Research Article

On a problem of Steinhaus

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Abstract
Let $N$ be a positive integer. A sequence $X = (x_1, x_2, ..., x_N)$ of points in the unit interval $[0,1)$ is piercing if $\{x_1, x_2, ..., x_n\} \cap \left[\frac{i}{n}, \frac{i+1}{n}\right) \neq \emptyset$ holds for every $n = 1, 2, ..., N$ and every $i = 0, 1, ..., n - 1$. In 1958, Steinhaus asked whether piercing sequences can be arbitrarily long. A negative answer was provided by Schinzel, who proved that any such sequence may have at most 74 elements. This was later improved to the best possible value of 17 by Warmus, and independently by Berlekamp and Graham. In this paper, we study a more general variant of piercing sequences. Let $f(n) \geq n$ be an infinite nondecreasing sequence of positive integers. A sequence $X = (x_1, x_2, ..., x_{f(n)})$ is $f$-piercing if $\{x_1, x_2, ..., x_{f(n)}\} \cap \left[\frac{i}{n}, \frac{i+1}{n}\right) \neq \emptyset$ holds for every $n = 1, 2, ..., N$ and every $i = 0, 1, ..., n - 1$. A special case of $f(n) = n + d$, with $d$ a fixed nonnegative integer, was studied by Berlekamp and Graham. They noticed that for each $d \geq 0$, the maximum length of any $(n + d)$-piercing sequence is finite. Expressing this maximum length as $s(d) + d$, they obtained an exponential upper bound on the function $s(d)$, which was later improved to $s(d) = O(d^3)$ by Graham and Levy. Recently, Konyagin proved that $2d \leq s(d) < 200d$ holds for all sufficiently big $d$. Using a different technique based on the Farey
fractions and stick-breaking games, we prove here that
the function \( s(d) \) satisfies
\[
\left\lfloor \frac{\ln 2}{1-\ln 2} \right\rfloor \approx 2.25 \text{ and } c_2 = \frac{1+\ln 2}{1-\ln 2} \approx 5.52. \]
We also prove that there exists an infinite \( f \)-piercing sequence
with \( f(n) = \gamma n + o(n) \) if and only if \( \gamma \geq \frac{1}{\ln 2} \approx 1.44. \)

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1 | INTRODUCTION

In his book *Sto zadání* [[14]] from 1958, in English translation *One Hundred Problems in Elementary Mathematics* [[15]], Steinhaus posed the following problem (Problems 6 and 7 in chapter 1):

Does there exist for every positive integer \( N \) a sequence \( X = (x_1, x_2, \ldots, x_N) \) of real numbers in \([0,1)\) such that \( \{x_1, x_2, \ldots, x_n\} \cap \left[ \frac{i}{n}, \frac{i+1}{n} \right) \neq \emptyset \) holds for every \( n = 1, 2, \ldots, N \) and every \( i = 0, 1, \ldots, n-1 \)?

We call such sequences *piercing* of order \( N \). The first solution was given by Schinzel (see [[15]]) who proved that no piercing sequence can have more than 74 elements. Then, Warmus [[16]] gave a complete solution by proving that the longest piercing sequence has 17 elements and constructing essentially all the possible solutions of which there are 768.

In the present paper, we study a more general variant of piercing sequences. Let \( f(n) \geq n \) be an infinite nondecreasing sequence of positive integers. A sequence \( X = (x_1, x_2, \ldots, x_{f(N)}) \) is called \( f \)-piercing of order \( N \) if \( \{x_1, x_2, \ldots, x_{f(n)}\} \cap \left[ \frac{i}{n}, \frac{i+1}{n} \right) \neq \emptyset \) holds for every \( n = 1, 2, \ldots, N \) and every \( i = 0, 1, \ldots, n-1 \). The original question of Steinhaus concerns the extremal case of \( f(n) = n \).

A natural class of \( f \)-piercing sequences with the function \( f(n) = n + d \), where \( d \) is a fixed non-negative integer, was introduced by Berlekamp and Graham [[1]]. They noticed that a fundamental result of Roth [[10]] on the discrepancy of sequences implies easily that for each \( d \geq 0 \), the maximum order of any \((n + d)\)-piercing sequence is bounded. Denoting this order by \( s(d) \), they proved (independently of Warmus) that \( s(0) = 17 \) and gave an exponential upper bound on \( s(d) \). Later, a proof of the polynomial upper bound \( s(d) \leq O(d^2) \) was sketched by Graham [[3]]. A full proof of this bound was recently completed by Levy [[6]], who also provided a lower bound \( s(d) \geq \Omega(\sqrt{d}) \). These results were very recently improved by Konyagin [[4]], who proved that \( s(d) \geq 2d \) for all \( d \geq 0 \), and \( s(d) < 200d \) for all \( d \geq 4 \cdot 10^{16} \).

Using different techniques, we obtain here a further improvement on the bounds for the function \( s(d) \). One of our main results reads as follows.

**Theorem 1.** The function \( s(d) \) satisfies \( |c_1d| \leq s(d) \leq c_2d + o(d) \) where \( c_1 = \frac{\ln 2}{1-\ln 2} \approx 2.25 \) and \( c_2 = \frac{1+\ln 2}{1-\ln 2} \approx 5.52 \).

A natural question arising from the problem of Steinhaus is to determine the minimum growth of a function \( f(n) \) allowing for arbitrarily long \( f \)-piercing sequences. It is not hard to see that \( f(n) = 2n \) is one such function. Actually, it allows even for an infinite \( f \)-piercing sequence \( X = (x_1, x_2, \ldots) \), defined by the natural condition that every prefix of \( X \) of the form \( (x_1, x_2, \ldots, x_{f(N)}) \) is an \( f \)-piercing sequence of order \( N \), for every \( N \geq 1 \). We prove the following general result.
Theorem 2. There exists an infinite \( f \)-piercing sequence with \( f(n) = \gamma n + o(n) \) if and only if \( \gamma \geq \frac{1}{\ln 2} \approx 1.44 \).

The proof uses the following more general idea of strongly piercing sequences. A sequence \( X = (x_1, x_2, \ldots, x_{f(N)}) \) is called strongly \( f \)-piercing of order \( N \) if \( \{x_1, x_2, \ldots, x_{f(n)}\} \cap [y, y + \frac{1}{n}) \neq \emptyset \) holds for every \( n = 1, 2, \ldots, N \) and every real \( 0 \leq y \leq 1 - \frac{1}{n} \), that is, \( \{x_1, x_2, \ldots, x_{f(n)}\} \) intersects every interval of length \( \frac{1}{n} \) that is contained in \([0,1)\). Clearly, any strongly \( f \)-piercing sequence is \( f \)-piercing in the former sense. Also, we may define analogously infinite sequences with this stronger piercing property.

We will prove in Section 2 that the assertion of Theorem 2 holds for strongly piercing sequences. Actually, the same result was obtained by de Bruijn and Erdős [2] already in 1948. Next, in Section 3 we explore connections between piercing and strongly piercing sequences more closely, using as a tool the well-known Farey fractions. This will allow for deriving the only if part of Theorem 2 and the proof of Theorem 1 in Section 4. In the last section, we pose some open problems.

