Scaling up Hybrid Probabilistic Inference with Logical and Arithmetic Constraints via Message Passing

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Skill matching system

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
Skill matching system

Each player has a certain skill

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
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Each player has a certain skill
Players can form teams

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
Skill matching system

- Each player has a certain skill
- Players can form teams

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
Skill matching system

Each player has a certain skill.
Players can form teams.
Each team’s skill is bounded by its players’ skills.

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
Skill matching system

- Each **player** has a certain skill
- Players can form **teams**
- Each team’s skill is bounded by its players’ skills
  ⇒ complex constraints!

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
Skill matching system

Each player has a certain skill
Players can form teams
Each team’s skill is bounded by its players’ skills
Good teams form a squad

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
Skill matching system

- Each player has a certain skill
- Players can form teams
- Each team’s skill is bounded by its players’ skills
- Good teams form a squad

$\Rightarrow$ discrete variables

---

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
“What is the probability of team $T_1$ to outperform team $T_2$, if $T_1$ is a squad but $T_2$ is not?”

Minka et al., “Trueskill 2: An improved bayesian skill rating system”, 2018
Continuous + discrete + constraints = ?
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Generative adversarial networks (GANs) [Goodfellow et al. 2014]
Variational Autoencoders (VAEs) [Kingma et al. 2013]
Continuous + discrete + constraints = ?

Generative adversarial networks (GANs) [Goodfellow et al. 2014]
Variational Autoencoders (VAEs) [Kingma et al. 2013]

⇒ limited inference capabilities, no constraints
Continuous + discrete + constraints = ?

Generative adversarial networks (GANs) [Goodfellow et al. 2014]
Variational Autoencoders (VAEs) [Kingma et al. 2013]

Hybrid Bayesian Networks (HBNs) [Heckerman et al. 1995; Shenoy et al. 2011]
Mixed Probabilistic Graphical Models (MPGMs) [Yang et al. 2014]
Continuous + discrete + constraints = ?

- Generative adversarial networks (GANs) [Goodfellow et al. 2014]
- Variational Autoencoders (VAEs) [Kingma et al. 2013]
- Hybrid Bayesian Networks (HBNs) [Heckerman et al. 1995; Shenoy et al. 2011]
- Mixed Probabilistic Graphical Models (MPGMs) [Yang et al. 2014]

⇒ strong distributional assumptions
Continuous + discrete + constraints = ?

Generative adversarial networks (GANs) [Goodfellow et al. 2014]
Variational Autoencoders (VAEs) [Kingma et al. 2013]
Hybrid Bayesian Networks (HBNs) [Heckerman et al. 1995; Shenoy et al. 2011]
Mixed Probabilistic Graphical Models (MPGMs) [Yang et al. 2014]
Tractable Probabilistic Circuits (PCs) [Molina et al. 2018; Vergari et al. 2019]
Continuous + discrete + constraints = ?

Generative adversarial networks (GANs) [Goodfellow et al. 2014]

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Tractable Probabilistic Circuits (PCs) [Molina et al. 2018; Vergari et al. 2019]

cannot deal with complex constraints
\textbf{Continuous} + \textbf{discrete} + \textbf{constraints} = \textbf{SMT}

\textit{Satisfiability Modulo Theories} of the linear arithmetic over the reals (\textit{SMT}(LRA)) delivers all these ingredients by design!

Widely used as a representation language for \textit{robotics, verification} and \textit{planning} [Barrett et al. 2010]
Continuous + discrete + constraints = SMT

Each player has a certain skill

Barrett et al., “Satisfiability modulo theories”, 2018
Continuous + discrete + constraints = SMT

$0 \leq X_{P_i} \leq 10$
for $i = 1, \ldots, N$

Barrett et al., “Satisfiability modulo theories”, 2018
Continuous + discrete + constraints = SMT

0 ≤ X_{P_i} ≤ 10
for i = 1, . . . , N

Each team’s skill is bounded by its players’ skills

Barrett et al., “Satisfiability modulo theories”, 2018
Continuous + discrete + constraints = SMT

$0 \leq X_{P_i} \leq 10$
for $i = 1, \ldots, N$

$|X_{T_j} - X_{P_i}| < 1$
for $j = 1, \ldots, M, i = 1, \ldots, |T_j|$

Barrett et al., “Satisfiability modulo theories”, 2018
Continuous + discrete + constraints = SMT

0 ≤ X_{P_i} ≤ 10
for i = 1, . . . , N

| X_{T_j} - X_{P_i} | < 1
for j = 1, . . . , M, i = 1, . . . , |T_j|

Good teams form a squad

Barrett et al., “Satisfiability modulo theories”, 2018
Continuous + discrete + constraints = SMT

0 ≤ X_{P_i} ≤ 10
for i = 1, \ldots, N

| X_{T_j} - X_{P_i} | < 1
for j = 1, \ldots, M, i = 1, \ldots, |T_j|

B_{S_j} \Rightarrow X_{T_j} > 2
for j = 1, \ldots, M, i = 1

Barrett et al., “Satisfiability modulo theories”, 2018
\[ \Delta = \bigwedge_i \ 0 \leq X_{P_i} \leq 10 \bigwedge_j \bigwedge_{i \in T_j} |X_{T_j} - X_{P_i}| < 1 \bigwedge_{j} (B_{S_j} \Rightarrow X_{T_j} > 2) \]

a single CNF SMT(\(\mathcal{LRA}\)) formula \(\Delta\)...

