PAINLEVÉ ANALYSIS FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Proceedings of the Cargèse school (3–22 June 1996)
La propriété de Painlevé, un siècle après
The Painlevé property, one century later

Abstract. The Painlevé analysis introduced by Weiss, Tabor and Carnevale (WTC) in 1983 for nonlinear partial differential equations (PDE's) is an extension of the method initiated by Painlevé and Gambier at the beginning of this century for the classification of algebraic nonlinear differential equations (ODE's) without movable critical points. In these lectures we explain the WTC method in its invariant version introduced by Conte in 1989 and its application to solitonic equations in order to find algorithmically their associated Bäcklund transformation. A lot of remarkable properties are shared by these so-called “integrable” equations but they are generically no more valid for equations modelising physical phenomena. Belonging to this second class, some equations called “partially integrable” sometimes keep remnants of integrability. In that case, the singularity analysis may also be useful for building closed form analytic solutions, which necessarily agree with the singularity structure of the equations. We display the privileged role played by the Riccati equation and systems of Riccati equations which are linearisable, as well as the importance of the Weierstrass elliptic function, for building solitary waves or more elaborate solutions.

The Painlevé property, one century later, ed. R. Conte, CRM series in mathematical physics (Springer–Verlag, Berlin, 1998)
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Chapter 1

Introduction

During the past thirty years, the interest for nonlinear phenomena has been growing in different fields of modern physics, such as optics, fluid dynamics, condensed matter, elementary particle physics, statistical mechanics, astrophysics. Although the manifestation of those phenomena varies according to the different fields, they present a common feature in their mathematical description. The link comes from their description by nonlinear evolution equations (i.e. PDEs) whose solutions represent the propagation of waves with a permanent profile. Moreover, the analytical methods for solving them are directly inspired by the works of the famous mathematicians L. Fuchs, H. Poincaré and P. Painlevé, as explained in Conte contribution, whose content is assumed known.

The propagation of a bell-shaped solitary wave on water has been approximately explained by the mathematical physicists J. Boussinesq [15] and Lord Rayleigh [114] only thirty years after its experimental discovery by Scott Russell in 1844 [117]. The full explanation was later given in 1895 by Korteweg and de Vries (KdV [74]) who derived the nonlinear dispersive equation

\[ u_t + u_{xxx} + 3(u^2)_x = 0, \tag{1.1} \]

possessing the two-parameter \((k, \tau)\) exact solution

\[ u_{sw}(k, \theta) = \frac{k^2}{2} \sech^2 \frac{\theta}{2}, \quad \sech = \frac{1}{\cosh}, \quad \theta = k\xi + \tau, \xi = x - ct, \ c = k^2. \tag{1.2} \]

The name soliton was introduced by Zabusky and Kruskal in 1965 [138] when they solved the initial value problem for KdV equation \((1.1)\) and discovered solutions describing the elastic collision of several waves \((1.2)\).

In these lectures we shall restrict our study by the method of singularities to nonlinear evolution equations possessing two different levels of integrability: complete integrability or partial integrability, including some chaotic PDE’s which possess explicit analytic solutions in very special circumstances.

Complete integrability means that
• either the nonlinear partial differential equation can be related to a linear partial differential equation by an explicit transformation,

• or the equation passes the Painlevé test and possesses the Painlevé property (PP) for PDEs, i.e.: firstly, on every noncharacteristic manifold its general solution has no movable critical singularities in the complex plane of an arbitrary function \( \varphi(x,t) \); secondly, the PDE possesses an auto-Bäcklund transformation or is related by a Bäcklund transformation to another PDE possessing the PP (PDEs passing the “weak Painlevé” test and related by a hodograph transformation to another equation possessing the PP are outside the scope of these lectures),

• or the equation possesses solitary waves, \( N \)-soliton solutions for arbitrary \( N \), an infinite number of conservation laws, bi-Hamiltonian structures, infinite-dimensional Lie algebras, . . .

• or the equation satisfies the Ablowitz-Ramani-Segur (ARS) conjecture on the relationship of all its reductions to ODE’s without movable critical points.

More explicit definitions concerning the properties of this first class of equations, as well as classical examples will be given in chapter 2.

Partial integrability means that some above listed properties are not satisfied (in particular the Painlevé test may be satisfied only with some constraints on the function \( \varphi \), or may never be satisfied whatever be \( \varphi \), and the ARS conjecture is no more valid) but the equation possesses explicit analytic solutions like for instance: degenerate solitary waves, \( N \)-shock solutions, \( N \)-soliton solutions with \( N \) bounded \([10, 47, 88]\), or retains some pieces of integrability like degenerate Bäcklund transformations, a finite number of conservation laws \([10, 106, 58]\). For equations belonging to this second class, methods for finding particular solutions, which must agree with the singularity structure of the equation, will be developed in chapter 4.

Chapter 3 contains the main subject of our lectures: it is devoted to the WTC \([13]\) method and its extensions in the invariant version introduced by Conte \([29]\), for finding algorithmically the auto-Bäcklund transformation of integrable nonlinear PDEs.
Chapter 2

Integrable equations

We present here a few classical examples of nonlinear partial differential equations either explicitly related to linear partial differential equations or characterised by the properties of complete integrability mentioned in the previous chapter.

2.1 Integration by direct linearisation

Some equations can be linearised by an explicit transformation:

Example 1. The Burgers equation

\[ u_t + (u_x + u^2)_x = 0 \]  

is linearised into the heat equation \[ u = (\log \varphi)_x, \quad ((\varphi_t + \varphi_{xx})/\varphi)_x = 0. \]

Example 2. The generalised Eckhaus equation \[ iu_t + u_{xx} + (\beta^2|u|^4 + 2\beta e^{i\gamma}(|u|^2)_x)u = 0, \quad (\beta, \gamma) \in \mathbb{R}, \]

is linearisable into the Schrödinger equation for \( \beta \cos \gamma \neq 0 \)

\[ iu_t + u_{xx} = 0, \quad u = \sqrt{\frac{1}{2\beta \cos \gamma}} \frac{\nu}{\sqrt{\varphi}} e^{-(i/2) \tan \gamma \log \varphi}, \quad \text{with} \quad \varphi_x = |\nu|^2 \]

and \( |u|^2 = \frac{1}{2\beta \cos \gamma} (\log \varphi)_x. \)

If \( \gamma = \pi/2, \) the Kundu \[ \[76, 77, 78 \] \] gauge transformation \( u = \nu e^{i\beta \varphi} \) transforms the more general higher order nonlinear Schrödinger equation (HNLS)

\[ iu_t + u_{xx} + \delta|u|^2 u + (\beta^2|u|^4 + 2i\beta|u|^2)_x)u = 0, \quad (\beta, \delta) \in \mathbb{R}, \]
into the nonlinear Schrödinger equation (NLS)

\[ \dot{i}u + u_{xx} + \delta |u|^2 u = 0. \]  

(2.7)

The natural question is then: where do these miraculous transformations from \( u \) to another field come from? This will be answered in section 3.2.1.

### 2.2 Reduction to ODEs with the Painlevé property

Ablowitz, Ramani and Segur [5, 3, 4], and McLeod and Olver [90] conjectured a link between integrable NLPDEs and the Painlevé ODEs [107] : for the integrable NLPDEs specially studied in these lectures, all known reductions to ODEs are singlevalued algebraic transforms of the Weierstrass or Painlevé equations. Some of them are listed in Table 2.1.

### 2.3 Construction of solitary wave solutions

In the integrable case, the solitary waves sech and sech² are degenerate elliptic functions, obtained by imposing boundary conditions to the general solution of the ODE defining the travelling wave reduction. In the partially or nonintegrable case, the general solution of the reduction may not exist. One then looks for particular solutions, taking advantage of the singularity structure of the ODE by the method of subequations in chapter 4.

**Example 1. KdV**

The reduction \( u(x, t) = U(\xi), \ \xi = x - ct, \) of Eq. (1.1) yields the ODE

\[ (-cU + U'' + 3U^2)' = 0 \]  

(2.8)

After two integrations, this equation becomes

\[-cU^2/2 + U'^3 + U'^2/2 + K_1U + K_2 = 0, \]  

(2.9)

which identifies to the Weierstrass elliptic equation

\[ \wp'^2 = 4\wp^3 - g_2\wp - g_3, \ (g_2, g_3) \text{ real constants}, \]  

(2.10)

\[ u = c/6 - 2\wp(x - ct - x_0, c^2/12 - K_1, K_2/2 + K_1c/12 - (c/6)^3). \]  

(2.11)

The solitary wave (1.2) is found by imposing the boundary conditions \( U(\xi) \to 0, \ U'(\xi) \to 0, \ U''(\xi) \to 0, \) when \( |\xi| \to \infty. \) Note that, for \( K_1 = K_2 = 0, \) equation (2.9) is a degenerate elliptic equation and

\[ \wp(x - ct - x_0, c^2/12, -(c/6)^3) = -(c/4) \text{sech}^2 \left( \sqrt{c}(x - ct - x_0)/2 \right) + c/12. \]  

(2.12)
Example 2. The generalised Tzitzéica equation

\[ u_{xt} + ae^u + a_1e^{-u} + ae^{-2u} = 0, \ a \neq 0 \]  \hspace{1cm} (2.13)

includes Liouville \((a_1 = a_0 = 0)\), sinh-Gordon \((a_1 \neq 0, a_0 = 0)\) or Tzitzéica \([123, 124]\) \((a_0 \neq 0, a_1 = 0)\) equations. It is polynomial in the variable \(v = e^u\)

\[ vv_{xt} - v_xv_t + av^3 + a_1v + a_0 = 0 \]  \hspace{1cm} (2.14)

Its reduction \((v, x, t) \rightarrow (V, \xi = x - ct)\) can be integrated once

\[-cV'^2 + 2aV^3 - 6KV^2 - 2a_1V - a_0 = 0, \ K \ arbitrary \]  \hspace{1cm} (2.15)

and possesses the general two-parameter solution \([32]\)

\[ aV = K + 2c \wp(\xi - \xi_0, (3K^2 + a_1a)/c^2, (4K^3 + 2a_1aK + a^2a_0)/(4c^3)) \]  \hspace{1cm} (2.16)

Moreover a linear superposition of two waves with opposite directions

\[ v(x, t) = Af(x - ct) + Bg(x + ct) \]  \hspace{1cm} (2.17)

is compatible with the Tzitzéica equation by assuming that \(f\) and \(g\) satisfy the following second order ODE with constant coefficients

\[ f'' = A_1f^2 + B_1, \ g'' = A_2g^2 + B_2 \]  \hspace{1cm} (2.18)

A particular solution of (2.13) for \(a_1 = 0\) is then \([101]\) :

\[ ae^u = 2c \wp(x - ct - x_1, g_2, K + a^2a_0/(8c^3)) - 2c \wp(x - ct - x_2, g_2, K - a^2a_0/(8c^3)), \]  \hspace{1cm} (2.19)

\[ c, x_1, x_2, g_2, K \ arbitrary \ constants. \]

Example 3. NLS

The reduction \(u(x, t) = \rho(\xi)e^{i[-\Omega t + \varphi(\xi)]}\) of (2.13) yields the coupled ODE’s

\[-c\rho' + 2\varphi' \rho' + \varphi'' \rho = 0, \]  \hspace{1cm} (2.20)

\[ \rho'' + (\Omega - (\varphi')^2 + c\varphi')\rho + 2c\rho'^3 = 0 \]  \hspace{1cm} (2.21)

Equation (2.20) admits the integrating factor \(\rho\)

\[ \varphi' = c + K_1/S, \ S = \rho^2. \]  \hspace{1cm} (2.22)

Then Eq. (2.21) admits the integrating factor \(\rho'\), hence

\[ S'^2 = -4\varepsilon S^3 - 4\alpha S^2 + 8K_2S - K_1^2, \ \alpha = \Omega + c^2/4, \]  \hspace{1cm} (2.23)

an elliptic equation for \(S\) with the general solution

\[ S = -\alpha/(3\varepsilon) - \varphi(x - ct - x_0, g_2, g_3)/\varepsilon \]  \hspace{1cm} (2.24)

\[ g_2 = 8\varepsilon(K_2 + \alpha^2/(6\varepsilon)), \ g_3 = (2\alpha/3)^3 + 8K_2\alpha\varepsilon/3 + \varepsilon^2K_1^2. \]  \hspace{1cm} (2.25)
The one-soliton solution is obtained for the values of $K_1, K_2$ making the Weierstrass elliptic function degenerate into a trigonometric function:

$$\wp(\xi, g_2, g_3) \to a_1 + a_2 \text{sech}^2 k\xi. \quad (2.26)$$

This happens in two cases:

1. $a_1 = K_1 = K_2 = 0$, $k^2 = -\alpha, \rho_2 = (k^2/\varepsilon) \text{sech}^2 k\xi$.

2. $a_1K_1K_2 \neq 0$, $a_1 = -(k^2 + \alpha)/(3\varepsilon)$, $\rho^2 = a_1 + (k^2/\varepsilon) \text{sech}^2 k\xi$.

They respectively correspond for equation (2.131) to the three-parameter $(c, k, x_0)$ solution ("bright" soliton) to equation (2.131) as

$$\varepsilon > 0: \quad u = \varepsilon^{-1/2}k \text{sech}(k(x - ct - x_0))e^{icx/2 + i(k^2 - (c/2)^2)t} \quad (2.27)$$

and the four-parameter $(c, k, K, x_0)$ solution ("dark" soliton) as

$$\varepsilon < 0: \quad u = (-\varepsilon)^{-1/2}[(k/2) \tanh(k(x - ct - x_0)/2) - i(K - c/2)]e^{iKx - 2ik^2/4 + (K - c/2)^2 + K^2/2t} \quad (2.28)$$

### 2.4 Conservation laws

**Definition.** Given a PDE $E(u; x, t) = 0$, a conservation law is a relation

$$T_t + X_x = 0, \quad (2.29)$$

where $T$ and $X$, respectively called density and flux, depend on $x, t, u$ and its derivatives. If the total variation of $X$ in the interval $a \leq x \leq b$ is zero, the quantity $\int_a^b T \, dx$ is a constant of the motion $I$ called conserved quantity. "Integrable" PDEs possess an infinite number of conservation laws. For example, the first three conservation laws are:

(a) for the KdV equation [13, 136, 13]

$$T_1 = u, \quad X_1 = 3u^2 + u_{xx}; \quad (2.30)$$

$$T_2 = u^2/2, \quad X_2 = 2u^3 + uu_{xx} - u_x^2/2; \quad (2.31)$$

$$T_3 = 2u^3 - u_x^2, \quad X_3 = 9u^4 + 6u^2u_{xx} - 12uu_x^2 - 2uu_xu_{3x} + u_{xx}^2; \quad (2.32)$$

(b) for the MKdV equation [2.93, 9]

$$T_1 = u, \quad X_1 = 2u^3 + u_{xx}; \quad (2.33)$$

$$T_2 = u^2/2, \quad X_2 = 3u^4/2 + uu_{xx} - u_x^2/2; \quad (2.34)$$

$$T_3 = u^4/4 - u_x^2/4, \quad X_3 = u^6 + u^3u_{xx} - 3u^2u_x^2 - u_xu_{3x}/2 + u_{xx}^2/4; \quad (2.35)$$

9
(c) for the sG equation \(2.118\), \(81, 121, 43\)

\[
T_1 = \frac{u_x^2}{2}, \quad X_1 = \cos u; \quad (2.36)
\]

\[
T_2 = \frac{u_x^4}{4} - u_x^2, \quad X_2 = u_x^2 \cos u; \quad (2.37)
\]

\[
T_3 = 3u_x^6 - 12u_x^2u_{xx}^2 + 16u_x^3u_{3x} + 72u_x^2, \quad X_3 = (2u_x^4 - 24u_{xx}^2) \cos u. \quad (2.38)
\]

(d) for the NLS equation \(2.131\), we reproduce three of the five conservation laws given by Zakharov and Shabat \(139, 140\) for \(\varepsilon = \pm 1\)

\[
\varepsilon = +1, \quad I_1 = \int_{-\infty}^{+\infty} |u|^2 \, dx, \quad I_2 = \int_{-\infty}^{+\infty} (\overline{u}u_x - u\overline{u}_x) \, dx,
\]

\[
I_3 = \int_{-\infty}^{+\infty} \left( |u_x|^2 - \frac{1}{2} |u|^4 \right) \, dx \quad (2.39)
\]

\[
\varepsilon = -1, \quad I_1 = \int_{-\infty}^{+\infty} (1 - |u|^2) \, dx, \quad I_2 = -\int_{-\infty}^{+\infty} (\overline{u}u_x - u\overline{u}_x) \, dx,
\]

\[
I_3 = \int_{-\infty}^{+\infty} \left( |u|^4 + |u_x|^2 - 1 \right) \, dx \quad (2.40)
\]

(where \(\overline{u}\) denotes the complex conjugate of \(u\)).

