A NOTE ON THE LOCALIZATION NUMBER OF RANDOM GRAPHS:
DIAMETER TWO CASE

ANDRZEJ DUDEK, ALAN FRIEZE, AND WESLEY PEGDEN

Abstract. We study the localization game on dense random graphs. In this game, a cop tries to locate a robber by asking for the graph distance of y from every vertex in a sequence of sets $W_1, W_2, \ldots, W_\ell$. We prove high probability upper and lower bounds for the minimum size of each $W_i$ that will guarantee that $x$ will be able to locate $y$.

1. Introduction

In this paper we consider the following Localization Game related to the well studied Cops and Robbers game; see Bonato and Nowakowski [2] for a survey on this game. A robber is located at a vertex $v$ of a graph $G$. In each round, a cop can ask for the graph distance between $v$ and vertices $W = \{w_1, w_2, \ldots, w_k\}$, where a new set of vertices $W$ can be chosen at the start of each round. The cop wins immediately if the $W$-signature of $v$, i.e. the set of distances $\text{dist}(v, w_i)$, $i = 1, 2, \ldots, k$ is sufficient to determine $v$. Otherwise, the robber will move to a neighbor of $v$ and the cop will try again with a (possibly) different test set $W$. Given $G$, the localization number $\zeta(G)$ is the minimum $k$ so that the cop can eventually locate the robber, that means, the cop determines the exact location of the robber from the test sets of size $k$. This game was introduced by Bosek et al. [3], who studied the localization game on geometric and planar graphs, and also independently, by Haslegrave et al. [6]. For some other related results see [4, 8, 9].

2. Results

The localization number is closely related to the metric dimension $\beta(G)$. This is the smallest integer $k$ such that the cop can always win the game in one round. Clearly, $\zeta(G) \leq \beta(G)$.

In this note we will study the localization number of the random graph $G_{n,p}$ with diameter two. Here and throughout the whole paper $\omega = \omega(n) = o(\log n)$ denotes a function tending arbitrarily slowly to infinity with $n$. We will also use the notation

$$q = 1 - p \quad \text{and} \quad \rho = p^2 + q^2.$$ 

We write $A_n \precsim B_n$ to mean that $A_n \leq (1 + O(1))B_n$ as $n$ tends to infinity. We further write $A_n \approx B_n$ if $A_n = (1 + O(1))B_n$ as $n$ tends to infinity. Finally, we say that an event $\mathcal{E}_n$ occurs asymptotically almost surely, or a.a.s. for brevity, if $\lim_{n \to \infty} \Pr(\mathcal{E}_n) = 1$.

The metric dimension of $G_{n,p}$ was studied by Bollobás et al. [1]. If we specialize their result to large $p$ then it can be expressed as:

The first author was supported in part by a grant from the Simons Foundation (522400, AD).
The second author was supported in part by NSF grant DMS1661063.
The third author was supported in part by NSF grant DMS136313.
Theorem 2.1. Suppose that
\[
\left( \frac{2 \log n + \omega}{n} \right)^{1/2} \leq p \leq 1 - \frac{3 \log \log n}{\log n}.
\]

Then,
\[
\frac{2 \log np}{\log 1/p} \approx \beta(G_{n,p}) \approx \frac{2 \log n}{\log 1/p} \text{ a.a.s.} \quad (1)
\]

Note that the upper and lower bounds in (1) are asymptotically equal if \( p \geq n^{-o(1)} \).

It is well-known (see, e.g., [5]) that if \( np^2 \geq 2 \log n + \omega \), then a.a.s. \( \text{diam}(G_{n,p}) \leq 2 \). We will condition on the diameter satisfying this. Graphs with diameter 2 enable some simplifications. Indeed, if a vertex \( v \) has \( W \)-signature \( \{d_1, \ldots, d_k\} \), where \( W = \{w_1, \ldots, w_k\} \),
\[
d_i = \begin{cases} 1 & \text{iff } \{v, w_i\} \in E \\ 2 & \text{iff } \{v, w_i\} \notin E. \end{cases}
\]

Consequently, the probability that two vertices \( u \) and \( v \) in \( G_{n,p} \) have the same \( W \)-signature,
\( W = \{w_1, \ldots, w_k\} \), such that \( u, v \notin W \) is equal to
\[
\prod_{i=1}^{k} \Pr(u, v \in N(w_i) \text{ or } u, v \notin N(w_i)) = \rho^k.
\]

The upper bound on \( p \) in the below theorem is determined by a result of [1] about the metric dimension of \( G_{n,p} \).

