Dynamical symmetries of generalized Taub-NUT and multi-center metrics

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Hidden symmetries of generalized Kaluza-Klein-type metrics are studied using van Holten’s systematic analysis [1] based on Killing tensors. Applied to generalized Taub-NUT metrics, Kepler-type symmetries with associated Runge-Lenz-type conserved quantities are constructed. In the multi-center case, the subclass of two-center metrics gives rise to a conserved Runge-Lenz-type scalar, while no Kepler-type constant of the motion does exist for non-aligned \((N \geq 3)\)-centers. We also investigated the diatomic molecule system of Wilczek et al. where “truly” non-Abelian gauge fields mimicking monopole-like fields arise. From the latter system we deduced a new conserved charge.

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I. INTRODUCTION TO KALUZA-KLEIN THEORIES

Kaluza-Klein (KK) theories have been extensively studied as schemes attempting to unify gravitation and gauge theory [2,3], through the physical assumption that the world admits, in addition to 4D space-time, an unobservable extra dimension. Thus, ordinary general relativity in five dimensions is considered to possess a local U(1) gauge symmetry arising from a “vertical” translation along the hidden extra dimension.

A perfect illustration of the KK framework has been given by the Sorkin, then Gross and Perry solution of the vacuum Einstein equation, involving an Abelian monopole potential [4, 5] which carries, unexpectedly, Kepler-type dynamical symmetries. Further examples are provided by the multi-center metrics [6–9] for which similar hidden symmetry properties have been revealed.

Let us consider the 5D KK metric tensor,

\[
g_{AB} = \left( \begin{array}{cc} \gamma_{\mu\nu} + VA_\mu A_\nu & A_\mu V \\ A_\nu V & V \end{array} \right),
\]

\(A, B = 0, \ldots, 4; \quad \mu, \nu = 0, \ldots, 3; \quad A_0 = 0,\)

where the 5D manifold can be viewed as a direct product of a 4D space-time (where \(x^0\) is the time coordinate) with an unobservable space-like loop, \(M^4 \otimes S^1\), and where all components of \(g_{AB}\) are independent of the extra coordinate \(x^4\). Here \(\gamma_{\mu\nu}\) is the metric of the 4D manifold \(M^4\). From [1], the dynamics of a classical point-like test particle of unit mass is given by the 5D geodesic motion,

\[
d^2 x^A \over d\tau^2 + \Gamma^A_{BC} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau} = 0,
\]

where \(\tau\) denotes the proper time. Using the effective theory [1] in [2], a routine calculation yields the equations of the motion,

\[
\begin{align*}
\frac{d}{d\tau} \left( V A_\mu \frac{dx^\mu}{d\tau} + V \frac{dx^4}{d\tau} \right) &= \frac{dq}{d\tau} = 0, \\
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} - q F^\kappa_\lambda \frac{dx^\lambda}{d\tau} - q^2 \frac{\partial^\nu V}{2V^2} &= 0.
\end{align*}
\]

Here the \(x^\mu\) are the coordinates on the 4D manifold \(M^4\) while \(x^4\) represents the “extra” coordinate. The first equation in (3) tells us that the “charge”,

\[
q = V \left( A_\mu \frac{dx^\mu}{d\tau} + \frac{dx^4}{d\tau} \right),
\]

is conserved along the 5D geodesics. The latter can also be viewed as being associated with translations in the “extra” direction, generated by the Killing vector \(\partial_{x^4}\). The second equation in (3) is a 4D geodesic equation involving an interaction with the scalar field \(V\) in addition to the Lorentz force. See [10–12] for further references.

The non-Abelian generalization of the 5D KK approach was given by Kerper [12] through generalizing the previous 5D manifold to a \((4+d)\)-dimensional one, namely \(M = M^4 \otimes S^d\) whose base space \(M^4\) is the usual space-time and where \(S^d\) represents an unobservable \(d\)-dimensional extra space. Here the \((4+d)\)D diffeomorphism symmetry is broken to \(4\)D infinitesimal coordinate transformations augmented with translations along the extra dimensions,

\[
x^\mu \rightarrow x^\mu + \delta x^\mu, \quad y^a \rightarrow y^a + f^i(x^{\nu})\xi^a_i(y),
\]

