On Collisions with Unlimited Energies in the Vicinity of Kerr and Schwarzschild Black Hole Horizons

A. A. Grib1,2*, Yu. V. Pavlov1,3**, and O. F. Piattella4,5***

1A. Friedmann Laboratory for Theoretical Physics, Griboedov kanal 30/32, St. Petersburg 191023, Russia
2Theoretical Physics and Astronomy Department, The Herzen University, Moika 48, St. Petersburg 191186, Russia
3Institute of Mechanical Engineering, Russian Acad. Sci., Bol’shoy pr. 61, St. Petersburg 199178, Russia
4Dep. de Física, Universidade Federal do Espírito Santo, av. F. Ferrari 514, 29075-910 Vitória, ES, Brazil
5INFN sezione di Milano, Via Celoria 16, 20133 Milano, Italy

Abstract—Two-particle collisions close to the horizon of a rotating non-extremal Kerr black hole and a Schwarzschild black hole are analyzed. For the case of multiple collisions, it is shown that high energy in the center-of-mass frame occurs due to a great relative velocity of two particles and a large Lorentz factor. We analyze the dependence of the relative velocity on the distance to the horizon and calculate the motion time from the point in the accretion disc to the point of scattering with large energy as well as the time of back motion to the Earth. It is shown that they have a reasonable order.

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1. INTRODUCTION

There is much interest today in high-energy processes in the ergosphere of the Kerr rotating black hole as a model for Active Galactic Nuclei (AGN). In [1], some of the authors of this paper put forward the hypothesis that, due to Penrose process and scattering in the vicinity of the horizon, superheavy dark matter particles, due to a large center-of-mass energy transfer, can become ordinary particles observed on Earth as ultra-high-energy cosmic rays (UHECR) by the AUGER group [2].

In [3], a resonance was found for the center-of-mass (CM) energy of two scattering particles close to the horizon of an extremal Kerr black hole. Let us call this effect the BSW effect.

In our papers [4–7], it was shown that the BSW effect can occur for a non-extremal black hole if one takes into account the possibility of multiple scattering of a particle: at the first scattering close to the horizon, the particle gets an angular momentum close to the critical one. At the second scattering close to the first one, the particles, due to the BSW effect, happen to be in the region of high-energy physics — Grand Unification or even Planckian physics.

In [8] (see also [9, 10], it has been shown that the BSW effect can be connected with a specific behavior of the Killing vector in the ergosphere and a large Lorentz factor for relative velocity of two particles. In this paper we continue our analysis of this process made in [4–7].

The system of units GR = c = 1 is used in the paper.

2. THE SCATTERING ENERGY IN THE CENTER-OF-MASS FRAME

The Kerr metric of a rotating black hole in the Boyer–Lindquist coordinates has the form

$$ds^2 = dt^2 - \frac{2Mr}{\rho^2} (dt - a \sin^2 \theta d\varphi)^2 - \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) - (r^2 + a^2) \sin^2 \theta d\varphi^2, \quad (1)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2, \quad (2)$$

$M$ is the black-hole mass, $aM$ is the angular momentum. The event horizon of a Kerr black hole corresponds to the value

$$r = r_H \equiv M + \sqrt{M^2 - a^2}. \quad (3)$$
The Cauchy horizon is
\[ r = r_C \equiv M - \sqrt{M^2 - a^2}. \]

For geodesics in Kerr’s metric (1) one obtains (see [11], Sec. 3.4.1)
\[ \rho^2 \frac{dt}{d\lambda} = -a(aE \sin^2 \theta - J) + \frac{r^2 + a^2}{\Delta} P, \quad (5) \]
\[ \rho^2 \frac{d\varphi}{d\lambda} = -\left( aE - \frac{J}{\sin^2 \theta} \right) + \frac{aP}{\Delta}, \quad (6) \]
\[ \rho^4 \left( \frac{dr}{d\lambda} \right)^2 = R, \quad \rho^4 \left( \frac{d\theta}{d\lambda} \right)^2 = \Theta. \quad (7) \]

