Doubly special relativity in de Sitter spacetime

S. Mignemi†

Dipartimento di Matematica, Università di Cagliari
viale Merello 92, 09123 Cagliari, Italy
and INFN, Sezione di Cagliari

Abstract

We discuss the generalization of Doubly Special Relativity to a curved de Sitter background. The model has three observer-independent scales, the velocity of light $c$, the radius of curvature of the geometry $\alpha$, and the Planck energy $\kappa$, and can be realized in a noncommutative position space. It is possible to construct a model exhibiting a duality for the interchange of positions and momenta together with the exchange of $\alpha$ and $\kappa$.

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† e-mail: smignemi@unica.it
1. Introduction

Since the early years of general relativity, de Sitter and anti-de Sitter spaces have acquired a fundamental importance, both theoretical and phenomenological, especially in the context of cosmology. Indeed, recent astrophysical observations seem to indicate that our universe has positive cosmological constant [1].

In spite of their relevance, there is not much literature about the extension of the kinematics of special relativity to de Sitter or anti-de Sitter backgrounds†. Geometrically, de Sitter space is defined as a space of constant positive curvature. Its isometries are generated by the de Sitter algebra, that can be considered as a deformation of the Poincaré algebra with a parameter $\alpha = 1/\sqrt{\Lambda}$ of dimension of length. Of course, several geometric and algebraic properties of de Sitter space differ from those of Minkowski space. For example, contrary to flat space, in de Sitter space the generators of the translations cannot be identified with the canonical momenta, as is obvious from the position dependence of the de Sitter Hamiltonian. Moreover, there is no natural parametrization of the space and, depending on the specific problem one is studying, different systems of coordinates can be more convenient.

A different kind of deformation of special relativity is given by the more recent proposal of deformed (or doubly) special relativity (DSR) [4]. This theory is based on the generalization of the standard energy-momentum dispersion law of particles $P^2 = m^2$. The deformation is achieved by modifying the action of the Lorentz group on momentum space by means of a new observer-independent constant $\kappa$, with the dimensions of energy (usually identified with the Planck energy). In this framework, the transformation laws of momenta become nonlinear, and that of positions momentum dependent. Special relativity is recovered in the limit $\kappa \to 0$. Different choices of the deformed dispersion law correspond to different DSR models and, even imposing suitable physical constraints, there exist in principle infinite inequivalent models. The physical motivations for the introduction of DSR are given by the possibility of explaining some anomalies observed in high-energy cosmic ray distribution [5] by means of deformed dispersion relations and by the theoretical requirement that the Planck energy, which sets the scale for quantum gravity, be invariant under Lorentz transformations‡.

Algebraically, DSR theories can be realized in two equivalent ways, either identifying the generators of the translations with the phase space momenta and then deforming the Poincaré group, as in the $\kappa$-Minkowski algebra approach [7], or by maintaining the form of the Poincaré algebra, but making it act nonlinearly on momentum space, as in the case of the MS model [8]. The second approach is especially convenient in the case of de Sitter algebra, where, as mentioned above, even classically the generators of the translations do not coincide with the canonical momenta.

From a physical perspective, the geometry of the spacetime on which DSR models operate has a fundamental importance. Unfortunately, however, DSR models are usually

† To our knowledge, this is discussed only in [2,3] for a specific choice of coordinates.
‡ More precisely, in most DSR models, the energy $\kappa$ is not left invariant by the deformed transformations, but sets an observer-independent limit on the energy or momentum of particles. An extreme example of this fact is given by the Snyder model [6], where Lorentz transformations act in the canonical (linear) way, and only the action of the translation generators is nonlinear.
defined only in momentum space and the spacetime geometry is not fixed uniquely from their postulates. Although it is possible to define DSR theories in ordinary spacetime, their most natural realization appears nevertheless to be in terms of noncommutative geometry, with momentum-dependent metric [9-11]. The momentum dependence of the metric has lead to a proposal for a generalization of general relativity that allows for the dependence of the geometry on the energy at which it is probed [12].

The simplest way to construct a DSR model starting from canonical special relativity was suggested in [13]: one can define the physical momenta as functions of auxiliary variables which transform in the standard way under Lorentz transformations. The deformed transformation laws and dispersion relations of the physical momenta then follow from this definition. More recently, it has been shown that also the definition of a suitable non-commutative position space can be obtained by an analogous procedure [14].

Algebraically, de Sitter space and the DSR momentum space have a very similar structure, both being realized by imposing a quadratic constraint on the coordinates of a five-dimensional space [15]. However, their physical interpretation is different: de Sitter space has a natural riemannian structure, and one can choose arbitrary coordinates on it; momentum space has no such structure, and different parametrizations cannot be interpreted as physically equivalent, unless further structure is added. In fact, they lead to inequivalent DSR models with different dispersion relations. Of course, a rigorous discussion of this topic requires a precise operational definition of momentum measurements.

In this paper, we extend DSR models to the case of de Sitter spacetime. The first example of a DSR deformation of the de Sitter algebra, limited to the momentum sector of phase space, was given in [16]. Later, the authors of [17] gave a different realization, extended to the full phase space. However, their approach was purely algebraic, since they did not define a metric structure on de Sitter space. This may lead to ambiguities in the interpretation of the spacetime structure.

It is interesting to remark that the deformed de Sitter algebra has two invariant scales, beyond the speed of light. These are the cosmological constant \( \Lambda \) and the Planck energy \( \kappa \) (or equivalently, the radius of curvature \( \alpha \sim 10^{25} \mathrm{m} \) and the Planck length \( 1/\kappa \sim 10^{-35} \mathrm{m} \)). The two scales differ by 60 orders of magnitude and are related to the opposite extrema of the range of observable physical phenomena. The origin of such difference is not explained by modern physics. One of the models discussed in this paper possesses a duality for the interchange of \( \alpha \) and \( \kappa \) together with the interchange of positions and momenta.

The paper is organized as follows: in section 2 we discuss the de Sitter algebra and different parametrizations of de Sitter space, realized as a hyperboloid embedded in five-dimensional spacetime. In section 3 we discuss the dynamics of a free particle in de Sitter space. Section 4 and 5 are devoted to the study of the generalization of the MS model to de Sitter space. In section 6, a different generalization of DSR in de Sitter space is considered, related to the Snyder model. An alternative realization is given in section 7. In section 8, some physical implications of our results are discussed.

Although we shall not consider this issue in detail, all our result can be straightforwardly extended to the anti-de Sitter case, by simply changing the sign of the cosmological constant.

