A new $F(R)$-gravity model

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Abstract

We suggest a new model of modified $F(R)$-gravity theory with the function $F(R) = (1/\beta) \arcsin(\beta R)$. Constant curvature solutions corresponding to the flat and de Sitter space-time are obtained. The Jordan and Einstein frames are considered; the potential and the mass of the scalar degree of freedom are found. We show that the flat space-time is stable and the de Sitter space-time is unstable. The slow-roll parameters $\epsilon$, $\eta$, and the $e$-fold number of the model are evaluated. Critical points of autonomous equations for the de Sitter phase and the matter dominated epoch are found and studied.

1 Introduction

It is possible to describe the inflation and the present time universe acceleration if one modifies the Einstein-Hilbert (EH) action of general relativity (GR). We suggest the particular case of the $F(R)$-gravity model with the help of replacing the Ricci scalar by the function $F(R) = (1/\beta) \arcsin(\beta R)$ in EH action, where $\beta$ is the parameter with the dimension of (length)$^2$. Thus, we introduce the fundamental length $\sqrt{\beta}$ which goes probably from quantum gravity. The $F(R)$-gravity models may describe the evolution of universe without introducing Dark Energy (DE) [1], [2], [3]. The cosmic acceleration occurs due to modified gravity in such models. Therefore, $F(R)$-gravity models can be an alternative to $\Lambda$-Cold Dark Matter ($\Lambda$CDM) model as new gravitational physics is considered. The $\Lambda$CDM model has a problem with the explanation of the smallness of the cosmological constant $\Lambda$.

It should be noted that the first successful models of $F(R)$-gravity were given in [1, 5, 6, 7]. Some $F(R)$-gravity models were introduced in [8, 2, 3, 9, 10, 11] and in many other publications. $F(R)$-gravity models

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are phenomenological models that may describe eras and evolution of universe. The first $F(R)$-gravity model was introduced in [4] that gives the self-consistent description of inflation.

The Minkowski metric $\eta_{\mu\nu}=$diag(-1, 1, 1, 1) is used and $c=\hbar=1$ is assumed.

2 The Model

Let us consider a new model of arcsin-gravity with the Lagrangian density

$$\mathcal{L} = \frac{\sqrt{-g}}{2\kappa^2} F(R) = \frac{\sqrt{-g}}{2\kappa^2} \left[ \frac{1}{\beta} \arcsin(\beta R) \right],$$

(1)

where, $g = \det g_{\mu\nu}$, $\kappa = M_{Pl}^{-1}$, $M_{Pl}$ is the reduced Planck mass, $\beta$ has the dimension of (length)$^2$, and the action without matter is given by $S = \int d^4x \mathcal{L}$. At $\beta R \ll 1$, we have $\arcsin(\beta R) \approx \beta R$, and Eq. (1) becomes the EH Lagrangian. The equation $F(0) = 0$ holds, corresponding to the flat space-time without cosmological constant. GR passes local tests and we imply that at present time low curvature regime occurs, $\beta R \ll 1$. We will describe the inflation and universe evolution in the model suggested. For the classical stability the inequality $F'(R) > 0$ is required [1] which is satisfied if $\beta R < 1$,

$$F'(R) = \frac{1}{\sqrt{1 - (\beta R)^2}} > 0.$$ 

(2)

Quantum stability claims the inequality $F''(R) > 0$ [1], that becomes in our model as follows:

$$F''(R) = \frac{\beta^2 R}{[1 - (\beta R)^2]^{3/2}} > 0,$$

(3)

and it is also satisfies at $0 < \beta R < 1$.