The original problem of Steinhaus is connected to some other, quite famous topics. Indeed, it is visibly related to the well-known “three gaps theorem” conjectured by him in the mid-1950s, and proved shortly thereafter by many authors, including Sós [11] and Świerczkowski [17]. The theorem says that for every real number \( \alpha > 0 \) and every natural \( N \), the set of points \( \{\alpha n \mod 1 : n = 1, 2, \ldots, N\} \) splits the unit interval into segments of at most three different lengths. In a less formal way, when you cut a circular cake by turning the knife around by the constant angle, you will always have pieces of at most three different sizes. This illustration resembles the celebrated problem of fair division, a topic of fundamental importance in game theory and economics, studied and popularized by Steinhaus [12] a decade earlier. Other possible inspirations can be perceived in his work [13] devoted to quite practical applications of mathematics in some problems of “measurement”.

## 2 STRONGLY PIERCING SEQUENCES VIA STICK-BREAKING GAMES

Consider the following stick-breaking game. At the beginning, we have a segment of unit length. In each subsequent round, we choose one of the existing segments and break it into two subsegments. Before the \( k \)th round we have exactly \( k \) segments with the total length always equal to one. Let \( M_k \) denote the maximum length of a segment before the \( k \)th round. Our goal in the game is to minimize the value of the limit \( \gamma = \lim \sup(kM_k) \) as \( k \to \infty \). Notice that \( kM_k \) is equal to the ratio \( \frac{M_k}{M} \), where \( M = \frac{1}{k} \) is the average length of a segment before the \( k \)th round.

The idea of studying uniform distribution of a sequence of points via stick-breaking games was explored earlier by Ramshaw [9]. Clearly, any infinite sequence \( X = (x_1, x_2, \ldots) \) of points in the unit segment determines uniquely a strategy in the stick-breaking game, and vice versa, any stick-breaking strategy gives rise to a unique sequence \( X \). Moreover, it is not hard to check that the sequence \( X \) is strongly \( (\gamma n + o(n)) \)-piercing if and only if \( \gamma \geq \lim \sup_{k \to \infty}(kM_k) \).

Consider now the following stick-breaking strategy. In each round, we choose one of the longest segments and break it into two in the ratio \( (1 - r) : r \), where \( 0 < r \leq \frac{1}{2} \) is a fixed number. We call this strategy nonchalant with parameter \( r \).
Example 3. Let $r = \frac{1}{2}$. In this case all segment lengths in every round are integer powers of 2, in particular before $k$th round, the longest one has length $M_k = 2^{-\lfloor \log_2 k \rfloor}$. For every natural $n$ the number $kM_k$ grows from 1 for $k = 2^n$ to $2 - 2^{-n}$ for $k = 2^{n+1} - 1$, hence $\gamma = \limsup_{k \to \infty} (kM_k) = 2$ and $\delta = \liminf_{k \to \infty} (kM_k) = 1$. Note that the rounds when there is exactly one longest segment remaining (e.g., one of length $\frac{1}{8}$ and 14 of length $\frac{1}{16}$ before round 15) are the ones that are important for computing the upper limit, while the immediately following rounds determine the lower limit (e.g., after round 15 there are 16 segments of length $\frac{1}{16}$).

To obtain lower values of $\gamma$ we will also consider strategies with a changing parameter. Starting with the segment of length 1 we will apply the nonchalant strategy with parameter $r_1$, but after some number of rounds (to be determined more precisely below, but rather large) we will change the parameter and begin to break the currently longest segment nonchalantly in the ratio $(1 - r_2) : r_2$ for some time, and so on. That will be called the lazy strategy corresponding to the sequence $(r_n)$.

Theorem 4. For every $0 < r \leq \frac{1}{2}$ there exists an increasing sequence $r_n \to r$ and a corresponding lazy strategy, such that

$$\limsup_{k \to \infty} (kM_k) = -\frac{1}{r \ln r + (1 - r) \ln(1 - r)}.$$

Proof. First we show the upper and lower limits of the ratio $\frac{M_k}{M}$ as $k \to \infty$ for a particular class of nonchalant strategies with $r < \frac{1}{2}$. For that purpose we are going to define a corresponding continuous process of cell splitting in time. In the original setting, every cell (segment) is characterized by its length $d$, and, as the number of round $k$ increases, the segments with the highest $d$ split. Instead of that, we will consider the number $-\ln d$ as the moment in time when the cell of length $d$ splits. Equivalently, considering the state of the game before round $k$, that is, at the moment when the longest segment has length $M_k$ and is just about to split, we see that the quantity $\tau = -\ln(\frac{d}{M_k})$ is the remaining lifetime of any cell of length $d$. Note that all cells with the same highest length $d = M_k$ have, consistently with the above, the remaining lifetimes $\tau = 0$, and we treat them as splitting one by one in consecutive rounds, although those rounds are played at the same moment of time.

The process thus begins with one cell that divides at time $t = 0$ and each cell, corresponding to a segment of length $d$, splits after the prescribed time into two parts, one of which can be called long-lived, as it corresponds to the subsegment of length $dr$, and will live for the time equal to $t_1 = -\ln r$, and the other one short-lived, analogously destined to survive the time $t_2 = -\ln(1 - r)$, where $t_2 < t_1$, as $r < \frac{1}{2}$.

The average length of the interval is currently the arithmetic mean of the expressions $e^{-\tau}$ over all existing cells under the condition that the oldest ones (meaning the longest intervals in the original process) have $\tau = 0$ and are in the process of splitting. Denoting the mean by $E$ we have thus $E(e^{-\tau}) = E(\frac{d}{M_k}) = \frac{M}{M_k}$.

Let us now assume that $t_2 \leq t_1$ is a rational number, equal to $\frac{q}{p}$ with $\gcd(p, q) = 1$, which lets us define $t = \frac{t_1}{p} = \frac{t_2}{q}$ as the unit of time. In other words, the lifetimes and the splitting moments of all cells are integer multiples of $t$. 
We will focus on the number $d_n$ of cells in the $n^{th}$ generation, by which we mean the cells that split at the moment $nt$. By the construction, these are exactly the short-lived cells born $q$ units earlier and the long-lived ones born $p$ units earlier, that is, $d_n = d_{n-q} + d_{n-p}$ for $n \geq 1$, which together with the initial conditions $d_0 = 1$ and $d_{-i} = 0$ for $1 \leq i \leq p-1$ gives us the generating function

$$D(x) = \sum d_n x^n = \frac{1}{1 - x^p - x^q}.$$  

We claim that $d_n = (C + o(1))\beta^{-n}$ where $C \neq 0$ and $\beta = e^{-\tau}$ (implying $\beta^p = r$, $\beta^q = 1-r$, and $\beta^p + \beta^q = 1$). To that end let us state several elementary properties of the complex polynomial $W(z) = z^p + z^q - 1$.

- $W$ has no rational roots. Indeed, by the rational root theorem the only candidates are $\pm 1$, but they are not roots of $W$.
- $W$ increases on $\mathbb{R}_+$ and therefore has exactly one positive real root, which is equal to $\beta$.
- $W$ has no multiple roots, either real or not. In fact, the equality $W(z) = W'(z) = 0$ implies that $z^p$ and $z^q$ are rational, which by $\gcd(p, q) = 1$ can only be true if $z$ is rational, and that is impossible.
- No complex number $z$ with $|z| \leq \beta$ except $z = \beta$ itself is a root of $W$. This follows from the fact that the inequalities in $1 = |z^p + z^q| \leq |z|^p + |z|^q \leq \beta^p + \beta^q = 1$ become equalities only if $|z| = \beta$ and the ratio $z^p/z^q$ is a positive real, which means that $z$ is a positive real.

