Iterative reconstruction of the wave speed for the wave equation with bounded frequency boundary data

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Received 24 July 2015, revised 10 December 2015
Accepted for publication 15 December 2015
Published 28 January 2016

Abstract
We study the inverse boundary value problem for the wave equation using measurements of waves emitted by band-limited boundary sources as the data. We prove how to choose classes of nonsmooth coefficient functions so that optimization formulations of inverse wave problems satisfy the prerequisites for the application of steepest descent and Newton-type iterative methods.

Keywords: inverse boundary value problems, wave equation, resolvent estimates, Landweber iteration, wave speed reconstruction

1. Introduction

In this paper, we study the inverse boundary value problem for the wave equation using measurements of waves emitted by band-limited boundary sources as the data. The mentioned inverse boundary value problem arises, for example, in reflection seismology [Sy1].

We show how to choose classes of nonsmooth coefficient functions so that optimization formulations of inverse wave problems satisfy the prerequisites for the application of steepest descent and Newton-type iterative methods. Indeed, we establish the existence of a misfit functional derived from the Hilbert–Schmidt norm and its gradient. The proof is based on resolvent estimates for the corresponding Helmholtz equation, exploiting the fact that the frequencies are contained in a bounded interval.

Via conditional Lipschitz stability estimates for the time-harmonic inverse boundary value problem, which we established in earlier work [BdHQ], we can then guarantee convergence of the iteration if it is initiated within a certain distance of the (unique) solution of the

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inverse boundary value problem. Indeed, such a convergence of a nonlinear projected steepest
descent iteration was obtained in [dHQs].

In our scheme we can allow approximate localization of the data in selected time win-
dows, with size inversely proportional to the maximum allowed frequency. This is of
importance to applications in the context of reducing the complexity of field data and thus of
the underlying coefficient functions. Note that no information is lost by cutting out a short
time window, since our source functions (and solutions), being compactly supported in
frequency, are analytic with respect to time.

The uniqueness of the mentioned inverse boundary value problem for the Helmholtz
equation, that is, using single-frequency data, was established by Sylvester and Uhlmann
[SyUh] assuming that the wave speed is a bounded measurable function. This inverse pro-
blem has also been extensively studied from an optimization point of view. We mention, in
particular, the work of [BOV].

This paper can be viewed as a counterpart of the work by Blazek, Stolk and Symes [BSS]
in the sense that we consider bounded frequency data. That is, we cannot allow arbitrarily
high frequencies in the data. Without this restriction Blazek et al observed that the adjoint
equation did not admit solutions: this problem does not appear in our formulation through the
use of resolvent estimates.

**Multi-frequency data**

The multi-frequency data are obtained from solutions to the corresponding boundary value
problem for the wave equation by applying a Fourier transform (see [LaTr] for the regularity
of hyperbolic equations in such settings). Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^3 \) and
\( c \in L^\infty(\mathbb{R}^3) \) be a strictly positive bounded measurable function, constant outside of \( \Omega \). We
consider the inhomogeneous problem for the wave equation

\[
\partial_t^2 u - c^2 \Delta u = f dS,
\]

(1.1)

where \( f \in L^2(\mathbb{R}; H^{-1/2}(\partial \Omega)) \) is compactly supported in frequency, and \( dS \) is the surface
measure on \( \partial \Omega \) inherited from \( \mathbb{R}^3 \).

Multi-frequency data have been exploited in so-called frequency progression in iterative
schemes for the purpose of regularization [BSZC, CDR, SiPr]. Frequency progression can be
realized in our approach by gradually enlarging the frequency interval of the data; see
also [YuSi].

**Reflection seismology and optimization**

Iterative methods for the inverse boundary value problem for the wave equation, in reflection
seismology, have been collectively referred to as full waveform inversion (FWI). (The term
‘full waveform inversion’ was presumably introduced by Pan, Phinney and Odom in [PPO]
with reference to the use of full seismograms information.) Lailly [La] and Tarantola
[Ta1, Ta2] introduced the formulation of the seismic inverse problem as a local optimization
problem with a least-squares \( L^2 \) minimization of a misfit functional. We also mention the
original work of Bamberger, Chavent and Lailly [BCL1, BCL2] in the one-dimensional case.
Since then, a range of alternative misfit functionals have been considered; we mention, here,
the criterion derived from the instantaneous phase used by Bozdag, Trampert and Tromp
[BTT]. The time-harmonic formulation was initially promoted by Pratt and collaborators in
[PrWo, PrGo]. Later the use of complex frequencies was studied in [HPYS, ShCh]. In FWI
one commonly applies a ‘nonlinear’ conjugate gradient method, a Gauss–Newton method, or
a quasi-Newton method (L-BFGS; for a review, see Brossier [Br] and Métivier et al [MBVO]).

**Single-layer potential operator as a Hilbert–Schmidt operator**

The introduction of the single-layer potential operator is motivated by what seismologists refer to as the process of source ‘blending’. This becomes clear upon introducing a Hilbert–Schmidt norm for this operator, which we justify in the development of a misfit criterion: basis functions of the underlying Hilbert space are viewed as blended sources. The use of ‘simultaneous’ sources in linearized inverse scattering was studied by Dai and Schuster [DaSc], and in FWI, for example, by Vigh and Starr [ViSt] (synthesizing source plane waves), Krebs et al [KAHNLBL] (random source encoding), Gao, Atle and Williamson [GAW] (deterministic source encoding), Symes [Sy2] (deterministic choice of extended source), and van Leeuwen and Herrmann [vLHe] (inversion using few effective sources).

Berkhout, Blacquire and Verschuur [Ber, BBV] considered simple time delays for the blending process, allowing the use of conventional sources in acquisition. The process of source blending has appeared in various acquisition (and imaging) strategies. Perhaps the most basic form involves synthesizing source plane waves from point source data in plane-wave migration [Wh]. So-called controlled illumination [RiBe] can also be viewed as a particular blending strategy. In blended acquisition, typically, time-overlapping point source experiments are generated in the field by using incoherent source arrays; for simultaneous source firing, see Beasley, Chambers and Jiang [BCJ] and for near simultaneous source firing, see Stefani, Hampson and Herkenhoff [SHH]. The use of simultaneous random sources has been proposed, further, by [NKKRDA] and others.

**Conditional Lipschitz stability estimates**

It is well-known that the logarithmic character of stability of the inverse boundary value problem for the Helmholtz equation [Al, No] cannot be avoided. In fact, in [Ma] Mandache proved that even with regularity or a priori assumptions of any order on the unknown wave speed, logarithmic stability is optimal. However, conditional Lipschitz stability estimates can be obtained: for example, accounting for discontinuities, such an estimate holds if the unknown wave speed is a finite linear combination of piecewise constant functions with an underlying known domain partitioning [BdHQ]. This estimate was obtained following an approach introduced by Alessandrini and Vessella [AlVe] and further developed by Beretta and Francini [BeFr] for electrical impedance tomography.

The relationship between the single-layer potential operator and the Dirichlet-to-Neumann map can be found in Nachman [Na]. Using this relationship, it follows that conditional Lipschitz stability using the Dirichlet-to-Neumann map as the data implies conditional Lipschitz stability using the single-layer potential operator as the data. Note that this stability result is the only place in our proof that we need the Dirichlet-to-Neumann map.

**Resolvent estimates**

We control the forward operator via resolvent estimates for the Helmholtz equation. In our low-regularity setting it is well-known that the resolvent norm may go to infinity exponentially in frequency as energy goes to infinity: a famous example is the square well potential which goes back to Gamow (see e.g. [DyZw, theorem 2.25]). It is known in very general smooth settings that this growth is the worst that may occur: see the results of Burq [Bu2, Bu1], Cardoso and Vodev [CaVo], and Rodnianski and Tao [RoTa]. In [Bel] and [Da]
this is proved in a lower regularity setting, and in section 3 below we give a generalization of this result to certain piecewise constant wave speeds. We also give an improved estimate by cutting off away from a sufficiently large compact set: this kind of improvement has been observed before in [Bu2, CaVo, RoTa].

Main result

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary, and let $\{D_0, \ldots, D_N\}$ be a finite collection of pairwise disjoint open subsets of $\Omega$ with Lipschitz boundaries, satisfying $\bigcup_{j=0}^N D_j = \Omega$ and $D_j \cap \partial \Omega = \emptyset \Rightarrow j = 0$.

