Angular quantization and form-factors in massive integrable models

Vadim Brazhnikov\(^1\) and Sergei Lukyanov\(^2,3\)

\(^1\)Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855-0849, USA
\(^2\)Newman Laboratory, Cornell University
Ithaca, NY 14853-5001, USA
and
\(^3\)L.D. Landau Institute for Theoretical Physics,
Chernogolovka, 142432, RUSSIA

Abstract

We discuss an application of the method of the angular quantization to reconstruction of form-factors of local fields in massive integrable models. The general formalism is illustrated with examples of the Klein-Gordon, sinh-Gordon and Bullough-Dodd models. For the latter two models the angular quantization approach makes it possible to obtain free field representations for form-factors of exponential operators. We discuss an intriguing relation between the free field representations and deformations of the Virasoro algebra. The deformation associated with the Bullough-Dodd models appears to be different from the known deformed Virasoro algebra.

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1. Introduction

The primary goal of Quantum Field Theory (QFT) is a reconstruction of a complete set of correlation functions satisfying a system of necessary requirements [1]. A way to achieve it is to express the correlation functions in terms of a representation of algebra of local fields in the physical Hilbert space. The canonical quantization of Lagrangian theories provides us with a method to build this representation. The approach suggests the following. First of all, we need to choose some $(D - 1)$-dimensional complete Cauchy hyper-surface $\Sigma$ in $D$-dimensional space-time. Solution of underlying classical system of relativistically invariant hyperbolic equations of motion is uniquely defined in the whole Minkowski space by Cauchy data specified on $\Sigma$. In the quantum theory the physical Hilbert space appears as a representation space for canonical commutation relations defined on $\Sigma$. Different choices of the complete Cauchy hyper-surface lead to unitary equivalent representations of the algebra of local fields and do not affect physically relevant quantities. The situation changes drastically if we try to quantize the theory using an incomplete hyper-surface. In this case either there are no solutions to the classical system of equations of motion at all or we can reconstruct a solution only in some region of the $D$-dimensional Minkowski space, the so called domain of dependence. The representation of the algebra of local fields associated with the incomplete hyper-surface differs significantly from the representation in the physical Hilbert space. It is not clear a priori what use it has for the purpose of calculation of physically interesting quantities. The best we might expect is that QFT can be formulated in terms of this “incomplete” representation with the use of some density matrix $\hat{w}$. It controls influence of regions outside the domain of dependence on the quantum dynamics. There is no general recipe to reconstruct $\hat{w}$. However in simple cases when the Cauchy surface is large enough the density matrix can be recovered from general QFT constraints on the correlation functions. The most studied example corresponds to $D = 2$ with a half-infinite line $t = 0, \ x > 0$ taken as the incomplete surface. In this case, solutions of the classical equations of motion can be defined only in the Rindler wedge,

$$x > |t| > 0.$$ \hfill (1.1)

The natural parameterization of this domain is provided by the angular coordinates

$$x = r \ \cosh(\alpha), \quad t = r \ \sinh(\alpha).$$ \hfill (1.2)
The Hamiltonian picture which appears most natural in this case is the one where the polar angle $\alpha$ is treated as time, and the angular Hamiltonian $K$ generates an infinitesimal shift along the “time” direction. The space of representation of the algebra of local fields associated with the equal-time slice $\alpha = 0$ is usually called by the angular quantization space. The correlation functions have to be single valued functions being continued into the Euclidean region. This uniquely determines the density matrix for the angular quantization $\hat{\omega} = e^{2\pi i K}$. (1.3)

The angular quantization turns out to be a useful tool to study various aspects of black hole evaporation $[2]$, $[3]$. There is strong evidence that it should be advantageous for massive integrable models too $[4]$. Typically these models describe a scaling behavior of exactly solvable 2d statistical systems. The angular quantization space can be viewed as the scaling limit of a space where a so called corner transfer matrix acts. Due to R.J. Baxter we know that the latter space possesses many remarkable features $[5]$. In the works $[6]$, $[7]$ a nice algebraic description of them was given. This has led to considerable progress in the calculation of lattice correlation functions $[8]$, $[7]$, $[9]$. Therefore, it is reasonable to expect that the angular quantization is a useful approach to massive integrable QFT.

In this paper we consider an application of the method of the angular quantization to reconstruction of form-factors in integrable QFT containing only one neutral particle in the spectrum – the Klein-Gordon, sinh-Gordon and Bullough-Dodd models.

Here is the layout of the paper. In Section 2 we briefly review basic features of two-body S-matrices and form-factors in integrable QFT. Heuristic construction $[10]$ for the angular quantization of massive integrable models is presented in Section 3. In Section 4 we illustrate the general formalism with an elementary example of the Klein-Gordon model. Section 5 and 6 are devoted to the angular quantization of the sinh-Gordon and Bullough-Dodd models where we obtain free field representations for form-factors of exponential fields. In Section 7 we discuss a relation between the free field representations and deformations of the Virasoro algebra. Finally we conclude with general remarks in Section 8.
2. Preliminaries

It is a common belief that the knowledge of the two-body $S$-matrix of integrable models \[11\] makes it possible, in principle, to compute correlation functions of local fields. In order to elaborate this problem, the so called form-factor approach \[12\], \[13\], \[14\] has been developed and successfully employed to study many interesting models. To simplify our discussion, we will consider QFT with the spectrum consisting of a single particle $B$ of mass $m$. It is convenient to parameterize the two-body $S$-matrix, describing $BB \to BB$ scattering, in terms of rapidity variables related to the two dimensional momenta by

\[ p_i^0 = m \cosh(\theta_i), \quad p_i^1 = m \sinh(\theta_i), \quad i = 1, 2. \] (2.1)

The amplitude $S = S(\theta_1 - \theta_2)$ satisfies general QFT conditions

\begin{align*}
(i) \text{ unitarity} : & \quad S^*(\theta^*) = S(-\theta) = [S(\theta)]^{-1} \\
(ii) \text{ crossing symmetry} : & \quad S(i\pi - \theta) = S(\theta). \quad (2.2)
\end{align*}

Due to (2.2) the amplitude $S(\theta)$ is a $2\pi i$-periodic function which is completely determined by the positions of its poles and zeros in the “physical strip” $0 < \theta < \pi$. The simple poles correspond to “bound state” particles either in the direct channel of $BB$ scattering or in the cross channel, depending on the sign of the residue. Since we assume that there is only one particle in the spectrum the bound state either does not exist at all or must coincide with the particle $B$ itself (“$\varphi^3$-property”). In the latter case the amplitude $S(\theta)$ develops the simple pole in the direct channel at $\theta = \frac{2i\pi}{3}$

\[ S(\theta) \to \frac{i \Gamma^2}{\theta - \frac{2i\pi}{3}} \] (2.3)

and satisfies the bootstrap equation

\[ S(\theta) = S(\theta - \frac{i\pi}{3}) S(\theta + \frac{i\pi}{3}). \] (2.4)

