SCHUR-CONVEXITY OF INTEGRAL ARITHMETIC MEANS OF CO-ORDINATED CONVEX FUNCTIONS IN $\mathbb{R}^3$

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Abstract. In this paper, we investigate Schur-convexity of some functions which are obtained from the co-ordinated convex functions on a rectangular box in $\mathbb{R}^3$.

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1. Introduction

The first study of Schur-convexity was done by Issai Schur in 1923. Since then numerous articles have been written about it, see for example [3, 4, 9, 10]. Schur-convexity has many important applications in analytic and geometric inequality, combinatorial analysis, combinatorial optimization, matrix theory, information theory, and other fields. We recall some definitions as follows:

Definition 1.1. [1] Suppose that $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. $x$ is said to be majorized by $y$ (in symbols $x \prec y$) if

$$\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i] \quad k = 1, 2, \ldots, n - 1,$$

and

$$\sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i],$$

where $x[i]$, denotes the i-th largest component in $x$.

Definition 1.2. [1] Let $E \subset \mathbb{R}^n$, $f : E \to \mathbb{R}$ is said to be Schur-convex function on $E$ if $x \prec y$ on $E$ implies $f(x) \leq f(y)$. $f$ is said to be Schur-concave if and only if $-f$ is Schur-convex.

Definition 1.3. [1, 8] (i) A set $E \subset \mathbb{R}^n$ is called symmetric, if $x \in E$ implies $Px \in E$ for every $n \times n$ permutation matrix $P$.

(ii) A function $f : E \to \mathbb{R}$ is is said to be a symmetric function if $f(Px) = f(x)$ for every permutation matrix $P$, and for every $x \in E$.

Recall that a $n \times n$ square matrix $P$ is said to be a permutation matrix if each row and column has a single unite entry, and all other entries are zero. The following theorem called the schur’s condition, is very useful for specifying Schur-convexity or Schur-concavity of functions.

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**Theorem 1.4.** [1] Let $E \subset \mathbb{R}^n$ be a symmetric convex set with nonempty interior ($E^o$ is the interior of $E$), and $f : E \to \mathbb{R}$ is a symmetric continuous function on $E$. If $f$ is differentiable on $E^o$, then $f$ is Schur-convex (Schur-concave) on $E^o$ if and only if
\[ (x_1 - x_2) \left( \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) \geq 0 \quad (\leq 0), \]
for every $x = (x_1, x_2, \ldots, x_n) \in E^o$.

In [5], S.S. Dragomir defined convex function on the co-ordinates (or co-ordinated convex functions) on the set $[a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows:

**Definition 1.5.** A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be convex on the co-ordinates on $[a, b] \times [c, d]$ if for every $y \in [c, d]$ and $x \in [a, b]$, the partial mappings,
\[ f_y : [a, b] \to \mathbb{R}, \quad f_y(u) = f(u, y), \]
and
\[ f_x : [c, d] \to \mathbb{R}, \quad f_x(v) = f(x, v), \]
are convex. This means that for every $(x, y), (z, w) \in [a, b] \times [c, d]$ and $t, s \in [0, 1],$
\[ f \left( tx + (1-t)z, sy + (1-s)w \right) \leq tf(x, y) + s(1-t)f(z, y) + t(1-s)f(x, w) + (1-t)(1-s)f(z, w). \]

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. The following Hermite-Hadamard type inequality for co-ordinated convex functions was also proved in [5].

**Theorem 1.6.** Suppose that $f : [a, b] \times [c, d] \to \mathbb{R}$ is convex on the co-ordinates on $[a, b] \times [c, d]$. Then,
\[
f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c + d}{2} \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy \right] \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \leq \frac{1}{4} f(a, c) + f(a, d) + f(b, c) + f(b, d).
\]
The above inequalities are sharp.

