A SMOOTH SPACE OF TETRAHEDRA

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ABSTRACT. We construct a smooth symmetric compactification of the space of all labeled tetrahedra in \( \mathbb{P}^3 \).

1. INTRODUCTION

1.1. Let \( \mathbb{P}^n \) be \( n \)-dimensional complex projective space, and let \( P \subset \mathbb{P}^n \) be a set of \( n + 1 \) labeled points in general position. By taking all possible linear spans of subsets of \( P \), one obtains a configuration of flats in \( \mathbb{P}^n \) arranged to form a simplex. The set \( X^o \) of all such configurations is naturally a quasi-projective variety with a canonical singular compactification \( X \). One is interested in the variety \( X \) for many reasons:

- For \( n = 2 \), the space \( X \) is the space of triangles in the plane. In \([14]\), Schubert described a desingularization of \( X \), and used it to study enumerative problems involving triangles \([13, 8, 3]\).
- The space \( X \) is a configuration variety in the sense of Magyar \([10, 18]\). Such spaces arise naturally in the study of generalized Schur modules. These are (reducible) GL\(_n\)-modules that generalize the classical Schur modules, and have been studied in various guises by many authors \([1, 4, 13, 8, 11, 12, 16]\). One hopes that configuration varieties will play a role in a Borel-Weil theory for these modules.
- Let \( \mathcal{B} \) be the Tits building for SL\(_{n+1}\)(\( \mathbb{C} \)), and let \( \mathcal{C} \) be the associated Coxeter complex \([17]\). Then \( X \) can be interpreted as the space of maps of \( \mathcal{C} \) into \( \mathcal{B} \). By considering other algebraic groups, one obtains a collection of natural configuration spaces related to the Bott-Samelson varieties of Demazure \([4]\). In particular, \( X \) can be regarded as a canonical Bott-Samelson variety associated to all reduced expressions of the longest word of the Weyl group of SL\(_{n+1}\).
- The space \( X \) is a natural generalization of the Fulton-MacPherson space \( \mathbb{P}^n[n+1] \). This variety adds data to an open set of the product \( \prod_{i=1}^{n+1} \mathbb{P}^n \) that records how points approach the (large) diagonals. In fact, \( \mathbb{P}^n[n+1] \) is a desingularization of the space of all \( 1 \)-skeleta of \( n \)-simplices in \( \mathbb{P}^n \).

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1.2. In this paper we consider the case $n = 3$, where $X$ is the space of tetrahedra in $\mathbb{P}^3$. We construct a symmetric compactification $\tilde{X}$ of $X^\circ$ that we call the space of complete tetrahedra. It is obtained by embedding $X^\circ$ into a large ambient variety $E_#$ and taking the closure of the image. The singular locus of the canonical compactification $X$ is contained in the subvariety consisting of “collapsed tetrahedra”—that is, configurations of flats where certain faces coincide (Figure 1)—and $E_#$ is constructed to capture the asymptotic behavior of a tetrahedron as it collapses. Our main theorem (Theorem 7.6) is that $\tilde{X}$ is nonsingular.

![Figure 1. A point in $X^\circ$ and a point in $X \setminus X^\circ$](image)

In a later paper [2] we will make a more detailed study of the geometry of $\tilde{X}$. We will show that the complement of $X^\circ$ in $\tilde{X}$ is a divisor with normal crossings and compute the cohomology ring of $\tilde{X}$.

1.3. Although this article considers the space of tetrahedra in $\mathbb{P}^3$, the definition of $\tilde{X}$ makes sense for all $n$. Many of the results of the paper (in particular, sections 4–6) hold for arbitrary $n$, but we have avoided this generality since we cannot complete the proof that $\tilde{X}$ is nonsingular in general (the combinatorial arguments in section 4 become infeasible when $n \geq 4$). However, we conjecture that $\tilde{X}$ provides a nonsingular compactification of $X^\circ$ for all $n$.

For $n = 2$, it is not hard to see that our variety $\tilde{X}$ is nonsingular and coincides with certain triangle varieties found in the literature. More precisely, it is isomorphic to the Fulton-MacPherson space $\mathbb{P}^2[3]$, which in turn coincides with an auxiliary compactification constructed by Roberts-Speiser [13]. It is not, however, isomorphic to Schubert’s compactification as a variety over $X^\circ$.

\footnote{The difference between Schubert’s space and Fulton-MacPherson-Roberts-Speiser’s space appears when one considers the torus action on them, cf. [18].}
1.4. We now give an overview of the definition of $\widetilde{X}$. The construction of $E_\#$ depends on the combinatorics of hypersimplices $[6]$, polytopes intimately related to the geometry of Grassmannians. For our considerations, the relevant polytopes are the 3-dimensional hypersimplices $\Delta_1$, $\Delta_2$, and $\Delta_3$ (Figure [2]). The vertices of these hypersimplices are in bijection with the labeled faces of a tetrahedron, and the edges of the hypersimplices correspond to certain pairs of faces of the same dimension.

For each edge $\alpha$ in a hypersimplex, we form a $(\mathbb{P}^1 \times \mathbb{P}^1)$-bundle $E_\alpha \to X$. The bundle $E_\alpha$ has a canonical section $u_\alpha$ and a diagonal subbundle $D_\alpha$. The section $u_\alpha$ tracks the subspaces corresponding to the vertices of $\alpha$ in such a way that $u_\alpha$ intersects $D_\alpha$ precisely when these subspaces coincide. In order to record the asymptotic behavior in $X$ near a collapsed tetrahedron, a natural idea is to form products of the $E_\alpha$’s and blow up the corresponding product of diagonals. The question is which products to take, and why.

1.5. Our main idea is that the relevant products are those indexed by the faces of dimension $\geq 2$ of the hypersimplices. The motivation is that a configuration of flats in $\mathbb{P}^3$ arranged to form a tetrahedron contains certain “sub-” and “quotient” configurations corresponding to proper faces of the hypersimplices. For example, the three points and three lines in a given face of a tetrahedron form a subconfiguration that corresponds to a triangular face in the hypersimplex $\Delta_1$, and the three lines and three planes containing a given point form a quotient configuration that corresponds to a triangular face in the hypersimplex $\Delta_3$. Our motivation is that a nonsingular compactification of $X^\circ$ should add data recording the “infinitesimal shapes” of these sub- and quotient configurations. Hence, each locus we blowup corresponds to the collapsing together of the subspaces labeled by some face of a hypersimplex.

More precisely, our definition is as follows. Let $\mathcal{H}$ be the set of faces of dimension $\geq 2$ of all the $\Delta_k$. For each $\beta \in \mathcal{H}$, let $\mathcal{E}(\beta)$ be the set of edges in $\beta$. Let $E_\beta$ be the product bundle

$$E_\beta := \prod_{\alpha \in \mathcal{E}(\beta)} E_\alpha,$$

and let $D_\beta$ be the corresponding product of the diagonals $D_\alpha$. The ambient variety $E_\#$ is then defined by blowing up each $E_\beta$ along $D_\beta$ and taking the product of the resulting blowups. The corresponding product of the sections $u_\alpha$, then determines an embedding $X^\circ \to E_\#$, and we define $\widetilde{X}$ to be the closure.

1.6. The paper is organized as follows. Section 2 sets up notation, defines $X$, and contains background on hypersimplices. Section 3 contains the construction of $E_\#$ and $\widetilde{X}$. In section 4 we describe a collection of affine open subvarieties that covers $X$ and give equations defining a typical element $U \subset X$ in this collection. In section 5 we restrict the construction of $E_\#$ to $U$ to obtain $\widetilde{U}$, a certain subvariety of $\widetilde{X}$. The
point of sections \[ \tilde{X} \] is that the nonsingularity of \( \tilde{X} \) follows from the nonsingularity of \( \tilde{U} \).