Theorem 2.2. Let
\[
\left( \frac{2 \log n + \omega}{n} \right)^{1/2} \leq p \leq 1 - \frac{3 \log \log n}{\log n} \quad \text{and} \quad \eta = \frac{\log(1/p)}{\log n}.
\]

and let \( c \) be a positive constant such that
\[
0 < c < \min \left\{ \frac{1}{2} \left( \frac{\log n - 3 \log \log n}{\log 1/p} - 1 \right), 1 \right\}.
\]

Then, a.a.s.
\[
\left( 1 - 2\eta - \frac{4 \log \log n}{\log n} \right) \frac{2 \log n}{\log 1/p} \leq \zeta(G_{n,p}) \leq (1 - cn) \frac{2 \log n}{\log 1/p}.
\]

2.1. Observations about Theorem 2.2.
First observe that if \( p \geq \frac{\log n}{n^{1/3}} \), then
\[
\frac{1}{2} \left( \frac{\log n - 3 \log \log n}{\log 1/p} - 1 \right) \geq 1
\]

and so \( c \) can be any positive constant less than 1. Furthermore, for any \( p \geq \left( \frac{2 \log n + \omega}{n} \right)^{1/2} \)
we have
\[
\frac{1}{2} \left( \frac{\log n - 3 \log \log n}{\log 1/p} - 1 \right) \geq \frac{1}{2} \left( \frac{\log n - 3 \log \log n}{\log (2 \log n + \omega)} - 1 \right) = \frac{1}{2} - o(1).
\]

Hence, we can always take \( c = \frac{1}{2} - o(1) \).
If \( p = 1/n^\alpha \) for some constant \( 0 < \alpha < 1/2 \), then,

\[
\eta = \alpha \quad \text{and} \quad c \leq \begin{cases} 1 - o(1) & \text{if } 0 < \alpha < \frac{1}{3} \\ \frac{1}{2\alpha} - \frac{1}{2} - o(1) & \text{otherwise.} \end{cases}
\]

Moreover,

\[
\rho = 1 - 2p + 2p^2 \quad \text{and so} \quad \log\frac{1}{\rho} = 2p + O(p^2) \approx \frac{2}{n^\alpha}.
\]

Hence, Theorem 2.2 implies the following corollary.

**Corollary 2.3.** Let \( p = 1/n^\alpha \), where \( 0 < \alpha < 1/2 \) is constant. Then, a.a.s.

\[
(1 - 2\alpha)n^\alpha \log n \lesssim \zeta(G_{n,p}) \lesssim \begin{cases} (1 - \alpha)n^\alpha \log n & \text{if } 0 < \alpha < \frac{1}{3} \\ \frac{(1+\alpha)^2}{2} n^\alpha \log n & \text{otherwise.} \end{cases}
\]

Notice that for \( 0 < \alpha < \frac{1}{3} \) the upper bound on \( \zeta(G_{n,p}) \) equals the lower bound from Theorem 2.1. Therefore, it is plausible to conjecture that \( \zeta(G_{n,p}) < \beta(G_{n,p}) \).

Now observe that if \( p = n^{-1/\omega} \), then

\[
2\eta = \frac{2\log(1/p)}{\log n} = \frac{2}{\omega} = o(1).
\]

Thus, Theorem 2.2 implies:

**Corollary 2.4.** Let \( p = n^{-1/\omega} \). Then,

\[
\zeta(G_{n,p}) \approx \frac{2\log n}{\log 1/\rho}.
\]

Clearly, this also holds for any constant \( p \). In particular, for \( p = 1/2 \), we get:

**Corollary 2.5.** For almost all graphs \( G \) we have

\[
\zeta(G) \approx \frac{2\log n}{\log 2} = 2\log_2(n).
\]

### 2.2. Proof of Theorem 2.2 — lower bound.

Since we will deal with “mostly independent” random variables, we will use the following form of Suen’s inequality (see, e.g. [7]).

