Classical mechanics of many particles defined on canonically deformed nonrelativistic space-time

Marcin Daszkiewicz$^1$, Cezary J. Walczyk$^2$

$^1$Institute of Theoretical Physics
University of Wroclaw pl. Maxa Borna 9, 50 – 206 Wroclaw, Poland
e-mail: marcin@ift.uni.wroc.pl

$^2$Department of Physics
University of Bialystok, ul. Lipowa 41, 15 – 424 Bialystok, Poland
e-mail: c.walczyk@alpha.uwb.edu.pl

Abstract

We provide the classical mechanics of many particles moving in canonically twist-deformed space-time. In particular, we consider two examples of such noncommutative systems - the set of $N$ particles moving in gravitational field as well as the system of $N$ interacting harmonic oscillators.
1 Introduction

The idea to use noncommutative coordinates is quite old - it goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, however, there were found new formal arguments based mainly on Quantum Gravity [2], [3] and String Theory models [4], [5], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. On the other side, the main reason for such considerations follows from many phenomenological considerations, which state that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincare invariance still remains valid at larger distances [6]-[9].

It is well-known that a proper modification of the Poincare and Galilei Hopf algebras can be realized in the framework of Quantum Groups [10], [11]. Hence, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [12], [13]), one can distinguish two simplest quantum spaces. First of them corresponds to the well-known canonical type of noncommutativity

\[ [x_\mu, x_\nu] = i\theta_{\mu\nu}, \]  

with antisymmetric constant tensor \( \theta_{\mu\nu} \). Its relativistic and nonrelativistic Hopf-algebraic realizations have been discovered with the use of twist procedure (see [14]) of classical Poincare [15], [16] and Galilei [17], [18] Hopf structures respectively.

The second class of deformations introduces the Lie-algebraic type of space-time noncommutativity

\[ [x_\mu, x_\nu] = i\theta_{\mu\nu}^\rho x_\rho, \]  

with particularly chosen constant coefficients \( \theta_{\mu\nu}^\rho \). The examples of corresponding Poincare quantum algebras have been introduced in [19], [20], while the suitable Galilei algebras - in [21], [17] and [18].

Recently, there appeared a lot of papers dealing with classical ([22]-[28]) and quantum ([29]-[33]) mechanics, Doubly Special Relativity frameworks ([34], [35]), statistical physics ([36], [37]) and field theoretical models (see e.g. [38]), defined on quantum space-times (1), (2). Particularly, there was investigated the impact of the mentioned above deformations on dynamics of basic classical and quantum systems. Consequently, in papers [21], [25], the authors considered classical particle moving in central gravitational field defined on canonically deformed space-time (1). They demonstrated, that in such a case there is generated Coriolis force acting additionally on the moving particle. Besides, in articles [32], [24] and [33] there was analyzed classical and quantum oscillator model formulated on canonically and Lie-algebraically deformed space-time respectively. Particulary, there has been found its deformed energy spectrum as well as the corresponding equation of motion. Interesting results have been also obtained in two papers [29], [30] concerning the hydrogen atom model defined on spaces (1) and (2). Besides, it should be noted that there appeared article [27], which provides the link between Pioneer anomaly phenomena [41] and classical mechanics defined on \( \kappa \)-Galilei quantum space. Preciously, there has been

\footnote{For earlier studies see [39] and [40].}
demonstrated that additional force term acting on moving satellite can be identified with the force generated by space-time noncommutativity. The value of deformation parameter \( \kappa \) has been fixed by comparison of obtained theoretical results with observational data.

Unfortunately, in all mentioned above articles there were analyzed only the one-particle relativistic and nonrelativistic dynamics in the field of forces. Here, we extend a such kind of investigations to the classical mechanics of many particles, which move in the modified canonically deformed space-time\(^2\)

\[
[t, x^i_a] = 0, \quad [x^i_a, x^j_b] = i \theta^{ij}; \quad i, j = 1, 2, 3,
\]

with indices \( a, b = 1, 2, \ldots, N \) labeling the particle. Further, we indicate that as in the case of one-particle system there appeared additional force terms generated by space-time noncommutativity. Of course, for \( N = 1 \) our results become the same as the results obtained in \([24],[32]\).

The motivations for present studies are manyfold. First of all we extend in natural way the results for one-particle model to the many-particle system. Secondly, such investigations permit to analyze the deformations of wide class of physical models, for example, one can applied the presented results to the studies on two deformed systems considered in Sect. 4. Finally, it gives a starting point for the construction of nonrelativistic quantum mechanics for many particles defined on modified space-time \([3]\).

The paper is organized as follows. In Sect. 2 we recall basic facts concerning the twisted canonically deformed Galilei Hopf algebra \( U_\theta(G) \) associated with space-time noncommutativity \([14]\) for \( \theta_{0i} = 0 \) and \( \theta_{ij} \neq 0 \). In Sect. 3 we provide the classical many-particle model defined on modified canonical space-time \([3]\). Section four includes two prominent\(^4\) examples of such deformed systems - the model of \( N \) particles moving in central gravitational field as well as the system of \( N \) coupling harmonic oscillators. The final remarks are presented in the last section.

