WORDS IN LINEAR GROUPS, RANDOM WALKS, AUTOMATA
AND P-RECURSIVENESS

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Abstract. Fix a finite set $S \subset \text{GL}(k, \mathbb{Z})$. Denote by $a_n$ the number of products of matrices in $S$ of length $n$ that are equal to 1. We show that the sequence $\{a_n\}$ is not always P-recursive. This answers a question of Kontsevich.

1. Introduction

An integer sequence $\{a_n\}$ is called polynomially recursive, or P-recursive, if it satisfies a non-trivial linear recurrence relation of the form

\[(*) \quad q_0(n)a_n + q_1(n)a_{n-1} + \ldots + q_k(n)a_{n-k} = 0,
\]

for some $q_i(x) \in \mathbb{Z}[x]$, $0 \leq i \leq k$. The study of P-recursive sequences plays a major role in modern Enumerative and Asymptotic Combinatorics, see e.g. [FS, Ges, Odl, SI]. They have D-finite (also called holonomic) generating series

\[A(t) = \sum_{n=0}^{\infty} a_n t^n,
\]

and various asymptotic properties (see Section 5 below).

Let $G$ be a group and $\mathbb{Z}[G]$ denote its group ring. For every $g \in G$ and $u \in \mathbb{Z}[G]$, denote by $[g]u$ the value of $u$ on $g$. Let $a_n = [1]u^n$, which denotes the value of $u^n$ at the identity element. When $G = \mathbb{Z}^k$ or $G = F_k$, the sequence $\{a_n\}$ is known to be P-recursive for all $u \in \mathbb{Z}[G]$, see [Ha]. Maxim Kontsevich asked whether $\{a_n\}$ is always P-recursive when $G \subseteq \text{GL}(k, \mathbb{Z})$, see [S2]. We give a negative answer to this question:

**Theorem 1.** There exists an element $u \in \mathbb{Z}[\text{SL}(4, \mathbb{Z})]$, such that the sequence $\{[1]u^n\}$ is not P-recursive.

We give two proofs of the theorem. The first proof is completely self-contained and based on ideas from computability. Roughly, we give an explicit construction of a finite state automaton with two stacks and a non-P-recursive sequence of accepting path lengths (see Section 3). We then convert this automaton into a generating set $S \subset \text{SL}(4, \mathbb{Z})$, see Section 4. The key part of the proof is a new combinatorial lemma giving an obstruction to P-recursiveness (see Section 2).

Our second proof of Theorem 1 is analytic in nature, and is the opposite of being self-contained. We interpret the problem in a probabilistic language, and use a number of advanced and technical results in Analysis, Number Theory, Probability and Group Theory to derive the theorem. Let us briefly outline the connection.

Let $S$ be a generating set of the group $G$. Denote by $p(n) = p_{G,S}(n)$ the probability of return after $n$ steps of a random walk on the corresponding Cayley graph $\text{Cay}(G, S)$. Finding
the asymptotics of \( p(n) \) as \( n \to \infty \) is a fundamental problem in probability, with a number of both classical and recent results (see e.g. [Peter, Woe]). In the notation above, we have:

\[
p(n) = \frac{a_n}{|S|}, \quad \text{where} \quad a_n = [1]u^n \quad \text{and} \quad u = \sum_{s \in S} s.
\]

Since P-recursiveness of \( \{a_n\} \) implies P-recursiveness of \( \{p(n)\} \), and much is known about the asymptotic of both \( p(n) \) and P-recursive sequences, this connection can be exploited to obtain non-P-recursive examples (see Section 5). See also Section 6 for final remarks and historical background behind the two proofs.

2. Parity of P-Recursive Sequences

In this section, we give a simple obstruction to P-recursiveness.

**Lemma 2.** Let \( \{a_n\} \) be a P-recursive integer sequence. Consider an infinite binary word \( w = w_1w_2 \ldots \) defined by \( w_n = a_n \mod 2 \). Then, there exists a finite binary word \( v \) which is not a subword of \( w \).