We will denote the physical Hilbert space of the QFT as $\pi_A$. A linear basis in $\pi_A$ is provided by a set of asymptotic states,

\[ |B(\theta_n) \ldots B(\theta_1)\rangle, \] (2.5)
where rapidities are ordered as $\theta_n > ... > \theta_1$. Any correlation function can be written as a spectral sum over all intermediate $n$-particle asymptotic states (2.3). For example,

$$
\langle O(x)O(y) \rangle = \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \frac{d\theta_1...d\theta_n}{n!(2\pi)^n} |F_O(\theta_1,...\theta_n)|^2 e^{-rm \sum_{k=1}^{n} \cosh \theta_k},
$$

$$
(x^\mu - y^\mu)^2 = -r^2 < 0.
$$

The spectral sum (2.3) involves the on-shell amplitudes of the local Hermitian field $O$ (form-factors)

$$
F_O(\theta_1,...\theta_n) = \langle \text{vac} | \pi_A(O) | B(\theta_n)...B(\theta_1) \rangle, \hspace{1cm} (2.7)
$$

where the matrix of $O$ in the basis of the asymptotic states is denoted by $\pi_A(O)$. The form-factors satisfy a set of requirements [12], [14]

(i) *Watson’s theorem,*

$$
F_O(\theta_1,...\theta_{j+1},\theta_j,...\theta_n) = S(\theta_j - \theta_{j+1}) \ F_O(\theta_1,...\theta_j,\theta_{j+1},...\theta_n). \hspace{1cm} (2.8)
$$

(ii) *The crossing symmetry condition,*

$$
F_O(\theta_2,...\theta_{n-1},\theta_n,\theta_1+2\pi i) = F_O(\theta_1,\theta_2,...\theta_n). \hspace{1cm} (2.9)
$$

(iii) *The kinematical pole condition.* $F_O(\theta_1,...\theta_n)$ being considered as a function of $\theta_n$ have a simple pole at the point $\theta_n = \theta_{n-1} + i\pi$ with the following residue

$$
F_O(\theta_1,...\theta_{n-1},\theta_n) \to \frac{1 - \prod_{j=1}^{n-2} S(\theta_{n-1} - \theta_j)}{\theta_n - \theta_{n-1} - i\pi} \ i \ F_O(\theta_1,...\theta_{n-2}). \hspace{1cm} (2.10)
$$

In presence of the bound state we must supplement the (i) – (iii) with

(iv) *The bound state pole condition.* The form-factor $F_O(\theta_1,...\theta_n)$ being considered as a function of $\theta_n$ have a simple pole at $\theta_n = \theta_{n-1} + \frac{2i\pi}{3}$,

$$
F_O(\theta_1,...\theta_{n-1},\theta_n) \to \frac{i \Gamma F_O(\theta_1,...\theta_{n-2},\theta_{n-1} + \frac{i\pi}{3})}{\theta_n - \theta_{n-1} - \frac{2i\pi}{3}}, \hspace{1cm} (2.11)
$$

where $\Gamma$ is defined by (2.3).

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1. Our convention for the normalization of the asymptotic states is

$$
\langle \text{vac} | \text{vac} \rangle = 1, \hspace{1cm} \langle B(\theta) | B(\theta') \rangle = 2\pi \delta(\theta - \theta').
$$
3. Heuristic construction

From the mathematical standpoint, the system of the form-factor requirements is a complicated Riemann-Hilbert problem. The free field representation approach provides a technique for its solution. It is based on the following heuristic construction [10].

To any factorizable scattering theory there corresponds a formal Zamolodchikov-Faddeev (ZF) algebra. Let us assume that we have some particular representation \( \pi_Z \) of the formal ZF algebra associated with the S-matrix \( S(\theta) \) (2.2). Denote the ZF operator, acting in \( \pi_Z \), by \( B(\theta) \)

\[
B(\theta_1)B(\theta_2) = S(\theta_1 - \theta_2) \ B(\theta_2)B(\theta_1) , \quad \Im (\theta_1 - \theta_2) = 0 .
\]

We require that the operator product \( B(\theta_2)B(\theta_1) \) has a simple pole at the point \( \theta_2 - \theta_1 = i\pi \) with \( c \)-number (not operator valued) residue

\[
B(\theta_2)B(\theta_1) \to \frac{i}{\theta_2 - \theta_1 - i\pi} .
\]

In the case of the scattering theory exhibiting the \( \varphi^3 \)-property it is also required that the operator product \( B(\theta_2)B(\theta_1) \) develops the simple pole at \( \theta_2 - \theta_1 = \frac{2i\pi}{3} \)

\[
B(\theta_2)B(\theta_1) \to \frac{i \Gamma \ B(\theta_1 + \frac{i\pi}{3})}{\theta_2 - \theta_1 - \frac{2i\pi}{3}} .
\]

Suppose also there exists an operator \( K \) acting in the space \( \pi_Z \) in the following manner

\[
B(\theta + \alpha) = e^{-\alpha K} \ B(\theta) e^{\alpha K} .
\]

Due to the \( S \) matrix properties (2.2), we can impose the conjugation condition

\[
B^+(\theta) = B(\theta + i\pi) , \quad K^+ = -K .
\]

Let \( \bar{\pi}_Z \) be a dual space to \( \pi_Z \). There exists an embedding of the linear space of asymptotic states \( \pi_A \) in the tensor product

\[
\pi_A \hookrightarrow \bar{\pi}_Z \otimes \pi_Z .
\]

In other words, we can identify an arbitrary vector \( |X\rangle \in \pi_A \) with some endomorphism (linear operator) \( X \) of the space \( \pi_Z \). To describe the embedding, we identify an arbitrary vector \( |B(\theta_n)\ldots B(\theta_1)\rangle \in \pi_A \) with an element of \( \text{End}[\pi_Z] \) as

\[
| B(\theta_n)\ldots B(\theta_1) \rangle \equiv B(\theta_n)\ldots B(\theta_1) \ e^{i\pi K} .
\]
The asymptotic states generate the basis in \( \pi_A \), therefore (3.7) unambiguously specifies the embedding of the linear space. As well as \( \pi_A \), the space \( \bar{\pi}_Z \otimes \pi_Z \) possesses a canonical Hilbert space structure with the scalar product given by

\[
\text{Tr}_{\pi_Z} \left[ Y^+ X \right]/\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right].
\]

We propose the basic conjecture:

Eq.(3.7) defines the embedding of the linear space \( \pi_A \hookrightarrow \bar{\pi}_Z \otimes \pi_Z \), which preserves the structure of the Hilbert spaces,

\[
\langle Y | X \rangle = \text{Tr}_{\pi_Z} \left[ Y^+ X \right]/\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right], \quad \text{if } | X \rangle \equiv X, \quad | Y \rangle \equiv Y. \quad (3.8)
\]

