Contextualist viewpoint to
Greenberger-Horne-Zeilinger paradox

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Abstract

We present probabilistic analysis of the Greenberger-Horne-Zeilinger (GHZ) scheme in the contextualist framework, namely under the assumption that distributions of hidden variables depend on settings of measurement devices. On one hand, we found classes of probability distributions of hidden variables for that the GHZ scheme does not imply a contradiction between the local realism and quantum formalism. On the other hand, we found classes of probability distributions of hidden variables for that the GHZ scheme still induce such a contradiction (despite variations of distributions). It is also demonstrated that (well known in probability theory) singularity/absolute continuity dichotomy for probability distributions is closely related to the GHZ paradox. Our conjecture is that this GHZ-coupling between singularity/absolute continuity dichotomy and incompatible/compatible measurements might be a general feature of quantum theory.

1 Introduction

Violations of Bell’s inequality [1] by quantum correlations may be interpreted as an evidence of the impossibility to use the local realism in quantum theory (see, for example, [2], [3]). Despite the general attitude to connect violations of Bell’s inequality with such problems as determinism and locality, there exists sufficiently strong opposition [4]-[7] to such a conclusion. We call such
an opposition the *probability opposition*. The general viewpoint of adherents of the probabilistic interpretation of violations of Bell’s inequality is that the derivation of this inequality is based on some (implicit) probabilistic assumptions. It seems that theoretical as well as experimental investigations of the EPR paradox (in particular, Bell’s inequality) must be at least partly reoriented to the investigation of probabilistic roots of this paradox.

A new strong argument in the favour of nonlocal (or nonreal) interpretation of violations of Bell’s inequality was given by so called Greenberger-Horne-Zeilinger (GHZ) paradox, [8]. The GHZ scheme is based on the probability $P = 1$ arguments. From the first point of view all probabilistic circumstances of the GHZ scheme are so straightforward that there is no more place for probabilistic counter arguments. There is rather general opinion that probability does not play any role in the GHZ scheme: *probability $P = 1$ statements are typically considered as deterministic statements*. However, the careful probabilistic analysis demonstrates that the GHZ paradox has even deeper connection to foundations of probability theory than Bell’s inequality.

In the present paper we study probabilistic structures induced by the combinations of quantum systems and measurement devices. We consider ‘integral hidden variables’ $w = (\lambda, \lambda^a, \lambda^b, \lambda^c)$, where $\lambda$ corresponds to a physical system (triple of photons in the GHZ framework) and $\lambda^a, \lambda^b, \lambda^c$ correspond to internal states of measurement devices. On the space of such hidden variables we introduce probability distribution $P_{\phi_1\phi_2\phi_3}$ corresponding to settings $\phi_1, \phi_2, \phi_3$ (phase shifts) of measurement apparatuses $A, B, C$. In fact, the GHZ used the implicit probabilistic assumption:

$I)$ Probability distribution $P_{\phi_1\phi_2\phi_3}$ does not depend on phase shifts $\phi_1, \phi_2, \phi_3$:

$P_{\phi_1\phi_2\phi_3} \equiv P$

It is demonstrated that if $I$ does not hold true, then the GHZ considerations do not imply a paradox. In fact, our study of the GHZ paradox is nothing than an application of so called contextualist approach to quantum mechanics. This approach was developed [6]. $^1$ However, we do not only apply the contextualist approach to the GHZ paradox. We did essentially more. On one hand, we found classes of probability distributions of hidden variables in that the GHZ scheme does not imply a contradiction between the local realism and quantum formalism. On the other hand, we found classes

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$^1$Contextualist approach is characterized by diversity of models, see [6]. We do not try to compare or discuss these models. In fact, my contextualist views were formed on the basis of reading of papers of W. De Muynck and S. Gudder.
of probability distributions of hidden variables in that the GHZ scheme still implies such a contradiction despite variations of probability distributions.

Notions of mutually singular and absolutely continuous distributions [9] play an important role in this framework. It is proved that if the transformation \( P_{\phi_1\phi_2\phi_3} \rightarrow P_{\phi'_1\phi'_2\phi'_3} \) is singular, then the GHZ scheme is destroyed; if this transformation is absolutely continuous, then the GHZ scheme still works. Therefore if (in the contextualist framework) we suppose that distributions of hidden variables in measurements of incompatible observables are absolutely continuous, then via the GHZ considerations we must again obtain a contradiction between the local realism and quantum formalism. To save the local realism, we have to assume that incompatible measurements produce probability distributions of hidden variables which are not absolutely continuous.

