Y-formalism and $b$ ghost in the Non-minimal Pure Spinor Formalism of Superstrings

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Abstract

We present the Y-formalism for the non-minimal pure spinor quantization of superstrings. In the framework of this formalism we compute, at the quantum level, the explicit form of the compound operators involved in the construction of the $b$ ghost, their normal-ordering contributions and the relevant relations among them. We use these results to construct the quantum-mechanical $b$ ghost in the non-minimal pure spinor formalism. Moreover we show that this non-minimal $b$ ghost is cohomologically equivalent to the non-covariant $b$ ghost.

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1 Introduction

Several years ago, a new formalism for the covariant quantization of superstrings was proposed by Berkovits [1]. Afterward, it has been recognized that this new formalism not only solves the longstanding problem of covariant quantization of the Green-Schwarz (GS) superstring, but also it is suitable to deal with problems that appear almost intractable in the Neveu-Schwarz-Ramond (NSR) approach, such as those involving space-time fermions and/or backgrounds with R-R fields.

In this approach, the GS superstring action (let us say in the left-moving sector) is replaced with a free action for the bosonic coordinates $X^a$ and their fermionic partners $\theta^\alpha$ with their conjugate momenta $p_\alpha$, plus an action for the bosonic ghosts $\lambda^\alpha$ and their conjugate momenta $\omega_\alpha$, where $\lambda^\alpha$ satisfy the "pure spinor constraint" $\lambda \Gamma^\alpha \lambda = 0$. The $\omega - \lambda$ action looks like a free action but is not really free owing to the pure spinor constraint, which is necessary to have vanishing central charge and correct level of the Lorentz algebra. This formulation is nowadays called "pure spinor formulation of superstrings" and many studies [2]-[24] were devoted to it in the recent years.

Another key ingredient in the pure spinor formulation is provided by the BRST charge $Q = \oint \lambda^\alpha d_\alpha$ where $d_\alpha \approx 0$ contains the constraints generating a fermionic $\kappa$ symmetry in the GS superstring and has the role of a spinorial derivative in superspace. The peculiar feature associated with this BRST charge is that $Q$ is nilpotent only when the bosonic spinor $\lambda^\alpha$ satisfies the pure spinor condition. This peculiar feature is in fact expected since the constraint $d_\alpha \approx 0$ in the GS approach involves both the first-class and the second-class constraints. Roughly speaking, the pure spinor condition is needed to handle the second-class constraint of the GS superstring, keeping the Lorentz covariance manifest.

Since the BRST charge $Q$ is nilpotent, one can define the cohomology and examine its physical content. Indeed, it has been shown that the BRST cohomology determines the physical spectrum which is equivalent to that of the RNS formalism and that of the GS formalism in the light-cone gauge [3]. Moreover, the BRST charge $Q$ of the pure spinor formalism was found to be transformed to that of the NSR superstring [4] as well as that of the GS superstring in the light-cone gauge [18, 19].

Even if the pure spinor formalism provides a Lorentz-covariant superstring theory with manifest space-time supersymmetry even at the quantum level, there are some hidden sources of possible violation of Lorentz covariance.

One of such sources is related to the $b$ field defined by $T = \{Q, b\}$ with $T$ being the stress-energy tensor, which is necessary to compute higher loop amplitudes. Since the pure spinor formulation is not derived from a diffeomorphism-invariant action and does not contain the $b - c$ ghosts of diffeomorphisms, the usual antighost $b$ is not present in this approach. In [3] a compound $b$ field whose BRST variation gives the stress energy tensor, was obtained. However this $b$ field is not Lorentz-covariant.

The same $b$ field follows from an attempt [10] to derive, at the classical level, the pure spinor

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3Alternative formalisms to remove the constraint were proposed in [25, 26].
formulation from a (suitably gauge-fixed and twisted) \( N = 2 \) superembedding approach. In this approach the \( b \) field is the twisted current of one of the two world-sheet (w.s.) supersymmetries whereas the integrand of the BRST charge \( Q \) is the twisted current of the other supersymmetry, suggesting an \( N = 2 \) topological origin of the pure spinor approach.

This \( b \) field turns out to be proportional to the quantity \( Y_\alpha = v_\alpha \) where \( v_\alpha \) is a constant pure spinor, such that \( b_Y = Y_\alpha C^\alpha \) where \( C^\alpha \) is a covariant, spinor-like compound field, so that \( b_Y \) is not only Lorentz non-covariant but also singular at \( v_\lambda = 0 \).

A way to overcome the problem of the non-covariance and singular nature of \( b_Y \) was given in [14] where a recipe to compute higher loop amplitudes was proposed, in terms of a picture-raised \( b \) field constructed with the help of suitable covariant fields \( G^\alpha \), \( H^{\alpha\beta} \), \( K^{\alpha\beta\gamma} \) and \( L^{\alpha\beta\gamma\delta} \) and some picture-changing operators \( Z \)'s and \( Y \)'s.  

Recently, a very interesting formalism called "non-minimal pure spinor formalism" has been put forward [27]. In this formalism, a non-minimal set of variables are added to that of the (minimal) pure spinor formulation. These non-minimal variables form a BRST quartet and have the role of changing the ghost-number anomaly from \(-8\) to \(+3\) without changing the central charge and the physical mass spectrum. A remarkable thing is that, in this formalism, one can define a Lorentz-covariant \( b \) ghost without the need of picture-changing operators. With the help of a suitable regulator, a recipe has been given to compute scattering amplitudes up to two-loop amplitudes. The OPE’s between the relevant operators that result in this approach show that the (non-minimal) pure spinor formulation is indeed a hidden, critical, \( N = 2 \) topological string theory. A significant improvement was obtained in [28]. Here a gauge invariant, BRST trivial regularization of the \( b \) field is proposed, that allows for a consistent prescription to compute amplitudes at any loop.

A further source of possible non-covariance arises at intermediate steps of calculations, since the solution of the pure spinor constraint in terms of independent fields implies the breaking of \( SO(10) \) to \( U(5) \).  

To be more precise, the space of (Euclidean) pure spinors in ten dimensions has the geometrical structure of a complex cone \( Q = \frac{SO(10)}{U(5)} \) [21]. This space has been studied by Nekrasov [29] and the obstructions to its global definition are analyzed. It was shown that the obstructions are absent if the tip of the cone is removed. Then this complex cone is covered by 16 charts, \( U^{(\alpha)}, (\alpha) = 1, \cdots, 16 \) and in each chart the local parametrization of the pure spinor, which breaks \( SO(10) \) to \( U(5) \), is taken such that the parameter that describes the generatrix of the cone is non-vanishing. This parametrization can be used to compute the relevant OPE’s [1, 3] (U(5)-formalism).

In a previous work [30], we have proposed a new formalism named "Y-formalism" for purposes of handling this unavoidable non-covariance stemming from the pure spinor condition. This Y-formalism is closely related to the \( U(5) \)-formalism, but has an advantage of treating all operators in a unified way without going back to the \( U(5) \)-decomposition. It is based on writing

\[ Y_\alpha = v_\alpha \]

The picture-lowering operators \( Y_C \), which are needed to absorb the zero modes of the ghost \( \lambda^\alpha \), break the Lorentz-covariance but this breaking is BRST trivial and then harmless.

\[ \lambda^\alpha \]

In the extended pure spinor formalism [26], the same non-covariance can be found in the ghost sector where the ghosts are invariant under only \( U(5) \) group, but not \( S(10) \) group.
the fundamental OPE between $\omega$ and $\lambda$ in a form that involves $Y_{a} = \frac{\omega}{v_{\lambda}}$. Strictly speaking, one needs 16, orthogonal, constant pure spinors $v^{(\alpha)}$ (and 16 $Y^{(\alpha)}$) for each chart, such that $U^{(\alpha)} (v^{(\alpha)} \lambda) \neq 0$ in each chart. However, for our purposes it is sufficient to work in a given chart.

Actually, it turned out that the Y-formalism is quite useful to find the full expression of $b$ ghost [30]. More recently, the Y-formalism was also utilized to construct a four-dimensional pure spinor superstring [31]. The $Y$-field also arises in the regularization prescription proposed in [28].

The aim of the present paper is to extend the Y-formalism to the non-minimal case and to discuss in the framework of this formalism the non-minimal, covariant $b$ field in addition to the fields $G^{\alpha}$, $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$ and $L^{\alpha\beta\gamma\delta}$, which are the building blocks of the $b$ field. This will be done not only at the classical but also at the quantum level, by taking into account the subtleties of normal ordering. The consistent results which we will get in this article, could be regarded as a good check of the consistency of the Y-formalism. Moreover we shall show that the non-minimal, covariant $b$ field is cohomologically equivalent to the non-covariant $b$ field $b_{Y}$, improved by the term coming from the non-minimal sector.

In section 2, we will review the Y-formalism for the minimal pure spinor formalism. In section 3, the operators $G^{\alpha}$, $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$ and $L^{\alpha\beta\gamma\delta}$, and their (anti-)commutation relations with the BRST charge, will be examined from the quantum-mechanical viewpoint. In section 4, we will construct the Y-formalism for the non-minimal pure spinor formalism. In section 5, based on the Y-formalism at hand, we will construct the Lorentz-covariant quantum $b$ ghost, which satisfies the defining equation $\{Q, b\} = T$. We shall also show that it is cohomologically equivalent to the non-covariant $b$ ghost $b_{Y}$ (improved by the term coming from the non-minimal sector). Section 6 is devoted to conclusion and discussions. Some appendices are added. Appendix A contains our notation, conventions and useful identities. In Appendix B, we will review the normal-ordering prescriptions, the generalized Wick theorem and the rearrangement theorem which we will use many times in this article. Finally in Appendix C we give some details of the main calculations.

\section{Review of the Y-formalism}

In this section, we start with a brief review of the (minimal) pure spinor formalism of superstrings [1], and then explain the Y-formalism [30]. For simplicity, we shall confine ourselves to only the left-moving (holomorphic) sector of a closed superstring theory. The generalization to the right-moving (anti-holomorphic) sector is straightforward.

The pure spinor approach is based on the BRST charge

$$Q = \int d\bar{z} \lambda^{\alpha} d_{\alpha},$$

and the action

$$I = \int d^{2}z \left( \frac{1}{2} \bar{\partial} X^{\alpha} \bar{\partial} X_{\alpha} + p_{\alpha} \bar{\partial} \theta^{\alpha} - \omega_{\alpha} \bar{\partial} \lambda^{\alpha} \right),$$
where $\lambda$ is a pure spinor

$$\lambda \Gamma^a \lambda = 0. \tag{2.3}$$

This action is manifestly invariant under (global) super-Poincaré transformations. It is easily shown that the action $I$ is also invariant under the BRST transformation generated by the BRST charge $Q$ which is nilpotent owing to the pure spinor condition (2.3). Notice that in order to use $Q$ as BRST charge it is implicit that the pure spinor condition is required to vanish in a strong sense.

Moreover, the action $I$ is invariant under the $\omega$-symmetry

$$\delta \omega_\alpha = \Lambda^a (\Gamma^a \lambda)_\alpha, \tag{2.4}$$

where $\Lambda^a$ are local gauge parameters. At the classical level the ghost current is

$$J_0 = \omega \lambda, \tag{2.5}$$

and the Lorentz current for the ghost sector is given by

$$N_{0b} = \frac{1}{2} \omega \Gamma^{ab} \lambda, \tag{2.6}$$

which together with $T_{0\lambda} = \omega \partial \lambda$ are the only super-Poincaré covariant bilinear fields involving $\omega$ and gauge invariant under the $\omega$-symmetry. From the field equations it follows that $p$, $\theta$, $\omega$ and $\lambda$ are holomorphic fields. At the quantum level, one obtains the following OPE’s $^6$ involving the superspace coordinates $Z^M = (X^a, \theta^\alpha)$ and their super-Poincaré covariant momenta $\bar{P}_M = (\Pi_a, p_\alpha)$:

$$< X^a(y) X^b(z) > = -\eta^{ab} \log(y - z),$$

$$< p_\alpha(y) \theta^\beta(z) > = \frac{1}{y - z} \gamma_\beta^\alpha,$$

so that

$$< d_\alpha(y) d_\beta(z) > = -\frac{1}{y - z} \Gamma_{\alpha\beta}^a \Pi_a(z),$$

$$< d_\alpha(y) \Pi^a(z) > = \frac{1}{y - z} (\Gamma^a \partial \theta)_\alpha(z), \tag{2.7}$$

where

$$d_\alpha = p_\alpha - \frac{1}{2} (\partial X^a + \frac{1}{4} \theta \Gamma^a \partial \theta)(\Gamma_a \theta)_\alpha,$$

$$\Pi^a = \partial X^a + \frac{1}{2} \theta \Gamma^a \partial \theta,$$

$$\bar{\Pi}^a = \bar{\partial} X^a + \frac{1}{2} \theta \Gamma^a \bar{\partial} \theta. \tag{2.8}$$

$^6$According to Appendix B, we should call them not the OPE’s but the contractions, but we have called "OPE’s" since the terminology is usually used in the references of the pure spinor formulation.
As for the ghost sector, the situation is a bit more complicated owing to the pure spinor condition (2.3). Namely, it would be inconsistent to assume a free field OPE between \( \omega \) and \( \lambda \). The reason is that since the pure spinor condition must vanish identically, not all the components of \( \lambda \) are independent: solving the condition, five of them are expressed nonlinearly in terms of the others. Accordingly, five components of \( \omega \) are pure gauge.

This problem is nicely resolved by introducing the Y-formalism. Let us first define the non-covariant object

\[
Y_\alpha = \frac{v_\alpha}{v\lambda},
\]

such that

\[
y_\alpha \lambda^\alpha = 1,
\]

where \( v_\alpha \) is a constant pure spinor \( YT^aY = 0 \). Then it is useful to define the projector

\[
K_\alpha^\beta = \frac{1}{2}(\Gamma^\alpha \lambda)_\alpha (YT_a)^\beta,
\]

which, since \( TrK = 5 \), projects on a 5 dimensional subspace of the 16 dimensional spinor space in ten dimensions. The orthogonal projector is \( (1 - K)_\alpha^\beta \). Now the pure spinor condition implies

\[
\lambda^\alpha K_\alpha^\beta = 0.
\]

Since \( K \) projects on a 5 dimensional subspace, Eq. (2.13) is a simple way to understand why a pure spinor has eleven independent components.

Then we postulate the following OPE between \( \omega \) and \( \lambda \):

\[
< \omega_\alpha(y)\lambda^\beta(z) > = \frac{1}{y-z}(\delta^\beta_\alpha - K^\beta_\alpha(z)).
\]

It follows from Eq. (2.14) that the OPE between \( \omega \) and the pure spinor condition vanishes identically. Moreover, the BRST charge \( Q \) is then strictly nilpotent even acting on \( \omega \). It is useful to notice that, with the help of the projector \( K \), one can obtain a non-covariant but gauge-invariant antighost \( \tilde{\omega} \) defined as

\[
\tilde{\omega}_\alpha = (1 - K)_\alpha^\beta \omega^\beta.
\]

In the framework of this formalism one can compute [30] the OPE’s among the ghost current, Lorentz current and stress energy tensor and one can obtain the quantum version of these operators. Indeed, it has been shown in [30] that all the non-covariant, \( Y \)-dependent contributions in the r.h.s. of the OPE’s among these operators disappear if the stress energy tensor, the Lorentz current for the ghost sector, and the ghost current at the quantum level, are improved by \( Y \)-dependent correction terms, those are

\[
T = -\frac{1}{2} \partial X^a \partial X_a - p_\alpha \partial \theta^\alpha + T_\lambda
\]

\[
= -\frac{1}{2} \Pi^a \Pi_a - d_\alpha \partial \theta^\alpha + \omega_\alpha \partial \lambda^\alpha + \frac{3}{2} \partial(Y \partial \lambda),
\]

5
\[ N^{ab} = \frac{1}{2} \left[ \omega \Gamma^{ab} \lambda - \frac{3}{2} \partial \Gamma^{ab} \lambda - 2 \Gamma^{ab} \partial \lambda \right], \quad (2.17) \]

\[ J = \omega \lambda + \frac{7}{2} Y \partial \lambda. \quad (2.18) \]

Then the OPE’s among \( T, N^{ab} \) and \( J \) read

\[ \langle T(y)T(z) \rangle = \frac{2}{(y-z)^2} T(z) + \frac{1}{y-z} \partial T(z), \quad (2.19) \]

\[ \langle T(y)J(z) \rangle = \frac{8}{(y-z)^3} + \frac{1}{(y-z)^2} J(z) + \frac{1}{y-z} \partial J(z), \quad (2.20) \]

\[ \langle T(y)N^{ab}(z) \rangle = \frac{1}{(y-z)^2} N^{ab}(z) + \frac{1}{y-z} \partial N^{ab}(z), \quad (2.21) \]

\[ \langle J(y)J(z) \rangle = -\frac{4}{(y-z)^2}, \quad (2.22) \]

\[ \langle J(y)N^{ab}(z) \rangle = 0, \quad (2.23) \]

\[ \langle N^{ab}(y)N^{cd}(z) \rangle = -\frac{3}{(y-z)^2} \eta^{d[a} \eta^{b]c} - \frac{1}{y-z} (\eta^{a[c} N^{d]b} - \eta^{b[c} N^{d]a}), \quad (2.24) \]

which are in full agreement with [1, 3]. Note that although the correction terms in the currents depend on the non-covariant \( Y \)-field explicitly, these can be rewritten as BRST-exact terms.

