A note on the von Neumann algebra underlying some universal compact quantum groups

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Abstract
We show that for $F \in GL(2, \mathbb{C})$, the von Neumann algebra associated to the universal quantum group $A_u(F)$ is a free Araki-Woods factor.

Introduction
It is a classical theorem that any compact Lie group is a closed subgroup of some $U(n)$. In [5], a class of quantum groups was introduced which plays the same rôle with respect to the compact matrix quantum groups (introduced in [8], but there called compact quantum pseudogroups). These universal quantum groups were denoted $A_u(F)$, where the parameter $F$ takes values in invertible matrices over $\mathbb{C}$. In [1], the representation theory of the $A_u(F)$ was investigated, and it was shown that the irreducible representations are naturally labeled by the free monoid with two generators. Also on the level of the ‘function algebra’ of $A_u(F)$, freeness manifests itself: it was shown in [1] that the (normalized) trace of the fundamental representation is a circular element w.r.t. the Haar state (in the sense of Voiculescu, see [6]). Furthermore, the von Neumann algebra associated to $A_u(I_2)$, where $I_2$ is the unit matrix in $GL(2, \mathbb{C})$, is actually isomorphic to the free group factor $\mathcal{L}(\mathbb{F}_2)$.

In this note, we generalize this last result by showing that for $0 < q \leq 1$, the von Neumann algebra underlying the universal quantum group $A_u(F)$ with $F = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$ is a free Araki-Woods factor ([4]), namely the one associated to the orthogonal representation

$\begin{pmatrix} \cos(t \ln q^2) & -\sin(t \ln q^2) \\ \sin(t \ln q^2) & \cos(t \ln q^2) \end{pmatrix}$

of $\mathbb{R}$ on $\mathbb{R}^2$. The proof of this fact uses a technique similar to the one of Banica for the case $F = I_2$, combined with results from [4] (which are based on the matrix model techniques from [4]). Since

$A_u(F) = A_u(\lambda U|F|U^*)$

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for any \( \lambda \in \mathbb{R}_0^+ \) and any unitary \( U \) (see [1]), we obtain that all \( A_u(F) \) with \( F \in GL(2, \mathbb{C}) \) have free Araki-Woods factors as their associated von Neumann algebras.

Remarks on notation: We denote by \( \odot \) the algebraic tensor product of vector spaces over \( \mathbb{C} \), and by \( \otimes \) the spatial tensor product between von Neumann algebras or Hilbert spaces. If \( M \) is a von Neumann algebra and \( x_1, x_2, \ldots \) are elements in \( M \), we denote by \( W^*(x_1, x_2, \ldots) \) the von Neumann subalgebra of \( M \) which is the \( \sigma \)-weak closure of the unital \( * \)-algebra generated by the \( x_i \).

1 Preliminaries

In this preliminary section, we will give, for the sake of economy, ad hoc definitions of the von Neumann algebras associated to the \( A_u(F) \) and \( A_o(F) \) quantum groups ([5]), and of the free Araki-Woods factors ([4]), for special values of their parameters.

Throughout this section, we fix a number \( 0 < q < 1 \).

Definition 1.1. We define the \( C^* \)-algebra \( C_u(H) \) as the universal enveloping \( C^* \)-algebra of the unital \( * \)-algebra generated by elements \( a \) and \( b \), with defining relations

\[
\begin{align*}
    a^*a + b^*b &= 1 \\
    ab &= qba \\
    aa^* + q^2bb^* &= 1 \\
    a^*b &= q^{-1}ba^* \\
    bb^* &= b^*b.
\end{align*}
\]

Remark: \( C_u(H) \) is the (universal) \( C^* \)-algebra associated with the quantum group \( H = SU_q(2) \). In [1], Proposition 5, it is shown that this equals the quantum group \( A_o\left( \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \right) \).

The following fact is found in [4].