2 Twisted Galilei Hopf algebra and corresponding canonically deformed space-time

In accordance with Drinfeld twist procedure \([14],[10],[11]\), the algebraic sector of arbitrary twisted Hopf algebra \( U(A) \) remains undeformed, while the coproducts and antipodes transform as follows

\[
\Delta_0(a) \longrightarrow \Delta(a) = F \circ \Delta_0(a) \circ F^{-1}, \quad S(a) = u S_0(a) u^{-1},
\]

\(^2x_0 = ct.\)

\(^3\)It should be noted that a such modification of relation \([11]\) (blind in \( a, b \) indices) is in accordance with the formal arguments proposed in \([32]\). Preciously, the relations \([3]\) are constructed with adopt so-called braided tensor algebra procedure, dictated by structure of quantum \( R \)-matrix for canonical deformation \([10],[11]\). Such a choice is compatible with Leibnitz rules for quantum algebra given by deformed coproduct \([9]\)-\([11]\).

\(^4\)Their classical (undeformed) versions are often discussed in the literature, see e.g. \([43]\).
with \(\Delta_0(a) = a \otimes 1 + 1 \otimes a\), \(S_0(a) = -a\) and \(u = \sum f_{(1)} S_0(f_{(2)})\) (we use Sweedler’s notation \(\mathcal{F} = \sum f_{(1)} \otimes f_{(2)}\)). Besides, present in the above formula twist factor \(\mathcal{F} \in \mathcal{U}(A) \otimes \mathcal{U}(A)\) satisfies the classical cocycle condition

\[
\mathcal{F}_{12} \cdot (\Delta_0 \otimes 1) \mathcal{F} = \mathcal{F}_{23} \cdot (1 \otimes \Delta_0) \mathcal{F},
\]

and the normalization condition

\[
(\epsilon \otimes 1) \mathcal{F} = (1 \otimes \epsilon) \mathcal{F} = 1,
\]

with \(\mathcal{F}_{12} = \mathcal{F} \otimes 1\) and \(\mathcal{F}_{23} = 1 \otimes \mathcal{F}\).

Consequently, in the case of the canonically deformed Galilei Hopf algebra \(\mathcal{U}_\theta(\mathcal{G})\) provided in [17], we have

\[
\mathcal{F}_\theta = \exp \left(-\frac{i}{4} \theta^{ij} \Pi_i \wedge \Pi_j\right),
\]

and, in accordance with (4), we get the following algebraic sectors.

\[
[K_{ij}, K_{kl}] = i (\delta_{il} K_{jk} - \delta_{jl} K_{ik} + \delta_{jk} K_{il} - \delta_{ik} K_{jl}),
\]

\[
[K_{ij}, V_k] = i (\delta_{jk} V_i - \delta_{ik} V_j), \quad [K_{ij}, \Pi_k] = i (\delta_{jk} \Pi_i - \delta_{ik} \Pi_j),
\]

\[
[K_{ij}, \Pi_0] = [V_i, V_j] = [V_i, \Pi_j] = 0, \quad [V_i, \Pi_0] = -i \Pi_i, \quad [\Pi_\mu, \Pi_\nu] = 0,
\]

and the coalgebraic

\[
\Delta_\theta(\Pi_\rho) = \Delta_0(\Pi_\rho), \quad \Delta_\theta(V_i) = \Delta_0(V_i),
\]

\[
\Delta_\theta(K_{ij}) = \Delta_0(K_{ij}) - \theta^{kl} [(\delta_{ki} \Pi_j - \delta_{kj} \Pi_i) \otimes \Pi_l + \Pi_k \otimes (\delta_{li} \Pi_j - \delta_{lj} \Pi_i)],
\]

sectors.

It is well-known (see e.g. [16]) that the deformed space-time corresponding to the Hopf algebra \(\mathcal{U}_\theta(\mathcal{G})\) is defined as the quantum representation space (Hopf module), with action of the deformed symmetry generators satisfying suitably deformed Leibnitz rules. The action of Galilei group \(\mathcal{U}_\theta(\mathcal{G})\) on a Hopf module of functions depending on space-time coordinates \((t, x_i)\) is given by

\[
\Pi_0 \triangleright f(t, \overline{x}) = i \partial_t f(t, \overline{x}), \quad \Pi_i \triangleright f(t, \overline{x}) = i \partial_i f(t, \overline{x}),
\]

\[
K_{ij} \triangleright f(t, \overline{x}) = i (x_i \partial_j - x_j \partial_i) f(t, \overline{x}), \quad V_i \triangleright f(t, \overline{x}) = it \partial_i f(t, \overline{x}),
\]

while the \(\ast\)-multiplication of arbitrary two functions is defined as follows

\[
f(t, \overline{x}) \ast_\theta g(t, \overline{x}) := \omega \circ (\mathcal{F}_\theta^{-1} \triangleright f(t, \overline{x}) \otimes g(t, \overline{x})) \quad ; \quad \omega \circ (a \otimes b) = a \cdot b.
\]