This coupling (via the GHZ framework) of singularity/absolute continuity dichotomy (see any advanced textbook in probability theory, for example, [10]) with incompatible/compatible measurements complementarity induce a conjecture:

**General splitting of physical measurements into quantum/classical measurements is an exhibition of singularity/absolute continuity dichotomy in probability theory.**

The singularity/absolute continuity dichotomy is one of fundamental facts of probability theory [10]: if the number of degrees of freedom \( N \rightarrow \infty \), then (under quite general assumptions) probability distributions are either singular or absolutely continuous.

In principle, distributions of hidden variables may fluctuate not only due to a change in the measurement arrangement, but also due to statistical variations corresponding to different runs of an experiment (fluctuations in preparation). This is a hypothesis on ensemble nonreproducibility, see W. De Baere [5], W. De Muynck, W. De Baere, H. Martens [6] and the author [5]. The ensemble reproducibility is used as an implicit probabilistic assumption both in Bell’s and GHZ’s frameworks:

**B** The probability distribution \( P \) of hidden variables \( \lambda \) (corresponding to physical systems) in an ensemble \( \Omega \) prepared for some measurement is uniquely determined by fixing of a quantum state \( \psi \).

So formally, if \( \Omega = \Omega_\psi \), then we can write \( P_\psi \) instead of \( P_{\Omega_\psi} \).

We demonstrate that if \( B \) is violated the GHZ considerations do not imply a contradiction between the local realism and quantum formalism.
2 Probabilistic analysis of the GHZ scheme

1. GHZ arguments. We briefly repeat GHZ arguments. We shall use an advanced variant of these arguments, see A. Shimony in [3]. The latter form of the ‘GHZ-paradox’ is based on rather sophisticated probabilistic considerations. It is really not easy to criticize these considerations from the viewpoint of hidden variables theory. On the other hand, the original GHZ arguments [8] can be easily destroyed by using the contextualist approach (and elementary probabilistic reasons), see Remark 1.2 at the end of this section.

There are three different phase shifts, \( \varphi_1, \varphi_2, \varphi_3 \). For each fixed setting of three shifts, there are three measurements (for three photons which are produced by the down-conversion process from a single photon). Results of these measurements are denoted \( A(\varphi_1), B(\varphi_2), C(\varphi_3) = \pm 1 \). Quantum formalism predicts that:

\[
P^{(\psi)}(A(\varphi_1)B(\varphi_2)C(\varphi_3) = 1|\varphi_1 + \varphi_2 + \varphi_3 = \frac{\pi}{2}, \mod 2\pi) = 1, \quad (1)
\]

\[
P^{(\psi)}(A(\varphi_1)B(\varphi_2)C(\varphi_3) = -1|\varphi_1 + \varphi_2 + \varphi_3 = \frac{\pi}{2}, \mod 2\pi) = 0, \quad (2)
\]

\[
P^{(\psi)}(A(\varphi_1)B(\varphi_2)C(\varphi_3) = 1|\varphi_1 + \varphi_2 + \varphi_3 = \frac{3\pi}{2}, \mod 2\pi) = 0, \quad (3)
\]

\[
P^{(\psi)}(A(\varphi_1)B(\varphi_2)C(\varphi_3) = -1|\varphi_1 + \varphi_2 + \varphi_3 = \frac{3\pi}{2}, \mod 2\pi) = 1. \quad (4)
\]

Let us describe the above experiment by using hidden variables. The GHZ used so called deterministic hidden variables model in that by fixing a value \( \lambda = \lambda_0 \) we fix values of (in general incompatible) physical observables.

It is assumed that there exists a Kolmorogov probability space \( (\Omega, \mathcal{F}, \mathbf{P}) \) where \( \Omega \) is the configuration space of hidden variables \( \lambda \) (it is typically denoted by \( \Lambda \) in papers on Bell or GHZ considerations); \( \mathcal{F} \) is a \( \sigma \)-field of subsets of \( \Omega \). \( \mathbf{P} \) is the probability distribution of hidden variables. Physical observables \( A(\varphi_1), B(\varphi_2), C(\varphi_3) \) are represented by random variables on \( \Omega : A = A(\varphi_1, \lambda), B = B(\varphi_2, \lambda), C = C(\varphi_3, \lambda) \). Thus \( (1), (2) \) and \( (3), (4) \) imply that if \( \varphi_1 + \varphi_2 + \varphi_3 = \frac{\pi}{2} \), then \( A(\varphi_1, \lambda)B(\varphi_2, \lambda)C(\varphi_3, \lambda) = 1 \) for \( \lambda \in \Omega^+_{\varphi_1,\varphi_2,\varphi_3} \in \mathcal{F}, \mathbf{P}(\Omega^+_{\varphi_1,\varphi_2,\varphi_3}) = 1 \), and if \( \varphi_1 + \varphi_2 + \varphi_3 = \frac{3\pi}{2} \), then \( A(\varphi_1, \lambda)B(\varphi_2, \lambda)C(\varphi_3, \lambda) = -1 \) for \( \lambda \in \Omega^-_{\varphi_1,\varphi_2,\varphi_3} \in \mathcal{F}, \mathbf{P}(\Omega^-_{\varphi_1,\varphi_2,\varphi_3}) = 1 \).

