GAUGE GRAVITY
AND CONSERVATION LAWS
IN HIGHER ORDER
ANISOTROPIC SPACES

Sergiu I. Vacaru

Institute of Applied Physics, Academy of Sciences,
5 Academy str., Chișinău 2028, Republic of Moldova
Fax: 011-3732-738149, E-mail: vacaru@lises.as.md

Abstract. We propose an approach to the theory of higher order anisotropic field interactions and curved spaces (in brief, ha–field, ha–interactions and ha–spaces). The concept of ha–space generalizes various types of Lagrange and Finsler spaces and higher dimension (Kaluza–Klein) spaces. This work consists from two parts. In the first we outline the theory of Yang–Mills ha–fields and two gauge models of higher order anisotropic gravity are analyzed. The second is devoted to the theory of nearly autoparallel maps (na–maps) of locally anisotropic spaces (la–spaces) and to the problem of formulation of conservation laws for la–field interactions. By defining invariants of na–map transforms we present a systematic classification of la–spaces.

PACS: 02.40.Vh; 02.90.+p; 04.20.Jb; 04.20.Fy; 04.50.+h; 04.50.+h; 04.62.+v; 04.90.+e; 11.10.Kk; 11.15.-q; 11.15.Kk; 11.30.Na; 12.10.-g

1991 Mathematics Subject Classification: 83E15 83C40 83D05 81T13 53B40 53B50

Keywords: Generalized Finsler and Kaluza–Klein spaces; higher order anisotropic field interactions


1 Introduction

Introductory remarks

This paper presents some applications of the higher order vector bundle formalism in physics: the theory of higher order anisotropic interactions of Yang Mills fields is formulated, there are developed two models of higher order anisotropic gauge gravity and found solutions of field equations describing anisotropic gravitational instantons and there are proposed two variants of definition the conservation laws for field interactions on locally anisotropic spaces.

This work as well fuses several areas of modern differential geometry, not all of which are familiar to theoretical and mathematical physicists. Here we cite our previous papers on nonlinear connections in vector (super)bundles [44], the Sinyukov’s theory of nearly geodesic maps [33] and the theory of...
nearly autoparallel maps (generalizing the class of conformal transforms) of Einstein–Cartan–Weyl spaces \[36, 37, 39\] and of generalized Lagrange and Finsler spaces \[56, 55, 43\]. We emphasize that we are inspired by the Yano–Ishihara \[61\] and Miron–Anastasiei–Atanasiu \[23, 24, 25\] geometric ideas and we consider that their constructions on modeling of locally anisotropic spaces on (in general higher order) tangent and vector bundles appears to be particularly promising in elaboration of some new divisions of quantum field theory and gravity.

We follow our program \[41, 51, 45, 46\] of formulation of self–consistent field theories incorporating various possible anisotropic, inhomogeneous and stochastic manifestations of classical and quantum interactions on locally anisotropic and higher order anisotropic spaces (respectively, in brief, la- and ha–spaces). To evolve this new type of physical theories we shall also use the geometric background and methods developed in monographs \[61, 22, 23, 24, 3, 10, 4, 5\] and papers \[25, 15\].

**Synopsis**

An approach to the theory of higher order anisotropic superspaces and superstrings and an analysis of low energy dynamics of supersymmetric nonlinear and anisotropic sigma models have been recently elaborated in our works \[46, 44, 45\]. This contribution is in a series of papers in which we more specifically focus on the (non supersymmetric) field theory of higher order anisotropic interactions. In our works \[49, 50\] we defined higher order anisotropic spinors on ha–spaces (in brief, ha–spinors) and presented a detailed study of the relationship between Clifford, spinor and nonlinear and distinguished connections structures on higher order extensions of vector bundles and tangent bundles and on prolongations of generalized Lagrange and Finsler spaces \[25\]. As a next step (in this paper) we consider the topics of higher order anisotropic gauge field interactions and of definition of conservation laws on locally anisotropic spaces.

By convention the contents of this paper can be divided into two parts: gauge ha–models and nearly autoparallel maps.

The aim of the first part is twofold. The first objective is to extend our results on locally anisotropic gauge theories \[51, 10, 12\] in order to consider Yang-Mills fields (with semisimple structural groups) on spaces with higher order anisotropy. The second objective is to propose a geometric formalism for gauge theories with nonsemisimple structural groups which permit a unique fiber bundle treatment for both higher order anisotropic Yang–Mills field and gravitational interactions. In general lines, we shall follow the ideas and methods proposed in Refs. \[34, 30, 31, 29, 12\], but we shall apply them in a form convenient for introducing into consideration higher order anisotropic physical theories.

There is a number of works on gauge models of interactions on Finsler spaces \[14, 13, 32\] and theirs extensions (see, for instance, \[4, 5, 6, 8, 24\]). One has introduced different variants of generalized gauge transforms, postulated corresponding Lagrangians for gravitational, gauge and matter
field interactions and formulated variational calculus (here we note the approaches developed by A. Bejancu \cite{9,11,10} and Gh. Munteanu and S. Ikeda \cite{27}). The main problem of such models is the dependence of the basic equations on chosen definition of gauge "compensation" symmetries and on type of space and field interactions anisotropy. In order to avoid the ambiguities connected with particular characteristics of possible ha–gauge theories we consider a "pure" geometric approach to gauge theories (on both locally isotropic and anisotropic spaces) in the framework of the theory of fiber bundles provided in general with different types of nonlinear and linear multiconnection and metric structures. This way developed by using global geometric methods holds also good for nonvariational, in the total spaces of bundles, gauge theories (in the case of gauge gravity one considers the Poincare or affine gauge groups); physical values and motion (field) equations have adequate geometric interpretation and do not depend on the type of local anisotropy of space–time background.

The elaboration of models with higher order anisotropic field interactions entails great difficulties because of problematical character of the possibility and manner of definition of conservation laws on ha–spaces. It will be recalled that, for instance, in special relativity the conservation laws of energy–momentum type are defined by the global group (the Poincare group) of automorphisms of the fundamental Mikowski spaces. For (pseudo)Riemannian spaces one has tangent space’s automorphisms and for particular cases there are symmetries generated by Killing vectors. No global or local automorphisms exist on generic ha–spaces and in result of this fact the formulation of ha–conservation laws is sophisticate and full of ambiguities. R. Miron and M. Anastasiei firstly pointed out the nonzero divergence of the matter energy–momentum d–tensor, the source in Einstein equations on la–spaces, and considered an original approach to the geometry of time–dependent Lagrangians \cite{2,23,24}. Nevertheless, the rigorous definition of energy–momentum values for locally anisotropic gravitational and matter fields and the form of conservation laws for such values have not been considered in present–day studies of the mentioned problem.

The aim of the second part of this paper is to develop a necessary geometric background (the theory of nearly autoparallel maps, in brief na–maps, and tensor integral formalism on multispaces) for formulation and a detailed investigation of conservation laws on locally isotropic and anisotropic curved spaces. For simplicity, the explicit constructions will be presented only for local anisotropies the transition to higher order anisotropies being considered as straightforward extensions. We shall develop for generic locally anisotropic spaces our previous results on the theory of na–maps for generalized affine spaces \cite{30,38,52}, Einstein-Cartan and Einstein spaces \cite{39,54,56}, fibre bundles \cite{37,57} and different subclasses of la–spaces \cite{58,55,43}.

The question of definition of tensor integration as the inverse operation of covariant derivation was posed and studied by A. Moór \cite{28}. Tensor–integral and bitensor formalisms turned out to be very useful in solving certain problems connected with conservation laws in general relativity.
In order to extend tensor–integral constructions we have proposed to take into consideration nearly autoparallel and nearly geodesic maps (in brief, we shall write ng–maps, ng–theory) which forms a subclass of local 1–1 maps of curved spaces with deformation of the connection and metric structures. A generalization of the Sinyukov’s ng–theory for spaces with local anisotropy was proposed by considering maps with deformation of connection for Lagrange spaces (on Lagrange spaces see and generalized Lagrange spaces. Tensor integration formalism for generalized Lagrange spaces was developed in. One of the main purposes of this work is to synthesize the results on na–maps and multisphere tensor integrals and to reformulate them for a very general class of locally anisotropic spaces. As the final step the problem of formulation of conservation laws on spaces with local anisotropy and definition of energy–momentum type value for la–gravity is considered.

Outline of the article

The paper is organized as follows: Section 2 contains a brief summary of the geometry of higher order anisotropic spaces. In Section 3 we formulate the theory of gauge (Yang-Mills) fields on generic ha–spaces and define a higher order anisotropic variant of Yang-Mills equations; the variational proof of gauge field equations is considered in connection with the "pure" geometrical method of definition of field equations. In Section 4 the ha–gravity is reformulated as a gauge theory for nonsemisimple groups. A model of nonlinear de Sitter gauge gravity with higher order anisotropy is formulated in Section 5. We study the gravitational gauge instantons with trivial local anisotropy in Section 6. Section 7 is devoted to the theory of nearly autoparallel maps of la–spaces. The classification of na–maps and corresponding invariant conditions are given in Section 8. In Section 9 we define the nearly autoparallel tensor–integral on locally anisotropic multispaces. The problem of formulation of conservation laws on spaces with local anisotropy is studied in Section 10. We present a definition of conservation laws for la–gravitational fields on na–images of la–spaces in Section 11. Concluding remarks are given in Section 12.

2 Higher Order Anisotropic Spaces

In this section we present the necessary results on higher order vector bundles provided with nonlinear and distinguished connections and metric structures which are used for modeling of spaces with higher order anisotropy. The denotations we are following are that from.

As a geometric background for our further constructions we use a locally trivial distinguished vector bundle, dv–bundle, $\mathcal{E}^{<z>} = (E^{<z>}, p, M, Gr, F^{<z>})$ where $F^{<z>} = \mathcal{R}^{m_1} \oplus ... \oplus \mathcal{R}^{m_z}$ (a real vector space of dimension $m = m_1 + ... + m_z$, $\dim F = m$, $\mathcal{R}$ denotes the real number field) is the typical fibre, the structural group is chosen to be the group of automorphisms of
\[ R^m \], i.e. \( Gr = GL(m, R) \), and \( p : E^{<z>} \to M \) (defined by intermediary projections

\[
p_{<z,z-1>} : E^{<z>} \to E^{<z-1>}, p_{<z-1,z-2>} : E^{<z-1>} \to E^{<z-2>}, ...
\]

\[ p : E^{<1>} \to M \]

is a differentiable surjection of a differentiable manifold \( E \) (total space, \( \dim E = n + m \)) to a differentiable manifold \( M \) (base space, \( \dim M = n \)). Local coordinates on \( E^{<z>} \) are denoted as

\[
u^{<\alpha>} = (x^i, y^{<\alpha>}) = (x^i = y^{a_0}, y^{a_1}, ..., y^{a_z})
\]

\[
= (..., y^{a_{(p)}}, ...) = \{ y^{a_{(p)}} \} = \{ y^{a_{(p)}} \},
\]

A local coordinate parametrization of \( E^{<z>} \) naturally defines a coordinate basis of the module of \( d \)-vector fields \( \Xi(\mathcal{E}^{<z>}) \),

\[
\partial_{<\alpha>} = (\partial_i, \partial_{<\alpha>}) = (\partial_i, \partial_{a_1}, ..., \partial_{a_p}, ..., \partial_{a_z}) = (1)
\]

\[
\frac{\partial}{\partial u^{<\alpha>}} = \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{<\alpha>}} \right) = \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{a_1}}, ..., \frac{\partial}{\partial y^{a_p}}, ..., \frac{\partial}{\partial y^{a_z}} \right),
\]

and the reciprocal to (1) coordinate basis

\[
d^{<\alpha>} = (d^i, d^{<\alpha>}) = (d^i, d^{a_1}, ..., d^{a_p}, ..., d^{a_z}) = (2)
\]

\[
du^{<\alpha>} = (dx^i, dy^{<\alpha>}) = (dx^i, dy^{a_1}, ..., dy^{a_p}, ..., dy^{a_z}),
\]

which is uniquely defined from the equations

\[
d^{<\alpha>} \circ \partial_{<\beta>} = \delta_{<\alpha>}, \delta_{<\beta>}
\]

where \( \delta_{<\alpha>}, \delta_{<\beta>} \) is the Kronecker symbol and by "\( \circ \)" we denote the inner (scalar) product in the tangent bundle \( T\mathcal{E}^{<z>} \).

A **nonlinear connection**, in brief, \( N \)-connection, (see a detailed study and basic references in [23], [24], [25]) in a \( dV \)-bundle \( \mathcal{E}^{<z>} \) can be defined as a distribution \( \{ N : E_u \to H_u E, T_u E = H_u E \oplus V^{(1)}_u E \oplus ... \oplus V^{(p)}_u E \oplus V^{(z)}_u E \} \) on \( E^{<z>} \) being a global decomposition, as a Whitney sum, into horizontal, \( \mathcal{H}E \), and vertical, \( \mathcal{V}E^{<p>} \), \( p = 1, 2, ..., z \) subbundles of the tangent bundle \( T\mathcal{E} : \mathcal{T}\mathcal{E} = H\mathcal{E} \oplus \mathcal{V}E^{<1>} \oplus ... \oplus \mathcal{V}E^{<p>} \oplus ... \oplus \mathcal{V}E^{<z>} \).

Locally a \( N \)-connection in \( \mathcal{E}^{<z>} \) is given by it components \( N_{<a_f>}(u), z \geq p > f \geq 0 \) (in brief we shall write \( N_{<a_f>}(u) \)) with respect to bases (1) and (2):

\[
N = N_{<a_f>}(u)\delta^{<a_f>} \oplus \delta_{<a_f>}, (z \geq p > f \geq 0),
\]

To coordinate locally geometric constructions with the global splitting of \( \mathcal{E}^{<z>} \) defined by a \( N \)-connection structure, we have to introduce a locally adapted basis ( \( la \)-basis, \( la \)-frame ),

\[
\delta_{<\alpha>} = (\delta_i, \delta_{<\alpha>}) = (\delta_i, \delta_{a_1}, ..., \delta_{a_p}, ..., \delta_{a_z}), (3)
\]

\[ 6 \]
with components parametrized as
\[
\delta_i = \partial_i - N_i^a \partial_a - \ldots - N_i^{a_1} \partial_{a_1}, \\
\delta_{a_1} = \partial_{a_1} - N_{a_1}^a \partial_a - \ldots - N_{a_1}^{a_2} \partial_{a_2}, \\
\ldots \\
\delta_{a_p} = \partial_{a_p} - N_{a_p}^{a_{p+1}} \partial_{a_{p+1}} - \ldots - N_{a_p}^{a_z} \partial_{a_z}, \\
\delta_{a_z} = \partial_{a_z}
\]
and its dual la–basis
\[
\delta^{<\alpha>} = (\delta^i, \delta^{<\alpha>}) = \left( \delta^i, \delta^{a_1}, \ldots, \delta^{a_p}, \ldots, \delta^{a_z} \right), \tag{4}
\]
\[
\delta x^i = dx^i, \\
\delta y^{a_1} = dy^{a_1} + M_i^{a_1} dx^i, \\
\delta y^{a_2} = dy^{a_2} + M_i^{a_2} dy^{a_1} + M_i^{a_2} dx^i, \\
\ldots \\
\delta y^{a_p} = dy^{a_p} + M_i^{a_p} dy^{a_{p-1}} + M_i^{a_p} dy^{a_{p-2}} + \ldots + M_i^{a_p} dx^i, \\
\ldots \\
\delta y^{a_z} = dy^{a_z} + M_i^{a_z} dy^{a_{z-1}} + M_i^{a_z} dy^{a_{z-2}} + \ldots + M_i^{a_z} dx^i.
\]

The interrelation between \(N\)– and \(M\)– coefficients from (3) and (4) is considered in [49, 50].

We emphasize that on a dv–bundle \(\mathcal{E}^{<z>}\) the higher order anisotropic operators (3) and (4) substitutes respectively the local operators of partial derivation (1) and of differentials (2).

The algebra of tensorial distinguished fields \(DT\left(\mathcal{E}^{<z>}\right)\) (d–fields, d–tensors, d–objects) on \(\mathcal{E}^{<z>}\) is introduced as the tensor algebra \(\mathcal{T} = \{\mathcal{T}_{q_1 \ldots q_s \ldots s_z}\}\) of the dv–bundle \(\mathcal{E}^{<z>}\),

\[
p_d : \mathcal{H}\mathcal{E}^{<z>} \oplus V^1 \mathcal{E}^{<z>} \oplus \ldots \oplus V^p \mathcal{E}^{<z>} \oplus \ldots \oplus V^z \mathcal{E}^{<z>} \to \mathcal{E}^{<z>},
\]

An element \(\mathbf{t} \in \mathcal{T}_{q_1 \ldots q_s \ldots s_z}\), d-tensor field of type \((p \quad r_1 \ldots r_p \ldots r_z \quad q \quad s_1 \ldots s_p \ldots s_z)\),
is written in local form as
\[
\mathbf{t} = t^{b_1 \ldots b_z}_{a_1 \ldots a_z}_{j_1 \ldots j_z} (u) \delta_{a_1} \otimes \ldots \otimes \delta_{a_p} \otimes d^{j_1} \otimes \ldots \otimes d^{j_z} \otimes
\]
\[
\delta_{a_1}^{(1)} \otimes \ldots \otimes \delta_{a_p}^{(1)} \otimes \delta_{a_1}^{(2)} \otimes \ldots \otimes \delta_{a_p}^{(2)} \otimes \ldots \otimes \delta_{a_1}^{(p)} \otimes \ldots \otimes \delta_{a_p}^{(p)} \otimes \ldots \otimes
\]
\[
\delta_{a_1}^{(p)} \otimes \ldots \otimes \delta_{a_p}^{(p)} \otimes \delta_{a_1}^{(z)} \otimes \ldots \otimes \delta_{a_p}^{(z)} \otimes \delta_{a_1}^{(z)} \otimes \ldots \otimes \delta_{a_p}^{(z)}.
\]

One use respectively denotations \(X(\mathcal{E}^{<z>})\) (or \(X(M)\)), \(\Lambda^p (\mathcal{E}^{<z>})\) (or \(\Lambda^p (M)\)) and \(\mathcal{F}(\mathcal{E}^{<z>})\) (or \(\mathcal{F}(M)\)) for the module of d-vector fields on \(\mathcal{E}^{<z>}\).
(or $M$), the exterior algebra of $p$-forms on $\mathcal{E}^{<z>}$ (or $M$) and the set of real functions on $\mathcal{E}^{<z>}(or M)$.

The geometric objects with various group and coordinate transforms coordinated with the $N$-connection structure on $\mathcal{E}^{<z>}$ are called in brief d–objects on $\mathcal{E}^{<z>}$. For a $N$-connection structure in $\mathcal{E}^{<z>}$ it is defined a corresponding decomposition of d-tensors into sums of horizontal and vertical parts, for example, for every d-vector $X \in \mathcal{X}(\mathcal{E}^{<z>})$ and 1-form $\tilde{X} \in \Lambda^1(\mathcal{E}^{<z>})$ we have respectively

$$X = hX + v_1X + ... + v_zX \quad \text{and} \quad \tilde{X} = h\tilde{X} + v_1\tilde{X} + ... v_z\tilde{X}.$$ 

In consequence, we can associate to every d-covariant derivation along a d-vector $X$, $D_X = X \circ D$, two new operators of $h$- and $v$-covariant derivations defined respectively as

$$D_X^{(h)}Y = D_{hX}Y$$

and

$$D_X^{(v_1)}Y = D_{v_1X}Y, ..., D_X^{(v_z)}Y = D_{v_zX}Y \quad \forall Y \in \mathcal{X}(\mathcal{E}^{<z>}),$$

for which the following conditions hold:

$$D_X Y = D_X^{(h)}Y + D_X^{(v_1)}Y + ... + D_X^{(v_z)}Y,$$

$$D_X^{(h)}f = (hX)f$$

and

$$D_X^{(v_p)}f = (v_pX)f, \quad X, Y \in \mathcal{X}(\mathcal{E}), f \in \mathcal{F}(M), p = 1, 2, ..., z.$$ 

A metric structure $G$ in the total space $E^{<z>}$ of a dv–bundle $\mathcal{E}^{<z>} = (E^{<z>}, p, M)$ over a connected and paracompact base $M$ is introduced as a symmetrical covariant tensor field of type $(0, 2)$, $G_{<\alpha><\beta>}$, being nondegenerate and of constant signature on $E^{<z>}$. Nonlinear connection $N$ and metric $G$ structures on $\mathcal{E}^{<z>}$ are mutually compatible it there are satisfied the conditions:

$$G \left( \delta_{af}, \delta_{ap} \right) = 0, \text{ or equivalently, } G_{afap}(u) = N_{af}^{<b>}(u) h_{af}^{<b>}(u) = 0,$$

where $h_{afbp} = G \left( \partial_{af}, \partial_{bp} \right)$ and $G_{bfap} = G \left( \partial_{bf}, \partial_{ap} \right), 0 \leq f < p \leq z$, which gives

$$N_{af}^{<b>}(u) = h_{<a><b>}(u) G_{<a><b>}(u)$$

(the matrix $h_{afbp}$ is inverse to $h_{afbp}$). With respect to la—basis (4) a d–metric is written as

$$G = g_{<\alpha><\beta>}(u) \delta^{<\alpha> \otimes \delta^{<\beta>}} = g_{ij}(u) d^i \otimes d^j + h_{<a><b>}(u) \delta^{<a> \otimes \delta^{<b>}}, \quad (5)$$

where $g_{ij} = G \left( \delta_i, \delta_j \right)$.

