Chern classes on differential $K$-theory

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Abstract

In this note we give a simple, model-independent construction of Chern classes as natural transformations from differential complex $K$-theory to differential integral cohomology. We verify the expected behaviour of these Chern classes with respect to sums and suspension.

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1 Statements

Complex $K$-theory and integral cohomology $H\mathbb{Z}$ are generalised cohomology theories which have a unique differential extensions $(\hat{K}, R, I, a, \int)$ and $(\hat{H}\mathbb{Z}, R, I, a, \int)$ with integration. Moreover, these extensions are multiplicative in a unique way. We refer to [BS] for a description of the axioms for differential extensions of cohomology theories and a proof of these statements.

The $i$’th Chern class is a natural transformation of set-valued functors

$$c_i : K^0 \to H\mathbb{Z}^{2i}$$

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1In our previous work instead of “differential cohomology” we used the term ”smooth cohomology”. We were convinced by D. Freed that differential cohomology is the better name.
on the category of topological spaces. The product $H^* := \prod_{i \geq 0} H^2i$ is a functor with values in commutative graded rings. We consider subfunctor $H^*_{ev} := 1 + \prod_{i \geq 1} H^2i \subseteq \prod_{i \geq 0} H^2i$ which takes values in the subgroup of units. The total Chern class

\[ c := 1 + c_1 + c_2 + \cdots : K^0 \to H^*_{ev} \]

is a natural transformation of \textit{group-valued} functors.

Let $\Omega^*_c(\ldots, K^*) \subseteq \Omega^*_e(\ldots, K^*)$ denote the graded ring valued functors on smooth manifolds of smooth differential forms with coefficients in $K^*$ and its subfunctor of closed forms. We use the powers of the Bott element in $K^2$ in order to identify the functors

\[ \Omega^0(\ldots, K^*) \cong \Omega^e(\ldots), \quad \Omega^{-1}(\ldots, K^*) \cong \Omega^{odd}(\ldots). \]

We therefore have natural transformations

\[ a : \Omega^{odd} \to \hat{K}^0, \quad R : \hat{K}^0 \to \Omega^{ev}_c, \]

where $a$ only preserves the additive structure, while $R$ is multiplicative.

We consider the symmetric formal power series in infinitely many variables

\[ \tilde{\text{ch}} := \sum_{i \geq 1} (e^{x_i} - 1) \in \mathbb{Q}[ [x_1, x_2, \ldots] ] . \]

We write $\text{ch}_i$ for the homogeneous component of degree $i$. Then there are polynomials

\[ C_i \in \mathbb{Q}[ s_1, s_2, \ldots ] \]

of degree $i$ (where $s_i$ has degree $i$) such that

\[ C_i(\text{ch}_1, \ldots, \text{ch}_i) = \sigma_i \]

is the $i$th elementary symmetric function in the $x_i$. The polynomial $C_i$ induces a natural transformation

\[ C_i : \Omega^{ev} \to \Omega^{2i} \]

which maps the even form $\omega = \omega_0 + \omega_2 + \omega_4 + \ldots, \omega_{2k} \in \Omega^{2k}(M)$ to

\[ C_i(\omega) := C_i(\omega_2, \ldots, \omega_{2i}) \in \Omega^{2i}(M) . \]

The following theorem states that the Chern classes have unique lifts to the differential extensions which are, in addition, compatible with the group structures.

**Theorem 1.1** 1. For every $i \geq 1$ there exists a unique natural transformation of set-valued functors on smooth manifolds

\[ \hat{c}_i : \hat{K}^0 \to \hat{H}^{2i} \]
such that the following diagram commutes:

\[
\begin{array}{c}
\Omega^{ev} \xrightarrow{c_i} \Omega^{2i} \\
\downarrow R \quad \downarrow R \\
\hat{K}^0 \xrightarrow{\hat{c}_i} \hat{H\Bbb Z}^{2i} \\
\downarrow I \quad \downarrow I \\
K^0 \xrightarrow{c_i} H\Bbb Z^i
\end{array}
\]

2. The total class

\[
\hat{c} = 1 + \hat{c}_1 + \cdots : \hat{K}^0 \to \hat{H\Bbb Z}_{ev,*}^1
\]

preserves the group structure.

