PERIODIC SYSTEMS FOR THE HIGHER-DIMENSIONAL LAPLACE TRANSFORMATION

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Abstract. We consider overdetermined systems of linear partial differential equations of the form

\[ y_{k\ell} + a_{k\ell} y_{,k} + a_{k\ell} y_{,\ell} + c_{k\ell} y = 0, \quad 1 \leq k \neq \ell \leq n, \]

where the coefficients are smooth functions satisfying certain integrability conditions. Generalizing the classical theory of second order linear hyperbolic partial differential equation in the plane, we consider higher-dimensional Laplace invariants of a system of the above class. These invariants are characterized as functions which must satisfy a set of differential equations. We establish a normal form for any system of the above class in terms of these invariants. Moreover, we solve the periodicity problem for the higher-dimensional Laplace transformation applied to such systems, generalizing a classical theorem of Darboux which shows that for \( n = 2 \) a 1-periodic equation is equivalent to the Klein-Gordon equation.

1. Introduction

The Laplace transformation, not to be confused with the Laplace transform in harmonic analysis, is a transformation of second-order linear scalar hyperbolic partial differential equations in the plane which constitutes the basis of a classical method of closed form integration. When it is applicable, the method of Laplace produces solutions depending on two arbitrary functions on one variable and finitely many of their derivatives. The classical works of Darboux [D], Goursat [G] and Forsyth [F] contain extensive treatments of this method of integration.

We have recently obtained a generalization of the classical Laplace transformation to \( n \) dimensions [KT] motivated by a geometric transformation of Chern [C1], [C2] for special submanifolds of \( n \)-dimensional projective space.

The \( n \)-dimensional Laplace transformation applies to overdetermined systems of \( n(n-1)/2 \) linear second-order partial differential equations, of the form

\[ y_{k\ell} + a_{k\ell} y_{,k} + a_{k\ell} y_{,\ell} + c_{k\ell} y + h_{k\ell} = 0, \quad 1 \leq k \neq \ell \leq n \] (1.1)

where \( y \) is a scalar function and where the coefficients are smooth functions of the independent variables \( x_1, \ldots, x_n \), satisfying certain conditions of compatibility.

We should mention that the systems of the form (1.1) play an important role in the analysis of the conserved quantities for semi-Hamiltonian, strongly hyperbolic systems of hydrodynamic type ([Tsa], [Ser]). Indeed, the densities for these conserved quantities are precisely governed by overdetermined systems of the type

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(1.1), which can be studied by the \( n \)-dimensional Laplace transformation. Such transformations also generate new systems which are rich in conservation laws from known ones (see [KT2]). For other applications of such systems see also [Ti] and [O].

In [KT], we showed that any system of the form (1.1) admits \( n(n-1)^2 \) higher-dimensional Laplace invariants and, generically, \( n(n-1) \) Laplace transformations which map (1.1) to an equation of the same form. We proved a fundamental reduction theorem which shows that the systems whose higher-dimensional Laplace invariants are all zero in one direction can be integrated by quadratures in terms of the solutions of an \((n-1)\)-dimensional system of the same type. These results give a multi-dimensional generalization of the classical method of Laplace, which has been successfully applied to obtain closed expressions for the conserved densities of some classical systems of hydrodynamic type such as the chromatography system and the electrophoresis system (see [KT2]).

One cannot use this method to integrate systems of equations whose Laplace invariants are all non-zero, and whose iterated Laplace transforms are also such that their Laplace invariants are all non-zero. Therefore, a basic question when one tries to integrate overdetermined systems (1.1) by the \( n \)-dimensional method of Laplace is that of periodicity. An equation is said to be \( r \)-periodic if it is equivalent to its \( r \)-th Laplace iterate under a rescaling of the dependent variable and a reparametrization of the independent variables. Periodic equations therefore cannot be integrated by applying the Laplace transformation in their direction of periodicity. It is a classical result of Darboux [D] that every 1-periodic equation in the plane is equivalent under one of the above transformations to the Klein-Gordon equation

\[
\frac{\partial^2 z}{\partial x \partial y} = \epsilon z, \quad \epsilon^2 = 1.
\]

In particular, if a planar equation is 1-periodic with respect to one of the Laplace transforms, then it is also 1-periodic with respect to the other. Darboux also showed that an equation is 2-periodic if and only if there exists a parametrization of the independent variables in which the two Laplace invariants are solutions of the hyperbolic sinh-Gordon equation.

Our goal in this paper is to generalize to \( n \)-dimensions part of the classical theory that we have reviewed above for second order linear hyperbolic equations in the plane. In this direction, we establish the normal form for any homogeneous system of type (1.1) in terms of its \( n(n-1)^2 \) invariants and we solve the 1-periodicity problem in \( n \) dimensions, characterizing such systems as an infinite family. For the 2-periodicity problem, although one may prove just as in the 2-dimensional case that a pair of invariants satisfies, after a change of variables, the hyperbolic sinh-Gordon equation, this does not characterize the 2-periodic systems in \( n > 2 \) dimensions. A complete study of the 2-periodicity problem remains to be done in this case.

In what follows, without loss of generality we will restrict ourselves to systems (1.1) which are homogeneous (\( h_{k\ell} = 0 \)). This is because the higher-dimensional Laplace invariants and the property of being reducible are independent of the inhomogeneous terms \( h_{k\ell} \).

In Section 2, we first define \((i, j)\)-higher-dimensional Laplace invariants \( m_{ij}, m_{ijk}, 1 \leq i, j, k \leq n, \quad i, j, k \) distinct, for the systems (1.1). In contrast with the 2-dimensional case, the higher-dimensional Laplace invariants cannot be prescribed freely in \( n \geq 3 \) dimensions. They are constrained by a set of non-linear differential equations which result from the compatibility conditions for the overdetermined
An important problem is therefore to characterize by a set of constraints those functions which are the higher-dimensional Laplace invariants of a system (1.1). This is precisely the content of Theorem 1, where we prove that for any collection of functions \( m_{ij}, m_{ijk} \), satisfying a set of five constraints, there exists, up to rescaling of \( y \), a unique overdetermined homogeneous system (1.1) whose higher-dimensional Laplace invariants are the given functions.

In Section 3, we consider the \( n(n-1) \) higher-dimensional Laplace transformations for (1.1). We then compute in Theorems 2 and 3 the transformation formulas for the higher-dimensional Laplace invariants under these Laplace transformations. We then state the reduction theorem for the systems whose higher-dimensional Laplace invariants vanish in one direction (Theorem 4), proved in [KT]. In Theorem 5, we prove that if one starts from a system (1.1), in the canonical form of Theorem 1, which is homogeneous and reducible in a given direction, then the reduced system is also necessarily homogeneous. We conclude Section 3 by proving a permutability theorem for the higher-dimensional transformation (Theorem 6). This is an important ingredient in the practical implementation of the method of Laplace, which does not appear in the 2-dimensional case. In Section 4 we first define the notion of periodicity in higher dimensions. We then obtain in Theorem 7, a set of necessary and sufficient conditions, in terms of the higher-dimensional Laplace invariants and their derivatives, for a system (1.1) to be 1-periodic in a given direction. In Theorem 8, we give a normal form for the most general system (1.1) which is 1-periodic, generalizing to \( n \) dimensions Darboux’s theorem on the Klein-Gordon equation in two dimensions. The special case of three dimensions is further examined in Theorem 9. We prove the remarkable fact that a three-dimensional system which is 1-periodic with respect to one of the transforms and transformable in all other directions will also be 1-periodic in all other directions.