**Proof.** Let \( \eta(n) \) denote the largest integer \( r \) such that \( 2^r | n \). By definition, there exist polynomials \( q_0, \ldots, q_k \in \mathbb{Z}[n] \), such that

\[
a_n = \frac{1}{q_0(n)}(a_{n-1}q_1(n) + \ldots + a_{n-k}q_k(n)), \quad \text{for all} \quad n > k.
\]

Let \( \ell \) be any integer such that \( q_i(\ell) \neq 0 \) for all \( i \). Similarly, let \( m \) be the smallest integer such that \( 2^m > k \), and \( m > \eta(q_i(\ell)) \) for all \( i \). Finally, let \( d > 0 \) be such that \( \eta(q_d(\ell)) \leq \eta(q_i(\ell)) \) for all \( i > 0 \).

Consider all \( n \) such that:

\((*)\) \quad \( n = \ell \mod 2^m \), \quad \( w_{n-d} = 1 \) and \( w_{n-i} = 0 \) for all \( i \neq 0, d \).

Note that \( \eta(q_i(n)) = \eta(q_i(\ell)) \) for all \( i \), since \( q_i(n) = q_i(\ell) \mod 2^m \) and \( \eta(q_i(\ell)) < m \). We have

\[
\eta(a_n) = \eta\left(a_{n-1}q_1(\ell) + \ldots + a_{n-k}q_k(\ell)\right) - \eta(q_0(\ell)).
\]

Since \( \eta(a_{n-d}q_d(\ell)) < \eta(a_{n-i}q_i(\ell)) \) for all \( i \neq d \), this implies that

\[
\eta(a_n) = \eta(a_{n-d}q_d(\ell)) - \eta(q_0(\ell)) = \eta(q_d(\ell)) - \eta(q_0(\ell)).
\]

Therefore, \( w_n = 1 \) if and only if \( \eta(q_d(\ell)) = \eta(q_0(\ell)) \). This implies that \( w_n \) is independent of \( n \), and must be the same for all \( n \) satisfying (*). In particular, this means that at least one of the words \( 0^{k-d}10^{d-1}1 \) and \( 0^{k-d}10^d \) cannot appear in \( w \) ending at a location congruent to \( \ell \) modulo \( 2^m \).

Consider the word \( v = (0^{k-d}10^{k-1}10^{d-1})^2 \). Note that \( 0^{k-d}10^{k-1}10^d \) has odd length, and contains both \( 0^{k-d}10^{d-1}1 \) and \( 0^{k-d}10^d \) as subwords. Therefore, the word \( v \) contains both \( 0^{k-d}10^{d-1}1 \) and \( 0^{k-d}10^d \) in every possible starting location modulo \( 2^m \). This implies that \( v \) cannot appear as a subword of \( w \). \( \square \)
3. Building an Automaton

In this section we give an explicit construction of a finite state automaton with the number of accepting paths given by a binary sequence which does not satisfy conditions of Lemma 2.

Let \( X \simeq F_3 \) be the free group generated by \( x, 1_x \), and \( 0_x \). Similarly, let \( Y \simeq F_3 \) be the free group generated by \( y, 1_y \), and \( 0_y \). We assume that \( X \) and \( Y \) commute.

Define a directed graph \( \Gamma \) on vertices \( \{ s_1, \ldots, s_8 \} \), and with edges as shown in Figure 1. Some of the edges in \( \Gamma \) are labeled with elements of \( X, Y \), or both. For a path \( \gamma \) in \( \Gamma \), denote by \( \omega_X(\gamma) \) the product of all elements of \( X \) in \( \gamma \), and by \( \omega_Y(\gamma) \) denote the product of all elements of \( Y \) in \( \gamma \). By a slight abuse of notation, while traversing \( \gamma \) we will use \( \omega_X \) and \( \omega_Y \) to refer to the product of all elements of \( X \) and \( Y \), respectively, on edges that have been traversed so far.

Finally, let \( b_n \) denote the number of paths in \( \Gamma \) from \( s_1 \) to \( s_8 \) of length \( n \), such that \( \omega_X(\gamma) = \omega_Y(\gamma) = 1 \). For example, the path \( \gamma : s_1 \xrightarrow{xy} s_1 \to s_2 \xrightarrow{1_y x^{-1}} s_4 \xrightarrow{1_x^{-1} 1_y} s_4 \xrightarrow{y^{-1}} s_5 \to s_6 \xrightarrow{1_x^{-1}} s_8 \) is the unique such path of length 7, so \( b_7 = 1 \).