Lifts of the Chern classes have previously been constructed in \[\text{Ber08}\]. The goal of the present paper is to give a much simpler, model-independent treatment. Further new, but not very deep, points of the present theorem are the assertions about uniqueness and the second statement. Our method of proof is different from \[\text{Ber08}\]. It is in fact a specialisation of a general principle already used in \[\text{BS}\] and \[\text{Bun09}\] for the construction of lifts of natural transformations between cohomology functors to their differential refinements. In the next two paragraphs we connect the differential Chern classes on differential \(K\)-theory with previous constructions of differential Chern classes in specific geometric situations.

If \(V := (V, h^V, \nabla^V)\) is a hermitian vector bundle with connection over a manifold \(M\), then we have the Cheeger-Simons classes

\[
\hat{c}_i^{CS}(V) \in \hat{H\Bbb Z}^{2i}(M)
\]

constructed in \[\text{CS85}\]. In the model of differential \(K\)-theory \[\text{BS07}\] the geometric bundle is a cycle for a differential \(K\)-theory class \([V] \in \hat{K}^0(M)\). We have

\[
\hat{c}_i([V]) = \hat{c}_i^{CS}(V).
\]

An even geometric family \(E\) over \(M\) (see \[\text{Bun02}\] for this notion) gives rise to a Bismut superconnection \(A(E)\) on an infinite-dimensional Hilbert space bundle \(H(E)\) over \(M\). This superconnection

\[
A(E) = D(E) + \nabla^{H(E)} + \text{higher terms}
\]

extends the family of Dirac operators \(D(E)\). If the kernel of \(D(E)\) is a vector bundle, then it has an induced metric \(h^{ker(D(E))}\) and connection \(\nabla^{ker(D(E))}\) obtained from \(\nabla^{H(E)}\) by projection. We thus get an induced geometric bundle

\[
H(E) = (\ker(D(E)), h^{ker(D(E))), \nabla^{ker(D(E))})
\]
and can define the class $c^CS_i(H(E)) \in \widehat{HZ}^{2i}(M)$. One of the original goals of [Bun02], which was not quite achieved there, was to extend this construction to the general case where we do not have a kernel bundle. Under the assumption that $\text{index}(D(E)) \in K^0(M)$ belongs to the $i$'th-step of the Atiyah-Hirzebruch filtration (i.e. vanishes after pull-back to any $i-1$-dimensional complex) in [Bun02, 4.1.19] we constructed a class $\hat{c}_i(E) \in \widehat{HZ}^{2i}(M)$ such that $I(\hat{c}_i(E)) = c_i(\text{index}(D(E)))$. On the other hand, the geometric family $E$ represents a differential $K$-theory class $[E, 0] \in \hat{K}^0(M)$ in the model [BS07], and we have $I([E, 0]) = \text{index}(D(E))$. The class $\hat{c}_i([E, 0]) \in \widehat{HZ}^{2i}(M)$ also satisfies $I(\hat{c}_i(E)) = c_i(\text{index}(D(E)))$ and thus gives a second differential refinement of the $i$'th Chern class of the index of $D(E)$. But in general the class $\hat{c}_i(E)$ differs from $\hat{c}_i([E, 0])$.

This can already be seen on the level of curvatures. Namely, we have

$$R(\hat{c}_i(E)) = R([E, 0])_{[2i]} , \quad R(\hat{c}_i([E, 0])) = C_i(R([E, 0])) ,$$

where $\omega_{[2i]}$ denotes the degree-$2i$ component of the form $\omega$. In a sense, the present note gives the right answer to the problem considered in [Bun02].

