RELATIVISTIC TODA CHAIN

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ABSTRACT. Investigated is the relativistic periodic Toda chain, to each site of which the ultra-local Weyl algebra is associated. Weyl’s $q$ we are considering here is restricted to be inside the unit circle. Quantum Lax operators of the model are intertwined by six vertex $R$-matrix. Both independent Baxter’s $Q$-operators are constructed explicitly as series over local Weyl generators. The operator-valued wronskian of $Q$-s is also calculated.

1. Introduction

Long ago, in his famous papers [1] R.Baxter has introduced the object, which is known now as $Q$-operator. This operator does satisfy the so-called Baxter, or $T - Q$, equation and besides has many interesting properties. Recently $Q$-operator was intensively discussed in the series of papers [2] in the connection with continuous quantum field theory. In [3, 4] it was pointed out the relation of $Q$-operator with quantum Bäklund transformations. In [12] was discovered the relation of $Q$-operator with Bloch solutions of quantum linear problem.

$Q$-operator was used initially for the solution of the eigenvalue problem of $XYZ$-spin chain, where usual Bethe ansatz fails. The reason is that $T - Q$ equation, together with an appropriate boundary conditions, provides an one-dimensional multiparameter spectral problem which allows one to determine the spectra both of the auxiliary transfer matrix $T$ and of the operator $Q$. In the case of the quantum mechanical integrable chains, e. g. the periodic Toda chain, the appropriate solution of the Baxter equation plays the prominent role in the functional Bethe ansatz and the quantum separation of variables.

In quite recent papers there obtained explicit constructions of $Q$-operators for several models, like the isotropic Heisenberg spin chain, [5], the periodic Toda chain and other models with the rational $R$-matrix, [12]. In these papers $Q$-operator (operator, but not the solution

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of the Baxter equation) was obtained as the trace of monodromies of the appropriate local operators. It is well known, that with free boundary conditions for $Q$, $T - Q$ equation provides an one-parametric family of solutions, so that one may extract two independent solutions with nonzero discrete wronskian, see [2] and [10]. In the papers [3, 12] both independent $Q$ operators were obtained for the models considered.

In this paper we investigate the exactly integrable model known as “the relativistic Toda chain”, [14, 15, 16]. Local $L$ operator for the model is constructed with the help of the Weyl algebra generators, commuting on $q$, and we deal with the case $|q| < 1$. In this paper we do not consider the Jacoby partners to the Weyl algebra, dealing thus with the compact $q$–dilogarithms. Quantum space of our model is a formal module of an enveloping of the tensor product of several copies of Weyl algebras. The only thing we suggest for the Weyl generators is their invertibility and a $q$-equidistant spectrum for one of them. Both independent operators $Q_+$ and $Q_-$ and their wronskian are calculated locally as the operators acting in the ultra-local Weyl algebra. Actually all our results are to be understood as the well defined series expansions for functions from the enveloping mentioned.

2. The model and the results

This section consists of two parts. We formulate the model at first, actually just defining the transfer matrix, and then we give the final formulae for $Q_{\pm}$ operators and their $q$–Wronskian. All the sections beyond the introduction are the QUISM-type derivation of these results.

2.1. Problem. First of all let us define the relativistic Toda chain $L$-operator as

\[
L_f(x) = \begin{pmatrix} xu_f - (x u_f)^{-1} & v_f \\ q^{-1/2} \lambda v_f^{-1} & 0 \end{pmatrix},
\]

where $\{u_f, v_f\}$ form the “half-integer” ultra-local Weyl algebra:

\[
u_f \cdot v_f = q^{1/2} v_f \cdot u_f,
\]

and elements with different $f$-s commute. As usual, the whole quantum space is the tensor product of some copies of Weyl modules, and $f$ marks the “number” of given Weyl algebra in this tensor product. Recall, we will always imply $|q| < 1$. 

The correspondence between the relativistic Toda chain and usual Toda chain may be established, for example, in the following parameterization

\[ q = e^{-i\epsilon}, \quad \lambda = -\epsilon^2, \quad x = e^{i\theta/2}, \quad u_f = e^{-\epsilon p_f/2}, \quad v_f = \epsilon e^{q_f}, \]

where

\[ [p, q] = i, \]

in the limit

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} L_f(x) = \begin{pmatrix} \theta - p_f & e^{q_f} \\ -e^{-q_f} & 0 \end{pmatrix}. \]

The right hand side of this relation is known as the Toda \( L \) operator.

In the \( L \)-operator as well as in all other objects the spectral parameter \( x \) will always couple with \( u_f \). So we introduce the useful notation:

\[ x^2 u_f^2 \overset{def}{=} q^{s_f}, \]

so that for any formal function \( g(s_f) \)

\[ g(s_f) \cdot v^n_f = v^n_f \cdot g(s_f + n) \quad \forall n. \]

We define the transfer matrix for the chain with \( F \) sites, \( f = 1, \ldots, F \), as

\[ T(x^2) = \left((-x)^F \prod_f u_f\right) \cdot \text{tr} \left(L_1(x) \cdot L_2(x) \cdots L_F(x)\right). \]

\( T(x^2) \) becomes a polynomial of \( x^2 \) with commutative coefficients:

\[ T(x^2) = \sum_{j=0}^{F} (-x^2)^{F-j} t_j. \]

Here it is implied \( t_F = 1 \) and

\[ t_0 = \prod_f u_f^2. \]

Note that apart from the trivial \( t_F = 1 \) all other \( F \) coefficients are independent. For given set \( \{t_j\} \) one can define another set \( \{\overline{t}_j\} \) by

\[ \overline{t}_j = t_0^{-1} t_{F-j}. \]

This means simply

\[ T(x^2) = (-x^2)^F t_0 \overline{T}(x^{-2}). \]

We shall fix now the coefficients in Baxter’s equation:

\[ T(x^2) \cdot Q(x^2) = \left((-\lambda x^2)^F t_0\right) Q(q x^2) + Q(q^{-1} x^2), \]
where $t_0$ is given by (10). In what follows we shall see that with this normalization of the coefficients in (13) the Baxter equation has a solution entire on $x^2$. We shall call this solution

$$Q_+(x^2) = J(x^2, \lambda, \{t\}).$$

(14)

**Proposition 1.** Entire on $x^2$ solution of (13) as a series on $\lambda$ is

$$J(x^2, \lambda, \{t\}) = \left( \prod_{k=1}^{\infty} T(q^k x^2) \right) \cdot \left( \sum_{k=0}^{\infty} (-, \lambda^{F} c_k(x^2)) \right),$$

(15)

where $c_{-1} \equiv 0, c_0 \equiv 1$ and recursively

$$c_k(x^2) = \sum_{j=1}^{\infty} \frac{(q^j x^2)^F c_{k-1}(q^{1+j} x^2)}{T(q^j x^2) T(q^{1+j} x^2)}.$$

(16)

Note, $J(x^2, \lambda, \{t\})$ is the entire function on all its arguments. The proof of this proposition is rather simple exercise.