The torsion $T$ of a d–connection $D$ in $\mathcal{E}^{<z>}$ is defined by the equation

$$T(X, Y) = XY^\lambda \circ T = D_X Y - D_Y X - [X, Y].$$
One holds the following $h$- and $v_{(p)}$-decompositions
\[
T(X, Y) = T(hX, hY) + T(hX, vY) + T(vX, hY) + T(vX, vY).
\]
We consider the projections: $hT(X, Y), v_{(p)} T(hX, hY), hT(hX, hY), ...$
and say that, for instance, $hT(hX, hY)$ is the $h(hh)$-torsion of $D$,
$v_{(p)} T(hX, hY)$ is the $v_{p}(hh)$-torsion of $D$ and so on.

A torsion $T(X, Y)$ is determined by local $d$-tensor fields, torsions, defined as
\[
T^i_{jk} = hT \left( \delta_k, \delta_j \right) \cdot d^i, \quad T^i_{jk} = v_{(p)} T \left( \delta_k, \delta_j \right) \cdot \delta^a_p,
\]
\[
P^i_{jb} = hT \left( \delta_{bp}, \delta_j \right) \cdot d^i, \quad P^i_{jb} = v_{(p)} T \left( \delta_{b}, \delta_j \right) \cdot \delta^a_p,
\]
\[
S^a_{bji} = v_{(p)} T \left( \delta_{cj}, \delta_{bi} \right) \cdot \delta^a_p.
\]
and with components computed in the form (see [25] and Paper I):
\[
T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{j < a> c} = C^i_{j < a> c}, \quad T^i_{< a> j c} = - C^i_{j < a> c},
\]
\[
T^i_{j < a> c} = 0, \quad T^i_{< a> b c} = S^i_{< a> b c} = C^a_{< b> c} - C^a_{< c> b},
\]
\[
P^i_{b c j} = \frac{\delta^a_{c}}{\partial y^a} \left( \delta_{b}, \delta_j \right) \delta^a, \quad P^i_{< a> b c} = \delta^a_{c} \left( \delta_{b}, \delta_j \right) \delta^a,
\]
\[
S^i_{< a> b c} = \delta^a_{c} \left( \delta_{b}, \delta_j \right) \delta^a, \quad S^i_{< a> b c} = \delta^a_{c} \left( \delta_{b}, \delta_j \right) \delta^a,
\]
\[
R^i_{h j k} = \delta^i \cdot R \left( \delta_k, \delta_j \right) \delta_h, \quad R^i_{< a> b j k} = \delta^a \cdot R \left( \delta_k, \delta_j \right) \delta_{b},
\]
\[
P^i_{j < a> c} = d^i \cdot R \left( \delta_{< c>}, \delta_{< k>} \right) \delta_j, \quad P^i_{j < a> b c} = \delta^a \cdot R \left( \delta_{< c>}, \delta_{< k>} \right) \delta_{b},
\]
\[
S^i_{j < a> b c} = d^i \cdot R \left( \delta_{< c>}, \delta_{< b>} \right) \delta_j, \quad S^i_{j < a> b c} = \delta^a \cdot R \left( \delta_{< d>}, \delta_{< c>} \right) \delta_{b},
\]
with components
\[
R^i_{h j k} = \frac{\delta L^i_{h j}}{\delta x^h} - \frac{\delta L^i_{h k}}{\delta x^j} + L^m_{h j} L^i_{m k} - L^m_{h k} L^i_{m j} + C^i_{h < a> c} R^a_{j k},
\]
\[
R^i_{< a> b j k} = \frac{\delta L^i_{< a> b j}}{\delta x^h} - \frac{\delta L^i_{< a> b k}}{\delta x^j} + L^c_{< a> b j} L^i_{c k} - L^c_{< a> b k} L^i_{c j} + C^i_{< a> b c} R^c_{j k},
\]
\[
P^i_{j < a> b c} = \frac{\delta L^i_{j < a> b c}}{\delta y^a} + C^i_{j < b> c} P^b_{j < a> c} -
(\partial C_{j<a>}^i + L_{ik}^j C_{j<a>}^d - L_{jk}^i C_{j<a>}^i - L_{<a>k}^i C_{j<a>}^i ),

P_{<b>k<a>} = \delta L_{<b>k}^i C_{<b<d>}^i P_{<d>k<a>} -

\left( \frac{\partial C_{<b<d>}^i}{\partial x^k} + L_{<d>k}^i C_{<b<d>}^d - L_{<b>k}^i C_{<b<d>}^i - L_{<a>k} C_{<b<d>}^i \right),

\delta C_{i<j<k}^a \partial y^a + C_{i<j<k}^a P_{<d>k}^i -

P_{<b>i} = - 2 P_{<a>i} = - P_{i<k}^i,

R_{<a><\beta>} = R_{<a><\beta>>} = R + S,

\bar{G}_{<a><\beta>} + \lambda g_{<a><\beta>} = \kappa E_{<a><\beta>},

where \bar{G}_{<a><\beta>} = R_{<a><\beta>} - \frac{1}{2} R g_{<a><\beta>}

is the Einstein d–tensor, \lambda and \kappa are correspondingly the cosmological and gravitational constants and by \(E_{<a><\beta>}\) is denoted the locally anisotropic energy–momentum d–tensor. By using the formulas (6)–(9) we can write down in explicit form the h- and v–components of the Einstein equations (10) (we shall not consider such formulas in this paper).

Because ha–spaces generally have nonzero torsions we shall add to (10) a system of algebraic d–field equations with the source \(S_{<a><\beta><\gamma>}\) being the locally anisotropic spin density of matter (if we consider a variant of higher order anisotropic Einstein–Cartan theory):

\[ T_{<\gamma><a><\beta>} + 2 S_{<\gamma><a><\beta><\delta>} T_{<\delta><a><\beta>} = \kappa S_{<a><\beta>} \]

In a more general case we must introduced some constraints on torsions and nonlinear connections induced from string theory [15], locally (and/or higher order) anisotropic supergravity [10]. A form of gauge dynamical field equations for torsions will be considered in the Section 5 of this paper.
3 Gauge Fields on Ha–Spaces

This section presents a geometrical background for gauge field theories on spaces with higher order anisotropy.

3.1 Bundles on ha–spaces

Let \((P, \pi, Gr, \mathcal{E}^{<z>})\) be a principal bundle \(\mathcal{E}^{<z>}\) (being a ha-space) with structural group \(Gr\) and surjective map \(\pi : P \rightarrow \mathcal{E}^{<z>}\). At every point \(u = (x, y_{(1)}, ..., y_{(z)}) \in \mathcal{E}^{<z>}\) there is a vicinity \(U \subset \mathcal{E}^{<z>}, u \in U\), with trivializing \(P\) diffeomorphisms \(f\) and \(\varphi\):

\[
f_U : \pi^{-1}(U) \rightarrow U \times Gr, \quad f(p) = (\pi(p), \varphi(p)),
\]

\[
\varphi_U : \pi^{-1}(U) \rightarrow Gr, \varphi(pq) = \varphi(p)q, \quad \forall q \in Gr, p \in P.
\]

We remark that in the general case for two open regions \(U, V \subset \mathcal{E}^{<z>}, U \cap V \neq \emptyset, f_{U|_p} \neq f_{V|_p}, \) even \(p \in U \cap V\).

Transition functions \(g_{UV}\) are defined as

\[
g_{UV} : U \cap V \rightarrow Gr, g_{UV}(u) = \varphi_U(p) (\varphi_V(p)^{-1}), \pi(p) = u.
\]

Hereafter we shall omit, for simplicity, the specification of trivializing regions of maps and denote, for example, \(f \equiv f_U, \varphi \equiv \varphi_U, s \equiv s_U, \) if this will not give rise to ambiguities.

Let \(\theta\) be the canonical left invariant 1-form on \(Gr\) with values in algebra \(\mathcal{G}\) of group \(Gr\) uniquely defined from the relation \(\theta(q) = q, \forall q \in \mathcal{G}\), and consider a 1-form \(\omega\) on \(U \subset \mathcal{E}^{<z>}\) with values in \(\mathcal{G}\). Using \(\theta\) and \(\omega\), we can locally define the connection form \(\Omega\) in \(P\) as a 1-form:

\[
\Omega = \varphi^*\theta + Ad \varphi^{-1}(\pi^*\omega)
\]  \hspace{1cm} (11)

where \(\varphi^*\theta\) and \(\pi^*\omega\) are, respectively, forms induced on \(\pi^{-1}(U)\) and \(P\) by maps \(\varphi\) and \(\pi\) and \(\omega = s^*\Omega\). The adjoint action on a form \(\lambda\) with values in \(\mathcal{G}\) is defined as

\[
(Ad \varphi^{-1} \lambda)_p = (Ad \varphi^{-1}(p)) \lambda_p
\]

where \(\lambda_p\) is the value of form \(\lambda\) at point \(p \in P\).

Introducing a basis \(\{\Delta_{\hat{a}}\}\) in \(\mathcal{G}\) (index \(\hat{a}\) enumerates the generators making up this basis), we write the 1-form \(\omega\) on \(\mathcal{E}^{<z>}\) as

\[
\omega = \Delta_{\hat{a}} \omega_{\hat{a}}(u), \quad \omega_{\hat{a}}(u) = \omega_{\hat{a}, \mu >}(u) \delta u^{<\mu>}
\]  \hspace{1cm} (12)

where \(\delta u^{<\mu>} = (dx^i, dy^{<a>})\) and the Einstein summation rule on indices \(\hat{a}\) and \(< \mu >\) is used. Functions \(\omega_{\hat{a}, \mu >}(u)\) from (12) will be called the components of Yang-Mills fields on ha-space \(\mathcal{E}^{<z>}\). Gauge transforms of \(\omega\) can be geometrically interpreted as transition relations for \(\omega_U\) and \(\omega_V\), when \(u \in U \cap V\),

\[
(\omega_U)_u = (g_{UV}^*\omega_U)_u + Ad g_{UV}^{-1}(\omega_V)_u.
\]  \hspace{1cm} (13)
To relate $\omega^\hat{\mu}_{\lambda\rho}$ with a covariant derivation we shall consider a vector bundle $\mathcal{Y}$ associated to $P$. Let $\rho : Gr \to GL(\mathcal{R}^s)$ and $\rho' : \mathcal{G} \to End(E^s)$ be, respectively, linear representations of group $Gr$ and Lie algebra $\mathcal{G}$ (in a more general case we can consider $\mathcal{C}^s$ instead of $\mathcal{R}^s$). Map $\rho$ defines a left action on $Gr$ and associated vector bundle $\mathcal{Y} = P \times \mathcal{R}^s/Gr$, $\pi_E : E \to \mathcal{E}^{<\omega}$.

Introducing the standard basis $\xi_{\lambda} = \{\xi_{\lambda_1}, \xi_{\lambda_2}, ..., \xi_{\lambda_s}\}$ in $\mathcal{R}^s$, we can define the right action on $P \times \mathcal{R}^s$, $(p, \xi) q = (pq, \rho(q^{-1})\xi), q \in Gr$, the map induced from $P$

$$p : \mathcal{R}^s \to \pi_E^{-1}(u), \quad (p(\xi) = (p\xi)Gr, \xi \in \mathcal{R}^s, \pi(p) = u)$$

and a basis of local sections $e_{\underline{i}} : U \to \pi_E^{-1}(U), \; e_{\underline{i}}(u) = s(u)\xi_{\underline{i}}$. Every section $\xi : \mathcal{E}^{<\omega} \to \mathcal{Y}$ can be written locally as $\xi = \xi^i e_i, \xi^i \in C^\infty(U)$. To every vector field $X$ on $\mathcal{E}^{<\omega}$ and Yang-Mills field $\omega^\hat{\mu}$ on $P$ we associate operators of covariant derivations:

$$\nabla_X\xi = e_{\underline{i}} \left[ X\xi^i + B(X)J^\underline{i}_{\underline{j}} \xi^j \right]$$

$$B(X) = (\rho'X)\omega^\hat{\mu}(X).$$

Transformation laws (13) and operators (14) are interrelated by these transition transforms for values $e_{\underline{i}}, \xi^i$, and $B_{<\mu}$:

$$e^V_{\underline{i}}(u) = [\rho_{\mu\nu}u]^{-1} e^V_{\underline{i}}, \; \xi^V_{\underline{i}}(u) = [\rho_{\mu\nu}u]^{-1} \xi^V_{\underline{i}},$$

$$B^{V}_{<\mu}(u) = [\rho_{\mu\nu}u]^{-1} \delta_{<\mu} [\rho_{\mu\nu}u] + [\rho_{\mu\nu}u]^{-1} B^{V}_{<\mu}(u) [\rho_{\mu\nu}u],$$

where $B^{V}_{<\mu}(u) = B^{<\mu}(\delta/du^{<\mu})(u)$.

Using (15), we can verify that the operator $\nabla^U_X$, acting on sections of $\pi_U : \mathcal{Y} \to \mathcal{E}^{<\omega}$ according to definition (14), satisfies the properties

$$\nabla^U_{f_1X + f_2Y} = f_1\nabla^U_X + f_2\nabla^U_X, \; \nabla^U_X (f\xi) = f\nabla^U_X \xi + (Xf)\xi,$$

$$\nabla^U_{X\xi} = \nabla^U_X \xi, \quad u \in U \cap \mathcal{V}, f_1, f_2 \in C^\infty(U).$$

So, we can conclude that the Yang–Mills connection in the vector bundle $\pi_U : \mathcal{Y} \to \mathcal{E}^{<\omega}$ is not a general one, but is induced from the principal bundle $\pi : P \to \mathcal{E}^{<\omega}$ with structural group $Gr$.

The curvature $\mathcal{K}$ of connection $\Omega$ from (11) is defined as

$$\mathcal{K} = D\Omega, \quad D = \mathcal{H} \circ d$$

where $d$ is the operator of exterior derivation acting on $\mathcal{G}$-valued forms as $d \left( \Delta_{\hat{a}} \otimes \chi^a \right) = \Delta_{\hat{a}} \otimes d \chi^a$ and $\mathcal{H}$ is the horizontal projecting operator actin, for example, on the 1-form $\lambda$ as $(\mathcal{H}\lambda)_P(X_p) = \lambda_p(H_pX_p)$, where $H_p$ projects on the horizontal subspace

$$H_p \in P_p [X_p ∈ H_p \text{ is equivalent to } \Omega_p(X_p) = 0].$$
We can express (16) locally as
\[ K = \text{Ad} \varphi_U^{-1} (\pi^* K_U) \] (17)
where
\[ K_U = d\omega_U + \frac{1}{2} [\omega_U, \omega_U]. \] (18)
The exterior product of \( G \)-valued form (18) is defined as
\[ \left[ \Delta \hat{a} \otimes \lambda \hat{a}, \Delta \hat{b} \otimes \xi \right] = \left[ \Delta \hat{a}, \Delta \hat{b} \right] \otimes \lambda \hat{a} \wedge \xi \hat{b}, \]
where the antisymmetric tensorial product is
\[ \lambda \hat{a} \wedge \xi \hat{b} = \lambda \hat{a} \xi \hat{b} - \xi \hat{b} \lambda \hat{a}. \]
Introducing structural coefficients \( f_{\hat{b} \hat{c}} \hat{a} \) of \( G \) satisfying
\[ \left[ \Delta \hat{b}, \Delta \hat{c} \right] = f_{\hat{b} \hat{c}} \hat{a} \Delta \hat{a}, \]
we can rewrite (18) in a form more convenient for local considerations:
\[ K_U = \Delta \hat{a} \otimes K_{\hat{a} <\mu> \hat{b} <\nu>} \delta u^{<\mu>} \wedge \delta u^{<\nu>} \] (19)
where
\[ K_{\hat{a} <\mu> \hat{b} <\nu>} = \frac{\delta \omega \hat{a} <\mu>}{\delta u^{<\mu>}} - \frac{\delta \omega \hat{a} <\nu>}{\delta u^{<\nu>}} + \frac{1}{2} f_{\hat{b} \hat{c}} \hat{a} \left( \omega \hat{b} <\mu>, \omega \hat{c} <\nu> - \omega \hat{b} <\nu>, \omega \hat{c} <\mu> \right). \]

This subsection ends by considering the problem of reduction of the local anisotropic gauge symmetries and gauge fields to isotropic ones. For local trivial considerations we can consider that the vanishing of dependencies on \( y \) variables leads to isotropic Yang-Mills fields with the same gauge group as in the anisotropic case, Global geometric constructions require a more rigorous topological study of possible obstacles for reduction of total spaces and structural groups on anisotropic bases to their analogous on isotropic (for example, pseudo-Riemannian) base spaces.

### 3.2 Yang-Mills equations on ha-spaces

Interior gauge (nongravitational) symmetries are associated to semisimple structural groups. On the principal bundle \( (P, \pi, Gr, E^{<z>}) \) with nondegenerate Killing form for semisimple group \( Gr \) we can define the generalized Lagrange metric
\[ h_p (X_p, Y_p) = G_{\pi(p)} (d\pi P X_p, d\pi P Y_p) + K (\Omega_p (X_p), \Omega_p (X_p)), \] (20)
where \( d\pi P \) is the differential of map \( \pi : P \to E^{<z>} \), \( G_{\pi(p)} \) is locally generated as the ha-metric (5), and \( K \) is the Killing form on \( G \):
\[ K \left( \Delta \hat{a}, \Delta \hat{b} \right) = f_{\hat{b} \hat{d}} \hat{a} f_{\hat{a} \hat{c}} \hat{d} = K_{\hat{a} \hat{b}}. \]
Using the metric $G_{<\alpha><\beta>}$ on $\mathcal{E}^{<z>}$, we can introduce operators $*_G$ and $\hat{\delta}_G$ acting in the space of forms on $\mathcal{E}^{<z>}$ ($*_H$ and $\hat{\delta}_H$ acting on forms on $\mathcal{E}^{<z>}$). Let $e_{<\mu>}$ be orthonormalized frames on $U \subset \mathcal{E}^{<z>}$ and $e^{<\mu>}$ the adjoint coframes. Locally

$$G = \sum_{<\mu>} \eta_{<\mu>}(e_{<\mu>} \otimes e^{<\mu>}),$$

where $\eta_{<\mu><\mu>} = \eta_{<\mu>}(<\mu>)$ for $<\mu> = 1, 2, ..., n_E, n_E = 1, ..., n + m_1 + \ldots + m_z$, and the Hodge operator $*_G$ can be defined as $*_G : \Lambda'(\mathcal{E}^{<z>}) \to \Lambda^{n+m_1+\ldots+m_z}(\mathcal{E}^{<z>})$, or, in explicit form, as

$$_{G} e^{<\mu_1>} \wedge \ldots \wedge e^{<\mu_r>} = \eta_{\nu_1} \ldots \eta_{\nu_{n_E-r}} \times \text{sign} 
\begin{pmatrix}
1 & 2 & \ldots & r & r+1 & \ldots & n_E \\
<\mu_1> & <\mu_2> & \ldots & <\mu_r> & <\nu_1> & \ldots & <\nu_{n_E-r}>
\end{pmatrix} \times
_e^{<\nu_1>} \wedge \ldots \wedge e^{<\nu_{n_E-r}>}.