Thus we have: \( A(\frac{\pi}{2}, \lambda)B(0, \lambda)C(0, \lambda) = 1 \), where \( \lambda \in \Omega^+_{\frac{\pi}{2},0} \); \( A(0, \lambda)B(\frac{\pi}{2}, \lambda)C(0, \lambda) = 1 \), where \( \lambda \in \Omega^+_{0,\frac{\pi}{2}} \), and \( A(0, \lambda)B(0, \lambda)C(\frac{\pi}{2}, \lambda) = 1 \), where \( \lambda \in \Omega^+_{0,0,\frac{\pi}{2}} \).
Set $\Sigma^+ = \Omega^+_{000} \cap \Omega^+_{0\pi 0} \cap \Omega^+_{\pi 00}$. As $P(\Omega^+_{000}) = P(\Omega^+_{0\pi 0}) = P(\Omega^+_{\pi 00}) = 1$, we obtain that $P(\Sigma^+) = 1$.

We have that, for each $\lambda \in \Sigma^+$, $A(\pi \frac{\pi}{2}, \lambda)B(\pi \frac{\pi}{2}, \lambda)C(\pi \frac{\pi}{2}, \lambda) = 1$. Thus $\Sigma^+ \subset \Omega^+_{000} \cap \Omega^+_{0\pi 0} \cap \Omega^+_{\pi 00}$. But $P(\Omega^+_{000}) = 1 - P(\Omega^+_{0\pi 0}) = 0$ and, hence, $P(\Sigma^+) = 0$. This is the GHZ 'paradox'. The standard inference is that we cannot use local hidden variables, because of (4).

2. Contextualist model. Here we could not suppose that by fixing a value $\lambda = \lambda_0$ of the hidden variable of a quantum system we fix values of (in general incompatible) physical observables. Internal states of measurement devices must be also taken into account (see, for example, J.Bell [11]).

Denote by $\Lambda, \Lambda_{\varphi_1}^a, \Lambda_{\varphi_2}^b, \Lambda_{\varphi_3}^c$ spaces of hidden variables, respectively, for triples of photons and measurement devices $A(\varphi_1), B(\varphi_2), C(\varphi_3)$.

Here $A(\varphi_1) = A(\varphi_1, \lambda, \lambda^a), B(\varphi_2) = B(\varphi_2, \lambda, \lambda^b), C(\varphi_3, \lambda, \lambda^c)$. Thus [11], [2] and [3], [4] imply that if $\varphi_1 + \varphi_2 + \varphi_3 = \pi$, then

$$A(\varphi_1, \lambda, \lambda^a)B(\varphi_2, \lambda, \lambda^b)C(\varphi_3, \lambda, \lambda^c) = 1 \quad (5)$$

for $w = (\lambda, \lambda^a, \lambda^b, \lambda^c) \in \Omega_{\varphi_1\varphi_2\varphi_3}^+ \in F_{\varphi_1\varphi_2\varphi_3}$ and

$$P_{\varphi_1\varphi_2\varphi_3}(\Omega_{\varphi_1\varphi_2\varphi_3}^+) = 1, \quad (6)$$

and if $\varphi_1 + \varphi_2 + \varphi_3 = \frac{3\pi}{2}$, then

$$A(\varphi_1, \lambda, \lambda^a)B(\varphi_2, \lambda, \lambda^b)C(\varphi_3, \lambda, \lambda^c) = -1 \quad (7)$$

for $w = (\lambda, \lambda^a, \lambda^b, \lambda^c) \in \Omega_{\varphi_1\varphi_2\varphi_3}^- \in F_{\varphi_1\varphi_2\varphi_3}$ and

$$P_{\varphi_1\varphi_2\varphi_3}(\Omega_{\varphi_1\varphi_2\varphi_3}^-) = 1. \quad (8)$$

The total space of hidden variables for the system of quantum particles and measurement apparatuses is the set $\Omega_{\varphi_1\varphi_2\varphi_3} = \Lambda \times \Lambda_{\varphi_1}^a \times \Lambda_{\varphi_2}^b \times \Lambda_{\varphi_3}^c$. We denote by the symbol $F_{\varphi_1\varphi_2\varphi_3}$ a $\sigma$-field of subsets of $\Omega_{\varphi_1\varphi_2\varphi_3}$. The $P_{\varphi_1\varphi_2\varphi_3}$ is the probability distribution of hidden variables $w = (\lambda, \lambda^a, \lambda^b, \lambda^c)$.

