MOTIVIC MEASURES AND $\mathbb{F}_1$-GEOMETRIES

LIEVEN LE BRUYN

Abstract. Right adjoints for the forgetful functors on $\lambda$-rings and bi-rings are applied to motivic measures and their zeta functions on the Grothendieck ring of $\mathbb{F}_1$-varieties in the sense of Lorscheid and Lopez-Pena (torified schemes). This leads us to a specific subring of $W(\mathbb{Z})$, properly containing Almkvist’s ring $W_0(\mathbb{Z})$, which might be a natural receptacle for all local factors of completed zeta functions.

1. Introduction

In [2] Jim Borger proposes to consider integral $\lambda$-rings as $\mathbb{F}_1$-algebras, with the $\lambda$-structure viewed as the descent data from $\mathbb{Z}$ to $\mathbb{F}_1$. Crucial is the fact that the functor of forgetting the $\lambda$-structure has the Witt-ring functor $W(\mathbb{Z})$ as its right adjoint. Recall that the $\lambda$-ring $W(\mathbb{Z}) = 1 + t\mathbb{Z}[[t]]$ has addition ordinary multiplication of power series, and a new multiplication induced functorially by demanding that $(1 - mt)^{-1} \ast (1 - nt)^{-1} = (1 - mnt)^{-1}$. We will view $W(\mathbb{Z})$ as a receptacle for motivic data, such as zeta-functions.

A counting measure is a ringmorphism $\mu : K_0(\text{Var}_\mathbb{Z}) \longrightarrow \mathbb{Z}$, with $K_0(\text{Var}_\mathbb{Z})$ the Grothendieck ring of schemes of finite type over $\mathbb{Z}$. A classic example being $\mu_{\mathbb{F}_p}([X]) = \# X_p(\mathbb{F}_p)$ where $X_p$ is the reduction of $X$ modulo $p$. The $\mathbb{F}_p$-counting measure $\mu_{\mathbb{F}_p}$ is exponentiable meaning that it defines a ringmorphism

$$\zeta_{\mathbb{F}_p} : K_0(\text{Var}_\mathbb{Z}) \longrightarrow W(\mathbb{Z})$$

$[X] \mapsto \zeta_{\mathbb{F}_p}(X_p, t) = \exp(\sum_{r \geq 1} \# X_p(\mathbb{F}_p^{2^r}) t^r/r)$

and is rational, meaning that $\zeta_{\mathbb{F}_p}$ factors through the Almkvist subring $W_0(\mathbb{Z})$ of $W(\mathbb{Z})$, consisting of all rational functions.

For a scheme $X$ of finite type over $\mathbb{Z}$, let $N(x)$ for every closed point $x \in |X|$ be the cardinality of the finite residue field at $x$, then the Hasse-Weil zeta function of $X$ decomposes as a product

$$\zeta_X(s) = \prod_{x \in |X|} \frac{1}{(1 - N(x)^{-s})} = \prod_p \zeta_{\mathbb{F}_p}(X_p, p^{-s})$$

over the non-archimedean local factors. If we take the product with the archimedean factors ($\Gamma$-factors) we obtain the completed zeta function $\hat{\zeta}_X(s)$.

One of the original motivations for constructing $\mathbb{F}_1$-geometries was to understand these $\Gamma$-factors, see the lecture notes [20] by Yuri I. Manin. For example, Manin conjectured that Deninger’s $\Gamma$-factor $\prod_{n \geq 0} \frac{e^{-n}}{2\pi} \zeta(\overline{\text{Spec}(\mathbb{Z})})$ at complex infinity
should be the zeta function of (the dual of) infinite dimensional projective space $\mathbb{P}_1^\infty$, see [19 4.3] and [21 Intro].

As a step towards this conjecture, we proposed in [14] to consider integral birings as $\mathbb{F}_1$-algebras, this time with the co-ring structure as the descent data from $\mathbb{Z}$ to $\mathbb{F}_1$. Here again, the forgetful functor has a right adjoint with assigns to $\mathbb{Z}$ the bi-ring $L(\mathbb{Z})$ of all integral recursive sequences equipped with the Hadamard product. These two approaches to $\mathbb{F}_1$-geometry are related, that is, we have a commuting diagram of (solid) ringmorphisms (dashed morphisms are explained below)

$$
\begin{array}{ccc}
\mathbb{W}_0(\mathbb{Z}) & \xrightarrow{\zeta_{\mathbb{F}_1}} & \mathbb{M}(\mathbb{Z}) \\
\downarrow & & \downarrow \\
\mathbb{Z}[L] & \xrightarrow{\zeta_{\mathbb{F}_1}} & L(\mathbb{Z}) \\
\downarrow & & \downarrow \\
& & \mathbb{Z}_1
\end{array}
$$

with the ghost-map $\delta = t \frac{d}{dt} \log(-)$ and $\mathbb{M}(\mathbb{Z})$ the pull-back of $\delta$ and the natural inclusion map $i$. One might speculate that the relevant counting measures $\mu : K_0(\text{Var}_{\mathbb{Z}}) \rightarrow \mathbb{Z}$ are those which determine a ring-morphism $\zeta_{\mu} : K_0(\text{Var}_{\mathbb{Z}}) \rightarrow \mathbb{M}(\mathbb{Z})$, with those factoring over $\mathbb{W}_0(\mathbb{Z})$ corresponding to the non-archimedean factors, and the remaining ones related to the $\Gamma$-factors.

This is motivated by our description of the $\mathbb{F}_1$-zeta function of Lieber, Manin and Marcolli in [15]. Here, one considers integral schemes with a decomposition into tori $G^n_m$ as $\mathbb{F}_1$-varieties and with morphisms respecting the decomposition and with all restrictions to tori being morphisms of group schemes. The corresponding Grothendieck ring $K_0(\text{Var}_{\mathbb{F}_1})$ can then be identified with the subring $\mathbb{Z}[L]$ of $K_0(\text{Var}_{\mathbb{F}_1})$. Kapranov’s motivic zeta function induces a natural $\lambda$-ring structure on $\mathbb{Z}[L]$ and we can also define a bi-ring structure on it by taking $D = L - 2$ to be a primitive generator. By right adjointness we then have natural one-to-one correspondences

$$
\text{comm}^+_{bi}(\mathbb{Z}[L], L(\mathbb{Z})) \leftrightarrow \text{comm}(\mathbb{Z}[L], \mathbb{Z}) \leftrightarrow \text{comm}^+_{bi}(\mathbb{Z}[L], \mathbb{W}(\mathbb{Z}))
$$

To a counting measure $\mathbb{L} \rightarrow m$ corresponds a $\lambda$-ring morphism $\zeta_{m} : \mathbb{Z}[L] \rightarrow \mathbb{W}(\mathbb{Z})$ which factors through $\mathbb{W}_0(\mathbb{Z})$ and coincides with $\zeta_{\mathbb{F}_1}$ when $m = p$. If $X$ is an integral scheme with toric decomposition, its $\mathbb{F}_1$-zeta function is defined to be the ringmorphism

$$
\zeta_{\mathbb{F}_1} : \mathbb{Z}[L] \rightarrow \mathbb{W}(\mathbb{Z}) \quad \zeta_{\mathbb{F}_1}(X, t) = \exp(\sum_{r \geq 1} \#X(\mathbb{F}_1^m) \frac{t^r}{r})
$$

with $\#X(\mathbb{F}_1^m)$ being the total number of $m$-th roots of unity in the tori making up $X$, see [15]. This $\zeta_{\mathbb{F}_1}$ is not a $\lambda$-ring morphism and does not factor through $\mathbb{W}_0(\mathbb{Z})$. However, the counting measure $\mathbb{L} \rightarrow 3$ corresponds to a bi-ring morphism $c_{\mathbb{F}_1} : \mathbb{Z}[L] \rightarrow L(\mathbb{Z})$ which factors through $\mathbb{M}(\mathbb{Z})$ and such that the composition with $\mathbb{M}(\mathbb{Z}) \rightarrow \mathbb{W}(\mathbb{Z})$ is the zeta-morphism $\zeta_{\mathbb{F}_1}$.
1.1. **Structure of this paper.** In section 2 we use right adjointness of the functor $W(-)$ to give quick proofs of the facts that the pre $\lambda$-structure on $K_0(Var_C)$ given by Kapranov’s motivic zeta function does not define a $\lambda$-ring structure, and that its universal motivic measure is not exponentiable.

In section 3 we relate the versions of $F_1$-geometry determined by $\lambda$-rings resp. biring-morphisms to the concrete resp. abstract Bost-Connes systems associated to cyclotomic Bost-Connes data as in [24]. This allows to have relative versions of $W_0(Z)$ and $L(Z)$ by imposing conditions on the eigenvalues of actions of Frobenii on (co)homology or on the roots and poles of zeta-polynomials.