Next, define the operator

$$_{G}^{-1} = \eta(1) \ldots \eta(n_E) (-1)^{r(n_E-r)} *_{G}$$

and introduce the scalar product on forms $\beta_1, \beta_2, \ldots \subset \Lambda^r(\mathcal{E}^{<z>})$ with compact carrier:

$$(\beta_1, \beta_2) = \eta(1) \ldots \eta(n_E) \int \beta_1 \wedge *_{G} \beta_2.$$

The operator $\hat{\delta}_G$ is defined as the adjoint to $d$ associated to the scalar product for forms, specified for $r$-forms as

$$\hat{\delta}_G = (-1)^r *_{G}^{-1} \circ d \circ *_{G}.$$

We remark that operators $*_H$ and $\delta_H$ acting in the total space of $P$ can be defined similarly to (21) and (22), but by using metric (20). Both these operators also act in the space of $G$-valued forms:

$$*_G \left(\Delta_{\hat{\alpha}} \otimes \varphi_{\hat{\alpha}}\right) = \Delta_{\hat{\alpha}} \otimes (*_{G} \varphi_{\hat{\alpha}}),$$

$$\hat{\delta} \left(\Delta_{\hat{\alpha}} \otimes \varphi_{\hat{\alpha}}\right) = \Delta_{\hat{\alpha}} \otimes (\hat{\delta} \varphi_{\hat{\alpha}}).$$

The form $\lambda$ on $P$ with values in $G$ is called horizontal if $\hat{H} \lambda = \lambda$ and equivariant if $R^*(q) \lambda = \text{Ad} q^{-1} \varphi, \forall g \in Gr, R(q)$ being the right shift on $P$. We can verify that equivariant and horizontal forms also satisfy the conditions

$$\lambda = \text{Ad} \varphi_{U}^{-1} (\pi^* \lambda), \quad \lambda_{U} = S_{U}^* \lambda,$$

$$(\lambda_{V})_{U} = \text{Ad} (g_{U} \lambda_{V}(u))^{-1} (\lambda_{U})_{u}.$$

Now, we can define the field equations for curvature (17) and connection (11):

$$\Delta K = 0.$$  

(23)
∇\mathcal{K} = 0, \quad (24)

where \Delta = \tilde{H} \circ \delta_H. Equations (23) are similar to the well-known Maxwell equations and for non-Abelian gauge fields are called Yang-Mills equations. The structural equations (24) are called Bianchi identities.

The field equations (23) do not have a physical meaning because they are written in the total space of bundle \( \Upsilon \) and not on the base anisotropic space-time \( \mathcal{E} \). But this difficulty may be obviated by projecting the mentioned equations on the base. The 1-form \( \Delta \mathcal{K} \) is horizontal by definition and its equivariance follows from the right invariance of metric (20). So, there is a unique form \( (\Delta \mathcal{K})_U \) satisfying

\[
\Delta \mathcal{K} = \text{Ad} \varphi_U^{-1} \pi^*(\Delta \mathcal{K})_U.
\]

Projection of (23) on the base can be written as \( (\Delta \mathcal{K})_U = 0 \). To calculate \( (\Delta \mathcal{K})_U \), we use the equality \[12, 31]\n
\[
d \left( \text{Ad} \varphi_U^{-1} \lambda \right) = \text{Ad} \varphi_U^{-1} d\lambda - \left[ \varphi_U^* \theta, \text{Ad} \varphi_U^{-1} \lambda \right]
\]

where \( \lambda \) is a form on \( P \) with values in \( G \). For r-forms we have

\[
\hat{\delta} \left( \text{Ad} \varphi_U^{-1} \lambda \right) = \text{Ad} \varphi_U^{-1} \hat{\delta} \lambda - (-1)^r \ast_H \{ \left[ \varphi_U^* \theta, \ast_H \text{Ad} \varphi_U^{-1} \lambda \right]
\]

and, as a consequence,

\[
\hat{\delta} \mathcal{K} = \text{Ad} \varphi_U^{-1} \{ \hat{\delta}_H \pi^* \mathcal{K}_U + \ast_H [\pi^* \omega_U, \ast_H \pi^* \mathcal{K}_U]\} - 
\ast_H \left[ \Omega, \text{Ad} \varphi_U^{-1} \ast_H (\pi^* \mathcal{K}) \right]. \quad (25)
\]

By using straightforward calculations in the adapted dual basis on \( \pi^{-1}(U) \) we can verify the equalities

\[
\left[ \Omega, \text{Ad} \varphi_U^{-1} \ast_H (\pi^* \mathcal{K}_U) \right] = 0, \quad \tilde{H} \delta_H (\pi^* \mathcal{K}_U) = \pi^* \left( \tilde{\delta}_G \mathcal{K} \right), 
\ast_H [\pi^* \omega_U, \ast_H (\pi^* \mathcal{K}_U)] = \pi^* \{ \ast_G [\omega_U, *_G \mathcal{K}_U] \}. \quad (26)
\]

From (25) and (26) it follows that

\[
(\Delta \mathcal{K})_U = \tilde{\delta}_G \mathcal{K}_U + \ast_G^{-1} [\omega_U, *_G \mathcal{K}_U]. \quad (27)
\]

Taking into account (27) and (22), we prove that projection on \( \mathcal{E} \) of equations (23) and (24) can be expressed respectively as

\[
\ast_G^{-1} \circ d \circ *_G \mathcal{K}_U + \ast_G^{-1} [\omega_U, *_G \mathcal{K}_U] = 0. \quad (28)
\]

\[
d \mathcal{K}_U + [\omega_U, \mathcal{K}_U] = 0.
\]

Equations (28) (see (27)) are gauge–invariant because

\[
(\Delta \mathcal{K})_U = \text{Ad} g^{-1}_{UV} (\Delta \mathcal{K})_V.
\]
By using formulas (19)-(22) we can rewrite (28) in coordinate form

\[ D_{\nu} \left( G^{\nu} G^{\lambda} K_{\alpha}^{\lambda} \right) + f_{\beta \gamma} G^{\nu} G^{\lambda} \omega_{\alpha}^{\lambda} K_{\alpha}^{\nu} = 0, \quad (29) \]

where \( D_{\nu} \) is, for simplicity, a compatible with metric covariant derivation on \( \mathcal{E}^{\nu} \).

We point out that for our bundles with semisimple structural groups the Yang-Mills equations (23) (and, as a consequence, their horizontal projections (28) or (29)) can be obtained by variation of the action

\[ I = \int K_{\gamma}^{\alpha} G^{\alpha} G^{\beta} K_{\alpha}^{\beta} \]

Equations for extremals of (30) have the form

\[ K_{\gamma}^{\alpha} G^{\alpha} G^{\beta} D_{\alpha} K_{\alpha}^{\beta} - K_{\alpha}^{\beta} G^{\alpha} G^{\beta} f_{\gamma \lambda} \omega_{\gamma}^{\lambda} K_{\alpha}^{\nu} = 0, \]

which are equivalent to ”pure” geometric equations (29) (or (28)) due to nondegeneration of the Killing form \( K_{\gamma}^{\alpha} \) for semisimple groups.

To take into account gauge interactions with matter fields (section of vector bundle \( \Upsilon \) on \( \mathcal{E} \)) we have to introduce a source 1–form \( J \) in equations (23) and to write them as

\[ \Delta K = J \quad (31) \]

Explicit constructions of \( J \) require concrete definitions of the bundle \( \Upsilon \); for example, for spinor fields an invariant formulation of the Dirac equations on la–spaces is necessary. We omit spinor considerations in this section (see [49, 50]).

4 Gauge Higher Order Anisotropic Gravity

A considerable body of work on the formulation of gauge gravitational models on isotropic spaces is based on using nonsemisimple groups, for example, Poincare and affine groups, as structural gauge groups (see critical analysis and original results in [59, 54, 21, 29]). The main impediment to developing such models is caused by the degeneration of Killing forms for nonsemisimple groups, which make it impossible to construct consistent variational gauge field theories (functional (30) and extremal equations are degenerate in these cases). There are at least two possibilities to get around the mentioned difficulty. The first is to realize a minimal extension of the nonsemisimple group to a semisimple one, similar to the extension of the Poincare group to the de Sitter group considered in [30, 31, 35] (in the next section we shall use this operation for the definition of locally anisotropic gravitational instantons). The second possibility is to introduce into consideration the bundle
of adapted affine frames on ha-space $\mathcal{E}^{<z>}$, to use an auxiliary nondegenerate bilinear form $a_{\alpha \beta}$ instead of the degenerate Killing form $K_{\alpha \beta}$ and to consider a ”pure” geometric method, illustrated in the previous section, of defining gauge field equations. Projecting on the base $\mathcal{E}^{<z>}$, we shall obtain gauge gravitational field equations on ha-space having a form similar to Yang-Mills equations.

The goal of this section is to prove that a specific parametrization of components of the Cartan connection in the bundle of adapted affine frames on $\mathcal{E}^{<z>}$ establishes an equivalence between Yang-Mills equations (31) and Einstein equations (10) on ha–spaces.

### 4.1 Bundles of linear ha–frames

Let $(X_{<\alpha>})_u = (X_1, X_{<\alpha>})_u = (X_1, X_{a_1}, ..., X_{a_z})_u$ be an adapted frame (see (14) at point $u \in \mathcal{E}^{<z>}$). We consider a local right distinguished action of matrices

$$A_{<\alpha>}^{<\alpha'>} = \begin{pmatrix} A_{p} & 0 & \cdots & 0 \\ 0 & B_{a_1}^{a_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{a_z}^{a_z} \end{pmatrix} \subset GL_{n_E} = GL(n, \mathcal{R}) \oplus GL(m_1, \mathcal{R}) \oplus ... \oplus GL(m_z, \mathcal{R}).$$

Nondegenerate matrices $A_{p}^{i}$ and $B_{a}^{j}$ respectively transforms linearly $X_{i|u}$ into $X_{i'|u} = A_{p}^{i} X_{i|u}$ and $X_{a|u}$ into $X_{a'|u} = B_{a}^{a'} X_{a|u}$, where $X_{<\alpha'>}^{<\alpha>}_u X_{<\alpha>} = A_{<\alpha'>}^{<\alpha>} X_{<\alpha>}^{(0)}$ is also an adapted frame at the same point $u \in \mathcal{E}^{<z>}$. We denote by $L a (\mathcal{E}^{<z>})$ the set of all adapted frames $X_{<\alpha>}$ at all points of $\mathcal{E}^{<z>}$ and consider the surjective map $\pi$ from $L a (\mathcal{E}^{<z>})$ to $\mathcal{E}^{<z>}$ transforming every adapted frame $X_{a|u}$ and point $u$ into point $u$. Every $X_{<\alpha'>}^{<\alpha>}_u$ has a unique representation as $X_{<\alpha'>}^{<\alpha>}_u = A_{<\alpha'>}^{<\alpha>} X_{<\alpha>}^{(0)}$, where $X_{<\alpha>}^{(0)}$ is a fixed distinguished basis in tangent space $T (\mathcal{E}^{<z>})$. It is obvious that $\pi^{-1} (U), U \subset \mathcal{E}^{<z>}$, is bijective to $U \times GL_{n_E} (\mathcal{R})$. We can transform $L a (\mathcal{E}^{<z>})$ in a differentiable manifold taking $(u^{<\beta>}, A_{<\alpha'>}^{<\alpha>})$ as a local coordinate system on $\pi^{-1} (U)$. Now, it is easy to verify that

$$L a (\mathcal{E}^{<z>}) = (L a (\mathcal{E}^{<z>}, \mathcal{E}^{<z>}, GL_{n_E} (\mathcal{R})))$$

is a principal bundle. We call $L a (\mathcal{E}^{<z>})$ the bundle of linear adapted frames on $\mathcal{E}^{<z>}$. The next step is to identify the components of, for simplicity, compatible d-connection $\Gamma_{<\alpha><\beta><\gamma>}$ on $\mathcal{E}^{<z>} :$

$$\Omega^{\hat{\alpha}}_{\hat{\nu} u} = \omega^{\hat{\alpha}} = \{ \omega^{\hat{\alpha}}_{<\lambda>,<\gamma>} = \Gamma_{<\lambda><\gamma>^{<\alpha>}} \}.$$  

Introducing (32) in (27), we calculate the local 1-form

$$(\Delta \mathcal{R}^{(\Gamma)})_{\hat{\nu} u} = \Delta^{\hat{\alpha}}_{\hat{\alpha} 1} \otimes (G^{<\nu><\lambda>} D_{<\lambda>\mathcal{R}^{<\hat{\alpha><\gamma>}^{<\nu><\mu>}}}. \quad (32)$$
\[ f^{\alpha<\nu}\beta<\delta}\lambda<\mu> G^{\nu<\lambda}\omega<\delta} R^{\gamma<\hat{\delta}}<\nu<\mu>) \delta u^{<\mu>}, \] (33)

where

\[
\Delta_{\alpha\beta} = \begin{pmatrix}
\Delta_{ij} & 0 & \ldots & 0 \\
0 & \Delta_{i1,1} & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \Delta_{a,b}
\end{pmatrix}
\]

is the standard distinguished basis in Lie algebra of matrices \( GL_{n_E}(\mathcal{R}) \) with \((\Delta_{ik})_{jl} = \delta_{ij}\delta_{kl} \) and \((\Delta_{\alpha\beta})_{\mu\nu} = \delta_{\mu\rho}\delta_{\nu\delta} \) being respectively the standard bases in \( GL(\mathcal{R}^{n_E}) \). We have denoted the curvature of connection (32), considered in (33), as

\[
\mathcal{R}^{(\Gamma)} = \Delta_{\alpha\beta}^{\Gamma} \otimes \mathcal{R}^{(\Gamma)}_{\alpha\beta} X^{<\nu>} \wedge X^{<\mu>},
\]

where \(\mathcal{R}^{(\Gamma)}_{\alpha\beta} X^{<\nu>} <\nu<\mu> = R_{<\alpha,\beta>}^{<\nu<\mu>} \) (see curvatures (7)).

### 4.2 Bundles of affine ha–frames and Einstein equations

Besides \( \mathcal{L}a(\mathcal{E}^{<\omega>}) \) with ha-space \( \mathcal{E}^{<\omega>} \), another bundle is naturally related, the bundle of adapted affine frames with structural group \( Af_{n_E}(\mathcal{R}) = GL_{n_E}(\mathcal{E}^{<\omega>}) \otimes \mathcal{R}^{n_E}. \) Because as linear space the Lie Algebra \( af_{n_E}(\mathcal{R}) \) is a direct sum of \( GL_{n_E}(\mathcal{R}) \) and \( \mathcal{R}^{n_E} \), we can write forms on \( \mathcal{A}a(\mathcal{E}^{<\omega>}) \) as \( \Theta = (\Theta_1, \Theta_2) \), where \( \Theta_1 \) is the \( GL_{n_E}(\mathcal{R}) \) component and \( \Theta_2 \) is the \( \mathcal{R}^{n_E} \) component of the form \( \Theta \). Connection (32), \( \Omega \) in \( \mathcal{L}a(\mathcal{E}^{<\omega>}) \), induces the Cartan connection \( \Omega \) in \( \mathcal{A}a(\mathcal{E}^{<\omega>}) \); see the isotropic case in [30, 31, 12]. This is the unique connection on \( \mathcal{A}a(\mathcal{E}^{<\omega>}) \) represented as \( i^*\Omega = (\Omega, \chi) \), where \( \chi \) is the shifting form and \( i : \mathcal{A}a \rightarrow \mathcal{L}a \) is the trivial reduction of bundles. If \( s_{U}^{(a)} \) is a local adapted frame in \( \mathcal{L}a(\mathcal{E}^{<\omega>}) \), then \( s_{U}^{(0)} = i \circ s_{U} \) is a local section in \( \mathcal{A}a(\mathcal{E}^{<\omega>}) \) and

\[
(\Omega_{U}) = s_{U} \Omega = (\Omega_{U}, \chi_{U}), \tag{34}
\]

\[
(\mathcal{R}_{U}) = s_{U} \mathcal{R} = (\mathcal{R}_{U}^{(\Gamma)}, T_{U}),
\]

where \( \chi = e_{\hat{\alpha}} \otimes \hat{\chi}_{<\mu<\nu>} X^{<\mu>} \), \( G_{<\alpha<\beta>}^{<\gamma>} = \chi_{<\alpha<\beta>} \chi_{<\beta<\gamma>} \eta_{\alpha<\beta<\gamma>} \) (\( \eta_{\hat{\alpha}<\hat{\beta}} \) is diagonal with \( \eta_{\hat{\alpha}<\hat{\beta}} = \pm 1 \)) is a frame decomposition of metric (5) on \( \mathcal{E}^{<\omega>} \), \( e_{\hat{\alpha}} \) is the standard distinguished basis on \( \mathcal{R}^{n_E} \), and the projection of torsion, \( T_{U} \), on base \( \mathcal{E}^{<\omega>} \) is defined as

\[
T_{U} = d\chi_{U} + \Omega_{U} \wedge \chi_{U} + \chi_{U} \wedge \Omega_{U} =
\]

\[
e_{\hat{\alpha}} \otimes \sum_{<\mu<\nu>} T^{\hat{\alpha}_{<\mu<\nu>} X^{<\mu>} \wedge X^{<\nu>}.
\]

For a fixed local adapted basis on \( U \subset \mathcal{E}^{<\omega>} \) we can identify components \( T^{\hat{\alpha}_{<\mu<\nu>} X^{<\mu>} \wedge X^{<\nu>} \) of torsion (35) with components of torsion (6) on \( \mathcal{E}^{<\omega>} \), i.e.

\[
T^{\hat{\alpha}_{<\mu<\nu>}} = T^{<\alpha<\mu<\nu>\nu>}. \tag{36}
\]

By straightforward calculation we obtain

\[
(\Delta \mathcal{R})_{U} = [(\Delta \mathcal{R})_{U}^{(\Gamma)}, (R_{T})_{U} + (R_{i})_{U}] \tag{36}
\]
where

\[(R_t)_{\mu} = \hat{\delta}_G T_{\mu} + *_{G}^{-1} [\Omega_{\mu}, *_{G} T_{\mu}], \quad (R_i)_{\mu} = *_{G}^{-1} [\chi_{\mu}, *_{G} R^{(\Gamma)}_{\mu}] .\]

Form \((R_i)_{\mu}\) from (36) is locally constructed by using components of the Ricci tensor (see (10)) as follows from decomposition on the local adapted basis \(X^{<\mu>} = \delta u^{<\mu>}\):

\[(R_i)_{\mu} = e_{\alpha} \otimes (-1)^{n_E+1} R_{<\lambda><\nu>} G^{\hat{\alpha}<\lambda>} \delta u^{<\mu>} .\]

We remark that for isotropic torsionless pseudo-Riemannian spaces the requirement that \((\Delta R)_{\mu} = 0\), i.e., imposing the connection (32) to satisfy Yang-Mills equations (23) (equivalently (28) or (29) we obtain [31, 31, 1] the equivalence of the mentioned gauge gravitational equations with the vacuum Einstein equations \(R_{ij} = 0\). In the case of ha–spaces with arbitrary given torsion, even considering vacuum gravitational fields, we have to introduce a source for gauge gravitational equations in order to compensate for the contribution of torsion and to obtain equivalence with the Einstein equations.

Considerations presented in this section constitute the proof of the following result

**Theorem 1** The Einstein equations (10) for ha–gravity are equivalent to Yang-Mills equations

\[(\Delta R) = J\] (37)

for the induced Cartan connection \(\Omega\) (see (32), (34)) in the bundle of local adapted affine frames \(\mathcal{A}_a (E)\) with source \(J_{\mu}\) constructed locally by using the same formulas (36) (for \((\Delta R)\)), where \(R_{<\alpha><\beta>}\) is changed by the matter source \(\tilde{E}_{<\alpha><\beta>} - \frac{1}{2} G_{<\alpha><\beta>} \tilde{E}\), where \(\tilde{E}_{<\alpha><\beta>} = k E_{<\alpha><\beta>} - \lambda G_{<\alpha><\beta>}\).

We note that this theorem is an extension for higher order anisotropic spaces of the Popov and Dikhin results [31] with respect to a possible gauge like treatment of the Einstein gravity. Similar theorems have been proved for locally anisotropic gauge gravity [10, 12, 51] and in the framework of some variants of locally (and higher order) anisotropic supergravity [44].