In [7], M.E. Özdemir defined convex function on a rectangular box $\Omega = [a, b] \times [c, d] \times [e, f]$ in $\mathbb{R}^3$ as follows: A function $f : \Omega \to \mathbb{R}$ is said to be convex on the co-ordinates on $\Omega$ if for every $(x, y) \in [a, b] \times [c, d], (x, z) \in [a, b] \times [e, f]$ and $(y, z) \in [c, d] \times [e, f]$, the partial mappings,
\[ f_z : [a, b] \times [c, d] \to \mathbb{R}, \quad f_z(u, v) = f(u, v, z), \quad z \in [e, f], \]
\[ f_y : [a, b] \times [e, f] \to \mathbb{R}, \quad f_y(u, w) = f(u, y, w), \quad y \in [c, d], \]
\[ f_x : [c, d] \times [e, f] \to \mathbb{R}, \quad f_x(v, w) = f(x, v, w), \quad x \in [a, b], \]
are convex. The following theorem is given in [7].
Theorem 1.7. Suppose that \( f : \Omega = [a, b] \times [c, d] \times [e, f] \rightarrow \mathbb{R} \) is convex on the co-ordinates on \( \Omega \). Then one has the inequalities:

\[
\begin{align*}
&f\left(\frac{a+b}{2}, \frac{c+d}{2}, \frac{e+f}{2}\right) \leq \frac{1}{(b-a)(d-c)(f-e)} \int \int \int_{\Omega} f(x, y, z) dy dz dx \\
&\leq \frac{1}{6} \left[ \frac{1}{(b-a)(d-c)} \int_{\Delta_1} f(x, y, e) dy dx \\
&\quad + \frac{1}{(b-a)(d-c)} \int_{\Delta_1} f(x, y, f) dy dx \\
&\quad + \frac{1}{(b-a)(f-e)} \int_{\Delta_2} f(x, c, z) dz dx \\
&\quad + \frac{1}{(b-a)(f-e)} \int_{\Delta_2} f(x, d, z) dz dx \\
&\quad + \frac{1}{(d-c)(f-e)} \int_{\Delta_3} f(a, y, z) dz dy \\
&\quad + \frac{1}{(d-c)(f-e)} \int_{\Delta_3} f(b, y, z) dz dy \right]
\end{align*}
\]

where \( \Delta_1 = [a, b] \times [c, d], \Delta_2 = [a, b] \times [e, f] \) and \( \Delta_3 = [c, d] \times [e, f] \).

In [6] Elezović and Pečarić investigated the Schur-convexity on the upper and the lower limit of the integral for the mean of convex function and proved the following important result by using the Hermite-Hadamard inequality.

Theorem 1.8. Let \( f \) be a continuous function on an interval \( I \), and

\[
F(x, y) = \begin{cases} 
\frac{1}{y-x} \int_x^y f(t) dt, & x, y \in I, \, x \neq y, \\
f(x), & x = y \in I.
\end{cases}
\]

Then \( F(x, y) \) is Schur-convex (Schur-concave) on \( I^2 \) if and only if \( f \) is convex (concave) on \( I \).

Let \( I \subset \mathbb{R} \) be an open interval and \( f \in C^2(I) \). In [3] Y. Chu et al. proved the following theorem.

Theorem 1.9. Let \( f : I \rightarrow \mathbb{R} \) be a continuous function. The function

\[
F(x, y) = \begin{cases} 
\frac{1}{y-x} \int_x^y f(t) dt - f\left(\frac{x+y}{2}\right), & x, y \in I, \, x \neq y, \\
0, & x = y \in I,
\end{cases}
\]

is Schur-convex (Schur-concave) on \( I^2 \) if and only if \( f \) is convex (concave) on \( I \).

We recall the following lemma from [2], which is known as Leibniz’s Formula.

Lemma 1.10. Suppose that \( f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R} \) and \( \frac{\partial f}{\partial t} : [a, b] \times [c, d] \rightarrow \mathbb{R} \) are continuous and \( \alpha_1, \alpha_2 : [c, d] \rightarrow [a, b] \) are differentiable functions. Then, the function \( \varphi : [c, d] \rightarrow \mathbb{R} \) defined by

\[
\varphi(t) = \int_{\alpha_1(t)}^{\alpha_2(t)} f(x, t) dx,
\]

is \( C^1 \)-continuous on \( [c, d] \).
has a derivative for each \( t \in [c,d] \), which is given by

\[
\varphi'(t) = f(\alpha_2(t), t)\alpha_2'(t) - f(\alpha_1(t), t)\alpha_1'(t) + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{\partial f}{\partial t}(x,t)dx.
\]

Moreover, we use the following lemma which will be useful in the sequel. A version of the following lemma proved in [9].