In section 4 we study counting measures on the Grothendieck ring of torified integral schemes, proving the results mentioned above. It turns out that the pull-back $M(Z)$ of $W(Z)$ and $L(Z)$ might be the appropriate receptacle for local factors of zeta functions of integral schemes. These results can be extended to other subrings of $K_0(Var_Z)$ which are $\lambda$-rings and admit a bi-ring structure.

In section 5 we introduce the category of all linear dynamical systems which plays the same role for $L(Z)$ as does the endomorphism category for $W_0(Z)$. To completely reachable systems we associate their transfer functions which are strictly proper rational functions. As such, these systems may be relevant in the study of zeta-polynomials, as introduced by Manin in [21].

**Acknowledgements** This paper owes much to recent work of Yuri I. Manin, Matilde Marcolli and co-authors, [23], [15] and [24]. Unconventional symbols are taken from the $\LaTeX$-package **halloweenmath** [25], befitting the current topic.

2. **Motivic measures on $K_0(Var_k)$**

Let $Var_k$ be the category of varieties over a field $k$. The Grothendieck ring $K_0(Var_k)$ is the quotient of the free abelian group on isomorphism classes $[X]$ of varieties by the relations $[X] = [Y] + [X - Y]$ whenever $Y$ is a closed subvariety of $X$, and multiplication is induced by products of varieties, that is, $[X][Y] = [X \times Y]$. As the structure of $K_0(Var_k)$ is fairly mysterious, we try to probe its properties via motivic measures.

**Definition 1.** A motivic measure on $K_0(Var_k)$ with values in a commutative ring $R$ is a ringmorphism

$$\mu : K_0(Var_k) \rightarrow R$$

The archetypical example of a motivic measure on the Grothendieck ring of varieties over a finite field $F_q$ is the **counting measure** with values in $\mathbb{Z}$

$$\mu_{\mathbb{Z}_q} : K_0(Var_{\mathbb{Z}_q}) \rightarrow \mathbb{Z} \quad [X] \mapsto \# X(F_q)$$

An example of a motivic measure on the Grothendieck ring of complex varieties $K_0(Var_C)$ with values in $\mathbb{Z}$ is the **Euler characteristic measure**

$$\chi_c : K_0(Var_C) \rightarrow \mathbb{Z} \quad [X] \mapsto \chi_c(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} H^i_c(X^{an}, \mathbb{Q})$$

There are plenty of motivic measures with values in other rings such as the **Hodge characteristic measure** $\mu_{H}$ with values in $\mathbb{Z}[u, v]$, see [10] §4.1, the **Poincaré characteristic measure** $P_X$ with values in $\mathbb{Z}[u]$, see [16] §4.1, the **Gillet-Soulé measure** $\mu_{GS}$ with values in the Grothendieck ring if Chow motives, see [6].
Of particular importance to us are the 'exotic' Larsen-Lunts measure $\mu_{LL}$ on $K_0(\text{Var}_k)$ with values in the quotient field of the monoid ring $\mathbb{Z}[C]$ with $C$ the multiplicative monoid of polynomials in $\mathbb{Z}[t]$ with positive leading coefficient, see [12], and the universal motivic measure, which is the identity morphism $id : K_0(\text{Var}_k) \to K_0(\text{Var}_k)$.

For a commutative ring $R$ let $\mathbb{W}(R)$ be the set $1+tR[[t]]$ of all formal power series over $R$ with constant term equal to one, and let multiplication of formal power series be the addition on $\mathbb{W}(R)$. We say that $R$ admits a pre $\lambda$-structure if there exists a morphism of additive groups

$$\lambda_t : R \to \mathbb{W}(R) = 1 + tR[[t]] \quad a \mapsto \lambda_t(a) = 1 + at + \ldots = \sum_{m \geq 0} \lambda^m(a)t^m$$

that is, it satisfies $\lambda_0(a) = 1$, $\lambda_1(a) = a$, and

$$\lambda_t(a + b) = \lambda_t(a)\lambda_t(b) \quad \text{that is} \quad \lambda^m(a + b) = \sum_{i+j=m} \lambda^i(a)\lambda^j(b)$$

Given a pre $\lambda$-structure $\lambda_t$ on $R$ we can define the Adams operations $\Psi_m$ on $R$ via

$$\frac{t}{d} \log(\lambda_t(a)) = t - \frac{1}{\lambda_t(a)} \frac{d\lambda_t(a)}{dt} = \sum_{m \geq 1} \Psi_m(a)t^m$$

and note that for all $m \in \mathbb{N}$ and all $a, b \in R$ we have $\Psi_m(a + b) = \Psi_m(a) + \Psi_m(b)$. We say that a pre $\lambda$-ring $R$ is a $\lambda$-ring if for all $m, n \in \mathbb{N}$ we have these conditions on the Adams operations

$$\Psi_m(a, b) = \Psi_m(a)\Psi_m(b) \quad \text{and} \quad \Psi_m \circ \Psi_n = \Psi_{mn} \circ \Psi_m$$

Equivalently, if we define a multiplication $\ast$ on $\mathbb{W}(R)$ induced by the functorial requirement that $(1-at)^{-1} \ast (1-bt)^{-1} = (1-abt)^{-1}$ for all $a, b \in R$, then the map $\lambda_t$ is a morphism of rings. For more on $\lambda$-rings, see [9], [11] and [33].

A morphism $\phi : (R, \lambda_t) \to (R', \lambda'_t)$ between two $\lambda$-rings is a ringmorphism such that for all $a \in R$ we have that $\lambda'_t(\phi(a)) = \mathbb{W}(\phi)(\lambda_t(a))$ where $\mathbb{W}(\phi)$ is the map on $\mathbb{W}(R) = 1 + tR[[t]]$ induced by $\phi$. With $\text{comm}_L^\pm$ we will denote the category of all (commutative) $\lambda$-rings. If $\text{comm}$ is the category of all commutative rings, then

$$\mathbb{W} : \text{comm} \to \text{comm}_L^\pm \quad A \mapsto \mathbb{W}(A)$$

is a functor, which is right adjoint to the forgetful functor $F : \text{comm}_L^\pm \to \text{comm}$. That is, for every $\lambda$-ring $(R, \lambda_t)$ and every commutative ring $A$ we have a natural one-to-one correspondence

$$\text{comm}_L^\pm(R, \mathbb{W}(A)) \leftrightarrow \text{comm}(R, A) \quad \phi \leftrightarrow \bar{\phi} \circ \phi$$

with the ghost components $\bar{\phi}_m : \mathbb{W}(A) \to A$ defined by

$$t \frac{1}{P} \frac{dP}{dt} = \sum_{m=1}^{\infty} \bar{\phi}_m(P)t^m \quad \text{for all} \ P \in \mathbb{W}(A) = 1 + tA[[t]]$$

Kapranov’s motivic zeta function $\zeta$ defines a natural pre $\lambda$-structure on $K_0(\text{Var}_k)$

$$\zeta : K_0(\text{Var}_k) \to \mathbb{W}(K_0(\text{Var}_k)) \quad [X] \mapsto \zeta_X(t) = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots$$

where $S^nX = X^n/S_n$ is the $n$-th symmetric product of $X$. 
Definition 2. A motivic measure $\mu : K_0(\text{Var}_k) \longrightarrow R$ with values in $R$ is said to be exponentiable if the uniquely determined map $\zeta_\mu : K_0(\text{Var}_k) \longrightarrow \mathbb{W}(R)$ by

$$\zeta_\mu([X]) = 1 + \mu([X])t + \mu([S^2X])t^2 + \mu([S^3X])t^3 + \ldots$$

is a ringmorphism.

Again, the archetypical example being the counting measure $\mu_{\mathbb{F}_q}$ on $K_0(\text{Var}_{\mathbb{F}_q})$ which is exponentiable, with corresponding zeta-function

$$\zeta_{\mu_{\mathbb{F}_q}} : K_0(\text{Var}_{\mathbb{F}_q}) \longrightarrow \mathbb{W}(\mathbb{Z}) \quad \zeta_{\mu_{\mathbb{F}_q}}([X]) = \sum_{m=1}^{\infty} \#X(\mathbb{F}_q^m)t^m = Z_{\mathbb{F}_q}(X, t)$$

the classical Hasse-Weil zeta function, see [26, Prop. 8] or [29, Thm. 2.1]. Also the Euler characteristic measure on $K_0(\text{Var}_C)$ is exponentiable with corresponding zeta function

$$\zeta_{\mu_c} : K_0(\text{Var}_C) \longrightarrow \mathbb{W}(\mathbb{Z}) \quad \zeta_{\mu_c}([X]) = \frac{1}{(1-t)^{\chi_c(X)}}$$

However, as shown in [30, §4] the Larsen-Luntz motivic measure $\mu_{\text{LL}}$ on $K_0(\text{Var}_C)$ is not exponentiable. For this would imply that

$$\zeta_{\mu_{\text{LL}}}(C_1 \times C_2) = \zeta_{\mu_{\text{LL}}}(C_1) * \zeta_{\mu_{\text{LL}}}(C_2)$$

for any pair of projective curves $C_1$ and $C_2$. Kapranov proved that $\zeta_c(C)$ is a rational function for every curve and every motivic measure, which would imply that $\mu_{\text{LL}}(C_1 \times C_2)$ would be rational too, by [30, Prop. 4.3], which contradicts [12, Thm 7.6] in case $C_1$ and $C_2$ have genus $\geq 1$.

