We introduce SketchySGD, a stochastic quasi-Newton method that uses sketching to approximate the curvature of the loss function. SketchySGD improves upon existing stochastic gradient methods in machine learning by using randomized low-rank approximations to the subsampled Hessian and by introducing an automated stepsize that works well across a wide range of convex machine learning problems. We show theoretically that SketchySGD with a fixed stepsize converges linearly to a small ball around the optimum. Further, in the ill-conditioned setting we show SketchySGD converges at a faster rate than SGD for least-squares problems. We validate this improvement empirically with ridge regression experiments on real data. Numerical experiments on both ridge and logistic regression problems show that SketchySGD can achieve comparable or better results to popular stochastic gradient methods with minimal hyperparameter tuning. The robustness of SketchySGD to hyperparameters is an advantage over other stochastic gradient methods, most of which require careful hyperparameter tuning (especially of the learning rate) to obtain good performance.

**Keywords** Stochastic optimization · Quasi-Newton methods · Preconditioning · Randomized numerical linear algebra

**1 Introduction**

Stochastic first-order methods such as SGD, SVRG, and Katyusha are the workhorses of large-scale machine learning due to their efficient iteration complexity. However, SGD and its variants alone typically do not suffice to ensure successful training. Indeed, in order to successfully reach a point with good test error, one must carefully tune the learning rate. The preceding issue is compounded in the presence of ill-conditioning, which can lead to painfully slow convergence of SGD and its relatives. As ill-conditioning is common in machine learning problems, SGD and its variants will often obtain sub-optimal performance if they are not carefully tuned. Unlike these competing methods, SketchySGD automatically selects its learning rate, and this automatic selection performs well across a wide selection of machine learning problems.

In optimization, the classic remedy for ill-conditioning is to use second-order optimizers based on the Hessian, such as Newton’s method and quasi-Newton methods. These methods converge at super-linear rates under mild assumptions, making them much faster than first-order methods. Alas, second-order methods have a much higher per-iteration cost than gradient based methods, which limits their use in large-scale machine learning.

The major innovation in this paper is the use of sketching to produce a new robust stochastic quasi-Newton framework called SketchySGD. Significantly, we provide a way to automatically select the learning rate for SketchySGD, which obviates the need for learning rate tuning. The procedure is based on estimating the preconditioned smoothness constant,
Figure 1: SketchySGD significantly outperforms standard stochastic gradient optimizers, even when they are tuned to yield their best performance. Each curve shows the trajectory for a given fixed learning rate; the dark curve shows the learning trajectory that achieves the best test loss for each optimizer.

using the largest eigenvalue of a preconditioned minibatch Hessian. Furthermore, we empirically verify the randomized quasi-Newton approximation may be updated infrequently, leading to a fast practical runtime, and avoiding the curse of scalability faced by standard quasi-Newton methods. Numerical experiments verify that SketchySGD yields comparable or superior performance to SGD, SVRG, stochastic L-BFGS [Moritz et al., 2016], and loopless Katyusha [Kovalev et al., 2020] equipped with tuned hyperparameters to attain their best performance. On the theoretical side, we establish SketchySGD converges linearly to a small ball around the optimum when the function being optimized is smooth and strongly convex. In addition, we prove that SketchySGD converges at a faster rate than SGD for the least-squares loss in the ill-conditioned setting.

1.1 SketchySGD

SketchySGD solves convex empirical risk minimization problems of the form

\[
\min_{w \in \mathbb{R}^p} f(w) := \frac{1}{n} \sum_{i=1}^{n} f_i(w)
\]

given access to a gradient oracle for each \( f_i \). Given iterate \( w_k \) at iteration \( k \), SketchySGD samples two independent batches \( S_k \) and \( B_k \subseteq [n] \) of data and computes a stochastic gradient \( g_{B_k}(w_k) \) and a randomized low-rank approximation \( \hat{H}_{S_k}(w_k) \) to the subsampled Hessian \( H_{S_k}(w_k) \). The approximate Hessian may be constructed from a sketch of the Hessian. Where the sketch is computed by calling an hvp oracle as detailed in Section 2 at a cost only a constant factor larger than computing a stochastic gradient. SketchySGD forms a regularized stochastic approximation to the Hessian \( (\hat{H}_{S_k} + \rho I) \) and updates the parameters as

\[
w_{k+1} = w_k - \eta_k (\hat{H}_{S_k} + \rho I)^{-1} g_{B_k}(w_k),
\]

(1)

where \( \eta_k \) is the learning rate and \( \rho > 0 \) is a regularization parameter. In particular, in the strongly convex case we require \( \rho \geq \mu \), where \( \mu \) is the strong convexity constant of \( f \). The SketchySGD update may be viewed as a preconditioned stochastic gradient step with Levenberg-Marquardt regularization [Levenberg, 1944, Marquardt, 1963] or as a stochastic quasi-Newton method with inexact gradients.

Contributions.

1. We develop a new robust stochastic quasi-Newton algorithm that is fast and generalizes well by accessing only a subsampled Hessian and stochastic gradient.
2. We devise an automated step size for this algorithm that works well in both ridge and logistic regression.
3. We show that SketchySGD converges linearly to a small ball around the optimum. Additionally, we show SketchySGD converges at a faster rate than SGD for ill-conditioned least-squares problems. We verify this improved rate of convergence in numerical experiments.
4. We present experiments showing that SketchySGD with little hyperparameter tuning can match or outperform SGD, SVRG, stochastic L-BFGS, and loopless Katyusha on ridge and logistic regression problems.

Outline. Section 2 describes the SketchySGD algorithm in detail, explaining how to compute \( \hat{H}_{S_k} \) and the update in (1) efficiently. Section 2.1 surveys previous work on stochastic quasi-Newton methods, particularly in the context of machine learning. Section 3 establishes convergence of SketchySGD in convex machine learning problems. Section 4 provides numerical experiments showing the superiority of SketchySGD relative to competing optimizers.
2 SketchySGD

Notation. Throughout the paper $B_k$ and $S_k$ denote subsets of $\{1, \ldots, n\}$ that are sampled independently and uniformly without replacement. The corresponding stochastic gradient and minibatch Hessian are given by

$$
g_{B_k}(w) = \frac{1}{b_{B_k}} \sum_{i \in B_k} g_i(w),$$

$$H_{S_k}(w) = \frac{1}{b_{S_k}} \sum_{i \in S_k} \nabla^2 f_i(w),$$

where $b_{B_k} = |B_k|$, $b_{S_k} = |S_k|$. For shorthand we often omit the dependence upon $w$ and simply write $g_{B_k}$ and $H_{S_k}$. We also define $H(w)$ as the Hessian of the objective $f$ at $w$, and we set $M(w) = \max_{1 \leq i \leq n} \|\nabla^2 f_i(w)\|$. Given any $\tau > 0$, we use the notation $H_{S_k}^{\tau}$ to denote $H_{S_k} + \tau I$. We abbreviate positive-semidefinite as psd. We denote the Loewner order on the convex cone of psd matrices by $\preceq$, where $A \preceq B$ means $B - A$ is psd. Given a psd matrix $A \in \mathbb{R}^{p \times p}$, we enumerate its eigenvalues in descending order, $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_p(A)$. Finally given a psd matrix $A$ and $\tau > 0$ we define the effective dimension by $d_{\text{eff}}(\tau) = \text{tr}(A(A + \tau I)^{-1})$, which provides a smoothed measure of the eigenvalues greater than or equal to $\tau$.

**Hessian vector product oracle.** SketchySGD relies on one main computational primitive, a (minibatch) Hessian vector product (hvp) oracle, to compute a low-rank approximation of the (minibatch) Hessian. Access to such an oracle naturally arises in machine learning problems. In the case of generalized linear models (GLMs), the Hessian is given by $H(w) = \frac{1}{n} A^T D(w) A$, where $A \in \mathbb{R}^{p \times n}$ is the data matrix and $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix. Accordingly, the minibatch Hessian is given by

$$H_{S_k}(w) = \frac{1}{b_{S_k}} \sum_{i \in S_k} d_i(w) a_i a_i^T.$$ 

Hence hvps with $H_{S_k}(w)$ may be computed by via the relation

$$H_{S_k}(w)v = \frac{1}{b_{S_k}} \sum_{i \in S_k} d_i(w) a_i (a_i^Tv).$$

For more complicated losses, an hvp can be computed by automatic differentiation (AD) [Pearlmutter 1994]. The general cost of $r$ hvps with $H_{S_k}(w)$ is $O(b_{S_k} pr)$. In contrast, explicitly instantiating a Hessian entails a heavy $O(p^2)$ storage and $O(np^2)$ computational cost. Further computational gains can be made when the subsampled Hessian enjoys more structure, such as sparsity. If $H_{S_k}(w)$ has $s$-sparse rows then the complexity of $r$ hvps enjoys a significant reduction from $O(b_{S_k} pr)$ to $O(b_{S_k} sr)$. Hence, computing hvps with $H_{S_k}(w)$ is extremely cheap in the sparse setting.