In the remaining sections we prove nonsingularity of \( \tilde{U} \). First, in section \[ 6 \] we show that \( \tilde{U} \) has the structure of a vector bundle over a certain (multi-) projective variety \( Z \) that we call the \textit{core}. We then study the \( \text{GL}_4 \)-action on \( \tilde{U} \) to show that nonsingularity of \( Z \) follows from its nonsingularity at points in a certain subvariety \( Z_{sp} \subset Z \). Finally in section \[ 7 \] we give equations that cut out \( Z \) from projective space and use a graphical description of these defining relations to show that \( Z_{sp} \) consists of nonsingular points of \( Z \); this proves Theorem \[ 7.6 \].

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2. Notation and the basic variety \( X \)

2.1. Let \( e_1, \ldots, e_4 \) be the standard basis of \( \mathbb{C}^4 \), and let \([4]\) be the set \( \{1, 2, 3, 4\} \). For any subset \( I \subset [4] \), let \( E_I \subset \mathbb{C}^4 \) be the subspace spanned by \( \{e_i \mid i \in I\} \). Let \( \mathbb{P}^3 \) be the projective space of lines in \( \mathbb{C}^4 \), and let \( G \) be the algebraic group \( \text{GL}_4(\mathbb{C}) \).

For \( k = 1, 2, 3 \), let \( \text{Gr}_k \) be the Grassmannian of \( k \)-dimensional subspaces of \( \mathbb{C}^4 \), and for each proper nonempty subset \( I \subset [4] \), let \( \text{Gr}_I := \text{Gr}_{|I|} \). (We use the notation \(|I|\) for the cardinality of \( I \).) Let \( Y \) be the product

\[
Y := \prod_{\emptyset \subsetneq I \subset [4]} \text{Gr}_I \cong (\text{Gr}_1)^4 \times (\text{Gr}_2)^6 \times (\text{Gr}_3)^4,
\]

and for each \( I \subset [4] \), let \( \pi_I \) be the projection to the \( I \)th factor. The group \( G \) acts on \( Y \) by left multiplication, and each \( \pi_I \) is \( G \)-equivariant.

**Definition 2.2.** Let \( p_0 \in Y \) be the point such that \( \pi_I(p_0) = E_I \) for all \( I \subset [4] \), and let \( X^\circ \subset Y \) be the \( G \)-orbit of \( p_0 \). Let \( X \subset Y \) be \( X^\circ \) (the bar denotes Zariski closure). The space \( X \) (respectively \( X^\circ \)) is called the \textit{space of tetrahedra} (resp. \textit{nondegenerate tetrahedra}).

Note that since the \( G \)-action preserves incidence relations among subspaces, for any \( p \in X \) we have \( \pi_I(p) \subset \pi_J(p) \) if \( I \subset J \). Hence the configuration of subspaces \( \{\pi_I(p) \mid I \subset [4]\} \) satisfies the incidence relations corresponding to the faces of a tetrahedron.

The symmetric group \( S_4 \) acts on \( Y \) by permuting the factors, and this clearly induces an action on \( X^\circ \) and \( X \): given \( \sigma \in S_4 \) and \( p \in X \), the point \( \sigma \cdot p \) is determined
by $\pi_I(\sigma \cdot p) = \pi_{\sigma^{-1}(I)}(p)$. This action can be viewed as “changing the labels” on the faces of a tetrahedron.

2.3. The construction of our resolution $\tilde{X} \to X$ is based on the combinatorics of hypersimplices, so we recall basic facts about them. More details can be found in [6].

Let $\varepsilon_1, \ldots, \varepsilon_4$ be the standard basis of $\mathbb{R}^4$, and for any subset $I \subset [4]$, let $\varepsilon_I := \sum_{i \in I} \varepsilon_i$. Then the hypersimplex (of rank $k$) $\Delta_k$ is defined by

$$\Delta_k := \text{Conv}\{\varepsilon_I \mid I \subset [4] \text{ and } |I| = k\},$$

where $\text{Conv}$ denotes convex hull. The hypersimplices $\Delta_1$ and $\Delta_3$ are 3-simplices, and $\Delta_2$ is an octahedron. Note that the vertices of a hypersimplex are indexed by proper nonempty subsets of $[4]$ (Figure 2). It will be convenient to fix a total ordering on the subsets of $[4]$, and thus on the vertices of the hypersimplices:

$\emptyset < 1 < 2 < 3 < 4 < 12 < 13 < 14 < 23 < 24 < 34 < 123 < 124 < 134 < 234 < 1234$.

If $I$ is a subset of $[4]$ with $|I| = k$, we let $I_0$ be the set 1, 12, or 123, depending on whether $k = 1$, 2, or 3, respectively.

2.4. We identify faces of the hypersimplices with their corresponding sets of vertices. Let $\mathcal{E}$ be the set of pairs $\{I, J\}$ with $I, J \subset [4]$ corresponding to edges of the hypersimplices, and for $k = 1, 2, 3$ let $\mathcal{E}_k \subset \mathcal{E}$ be the subset corresponding to edges of $\Delta_k$. Let $\mathcal{H}$ be the set of vertex sets of all faces of dimension $\geq 2$. Hence $\mathcal{H}$ contains the maximal 3-dimensional faces $\{1, 2, 3, 4\}, \{12, 13, 14, 23, 24, 34\}$, and $\{123, 124, 134, 234\}$, as well as 16 triangular faces. These last faces can be oriented as follows. Let $\beta = \{I, J, K\} \in \mathcal{H}$ be a triangular face, and let $(JK, IK, IJ)$ be the corresponding triple of edges. We call this triple an ordered triangle if $I < J < K$.

We shall need notation for the edges of a given face in $\mathcal{H}$. For each $\beta \in \mathcal{H}$, we let $\mathcal{E}(\beta) \subset \mathcal{E}$ be the subset corresponding to the edges of $\beta$. For example, if $\beta$ is the triangular face $\{12, 13, 23\}$ of the octahedron, then $\mathcal{E}(\beta) = \{\{12, 13\}, \{12, 23\}, \{13, 23\}\}$.

![Figure 2](image)

3. The resolution $\tilde{X}$

3.1. As the first step towards defining $\tilde{X}$, we establish a correspondence between edges of the hypersimplices and certain $\mathbb{P}^1$-bundles over $X$. For each nonempty subset
I \subset \binom{[4]}{I}$, let $F_I \to X$ be the pullback of the tautological $|I|$-plane bundle on $\text{Gr}_{|I|}$ via the composition

$$X \longrightarrow Y \xrightarrow{\pi_I} \text{Gr}_{|I|}.$$ 

Thus the fiber of $F_I$ over a point $p \in X$ can be identified with the $k$-dimensional subspace $\pi_I(p) \subset \mathbb{C}^4$. The incidence conditions on $X$ imply that if $I \subset J$, then $F_I \subset F_J$ is a subbundle.

For each $\alpha \in \mathcal{E}$, let $F_\alpha$ be the quotient $F_{I \cup J}/F_{I \cap J}$ where $\alpha = \{I, J\}$; this is a rank 2 vector bundle since $|I \cup J \setminus I \cap J| = 2$. Let $P_\alpha$ be the projectivized bundle

$$P_\alpha = \mathbb{P}(F_\alpha).$$

3.2. The bundle $P_\alpha$ has canonical sections $u^-_\alpha, u^+_\alpha : X \to P_\beta$, defined geometrically as follows. The fiber of $P_\alpha$ over $p$ can be identified with the set of lines in the 2-dimensional vector space $\pi_{I \cup J}(p)/\pi_{I \cap J}(p)$. We assume that $I < J$, and define $u^-_\alpha(p)$ to be the line $\pi_I(p)/\pi_{I \cap J}(p)$ and $u^+_\alpha(p)$ to be the line $\pi_J(p)/\pi_{I \cap J}(p)$. Since we will want to keep track of both sections simultaneously, we introduce the product bundle

$$E_\alpha = P_\alpha \times_X P_\alpha,$$

and let $u_\alpha : X \to E_\alpha$ be the product $u^-_\alpha \times u^+_\alpha$.