**Figure 1.** The graph \( \Gamma \).

**Lemma 3.** For every \( n \geq 1 \) we have \( b_n \in \{0,1\} \). Moreover, every finite binary word is a subword of \( b = b_1 b_2 \ldots \)

**Proof.** To simplify the presentation, we split the proof into two parts.

(a) **The structure of paths.** Let \( \gamma \) be a path from \( s_1 \) to \( s_8 \). Denote by \( k \) the number of times \( \gamma \) traverses the loop \( s_1 \xrightarrow{xy} s_1 \). The value of \( \omega_X \) after traversing these \( k \) loops is \( x^k \), and the value of \( \omega_Y \) is \( y^k \).

There must be \( k \) instances of the edge \( s_4 \xrightarrow{y^{-1}} s_5 \) in \( \gamma \) to cancel out the \( y^k \). Further, any time the path traverses this edge, the product \( \omega_Y \) must change from some \( y^j \) to \( y^{j-1} \), with no \( 0_y \) or \( 1_y \) terms. Therefore, every time \( \gamma \) enters the vertex \( s_4 \), it must traverse the two loops \( s_4 \xrightarrow{1_y^{-1}} s_4 \) and \( s_4 \xrightarrow{0_y^{-1}} s_4 \) enough to replace any \( 0_y \) and \( 1_y \) terms in \( \omega_Y \) with \( 0_x \) and \( 1_x \).
terms in $\omega_X$. This takes the binary word at the end of $\omega_Y$, and moves it to the end of $\omega_X$ in the reverse order.

Similarly, any time $\gamma$ traverses the edge $s_3 \xrightarrow{x^{-1}} s_4$ or $s_2 \xrightarrow{1+x^{-1}} s_4$, the product $\omega_X$ must change from some $x^j$ to $x^{j-1}$, with no $0_x$ or $1_x$ terms. Every time $\gamma$ enters the vertex $s_2$, it must remove all $0_x$ and $1_x$ terms from $\omega_X$ before transitioning to $s_4$. The $s_2$ and $s_3$ vertices ensure that as this binary word is deleted from $\omega_X$, another binary word is written at the end of $\omega_Y$ such that the reverse of the binary word written at the end of $\omega_Y$ is one greater as a binary integer than the word removed from the end of $\omega_X$.

Every time $\gamma$ traverses the edge $s_4 \xrightarrow{y^{-1}} s_5$, the number written in binary at the end of $\omega_X$ is incremented by one. Thus, after traversing this edge $k$ times, the $X$ word will consist of $k$ written in binary, and $\omega_Y$ will be the identity. At this point, $\gamma$ will traverse the edge $s_5 \xrightarrow{y^{-1}} s_6$.

After entering the vertex $s_6$, all of the $0_x$ and $1_x$ terms from $\omega_X$ will be removed. Each time a $1_x$ term is removed, $\gamma$ can move to the vertex $s_8$. From $s_8$, the $0_x$ and $1_x$ terms will continue to be removed, but $\gamma$ will traverse two edges for every term removed, thus moving at half speed. After all of these terms are removed, the products $\omega_X(\gamma)$ and $\omega_Y(\gamma)$ are equal to identity, as desired.

(b) The length of paths. Now that we know the structure of paths through $\Gamma$, we are ready to analyze the possible lengths of these paths. There are only two choices to make in specifying a path $\gamma$: first, the number $k = k(\gamma)$ of times the loop from $s_1$ to itself is traversed, and second, the number $j = j(\gamma)$ of digits still on $\omega_X(\gamma)$ immediately before traversing the edge from $s_6$ to $s_8$. The number $j$ must be such that the $j$-th binary digit of $k$ is a 1.