The other solution $Q_-(x^2)$ must contain a cut with respect to $x$, and up to this cut we guess $Q_-(x^2)$ to be entire on $x^{-2}$. More exactly, with the $s_f$ notation introduced by eq. (6),

$$Q_-(x^2) = \lambda^{-\sum s_f} \cdot J(x^{-2}, \lambda, \{t\}).$$

(17)

The last definition we need is the $q$-Wronskian of these two solutions:

$$W(x^2) \overset{\text{def}}{=} Q_+(q^{-1} x^2) Q_-(x^2) - Q_+(x^2) Q_-(q^{-1} x^2).$$

(18)

2.2. **Solution.** In this paper we’ll give explicit expressions for both functions $Q_{\pm}$. The natural question arises: we’ve got yet the form (13) and (10), what one may do else. The aim of this paper is to investigate the the relativistic Toda chain by QUISM method, to construct local operators $M_f(x^2)$ such that a trace of their monodromy gives $Q_{\pm}(x^2)$, to prove the commutativity of the transfer matrices and $Q_{\pm}$ and to calculate the Wronskian. Note that in QUISM approach we’ll construct $Q_{\pm}(x^2)$ not as functions of $\{t\}$, but as functions of local $u_f, v_f$. This is in some sense a factorization, the simplest analogue of this is well known $q$-exponential formula

$$(x + y; q)_{\infty} = (x; q)_{\infty} \cdot (y; q)_{\infty}; \quad x y = q y x,$$

where conventionally

$$(x; q)_{n} \overset{\text{def}}{=} \prod_{k=0}^{n-1} (1 - q^k x), \quad (x; q)_{\infty} \overset{\text{def}}{=} \prod_{n=0}^{\infty} (1 - q^k x),$$

(19)

(20)
and as the series expansions

\[ (x; q)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{(-x)^n}{(q; q)_n}, \quad (x; q)^{-1}_\infty = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n}, \]

The right hand side of eq. (19) we call the local form of its “global” left hand side.

Now we describe the local form of all solutions. First of all we introduce an universal function

\[ g_{\alpha, \beta}(n, m) \overset{\text{def}}{=} q^n \alpha^m \frac{(q^{1+n}; q)_\infty (q^{1+m}; q)_\infty}{(q; q)_\infty}, \]

where \( \alpha \) and \( \beta \) are complex numbers, and elements \( q^n \) and \( q^m \) commute.

**Proposition 2.** Operator \( Q_+(x^2) \), defined by eqs. (14,15,16), in the local form is

\[ Q_+(x^2) = \sum_{\{n_f \geq 0\}} \left( \prod_f g_{1, \lambda}(n_f + s_f, n_f) \right) \cdot \left( \prod_f (uv)^{n_f+1-n_f} \right), \]

and operator \( Q_-(x^2) \), defined by eqs. (17,15,16), in the local form is

\[ Q_-(x^2) = \sum_{\{n_f \geq 0\}} \left( \prod_f g_{1, \lambda}(n_f, n_f-s_f) \right) \cdot \left( \prod_f (uv)^{n_f-n_f-1} \right). \]

Their Wronskian, defined by eq. (18), is

\[ W(x^2) = \left( \prod_f (q^{s_f}; q)_\infty (q^{1-s_f}; q)_\infty \lambda^{-s_f} \right) \cdot \left( \prod_f \left( \frac{\lambda (uv)^{f+1}}{(uv)_f}; q \right)_\infty \right). \]

This paper is actually the proof of the second proposition.

3. **Intertwiners**

3.1. **Integrability.** First of all, from what follows the integrability of the the relativistic Toda chain, namely the commutativity of the transfer matrices (8). The origin of it is the famous six-vertex \( R \) matrix. The following relation holds:

\[ R_{1,2}(x/y) \cdot L_{1,f}(x) \cdot L_{2,f}(y) = L_{2,f}(y) \cdot L_{1,f}(x) \cdot R_{1,2}(x/y), \]
where \( L_{1,f}(x) = L_f(x) \otimes 1, L_{2,f}(y) = 1 \otimes L_f(y) \) etc., the cross product implies the tensor product of the \( 2 \times 2 \) matrices, and the six-vertex \( R \) matrix has the form
\[
R(x) = \begin{pmatrix}
1 - x^{-2}q & 0 & 0 & 0 \\
0 & q^{1/2}(1 - x^{-2}) & x^{-1}(1 - q) & 0 \\
0 & x^{-1}(1 - q) & q^{1/2}(1 - x^{-2}) & 0 \\
0 & 0 & 0 & 1 - x^{-2}q
\end{pmatrix}.
\]

The Yang-Baxter relation (26) provides the commutativity of the traces of the monodromies for \( L_{1,f}(x) \) and \( L_{2,y}(x) \), and as the extra multiplier in our definition of the transfer matrix (8) is the total shift operator, our modified transfer matrices (8) also form the commutative family:
\[
T(x^2) \cdot T(y^2) = T(y^2) \cdot T(x^2).
\]

The appearance of the six-vertex \( R \) matrix is the criterion of the existence of Baxter’s “\( TQ = Q' + Q'' \)” relation for our transfer matrix. Let \( M_{h,f}(x^2) \) be an operator, acting in tensor product of \( f \)-th quantum Weyl algebra and its auxiliary space “\( h \)”, such that the trace over this auxiliary space of the monodromy of \( M \) operators gives \( Q \)-operator:
\[
Q(x^2) = \text{tr}_h (M_{h,1}(x) \cdot M_{h,2}(x) \cdots M_{h,F}(x)).
\]

The commutativity of \( Q \) with \( T \) must provide the intertwining relation for \( M_{h,f} \) and \( L_f \):
\[
\tilde{L}_h(x/y) * L_f(x) \cdot M_{h,f}(y) = M_{h,f}(y) \cdot L_f(x) * \tilde{L}_h(x/y),
\]
where “\(*\)” means the \( 2 \times 2 \) matrix multiplication, and \( \tilde{L}_h(z) \) is an auxiliary \( L \)-operator. Below we will give the explicit form of this \( \tilde{L} \), but now we investigate the triangle relations following from eq. (30) and providing Baxter’s \( TQ = Q' + Q'' \) equation.