Remark 1.1. It is natural that the distribution of hidden variables $w$ depends on the configuration $(\varphi_1, \varphi_2, \varphi_3)$ of phase shifts. In fact, we should use the symbol $w_{\varphi_1\varphi_2\varphi_3} = (\lambda, \lambda_{\varphi_1}^a, \lambda_{\varphi_2}^b, \lambda_{\varphi_3}^c)$ to denote this hidden multivariable. Thus we have

$$A(\pi \frac{\pi}{2}, \lambda, \lambda^a)B(0, \lambda, \lambda^b)C(0, \lambda, \lambda^c) = 1, \; w \in \Omega_{000}^+, \quad (9)$$
\[ A(0, \lambda, \lambda)B(\frac{\pi}{2}, \lambda, \lambda)C(0, \lambda, \lambda) = 1, \quad w \in \Omega_{00}^+; \quad (10) \]

\[ A(0, \lambda, \lambda)B(0, \lambda, \lambda)C(\frac{\pi}{2}, \lambda, \lambda) = 1, \quad w \in \Omega_{00}^+; \quad (11) \]

\[ A(\frac{\pi}{2}, \lambda, \lambda)B(\frac{\pi}{2}, \lambda, \lambda)C(\frac{\pi}{2}, \lambda, \lambda) = -1, \quad w \in \Omega_{00}^-; \quad (12) \]

\[ P_{\frac{\pi}{2}00}(\Omega_{\frac{\pi}{2}00}) = 1, \quad P_{0\frac{\pi}{2}0}(\Omega_{0\frac{\pi}{2}0}) = 1, \quad P_{00\frac{\pi}{2}}(\Omega_{00\frac{\pi}{2}}) = 1; \quad (13) \]

\[ P_{\frac{\pi}{2}00}(\Omega_{\frac{\pi}{2}00}) = 1. \quad (14) \]

The following two assumptions will play an important role in our further considerations:

(A) The space of hidden variables \( \Omega_{\varphi_1, \varphi_2, \varphi_3} \) does not depend on shifts \( \varphi_1, \varphi_2, \varphi_3 \) (the sets of possible microstates of apparatuses do not depend on shifts).

(B) The distribution of hidden variables \( P_{\varphi_1, \varphi_2, \varphi_3} \) does not depend on shifts \( \varphi_1, \varphi_2, \varphi_3 \)

Under assumption (A) we can set \( \Omega = \Omega_{\varphi_1, \varphi_2, \varphi_3} \). Here we can define the set \( \Sigma^+ = \Omega_{\frac{\pi}{2}00}^+ \cap \Omega_{0\frac{\pi}{2}0}^+ \cap \Omega_{00\frac{\pi}{2}}^+ \).

It is evident that \( \Sigma^+ \subset \Omega_{\frac{\pi}{2}2\frac{\pi}{2}}^+ \). Thus (14) implies that

\[ P_{\frac{\pi}{2}2\frac{\pi}{2}2}(\Omega_{\frac{\pi}{2}2\frac{\pi}{2}2}) = 0. \quad (15) \]

Under assumption (A)+(B) we can set \( \Omega = \Omega_{\varphi_1, \varphi_2, \varphi_3} \) and \( P = P_{\varphi_1, \varphi_2, \varphi_3} \). We can omit indexes of probability distributions in (13) and (15) and obtain \( P(\Sigma^+) = 1 \) and \( P(\Sigma^+) = 0 \). This is the GHZ paradox.

To obtain the GHZ paradox, we must assume (A) and (B). The assumption (A) seems to be quite natural: even if some hidden parameters \( w \) for shifts configuration \( \varphi_1, \varphi_2, \varphi_3 \) are eliminated by other shifts configuration \( \varphi_1', \varphi_2', \varphi_3' \), we can still assume that they belong to the space \( \Omega_{\varphi_1', \varphi_2', \varphi_3'} \), by setting \( P_{\varphi_1', \varphi_2', \varphi_3'}(w) = 0 \). However, we have to recognize that the assumption (B) has no physical justification at the present level of quantum experiments:

1). It seems that by changing the experimental arrangement (configuration of phase shifts) we change the distribution of hidden variables (corresponding to quantum particles+measurement devices), so \( P_{\varphi_1, \varphi_2, \varphi_3} \neq P_{\varphi_1', \varphi_2', \varphi_3'} \).