Now a remark is in order. It appears at first sight that, due to the correction terms, the operators \( J, N^{ab} \) and \( T \) are singular at \( v \lambda = 0 \) but the opposite is in fact true: it is clear from Eqs. (2.19)-(2.24) that the \( Y \)-dependent correction terms have just the role of cancelling the singularities which are present in the operators \( T_0, N_0^{ab} \) and \( J_0 \), owing to the singular nature of the OPE (2.14) between \( \omega \) and \( \lambda \). \(^7\)

It will be convenient to rewrite (2.17), (2.18) and \( T_\lambda \) as

\[ N^{ab} = \frac{1}{2} [\Omega \Gamma^{ab} \lambda - 2 \Gamma^{ab} \partial \lambda], \quad (2.25) \]

\[ J = \Omega \lambda + 2 \Gamma \partial \lambda, \quad (2.26) \]

\(^7\)As anticipated in the notation, we will append a suffix "0" when we refer to compound fields at the classical level, that is, given in terms of \( T_0, N_0^{ab} \) and \( J_0 \), and we will reserve the notation without suffix "0" in denoting the corresponding quantities at the quantum level, given in terms of \( T, N^{ab} \) and \( J \).
\[ T_\lambda = \Omega \partial \lambda + 3 \partial Y \partial \lambda + \frac{3}{2} Y \partial^2 \lambda, \quad (2.27) \]

where we have introduced the quantity

\[ \Omega_\alpha = \omega_\alpha - \frac{3}{2} \partial Y_\alpha. \quad (2.28) \]

The \( \Lambda \)-formalism explained thus far is also useful to deal with the \( b \) field which plays an important role in computing higher loop amplitudes. Its main property is

\[ \{ Q, b(z) \} = T(z), \quad (2.29) \]

where \( T \) is the stress energy tensor. Since in the pure spinor formulation the reparametrization ghosts do not exist, \( b \) must be a composite field. Moreover, since the \( b \) ghost has ghost number \(-1\) and the covariant fields, which include \( \omega_\alpha \) and are gauge invariant under the \( \omega \)-symmetry, always have ghost number zero or positive, one must use \( Y_\alpha \) (which also has ghost number \(-1\)) to construct the \( b \) ghost. Therefore \( b \) is not super-Poincaré invariant. The \( b \) ghost has been constructed for the first time in [3] in the \( U(5) \)-formalism in such a way that it satisfies Eq. (2.29). In the \( \Lambda \)-formalism at hand, at the classical level it takes the form

\[ b_0 Y = \frac{1}{2} \Pi^a Y \Gamma_d + \omega(1 - K) \partial \theta = Y_\alpha G_\alpha^0, \quad (2.30) \]

where

\[ G_\alpha^0 = \frac{1}{2} \Pi_a (\Gamma^d)_b - \frac{1}{4} N_{ab} (\Gamma_{ab} \partial \theta)^\alpha - \frac{1}{4} J_0 \partial \theta^\alpha. \quad (2.31) \]

The last equality in (2.30) follows from the identity (A.3). The expression of \( b Y \) at the quantum level will be derived in section 5.

The non-covariance of \( b Y \) is not dangerous since, as we shall show in section 5, the Lorentz variation of \( b Y \) (or of its improvement at the non-minimal level) is BRST-exact. However, this operator cannot be accepted as insertion to compute higher loop amplitudes. Indeed, contrary to the operators \( T, N_{ab} \) and \( J \), it has a true singularity at \( v \lambda = 0 \) of the form \( (v \lambda)^{-1} \). The point is that there exists an operator \( \xi = Y \theta \), singular with a pole at \( v \lambda \to 0 \), such that \( \{ Q, \xi \} = 1 \) and the cohomology would become trivial if this operator is allowed in the Hilbert space, since for any closed operator \( V \), \( V = \{ Q, \xi V \} \). Then, for consistency, operators singular at \( v \lambda \to 0 \) must be excluded from the Hilbert space.

### 3 Fundamental operators and normal-ordering effects

When we attempt to construct a \( b \) ghost covariantly, either a picture-raised \( b \) ghost [14, 30] or a covariant \( b \) ghost in the framework of the non-minimal approach [27], we encounter several fundamental operators, \( G^\alpha, H^{\alpha\beta}, K^{\alpha\beta\gamma} \) and \( L^{\alpha\beta\gamma\delta} [14, 30] \), which are a generalization of the constraints introduced by Siegel some time ago in [32]. Thus, in this section, we will consider
those operators in order. We will pay a special attention to a consistent treatment of the normal-ordering effects.

Let us notice that in addition to $G^{\alpha}$, the totally antisymmetrized operators $H^{[\alpha\beta]}$, $K^{[\alpha\beta\gamma]}$ and $L^{[\alpha\beta\gamma\delta]}$ are the more fundamental objects and are of particular interest since they are involved in the construction of the $b$ field in the non-minimal formulation. At the classical level, $G^{\alpha}$ is defined in (2.31) and $H^{[\alpha\beta]}$, $K^{[\alpha\beta\gamma]}$ and $L^{[\alpha\beta\gamma\delta]}$ are given by

\begin{align*}
H_0^{[\alpha\beta]} &= \frac{1}{384} \Gamma_{\alpha\beta}^{\gamma} d^{\gamma} + 24 N_0^{\alpha\beta}, \\
K_0^{[\alpha\beta\gamma]} &= -\frac{1}{96} \Gamma_{\alpha\beta\gamma}^{\delta} N_0^{\delta}, \\
L_0^{[\alpha\beta\gamma\delta]} &= -\frac{1}{3072} (\Gamma_{\alpha\beta\gamma\delta})^a_b \Gamma_{b}^{a\delta} N_0^{\delta},
\end{align*}

(3.1)

They satisfy the following recursive relations:

\begin{align*}
\{Q, G_0^{\alpha}\} &= \lambda^\alpha T_0, \\
[Q, H_0^{[\alpha\beta]}] &= \lambda^\beta G_0^{\alpha}, \\
\{Q, K_0^{[\alpha\beta\gamma]}\} &= \lambda^\gamma H_0^{[\alpha\beta\gamma]}, \\
[Q, L_0^{[\alpha\beta\gamma\delta]}] &= \lambda^\delta K_0^{[\alpha\beta\gamma\delta]}, \\
\lambda^\alpha L_0^{[\beta\gamma\delta\rho]} &= 0,
\end{align*}

(3.2)

which one can verify easily. The full fields $H_0^{[\alpha\beta]}$, $K_0^{[\alpha\beta\gamma]}$ and $L_0^{[\alpha\beta\gamma\delta]}$, which are involved in the construction of the picture-raised $b$ ghost, can be obtained by adding new terms symmetric with respect to at least a couple of adjacent indices, and they satisfy the recursive relations

\begin{align*}
[Q, H_0^{[\alpha\beta]}] &= \lambda^\alpha G_0^{\beta} + \cdots, \\
\{Q, K_0^{[\alpha\beta\gamma]}\} &= \lambda^\alpha H_0^{[\beta\gamma]} + \cdots, \\
[Q, L_0^{[\alpha\beta\gamma\delta]}] &= \lambda^\alpha K_0^{[\beta\gamma\delta]} + \cdots, \\
\lambda^\alpha L_0^{[\beta\gamma\delta\rho]} &= 0 + \cdots,
\end{align*}

(3.3)

where the dots denote "$\Gamma_1$-traceless terms", i.e. terms that vanish if saturated with a $\Gamma_{\alpha_1\alpha_{i+1}}^{\alpha_i}$ between two adjacent indices. The fields $H_0^{[\alpha\beta]}$, $K_0^{[\alpha\beta\gamma]}$ and $L_0^{[\alpha\beta\gamma\delta]}$ are defined modulo $\Gamma_1$-traceless terms.

In this section we wish to discuss these operators and their recursive relations at the quantum level. A remark is in order. At the quantum level, in dealing with holomorphic operators composed of fields with singular OPE’s, a normal-ordering prescription is needed for their definition. As a rule, for the normal ordering of two operators $A$ and $B$ we shall adopt in this paper the generalized normal-ordering prescription, denoted by $(AB)$ in [33] since it is convenient in carrying out explicit calculations. As explained in Appendix B, this prescription consists in subtracting the singular poles, evaluated at the point of the second entry and it is given by the contour integration

\[(AB)(z) = \oint \frac{dw}{w - z} A(w) B(z).\]
Often, for simplicity, in dealing with this prescription the outermost parenthesis is suppressed and the normal ordering is taken from the right so that, in general, \(A_1A_2A_3\ldots A_n\) means \(A_1(A_2(A_3(\cdots A_n))\cdots))\).

A different prescription denoted as : \(AB\); that we shall call ”improved”, consists in subtracting the full contraction \(< A(y)B(z) >\), included a possible finite term, as computed from the canonical OPE’s (2.7) and (2.14). In many cases the two prescriptions coincide but when they are different, it happens, as we shall see, that the final results look more natural if expressed in the improved prescription.

### 3.1 \(G^\alpha\)

\(G^\alpha\) is obtained from (2.31) by replacing \(N_0^{ab}\) and \(J_0\) with \(N^{ab}\) and \(J\) as defined in Eqs. (2.17) and (2.18) and adding a normal-ordering term parametrized by a constant \(c_1\)

\[
G^\alpha = \frac{1}{2}\Pi^a(\Gamma_a d)^\alpha - \frac{1}{4}N_{ab}(\Gamma^{ab}\partial\theta)^\alpha - \frac{1}{4}J\partial\theta^\alpha + c_1\partial^2\theta^\alpha
\]

\[
\equiv G_1^\alpha + G_2^\alpha + G_3^\alpha + G_4^\alpha. \tag{3.5}
\]

The constant \(c_1\) will be determined from the requirement that \(G^\alpha\) should be a primary field of conformal weight 2. Then we have to compute the OPE \(< T(y)G^\alpha(z) >\). The three terms \(G_1^\alpha \equiv \frac{1}{2}\Pi^a(\Gamma_a d)^\alpha\), \(G_2^\alpha \equiv -\frac{1}{4}N_{ab}(\Gamma^{ab}\partial\theta)^\alpha\) and \(G_3^\alpha \equiv -\frac{1}{4}J\partial\theta^\alpha\) are all products of two operators of conformal weight 1 so that their OPE’s with the stress energy tensor can be easily calculated. One finds that only \(G_2^\alpha\) is a primary field. \(G_1^\alpha\) has a triple pole with residuum \(-5\partial\theta^\alpha\) and \(G_3^\alpha\) has a triple pole with residuum \(-2\partial\theta^\alpha\). Moreover, the normal-ordering term \(G_4^\alpha \equiv c_1\partial^2\theta^\alpha\) also has a triple pole with residuum \(2c_1\partial\theta^\alpha\). Therefore, putting them together, one has

\[
<T(y)G^\alpha(z)> = \frac{-5 - 2 + 2c_1}{(y - z)^3}\partial\theta^\alpha(z) + \frac{2}{(y - z)^2}G^\alpha(z) + \frac{1}{y - z}\partial G^\alpha(z). \tag{3.6}
\]

Hence, the requirement that \(G^\alpha\) must be a primary field of conformal weight 2 is satisfied when we select the constant \(c_1\) to be \(\frac{7}{2}\).

In spite of the appearance, it turns out that this figure is in agreement with the result of [14] where the value \(-\frac{1}{4}\) is indicated as the coefficient in front of the normal-ordering term \(\partial^2\theta^\alpha\) in \(G^\alpha\). The difference is an artifact of the different normal-ordering prescriptions, the generalized normal-ordering prescription in (3.5) and the improved one. Whereas the two prescriptions coincide for \(G_2^\alpha\) and \(G_3^\alpha\), there appears a difference in \(G_1^\alpha\). Indeed, since

\[
\Pi^a(x)d_\alpha(z) = \frac{1}{2}\frac{1}{(x - z)^2}[(\Gamma^a\theta)_\alpha(z) - (\Gamma^a\theta)_\alpha(x)]
\]

\[
- \frac{1}{2}\frac{1}{x - z}(\Gamma^a\partial\theta)_\alpha(x) + : \Pi^a(z)d_\alpha(z) : + \cdots, \tag{3.7}
\]

we obtain

\[
\frac{1}{2}\Pi^a(\Gamma_a d)^\alpha = -\frac{15}{4}\partial^2\theta^\alpha + \frac{1}{2} :\Pi^a d_\alpha:. \tag{3.8}
\]
Substituting this result into Eq. (3.5), setting \( c_1 = \frac{7}{2} \), we have

$$G^\alpha =: \frac{1}{2} \Pi^a (\Gamma_a d)^\alpha - \frac{1}{4} N_{ab} (\Gamma^{ab} \partial \theta)^\alpha - \frac{1}{4} J \partial \theta^\alpha - \frac{1}{4} \partial^2 \theta^\alpha,$$

which precisely coincides with the expression given in [14].

Next, we want to derive the quantum counterpart of the first (classical) recursive relations in (3.2) and, for that, we need to compute \( \{Q, G^\alpha\} \). In doing this calculation, one must be careful to deal with the order of the factors in the terms coming from the (anti)commutator among \( Q \) and \( G^\alpha \) and use repeatedly the rearrangement theorem, reviewed in Appendix B, in order to recover the operator \( \lambda^\alpha T \). The details of this calculation are presented in Appendix C. As expected from the covariance of \( \{Q, G^\alpha\} \), the terms involving \( Y \), coming from the rearrangement procedure, cancel exactly those coming from the \( Y \)-dependent correction terms of the operators \( N_{ab} \) and \( J \) (see (2.17) and (2.18)) present in the definition of \( G^\alpha \). The final result is

$$\{Q, G^\alpha\} = \lambda^\alpha T - \frac{1}{2} \partial^2 \lambda^\alpha.$$

The normal-ordering term \(-\frac{1}{2} \partial^2 \lambda^\alpha\) in (3.10) might appear to be strange at first sight, but it is indeed quite reasonable. The point is that it is not \( \lambda^\alpha T \) but \( \lambda^\alpha T - \frac{1}{2} \partial^2 \lambda^\alpha \) that is a primary field of conformal weight 2 when we take account of the normal-ordering effects. In fact, since

$$< \lambda^\alpha(y)T(z) > = R^\alpha_2(z) \equiv \frac{R^\alpha_2(z)}{y-z} = \frac{-\partial \lambda^\alpha(z)}{y-z},$$

has a triple pole with residuum \(+\partial^2 \lambda\), and \( \frac{1}{2} \partial^2 \lambda \) has the same triple pole, it follows that

$$B^\alpha_1 = \lambda^\alpha T - \frac{1}{2} \partial^2 \lambda^\alpha,$$

is a BRST-closed primary operator of conformal weight 2. From now on, it is convenient to define

$$\hat{G}^\alpha = G^\alpha + \frac{1}{2} \partial^2 \theta^\alpha,$$

so that (3.10) becomes

$$\{Q, \hat{G}^\alpha\} = \lambda^\alpha T.$$

Now we would like to study the operator \( \lambda^\alpha G^\beta \), that is expected to arise in the quantum counterpart of the second recursive relations in (3.2). As before, \( \lambda^\alpha G^\beta \) is not primary since

$$< \lambda^\alpha(y)G^\beta(z) > \neq 0,$$

is different from zero. Indeed,

$$< \lambda^\alpha(y)G^\beta(z) > = R^{\alpha\beta}_2(z) \equiv \frac{R^{\alpha\beta}_2(z)}{y-z},$$

so that (3.10) becomes
where

\[ R_2^{\alpha \beta} = -\partial \theta^{\alpha} \lambda^{\beta} + \frac{1}{2} \Gamma^{\alpha \beta}_{\alpha}(\partial \theta \Gamma^{\alpha} \lambda). \]  

(3.16)

Note that since \( \partial \lambda^{\alpha} \partial \theta^{\beta} \) is also primary, there is an ambiguity in defining a primary operator, say \( B_2^{\alpha \beta} \), associated to \( \lambda^{\alpha} \Gamma^{\beta} \). Given (3.16), for the symmetric one, one has

\[ B_2^{(\alpha \beta)} = \frac{1}{16} \Gamma^{\alpha \beta}_{\alpha}(\lambda \Gamma^{\alpha} \lambda G^{\beta}) + \frac{7}{2} \partial(\lambda^{\alpha} \partial \theta^{\beta}) + c_+ (\partial \lambda^{\alpha} \partial \theta^{\beta}), \]  

(3.17)

while, for the antisymmetric one, one has

\[ B_2^{[\alpha \beta]} = \lambda^{[\alpha} \Gamma^{\beta]} + \frac{1}{2} \partial(\lambda^{[\alpha} \partial \theta^{\beta]}) + c_- \partial \lambda^{[\alpha} \partial \theta^{\beta]} \]  

(3.18)

Let us remark that \( \lambda^{\alpha} \Gamma^{\beta} \) is not BRST-closed. Indeed \( \{Q, \lambda^{\alpha} \Gamma^{\beta}\} = \lambda^{\alpha} \lambda^{\beta} T - \frac{1}{2} \lambda^{\alpha} \partial^2 \lambda^{\beta} \). Whereas \( \langle T(\lambda^{[\alpha} \Gamma^{\beta)} T) \rangle = (T(\lambda^{[\alpha} \lambda^{\beta])) = (T(\lambda^{\alpha} \lambda)\rangle - 2(\lambda^{\alpha} \partial^2 \lambda) \). Therefore the requirement that \( B_2^{(\alpha \beta)} \) and \( B_2^{[\alpha \beta]} \) are BRST-closed implies \( c_+ = -\frac{5}{2} \) and \( c_- = -\frac{1}{2} \) so that

\[ B_2^{(\alpha \beta)} = \frac{1}{16} \Gamma^{(\alpha \beta}_{\alpha}(\lambda \Gamma^{\alpha} \lambda G^{\beta}) + \frac{5}{2} \lambda^{\alpha} \partial^2 \theta^{\beta} + \partial(\lambda^{\alpha} \partial \theta^{\beta}), \]  

(3.19)

\[ B_2^{[\alpha \beta]} = \lambda^{[\alpha} \Gamma^{\beta]} + \frac{1}{2} \lambda^{[\alpha} \partial^2 \theta^{\beta]} = \lambda^{[\alpha} \hat{\Gamma}^{\beta]}. \]  

(3.20)

### 3.2 \( H^{\alpha \beta} \)

A minimal choice for \( H^{\alpha \beta} \) is

\[ H^{\alpha \beta} = H^{(\alpha \beta)} + H^{[\alpha \beta]}, \]  

(3.21)

where

\[ H^{(\alpha \beta)} = \frac{1}{16} \Gamma^{(\alpha \beta}_{\alpha}(N^{ab} \Pi_b - \frac{1}{2} J \Pi^b + c_2 \partial \Pi^b), \]  

(3.22)

\[ H^{[\alpha \beta]} = \frac{1}{96} \Gamma^{(\alpha \beta}_{\alpha}(\frac{1}{4} d \Gamma^{abc} d + 6 N^{ab} \Pi^c). \]  

(3.23)

First, we shall evaluate \( < T(y)H^{\alpha \beta}(z) > \) in order to fix the normal-ordering term. We can easily show that \( H^{[\alpha \beta]} \) and the first term in \( H^{(\alpha \beta)} \) are primary fields whereas \( -\frac{1}{32} \Gamma^{\alpha \beta}_{\alpha} J \Pi^a \) and \( \frac{2}{16} \Gamma^{\alpha \beta}_{\alpha} \partial \Pi^a \) have a triple pole with residua \( -4 \frac{1}{16} \Gamma^{\alpha \beta}_{\alpha} \Pi^a \) and \( 2c_2 \frac{1}{16} \Gamma^{\alpha \beta}_{\alpha} \Pi^a \), respectively. Thus, we obtain

\[ < T(y)H^{\alpha \beta}(z) > = \frac{-4 + 2c_2}{(y - z)^3} \frac{1}{16} \Gamma^{\alpha \beta}_{\alpha} \Pi^a(z) + \frac{2}{(y - z)^2} H^{\alpha \beta}(z) + \frac{1}{y - z} \partial H^{\alpha \beta}(z), \]  

(3.24)
thereby taking $c_2 = 2$ makes $H^{\alpha\beta}$ a primary field of conformal weight 2. This value agrees with the value in the Berkovits’ paper [14]. Next, we wish to evaluate $[Q, H^{\alpha\beta}]$:

$$[Q, H^{(\alpha\beta)}] = \frac{1}{16} \Gamma^{\alpha\beta}_a \left[ \frac{1}{2}(\lambda \Gamma^{ab} d) \Pi_b + N^{ab}(\lambda \Gamma \partial \theta) + \frac{1}{2}(\lambda d) \Pi a \right.$$

$$\left. - \frac{1}{2} J(\lambda \Gamma \partial \theta) + c_2 \partial(\lambda \Gamma \partial \theta) \right], \quad (3.25)$$

and

$$[Q, H^{[\alpha\beta]}] = \frac{1}{96} \Gamma^{\alpha\beta}_{abc} \left[ - \frac{1}{4} ((\Gamma^d \lambda)_\rho \Pi_d) (\Gamma^{abc} d)\rho - \frac{1}{4} (\Gamma^{abc} d)(\Gamma^d \lambda)_\rho \Pi d \right.$$

$$\left. + 3(\lambda \Gamma^{ab} d) \Pi c + 6 N^{ab}(\lambda \Gamma c \partial \theta) + 2 c_3 (\partial \lambda \Gamma abc \partial \theta) \right]. \quad (3.26)$$

Then, after some algebra and taking into account the normal-ordering terms by the rearrangement formula, we get for the symmetric part of $H^{\alpha\beta}$

$$[Q, H^{(\alpha\beta)}] = \frac{1}{16} \Gamma^{\alpha\beta}_a \left[ \lambda \Gamma^a G + \frac{5}{2} \lambda \Gamma^a \partial^2 \theta + \partial(\lambda \Gamma^a \partial \theta) \right], \quad (3.27)$$

and for the more interesting antisymmetric part $H^{[\alpha\beta]}$

$$[Q, H^{[\alpha\beta]}] = \lambda^{[\alpha} G^{\beta]} + \frac{1}{2} \lambda^{[\alpha} \partial^2 \theta^{\beta]} = \lambda^{[\alpha} \hat{G}^{\beta]}, \quad (3.28)$$

in agreement with (3.19) and (3.20). Notice that the $Y$-dependent contributions coming from rearrangement theorem cancel exactly those coming from the definitions (2.17) and (2.18) of $N^{ab}$ and $J$ (For details see Appendix C).