For the Tzitzéica equation \(2.62\), Dodd and Bullough \(43\) first obtained two nontrivial conservation laws, then Mikhailov \(74\) gave a recursion formula for an infinite set of nontrivial polynomial conserved densities.

### 2.5 Bäcklund transformations

#### 2.5.1 Definition

A Bäcklund \(8\) transformation (BT) between two given PDEs

\[
E_1(u; x, t) = 0, \quad E_2(v; x', t') = 0 \quad (2.41)
\]

is a set of four relations (\[12\] vol. III chap. XII)

\[
F_j(u, v, u_x, v_x, u_t, v_t, \ldots; x, t, x', t') = 0, \quad j = 1, 2 \quad (2.42)
\]

\[
x' = X(x, t, u, u_x, u_t, v), \quad t' = T(x, t, u, u_x, u_t, v) \quad (2.43)
\]

such that the elimination of \(u\) (resp. \(v\)) between \((F_1, F_2)\) implies

\[
E_2(v; x', t') = 0 \quad (\text{resp. } E_1(u; x, t) = 0). \quad (2.44)
\]

In case the two PDEs are the same, the BT is called an auto-BT.
Bäcklund theory originates from the work of Lie and Bäcklund for the study of surfaces in differential geometry. The subject was subsequently developed by Goursat [57] and Clairin [24]. Bäcklund transformations represent an extension of Lie contact transformations. They were first obtained for second order PDEs in two independent variables, linear in the highest derivatives (i.e., a special type of Monge-Ampère equation).

For more details on BTs, the reader is advised to consult the book by Rogers and Shadwick [115] and the classical book of Goursat [56].

2.5.2 Examples: second order PDEs

Burgers and heat equations

Given the two equations

\[ E_1 \equiv u_t + (u_x + u^2)_x = 0, \quad E_2 \equiv v_t + v_{xx} = 0. \] (2.44)

the two relations defining the BT are:

\[ F_1 \equiv v_x - uv = 0, \quad F_2 \equiv v_t + u^2v + vu_x = 0. \] (2.45)

Indeed, the elimination of \( v \) (resp. \( u \)) yields the identities

\[ (F_2/v)_x - (F_1/v)_t \equiv E_1, \quad v \neq 0, \quad \text{and} \quad F_2 + F_{1,x} + uF_1 \equiv E_2. \] (2.46)

Liouville and d’Alembert

Given the two equations

\[ E_1 \equiv u_{xt} - e^u = 0, \quad E_2 \equiv v_{xt} = 0, \] (2.47)

the two relations

\[ F_1 \equiv u_x - v_x + \lambda e^{(u+v)/2} = 0 \] (2.48)

\[ F_2 \equiv u_t + v_t + (2/\lambda) e^{(u-v)/2} = 0, \] (2.49)

where \( \lambda \) is an arbitrary real constant called Bäcklund parameter, define a Bäcklund transformation as shown by the elimination of \( v \) (resp. \( u \))

\[ F_{1,t} + F_{2,x} - (1/\lambda)e^{(u-v)/2}F_1 - (\lambda/2)e^{(u+v)/2}F_2 \equiv 2E_1 \] (2.50)

\[ F_{1,t} - F_{2,x} + (1/\lambda)e^{(u-v)/2}F_1 - (\lambda/2)e^{(u+v)/2}F_2 \equiv -2E_2. \] (2.51)

Thus, the general solution of d’Alembert equation

\[ v = f(x) + g(t), \quad (f,g) \text{ arbitrary functions}, \] (2.52)
provides, by integration of the ODEs (2.48)–(2.49), a solution of (2.47)
\[ e^u = 2\varphi_x\varphi_t/\varphi^2, \varphi = (\lambda/2) \int^x e^f dx + (1/\lambda) \int^t e^{-g} dt \] (2.53)
which is the general solution. Travelling waves are built by the choice
\[ \varphi = \coth(\alpha x) - \tanh(\beta t) \Rightarrow e^u = 2\alpha\beta/\cosh^2(\alpha x - \beta t). \] (2.54)

**Sine-Gordon**

Given two solutions \( u \) and \( U \) of the sine-Gordon equation
\[ E_1 \equiv u_{xt} - \sin u = 0, \ E_2 \equiv U_{xt} - \sin U = 0, \] (2.55)
the auto-Bäcklund transformation is defined by
\[ F_1 \equiv (u + U)_x - 2\lambda\sin((u - U)/2) = 0 \] (2.56)
\[ F_2 \equiv (u - U)_t - (2/\lambda)\sin((u + U)/2) = 0, \ \lambda \text{ arbitrary constant} \] (2.57)
as can easily be checked quite similarly to the Liouville and d’Alembert case, by elimination of \( U \) (resp. \( u \)) between these two relations
\[ F_{1,t} + F_{2,x} + (1/\lambda)\cos((u + U)/2)F_1 + \lambda\cos((u - U)/2)F_2 \equiv 2E_1 \] (2.58)
\[ F_{1,t} - F_{2,x} - (1/\lambda)\cos((u + U)/2)F_1 + \lambda\cos((u - U)/2)F_2 \equiv 2E_2. \] (2.59)
Lamb [80] built from (2.56)–(2.57) infinite families of solutions, e.g. the \( N \)-soliton solution: at the first iteration, one starts from the solution \( U = 0 \) (“vacuum”), and the integration of the ODEs (2.56)–(2.57) yields
\[ \tan(u/4) = e^{\lambda x + \lambda^{-1} t + \delta}, \ \delta \text{ arbitrary constant}, \] (2.60)
i.e. the one-soliton solution
\[ u_x = 2\lambda\sech(\lambda x + \lambda^{-1} t + \delta), \ u_t = 2\lambda^{-1}\sech(\lambda x + \lambda^{-1} t + \delta). \] (2.61)

**Tzitzéica**

For the Tzitzéica equation (Tzi)
\[ u_{xt} = e^u - e^{-2u} \] (2.62)
there exists a complicated auto-BT [118, 13], and another, much simpler one will be published soon [36]. We only report here the classical, well established results.
The Lax pair given by Tzitzéica \[123, 124\] and rediscovered by Mikhailov \[93, 94\] consists in the following matricial system of linear PDE’s:

\[
\begin{align*}
\frac{\partial}{\partial x} \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_t \varphi \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & U_x & \lambda e^{-U} \\ e^U & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_t \varphi \end{pmatrix} \\
\frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_t \varphi \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ e^U & 0 & 0 \\ 0 & \lambda^{-1} e^{-U} & U_t \end{pmatrix} \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_t \varphi \end{pmatrix}
\end{align*}
\]

(2.63)

(2.64)

while the Moutard \[97\] transformation between two solutions \(U, u\) writes \[124\] :

\[e^u = -e^U + 2 \varphi_x \varphi_t / \varphi^2.\]

(2.65)

Since 1973, BTs have been found for PDEs of order greater than two. Different approaches have been used for deriving those transformations:

1. the method of Clairin \[24, 82\],
2. the method of differential forms developed by Wahlquist and Estabrook \[129, 130, 46\],
3. the method of bilinear transformations of Hirota \[60, 62, 91\],
4. the method of gauge transformations developed by Boiti \textit{et al.} \[12, 13\] and Levi \textit{et al.} \[84, 85\].

In the last two methods, the BT results from the elimination of the wave function between the Lax pair and the DT. In next sections, these two main concepts of complete integrability are briefly recalled; then the principle of the method of gauge transformations is presented. Lax pairs and DTs are explicitly given for the PDE’s of the AKNS scheme (KdV, MKdV, sine-Gordon, NLS) and for some fifth order PDEs, respectively in sections \[2.6.3\] and \[2.6.4\]. But, let us first give some definitions.

## 2.6 Darboux transformation and Lax pair

### 2.6.1 Definitions

\textbf{Crum-Darboux transformation}

This transformation is a key in the theory of nonlinear integrable evolution equations for building soliton solutions and understanding their “asymptotically
linear" superposition rules. It is based on a result obtained by the French mathematician Gaston Darboux in the special case of the Sturm-Liouville equation (also called Schrödinger equation in quantum mechanics). We briefly recall this old theorem \[41\] and its generalisation due to Crum \[40\].

**Theorem 1.** (Darboux) The linear Schrödinger equation

\[
\psi_{xx} + (u + \lambda)\psi = 0 \tag{2.66}
\]

is invariant under

\[
\psi \rightarrow \tilde{\psi} = (\partial_x - \frac{\psi_0 x}{\psi_0})\psi \tag{2.67}
\]

\[
u \rightarrow \tilde{\nu} = u + 2(\log \psi_0)_{xx} \tag{2.68}
\]

where \(\psi_0 \equiv \psi(x, \lambda_0)\) is an eigenfunction of (2.66) with parameter \(\lambda_0\). The essential point is that the new potential \(\tilde{\nu}\) depends only on \(\psi_0\) and not on \(\psi\). This transformation can then be iterated to obtain

**Theorem 2.** (Crum) The function

\[
\tilde{\psi} = \frac{W(\psi_1, \psi_2, \ldots, \psi_N, \psi)}{W(\psi_1, \psi_2, \ldots, \psi_N)} \tag{2.69}
\]

where \(\psi_1, \psi_2, \ldots, \psi_N\) are eigenfunctions of (2.66) associated with parameters \(\lambda_1, \lambda_2, \ldots, \lambda_N\) and the symbol \(W\) represents the Wronskian determinant, solves the equation (2.66) for the potential

\[
\tilde{\nu} = u + 2(\log W(\psi_1, \psi_2, \ldots, \psi_N))_{xx}. \tag{2.70}
\]

**Lax pair**

In 1968 Lax \[83\] explained in a very transparent way the greater part of the result of Gardner \textit{et al.} \[55\] by introducing the following operators

\[
L = -\partial_x^2 - u(x, t), \quad A = -4\partial_x^3 - 6u\partial_x - 3u_x \tag{2.71}
\]

such that the KdV equation (1.1) may be represented in the following way

\[
\partial_t L = [A, L] \tag{2.72}
\]

called the Lax representation. Equation (2.72) expresses the compatibility between the two partial differential equations of the system

\[
\begin{align*}
L\psi &= \lambda \psi \\
\psi_t &= A\psi \end{align*} \iff \begin{cases}
\psi_{xx} + (u + \lambda)\psi = 0 \\
\psi_t + (2u - 4\lambda)\psi_x - u_x \psi = 0
\end{cases} \tag{2.73}
\]

called Lax pair. This equivalence results from the identity

\[
\psi_{xxx} - \psi_{txx} \equiv \text{KdV}(u)\psi \tag{2.74}
\]
The system (2.73) is invariant under the Darboux transformation (2.67)–(2.68) with the compatibility condition

\[(\partial_t \tilde{L})\tilde{\psi} = [\tilde{A}, \tilde{L}]\tilde{\psi}\]  
(2.75)

where \((\tilde{L}, \tilde{A})\) results from the substitution of \(u\) by \(\tilde{u}\) in \((L, A)\).

### 2.6.2 Bäcklund gauge transformation

A general procedure to obtain BT for nonlinear PDE’s derived as compatibility conditions between a given generalised Lax pair of operators was simultaneously considered by Boiti et al. and Levi et al. in 1982. It has provided new results for multidimensional nonlinear PDE’s. Here, we only report the principle of the method. Let us consider the Lax pair

\[\psi_x = L\psi, \quad \psi_t = M\psi\]  
(2.76)

where \(\psi\) is an \(N \times N\) matrix as well as \(L, M\) which have a preassigned dependence on a matrix “potential” \(Q(x, t)\) and on a constant parameter \(\lambda\). The compatibility condition between the two equations of the system (2.76) implies the following nonlinear equation

\[L_t - M_x + [L, M] = 0\]  
(2.77)

To construct the BT for this nonlinear partial differential equation one has to consider two different systems of type (2.76) corresponding to two different “potentials”, say \(Q(x, t)\) and \(\tilde{Q}(x, t)\) :

\[
\begin{align*}
\psi_x &= L(Q(x, t); \lambda)\psi, & \psi_t &= M(Q(x, t); \lambda)\psi \\
\tilde{\psi}_x &= \tilde{L}(\tilde{Q}(x, t); \lambda)\tilde{\psi}, & \tilde{\psi}_t &= \tilde{M}(\tilde{Q}(x, t); \lambda)\tilde{\psi}
\end{align*}
\]  
(2.78)

One assumes that the following generalised DT holds between the wave functions \(\psi\) and \(\tilde{\psi}\) :

\[\tilde{\psi} = B\psi\]  
(2.80)

where \(B\) is a matrix function of \(Q, \tilde{Q}, x, t\) and \(\lambda\). The compatibility between (2.80) and the system (2.78)–(2.79) gives the auto-BT

\[B_x = \tilde{L}B - BL, \quad B_t = \tilde{M}B - BM\]  
(2.81)

By cross-differentiating these two relations one gets :

\[(\tilde{L}_t - \tilde{M}_x + [\tilde{L}, \tilde{M}]B - B(L_t - M_x + [L, M]) = 0\]  
(2.82)

which implies that if \(Q(x, t)\) satisfies the nonlinear PDE (2.77) then \(\tilde{Q}(x, t)\) satisfies the same equation. This exactly coincides with the definition of the BT previously given in section 2.5.1.
Let us also mention the book of Matveev and Salle [92] as a basis reference on Darboux transformation and its development in soliton theory.

In the extension of Painlevé analysis to NLPDEs [135], if a PDE fulfills the necessary conditions of integrability (“Painlevé test”), one tries to determine a Lax pair and a Darboux transformation relating two solutions of the same PDE in order to constructively prove the sufficiency of these conditions. A method (truncation procedure) leading to such a Lax pair and DT will be explained in section 3.2.2. In this formalism, the link with the notion of “general solution” is that the knowledge of the BT a priori allows to build wide classes of solutions. In one space dimension the “good” Lax pair of a given nonlinear PDE must depend on the solution of this equation and an arbitrary constant \( \lambda \). In the next section we show on examples how to derive the Lax pair and DT from the associated BT. In each case, it will be the aim of these lectures to show in chapter 3 how these two informations can be found algorithmically by singularity analysis.

