Hard thermal loops for soft or collinear external momenta

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Abstract

We consider finite temperature 1-loop diagrams with hard loop momenta and an arbitrary number of external gauge fields when the external momenta are either soft, or near the light cone and nearly collinear with the loop momentum. We obtain a recursion relation for these diagrams which we translate into an equation for their generating functional. By integrating out the soft fields while keeping two collinear ones we find an integral equation, originally due to Arnold, Moore, and Yaffe, which sums the bremsstrahlung and pair annihilation contribution to the thermal photon production rate.

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1 Introduction

Particle production in thermal and non-thermal systems plays an important role in relativistic heavy ion physics and in cosmology. Typically there is some weakly interacting particle species with a low abundance in a hot medium, from which these particles are then produced. Examples are photons produced in a quark–gluon plasma, or dark matter candidates such as gravitinos, axions, or axinos produced in an otherwise thermal universe and at reheating after inflation.

The production of photons from a thermal quark-gluon plasma is remarkably complicated. At leading logarithmic order, that is, at leading order not just in the strong coupling $\alpha_S$, but rather in $\alpha_S \log(1/\alpha_S)$ only $2 \rightarrow 2$ scattering processes contribute and the rate has been computed by Kapusta et al. \cite{1} and by Baier et al. \cite{2}. However, already for the complete leading order contribution the production cannot simply be understood in terms of scattering processes involving only a handful of particles. Bremsstrahlung and pair annihilation involve multiple interactions via soft gluon exchange. Such processes are not suppressed, despite the large number of interactions \cite{3, 4}. The emission occurs almost collinearly so that internal lines are nearly on-shell and compensate the suppression. Viewed in position space the radiated photon and its source overlap over large distances, and the interference of different interactions cannot be neglected. This leads to the so-called Landau-Pomeranchuk-Migdal (LPM) effect \cite{4, 5, 6, 7}. The complete leading order photon production rate has been computed at leading order by Arnold, Moore and Yaffe by summing all relevant diagrams.

In this paper we present a different approach. We consider hard particles with momenta of order $T$ which propagate through a gauge field background. The gauge field momenta are either soft ($k \sim gT$) or are almost on-shell with virtuality $k^2 \sim g^2 T^2$ and almost collinear with the loop momentum. Formally we integrate out the hard modes in this soft and collinear background, leaving us with an effective theory for soft and collinear modes. The effective theory is described by an equation which has a similar structure as the non-abelian Vlasov equations \cite{8} describing Hard Thermal Loops \cite{9}. Then we distinguish the soft and collinear fields and identify them with the soft gluon field and the photon field, respectively. In a second step we integrate out the soft gluons to obtain an effective theory for the collinear gauge fields only. \cite{3} The effective theory takes the form of an integral equation which has been obtained

\footnote{This approach is similar to the one used in Ref. \cite{10}, where soft gauge fields were integrated out to obtain an effective theory for ultrasoft ($k \sim g^2 T$) fields.}
previously in the calculation of the photon production rate \[4\].

This paper is organized as follows. In Sec. 2 we briefly recall the relation between thermal photon production rate and the finite temperature polarization tensor. The main part of the paper is contained in Sec. 3 with the calculation of the 1-loop diagrams in a soft or collinear gauge field background. The key steps of our calculation are the approximation (7) and the partial fractioning (10). These are used both to compute the 2-point function (Sec. 3.2) and then to obtain the recursion relation for the \(n\)-point functions in terms of \((n - 1)\)-point functions (Sec. 3.3). Then all \(n\)-point functions are put together in an effective action (Sec. 3.4). In Sec. 4 we integrate out the gluon fields and in Sec. 5 we discuss the generalization from scalar quarks to spin-1/2 quarks. We summarize and conclude in Sec. 6. Finally, in Appendix A we show that the connected pieces which we encounter in Sec. 4 vanish.

Note and conventions: We use the metric with signature \(+\, -\, -\, -\).
at some point emits a bremsstrahlung photon or annihilates with an antiquark into a photon.

Therefore we want to evaluate $\Pi^{\mu\nu}$ with a quark interacting via soft gluons only. For simplicity we first consider scalar quarks, the generalization of our method to spin-1/2 quarks is straightforward and is described in Sec.5. The first step we will take is to integrate out the scalar quark fields in a soft or collinear gauge field background. That means that we have to calculate diagrams with two external photon lines and an arbitrary number of soft external gluon lines.

## 3 1-loop diagrams with soft or collinear external gauge fields

We consider thermal 1-loop diagrams with an arbitrary number of external gauge field legs. For the bremsstrahlung and pair annihilation contribution to photon production, two of the gauge field momenta are hard ($k \sim T$) and correspond to the produced photon. The remaining ones are soft ($k \sim gT$) gluons. The particle in the loop corresponds to a quark which suffers soft scattering via gluon exchange and radiates the photon.