**Randomized low-rank approximation.** The hvp primitive allows for efficient randomized low-rank approximation to the minibatch Hessian by sketching. Sketching reduces the cost of fundamental numerical linear algebra operations without much loss in accuracy [Woodruff 2014, Martinsson and Tropp 2020] by computing the quantity of interest from a sketch, or randomized linear image, of a matrix. In particular, sketching enables efficient computation of a near-optimal low-rank approximation to $H_{S_k}$ [Halko et al. 2011, Cohen et al. 2015, Tropp et al. 2017d]. SketchySGD computes a sketch of the subsampled Hessian using hvps and returns a randomized low-rank approximation $\hat{H}_{S_k}$ of $H_{S_k}$ in the form of an eigendecomposition $\hat{V} \hat{\Lambda} \hat{V}^T$, where $\hat{V} \in \mathbb{R}^{p \times r}$ and $\hat{\Lambda} \in \mathbb{R}^{r \times r}$. Many algorithms are available to construct a low-rank approximation from the sketch, including the randomized SVD [Halko et al. 2011]. In this paper, we use the randomized Nyström approximation, following the stable implementation in [Tropp et al. 2017d]. The resulting algorithm RandNysApprox appears in Appendix [A]. The cost of forming the Nyström approximation is $O(b_{S_k} pr + pr^2)$, as we need to perform $r$ minibatch hvps to compute the sketch, and we must perform a skinny SVD at a cost of $O(pr^2)$. The procedure is extremely cheap, as we find empirically we can take $r$ to be 5 or less, so constructing the low-rank approximation has negligible cost.

**Remark 1.** If the objective for $f$ includes an $\ell^2$-regularizer $\nu$, so that the subsampled Hessian has the form $H_{S_k}(w) = \frac{1}{b_{S_k}} \sum_{i \in S_k} \nabla^2 f_i(w) + \nu I$, we do not include $\nu$ in the computation of the sketch in algorithm [1]. The sketch is only computed using minibatch hvps with $\frac{1}{b_{S_k}} \sum_{i \in S_k} \nabla^2 f_i(w)$. SketchySGD. As shown in Algorithm [1] the SketchySGD method uses a low-rank approximation of a minibatch Hessian to perform a stochastic quasi-Newton update with the minibatch gradient. SketchySGD reestimates the
Algorithm 1 SketchySGD

Input: initialization $w_0$, hvp oracle $O_H$, ranks $\{r_j\}$, regularization $\rho$, preconditioner update frequency $u$
repeat
   Sample independent batches $S_k$ and $B_k$
   Compute stochastic gradient $g_{B_k}(w_k)$
   if $k \equiv 0 \pmod{u}$ then
      $\Phi = \text{randn}(p, r_{k/u})$
      $Q = \text{qr}_\text{econ}(\Phi)$
      Compute sketch $Y = H_{S_k}(w_k)Q$
      $[\hat{V}, \hat{\Lambda}] = \text{RandNysApprox}(Y, Q, r_{k/u})$
      $\eta = \text{get\_learning\_rate}(O_{H_{S_k}}, \hat{V}, \hat{\Lambda}, \rho)$
      Compute $v_k = (\hat{H}_{S_k} + \rho I)^{-1}g_{B_k}(w_k)$ via (2)
      $w_{k+1} = w_k - \eta v_k$
   else
      $\Phi = \text{randn}(p, 1)$
      $Q = \text{qr\_econ}(\Phi)$
      Compute sketch $Y = H_{S_k}(w_k)Q$
      $[\hat{V}, \hat{\Lambda}] = \text{RandNysApprox}(Y, Q, r_{k/u})$
      $\eta = \text{get\_learning\_rate}(O_{H_{S_k}}, \hat{V}, \hat{\Lambda}, \rho)$
      Compute $v_k = (\hat{H}_{S_k} + \rho I)^{-1}g_{B_k}(w_k)$ via (2)
      $w_{k+1} = w_k - \eta v_k$
   end
until convergence
Return: $\eta$

Algorithm 2 get\_learning\_rate

Input: hvp oracle $O_H$, Nyström approximation factors $\hat{V}$, $\hat{\Lambda}$, regularization $\rho$, maximum number of iterations $q$
$z = \text{randn}(p, 1)$
$y_0 = z/\|z\|$ 
for $i = 1, \ldots, q$ do
   Compute $v = (\hat{H}_{S_k} + \rho I)^{-1/2}y_{i-1}$
   Compute $v' = H_{S_k}u$ by calling oracle $O_{H_{S_k}}$
   Compute $y_i = (\hat{H}_{S_k} + \rho I)^{-1/2}v'$
   $\lambda_i = y_i^T y_i$
   $y_i = y_i/\|y_i\|$ 
   Set $\eta = 1/\lambda_q$
Return: $\eta$

One limitation of using the randomized Nyström approximation to construct the preconditioner, is that in the case $g_{B_k}$ is sparse, (2) still costs $O(pr)$. Empirically, we have found SketchySGD still runs fast in the sparse setting, see section 4. Nevertheless, there are sketching methods that preserve sparsity. We consider it an interesting direction for future work to take advantage of such methods to accelerate performance in the sparse data setting.
2.1 Comparison to previous work

Many authors have sought to scale Newton’s method and its variants to large-scale machine learning tasks. Classical approaches include BFGS, its low-memory variant L-BFGS, and inexact Newton methods, where the sub-problem is solved approximately [Broyden 1970, Liu and Nocedal 1989, Dembo et al., 1982]. Given a function and gradient oracle, BFGS and L-BFGS yield satisfactory solutions to most optimization problems. However, for very large-scale problems, the cost of computing an exact gradient or function evaluation at each iteration is too large; hence methods for these problems must work with sampled function values and stochastic gradients.

Stochastic quasi-Newton methods address both problems by taking a generalized Newton step using a stochastic approximation to the Hessian (and, sometimes, to the gradient as well). Stochastic quasi-Newton methods that use exact gradients with a stochastic Hessian approximation constructed via sketching or subsampling include [Byrd et al., 2011, Erdogdu and Montanari 2015, Pilanci and Wainwright 2017, Gower et al. 2019a]. Others, such as [Moritz et al. 2016, Roosta-Khorasani and Mahoney 2019, Bollapragada et al. 2019, Meng et al. 2020], can handle stochastic approximations to both the gradient and Hessian.

SketchySGD improves on these stochastic quasi-Newton methods and generalizes many of them. In particular, subsampled Newton methods [Roosta-Khorasani and Mahoney 2019, Bollapragada et al. 2019, Meng et al. 2020] may be viewed as a special case of SketchySGD that uses the full minibatch-Hessian instead of a low-rank approximation. We recover the method of [Erdogdu and Montanari 2015] by using exact gradients and a regularized exact low-rank approximation to the subsampled Hessian. SketchySGD enjoys much faster iteration complexity relative to the standard subsampled Newton method. One iteration of subsampled Newton costs $O(b_hn^2 + p^3)$ using a direct method to compute the Newton step, which is very expensive when $p$ is large, and it further incurs an $O(p^2)$ storage cost.

Similarly, using an iterative method such as conjugate gradients (CG) to compute the Newton step is also expensive, costing $O(b_hn^2p^2 + np^3)$ per iteration, and this is based on assuming the worst-case update frequency, where $H_{S_n}$ is updated every iteration. Thus, SketchySGD is significantly less expensive than the vanilla subsampled Newton method.

3 Theory

We now prove convergence of SketchySGD when $f$ is smooth and strongly convex, as formally elucidated by assumption [1] and assumption [2].

**Assumption 1** (Differentiability and smoothness). The function $f$ is twice differentiable and $L$-smooth. Further, each $f_i$ is $L_i$-smooth with $L_i \leq L_{\max}$ for every $i = 1, \ldots, n$.