Let us define \( \pi_Z(\mathcal{O}) \in \text{End}[\pi_Z] \) associated with the state \( \pi_A(\mathcal{O})|\text{vac}\rangle \) in the following way

\[
\pi_A(\mathcal{O})|\text{vac}\rangle \equiv \pi_Z(\mathcal{O}) e^{|2\pi i K}. \quad (3.9)
\]

Notice that for a Hermitian operator \( \mathcal{O} \)

\[
\left[ \pi_Z(\mathcal{O}) \right]^+ = \pi_Z(\mathcal{O}), \quad (3.10)
\]

as a consequence of our conjecture. Using (3.8) the form-factors can be written as traces over the space \( \pi_Z \),

\[
F_\mathcal{O}(\theta_1, \ldots \theta_n) = \text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \pi_Z(\mathcal{O}) B(\theta_n) \ldots B(\theta_1) \right]/\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right]. \quad (3.11)
\]

The functions (3.11) satisfy Watson’s theorem (2.8) due to the ZF commutation relation (3.1). Using the cyclic property of the trace and the commutation relation (3.5), we can transform (3.11)

\[
(3.11) = \text{Tr}_{\pi_Z} \left[ B(\theta_1) e^{2\pi i K} \pi_Z(\mathcal{O}) B(\theta_n) \ldots B(\theta_2) \right]/\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right] = \text{Tr}_{\pi_Z} \left[ e^{2\pi i K} B(\theta_1 + 2\pi i) \pi_Z(\mathcal{O}) B(\theta_n) \ldots B(\theta_2) \right]/\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right].
\]

According to the crossing symmetry condition (2.9), it should coincide with

\[
F_\mathcal{O}(\theta_2, \ldots \theta_n, \theta_1 + 2\pi i) = \text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \pi_Z(\mathcal{O}) B(\theta_1 + 2\pi i) B(\theta_n) \ldots B(\theta_2) \right]/\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right].
\]

Hence we conclude that \( \pi_Z(\mathcal{O}) \in \text{End}[\pi_Z] \) associated with a local Hermitian field must commute with \( B(\theta) \),

\[
\left[ \pi_Z(\mathcal{O}), B(\theta) \right] = 0. \quad (3.12)
\]
Similarly, basing on the requirements (3.1)-(3.4) and the cyclic property of a matrix trace, one can check that the functions (3.11) obey the kinematical pole (2.10) and the bound state pole conditions (2.11).

Of course our construction is not a rigorous method to solve the Riemann-Hilbert problem. We did not give a satisfactory definition of the representation $\pi_Z$ of the ZF algebra and a proof of the basic conjecture (3.8). As a result we can not prove that the traces (3.11) are well defined and satisfy the necessary analytical requirements imposed upon form-factors. In fact, a careful analysis suggests that the traces (3.11) suffer from ultraviolet divergences and need to be regularized. Fortunately, these divergences can be extracted by rewriting (3.11) in the form

$$F_O(\theta_1, ..., \theta_n) = \langle O \rangle \frac{\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \pi_Z(O) B(\theta_n)...B(\theta_1) \right]}{\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \pi_Z(O) \right]},$$

(3.13)

with

$$\langle O \rangle = \frac{\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \pi_Z(O) \right]}{\text{Tr}_{\pi_Z} \left[ e^{2\pi i K} \right]}.$$  (3.14)

Then the second factor in (3.13) is well defined and all divergences are absorbed in the $\theta$-independent constant $\langle O \rangle$. The latter is a vacuum expectation value of the local field $O$ [15]. Standard arguments suggest that

$$\langle O \rangle \sim (m\epsilon)^{d_O}$$

(3.15)

up to some finite dimensionless constant. Here $\epsilon$ is a parameter of the short distance cut-off (lattice scale) and $d_O$ is the ultraviolet scaling dimension of the field $O$. Handling of the divergences (3.15) demands a proper normalization of the local fields [15], [16]. We will not discuss this problem here.

4. Klein-Gordon model

Before proceeding to analysis of a nontrivial QFT let us demonstrate how the formal construction from the previous section works for the Klein-Gordon model. In this case the scattering theory is trivial

$$S_{KG}(\theta) = 1$$

(4.1)

and it is easy to find the proper representation $\pi_Z$ of the ZF algebra [17]. Let us introduce a set of oscillators $b_{\nu}$, satisfying the commutation relations

$$[b_{\nu}, b_{\nu^\prime}] = 2 \sinh(\pi\nu) \delta(\nu + \nu^\prime).$$

(4.2)
The algebra admits a representation in the Fock space

\[ \mathcal{F} : \oplus b_{-\nu_1}...b_{-\nu_n}|0\rangle, \quad \nu_k > 0. \]  

(4.3)

The highest vector \(|0\rangle\) (not to be confused with the physical vacuum \(|\text{vac}\rangle\)) obeys the equations \(b_\nu|0\rangle = 0, \quad \nu > 0\). The Heisenberg algebra (4.3) is compatible with the conjugation

\[ b^+_\nu = b_{-\nu}. \]  

(4.4)

It is easy to check that the operators

\[ B(\theta) = \int_{-\infty}^{+\infty} d\nu \ b_\nu \ e^{i\nu(\theta-i\pi)}, \]  

(4.5)

\[ K = i \int_0^{+\infty} \frac{d\nu}{2} \frac{\nu}{\sinh(\pi\nu)} b_{-\nu} b_\nu \]  

(4.6)

satisfy the necessary requirements. For example,

\[
\left[ B(\theta_1), B(\theta_2) \right] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu d\nu' \ [b_\nu, b_{\nu'}] e^{i\nu\theta_1+i\nu'\theta_2} = \\
\int_{-\infty}^{+\infty} d\nu \ (e^{\pi\nu} - e^{-\pi\nu}) \ e^{i\nu(\theta_1-\theta_2)} = 2\pi \ \{ \delta(\theta_1-\theta_2-i\pi) - \delta(\theta_1-\theta_2+i\pi) \}. 
\]

Therefore for real \(\theta\)

\[ B(\theta_1)B(\theta_2) = B(\theta_2)B(\theta_1). \]  

(4.7)

Similar calculations allow one to check that

\[ B(\theta_2)B(\theta_1) \rightarrow \frac{i}{\theta_2 - \theta_1 - i\pi} \quad \text{as} \quad \theta_2 \rightarrow \theta_1 + i\pi, \]

\[ B(\theta + \alpha) = e^{-\alpha K} B(\theta) e^{\alpha K} \]

and the conjugation (4.4) leads to

\[ B^+(\theta) = B(\theta + i\pi), \quad K^+ = -K. \]

Therefore \(\pi_Z\) coincides with the Fock space \(\mathcal{F}\) (4.3). We expect that this representation corresponds to the Klein-Gordon model,

\[ (\partial_t^2 - \partial_x^2) \varphi + m^2 \varphi = 0. \]  