2.1 Constant Curvature Solutions

If the Ricci scalar $R$ is a constant, $R = R_0$, equations of motion [12] become

$$2F(R_0) = R_0 F'(R_0).$$

(4)

and are given, in the model (1), by

$$2\sqrt{1 - (\beta R_0)^2} \arcsin(\beta R_0) = \beta R_0.$$ 

(5)
We note that constant curvature solutions corresponds to the extremum of the effective potential. Eq. (5) possesses two solution, \(R_0 = 0\) corresponding to the flat space-time, and non-trivial solution \(\beta R_0 \approx 0.919\). We will show that the last solution goes with the Schwarzschild-de Sitter space-time and with the maximum of the effective potential in the Einstein’s frame. The constant curvature solutions describe DE which is future stable if inequality \(F'(R_0)/F''(R_0) > R_0\) occurs [13], and we have from Eq. (2),(3)

\[
1 - (\beta R)^2 > (\beta R)^2,
\]

which is equivalent to \(\beta R < 1/\sqrt{2} \approx 0.707\). Thus, the solution \(R_0 = 0\) obeys Eq. (6) and the flat space-time is stable. The second constant curvature solution \(\beta R_0 \approx 0.919\) does not satisfy Eq. (6), leads to unstable de Sitter space-time, and describes inflation.

### 3 The Scalar-Tensor Formulation

Now we investigate the model in Einstein’s frame performing the conformal transformation of the metric [14]

\[
\bar{g}_{\mu\nu} = F'(R)g_{\mu\nu} = \frac{1}{\sqrt{1 - (\beta R)^2}}g_{\mu\nu}.
\]

Then the Lagrangian density in Einstein’s frame becomes

\[
\mathcal{L} = \sqrt{-\bar{g}} \left( \frac{\bar{R}}{2\kappa^2} - \frac{1}{2} \bar{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right).
\]

Here Ricci scalar \(\bar{R}\) in the Einstein frame is calculated in new metric (7) and the scalar field \(\phi\) is

\[
\phi(R) = -\frac{\sqrt{3}}{2\kappa} \ln F'(R) = \frac{\sqrt{3}}{2\kappa} \ln \sqrt{1 - (\beta R)^2}.
\]

The plot of the functions \(\kappa \phi\) is presented in Fig[1]. The potential \(V(\phi)\) is given by

\[
V(R) = \frac{RF'(R) - F(R)}{2\kappa^2 F^2(R)}
\]
Figure 1: The function $\kappa \phi$ versus $\beta R$.

\[
\frac{1}{2} \beta \kappa^2 \left[ \beta R \sqrt{1 - (\beta R)^2} - \left( 1 - (\beta R)^2 \right) \arcsin(\beta R) \right].
\]

The plot of the function $\beta \kappa^2 V$ versus $\beta R$ is given in Fig. 2 and the plot of function $\beta \kappa^2 V$ versus $\kappa \phi$ is represented by Fig. 3. The extremum of the potential, $V'(R) = 0$, with the help of Eq. (10) leads to Eq. (4). The potential (10) possesses the minimum at $R = 0$ and the maximum at $\beta R_0 \approx 0.919$. The flat space-time ($R = 0$) is the stable state and the state with the curvature $R_0 \approx 0.919/\beta$ is unstable.

We obtain the mass squared of a scalaron (scalar degree of freedom) from Eq. (10),

\[
m^2_\phi = \frac{d^2V}{d\phi^2} = \frac{1}{3} \left( \frac{1}{F''(R)} + \frac{R}{F'(R)} - \frac{4F(R)}{F''(R)} \right)
= \frac{1}{3\beta} \left[ \frac{(1 - x^2)^{3/2}}{x} + x\sqrt{1 - x^2} - 4(1 - x^2) \arcsin x \right], \tag{11}
\]
where $x = \beta R$. The plot of the function $\beta \kappa^2 V$ versus $x = \beta R$ is given by Fig. 2. One can verify that $m_\phi^2 < 0$ for the constant curvature solution $R_0 \approx 0.919/\beta$, and, therefore, this solution corresponds to unstable state as it was mentioned before. It follows from Eq. (11) that at $0.529 < \beta R < 1$ we have non-stable states, $m_\phi^2 < 0$. The stability of the de Sitter solution in F(R)-gravity models was first studied in [13]. If the value $\beta R$ is small the mass $m_\phi$ is big according to Fig. 2 and corrections to the Newton law are negligible.