**Assumption 2** (Strong convexity). The function $f$ is $\mu$-strongly convex for some $\mu > 0$.

Our analysis of the convex case requires the notions of relative smoothness and relative convexity from [Gower et al. 2019a].

**Definition 1** (Relative smoothness and relative convexity). Let $f$ be a twice-differentiable function. Then the relative smoothness constant is defined by,

$$
\hat{L} := \sup_{w,w' \in \mathbb{R}^p} \int_0^1 2(1-t) \left\| w' - w \right\|^2_{H(w + t(w' - w))} dt.
$$

Similarly, the relative convexity constant is defined by

$$
\hat{\mu} := \inf_{w,w' \in \mathbb{R}^p} \int_0^1 2(1-t) \left\| w' - w \right\|^2_{H(w + t(w' - w))} dt.
$$

We say $f$ is relatively smooth (relatively convex) if $\hat{L} < \infty$ ($\hat{\mu} > 0$).

The following result easily follows from definition [1] see [Gower et al. 2019a].

**Proposition 3.** Let $f$ be relatively smooth and relatively convex. Then for all $w, w' \in \mathbb{R}^p$, the following inequalities hold:

$$
f(w') \leq f(w) + \langle g(w), w' - w \rangle + \frac{\hat{L}}{2} \left\| w' - w \right\|^2_{H(w)},
$$

$$
f(w') \geq f(w) + \langle g(w), w' - w \rangle + \frac{\hat{\mu}}{2} \left\| w' - w \right\|^2_{H(w)}.
$$

5
Relative smoothness and relative convexity were introduced in [Gower et al., 2019a]. These assumptions are weaker than assuming \( f \) is \( L \)-smooth and \( \mu \)-strongly convex. Indeed, it is easy to show that if \( f \) is \( L \)-smooth and \( \mu \)-strongly convex then \( \hat{f} \) is relatively smooth and relatively convex, see [Gower et al., 2019a] for details. More importantly, (5) and (6) hold with non-vacuous values of \( \hat{L} \) and \( \hat{\mu} \). In the case of least-squares \( \hat{L} = \hat{\mu} = 1 \), and for generalized linear models \( L \) and \( \mu \) are \( O(1) \) and independent of the condition number \( \kappa = L/\mu \) [Gower et al., 2019a]. Thus, for many popular machine learning problems the relative condition number \( \hat{\kappa} = \hat{L}/\hat{\mu} \) is a constant independent of the conditioning of the data.

The worst case \((r+1)\)th eigenvalue of the Hessian plays an important role in our convergence result.

**Definition 2.** Fix \( 1 \leq r \leq p \), and set
\[
\lambda_{r+1}^* = \sup_{w \in \mathbb{R}^p} \lambda_{r+1}(H(w)).
\]
Observe \( \lambda_{r+1}^* \leq L \) and is often significantly smaller. For the least-squares loss, \( \lambda_{r+1}^* = \lambda_{r+1}(H) \) and \( L = \lambda_1(H) \). As most machine learning data matrices exhibit polynomial or exponential spectral decay, [Derezinski et al., 2020] Zhao et al., 2022 we generally have \( \lambda_{r+1}(H) \ll \lambda_1(H) \) and hence \( \lambda_{r+1}^* \ll L \).

**Theorem 4 (SketchySGD convex convergence – informal).** Instate Assumptions [7][2] Run Algorithm [7] with batch sizes \( b_g \), \( b_h = \tilde{O}\left(\frac{M(w_0)}{\lambda_{r+1}^*}\right) \), stepsize \( \eta = O\left(\frac{1}{\kappa} \right) \), regularization \( \rho = \lambda_{r+1}^* \), and at each iteration constructing the randomized Nyström approximation with rank \( r_k = O\left(d_{\text{eff}}(\zeta \lambda_{r+1}^*)\right) \) where \( r > 0 \) is fixed. Then with high probability, for sufficiently small \( \epsilon \),
\[
\mathbb{E}[f(w_T)] - f(w_*) \leq \epsilon
\]
after \( T = O\left(\frac{\lambda_{r+1}^* \kappa \log(\frac{1}{\epsilon})}{\rho} \right) \) iterations.

**Remark 2.** The formal version of Theorem 4 appears in the supplement.

We have the following corollary that gives the improvement ratio of SketchySGD relative to SGD. The complexity for SGD may be found in theorem [14] in the appendix.

**Corollary 1.** Let \( T_{\text{SketchySGD}} \) denote the iteration complexity of SketchySGD from Theorem 4 and \( T_{\text{SGD}} \) denote the iteration complexity of SGD given from theorem [14] Then
\[
\frac{T_{\text{SGD}}}{T_{\text{SketchySGD}}} \geq \frac{\hat{\mu}}{\hat{\kappa}} \frac{L}{30 \lambda_{r+1}^*}.
\]
In particular, in the case of the least-squares loss we have
\[
\frac{T_{\text{SGD}}}{T_{\text{SketchySGD}}} \geq \frac{L}{30 \lambda_{r+1}^*} = \frac{\lambda_1(H)}{30 \lambda_{r+1}(H)}.
\]
For ill-conditioned problems, we expect the ratio \( L/\lambda_{r+1}^* \) to satisfy \( L/\lambda_{r+1}^* \gg 1 \), in which case SketchySGD will be significantly faster than SGD. We shall see in the ridge-regression experiments in section [4] that the gap predicted corollary 1 is realized in practice.

4 **Numerical experiments**

In this section, we evaluate the performance of SketchySGD through three groups of experiments. The first and second group of experiments are in the ridge regression (Section [4.1]) and logistic regression settings (Section [4.2]), where we compare the performance of SketchySGD to SGD, SVRG, stochastic L-BFGS (SLBFGS), and loopless Katyusha (L-Katyusha). The third group of experiments is an ablation study showing the effect of changing the rank and preconditioner update frequency on the performance of SketchySGD.

In both regression settings, we run SketchySGD without tuning. For SketchySGD, we set \( \rho = 10^{-3} \), rank \( r = 1 \), and Hessian batch size \( b_h = 256 \). In ridge regression, the Hessian of the loss \( f \) is a constant, so we would expect the first preconditioner computed by SketchySGD to be effective throughout training. Since logistic regression and ridge regression are both GLMs, we would hope that leaving the preconditioner fixed in logistic regression leads to good results. Consequently, the preconditioner update frequency \( u \) is set to \( \infty \), i.e., we compute the low-rank approximation \( \hat{H}_{S_k} \) to the minibatch Hessian on the very first iteration, and set \( \hat{H}_{S_k} = \hat{H}_{S_0} \) for all \( k \).
We run SGD, SVRG, and SLBFGS with 10 learning rates selected by random search. We run L-Katyusha with 10 values for the smoothness parameter $L$ selected by random search. Furthermore, all of the optimizers are initialized at $w_0 = 0$ and run for 20 epochs. All results are averaged across 3 random seeds to reduce variability in the outcomes.

The figures we show are plotted with respect to wall-clock time. We plot up to the point where the second-fastest optimizer terminates, since we would like to determine which methods provide good results while also being time-efficient.

Additional details appear in Appendix D and code to reproduce our experiments may be found at the anonymous git repo https://anonymous.4open.science/r/SketchySGD

### 4.1 Ridge regression

We perform ridge regression on the YearPredictionMSD [Dua and Graff 2017], E2006-tfidf [Kogan et al. 2009], and w8a [Platt 1998] datasets from LIBSVM. All datasets are normalized to have unit row norm. We perform further preprocessing on YearPredictionMSD and w8a. In the case of YearPredictionMSD, we transform the data via ReLU random features [Mei and Montanari 2022], where the number of features is 1% of $n_{tr}$, where $n_{tr}$ is the size of the training set. In the case of w8a, we transform the data via random features [Rahimi and Recht 2007], where the number of features is 5% of $n_{tr}$, and the bandwidth is 0.05. We use a $\ell^2$-regularization of $10^{-2}/n_{tr}$.

We assess test accuracy on w8a by computing the signs of the predictions given by the model. For a given datapoint $a_i$, if the sign of the prediction matches the label $b_i$, we identify $a_i$ as being classified correctly by the model.