Here \( m \) is the rest mass of the test particle, \( \lambda = \tau/m \), where \( \tau \) is the proper time for a massive particle, \( E \) is the conserved energy of the test particle, \( J \) is the conserved angular momentum projection on the rotation axis of the black hole,
\[ P = (r^2 + a^2)E - aJ, \quad (8) \]
\[ R = P^2 - \Delta[m^2r^2 + (J - aE)^2 + Q], \quad (9) \]
\[ \Theta = Q - \cos^2 \theta \left[ a^2(m^2 - E^2) + \frac{J^2}{\sin^2 \theta} \right], \quad (10) \]

\( Q \) is the Carter constant. For a massless particle one must take the limit \( m \to 0 \) in Eqs. (9), (10). For equatorial \( (\theta = \pi/2) \) geodesics, the Carter constant is zero, and also \( \Theta = 0 \).

Let us find the energy \( E_{\text{c.m.}} \) in the center-of-mass frame of two colliding particles with rest masses \( m_1 \) and \( m_2 \) in an arbitrary gravitational field. It can be obtained from
\[ (E_{\text{c.m.}}, 0, 0, 0) = p_{(1)}^i + p_{(2)}^i, \quad (11) \]
where \( p_{(n)}^i \) is the 4-momentum of a particle number \( n \). Taking the squared Eq. (11) and due to \( p_{(n)}^i p_{(n)i} = m_n^2 \) one obtains
\[ E_{\text{c.m.}}^2 = m_1^2 + m_2^2 + 2p_{(1)}^i p_{(2)i}. \quad (12) \]
The scalar product does not depend on the choice of the coordinate frame, so (12) is valid in an arbitrary coordinate system and for an arbitrary gravitational field.

Note that the transformation to the center-of-mass coordinate system is always possible, excluding the case of two massless particles with identically directed momenta. But in this case the particles cannot collide.

Let us find an expression for the collision energy of two particles freely falling in the equatorial plane of a rotating black hole. We denote \( x = r/M, A = a/M, \) \( j_n = J_n/M, \) \( x_H = 1 + \sqrt{1 - A^2}, \) \( x_C = 1 - \sqrt{1 - A^2} \), \( \Delta_x = x^2 - 2x + A^2 = (x - x_H)(x - x_C). \)

Using (5)–(7), one obtains:
\[ p_{(1)}^i p_{(2)i} = \frac{1}{x \Delta_x} \left\{ E_1 E_2 \left( x^3 + A^2(x + 2) \right) - 2A(j_1 E_2 + j_2 E_1) + j_1 j_2(2 - x) \right. \]
\[ - \left[ \left( 2E_1^2 x^2 + 2(j_1 - E_1 A)^2 - j_1^2 x \right) + (E_1^2 - m_1^2)x \Delta_x \right] \left( 2E_2^2 x^2 + 2(j_2 - E_2 A)^2 - j_2^2 x + (E_2^2 - m_2^2)x \Delta_x \right) \right\}^{1/2}. \quad (15) \]

The value of \( \Delta_x \) tends to zero on the event horizon and, as is evident from (15), the scalar product of four vectors \( u_{(1)}^i u_{(2)i} \) and the collision energy of particles on the horizon can be divergent depending on the behavior of the denominator of the formula. To find the limit \( r \to r_H \) for a black hole with a given angular momentum \( A \), one must take in (15) \( x = x_H + \alpha \) with \( \alpha \to 0 \) and do calculations up to the order \( \alpha^2 \).