### 2.6.3 Examples : AKNS scheme

**Korteweg-de Vries**

Its conservative form is (1.1) and we define the potential form as

\[
    u = w_x, \quad F(w) \equiv w_t + w_{xxx} + 3w_x^2 = 0. \tag{2.83}
\]

Given two solutions \( w \) and \( W \) of (2.83), the auto-BT is defined by [82]

\[
    (w + W)_x = 2\lambda - (w - W)^2 / 2 \tag{2.84}
\]

\[
    (w + W)_t = -2(w_x^2 + w_x W_x + W_x^2) - (w - W)(w - W)_{xx}, \tag{2.85}
\]

where \( \lambda \) is the Bäcklund parameter. After changing variables \( w,W \) to \( W,Y = (w - W)/2 \), the gradient of \( Y \) is defined by the Riccati equations

\[
    Y_x = \lambda - U - Y^2, \quad U = W_x \tag{2.86}
\]

\[
    Y_t = (U_x - (2U - 4\lambda)Y)_x. \tag{2.87}
\]

The transformation

\[
    Y = \partial_x \log \psi \tag{2.88}
\]

linearises these Riccati equations into one second order ODE and one first order PDE

\[
    \psi_{xx} + (U - \lambda)\psi = 0 \tag{2.89}
\]

\[
    \psi_t + (2U + 4\lambda)\psi_x - (U_x + G(t))\psi = 0, \quad G \text{ arbitrary function.} \tag{2.90}
\]

The Lax pair of KdV is defined by these two linear equations, which satisfy the compatibility condition

\[
    \psi_{xxt} - \psi_{txx} = E(U)\psi, \tag{2.91}
\]
while the DT for KdV is defined by the $x-$derivative of Eq. (2.88)

$$u - U = 2\partial^2_x \log \psi.$$  \hfill (2.92)

**Modified Korteweg-de Vries**

Its conservative form is

$$E(u) \equiv u_t + u_{xxx} - 2a^{-2}(u^3)_x = 0$$  \hfill (2.93)

and we define the potential form as

$$u = w_x, \quad F(w) \equiv w_t + w_{xxx} - 2a^{-2}w^3_x = 0.$$  \hfill (2.94)

Given two solutions $w$ and $W$ of (2.94), the auto-BT is given by \[82\]

$$(w + W)_x = -2a\lambda \sinh((w - W)/a)$$  \hfill (2.95)

$$(w + W)_t = 8\lambda^2 W_x - 4\lambda W_{xx} \cosh((w - W)/a) + 4a(2\lambda^3 - \lambda W^2_x/a^2) \sinh((w - W)/a),$$  \hfill (2.96)

where $\lambda$ is the Bäcklund parameter. The change of variables

$$(w, W) \rightarrow (W, Y = e^{(w-W)/a})$$  \hfill (2.97)

maps these equations into the two Riccati equations for $Y$,

$$Y_x = -2(U/a)Y + \lambda(1 - Y^2), \quad U = W_x$$  \hfill (2.98)

$$Y_t = 2A_1 Y + B_1(1 + Y^2) + C_1(1 - Y^2)$$  \hfill (2.99)

$$= (-4U/a + (2(U/a)^2 - 4\lambda^2 + 2(U_x/a))Y)_x$$  \hfill (2.100)

$$A_1 = \frac{U_{xx}}{a} - 2\frac{U^3}{a^3} + 4\lambda^2 U/a, \quad B_1 = -2\lambda \frac{U_x}{a}, \quad C_1 = 2\lambda U^2/a^2 - 4\lambda^3.$$  \hfill (2.101)

The compatibility condition of this “Riccati pseudopotential” $Y$ is

$$Y_{xt} - Y_{tx} = -(2/a)E(U)Y.$$  \hfill (2.102)

The Lax pair is obtained by linearising these two Riccati equations by the transformation

$$Y = \psi_1/\psi_2$$  \hfill (2.103)

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -U/a & \lambda \\ \lambda & U/a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$  \hfill (2.104)

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = \begin{pmatrix} A_1 & B_1 + C_1 \\ B_1 - C_1 & -A_1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$  \hfill (2.105)
while the Darboux transformation is defined by
\[ u - U = a \partial_x \log Y, \]
which, by elimination of \( Y \) with (2.98), is identical to
\[ u + U = a \lambda (Y^{-1} - Y). \]
The homographic transformation with \( \alpha = U/a \)
\[ Y = \lambda \chi / (1 + \alpha \chi), \]
maps the Riccati system (2.98)–(2.99) into the simpler form:
\[
\begin{align*}
\chi_x &= 1 + (S/2)\chi^2, \\
\chi_t &= -C + C_x \chi - (1/2)(CS + C_{xx})\chi^2, \\
S &= 2 \left( \frac{U_x}{a} - \left( \frac{U}{a} \right)^2 - \lambda^2 \right), \\
C &= 2 \left( \frac{U_x}{a} - \left( \frac{U}{a} \right)^2 + 2\lambda^2 \right).
\end{align*}
\]
We shall see that the relation between the two functions \( S \) and \( C \):
\[ S - C + 6\lambda^2 = 0 \]
corresponds to the singular manifold (SM) equation of the KdV equation \cite{133} and can be found algorithmically \cite{110} when one performs the Painlevé analysis of the MKdV equation. In the variable
\[ f = a(Y - 1)/(Y + 1), \]
the system (2.98)–(2.99) and the DT (2.106) or (2.107) become \cite{126, 128}
\[
\begin{align*}
u - U &= 2a^2 f_x / (a^2 - f^2), \\
u + U &= -4a^2 \lambda f / (a^2 - f^2), \\
a f_x &= -\left( U/a \right)(a^2 - f^2) - 2\lambda af \\
a f_t &= A_1(a^2 - f^2) + B_1(a^2 + f^2) - 2C_1 af.
\end{align*}
\]
\textbf{Sine-Gordon}
\[
E(u) = u_{xt} - \sin u = 0.
\]
Given two solutions \( u \) and \( U \) of (2.118), the auto-BT is given by \cite{80}
\[
\begin{align*}
(u + U)_x &= -4\lambda \sin((u - U)/2), \\
(u - U)_t &= -\lambda^{-1} \sin((u + U)/2),
\end{align*}
\]
where $\lambda$ is the Bäcklund parameter. The change of variables

$$(u, U) \rightarrow (U, Y = e^{-i(u-U)/2})$$

maps these equations into the two Riccati equations for $Y$

$$Y_x = iU_x Y + \lambda(1 - Y^2)$$
$$Y_t = ((1 - Y^2) \cos U + i(1 + Y^2) \sin U)/(4\lambda).$$

The compatibility condition of the Riccati pseudopotential $Y$ is

$$Y_{xt} - Y_{tx} = iE(U)Y.$$ 

The Lax pair is obtained by linearising the Riccati system

$$Y = \frac{\psi_1}{\psi_2}$$

while the Darboux transformation for sG is defined by

$$u - U = 2i \log Y,$$
$$\quad (u + U)_x = 2i\lambda(Y^{-1} - Y).$$

The homographic transformation (2.108) with $\alpha = -iU_x/2$ maps the Riccati system (2.122)–(2.123) into the simpler form (2.109)–(2.110), with

$$S = -iU_{xx} + U_x^2/2 - 2\lambda^2, \quad C = -e^{iU}/(4\lambda^2).$$

The relation between $S$ and $C$

$$S + C_{xx}/C - (1/2)(C_x/C)^2 + 2\lambda^2 = 0$$

represents the SM equation obtained by Conte [29] when performing the invariant Painlevé analysis of the sine-Gordon equation.

**Nonlinear Schrödinger**

$$E(u) \equiv iu_t + u_{xx} + 2\varepsilon |u|^2 u = 0, \ \varepsilon = \pm 1.$$  

Given two solutions $u$ and $u'$ of (2.131), the auto-BT can be written as [23, 22, 73, 86]

$$(u + U)_x = (u - U)\sqrt{4\lambda^2 - \varepsilon|u + U|^2}$$
$$(u + U)_t = i(u - U)_x\sqrt{4\lambda^2 - \varepsilon|u + U|^2} + i\varepsilon(u + U)(|u + U|^2 + |u - U|^2)/2.$$
The extension to NLS of the transformation (2.114) is
\[ u + U = -4\lambda f / (1 + \varepsilon |f|^2). \] (2.134)
Therefore, the change of variables \((u, U) \to (U, f)\) transforms (2.132) into
\[ -f_x + \varepsilon f^2 \overline{T}_x = (1 - \varepsilon |f|^2)(U(1 + \varepsilon |f|^2) + 2\lambda f). \] (2.135)
The elimination of \(f\) between this equation and its c.c., assuming \(1 - |f|^4 \neq 0\), provides
\[ f_x = -2\lambda f - U - \varepsilon U f^2. \] (2.136)
while the \(t\)-part is
\[ f_t = (\lambda U + U_x) + (\varepsilon U \overline{U} + \lambda^2) f + (\lambda U - \overline{U}_x) f^2 \] (2.137)
with the identity
\[ f_{xt} - f_{tx} = E + \varepsilon E f^2. \] (2.138)
Equations (2.132) and (2.134) imply
\[ u - U = 2(f_x - f^2 \overline{T}_x)/(1 - |f|^4). \] (2.139)
In all the above examples (KdV, MKdV, sG, NLS), the DT is defined with one (for KdV) or two (for the others) entire functions \(\psi\). This distinction is the only relevant feature needed to obtain in an algorithmic way the Lax pair by methods linked to the singularity structure of these equations.

### 2.6.4 Higher order KdV-type equations

Among the fifth order nonlinear evolution equations
\[ u_t + (u_{xxxxx} + (8\alpha - 2\beta)uu_{xx} - 2(\alpha + \beta)u_x^2 - (20/3)\alpha \beta u^3)_x = 0 \] (2.140)
only three cases are integrable:
\[ \beta/\alpha = -1 : u_t + (u_{xxxxx} + 10\alpha uu_{xx} + 20\alpha^2 u_x^3/3)_x = 0 \] (2.141)
\[ \beta/\alpha = -6 : u_t + (u_{xxxxx} + 20\alpha uu_{xx} + 10\alpha u_x^2 + 40\alpha^2 u^3)_x = 0 \] (2.142)
\[ \beta/\alpha = -16 : u_t + (u_{xxxxx} + 40\alpha uu_{xx} + 30\alpha u_x^2 + 320\alpha^2 u^3/3)_x = 0 \] (2.143)
respectively named Sawada-Kotera (SK) or Caudrey-Dodd-Gibbon [120, 21], Lax's 5th order KdV (KdV5) [83] and Kaup-Kupershmidt (KK) [69].

Their respective Lax representation (2.72) is
(SK) \(\alpha = 3, \ L = \partial_x^3 + 6u \partial_x, \ A = 9\partial_x^5 + 90u \partial_x^3 + 90u_x \partial_x^2 + (60u_{xx} + 180u^2) \partial_x;\)
(KdV5) \(\alpha = 1/2, \ L = \partial_x^3 + u, \ A = 16\partial_x^5 + 40u \partial_x^3 + 60u_x \partial_x^2 + (50u_{xx} + 30u^2) \partial_x + 15u_{xxx} + 30uu_x;\)
(KK) \(\alpha = 3/4, \ L = \partial_x^3 + 6u \partial_x + 3u_x, \ A = 3(3\partial_x^5 + 30w \partial_x^3 + 45u_x \partial_x^2 + (35u_{xx} + 60u^2) \partial_x + 10u_{xxx} + 30uu_x).\)
We discard the equation (2.142) for it has the same second order scattering problem as the KdV equation, and we restrict to the two equations (2.141)–(2.143) possessing two different third order scattering problems \( L\psi = \lambda \psi \).

The SK equation possesses the Darboux transformation \( [119] \)

\[
    u = U + \partial^2_x \log \psi \tag{2.146}
\]

while for the KK equation this transformation writes \( [87] \)

\[
    u = U + (1/2) \partial^2_x \log \varphi, \quad \varphi = \psi\psi_{xx} - (1/2)\psi^2_x + 3U \psi^2. \tag{2.147}
\]

In the notation \( w_x = u, W_x = U \), the \( x \)-part of the BT for SK \( [44, 119] \) is

\[
    (w - W)_{xx} + 3(v - W)(w + W)_x + (w - W)^3 = \lambda, \tag{2.148}
\]

while for the KK equation it writes

\[
    (w - W)_{xx} + 3(w - W)(w + W)_x - (3/4)(w - W)^2_x/(w - W) + (w - W)^3 = \lambda. \tag{2.149}
\]

This last expression was obtained for the first time by Rogers and Carillo \( [116] \) in the particular case \( \lambda = 0 \).
Table 2.1: Some reductions of a PDE to an ODE and their solutions. The PDEs $E(u, x, t) = 0$ (KdV, MKdV, sG, Bq, NLS, Tzi) are respectively defined by the equations (1.1), (2.93), (2.118), (3.50), (2.131), (2.62). The reduction to an ODE for $U(ξ)$ is defined by the two expressions of $u$ in terms of $(U, x, t)$ and of $ξ$ in terms of $(x, t)$. The letter $K$, with or without subscript, denotes an arbitrary constant. Last column indicates the elementary function $(φ, (P1)–(P6))$ whose general solution of the ODE is a singlevalued algebraic transform.

| PDE  | $u$   | $ξ$   | ODE                                      | $φ$, $(Pn)$ |
|------|-------|-------|------------------------------------------|-------------|
| KdV  | $U$   | $x - ct$ | $U'' + 2U^3 - cU^2 + 2K_1U + 2K_2 = 0$ | $φ$         |
| KdV  | $U - λt$ | $x + 3λt^2$ | $U'' + 3U^2 - λξ + K = 0$ | $(P1)$      |
| MKdV | $U$   | $x - ct$ | $U'' - U^4 - cU^2 + K_1U + K_2 = 0$     | $φ$         |
| MKdV | $(3t)^{-1/3}U$ | $(3t)^{-1/3}$ | $U'' - 2U^3 - ξU + K = 0$                  | $(P2)$      |
| sG   | $-i \log U$ | $x - ct$ | $cU'' + U^3 + KU^2 + U = 0$             | $φ$         |
| sG   | $-i \log U$ | $xt$ | $U'' - U^2/U + U'/ξ + (1 - U^2)/(2ξ) = 0$ | $(P3)$      |
| Bq   | $U$   | $x - ct$ | $(U''/3) + U^2 + c^2U + K_1ξ + K_2 = 0$ | $φ$, $(P1)$ |
| Bq   | $2(U' + ξ - t^2)$ | $x - t^2 + K_1$ | $U''/2 + U^3 + 12(U'' - U')U'' + K_2U' + K_3 = 0$ | $(P2)$ $[11, 22]$ |
| Bq   | $(U' - ξ^2/2)/t$ | $xt^{-1/2}$ | $-(9/8)(U - ξU)^2 + K_1(U - ξU') + K_2U' + K_3 = 0$ | $(P4)$ $[13, 22]$ |
| NLS  | $(2.20)$ | $x - ct$ | $(2.23)$                                      | $φ$         |
| NLS  | $e^{i(xt-4t^3/3)U}$ | $x - t^2$ | $U'' + 2ξU^3 - 2ξU = 0$                      | $(P2)$ $[12, 23]$ |
| NLS  | $t^{-1/2}\sqrt{U''}e^{iφ}$ | $xt^{-1/2}$ | $4U''^2 + 4ξU^3 + KU'' + (ξU'' - U'')^2/4 = 0$ | $(P4)$ $[13, 22, 19]$ |
| Tzi  | $\log U$ | $x - ct$ | $-cU'' + 2aU^3 + KU^2 - a_0 = 0$                  | $φ$         |
| Tzi  | $\log U$ | $xt$ | $(ξU'/U) + aU + a_0U^{-2} = 0$                   | $(P3)$ $[27]$ |
Chapter 3

Painlevé analysis for PDEs

The WTC extension [133] of Painlevé analysis to partial differential equations consists of two parts

1. generation of necessary conditions (Painlevé test) for the absence of movable critical singularities in the "general solution",

2. explicit proof of sufficiency by finding the transformation which linearises the PDE or yields an auto-BT or a BT to another PDE with the PP.