We use the following notations: \( A, B = 0, \ldots, 4; \mu, \nu = 0, \ldots, 3; i,j = 1, \ldots, 3 \). Ex-
cept when dealing explicitly with the spacetime metric $g_{\mu\nu}$, we always use lower indices, for example $X_\mu \equiv \eta_{\mu\nu}X^\nu$, where $X^\mu$ are the natural (contravariant) coordinates. The manipulation of indices are always performed with the flat metric $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$, and not with the metric $g_{\mu\nu}$. The product between two 4-vectors $\eta_{\mu\nu}V^\mu W^\nu = \eta^{\mu\nu}V_\mu W_\nu$ is denoted by $V \cdot W$, and if $V = W$ by $V^2$. For 5-vectors, we write the indices explicitly. We also use coordinates without superscripts when dealing with expressions that do not depend on the specific choice of coordinates.

2. de Sitter space

We review some properties of de Sitter space and its symmetry group, which are not easily found in the literature. In particular, we discuss some coordinate systems that will be useful in the following. Unfortunately, contrary to Minkowski space, de Sitter space does not admit a natural choice of coordinates. In particular, as we shall see, different quantities have simpler expressions in different coordinate systems. We shall therefore alternate between them, depending on the subject under consideration.

2.1. Generalities

It is well known that de Sitter space can be realized as a hyperboloid of equation $\xi_A^2 = -\alpha^2$ embedded in 5-dimensional flat space, with coordinates $\xi_A$ and metric tensor $\eta_{AB} = \text{diag}(1,-1,-1,-1,-1)$. In the following, we shall often use the traditional notation $\Lambda = 1/\alpha^2$ for the cosmological constant.

The isometries of de Sitter space are generated by the de Sitter algebra. This can be identified with the Lorentz algebra $so(1,4)$ of the 5-dimensional space, which leaves invariant the hyperboloid. The generators $J_{AB}$ of the Lorentz algebra read, in terms of the 5-dimensional canonical positions $\xi_A$ and momenta $\pi_A$, $J_{AB} = \xi_A \pi_B - \xi_B \pi_A$, and obey the Poisson brackets

$$\{J_{AB}, J_{CD}\} = \eta_{BC}J_{AD} - \eta_{BD}J_{AC} + \eta_{AD}J_{BC} - \eta_{AC}J_{BD}. \quad (2.1)$$

Their interpretation as generators of the de Sitter algebra is obtained by splitting them into Lorentz generators $J_{\mu\nu}$ and translation generators $T_\mu = \sqrt{\Lambda} J_{4\mu}$. The de Sitter algebra can then be written as

$$\{J_{\mu\nu}, J_{\rho\sigma}\} = \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho},$$
$$\{J_{\mu\nu}, T_\lambda\} = \eta_{\mu\lambda}T_\nu - \eta_{\nu\lambda}T_\mu, \quad \{T_\mu, T_\nu\} = -\Lambda J_{\mu\nu}. \quad (2.2)$$

The Lorentz subalgebra of the 4-dimensional de Sitter algebra is identical to the flat space Lorentz algebra, and therefore its generators can be realized in the standard way in terms of the 4-dimensional coordinates $X_\mu$ and their canonically conjugate momenta $P_\mu$, as $J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$. Thus the positions and the momenta obey the usual transformation laws under Lorentz transformations,

$$\{J_{\mu\nu}, X_\lambda\} = \eta_{\mu\lambda}X_\nu - \eta_{\nu\lambda}X_\mu, \quad \{J_{\mu\nu}, P_\lambda\} = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu. \quad (2.3)$$

As we shall see, the realization of the translation generators $T_\mu$ depends instead on the specific choice of coordinates on the hyperboloid.
2.2. Natural coordinates

The de Sitter hyperboloid can be parametrized by arbitrary coordinates and, contrary to the case of flat space, there is no privileged system of coordinates for de Sitter space. The systems commonly used in the applications to general relativity single out the time coordinate, while the most interesting for our purposes are isotropic in space and time. Since these systems of coordinates are not very well known, we shortly review their properties.

We start by considering the natural parametrization, given by $\hat{X}_\mu = \xi_\mu$. The metric induced on the hyperboloid by the five-dimensional flat metric reads

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} - \frac{\Lambda \hat{X}_\mu \hat{X}_\nu}{1 + \Lambda \hat{X}^2}, \quad \hat{g}^{\mu\nu} = \eta^{\mu\nu} + \Lambda \hat{X}^\mu \hat{X}_\nu. \quad (2.4)$$

In these coordinates no cosmological horizon arises at finite distance.

From the definition of $T_\mu = \sqrt{\Lambda} J_\mu$, it is easy to see that under translations

$$\{T_\mu, \hat{X}_\nu\} = -\sqrt{1 + \Lambda \hat{X}^2} \eta_{\mu\nu}. \quad (2.5)$$

The nontrivial effect of translations is of course due to the curvature of the space.

From (2.5) it is evident that the translation generators $T_\mu$ do not coincide with the momenta $\hat{P}_\mu = \pi_\mu$ canonically conjugate to $\hat{X}_\mu$. In fact,

$$T_\mu = \sqrt{1 + \Lambda \hat{X}^2} \hat{P}_\mu, \quad (2.6)$$

and the momenta transform as

$$\{T_\mu, \hat{P}_\nu\} = \frac{\Lambda \hat{X}_\nu \hat{P}_\mu}{\sqrt{1 + \Lambda \hat{X}^2}}. \quad (2.7)$$

2.3. Conformal coordinates

The parametrization that yields the simplest form of the metric is given by conformal coordinates $\tilde{X}_\mu = 2\xi_\mu/(1 + \sqrt{\Lambda} \xi_4)$, with inverse $\xi_\mu = \tilde{X}_\mu/(1 - \Lambda \hat{X}^2/4)$. In these coordinates the metric takes the diagonal form

$$\tilde{g}_{\mu\nu} = \frac{\eta_{\mu\nu}}{(1 - \Lambda \hat{X}^2/4)^2}, \quad \tilde{g}^{\mu\nu} = \left(1 - \frac{\Lambda \hat{X}^2}{4}\right)^2 \eta^{\mu\nu}, \quad (2.8)$$

and displays a cosmological horizon at $\tilde{X}^2 = 4/\Lambda$.