Given positive real numbers $b_1, \ldots, b_N$, define a wave speed $c$ given by $b_j$ on $D_j$ and 1 otherwise, and consider the following forward solution to \((1.1)\):

$$u(t) := U_c f(t) := \frac{1}{2\pi} \int e^{-i\langle \lambda \rangle} (-c^2 \Delta - (\lambda + i0)^2)\hat{f}(\lambda) d\lambda,$$

$$\hat{f}(\lambda) := \int e^{-i\langle \lambda \rangle} f(t) dt,$$

(1.2)

where $\hat{f} \in L^2(\mathbb{R}; H^{-1/2}(\partial \Omega))$ vanishes for $\lambda \not\in [-\lambda_0, \lambda_0]$ (see (4.4)). Throughout the paper we take $\lambda_0 \geq 1$.

Letting $\tau_{\partial \Omega}$ denote restriction to the boundary $\partial \Omega$, we take as our data knowledge of the operator

$$F : X_0 \rightarrow Y, \quad F(b_1, \ldots, b_N) = \tau_{\partial \Omega} (U_c - U_1),$$

where $X_0 := (0, \infty)^N \subset X := \mathbb{R}^N$ and $Y$ is a tensor product of $L^2$ functions on a time window with a Hilbert–Schmidt operators. More specifically

$$\|c\|^2_2 = \sum_{j=1}^N |b_j|^2, \quad \|F(c)\|^2_2 = \sum_{j=1}^N \int_0^\infty \|\tau_{\partial \Omega} (U_c - U_1) \psi_j \|^2_{L^2(\partial \Omega)} dt.$$

Here $U_1$ is the free wave evolution (i.e. the operator $U_c$ defined in (1.2) with $c \equiv 1$), $\{\psi_j\}_{j=1}^\infty$ is any orthonormal basis of the space of sources $f$, and $I$ is any interval of length $\pi/\lambda_0$ (see section 5). Let $c^1 \in X_0$ be the ‘true wave speed’ which we wish to recover, and define Landweber iterates by

$$c_{m+1} = c_m - \mu DF(c_m)^*(F(c_m) - F(c^1)),$$

(1.3)

for some $c_0 \in X_0$ and $\mu > 0$ sufficiently small, where $DF(c_m)^*$ is the gradient of $F$ at $c_m$ (see (5.5)).

**Theorem.** Locally, the Landweber iteration converges exponentially fast. More specifically:

1. If $\|c_0 - c^1\|_X \leq C_0$ and $C_\mu \mu > 0$ are sufficiently small, then there is $C_1$ such that $\|c_m - c^1\|_X \leq C_0 e^{-m/C_\mu}$.
2. If the boundaries of the subdomains $D_j$ respect polar coordinates, in the sense that the radial derivative of any wave speed under consideration is supported on a union of spheres, then $C_0^{-1}$, $\mu^{-1}$, and $C_1$, all grow at most exponentially in $\lambda_0$.

**Remarks.**

Note that the forward solution is not defined by an inverse Fourier transform: that would give the backward solution $\tilde{u}(t) := u(-t)$ (to which our results below also apply, as can be seen using the change of variables $t \mapsto -t$).
1. Note that the convergence here is local in the sense that the initial wave speed \( c_0 \) must be close to the true one \( c^\dagger \).

2. In (2) of the theorem, the condition that the \( D_j \) respects the polar coordinates is equivalent to the condition that each \( \partial D_j \) for \( j \in \{1, \ldots, N\} \) is a union of radial line segments and open subsets of spheres centered at the origin: see figure 1.

3. Since the residual is a Hilbert–Schmidt operator, we can evaluate the norm in the misfit via any basis \( \psi_j \) at all. This is important in the iteration formula (1.3) (see also section 5.2 for a more detailed formula for the gradient \( DF^* \), including the functions \( \psi_j \)). These basis functions \( \psi_j \) can be viewed as the ‘simultaneous’ sources in ‘blended’ data acquisition; we should perhaps emphasize that there is no need to extract the single-layer potential operator, that is to say no need for ‘deblending’.

4. On the other hand, we require data on the whole boundary \( \partial \Omega \); it would be interesting to know whether this requirement could be relaxed. Although the stability result for the Dirichlet-to-Neumann map [BdHQ] requires data on only a part of the boundary, it is not clear how to translate this into a partial data requirement for the single-layer potential operator as in section 2 below.

5. It would also be interesting to consider the case where \( \Delta \) is replaced by a more general operator, such as \( \text{div} \rho^{-1} \nabla \), or to consider more general elasto-dynamic equations: it is very possible that our methods could be adapted to such problems.

Outline of the paper

In section 2 we review some known results about the Helmholtz equation with a bounded measurable potential function, including the relationship between the Dirichlet-to-Neumann map and the single-layer potential operator. In section 3 we give high-energy resolvent estimates for certain classes of wave speeds. In section 4 we study the resolvent for a bounded measurable wave speed, we introduce a forward solution operator for the wave equation with ‘band-limited’ data, and we compute its Fréchet derivative. In section 5 we give an abstract setting for a Landweber iteration, which reconstructs the wave speed from the forward operator. We build in a time localization, and control the forward operator in a Hilbert–
Schmidt sense. We compute the misfit functional and gradient in terms of the corresponding weighted $L^2$ inner product in time and a Hilbert–Schmidt inner product in space. In section 6 we apply the results of the previous sections to prove the theorem.

For most of the paper we work in greater generality than the setting described above in the statement of the theorem: see the beginning of each section for the assumptions used in that section.

This research was supported in part by members: BGP, ExxonMobil, PGS, Statoil and Total, of the Geo-Mathematical Imaging Group, and by a Simons Foundation Collaboration Grant for Mathematicians.

The authors are grateful to Maciej Zworski for helpful discussions about resolvent estimates, and also to the two anonymous referees for their many helpful comments and suggestions.

2. Modeling time-harmonic data: Dirichlet-to-Neumann map versus single-layer potential operator

In this section we review the relationship between the Dirichlet-to-Neumann map and the single-layer potential operator. In this paper we are principally concerned with the latter, and use the former only for stability estimates. It is only because we use the Dirichlet-to-Neumann map at all that, at a few places in our proof, we must assume we are away from the Dirichlet spectrum of certain operators on $\Omega$. We expect stability estimates to hold for the single-layer potential operator itself without reference to the Dirichlet-to-Neumann map, and with such estimates in hand one would be able to remove all reference to Dirichlet spectra from the proof.

**Dirichlet-to-Neumann map**

Here, we consider time-harmonic waves, described by solutions, $u$ say, of the Helmholtz equation on a bounded open domain $\Omega \subset \mathbb{R}^n$. We write

$$\tilde{q}(\tilde{x}) = -\lambda^2 e^{-2}(\tilde{x}),$$

where $\tilde{x}$ is a coordinate on $\mathbb{R}^n$ (we will use the notation $x$ without the tilde for an element of a Hilbert space later in section 5), and keep $\lambda \in \mathbb{R}$ fixed; we assume that $c \in L^\infty(\Omega)$ and that $c$ is bounded below by a positive constant. Note that in the rest of the paper we use the resolvent $R_c(\lambda) = (-c(\tilde{x})^2 \Delta - \lambda^2)$, whose integral kernel differs from the Green’s function of this section by a factor of $c(\tilde{x})^2$. We have the general formulation

$$\begin{cases}
\Delta \tilde{u} + \tilde{q}(\tilde{x})u = 0, & \tilde{x} \in \Omega, \\
u \cdot \nabla u = g, & \tilde{x} \in \partial \Omega.
\end{cases}$$

(2.1)

Here, $g = g(\tilde{x}, \lambda)$ is a boundary source.

The data generated by the boundary sources, $g$, represent the Dirichlet-to-Neumann map $\Lambda_q$ such that

$$\Lambda_q : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega), \quad g \mapsto \frac{\partial u}{\partial \nu} \bigg|_{\partial \Omega},$$

where $\nu$ represents the outward unit normal to $\partial \Omega$. We assume that the boundary $\partial \Omega$ is in $C^{1,1}$, and that $\Omega' = \mathbb{R}^n \setminus \overline{\Omega}$ is connected. We also assume that 0 is not a Dirichlet eigenvalue for $-\Delta + \tilde{q}$ in $\Omega$. 