5 Nonlinear De Sitter Gauge Ha–Gravity

The equivalent reexpression of the Einstein theory as a gauge like theory implies, for both locally isotropic and anisotropic space–times, the non-semisimplicity of the gauge group, which leads to a nonvariational theory in the total space of the bundle of locally adapted affine frames. A variational gauge gravitational theory can be formulated by using a minimal extension of the affine structural group \(\mathcal{A}_{n_E} (R)\) to the de Sitter gauge group \(S_{n_E} = SO (n_E)\) acting on distinguished \(R^{n_E+1}\) space.
5.1 Nonlinear gauge theories of de Sitter group

Let us consider the de Sitter space \( \Sigma^{n_E} \) as a hypersurface given by the equations \( \eta_{AB} u^A u^B = -l^2 \) in the \((n+m)\)-dimensional spaces enabled with diagonal metric \( \eta_{AB}, \eta_{AA} = \pm 1 \) (in this subsection \( A, B, C, \ldots = 1, 2, \ldots, n_E + 1 \), \( n_E = n + m_1 + \ldots + m_z \)), where \( \{u^A\} \) are global Cartesian coordinates in \( \mathcal{R}^{n_E+1}; l > 0 \) is the curvature of de Sitter space. The de Sitter group \( S_\eta = SO_\eta(n_E + 1) \) is defined as the isometry group of \( \Sigma^{n_E} \)-space with \( \frac{n_E}{2}(n_E + 1) \) generators of Lie algebra \( so_\eta(n_E + 1) \) satisfying the commutation relations

\[
[M_{AB}, M_{CD}] = \eta_{AC} M_{BD} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC} + \eta_{BD} M_{AC}.
\]

(38)

Decomposing indices \( A, B, \ldots \) as \( A = (\hat{\alpha}, n_E + 1) \), \( B = (\hat{\beta}, n_E + 1) \), ..., the metric \( \eta_{AB} \) as \( \eta_{AB} = (\eta_{\hat{\alpha}\hat{\beta}}, \eta((n_E+1)(n_E+1)) \), and operators \( M_{AB} \) as \( M_{\hat{\alpha}\hat{\beta}} = F_{\hat{\alpha}\hat{\beta}} \) and \( P_{\hat{\alpha}} = l^{-1} M_{\hat{\alpha}n_E+1} \hat{\alpha} \); we can write (38) as

\[
\begin{align*}
[F_{\hat{\alpha}\hat{\beta}}, F_{\gamma\delta}] &= \eta_{\gamma\delta} F_{\hat{\alpha}\hat{\beta}} - \eta_{\delta\gamma} F_{\hat{\alpha}\hat{\beta}} + \eta_{\beta\delta} F_{\hat{\alpha}\gamma} - \eta_{\beta\gamma} F_{\hat{\alpha}\delta}, \\
[P_{\hat{\alpha}}, P_{\hat{\beta}}] &= -l^{-2} F_{\hat{\alpha}\hat{\beta}}, \quad [P_{\hat{\alpha}}, F_{\gamma\delta}] = \eta_{\alpha\beta} P_{\gamma} - \eta_{\gamma\alpha} P_{\delta},
\end{align*}
\]

where we have indicated the possibility to decompose \( so_\eta(n_E + 1) \) into a direct sum, \( so_\eta(n_E + 1) = so_\eta(n_E) \oplus V_{n_E}, \) where \( V_{n_E} \) is the vector space stretched on vectors \( P_{\hat{\alpha}} \). We remark that \( \Sigma^{n_E} = S_\eta/L_\eta \), where \( L_\eta = SO_\eta(n_E) \). For \( \eta_{AB} = \text{diag}(1, -1, -1, -1) \) and \( S_{10} = SO(1, 4), L_6 = SO(1, 3) \) is the group of Lorentz rotations.

Let \( W \left( \mathcal{E}, \mathcal{R}^{n_E+1}, S_\eta, P \right) \) be the vector bundle associated with principal bundle \( P \left( S_\eta, \mathcal{E} \right) \) on la-spaces. The action of the structural group \( S_\eta \) on \( E \) can be realized by using \( (n_E) \times (n_E) \) matrices with a parametrization distinguishing subgroup \( L_\eta \) :

\[
B = b B_L,
\]

(39)

where

\[
B_L = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix},
\]

\( L \in L_\eta \) is the de Sitter bust matrix transforming the vector \((0, 0, ..., \rho) \in \mathcal{R}^{n_E+1} \) into the arbitrary point \((V^1, V^2, ..., V^{n_E+1}) \in \Sigma^{n_E} \subset \mathcal{R}^{n_E+1} \) with curvature \( \rho \) (\( V_A V^A = -\rho^2, V^A = t^A \rho \)). Matrix \( b \) can be expressed as

\[
b = \begin{pmatrix} \delta_{\hat{\alpha}} & \hat{\omega}_{\hat{\alpha}} \\ \hat{\theta}_{\hat{\beta}} & \hat{\rho}_{\hat{\beta}} \end{pmatrix} + \begin{pmatrix} \hat{\omega}_{\hat{\alpha}} \\ \hat{\rho}_{\hat{\beta}} \end{pmatrix} t^{n_E+1}.
\]

The de Sitter gauge field is associated with a linear connection in \( W \), i.e., with a \( so_\eta(n_E + 1) \)-valued connection 1–form on \( \mathcal{E}^{<z>} \):

\[
\tilde{\Omega} = \begin{pmatrix} \omega_{\hat{\alpha}} & \hat{\omega}_{\hat{\alpha}} \\ \hat{\theta}_{\hat{\beta}} & \hat{\rho}_{\hat{\beta}} \end{pmatrix},
\]

(40)
where $\omega^\alpha_\beta \in so(n_E)_{(\eta)}$, $\theta^\alpha_\beta \in R^{n_E}$, $\hat{\theta}^\alpha_\beta \in \eta_{\beta \alpha} \hat{\theta}^\alpha$.

Because $S_{(\eta)}$-transforms mix $\omega^\alpha_\beta$ and $\theta^\alpha$ fields in (40) (the introduced parametrization is invariant on action on $SO_{(\eta)}(n_E)$ group we cannot identify $\omega^\alpha_\beta$ and $\theta^\alpha$, respectively, with the connection $\Gamma^{<\alpha><\beta><\gamma}$ and the fundamental form $\chi^{<\alpha>}$ in $\mathcal{E}^{<\gamma>}$ (as we have for (32) and (34)). To avoid this difficulty we consider $[35, 29]$ a nonlinear gauge realization of the de Sitter group $S_{(\eta)}$, namely, we introduce into consideration the nonlinear gauge field

$$\Omega = b^{-1} \Omega b + b^{-1} db = \begin{pmatrix} \Gamma^\alpha_\beta & \hat{\theta}^\alpha \\ \hat{\theta}^\alpha_\beta & 0 \end{pmatrix}, \quad (41)$$

where

$$\Gamma^\alpha_\beta = \omega^\alpha_\beta - \left( t^\alpha D^\beta - t^\beta D^\alpha \right) / (1 + t^{n_E + 1}),$$

$$\theta^\alpha = t^{n_E + 1} \hat{\theta}^\alpha + D t^\alpha - t^\alpha \left( dt^{n_E + 1} + \hat{\theta}^\alpha t^\beta \right) / (1 + t^{n_E + 1}),$$

$$D t^\alpha = dt^\alpha + \omega^\alpha_\beta \hat{\theta}^\alpha t^\beta.$$

The action of the group $S(\eta)$ is nonlinear, yielding transforms $\Gamma' = L' \Gamma (L')^{-1} + L' d (L')^{-1}$, $\theta' = L \theta$, where the nonlinear matrix-valued function $L' = L' (t^{<\alpha>}, b, B_T)$ is defined from $B_b = b B_{L'}$ (see parametrization (39)).

Now, we can identify components of (41) with components of $\Gamma^{<\alpha><\beta><\gamma}$ and $\chi^{<\alpha>}$ on $\mathcal{E}^{<\gamma>}$ and induce in a consistent manner on the base of bundle $W \left( \mathcal{E}, R^{n_E + 1}, S_{(\eta)}, P \right)$ the ha–gravity.

### 5.2 Dynamics of the nonlinear $S(\eta)$ ha–gravity

Instead of the gravitational potential (32), we introduce the gravitational connection (similar to (41))

$$\Gamma = \begin{pmatrix} \hat{\Gamma}^\alpha_\beta & l_0^{-1} \chi^\alpha \\ l_0^{-1} \chi^\alpha & 0 \end{pmatrix}, \quad (42)$$

where

$$\hat{\Gamma}^\alpha_\beta = \Gamma^{<\alpha><\beta><\gamma} \delta^{<\mu>},$$

$$\Gamma^{<\alpha><

\chi^\alpha = \chi^\alpha \mu \delta u^\mu, \text{ and } G_{\alpha \beta} = \chi^\alpha \chi^\beta \eta_{\alpha \beta}, \text{ and } \eta_{\alpha \beta} \text{ is parametrized as}$$

$$\eta_{\alpha \beta} = \begin{pmatrix} \eta_{ij} & 0 & ... & 0 \\ 0 & \eta_{a_1 b_1} & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \eta_{a_n b_n} \end{pmatrix},$$

$\eta_{ij} = (1, -1, ..., -1), ... \eta_{ij} = (\pm 1, \pm 1, ..., \pm 1), ..., l_0 \text{ is a dimensional constant.}$
The curvature of (42), $\mathcal{R}^{(\Gamma)} = d\Gamma + \Gamma \wedge \Gamma$, can be written as

$$\mathcal{R}^{(\Gamma)} = \begin{pmatrix} R^\alpha_\beta + l_0^{-1} \pi^\alpha_\beta & l_0^{-1} T^\alpha_\beta \\ l_0^{-1} T^\beta_\alpha & 0 \end{pmatrix},$$

(43)

where

$$\pi^\alpha_\beta = \chi^\alpha_\beta \wedge \chi^\beta_\alpha, \quad R^\alpha_\beta = \frac{1}{2} R^\alpha_\beta \delta u^\alpha <\mu> \wedge \delta u^\nu <\nu>,$$

and

$$R^\alpha_\beta \delta u^\alpha <\mu> \wedge \delta u^\nu <\nu>$$

(see (7), the components of d-curvatures). The de Sitter gauge group is semisimple and we are able to construct a variational gauge gravitational locally anisotropic theory (bundle metric (20) is nondegenerate). The Lagrangian of the theory is postulated as

$$L = L_{(G)} + L_{(m)}$$

where the gauge gravitational Lagrangian is defined as

$$L_{(G)} = \frac{1}{4\pi} Tr \left( \mathcal{R}^{(\Gamma)} \wedge \star_G \mathcal{R}^{(\Gamma)} \right) = \mathcal{L}_{(G)} |G|^{1/2} \delta^{nE} u,$$

(44)

and

$$\mathcal{L}_{(G)} = \frac{1}{2l^2} T^\alpha_< \delta u^\alpha_\nu > T^\alpha_\nu <\nu> + \frac{1}{8\lambda} \mathcal{R}^\alpha_\beta \delta u^\alpha_\nu <\nu> - \frac{1}{l^2} \left( \bar{R} (\Gamma) - 2\lambda_1 \right),$$

where

$$T^\alpha_< \delta u^\alpha_\nu > = \chi^\alpha_< \delta u^\alpha_\nu >$$

the gravitational constant $l^2$ in (44) satisfies the relations $l^2 = 2l_0^2 \lambda, \lambda_1 = \lambda - 3/l_0$, $Tr$ denotes the trace on $\hat{\alpha}, \hat{\beta}$ indices, and the matter field Lagrangian is defined as

$$L_{(m)} = -\frac{1}{2} Tr \left( \Gamma \wedge \star_G I \right) = \mathcal{L}_{(m)} |G|^{1/2} \delta^{nE} u,$$

(45)

and

$$\mathcal{L}_{(m)} = -\frac{1}{2} \Gamma^\alpha_\beta \delta u^\alpha_\nu <\nu> - t^\alpha_\mu \delta u^\alpha_\mu <\nu>.$$

The matter field source $I$ is obtained as a variational derivation of $\mathcal{L}_{(m)}$ on $\Gamma$ and is parametrized as

$$I = \left( \begin{array}{cc} S^\alpha_\beta & -l_0 t^\alpha_\beta \\ -l_0 t^\beta_\alpha & 0 \end{array} \right),$$

(46)

with $t^\alpha_\mu = t^\alpha_\mu \delta u^\alpha_\nu$ and

$$S^\alpha_\beta = S^\alpha_\beta \delta u^\alpha_\mu <\nu>$$

being respectively the canonical tensors of energy-momentum and spin density. Because of the contraction of the "interior" indices $\hat{\alpha}, \hat{\beta}$ in (44) and (45) we used the Hodge operator $\star_G$ instead of $\star_H$ (hereafter we consider $\star_G = \star$).

Varying the action

$$S = \int |G|^{1/2} \delta^{nE} u \left( \mathcal{L}_{(G)} + \mathcal{L}_{(m)} \right)$$
on the \( \Gamma \)-variables (36), we obtain the gauge–gravitational field equations:

\[
d (\ast \mathcal{R}^{(\Gamma)}) + \Gamma \wedge (\ast \mathcal{R}^{(\Gamma)}) - (\ast \mathcal{R}^{(\Gamma)}) \wedge \Gamma = -\lambda (\ast I).
\] (47)

Specifying the variations on \( \Gamma \hat{\alpha} \hat{\beta} \) and \( l\hat{\alpha} \)-variables, we rewrite (47) as

\[
\hat{D} (\ast \mathcal{R}^{(\Gamma)}) + \frac{2\lambda}{l^2} (\hat{D} (\ast \pi) + \chi \wedge (\ast T^T) - (\ast T) \wedge \chi^T) = -\lambda (\ast S),
\] (48)

\[
\hat{D} (\ast T) - (\ast \mathcal{R}^{(\Gamma)}) \wedge \chi - \frac{2\lambda}{l^2} (\ast \pi) \wedge \chi = \frac{l^2}{2} \left( \ast t + \frac{1}{\lambda} \ast \tau \right),
\] (49)

where

\[
T^t = \{ T_{\hat{\alpha}} = \eta_{\hat{\alpha} \hat{\beta}} T_{\hat{\beta}}, T_{\hat{\beta}} = \frac{1}{2} T_{\hat{\alpha} \hat{\beta}} \delta u_{\hat{\alpha}} \delta u_{\hat{\beta}} \},
\]

\[
\chi^T = \{ \chi_{\hat{\alpha}} = \eta_{\hat{\alpha} \hat{\beta}} \chi_{\hat{\beta}}, \chi_{\hat{\beta}} = \chi_{\hat{\alpha}} \delta u_{\hat{\alpha}} \}, \quad \hat{D} = d + \hat{\Gamma}
\]

(\( \hat{\Gamma} \) acts as \( \Gamma \hat{\alpha} \hat{\beta} \) on indices \( \hat{\gamma}, \hat{\delta}, \ldots \) and as \( \Gamma^{<\alpha><\beta><\mu>} \) on indices \( <\gamma>, <\delta>, \ldots \)). In (49), \( \tau \) defines the energy–momentum tensor of the \( S_{(\eta)} \)-gauge gravitational field \( \hat{\Gamma} \):

\[
\tau_{<\mu><\nu>} (\hat{\Gamma}) = \frac{1}{2} Tr \left( R_{<\mu><\alpha>} R_{<\nu><\alpha>} - \frac{1}{4} R_{<\alpha><\beta><\nu><\mu>} G_{<\nu><\mu>} \right).
\] (50)

Equations (47) (or equivalently (48),(49)) make up the complete system of variational field equations for nonlinear de Sitter gauge gravity with higher order anisotropy. They can be interpreted as a generalization of gauge like equations for ha–gravity [51] (equivalently, of gauge gravitational equations (37)) to a system of gauge field equations with dynamical torsion and corresponding spin-density source.

A. Tseytlin [35] presented a quantum analysis of the isotropic version of equations (48) and (49). Of course, the problem of quantizing gravitational interactions is unsolved for both variants of locally anisotropic and isotropic gauge de Sitter gravitational theories, but we think that the generalized Lagrange version of \( S_{(\eta)} \)-gravity is more adequate for studying quantum radiational and statistical gravitational processes. This is a matter for further investigations.

Finally, we remark that we can obtain a nonvariational Poincare gauge gravitational theory on ha–spaces if we consider the contraction of the gauge potential (42) to a potential with values in the Poincare Lie algebra

\[
\Gamma = \left( \begin{array}{ll}
\Gamma_{\hat{\alpha}}^{\hat{\beta}} & l_0^{-1} \chi_{\hat{\alpha}} \\
l_0^{-1} \chi_{\hat{\beta}} & 0
\end{array} \right) \rightarrow \Gamma = \left( \begin{array}{ll}
\Gamma_{\hat{\alpha}}^{\hat{\beta}} & l_0^{-1} \chi_{\hat{\alpha}} \\
l_0^{-1} \chi_{\hat{\beta}} & 0
\end{array} \right).
\]

Isotropic Poincare gauge gravitational theories are studied in a number of papers (see, for example, [59, 35, 21, 24]). In a manner similar to considerations presented in this work, we can generalize Poincare gauge models for spaces with local anisotropy.
6 Ha–Gravitational Gauge Instantons

The existence of self-dual, or instanton, topologically nontrivial solutions of Yang-Mills equations is a very important physical consequence of gauge theories. All known instanton-type Yang-Mills and gauge gravitational solutions (see, for example, [35, 29]) are locally isotropic. A variational gauge-gravitational extension of la-gravity makes possible a straightforward application of techniques of constructing solutions for first order gauge equations for the definition of locally anisotropic gravitational instantons. This section is devoted to the study of some particular instanton solutions of the gauge gravitational theory on la-space.

Let us consider the Euclidean formulation of the $S_{(η)}$-gauge gravitational theory by changing gauge structural groups and flat metric:

$$SO_{(η)}(n_E + 1) \rightarrow SO(n_E + 1), SO_{(η)}(n_E) \rightarrow SO(n_E), η_{AB} \rightarrow −δ_{AB}.$$  

Self-dual (anti-self-dual) conditions for the curvature (43)

$$R^{αβ} = R^{αβ} (−R^{αβ})$$

can be written as a system of equations

$$\left( R^{α} − l_0^2 π^{α}_β \right) = ± \left( R^{α} − l_0^2 π^{α}_β \right)$$

(51)

$$T^{α} = ± T^{α}$$

(52)

(the “−” refers to the anti-self-dual case), where the “−” before $l_0^2$ appears because of the transition of the Euclidean negatively defined metric $−δ_{αβ}$, which leads to $χ^{α}_β (−) \rightarrow iχ^{α}_β (E), π \rightarrow −π (E)$ (we shall omit the index $(E)$ for Euclidean values).

For solutions of (51) and (52) the energy–momentum tensor (50) is identically equal to zero. Vacuum equations (47) and (48), when source is $I ≡ 0$ (see (46)), are satisfied as a consequence of generalized Bianchi identities for the curvature (43). The mentioned solutions of (51) and (52) realize a local minimum of the Euclidean action

$$S = \frac{1}{8λ} \int \left| G^{1/2} \right| δ^{ne} \left\{ (R (Γ) − l_0^2 π)^2 + 2T^2 \right\},$$

where $T^2 = T^{α <μ><ν} T_{α <μ><ν}$ is extremal on the topological invariant (Pontryagin index)

$$p_2 = −\frac{1}{8π^2} \int Tr \left( R^{(Γ)} \land R^{(Γ)} \right) = −\frac{1}{8π^2} \int Tr \left( ̂R \land ̂R \right).$$

For the Euclidean de Sitter spaces, when

$$R = 0 \quad \{ T = 0, ̂R^{αβ}_{<μ><ν>} = −\frac{2}{l_0^2} χ^{[α}_{<μ>χ^{β]}_{<ν>} } \}$$

(53)

we obtain the absolute minimum, $S = 0$. 

24
We emphasize that for \( R^{<\alpha><\mu><\beta><\nu>} = (2/l_0^2) \delta^{<\alpha><\mu>} G^{<\nu><\beta>} \) torsion vanishes. Torsionless instantons also have another interpretation. For \( T^{<\alpha><\beta><\gamma>} = 0 \) contraction of equations (51) leads to Einstein equations with cosmological \( \lambda \)-term (as a consequence of generalized Ricci identities):

\[
R^{<\alpha><\beta><\mu><\nu>} - R^{<\mu><\nu><\alpha><\beta>} = \frac{3}{2} \{ R^{<\alpha><\beta><\mu><\nu>} - R^{<\nu><\mu><\alpha><\beta>} \}
\]

So, in the Euclidean case the locally anisotropic vacuum Einstein equations are a subset of instanton solutions.

Now, let us study the \( SO(n_E) \) solution of equations (51) and (52). We consider the spherically symmetric ansatz (in order to point out the connection between high-dimensional gravity and ha–gravity the \( N \)–connection structure is chosen to be trivial, i.e. \( N^a_j(u) \equiv 0 \)):

\[
\Gamma^\alpha_\beta^\mu = a(u) (u^\alpha \delta^\beta_\mu - u^\beta \delta^\alpha_\mu) + q(u) \epsilon^\alpha_\beta^\mu u^\nu,
\]

\[
\chi^\alpha_{<\alpha>} = f(u) \delta^\alpha_{<\alpha>} + n(u) u^\alpha u_\alpha,
\]

where \( u = u^{<\alpha>} u^{<\beta>} G^{<\alpha><\beta>} = \hat{x}_i \hat{x}_i + \hat{y}_a \hat{y}_a \), and \( a(u), q(u), f(u) \) and \( n(u) \) are some scalar functions. Introducing (54) into (51) and (52), we obtain, respectively,

\[
u \left( \pm \frac{dq}{du} - a^2 - q^2 \right) + 2 (a \pm q) + l_0^{-1} f^2, \quad (55)
\]

\[
2d (a \mp q)/du + (a \pm q)^2 - l_0^{-1} fn = 0, \quad (56)
\]

\[
2 \frac{df}{du} + f (a \mp 2q) + n (au - 1) = 0. \quad (57)
\]

The traceless part of the torsion vanishes because of the parametrization (54), but in the general case the trace and pseudo-trace of the torsion are not identical to zero:

\[
T^\mu = q^{(0)} u^\mu (-2df/du + n - a (f + vu)),
\]

\[
\Gamma^\mu = q^{(1)} u^\mu (2qf),
\]

\( q^{(0)} \) and \( q^{(0)} \) are constant. Equation (52) or (57) establishes the proportionality of \( T^\mu \) and \( \Gamma^\mu \). As a consequence we obtain that the \( SO(n + m) \) solution of (52) is torsionless if \( q(u) = 0 \) of \( f(u) = 0 \).