The symbols \(K_{ij}, V_i\) and \(\Pi_\mu\) denote rotations, boosts and space-time translation generators respectively.
Consequently, in the representation (12), (13) the twist factor (7), and the corresponding nonrelativistic space-time take respectively the forms
\[ F_\theta = \exp \left( \frac{i}{4} \theta^{ij} \partial_i \wedge \partial_j \right), \] (15)
and
\[ [t, x_i]_{\star_\theta} = 0, \quad [x_i, x_j]_{\star_\theta} = i\theta^{ij}, \] (16)
with \([a, b]_{\star_\theta} := a \star_\theta b - b \star_\theta a\). Obviously, for deformation parameter \(\theta^{ij}\) approaching zero the above quantum space becomes the classical one.

3 Classical mechanics of many particles moving in canonically deformed space-time

In this section we provide the classical model of \(N\) nonrelativistic particles moving in canonically deformed space-time (3), which for \(N = 1\) reproduces commutation relations (16). As it was mentioned in Introduction, similar constructions have been performed in the case of one-particle system in the series of papers [24], [25] and [32].

In a first step of our investigation we start with the following phase space\(^6\)
\[ \{ x^i_a, x^j_b \} = \theta^{ij}, \] (17)
\[ \{ p^i_a, p^j_b \} = 0, \quad \{ x^i_a, p^j_b \} = \delta^{ij}\delta_{ab}. \] (18)
which satisfies the Jacobi identity and for \(N = 1\) becomes the same as phase space provided in [24].

Next, following [43], for arbitrary two functions \(F(\zeta^A)\) and \(G(\zeta^A)\) we define Poisson bracket as follows
\[ \{ F, G \} = \sum_{A,B=1}^{6N} \{ \zeta^A, \zeta^B \} \frac{\partial F}{\partial \zeta^A} \frac{\partial G}{\partial \zeta^B}, \] (19)
with \(\zeta^A = (x^i_a, p^i_a)\).

In terms of the above structure and given Hamiltonian \(H = H(\zeta^A)\) one can write the equations of motion as
\[ \dot{\zeta}^A = \{ \zeta^A, H \} ; \quad \dot{\zeta}^A := \frac{d\zeta^A}{dt}. \] (20)
Moreover, in general case, i.e. for any function \(F\) depending on \(\zeta^A\), we have
\[ \dot{F} = \{ F, H \}. \] (21)

Let us now introduce the standard Hamiltonian function describing the set of \(N\) particles
\[ H(\vec{p}_1, \ldots, \vec{p}_N, \vec{r}_1, \ldots, \vec{r}_N) = \sum_{a=1}^{N} \frac{\vec{p}_a^2}{2m_a} + V(\vec{r}_1, \ldots, \vec{r}_N), \] (22)
\(^6\text{We use the correspondence relation } \{ a, b \} = \frac{1}{\hbar} [\hat{a}, \hat{b}] (\hbar = 1).\)
with \( \vec{p}_a = [p_{a1}^1, p_{a1}^2, p_{a1}^3] \) and \( \vec{r}_a = [x_1^a, x_2^a, x_3^a] \). Then, in accordance with (20), we get the following 2N equations of motion

\[
\dot{x}_i^a = \frac{p_i^a}{m_a} + \sum_{b=1}^{N} \sum_{j=1}^{3} \theta^{ij} \frac{\partial V}{\partial x_b^j}, \quad \ddot{p}_i^a = -\frac{\partial V}{\partial x_i^a},
\]

which lead to N Newton equations

\[
m_a \ddot{x}_i^a = -\frac{\partial V}{\partial x_i^a} + m_a \sum_{b,c=1}^{N} \sum_{j,k=1}^{3} \theta^{ij} \frac{\partial^2 V}{\partial x_b^j \partial x_c^k} \dot{x}_c^k.
\]

It should be noted that in the case of one-particle system (\( N = 1 \)) the above equation becomes the same as the one derived in [24].

Further, we consider the multiparticle potential function given by

\[
V(\vec{r}_1, \ldots, \vec{r}_N) = \sum_{a=1}^{N} V_{a}^{\text{out}}(|\vec{r}_a|) + \frac{1}{2} \sum_{a=1}^{N} \sum_{a \neq b} V_{ab}^{\text{inn}}(|\vec{r}_a - \vec{r}_b|),
\]

where in the above formula symbol \( V_{a}^{\text{out}} \) denotes the ”outer” potential acting on \( a \)-th particle, and \( V_{ab}^{\text{inn}} = V_{ba}^{\text{inn}} \) corresponds to the so-called ”inner” potential describing interactions between particles \( a \) and \( b \). Besides, we have

\[
r_a = |\vec{r}_a| = \sqrt{\sum_{i=1}^{3} x_i^a \cdot x_i^a}, \quad r_{ab} = |\vec{r}_a - \vec{r}_b|.
\]