The above properties imply that the partial fraction decomposition of $D(x)$ has the form:

$$D(x) = \frac{A}{x-\beta} + \frac{A_2}{x-z_2} + \cdots + \frac{A_p}{x-z_p},$$

where the $z_i$’s are the remaining roots of $W$, and that $d_n = C\beta^{-n} + o(\beta^{-n})$. The last part of the claim, stating that $C \neq 0$ can be proved by contradiction: if $C = 0$ then $\lim_{n \to \infty} d_n \beta^n = 0$. In other words, the total length of the segments of generation $n$ would tend to 0 as $n \to \infty$. That is, however, not possible because at most $p+1$ generations of cells/segments are alive at any particular moment, while the total length of all segments is constantly equal to the positive length of the original segment, hence it does not tend to 0.

It should be noted that we actually proved a little more, namely that the asymptotics $d_n \sim \beta^{-n}$ of the solution of $d_n = d_{n-q} + d_{n-p}$ holds for all nonnegative initial conditions, provided that $d_n$ is not the constant 0 sequence.

To compute the average value of $e^{-\tau}$ at the moment when the last cell belonging to the generation splitting at the time $nt$ is about to disappear (cf. Example 3), it suffices to count how many cells belong to each generation. Here is the detailed bookkeeping: beside the last cell mentioned above, whose remaining lifetime is 0, we have $d_n -1$ newborn long-lived cells with remaining lifetime $pt$, $d_{n-1}$ long-lived cells born one unit earlier with remaining lifetime $(p-1)t$, and so on, up to $d_{n-p+1}$ long-lived cells that are going to split next time, their remaining lifetime being $t$, and $q$ generations of short-lived cells with analogous cardinalities and lifetimes. The result is:

$$E_n(e^{-\tau}) = \frac{1 + \sum_{i=0}^{p-1} \beta^{p-i}d_{n-i} - \beta^p + \sum_{i=0}^{q-1} \beta^{q-i}d_{n-i} - \beta^q}{1 + \sum_{i=0}^{p-1} d_{n-i} - 1 + \sum_{i=0}^{q-1} d_{n-i} - 1}.$$
Knowing that $\frac{d_{n-i}}{d_n} \to \beta^i$ as $n \to \infty$ and recalling that $\beta^p + \beta^q = 1$, we can write

$$
\lim_{n \to \infty} E_n(e^{-t}) = \frac{\sum_{i=0}^{p-1} \beta^{p-i} \beta^i + \sum_{i=0}^{q-1} \beta^{q-i} \beta^i}{\sum_{i=0}^{p-1} \beta^i + \sum_{i=0}^{q-1} \beta^i} = \frac{(1 - \beta)(p \beta^p + q \beta^q)}{1 - \beta^p + 1 - \beta^q} = (1 - \beta)(p \beta^p + q \beta^q)
$$

$$
= -\frac{1 - e^{-t}}{t}(r \ln r + (1 - r) \ln(1 - r)).
$$

As $E(e^{-t}) = \frac{M}{M_k} = kM_k$, we have

$$
\gamma(r) = \limsup_{k \to \infty} (kM_k) = \frac{1}{r \ln r + (1 - r) \ln(1 - r)} \frac{t}{1 - e^{-t}}.
$$

In an analogous fashion, we obtain the formula

$$
\delta(r) = \liminf_{k \to \infty} (kM_k) = \frac{1}{r \ln r + (1 - r) \ln(1 - r)} \frac{t}{e^t - 1}.
$$

This ends our analysis of nonchalant strategies with parameter $r$ satisfying $\frac{\ln(1-r)}{\ln r} \in \mathbb{Q}$. To finish the proof note that for any positive value of $r \leq \frac{1}{2}$, no matter whether $\frac{\ln(1-r)}{\ln r}$ is rational or not, we can construct such a sequence $r_n \to r$ of numbers less than $\frac{1}{2}$, that the sequence $p_n \ln(1-r_n) \ln r_n$ is an increasing sequence of rational numbers approximating $\frac{\ln(1-r)}{\ln r}$ with $p_n \to \infty$. To be more specific we can use the sequence of binary approximations, meaning that the numbers $p_n$ will be powers of 2.

If we now apply a lazy strategy in the following way: first use $r = r_1$ for such a long time that $kM_k$ will be as close to its lower limit

$$
-\frac{1}{r_1 \ln r_1 + (1 - r_1) \ln(1 - r_1)} \frac{t_1}{e^{t_1} - 1}
$$

as we wish (e.g., closer than $\frac{1}{10}$), then switch to $r = r_2$ and continue till $kM_k$ becomes less than

$$
-\frac{1}{r_2 \ln r_2 + (1 - r_2) \ln(1 - r_2)} \frac{t_2}{e^{t_2} - 1} + \frac{1}{100},
$$

and so on. It has to be stressed that the nonchalant strategy with parameter $r_n$ uses as its starting point the set of cells (segments) generated by the strategy with parameter $r_{n-1}$. It is true that the units of time $t_n$ and $t_{n-1}$ are different, and that the expected times of splits may be noninteger numbers when recalculated in the new units, however that does not pose any special problems; we can say that cells with different fractional parts of their splitting times form separate nonchalant subprocesses with parameter $r_n$ that are activated in cyclic order. The average length of a segment is then a weighted mean of averages in those processes, and accordingly both $\limsup$ and $\liminf$ of $kM_k$ belong to the interval $[\delta(r_n), \gamma(r_n)]$. As the sequence $p_n$ tends to infinity, the sequence of time units $t_n = -\frac{\ln r_n}{p_n}$ tends to 0, which implies $\frac{t_n}{(1-e^{-t_n})} \to 1$ and $\frac{t_n}{(e^{t_n-1})} \to 1$, and thus in the limit.
we obtain
\[ \limsup_{k \to \infty} (kM_k) = \liminf_{k \to \infty} (kM_k) = -\frac{1}{r \ln r + (1 - r) \ln(1 - r)}, \]
as asserted. □

The following result due to de Bruijn and Erdős [2] can now be seen as the special case of
Theorem 4 for \( r = \frac{1}{2} \).

**Corollary 5** (de Bruijn and Erdős [2]). There exists a strongly \( f \)-piercing sequence with \( f(n) = \frac{1}{\ln 2} n + o(n) \).

The original proof of this statement in [2] provides an elegant explicit example of the desired
sequence. Namely, it is proved there that the sequence of fractional parts of the numbers \( x_i = \log_2(2i - 1) \), \( i = 1, 2, \ldots \), has the claimed strong piercing property. It is also proved in [2] that the constant \( \frac{1}{\ln 2} \) in this result is optimal. For the sake of completeness, we present the proof of de
Bruijn and Erdős of this fact. The key Lemma 6 implying it will also play a role in our proof of
Theorem 1.

For any positive integer \( n \), let \( H_n \) denote the \( n \)th harmonic number, that is, \( H_n = \sum_{i=1}^{n} \frac{1}{i} \).