Finally we discuss odd Chern classes. In topology, the odd Chern classes $c_i^{\text{odd}} : K^{-1} \to H\mathbb{Z}^{i}$ are related with the even Chern classes by suspension

$$\begin{align*}
\tilde{K}^0(\Sigma M_+) & \xrightarrow{c_i^{\text{even}}} \tilde{H}\mathbb{Z}^{i+1}(\Sigma M_+) \\
\cong & \quad \cong \\
K^{-1}(M) & \xrightarrow{c_i^{\text{odd}}} H\mathbb{Z}^{i}(M)
\end{align*}$$

In the smooth context the suspension isomorphism is replaced by the integration $\int$ along $S^1 \times M \to M$. We have the following odd counterpart of Theorem 1.1.

**Theorem 1.2** For odd $i \in \mathbb{N}$ there are unique natural transformations $\hat{c}_i^{\text{odd}} : \tilde{K}^{-1} \to \tilde{H}\mathbb{Z}^{i}$ such that

$$\begin{align*}
\tilde{K}^0(S^1 \times M) & \xrightarrow{\hat{c}_i^{\text{odd}}} \tilde{H}\mathbb{Z}^{i+1}(S^1 \times M) \\
\int & \quad \int \\
\tilde{K}^{-1}(M) & \xrightarrow{\hat{c}_i^{\text{odd}}} H\mathbb{Z}^{i}(M)
\end{align*}$$

commutes. The transformation in addition satisfies

$$I \circ \hat{c}_i^{\text{odd}} = \hat{c}_i^{\text{odd}} \circ I .$$

\footnote{Note that in [Bun02] we index the Chern classes by their degree, where in the present note we adopt the usual convention.}
Let $\pi : W \to B$ be a proper $K$-oriented map between manifolds. Then we have an Umkehr map $\pi^! : K^*(W) \to K^{*-n}(B)$, where $n = \dim(W) - \dim(B)$. An integral index theorem is an assertion about the Chern classes $c_*(\pi^!(x))$, or $c_{*\text{odd}}(\pi^!(x))$ for $x \in K^*(W)$, e.g. an expression of these classes in terms of the classes $c_*(x)$ or $c_{*\text{odd}}(x)$, respectively. A prototypical example is given in [Mad09]. The construction of differential lifts of Chern classes makes it possible to ask for geometric refinements of these kinds of results. An example of such a theorem related to the Pfaffian bundle will be discussed in a forthcoming paper.

2 Proofs

Proof. Let $K_0 \simeq \mathbb{Z} \times BU$ be a representative of the homotopy type of the classifying space of the functor $K^0$. We choose by [BS, Prop 2.1] a sequence of manifolds $(\mathcal{K}_k)_{k \geq 0}$ together with maps $x_k : \mathcal{K}_k \to K_0$, $\kappa_k : \mathcal{K}_k \to \mathcal{K}_{k+1}$ such that

1. $\mathcal{K}_k$ is homotopy equivalent to an $i$-dimensional CW-complex,
2. $\kappa_k : \mathcal{K}_k \to \mathcal{K}_{k+1}$ is an embedding of a closed submanifold,
3. $x_k : \mathcal{K}_k \to K_0$ is $k$-connected,
4. $x_{k+1} \circ \kappa_k = x_k$.

Let $u \in K^0(K_0)$ the universal class represented by the identity map $K_0 \to K_0$. By [BS, Prop. 2.6] we can further choose a sequence $\hat{u}_k \in \hat{K}^0(\mathcal{K}_k)$ such that $I(\hat{u}_k) = x^*_k u$ and $\kappa_k^* \hat{u}_{k+1} = \hat{u}_k$ for all $k \geq 0$. By [BS, Lem. 3.8] and $2j - 1 < k$ we have that $H^{2j-1}(\mathcal{K}_k, \mathbb{R}) = 0$. We consider the canonical natural transformation $\iota_\mathbb{R} : H\mathbb{Z}^* \to H\mathbb{R}^*$ and the de Rham map $\text{Rham} : \Omega^*_{cl} \to H\mathbb{R}^*$. Since $\text{Rham}$ is multiplicative we have

$$\iota_\mathbb{R}(c_i(I(\hat{u}_k))) = C_i(\text{ch}(I(\hat{u}_k))) = C_i(\text{Rham}(R(\hat{u}_k))) = \text{Rham}(C_i(R(\hat{u}_k))).$$