Taking into account that \( A^2 = x_H x_C, x_H + x_C = 2 \), after resolution of uncertainties in the limit \( \alpha \to 0 \) one obtains
\[ E_{\text{c.m.}}^2(r \to r_H) = \frac{(j_{1H} J_2 - j_{2H} J_1)^2}{4M^2 (J_{1H} - J_1)(J_{2H} - J_2)} \]
\[ + m_1^2 \left[ 1 + \frac{J_{2H} - J_2}{J_{1H} - J_1} \right] \]
\[ + m_2^2 \left[ 1 + \frac{J_{1H} - J_1}{J_{2H} - J_2} \right], \quad (16) \]

where
\[ J_{nH} = \frac{2E_n r_H}{A} = \frac{E_n}{\Omega_H}. \quad (17) \]

\( \Omega_H = A/2r_H \) is the horizon angular velocity [11]. Eq. (16) can be used when one or both particles become massless. The BSW effect can also be considered in this case. In [7, 12], the expression of \( E_{\text{c.m.}} \) is written in other forms.

For a Schwarzschild black hole \( (A = 0) \), the collision energy in the center-of-mass frame is
\[ E_{\text{c.m.}}^2(r \to r_H) = \frac{(E_1 J_2 - E_2 J_1)^2}{4M^2 E_1 E_2} \]
\[ + m_1^2 \left( 1 + \frac{E_2}{E_1} \right) + m_2^2 \left( 1 + \frac{E_1}{E_2} \right) \quad (18) \]

As can be seen from (16), the collision energy of a particle in the center-of-mass frame tends to infinity on the horizon if the angular momentum of
one of the freely falling particles has the value \( J_{nH} \). Is it possible that a particle with such a value of the angular momentum falls to the horizon? For the case of free fall from infinity on the non-extremal \( (A < 1) \) rotating black hole it is impossible. This can be seen from the fact that the expression (9) tends to zero at the horizon \( J \to J_H \), but its derivative in \( r \) is negative for \( A < 1 \).

Consider a collision of two massive particles. For massive particles, \( p_i^{(n)} = m_i^{(n)}u_i^{(n)} \), where \( u^i = dx^i/d\tau \). In this case, as can be seen from (12), the energy \( E_{\text{c.m.}} \) has a maximum value for given \( u_1 \) and \( u_2 \) and \( m_1 + m_2 \) if the particle masses are equal: \( m_1 = m_2 \).

Let us find the expression of the energy in the center-of-mass frame through the relative velocity \( v_{\text{rel}} \) of particles at the moment of collision [13]. In the reference frame of the first particle one has for the components of particle 4-velocities at this moment

\[
u_i^{(1)} = (1, 0, 0, 0), \quad \nu_i^{(2)} = \frac{(1, v_{\text{rel}})}{\sqrt{1 - v_{\text{rel}}^2}}.\]

Thus \( \nu_i^{(1)}u_i^{(2)i} = 1/\sqrt{1 - v_{\text{rel}}^2} \), and

\[
v_{\text{rel}} = \sqrt{1 - (u_i^{(1)}u_i^{(2)i})^2}.\]

These expressions evidently do not depend on the coordinate system.

From (12) and (20) one obtains

\[
E_{\text{c.m.}}^2 = m_1^2 + m_2^2 + \frac{2m_1m_2}{\sqrt{1 - v_{\text{rel}}^2}},
\]

and an unlimited growth of the collision energy in the center-of-mass frame occurs due to growth of the relative velocity to the velocity of light [10].

The massive particle free fall in a black hole with the dimensionless angular momentum \( A \), being non-relativistic at infinity \( (E = m) \), to achieve the black hole horizon must have an angular momentum in the range \((l_L, l_R)\), where

\[
l_L = -2[1 + \sqrt{1 + A}], \quad l_R = 2[1 + \sqrt{1 - A}]
\]

(22)

(here the notation \( l = J/mM \) is used). For \( A < 1 \) one has \( l_R < l_H \), and even close to the extremal \((A = 1)\) rotating black hole \( E_{\text{c.m.}}^\max /\sqrt{m_1m_2} \) can be not very large, as mentioned in [14, 15] for the case \( m_1 = m_2 \). So for \( A_{\max} = 0.998 \) considered as the maximal possible dimensionless angular momentum of astrophysical black holes (see [16]) one obtains \( E_{\text{c.m.}}^\max /\sqrt{m_1m_2} \approx 18.97 \). However, this estimate is sufficient for collisions of superheavy particles of dark matter with masses close to the Grand Unification scale to occur in the region of Grand Unification interaction physics, so that these particles can decay into quarks and be observed as UHECR [1].