**Lemma 6** (de Bruijn and Erdős [2]). For every integer \( N \geq 2 \), define \( \gamma_N = \frac{1}{H_{2N} - H_{N-1}} \). For every
sequence \( X = (x_1, x_2, \ldots, x_{2N}) \) of reals in \([0,1)\), for every \( N \leq n \leq 2N \), let \( Y^n \) be the sequence
of numbers \( 0, 1, x_1, x_2, \ldots, x_n \) arranged in increasing order, that is, \( Y^n = (y^n_1, y^n_2, \ldots, y^n_{n+2}) = \)
\text{sorted}(0, 1, x_1, x_2, \ldots, x_n), and \( b_n \) be the maximum difference between any two consecutive elements of \( Y^n \), that is, \( b_n = \max_{i=1,2,\ldots,n+1} (y^n_{i+1} - y^n_i) \). Then, for at least one \( N \leq n \leq 2N \), we have
\( b_n \geq \gamma_N \cdot \frac{1}{n} \).

**Proof.** Assume to the contrary that \( b_n < \gamma_N \cdot \frac{1}{n} \) holds for every \( N \leq n \leq 2N \). Let \( d_1 \leq d_2 \leq \cdots \leq d_{N+1} \) be the sequence of differences between consecutive elements of \( Y^n \) arranged in increasing order, that is, \( (d_1, d_2, \ldots, d_{N+1}) = \text{sorted}(y^n_2 - y^n_1, y^n_3 - y^n_2, \ldots, y^n_{N+2} - y^n_{N+1}) \). First, observe that \( d_1 + d_2 + \cdots + d_{N+1} = 1 \). Second, as adding one point to the sequence splits only one difference
between two consecutive elements, we have \( b_n = d_{N+1}, b_{N+1} \geq d_N, b_{N+2} \geq d_{N-1}, \ldots, b_{2N} \geq d_1 \). Thus, we get
\[ 1 = \sum_{i=1}^{N+1} d_i \leq \sum_{n=N}^{2N} b_n < \gamma_N \sum_{n=N}^{2N} \frac{1}{n} = \gamma_N(H_{2N} - H_{N-1}), \]
which contradicts the choice of \( \gamma_N \). □

**Theorem 7** (de Bruijn and Erdős [2]). There does not exist a strongly \( f \)-piercing sequence with \( f(n) = \gamma n + o(n) \) for any real number \( \gamma < \frac{1}{\ln 2} \).

**Proof.** Let \( X \) be an infinite strongly \( f \)-piercing sequence with \( f(n') = \gamma n' + o(n') \) and \( \gamma < \frac{1}{\ln 2} \). As
\( H_{2N} - H_{N-1} \to \ln 2 \) when \( N \to \infty \), we can choose \( N_0 \) large enough, so that
\[ f(n') \leq \frac{n'}{H_2 f(N_0) - H f(N_0) - 1 + \frac{1}{N_0}} \]

for every \( n' \geq N_0 \). For every \( f(N_0) \leq n \leq 2f(N_0) \) we have that \( Z_n = (X_1, X_2, \ldots, X_n) \) contains as prefix a strongly \( f \)-piercing sequence of order

\[ \left\lfloor n \left( H_2 f(N_0) - H f(N_0) - 1 + \frac{1}{N_0} \right) \right\rfloor > n(H_2 f(N_0) - H f(N_0) - 1) . \]

Lemma 6 states that for some \( f(N_0) \leq n \leq 2f(N_0) \) we have that the maximum distance between consecutive elements of \( \text{sorted}(0, 1, X_1, X_2, \ldots, X_n) \) equals at least

\[ \frac{1}{(H_2 f(N_0) - H f(N_0) - 1)n} , \]

which gives a contradiction. \( \square \)

3 | PIERCING SEQUENCES AND FAREY FRACTIONS

Our aim in this section is to prove that, given an \( f \)-piercing sequence with \( f(n) = \gamma n + o(n) \), one may construct a strongly \( g \)-piercing sequence with \( g(n) = \gamma' n + o(n) \), where the constant \( \gamma' \) is arbitrarily close to \( \gamma \). By Theorem 7, this proves Theorem 2.

Our main tool are the well-known sequences of Farey fractions. Recall that the sequence of Farey fractions of order \( p \) consists of all irreducible fractions in the unit interval \([0,1]\) whose denominators do not exceed \( p \) (see [7]). We use the term Farey points for the points on the real line corresponding to Farey fractions.

We need the following notation and terminology. For any integers \( 0 < n \leq m \), let

\[ \text{FP}_m^n = \left\{ \frac{a}{p} : n \leq p \leq m, 0 \leq a \leq p \right\} \]

denote the set of Farey points that can be expressed by a fraction with denominator between \( n \) and \( m \). We say that two points \( \frac{a}{p}, \frac{b}{q} \in \text{FP}_m^n \) are consecutive in \( \text{FP}_m^n \), \( \frac{a}{p} \) is previous for \( \frac{b}{q} \) in \( \text{FP}_m^n \), and \( \frac{b}{q} \) is next for \( \frac{a}{p} \) in \( \text{FP}_m^n \), when \( \frac{a}{p} < \frac{b}{q} \) and there is no other \( \frac{c}{r} \in \text{FP}_m^n \) with \( \frac{a}{p} < \frac{c}{r} < \frac{b}{q} \). Similarly, let

\[ \text{FI}_m^n = \left\{ \left[ \frac{a}{p}, \frac{a+1}{p} \right) : n \leq p \leq m, 0 \leq a \leq p - 1 \right\} \]

denote the set of Farey intervals defined by Farey points in \( \text{FP}_m^n \). Finally, let

\[ \text{AI}^n = \left\{ \left[ y, y + \frac{1}{n} \right) : 0 \leq y \leq \frac{n-1}{n} \right\} \]

denote the set of all intervals of length \( \frac{1}{n} \) that are contained in \([0, 1]\).

We begin our investigation with a lemma, that captures the following intuition. Given any two reals \( \alpha > \beta > 1 \) and \( n \) big enough, a large majority of intervals of length \( \frac{1}{n} \) in \([0, 1]\) contain one
of the Farey intervals with denominator between \( \beta n \) and \( \alpha n \). There are only some exceptional regions in \([0,1)\), but not too many.

**Lemma 8.** Let \( W \geq 2 \) be a fixed integer, \( \alpha = \frac{W+1}{W-1} \), and \( \beta = \frac{W}{W-1} \). For every \( N > N_0 = N_0(W) = 2W^3 \), and every interval \([y, y + \frac{1}{N}) \in \mathbb{A}I_N\), we have that either

- \([y, y + \frac{1}{N})\) contains one of Farey intervals \([\frac{c}{r}, \frac{c+1}{r}) \in F_I^{[\alpha N]}\), or
- \( |y - \frac{b}{q}| < \frac{W+1}{W} \) for some Farey point \( \frac{b}{q} \in FP^{W-1}_1\).

**Proof.** For any \( y \in [0, \frac{N-1}{N})\), we say that a Farey point \( \frac{c}{r} \) in \( FP^{[\alpha N]} \) is a valid cover of \( y \) when \( y < \frac{c}{r} \leq y + \frac{1}{WN} \). Observe that for any valid cover \( \frac{c}{r} \) of \( y \), we get

\[
y < \frac{c}{r} < \frac{c+1}{r} \leq y + \frac{1}{WN} + \frac{1}{r} \leq y + \frac{1}{WN} + \frac{1}{\beta N} = y + \frac{1}{WN} + \frac{1 + (W-1)}{WN} = y + \frac{1}{N},
\]

and the interval \([y, y + \frac{1}{N})\) contains Farey interval \([\frac{c}{r}, \frac{c+1}{r})\). Further, if \( \frac{c}{r} \) is a valid cover of \( \frac{a}{p} \), then \( \frac{c}{r} \) is a valid cover of all \( y \) in \([\frac{a}{p}, \frac{c}{r})\). Thus, we are interested in finding all pairs of Farey points consecutive in \( FP^{[\alpha N]} \) that are at distance larger than \( \frac{1}{WN} \). We show that for every \( \frac{a}{p} \in FP^{[\alpha N]} \) either

- there is a valid cover \( \frac{c}{r} \) of \( \frac{a}{p} \), or
- there is a Farey point \( \frac{b}{q} \in FP^{W-1}_1 \) with \( |\frac{a}{p} - \frac{b}{q}| < \frac{W}{N} \).