If we choose $k \geq 2i$, then the diagram

$$
\begin{array}{ccc}
\hat{H\mathbb{Z}}^{2i}(\mathcal{K}_k) & \xrightarrow{I} & H\mathbb{Z}^{2i}(\mathcal{K}_k) \\
\downarrow R & & \downarrow \iota_\mathbb{R} \\
\Omega^{2i}_{cl}(\mathcal{K}_k) & \xrightarrow{\text{Rham}} & H\mathbb{R}^{2i}(\mathcal{K}_k)
\end{array}
$$

is cartesian. Hence for $k \geq 2i$ there exists a unique class $\hat{z}_{i,k} \in \hat{H\mathbb{Z}}^{2i}(\mathcal{K}_k)$ such that

$$I(\hat{z}_{i,k}) = c_i(I(\hat{u}_k)), \quad R(\hat{z}_i) = C_i(R(\hat{u}_k)).$$
Furthermore, we have $\kappa_k \tilde{z}_{i,k+1} = \tilde{z}_{i,k}$. For $k < 2i$ we define $z_{i,k} := (\kappa_k^* \circ \cdots \circ \kappa_{2k-1}^*) z_{i,2i}$.

We now define the natural transformation $\hat{\iota}_i$. We start with the observation that if $\hat{\iota}_i$ exists, then it satisfies

$$\hat{\iota}_i(\hat{u}_k) = \tilde{z}_{i,k}.$$ 

Let $\hat{w} \in \hat{K}^0(M)$. By [BS, Prop. 2.6] we have $K^0(M) \cong \colim_k [M, K_k]$, and the underlying class $I(\hat{w}) \in K^0(M)$ can be written as $I(\hat{w}) = f^* x_k^* u$ for some $k$ and $f : M \to K_k$. We choose a form $\rho \in \Omega^{\text{odd}}(M)$ such that

$$\hat{w} = f^* \hat{u}_k + a(\rho).$$

We consider a form $\tilde{\rho} \in \Omega^{\text{odd}}([0,1] \times M)$ which restricts to $\rho$ on $\{1\} \times M$ and to 0 on $\{0\} \times M$. We get a class $\tilde{w} = \pr^*_M \hat{w} + a(\tilde{\rho}) \in \hat{K}^0([0,1] \times M)$. Note that

$$\tilde{w}|_{[0]} \times M = f^* \hat{u}_k, \quad \tilde{w}|_{[1]} \times M = \hat{w}.$$ 

If $\hat{\iota}_i$ exists, then we must have by naturality and the homotopy formula [BS, (1)]

$$\hat{\iota}_i(\tilde{w}|_{[0]} \times M) = f^* \hat{z}_{i,k}, \quad \hat{\iota}_i(\tilde{w}|_{[1]} \times M) - \hat{\iota}_i(\tilde{w}|_{[0]} \times M) = a(\int_{[0,1] \times M/M} R(\hat{\iota}_i(\tilde{w}))).$$

Furthermore, by the commutativity of the upper square in (1) we must require

$$R(\hat{\iota}_i(\tilde{w})) = C_i(R(\tilde{w})).$$

Therefore we are forced to define

$$\hat{\iota}_i(\tilde{w}) := f^* \tilde{z}_{i,k} + a(\int_{[0,1] \times M/M} C_i(R(\tilde{w}))). \quad (2)$$

We see that if $\hat{\iota}_i$ exists, then it is automatically unique.

**Lemma 2.1** The definition of $\hat{\iota}_i(\tilde{w})$ by (2) is independent of the choices of $\tilde{\rho}$, $\rho$ and $f : M \to K_k$.