The methods relative to both parts are different.

In the first part, for every noncharacteristic manifold ($\varphi(x, t) = 0, \varphi_x \neq 0$), one tests the existence of all possible local representations of the "general solution" by a Laurent series in the neighbourhood of $\varphi = 0$. This test may

- pass whatever be $\varphi$; the PDE may then have the PP,
- fail whatever be $\varphi$; this is typical of chaotic PDEs,
- pass with some constraints on $\varphi$; then there exists particular Laurent series and the PDE is called "partially integrable".

In the second part, the Weiss truncation procedure [133], using only the singular part of the Laurent series, may yield constructive results like

- the linearising transformation or the BT, in case the PDE passes the Painlevé test for every $\varphi$,
- particular solutions, necessarily compatible with the singularity structure of the PDE, in case the Painlevé test is conditionally or not satisfied (see chapter 4).
3.1 Necessary conditions (Painlevé test)

Contrary to the case of ODEs, the singularities in the complex domain of \((x, t)\) are not isolated. Given a PDE \(E(u, x, t) = 0\) of order \(N\) polynomial in \(u\) and its partial derivatives (maybe after a preliminary change of variables), we consider the associated equation \(\varphi(x, t) = 0\) of the movable SM and an expansion of \(u\) and \(E\) as a Laurent series in \(\chi\) in the neighborhood of \(\varphi = 0\).

We distinguish between \(\varphi\) and the expansion variable \(\chi\) and only require \(\chi\) to vanish as

\[
 u(x, t) = \sum_{j=0}^{\infty} u_j(x, t) \chi^{j+p}, \quad E(u, x, t) = \sum_{j=0}^{\infty} E_j(x, t) \chi^{j+q},
\]

where \((p, q)\) are two negative integers with \(q \leq p - 1\), and \((u_j, E_j)\) the Laurent series coefficients. The result of the Painlevé test (necessary conditions) is independent of the explicit expression for \(\chi\) but some particular choices are better than others during the second part (sufficient conditions) when one looks for the Lax pair or tries to linearise the equation.

The main choices (gauges) for the expansion variable \(\chi\) are

- i) \textbf{WTC gauge} \([35]\) \(\chi = \varphi\), hence coefficients \((u_j, E_j)\) rational in the derivatives \(D\varphi\) of \(\varphi\)

- ii) \textbf{dimensionless WTC gauge} \(\chi = \varphi/\varphi_x\), hence coefficients \((u_j, E_j)\) rational in the derivatives \(D\varphi\) of \(\varphi\) of homogeneity degree zero,

- iii) \textbf{Kruskal gauge} \([38]\) \(\chi = x - f(t)\), \(f\) arbitrary, hence coefficients \((u_j, E_j)\) independent of \(x\) and rational in the derivatives of \(f\). This is the simplest choice for the test, but it cannot be used to obtain the Lax pair or particular solutions.

- iv) \textbf{Conte gauge} \([23]\) \(\chi = \varphi/(\varphi_x - \varphi_{xx}\varphi/(2\varphi_x)) \sim_{\varphi \to 0} \varphi/\varphi_x\), hence coefficients \((u_j, E_j)\) rational in the derivatives of \(\varphi\) invariant under the group of homographic transformations \(\varphi \rightarrow (a\varphi + b)/(c\varphi + d)\), \((a, b, c, d)\) arbitrary complex constants.

In this last case, the Riccati system satisfied by \(\chi\) is

\[
 \chi_x = 1 + (S/2)\chi^2 \\
 \chi_t = -C + C_x\chi - (1/2)(CS + C_{xx})\chi^2 \\
 2((\chi_t^{-1})_x - (\chi_x^{-1})_t) = S_t + C_{xxx} + 2C_xS + CS_x = 0
\]

with

\[
 S = \{\varphi; x\} = (\varphi_{xx}/\varphi_x)x - (1/2)(\varphi_{xx}/\varphi_x)^2, \quad C = -\varphi_t/\varphi_x.
\]
The transformation $\chi = \psi/\psi_x$ linearises this Riccati system into

$$\psi_{xx} + (S/2)\psi = 0$$

(3.6)

$$\psi_t + C\psi_x - (C_x/2 + g(t))\psi = 0, \ g \text{ arbitrary function.}$$

(3.7)

This choice of gauge is equivalent to the expansion of $(u, E)$ as

$$u = \sum_{j=0}^{+\infty} u_j(\psi/\psi_x)^j, \ E = \sum_{j=0}^{+\infty} E_j(\psi/\psi_x)^j,$$

(3.8)

where the function $\psi$ satisfies a second order linear ODE in the $x$ variable.

To obtain the couples $(u_0, p)$ one substitutes in the polynomial PDE

$$u \rightarrow u_0\chi^p, \ \chi_x \rightarrow 1, \ \chi_t \rightarrow -C, \ Du \rightarrow u_0D(\chi^p).$$

(3.9)

One then determines the balance between the different terms of this polynomial expression. Each different solution $(u_0, p)$ defines a family. For every $j \geq 1$ the recurrence relation determining $u_j$ is

$$\forall j \geq 1 : \ P(j)u_j = Q_j(\{u_k, Du_k, \ k \in [0, j-1]\}),$$

(3.10)

where $P$ is a polynomial of degree at most $N$.

The main requirements of the Painlevé test are

- the zeros of $P$ (Fuchs indices, also named Painlevé resonances) are distinct integers,
- for every index $i$ and every $\varphi$, the compatibility condition $Q_i = 0$ holds.

### 3.2 Methods for proving sufficiency

One distinguishes two main methods:

1. **the singular part transformation** which may provide the explicit transformation linearising the nonlinear PDE. If this is not the case, the transformation may yield an equation in a form more convenient than the original one to search for explicit solutions,

2. **the truncation procedure** of Weiss and its extensions for obtaining the BT and thus proving that the nonlinear PDE possesses the PP.

#### 3.2.1 Singular part transformation

The method consists of transforming the PDE for $u$ into an equation for $\varphi$ by the nonlinear transformation...
\[ u = D \log \varphi, \quad (3.11) \]

where \( D \) is the singular part operator associated with one of the families defined in the Painlevé test.

**Example 1 (linearisation).** **Burgers** equation

\[ u_t + u_{xx} + (u^2)_x = 0 : \quad u = \varphi_x \varphi^{-1} \quad (3.12) \]

\[ u = D \log \varphi = \partial_x \log \varphi ; \quad \varphi_t + \varphi_{xx} + K(t) \varphi = 0, \quad K(t) \text{arbitrary function}. \quad (3.13) \]

**Example 2 (linearisation).** **Liouville** equation

\[ \nu_t = e^v \quad (3.14) \]

\[ e^v = u, \quad uu_{xt} - u_x u_t - u^4 = 0 : \quad u = 2(\varphi_x \varphi_t \varphi^{-2} - \varphi_{xt} \varphi^{-1}) \quad (3.15) \]

\[ u = D \log \varphi = -2 \partial^2_{xx}(\log \varphi), \quad \varphi_{xt} = 0. \quad (3.16) \]

**Example 3 (linearisation).** **Eckhaus** equation \[ [20, 26, 34] \]

\[ iu_t + u_{xx} + q_r(|u|^4 + 2a(|u|^2)_x)u = 0, \quad a^2 = 1/q_r, \quad q_r \in \mathcal{R}. \quad (3.17) \]

In the variables \((w, \theta)\) defined by \(\theta = \arg u, w_x = |u|^2\), the equation (3.17) is equivalent to the system

\[ \theta_x = -\frac{1}{2} \frac{w_t}{w_x}, \quad \theta_t = \frac{1}{4} \left( 2 \frac{w_{xxx}}{w_x} - \frac{w^2_{xx}}{w^2_x} \right) - \frac{1}{4} \frac{w^2}{w^2_x} + q_r (w_x^2 + 2aw_{xx}) \quad (3.18) \]

whose compatibility condition is

\[ \theta_{xt} - \theta_{tx} = \left( w_t w_x^2 + w_{xx} w_{xx} \right)/2 + \left( w_{xxxx} w_x^2 + w_{xx}^3 \right)/2 - w_{xx} w_{xxxx} \]

\[ - w_t w_x w_{xx} + 2q_r (w_x^4 w_{xx} + aw_x^3 w_{xxx}) = 0. \quad (3.19) \]

Under the transformation \(w = (a/2) \log \varphi\) defined by the singular part operator \(D\) of the equation for \(w\), these three equations become

\[ \theta_x = -\varphi/(2\varphi_x), \quad \theta_t = \varphi_{xxx}/(2\varphi_x) - \varphi^2_{xx}/(4\varphi^2_x) - \varphi^2/(4\varphi^2_x) \quad (3.20) \]

\[ \theta_{xt} - \theta_{tx} = \varphi_{xx}^2 \varphi_{xxx} - \varphi_{x} \varphi_{xx} \varphi_{x} = 0. \quad (3.21) \]

The three equations (3.20), (3.20), (3.21) are deduced from the three previous ones (3.18), (3.18), (3.19) by the following simple operation : change \(w\) to \(\varphi\) and assign \(q_r\) to zero. Thus the transformation has linearised the Eckhaus equation (3.17) into the Schrödinger equation

\[ i\nu_t + \nu_{xx} = 0, \quad \varphi_x = |\nu|^2, \quad \arg \nu = \arg u. \quad (3.22) \]
Because of the conservation of the phase, one finally has
\[ u = \left( \sqrt{\frac{a}{2}} \right) \nu \int \sqrt{\nu^2} dx, \quad |u|^2 = (a/2) \partial_x \log \varphi. \quad (3.23) \]

Example 4 (bilinearisation). Korteweg-de Vries equation
\[ u_t + u_{xxx} + 3(u^2)_x = 0 : \quad u = -2\varphi_x^2 \varphi^{-2} + 2\varphi_{xx}/\varphi^{-1} \quad (3.24) \]
\[ u = D \log \varphi = 2 \partial_x \log \varphi; \quad (D_x D_t + D_x^2)(\varphi \cdot \varphi) = 0 \quad (3.25) \]

The transformed equation, quadratic in \( \varphi \) (see the numerous papers of Hirota [59, 60] for the definition of the bilinear operators \( D_x, D_t \)) is convenient to look for \( N \)-soliton solutions, auto-BTs, Miura transformations.

3.2.2 Weiss method and its limitations

If a nonlinear PDE passing the Painlevé test is not linearisable, the idea of Weiss [132, 133] is that the principal part of this local Laurent series contains all the information for proving that the PDE possesses the Painlevé property through the knowledge of its BT (i.e. its DT and Lax pair). This method consists of truncating the Laurent series for \( u \) and \( E(u) \) to their nonpositive powers in \( \chi \)
\[ u_T = \sum_{j=0}^{-p} u_j \chi^{j+p}, \quad E_T = \sum_{j=0}^{-q} E_j \chi^{j+q} \quad (3.26) \]

and identifying to zero the coefficients \( E_j \) of the \( \chi \)−polynomial \( \chi^{-q} E_T(\chi) \).

Equations \( E_j = 0 \) for \( j = 0, \ldots, -p \) determine the \( p + 1 \) coefficients \( u_j \) as equal to those of the infinite expansion. After replacement of \( u_j \) by these values, the remaining equations are
\[ E_j(D \varphi, u_i) = 0 \quad j \in \{-p + 1, \ldots, -q\} \quad (3.27) \]
\[ j \neq \text{compatible indices} \quad i = \text{indices} \in \{0, \ldots, -p\} \]

In the Conte gauge, the coefficients \( u_j, E_j \) depend on the derivatives of \( \varphi \) through the homographic invariants \( (S, C) \) and their derivatives. As the variable \( \chi^{-1} = \psi_x/\psi \) satisfies a Riccati equation one can connect the monomial \( (\psi_x/\psi)^n \) with the derivatives \( (\log \psi)_j x, (j \leq n \in \mathbb{N}^+) \) and show that
\[ \sum_{j=0}^{p-1} u_j(S, C)(\psi/\psi_x)^{j+p} \equiv \sum_{j=1}^{-p} \tilde{u}_j(S, C)(\log \psi)_j x + f(S, C). \quad (3.28) \]

Then the relation
\[ u_T - \tilde{u} = \sum_{j=1}^{-p} \tilde{u}_j(S, C)(\log \psi)_j x = D \log \psi, \quad (3.29) \]
where $D$ is the singular part operator and $\tilde{u} = u - p(S, C) + f(S, C)$, defines a Darboux transformation if $E(\tilde{u}) = 0$. For this reason, we call equations (3.27) Painlevé-Darboux equations. The elimination of the arbitrary functions $u_i$ among this set must produce only one “independent” equation

$$F(S, C) = 0$$

(3.30)
called the singular manifold equation, modulo the ever present link between $S$ and $C$ given by equation (3.4).

The next step consists of finding a parametric representation for equation (3.31) under the form $(S, C)$ depending on a function $U$ and an arbitrary constant $\lambda$ such that the cross-derivative condition (3.4) is identical to the original equation $E(U) = 0$ for $U$. If this is indeed the case and if $U$ can be identified with $\tilde{u}$, the truncation will provide the DT (as a consequence of equation (3.29)), the Lax pair (as consequences of the linear system (3.6)–(3.7) and the parametrisation of $S$ and $C$) and thus the BT.

