Some Aspects of the Electromagnetic Multipole Expansions

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Abstract. Various procedures for expressing the multipolar expansion of the electromagnetic field are considered with application to the calculation of the radiated power. Some results from the literature are discussed and perspective of developing the subject is pointed out.

1. Introduction

The multipole expansion of the electromagnetic field is an useful tool in physics and can be found in any book on electrodynamics or on the theory of atomic and nuclear transitions. In most of the textbooks (see, e.g., [1, 2]) a full and systematic treatment is given in spherical coordinates, while for Cartesian coordinates the problem is presented fully only for the static case; the dynamic case is given only for the lowest order multipoles. A general procedure for the reduction of the multipole tensors represented by Cartesian coordinate components to fully symmetric traceless ones in the static case is given in [3, 4] and it is generalized in [5] to the dynamic case. This method is applied in [6, 7] to the radiation field.

In the present paper different procedures are applied for calculating the total power radiated by a confined system of charges. A first one represents the traditional method of expressing the field expansions by the multipole Cartesian tensors, applying finally the reduction of these tensors. The second one uses the reduction technique done in [5] but more systematically considered here. The results of [6, 7] are analysed and compared to those of other methods from the literature [9, 10]. The advantages of the Cartesian coordinates are emphasized firstly by the simplicity of the formalism: only algebraic manipulations and combinatorics are implied, no special functions being required. Secondly, the procedure initiated in [5] and used here leads to a nontrivial grouping of different multipolar contributions standing out the toroidal multipole contributions [11, 12, 13].

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In section 2 some basic formulae for multipole expansions are presented. Section 3 deals with the radiation field as well as with the expression and expansion of the total radiated power. In Section 4 the procedure of reduction of a tensor to a symmetric traceless one is given. The total radiated power is then treated in Section 5 using these reduced moments and a comparison with literature is made. Section 6 presents, in a more systematic and concise way, the results from [6, 7]. Then the total radiated power is expressed in Section 7 by the transformed moments. The conclusions are given in Section 8. In Appendix the proof of some formulas used throughout this paper is given, as well as the reduction scheme and a justification of the reduction procedure using the charge and current expansions.

2. Basic Formulae for the Multipole Expansions

Let us consider charge \( \rho(r, t) \) and current \( j(r, t) \) distributions having supports included in a finite domain \( D \). Choosing the origin \( O \) of the Cartesian coordinates in \( D \), and using the notation \( e_i \) for the orthogonal unit vectors along the axes, the retarded scalar and vector potentials at a point outside \( D \), \( r = x_i e_i \), are

\[
\Phi(r, t) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\xi, t - R/c)}{R} d^3\xi, \quad A(r, t) = \frac{\mu_0}{4\pi} \int \frac{j(\xi, t - R/c)}{R} d^3\xi
\]

where \( R = r - \xi \). The Taylor series expansion of the function \( f(R) \) is

\[
f(R) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \xi_{i_1} \cdots \xi_{i_n} \partial_{i_1 \cdots i_n} f(r) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \xi^n || \nabla^n f(r)
\]

where

\[
\partial_{i_1 \cdots i_n} = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}}
\]

and \( a^n \) is the \( n \)-fold tensorial product \( a \otimes \cdots \otimes a : (a \otimes \cdots \otimes a)_{i_1 \cdots i_n} = a_{i_1} \cdots a_{i_n} \). Denoting by \( T^n \) a \( n \)-th order tensor, \( A^{(n)}||B^{(m)} \) is a \( |n - m| \)-th order tensor with the components:

\[
(A^{(n)}||B^{(m)})_{i_1 \cdots i_{|n-m|}} = \begin{cases} A_{i_1 \cdots i_{n-m}j_1 \cdots j_m} B_{j_1 \cdots j_m} & , n > m \\ A_{j_1 \cdots j_n} B_{j_1 \cdots j_n} & , n = m \\ A_{i_1 \cdots j_n} B_{j_1 \cdots j_n i_1 \cdots i_{m-n}} & , n < m \end{cases}
\]

By applying the formula for the Taylor series expansion to the scalar potential we get:

\[
\Phi(r, t) = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho(\xi, t - R/c)}{r} d^3\xi \left[ \frac{P^{(n)}(t - r/c)}{r} \right], \quad P^{(n)}(t) = \int_\mathcal{D} \rho(r, t) d^3x,
\]

\( P^{(n)} \) being the \( n \)-th order electric multipole tensor.

For the vector potential we obtain the expression:

\[
A(r, t) = \frac{\mu_0}{4\pi} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \nabla^n || \left[ \frac{\mu^{(n+1)}(t - r/c)}{r} \right] \]

\[
= \frac{\mu_0}{4\pi} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \epsilon_i \partial_{i_1} \cdots \partial_{i_n} \left[ \frac{\mu_{i_1 \cdots i_n}(t_0)}{r} \right], \quad t_0 = t - r/c
\]

(3)
In the previous equation the magnetic multipole tensor was introduced by its Cartesian components:

\[ \mu_{i_1 \ldots i_n} = \int_D x_{i_1} \ldots x_{i_n} j_i(r, t) d^3 x \]

### 3. The Radiation Field

The formula for the power radiated by a charged system described by the charge \( \rho \) and current \( j \) densities with supports included in a finite domain \( D \) is well known [2]:

\[
J(\nu) = \frac{dP}{d\Omega}(\nu, t) = \frac{r^2}{\mu_0 c} \left[ \nu \times \frac{\partial}{\partial t} A_{rad}(r, t) \right]^2
\]

(4)

Here, \( \nu = r/r \) and \( dP/d\Omega \) is related to the flow of the energy detected in the observation point \( r \) at large distance \( r \) compared with the dimensions of the given charged system. The vector \( A_{rad} \) is obtained from the retarded potential (1) by retaining only the dominant terms at large distances.

In the following the expansion of \( A_{rad} \) is derived [4, 6]. Starting from:

\[
A(r, t) \approx \frac{\mu_0}{4\pi r} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \nabla^n |\mu^{(n+1)}(t - r/c)\|
\]

\[
\approx \frac{\mu_0}{4\pi} e_i \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \partial_{i_1} \ldots \partial_{i_n} \mu_{i_1 \ldots i_n}(t - r/c)
\]

and considering:

\[
\partial_{i_1} \ldots \partial_{i_n} f(t - r/c) = \frac{(-1)^n}{c^n} \nu_{i_1} \ldots \nu_{i_n} \frac{d^n}{dt^n} f(t - r/c) + O(1/r)
\]

one obtains the following expression for the part of \( A \) contributing to the radiation:

\[
A_{rad}(r, t) = \frac{\mu_0}{4\pi} e_i \sum_{n=0}^{\infty} \frac{1}{n!c^n} \nu_{i_1} \ldots \nu_{i_n} \frac{d^n}{dt^n} \mu_{i_1 \ldots i_n}(t - r/c)
\]

\[
= \frac{\mu_0}{4\pi} \sum_{n \geq 0} \frac{1}{n!c^n} [\nu^n |\mu^{(n+1)}_n|]
\]

(5)

with the notation:

\[
\mu^{(n+1)}_n = \frac{d^n}{dt^n} \mu^{(n+1)}
\]

So, considering the angular distribution of the radiation given by equation (4) and applying the expansion of \( A_{rad} \), one gets:

\[
J(\nu) = \frac{1}{16\pi^2 \varepsilon_0 c^2} \left[ \nu \times \sum_{n \geq 0} \frac{1}{n!c^n} [\nu^n |\mu^{(n+1)}_n|] \right]^2
\]

(6)

Finally the following result for the angular distribution of the radiation is obtained:

\[
16\pi^2 \varepsilon_0 c^2 J(\nu) = \sum_{n \geq 0, m \geq 0} \frac{1}{n!m!c^{n+m}} [\varepsilon_{ijk} \varepsilon_{ij'k'} \nu_{j'} \nu_{i_1} \ldots \nu_{i_n} \nu_{j_1} \ldots \nu_{j_m} \frac{d^{n+1}}{dt^{n+1}} \mu_{i_1 \ldots i_n k'}]
\]
from the values of the field in the zone where $\lambda >> d$

In the previous equation it is useful an averaging formula introduced in $[6, 7]$ and justified in $[8]$:

$$J = \frac{1}{4\pi\varepsilon_0 c^3} \sum_{n\geq 0, m\geq 0} \frac{1}{n! m! c^{n+m}} \left[ \langle \nu_1 \cdots \nu_n \nu_{n+1} \cdots \nu_{n+m} \frac{d^{n+1}}{dt^{n+1}} \mu_{i_1 \cdots i_n} \rangle \sum_{D(i)} \delta_{i_1 i_2 \cdots i_{n+2}} \delta_{i_{n+1} i_{n+2}} \right]$$

In the previous equation it is useful an averaging formula introduced in $[6, 7]$ and justified in $[8]$:

$$\langle \nu_1 \cdots \nu_{i_0} \cdots \nu_{i_0} \rangle = \frac{1}{(2n + 1)!! \sum_{D(i)} \delta_{i_1 i_2 \cdots i_{n+2}} \delta_{i_{n+1} i_{n+2}}$$

When working with the expansion $[7]$ of the radiated power, one must have a strict criterion regarding the comparison of the different terms contributions. This criterion can be easily obtained if one refers, particularly, to monochromatic sources. The possibility to represent any type of variation in time as a superposition of functions corresponding to the monochromatic sources is taken into account, the conclusions will be general. In the case of the variation characterized by the pulsation $\omega$, the term from the expansion indexed with the pair $(n, m)$ contributes with an order of magnitude equal to $(d/\lambda)^{n+m}$ to the radiated power. Since the expression of the radiated power comes from the values of the field in the zone where $\lambda >> d$, it is obvious that the order of magnitude of a term is given by the sum $n + m$. Therefore, a consistent way of using a finite number of terms in the expansion $[7]$, terms that accurately characterize the radiation up to a given order $M$, is to retain all the terms corresponding to the values $n + m$ between 0 and $M$. Apparently, in literature such a procedure was not consistently used everywhere, as it will be shown later on.

So, considering $J^{(M)}$ the $M$ order term from the expansion $[7]$, we have

$$J^{(M)} = \sum_{n+m=M} J^{(n,m)}.$$  

$$4\pi\varepsilon_0 c^3 J^{(n,m)} = \frac{1}{n! m! c^{n+m}} \left( \langle \nu^{n} \rangle \langle \mu_{n+1} \rangle \right) \sum_{D(i)} \langle \nu_{n+1} \rangle \langle \mu_{m+1} \rangle .$$

For particular cases,

$$J^{(0)} = J^{(0,0)}, J^{(1)} = J^{(0,1)} + J^{(1,0)} = 0,$$

$$J^{(2)} = J^{(0,2)} + J^{(2,0)} + J^{(1,1)}, J^{(3)} = 0,$$

$$J^{(4)} = J^{(2,2)} + J^{(1,3)} + J^{(3,1)} + J^{(0,4)} + J^{(4,0)}.$$
where the fact that the averaged terms with $n + m$ odd are zero is considered.