The results are presented in Figures 2 and 3. On all three datasets, SketchySGD converges to models with much better test loss than SGD, SVRG, SLBFGS, and L-Katyusha. SketchySGD converges to a model with good training loss at a faster rate than SGD. This improved convergence is an empirical confirmation that SketchySGD converges at a faster rate than SGD on the least-squares loss, as shown in Corollary 1.

| Method | Dataset       | YearPredictionMSD | E2006-tfidf | w8a      |
|--------|---------------|-------------------|-------------|----------|
| SGD    | 4.72·10^{-1} | 1.57·10^{-9}     | 1.25·10^{-9}|
| SVRG   | 4.39·10^{-3} | 3.64·10^{-9}     | 1.46·10^{-9}|
| SLBFGS | 1.97·10^{-2} | 2.54·10^{-9}     | 2.39·10^{-9}|
| L-Katyusha | 7.92·10^{-2} | 6.22·10^{-9}     | 3.61·10^{-9}|

We report the best hyperparameters in ridge regression (in terms of best test loss/accuracy attained during training) for SGD, SVRG, SLBFGS, and L-Katyusha in Table 1. We observe that the best hyperparameter varies significantly across different datasets, showing the difficulty of tuning all four of these optimizers in ridge regression.

### 4.2 Logistic regression

We train logistic regression classifiers on the rcv1 [Lewis et al. 2004], news20 [Lang 1995], and real-sim datasets from LIBSVM. All datasets are normalized to have unit row norm. We use a $\ell^2$-regularization of $10^{-2}/n_{tr}$, where $n_{tr}$ is the number of training samples.

| Method | Dataset      | rcv1       | news20     | real-sim   |
|--------|--------------|------------|------------|------------|
| SGD    | 7.07·10^{-1} | 1.24·10^{-1} | 4.31·10^{-1} |
| SVRG   | 7.37·10^{-1} | 5.68·10^{-1} | 1.12·10^{-1} |
| SLBFGS | 1.57·10^{-1} | 2.02·10^{-1} | 1.87·10^{-1} |
| L-Katyusha | 1.14·10^{-4} | 4.48·10^{-9} | 7.55·10^{-8} |

The results are presented in Figure 4. On all three datasets, SketchySGD converges to models with better or comparable test accuracy than SGD, SVRG, SLBFGS, and L-Katyusha. We report the best hyperparameters (in terms of best test accuracy attained during training) for SGD, SVRG, SLBFGS, and L-Katyusha in Table 2. For SGD, SVRG, and L-Katyusha, we observe that the best hyperparameter varies significantly across different datasets, showing the difficulty of tuning these three optimizers in logistic regression. Although the best learning rates for SLBFGS do not vary significantly over the datasets, Figure 4 shows that the performance of SLBFGS can vary wildly with the learning rate.
Figure 2: Performance of SketchySGD, SGD, SVRG, sLBFGS, and L-Katyusha on test sets in the ridge regression setting. The translucent curves correspond to the suboptimal runs of SGD, SVRG, SLBFGS, and L-Katyusha. The opaque curves correspond to the best runs of SGD, SVRG, SLBFGS, and L-Katyusha.

Figure 3: A comparison of training loss between SketchySGD, SGD, SVRG, sLBFGS, and L-Katyusha in the ridge regression setting. The curves for SGD, SVRG, SLFBGS, and L-Katyusha are selected based on the lowest training loss attained during training.

Figure 4: Performance of SketchySGD, SGD, SVRG, SLBFGS, and L-Katyusha in the logistic regression setting. The translucent curves correspond to the suboptimal runs of SGD, SVRG, SLBFGS, and L-Katyusha. The opaque curves correspond to the best runs of SGD, SVRG, SLBFGS, and L-Katyusha.
4.3 Ablation study

We investigate how changing the preconditioner update frequency $u$ and rank $r$ affects the performance of SketchySGD in both ridge and logistic regression. In the first set of ablation experiments, we select update frequencies $u \in \{0.5, 1, 2, 5, \infty\}$ epochs while holding the rank fixed at $r = 1$. In the second set of ablation experiments, we select ranks $r \in \{1, 2, 5, 10, 20, 50\}$ while holding the update frequency fixed at $u = 1$ epoch. We set $\rho = 10^{-3}$, Hessian batch size $b_{h_k} = 256$, and run SketchySGD for 20 epochs on each dataset in the regression experiments.

![Figure 5: Sensitivity of SketchySGD to preconditioner update frequency $u$ and rank $r$ on E2006-tfidf in ridge regression.](image1)

![Figure 6: Sensitivity of SketchySGD to preconditioner update frequency $u$ and rank $r$ on rcv1 in logistic regression.](image2)

Figures 5 and 6 show the results of the ablation study on the E2006-tfidf and rcv1 datasets. As expected, increasing the rank leads to a longer runtime, and increasing the update frequency leads to a shorter runtime. We see that larger update frequencies ($5$ and/or $\infty$) lead to the best performance when the rank is held fixed. In addition, increasing the rank does not appear to improve performance. For both datasets, setting the rank equal to $1$, $2$, or $5$ leads to similar performance, while increasing the rank beyond $5$ actually degrades performance. These trends generally hold on the other datasets; further results are provided in Appendix E.

5 Conclusion

In this paper, we have presented SketchySGD, a fast and robust stochastic quasi-Newton method for convex machine learning problems. SketchySGD uses subsampling and randomized low-rank approximation to improve conditioning by approximating the curvature of the loss. Furthermore, SketchySGD relies on a novel, automated learning rate which reduces hyperparameter tuning.

SketchySGD has strong benefits both in theory and in practice. In the least-squares setting, our theory shows that SketchySGD converges to a neighborhood of the optimum at a faster rate than SGD, and our experiments validate this improved convergence rate. SketchySGD outperforms or matches the performance of SGD, SVRG, SLBFGS, and L-Katyusha (the last three of which use variance reduction) with minimal hyperparameter tuning, even when optimizing the learning rate for the competing methods using random search.

\footnote{If we set $u = \infty$, which fixes the preconditioner throughout the run of SketchySGD, we may not see the full impact of varying $r$ due to a poor Hessian approximation at initialization.}
Given the great promise of SketchySGD, we believe there are several interesting directions for improving this algorithm. First, we can investigate sketching methods that preserve sparsity, which could improve the performance of SketchySGD in sparse problems. Second, we could investigate combining SketchySGD with variance-reduced methods like SVRG and L-Katyusha, which could lead to improved theoretical (e.g., linear convergence to the optimum at improved rates) and empirical results.

A RandNysApprox

In this section we provide the pseudocode for the RandNysApprox algorithm mentioned in Section 2, which SketchySGD uses to construct the low-rank approximation $\hat{H}_{S_k}$.

\begin{algorithm}[h]
\caption{RandNysApprox}
\begin{algorithmic}
\Require sketch $Y$ of $H_{S_k}$, orthogonalized test matrix $Q$, rank $r_k$
\State $\nu = \sqrt{\text{eps}(\|Y, 2\|)}$
\State $Y_\nu = Y + \nu Q$
\State $C = \text{chol}(Q^T Y_\nu)$
\State $B = Y/C$
\State $[\hat{V}, \Sigma, \sim] = \text{svd}(B, 0)$
\State $\hat{\Lambda} = \max\{0, \Sigma^2 - \nu I\}$
\Ensure $\hat{V}, \hat{\Lambda}$
\end{algorithmic}
\end{algorithm}

Algorithm 3 follows Tropp et al. [2017b]. eps(x) is defined as the positive distance between $x$ and the next largest floating point number of the same precision as $x$. The test matrix $Q$ is the same test matrix used to generate the sketch $Y$ of $H_{S_k}$. The resulting Nyström approximation $\hat{H}_{S_k}$ is given by $\hat{V}\hat{\Lambda}\hat{V}^T$. The resulting Nyström approximation is psd but may have eigenvalues that are equal to 0. In our algorithms, this approximation is always used in conjunction with a regularizer to ensure positive definiteness.

B Controlling minibatch gradient variance

Central to our analysis is the ability to control the variance of a gradient minibatch. A key quantity in this regards is the variance of the gradient at the optimum $\nu$:

$$\sigma_\nu^2 = \frac{1}{n} \sum_{i=1}^{n} \|g_i(\nu)\|^2.$$  

We may view $\sigma^2$ as measure of how well the model fits the training set. Indeed, in the extreme case when the model interpolates the dataset, $\sigma^2 = 0$. Thus the smaller $\sigma^2$, the better the model fit. The fact that $\sigma^2$ helps control the variance of minibatch gradient is shown in the following proposition due to Gower et al. [2019b].

