The Hardness of Approximating the Threshold Dimension, Boxicity and Cubicity of a Graph

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Abstract. The threshold dimension of a graph $G(V, E)$ is the smallest integer $k$ such that $E$ can be covered by $k$ threshold spanning subgraphs of $G$. A $k$-dimensional box is the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where each $R_i$ is a closed interval on the real line. The boxicity of a graph $G$, denoted as box($G$), is the minimum integer $k$ such that $G$ can be represented as the intersection graph of a collection of $k$-dimensional boxes. A unit cube in $k$-dimensional space or a $k$-cube is defined as the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where each $R_i$ is a closed interval on the real line of the form $[a_i, a_i+1]$. The cubicity of $G$, denoted as cub($G$), is the minimum integer $k$ such that $G$ can be represented as the intersection graph of a collection of $k$-cubes. In this paper we will show that there exists no polynomial-time algorithm to approximate the threshold dimension of a graph on $n$ vertices with a factor of $O(n^{0.5-\epsilon})$ for any $\epsilon > 0$, unless $\mathsf{NP} = \mathsf{ZPP}$. From this result we will show that there exists no polynomial-time algorithm to approximate the boxicity and the cubicity of a graph on $n$ vertices with factor $O(n^{0.5-\epsilon})$ for any $\epsilon > 0$, unless $\mathsf{NP} = \mathsf{ZPP}$. In fact all these hardness results hold even for a highly structured class of graphs namely the split graphs. We will also show that it is NP-complete to determine if a given split graph has boxicity at most 3.

Keywords: Threshold dimension, Partial order dimension, Boxicity, Cubicity, Split graph, NP-completeness, Approximation hardness

1 Introduction

In [14] Yannakakis studied the complexity of the partial order dimension problem and its consequences on various graph parameters. He proved that it is NP-complete to determine whether the dimension of a partial order is at most 3 and reduced it to the problems of determining the threshold dimension, boxicity and cubicity of graphs.

Recently, Hegde and Jain [8] showed that it is hard to even approximate the dimension of a partial order. To state more precisely,

Theorem 1. [8] There exists no polynomial-time algorithm to approximate the poset dimension on an $N$-element set with a factor of $O(N^{0.5-\epsilon})$ for any $\epsilon > 0$, unless $\mathsf{NP} = \mathsf{ZPP}$. 

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1.1 Our Results

In this paper we will show that

1. There exists no polynomial-time algorithm to approximate the threshold dimension of a graph on \( n \) vertices with a factor of \( O(n^{0.5-\epsilon}) \) for any \( \epsilon > 0 \), unless \( \text{NP} = \text{ZPP} \).
2. There exists no polynomial-time algorithm to approximate the boxicity of a graph on \( n \) vertices with a factor of \( O(n^{0.5-\epsilon}) \) for any \( \epsilon > 0 \), unless \( \text{NP} = \text{ZPP} \).
3. There exists no polynomial-time algorithm to approximate the cubicity of a graph on \( n \) vertices with a factor of \( O(n^{0.5-\epsilon}) \) for any \( \epsilon > 0 \), unless \( \text{NP} = \text{ZPP} \).
4. If \( G \) is a split graph then it is NP-complete to determine whether \( \text{box}(G) \leq 3 \).

Notations

For a positive integer \( k \), let \( [k] \) denote the set \( \{1, 2, \ldots, k\} \). Throughout this paper we will consider only simple undirected graphs. Let \( V(G) \) and \( E(G) \) denote the vertex set and the edge set of graph \( G \) respectively. For each vertex \( v \in V(G) \) let \( N(v, G) \) denote the set of vertices in \( V(G) \) to which \( v \) is adjacent. Whenever there is no ambiguity regarding the graph under consideration, we will use the abbreviated notation \( N(v) \).

A graph \( H \) is said to be a subgraph of \( G \) if and only if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). In this paper we will use the notation \( H \subseteq G \) to denote \( H \) is a subgraph of \( G \).

\( G[V'] \) denotes the induced subgraph of \( G \) on the vertex set \( V' \).

1.2 Posets

A partially ordered set (or poset) \( P = (S, \leq_P) \) consists of a non empty set \( S \) and a reflexive, antisymmetric and transitive binary relation \( \leq_P \) on \( S \). \( S \) is called the ground set of \( P \). If \( x \leq_P y \) or \( y \leq_P x \) then \( x \) and \( y \) are said to be comparable. Otherwise we say that they are incomparable and we denote this relation as \( x \parallel_P y \). We write \( x <_P y \) when \( x \leq_P y \) and \( x \neq y \).

A totally ordered set is a poset in which every two elements are comparable. A linear extension \( L \) of a poset \( P \) is a totally ordered set \( (S, \leq_L) \) which satisfies: \( x \leq_P y \implies x \leq_L y \). Let \( L(u) = |\{v|v \leq_L u\}| \) denote the index of the element \( u \) in the totally ordered set \( L \).