Below we will present the first terms of the expansion:

$$4\pi\varepsilon_0 c^3 J^{(0,0)} = \langle \hat{\mu}_i \hat{\mu}_i - (\mathbf{v} \cdot \hat{\mu}) \rangle_i = \hat{\mu}_i^2 = \nu_i \nu_j - \hat{\mu}_i \hat{\mu}_i = \frac{2}{3} \hat{\mu}_i^2$$

$$4\pi\varepsilon_0 c^3 J^{(1,1)} = \frac{1}{c^2} \left( \langle \mathbf{v} \| \mathbf{\hat{\mu}} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i - \langle \mathbf{v} \| \hat{\mu} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i \right)$$

$$= \frac{1}{c^2} \left[ \nu_i \nu_j - \hat{\mu}_i \hat{\mu}_j \right]$$

$$4\pi\varepsilon_0 c^3 J^{(0,2)} = \frac{1}{2c^2} \left( \langle \mathbf{v} \| \mathbf{\hat{\mu}} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i - \langle \mathbf{v} \| \hat{\mu} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i \right)$$

$$= \frac{1}{2c^2} \left[ 2 \hat{\mu}_i \hat{\mu}_j - \hat{\mu}_i \hat{\mu}_j \right]$$

$$4\pi\varepsilon_0 c^3 J^{(2,2)} = \frac{1}{4c^4} \left( \langle \mathbf{v} \| \mathbf{\hat{\mu}} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i - \langle \mathbf{v} \| \hat{\mu} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i \right)$$

$$= \frac{1}{4c^4} \left[ \nu_i \nu_j - \hat{\mu}_i \hat{\mu}_j \right]$$

$$4\pi\varepsilon_0 c^3 J^{(1,3)} = \frac{1}{6c^6} \left( \langle \mathbf{v} \| \mathbf{\hat{\mu}} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i - \langle \mathbf{v} \| \hat{\mu} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i \right)$$

$$= \frac{1}{6c^6} \left[ \nu_i \nu_j - \hat{\mu}_i \hat{\mu}_j \right]$$

$$4\pi\varepsilon_0 c^3 J^{(0,4)} = \frac{1}{24c^8} \left( \langle \mathbf{v} \| \mathbf{\hat{\mu}} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i - \langle \mathbf{v} \| \hat{\mu} \| (\mathbf{v} \cdot \hat{\mu}) \rangle_i \right)$$

$$= \frac{1}{24c^8} \left[ \nu_i \nu_j - \hat{\mu}_i \hat{\mu}_j \right]$$

(10)

For $J^{(2,2)}$, $J^{(1,3)}$, $J^{(0,4)}$, the results of the contractions with the $\delta$-tensors will be given bellow by a simpler method.

The expressions for $J^{(0,0)}$, $J^{(1,1)}$, $J^{(0,2)}$, $J^{(2,0)}$ are given by Bellotti and Bornatici in [9]. They go further, introducing the reduced multipole moments and finding a new term, as we will see in the following.

Sometimes it is easier to use the magnetic moments defined as:

$$M^{(n)} = \frac{n}{n + 1} \int_D \mathbf{\xi}^n \times \mathbf{j} d^3 \mathbf{\xi} : M_i \ldots i_n = \frac{n}{n + 1} \int_D \mathbf{\xi}_i \ldots \mathbf{\xi}_{i_{n-1}} (\mathbf{\xi} \times \mathbf{j})_{i_n} d^3 \mathbf{\xi}$$

(11)

It can be shown that instead of the expansion [3] one can use an expansion obtained from this one by performing the substitution:

$$\mu_{i_1 \ldots i_n} \rightarrow -\varepsilon_{i_{n-1} i_n} M_{i_1 \ldots i_n} - k + \frac{1}{n} \hat{\mu}_{i_1 \ldots i_n}$$

(12)

The result for the vector potential is given in [4] [5]:

$$A(r, t) = \frac{\mu_0}{4\pi} \nabla \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \left[ \frac{1}{r} M^{(n)}(t - r/c) \right]$$

$$+ \frac{\mu_0}{4\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \left[ \frac{1}{r} P^{(n)}(t - r/c) \right]$$

(13)
The transformation (12) is used in the equations (10):

\[4\pi\varepsilon_0c^3J^{(0,0)} = \mu_t\mu_i - \nu_i\nu_t > \mu_t\mu_i = \frac{2}{3}p^2\]

\[4\pi\varepsilon_0c^3J^{(1,1)} = \frac{1}{c^2} < \nu_i\nu_t > (-\varepsilon_{ii1k}\tilde{m}_k + \frac{1}{2}\tilde{P}_{i1i})(-\varepsilon_{i1k}\tilde{m}_k + \frac{1}{2}\tilde{P}_{i1i})\]

\[-<\nu_i\nu_{i2}\nu_{i3}\nu_{i4}> (-\varepsilon_{i21k}\tilde{m}_k + \frac{1}{2}\tilde{P}_{i1i})(-\varepsilon_{i23k}\tilde{m}_k + \frac{1}{2}\tilde{P}_{i1i})\]

\[= \frac{1}{c^2} \left[ <\nu_i\nu_t > \left( \varepsilon_{ii1k}\varepsilon_{i1k}\tilde{m}_k\tilde{m}_k - \frac{1}{2}\varepsilon_{ii1k}\tilde{m}_k\tilde{P}_{i1i} - \frac{1}{2}\varepsilon_{ii1k}\tilde{m}_k\tilde{P}_{i1i}\right)\right] + \frac{1}{4}\tilde{P}_{i1i}\tilde{P}_{i1i} - \frac{1}{4} <\nu_i\nu_t > \tilde{P}_{i1i}\tilde{P}_{i1i}\]

Here the fact that the contractions of symmetric and antisymmetric pairs of indices cancel is considered. Because \(<\nu_i\nu_t > = (1/3)\delta_{ij}, <\nu_i\nu_{i2} > \varepsilon_{ii1k}\tilde{m}_k\tilde{P}_{i1i} = (1/3)\varepsilon_{ii1k}\tilde{m}_k\tilde{P}_{i1i} = 0 \quad \text{etc} \quad \text{and} \quad \varepsilon_{ii1k}\varepsilon_{i1k} = 2\delta_{kl}, \quad \text{so what is left is:}\]

\[4\pi\varepsilon_0c^3J^{(1,1)} = \frac{2}{3}\tilde{m}^2 + \frac{1}{20c^2}\tilde{P}_{ij}\tilde{P}_{ij} - \frac{1}{60c^2}\tilde{P}_{ij}\tilde{P}_{ij}\]

\[4\pi\varepsilon_0c^3J^{(2,0)} = 4\pi\varepsilon_0c^3J^{(2,0)} = \frac{1}{2c^2} \left( \mu_t(\nu^2||\mu^3) - (\tilde{v} \cdot \tilde{\mu})(\nu^3||\mu^3) \right)\]

\[= \frac{1}{2c^2} \left[ <\nu_i\nu_t > \mu_t \tilde{\mu}_{i12} - <\nu_i\nu_{i2} > \mu_t \tilde{\mu}_{i23}\right]\]

\[= \frac{1}{2c^2} \left[ <\nu_i\nu_t > \tilde{\mu}_i(-\varepsilon_{ii1k}\tilde{M}_{i1k} + \frac{1}{3}\tilde{P}_{i1i}) - <\nu_i\nu_{i2} > \tilde{\mu}_i\right]\]

\[<\nu_i\nu_{i3}\nu_{i4} > \tilde{\mu}_{4i3}(1 + \frac{1}{3}\tilde{P}_{i1i}) = \frac{1}{2c^2} \left[ <\nu_i\nu_t > \tilde{\mu}_i + \frac{1}{3}\tilde{P}_{i1i} \tilde{P}_{i1i} \right]\]

where

\[N_i = \varepsilon_{ips}M_{ps} = \frac{2}{d}\int [\xi \times (\xi \times j)]d^3\xi\]

\[4\pi\varepsilon_0c^3J^{(2,2)} = \frac{1}{4c^4} \left[ <\nu_i\nu_{i4} > \tilde{\mu}_{i12}	ilde{\mu}_{i12} - <\nu_i\nu_{i6} > \tilde{\mu}_{i12} \tilde{\mu}_{i12} \right]\]

\[= \frac{1}{4c^4} \left[ <\nu_i\nu_{i4} > \left( \varepsilon_{ii1k}\varepsilon_{i1k}\tilde{M}_{i1k}\tilde{M}_{i1k} - \frac{2}{3}\varepsilon_{ii1k}\tilde{M}_{i1k}\tilde{P}_{i1i}\right)\right] + \frac{1}{9}\tilde{P}_{i1i}\tilde{P}_{i1i} - \frac{1}{9} <\nu_i\nu_{i6} > \tilde{P}_{i1i}\tilde{P}_{i1i}\]

\[= \frac{1}{105} \left( \tilde{P}_{i1i}\tilde{P}_{i1i} \right) + \frac{2}{3}\tilde{P}_{i1i}\tilde{P}_{i1i}\]

\[= \frac{1}{105} \left( \tilde{P}_{i1i}\tilde{P}_{i1i} \right) + \frac{2}{3}\tilde{P}_{i1i}\tilde{P}_{i1i}\]

\[4\pi\varepsilon_0c^3J^{(1,3)} = 4\pi\varepsilon_0c^3J^{(3,1)}\]

\[= \frac{1}{105} \left( \tilde{P}_{i1i}\tilde{P}_{i1i} \right) + \frac{2}{3}\tilde{P}_{i1i}\tilde{P}_{i1i}\]

\[-\frac{1}{2}\varepsilon_{ii1k}\varepsilon_{i1k}\tilde{M}_{i1k}\tilde{P}_{i1i} + \frac{1}{8}\tilde{P}_{i1i}\tilde{P}_{i1i}\]

\[-\frac{1}{8} <\nu_i\nu_{i4} > \tilde{P}_{i1i}\tilde{P}_{i1i}\]
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$$= \frac{1}{64c^4} \left[ \frac{4}{15} \tilde{M}_{qqk} - \frac{1}{15} N_{ij}^{(3,1)} \tilde{P}_{li} - \frac{3}{8 \times 105} \tilde{P}_{qij} + \frac{9}{8 \times 105} \tilde{P}_{ij} \tilde{P}_{qqij} \right]$$