Since the term $+ \partial(\lambda \Gamma^a \partial \theta)$ in (3.27) is the BRST variation of $\partial \Pi^a$, (3.27) can be rewritten as

$$[Q, \hat{H}^{(\alpha\beta)}] = \frac{1}{16} \Gamma^{\alpha\beta}_a \left[ \lambda \Gamma^a G + \frac{5}{2} \lambda \Gamma^a \partial^2 \theta \right], \quad (3.29)$$

where we have defined as $\hat{H}^{(\alpha\beta)} = H^{(\alpha\beta)} - \frac{1}{16} \Gamma^{\alpha\beta}_a \partial \Pi^a$.

Now let us consider the composite operator $\lambda^\alpha H^{\beta\gamma}$. Since $H^{\alpha\beta}$ has conformal weight 2 but its contraction with $\lambda^\alpha$ does not vanish, one can expect that $\lambda^\alpha H^{\beta\gamma}$ is not primary. Actually, using the fact

$$< \lambda^\alpha (y) H^{\beta\gamma}(z) > \equiv \frac{R_{3}^{\alpha\beta\gamma}(z)}{y - z}, \quad (3.30)$$

with $R_{3}^{\alpha\beta\gamma}$ being given by

$$R_{3}^{\alpha\beta\gamma} = - \frac{1}{32} \Gamma^{\beta\gamma}_b [(\Gamma^{ab} \lambda)^\alpha \Pi_b - \lambda^\alpha \Pi^a] - \frac{1}{32} \Gamma^{\alpha\beta\gamma}_b (\Gamma^{ab} \lambda)^\alpha \Pi^c, \quad (3.31)$$

it turns out that a primary field of conformal weight 2 related to $\lambda^\alpha H^{\beta\gamma}$ is

$$B_{3}^{\alpha\beta\gamma} \equiv \lambda^\alpha H^{\beta\gamma} + \frac{1}{2} \partial R_{3}^{\alpha\beta\gamma}. \quad (3.32)$$
Again there is an arbitrariness in choosing the primary field related to $\lambda^\alpha H^\beta\gamma$ since $\partial \lambda^\alpha \Pi^\alpha$ is primary.

As in previous cases we are especially interested in the antisymmetric part $B_3^{[\alpha\beta\gamma]}$ of $B_3^{\alpha\beta\gamma}$. Since, in $D = 10$, a field which is totally antisymmetric in its three, spin-or-like indices contains only the $SO(10)$ irreducible representation (irrep.) 560 and $R_3^{[\alpha\beta\gamma]}$ in Eq. (3.31) does not contain such an irrep., it follows that

$$R_3^{[\alpha\beta\gamma]} = 0,$$

so that $B_3^{[\alpha\beta\gamma]}$ simply becomes

$$B_3^{[\alpha\beta\gamma]} = \lambda^{[\alpha} H^{\beta\gamma]}.$$

From Eqs. (3.28), (3.15) and (3.16), it is then easy to show that $\lambda^{[\alpha} H^{\beta\gamma]}$ is BRST-closed. Indeed, one finds

$$[Q, \lambda^{[\alpha} H^{\beta\gamma]}] = \lambda^{[\alpha} (\lambda^\beta \hat{G}^{\gamma]})$$

$$= \hat{G}^{[\gamma} \lambda^\alpha \lambda^\beta] + \lambda^{[\alpha} \partial (\lambda^\beta \partial \theta^{\gamma]}) + \partial (\lambda^{[\alpha} \partial \theta^{\gamma]} \lambda^\beta]$$

$$= 0.$$

### 3.3 $K^{\alpha\beta\gamma}$

A covariant expression of $K^{\alpha\beta\gamma}$ is

$$K^{\alpha\beta\gamma} = -\frac{1}{48} \Gamma^\alpha_a (\Gamma_b d)^\gamma N^{ab} - \frac{1}{192} \Gamma^\alpha_{abc} (\Gamma^a d)^\gamma N^{bc}$$

$$+ \frac{1}{192} \Gamma^\gamma_{a [b} (\Gamma_c d)^\alpha N^{ab} + \frac{3}{2} (\Gamma^a d)^\alpha J + c_3 (\Gamma^a d)^\alpha)$$

$$- \frac{1}{192} \Gamma^\gamma_{abc} (\Gamma^a d)^\alpha N^{bc}$$

$$\equiv K_1^{\alpha\beta\gamma} + K_2^{\alpha\beta\gamma} + K_3^{\alpha\beta\gamma} + K_4^{\alpha\beta\gamma} + K_5^{\alpha\beta\gamma} + K_6^{\alpha\beta\gamma},$$

whereas the totally antisymmetric part is given by

$$K^{[\alpha\beta\gamma]} = -\frac{1}{96} \Gamma^{[\alpha\beta}_{abc} (\Gamma^a d)^\gamma N^{bc}.$$ (3.37)

The term including a constant $c_3$ describes the normal-ordering contribution. As before, we will calculate $\langle T(y) K_4^{\alpha\beta\gamma}(z) \rangle$ in order to fix the normal-ordering term. One finds that all the terms $K_i^{\alpha\beta\gamma}$ are primary with conformal weight 2, except $K_4^{\alpha\beta\gamma} \equiv \frac{1}{192} \Gamma^\gamma_{a [b} (\Gamma^a d)^\alpha J$ and $K_5^{\alpha\beta\gamma} \equiv c_3 \frac{1}{192} \Gamma^\gamma_{a [b} (\Gamma^a d)^\alpha \partial d)^\alpha$ which have triple poles in their OPE’s with $T$. In fact,

$$\langle T(y) K_4^{\alpha\beta\gamma}(z) \rangle = \frac{1}{192} \Gamma^\gamma_{a [b} \frac{12}{(y-z)^3} (\Gamma^a d)^\alpha(z) + \frac{2}{(y-z)^2} K_4^{\alpha\beta\gamma}(z) + \frac{1}{y-z} \partial K_4^{\alpha\beta\gamma}(z),$$

$$\langle T(y) K_5^{\alpha\beta\gamma}(z) \rangle = \frac{1}{192} \Gamma^\gamma_{a [b} \frac{2c_3}{(y-z)^3} (\Gamma^a d)^\alpha(z) + \frac{2}{(y-z)^2} K_5^{\alpha\beta\gamma}(z) + \frac{1}{y-z} \partial K_5^{\alpha\beta\gamma}(z).$$ (3.38)
Therefore, one obtains
\[
<T(y)K_{\alpha\beta\gamma}(z) >= \frac{1}{192} \Gamma_\alpha^{\beta\gamma} \frac{12 + 2c_3}{(y - z)^3}(\Gamma^a d)^\alpha(z) + \frac{2}{(y - z)^2}K_{\alpha\beta\gamma}(z) + \frac{1}{y - z}\partial K_{\alpha\beta\gamma}(z),
\]
so that the condition of a primary operator of conformal weight 2 requires us to take \( c_3 = -6 \), which is a new result.

As for \( \{Q, K_{\alpha\beta\gamma}\} \), we will limit ourselves to considering only the antisymmetric part \( K_{[\alpha\beta\gamma]} \)
\[
\{Q, K_{[\alpha\beta\gamma]}\} = \frac{1}{96} \Gamma^\alpha_{abc} \left[ (\Gamma^a \Gamma^d \lambda)^\gamma \Pi_d N^{bc} + \frac{1}{2}(\Gamma^a d)^\gamma(\lambda \Gamma^{bc} d) \right].
\]
(3.40)

As before, the \( Y \)-dependent contributions coming from rearrangement theorem are exactly cancelled by those coming from the definition (2.17) of \( N^{ab} \), as expected from the covariance of the l.h.s of (3.40). Then from the rearrangement theorem and with a few algebra one gets
\[
\{Q, K_{[\alpha\beta\gamma]}\} = \lambda^{[\alpha H^{\beta\gamma]}},
\]
(3.41)

Given that the \( Y \)-dependent terms are absent, (3.41) can also been argued as follows: cohomology arguments based on Eq. (3.35) and the classical equivalence between \( \{Q, K_{[\alpha\beta\gamma]}\} \) and \( \lambda^{[\alpha H^{\beta\gamma]} \} \) imply \( \{Q, K_{[\alpha\beta\gamma]}\} = \lambda^{[\alpha H^{\beta\gamma]} + \Lambda_{3\beta\gamma}] \), where \( \Lambda^{[\alpha\beta\gamma]} \) is a primary field of conformal weight 2 satisfying \( [Q, \Lambda^{[\alpha\beta\gamma]}] = 0 \). Then, notice that \( \Lambda^{[\alpha\beta\gamma]} \) has ghost number +1 and involves \( \partial \lambda^\alpha \) and \( \Pi^a \) or \( \partial \Pi^a \) and \( \lambda^\alpha \). However, using these fields, it is impossible to construct a 560 irrep. of \( SO(10) \), so \( \Lambda^{[\alpha\beta\gamma]} \) vanishes identically.

As before, let us construct a primary field of conformal weight 2 from \( \lambda^\alpha K_{[\beta\gamma]} \). We define \( R_4^{\alpha\beta\gamma\delta} \) by
\[
<\lambda^\alpha(y)K_{[\beta\gamma]}(z)>= \frac{R_4^{\alpha\beta\gamma\delta}(z)}{y - z},
\]
(3.42)
where \( R_4^{\alpha\beta\gamma\delta} \) takes the form
\[
R_4^{\alpha\beta\gamma\delta} = \frac{1}{96}(\Gamma^a b \lambda)^\alpha \Gamma^\beta\gamma^\delta \Gamma_b d + \frac{1}{384}(\Gamma^a b \lambda)^\alpha \Gamma^\beta\gamma^\delta \Gamma_a d \Gamma^c d + \frac{1}{384}((\Gamma^a b \lambda)^\alpha \Gamma_b d)^\beta + 3\lambda^\alpha(\Gamma^a d)^\beta \Gamma_a^\gamma\delta + \frac{1}{384}(\Gamma^a b \lambda)^\alpha (\Gamma^c d)^\beta \Gamma_a^\gamma\delta.
\]
(3.43)

Provided that we define \( B_4^{\alpha\beta\gamma\delta} \) as
\[
B_4^{\alpha\beta\gamma\delta} \equiv \lambda^\alpha K_{[\beta\gamma]} + \frac{1}{2}\partial R_4^{\alpha\beta\gamma\delta},
\]
(3.44)

it is easy to get
\[
<T(y)B_4^{\alpha\beta\gamma\delta}(z)> = \frac{2}{(y - z)^2}B_4^{\alpha\beta\gamma\delta}(z) + \frac{1}{y - z}\partial B_4^{\alpha\beta\gamma\delta}(z),
\]
(3.45)
which means that $B_4^{\alpha\beta\gamma\delta}$ is a primary field of conformal weight 2 as expected. As before, there is an arbitrariness in the choice of the primary field related to $\lambda^\alpha K^{\beta\gamma\delta}$ since the field $\partial \lambda^\alpha d_\beta$ is primary.

If one considers the completely antisymmetric component $B_4^{[\alpha\beta\gamma\delta]}$, one can notice that, in $D = 10$, a field antisymmetric in its four, spinor-like indices contains only the irreps. 770 and 1050 which are absent in the expression (3.43) of $R_4^{[\alpha\beta\gamma\delta]}$ so that one obtains

$$R_4^{[\alpha\beta\gamma\delta]} = 0. \quad (3.46)$$

Consequently, we have

$$B_4^{[\alpha\beta\gamma\delta]} = \lambda^{[\alpha} K^{\beta\gamma\delta]}.$$  

Furthermore, Eq. (3.41) together with (3.33) gives us the equation

$$\{Q, \lambda^{[\alpha} K^{\beta\gamma\delta]}\} = 0. \quad (3.48)$$

### 3.4 $L^{\alpha\beta\gamma\delta}$

In this final subsection, we wish to consider $L^{\alpha\beta\gamma\delta}$. In our previous paper [30], at the classical level, the form of $L^{\alpha\beta\gamma\delta}$ was fixed to be

$$L_0^{\alpha\beta\gamma\delta} = \frac{-1}{24} \lambda^\alpha (\bar{\omega} \Gamma^a)^\beta [\lambda^\gamma (\bar{\omega} \Gamma_a)^\delta - \frac{1}{4} (\Gamma_b \Gamma_a)^\gamma (\bar{\omega} \Gamma^b)^\delta], \quad (3.49)$$

where $\bar{\omega}$ is defined in (2.15). One subtle point associated with this expression is that $L_0^{\alpha\beta\gamma\delta}$ cannot be entirely expressed in terms of $N_0^{ab}$ and $J_0$. However, we have found that the dangerous terms involving $\bar{\omega} \Gamma^{a_1 a_2 a_3 a_4} \lambda$ cancel exactly in constructing the picture raised $b$ ghost.

On the other hand, when we consider the totally antisymmetrized part of $L_0^{\alpha\beta\gamma\delta}$, these dangerous terms never appear. In order to show that, let us notice that, given Eq. (3.49), one can write:

$$L_0^{\alpha\beta\gamma\delta} + L_0^{\gamma\delta\alpha\beta} = \frac{-1}{144} (\bar{\omega} \Lambda^\alpha_{\gamma\delta}) (\bar{\omega} \Lambda^\gamma_{\delta\alpha}) - \frac{1}{24} \lambda^\alpha (\bar{\omega} \Gamma^a)^\beta \lambda^\gamma (\bar{\omega} \Gamma_a)^\delta, \quad (3.50)$$

where we have defined

$$\bar{\omega} \Lambda^\alpha_{\gamma\delta} = (\bar{\omega} \Gamma_c)^\alpha \lambda^\beta - \frac{1}{4} (\bar{\omega} \Gamma_a)^\alpha (\Gamma^a \Gamma^b_\gamma \lambda)^\beta. \quad (3.51)$$

Then, taking the totally antisymmetrized part of Eq. (3.50) one gets

$$L_0^{[\alpha\beta\gamma\delta]} = \frac{-1}{48} (\bar{\omega} \Lambda^{[\alpha\beta]}_{\gamma\delta}) (\bar{\omega} \Lambda^{\gamma\delta]_{\alpha\beta}}). \quad (3.52)$$

Using (3.51) and (A.3), we can rewrite $\bar{\omega} \Lambda^{[\alpha\beta]}_{\gamma\delta}$ as

$$\bar{\omega} \Lambda^{[\alpha\beta]}_{\gamma\delta} = \frac{1}{16} \Gamma^{[\alpha\beta}_{abc} (\omega \Gamma^{ab}_c \lambda)$$

$$= \frac{1}{8} \Gamma^{[\alpha\beta}_{abc} \lambda_{bc}^0. \quad (3.53)$$
Hence, we have shown that $L_{[0}^{[\alpha\beta\gamma\delta]}$ is in fact expressed by $N_{0}^{ab}$.

In order to have a covariant expression for $L^{[\alpha\beta\gamma\delta]}$, at the quantum level, the classical Lorentz generator $N_{0}^{bc}$ must be replaced with $N^{bc}$ as given in (2.17) so that $L^{[\alpha\beta\gamma\delta]}$ is

$$L^{[\alpha\beta\gamma\delta]} = -\frac{1}{3072}(\Gamma_{abc}^{[\alpha\beta}(\Gamma^{ade})^{\gamma\delta]}N^{bc}N_{de}).$$

(3.54)

From the OPE’s $<T(y)N^{ab}(z)>$ and $<N^{ab}(y)N^{cd}(z)>$, one can easily verify that $L^{[\alpha\beta\gamma\delta]}$ is a covariant, primary field of conformal weight 2.

At the classical level one has the identities

$$[Q, L_{0}^{[\alpha\beta\gamma\delta]}] = \lambda^{[\alpha}K_{0}^{\beta\gamma\delta]},
\lambda^{[\alpha}L_{0}^{\beta\gamma\delta]\rho] = 0,$$

(3.55)

where the last identity follows by noting that $L_{0}^{[\alpha\beta\gamma\delta]}$ is proportional to $\lambda^{[\alpha}(\omega\Gamma_{\alpha})^{\beta}(\omega\Gamma_{\beta})^{\gamma}(\Gamma^{ab}\lambda)^{\delta]}$. Since $L^{[\alpha\beta\gamma\delta]}$ and $\lambda^{[\alpha}L_{0}^{\beta\gamma\delta]}$ are covariant tensors and a possible quantum failure of these identities would involve $Y_{\alpha}$, thereby inducing violation of Lorentz covariance, one should expect that these identities hold at the quantum level as well. It is worthwhile to verify this result directly as a nice check of the consistency of the Y-formalism.