Even though the photons and the gluons play very different roles, our approach allows for a unified treatment, and in this section we do not have to distinguish these two. First we review the relevant kinematics which has been extensively discussed in Ref. [4]. It allows us to simplify propagators and vertices. We obtain a recursion relation between a diagram with $n$ external gauge field lines and the difference of two diagrams with $n-1$ external lines. Then we consider the current induced by the gauge fields which is the first derivative of the generating functional of all $n$-point functions. The recursion relation turns into a generalized kinetic equation. It can be viewed as a generalized Vlasov equation which contains a convective term and a force term.

### 3.1 Kinematics and power counting

The hard momenta we are considering are all almost collinear. Up to higher orders they all point into the same direction which we denote by $v$ with $v^2 = 1$. The 3-momentum components in the $v$-direction are denoted by

$$p_{\parallel} \equiv p \cdot v$$

(4)
We define the light-like vector \( v \equiv (1, v) \). One has to account for three distinct momentum scales.

1. The emitting charged particle, which corresponds to the particle in the loop, and the emitted particle both have \( p_\parallel \) of order \( T \), which is our hard scale.

2. All 3-momenta perpendicular to \( v \) are soft, \( p_\perp \sim gT \). Furthermore, all momentum components of the gluons are soft, \( k_\mu \sim gT \).

3. Finally, all 4-momenta \( k \) are ‘collinear’, \( v \cdot k = (k_0 - k_\parallel) \sim g^2T \).

This includes the case that the emitted photon is off-shell by an amount \( k^2 \sim g^2T^2 \) which is relevant for dilepton production.

All momenta in the loop have \( k^2 \sim g^2T^2 \). Therefore the propagators are sensitive to the so-called asymptotic mass \( m \sim gT \), which is given by the real part of the thermal self-energy, with hard loop momentum, of a light-like hard particle \([11]\). It is thus generated by integrating out the gluons with hard momenta. This procedure does not yet yield a thermal width which is only generated by integrating out the gluons with soft momenta. \([13]\) For scalars and fermions in the representation \( r \) of the gauge group

\[
m^2 = \frac{1}{4} C_2(r) g^2 T^2
\]

with the quadratic Casimir of the representation \( r \). In a SU(\( N \)) gauge theory \( C_2(r) = (N^2 - 1)/(2N) \).

### 3.2 2-point function

In this section we explicitly compute the 2-point function. It turns out that all \( n \)-point functions with \( n > 2 \) can be obtained from \( (n - 1) \)-point functions through a simple recursion relation. Furthermore, all kinematic approximations which are needed for the general case already appear for the 2-point function.

We work in the imaginary time formalism, where the loop integral consists of a sum over imaginary Matsubara frequencies and an integral over 3-momenta. The external momenta must also be taken imaginary and can be continued to real values only after

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\( \footnote{It is oftentimes referred to as \( m_\infty \).} \)

\( \footnote{Therefore, in contrast to \([13]\) we never include the (IR divergent) thermal width in the quark propagators, only the asymptotic mass.} \)
the sum has been performed. The Matsubara sum is performed as usual by writing it as a contour integral

\[ T \sum_{p_0 = -in2\pi T} h(p_0) = \int_C \frac{dp_0}{2\pi i} \left[ \frac{1}{2} + f_B(p_0) \right] h(p_0) \tag{6} \]

where the integration contour goes up on the right of the imaginary axis and goes down on the left of the imaginary axis. Then we close the contour around the poles of the propagators. After evaluating the integral (6), we can continue \( k_0 \) from a Matsubara frequency towards the real axis. The retarded polarization tensor is obtained by taking \( k_0 = \text{Re}(k_0) + i\epsilon \).

Now we describe the two main steps of our calculation. In the imaginary time formalism the scalar propagator appearing in Fig. 1 is

\[ \Delta(p) \equiv -\frac{1}{p^2 - m^2} = -\frac{1}{\bar{v} \cdot p v \cdot p - p_\perp^2 - m^2} \tag{7} \]

with \( \bar{v} = (1, -v) \). The other propagator in Fig. 1(a) will only be parametrically large, of order \( g^{-2}T^{-2} \), when \( v \cdot p \sim v \cdot k \) because \( v \cdot k \sim g^2T \). In this case \( p_0 \) approximately equals \( p_\parallel \), and \( \bar{v} \cdot p \approx 2p_\parallel \). Therefore we can approximate

\[ \Delta(p) \simeq \frac{1}{2p_\parallel} D(p) \tag{8} \]

with

\[ D(p) = \frac{-1}{v \cdot p - (p_\perp^2 + m^2)/(2p_\parallel)} \tag{9} \]