**Proof.** Let us start with a second choice $\tilde{\rho}'$ and write $\tilde{w}' := \pr^*_M \hat{w} + a(\tilde{\rho}')$. Then we can connect $\tilde{\rho}$ with $\tilde{\rho}'$ by a family of such forms, e.g. the linear path. This path can be considered as a form $\tilde{\rho}$ on $[0,1] \times [0,1] \times M$. By construction $\tilde{\rho}|_{[0,1] \times [j]} \times M$ is constant and has no component in the direction of the first variable for $j = 0, 1$. This implies that

$$R(\tilde{w}')|_{[0,1] \times [j]} \times M = 0. \quad (3)$$

We set $\tilde{w} := \pr^*_M \hat{w} + a(\tilde{\rho}) \in \hat{K}^0([0,1] \times [0,1] \times M)$. By Stokes theorem we have

$$d \int_{[0,1] \times [0,1] \times M/M} C_i(R(\tilde{w})) = \int_{[0,1] \times M/M} C_i(R(\tilde{w}')) - \int_{[0,1] \times M/M} C_i(R(\tilde{w})).$$

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(these are the contributions of the faces \{j\} × [0, 1] × M) since the integral over the other two faces [0, 1] × \{j\} × M vanishes by (3). Since \(a\) annihilates exact forms this implies that

\[ a\left( \int_{[0,1] \times M/M} C_i(R(\tilde{w})) \right) = a\left( \int_{[0,1] \times M/M} C_i(R(\tilde{w}')) \right). \]

Assume now that we have chosen a different \(\rho'\). Then \(a(\rho' - \rho) = 0\) so that by the exactness axiom \([BS, (2)]\) there exists a class \(\hat{\nu} \in \hat{K}^1(M)\) with \(R(\hat{\nu}) = \rho' - \rho\). Let \(\hat{e} \in \hat{K}^1(S^1)\) be a lift of the generator of \(K^1(S^1) \cong \mathbb{Z}\) with \(R(\hat{e}) = dt\). We consider the form \(\tilde{\sigma} \in \Omega^{\text{odd}}([0, 1] \times M)\) with no \(dt\)-component given by

\[ \tilde{\sigma}|_{\{0\} \times M} = 0, \quad \tilde{\sigma}|_{\{1\} \times M} = \rho' - \rho, \quad d\tilde{\sigma} = dt \wedge \text{pr}_M^*R(\hat{\nu}) = R(\hat{e} \times \hat{\nu}). \]

We now consider

\[ \tilde{\nu} := \text{pr}_M^*\dot{w} + \text{pr}_M^*a(\rho) + a(\tilde{\sigma}) \in \hat{K}^0([0, 1] \times M) \]

and calculate modulo the image of \(d\)

\[ \int_{[0,1] \times M/M} C_i(R(\tilde{\nu})) \equiv \int_{S^1 \times M/M} C_i(R(\text{pr}_M^*\dot{w})) + \text{pr}_M^*d\rho + R(\hat{e} \times \hat{\nu}) \]

\[ \equiv \int_{S^1 \times M/M} C_i(R(\text{pr}_M^*\dot{w})) + R(\hat{e} \times \hat{\nu}) \]

\[ \equiv \int_{S^1 \times M/M} C_i(R(\text{pr}_M^*\dot{w} + \hat{e} \times \hat{\nu})). \]

It follows that

\[ \text{Rham}(\int_{[0,1] \times M/M} C_i(R(\tilde{\nu}))) = \text{Rham}(\int_{S^1 \times M/M} C_i(R(\text{pr}_M^*\dot{w} + \hat{e} \times \hat{\nu}))) \]

\[ = \int_{S^1 \times M/M} \text{Rham}(C_i(R(\text{pr}_M^*\dot{w} + \hat{e} \times \hat{\nu}))) \]

\[ = \int_{S^1 \times M/M} \iota_\mathbb{R}(c_i(I(\text{pr}_M^*\dot{w} + \hat{e} \times \hat{\nu}))). \]