A realizer of a poset \( P \) is a set of linear extensions of \( P \), say \( \mathcal{L} : L_1, L_2, \ldots, L_k \) which satisfy the following condition: if \( x \parallel_P y \) then there exists two linear extensions \( L_i, L_j \in \mathcal{L} \) such that \( x <_{L_i} y \) and \( y <_{L_j} x \). The poset dimension of \( P \) denoted by \( \text{dim}(P) \) is the minimum integer \( k \) such that there exists a realizer of \( P \) of cardinality \( k \). Poset dimension was introduced by Dushnik and Miller [5]. The poset dimension problem is to decide for a given poset and integer \( d \) whether the dimension of the poset is at most \( d \). It was shown to be NP-complete by Yannakakis [14]. For more references and survey on dimension theory of posets see Trotter’s monograph [11] or survey paper [12]. In [8] Hegde and Jain reduced the fractional chromatic number problem to the poset dimension problem to show the approximation hardness of computing the dimension of a given poset.
1.3 Split Graphs and the Threshold Dimension Problem

A graph $G(V, E)$ is a split graph if its vertex set can be partitioned into a clique and an independent set. We will denote the clique by $C(G)$ and independent set by $I(G)$. Note that this partition need not be unique. But whenever we refer to $C(G)$, the set $V \setminus C(G)$ is an independent set and is denoted by $I(G)$. Split graphs were first studied by Földes and Hammer in [6,2], and independently introduced by Tyshkevich and Chernyak [13]. For other characterizations and properties of split graphs one can refer to Golumbic [7].

**Fact 1.** Complement of a split graph is a split graph.

*Threshold graphs:* A graph is a threshold graph if there is a real number $S$ and a weight function $w : V \rightarrow \mathbb{R}$ such that for any two vertices $u, v \in V(G)$, $(u, v)$ is an edge if and only if $w(u) + w(v) \geq S$. Chvátal and Hammer [2] introduced these graphs for their application in set-packing problems. We will use the following property frequently in later sections.

**Fact 2.** A graph $G(V, E)$ is a threshold graph if and only if it is a split graph and for every pair of vertices $u, v \in I(G)$, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. Equivalently, a threshold graph can be defined as a split graph without an induced $P_4$ (i.e. a path on 4 vertices).

**Fact 3.** Complement of a threshold graph is a threshold graph.

**Definition 1.** A threshold cover of a graph $G$ is a set of threshold graphs $G_i, i = 1, 2, \ldots, k$ on the same vertex set as $G$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$. The threshold dimension $t(G)$ is the least integer $k$ such that a threshold cover of size $k$ exists.

Chvátal and Hammer [2] introduced the concept of threshold dimension.

For a graph $G$ let $G_i, 1 \leq i \leq k$ be graphs on the same vertex set as $G$ such that $E(G) = E(G_1) \cap E(G_2) \cap \cdots \cap E(G_k)$. Then we say that $G$ is the intersection graph of $G_i$ s for $1 \leq i \leq k$ and denote it as $G = \bigcap_{i=1}^k G_i$.

**Fact 4.** From Fact 3 it is easy to see that threshold dimension of a graph $G$ is the smallest integer $k$ such that the complement graph $\overline{G}$ can be represented as the intersection of $k$ threshold graphs. Also, if $G = G_1 \cap G_2 \cap \cdots \cap G_k$, then $t(\overline{G}) \leq \sum_{i=1}^k t(\overline{G_i})$.

**Lemma 1.** Let $G$ be a split graph. Let $G'$ be a threshold supergraph of $G$. Then we can construct another threshold graph $H$ such that $G \subseteq H \subseteq G'$ and $\mathcal{I}(H) = \mathcal{I}(G)$.

See Appendix for proof.
1.4 Interval Graphs

A graph $G$ is an interval graph if and only if $G$ has an interval representation: i.e. each vertex of $G$ can be associated with an interval on the real line such that two intervals intersect if and only if the corresponding vertices are adjacent. An interval graph $G$ is said to be a unit interval graph if and only if there is some interval representation of $G$ in which all the intervals are of the same length.

Suppose $G$ is an interval graph. Let us consider an interval representation of $G$. Without loss of generality we can assume that the endpoints of each interval are integers. For any vertex $u$, let $l(u)$ and $r(u)$ denote the integers corresponding to the left endpoint and right endpoint respectively of the interval corresponding to $u$.

Property 1. Helly property of intervals: Suppose $A_1, A_2, \ldots, A_k$ is a finite set of intervals on the real line with pairwise non-empty intersection. Then there exists a common point of intersection for all the intervals i.e. $\bigcap_{i=1}^{k} A_i \neq \emptyset$.

Definition 2. A split interval graph is a graph which is both a split graph and an interval graph.

Note that threshold graphs are interval graphs (Can be easily seen from Fact 2).

1.5 Boxicity and Cubicity

A $d$-dimensional box is a Cartesian product $R_1 \times R_2 \times \cdots \times R_d$ where each $R_i$ (for $1 \leq i \leq d$) is a closed interval of the form $[a_i, b_i]$ on the real line. A $k$-box representation of a graph $G$ is a mapping of the vertices of $G$ to $k$-boxes such that two vertices in $G$ are adjacent if and only if their corresponding $k$-boxes have a non-empty intersection. The boxicity of a graph denoted box($G$), is the minimum integer $k$ such that $G$ can be represented as the intersection graph of $k$-dimensional boxes. Clearly, graphs with boxicity at most 1 are precisely the interval graphs. A $d$-dimensional cube is a Cartesian product $R_1 \times R_2 \times \cdots \times R_d$ where each $R_i$ (for $1 \leq i \leq d$) is a closed interval of the form $[a_i, a_i + 1]$ on the real line. A $k$-cube representation of a graph $G$ is a mapping of the vertices of $G$ to $k$-cubes such that two vertices in $G$ are adjacent if and only if their corresponding $k$-cubes have a non-empty intersection. The cubicity of $G$ is the minimum integer $k$ such that $G$ has a $k$-cube representation. Clearly, graphs with cubicity at most 1 are precisely the unit interval graphs.