Here, again, there is a notation

$$N_{ij}^{(3,1)} = \varepsilon_{jps} M_{ips}.$$  \hfill (15)

$$4\pi\varepsilon_0 c^3 J^{(0,4)} = 4\pi\varepsilon_0 c^3 J^{(4,0)}$$

$$= \frac{1}{24c^4} \left[ < \nu_1 \ldots \nu_4 > \tilde{p}_i (-\varepsilon_{i4k} \tilde{M}_{i1i2i3i4} + \frac{1}{5} \tilde{p}_{i1i2i3i4}) \right.$$

$$\left. - < \nu_1 \ldots \nu_6 > \tilde{p}_i (-\varepsilon_{i4k} \tilde{M}_{i1i2i4i6} + \frac{1}{5} \tilde{p}_{i2i3i4i5i6}) \right]$$

The terms where the symmetric-antisymmetric tensor contractions are present cancel again, giving the final result for $J^{(0,4)}$:

$$4\pi\varepsilon_0 c^3 J^{(0,4)} = 4\pi\varepsilon_0 c^3 J^{(4,0)} = \frac{1}{24c^4} \left[ -\frac{1}{5} \varepsilon_{ijk} \tilde{M}_{qqjk} \tilde{p}_i + \frac{2}{5 \times 35} \tilde{p}_i \tilde{P}_{qqij} \right]$$

4. Symmetrising and ”Detracing” the Tensors

In this section the symmetrisation and detracing method specific for the electromagnetic moments is presented. The procedure and notations from [14] and [3] are used, as well as a theorem introduced by Applequist in [14]. The first step is to symmetrise the magnetic tensors $M^{(n)}$, already symmetric in the first $n-1$ indices:

$$M_{(sym)i_1 \ldots i_n} = \frac{1}{n!} \left[ M_{i_1 \ldots i_n} + M_{i_2 \ldots i_1} + M_{i_3 \ldots i_2} + \ldots + M_{i_1 \ldots i_{n-1} i_n} \right]$$

$$\equiv \sum_{D(i)} M_{i_1 \ldots i_n}$$

where $\sum_{D(i)}$ represents the sum over the independent terms only, from all the permutations of the $n$ indices. It can also be written as:

$$M_{(sym)i_1 \ldots i_n} = M_{i_1 \ldots i_n} - \frac{1}{n} \sum_{i=1}^{n-1} \varepsilon_{i_1 i_2 \ldots i_n} N_{i_1 \ldots i_{n-1} i_n}^{(n,1)}$$

where $T_{i_1 \ldots i_n}$ is the component which does not have the $i_\lambda$ index, while:

$$N_{i_1 \ldots i_{n-1}}^{(n,1)} = \varepsilon_{i_1 \ldots i_{n-1} p s} M_{i_1 \ldots i_{n-2} p s}$$

is a $n-1$ order tensor. Further, the correspondence:

$$T^{(n)} \rightarrow N[T^{(n)}]$$

is introduced, where $N[T^{(n)}]$ is a $n-1$ order tensor:

$$N[T^{(n)}]_{i_1 \ldots i_{n-1}} = \varepsilon_{i_1 \ldots i_{n-1} p s} T_{i_1 \ldots i_{n-2} p s}$$

and $N^k[M^{(n)}] \equiv N^{(n,k)}$ is a tensor of rank $n-k$. Here are some examples:

$$N_{i_1 \ldots i_{n-1}}^{(n,1)} = \frac{n}{n+1} \int \xi_{i_1} \ldots \xi_{i_{n-2}} [\xi \times (\xi \times j)]_{i_{n-1}} d^3 \xi$$
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\[ N_{i_1 \ldots i_{n-1}}^{(n,2)} = - \frac{n}{n+1} \int \xi^2 \xi_{i_1} \ldots \xi_{i_{n-3}} (\xi \times j)_{i_{n-2}} d^3 \xi \]

The following relations are also to be considered:

\[ N^{2k}[M^{(n)}] \equiv N^{(n;2k)} = \frac{(-1)^k n}{n+1} \int \xi^{2k} \xi^{n-2k} \times j d^3 \xi = (-1)^k M^{(n;k)}, \]

\[ N^{2k+1}[M^{(n)}] \equiv N^{(n;2k+1)} = \frac{(-1)^k n}{n+1} \int \xi^{2k} \xi^{n-2k-1} \times (\xi \times j) d^3 \xi = (-1)^k N^{(n;1;k)}, \]

\[ k = 0, 1, \ldots \]

where \( T^{(n-\cdot);k} \) is the tensor obtained from the contraction of \( k \) pairs of indices. Particularly, the tensors introduced by equations (14) and (15) are

\[ N_i = N_i^{(2,1)}, \quad N_{ij} = N_{ij}^{(3,1)}. \]

For the reduction of a totally symmetric tensor to a traceless one we write as in [3, 4]:

\[ \tilde{S}_{i_1 \ldots i_n} = S_{i_1 \ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda[S^{(n)}]_{i_3 i_4 \ldots i_n} \]

where \( S \) is a totally symmetric tensor and \( \Lambda[S^{(n)}] \) is a totally symmetric tensor of rank \( n - 2 \). Further the following notations are used:

\[ \Lambda[M^{(n)}] = \Lambda^{(n-2)}, \quad \Lambda[P^{(n)}] = \Pi^{(n-2)}. \]

Moreover, sometimes, for writing some equations in a simpler form, we will also use the notation \( \Lambda[T^{(n)}] \) for \( T^{(n)} \) an arbitrary tensor but by this notation we will suppose that the symmetrization of \( T^{(n)} \) is implied i.e.

\[ \Lambda[T^{(n)}] \equiv \Lambda[T^{(n)}_{sym}]. \]

In [14] a general procedure of detracing a symmetric tensor is presented. In our case the result of this procedure may be written as:

\[ \Lambda[S^{(n)}]_{i_3 \ldots i_n} = \sum_{m=1}^{[n/2]} \frac{(-1)^{m-1}(2n-1-2m)!!}{(2n-1)!!m} \times \]
\[ \times \sum_{D(i)} \delta_{i_3 i_4} \ldots \delta_{i_{2m-1} i_{2m}} S_{i_{2m+1} \ldots i_n}^{mn} \]

Using the definitions and notations presented above it is easy to obtain the following useful results:

- detracing the electric moment tensors:

\[ \Pi = \Lambda[P^{(2)}] : \quad \Pi = \frac{1}{3} P_{ii} \quad \text{(16)} \]

\[ \Pi^{(1)} = \Lambda[P^{(3)}] : \quad \Pi_{ii} = \frac{1}{3} P_{qii} \quad \text{(17)} \]

\[ \Pi^{(2)} = \Lambda[P^{(4)}] : \quad \Pi_{ij} = \frac{1}{7} P_{qij} - \frac{1}{70} P_{qij} \delta_{ij} \quad \text{(18)} \]
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\[ \Pi^{(3)} = \Lambda[M^{(5)}] : \Pi_{ijk} = \frac{1}{11}P_{qqijk} - \frac{1}{11 \times 18} \sum_{D(i)} \delta_{ij} P_{qllk} \]  \hspace{1cm} (19)

- detracing the magnetic moments tensors:

\[ \Lambda = \Lambda[M^{(2)}_{sym}] = 0 \]  \hspace{1cm} (20)
\[ \Lambda^{(1)} = \Lambda[M^{(3)}_{sym}] : \Lambda_i = \frac{1}{15}M_{qqi} \]  \hspace{1cm} (21)
\[ \Lambda^{(2)} = \Lambda[M^{(4)}_{sym}] : \Lambda_{ij} = \frac{1}{28}(M_{qqij} + M_{qiqj}) \]  \hspace{1cm} (22)
\[ \Lambda^{(3)} = \Lambda[M^{(5)}_{sym}] : \Lambda_{ijk} = \frac{1}{45} \sum_{D} M_{qqijk} - \frac{1}{14 \times 45} \sum_{D} \delta_{ij} M_{qllk} \]  \hspace{1cm} (23)

5. The Total Radiated Power in Terms of Reduced Moments

The usual procedure is to emphasize, in the expression of \( J \), the reduced tensors by utilizing the reduction relations given in the previous section. The well-known notations for dipole moments are used:

\[ P_i = p_i : \vec{p}, \quad M_i = m_i : \vec{m} \]

The static expressions of the reduced tensors of electric and magnetic polarizations are \( P^{(n)} \) and \( M^{(n)} \) with :

\[ P_{i_1...i_n} = \frac{(-1)^n}{(2n - 1)!!} \int_D \rho(r, t) r^{2n+1} \nabla^n \frac{1}{r} \, d^3 x \]

and [3],

\[ M_{i_1...i_n} = \frac{(-1)^n}{(n + 1)(2n - 1)!!} \sum_{\lambda=1}^{n} \int_D r^{2n+1} [j(r, t) \times \nabla]_{i_k} \delta^{(\lambda)}_{i_1...i_n} \frac{1}{r} \, d^3 x \]

Below there are the results of the reduction of the first terms in the expansion of \( J \), which correspond to the limitation to the order \( (d/\lambda)^4 \):

\[ 4\pi\varepsilon_0 c^3 J^{(0,0)} = \frac{2}{3} \vec{\rho}^2 \]
\[ 4\pi\varepsilon_0 c^3 (J^{(0,2)} + I^{(2,0)}) = -\frac{4}{3c^2} \vec{p}_i \vec{T}_i, \quad T_i = \frac{1}{4}N_i - \frac{1}{6} \vec{\Pi}_i \]
\[ 4\pi\varepsilon_0 c^3 I^{(1,1)} = \frac{2}{3c^2} \vec{m}^2 + \frac{1}{20c^2} \vec{\Pi}_{ij} \vec{\Pi}_{ij} \]
\[ 4\pi\varepsilon_0 c^3 J^{(2,2)} = \frac{1}{4c^4} \left[ \frac{1}{5} \vec{\Pi}_{ij} \vec{\Pi}_{ij} + \frac{1}{6} \vec{\Pi}_i \vec{\Pi}_i + \frac{2}{9} \vec{\Pi}_i \vec{\Pi}_i + \frac{2}{27} \vec{\Pi}_i \vec{\Pi}_i + \frac{3}{15} \vec{\Pi}_{ij} \vec{\Pi}_{ij} \right] \]
\[ 4\pi\varepsilon_0 c^3 (J^{(1,3)} + J^{(3,1)}) = \frac{1}{3c^4} \left[ \frac{4}{15} \vec{\Pi}_k \vec{\Pi}_{kk} - \frac{1}{15} \vec{\Pi}_{ij} \vec{\Pi}_{ij} + \frac{8}{45} \vec{\Pi}_{ij} \vec{\Pi}_{ij} \right] \]
\[ 4\pi\varepsilon_0 c^3 (J^{(0,4)} + J^{(4,0)}) = \frac{1}{12c^4} \left[ -\frac{1}{5} \vec{p}_i \vec{n}_{qqi} + \frac{4}{25} \vec{p}_i \vec{\Pi}_{jjj} \right] \]  \hspace{1cm} (24)
The toroid dipole moment \[11, 12, 13\] was introduced, with the definition:

\[
T_i = \frac{1}{4} N_i - \frac{1}{6} \dot{\Pi}_i
\]