(4.8)
According to the discussion from the Section 3, we should identify the asymptotic states in \((4.8)\) with the following endomorphisms of the Fock space,

\[ |B(\theta_n)...B(\theta_1)\rangle \equiv B(\theta_n)...B(\theta_1) e^{i\pi K}. \]

At this point we need to find the endomorphism \(\pi_Z(\varphi(x,t))\) corresponding to the state \(\pi_A(\varphi(x,t))|\text{vac}\rangle\). It can be easily done since

\[
\pi_A(\varphi(x,t))|\text{vac}\rangle = \int_{-\infty}^{+\infty} \frac{d\theta}{\sqrt{\pi}} e^{-im(x \sinh \theta - t \cosh \theta)} |B(\theta)\rangle. \tag{4.9}
\]

Then according to our rules \((3.7)\), \((3.9)\)

\[
\pi_A(\varphi(x,t))|\text{vac}\rangle \equiv \pi_Z(\varphi(x,t)) e^{i\pi K} = \int_{-\infty}^{+\infty} \frac{d\theta}{\sqrt{\pi}} e^{-imr \sinh(\theta - \alpha)} B(\theta) e^{i\pi K}. \tag{4.10}
\]

Here we assume, that \(x > |t| > 0\) and introduce the angular coordinates \((1.2)\) in the Rindler wedge. Now, let us combine \((1.5)\) with \((4.10)\). Deforming the integration contour over \(\theta\), we obtain

\[
\pi_Z(\varphi(x,t)) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} d\nu b_\nu e^{i\nu\alpha} K_{i\nu}(mr), \tag{4.11}
\]

where

\[
K_{i\nu}(mr) = \frac{1}{2} \int_{-\infty}^{+\infty} d\theta e^{-mr \cosh(\theta)} e^{i\nu\theta}
\]

is the Macdonald (modified Bessel) function of the imaginary order. As was expected

\[
\left[\pi_Z(\varphi(x,t))\right]^+ = \pi_Z(\varphi(x,t)), \quad [\pi_Z(\varphi(x,t)), B(\theta)] = 0. \tag{4.12}
\]

The formula \((4.11)\) gives a general solution to the Klein-Gordon equation in the Rindler wedge \((1.1)\), satisfying the canonical commutation relation,

\[
\pi(r) = \partial_\alpha \varphi(x,t)|_{\alpha=0}, \quad \varphi(r) = \varphi(x,t)|_{\alpha=0}, \quad [\pi(r), \varphi(r')] = -8\pi i r \delta(r - r'). \tag{4.13}
\]

In this way we arrive at the interpretation of the space \(\pi_Z\) as the angular quantization space of the model. In this quantization procedure the angle \(\alpha\) is treated as the “time”, and the space \(\pi_Z\) is associated with the “equal time” slice \(\alpha = 0\). It is important that this is a model independent interpretation of \(\pi_Z\). Moreover the identification of the physical
vacuum state $|\text{vac}\rangle$ with $e^{\pi i K} \in \text{End}[\pi_Z]$ do not suggest any integrability condition and has long been known [2], [3].

Finishing the elementary example, it is instructive to illustrate the formula for form-factors (3.11). According to (3.11),

$$\langle \text{vac} | \pi_A(\varphi(x,t)) | B(\theta) \rangle = \text{Tr}_F \left[ e^{2\pi i K} \pi_Z(\varphi(x,t)) B(\theta) \right] / \text{Tr}_F \left[ e^{2\pi i K} \right] =$$

$$\frac{2}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu d\nu' e^{i\nu + i \nu' (\theta + \frac{i \pi}{2})} K_{i\nu}(mr) \langle \langle b_{\nu} b_{\nu'} \rangle \rangle,$$

where

$$\langle \langle b_{\nu} b_{\nu'} \rangle \rangle = \text{Tr}_F \left[ e^{2\pi i K} b_{\nu} b_{\nu'} \right] / \text{Tr}_F \left[ e^{2\pi i K} \right].$$

The trace $\langle \langle b_{\nu} b_{\nu'} \rangle \rangle$ can be derived only from the cyclic property and commutation relations for oscillators given in (1.2)

$$\langle \langle b_{\nu} b_{\nu'} \rangle \rangle = e^{2\pi \nu} \langle \langle b_{\nu'} b_{\nu} \rangle \rangle = e^{2\pi \nu} \left\{ \langle \langle b_{\nu} b_{\nu'} \rangle \rangle - 2 \sinh(\pi \nu) \right\}.$$

Hence

$$\langle \langle b_{\nu} b_{\nu'} \rangle \rangle = e^{\pi \nu} \delta(\nu + \nu')$$

and

$$\langle \text{vac} | \pi_A(\varphi(x,t)) | B(\theta) \rangle = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} d\nu e^{-i\nu (\theta - \alpha + \frac{i \pi}{2})} K_{i\nu}(mr) = 2\sqrt{\pi} e^{-i mr \sinh(\theta - \alpha)},$$

which is in agreement with (1.9). It is important that Wick’s theorem can be used for calculations of traces over the free Fock space. The trace (4.13) plays the role of Wick’s pairing. By this means the calculation of form-factors of an arbitrary local operator is reduced to a combinatoric procedure.

5. Sinh-Gordon model

At the first glance the free field representation for the angular quantization space appeared during discussion of the Klein-Gordon model is an artifact of this particular QFT. As a matter of fact, the similar free field representations exist for nontrivial integrable models. We would like to demonstrate this important phenomena on the examples of the sinh-Gordon and Bullough-Dodd models.
The sinh-Gordon QFT describes dynamics of the real scalar field $\varphi$ governed by the Euclidean action,

$$\mathcal{A}_{shG} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \varphi)^2 + 2\mu \cosh (b\varphi) \right\} . \quad (5.1)$$

There is only one particle in the spectrum which does not form any bound states. The corresponding two-body $S$-matrix does not have singularities in the physical strip. It was found in the work [18] that

$$S_{shG}(\theta) = \frac{\tanh(\frac{\theta}{2} - \frac{i\pi b}{2Q})}{\tanh(\frac{\theta}{2} + \frac{i\pi b}{2Q})} , \quad (5.2)$$

where

$$Q = b^{-1} + b . \quad (5.3)$$

To describe the free field representation of the angular quantization space $\pi_Z$ of (5.1), let us introduce the Heisenberg algebra

$$[\lambda_\nu, \lambda_{\nu'}] = \frac{2 \sinh \left( \frac{\pi b \nu}{2Q} \right) \sinh \left( \frac{\pi \nu}{2Q} \right)}{\nu \cosh \left( \frac{\pi \nu}{2} \right)} \delta(\nu + \nu') \quad (5.4)$$

and the vertex operator

$$\Lambda(\theta) = : \exp \left\{ -i \int_{-\infty}^{+\infty} d\nu \; \lambda_\nu \; e^{i\nu(\theta-i\frac{\pi}{2})} \right\} : \quad (5.5)$$