The concept of boxicity was introduced by Roberts [10]. Cozzens [3] showed that computing the boxicity of a graph is NP-hard. This was later strengthened by Yannakakis [13] and finally by Kratochvil [9] who showed that determining whether boxicity of a graph is at most two is NP-complete. In [14] Yannakakis has showed that it is NP-complete to determine whether the cubicity of a given graph is at most 3. Boxicity can be stated in terms of intersection of interval graphs as follows:
Lemma 2. Roberts [10] The boxicity of a graph $G$ is the minimum positive integer $b$ such that $G$ can be represented as the intersection of $b$ interval graphs. Moreover, if $G = \bigcap_{i=1}^{m} G_i$ for some graphs $G_i$ then $\text{box}(G) \leq \sum_{i=1}^{m} \text{box}(G_i)$.

Similarly cubicity can be stated in terms of intersection of unit interval graphs as follows:

Lemma 3. Roberts [10] The cubicity of a graph $G$ is the minimum positive integer $b$ such that $G$ is the intersection of $b$ unit interval graphs. Moreover, if $G = \bigcap_{i=1}^{m} G_i$ for some graphs $G_i$ then $\text{cub}(G) \leq \sum_{i=1}^{m} \text{cub}(G_i)$.

The boxicity problem is defined to be the problem of computing the boxicity for a given graph $G$.

2 Characteristic Poset of a Split Graph

In this section, we will introduce the concept of the characteristic poset of a split graph and we will relate the threshold dimension and the boxicity of split graphs to the dimension of this poset.

Definition 3. Let $G$ be a split graph with $\mathcal{I}(G)$ and $\mathcal{C}(G)$ being the independent set and clique respectively. Let $\mathcal{X}(G) = \{N(u,G)|u \in \mathcal{I}(G)\}$. The characteristic poset of $G$ is $P = (\mathcal{X}(G), \subseteq)$, i.e. the set of neighbourhoods of the independent set vertices ordered by inclusion.

Note that the characteristic poset is unique to a split graph and by Fact 2 we can infer that the characteristic poset is a totally ordered set if and only if the split graph is a threshold graph.

Theorem 2. Let $P$ be the characteristic poset of the split graph $G$. Then, $\dim(P) \leq t(G)$.

Proof. Let $t(G) = k$. Suppose $\mathcal{T} : T_1, T_2, \ldots, T_k$ is a set of threshold graphs such that $\bigcap_{i=1}^{k} T_i = G$. From each $T_i$, we will construct linear extension $L_i$ of $P$ such that $L_i$s form a realizer of $P$.

From Lemma 1 we can assume that $\mathcal{I}(T_i) = \mathcal{I}(G)$ for $1 \leq i \leq k$. For each $T_i$ let $\mathcal{X}(T_i) = \{N(u,T_i)|u \in \mathcal{I}(G)\}$. Consider the function $f_i : \mathcal{X}(G) \rightarrow \mathcal{X}(T_i)$ where, for $X \in \mathcal{X}(G)$, $f_i(X)$ is the smallest subset in $\mathcal{X}(T_i)$ containing $X$. Note that $f_i$ is well-defined: For each $X \in \mathcal{X}(G)$, there exists an $X' \in \mathcal{X}(T_i)$ such that $X \subseteq X'$ since $T_i$ is a supergraph of $G$. Moreover, the smallest subset $f_i(X)$ is unique since $\mathcal{X}(T_i)$ is a totally ordered set with respect to set inclusion. We define $L_i$ as follows: For any two distinct elements $X, Y \in \mathcal{X}(G)$,

1. If $f_i(X) \subset f_i(Y)$, then, $X <_{L_i} Y$. 

Lemma 6. If \( f_i(X) = f_i(Y) \) and \( X \prec_P Y \), then, \( X \lesssim_{L_i} Y \).

3. If \( f_i(X) = f_i(Y) \) and \( X\|_P Y \), then, we either make \( X \lesssim_{L_i} Y \) or \( Y \lesssim_{L_i} X \).

We observe that

\[
X \subseteq Y \implies f_i(X) \subseteq f_i(Y)
\]

\[
\implies X \lesssim_{L_i} Y
\]

Hence, \( L_i \)'s are linear extensions of \( P \). Suppose \( X\|_P Y \), then there exist \( u, v \in \mathcal{I}(G) \) such that \( N(u, G) = X \) and \( N(v, G) = Y \) and therefore there exist \( u', v' \in \mathcal{C}(G) \) such that \( u' \in N(u, G) \setminus N(v, G) \) and \( v' \in N(v, G) \setminus N(u, G) \). Since \( \bigcap_{i=1}^{k} T_i = G \), there exist two threshold graphs \( T_j, T_l \in \mathcal{T} \) such that \( u' \notin N(v, T_j) \) and \( v' \notin N(u, T_l) \). This implies that \( f_j(Y) \subseteq f_j(X) \) and \( f_l(X) \subseteq f_l(Y) \). Therefore, \( Y \lesssim_{L_j} X \) and \( X \lesssim_{L_l} Y \). Hence, we have proved that \( L_i \)'s form a realizer of \( P \).

\[\square\]

**Lemma 4.** Let \( G \) be a split graph. Let \( G' \) be an interval supergraph of \( G \). Then we can construct a split interval graph \( H \) such that \( G \subseteq H \subseteq G' \) and \( \mathcal{I}(H) = \mathcal{I}(G) \).

See Appendix for proof.