2. The Higher-Dimensional Laplace Invariants. We consider linear systems of second-order partial differential equations, of the form

\[
y_{k\ell} + a^k_{k\ell} y_{,k} + a^\ell_{k\ell} y_{,\ell} + c_{k\ell} y = 0, \quad 1 \leq k \neq \ell \leq n
\]  

(2.1)

where \( y \) is a scalar function of the independent variables \( x_1, \ldots, x_n \) and the coefficients \( a \) and \( c \) are smooth functions of \( x_1, \ldots, x_n \) which are symmetric in the pair of lower indices. One could also study systems with a non-homogeneous term \( h_{k\ell} \) added to the left-hand side of (2.1), but this will not be needed here.

Our goal for this section is to prove the fundamental fact that the system (2.1) is determined in an essentially unique way by its higher-dimensional Laplace invariants. This result is analogous to the one proved by Darboux [Da] in the two-dimensional case. In contrast with the planar case, the higher-dimensional invariants cannot be prescribed arbitrarily. They must satisfy a set of non-linear differential constraints which are equivalent to the compatibility conditions for overdetermined systems of the form (2.1) for \( n \geq 3 \).

We shall consider characteristic initial data given by

\[
y(x_1^0, \ldots, x_{\ell}^0, \ldots, x_n^0) = f_\ell(x_\ell), \quad 1 \leq \ell \leq n.
\]  

(2.2)

Any smooth solution of (2.1) must of course satisfy

\[
y_{,k\ell j} = y_{,k\ell j},
\]  

(2.3)
for $k, j$ and $l$ distinct. Combined with (2.2), these conditions imply that the coefficients $a$ and $c$ in (2.1) must satisfy the following set of compatibility conditions, for $k, j$ and $\ell$ distinct,

\[
a'_{jk} - a'_{kj} = 0, \\
c_{\ell j} = a_{\ell k,j} - a_{\ell k,j} + a_{\ell j}a_{\ell k} + a_{\ell j}a_{\ell k}, \\
c_{\ell k,j} - c_{\ell j,k} + (a'_{\ell j} - a'_{\ell k})c_{\ell k} - a'_{\ell k}c_{\ell j} = 0.
\]

(2.4)

We observe that the first equation of (2.4) is also a consequence of the second one.

The following example of an overdetermined system (2.1) whose coefficients satisfy the compatibility conditions (2.4) appears in a treatise of Darboux on the classification of orthogonal systems ([Da2], p. 312):

\[
y_{12} = \left( \frac{H_{1,2}}{H_1} \right) y_{1} + \left( \frac{H_{2,1}}{H_2} \right) y_{2}, \\
y_{23} = \left( \frac{H_{2,3}}{H_2} \right) y_{2} + \left( \frac{H_{3,2}}{H_3} \right) y_{3}, \\
y_{13} = \left( \frac{H_{1,3}}{H_1} \right) y_{1} + \left( \frac{H_{3,1}}{H_3} \right) y_{3},
\]

where the coefficients $H_1, H_2$ and $H_3$ are assumed to satisfy the nonlinear system

\[
H_{1,23} = \left( \frac{H_{2,3}}{H_2} \right) H_{1,2} + \left( \frac{H_{3,2}}{H_3} \right) H_{1,3}, \\
H_{2,13} = \left( \frac{H_{1,3}}{H_1} \right) H_{2,1} + \left( \frac{H_{3,1}}{H_3} \right) H_{2,3}, \\
H_{3,12} = \left( \frac{H_{1,2}}{H_1} \right) H_{3,1} + \left( \frac{H_{2,1}}{H_2} \right) H_{3,2}.
\]

Upon setting $H_1 = \cos \theta$, $H_2 = \sin \theta$, $H_3 = \theta, \lambda$, the constraints reduce to the single equation

\[
\theta_{123} + \theta_{2}\theta_{13} \tan \theta - \theta_{1}\theta_{23} \cot \theta = 0.
\]

The general form of the system (2.1) is preserved under the admissible transformations

\[
y = \lambda(x_1, \cdots, x_n)\varphi, \\
x_i = f_i(\varphi_i), \quad 1 \leq i \leq n,
\]

where $\lambda$ is smooth and non-vanishing and the $f_i$’s are smooth and have non-vanishing derivatives. It is easily verified that under an admissible transformation, the coefficients $a$ and $c$ transform according to, for $1 \leq k \neq \ell \leq n$

\[
\sigma_{\ell k} = f^i_{\ell} \left( a_{\ell k}^i + (\log \lambda)_{,k} \right), \\
\tau_{\ell k} = \frac{1}{f_{\ell}} \left( c_{\ell k} + a_{\ell k}^i (\log \lambda)_{,\ell} + a_{\ell k}^i (\log \lambda)_{,k} + \frac{\lambda_{,\ell}}{\lambda} \right).
\]

(2.7)

We define the higher-dimensional Laplace invariants of (2.1) to be the $n(n - 1)^2$ functions given by [KT]

\[
m_{ij} = a_{ij,i}^i + a_{ij,j}^i - c_{ij}, \quad m_{ijk} = a_{k,j}^i - a_{d,k}^i, \quad k \neq i, j.
\]

(2.8)
for all ordered pairs \((i, j)\), \(1 \leq i \neq j \leq n\). We observe that in (2.8) we are using a notation which is slightly different from the one used in [KT]. This is because we need to use the set of invariants for all pairs \((i, j)\). It is readily checked, using (2.7), that under an admissible transformation, we have

\[
\overline{m}_{ij} = f'_i f'_j m_{ij}, \quad \overline{m}_{ijk} = f'_j m_{ijk}.
\] (2.9)

In particular, the functions \(m_{ij}\) \(m_{ijl}\)'s are invariant under pure rescalings (2.5).

Note that in the classical case \(n = 2\), the compatibility conditions (2.4) are vacuous, the formulas (2.7) reduce to the ones given by Darboux [Da1], the \(m_{ijk}\)'s are not defined and the \(m_{ij}\)'s correspond to the classical Laplace invariants \(h\) and \(k\) of the equation. Indeed, if the equation is given by

\[
z_{xy} + a(x, y)z_x + b(x, y)z_y + c(x, y)z = 0,
\] (2.10)

then we have

\[
m_{12} = h = a_x + ab - c, \quad m_{21} = k = b_y + ab - c,
\] (2.11)

According to a fundamental result of Darboux [Da1], given any two functions \(h\) and \(k\) of \(x, y\), there exists a linear p.d.e. (2.10) such that \(h\) and \(k\) are its Laplace invariants. Any such p.d.e. is of course defined up to a rescaling

\[
z = \lambda(x, y) \overline{\varphi}, \quad \lambda(x, y) \neq 0.
\] (2.12)

The p.d.e. is uniquely determined if we choose \(\lambda\) such that upon the rescaling (2.12), we have \(ab - c\) identically zero, \(a\) identically zero on a characteristic curve \(x = x_0\) and \(b\) identically zero on a characteristic curve \(y = y_0\). In this case, the p.d.e. is given by

\[
z_{xy} + \left( \int_{x_0}^x h dx \right) z_x + \left( \int_{y_0}^y k dy \right) z_y + \left( \int_{x_0}^x h dx \int_{y_0}^y k dy \right) z = 0.
\]

We refer to [DOV] for a discussion of the equivalence problem for the p.d.e.s (2.10) under the pseudo-group of rescalings (2.12) and reparametrizations

\[
\overline{\varphi} = \varphi(x), \quad \overline{\psi} = \psi(y), \quad \varphi' \neq 0, \quad \psi' \neq 0.
\]

In order to establish the \(n\)-dimensional analogue of Darboux’s result, we need to obtain a set of differential relations constraining the higher-dimensional Laplace invariants. In the following Lemma, we derive a set of necessary conditions. These will be shown to be sufficient in the proof of Theorem 1 below.