Does it mean that in processes of usual particles (protons, electrons) scattering in the vicinity of a rotating non-extremal black holes the scattering energy is limited, so that no Grand Unification or even Planckian energies can be obtained? As was shown for the first time in [4], if one takes into account the possibility of multiple scattering, so that a particle falling from infinity to a black hole with some fixed angular momentum, changes its momentum as a result of interaction with particles in the accreting disc and after that is again scattered close to the horizon, then the scattering energy can be unlimited.

Note that the critical values of \( J_H \) are different for different values of the energies \( E \) (see (17)). That is why there is not only one critical trajectory for collision with an unlimited energy but a set of trajectories corresponding to a range of specific energies of particles falling to a black hole.

Let us obtain an expression for the permitted interval of \( r \) for particles with the angular momentum \( l = l_H - \delta \) close to the horizon. From (7), (8), and (9) one has for massive particle motion in the equatorial plane:

\[
\left( \frac{dr}{d\tau} \right)^2 = \varepsilon^2 + \frac{2}{x^3}(A\varepsilon - l)^2
\]

\[+ \frac{A^2\varepsilon^2 - l^2}{x^2} - \frac{\Delta_x}{x^2},\]

(23)

where \( \varepsilon = E/m \) is the specific energy of a particle. As shown in [17], Sec. 88, the specific energy in the static gravitational field is equal to

\[
\varepsilon = \frac{900}{\sqrt{1 - \nu^2}},
\]

(24)

where \( \nu \) is the particle velocity measured by an observer at rest at the point of the passing particle.

To find the ends of the permitted interval of \( x \), one must put the left hand side of (23) to zero and find the root. In the second order in \( \delta \) close to the horizon one obtains

\[
x_\delta \approx x_H + \frac{\delta^2 x_H^2}{x_H(x_H - x_C)(\varepsilon^2 x_H + x_C)}.\]

(25)

Note that the smallness of \( x_\delta - x_H \) does not mean the smallness of the “physical distance” from \( r_\delta \) to the horizon (see [17], Sec. 84).

From (15) and (25) one obtains for values of function \( u_i^{(1)}u_i^{(2)i} \), for \( l_1 = l_{1H} - \delta \) after the first scattering at the points \( x_\delta \) and at the horizon:

\[
u_i^{(1)}u_i^{(2)i}(x_\delta) \approx \frac{(l_{2H} - l_2)(\varepsilon^2 x_H + x_C)}{\delta \cdot x_C},\]

(26)
Therefore the function $u_{(1)}^i u_{(2)i}(x_H)$ and hereby the energy of collisions decrease near the horizon! The dependence of $u_{(1)}^i u_{(2)i}$ on the coordinate $r$ is shown in Fig. 1.

The left panel shows that the collision energy can be very large immediately after acquiring the angular momentum $l_H - \delta$. But, for $A_{\text{max}} = 0.998$, $l_H - l_R \approx 0.04$, and so the collision energy of particles with the angular momentum $l_R$ is not large. This means that a decrease in the collision energy with fixed angular momentum value does not imply an extremely large energy of particles in the center-of-mass frame needed to get the value $l_H - \delta$ in the intermediate collision.