It is a natural to ask whether the pre $\lambda$-structure on $K_0(\text{Var}_k)$ defined by Kapranov’s motivic zeta function defines a $\lambda$-ring structure on $K_0(\text{Var}_k)$, see [29, §3 Questions] or [7, §2.2]. The following is well-known to the experts, but we cannot resist including the short proof.

Proposition 1. If Kapranov’s motivic zeta function makes $K_0(\text{Var}_k)$ into a $\lambda$-ring, then every motivic measure $\mu : K_0(\text{Var}_k) \longrightarrow R$ is exponentiable.

As a consequence, Kapranov’s zeta function does not define a $\lambda$-ring structure on $K_0(\text{Var}_C)$.

Proof. If $K_0(\text{Var}_k)$ is a $\lambda$-ring, then by right adjunction of $\mathbb{W}(-)$ with respect to the forgetful functor, we have a natural one-to-one correspondence

$$\text{comm}(K_0(\text{Var}_k), R) \leftrightarrow \text{comm}_\lambda^+(K_0(\text{Var}_k), \mathbb{W}(R))$$

and under this correspondence the motivic measure $\mu$ maps to a unique $\lambda$-ring morphism $\zeta_\mu : K_0(\text{Var}_k) \longrightarrow \mathbb{W}(R)$.

Because the Larsen-Luntz motivic measure $\mu_{\text{LL}}$ on $K_0(\text{Var}_C)$ is not exponentiable, it follows that $K_0(\text{Var}_C)$ cannot be a $\lambda$-ring. □

Another immediate consequence is this negative answer to [29, §3 Questions].

Proposition 2. The universal motivic measure on $K_0(\text{Var}_C)$ is not exponentiable.
Proof. By functoriality, any motivic measure $\mu : K_0(\text{Var}_C) \to R$ gives rise to a morphism of $\lambda$-rings $\mathbb{W}(\mu) : \mathbb{W}(K_0(\text{Var}_C)) \to \mathbb{W}(R)$.

If the universal measure would be exponentiable, this would give a ringmorphism $\zeta : K_0(\text{Var}_C) \to \mathbb{W}(K_0(\text{Var}_C))$ and composition $\mathbb{W}(\mu) \circ \zeta : K_0(\text{Var}_C) \to \mathbb{W}(R)$ would then imply that $\mu$ is exponentiable, which cannot happen for $\mu_{LL}$. □

An important condition on a motivic measure $\mu : K_0(\text{Var}_k) \to R$ is its rationality. In order to define this, we need to recall the endomorphism category and its Grothendieck ring, see [1] and [8].

For a commutative ring $R$ consider the category $\mathcal{E}_R$ consisting of pairs $(E,f)$ where $E$ is a projective $R$-module of finite rank and $f$ is an endomorphism of $E$. Morphisms in $\mathcal{E}_R$ are module morphisms compatible with the endomorphisms. There is a duality $(E,f)\leftrightarrow (E^*,f^*)$ on $\mathcal{E}_R$ and we have $\oplus$ and $\otimes$ operations

$$(E_1,f_1) \oplus (E_2,f_2) = (E_1 \oplus E_2,f_1 \oplus f_2) \quad (E_1,f_1) \otimes (E_2,f_2) = (E_1 \otimes E_2,f_1 \otimes f_2)$$

with a zero object $0 = (0,0)$ and a unit object $1 = (R,1)$. These operations turn the Grothendieck ring $K_0(\mathcal{E}_R)$ into a commutative ring, having an ideal consisting of the pairs $(E,0)$, with quotient ring $\mathbb{W}_0(R)$.

The ring $\mathbb{W}_0(R)$ comes equipped with Frobenius ring endomorphisms $Fr_n(E,f) = (E,f^n)$, Verschiebung additive maps

$$V_n(E,f) = (E^{\oplus n}, \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & f \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix})$$

and ghost ringmorphisms $\delta_n(E,f) = Tr(f^n) : \mathbb{W}_0(R) \to R$. For various relations among the maps $Fr_n, V_n$ and $\delta_n$, see for example [1] Prop. 2.2.

The connection between Almkvist’s functor $\mathbb{W}_0(-)$ and $\mathbb{W}(-)$ is given by the ringmorphisms

$$L_R : \mathbb{W}_0(R) \to \mathbb{W}(R) \quad L_R(E,f) = \frac{1}{\det(1-tM_{\tilde{f}})}$$

where $M_{\tilde{f}}$ is the matrix associated to $f$ (that is, if $f = \sum_i x_i^* \otimes x_i \in \text{End}_R(E) = E^* \otimes E$, then $M_{\tilde{f}} = (a_{ij})_{i,j}$ with $a_{ij} = x_i^*(x_j)$). By [1] Thm 6.4] we know that $L_R$ is injective with image all rational formal power series of the form

$$\frac{1 + a_1 t + \ldots + a_n t^n}{1 + b_1 t + \ldots + b_m t^m} \quad a_i, b_i \in R, m, n \in \mathbb{N}.$$ 

**Definition 3.** We say that a motivic measure $\mu : K_0(\text{Var}_k) \to R$ is rational if it is exponentiable and if the corresponding zeta-function $\zeta_\mu$ factors through $\mathbb{W}_0(RT)$. That is, there is a unique ringmorphism

$$r_\mu : K_0(\text{Var}_k) \to \mathbb{W}_0(R)$$

such that $\zeta_\mu = L_R \circ r_\mu$. 

By a classic result of Dwork we know that the counting measure $\mu_{F_q}$ is rational, as is the Euler characteristic measure $\mu_c$.

3. Cyclotomic Bost-Connes data

Let $R$ be an integral domain with field of fractions $K$ of characteristic zero and with algebraic closure $\overline{K}$. Let $\mathbb{K}_\times$ be the multiplicative group of all non-zero elements and $\mu_\infty$ the subgroup consisting of all roots of unity. The power maps $\sigma_n : x \mapsto x^n$ for $n \in \mathbb{N}_+$ form a commuting family of endomorphisms of $\mathbb{K}_\times$ and its subgroups. Following M. Marcolli en G. Tabuada in [24] we define:

**Definition 4.** A cyclotomic Bost-Connes datum is a divisible subgroup $\Sigma \subseteq \mathbb{K}_\times$ stable under the action of the Galois group $G = \text{Gal}(\overline{K}/K)$.

The subgroup $\Sigma$ should be considered as ‘generalised’ Weil numbers (recall that for each prime power $q = p^r$ the Weil $q$-numbers are an instance, see [24, Example 4]).

Observe that cyclotomic Bost-Connes data are special cases of concrete Bost-Connes data as in [24, Def. 2.3] with the endomorphisms $\sigma_n$ the $n$-th power maps $\sigma_n(x) = x^n$ and $\rho_n(x) = \mu_n \sqrt[n]{x} \subseteq \Sigma$. In [24, §4] Marcolli and Tabuada associate to a cyclotomic Bost-Connes system with $K = \overline{Q}$ a quantum statistical mechanical system. Further, in [24, §2] both concrete and abstract Bost-Connes systems are associated to a cyclotomic Bost-Connes datum $\Sigma$. We will relate these to $F_1$-geometries.

A powerful idea, due to Jim Borger [2] and [3], to construct ‘geometries’ under $\text{Spec}(\mathbb{Z})$ is to consider a subcategory $\text{comm}^\lambda_{\mathbf{X}}$ of commutative rings $\text{comm}$ which allows a right adjoint $R$ to the forgetful functor $F : \text{comm} \longrightarrow \text{comm}$.

The additional structure $\mathbf{X}$ should be thought of as descent data from $\mathbb{Z}$ to $F_1$, the elusive field with one element. As a consequence, the commutative ring $F(R(\mathbb{Z}))$ can then be considered to be the coordinate ring of the arithmetic square $\text{Spec}(\mathbb{Z}) \times_{\text{Spec}(F_1)} \text{Spec}(\mathbb{Z})$.

We propose to view the object $R(\mathbb{Z}) \in \text{comm}^\lambda_{\mathbf{X}}$ as a receptacle for motivic data. That is, (co)homology groups with actions of Frobenii and zeta-functions determine elements in $R(\mathbb{Z})$ and the subobject in $\text{comm}^\lambda_{\mathbf{X}}$ they generate can then be seen as its representative in the corresponding version of $F_1$-geometry.