Lemma 1.2. Let \( \mathcal{H} \) be the Hilbert space \( l^2(\mathbb{N}) \otimes l^2(\mathbb{Z}) \), whose canonical basis elements we denote as \( \xi_{n,k} \) (and with the convention \( \xi_{n,k} = 0 \) when \( n < 0 \)). Then there exists a faithful unital \( * \)-representation of \( C_u(H) \) on \( \mathcal{H} \), determined by

\[
\begin{align*}
    \pi(a) \xi_{n,k} &= \sqrt{1 - q^{2n}} \xi_{n-1,k}, \\
    \pi(b) \xi_{n,k} &= q^n \xi_{n,k+1}.
\end{align*}
\]

Definition 1.3. In the notation of the previous lemma, denote by \( \psi \) the state

\[
\psi(x) = (1 - q^2) \sum_{n \in \mathbb{N}} q^{2n} \langle \pi(x) \xi_{n,0}, \xi_{n,0} \rangle
\]

on \( C_u(H) \). Then \( \psi \) is called the Haar state on \( C_u(H) \).

Of course, this name is motivated by the further compact quantum group structure on \( C_u(H) \), which we will however not need in the following.

Definition 1.4. The von Neumann algebra \( L^\infty(H) \) is defined to be the \( \sigma \)-weak closure of \( C_u(H) \) in its GNS-representation with respect to the Haar state \( \psi \).

We then continue to write \( \psi \) for the extension of \( \psi \) to a normal state on \( L^\infty(H) \).
Notation 1.5. We will further use the following notations:

- The matrix units of $B(l^2(\mathbb{N}))$ w.r.t. the canonical basis of $l^2(\mathbb{N})$ are written $e_{ij}$.
- We denote $\omega$ for the normal state $\omega(e_{ij}) = \delta_{ij}(1-q^2)q^{2i}$ on $B(l^2(\mathbb{N}))$.
- We denote by $S \subseteq \mathcal{L}(\mathbb{Z})$ the shift operator $\xi_k \rightarrow \xi_{k+1}$ on $l^2(\mathbb{Z})$.
- We denote by $\tau$ the state on $\mathcal{L}(\mathbb{Z})$ which makes $S$ into a Haar unitary with respect to $\tau$.

This last fact simply means that $\tau(S^n) = 0$ for $n \in \mathbb{Z}_0$.

We will use the terminology ‘$W^*$-probability space’ when talking about a von Neumann algebra with some fixed normal state on it. An isomorphism between two $W^*$-probability spaces is then a $*$-isomorphism between the underlying von Neumann algebras, preserving the associated fixed states.

Lemma 1.6. There is a natural isomorphism

$$(\mathcal{L}^\infty(H), \psi) \rightarrow (B(l^2(\mathbb{N})) \otimes \mathcal{L}(\mathbb{Z}), \omega \otimes \tau)$$

of $W^*$-probability spaces.

Proof. By the construction of $\psi$, we may identify $\mathcal{L}^\infty(H)$ with $\pi(C_u(H))''$, and it is then sufficient to prove that this last von Neumann algebra equals $B(l^2(\mathbb{N})) \otimes \mathcal{L}(\mathbb{Z})$. Clearly, $\pi(C_u(H))'' \subseteq B(l^2(\mathbb{N})) \otimes \mathcal{L}(\mathbb{Z})$. By functional calculus on $a$ and $b$, we have $e_{ij} \otimes S^n \in \pi(C_u(H))''$ for all $i,j \in \mathbb{N}$ and $n \in \mathbb{Z}$, so in fact equality holds. \hfill \qed

We will always write $(1 \otimes S)$ for the copy of $S \in \mathcal{L}(\mathbb{Z})$ inside $\mathcal{L}^\infty(H)$. Hence there should be no notational confusion in the following definition.

