Elasto-limited plastic analysis of structures for probabilistic conditions

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Abstract. With applying plastic analysis and design methods, significant saving in material can be obtained. However, as a result of this benefit excessive plastic deformations and large residual displacements might develop, which in turn might lead to unserviceability and collapse of the structure. In this study, for deterministic problem the residual deformation of structures is limited by considering a constraint on the complementary strain energy of the residual forces. For probabilistic problem the constraint for the complementary strain energy of the residual forces is given randomly and critical stresses updated during the iteration. Limit curves are presented for the plastic limit load factors. The results show that these constraints have significant effects on the load factors. The formulations of the deterministic and probabilistic problems lead to mathematical programming which are solved by the use of nonlinear algorithm.

1. Introduction
The basic aim of plastic limit analysis is to calculate the plastic limit load factors and the stresses, strain rates at the plastic limit condition of the structures. The strong preference of this principle is due to the fact that, without a study of the entire loading history, the principal directly determine the value of the upper and lower boundaries of the plastic limit load factors. At the plastic limit analysis stresses can be solved using a static equilibrium equation with the limited plastic load and, meanwhile, satisfy the yield condition at any part in the structure.

2. Loading and limit state of elasto-plastic structures
Let us consider a body what is subjected to multiparameter quasi-static load on its surfaces. This loading can be given in separable form:

$$Q_i(x_i, t) = \sum_j m_i(t) Q_{j0}(x_i). \quad (j = 1, 2, ..., n) \quad (1)$$

Here the independent group of forces $Q_{j0}(x_i); (j = 1, 2, ..., n)$ do not depend on time. The time dependency is given by the associated load multipliers $m_i(t)$. The loading process is represented by a curve in loading space $m_1, m_2, ..., m_j$. These multipliers can vary independently, but they are limited by the following equation:

$$L(m_j) = 0, \quad (2)$$

Equation (2) defines all the possible combinations of the external loads which can occur during the
loading process. In case of two-parameter loading the loading process and the loading domain are presented in figure 1. Here the formulation \( L(m_1, m_2) = 0 \) means all the combined values of the load intensities \( m_1 \) and \( m_2 \) where the structure is in plastic limit state. This equation is represented by a closed curve in coordinate system \( m_1, m_2 \) and it can be proved that it is convex Kaliszky [1].

![Figure 1. Loading process and loading domain.](image)

In the literature there are several computational methods which provide appropriate tools to determine the limit of the accumulated permanent deformations (see e.g. Maier [2]; Polizzotto [3]; König [4]; Weichert and Maier [5]; Levy et al [6]; Simon and Weichert [7]; Ponter [8]; Corradi [9]; Capurso et al [10]; Kaneko and Maier [11]; Tin-Loi [12] and Liepa et al [13]). Also an appropriate computational procedure is presented which based on the fact that the permanent deformations and residual stresses are not independent, the complementary strain energy of the residual stresses can approximately describe the plastic behaviour of structure and the amount of the plastic deformations, respectively (Kaliszky and Lógó [14]; Movahedi and Lógó [15]; Lógó et al [16] and Movahedi [17]). Therefore if the complementary strain energy of the residual stresses is bounded, in the same time it can limit the amount of the permanent deformations of the structure.

In the case of truss element the complementary strain energy of the residual stresses can be expressed as follow:

\[
C_p = \frac{1}{2E} \sum_{k=1}^{n} \frac{l_k}{A_k} (S_k^r)^2; \tag{3}
\]

in the case of residual moments:

\[
C_p = \frac{1}{2E} \sum_{k=1}^{n} \frac{1}{l_k} \int_0^{l_k} (M_k^r(s))^2 \, ds. \tag{4}
\]

Here \( l_k, (k = 1, 2, ..., n) \) denotes the length of the bars or beam members, \( A_k \) and \( l_k \) are the area and moment of inertia of the bar and beam elements, respectively, \( S_k^r \): the residual normal force of the bar members, \( M_k^r(s) \): the residual moment of the beam elements, \( E \): the Young’s modulus. The integral expression in equation (4) can be defined in case of the residual moments \( M_{k1}^r(s) \) and \( M_{k2}^r(s) \) acting at the ends of the beam members as:

\[
R_k^r = \frac{1}{3} [(M_{k1}^r)^2 + (M_{k1}^r)(M_{k2}^r) + (M_{k2}^r)^2], \tag{5}
\]

for sake of simplicity, \( R_k^r \) denotes the residual force of the element, \( D_k \) is the area or the moment of inertia for each member. The general form of equation(3) and equation (4) can be written as follows:

\[
C_p = \frac{1}{2E} \sum_{k=1}^{n} \frac{l_k}{A_k} (R_k^r)^2. \tag{6}
\]

