Computer modelling of architectural forms based on ruled surfaces with imaginary axes

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Abstract. One of the methods for constructing ruled surfaces is to isolate them from a linear congruence by immersing a guide line into the congruence body. The article considers a linear congruence defined by four intersecting lines. A theorem on the existence of a beam of planes intersecting a ruled algebraic surface of the order \( k+2 \) along algebraic curves of the order \( k \) (Theorem 1) is proved. The power frame of such surfaces is formed by rectilinear rods and arcs of second-order curves, which is their technological advantage. An algorithm is proposed for the transition from a linear congruence defined by four lines to an identical congruence defined by collinear fields \( \Pi \leftrightarrow \Pi' \). Such a transition makes it possible to solve the practically important problem of constructing a ruled surface passing through two conical sections. The theorem of an existing congruence defined by collinear fields with second-order curves drawn in them (Theorem 2) is proved. Based on Theorem 2, the construction of a ruled surface passing through predetermined conic sections \( r, r' \) is performed. Examples of constructing biaxial ruled surfaces with real and imaginary axes are considered. Architectural forms based on ruled surfaces with imaginary axes are proposed. The practical use of ruled surfaces with imaginary axes allows expanding the scope of ruled structures in architectural design. Keywords: ruled surfaces, computer modeling, architectural forms.

1. Introduction
One of the methods for constructing ruled surfaces is to isolate them from a linear congruence by immersing a directional line in the congruence body [1,2]. A linear congruence is a two-parameter set of lines intersecting in the advance given axes \( u, v \). The axes can be real different (hyperbolic linear congruence, HLC), coinciding (parabolic linear congruence, PLC) or imaginary (elliptic linear congruence, ELC) [3].

The set of lines of a linear congruence intersecting an algebraic curve in the order \( k \) forms a ruled algebraic surface \( \Theta \) of the order \( k+2 \) [4,5]. For example, when immersing the circle \( r \) in a congruence with the real axes \( u, v \), we obtain a fourth-order algebraic ruled surface (Figure 1).

2. Formulation of the problem
A constructive method for isolating ruled surfaces from HLC, based on well-known methods of descriptive geometry, is widely used in engineering practice [2]. If it is required to isolate the surface from the ELC, then the problem reduces to constructing a one-parameter set of real lines intersecting a pair of imaginary axes \( u\sim v \). In this case, the classical methods of descriptive geometry are not directly
applicable [6–9]. To eliminate this drawback, it is necessary to develop a graphical algorithm for constructing and visualizing ruled surfaces isolated from a linear congruence with imaginary axes.

3. Isolation of a surface from a congruence with imaginary axes

The linear congruence $K_r$ can be defined both by its axes $K_r(u, v)$ and by indicating four intersecting straight lines $K_r(a, b, c, d)$. In this case, to construct the axes, it is necessary to introduce the auxiliary ruled quadric $\varphi$ given by any three of the indicated lines (for example, $a, b, c$), and to find the intersection points of $U, V$ of the fourth straight line $d$ with the quadric $\varphi$. The required lines $u, v$ are determined from the conditions of their incidence to the points $U, V$ and the quadric $\varphi$ [10,11].

**Theorem 1.** If the algebraic ruled surface $\Theta$ of order $k+2$ is given by the guiding curve $e^k$ of the order $k$ and two intersecting rectilinear guides $u, v$ (real or imaginary conjugate) intersecting the plane of the curve $e^k$ at points $U, V$ (real or imaginary conjugate), then the line $UV$ (always real) is the axis of the beam of planes intersecting the surface $\Theta$ along the algebraic curves of the order $k$.

**Proof.** The straight line $UV$ intersects the guiding curve $e^k$ at two points (real or imaginary conjugate), and also intersects the rectilinear guides $u, v$; therefore, $UV$ represents two coincident generatrix surfaces $\Theta$ [12,13,14]. The ruled surface $\Theta$ is an algebraic surface of the order $k+2$, in the cross section of which, by the arbitrary plane passing through $UV$, we obtain a curve decomposed into the considered straight line $UV$ and a curve of the order $k$. The theorem is proved.

Based on Theorem 1, we construct the ruled surface with imaginary axes. The congruence $K_r(a, b, c, d)$ defined by the intersecting lines $a, b, c, d$ can be identically defined by the congruence $K_r(\Pi \leftrightarrow \Pi')$, consisting of lines connecting the corresponding points of collinear fields $\Pi \leftrightarrow \Pi'$. The congruence defined by collinear fields allows solving the practically important problem of constructing a ruled surface passing through a pair of conical sections given in advance [15-19].