3.3. For each $\beta \in \mathcal{H}$, let $E_\beta$ be the product bundle

$$E_\beta = \prod_{\alpha \in \mathcal{E}(\beta)} E_\alpha.$$ 

This is a $(\mathbb{P}^1 \times \mathbb{P}^1)^3$-bundle over $X$ when $\beta$ is a triangular face; for the maximal faces, $P_{\Delta_k}$ is a $(\mathbb{P}^1 \times \mathbb{P}^1)^6$-bundle for $k = 1, 3$ and a $(\mathbb{P}^1 \times \mathbb{P}^1)^{12}$-bundle for $k = 2$. Let $p_\beta : E_\beta \to X$ be the projection, and let $u_\beta : X \to E_\beta$ be the section obtained by taking the product of the sections $u_\alpha$ for all $\alpha \in \mathcal{E}(\beta)$.

3.4. We define the ambient variety $E$ to be the product bundle

$$E = \prod_{\beta \in \mathcal{H}} E_\beta.$$ 

By Figure 2, there are 3 maximal elements of $\mathcal{H}$ with 6, 12, and 6 edges each, and there are 16 triangular faces with 3 edges each; thus, $E \to X$ is a locally trivial bundle with fiber isomorphic to

$$(\mathbb{P}^1 \times \mathbb{P}^1)^6 \times (\mathbb{P}^1 \times \mathbb{P}^1)^{12} \times (\mathbb{P}^1 \times \mathbb{P}^1)^6 \times ((\mathbb{P}^1 \times \mathbb{P}^1)^3)^{16}.$$ 

Let $p : E \to X$ be the projection, and let $u : X \to E$ be the section obtained by taking the product of the sections $u_\beta$, $\beta \in \mathcal{H}$. 
3.5. To build $\widetilde{X}$, we keep track of “limiting configurations” of the subspaces $\{\pi_I(p)\}$ as certain collections of them coincide. The relevant collections turn out to correspond to the faces $\mathcal{H}$ of the hypersimplices.

For each $\alpha \in \mathcal{E}$, let $D_\alpha \subset E_\alpha$ be the diagonal subbundle, and for each $\beta \in \mathcal{H}$, let $D_\beta \subset E_\beta$ be the subbundle

$$D_\beta = \prod_{\alpha \in \mathcal{E}(\beta)} D_\alpha.$$ 

The geometric significance of $D_\beta$ is that the set of points $p \in X$ such that $u_\beta(p) \in D_\beta$ is precisely the set of $p$ such that $\pi_I(p) = \pi_J(p)$ for all $I, J \in \beta$.

Let

$$b_\beta : (E_\beta)_{\#} \longrightarrow E_\beta$$

be the blowup of $E_\beta$ along $D_\beta$. Since $E_\beta$ is locally trivial over $X$, as is the subbundle $D_\beta$, the blowup $(E_\beta)_{\#}$ is also locally trivial over $X$; the fiber of this last bundle is isomorphic to the blowup of $(\mathbb{P}^1 \times \mathbb{P}^1)^n$ along the product of diagonals (where $n = 3, 6, \text{ or } 12$ depending on $\beta$).

3.6. Since for any $p \in X^\circ$ the image $u_\beta(p)$ avoids the blowup center $D_\beta$, we have a regular map

$$b_\beta^{-1} \circ u_\beta : X^\circ \longrightarrow (E_\beta)_{\#}.$$ 

We define the complete ambient variety $E_\#$ to be the product

$$E_\# = \prod_{\beta \in \mathcal{H}} (E_\beta)_{\#},$$ 

and let $b : E_\# \rightarrow E$ be the product of the blowup maps $b_\beta, \beta \in \mathcal{H}$.

**Definition 3.7.** Let $\widetilde{X}^\circ$ be the image of the embedding $X^\circ \longrightarrow E_\#$ obtained by taking a product of the maps $b_\beta^{-1} \circ u_\beta$ for all $\beta \in \mathcal{H}$. The complete space of tetrahedra, denoted $\widetilde{X}$, is the closure of $\widetilde{X}^\circ$ in $E_\#$.

The composition $p \circ b : E_\# \rightarrow X$ restricts to a surjective birational morphism $\rho : \widetilde{X} \rightarrow X$.

**Remark 3.8.** The map $b : E_\# \rightarrow E$ can be realized as an iterated blowup along regularly embedded subschemes. In this setting, the complete space of tetrahedra $\widetilde{X}$ is the (iterated) proper transform of $u(X) \subset E$. 

3.9. Since the bundles $E_\beta$ are constructed from tautological bundles, they admit natural $G$-actions lifting the action on $X$. Since the diagonals are preserved by these actions, the blown-up bundles $(E_\beta)_\#$ also admit natural $G$-actions that lift the action on $X$, and the blowdown maps $b_\beta$ are equivariant. It follows that there are natural $G$-actions on $E$ and $E_\#$, and that $b: E_\# \to E$ is equivariant. One can check that the section $u$ is also equivariant, and thus $\tilde{X}^o$ is $G$-stable. It follows that the action on $E_\#$ restricts to an action on $\tilde{X}$ and that $\rho: \tilde{X} \to X$ is $G$-equivariant.

Similar remarks apply to the $S_4$-action. This action also lifts to actions on $E$ and $E_\#$ that permute the various factors of these product bundles. The map $b: E_\# \to E$ and the section $u: X \to E$ are both equivariant, so $\tilde{X}^o$ is $S_4$-stable. Hence, the $S_4$-action on $E_\#$ restricts to an action on $\tilde{X}$, and $\rho: \tilde{X} \to X$ is $S_4$-equivariant.

4. The local variety $U$

4.1. Let $\text{Fl}$ be the flag variety of full flags in $\mathbb{C}^4$, and let $V_* \in \text{Fl}$ correspond to a flag

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq V^4 = \mathbb{C}^4,$$

where $V_k$ is a subspace of dimension $k$. Let $U(V_*)$ be the set of all $p \in X$ in general position to $V_*$. That is, $U(V_*)$ consists of all $p$ such that the $|I|$-plane $\pi_I(p)$ is transverse to $V_k$ for all $1 \leq k \leq 3$ and all proper nonempty subsets $I \subset [4]$.

The subset $U(V_*)$ can be described in terms of Schubert cells in the factors $\text{Gr}_I$ of $Y$ as follows. For each $k = 1, 2, 3$, let $U_k$ be the open cell in $\text{Gr}_k$ consisting of $k$-planes in general position to the fixed flag $V_*$. For each proper nonempty subset $I \subset [4]$, let $U_I := U_{[I]}$. Then $\prod U_I$ is an open subvariety of $Y$ isomorphic to an affine space. The variety $U(V_*)$ is the intersection of this open set and the subvariety $X$ of $Y$. In particular, $U(V_*)$ is an affine open subset of $X$.

4.2. Let $E_{\#}|_{U(V_*)}$ be the restriction of the ambient bundle to $U(V_*)$. Let $U^o(V_*) = U(V_*) \cap X^o$, and let $\tilde{U}^o(V_*)$ be the image of $U^o(V_*)$ under the embedding $U^o(V_*) \to E_{\#}|_{U(V_*)}$ of 3.6. Take $\tilde{U}(V_*)$ to be the closure of $\tilde{U}^o(V_*)$ in $E_{\#}|_{U(V_*)}$.

Lemma 4.3. The collection $\{U(V_*) \mid V_* \in \text{Fl}\}$ (respectively, $\{\tilde{U}(V_*) \mid V_* \in \text{Fl}\}$) is an affine open cover of $X$ (resp., $\tilde{X}$). The group $G$ acts transitively on both of these covers.