This method only succeeds for a few equations, like KdV [135], KdV5, AKNS [98], all belonging to the same hierarchy. Let us describe it for the KdV equation (1.1). This equation, which passes the Painlevé test, admits the single family

$$u \sim -2\chi^{-2}$$

with Fuchs indices $-1, 4, 6$

$$u = -2\chi^{-2} + (C - 4S)/6 - (1/6)(C - S)\chi + O(\chi^2).$$

(3.31)

The algorithmic results of the Painlevé analysis for KdV are given in Table 3.1. They yield the SM equation

$$C - S + 6\lambda = 0, \quad \lambda = \text{arbitrary constant}.$$

(3.32)

Its parametric representation

$$S = 2(U + \lambda), \quad C = 2(U - 2\lambda)$$

(3.33)

provides the second order linear system (3.6)–(3.7)

$$\psi_{xx} + (U + \lambda)\psi = 0$$

(3.34)

$$\psi_t + 2(U - 2\lambda)\psi_x - U_x\psi = 0$$

(3.35)

satisfying the cross-derivative condition $\psi_{xxt} - \psi_{txx} \equiv 2 \text{KdV}(U)\psi = 0$. The map between two solutions of KdV coming out of the truncation is

$$u_T = 2(\text{Log } \psi)_{xx} + (C + 2S)/6 = 2(\text{Log } \psi)_{xx} + U$$

(3.36)

Thus, the Weiss truncation yields both the Lax pair (2.89)–(2.90) and the DT (2.93) of the KdV equation. The auto-BT (2.84)–(2.85) is obtained by substitution of the DT (3.31), i.e. $\psi_x/\psi = (w - W)/2$, into the couple (3.34) and (3.33) (notation $u_T = u = u_x, U = U_x$).
It happens that, for other equations possessing either one family of movable singularities or several families with nonopposite residues like Boussinesq, Sawada-Kotera, Hirota-Satsuma \([63]\) equations, the parametrization of \((S,C)\) yields a condition (3.4) for \(U\) different from the original equation, this defines a transformation between \(u_T\) and \(U\), called Miura transformation, obtained by the elimination of \((\chi,S,C)\) between the four equations : \(u_T = \text{the truncation}\), the two equations of the parametric representation \((S,C) = f(U)\) and anyone of the two (nonindependent) equations (3.2), (3.3). Then in order to obtain the auto-BT, one requires that the function \(\psi\) in equation (3.29) satisfies a linear third order system whose coefficients are to be determined as functions of \(\lambda\) and another solution \(U\) of the analyzed PDE linked to \(u_T\) through the Darboux transformation.

### 3.2.3 Method for third order Lax pair

Let us denote \((a,b,c,d,e)\) the five unknown coefficients defining a third order linear system for \(\psi\)

\[
\begin{align*}
\psi_{xxx} &= a\psi_x + b\psi \quad (3.37) \\
\psi_t &= c\psi_{xx} + d\psi_x + e\psi \quad (3.38)
\end{align*}
\]

whose compatibility condition is

\[
(\psi_t)_{xxx} - (\psi_{xxx})_t \equiv X_0\psi + X_1\psi_x + X_2\psi_{xx} = 0 \quad (3.39)
\]

\[
\begin{align*}
X_0 &\equiv -b_t - ac_x + e_{xxx} + b_x c + 3bc_{xx} + 3bd_x + b_x d = 0 \quad (3.40) \\
X_1 &\equiv -a_t + 3e_{xx} + 2b_x c + a_{xx}c + d_{xxx} + 3ac_{xx} + 2ad_x + 3acx + 3bcx + axd = 0 \quad (3.41) \\
X_2 &\equiv (2ac + cxx + 3dx + 3e)x = 0. \quad (3.42)
\end{align*}
\]

In the two independent components \(Z_1 = \psi_x/\psi, Z_2 = \psi_{xx}/\psi\), the linear system (3.37)–(3.38) is equivalent to the projective Riccati system [3]

\[
\begin{align*}
Z_{1,x} &= (-Z_1)Z_1 + Z_2 \quad (3.44) \\
Z_{2,x} &= (-Z_1)Z_2 + aZ_1 + b \quad (3.45) \\
Z_{1,t} &= (-dZ_1 - cZ_2)Z_1 + (ac + dx)Z_1 + (c_x + d)Z_2 + e_x + bc \quad (3.46) \\
Z_{2,t} &= (-dZ_1 - cZ_2)Z_2 + (2acx + axc + bc + dx + ad + 2ex)Z_1 + (c_{xx} + 2dx + ac)Z_2 + 2bcx + bx + bd + e_{xx}. \quad (3.47)
\end{align*}
\]

The determining equations for the coefficients \((a,b,c,d,e)\) of the Lax pair are generated by the expansion of \(E_T = E(u_T)\) on the basis \((Z_1,Z_2)\)

\[
E_T = \sum_{i,m} C_{i,m} Z_1^i Z_2^m, \quad (3.48)
\]
\[ C_{l,m} \equiv C_{l,m}(a,b,c,d,e,U) = 0. \] (3.49)

In case the solution of the determining equations does not lead to the expected solution, for a reason like the absence of a spectral parameter, the assumption to be changed is the order of the underlying scattering problem.

Let us give more details on the procedure [99, 100] for finding the BT of the Boussinesq and Sawada-Kotera equations.

**First example: Boussinesq equation**

Let us consider the Boussinesq (Bq) equation [125, 142]

\[ E(u) \equiv u_{tt} + \varepsilon^2 \left( (u + \alpha)^2 + (\beta^2 / 3)u_{xx} \right)_{xx} = 0, \] (3.50)

with \((\alpha, \beta, \varepsilon)\) constant. The algorithmic results of the Painlevé analysis are:

\[ p = -2, \quad q = -6, \quad \text{indices } -1,4,5,6 \text{ compatible} \]

\[ u_T = -(2/3)\beta^2 \chi^{-2} - (1/2)\varepsilon^{-2}C^2 - (4/3)\beta^2(\chi^{-1})_x. \] (3.51)

The set of Painlevé-Darboux equations reduces to the single equation:

\[ E_3 \equiv (1/3)\beta^2 \varepsilon^2 S_x - C_t + CC_x = 0, \] (3.52)

which is the SM equation for the Bq equation [33] in the invariant form.

This is a conservation law, which can be parametrised as

\[ C = (\beta \varepsilon)^2 z_x, \quad S = 3z_t - (3/2)(\beta \varepsilon)^2 z_x^2. \] (3.53)

The compatibility condition of the system (3.6)–(3.7) reads

\[ 3z_{tt} + (\beta \varepsilon)^2 z_{xxxx} + 6(\beta \varepsilon)^2 z_t z_{xx} - 6(\beta \varepsilon)^4 z_x^2 z_{xx} = 0, \] (3.54)

which is not the Bq equation but another PDE called modified Bq equation [64, 50]. The elimination of \(S\) between (3.51) and (3.2) yields the Miura transformation between the Bq and the modified equation

\[ u_T = -(2/3)\beta^2 \chi^{-2} - (1/2)\varepsilon^{-2}C^2 + (4/3)\beta^2(\chi^{-1})_x. \] (3.55)

while the assumption for a DT like (3.29) leads to

\[ \ddot{u} = -(2/3)\beta^2 \chi^{-2} - (1/2)\varepsilon^{-2}C^2 - (2/3)\beta^2(\chi^{-1})_x, \] (3.56)

which does not coincide with (3.55). We then conclude that a second order linear system is not convenient to represent the Lax pair of the Bq equation.

So, let us assume an underlying scattering problem of the third order for \(\psi\) and the existence of a DT given by the singular part operator

\[ v_T = 2\beta^2 \log \psi + V, \quad \text{Bq}(v_T,xx) = 0, \quad \text{Bq}(V_{xx}) = 0. \] (3.57)
Defining the “second potential Bq” equation

\[ F(v) \equiv v_{tt} + \varepsilon^2 (v_{xx} + \alpha)^2 + (\beta^2 / 3)v_{xxxx} = 0, \quad (3.58) \]

\( F(v_T) \) is a second degree polynomial in \((Z_1, Z_2)\):

\[ F(v_T) \equiv C_{02}Z_2^2 + C_{11}Z_1Z_2 + C_{20}Z_1^2 + C_{01}Z_2 + C_{10}Z_1 + C_{00} = 0 \quad (3.59) \]

which we identify to zero. This provides

\[
\begin{align*}
C_{02} &\equiv 2((\beta\varepsilon)^2 - c^2) = 0, \quad \Rightarrow c^2 = (\beta\varepsilon)^2 \quad (3.60) \\
C_{11} &\equiv -4cd = 0, \quad \Rightarrow d = 0 \quad (3.61) \\
C_{20} &\equiv V_{xx} + \alpha + 2\beta^2a/3 = 0, \quad \Rightarrow a = -3(V_{xx} + \alpha)/(2\beta^2) \quad (3.62) \\
C_{01} &\equiv 2(\beta\varepsilon^{-1}ac + 2(V_{xx} + \alpha) + \beta^2a/3 = 0, \quad \Rightarrow c = \beta\varepsilon \quad (3.63) \\
C_{10} &\equiv 8(\beta\varepsilon)^2a_x/3 + 4e_xc = 0, \quad \Rightarrow e_x = \beta^{-1}\varepsilon V_{xxx}, \quad (3.64) \\
C_{00} &\equiv 2(\beta^{-2}V_{xxx} + (4/3)\beta\varepsilon b_x + e_{xx}) = 0 \quad (3.65) \\
&\Rightarrow b = g(t) - (3/4)(\beta^{-2}V_{xxx} + \beta^{-3}\varepsilon^{-1}V_{xt}). \quad (3.66)
\end{align*}
\]

Finally, the compatibility condition \(X_0 = 0\) implies that \(g(t)\) is an arbitrary constant denoted \(\lambda\). The coefficients \(a, b, c, d, e\) are

\[
a = -(3/2)\beta^{-2}(U + \alpha), \quad c = \beta\varepsilon, \quad d = 0, \quad b = \lambda - (3/4)\beta^{-2}U_x - (3/4)\beta^{-3}\varepsilon^{-1}V_{xt}, \quad e = \beta^{-2}c(U + \alpha), \quad (3.67)
\]

i.e. the associated third order Lax pair \[141, 142, 96\] of the derivative of \(3.58\) (notation \(U = V_{xx}\)).

Since \(d = c_x = 0\), the BT obtained by eliminating \(Z_2\) between \[3.44\]–\[3.46\] writes

\[
\begin{align*}
Z_{1,xx} + 3Z_1Z_{1,xx} + Z_1^3 - aZ_1 - b & = 0, \quad (3.68) \\
Z_{1,t} + c(Z_1Z_{1,xx} + Z_1^3 - aZ_1 - \beta^{-2}U_x - b) & = 0, \quad (3.69) \\
(Z_{1,xx})_t - (Z_{1,t})_{xx} & = -\frac{3}{4}\beta^{-3}\varepsilon^{-1}(F(V))_x \quad (3.70)
\end{align*}
\]

or equivalently, with \(U = W_x = V_{xx}\) and \(Z_1 = (w - W)/(2\beta^2)\),

\[
(w - W)_{xx} + 3\beta^{-2}(w - W)((w + W)_x + 2\alpha) + \beta^{-4}(w - W)^3 \\
+ 3\beta^{-1}\varepsilon^{-1}(w + W)_t - 8\beta^2\lambda = 0, \quad (3.71) \\
(w + W)_{xx} + \beta^{-2}(w - W)(w - W)_x - \beta^{-1}\varepsilon^{-1}(w - W)_t = 0, \quad (3.72)
\]

an extension to \(\lambda \neq 0\) of the bilinear BT of Hirota and Satsuma \[24\] \[55\].
Second example : Sawada-Kotera equation

In the same way, we can easily find the coefficients of the third order Lax pair by processing the fifth order potential equation

\[ \text{pSK}(v) \equiv v_t + v_{5x} + 30v_x v_{3x} + 60v_x^3 + F(t) = 0, \quad F(t) \text{ arbitrary} \quad (3.73) \]

The algorithmic results of the Painlevé analysis are the following : the equation (3.73) possesses two families, each with five compatible indices. For the “principal” family

\[ p = -1, \quad q = -6, \quad u_0 = 1, \quad \text{indices} \quad -1, 1, 2, 3, 10 \text{ compatible} \quad (3.74) \]

the truncation is

\[ v_T = \chi^{-1} + v_1. \quad (3.75) \]

The assumption \( \chi = \psi/\psi_x \) with \( \psi \) solution of the second order Lax pair (3.6)–(3.7) generates the Painlevé-Bäcklund equations [30]

\[ E_4 \equiv C - 4S^2 + 9S_{xx} + 60Sv_{1x} - 180v_{1,x}^2 - 30v_{1,xxx} = 0 \]
\[ E_5 \equiv -C_x - 2SS_x + S_{xxx} + 30S_x v_{1x} = 0 \]
\[ E_6 \equiv \text{pSK}(v_1) + (SE_4 - E_5,x)/2 + (5/2)S_x(6v_{1,x} - S_x) = 0. \quad (3.78) \]

Demanding that \( v_1 \) be another solution of pSK implies \( v_{1,xx} = S_x/6 \) and, after computation, provides a nongeneric solution. Note that, however, a particular solution of the truncation is [132]

\[ S_{xx} + 4S^2 - C = 0, \quad v_{1,x} = S/3, \quad \text{KK}(v_1) = 0, \quad (3.79) \]

which defines a Miura transformation between SK and KK equations.

As in the preceding example, the hypothesis of the DT

\[ v = (\log \psi)_x + V, \quad (3.80) \]

with \( V \) another solution of pSK and \( \psi \) a solution of the third order linear system (3.37)–(3.38), makes pSK(\( v \))–pSK(\( V \)) a second degree polynomial in \( (Z_1, Z_2) \) like (3.59). The six determining equations \( C_{lm} = 0 \), added to the three compatibility conditions (3.40)–(3.43), have the unique solution depending on an arbitrary constant \( \lambda \)

\[ a = -6V_x, \quad b = \lambda, \quad c = 9\lambda - 18V_{xx}, \quad d = -36V_x^2 + 6V_{3x}, \quad e_x = 36\lambda V_{xx}, \quad (3.81) \]

a result which coincides with the Lax pair (2.144).

The \( x \)-part of the BT (2.144) is obtained by eliminating \( Z_2 \) between (3.44) and (3.45), then substituting \( Z_1 = v - V \) as results from (3.80).
Some results for Kaup-Kupershmidt equation

In the case of potential KK equation, the hypothesis of the differential operator $D = \partial_x$ for the DT, associated to the linear system (3.37)–(3.38), yields neither the Lax pair (2.145) nor the BT (2.149). This problem has been recently solved \[102\] by remarking that in his classification of second order first degree nonlinear ODEs possessing the Painlevé property, Gambier \[54\] mentions that the following equations:

\begin{align*}
\text{(G.5)}: \quad Y_{1,xx} + 3Y_1Y_{1,x} + Y_1^3 + 6UY_1 - \lambda &= 0 \quad (3.82) \\
\text{(G.25)}: \quad Y_{2,xxx} - \frac{3}{4}Y_{2,x}^2 + \frac{3}{2}Y_2^2Y_{2,x} + \frac{1}{4}Y_2^4 + 6UY_2^2 - 2\lambda Y_2 &= 0, \quad (3.83)
\end{align*}

are linearisable into third order equations

\begin{align*}
\text{(G.5)}: \quad Y_1 &= \psi_x/\psi, \quad \psi_{xxx} + 6U\psi_x - \lambda\psi = 0 \quad (3.84) \\
\text{(G.25)}: \quad Y_2^{-1} &= \lambda^{-1}[(\psi_x/\psi)_x + (1/2)(\psi_x/\psi)^2 + 3U], \\
& \psi_{xxx} + 6U\psi_x + (3U_x - \lambda)\psi = 0 \quad (3.85)
\end{align*}

corresponding to the scattering problem of, respectively, the SK equation for $U$ and the KK equation for $U$. It can then be shown that the DTs

\begin{align*}
Y_1 &= w - W, \quad \text{with SK}(w_x) = \text{SK}(W_x) = 0 \quad (3.86) \\
Y_2 &= 2(w - W), \quad \text{with KK}(w_x) = \text{KK}(W_x) = 0 \quad (3.87)
\end{align*}

leading to the BTs (2.148) and (2.149), can be found by singularity analysis.