3.1. Concrete Bost-Connes systems and $\text{comm}^\lambda_{\mathbf{X}}$.

Following [24, Def. 2.6] one associates to $\Sigma$ the concrete Bost-Connes system which consists of the integral group ring $\mathbb{Z}[\Sigma]$ equipped with

1. the induced $G = \text{Gal}(\overline{K}/K)$-action,
2. $G$-equivariant ring endomorphisms $\bar{\sigma}_n$ induced by $\bar{\sigma}_n(x) = x^n$ for all $x \in \Sigma$,
3. $G$-equivariant $\mathbb{Z}$-module maps $\bar{\rho}_n$ induced by $\bar{\rho}_n(x) = \sum_{x' \in \rho_n(x)} x'$ for all $x \in \Sigma$.

**Proposition 3.** For a cyclotomic Bost-Connes datum $\Sigma$, the concrete Bost-Connes system $(\mathbb{Z}[\Sigma], \bar{\sigma}_n, \bar{\rho}_n)$ is a sub-system of $(\mathcal{W}_0(\overline{K}), \text{Fr}_n, V_n)$. 
Proof. From [4, Prop. 2.3] we recall that $\mathcal{W}_0(\mathcal{K})$ is isomorphic to the integral group ring $\mathbb{Z}[\mathcal{K}]$ via the map that assigns to $(E, f)$ the divisor of non-zero eigenvalues of $f$ (with multiplicities).

Under this isomorphism the Frobenius maps $Fr_n$ become $\tilde{\sigma}_n$ and the Ver- 

schiebung $V_n$ the map $\tilde{\rho}_n$ for the cyclotomic Bost-Connes datum $\mathcal{K}_\infty$. 

Definition 5. For a cyclotomic Bost-Connes datum $\Sigma$, let $\mathcal{E}_{\Sigma,R}$ be the full sub-category of $\mathcal{E}_R$ consisting of pairs $(E, f)$ with $E$ a projective $R$-module and $M_f$ a $\mathcal{K}$-diagonalisable matrix having all its eigenvalues in $\Sigma$. With $\mathcal{W}_0(\Sigma, R)$ we denote the subring of $\mathcal{W}_0(R)$ generated by $\mathcal{E}_{\Sigma,R}$.

Example 1. Consider Yuri I. Manin’s idea to replace the action of the Frobenius map on étale cohomology of an $\mathbb{F}_q$-variety at $q = 1$ by pairs $(H_k(M, \mathbb{Z}), f_{sk})$ where $f_{sk}$ is the action of a Morse-Smale diffeomorphism $f$ on a compact manifold $M$ upon its homology $H_k(M, \mathbb{Z})$, [13, §1.2]. This implies that each $f_{sk}$ is quasi-unipotent, that is all its eigenvalues are roots of unity. This fits in with Manin’s view that $1$-Frobenius morphisms acting upon their (co)homology have eigenvalues which are roots of unity.

In [23, §2], Manin and Matilde Marcolli assign an object in $\text{comm}_\Sigma^+$ to the Morse-Smale setting $(M, f)$ as follows. Each $H_k(M, \mathbb{Z})$ is viewed as a $\mathbb{Z}[t, t^{-1}]$-module by letting $t$ act as $f_{sk}$. Next, they consider the minimal category $C_M$ of $\mathbb{Z}[t, t^{-1}]$-modules, containing all $H_k(M, \mathbb{Z})$, and closed with respect to direct sums, tensor products and exterior products. Then, its Grothendieck ring $K_0(C_M)$ comes equipped with a $\lambda$-ring structure coming from the exterior products, which is then said to be the representative of $\{(H_k(M, \mathbb{Z}), f_{sk}); k\}$ in $\mathbb{F}_1$-geometry, see [23, Def. 2.1.2].

Alternatively, one can assign to each $(H_k(M, \mathbb{Z}), f_{sk})$ the element $\det(1 - t(f_{sk}|H_k(M, \mathbb{Z})))^{-1} \in 1 + t\mathbb{Z}[t] = \mathcal{W}(\mathbb{Z})$ and consider the $\lambda$-subring of $\mathcal{W}(\mathbb{Z})$ generated by these elements. Clearly, all $(H_k(M, \mathbb{Z}), f_{sk})$ lie in $\mathcal{E}_{\Sigma, \mathbb{Z}}$.

3.2. Abstract Bost-Connes systems and $\text{comm}_\Sigma^+$. Following [23, Def. 2.5] one can associate to a cyclotomic Bost-Connes datum $\Sigma$ the abstract Bost-Connes system which consists of the Galois-invariants of the group ring of $\Sigma$ over $\mathcal{K}$, that is,

(1) the $K$-algebra $\mathcal{K}[\Sigma]^{\text{Gal}(\mathcal{K}/K)}$, equipped with

(2) $K$-algebra morphisms $\tilde{\sigma}_n$ induced by $x \mapsto x^n$ for all $x \in \Sigma$, and

(3) $K$-linear maps $\tilde{\rho}_n$ induced by $x \mapsto x' \in \rho_n(x)$ for all $x \in \Sigma$.

Clearly, $\mathcal{K}[\Sigma]^G$ is a Hopf-algebra and from [23, Thm. 1.5,(iv)] we recall that the affine group $K$-scheme $\text{Spec}(\mathcal{K}[\Sigma]^G)$ agrees with the Galois group of the neutral Tannakian category $\mathcal{E}_\Sigma^+$ consisting of pairs $(V, \Phi)$ with $V$ a finite dimensional $K$-vectorspace and

$$\Phi : V \otimes \mathcal{K} \longrightarrow V \otimes \mathcal{K}$$

a $G$-equivariant diagonalisable automorphism all of whose eigenvalues belong to $\Sigma$, that is, the category $\mathcal{E}_{\Sigma,K}$ introduced above.

In [13] we proposed to consider the category $\text{comm}_\Sigma^+$ of all (torsion free) commutative and co-commutative $\mathbb{Z}$-birings. This time, the forgetful functor
For a commutative domain $R$, consider the polynomial ring $R[t]$ with coring structure defined by letting $t$ be a group-like element, that is, $\Delta(t) = t \otimes t$ and $\epsilon(t) = 1$.

The full linear dual $R[t]^\ast$ can be identified with the module of all infinite sequences $f = (f_n)_{n=0}^{\infty} \in R^{\infty}$ with $f(t^n) = f_n$. $L(R)$ will be $R[t]^\ast$, that is, the submodule of all sequences $f$ such that $\text{Ker}(f) = (m(t))$ with $m(t) = t^r - a_1 t^{r-1} - \ldots - a_r$ is a monic polynomial. As $f(t^r m(t)) = 0$ it follows that $f$ is a linear recursive sequence, that is, for all $n \geq r$ we have $f_n = a_1 f_{n-1} + a_2 f_{n-2} + \ldots + a_r f_{n-r}$. Therefore,

$$L(R) = R[t]^\ast = \lim_{\to} \left( \frac{R[t]}{(m(t))} \right)^*$$

where the limit is taken over the multiplicative system of monic polynomials with coefficients in $R$.

We define a coring structure on $L(R)$ dual to the ring structure on $R[t]/(m(t))$. With this coring structure, $L(R)$ becomes an integral biring if we equip $L(R)$ with the Hadamard product of sequences, that is, componentwise multiplication ($R$ coefficients in $L$).

If $K$ is a field of characteristic zero, one can describe the co-algebra structure on $L(K)$ explicitly, see [28] for more details.

On the linear recursive sequence $f = (f_n)_{n=0}^{\infty} \in K^{\infty}$ the counit acts as $\epsilon(f) = f_0$, projection on the first component. To define the co-multiplication recall that the Hankel matrix $M(f)$ of the sequence $f$ is the symmetric $k \times k$ matrix

$$H(f) = \begin{bmatrix} f_0 & f_1 & f_2 & \ldots & f_{k-1} \\ f_1 & f_2 & f_3 & \ldots & f_k \\ f_2 & f_3 & f_4 & \ldots & f_{k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{k-1} & f_k & f_{k+1} & \ldots & f_{2k-2} \end{bmatrix}$$

with $k$ maximal such that $H(f)$ is invertible. If $H(f)^{-1} = (s_{ij})_{i,j} \in M_n(K)$ then we have in $L(K)$

$$\Delta(f) = \sum_{i,j=0}^{k-1} s_{ij} (D^i f) \otimes (D^j f)$$

where $D$ is the shift operator $D(f_0, f_1, f_2, \ldots) = (f_1, f_2, \ldots)$. Clearly, if $K$ is the fraction field of $R$, and if a sequence $f \in L(R)$ has Hankel matrix $H(f)$ with determinant a unit in $R$, the same formula applies for $\Delta(f)$ as $L(R)$ is a sub-biring of $L(K)$. In general however, $\Delta(f)$ cannot be diagonalized in terms of $f, Df, D^2 f, \ldots$ with $R$-coefficients and we have no other option to describe the comultiplication than as the direct limit of linear duals of the ringstructures on $R[t]/(m(t))$.