**Lemma 5.** If \( G \) is a split interval graph, then \( t(\overline{G}) \leq 2 \).

**Proof.** Let us consider an interval representation of \( G \). We will construct two threshold graphs \( G_l \) and \( G_r \) as follows. Let \( l = \min_{u \in V(G)} l(u) \) and \( r = \max_{u \in V(G)} r(u) \) be the leftmost and the rightmost points respectively, in the interval representation of \( G \). Now, to define \( G_l \), we change the intervals corresponding to \( u \in \mathcal{C}(G) \) by redefining their left end points: \( l(u) = l, \forall u \in \mathcal{C}(G) \). We do not disturb the intervals corresponding to the vertices in \( \mathcal{I}(G) \). Now we claim that \( G_l \) is a threshold graph: Clearly \( \mathcal{I}(G) \) induces an independent set in \( G_l \) also. Therefore let \( \mathcal{I}(G_l) = \mathcal{I}(G) \). Let \( u, v \in \mathcal{I}(G_l) \). It is easy to see that \( N(u, G_l) \supseteq N(v, G_l) \) if \( l(u) \leq l(v) \) and therefore, for every \( u, v \in \mathcal{I}(G_l) \), we have either \( N(u, G_l) \supseteq N(v, G_l) \) or \( N(v, G_l) \supseteq N(u, G_l) \).

Similarly, let \( G_r \) be obtained by letting \( r(u) = r, \forall u \in \mathcal{C}(G) \), while keeping other end points unchanged. Again by construction, \( G_r \) is a threshold graph. It is easy to see that \( G_l \cap G_r = G \): By construction, \( G_l \supseteq G \) and \( G_r \supseteq G \) and if \( (u, v) \notin E(G) \), it is clear that either in \( G_l \) or in \( G_r \), the intervals corresponding to \( u \) and \( v \) are disjoint. \[\square\]

**Lemma 6.** If \( G \) is a split graph, then \( t(\overline{G}) \leq 2 \text{box}(G) \).

**Proof.** Let \( \text{box}(G) = k \) and \( G_1, G_2, \ldots, G_k \) be interval graphs on the same vertex set as \( G \) such that \( \bigcap_{i=1}^{k} G_i = G \). By Lemma 4, we can assume that all the \( G_i \)'s are split interval graphs. By Lemma 5, corresponding to each \( G_i \), we can construct two threshold graphs \( T_{2i-1} \) and \( T_{2i} \) such that \( G_i = T_{2i-1} \cap T_{2i} \). Therefore, we have \( 2k \) threshold graphs whose intersection gives \( G \). Hence, proved. \[\square\]
Combining the above Lemma and Theorem 2 we have:

**Theorem 3.** Let \( P = (S, \leq_P) \) be a characteristic poset of the split graph \( G \). Then \( \dim(P) \geq 2 \text{box}(G) \).

**Remark 1.** We observe that the constructions in Theorem 2 and Lemmas 4, 5 and 6 can be achieved in polynomial time.

## 3 Hardness of Approximation

Given poset \( P \), we will construct a split graph \( G_P \) such that \( P \) is the characteristic poset of \( G_P \). Consider a poset \( P = (S, \leq_P) \) where \( |S| = n \). Let \( g : S \rightarrow [n] \) be a bijective map. For convenience, we will assume that \( S \) and \( [n] \) are disjoint sets. We define a split graph \( G_P \) as follows: \( V(G_P) = S \cup [n] \), \( C(G_P) = [n] \) and \( I(G_P) = S \). For any \( u \in S \) and \( v \in [n] \), \( (u, v) \in E(G_P) \) if and only if \( g^{-1}(v) \leq_P u \). Therefore, \( g^{-1}(N(u, G_P)) = \{ x \in S | x \leq_P u \} \). It is easy to see that \( P \) is the characteristic poset of \( G_P \).

**Theorem 4.** \( \dim(P) \geq t(G_P) \).

**Proof.** Let \( \dim(P) = k \). Suppose \( L_1, L_2, \ldots, L_k \) form a realizer of \( P \). We will construct threshold graphs \( G_i \) corresponding to each \( L_i \) for \( 1 \leq i \leq k \) such that \( \bigcap_{i=1}^{k} G_i = G_P \).

The \( G_i \)'s are defined as follows: \( V(G_i) = S \cup [n] \) with \( C(G_i) = [n] \) and \( I(G_i) = S \). For any \( u \in S \) and \( v \in [n] \), \( (u, v) \in E(G_i) \) if and only if \( g^{-1}(v) \leq_{L_i} u \). \( G_i \) is a threshold graph because \( L_i \) (a totally ordered set) is the characteristic poset of \( G_i \).

Now, we will show that if \( (u, v) \in E(G_P) \) then \( (u, v) \in E(G_i) \) \( \forall i \in [k] \). Since \( C(G_i) = C(G_P) \), any \( u, v \in C(G_i) \) are adjacent in \( G_i \). Suppose \( u \in I(G_P) \) and \( v \in C(G_P) \),

\[
(u, v) \in E(G_P) \implies g^{-1}(v) \leq_P u \\
\implies g^{-1}(v) \leq_{L_i} u, \forall i \in [k] \\
\implies (u, v) \in E(G_i), \forall i \in [k]
\]

Hence, each \( G_i \) is a supergraph of \( G_P \). Next we will show that if \( (u, v) \notin E(G_P) \) then there exists \( G_j \) such that \( (u, v) \notin E(G_j) \). If \( (u, v) \notin E(G_P) \) then either \( u <_{G} g^{-1}(v) \) or \( u \geq_{G} g^{-1}(v) \). In either case, there exists \( L_j \) such that \( u <_{L_j} g^{-1}(v) \). By definition of \( G_j, (u, v) \notin E(G_j) \). Hence, proved.

Combining Theorems 2 and 4 we have the following result.

**Corollary 1.** \( \dim(P) = t(G_P) \).

Cozzens and Halsey [4] proved that the boxicity of any graph \( G(V, E) \) is not more than the threshold dimension of its complement \( \overline{G} \), i.e. \( \text{box}(G) \leq t(\overline{G}) \). Hence,

**Corollary 2.** \( \dim(P) \geq \text{box}(G_P) \).
Remark 2. We note that the construction in Theorem 4 can be achieved in polynomial time.