**Proposition 5** (Proposition 3.8 [Gower et al., 2019b]). Suppose each $f_i$ is $L_i$-smooth and convex and that we form the gradient sample $g_{B_k}$ with batch-size $b_k$. Then for every $w \in \mathbb{R}^p$,

$$\mathbb{E}_{B_k} \|g_{B_k}(w)\|^2 \leq 4L(f(w) - f(\nu)) + 2\sigma^2,$$

where $L \leq n(b_0(n-1)L + \sum_{i=1}^{n-b_0}L_{\max} + \sigma^2 = \frac{1}{b_0} \frac{n-b_0}{n-1} \sigma_\nu^2$.

Proposition 5 shows that the variance of a minibatch gradient is controlled by the distance to the optimum, the batch-size $b_0$, and $\sigma_\nu^2$. We see that as we increase the batch-size $b_0$, the variance of the minibatch gradient decreases. More interestingly, as $\sigma_\nu^2$ decreases the minibatch gradient variance decreases, and the variance vanishes when the model class perfectly fits the data. As items 1. or 2. in Proposition 5 always hold in this paper, we may use this proposition to control the minibatch gradient variance, which is crucial for establishing Theorem 4.

C SketchySGD convergence

C.1 Preliminary technical lemmas

In this subsection, we set $P_k = \hat{H}_{S_k}^\beta$ in order to avoid notational clutter. Recall the SketchySGD update is given by

$$w_{k+1} = w_k - \eta_k P_k^{-1} g_{B_k},$$
where \( \mathbb{E}_{B_k}[g_{B_k}] = g_k \). We start by making the following observation about the SketchySGD update.

**Lemma 6** (SketchySGD is SGD in precond. space). At iteration \( k \) define \( f_{P_k}(z) = f(P_k^{-1/2}z) \), that is define the change of variable \( w = P_k^{-1/2}z \). Then,

\[
\begin{align*}
g_{P_k}(z) &= P_k^{-1/2}g(P_k^{-1/2}z) \\
H_{P_k}(z) &= P_k^{-1/2}H(w)P_k^{-1/2}.
\end{align*}
\]

Hence the SketchySGD update may be realized as

\[
\begin{align*}
z_{k+1} &= z_k - \eta_k \bar{g}_{P_k}(z_k) \\
w_{k+1} &= P_k^{-1/2}z_{k+1},
\end{align*}
\]

where \( \bar{g}_{P_k}(z_k) = P_k^{-1/2}g_{P_k}(P_k^{-1/2}z_k) \).

**Proof.** The first display of equations follow from the definition of the change of variable and the chain rule, while the last display follows from definition of the SketchySGD update and the first display. \( \square \)

The preceding lemma shows SketchySGD update may be realized by performing a stochastic gradient step in precond. space, and then converting back to the original space. This view will help in establishing the convergence rate of SketchySGD in the convex case.

**Definition 3** (Preconditioned gradient variance). We define \( \sigma^2_P := \frac{1}{b_n} \sup_{k \geq 0} \frac{1}{\eta} \sum_{i=1}^{n} \| P_k^{-1/2} \nabla f_i(w_*) \|^2 \).

The quantity \( \sigma^2_P \) is the analogue of \( \sigma^2 \) from Proposition 5 in preconditioned space. \( \sigma^2_P \) will be a critical quantity in showing convergence of SketchySGD in the convex case.

We will also need the following quantity that measures how the preconditioner acts on the Hessian of one example relative to the full Hessian.

**Definition 4** (Relative preconditioning constant). We define the relative preconditioning constant to be

\[
\gamma_{P_k} := \sup_{1 \leq i \leq n} \frac{\lambda_1(P_k^{-1/2} \nabla^2 f_i(w) P_k^{-1/2})}{\lambda_1(P_k^{-1/2} H(w) P_k^{-1/2})}.
\]

Finally, as the preconditioner \( P_k \) changes at each iteration, we define the following quantity that gives a uniform bound over the \( \gamma_{P_k} \)'s.

**Definition 5** (Uniform relative preconditioning constant). We define the uniform relative preconditioning constant to be

\[
\gamma_P := \sup_{k \geq 0} \gamma_{P_k}.
\]

We now show under assumption 1 and assumption 2 that the quantities \( \gamma_P \) and \( \sigma^2_P \) are finite, and hence are non-vacuous quantities.

**Lemma 7.** Under assumption 1 and assumption 2 the quantities \( \sigma^2_P \) and \( \gamma \) in definition 3 and definition 5 are finite.

**Proof.** For \( \sigma^2_P \), the claim follows as for every \( k \geq 0 \), \( \lambda_1(P_k^{-1/2}) \leq 1/\sqrt{p} \). So, by Cauchy-Schwarz we reach

\[
\sigma^2_P \leq \frac{\sigma^2}{\rho} < \infty.
\]

The reasoning for \( \gamma_P \) is similar. We have for any \( k \) and \( i \in \{1, \cdots, n\} \) that

\[
\begin{align*}
\frac{\lambda_1(P_k^{-1/2} \nabla^2 f_i(w) P_k^{-1/2})}{\lambda_1(P_k^{-1/2} H(w) P_k^{-1/2})} \leq \frac{\lambda_1(P_k^{-1/2} \nabla^2 f_i(w) P_k^{-1/2})}{\lambda_1(P_k^{-1/2} H(w) P_k^{-1/2})} \leq \frac{L_i/\rho}{\mu/\rho} \leq \frac{L_{\max}/\rho}{\mu/\rho}.
\end{align*}
\]
where (1) uses matrix similarity, (2) uses the identity \( \lambda_1(AB) \geq \lambda_1(A)\lambda_p(B) \) for symmetric positive definite matrices \( A, B \), and (3) uses strong convexity of \( f \) and \( L_i \leq L_{\text{max}} \) for all \( i \). Hence for every \( k \)

\[
\gamma_{P_k} \leq \frac{L_{\text{max}}}{\mu},
\]

from which we immediately conclude

\[
\gamma_P \leq \frac{L_{\text{max}}}{\mu} < \infty.
\]

When we go to preconditioned space induced by the preconditioner \( P_k \), the expected smoothness constant in proposition \( \mathbb{P} \) goes from \( \mathcal{L} \) to \( \mathcal{L}_{P_k} \). As we need to be able to control the mini-batch gradient variance in preconditioned space, the notion uniform preconditioned expected smoothness constant is useful, which leads to the following definition.

**Definition 6** (Uniform preconditioned expected smoothness constant). Let \( \mathcal{L}_{P_k} \) denote the preconditioned expected smoothness constant in proposition \( \mathbb{P} \) for iteration \( k \). Then the uniform preconditioned smoothness constant is defined by

\[
\mathcal{L}_P := \sup_{k \geq 0} \mathcal{L}_{P_k}.
\]

Similar to \( \gamma_P \) and \( \sigma_P^2 \), \( \mathcal{L}_P \) is also always finite, as shown by the following lemma.

**Lemma 8.** Under assumption \( \mathbb{P} \) and assumption \( \mathbb{Q} \) we have

\[
\mathcal{L}_P < \infty.
\]

**Proof.** For any \( k \), proposition \( \mathbb{P} \) the definition of \( \gamma_P \), and lemma \( \mathbb{Q} \) yield

\[
\mathcal{L}_{P_k} \leq \left( \frac{n(b_g - 1)}{b_g(n - 1)} + \frac{n - b_g}{b_g(n - 1)} \gamma_P \right) \mathcal{L}_{P_k},
\]

where \( \mathcal{L}_{P_k} = \sup_{w \in \mathbb{R}^p} \lambda_1(P_k^{-1/2}H(w)P_k^{-1/2}) \). Now any \( w \)

\[
\lambda_1(P_k^{-1/2}H(w)P_k^{-1/2}) \leq \frac{L}{\rho}.
\]

Hence for any \( k \), \( \mathcal{L}_{P_k} \leq L/\rho \), and so

\[
\mathcal{L}_P \leq \left( \frac{n(b_g - 1)}{b_g(n - 1)} + \frac{n - b_g}{b_g(n - 1)} \gamma_P \right) \frac{L}{\rho} < \infty.
\]

**Controlling the quality of the preconditioner** In what follows we show that \( \hat{H}^{\delta}_{S_k} \) is close to the exact Hessian \( H \) in the Loewner ordering, which shows it makes excellent preconditioner. Below, we state Lemma 12 from [Ye et al. 2021] that shows if we sample appropriately then the subsampled Hessian is close to the full Hessian.