The decrease in the collision energy in the center-of-mass frame of a freely falling particle and a particle with the critical angular momentum $l = l_H - \delta$ as they move to the horizon is explained by a decrease of the relative velocity of these particles when going from the point $x_\delta$. Due to the definition (25), the radial velocity of a particle with the angular momentum $l = l_H - \delta$ is equal to zero: $dr/d\tau = 0$ and $dr/dt = 0$. But another non-critical particle colliding with a critical one has a large radial velocity. The angular velocities $d\varphi/dt$ of both particles tend to the same value $\Omega_H$. So a large relative velocity of two particles occurs due to a critical particle “stopping” in the radial direction. But after that both particles increase their radial velocities $dr/d\tau$, and the relative velocity is decreasing: the critical particle is “running down” the freely falling one. From (20), (26), and (27) one obtains for the relative velocity of particles colliding at the point $x_\delta$ and at the horizon:

$$1 - v_{\text{rel}}(x_\delta) = \frac{\delta^2 x_C^2}{2 \left(l_H - l_2 - l_l^2\right)^2 \left(\varepsilon_1^2 x_H + x_C\right)^2},$$

$$1 - v_{\text{rel}}(x_H) = \frac{2 \delta^2 x_C^2}{\left(l_H - l_2 - l_l^2\right)^2 \left(\varepsilon_1^2 x_H + x_C\right)^2}.\quad (28)$$

So the physical reason for the unlimitedly great energy of the collision in the center-of-mass frame for particles falling into a black hole is the increase in the relative velocity of particles at the moment of collision up to the velocity of light. One can expect a very large collision energy in the case where one of the particles, due to multiple intermediate collisions in the accretion disc, strongly diminishes its energy, so that its velocity becomes small near the horizon. Indeed, from (16) it is easy to obtain that

$$E_{\text{c.m.}}^{i, j} \sim \sqrt{m_1 m_2} \frac{l_2 - l_1}{l_H - l_1} \rightarrow \infty, \quad \varepsilon_1, l_1 \rightarrow 0.\quad (30)$$

From the same considerations one can conclude that for the case of a Schwarzschild non-rotating black hole the collision energy of a freely falling particle with a particle at rest close to the horizon is also great and unlimited. Using (12), (15), and (24) for $A = 0$ and a particle at rest at a point with the radial coordinate $r_0$ (so that $l_1 = 0$, $\varepsilon_1 = 1$, $l_2 = l_l$), from (12) one obtains for the energy of its collision with a particle with $\varepsilon_2$, $l_2$ one obtains in the center-of-mass frame

$$E_{\text{c.m.}}^{i, j} = m_1^2 + m_2^2 + 2m_1 E_2 \sqrt{\frac{r_0}{r_0 - r_g}},\quad (31)$$

which evidently infinitely grows as $r_0 \rightarrow r_g$.

Note that stopping of a particle at the horizon of a nonrotating charged black hole takes place at its critical charge, and then one should expect an infinity energy of collisions, as shown in [18].

If a particle at a point with the radial coordinate $r_0$ has $dr/d\tau = 0$ but $l_1 \neq 0$, then from (12), (15), and (23) one has

$$E_{\text{c.m.}}^{i, j} = m_1^2 + m_2^2 + 2m_1 E_2 \left[\varepsilon_2 \sqrt{\frac{l_1^2 + x_0^2}{(x_0 - 2)x_0} - \frac{l_1 l_2}{x_0^2}}\right],\quad (32)$$

which also infinitely grows as $x_0 \rightarrow x_H = 2$.

Note that for particles nonrelativistic at infinity, with $m_1 = m_2 = m$, freely falling into a Schwarzschild black hole, the limiting energy of collisions is only $2\sqrt{5}m$ (see [19]).