**Theorem 5.** There exists no polynomial-time algorithm to approximate the threshold dimension of a split graph on \( n \) vertices with a factor of \( O(n^{0.5-\epsilon}) \) for any \( \epsilon > 0 \) unless \( NP = ZPP \).

**Proof.** Suppose there exists an algorithm to compute the boxicity of a split graph on \( n \) vertices with approximation factor \( O(n^{0.5-\epsilon}) \). As we have seen for any poset \( P \) on \( N \) elements we can construct a split graph \( G_P \) on \( n = 2N \) vertices such that \( t(G_P) = \dim(P) \) by Corollary 1. This immediately implies that \( \dim(P) \) can be approximated within factor \( O(n^{0.5-\epsilon}) \). But, from Theorem 1 we know that there exists no polynomial-time algorithm to approximate the poset dimension problem with a factor \( O(n^{0.5-\epsilon}) \) for any \( \epsilon > 0 \), a contradiction. \( \square \)

**Theorem 6.** There exists no polynomial-time algorithm to approximate the boxicity of a split graph on \( n \) vertices with a factor of \( O(n^{0.5-\epsilon}) \) for any \( \epsilon > 0 \) unless \( NP = ZPP \).

**Proof.** The proof is similar to that of Theorem 5. From Theorem 3 and Corollary 2 we have \( \text{box}(G_P) \leq \dim(P) \leq 2\text{box}(G_P) \). The rest follows from Theorem 1. \( \square \)

**Corollary 3.** There exists no polynomial-time algorithm to approximate the cubicity of a split graph on \( n \) vertices with a factor of \( O(n^{0.5-\epsilon}) \) for any \( \epsilon > 0 \) unless \( NP = ZPP \).

**Proof.** In [1] it is shown that for any graph \( G \) on \( n \) vertices, \( \text{cub}(G) \leq \text{box}(G) \lceil \log_2 n \rceil \). Since any representation of \( G \) as the intersection of cubes also serves as an intersection of boxes, it follows that \( \text{cub}(G) \geq \text{box}(G) \). Hence, given a poset \( P \) and the corresponding split graph \( G_P \) as constructed in Section 3, we have \( \text{cub}(G_P)/ \lceil \log_2 n \rceil \leq \dim(P) \leq 2\text{cub}(G_P) \). The rest follows as in Theorem 5. \( \square \)

### 4 NP-Completeness of Boxicity of Split Graph

The following Theorem was proved by Yannakakis in [14].

**Theorem 7.** [14] It is NP-complete to determine if a given split graph has threshold dimension at most 3.

We will reduce the threshold dimension problem of split graphs to the problem of computing boxicity of a split graph. Let \( H \) be any split graph. Let \( |V(H)| = n \). We will construct another split graph \( G' \) in polynomial time such that \( \text{box}(G') = t(H) \). A split graph \( G \) is said to be a complete split graph if for all \( u \in \mathcal{I}(G) \) and \( v \in \mathcal{C}(G) \), \( (u,v) \in E(G) \). If \( H \) is a complete split graph then we take \( G' = H \) since \( \text{box}(H) = t(H) = 1 \). So for the rest of the proof we will assume that \( H \) is not a complete split graph. Let \( G = \overline{\overline{P}} \)
and $G_1$, $G_2$ be copies of $G$. Let $V(G_1) = C(G_1) \cup I(G_1)$ and $V(G_2) = C(G_2) \cup I(G_2)$. 
$V(G') = V(G_1) \cup V(G_2)$ and $E(G') = E(G_1) \cup E(G_2) \cup \{(u, v) | u \in C(G_1), v \in C(G_2)\} \cup \{(u, v) | u \in C(G_1), v \in I(G_2)\} \cup \{(u, v) | u \in C(G_2), v \in I(G_1)\}$. Clearly, $G'$ is a split graph with $C(G') = C(G_1) \cup C(G_2)$.

4.1 box($G'$) $\leq t(H)$

Let $t(H) = k$ and $T_1, T_2, \ldots, T_k$ be a set of threshold graphs such that $\bigcap_{i=1}^{k} T_i = G$. Due to Lemma [1] we can assume that $I(T_i) = I(G)$. Now we construct interval graphs $H_i$ corresponding to each $T_i$ as follows: Let $T_1$ and $T_2$ be two copies of $T_i$. We assume that $V(G_1) = V(T_1)$ and $V(G_2) = V(T_2)$. Let $V(H_i) = V(G_1) \cup V(G_2)$. Let $g_i^j : I(T_i^j) \rightarrow [n]$, $j = 1, 2$ be functions which assign to each vertex in the independent set of $T_i^j$ a distinct number satisfying: $u, v \in I(T_i^j)$, $N(u, T_i^j) \subset N(v, T_i^j) \implies g_i^j(u) > g_i^j(v)$. We define another function $h_i^j : C(T_i^j) \rightarrow [n]$, $j = 1, 2$, as: $\forall u \in C(T_i^j)$

$$h_i^j(u) = \begin{cases} 
0, & \text{if } N(u, T_i^j) \cap I(T_i^j) = \emptyset, \\
\max_{v \in N(u, T_i^j) \cap I(T_i^j)} g_i^j(v), & \text{otherwise.}
\end{cases}$$

Each $u \in I(T_i^1)$ is associated with the single point interval $[g_i^1(u), g_i^1(u)]$ and $u \in C(T_i^1)$ with interval $[-n, h_i^1(u)]$. Each $u \in I(T_i^2)$ is associated with the single point interval $[-g_i^2(u), -g_i^2(u)]$ and $u \in C(T_i^2)$ with interval $[-h_i^2(u), n]$. Now $H_i$ is defined to be the intersection graph of this family of intervals which corresponds to $V(G_1) \cup V(G_2)$.