**Lemma 9** (Closeness in Loewner ordering between \( H \) and \( H_S \)). Let \( H \) be the Hessian of \( f \), \( M(w) = \max_{1 \leq i \leq n} \| \nabla^2 f_i(w) \| \) and let \( \xi > 0 \) and \( \delta > 0 \) be given. Suppose we construct the subsampled Hessian \( H_S \) with \( |S| = O(\frac{M(w)}{\xi} \log(\frac{1}{\delta})) \) samples. Then for \( \zeta_0 = \max \{ \frac{3\xi + \mu}{2\xi + 4\mu}, \frac{1}{2} \} \in (0, 1) \) the event

\[
\mathcal{E}^{(1)} = \{(1 - \zeta_0)(H_S + \xi I) \preceq H \preceq (1 + \zeta_0)(H_S + \xi I)\}
\]

holds with probability at least \( 1 - \delta \).

Similarly, we may control the approximation quality of the randomized low-rank approximation used in SketchySGD via the following technical result from [Zhao et al. 2022].

**Lemma 10** (Controlling low-rank approximation error). Let \( \tau > 0 \) and \( E_k = H_{S_k} - \hat{H}_{S_k} \). Construct a randomized Nyström approximation from a standard Gaussian random matrix \( \Omega \) with rank \( r_k = O(d_{\text{eff}}(\tau) + \log(\frac{1}{\delta})) \). Then the event

\[
\mathcal{E}_k^{(2)} = \{ \| E_k \| \leq \tau \}
\]

holds with probability at least \( 1 - \delta \).
Lemma 11 allows us to establish the following result which shows our low-rank approximation is close to the regularized subsampled Hessian in the Loewner ordering.

**Lemma 11 (Closeness in Loewner ordering between $H_{S_k}^\xi$ and $\hat{H}_{S_k}^\xi$).** Let $0 < \zeta < 1$. Suppose at iteration $k$ SketchySGD uses a low-rank approximation $\hat{H}_{S_k}$ to $H_{S_k}$ with rank $r_k = \mathcal{O}(d_{\text{eff}}(\zeta\xi) + \log(\frac{1}{\delta}))$. Then with probability at least $1 - \delta$,

\[
(1 - \zeta_0)\hat{H}_{S_k}^\xi \preceq H_{S_k}^\xi \preceq (1 + \zeta_0)(1 + \zeta)\hat{H}_{S_k}^\xi
\]  

(9)

**Proof.** Let $E_k = H_{S_k} - \hat{H}_{S_k}$, and note by the properties of the Nyström approximation that $E_k \geq 0$ [Tropp et al., 2017b]. Now, notice the event

\[
\mathcal{E}_k^{(2)} = \{ \|E_k\| \leq \zeta \xi \}
\]

holds with probability at least $1 - \delta$ by Lemma 10. Henceforth, all analysis is conditioned on $\mathcal{E}_k$, and therefore the conclusions hold with probability at least $1 - \delta$. Now, the regularized Hessian and approximate Hessian satisfy

\[
H_{S_k}^\xi = \hat{H}_{S_k}^\xi + E_k,
\]

(10)

Let $P = \hat{H}_{S_k}^\xi$. Then combining (10) with Weyl's inequalities yields

\[
\lambda_1(P^{-1/2}H_{S_k}^\xi P^{-1/2}) \leq \lambda_1 \left( P^{-1/2}\hat{H}_{S_k}^\xi P^{-1/2} \right) + \lambda_1 \left( P^{-1/2}E_k P^{-1/2} \right)
\]

\[
= 1 + \|P^{-1/2}E_k P^{-1/2}\| \leq 1 + \|P^{-1/2}\|E_k\| \leq 1 + \frac{\|E_k\|}{\xi} \leq 1 + \zeta.
\]

To bound the smallest eigenvalue, observe that

\[
\hat{H}_{S_k}^\xi \succeq H_{S_k}^\xi \implies H_{S_k}^\xi \succeq P.
\]

Thus, conjugating the preceding relation by $P^{-1/2}$ we obtain

\[
P^{-1/2}H_{S_k}^\xi P^{-1/2} \preceq I_p,
\]

where $I_p$ is the $p \times p$ identity matrix. The preceding inequality immediately yields

\[
\lambda_1(P^{-1/2}H_{S_k}^\xi P^{-1/2}) \geq 1.
\]

Hence,

\[
1 \leq \lambda_p(P^{-1/2}H_{S_k}^\xi P^{-1/2}) \leq \lambda_1(P^{-1/2}H_{S_k}^\xi P^{-1/2}) \leq 1 + \zeta.
\]

As an immediate consequence, we obtain the Loewner ordering relation

\[
I_p \preceq P^{-1/2}H_{S_k}^\xi P^{-1/2} \preceq (1 + \zeta)I_p
\]

which we conjugate by $P^{1/2}$ to show

\[
\hat{H}_{S_k}^\xi \preceq H_{S_k}^\xi \preceq (1 + \zeta)\hat{H}_{S_k}^\xi.
\]

The claimed result follows. \hfill \square

Combining Lemma 9 and Lemma 11 immediately yields the following corollary. This corollary shows our regularized low-rank approximation is close to the true Hessian and so makes an excellent preconditioner.

**Corollary 2 (Closeness in Loewner ordering between $H$ and $\hat{H}$).** Instate the hypotheses of Lemma 9 and Lemma 11 with $\delta' = \frac{\delta}{2}$, where $\delta \in (0,1)$ is given. Then with probability at least $1 - \delta$

\[
(1 - \zeta_0)\hat{H}_{S_k}^\xi \preceq H \preceq (1 + \zeta_0)(1 + \zeta)\hat{H}_{S_k}^\xi.
\]

Hence

\[
1 - \zeta_0 \leq \lambda_p \left( (\hat{H}_{S_k}^\xi)^{-1/2}H(\hat{H}_{S_k}^\xi)^{-1/2} \right) \leq \lambda_1 \left( (\hat{H}_{S_k}^\xi)^{-1/2}H(\hat{H}_{S_k}^\xi)^{-1/2} \right) \leq (1 + \zeta_0)(1 + \zeta).
\]

**Corollary 3 (Union bound).** Let $\mathcal{E} = \bigcap_{k=0}^t \mathcal{E}_k$, where $\mathcal{E}_k = \mathcal{E}_k^{(1)} \cap \mathcal{E}_k^{(2)}$ and suppose that at iteration $k$ we construct $\hat{H}_{S_k}$ with rank $r_k = \tilde{O}(d_{\text{eff}}(\zeta\rho_k))$, where $\{\rho_k\}$ is the regularization sequence. Then

\[
\mathbb{P}(\mathcal{E}) \geq 1 - \delta.
\]
With these notational preliminaries out the way, we may now begin the proof.

Theorem 13

Nyström approximation with rank $r$.

The desired inequality now follows from the preceding display and Lemma 6.

For convenience in the proof below we define the following quantities:

Proof.

Rearranging, we find

$\text{Lemma 12 (Gradient-based suboptimality bound). Assume the conclusion of Corollary 2 holds. Then} $

$2(1 - \zeta_0)\hat{\mu}(f(w_k) - f(w_*)) \leq \|P_k^{-1/2}g_k\|^2.$

Proof.

Recall $f_{P_k}(z) = f(P_k^{-1/2}w)$ so Lemma 6 and relative convexity of $f(L)$ yield

$\min_{z'\in\{g_{P_k}(z_k), z_* - z_k\}} f_{P_k}(z_k) + \langle g_{P_k}(z_k), z' - z_k \rangle = f_{P_k}(z_k) - \frac{1}{2(1 - \zeta_0)\hat{\mu}}\|g_{P_k}(z_k)\|^2.$

The desired inequality now follows from the preceding display and Lemma 6.

Proof. Before giving the proof we provide a more detailed statement of Theorem 4, which includes the exact value of $\eta$.