6
Green’s functions

Seismic reflection data are generated by point sources on \( \partial \Omega \) and observed at points on \( \partial \Omega \). In preparation of a description of the data in terms of fundamental solutions in \( \mathbb{R}^n \), we extend \( \tilde{q}(\tilde{x}) \) to a function with value

\[
-k^2 = -\lambda^2 c_0^{-2}
\]

in \( \Omega' \). Let \( G^+_k(\tilde{x}, \tilde{y}) \) be the outgoing Green’s function for the Helmholtz equation with constant coefficient, \( c_0^{-2} \), in \( \mathbb{R}^n \), which is given by

\[
G^+_k(\tilde{x}, \tilde{y}) = \frac{1}{(2\pi)^n} \int \frac{e^{i(\tilde{x} - \tilde{y})\xi}}{\xi^2 - k^2 - i0} d\xi = \frac{i}{4} \left( \frac{|k|}{2\pi|\tilde{x} - \tilde{y}|} \right)^{(n-2)/2} H_{(n-2)/2}^{(1)}(k|\tilde{x} - \tilde{y}|).
\]

(2.3)

We set

\[
q(\tilde{x}) = q(\tilde{x}) + k^2,
\]

which is compactly supported. We assume that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta + q \) or \( -\Delta \) in \( \Omega \). We let \( G_{q,k}(\tilde{x}, \tilde{y}) \) be the solution of

\[
(-\Delta + q - k^2) G_{q,k}(\tilde{x}, \tilde{y}) = \delta(\tilde{x} - \tilde{y}), \quad \tilde{x}, \tilde{y} \in \mathbb{R}^n,
\]

(2.4)

satisfying the Sommerfeld radiation condition as \( |\tilde{x}| \to \infty \). Restricting \( \tilde{x} \) and \( \tilde{y} \) to \( \partial \Omega \) then yields the seismic reflection data:

\[
A = \{ G_{q,k}(\tilde{x}, \tilde{y}) | \tilde{x}, \tilde{y} \in \partial \Omega, \tilde{x} \neq \tilde{y} \},
\]

if \( q^1(\tilde{x}) \) signifies the ‘true’ model.

Single-layer potential operator

In the constant ‘reference’ model with wave speed \( c_0 \), we introduce the operator,

\[
S^+_{k}: H^{1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega),
\]

by

\[
S^+_k w(\tilde{x}) = \int_{\partial \Omega} G^+_k(\tilde{x}, \tilde{y}) w(\tilde{y}) dS(\tilde{y}), \quad \tilde{x} \in \partial \Omega,
\]

(2.5)

which is bounded. Here, \( dS \) is the natural area element on \( \partial \Omega \). In a general heterogeneous model, we introduce

\[
S_{q,k}: H^{1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega),
\]

with

\[
S_{q,k} w(\tilde{x}) = \int_{\partial \Omega} G_{q,k}(\tilde{x}, \tilde{y}) w(\tilde{y}) dS(\tilde{y}), \quad \tilde{x} \in \partial \Omega,
\]

(2.6)

which is bounded also [Na, theorem 1.6]. Data generated by or synthesized with simultaneous sources are then represented by

\[
B_k = \{ S_{q,k} w | w \in H^{1/2}(\partial \Omega) \}.
\]

This is the set of single-frequency data. The single-layer potential operator, \( S_{q,k} \), or data \( B_k \), are equivalent to the Dirichlet-to-Neumann map, \( \Lambda_{q,k}^{1/2} \), in the sense that they contain the same information about \( q^1 \). Indeed, from \( S^+_k \), we can build the relation between \( S_{q,k} \) and the
Dirichlet-to-Neumann map. We have
\[ \Lambda_{q-k^2} = \Lambda_{-k^2} + S_{q,k}^{-1} = (S_{k}^+)^{-1}, \] (2.7)
or
\[ S_{q,k} - S_{k}^+ = -S_{k}^+(\Lambda_{q-k^2} - \Lambda_{-k^2})S_{q,k}. \] (2.8)
This identity is defined on \( H^{1/2}(\partial\Omega) \), and can be derived from the resolvent equation,
\[ G_{q,k}(\vec{x}, \vec{y}) = G_k^+(\vec{x} - \vec{y}) - \int_{\Omega} G_k^+(\vec{x} - \vec{z})q(\vec{z})G_{q,k}(\vec{z}, \vec{y}) d\vec{z}. \] (2.9)
For \( w \in H^{1/2}(\partial\Omega) \), we then find that
\[ S_{q,k}w(\vec{x}) - S_k^+w(\vec{x}) = -\int_{\Omega} G_k^+(\vec{x} - \vec{z})q(\vec{z})(S_{q,k}w)(\vec{z}) d\vec{z}. \] (2.10)
From (2.7) we straightforwardedly obtain
\[ \Lambda_{q-k^2} - \Lambda_{q'-k^2} = S_{q,k}^{-1} - S_{q',k}^{-1}, \] (2.11)
or
\[ \Lambda_{q-k^2} - \Lambda_{q'-k^2} = -S_{q',k}^{-1}(S_{q,k} - S_{q',k})S_{q',k}^{-1}. \]

**Conditional Lipschitz stability**

The convergence rate and convergence radius of our iterative scheme are based on a conditional Lipschitz-type stability estimate for the inverse problem:
\[ \|c^{-2} - c'^{-2}\| \leq C_{\delta} \|S_{\delta^{2}} - S_{\delta^{2}+\varepsilon}\|, \] (2.12)
where \( c^\dagger \) is the true wave speed. In case the wave speed is piecewise constant [BdHQ], the above holds using the Dirichlet-to-Neumann map as the data. However, if the inverse boundary value problem with the Dirichlet-to-Neumann map as the data is Lipschitz stable, then the inverse problem with the single-layer potential operator as the data is Lipschitz stable.

Indeed, assume that \( k^2 \) is not a Dirichlet eigenvalue of \(-\Delta + q\) or of \(-\Delta\) in \( \Omega \). Then the inverse, \( (S_{k}^+)^{-1} \), of operator \( S_{k}^+ \) exists and is bounded, \( H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega) \). Moreover, the inverse, \( S_{q,k}^{-1} \), of operator \( S_{q,k} \) exists and is bounded, \( H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega) \). For a proof of this proposition, see [Na, section 6]. Essentially, it follows that
\[ S_{q,k} = S_{q,k}^{+}[I + (S_{k}^+)^{-1}(S_{q,k} - S_{k}^+)] \]
is invertible by showing that \(-1\) cannot be an eigenvalue of \((S_{k}^+)^{-1}(S_{q,k} - S_{k}^+)\). It is possible to express \( S_{q,k}^{-1} \) in terms of the difference of an interior and an exterior Dirichlet-to-Neumann map, which are both bounded.

It is immediate that
\[ \|S_{q,k} - S_{q',k}\| \leq \|S_{q,k}\| \|S_{q',k}\| \|\Lambda_{q-k^2} - \Lambda_{q'-k^2}\|, \]
while, using the statement above, it also follows that
\[ \|\Lambda_{q-k^2} - \Lambda_{q'-k^2}\| \leq \|S_{q,k}^{-1}\| \|S_{q',k}^{-1}\| \|S_{q,k} - S_{q',k}\|. \] (2.13)
As a consequence, (conditional) Lipschitz stability for the Dirichlet-to-Neumann map implies (conditional) Lipschitz stability for the single-layer potential operator.
3. Resolvent estimates

Let \( n \geq 3 \), and let \( c \in L^2(\mathbb{R}^n) \) be bounded below by a positive constant and be constant outside of a compact set. Then \( -e^2 \Delta \) is self-adjoint and nonnegative on \( L^2(\mathbb{R}^n) \) with domain \( H^2(\mathbb{R}^n) \), with respect to the inner product \( \langle u, v \rangle = \int u \overline{v} e^{-2} \). Define the resolvent

\[
R_c(\lambda) := (-e^2 \Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n), \quad \text{Im} \lambda > 0.
\]  

Proposition 3.1. Let \( c \in L^\infty \) be bounded below by a positive constant, and suppose \( \partial_c \) is a compactly supported measure which is bounded above by a radial measure, where \( \partial_r \) is the radial vector field. There is a compact set \( \mathcal{K} \subset \mathbb{R}^n \) such that, for any \( \chi_0, \chi_1 \in C^\infty_c(\mathbb{R}^n) \) with \( \text{supp} \chi_1 \cap \mathcal{K} = \emptyset \), there are \( C \) and \( \lambda_\ell > 0 \) such that

\[
\| \chi_0 R_c(\lambda) \chi_0 \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C e^{CRe},
\]

and

\[
\| \chi_1 R_c(\lambda) \chi_1 \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C/Re \lambda,
\]

for all \( \lambda \) with \( \text{Im} \lambda > 0 \) and \( \text{Re} \lambda \geq \lambda_\ell \).