After this approximation the propagator, viewed as a function of the energy variable, has only one pole. For the following it turns out to be very convenient to partial fraction the product of the two propagators,

\[ D(p)D(p - k) = \frac{1}{\epsilon(k, p)} [D(p) - D(p - k)] \tag{10} \]

\[ ^6 \text{Note that we have made the approximation before the Matsubara summation, that is, when } p_0 \text{ is still purely imaginary and of order } T, \text{ even though } \text{is valid only for real frequencies. This is justified, however, because the second pole in } p_0 \text{ of the propagator, which gets lost through the approximation, would give } v \cdot p \approx -2p_\parallel \sim T. \text{ With this value of } v \cdot p \text{ the second propagator } \Delta(k - p) \text{ would be of order } T^{-2}. \text{ This is suppressed relative to the contribution we keep, for which we have } \Delta(k - p) \sim g^{-2}T^{-2}. \text{ The same type of argument applies to the poles of } \Delta(k - p). \]
where
\[ \epsilon(k, p) \equiv v \cdot k + \frac{(p - k)^2 + m^2}{2(p - k)} - \frac{p^2 + m^2}{2p} \] (11)
denotes the difference of the poles of the two propagators which is of order \( g^2 T \).

Both vertices are proportional to \((2p - k)_\mu\). We associate the factor \((2p)_\parallel^{-1}\) of Eq. (8) with the left vertex in Fig. (a) rather than with the propagator itself and define the resulting vertex factor
\[ V^\mu(p, p - k) \equiv \frac{1}{2p_\parallel}(2p - k)^\mu \] (12)
The different components of \( V \) have different orders of magnitude. The largest component is in the direction of \( v \) and is \( O(1) \). The transverse components are of order \( g \), and the \( \bar{v} \)-component is \( O(g^2) \). For computing the production rate of real photons one needs the transverse components. It is therefore not sufficient to take into account only the leading order piece. We also include the transverse components, and the approximation
\[ V^\mu(p, p - k) \simeq \frac{1}{2p_\parallel} [(2p_\parallel - k_\parallel)\nu^\mu + (2p - k)_\perp^\mu] \] (13)
is implicitly understood in the following.

We write the polarization tensor as
\[ \Pi^b_{\nu}(k, p) = \int \frac{d^3p}{(2\pi)^3} V_\nu(p, p - k) t^b \tilde{\Pi}_\nu^b(k, p) \] (14)
which turns out to be convenient for discussing the general \( n \)-point function, and define
\[ \mathcal{F}(p_\parallel, k_\parallel) \equiv f_B(p_\parallel) - f_B(p_\parallel - k_\parallel) \] (15)
Our result for the 2-point function is then
\[ \tilde{\Pi}_\nu^b(k, p) \equiv \frac{1}{\epsilon(k, p)} \mathcal{F}(p_\parallel, k_\parallel) V_\nu(p - k, p) t^b \] (16)
As we already stated below Eq. (2), we do not include the gauge coupling in \( \Pi_{\mu\nu} \).

There is also the tadpole diagram containing the 4-point vertex which contributes to the 1-loop polarization tensor. It does not have the collinear enhancement since it does not depend on the external momentum. But its transverse components are of the same order as those of the diagram in Fig. (a). However, due to the independence on the external momentum the tadpole diagram has no discontinuity and therefore does not contribute to the production rate. Therefore we do not consider it here.
Figure 1: 1-loop diagrams with soft or collinear external gauge field lines. Only the 2-point function (a) needs to be calculated explicitly. The \( n \)-point functions (b) are related to the \((n - 1)\)-point functions by a recursion relation. All external momenta are outgoing. 4-momentum conservation implies \( k = \sum_{j=1}^{n-1} k_j \).

### 3.3 Recursion relation for \( n \)-point diagrams

Now we consider the diagram with \( n > 2 \) external gauge field lines in Fig. 1(b). We will find that it can be recursively related to diagrams with \( n - 1 \) external lines. To obtain this relation we pick out one vertex, the leftmost one in Fig. 1(b) which carries momentum \( k \). The remaining ones have incoming momenta \( k_j \) with \( j = 1, \ldots, n - 1 \). As for the 2-point function we can use the approximation (8) for the propagators. After the partial fractioning (10) this diagram is proportional to the difference of the diagrams in which either the vertex with \( k_1 \) or the vertex with \( k_{n-1} \) is omitted.