Theorem 13 (SketchySGD convergence). Instate Assumptions 1 and 2. Let $\zeta_0 = \frac{3\lambda_r^{\mu+1} + \mu}{3\lambda_r^{\mu+1} + 4\mu}$ and let $\sigma_P^2$, $L_P$ be as in definition 3 and definition 6. Run Algorithm 1 with batch sizes $b_g$, $b_h = O\left(\frac{M(w_k)}{\lambda_r^{\mu+1}} \log(\frac{B}{\delta})\right)$, stepsize $\eta = \frac{(1 - \zeta_0)\hat{\mu}}{(1 + \zeta_0)(1 + \zeta_0)\hat{\mu}L_P}$, regularization $\rho = \lambda_r^{\mu+1}$, and at each iteration constructing the randomized Nyström approximation with rank $r_k = O(d_{\text{dim}}(\lambda_r^{\mu+1} + \log(\frac{B}{\eta}))$ where $r > 0$ is fixed. Further, condition on the event $\mathcal{E} = \cap_{t=1}^T \mathcal{E}_t$, which holds with probability at least $1 - \delta$. Additionally, suppose $\epsilon \leq \sigma_P^2 / ((1 - \zeta_0)L_P\hat{\mu})$. Then after $t \geq (1 + \zeta)(2 + \frac{3\lambda_r^{\mu+1} + \mu}{\mu}) \left(\frac{2(1 + \zeta_0)\hat{\mu}}{\mu}\log\left(\frac{2(f(w_k) - f(w_*))}{\epsilon}\right)\right)$ iterations,

$\mathbb{E}[f(w_t)] - f(w_*) \leq \epsilon.$

Proof. For convenience in the proof below we define the following quantities:

$P_k := H_b^{\lambda_r^{\mu+1}}$, 

$\hat{L}_P := (1 + \zeta)(1 + \zeta_0)\hat{L}$, 

$\hat{\mu}_P := (1 - \zeta_0)\hat{\mu}$.

With these notational preliminaries out the way, we may now begin the proof. Observe that relative smoothness implies the inequality

$f(w_{k+1}) \leq f(w_k) - \eta(g_k, P_k^{-1}g_{B_k}) + \frac{\eta^2\hat{L}}{2}\|P_k^{-1}g_{B_k}\|^2_{H_k}$.
Hence

\[
f(w_{k+1}) \leq f(w_k) - \eta(g_k, \nabla f_k) + \frac{\eta^2 \hat{L}}{2} \|P_k^{-1/2} g_{B_k}\|^2 + \frac{\eta^2 (1 + \zeta_0)(1 + \zeta) \hat{L}}{2} \|P_k^{-1/2} g_{B_k}\|^2
\]

\[(i)\]

\[
f(w_k) - \eta(g_k, \nabla f_k) + \frac{\eta^2 \hat{L}}{2} \|P_k^{-1/2} g_{B_k}\|^2 = f(w_k) - \eta(g_k, \nabla f_k) + \frac{\eta^2 \hat{L}}{2} \|P_k^{-1/2} g_{B_k}\|^2
\]

\[(ii)\]

\[
f(w_k) - \eta(g_k, \nabla f_k) + \frac{\eta^2 \hat{L}}{2} \|P_k^{-1/2} g_{B_k}\|^2 = f(w_k) - \eta(g_k, \nabla f_k) + \frac{\eta^2 \hat{L}}{2} \|P_k^{-1/2} g_{B_k}\|^2
\]

where (i) follows from Corollary 2 whose conclusions hold as we are conditioned on \( E \), and (ii) follows from definition of \( \hat{L}_P \). Next we take the expectation with respect to \( B_k \) to obtain

\[
\mathbb{E}_{B_k}[f(w_{k+1})] \leq f(w_k) - \eta \|P_k^{-1/2} g_k\|^2 + \frac{\hat{L}_P \eta^2}{2} \mathbb{E}_{B_k}[\|P_k^{-1/2} g_{B_k}\|^2].
\]

We now bound \( \mathbb{E}_{B_k}[\|P_k^{-1/2} g_{B_k}\|^2] \). By Lemma 6 we have \( \mathbb{E}_{B_k}[\|P_k^{-1/2} g_{B_k}\|^2] = \mathbb{E}_{B_k}[\|g_{P_k}(z_k)\|^2] \). Thus, we may apply the expected smoothness condition (Proposition 5) on the stochastic gradient of the preconditioned objective to reach

\[
\mathbb{E}_{B_k}[\|P_k^{-1/2} g_{B_k}\|^2] \leq 4\mathcal{L}_P(f_{P_k}(z_k) - f_{P_k}(z_*)) + 2\sigma_P^2.
\]

Hence

\[\begin{align*}
\mathbb{E}_{B_k}[f(w_{k+1})] &\leq f(w_k) - \eta \|P_k^{-1/2} g_k\|^2 + \frac{\hat{L}_P \eta^2}{2} (4\mathcal{L}_P(f_{P_k}(z_k) - f_{P_k}(z_*)) + 2\sigma_P^2) \\
&\leq f(w_k) - 2\eta \hat{\mu} (1 - \eta \hat{L}_P \mathcal{L}_P)(f(w_k) - f(w_*)) + \frac{\hat{L}_P \eta^2}{2} 2\sigma_P^2 \\
&\leq (1 - \eta \hat{\mu} P)(f(w_k) - f(w_*)) + \hat{L}_P \eta^2 \sigma_P^2. \tag{3}
\end{align*}\]

Where (1) inserts the expected smoothness bound, (2) applies Lemma 12 and combines like terms, and (3) uses that \( \eta \leq 1/(2\mathcal{L}_P \hat{L}_P) \). Now, taking expectations over all time-steps and recursing we find

\[\mathbb{E}[f(w_{k+1})] - f(w_*) \leq (1 - \eta \hat{\mu} P)^k (f(w_0) - f(w_*)) + \frac{\hat{L}_P \eta^2 \sigma_P^2}{\hat{\mu} P}.\]

Using that \( \eta \leq \hat{\mu} P / (2\hat{L}_P \sigma_P^2) \), the last display simplifies to

\[\mathbb{E}[f(w_t)] - f(w_*) \leq (1 - \eta \hat{\mu} P)^t (f(w_0) - f(w_*)) + \frac{\epsilon}{2}.
\]

As \( \eta = \min\{1/(2\mathcal{L}_P \hat{L}_P), \hat{\mu} P / (2\hat{L}_P \sigma_P^2)\} \), it follows immediately from the last display that

\[\mathbb{E}[f(w_t)] - f(w_*) \leq \epsilon
\]

after \( t \geq (1 + \zeta) \frac{1 + \zeta_0}{1 - \zeta_0} \max\left\{2\mathcal{L}_P, \frac{2\sigma_P^2}{(1 - \zeta_0) \hat{\mu} P}\right\} \log \left(\frac{2(f(w_0) - f(w_*))}{\epsilon}\right) \) iterations. After some algebra we find

\[
1 + \zeta_0 - \frac{\zeta_0}{1 - \zeta_0} = 2 + 3\lambda_{r+1}^0
\]

Now if \( \epsilon \leq \sigma_P^2 / ((1 - \zeta_0) \hat{\mu} \mathcal{L}_P) \), the maximum in the bound on the iteration complexity is attained by the second term. Thus, after \( t \geq (1 + \zeta) \left(2 + 3\lambda_{r+1}^0\right) \frac{\hat{L}_P \sigma_P^2}{(1 - \zeta_0) \hat{\mu} P} \log \left(\frac{2(f(w_0) - f(w_*))}{\epsilon}\right) \) iterations

\[\mathbb{E}[f(w_t)] - f(w_*) \leq \epsilon,
\]

yielding the complexity statement in the theorem. \( \square \)
C.3 Proof of Corollary

Assuming that $f$ is $L$-smooth and $\mu$-strongly convex, using the same logic (only simpler) as in the proof of theorem, it is not hard to show the following theorem.

**Theorem 14.** Let $f$ satisfy assumption[1] and assumption[2] and let $\mathcal{L}$ and $\sigma^2$ be defined as in Proposition[5]. Let $\epsilon > 0$ and suppose we run SGD for $T$ iterations with step size $\gamma = \frac{\mu}{\epsilon} \min\{\frac{r}{2\sigma^2}, 1/(2\mathcal{L}\mu)\}$. Then

$$T \geq \frac{L}{\mu} \max \left\{ \frac{2\sigma^2}{\mu\epsilon}, 2\mathcal{L} \right\} \log \left( \frac{2(f(w_0) - f(w_*))}{\epsilon} \right) \implies \mathbb{E}[f(w_T)] - f(w_*) \leq \epsilon.$$

Now we provide the proof of Corollary. The idea of the proof is to compare the number of iterations that guarantee SketchySGD and SGD to reach an $\epsilon$-suboptimal point in expectation.