**Lemma:** The higher dimensional Laplace invariants of a compatible system (2.1) satisfy the following relations: for \(1 \leq i, j, k, \ell \leq n\), \(i, j, k, \ell\) distinct,

\[
\begin{align*}
& m_{ijk} + m_{kji} = 0, \\
& m_{ijk,k} - m_{jk} m_{ijk} - m_{kji} = 0, \\
& m_{ij,k} + m_{ij,k} m_{ik} + m_{ikj} m_{ij} = 0, \\
& m_{i,jk} - m_{ij} m_{k} - m_{ijk} m_{ij} = 0, \\
& m_{ijk,j} + m_{ij} m_{kij} + m_{ij} m_{kij} = 0,
\end{align*}
\] (2.14)
Proof. All the relations in (2.14), except for the third one, are established using the definition of the higher dimensional Laplace invariants and the first two relations of (2.4). In order to complete the proof of the Lemma, we observe that as a consequence of the second equation of (2.4), we get

$$a_{ij,ik} = a_{ij,ki} = c_{k,i} - a_{ik,i}m_{ijk} - a_{ij,i}m_{ikj} - a_{ik}a_{k,j,i} - a_{ij}a_{k,j,i}.$$  

As a consequence of this equation and the last two equations of (2.4) we obtain the third relation of (2.14)

**Remark 1.** The theory of the higher-dimensional Laplace transformation was motivated by the geometric construction of the Laplace transformation for Cartan manifolds (see [KT1]). Such submanifolds are locally given by immersions $X: U \subset \mathbb{R}^n \to \mathbb{R}^{2n}$, whose mixed partial derivatives are of the form

$$X_{,rs} = \Gamma_{rs}^r X_r + \Gamma_{rs}^s X_s \quad 1 \leq r \neq s \leq n,$$

where $\Gamma_{rs}^r$ are the Christoffel symbols. Moreover, the structure equations of such submanifolds are satisfied

$$\Gamma_{ij,k}^i = \Gamma_{ik,j}^i \quad 1 \leq i \neq k \neq \ell \leq n$$

$$\Gamma_{ik,\ell}^k + \Gamma_{ik}^k \Gamma_{\ell k}^k - \Gamma_{i\ell}^i \Gamma_{ik}^k - \Gamma_{\ell k}^\ell \Gamma_{ik}^k = 0. \quad (2.15)$$

Since each coordinate function of $X$ satisfies an overdetermined system of the form (2.1), we can define the higher-dimensional Laplace invariants $m_{ij}$ and $m_{ijk}$ as in (2.8) in terms of the Christoffel symbols. The relations (2.14), which are satisfied by these invariants, are either identities or equivalent to the structure equation (2.15).

**Remark 2.** As a consequence of the relations (2.14), we obtain the following equalities, which will be useful later. Since the last two terms of the third relation of (2.14) are symmetric with respect to the indices $k$ and $j$, we conclude that

$$m_{ij,k} - m_{ik,j} = 0.$$

Moreover, it follows from the first and last relations of (2.14) that

$$m_{\ell k,j} = -m_{k\ell,j} = m_{jk}m_{\ell k} + m_{kj}m_{\ell j}.$$

Therefore, using again the first and last relations of (2.14), we obtain $m_{\ell k,j} = m_{\ell j,k}.$

The expression of a system (2.1) in terms of its higher-dimensional Laplace invariants is established in the following theorem.

**Theorem 1:** Given any collection of $n(n-1)/2$, $n \geq 3$ smooth functions of $x_1, \cdots, x_n$,

$$m_{ij}, \quad 1 \leq i, j, k \leq n, \quad i, j, k, \text{ distinct},$$

satisfying, the constraints (2.14) there exists a linear system (2.1) whose higher-dimensional Laplace invariants are the given functions $m_{ij}$ and $m_{ijk}$. Any such system is defined up to a rescaling (2.5). A representative is given by

$$y_{ij} + A y_{j} - m_{ij}y = 0,$$

$$y_{ik} + (m_{ijk} + A) y_{k} - m_{ik}y = 0,$$

$$y_{jk} + m_{ik}y_{j} + m_{jk}y_{k} = 0,$$

$$y_{\ell k} + m_{k\ell}y_{\ell} + m_{\ell k}y_{k} = 0. \quad (2.16)$$
where \((i, j)\) is a fixed (ordered) pair, \(1 \leq i, j, k, \ell \leq n\) are distinct and \(A\) is a function which satisfies the following:

\[
A_{ij} = m_{ji} - m_{ij}, \quad A_{ik} = -m_{jki}.
\] (2.17)

**Proof.** The first step in the proof is to solve for the coefficients \(a\) and \(c\) of the system (2.1) in terms of the higher-dimensional Laplace invariants \(m_{ij}\) and \(m_{ijk}\), using the definition (2.8) and the hypothesis (2.14) on the \(m\)'s. Since \(n \geq 3\), this is an overdetermined system of \(n(n - 1)^2\) equations and \(3n(n - 1)/2\) unknowns.

The second step is to check that the compatibility conditions (2.4) are satisfied as a consequence of (2.14). Throughout the proof, the range of the indices \(i, j, k, \ell\) will be the same as in the statement of Theorem 1.

Because of the scaling freedom (2.5) and the transformation laws (2.7), we know that we can choose \(\lambda(x_1, \cdots, x_n)\) so as to have, for an ordered pair \((i, j)\),

\[
a_{ij} = 0.
\] (2.18)

This gives immediately

\[
c_{ij} = -m_{ij},
\] (2.19)

on account of (2.8). Using (2.8), (2.18), (2.19) and the second of the constraints (2.14), we obtain

\[
a_{ji,j} = m_{ji} - m_{ij}.
\] (2.20)

From (2.8) and (2.18), we get

\[
a_{k}^{j} = m_{ijk}.
\] (2.21)

From (2.8), (2.20) and the first of the constraints (2.14), it follows that

\[
m_{ijk} = a_{ki}^{k} = \int (m_{ji} - m_{ij}) dx_{j},
\]

and therefore that

\[
a_{ki}^{k} = m_{ijk} + \int (m_{ji} - m_{ij}) dx_{j},
\] (2.22)

Using (2.8), (2.21) and the second of the constraints (2.14), we have

\[
m_{jk} - m_{kj} = a_{jk,j}^{j} - a_{kj,k}^{k} = a_{jk,j}^{j} - m_{ijk,k},
\]

and

\[
m_{jk} - m_{kj} = m_{jk,j} - m_{ijk,k},
\]

so that

\[
a_{jk}^{j} = m_{jk} + f^{k}(x_1, \cdots, \hat{x}_j, \cdots, x_n),
\] (2.23)

where \(f^{k}\) is an arbitrary function of all the variables \(x_1, \cdots, x_n\) except \(x_j\).

From (2.8), (2.21), (2.23) and the first and second of the constraints (2.14), we obtain

\[
c_{jk} = m_{jk,j} + m_{kj} m_{ijk} + f^{k} m_{ijk} - m_{jk}
= m_{ijk} m_{jk} + m_{ik} m_{ijk} + f^{k} m_{ijk} = f^{k} m_{ijk}.
\] (2.24)
Using (2.8) and (2.23), we have
\[ a_{jk}^i - a_{ik}^j = m_{ikj} + f^k - a_{ik}^i, \]
so that
\[ a_{ik}^i = f^k. \]  

(2.25)

It now follows from (2.8) and (2.25) that
\[ c_{ik} = f^k_m - m_{ik} + f^k m_{jik} + f^k \int (m_{ji} - m_{ij}) dx_j \]  

(2.26)

From (2.8) and (2.25), it follows that
\[ a_{ik}^i = f^k_m, \]
and
\[ a_{ik}^i = f^k. \]  

(2.27)

We obtain, using (2.8), (2.27) and the first two constraints of (2.14), that
\[ c_{ik} = f^k_m + m_{ikl} f^l + m_{ikl} f^k + f^k f^l. \]  