To prove that $\tilde{X}$ is nonsingular, it suffices by Lemma 4.3 to prove that $\tilde{U}(V_*)$ is nonsingular for one particular choice of the flag $V_*$. We fix $V_*$ to be the flag at infinity:

$$E_4 \subset E_{34} \subset E_{234},$$

and define $U$, $U^o$, and $\tilde{U}$ to be the varieties $U(V_*)$, $\tilde{U}^o(V_*)$, and $\tilde{U}(V_*)$ (respectively).
4.4. We put coordinates on $U$ using Plücker coordinates on the Grassmannians $\text{Gr}_I$. For each $k = 1, 2, 3$, we have the Plücker embedding $\text{Gr}_k \to \mathbb{P}(\wedge^k \mathbb{C}^4)$ with its usual coordinates $\{f_I \mid I \subset [4], |I| = k\}$. The ratios $\{f_I/f_{I_0} \mid |I| = k\}$ provide coordinates on $U_k$. For any pair $I, J \subset [4]$ with $|I| = |J|$, let $f_{I,J}$ be the regular function on $U$ defined by

$$f_{I,J} := \pi_I^*(f_J/f_{I_0}).$$

It is clear that these functions generate the ring $\mathcal{O}_X(U)$, and that $f_{I,J_0} = 1$.

4.5. There is a more symmetric set of generators for $\mathcal{O}_X(U)$, which arises from the observation that $p \in U$ can be constructed from functions on $F_l$ and functions that measure the “difference” between the planes $\pi_I(p)$ and $\pi_J(p)$ for each edge $\{I, J\}$ of the appropriate hypersimplex.

The functions on the flag variety are defined as follows. There is a natural map $X \to F_l$ given by $p \mapsto \{\pi_{I_0}(p)\}$. Let $U_{o_\alpha}$ be the open cell in $F_l$ consisting of flags in general position to the flag at infinity. This cell has local coordinates

$$f_2/f_1, f_3/f_1, f_4/f_1, f_{13}/f_{12}, f_{14}/f_{12}, f_{123}/f_{123}.$$

The corresponding functions

$$f_{1,2}, f_{1,3}, f_{1,4}, f_{12,13}, f_{12,14}, f_{123,124}$$

on $U$ will be called flag coordinates on $U$.

4.6. The functions on $U$ corresponding to edges in the hypersimplices are easiest to describe using certain local sections of the bundles of $\mathbb{C}^4$.

For each nonempty $I \subset [4]$, let $\mathcal{F}_I$ be the sheaf of sections of the bundle $F_I \to X$. We define local sections $s_I \in \mathcal{F}_I(U)$ as follows. Let $k = |I|$. Then for each $p \in U$, the fiber of $F_I$ over $p$ can be identified with the $k$-dimensional subspace $\pi_I(p) \subset \mathbb{C}^4$. This subspace intersects the subspace $V_{5-k}$ of our flag at infinity in a 1-dimensional subspace, and intersects $V_{4-k}$ in the zero subspace. It follows that there is a unique vector $s_I(p) \in \pi_I(p) \cap V_{5-k}$ whose $k$th coordinate (with respect to the standard basis) is 1. This defines the section $s_I: U \to F_I|_U$.

In terms of Plücker coordinates, these sections can be expressed as

$$s_i = e_1 + f_{i,2}e_2 + f_{i,3}e_3 + f_{i,4}e_4,$$

$$s_{ij} = e_2 + f_{ij,13}e_3 + f_{ij,14}e_4,$$

$$s_{ijk} = e_3 + f_{ijk,124}e_4,$$

$$s_{1234} = e_4.$$

A priori, these are all sections of the trivial bundle $U \times \mathbb{C}^4$, but a simple verification shows that their images are contained in $F_I|_U$. The following lemma describes a crucial relation among these sections. We omit the straightforward proof.

**Lemma 4.7.** Let $k = 1, 2, 3$. For each edge $\alpha = \{I, J\} \in \mathcal{E}_k$, we have

$$s_J - s_I = (f_{J,K} - f_{I,K})s_{I,J},$$
where $K$ is the subset $2$, $13$, or $124$ depending on whether $k$ is $1$, $2$, or $3$, respectively.

For $k = 1, 2, 3$ and each edge $\alpha = \{I, J\} \in \mathcal{E}_k$ with $I < J$, we define the edge coordinate $x_\alpha$ by

$$x_\alpha = f_{J,K} - f_{I,K},$$

where $K$ is the subset $2$, $13$, or $124$ depending on whether $k$ is $1$, $2$, or $3$, respectively.

**Lemma 4.8.** The ring $\mathcal{O}_X(U)$ is generated by the flag coordinates and the edge coordinates.

**Proof.** We show that the functions $f_{I,J}$ can be expressed in terms of the flag coordinates and the $x_\alpha$'s. The proof is by induction on $I$, using the total ordering $1 \prec 12 \prec 123 \prec 2 \prec 13 \prec 3 \prec 124 \prec 4 \prec 24 \prec 134 \prec 34 \prec 234$.

The key property of the ordering $\prec$ is that for each $J \succ 123$, there exists an edge $\{I, J\} \in \mathcal{E}$ such that $I, I \cap J, I \cup J \prec J$.

Using the formulas of 4.6, we can express the sections $s_1$, $s_{12}$, and $s_{123}$ entirely in terms of the flag coordinates (and the basis $e_1, e_2, e_3, e_4$). Since $s_1$ determines the line $\pi_1$, $s_1 \cap s_{12}$ determines the plane $\pi_{12}$, and $s_1 \cap s_{12} \cap s_{123}$ determines the 3-plane $\pi_{123}$, we can express all of the corresponding functions $f_{1,J}, f_{12,J},$ and $f_{123,J}$ in terms of the flag coordinates.

For the case $I = 2$, since $s_2 = s_1 + x_{1,2}s_{12}$ (by Lemma 4.7) and $s_2$ determines $\pi_2$, we can express the functions $f_{2,J}$ in terms of $x_{1,2}$ and the flag coordinates. For $I = 13$, $s_{13} = s_{12} + x_{12,13}s_{123}$ and $s_1 \cap s_{13}$ determines $\pi_{13}$, so we can express the functions $f_{13,J}$ in terms of $x_{12,13}$ and the flag coordinates. Expressions for the remaining functions are obtained similarly.

**Lemma 4.10.** Define polynomial rings

$$R_{op} := \mathbb{C}[f_{1,2}, f_{1,3}, f_{1,4}, f_{12,13}, f_{12,14}, f_{123,124}],$$

$$R_\mathcal{E} := \mathbb{C}[x_\alpha \mid \alpha \in \mathcal{E}],$$

and let $\mathcal{A}_{op} = \text{Spec } R_{op}, \mathcal{A}_\mathcal{E} = \text{Spec } R_\mathcal{E}$. Lemma 4.8 says that the natural homomorphism $R_{op} \otimes R_\mathcal{E} \to \mathcal{O}_X(U)$ is surjective, so $U \subset \mathcal{A}_{op} \times \mathcal{A}_\mathcal{E}$. We now describe the ideal that set-theoretically cuts out $U$. This ideal is generated by linear, quadric, cubic, and quartic polynomials in the flag and edge coordinates.

First consider the flag coordinates. The map $X \to \text{Fl}$ is actually a locally trivial fibration, and our coordinates define a trivialization over $U_{op}$. Since $U_{op}$ is nonsingular and the flag coordinates on $U$ are pulled back from a system of local parameters on $U_{op}$, there are no relations among the flag coordinates holding on $U$.

Now consider the edge coordinates. Recall that a triple of edges $(JK, IK, IJ)$ is an ordered triangle if $\{I, J, K\}$ is a triangular face and $I < J < K$.