**Theorem 2.6** (Suen’s Inequality). Let \( \theta_i, i \in I \) be indicator random variables which take value 1 with probability \( p_i \). Let \( L \) be a dependency graph. Let \( X = \sum_{i \in I} \theta_i \), and \( \mu = E(X) = \sum_{i \in I} p_i \). Moreover, write \( i \sim j \) if \( ij \in E(L) \), and let \( \Delta = \frac{1}{2} \sum_{i \sim j} E(\theta_i \theta_j) \) and \( \delta = \max_i \sum_{j \sim i} p_j \). Then,

\[
\Pr(X = 0) \leq \exp \left\{ - \min \left\{ \frac{\mu^2}{8\Delta}, \frac{\mu}{2}, \frac{\mu}{2\delta} \right\} \right\}.
\]

We will also use the following simple fact.

**Lemma 2.7.** Let \( 0 < p < 1 \) and \( p + q = 1 \). Then,

\[
\frac{\log(p^3 + q^3)}{\log \rho} \geq \frac{3}{2}.
\]
Proof. This inequality is equivalent to
\[
\log(p^3 + q^3)^2 \leq \log(p^2 + q^2)^3
\]
and so to
\[
(p^3 + q^3)^2 \leq (p^2 + q^2)^3.
\]
The latter is equivalent to
\[
2p^3q^3 \leq 3p^4q^2 + 3p^2q^4 = 3p^2q^2(p^2 + q^2) = 3p^2q^2(1 - 2pq)
\]
and consequently to
\[
2pq \leq 3(1 - 2pq)
\]
which is equivalent to
\[
pq \leq \frac{3}{8}.
\]
But this is always true since \(pq \leq \frac{1}{4}\). \(\square\)

The lower bound in Theorem 2.2 will follow from the following result.

Lemma 2.8. Let
\[
\frac{\log^2 n}{n^{1/2}} < p \leq 1 - \frac{1}{\log n} \quad \text{and} \quad \varepsilon = \frac{2\log \left( \frac{\log^2 n}{p} \right)}{\log n} \quad \text{and} \quad k = \frac{2(1 - \varepsilon) \log n}{\log 1/\rho}.
\]
Then a.a.s.,
\[
\zeta(G_{n,p}) \geq k.
\]
First observe that \(\varepsilon = 2\eta + \frac{4\log \log n}{\log n}\) and so the lower bound in Theorem 2.2 holds.

Proof. For a fixed vertex \(u\) and \(k\)-set \(S\) let \(X_{u,S}\) count the number of unordered pairs \(w, v \in N(u)\) with the same signature induced by \(S\). We prove that the probability that there is a vertex \(u\) and a \(k\)-set \(S\) such that \(X_{u,S} = 0\) is \(o(1)\). Consequently, this will imply that a.a.s. for every vertex \(u\) and \(k\)-set \(S\) there are at least two neighbors of \(u\) with the same signature in \(S\). Hence, a.a.s. the localization number is at least \(k\).