Under translations the position coordinates transform as

$$\{T_\mu, \tilde{X}_\nu\} = -\left(1 + \frac{\Lambda \tilde{X}^2}{4}\right) \eta_{\mu\nu} + \frac{\Lambda}{2} \tilde{X}_\mu \tilde{X}_\nu, \quad (2.9)$$
and hence, in terms of the canonical momenta \( \tilde{P}_\mu = (1 + \sqrt{\Lambda} \xi_4) \pi_\mu / 2 \),

\[
T_\mu = \left( 1 + \frac{\Lambda \tilde{X}^2}{4} \right) \tilde{P}_\mu - \frac{\Lambda}{2} \tilde{X} \cdot \tilde{P} \tilde{X}_\mu. \tag{2.10}
\]

The momenta transform as

\[
\{ T_\mu, \tilde{P}_\nu \} = -\frac{\Lambda}{2} (\tilde{X} \cdot \tilde{P} \eta_{\mu\nu} + \tilde{X}_\mu \tilde{P}_\nu - \tilde{X}_\nu \tilde{P}_\mu). \tag{2.11}
\]

### 2.4. Beltrami coordinates

Another useful parametrization of the de Sitter hyperboloid is given by Beltrami coordinates \([2,3]\), \( \tilde{X}_\mu = \xi_\mu / \sqrt{\Lambda} \xi_4 \), with inverse \( \xi_\mu = \tilde{X}_\mu / \sqrt{1 - \Lambda \tilde{X}^2} \). In these coordinates, the metric has the form

\[
\bar{g}_{\mu\nu} = (1 - \Lambda \tilde{X}^2) \eta_{\mu\nu} + \Lambda \tilde{X}_\mu \tilde{X}_\nu, \quad \bar{g}^{\mu\nu} = (1 - \Lambda \tilde{X}^2) (\eta^{\mu\nu} - \Lambda \tilde{X}_\mu \tilde{X}^\nu). \tag{2.12}
\]

A cosmological horizon is present at \( \tilde{X}^2 = 1 / \Lambda \).

Under translations, the coordinates \( \tilde{X}_\mu \) transform as

\[
\{ T_\mu, \tilde{X}_\nu \} = -\eta_{\mu\nu} + \Lambda \tilde{X}_\mu \tilde{X}_\nu. \tag{2.13}
\]

In terms of the canonical momenta \( \tilde{P}_\mu = \xi_4 \pi_\mu / \sqrt{\Lambda} \), the translation generators read

\[
T_\mu = \tilde{P}_\mu - \Lambda \tilde{X} \cdot \tilde{P} \tilde{X}_\mu, \tag{2.14}
\]

and

\[
\{ T_\mu, \tilde{P}_\nu \} = -\Lambda (\tilde{X} \cdot \tilde{P} \eta_{\mu\nu} + \tilde{X}_\mu \tilde{P}_\nu). \tag{2.15}
\]

### 3. Motion in de Sitter space

We consider now the motion of a free particle in de Sitter space in the coordinate systems introduced in the previous section, using the hamiltonian formalism. Equivalent results could be obtained with greater effort using the Dirac theory of constrained systems.

#### 3.1. Generalities*

The lagrangian of a free particle of mass \( m \), invariant under de Sitter transformations, is given by

\[
L = \frac{m}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu, \tag{3.1}
\]

where a dot denotes derivative with respect to an evolution parameter \( \tau \).

* In this subsection we restore the difference between upper and lower indices.
Varying with respect to $X_\mu$ one obtains the geodesics equations. Alternatively, defining the canonically conjugate momenta

$$P_\mu \equiv \frac{\partial L}{\partial \dot{X}^\mu} = m g_{\mu\nu} \dot{X}^\nu,$$

(with inverse $\dot{X}^\mu = \frac{1}{m} g^{\mu\nu} P_\nu$), one can define the Hamiltonian as

$$H = P_\mu \dot{X}^\mu - L = \frac{1}{2m} g^{\mu\nu} P_\mu P_\nu. \quad (3.3)$$

The equations of motion in hamiltonian form are

$$\dot{X}^\mu = \{X^\mu, H\} = g^{\mu\nu} P_\nu, \quad \dot{P}_\mu = \{P_\mu, H\} = -\frac{\partial g^{\lambda\nu}}{\partial X^\mu} P_\lambda P_\nu, \quad (3.4)$$

and can be derived by varying the action

$$\int d\tau (\dot{X}^\mu P_\mu - H). \quad (3.5)$$

An equivalent way to obtain the hamiltonian is to identify it with the quadratic Casimir invariant of the de Sitter group, $J_{AB}J_{AB} = J_{\mu\nu}J_{\mu\nu} - \frac{\Lambda}{g} T_\mu T_\mu$, written in terms of the four-dimensional phase space variables. It is easy to see that in this way one recovers the previous results.

### 3.2. Conformal coordinates

These coordinates give the simplest relation between velocity and momentum. The hamiltonian takes the form (from now on we put $m = 1$),

$$\tilde{H} = \frac{1}{2} \left(1 - \frac{\Lambda \tilde{X}^2}{4}\right)^2 \tilde{P}^2, \quad (3.6)$$

with field equations

$$\dot{\tilde{X}}_\mu = \left(1 - \frac{\Lambda \tilde{X}^2}{4}\right)^2 \tilde{P}_\mu, \quad \dot{\tilde{P}}_\mu = \frac{\Lambda}{2} \left(1 - \frac{\Lambda \tilde{X}^2}{4}\right) \tilde{P}^2 \tilde{X}_\mu. \quad (3.7)$$

In these coordinates, the 3-velocity $v_i$ is given by

$$v_i \equiv \frac{\dot{X}_i}{\ddot{X}_0} = \frac{\ddot{P}_i}{\ddot{P}_0}, \quad (3.8)$$

and hence the relation between 3-velocity and momenta is the same as in flat space.
The field equations may be integrated by substituting the first equation (3.7) into the second. However, it is more convenient to use the conservation law associated with the translations, which gives a first integral \( T_\mu = A_\mu \), with \( A_\mu \) a constant vector. From (2.10) and (3.7),

\[
\left(1 + \frac{\Lambda \tilde{X}^2}{4}\right) \ddot{P}_\mu - \frac{\Lambda}{2} \tilde{X} \cdot \dot{P} \tilde{X}_\mu = \frac{(1 + \Lambda \tilde{X}^2/4) \dot{X}_\mu - \Lambda \tilde{X} \cdot \dot{X} \tilde{X}_\mu/2}{(1 - \Lambda \tilde{X}^2/4)^2} = A_\mu.
\]

(3.9)

Inverting, one obtains

\[
\dot{\tilde{X}}_\mu = \frac{1 - \Lambda \tilde{X}^2/4}{1 + \Lambda \tilde{X}^2/4} \left[ \left(1 - \frac{\Lambda \tilde{X}^2}{4}\right) A_\mu - \frac{\Lambda}{2} A \cdot \tilde{X} \tilde{X}_\mu \right],
\]