3.2. Triangle Relations. Triangle relations should appear when we choose such \( x/y \) in eq. (30) that \( \tilde{L} \) becomes degenerate as a \( 2 \times 2 \) matrix. Without lost of generality we may put \( \det \tilde{L}(1) = 0 \). We imply \( \tilde{L}^{-1} \) to be normalized to the determinant of \( \tilde{L} \), i.e. \( \tilde{L}(1) \) and \( \tilde{L}^{-1}(1) \) must be orthogonal. Thus introducing the notations \( a \) and \( a^+ \) for two appropriate elements for \( h \)-algebra, one may write:
\[
\tilde{L}_h(1) = \begin{pmatrix}
-a^+ \\
1
\end{pmatrix} k_1 (-a, 1) \quad \text{and} \quad \tilde{L}^{-1}_h(1) = \begin{pmatrix}1 \\
a\end{pmatrix} k_2 (1, a^+),
\]
Writing this decomposition, we do not impose no conditions on \(a, a^+\) and unknown factors \(k_1\) and \(k_2\). The experience of usual Toda chain \cite{12} says that when \(q \rightarrow 1\), \(a\) and \(a^+\) become usual bosonic annihilation and creation operators, this inspires our notations. In the next derivations we will suggest the invertibility of \(a\) and \(a^+\), so that the decompositions \cite{31} are written without loss of generality. Still we know nothing about algebra of \(a\) and \(a^+\). To get something applicable, let us introduce an element \(N\), such that a sort of \(q\)-oscillator relations hold: firstly, for any function \(g\) let
\[
a \cdot g(N) = g(N+1) \cdot a,
\]
and secondly let there exists a function \([N]\):
\[
a^+ \cdot a = [N], \quad a \cdot a^+ = [N+1].
\]
Element \(N\) is introduced without loss of generality as a pair to \(a\). Operators \(\tilde{L}\) must form an integrable chain, this provides relations \cite{33}, and therefore \(k_1 = k_1(N)\) and \(k_2 = k_2(N)\) in eq. \cite{31}. Thus, parameterization \cite{31} is the general one. The degenerate \(\tilde{L}^\pm(1)\) matrices becomes the orthogonal projectors when one choose the unknowns \(k_1 = 1/(1 + [N+1])\) and \(k_2 = 1/(1 + [N])\).

Now let us write the explicit form of the triangle relations:
\[
(-a, 1) L(x) M(x) = M'(x) (-a, 1),
\]
\[
L(x) M(x) \begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix} M''(x),
\]
and
\[
M(x) L(x) \begin{pmatrix} -a^+ \\ 1 \end{pmatrix} = \begin{pmatrix} -a^+ \\ 1 \end{pmatrix} M'(x),
\]
\[
(1, a^+) M(x) L(x) = M''(x) (1, a^+).
\]
Inserting the projectors into the product of \(L(x)\) and \(M(x)\) monodromies and taking the traces, one obtains in usual way the consequence of eqs. \cite{34} of \cite{35} Baxter’s equation
\[
T(x^2) Q(x^2) = \left( (-x)^F \prod_f u_f \right) \left( Q'(x^2) + Q''(x^2) \right),
\]
where extra multiplier appears due to our normalization of the transfer matrix,(see definition \cite{8}). Note, both \cite{34} and \cite{35} have to provide the same Baxter’s equation, hence the traces of \(M^\#\) and \(\overline{M}^\#\) monodromies should be the same.
The spectral parameter $x$ in $L$ operator always stay in the combination $x u$ therefore the shift of the spectral parameter thus may appear as

$$
\begin{align*}
g(q^{1/2} x u_f) & \equiv v_f^{-1} g(x u_f) v_f .
\end{align*}
$$

Due to this property we can put $x = 1$ for the shortness and omit the spectral parameter in our formulae, the $x$ may be restored subsequently in all equations by shift $u_f \mapsto x u_f$.

The triangle equations (34,35) are equivalent to two systems:

$$
\begin{align*}
\begin{cases}
M' &= -a v M , \\
M'' &= q^{-1/2} \lambda (a v)^{-1} M ,
\end{cases}
\end{align*}
$$

and

$$
\begin{align*}
\begin{cases}
M' &= -M q^{-1/2} \lambda v^{-1} a^+ , \\
M'' &= M v (a^+)^{-1} , \\
- q^{-1/2} \lambda a^+ M &= M (u - u^{-1} - (a^+)^{-1} v) v .
\end{cases}
\end{align*}
$$

In Baxter’s equation it is implied $Q'(x^2) \sim Q(q^{-1} x^2)$ and $Q''(x^2) \sim Q(q x^2)$ up to some operator-valued multipliers. These multipliers are to be integrals of motion in a form of pure product over $f$. There is only one such integral of motion, it is $t_0$, and hence there must exist a monomial function $\phi(u) \sim u^c$ such that

$$
\begin{align*}
M' &= v M \phi(u) v^{-1} , \\
M' &= v \phi(u) M v^{-1} ,
\end{align*}
$$

and therefore

$$
\begin{align*}
M'' &= -q^{-1/2} \lambda v^{-1} M \phi^{-1}(u) , \\
M'' &= -q^{-1/2} \lambda \phi^{-1}(u) v^{-1} M v .
\end{align*}
$$

The same $\phi(u)$ is used for $M$ and $M'$ because of $Q$ must be the same. It is important that in equations (40) the multiplier $\phi(u)$ stands from the right of to $M$ for $M'$ and from the left of $M$ for $M'$. The order of multipliers is governed by Yang-Baxter equation (30).