One can check, that in the case of hamiltonian function (22) the equations of motion (23) look as follows

\[
\dot{x}_i^a = \frac{p_i^a}{m_a} + \sum_{b=1}^{N} \sum_{j=1}^{3} \theta^{ij} \frac{1}{r_b} \frac{\partial V_{b}^{\text{out}}}{\partial r_b} x_b^j, \quad \dot{p}_i^a = -\frac{1}{r_a} \frac{\partial V_{a}^{\text{out}}}{\partial x_a^i} x_a^i - \sum_{b \neq a} \frac{1}{r_{ab}} \frac{\partial V_{ab}^{\text{inn}}}{\partial r_{ab}} (x_a^i - x_b^i),
\]

while the Newton equations (24) take the form

\[
m_a \ddot{x}_i^a = -\frac{x_i^a}{r_a} \frac{\partial V_{a}^{\text{out}}}{\partial r_a} + \sum_{b \neq a} \frac{(x_i^a - x_b^i)}{r_{ab}} \frac{\partial V_{ab}^{\text{inn}}}{\partial r_{ab}} + \sum_{b=1}^{N} \sum_{j,k=1}^{3} \epsilon^{ijk} \left( \dot{\Omega}_{k}^{b}(x, \dot{x}) \dot{x}_b^j + \Omega_{k}^{b}(x) \ddot{x}_b^j \right),
\]

with

\[
\theta^{ij} = \sum_{k=1}^{3} \epsilon^{ijk} \theta_k,
\]
\[\Omega^a_i(x) = \frac{1}{r_a} \frac{\partial V^\text{out}_a}{\partial r_a} \theta_i,\]  
(30)

\[\dot{\Omega}^a_i(x, \dot{x}) = \frac{\theta_i}{r_a^2} \left( \frac{\partial^2 V^\text{out}_a}{\partial r_a^2} - \frac{1}{r_a} \frac{\partial V^\text{out}_a}{\partial r_a} \right) \vec{x}_a \cdot \dot{x}_a.\]  

Finally, it should be noted that the following position and momentum dependent function

\[L_\theta(x, p) = \sum_{a=1}^N \sum_{i,j=1}^3 \theta^{ij} x^i_a p^j_a + \frac{1}{2} \sum_{i,j,k=1}^3 \theta^{ij} P^j \theta^{ik} P^k; \quad P^j = \sum_{a=1}^N p^j_a,\]  
(31)

plays the role of constant of motion, i.e. it satisfies the equation

\[\dot{L}_\theta = \{ L_\theta, H \} = 0,\]  
(32)

and it transforms the phase space variables as follows\footnote{One can notice, that for \(\theta^{ij} = \sum_{k=1}^3 \epsilon^{ijk} \theta_k\) the function \(L_\theta\) generates rotations in \(\theta_i\)-directions.}

\[\{ x^i_a, L_\theta \} = \sum_{j=1}^3 \theta^{ij} x^j_a, \quad \{ p^i_a, L_\theta \} = - \sum_{j=1}^3 \theta^{ij} p^j_a.\]  
(33)

Besides, one should observe that the standard total momentum

\[\vec{P} = \sum_{a=1}^N \vec{p}_a,\]  
(34)

as well as the standard total angular momentum

\[\vec{L} = \sum_{a=1}^N \vec{L}_a; \quad \vec{L}_a = \sum_{a=1}^N \vec{r}_a \times \vec{p}_a,\]  
(35)

are not conserved in time due to the presence of ”outer” potential \(V^\text{out}_a\), i.e.

\[\dot{\vec{P}} = \{ \vec{P}, H \} = - \sum_{a=1}^N \frac{\vec{x}_a}{r_a} \frac{\partial V^\text{out}_a}{\partial r_a},\]  
(36)

and

\[\dot{\vec{L}} = \{ \vec{L}, H \} = \sum_{a,b=1}^N \frac{1}{r_b} \frac{\partial V^\text{out}_b}{\partial r_b} \vec{p}_a \times \left( \vec{\theta} \times \vec{r}_b \right),\]  
(37)

respectively, with \(\vec{\theta} = [\theta_1, \theta_2, \theta_3]\).

Obviously, for deformation parameter \(\theta^{ij}(\theta^k)\) approaching zero the above model becomes undeformed, i.e. we recover the ordinary Newton mechanics for set of \(N\) particles \[43]\.
4 Two examples of canonically deformed many-particle models

In this section we discuss two selected examples (see e.g. [43]) of many-particle systems defined on noncommutative space-time (3), namely the system of $N$ particles moving in gravitational field, and the set of $N$ interacting harmonic oscillators.