**Proposition 4.** For a cyclotomic Bost-Connes datum $\Sigma$, the Hopf-algebra $\mathbb{K}[\Sigma]^G$ describing the abstract Bost-Connes system is a sub-bialgebra of $L(K)$.
We can describe the bialgebra $\mathbb{L}(\mathbb{K})$ of linear recursive sequences over $\mathbb{K}$ using the structural results for commutative and co-commutative Hopf algebras over an algebraically closed field of characteristic zero, see [13].

Let $T$ be the set of all sequences over $\mathbb{K}$ which are zero almost everywhere, then $T$ is a bialgebra ideal in $\mathbb{L}(\mathbb{K})$ and we have a decomposition

$$\mathbb{L}(\mathbb{K}) = \mathbb{K}[t]^\circ \simeq \mathbb{K}[t, t^{-1}]^\circ \oplus T$$

One verifies that in the Hopf-dual $\mathbb{K}[t, t^{-1}]^\circ$ the group of group-like elements is isomorphic to the multiplicative group $\mathbb{K}^*$, with $s \in \mathbb{K}^*$ corresponding to the geometric sequence $(1, s, s^2, s^3, \ldots)$. Further, there is a unique primitive element corresponding to the sequence $d = (0, 1, 2, 3, \ldots)$. Then, the structural result implies that, as bialgebras, we have an isomorphism

$$\mathbb{L}(\mathbb{K}) \simeq (\mathbb{K}[\mathbb{K}^*] \otimes \mathbb{K}[d]) \oplus T$$

As the Galois group $G = Gal(\mathbb{K}/\mathbb{K})$ acts on this bialgebra and as $\mathbb{L}(\mathbb{K}) = \mathbb{L}(\mathbb{K})^G$, the claim follows.

**Example 2.** Continuing Example 1 on Morse-Smale diffeomorphism, as anticipated in [23, remark 2.4.3], in the comm$^+$-proposal, one can associate to each $(H_k(M, \mathbb{Z}), f^*|_{H_k(M, \mathbb{Z})})$ the element $(Tr(f^*|_{H_k(M, \mathbb{Z})}), Tr(f^2|_{H_k(M, \mathbb{Z})}), Tr(f^3|_{H_k(M, \mathbb{Z})}), \ldots) \in \mathbb{L}(\mathbb{Z})$ and considers the sub-biring of $\mathbb{L}(\mathbb{Z})$ generated by these elements.

### 3.3. Motivic measures and $\mathbb{L}(\mathbb{R})$

By taking the trace of the Cayley-Hamilton polynomial we have a ghost ringmorphism $\Delta : \mathbb{W}_0(\mathbb{R}) \longrightarrow \mathbb{L}(\mathbb{R})$

$$(E, f) \mapsto (\Delta_1(E, f), \Delta_2(E, f), \ldots) = (Tr(Mf), Tr(Mf^2), \ldots)$$

Further, we have a traditional ghost morphism $\mathbb{W} : \mathbb{W}(\mathbb{R}) \longrightarrow \mathbb{R}^\infty$ determined by $t \frac{d}{dt} \log(\mathbb{W})$ on $\mathbb{W}(\mathbb{R}) = 1 + t \mathcal{R}[[t]]$

$$\mathbb{W}(f(t)) = (a_1, a_2, \ldots) \quad \text{where} \quad t \frac{d}{dt} \log(f(t)) = \sum_{m=1}^\infty a_m t^m$$

**Proposition 5.** Let $\mathbb{R}$ be a commutative ring and $\mu : K_0(\text{Var}_k) \longrightarrow \mathbb{R}$ a motivic measure. The measure $\mu$ is exponentiable if there exists a ringmorphism $\zeta_\mu$, and is rational if there is a ringmorphism $r_\mu$, making the diagram below commute

$$\begin{array}{ccc}
K_0(\text{Var}_k) & \xrightarrow{\mu} & \mathbb{R} \\
\downarrow r_\mu & & \\
\mathcal{E}_R \xrightarrow{\Delta} & \mathbb{W}_0(\mathbb{R}) \xrightarrow{L_R} & \mathbb{W}(\mathbb{R}) \\
\downarrow \Delta & & \downarrow \Delta \\
S_{CR} \xrightarrow{i} & \mathbb{L}(\mathbb{R}) \xrightarrow{\iota} & \mathbb{R}^\infty
\end{array}$$

The left-most maps are additive and multiplicative from the endomorphism category, resp. the category of completely reachable systems, to be defined in §5.

**Proof.** This follows from the definitions above and the fact that $\log(L_R(E, f)) = \sum_{m \geq 1} Tr(Mf^m) \frac{t^m}{m}$. □
Example 3. As a consequence, an exponentiable motivic measure \( \mu \) assigns to a \( k \)-variety \( X \) the element \( \zeta_\mu([X]) \in \mathbb{W}(R) \), and a rational motivic measure \( \mu \) assigns to \( X \) elements \( \hat{\xi}(r_\mu([X])) \in \mathbb{L}(R) \) and \( L_R(r_\mu([X])) \in \mathbb{W}(R) \).

4. Motivic measures on \( K_0(\text{Var}_{\mathbb{F}_1}^{tor}) \)

In this section we consider yet another approach to \( \mathbb{F}_1 \)-geometry based on the notion of torifications as introduced by Lorscheid and Lopez Pena in [17] and generalized by Manin and Marcolli in [22].

A torification of a complex algebraic variety, defined over \( \mathbb{Z} \), is a decomposition into algebraic tori

\[
X = \sqcup_{i \in I} T_i \quad \text{with} \quad T_i \simeq \mathbb{G}_m^n
\]

We consider here strong morphisms between torified varieties (see [15, §5.1] for weaker notions), that is a morphism of varieties, defined over \( \mathbb{Z} \),

\[
f : X = \sqcup_{i \in I} T_i \longrightarrow Y = \sqcup_{j \in J} T_j'
\]

together with a map \( h : I \longrightarrow J \) of the indexing sets such that the restriction of \( f \) to any torus

\[
f_i = f|_{T_i} : T_i \longrightarrow T'_h(i)
\]

is a morphism of algebraic groups. With \( K_0(\text{Var}_{\mathbb{F}_1}^{tor}) \) we denote the Grothendieck ring generated by the strong isomorphism classes \([X = \sqcup_i T_i]\) of torified varieties, modulo the scissor relations

\[
[X = \sqcup_i T_i] = [Y = \sqcup_j T'_j] + [X \setminus Y = \sqcup_k T''_k]
\]

whenever the decomposition in tori in the torifications of \( Y \) and \( X \setminus Y \) is a union of tori of the torification of \( X \). This condition is very strong and implies that the class of any torified variety in \( K_0(\text{Var}_{\mathbb{F}_1}^{tor}) \) is of the form

\[
[X = \sqcup_i T_i] = \sum_{n \geq 0} a_n T^n \quad \text{with} \quad a_n \in \mathbb{N}_+ \quad \text{and} \quad T = [\mathbb{G}_m] = \mathbb{L} - 1 \in K_0(\text{Var}_{\mathbb{C}})
\]

That is,

\[
K_0(\text{Var}_{\mathbb{F}_1}^{tor}) = \mathbb{Z}[T] = \mathbb{Z}[\mathbb{L}] \subset K_0(\text{Var}_{\mathbb{C}})
\]

with \( \mathbb{L} = [\mathbb{A}^1] \) the Lefschetz motive. Whereas Kapranov’s motivic zeta function does not make \( K_0(\text{Var}_{\mathbb{C}}) \) into a \( \lambda \)-ring, it does define a \( \lambda \)-structure on certain subrings, including \( \mathbb{Z}[\mathbb{L}] \), see [7, §2.2 Example], with \( S^n(\mathbb{L}) = \mathbb{L}^n \)

Proposition 6. Any motivic measure \( \mu : K_0(\text{Var}_{\mathbb{F}_1}^{tor}) \longrightarrow R \) with values in a commutative ring \( R \) is exponentiable and rational.

Proof. Because \( K_0(\text{Var}_{\mathbb{F}_1}^{tor}) = \mathbb{Z}[\mathbb{L}] \) is a \( \lambda \)-ring, we have by right adjointness of \( \mathbb{W}(\_\_\_\_) \) a natural one-to-one correspondence

\[
\text{comm}(K_0(\text{Var}_{\mathbb{F}_1}^{tor}), R) \leftrightarrow \text{comm}_\lambda^+(K_0(\text{Var}_{\mathbb{F}_1}^{tor}), \mathbb{W}(R))
\]

with \( \mu \) corresponding to a unique \( \lambda \)-ring morphism

\[
\zeta_\mu : K_0(\text{Var}_{\mathbb{F}_1}^{tor}) \longrightarrow \mathbb{W}(R) = 1 + tR[[t]] \quad \text{with} \quad \mathbb{L} \mapsto 1 + rt + r^2t^2 + \ldots = \frac{1}{1 - rt}
\]

with \( r = \mu(\mathbb{L}) \). That is, \( \mu \) is exponentiable and rational as it factors through the ringmorphism \( r_\mu : K_0(\text{Var}_{\mathbb{F}_1}^{tor}) \longrightarrow \mathbb{W}_0(R) \) defined by \( \mathbb{L} \mapsto [R, r] \). \( \square \)
If we equip $K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_1}) = \mathbb{Z}[L]$ with the bi-ring structure induced by letting $L$ be a group-like generator, that is $\Delta(L) = L \otimes L$ and $\epsilon(L) = 1$, we have a bi-ring morphism $c_\mu : K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_1}) \longrightarrow \mathbb{L}(R)$ defined by $L \mapsto (1, r, r^2, \ldots)$ making the diagram below commutative.