There was also used the simplified notation \(N_i = N_i^{(2,1)}\), while \(\tilde{\Pi}^{(2)}\) is a traceless tensor (that is \(\Pi^{(2)}\) “detraced”):

\[
\tilde{\Pi}_{ik} = \Pi_{ik} - \frac{1}{3} \Pi_{qq} \delta_{ik} = \frac{1}{7} P_{qqik} - \frac{1}{21} P_{qll} \delta_{ik}
\]

The sum of terms from the equation (24) represents the total radiated power expanded until the fourth order with respect to \(d/\lambda\), that is:

\[
\sum_{n+m \leq 4} J^{(n,m)}
\]

It is verified the relation:

\[
\frac{2}{3} \dot{p}^2 - \frac{4}{3c^2} \hat{p}_i T_i + \frac{1}{24c^4} \tilde{N}_i \tilde{N}_i - \frac{1}{18c^4} \tilde{N}_i \tilde{\Pi}_i + \frac{1}{54} \tilde{\Pi}_i \tilde{\Pi}_i = \frac{2}{3} \left( \frac{\dot{p} - \frac{1}{c^2} \ddot{T}}{c^2} \right)^2
\]

By eliminating the higher order derivatives, we may write:

\[
\frac{1}{15} N_{ik} P_{ik} - \frac{3}{40} \tilde{\Pi}_{ik} P_{ik} = \frac{1}{15} \tilde{N}_{ik} P_{ik} - \frac{3}{40} \tilde{\Pi}_{ik} P_{ik} = \frac{3}{10} \left( \frac{1}{9} N_{ik} - \frac{1}{4} \tilde{\Pi}_{ik} \right) P_{ik}
\]

with the notation: \(N_{ik} = N_{ik}^{(3,1)}\). The so-called quadrupole toroid moment \[11, 12, 13, 15\] is also defined by:

\[
T_{ik} = \frac{1}{9} \tilde{N}_{ik} - \frac{1}{4} \tilde{\Pi}_{ik}
\]

Explicitly, the two toroid moments are written as:

\[
T_i = \frac{1}{6} \int_D [\xi \times (\xi \times j)]_i d^3 \xi - \frac{1}{30} \dot{P}_{qqi}
\]

\[
= \frac{1}{6} \int_D [\xi \cdot j] \xi_i - \xi^2 j_i] d^3 \xi - \frac{1}{30} \int_D \xi^2 \xi_i \dot{\rho} d^3 \xi
\]

but since:

\[
\int_D \xi^2 \xi_i \dot{\rho} d^3 \xi = - \int_D \xi^2 \xi_i \nabla \cdot j d^3 \xi = \int_D j \cdot (\xi^2 \xi_i) d^3 \xi = \int_D [2(\xi \cdot j) \xi_i + \xi^2 j_i] d^3 \xi
\]

it results:

\[
T_i = \frac{1}{10} \int_D [\xi \cdot j] \xi_i - 2 \xi^2 j_i] d^3 \xi
\]

In a similar way:

\[
T_{ik} = \frac{1}{42} \int_D [4(\xi \cdot j) \xi_i \xi_k - 5 \xi^2 (\xi_i j_k + \xi_k j_i) + 2 \xi^2 (\xi \cdot j) \delta_{ik}] d^3 \xi
\]
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After regrouping the terms and considering the above presented relations, one gets:

\[ 4\pi\varepsilon_0 c^3 J^{(4)} = \frac{2}{3} \left( \ddot{p} - \frac{1}{c^2} \dddot{T} \right)^2 + \frac{2}{3c^2} \dddot{m}^2 + \frac{1}{20c^2} \dddot{p}_{ij} \dddot{p}_{ij} - \frac{1}{10c^4} \dddot{T}_{ij} \dddot{p}_{ij} \]
\[ + \frac{4}{45c^4} \dddot{m}_k \dddot{M}_{qqk} - \frac{1}{60c^4} \dddot{p}_i \left( \dddot{N}_{qqi} - \frac{4}{5} \dddot{P}_{qqi} \right) + \frac{1}{20c^4} \dddot{M}_{ij} \dddot{M}_{ij} + \frac{2}{945c^4} \dddot{P}_{ijk} \dddot{P}_{ijk} \] (25)

If the expansion of the radiated power up to the second order with respect to \( d/\lambda \) is considered, only the following sum will be kept:

\[ J^{(2)} = J^{(0,0)} + J^{(1,1)} + 2T^{(0,2)} \]
\[ = \frac{1}{4\pi\varepsilon_0 c^3} \left[ \frac{2}{3} \dddot{p} + \frac{2}{3c^2} \dddot{m}^2 - \frac{4}{3c^2} \dddot{p}_i \dddot{T}_i + \frac{1}{20c^2} \dddot{p}_{ij} \dddot{p}_{ij} \right] \]

This is, actually, a result obtained by Bellotti and Bornatici [9]. They observe that, compared to the expression for the dipolar magnetic-quadrupolar electric radiation from the books of Jackson and Landau [11, 2], as well as from many other electrodynamics books (including the one by C. Vrejoiu [4]), this result contains a supplementary term represented by the contribution of the vector \( \dddot{T} \). We point out that in [11, 2] the goal is only to calculate the isolated contributions of the electric dipole, electric 4-pole and magnetic dipole to the total radiated power without regarding it as a result of an expansion. If one associates a multipole moment to an elementary system, one obtains an isolated contribution of this multipole, but when one considers a composite system, all multipoles, giving contributions of the same order of magnitude, must be considered. This is the reason for which the result from [11, 2] should not to be considered erroneous.

Without knowing, probably, the results published ever since 1974 by Dubovik et al. [11] and all the other publications referring to toroid moments, they consider this term as being something new. On the other hand, Bellotti and Bornatici give a correct result, if one considers how far they go with the expansion of the radiated power.

The impression is that in general in literature similar results contain a series of confusions due to the inconsistent application of the criteria according to which different terms of this expansion must be compared. For instance, in [13] a result which is supposed to represent the expansion including the fourth order terms is presented:

\[ 4\pi\varepsilon_0 c^3 J_{Dubovik} = \frac{2}{3} \left( \dddot{p} - \frac{1}{c^2} \dddot{T} \right)^2 + \frac{2}{3c^2} \dddot{m}^2 \]
\[ + \frac{1}{20c^2} \left( \dddot{p}_{ik} - \frac{1}{c^2} \dddot{T}_{ik} \right) \left( \dddot{p}_{ik} - \frac{1}{c^2} \dddot{T}_{ik} \right) \]
\[ + \frac{2}{945c^4} \left( \dddot{P}_{ijk} \dddot{P}_{ijk} + \dddot{M}_{ijk} \dddot{M}_{ijk} \right) + \ldots \] (26)

Apart from the mistake (of course, typo error) that \( T_{ik} \) is derived only three times, there are certain things that must be observed:

- the term \( M_{ijk} M_{ijk} \) from this expression is of the sixth order (with the 1/\( \lambda \) criterion);
- the term \( T_{ik} T_{ik} \) is also of the sixth order;
• these terms should not be present if in the expansion of $\mathcal{J}$ the terms up to the sixth order are not consiered;
• the term $\mathcal{M}_{ij}\mathcal{M}_{ij}$ is missing, as well as other terms that should be present, at least up to the fourth order;
• the sign three points ". . . " at the end of the expression is losing its usual sense when a finite number of terms is considered in a series expansion.

In some more recent papers, as for example in [10], it is claimed that, as an application of more general formulae, the expansion up to the orde r $1/c^5$ is given (that is, exactly how it is considered in [9]). The $1/c$ criterion is correlated with the wavelength criterion if one takes into account the powers of the parameter $1/c$ together with the orders of the partial temporal derivatives in the expansion of the radiated power. The expression given in [10] (written with our notations) is:

$$4\pi\varepsilon_0c^3 J_{R-V} = \frac{2}{3} \left( \vec{\hat{r}} - \frac{1}{c^2} \vec{\hat{T}} \right)^2 + \frac{4}{45c^4} \vec{\hat{r}} m_{kk} \vec{\hat{r}} + \frac{1}{20c^2} \vec{\hat{r}} P_{ij} \vec{\hat{r}}_{ij} + \frac{1}{20c^4} \mathcal{M}_{ij} \mathcal{M}_{ij}$$

(27)

where, in order to identify it with the equation (4.12) from [10], one must consider the definitions of the quadrupol moments and the identity:

$$M_{qqk} = \frac{3}{4} \int \xi^2 (\xi \times \vec{j})_k d^3\xi = \frac{1}{2} \vec{\rho}^2$$

this last parameter being the one used in the equation (27).

It is easy to observe that if one tries to interpret equation (27) as the expansion of $\mathcal{J}$ up to the second order, the fourth order term $T^2$ is present without justification. On the other hand, if the same equation is considered as an expansion up to the fourth order, many terms are missing.

6. Reduction of the Multipole Tensors and Gauge Invariance

In this section the results from [5, 7] will be presented in a more systematic and concise way. A different explanation of these results is given in Appendix A using the formalism of Dirac’s $\delta-$ function. These results present the possibility of expressing the electromagnetic potentials, as well as the field, exclusively in terms of reduced moments, that is moments represented by reduced tensors (symmetric and traceless). This procedure allows, particularly, to express the radiated power in a very simple general form, as it will be shown below.

Let us consider the expansion of the potentials with the moments $M^{(n)}$ (see equation (11)) and $P^{(n)}$ (see equation (2)):

$$A(r, t) = \frac{\mu_0}{4\pi} \nabla \times \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \left| \left[ \frac{1}{r} M^{(n)}(t - r/c) \right] \right|$$
$$+ \frac{\mu_0}{4\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \nabla^{n-1} \left| \left[ \frac{1}{r} P^{(n)}(t - r/c) \right] \right|$$

(28)
\[ \Phi(r, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \left[ \frac{p^{(n)}(t - r/c)}{r} \right] \]  

(29)

The basic idea is that the tensors \( M^{(n)} \) and \( P^{(n)} \) can be replaced by reduced tensors. This leads to a modification of some of the inferior order moments, such that finally all the moments up to (and including) the \( n \)th order moment must be expressed by reduced tensors. Not all of them, though, reduce to the static expressions \( \mathcal{M} \) and \( \mathcal{P} \). The final results for these moments are \( \tilde{M} \) and \( \tilde{P} \). Formally, as a result of this procedure, the expressions for the potentials expansions up to the \( n \)th order will be given by the equations (28) and (29) with the substitutions \( P \rightarrow \tilde{P} \) and \( M \rightarrow \tilde{M} \).