In general one may put $\phi(u)$ to the other sides, this would give another system for $M$ with another solution. We will not investigate such case separately, because of there exists an involutive automorphism $\tau$, defined as

$$
\begin{align*}
v^\tau &= v , \\
u^\tau &= u^{-1} , \\
q^\tau &= q ,
\end{align*}
$$

such that $L$-operator is invariant with respect to $\tau$-involution:

$$
\begin{align*}
L(1)^\tau &= -\sigma_3 L(1) \sigma_3 .
\end{align*}
$$
Also important is that \( \tau \) does not change \( q \). Therefore \( T^\tau (x^2) = (-)^F T(x^2) \), and the another case of positions of \( \phi \) just corresponds to the consideration of \( M^\tau \).

With these expressions for \( M' \) and \( M'' \) systems (38,39) are equivalent to

\[
(i) \quad -a M = M \phi(u) v^{-1},
(ii) \quad -q^{-1/2} \lambda M a^+ = v \phi(u) M,
(iii) \quad M a = v^{-1} \left( u^{-1} - u + q^{-1/2} \lambda a^{-1} v^{-1} \right) M,
(iv) \quad -q^{-1/2} \lambda a^+ M = M \left( u - u^{-1} - (a^+)^{-1} v \right) v.
\]

It is useful to complement system (44) by equations with \( k_1, k_2 \) following from (30):

\[
(v) \quad M \phi^{-1}(u) k_1(N) = \phi^{-1}(u) k_1(N) M,
(vi) \quad M \phi^{-1}(q^{-1/2} u) k_2(N) = \phi^{-1}(q^{-1/2} u) k_2(N) M.
\]

This is the final set of equations that we are going to solve. We will give the solution of it in two forms. The fist one is a formal series solution that admits an interpretation of \( a \) and \( a^+ \) as \( q \)-oscillator (spectrum of \( N \) is non-negative integers, and there exists the vacuum vector for \( a \)). Another form implies the Weyl algebra parameterization of \( a, a^+ \), when the permutation between \( h \) and \( f \) spaces pays the significant role. Actually these two forms differ by the notion of the trace in \( h \)-space, and \( q \)-oscillator trace will give \( Q_- \) while the Weyl trace will give \( Q_+ \).

3.3. Series solution. First, let us test system (44) for the formal operator arguments of \( M \). Just considering the expressions \( M x M^{-1} x^{-1} \) for several \( x \), one may conclude

\[
M = M(a v, u, N).
\]

Hence

\[
M \cdot q^N u^2 = u^2 q^N \cdot M,
\]

this trivializes two equivalent relations \((v)\) and \((vi)\) of system (44).

The further analysis of (44) we start from the permutation-like relation \((i)\). The relations like

\[
x \cdot M = M \cdot y
\]

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are to be solved as
\[ M = \sum_{n \in \mathbb{Z}} x^n \cdot G \cdot y^{-n}, \] (49)
and in the case of (44(i)) this gives
\[ M = \sum_{n \in \mathbb{Z}} a^n \cdot G(N, u^2) \cdot (-v \phi^{-1}(u))^n. \] (50)

\( G \) does not depend on \( aw \), because any such dependence may be extracted to \( a^n \). Now all other relations from (44) must give recursion relations for \( G \). Equation (iii) of (44) is equivalent to
\[ (u - u^{-1}) G(N, u^2) = \]
\[ = -q^{-1/2} \lambda \phi^{-1}(u) G(N, qu^2) + \phi(q^{-1/2}u) G(N - 1, q^{-1}u^2). \] (51)

Eq. (ii) of (44) gives another permutation-like structure, but with the formal correspondence \( a^+ = [N] a^{-1} \) it gives
\[ \frac{G(N - 1, u^2)}{G(N, qu^2)} = q^{1/2} \lambda \frac{[N]}{\phi^2(u)}. \] (52)

Due to (52) \( M \) may be rewritten in the form of the other permutation-like structure:
\[ M = \sum_{n \in \mathbb{Z}} (-q^{1/2} \lambda \phi(u))^n \cdot G(N, u^2) \cdot (a^+)^{-n}. \] (53)

Moreover, this allows one to write \( M \) without negative powers of \( a \) or \( a^+ \):
\[ M = G(N, u^2) + \]
\[ + \sum_{n=1}^{\infty} a^n \cdot G(N, u^2) \cdot (-v \phi^{-1}(u))^n + \]
\[ + \sum_{n=1}^{\infty} (-q^{1/2} \lambda \phi(u))^{-n} \cdot G(N, u^2) \cdot (a^+)^n. \] (54)

Apparently, this form is good for \( q \)-oscillator representation.

The equation (iv) of (44) coincides with (51) if one uses the series (53). But it is important to note that in general (51) and (52) are not compatible. Their compatibility (i.e. zero curvature) condition is the
following functional relation for $\phi(u)$ and $[N]$:

$$q^{-1/2} \lambda \left( \frac{[N]}{\phi(q^{1/2}u)} - \frac{[N - 1]}{\phi(q^{-1/2}u)} \right) =$$

$$(55)$$

$$= u^{-1} \left( 1 - q^{-1/2} \frac{\phi(u)}{\phi(q^{1/2}u)} \right) - u \left( 1 - q^{1/2} \frac{\phi(u)}{\phi(q^{1/2}u)} \right).$$

Here we used that $\phi(u) \sim u^\alpha$.

Eq. (55) has only two solutions for $\phi(u)$ and $[N]$, corresponding to $|q| < 1$ and $|q| > 1$. In our case $|q| < 1$

$$(56) \quad \phi(u) = -q^{-1/2} \alpha u^{-1}, \quad [N] = -q^{1/2} \frac{\alpha}{\lambda} (1 - q^{-N}),$$

where $\alpha$ is a complex parameter, $[N]$ is normalized so as $[0] = 0$. With these $\phi(u)$ and $[N]$ eqs. (51) and (52) may be solved easily giving

$$(57) \quad G_{|q|<1}(N, u^2) = g_{\alpha,\lambda/\alpha}(N, N - s),$$

where $u^2 \equiv q^s$ (see eq. (5)), and $g_{\alpha,\beta}(n, m)$ is defined by (22). Parameter $\alpha$ is an avoidable scale of $u$ and it is convenient to put it to unity, $\alpha \equiv 1$. Note that expressions like $(x; q)_\infty$ in $g$-function appear as the appropriate solutions of difference relations

$$(58) \quad (x; q)_\infty = (1 - x) (qx; q)_\infty,$$

and separation between $|q| < 1$ and $|q| > 1$ is originated from the unavoidable sign of the quadratic exponent $q^{\pm N(N-s)}$. With $\phi(u)$ defined, the final expressions for $M$ are: the short two

$$M = \sum_{n \in \mathbb{Z}} a^n g_{1,\lambda}(N, N - s) (uv)^n \equiv$$

$$(59) \quad \equiv \sum_{n \in \mathbb{Z}} (\lambda uv^{-1})^{-n} g_{1,\lambda}(N, N - s) (a^+)^{-n},$$