Notice that the set of oscillators, which appear during consideration of the Klein-Gordon model, is simply related to $\lambda_\nu$ in the limit $b \to 0$,

$$b_\nu = \frac{2}{\sqrt{\pi}} \cosh \left( \frac{\pi \nu}{2} \right) \lim_{b \to 0} \left( b^{-1} \lambda_\nu \right) . \quad (5.6)$$

The algebra (5.4) is represented in the Fock space in much the same way as (4.3) is. We extend the construction and introduce also the pair of the canonical conjugate operators (“zero modes”), commuting with the oscillators $\lambda_\nu$,

$$[P, Q] = -i . \quad (5.7)$$

The extended Heisenberg algebra (5.4), (5.7) admits the representation in the direct sum of the Fock spaces,

$$\bigoplus_p \mathcal{F}_p , \quad \text{where} \quad P \mathcal{F}_p = p \mathcal{F}_p . \quad (5.8)$$
In this space we define the action of operators $B(\theta)$ and $K$ in the following manner [19], [20], [21]

$$B(\theta) = -iC_{shG} \left\{ e^{i\frac{2\theta}{Q}} \Lambda(\theta + i\frac{\pi}{2}) - e^{-i\frac{2\theta}{Q}} \Lambda^{-1}(\theta - i\frac{\pi}{2}) \right\}, \quad (5.9)$$

$$K = i \int_{0}^{+\infty} d\nu \frac{2\nu^2 \cosh(\frac{\pi\nu}{2})}{\sinh(\frac{\pi\nu}{2Q}) \sinh(\frac{\pi\nu}{2Q})} \lambda_{-\nu} \lambda_{\nu}. \quad (5.10)$$

One can check that the operators (5.9) satisfy the requirements (3.1), (3.2), (3.4). The real constant $C_{shG}$ in (5.9) should be chosen in order to match the normalization of the residue in (3.2). We will not use its explicit form here. The conjugation

$$\lambda_{\nu}^+ = \lambda_{-\nu}, \quad P^+ = P \quad (5.11)$$

leads to the condition (3.3). Hence we identify the space of the angular quantization $\pi_Z$ of the sinh-Gordon model with the direct sum of the free Fock spaces (5.8).

Now, we can take advantage of the proposed free field representation to obtain form-factors of some local fields in the model (5.1). In order to do this we need to find endomorphisms in $\pi_Z = \bigoplus_p F_p$, commuting with the action of the ZF operator. Since $[P, \Lambda(\theta)] = 0$, an obvious candidate for the proper endomorphism is a projector on the Fock space $F_{p_a}$ with a given value

$$p_a = a.$$ 

Let us denote the local field corresponding to the projector on $F_{p_a}$ as $O_a$. Then, we have

$$\langle vac | \pi_A(O_a) | B(\theta_n)...B(\theta_1) \rangle = \langle O_a \rangle \frac{\text{Tr}_{F_{p_a}}[e^{2\pi iK} B(\theta_n)...B(\theta_1)]}{\text{Tr}_{F_{p_a}}[e^{2\pi iK}]} \quad (5.12)$$

As was mentioned already, Wick’s theorem can be applied to calculation of the traces over the free Fock space. If we know the elementary Wick’s pairing, the formulae (5.9), (5.12) reduce the calculating of the form-factors to a combinatoric procedure. The pairing can be evaluated in the same fashion as in the case of the Klein-Gordon model. The rules for reconstruction of the form-factors appear to be as following [21]:

$$\langle vac | \pi_A(O_a) | B(\theta_n)...B(\theta_1) \rangle = \langle O_a \rangle \rho^n \times$$

$$\sum_{\{\sigma_j=\pm\}^{n}_{j=1}} e^{i\pi(\frac{1}{2} - \frac{n}{2})(\sigma_1 + \cdots + \sigma_n)} \langle \langle \Lambda^{\sigma_n}(\theta_n - \sigma_n \frac{i\pi}{2}) \cdots \Lambda^{\sigma_1}(\theta_n - \sigma_1 \frac{i\pi}{2}) \rangle \rangle. \quad (5.13)$$

The averaging

$$\langle \langle \ldots \rangle \rangle = \frac{\text{Tr}_{F}[e^{2\pi iK} \ldots]}{\text{Tr}_{F}[e^{2\pi iK}]}.$$
is performed with use of the formula

\[
\langle \Lambda^{\sigma_n}(\theta_n - \sigma_n \frac{i\pi}{2}) \ldots \Lambda^{\sigma_1}(\theta_1 - \sigma_1 \frac{i\pi}{2}) \rangle = \prod_{1 \leq k < j \leq n} R(\theta_{kj}) \left[ 1 + (\sigma_j - \sigma_k) \frac{i}{2} \sinh(\frac{\pi b}{Q}) \right],
\]

(5.14)

where \( \theta_{kj} = \theta_k - \theta_j \) and the function \( R(\theta) \) for \( -2\pi < \Im \theta < 0 \) reads

\[
R(\theta) = \exp \left\{ -2 \int_0^\infty \frac{dt}{t} \frac{\sinh(\frac{tb}{2Q}) \sinh(\frac{t}{2Q})}{\sinh(t) \cosh(\frac{t}{2})} \cos(t(1 - \frac{i\theta}{\pi})) \right\}.
\]

(5.15)

It is the so called minimal form-factor [12], [14]. We should choose the constant \( \rho \) in (5.13) according to the kinematical pole residue normalization (2.10),

\[
\rho = \left[ \sin\left(\frac{\pi b}{Q}\right) \right]^{-\frac{1}{2}} \exp \left\{ \int_0^\infty \frac{dt}{t} \frac{\sinh(\frac{tb}{2Q}) \sinh(\frac{t}{2Q})}{\sinh(t) \cosh(\frac{t}{2})} \right\}.
\]

(5.16)

One can simplify the expressions of the first form-factors,

\[
\langle \text{vac} | \pi_A(\mathcal{O}_a) | B \rangle = \langle \mathcal{O}_a \rangle h \left[ a \right],
\]

\[
\langle \text{vac} | \pi_A(\mathcal{O}_a) | B(\theta_2)B(\theta_1) \rangle = \langle \mathcal{O}_a \rangle h^2 \left[ a \right]^2 R(\theta_{12})
\]

\[
\langle \text{vac} | \pi_A(\mathcal{O}_a) | B(\theta_3)B(\theta_2)B(\theta_1) \rangle = \langle \mathcal{O}_a \rangle h^3 \left[ a \right] \prod_{1 \leq k < j \leq 3} R(\theta_{kj}) \times \left\{ \left[ a \right]^2 + \frac{x_1x_2x_3}{(x_1 + x_2)(x_2 + x_3)(x_1 + x_3)} \right\},
\]