(2.28)

Moreover, since \( c_{ik} = c_{ki} \) we get
\[ f^k_m = f^k_m. \]  

(2.29)

Now we consider \( m_{ki} \) given by (2.8). It follows from (2.22), (2.25), (2.26) and the relations (2.14), that \( f^k \) satisfies
\[ f^k_m = m_{jki,i} + \int (m_{ji} - m_{ij}) k dx_j. \]  

(2.30)

As a consequence of (2.14), the remaining equations of (2.8), given by \( m_{kk}, m_{iik}, m_{ikl}, m_{jk}, \) and \( m_{rk} \), are satisfied with the coefficients \( a \) and \( c \) defined above. We have now solved for the coefficients of the system (2.1) in terms of the \( m \)'s as follows.

\[ y_{ij} + A y_{ij} - m_{ij} y = 0, \]
\[ y_{ik} + f^k y_i + (m_{ijk} + A) y_k + (f^k_i - m_{ik} + f^k (m_{jik} + A)) y = 0, \]
\[ y_{jk} + (m_{ikj} + f^k) y_j + m_{iij} y_k + m_{ij} f^k y = 0, \]
\[ y_{k} + (m_{ik} + f^k) y_i + (m_{iik} + f^k) y_j + (f^k + m_{iik} f^l + m_{iik} f^k + f^k f^l) y = 0, \]  

(2.31)

where \( A \) satisfies
\[ A_j = m_{ji} - m_{ij}, \quad A_k = - m_{jki,i} + f^k_i. \]  

(2.32)

It is now a straightforward calculation to check that the coefficients of (2.31) satisfy the compatibility conditions (2.4) as a consequence of the hypotheses (2.14) on the \( m \)'s.

Our next step is to show that any two such systems are equivalent under an admissible transformation. In fact, assume that \( f^k \) and \( \hat{f}^k \) are two functions independent of \( x_j \), for which (2.29) and (2.30) hold. If \( A \) and \( \hat{A} \) are also two distinct functions satisfying (2.32), it is easy to see that the corresponding systems (2.31) are equivalent by rescaling as in (2.5), where \( \lambda \) is such that
\[ \hat{f}^k = f^k + (\log \lambda)_k, \quad \hat{A} = A + (\log \lambda)_i. \]
Therefore, without loss of generality we may choose $f^k = 0$. Therefore $A$ satisfies (2.17) and the system (2.31) reduces to (2.16).

We observe that (2.17) is compatible. In fact, it follows from the second equation of (2.14) that

$$m_{jki,j} = ((m_{kij}m_{jki} + m_{ik})_j = ((m_{ikj}m_{kij} + m_{ij})m_{jki} - m_{kij}(m_{kij}m_{ikj} + m_{jk}) + m_{ik,j}.$$ 

Therefore, using the first equation of (2.14) we obtain that

$$m_{jki,ij} = m_{ji} - m_{ij}(m_{kij}m_{jk} + m_{kij}).$$

Hence, we conclude using Remark 2 and again the first equation of (2.14) that

$$m_{ji,k} - m_{ij,k} = -m_{jki,ij}.$$ 

□

3. The Higher-Dimensional Laplace Transformation. Consider a system of p.d.e.’s (1.1) for $y$ and let $(i, j)$, $1 \leq i, j \leq n$, denote an ordered pair. We say that the system is $(i, j)$-transformable if

$$\Omega_{ij} := m_{ij} \prod_{k \neq i, j} m_{ijk} \neq 0. \quad (3.1)$$

Observe that the property of $(i, j)$-transformability is invariant under the admissible transformations (2.5) and (2.6). If the system is $(i, j)$-transformable, then we define

$$\tilde{y} = y_{,j} + \alpha_{ij} y, \quad (3.2)$$

to be the $(i, j)$-Laplace transform of $y$, which we denote by

$$\tilde{y} = \mathcal{L}_{(i,j)}(y). \quad (3.3)$$

The transformation $\mathcal{L}_{(i,j)}$ is the higher-dimensional generalization of the classical Laplace transformation for linear scalar second-order p.d.e.’s in the plane [Da1]. It is based on the Euclidean version of a geometric transformation first obtained by Chern [C1], [C2], for special submanifolds in projective space. If (2.1) is $(i, j)$-transformable, then $\tilde{y}$ will satisfy a system of differential equations of the same type as (2.1) whose coefficients are expressible in terms of those of the original system and its higher-dimensional Laplace invariants:

**Theorem 2**: Consider a system (2.1) for $y$, whose coefficients satisfy the compatibility conditions (2.4). If the system is $(i, j)$-transformable, then $\tilde{y} = \mathcal{L}_{(i,j)}(y)$ satisfies a system of the same type, whose coefficients are given by

$$\tilde{a}_{ij} = a_{ij} - (\log m_{ij})_j, \quad \tilde{a}_{ij} = a_{ij} + m_{ij} - m_{ij} - a_{ij}(\log m_{ij})_j, \quad (3.4)$$

for each $k$, $k \neq i, k \neq j$, we have

$$\tilde{a}_{ik} = a_{ik} - (\log m_{ij})_k, \quad \tilde{a}_{ik} = a_{ik} + \frac{m_{ij}}{m_{ijk}},$$

$$\tilde{c}_{ik} = c_{ik} - a_{ij}(\log m_{ij})_k + a_{ik} + \frac{m_{ij} - a_{ij}(\log m_{ij})_j}{m_{ijk}}, \quad (3.5)$$

$$\tilde{c}_{jk} = c_{jk} + m_{ijk}m_{jki} + m_{jk} - a_{jk}(\log m_{ijk})_j.$$
and when \( n \geq 4 \), we have for each \( k \) and \( \ell \) distinct from \( i \) and \( j \)

\[
\tilde{a}^k_{\ell} = a^k_{\ell} - (\log m_{ijk})_\ell, \quad \tilde{a}^j_k = a^j_k - (\log m_{ij})_k, \\
\tilde{c}_{\ell k} = c_{\ell k} - a^j_{jk} (\log m_{ijk})_\ell - a^j_k (\log m_{ij})_k.
\]

(3.6)

These coefficients also satisfy the compatibility conditions (2.4).

**Proof.** We shall only give the proof for some of the transformation laws. The other ones proceed along the same lines. From Eq. (54) of [KT], equation (2.4) and the definition (2.8) of the higher-dimensional Laplace invariants, we have

\[
\tilde{c}_{jk} = m_{ijk} m_{kji} + a^j_{jk} a^k_{ij} - a^j_{jk} (\log m_{ijk})_j,
\]

Likewise, from Eq. (57) of [KT], (2.4) and from (2.8), we obtain

\[
\tilde{c}_{\ell k} = -a^j_{\ell k} m_{\ell kj} + c_{\ell k} - a^j_{jk} (\log m_{ijk})_\ell + a^j_{\ell k} (\log m_{ij})_k.
\]

We are ready to compute the transformation laws under the \((i,j)\)-transform for the higher-dimensional Laplace invariants of any \((i,j)\)-transformable system (2.1).