Proof. Let \( h = 1/\text{Re} \lambda \). Then \( R_c(\lambda) = h^2 (h^2 \Delta - c^{-2})^{-1} c^{-2} \) and it is enough to show that

\[
\| \chi_j (h^2 \Delta - c^{-2} - ie)^{-1} \chi_j \|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq \begin{cases} e^{C/h}, & j = 0, \\ C/h, & j = 1, \end{cases}
\]

for all \( h > 0 \) sufficiently small and for all \( \varepsilon > 0 \).

To simplify notation, in the remainder of the proof we identify radial functions on \( \mathbb{R}^n \) with functions on \( [0, \infty) \). Fix \( E > 0 \) such that \( V := E - c^{-2} \) is compactly supported. Arguing as in [Da, section 2], it suffices to construct \( \varphi = \varphi_h : [0, \infty) \rightarrow [0, \infty) \) such that \( \varphi' \) is nonnegative with \( \text{max} \varphi' \) and \( \text{supp} \varphi' \) uniformly bounded in \( h \), \( \varphi'' \) is a measure, and such that

\[
-E w'/2 \leq \partial_t (w (\varphi'^2 - h \varphi'' - V)), \quad w := 1 - (1 + r)^{-\delta},
\]

for some \( \delta > 0 \) sufficiently small (and independent of \( h \)). Indeed, once we have established (3.5), we may follow [Da, section 2] word by word, except that we replace [Da, (2.1)] with (3.5).

We will first construct \( \psi = \psi_h (r) : [0, \infty) \rightarrow [0, \infty) \) with \( \psi' \) a measure such that

\[
-E w'/2 \leq \partial_t (w (\psi' - V)).
\]

Fix \( R > 0 \) such that \( \text{supp} V \) is contained in the open ball centered at zero of radius \( R \). Let \( \mu \) be a nonnegative, compactly supported, radial measure with \( \partial_t V \leq \mu \), and let

\[
\psi = \psi_h (r) := \begin{cases} \mu((0, r)) + \max V, & r \leq R, \\ \frac{B}{w} - \frac{E}{2}, & R < r \leq R_0, \\ 0, & r > R_0, \end{cases}
\]

where \( B := w(R)((\psi(R)) + E/2) \) and \( R_0 := w^{-1}(2B/E) \) are taken so as to make \( \psi \) continuous at \( r = R, R_0 \). Note that for the latter definition to make sense we must have \( 2B/E < 1 \) since \( w \)
takes values in $(0, 1)$, but since $w(R) \to 0$ as $\delta \to 0^+$, we have $B \to 0$ then as well so it suffices to take $\delta > 0$ sufficiently small. Then $0 \leq \partial_t(w(\psi - V))$ for $r \in (0, R) \cup (R_0, \infty)$, and $-\mathcal{E}w'/2 = (w\psi)'$ for $r \in (R, R_0)$, giving (3.6).

It now remains to construct $\varphi$ as above with

$$\varphi'^2 - h\varphi'' = \psi.$$  

For this, we consider the solution to the initial value problem

$$u' = (u^2 - \psi)/h, \quad u(R_0) = 0. \quad (3.7)$$

A solution exists and is absolutely continuous in the neighborhood of $R_0$ by Carathéodory’s theorem (see e.g. [CoLe, chapter 2, theorem 1.1]), and it is unique because if $u_1$ and $u_2$ are two such solutions then the difference $\bar{u} = u_1 - u_2$ solves $\bar{u}' = (u_1 + u_2)\bar{u}, \bar{u}(R_0) = 0$, and hence vanishes identically.

Observe that since $\psi(r) = 0$ for all $r \geq R_0$, it follows that $u(r) = 0$ there. We will prove that $0 \leq u \leq \sqrt{\psi(R)}$ wherever $u$ is defined. It then follows (see e.g. [CoLe, chapter 2, theorem 1.3]) that $u$ can be extended to $[0, \infty)$, where it obeys the same bounds, and we may put $\varphi' := u$. It remains to show that $0 \leq u(r) \leq \sqrt{\psi(R)}$ for $r < R_0$.

That $u(r) \geq 0$ for $r < R_0$ follows from $u' \leq u^2/h$. Indeed if there existed $r_0 < R_0$ with $u(r_0) < 0$ then nearby we would have $u'/u^2 \leq 1/h$ and hence

$$u(r_0)^{-1} - u(r)^{-1} \leq (r - r_0)/h. \quad (3.8)$$

As $r$ increases from $r_0$ this must remain true until $u(r)$ vanishes, but as $r$ approaches the first point where $u(r)$ vanishes (and such a point must exist since $u(R_0) = 0$), the left hand side of (3.8) increases without bound, which is a contradiction.

That $u \leq \sqrt{\psi(R)}$ for $r < R_0$ follows from $u' \leq (u^2 - \psi(R))/h$ by a similar argument. Indeed, let $v$ be the solution to

$$v' = (v^2 - \psi(R))/h, \quad v(R_0) = 0,$$

and observe that $v$ is defined on $\mathbb{R}$ and obeys $0 < v(r) < \sqrt{\psi(R)}$ for $r < R_0$. Suppose there existed $r_0 < R_0$ with $u(r_0) > v(r_0)$. Let $z = u - v$, so that

$$z' \geq (u^2 - v^2)/h = z(u + v)/h. \quad (3.9)$$

Since $z(r_0) > 0$ and $z(R_0) = 0$ there must be a point $r' \in (r_0, R_0)$ such that $z(r') > 0$ and $z'(r') < 0$ by the mean value theorem, but this contradicts (3.9), proving $u \leq v < \sqrt{\psi(R)}$.

In this paper we only use the bound (3.2). It would be interesting to see if the improvement (3.3) can be used to get better estimates at high frequencies below, possibly improving part (2) of the theorem.

### 4. Forward operator

Beginning in this section we take the dimension $n = 3$. The results of this section generalize almost without changes to the case of arbitrary odd dimensions $\geq 3$ and to wave speeds $c$ which are any constant $c_0 > 0$ outside of a compact set—only the notation is a little more complicated then. We expect even dimensions to also be manageable, once the behavior of the resolvent near 0 is analyzed, e.g. in the manner of [Bu1].

Let $\Omega \Subset \mathbb{C}^3$ be a bounded domain with a smooth boundary. For every $\varepsilon > 0$, let $\Omega_\varepsilon$ be the set of points in $\Omega$ of distance greater than $\varepsilon$ to $\partial\Omega$, and $L_{1,\varepsilon}^\infty$ be the set of functions
\( c \in L^\infty(\mathbb{R}^3) \) which are bounded below by a positive constant and which are identically 1 outside of \( \Omega_\varepsilon \). For \( c \in L^\infty(\mathbb{R}^3) \), let \( R_c(\lambda) \) be the resolvent as defined in (3.1). In the following lemma we review some resolvent bounds which are essentially well-known.

**Lemma 4.1.** Let \( \varepsilon > 0 \), let \( c \in L^\infty(\mathbb{R}^3) \) and fix \( \chi_0 \in C^\infty(\mathbb{R}^3) \) which is identically 1 near \( \overline{\Omega} \).

1. The cutoff resolvent \( \chi_0 R_c(\lambda) \chi_0 : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3) \) extends continuously from \( \{ \lambda \in \mathbb{C} | \text{Im} \lambda > 0 \} \) to \( \mathbb{R} \).