(3.10)

and therefore

\[
v_i = \frac{(1 - \Lambda \tilde{X}^2/4) A_i - \Lambda A \cdot \tilde{X} \tilde{X}_i/2}{(1 - \Lambda \tilde{X}^2/4) A_0 - \Lambda A \cdot \tilde{X} \tilde{X}_0/2}.
\]

(3.11)

**3.3. Beltrami coordinates**

These coordinates have the nice property that 3-dimensional geodesics are straight lines. The Hamiltonian takes the form

\[
\bar{H} = \frac{1}{2} \left(1 - \Lambda \tilde{X}^2\right) [\bar{P}^2 - \Lambda (\bar{X} \cdot \bar{P})^2],
\]

(3.12)

with equations of motion

\[
\dot{\bar{X}}_\mu = (1 - \Lambda \tilde{X}^2)(\bar{P}_\mu - \Lambda \bar{X} \cdot \bar{P} \bar{X}_\mu),
\]

\[
\dot{\bar{P}}_\mu = \Lambda \left[(\bar{P}^2 - \Lambda (\bar{X} \cdot \bar{P})^2) \bar{X}_\mu + (1 - \Lambda \tilde{X}^2) \bar{X} \cdot \bar{P} \bar{P}_\mu\right].
\]

(3.13)

Hence the 3-velocity can be written as

\[
v_i \equiv \frac{\dot{\bar{X}}_i}{\bar{X}_0} = \frac{\ddot{X}_i - \Lambda \bar{X} \cdot \dot{\bar{P}} \bar{X}_i}{\ddot{X}_0 - \Lambda \bar{X} \cdot \dot{\bar{P}} \bar{X}_0}.
\]

(3.14)

Its form no longer coincides with its flat space analogous.

The equations (3.13) are rather involved, but one can exploit the conservation law \( \bar{T}_\mu = 0 \) to obtain a first integral,

\[
\ddot{P}_\mu - \Lambda \tilde{X} \cdot \ddot{P} \tilde{X}_\mu = \frac{\dot{X}_\mu}{1 - \Lambda \tilde{X}^2} = A_\mu,
\]

(3.15)

for constant \( A_\mu \). Inverting, one obtains

\[
\dot{X}_\mu = (1 - \Lambda \tilde{X}^2) A_\mu,
\]

(3.16)
and hence
\[ v_i = \frac{A_i}{A_0}. \]  
(3.17)

Therefore, free particles have constant 3-velocity and their trajectories in 3-space are straight lines.

3.4. Natural coordinates

These coordinates do not give rise to particularly simple expressions. Therefore, we just summarize the main results. The Hamiltonian has the form
\[ \hat{H} = \frac{1}{2} [\hat{P}^2 + \Lambda (\hat{X} \cdot \hat{P})^2], \]  
(3.18)

and yields the equations of motion
\[ \hat{X}_\mu = \hat{P}_\mu + \Lambda \hat{X} \cdot \hat{P} \hat{X}_\mu, \quad \hat{P}_\mu = -\Lambda \hat{X} \cdot \hat{P} \hat{P}_\mu, \]  
(3.19)

with 3-velocity
\[ v_i = \frac{\hat{P}_i + \Lambda \hat{X} \cdot \hat{P} \hat{X}_i}{\hat{P}_0 + \Lambda \hat{X} \cdot \hat{P} \hat{X}_0}. \]  
(3.20)

One can again exploit the conservation law \( \dot{T}_\mu = 0 \) to obtain a first integral,
\[ \hat{P}_\mu + \Lambda \hat{X} \cdot \hat{P} \hat{X}_\mu = \frac{(1 + \Lambda \hat{X}^2) \hat{X}_\mu - \Lambda \hat{X} \cdot \hat{X} \hat{X}_\mu}{\sqrt{1 + \Lambda \hat{X}^2}} = A_\mu. \]  
(3.21)

Inverting, one obtains \( \hat{X}_\mu \) and then
\[ v_i = \frac{A_i - \Lambda A \cdot \hat{X} \hat{X}_i}{A_0 - \Lambda A \cdot \hat{X} \hat{X}_0}. \]  
(3.22)

4. The MS model in de Sitter space

DSR theories in flat space can be implemented in two different ways. One can either deform the Poincaré algebra [7,9], imposing nonlinear Poisson brackets between the generators, or maintain the canonical form of the algebra, but modify its action on the momentum variables [8,13]. The first approach has been considered in [16,17] in order to derive a deformed de Sitter algebra. However, for the discussion of the extension of DSR models to the full phase space, especially in the case of a de Sitter background, the second approach appears to be more useful.

The MS model was introduced in [8] and is characterized by a deformed dispersion relation \( p^2/(1 - p_0/\kappa)^2 = m^2 \). A remarkable property of this model is that the Planck energy \( \kappa \) is left invariant under the deformed Lorentz transformations. The covariant realization of the model in a noncommutative position space was discussed in [10,11].
In [14] it was observed that the representation of the MS algebra in phase space can be obtained in a straightforward way from the Poincaré algebra acting canonically on a space of coordinates $X_\mu, P_\mu$, by performing the substitution

$$X_\mu = (1 - p_0/\kappa) x_\mu, \quad P_\mu = \frac{p_\mu}{1 - p_0/\kappa};$$  \hspace{1cm} (4.1)

with inverse

$$x_\mu = (1 + P_0/\kappa) X_\mu, \quad p_\mu = \frac{P_\mu}{1 + P_0/\kappa}. \hspace{1cm} (4.2)$$

Here $x_\mu, p_\mu$ are interpreted as physical observables, in contrast with the auxiliary variables $X_\mu, P_\mu$.

The symplectic structure of phase space is then deformed and takes the form [10,11],

$$\{x_0, x_i\} = \frac{x_i}{\kappa}, \hspace{0.5cm} \{x_i, x_j\} = 0, \hspace{0.5cm} \{p_0, p_i\} = \{p_i, p_j\} = 0,$$

$$\{x_0, p_0\} = 1 - \frac{p_0}{\kappa}, \hspace{0.5cm} \{x_i, p_j\} = -\delta_{ij},$$

$$\{x_0, p_i\} = -\frac{p_i}{\kappa}, \hspace{0.5cm} \{x_i, p_0\} = 0. \hspace{1cm} (4.3)$$

In particular, the coordinates $x_\mu$ do not commute.