where \(f^i(x^{\nu})\) are infinitesimal functions and \(\xi^a_i\) denoting the isometry generators on the compact manifold \(S^d\). The \((4+d)\)D generalized metric, invariant under (3), then reads

\[
\tilde{g}_{CD} = \left( \begin{array}{cc} \gamma_{\mu\nu} + \kappa_{ab} B^a_\mu B^b_\nu & B^b_\mu \kappa_{ab} \\
\kappa_{ab} B^a_\nu & \kappa_{ab} \end{array} \right),
\]
where $\kappa_{ab}$ represents the $SU(d-1)$ invariant metric on $S^d$, and $B_a^b = A_a^b \xi_b^0$ includes the $SU(d-1)$ Lie algebra valued one-form $A_a^b$, identified as a Yang-Mills field.

We can now deduce the geodesic equations which yield the equations of motion of an isospin-carrying particle in a curved space plus a Yang-Mills field,

\[
\begin{align*}
D_a^c & = \frac{dx^c}{dt} - \mathcal{I}_a e^a_{bc} A^b_{\mu} \frac{dx^\mu}{dt} = 0, \\
\frac{d^2 x^\beta}{dt^2} + \Gamma^\beta_{\mu \nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} + \gamma^\beta_{\mu \nu} F^b_{\mu \nu} \frac{dx^\mu}{dt} = 0.
\end{align*}
\]

Analogously to the Abelian case, the first equation in the pair identifies the classical isospin variable,

\[
\mathcal{I}_a = \kappa_{ab} \left( \frac{dy^b}{dt} + A^b_a \frac{dx^a}{dt} \right),
\]

which is parallel transported and describes the motion in non-Abelian internal space. The isospin is analogous to the electric charge which is parallel transported and describes the motion in 4D real space. Note here the generalized Lorentz force

\[
\gamma^\beta_{\mu \nu} F^b_{\mu \nu} \mathcal{I}_b \left( \frac{dx^\mu}{dt} \right)
\]
due to the Yang-Mills field and with the electric charge replaced by the isospin.

The equations (6) are known as the Kerner-Wong equations, as they were also obtained by Wong by “dequantizing” the Dirac equation. They can be derived from a variational principle; alternatively, they can be studied using a symplectic approach.

II. VAN HOLTEN’S METHOD TO DERIVE THE CONSTANTS OF THE MOTION

Now, inquiring about the symmetries of the KK-type metrics, we recall that constants of the motion denoted as $Q$, which are polynomial in the momenta, can be derived following van Holten’s algorithm. The recipe is to expand $Q$ into a power series of the covariant momentum,

\[
Q = C + C^i \Pi_i + \frac{1}{2!} C^{ij} \Pi_i \Pi_j + \frac{1}{3!} C^{ijk} \Pi_i \Pi_j \Pi_k + \cdots
\]

and to require $Q$ to Poisson-commute with the Hamiltonian augmented with an effective potential, \( \{ Q, \mathcal{H} = \Pi^2/2 + G(x) \} = 0 \). This yields the series of constraints,

\[
\begin{align*}
C^m \partial_m G &= 0 \quad \text{o(0)} \\
\partial_n C &= q F_{nm} C^m + C^n \partial_m G \quad \text{o(1)} \\
D_i (C_l) &= q (F_{im} C^m_l + F_{jm} C^m_i) + C_{ik} \partial_k G \quad \text{o(2)} \\
D_{ij} (C_l) &= q F_{im} C^m_{lj} + q F_{jm} C^m_{li} + q F_{im} C^m_{jl} + C_{ijl} \partial_k G \quad \text{o(3)} \\
\ldots \ldots 
\end{align*}
\]

Here the zeroth-order constraint can be interpreted as a consistency condition for the effective potential. It is worth noting that the expansion can be truncated at a finite order provided some higher-order constraint reduces to a Killing equation,

\[
D_i (C_{i_2 \ldots i_n}) = 0,
\]

where the covariant derivative is constructed with the Levi-Civita connection so that

\[
D_i C^{ij} = \partial_i C^{ij} + \Gamma^{ij}_{\ k} C^k.
\]

Then $C_{i_2 \ldots i_n} = 0$ for all $p \geq n$ and the constant of the motion takes the polynomial form,

\[
Q = \sum_{k=0}^{p-1} \frac{1}{k!} C^{i_1 \ldots i_k} \Pi_{i_1} \cdots \Pi_{i_k}.
\]

It is worth noting that apart from zeroth-order conserved charges which are independent of the covariant momenta, all order-$n$ invariants are deduced by the van Holten method involving rank-$n$ Killing tensors of the curved manifold. Generating symmetries using Killing tensors was first advocated by Carter in the context of the Kerr metric.