Now, fix any \( \frac{b}{q} \in FP^{W-1}_1 \) and observe that for an \( \frac{a}{p} \in FP^{[\alpha N]} \),

- if \( \frac{a}{p} < \frac{b}{q} \) and \( a \geq \frac{b(W-1)p-p-q}{q(W-1)} \), then for \( \frac{c}{r} = \frac{a+b}{p+q} \) we have:
  \[
  b(W-1)p - p - q \leq a(W-1)q + +a(W-1)p
  \]
  \[
  (a+b)(W-1)p \leq a(W-1)(p+q) + (p+q) \quad / \div (W-1)p(p+q)
  \]
  \[
  \frac{a+b}{p+q} \leq \frac{a}{p} + \frac{1}{(W-1)p},
  \]
  and thus:
  \[
  \frac{a}{p} < \frac{c}{r} = \frac{a+b}{p+q} \leq \frac{a}{p} + \frac{1}{(W-1)p} \leq \frac{a}{p} + \frac{1}{(W-1)\beta N} = \frac{a}{p} + \frac{1}{WN}.
  \]
  Hence, \( \frac{c}{r} \) is a valid cover of \( \frac{a}{p} \), if \( p + q = r \leq [\alpha N] \).

- Similarly, if \( \frac{a}{p} > \frac{b}{q} \) and \( a \leq \frac{b(W-1)p-p-q}{q(W-1)} \), then for \( \frac{c}{r} = \frac{a-b}{p-q} \) we have:
  \[
  \frac{a}{p} < \frac{c}{r} = \frac{a-b}{p-q} \leq \frac{a}{p} + \frac{1}{(W-1)p} \leq \frac{a}{p} + \frac{1}{WN}.
  \]

Thus, \( \frac{c}{r} \) is a valid cover of \( \frac{a}{p} \), if \( p - q = r \geq [\beta N] \).
FIGURE 1  $W = 5, N = 60, \alpha N = 90, \beta N = 75$. Farey points in $FP_{1}^{W} = \{0, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \frac{2}{7}, \frac{3}{5}, \frac{1}{1}\}$ provide cover of regions of $FP_{75}^{90}$. Arrows represent the direction of $\pm \frac{b}{q}$. For every point $\frac{a}{p}$ in the blue region to the right of $\frac{1}{3}$ arrow, the point $\frac{a-1}{p-3}$ is a valid cover of $\frac{a}{p}$.

Inspired by these observations, we say that a Farey point $\frac{b}{q} \in FP_{1}^{W-1}$ provides cover of a Farey point $\frac{a}{p} \in FP_{1}^{\lfloor \alpha N \rfloor \lceil \beta N \rceil}$, if either

- $\frac{a}{p} < \frac{b}{q}$, $p + q \leq \lfloor \alpha N \rfloor$, and $a \geq \frac{b(W-1)p-p-q}{q(W-1)}$, or
- $\frac{a}{p} > \frac{b}{q}$, $p - q \geq \lceil \beta N \rceil$, and $a \leq \frac{b(W-1)p+p-q}{q(W-1)}$.

See Figure 1 for an example, where for $W = 5, N = 60$ (which does not satisfy the condition $N > 2W^3$, but allows for clearer drawing), and each $\frac{b}{q} \in FP_{1}^{W-1}$, there are depicted regions of those $\frac{a}{p}$ for which $\frac{b}{q}$ provides cover. It is not a coincidence that in Figure 1, regions covered by any two consecutive Farey points overlap.

Consider any two consecutive Farey points $\frac{b_1}{q_1} < \frac{b_2}{q_2}$ in $FP_{1}^{W-1}$. As it is well-known that $q_1 + q_2 \geq W$ and $b_2q_1 - b_1q_2 = 1$ (see [7]), for any integer $p \geq \beta N > 2W^3 > 2(W-1)q_1q_2$, we have:

\[
q_2 + q_1 > (b_2q_1 - b_1q_2)(W-1) + 1 \quad \text{//p}.
\]

\[
b_1(W-1)pq_2 + pq_2 \geq b_2(W-1)pq_1 - pq_1 + p \quad \text{//−q_1q_2}.
\]

\[
b_1(W-1)pq_2 + pq_2 - q_1q_2 > b_2(W-1)pq_1 - pq_1 - q_1q_2 + 2(W-1)q_1q_2 \quad \text{//+(W-1)q_1q_2}.
\]

\[
\frac{b_1(W-1)p + p - q_1}{(W-1)q_1} > \frac{b_2(W-1)p - p - q_2}{(W-1)q_2} + 2.
\]

(2)

Thus, there is at least one integer $a$ such that $\frac{b_2(W-1)p - p - q_2}{q_2(W-1)} \leq a \leq \frac{b_1(W-1)p + p - q_1}{q_1(W-1)}$. We say then that $\frac{a}{p}$ glues $\frac{b_1}{q_1}$ with $\frac{b_2}{q_2}$. Note its existence in particular implies that for every $\frac{a}{p} \in FP_{1}^{\lceil \alpha N \rceil \lceil \beta N \rceil} \cap \left\{ \frac{b_1}{q_1}, \frac{b_2}{q_2} \right\}$ at least one of the inequalities: $a \geq \frac{b_2(W-1)p - p - q_2}{q_2(W-1)}$ or $a \leq \frac{b_1(W-1)p + p - q_1}{q_1(W-1)}$ holds. Thus, if only $p \in \lceil \beta N \rceil + q_1, \lfloor \alpha N \rfloor - q_2 \}$ or $[\lfloor \beta N \rfloor + W - 1, \lfloor \alpha N \rfloor - W + 1]$, we are certain that there is at least one $\frac{b}{q} \in FP_{1}^{W-1}$ that provides cover of $\frac{a}{p}$, which implies that $\frac{a}{p}$ has a valid cover.
The last remaining problem is to find valid covers of Farey points \( \frac{a}{p} \in \text{FP}_{\rho}^{[\alpha N]} \) whose denominators are near \( \beta N \) or \( \alpha N \) and which are not too close to \( \text{FP}_{W-1}^{1} \) themselves, that is, \( \frac{a}{p} \in \left[ \frac{b'}{q'} + \frac{W}{N}, \frac{b}{q} - \frac{W}{N} \right] \) for some consecutive \( \frac{b'}{q'}, \frac{b}{q} \) in \( \text{FP}_{W-1}^{1} \).