**Theorem 2.** If in flat fields $\Pi, \Pi'$ the conic sections $r, r'$ are drawn with the points $A, A'$ marked on them, then there are only two variants of collineations $\Pi \leftrightarrow \Pi'$ with a self-consistent line $l=\Pi \cap \Pi'$, in which $r \leftrightarrow r'$, $A \leftrightarrow A'$.

**Proof.** Draw the tangents $t_{\alpha}, t_{\alpha}'$ to the given conics at the points $A, A'$ (Figure 2). From the points $1=t_\alpha \cap l, 1'=t_{\alpha}' \cap l$ we draw the tangents $t_\beta, t_\beta'$. Having marked the points $2=AB \cap l, 2'=A'B' \cap l$, we draw from these points the tangents $t_{\gamma}, t_{\gamma}', t_{\gamma}'', t_{\gamma}''$ to the conics $r, r'$. We get the quadrilaterals $(t_{\alpha}, t_\beta, t_{\gamma}, t_{\gamma}'')$ and $(t_{\alpha}', t_{\beta}', t_{\gamma}', t_{\gamma}'')$ with the vertices $(1, 2, P, Q, M, N)$ and $(1', 2', P', Q', M', N')$ described near the conics $r, r'$. According to Brianchon’s theorem, the lines $AB, CD$ and $A'B', CD'$ connecting the points of contact pass through the points $3=QP \cap MN, 3'=Q'P' \cap M'N'$ of the intersection of the diagonals of these quadrilaterals. The straight lines $CD, AB$ and $C'D', A'B'$, according to the theory of poles and polars,
are incident to points 1, 2 and 1', 2', therefore, in the planes Π and Π' full quadrangles $MNPQ$ and $M'N'P'Q'$ are obtained, bearing on their sides harmonic groups of points [14].

Thus, the correspondence of the flat fields Π, Π', given by the tangents $(t_A, t_B, t_C, t_D)$ and $(t'_A, t'_B, t'_C, t'_D)$ or the points $(M, N, P, Q), (M', N', P', Q')$, s a collineation corresponding to the conditions of the theorem. In this collineation, the conics $r, r'$ are mutually corresponding, and the straight line $l=12$ becomes the straight line $l'=1'2'$ coinciding with it. Only two collineation options are possible: $(t_A, t_B, t_C, t_D) \leftrightarrow (t'_A, t'_B, t'_C, t'_D)$ or $(t_A, t_B, t_C, t_D) \leftrightarrow (t'_A, t'_B, t'_C, t'_D)$. The theorem is proved.

Based on Theorem 2, construct a ruled surface passing through the conical sections $r, r'$ [20, 21]. Assuming that the collineation $\Pi \leftrightarrow \Pi'$ is given by the drawn conics $r, r'$ and the pair of corresponding points $A \leftrightarrow A'$ on these conics, we obtain, by Theorem 2, the congruence $\mathcal{K}r \Pi \leftrightarrow \Pi'$. To select the desired surface from it, it is necessary to indicate several $T_i$ points on one of the conical sections and find the $T_i$ points corresponding to the collineation $\Pi \leftrightarrow \Pi'$. Mark the point $T$ on the curve $r$ and consider the algorithm for constructing the point $T'$ (Figure 3).

**Figure 3.** Isolation of the generator $t$ from the congruence $\mathcal{K}r(\Pi \leftrightarrow \Pi')$

**Step 1.** Draw the tangents $t_A, t'_A$ (Figure 3, left). From the points $1=t_A \cap l, 1'=t'_A \cap l$ draw the tangents $t_B, t'_B$. Narking the points $2=AB \cap l, 2'=A'B' \cap l$, draw the tangents $t_C, t'_C$ from these points. According to Theorem 2, the collinear field matching $\Pi \leftrightarrow \Pi'$ is well defined by fours of lines $(t_A, t_B, t_C, AB) \leftrightarrow (t'_A, t'_B, t'_C, A'B')$. Another possible collineation is generated by the corresponding lines $(t_A, t_B, t_C, AB) \leftrightarrow (t'_A, t'_B, t''C, A'B')$. Moreover, the correspondence $C \leftrightarrow C'$ is replaced by the correspondence $C \leftrightarrow C''$.