In our set of constraints, a given Killing tensor is the highest-order coefficient of the expansion, allowing us to solve the truncated series of constraints and thus generating a conserved quantity. The intermediate-order constraints then determine the other coefficient-terms of the invariant.

In what follows, our strategy will be to find conditions for lifting to the “Kaluza-Klein” 4-space those Killing tensors which generate, in flat space, the conserved angular momentum and the Runge-Lenz vector of planetary motion, respectively.

III. HIDDEN SYMMETRIES OF GENERALIZED TAUB-NUT METRICS

Let us first investigate the symmetries of the [Abelian] Kaluza-Klein monopole. The latter, obtained by imbedding the Taub-NUT gravitational instanton into Kaluza-Klein theory, provides us with an exact solution of four-dimensional Euclidean gravity approaching the vacuum solution at spatial infinity,

\[
\begin{align*}
\frac{dx^2}{g_{ij}(\bar{x})} dx^i dx^j + h(\bar{x}) (dx^4 + A_k dx^k)^2 \\
g_{ij}(\bar{x}) &= f(\bar{x}) \delta_{ij}(\bar{x})
\end{align*}
\]

where $f(\bar{x})$, $h(\bar{x})$ and $A_k$ are real functions and the gauge potential of an Abelian magnetic field, respectively. The Lagrangian function associated reads

\[
\mathcal{L} = \frac{1}{2} f(\bar{x}) \dot{\bar{x}}^2 + \frac{1}{2} f^{-1}(\bar{x}) \left( \frac{dx^4}{dt} + A_k \frac{dx^k}{dt} \right)^2 - U(r),
\]
where \( U(r) \) is an external scalar potential.

Inspired by Kaluza’s hypothesis, as the fourth dimension \( \tilde{x}^4 \) is considered to be cyclic, we use the conservation of the “vertical” component of the momentum interpreted as a conserved electric charge,

\[
p_4 = h(\tilde{x}) (dx^4/dt + A_k dx^k/dt) = q ,
\]
to reduce the four-dimensional problem to one in three dimensions, where we have strong candidates for the way dynamical symmetries act \[22\]. Then, the lifting problem can be conveniently solved using the Van Holten technique \[17\] based on Killing tensors. The geodesic motion on the 4-manifold therefore projects onto the curved 3-manifold with the metric \( g_{ij}(\tilde{x}) \), augmented with an effective potential,

\[
V(\tilde{x}) = q^2 / 2h(\tilde{x}) + U(r) ,
\]
such as

\[
\begin{align*}
\dot{x}^i &= g^{ij} \Pi_j , \\
\Pi_i &= p_i - q A_i ,
\end{align*}
\]
(12)

Note that the Lorentz equation in \[12\] involves also in addition to the monopole and potential terms, a curvature term which is quadratic in the velocity.

Let us now focus Kaluza-Klein-type metrics \[11\] whose hidden symmetries have been extensively investigated \[6–8\] \[21\]. For geodesic motion on hyperbolic space, for instance, Gibbons and Warnick \[6\] found a large class of systems admitting such hidden symmetries.

We assume that the metric \[11\] is radial,

\[
f(\tilde{x}) = f(r) , \quad h(\tilde{x}) = h(r) ,
\]

and the magnetic field is that of a Dirac monopole of charge \( g \). In that event, the generator of spatial rotations,

\[
C_i = g_{ij}(r) e^j_{kl} n^k x^l ,
\]
(13)

applied to \[11\] provides us with the conserved angular momentum involving the typical monopole term,

\[
\tilde{J} = \tilde{x} \times \tilde{\Pi} - q g \tilde{r} / r .
\]
(14)

Turning to second order Runge-Lenz-type conserved quantities, we use the rank-2 Killing tensor,

\[
C_{ij} = 2g_{ij}(r)n_k x^k - g_{ik}(r)n_j x^k - g_{jk}(r)n_i x^k ,
\]
(15)
influenced by its form in the Kepler problem. We therefore deduce from \[17\] that:

1. For the original Taub-NUT case \[4, 5\] with no external scalar potential,

\[
f(r) = \frac{1}{h(r)} = 1 + \frac{4m}{r} ,
\]
(16)

where \( m \) is real \[4, 7, 8\] we obtain, for the energy \( \mathcal{E} \) and the charge \( g = \pm 4m \), the conserved Runge-Lenz vector,

\[
\vec{K} = \vec{\Pi} \times \vec{J} - 4m (\mathcal{E} - q^2) \frac{\vec{r}}{r} .
\]
(17)

2. Lee and Lee \[38\] argued that for monopole scattering with independent Higgs expectation values, the geodesic Lagrangian derived from \[11\] should be replaced by \( \mathcal{L} \to \mathcal{L} - W(r) \), where

\[
W(r) = \frac{1}{2} \frac{a_0^2}{1 + \frac{4m}{r}} .
\]
(18)

It is easy to see that this addition merely shifts the value of the energy by a constant \( a_0^2 / 2 \), so the previously found Runge-Lenz vector \[17\] is still valid.

3. The metric associated with winding strings \[39\],

\[
f(r) = 1 , \quad h(r) = \frac{1}{(1 - \frac{3}{r})^2} ,
\]
(19)

with charge \( g = \pm 1 \), leads to the conserved Runge-Lenz vector,

\[
\vec{K} = \vec{\Pi} \times \vec{J} - q^2 \frac{\vec{r}}{r} .
\]
(20)

4. The extended Taub-NUT metric \[22, 26\]

\[
f(r) = b + \frac{a}{r} , \quad h(r) = \frac{a r + b r^2}{1 + d r + c r^2} ,
\]
(21)

with \((a, b, c, d)\) real. With no external scalar potential and charge \( g = \pm 1 \), we have the conserved Runge-Lenz vector,

\[
\vec{K} = \vec{\Pi} \times \vec{J} - (a \mathcal{E} - \frac{1}{2} d q^2) \frac{\vec{r}}{r} .
\]
(22)

5. Considering the oscillator-type metric discussed by Iwai and Katayama \[22\], the functions \( f(r) \) and \( h(r) \) take the particular form

\[
f(r) = b + a r^2 , \quad h(r) = \frac{a r + b r^2}{1 + c r^2 + d r^4} .
\]
(23)

A direct calculation leads to the following Runge-Lenz-type vector \[25\] ,

\[
\vec{K} = (b + a r^2) \frac{\dot{\tilde{x}} \times \vec{J} + \beta \frac{\vec{r}}{r}}{f(r)} ,
\]
(24)

conserved only for a scalar potential of the form

\[
W(r) = \frac{(q^2 \mathcal{E}^2 + \beta)}{2 r^2} \frac{1}{f(r)} - \frac{q^2}{h(r)} .
\]
(25)

1. Monopole scattering corresponds to \( m = -1/2 \) \[21\].
IV. N-CENTER METRICS WITH SCALAR CONSTANTS OF THE MOTION

The multi-center metrics in which we are interested here are Euclidean vacuum solutions of the Einstein equations with self-dual curvature; they can be viewed as generalizations of the previously investigated Taub-NUT metrics.

Let us consider a particle moving in Gibbons-Hawking space \([10]\). The Lagrangian function associated with this dynamical system, derived from \([11]\), is

\[
\mathcal{L} = \frac{1}{2} f(\vec{x}) \dot{\vec{x}}^2 + \frac{1}{2} f^{-1}(\vec{x}) \left( \frac{dx^i}{dt} + A_k \frac{dx^k}{dt} \right)^2 - U(\vec{x}),
\]

where the functions \(f(\vec{x})\) obey the “self-dual” condition \(\nabla \cdot f(\vec{x}) = \pm \nabla \times \vec{A}\). \(f(\vec{x})\) satisfies therefore the three-dimensional Laplace equation,

\[
\Delta f(\vec{x}) = 0,
\]

whose most general solution is given by

\[
f(\vec{x}) = f_0 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|}, \quad (f_0, m_i) \in \mathbb{R}^{N+1}.
\]