Each vertex carries a generator \( t^a \) of some gauge group. As before we do not include the gauge couplings in the vertices. From the calculation of the 2-point function we know that in the vertices we have \( p^0 \simeq p_{\parallel} \). In analogy with Eq. (14) we write

\[
\Pi^{(n)\alpha_1 \ldots \alpha_{n-1}}_{\mu_1 \ldots \mu_{n-1}}(k_1, \ldots, k_{n-1}) = \int \frac{d^3 p}{(2\pi)^3} V_{\mu}(p, p - k) \text{tr} \left[ t^a \Pi^{(n)\alpha_1 \ldots \alpha_{n-1}}_{\mu_1 \ldots \mu_{n-1}}(k_1, \ldots, k_{n-1}, p) \right]
\]

(17)

We only have to consider the two propagators \( D(p) \) and \( D(p - k) \) which are connected to the left vertex in Fig. 1(b). We apply the same approximations as for the 2-point function including the partial fractioning (10). The two terms in Eq. (10) are then proportional to diagrams in which either the propagator with momentum \( p \) or the one
with momentum $p - k$ has been omitted from Fig. 1(b). Thus, if one also leaves out the vertex factors connected to these propagators each of the two terms gives a $(n - 1)$-point function. For the second term in Eq. (10) we obtain a contribution in which the propagator with momentum $p$ has been omitted. Here we perform a shift in the summation variable, $p^0 	o p^0 + k_1^0$. Then the remaining propagators are the same which appear in the $(n - 1)$-point function with incoming momenta $k_2, \ldots, k_{n-1}$, but with the loop 3-momentum $p$ replaced by $p - k_1$. Therefore we can write $\hat{\Pi}^{(n)}(k_1, \ldots, k_{n-1}, p)$ in terms of the difference of $\hat{\Pi}^{(n-1)}$ with either $k_1$ or $k_{n-1}$ omitted,

$$
\epsilon(k, p)\hat{\Pi}^{(n)\mu_1 \cdots \mu_{n-1}}(k_1, \ldots, k_{n-1}, p)
= -\hat{\Pi}^{(n-1)\mu_1 \cdots \mu_{n-2}}(k_1, \ldots, k_{n-2}, p)V_{\mu_{n-1}}(p - k, p - k + k_{n-1})t^{a_{n-1}}
+ V_{\mu_1}(p - k_1, p)t^{a_1}\hat{\Pi}^{(n-1)\mu_2 \cdots \mu_{n-1}}(k_2, \ldots, k_{n-1}, p - k_1)
$$

(18)

### 3.4 The induced current

Now we attach a gauge field $W_\mu \equiv t^a W_\mu^a$ to each vertex in Fig. 1(b) except to the one with momentum $k$, and then sum over all $n$. The result can be interpreted as the current $J_\mu$ which is induced by the gauge field background. It can also be viewed as the first functional derivative of the effective action or generating functional of our diagrams. Working with the induced current is more convenient than with individual diagrams because one does not have to worry about summing over all permutations of external lines. Furthermore, it can be used for integrating out the soft gauge fields. As in Eqs. (14) and (17) we write

$$
J_\mu^a(k) = \int \frac{d^3p}{(2\pi)^3} V_\mu(p, p - k) \text{tr} \left( t^a \hat{J}(k, p) \right)
$$

(19)

The “unintegrated” current $\hat{J}$ is given by

$$
\hat{J}(k, p) = \sum_{n=2}^{\infty} \prod_{i=1}^{n-1} \left( \int \frac{d^4k_i}{(2\pi)^4} W^\mu_{ai}(k_i) \right) (2\pi)^4 \delta \left( k - \sum_{j=1}^{n-1} k_j \right)
\times \hat{\Pi}^{(n)\mu_1 \cdots \mu_{n-1}}(k_1, \ldots, k_{n-1}, p)
$$

(20)
Using Eqs. (16) and (18) one obtains a relation\(^7\) for \(\hat{J}(k,p)\),
\[
\begin{align*}
\epsilon(k,p)\hat{J}(k,p) & = \mathcal{F}(p_\parallel,k_\parallel)V(p-k,p) \cdot W(k) \\
& - \int \frac{d^4q}{(2\pi)^4} \left[ (k-q,p)V(p-k,p-k+q) \cdot W(q) \\
& - V(p-q,p) \cdot W(q)\hat{J}(k-q,p-q) \right] 
\end{align*}
\]
(21)

4 Integrating out soft gluons

The photon polarization tensor which enters the production rate \([11]\) can be obtained from the diagrams in Fig. 1 by identifying two external lines with photons and the remaining ones with gluons. Connecting the gluon vertices with propagators and integrating over the gluon momenta will generate precisely the ladder diagrams studied in \([4]\). In addition, it generates quark self-energy insertions with soft gluon loops, which at LO are purely imaginary and correspond to a thermal width. These two contributions by themselves would be IR divergent, but their sum is IR convergent.

In terms of the current \([19]\) it means that one of the background gauge fields is the photon field and all others are gluons. The gluon fields are integrated out, leaving only the photon. Then the current is just \(\Pi^{\mu\nu}A_\nu\), from which one can read off the polarization tensor \(\Pi^{\mu\nu}\).

We therefore distinguish between external photon and gluon fields \(A^\mu\) and \(G^\mu\), and write
\[
W^\mu = A^\mu + G^\mu
\]
(22)

The photon carries \(k_\parallel\) of order \(T\), while the gluon field has \(q_\parallel \sim gT\). In order to compute the photon polarization tensor we consider one external photon field. In \(\hat{J}\) we only need to consider terms zeroth and first order in \(A^\mu\), \(\hat{J} = \hat{J}_0 + \hat{J}_1\), and what we need to compute is \(\hat{J}_1\).