**Theorem 3:** Under the hypotheses of Theorem 2, the higher-dimensional Laplace invariants transform according to the following transformation laws

\[
\tilde{m}_{ijk} = -\tilde{m}_{kji} = m_{ijk} + \left( \frac{\log m_{ij}}{m_{ijk}} \right)_j, \quad \tilde{m}_{ij} = 2m_{ij} - m_{ji} - (\log m_{ij})_{ij},
\]

\[
\tilde{m}_{ji} = m_{ij}, \quad \tilde{m}_{jik} = -\tilde{m}_{kij} = \frac{m_{ij}}{m_{ijk}},
\]

\[
\tilde{m}_{kj} = m_{kj} - m_{jk} - (\log m_{ijk})_{jk} - m_{ijk} m_{kji},
\]

\[
\tilde{m}_{kji} = m_{kji} + \left( \frac{\log m_{ijk}}{m_{ij}} \right)_j, \quad \tilde{m}_{jki} = m_{jki} - \frac{m_{ij} m_{kij}}{m_{ijk}},
\]

\[
\tilde{m}_{ijk} = m_{ijk} - (\log m_{ij})_{ik} - m_{jki} (\log m_{ijk})_i - \frac{(m_{ij})_k}{m_{ijk}},
\]

\[
\tilde{m}_{ijk} = -\tilde{m}_{kji} = m_{ijk} + \left( \frac{\log m_{ij}}{m_{ijk}} \right)_j,
\]

\[
\tilde{m}_{k\ell} = m_{k\ell} + m_{kj} (\log m_{ij})_\ell + m_{kj} (\log m_{ijk})_\ell - (\log m_{ij})_k (\log m_{ijk})_\ell - (\log m_{ij})_{k\ell},
\]

\[
\tilde{m}_{k\ell r} = m_{k\ell r} - \left( \frac{\log m_{ijr}}{m_{ijk}} \right)_\ell,
\]

(3.7)
where the indices \( i, j, k, \ell, r \) are distinct.

**Proof.** Again, we shall only give the proof for some of the more elaborate transformation laws. Using the formulas established in Theorem 2, equations (2.4), (2.14) and the definitions (2.8), we obtain

\[
\begin{align*}
\tilde{m}_{ik} &= a_{ij, k}^i + (m_{ij}/m_{ijk})_k + \left( a_{ij}^j + m_{ij}/m_{ijk} \right) (a_{ij, k} - (\log m_{ij}, k)) \\
&= c_{ki} + a_{ij, \log m_{ij}} + a_{ij, \log m_{ijk}} + a_{ij}^j m_{ijk} \\
&= m_{ij} (m_{ijk}, k)/(m_{ijk})^2 + m_{ij} m_{kij}/m_{ijk} \\
&= -m_{ij} m_{kij}/(m_{ijk})^2,
\end{align*}
\]

\[
\begin{align*}
\tilde{m}_{ik} &= a_{ik, \log m_{ij}} - (\log m_{ij}, k) + (a_{ij}^i + m_{ij}/m_{ijk}) \\
&= c_{ik} + a_{ij, \log m_{ij}} + a_{ij, \log m_{ijk}} + a_{ij, \log m_{ijk}}^j \\
&= m_{ik} (\log m_{ij}, k) - m_{kj} (\log m_{ijk})_i - (m_{ij})_k/m_{ijk},
\end{align*}
\]

\[
\begin{align*}
\tilde{m}_{kt} &= a_{kt}^k + (m_{ijk}, k) + (a_{ij}^j - (\log m_{ij}), k) \\
&= c_{kt} + a_{ij, \log m_{ij}}^j + a_{jk, \log m_{ijk}}^i + a_{ij, \log m_{ijk}}^j \\
&= m_{kt} + m_{kj} (\log m_{ijk})_k + m_{kt} (\log m_{ijk})_k \\
&= (\log m_{ijk})_k (\log m_{ijk})_k - (\log m_{ijk})_k.
\end{align*}
\]

We recall from [KT] that the \((i, j)\) transform can be generically inverted. Indeed, consider a system (2.1) for \( y \) and let \( \tilde{y} = L_{(i, j)}(y) \). It is easy to prove using Theorem 2 that if \( m_{ij} \neq 0 \), then the inverse of the \((i, j)\) transform exists and is given by

\[
y = \left[ L_{(i, j)}(\tilde{y}) \right]/m_{ij}. \tag{3.8}
\]

It is well known [Da1] that in the planar case \((n = 2)\), the vanishing of one of the Laplace invariants \( b = m_{12} \), or \( k = m_{21} \) implies that the p.d.e. (2.10) factors into two parametrized first-order o.d.e.s in \( x \) and \( y \) respectively, so that it can be integrated by quadratures in terms of two arbitrary functions of one variable. The principle of the classical method of integration of Laplace for (2.10) is to iterate the transforms \( L_{(1, 2)} \) or \( L_{(2, 1)} \) until one possibly obtains a transformed p.d.e. with one of its Laplace invariants equal to zero. One then transforms back using the inversion formula (3.8) to obtain solutions of the original equation.

The generalization to higher dimensions of the reduction to parametrized o.d.e.s which occurs for (2.10) when either \( b \) or \( k \) is identically zero was obtained in [KT]. We now briefly recall the main content of this result. A system of p.d.e.s (2.1), whose coefficients \( a \) and \( c \) satisfy (2.4), is said to be \((i, j)\)-reducible, for an ordered pair \((i, j)\), \( 1 \leq i, j \leq n \), if

\[
\Delta^{ij} := (m_{ij})^2 + \sum_{k \neq i, j} (m_{ijk})^2 = 0. \tag{3.9}
\]

We see from (2.9) that the condition of \((i, j)\) reducibility is invariant under (2.5),(2.6). In the case \( n = 2 \), any p.d.e. (2.1) is either \((i, j)\)-transformable or \((i, j)\)-reducible. This is no longer true in the overdetermined case \( n \geq 3 \), where a system (2.1) could be neither \((i, j)\)-transformable nor \((i, j)\) reducible.
In [KT], we considered systems with a non-homogeneous term $h_{kl}$ as in (1.1), where the coefficients satisfy (2.4) and $h_{kl}$ satisfy

$$h_{tk,j} - h_{tj,k} + a_{tk}^j h_{tk} + (a_{tk}^j - a_{tk}^k) h_{kj} - a_{tk}^j h_{tj} = 0.$$  

For such systems, we proved the following reduction theorem which gives a generalization of the classical result for p.d.e.s in the plane:

**Theorem 4:** Consider a system (1.1) for $y$ whose coefficients $a$, $c$ and $h$ satisfy (2.4) and (3.10). If the system is $(i,j)$-reducible, then the general solution of the system is given by

$$y = Q + e^{-t} G(x_j),$$

where $Q = -e^{-t} \int e^{J-t} [\int e^t h_{ij} dx_i - F(x_j)] dx_j$, $I = \int a_{ij}^t dx_i$, $J = \int a_{ij}^t dx_j$, where $F$ is an arbitrary function of $x_j$, $G(x_1, \cdots, x_j, \cdots, x_n)$ does not depend on $x_j$ and where the antiderivative $I$ is such that $I_k = a_{ij}^k$ for $k \neq i, k \neq j$. Then $G$ satisfies a linear system in $n-1$ independent variables $x_1, \cdots, \hat{x}_j, \cdots, x_n$ of the form

$$G_{,kl} + g_{kl}^j G_{,k} + g_{kl}^l G_{,l} + b_{kl} G + r_{kl} = 0, \quad k \neq \ell \text{ distinct from } j.$$  

where

$$g_{kl}^j = a_{ik}^j - J_{ik}, \quad g_{kl}^l = a_{ik}^l - J_{ik},$$

$$b_{ik} = c_{ik} + J_{ik} J_{jk} - J_{ik} a_{ik}^j - a_{ik}^j J_{ik}$$

and

$$r_{ik} = e^{J} (h_{ik} + Q_{ik} + a_{ik}^j Q_{,j} + a_{ik}^l Q_{,l} + c_{ik} Q).$$

The proof of Theorem 3 is given in Theorem 2 of [KT].