One can apply the same procedure in the de Sitter case. In this context it is useful to rewrite the de Sitter algebra in the form

$$\{N_i, N_j\} = \epsilon_{ijk} M_k, \hspace{0.5cm} \{M_i, N_j\} = \epsilon_{ijk} N_k, \hspace{0.5cm} \{M_i, M_j\} = \epsilon_{ijk} M_k,$$

$$\{T_i, T_j\} = -\Lambda \epsilon_{ijk} M_k, \hspace{0.5cm} \{T_0, T_j\} = -\Lambda N_j,$$

$$\{M_i, T_j\} = \epsilon_{ijk} T_k, \hspace{0.5cm} \{M_i, T_0\} = 0,$$

$$\{N_i, T_j\} = \delta_{ij} T_0, \hspace{0.5cm} \{N_i, T_0\} = T_i. \hspace{1cm} (4.4)$$

where $M_k = \frac{1}{2} \epsilon_{ijk} J_{ij}$ are the generators of rotations and $N_i = J_{0i}$ the generators of boosts.

The Poisson brackets between phase space variables maintain the form (4.3). Also the deformed action of the Lorentz subalgebra on coordinates and momenta is the same as in the flat space MS model [11],

$$\{M_i, x_j\} = \epsilon_{ijk} x_k, \hspace{0.5cm} \{M_i, x_0\} = 0,$$

$$\{N_i, x_j\} = \delta_{ij} x_0 + p_i x_j/\kappa, \hspace{0.5cm} \{N_i, x_0\} = x_i + p_i x_0/\kappa.$$

$$\{M_i, p_j\} = \epsilon_{ijk} p_k, \hspace{0.5cm} \{M_i, p_0\} = 0,$$

$$\{N_i, p_j\} = \delta_{ij} p_0 - p_i p_j/\kappa, \hspace{0.5cm} \{N_i, p_0\} = p_i - p_i p_0/\kappa. \hspace{1cm} (4.5)$$

The action of translations on coordinates and momenta depends instead on the specific coordinates chosen for de Sitter space. For example, in the natural parametrization,

$$\{T_\mu, \hat{x}_\nu\} = -\sqrt{(1 - \hat{p}_0/\kappa)^{-2} + \Lambda \hat{x}^2} \left[ \eta_{\mu\nu} - \frac{\Lambda}{\kappa} \frac{\hat{x}_0 \hat{x}_\nu \hat{p}_\mu}{(1 - \hat{p}_0/\kappa)^{-2} + \Lambda \hat{x}^2} \right],$$

$$\{T_\mu, \hat{p}_\nu\} = \frac{\Lambda (\hat{x}_\nu - \hat{p}_\nu \hat{x}_0/\kappa) \hat{p}_\mu}{\sqrt{(1 - \hat{p}_0/\kappa)^{-2} + \Lambda \hat{x}^2}}. \hspace{1cm} (4.6)$$
An interesting physical implication of this model is that the cosmological constant becomes effectively energy dependent. Consider for example natural coordinates and define, in analogy with (4.2), \[ \hat{x}_4 = (1 + \hat{P}_0/\kappa) \hat{X}_4, \] with \( \hat{X}_4 = \xi_4 \). Then \( \hat{x}_4^2 = -\alpha^2/(1 - \hat{p}_0/\kappa)^2 \equiv -1/\Lambda(\hat{p}_0) \). In particular, for \( \hat{p}_0 \to \kappa \), \( \Lambda(\hat{p}_0) \to 0 \), i.e. particles with energy close to the Planck energy do not experience the curvature of spacetime.

5. Dynamics of the MS model in de Sitter space

Also the hamiltonian of a free particle can be obtained by substituting (4.1) into the undeformed hamiltonian [14]. The equations of motion can then be obtained by taking into account the deformed symplectic structure (4.4), namely,

\[
\begin{align*}
\dot{x}_0 &= \{x_0, H\} = \left(1 - \frac{p_0}{\kappa}\right) \frac{\partial H}{\partial p_0} - \frac{p_i}{\kappa} \frac{\partial H}{\partial p_i} + \frac{x_i}{\kappa} \frac{\partial H}{\partial x_i}, \\
\dot{x}_i &= \{x_i, H\} = -\frac{\partial H}{\partial p_i} - \frac{x_i}{\kappa} \frac{\partial H}{\partial x_0},
\end{align*}
\] (5.1)

and

\[
\begin{align*}
\dot{p}_0 &= \{p_0, H\} = -\left(1 - \frac{p_0}{\kappa}\right) \frac{\partial H}{\partial x_0}, \\
\dot{p}_i &= \{p_i, H\} = \frac{\partial H}{\partial x_i} + \frac{p_i}{\kappa} \frac{\partial H}{\partial x_0}.
\end{align*}
\] (5.2)

Equivalently, the Hamilton equations can be obtained by varying the action in which the substitution (4.1) has been done.

For example, in conformal coordinates the hamiltonian is given by

\[ \bar{H} = \bar{\Delta}^2 \bar{p}^2, \] (5.3)

where

\[ \bar{\Delta} = \frac{1}{1 - \bar{p}_0/\kappa} - \frac{\Lambda}{4} (1 - \bar{p}_0/\kappa) \bar{x}^2. \] (5.4)

The Hamilton equations then read

\[
\begin{align*}
\dot{x}_\mu &= \bar{\Delta}^2 \bar{p}_\mu + \frac{\Lambda}{2\kappa} (1 - \bar{p}_0/\kappa) \bar{\Delta} \bar{p}^2 \bar{x}_0 \bar{x}_\mu, \\
\end{align*}
\] (5.5)

and

\[
\begin{align*}
\dot{p}_\mu &= \frac{\Lambda}{2} (1 - \bar{p}_0/\kappa) \bar{\Delta} \bar{p}^2 (\bar{x}_\mu - \bar{x}_0 \bar{p}_\mu/\kappa).
\end{align*}
\] (5.6)

They can also be recovered from the action

\[
\begin{align*}
I &= \int d\tau \left[ \frac{\dot{X}_\mu \dot{P}_\mu}{1 - \bar{p}_0/\kappa} - \frac{1}{2} (1 - \Lambda \bar{X}^2/4)^2 \bar{P}^2 \right] \\
&= \int d\tau \left[ \frac{\bar{p}_\mu}{1 - \bar{p}_0/\kappa} \frac{d}{d\tau} \left[(1 - \bar{p}_0/\kappa) \bar{x}_\mu\right] - \frac{1}{2} \bar{\Delta}^2 \bar{p}^2 \right].
\end{align*}
\] (5.7)
The Hamilton equations (5.5) have acquired complicated terms proportional to $\Lambda/\kappa$, and are no longer linear in the momentum, so that it is not easy to invert them in order to obtain $\tilde{p}_\mu$ in terms of $\dot{\tilde{x}}_\mu$. Because of this, it is difficult to obtain the equations of motion in second order form, even using the conservation law for $T_\mu$.