2). Distributions used in GHZ considerations are induced by four different runs of the experiment. It may be that distributions of hidden variables (even for \( \lambda \)) fluctuate from run to run. This is the hypothesis of nonreproducibility.
(see [5]). At the moment we have neither arguments against this hypothesis nor in favour of this hypothesis.

Further considerations will be performed under assumption (A).

**Remark 1.2.** (The original GHZ arguments). The original GHZ scheme [8] was based on the consideration of angles, $\alpha, \beta, \gamma, \delta$. We follow this scheme. We set $\Pi(\alpha, \beta, \gamma, \delta) = A(\alpha, \lambda)B(\beta, \lambda)C(\gamma, \lambda)D(\delta, \lambda)$, where $A, B, C, D$ are physical observables considered in [8]. Greenberger, Horne and Zeilinger obtained the following conditions:

$$\Pi(\alpha, \beta, \gamma, \delta) = 1, \quad \alpha + \beta + \gamma + \delta = 0,$$

and

$$\Pi(\alpha, \beta, \gamma, \delta) = -1, \quad \alpha + \beta + \gamma + \delta = \pi.$$  

Then they remarked: "But it turns out that there is no way to satisfy this condition. It is too restrictive, because we can continuously vary two of the parameters while keeping the other two constant. This leads to the conclusion that $A = B = C = D = constant$. But it is impossible, since the product sometimes equals +1 and sometimes equals -1. This is true for any value of $\lambda$ so that there is no need to integrate over it," [8], p.72.

Unfortunately these considerations are based on a rather elementary misunderstanding of the notion of probability 1, see, for example, [9], [10]. In fact, the probability 1 arguments need not imply that something "is true for any value of $\lambda." Such arguments only imply that, for example, (16) holds true for $\lambda$ belonging to a set $\Omega_{\alpha,\beta,\gamma,\delta}^+$ which has the probability measure 1. $\Omega_{\alpha,\beta,\gamma,\delta}^+$ may depend on the parameters $\alpha, \beta, \gamma, \delta$. Moreover, this is the typical situation even in ‘classical probabilistic models’, see, for example, [10]. Therefore we could not vary the parameters $\alpha, \beta, \gamma, \delta$ for a fixed value of $\lambda$. Thus there are no reasons to suppose (as it was done by Greenberger, Horne and Zeilinger) that $A = B = C = D = constant$. Finally, we remark that the assumption that the set $\Omega_{\alpha,\beta,\gamma,\delta}^+$ depends on the experimental settings $\alpha, \beta, \gamma, \delta$ is nothing than a contextualist assumption.

### 3 GHZ scheme for absolutely continuous and singular variations of probability distributions of hidden variables

1. Absolutely continuous and singular probability distributions.
Let $P'$ and $P''$ be two probability measures. $P''$ is absolutely continuous with respect to $P'$ if $P''(E) = 0$ whenever $P'(E) = 0$, $E \in F$ ($P'' << P'$). $P''$ is singular with respect to $P'$ if there is a set $E \in F$ such that $P''(E) = 1$ and $P'(E) = 0$ ($P'' \perp P'$). If $P''$ and $P'$ are mutually absolutely continuous, they are called equivalent.

2. GHZ paradox for equivalent probability distributions of hidden variables. Suppose that different settings $\varphi_1, \varphi_2, \varphi_3$ (and different runs of the experiment) produce in general different probability distributions $P_{\varphi_1, \varphi_2, \varphi_3}$, but they are absolutely continuous with respect to each other: $P_{\varphi_1, \varphi_2, \varphi_3}$ is equivalent to $P_{\varphi_1', \varphi_2', \varphi_3'}$. Thus

$$P_{\varphi_1', \varphi_2', \varphi_3'}(dw) = f(w; \varphi_1' \varphi_2' \varphi_3') P_{\varphi_1, \varphi_2, \varphi_3}(dw),$$

where $f$ is the density function.

'Theorem'. Suppose that probability distributions of hidden variables $P_{\varphi_1, \varphi_2, \varphi_3}$ and $P_{\varphi_1', \varphi_2', \varphi_3'}$ are equivalent for all possible settings of measurement devices in the GHZ scheme. Then GHZ arguments imply that quantum formalism and local realism are incompatible.