\(^7\) Fourier transformed with respect to \(k\) this relation has a similar structure as the non-abelian Vlasov \([8]\) equations from which one obtains the generating functional of Hard Thermal Loops \([9]\). The linear part of the convective term \(v \cdot k\) in the Vlasov equation got replaced by \(\epsilon(k,p)\). The force term corresponds to the first term on the RHS. Finally, the non-linear part of the covariant convective derivative is replaced by the integral on the RHS. In fact, if one could neglect \(k\) and \(q\) relative to \(p\), then the integral would be proportional to the Fourier transform of the commutator \([v \cdot W, \hat{J}]\). Unlike the Vlasov equation our equation is non-local with two sources of non-locality. One is the terms with \(k_\parallel\) in the denominator, and the other is due to the fact that the second \(\hat{J}\) in the integral depends on \(p - q\).
In the equation for $\hat{J}_0$ the function $\mathcal{F}$ vanishes at leading order. Therefore $\hat{J}_0$ is suppressed compared to $\hat{J}_1$ and it can be neglected in the equation for $\hat{J}_1$. Thus the equation for $\hat{J}_1$ takes exactly the same form as Eq. (21) for $\hat{J}$, with $W$ replaced by $A$ in the inhomogeneous term, and with the two $W$’s inside the integral replaced by $G$. We keep only the leading order piece of the gluon vertex factors, so that $V(p-k, p-k+q)\simeq V(p-q, p)\simeq v$ and thus

$$
\epsilon(k, p)\hat{J}_1(k, p) = \mathcal{F}(p_\parallel, k_\parallel)V(p-k, p) \cdot A(k) - \int \frac{d^4q}{(2\pi)^4} \left[ \hat{J}_1(k-q, p)v \cdot G(q) - v \cdot G(q)\hat{J}_1(k-q, p_\parallel, p_\perp - q_\perp) \right]
$$

Now we iterate it once to obtain

$$
\hat{J}_1 = \int \frac{d^4q}{(2\pi)^4} \left[ \hat{J}_1(k-q, p)v \cdot G(q) - v \cdot G(q)\hat{J}_1(k-q, p_\parallel, p_\perp - q_\perp) \right]
$$

In the equation for $\hat{J}_1$ the function $\mathcal{F}$ vanishes at leading order. Therefore $\hat{J}_0$ is suppressed compared to $\hat{J}_1$ and it can be neglected in the equation for $\hat{J}_1$. Thus the equation for $\hat{J}_1$ takes exactly the same form as Eq. (21) for $\hat{J}$, with $W$ replaced by $A$ in the inhomogeneous term, and with the two $W$’s inside the integral replaced by $G$. We keep only the leading order piece of the gluon vertex factors, so that $V(p-k, p-k+q)\simeq V(p-q, p)\simeq v$ and thus

$$
\epsilon(k, p)\hat{J}_1(k, p) = \mathcal{F}(p_\parallel, k_\parallel)V(p-k, p) \cdot A(k) - \int \frac{d^4q}{(2\pi)^4} \left[ \hat{J}_1(k-q, p)v \cdot G(q) - v \cdot G(q)\hat{J}_1(k-q, p_\parallel, p_\perp - q_\perp) \right]
$$

Now we would like to integrate out the gluon field. We denote the resulting current by $\langle \hat{J}_1 \rangle$. In order to see how it works we write Eq. (23) schematically as $\hat{J}_1 \sim A + G\hat{J}_1$, leaving out all terms and all factors which are not relevant for the present discussion. Now we iterate it once to obtain $\hat{J}_1 \sim A + GA + GG\hat{J}_1$. Integrating out the gluons then gives $\langle \hat{J}_1 \rangle \sim A + \langle GG\hat{J}_1 \rangle$. The two gluon fields can either be contracted with each other, or with the other gluon fields in $\hat{J}_1$, that is, $\langle GG\hat{J}_1 \rangle \sim \langle GG \rangle \langle \hat{J}_1 \rangle + \langle GG\hat{J}_1 \rangle_{\text{connected}}$. In Appendix A we show that the connected part vanishes at leading order and can therefore be dropped. Thus, by integrating out the gluons one obtains a closed equation for $\langle \hat{J}_1 \rangle$ of the form $\langle \hat{J}_1 \rangle \sim A + \langle GG \rangle \langle \hat{J}_1 \rangle$.