4.1 The system of $N$ particles moving in gravitational field

Let us consider the system of $N$ particles moving in the presence of mass $M$ located in the origin of the coordinate system. Then, the "outer" potential takes the form

$$ V_{\text{out}}^a(|\vec{r}_a|) = -\frac{GMm_a}{|\vec{r}_a|} , $$

while the "inner" potentials look as follows

$$ V_{\text{inn}}^{ab}(|\vec{r}_a - \vec{r}_b|) = -\frac{Gm_a m_b}{|\vec{r}_a - \vec{r}_b|} . $$

One can check that for potential functions (38) and (39) we get the following canonical equation of motion

$$ \dot{x}_i^a = \frac{p_i^a}{m_a} + \sum_{b=1}^{N} \sum_{j=1}^{3} \theta^{ij} \frac{GMm_b}{r^3_b} x_j^b , $$

$$ \dot{p}_i^a = -\frac{GMm_a}{r^3_a} x_i^a - \sum_{b \neq a} Gm_a m_b \left( x_i^a - x_i^b \right) \frac{1}{r_{ab}^3} , $$

leading to the set of Newton equations

$$ m_a \ddot{x}_i^a = -Gm_a \left( \frac{x_i^a M}{r_a^2} + \sum_{b \neq a} \frac{(x_i^a - x_i^b) m_b}{r_{ab}^2} \right) + $$

$$ -m_a \sum_{b=1}^{N} \sum_{j,k=1}^{3} \epsilon^{ijk} \left( \hat{\Omega}_k^b(x, \dot{x}) x_j^b + \Omega_k^b(x) \dot{x}_j^b \right) , $$

with

$$ \Omega_k^b(x) = G \frac{M m_b}{r_b^4} \theta_k , $$

and

$$ \dot{\hat{\Omega}}_k^b(x, \dot{x}) = -3G \theta_k \frac{M m_b}{r_b^5} \ddot{x}_b \dot{x}_b . $$
Obviously, for mass $M$ equal zero, i.e. for vanishing external potential $V^\text{out}_a$, the equation of motion (42) takes the standard form

$$m_a \ddot{x}_a^i = - \sum_{b \neq a} \frac{(x_a^i - x_b^i) G m_a m_b}{r_{ab}^2}.$$  \hspace{1cm} (45)

Besides, one can observe that for $N = 1$ the above system reproduces the well-known model for single particle moving in gravitational field [43].

In order to analyze the above system in terms of commutative variables $(t, x_{\text{com}})$, by analogy to the star multiplication (14), one should replace the product of two arbitrary functions defined on quantum space (3), by the following $\hat{\star}_\theta$-product (see [23])

$$f(t, \mathbf{x}) \cdot g(t, \mathbf{x}) \longrightarrow f(t, \mathbf{x}_{\text{com}}) \hat{\star}_\theta g(t, \mathbf{x}_{\text{com}}) := \omega \circ (\mathcal{O}_\theta \triangleright f(t, \mathbf{x}) \otimes g(t, \mathbf{x})) ,$$  \hspace{1cm} (46)

with

$$\mathcal{O}_\theta = \exp \left( \frac{1}{4} \sum_{a,b=1}^{N} \sum_{i,j=1}^{3} \theta_{ij} \partial^i_{a,\text{com}} \wedge \partial^j_{b,\text{com}} \right).$$  \hspace{1cm} (47)

Then, we get

$$m_a \ddot{x}_{a,\text{com}}^i = -G m_a \left( M x_a^i \hat{\star}_\theta \left( \frac{1}{r_{a,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{a,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{a,\text{com}}} \right) + + \sum_{b \neq a} m_b \left( x_{a,\text{com}}^i - x_{b,\text{com}}^i \right) \hat{\star}_\theta \left( \frac{1}{r_{ab,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{ab,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{ab,\text{com}}} \right) + + m_a \sum_{b=1}^{N} \sum_{j,k=1}^{3} \varepsilon^{ikj} \left( \hat{\Omega}_{k}^b (x_{\text{com}}, \dot{x}_{\text{com}}) \hat{\star}_\theta x_{b,\text{com}}^j + \hat{\Omega}_{k}^b (x_{\text{com}}) \hat{\star}_\theta \dot{x}_{b,\text{com}}^j \right) ,$$  \hspace{1cm} (48)

where

$$\hat{\Omega}_{k}^b (x_{\text{com}}) = G M m_b \theta_k \left( \frac{1}{r_{b,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{b,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{b,\text{com}}} \right) ,$$  \hspace{1cm} (49)

$$\hat{\Omega}_{k}^b (x_{\text{com}}, \dot{x}_{\text{com}}) = -3G \theta_k M m_b \left( \frac{1}{r_{b,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{b,\text{com}}} \right) \hat{\star}_\theta \left( \frac{1}{r_{b,\text{com}}} \right) \hat{\star}_\theta \dot{x}_{b,\text{com}} \hat{\star}_\theta \dot{x}_{b,\text{com}} ,$$  \hspace{1cm} (50)

and

$$r_{a,\text{com}} = \sqrt{\sum_{i=1}^{3} x_{a,\text{com}}^i \hat{\star}_\theta x_{a,\text{com}}^i} = \sqrt{\sum_{i=1}^{3} x_{a,\text{com}}^i \cdot x_{a,\text{com}}^i} .$$  \hspace{1cm} (51)