### 3.2.4 Two-singular manifold method

For equations with two families of movable poles with opposite residues, the truncation procedure which considers only one family of singularities does not yield the auto-BT. An extension of the Weiss method consists of considering two distinct functions $\psi_1, \psi_2$, assuming now a DT of the form

\[ w_T - U = D\log \psi_1 - D\log \psi_2, \quad E(w_T) = E(U) = 0 \quad (3.88) \]

and, assuming that $Y = \psi_1/\psi_2$ satisfies the most general Riccati system

\begin{align*}
Y_x &= R_0 + R_1Y + R_2Y^2 \quad (3.89) \\
Y_t &= S_0 + S_1Y + S_2Y^2 \quad (3.90) \\
Y_{xt} - Y_{tx} &= X_0 + X_1Y + X_2Y^2, \quad (3.91) \\
X_0 &= R_{0,t} - S_{0,x} + R_1S_0 - R_0S_1, \quad (3.92) \\
X_1 &= R_{1,t} - S_{1,x} + 2(R_2S_0 - R_0S_2), \quad (3.93) \\
X_2 &= R_{2,t} - S_{2,x} + R_2S_1 - R_1S_2, \quad (3.94)
\end{align*}
eliminating derivatives of \( Y \) and identifying \( E_T \) to the null polynomial in \( Y \).

The determining equations so generated for the six unknowns \((R_i, S_i)\) must have a solution such that each \( R_i \) is a linear function of \( U \) and an arbitrary constant \( \lambda \) and such that at least one of the three cross-derivative conditions \( X_i = 0 \) is identical to the original equation for \( U \). In such a case, one has found the DT and the Lax pair, i.e. the BT. In the case of two simple poles with constant opposite residues \( \pm u_0 \) and opposite singular part operators \( \pm u_0 \partial_x \), the truncation

\[
 u_T = u_0(\text{Log } Y)_x + U \tag{3.95}
\]

becomes, by elimination of \( Y_x \) from (3.89)

\[
 u_T = u_0(R_0 Y^{-1} + R_1 + R_2 Y) + U. \tag{3.96}
\]

This represents an extension of the Weiss truncation to the positive powers of \( Y \). A similar extension was previously made \[109\] in the variable \( \chi = \psi/\psi_x \) for obtaining particular solutions of nonlinear PDEs. The “two-singular manifold” method is successful for finding the auto-BT of MKdV and sine-Gordon equations \[101\] but only partially for the NLS equation. Before detailing this result, let us first reproduce in Table 3.1 the algorithmic results of the Painlevé analysis for the four equations belonging to the AKNS scheme; these include the SM equation associated with these well known NLPDEs which pass the Painlevé test.

Table 3.1: Algorithmic results of the Painlevé analysis. The integers \((p, q)\) are defined in (3.26); for sG, the polynomial PDE is (3.108). Next column lists the indices, excepted \(-1\). Column “PD equations” lists the subscripts of the non identically zero Painlevé-Darboux equations; in the sG and NLS cases, they depend on the arbitrary coefficients introduced at the index 2 (sG) and 0 (NLS).

| Name | \( p \) | \( q \) | Indices | PD eq. | Singular manifold equation |
|------|-------|-------|---------|-------|---------------------------|
| KdV  | \(-2\) | \(-5\) | \(4, 6\) | 3.5 | \( S - C = 6\lambda \) |
| MKdV | \(-1\) | \(-4\) | \(3, 4\) | 2   | \( S - C = 0 \) |
| sG   | \(-2\) | \(-6\) | \(2\)   | 3.4, 5, 6 | \( S + C^{-1}C_{xx} \) |
|      |       |       |         |       | \( -C^{-2}C_x^2/2 \)     |
|      |       |       |         |       | \( +2\lambda = 0 \)      |
| NLS  | \((-1, -1)\) | \((-3, -3)\) | \(0, 3, 4\) | 2, 2, 3 | \( C_t + 3CC_x - S_x \) |
|      |       |       |         |       | \( +8\lambda C_x = 0 \)   |

Let us now use the information contained in the SME.
Modified Korteweg-de Vries

The equation (2.93) has two families \( u \sim \pm a \chi^{-1} \), denoted \( u \sim a \chi^{-1} \) since \( a \) is defined by its square. The truncated expansion of a family is

\[ u_T = a \chi^{-1}. \]  

(3.97)

The SME \( S - C = 0 \) is parametrised as

\[ S = 2v, \ C = 2v, \ \text{KdV}(v) = 0, \]  

(3.98)

and the precise relation between \( u \) and \( v \) (Miura transformation) is obtained by eliminating \( \chi \) between (3.97) and (3.2)

\[ (u_T/a)_x + (u_T/a)^2 = -v. \]  

(3.99)

In fact there are two such Miura transformations, one for each sign of \( a \), i.e. one for each family.

Let us first obtain the Darboux transformation for MKdV from that of KdV and show that it involves two SMs. The Darboux transformation for KdV has been obtained in section 3.2.2, eq. (3.36). The two Miura transformations (3.99) and the parametrisation (3.98) imply:

\[ -\frac{S_1}{2} = \left( \frac{u_T}{a} \right)^2 + \left( \frac{u_T}{a} \right)_x = 2(\log \psi_1)_{xx} + \left( \frac{U}{a} \right)^2 + \left( \frac{U}{a} \right)_x \]  

(3.100)

\[ -\frac{S_2}{2} = \left( \frac{u_T}{a} \right)^2 - \left( \frac{u_T}{a} \right)_x = 2(\log \psi_2)_{xx} + \left( \frac{U}{a} \right)^2 - \left( \frac{U}{a} \right)_x \]  

(3.101)

and the elimination of the nonlinear terms leads to

\[ u_{T,x} = a(\log(\psi_1/\psi_2))_{xx} + U_x, \]  

(3.102)

which after one integration yields the Darboux transformation for MKdV

\[ u_T = a(\log(\psi_1/\psi_2))_x + U. \]  

(3.103)

With this DT, the Lax pair is obtained as explained in the introduction of this section. Setting \( Y = \psi_1/\psi_2 \) and taking account of (3.89)–(3.90), every derivative of \( Y \) can be replaced by a polynomial in \( Y \). Consequently, the Darboux transformation (3.103) becomes identical to

\[ u_T = a(R_0 Y^{-1} + R_1 + R_2 Y) + U \]  

(3.104)

and one must identify to zero the polynomial in \( Y \)

\[ E(u_T) \equiv E_T = \sum_{j=0}^{8} E_j Y^{j-4}. \]  

(3.105)
Among the nine Painlevé-Darboux equations $E_j = 0$, only four ($j = 1, 2, 6, 7$) are not identically zero. Their resolution, as detailed in [101], yields the following parametric representation of the six unknowns $R_i, S_i$ in which $R_i$ is linear in $U$ and the spectral parameter $\lambda$

$$
R_0 = \lambda, \quad R_2 = -\lambda, \quad R_1 = -2U/a \quad \text{(3.106)}
$$

$$
S_0 = -4\lambda^2 + 2(U/a)^2 - 2U_x/a, \quad S_1 = 8\lambda^2 U/a - 4(U/a)^3 + 2U_{xx}/a
$$

$$
S_2 = -4\lambda^2 + 2(U/a)^2 + 2U_x/a. \quad \text{(3.107)}
$$

This solution associated with the Riccati equations (3.89)–(3.90) reproduces the equations (2.98)–(2.99) for the pseudopotential of MKdV.

### Sine-Gordon

The sine-Gordon equation (2.118), invariant by parity on $u$, is first transformed into a polynomial equation for $v = e^{iu}$, invariant under $v \rightarrow 1/v$.

$$
\text{PsG}(v) = 2vv_{xt} - 2v_xv_t - v^3 + v = 0, \quad v = e^{iu}. \quad \text{(3.108)}
$$

This PDE has two families of movable singularities $v = v_1 \sim -4C_1\chi^{-2}$ and $v = v_2^{-1} \sim -4C_2\chi^{-2}$. The truncation equations have the following general solution [133, 29]. For the first family ($v, S, C, \psi$) subscripted with 1

$$
S_1 = -v_{1,xx}/v_1 + v_{1,x}^2/(2v_1^2) - 2\lambda = -iu_{xx} + u_x^2/2 - 2\lambda \quad \text{(3.109)}
$$

$$
C_1 = -v_1/(4\lambda) = -e^{iu}/\lambda \quad \text{(3.110)}
$$

$$
v_1 = -4(\log \psi_1)_{xt} + V_1, \quad \text{PsG}(V_1) = 0. \quad \text{(3.111)}
$$

For the second family $e^{-iu} = v_2 \sim -4C_2\chi^{-2}$

$$
S_2 = -v_{2,xx}/v_2 + v_{2,x}^2/(2v_2^2) - 2\lambda = iu_{xx} + u_x^2/2 - 2\lambda \quad \text{(3.112)}
$$

$$
C_2 = -v_2/(4\lambda) = -e^{-iu}/\lambda \quad \text{(3.113)}
$$

$$
v_2 = -4(\log \psi_2)_{xt} + V_2, \quad \text{PsG}(V_2) = 0. \quad \text{(3.114)}
$$

If one considers only one of these two equivalent SMs, the Schwarzian $S_i, i = 1 \text{ or } 2$, does depend on an arbitrary constant $\lambda$ but it has two drawbacks: it is not invariant under parity on $u$, it is not linear in the physical field $u$ as requested for the Lax pair (3.49)–(3.50) to be a “good” one.

Since $v_1 - v_2 = 2i \sin u$, the difference of (3.111) and (3.114) reads

$$
\sin u = 2i(\log(\psi_1/\psi_2))_{xt} + \sin U, \quad \text{sG}(U) = 0, \quad \text{(3.115)}
$$

i.e., from the definition of the equation

$$
u_{xt} = 2i(\log(\psi_1/\psi_2))_{xt} + U_{xt}. \quad \text{(3.116)}
$$
Integrating twice, we finally obtain the Darboux transformation of $sG$

$$u = 2i \log(\psi_1/\psi_2) + U,$$  

(3.117)

defined in terms of both families. For the solution of the polynomial PDE

$$PsG(v) = 0$$  

associated to the $sG$ equation by $v = e^{iu}$, the DT is:

$$v = vY^{-2}, \ Y = \psi_1/\psi_2, \ \text{PsG}(V) = 0,$$  

(3.118)

and one must identify

$$E(v) = V^2 \sum_{i=0}^{4} E_i Y^{i-6}$$  

(3.119)

to the null polynomial in $Y$. Among the five Painlevé-Darboux equations, the equation $E_2$ is functionally dependent on $(E_0, E_1)$, a consequence of the compatibility of the index 2. Their resolution yields the Riccati pseudopotential

$$Y_x = \lambda(1 - Y^2) + iU_x Y,$$  

(3.120)

$$Y_t = ((1 - Y^2) \cos U + i(1 + Y^2) \sin U)/(4\lambda),$$  

(3.121)

$$\frac{Y_{xt} - Y_{tx}}{Y} = sG(U),$$  

(3.122)

where the $x$-part is now linear in the spectral parameter $\lambda$ and the field $U$ associated with the DT (3.117).

### 3.2.5 Weiss method plus homography

Pickering [110] remarks three drawbacks in the previous method:

(i) any explicit relationship is given between the variable $Y \equiv \psi_1/\psi_2$ and the variable $\chi$ of the invariant Painlevé analysis while the one between $\chi$ and $\varphi$ is well defined by the homographic transformation

$$\chi = \frac{\varphi}{\varphi_x - \varphi_xx \varphi/(2\varphi_x)},$$  

(3.123)

(ii) the result of the previous truncation for MKdV does not reveal any relationship between the MKdV and KdV equations as it would be,

(iii) the knowledge of the DT is required in advance.

He notices that for finding the BT of MKdV and sine-Gordon equations it is sufficient to consider a Riccati system constructed from the nonlinearisation of the following second order scalar linear system

$$\eta_{xx} = 2A\eta_x + B\eta,$$  

(3.124)

$$\eta_t = -C\eta_x + \left( \int_{x}^{x'} D \ dx' \right) \eta$$  

(3.125)
by the transformation $Z^{-1} = \eta_x/\eta$. The corresponding nonlinear system
\begin{align}
Z_x &= 1 - 2A - BZ^2 \\
Z_t &= -C + (C_x + 2AC)Z - (D - BC)Z^2
\end{align}
(3.126)

depends on four functions $A, B, C, D$ in place on six like the system (3.89)–(3.90) considered in the 2-SM method. Its compatibility condition is
\begin{align}
Z_{xt} - Z_{tx} &= X_1 Z + X_2 Z^2 = 0 \\
X_1 &= 2(D - (A_t + (AC)_x + C_x))/2 = 0 \\
X_2 &= D_x - B_t - 2BC_x - B_zC - 2AD = 0
\end{align}
(3.128)

The solution $\eta$ of the linear ODE (3.124) is related to the solution $\psi$ of (3.6) by the gauge transformation
\begin{equation}
\eta = \left(e^{\int^x A dx'}\right) \psi
\end{equation}
(3.131)

with
\begin{equation}
S = -2(B + A^2 - A_x)
\end{equation}
(3.132)

Then, computing the $x$–derivative of $\text{Log}\, \eta$, one gets the transformation
\begin{equation}
Z^{-1} = \chi^{-1} + A
\end{equation}
(3.133)

which means that the new expansion variable $Z$ is related to $\chi$ by an homographic transformation (as suggested in [93], formula (17)) such that in the neighbourhood of $\chi = 0$, one has $Z \sim \chi$. Then the system (3.126)–(3.127) combined with the truncation in $Z$
\begin{equation}
u_T = a\partial_x \text{Log}\, Z + U
\end{equation}
(3.134)

($U$ function of $(x,t)$ and $a$ constant) such that $u_T \sim a\chi^{-1}$ as $\chi \to 0$, extends the Weiss truncation to positive powers of $Z$. Solving the Painlevé-Darboux equations associated with $E(u_T) = 0$ Pickering obtains for the MKdV equation (2.93) the following results:
\begin{align}
A &= U/a, \quad B = \lambda^2, \quad C = 2(U_x/a - (U/a)^2 + 2\lambda^2) \\
D &= 4\lambda^2 U_x/a, \quad \lambda = \lambda(t) \text{ arbitrary integration function}
\end{align}
(3.135)

and the compatibility conditions (3.129)–(3.130) yield
\begin{align}
X_1 &= -(2/a)(U_t + U_{xxx} - 2a^{-2}(U^3)_x) = 0 \\
X_2 &= -(\lambda^2)_x = 0
\end{align}
(3.137)

From (3.133) and (3.135) one gets
\begin{align}
S &= 2(U_x/a - (U/a)^2 - \lambda^2) \\
S - C + 6\lambda^2 &= 0
\end{align}
(3.139)

38
the latter equation being the SM equation of the KdV equation.

Let us remark that the expressions of $S$ and $C$ in function of the solution $U$ of the MKdV equation and the constant parameter $\lambda$ coincide with the relation (2.111) obtained previously. The expression (2.108) for $Y$ with $\alpha = U/a$ implies the identification $Z = \lambda^{-1}Y$.