**Lemma 1.11.** Let \( F(u,v) = \int_u^v \int_u^v f(x,y,z)dx dy dz \), where \( f(x,y,z) \), \( \frac{\partial}{\partial b} \int_u^v f(x,y,z)dx \), and \( \frac{\partial}{\partial b} \int_u^v \int_u^v f(x,y,z)dydz \) are continuous on the cube \( \Omega = [a,p] \times [a,p] \times [a,p] \), \( u = u(b) \) and \( v = v(b) \) are differentiable with \( a \leq u(b) \leq p \) and \( a \leq v(b) \leq p \). Then,

\[
\frac{\partial F}{\partial b} = \left( \int_u^v \int_u^v f(x,y,v)dx dy + \int_u^v \int_u^v f(x,v,z)dx dz \right) v'(b) - \left( \int_u^v \int_u^v f(x,y,u)dx dy + \int_u^v \int_u^v f(u,y,z)dy dz \right) u'(b).
\]

**Proof.** If \( G(u,v,y,z) = \int_u^v f(x,y,z)dx \) and \( H(u,v,y,z) = \int_u^v G(u,v,y,z)dy \) then \( F(u,v) = \int_u^v H(u,v,y,z)dz \). Therefore by Lemma 1.10, we have

\[
\frac{\partial F}{\partial b} = H(u,v,v) v'(b) - H(u,v,u) u'(b) + \int_u^v \frac{\partial H(u,v,y,z)}{\partial b} dy,
\]

\[
\frac{\partial H(u,v,y,z)}{\partial b} = G(u,v,v,z) v'(b) - G(u,v,u,z) u'(b) + \int_u^v \frac{\partial G(u,v,y,z)}{\partial b} dy,
\]

\[
\frac{\partial G(u,v,y,z)}{\partial b} = f(v,y,z) v'(b) - f(u,y,z) u'(b).
\]

By replacing (1.3) and (1.4) in (1.2) we obtained required result in (1.1). \( \square \)

## 2. Main Results

In this section we prove new theorems like those Theorem 1.8 and Theorem 1.9 for co-ordinated convex functions.

To reach our main results, we need the following two lemmas.

**Lemma 2.1.** Let \( \Omega := [a_1,b_1] \times [a_1,b_1] \times [a_1,b_1] \) be a cube in \( \mathbb{R}^3 \) with \( a_1 < b_1 \), and the function \( f : \Omega \to \mathbb{R} \) is continuous, and has continuous second order partial derivatives on \( \Omega^o \) (
Then, by using the L’Hospital’s rule, and Lemmas 1.10, 1.11 we see that

\[
\text{the interior of } \Omega \}. \text{ Choose } a, b \in (a_1, b_1), \text{ with } a < b, \text{ and let } D := [a, b] \times [a, b]. \text{ Suppose that }
\]

the function \( F : D \to \mathbb{R} \) is defined by

\[
F(x, y) := \begin{cases} 
\frac{1}{(y-x)^2} \int_x^y \int_x^y f(r, s, t) dr ds dt, & x \neq y, \quad x, y \in [a, b], \\
\int f(x, x, ), & x = y, \quad x, y \in [a, b].
\end{cases}
\]

Then,

\[
\frac{\partial F}{\partial x}(t_0, t_0) = \frac{\partial F}{\partial y}(t_0, t_0) = \frac{1}{24} \left[ 6 \frac{\partial f}{\partial t}(t, t) \bigg|_{t_0} + 2 \left( g_1(t_0, t_0) + g_2(t_0, t_0) + g_3(t_0, t_0) + f_1(t_0, t_0) + f_2(t_0, t_0) + f_3(t_0, t_0) \right) \right],
\]

for all \( t_0 \in [a, b] \), where

\[
f_1(u, v, t_0 + t) = \frac{\partial f}{\partial t}(u, v, t_0 + t),
\]

\[
f_2(u, t_0 + t, w) = \frac{\partial f}{\partial t}(u, t_0 + t, w),
\]

\[
f_3(t_0 + t, v, w) = \frac{\partial f}{\partial t}(t_0 + t, v, w),
\]

and

\[
g_1(u, t_0 + t_0 + t) = \frac{\partial f}{\partial t}(u, t_0 + t_0 + t),
\]

\[
g_2(t_0 + t, v, t_0 + t) = \frac{\partial f}{\partial t}(t_0 + t, v, t_0 + t),
\]

\[
g_3(t_0 + t_0 + t, w) = \frac{\partial f}{\partial t}(t_0 + t_0 + t, w).
\]