2. For \( \lambda_0 \geq 1 \), put
   \[
   a_c(\lambda_0) \triangleq 1 + \max_{\lambda \in [-\lambda_0, \lambda_0]} \| \chi_0 R_c(\lambda) \chi_0 \|_{L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)}.
   \]
   If \( c' \in L^\infty(\mathbb{R}^3) \) obeys \( c'^2 - c^2 \|_{L^\infty} \leq 1/2 a_c(\lambda_0) \), then
   \[
   a_{c'}(\lambda_0) \leq (1 + C \| c'^2 - c^2 \|_{L^\infty}) a_c(\lambda_0).
   \]

3. For every \( \chi \in C^\infty_c(\mathbb{R}^3) \) with \( \text{supp} \chi \cap \text{supp}(1 - c) = \emptyset \) and \( \chi \chi_0 = \chi_c \), \( \chi_0 R_c(\lambda) \chi \) extends to a bounded family of operators \( H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3) \) for every \( s \in [-2, 0] \) and \( \lambda \in \mathbb{R} \), and
   \[
   \max_{\lambda \in [-\lambda_0, \lambda_0]} \| \chi_0 R_c(\lambda) \chi \|_{H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)} \leq C \lambda_0^s a_c(\lambda_0).
   \]

4. For every \( \chi_1 \in C^\infty_c(\mathbb{R}^3) \) with \( \text{supp} \chi_1 \cap \text{supp} \chi = \emptyset \), \( \chi_0 R_c(\lambda) \chi_1 \) extends to a bounded family of operators \( L^2(\mathbb{R}^3) \to H^s(\mathbb{R}^3) \) for every \( N \in \mathbb{N} \), and
   \[
   \max_{\lambda \in [-\lambda_0, \lambda_0]} \| \chi_1 R_c(\lambda) \chi_1 \|_{L^2(\mathbb{R}^3) \to H^s(\mathbb{R}^3)} \leq C \lambda_0^{N+2} a_c(\lambda_0).
   \]

**Proof.** To prove (1), we observe that \( -\Delta \) is a black box operator in the sense of Sjöstrand and Zworski [SjZw], so \( R_c(\lambda) : L^2_{\text{comp}}(\mathbb{R}^3) \to L^2_{\text{loc}}(\mathbb{R}^3) \) continues meromorphically to \( \lambda \in \mathbb{C} \) ([Sj, theorem 2.2] or [DyZw, theorem 4.4]). We may replace \( L^2_{\text{loc}}(\mathbb{R}^3) \) by \( H^2_{\text{loc}}(\mathbb{R}^3) \) thanks to the identity \( \Delta R_c(\lambda) = -c^2(\lambda^2 R_c(\lambda) + \chi) \), so to prove (1) it remains to show that there are no poles in \( \mathbb{R} \). Indeed, suppose by way of contradiction that \( \lambda' \in \mathbb{R} \) is such a pole. Then, by [Sj, section 2.4] or [DyZw, section 4.2] there is a corresponding outgoing resonant state, that is an outgoing solution \( u_0 \) to \((\Delta - c^2 \lambda^2 \chi_0)u_0 = 0 \) which is not identically zero. If \( \lambda_0 = 0 \), then \( u_0 \) is a bounded harmonic function and must vanish. If \( \lambda_0 \neq 0 \), then by [Sj, theorem 2.4] or [DyZw, theorem 3.32] \( u_0 \) is compactly supported and hence must vanish by Aronszajn’s unique continuation theorem [Ar].

To prove (2) we observe that, multiplying by \( \chi_0 \) on the right and solving for \( R_c(\lambda) \chi_0 \) in the resolvent identity

\[
R_c(\lambda) - R_c(\lambda) = R_c(\lambda)(c'^2 - c^2) \Delta R_c(\lambda)
\]

(4.1) gives, using the fact that \( \chi_0 = 1 \) near \( \text{supp}(c' - c) \),

\[
R_c(\lambda) \chi_0 = R_c(\lambda) \chi_0 \sum_{k=0}^{\infty} (c'^2 - c^2) \Delta \chi_0 R_c(\lambda) \chi_0^k,
\]

(4.2)
where the sum is a Neumann series in the sense of operators $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$. Hence
\[
\|\chi_0 R_c'(\lambda)\chi_0\|_{L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)} \leq (1 + C\|e^{\lambda^2} - e^{\lambda^2}\|_{L^\infty}) \|\chi_0 R_c(\lambda)\chi_0\|_{L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)},
\]
and (2) follows.

To prove (3), we use the resolvent identity
\[
R_c(\lambda) = R(\lambda) + R_c(\lambda)(c^2 - 1)\Delta R(\lambda) + R_c(\lambda)(1 - c^2)R(\lambda),
\]
where $R(\lambda) = (-\Delta - \lambda^2)^{-1}$. Since $(1 - c)\chi = 0$, this implies
\[
R_c(\lambda)\chi = R(\lambda)\chi + \lambda^2 R_c(\lambda)(1 - c^2)R(\lambda)\chi.
\]
(4.3)
Then (3) follows from the fact that $\chi_0 R(\lambda)\chi_0$ is a continuous family of operators $H^s(\mathbb{R}^3) \to H^{s+2}(\mathbb{R}^3)$ for every $\chi_0$.

Finally, (4) has already been established for $N = 2$. It follows for larger $N$ by induction, since
\[
\Delta R_c(\lambda)\chi_1 = -\lambda^2 R_c(\lambda)\chi_1 + [\Delta, \chi]R_c(\lambda)\chi_1.
\]
\]

Denote a Fourier transform in time by
\[
\hat{f}(\lambda) = \int e^{-i\lambda t}f(t)dt.
\]
For $\lambda_0 \geq 1$, let $L_{\lambda_0}$ be the set of functions $f = f(t, x) \in L^2(\mathbb{R}; H^{-1/2}(\partial\Omega))$, such that $\hat{f}(\lambda, x) \equiv 0$ when $|\lambda| \geq \lambda_0$. Let $\tau_{\partial\Omega}$ denote the trace map to $\partial\Omega$, i.e. the map which restricts a function on $\mathbb{R}^3$ to $\partial\Omega$. Recall that $\tau_{\partial\Omega}$ is bounded from $H^{s+1/2}(\mathbb{R}^3)$ to $H^s(\partial\Omega)$ for $s > 0$, and hence by duality $f \in L^2(\mathbb{R}; H^{-1/2}(\partial\Omega)) \Rightarrow f\tau_{\partial\Omega} \in L^2(\mathbb{R}; H^{-1}(\mathbb{R}^3))$.

To simplify the formulas below we write $f$ for $f\tau_{\partial\Omega}$ below.

In the next proposition, we apply the resolvent bounds of lemma 4.1 to study solutions \(^4\) to the wave equation (1.1).

**Proposition 4.1.** Fix $\varepsilon > 0$.

1. For every $c \in L_{\lambda_0}^{\varepsilon, c}$, the formula
\[
U_c f(t) = \frac{1}{2\pi} \int e^{-i\lambda t}R_c(\lambda)\hat{f}(\lambda)\,d\lambda d\lambda,
\]
defines a bounded linear operator $U_c : L_{\lambda_0} \to H^m(\mathbb{R}; H^{1-\varepsilon}(\mathbb{R}^3))$ for every $\lambda_0 > 0, m \in \mathbb{N}$.

For every $f \in L_{\lambda_0}$ we have
\[
(\partial_t^2 - c^2\Delta)U_c f = f.
\]
There is a constant $C$, depending on $m$, such that
\[
\|\tau_{\partial\Omega} U_c f\|_{H^m(\mathbb{R}; H^{1-\varepsilon}(\partial\Omega))} \leq C\lambda_0^{m+2} a_c(\lambda_0) \|f\|_{L^2(\mathbb{R}; H^{-1/2}(\partial\Omega))},
\]
for all $f \in L_{\lambda_0}$.

2. If $c, c' \in L_{\lambda_0}^{\varepsilon, c}$, then for every $m \in \mathbb{N}, N \in 2\mathbb{N}$, there is a constant $C$ such that, for every $\lambda_0 > 0$ and $f \in L_{\lambda_0}$ we have

\[^4\) In this paper we study *forward* solutions; see also the footnote following (1.2).
\[ \| \tau_{\Omega} (U_{c'} - U_{c}) f \|_{H^m(\mathbb{R}; H^{N/2} (\partial \Omega))} \leq C \lambda_0^{m+N+2} a_c(\lambda_0) a_{c'}(\lambda_0) \| f \|_{L^2(\mathbb{R}; H^{1/2} (\partial \Omega))}. \]  

(4.6)

3. If \( c \in L^\infty_{\text{loc}}, \) then for every \( m \in \mathbb{N}, N \in 2\mathbb{N}, \) there is a constant \( C \) such that, for every \( \lambda_0 > 0 \) and \( f \in L_{\lambda_0} \) and \( c' \in L^\infty_{\text{loc}} \) such that \( \| c/c' - c \|_{L^\infty} \leq 1/2a_c(\lambda_0) \) we have

\[ U_{c'} f(t) - U_c f(t) = G_{c,c'} f(t) + E_{c,c'} f(t), \]  

(7.4)

where

\[ G_{c,c'} f(t) = \frac{1}{2\pi} \int e^{-it\lambda} R_e(\lambda)(1 - c' c - 2) \lambda^2 R_e(\lambda) f(\lambda) d\lambda, \]

and

\[ \| \tau_{\Omega} G_{c,c'} f \|_{H^m(\mathbb{R}; H^{N/2} (\partial \Omega))} \leq C \lambda_0^{m+N+2} a_c(\lambda_0) \| c' c - 2 \|_{L^\infty(\mathbb{R})} \| f \|_{L^2(\mathbb{R}; H^{1/2} (\partial \Omega))}, \]

\[ \| \tau_{\Omega} E_{c,c'} f \|_{H^m(\mathbb{R}; H^{N/2} (\partial \Omega))} \leq C \lambda_0^{m+N+2} a_c(\lambda_0) \| c' c - 2 \|_{L^\infty(\mathbb{R})} \| f \|_{L^2(\mathbb{R}; H^{1/2} (\partial \Omega))}. \]  