$^1$ The other solution of zero curvature condition is

$$\phi(u) = \alpha u, \quad [N] = -\frac{\alpha}{\lambda} (1 - q^N).$$

This gives

$$G(N, q^s) = q^{-N(N-s)} \left( q^{-1/2} \frac{\lambda}{\alpha} \right)^{N-s} \left( q^{1/2} \alpha \right)^N \times$$

$$\times \left( q^{1-N+s}; q^{-1}_\infty \right) (q^{1-N}; q^{-1}_\infty) \times$$

$$(q^{-1}; q^{-1}_\infty).$$
and \( q \)-oscillator-type

\[
M = g_{1,\lambda}(N, N - s) + \\
+ \sum_{n=1}^{\infty} a^n g_{1,\lambda}(N, N - s) (uv)^n + \\
+ \sum_{n=1}^{\infty} (\lambda uv^{-1})^n g_{1,\lambda}(N, N - s) (a^+)^n ,
\]

(60)

where, recall eq. (5), \( u^2 = q^s \).

Substituting \( \phi(u) = -q^{-1/2}u^{-1} \) into the expressions for \( M' \) and \( M'' \), (40), and using our definition of the transfer matrix (8), we obtain the Baxter equation exactly in the form (13).

Existence of the form (60) allows one to interpret \( a, a^+ \) exactly as \( q \)-oscillator generators, such that the spectrum of \( N \) is 0, 1, 2, ... (we have normalized \( [N] \) so that \( [0] = 0 \)), and the state \( |N = 0> = 0 \). Thus one may define the \( q \)-oscillator trace of any operator \( F = F(a, a^+, N) \),

(61)

\[
F = f_0(N) + \sum_{n \geq 1} f_n(N) a^n + \sum_{n \geq 1} f_n^+(N) (a^+)^n ,
\]

taking such trace one has to take \( a^0 \) and \( (a^+)^0 \)-th components and then take the sum over \( N = 0, 1, 2, ...: \)

(62)

\[
\text{tr}_{q\text{-osc}} F(a, a^+, N) \overset{\text{def}}{=} \sum_{n \geq 0} f_0(n) .
\]

Being applied to the monodromy (29) of \( M \), this trace definition gives exactly \( Q_-(x^2) \), eq. (24).

In general one may obtain \( Q_+ \) immediately, considering the \( \tau \) - involution applied to \( M \) and to \( Q_- \):

(63)

\[
M^\tau = \sum_{n \in \mathbb{Z}} (avu^{-1})^n g_{1,\lambda}(N, N + s) .
\]

But there are two objections to consider this case: firstly, \( \tau \)-involution changes a little the Baxter equation, and secondly, \( M^\tau \) is the degenerate operator,

(64)

\[(avu^{-1} - 1) \cdot M^\tau = 0 ,\]

and hence we will look for another way to obtain \( Q_+ \) operator.
3.4. **Extraction of a permutation.** Solving eqs. (44), we mentioned the permutation-like relations. In this section let us suppose that the quantum space $f$ and the auxiliary one $h$ are isomorphic. Our aim is to extract the permutation operator, giving eq. (i) of (44) “by hands”. As previously, we deal with the case $|q| < 1$, $x = 1$, $u^2 = q^s$, and

$$\phi(u) = -q^{-1/2} u^{-1}, \quad [N] = -q^{1/2} \lambda^{-1} (1 - q^{-N}),$$

so that we are looking for another realization of the same operator $M$. We will search for $M$ in the form

$$M = \mathcal{M} \cdot P_{h,f} ,$$

where

$$a \ P_{h,f} = P_{h,f} (uv)^{-1}, \quad N \ P_{h,f} = P_{h,f} s, \quad P_{h,f}^2 = 1 .$$

Here the first relation is exactly eq. (i) of (44), the second one is the consequence of eq. (47), and the last one is the definition of the permutation. System (44) for operator $\mathcal{M}$ can be rewritten as follows:

$$(66)$$

$$(i) : a \cdot \mathcal{M} = \mathcal{M} \cdot a ,$$

$$(ii) : u^{-1}v \cdot \mathcal{M} = \mathcal{M} \cdot (1 - u^2) u^{-1}v ,$$

$$(iii) : \mathcal{M} \cdot (uv)^{-1} = (uv)^{-1} \left( 1 - u^2 + q^{-1} \lambda a^{-1} v^{-1} u \right) \cdot \mathcal{M} ,$$

$$(iv) : q^{-N} \cdot \mathcal{M} = \mathcal{M} \cdot \left( 1 + q^{-1} \lambda \left( 1 - q u^2 \right)^{-1} a^{-1} v^{-1} u \right) q^{-N} .$$

Solution of it is given by

$$\mathcal{M} = (-\lambda a^{-1} v^{-1} u; q)_\infty (q u^2; q)_\infty .$$

Operator (66) with the definitions (67,69) does solve the system of the relations (44). Using the series decomposition for the compact quantum dilogarithms, one may obtain the series representation for $M$ (66):

$$(70) \quad M = \sum_{n \geq 0} \frac{q^n}{(q;q)_n} \lambda^n u^{2n} (q^{1+n} u^2; q)_\infty (uv)^{-n} P_{h,f} (uv)^n .$$

Note, function $g$, eq. (22), appears in this decomposition:

$$(71) \quad M = \sum_{n \geq 0} g_{1,\lambda}(n + s, n) (uv)^{-n} P_{h,f} (uv)^n .$$