**Lemma 4.10.** The subvariety $U \subset \mathcal{A}_{op} \times \mathcal{A}_\mathcal{E}$ is defined set-theoretically by the following polynomials:
1. The linear functions
\[ x_{\alpha_1} - x_{\alpha_2} + x_{\alpha_3}, \]
for all ordered triangles \((\alpha_1, \alpha_2, \alpha_3)\) (Figure 3).

2. The quadric functions
\[ x_{\alpha_1} x_{\alpha_2}^* - x_{\alpha_2} x_{\alpha_1}^*, \]
where \(\alpha_1 = \{i, j\}, \alpha_2 = \{j, k\}, \alpha_1^* = \{ik, jk\}, \alpha_2^* = \{ij, ik\} \) or \(\alpha_1 = \{il, jl\}, \alpha_2 = \{jl, kl\}, \alpha_1^* = \{ikl, jkl\}, \alpha_2^* = \{ijl, ikl\}\) (Figure 4).

3. The cubic functions
\[ x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} - x_{\alpha_1}^* x_{\alpha_2}^* x_{\alpha_3}^*, \]
where \(\alpha_1 = \{ij, il\}, \alpha_2 = \{ik, kl\}, \alpha_3 = \{jk, jl\}, \alpha_1^* = \{jk, jl\}, \alpha_2^* = \{ij, il\}, \alpha_3^* = \{ik, il\}\) (Figure 3).

4. The quartic functions
\[ x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} x_{\alpha_4}^* - x_{\alpha_2} x_{\alpha_4} x_{\alpha_1}^* x_{\alpha_3}^*, \]
where \(\alpha_1 = \{i, j\}, \alpha_2 = \{j, k\}, \alpha_3 = \{k, l\}, \alpha_4 = \{l, i\}, \alpha_1^* = \{ikl, jkl\}, \alpha_2^* = \{ijkl, ikl\}, \alpha_3^* = \{ijk, ijl\}, \alpha_4^* = \{jkl, ijk\}\) (Figure 4).

**Figure 3.** Edges in the linear and quadric relations

**Figure 4.** Edges in the cubic and quartic relations

Proof. The vanishing of the linear polynomials follows from the definition of the edge coordinates. The quadric relations follow from this definition and Lemma 4.7. The cubic (resp., quartic) relations can be obtained from the quadric relations by eliminating coordinates corresponding to edges in \(\mathcal{E}_1\) and \(\mathcal{E}_3\) (resp., \(\mathcal{E}_2\)). It follows that \(U\) is a subvariety of the variety defined by the given polynomials.

To see that \(U\) coincides with this variety, it suffices to show that they are both irreducible and have the same dimension. Since \(U\) is the closure of the connected
12-dimensional nonsingular variety $U^0$, it is a 12-dimensional irreducible variety. Let $U_{\text{inc}}$ be the variety defined by the vanishing of the linear and quadric polynomials. A simple computation using Macaulay2 \cite{M2} shows that $U_{\text{inc}}$ has three irreducible components, two of which correspond to the ideals

$$\langle x_\alpha \mid \alpha \in \mathcal{E}_2 \rangle \quad \text{and} \quad \langle x_\alpha \mid \alpha \in \mathcal{E}_1 \cup \mathcal{E}_3 \rangle.$$ 

The remaining component $U_1$ is 12-dimensional. Since $U$ contains points where all $x_\alpha$ are nonzero, we have $U = U_1$. \hfill \Box

Remark 4.11. Let $X_{\text{inc}}$ be the incidence variety consisting of all $p \in Y$ such that $\pi_I(p) \subset \pi_J(p)$ whenever $I \subset J$. Then one can show $U_{\text{inc}} = X_{\text{inc}} \cap \prod_I U_I$, where $\prod_I U_I$ is the affine cell of \cite{1}, although we will not need this here. Hence the linear and quadric relations provide a very simple description of the incidence variety.

In the case under study, Lemma 4.10 shows that $X_{\text{inc}}$ has two other components besides $X$, corresponding to the following types of configurations in $\mathbb{C}^4$:

1. Four 3-planes containing a 2-plane that contains four lines.
2. Six 2-planes containing a line and contained in a 3-plane.

Figure 5 shows general points in these components as configurations in $\mathbb{P}^3$. For the general case of configurations in $\mathbb{C}^n$, the components of $X_{\text{inc}}$ are unknown.

![Figure 5. Other components of the incidence variety.](image)

5. The local resolution $\tilde{U}$

5.1. In this section we describe an open subbundle $W$ (with affine space fibers) of the restricted ambient bundle $E|_U$. The section $u$ restricts to a section of $W$, thus we can apply the blowup construction to $W$, obtaining an open subbundle $W_\#$ of $E_\#|_U$ that contains the local resolution $\tilde{U}$. To get functions on $\tilde{U}$, we then show that $\tilde{U}$ is contained in a certain closed subvariety of $W_\#$ defined in terms of the edge coordinates.

To describe $W$, we first need to describe some trivializations of the various bundles restricted to $U$. 
5.2. For each $\alpha = \{I, J\} \in \mathcal{E}$, let $\mathcal{F}_\alpha$ be the sheaf of sections of the quotient bundle $F_\alpha = F_{I \cup J}/F_{I\cap J}$. Letting $\mathcal{F}_\alpha^*$ be the dual module, and restricting to $U$, we then have (by definition)

$$F_\alpha|_U = \text{Spec}\mathcal{O}_X(U) S_\alpha$$

where $S_\alpha$ is the symmetric $\mathcal{O}_X(U)$-algebra $\text{Sym}\mathcal{F}_\alpha^*(U)$.

There are natural maps $F_I \rightarrow F_\alpha$, $F_J \rightarrow F_\alpha$, and $F_{I \cup J} \rightarrow F_\alpha$ induced by the corresponding inclusions of $F_I$, $F_J$, and $F_{I \cup J}$ (respectively) into $F_{I \cup J}$. With respect to these maps, the local sections $s_I \in \mathcal{F}_I(U)$, $s_J \in \mathcal{F}_J(U)$, and $s_{I \cup J} \in \mathcal{F}_{I \cup J}(U)$ all have images in $\mathcal{F}_\alpha(U)$, which we denote by $s_I$, $s_J$, and $s_{I \cup J}$, respectively. It follows from the explicit descriptions in 4.6 that $s_I$ and $s_J$ are nonzero and linearly independent.

Since $F_\alpha$ is a rank-2 vector bundle, these sections determine dual sections $g_\alpha, h_\alpha \in \mathcal{F}_\alpha^*(U)$, giving an isomorphism

$$F_\alpha|_U \cong U \times \text{Spec}\mathbb{C}[g_\alpha, h_\alpha].$$

Passing to the projectivized bundle $P_\alpha = \mathbb{P}(F_\alpha)$ restricted to $U$, we obtain

$$P_\alpha|_U = \text{Proj}\mathcal{O}_X(U) S_\alpha,$$

which becomes

$$P_\alpha|_U \cong U \times \text{Proj}\mathbb{C}[g_\alpha, h_\alpha].$$

5.3. Let $D(g_\alpha) \subset P_\alpha|_U$ be the divisor determined by $g_\alpha$, and let $V_\alpha$ be the corresponding open set $P_\alpha|_U - D(g_\alpha)$. Then $V_\alpha$ is an affine line bundle over $U$, and we have an isomorphism

$$V_\alpha \cong U \times \text{Spec}\mathbb{C}[h_\alpha/g_\alpha].$$

5.4. For each $\alpha \in \mathcal{E}$, let $W_\alpha$ be the product $V_\alpha \times_U V_\alpha$. To distinguish the two factors of $W_\alpha$, we denote the first by $V^-_\alpha$ and the second by $V^+_\alpha$. For any $\beta \in \mathcal{H}$, let $W_\beta$ be the product of the $W_\alpha$ as $\alpha$ runs over all edges in $\mathcal{E}(\beta)$. Finally, let $W$ be the product of the $W_\beta$ as $\beta$ ranges over all faces in $\mathcal{H}$. Combining the previous sections, we then have the following:

Lemma 5.5. Let $R$ be the polynomial ring

$$R := \mathbb{C}[x^+_{\alpha,\beta}, x^-_{\alpha,\beta} | \beta \in \mathcal{H}, \alpha \in \mathcal{E}(\beta)],$$

and let $\mathbb{A}_{amb} = \text{Spec} R$. Then $W$ is an open subbundle of $E|_U$, and is isomorphic to the product $U \times \mathbb{A}_{amb}$. The indeterminate $x^-_{\alpha,\beta}$ (resp., $x^+_{\alpha,\beta}$) corresponds to the function $h_\alpha/g_\alpha$ on the factor $V^-_\alpha$ (resp., $V^+_\alpha$) of $W_\beta$. 

