Comments on “Iteratively Re-Weighted Algorithm for Fuzzy c-Means”

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Abstract—In this comment, we present a simple alternate derivation to the IRW-FCM algorithm presented in [1] for Fuzzy c-Means problem. We show that the iterative steps derived in [1] are nothing but steps of the popular Majorization Minimization (MM) algorithm. The derivation presented in this note is much simpler and straightforward and, unlike the derivation in [1], the derivation here does not involve introduction of any auxiliary variable. Moreover, by showing the steps of IRW-FCM as the MM algorithm, the inner loop of the IRW-FCM algorithm can be eliminated and the algorithm can be effectively run as a “single loop” algorithm. More precisely, the new MM-based derivation deduces that a single inner loop of IRW-FCM is sufficient to decrease the Fuzzy c-means objective function, which speeds up the IRW-FCM algorithm.

Index Terms—Fuzzy c-Means, IRW-FCM, MM Algorithm.

I. INTRODUCTION AND IRW-FCM ALGORITHM

Following the notations used in [1], let \( x_1, x_2, \ldots, x_n \) denote the data points that have to be divided into \( c \) clusters with centers \( \{m_j\}_{j=1}^c \), and \( f_{ij} \) denote the \((i,j)\)th element of the membership matrix \( F \) which indicates the membership of the \( i \)th data point to the \( j \)th cluster. It is obvious that the elements of \( F \) have to be non-negative and unlike the K-Means clustering technique, which requires the elements of \( F \) to be either 0 or 1, in the case of fuzzy clustering the elements of \( F \) can be between 0 and 1 but the sum of elements in any row of \( F \) should be equal to 1, i.e., \( \sum_j f_{ij} = 1 \). With this, the Fuzzy c-Means problem can be stated as follows:

\[
\begin{align*}
\min_{F_1=1,F\geq0,\{m_j\}} & \sum_{i=1}^n \sum_{j=1}^c f_{ij} \| x_i - m_j \|^2_2, \\
\min_{F_1=1,F\geq0,\{m_j\}} & \sum_{i=1}^n \sum_{j=1}^c f_{ij} (x_i^T x_i - 2x_i^T m_j + m_j^T m_j).
\end{align*}
\]

The parameter \( r \) in (1) indicates the level of fuzziness in the clustering. A usual approach to solve (1) is to use alternating minimization, i.e., for a fixed \( F \), update \( \{m_j\} \) and vice-versa. For a fixed \( F \), the minimizer over \( \{m_j\} \) is given by:

\[
m_j = \frac{\sum_{i=1}^n f_{ij} x_i}{\sum_{i=1}^n f_{ij}} = \frac{\sum_{i=1}^n g_{ij} x_i}{\sum_{i=1}^n g_{ij}} = \frac{Xg_j}{g_j^T 1} \tag{2}
\]

where \( g_{ij} = f_{ij} \). With fixed \( \{m_j\} \), solving for \( F \) via the Lagrange Multiplier method with the constraint \( \sum_j f_{ij} = 1 \) gives:

\[
f_{ij} = \frac{d_{ij}^2}{\sum_k d_{ik}^2}, \tag{3}
\]

where \( d_{ij} = \| x_i - m_j \|_2 \). The alternating steps of (2) and (3) is known as FCM algorithm [2]. In [1], the authors have stated that the FCM algorithm suffers from convergence to sub-optimal minimum by blaming the alternating approach of FCM, and they proposed a new algorithm named IRW-FCM, that minimizes an equivalent problem of (1). More precisely, they obtained a problem only in \( F \) by substituting for the optimal minimizer over \( \{m_j\} \) (from (2)) in (1) and obtained the equivalent problem as shown below:

\[
\begin{align*}
\min_{F_1=1,F\geq0} & \phi(\{g_j\}) := \sum_{j=1}^c \sum_{i=1}^n g_{ij} x_i^T x_i - \sum_{j=1}^c \left( \frac{g_j^T X^T X g_j}{g_j^T 1} \right), \tag{4}
\end{align*}
\]

which they solved as follows. First, they introduced an auxiliary variable \( s \in R^c \times 1 \) and came up with a problem in \( \{g_j\} \) and \( s \), which is equivalent to (4):

\[
\begin{align*}
\min_{F_1=1,F\geq0,s} & \psi(\{g_j\},s) := \sum_{j=1}^c \sum_{i=1}^n g_{ij} x_i^T x_i + \sum_{j=1}^c (s_j^2 g_j^T 1 - 2s_j \sqrt{g_j^T X^T X g_j}). \tag{5}
\end{align*}
\]

For a fixed \( F \), the optimal minimizer over \( s \) is given by:

\[
s_j = \sqrt{g_j^T X^T X g_j} \frac{1}{g_j^T 1}, \tag{6}
\]

and substituting it back in (5), we get (4) and this proves the equivalence of (4) and (5). Then they proposed to solve (5) via alternating minimization: For fixed \( s \), update \( F \) and vice-versa. For a fixed \( s \) (say \( s^{t+1} \) computed using \( F^t \)), the minimization problem over \( F \) is given by:

\[
\begin{align*}
\min_{F_1=1,F\geq0} & \sum_{j=1}^c \sum_{i=1}^n g_{ij} x_i^T x_i + \sum_{j=1}^c (s_j^{t+1})^2 g_j^T 1 - 2s_j^{t+1} \sqrt{g_j^T X^T X g_j}. \tag{6}
\end{align*}
\]