(4.8)

So in particular

\[ \| \tau_{\Omega} (U_{c'} - U_{c}) f \|_{H^m(\mathbb{R}; H^{N/2} (\partial \Omega))} \leq C \lambda_0^{m+N+2} a_c(\lambda_0) \| c' c - 2 \|_{L^\infty(\mathbb{R})} \| f \|_{L^2(\mathbb{R}; H^{1/2} (\partial \Omega))}. \]  

(4.9)

**Proof.** Fix \( \chi_0, \chi, \chi_1 \in C_c^\infty(\mathbb{R}^3), \) such that \( \chi \) is 1 near \( \partial \Omega, \) \( \text{supp} \chi \cap \overline{\Omega} = \emptyset, \) and \( \lambda_0 = 1 \) near \( \overline{\Omega} \cup \text{supp} \chi, \chi_1 = 1 \) near \( \overline{\Omega} \) and \( \text{supp} \chi_1 \cap \text{supp} \chi = \emptyset. \)

1. Since \( f \in H^{-1}(\mathbb{R}^3), \) and \( \chi f = f, \) for any \( t \in \mathbb{R} \) (4.4), defines a function \( U_t f(t) \in H^1_{\text{loc}}(\mathbb{R}^3) \) by lemma 4.1 (3). Moreover for any \( m \geq 0 \) there is a constant \( C \) such that

\[ \| \chi_0 U_t f \|_{H^m(\mathbb{R}; H^{N/2})} \leq C \lambda_0^{m+2} a_c(\lambda_0) \| f \|_{L^2(\mathbb{R}; H^{1/2} (\partial \Omega))}, \]

for all \( f \in L_{\lambda_0}. \) Applying \( \tau_{\Omega} \) gives (4.5).

2. Multiplying the resolvent identity (4.1) on the right by \( \chi, \) and arguing as we did to obtain (4.3), we have

\[ (R_{c'}(\lambda) - R_e(\lambda)) \chi = R_{c'}(\lambda)(1 - c' c - 2) \lambda^2 R_e(\lambda) \chi, \]  

(4.10)

and hence

\[ (U_{c'} - U_c) f(t) = \frac{1}{2\pi} \int e^{-it(\lambda-\lambda)} R_{c'}(\lambda)(1 - c' c - 2) \lambda^2 R_e(\lambda) f(\lambda) d\lambda, \]

and in particular

\[ \tau_{\Omega} (U_{c'} - U_c) f(t) = \tau_{\Omega} \frac{1}{2\pi} \int e^{-it\lambda} R_{c'}(\lambda) \chi_1(1 - c' c - 2) \lambda^2 R_e(\lambda) f(\lambda) d\lambda, \]

Then lemma 4.1 (3) and (4) give (4.6).
3. Starting with (4.2) and arguing as we did to obtain (4.3), we have
\[
(R_c^t(\lambda) - R_c(\lambda))\chi = R_c(\lambda) \sum_{k=1}^{\infty} ((c^{t^2} - c^2) \Delta R_c(\lambda))^k \chi
\]
and in particular
\[
(R_c^t(\lambda) - R_c(\lambda))f = R_c(\lambda) \sum_{k=0}^{\infty} ((c^{t^2} - c^2) \Delta R_c(\lambda))^k \left(1 - c^2 e^{-2} \lambda^2 R_c(\lambda)\right) \chi f(\lambda).
\]
This implies that in the decomposition (4.7) we have
\[
E_{c,t} f(t) = \frac{1}{2\pi} \int e^{-it\lambda} R_c(\lambda) \sum_{k=1}^{\infty} ((c^{t^2} - c^2) \Delta R_c(\lambda))^k \left(1 - c^2 e^{-2} \lambda^2 R_c(\lambda)\right) \chi f(\lambda) d\lambda.
\]
On the other hand
\[
\tau_{0(t)} G_{c,t} f(t) = \tau_{0(t)} \frac{1}{2\pi} \int e^{-it\lambda} R_c(\lambda) \chi_0(1 - c^2 e^{-2} \lambda^2 R_c(\lambda)) \chi f(\lambda) d\lambda.
\]
Then lemma 4.1 (3) and (4) gives the first of (4.8). The proof of the second of (4.8) (which, together with the first of (4.8), implies (4.9)) is very similar. □

5. Landweber iteration

Let $X$ be a Hilbert space\(^5\) and let $c : X \to L^\infty(\mathbb{R}^3)$ be Fréchet differentiable with locally Lipschitz derivative, and weakly sequentially continuous (in the sense that it sends weakly convergent sequences to weakly convergent sequences). Fix an open set $X_0 \subset X$ such that $c(X_0) \subset L^\infty(\mathbb{R}^3)$. Note that in the theorem $c$ is the inclusion map from a finite dimensional subspace of $L^\infty(\mathbb{R}^3)$; in the statement there we identify $X_0$ and $c(X_0)$, and write simply $c \in X_0$.

Fix $\lambda_0 > 0$, $r > 0$, and $w \in L^1(\mathbb{R})$ which is nonnegative, and continuous at 0 with $w(0) = 1$. For each $T > 0$ and $t_0 \in \mathbb{R}$, let
\[
w_T(t) = w((t - t_0)/T),
\]
and let $Y = L^2(\mathbb{R}, w_T dt; HS(L_{\lambda_0} \to H'(\partial\Omega)))$, that is the Hilbert space of functions in the $w_T$-weighted $L^2$ space on $\mathbb{R}$ with values in the space of Hilbert–Schmidt operators from $L_{\lambda_0}$ to $H'(\partial\Omega)$. In the theorem, $w$ is the characteristic function of $[-1, 1], T = \pi/2\lambda_0$, and $r = 1/2$.

Define $F : X_0 \to Y$ by
\[
F(x)f(t) = \tau_{0(t)}(U_{(-1)} - U_{(0)})f(t), \quad x \in X_0, f \in L_{\lambda_0}, t \in \mathbb{R}.
\] (5.1)
To see that $F(x) \in Y$, observe that (4.6) implies that $f \mapsto F(x)f(t)$ is bounded from $L_{\lambda_0}$ to $H^M(\partial\Omega)$ for every $M \in \mathbb{N}$, and hence it is a Hilbert–Schmidt operator from $L_{\lambda_0}$ to $H'(\partial\Omega)$ since the inclusion operator $H^M(\partial\Omega) \hookrightarrow H'(\partial\Omega)$ is Hilbert–Schmidt for $M - r > 3/2 = \dim \partial\Omega/2$ (see e.g. the proof of [DyZw, proposition B.20]). More precisely, fix $N \in 2\mathbb{N}$ with $N > r + 2$ and put

\(^5\) Following the methods of [DHQS], one could also allow $X$ to be a Banach space satisfying the convexity and smoothness conditions of that paper.
so that

\[ \| F(x) \|_2^2 = \int \| F_t(x) \|_{L^2(L^2(\mathbb{R^d}))}^2 \, dt \leq C \int \| F_t(x) \|_{L^2(L^2(\mathbb{R^d}))}^2 \, dt \]

\[ \leq CT \sup \left\{ \int \tau \left( U_c(x) - U(t) \right) \right\} \| f \|_{H^p(\mathbb{R}; \mathbb{R}^{n \times n})} \| f \|_{H^{-1/2}(\mathbb{R}^d)}^2 \]

\[ \leq CT \lambda_0^{2N+6} a_c(\lambda_0)^4, \]

(5.2)

where in the second inequality we used \( \| \cdot \|_{L^2(\mathbb{R}^d)} \leq \| \cdot \|_{H^p(\mathbb{R})} \), and in the third we used (4.6).

5.1. Fréchet derivative

By proposition 4.1(3), the Fréchet derivative of \( F \) is given by

\[ DF(x)(\xi - x)f = \frac{1}{\pi} \tau \int e^{-it\lambda} R_{c(x)}(\lambda) [Dc(x)(\xi - x)] c(x)^{-1} \lambda^2 R_{c(x)}(\lambda) \tilde{f}(\lambda) d\lambda, \]

where \( Dc \) is the Fréchet derivative of \( c \).