The quantum counterpart of the former equation in Eq. (3.55) reads

$$[Q, L^{[\alpha\beta\gamma\delta]}] = \lambda^{[\alpha}K^{\beta\gamma\delta]}.$$

(3.56)

In this case there are no contributions from the rearrangement theorem and using (3.37) and (3.54) one finds that both sides of Eq. (3.56) are equal to $\frac{1}{768}(\Gamma_{abc}^{[\alpha\beta}(\Gamma^{ade})^{\gamma\delta]}(d^{bc}\lambda)N_{de}$, thus showing that (3.56) is true. It is a little more cumbersome to verify the quantum analog of the latter equation in Eq. (3.55), which is given by

$$\lambda^{[\alpha}L^{\beta\gamma\delta]} = 0.$$

(3.57)

To do that it is convenient to introduce the following notation that extends that in Eq. (3.51): if $\Psi_{[\alpha}$ and $\Phi_{\beta]}$ are two spinors that (by the conventions which we adopt) belong to the 16 and the 16 of $SO(10)$, respectively, we define

$$\Psi\Lambda_{c}^{[\alpha\beta]}\Phi = (\Psi\Gamma_{c})^{[\alpha}(\Phi\Gamma_{c})^{\beta]} - \frac{1}{4}(\Psi\Gamma_{b})^{[\alpha}(\Gamma_{b}\Gamma_{c}\Phi)^{\beta]}.$$

(3.58)

Then, from Eqs. (2.17) and (3.54), $L^{[\alpha\beta\gamma\delta]}$ can be rewritten as

$$L^{[\alpha\beta\gamma\delta]} = -\frac{1}{48}N_{c}^{[\alpha\beta}\Lambda_{c}^{\gamma\delta]},$$

(3.59)

where

$$N_{c}^{[\alpha\beta]} \equiv \frac{1}{8}\Gamma_{abc}^{\alpha\beta}N^{ab} = \Omega\Lambda_{c}^{[\alpha\beta]}\lambda - 2Y\Lambda_{c}^{[\alpha\beta]}\partial\lambda,$$

(3.60)
and Ω is defined in (2.28).

Using Eqs. (3.59) and (3.60), the l.h.s. of Eq. (3.57) splits in three parts:

\[ \lambda^\epsilon [L_1^{\alpha\beta\gamma\delta}] = -\frac{1}{48} \left[ \lambda^\epsilon L_1^{\alpha\beta\gamma\delta} + \lambda^\epsilon L_2^{\alpha\beta\gamma\delta} + \lambda^\epsilon L_3^{\alpha\beta\gamma\delta} \right], \quad (3.61) \]

where we have defined

\[ \lambda^\epsilon L_1^{\alpha\beta\gamma\delta} = \lambda^\epsilon \left( \Omega \Lambda_c^{\alpha\beta\lambda} \right) \left( \Omega \Lambda_c^{\gamma\delta\epsilon} \lambda \right), \quad (3.62) \]

\[ \lambda^\epsilon L_2^{\alpha\beta\gamma\delta} = -2 \left[ \lambda^\epsilon \left( \Omega \Lambda_c^{\alpha\beta\lambda} \right) \left( Y \Lambda_c^{\gamma\delta\epsilon} \partial \lambda \right) \lambda + \lambda^\epsilon \left( \Omega \Lambda_c^{\alpha\beta\partial\lambda} \right) \left( \Omega \Lambda_c^{\gamma\delta\epsilon} \lambda \right) \right], \quad (3.63) \]

\[ \lambda^\epsilon L_3^{\alpha\beta\gamma\delta} = 4 \lambda^\epsilon \left( Y \Lambda_c^{\alpha\beta\partial\lambda} \right) \left( Y \Lambda_c^{\gamma\delta\epsilon} \partial \lambda \right). \quad (3.64) \]

To compute the l.h.s. of Eq. (3.57), one must shift the fields Ω to the left using the rearrangement formula. Then

\[ \lambda^\epsilon [L_1^{\alpha\beta\gamma\delta}] = \Omega_\sigma \left( \Omega_\tau \left( \lambda^\epsilon \left( \Lambda_c^{\alpha\beta\lambda} \lambda \right) \left( \Lambda_c^{\gamma\delta\epsilon} \lambda \right) \right) \right) + \Omega_\sigma A_1^{[\alpha\beta\gamma\delta] \sigma} + A_0^{[\alpha\beta\gamma\delta]}, \quad (3.65) \]

\[ \lambda^\epsilon [L_2^{\alpha\beta\gamma\delta}] = \Omega_\sigma B_1^{[\alpha\beta\gamma\delta] \sigma} + B_0^{[\alpha\beta\gamma\delta]}, \quad (3.66) \]

where \( A_1, A_0, B_1 \) and \( B_0 \) are Ω-independent, Y-dependent fields.

The term quadratic in Ω in the r.h.s. of Eq. (3.65) vanishes since it contains the factor \( \lambda^\epsilon \lambda^\beta \left( \Gamma_{\alpha\beta\delta} \right) \). An explicit calculation shows that the terms linear in Ω in (3.65) and (3.66) cancel each other:

\[ \Omega_\sigma A_1^{[\alpha\beta\gamma\delta] \sigma} + \Omega_\sigma B_1^{[\alpha\beta\gamma\delta] \sigma} = 0, \quad (3.67) \]

and that the sum of the terms of zero-order in Ω in (3.65), (3.66) and (3.64) vanishes

\[ A_0^{[\alpha\beta\gamma\delta]} + B_0^{[\alpha\beta\gamma\delta]} + \lambda^\epsilon [L_3^{\alpha\beta\gamma\delta}] = 0, \quad (3.68) \]

so that (3.57) is proved.

The details of this calculation are given in Appendix C.

### 4 Y-formalism for the non-minimal pure spinor formalism

In this section, we would like to construct the Y-formalism for the non-minimal pure spinor formalism which has been recently proposed by Berkovits [27]. Before doing that, we will first review the non-minimal pure spinor formalism briefly. The main idea is to add to the fields
involved in the *minimal* formalism a BRST quartet of fields \( \bar{\lambda}_\alpha, \bar{\omega}^\alpha, r_\alpha \) and \( s^\alpha \) in such a way that their BRST variations are \( \delta \bar{\lambda}_\alpha = r_\alpha, \delta r_\alpha = 0, \delta s^\alpha = \bar{\omega}^\alpha \) and \( \delta \bar{\omega}^\alpha = 0 \). Here, \( \bar{\lambda}_\alpha \) is a bosonic field, \( r_\alpha \) is a fermionic field, and \( \bar{\omega}^\alpha \) and \( s^\alpha \) are the conjugate momenta of \( \bar{\lambda}_\alpha \) and \( r_\alpha \), respectively. These fields are required to satisfy the pure spinor conditions

\[
\bar{\lambda} \Gamma^a \bar{\lambda} = 0, \\
\bar{\lambda} \Gamma^a r = 0. 
\] (4.1)

The action is then obtained by adding to the conventional pure spinor action \( I \) in Eq. (2.2), \( \bar{I} \) given by the BRST variation of the "gauge fermion" \( F = - \int (s \bar{\partial} \bar{\lambda}) \) so that

\[
I_{nm} \equiv I + \bar{I} = \int d^2 z(\frac{1}{2} \partial X^a \bar{\partial} X_a + p_a \bar{\partial} \theta^a - \omega_a \bar{\partial} \lambda^a + s^a \bar{\partial} r_a - \bar{\omega}^a \bar{\partial} \bar{\lambda}_a). 
\] (4.2)

In addition to the \( \omega \)-symmetry Eq. (2.4), due to the conditions Eq. (4.1), this action is invariant under new gauge symmetries involving \( \bar{\omega} \) and \( s \),

\[
\delta \bar{\omega}^\alpha = \Lambda^{(1)}_a (\Gamma^a \bar{\lambda})^\alpha - \Lambda^{(2)}_a (\Gamma^a r)^\alpha, \\
\delta s^\alpha = \Lambda^{(2)}_a (\Gamma^a \bar{\lambda})^\alpha, 
\] (4.3)

where \( \Lambda^{(1)}_a \) and \( \Lambda^{(2)}_a \) are local gauge parameters. Let us note that the conditions Eq. (4.1) and these symmetries reduce the independent components of each field in the quartet to eleven components. It is easy to show that the action \( I_{nm} \) is invariant under the new BRST transformation with BRST charge

\[
Q_{nm} = \oint dz (\lambda^a d_a + \bar{\omega}^a r_a). 
\] (4.4)

Of course the quartet does not contribute to the central charge and has trivial cohomology with respect to the (new) BRST charge.

As a final remark, it is worthwhile to recall that this new formalism can be interpreted [27] as a critical topological string with \( \hat{c} = 3 \) and (twisted) \( N = 2 \) supersymmetry. Then it is possible to apply topological methods to the computation of multiloop amplitudes where a suitable regularization factor replaces picture-changing operators to soak up zero modes. The covariant \( b \) field and the regulator proposed in [27] allow to compute loop amplitudes up to \( g = 2 \). A more powerful regularization of \( b \) that allows to compute loop amplitudes at any \( g \) loop has been presented in [28]. This regularization is gauge invariant but Lorentz non-covariant since it involves the \( Y \)-field. However, this non-covariance is harmless since the regularized \( b \) field differs from the covariant one by BRST-exact terms.

Now we are ready to present the \( Y \)-formalism for the non-minimal pure spinor quantization. As in Eqs. (2.10) and (2.12), we first introduce the non-covariant object

\[
\bar{Y}^\alpha = \bar{\theta}^\alpha / \bar{\lambda}, 
\] (4.5)

and the projector

\[
\bar{K}^\alpha \beta = \frac{1}{2} (\Gamma^a \bar{\lambda})^\alpha (\bar{Y} \Gamma_a)\beta, 
\] (4.6)
where $\bar{v}^\alpha$ is a constant pure spinor so that we have

$$\bar{Y}\Gamma^a\bar{Y} = 0.$$  \hspace{1cm} (4.7)

Let us note that the conditions (4.1) lead to relations $\bar{\lambda}_\alpha K^\alpha_{\beta} = r_\alpha \bar{K}^\alpha_{\beta} = 0$, which imply that $\bar{\lambda}_\alpha$ and $r_\alpha$ have respectively eleven independent components.

Next we postulate the following OPE’s among $\bar{\omega}^\alpha$, $\bar{\lambda}_\alpha$, $s^\alpha$ and $r_\alpha$:

$$< \bar{\omega}^\alpha(y)\bar{\lambda}_\beta(z) > = \frac{1}{y-z}(\bar{\delta}^\alpha_\beta - \bar{K}^\alpha_{\beta}(z)), \hspace{1cm} (4.8)$$

$$< s^\alpha(y)r_\beta(z) > = \frac{1}{y-z}(\delta^\alpha_\beta - \bar{K}^\alpha_{\beta}(z)), \hspace{1cm} (4.9)$$

$$< \bar{\omega}^\alpha(y)r_\beta(z) >= \frac{1}{y-z}[K^\alpha_{\beta}(z)(\bar{Y}r)(z) - \frac{1}{2} (\Gamma^a_{\beta})^\alpha(z)(\bar{Y}\Gamma_a)(z)], \hspace{1cm} (4.10)$$

$$< s^\alpha(y)\bar{\lambda}_\beta(z) >= 0. \hspace{1cm} (4.11)$$

Then, with these OPE’s it is easy to check that the OPE’s between the conjugate momenta $\bar{\omega}^\alpha$ and $s^\alpha$, and the conditions (4.1) vanish identically:

$$< \bar{\omega}^\alpha(y)(\bar{\lambda}\Gamma^a\lambda)(z) > = 0,$$

$$< \bar{\omega}^\alpha(y)(\bar{\lambda}\Gamma^a r)(z) > = 0,$$

$$< s^\alpha(y)(\bar{\lambda}\Gamma^a\lambda)(z) > = 0,$$

$$< s^\alpha(y)(\bar{\lambda}\Gamma^a r)(z) > = 0. \hspace{1cm} (4.12)$$

Notice that (4.10) follows for consistency by acting with the BRST charge $Q_{nm}$ on (4.8) (or (4.9)).

Following [27], the only holomorphic currents involving $\bar{\omega}$ and $s$ and gauge invariant under (4.3) are:

- i) the bosonic currents

$$\bar{N}_{ab} = \frac{1}{2}(\bar{\omega}\Gamma_{ab}\bar{\lambda} - s\Gamma_{ab}r),$$

$$\bar{J}_\lambda = \bar{\omega}\bar{\lambda} - sr,$$

$$T_\lambda = \bar{\omega}\partial\bar{\lambda} - s\partial r, \hspace{1cm} (4.13)$$

those are, the Lorentz current, the ghost current and the stress energy tensor of the non-minimal fields, respectively.
- ii) the fermionic currents

\[ S_{ab} = \frac{1}{2} s \Gamma_{ab} \bar{\lambda}, \]
\[ S = s \lambda, \]
\[ S_{(b)} = s \partial \bar{\lambda}. \]  

(4.14)

- iii) the doublet

\[ J_0 = rs, \]
\[ \Phi_0 = \bar{\omega} r. \]  

(4.15)

Using the fundamental OPE’s (4.8)-(4.11), one can compute the OPE’s among these operators. The OPE’s of \( \bar{N}^{ab}, T_\lambda \) and \( \bar{J}_\lambda \) with \( \lambda \) and \( r \) and the ones among themselves are canonical, namely

\[ < \bar{N}_{ab}(y) \bar{\lambda}_\alpha(z) > = \frac{1}{2 y - z} \left( \Gamma_{ab} \bar{\lambda}_\alpha(z), < \bar{N}_{ab}(y) r_\alpha(z) > = \frac{1}{2 y - z} \left( \Gamma_{ab} r_\alpha(z), \right. \right. \]
\[ \left. \left. < \bar{J}_\lambda(y) \bar{\lambda}_\alpha(z) > = \frac{1}{y - z} \bar{\lambda}_\alpha(z), < \bar{J}_\lambda(y) r_\alpha(z) > = \frac{1}{y - z} r_\alpha(z), \right. \right. \]
\[ < T_\lambda(y) \bar{\lambda}_\alpha(z) > = \frac{1}{y - z} \partial \bar{\lambda}_\alpha(z), < T_\lambda(y) r_\alpha(z) > = \frac{1}{y - z} \partial r_\alpha(z), \]  

(4.16)

and

\[ < \bar{N}_{ab}(y) \bar{N}_{cd}(z) > = -\frac{1}{y - z} \left( \eta_{[c|b} \bar{N}_{a]d} - \eta_{[d|b} \bar{N}_{a]c} \right)(z), \]
\[ < \bar{N}_{ab}(y) \bar{J}_\lambda(z) > = 0, < \bar{N}_{ab}(y) T_\lambda(z) >= \frac{1}{(y - z)^2} \bar{N}_{ab}(z), \]
\[ < \bar{J}_\lambda(y) \bar{J}_\lambda(z) > = 0, < \bar{J}_\lambda(y) T_\lambda(z) >= \frac{1}{(y - z)^2} \bar{J}_\lambda(z), \]
\[ < T_\lambda(y) T_\lambda(z) > = \frac{2}{(y - z)^2} T_\lambda(z) + \frac{1}{y - z} \partial T_\lambda(z). \]  

(4.17)

Notice that in contrast with the operators \( T, N^{ab} \) and \( J \) in (2.16)-(2.18), the operators \( \bar{N}_{ab}, T_\lambda \) and \( \bar{J}_\lambda \) do not involve \( \bar{Y} \)-correction terms since the \( \bar{Y} \)-dependent terms which arise in their OPE’s are absent or cancel in the combinations (4.13). It is instructive to see explicitly how this cancellation arises. Let us write \( \bar{N}_{ab}^{(\bar{\omega} \lambda)} = \frac{1}{2} \bar{\omega} \Gamma_{ab} \bar{\lambda} \) and \( \bar{N}_{ab}^{(sr)} = \frac{1}{2} s \Gamma_{ab} r \) and consider for instance the OPE between \( \bar{N}_{ab} = \bar{N}_{ab}^{(\bar{\omega} \lambda)} - \bar{N}_{ab}^{(sr)} \) and \( r_\alpha \). From Eq. (4.9), one obtains

\[ < \bar{N}_{ab}^{(sr)}(y) r_\alpha(z) > = \frac{1}{2 y - z} \left( \Gamma_{ab} r_\alpha(z), + \frac{1}{4 y - z} \left( \Gamma_{ab} r_\alpha(z) \right. \right. \]
\[ \left. \left. \left( \Gamma_{ab} r_\alpha(z) \right) \right) \right). \]  

(4.18)

Then, the second term in the r.h.s. of (4.18) is exactly cancelled by the contribution of the OPE \( < \bar{N}_{ab}^{(\bar{\omega} \lambda)}(y) r_\alpha(z) > \) in terms of Eq. (4.10). As a second example, consider the OPE
\( < N_{ab}(y) N_{cd}(z) > \). The double poles coming from \( < N_{ab}^{(z \lambda)}(y) N_{cd}^{(z \lambda)}(z) > \) are cancelled by those coming from \( < N_{ab}^{(sr)}(y) N_{ab}^{(sr)}(z) > \). As for the simple poles, one has

\[
< N_{ab}^{(z \lambda)}(y) N_{cd}^{(z \lambda)}(z) > + < N_{ab}^{(sr)}(y) N_{cd}^{(sr)}(z) > = -\frac{1}{y-z} (\eta_{[b} N_{a]d} - \eta_{[d} N_{a]c}) + \frac{1}{8} [(s \Gamma_{ab} \Gamma_f \bar{Y})(r \Gamma_f \Gamma_c \bar{\lambda}) + (s \Gamma_{cd} \Gamma_f \bar{Y})(r \Gamma_f \Gamma_{ab} \bar{\lambda})],
\]

(4.19)

but the terms, which are independent of \( \bar{N}_{ab} \) in the r.h.s. of (4.19), are just cancelled by the contributions stemming from \(-( < N_{ab}^{(z \lambda)}(y) N_{cd}^{(sr)}(z) > + < N_{ab}^{(sr)}(y) N_{cd}^{(sr)}(z) >)\). For all the remaining OPE’s in both (4.16) and (4.17), the spurious, \( \bar{Y} \)-dependent terms are absent or cancelled in a similar way. Moreover, the OPE’s among \( N_{ab}, J_{\lambda} \) and \( T_{\lambda} \) with \( S_{ab} \), \( S \) and \( S_{(b)} \) are regular and those of \( N_{ab}, J_{\lambda} \) and \( T_{\lambda} \) with \( S_{ab} \), \( S \) and \( S_{(b)} \) are canonical so that \( S_{ab} \), \( S \) and \( S_{(b)} \) are covariant primary fields with weight 1 and ghost number 2 with respect to the ghost current \( J_{\lambda} \). Thus, as for \( \bar{N}_{ab}, J_{\lambda} \) and \( T_{\lambda} \), they do not have to include \( \bar{Y} \)-dependent corrections.