Suppose first that \( p \geq \lfloor \alpha N \rfloor - q + 1 \). Let \( \frac{c}{r} \in \text{FP}_{\rho}^{[\alpha N]} \) be such that \( \lfloor \beta N \rfloor \leq r < \lfloor \beta N \rfloor + q, r \equiv p \pmod{q} \), and \( c = \lfloor \frac{ar}{p} \rfloor \). If \( \frac{b'}{q'} \) provides cover of \( \frac{a}{p} \), we are done. We may thus assume this is not the case, in particular \( \frac{a}{p} \) does not glue \( \frac{b'}{q'} \) with \( \frac{b}{q} \), and hence by (2),

\[
\frac{a}{p} > \frac{b(W - 1)p - p - q}{q(W - 1)} + 2 = \left( \frac{b(W - 1)p - p - \frac{qP}{r}}{q(W - 1)} + \frac{P}{r} \right) + \frac{P(2 - W) + 2W - 3}{W - 1} > \frac{b(W - 1)p - p - \frac{qP}{r}}{q(W - 1)} + \frac{P}{r},
\]

(3)

where the last inequality follows by the fact that \( \frac{P}{r} \leq \frac{\alpha}{\beta} = \frac{W + 1}{W} < 2 \). Note that \( \frac{c}{r} \leq \frac{a}{p} < \frac{c + 1}{r} \). Thus, by (3),

\[
\frac{c}{r} > \frac{ar}{p} - 1 > \frac{r}{p} \left( \frac{b(W - 1)p - p - \frac{qP}{r}}{q(W - 1)} + \frac{P}{r} \right) - 1 = \frac{b(W - 1)r - r - q}{q(W - 1)},
\]

and hence \( \frac{b}{q} \) provides cover of \( \frac{c}{r} \).

Now, let \( t = \frac{p-r}{q} \), and consider the sequence \( \frac{c_i}{r_i} = \frac{c + bi}{r + qi} \), for \( i = 0, 1, ..., p - r = t \) and observe that for each \( i \geq 1 \), \( \frac{c_i}{r_i} > \frac{c_{i-1}}{r_{i-1}} \) and, as \( \frac{b}{q} \) provides cover of \( \frac{c}{r} \), then by (1),

\[
\frac{1}{WN} \geq \frac{c + b}{r + q} - \frac{c}{r} = \frac{rb - qc}{(r + q)r} \geq \frac{rb - qc}{(r + qi)(r + q(i - 1))} = \frac{c_i}{r_i} - \frac{c_{i-1}}{r_{i-1}},
\]

and hence \( \frac{c_i}{r_i} \) is a valid cover of \( \frac{c_{i-1}}{r_{i-1}} \). See Figure 2 for an example with \( W = 5, N = 300, \frac{a}{b} = \frac{2}{3}, \frac{a}{p} = \frac{285}{448} \), and \( \frac{c_i}{r_i} = \left( \frac{238}{376}, \frac{240}{379}, ..., \frac{285}{448} \right) \).

For \( i = 0 \), we have \( \frac{c_0}{r_0} = \frac{\frac{c}{r}}{1} \leq \frac{a}{p} \). For \( i = t = \frac{p-r}{q} \), we have \( \frac{c_t}{r_t} = \frac{c + bt}{r + pt} \). Observe that for \( N > 2W^3 \), we have

\[
p - r > (\alpha N - q) - (\beta N + q) = \frac{1}{W - 1}N - 2q \geq \frac{1}{W - 1}N - 2(W - 1) > \frac{N}{W},
\]

(4)

and therefore:

\[
\frac{a}{p} \leq \frac{b}{q} - \frac{W}{N},
\]

\[
\frac{a}{p} < \frac{b}{q} - \frac{1}{p - r}
\]
\[ \frac{aq(p - r)}{p} < \frac{bp(p - r)}{p} - pq \]
\[ aq < arq - pq + b(p - r)p \]
\[ a < \frac{ar - p}{p} + b \cdot \frac{p - r}{q} < c + bt = c_t \]
\[ \frac{a}{p} < \frac{c_t}{r_t} \cdot \frac{c_t}{r_t} \]

So, \( \frac{c_0}{r_0} \leq \frac{a}{p} < \frac{c_1}{r_1} \), and hence there is some \( i \in \{1, 2, ..., t\} \), such that \( \frac{c_{i-1}}{r_{i-1}} \leq \frac{a}{p} < \frac{c_i}{r_i} \). As \( \frac{c_i}{r_i} \) is a valid cover of \( \frac{c_{i-1}}{r_{i-1}} \), it is also a valid cover of \( \frac{a}{p} \).

It remains to discuss the situation when \( p \) is close to \( \beta N \). By (2) and the case discussed above we may thus assume that \( p \leq \lfloor \beta N \rfloor + q' - 1 \). This time we choose \( \frac{c}{r} \in \text{FP}_{\lfloor \beta N \rfloor} \) with \( \lfloor \alpha N \rfloor - q' < r \leq \lfloor \alpha N \rfloor \), \( r \equiv p \pmod{q'} \), and \( c = a + 1 + b' \cdot \frac{r-p}{q'} \). Similarly as in (4), \( r - p > \frac{N}{W} \). As additionally \( \frac{a}{p} - \frac{b'}{q'} \geq \frac{W}{N} \), then we have:

\[ 1 < \left( \frac{a}{p} - \frac{b'}{q'} \right)(r - p) \]
\[ 1 + \frac{b'}{q'}(r - p) < \frac{a}{p}(r - p) \]
\[ p + b' \frac{r-p}{q'} < a(r - p) \]
\[ pa + p + b' \frac{r - p}{q'} < ar \quad \text{/} (pr) \]

\[ c = \frac{a + 1 + b' \frac{r - p}{q'}}{r} < \frac{a}{p} \] \,(5)\]

and moreover, as \( \frac{a}{p} > \frac{b'}{q'} = \frac{b' \frac{r - p}{q'}}{q'} \), then

\[ \frac{b'}{q'} < \frac{a + b' \frac{r - p}{q'}}{p + q' \frac{r - p}{q'}} < \frac{a + 1 + b' \frac{r - p}{q'}}{r} = \frac{c}{r}. \]

Now, consider the sequence \( c_i = c_{i-1} - \frac{b'}{q'} \), for \( i = 0, 1, \ldots, t = \frac{r - p}{q'} \), and note that \( \frac{a}{p} = \frac{a + 1 + b' \frac{r - p}{q'}}{r} \), while by (5), \( c_0 = c < \frac{a}{p} \). Thus for some \( i \in \{1, 2, \ldots, t\} \), we have \( \frac{a_{i-1}}{r_{i-1}} < \frac{a}{p} < \frac{a_i}{r_i} \). Analogously as in the previous case, by (2), we may assume that \( a < \frac{b'(W-1)p + p - q'}{q'(W-1)} - 2 < \frac{b'(W-1)p + p - q'}{q'(W-1)} - 1 \), and thus, as \( r_{i-1} - q' \geq p \geq \lceil \frac{\beta N}{W} \rceil \),

\[ \frac{c_{i-1}}{r_{i-1}} < \frac{a + 1}{p} < \frac{b'(W - 1) + 1 - \frac{q'}{p}}{q'(W - 1)} < \frac{b'(W - 1) + 1 - \frac{q'}{r_{i-1}}}{q'(W - 1)}, \]

and thus \( \frac{b'}{q'} \) provides cover of \( \frac{c_{i-1}}{r_{i-1}} \), which means that \( \frac{1}{NW} \geq \frac{a_{i-1} - \frac{b'}{r_{i-1} - q'}}{c_{i-1} - \frac{a_i}{r_i}} = \frac{c_i}{r_i} - \frac{c_{i-1}}{r_{i-1}} \), and thus \( \frac{c_i}{r_i} \) is a valid cover of \( \frac{a}{p} \).