When $\gamma$ reaches $s_5$ for the first time, it has traversed $k + 4$ edges. In moving from the $i$-th instance of $s_5$ along $\gamma$ to the $(i + 1)$-st instance of $s_5$, the number of edges traversed is $3 + \lfloor 1 + \log_2(i) \rfloor + \lfloor 1 + \log_2(i + 1) \rfloor$, three more than the sum of the number of binary digits in $i$ and $i + 1$. Therefore, the number of edges traversed by the time $\gamma$ reaches $s_6$ is equal to

$$k + 5 + \sum_{i=1}^{k-1} (3 + \lfloor 1 + \log_2(i) \rfloor + \lfloor 1 + \log_2(i + 1) \rfloor).$$

If $j = 1$, the edge from $s_6$ to $s_8$ is traversed at the last possible opportunity and $\lfloor 1 + \log_2(k) \rfloor$ more edges are traversed. However, if $j > 1$, there are an additional $j - 1$ edges traversed, since the $s_7$ and $s_8$ states do not remove $\omega_X$ terms as efficiently as $s_6$. In total, this gives $|\gamma| = L(k(\gamma), j(\gamma))$, where

$$L(k, j) = j - 1 + \lfloor 1 + \log_2(k) \rfloor + k + 5 + \sum_{i=1}^{k-1} \left(3 + \lfloor 1 + \log_2(i) \rfloor + \lfloor 1 + \log_2(i + 1) \rfloor\right).$$

This simplifies to

$$L(k, j) = j + 6k + 2 \sum_{i=1}^{k} \lfloor \log_2 i \rfloor.$$

Since $1 \leq j \leq \lfloor 1 + \log_2(k) \rfloor$, we have $L(k + 1, 1) > L(k, j)$ for all possible values of $j$. Thus, there are no two paths of the same length, which proves the first part of the lemma.

Furthermore, we have $b_n = 1$ if and only if $n = L(k, j)$ for some $k \geq 1$ and $j$ such that the $j$-th binary digit of $k$ is a 1. Thus, the binary subword of $b$ at locations $L(k, 1)$ through $L(k, \lfloor 1 + \log_2(k) \rfloor)$ is exactly the integer $k$ written in binary. This is true for every positive integer $k$, so $b$ contains every finite binary word as a subword. □
Example 4. For \( k = 3 \) and \( j = 2 \), we have \( L(k, j) = 24 \). This corresponds to the unique path in \( \Gamma \) of length 24:

\[
\begin{align*}
1_\omega & \rightarrow s_2 \rightarrow s_4 \rightarrow s_5 \rightarrow s_2 \\
1_x & \rightarrow s_3 \rightarrow s_4 \rightarrow s_5 \rightarrow s_2 \\
x & \rightarrow s_6 \rightarrow s_5 \rightarrow s_6 \\
\end{align*}
\]

4. Proof of Theorem

4.1. From automata to groups. We start with the following technical lemma.

Lemma 5. Let \( G = F_{11} \times F_3 \). Then there exists an element \( u \in \mathbb{Z}[G] \), such that \( [1]u^{2n+1} \) is always even, and \( w = w_1w_2 \ldots \) given by \( w_n = (\frac{1}{2})[1]u^{2n+1} \) mod 2, is an infinite binary word that contains every finite binary word as a subword.

Proof. We suggestively label the generators of \( F_{11} \) as \( \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, x, 0_x, 1_x\} \) and label the generators of \( F_3 \) as \( \{y, 0_y, 1_y\} \). Consider the following set \( S \) of 19 elements of \( G \):

\[
\begin{align*}
(1) \ z_1 &= s_1^{-1}xy, \\
(2) \ z_2 &= s_1^{-1}s_2, \\
(3) \ z_3 &= s_2^{-1}x^{-1}0_y, \\
(4) \ z_4 &= s_2^{-1}0_y^{-1}1_y, \\
(5) \ z_5 &= s_3^{-1}1_x^{-1}1_y, \\
(6) \ z_6 &= s_3^{-1}0_x^{-1}0_y, \\
(7) \ z_7 &= s_3^{-1}x^{-1}s_4, \\
(8) \ z_8 &= s_2^{-1}1_y^{-1}s_4, \\
(9) \ z_9 &= s_4^{-1}1_x^{-1}s_4, \\
(10) \ z_{10} &= s_4^{-1}0_y^{-1}s_4, \\
(11) \ z_{11} &= s_4^{-1}y^{-1}s_5, \\
(12) \ z_{12} &= s_5^{-1}s_2, \\
(13) \ z_{13} &= s_5^{-1}s_6, \\
(14) \ z_{14} &= s_6^{-1}1_x^{-1}s_6, \\
(15) \ z_{15} &= s_6^{-1}0_x^{-1}s_6, \\
(16) \ z_{16} &= s_6^{-1}1_x^{-1}s_8, \\
(17) \ z_{17} &= s_7^{-1}s_8, \\
(18) \ z_{18} &= s_8^{-1}1_x^{-1}s_7, \\
(19) \ z_{19} &= s_8^{-1}0_x^{-1}s_7.
\end{align*}
\]