\footnote{One can derive the commutation relations (3) using definition (16) with inserted $\hat{\star}_\theta$-product instead $\star_\theta$-multiplication (see also footnote 6).}
Further, one can check that linear in deformation parameter $\theta^k$ corrections, which appear in the equation (48), look as follows

$$m_a \dddot{x}_a = -Gm_a \left( \frac{M x_a \theta}{r_a^3} \right) - \frac{3}{2} \sum_{j=1}^{3} \theta_{ij} \frac{M x_j}{r_a^3} + \sum_{b \not= a} \frac{m_b (x_a - x_b \theta)}{r_{ab}^3} \right) +$$

$$- m_a \sum_{b=1}^{N} \sum_{j,k=1}^{3} \epsilon_{ijk} \left( \ddot{\Omega}_b^k(x) \right) x_{b}^j,$$

with

$$\ddot{\Omega}_b^k(x) = \theta_k \frac{G M m_b}{r_{bc}^3},$$

and

$$\ddot{\Omega}_k^b(x) = -3 \theta_k \frac{G M m_b}{r_{bc}^3} \ddot{x}_b \cdot \dot{x}_b.$$

Unfortunately, due to the complicated form of the above Newton equation, its solution can be studied by using only numerical methods. Such an investigation will be omitted in present article.

### 4.2 The system of $N$ coupled harmonic oscillators

Let us now turn to the second example of multiparticle system - the model of $N$ coupled harmonic oscillators. In such a case the "outer" and "inner" potentials take the form

$$V_a^{\text{ext}}(|r_a|) = \frac{m_a \omega_a^2}{2} |r_a|^2,$$

and

$$V_{ab}^{\text{inn}}(|r_a - r_b|) = \frac{\lambda_{ab}^2}{2} |r_a - r_b|^2,$$

with $\omega_a$ and $\lambda_{ab}$ denoting the frequency and coupling constant respectively. The corresponding Newton equation looks as follows

$$m_a \dddot{x}_a = - \left( m_a \omega_a^2 + \sum_{b \not= a} \lambda_{ab}^2 \right) x_a^i + \sum_{b \not= a} \lambda_{ab}^2 x_b^i - m_a \sum_{b=1}^{N} \sum_{j,k=1}^{3} \epsilon_{ijk} \ddot{\Omega}_b^k x_b^j,$$

where

$$\dot{\Omega}_a^b = m_a \omega_a^2 \theta_i = \text{const.} \quad (\dot{\Omega}_a^a = 0).$$

Of course, in the case of single oscillator model ($N = 1, \lambda_{ab} = 0$) one recovers the equation of motion proposed in [24].
As an illustration let us consider the case of \( m_a = m, \ \omega_a = \omega \) and \( \lambda_{ab} = \lambda \) for \( a, b = 1, 2, \ldots N \). Then, it follows from (58) that the corresponding equation of motion takes the form

\[
\ddot{x}_a^i = - \left( \frac{\omega^2}{m} + \frac{\lambda^2}{m} \sum_{b \neq a}^N \right) x_a^i + \frac{\lambda^2}{m} \sum_{b \neq a}^N \ddot{x}_b^i - m \omega^2 \sum_{b = 1}^N \sum_{j, k = 1}^3 \epsilon^{ijk} \theta_k \dot{x}_b^j, \tag{60}
\]

and, due to linearity of the above formula with respect \( x \)-variables, its form remains the same on commutative space-time as (60).

The above equations can be solved in few steps. Firstly, we introduce the following "relative" position variables

\[
\mathbf{\hat{x}}_{com} = \mathbf{x}_{a+1} - \mathbf{x}_a \quad ; \quad a = 1, 2, \ldots, N - 1,
\]

and then, we have

\[
x_{n, com}^i = \sum_{a=1}^{n-1} \hat{x}_{a, com}^i + x_{1, com}^i, \tag{62}
\]

for \( 2 \leq n \leq N \).

Next, we rewrite the Newton equation (60) in terms of variables (61) as follows

\[
\ddot{x}_{com}^1 = -\omega^2 \mathbf{x}_{1, com} + \omega M T \mathbf{\dot{x}}_{com} + \mathbf{h}, \tag{63}
\]

\[
\ddot{x}_{com}^a = - \left( \frac{\omega^2}{m} + \frac{N \lambda^2}{m} \right) \mathbf{x}_{com}^a, \tag{64}
\]

where

\[
M = N m, \quad T^{ij} = \omega \sum_{k=1}^3 \theta_k e^{kij}, \tag{65}
\]

\[
h_i = \frac{1}{m} \sum_{b=2}^N \sum_{a=1}^{b-1} \left( \lambda^2 \hat{x}_{a, com}^i + m^2 \omega \sum_{j=1}^3 T^{ij} \hat{x}_j^a \right), \tag{66}
\]

The solution of equation (64) can be found easily, and it takes the form

\[
\hat{x}_{com}^i(t) = \hat{x}_{a, com}^i(0) \cos(\Omega t) + \frac{\dot{\hat{x}}_{com}^i(0)}{\Omega} \sin(\Omega t) \quad ; \quad \Omega = \sqrt{\omega^2 + \frac{N \lambda^2}{m}}, \tag{67}
\]

\[
h_i(t) = h_s^i \sin(\Omega t) + h_c^i \cos(\Omega t), \tag{68}
\]

with

\[
h_s^i = \frac{1}{\Omega m} \sum_{b=2}^N \sum_{a=1}^{b-1} \left( \lambda^2 \hat{x}_{a, com}^i(0) - m^2 \omega \Omega \sum_{j=1}^3 T^{ij} \hat{x}_j^a \right), \tag{69}
\]

\[
h_c^i = \frac{1}{m} \sum_{b=2}^N \sum_{a=1}^{b-1} \left( \lambda^2 \hat{x}_{a, com}^i(0) + m^2 \omega \sum_{j=1}^3 T^{ij} \hat{x}_j^a \right), \tag{70}
\]
and with symbols $\hat{x}_a^{i\text{com}}(0), \hat{v}_a^{i\text{com}}(0)$ denoting the initial "relative" positions and velocities respectively.