The story is completely different for the currents \( J_{r} \) and \( \Phi \). Indeed, using the OPE’s (4.8)-(4.11), one finds

\[
< (rs)(y) \bar{N}_{ab}(z) > = \frac{3}{2} \frac{1}{(y-z)^2} \bar{Y} \Gamma^{ab} \bar{\lambda}.
\]

(4.20)

And since one has

\[
< (\bar{Y} \partial \bar{\lambda})(y) \bar{N}_{ab}(z) > = \frac{1}{2} \frac{1}{(y-z)^2} \bar{Y} \Gamma^{ab} \bar{\lambda},
\]

(4.21)

the \( \bar{Y} \)-dependent term in \( < J_{r}(y) \bar{N}_{ab}(z) > \) disappears if one assumes, as definition of \( J_{r} \) at quantum level,

\[
J_{r} = rs - 3 \bar{Y} \partial \bar{\lambda}.
\]

(4.22)

With this definition, the OPE’s of \( J_{r} \) with \( N_{ab}, J_{\lambda}, T_{\lambda}, S_{ab}, S \) and \( S_{(b)} \) read

\[
< J_{r}(y) J_{r}(z) > = \frac{11}{(y-z)^2},
\]

\[
< J_{r}(y) \bar{N}_{ab}(z) > = 0,
\]

\[
< J_{\lambda}(y) J_{r}(z) > = \frac{8}{(y-z)^2},
\]

\[
< J_{r}(y) T_{\lambda}(z) > = \frac{11}{(y-z)^3} + \frac{1}{(y-z)^2} J_{r},
\]

\[
< J_{r}(y) S_{ab}(z) > = \frac{1}{y-z} S_{ab},
\]

\[
< J_{r}(y) S(z) > = \frac{1}{y-z} S,
\]

\[
< J_{r}(y) S_{(b)}(z) > = \frac{1}{y-z} S_{(b)}. \]

(4.23)
In particular, note that the coefficient 8 of the double pole in the contraction \( \langle \bar{\bar{J}}_\lambda(y)J_r(z) \rangle \) emerges from the arithmetic 8 = 11 - 3 where 11 comes from the first term and -3 from the second term in (4.22).

In a similar manner, for \( \Phi \) one has

\[
\langle (\bar{\bar{\omega}}r)(y)\bar{N}^{ab}(z) \rangle = -\frac{3}{2(y-z)^2}[\bar{Y}\Gamma^{ab}r - (\bar{Y}r)(\bar{Y}\Gamma^{ab}\bar{\lambda})],
\]

\[
\langle (\bar{\bar{\omega}}r)(y)S^{ab}(z) \rangle = -\frac{3}{2(y-z)^2}\bar{Y}\Gamma^{ab}\bar{\lambda}.
\] (4.24)

Therefore, at quantum level \( \Phi \) must be defined as

\[
\Phi = \bar{\bar{\omega}}r + 3[\bar{Y}\partial r - (\bar{Y}r)(\bar{Y}\partial\bar{\lambda})] = \bar{\bar{\omega}}r + 3\partial(\bar{Y}r),
\] (4.25)

in order to avoid spurious \( \bar{Y} \)-dependent terms. With this new definition, one can also derive

\[
\langle \Phi(y)\bar{N}^{ab}(z) \rangle = 0,
\]

\[
\langle \Phi(y)\bar{J}_\lambda(z) \rangle = 0,
\]

\[
\langle \Phi(y)T_\lambda(z) \rangle = \frac{1}{(y-z)^2}\Phi,
\]

\[
\langle \Phi(y)S^{ab}(z) \rangle = \frac{1}{y-z}\bar{N}^{ab},
\]

\[
\langle \Phi(y)S(z) \rangle = \frac{8}{(y-z)^2} + \frac{1}{y-z}\bar{J}_\lambda,
\]

\[
\langle \Phi(y)S(b)(z) \rangle = \frac{11}{(y-z)^3} + \frac{1}{(y-z)^2}J_r + \frac{1}{y-z}T_\lambda,
\]

\[
\langle \Phi(y)J_r(z) \rangle = \frac{1}{y-z}\Phi.
\] (4.26)

The operator \( \Phi \) is part of the BRST current and \( S(b) \) is a contribution of the \( b \) ghost as will be seen in the next section.

From the definitions (4.13) and (4.14), one finds that the operators \( \bar{N}^{ab}, \bar{J}_\lambda, \) and \( T_\lambda \) are the BRST variations of the operators \( S^{ab}, S, \) and \( S(b) \), respectively. Moreover, contrary to what happens for the operators in (2.16)-(2.18), the correction term of \( J_r \) in (4.22) is not BRST-exact but its BRST variation is just the correction term for \( -\Phi \) in (4.25), so that \( \Phi \) is just the BRST variation of \( -J_r \). These properties are fully consistent with the OPE’s we have computed thus far. \(^8\)

As a final remark, let us note that in all the derivations of this section (but the second equality of (4.25)) we have never used the fact that \( \bar{v} \) in (4.5) is constant and therefore all the equations in this section remain true even if one replaces \( \bar{Y}^\alpha \) with \( \bar{Y}^\alpha \equiv \frac{\bar{\alpha}^\alpha}{\lambda^\lambda} \).

\(^8\)Apart from a difference in the OPE \( \langle \Phi S \rangle \) where we find a double pole with residuum 8, not present in [27] (perhaps a misprint in [27]), our results agree with those computed in [27] by using the \( U(5) \)-formalism.
5 A quantum $b$ ghost in the non-minimal pure spinor formalism

In Ref. [27], Berkovits has obtained an expression for a covariant $b$ ghost in the framework of non-minimal formalism. His idea was triggered by the observation that in this formalism the non-covariant $Y_\alpha$ field can be replaced by a covariant field $\tilde{\lambda}_\alpha$ (which will be defined soon) and then one can look for a new, covariant $b$ ghost satisfying the defining equation

$$\{Q_{nm}, b_{nm}(z)\} = T(z) + T_\lambda(z) \equiv T_{nm}(z),$$

by starting with $b_{nm} = \tilde{\lambda}_\alpha G^\alpha + s^\alpha \partial \tilde{\lambda}_\alpha + \cdots$. The result, given in [27], is

$$b_{nm} = s^\alpha \partial \tilde{\lambda}_\alpha + \tilde{\lambda}_\alpha G^\alpha - 2\tilde{\lambda}_\beta \tilde{r}_\alpha H^{[\alpha\beta]} + 6\tilde{\lambda}_\gamma \tilde{r}_\beta \tilde{r}_\alpha K^{[\alpha\beta\gamma]} - 24\tilde{\lambda}_\delta \tilde{r}_\beta \tilde{r}_\gamma \tilde{r}_\alpha L^{[\alpha\beta\gamma\delta]},$$

where we have defined

$$\tilde{\lambda}_\alpha = \frac{\bar{\lambda}_\alpha}{(\lambda\lambda)},$$

$$\tilde{r}_\alpha = \frac{r_\alpha}{(\lambda\lambda)}.$$  

Note that $\tilde{\lambda}_\alpha$ and $\tilde{r}_\alpha$ are primary fields of conformal weight 0 with respect to $T_{nm}$.

In this section, we will construct a covariant, quantum-mechanical $b$ ghost in the non-minimal pure spinor formalism on the basis of our Y-formalism, taking care of normal-ordering effects. Furthermore, we shall show that this covariant $b$ ghost is cohomologically equivalent to the non-covariant $\tilde{b}_Y$ ghost, improved by the non-minimal term $s^\alpha \partial \tilde{\lambda}_\alpha$ which takes the form at the classical level

$$\tilde{b}_0Y = Y_\alpha G^\alpha_0 + s^\alpha \partial \tilde{\lambda}_\alpha.$$  

It is now convenient to consider the following operators:

$$\frac{1}{2} \rho^{[\alpha\beta]} \equiv \frac{1}{2} (\tilde{r}_\alpha \tilde{\lambda}_\beta - \tilde{r}_\beta \tilde{\lambda}_\alpha) \equiv \tilde{r}_{[\alpha\beta]} \tilde{\lambda},$$

$$\frac{1}{3!} \rho^{[\alpha\beta\gamma]} \equiv -\tilde{r}_{[\alpha\beta\gamma]} \tilde{\lambda},$$

$$\frac{1}{4!} \rho^{[\alpha\beta\gamma\delta]} \equiv -\tilde{r}_{[\alpha\beta\gamma\delta]} \tilde{\lambda},$$

$$\frac{1}{5!} \rho^{[\alpha\beta\gamma\delta\epsilon]} \equiv \tilde{r}_{[\alpha\beta\gamma\delta\epsilon]} \tilde{\lambda},$$

that satisfy the recursive relations

$$[Q_{nm}, \tilde{\lambda}_\alpha] = \lambda^\beta \rho^{[\alpha\beta]},$$

$$\{Q_{nm}, \rho^{[\alpha\beta]}\} = \lambda^\gamma \rho^{[\alpha\beta\gamma]},$$

$$[Q_{nm}, \rho^{[\alpha\beta\gamma]}] = \lambda^\delta \rho^{[\alpha\beta\gamma\delta]},$$

$$\{Q_{nm}, \rho^{[\alpha\beta\gamma\delta]}\} = \lambda^\epsilon \rho^{[\alpha\beta\gamma\delta\epsilon]}.$$  

(5.6)
Next, let us also recall the results which were obtained in section 3 and hold at the quantum level:

\begin{align*}
\{Q, \hat{G}^\alpha\} &= \lambda^\alpha T, \\
[Q, H^{[\alpha\beta]}] &= \lambda^{[\alpha} \hat{G}^{\beta]}, \\
\{Q, K^{[\alpha\beta\gamma]}\} &= \lambda^{[\alpha} H^{\beta\gamma]}, \\
[Q, L^{[\alpha\beta\gamma\delta]}] &= \lambda^{[\alpha} K^{\beta\gamma\delta]}, \\
\lambda^{[\alpha} L^{\beta\gamma\delta]} &= 0, \tag{5.7}
\end{align*}

where \(\hat{G}^\alpha\) is defined in (3.13).

It is also useful to compute the contractions:

\begin{align*}
< \hat{G}^\beta(y) \rho_{[\beta\alpha]}(z) > &= \frac{R_{1\alpha}}{y-z}, \\
< H^{[\beta\gamma]}(y) \rho_{[\gamma\alpha]}(z) > &= \frac{R_{2\alpha}}{y-z}, \\
< K^{[\beta\gamma\delta]}(y) \rho_{[\delta\alpha]}(z) > &= \frac{R_{3\alpha}}{y-z}, \\
< L^{[\beta\gamma\delta\epsilon]}(y) \rho_{[\epsilon\delta\gamma\alpha]}(z) > &= \frac{\check{R}_{4\alpha}}{(y-z)^2} + \frac{R_{4\alpha}}{y-z}. \tag{5.8}
\end{align*}

After a simple calculation, it turns out that \(R_{1\alpha}\) is given by

\[ R_{1\alpha} = -2 \rho_{[\alpha\beta]} \lambda^\beta (\tilde{\lambda} \partial \theta) - \frac{1}{2} (\Gamma^a \tilde{\lambda} \lambda^\beta (\lambda \Gamma_a \partial \theta)), \tag{5.9} \]

but the second term in the square bracket vanishes when contracted with \(\rho_{[\alpha\beta]}\) due to the conditions (4.1). As for \(R_{2\alpha}, R_{3\alpha}, R_{4\alpha}\) and \(\check{R}_{4\alpha}\), they all contain (at least) a factor \(\frac{1}{16} \Gamma^{[\alpha\beta\gamma]}(\tilde{\lambda} \Gamma^{abc} \lambda) \equiv \tilde{\lambda} \lambda^{[\alpha\beta\gamma]}\lambda\) and therefore vanish when contracted with \(\rho_{[\beta\gamma\delta]}\) by taking into account (5.5), (3.58) and (4.1). \(^9\) To summarize, we have the following results:

\begin{align*}
R_{1\alpha} &= -2 \rho_{[\alpha\beta]} \lambda^\beta (\tilde{\lambda} \partial \theta), \\
R_{2\alpha} &= R_{3\alpha} = R_{4\alpha} = \check{R}_{4\alpha} = 0. \tag{5.10}
\end{align*}

As already noted, the non-minimal \(b\) field is expected to be of the form:

\[ b_{nm} = S_{(b)} + \tilde{\lambda} G + \cdots. \tag{5.11} \]

The anticommutator of \(Q_{nm}\) with \(S_{(b)} = s \partial \tilde{\lambda}\) is

\[ \{Q_{nm}, S_{(b)}\} = T_{\tilde{\lambda}}. \tag{5.12} \]

\(^9\)In \(\check{R}_{4\alpha}\), there is also a term proportional to \(\rho_{[\alpha\beta\gamma]}(\Gamma_{abc})^{\alpha\beta}(\Gamma^{def})^{\gamma\delta}(\tilde{\lambda} \Gamma^{abc} \Gamma_{def} \lambda)\) that vanishes for the same reason.
Now let us compute the anticommutator \( \{ Q_{nm}, (\tilde{\lambda}_\alpha \hat{G}^\alpha) \} \)

\[
\{ Q_{nm}, (\tilde{\lambda}_\alpha \hat{G}^\alpha) \} = \tilde{\lambda}_\alpha (\lambda^\alpha T) + (\lambda^{\beta} \rho_{[\alpha\beta]}) \hat{G}^\alpha.
\] (5.13)

Using the rearrangement theorem and some algebra, (5.13) can be rewritten as

\[
\{ Q_{nm}, (\tilde{\lambda}_\alpha \hat{G}^\alpha) \} = T + \rho_{[\alpha\beta]}(\lambda^{\beta} \hat{G}^\alpha) + \{ Q_{nm}, \partial \tilde{\lambda} \partial \theta - (\tilde{\lambda} \partial \lambda)(\tilde{\lambda} \partial \theta) \}.
\] (5.14)

Here it is of interest to remark that the term \( \partial \tilde{\lambda} \partial \theta - (\tilde{\lambda} \partial \lambda)(\tilde{\lambda} \partial \theta) \) that arises in the r.h.s. of (5.14) is just the difference between the generalized normal ordering (\( \cdots \)) in [33] and the improved one (\( \cdots \)) of \( \tilde{\lambda}_\alpha \hat{G}^\alpha \), that is

\[
(\tilde{\lambda}_\alpha \hat{G}^\alpha) =: \tilde{\lambda}_\alpha \hat{G}^\alpha : + \partial \tilde{\lambda} \partial \theta - (\tilde{\lambda} \partial \lambda)(\tilde{\lambda} \partial \theta),
\] (5.15)

so that (5.14) becomes

\[
\{ Q_{nm}, : \tilde{\lambda}_\alpha \hat{G}^\alpha : \} = T + \rho_{[\alpha\beta]}(\lambda^{\beta} \hat{G}^\alpha).
\] (5.16)

With the help of the second recursive equations in (5.6) and (5.7) the last term in the r.h.s. of Eq. (5.16) reads

\[
\rho_{[\alpha\beta]}(\hat{\lambda}^{[\beta} \hat{G}^{\alpha]}) = \rho_{[\alpha\beta]}([Q_{nm}, H^{[\beta\alpha]}]) = \{ Q_{nm}, \rho_{[\alpha\beta]} H^{[\alpha\beta]} \} - \frac{1}{3} \rho_{[\alpha\beta\gamma]}(\lambda^{[\alpha} H^{\beta\gamma]}).
\] (5.17)

In this case, the rearrangement theorem does not give extra contributions since

\[
(\rho_{[\alpha\beta\gamma]} \lambda^{[\alpha} H^{\beta\gamma]}) - \rho_{[\alpha\beta\gamma]}(\lambda^{[\alpha} H^{\beta\gamma]}) = R_{2\gamma} \partial \lambda^{\gamma} + \partial \rho_{[\alpha\beta\gamma]} R_{3[\alpha\beta\gamma]}^{[\alpha\beta\gamma]},
\] (5.18)

and the r.h.s. vanishes from (5.10) and (3.33). Therefore, Eq. (5.17) can be rewritten as

\[
\rho_{[\alpha\beta]}(\hat{\lambda}^{[\beta} \hat{G}^{\alpha]}) = \{ Q_{nm}, \rho_{[\alpha\beta]} H^{[\alpha\beta]} \} - \frac{1}{3} \rho_{[\alpha\beta\gamma]}(\lambda^{[\alpha} H^{\beta\gamma]}).
\] (5.19)

For the last term in the r.h.s. of this equation, one can repeat the same procedure using the third recursive equations in (5.6) and (5.7). Again the contributions from the rearrangement theorem are absent since they involve the operators \( R_{4}^{[\alpha\beta\gamma\delta]} \) and \( R_{3\alpha} \) that vanish according to (3.46) and (5.10). As a result, one obtains

\[
\rho_{[\alpha\beta\gamma]}(\lambda^{[\alpha} H^{\beta\gamma]}) = \{ Q_{nm}, \rho_{[\alpha\beta\gamma]} K^{[\alpha\beta\gamma]} \} + \frac{1}{4} \rho_{[\alpha\beta\gamma\delta]}(\lambda^{[\alpha} K^{\beta\gamma\delta]}).
\] (5.20)

As a last step, one can express the last term in (5.20) in terms of \( \{ Q_{nm}, \rho_{[\alpha\beta\gamma\delta]} L^{[\alpha\beta\gamma\delta]} \} \) by using the fourth recursive equations in (5.6) and (5.7). Again the contributions from the rearrangement theorem are absent as before, so we have

\[
(\rho_{[\alpha\beta\gamma\delta]} \lambda^{[\alpha} K^{\beta\gamma\delta]}) = - \{ Q_{nm}, (\rho_{[\alpha\beta\gamma\delta]} L^{[\alpha\beta\gamma\delta]} \})
\] (5.21)
where we have disregarded the term $\rho_{[\alpha_\beta\gamma_\delta]} \lambda^\epsilon L^{\alpha\beta\gamma\delta}$ that vanishes according to (3.57).

Finally, using (5.12) and (5.16)-(5.21) we arrive at the result
\[
\{Q_{nm}, b_{nm}\} = T_{nm},
\]  
(5.22)

where
\[
b_{nm} = s^\alpha \partial \tilde{\lambda}_\alpha + :\tilde{\lambda}_\alpha \tilde{G}^\alpha : - 2(\tilde{\lambda}_\beta \tilde{r}_\alpha) H^{[\alpha\beta]} + 6(\tilde{\lambda}_\gamma \tilde{r}_\beta \tilde{r}_\alpha) K^{[\alpha\beta\gamma]} - 24(\tilde{\lambda}_\delta \tilde{r}_\gamma \tilde{r}_\beta \tilde{r}_\alpha) L^{[\alpha\beta\gamma\delta]}.
\]  
(5.23)

In conclusion, we have confirmed Eq. (5.2) provided that one interprets the compound field $\tilde{\lambda}_\alpha G^\alpha$ as the operator $:\tilde{\lambda}_\alpha \tilde{G}^\alpha :$ which is normal-ordered according to the improved prescription (For the other terms in (5.23) the generalized and the improved normal-ordering prescriptions coincide). Incidentally, we have also checked that this $b_{nm}$ possesses conformal weight 2.