As a basis for this procedure one could take the following observations resuming systematically the results from [5]:

- Let \( L^{(n)} \) be a \( M^{(n)} \)-type tensor \( i.e. \)

\( \) is fully symmetric in the first \( n - 1 \) indices, \( L_{i_1...i_{n-1}i_k} = 0 \) if \( k \leq n - 1 \).  

(30)

The symmetrisation of \( L \) is realised with the relation:

\[ L^{(sym)}_{i_1...i_n} = L_{i_1...i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1i_2q} \mathcal{N}^{(\lambda)}_{i_1...i_{n-1}q} [L^{(n)}] \]  

(31)

with:

\[ \mathcal{N}_{i_1...i_{n-1}}[L^{(n)}] = \varepsilon_{i_{n-1}ps} L_{i_1...i_{n-2}ps} \]

In this case:

a) the substitution:

\[ M_{i_1...i_n} \rightarrow M^{(L)}_{i_1...i_n} = M_{i_1...i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1i_2q} \mathcal{N}^{(\lambda)}_{i_1...i_{n-1}q} [L^{(n)}] \]  

(32)

produces changes of the potentials which, up to a gauge transformation, are compensated by the following transformation:

\[ p^{(n-1)} \rightarrow p^{(n-1)} - \frac{n-1}{\varepsilon n^2} \mathcal{N}[L^{(n)}]; \]  

\( (I) \)  

(33)

b) the substitution:

\[ P_{i_1...i_n} \rightarrow P^{(L)}_{i_1...i_n} = P_{i_1...i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1i_2q} \mathcal{N}^{(\lambda)}_{i_1...i_{n-1}q} [L^{(n)}] \]  

(34)

produces changes of the potentials which are compensated by the following transformation:

\[ M^{(n-1)} \rightarrow M^{(n-1)} + \frac{n-1}{n^2} \mathcal{N}[L^{(n)}]; \]  

\( (II) \)  

(35)

- Let \( S^{(n)} \) be a tensor and the detracing operation for it:

\[ \tilde{S}_{i_1...i_n} = S_{i_1...i_n} - \sum_{D(i)} \delta_{i_1i_2} \Lambda_{i_3...i_n} [S^{(n)}] \]  

(36)

Then:
c) the substitution:

\[ M_{i_1...i_n} \rightarrow M_{(S)i_1...i_n} - \sum_{D(i)} \delta_{i_1i_2} \Lambda_{i_3...i_n} [S^{(n)}] \] (37)

produces changes of the potentials which, up to a gauge transformation, are compensated by the following transformation:

\[ M^{(n-2)} \rightarrow M^{(n-2)} + \frac{n-2}{2c^2 n} \hat{\Lambda} [S^{(n)}] \] (III) (38)

d) the substitution:

\[ P_{i_1...i_n} \rightarrow P_{(S)i_1...i_n} - \sum_{D(i)} \delta_{i_1i_2} \Lambda_{i_3...i_n} [S^{(n)}] \] (39)

produces changes of the potentials which, up to a gauge transformation, are compensated by the following transformation:

\[ P^{(n-2)} \rightarrow P^{(n-2)} + \frac{n-2}{2c^2 n} \hat{\Lambda} [S^{(n)}] \] (IV) (40)

These four transformation relations of the electromagnetic potentials are sufficient for the development of a scheme in which the replacement of the multipole moments tensors by symmetric and traceless tensors is presented.

Such a scheme, valid for expansion up to the fourth order with respect to \( d/\lambda \), can be found in Appendix B with results written in the Tables from Appendix C. It can be easily continued for higher orders. It is possible also to give results for the reduced tensors \( \tilde{P}^{(n)} \) and \( \tilde{M}^{(n)} \) for arbitrary \( n \) which will be given elsewhere [16].

7. Radiated Power Expressed by Transformed Moments

By applying the transformations [12] to the expansion of the vector potential of the radiated field, one gets:

\[ A_{rad}(r, t) = \frac{\mu_0}{4\pi r} \sum_{n=1}^{\infty} \frac{1}{n! c^n} \nu^{(n-1)} ||M^{(n)}_{,n}(t - r/c)|| \times \nu \]

\[ + \frac{\mu_0 c}{4\pi r} \sum_{n=1}^{\infty} \frac{1}{n! c^n} \nu^{(n-1)} ||P^{(n)}_{,n}(t - r/c)|| \]

The radiated power (angular distribution) will be given by:

\[ 4\pi \varepsilon_0 J(\nu) = \]

\[ = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n! m! c^{m+n}} \left[ (\nu^{(n-1)} ||M^{(n)}_{,n+1}||, (\nu^{(m-1)} ||M^{(m)}_{,m+1}||) \right. \]

\[ - (\nu^{(n)} ||M^{(n)}_{,n+1}||, (\nu^{(m)} ||M^{(m)}_{,m+1}||) \right] \]

\[ + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{c^2}{n! m! c^{m+n}} \left[ (\nu^{(n-1)} ||P^{(n)}_{,n+1}||, (\nu^{(m-1)} ||P^{(m)}_{,m+1}||) \right. \]

\[ - (\nu^{(n)} ||P^{(n)}_{,n+1}||, (\nu^{(m)} ||P^{(m)}_{,m+1}||) \right] \]

\[ + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{c}{n! m! c^{m+n}} \left\{ (\nu^{(n-1)} ||M^{(n)}_{,n+1}||, \nu \times (\nu^{(m-1)} ||P^{(m)}_{,m+1}||) \right\} \]

\[ + (\nu^{(n-1)} ||M^{(n)}_{,n+1}||, \nu \times (\nu^{(m-1)} ||P^{(m)}_{,m+1}||) \right] \} \]
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Considering the procedure of reduction of the moments tensors from the expansion of the vector potential applied up to the \( \mu \)th order for the magnetic and to the \( \varepsilon \)th order for the electric moments, the sum of the terms from the expansion of the radiated power which contain exclusively magnetic moments reduced up to \( \mu \) and electric moments up to \( \varepsilon \) is:

\[
4\pi\varepsilon_0J_{\mu,\varepsilon}(\nu) = 
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n!m!c^{n+m}} \left[ (\nu^{(n-1)}||M^{(n)}_{n+1}) (\nu^{(m-1)}||M^{(m)}_{m+1}) 
- (\nu^{(n)}||M^{(n)}_{n+1}) (\nu^{(m)}||M^{(m)}_{m+1}) \right] 
+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{c^2}{n!m!c^{n+m}} \left[ (\nu^{(n-1)}||P^{(n)}_{n+1}) (\nu^{(m-1)}||P^{(m)}_{m+1}) 
- (\nu^{(n)}||P^{(n)}_{n+1}) (\nu^{(m)}||P^{(m)}_{m+1}) \right] 
+ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{c}{n!m!c^{n+m}} \left\{ (\nu^{(n-1)}||\tilde{M}^{(n)}_{n+1}) \cdot [\nu \times (\nu^{(m-1)}||\tilde{P}^{(m)}_{m+1})] \right\} 
+ \left( \nu^{(n-1)}||\tilde{M}^{(n)}_{n+1}) \cdot [\nu \times (\nu^{(m-1)}||\tilde{P}^{(m)}_{m+1})] \right) \right) \] (41)

It is easy to understand that the above sum cannot be identified with the expansion of the radiated power containing the magnetic moments \( M^{(k)} \), \( k \leq \mu \) and the electric moments \( P^{(l)} \), \( l \leq \varepsilon \). This happens because some of the magnetic moments \( \tilde{M}^{(k)} \) contain tensorial expressions built with magnetic and electric tensors of superior orders in \( k \) and similar for the electric case. But, once \( J_{\mu,\varepsilon} \) is settled, the superior orders contributions can be eliminated in order to obtain the correct expansion with the \( d/\lambda \) criterion.

Returning to the expression of the total radiated power as a function of the magnetic moments \( \mu^{(n)} \) and considering the expansion (4), it results that this one corresponds to the expansion (41) for \( \mu = M \) and \( \varepsilon = M + 1 \) when only the contributions of order not higher than \( M \) are retained.

According to the results from (4), expressing the total radiated power is easier if one uses the averaging formula (8) and the symmetric and traceless character of the reduced tensors. Hence, the following properties are valid:

a) let two symmetric traceless tensors \( A^{(n)} \) and \( B^{(n)} \) and their averaged contraction:

\[
\langle (\nu^k||A^{(n)}) || (\nu^{k'}||B^{(m)}) \rangle = \langle \nu_{i_1} \ldots \nu_{i_k} \nu_{j_1} \ldots \nu_{j_{k'}} A_{i_1 \ldots i_k i_{k+1} \ldots i_n} B_{j_1 \ldots j_{k'} j_{k'+1} \ldots j_m} \rangle
\]

This average is different from zero only for products \( \delta_{ij} \) with \( p = 1, \ldots, k \) and \( q = 1, \ldots, k' \) and the following relation is valid:

\[
\langle (\nu^k||A^{(n)}) || (\nu^{k'}||B^{(m)}) \rangle = \frac{k!}{(2k + 1)(2k')!} [A^{(n)}||B^{(m)}] \delta_{kk'}
\]

b) The terms of the last sums, containing mixed moments products, in equation (41), give contributions of the type:

\[
\langle \nu_{i_1} \ldots \nu_{i_{n-1}} \nu_{j_1} \ldots \nu_{j_{m-1}} \nu_p > \varepsilon_{inpq} A_{i_1 \ldots i_n} B_{j_1 \ldots j_{m-1} q}
\]
but all the terms from the $\delta$ products, representing the averages of the $\nu$ components products, contain either $\delta_{ikp}$ or $\delta_{pjl}$, $k = 1, \ldots, n - 1$, $l = 1, \ldots, m - 1$ such that, because of $\varepsilon_{i\nu pq}$ and of the symmetry of $A$ and $B$, the result is zero. Using these properties one can write the result:

$$J_{\mu \nu} = \frac{1}{4\pi \varepsilon_0 c^3} \left[ \sum_{n=1}^{\infty} \frac{n+1}{nn!(2n+1)c^{2n}} \left( \bar{M}_{n+1}^{(n)} \bar{M}_{n+1}^{(n)} \right) 
+ \sum_{n=1}^{\infty} \frac{n+1}{nn!(2n+1)c^{2n-2}} \left( \bar{P}_{n+1}^{(n)} \bar{P}_{n+1}^{(n)} \right) \right].$$

Let us consider now different expressions of $J^{(M)}$, calculated this time starting from the formula given by the equation written above. As settled, for a given value of $M$ one must consider this formula for $\mu = M$, $\varepsilon = M + 1$.