Now we can become more explicit. After iterating the integral equation once and integrating out the soft gluons, they have disappeared as external particles and appear only in terms of their propagator,

$$
\langle G_{\mu}^{a}(q)G_{\nu}^{b}(q') \rangle = g^2\delta^{ab}\delta_{\mu\nu}(q)(2\pi)^4\delta(q + q')
$$

Since we are interested in the electromagnetic current we put $t^a = 1$ in Eq. (19). Using $\text{tr}1 = d(r)$ and $t^a t^a = C_2(r)1$, we obtain

$$
\epsilon(k, p) \text{tr} \left[ \hat{J}_1(k, p) \right] = d(r)\mathcal{F}(p_\parallel, k_\parallel)V(p-k, p) \cdot A(k) - 2C_2(r)g^2 \int \frac{d^4q}{(2\pi)^4} v^\mu v'^\nu \Delta_{\mu\nu}(q) \text{tr} \left[ \hat{J}_1(k, p) \right] - \left[ \langle \hat{J}_1(k, p) \rangle - \langle \hat{J}_1(k, p_\parallel, p_\perp - q_\perp) \rangle \right]
$$

The square bracket neither depends on $q_0$ nor on $q_\parallel$, and the integrals over $q_0$ and $q_\parallel$ can be performed. Eq. (25) is thus an integral equation which determines the transverse momentum dependence $\text{tr} \langle \hat{J}_1 \rangle$. Inside the integral we have approximated $\epsilon(k-q, p') \simeq v \cdot (k-q)$, i.e., we have neglected the terms containing transverse momenta and

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thermal masses, even though they are of the same order as the term we kept. This is possible because the gluon propagator separately depends on $q_0$ and $q_\parallel$, which are both of order $gT$, and not on their difference $v \cdot q$, which is of order $g^2T$. Therefore the terms we have omitted only contribute to a higher order shift of the integration variable $q_\parallel$.

To perform the integrals over $q_0$ and $q_\parallel$ in Eq. (25) we use again the imaginary time formalism. It means that we replace the integral $(2\pi)^{-1} \int dq_0$ by a sum over Matsubara frequencies like in Eq. (6). Furthermore, it means that $k_0$ in Eq. (25) has to be an (imaginary) Matsubara frequency. Only after performing the sum over $q_0$ one can analytically continue $k_0$ towards the real axis. We are interested in the production rate which is proportional to the imaginary part of the retarded propagator. Therefore we have to give $k_0$ a small imaginary part, i.e., $k_0 = \text{Re}(k_0) + i\varepsilon$. In Appendix A we show how we perform the Matsubara sum. Using standard results for the HTL resummed propagators [12] and the sum rule of Ref. [13] one then obtains

$$T \sum_{q_0} \int \frac{dq_\parallel}{2\pi} \frac{v^\mu v^\nu \Delta_{\mu\nu}(q)}{v \cdot (k - q)} \simeq \frac{i}{4} T \left[ -\frac{1}{q_\perp^2} + \frac{1}{q_\perp^2 + m_D^2} \right]$$

where we were able to neglect the dependence on $v \cdot k \sim g^2T$. That is, the only dependence on $v \cdot k$ enters through the imaginary part of $k_0$. The Debye mass squared enters through the gluon propagators and is given by

$$m_D^2 = \frac{g^2T^2}{6} (2N + N_s + N_f)$$

for a SU($N$) gauge theory with $N_s$ complex scalars and $N_f$ Dirac fermions.

For the production rate (1) we need the polarization tensor which we write as

$$\Pi_{\mu\nu}(k) = \int \frac{d^3p}{(2\pi)^3} V_\mu(p, p - k) \hat{\Pi}_\nu(k, p)$$

The reduced polarization tensor $\hat{\Pi}_\nu$ is related to $\hat{J}_\nu$ through

$$\text{tr}(\hat{J}_\nu(k, p)) = \hat{\Pi}_\nu(k, p) A^\nu(k)$$

Therefore it satisfies the integral equation

$$\epsilon(k, p) \hat{\Pi}_\nu(k, p) = d(r) F(p_\parallel, k_\parallel) V_\nu(p - k, p)$$

$$+ iC_2(r) g^2T \int \frac{d^2q_\perp}{(2\pi)^2} \left[ \frac{1}{q_\perp^2} - \frac{1}{q_\perp^2 + m_D^2} \right] \left[ \hat{\Pi}_\nu(k, p) - \hat{\Pi}_\nu(k, p_\parallel, p_\perp - q_\perp) \right]$$
We finally want to show that our integral equation (30) can be reduced to the one in Ref. [4] for the production of real photons. We choose \( v \) in the direction of \( k \) so that \( k_\perp = 0 \). Then we have \( v \cdot k = 0 \), and thus \( \epsilon(k, p) = (p_\perp^2 + m^2)k_\parallel/[2p_\parallel(p_\parallel - k_\parallel)] \). Only the transverse components of \( \hat{\Pi}_\nu \) contribute since the polarization vectors in (1) are purely transverse. If we define a new function \( f \) via

\[
\hat{\Pi}_\perp(k, p) = \frac{i d(r) F(p_\parallel, k_\parallel)}{2(p_\parallel - k_\parallel)} f(k, p)
\]

we obtain the equation of the same form as in [4]

\[
2p_\perp = i\epsilon(k, p) f(k, p)
\]

\[
+ g^2C_2(r)T \int \frac{d^2q_\perp}{(2\pi)^2} \left[ \frac{1}{q_\perp^2} - \frac{1}{q_\perp^2 + m_D^2} \right] \left[ f(k, p) - f(k, p_\parallel, p_\perp - q_\perp) \right]
\]