If one applies Theorem 3 to a homogeneous reducible system, then the reduced system that one obtains is also homogeneous. This is what we prove in the following theorem:

**Theorem 5:** Consider a system (2.16) for $y$, in terms of its higher Laplace invariants. If the system is $(i,j)$-reducible, then its solution is given by

$$y = Q + G(x_1, \cdots, \hat{x}_j, \cdots, x_n)$$

where

$$Q = \int e^{-t} F(x_j) dx_j, \quad I = \int A dx_i,$$

where $F$ is an arbitrary function of $x_j$, $G$ does not depend on $x_j$ and where the antiderivative $I$ is such that

$$I_k = A, \quad I_k = m_{ikj}, \quad k \neq i, k \neq j.$$  

Then $G$ satisfies a linear system in $n-1$ independent variables $x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n$ of the form

$$G_{,kl} + g_{kl}^j G_{,k} + g_{kl}^l G_{,l} + b_{kl} G = 0, \quad k \neq \ell \text{ distinct from } j.$$  

Proof. By hypothesis the system is \((i,j)\)-reducible, i.e.

\[ m_{ij} = m_{ijk} = 0. \] (3.17)

Therefore, the system (2.16) is given by

\[
\begin{align*}
y_{ij} + A_j y_j &= 0, \\
y_{ik} + (m_{jik} + A) y_k - m_{ik} y &= 0, \\
y_{jk} + m_{skj} y_j &= 0, \\
y_{tk} + m_{ikj} + m_{ik} y_k &= 0,
\end{align*}
\] (3.18)

where \(1 \leq i, j, k, \ell \leq n\) are distinct and from (2.17) \(A\) satisfies the following:

\[ A_j = m_{ji}, \quad A_k = -m_{jki}. \] (3.19)

Using (3.17) and the two first relations of (2.14), we get

\[-m_{jik,j} = m_{kij,j} = m_{ji}. \] (3.20)

Therefore, it follows from (3.19) that \(A_j = -m_{jik,j}\), and hence,

\[ A = -m_{jik} + T(x_1, ..., \hat{x}_j, ..., x_n), \] (3.21)

where \(T\) does not depend on \(x_j\). Moreover, it follows from (3.15), the third and last equations of (2.14), that

\[ m_{ik,j} = 0, \quad m_{ikt,j} = 0. \] (3.22)

When Theorem 4 is applied to the system (3.18), it gives the solution \(y\) as in (3.13), where \(Q\) and \(I\) are given by (3.14). In order to show that the system corresponding to (3.11) is homogeneous, we need to show that the coefficients \(r_{ik}\) and \(r_{tk}\), given by the formula (3.12), vanish. We start by computing the derivatives of \(Q\). Using (3.14) and (3.15), we get

\[ Q_{jk} = -\int e^{-t} m_{ikj} F dx_j, \quad Q_{ik} = -\int e^{-t} A F dx_j, \quad Q_{\ell k} = -\int e^{-t} m_{ik} F dx_j. \] (3.23)

From (3.15) and (2.14), we obtain

\[ A_k = m_{ikj,i} = -m_{jki} = m_{jki} - m_{ik}. \] (3.24)

Therefore, we get

\[ Q_{ik} = \int e^{-t} I_k A F dx_j - \int e^{-t} A_k F dx_j = \int e^{-t} F(m_{ik} T + m_{ik}) dx_j. \]

where we have used (3.15), (3.20), (3.23) and the first of the relations (2.14). Similarly, from (3.15), (3.22) and (2.14) we get

\[ Q_{tk} = \int e^{-t} m_{ikj} m_{ij} F dx_j + \int e^{-t} (m_{ikj} m_{jti} + m_{ikt} m_{jtk}) F dx_j. \]
From (3.12), (3.18) and (3.20), it follows that
\[ r_{ik} = Q_{ik} + TQ_{ik} - m_{ik}Q = \int e^{-I} F m_{i\ell} dx_j - m_{ik} \int e^{-I} F dx_j = 0, \]
where the last equality is a consequence of (3.21). Finally, by using the expressions given by (3.12), (3.18) and the fact that \( m_{ik\ell} \) does not depend on \( x_j \), we obtain
\[ r_{\ell k} = Q_{\ell k} + m_{ik\ell} Q_{i\ell} + m_{i\ell k} Q_{k} = \int e^{-I} F m_{ikj}(m_{i\ell j} - m_{k\ell j} - m_{i\ell k}) dx_j. \]

The integrand of the above expression vanishes as a consequence of the fourth equation of (2.14).

The coefficients of the reduced system (3.16) are given in Theorem 4.

We conclude by proving a permutability theorem for the higher-dimensional Laplace transformation. This property, which does not appear in the 2-dimensional case, is useful in practical implementation of the method of Laplace.

**Theorem 6:** Suppose that we are given a system (2.1) which is \((i, j)\)-transformable and \((k, j)\)-transformable for some \(i, j, k\) distinct. Then for any smooth function \(z(x_1, \ldots, x_n)\), we have
\[(L_{kj} \circ L_{ij})(z) = (L_{ij} \circ L_{kj})(z).\]

**Proof.** Using (3.5), we have
\[(L_{kj} \circ L_{ij})(z) = z_{j,j} + [a_{ij}^i + a_{kj}^k - m_{ijkj}^i]z_{j,j} + [a_{ij,j}^i + (a_{kj}^k - m_{ijkj}^k)a_{ij}^i]z,
from which it follows, using (2.8) and the first of the relations (2.14)
\[(L_{kj} \circ L_{ij})(z) - (L_{ij} \circ L_{kj})(z) = \left( m_{kjij,j} - \frac{m_{ijkj}}{m_{ijkj}} m_{kjij} \right) z = 0.\]

4. Periodic Systems. Suppose that we are given a system \(S\) of the form (2.1) whose coefficients \(a\) and \(c\) satisfy (2.4). If the system is \((i, j)\)-reducible for some ordered pair \((i, j)\), then we may apply Theorem 3 to reduce the number of independent variables form \(n\) to \(n-1\). If the system \(S\) is not \((i, j)\)-reducible for any ordered pair \((i, j)\) and it is \((k, \ell)\)-transformable for some direction \((k, \ell)\), then the application of the \((k, \ell)\) transformation to \(S\) will lead to a new system which could again be considered for reducibility. We will thus say that \(S\) is reducible after \(r\) steps if there exists a path of ordered pairs \((I, J) = ((i_1, j_1), \ldots, (i_r, j_r), (i, j))\) such that the composition \(\tilde{S} = L_{(i_r, j_r)} \circ \cdots \circ L_{(i_1, j_1)}(S)\) exists and is \((i, j)\)-reducible. Given that there are a priori \(n(n-1)\) directions in which to transform at each stage, it is important to have certain criteria by which one can eliminate certain pairs \((i_r, j_r)\) in the sequence \((I, J)\). One such criterion is provided by the notion of periodicity. Given a system \(S\) which is \((i, j)\)-transformable for some pair \((i, j)\), we say that \(S\) is 1-periodic in \(L_{(i, j)}\) if \(S\) and \(L_{(i, j)}(S)\) are equivalent under an admissible transformation (2.5), (2.6). Thus, one should never transform a system in a direction in which it is 1-periodic when constructing a path which is to lead to an \((i, j)\)-reducible system.
In the case \( n = 2 \), we only have two transforms \( L(1,2) \) and \( L(2,1) \), any p.d.e. \( (2.10) \) which is 1-periodic in the (1,2) direction will also be periodic in the (2,1) direction (because of (3.8)), and a p.d.e. which is 1-periodic is not reducible after any number of steps. Darboux [Da1] proved that every 1-periodic p.d.e. \( (2.10) \) is equivalent under an admissible transformation \( (2.5), (2.6) \) to the Klein-Gordon equation

\[
 z_{xy} = \varepsilon z, \quad \varepsilon^2 = 1. \tag{4.1}
\]

In what follows, we solve the 1-periodicity problem in \( n \) dimensions providing a generalization of Darboux’s theorem.

We begin by establishing necessary and sufficient conditions for the 1-periodicity of a system \( (2.1) \) in terms of its higher-dimensional Laplace invariants.