Remark 3. $C(T_i^j) = C(G_j)$ and $I(T_i^j) = I(G_j)$ for $1 \leq i \leq k$ and $j = 1, 2$.

Lemma 7. $H_i$ is a split graph with $C(H_i) = C(G')$ and $I(H_i) = I(G')$ for $1 \leq i \leq k$.

Proof. In view of the construction of $H_i$ clearly, 0 is a common point for intervals corresponding to all vertices $u \in C(T_i^1) \cup C(T_i^2)$. Also, by definition of $g_i^j$, it follows that intervals corresponding to all vertices $u \in I(T_i^1) \cup I(T_i^2)$ are mutually disjoint. Hence, $C(H_i) = C(G')$ and $I(H_i) = I(G')$. Therefore, $H_i$ is a split graph. \qed

Lemma 8. $H_i[V(G_1)] = T_i^1$ and $H_i[V(G_2)] = T_i^2$ for $1 \leq i \leq k$.

Proof. Clearly $H_i[V(G_1)]$ is a split graph with $I(H_i[V(G_1)]) = I(T_i^1)$ and $C(H_i[V(G_1)]) = C(T_i^1)$. By construction it is easy to see that $E(H_i[V(G_1)]) \supseteq E(T_i^1)$. Let $x \in I(T_i^1)$ and $y \in C(T_i^1)$ such that $(y, x) \notin E(T_i^1)$. Let $z \in I(T_i^1)$ be such that $(y, z) \in E(T_i^1)$. According to Fact 2 we have either $N(x, T_i^1) \subseteq N(z, T_i^1)$ or $N(x, T_i^1) \supseteq N(z, T_i^1)$. But since $y \notin N(x, T_i^1)$ and $y \in N(z, T_i^1)$ we can infer that $N(x, T_i^1) \subset N(z, T_i^1)$. It follows that $g_i^1(x) > g_i^1(z)$. Clearly $h_i^1(y) \leq g_i^1(z) < g_i^1(x)$. Therefore $(x, y) \notin E(H_i[V(G_1)])$ and therefore $H_i[V(G_1)] = T_i^1$. A similar proof shows that $H_i[V(G_2)] = T_i^2$. \qed

Lemma 9. box($G'$) $\leq t(H)$.
\textbf{Proof.} According to Lemma 7, $C(H_i) = C(G')$ and $I(H_i) = I(G')$ for $1 \leq i \leq k$. Let $u \in C(G')$ and $v \in I(G')$. We consider the following cases:

1. $u \in C(G_1)$ and $v \in I(G_2)$: Then $(u, v) \in E(G')$ by construction of $G'$. According to Remark 3 and by construction of $H_i$, the interval corresponding $u \in C(T^1_i)$ contains $[-n, 0]$ and $v \in I(T^2_i)$ corresponds to a single point interval on the negative x-axis.

It follows that $(u, v) \in E(H_i)$ for $1 \leq i \leq k$.

2. $u \in C(G_2)$ and $v \in I(G_1)$: Similar to case 1.

3. $u \in C(G_1)$ and $v \in I(G_1)$: Note that $G'[V(G_1)] = G_1$ and by Lemma 8, $H_i[V(G_1)] = T^1_i$ for $1 \leq i \leq k$. Since $\bigcap_{i=1}^k T^1_i = G_1$ we have $\bigcap_{i=1}^k H_i[V(G_1)] = \bigcap_{i=1}^k T^1_i = G_1 = G'[V(G_1)]$.

4. $u \in C(G_2)$ and $v \in I(G_2)$: Similar to case 3. We can show that $\bigcap_{i=1}^k H_i[V(G_2)] = \bigcap_{i=1}^k T^2_i = G_2 = G'[V(G_2)]$.

From the above points we can infer that if $(u, v) \in E(G')$ then $(u, v) \in E(H_i)$ for $1 \leq i \leq k$ and if $(u, v) \notin E(G')$ then $(u, v) \notin E(H_i)$ for some $l \in [k]$. Therefore $\bigcap_{i=1}^k H_i = G'$ and hence $\text{box}(G') \leq k = t(H)$. \hfill \square

\textbf{4.2 box}(G') \geq t(H)

Let box(G') = $l$ and $I_1, I_2, \ldots, I_l$ be interval graphs such that $\bigcap_{i=1}^l I_i = G'$. From Lemma 4 we can assume that each $I_i$ is a split graph with $I(I_i) = I(G')$. Moreover,

Remark 4. $I_i[V(G_1)]$ and $I_i[V(G_2)]$ are split graphs with $I(I_i[V(G_1)]) = I(G_1)$ and $I(I_i[V(G_2)]) = I(G_2)$ respectively for $1 \leq i \leq l$.

A threshold graph $G(V, E)$ is said to be a complete threshold graph if for all $u \in I(G)$ and $v \in C(G)$, $(u, v) \in E(G)$. We shall use the notation $T_C$ to denote a complete threshold graph.

\textbf{Lemma 10.} With respect to an interval representation of $I_i$, let $u_t$ and $u_r$ be the vertices corresponding to the leftmost and rightmost intervals respectively, among the vertices in $I(I_i)$.