It might appear from (5.23) and the definition of $\tilde{\lambda}$ and $\tilde{r}$ that $b_{nm}$ is singular at $\bar{\lambda}\lambda \to 0$ with poles up to fourth order. However, as explained in [28], this singularity is not dangerous. Indeed in this case, the analogous of the operator $\xi = Y \theta$ that would trivialize the cohomology, is
\[
\xi_{nm} = \tilde{\theta} \lambda \lambda + \tilde{r} \theta = \lambda \theta \sum_{n=1}^{11} \frac{(-r \theta)^{n-1}}{(\lambda \lambda)^n},
\]  
(5.24)

since $\{Q_{nm}, \xi_{nm}\} = 1$. However, $\xi_{nm}$ diverges like $(\lambda \lambda)^{-11}$ and to have a nontrivial cohomology it is sufficient to exclude from the Hilbert space operators that diverge like $\xi_{nm}$ or stronger. Therefore $b_{nm}$ is allowed as insertion to compute higher loop amplitudes. To do actual calculations at more than two loops [28], $b_{nm}$ must be regularized properly. In fact, in [28] a consistent regularization has been proposed.

Now let us come back to the non-covariant $b$ ghost $\tilde{b}_{Y\theta}$ in (5.4). As a first step, let us derive a quantum counterpart of $\tilde{b}_{Y\theta}$, which is denoted as $\tilde{b}_Y$. From the first equation in (5.7), one has
\[
\{Q_{nm}, Y_\alpha \hat{G}^\alpha \} = Y_\alpha (\lambda^\alpha T).
\]  
(5.25)

Moreover, since $Y_\alpha (\lambda^\alpha T) - (Y_\alpha \lambda^\alpha) T = 2 \partial Y \partial \lambda$ from the rearrangement theorem, one obtains
\[
\{Q_{nm}, Y_\alpha \hat{G}^\alpha - 2 \partial Y \partial \theta \} = T.
\]  
(5.26)

As before, the term $2 \partial Y \partial \theta$ is just the difference between $(Y_\alpha \hat{G}^\alpha)$ and $:Y_\alpha \hat{G}^\alpha :$ and therefore the quantum non-covariant $b$ ghost takes the form
\[
\tilde{b}_Y = Y_\alpha \hat{G}^\alpha + (s \partial \lambda),
\]  
(5.27)

and it satisfies
\[
\{Q_{nm}, \tilde{b}_Y \} = T_{nm}.
\]  
(5.28)
Even if $b_Y$ is non-covariant, its Lorentz variation is BRST-exact. Actually, one has
\[
\delta_L b_Y = [Q_{nm}, 2(L^\beta Y_\gamma H^{[\gamma\alpha]}],
\]  
(5.29)
where $L_\alpha^\beta$ are (global) Lorentz parameters.

From (5.22) and (5.28), it follows that $\bar{b}_Y - b_{nm}$ is BRST-closed and then it is plausible that it is also exact. Indeed in [34], we have shown that, at the classical level, the covariant non-minimal $b$ ghost (5.2) and the non-covariant one (5.4) are cohomologically equivalent. In this respect, we wish to verify the cohomological equivalence between $b_{nm}$ and $\bar{b}_Y$ even at the quantum level
\[
b_{nm} = \bar{b}_Y + [Q_{nm}, W],
\]  
(5.30)
where
\[
W = 2(\bar{\lambda}_\beta Y_\alpha) H^{[\alpha\beta]} + 3!(\bar{\lambda}_\beta \bar{r}_\gamma Y_\alpha) K^{[\alpha\beta\gamma]} + 4!(\bar{\lambda}_\beta \bar{r}_\gamma \bar{r}_\delta Y_\alpha) L^{[\alpha\beta\gamma\delta]} + W_R,
\]  
(5.31)
with $W_R$ being a quantum contribution coming from the rearrangement theorem, which will be determined later.

In order to verify (5.30), let us compute the (anti)-commutators of $Q_{nm}$ with the first three terms in the r.h.s. of (5.31). We have
\[
2[Q_{nm}, (\bar{\lambda}_\beta Y_\alpha) H^{[\alpha\beta]}] = -\rho_\alpha^\beta H^{[\alpha\beta]} + (Y_\gamma \rho_\alpha^\beta \lambda^\gamma) H^{[\alpha\beta]} + (\bar{\lambda}_\alpha Y_\alpha)(\lambda^\beta \hat{G}^{\beta})
\]  
(5.32)
\[
= -2(\bar{\lambda}_\beta \bar{r}_\alpha) H^{[\alpha\beta]} - 3!(Y_\gamma \bar{r}_\alpha \bar{\lambda}_\beta)(\lambda^\gamma H^{[\alpha\beta]} + (\bar{\lambda}_\alpha \hat{G}^{\alpha}) - (Y_\alpha \hat{G}^{\alpha}) + R_H + R_G,
\]  
where $R_H$ and $R_G$ are the contributions coming from the rearrangement theorem of the last two terms in the first row of this equation. Then
\[
3![Q_{nm}, (Y_\alpha \bar{r}_\beta \bar{\lambda}_\gamma) K^{[\alpha\beta\gamma]}] = 3!(\bar{r}_\alpha \bar{r}_\beta \bar{\lambda}_\gamma) K^{[\alpha\beta\gamma]} - 4!(Y_\alpha \bar{r}_\beta \bar{r}_\gamma \bar{\lambda}_\delta)(\lambda^\delta K^{[\alpha\beta\gamma]}),
\]  
(5.33)
\[
+ 3!(Y_\alpha \bar{r}_\beta \bar{\lambda}_\gamma)(\lambda^\alpha \hat{H}^{[\beta\gamma\delta]}) + R_K,
\]  
where $R_K$ arises from rearrangement theorem. Finally, we have
\[
4![Q_{nm}, (Y_\alpha \bar{r}_\beta \bar{r}_\gamma \bar{\lambda}_\delta L^{[\alpha\beta\gamma\delta]})] = 4!(\bar{r}_\alpha \bar{r}_\beta \bar{r}_\gamma \bar{\lambda}_\delta L^{[\alpha\beta\gamma\delta]} + 4!(Y_\alpha \bar{r}_\beta \bar{r}_\gamma \bar{\lambda}_\delta)(\lambda^\alpha K^{[\beta\gamma\delta]}),
\]  
(5.34)
where $R_L$ comes from rearrangement formula. The quantum contributions $R_G, R_H, R_K$ and $R_L$ are explicitly given by
\[
R_G = -[\partial \bar{\lambda} \partial \theta - (\bar{\lambda} \partial \lambda)(\bar{\lambda} \partial \theta) - 2\partial Y \partial \theta] + 2[Q_{nm}, (Y_\alpha \bar{\lambda}_beta) W_{R1}^{[\alpha\beta]}] - \frac{1}{2} A^a_G \Pi_a,
\]  
(5.35)
\[
R_H = 3![Q_{nm}, (Y_\alpha \bar{r}_\beta \bar{\lambda}_gamma) W_{R2}^{[\alpha\beta\gamma]}] - \frac{1}{4} A^a_{Ha} (d \Gamma_a)^a + \frac{1}{2} A^a_G \Pi_a,
\]  
(5.36)
\[ R_K = 4! \{Q_{nm}, (Y_{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\lambda})_G W^{\hat{\alpha}\beta\gamma\delta}_{R3} \} + \frac{1}{4} A^a_{H\alpha} (d \Gamma_a) \alpha + \frac{1}{12} A^c_{K\alpha\beta} N_c^{\alpha\beta}, \] (5.37)

\[ R_L = -\frac{1}{12} A^c_{K\alpha\beta} N_c^{\alpha\beta} + B_L, \] (5.38)

where

\[ W^{[\alpha\beta]}_{R1} = \frac{1}{2}((Y + \tilde{\lambda}) \Gamma_a)^{[\alpha} \partial^{\beta]} \Pi^a, \]
\[ W^{[\alpha\beta\gamma]}_{R2} = \frac{1}{8}((Y + 2\tilde{\lambda}) \Gamma_a)^{[\alpha} \partial^{\beta} (\Gamma^a d)^{\gamma]}, \]
\[ W^{[\alpha\beta\gamma\delta]}_{R3} = \frac{1}{12}((Y + 3\tilde{\lambda}) \Gamma_a)^{[\alpha} \partial^{\beta} N^{\gamma\delta]a}, \] (5.39)

and

\[ A^a_C = 3! Y_{[a \tilde{\beta} \tilde{\gamma} \tilde{\lambda}]}(Y + 2\tilde{\lambda}) \Gamma^a, \]
\[ A^a_{H\alpha} = 4! Y_{[a \tilde{\beta} \tilde{\gamma} \tilde{\lambda}]}(Y + 3\tilde{\lambda}) \Gamma^a, \]
\[ A^a_{K\alpha\beta} = 5! Y_{[a \tilde{\beta} \tilde{\gamma} \tilde{\lambda}]}(Y + 4\tilde{\lambda}) \Gamma^a. \] (5.40)

The Y-dependent operators \( A^a_C, A^a_{H\alpha} \) and \( A^a_{K\alpha\beta} \) cancel when (5.35)-(5.38) are summed up.

As for \( B_L \), it turns out that it is BRST-exact:

\[ B_L = 4! \{Q_{nm}, (Y_{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\lambda})_G W^{[\alpha\beta\gamma\delta]}_{R4} \} + 4! \{Q_{nm}, \partial((Y_{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\lambda})_G W^{[\alpha\beta\gamma\delta]}_{R5} \} \],

where

\[ W^{[\alpha\beta\gamma\delta]}_{R4} = \frac{1}{96}[(\Gamma^c Y)^{[a} (\Gamma^b (Y + 3\tilde{\lambda}))^\beta (\Gamma_b \partial \lambda)^\gamma \partial \lambda^\delta] - (\Gamma^c Y)^{[a} (\Gamma^b (Y + 3\tilde{\lambda}))^\beta (\Gamma_b \partial \lambda)^\gamma \partial^2 \lambda^\delta] + 3(\tilde{\lambda} \Gamma^c Y) (\Gamma^b (Y + 2\tilde{\lambda}))^{[a} \partial^{\beta} (\Gamma_c \Gamma^a \lambda)^\gamma (\Gamma_b \Gamma_a \partial \lambda)^\delta], \] (5.42)

and

\[ W^{[\alpha\beta\gamma\delta]}_{R5} = \frac{1}{96} (\Gamma^c Y)^{[a} (\Gamma^b (Y + 3\tilde{\lambda}))^\beta [(\Gamma_b \partial \lambda)^\gamma \partial \lambda^\delta] - \frac{1}{2} (\Gamma_c \Gamma^a \lambda)^\gamma (\Gamma_b \Gamma_a \partial \lambda)^\delta]. \] (5.43)

Some details on the derivations of these results will be given in Appendix C. From (5.15), one finds that the term \(-\partial \tilde{\lambda} \partial \theta + (\tilde{\lambda} \partial \lambda)(\tilde{\lambda} \partial \theta) - 2\partial Y \partial \theta \) transforms \((\tilde{\lambda}_\alpha \tilde{G}^\alpha) - (Y_\alpha \tilde{G}^\alpha)\) to \( : \lambda_\alpha \tilde{G}^\alpha : \).

Collecting Eqs. (5.32)-(5.43), one recovers (5.30) where \( b_{nm} \) and \( \tilde{b}_Y \) are given in (5.23) and (5.27), respectively and

\[ W = 2(\tilde{\lambda}_\beta Y_\alpha)(H^{[\alpha\beta]} + W^{[\alpha\beta]}_{R1}) + 3!(\tilde{\lambda}_\gamma \tilde{\beta} Y_\alpha)(K^{[\alpha\beta\gamma]} + W^{[\alpha\beta\gamma]}_{R2}) + 4!(\tilde{\lambda}_\delta \tilde{\gamma} \tilde{\beta} Y_\alpha)(L^{[\alpha\beta\gamma\delta]} + W^{[\alpha\beta\gamma\delta]}_{R3} + W^{[\alpha\beta\gamma\delta]}_{R4}) + 4!(\tilde{\lambda}_\delta \tilde{\gamma} \tilde{\beta} \tilde{\gamma} Y_\alpha) W^{[\alpha\beta\gamma\delta]}_{R5}. \] (5.44)
6 Conclusion

In this article, using the Y-formalism [30], we have calculated the normal-ordering contributions existing in various composite operators in the pure spinor formalism of superstrings. These operators naturally appear when we try to construct a b ghost. Moreover, we have constructed the Y-formalism for the non-minimal sector. Using these information, we have presented a quantum-mechanical expression of the b ghost, $b_{nm}$, in the non-minimal formulation and we have shown, in this case, that the non-covariant b field $b_Y$ and $b_{nm}$, are equivalent in cohomology.

The consistent results we have obtained in this article could be regarded as a consistency check of the Y-formalism in both minimal and non-minimal pure spinor formulation of superstrings.

In the case of the non-minimal formulation, due to its field content and structure, it is natural to ask if it is possible to reach a fully covariant system of rules for the OPE’s in the minimal and non-minimal ghost sectors, by replacing the non-covariant fields $Y_\alpha$ and $\bar{Y}^\alpha$ with the covariant ones $\bar{\lambda} = \bar{\lambda}_\lambda$ and $\bar{Y}_{\lambda} = \frac{\lambda^{\alpha}}{\lambda^\lambda}$, respectively. As for the replacement of $\bar{Y}^\alpha$ with $\bar{Y}_{\lambda}$, that is of $\bar{v}^\alpha$ with $\lambda^{\alpha}$ for the non-minimal sector, we do not see any problem, as noted at the end of section 4 because $\bar{v}^\alpha$ and $\lambda^{\alpha}$ are both BRST invariant and all the OPE’s among the currents of the non-minimal sector remain unchanged.

On the contrary, a naive, straightforward replacement of $Y_\alpha$ with $\bar{\lambda}_\lambda$ looks problematic. Indeed, even if the OPE’s among the Lorentz current $N_{ab}$, the ghost current $J$, and the stress energy tensor $T_{\lambda}$ of the minimal ghost sector are unchanged, those among these operators and that of the non-minimal sector become different from zero, since the correction terms in (2.16)-(2.18) now acquire a dependence from $\bar{\lambda}$. Therefore the OPE’s among the total Lorentz current, ghost current and stress energy tensor of the (minimal and non-minimal) ghost sector do not close correctly. Moreover, the BRST variation of (2.14) appears to be inconsistent. We cannot exclude a possibility that these problems could be overcome by a smart modification of the basic OPE’s, but it is far from obvious that a consistent modification could be found. Thus, in this paper, we have refrained from exploring this possibility further and we hope to come back to this question in future.

Acknowledgements

The work of the first author (I.O.) was partially supported by the Grant-in-Aid for Scientific Research (C) No.14540277 from the Japan Ministry of Education, Science and Culture. The work of the second author (M.T.) was supported by the European Community’s Human Potential Programme under contract MRTN-CT-2004-005104 “Constituents, Fundamental Forces and Symmetries of the Universe”.

A Notation, Conventions and Useful identities

In this appendix, we collect our notation, conventions and some useful formulae employed in this paper.
As usual, in ten space-time dimensions, $\Gamma^a$ are the Dirac matrices $\gamma^a$ times the charge conjugation matrix $C$, that is, $(\Gamma^a)^{\beta\alpha} = (\gamma^a C)^{\alpha\beta}$ and $(\Gamma^a)_{\beta\alpha} = (C^{-1}\gamma^a)_{\alpha\beta}$; they are 16 $\times$ 16 symmetric matrices with respect to the spinor indices, and satisfy the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$. Our metric convention is $\eta^{ab} = (-, +, \cdots, +)$.

The square bracket and the brace respectively denote the antisymmetrization and the symmetrization of $p$ indices, normalized with a numerical factor $\frac{1}{p!}$ so that, for instance $A_{\mu}B_{\nu} = \frac{1}{2}(A_{[\mu}B_{\nu]} - A_{\nu}B_{\mu})$. As for the products of $\Gamma^a$, $\Gamma^{a_1\cdots a_p} = \Gamma^{[a_1\cdots a_p]}$. These antisymmetrized products of $\Gamma$ have definite symmetry properties, which are given by $(\Gamma^{ab})_{\alpha\beta} = -(\Gamma^{ab})_{\beta\alpha}$, $(\Gamma^{abc})_{\alpha\beta} = -(\Gamma^{abc})_{\beta\alpha}$, $(\Gamma^{abcd})_{\alpha\beta} = (\Gamma^{abcd})_{\beta\alpha}$, $(\Gamma^{abcde})_{\alpha\beta} = (\Gamma^{abcde})_{\beta\alpha}$, etc.

The product of generic spinors $f_\alpha$ and $g_\beta$ can be expanded in terms of the complete set of gamma matrices as

$$f_\alpha g_\beta = \frac{1}{16} \Gamma^{a\beta}_\alpha (f \Gamma_a g) + \frac{1}{16 \times 3!} \Gamma^{abc\beta}_\alpha (f \Gamma_{abc} g) + \frac{1}{16 \times 5!} \Gamma^{abcde\beta}_\alpha (f \Gamma_{abcde} g). \quad (A.1)$$

Similarly, for spinors $f_\alpha$ and $g_\beta$ we have

$$f_\alpha g_\beta = \frac{1}{16} \delta^{\beta}_\alpha (fg) + \frac{1}{16 \times 2!} (\Gamma^{ab})^{\beta}_\alpha (f \Gamma_{ab} g) + \frac{1}{16 \times 4!} (\Gamma^{abcd})^{\beta}_\alpha (f \Gamma_{abcd} g). \quad (A.2)$$

A useful identity, involving three spinor-like operators $A_\alpha$, $B^\beta$ and $C^\gamma$ is

$$-\frac{1}{8} (B \Gamma^{ab} A)(\Gamma_{ab} C)^\alpha - \frac{1}{4} (BA) C^\alpha = (B\beta A^\alpha)C^\beta - \frac{1}{2} ((\Gamma^a B)^\alpha A^\beta)(\Gamma_a C)^\beta. \quad (A.3)$$

\section{Normal ordering, the generalized Wick theorem and rearrangement theorem}

In this appendix, we explain the prescription of normal ordering, the generalized Wick theorem and rearrangement theorem, which are used in this paper. The more detail of them can be seen in the texbook of conformal field theory [33].