(32)

## 5 Spin-1/2 quarks

So far we have always been dealing with scalar quarks to avoid technical complications. There is no need to redo the entire calculation for spin-1/2 quarks. The results above are still valid up to a few modifications which we now describe.

We now have to deal with the resummed fermion propagator

\[
S(p) = -\frac{1}{p - \Sigma(p)}
\]

(33)

in the high temperature limit when the zero temperature mass can be neglected. In the plasma rest frame the self-energy \( \Sigma(p) \) takes the general form [14]

\[
\Sigma(p) = a(p)\gamma^0 + b(p)\gamma^0 \gamma^0 \gamma^0
\]

(34)

Therefore chiral symmetry is not broken by thermal effects, and the left- and right-chiral fermions propagate independently. Since \( \Sigma(p) \sim g^2T \), we have \( a(p) \sim g^2 \). Thus \( a(p)\gamma^0 \) is small compared to the tree level contribution and may be neglected. For \( p \sim T \), \( p^2 \sim g^2T^2 \) the propagator can then be written as

\[
S(p) \simeq -\frac{p + b(p)\gamma^0}{p^2 - 2b(p)p^0} \simeq -\frac{p}{p^2 - m^2}
\]

(35)

Here we have neglected terms of order \( g^2T \) in the numerator and we have identified

\[
b(p) = \frac{m^2}{2p^0} \]

(36)
at $p^2 = m^2$, where $m$ is the asymptotic thermal mass in Eq. (5). Since the left and right-handed fermions propagate independently, one may consider the photon production from left-handed quarks only. The complete rate is then twice as large. Thus one can deal with Weyl instead of Dirac spinors, the vertices contain $\bar{\sigma}^\mu$ instead of $\gamma^\mu$, and the propagator (35) contains $\sigma \cdot p$ instead of $p$. Up to terms of order $g^2T$, which we neglect in the numerator, $p^\mu$ is light-like. Therefore we can write

$$\sigma \cdot p \simeq 2p_\parallel \eta(p)\eta^\dagger(p)$$

(37)

where $\eta(p)$ is a normalized eigenvector of $p \cdot \sigma/p_\parallel$ with negative eigenvalue. Thus we find, similarly to Ref. [4],

$$S_L(p) \simeq \eta(p)\eta^\dagger(p)D(p)$$

(38)

with the same $D(p)$ as in Eq. (9). Note that, unlike in Ref. [4], our result is valid for both signs of $p_\parallel$. We associate the spinors $\eta(p)$ and $\eta^\dagger(p)$ with the vertices on either side of the propagator rather than with the propagator itself. Therefore the vertex factor now reads

$$V^\mu(p, p - k) = \eta^\dagger(p - k)\bar{\sigma}^\mu\eta(p)$$

(39)

instead of Eq. (12). For real photon production one needs $V$ up to order $g$,

$$V^\mu(p, p - k) = v^\mu + V_\perp^\mu + O(g^2)$$

(40)

In the helicity basis the transverse components $V_\perp^\mu$ are particularly simple. We choose $v$ as the 3-direction. Then for $V^\pm \equiv (V^1 \pm iV^2)/\sqrt{2}$ one finds

$$V^+ = \frac{p^+}{p_\parallel - k_\parallel} + O(g^2), \quad V^- = \frac{p^-}{p_\parallel} + O(g^2)$$

(41)

The other difference compared to scalar quarks is due to the Fermi-Dirac statistics, the Bose-Einstein functions get replaced by Fermi-Dirac distributions, so that $F$ becomes

$$F(p_\parallel, k_\parallel) = f_F(p_\parallel) - f_F(p_\parallel - k_\parallel)$$

(42)

6 Summary and Conclusions

In this paper we have obtained an integral equation (Eq. [21]) which sums all thermal 1-loop diagrams with an arbitrary number of soft or collinear external gauge fields. We
have applied it to compute the rate for real photon production by bremsstrahlung and pair annihilation in a hot QCD plasma.

Compared to the original calculation of the photon production rate in Ref. [4] our approach is significantly simplified by the fact the calculation is done in two steps. In the first step we have integrated out the hard momentum modes at one loop. The resulting effective theory is summarized by Eq. (21). It has a similar structure as the non-abelian Vlasov equation which describes the Hard Thermal Loops for soft external gauge fields. In a second step we have integrated out the soft gauge fields corresponding to gluons. This results in the integral equation obtained earlier in Ref. [4] which sums all leading order ladder and self energy contributions to the photon polarization tensor for hard on-shell photons, and which thus describes the Landau-Pomeranchuk-Migdal effect on thermal photon production.