(5.17)

here \( x_k = e^{\theta_k} (k = 1, 2, 3) \) and

\[
\left[ a \right] = \frac{\sin\left(\frac{\pi a}{Q}\right)}{\sin\left(\frac{\pi b}{Q}\right)},
\]

\[
h = 2\rho \sin\left(\frac{\pi b}{Q}\right).
\]

The functions (5.13), (5.17) coincide with the form-factors of exponential operators in the sinh-Gordon model proposed in the work [22]. Thus we conclude

\[
\mathcal{O}_a = e^{a\phi}.
\]

(5.18)

6. Bullough-Dodd model

The construction discussed in this section is very similar to the sinh-Gordon case. In order to avoid additional subscripts we will denote analogous quantities in this section by the same symbols as in the previous one.
As an example of QFT possessing the $\varphi^3$-property we consider the Bullough-Dodd model [23], [24]. It is an integrable QFT defined by the Euclidean action,

$$A_{BD} = \int d^2x \left\{ \frac{1}{16\pi} (\partial_\nu \varphi)^2 + \mu \left( e^{\sqrt{2b}\varphi} + 2e^{\frac{\sqrt{2b}}{2}\varphi} \right) \right\}. \quad (6.1)$$

The spectrum of the model consists of one particle that appears to be the bound state of itself. The corresponding two-body S-matrix was proposed in the work [25]

$$S_{BD}(\theta) = \frac{\tanh\left(\frac{\theta}{2} + \frac{i\pi}{3}\right) \tanh\left(\frac{\theta}{2} - \frac{i\pi}{3}\right) \tanh\left(\frac{\theta}{2} + \frac{i\pi}{3}\right) \tanh\left(\frac{\theta}{2} - \frac{i\pi}{3}\right)}{\tanh\left(\frac{\theta}{2} - \frac{i\pi}{3}\right) \tanh\left(\frac{\theta}{2} + \frac{i\pi}{3}\right) \tanh\left(\frac{\theta}{2} + \frac{i\pi}{3}\right) \tanh\left(\frac{\theta}{2} - \frac{i\pi}{3}\right)}, \quad (6.2)$$

As before we use the notation $Q = b^{-1} + b$. Notice that for $b = 1$ the bound state pole disappears and (6.2) reduces to the sinh-Gordon S-matrix with the coupling constant $b_{shG} = \frac{1}{\sqrt{2}}$.

The space of the angular quantization of the Bullough-Dodd model is almost identical in construction to the sinh-Gordon one. Introduce the set of oscillators obeying the commutation relations

$$[\lambda_\nu, \lambda_{\nu'}] = \frac{4 \sinh\left(\frac{\pi b\nu}{3Q}\right) \sinh\left(\frac{\pi \nu}{3Qb}\right) \cosh\left(\frac{\pi \nu}{6}\right)}{\nu \cosh\left(\frac{\pi \nu}{2}\right)} \delta(\nu + \nu') \quad (6.3)$$

and the zero modes $P, Q$ (5.7). As in the previous case we build $\pi_Z$ as a direct sum of the Fock space $\bigoplus_p F_p$. The ZF operators read

$$B(\theta) = -iC_{BD} \left\{ e^{i\frac{2\pi P}{2}} \Lambda(\theta + \frac{i\pi}{2}) - e^{-i\frac{2\pi P}{2}} \Lambda^{-1}(\theta - \frac{i\pi}{2}) + i \frac{1 - b^2}{1 + b^2} : \Lambda(\theta + \frac{i\pi}{6}) \Lambda^{-1}(\theta - \frac{i\pi}{6}) : \right\}, \quad (6.4)$$

with

$$\Lambda(\theta) = : \exp \left\{ -i \int_{-\infty}^{+\infty} d\nu \, \lambda_\nu \, e^{i\nu(\theta - i\frac{\pi}{2})} \right\} :.$$ \hspace{1cm} C_{BD} \text{ is a real constant providing the normalization (5.2). The operator } K \text{ is given by the formula which is similar to (5.10).}

As well as in the sinh-Gordon model the projectors on the Fock space with a given eigenvalue of the zero mode $P$ commute with the ZF operator. Let us denote again by $O_a$ the local fields which correspond to the projector on the space $F_{p_a}$ with

$$p_a = \frac{1}{6} \left( 4\sqrt{2}a - b^{-1} + b \right).$$
Thus we have the one parametric family of the form-factors given by (5.12). The trace calculation is similar to the one discussed before. It is convenient to formulate the result as the following prescription,

\[ \langle \text{vac} | \pi_A(\mathcal{O}_a) | B(\theta_n) \ldots B(\theta_1) \rangle = \langle \mathcal{O}_a \rangle \langle \langle B(\theta_n) \ldots B(\theta_1) \rangle \rangle , \tag{6.5} \]

where

\[ B(\theta) = \rho \left\{ \gamma \Lambda(\theta + i\frac{\pi}{2}) + \gamma^{-1} \Lambda^{-1}(\theta - i\frac{\pi}{2}) + \kappa : \Lambda(\theta + i\frac{\pi}{6}) \Lambda^{-1}(\theta - i\frac{\pi}{6}) : \right\} , \tag{6.6} \]

and

\[ \gamma = -i \exp \left( \frac{i\pi}{6Q} (4\sqrt{2}a - b^{-1} + b) \right) , \]

\[ \kappa = 2 \sin \left( \frac{\pi}{6Q} (b^{-1} - b) \right) , \]

\[ \rho = \left[ \frac{\sin \left( \frac{\pi}{3} \right)}{\sin \left( \frac{2\pi b}{3Q} \right) \sin \left( \frac{2\pi b}{3Qb} \right)} \right]^{\frac{1}{2}} \exp \left\{ 2 \int_0^\infty \frac{dt}{t} \cosh \left( \frac{t}{6} \right) \sinh \left( \frac{tb}{3Q} \right) \sinh \left( \frac{tb}{3Qb} \right) \sinh(t) \cosh \left( \frac{t}{2} \right) \right\} . \tag{6.7} \]