\subsection{Normal ordering}

In conformal field theory, we usually consider normal ordering for free fields where the OPE contains only one singular term with a constant coefficient. Then, normal ordering is defined as the subtraction of this singular term. This definition of normal ordering is found to be equivalent to the conventional normal ordering in the mode expansion where the annihilation operators are placed at the rightmost position. However, we sometimes meet the case for which the fields are not free in this sense. One of the well-known examples happens when we try to regularize the OPE between two stress energy tensors $T(y)T(z)$. In this case, we have two singular terms where one singular term contains the quartic pole whose coefficient is proportional to the central charge while the other singular term contains the quadratic pole.
whose coefficient is not a constant but \((2\times)\) stress energy tensor itself. The usual normal ordering prescription amounts to subtraction of the former, most singular term, but the latter singular term is still remained. Let us note that in the present context, the OPE between \(\omega\) and \(\lambda\) is not free owing to the existence of the projection \(K\) reflecting the pure spinor constraint. From the physical point of view, we want to subtract all the singular terms in the OPE’s, so we have to generalize the definition of normal ordering.

To this end, we introduce the generalized normal ordering which is usually denoted by parentheses, that is, explicitly, the generalized normal ordering of operators \(A\) and \(B\) is written as \((AB)(z)\). A definition of the generalized normal ordering is given by the contour integration [33]

\[
(AB)(z) = \oint \frac{dw}{w-z} A(w)B(z). \tag{B.1}
\]

Then the OPE of \(A(z)\) and \(B(w)\) is described by

\[
A(z)B(w) = \langle A(z)B(w) \rangle + (A(z)B(w)), \tag{B.2}
\]

where \(\langle A(z)B(w) \rangle\) denotes the contraction containing all the singular terms of the OPE and \((A(z)B(w))\) stands for the complete sequence of regular terms whose explicit forms can be extracted from the Taylor expansion of \(A(z)\) around \(w\):

\[
(A(z)B(w)) = \sum_{k\geq 0} \frac{(z-w)^k}{k!} (\partial^k A \cdot B)(w). \tag{B.3}
\]

Another definition of the generalized normal ordering is provided by the mode expansion. If the OPE of \(A\) and \(B\) is written as

\[
A(z)B(w) = \sum_{k=-\infty}^{N} \frac{\{AB\}_k(w)}{(z-w)^k}, \tag{B.4}
\]

where \(N\) is some positive integer, the definition of the generalized normal ordering reads

\[
(AB)(z) = \{AB\}_0(z). \tag{B.5}
\]

Incidentally, in this context, the contraction is expressed by

\[
\langle A(z)B(w) \rangle = \sum_{k=1}^{N} \frac{\{AB\}_k(w)}{(z-w)^k}, \tag{B.6}
\]

In this paper, we adopt the definition of the contour integration (B.1). Moreover, for simplicity, we do not write explicitly the outermost parenthesis representing the generalized normal ordering whenever we can easily judge from the context whether some operators are normal-ordered or not.
B.2 The generalized Wick theorem

Relating to the generalization of the normal-ordering prescription, we also have to reformulate the Wick theorem for interacting fields. In general, the Wick theorem relates the time-ordered product to the normal-ordered product of free fields. However, such a relation cannot be generalized to interacting fields in a straightforward manner. Hence, the generalized Wick theorem is defined by generalizing a special form of the Wick theorem for the contraction of free fields. More explicitly, the generalized Wick theorem is simply defined as

\[ \langle A(z)(BC)(w) \rangle = \oint_w \frac{dx}{x-w}[\langle A(z)B(x)C(w) + B(x)A(z)C(w) \rangle]. \]  

(B.7)

From this definition, it is important to notice that the first regular term of the various OPE’s always contributes. If we would like to calculate \( \langle (BC)(z)A(w) \rangle \), we first calculate \( \langle A(z)(BC)(w) \rangle \), then interchange \( w \) and \( z \), and finally expand the fields evaluated at \( z \) in the Taylor series around \( w \).

B.3 Rearrangement theorem

We often encounter the situation where many of operators are normal-ordered, e.g., \( (A(BC))(z) \). With the generalized normal ordering, some complication occurs since there is no associativity in such normal-ordered operators

\[ (A(BC))(z) \neq ((AB)C)(z). \]  

(B.8)

To deal with normal ordering of such composite operators, we make use of the rearrangement theorem. The useful formulae are given by

\[
\begin{align*}
(AB) &= (BA) + ([A, B]), \\
(A(BC)) &= (B(AC)) + ([A, B]C), \\
((AB)C) &= (A(BC)) + (A([C, B])) + ([C, A]B) + ([AB], C),
\end{align*}
\]

(B.9)  

(B.10)  

(B.11)

where \( A, B, \) and \( C \) are all the Grassmann-even quantities. Note that if the Grassmann-odd quantities are involved, we must change the sign and the commutator in a suitable manner. For instance, for the Grassmann-even \( A \) and the Grassmann-odd \( B \) and \( C \), the last rearrangement theorem is modified as

\[
((AB)C) = (A(BC)) - (A\{B, C\}) - ([C, A]B) + \{(AB), C\}.
\]  

(B.12)

In making use of these rearrangement theorems, we are forced to evaluate the generalized normal ordering of the (anti-)commutator \( ([A, B]) \). Then, we rely on the useful formula

\[
([A, B])(z) = \sum_{k=1} \frac{(-1)^{k+1}}{k!} \partial^k \{AB\}_k(z).
\]  

(B.13)

Note that field-dependent singular terms contribute to the normal-ordering (anti-)commutator while the non-singular term \( \{AB\}_0 \) does not. In this paper, we make heavy use of these formulae in evaluating various normal-ordered products of operators.
C Some details about the calculations

C.1 BRST variation of $G^\alpha$

To compute the BRST variation of $G^\alpha$ it is convenient to use the following notation

$$g^\alpha(B, A, C) = -\frac{1}{8}(B\Gamma^{ab}A)(\Gamma_{ab}C)^\alpha - \frac{1}{4}(BA)C^\alpha = (B_\beta A^\alpha)C^\beta - \frac{1}{2}((\Gamma^a B^\alpha A^\beta)(\Gamma_a C)_\beta, \tag{C.1}$$

where $A^\alpha, B_\beta,$ and $C^\gamma$ are generic spinors and the last step is the identity (A.3). Then, given (3.5), one has

$$\{Q, G^\alpha\} = -\frac{1}{2}\lambda^\alpha(\Pi^a\Pi_a) + \frac{1}{2}(\lambda\Gamma_a\partial\theta)(\Gamma^a d)^\alpha. \tag{C.2}$$

Moreover,

$$\{Q, G^\alpha + G^\alpha_3\} = -g^\alpha(d, \lambda, \partial\theta) + g^\alpha(\Omega, \lambda, \partial\lambda) - 2g^\alpha(Y, \partial\lambda, \partial\lambda) - (Y\partial\lambda)\partial\lambda^\alpha. \tag{C.3}$$

The last three terms come from the definitions (2.25) and (2.26) of $N_{ab}$ and $J$.

Using the rearrangement formula (cf. (B.12)), one has

$$g^\alpha(d, \lambda, \partial\theta) = \lambda^\alpha(d\partial\theta) + 8\partial^2\lambda^\alpha + \frac{1}{2}(\lambda\Gamma_a\partial\theta)(\Gamma^a d)^\alpha, \tag{C.4}$$

and

$$g^\alpha(\Omega, \lambda, \partial\lambda) = (\Omega_\beta\lambda^\alpha)\partial\lambda^\beta - \frac{1}{2}((\Gamma^a\Omega)^a\lambda^\beta)(\Gamma_a\partial\lambda)_\beta. \tag{C.5}$$

Using the rearrangement theorem, the first term in the r.h.s. of Eq. (C.5) becomes

$$(\Omega_\beta\lambda^\alpha)\partial\lambda^\beta = \lambda^\alpha(\Omega\partial\lambda) - \frac{1}{2}(Y\Gamma^a)^a(\partial\lambda\Gamma_a\partial\lambda) + \frac{11}{2}\partial^2\lambda^\alpha, \tag{C.6}$$

whereas the second term can be rewritten as

$$-\frac{1}{2}((\Gamma^a\Omega)^a\lambda^\beta)(\Gamma_a\partial\lambda)_\beta = -\frac{3}{2}\partial^2\lambda^\alpha + \frac{3}{2}(Y\partial^2\lambda)\lambda^\alpha + (Y\partial\lambda)\partial\lambda^\alpha
+ 3(\partial Y\partial\lambda)\lambda^\alpha + 2g^\alpha(Y, \partial\lambda, \partial\lambda) + \frac{1}{2}(Y\Gamma^a)^a(\partial\lambda\Gamma_a\partial\lambda), \tag{C.7}$$

so that from (C.5)-(C.7), one obtains

$$g^\alpha(\Omega, \lambda, \partial\lambda) = \lambda^\alpha(\Omega\partial\lambda) + 4\partial^2\lambda^\alpha + \frac{3}{2}(Y\partial^2\lambda)\lambda^\alpha
+ (Y\partial\lambda)\partial\lambda^\alpha + 3(\partial Y\partial\lambda)\lambda^\alpha + 2g^\alpha(Y, \partial\lambda, \partial\lambda). \tag{C.8}$$

Adding Eqs. (C.2), (C.3) and $\{Q, G^\alpha_4\} = c_1\partial^2\lambda^\alpha$ with $c_1 = \frac{7}{2}$, taking into account (C.4), (C.8) and using the definition (2.16) of the stress energy tensor $T$, we finally obtain

$$\{Q, G^\alpha\} = \lambda^\alpha T - \frac{1}{2}\partial^2\lambda^\alpha. \tag{C.9}$$
C.2 BRST variation of $H^{\alpha\beta}$

Now let us consider the BRST variation of $H^{\alpha\beta}$. Eq. (3.25) can be rewritten as

$$[Q, H^{(\alpha\beta)}] = \frac{1}{16} \Gamma^a_{\alpha\beta} h^a, \quad (C.10)$$

where

$$h^a = \frac{1}{2} (\lambda \Gamma^a \Gamma_b d) \Pi^b + N^{ab} (\lambda \Gamma_b \partial \theta) - \frac{1}{2} J (\lambda \Gamma^a \partial \theta) + 2 \partial (\lambda \Gamma^a \partial \theta). \quad (C.11)$$

The first term in the r.h.s. of this equation can be rewritten as

$$\frac{1}{2} (\lambda \Gamma^a \Gamma_b d) \Pi^b = \frac{1}{2} (\lambda \Gamma^a \Gamma_b \Pi^b d) + 5 \partial (\lambda \Gamma^a \partial \theta). \quad (C.12)$$

With the notation

$$\Lambda^{\alpha\beta} \equiv \frac{1}{2} \partial \lambda^{[\alpha \lambda \beta]}, \quad (C.13)$$

the vector

$$V^a = N^{ab} (\lambda \Gamma_b \partial \theta) - \frac{1}{2} J (\lambda \Gamma^a \partial \theta), \quad (C.14)$$

becomes

$$V^a = \frac{1}{2} (\Omega \Gamma^a \Gamma^b \lambda) (\lambda \Gamma_b \partial \theta) - J (\lambda \Gamma^a \partial \theta) + 4 (Y \Lambda \Gamma^a \partial \theta) + 4 (Y \Gamma^a \tilde{\Lambda} \partial \theta) + 2 (\partial \lambda \Gamma^a \partial \theta). \quad (C.15)$$

The first term in the r.h.s. of (C.15) vanishes modulo a rearrangement contribution:

$$\frac{1}{2} (\Omega \Gamma^a \Gamma^b \lambda) (\lambda \Gamma_b \partial \theta) = -4 (Y \Gamma^a \tilde{\Lambda} \partial \theta) - 4 (Y \Lambda \Gamma^a \partial \theta) + 4 (\partial \lambda \Gamma^a \partial \theta), \quad (C.16)$$

so that $h^a$ becomes

$$h^a = \frac{1}{2} (\lambda \Gamma^a \Gamma_b \Pi^b d) + 5 \partial (\lambda \Gamma^a \partial \theta) - J (\lambda \Gamma^a \partial \theta) - 2 (\partial \lambda \Gamma^a \partial \theta) + 2 \partial (\lambda \Gamma^a \partial \theta). \quad (C.17)$$

On the other hand,

$$\lambda \Gamma^a G = \frac{1}{2} (\lambda \Gamma^a \Gamma_b \Pi^b d) + \frac{7}{2} (\lambda \Gamma^a \partial^2 \theta) + \tilde{V}^a, \quad (C.18)$$

where $\tilde{V}^a = -\frac{1}{4} (\tilde{\Lambda} \Gamma^a N^{bc} \Gamma_c \Gamma^b \partial \theta) - \frac{1}{4} (\lambda \Gamma^a J \partial \theta)$. Then, using (2.25) and (2.26)

$$\tilde{V}^a = -\frac{1}{2} (\lambda \Gamma_c (\Omega \Gamma^a \Gamma^c \lambda) \partial \theta) + \frac{1}{8} (\lambda \Gamma_c \Gamma_b (\Omega^b \Gamma^c \lambda) \Gamma^a \partial \theta) - (\lambda \Gamma^a (\Omega \lambda) \partial \theta) - 4 (Y \Gamma^a \tilde{\Lambda} \partial \theta) - 4 (Y \Lambda \Gamma^a \partial \theta) + 2 (Y \Lambda \Gamma^a \partial \theta) + (\partial \lambda \Gamma^a \partial \theta). \quad (C.19)$$
But the first two terms in the r.h.s. of (C.19) vanish modulo the Y-dependent term $4[\mathcal{Y} \Gamma^a \bar{\Lambda} \partial \theta] - 6(\mathcal{Y} \Lambda \Gamma^a \partial \theta)$ coming from rearrangement theorem, so that we have

$$\tilde{V}^a = -(\Lambda \Gamma^a (\Omega \lambda) \partial \theta) - 4(\mathcal{Y} \Lambda \Gamma^a \partial \theta) + (\partial \Lambda \Gamma^a \partial \theta)$$

$$= -J(\Lambda \Gamma^a \partial \theta) + 4 \partial(\Lambda \Gamma^a \partial \theta) - 4(\Lambda \Gamma^a \partial^2 \theta), \quad (C.20)$$

and therefore

$$\lambda \Gamma^a G = \frac{1}{2}(\lambda \Gamma^a \Gamma_b \Pi^b d) + \frac{7}{2}(\lambda \Gamma^a \partial^2 \theta) - J(\lambda \Gamma^a \partial \theta) - 4(\lambda \Gamma^a \partial^2 \theta) + 4 \partial(\lambda \Gamma^a \partial \theta). \quad (C.21)$$

Then comparing (C.17) with (C.21), one gets Eq. (3.27).

Next let us consider the BRST variation of $H_{[\alpha \beta]}$. Eq. (3.26) can be rewritten as

$$[Q, H_{[\alpha \beta]}] = \frac{1}{96} \Gamma_{abc} \left[ \frac{1}{2}(\lambda \Gamma^{abc} \Gamma^d \Pi_d d) + 6(\lambda \mathcal{N}^{[ab} \Gamma^c \partial \theta) + 4(\lambda \Gamma^{abc} \partial^2 \theta) + (\partial \lambda \Gamma^{abc} \partial \theta) \right], \quad (C.22)$$

where the last two terms in the r.h.s. of this equation come from normal ordering.

On the other hand,

$$\lambda \Gamma^{abc} \hat{G} = \frac{1}{2}(\lambda \Gamma^{abc} \Gamma^d \Pi_d d) + 4(\lambda \Gamma^{abc} \partial^2 \theta) + 6(\lambda \mathcal{N}^{[ab} \Gamma^c \partial \theta) + 3(\lambda \mathcal{F}^{f[a} \Gamma^{bc]} \partial \theta) + \frac{1}{4}(\lambda \mathcal{F}^f \Gamma_g \Gamma^{abc} \mathcal{N}^{fg} \partial \theta) - \frac{1}{4}(\lambda \Gamma^{abc} \mathcal{J} \partial \theta). \quad (C.23)$$

Using (2.30), (2.31) and the notation introduced in (C.13) the quantity in the second row of (C.23), that is,

$$h^{[abc]} = +3(\lambda \mathcal{F}^f \Gamma_g \Gamma^{abc} \mathcal{N}^{fg} \partial \theta) + \frac{1}{4}(\lambda \mathcal{F}^f \Gamma_g \Gamma^{abc} \mathcal{N}^{fg} \partial \theta) - \frac{1}{4}(\lambda \Gamma^{abc} \mathcal{J} \partial \theta), \quad (C.24)$$

can be rewritten as

$$h^{[abc]} = -\frac{3}{2}(\lambda \mathcal{F}^f \Gamma^{[ab}(\Omega \Gamma^c) \Gamma^f \lambda) \partial \theta) + \frac{1}{8}(\lambda \mathcal{F}^f \Gamma_g \Gamma^{abc}(\Omega \Gamma^f \Gamma^g \lambda) \partial \theta) - 12(\mathcal{Y} \Gamma^a \bar{\Lambda} \Gamma^{bc} \partial \theta)$$

$$- 6(\mathcal{Y} \Lambda \Gamma^{abc} \partial \theta) + (\partial \Lambda \Gamma^{abc} \partial \theta). \quad (C.25)$$

On the other hand, by reordering, the sum of the first two, $\Omega$-dependent terms in (C.25) yields

$$12(\mathcal{Y} \Gamma^a \bar{\Lambda} \Gamma^{bc} \partial \theta) + 6(\mathcal{Y} \Lambda \Gamma^{abc} \partial \theta)$$

so that $h^{[abc]} = \partial \Lambda \Gamma^{abc} \partial \theta$ and (C.23) becomes

$$\lambda \Gamma^{abc} \hat{G} = \frac{1}{2}(\lambda \Gamma^{abc} \Gamma^d \Pi_d d) + 4(\lambda \Gamma^{abc} \partial^2 \theta) + 6(\lambda \mathcal{N}^{[ab} \Gamma^c \partial \theta) + (\partial \Lambda \Gamma^{abc} \partial \theta). \quad (C.26)$$

By comparing (C.22) with (C.26) one gets Eq. (3.28).
C.3 BRST variation of $K^{[\alpha\beta\gamma]}$

Now let us check (3.41). Let us rewrite (3.40) as

$$\{Q, K^{[\alpha\beta\gamma]}\} = k_1^{[\alpha\beta\gamma]} + k_2^{[\alpha\beta\gamma]},$$  \hspace{1cm} (C.27)

where

$$k_2^{[\alpha\beta\gamma]} = -\frac{1}{12}(\Gamma^a d)^{[\alpha} \left[ \frac{3}{4}(\Gamma^b d)^\beta \lambda^\gamma - \frac{1}{4}(\Gamma^b d)^\beta \Gamma^{ba} \lambda^\gamma \right]$$

$$= \frac{1}{384} \lambda^{[\alpha} \Gamma_{abc]} (d\Gamma^{abc} d),$$  \hspace{1cm} (C.28)

and

$$k_1^{[\alpha\beta\gamma]} = \frac{1}{12} \Pi_d (\Gamma^a \Gamma^d \lambda)^{[\alpha} N_a^\beta\gamma]$$

$$= \frac{1}{6} \Pi_d (\Gamma^a \Gamma^d \lambda)^{[\alpha} \left[ \frac{1}{2} (\Omega \Lambda_a^\lambda) \lambda - (Y \Lambda_a^\lambda \partial \lambda) \right].$$  \hspace{1cm} (C.29)