Moreover, the property that the velocity has the same expression as in the undeformed case, valid for the MS model, does not extend to the de Sitter case. In fact, this property was proven in [18] to hold for position-independent hamiltonians. If one wishes to maintain its validity, one should look for a different deformation of the symplectic structure. For the same reason, contrary to flat space, the evolution parameter $d\tau$ cannot be identified with the line element invariant under the deformed transformations, which reads

$$ds^2 = \frac{d\tilde{x}^2}{\tilde{\Delta}^2} = \frac{(1 - \tilde{p}_0/\kappa)^2 d\tilde{x}^2}{[1 - \frac{1}{\Lambda^2} (1 - \tilde{p}_0/\kappa)^2 \tilde{x}^2]^2}. \tag{5.8}$$

It is interesting to notice that the metric (5.8) exhibits a momentum-dependent cosmological horizon at $\Lambda \tilde{x}^2 = 4(1 - p_0/\kappa)^{-2}$. The dependence of the horizon on the momentum is of course related to the momentum dependence of the cosmological constant discussed at the end of previous section, and (5.8) can be considered an example of rainbow metric [12].

An analogous calculation can be performed in natural coordinates. The deformed hamiltonian is

$$\hat{H} = \frac{1}{2} \left[ \frac{\hat{p}^2}{(1 - \hat{p}_0/\kappa)^2} + \Lambda(\hat{x} \cdot \hat{p})^2 \right], \tag{5.9}$$

with Hamilton equations

$$\dot{\hat{x}}_\mu = (1 - \hat{p}_0/\kappa)^{-2} \hat{p}_\mu + \Lambda(1 - \hat{p}_0/\kappa) \hat{x} \cdot \hat{p} \hat{x}_\mu,$$

$$\dot{\hat{p}}_\mu = -\Lambda(1 - \hat{p}_0/\kappa) \hat{x} \cdot \hat{p} \hat{p}_\mu. \tag{5.10}$$

Also in this case one finds the same problems as with conformal coordinates. The same problems hold for Beltrami coordinates as well, in which the equations of motion are even more involved. In particular, it does not seem that the three-dimensional geodesics are still straight lines in the deformed theory.

Finally, we notice that in the limits $\kappa \to \infty$ and $\Lambda \to 0$ one recovers the ordinary de Sitter space and the flat space MS model, respectively, while the limit $p_0 \to \kappa$ is analogous to that of the MS model [8].

6. DSR in de Sitter space in a Snyder-like basis

It is known that DSR theories can be realized in several different ways. An interesting realization is given by the so-called Snyder basis [6], which is characterized by the dispersion relation $P^2/(1 - P^2/\kappa^2) = m^2$, that implies that the rest mass of particles must always be less than $\kappa$. Another important property of this basis is that only the action of the translations is deformed, while that of the Lorentz group is not affected. This example illustrates the fact that the most relevant characteristic for the implementation of DSR
is the deformation of the action of translations (and hence a modified composition law of momenta) and not that of Lorentz transformations, as usually postulated.

6.1 Minkowski space

Let us briefly review the case of flat spacetime. It is easy to see that, in analogy with our previous treatment of the MS model, the easiest way to obtain the Snyder realization of DSR is to define new coordinates from the canonical $X_\mu, P_\mu$, which are thus interpreted as auxiliary variables,

$$\begin{align*}
 X_\mu &= \sqrt{1 + \Omega P^2} X_\mu, \\
 P_\mu &= \frac{P_\mu}{\sqrt{1 + \Omega P^2}},
\end{align*}$$

(6.1)

where $\Omega = 1/\kappa^2$ is the Planck area\(^\ddagger\). The inverse transformations are

$$\begin{align*}
 X_\mu &= \sqrt{1 - \Omega P^2} X_\mu, \\
 P_\mu &= \frac{P_\mu}{\sqrt{1 - \Omega P^2}},
\end{align*}$$

(6.2)

One has then,

$$\begin{align*}
 \{X_\mu, X_\nu\} &= -\Omega (X_\mu P_\nu - X_\nu P_\mu), \\
 \{P_\mu, P_\nu\} &= 0, \\
 \{X_\mu, P_\nu\} &= \eta_{\mu\nu} - \Omega P_\mu P_\nu.
\end{align*}$$

(6.3)

The Lorentz transformations acting on $X_\mu, P_\mu$ maintain the canonical form (2.3). The translation generators $T_\mu$ must instead be identified with $P_\mu = P_\mu/\sqrt{1 - \Omega P^2}$. Their action changes accordingly,

$$\begin{align*}
 \{T_\mu, X_\nu\} &= \frac{\eta_{\mu\nu}}{\sqrt{1 - \Omega P^2}}, \\
 \{T_\mu, P_\nu\} &= 0.
\end{align*}$$

(6.4)

The invariant hamiltonian for a free particle can be written as

$$H = \frac{P^2}{2} = \frac{1}{2} \frac{P^2}{1 - \Omega P^2},$$

(6.5)

with equations of motion

$$\begin{align*}
 \dot{X}_\mu &= \frac{P_\mu}{1 - \Omega P^2}, \\
 \dot{P}_\mu &= 0.
\end{align*}$$

(6.6)

It follows that $\dot{X}_\mu = A_\mu$ is constant. The 3-velocity is then given by

$$v_i = \frac{P_i}{P_0} = \frac{A_i}{A_0},$$

(6.7)

and the 3-dimensional geodesics are straight lines. Moreover, it is easy to verify that the invariant line element $ds^2 = (1 - \Omega P^2) dX^2$ can be identified with $d\tau^2$, with $\tau$ the evolution parameter.

\(^\ddagger\) In principle one may choose a negative sign for $\Omega$, obtaining an inequivalent model with rather different properties [19].
6.2 de Sitter space

Let us now extend the above construction to the case of de Sitter space in the Beltrami coordinates of section 2.4. The substitution (6.1) yields

$$\bar{X}_\mu = \sqrt{1 + \Omega \bar{P}^2} \bar{X}_\mu = \sqrt{\Omega \pi^2 + (1 + \Lambda \xi^2)^{-1}} \xi_\mu,$$
$$\bar{P}_\mu = \frac{\bar{P}_\mu}{\sqrt{1 + \Omega \bar{P}^2}} = \frac{\pi_\mu}{\sqrt{\Omega \pi^2 + (1 + \Lambda \xi^2)^{-1}}},$$

(6.8)

where $\xi_\mu$ are as usual the coordinates of the five-dimensional embedding space. Inverting,

$$\xi_\mu = \frac{\bar{X}_\mu}{\Phi}, \quad \pi_\mu = \Phi \bar{P}_\mu,$$

(6.9)

where

$$\Phi = \sqrt{(1 - \Omega \bar{P}^2)^{-1} - \Lambda \bar{X}^2}.$$  

(6.10)

The phase space coordinates $\bar{X}_\mu, \bar{P}_\mu$ satisfy the Poisson brackets (6.3).