Definition 1.7. The $W^*$-probability space $(\mathcal{L}^\infty(G), \varphi)$ is defined as

$$(W^*(Sa, Sb, Sa^*, Sb^*), (\tau \ast \psi)_{|\mathcal{L}^\infty(G)}) \subseteq (\mathcal{L}(\mathbb{Z}), \tau) \ast (\mathcal{L}^\infty(H), \psi).$$

Remark: By [1], Théorème 1.(iv), the von Neumann algebra $\mathcal{L}^\infty(G)$ will coincide with the von Neumann algebra associated with the universal quantum group $A_u\left(\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}\right)$, and $\varphi$ with its Haar state.

Recall that the state $\omega$ was introduced in Notation 1.5.

Definition 1.8. ([3], Corollary 4.9) By a free Araki-Woods factor (at parameter $q^2$), we mean a $W^*$-probability space $(N, \phi)$ isomorphic to the free product $(\mathcal{L}(\mathbb{Z}), \tau) \ast (B(l^2(\mathbb{N})), \omega)$.

2 \hspace{1cm} $\mathcal{L}^\infty(G)$ is free Araki-Woods

Throughout this section, we again fix a number $0 < q < 1$. We also continue to use the notations introduced in the previous section.

We proceed to prove the following theorem.

Theorem 2.1. The $W^*$-probability space $(\mathcal{L}^\infty(G), \varphi)$ is a free Araki-Woods factor at parameter $q^2$.

By the remark after Definition 1.7 and the remarks in the introduction, this will imply that if $F \in GL(2, \mathbb{C})$, then the von Neumann algebra associated to $A_u(F)$ is the free Araki-Woods factor at parameter $\frac{\lambda_1}{\lambda_2}$, where $\lambda_1 \leq \lambda_2$ are the eigenvalues of $F^*F$ (where we take $\mathcal{L}(\mathbb{F}_2)$ to be the free...
Araki-Woods factor at parameter 1).

The proof of Theorem 2.1 will be preceded by three lemmas. Consider the following von Neumann subalgebras of \((L(Z), \tau) \ast (L^\infty(H), \psi)\):

\[(M_1, \varphi_1) = (W^*(S(1 \otimes S)), (\tau \ast \psi)|_{M_1})\]

and

\[(M_2, \varphi_2) = (W^*((1 \otimes S^*)a, (1 \otimes S^*)b, (1 \otimes S^*)a^*, (1 \otimes S^*)b^*), (\tau \ast \psi)|_{M_2}).\]

**Lemma 2.2.** The von Neumann algebras \(M_1\) and \(M_2\) are free with respect to each other, and \(L^\infty(G)\) is the smallest von Neumann subalgebra of \(L(Z) \ast L^\infty(H)\) which contains them.

**Proof.** The proof is entirely similar to the one of Théorème 6 in [1]. First of all, remark that \(S(1 \otimes S)\) is the unitary part in the polar decomposition of \(Sb\), so that \(S(1 \otimes S)\) is in \(L^\infty(G)\). Then of course

\[(1 \otimes S^*)a = (1 \otimes S^*)S^* \cdot Sa\]

is in \(L^\infty(G)\), and similarly for the other generators of \(M_2\). Hence \(M_1\) and \(M_2\) indeed generate \(L^\infty(G)\).

The proof of the freeness of \(M_1\) w.r.t. \(M_2\) is based on a small alteration of Lemme 8 of [1].

**Lemma.** Let \((A, \phi)\) be a unital \(*\)-algebra together with a functional \(\phi\) on it. Let \(B \subseteq A\) be a unital sub-*-algebra, and \(d \in B\) a unitary in the center of \(B\) such that \(\phi(d) = \phi(d^*) = 0\). Let \(u \in A\) be a Haar unitary which is \(*\)-free from \(B\) w.r.t. \(\phi\). Then \(ud\) is a Haar unitary which is \(*\)-free from \(B\) w.r.t. \(\phi\).

**Proof.** This is precisely Lemme 8 of [1], with the condition ‘\(\phi\) is a trace’ replaced by ‘\(d\) is in the center of \(B\)’. However, the proof of that lemma still applies ad verbam.