The Wick’s averaging of the products of the vertex operators should be performed using the rules

\[ \langle \langle \Lambda(\theta) \rangle \rangle = 1 , \]

\[ \langle \langle \Lambda^{\sigma_2}(\theta_2)\Lambda^{\sigma_1}(\theta_1) \rangle \rangle = \left[ R(\theta_1 - \theta_2) \right]^{\sigma_1\sigma_2} , \quad \sigma_i = \pm 1 , \tag{6.8} \]

where

\[ R(\theta) = \exp \left\{ -4 \int_0^\infty \frac{dt}{t} \cosh \left( \frac{t}{6} \right) \sinh \left( \frac{tb}{3Q} \right) \sinh \left( \frac{tb}{3Qb} \right) \sinh(t) \cosh \left( \frac{t}{2} \right) \cosh \left( t(1 - \frac{i\theta}{\pi}) \right) \right\} \]

is the minimal form-factor for the Bullough-Dodd model [26]. The dots : ... : in (6.6) mean that we do not need to pair the vertex operators within the normally ordered group under Wick’s averaging. From (6.5)-(6.8), one can easily derive the first two form-factors

\[ \langle \text{vac} | \pi_A(\mathcal{O}_a) | B \rangle = \langle \mathcal{O}_a \rangle \rho (\gamma + \gamma^{-1} + \kappa) , \]

\[ \langle \text{vac} | \pi_A(\mathcal{O}_a) | B(\theta_2)B(\theta_1) \rangle = \langle \mathcal{O}_a \rangle \rho^2 \left[ R(\theta_{12}) \left( \gamma^2 + \gamma^{-2} + G(\theta_{12} - \frac{i\pi}{2}) + G(\theta_{12} + \frac{i\pi}{2}) + \kappa^2 F(\theta_{12}) \right) \right] . \tag{6.9} \]

Here we use the notations,

\[ F(\theta) = \frac{R(\theta)}{R(\theta - \frac{i\pi}{3} + i\frac{\pi}{3})}, \]

\[ G(\theta) = \frac{1}{R(\theta - \frac{i\pi}{2} + i\frac{\pi}{2})} . \tag{6.10} \]
The functions $F$ and $G$ read explicitly,

$$
F(\theta) = \frac{\sinh \left( \frac{\theta}{2} + \frac{i\pi}{6Q} (b^{-1} - b) \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi}{6Q} (b^{-1} - b) \right)}{\sinh \left( \frac{\theta}{2} + \frac{i\pi}{6} \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi}{6} \right)},
$$

$$
G(\theta) = \frac{\sinh \left( \frac{\theta}{2} - \frac{i\pi}{4} + \frac{i\pi b}{3Q} \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi}{4} - \frac{i\pi b}{3Q} \right) \sinh \left( \frac{\theta}{2} + \frac{i\pi}{3Q} \right) \sinh \left( \frac{\theta}{2} + \frac{i\pi}{3Q} \right)}{\sinh \left( \frac{\theta}{2} + \frac{i\pi}{4} \right) \sinh \left( \frac{\theta}{2} + \frac{i\pi}{4} \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi}{6} \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi}{6} \right)}.
$$

(6.11)

We can simplify (6.9) and present in the form

$$
\langle \text{vac} | \pi_A(\mathcal{O}_a) | B \rangle = \langle \mathcal{O}_a \rangle \ h \{a\},
$$

$$
\langle \text{vac} | \pi_A(\mathcal{O}_a) | B(\theta_2) B(\theta_1) \rangle = \langle \mathcal{O}_a \rangle \ h^2 \{a\} \ R(\theta_{12}) \left( \{a\} - \frac{x_1 x_2}{x_1^2 + x_2^2 + x_1 x_2} \right),
$$

(6.12)

where $x_k = e^{\theta_k}$ ($k = 1, 2$) and

$$
\{a\} = \frac{\sin (\frac{\pi \sqrt{2} a}{3Q}) \cos (\frac{\pi}{6Q} (2\sqrt{2} a - b^{-1} + b))}{2 \sin (\frac{\pi}{6Q} (b^{-1} - b)) \sin (\frac{\pi b}{3Q}) \sin (\frac{\pi}{3Q} b)},
$$

$$
h = 8 \rho \sin (\frac{\pi}{6Q} (b^{-1} - b)) \sin (\frac{\pi b}{3Q}) \sin (\frac{\pi}{3Q} b).
$$

The functions (6.12) coincide with the form-factors of the exponential fields $e^{a \varphi}$ found in the work [27]. We have also checked that (6.5) reproduces the three particle form-factors presented in [27]. In all likelihood $\mathcal{O}_a = e^{a \varphi}$ again.

7. Deformations of Virasoro algebra

In this section we would like to discuss an intriguing relation between the ZF operators for the sinh-Gordon and Bullough-Dodd models and deformations of the Virasoro algebra. Let us introduce the real parameter

$$
0 < x < 1.
$$

The ZF operator $B_{shG}(\theta)$ (5.9) can be written as $x \to 1$ (“scaling”) limit,

$$
B_{shG}(\theta) = C_{shG} \lim_{x \to 1} \ U \left( x^{\frac{2i\theta}{\pi}} \right),
$$

(7.1)

where

$$
U(z) = -i \ \left\{ e^{i \frac{2\varphi}{\pi}} \ Λ_x (zx^{-1}) - e^{-i \frac{2\varphi}{\pi}} \ Λ_x^{-1} (zx) \right\},
$$

(7.2)

$$
Λ_x(z) = : \exp \left\{ -i \sum_{n \neq 0} \ λ_n (zx)^{-n} \right\} :.
$$
The oscillators $\lambda_n, \lambda_m$, 

$$[\lambda_n, \lambda_m] = \frac{(\frac{a}{Q} - x^{\frac{a}{Q}})(x^{\frac{a}{Q}} - x^{-\frac{a}{Q}})}{n (x^n + x^{-n})} \delta_{n+m,0}, \quad (7.3)$$

are specified by an integer $n$ rather than a continuous parameter $\nu \rightarrow -\frac{2n}{\pi} \log(x)$ ($x \rightarrow 1$) as it was in (5.3). In other words, we treat the Fourier integral in (5.5) as a limiting value of the proper Fourier series. The “deformed” ZF operator generates a quadratic algebra with the following commutation relation [28], [29], [30]

$$f(\zeta z^{-1}) U(z)U(\zeta) - f(\zeta^{-1} x) U(\zeta)U(z) =
\frac{\delta(z \zeta^{-1} x^{-2}) - \delta(z \zeta^{-1} x^2)}{(x-x^{-1})} \bigg[ b \bigg[ \frac{b}{Q} \bigg] , \frac{1}{Q} \bigg] x \bigg[ b \bigg[ \frac{1}{Qb} \bigg] x \bigg], \quad (7.4)$$

with

$$f(z) = \frac{(z x^{\frac{a}{Q}}; x^4)_{\infty} (z x^{-\frac{a}{Q}}; x^4)_{\infty}}{(1-z) (z x^{2+\frac{a}{Q}}; x^4)_{\infty} (z x^{2-\frac{a}{Q}}; x^4)_{\infty}}. \quad (7.5)$$