Let first analyze the torsionless instantons, \( T^{<\alpha><\beta>} = 0 \). If \( f = 0 \), then from (56) one has two possibilities: (a) \( n = 0 \) leads to nonsense because \( \chi^\alpha_{<\alpha>} = 0 \) or \( G_{\alpha\beta} = 0 \). b) \( a = u^{-1} \) and \( n(u) \) is an arbitrary scalar function; we have from (56) \( a \mp q = 2/(a + C^2) \) or \( q = \pm 2/u (u + C^2) \), where \( C = const. \) If \( q(u) = 0 \), we obtain the de Sitter space (53) because equations
(55) and (56) impose vanishing of both self-dual and anti-self-dual parts of
\( R^\hat{\alpha} /_{\hat{\beta}} - l_0^2 \pi^\hat{\alpha} /_{\hat{\beta}} \), so, as a consequence, \( R^\hat{\alpha} /_{\hat{\beta}} - l_0^2 \pi^\hat{\alpha} /_{\hat{\beta}} \equiv 0 \). There is an
infinite number of \( SO(n_E) \)-symmetrical solutions of (53):
\[
f = l_0 [a (2 - au)]^{1/2}, \quad n = l_0 \{2 \frac{da}{du} + \frac{a^2}{[a (2 - au)]^{1/2}} \},
\]
a\((u)\) is a scalar function.

To find instantons with torsion, \( T^\alpha /_{\beta \gamma} \neq 0 \), is also possible. We present
the \( SO(4) \) one-instanton solution, obtained in\(^{[29]}\) (which in the case of \( H^4 \)-space parametrized by local coordinates \( (x^1, x^2, y^1, y^2) \), with \( u = x^1 x^1 + x^2 x_2 + y^1 y^1 + y^2 y_2 \) :
\[
a = a_0 (u + c^2)^{-1}, \quad q = \mp q_0 (u + c^2)^{-1}
\]
\[
f = l_0 (\alpha u + \beta)^{1/2} / (u + c^2), \quad n = c_0 / (u + c^2)(\gamma u + \delta)^{1/2}
\]
where
\[
a_0 = -1/18, \quad q_0 = 5/6, \quad \alpha = 266/81, \quad \beta = 8/9,
\]
\[
\gamma = 10773/11858, \quad \delta = 1458/5929.
\]
We suggest that local regions with \( T^\alpha /_{\beta \gamma} \neq 0 \) are similarly to Abrikosov
vortices in superconductivity and the appearance of torsion is a possible
realization of the Meisner effect in gravity (for details and discussions on
the superconducting or Higgs-like interpretation of gravity see\(^{[35, 29]}\)).

7 Nearly Autoparallel Maps of La–Spaces

The aim of the section is to present a generalization of the nearly geodesic
map (ng–map) theory\(^{[33]}\) and nearly autoparallel map (na–map) theory
\(^{[34, 37, 39, 52, 53, 57]}\) by introducing into consideration maps of vector
bundles provided with compatible N–connection, d–connection and metric
structures. For simplicity, the basic definitions and theorems will be formulat-
ed only for locally anisotropic spaces. The transition to higher order
anisotropies can be made in a straightforward manner by introducing higher
order distinguishing of indices, \( e \alpha \rightarrow < \alpha > \), corresponding to a higher order
distinguishing of nonlinear connection and of basic geometric objects.

Our geometric arena consists from pairs of open regions \((U, \bar{U})\) of la–spaces,
\( U \subseteq \xi \), \( \bar{U} \subseteq \bar{\xi} \), and 1–1 local maps \( f : U \rightarrow \bar{U} \) given by functions \( f^\alpha (u) \)
of smoothly class \( C^r(U) (r > 2) \), or \( r = \omega \) for analytic functions) and
their inverse functions \( f^\mu (v) \) with corresponding non–zero Jacobians in every
point \( u \in U \) and \( v \in \bar{U} \).

We consider that two open regions \( U \) and \( \bar{U} \) are attributed to a common
for f–map coordinate system if this map is realized on the principle of
coordinate equality \( q(u^\alpha) \rightarrow \bar{q}(u^\alpha) \) for every point \( q \in \bar{U} \) and its f–image \( \bar{q} \in \bar{U} \).
We note that all calculations included in this work will be local in nature
and taken to refer to open subsets of mappings of type $\xi \supset U \xrightarrow{f} U \subset \xi$. For simplicity, we suppose that in a fixed common coordinate system for $U$ and $U$ spaces $\xi$ and $\xi$ are characterized by a common N–connection structure (in consequence of $(5)$ by a corresponding concordance of d–metric structure), i.e.

$$N^a_j(u) = N^a_j(u) = N^a_j(u),$$

which leads to the possibility to establish common local bases, adapted to a given N–connection, on both regions $U$ and $U$. We consider that on $\xi$ it is defined the linear d–connection structure with components $\Gamma^\alpha_{\beta\gamma}$. On the space $\xi$ the linear d–connection is considered to be a general one with torsion

$$T^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} + w^\alpha_{\beta\gamma},$$

and nonmetricity

$$K^\alpha_{\beta\gamma} = G^\alpha_{\beta\gamma}.$$

Geometrical objects on $\xi$ are specified by underlined symbols (for example, $A^\alpha, B^\alpha_{\beta}$) or underlined indices (for example, $A^a, B^a_{\beta}$).

For our purposes it is convenient to introduce auxiliary symmetric d–connections, $\gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\gamma\beta}$ on $\xi$ and $\gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\gamma\beta}$ on $\xi$ defined, correspondingly, as

$$\Gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\beta\gamma} + T^\alpha_{\beta\gamma} \quad \text{and} \quad \Gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\beta\gamma} + T^\alpha_{\beta\gamma}.$$

We are interested in definition of local 1–1 maps from $U$ to $U$ characterized by symmetric, $P^\alpha_{\beta\gamma}$, and antisymmetric, $Q^\alpha_{\beta\gamma}$, deformations:

$$\gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\beta\gamma} + P^\alpha_{\beta\gamma},$$

and

$$T^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + Q^\alpha_{\beta\gamma}. (60)$$

The auxiliary linear covariant derivations induced by $\gamma^\alpha_{\beta\gamma}$ and $\gamma^\alpha_{\beta\gamma}$ are denoted respectively as $(\gamma)D$ and $(\gamma)\bar{D}$.

Let introduce this local coordinate parametrization of curves on $U$:

$$u^\alpha = u^\alpha(\eta) = (x^i(\eta), y^j(\eta)), \; \eta_1 < \eta < \eta_2,$$

where corresponding tangent vector field is defined as

$$v^\alpha = \frac{du^\alpha}{d\eta} = \left(\frac{dx^i(\eta)}{d\eta}, \frac{dy^j(\eta)}{d\eta}\right).$$

**Definition 1** Curve $l$ is called auto parallel, $a$–parallel, on $\xi$ if its tangent vector field $v^\alpha$ satisfies $a$–parallel equations:

$$vDv^\alpha = v^\beta(\gamma)D_\beta v^\alpha = \rho(\eta)v^\alpha,$$

where $\rho(\eta)$ is a scalar function on $\xi$.  27
Let curve \( l \subset \xi \) is given in parametric form as \( u^\alpha = u^\alpha(\eta) \), \( \eta_1 < \eta < \eta_2 \) with tangent vector field \( v^\alpha = \frac{du^\alpha}{d\eta} \neq 0 \). We suppose that a 2-dimensional distribution \( E_2(l) \) is defined along \( l \), i.e. in every point \( u \in l \) is fixed a 2-dimensional vector space \( E_2(l) \subset \xi \). The introduced distribution \( E_2(l) \) is coplanar along \( l \) if every vector \( \frac{\partial}{\partial u^b(0)}(u^\alpha(\eta)) \subset E_2(l), u^\beta(0) \subset l \) rests contained in the same distribution after parallel transports along \( l \), i.e. \( \frac{\partial}{\partial u^b(\eta)}(u^\alpha(\eta)) \subset E_2(l) \).

**Definition 2** A curve \( l \) is called nearly autoparallel, or in brief an na–parallel, on space \( \xi \) if a coplanar along \( l \) distribution \( E_2(l) \) containing tangent to \( l \) vector field \( v^\alpha(\eta) \), i.e. \( v^\alpha(\eta) \subset E_2(l) \), is defined.

We can define nearly autoparallel maps of la–spaces as an anisotropic generalization (see also [58, 55] and na–maps [36, 52, 57, 53]):

**Definition 3** Nearly autoparallel maps, na–maps, of la–spaces are defined as local 1–1 mappings of v–bundles, \( \xi \rightarrow \xi \), changing every a–parallel on \( \xi \) into a na–parallel on \( \xi \).

Now we formulate the general conditions when deformations (59) and (60) characterize na–maps : Let a–parallel \( l \subset U \) is given by functions \( u^\alpha = u^\alpha(\eta) \), \( v^\alpha = \frac{du^\alpha}{d\eta} \), \( \eta_1 < \eta < \eta_2 \), satisfying equations (61). We suppose that to this a–parallel corresponds a na–parallel \( l \subset U \) given by the same parameterization in a common for a chosen na–map coordinate system on \( U \) and \( U \). This condition holds for vectors \( v_1^\alpha = v D v^\alpha \) and \( v_2^\alpha = v D v^\alpha(1) \) satisfying equality

\[
v_2^\eta = a(\eta) v^\alpha + b(\eta) v_1^\alpha
\]

for some scalar functions \( a(\eta) \) and \( b(\eta) \) (see Definitions 2 and 3). Putting splittings (59) and (4.3) into expressions for \( v_1^\alpha \) and \( v_2^\alpha \) in (62) we obtain:

\[
v^\beta v^\gamma v^\delta(D_{\beta} P^\alpha_{\gamma\delta} + P^\alpha_{\beta\tau} P^\tau_{\gamma\delta} + Q^\alpha_{\beta\tau} P^\tau_{\gamma\delta}) = bv^\gamma v^\delta P^\alpha_{\gamma\delta} + av^\alpha,
\]

where

\[
b(\eta, v) = b - 3\rho, \quad a(\eta, v) = a + b\rho - v^b \partial_v \rho - \rho^2
\]

are called the deformation parameters of na–maps.

The algebraic equations for the deformation of torsion \( Q^\alpha_{\beta\tau} \) should be written as the compatibility conditions for a given nonmetricity tensor \( K_{\alpha\beta\gamma} \) on \( \xi \) (or as the metricity conditions if d–connection \( D_\alpha \) on \( \xi \) is required to be metric):

\[
D_\alpha G_{\beta\gamma} - P^\delta_{\alpha(\beta G_\gamma)\delta} - K_{\alpha\beta\gamma} = Q^\delta_{\alpha(\beta G_\gamma)\delta};
\]

where \( (\ldots) \) denotes the symmetrical alternation.

So, we have proved this
Theorem 2  The na–maps from la–space $\xi$ to la–space $\xi$ with a fixed common nonlinear connection $N^\alpha_j(u) = \sum_j^\alpha(u)$ and given d–connections, $\Gamma^\alpha_{\beta\gamma}$ on $\xi$ and $\sum^\alpha_{\beta\gamma}$ on $\xi$ are locally parametrized by the solutions of equations (63) and (65) for every point $u^\alpha$ and direction $v^\alpha$ on $U \subset \xi$.

We call (63) and (65) the basic equations for na–maps of la–spaces. They generalize the corresponding Sinyukov’s equations [33] for isotropic spaces provided with symmetric affine connection structure.

8 Classification of Na–Maps of La–Spaces

Na–maps are classed on possible polynomial parametrizations on variables $v^\alpha$ of deformations parameters $a$ and $b$ (see (63) and (64)).

Theorem 3  There are four classes of na–maps characterized by corresponding deformation parameters and tensors and basic equations:

1. for $\text{na}_0$–maps, $\pi_0$–maps,
   \[ P^\alpha_{\beta\gamma}(u) = \psi(\delta^\alpha_{\beta\gamma}) \]
   (where $\delta^\alpha_{\beta\gamma}$ is Kronecker symbol and $\psi(\beta) = \psi(\beta(u))$ is a covariant vector field);

2. for $\text{na}_1$–maps
   \[ a(u,v) = a_{\alpha\beta}(u)v^\alpha v^\beta, \quad b(u,v) = b_{\alpha}(u)v^\alpha \]
   and $P^\alpha_{\beta\gamma}(u)$ is the solution of equations
   \[ D(\alpha\beta\gamma) + P^\tau_{\alpha\beta\gamma} + P^\tau_{\beta\gamma} = b_{\alpha}(u)v^\alpha + a_{\alpha\beta\gamma} \]

3. for $\text{na}_2$–maps
   \[ a(u,v) = a_{\beta}(u)v^\beta, \quad b(u,v) = b_{\alpha}(u)v^\alpha \]
   and $F^\alpha_{\beta}(u)$ is the solution of equations
   \[ D(\alpha\beta\gamma) = \psi(\delta^\alpha_{\beta\gamma}) + \sigma_{\alpha}(F^\gamma_{\beta}) \]

4. for $\text{na}_3$–maps
   \[ b(u,v) = b_{\alpha\beta\gamma}(u)v^\alpha v^\beta v^\gamma \]
   \[ P^\alpha_{\beta\gamma}(u) = \psi(\delta^\alpha_{\beta\gamma}) + \sigma_{\alpha\beta\gamma}(\varphi^\alpha) \]
   where $\varphi^\alpha$ is the solution of equations
   \[ D(\alpha\beta\gamma) = \nu\delta^\alpha_{\beta\gamma} + \mu_{\beta\gamma}(\varphi^\alpha) + \varphi^\gamma Q^\alpha_{\beta\gamma} \]

$\alpha_{\beta\gamma}(u), \sigma_{\alpha\beta\gamma}(u), \psi_{\beta}(u), \nu(u)$ and $\mu_{\beta}(u)$ are d–tensors.
Proof. We sketch the proof respectively for every point in the theorem:

1. It is easy to verify that a–parallel equations (61) on ξ transform into similar ones on ξ if and only if deformations (4.2) with deformation d–tensors of type \( P^\alpha_{\beta \gamma} (u) = \psi(\beta \delta^\alpha_{\gamma}) \) are considered.

2. Using corresponding to \( na_{(1)} \)–maps parametrizations of \( a(u, v) \) and \( b(u, v) \) (see conditions of the theorem) for arbitrary \( v^\alpha \neq 0 \) on \( U \in \xi \) and after a redefinition of deformation parameters we obtain that equations (63) hold if and only if \( P^\alpha_{\beta \gamma} \) satisfies (60).

3. In a similar manner we obtain basic \( na_{(2)} \)–map equations (67) from (63) by considering \( na_{(2)} \)–parametrizations of deformation parameters and d–tensor.

4. For \( na_{(3)} \)–maps we must take into consideration deformations of torsion (4.3) and introduce \( na_{(3)} \)–parametrizations for \( b(u, v) \) and \( P^\alpha_{\beta \gamma} \) into the basic \( na \)–equations (63). The last ones for \( na_{(3)} \)–maps are equivalent to equations (68) (with a corresponding redefinition of deformation parameters).

We point out that for \( \pi_{(0)} \)–maps we have not differential equations on \( P^\alpha_{\beta \gamma} \) (in the isotropic case one considers a first order system of differential equations on metric \([33]\); we omit constructions with deformation of metric in this section).

To formulate invariant conditions for reciprocal \( na \)–maps (when every a–parallel on \( \xi \) is also transformed into \( na \)–parallel on \( \xi \) ) it is convenient to introduce into consideration the curvature and Ricci tensors defined for auxiliary connection \( \gamma^\rho_{\alpha \beta \gamma} : \)

\[
\gamma^\delta_{\alpha \beta \gamma} = \delta_{\beta \gamma} \alpha + \gamma^\delta_{\rho \beta \gamma} \alpha + \gamma^\delta_{\alpha \phi} w^\phi_{\beta \gamma}
\]

and, respectively, \( r_{\alpha \tau} = r^{\gamma}_{\alpha \gamma \tau} \), where \([ \ ]\) denotes antisymmetric alternation of indices, and to define values:

\[
(0) T^\mu_{\alpha \beta} = \Gamma^\mu_{\alpha \beta} - T^\mu_{\alpha \beta} - \frac{1}{(n + m + 1)}(\delta^\mu_{\alpha} \Gamma^\delta_{\beta \gamma} - \delta^\mu_{\alpha} T^\delta_{\beta \gamma}),
\]

\[
(0) W^\tau_{\alpha \beta \gamma} = r^\tau_{\alpha \beta \gamma} + \frac{1}{n + m + 1} \left[ \gamma^\tau_{\varphi \tau} \delta^\tau_{\alpha \beta \gamma} w^\varphi_{\beta \gamma} - (\delta^\tau_{\alpha} r^\tau_{\beta \gamma} + \delta^\tau_{\gamma} r^\tau_{\alpha \beta} - \delta^\tau_{\beta} r^\tau_{\alpha \gamma}) \right] - \frac{1}{(n + m + 1)^2} \left[ \delta^\tau_{\alpha} (2 \gamma^\tau_{\varphi \tau} w^\varphi_{\beta \gamma} - \gamma^\tau_{\beta \gamma} w^\varphi_{\beta \gamma}) + \delta^\tau_{\gamma} (2 \gamma^\tau_{\varphi \tau} w^\varphi_{\beta \alpha} - \gamma^\tau_{\alpha \beta} w^\varphi_{\gamma \tau}) - \delta^\tau_{\beta} (2 \gamma^\tau_{\varphi \tau} w^\varphi_{\alpha \gamma} - \gamma^\tau_{\alpha \beta} w^\varphi_{\gamma \tau}) \right],
\]

\[
(3) T^\delta_{\alpha \beta} = \gamma^\delta_{\alpha \beta} + \epsilon \varphi^\tau_{\delta} D_{\beta} q_{\tau} + \frac{1}{n + m} \left( \delta^\alpha_{\beta} - \epsilon \varphi^\delta q_{\alpha} \right) [\gamma^\tau_{\beta \tau} + \epsilon \varphi^\tau_{\gamma} D_{\gamma} q_{\lambda} - \frac{1}{n + m} (\delta^\delta_{\beta} - \epsilon \varphi^\delta q_{\beta} [\gamma^\tau_{\alpha \tau} +
\]

30
\[
\begin{align*}
&\epsilon \varphi^{\gamma(\gamma)} D_\alpha q_\tau + \frac{1}{n + m - 1} q_\alpha \left(\epsilon \varphi^{\gamma(\gamma)} \gamma_{\lambda} + \varphi^{\lambda} \varphi^{\gamma(\gamma)} D_\tau q_\lambda\right), \\
&\left(3\right) \hat{W}_{\beta\gamma} = \rho_{\beta\gamma\delta} + \epsilon \varphi^{\alpha} q_\tau \rho_{\beta\gamma\delta} + \left(\Delta_\delta - \epsilon \varphi^{\alpha} q_\delta\right) p_{\beta\gamma} - \left(\Delta_\delta - \epsilon \varphi^{\alpha} q_\delta\right) p_{\gamma\delta}, \\
&\left(n + m - 2\right) \rho_{\alpha\beta} = -\rho_{\alpha\beta} - \epsilon q_\gamma \varphi^{\tau(\gamma)} + \frac{1}{n + m} \left[\rho_{\tau,\alpha\beta}^{\tau} - \epsilon q_\gamma \varphi^{\tau(\gamma)} + \epsilon q_\gamma \varphi^{\tau(\gamma)} + \epsilon q_\gamma \varphi^{\tau(\gamma)} \rho_{\gamma\beta}\right]. \\
\end{align*}
\]

where \( q_\alpha \varphi^{\alpha} = \epsilon = \pm 1 \),

\[
\rho_{\beta\gamma\delta} = r_{\beta\gamma\delta} + \frac{1}{2} \left(\psi(\beta(\varphi)) + \sigma_{\beta\varphi} \varphi^{\tau}\right) w^{\tau}_{\gamma\delta}
\]

( for a similar value on \( \xi \) we write \( \hat{W}_{\beta\gamma} = \hat{W}_{\beta\gamma}^{\tau} - \frac{1}{2} \left(\psi(\beta(\varphi)) - \sigma_{\beta\varphi} \varphi^{\tau}\right) w^{\tau}_{\gamma\delta} \) )

and \( \rho_{\alpha\beta} = \rho_{\alpha\beta}^{\tau} .