Proof. As $P_{\varphi_1, \varphi_2, \varphi_3}$ is absolutely continuous with respect to $P_{\varphi_1', \varphi_2', \varphi_3}$, we obtain that $P_{\varphi_1, \varphi_2, \varphi_3}(\Omega^+_{\varphi_1, \varphi_2, \varphi_3}) = 1 \rightarrow P_{\varphi_1, \varphi_2, \varphi_3}(\Omega^+_{\varphi_1, \varphi_2, \varphi_3}) = 1 \rightarrow P_{\varphi_1, \varphi_2, \varphi_3}(\Omega^+_{\varphi_1, \varphi_2, \varphi_3}) = 1$. Thus $P_{\varphi_1, \varphi_2, \varphi_3}(\sum^+) = 1$. On the other hand, as usual, we have $P_{\varphi_1', \varphi_2', \varphi_3}(\sum^+) = 0$.

3. No GHZ inference for singular probability distributions of hidden variables. Suppose that ensemble fluctuations produce singular distributions of hidden variables. Thus $P_{\varphi_1, \varphi_2, \varphi_3} \perp P_{\varphi_1', \varphi_2', \varphi_3}$.

Suppose that $\Omega^+_{\varphi_1, \varphi_2, \varphi_3}, \Omega^+_{\varphi_1', \varphi_2', \varphi_3}$ play the role of the set $E$ in the definition of singularity for distributions $P_{\varphi_1, \varphi_2, \varphi_3}$ and $P_{\varphi_1', \varphi_2', \varphi_3}$, $P_{\varphi_1, \varphi_2, \varphi_3}$ and $P_{\varphi_1', \varphi_2', \varphi_3}$, respectively. Then we have:

- $P_{\varphi_1, \varphi_2, \varphi_3}(\Omega^+_{\varphi_1, \varphi_2, \varphi_3}) = 1$ and $P_{\varphi_1, \varphi_2, \varphi_3}(\Omega^+_{\varphi_1, \varphi_2, \varphi_3}) = 1$.
- $P_{\varphi_1', \varphi_2', \varphi_3}(\Omega^+_{\varphi_1', \varphi_2', \varphi_3}) = 1$ and $P_{\varphi_1', \varphi_2', \varphi_3}(\Omega^+_{\varphi_1', \varphi_2', \varphi_3}) = 0$.

Therefore $P_{\varphi_1, \varphi_2, \varphi_3}(\sum^+) = 0$. Thus there is no GHZ ‘paradox’.

4. Infinite-dimensional spaces of hidden variables and GHZ paradox. Let $\Omega$ be an infinite dimensional linear space. Singularity of probability measures on $\Omega$ is quite typical. We shall consider the example of Gaussian distributions on a Hilbert space $\Omega$ of hidden variables (here $F$ is a $\sigma$-field of Borel sets). We shall demonstrate that singularity of Gaussian
probabilities can be induced by negligibly small perturbations of parameters of these distributions. Thus, in principle, we may obtain singular distributions of hidden variables in different runs of the experiment due to negligibly small fluctuations of parameters in the preparation device (as well as measurement devices). Let $\xi(\omega)$ and $\xi'(\omega)$ be Gaussian random variables in the Hilbert space $\Omega$ with mean values $a$ and $a'(\in \Omega)$ and covariation operators $B$ and $B'$, respectively (see, for example, [12]).

First we consider the case in that ensemble fluctuations can change only mean values: $\delta a = a' - a$. Let $\{e_j\}_{j=1}^\infty$ be the orthonormal basis in $\Omega$ consisting of eigenvectors of the covariation operator $B : Be_j = b_j e_j, j = 1, \ldots, \infty$. We suppose that $B > 0$, so $b_j > 0$. Let $P = P_\xi$ and $P' = P_{\xi'}$ be probability distributions of Gaussian random variable $\xi$ and $\xi'$. They are singular if

$$\sum_{j=1}^\infty \frac{(\delta a_j)^2}{b_j} = \infty, \quad \delta a = (\delta a_1, \ldots, \delta a_N, \ldots).$$

(19)

For example, suppose that $\delta a_j = \epsilon \sqrt{\frac{b_j}{j}}$, where $\epsilon > 0$ is an arbitrary small constant. Then $P \perp P'$. We remark that the covariation operator $B$ of a Gaussian measure is a nuclear operator in the Hilbert space $\Omega$. Thus $\sum_{j=1}^\infty b_j < \infty$. So $\delta a_j \to 0, j \to \infty$. For example, let $b_j = \frac{1}{j^2}$ and $\epsilon = 10^{-100}$. The perturbation $\delta a_j = 10^{-100}/j^{3/2}, j = 1, 2, \ldots$, would imply singularity. Therefore, to escape singularity in different runs, we must have extremely good statistical reproducibility.

