, the diversity values of different algorithms slightly grow with $n$ but are always close to each other for different values of $n$ when $m = 2$. When $m = 10$, the advantage of SFDM2 over FairFlow in solution quality becomes larger with $n$. Finally, the running time (as well as memory usage) of offline algorithms are linear to $n$. But the running time and memory usage of SFDM1 and SFDM2 are independent of $n$, as analyzed in Section IV.

VI. Conclusion

In this paper, we studied the diversity maximization problem with fairness constraints in the streaming setting. First of all, we proposed a $\frac{1-\epsilon}{m+\epsilon}$-approximation streaming algorithm for this problem when there were two groups in the dataset. Furthermore, we designed a $\frac{1-\epsilon}{2m+\epsilon}$-approximation streaming algorithm that could deal with an arbitrary number $m$ of groups in the dataset. Extensive experiments on real-world and synthetic datasets confirmed the efficiency, effectiveness, and scalability of our proposed algorithms.

In future work, we would like to improve the approximation ratios of the proposed algorithms and consider diversity maximization problems with fairness constraints in more general settings, e.g., the sliding-window model and fairness constraints defined on multiple sensitive attributes.

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REFERENCES

[1] Z. Abbassi, V. S. Mirrokni, and M. Thakur, “Diversity maximization under matroid constraints,” in KDD, 2013, pp. 32–40.
[2] S. Aghamolaei, M. Farhadi, and H. Zarrabi-Zadeh, “Diversity maximization via composable coresets,” in CCCG, 2015, pp. 38–48.
[3] M. Ahmed, “Data summarization: a survey,” Knowl. Inf. Syst., vol. 58, no. 2, pp. 249–273, 2019.
[4] C. Bauckhage, R. Sifa, and S. Wrobel, “Adiabatic quantum computing for max-sum diversification,” in SDM, 2020, pp. 343–351.
[5] A. Bhaskara, M. Ghadiri, V. S. Mirrokni, and O. Svensson, “Linear relaxations for finding diverse elements in metric spaces,” in NIPS, 2016, pp. 4098–4106.
[6] D. M. Blei, A. Y. Ng, and M. I. Jordan, “Latent dirichlet allocation,” J. Mach. Learn. Res., vol. 3, pp. 993–1022, 2003.
[7] M. Borassi, A. Epsito, S. Lattanzi, S. Vassilvitskii, and M. Zadimoghaddam, “Better sliding window algorithms to maximize subadditive and diversity objectives,” in PODS, 2019, pp. 254–268.
[8] A. Borodin, H. C. Lee, and Y. Ye, “Max-sum diversification, monotone submodular functions and dynamic updates,” in PODS, 2012, pp. 155–166.
[9] M. Ceccarello, A. Pietracaprina, and G. Pucci, “Fast coreset-based diversity maximization under matroid constraints,” in WSDM, 2018, pp. 81–89.
[10] M. Ceccarello, A. Pietracaprina, G. Pucci, and E. Upfal, “Mapreduce and streaming algorithms for diversity maximization in metric spaces of bounded doubling dimension,” Proc. VLDB Endow., vol. 10, no. 5, pp. 469–480, 2017.
[11] E. C. Celis, K. Keswani, D. Straszak, A. Deshpande, T. Kathuria, and N. K. Vishnoi, “Fair and diverse DPP-based data summarization,” in ICML, 2018, pp. 715–724.
[12] A. Cevallos, F. Eisenbrand, and R. Zenklusen, “Max-sum diversification via convex programming,” in SoCG, 2016, pp. 26:1–26:14.
[13] ———, “Local search for max-sum diversification,” in SODA, 2017, pp. 130–142.
[14] D. Chakrabarty, Y. T. Lee, A. Sidford, S. Singla, and S. C. Wong, “Faster matroid intersection,” in FOCS, 2019, pp. 1146–1168.
[15] B. Chandra and M. M. Halldórsson, “Approximation algorithms for dispersion problems,” *J. Algorithms*, vol. 38, no. 2, pp. 438–465, 2001.

[16] D. Z. Chen, J. Li, H. Liang, and H. Wang, “Matroid and knapsack center problems,” *Algorithmica*, vol. 75, no. 1, pp. 27–52, 2016.

[17] A. Chiplunkar, S. Kale, and S. N. Ramamooorthy, “How to solve fair k-center in massive data models,” in *ICML*, 2020, pp. 1877–1886.