1. If $u_t \in I(G_1)$ and $u_r \in I(G_2)$ then $t(I_i[V(G_1)]) = 1$ and $t(I_i[V(G_2)]) = 1$.
2. If $u_t \in I(G_2)$ and $u_r \in I(G_1)$ then $t(I_i[V(G_1)]) = 1$ and $t(I_i[V(G_2)]) = 1$.
3. If $u_t, u_r \in I(G_1)$ then $t(I_i[V(G_1)]) \leq 2$ and $I_i[V(G_2)] = T_C$.
4. If $u_t, u_r \in I(G_2)$ then $I_i[V(G_1)] = T_C$ and $t(I_i[V(G_2)]) \leq 2$.

\textbf{Proof(1):} First we will prove that $t(I_i[V(G_1)]) = 1$. By assumption $r(u) < r(u_r)$ for all $u \in I(I_i)$, $u \neq u_r$. Since $I(G_1) \cup I(G_2)$ induces an independent set in $I_i$ we have $r(u) < l(u_r)$ for all $u \in I(G_1)$ because otherwise $l(u_r) \leq r(u) < r(u_r)$ and hence intervals corresponding to $u$ and $u_r$ intersect in the interval representation of $I_i$. For any $v \in C(G_1)$, $v \geq l(u_r)$ since by construction of $G'$, $(v, u_r) \in E(G')$ and $G' \subseteq I_i$. 

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Combining these two observations, we get \( r(u) < l(u_r) \leq r(v) \) and thus \( r(u) < r(v) \) for all \( u \in \mathcal{I}(G_1), v \in \mathcal{C}(G_1) \). Suppose \( u_1, u_2 \in \mathcal{I}(G_1) \) such that \( r(u_1) \leq r(u_2) \). Now for all \( v \in \mathcal{C}(G_1), r(u_1) \leq r(u_2) < r(v) \). If \( (u_1, v) \in E(\mathcal{I}_i[V(G_1)]) \) then \( l(v) \leq r(u_1) \leq r(u_2) \). Hence \( (u_2, v) \in E(\mathcal{I}_i[V(G_1)]) \) also. From this and Remark 4, it is clear that Fact 2 holds for \( I_i[V(G_1)] \). Therefore \( I_i[V(G_1)] \) is a threshold graph and by Fact 3, \( t(I_i[V(G_1)]) = 1 \).

Now we want to prove that \( t(\overline{I}_i[V(G_2)]) = 1 \). The arguments are similar to the previous case. By assumption \( l(u_i) < l(u) \) for all \( u \in \mathcal{I}(I_i), u \neq u_i \). Since \( \mathcal{I}(G_1) \cup \mathcal{I}(G_2) \) induces an independent set in \( I_i \) we have \( l(u) > r(u_i) \) for all \( u \in \mathcal{I}(G_2) \) because otherwise \( l(u_i) < l(u) \) and hence intervals corresponding to \( u \) and \( u_i \) intersect in the interval representation of \( I_i \). For all \( v \in \mathcal{C}(G_2), l(v) \leq r(u_i) \) since by construction of \( G' \), \( (v, u_i) \in E(G') \) and \( G' \subseteq I_i \). Combining these two observations, we get \( l(v) \leq r(u_i) \) for all \( u \in \mathcal{I}(G_2), v \in \mathcal{C}(G_2) \). Suppose \( u_1, u_2 \in \mathcal{I}(G_2) \) such that \( l(u_1) \leq l(u_2) \). Now for any \( v \in \mathcal{C}(G_2), l(v) < l(u_1) \). If \( (u_2, v) \in E(\mathcal{I}_i[V(G_2)]) \) then \( l(v) < l(u_1) \). Hence, \( (u_1, v) \in E(\mathcal{I}_i[V(G_2)]) \). From this and Remark 4, it is clear that Fact 2 holds for \( I_i[V(G_2)] \). Therefore \( I_i[V(G_2)] \) is a threshold graph and by Fact 3, \( t(I_i[V(G_2)]) = 1 \).

**Proof (2):** Similar to Proof of (1).

**Proof (3):** Since \( \mathcal{I}(G_1) \cup \mathcal{I}(G_2) \) induces an independent set in \( I_i \), we have for all \( u \in \mathcal{I}(G_2), l(u) > r(u_i) \) and \( r(u) < l(u_r) \). Since by construction of \( G' \) for all \( v \in \mathcal{C}(G_2), (v, u_i) \in E(G'), (v, u_r) \in E(G') \) and \( G' \subseteq I_i \), we have \( l(v) \leq r(u_i) \) and \( r(v) \geq l(u) \). This implies \( l(v) < l(u) \leq r(u) < r(v) \) for all \( u \in \mathcal{I}(G_2), v \in \mathcal{C}(G_2) \). Hence all vertices in \( \mathcal{I}(G_2) \) are adjacent to all vertices in \( \mathcal{C}(G_2) \). Now \( I_i[V(G_2)] \) is a complete threshold graph and hence \( I_i[V(G_2)] = T_C \). On the other hand by Remark 4, \( I_i[V(G_1)] \) is a split interval graph. Hence from Lemma 5, \( t(\overline{I}_i[V(G_1)]) \leq 2 \).

**Proof (4):** Similar to Proof of (3).

**Remark 5.** Suppose \( G \) is a split graph with \( t(G) = k \). Let \( T : T_1, T_2, \ldots, T_k \) be a set of threshold graphs such that \( \bigcap_{i=1}^{k} T_i = G \). It is easy to see that there does not exist a pair of graphs \( T_i, T_j \in T \) such that \( T_i \subseteq T_j \). Suppose this was not the case, then, \( G = \bigcap_{i=1,i \neq j}^{k} T_i \), i.e. we could discard \( T_j \), thus contradicting the minimality of \( k \).