---

2 There is used

$$(v^{-1}u)^n = q^{n(n+1)/2} u^{2n} (uv)^{-n} .$$
In this form all the $h$-space operators $a, a^+$ and $N$ are hidden into the permutation symbol. The permutation operator allows one to calculate the trace in the auxiliary space $h$ in the invariant way via

$$\text{tr}_{\text{inv}}(P_{h,1} P_{h,2} \cdots P_{h,F}) = P,$$

where $P$ is the cyclic shift operator for the chain $f = 1, 2, \ldots, F, F+1 \sim 1$:

$$u_f P = P u_{f+1}, \quad v_f P = P v_{f+1}, \quad f \sim f + F.$$  

The shift is one of the integrals of motion. Now using eq. (71) and the definition of the shift operator, one obtains exactly $Q_+$ (23) for the trace of $M$-monodromy up to the shift:

$$Q_+ P = P Q_+ = \text{tr}_{\text{inv}}(M_1 M_2 \cdots M_F).$$

Now both forms of $M$-operators have been obtained, eqs. (54) and (71), actually coincide. To show it, let us represent $P_{h,f}$ in the following form:

$$P_{h,f} = \sum_{n \in \mathbb{Z}} \delta(N - s = n) (auv)^n,$$

where $\delta(N - s = n)$ is the projector of $N - s$ into a state with the eigenvalue $n$:

$$\delta(N - s = n) = \delta(N - s = n) (N - s) = n \delta(N - s = n).$$

With this form of $P_{h,f}$ eq. (71) could be written as follows:

$$M = \sum_{n,k} a^k g_{1,\lambda}(n + s, n) \delta(N - s = n) (uv)^k.$$

Now one may take the sum over $n$ using the projectors as the delta symbols, and exactly eq. (59) appears:

$$M = \sum_{k} a^k g_{1,\lambda}(N, N - s) (uv)^k.$$

Such exercises with the projector decomposition of operators are rather formal. One may consider projectors and spectral decompositions of many types, imposing some extra conditions for the spectra of the operators involved. What is actually the difference between both $Q$ operators: the difference is the conjecture about the spectrum of $N$. Due to the Weyl algebra relations, the spectrum of $N$ must be equidistant:

$$N \in Z + \zeta.$$

In the case when $\zeta = 0$ we get q-oscillator representation. In the case when $\zeta$ is the same as for $s \in \mathbb{Z} + \zeta$, we get the isomorphism between $h$ and $f$ spaces and the permutation extracted representation. In general one may generalize both $Q_+$ and $Q_-$ into $Q_\zeta$, dealing with arbitrary characteristics of $N$.

\begin{equation}
Q_\zeta = \sum_{\{n_f \in \mathbb{Z} + \zeta\}} \left( \prod_f g_{1,\lambda}(n_f, n_f - s_f) \right) \cdot \left( \prod_f (uv)^{n_f-n_f-1}_f \right).
\end{equation}

A summation over $n \in \mathbb{Z}$ may be restricted in q-hypergeometry by the factor

\begin{equation}
\frac{1}{(q;q)_n} = \frac{(q^{1+n};q)_\infty}{(q;q)_\infty} = 0 \quad \text{for} \quad n < 0.
\end{equation}

Such restrictions in eq. (80) appear when $\zeta = 0$ and when $s \in \mathbb{Z} + \zeta$, these are exactly the cases of $Q_-$ and $Q_+ P$.

Similarly to the spectral decomposition of the permutation operator, one may write down the spectral decomposition of the shift operator:

\begin{equation}
P = \sum_{\{n_f \in \mathbb{Z}\}} \left( \prod_f \delta(s_f = n_f + \zeta) \right) \left( \prod_f (uv)^{n_f-n_f-1}_f \right).
\end{equation}

In this formula it is implied that $\zeta$ is the characteristics of $s_f$. An example of application of such formula, i.e. explicit extraction of the shift operator, is the following summation, where the shift $n_f \mapsto n_f + s_f$ is done:

\begin{equation}
\sum_{\{n_f \in \zeta + \mathbb{Z}\}} G(\{n_f, n_f - s_f\}) \prod_f (uv)^{n_f-n_f-1}_f
\end{equation}

To obtain it, one has to apply the spectral decomposition of each $s_f$, and then make the re-summation. This trick gives $Q_\zeta = Q_+(x^2) P$ when $s_f \in \zeta + \mathbb{Z}$.

4. Properties of $M$ operators

4.1. Auxiliary $L$-operator.

**Proposition 3.** Equation (80), provided by eqs. (44,47,65), holds for

\begin{equation}
\tilde{L}(x) = \begin{pmatrix}
x q^{N/2} - x^{-1} q^{-N/2} & \lambda a^+ q^{N/2} \\
\lambda q^{N/2} a & -\lambda x^{-1} q^{N/2}
\end{pmatrix}.
\end{equation}
To be exact, in our normalization $M = M(1)$, for which eqs. (44, 47, 63) are written down, eq. (20) looks like

$$(85) \quad M \cdot L(x) \cdot \tilde{L}(x) = \tilde{L}(x) \cdot L(x) \cdot M,$$

and $M$ must intertwine each power of $x$. Note, as far the quantum Lax operator is called “the relativistic Toda chain L-operator”, then operator (84) is to be called “the relativistic Dual self-trapping L-operator”, see e.g. [4].

4.2. Intertwining. Now let us consider the commutation relations of different $Q$-operators. Let the operators $Q_1(y)$ and $Q_2(x)$ are constructed with the help of different local $M_{h_1,f}(y)$ and $M_{h_2,f}(x)$ (here we imply different characteristics of $h_1$ and $h_2$).