$J^{(0)}$: $\mu = 0, \varepsilon = 1$, only the electric dipole moment has a contribution:

$$4\pi \varepsilon_0 c^3 J^{(0)} = \frac{2}{3} \frac{2\varepsilon^2}{p^2}$$

$J^{(2)}$: $(\mu, \varepsilon) = (2, 3)$; in this case the results from the table in Appendix A are used, considering in the end only the terms corresponding to the order $M$:

$$4\pi \varepsilon_0 c^3 J^{(2)} = \left\{ \frac{2}{3c^2} \tilde{M} \tilde{M} + \frac{1}{20c^4} \tilde{M}_{ij} \tilde{M}_{ij} + \frac{2}{3} \tilde{P} \tilde{P} + \frac{1}{20c^2} \tilde{P}_{ij} \tilde{P}_{ij} \right\}_2$$

where the index of the bracket is the maximum order in $d/\lambda$ which has to be retained in this bracket. It follows that:

$$4\pi \varepsilon_0 c^3 J^{(2)} = \left[ \frac{2}{3} \frac{2\varepsilon^2}{p^2} + \frac{2}{3c^2} \frac{\varepsilon^2}{m^2} - \frac{4}{3c^2} \tilde{p} \tilde{T} + \frac{1}{20c^2} \tilde{P}_{ij} \tilde{P}_{ij} \right].$$

$J^{(4)}$: $(\mu, \varepsilon) = (4, 5)$

$$4\pi \varepsilon_0 c^3 J^{(4)} = \left\{ \frac{2}{3c^2} (\tilde{M} + \frac{1}{6c^2} \tilde{A} - \frac{1}{18c^2} \tilde{N})_i \right\},$$

$$+ \frac{1}{20c^4} (\tilde{M} + \frac{1}{12c^2} \tilde{A} - \frac{1}{24c^2} \tilde{N})_{ij} (\tilde{M} + \frac{1}{6c^2} \tilde{A} - \frac{1}{24c^2} \tilde{N})_{ij}$$

$$+ \frac{2}{945c^6} \tilde{M}_{ijk} \tilde{M}_{ijk} + \frac{1}{18144c^8} \tilde{M}_{ijkl} \tilde{M}_{ijkl}$$

$$+ \frac{2}{3} (\tilde{P} - \frac{1}{c^2} \tilde{T} - \frac{1}{32c^4} \tilde{A} [\tilde{N}^{(4,1)}_{sym}] + \frac{1}{20c^4} \tilde{A} [\tilde{P}^{(4)}_{sym}] + \frac{1}{96c^4} \tilde{N}^{(4,4)}_{sym} )$$

$$+ \frac{1}{20c^4} (\tilde{P} - \frac{1}{c^2} \tilde{T})_{ij} (\tilde{P} - \frac{1}{c^2} \tilde{T})_{ij} + \frac{1}{20c^4} (\tilde{P} - \frac{1}{c^2} \tilde{T})_{ijk} (\tilde{P} - \frac{1}{c^2} \tilde{T})_{ijk}$$

$$+ \frac{1}{25 \times 3^3 \times 7} \tilde{P}_{ijkl} \tilde{P}_{ijkl} + \frac{1}{4 \times 3^3 \times 5^3 \times 77} \tilde{P}_{ijkl} \tilde{P}_{ijkl} \right\}_4$$. 

Only the fourth order terms must be kept from this expansion. This could be done by eliminating the useless terms, but, for obvious reasons, a detailed description is preferred.
here. The table with reduced moments is examined and the terms corresponding to the considered approximation are retained from the expressions representing the squares of these tensors. The result is:

\[
4\pi\varepsilon_0 c^3 \mathcal{J}^{(4)} = \frac{2}{3} \left( \frac{\vec{p}^2}{c^2} - \frac{1}{c^2} \vec{T} \right)^2 + \frac{4}{3} \hat{p}_i \left( -\frac{1}{32c^4} \vec{\Lambda}[N^{(4,1)}]_{\text{sym}} + \frac{1}{20c^4} \vec{\Lambda} [\Pi^{(3)}] + \frac{1}{96c^4} \vec{N}^{(4,3)} \right)_i \\
+ \frac{1}{20c^2} \vec{P}_{ij} \vec{P}_{ij} - \frac{1}{10c^4} \vec{P}_{ij} \vec{T}_{ij} + \frac{2}{945c^4} \vec{P}_{ijk} \vec{P}_{ijk} \\
+ \frac{2}{3c^2} \vec{m}^2 + \frac{4}{3c^2} \vec{m}_i \left( \frac{1}{6c^2} \vec{\Lambda} - \frac{1}{18c^2} \vec{N}^{(3,2)} \right)_i + \frac{1}{20c^4} \vec{M}_{ij} \vec{M}_{ij}
\]

Because

\[
N^{(4,1)}_{qqi} - \frac{4}{5} \Pi_{qqi} = \frac{5}{2} \Lambda_i [N^{(4,1)}_{\text{sym}}] - 4\Lambda_i [\Pi^{(3)}] - \frac{5}{6} N^{(4,3)}_i = \frac{2}{3} \int_D \left[ (2\xi^2 (\xi \cdot \vec{j}) \xi_i - 3\xi^4 j_i) \right] d^3\xi,
\]

and

\[
\Lambda_i - \frac{1}{3} N^{(3,2)}_i = \frac{2}{5} M_{qqi},
\]

the last expression of \( \mathcal{J}^{(4)} \) is the same as the expression given in equation (25).

8. Conclusions

The multipole expansions in Cartesian coordinates are not largely treated in literature or, if they are, many inaccuracies and uncertainties appear. In papers like [5] and [7] the expansion in Cartesian coordinates of the electromagnetic field is presented and applied to radiation. Their results were presented here as well, in a more systematic and concise way. They were compared to other results from literature, usually obtained by expansion in spherical coordinates. It was underlined the fact that the problem is not accurately treated everywhere, this fact leading to some confusions.

The main difficulty in the case of Cartesian coordinates is the procedure of reduction of the \( n \)th-order multipole tensors to symmetric traceless ones to obtain in this way a description of multipoles in terms of irreducible rotation group representations. This procedure was presented here, in Chapter 3. The transformations implied in the reduction procedure were defined such that the electromagnetic potentials are altered only by gauge transformations. This implied a specific feature of the dynamic case: the redefinitions of the multipole tensors in the lower \( n < N \) orders, induced by the reduction of tensors in a given order \( N \).

The following problems are left for a future discussion and they could be some possible research topics:

- completely systemizing all types of moments grouped, in the different orders, in reduced moments;
- defining the singular distributions associated to toroid moments and their physical meaning;
• study of the implication of the existence of interactions associated to toroid
moments and the setting, according to correct criteria, of the different terms
contributions;
• making symbolic programs which realize a reduction scheme of the type presented
in Appendix for the general case.

Appendix A. Equivalent multipole expansions

Following the basic ideas from [5], we present in this Appendix a somehow different
procedure for justifying the reduction scheme presented in Section 7.

Let the Taylor expansion of the \( \delta - \) function:

\[
\delta(r - r') = \sum_{n \geq 0} \frac{(-1)^n}{n!} r_i \cdots r_i \partial_{r_i} \cdots \partial_{r_i} \delta(r).
\]

Applied to the current and charge distributions, it gives the following results:

\[
j(r, t) = \int j(r', t)\delta(r - r') d^3x' = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int x_i \cdots x_i j(r', t) d^3x' \partial_{r_i} \cdots \partial_{r_i} \delta(r),
\]

\[
\rho(r, t) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int x_i \cdots x_i \rho(r', t) d^3x' \partial_{r_i} \cdots \partial_{r_i} \delta(r).
\]

The electric multipole moments are introduced by

\[
P_{i_1 \cdots i_n} = \int x_i \cdots x_i \rho(r', t) d^3x'
\]

such that:

\[
\rho(r, t) = \sum_{n \geq 0} \frac{(-1)^n}{n!} P_{i_1 \cdots i_n}(t) \partial_{i_1} \cdots \partial_{i_n} \delta(r) = \sum_{n \geq 0} \frac{(-1)^n}{n!} P^{(n)}_{i} \| \nabla^n \delta(r).
\]

Considering the continuity equation for electric charge, we may write the equation

\[
j_i = \nabla'(x_i j) + x_i \frac{\partial \rho}{\partial t}.
\]

Let

\[
a_i^{(n)} = \int x_i \cdots x_i j_i \partial_{i_1} \cdots \partial_{i_n} \delta(r)
\]

\[
= \int x_i \cdots x_i \nabla'(x_i j) d^3x' \partial_{i_1} \cdots \partial_{i_n} \delta(r) + \left[ \hat{\rho}^{(n+1)}(t) \| \nabla^n \right]_i \delta(r)
\]

\[
= - \int x_i j \cdot \nabla'(x_i \cdots x_i) \partial_{i_1} \cdots \partial_{i_n} \delta(r) + \left[ \hat{\rho}^{(n+1)}(t) \| \nabla^n \right]_i \delta(r)
\]

\[
= - n \int x_i \cdots x_i \partial_{i_1} \cdots x_i j_i d^3x' \partial_{i_1} \cdots \partial_{i_n} \delta(r) + \left[ \hat{\rho}^{(n+1)}(t) \| \nabla^n \right]_i \delta(r)
\]

\[
= - n \int x_i \cdots x_i \partial_{i_1} \cdots x_i j_i \left[ (x_i j_i - x_i j_i) \right] d^3x' \partial_{i_1} \cdots \partial_{i_n} \delta(r) - a_i^{(n)} + \left[ \hat{\rho}^{(n+1)}(t) \| \nabla^n \right]_i \delta(r).
\]
One obtains:
\[
a_i^{(n)} = -\frac{n}{n+1}\varepsilon_{i_1, k} \int x_i' \ldots x_{i_{n-1}}' (r' \times j)_k \, d^3x' \partial_{i_1} \ldots \partial_{i_n} \delta(r)
+ \frac{1}{n+1} \left[ \hat{P}^{(n+1)}(t) || \nabla^n \right]_i \delta(r).
\]
The magnetic multipole tensor is introduced by:
\[
M_{i_1 \ldots i_n} = \frac{n}{n+1} \int x_i' \ldots x_{i_{n-1}}' (r' \times j)_{i_n} \, d^3x',
\]
and one may write
\[
a_i^{(n)} = -\varepsilon_{i_1 k} \partial_i M_{i_1 \ldots i_{n-1} k} \partial_{i_1} \ldots \partial_{i_{n-1}} \delta(r) + \frac{1}{n+1} \left[ \hat{P}^{(n+1)}(t) || \nabla^n \right]_i \delta(r)
\]
with the result
\[
j_i(r, t) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \varepsilon_{i k l} \partial_{i} [M^{(n)} || \nabla^{n-1}]_k \delta(r)
+ \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left[ \hat{P}^{(n)}(t) || \nabla^{n-1} \right]_i \delta(r).
\]

Some invariance properties of the field are established in the following.