**Theorem 7**: An \((i,j)\)-transformable system \( (2.1) \) is 1-periodic in \( L(i,j) \) if and only if its higher-dimensional Laplace invariants \( m_{ij}, m_{ijk}, k \neq i, j \) satisfy

\[
m_{ij} = m_{ji} = m_{ijkm_{jki}} \tag{4.2}
\]

\[
m_{ij,k} = 0, \quad (\log m_{ij})_{ij} = 0, \quad \left( \log \frac{m_{ij}}{m_{jki}} \right)_{j} = 0 \tag{4.3}
\]

\[
m_{jk} + m_{jki}m_{jk} = 0, \quad k \neq i, j \tag{4.4}
\]

\[
m_{jik}m_{jk} + m_{kli}m_{ijkl} = 0, \tag{4.5}
\]

and whenever \( n \geq 4 \),

\[
m_{ij, k} = 0, \quad m_{ij}m_{ijkl} - m_{kli}m_{ijkl} = 0, \quad i, j, k, \ell \text{ distinct}. \tag{4.6}
\]

**Proof.** In view of Theorem 1, we know that a system \( S \) will be 1-periodic in \( L(i,j) \) if and only if the higher-dimensional Laplace invariants \( \tilde{m}_{kl} \) and \( \tilde{m}_{klr}, r \neq k, \ell \) of \( \tilde{L}_{ij}(S) \) satisfy, for each ordered pair \((k, \ell)\),

\[
\tilde{m}_{k\ell} = m_{k\ell}, \quad \tilde{m}_{k\ell r} = m_{k\ell r}, \quad r \neq k, \ell. \tag{4.7}
\]

The necessity of conditions \((4.2)\) to \((4.6)\) is easy to establish as a consequence of the periodicity conditions \((4.7)\) and the transformation laws \((3.7)\). We now prove sufficiency. The 1-periodicity conditions

\[
\tilde{m}_{ijk} = m_{ijk}, \quad \tilde{m}_{ij} = m_{ij}, \quad \tilde{m}_{ji} = m_{ji},
\]

\[
\tilde{m}_{jik} = m_{jik}, \quad \tilde{m}_{jki} = m_{jki}, \quad \tilde{m}_{jk} = m_{jk},
\]

\[
\tilde{m}_{jk, k} = m_{jk, k}, \quad k \neq i, j, \tag{4.8}
\]

\[
\tilde{m}_{k\ell} = m_{k\ell}, \quad \tilde{m}_{k\ell r} = m_{k\ell r}, \quad i, j, k, \ell, r \text{ distinct}
\]

follow directly from \((4.2), (4.3), (4.4), (4.5)\) and \((4.6)\). We now consider the remaining 1-periodicity conditions in turn. Using \((3.7)\) and \((4.4)\), the condition \( \tilde{m}_{kj} = m_{kj}, \quad k \neq i, j \), reduces to

\[
(\log m_{ijkl})_{jk} = 0. \tag{4.9}
\]
The third of the equations (4.3) differentiated with respect to $x^k$ gives

$$\left(\log m_{ijk}\right)_{jk} = \left(\log m_{ij}\right)_{jk}.$$  \hspace{1cm} (4.10)

The right-hand side of (4.10) is identically zero by the first equation in (4.3), so that (4.9) is indeed satisfied.

The condition $\tilde{m}_{kj\ell} = m_{kj\ell}, \; i, j, k, \ell$ distinct, is equivalent due to (3.7) to

$$\left(\log m_{ij}\right)_j = \left(\log m_{ij\ell}\right)_j,$$ \hspace{1cm} (4.11)

which is an identity by the third of the equations (4.3). $\tilde{m}_{ki} = m_{ki}$ is equivalent, using (3.7), to

$$m_{ki} (m_{ijk})^2 = -m_{ij} m_{kj}.$$ \hspace{1cm} (4.12)

By (4.5) and (4.2), we have

$$m_{ki} (m_{ijk})^2 = -m_{jk} m_{ij} m_{ijk} = -m_{ij} m_{kj},$$

so that (4.12) is indeed an identity.

Next, we turn to the condition $\tilde{m}_{ki\ell} = m_{ki\ell}, \; i, j, k, \ell$ distinct, which, on account of (3.7), is equivalent to

$$m_{i\ell} m_{ij} m_{ijk} = m_{ij} m_{\ell j}.$$ \hspace{1cm} (4.13)

This relation is obtained by multiplying (4.2) by $m_{ij}$.

Finally, by (3.7) and the first of the conditions (4.3), the condition $\tilde{m}_{ik} = m_{ik}, \; k \neq i, j$, reduces to

$$m_{ki} \left(\log m_{ijk}\right)_i = 0.$$ \hspace{1cm} (4.14)

To prove that (4.14) is an identity under the hypotheses of Theorem 5, we observe using (2.8) that the first of the equations (4.2) is equivalent to

$$a^i_{ij,i} = a^j_{ij,j}.$$ \hspace{1cm} (4.15)

Taking the logarithmic derivative with respect to $x^j$ of the second equation in (4.2), we obtain

$$\left(\log m_{ij}\right)_j = \left(\log m_{ij\ell}\right)_j + \left(\log m_{ijk}\right)_j,$$

which implies, using the third equation (4.3), that

$$m_{ijk,j} = 0.$$ \hspace{1cm} (4.16)

Now we obtain from (2.8), (2.4), (4.15) and (4.16) that

$$m_{ijk,i} = a^k_{kji} - a^i_{ij,i} = a^k_{ki,j} - a^j_{ij,j} = m_{ijk,j} = 0,$$

so that (4.14) is indeed an identity. \hfill \square

We now have all the tools we need to state and prove our $n$-dimensional generalization of Darboux’s result on the Klein-Gordon equation.

**Theorem 8**: An $(i, j)$-transformable system (2.1) is 1-periodic in $L_{(i, j)}$ if and only if it is equivalent under an admissible transformation to a system of the form

$$y_{ij} - \varepsilon_i \varepsilon_j y = 0$$

$$y_{ki} + \varepsilon_k \varepsilon_k y_{ik} + a^k_{kji} y_{ij} + \varepsilon_k \varepsilon_k a^k_{kj} y = 0, \; 1 \leq k \neq i, j \leq n$$

$$y_{kj} + \varepsilon_k \varepsilon_k y_{jk} + a^k_{kj} y_{ij} + \varepsilon_k \varepsilon_k a^k_{ij} y = 0, \; 1 \leq k \neq i, j \leq n$$

$$y_{,k\ell} + a^k_{k\ell} y_{,k} + a^k_{k\ell} y_{,\ell} + c_{k\ell} y = 0, \; 1 \leq k, \ell \leq n, \; i, j, k, \ell \text{ distinct}.$$ \hspace{1cm} (4.17)
where
\[ a^j_{k,j} = 0, \quad a^j_{k,j,i} = 0, \quad a^i_{k,i,j} = 0, \quad a^i_{k,i} = 0, \quad \varepsilon_i^2 = \varepsilon_j^2 = \varepsilon_k^2 = 1. \] (4.18)

**Proof.** The principle of the proof is to integrate the 1-periodicity conditions given in Theorem 7 in terms of the higher-dimensional Laplace invariants while keeping account of the compatibility conditions (2.14) of Theorem 1, and to use the freedom of admissible transformations to normalize the coefficients of the periodic system.

From the transformation laws (2.7), we see that by choosing \( \lambda \) such that
\[ \log \lambda = -\int a^i_{ij} dx_j + f(x_1, \ldots, x_j, \ldots, x_n), \] (4.19)
where \( f \) is an arbitrary function of all the \( x \)'s except for \( x_j \), we can set
\[ a^i_{ij} = 0 \] (4.20)
under the admissible transformation (2.5) corresponding to (4.19). We can now choose the arbitrary function \( f \) in (4.19) so as to have \( \pi^i_{ij} = 0 \). Indeed from the transformation laws (2.7) and from (4.19), we have
\[ \pi^i_{ij} = a^i_{ij} - \int a^i_{ij,i} dx_j + f, \] (4.21)
and the 1-periodicity condition \( m_{ij} = m_{ji} \) (see (4.2)) implies that \( a^i_{ij,i} = a^j_{ji,j} \).