5.6. We now restrict the blowup construction of 5.3 to the affine space bundle $W \to U$. For each $\beta \in \mathcal{H}$, we let $(W_\beta)_#$ be the blowup of $W_\beta$ along the product of diagonals $D_\beta \cap W_\beta$. In terms of the coordinates in Lemma 5.5, the ideal defining this product of diagonals is $\langle x_{\alpha,\beta}^+ - x_{\alpha,\beta}^- | \alpha \in \mathcal{E}(\beta) \rangle$. Thus, if $R_\beta$ is the polynomial ring $\mathbb{C}[y_{\alpha,\beta} | \alpha \in \mathcal{E}(\beta)]$ and $\mathbb{P}_\beta := \text{Proj} R_\beta$, we have a closed embedding

$$(W_\beta)_# \to W_\beta \times \mathbb{P}_\beta,$$

where the ideal defining the image is

$$\langle y_{\alpha,\beta}(x_{\alpha,\beta}^+ - x_{\alpha,\beta}^-) - y_{\alpha^*,\beta}(x_{\alpha,\beta}^+ - x_{\alpha,\beta}^-) | \alpha, \alpha^* \in \mathcal{E}(\beta) \rangle.$$  

5.7. Now we take products over $\mathcal{H}$. Let $W_#$ be the product over $U$ of the blowups $\{(W_\beta)_# | \beta \in \mathcal{H}\}$. We then have an embedding

$$W_# \to W \times \prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta,$$

and the blowdown map $b : W_# \to W$ is simply the restriction of the projection to the first factor. Moreover, in terms of coordinates in Lemma 5.3, the image of this embedding is cut out by the multihomogeneous polynomials

$$y_{\alpha,\beta}(x_{\alpha,\beta}^+ - x_{\alpha,\beta}^-) - y_{\alpha^*,\beta}(x_{\alpha,\beta}^+ - x_{\alpha,\beta}^-)$$

for all $\beta \in \mathcal{H}$ and $\alpha, \alpha^* \in \mathcal{E}(\beta)$.

5.8. We now consider the subvariety $\widetilde{U} \subset W#$. This is, by definition, the closure of $\widetilde{U}^\circ = b^{-1}(u(U^\circ))$.

**Lemma 5.9.** The image of the section $u : U \to E|_U$ is contained in the open subvariety $W$. In terms of the coordinates of Lemma 5.3 the section $u : U \to W$ is defined by the $\mathcal{E}(U)$-module homomorphism defined by $x_{\alpha,\beta}^\pm \mapsto \pm x_\alpha$.

**Proof.** Let $\alpha = \{I, J\} \in \mathcal{E}$ with $I < J$. Then the section $u_\alpha^-$ (respectively, $u_\alpha^+$) is defined, at each point $p \in U$, to be the linear span of the nonzero vector $\overline{s}_I(p)$ (resp., $\overline{s}_J(p)$). With $g_\alpha$ as in 4.4, it follows from the formulas in 4.6 that $g_\alpha(\overline{s}_I)$ and $g_\alpha(\overline{s}_J)$ are nonzero on $U$. Hence the image $u_\alpha(U)$ is contained in $W_\alpha$. Taking suitable products of these sections, we then have $u(U) \subset W$.

Using the various bundle trivializations above, one can show that the section $u$ is given by $x_{\alpha,\beta}^- \mapsto h_\alpha(\overline{s}_I)/g_\alpha(\overline{s}_I)$ and $x_{\alpha,\beta}^+ \mapsto h_\alpha(\overline{s}_J)/g_\alpha(\overline{s}_J))$. By Lemma 4.7, we have

$$\overline{s}_I - f_{I,K} \overline{s}_{I\cup J} = \overline{s}_J - f_{J,K} \overline{s}_{I\cup J}.$$

Applying $g_\alpha$ to this equation, and using $g_\alpha(\overline{s}_I + \overline{s}_J) = 1$, we have $g_\alpha(\overline{s}_I) = g_\alpha(\overline{s}_J) = 1/2$. Applying $h_\alpha$ to this equation, and using $h_\alpha(\overline{s}_I + \overline{s}_J) = 0$, we have $h_\alpha(\overline{s}_J) = -h_\alpha(\overline{s}_I) = (f_{J,K} - f_{I,K})/2 = x_\alpha/2$. Thus $u : U \to W$ is given by $x_{\alpha,\beta}^\pm \mapsto \pm x_\alpha$. \qed
5.10. Combining the trivialization of Lemma 5.5 with the embedding of Lemma 4.10, we can view $W$ as a subvariety of the affine space $A_{op} \times A_E \times A_{amb}$. It follows from Lemma 5.9 that the section $u: U \to W$ is the restriction of the inclusion $A_{op} \times A_E \to A_{op} \times A_E \times A_{amb}$ defined by $x_{\alpha, \beta}^\pm \mapsto \pm x_\alpha$; thus, $u(U)$ is defined set theoretically by the polynomials of Lemma 4.10 together with the linear polynomials $x_{\alpha, \beta}^+ - x_\alpha$ and $x_{\alpha, \beta}^- + x_\alpha$ for all $\beta \in \mathcal{H}$ and $\alpha \in \mathcal{E}(\beta)$.

5.11. By combining the embedding of 5.7 with 5.10, the blowup $W_\#$ (and hence $\tilde{U}$) can be regarded as a subvariety of $A_{op} \times A_E \times A_{amb} \times \prod_{\beta \in \mathcal{H}} P_{\beta}$.

It follows from the relations in 5.10 that $\tilde{U}$ (and hence its closure $\tilde{U}$) will be contained in the subvariety defined by $x_{\alpha, \beta}^\pm = \pm x_\alpha$ for all $\beta \in \mathcal{H}$ and $\alpha \in \mathcal{E}(\beta)$. Since the projection

$$A_{op} \times A_E \times A_{amb} \times \prod_{\beta \in \mathcal{H}} P_{\beta} \longrightarrow A_{op} \times A_E \times \prod_{\beta \in \mathcal{H}} P_{\beta}$$

is an isomorphism when restricted to this subvariety, the further restriction to $\tilde{U}$ defines a closed embedding $\tilde{U} \longrightarrow A_{op} \times A_E \times \prod_{\beta \in \mathcal{H}} P_{\beta}$.

The image of this embedding is cut out by the polynomials of Lemma 4.10 together with the multihomogeneous polynomials $y_{\alpha, \beta} x_{\alpha}^* - y_{\alpha^*, \beta} x_\alpha$, $\beta \in \mathcal{H}$, $\alpha, \alpha^* \in \mathcal{E}(\beta)$.

6. The core $Z$

6.1. In this section we use the embedding of 5.11 to show that $\tilde{U}$ is isomorphic to a 9-dimensional vector bundle over a certain 3-dimensional multi-projective variety.