The multicenter metric admits multi-NUT singularities so that the position of the \(i\)th NUT singularity with the charge \(m_i\) is \(\vec{a}_i\). These singularities can be removed provided to have all NUT charges equal. In this case, the cyclic variable \(x^4\) is periodic with range \(0 \leq x^4 \leq 4\pi m\). We are thus interested in the projection of the motion on the curved 3-manifold whose metric is,

\[
g_{jk}(\vec{x}) = (f_0 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|}) \delta_{jk}.
\]

For simplicity, we limit ourselves to two-center metrics,

\[
f(\vec{x}) = f_0 + \frac{m_1}{|\vec{x} - \vec{a}|} + \frac{m_2}{|\vec{x} + \vec{a}|}.
\]

Turning to rotational symmetry, the rank-1 Killing tensor satisfying the second-order equation in \([7]\),

\[
C_l = g_{lm} \epsilon^{mn}_{\quad ik} \hat{a}^l x^k, \quad \hat{a} = \hat{a}/a,
\]

generates rotational symmetry around the axis through the two centers. The corresponding conserved quantity,

\[
\mathcal{J}_a = (\vec{x} \times \vec{a}) \cdot \hat{a} - q(f(\vec{x}) - f_0) \vec{x} \cdot \hat{a} + \frac{q \hat{a} m_1}{|\vec{x} - \vec{a}|} - \frac{q \hat{a} m_2}{|\vec{x} + \vec{a}|},
\]

is the projection of the angular momentum onto the axis of the two centers.

Now we study quadratic conserved quantities by considering the reducible rank-2 Killing tensor,

\[
g_{ij}(\vec{x}) = f(\vec{x}) \delta_{ij} - f_0 \delta_{ij} + \sum_{m=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|} \delta_{ij},
\]

which is a symmetrized product of the Killing-Yano tensors, \(C_l = g_{lm} \epsilon^{mn}_{\quad ik} \hat{a}^l x^k\) generating rotations around the axis of the two centers and \(C_l = g_{lm} \hat{a}^m\) generating spatial translation along the axis of the two centers. Injecting \([20]\) into the system of constraint \([4]\) yields, for vanishing effective potential \(U = 0\), the Casimir, which combines the square of the projected angular momentum with the square of the component of the covariant momentum along the axis,

\[
Q = \mathcal{J}_a^2 + \Pi_a^2.
\]

As expected from the construction of the associated Killing tensor \([8]\), the obtained conserved quantity is a combination of two constant of the motion \([27]\).

Now introducing into \([4]\) the rank two Killing tensor generating Kepler-type dynamical symmetry provides us with the conserved scalar,

\[
K_a = \left(\vec{1} \times \vec{J}\right) \cdot \hat{a} + \frac{\beta}{q} \left(\mathcal{L}_a - \mathcal{J}_a\right),
\]

where \(\beta \in \mathbb{R}, K_a\) therefore represents, for the two-center metrics \([20]\), a conserved Runge-Lenz-type scalar for particle motion confined onto the “Appollonius” two-sphere \([22]\) of center at \(\hat{a}_\rho\) and with radius \(R = a \sqrt{\rho^2 - 1}\), provided the effective potential is of the form,

\[
W = \frac{q^2}{2} (f(\vec{x}) - f_0)^2 + \beta (f(\vec{x}) - f_0) + \gamma
\]

\(\gamma\) a constant, which satisfies the consistency condition given by the zeroth-order constraint of \([7]\).

It is worth mentioning that a scalar Runge-Lenz-type conserved quantity does exist only for a particle moving along the axis of the two centers, or for motions confined onto the “Appollonius” two-sphere defined above. In the Eguchi-Hanson case \(m_1 = m_2\), and the 2-sphere is replaced by the median plane of the two centers \([22]\).