To conclude this section, note that the probability of a collision of a relativistic particle with a particle at rest close to the Schwarzschild horizon is very small. So it is the main difference with the situation where the BSW resonance occurs. This can be seen from evaluation of the the proper time of falling from the
point \( r_0 \) where \( dr/d\tau = 0 \) to the horizon. Define the effective potential using the right-hand side of (23):

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}} = 0. \tag{33}
\]

Then

\[
\frac{dr}{d\tau} \bigg|_{r_0} = 0 \Rightarrow r \approx r_0 - \frac{\Delta \tau^2}{2} \frac{dV_{\text{eff}}}{dr}. \tag{34}
\]

So the proper time of particle falling to the horizon is

\[
\Delta \tau \approx M \sqrt{\frac{2(x_0 - x_H)}{4 + l^2}}, \quad x_0 \to x_H. \tag{35}
\]

For the Schwarzschild black hole one obtains

\[
\Delta \tau \approx 4M \sqrt{\frac{2(x_0 - x_H)}{4 + l^2}}, \quad x_0 \to x_H. \tag{36}
\]

For the Kerr black hole, taking \( l \) close to \( l_H \), from (23), (33), (35) one obtains

\[
\Delta \tau \approx MA \sqrt{\frac{2\xi_H(x_0 - x_H)}{\sqrt{1 - \xi^2} (\xi x_H + 1)}}, \quad x_0 \to x_H, \tag{37}
\]

which is evidently much larger than (36) for \( A \to 1 \).

3. ESTIMATE OF THE FALLING TIME BEFORE A COLLISION LEADING TO A LARGE ENERGY

In [4–6] we have shown that, in order to get an unboundedly growing energy for the extremal case, one should have the time interval (both in the coordinate and proper time) from the beginning of the falling inside the black hole to the moment of collision also growing infinitely. Quantitative estimations have been given for the case of an extremely rotating black hole, \( A = 1 \). [5, 7]. Here we consider the case \( A < 1 \).

From Eqs. (5) and (23) one gets

\[
\frac{[dt/dx]}{\Delta x} = \frac{M \sqrt{2} ((x^3 + A^2 x + 2A^2 \varepsilon - 2A l)}{2\varepsilon^2 x^2 - l^2 x + 2(A\varepsilon - l)^2 + (\varepsilon^2 - 1)\Delta x} \tag{38}
\]

For \( A < 1 \), from (14) and (38) one obtains that the time interval measured by a clock of a distant observer, necessary to achieve the horizon, is logarithmically divergent. From (38) one has for \( l < l_H = 2\varepsilon x_H/A, A < 1 \) and \( x_0 \) close to \( x_H \) (e.g., \( x_0 = 2x_H \))

\[
\Delta t \sim \frac{2Mx_H}{x_H - x_C} \log(x_f - x_H), \quad x_f \to x_H. \tag{39}
\]

Recall that for an extremal black hole and the critical value of the angular momentum of the falling particle this interval diverges as \( 1/(r - r_H) \).

From Eqs. (12), (25), (26), (27), and (39) it is easy to obtain for collisions of two particles with \( l_1 = l_H - \delta \) close to the horizon at the point \( x_\delta \)

\[
\Delta t \sim \frac{8Mx_H}{x_H - x_C} \log \frac{E_{\text{c.m.}}}{\sqrt{m_1 m_2}} \tag{40}
\]

So for \( A = 0.998 \),

\[
\Delta t \sim 3.2 \times 10^{-4} \frac{M}{M_\odot} \log \frac{E_{\text{c.m.}}}{\sqrt{m_1 m_2}} \text{s}. \tag{41}
\]

Taking the value of the Grand Unification energy \( E_{\text{c.m.}}/\sqrt{m_1 m_2} = 10^{14} \) and the mass of the black hole \( 10^8 M_\odot \) typical of active galactic nuclei, one obtains \( \Delta t \sim 10^6 \text{ s} \), i.e. of the order of 12 days. So in the case of a non-extremal rotating black hole, the mechanism of an intermediate collision to get the additional angular momentum with the following collision with another relativistic particle leading to a large collision energy, which we have suggested in [4], needs a reasonable time, much smaller than that in the extremal case.

One can ask about the time of back motion of a particle after collision with a very high energy from the vicinity of the horizon to the Earth. Due to time reversibility of the equations of motion, it is easy to see that this time is equal to a sum of the same 12 days to the accretion disc and some 10–100 megaparsec ”— the AGN distance from the Earth.

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