We can then apply this lemma to get that \(S(1 \otimes S)\) is \(*\)-free w.r.t. \(L^\infty(H)\), by taking \((A, \phi) = (L(Z), \tau) \ast (L^\infty(H), \psi), B = L^\infty(H), d = 1 \otimes S\) and \(u = S\). A fortiori, we will then have \(M_1\) free w.r.t. \(M_2\).

**Lemma 2.3.** We have

\[(M_1, \varphi_1) \cong (L(Z), \tau)\]

and

\[(M_2, \varphi_2) \cong (B(l^2(\mathbb{N})) \otimes L(Z), \omega \otimes \tau).\]

**Proof.** The fact that \((M_1, \varphi_1) \cong (L(Z), \tau)\) is of course trivial. We want to show that \((M_2, \varphi_2) \cong (B(l^2(\mathbb{N})) \otimes L(Z), \omega \otimes \tau)\).

We have that \(1 \otimes S^2\) is in \(M_2\), since this is the adjoint of the unitary part of the polar decomposition of \((1 \otimes S^*)b^*\). Also all \(e_{ii} \otimes 1\) are in \(M_2\), by functional calculus on the positive part of this polar decomposition. Hence, by multiplying \((1 \otimes S^*)a\) or \((1 \otimes S^*)a^*\) to the left with the left of \(e_{ii} \otimes 1\), and possibly multiplying with \(1 \otimes S^2\), we conclude that \(e_{ij} \otimes S^{i-j}\) with \(|i - j| = 1\) are in \(M_2\). But then also all \(f_{ij} = e_{ij} \otimes S^{i-j}\) with \(i, j \in \mathbb{N}\) are in \(M_2\), and it is not hard to see that in fact \(M_2 = W^*(f_{ij}, (1 \otimes S^2))\). Since \(\psi(f_{ij}(1 \otimes S^2)v = (\omega \otimes \tau)(e_{ij} \otimes S^n)\) by an easy calculation, we are done.

**Lemma 2.4.** The \(W^*\)-probability space \((N, \phi) := (L(Z), \tau) \ast (B(l^2(\mathbb{N})) \otimes L(Z), \omega \otimes \tau)\) is a free Araki-Woods factor at parameter \(q^2\).
Proof. The proof is completely similar to the one of Theorem 3.1 of [3]. Denote \((N, \theta) = (L(\mathbb{Z}), \tau) \ast (B(l^2(\mathbb{N})), \omega)\), and denote \(\phi_0 = \frac{1}{1-q^2} \phi\) and \(\theta_0 = \frac{1}{1-q^2} \theta\). Then by Proposition 3.10 of [3], we will have that
\[
(e_{00}Me_{00}, \phi_0) \sim (L(\mathbb{Z}), \tau) \ast (e_{00}Ne_{00}, \theta_0).
\]
By Proposition 2.7 in [3] (which is based on the proof of Theorem 5.4 and Proposition 6.3 in [4]) and the remark before it, we know that \((e_{00}Ne_{00}, \theta_0)\) as well as \((N, \theta) \cong (e_{00}Ne_{00}, \theta_0) \otimes (B(l^2(\mathbb{N})), \omega)\) are free Araki-Woods factors at parameter \(q^2\). By the free absorption property ([4], Corollary 5.5), \((e_{00}Me_{00}, \phi_0)\) is a free Araki-Woods factor at parameter \(q^2\), and hence also \((M, \phi) \cong (e_{00}Me_{00}, \phi_0) \otimes (B(l^2(\mathbb{N})), \omega)\) is.

Proof (of Theorem 2.1). By the first two lemmas, \((L^\infty(G), \varphi)\) is isomorphic to the free product of \((L(\mathbb{Z}), \tau)\) with \((B(l^2(\mathbb{N})) \otimes L(\mathbb{Z}), \omega \otimes \tau)\), which by the third lemma is a free Araki-Woods factor at parameter \(q^2\).

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