Barrett et al., “Satisfiability modulo theories”, 2018
Continuous + discrete + constraints = SMT

a single CNF SMT(ŁRA) formula \( \Delta \) ...and its **primal graph**

Barrett et al., “Satisfiability modulo theories”, 2018
\[\bigwedge_{i} 0 \leq X_{P_i} \leq 10\]
\[\bigwedge_{j} \bigwedge_{i \in T_j} |X_{T_j} - X_{P_i}| < 1\]
\[\bigwedge_{j} (B_{S_j} \Rightarrow X_{T_j} > 2)\]

\[\sum \begin{cases} w(X_{P_i}), & \text{if } 0 \leq X_{P_i} \leq 10 \\ w(X_{T_j}, X_{P_i}), & \text{if } |X_{T_j} - X_{P_i}| < 1 \\ w(B_{S_j}, X_{T_j}), & \text{if } B_{S_j} \Rightarrow X_{T_j} > 2 \end{cases}\]

**SMT formula** \(\triangleleft\) **weight functions** \(\triangleright\)

Belle et al., “Probabilistic inference in hybrid domains by weighted model integration”, 2015
SMT + weights = Weighted Model Integration

\[
\begin{align*}
\bigwedge_i & \ 0 \leq X_{P_i} \leq 10 \\
\bigwedge_j \bigwedge_{i \in T_j} & \ |X_{T_j} - X_{P_i}| < 1 \\
\bigwedge_j & \ (B_{S_j} \Rightarrow X_{T_j} > 2)
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  w(X_{P_i}), & \text{if } 0 \leq X_{P_i} \leq 10 \\
  w(X_{T_j}, X_{P_i}), & \text{if } |X_{T_j} - X_{P_i}| < 1 \\
  w(B_{S_j}, X_{T_j}), & \text{if } B_{S_j} \Rightarrow X_{T_j} > 2
\end{cases}
\end{align*}
\]

complex support + densities = \(\text{(unnormalized)}\) \(\text{Pr}_\Delta(X, B)\)

Belle et al., “Probabilistic inference in hybrid domains by weighted model integration”, 2015
Given an SMT(\(\mathcal{LRA}\)) formula \(\Delta\) over continuous vars \(\mathbf{X}\) and discrete ones \(\mathbf{B}\), and weight function \(\mathcal{W}\), the \textit{weighted model integral} (WMI) is

\[
\text{WMI}(\Delta, \mathcal{W}; \mathbf{X}, \mathbf{B}) \triangleq \sum_{b \in \mathbf{B}} \int_{(x,b) \models \Delta} w(x, b) \, dx.
\]

i.e., computing the \textit{partition function} of the unnormalized distribution \(\Pr_{\Delta}\)

\[\implies \text{i.e., integrating the weighted volumes of the feasible regions of } \Delta! \]
Advanced probabilistic reasoning

“What is the probability of team $T_1$ to outperform team $T_2$, if $T_1$ is a squad but $T_2$ is not?”

Belle et al., “Probabilistic inference in hybrid domains by weighted model integration”, 2015
\[ \Phi_S : (B_{S_1} = 1 \land B_{S_2} = 0) \implies T_1 \text{ is a squad, } T_2 \text{ is not} \]
\[ \Phi_T : (X_{T_1} > X_{T_2}) \implies T_1 \text{ outperforms } T_2 \]
\[ \Phi_S : (B_{S_1} = 1 \land B_{S_2} = 0) \implies T_1 \text{ is a squad, } T_2 \text{ is not} \]
\[ \Phi_T : (X_{T_1} > X_{T_2}) \implies T_1 \text{ outperforms } T_2 \]

\[ \Pr_\Delta(\Phi_T \mid \Phi_S) = \frac{\text{WMI}(\Delta \land \Phi_T \land \Phi_S, \mathcal{W})}{\text{WMI}(\Delta \land \Phi_S, \mathcal{W})} = \frac{4,206}{7,225} \approx 58.22\% \]

\[ \implies \text{conditional probabilities as a ratio of two weighted model integrals} \]

Belle et al., “Probabilistic inference in hybrid domains by weighted model integration”, 2015
Tractable WMI