Applying the expressions above the amount of the plastic deformations can be limited if a permissible energy value \( C_{p0} \) is introduced. Permissible energy value can not exceed by the complementary strain energy of the residual stresses, the inequality can be written as follows:
\[ \frac{1}{2E} \sum_{k=1}^{n} \frac{1}{D_k} (R_k^r)^2 \leq C_{p0}. \] (7)

### 2.1. Probabilistic problem

Reliability methods aim at evaluating the probability of failure of a system whose modelling takes into account randomness. The probability function for a random variable \( X \) can be defined by the following integral:

\[ P_f = \int f(X)dx. \] (8)

Since uncertainties have significant role in engineering, the constraints on the complementary strain energy of the residual forces are considered randomly and for the reason of simplicity normal distribution function applied. Mean value: \( C_{p0} \) and standard deviation: \( S \). The probability of the failure function can be constructed as follow:

\[ P_{f,\text{target}} = \int f(C_{p0},S)dx. \] (9)

Probabilistic bound can be defined with introducing reliability index \( \beta \):

\[ \beta_{\text{target}} - \beta_{\text{calc}} \leq 0; \] (10)

where \( \beta_{\text{target}} \) is working as a constraint for \( \beta_{\text{calc}} \). Reliability indexes can be defined as follows:

\[ \beta_{\text{target}} = -\Phi^{-1}(P_{f,\text{target}}); \] (11)

\[ \beta_{\text{calc}} = -\Phi^{-1}(P_{f,\text{calc}}); \] (12)

in equations (11) and (12) \( \Phi \) is cumulative distribution function of the normal distribution.

### 3. Elasto-limited plastic analysis

Consider any statically admissible internal force \( Q^s \) fields which satisfy equilibrium equations and define to them a statically admissible load multiplier \( m_s \), the following equilibrium equation can be constructed:

\[ GQ^s + m_s P_0 = 0; \] (13)

\( G \): equilibrium matrix and \( P_0 \): base load vector.

Equation (14) defines the yields condition:

\[ -Q^p \leq Q^s \leq Q^p; \] (14)

\( Q^p \): limit plastic force. The following equation defines elastic fictitious internal force:

\[ Q^e = T^{-1}GK^{-1}m_s P_0; \] (15)

Here \( T \): flexibility matrix and \( K \): stiffness matrix. The static principle of limit analysis states that: any statically admissible stable load multiplier \( m_s \) of the elasto-plastic structures is less than or equal to the collapse load multiplier \( m_p \), i.e.:

\[ m_s \leq m_p. \] (16)

Since the exact value of \( m_p \) is equal to the maximum value of \( m_s \), the problem based on the static principle can be define as follows:

For deterministic problems residual plastic behavior of structures are limited with applying permissible energy value \( C_{p0} \):
In the case of probabilistic problems bound can be defined with introducing reliability index \( \beta \):

\[
\begin{align*}
    m_s &= \max \\
    GQ^s + m_s P_0 &= 0 \\
    -Q^p \leq Q^s \leq Q^p \\
    Q^e &= T^{-1} G K^{-1} m_s P_0 \\
    \frac{1}{2E} \sum_{k=1}^{n} \frac{l_k}{D_k} (R_k^0)^2 &\leq C_{p0}
\end{align*}
\]

(17)

4. Numerical examples and results

4.1. Example 1
Consider the skeletal frame illustrated in figure 2. The structure is subjected to horizontal loads \( P_1 \geq 0 \) and vertical loads \( P_2 \geq 0 \). The elastic modulus \( E = 21 \cdot 10^4 \text{ N/mm}^2 \) and allowable stress \( f_y = 210 \text{ N/mm}^2 \). The moment of inertia of the beams and columns \( I = 19.43 \cdot 10^6 \text{ mm}^4 \) and \( 38.91 \cdot 10^6 \text{ mm}^4 \), respectively.

The results of the numerical example in case of two-parameter \((m_1, m_2)\) loading domains for different permissible energy value \( C_{p0} \) are presented in figure 3 for probabilistic conditions.

4.2. Example 2
The second numerical example shows a long steel pile structure in cohesion less soil. In this example two loading parameter \((m_1, m_2)\) are taken into consideration for different cross-sections (figure 4). The elastic modulus \( E = 21 \cdot 10^4 \text{ N/mm}^2 \) and allowable stress \( f_y = 210 \text{ N/mm}^2 \).

Figure 5 shows loading domains in case of two-parameter \((m_1, m_2)\) for different cross-sections. Permissible energy value \( C_{p0} = 50 \). In this example elasto-limited plastic analysis are considered for deterministic and probabilistic problems. Standard deviation \( S = 3 \) for probabilistic problems.
5. Conclusions
In this research elasto-limited plastic analysis of two different typical type of steel structures are taken into consideration for deterministic and probabilistic solutions. In both examples, residual plastic behavior of structures are restricted with applying permissible energy value. Loading domains are presented for different given permissible energy values. The numerical examples shows that the bounds on the residual plastic deformation of structures and expected probability of failure have significant effects on the load parameters and loading domains.

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