[18] W. H. Cunningham, “Improved bounds for matroid partition and intersection algorithms,” *SIAM J. Comput.*, vol. 15, no. 4, pp. 948–957, 1986.

[19] M. Drosou and E. Pitoura, “Disc diversity: result diversification based on dissimilarity and coverage,” *Proc. VLDB Endow.*, vol. 6, no. 1, pp. 13–24, 2012.

[20] ——, “Diverse set selection over dynamic data,” *IEEE Trans. Knowl. Data Eng.*, vol. 26, no. 5, pp. 1102–1116, 2014.

[21] M. El Halabi, S. Mitrović, A. Norouzi-Fard, J. Tardos, and J. M. Tartanowski, “Fairness in streaming submodular maximization: Algorithms and hardness,” in *NeurIPS*, 2020, pp. 13 609–13 622.

[22] E. Erkut, “The discrete p-dispersion problem,” *Eur. J. Oper. Res.*, vol. 46, no. 1, pp. 48–60, 1990.

[23] S. Gollapudi and A. Sharma, “An axiomatic approach for result diversification,” in *WWW*, 2009, pp. 381–390.

[24] T. F. Gonzalez, “Clustering to minimize the maximum intercluster distance,” *Theor. Comput. Sci.*, vol. 38, pp. 293–306, 1985.

[25] R. Hassin, S. Rubinstein, and A. Tamir, “Approximation algorithms for maximum dispersion,” *Oper. Res. Lett.*, vol. 21, no. 3, pp. 133–137, 1997.

[26] L. Huang, S. H. Jiang, and N. K. Vishnoi, “Coresets for clustering with fairness constraints,” in *NeurIPS*, 2019, pp. 7587–7598.

[27] P. Indyk, S. Mahabadi, M. Mahdian, and V. S. Mirrokni, “Composable core-sets for diversity and coverage maximization,” in *PODS*, 2014, pp. 100–108.

[28] M. Jones, H. Nguyen, and T. Nguyen, “Fair k-centers via maximum matching,” in *ICML*, 2020, pp. 4940–4949.

[29] M. Klein, S. P. Awasthi, and J. Morgenstern, “Fair k-center clustering for data summarization,” in *ICML*, 2019, pp. 3448–3457.

[30] B. Korte and J. Vygen, *Combinatorial Optimization: Theory and Algorithms*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012.

[31] M. Kunaver and T. Pozl, “Diversity in recommender systems - A survey,” *Knowl. Based Syst.*, vol. 123, pp. 154–162, 2017.

[32] Z. Moumoulidou, A. McGregor, and A. Meliou, “Diverse data selection under fairness constraints,” in *ICDT*, 2021, pp. 13:1–13:25.

[33] H. L. Nguyen, “A note on cunningham’s algorithm for matroid intersection,” arXiv:1904.04129 [cs.DS], 2019.

[34] L. Qin, J. X. Yu, and L. Chang, “Diversifying top-k results,” *Proc. VLDB Endow.*, vol. 5, no. 11, pp. 1124–1135, 2012.

[35] D. Rafiei, K. Bharat, and A. Shukla, “Diversifying web search results,” in *WWW*, 2010, pp. 781–790.

[36] S. S. Ravi, D. J. Rosenkrantz, and G. K. Tayi, “Heuristic and special case algorithms for dispersion problems,” *Oper. Res.*, vol. 42, no. 2, pp. 299–310, 1994.

[37] M. Schmidt, C. Schwiegelshohn, and C. Sohler, “Fair coresets and streaming algorithms for fair k-means,” in *WAOA*, 2019, pp. 232–251.

[38] Y. Wang, F. Fabbri, and M. Mathioudakis, “Fair and representative subset selection from data streams,” in *WWW*, 2021, pp. 1340–1350.

[39] S. A. Zadeh, M. Ghadiri, V. S. Mirrokni, and M. Zadimoghaddam, “Scalable feature selection via distributed diversity maximization,” in *AAAI*, 2017, pp. 2876–2883.

[40] G. Zhang and A. Gionis, “Maximizing diversity over clustered data,” in *SDM*, 2020, pp. 649–657.

[41] K. Zheng, H. Wang, Z. Qi, J. Li, and H. Gao, “A survey of query result diversification,” *Knowl. Inf. Syst.*, vol. 51, no. 1, pp. 1–36, 2017.