For example, any motivic measure with values in $\mathbb{Z}$ is of the form $\mu_m : K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_1}) = \mathbb{Z}[L] \longrightarrow \mathbb{Z}$ $L \mapsto m + 1$ and if $m + 1 = p$ with $p$ a prime number, the corresponding zeta function $\zeta_{\mu_m}(X, t)$ coincides with the Hasse-Weil zeta function of the reduction mod $p$ of the torified variety $X$. The reason for choosing $m + 1$ rather than $m$ will be explained in 4.1 below.

Similarly, we can define $\mathbb{F}_{1m}$-varieties to be torified varieties $X = \sqcup T_i$ with the natural action of the group of $m$-th roots of unity $\mu_m$ on each torus $T_i$. As a consequence we have $K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_{1m}}) = \mathbb{Z}[T] = \mathbb{Z}[L]$ and the previous result holds also for $K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_{1m}})$.

4.1. Counting $\mathbb{F}_{1m}$-points. The motivic measure $\mu_{2m}$ can be interpreted as a 'counting measure' associated to the $\mathbb{F}_{1}$-extension $\mathbb{F}_{1m}$.

Indeed, in [15, Lemma 5.6] Joshua Lieber, Yuri I. Manin and Matilde Marcolli define for a torified variety $X$ with Grothendieck class $[X] = \sum_{i=0}^{N} a_i T_i \in K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_1})$ that

$$\#X(\mathbb{F}_{1m}) = \sum_{i=0}^{N} a_i m^i$$

That is, $\#X(\mathbb{F}_1)$ counts the number of tori in the torified variety $X$, and $\#X(\mathbb{F}_{1m})$ counts the number of $m$-th roots of unity in the tori-decomposition of $X$. Therefore, $\mu_m = \mu_{\mathbb{F}_{1m}}$.

In analogy with this Hasse-Weil zeta function of varieties over $\mathbb{F}_q$, Lieber, Manin and Marcolli then define the $\mathbb{F}_{1}$- zeta function to be the ring morphism, by [15, Prop. 6.2]

$$\zeta_{\mathbb{F}_1} : K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_1}) \longrightarrow \mathbb{W}(\mathbb{Z}) \quad [X] = \sum_{k=0}^{N} a_k T^k \mapsto \exp(\sum_{k=0}^{N} a_k Li_{1-k}(t))$$

where $Li_s(t)$ is the polylogarithm function, that is, $Li_{1-k}(t) = \sum_{i>1} t^{k-1} \ln t$. This gives us a motivic measure on $K_0(\text{Var}^{\text{tor}}_{\mathbb{F}_1})$ with values in $\mathbb{W}(\mathbb{Z})$, but it does not correspond to any of the zeta-functions $\zeta_{\mu_k}$ corresponding to the motivic measure $\mu_k$. In particular, $\zeta_{\mathbb{F}_1}$ is not a morphism of $\lambda$-rings.
Mutatis mutandis we can define similarly the $\mathbb{F}_1$-zeta function, for the field extension $\mathbb{F}_1^m$ of $\mathbb{F}_1$, to be the ring morphism

$$\zeta_{\mathbb{F}_1^m} : K_0(Var^\text{tor}_{\mathbb{F}_1^m}) \longrightarrow \mathbb{W}(\mathbb{Z}) \quad [X] = \sum_{k=0}^{N} a_k T^k \mapsto \exp(\sum_{k=0}^{N} a_k m^k L_{1-k}(t))$$

and again, this zeta function does not come from any of the motivic measures $\mu_k$ on $K_0(Var^\text{tor}_{\mathbb{F}_1^m})$.

However, we can define another bi-ring (actually, Hopf-ring) structure on $K_0(Var^\text{tor}_{\mathbb{F}_1^m}) = \mathbb{Z}[T]$ induced by taking $\mathbb{D} = T - m$ (observe that $\#\mathbb{D}(\mathbb{F}_1^m) = 0$) to be the primitive generator, that is,

$$\Delta(\mathbb{D}) = \mathbb{D} \otimes 1 + 1 \otimes \mathbb{D} \quad \text{and} \quad \epsilon(\mathbb{D}) = 0$$

We will call this the Lie algebra structure on $K_0(Var^\text{tor}_{\mathbb{F}_1^m})$.

**Proposition 7.** If we equip $K_0(Var^\text{tor}_{\mathbb{F}_1^m}) = \mathbb{Z}[T]$ with the Lie-algebra structure, then under the natural one-to-one correspondence

$$\text{comm}(K_0(Var^\text{tor}_{\mathbb{F}_1^m}), \mathbb{Z}) \leftrightarrow \text{comm}^+(K_0(Var^\text{tor}_{\mathbb{F}_1^m}), \mathbb{L}(\mathbb{Z}))$$

the motivic measure $\mu_{2m} : K_0(Var^\text{tor}_{\mathbb{F}_1^m}) \longrightarrow \mathbb{Z}$ corresponds to a unique bi-ring morphism $c_{\mu_{2m}} : K_0(Var^\text{tor}_{\mathbb{F}_1^m}) \longrightarrow \mathbb{L}(\mathbb{Z})$, making the diagram below commutative

![Diagram](attachment:diagram.png)

**Proof.** By definition we have that $\zeta_{\mathbb{F}_1^m}(T^i) = \exp(\sum_{k \geq 1} m^i k^{i-1} t^k)$, and therefore, because $\Delta$ corresponds to $\frac{d^i}{dT^i} \log(-)$, we have that

$$\Delta(\zeta_{\mathbb{F}_1^m}(T)) = (m^i, m^i 2^i, m^i 3^i, \ldots) = \mathbb{D}(\zeta_{\mathbb{F}_1^m}(T))$$

To enforce commutativity with a ringmorphism $c_\mu$ we must have that

$$c_\mu(T) = (m, 2m, 3m, \ldots) = m d + m.1$$

for the primitive element $d = (0, 1, 2, \ldots) \in \mathbb{L}(\mathbb{Z})$, that is, $\Delta(d) = d \otimes 1 + 1 \otimes d$ and $\epsilon(d) = 0$ and with $1 = (1, 1, 1, \ldots) \in \mathbb{L}(\mathbb{Z})$.

But then, for the Lie algebra structure on $K_0(Var^\text{tor}_{\mathbb{F}_1^m})$ we have that $c_\mu(\mathbb{D})$ is the primitive element $m d \in \mathbb{L}(\mathbb{Z})$, and therefore $c_\mu$ is the unique bi-ring morphism $K_0(Var^\text{tor}_{\mathbb{F}_1^m}) \longrightarrow \mathbb{L}(\mathbb{Z})$ corresponding to the motivic measure $\mu_{2m} : K_0(Var^\text{tor}_{\mathbb{F}_1^m}) \longrightarrow \mathbb{Z}$ because the second component of $c_\mu(T) = 2m$.

Suppose there would be a ringmorphism $r : K_0(Var^\text{tor}_{\mathbb{F}_1^m}) \longrightarrow \mathbb{W}(\mathbb{Z})$, then we must have that $\mathbb{D}(r(T - m)) = m d \in \mathbb{L}(\mathbb{Z})$. By functoriality we have a commuting
square
\[
\begin{align*}
\mathcal{W}_0(\mathbb{Z}) & \longrightarrow \mathcal{W}_0(\mathbb{Q}) = \mathbb{Z}[\mathbb{Q}_\times] \\
\mathbb{L}(\mathbb{Z}) & \longrightarrow \mathbb{L}(\mathbb{Q}) = (\mathbb{Q}[\mathbb{Q}_\times] \otimes \mathbb{Q}[d]) \oplus K \\
\end{align*}
\]
and \(d\) does not lie in the image of the rightmost map. \(\square\)

Because \(K_0(\var_{\mathbb{F}_1}^{tor})\) is both a \(\lambda\)-ring (with \(\Psi_k(\mathbb{L}^L) = \mathbb{L}^{k_1}\)) and a bi-ring (with the Lie algebra structure with primitive element \(\mathcal{D} = \mathbb{T} - 1\)) we have natural one-to-one correspondences

\[
\text{comm}^+_{bi}(K_0(\var_{\mathbb{F}_1}^{tor}), \mathbb{L}(\mathbb{Z})) \leftrightarrow \text{comm}(K_0(\var_{\mathbb{F}_1}^{tor}), \mathbb{Z}) \leftrightarrow \text{comm}^+_{\lambda}(K_0(\var_{\mathbb{F}_1}^{tor}), \mathcal{W}(\mathbb{Z}))
\]

Under the left correspondence, the motivic measure \(\mu_m\) defined by \(\mu_m(T) = m\) corresponds to the bi-ring morphism

\[
b_m : K_0(\var_{\mathbb{F}_1}^{tor}) = \mathbb{Z}[\mathbb{D}] \longrightarrow \mathbb{L}(\mathbb{Z}) \quad \mathbb{D} \mapsto (m-1)d = (0, m-1, 2(m-1), \ldots)
\]
as \(b_m(T) = (1, m, 2m-1, \ldots)\) and the corresponding ring-morphism to \(\mathbb{Z}\) is composing with projection on the second factor.