**Proof.** From the proof of Theorem[13] for sufficiently small $\epsilon$, after

$$(1 + \zeta) \frac{1 + \zeta_0}{1 - \zeta_0} \mathcal{L} \frac{2\sigma^2}{\mu\epsilon} \log \left( \frac{2(f(w_0) - f(w_*))}{\epsilon} \right) =: T_{\text{SketchySGD}}$$

iterations, SketchySGD is guaranteed to reach an $\epsilon$-suboptimal point in expectation. Similarly, Theorem[14] guarantees that for sufficiently small $\epsilon$, SGD will reach an $\epsilon$-suboptimal point in expectation after

$$\frac{L}{\mu} \frac{2\sigma^2}{\mu\epsilon} \log \left( \frac{2(f(w_0) - f(w_*))}{\epsilon} \right) =: T_{\text{SGD}}$$

iterations. Taking the ratio of $T_{\text{SGD}}$ to $T_{\text{SketchySGD}}$, we find

$$\frac{T_{\text{SGD}}}{T_{\text{SketchySGD}}} = \frac{\sigma^2}{\mathcal{L}} \frac{\mu}{\mu_{\epsilon}} \frac{1 - \zeta_0}{1 + \zeta_0} \left( \frac{3\lambda^*_r + 1}{\mu} + 2 \right)^{-1},$$

where we used $\kappa = \frac{L}{\mu}, \hat{\kappa} = \frac{L}{\mu}$, and $\frac{1 + \zeta_0}{1 - \zeta_0} = \frac{3\lambda^*_r + 1}{\mu} + 2$. Since $\sigma^2 \leq \frac{1}{\lambda^*_r + 1} \sigma^2$, we have

$$\frac{T_{\text{SGD}}}{T_{\text{SketchySGD}}} \geq \lambda^*_r \frac{\mu}{\mu_{\epsilon}} \frac{1 - \zeta_0}{1 + \zeta_0} \left( \frac{3\lambda^*_r + 1}{\mu} + 2 \right)^{-1}.$$

We now develop a lower bound on the term $\frac{1 - \zeta_0}{1 + \zeta_0}$. Since $0 < \zeta < 1$, $\frac{1 - \zeta_0}{1 + \zeta_0} > \frac{1}{2}$. In addition, $1 - \zeta_0 = \frac{2\mu}{3\zeta_{r+3}} > \frac{\mu}{3\lambda^*_r + 1}$, since $\lambda^*_{r+1} \geq \mu$. Therefore

$$\frac{T_{\text{SGD}}}{T_{\text{SketchySGD}}} \geq \lambda^*_r \frac{\mu}{\mu_{\epsilon}} \frac{1 - \zeta_0}{1 + \zeta_0} \left( \frac{3\lambda^*_r + 1}{\mu} + 2 \right)^{-1} = \frac{\hat{\mu}}{\hat{\kappa}} \frac{\mu}{\mu_{\epsilon}} \frac{1 - \zeta_0}{1 + \zeta_0} \left( \frac{3\lambda^*_r + 1}{\mu} + 2 \right)^{-1}.$$

Since $\lambda^*_r \geq \mu, \frac{3\lambda^*_r + 1}{\mu} + 2 \leq \frac{3\lambda^*_r + 1}{\mu} + \frac{2\lambda^*_r + 1}{\mu} = \frac{5\lambda^*_r + 1}{\mu}$. Substituting this inequality into the previous display gives

$$\frac{T_{\text{SGD}}}{T_{\text{SketchySGD}}} \geq \lambda^*_r \frac{\mu}{\mu_{\epsilon}} \frac{1 - \zeta_0}{1 + \zeta_0} \left( \frac{5\lambda^*_r + 1}{\mu} \right)^{-1} = \frac{\mu}{\mu_{\epsilon}} L \frac{5\lambda^*_r + 1}{\mu},$$

which is the claimed ratio. For least-squares problems, $\hat{\mu} = \hat{\kappa} = 1$, implying $\frac{T_{\text{SGD}}}{T_{\text{SketchySGD}}} \geq \frac{L}{30\lambda^*_r} = \lambda^*_r (H) / 30\lambda^*_r (H)$. \qed

**D Experimental details**

Here we provide more details regarding the ridge (Section 4.1) and logistic regression (Section 4.2) experiments.
Additional hyperparameters For SVRG and SLFBGS we perform a full gradient computation at every epoch. For SLFBGS we update the inverse Hessian approximation every epoch and set the Hessian batch size to 256. In addition, we follow [Moritz et al., 2016] and set the memory size of SLFBGS to 10. For L-Katyusha, we initialize the update probability \( p = \frac{b_g}{n} \) to ensure the average distance between full gradient computations is one epoch. We follow [Kovalev et al., 2020] and set \( \mu = \text{ridge parameter} \), \( \sigma_L = \mu \), \( \theta_1 = \min\left\{ \sqrt{\frac{2\sigma n}{3}}, \frac{1}{2} \right\} \), and \( \theta_2 = \frac{1}{2} \). All algorithms use a batch size of 256 for computing stochastic gradients.

Random search ranges After row normalization, the smoothness constant \( L_f \) of the loss function \( f \) is less than or equal to 1 in the case of ridge regression. As a result, a step size \( \eta < \frac{1}{L_f} < 1 \) is guaranteed to converge in the case of full gradient descent. However, this theoretical guarantee can be conservative. Consequently, we set \( [10^{-3}, 10^2] \) as the search range for the step size in SGD and SVRG. Following a similar idea, we set \( [10^{-5}, 10^0] \) as the search range for the smoothness parameter \( L \) in L-Katyusha. For SLBFGS, we set the search range to be \( [10^{-5}, 10^0] \) to have the same log-width as the search range for SGD and SVRG. Similarly, the smoothness constant \( L_f \) is less than or equal to 0.25 in logistic regression. The search ranges for SGD/SVRG, L-Katyusha, and SLBFGS become \( [4 \cdot 10^{-3}, 4 \cdot 10^2], [2.5 \cdot 10^{-3}, 2.5 \cdot 10^{-1}], \) and \( [4 \cdot 10^{-5}, 4 \cdot 10^0] \), respectively.

Ridge regression datasets The ridge regression experiments are run on the datasets described in the main text. For YearPredictionMSD we use a ReLU random features transformation that gives us 4367 features in total. For w8a we use a random features transformation with bandwidth \( 0.05 \) that gives us 2487 features in total. In Table 3, we provide the dimensions of the datasets, where \( n_{tr} \) is the number of training samples, \( n_{test} \) is the number of testing samples, and \( d \) is the number of features.

| Dataset          | \( n_{tr} \) | \( n_{test} \) | \( d \) |
|------------------|--------------|----------------|-------|
| YearPredictionMSD| 463715       | 41630          | 4367  |
| E2006-tfidf      | 16087        | 3308           | 150360|
| w8a              | 49749        | 14951          | 2487  |

Logistic regression datasets The logistic regression experiments are run on the datasets described in the main text. For news20 and real-sim, we use a random 80-20 split to form a training and test set. In Table 4, we provide the dimensions of the datasets, where \( n_{tr} \) is the number of training samples, \( n_{test} \) is the number of testing samples, and \( d \) is the number of features.

| Dataset | \( n_{tr} \) | \( n_{test} \) | \( d \) |
|---------|--------------|----------------|-------|
| rcv1    | 20242        | 667399         | 47236 |
| news20  | 15997        | 3999           | 1355191|
| real-sim| 57847        | 14462          | 20958 |

E Additional ablation experiments

Figures 7 to 10 show how changing the preconditioner update frequency \( u \) and rank \( r \) impacts the performance of SketchySGD. We observe that increasing the rank does not lead to improvements in performance; in fact, increasing the rank can degrade performance. In most cases, holding the preconditioner fixed throughout leads to the best results. However, an update frequency of 5 epochs works better for w8a and news20.

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Figure 9: Sensitivity of SketchySGD to preconditioner update frequency $u$ and rank $r$ on news20 in logistic regression.

Figure 10: Sensitivity of SketchySGD to preconditioner update frequency $u$ and rank $r$ on real-sim in logistic regression.

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