#P-hard in general
\[ w(X_{P_i}) = X_{P_i} \]
\[ w(X_{T_j}, X_{P_i}) = X_{T_j} X_{P_i} \]
\[ w(B_{S_j}, X_{T_j}) = X_{T_j}^2 \]

\[ \text{treeMI} = [\text{Zeng et al. 2019}] \]

**tree-shaped primal graph**

**constrained monomials \( \bigwedge \)**

**polytime WMI inference**

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Zeng et al., “Efficient Search-Based Weighted Model Integration”, 2019
Tractable WMI

WMI

treeMI

#P-hard in general

largest tractable class known so far

Zeng et al., “Efficient Search-Based Weighted Model Integration”, 2019
Tractable WMI

- #P-hard in general
- largest tractable class known so far
- still #P-hard!

Zeng et al., “Efficient Search-Based Weighted Model Integration”, 2019
Tractable WMI

- #P-hard in general
- largest tractable class known so far
- still #P-hard!
- can we do better?

Zeng et al., “Efficient Search-Based Weighted Model Integration”, 2019
We frame tractable WMI inference at scale as a message passing scheme...

...on primal graphs...
We frame tractable WMI inference at scale as a *message passing* scheme...

...on primal graphs turned into *factor graphs*
We frame tractable WMI inference at scale as a *message passing* scheme...

...on primal graphs turned into *factor graphs*

- comprising an *upward*
We frame tractable WMI inference at scale as a *message passing* scheme...

...on primal graphs turned into *factor graphs*

- comprising an *upward* and a *downward* pass
We frame tractable WMI inference at scale as a *message passing* scheme...

...on primal graphs turned into *factor graphs*

- comprising an **upward** and a **downward** pass
- exchanging messages from **node to factors**

\[
m_{x_i \rightarrow f_S}(x_i) = \prod_{f_{S'} \in \text{neigh}(x_i) \setminus f_S} m_{f_{S'} \rightarrow x_i}(x_i)
\]
We frame tractable WMI inference at scale as a *message passing* scheme...

...on primal graphs turned into *factor graphs*

- comprising an *upward* and a *downward* pass
- exchanging messages from *node to factors*
- and from *factors to nodes*

\[
m_{f_{ij} \rightarrow x_i}(x_i) = \int f_{ij}(x_i, x_j) \cdot m_{x_j \rightarrow f_{ij}}(x_j) \, dx_j
\]
Tractable Weight Conditions

Which parametric family $\Omega$ for weights to guarantee tractable WMI inference?
Tractable Weight Conditions

Which parametric family $\Omega$ for weights to guarantee tractable WMI inference?

$$m_{f_{ij} \rightarrow x_i}(x_i) = \int \prod_{\Gamma \in \Delta_S} \mathbb{I}_{x_S \models \Gamma} \prod_{\ell \in \mathcal{L}_\Gamma} w_\ell(x_S)^{x_S \models \ell} \cdot m_{x_j \rightarrow f_{ij}}(x_j) \, dx_j$$

$$m_{x_i \rightarrow f_S}(x_i) = \prod_{f_{S'} \in \text{neigh}(x_i) \setminus f_S} m_{f_{S'} \rightarrow x_i}(x_i)$$
Tractable Weight Conditions

Which parametric family $\Omega$ for weights to guarantee tractable WMI inference?

$$m_{f_{ij} \rightarrow x_i}(x_i) = \int \prod_{\Gamma \in \Delta_S} \mathbb{I}_{x_S \models \Gamma} \prod_{\ell \in \mathcal{L}_\Gamma} w_\ell(x_S) \mathbb{I}_{x_S \models \ell} \cdot m_{x_j \rightarrow f_{ij}}(x_j) \, dx_j$$

$$m_{x_i \rightarrow f_S}(x_i) = \prod_{f_{S'} \in \text{neigh}(x_i) \setminus f_S} m_{f_{S'} \rightarrow x_i}(x_i)$$

Weights $\mathcal{W} \in \Omega$ should be closed under product...
Tractable Weight Conditions

Which parametric family $\Omega$ for weights to guarantee tractable WMI inference?

$$m_{f_{ij} \rightarrow x_i}(x_i) = \int \prod_{\Gamma \in \Delta_S} \left[ x_S \models \Gamma \right] \prod_{\ell \in L_{\Gamma}} w_{\ell}(x_S)^{[x_S \models \ell]} \cdot m_{x_j \rightarrow f_{ij}}(x_j) \, dx_j$$

$$m_{x_i \rightarrow f_S}(x_i) = \prod_{f_{S'} \in \text{neigh}(x_i) \setminus f_S} m_{f_{S'} \rightarrow x_i}(x_i)$$

Weights $\mathcal{W} \in \Omega$ should be closed under product, closed under integration, and tractable for symbolic integration.
Tractable Weight Conditions