Clearly,
\[
\mu = E(X_{u,S}) = \binom{n - k - 1}{2} \rho^k p^2 \geq \frac{p^2}{4} \exp\{k \log \rho + 2 \log n\}
\]
\[
= \frac{p^2}{4} \exp\{-2(1 - \varepsilon) \log n + 2 \log n\} = \frac{p^2}{4} n^{2\varepsilon}.
\]
Furthermore, since every triple of vertices in \(N(u)\) with the same signature contributes three unordered pairs of variables to \(\Delta\), we get
\[
\Delta \leq 3 \left( \binom{n}{3} (p^3 + q^3)^k p^3 \right)
\]
\[
\leq \frac{p^3}{2} \exp\{k \log(p^3 + q^3) + 3 \log n\}
\]
\[
= \frac{p^3}{2} \exp\{ -2(1 - \varepsilon)(\log n) \frac{\log(p^3 + q^3)}{\log \rho} + 3 \log n\}.
\]
Now, by Lemma 2.7
\[ \Delta \leq \frac{p^3}{2} \exp \left\{ -2(1 - \varepsilon)(\log n) \cdot \frac{3}{2} + 3 \log n \right\} = \frac{p^3}{2} n^{3\varepsilon}. \]
Similarly
\[ \delta \leq 2n\rho^k p^2 = 2p^2 \exp (k \log \rho + \log n) = 2p^2 n^{-1+2\varepsilon}. \]
Thus,
\[ \frac{\mu^2}{8\Delta} \geq \frac{1}{64} pn^\varepsilon, \quad \frac{\mu}{2} \geq \frac{1}{8} (pn^\varepsilon)^2 \quad \text{and} \quad \frac{\mu}{6\delta} \geq \frac{1}{48} n. \]
Since $0 < \varepsilon < 1$ and $pn^\varepsilon \to \infty$ (due to our choice of $\varepsilon$) the lower bound in the first inequality is the smallest. Hence, by Theorem 2.6
\[ \Pr(X_{u,S} = 0) \leq \exp \left\{ -\frac{1}{64} pn^\varepsilon \right\}. \]
Now we use the union bound to show that the probability that there is a vertex $u$ and a $k$-set $S$ such that $X_{u,S} = 0$ is $o(1)$. Indeed, this probability is at most
\[ n \binom{n}{k} \exp \left\{ -\frac{1}{64} pn^\varepsilon \right\} \leq \exp \left\{ (k+1) \log n - \frac{1}{64} pn^\varepsilon \right\}. \tag{2} \]
Now observe that $\rho = (p+q)^2 - 2pq = 1 - 2pq$ and so
\[ k = \frac{2(1 - \varepsilon) \log n}{\log 1/\rho} = -\frac{2(1 - \varepsilon) \log n}{\log(1 - 2pq)} \leq -\frac{2 \log n}{\log(1 - 2pq)}. \]
Since $1 - x \leq e^{-x}$ and $2pq < 1$, we get that
\[ k \log n \leq \frac{(\log n)^2}{pq}. \]
Furthermore, since by assumption $p \leq 1 - \frac{1}{\log n}$, we obtain $q \geq \frac{1}{\log n}$ and so
\[ k \log n \leq \frac{(\log n)^3}{p}. \]
Also
\[ pn^\varepsilon = pe^{\varepsilon \log n} = \frac{(\log n)^4}{p}. \]
Thus, the exponent in (2) tends to $-\infty$. This completes the proof of Lemma 2.8. \[\square\]

2.3. Proof of Theorem 2.2 – upper bound.

We will need the following auxiliary result:

**Proposition 2.9.** Let
\[ \sqrt{\frac{2 \log n + \omega}{n}} \leq p \leq 1 - \frac{1}{\log n} \quad \text{and} \quad \eta = \frac{\log 1/p}{\log n} \quad \text{and} \quad k = \frac{2(1 - c\eta) \log n}{\log 1/\rho}, \]
where
\[ 0 < c < \min \left\{ \frac{1}{2} \left( \frac{\log n - 3 \log \log n}{\log 1/p} - 1 \right), 1 \right\}. \]
Let $G = G_{n,p} = (V,E)$ and let $U \subseteq V$ and $S \subseteq V$ be disjoint subsets such that $|U| = O(k)$ and $|S| = k$. Then, a.a.s. there is no pair $u \in U$ and $v \in V \setminus S$ such that $u$ and $v$ have the same signature induced by $S$.