In terms of the variables $\bar{X}_\mu, \bar{P}_\mu$, the Lorentz generators of the de Sitter algebra (2.2) have canonical form, while the translation generators read

$$T_\mu = \frac{1}{\sqrt{1 - \Omega \bar{P}^2}} [\bar{P}_\mu - \Lambda (1 - \Omega \bar{P}^2) \bar{X} \cdot \bar{P} \bar{X}_\mu],$$

(6.11)

and

$$\{T_\mu, \bar{X}_\nu\} = -\frac{1}{\sqrt{1 - \Omega \bar{P}^2}} [\eta_{\mu\nu} - \Lambda (1 - \Omega \bar{P}^2) \bar{X}_\mu \bar{X}_\nu + \Lambda \Omega (1 - \Omega \bar{P}^2) \bar{X} \cdot \bar{P} \bar{P}_\mu \bar{X}_\nu],$$

$$\{T_\mu, \bar{P}_\nu\} = -\Lambda \sqrt{1 - \Omega \bar{P}^2} [\bar{X} \cdot \bar{P} (\eta_{\mu\nu} - \Omega \bar{P}_\mu \bar{P}_\nu) - (1 - \Omega \bar{P}^2) \bar{X}_\mu \bar{P}_\nu].$$

(6.12)

One can also define a hamiltonian, invariant under the full deformed de Sitter group,

$$H = \frac{1}{2} \Phi^2 [(\bar{P}^2 - \Lambda (1 - \Omega \bar{P}^2)(\bar{X} \cdot \bar{P})^2].$$

(6.13)

Unfortunately, the Hamilton equations take an extremely involved form and we shall not report them here. The invariant metric for this model is

$$g_{\mu\nu} = \frac{\Phi^2 \eta_{\mu\nu} + \Lambda \bar{X}_\mu \bar{X}_\nu}{\Phi^4}.$$  

(6.14)

In the limits $\Omega \to 0$ and $\Lambda \to 0$ one recovers the ordinary de Sitter space and the flat space Snyder model of previous section, respectively. Also interesting is the presence of a cosmological horizon at $\Lambda \bar{X}^2 = 1 - \Omega \bar{P}^2$ in the metric (6.14), whose location is momentum dependent.
7. A different Snyder-like realization

The Snyder realization of DSR in de Sitter space given in the previous section is rather awkward. In this section, we consider a slightly different realization, which takes a more symmetric form and gives rise to more elegant formulas. The algebra of this model displays some similarities with that proposed in [17].

We define

\[ \mathcal{X}_\mu = \sqrt{\frac{1 + \Omega}{1 + \Lambda}} \xi \mu, \quad \mathcal{P}_\mu = \sqrt{\frac{1 + \Lambda}{1 + \Omega}} \xi \mu, \quad (7.1) \]

with inverse

\[ \xi_\mu = \sqrt{\frac{1 - \Omega}{1 - \Lambda}} \mathcal{X}_\mu, \quad \pi_\mu = \sqrt{\frac{1 - \Lambda}{1 - \Omega}} \mathcal{P}_\mu. \quad (7.2) \]

The coordinates (7.1) satisfy the Poisson brackets

\[ \{ \mathcal{X}_\mu, \mathcal{X}_\nu \} = -\frac{\Omega}{1 - \Lambda \mathcal{X}^2} (\mathcal{X}_\mu \mathcal{P}_\nu - \mathcal{X}_\nu \mathcal{P}_\mu), \]
\[ \{ \mathcal{P}_\mu, \mathcal{P}_\nu \} = -\frac{\Lambda}{1 - \Omega \mathcal{X}^2} (\mathcal{X}_\mu \mathcal{P}_\nu - \mathcal{X}_\nu \mathcal{P}_\mu), \]
\[ \{ \mathcal{X}_\mu, \mathcal{P}_\nu \} = \eta_{\mu\nu} - \frac{\Lambda}{1 - \Omega \mathcal{X}^2} \mathcal{X}_\mu \mathcal{X}_\nu + \Omega (1 - \Lambda \mathcal{X}^2) \mathcal{X}_\mu \mathcal{X}_\nu + \Omega (1 - \Lambda \mathcal{X}^2) \mathcal{X}_\mu \mathcal{X}_\nu + \Omega (1 - \Lambda \mathcal{X}^2) \mathcal{X}_\mu \mathcal{X}_\nu + \Omega (1 - \Lambda \mathcal{X}^2) \mathcal{X}_\mu \mathcal{X}_\nu. \quad (7.3) \]

The Lorentz generators of the de Sitter algebra have canonical form, while the dilatation generators are

\[ T_\mu = \sqrt{\frac{1 - \Lambda \mathcal{X}^2}{1 - \Omega \mathcal{P}^2}} \mathcal{P}_\mu, \quad (7.4) \]

and their action is given by

\[ \{ T_\mu, \mathcal{X}_\nu \} = -\sqrt{\frac{1 - \Lambda \mathcal{X}^2}{1 - \Omega \mathcal{P}^2}} \left[ \eta_{\mu\nu} - \frac{\Lambda}{1 - \Omega \mathcal{X}^2} \left( \mathcal{X}_\mu \mathcal{P}_\nu + \Omega \frac{(1 - \Lambda \mathcal{X}^2) \mathcal{X}_\mu \mathcal{P}_\nu}{1 - \Lambda \mathcal{X}^2} \right) \right], \]
\[ \{ T_\mu, \mathcal{P}_\nu \} = -\Lambda \sqrt{\frac{1 - \Omega \mathcal{P}^2}{1 - \Lambda \mathcal{X}^2}} \left[ \mathcal{X}_\mu \mathcal{P}_\nu - \mathcal{X}_\nu \mathcal{P}_\mu + \Omega \frac{(1 - \Lambda \mathcal{X}^2) \mathcal{X}_\mu \mathcal{P}_\nu}{1 - \Lambda \mathcal{X}^2} \mathcal{P}_\mu \mathcal{P}_\nu \right]. \quad (7.5) \]