Further, using the formula (68) we get the following explicite solution of the equation (63)

\[
x_1^{\text{com}}(t) = \frac{1}{\omega \sqrt{4 + M^2 \omega^2 \theta^2}} \left( (v_1^{\text{com}}(0) + \Omega_3 x_1^{\text{com}}(0)) \sin(\Omega_2 t) \right.
\]
\[
+ (v_1^{\text{com}}(0) - \Omega_3 x_1^{\text{com}}(0)) \cos(\Omega_2 t) + \frac{1}{(\Omega_2^2 - \Omega^2) \sqrt{4 + M^2 \omega^2 \theta^2 \omega}} \left( (h^2 \Omega - h^4 \Omega_3) (\cos(\Omega_3 t) - \cos(\Omega t)) 
\right.
\]
\[
- (h^2 \Omega_2 + h^4 \Omega) \sin(\Omega_2 t) + (h^4 \Omega_2 - h^2 \Omega) \sin(\Omega t) \right) 
\]
\[
+ \frac{1}{(\Omega_3^2 - \Omega^2) \sqrt{4 + M^2 \omega^2 \theta^2 \Omega}} \left( (h^4 \Omega - h^2 \Omega_3) (\cos(\Omega_3 t) - \cos(\Omega t)) 
\right. 
\]
\[
- (h^4 \Omega_3 + h^2 \Omega) \sin(\Omega_3 t) + (h^2 \Omega_3 + h^4 \Omega) \sin(\Omega t) \right),
\]
\[
x_2^{\text{com}}(t) = \frac{1}{\omega \sqrt{4 + M^2 \omega^2 \theta^2}} \left( (v_1^{\text{com}}(0) + \Omega_3 x_2^{\text{com}}(0)) \cos(\Omega_2 t) 
\right.
\]
\[
+ (v_1^{\text{com}}(0) - \Omega_3 x_2^{\text{com}}(0)) \sin(\Omega_2 t) + \frac{1}{(\Omega_2^2 - \Omega^2) \sqrt{4 + M^2 \omega^2 \theta^2 \omega}} \left( (h^4 \Omega - h^2 \Omega_2) (\cos(\Omega_2 t) - \cos(\Omega t)) 
\right.
\]
\[
+ (h^4 \Omega_2 - h^2 \Omega) \sin(\Omega_2 t) + (h^2 \Omega_2 - h^4 \Omega) \sin(\Omega t) \right),
\]
\[
x_3^{\text{com}}(t) = \frac{1}{\omega} v_1^{\text{com}}(0) \sin(\omega t) + x_3^{\text{com}}(0) \cos(\omega t) + \frac{1}{\omega (\omega^2 - \Omega^2)} \left( h^3 \omega (\cos(\omega t) - \cos(\Omega t)) + h^3 \omega (\sin(\omega t) - \omega \sin(\Omega t)) \right),
\]

with the assumption $\tilde{\theta} = [0, 0, \theta]$, symbols $x_1^{\text{com}}(0)$ and $v_1^{\text{com}}(0)$ denoting the initial data for first particle, and $\Omega_2, \Omega_3$ given by

\[
\Omega_2 = \frac{\omega}{2} \left( \omega M \theta + \sqrt{4 + \omega^2 M^2 \theta^2} \right), \quad \Omega_3 = \frac{\omega}{2} \left( \sqrt{4 + \omega^2 M^2 \theta^2} - \omega M \theta \right).
\]

The remaining trajectories $\vec{x}_{\text{com}}^2(t), \ldots, \vec{x}_{\text{com}}^N(t)$ can be easily obtained by use of the formulas (62).
5 Final remarks

In this article we construct the classical model of \( N \) nonrelativistic particles moving in noncommutative space-time (3). The corresponding equation of motion for arbitrary spherically symmetric potential (25) are provided. In particular, there are analyzed two distinguished examples of such systems - the set of \( N \) coupled oscillators as well as the system of \( N \) particles moving in the presence of gravitational field provided by massive point-like source.

It should be noted that the presented considerations can be extended at least in two directions. First of all, one can consider the multiparticle system associated with the Lie-algebraically deformed space-time (2). Secondly, one can quantize the analyzed above (classical) model by introducing the Schrödinger equation defined on deformed space-time (3). The studies in these directions already started and are in progress.