In other words, \(\text{Rham}(\int_{[0,1] \times M/M} C_i(R(\tilde{\nu})))\) is an integral class, and this implies

\[ a(\int_{[0,1] \times M/M} C_i(R(\tilde{\nu}))) = 0 \]
by \cite[2]{BS}. If \( \tilde{\rho} \) was the path connecting \( \rho \) with 0, then we construct the path \( \tilde{\rho}' \) from \( \rho' \) to 0 by concatenating \( \tilde{\rho} \) with \( \tilde{\sigma} \) (in order to concatenate smoothly we can change \( \tilde{\rho} \)). Then get \( \tilde{w}' := \text{pr}_M^* \tilde{w} + a(\tilde{\rho}') \in \tilde{K}^0([0,1] \times M) \) and

\[
\begin{align*}
    a(\int_{[0,1] \times M/M} C_i(R(\tilde{w}'))) & = a(\int_{[0,1] \times M/M} C_i(R(\tilde{w}))) \\
    & \quad + a(\int_{[0,1] \times M/M} C_i(R(\tilde{\rho}))) \\
    & = a(\int_{[0,1] \times M/M} C_i(R(\tilde{w})))
\end{align*}
\]

This finishes the verification that our construction of \( c_i \) is independent of the choice of \( \rho \). Finally we verify that \( \hat{c}_i(\tilde{w}) \) is independent of the choice of \( f : M \to \mathcal{K}_k \). If we replace \( k \) by \( k + 1 \) and \( f \) by \( \kappa_k \circ f \), then we obviously get the same result. For two choices \( f : M \to \mathcal{K}_k \) and \( f' : M \to \mathcal{K}_{k'} \) there exists \( k'' \geq \max\{k,k'\} \) such that \( \kappa_{k''} \circ f \) and \( \kappa_{k''} \circ f' \) are homotopic. Here \( \kappa^j_i : \mathcal{K}_i \to \mathcal{K}_j \) denotes for \( j > i \) the composition \( \kappa^j_i := \kappa_{j-1} \circ \cdots \circ \kappa_i \). Therefore it remains to show that a choice \( f' : M \to \mathcal{K}_k \) homotopic to \( f : M \to \mathcal{K}_k \) gives the same result for \( \hat{c}_i(\tilde{w}) \). Let \( H : [0,1] \times M \to \mathcal{K}_k \) be a homotopy from \( f \) to \( f' \). Then we use \( H \) in the construction of \( \hat{c}_i(\text{pr}_M^* \tilde{w}) \in \tilde{H}\tilde{Z}^{2i}_i([0,1] \times M) \). If we let \( \tilde{c}_i' \) denote the result of the construction based on the choice of \( f' \) we have by the homotopy formula

\[
\tilde{c}_i'(\tilde{w}) - \tilde{c}_i(\tilde{w}) = a(\int R(\hat{c}_i(\text{pr}_M^* \tilde{w}))) = a(\int \text{pr}_M^* C_i(\tilde{w})) = 0 .
\]

\[
\square
\]

Lemma 2.2 The construction of \( \hat{c}_i \) defines a natural transformation \( \hat{c}_i : \tilde{K} \to \tilde{H}\tilde{Z}^{2i}_i \) of set-valued functors on smooth manifolds.

Proof. Let \( g : N \to M \) be a smooth map between manifolds. Let \( \tilde{w} \in \tilde{K}^0(M) \) and assume that we have constructed \( \hat{c}_i(\tilde{w}) \) using the choices of \( f : M \to \mathcal{K}_k, \rho \in \Omega^{\text{odd}}(M) \) and \( \tilde{\rho} \in \Omega^{\text{odd}}([0,1] \times M) \). Then we construct \( \hat{c}_i(g^* \tilde{w}) \) using the choices \( f \circ g : N \to \mathcal{K}_k \) and \( g^* \tilde{\rho} \in \Omega^{\text{odd}}(N), (\text{id} \times g)^* \tilde{\rho} \in \Omega^{\text{odd}}([0,1] \times N) \). With these choices we have \( (\text{id} \times g)^* \tilde{w} = g^* \tilde{w} \in \tilde{K}^0([0,1] \times N) \) and

\[
\begin{align*}
g^* \hat{c}_i(\tilde{w}) & = g^* f^* \hat{z}_{i,k} + g^* a(\int_{[0,1] \times M/M} C_i(R(\tilde{w}))) \\
& = (f \circ g)^* \hat{z}_{i,k} + a(\int_{[0,1] \times M/M} C_i(R((\text{id} \times g)^* \tilde{w}))) \\
& = (f \circ g)^* \hat{z}_{i,k} + a(\int_{[0,1] \times M/M} C_i(R(g^* \tilde{w}))) \\
& = \hat{c}_i(g^* \tilde{w}) .
\end{align*}
\]
This finishes the proof of Assertion 1 of Theorem 1.1.