**Step 2.** Through the point $T$ draw some auxiliary line (for example, the line $TC$) and mark the point $3=TC \cap AB$ on it (Figure 3, right). From the condition of conservation of the complex relation $(2AB3)=(2'A'B'3')$ find point $3'$.

**Step 3.** On the lines $B5$ and $B'5'$, mark points 4 and 4'. From the condition for maintaining the complex relation $(CT43)=(CT4'3')$ find the point $T'$. By a direct check on a computer layout, verify that $T'$ is incident to the conic $r'$. Connecting the points $T$ and $T'$, we obtain the generator $t$.

Repeated application of the algorithm allows finding any number of rectilinear generators of the desired surface. According to Theorem 1, a beam of planes passing through $l$ intersects this surface along the second-order curves [22, 23].

**Example.** Construct a fourth-order ruled algebraic surface passing through the conical sections $r, r'$ and having the generator $AA'$ (Figure 4).

According to Theorem 2, obtain two variants of the collineation: $\Pi(ABCK) \leftrightarrow \Pi'(A'B'C'K')$ (Figure 4, left) and $\Pi(ABCK) \leftrightarrow \Pi'(A'B'C''K')$ (Figure 4, right). According to Theorem 1, both surfaces carry a family of conic sections (Figure 5).
Special case. If the planes $\Pi$, $\Pi'$ are parallel, the line $l$ becomes infinitely remote. In this case, we obtain the affine correspondence of the point fields $\Pi \leftrightarrow \Pi'$.

Let the surface framework be defined by the rectilinear generatrix $AA'$ and the conics $r, r'$, lying in the parallel planes $\Pi(zx)$ and $\Pi'(z'x')$ (Figure 6). Marking in the fields $\Pi$, $\Pi'$ three pairs of the corresponding tangents $(t_A, t_B, t_C) \leftrightarrow (t'_A, t'_B, t'_C)$, we obtain the affinity $\Pi(r, A) \leftrightarrow \Pi'(r', A')$ (Figure 6, left). The straight lines connecting the corresponding points of the affine fields $\Pi$, $\Pi'$ form the body of the linear congruence $Kr(\Pi \leftrightarrow \Pi')$. Selecting several generators of the desired surface from $Kr(\Pi \leftrightarrow \Pi')$ and using Theorem 1, we obtain a fourth-order algebraic surface framework consisting of a family of rectilinear generators and a family of second-order curves lying in parallel planes (Figure 6, right). If the conics $r$, $r'$ are similar and similarly arranged, then the fourth-order algebraic surface degenerates into a linear quadric.

4. Architectural forms based on ruled surfaces with imaginary axes

Example 1. The bearing frame of the surface is defined by the circle $r$ and four lines $a$, $b$, $c$, $d$ (Figure 7, a). It is required to “pull” the ruled surface onto the given frame.
Figure 7. Surface selected from the congruence $K_r (a, b, c, d)$: $a$ – bearing frame; $b$ – modeling result; $c$ – upper and lower cavities; $d$ – architectural form.

**Solution.** The lines $a, b, c, d$ define the linear congruence with imaginary axes. The circle $r$ selects from the set of straight lines of congruence the fourth-order algebraic ruled surface, which, according to Theorem 1, carries a family of second-order curves (Figure 7, b). The resulting surface breaks up into upper and lower cavities, which are connected in the circular section $r$ (Figure 7, c). Both cavities can be used to design new architectural forms (Figure 7, d).

**Example 2.** The bearing frame of the surface is defined by two hyperbolas $g, g'$ and the rectilinear generator $AA'$ (Figure 8, left). It is required to “pull” the ruled surface onto the given frame.

**Solution.** Using the three-step algorithm considered above, we find several rectilinear generators of the constructed surface. According to Theorem 1, the surface carries a family of conical sections (in this example, hyperbolas) whose planes $\Pi_1 \ldots \Pi_4$ pass through $l$ (Figure 8, right). The visualization of the surface compartment is shown in Figure 9.

Figure 8. For Example 2: initial data (left); surface frame (right).

Figure 9. Visualization (for Example 2).

5. **Conclusion**

The practical use of ruled surfaces with imaginary axes allows constructing architectural forms, floors and shells that differ from traditional surfaces based on material axes, which expands the scope of ruled structures in architectural and construction design [24, 25].

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