**Proposition 4.** Two products: $M_{h_1,f}(y) \cdot M_{h_2,f}(x)$ and $M_{h_2,f}(x) \cdot M_{h_1,f}(y)$, are connected by a canonical mapping $K_{h_1,h_2}(y/x)$ of the pair of Weyl algebras $h_1$ and $h_2$:

$$(86) \quad K_{h_1,h_2} \left( \frac{y}{x} \right) M_{h_1,f}(y) M_{h_2,f}(x) = M_{h_2,f}(x) M_{h_1,f}(y) K_{h_1,h_2} \left( \frac{y}{x} \right),$$

where $K$ acts as follows:

$$K(z) a_1^\pm = z^{-1} a_2^\pm K(z), \quad K(z) q^{N_1} = \frac{1 + q^{1/2} z a_1 a_2^+}{1 + q^{1/2} z^{-1} a_1 a_2^+} q^{N_2} K(z),$$

$$(87) \quad K(z) a_2 = z a_1 K(z), \quad K(z) q^{N_2} = q^{N_1} \frac{1 + q^{1/2} z^{-1} a_1 a_2^+}{1 + q^{1/2} z a_1 a_2^+} K(z).$$

As an example we give the realization of $K(z)$ with the permutation extracted:

$$(88) \quad K_{h_1,h_2}(z) = \left( -q^{1/2} z^{-1} a_1 a_2^+ ; q \right) \left( -q^{1/2} z a_1 a_2^+ ; q \right) z^{-N_1-N_2} P_{h_1,h_2},$$

where $P_{h_1,h_2}$ is usual external permutation of the spaces $h_1$ and $h_2$. This permutation may be canceled from $KMM$ equation, and following relation for the $Q$-monodromies appears:

$$(89) \quad \tilde{K}_{h_1,h_2} \left( \frac{y}{x} \right) \tilde{Q}_{h_1}(y^2) \tilde{Q}_{h_2}(x^2) = \tilde{Q}_{h_1}(x^2) \tilde{Q}_{h_2}(y^2) \tilde{K}_{h_1,h_2} \left( \frac{y}{x} \right),$$

\footnote{Useful relations following from eqs. (44) are

$$M \cdot (uv)^{-1} = a \cdot M, \quad M \cdot uv^{-1} = q^N (\lambda uv^{-1} + q^{1/2} a) \cdot M,$$

and

$$vu^{-1} \cdot M = M \cdot \lambda a^+, \quad uv \cdot M = M \cdot \lambda q^N (uv + q^{-1/2} a^+).$$}
where

\begin{equation}
\hat{K}_{h_1 h_2} \left( \frac{y}{x} \right) = P_{h_1 h_2} K_{h_1 h_2} \left( \frac{y}{x} \right) = \left( \frac{x}{y} \right)^{N_1 + N_2} \frac{(-q^{1/2} y a_1^+ a_2^+; q)_\infty}{(-q^{1/2} y a_1^+ a_2; q)_\infty},
\end{equation}

and \( \hat{Q} \) – the monodromy of \( M \) operators, \( Q(x^2) = \text{tr}_h \hat{Q}_h \). Eq. \((94)\) leads to the pseudo-commutation of the pair of \( Q \) matrices with different \( \zeta \)-characteristics and allows one to calculate the wronskian.

4.3. **Wronskian.** To calculate the wronskian, it is necessary to consider eq. \((90)\) with \( \frac{x}{y} = q^{1/2} \). Then \( \hat{K}(q^{-1/2}) = q^{(N_1 + N_2)/2} (1 + a_1^+ a_2) \), and

\begin{equation}
q^{(N_1 + N_2)/2} (1 + a_1^+ a_2) \hat{Q}_1(q^{-1}x^2) \hat{Q}_2(x^2) = \hat{Q}_1(x^2) \hat{Q}_2(q^{-1}x^2) (1 + a_1^+ a_2) q^{(N_1 + N_2)/2}.
\end{equation}

Let \( \delta_W \) be a projector to the subspace \( a_1^+ a_2 = -1 \), i.e.

\begin{equation}
\delta_W \cdot (a_1^+ a_2 + 1) = (a_1^+ a_2 + 1) \cdot \delta_W = 0.
\end{equation}

Then the pseudo-commutation relation provides the following triangle structure:

\begin{equation}
\hat{Q}_1(q^{-1}x^2) \hat{Q}_2(x^2) \delta_W = \delta_W \hat{Q}_1(q^{-1}x^2) \hat{Q}_2(x^2) \delta_W,
\end{equation}

\begin{equation}
\delta_W \hat{Q}_1(x^2) \hat{Q}_2(q^{-1}x^2) = \delta_W \hat{Q}_1(x^2) \hat{Q}_2(q^{-1}x^2) \delta_W.
\end{equation}

Locally we consider products:

\begin{equation}
M_1(y) M_2(x) \delta_W = \sum_{n \in \mathbb{Z}} (\lambda yuv^{-1})^n \mathcal{F}_{y,x}(N_1, N_2, u^2) (a_1^+)^n \delta_W,
\end{equation}

\begin{equation}
\delta_W M_1(x) M_2(y) = \delta_W \sum_{m \in \mathbb{Z}} a_2^m \tilde{\mathcal{F}}_{x,y}(N_1, N_2, u^2) (yuv)^n,
\end{equation}

where the sum simplified due to

\begin{equation}
\delta_W \cdot (a_1^+)^n a_2^{m-n} \equiv \delta_W \cdot (-)^n a_2^m.
\end{equation}

Triangle structure means that when \( y^2 = q^{-1}x^2 \), both \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) depend actually only on \( N_1 + N_2 \) and \( u^2 \).
For $x$ and $y$ arbitrary, one has

\[ F_{y,x}(N_1, N_2, u^2) \overset{\text{def}}{=} \sum_{m \in \mathbb{Z}} (-\lambda xy u^2)^m q^{n^2/2} g_{1,\lambda}(N_1, N_1 - s_y - m) g_{1,\lambda}(N_2 + m, N_2 - s_x), \]

\[ \tilde{F}_{x,y}(N_1, N_2, u^2) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} (-\lambda xy u^2)^n q^{n^2/2} g_{1,\lambda}(N_1 + n, N_1 - s_x) g_{1,\lambda}(N_2, N_2 - s_y - n). \]

Here

\[ q^{s_x} = x^2 u^2, \quad q^{s_y} = y^2 u^2. \]  

One may see,

\[ \tilde{F}_{x,y}(N_1, N_2, u^2) \equiv F_{y,x}(N_2, N_1, u^2). \]