For the sine-Gordon equation (2.118), considering the truncation
\[
u_T = 2i \log Z + U \]
(3.141)
such that $u_T \sim 2i \log \chi$ as $\chi \to 0$, Pickering obtains:
\[
A = -(i/2)u_x, \quad B = \lambda^2, \quad D = -(i/2)\sin U, \quad \lambda \text{ arbitrary constant, (3.142)}
\]
\[
C = -\lambda^{-2}e^{iU}/4, \quad S = -iu_{xx} + (1/2)U_x^2 - 2\lambda^2, \quad sG(U) = 0, (3.143)
\]
\[
S + C_{xx}/C - (1/2)(C_x/C)^2 + 2\lambda^2 = 0 \quad (3.144)
\]
Again the expressions for $S, C$ in function of $U$ and $\lambda$ and the equation (3.144) coincide with the relations (2.124) and (2.130). The identification $Z = \lambda^{-1}Y$ is also easy to find taking account that, for $sG, \alpha = -iu_x/2$.

### 3.2.6 Weiss method plus involutions

The AKNS system [139, 2]
\[
E^{(1)} = iu_t + p_r u_{xx} + q_r u^2v = 0, \quad E^{(2)} = -iv_t + p_r v_{xx} + q_r uv^2 = 0 \quad (3.145)
\]
has the BT [22, 73, 80] $(a^2 = -2p_r/q_r, \quad R^2 = (u + U)(v + V)/a^2 - (\lambda - \mu)^2)$
\[
(u + U)_x = -(u - U)R - i(\lambda + \mu)(u + U) + ip_r^{-1}(u + U)_t = (u - U)_xR + (u + U)M + i(\lambda + \mu)(u + U)_x \]
\[
(v + V)_x = -(v - V)R + i(\lambda + \mu)(v + V) - ip_r^{-1}(v + V)_t = (v - V)_xR + (v + V)M - i(\lambda + \mu)(v + V)_x \]
\[
M = (uv + UV)/a^2 \quad (3.146)
\]
with $\lambda, \mu$ arbitrary complex constants. Galilean invariance $(x, t, u, v) \to (x - 2p_r^t c_t + t, e^{i(\lambda - \mu)c^2t}u, e^{-i(\lambda - \mu)c^2t}v)$ allows to choose $c = \lambda + \mu = 0$ [73].

The above BT cannot be found neither by the one–SM method [134], nor by the two–SM method [101], nor by the one–SM method plus homography [110]. The challenge of the Painlevé approach to find this BT by singularity analysis only is solved in [35] as follows.

As the one–SM method only provides some partial result $T(\chi, u, \lambda)$ for the truncation, one then considers all transformations on $u$ conserving the equation $E(u) = 0$ in order to uncover a second solution $U$, see Table 3.2.

For the AKNS system (3.145), the one–family truncation
\[
u = u_0 \chi^{-1} + u_1, \quad v = v_0 \chi^{-1} + v_1 \quad (3.147)
\]
Table 3.2: Transformations of the dependent variable(s) conserving the equation(s), for the AKNS group PDEs (complex conjugation, phase shift, parity).

| PDE      | Transformation(s)                                                                 |
|----------|-----------------------------------------------------------------------------------|
| AKNS system | \((u, v, i) \rightarrow (v, u, -i); \forall k : (u, v) \rightarrow (ku, v/k)\) |
| Sine-Gordon | \(u \rightarrow -u\)                                                            |
| MKdV     | \(u \rightarrow -u\)                                                            |
| KdV      | none                                                                              |

which has the general solution \[134, 101\] \((\lambda \text{ arbitrary complex constant})\)

\[
\begin{align*}
  u &= a(\chi^{-1} - f_x/(2f) - i\lambda)f, \\
  v &= a(\chi^{-1} + f_x/(2f) + i\lambda)/f, \\
  f_x/f &= -2i\lambda - (u/a)f^{-1} + (v/a)f, \\
  ip^{-1}_r f_t/f &= 2uv/a^2 + 4\lambda^2 + (ux - 2i\lambda u)/f + (vx + 2i\lambda v)f/a, \\
  (f_xt - f_xt)/f &= (f^{-1}E^{(1)} + f E^{(2)})/a
\end{align*}
\]

fails to introduce a second solution \((U, V)\), see details in appendix C of Ref. [101]. This is done by applying the two point transformations of Table 3.2 to the above truncation \((3.148)–(3.151)\):

\[
T_1 \begin{cases} T_2 \end{cases} T_3 \begin{cases} T_4 \end{cases} : \begin{cases} \chi_1 \ u \ v \ i \ f \ \lambda \ (\text{identity}) \\
\chi_2 \ v \ u \ -i \ g \ \mu \ (\text{conjugation}) \\
\chi_3 \ kU \ k^{-1}V \ i \ f \ \lambda' \ (\text{phase shift}) \\
\chi_4 \ k^{-1}V \ kU \ -i \ g \ \mu' \ (\text{both}) \end{cases}
\]

These transformations act on \((u, v, f, \lambda)\) like in Chen [23]. This is equivalent to successively process the four families of the AKNS system by the one–SM method. In order that \((u, v)\) and \((kU, V/k)\) be distinct, one must have \(\lambda' = \mu, \mu' = \lambda\).

The four sets \((3.148)–(3.149)\) define a system of eight equations in the eight unknowns \((\chi_1^{-1}, \chi_2^{-1}, \chi_3^{-1}, \chi_4^{-1}, u, v, kU, V/k)\). This system is linear with determinant \(fg - 1/(fg)\) and it provides the DT straightforwardly (with the nonrestricive choice \(k = -1\)):

\[
\begin{align*}
  u - U &= 2a[\partial_x \log(g - 1/f) - i(\lambda + \mu)]/(g + 1/f) \quad (3.154) \\
  v - V &= 2a[\partial_x \log(f - 1/g) + i(\lambda + \mu)]/(f + 1/g) \quad (3.155) \\
  u + U &= 2ia(\lambda - \mu)/(g - 1/f) \quad (3.156) \\
  v + V &= 2ia(\lambda - \mu)/(f - 1/g) \quad (3.157)
\end{align*}
\]
(to stick to our definition, the DT is made of two equations, either \((3.154)–(3.155)\) or \((3.156)–(3.157)\)). The nonconstant factor of the logarithmic derivatives is similar to that of \((P3), (P5), (P6)\), see section 7.1 in the Conte lecture while \(\lambda + \mu\) is chosen as a real constant.

The Lax pair in its Riccati form is made of the four equations resulting from the action of \(T_3\) and \(T_4\) on \((3.150)–(3.151)\). The BT is made of the four equations resulting from the elimination of the pseudopotentials \((f, g)\) between the six equations defining the DT and the Lax pair, and these are precisely \((3.146)\). This elimination is quite easy since equations \((3.156)–(3.157)\) are algebraic in \((f, g)\):

\[
f = ia(\lambda - \mu + R)/(v + V),\quad g = ia(\lambda - \mu + R)/(u + U).
\] (3.158)

Remark. The system \([23]\) of two equations for \((f, g)\), obtained by eliminating \((u, v, U, V)\) between \((3.154)–(3.157)\) and the PDE, is invariant under \((\lambda, \mu) \to (\mu, \lambda)\). The elimination of \(g\) between this system provides the Broer-Kaup equation for \(w = -i \log f\), a result also obtainable by the Weiss truncation \([10]\)

\[
p_{r}^{-1}w_{tt} + 4w_{x}w_{xt} + 2w_{x}w_{xx} + p_{r}(6w_{x}^{2}w_{xx} + w_{xxxx}) = 0.
\] (3.159)

### 3.2.7 Reductions of the DT of AKNS system

The \(x\)–part of the AKNS spectral problem admits the three reductions \(v = \pi, u = \pm u, v = 1,\) and the DT, obtained only from the \(x\)–part, must admit them. This is indeed the case: equations \((3.154)–(3.157)\) admit the two reductions \((u, v, g, \mu) = (\pi, \epsilon u, \epsilon f, -\lambda), \epsilon^{2} = 1,\) and one must add the case \(g = \epsilon/f\) when the determinant vanishes. Table 3.3 summarises these reductions and the homographic link between \(f\) and the \(\chi\) of the invariant analysis.
Table 3.3: Reductions of the Darboux transformation of the AKNS system.

| PDE   | v, g, μ | χ⁻¹ | (u − U)/a | (u + U)/a |
|-------|--------|-----|-----------|-----------|
| NLS   | w, f, λ|      | [3.154]   | 4(Im λ)/(1/f − f) |
| sG    | εu, εf, −λ | ε² = ε | -4(Im λ)/f | ε² = ε |
| MKdV  | εu, εf, −λ | ε² = ε | (4/ε)Y | ε² = ε |
| KdV   | 1, ε/f, −λ |       | -2fz = 2(f² − 2iλf) | 2(χ² + λ²) |

3.3 Cosgrove classification for semilinear PDEs of second order

In two papers [37, 38], Cosgrove classifies two cases of Painlevé type semilinear PDEs of second order. The necessary conditions for the Painlevé property which he establishes combine the criteria of Painlevé and Gambier for ODEs and the WTC ones for PDEs.

For hyperbolic PDEs in two independent variables of the type

\[ u_{xt} = F(x, t, u, u_x, u_t) \]  

(3.160)

the classes of equivalence are defined by the H-transformation

\[ \tilde{u} = \frac{\alpha(x, t)u + \beta(x, t)}{\gamma(x, t)u + \delta(x, t)} \]  

(3.161)

where

\[ u = u(\tilde{x}, \tilde{t}), \quad \tilde{x} = X(x), \quad \tilde{t} = T(t), \quad \alpha\delta - \beta\gamma \neq 0. \]  

(3.162)

and the necessary conditions are

1. the dependence in \( u_x, u_t \) of \( F \) must be of the form

\[ u_{xt} = A(x, t, u)u_xu_t + B(x, t, u)u_x + C(x, t, u)u_t + D(x, t, u), \]  

(3.163)

2. as a function of \( u \), the term \( A \) is the sum of at most three simple poles, at locations set to \( u = 0, 1, H(x, t) \) (\( H \) arbitrary function of \( x, t \)) while \( B, C, D \) cannot grow faster than, respectively, \( u, u, u^3 \),

3. the equation must pass the WTCK Painlevé test, i.e. all Fuchs indices are distinct integer and all positive indices are compatible in order to guarantee the existence of local Laurent series in the Kruskal variable \( \varphi(x, t) = x ± f_1(t) \) or \( \varphi(x, t) = t ± f_2(x) \).
At the end, he obtains 22 canonical equations, reducible to

\[ u_{xt} = \sin u \text{ (sine-Gordon)} \text{ or } u_{xt} = ae^{2u} + be^{-u} \text{ (Tzitzéica)} \]  

(3.164)

or linearisable by the singular part transformation.

For parabolic PDEs the class of equivalence is a little bit larger than in the previous case in the sense that the new independent variables in the transformation (3.161) may be related to \((x,t)\) as

\[ \tilde{x} = X(x,t), \quad \tilde{t} = T(t) \]  

(3.165)

The successive necessary conditions for having the Painlevé property yield the following results

1. in two independent variables, the sole equation is

\[ u_t + u_{xx} + 2uu_x = F(x,t), \quad F(x,t) \text{ arbitrary function} \]  

(3.166)

i.e. the Forsyth-Burgers equation linearisable into the heat equation,

2. in more than two independent variables, only trivial soliton equation are obtained, i.e. nonlinear PDEs related to linear ones.

### 3.4 PDEs with variable coefficients

The Painlevé test can be applied to nonlinear PDEs with variable coefficients in order to determine the conditions under which the equation might be integrable. The sufficient part of the analysis entails the determination of the DT and the Lax pair, as well as the transformation which could relate the equation to its autonomous integrable counterpart. Many authors have considered the generalised KdV and NLS equations with variable coefficients due to their interest in many physical systems. The results of Brugarino for the generalised variable coefficient KdV equation (VCKdV)

\[ u_t + a(t)u + (b(x,t)u)_x + c(t)uu_x + d(t)u_{xxx} + e(x,t) = 0, \]  

(3.167)

are

(i) the equation passes the Painlevé test under the condition

\[ b_t + (a - Lc)b + bb_x + db_{xxx} = 2ah + hL(d/c^2) + h' + ce 
+ x(2a^2 + aL(d^3/c^4) + a' + L(d/c)L(d/c^2) + (L(d/c))') \]  

(3.168)

with \( L = (d/dt) \log \) and \( h(t) \) arbitrary,

(ii) the solution of the Weiss truncation yields the DT and the Lax pair,
(iii) with the transformation from \((u, x, t)\) to \((\Theta, \xi, t)\)

\[
u = ((a + L(d/c))\xi + g - b + \Theta)/c, \quad \xi = x - \int^t g(T) dT \tag{3.169}
\]

and the condition \([3.168]\), he gets the equation

\[
\Theta_t + (2a + L(d/c^2)) \Theta + \xi (a + L(d/c)) \Theta + \Theta \Theta_t + d\Theta_{\xi\xi} = 0, \tag{3.170}
\]

equivalent to \([31, 17, 66]\) the KdV equation with constant coefficients. In case \(a = b = c = 0\), the condition \([3.168]\) becomes simply

\[
d = c \left(K_1 \int^t c(T) dT + K_2\right), \quad K_1, K_2 \text{ arbitrary constants}, \tag{3.171}
\]

i.e. the one given by Joshi \([67]\) when performing the Painlevé test on this particular VCKdV equation. The same relation was obtained by Winternitz and Gazeau \([137]\) using the symmetry group,

(iii) several equations of physical interest which satisfy \([3.168]\) are presented.