Similar values,

\[
\left(0\right) T_{\alpha\beta\gamma} = \left(0\right) \hat{T}_{\alpha\beta\gamma}, \quad \left(0\right) \hat{W}_{\alpha\beta\gamma}, \quad \left(3\right) \hat{W}_{\alpha\beta\gamma}, \quad \left(3\right) \hat{W}_{\alpha\beta\gamma}^{\tau},
\]

and \( \left(3\right) \hat{W}_{\beta\gamma\delta} \) are given, correspondingly, by auxiliary connections \( \Gamma_{\alpha\beta\gamma}^{\mu} \)

\[
\hat{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} + \epsilon F_{\alpha(\gamma)} D_{(\beta} F_{\lambda)}^{\gamma}, \quad \hat{\gamma}_{\alpha\beta}^\gamma = \gamma_{\alpha\beta}^\gamma + \epsilon F_{\alpha(\gamma)} D_{(\beta} F_{\lambda)}^{\gamma},
\]

\[
\hat{\gamma}_{\alpha\beta}^\gamma = \gamma_{\alpha\beta}^\gamma + \sigma_{\beta} F_{\alpha(\gamma)}^{\gamma}, \quad \hat{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} + \sigma_{\beta} F_{\alpha(\gamma)}^{\gamma},
\]

where \( \sigma_{\beta} = \sigma_{\alpha} F_{\alpha\beta}^{\tau} \).

**Theorem 4** Four classes of reciprocal na–maps of la–spaces are characterized by corresponding invariant criterions:

1. for a–maps \( \left(0\right) T_{\alpha\beta} = \left(0\right) \hat{T}_{\alpha\beta}, \)

\[
\left(0\right) \hat{W}_{\alpha\beta\gamma} = \left(0\right) \hat{W}_{\alpha\beta\gamma}, \tag{69}
\]

2. for na\(_{1}\)–maps

\[
3 \left(\gamma\right) D_{\lambda} P_{\alpha\beta}^{\delta} + P_{\tau(\gamma)}^{\delta} P^{\tau}_{\alpha\beta} = r_{(\alpha\beta)\lambda}^{\delta} - \hat{L}_{(\alpha\beta)\lambda}^{\delta} + \left[T_{(\alpha\beta)\lambda}^{\delta} - Q_{(\alpha\beta)\lambda}^{\delta} + b_{(\alpha\beta)\lambda}^{\delta} + \delta_{(\alpha\beta)\lambda}^{\delta}\right], \tag{70}
\]

3. for na\(_{2}\)–maps \( \hat{T}_{\alpha\beta\gamma} = \ast \hat{T}_{\alpha\beta\gamma}, \)

\[
\hat{W}_{\alpha\beta\gamma} = \ast \hat{W}_{\alpha\beta\gamma}, \tag{71}
\]

4. for na\(_{3}\)–maps \( \left(3\right) T_{\beta\gamma} = \left(3\right) \hat{T}_{\beta\gamma}, \)

\[
\left(3\right) \hat{W}_{\beta\gamma\delta} = \left(3\right) \hat{W}_{\beta\gamma\delta}. \tag{72}
\]

**Proof.**

}\]
1. Let us prove that $a$–invariant conditions (69) hold. Deformations of $d$–connections of type

\[ (0)\gamma^\mu_{\alpha\beta} = \gamma^\mu_{\alpha\beta} + \psi^{(\alpha}_\beta \delta^\mu_{\beta)} \]  

(73)
define $a$–applications. Contracting indices $\mu$ and $\beta$ we can write

\[ \psi^\alpha = \frac{1}{m + n + 1} (\gamma^\beta_{\alpha\beta} - \gamma^\beta_{\alpha\beta}). \]

(74)
Introducing $d$–vector $\psi^\alpha$ into previous relation and expressing $\gamma^\alpha_{\beta\tau} = -T^\alpha_{\beta\tau} + \Gamma^\alpha_{\beta\tau}$ and similarly for underlined values we obtain the first invariant conditions from (69).

Putting deformation (73) into the formula for

\[ r^\cdot_{\alpha\cdot\beta\gamma} \quad \text{and} \quad r_{\alpha\beta} = r^\cdot_{\alpha\tau\beta\tau} \]

we obtain respectively relations

\[ L^\cdot_{\alpha\cdot\beta\gamma} - r^\cdot_{\alpha\cdot\beta\gamma} = \delta^\tau_{\alpha} \psi^\cdot_{[\gamma\beta]} + \psi^\cdot_{[\beta\delta]} \delta^\tau_{\gamma} + \delta^\tau_{(\alpha} \psi_{\cdot\delta)} w^\cdot_{\cdot\beta\gamma} \]  

(75)
and

\[ L_{\alpha\beta} - r_{\alpha\beta} = \psi^\cdot_{[\alpha\beta]} + (n + m - 1) \psi^\cdot_{[\alpha\beta]} + \psi_{\cdot\phi\cdot\psi} w^\cdot_{\cdot\beta\phi} + \psi_{\cdot\psi\cdot\phi} w^\cdot_{\cdot\beta\phi}, \]

(76)
where

\[ \psi_{[\alpha\beta]} = (\gamma) D_\beta \psi^\cdot_{\gamma} - \psi^\cdot_{\alpha} \psi^\cdot_{\beta}. \]

Putting (73) into (75) and (76) we can express $\psi^\cdot_{[\alpha\beta]}$ as

\[ \psi^\cdot_{[\alpha\beta]} = \frac{1}{n + m + 1} [L_{\alpha\beta} + \frac{2}{n + m + 1} \gamma^\cdot_{\cdot\phi\cdot\tau} w^\cdot_{\cdot\alpha\beta}] - \frac{1}{n + m + 1} \gamma^\cdot_{\cdot\tau(\alpha} w^\cdot_{\cdot\beta\cdot\phi)} \]

(77)

To simplify our consideration we can choose an $a$–transform, parametrized by corresponding $\psi$–vector from (73), (or fix a local coordinate chart) the antisymmetrized relations (77) to be satisfied by $d$–tensor

\[ \psi^\cdot_{\alpha\beta} = \frac{1}{n + m + 1} [L_{\alpha\beta} + \frac{2}{n + m + 1} \gamma^\cdot_{\cdot\phi\cdot\tau} w^\cdot_{\alpha\cdot\beta}] - \frac{1}{n + m + 1} \gamma^\cdot_{\cdot\tau(\alpha} w^\cdot_{\cdot\beta\cdot\phi)} - \]

\[ r_{\alpha\beta} - \frac{2}{n + m + 1} \gamma^\cdot_{\cdot\phi\cdot\tau} w^\cdot_{\alpha\beta} \]

(78)
Introducing expressions (73), (77) and (78) into deformation of curvature (74) we obtain the second conditions (69) of $a$-map invariance:

\[ (0) W^\cdot_{\alpha\cdot\beta\gamma} = (0) W^\cdot_{\cdot\alpha\cdot\beta\cdot\gamma}, \]
where the Weyl d–tensor on $\xi$ (the extension of the usual one for geodesic maps on (pseudo)–Riemannian spaces to the case of v–bundles provided with N–connection structure) is defined as

$$(0) W_{\alpha\beta\gamma} =$$

$$L_{\alpha\beta\gamma} + \frac{1}{n + m + 1} [\gamma_{\alpha\beta\gamma} - (\delta_{\alpha\beta} L_{\gamma\beta} + \delta_{\gamma\beta} L_{\alpha\beta} - \delta_{\beta\gamma} L_{\alpha\beta})] -$$

$$\frac{1}{(n + m + 1)^2} \left[ \delta_{\alpha\beta} (2\gamma_{\tau\phi} w_{\phi\gamma}) + \frac{1}{m} \left( \gamma_{\tau\phi} w_{\phi\beta\gamma} - \gamma_{\tau\phi} w_{\phi\beta\phi} \right) + \delta_{\gamma} (2\gamma_{\tau\phi} w_{\phi\alpha\beta} - 2\gamma_{\tau\phi} w_{\phi\beta\phi}) - \right.$$  

$$\delta_{\beta} (2\gamma_{\tau\phi} w_{\phi\alpha\gamma} - 2\gamma_{\tau\phi} w_{\phi\beta\phi}).$$

2. To obtain $na_{(1)}$–invariant conditions we rewrite $na_{(1)}$–equations (66) as to consider in explicit form covariant derivation $^{(n)} D$ and deformations (4.2) and (4.3):

$$2^{(n)} D_{\alpha} P_{\beta\gamma}^{\delta} + ^{(n)} D_{\beta} P_{\alpha\gamma}^{\delta} + ^{(n)} D_{\gamma} P_{\alpha\beta}^{\delta} + P_{\tau\alpha}^{\delta} P_{\beta\gamma}^{\tau} +$$

$$P_{\tau\beta}^{\delta} P_{\alpha\gamma}^{\tau} + P_{\tau\gamma}^{\delta} P_{\tau\beta}^{\alpha} + T_{\tau}^{\delta} (\alpha P_{\beta\gamma}^{\tau}) +$$

$$H_{\tau}^{\delta} (\alpha P_{\beta\gamma}^{\tau}) + b_{\alpha\beta} P_{\beta\gamma}^{\delta} + a_{\alpha\beta} \delta_{\gamma}. \quad (79)$$

Alternating the first two indices in (79) we have

$$2^{(n)} D_{\alpha} P_{\beta\gamma}^{\delta} - ^{(n)} D_{\alpha\beta} P_{\beta\gamma}^{\delta} = 2^{(n)} D_{\alpha} P_{\beta\gamma}^{\delta} +$$

$$^{(n)} D_{\beta} P_{\alpha\gamma}^{\delta} - 2^{(n)} D_{\gamma} P_{\alpha\beta}^{\delta} + P_{\tau\alpha}^{\delta} P_{\beta\gamma}^{\tau} + P_{\tau\beta}^{\delta} P_{\tau\alpha}^{\gamma} + 2 P_{\tau\gamma}^{\delta} P_{\tau\beta}^{\alpha}.$$  

Substituting the last expression from (79) and rescaling the deformation parameters and d–tensors we obtain the conditions (66).

3. Now we prove the invariant conditions for $na_{(0)}$–maps satisfying conditions

$$\epsilon \neq 0 \quad \text{and} \quad \epsilon - F_{\beta}^{\alpha} F_{\alpha}^{\beta} \neq 0$$

Let define the auxiliary d–connection

$$\tilde{\gamma}_{\beta\gamma}^{\alpha} = \gamma_{\beta\gamma}^{\alpha} - \psi_{\beta}^{\alpha} = \gamma_{\beta\gamma}^{\alpha} + \sigma_{\beta} F_{\beta}^{\alpha} \quad (80)$$

and write

$$\tilde{D}_{\gamma} = ^{(n)} D_{\gamma} F_{\beta}^{\alpha} + \sigma_{\gamma} F_{\beta}^{\alpha} - \epsilon \sigma_{\beta} \delta_{\gamma},$$

where $\sigma_{\beta} = \sigma_{\alpha} F_{\beta}^{\alpha}$, or, as a consequence from the last equality,

$$\sigma_{\alpha} F_{\beta}^{\gamma} = \epsilon F_{\tau}^{\gamma} \left( D_{\alpha} F_{\beta}^{\gamma} - \tilde{D}_{\alpha} F_{\beta}^{\gamma} \right) + \tilde{\sigma}_{\alpha} \delta_{\beta}.$$

Introducing auxiliary connections

$$\star_{\beta\gamma}^{\alpha} = \gamma_{\beta\gamma}^{\alpha} + \epsilon F_{\tau}^{\alpha} (\gamma_{\beta} D_{\tau} F_{\beta}^{\gamma})$$

and

$$\tilde{\gamma}_{\beta\gamma}^{\alpha} = \gamma_{\beta\gamma}^{\alpha} + \epsilon F_{\tau}^{\alpha} (\tilde{D}_{\beta} F_{\beta}^{\gamma}).$$
we can express deformation (80) in a form characteristic for a–maps:
\[ \hat{\gamma}_{\beta\gamma} = \ast \gamma_{\beta\gamma} + \sigma_{\beta\gamma} \delta_{\lambda}^\alpha. \]

Now it’s obvious that \( na_{(2)} \)–invariant conditions (81) are equivalent with a–invariant conditions (69) written for d–connection (81). As a matter of principle we can write formulas for such \( na_{(2)} \)–invariants in terms of ”underlined” and ”non–underlined” values by expressing consequently all used auxiliary connections as deformations of ”prime” connections on \( \xi \) and ”final” connections on \( \xi \). We omit such tedious calculations in this work.

4. Finally, we prove the last statement, for \( na_{(3)} \)–maps, of this theorem. Let
\[ q_\alpha \varphi^\alpha = e = \pm 1, \]
where \( \varphi^\alpha \) is contained in
\[ \hat{\gamma}_{\beta\gamma} = \gamma_{\beta\gamma} + \psi_{(\beta} \delta_{\gamma)}^\alpha + \sigma_{\beta\gamma} \varphi^\alpha. \]

Acting with operator \((\gamma) D_\beta \) on (82) we write
\[ (\gamma) D_\beta q_\alpha = (\gamma) D_\beta q_\alpha - \psi_{(\alpha} q_{\beta)} - e \sigma_{\alpha\beta}. \]

Contracting (84) with \( \varphi^\alpha \) we can express
\[ e \varphi^\alpha \sigma_{\alpha\beta} = \varphi^\alpha (\gamma) D_\beta q_\alpha - (\gamma) D_\beta q_\alpha - \varphi_\alpha q^\alpha q_{\beta} - e \psi_\beta. \]

Putting the last formula in (83) contracted on indices \( \alpha \) and \( \gamma \) we obtain
\[ (n + m) \psi_\beta = \Delta^\alpha_{\alpha\beta} - \gamma^\alpha_{\alpha\beta} + e \psi_\alpha \varphi^\alpha q_{\beta} + e \varphi^\alpha \varphi^\beta (\gamma) D_\beta \] 
\[ - (\gamma) D_\beta). \]

From these relations, taking into consideration (82), we have
\[ (n + m - 1) \psi_\alpha \varphi^\alpha = \]
\[ \varphi^\alpha (\Delta^\alpha_{\alpha\beta} - \gamma^\alpha_{\alpha\beta}) + e \varphi^\alpha \varphi^\beta (\gamma) D_\beta q_\alpha - (\gamma) D_\beta q_\alpha. \]

Using the equalities and identities (84) and (85) we can express deformations (83) as the first \( na_{(3)} \)–invariant conditions from (72).

To prove the second class of \( na_{(3)} \)–invariant conditions we introduce two additional d–tensors:
\[ \rho^\alpha_{\beta\gamma\delta} = r^\alpha_{\beta\gamma\delta} + \frac{1}{2} (\varphi^\alpha \varphi^T ) w^\alpha_{\gamma\delta} \]
\[ \rho^\alpha_{\beta\gamma\delta} = r^\alpha_{\beta\gamma\delta} - \frac{1}{2} (\varphi^\alpha \varphi^T ) w^\alpha_{\gamma\delta}. \]

(86)
Using deformation (83) and (86) we write relation
\[ \tilde{\sigma}_{\beta \gamma \delta} = \rho_{\beta \gamma \delta}^\alpha - \rho_{\beta \gamma \delta}^\alpha = \psi_{\beta [\delta \gamma]}^\alpha - \psi_{[\beta \delta \gamma]}^\alpha - \sigma_{\beta \gamma \delta} \varphi^\alpha, \]  
(87)
where
\[ \psi_{\alpha \beta} = (\gamma) D_{\beta} \psi_{\alpha} + \psi_{\alpha} \psi_{\beta} - (\nu + \varphi^\gamma \psi_\tau) \sigma_{\alpha \beta}, \]
and
\[ \sigma_{\alpha \beta \gamma} = (\gamma) D_{\gamma} \sigma_{\beta \alpha} + \mu_{\gamma} \sigma_{\beta \alpha} - \sigma_{\alpha \beta \gamma} \varphi^\gamma. \]
Let multiply (87) on \( q_\alpha \) and write (taking into account relations (82)) the relation
\[ e \sigma_{\alpha \beta} = -q_\tau \tilde{\sigma}_{\alpha \beta \delta} + \psi_{\alpha \beta} q_\gamma - \psi_{\gamma} q_\alpha, \]  
(88)
The next step is to express \( \psi_{\alpha \beta} \) through \( d \)-objects on \( \xi \). To do this we contract indices \( \alpha \) and \( \beta \) in (87) and obtain
\[ (n + m) \psi_{[\alpha \beta]} = -\sigma_{\tau \alpha \beta}^\tau + e q_\tau \varphi^\lambda \sigma_{\lambda \alpha \beta}^\tau - e \tilde{\psi}_{[\alpha \beta]} q_\gamma. \]
Then contracting indices \( \alpha \) and \( \delta \) in (87) and using (88) we write
\[ (n + m - 2) \psi_{\alpha \beta} = \tilde{\sigma}_{\alpha \beta \tau}^\tau - e q_\tau \varphi^\lambda \sigma_{\lambda \alpha \beta}^\tau + \psi_{[\beta \alpha]} + e (\tilde{\psi}_{\beta} q_\alpha - \tilde{\psi}_{[\alpha \beta]} q_\gamma), \]  
(89)
where \( \tilde{\psi}_{\alpha} = \varphi^\gamma \psi_\tau \). If the both parts of (89) are contracted with \( \varphi^\alpha \), it results that
\[ (n + m - 2) \tilde{\psi}_{\alpha} = \varphi^\gamma \sigma_{\tau \alpha \lambda}^\lambda - e q_\tau \varphi^\lambda \varphi^\delta \sigma_{\lambda \alpha \delta}^\tau - e q_\alpha, \]
and, in consequence of \( \sigma_{\beta (\gamma \delta)}^\alpha = 0 \), we have
\[ (n + m - 1) \varphi = \varphi^\beta \varphi^\gamma \sigma_{\beta \gamma \alpha}^\alpha. \]  
By using the last expressions we can write
\[ (n + m - 2) \tilde{\psi}_{\alpha} = \varphi^\gamma \sigma_{\tau \alpha \lambda}^\lambda - e q_\tau \varphi^\lambda \varphi^\delta \sigma_{\lambda \alpha \delta}^\tau - e (n + m - 1)^{-1} q_\alpha \varphi_\tau \varphi^\lambda \sigma_{\tau \lambda \delta}^\tau. \]  
(90)
Contracting (89) with \( \varphi^\beta \) we have
\[ (n + m) \tilde{\psi}_{\alpha} = \varphi^\gamma \sigma_{\alpha \tau \lambda}^\lambda + \tilde{\psi}_{\alpha} \]
and taking into consideration (90) we can express \( \tilde{\psi}_{\alpha} \) through \( \sigma_{\beta \gamma \delta}^\alpha \). As a consequence of (88)–(90) we obtain this formulas for \( d \)-tensor \( \psi_{\alpha \beta} : \)
\[ (n + m - 2) \psi_{\alpha \beta} = \sigma_{\alpha \beta \tau}^\tau - e q_\tau \varphi^\lambda \sigma_{\alpha \beta \lambda}^\lambda + \]
\[ \frac{1}{n + m} \left\{ -\sigma_{\tau \beta \alpha}^\tau + e q_\tau \varphi^\lambda \sigma_{\lambda \beta \alpha}^\tau - q_\beta (e \varphi^\tau \sigma_{\alpha \tau \lambda}^\lambda - q_\tau \varphi^\lambda \varphi^\delta \sigma_{\lambda \alpha \delta}^\tau) + e q_\alpha \times \right\} \]
\[ \left[ \varphi^\lambda \sigma_{\tau \beta \alpha}^\tau - e q_\tau \varphi^\lambda \varphi^\delta \sigma_{\lambda \alpha \delta}^\tau - \frac{e}{n + m - 1} q_\beta (\varphi^\tau \varphi^\delta \varphi^\sigma \sigma_{\tau \gamma \delta}^\delta - e q_\tau \varphi^\lambda \varphi^\delta \varphi^\sigma \sigma_{\tau \lambda \delta}^\delta) \right]. \]
Finally, putting the last formula and (88) into (87) and after a re-arrangement of terms we obtain the second group of \( na(3) \)-invariant conditions (72). If necessary we can rewrite these conditions in terms of geometrical objects on \( \xi \) and \( \xi \). To do this we must introduce splittings (86) into (72). \( \Box \)
For the particular case of \( na(3) \)-maps when
\[
\psi_\alpha = 0, \varphi_\alpha = g_{\alpha\beta} \varphi^\beta = \frac{\delta}{\delta u^\alpha} (\ln \Omega), \Omega(u) > 0
\]
and
\[
\sigma_{\alpha\beta} = g_{\alpha\beta}
\]
we define a subclass of conformal transforms \( g_{\alpha\beta}(u) = \Omega^2(u) g_{\alpha\beta} \) which, in consequence of the fact that \( d \)-vector \( \varphi_\alpha \) must satisfy equations (68), generalizes the class of concircular transforms (see [33] for references and details on concircular mappings of Riemannian spaces).