Proof. Assume that $\ell$ is a positive constant and $|U| = \ell k$. The probability that there is a pair $u \in U$ and $v \in V \setminus S$ such that $u$ and $v$ have the same signature induced by $S$ is at most

$$|U| \cdot |V| \cdot \rho^k \leq \ell k \cdot \exp \left\{ \log n + k \log \rho \right\} = \ell k \cdot n^{2\eta - 1}.$$ 

But

$$k \leq \frac{2 \log n}{\log 1/\rho} = \frac{2 \log n}{\log(1 - 2pq)} \leq \frac{\log n}{pq},$$

since $1 - x \leq e^{-x}$ and $2pq < 1$. Furthermore, since $q \geq \frac{1}{\log n}$ we get

$$k \leq \frac{(\log n)^2}{p}.$$ 

Similarly,

$$n^{2\eta - 1} = \frac{\exp\{2c\eta \log n\}}{p} = \frac{1}{p^{2c\eta}}.$$ 

Thus,

$$\ell k \cdot n^{2\eta - 1} \leq \ell \frac{(\log n)^2}{p} \cdot \frac{1}{p^{2c\eta}} = \ell \frac{(\log n)^2}{p^{1+2c\eta}} \leq \frac{\ell}{\log n} = o(1),$$

where the latter inequality follows from the choice of $c$. □

Lemma 2.10.

(i) Let

$$e^{-\frac{\log n}{\omega}} \leq p \leq 1 - \frac{3 \log \log n}{\log n}.$$ 

Then, a.a.s.

$$\zeta(G_{n,p}) \leq \frac{2 \log n}{\log 1/\rho}.$$ 

(ii) Let

$$\left(\frac{2 \log n + \omega}{n}\right)^{1/2} \leq p \leq e^{-O(\log n)} \quad \text{and} \quad \eta = \frac{\log 1/p}{\log n} \quad \text{and} \quad k = \frac{2(1 - c\eta) \log n}{\log 1/\rho},$$

where

$$0 < c < \min \left\{ \frac{1}{2} \left(\frac{\log n - 3 \log \log n}{\log 1/p} - 1\right), 1 \right\}.$$ 

Then, a.a.s.

$$\zeta(G_{n,p}) \leq k.$$ 

Proof. Part (i) follows immediately from Theorem 2.1.

Here we prove (ii). Let $S_1, \ldots, S_\ell$ be pairwise disjoint subsets of $V$ such that $|S_i| = k$ and $\ell = O(1)$ and let $T_1 = V$. Now we reveal all edges between $S_1$ and $V \setminus S_1$. Let $X_1$ be the number of pairs with the same signature in $S_1$. Then,

$$E(X_1) \leq n^2 \rho^k = \exp\{2 \log n + k \log \rho\} = n^{2\eta}.$$
and by the Markov inequality we have \(X_i \leq \omega n^{2c\eta}\) a.a.s.. Thus, the set \(R\) of vertices with exactly the same signature in \(S\) as the robber is a.a.s. of size at most \(\omega^{1/2}n^{c\eta}\). Also it is well known (see e.g. [5]) that each vertex a.a.s. has \(\leq pn\) neighbors. Let \(T_2\) consist of \(R\) and the set of neighbors of \(R\). The robber can move to somewhere in \(T_2\). Clearly, \(|T_2| \leq 2\omega^{1/2}n^{c\eta}pn\) a.a.s.

Now we start the second round by revealing the edges between \(S_2\) and \(V\setminus(S_1 \cup S_2)\). Let \(X_2\) be the number of pairs with the same signature in \(S_2\). By Proposition 2.9 we can assume that the only pairs with the same signature induced by \(S_2\) are in \(V\setminus(S_1 \cup S_2)\). Thus,

\[
E(X_2) \leq (2\omega^{1/2}n^{c\eta}pn)^2 \rho^k = (2\omega^{1/2}p)^2 \exp((2 + 2c\eta)(\log n) + k \log \rho) = 4\omega p^2 n^{4c\eta}
\]

and by the Markov inequality we get that a.a.s we have \(X_2 \leq \omega^2 p^2 n^{4c\eta}\). Thus, the number of vertices with exactly the same signature as the robber in \(S_2\) is at most \(\omega pn^{2c\eta}\). Let \(T_3\) consist of these vertices together with their neighbors. Clearly, \(|T_3| \leq 2\omega^2 p^2 n^{2c\eta+1}\).