- By introducing in the expression of \( j \) the tensor \( M^{(n)}(L) \) defined by the equation (32), and using the relation \( \varepsilon_{i i k} \varepsilon_{i q k} = -\delta_{i k} \delta_{i q} + \delta_{i q} \delta_{i k} \), one obtains

\[
j_i = j_i \left( M^{(n)} \rightarrow M^{(n)}(L) \right) + \frac{(-1)^{n-1}}{n!} \sum_{\lambda = 1}^{n-1} \varepsilon_{i k l} \varepsilon_{i q l} N^{(\lambda)}_{i_1 \ldots i_{n-1} q} L^{(n)} \partial_{i_1} \ldots \partial_{i_{n-1}} \delta(r)
\]

\[
= j_i \left( M^{(n)} \rightarrow M^{(n)}(L) \right) + \frac{(-1)^n (n-1)}{n!} \left[ N^{(n)} || \nabla^{n-1} \right]_i \partial_i \delta(r)
+ \frac{(-1)^n (n-1)}{n!} \left[ N^{(n)} || \nabla^{n-2} \right]_i \Delta \delta(r).
\]

Considering the corresponding results for the potentials, the presence of a term like \( f(t) \Delta \delta(r) \) in the expansion of \( j \) leads to a term in \( A \) or in \( \Phi \) which contains the expression
\[
\Delta \frac{f(t - R/c)}{r} = \frac{1}{c^2} \frac{\ddot{f}(t)}{r}, \quad r \neq 0
\]

Therefore, regarding the contribution to the vector potential, the \( j \) expansion is equivalent to:
\[
j_i' = j_i \left( M^{(n)} \rightarrow M^{(n)}(L) \right) - \frac{(-1)^n (n-1)}{n! c^2} \left[ N^{(n)} || \nabla^{n-2} \right]_i \delta(r)
+ \frac{(-1)^n (n-1)}{n!} \left[ N^{(n)} || \nabla^{n-2} \right]_i \partial_i \delta(r).
\]

The last term in the precedent expression produces in \( A \) an additional term like \( \nabla \Psi(r, t) \) which, as one sees bellow, is a contribution from a gauge transformation of the potentials.

Let the transformation
\[
P_{i_1 \ldots i_{n-1}} \rightarrow P'_{i_1 \ldots i_{n-1}} = P_{i_1 \ldots i_{n-1}} - \frac{n-1}{c^2 n^2} \hat{N}_{i_1 \ldots i_{n-1}} L^{(n)}.
\]
By substituting \( P_{i_1 \ldots i_{n-1}} \) in the \( j \) and \( \rho \) expansions by
\[
P'_{i_1 \ldots i_{n-1}} = P'_{i_1 \ldots i_{n-1}} + \frac{n-1}{c^2 n^2} \tilde{N}_{i_1 \ldots i_{n-1}},
\]
one gets:
\[
j'_i = j_i \left( \frac{M^{(n)} \rightarrow M^{(n)}_{(L)}}{P^{(n-1)} \rightarrow P^{(n-1)}} \right) + \frac{(-1)^n (n-1)}{n!} \left[ \mathcal{N}[L^{(n)}] || \nabla^{n-1} \right] \partial_i \partial \delta(r)
\]
\[
\rho = \rho \left( P^{(n-1)} \rightarrow P'^{(n-1)} \right) - \frac{(-1)^n (n-1)}{n! n c^2} \left[ \mathcal{N}[L^{(n)}] || \nabla^{n-1} \right] \delta(r).
\]

It is easy to see that the potentials \( (\tilde{\Phi}, \tilde{A}) \) corresponding to the new densities \( (\tilde{\rho}, \tilde{j}) \) with
\[
\tilde{\rho} = \rho \left( P^{(n-1)} \rightarrow P'^{(n-1)} \right), \quad \tilde{j} = j \left( \frac{M^{(n)} \rightarrow M^{(n)}_{(L)}}{P^{(n-1)} \rightarrow P^{(n-1)}} \right)
\]
differ from the original one by a gauge transformation.

We point out that the physical equivalence of the two distributions \( (\rho, j) \) and \( (\tilde{\rho}, \tilde{j}) \) is true only for the field in the exterior of the domain \( D \) including the support of the charges and currents.

- Let
\[
P_{(L)i_1 \ldots i_n} = P_{i_1 \ldots i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1 i_2 \ldots i_n} \delta \left( \tilde{N}_{i_1 \ldots i_{n-1} q} [L^{(n)}] \right).
\]

It is a simple matter to see that the transformation \( P^{(n)} \rightarrow P_{(n)}' \) does not alter the density \( \rho \) but
\[
j_i = j_i \left( P^{(n)} \rightarrow P_{(L)}^{(n)} \right) - \frac{(-1)^{n-1} (n-1)}{n!} \varepsilon_{i_1 i_2 \ldots i_n} \delta \left( \tilde{N}_{i_1 \ldots i_{n-1} q} [L] || \nabla^{n-2} \right) \delta(r).
\]

The difference between \( j_i [P^{(n)}] \) and \( j_i [P^{(n)} \rightarrow P_{(L)}^{(n)}] \) may be eliminated by the transformation
\[
M_{i_1 \ldots i_{n-1}} \rightarrow M_{i_1 \ldots i_{n-1}}' M_{i_1 \ldots i_{n-1}} + \frac{n-1}{n^2} \tilde{N}_{i_1 \ldots i_{n-1} q} [L]
\]
and so
\[
j_i = j_i \left( P^{(n)} \rightarrow P_{L}^{(n)} \right).
\]

- Let the transformation \( (37) \) of the magnetic tensor
\[
M_{i_1 \ldots i_n} \rightarrow M_{(S)i_1 \ldots i_n} = M_{i_1 \ldots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3 \ldots i_n} [S^{(n)}]
\]
where \( S^{(n)} \) is a fully symmetric tensor and the operator \( \Lambda \) is used as in equation \( (36) \) for detracing an arbitrary fully symmetric tensor.

Introducing the tensor \( M_{(S)}^{(n)} \) in the expansion of \( j \) we may write
\[
j_i = j_i (M^{(n)} \rightarrow M_{(S)}^{(n)}) + \frac{(-1)^{n-1}}{n!} \varepsilon_{i_1 i_2} \delta_{i_3 \ldots i_{n-1} k} [S^{(n)}] \partial_{i_1} \ldots \partial_{i_{n-1}} \delta(r)
\]
A being a fully symmetric tensor. In the sum there are \( n - 1 \) terms with \( \delta_{ki,j} \), \( j = 1 \ldots n - 1 \) but \( \delta_{ki,j} \partial_{ij} = \partial_k \) and \( \varepsilon_{ijk} \partial_k \partial_l = 0 \) such that these terms give a null result. In the remaining terms we have the factors like \( \delta_{ij,q} \), \( j,q = 1 \ldots n - 1 \) resulting \( \delta_{ij,q} \partial_{ij} \partial_{iq} = \Delta \).

The number of these terms is \( C_2^{n-1} = (n - 1)(n - 2)/2 \), such that

\[
\frac{(-1)^{n-1}(n-1)(n-2)}{2n!} \varepsilon_{ijk} \partial_k \Lambda_{i_1 \ldots i_{n-3}k} [S^{(n)}] \partial_{i_1} \ldots \partial_{i_{n-3}} \Delta \delta(r)
\]

\[
=j_i(M^{(n)} \rightarrow M^{(n)}_{(S)}) + \frac{(-1)^{n-1}(n-1)(n-2)}{2n!} \{ \nabla \times [\Lambda[S^{(n)}]|\nabla|^{-3}] \}_i \Delta \delta(r).
\]

This relation allows the introduction of an equivalent current density regarding the vector potential:

\[
j_i'' = j_i(M^{(n)} \rightarrow M^{(n)}_{(S)}) + \frac{(-1)^{n-1}(n-1)(n-2)}{2n!c^2} \{ \nabla \times [\Lambda[S^{(n)}]|\nabla|^{-3}] \}_i \delta(r).
\]

The suplimentary term is eliminated by the transformation

\[
M^{(n-2)} \rightarrow M^{(n-2)}' = M^{(n-2)} + \frac{n-2}{2nc^2} \Lambda[S^{(n)}]
\]

such that

\[
\tilde{j}'' = \tilde{j} \left( \begin{array}{c} M^{(n)} \rightarrow M^{(n)}_{(S)} \cr M^{(n-2)} \rightarrow M^{(n-2)} \end{array} \right).
\]

Let

\[
P_{(S)i_1 \ldots i_n} = P_{i_1 \ldots i_n} - \sum_{D(i)} \delta_{i_1i_2} \Lambda_{i_3 \ldots i_n} [S^{(n)}].
\]

One gets

\[
j_i = j_i (P^{(n)} \rightarrow P^{(n)}_S) + \frac{(-1)^{n-1}}{n!} \left[ \sum_{D(i)} \delta_{i_1i_2} \tilde{\Lambda}_{i_3 \ldots i_{n-1} i} [S^{(n)}] \right] \partial_{i_1} \ldots \partial_{i_{n-1}} \delta(r)
\]

\[
= j_i (P^{(n)} \rightarrow P^{(n)}_S) + \frac{(-1)^{n-1}(n-1)}{n!} \left[ \hat{\Lambda}[S^{(n)}]|\nabla|^{-2} \right] \partial_i \delta(r)
\]

\[\quad + \frac{(-1)^{n-1}(n-1)(n-2)}{2n! c^2} [\hat{\Lambda}|S^{(n)}]|\nabla|^{-3} \}_i \Delta \delta(r)'.
\]

So, we obtain an equivalent current density

\[
j_i' = j_i (P^{(n)} \rightarrow P^{(n)}_S) + \frac{(-1)^{n-1}(n-1)}{n!} \left[ \hat{\Lambda}[S^{(n)}]|\nabla|^{-2} \right] \partial_i \delta(r)
\]

\[\quad + \frac{(-1)^{n-1}(n-1)(n-2)}{2n! c^2} [\hat{\Lambda}|S^{(n)}]|\nabla|^{-3} \}_i \delta(r).
\]

For the charge density we have

\[
\rho = \rho (P^{(n)} \rightarrow P^{(n)}_S) + \frac{(-1)^{n}}{n!} \sum_{D(i)} \delta_{i_1i_2} \Lambda_{i_3 \ldots i_n} [S^{(n)}] \partial_{i_1} \ldots \partial_{i_n} \delta(r).
\]

Because in the last sum there are \( C_n^2 \) terms like \( \delta_{i_1i_2} \) with equal contributions, one obtains

\[
\rho = \rho (P^{(n)} \rightarrow P^{(n)}_S) + \frac{(-1)^n n(n-1)}{2n!} [\Lambda[S^{(n)}]|\nabla|^{-2} \Delta \delta(r)
\]
and we may introduce the equivalent density
\[
\rho' = \rho \left( P^{(n)} \rightarrow P_S^{(n)} \right) + \frac{(-1)^n n (n-1)}{2n!c^2} \left[ \dddot{\Lambda}[S^{(n)}] || \nabla^{n-2} \right] \delta(r)
\]
Let us define
\[
P''^{(n-2)} = P^{(n-2)} + \frac{n - 2}{2nc^2} \dddot{\Lambda}[S^{(n)}].
\]
It is easy to see that
\[
j'_i = j_i \left( P^{(n)} \rightarrow P_S^{(n)} \right) + \frac{(-1)^{n-1} (n-1)}{n!} \left[ \dddot{\Lambda}[S^{(n)}] || \nabla^{n-2} \right] \partial_i \delta(r)
\]
and
\[
\rho' = \rho \left( P^{(n)} \rightarrow P_S^{(n)} \right) + \frac{(-1)^{n} (n-1)}{n!c^2} \left[ \dddot{\Lambda}[S^{(n)}] || \nabla^{n-2} \right] \delta(r).
\]
So, the densities \((\rho', j')\) and
\[
\left\{ \rho \left( P^{(n)} \rightarrow P_S^{(n)} \right) \right\}, \left\{ \dddot{j} \left( P^{(n)} \rightarrow P_S^{(n)} \right) \right\}
\]
are equivalent because the corresponding potentials differ only by a gauge transformation.