Let \( \Gamma \) be as defined in the previous section. For every edge from \( s_i \rightarrow s_j \) in \( \Gamma \), there is one element of \( S \) equal to \( s_i^{-1}r_{s_j} \). We show that the number of ways to multiply \( n \) terms from \( S \) to get \( s_i^{-1}s_8 \) is exactly \( b_n \).

First, we show that there is no product of terms in \( S \) whose \( F_{11} \) component is the identity. Assume that such a product exists, and take one of minimal length. If there are two consecutive terms in this product such that \( s_i \) at the end of one term does not cancel the \( s_j^{-1} \) at the start of the following term, then either the \( s_i \) must cancel with a \( s_i^{-1} \) before it or the \( s_j^{-1} \) must cancel with a \( s_j \) after it. In both cases, this gives a smaller sequence of terms whose product must have \( F_{11} \) component equal to the identity. If the \( s_i \) at the end of each term cancels the \( s_j^{-1} \) at the beginning of the next term, then this product corresponds to a cycle \( \gamma \in \Gamma \) such that \( \omega_X(\gamma) \) is the identity. Straightforward analysis of \( \Gamma \) shows that no such cycle exists, so there is no product of terms in \( S \) whose product \( F_{11} \) component equal to the identity.

This also means that the \( s_i \) at the end of each term must cancel the \( s_j^{-1} \) at the start of the following term, since otherwise either the \( s_i \) must cancel with a \( s_i^{-1} \) before it or the \( s_j^{-1} \) must cancel with a \( s_j \) after it, forming a product of terms in \( S \) whose \( F_{11} \) component is equal to the identity.

Since each \( s_i \) cancels with an \( s_i^{-1} \) at the start of the following term, the product must correspond to a path \( \gamma \in \Gamma \). If \( \gamma \) is from \( s_i \) to \( s_j \), the product will evaluate to \( s_i^{-1}\omega_X(\gamma)\omega_Y(\gamma)s_j \). Therefore, the number of ways to multiply \( n \) terms from \( S \) to get \( s_i^{-1}s_8 \) is equal to \( b_n \).

We can now define \( u \in \mathbb{Z}[G] \) as

\[
u = 2s_8^{-1}s_1 + \sum_{z_i \in S} z_i.
\]
We claim that \( \frac{1}{2} [1] u^{2n+1} = b_{2n} \mod 2 \). We already showed that one cannot get 1 by multiplying only elements of \( S \), so the \( 2s_8^{-1}s_1 \) term must be used at least once. If this term is used more than once, then the contribution to \([1] u^{2n+1}\) will be \( 0 \mod 4 \). Therefore, we need only consider the cases where this term is used exactly once, so \( \frac{1}{2} [1] u^{2n+1} \) is equal modulo 2 to the number of products of the form

\[
(\ast\ast) \quad 2 = z_{i_1} \ldots z_{i_{k-1}} (2s_8^{-1}s_1) z_{i_k+1} \ldots z_{i_{2n+1}}.
\]

This condition holds if and only if

\[
z_{i_{k+1}} \ldots z_{i_{2n+1}} z_{i_1} \ldots z_{i_{k-1}} = s_1^{-1}s_8,
\]

which can be achieved in \( b_{2n} \) ways.

There are \( 2n + 1 \) choices for the location \( k \) of the \( 2s_8^{-1}s_1 \) term, and for each such \( k \), there are \( b_{2n} \) solutions to \((\ast\ast)\). This gives

\[
\frac{1}{2} [1] u^{2n+1} = (2n + 1)b_{2n} = b_{2n} \mod 2,
\]

which implies \( w_n = b_{2n} \). By Lemma 5 we conclude that \( w \) is an infinite binary word which contains every finite binary word as a subword. \( \square \)

4.2. **Counting words mod 2.** We first deduce the main result of this paper and then give a useful minor extension.