Acknowledgments

The authors would like to thank J. Lukierski for valuable discussions.

This paper has been financially supported by Ministry of Science and Higher Education grant NN202318534.

References

[1] H.S. Snyder, Phys. Rev. 72, 68 (1947)
[2] S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. B 331, 39 (1994)
[3] A. Kempf and G. Mangano, Phys. Rev. D 55, 7909 (1997); hep-th/9612084
[4] A. Connes, M.R. Douglas, A. Schwarz, JHEP 9802, 003 (1998)
[5] N. Seiberg and E. Witten, JHEP 9909, 032 (1999); hep-th/9908142
[6] S. Coleman, S.L. Glashow, Phys. Rev. D 59, 116008 (1999)
[7] R.J. Protheore, H. Meyer, Phys. Lett. B 493, 1 (2000)
[8] F.W. Stecker, S.L. Glashow, Astroparticle Phys. 16, 97 (2001)
[9] G. Amelino-Camelia, T. Piran, Phys. Lett. B 497, 265 (2001)
[10] V. Chari, A. Pressley, "A Guide to Quantum Groups", Cambridge University Press, Cambridge, 1994
[11] L.A. Takhtajan, "Introduction to Quantum Groups"; in Clausthal Proceedings, Quantum groups 3-28 (see High Energy Physics Index 29 (1991) No. 12256)
[12] S. Zakrzewski, "Poisson Structures on the Poincare group"; q-alg/9602001
[13] Y. Brihaye, E. Kowalczyk, P. Maslanka, "Poisson-Lie structure on Galilei group"; math/0006167
[14] V.G. Drinfeld, Soviet Math. Dokl. 32, 254 (1985); Algebra i Analiz (in Russian), 1, Fasc. 6, p. 114 (1989)
[15] R. Oeckl, Nucl. Phys. B 581, 559 (2000)
[16] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, Phys. Lett. B 604, 98 (2004)
[17] M. Daszkiewicz, Mod. Phys. Lett. A 23, 505 (2008); arXiv: 0801.1206 [hep-th]
[18] M. Daszkiewicz, Mod. Phys. Lett. A 23, 1757 (2008); arXiv: 0807.0133 [hep-th]
[19] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, J. Phys. A 27, 2389 (1994)
[20] J. Lukierski and M. Woronowicz, Phys. Lett. B 633, 116 (2006); hep-th/0508083
[21] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz, Phys. Lett. B 286, 57 (1992)
[22] A. Deriglazov, JHEP 0303, 021 (2003); hep-th/0211105
[23] A.E.F. Djemaï, "On noncommutative classical mechanics"; hep-th/0309034
[24] J.M. Romero and J.D. Vergara, Mod. Phys. Lett. A 18, 1673 (2003); hep-th/0303064
[25] J.M. Romero, J.A. Santiago, J.D. Vergara, Phys. Lett. A 310, 9 (2003); hep-th/0211165
[26] Y. Miao, X. Wang, S. Yu, "Classical mechanics on noncommutative space with Lie-algebraic structure"; arXiv: 0911.5227 [math-ph]
[27] E. Harikumar, A.K. Kapoor, "Newton equation on the kappa space-time and the Kepler problem"; arXiv: 1003.4603 [hep-th]
[28] M. Daszkiewicz, C.J. Walczyk, Phys. Rev. D 77, 105008 (2008); 0802.3575 [mat-ph], [hep-th]
[29] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001); hep-th/0010175
[30] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Euro Phys. J. C 36, 251 (2004); hep-th/0212259
[31] M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari, A. Tureanu, Nucl. Phys. B 11, 383 (2001)
[32] A. Kijanka, P. Kosinski, Phys. Rev. D 70, 127702 (2004); hep-th/0407246

[33] M. Daszkiewicz, C.J. Walczyk, Acta Phys. Pol. B 40, 293 (2009); arXiv: 0812.1264 [hep-th]

[34] G. Amelino-Camelia, Phys. Lett. B 510, 255 (2001)

[35] G. Amelino-Camelia, Mod. Phys. Lett. A 17, 899 (2002); gr-qc/0204051

[36] M. Maggiore, Phys. Rev. D 49, 5182 (1994); hep-th/9305163

[37] S.M. Rama, Phys. Lett. B 519, 103 (2001); hep-th/0107255

[38] R.J. Szabo, Phys. Rept. 378, 207 (2003); hep-th/0109162

[39] J. Lukierski, H. Ruegg, W.J. Zakrzewski, Ann. Phys. 243, 90 (1995)

[40] J. Lukierski, P. Stichel, W.J. Zakrzewski, Ann. Phys. 260, 224 (1997)

[41] J.D. Anderson, P.A. Laing, E.L. Lau, A.S. Lin, M.M. Nieto, S.G. Turyshhev, Phys. Rev. Lett. 81, 2858 (1998); gr-qc/9808081

[42] G. Fiore, J. Wess, Phys. Rev. D 75, 105022 (2007); hep-th/0701078

[43] J.E. Marsden and T.S. Ratiu, ”Introduction to mechanics and symmetry”, Springer-Verlag (1999)