Appendix B. The Reduction Scheme

\[
(\mu, \varepsilon) = (1, 2)
\]

\[
\begin{array}{c}
M^{(1)} \\
\downarrow \\
P^{(2)} \\
\downarrow \\
P^{(2)}
\end{array}
\]

Note. \(P_{ij} = P_{ij} - \frac{1}{3} P_{qq} \delta_{ij}\),
no extra-gauge changes for \(A\) and \(\Phi\) for \(P^{(2)} \rightarrow P^{(2)}\)
\((\mu, \varepsilon) = (2, 3)\)

\[ M^{(2)} \]

\[ N^{(2,1)} \]

\[ \mathcal{M}^{(2)} = M^{(2)}_{\text{sym}} \]

\[ P^{(3)} \]

\[ \Pi^{(1)} \]

\[ \mathcal{P}^{(3)} \]

Note: \( T_i = \frac{1}{4} N_i^{(2,1)} - \frac{1}{6} \Pi_i \)
\((\mu, \epsilon) = (3, 4)\)

\[\mathbf{M}^{(3)}\]

\[N_{\text{(3,1)}}\]

\[\mathbf{M}_{\text{(sym)}}^{(3)}\]

\[\Lambda^{(1)}\]

\[\mathbf{M}^{(3)}\]

\[\mathbf{P}^{(4)}\]

\[\Pi^{(2)}\]

\[\mathbf{P}^{(4)}\]

\[\mathbf{P}^{(2)} - \frac{2}{9c^2} \mathbf{N}[\mathbf{M}^{(3)}] + \frac{1}{4c^2} \tilde{\Lambda}[\mathbf{P}^{(4)}]\]

\[\dot{N}_{\text{sym}}^{(3,1)} = \mathbf{\dot{N}}_{\text{sym}}^{(3,1)} = \text{traceless}\]

\[T_{ij} = \frac{2}{9} N_{\text{sym}}^{(3,1)ij} - \frac{1}{4} \tilde{\Pi}_{ij}\]

Transf. \(\Pi^{(2)} \rightarrow \tilde{\Pi}^{(2)}\)

produces only a gauge transformation
\[(\mu, \varepsilon) = (4, 5)\]

\[M^{(4)} \]

\[N^{(4,1)} \downarrow \quad \text{Eq. (I), } n = 4 \rightarrow P^{(3)} - \frac{3}{16c^2} \hat{N}^{(4,1)} + \frac{3}{10c^2} \hat{\Pi}^{(3)} \]

\[M^{(4)}_{\text{(sym)}} \]

\[\Lambda^{(2)} \downarrow \quad \text{Eq. (III), } n = 4 \rightarrow M^{(2)} + \frac{1}{4c^2} \hat{\Lambda}[M^{(4)}_{\text{sym}}] \equiv M^{(2)} + \frac{1}{16c^2} \hat{\Lambda}^{(2)} \]

\[P^{(5)} \]

\[\Pi^{(3)} \downarrow \quad \text{Eq. (IV), } n = 5 \rightarrow P^{(3)} - \frac{3}{16c^2} \hat{N}^{(4,1)} + \frac{3}{10c^2} \hat{\Pi}^{(3)} \]

\[P^{(3)} - \frac{3}{16c^2} \hat{N}^{(4,1)} + \frac{3}{10c^2} \hat{\Pi}^{(3)} \]

\[\hat{N}^{(4,2)} \downarrow \quad \text{Eq. (II), } n = 4 \rightarrow M^{(2)} + \frac{1}{4c^2} \hat{\Lambda}[M^{(4)}_{\text{sym}}] - \frac{1}{24c^2} \hat{\Lambda}^{(2)} - \frac{1}{16c^2} \hat{N}^{(4,2)} \]

\[\hat{N}^{(4,1)} \hat{\Lambda}[N^{(4,1)}_{\text{sym}}], \hat{\Lambda}[\Pi^{(3)}] \downarrow \quad \text{Eq. (IV), } n = 3 \rightarrow P^{(1)} - \frac{1}{c^2} \hat{T}^{(1)} - \frac{1}{32c^2} \hat{\Lambda}[\hat{N}^{(4,1)}_{\text{sym}}[M^{(4)}]] + \frac{1}{20c^4} \hat{\Lambda}[P^{(5)}] \equiv P^{(1)} - \frac{1}{c^2} \hat{T}^{(1)} - \frac{1}{32c^2} \hat{\Lambda}[P^{(5)}] + \frac{1}{20c^4} \hat{\Lambda}[P^{(5)}] \]

\[P^{(3)} - \frac{3}{16c^2} \hat{N}^{(4,1)} + \frac{3}{10c^2} \hat{\Pi}^{(3)} \equiv P^{(3)} - \frac{3}{16c^2} \hat{N}^{(4,1)} + \frac{3}{10c^2} \hat{\Pi}^{(3)} \]

\[\hat{\Pi}^{(3)} \quad \text{no effect} \]

\[\hat{N}^{(4,3)} \downarrow \quad \text{Eq. (I), } n = 2 \rightarrow P^{(1)} - \frac{1}{c^2} \hat{T}^{(1)} - \frac{1}{32c^4} \hat{\Lambda}[\hat{N}^{(4,1)}_{\text{sym}}[M^{(4)}]] + \frac{1}{20c^4} \hat{\Lambda}[\hat{N}^{(4,1)}_{\text{sym}}[M^{(4)}]] + \frac{1}{96c^6} \hat{\Lambda}^{(3)}[M^{(4)}] \equiv P^{(1)} - \frac{1}{c^2} \hat{T}^{(1)} - \frac{1}{32c^2} \hat{\Lambda}[P^{(5)}] + \frac{1}{96c^6} \hat{\Lambda}[\Pi^{(3)}] + \frac{1}{96c^6} \hat{\Lambda}[\Pi^{(3)}] \]

\[\hat{\Lambda}^{(2)} \downarrow \quad \text{no effect} \]
Appendix C. Results

Table I

| \((\mu, \varepsilon) = (1, 2)\) | \((\mu, \varepsilon) = (2, 3)\) | \((\mu, \varepsilon) = (3, 4)\) |
|---|---|---|
| \(\tilde{P}^{(1)}\) | \(P^{(1)} : (\tilde{p})\) | \(P^{(1)} - \frac{1}{c^2} \tilde{T}^{(1)}\) | \(P^{(1)} - \frac{1}{c^2} \tilde{T}^{(1)}\) |
| \(\tilde{P}^{(2)}\) | \(P^{(2)} : P_{ij} = P_{ij} - \frac{1}{2} P_{qq} \delta_{ij}\) | \(P^{(2)} - \frac{1}{c^2} \tilde{T}^{(2)}\) |
| \(\tilde{P}^{(3)}\) | \(P^{(3)}\) | \(P^{(3)}\) |
| \(\tilde{P}^{(4)}\) | \(P^{(4)}\) |
| \(\tilde{P}^{(5)}\) | \(P^{(5)}\) |
| \(\tilde{M}^{(1)}\) | \(M^{(1)} : (\tilde{m})\) | \(M^{(1)}\) | \(M^{(1)} + \frac{1}{16c^2} \tilde{\Lambda}^{(1)} - \frac{1}{18c^2} \tilde{N}^{(3,2)}\) |
| \(\tilde{M}^{(2)}\) | \(\mathcal{M}^{(2)}\) | \(\mathcal{M}^{(2)}\) |
| \(\tilde{M}^{(3)}\) | \(\mathcal{M}^{(3)}\) |
| \(\tilde{M}^{(4)}\) | \(\mathcal{M}^{(4)}\) |

Table II

| \((\mu, \varepsilon) = (4, 5)\) |
|---|
| \(\tilde{P}^{(1)}\) | \(P^{(1)} - \frac{1}{c^2} \tilde{T}^{(1)} - \frac{1}{32c^4} \tilde{\Lambda} [N^{(4,1)}_{sym}] + \frac{1}{20c^2} \tilde{\Lambda} [\mathcal{M}^{(3)}] + \frac{1}{36c^2} \tilde{N}^{(4,3)}\) |
| \(\tilde{P}^{(2)}\) | \(P^{(2)} - \frac{1}{c^2} \tilde{T}^{(2)}\) |
| \(\tilde{P}^{(3)}\) | \(P^{(3)} - \frac{1}{c^2} \tilde{T}^{(3)}\) |
| \(\tilde{P}^{(4)}\) | \(P^{(4)}\) |
| \(\tilde{P}^{(5)}\) | \(P^{(5)}\) |
| \(\tilde{M}^{(1)}\) | \(M^{(1)} + \frac{1}{6c^2} \tilde{\Lambda}^{(1)} - \frac{1}{18c^2} \tilde{N}^{(3,2)}\) |
| \(\tilde{M}^{(2)}\) | \(\mathcal{M}^{(2)} + \frac{1}{4c^2} \tilde{\Lambda}^{(2)} - \frac{1}{24c^2} \tilde{N}^{(4,2)}\) |
| \(\tilde{M}^{(3)}\) | \(\mathcal{M}^{(3)}\) |
| \(\tilde{M}^{(4)}\) | \(\mathcal{M}^{(4)}\) |

In the above formulas it was introduced the electric toroid moment of undefined order, as it results from the algorithm presented before:

\[
T^{(n)} = \frac{n}{(n + 1)^2} \tilde{N}^{(n+1,1)} - \frac{n}{2(n + 2)} \tilde{\Lambda}^{(n)}
\]

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