In order to show the second Assertion 2 we consider the natural transformation

\[ \hat{B} : \hat{K}^0 \times \hat{K}^0 \to \hat{HZ}^{ev} \]

given by

\[ \hat{B}(\hat{w}, \hat{v}) := \hat{c}(\hat{w}) \cup \hat{c}(\hat{v}) - \hat{c}(\hat{w} + \hat{v}) \in \hat{HZ}^{ev}(M), \quad \hat{w}, \hat{v} \in \hat{K}^0(M). \]

If we apply \( I \) we get

\[ I(\hat{B}(\hat{w}, \hat{v})) = I(\hat{c}(\hat{w}) \cup \hat{c}(\hat{v})) - I(\hat{c}(\hat{w} + \hat{v})) \]
\[ = I(\hat{c}(\hat{w})) \cup I(\hat{c}(\hat{v})) - I(\hat{c}(\hat{w} + \hat{v})) \]
\[ = c(I(\hat{w})) \cup c(I(\hat{v})) - c(I(\hat{w} + \hat{v})) \]
\[ = 0. \]

Let \( C = 1 + C_1 + C_2 + \cdots \in \mathbb{Q}[[s_0, s_1, \ldots]] \). Then we have the identity

\[ C(s_0 + s'_0, s_1 + s'_1, \ldots) = C(s_0, s_1, \ldots)C(s'_0, s'_1, \ldots). \]

Indeed, if

\[ \tilde{c}h = \sum_{i \geq 1} (e^{x_i} - 1), \]

then

\[ C(\tilde{c}h, \ldots) = \prod_{i \geq 1} (1 + x_i). \]

If we introduce another set of variables \( x'_i \) and set \( \tilde{c}h' = \sum_{i \geq 1} (e^{x'_i} - 1), \) then

\[ C(\tilde{c}h_1, \tilde{c}h'_1, \tilde{c}h_2, \tilde{c}h'_2, \ldots) = \prod_{i \geq 1} (1 + x_i)(1 + x'_i) \]
\[ = C(\tilde{c}h_1, \tilde{c}h_2, \ldots)C(\tilde{c}h'_1, \tilde{c}h'_2, \ldots). \]

We now calculate

\[ R(\hat{B}(\hat{w}, \hat{v})) = R(\hat{c}(\hat{w}) \cup \hat{c}(\hat{v})) - R(\hat{c}(\hat{w} + \hat{v})) \]
\[ = R(\hat{c}(\hat{w})) \cup R(\hat{c}(\hat{v})) - R(\hat{c}(\hat{w} + \hat{v})) \]
\[ = C(R(\hat{w})) \wedge C(R(\hat{v})) - C(R(\hat{w}) + R(\hat{v})) \]
\[ = 0. \]
It follows that \( \hat{B} \) factorises over the subfunctor
\[
H\mathbb{R}^{\text{odd}} / H\mathbb{Z}^{\text{odd}} \subset H\mathbb{R} / H\mathbb{Z}^{\text{odd}} \subset \widetilde{H\mathbb{Z}}^{\text{ev}}
\]
where the inclusion is induced by \( a \). Let \( \rho \in \Omega^{\text{odd}}(M) \) and consider \( \hat{\rho} := \text{tr}_{M}^{\ast} \rho \in \Omega^{\text{odd}}([0, 1] \times M) \). Then we have
\[
\hat{B}(\hat{w} + a(\rho), \hat{v}) - \hat{B}(\hat{w}, \hat{v}) = \hat{B}(\text{pr}_{M}^{\ast} \hat{w} + a(\hat{\rho}), \hat{v}) |_{\{1\} \times M} - \hat{B}(\text{pr}_{M}^{\ast} \hat{w} + a(\hat{\rho}), \hat{v}) |_{\{0\} \times M}.
\]
Since \( \hat{B} \) takes values in the homotopy invariant subfunctor \( H\mathbb{R}^{\text{odd}} / H\mathbb{Z}^{\text{odd}} \) we conclude that \( \hat{B}(\hat{w} + a(\rho), \hat{v}) = \hat{B}(\hat{w}, \hat{v}) \). In a similar manner we see that \( \hat{B}(\hat{w}, \hat{v} + a(\rho)) = \hat{B}(\hat{w}, \hat{v}) \).