We have found a valid cover of every Farey point in \( FP_{\frac{aN}{\lfloor \beta N \rfloor}} \) except those that are at distance smaller than \( \frac{W}{N} \) from some Farey point in \( FP_{\frac{1}{W-1}} \). As for every \( \frac{b}{q} \in FP_{\frac{1}{W-1}} \) there must be some Farey point in \( FP_{\frac{aN}{\lfloor \beta N \rfloor}} \) at distance at most \( \frac{1}{\lfloor aN \rfloor} < \frac{1}{N} \) (in any direction) from \( \left( \frac{b}{q} - \frac{W}{N} + \frac{b}{q} + \frac{W}{N} \right) \), this yields a valid cover of every real point \( y \in [0, 1) \) except possibly those that are at distance smaller than \( \frac{W+1}{N} \) from some Farey point in \( FP_{\frac{1}{W-1}} \) and ends the proof.

Observe that there are only \( O(W^2) \) Farey points in \( FP_{\frac{1}{W-1}} \), and the length of the union of all intervals in \( AI^n \) that do not include any interval from \( FI_{\lfloor aN \rfloor} \) is \( O(W^3) \). This length tends to zero as \( N \) tends to infinity. The next lemma exploits this fact to construct a strongly piercing sequence from a piercing one having similar parameters.

**Lemma 9.** Let \( W \geq 2 \) be a fixed integer, and let \( \alpha = \frac{W+1}{W-1} \). For every function \( f(n) = \gamma n + o(n) \), there exist \( N_0 \) and a function \( g(n) = \alpha^2 \gamma n + o(n) \) such that for every \( N \geq N_0 \) if there is an \( f \)-piercing sequence \( X \) of order \( \lfloor aN \rfloor \), then there is a strongly \( g \)-piercing sequence \( Z \) of order \( \alpha N \).

**Proof.** Let \( \beta = \frac{W}{W-1} \), and choose \( N_0 > 2W^3 \) so that \( f(N) < \alpha \gamma N \) for all \( N \geq N_0 \). We define \( g(n) = [\alpha^2 \gamma n + N_0 + 5W^3 \log_2 n + 1] \). Assume that \( X = (x_1, x_2, \ldots, x_{f(\lfloor aN \rfloor)}) \) is an \( f \)-piercing sequence of order \( \lfloor aN \rfloor \) for some \( N \geq N_0 \).

By Lemma 8, for any \( N_0 \leq n \leq N \) and any interval \( [y, y + \frac{1}{n}] \in AI^n \), we have that either \( |y - \frac{b}{q}| < \frac{W+1}{n} \) for some Farey point \( \frac{b}{q} \in FP_{\frac{1}{W-1}} \), or \( [y, y + \frac{1}{n}] \) contains one of the intervals from
As the first $f(\lfloor \alpha n \rfloor)$ elements of $X$ is an $f$-piercing sequence of order $\lfloor \alpha n \rfloor$, we have that for each interval $I$ in $\text{Fl}_{\lfloor \beta n \rfloor}$, there is at least one $x_i \in I$ for some $i = 1, 2, \ldots, f(\lfloor \alpha n \rfloor)$.

Our goal is to construct a strongly $g$-piercing sequence $Z$ of order $N$. The sequence $Z$ will include $X$ as a subsequence, and already thereby will have a point in large majority of intervals of length $\frac{1}{n}$ from $[0,1)$. Intuitively, it remains to appropriately fill in the constructed $Z$ with relatively small number of points which will handle intervals $[y, y + \frac{1}{n})$ with $n < N_0$ and those with $n \geq N_0$ and $y$ close to some Farey point in $\text{FP}_{1}^{W-1}$.

Now, for any nonnegative integer $r$ we define the set

$$H_r^W = \left\{ \frac{b}{q} + \frac{a}{2^r} : \frac{b}{q} \in \text{FP}_{1}^{W-1}, -2(W+1) \leq a < 2(W+1) \right\} \cap [0,1).$$

As there are less than $(W - 1)^2$ points in $\text{FP}_{1}^{W-1}$, there are less than $5W^3$ elements in any $H_r^W$. Observe, that for every $2^{r-1} \leq n < 2^r$, every $\frac{b}{q} \in \text{FP}_{1}^{W-1}$, and every $y$ with $|y - \frac{b}{q}| < \frac{W+1}{n}$, the interval $[y, y + \frac{1}{n})$ contains at least one of the points in $H_r^W$. In what follows, we treat $H_r^W$ as a sequence of points arranged in any order.

We are ready to construct a sequence $Z$ that combines sequences $H_r^W$ with $X$. We construct $Z$ incrementally by adding some sequences to the end. For two finite sequences $P$ and $Q$, we write $P \odot Q$ to denote their concatenation.

First, we explicitly add points piercing all intervals in $A_1^1, A_2^1, \ldots, A_{N_0}^1$, and elements $x_1, \ldots, x_{f(1)}$. Let

$$Z_0 = \left( \frac{1}{N_0}, \frac{2}{N_0}, \ldots, \frac{N_0 - 1}{N_0} \right) \odot (x_1, \ldots, x_{f(1)}).$$

Next, let $l = \lfloor \log_2 \lfloor \alpha N \rfloor \rfloor$, and for each $r = 1, \ldots, l$, we extend $Z$ with points from $H_r^W$ and part of $X$. Let

$$Z_r = Z_{r-1} \odot H_r^W \odot (x_{f(2^{r-1})+1}, \ldots, x_{f(2^r)}).$$

Finally, let

$$Z = Z_l \odot H_{l+1}^W \odot (x_{f(2^l)+1}, \ldots, x_{f(\lfloor \alpha N \rfloor)}).$$

Now we show that $Z$ is strongly $g$-piercing of order $N$. For every $1 \leq n \leq N_0$ we have $g(n) \geq N_0$, and every interval of length $\frac{1}{n}$ within $[0,1)$ contains one of the points from $Z_0$. For every $N_0 < n \leq N$, the first $g(n)$ elements of $Z$ contain:

- $\left\{ \frac{i}{N_0} : 1 \leq i < N_0 \right\}$ (first $N_0$ elements),
- $H_1^W \cup H_2^W \cup \ldots \cup H_{\lfloor \log_2 n \rfloor}^W$ (fewer than $5W^3 \lfloor \log_2 n \rfloor$ elements),
- $\left( x_1, x_2, \ldots, x_{f(\lfloor \alpha n \rfloor)} \right)$ (as $f(\lfloor \alpha n \rfloor) < \alpha^2 \gamma n$),

and hence, for each interval $I = [y, y + \frac{1}{n})$ within $[0,1)$, if $|y - \frac{b}{q}| < \frac{W+1}{n}$ for some $\frac{b}{q} \in \text{FP}_{1}^{W-1}$, then $I$ contains one of the points in $H_{\lfloor \log_2 n \rfloor}^W$. Otherwise, $I$ contains one of the Farey intervals in $\text{Fl}_{\lfloor \beta n \rfloor}$, which in turn includes one of the elements in $(x_1, x_2, \ldots, x_{f(\lfloor \alpha n \rfloor)})$.

This ends the proof. □
4 | PROOFS OF THE MAIN RESULTS

We are ready to prove our two main theorems stated in the introduction.

Theorem 2. There exists an infinite \( f \)-piercing sequence with \( f(n) = γn + o(n) \) if and only if \( γ ≥ \frac{1}{\ln 2} ≈ 1.44 \).

Proof. The if part follows immediately from Corollary 5.