However, (6) does not have any closed form solution, so they resorted to another optimization strategy named IRW [3], which approximates the negative square root function \(-\sqrt{g_j^T X^T X g_j}\) in (6) with a linear function at some given \( F^t \), and solved the following approximated problem:

\[
\begin{align*}
\min_{F_1=1,F\geq0} & \sum_{j=1}^c \sum_{i=1}^n g_{ij} x_i^T x_i + \sum_{j=1}^c ((s_j^{t+1})^2 1 + 2s_j^{t+1} n_j^T g_j), \tag{7}
\end{align*}
\]
where \( a_j = \frac{x_j^T x_j}{\sqrt{(g_j^T x_j)^2 + \epsilon}} \). The problem in (7) has a closed form solution given by:

\[
f_{ij}^{t+1} = \left( x_j^T x_j + (s_j^{t+1})^2 - 2s_j^{t+1} a_j \right) \frac{1}{\sum_{k=1}^{n} (x_k^T x_k + (s_k^{t+1})^2 - 2s_k^{t+1} a_k)}.
\]

Thus, the IRW-FCM approach which solves (4) is iterative, and effectively it is a “double loop” algorithm (one loop for alternating minimization and second inner loop for IRW procedure) as summarized in Algorithm 1.

**Algorithm 1** IRW-FCM Algorithm

- **Input** \((X, c)\)
- **Initialize** \(F^0\)
- repeat
  - Calculate \(\{s_j^{t+1}\}\)
  - Calculate \(a_j\)
  - Calculate \(F_j\) using equation (8)
  until convergence, **Output** \(F^{t+1}\)
- until convergence
- **Output** \(F^*, F_t^*\) denotes the optimal minimizer.

### II. MM BASED DERIVATION

The authors of (11) criticised the alternating minimization approach of FCM by saying that it suffers from convergence to sub-optimal minimum and motivated to solve the equivalent problem in (4) to avoid any such issues. However, in the process of solving (4) they “reversed back” and introduced a new auxiliary variable and again relied on the alternating minimization approach to solve the equivalent problem in (5).

Moreover, they had to resort to a separate iterative procedure of IRW to solve one of the subproblems of the alternating minimization, which results in a “double-loop” algorithm.

In the following, we present a much straightforward procedure in which we try and derive an iterative algorithm to solve (4) by working only with the variable of interest \(F\). We will employ a popular approach named Majorization-Minimization (MM) (4) to accomplish this. Before we venture into solving the problem in (4), we first briefly introduce the MM approach. Let us say that we are interested in minimizing a function \(f(x)\) over a domain \(\chi\), MM does the job in two steps: In the first step, given some \(x^t (x^t \in \chi)\), it constructs a tighter upperbound \(h(x|x^t)\) to the cost function \(f(x)\), and in the second step it minimizes the upperbound \(h(x|x^t)\) w.r.t. \(x\) and obtains the next iterate \(x^{t+1}\) and this is repeated till convergence. The upperbound \(h(x|x^t)\) should satisfy the following properties:

\[
f(x^t) = h(x^t|x^t), \text{ and } f(x) \leq h(x|x^t) \quad \forall x \in \chi.
\]

It is easy to verify that the series of iterates obtained via MM will monotonically decrease the objective as shown below:

\[
f(x^{t+1}) \leq h(x^{t+1}|x^t) \leq h(x^t|x^t) = f(x^t).
\]

The first inequality in (10) is due to (9) and the second inequality is due to fact that \(x^{t+1}\) is obtained by minimizing \(h(x|x^t)\), and the last equality is also due to (9). Now, coming back to problem in (4), it is easy to see that the second term (without the negative sign) in (4) is a quadratic over linear function (with \(g_j^T 1 > 0\)), which is a convex function (see section 3.1.5 in [3] or, in other words, the function \(-\frac{g_j^T X x_j}{g_j^T 1}\) is a concave function in ‘\(g_j\)’. Thus a tighter upperbound (at a given \(F^t\)) can be obtained via a tangent hyperplane passing through \(F^t\), which of course can be obtained via first order Taylor series expansion as shown below:

\[
-\frac{g_j^T X x_j}{g_j^T 1} \leq -\frac{(g_j^t)^T X x_j}{(g_j^t)^T 1} - \left[ 2(1)^T X x_j - (g_j^t)^T X x_j 1^T \right] \left( g_j - g_j^t \right) \frac{1}{(g_j^t)^T 1},
\]

with equality achieved at \(g_j = g_j^t\). The MM upperbound for the objective in (4) at some given \((F^t)\) is then given by:

\[
\Phi(\{g_j\}) \leq \sum_{j=1}^{c} \sum_{i=1}^{n} g_{ij} x_i^T x_i - \sum_{j=1}^{c} \left[ (g_j^t)^T X x_j + \frac{1}{(g_j^t)^T 1} \right] + \sum_{j=1}^{c} \left[ 2(1)^T X x_j - (g_j^t)^T X x_j 1^T \right] \left( g_j - g_j^t \right) \frac{1}{(g_j^t)^T 1}.
\]

\[
\triangleq h(\{g_j\}||j^t). \tag{12}
\]