Here we use the conventional notations,

$$\left(z; q\right)_{\infty} = \prod_{n=0}^{+\infty} (1 - z q^n),$$

$$[a]_x = \frac{x^{\frac{a}{Q}} - x^{-\frac{a}{Q}}}{x - x^{-1}},$$

$$\delta(z) = \sum_{n=-\infty}^{+\infty} z^n.$$ 

The operator $U(z)$ admits the power series expansion,

$$U(z) = \sum_{n=-\infty}^{+\infty} U_n z^{-n}, \quad (7.6)$$

and Eq.(7.4) can be equivalently rewritten in terms of the modes $U_n$

$$\sum_{k=0}^{+\infty} f_k \left(U_n - U_{n+k} - U_{n-k} U_{n+k}\right) = (x - x^{-1})^2 \left[ b \right], \left[ b \right], \left[ 2n \right] \delta_{n+m,0}. \quad (7.7)$$

where the coefficients $f_k$ are defined by the formula

$$f(z) = \sum_{k=0}^{+\infty} f_k z^k.$$
The algebra (7.7) is a deformation of the famous Virasoro algebra. Indeed, assuming

\[ U_n = 2\delta_{n,0} - (x - x^{-1})^2 \left(Q^{-2} L_n - \frac{\delta_{n,0}}{4}\right) + O\left((x - x^{-1})^4\right), \]  

(7.8) leads to the well known commutation relations for the generators \( L_n \),

\[ [L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n), \]  

(7.9)

where \( c = c(b) \),

\[ c(b) = 1 + 6 (b + b^{-1})^2. \]  

(7.10)

The ZF operator corresponding to the Bullough-Dodd model (6.4) can be deformed in analogous way,

\[ B_{BD}(\theta) = C_{BD} \lim_{\theta \to 1} V(x^{2/b}) \]  

(7.11)

The “current” \( V(z) \) also generate the quadratic algebra,

\[ g(\zeta^{-1}) V(z)V(\zeta) - g(z\zeta^{-1}) V(\zeta)V(z) = \]

\[ (x - x^{-1}) \left\{ \delta(z\zeta^{-1} - x^{-2}) - \delta(z\zeta^{-1} - x^2) \right\} \frac{2b}{3Q} x \frac{2}{3Q} x \frac{1}{3} + \frac{2b}{3Q} x \frac{1}{3} + \frac{2}{3Q} x \]

(7.12)

where

\[ g(z) = \frac{(x^{10}/z; x^4)_{\infty}}{(1 - z)(x^{1/3}; x^4)_{\infty} (x^{2/3}; x^4)_{\infty}} \times \frac{(x^{2/3}; x^4)_{\infty} (x^{2/3}; x^4)_{\infty}}{(x^{2/3}; x^4)_{\infty} (x^{2/3}; x^4)_{\infty}} \]

(7.13)

In terms of the Laurent modes, \( V(z) = \sum_{n=-\infty}^{+\infty} V_n z^{-n} \), the commutation relation (7.12) reads as

\[ \sum_{k=0}^{+\infty} g_k \left( V_{n-k} V_{m+k} - V_{m-k} V_{n+k} \right) = \]

\[ (x - x^{-1})^2 \left[ \frac{2b}{3Q} x \frac{2}{3Q} x \frac{1}{3} + \frac{2b}{3Q} x \frac{1}{3} + \frac{2}{3Q} x \right] [2n] x \delta_{n+m,0} + \]

(7.14)

\[ (x - x^{-1})^2 \left[ \frac{2b}{3Q} x \frac{2}{3Q} x \frac{b^{-1} - b}{3Q} x \right] \frac{2}{3} [2(n-m)] x V_{n+m}. \]

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Here $g_k$ are generated by the function $g(z)$

$$g(z) = \sum_{k=0}^{+\infty} g_k z^k.$$ 

Again, assuming that

$$V_n = \pm v(b^{\pm1}) \delta_{n,0} \mp (x - x^{-1})^2 \left( \frac{8}{9Q^2} L_n^{\pm} - s(b^{\pm1}) \delta_{n,0} \right) + O((x - x^{-1})^4), \quad (7.15)$$

where

$$v(b) = \frac{3 + b^2}{Qb},$$

$$s(b) = \frac{(3 + b^2)(8 + 13b^2 + 6b^4)}{54Q^3 b^3},$$

one can check that $L_n^{\pm}$ obey (7.9) with the central charges

$$c^{\pm} = c(\sqrt{2b^{\mp1}}).$$

The function $c(b)$ is given by (7.10). We conclude that both (7.7) and (7.14) provide associative deformations of the Virasoro algebra. They look very different. It would be interesting to analyze an equivalence (or nonequivalence) of these deformations.

8. Conclusion

Our study of the angular quantization spaces suggests that the following observations are quite common features of integrable QFT models:

(i) Form-factors of local operators can be represented as traces over the space of angular quantization.

(ii) The space of the angular quantization for a massive integrable model admits description in terms of free Fock spaces. Therefore the traces can be easily evaluated.

(iii) The space of the angular quantization for a massive integrable model can be treated as a “scaling” limit of a representation of some “deformed” algebra. The ZF operators corresponding to a diagonal scattering theory are the scaling limits of currents of the deformed algebra.

The construction, similar to the one discussed in the body of the paper, has been recently developed for the affine $A^{(1)}_{N-1}$ Toda QFT, which contains $N - 1$ particle in the
spectrum \[31\]. In this case the ZF operators are the “scaling” limits of currents of the deformed \(W_{A_{N-1}}\) algebra \[30\], \[32\].

We can not resist the temptation to mention here a remarkable similarity of the free field representations of ZF operators and famous Baxter \(T - Q\) equations \[5\]. It can be illustrated with the example of the Bullough-Dodd model. The model belongs to the class of the affine Toda QFT corresponding to the \(A_2^{(2)}\) root system \[33\] while the Baxter equation associated with this root system reads \[34\], \[35\]

\[
\mathcal{T}(\theta) \, Q(\theta + \frac{i\pi}{6}) \, Q(\theta - \frac{i\pi}{6}) = \gamma(\theta) \, Q(\theta + \frac{5i\pi}{6}) \, Q(\theta - \frac{i\pi}{6}) + \\
\gamma^{-1}(\theta) \, Q(\theta - \frac{5i\pi}{6}) \, Q(\theta + \frac{i\pi}{6}) + \kappa(\theta) \, Q(\theta + \frac{i\pi}{2}) \, Q(\theta - \frac{i\pi}{2}),
\]

where \(\mathcal{T}(\theta)\) and \(Q(\theta)\) are the fundamental transfer matrix and Baxter \(Q\)-operator, respectively. The numerical functions \(\gamma(\theta)\) and \(\kappa(\theta)\) are non universal and depend on a specific model from the \(A_2^{(2)}\) class. If we introduce

\[
\Lambda(\theta) = \frac{Q(\theta + \frac{i\pi}{3})}{Q(\theta - \frac{i\pi}{3}),}
\]

the form of the \(T - Q\) equation will essentially coincide with the formula \((6.6)\). We expect that the numerous results on the Baxter equations associated with the different root systems \[36\], \[37\] will be extremely useful to develop the angular quantization of general affine Toda QFT.

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\[2\] The resemblance of the deformed W-algebras and Baxter equations was already noted by E. Frenkel and N. Reshetikhin \[28\].

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