**Proof.** Fix \( t_0 \in [a, b] \). We put

\[
h_1(u, v, t_0 + t) = \frac{\partial f_1}{\partial t}(u, v, t_0 + t),
\]

\[
h_2(u, t_0 + t, w) = \frac{\partial f_2}{\partial t}(u, t_0 + t, w),
\]

\[
h_3(t_0 + t, v, w) = \frac{\partial f_3}{\partial t}(t_0 + t, v, w).
\]

By using the L’Hospital’s rule, and Lemmas 1.10, 1.11 we see that

\[
\frac{\partial F}{\partial x}(t_0, t_0) = \lim_{t \to 0} \frac{F(t_0 + t, t_0) - F(t_0, t_0)}{t} = \lim_{t \to 0} \frac{1}{t^4} \left[ \int_{t_0}^{t_0+t} \int_{t_0}^{t_0+t} f(u, v, w) dw dv - t^3 f(t_0, t_0) \right] = \lim_{t \to 0} \frac{1}{4t^4} \left[ \int_{t_0}^{t_0+t} \int_{t_0}^{t_0+t} f(u, v, t_0 + t) dw dv + \int_{t_0}^{t_0+t} \int_{t_0}^{t_0+t} f(u, t_0 + t, w) dw + \int_{t_0}^{t_0+t} \int_{t_0}^{t_0+t} f(t_0 + t, v, w) dw dv - 3t^2 f(t_0, t_0) \right].
\]
By changing the role of $x$ by $y$ in (2.2), we obtain required results in (2.1).

The proof of the following lemma is similar to once in lemma 2.1 hence we omit it.

**Lemma 2.2.** Let $\Omega := [a_1, b_1] \times [a_1, b_1] \times [a_1, b_1]$ be a cube in $\mathbb{R}^3$ with $a_1 < b_1$, and the function $f : \Omega \rightarrow \mathbb{R}$ is continuous, and has continuous four order partial derivatives on $\Omega^6$. Choose

...
a, b ∈ (a₁, b₁), with a < b, and let D := [a, b] × [a, b]. Suppose that the function G : D → ℝ is defined by
\[
G(x, y) := \begin{cases}
\frac{1}{(y-x)^3} \int_x^y \int_x^y f(r, s, t) dr ds dt \\
-f(\frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}), & x \neq y, x, y \in [a, b], \\
0, & x = y, x, y \in [a, b].
\end{cases}
\]
Then,
\[
\frac{\partial G}{\partial x} \bigr|_{(t_0, t_0)} = \frac{\partial G}{\partial y} \bigr|_{(t_0, t_0)} = \frac{1}{24} \left[ -3 \frac{\partial f}{\partial t}(t, t, t) \big|_{t_0} + 2 \left( g_1(t_0, t_0, t_0) + g_2(t_0, t_0, t_0) + g_3(t_0, t_0, t_0) + f_1(t_0, t_0) + f_2(t_0, t_0) + f_3(t_0, t_0) \right) \right],
\]
for all \( t_0 \in [a, b] \), where
\[
\begin{align*}
f_1(u, v, t_0 + t) &= \frac{\partial f}{\partial t}(u, v, t_0 + t), \\
f_2(u, t_0 + t, w) &= \frac{\partial f}{\partial t}(u, t_0 + t, w), \\
f_3(t_0 + t, v, w) &= \frac{\partial f}{\partial t}(t_0 + t, v, w),
\end{align*}
\]
and
\[
\begin{align*}
g_1(u, t_0 + t, t_0 + t) &= \frac{\partial f}{\partial t}(u, t_0 + t, t_0 + t), \\
g_2(t_0 + t, v, t_0 + t) &= \frac{\partial f}{\partial t}(t_0 + t, v, t_0 + t), \\
g_3(t_0 + t, t_0 + t, w) &= \frac{\partial f}{\partial t}(t_0 + t, t_0 + t, w).
\end{align*}
\]

We now derive the next results for co-ordinates convex functions.