Then, the upperbound minimization problem to be solved is given by:

\[
\min_{F_1=1, F_2 \geq 0} h(\{g_j\}||j^t) = \sum_{j=1}^{c} \sum_{i=1}^{n} g_{ij} x_i^T x_i - \sum_{j=1}^{c} \left[ 2(1)^T X x_j - (g_j^t)^T X x_j 1^T \right] \left( g_j - g_j^t \right) \frac{1}{(g_j^t)^T 1} + \text{const.}
\]

Leaving out the constant terms, we get

\[
\min_{F_1=1, F_2 \geq 0} \sum_{j=1}^{c} \sum_{i=1}^{n} g_{ij} x_i^T x_i - \sum_{j=1}^{c} \left[ 2(1)^T X x_j - (g_j^t)^T X x_j 1^T \right] \left( g_j - g_j^t \right) \frac{1}{(g_j^t)^T 1}, \tag{13}
\]

which has a simple closed form solution given by:

\[
f_{ij}^{t+1} = \left[ x_j^T x_j + \frac{(g_j^t)^T X x_j}{(g_j^t)^T 1} - \frac{2x_j^T x_j}{(g_j^t)^T 1} \right] \frac{1}{\sum_{k=1}^{n} \left[ x_k^T x_k + \frac{(g_k^t)^T X x_k}{(g_k^t)^T 1} - \frac{2x_k^T x_k}{(g_k^t)^T 1} \right]} \quad \forall i, j.
\]

(15)

If one takes a closer look at the problem in (14), it is nothing but the problem in (7) with choice of \(s_j^{t+1} = \sqrt{(g_j^t)^T X x_j} / (g_j^t)^T 1\), \(a_j = \frac{x_j^T x_j}{\sqrt{(g_j^t)^T X x_j}}\). This in a way proves that the iterative steps of IRW-FCM can be easily derived via the MM approach, and more importantly, the inner loop of IRW-FCM algorithm can be run only for one time and that is sufficient to decrease the cost function in (4).
Double loop algorithm.

Faster than FCM-IRW

Consider five different datasets from the UCI ML Repository ¹ with features mentioned in Table I. In Figure 1 and Figure 2, we show the objective vs iteration and objective vs runtime (in sec) of the IRW-FCM and FCM-MM algorithms for the five datasets. The initialization (F₀⁵) for both the algorithms is kept the same and in the case of IRW-FCM, the objective is plotted with respect to the outer iterations and the inner loop is run till convergence. It can be noticed from figure 1 that IRW-FCM seem to be converging faster in terms of iterations, however, in terms of runtime FCM-MM is generally faster than IRW-FCM algorithm - this clearly indicates that running the inner loop of IRW-FCM till convergence is not necessary and can be run only one time to attain the same minimum at a much faster runtime.

IV. CONCLUSION

In this note, we present a simple alternate derivation to the IRW-FCM algorithm presented in [1] to solve the Fuzzy c-Means problem. The derivation presented in this note is based on the popular Majorization-Minimization approach which enjoys nicer convergence properties. The derivation presented in this note shows that the inner loop in the IRW-FCM algorithm can be eliminated (by running it only one time) and yet convergence can be guaranteed.

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¹The MATLAB codes for IRW-FCM are obtained from https://github.com/Sara-Jingjing-Xue
²http://archive.ics.uci.edu/ml/index.php

| Sr. No. | FCM-MM | FCM-IRW |
|---------|--------|---------|
| 1.      | Single loop algorithm. | Double loop algorithm. |
| 2.      | Does not involve introduction of any auxiliary variable therefore has less space complexity. | Requires auxiliary variables and therefore requires more storage. |
| 3.      | Faster than FCM-IRW as it is equivalent to FCM-IRW with the inner loop run only one time. | Slower due to the redundant inner loop. |

Table I: DIFFERENCES AND SIMILARITIES BETWEEN FCM-MM AND FCM-IRW ALGORITHMS

III. NUMERICAL SIMULATION RESULTS

In this section, we present some simulation results comparing the double loop IRW-FCM² from [1] and the MM based algorithm derived in this note (we named it as FCM-MM). We consider five different datasets from the UCI ML Repository with features mentioned in Table I. In Figure 1 and Figure 2 we show the objective vs iteration and objective vs runtime (in sec) of the IRW-FCM and FCM-MM algorithms for the five datasets. The initialization (F₀⁵) for both the algorithms is kept the same and in the case of IRW-FCM, the objective is plotted with respect to the outer iterations and the inner loop