The \((N \geq 3)\)-centers metrics of the form

\[
g_{ij}(\vec{x}) = f(\vec{x}) \delta_{ij} - f_0 \delta_{ij} + \sum_{m=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|} \delta_{ij}
\]

can also be investigated, but they carry no Runge-Lenz-type symmetry for \(N \geq 3\) non aligned centers. To obtain this result, let us first generalize the rank-2 Killing tensor \([11]\) generating the Runge-Lenz vector as

\[
C_{ij} = 2g_{ij}(\vec{x}) n_k x^k - g_{ik}(\vec{x}) n_j x^k - g_{jk}(\vec{x}) n_i x^k.
\]

Requiring the Killing equation to be satisfied as

\[
\mathcal{D}(k) C_{ij} = 0,
\]

\(^2\) This sphere was known already by Apollonius of Perga in the 2th century BC.
a tedious calculation then provides us with the condition
\[ \vec{n} \times (\vec{\partial} \times \vec{\nabla} f(\vec{x})) = 0. \] (36)

For the metric \[ \text{(33)} \] above this requires
\[ \sum_{i=1}^{N} \left( \vec{n} \cdot \vec{x} \right) \vec{a}_i - \left( \vec{n} \cdot \vec{a}_i \right) \vec{x} \left| \vec{x} - \vec{a}_i \right|^3 = 0, \] (37)
which cannot be satisfied for more than two non-aligned centers. The only possibility we get for \[ \text{(37)} \] to be satisfied by more than two centers is to have all of them in the same alignment describing a straight line of centers.

V. THE NON-ABELIAN CASE : THE DIATOMIC MOLECULE

In Ref. [41] Moody, Shapere and Wilczek have shown that nuclear motion in a diatomic molecule can be described, in the Born-Oppenheimer approximation, by an effective non-Abelian gauge field of “hedgehog” form,
\[ A_i^a = (1 - \kappa) \epsilon_{aij} \frac{x^j}{r^2}, \quad F_{ij}^a = (1 - \kappa^2) \epsilon_{ijk} \frac{x^k}{r^4}. \] (38)

This gauge field mimics the structure of that of a non-Abelian monopole \[ \text{[42][43]} \] Note here the unquantized constant real factor \[ 1 - \kappa^2 \]. The potential \[ \text{(38)} \] becomes that of a Wu-Yang [i.e., an imbedded Dirac] monopole of unit charge when \( \kappa = 0 \); for other values of \( \kappa \), it is a truly non-Abelian configuration except for \( \kappa = \pm 1 \), when it is a gauge transform of the vacuum.

Now we investigate the symmetries of an isospin-carrying particle carrying unit charge, evolving in the monopole-like field of the diatom \[ \text{(38)} \] plus a scalar potential. The Hamiltonian describing the dynamics of this particle is finally expressed as
\[ \mathcal{H} = \frac{\vec{p}^2}{2} + V(\vec{x}, \vec{\pi}, \vec{T}), \quad \pi_i = p_i - A_i^a T^a. \]

Defining the covariant Poisson-brackets as
\[ \left\{ M, N \right\} = D_j M \frac{\partial N}{\partial \pi_j} - D_j N \frac{\partial M}{\partial \pi_j} + \mathcal{T}^a F_{ij}^a \frac{\partial M}{\partial \pi_j} \frac{\partial N}{\partial \pi_k} - \epsilon_{abc} \mathcal{T}^a \mathcal{T}^b \mathcal{T}^c, \]

the non-vanishing brackets are
\[ \{ x^i, \pi_j \} = \delta^i_j, \quad \{ \pi_i, \pi_j \} = \mathcal{T}^a F_{ij}^a, \quad \{ \mathcal{T}^a, \mathcal{T}^b \} = -\epsilon_{abc} \mathcal{T}^c. \]

The equations of motion governing an isospin-carrying particle in the static non-Abelian gauge field \[ \text{(38)} \] read
\[ \begin{cases} \ddot{x}_i - \mathcal{T}^a F_{ij}^a \pi^j + D_i V = 0, \\ \dot{\mathcal{T}}^a + \epsilon_{abc} \mathcal{T}^b (A_i^c \dot{x}^j - \frac{\partial V}{\partial \mathcal{T}^c}) = 0. \end{cases} \] (39)

These equations generalize the Kerner-Wong equations \[ \text{(36)} \] to an additional scalar potential.

Turning to the conserved quantities constructed with the van Holten algorithm, the zeroth-order conserved charge which used-to-be interpreted as electric charge for \( \kappa \neq 0 \),
\[ Q = \frac{\vec{x} \cdot \vec{a}}{r}, \] (40)
is not more covariantly conserved in general,
\[ \{ Q, \mathcal{H} \} = \vec{\pi} \cdot \vec{D} Q, \quad D_j Q = \frac{\kappa}{r} (\mathcal{T}^j - Q \mathcal{F}^j), \] (41)
Detailed calculation shows that the equation \( D_j Q = 0 \) can only be solved, for an imbedded Abelian monopole field, when \( \kappa = 0, \pm 1 \).