**Proof of Theorem 7.** The group \( \text{SL}(4, \mathbb{Z}) \) contains \( \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \) as a subgroup. The group \( \text{SL}(2, \mathbb{Z}) \) contains Sanov’s subgroup isomorphic to \( F_2 \), and thus every finitely generated free group \( F_t \) as a subgroup (see e.g. [HIH]). Therefore, \( F_{11} \times F_3 \) is a subgroup of \( \text{SL}(4, \mathbb{Z}) \), and the element \( u \in \mathbb{Z}[F_{11} \times F_3] \) defined in Lemma 5 can be viewed as an element of \( \mathbb{Z}[\text{SL}(4, \mathbb{Z})] \).

Let \( a_n = [1]u^n \). By Lemma 5 the number \( a_{2n+1} \) is always even, and the word \( w = w_1w_2 \ldots \) given by \( w_n = \frac{1}{2} a_{2n+1} \mod 2 \) is an infinite binary word which contains every finite binary word as a subword. Therefore, by Lemma 2 the sequence \( \{\frac{1}{2} a_{2n+1}\} \) is not P-recursive. Since P- recursivity is closed under taking a subsequence consisting of every other term, the sequence \( \{a_n\} \) is also not P-recursive. \( \square \)

**Theorem 6.** There is a group \( G \subset \text{SL}(4, \mathbb{Z}) \) and two generating sets \( \langle S_1 \rangle = \langle S_2 \rangle = G \), such that for the elements

\[
u_1 = \sum_{s \in S_1} s, \quad u_2 = \sum_{s \in S_2} s,
\]

we have the sequence \( \{[1]u_1^n\} \) is P-recursive, while \( \{[1]u_2^n\} \) is not P-recursive.

**Proof.** Let \( G = F_{11} \times F_3 \) be as above. Denote by \( X_1 \) and \( X_2 \) the standard generating sets of \( F_{11} \) and \( F_3 \), respectively. Finally, let \( S_1 = (X \times 1) \cup (1 \times Y) \),

\[
w_1 = \sum_{x \in X_1} x, \quad w_2 = \sum_{x \in X_2} x.
\]

Recall that if \( \{c_n\} \) is P-recursive, then so is \( \{c_n/n!\} \) and \( \{c_n \cdot n!\} \). Observe that

\[
\sum_{n=0}^{\infty} [1]u_1^n t^n/n! = \left( \sum_{n=0}^{\infty} [1]u_1^n t^n/n! \right) \left( \sum_{n=0}^{\infty} [1]u_2^n t^n/n! \right).
\]

and that \( \{[1]u_1^n\} \) and \( \{[1]u_2^n\} \) are P-recursive by Haiman’s theorem [Hal]. This implies that \( \{[1]u_1^n\} \) is also P-recursive, as desired.

Now, let \( S_2 = 2S_1 \cup S \), where \( S \) is the set constructed in the proof of Lemma 5 and \( 2S_1 \) means that each element of \( S_1 \) is taken twice. Observe that \([1]u_2^n \equiv [1]u^n \mod 2\), where \( u \) is as in the proof of Theorem 1. This implies that \( \{[1]u_1^n\} \) is not P-recursive, and finishes the proof. \( \square \)
5. Asymptotics of P-recursive sequences and the return probabilities

5.1. Asymptotics. The asymptotics of general P-recursive sequences is understood to be a finite sum of the terms

\[ A(n)^s \lambda^n e^{Q(n)} n^\alpha (\log n)^\beta, \]

where \( s, \gamma \in \mathbb{Q}, \alpha, \lambda \in \mathbb{Q}, \beta \in \mathbb{N}, \) and \( Q(\cdot) \) is a polynomial. This result goes back to Birkhoff and Trjitzinsky (1932), and also Turrittin (1960). Although there are several gaps in these proofs, they are closed now, notably in [Imm]. We refer to [FS, §VIII.7], [Odl] §9.2 and [Pak] for various formulations of general asymptotic estimates, an extensive discussion of priority issues and further references.

For the integer P-recursive sequences which grow at most exponentially, the asymptotics have further constraints summarized in the following theorem.