Definition 6.2. Let $\tilde{U} \to \prod_{\beta} P_{\beta}$ be the composition of the embedding of 5.11 with the projection to the projective spaces. The image of this map will be called the core, and denoted $Z$. We let $\eta : \tilde{U} \to Z$ denote the induced map.
6.3. For each $k = 1, 2, 3$, we consider the projection $Z \to \mathbb{P}_{\Delta_k}$, and let $L_k \to Z$ be the pullback of the tautological line bundle. Since $\mathbb{P}_{\Delta_k} = \text{Proj} \, R_{\Delta_k}$ (see 5.6), $L_k$ is naturally a subvariety of $\text{Spec} \, R_{\Delta_k} \times Z$. By identifying the ring $R_\xi$ with $R_{\Delta_1} \otimes R_{\Delta_2} \otimes R_{\Delta_3}$ (via $y_{\alpha, \Delta_k} \mapsto x_\alpha$), we can identify the 3-dimensional bundle $L_1 \times_Z L_2 \times_Z L_3$ with a subvariety of $A_\xi \times Z$. Let $N \to Z$ be the 9-dimensional vector bundle obtained by taking the product (over $Z$) of the trivial bundle $A_{\text{op}} \times Z$ and the bundle $L_1 \times_Z L_2 \times_Z L_3$. There is a natural embedding

$$N \longrightarrow A_{\text{op}} \times A_\xi \times \prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta,$$

and it follows from the equations in 5.11 that $\tilde{U}$ is contained in the image. Thus, we have an embedding

$$\tilde{U} \longrightarrow N$$

whose composition with the bundle projection to $Z$ coincides with the map $\eta$.

6.4. To prove that $\tilde{U}$ coincides with $N$, we use the $G$-action on $X$. The stabilizer of the flag at infinity is the subgroup $B$ of $G$ consisting of lower triangular matrices. The group $B$ acts on the varieties $U^\circ$ and $U$ by the usual action on the Plücker coordinates. In this section, we describe a $B$-action on the bundle $N \subset A_{\text{op}} \times A_\xi \times Z$, with the property that the embedding $\tilde{U}^\circ \to N$ is $B$-equivariant.

The action on $Z$ is trivial. The action on $A_{\text{op}}$ is the usual action of the Borel on the corresponding big cell $U_{\text{op}}$ in the flag variety. The action on $A_\xi$ is given in terms of the characters $t_k : B \to \mathbb{C}^\times$ defined by $t_1(b) = b_{22}/b_{11}$, $t_2(b) = b_{33}/b_{22}$, and $t_3(b) = b_{44}/b_{33}$ where $b$ is the matrix $(b_{ij})$. For each $\alpha \in E_k$, the action of $b$ on $x_\alpha$ is then the diagonal action $x_\alpha \mapsto t_k(b)x_\alpha$. It is clear that $N$ is a $B$-stable subvariety of $A_{\text{op}} \times A_\xi \times Z$.

**Lemma 6.5.** The embedding $\tilde{U} \to N$ is a $B$-equivariant isomorphism.

**Proof.** Since the flag coordinates on $U$ are pulled back from the coordinates on the flag variety, the composition $\tilde{U}^\circ \to N \to A_{\text{op}}$ is equivariant. An explicit calculation using the sections of $L_\xi$ and the definition of the edge coordinates shows that the composition $\tilde{U}^\circ \to N \to A_\xi$ is equivariant. And finally, since for each $\beta \in \mathcal{H}$, the group $B$ acts via the same character on $x_\alpha$, for all $\alpha \in E(\beta)$, the induced action on each $\mathbb{P}_\beta$ will be trivial. It follows that $\tilde{U}^\circ$ embeds equivariantly into $N$; hence, so does its closure.

To see that the embedding is an isomorphism, we let $Z^\circ = \eta(\tilde{U}^\circ)$. Since $\tilde{U}$ is the closure of $\tilde{U}^\circ$ in $W_\sharp$, $Z^\circ$ is dense in $Z$. It follows from the description of the $B$-action that the unipotent subgroup of $B$ acts freely and transitively on $A_{\text{op}}$, and that the diagonal subgroup of $B$ acts fiberwise on the product of the complements of the zero sections in $L_1 \times_Z L_2 \times_Z L_3$. Thus $B$ acts with dense orbit on each fiber of $N$. Since each $x_\alpha$ is nonzero on the image of $\tilde{U}^\circ$ in $N$, the image of $\tilde{U}^\circ$ intersects this $B$-orbit
for every fiber of $N|_{Z^o} \to Z^o$. It follows that the image of $\tilde{U}^o$ is dense in $N$, so the image of its closure $\tilde{U}$ coincides with $N$. □

6.6. By Lemma 5.3 we know that $\tilde{U}$ is isomorphic to a vector bundle over $Z$, hence $\tilde{U}$ will be smooth if and only if $Z$ is smooth. We next show that nonsingularity of the core $Z$ follows from nonsingularity along a certain subvariety, called the locus of special points. We begin with some notation.

For any point $\tilde{p} \in \tilde{X}$, let $p$ be its image in $X$. We define the number of $k$-planes in $\tilde{p}$ by

$$n_k(\tilde{p}) := \text{Card}\{\pi_I(p) \mid I \subset \{1, 2, 3, 4\}, |I| = k\}.$$ 

The point $\tilde{p}$ is called split (resp. minimally split) if $n_k(\tilde{p}) \geq 2$ (resp. $n_k(\tilde{p}) = 2$) for all $k \leq 3$. A point $z \in Z$ is called special if there exists a minimally split point $\tilde{p} \in \tilde{U}$ with $\eta(\tilde{p}) = z$. We let $Z_{sp} \subset Z$ be the subvariety of special points.

**Proposition 6.7.** The split points are open in each fiber of $\eta: \tilde{U} \to Z$.

**Proof.** For any fiber, we can choose $\tilde{p} \in \tilde{U}$ whose $B$-orbit is open in that fiber. The description of the $B$-action in 6.4 therefore implies that for any $k \leq 3$, there will be some $\alpha \in E_k$ such that $x_\alpha \neq 0$ on the image of $\tilde{p}$ in $N$. But $x_\alpha \neq 0$ implies $\pi_I(p) \neq \pi_J(p)$, where $\alpha = \{I, J\}$. Therefore $\tilde{p}$ is split, and the result follows since $n_k$ is constant on $B$-orbits. □

**Proposition 6.8.** If $\tilde{p} \in \tilde{U}$ is split, then $\overline{G \cdot \tilde{p}} \cap \tilde{U}$ contains a minimally split point.

To prove the proposition we require some lemmas.

**Lemma 6.9.** Let $k \in \{1, 2, 3\}$ and let $F_1, F_2, F_3 \in \text{Gr}_k$ be three distinct points. Then there exists a one-parameter subgroup $\mu: \mathbb{C}^k \to G$ such that

$$\lim_{t \to 0} \mu(t) \cdot F_2 = F_1$$

and

$$\lim_{t \to 0} \mu(t) \cdot F_3 \neq F_1.$$

**Proof.** We can find a subspace $F_4 \subset \mathbb{C}^4$ such that $F_4 \oplus F_1 = F_4 \oplus F_2 = \mathbb{C}^4$, and such that $\dim F_4 \cap F_3 > 0$. Then for $\mu$ we can take any one-parameter subgroup that scales in $F_1$ with a negative weight and scales in $F_4$ with a positive weight. □

**Lemma 6.10.** Let $\tilde{p}, \tilde{s} \in \tilde{U}$ with $\tilde{p}$ split and $\tilde{s} \in G \cdot \tilde{p}$. Then there exists a split point $\tilde{r} \in G \cdot \tilde{p} \cap \tilde{U}$ such that $\tilde{s} \in B \cdot \tilde{r}$. Moreover, if $n_k(\tilde{s}) > 1$, then $n_k(\tilde{s}) = n_k(\tilde{r})$.