**Theorem 2.3.** Let D := [a₁, b₁] × [a₁, b₁] × [a₁, b₁] be a cube in ℝ³ with a₁ < b₁, and the function f : D → ℝ is continuous, and has continuous second order partial derivatives on D°. Choose a, b ∈ (a₁, b₁), with a < b, and let \( \Delta := [a, b] \times [a, b] \times [a, b] \). Suppose that f is convex on the co-ordinates in \( \Delta \), then the function F : [a, b] × [a, b] → ℝ defined by
\[
(2.3) \quad F(x, y) := \begin{cases}
\frac{1}{(y-x)^3} \int_x^y \int_x^y f(r, s, t) dr ds dt, & x \neq y, x, y \in [a, b], \\
f(x, x), & x = y, x, y \in [a, b].
\end{cases}
\]
is Schur-convex on [a, b] × [a, b].

**Proof.** Case 1: If x, y ∈ [a, b], with x = y. Then Lemma 2.1 implies that
\[
(y - x) \left( \frac{\partial F}{\partial y} - \frac{\partial F}{\partial x} \right) = 0.
\]
Case 2: If \( x, y \in [a, b] \), with \( x \neq y \). Then by Lemma 1.11 we have
\[
\frac{\partial F}{\partial y} = \frac{-3}{(y-x)^3} \int_x^y \int_x^y f(r, s, t) dr ds dt \\
+ \frac{1}{(y-x)^3} \left[ \int_x^y \int_x^y f(r, s, y) dr ds \\
+ \int_x^y \int_x^y f(r, y, t) dr dt + \int_x^y \int_x^y f(y, s, t) ds dt \right],
\]
and
\[
\frac{\partial F}{\partial x} = \frac{3}{(y-x)^3} \int_x^y \int_x^y f(r, s, t) dr ds dt \\
- \frac{1}{(y-x)^3} \left[ \int_x^y \int_x^y f(r, s, x) dr ds \\
+ \int_x^y \int_x^y f(r, x, t) dr dt + \int_x^y \int_x^y f(x, s, t) ds dt \right].
\]
Thus,
\[
\left( \frac{\partial F}{\partial y} - \frac{\partial F}{\partial x} \right) = \frac{-6}{(y-x)^4} \int_x^y \int_x^y f(r, s, t) dr ds dt \\
+ \frac{1}{(y-x)^3} \left[ \int_x^y \int_x^y (f(r, s, x) + f(r, s, y)) dr ds \\
+ \int_x^y \int_x^y (f(r, x, t) + f(r, y, t)) dr dt \\
+ \int_x^y \int_x^y (f(x, s, t) + f(y, s, t)) ds dt \right].
\]
Then \((y-x)\left( \frac{\partial F}{\partial y} - \frac{\partial F}{\partial x} \right)\) is nonnegative if
\[
\frac{1}{y-x} \int_x^y \int_x^y f(r, s, t) dr ds dt \\
\leq \frac{1}{6} \left[ \int_x^y \int_x^y (f(r, s, x) + f(r, s, y)) dr ds \\
+ \int_x^y \int_x^y (f(r, x, t) + f(r, y, t)) dr dt \\
+ \int_x^y \int_x^y (f(x, s, t) + f(y, s, t)) ds dt \right].
\]
Since \( f \) is convex on the co-ordinates the last inequality holds by Theorem 1.7. Therefore by Theorem 1.4 the function \( F \) is Schur-convex. \( \square \)

The following theorem also holds:

**Theorem 2.4.** Let \( D := [a_1, b_1] \times [a_1, b_1] \times [a_1, b_1] \) be a cube in \( \mathbb{R}^3 \) with \( a_1 < b_1 \), and the function \( f : D \to \mathbb{R} \) is continuous, and has continuous four order partial derivatives on \( D^o \). Choose \( a, b \in (a_1, b_1) \), with \( a < b \), and let \( \Delta := [a, b] \times [a, b] \times [a, b] \). Suppose that \( f \) is convex
on the co-ordinates on $\Delta$, then the function $G : [a, b] \times [a, b] \rightarrow \mathbb{R}$ defined by

\begin{equation}
G(x, y) := \begin{cases} 
\frac{1}{(y-x)^3} \int_x^y \int_x^y \int_x^y f(r, s, t) \, dr \, ds \, dt, & x \neq y, \ x, y \in [a, b], \\
-f\left(\frac{x+y}{2}, \frac{x+y}{2}, \frac{x+y}{2}\right), & x = y, \ x, y \in [a, b].
\end{cases}
\end{equation}

is Schur-convex on $[a, b] \times [a, b]$.