Now is \( Q^2 \) conserved, \( \{ Q^2, \mathcal{H} \} = 2\kappa Q (\vec{\pi} \cdot \vec{D} Q) \cdot \). But note that, unlike \( Q^2 \), the length of the isospin, \( \mathcal{T}^2 \), is conserved,
\[ \{ \mathcal{H}, \mathcal{T}^2 \} = 0. \] (42)
These results are consistent with those in [41].

Turning to linear conserved quantities, we use the Killing vector generating spatial rotations,
\[ \vec{C} = \vec{n} \times \vec{x}, \] (43)
to build up the conserved angular momentum,
\[ \vec{J} = \vec{x} \times \vec{\pi} - (1 - \kappa) Q \frac{\vec{x}}{r} - \kappa \vec{a}. \] (44)

Moody, Shapere and Wilczek [41] did find this expression for \( \kappa = 0 \) but, as they say it, “they are not aware of a canonical derivation when \( \kappa \neq 0 \)”. Our construction here is an alternative to that of Jackiw [45], who obtained it using the method of Ref. [46].

We now consider the rank-2 Killing tensor,
\[ C_{ij} = 2\delta_{ij} x^2 - 2x_i x_j, \] (45)
which satisfies the third-order constraint in [47]. The Killing tensor \[ \text{[45]} \] thus yields the conserved Casimir,
\[ L^2 = (\vec{x} \times \vec{\pi})^2 = x^2 \pi^2 - (\vec{x} \cdot \vec{\pi})^2, \] (46)
which is the square of non-conserved orbital angular momentum, \( \vec{L} = \vec{x} \times \vec{\pi} \). Since \( J^2 \) and \( L^2 \) are both conserved, the new charge,
\[ \Gamma = J^2 - L^2 = (1 - \kappa)^2 Q^2 - \kappa^2 \mathcal{T}^2 - 2\kappa \vec{J} \cdot \vec{a}, \] (47)
is conserved for motion in the monopole-like field of diatomic molecule. It is worth noting that the charge \( \Gamma \) becomes, in the Abelian limit \( \kappa = 0 \), the square of conserved electric charge.

Just like \( \vec{J} \), \( J^2 \) and \( L^2 \), the charge \( \Gamma \) is conserved for any radially symmetric potential, \( V(r) \). \( \Gamma \) can also be obtained by using the Killing vector,
\[ \vec{C} = 2\kappa (\vec{x} \times \vec{a}), \] (48)
in the van Holten algorithm [47]. We note at last, that no Runge-Lenz-type conserved quantity could be found in this case except in the Abelian case, cf. [18].
VI. CONCLUSION

We studied the geodesic motion of a particle in Kaluza-Klein-type monopole spaces and in its Gibbons-Hawking generalization. As illustrations, we treated in detail the generalized Taub-NUT metrics, for which we derived Runge-Lenz-type vectors. We considered the subclass of two-center metrics into which a conserved Runge-Lenz-type scalar has been revealed in the special case of motions confined to a particular “Apollonius” sphere. For the \((N \geq 3)\)-center metrics, we demonstrated that no symmetry of the Kepler-type occurs for non-aligned centers. We also treated the case of the effective “truly” non-Abelian monopole-like field generated by nuclear motion in a diatomic molecule. This system is due to Wilczek et al. where despite the non-conservation of the electric charge, we surprisingly constructed, in addition to the “unusual” angular momentum, a new conserved charge.

It is worth mentioning that apart from the generic importance of constructing conserved quantities to confine classical trajectories to conic sections as in the case of Kepler-type systems, the existence of quadratic conserved quantity like Runge-Lenz vector yields, in particular, the separability of the Hamilton-Jacobi equation for the generalized Taub-NUT and two-center metrics.

“Hidden” symmetries also play a rôle in quantum mechanics. The system may be quantized by the usual procedure of replacing Poisson brackets with commutators and in this way the energy levels and degeneracies of Kepler-type systems may be found using dynamical symmetries.

We mention in conclusion that the van Holten’s method can also be extended to study supersymmetries, cf. [17].

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