We recall that $b_j = \mathbb{E}(\xi_j - a_j)^2 = \int (e_j, \lambda - a)^2 P(d\lambda)$ is dispersion of the (Gaussian) random variable $\xi_j = (e_j, \xi)$ (here $a_j = (e_j, a)$). Relation (19) implies that if we increase the sharpness of the distribution of hidden variables $b_j \to \gamma_j b_j, j = 1, \ldots, \gamma_j < 1$, then we must decrease the ranges of perturbations $\delta a_j$ to escape singularity. Hence, if we approach the domain of eigenstates for hidden variables $\lambda : b_j \approx 0$, then we have to have 100% reproducibility of statistical distributions of hidden variables ($\delta a_j \approx 0$) to escape the singularity.

**Conclusion.** A sharp preparation of hidden variables practically definitely implies the singularity of probability distributions of hidden variables for different runs of an experiment.

We now consider the effect of fluctuations of the parameter $B : B' = B + \delta B$, where $\delta B$ is a perturbation of the covariance operator $\delta B$. We
study the simplest case in that the operator $\delta B$ is diagonal in the basis $\{e_j\}_{j=1}^{\infty}$ (consisting of eigenvectors of $B$). We exclude from considerations the case $[B, B'] \neq 0$ (which may be interesting). So let $\delta Be_j = \delta b_j e_j, j = 1, \ldots, \infty, \delta b_j \geq 0$. It is assumed that $a' = a$. Gaussian distributions $P$ and $P'$ are singular if (see [12]):

$$\sum_{j=1}^{\infty} \left( \frac{\delta b_j}{b_j^2} \right)^2 = \infty.$$  

For example, suppose that $\delta b_j = \frac{\epsilon b_j}{\sqrt{j}}$: where $\epsilon > 0$ is an arbitrary small constant. Then $P \perp P'$. Thus singularity of distributions corresponding for different runs can be induced by negligibly small fluctuations of dispersion parameters. We again observe that if $\xi(\omega)$ gives a sharp distribution of hidden variables, namely $b_j \approx 0$ for all $j$, then, to escape singularity, we need to have practically precise reproducibility: $\delta b_j \approx 0$ for all $j$.

5. Physical meaning of infinite dimensional spaces of hidden variables. There are a few possible sources of the infinite dimension of the space of hidden variables $\Omega$: (1) Extremely complex structure (from the micro viewpoint) of measurement apparatuses. (2) It may be that physical observables have to be described as functions of the whole trajectories of hidden variables (for quantum systems and measurement apparatuses) in the process of interaction: $A = A(\lambda(\cdot), \lambda^a(\cdot)), B = B(\lambda(\cdot), \lambda^b(\cdot)), C = C(\lambda(\cdot), \lambda^c(\cdot))$. These trajectories are nothing than infinite dimensional hidden variables (compare with De Muynck and Stekelenborg in [6]). (3)(Bohm-Hiley conjecture, [13]) Quantum particles might have an extremely complex internal structure. Such a complexity can be described by infinite dimensional spaces of hidden variables.

In this paper we consider assumption (1). Each measurement apparatus consists of a huge number of quantum systems. If each quantum system can be described by a hidden variable $\lambda^a_j \in \mathbb{R}$, then the hidden variable of the whole apparatus $\lambda^a = (\lambda^a_j)_{j=1}^{N}, N \rightarrow \infty$. The assumption on the Gaussian distribution of hidden variables for an apparatus is quite natural: the concrete setting of an apparatus is created by the concentration of parameters to some mean value $\lambda^a_0 = (\lambda^a_{j0})$ (corresponding to this setting).
4 Singularity/equivalence dichotomy, the GHZ paradox, the principle of complementarity.

Two Gaussian measures on an infinite dimensional space is either singular or equivalent (Hajek-Feldman dichotomy, [12]). Our considerations demonstrated that this mathematical fact has the close relation to foundations of quantum mechanics. If we use the GHZ scheme, but do not apply to non-locality or determinism, then it seems that to escape the GHZ paradox we have to assume that quantum measurement/preparation procedure generates singular distributions of hidden parameters for incompatible measurements; classical measurement/preparation procedure generate only equivalent distributions of hidden parameters.

We can speculate that the principle of complementarity is nothing than the exhibition of singularity of probability distributions of hidden variables for incompatible measurement.

Thus in classical measurements we always obtain equivalent probability distributions, in quantum measurements there are settings having singular probability distributions.

In fact, singularity/equivalence dichotomy is not a property of only Gaussian distributions on infinite-dimensional spaces. We have the general Kakutani dichotomy [10]:

**Theorem.** Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n, \ldots)$ be sequences of independent random variables for which $P(\eta_1, \ldots, \eta_n) \ll P(\xi_1, \ldots, \xi_n)$ for $n \geq 1$. Then either $P_\eta \ll P_\xi$ or $P_\eta \perp P_\xi$.