**Proof.** Case 1: If $x, y \in [a, b]$, with $x = y$. Then Lemma 2.2 implies that

\[(y - x) \left( \frac{\partial G}{\partial y} - \frac{\partial G}{\partial x} \right) = 0\]

Case 2: If $x, y \in [a, b]$, with $x \neq y$. Then by Lemma 1.11 we have

\[(y - x) \left( \frac{\partial G}{\partial y} - \frac{\partial G}{\partial x} \right) \geq 0,\]

if

\[
\frac{1}{y-x} \int_x^y \int_x^y \int_x^y f(r, s, t) \, dr \, ds \, dt \\
\leq \frac{1}{6} \left[ \int_x^y \int_x^y \int_x^y \left( f(r, s, x) + f(r, s, y) \right) \, dr \, ds \\
+ \int_x^y \int_x^y \int_x^y \left( f(r, x, t) + f(r, y, t) \right) \, dr \, dt \\
+ \int_x^y \int_x^y \int_x^y \left( f(x, s, t) + f(y, s, t) \right) \, ds \, dt \right].
\]

The result follows from Theorem 1.7 and Theorem 1.4.

In the following examples we show that the converses of theorems 2.3 and 2.4 are not true in general.

**Example 2.5.** Consider the non co-ordinates convex function,

\[f(r, s, t) := r^2 - \frac{1}{2}s^2 + t^2, \ r, t, s \in [1, 2].\]

It is easy to see that for the function $F$ was defined in (2.3) we have $F(x, x) = \frac{3}{2}x^2$, for every $x \in [1, 2]$, and $F(x, y) = \frac{1}{(y-x)^3} \int_x^y \int_x^y \int_x^y \left( r^2 - \frac{1}{2}s^2 + t^2 \right) \, dr \, ds \, dt = \frac{1}{2}(x^2 + y^2 + xy)$, for every $x, y \in [1, 2]$, with $x \neq y$. Thus,

\[F(x, y) = \frac{1}{2}(x^2 + y^2 + xy),\]

for every $x, y \in [1, 2]$. Clearly $F$ is symmetric, continuous and differentiable on $[1, 2] \times [1, 2]$. If $x, y \in [1, 2]$, we have

\[(y - x) \left( \frac{\partial F}{\partial y} - \frac{\partial F}{\partial x} \right) = \frac{1}{2}(y - x)^2 \geq 0.
\]

Therefore by Theorem 1.4 $F$ is Schur-convex.
Remark 2.6. It is easy to see that for the function $f$ was defined in example 2.5 each of the inequalities in theorem 1.7 is valid while $f$ is not convex on the co-ordinates. This means that the converse of theorem 1.7 is not valid in general.

Example 2.7. Consider the non co-ordinated convex function:

$$f(r, s, t) := 2r^2 - s^2 + t^2, \quad r, t, s \in [1, 2].$$

It is easy to see that for the function $G$ was defined in (2.4) we have $G(x, x) = 0$, for every $x \in [1, 2]$, and

$$G(x, y) = \frac{1}{(y-x)^3} \int_x^y \int_x^y \int_x^y (2r^2 - s^2 + t^2) drdsdt - \frac{(x+y)^2}{2}$$

$$= \frac{2}{3} (x^2 + y^2 + xy) - \frac{(x+y)^2}{2},$$

for every $x, y \in [1, 2]$, with $x \neq y$. Thus,

$$G(x, y) = \frac{2}{3} (x^2 + y^2 + xy) - \frac{(x+y)^2}{2}.$$

Clearly $G$ is symmetric, continuous and differentiable on $[1, 2] \times [1, 2]$. If $x, y \in [1, 2]$, we have

$$(y-x) \left( \frac{\partial G}{\partial y} - \frac{\partial G}{\partial x} \right) = \frac{2}{3} (y-x)^2 \geq 0.$$

Therefore by Theorem 1.4 the function $G$ is Schur-convex.

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