The Hamiltonian of a free particle can be obtained from the Casimir invariant of the de Sitter algebra, and takes the form

\[ H = \frac{1}{2} \left[ \frac{1 - \Lambda \mathcal{X}^2}{1 - \Omega \mathcal{P}^2} \mathcal{P}^2 + \Lambda (\mathcal{X} \cdot \mathcal{P})^2 \right]. \quad (7.6) \]
Taking into account the symplectic structure (7.3), the equations of motion ensuing from the hamiltonian are

\[
\dot{\mathcal{X}}_\mu = (1 - \Lambda \mathcal{X}^2) \left[ \frac{\mathcal{P}_\mu}{1 - \Omega \mathcal{P}^2} - \frac{\Lambda \mathcal{X} \cdot \mathcal{P} \mathcal{P}^2 \mathcal{X}_\mu}{1 - \Lambda \mathcal{X}^2 \mathcal{P}^2} \right],
\]

\[
\dot{\mathcal{P}}_\mu = \frac{\Lambda \Omega (1 - \Lambda \mathcal{X}^2)}{1 - \Lambda \mathcal{X}^2 \mathcal{P}^2} \mathcal{X} \cdot \mathcal{P} \mathcal{P}^2 \mathcal{P}_\mu.
\] (7.7)

Also in this case, there does not seem to exist a simple relation between velocity and momentum.

The 4-dimensional metric can be derived in the usual way from the 5-dimensional flat metric subject to the constraint

\[
\xi_4 = \sqrt{1 + \Lambda \mathcal{X}^2} = \sqrt{\frac{1 - \Lambda \Omega \mathcal{X}^2 \mathcal{P}^2}{1 - \Lambda \mathcal{X}^2}},
\] (7.8)

and reads

\[
g_{\mu\nu} = \frac{1 - \Omega \mathcal{P}^2}{1 - \Lambda \mathcal{X}^2} \left[ \eta_{\mu\nu} + \Lambda \frac{1 - \mathcal{X}^2 \mathcal{P}^2}{(1 - \Lambda \mathcal{X}^2)(1 - \Lambda \Omega \mathcal{X}^2 \mathcal{P}^2)} \mathcal{X}_\mu \mathcal{X}_\nu \right].
\] (7.9)

Also in this case there is no evident relation between the metric and the differential \(d\tau\) of the evolution parameter. It is interesting to notice that, in addition to the cosmological horizon at \(\mathcal{X}^2 = 1/\Lambda\), the metric (7.9) presents a second momentum-dependent coordinate singularity at \(\Lambda \mathcal{X}^2 = 1/\Omega \mathcal{P}^2\), or better \(\mathcal{X}^2 \mathcal{P}^2 = 1/\Lambda \Omega\). However, for such values of \(\mathcal{X}\) and \(\mathcal{P}\) the model is ill-defined (see (7.3)): this region is also far beyond the range of physically observable phenomena, since \(1/\Lambda \Omega \sim 10^{120}\).

In the limit \(\Lambda \to 0\) one of course recovers the flat-space Snyder model of previous section, while in the limit \(\Omega \to 0\) one gets the standard de Sitter space, although with noncanonical Poisson brackets between positions and momenta (since the momenta are identified with the translation generators in this limit). More interesting are the limits \(\mathcal{X} \to \alpha\) and \(\mathcal{P} \to \kappa\). For \(\mathcal{X} \to \alpha\), one is close to the cosmological horizon, and the symplectic structure reduces to the undeformed one obtained in the limit \(\Omega = 0\). The limit \(\mathcal{P} \to \kappa\) corresponds instead to the extremal value of the momentum. In this limit, the symplectic structure is that of the flat Snyder model, \(\Lambda = 0\), and the metric and the hamiltonian are singular.

It is also interesting to notice that the Poisson brackets (7.3) lead after quantization to generalized commutation relations of the most general kind proposed in [20] that, in case of negative \(\Lambda\) and \(\Omega\), imply the existence of both a minimal length and momentum.

Another interesting property of this model is the existence of a duality for the exchange of \(\mathcal{X} \leftrightarrow \mathcal{P}\), together with \(\Lambda \leftrightarrow \Omega\). This duality connects the high-energy/short-distance regime, governed by the Planck area \(\Omega\), with the low-energy/long-distance regime, governed by the cosmological constant \(\Lambda\).
8. Conclusions

It is known that DSR models can be derived from a 5-dimensional momentum space of coordinates \( \pi_A \), subject to the constraint \( \pi^2_A = -\kappa^2 \) \([15]\). This is similar to the de Sitter constraint for the spacetime coordinates. However, the physical interpretation is quite different. First of all, de Sitter spacetime inherits a metric structure from the 5-dimensional space and this allows one to define a curvature. Different systems of coordinates are physically equivalent. The momentum space, instead, does not possess a metric structure and its coordinates cannot be considered physically equivalent, unless one adds further structure. In fact, different realizations of DSR lead to different physical theories. Moreover, the mere existence of a de Sitter group of transformations on a four-dimensional manifold does not automatically imply that this can be identified with de Sitter space.

With these remarks in mind, one may try to construct a realization of a deformed de Sitter relativity starting from five-dimensional space, similarly to what has been done for flat space \([21]\). Unfortunately, however, it is not possible to impose contemporary constraints on the five-dimensional positions and momenta, and one is forced to start from a six-dimensional space. The construction of a Hamiltonian formalism in six-dimensional phase space with coordinates \( \Xi_M \) and momenta \( \Pi_M \), subject to the constraints \( \Xi^2_M = -\alpha^2 \), \( \Pi^2_M = -\kappa^2 \) will be the subject of a separate paper \([19]\).

From the study of DSR in de Sitter space one can also learn some lessons concerning the flat space limit. First of all, it is useful to distinguish the translation generators, that dictate the conservation laws for the momentum, from the physical momentum, identified with the phase space momentum variables. This observation also gives a physical meaning to the auxiliary variables obeying canonical transformation laws introduced in ref. \([13]\), whose interpretation was unclear: they are simply the generators of translations. Moreover, it appears that the distinguishing feature of DSR is not the deformation of the Lorentz symmetry, as usually postulated, but rather that of the translation symmetry, as shown by the Snyder model discussed in section 6 and 7. Of course, a complete discussion of this topic requires an operational definition of the momentum of a particle.

Although we have not considered this subject in detail, it is also important to stress that all our considerations can be easily extended to the case of anti-de Sitter space, by simply changing the sign of the cosmological constant \( \Lambda \).
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