We proceed inductively. Assume that \(|T_i| \leq 2(\omega^{1/2}p)^{i-1}n^{(i-1)c\eta+1}\). Now

\[
E(X_{i+1}) \leq 2((\omega^{1/2}p)^{i-1}n^{(i-1)c\eta+1})^2 \rho^k = 2(\omega^{1/2}p)^{2(i-1)}n^{2ic\eta}
\]

and so by the Markov inequality,

\[
X_{i+1} \leq \omega(\omega^{1/2}p)^{2(i-1)}n^{2ic\eta}\text{ a.a.s..}
\] (3)

Thus, the number of vertices with exactly the same signature in \(S_{i+1}\) is at most \(\omega^{1/2}(\omega^{1/2}p)^{i-1}n^{ic\eta}\). Hence,

\[
|T_{i+1}| \leq 2\omega^{1/2}(\omega^{1/2}p)^{i-1}n^{ic\eta}pn = 2(\omega^{1/2}p)^i n^{ic\eta+1},
\]

completing the induction.

After \(\ell\) rounds we get that with probability at least \(1 - \ell \omega^{-1}\) we have, using (3),

\[
|X_\ell| \leq \omega(\omega^{1/2}p)^{2(\ell-2)}n^{2(\ell-1)c\eta} = \omega^{\ell-1} \exp \{2(\ell - 2) \log p + 2(\ell - 1)c\eta \log n\}
\]

\[
= \omega^{\ell-1} \exp \{-2(\ell - 2 - c(\ell - 1)) \log(1/p)\}.
\] (4)

Clearly, (4) is \(o(1)\) for sufficiently large constant \(\ell\), since by assumption \(\log(1/p) = \Omega(\log n)\).

\[\square\]

3. Summary

We have separated the localization value \(\zeta(G_{n,p})\) from the metric dimension \(\beta(G_{n,p})\) in the range where the diameter of \(G_{n,p}\) is two a.a.s.. It would be interesting to continue the analysis in the range of \(p\) for which the diameter of \(G_{n,p}\) is at least 3. It would also be of interest to examine the localization game on random regular graphs.

Acknowledgment We are grateful to all referees for their detailed comments on an earlier version of this paper.

References

[1] B. Bollobás, P. Prałat and D. Mitsche, Metric dimension for random graphs, \textit{The Electronic Journal of Combinatorics} 20 (2013).

[2] A. Bonato and R. Nowakowski. \textit{The game of cops and robbers on graphs} American Mathematical Society, 2011.

[3] B. Bosek, P. Gordinowicz, J. Grytczuk, N. Nisse, J. Sokół and M. Śleszyńska-Nowak, \textit{Localization game on geometric and planar graphs} \texttt{arXiv:1709.05904}.
8

ANDRZEJ DUDEK, ALAN FRIEZE, AND WESLEY PEGDEN

[4] J. Carraher, I. Choi, M. Delcourt, L. H. Erickson, and D. B. West, Locating a robber on a graph via distance queries, *Theoretical Computer Science* 463, pp. 54–61 (2012).

[5] A.M. Frieze and M. Karoński, Introduction to Random Graphs, *Cambridge University Press*, 2015.

[6] J. Haslegrave, R. Johnson and S. Koch, Locating a robber with multiple probes, *Discrete Mathematics* 341 (2018), no. 1, 184–193.

[7] S. Janson, T. Łuczak and A. Ruciński, Random Graphs, *Wiley*, 2000.

[8] S. Seager, Locating a robber on a graph, *Discrete Mathematics* 312, pp. 3265–3269 (2012).

[9] S. Seager, Locating a backtracking robber on a tree, *Theoretical Computer Science* 539, pp. 28–37 (2014).

Department of Mathematics, Western Michigan University, Kalamazoo, MI

E-mail address: andrzej.dudek@wmich.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA

E-mail address: alan@random.math.cmu.edu

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA

E-mail address: wes@math.cmu.edu