To assure that corrections of $F(R)$-gravity model are small as compared to GR for $R \gg R_1$, where $R_1$ is a curvature at the present time, the relations

$$|F(R) - R| < R, \quad |F'(R) - 1| < 1, \quad|RF''(R)| < 1$$

should hold [11]. As arcsin $x > x$ at $1 > x > 0$ ($x = \beta R$), the first inequality in Eq. (12) becomes arcsin $x < 2x$, and it is satisfied at $1 > x > 0$. The second
inequality in Eq. (12) is equivalent to \( x < \sqrt{3}/2 \approx 0.866 \) (as \( F'(R) > 1 \) for \( 0 < x < 1 \)). The third inequality in Eq. (12) holds at \( 0 < x < 0.655 \). As a result all Eqs. (12) are satisfied if \( 0 < x < 0.655 \).

4 Slow-Roll Cosmological Parameters

The slow-roll parameters are given by [15],

\[
\epsilon(\phi) = \frac{1}{2} M_{Pl}^2 \left( \frac{V''(\phi)}{V(\phi)} \right)^2, \quad \eta(\phi) = M_{Pl}^2 \frac{V''(\phi)}{V(\phi)}.
\] (13)

When conditions \( |\eta(\phi)| \ll 1, \epsilon(\phi) \ll 1 \) hold the slow-roll approximation takes place. From Eqs. (10),(11) we find the slow-roll parameters as follows:

\[
\epsilon = \frac{1}{3} \left[ \frac{RF'(R) - 2F(R)}{RF'(R) - F(R)} \right]^2
\]
The plots of the functions $\epsilon$, $\eta$ are given in Fig. 5, Fig. 6. The equation $\epsilon = 1$ has the solution $x \approx 0.766$. It follows from Fig. 5 that at $1 > \beta R > 0.766$ the inequality $\epsilon < 1$ holds. The equation $|\eta| = 1$ is satisfied at $x \approx 0.516$, $x \approx 0.544$ and $x \approx 0.925$. At $0.544 > \beta R > 0.516$ and at $1 > \beta R > 0.925$, we have the result $|\eta| < 1$. Therefore, the slow-roll approximation, $\epsilon < 1$ and $|\eta| < 1$, takes place at $1 > \beta R > 0.925$. 

Figure 4: The function $\beta m_{\phi}^2$ versus $\beta R$. 

$$
\epsilon \approx \frac{1}{3} \left( \frac{2\sqrt{1 - x^2} \arcsin x - x}{\sqrt{1 - x^2} \arcsin x - x} \right)^2,
$$

(14)

$$
\eta = \frac{2}{3} \left[ \frac{F'(R) + F''(R)[RF'(R) - 4F(R)]}{F''(R)[RF'(R) - F(R)]} \right] 
= \frac{2 \left( 1 - 4x\sqrt{1 - x^2} \arcsin x \right)}{3x \left( x - \sqrt{1 - x^2} \arcsin x \right)},
$$

(15)
Figure 5: The function $\epsilon$ versus $\beta R$.

The age of the inflation can be obtained by calculating the $e$-fold number

$$N_e \approx \frac{1}{M_{Pl}^2} \int_{\phi_{\text{end}}}^{\phi} \frac{V(\phi)}{V'(\phi)} d\phi.$$  \hspace{1cm} (16)

We find, from Eqs. (9), (10), the number of $e$-foldings

$$N_e \approx \frac{3}{2} \int_{x_{\text{end}}}^{x_0} \frac{x \left( \sqrt{1-x^2} \arcsin x - x \right) dx}{(1-x^2) \left( 2\sqrt{1-x^2} \arcsin x - x \right)},$$  \hspace{1cm} (17)

were $x_{\text{end}} = \beta R_{\text{end}}$ corresponds to the time of the end of inflation when $\epsilon$ or $|\eta|$ are close to 1. Thus, inflation ends when slow-roll conditions are violated. We obtain the amount of inflation $N_e \approx 9.7$ at $x_0 = 0.9999$ and $x_{\text{end}} = 0.92$, and therefore the model can describe the inflation. It should be noted that it is required around 70 $e$-foldings of inflation to solve the flatness and horizon problems.
5 Critical Points of Autonomous Equations