Hence \( \hat{B} \) has a factorisation over a natural transformation
\[
K^{0} \times K^{0} \to H\mathbb{R}^{\text{odd}} / H\mathbb{Z}^{\text{odd}} \subset H\mathbb{R} / H\mathbb{Z}^{\text{odd}}.
\]
Such a natural transformation between homotopy invariant functors on manifolds must be represented by a map of classifying spaces
\[
K_{0} \times K_{0} \to K(\mathbb{R} / \mathbb{Z}, \text{odd})
\]
where \( K(\mathbb{R} / \mathbb{Z}, \text{odd}) := \bigvee_{i \geq 0} K(\mathbb{R} / \mathbb{Z}, 2i + 1) \) is a wedge of Eilenberg-MacLane spaces, i.e. by a class in \( B \in H^{\text{odd}}(K_{0} \times K_{0}; R / Z) \). Since \( K_{0} \) and therefore \( K_{0} \times K_{0} \) are even spaces we know that \( H^{\text{odd}}(K_{0} \times K_{0}; Z) = 0 \). It follows by the universal coefficient formula that \( H^{\text{odd}}(K_{0} \times K_{0}; R / Z) \cong \text{Hom}(H^{\text{odd}}(K_{0} \times K_{0}; Z), R / Z) = 0 \). We see that \( B = 0 \) and therefore \( \hat{B} = 0 \). This finishes the proof of Assertion 2 of Theorem 1.1. \( \square \)

We now show Theorem 1.2. We let \( \hat{e} \in K^{1}(S^{1}) \) be, as above, the unique element with \( R(\hat{e}) = dt, I(\hat{e}) = e \in K^{1}(S^{1}) \) the canonical generator, and \( \hat{e}_{1} = 0 \) for a basepoint \( * \in S^{1} \).

Then we define for odd \( i \in \mathbb{N} \) and \( \hat{x} \in \hat{K}^{-1}(M) \)
\[
c_{i}^{\text{odd}}(\hat{x}) := \int \hat{c}_{i+1}(\hat{e} \times \hat{x})
\]
Note that
\[
I\left( \int \hat{c}_{i+1}(\hat{e} \times \hat{x}) \right) = \int c_{i+1}(e \times I(\hat{x})).
\]
We have a natural inclusion \( \widetilde{H\mathbb{Z}}^{*}(\Sigma M_{+}) \subset H\mathbb{Z}^{*}(S^{1} \times M) \) as the subspace of classes whose restriction to \( \{ * \} \times M \) vanishes. Since \( e_{1} = 0 \) we see that \( e \times I(\hat{x}) \) belongs to this subspace. The restriction of \( \int \) to this subspace coincides with the suspension isomorphism \( \widetilde{H\mathbb{Z}}^{*+1}(\Sigma M_{+}) \cong H\mathbb{Z}^{*}(M), \int (e \times x) = x \) with inverse \( x \mapsto e \times x \). Therefore
\[
\int c_{i}^{\text{odd}}(e \times I(\hat{x})) = c_{i}^{\text{odd}}(I(\hat{x})�) \].
In this way we get a natural transformation which has the required property. Since $\int : \hat{K}^0(S^1 \times M) \to \hat{K}^{-1}(M)$ is surjective it is clear that $\hat{c}_i^{\text{odd}}$ is unique. \qed

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