**Lemma 11.** \( \text{box}(G') \geq t(H) \).

**Proof.** Based on Lemma 10, we can infer that \( I_i[V(G_1)] \) belongs to exactly one of the following 3 cases: 1) \( t(I_i[V(G_1)]) = 1 \) and \( I_i[V(G_1)] \neq T_C \). 2) \( t(I_i[V(G_1)]) \leq 2 \). 3) \( I_i[V(G_1)] = T_C \). Let \( l_1, l_2, l_3 \) be such that \( l_j \) denotes the number of times \( I_i[V(G_1)] \) belongs to case \( j \) for \( 1 \leq i \leq 1 \) and \( 1 \leq j \leq 3 \). Clearly \( l_1 + l_2 + l_3 = l \). Recall that \( H \) is not a complete split graph. Therefore there exists some \( i \in [l] \) such that \( I_i \neq T_C \). Note that \( G_1 = \bigcap_{i=1}^{l} I_i[V(G_1)] \) and therefore \( t(G_1) \leq \sum_{i=1}^{l} t(I_i[V(G_1)]) \leq l_1 + 2l_2 + 3l_3 \). Since any threshold graph \( T \) which is a supergraph of \( \overline{H} \) is a subgraph of \( T_C \), by Remark 5, \( T_C \)
can be discarded and therefore, we can ignore the term \(l_3t(T_C)\) in the above expression. Hence we get \(\overline{t(G_1)} \leq l_1 + 2l_2\).

We can get 3 similar cases for \(I_i[V(G_2)]\). Let \(l'_j\) denotes the number of times \(I_i[V(G_2)]\) belongs to case \(j\) for \(1 \leq i \leq l\) and \(1 \leq j \leq 3\). Clearly \(l'_1 + l'_2 + l'_3 = l\). From Lemma [10], it is easy to see that \(l'_1 = l_1, l'_2 = l_3\) and \(l'_3 = l_2\). Therefore \(\overline{t(G_2)} \leq \sum_{i=1}^{l} t(I_i[V(G_2)])\) \(\leq l_1 + 2l_3\). Hence realizing that \(G_1\) and \(G_2\) are isomorphic to \(H\),

\[2t(H) = t(G_1) + t(G_2) \leq 2(l_1 + l_2 + l_3) = 2l.\]

Hence, we get \(t(H) \leq l = \text{box}(G')\).

**Theorem 8.** It is NP-complete to determine if a given split graph has boxicity at most 3.

**Proof.** We reduce the problem of determining the threshold dimension of a split graph to this problem. Given a split graph \(H\) we can construct another split graph \(G'\) in polynomial time such that \(\text{box}(G') = t(H)\) by Lemma [9] and Lemma [11]. The rest follows from Theorem [7]. \(\square\)

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Appendix

Proof of Lemma 1

First we observe that $C(G) \subseteq C(G')$. The graph $H$ is obtained as follows: $C(H) = C(G)$ and $I(H) = I(G)$. For each $u \in I(H)$, $N(u, H) = N(u, G') \cap C(G)$. By definition, $N(u, G) \subseteq N(u, H) \subseteq N(u, G')$. Therefore $G \subseteq H \subseteq G'$.

Now we will show that $H$ is a threshold graph. Suppose there exist $u, v \in I(H)$, such that neither $N(u, H) \subseteq N(v, H)$ nor $N(v, H) \subseteq N(u, H)$. There exist two vertices $u', v' \in C(H)$ such that $u' \in N(u, H) \setminus N(v, H)$ and $v' \in N(v, H) \setminus N(u, H)$. This implies $u' \in N(u, G') \setminus N(v, G')$ and $v' \in N(v, G') \setminus N(u, G')$, which in turn implies that $u'uvv'$ forms an induced $P_4$ in $G'$. But, by Fact 2 this is a contradiction since $G'$ is a threshold graph.

Proof of Lemma 4

Consider an interval representation of $G'$ such that it satisfies the following two properties: (1) None of the intervals used is a single point interval. (2) No two intervals share a common end point. It is easy to see that such an interval representation can be constructed from any given interval representation in polynomial time. Now let $x \in I(G)$.

Clearly $\{x\} \cup N(x, G)$ induces a clique in $G$ and therefore in $G'$. Let $f'(v)$ denote the interval assigned to the vertex $v$ in the interval representation chosen for $G'$. By Helly property of the intervals, $\bigcap_{v \in \{x\} \cup N(x, G)} f'(v) \neq \emptyset$. From properties (1) and (2) we can easily infer that $\bigcap_{v \in \{x\} \cup N(x, G)} f'(v)$ is not a single point interval. Now we define the interval graph $H$ on the vertex set $V(G)$, by assigning the interval $f(v)$ to each vertex $v \in V(G)$, defined as follows

$$f(v) = \begin{cases} f'(v) \forall v \in C(G), \\ P(v) \forall v \in I(G), \end{cases}$$

where $P(v)$ is a point in $\bigcap_{x \in \{v\} \cup N(v, G)} f'(x)$. Note that since $\bigcap_{x \in \{v\} \cup N(v, G)} f'(x)$ is not a single point we can assume that $P(v) \neq P(u)$ for all distinct $u, v \in I(G)$. Also note that for each $v \in I(G)$, $N(v, G) \subseteq N(v, H)$ by the construction. Since we have only changed the intervals corresponding to the vertices in $I(G)$, we infer that $G \subseteq H$. On the other hand $f'(v) \supseteq f(v)$ for all $v \in V(G)$ and therefore $H \subseteq G'$, as required. Moreover it is easy to see that $I(G)$ induces an independent set in $H$. Hence, $H$ is split graph with same partition as $G$. Therefore, $H$ is a split interval graph.