Lemma 5.1. Fix \( x_0 \in X_0 \). There are constants \( C_{c} \) and \( C_{l} \) and a small closed ball \( B \subset X_0 \) with \( x_0 \in B \), such that the following hold for all \( x, \xi \in B \), and \( \lambda_0 > 0 \):

\[ \| DF(x) \|_{X^\ast - Y} \leq \hat{L} := C_{c} T \lambda_0^{N+3} a_c(\lambda_0)^2, \]  

(5.3)

\[ \| DF(x) - DF(\xi) \|_{X^\ast - Y} \leq L \| x - \xi \|, \quad L := C_{l} T \lambda_0^{N+3} a_c(\lambda_0)^3. \]  

(5.4)

Also, \( F \) is weakly sequentially closed, in the sense that if \( x_n \to x \) weakly and \( F(x_n) \to y \) in \( Y \), then \( F(x) = y \).

Proof. Arguing as in (5.2), but using (4.8) in place of (4.6), we have

\[ \| DF(x)(x - \xi) \|_{L^2}^2 \]

\[ \leq CT \sup_f \left\{ \int \tau \int e^{-it\lambda} R_{c(x)}(\lambda) [Dc(x)(\xi - x)] c(x)^{-1} \lambda^2 R_{c(x)}(\lambda) \tilde{f}(\lambda) d\lambda \right\} \]

\[ \leq CT \lambda_0^{2N+6} a_c(\lambda_0)^4 \| Dc(x)(\xi - x) \|_{L^2}^2 \leq CT \lambda_0^{2N+6} a_c(\lambda_0)^4 \| x - \xi \|_Y^2, \]

where the sup is again taken over \( f \in L_{\lambda_0} \) with \( \| f \|_{L^2(\mathbb{R}; H^{1/2}(\mathbb{R}^d))} = 1 \). This implies (5.3).

Let \( h \in X \) have \( \| h \|_X = 1 \) and \( f \in L_{\lambda_0} \) have \( \| f \|_{L^2(\mathbb{R}; H^{1/2}(\mathbb{R}^d))} = 1 \). To save space, write \( c \) for \( c(x) \) and \( \xi \) for \( c(\xi) \). Then

\[ (DF(x) - DF(\xi))(h)f = \frac{1}{\pi} \tau \int e^{-it\lambda} [R_{c}(\lambda) [Dc(x)h] e^{-1} R_{c}(\lambda) - R_{c}(\lambda) [Dc(h)] e^{-1} R_{c}(\lambda)] \tilde{f}(\lambda) d\lambda, \]

where \( \tilde{f}(\lambda) \) denotes the Fourier transform of \( f \) with respect to \( \lambda \).

Adding and subtracting \( \frac{1}{\pi} \tau \int e^{-it\lambda} R_{c}(\lambda) [(Dc(h)] e^{-1} R_{c}(\lambda) \tilde{f}(\lambda) d\lambda \), we estimate this in pieces. First, using \( \| (Dc(x) - Dc(\xi))h \|_{L^\infty} \leq C \| x - \xi \|_X \) (since \( c \) is Fréchet differentiable with locally Lipschitz derivative), we have
Second
\[ R_c(\lambda)[(D\bar{c})h]c^{-1}R_c(\lambda) - R_c(\lambda)[(D\bar{c})h]\tilde{c}^{-1}R_c(\lambda) \]
can be written as a sum of three differences similarly (using (4.1) and (4.10)), giving
\[ \|Df(x) - Df(\tilde{x})\|_F \leq C\lambda_0^{N+3}a_c(\lambda)\|x - \tilde{x}\|_X. \]
To prove that \(F\) is weakly sequentially closed, it is enough to show that if \(x_n \to x\) weakly, then \(F(x_n)\) tends to \(F(x)\) in the sense of distributions on \(\mathbb{R} \times \partial \Omega\). As in the proof of proposition 4.1(2), we write
\[ F(x_n) - F(x)_f = \tau_{0\Omega} \int e^{-i\lambda}R_{c(x_n)}(\lambda)(c(x_n)^2 - c(x)^2)g(\lambda)d\lambda, \]
and observe that \(\|g(\lambda)\|_{L^\infty(\Omega)}\) is uniformly bounded for \(\lambda \in [-\lambda_0, \lambda_0]\). Pairing with \(\varphi \in C_c^\infty(\mathbb{R} \times \partial \Omega)\) gives
\[ \langle F(x_n)f - F(x)f, \varphi \rangle = \int_{-\lambda_0}^{\lambda_0} \int_{\Omega} (c(x_n)^2 - c(x)^2)g(\lambda)\psi_n(\lambda)d\xi d\lambda, \]
where \(\xi\) is a coordinate in \(\mathbb{R}^3\). By lemma 4.1(4), for each \(M \in \mathbb{N}\) the norm \(\|\psi_n(\lambda)\|_{H^M(\Omega)}\) is uniformly bounded for \(\lambda \in [-\lambda_0, \lambda_0]\) and \(n \in \mathbb{N}\). Passing to a subsequence, \(\psi_n(\lambda)\) converges uniformly in \(H^M(\Omega)\) to a limit, which we denote \(\psi(\lambda)\). Then, as \(n \to \infty\),
\[ \left| \int_{-\lambda_0}^{\lambda_0} \int_{\Omega} (c(x_n)^2 - c(x)^2)g(\lambda)(\psi_n(\lambda) - \psi(\lambda))d\xi d\lambda \right| \leq C(\lambda_0)\sup_{\lambda} \|\psi_n(\lambda) - \psi(\lambda)\|_L^\infty \to 0, \]
and
\[ \int_{-\lambda_0}^{\lambda_0} \int_{\Omega} (c(x_n)^2 - c(x)^2)g(\lambda)\psi(\lambda)d\xi d\lambda \to 0 \]
by the dominated convergence theorem, since \(\int_{\Omega} (c(x_n)^2 - c(x)^2)g(\lambda)\psi(\lambda)d\xi\) is uniformly bounded in \(\lambda\) and \(n\) and tends to 0 for each \(\lambda\) since \(c(x_n) \to c(x)\) weakly (this follows from \(x_n \to x\) weakly since \(c\) is weakly sequentially continuous). This shows that every subsequence of \((F(x_n)f - F(x)f, \varphi)\) has a subsequence which tends to 0, proving that the original sequence tends to 0.

5.2. Hilbert–Schmidt misfit functional and gradient
We use the misfit functional
\[ \|F(x) - F(x^\dagger)\|_F^2 = \sum_{j=1}^\infty \int_{\partial \Omega} \|U_{c(x^\dagger)} - U_{c(x)}\|_{H^1(\partial \Omega)}^2 w_T d\xi, \]
where \(x^\dagger \in B\) is the ‘true’ model, and where \(\{\psi_j\}_{j=1}^\infty\) is any orthonormal basis of \(L_{\lambda_0}\).
Writing \( c \) for \( c(x) \), the gradient \( DF(x)^\ast \) is given by

\[
\langle DF(x)^\ast(y - \bar{y}), (x - \bar{x}) \rangle_X
= \langle (y - \bar{y}), DF(x)(x - \bar{x}) \rangle_Y
= \frac{1}{\pi} \sum_{j=1}^{\infty} \int \left( \langle y - \bar{y}, \psi_j \rangle_{\partial \Omega} \right) e^{-it\lambda} R_c(\lambda) [Dc(x - \bar{x})] e^{-i\lambda^2 - \lambda} \psi_j(\lambda) d\lambda dT
\]

\[
= \sum_{j=1}^{\infty} \int \frac{(Dc)(x - \bar{x})}{\pi e} \lambda^2 R_c(\lambda) [1 - \Delta_{\partial \Omega}]^\prime w_T(y - \bar{y}) \psi_j(\lambda) d\lambda dT,
\]

where \( \bar{x} \) is a coordinate on \( \mathbb{R}^2 \) and \( \Delta_{\partial \Omega} \) is the nonpositive Laplacian on \( \partial \Omega \). Writing \( c_m \) for \( c(x_m) \), we have

\[
DF(x_m)^\ast(F(x_m) - F(x)) = (Dc_m)^\ast \sum_{j=1}^{\infty} \frac{1}{\pi c_m} \int \lambda^2 [R_{c_m}(\lambda)] [(1 - \Delta_{\partial \Omega})^\prime w_T(F(x_m) - F(x)) \psi_j(\lambda)] d\lambda dT
\]

Note that in the theorem \( c \) is an inclusion map, so \((Dc_m)^\ast\) is a projection (onto piecewise constant wave speeds).