**Theorem 7.** Let \( \{a_n\} \) be an integer P-recursive sequence defined by \((\ast)\), and such that \( a_n < C^n \) for some \( C > 0 \) and all \( n \geq 1 \). Then

\[ a_n \sim \sum_{i=1}^{m} A_i \lambda_i^n n^{\alpha_i} (\log n)^{\beta_i}, \]

where \( \alpha_i \in \mathbb{Q}, \lambda_i \in \mathbb{Q} \) and \( \beta_i \in \mathbb{N} \).

The theorem is a combination of several known results. Briefly, the generating series \( A(t) \) is a \( G \)-functions in a sense of Siegel (1929), which by the works of André, Bombieri, Chudnovsky, Dwork and Katz, must satisfy an ODE which has only regular singular points and rational exponents (see a discussion on [And] p. 719 and an overview in [Bou]). We then apply the Birkhoff–Trjitzinsky theorem, which in the regular case has a complete and self-contained proof (see Theorem VII.10 and subsequent comments in [FS]). We refer to [Pak] for further references and details.

5.2. Probability of return. Let \( G \) be a finitely generated group. A generating set \( S \) is called symmetric if \( S = S^{-1} \). Let \( H \) be a subgroup of \( G \) of finite index. It was shown by Pittet and Saloff-Coste [PS2], that for two symmetric generating sets \( \langle S \rangle = G \) and \( \langle S' \rangle = H \) we have

\[ C_1 \rho_{G,S}(\alpha_1 n) < p_{G,S}(n) < C_2 \rho_{G,S}(\alpha_2 n), \]

for all \( n > 0 \) and fixed constants \( C_1, C_2, \alpha_1, \alpha_2 > 0 \). For \( G = H \), this shows, qualitatively, that the asymptotic behavior of \( p_{G,S}(n) \) is a property of a group. The following result gives a complete answer for a large class of groups.

**Theorem 8.** Let \( G \) be an amenable subgroup of \( GL(k, \mathbb{Z}) \) and \( S \) is a symmetric generating set. Then either \( G \) has polynomial growth and polynomial return probabilities:

\[ A_1 n^{-d} < p_{G,S}(2n) < A_2 n^{-d}, \]

or \( G \) has exponential growth and mildly exponential return probabilities:

\[ A_1 \rho_1^{\sqrt{\pi}} < p_{G,S}(2n) < A_2 \rho_2^{\sqrt{\pi}}, \]

for some \( A_1, A_2 > 0, 0 < \rho_1, \rho_2 < 1, \) and \( d \in \mathbb{N} \).

The theorem is again a combination of several known results. Briefly, by the Tits alternative, group \( G \) must be virtually solvable, which implies that it either has a polynomial or exponential growth (see e.g. [dH]). By the quasi-isometry \((\ast)\), we can assume that \( G \) is solvable. In the polynomial case, the lower bound follows from the CLT by Crépel and Raugi [CR], while the upper bound was proved by Varopoulos using the Nash inequality [V1] (see also [V3]). For the more relevant to us case of exponential growth, recall Mal’tsev’s theorem, which says that all solvable subgroups of \( SL(n, \mathbb{Z}) \) are polycyclic (see e.g. [Sup, Thm. 22.7]). For polycyclic groups of exponential growth, the upper bound is due to Varopoulos [V2] and the lower bound is due
5.3. Applications to P-recursiveness. We can now show that non-P-recursiveness for amenable linear groups of exponential growth.

**Theorem 9.** Let $G$ be an amenable subgroup of $GL(k, \mathbb{Z})$ of exponential growth, and let $S$ be a symmetric generating set. Then the probability of return sequence $\{p_{G,S}(n)\}$ is not P-recursive.