For the only if part, suppose that there exists some infinite \( f \)-piercing sequence \( X \) with \( f(n) = γn + o(n) \) and \( γ < \frac{1}{\ln 2} \). Choose \( W > 2 \) large enough, so that for \( α = \frac{W + 1}{W - 1} \) we have \( α^2γ < \frac{1}{\ln 2} \). Let \( N_0 \) and \( g(n) = α^2γn + o(n) \) be the constant and the functions guaranteed by Lemma 9. For every \( N > N_0 \) we can apply Lemma 9 to the prefix of the first \([ αN ]\) elements of \( X \) to obtain a sequence \( Z_N \). It is obvious from the construction used in Lemma 9 that the sequence of sequences \( Z_N \) for all \( N > N_0 \) converges to an infinite sequence \( Z \), and that the sequence \( Z \) is strongly \( g \)-piercing. This contradicts Theorem 7. \( \Box \)

Theorem 1. The function \( s(d) \) satisfies \( |c_1d| ≤ s(d) ≤ c_2d + o(d) \) where \( c_1 = \frac{\ln 2}{1 - \ln 2} ≈ 2.25 \) and \( c_2 = \frac{1 + \ln 2}{1 - \ln 2} ≈ 5.52 \).

Proof. For the lower bound, we may use the strongly \( (\lceil \frac{n}{\ln 2} \rceil) \)-piercing sequence \( X \) given by de Bruijn and Erdős, defined by \( x_i = \log_2(2i + 1) \) (mod 1), for \( i = 1, 2, ..., \lceil \frac{N}{\ln 2} \rceil \). For \( N ≤ \lceil \frac{\ln 2}{1 - \ln 2} - d \rceil \), we have that \( N + d ≥ \lceil \frac{N}{\ln 2} \rceil \) and \( X \) is strongly \((n + d)\)-piercing of order \( N \).

For the upper bound, assume to the contrary that \( s(d) \) is not \( \frac{1 + \ln 2}{1 - \ln 2} d + o(d) \). Then there exists a constant \( c > \frac{1 + \ln 2}{1 - \ln 2} \) such that \( s(d) > cd \) for infinitely many values of \( d \). Choose an integer \( W > 10 \) large enough so that for \( α = \frac{W + 1}{W - 1} \) we have \( \frac{α^2}{1 - α^2} < c \).

Then, choose \( d \) large enough so that

\[
s(d) > \left[ \frac{1 + α^2 \ln 2}{1 - α^2 \ln 2} \cdot d + 500W^3 \right],
\]

and for \( N_0 = \lceil \frac{s(d)}{α} \rceil \) we additionally have

\[
\frac{1}{H_2 \left[ \frac{N_0 - 1}{2} \right]} - H_2 \left[ \frac{N_0 - 1}{2} \right] > \frac{1}{α \ln 2}.
\]

Observe that

\[
N_0 = \left\lfloor \frac{s(d)}{α} \right\rfloor > \frac{1 + α^2 \ln 2}{α(1 - α^2 \ln 2)}d + 400W^3
\]

and let \( X = (x_1, x_2, ..., x_{[αN_0]}) \) be an \((n + d)\)-piercing sequence of order \([αN_0]\).

Now, as in the proof of Lemma 9, for any nonnegative integer \( r \) we define the set

\[
H_r^W = \left\{ \frac{b}{q} + \frac{a}{2^r} : \frac{b}{q} \in \text{FP}_{2}^{W-1}, -2(W + 1) ≤ a ≤ 2(W + 1) \right\} \cap [0,1).
\]
Let 
\[
Z = H^W_{\lceil \log_2 N_0 \rceil} \odot H^W_{\lceil \log_2 N_0 \rceil} \odot H^W_{\lceil \log_2 N_0 \rceil} \odot X
\]
be a sequence of length $2L$ (as it will be clear that we can remove its last element if necessary). We have $2L > \alpha N_0$ and $L > \frac{N_0}{2}$. Observe that for every $L \leq l \leq 2L$, we have that the first $l$ elements of $Z$ includes sequences $H^W_r$, and at least first $l - 20W^3 - 1$ elements of $X$. Observe that for $M = l - 20W^3 - 1 - d$, as $X$ is $(n + d)$-piercing, we have that the first $l - 20W^3 - 1$ elements of $X$ include a point in every interval $[\frac{i}{m}, \frac{i+1}{m}]$ for every $m = 1, 2, ..., M$, and $i = 0, 1, ..., m - 1$. Further, $\frac{N_0}{2} < \frac{M}{\alpha} \leq N_0$, and thus Lemma 8 implies that the maximum distance between consecutive first $l$ elements of $Z$ is at most $\frac{\alpha}{M}$.

On the other hand, applying Lemma 6 to $Z$ we get that for some $L \leq l \leq 2L$ we have that the maximum distance between consecutive elements of $\{0, 1, Z_1, Z_2, ..., Z_l\}$ equals at least
\[
\frac{1}{(H_{2L} - H_{L-1})l} > \frac{1}{(H_{\lceil \frac{N_0}{2} \rceil} - H_{\lceil \frac{N_0}{2} \rceil - 1})l} > \frac{1}{\alpha l \ln 2}.
\]

It remains to prove that $\frac{\alpha}{M} \leq \frac{1}{s(1) \ln 2}$ to get a contradiction.

Starting with the assumption on the order of $s(d)$, we get the following:

\[
s(d) \geq \frac{d(1 + \alpha^2 \ln 2) + 2 + 40W^3}{1 - \alpha^2 \ln 2} + d
\]

\[
2L \geq s(d) + d \geq \frac{2d + 2 + 40W^3}{1 - \alpha^2 \ln 2} + d
\]

\[
l \geq L \geq \frac{d + 1 + 20W^3}{1 - \alpha^2 \ln 2}
\]

\[
l(1 - \alpha^2 \ln 2) \geq d + 1 + 20W^3
\]

\[
M = l - 20W^3 - 1 - d \geq \alpha^2 l \ln 2
\]

\[
\frac{M}{\alpha} \geq \alpha l \ln 2.
\]

\[\square\]

## 5 FINAL REMARKS

Let us conclude the paper with some discussion on possible future directions of studies in this topic.

Perhaps the most natural and challenging one concerns the determination of the concrete values of the function $s(d)$. We know only that $s(0) = 17$, by the result of Warmus [16], and that $s(1) \geq 31$, by an explicit example found by Oliveira e Silva [8], who also claims that $s(1) = 31$ was verified by computer search. Some computational experiments were also made by Levy in his PhD thesis [5], but no conjecture is stated there. Even now, though we know of quite close linear bounds from both sides for the function $s(d)$, a more exact formula for $s(d)$ seems rather elusive. However, it would be interesting to determine exactly the multiplicative constant in the presumed asymptotics of $s(d)$. We conjecture that our lower bound is the correct value.
Conjecture 10. The limit \( \lim_{d \to \infty} \frac{s(d)}{d} = \frac{\ln 2}{1 - \ln 2} \).

Let us stress, however, that it is not known whether the above limit actually exists.

Going back to the original Steinhaus’ problem, one may ask for possible multidimensional versions. Here is one of them for sequences of points in the plane. Imitating the original piercing property, we say that a sequence of points \( X \) in the unit square is piercing if for every finite prefix of \( X \) of length \( n \) there exists a tiling of the square into \( n \) rectangles, each of area \( \frac{1}{n} \), such that every rectangle contains exactly one point of the prefix. Is it true that there exist arbitrarily long piercing sequences in the unit square?

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