**Adjoint state equation.** The first factor in the cross correlation integral above can be written as the solution to an inhomogeneous wave equation solved backwards in time, namely

\[
u(t) = \frac{1}{2\pi} \int e^{-i\lambda t} R_c(x_m)(\lambda) [(1 - \Delta_{\partial \Omega})^\prime w_T(F(x_m) - F(x)) \psi_j(\lambda)] d\lambda dT
\]

is the backwards in time solution to

\[
(\partial^2_t - c^2 \Delta) u = [(1 - \Delta_{\partial \Omega})^\prime w_T(F(x_m) - F(x)) \psi_j(\lambda)](-t).
\]

In [BSS], Blazek, Stolk and Symes show that the analogous equation for a problem without frequency ‘bandlimitation’ does not have a solution.

**5.3. Landweber iteration**

We define the Landweber iterates by the equation

\[
x_{m+1} = x_m - \mu DF(x_m)^\ast(F(x_m) - F(x)),
\]

where the step size \( \mu \) is sufficiently small (see [dHQS, (3.5)]).

Suppose the inversion has uniform Hölder-type stability in the sense that there is a constant \( C_F \) such that for every \( x, \bar{x} \in B \) we have

\[
\frac{1}{\sqrt{2}} \|x - \bar{x}\|_X \leq C_F \|F(x) - F(\bar{x})\|_Y^{1/2},
\]

for some \( \epsilon \in (0, 1] \). Note that below we will only use the case \( \epsilon = 1 \). Then [dHQS, theorem 3.2] applies and the Landweber iteration converges: see the next section for details.
6. Proof of theorem and discussion: convergence

In this section we work under the assumptions of section 5 and the additional assumptions of the theorem; in particular \( X = \mathbb{R}^N \), \( w \) is the characteristic function of \([-1, 1] \), \( T = \pi/2\lambda_0 \), and \( r = 1/2 \). We apply the following convergence result, a consequence of [dHQS, theorem 3.2].

**Proposition 6.1.** ([dHQS, theorem 3.2]) Let \( X \) and \( Y \) be Hilbert spaces, \( B \) a closed ball in \( X \), and \( F : B \to Y \) continuous and Fréchet differentiable with Lipschitz derivative. Suppose further that \( F \) is weakly sequentially closed and that there are constants \( L, \hat{L}, C_F \geq 1 \) such that

\[
\|DF(x)\|_{X \to Y} \leq \hat{L}, \quad \|DF(x) - DF(\hat{x})\|_{X \to Y} \leq L,
\]

and (5.6) hold for all \( x, \hat{x} \in B \), with \( \epsilon = 1 \) in the case of (5.6). Let \( \mu \in (0, \min\{1/2\hat{L}^2, 4C_F^2\}) \) and \( \mathcal{B}_1 \subset \mathcal{B} \) be a closed ball of radius \( R \leq 1/2C_F\sqrt{\hat{L}} \). Then for any \( x_0, x^1 \in \mathcal{B}_1 \), the sequence of Landweber iterates, defined by (5.5), converges to \( x^1 \) at the following exponential rate:

\[
\|x_k - x^1\|_X \leq R \left(1 - \frac{\mu}{4C_F^2}\right)^{k/2}.
\]

Now the theorem follows from proposition 6.1. All the assumptions apart from (5.6) follow from lemma 5.1, and (5.6) is deduced below from [BdHQ, theorem 2.7]. To prove part (2) of the theorem, we observe that (3.2) implies that \( L, \hat{L} \) grow at most exponentially in \( \lambda_0 \), while the proof of (5.6) below shows that \( C_F \) grows at most like \( \sqrt{\lambda_0} \).

**Proof of (5.6).** To save space we write \( c = c(x), \hat{c} = c(\hat{x}) \). Fix \( \lambda_0 \in (0, 1) \) such that \( \lambda_0^2 \) is not a Dirichlet eigenvalue of \(-\Delta\) or of \(-\Delta + \lambda_0^2(1 - \hat{c}^2)\) or of \(-\Delta + \lambda_0^2(1 - \hat{c}^2)\) for any \( x, \hat{x} \in \mathcal{B} \) (this can be arranged by taking the ball \( \mathcal{B} \) small enough).

As an orthonormal basis of \( L_{\lambda_0} \), take \( \{\psi_j\}_{j=1}^{\infty} \) such that \( \{\hat{\psi}_j\}_{j=1}^{\infty} = \{\hat{a}_j b_{m}\}_{(m, n) \in \mathbb{Z} \times \mathbb{N}} \), where each \( \hat{a}_j \) is the characteristic function of \([-\lambda_0, \lambda_0] \) multiplied by \( e^{-\delta j^2/\lambda_0}/\sqrt{\lambda_0} \), and \( \{b_{m}\}_{m=1}^{\infty} \) is an orthonormal basis of \( H^{-1/2}(\partial \Omega) \) with \( b_1 \) chosen such that

\[
\frac{\|S_{\lambda_0}(1 - \hat{c}^2, \lambda_0) - S_{\lambda_0}(1 - \hat{c}^2, \lambda_0) b_1\|_{H^{1/2}(\partial \Omega)}}{\|S_{\lambda_0}(1 - \hat{c}^2, \lambda_0) - S_{\lambda_0}(1 - \hat{c}^2, \lambda_0)\|_{H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)}} \geq 1/2.
\]

By (2.13) we have

\[
\frac{\|S_{\lambda_0}(1 - \hat{c}^2, \lambda_0) - S_{\lambda_0}(1 - \hat{c}^2, \lambda_0)\|_{H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)}}{\|\Lambda - \lambda_0^2 e^{-2} - \Lambda - \lambda_0^2 e^{-2}\|_{H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)}} \geq 1/C,
\]

where we used the fact that if the ball \( \mathcal{B} \) is small enough then \( \|S_{\lambda_0}(1 - \hat{c}^2, \lambda_0)\| / 2 \leq \|S_{\lambda_0}(1 - \hat{c}^2, \lambda_0)\| \leq 2\|S_{\lambda_0}(1 - \hat{c}^2, \lambda_0)\| \) by (4.2). By [BdHQ, theorem 2.7], we have

\[
\|\Lambda - \lambda_0^2 e^{-2} - \Lambda - \lambda_0^2 e^{-2}\|_{H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)} \geq \|x - \hat{x}\|_X / C.
\]

On the other hand, if \( J \) is a sufficiently small open interval containing \( \lambda_1 \), we have

\[
\min_{\lambda \in J} \|\tau_\Omega(R_c(\lambda) - R_c(\lambda)) b_1\|_{H^{1/2}(\partial \Omega)} \geq \|\tau_\Omega(R_c(\lambda) - R_c(\lambda)) b_1\|_{H^{1/2}(\partial \Omega)}/2.
\]
Meanwhile, since \( \hat{a}_t(\lambda) = e^{-it\pi\lambda/\lambda_0} \hat{a}_0(\lambda) \), we have \( U_t a_t b_m(t) = U_t a_0 b_m(t + t\pi/\lambda_0) \), so that
\[
\| F(x) - F(\bar{x}) \|^2 \geq \sum_{\ell \in \mathbb{Z}} \int_{\partial\Omega} \| \tau_{\partial\Omega}(U_t - U_{\ell}) a_\ell b_1 \|_{H^{1/2}(\partial\Omega)}^2 \, w_T \, dt
\]
\[
= \int_{\partial\Omega} \| \tau_{\partial\Omega}(U_t - U_{\ell}) a_0 b_1 \|_{H^{1/2}(\partial\Omega)}^2 \, dt.
\]
Putting these estimates together gives
\[
\| F(x) - F(\bar{x}) \|^2 \geq \frac{1}{8\pi^2\lambda_0} \int_{-\lambda_0}^{\lambda_0} \| \tau_{\partial\Omega}(R_x(\lambda) - R_x(\lambda_0)) b_1 \|_{H^{1/2}(\partial\Omega)}^2 \, d\lambda
\]
\[
\geq \| \tau_{\partial\Omega}(R_x(\lambda_0)) b_1 \|_{H^{1/2}(\partial\Omega)}^2 / C\lambda_0
\]
\[
= \| (S_{\lambda_0}(1-e^{-2\lambda_0}) - S_{\lambda_0}(1-e^{-2\lambda_0})) b_1 \|_{H^{1/2}(\partial\Omega)}^2 / C\lambda_0
\]
\[
\geq \| x - \bar{x} \|^2 / C\lambda_0.
\]

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