To investigate critical points of equations of motion, it is useful to introduce the dimensionless parameters \[16\] which become

\[
x_1 = -\frac{\dot{F}'(R)}{HF''(R)} = -\frac{x\dot{x}}{H(1-x^2)},
\]

\[
x_2 = -\frac{F(R)}{6F'(R)H^2} = -\frac{\sqrt{1-x^2}\arcsin x}{6\beta H^2},
\]

\[
x_3 = \frac{\dot{H}}{H^2} + 2, \tag{18}
\]

\[
m = \frac{RF''(R)}{F'(R)} = \frac{x^2}{(1-x^2)}.
\]
\[
\frac{d}{dt} = -\frac{RF'(R)}{F(R)} = \frac{x_3}{x_2} = -\frac{x}{\sqrt{1 - x^2 \arcsin x}},
\]

where \( H \) is a Hubble parameter, \( x = \beta R \), and the dot means the derivative with respect to the time. The deceleration parameter \( q \) is given by \( q = 1 - x_3 \). The critical points for the system of equations can be studied by the investigation of the function \( m(r) \). Equations of motion in the absence of the radiation, \( \rho_{\text{rad}} = 0 \), with the help of Eqs. (18), (19) can be written in the form of autonomous equations \[16\]. One can investigate the critical points of the system of equations by the study of the function \( m(r) \) which shows the deviation from the \( \Lambda \)CDM model. The plot of the function \( m(r) \) is presented by Fig.7. The de Sitter point \( P_1 \) \[16\], in the absence of radiation, \( x_4 = 0 \),

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{plot.png}
\caption{The function \( m(r) \)}
\end{figure}

corresponds to the parameters \( x_1 = 0, x_2 = -1, x_3 = 2 \) (\( \dot{H} = 0, H^2 = R/12, r = -2 \)). The point \( P_1 \) corresponds to the constant curvature solutions that may be verified using Eqs. (5),(18). The effective equation of state (EoS)
parameter, $w_{\text{eff}}$, and the parameter of matter energy fraction, $\Omega_m$, are given for this point by

$$w_{\text{eff}} = -1 - 2\frac{\dot{H}}{(3H^2)} = -1, \quad \Omega_m = 1 - x_1 - x_2 - x_3 = 0.$$  \hfill (20)

which correspond to DE. This point mimics a cosmological constant and the deceleration parameter becomes $q = -1$. The constant curvature solution $x \approx 0.919$ corresponds to unstable de Sitter space as $1 < m(r = -2) \approx 5.4$ \[^{[16]}\].

For the critical point $P_5 \ (x_3 = 1/2)$, $m \approx 0$, $r \approx -1$, and EoS of a matter era is $w_{\text{eff}} = 0 \ (a = a_0 t^{2/3})$. Then we have a viable matter dominated epoch prior to late-time acceleration \[^{[16]}\]. The equation $m = -r - 1$ has the solution $m = 0$, $r = -1$, $R = 0$, corresponding to the point $P_5$. One can verify with the help of Eq. (19) (see Fig.7) that $m'(r = -1) = 0$. As a result, the condition $m'(r = -1) > -1$ holds and we have the standard matter era \[^{[16]}\]. Therefore, the correct description of the standard matter era occurs in the model under consideration.

### 6 Conclusion

We suggest a new model of modified $F(R)$-gravity representing the effective gravity model which can describe the evolution of universe. The constant curvature solutions, $\beta R_0 = 0$, $\beta R_0 = 0.919$ were obtained that correspond to the flat space-time and the de Sitter space-time, correspondingly. The de Sitter space-time gives the acceleration of universe and corresponds to inflation. The flat space-time is stable but the de Sitter space-time is unstable in the model and it goes with the maximum of the effective potential. The Jordan and Einstein frames were considered and we have obtained the potential and the mass of the scalar degree of freedom. The slow-roll parameters $\epsilon$, $\eta$ and the $e$-fold number of the model were evaluated. The model gives $e$-fold number $N_e \approx 9.7$ characterizing the age of inflation. We show by the analysis of critical points of autonomous equations that the standard matter era exists and the standard matter era conditions are satisfied. The model may be alternative to GR, and can describe early-time inflation.
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