Our approach should easily allow for generalizations. The method can be adopted to the production of other particles than photons, e.g. the production of spin-1/2-fermions. We also hope for a possible generalization to non-equilibrium situations, as they occur e.g. in heavy ion collision.

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### A Connected pieces

Here we show that only the disconnected parts contribute in the calculation of the photon production rate at leading order, as claimed in Sec.4. To simplify the discussion we can drop the dependence on spatial momenta and on $m^2$, and replace $\epsilon(k,p)$ by $v \cdot k$. Furthermore we leave out the vector indices of $A$ and $G$. All these simplifications do not affect our argument.

Therefore, for the present discussion, we may study, instead of the full integral equation (23), the simplified version

$$v \cdot k \tilde{J}_1(k) = A(k) + \int_{k_1} G(k_1) \tilde{J}_1(k - k_1)$$

(A.1)

where we have introduced the compact notation

$$\int_k \equiv T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3}$$

(A.2)
One can easily write down the solution to Eq. (A.1)

$$\hat{J}_1(k) = \frac{1}{v \cdot k} \sum_{N=0}^{\infty} \prod_{n=1}^{N} \left( \int_{k_n} G(k_n) \frac{1}{v \cdot (k - \sum_{l=1}^{n} k_l)} \right) A \left( k - \sum_{l=1}^{N} k_l \right)$$

(A.3)

In the connected part \(\langle G G \hat{J}_1 \rangle_{\text{connected}}\), which was dropped in Sec. 4, the gluon field \(G(k_1)\) is contracted with some \(G(k_M)\) with \(M \geq 3\). Thus \(\langle G G \hat{J}_1 \rangle_{\text{connected}}\) contains the Matsubara sum

$$T \sum_{k_1^0} \Delta(k_1) \frac{1}{v \cdot (k - k_1)} \frac{1}{v \cdot (k - k_1 - k_2)} \cdots \frac{1}{v \cdot (k - k_1 \cdots - k_{M-1})}$$

(A.4)

We use the spectral representation of the propagators

$$\Delta(k_1) = -\int \frac{d\omega}{2\pi i} \frac{1}{k^0_1 - \omega \text{ Disc} \Delta(\omega, k_1)}$$

(A.5)

The thermal sum can now easily be performed using (6). Out of the \(M\) poles, only the one at \(k_1^0 = \omega\) which gives \(f_B(\omega) \simeq T/\omega = \mathcal{O}(1/g)\) contributes at leading order because the gluons are soft. At all other poles the Bose distribution function would be \(\mathcal{O}(1)\) and the corresponding contributions can be neglected.

After performing all Matsubara sums, we may analytically continue \(k^0_1\) towards the real axis and replace it by \(k^0_1 + i\varepsilon\) where \(k^0_1\) is now real. Then (A.4) turns into

$$\int \frac{dk_1^0}{2\pi i} \frac{T}{k_1^0} \text{ Disc} \Delta(k_1) \frac{1}{v \cdot (k - k_1) + i\varepsilon} \frac{1}{v \cdot (k - k_1 - k_2) + i\varepsilon} \cdots \frac{1}{v \cdot (k - k_1 \cdots - k_{M-1}) + i\varepsilon}$$

(A.6)

Here we have to integrate over the region where \(v \cdot k_1\) is of order \(g^2 T\). That means that \(k^0_1\) is equal to \(k_1||\) up to terms of order \(g^2 T\) and that we may replace

$$\frac{1}{k_1^0} \text{ Disc} \Delta(k_1) \rightarrow \frac{1}{k_{1||}} \text{ Disc} \Delta(k_{1||}, k_1)$$

(A.7)

without changing the leading order result. Now we are done because all poles in the integrand lie above the real \(k^0_1\)-axis. Therefore we can close the integration contour at \(-i\infty\) and we obtain zero. This proves that at leading order only the disconnected part contributes.

We finally remark that if we contract \(G(k_1)\) with \(G(k_2)\), which corresponds to the disconnected contribution (and to \(M = 2\) in the calculation above), this argument fails
since the integrand does not fall off rapidly enough at infinity and the integral over a
closed loop would give a non-vanishing contribution. In fact, in that case we obtain
the result (26).

In Ref. [4] the gluons where integrated out by (i) including the width in the quark
propagators, and (ii) using Feynman diagrams for the remaining contributions. There
it was shown that the leading order contributions are due to ladder diagrams with
uncrossed rungs, and that ladder diagrams with crossed rungs or vertex corrections
vanish at leading order. Even though we have not checked it explicitly, it seems to be
clear that these diagrams are part of the connected pieces.

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