Which parametric family $\Omega$ for weights to guarantee tractable WMI inference?

$$m_{f_{ij} \rightarrow x_i}(x_i) = \int \prod_{\Gamma \in \Delta_S} \mathbb{1}_{x_S \models \Gamma} \prod_{\ell \in \mathcal{L}_\Gamma} w_\ell(x_S)_{x_S \models \ell} \cdot m_{x_j \rightarrow f_{ij}}(x_j) \, dx_j$$

$$m_{x_i \rightarrow f_S}(x_i) = \prod_{f_{S'} \in \text{neigh}(x_i) \setminus f_S} m_{f_{S'} \rightarrow x_i}(x_i)$$

Weights $W \in \Omega$ should be closed under product, closed under integration, and tractable for symbolic integration $\Rightarrow$ e.g., arbitrary polynomials, exponentiated linear polynomials, etc.
An SMT formulation induces a **piecewise weight representation** strikingly different from message passing for classical PGMs!

\[
\begin{align*}
\mathbf{m}_{f_{ij} \rightarrow x_i} (x_i) &= \int \prod_{\Gamma \in \Delta_S} [x_S \models \Gamma] \prod_{\ell \in \mathcal{L}_\Gamma} w_\ell(x_S)_{[x_S = \ell]} \cdot \mathbf{m}_{x_j \rightarrow f_{ij}} (x_j) \, dx_j \\
\mathbf{m}_{x_i \rightarrow f_S} (x_i) &= \prod_{f_{S'} \in \text{neigh}(x_i) \setminus f_S} \mathbf{m}_{f_{S'} \rightarrow x_i} (x_i)
\end{align*}
\]
An SMT formulation induces a **piecewise weight representation** strikingly different from message passing for classical PGMs!
An SMT formulation induces a piecewise weight representation \( \Rightarrow \) strikingly different from message passing for classical PGMs!

The number of all pieces in MP-WMI is \( \mathcal{O}(4nc)^{2d+2} \), where \( d \) is the graph diameter \( \Rightarrow \) the primal graph should have a bounded diameter!
#P-hard in general

the largest tractable class known before

still #P-hard

new largest class!

Zeng et al., “Efficient Search-Based Weighted Model Integration”, 2019
Scaling-up inference

Large set of synthetic benchmarks up to $N = 100$ vars, 5 trials, different primal graphs

**STAR**

treewidth: 1
diameter: 2

MP-WMI takes a *fraction of the time* of other exact WMI solvers like PA [Morettin et al. 2017] and F-XSDD [Zuidberg Dos Martires et al. 2019]
Scaling-up inference

Large set of synthetic benchmarks up to $N = 100$ vars, 5 trials, different primal graphs

SNOW

treewidth: 1
diameter: $\log(N)$

MP-WMI takes a fraction of the time of other exact WMI solvers like PA [Morettin et al. 2017] and F-XSDD [Zuidberg Dos Martires et al. 2019]
Scaling-up inference

Large set of synthetic benchmarks up to $N = 100$ vars, 5 trials, different primal graphs

**PATH**

- Treewidth: 1
- Diameter: $N$

MP-WMI takes a *fraction of the time* of other exact WMI solvers like PA [Morettin et al. 2017] and F-XSDD [Zuidberg Dos Martires et al. 2019]
Query amortization

A single message exchange allows to *amortize univariate and bivariate queries* ⇒ also *all marginals and all moments*!

MP-WMI answers 100 WMI queries faster than competitors solving 10 [Zeng et al. 2019]
Real-world data is *noisy*...
Real-world data is noisy, complex...
Real-world data is \textit{noisy, complex} and \textit{mixed continuous-discrete}...
Conclusions

Real-world data is *noisy, complex* and *mixed continuous-discrete*...

*The WMI framework* is very appealing for probabilistic inference in the real-world!
Conclusions

Real-world data is noisy, complex and mixed continuous-discrete...

*The WMI framework* is very appealing for probabilistic inference in the real-world!

MP-WMI delivers fast inference and defines the *largest class of tractable WMI models*
Conclusions

Real-world data is noisy, complex and mixed continuous-discrete... The WMI framework is very appealing for probabilistic inference in the real-world! MP-WMI delivers fast inference and defines the largest class of tractable WMI models.

Next

However, MP-WMI requires tree-shaped bounded diameter primal graphs ⇒ we can build approximate inference schemes on it!
Conclusions

Real-world data is *noisy, complex* and *mixed continuous-discrete*...

The WMI framework is very appealing for probabilistic inference in the real-world! MP-WMI delivers fast inference and defines the *largest class of tractable WMI models*

Next

However, MP-WMI requires tree-shaped bounded diameter primal graphs

⇒ *we can build approximate inference schemes on it!*

Code

[github.com/UCLA-StarAI/mpwmi](http://github.com/UCLA-StarAI/mpwmi)
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