**Proof.** It is easy to see that $H$ has exponential growth, so Theorem 8 applies. Let $a_n = |S|^n p_{G,S}(n) \in \mathbb{N}$ as in the introduction. If $\{p_{G,S}(n)\}$ is P-recursive, then so is $\{a_{2n}\}$. On the other hand, Theorem 7 forbids mildly exponential terms $\rho^{2n}$ in the asymptotics of $a_{2n}$, giving a contradiction. □

To obtain Theorem 1 from here, consider the following linear group $H \subset SL(3, \mathbb{Z})$ of exponential growth:

$H = \left\{ \begin{pmatrix} x_{11} & x_{12} & y_1 \\ x_{21} & x_{22} & y_2 \\ 0 & 0 & 1 \end{pmatrix} \right\} \text{ s.t. } \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^k, k \in \mathbb{Z}$

(see e.g. [Woe §15.B]). Observe that $H \simeq \mathbb{Z} \times \mathbb{Z}^2$, and therefore solvable. Thus, $H$ has a natural symmetric generating set

$E = \left\{ \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pm 1, \begin{pmatrix} 1 & 0 & \pm 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

By Theorem 9, the probability of return sequence $\{p_{H,E}(n)\}$ is not P-recursive, as desired.

6. Final Remarks

6.1. Kontsevich’s question was originally motivated by related questions on the “categorical entropy” [DHKK]. In response to the draft of this paper, Ludmil Katzarkov, Maxim Kontsevich and Richard Stanley asked us if the examples we construct satisfy algebraic differential equations (ADE), see e.g. [SI, Exc. 6.63]. We believe that the answer is No, and plan to explore this problem in the future.

6.2. The motivation behind the proof of Theorem 1 lies in the classical result of Mihailova that $G = F_2 \times F_2$ has an undecidable group membership problem [Mih]. In fact, we conjecture that the problem whether $\{[1] u^n\}$ is P-recursive is undecidable. We refer to [Hal] for an extensive survey of decidable and undecidable matrix problems.

6.3. Following the approach of the previous section, Theorem 9 can be extended to all polycyclic groups of exponential growth and solvable groups of finite Prüfer rank [PS4]. It also applies to various other specific groups for which mildly exponential bounds on $p(n)$ are known, such as the Baumslag–Solitar groups $BS_q \subset GL(2, \mathbb{Q})$, $q \geq 2$, and the lamplighter groups $L_d = \mathbb{Z}_d \mathbb{Z}$, $d \geq 1$, see e.g. [Woe §15]. Let us emphasize that P-recursiveness fails for all symmetric generating sets in these cases. In view of Theorem 8 the P-recursiveness fails for some generating sets of non-amenable groups containing $F_2 \times F_2$. This suggests that P-recursiveness of all generating sets is a rigid property which holds for very few classes of group. We conjecture that it holds for all nilpotent groups.
6.4. Lemma 2 can be rephrased to say that the subword complexity function $c_w(n) < 2^n$ for some $n$ large enough (see e.g. [AS, BLRS]). This is likely to be far from optimal. For example, for the Catalan numbers $C_n = \frac{1}{n+1} {2n \choose n}$, we have $w = 10100010000001 \ldots$ In this case, it is easy to see that the word complexity function $c_w(n) = \Theta(n)$, cf. [DS]. It would be interesting to find sharper upper bounds on the maximal growth of $c_w(n)$, when $w$ is the infinite parity word of a P-recursive sequence. Note that $c_w(n) = \Theta(n)$ for all automatic sequences [AS, §10.2], and that the exponentially growing P-recursive sequences modulo almost all primes are automatic provided deep conjectures of Bombieri and Dwork, see [Chr].

6.5. The integrality assumption in Theorem 7 cannot be removed as the following example shows. Denote by $a_n$ the number of fragmented permutations, defined as partitions of \{1, \ldots, n\} into ordered lists of numbers (see sequence A000262 in [OEIS]). It is P-recursive since

$$a_n = (2n-1)a_{n-1} - (n-1)(n-2)a_{n-2} \quad \text{for all} \quad n > 2.$$ 

The asymptotics is given in [FS, Prop. VIII.4]:

$$\frac{a_n}{n!} \sim \frac{1}{2 \sqrt{\pi} n^{3/2}} e^{2 \sqrt{n}} n^{-3/4}.$$

This implies that the theorem is false for the rational, at most exponential P-recursive sequence \{a_n/n!\}, since in this case we have mildly exponential terms. To understand this, note that $\sum_n a_n t^n/n!$ is not a $G$-function since the lcm of denominators of $a_n/n!$ grow superexponentially.

6.6. Proving that a combinatorial sequence is not P-recursive is often difficult even in the most classical cases. We refer to [B+, BRS, BP, FGS, Kla, MR] for various analytic arguments. As far as we know, this is the first proof by a computability argument.

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