We emphasize that basic \( na \)-equations (66)–(68) are systems of first order partial differential equations. The study of their geometrical properties and definition of integral varieties, general and particular solutions are possible by using the formalism of Pfaff systems [57]. Here we point out that by using algebraic methods we can always verify if systems of \( na \)-equations of type (66)–(68) are, or not, involute, even to find their explicit solutions it is a difficult task (see more detailed considerations for isotropic \( ng \)-maps in [33] and, on language of Pfaff systems for \( na \)-maps, in [37]). We can also formulate the Cauchy problem for \( na \)-equations on \( \xi \) and choose deformation parameters (64) as to make involute mentioned equations for the case of maps to a given background space \( \xi \). If a solution, for example, of \( na(1) \)-map equations exists, we say that space \( \xi \) is \( na(1) \)-projective to space \( \xi \). In general, we have to introduce chains of \( na \)-maps in order to obtain involute systems of equations for maps (superpositions of \( na \)-maps) from \( \xi \) to \( \tilde{\xi} \):
\[
U \xrightarrow{ng<_{i_1}>} U_1 \xrightarrow{ng<_{i_2}>} \ldots \xrightarrow{ng<_{i_k-1}>} U_{k-1} \xrightarrow{ng<_{i_k}>} U
\]
where \( U \subset \xi, U_1 \subset \xi_1, \ldots, U_{k-1} \subset \xi_{k-1}, U \subset \xi_k \) with corresponding splittings of auxiliary symmetric connections
\[
\gamma^\alpha_{\beta\gamma} = <_{i_1}> P^\alpha_{\beta\gamma} + <_{i_2}> P^\alpha_{\beta\gamma} + \cdots + <_{i_k}> P^\alpha_{\beta\gamma}
\]
and torsion
\[
T^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + <_{i_1}> Q^\alpha_{\beta\gamma} + <_{i_2}> Q^\alpha_{\beta\gamma} + \cdots + <_{i_k}> Q^\alpha_{\beta\gamma}
\]
where cumulative indices \( <_{i_1} > = 0, 1, 2, 3 \), denote possible types of \( na \)-maps.

**Definition 4** Space \( \xi \) is nearly conformally projective to space \( \xi \), \( nc : \xi \rightarrow \tilde{\xi} \) if there is a finite chain of \( na \)-maps from \( \xi \) to \( \tilde{\xi} \).

For nearly conformal maps we formulate:

**Theorem 5** For every fixed triples \( (N^\alpha_0, \Gamma^\alpha_{\beta\gamma}, U \subset \xi) \) and \( (N^\alpha_j, \Gamma^\alpha_{\beta\gamma}, U \subset \xi) \), components of nonlinear connection, \( d \)-connection and \( d \)-metric being of class \( C^r(U), C^r(U) \), \( r > 3 \), there is a finite chain of \( na \)-maps \( nc : U \rightarrow U \).
Proof is similar to that for isotropic maps [36, 52, 39] (we have to introduce a finite number of na-maps with corresponding components of deformation parameters and deformation tensors in order to transform step by step coefficients of d-connection $\Gamma^\alpha_{\beta\gamma}$ into $\Gamma^\alpha_{\beta\gamma}$).

Now we introduce the concept of the Category of la–spaces, $\mathcal{C}(\xi)$. The elements of $\mathcal{C}(\xi)$ consist from $\text{Ob}\mathcal{C}(\xi) = \{\xi, \xi_{<i_1>}, \xi_{<i_2>}, \ldots\}$ being la–spaces, for simplicity in this work, having common N–connection structures, and $\text{Mor}\mathcal{C}(\xi) = \{\text{nc}(\xi_{<i_1>}, \xi_{<i_2>})\}$ being chains of na–maps interrelating la–spaces. We point out that we can consider equivalent models of physical theories on every object of $\mathcal{C}(\xi)$ (see details for isotropic gravitational models in [36, 39, 57, 37, 52, 53] and anisotropic gravity in [13, 53, 58]). One of the main purposes of this section is to develop a d–tensor and d–variational formalism on $\mathcal{C}(\xi)$, i.e. on la–multispaces, interrelated with nc–maps. Taking into account the distinguished character of geometrical objects on la–spaces we call tensors on $\mathcal{C}(\xi)$ as distinguished tensors on la–space Category, or dc–tensors.

Finally, we emphasize that presented in this section definitions and theorems can be generalized for v–bundles with arbitrary given structures of nonlinear connection, linear d–connection and metric structures. Proofs are similar to those from [38, 33].

9 Na-Tensor-Integral on La-Spaces

The aim of this section is to define tensor integration not only for bitensors, objects defined on the same curved space, but for dc–tensors, defined on two spaces, $\xi$ and $\xi_0$, even it is necessary on la–multispaces. A. Moór tensor–integral formalism having a lot of applications in classical and quantum gravity [34, 60, 16] was extended for locally isotropic multispaces in [57, 52]. The unispacial locally anisotropic version is given in [10, 18].

Let $T_u\xi$ and $T_u\xi_0$ be tangent spaces in corresponding points $u \in U \subset \xi$ and $u \in U_0 \subset \xi_0$ and, respectively, $T_u^*\xi$ and $T_u^*\xi_0$ be their duals (in general, in this section we shall not consider that a common coordinatization is introduced for open regions $U$ and $U_0$). We call as the dc–tensors on the pair of spaces $(\xi, \xi_0)$ the elements of distinguished tensor algebra

$$(\otimes^\alpha T_u\xi) \otimes (\otimes^\beta T_u^*\xi) \otimes (\otimes^\gamma T_{u_0}\xi) \otimes (\otimes^\delta T_{u_0}^*\xi)$$

defined over the space $\otimes^\alpha_\xi \otimes^\beta_\xi$, for a given nc : $\xi \rightarrow \xi_0$.

We admit the convention that underlined and non–underlined indices refer, respectively, to the points $u$ and $u_0$. Thus $Q^\alpha_{\beta\gamma}$, for instance, are the components of dc–tensor $Q \in T_u\xi \otimes T_{u_0}\xi$.

Now, we define the transport dc–tensors. Let open regions $U$ and $U_0$ be homeomorphic to sphere $\mathcal{R}^{2n}$ and introduce isomorphism $\mu_{u,u_0}$ between $T_u\xi$ and $T_{u_0}\xi$ (given by map nc : $U \rightarrow U_0$). We consider that for every d–vector $v^\alpha \in T_u\xi$ corresponds the vector $\mu_{u,u_0}(v^\alpha) = v_\alpha \in T_{u_0}\xi$, with components $v_\alpha$ being linear functions of $v^\alpha$:

$$v_\alpha = h^\alpha_{\alpha}(u, u_0)v^\alpha, \quad v^\alpha = h^\alpha_{\alpha}(u, u)v_\alpha.$$
where \( h^\alpha_\alpha(u,u) \) are the components of dc–tensor associated with \( \mu^{-1}_\mu \). In a similar manner we have

\[
v^\alpha = h^\alpha_\alpha(u,u)v^\alpha, \quad v_\alpha = h^\alpha_\alpha(u,u)v_\alpha.
\]

In order to reconcile just presented definitions and to assure the identity for trivial maps \( \xi \to \xi, u = u \), the transport dc-tensors must satisfy conditions:

\[
h^{\alpha}(u,u)h^{\beta}(u,u) = \delta^{\beta}_{\alpha}, \quad h^{\alpha}(u,u)h^{\alpha}(u,u) = \delta^{\alpha}_{\alpha},
\]

and

\[
\lim_{u \to u} h^{\alpha}(u,u) = \delta^{\alpha}_{\alpha}, \quad \lim_{u \to u} h^{\alpha}(u,u) = \delta^{\alpha}_{\alpha}.
\]

Let \( S_p \subset U \subset \xi \) is a homeomorphic to \( p \)-dimensional sphere and suggest that chains of na–maps are used to connect regions:

\[
U \xrightarrow{n(1)} S_p \xrightarrow{n(2)} U.
\]

**Definition 5** The tensor integral in \( \pi \in S_p \) of a dc–tensor \( N_{\gamma \alpha \beta \ldots \lambda}(\pi, u), \) completely antisymmetric on the indices \( \alpha, \ldots, \lambda \), over domain \( S_p \), is defined as

\[
N_{\gamma \alpha \beta \ldots \lambda}(\pi, u) = \int_{S_p} N_{\gamma \alpha \beta \ldots \lambda}(\pi, u) dS_{\pi1} \ldots dS_{\pi p},
\]

where \( dS_{\pi1} \ldots dS_{\pi p} = \delta u_{\pi1} \wedge \cdots \wedge \delta u_{\pi p} \).

Let suppose that transport dc–tensors \( h^{\alpha}_\alpha \) and \( h^\alpha_\alpha \) admit covariant derivations of order two and postulate existence of deformation dc–tensor \( B_{\alpha \beta}(u,u) \) satisfying relations

\[
D_\alpha h^\beta_\beta(u,u) = B_{\alpha \beta}(u,u)h^\beta_\gamma(u,u),
\]

and, taking into account that \( D_\alpha \delta^\beta_\gamma = 0 \),

\[
D_\alpha h^\beta_\beta(u,u) = -B_{\alpha \gamma}(u,u)h^\gamma_\beta(u,u).
\]

By using formulas for torsion and, respectively, curvature of connection \( \Gamma^\alpha_{\beta \gamma} \) we can calculate next commutators:

\[
D_\alpha D_\beta h^\gamma_\gamma = -(R^\lambda_{\gamma \alpha \beta} + T^\tau_{\alpha \beta} B^\lambda_{\tau \gamma})h^\gamma_\lambda,
\]

On the other hand from (92) one follows that

\[
D_\alpha D_\beta h^\gamma_\gamma = (D_\alpha B^\lambda_{\beta \gamma} + B_{\alpha \beta \gamma}^\lambda)h^\gamma_\lambda,
\]

where \( |\tau| \) denotes that index \( \tau \) is excluded from the action of antisymmetrization \( [ \ ] \). From (93) and (94) we obtain

\[
D_\alpha B^\lambda_{\beta \gamma} = (R^\lambda_{\gamma \alpha \beta} + T^\tau_{\alpha \beta} B^\lambda_{\tau \gamma}).
\]
Let $\overline{S}_p$ be the boundary of $\overline{S}_{p-1}$. The Stoke's type formula for tensor–integral (91) is defined as

$$I_{\overline{S}_p} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} dS^{\pi_1 \ldots \pi_p} = I_{\overline{S}_{p+1}}^{(p)} D_{\pi_1 \ldots \pi_p} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} dS^{\pi_1 \ldots \pi_p},$$

where

$$^{(p)} D_{\pi_1 \ldots \pi_p} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} = D_{\pi_1 \ldots \pi_p} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} + \rho T^{\gamma \pi}_{\pi_1 \ldots \pi_p} N^{\gamma \pi}_{\varphi \tau \varphi_2 \ldots \varphi_p} - B_{\pi_1 \ldots \pi_p}^{\gamma \pi} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} + B_{\pi_1 \ldots \pi_p}^{\gamma \pi} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p}.$$

We define the dual element of the hypersurfaces element $dS^{\gamma_1 \ldots \gamma_p}$ as

$$dS_{\gamma_1 \ldots \gamma_p} = \frac{1}{p!} \varepsilon_{\gamma_1 \ldots \gamma_p \alpha_1 \ldots \alpha_p} dS^{\alpha_1 \ldots \alpha_p},$$

where $\varepsilon_{\gamma_1 \ldots \gamma_p}$ is completely antisymmetric on its indices and

$$\varepsilon_{12 \ldots (n+m)} = \sqrt{|G|}, G = det[G_{\alpha \beta}],$$

$G_{\alpha \beta}$ is taken as the d–metric (5). The dual of dc–tensor $N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p}$ is defined as the dc–tensor $N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_{n+m-p}}$ satisfying

$$N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_{n+m-p}} = \frac{1}{p!} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_{n+m-p}} \varepsilon_{\gamma_1 \ldots \gamma_p \alpha_1 \ldots \alpha_p}.$$

Using (73), (96) and (97) we can write

$$I_{\overline{S}_p} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} dS^{\pi_1 \ldots \pi_p} = \int_{\overline{S}_{n+1}} \mathcal{F}_{\tau_1 \ldots \tau_{n+m-p-1}} D_{\tau_1 \ldots \tau_{n+m-p-1}} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} dS_{\gamma_1 \ldots \gamma_{n+m-p-1}},$$

where

$$\mathcal{F} D_{\tau_1 \ldots \tau_{n+m-p-1}} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} = D_{\tau_1 \ldots \tau_{n+m-p-1}} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} + (-1)^{(n+m-p)(n+m-p+1)} T^{\gamma \pi}_{\tau_1 \ldots \tau_{n+m-p-1}} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} + B_{\tau_1 \ldots \tau_{n+m-p-1}}^{\gamma \pi} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p} + B_{\tau_1 \ldots \tau_{n+m-p-1}}^{\gamma \pi} N^{\gamma \pi}_{\varphi \tau \varphi_1 \ldots \varphi_p}.$$

To verify the equivalence of (97) and (98) we must take in consideration that

$$D_{\gamma} \varepsilon_{\alpha_1 \ldots \alpha_k} = 0 \text{ and } \varepsilon_{\beta_1 \ldots \beta_{n+m-p} \alpha_1 \ldots \alpha_p} \varepsilon^{\beta_1 \ldots \beta_{n+m-p} \gamma_1 \ldots \gamma_p} = p! (n+m-p)! \delta^{(n+m)}_{\alpha_1 \ldots \alpha_p}.$$
10.1 Nonzero divergence of the energy–momentum d–tensor

R. Miron and M. Anastasiei [23, 24] pointed to this specific form of conservation laws of matter on la–spaces: They calculated the divergence of the energy–momentum d–tensor on la–space \( \xi \),

\[
D_\alpha E^\alpha_\beta = \frac{1}{\kappa_1} U_\alpha, \tag{99}
\]

and concluded that d–vector

\[
U_\alpha = \frac{1}{2} (G^\beta\phi R^\gamma_\phi T^\phi_\beta \cdot \alpha - G^\beta R^\gamma_\phi T^\phi_\beta \cdot \alpha + R^\beta_\phi T^\phi_\beta \gamma)
\]

vanishes if and only if d–connection \( D \) is without torsion.

No wonder that conservation laws, in usual physical theories being a consequence of global (for usual gravity of local) automorphisms of the fundamental space–time, are more sophisticated on the spaces with local anisotropy. Here it is important to emphasize the multiconnection character of la–spaces. For example, for a d–metric (5) on \( \xi \) we can equivalently introduce another metric linear connection \( \tilde{D} \). The Einstein equations

\[
\tilde{R}_{\alpha\beta} - \frac{1}{2} G_{\alpha\beta} \tilde{R} = \kappa_1 \tilde{E}_{\alpha\beta}\tag{100}
\]

constructed by using connection (80) have vanishing divergences

\[
\tilde{D}^\alpha (\tilde{R}_{\alpha\beta} - \frac{1}{2} G_{\alpha\beta} \tilde{R}) = 0 \quad \text{and} \quad \tilde{D}^\alpha \tilde{E}_{\alpha\beta} = 0,
\]

similarly as those on (pseudo)Riemannian spaces. We conclude that by using the connection \( \gamma^\alpha_\beta \gamma \) we construct a model of la–gravity which looks like locally isotropic on the total space \( E \). More general gravitational models with local anisotropy can be obtained by using deformations of connection \( \tilde{\Gamma}^\alpha_\beta \gamma \),

\[
\Gamma^\alpha_\beta \gamma = \tilde{\Gamma}^\alpha_\beta \gamma + P^\alpha_\beta \gamma + Q^\alpha_\beta \gamma,
\]

were, for simplicity, \( \Gamma^\alpha_\beta \gamma \) is chosen to be also metric and satisfy Einstein equations (100). We can consider deformation d–tensors \( P^\alpha_\beta \gamma \) generated (or not) by deformations of type (66)–(68) for na–maps. In this case d–vector \( U_\alpha \) can be interpreted as a generic source of local anisotropy on \( \xi \) satisfying generalized conservation laws (99).

10.2 Deformation d–tensors and tensor–integral conservation laws

From (91) we obtain a tensor integral on \( C(\xi) \) of a d–tensor:

\[
N^{\alpha_1 \ldots \alpha_p}_\xi (\underline{u}) = I_{S_p} N^{\overline{T}_1 \ldots \overline{T}_p}_\tau (\underline{\overline{u}}) h^{T}_\tau (\underline{u}, \overline{u}) h^{\overline{T}} (\overline{u}, \underline{u}) dS^{\overline{T}_1 \ldots \overline{T}_p}.
\]
We point out that tensor–integral can be defined not only for dc–tensors but and for d–tensors on $\xi$. Really, suppressing indices $\varphi$ and $\gamma$ in (97) and (98), considering instead of a deformation dc–tensor a deformation tensor

$$B_{\alpha\beta}^{\gamma}(u, u) = B_{\alpha\beta}^{\gamma}(u) = P_{\alpha\beta}^{\gamma}(u)$$  (101)

(we consider deformations induced by a nc–transform) and integration $I_{S_{p}}...dS^{\alpha_{1}...\alpha_{p}}$ in la–space $\xi$ we obtain from (91) a tensor–integral on $\mathcal{C}(\xi)$ of a d–tensor:

$$N_{\alpha}^{\beta}(u) = I_{S_{p}}N_{\tau,\alpha_{1}...\alpha_{p}}^{\kappa}(u)h_{\alpha}^{\tau}(u, u)h_{\alpha}^{\mu}(u, u)dS^{\alpha_{1}...\alpha_{p}}.$$  

Taking into account (95) we can calculate that curvature $R^{\lambda}_{\gamma,\alpha\beta}(u) = D_{[\beta}B^{\lambda}_{\alpha\gamma]} + B^{\nu}_{[\alpha]\gamma}|B^{\tau}_{\beta]} + T_{\tau,\alpha\beta}B^{\tau}_{\gamma}$ of connection $\Gamma^{\gamma}_{\alpha\beta}(u) = \Gamma^{\gamma}_{\alpha\beta}(u) + B^{\tau}_{\alpha\beta}(u)$, with $B^{\tau}_{\alpha\beta}(u)$ taken from (101), vanishes, $R^{\lambda}_{\gamma,\alpha\beta} = 0$. So, we can conclude that la–space $\xi$ admits a tensor integral structure on $\mathcal{C}(\xi)$ for d–tensors associated to deformation tensor $B_{\alpha\beta}^{\gamma}(u)$ if the nc–image $\xi$ is locally parallelizable. That way we generalize the one space tensor integral constructions in [16, 18, 40], were the possibility to introduce tensor integral structure on a curved space was restricted by the condition that this space is locally parallelizable. For $q = n+m$ relations (98), written for d–tensor $N_{\alpha}^{\beta\gamma}(we change indices $\alpha, \beta,...$ into $\alpha, \beta,...$) extend the Gauss formula on $\mathcal{C}(\xi)$:

$$I_{S_{q-1}}N_{\alpha}^{\beta\gamma}dS_{\alpha} = I_{S_{q-1}}g^{-1}D_{\alpha}N_{\alpha}^{\beta\gamma}dV.$$  (102)

where $dV = \sqrt{|G_{\alpha\beta}|}du^{1}...du^{q}$ and

$$g^{-1}D_{\alpha}N_{\alpha}^{\beta\gamma} = D_{\alpha}N_{\alpha}^{\beta\gamma} - T_{\alpha}^{\tau}N_{\alpha}^{\beta\gamma} - B_{\alpha\tau}N_{\alpha}^{\beta\gamma} + B_{\alpha\tau}N_{\alpha}^{\beta\gamma}.$$  (103)

Let consider physical values $N_{\alpha}^{\beta}$ on $\xi$ defined on its density $N_{\alpha}^{\beta\gamma}$, i. e.

$$N_{\alpha}^{\beta} = I_{S_{q-1}}N_{\alpha}^{\beta\gamma}dS_{\alpha}$$  (104)

with this conservation law (due to (102)):

$$g^{-1}D_{\alpha}N_{\alpha}^{\beta} = 0.$$  (105)

We note that these conservation laws differ from covariant conservation laws for well known physical values such as density of electric current or of energy–momentum tensor. For example, taking density $E_{\beta}^{\gamma}$, with corresponding to (103) and (105) conservation law,

$$g^{-1}D_{\alpha}E_{\alpha}^{\gamma} = D_{\alpha}E_{\alpha}^{\gamma} - T_{\alpha}^{\tau}E_{\alpha}^{\gamma} - B_{\alpha\tau}E_{\alpha}^{\gamma} = 0,$$  (106)

we can define values (see (102) and (104))

$$P_{\alpha} = I_{S_{q-1}}E_{\alpha}^{\gamma}dS_{\gamma}.$$

41
Defined conservation laws (106) for $E^\beta_\alpha$ have nothing to do with those for energy–momentum tensor $E^\gamma_\alpha$ from Einstein equations for the almost Hermitian gravity \[23, 24\] or with $\tilde{E}^\alpha_\beta$ from (100) with vanishing divergence $D_\gamma E^\gamma_\alpha = 0$. So $E^\beta_\alpha \neq E^\gamma_\alpha$. A similar conclusion was made in [18] for unispacial locally isotropic tensor integral. In the case of multispatial tensor integration we have another possibility (firstly pointed in [57, 40] for Einstein-Cartan spaces), namely, to identify $E^\beta_\alpha$ from (106) with the na-image of $E^\gamma_\alpha$ on la–space $\xi$. We shall consider this construction in the next section.