Under the right correspondence, the motivic measure \(\mu_m\) corresponds to the \(\lambda\)-ring morphism \(l_m : K_0(\var_{\mathbb{F}_1}^{tor}) = \mathbb{Z}[\mathbb{L}] \longrightarrow \mathcal{W}(\mathbb{Z})\)

\[
\mathbb{L} \mapsto \frac{1}{1- (m+1)t} = 1 + (m+1)t + (m+1)^2t^2 + \ldots
\]
as \(l_m(T) = (1-t)L_1 = 1 + mt + m(m+1)t^2 + \ldots\) and the corresponding ring morphism to \(\mathbb{Z}\) is \(l_m(T) = m\).

It follows from propositions 6 and 7 that these morphisms factor through the pull-back \(M(\mathbb{Z})\).

Motivated by this, one might view \(M(\mathbb{Z})\) as the correct receptacle for ringmorphisms \(K_0(\var_\mathbb{F}_1) \longrightarrow \mathcal{W}(\mathbb{Z})\) determined by a counting measure \(K_0(\var_\mathbb{F}_1) \longrightarrow \mathbb{Z}\). Here, local factors corresponding to non-archimedean places can be distinguished from the \(\Gamma\)-factors by the fact that they factor through \(\mathcal{W}_0(\mathbb{Z})\).

5. Linear systems and zeta-polynomials

The original motivation for proposing bi-rings as \(\mathbb{F}_1\)-algebras was to give a potential explanation of Manin’s interpretation of Deninger’s \(\Gamma\)-factor \(\prod_{n \geq 0} \frac{x^n}{n!}\) at complex infinity as the zeta function of (the dual of) infinite dimensional projective space \(\mathbb{P}_{\mathbb{F}_1}^\infty\), see [19, 4.3] and [21, Intro]. In [13] a noncommutative moduli space was constructed using linear dynamical systems having the required motive. This
suggests the introduction of the category $\mathcal{S}_R$ of discrete $R$-linear dynamical systems, which plays a similar role for $\mathbb{L}(R)$ as does the endomorphism category $\mathcal{E}_R$ for $\mathbb{W}_0(R)$ and $\mathbb{W}(R)$.

For $R$ a commutative ring consider the category $\mathcal{S}_R$ with objects quadruples $(E, f, v, c)$ with $E$ a projective $R$-module of finite rank, $f \in \text{End}_R(E)$, $v \in E$ and $c \in E^*$ and with morphisms $R$-module morphisms $\phi : E \rightarrow E'$ such that $\phi \circ f = f' \circ \phi$, $\phi(v) = v'$ and $c = c' \circ \phi$. A quadruple $(E, f, v, c)$ can be seen as an $R$-representation of the quiver

![Quiver Diagram](image)

and morphisms correspond to quiver-morphisms.

Again, there is a duality $S = (E, f, v, c) \leftrightarrow S^* = (E^*, f^*, c^*, v^*)$ on $\mathcal{S}_R$ and we have $\oplus$ and $\otimes$ operations

\[
\begin{align*}
(E_1, f_1, v_1, c_1) \oplus (E_2, f_2, v_2, c_2) &= (E_1 \oplus E_2, f_1 \oplus f_2, v_1 \oplus v_2, c_1 \oplus c_2) \\
(E_1, f_1, v_1, c_1) \otimes (E_2, f_2, v_2, c_2) &= (E_1 \otimes E_2, f_1 \otimes f_2, v_1 \otimes v_2, c_1 \otimes c_2)
\end{align*}
\]

with a zero object $0 = (0, 0, 0, 0)$ and a unit object $1 = (R, 1, 1, 1)$.

We will call a quadruple $S = (E, f, v, c)$ a discrete $R$-linear dynamical system. Borrowing terminology from system theory, see for example [32, VI.§5], we define:

**Definition 6.** For $S = (E, f, v, c) \in \mathcal{S}_R$ with $E$ of rank $n$, we say that

1. $S$ is completely reachable if $E$ is generated as $R$-module by the elements $\{v, f(v), f^2(v), \ldots\}$.
2. $S$ is completely observable if the $R$-module morphism $\phi : E \rightarrow R^n$ given by $\phi(x) = (c(x), c(f(x)), \ldots, c(f^{n-1}(x)))$ is injective.
3. $S$ is a canonical system if $S$ is both completely reachable and completely observable.
4. $S$ is a split system if both $S$ and $S^*$ are completely reachable.

**Definition 7.** There is an additive and multiplicative bat-map

$\bullet \bullet \bullet_R : \mathcal{S}_R \rightarrow \mathbb{L}(R)$ \quad $(E, f, v, c) \mapsto (c(v), c(f(v)), c(f^2(v)), c(f^3(v)), \ldots)$

sending a linear dynamical system to its input-output or transfer sequence. We say that a linear recursive sequence $s = (s_0, s_1, s_2, \ldots) \in \mathbb{L}(R)$ is realisable by the system $(E, f, v, c) \in \mathcal{S}_R$ if $\bullet \bullet \bullet_R(E, f, v, c) = s$.

**Remark 1.** In system theory, see for example [32, VI.§5], one relaxes the condition on the state-space $E$ which is merely an $R$-module and replaces the $\text{rk}(E) = n$ condition by the requirement that $E$ is generated by $n$ elements.

We will now prove that every element $s \in \mathbb{L}(R)$ is realisable by a completely reachable system and verify when this system is in addition canonical, respectively split.

For $s = (s_0, s_1, s_2, \ldots) \in \mathbb{L}(R)$ satisfying the recurrence relation $s_n = a_1 s_{n-1} + a_2 s_{n-2} + \ldots + a_r s_{n-r}$ of depth $r$, valid for all $n \in \mathbb{N}$ with the $a_i \in R$. Consider the
system $S_s = (E_s, f_s, v_s, c_s) \in \mathcal{S}_R$ with

$$E_s = \frac{R[x]}{(x^r - a_1x^{r-1} - \ldots - a_r)}.$$  

and consider the $r \times r$ matrix, with $r$ the depth of the recurrence relation

$$H_i(s) = \begin{bmatrix}
  s_0 & s_1 & s_2 & \ldots & s_{r-1} \\
  s_1 & s_2 & s_3 & \ldots & s_r \\
  s_2 & s_3 & s_4 & \ldots & s_{r+1} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  s_{r-1} & s_r & s_{r+1} & \ldots & s_{2r-2}
\end{bmatrix}$$

Proposition 8. With notations as above, $s \in \mathbb{L}(R)$ is realisable by the system $S_s = (E_s, f_s, v_s, c_s) \in \mathcal{S}_R$, and

1. $S_s$ is completely reachable,
2. $S_s$ is canonical if and only if $\det(H_r(s)) \neq 0$,
3. $S_s$ is split if and only if $\det(H_r(s)) \in R^*$.

Proof. Clearly, $E_s$ is a free $R$-module of rank $r$ and one verifies that $\bullet \bullet \bullet \bullet \bullet (S_s) = s$. Further, $\{v_s, f_s(v_s), f_s^2(v_s), \ldots, f_s^{r-1}(v_s)\} = \{1, x, x^2, \ldots, x^{r-1}\}$ and these elements generate $E_s$ whence $S_s$ is completely observable if and only if $\det(H_r(s)) \neq 0$.