Gagnon and Winternitz \([53]\) have analysed from the point of view of symmetries a variable coefficient nonlinear Schrödinger (VCNLS) equation

\[
iu_t + f(x, t)u_{xx} + g(x, t)|u|^2 + h(x, t)u = 0 \tag{3.172}
\]

involving the three complex functions \(f, g\) and \(h\) of the variables \(x, t\). The symmetry group is shown to be five-dimensional iff the equation \([3.172]\) is equivalent to NLS itself or to CGL3, and at most four-dimensional in all other cases. In this framework, they give the allowed transformations of \([3.172]\), in case \(f = 1 + if_2\), to the equation with constant coefficients

\[
iu_{\tilde{t}} + (1 + if_2)u_{\tilde{x}\tilde{x}} + (\tilde{g}_1 + if_2)u|u|^2 + (\tilde{h}_1 + if_2)\tilde{u} = 0 \tag{3.173}
\]

which are, in the special case \(f_2 = 0\) \((I, J, K, L \text{ arbitrary functions of } t)\),

\[
g = (\tilde{g}_1 + if_2)\tilde{T}I^{-2} \tag{3.174}
\]

\[
\text{Re } h = (\tilde{K} + 4\tilde{K}^2)x^2 + (\tilde{L} + 4\tilde{K}L)x + \tilde{J} + \tilde{L}^2 + \tilde{h}_1 \tilde{T} \tag{3.175}
\]

\[
\text{Im } h = -\tilde{T}I^{-1} - 2K + \tilde{h}_2 \tilde{T} \tag{3.176}
\]

\[
\tilde{t} = T, \quad \tilde{T} = T_0 e^{-8 \int K dt}, \quad \tilde{x} = \sqrt{T} x + \xi, \quad \tilde{\xi} = -2\sqrt{T} L \tag{3.177}
\]

\[
u = \tilde{u}(\tilde{x}, \tilde{t}) I e^{i(K x^2 + L x + J)}. \tag{3.178}
\]

One example given by the authors leading to the NLS equation \((\tilde{g}_2 = \tilde{h}_1 = \tilde{h}_2 = 0)\) corresponds to the following choice of the arbitrary functions involved in the transformation

\[
\tilde{T}I^{-1} = -2K, \quad J = L = 0, \quad \tilde{K} + 4\tilde{K}^2 = K_0/4, \quad K_0 \text{ constant} \tag{3.179}
\]
and yields the VCNLS equation

\[ iu_t + u_{xx} + \tilde{g}_1 T_0 \text{sech} (\sqrt{K_0} t) u |u|^2 + (K_0/4)x^2 u = 0. \] (3.180)

This equation is related to NLS by the change of variables

\[ \tilde{x} = \sqrt{T_0} x \text{sech} (\sqrt{K_0} t) + x_0, \quad \tilde{t} = T_0 \tanh (\sqrt{K_0} t) / \sqrt{K_0} \] (3.181)
\[ u(x, t) = \tilde{u}(\tilde{x}, \tilde{t}) \text{sech}^{1/2} (\sqrt{K_0} t) e^{(1/4)i \sqrt{K_0} x^2 \tanh (\sqrt{K_0} t)} \] (3.182)

Considering the other choice: \( \dot{T} I^{-2} = 1 \), one obtains

\[ iu_t + u_{xx} + \tilde{g}_1 u |u|^2 + \left( (4K^2 + \tilde{K})x^2 + (\dot{L} + 4KL)x + L^2 + \dot{J} + 2iK \right) u = 0 \] (3.183)

equivalent to NLS by the transformation

\[ \tilde{t} = T_0 \int_t^\infty dt' e^{-8 \int t' K ds} \quad \tilde{x} = x \sqrt{T_0} e^{-4 \int t' K dt'} + \xi \] (3.184)
\[ \xi = \xi_0 - 2 \sqrt{T_0} \int_t^\infty dt' L(t') e^{-4 \int t' K ds} \]
\[ u(x, t) = \tilde{u}(\tilde{x}, \tilde{t}) e^{-4 \int t' K dt'} e^{i(Kx^2 + Lx + J)}. \]

For \( K(t) = -\beta(t)/2 \), Equ. (3.183) and the transformation (3.184) coincide with the results of Clarkson [28] in the analysis of the PDE

\[ iu_t + u_{xx} - 2u |u|^2 = a(x, t) u + b(x, t) \] (3.185)

which passes the Painlevé test iff there exist functions \( (\beta(t), \alpha_1(t), \alpha_0(t)) \) such that

\[ a = i \beta + x^2 [\beta'/2 - \beta^2] + x \alpha_1 + \alpha_0, \quad b = 0. \] (3.186)

As noticed by Clarkson, this result proves that the equation

\[ iu_t + u_{xx} - 2u |u|^2 = \beta x^2 u, \quad \beta \text{ constant}, \quad \beta \neq 0, \] (3.187)

which does not satisfy the condition (3.186), is not integrable while

\[ iu_t + u_{xx} - 2u |u|^2 = (i \kappa - \kappa^2 x^2) u, \quad \kappa \text{ real}, \] (3.188)

can be transformed into NLS and hence is integrable.
Chapter 4

Partially and non integrable equations

Let us consider a polynomial PDE (depending on a parameter $\mu$) which does not pass the Painlevé test but possesses a singularity structure compatible with singlevalued solutions. The reason for the failure of the Painlevé test are the following ones:

(i) the Fuchs indices are all integers (possibly for particular values of $\mu$) but some of them do not satisfy the compatibility condition, therefore the existence of a Laurent series is submitted to conditions on the SM,

(ii) whatever be $\mu$, some indices are irrational and thus unable to generate compatibility conditions, therefore the number of arbitrary functions which can be introduced in the Laurent series is lower than the order of the equation; this generally characterises equations with chaotic behaviour.

In the first case, we classify the equation as being partially integrable while in the second case as being nonintegrable. We provide methods for finding particular closed form solutions and illustrate them on some examples. In each case one looks for solutions related by a rational transformation to the general solutions of first order nonlinear ODEs like Riccati, Weierstrass or Jacobi possessing the PP, or solutions related by a nonlinear transformation like the logarithmic derivative to a second or third order linear system with constant coefficients.

4.1 Partially integrable equations

4.1.1 KPP equation
The Kolmogorov, Petrovskii and Piskunov equation (KPP) \[71, 48, 103\]

\[E \equiv u_t - u_{xx} + \frac{2}{d^2}(u - e_1)(u - e_2)(u - e_3) = 0, \quad e_j \text{ distinct}, \quad (4.1)\]

possesses two opposite families of singularities

\[u = d\chi^{-1} + s_1/3 - dC/6 + O(\chi), \quad \text{indices:} \quad -1, 4, \quad D = d\partial_x \quad (4.2)\]

\[s_1 = e_1 + e_2 + e_3, \quad s_2 = e_2e_3 + e_3e_1 + e_1e_2, \quad s_3 = e_1e_2e_3. \quad (4.3)\]

with the condition on the SM coming from the index 4

\[Q_4 \equiv C[-3(C_1 + CC_x) + \prod_{k=1}^{3}(C + s_1 - 3e_k)] = 0 \quad (4.4)\]

Denoting \((j, l, m)\) any permutation of \((1, 2, 3)\) and \(k_i = (3e_i - s_1) / (3d), \) the one-family truncation \(u = d\chi^{-1} + s_1/3 - dC/6\) yields the moving front solution:

\[u = \frac{e_l + e_m}{2} + d\frac{k_2}{2} \tanh \frac{k_2}{2}(x - ct - x_0) \quad (4.5)\]

\[k_2^2 = (k_l - k_m)^2, \quad c = -3(k_l + k_m)/2. \quad (4.6)\]

The two–family truncation like for MKdV or sG yields the stationary pulse

\[u = e_1 + \frac{e_2 - e_3}{\sqrt{2}} \text{sech} \left[ \frac{e_2 - e_3}{d\sqrt{2}}(x - x_0), \right] 2e_1 - e_2 - e_3 = 0. \quad (4.7)\]

A third very interesting solution representing the collision of two fronts \[70\] with different velocities can be easily found \[33\] with the assumption

\[u = \frac{s_1}{3} + u_0 \partial_x \log \psi, \quad (4.8)\]

where \(\psi\) is the general solution of a linear system with constant coefficients

\[
\psi = \sum_{n=1}^{3} C_n \exp \left[ k_n(x + b_n t) + k_n^2 b_n t \right], \quad C_n \text{ arbitrary, } C_1C_2C_3 \neq 0 \quad (4.9)
\]

\[
\psi_{xxx} - a_1 \psi_x - a_2 \psi = 0, \quad \psi_t - b_1 \psi_{xx} - b_2 \psi_x = 0 \quad (4.10)
\]

with the following values of the coefficients and the constant \(k_n\)

\[a_1 = (s_1^2 - 3s_2)/(3d^2), \quad a_2 = (2s_1^3 - 9s_1 s_2 + 27s_3)/(27d^3), \quad b_1 = -3, \quad b_2 = 0, \quad k_n = (3e_n - s_1)/(3d). \quad (4.11)\]

### 4.2 Nonintegrable equations
4.2.1 Kuramoto-Sivashinsky equation

The Kuramoto-Sivashinsky (KS) equation

$$E \equiv u_t + uu_x + u_{xx} + u_{xxxx} = 0,$$  \hspace{1cm} (4.12)

describes, for instance, the fluctuation of the position of a flame front, or the motion of a fluid going down a vertical wall, or a spatially uniform oscillating chemical reaction in a homogeneous medium. For a review, see ref. [8 9].

This equation possesses only one family of singularities [31]

$$u = 120 \chi^{-3} + 60(S + 1/19)\chi^{-1} + (C - 15Sx) + O(\chi),$$  \hspace{1cm} (4.13)

indices : $-1, 6, (13 \pm i\sqrt{71})/2, \ D = 60\partial_x^3 + (60/19)\partial_x$  \hspace{1cm} (4.14)

Due to the existence of the two complex irrational Fuchs indices the Laurent series depends only on two arbitrary functions ($u_6$ and the arbitrary function in the expansion variable $\chi$) whatever be the SM. The WTC truncation

$$u = 120 \chi^{-3} + 60(S + 1/19)\chi^{-1} + (C - 15Sx)$$  \hspace{1cm} (4.15)

generates three determining equations $E_4 = 0, E_5 = 0, E_7 = 0$ whose general solution is $C = \text{arbitrary $c$, $S = -11/38, 1/38$. This corresponds to the well-known travelling wave solution of Kuramoto and Tsuzuki [79] only existing for two values $k^2 = -1/19$ or $11/19$ :}

$$u = c + \left(\frac{30}{19}k - 15k^3\right) \tanh\frac{k}{2}(\xi - x_0) + 15k^3 \tanh^3\frac{k}{2}(\xi - x_0)$$  \hspace{1cm} (4.16)

where $\xi = x - ct$ and $c, x_0$ are arbitrary constants.

This solution can also be retrieved with the assumption that $u = c + D \log \psi$ with $\psi$ the general solution of a linear system with constant coefficients

$$\psi_{xx} - (k^2/4)\psi = 0 \ , \ \psi_t - c\psi_x = 0.$$ \hspace{1cm} (4.17)

Let us remark that the reduction $u(x,t) \rightarrow c + U(x - ct)$ of the PDE (4.12) yields the nonintegrable ODE

$$U''' + U' + U^2/2 + K = 0, \ K \ \text{arbitrary}$$ \hspace{1cm} (4.18)

for which we have found, in the case $K = -450k^2/19^2$, a Riccati sub-equation linearisable into the system (4.17). The challenge not yet solved is to find for every $K$ a closed form particular solution depending on one arbitrary constant.

Let us also mention the interesting work of Porubov [111, 112] who has found for a large class of nonlinear PDEs like

$$\eta_t + a_1\eta_x + a_2\eta\eta_x + a_3(\eta\eta_x)_x + a_4\eta_{xx} + a_5\eta_{xxx} + a_6\eta_{xxxx} = 0$$ \hspace{1cm} (4.19)
and for some particular values of the constant parameters \( \{a_i\} \), travelling wave solutions in terms of Weierstrass elliptic functions or its logarithmic derivative.

For the KS equation with an additional dispersive term

\[
E \equiv u_t + uu_x + u_{xx} + bu_{xxxx} + u_{xxxx} = 0, \quad (4.20)
\]

which belongs to the previous class \( (4.19) \). Kudryashov \( [75] \) has found a particular solution depending on three arbitrary constants in terms of the Weierstrass elliptic function and its derivative in the single case \( b^2 = 16 \). An easy way to find it is to assume the existence of a solution of the form

\[
u = a_0 + a_2 \wp(x - ct - x_0, g_2, g_3) + a_3 \wp'(x - ct - x_0, g_2, g_3), \quad (4.21)
\]

compatible with the singularity structure of its Laurent series expansion. Enforcing in the LHS \( E \) the conditions \( \wp'^2 = 4 \wp^3 - g_2 \wp - g_3 \) and \( \wp'' = 6 \wp^2 - (g_2/2) \), one identifies to zero a polynomial of two variables \( (\wp, \wp') \) of degree one in \( \wp' \). This similarly generates six equations in the five unknowns \( a_j, g_2, g_3 \) and yields a nondegenerate elliptic solution only if \( b^2 = 16 \) and

\[
a_0 = c - 4/b, \quad a_2 = -15b, \quad a_3 = -60, \quad g_2 = 1/12, \quad (g_3, c, x_0) \text{ arbitrary.} \quad (4.22)
\]

### 4.2.2 Complex Ginzburg-Landau equation CGL3

The one-dimensional cubic complex Ginzburg-Landau (CGL3) equation

\[
E \equiv iu_t + pu_{xx} + q|u|^2u - i\gamma u = 0, \quad pq \neq 0, \ (u, p, q) \in \mathbb{C}, \ \gamma \in \mathbb{R}, \quad (4.23)
\]

with \( p, q, \gamma \) constants, describes pattern formation and coherent structures in many different domains: Taylor-Couette flows between coaxial rotating cylinders, wave propagation in optical fibers and chemical reactions. For a review see \( [39] \). This PDE is physically strongly connected to the KS equation and it also possesses two irrational complex conjugate indices. As is the case for the AKNS system, CGL3 equation possesses four families of singularities

\[
u = A_0 \chi^{-1+ia}(1 + A_1 \chi + O(\chi^2)), \quad \pi = B_0 \chi^{-1-ia}(1 + B_1 \chi + O(\chi^2)), \quad (4.24)
\]

\[
A_0 B_0 = 3|p|^2/\alpha/D_1, \quad \alpha = D_r \pm \sqrt{D_r^2 + 8D_1^2/9}, \quad D_1 = p_r q_i - p_i q_r, \quad (4.25)
\]

\[
D_r = (p_r q_i + p_i q_r), \quad \text{Fuchs indices } : -1, 0, 7/2 \pm \sqrt{1 - 24\alpha^2/2}. \quad (4.26)
\]

The important information we get from the singularity analysis is that neither \((u, \pi)\), nor \(|u|, \arg u)\) nor \((\Re u, \Im u)\) have a simple singularity structure. In this framework, better variables are \((Z, \theta)\) defined as

\[
u = Ze^{i\theta}, \quad \theta = \alpha \log \psi + \theta_0, \quad \psi/\psi_x = \chi, \quad (4.27)
\]
where $\theta_0$ is an arbitrary function representing the index 0. Then $Z$ and $\text{grad} \theta$ behave like simple poles, so that the usual methods are applicable. With $\psi$ the general solution of a second order or third order system and with a one–family or two–family truncation for $(Z, \theta)$ one finds all the known closed form solutions of this equation. Among them, a very interesting solution representing a “collision of two shocks” is easily found by assuming that $\psi$ satisfies a third order linear system. For $q_r = 0$ (hence $\alpha = 0$) this solution degenerates to the “collision of two fronts” solution of KPP previously considered. The Table 4.1 summarises the known solutions of CGL3 and their degeneracies to NLS and KPP equations. An open problem is to find the solution of CGL3 which degenerates for $p_i = q_i = \gamma = 0$ to the bright soliton of NLS.

$$u = A_0 \frac{k \sinh \frac{kx}{2}}{2 \cosh \frac{kx}{2} + e^{-2\gamma(t-t_0)/2}} e^{i\alpha \frac{\log(1+e^{3\gamma(t-t_0)/2} \cosh kx/2)}},$$

$$k^2 = -2\gamma/p, \quad p_r = 0,$$  \hspace{1cm} (4.28)

is easily found by assuming that $\psi$ satisfies a third order linear system. For $q_r = 0$ (hence $\alpha = 0$) this solution degenerates to the “collision of two fronts” solution of KPP previously considered. The Table 4.1 summarises the known solutions of CGL3 and their degeneracies to NLS and KPP equations. An open problem is to find the solution of CGL3 which degenerates for $p_i = q_i = \gamma = 0$ to the bright soliton of NLS.

Table 4.1: Degeneracies of the known solutions of CGL3.

| CGL3                | NLS ($p_i = q_i = \gamma = 0$) | KPP ($p_r = q_r = 0$) |
|---------------------|---------------------------------|------------------------|
| propagating hole    | [4] [8]                         | dark soliton (2.28)    |
| shock or front      | [105] [83]                     | front (4.5)            |
| pulse or solitary wave | [105] [83]                   | stationary pulse (4.7) |
| collision of two shocks ($p_r = 0$) | (4.28) [104] [105] [83] | collision of two fronts (4.8) |
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