These sums may be calculated with the help of the Rogers-Ramanujan summation formula. Auxiliary relations for this calculations are:

\[ (\lambda xy uv - 1)^n (yuv)^n = q^{n^2/2} (\lambda xy u^2)^n, \]

\[ \frac{g_{1,\lambda}(N + n, N - s)}{g_{1,\lambda}(N, N - s)} = q^{n(N-s)} \frac{1}{(q^{1+N}; q)_n}, \]

\[ \frac{g_{1,\lambda}(N, N - s - n)}{g_{1,\lambda}(N, N - s)} = q^{-n^2/2+n/2} (-\lambda q^n)^{-n} (q^{s-N}; q)_n, \]

and the Rogers-Ramanujan celebrated identity is

\[ 1\Psi_1(x, y; z) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} \frac{(x; q)_n}{(y; q)_n} z^n = \]

\[ = \frac{(q; q)_\infty (y/x; q)_\infty (xz; q)_\infty (q/xz; q)_\infty }{(y; q)_\infty (q/x; q)_\infty (z; q)_\infty (y/xz; q)_\infty }, \]

where the series for $1\Psi_1$ is convergent in

\[ \left| \frac{y}{x} \right| < |z| < 1. \]
The results of summations are:
\[ F_{y,x}(N_1, N_2, u^2) = g_{1,\lambda}(N_1, N_1 - s_y) \times \times \times \times 1 \psi_1(q^{s_y-N_1}, q^{1+N_2}; q^{1/2+N_2-s_y} y) \times \times \times \times 1 \psi_1(q^{s_y-N_1}, q^{1+N_2}; q^{1/2+N_2-s_y} y) . \]
(104)
and
\[ \tilde{F}_{x,y}(N_1, N_2, u^2) = g_{1,\lambda}(N_1, N_1 - s_x) \times \times \times \times 1 \psi_1(q^{s_y-N_1}, q^{1+N_2}; q^{1/2+N_2-s_y} y) . \]
(105)
Put now \( y^2 = q^{-1} x^2 \), then it appeared
\[ F_{1,2}(N_1 + N_2, s_x) \overset{def}{=} F_{y,x}(N_1, N_2, u^2) = - \tilde{F}_{x,y}(N_1, N_2, u^2) , \]
where \( y^2 = q^{-1} x^2 \), and
\[ F_{1,2}(N_1 + N_2, s_x) = q^{N_2(N_2-s_x)+N_1(N_1-s_x+1)} \times \times \times \times 1 \psi_1(q^{s_y-N_1}, q^{1+N_2}; q^{1/2+N_2-s_y} y) . \]
(106)
Here it is used the \( \theta \)-function notation
\[ \Theta(x) = (x; q) \infty (q x^{-1}; q) \infty (q; q) \infty = \sum_{n \in \mathbb{Z}} (-x)^n q^{n(n-1)/2} , \]
(107)
such as
\[ \Theta(q^k x) = (-x)^{-k} q^{-k(k-1)/2} \Theta(x) , \ \Theta(x^{-1}) = -x^{-1} \Theta(x) , \]
(108)
Indeed, due to the equidistance of \( N_1 \) and \( N_2 \), the \( F_{1,2} \) depends only on \( N_1 + N_2 \).
Now we may calculate the wronskian. By definition, it is
\[ W(x^2)_{1,2} = Q_1(q^{-1} x^2) Q_2(x^2) - Q_1(x^2) Q_2(q^{-1} x^2) . \]
(109)
Considering the monodromies of \( Q_1 \) and \( Q_2 \), standing in the definition of the wronskian and using eq. (89), one may see that the most parts in the subtraction (109) are canceled. Only possible exception is the subspace \( \delta w : a_1^+ a_2^- = -1 \). So to calculate the wronskian, one has to take a trace only over this subspace. In general, let \( \zeta_1 \) and \( \zeta_2 \) be the characteristics of \( N_1 \) and \( N_2 \) respectively. Then using the definition of
\( \mathcal{F} \) and \( \tilde{\mathcal{F}} \), eqs. (106) equivalence of \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \), one may conclude:

\[
W_{1,2} = \Xi_{1,2} \sum_{n_f \in \zeta_1 + \zeta_2 + \mathbb{Z}} \left( \prod_f \mathcal{F}_{1,2}(n_f, s_f) \right) \left( \prod_f (uv)^{n_f-n_f-1} \right),
\]

where \( \Xi_{1,2} \) is an extra multiplier that may come from the subtraction, \( \delta_W \)-trace definition and so on. Nevertheless, considering the case \( s_x = \zeta_1 \) modulo \( \mathbb{Z} \) and \( \zeta_2 = 0 \), one obtains the following useful form of \( \mathcal{F}_{1,2} \):

\[
\mathcal{F}_{1,2}(n+s-1, s) = (-)^n q^{n(n-1)/2} \frac{\lambda^n}{(q; q)_n} \frac{\lambda^{-s} \Theta(q^s)}{(q; q)_\infty}.
\]

Extracting now the shift operator as it is described in eq. (83), one obtains relation (25). Extra multiplier is equal to unity, this we have checked by a series expansion with respect to \( \lambda \).

5. Discussion

The technique and results, given in this paper, are rather formal. We have dealt with the single Weyl pair in each site of the lattice, and \( q \) is an arbitrary complex number inside the unit circle. It is well known, this regime is absolutely non-physical, and thus the results presented are to be considered as just an exercise in the field of \( q \)-combinatorial analysis. But, nevertheless, some applications of the results and technique presented may be found.

Talking about the Weyl algebra, people usually keep in mind two aspects: the first one implies the dualization and \( q = \exp\{i \pi e^{i\theta}\} \), and the second one is the finite state \( q = e^{2\pi i/N} \). Our experience in the Weyl algebra exercises says that most our results, especially contained the \( q \)-dilogarithms and permutations, may be immediately rewritten in the dualized form. In this way the results may be applied to the physical relativistic Toda chain, \([17]\). It will be done in a separate paper.

The second aspect is also valid, especially in the part of the technique derived. Preliminary considerations show that at the root of unity the model contains the Baxter curve for the Chiral Potts model, the point on Baxter’s curve is the spectral parameter of \( Q \)-operator, our constant parameter \( \lambda \) is connected with the modulus of Baxter’s curve.

\[\text{This paper suggests the third aspect, applied in the backward direction yet: several Toda-chain-type models, physical as well, may be obtained from a model with arbitrary } q \text{ in the limit } q \rightarrow 1 + \hbar, \text{ regarded in a special way, such that a rational Weyl algebra mapping is linearized with respect to one of Weyl generators in the first order of } \hbar.\]
Remarkable is that in the relativistic Toda chain at the root of unity there appears only one point at Baxter’s curve, while in the Chiral Potts model such point lives at each site on the spin chain. This fact makes the relativistic Toda chain much more simple than CPM itself. This model will be considered in a separate paper.

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