11 Na–Conservation Laws in La–Gravity

Let us consider a fixed background la–space $\xi$ with given metric $G_{\alpha\beta} = (g_{ij}, h_{ab})$ and d–connection $\tilde{\Gamma}_{\alpha}^{\cdot \beta \gamma}$. For simplicity, we suppose that metric is compatible and that connections are torsionless and with vanishing curvatures. Introducing an nc–transform from the fundamental la–space $\xi$ to an auxiliary one $\xi$ we are interested in the equivalents of the Einstein equations (100) on $\xi$.

We suppose that a part of gravitational degrees of freedom is "pumped out" into the dynamics of deformation d–tensors for d–connection, $P^\alpha_{\beta\gamma}$, and metric, $B^{\alpha\beta} = (b^{ij}, b^{ab})$. The remained part of degrees of freedom is coded into the metric $G_{\alpha\beta}$ and d–connection $\tilde{\Gamma}_{\alpha}^{\beta \gamma}$.

Following [19, 39] we apply the first order formalism and consider $B^{\alpha\beta}$ and $P^\alpha_{\beta\gamma}$ as independent variables on $\xi$. Using notations

$$P_\alpha = P^\beta_\beta \alpha, \quad \Gamma_\alpha = \Gamma^\beta_\beta \alpha,$$

$$\hat{B}^{\alpha\beta} = \sqrt{|G|} B^{\alpha\beta}, \quad \hat{G}^{\alpha\beta} = \sqrt{|G|} G^{\alpha\beta}, \quad \hat{\Gamma}^{\alpha\beta} = \sqrt{|G|} \hat{G}^{\alpha\beta}$$

and making identifications

$$\hat{B}^{\alpha\beta} + \hat{G}^{\alpha\beta} = \hat{G}^{\alpha\beta}, \quad \Gamma^{\alpha}_{\beta \gamma} - P^\alpha_{\beta \gamma} = \Gamma^{\alpha}_{\beta \gamma},$$

we take the action of la–gravitational field on $\xi$ in this form:

$$S^{(g)} = -(2c\kappa_1)^{-1} \int \delta^{4u} \mathcal{L}^{(g)},$$

(107)

where

$$\mathcal{L}^{(g)} = \hat{B}^{\alpha\beta}(D_\beta P_\alpha - D_\gamma P^\tau_{\alpha \beta}) + (\hat{G}^{\alpha\beta} + \hat{B}^{\alpha\beta})(P_\tau P^\tau_{\alpha \beta} - P^\alpha_{\alpha \kappa} P^\kappa_{\beta \tau})$$

and the interaction constant is taken $\kappa_1 = \frac{4\pi}{c^2} k$, \quad ($c$ is the light constant and $k$ is Newton constant) in order to obtain concordance with the Einstein theory in the locally isotropic limit.

We construct on $\xi$ a la–gravitational theory with matter fields (denoted as $\varphi_A$ with $A$ being a general index) interactions by postulating this Lagrangian density for matter fields

$$\mathcal{L}^{(m)} = \mathcal{L}^{(m)}[\hat{G}^{\alpha\beta} + \hat{B}^{\alpha\beta}; \frac{\delta}{\delta u^\gamma}(\hat{G}^{\alpha\beta} + \hat{B}^{\alpha\beta}); \varphi_A; \frac{\delta \varphi_A}{\delta u^\tau}].$$

(108)
Starting from (107) and (108) the total action of la–gravity on $\xi$ is written as
\[
S = (2c\kappa_1)^{-1} \int \delta^q u L^{(g)} + c^{-1} \int \delta^{(m)} L^{(m)}.
\] (109)

Applying variational procedure on $\xi$, similar to that presented in [19] but in our case adapted to N–connection by using derivations (3) instead of partial derivations (1), we derive from (109) the la–gravitational field equations
\[
\Theta_{\alpha\beta} = \kappa_1 (t_{\alpha\beta} + T_{\alpha\beta})
\] (110)
and matter field equations
\[
\Delta \frac{L^{(m)}}{\Delta \phi_A} = 0,
\] (111)
where $\frac{\Delta}{\Delta \phi_A}$ denotes the variational derivation.

In (109) we have introduced these values: the energy–momentum d–tensor for la–gravitational field
\[
\kappa_1 t_{\alpha\beta} = (\sqrt{|G|})^{-1} \frac{\Delta L^{(g)}}{\Delta G^{\alpha\beta}} = K_{\alpha\beta} + P^\gamma_{\alpha\beta} P_\gamma - P^\gamma_{\alpha \tau} P^\tau_{\beta \gamma} + \frac{1}{2} G_{\alpha \beta} G^{\gamma \tau} (P^\phi_{\gamma \tau} P_\phi - P^\phi_{\gamma \tau} P^\phi_{\tau \phi}),
\] (112)

where
\[
K_{\alpha\beta} = D_\gamma K^\gamma_{\alpha\beta},
\]
\[
2K^\gamma_{\alpha\beta} = -B^\gamma_\tau P^\tau_{\alpha (G^\gamma_{\beta} - G^\gamma_\tau B_{\beta \tau} - G^\gamma_{\tau \beta} B^\tau_\tau - B_{\alpha \beta} P^\gamma) + G^\gamma_\phi p_{(\alpha} P_{\beta)} + G^\gamma_\tau G^{\phi \psi} P^\phi_{\gamma \tau} G^{\tau \phi}_{(\alpha} B_{\beta)\phi} + G^\gamma_{\alpha \beta} B^\tau_\tau P^\gamma_{\tau \epsilon} - B_{\alpha \beta} P^\gamma),
\]
and the energy–momentum d–tensor of matter
\[
T_{\alpha\beta} = 2 \frac{\Delta L^{(m)}}{\Delta G^{\alpha\beta}} - G_{\gamma \delta} G^{\gamma \delta} \Delta L^{(m)} - \frac{\Delta G^{\alpha\beta}}{\Delta G^{\gamma \delta}}.
\] (113)

As a consequence of (111)–(113) we obtain the d–covariant on $\xi$ conservation laws
\[
D_\alpha (t_{\alpha\beta} + T_{\alpha\beta}) = 0.
\] (114)

We have postulated the Lagrangian density of matter fields (108) in a form as to treat $t_{\alpha\beta} + T_{\alpha\beta}$ as the source in (110).

Now we formulate the main results of this section:

**Proposition 1** The dynamics of the Einstein la–gravitational fields, modeled as solutions of equations (100) and matter fields on la–space $\xi$, can be equivalently locally modeled on a background la–space $\xi$ provided with a trivial d–connection and metric structures having zero d–tensors of torsion and curvature by field equations (110) and (111) on condition that deformation tensor $P^\alpha_{\beta \gamma}$ is a solution of the Cauchy problem posed for basic equations for a chain of na–maps from $\xi$ to $\xi$. 

43
Proposition 2 Local, \( d \)-tensor, conservation laws for Einstein la–gravitational fields can be written in form (114) for la–gravitational (112) and matter (113) energy–momentum \( d \)-tensors. These laws are \( d \)-covariant on the background space \( \xi \) and must be completed with invariant conditions of type (69)–(72) for every deformation parameters of a chain of \( na \)-maps from \( \xi \) to \( \xi \).

The above presented considerations consist proofs of both propositions.

We emphasize that nonlocalization of both locally anisotropic and isotropic gravitational energy–momentum values on the fundamental (locally anisotropic or isotropic) space \( \xi \) is a consequence of the absence of global group automorphisms for generic curved spaces. Considering gravitational theories from view of multispaces and their mutual maps (directed by the basic geometric structures on \( \xi \) such as \( N \)-connection, \( d \)-connection, \( d \)-torsion and \( d \)-curvature components, see coefficients for basic \( na \)–equations (66)–(68)), we can formulate local \( d \)-tensor conservation laws on auxiliary globally automorphic spaces being related with space \( \xi \) by means of chains of \( na \)-maps. Finally, we remark that as a matter of principle we can use \( d \)-connection deformations in order to modelate the la–gravitational interactions with nonvanishing torsion and nonmetricity. In this case we must introduce a corresponding source in (114) and define generalized conservation laws as in (99) (see similar details for locally isotropic generalizations of the Einstein gravity in Refs [53, 57, 37]).

12 Concluding Remarks

In this paper we have reformulated the fiber bundle formalism for both Yang-Mills and gravitational fields in order to include into consideration space-times with higher order anisotropy. We have argued that our approach has the advantage of making manifest the relevant structures of the theories with local anisotropy and putting greater emphasis on the analogy with anisotropic models than the standard coordinate formulation in Finsler geometry and on higher dimension (Kaluza–Klein) spaces.

Our models of higher order anisotropic gauge and gravitational interactions are refined in such a way as to shed light on some of the more specific properties and common and distinguishing features of the Yang-Mills and Einstein higher order anisotropic fields. As we have shown, it is possible a gauge like treatment for both models with local anisotropy (by using correspondingly defined linear connections in bundle spaces with semisimple structural groups, with variants of nonlinear realization and extension to semisimple structural groups, for gravitational fields).

Another main results of this paper are the formulation of the theory of nearly autoparallel maps of locally anisotropic spaces and a corresponding classification of such type spaces by using chains of nearly autoparallel maps (generalizing the class of conformal transforms). We have also analyzed in detail two variants of solution of the problem of formulation of conservation laws for field interactions with local anisotropy.
Finally, we emphasize that there are various possible developments of the ideas presented here. For instance, we point to a possible extension of the Ashtekar approach to gravity for higher order anisotropic Kaluza–Klein models, possible applications of exact solutions for la–gravity in modern cosmology and astrophysics, as well to a study of anisotropic low energy limits of string theories and a generalization functional integration formalism for locally anisotropic theories. Such problems will require our attention in the future.

Acknowledgments

The author is grateful to Igor Kanatchikov for valuable discussions and support during his visit to Warsaw.
Bibliography

[1] Aldovandi R and Stedile E 1984 *Int. J. Theor. Phys.* 23 301

[2] Anastasiei M and Kawaguchi H 1990 *Tensor, N. S.* 49 296

[3] Antonelli P L and Miron R (eds) 1996 *Lagrange and Finsler Geometry, Applications to Physics and Biology* (Dordrecht, Boston, London: Kluwer Academic Publishers)

[4] Asanov G S 1985 *Finsler Geometry, Relativity and Gauge Theories* (Boston: Reidel)

[5] Asanov G S and Ponomarenko S F 1988 *Finsler Bundle on Space-Time. Associated Gauge Fields and Connections* (Chişinău, Știinţa) [in Russian]

[6] Asanov G S 1989 *Fibered Generalization of the Gauge Field Theory. Finslerian and Jet Gauge Fields* (Moscow: Moscow University Press) [in Russian]

[7] Ashtekar A 1995 *Recent mathematical developments in quantum general gravity*, in *The Proceedings of the VIIth Marcel Grossmann Conference*, eds R. Ruffini and M. Keiser (Singapore: World Scientific), gr-qc/9411055

[8] Beil G G 1982 *Int. J. Theor. Phys.* 31 1025

[9] Bejancu A 1989 *Generalized Gauge Theories*, in *Colloquia Mathematica Societatis János Bolyai*. 56. Differential geometry (EGER, Hungary, 1989) pp 101

[10] Bejancu A 1990 *Finsler Geometry and Applications* (Chichester, England: Ellis Horwood)

[11] Bejancu A 1991 *An. Șt. Univ. "Al. I. Cuza" Iași (new ser., fasc.2)* 36 123

[12] Bishop R D and Crittenden R J 1964 *Geometry of Manifolds* (New York, Academic Press)

[13] Cartan E 1935 *Les Espaces de Finsler* (Paris: Hermann)
[14] Finsler P 1918 Über Kurven und Flächen in Allgemeiner Räumen, Dissertation (Göttingen); reprinted 1951 (Basel: Birkhäuser)

[15] Ikeda S 1994 Tensor N. S. 55 39

[16] Gottlieb I, Oproiu V and Zet G 1974 An. Ști. Univ. ”Al. I. Cuza” Iasi, Sect. Ia. Mat. (N.S.) 20 123

[17] Gottlieb I and Vacaru S 1994 in Colloquium on Differential Geometry, 25–30 July, 1994, Debrecen, Hungary (Debrecen: Lajos Kossuth University) pp 9

[18] Gottlieb I and Vacaru S 1996 Lagrange and Finsler Geometry, Applications to Physics and Biology, eds. P. L. Antonelli and Radu Miron (Dordrecht, Boston, London: Kluwer Academic Publishers), pp 209

[19] Grishchuk L P, Petrov A N and Popova A D 1984 Commun. Math. Phys. 94 379

[20] Kern J 1974 Arch. Math. 25 438

[21] Luehr C P and Rosenbaum M 1980 J. Math. Phys. 21 1432

[22] Matsumoto M 1986 Foundations of Finsler Geometry and Special Finsler Spaces (Kaisisha: Shigaken)

[23] Miron R and Anastasiei M 1987 Vector Bundles. Lagrange Spaces. Application in Relativity (Academiei, Romania 1987) [in Romanian]; English translation 1996 (Bucharest: Balkan Press)

[24] Miron R and Anastasiei M 1994 The Geometry of Lagrange Spaces: Theory and Applications (Dordrecht, Boston, London: Kluwer Academic Publishers)

[25] Miron R and Atanasiu Gh 1994 Compendium sur les Espaces Lagrange D’ordre Supérieur, Seminarul de Mecanică. Universitatea din Timișoara. Facultatea de Matematică; Miron R and Atanasiu Gh 1996 Revue Roumaine de Mathematiques Pures et Appliquees XLI Nos 3–4 205; 237; 251

[26] Miron R, Tavakol R K, Balan V and Roxburgh I 1993 Geometry of Space-Time and Generalized Lagrange Gauge Theory, Publicationes Mathematicae, Debrecen, Hungary 42 215

[27] Munteanu Gh and Ikeda S 1995 Tensor N. S. 56 (1995) 166

[28] Moór A 1951 Acta Math. 86 71

[29] Ponomarev V N, Barvinsky A O and Obukhov Yu N 1985 Geometro-dynamical Methods and Gauge Approach to Gravity Theory (Moscow: Energoatomizdat) [in Russian]
[30] Popov D A 1975 *Theor. Math. Phys.* **24** 347 [in Russian]

[31] Popov D A and Dikhin L I 1975 *Doklady Akademii Nauk SSSR* **225** 347 [in Russian]

[32] Rund H 1959 *The Differential Geometry of Finsler Spaces* (Berlin: Springer–Verlag)

[33] Sinyukov N S 1979 *Geodesic Maps of Riemannian Spaces* (Moscow: Nauka) [in Russian]

[34] Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland Publishing Company)

[35] Tseytlin A A 1982 *Phys. Rev.* **D26** 3327

[36] Vacaru S 1992 *Contr. Int. Conf. ”Lobachevski & Modern Geometry”,* Part II, ed. V. Bajanov et all (Kazani; University Press) pp 64

[37] Vacaru S 1993 *Buletinul Academiei de Ştiinţe a Republicii Moldova, Fizica şi Tehnica* (Izvestya Akademii Nauk Respubliky Moldova, fizika i tehnika) **3**(12) 17

[38] Vacaru S 1993 *Applications of Nearly Autoparallel Maps and Twistor–Gauge Methods in Gravity and Condensed States*, Ph D Thesis (Iaşi, România: ”Al. I. Cuza” University) [in Romanian]

[39] Vacaru S 1994 *Romanian Journal of Physics* **39** 37

[40] Vacaru S 1995 *Buletinul Academiei de Ştiinţe a Republicii Moldova, Fizica şi Tehnica* (Izvestya Akademii Nauk Respubliki Moldova, fizika i tehnika) **1** 54

[41] Vacaru S 1996 *J. Math. Phys* **37** 508

[42] Vacaru S 1996 *Buletinul Academiei de Ştiinţe a Republicii Moldova, Fizica şi Tehnica* (Izvestya Akademii Nauk Respubliki Moldova, fizika i tehnika) **1** 62

[43] Vacaru S 1996 *Nearly autoparallel maps, tensor integral and conservation laws on locally anisotropic spaces*; E–print: gr-qc/9604017

[44] Vacaru S 1996 *Nonlinear Connections in Superbundles and Locally Anisotropic Supergravity*, E–print: gr-qc/9604016; Vacaru S 1996 *Locally Anisotropic Interactions: I. Nonlinear Connections in Higher Order Anisotropic Superspaces*, E–print: hep-th/9607194; II. *Torsions and Curvatures of Higher Order Anisotropic Superspaces*, E–print: hep-th/9607195; III. *Higher Order Anisotropic Supergravity*, E–print: hep-th/9607196

[45] Vacaru S 1997 *Ann. Phys. (N.Y.)*, **256** 39; E–print: gr-qc/9604013
[46] Vacaru S 1997  *Nucl. Phys. B* **424** 590; E–prints: hep–th/9611034; hep–th/9607196; hep–th/9607195; hep–th/9607194

[47] Vacaru S 1998  *Miron’s generalization of Lagrange and Finsler geometries: a self–consistent approach to locally anisotropic gravity*, to be published in the collection of works in the honor of 70th birthday of Academician Radu Miron; E–print: physics/9801010

[48] Vacaru S 1998  *Exact solutions in locally anisotropic gravity and strings*, to be published by the American Institute of Physics as Proceedings of the Conference “Particles, Fields and Gravitation”, Lodz, Poland, April 15-18, 1998; E–print: gr-qc/9806080

[49] Vacaru S 1998  *Spinors and Field Interactions in Higher Order Anisotropic Spaces*, *J. High Energy Phys.* **09**(1998)011; E–print: hep–th/9807214

[50] Vacaru S 1998  *Interactions, Strings, and Isotopies in Higher Order Anisotropic Superspaces* (Palm Harbor: Hadronic Press); summary in E–print: physics/9706038

[51] Vacaru S and Goncharenko Yu 1995  *Int. J. Theor. Phys.* **34** 1955

[52] Vacaru S and Ostaf S 1993  *Buletinul Academiei de Științe a Republicii Moldova, Fizica și Tehnica* (Izvestya Akademii Nauk Respubliki Moldova, fizika i tehnika) **3** 4

[53] Vacaru S and Ostaf S 1994  *Buletinul Academiei de Științe a Republicii Moldova, Fizica și Tehnica* (Izvestia Akademii Nauk Respubliki Moldova, fizika i tehnika) **1** 64

[54] Vacaru S and Ostaf S 1994  *Colloquium on Differential Geometry*, 25-30 July 1994 (Debrecen, Hungary: Lajos Kossuth University) pp 56

[55] Vacaru S and Ostaf S 1996  *Lagrange and Finsler Geometry*, eds. P. L. Antonelli and R. Miron, (Dordrecht, Boston, London: Kluwer Academic Publishers) 241

[56] Vacaru S and Ostaf S 1996  *Rep. Math. Phys.* **37** 309; E-print: gr–qc/9602010

[57] Vacaru S, Ostaf S and Goncharenko Yu 1994  *Romanian J. Physics* **39** 199

[58] Vacaru S, Ostaf S, Goncharenko Yu and Doina A 1996  *Buletinul Academiei de Științe a Republicii Moldova, Fizica și Tehnica* (Izvestya Akademii Nauk Respubliki Moldova, fizika i tehnika) **3** 42

[59] Walner R P 1985  *General Relativity and Gravitation* **17** 1081

[60] DeWitt B S 1965  *Dynamical Theory of Groups and Fields* (New York: Gordon and Breach)
[61] Yano K and Ishihara S I 1973  *Tangent and Cotangent Bundles. Differential Geometry* (New York: Marcel Dekker)