**Conclusion.** The rigorous hidden variables description of the GHZ measurements demonstrates that there are two classes of preparation/measurement procedures: quantum (which may produce singular probability distributions) and classical (which always produce absolutely continuous probability distributions).\(^2\)

Despite the general opinion, the hidden variables description need not imply the reduction of ‘quantum reality’ to ‘classical reality’. Although both realities can be described by deterministic hidden variables, there is the crucial difference in behaviour of probability distributions. It seems that we have found the origin of this difference: This is singularity/equivalence dichotomy which is a general property of the large class of distributions of

\(^2\)We assume that the preparation/measurement procedure depends on a huge (practically infinite) number of hidden parameters.
random variables.

5 GHZ scheme in the presence of ensemble fluctuations.

In this section we obtain the estimate of the measure of ensemble fluctuations $\epsilon$ which is induced by the GHZ scheme. The following considerations can be interesting only for models of hidden variables which do not have singularity/equivalence dichotomy.

1. **Metric on the space of measures.** Let $\mu$ be a signed measure defined on a $\sigma$-algebra $\mathcal{F}$ (of subsets of $\Omega$). Let $\mu = \mu^+ - \mu^-$, where $\mu^+, \mu^-$ are positive measures, be the Jordan decomposition of $\mu$, see, for example, [9]. The total variation of $\mu$ is defined as $\|\mu\| = \mu^+(\Omega) + \mu^-(\Omega)$. Let $\mu$ be a discrete signed measure which is concentrated on a sequence of points $\{\lambda_j\}_{j=1}^\infty$, namely $\mu(A) = \sum_{\lambda_j \in A} \mu(\lambda_j)$. Here $\|\mu\| = \sum_{j=1}^\infty |\mu(\lambda_j)|$. Let $\mu$ be a signed measure that is absolutely continuous with respect to a positive measure $\nu$: $\mu(d\omega) = f(\omega)\nu(d\omega)$, where $f: \Omega \to \mathbb{R}$ is a $\nu$-integrable function. Here $\|\mu\| = \int_{\Omega} |f(\omega)|\nu(d\omega)$.

Denote the space of all (signed) measures on $\mathcal{F}$ by the symbol $\mathcal{M}$. Set $\rho(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|$. This is a metric on the space $\mathcal{M}$ (and $\mathcal{M}$ is complete).

2. **Probability invariant.** Let $\psi$ be a quantum state. Denote the family of all probability distributions of hidden variables corresponding to $\psi$ by the symbol $\mathcal{T}_\psi$. Thus, for different runs corresponding to $\psi$, we prepare in general distinct elements of $\mathcal{T}_\psi$ (if Bell’s implicit assumption $\mathcal{B}$ is true, then $\mathcal{T}_\psi$ must be a singleton). Set

$$\epsilon_\psi = \sup \{\rho(P_1, P_2) : P_1, P_2 \in \mathcal{T}_\psi\}.$$  \hfill (21)

This is the probability invariant of the quantum state $\psi$.

3. **Influences of ensemble fluctuations.** We have

$$0 = P_{0000}(\bar{\Sigma}^+) = 1 - P_{0000}(\bar{\Sigma}^+) \geq 1 - P_{0000}(\bar{\Omega}^+_{000} - P_{0000}(\bar{\Omega}^+_{000} - P_{0000}(\bar{\Omega}^+_{000}) - P_{0000}(\bar{\Omega}^+_{000}) - P_{0000}(\bar{\Omega}^+_{000}) - 3\epsilon = 1 - 3\epsilon.$$  \hfill (22)

For a set $D$, the symbol $\bar{D}$ denotes the complement of $D$. So $\epsilon \geq 1/3$. Thus if the measure of ensemble fluctuation $\epsilon$ is larger than $1/3$, the GHZ scheme does not imply a contradiction between quantum formalism and local realism.
Example. Let $\Omega = \{w_1, \ldots, w_N\}$ and let $P$ and $P'$ be two discrete probability distributions: $P(w_j) = P_j$ and $P'(w_j) = P'_j$. Let $|P_j - P'_j| = \delta$. Then $\rho(P, P') = N\delta$. If $\delta N \geq 1/3$, i.e., $\delta \geq \frac{1}{3N}$, then there is no contradiction (via the GHZ scheme) between quantum formalism and local realism. If $N \gg 1$, then the presence of negligibly small ensemble fluctuations destroys the GHZ arguments.

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