Therefore, it follows from (4.21) that \( f \) can be chosen so that
\[ \pi^i_{ij} = 0. \] (4.22)

From (4.20), (4.22) and (2.8), we therefore have, dropping bars,
\[ a^i_{ij} = 0, \quad a^j_{ij} = 0, \quad m_{ij} = -c_{ij} \] (4.23)
The first and second of the 1-periodicity conditions (4.3) then imply that
\[ c_{ij} = g(x_i)h(x_j), \]
where both \( g \) and \( h \) are non-zero since the system is assumed to be \((i, j)\)-transformable. The \((i, j)\) equation in the system therefore reads
\[ y_{ij} + g(x_i)h(x_j)y = 0, \]
and since \( g \) and \( h \) are non-zero, it is equivalent under an admissible transformation of the form (2.6) on \( x_i \) and \( x_j \) to
\[ y_{ij} - \varepsilon_i \varepsilon_j y = 0, \] (4.24)
where \( \varepsilon_i^2 = \varepsilon_j^2 = 1 \). Note that we still have the freedom to perform admissible transformations of the form (2.6) on \( x_k \) for \( k \neq i, j \).

From (2.8), the third of the conditions (4.3) and the first of the conditions (4.6) we obtain
\[ a^k_{k,j,\ell} = 0, \quad a^k_{k,j,j} = 0, \]
where \( i, j, k \) and \( \ell \) are distinct, so that \( a^k_{k,j} = \ell(x_k) \),
where \( \ell \) is non-zero since the system is \((i, j)\)-transformable. By an admissible transformation (2.6) on \( x_k \), we can therefore normalize \( a^k_{kj} \),

\[
a^k_{kj} = \varepsilon_j \varepsilon_k, \quad \varepsilon_k^2 = 1. \tag{4.25}
\]

The second of the 1-periodicity conditions (4.2) combined with (4.24) and (4.25) now gives

\[
a^k_{ki} = \varepsilon_i \varepsilon_k \tag{4.26}
\]

It is now straightforward to check, using (2.8), (4.25) and (4.26), that the second of the 1-periodicity conditions (4.6) and the first, fourth, and fifth of the compatibility conditions (2.14) are identically satisfied. We thus still have to impose the 1-periodicity conditions (4.4) and (4.5) and the remaining compatibility conditions in (2.14), namely the second and third. The second condition in (2.14) gives

\[
c^3_{kj} = \varepsilon_k \varepsilon_j a^i_{ik}, \quad k \neq i, j. \tag{4.27}
\]

while the third condition in (2.14) gives

\[
c^3_{ik} = a^i_{ik,j} + \varepsilon_i \varepsilon_k a^j_{jk}, \quad k \neq i, j. \tag{4.28}
\]

The substitution of (2.8), (4.23), (4.25), (4.26), (4.27) and (4.28) into (4.4) and (4.5) gives respectively

\[
a^j_{ik,j} = 0, \quad a^i_{ik,j} = 0, \quad k \neq i, j, \tag{4.29}
\]

and (4.28) therefore becomes

\[
c^3_{ik} = \varepsilon_i \varepsilon_k a^j_{jk}. \tag{4.30}
\]

The normal form (4.17) follows then from (4.24), (4.25), (4.26), (4.27), (4.28), and (4.30).

In dimension three, we have the remarkable fact that every system which is 1-periodic with respect to one of the higher-dimensional Laplace transforms and transformable in all other directions is also 1-periodic in all other directions. More precisely, we have the following result

**Theorem 9:** A compatible system of type (2.1) and \( n = 3 \) is 1-periodic with respect to one of its higher dimensional Laplace transforms if and only if, after a relabelling of independent variables, it is equivalent under an admissible transformation to a system of the form

\[
\begin{align*}
y_{12} - \varepsilon_1 \varepsilon_2 y &= 0, \\
y_{31} + \varepsilon_1 \varepsilon_3 y_{13} + f y_{11} + \varepsilon_1 \varepsilon_3 \varepsilon_3 g y &= 0, \\
y_{32} + \varepsilon_2 \varepsilon_3 y_{13} + g y_{22} + \varepsilon_2 \varepsilon_3 f y &= 0,
\end{align*} \tag{4.31}
\]

where \( \varepsilon_i^2 = 1, 1 \leq i \leq 3, \) \( f \) and \( g \) are differentiable functions of \( x_3 \) only. If \( f \neq g \), then the system (4.31) is 1-periodic with respect to all higher-dimensional Laplace transforms.

If \( f = g \), then the general solution of (4.31) is given, in terms of the solutions of the Klein-Gordon equation, by

\[
y = e^{-h - \varepsilon_3 (\varepsilon_1 x_1 + \varepsilon_2 x_2)} \int e^{h} F(x_3) \, dx_3 + e^{-h} z(x_1, x_2)
\]
where \( h(x_3) \) is such that \( h' = f \), \( F \) is an arbitrary function of \( x_3 \) and \( z(x_1, x_2) \) is a solution of the equation
\[
z_{,12} - \varepsilon_1 \varepsilon_2 z = 0.
\]

**Proof.** Without loss of generality we will consider 1-periodicity in \( L(1,2) \). As a consequence of Theorem 6 we obtain (4.31), whose higher Laplace invariants are given by
\[
m_{12} = m_{21} = \varepsilon_1 \varepsilon_2, \quad m_{213} = -m_{312} = \varepsilon_1 \varepsilon_3, \quad m_{123} = -m_{321} = \varepsilon_2 \varepsilon_3,
\]
\[
m_{13} = m_{31} = \varepsilon_1 \varepsilon_3 (f - g), \quad m_{23} = m_{32} = -\varepsilon_2 \varepsilon_3 (f - g), \quad m_{231} = -m_{132} = f - g.
\]

It is easy to check, using Theorem 5, that if \( f - g \) does not vanish then (4.31) is 1-periodic with respect to all higher-dimensional Laplace transforms. If \( f = g \), then the system is (1,3)-reducible and it follows from Theorem 3 that its general solution is given by
\[
y = Q + e^{-J} G(x_1, x_2) \quad (4.32)
\]
where
\[
Q = e^{-J} \int e^{J} F(x_3) dx_3, \quad I = \int \varepsilon_1 \varepsilon_3 dx_1, \quad J = \int f(x_3) dx_3, \quad (4.33)
\]
where \( F \) is an arbitrary function of \( x_3 \), \( G \) does not depend on \( x_3 \) and where the antiderivative \( I \) is chosen so that \( I_{,2} = \varepsilon_2 \varepsilon_3 \). Then \( G \) satisfies the system
\[
G_{,12} + g_{12}^1 G_{,1} + g_{12}^2 G_{,2} + b_{12} G = 0, \quad (4.34)
\]
where
\[
g_{12}^1 = -J_{,2}, \quad g_{12}^2 = -J_{,1}, \quad b_{12} = -\varepsilon_1 \varepsilon_2 + J_{,1} J_{,2} - J_{,12}. \quad (4.35)
\]
Let \( h(x_3) \) be an antiderivative of \( f \), i.e.
\[
J = h(x_3) + t(x_1, x_2).
\]

We introduce a new function \( z(x_1, x_2) \) by \( z = e^{-t} G \). Then using (4.35), equation (4.34) reduces to the Klein-Gordon equation for \( z \) and we conclude the proof by using (4.32) and (4.33).

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