The first term in the r.h.s. of (C.29) can be elaborated as follows:

$$\frac{1}{12} \Pi_d (\Gamma^a \Gamma^d \lambda)^{[\alpha} (\Omega \Lambda_a^\lambda) \lambda = \frac{1}{8} \Pi^d (\Omega \Gamma^\lambda)^{[\alpha} \lambda^\beta (\Gamma_b \Gamma_d \lambda)^\gamma + \Delta^{[\alpha\beta\gamma]}$$

$$= \frac{1}{2} \Pi^d \lambda^{[\alpha} \Omega \Lambda_d^\lambda \lambda + \Delta^{[\alpha\beta\gamma]} + \hat{\Delta}^{[\alpha\beta\gamma]},$$  \hspace{1cm} (C.30)

where $\Delta^{[\alpha\beta\gamma]}$ and $\hat{\Delta}^{[\alpha\beta\gamma]}$ are the contributions of rearrangement theorem and are given by

$$\Delta^{[\alpha\beta\gamma]} = \frac{1}{192} \Pi^d \Gamma^{abc}_{\alpha\beta\gamma} (\Gamma^a \Gamma_d \partial K \Gamma^b c \lambda)^\gamma$$

$$= \frac{1}{24} \Pi^f (\Gamma^a Y)^{[\alpha} \partial \lambda^\beta (\Gamma_a \Gamma_f \lambda)^\gamma + (\Gamma_a \Gamma_f \partial \lambda)^\beta \lambda^\gamma - \frac{1}{2} (\Gamma_f \Gamma_b \partial \lambda)^\beta (\Gamma_a \Gamma^b \lambda)^\gamma]$$

$$- \frac{1}{2} \Pi^d (\Gamma_a Y)^{[\alpha} \partial \lambda^\beta \lambda^\gamma],$$  \hspace{1cm} (C.31)

and

$$\hat{\Delta}^{[\alpha\beta\gamma]} = \frac{1}{4} \Pi^d (\Gamma^j Y)^{[\alpha} (\partial \lambda \Gamma_f \Lambda_d^\beta\gamma) \lambda).$$  \hspace{1cm} (C.32)

Therefore, $k_1^{[\alpha\beta\gamma]}$ becomes

$$k_1^{[\alpha\beta\gamma]} = \frac{1}{2} \Pi^d \lambda^{[\alpha} N_d^\beta\gamma] + \{ \Pi^d (\lambda^{[\alpha} Y \Lambda_d^\beta\gamma) \partial \lambda) - \frac{1}{6} \Pi_d (\Gamma^a \Gamma^d \lambda)^{[\alpha} (Y \Lambda_d^\lambda \partial \lambda) \}$

$$+ \hat{\Delta}^{[\alpha\beta\gamma]} + \Delta^{[\alpha\beta\gamma]} \right).$$  \hspace{1cm} (C.33)

With a little algebra, it is easy to show that the terms in the curly bracket in the r.h.s. of (C.33) vanish so that (C.33) becomes

$$k_1^{[\alpha\beta\gamma]} = \frac{1}{2} \lambda^{[\alpha} \Pi^d N_d^\beta\gamma].$$  \hspace{1cm} (C.34)

Then, (C.27), together with (C.28), (C.34) and (3.23), reproduces Eq. (3.41).
C.4 The vanishing of $\lambda^e L^{\alpha\beta\gamma\delta}$

Now let us consider $\lambda^e L^{\alpha\beta\gamma\delta}$ in order to verify that it vanishes. As discussed at the end of section 3, it consists of three terms:

$$\lambda^e L_1^{\alpha\beta\gamma\delta} = \lambda^e (\Omega^e \sigma^\alpha \lambda)(\Omega^e \sigma^\gamma \lambda)$$

$$= \Omega_{\sigma} A_1^{[\alpha\beta\gamma\delta]e} + A_0^{[\alpha\beta\gamma\delta]} \tag{C.35}$$

$$\lambda^e L_2^{\alpha\beta\gamma\delta} = \Omega_{\sigma} B_1^{[\alpha\beta\gamma\delta]e} + B_0^{[\alpha\beta\gamma\delta]} \tag{C.36}$$

and

$$\lambda^e L_3^{\alpha\beta\gamma\delta} = 4 \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(Y \Lambda^e \sigma^\gamma \sigma^\delta \partial \lambda) \tag{C.37}$$

where $\Omega_{\sigma} B_1^{[\alpha\beta\gamma\delta]e}$ is

$$\Omega_{\sigma} B_1^{[\alpha\beta\gamma\delta]e} = -4 \Omega_{\sigma} \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e, \tag{C.38}$$

and $\Omega_{\sigma} A_1^{[\alpha\beta\gamma\delta]e}$ gives

$$\Omega_{\sigma} A_1^{[\alpha\beta\gamma\delta]e} = \Omega_{\sigma} (\Lambda^e \sigma^\alpha (\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e \Gamma^e Y)^e \phi + \Omega_{\sigma} (\Lambda^e \sigma^\alpha \phi \Gamma^e Y)^e \phi (\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e \phi \tag{C.39}$$

The first term in the r.h.s. of this equation can be rewritten as $4 \Omega_{\sigma} \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e - 2 \Omega_{\sigma} (\Lambda^e \sigma^\alpha \phi \Gamma^e Y)^e \phi \partial \lambda \sigma^\gamma \lambda \phi$ and the second one as $2 \Omega_{\sigma} (\Lambda^e \sigma^\alpha \phi \Gamma^e Y)^e \phi \partial \lambda \sigma^\gamma \lambda \phi$ so that we have

$$\Omega_{\sigma} A_1^{[\alpha\beta\gamma\delta]e} = 4 \Omega_{\sigma} \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e. \tag{C.40}$$

Then, using (C.38) and (C.40), Eq. (3.67) can be derived.

As for $A_0^{[\alpha\beta\gamma\delta]}$ and $B_0^{[\alpha\beta\gamma\delta]}$, the explicit calculation gives

$$A_0^{[\alpha\beta\gamma\delta]} = \frac{1}{2} (\Gamma^e Y)^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e (Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e - \frac{3}{2} (\Gamma^e Y)^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e (Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e$$

$$+ \frac{1}{2} \partial \{((\Gamma^e Y)^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e (Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e) \}, \tag{C.41}$$

and

$$B_0^{[\alpha\beta\gamma\delta]} = -2 (\Gamma^e Y)^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e (Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e + 4 \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e$$

$$+ 2 \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e - 2 \partial \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e, \tag{C.42}$$

so that we obtain

$$A_0^{[\alpha\beta\gamma\delta]} + B_0^{[\alpha\beta\gamma\delta]} = \frac{1}{2} (\Gamma^e Y)^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e (Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e - \frac{7}{2} (\Gamma^e Y)^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e (Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e$$

$$+ 4 \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e + 2 \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e$$

$$+ \frac{1}{2} \partial \{((\Gamma^e Y)^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(\partial \lambda \Gamma_{\sigma^\gamma \lambda}) (\Gamma^e Y)^e (Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e) - 4 \lambda^e (Y \Lambda^e \sigma^\alpha \sigma^\beta \partial \lambda)(Y \Lambda^e \sigma^\gamma \sigma^\delta \lambda)^e \}. \tag{C.43}$$
In order to verify (3.68), one needs three useful identities:

\[(\Gamma^f Y)^{[c}(Y \Lambda_s^\alpha \beta \partial \lambda)(\partial \lambda \Gamma_f \Lambda^{\gamma \delta]} \partial \lambda) = 4 \lambda^k(Y \Lambda_s^\alpha \beta \partial \lambda)(Y \Lambda^{\gamma \delta]} \partial \lambda), \quad (C.44)\]

\[(\Gamma^f Y)^{[c}(Y \Lambda_s^\alpha \beta \lambda)(\partial \lambda \Gamma_f \Lambda^{\gamma \delta]} \partial \lambda) = 5 \lambda^k(Y \Lambda_s^\alpha \beta \partial \lambda)(Y \Lambda^{\gamma \delta]} \partial \lambda) + 10 \partial \lambda^k(Y \Lambda_s^\alpha \beta \lambda)(Y \Lambda^{\gamma \delta]} \partial \lambda), \quad (C.45)\]

\[(\Gamma^f Y)^{[c}(Y \Lambda_s^\alpha \beta \partial \lambda)(\partial \lambda \Gamma_f \Lambda^{\gamma \delta]} \partial \lambda) = 3 \lambda^k(Y \Lambda_s^\alpha \beta \partial \lambda)(Y \Lambda^{\gamma \delta]} \partial \lambda) + 2 \partial \lambda^k(Y \Lambda_s^\alpha \beta \lambda)(Y \Lambda^{\gamma \delta]} \partial \lambda). \quad (C.46)\]

From the identity (C.44), the derivative term in the last row of the r.h.s. of (C.43) vanishes. Then, removing the first two terms in the r.h.s. of (C.43) by means of the two identities (C.45) and (C.46), one gets

\[A_0^{[\alpha \beta \gamma \delta]} + B_0^{[\alpha \beta \gamma \delta]} = -4 \lambda^k(Y \Lambda_s^\alpha \beta \partial \lambda)(Y \Lambda^{\gamma \delta]} \partial \lambda), \quad (C.47)\]

which is nothing but Eq. (3.68). In this way, we have succeeded in proving Eq. (3.57).

C.5 Equivalence in cohomology of \(b_Y\) and \(b_{nm}\)

As a last remark, let us report briefly about the derivation of the rearrangement terms \(R_G, R_H, R_L, R_K\) and \(B_L\), which appear at the end of section 5. In particular we shall show that \(B_L\) is BRST-exact.

From the recipe given in Appendix B.3 and using the OPE (3.15), one can compute \(R_G = (\lambda_\beta Y_\alpha)(\lambda^\alpha \hat{G}^{\beta}) - (\lambda_\beta Y_\alpha \lambda^\alpha \hat{G}^{\beta})\) with the result

\[R_G = -[\partial \lambda \partial \theta - (\lambda \partial \lambda)(\lambda \partial \theta) - 2 \partial \lambda \partial \theta] + \hat{R}_G, \quad (C.48)\]

where

\[\hat{R}_G = Y_\alpha \hat{\lambda}_\beta((Y + \lambda) \Gamma_a)^a \partial \lambda^\beta(\lambda \Gamma^a \partial \theta). \quad (C.49)\]

With the replacement

\[\lambda \Gamma^a \partial \theta = [Q_{nm}, \Pi^a], \quad (C.50)\]

and some simple algebra, \(\hat{R}_G\) can be rewritten as

\[\hat{R}_G = 2[Q_{nm}, (Y_\alpha \hat{\lambda}_\beta)W_{R_1}^{[\alpha \beta]}] - \frac{1}{2} A_G^a \Pi_a, \quad (C.51)\]

where \(W_{R_1}^{[\alpha \beta]}\) is defined in (5.39) and \(A_G^a\), defined in (5.40), comes from

\[\frac{1}{2} A_G^a \Pi_a = [Q_{nm}, Y_\alpha \hat{\lambda}_\beta((Y + \lambda) \Gamma_a)^a \partial \lambda^\beta] \Pi^a, \quad (C.52)\]
by using a simple algebra.

In a similar way, $R_H$ is given by

$$R_H = -\frac{1}{2}A^a_\alpha \Pi_a + \hat{R}_H,$$

where $\hat{R}_H$ contains the factor $(\Gamma^c \Gamma_a \Pi^a \lambda)^\gamma$ which can be replaced by $-\{Q_{nm}, (\Gamma^c d)^\gamma\}$ and then, working as before, one arrives at

$$\hat{R}_H = 3!\left[ Q_{nm}, Y_{[a} \tilde{r}_\beta \tilde{\lambda}_\gamma W^{[\alpha\beta\gamma]}_{R_2} \right] - \frac{1}{4}A^a_\alpha (d\Gamma_a)^\alpha,$$

(C.54)

where $W^{[\alpha\beta\gamma]}_{R_2}$ and $A^a_\alpha$ are defined in (5.39) and (5.40), respectively. Moreover,

$$R_K = \frac{1}{4}A^a_\alpha (d\Gamma_a)^\alpha + \hat{R}_K,$$

(C.55)

where

$$\hat{R}_K = \frac{4!}{12} (Y_{[a} \tilde{r}_\beta \tilde{r}_\gamma \tilde{\lambda}_\delta]\left( (Y + 3\tilde{\lambda}) \Gamma^c \right)^\alpha \partial \lambda^\beta [Q_{nm}, N^{\gamma\delta}]$$

(C.56)

$$= 4!\left[ Q_{nm}, (Y_{[a} \tilde{r}_\beta \tilde{r}_\gamma \tilde{\lambda}_\delta] W^{[\alpha\beta\gamma\delta]}_{R_3} \right] + A^a_{[\alpha\beta} N^{\gamma\delta]}_c,$$

where again $W^{[\alpha\beta\gamma\delta]}_{R_3}$ and $A^a_{[\alpha\beta}$ are defined in (5.39) and (5.40), respectively. Moreover,

Now let us move on to $R_L$ which, according to (5.34), is

$$R_L = 5! \left( Y_{[a} \tilde{r}_\beta \tilde{r}_\gamma \tilde{r}_\delta \tilde{\lambda}_\epsilon] (N^{\alpha\beta} N^{\gamma\delta}) \right),$$

(C.57)

With rearrangement formula and using (3.58), $R_L$ becomes

$$R_L = 5! \left( \frac{1}{48} (Y_{[a} \tilde{r}_\beta \tilde{r}_\gamma \tilde{r}_\delta \tilde{\lambda}_\epsilon] (Y + 4\tilde{\lambda}) (\Sigma^{\gamma\delta}) \lambda^\epsilon) \right) - (K_{[\alpha\beta\gamma\delta]} N^{\gamma\delta}) \lambda^\epsilon$$

(C.58)

where we have defined $K_{[\alpha\beta\gamma\delta]} = Y_{[a} \tilde{r}_\beta \tilde{r}_\gamma \tilde{r}_\delta \tilde{\lambda}_\epsilon]$. (The expression of $\hat{R}_L$ will be given below.) To the first term in the last row of Eq. (C.58), adding and subtracting the term defined by

$$R_0 = \frac{5!}{24} \left( (Y_{[a} \tilde{r}_\beta \tilde{r}_\gamma \tilde{r}_\delta \tilde{\lambda}_\epsilon] (Y + 4\tilde{\lambda}) (\Sigma^{\gamma\delta}) \lambda^\epsilon) \right) N^{\alpha\beta},$$

(C.59)

where we have also defined

$$\tilde{Y}_\Sigma^{[\alpha\beta]} \lambda = \tilde{Y} \Gamma_c^{[\alpha \lambda \beta]} + \frac{1}{4} (\tilde{Y} \Gamma)^{[\alpha} (\Gamma^b \Gamma_c^{\beta \lambda})^{\beta]},$$

(C.60)

and

$$\tilde{Y}_\alpha = Y_\alpha + 4\tilde{\lambda}_\alpha,$$  

(C.61)
$R_L$ is then reduced to

$$R_L = -\frac{1}{12} A^\alpha_{\alpha\beta} N^{\alpha\beta}_{c} + R_0 + \hat{R}_L. \quad (C.62)$$

Here we have introduced the quantity

$$\hat{R}_L = 5! \frac{1}{48} (\partial R_1 + R_2 + R_3), \quad (C.63)$$

where $R_1$, $R_2$ and $R_3$ are defined by

$$
R_1 = \frac{1}{4} Y_{[\alpha \beta \delta]}(\Gamma^a Y)^{\alpha}(\Gamma^b \tilde{Y})^{\beta} (\Gamma^c \Gamma^d \lambda)^{\gamma} [(\Gamma_a \Gamma_c \partial \lambda)^{\delta} + 2 \delta_{ac} \partial \lambda^{\delta}], \\
R_2 = -\frac{1}{2} Y_{[\alpha \beta \delta]}(\Gamma^c Y)^{\alpha}(\Gamma^b \tilde{Y})^{\beta} \partial [(\Gamma_b \lambda)^{\gamma} \partial \lambda^{\delta}], \\
R_3 = Y_{[\alpha \beta \delta]}(\bar{\lambda} \Gamma^f Y)(\Gamma^c (2Y + 5\tilde{\lambda}))^{\alpha} (\Gamma_d \Gamma^b \lambda)^{\beta} (\Gamma_f \Gamma_b \partial \lambda)^{\gamma} \partial \lambda^{\delta}. \quad (C.64)
$$

It is of importance that $R_1$, $R_2$ and $R_3$ are all BRST-exact:

$$
R_1 = \frac{1}{20} [Q_{nm}, Y_{[\alpha \beta \delta]}(\Gamma^a Y)^{\alpha}(\Gamma^b (Y + 3\tilde{\lambda}))^{\beta} (\Gamma^c \Gamma^d \lambda)^{\gamma} [(\Gamma_a \Gamma_c \partial \lambda)^{\delta} + 2 \delta_{ac} \partial \lambda^{\delta}], \\
R_2 = -\frac{1}{10} [Q_{nm}, Y_{[\alpha \beta \delta]}(\Gamma^c Y)^{\alpha}(\Gamma^b (Y + 3\tilde{\lambda}))^{\beta} \partial [(\Gamma_b \lambda)^{\gamma} \partial \lambda^{\delta}], \\
R_3 = \frac{3}{10} [Q_{nm}, Y_{[\alpha \beta \delta]}(\bar{\lambda} \Gamma^f Y)(\Gamma^c (Y + 2\tilde{\lambda}))^{\alpha} (\Gamma_d \Gamma^b \lambda)^{\beta} (\Gamma_f \Gamma_b \partial \lambda)^{\gamma} \partial \lambda^{\delta}. \quad (C.65)
$$

On the other hand, by rearrangement theorem, $R_0$ can be rewritten as

$$
R_0 = \frac{5!}{24} \Omega_{\sigma} (Y_{[\alpha \beta \delta]}(\tilde{Y} \Sigma^\delta_{c} \lambda) \partial \lambda^{\alpha} (\Lambda^{\alpha\beta\gamma})^{\delta}) + \quad (C.66)
$$

The first term in the r.h.s. of (C.66) vanishes and the second one is BRST-exact. Indeed, one has

$$
R_0 = \frac{1}{4} [Q_{nm}, Y_{[\alpha \beta \delta]}(\Gamma^c (Y + 3\tilde{\lambda}))^{\alpha} \partial \lambda^{\beta} (\Gamma^b Y)^{\gamma} (\Gamma_b \Gamma_c \partial \lambda)^{\delta}] \quad (C.67)
$$

Eq. (C.62) is just Eq. (5.38) with $B_L = R_0 + \hat{R}_L$. Then, from Eqs. (C.63), (C.65) and (C.67), one can reproduce Eqs. (5.41)-(5.43).

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