The dual module, $E_s^* = Re_0 \oplus \ldots \oplus Re_{r-1}$ where $e_i(x^j) = \delta_{ij}$. With respect to this basis we have $f_s^*(e_i) = e_{i-1} + a_{r-i}e_{r-1}$ for $i \geq 1$ and $f_s^*(e_0) = a_re_{r-1}$, that is

$$M_{f_s^*} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\
  & \ddots & \ddots & \vdots \\
  & & 0 & 1 \\
  a_r & a_{r-1} & \ldots & a_1 \end{bmatrix}, \quad c_s^* = \begin{bmatrix} s_0 \\
  s_1 \\
  \vdots \\
  s_{r-1} \end{bmatrix}$$

and $v_s^* = (1, 0, \ldots, 0)$. It follows that $\{c_s^*, f_s^*(c_s^*), f_s^{*2}(c_s^*), \ldots, f_s^{*m}(c_s^*)\}$ generate $E_s^*$ if and only if $H_r(s) \in GL_r(R)$. \hfill \square

Example 4. Consider the sequence $s = (1, 2, 3, \ldots)$ which we encountered in our study of the $\mathbb{F}_1$-zeta function. We have

$$\begin{bmatrix} 1 & 2 \\
  2 & 3 \end{bmatrix} \in GL_2(\mathbb{Z}) \quad \text{and} \quad \det \begin{bmatrix} 1 & 2 & 3 \\
  2 & 3 & 4 \\
  3 & 4 & 5 \end{bmatrix} = 0$$

leading to the (minimal) recurrence relation $x^2 - 2x + 1 = (x - 1)^2$. The corresponding system $S_s = (E_s, f_s, v_s, c_s)$ is split and determined by

$$E_s = \frac{\mathbb{Z}[x]}{(x - 1)^2}, \quad f_s = \begin{bmatrix} 0 & -1 \\
  1 & 2 \end{bmatrix}, \quad v_s = \begin{bmatrix} 1 \\
  0 \end{bmatrix}, \quad \text{and} \quad c_s = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$
Clearly, if $S = (E, f, v, c)$ is split, it is a canonical system. Over a field $K$ the converse is also true. Note that the difference between canonical and split systems over $R$ is also important for the co-multiplication on $\mathbb{L}(R)$.

Over a field $K$ every recursive sequence $s = (s_0, s_1, \ldots) \in \mathbb{L}(K)$ has a minimal canonical realisation, that is, one with the dimension of the state-space $E$ minimal. To find it, start with a recursive relation $s_t = a_1 s_{n-1} + a_2 s_{n-2} + \ldots + a_r s_{n-r}$ of depth $r$ and form as above the matrix $H_r(s)$ with columns $H_0, H_1, \ldots, H_{r-1}$. Let $t$ be the largest integer such that the columns $H_0, H_1, \ldots, H_{r-1}$ are linearly independent. If $t = r$ then the previous lemma gives a minimal canonical realisation. If $t < r$ then we have unique coefficients $\alpha_i \in K$ such that $H_t = \alpha_1 H_{t-1} + \alpha_2 H_{t-2} + \ldots + \alpha_t H_0$. But then, it follows that

$$s_n = \alpha_1 s_{n-1} + \alpha_2 s_{n-2} + \ldots + \alpha_t s_{n-t}$$

is a recursive relation for $s$ of minimal depth $t$. Using this recursive relation we can then construct a canonical realisation as in the previous lemma, with this time a state-space of minimal dimension. Over a Noetherian domain $R$ one always has a canonical realisation (in the weak sense that the state module $E$ need not be canonical in general. Still, we can consider its input-output sequence a recursive sequence has a minimal canonical realisation, with free state module, see [32] VI.5.8.iii].

Over a field $K$ we know that canonical systems $S_K = (E_K, f_K, v_K, c_K)$, with $\text{dim}(E_K) = n$ are also classified up to isomorphism by their transfer function

$$T_{S_K}(z) = c_K(z I - M_{f_K})^{-1} v_K = \frac{Y(z)}{X(z)} = \frac{c_n z^{n-1} + \ldots + c_1 z + c_0}{z^n + d_{n-1} z^{n-1} + \ldots + d_1 z + d_0}$$

which are strictly proper rational functions of McMillan degree $n$, that is, $\text{deg}(Y(z)) < \text{deg}(X(z)) = n$ (this is immediate from Cramer’s rule) and $(Y(z), X(z)) = 1$, see for example [32] II.§5).

**Proposition 9.** Let $T(z) = \frac{Y(z)}{X(z)}$ be a strictly proper rational $K$-function with $Y(z), X(z) \in R[z]$, then there is a completely reachable $R$-linear system $S = (E, f, v, c)$ such that $T(z) = c(z I - M_f)^{-1} v$. If $R$ is a principal ideal every linear recursive sequence has a minimal canonical realisation, with free state module, see [32] VI.5.8.iii].

Proof. We can always find an $R$-system $S' = (E', f', v', c')$ with transfer function $T(z) = c'.(z I - M_{f'})^{-1} v'$, with $E' = R^n$

$$f' = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -d_0 & -d_1 & -d_2 & \ldots & -d_{n-1} \end{bmatrix}, \quad v' = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad c' = \begin{bmatrix} c_0 & c_1 & \ldots & c_{n-1} \end{bmatrix}$$

and this system is completely reachable as $\{v', f'(v'), f'^2(v'), \ldots\}$ generate $R^n$. However, it need not be canonical in general. Still, we can consider its input-output sequence

$$\mathbb{L}(S') = (c'.v', c'.M_{f'}v', c'.M_{f'}^2v', \ldots) \in \mathbb{L}(R)$$
By surjectivity on canonical systems in case $R$ is a principal ideal domain, there is a canonical $R$-system $S = (E, f, v, c)$ with $\text{R}(S) = \text{R}(S')$, that is,

$$c'.v' = c.v, \quad c'.M'_f.v' = c.M_f.v, \quad c'.M^2_f.v' = c.M^2_f.v, \ldots$$

But, as $T(z) = c'(zI - M_f)^{-1}.v' = c'.v'z^{-1} + c'.M_f.v'z^{-2} + c'.M^2_f.v'.z^{-3} + \ldots$ we see that $T(z)$ is also the transfer function of the canonical $R$-system $S$, proving the claim.

**Definition 8.** For a cyclotomic Bost-Connes datum $\Sigma$, let $S_{cr, R}$ be the full subcategory of $S_R$ consisting of all completely reachable systems $S = (E, f, v, c)$ such that all zeroes and poles of the transfer function

$$T_S(z) = c.(zI - M_f)^{-1}.v$$

are in $\Sigma$.

**Example 5.** Continuing example 4, we have for $T_S$,

$$\begin{bmatrix} 1 & 2 \\ z & 1 \\ -1 & z - 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{z}{(z - 1)^2} = Li_{-1}$$

### 5.1. Zeta polynomials.

An interesting class of strictly proper rational functions is associated to Manin’s ‘zeta polynomials’ introduced in [21, §1] and generalized in [10] and [27], see also [23, §2.5]. The terminology comes from a result of F. Rodriguez-Villegas [31]. Let $U(z)$ be a polynomial of degree $e$ with $U(1) \neq 0$ and consider the strictly proper rational function

$$P(z) = \frac{U(z)}{(1 - z)^{e+1}}$$

There is a polynomial $H(z)$ of degree $e$ such that the power series expansion of $P(z)$ is

$$P(z) = \sum_{n=0}^{\infty} H(n)z^n$$

If all roots of $U(z)$ lie on the unit circle, Rodriguez-Villegas proved that the polynomial $Z(z) = H(-z)$ has zeta-like properties: all roots of $Z(z)$ lie on the vertical line $Re(z) = \frac{1}{2}$ and if all coefficients of $U(z)$ are real then $Z(z)$ satisfies the functional equation

$$Z(1 - z) = (-1)^e Z(s)$$

In [21, §1] Yuri I. Manin associates such a zeta-polynomial to each cusp $f$ form of $\Gamma = PSL_2(\mathbb{Z})$ which is an eigenform for all Hecke operators, and views this polynomial as ‘the local zeta factor in characteristic one’. The corresponding numerator $U_f(z)$ of the strictly proper rational function comes from the period polynomial divided by the real zeroes and by [5] the remaining zeros all lie on the unit circle.

In [10] this construction was generalised to the case of cusp newforms of even weight for the congruence subgroups $\Gamma_0(N)$, where this time the zeroes of period polynomials all lie on the circle with radius $\frac{1}{\sqrt{N}}$.

Let $Z_i(z)$ be a suitable collection of zeta-polynomials determined by strictly proper rational functions $P_i(z) = \frac{U_i(z)}{(1 - z)^{e+1}}$, with $U_i(z) \in \mathbb{R}[z]$ then we can view the sub bi-ring of $\mathbb{L}(\mathbb{Z})$ generated by the elements $\text{R}(S_i) \subseteq \mathbb{L}(R)$, where $S_i$ is a
completely reachable or minimal canonical system realizing \( P_i(z) \), as a representative for the collection of zeta-polynomials in the \( \text{comm}_{\mathbb{F}_1} \)-version of \( \mathbb{F}_1 \)-geometry. Again, we can define similarly versions relative to a cyclotomic Bost-Connes datum \( \Sigma \) by imposing that the zeroes of the zeta-polynomials must lie in \( \Sigma \).

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**Department Mathematics, University of Antwerp**, Middelheimlaan 1, B-2020 Antwerp (Belgium) lieven.lebruyn@uantwerpen.be