**Proof.** Let $W$ be the set of split points in $G \cdot \tilde{p}$. This set is open, and thus $\tilde{s}$ is in its closure. But since the $B$-orbit of any point in $W$ lies in $W$, the entire fiber $\eta^{-1}(\eta(\tilde{s}))$ must also be in the closure of $W$. Letting $\tilde{r}$ be a split point in this fiber completes the proof of the first statement. For the second statement, a look at the $B$-action shows that passing to a point in $\tilde{U}$ that is in an orbit closure either preserves $n_k$ or drops it down to 1. □
Proof of Proposition 6.8. We use the lemmas above to collapse the configuration associated to \( \tilde{p} \) so that only two subspaces of each dimension remain. We use implicitly that in passing to a point in the closure of a \( G \)-orbit, the number of planes in any given dimension cannot increase.

We begin with the subspaces of dimension 1. By assumption, \( n_1(\tilde{p}) > 1 \). If \( n_1(\tilde{p}) > 2 \), then we can use Lemma 6.3 to find \( \tilde{o} \in \overline{G \cdot \tilde{p}} \) such that \( n_1(\tilde{p}) = n_1(\tilde{o}) \geq 2 \). The orbit \( G \cdot \tilde{o} \) must lie in \( \overline{G \cdot \tilde{p}} \), and since \( G \) acts transitively on our charts that cover \( \tilde{X} \) (see 4.3), we can find a \( G \)-translate \( \tilde{s} \) of \( \tilde{o} \) such that \( \tilde{s} \in \overline{G \cdot \tilde{p} \cap \tilde{U}} \). Since the functions \( n_k \) are constant on \( G \)-orbits, we have \( n_1(\tilde{s}) = n_1(\tilde{o}) \).

Lemma 6.10 implies that we can find a split point \( \tilde{r} \in \overline{G \cdot \tilde{p}} \) with \( \tilde{s} \in B \cdot \tilde{r} \cap \tilde{U} \), and such that \( n_1(\tilde{r}) = n_1(\tilde{s}) \) is \( \geq 2 \) and is \( < n_1(\tilde{p}) \). Since any point in \( \overline{G \cdot \tilde{r}} \) is also in \( \overline{G \cdot \tilde{p}} \), we can repeat this procedure until we find a split point \( \tilde{p}_1 \in \overline{G \cdot \tilde{p} \cap \tilde{U}} \) with \( n_1(\tilde{p}_1) = 2 \).

Now we induct on \( k \) to produce points \( \tilde{p}_2 \) and \( \tilde{p}_3 \). The key point is that we can apply the lemmas to reduce \( n_k \) while preserving \( n_l \) for \( l \neq k \). Since the final point \( \tilde{p}_3 \in \overline{G \cdot \tilde{p} \cap \tilde{U}} \) is minimally split, this completes the proof.

6.11. Propositions 6.7 and 6.8 imply that nonsingularity of \( \tilde{U} \) follows from nonsingularity at minimally split points. Since \( \tilde{U} \) is a vector bundle over \( Z \), nonsingularity at the minimally split points follows from the nonsingularity of \( Z_{sp} \).

7. Nonsingularity

7.1. Recall that the core \( Z \) is a subvariety of

\[
\prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta \cong \mathbb{P}^5 \times \mathbb{P}^{11} \times \mathbb{P}^5 \times (\mathbb{P}^2)^4 \times (\mathbb{P}^2)^8 \times (\mathbb{P}^2)^4,
\]

where the index set \( \mathcal{H} \) corresponds to faces of the hypersimplices of dimension \( \geq 2 \). Each factor \( \mathbb{P}_\beta \) has homogeneous coordinates \( \{ y_{\alpha,\beta} \mid \alpha \in \mathcal{E}(\beta) \} \), corresponding to the edges of the face \( \beta \). By combining the polynomials of 6.11 with the polynomials defining \( U \) in 4.10, we obtain polynomials defining \( Z \).

Lemma 7.2. The subvariety \( Z \) of \( \prod_{\beta} \mathbb{P}_\beta \) is defined set-theoretically by the following multihomogeneous polynomials:

1. The linear polynomials

\[
y_{\alpha_1,\beta} - y_{\alpha_2,\beta} + y_{\alpha_3,\beta},
\]

where \( \alpha_1, \alpha_2, \alpha_3 \) are as in Lemma 4.10(item 1) and \( \beta \in \mathcal{H} \) is such that \( \alpha_1, \alpha_2, \alpha_3 \in \mathcal{E}(\beta) \).

2. The quadric polynomials

\[
y_{\alpha_1,\beta} y_{\alpha_2,\beta^*} - y_{\alpha_2,\beta} y_{\alpha_1,\beta^*},
\]

where \( \alpha_1 \) and \( \alpha_2 \) are any two edges that share a vertex and \( \beta, \beta^* \in \mathcal{H} \) are such that \( \alpha_1, \alpha_2 \in \mathcal{E}(\beta) \) and \( \alpha_1, \alpha_2 \in \mathcal{E}(\beta^*) \).
3. The quadric polynomials

\[ y_{\alpha_1, \beta} y_{\alpha_2, \beta} - y_{\alpha_2, \beta} y_{\alpha_1, \beta^*}, \]

where \( \alpha_1, \alpha_2, \alpha_1^*, \alpha_2^* \) are as in Lemma 4.10(item 2) and \( \beta, \beta^* \in \mathcal{H} \) are such that \( \alpha_1, \alpha_2 \in E(\beta) \) and \( \alpha_1^*, \alpha_2^* \in E(\beta^*) \).

4. The cubic polynomials

\[ y_{\alpha_1, \beta} y_{\alpha_2, \beta} y_{\alpha_3, \beta} - y_{\alpha_1^*, \beta} y_{\alpha_2^*, \beta} y_{\alpha_3^*, \beta}, \]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_1^*, \alpha_2^*, \alpha_3^* \) are as in Lemma 4.10(item 3) and \( \beta \) is the hypersimplex \( \Delta_2 \).

5. The quartic polynomials

\[ y_{\alpha_1, \beta} y_{\alpha_2, \beta} y_{\alpha_3, \beta} y_{\alpha_4, \beta} - y_{\alpha_1^*, \beta} y_{\alpha_2^*, \beta} y_{\alpha_3^*, \beta} y_{\alpha_4^*, \beta}, \]

where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^* \) are as in Lemma 4.10(item 4) and \( \beta, \beta^* \) are the hypersimplices \( \Delta_1, \Delta_3 \), respectively.

7.3. We represent points in \( \prod_{\beta} \mathbb{P}_\beta \) combinatorially using the graph \( \Gamma \) in Figure 6. The edges of \( \Gamma \) are in bijection with the variables \( y_{\alpha, \beta} \), and we encode a point in \( \prod_{\beta} \mathbb{P}_\beta \) by assigning values to the edges modulo \((\mathbb{C}^\times)^{19}\) (since \( \Gamma \) has 19 connected components). It will be convenient to abuse language slightly by identifying points in \( \prod_{\beta} \mathbb{P}_\beta \) with \( \Gamma \). In doing so we shall always assume that values have been assigned to the variables \( y_{\alpha, \beta} \).

![Figure 6. The graph \( \Gamma \)](image_url)

Let \( T \) and \( T^* \) be two triangular subgraphs of \( \Gamma \). Then \( T \) (respectively, \( T^* \)) corresponds to a choice of an ordered triangle and a choice of a face in \( \mathcal{H} \) to designate the component of \( \Gamma \) that contains \( T \) (resp., \( T^* \)). We shall say that \( T \) and \( T^* \) are related if one of the following two conditions holds:

1. The two ordered triangles for \( T \) and \( T^* \) coincide, or
2. The faces in Figure 2 that correspond to the two ordered triangles for \( T \) and \( T^* \) are in adjacent hypersimplices and one is a 180° rotated copy of the other.

In either case, there is a natural correspondence between the three edges of \( T \) and the three edges of \( T^* \), and we say that \( T \) and \( T^* \) have the same shape if the corresponding triples of values are proportional. Our calculations will involve the use of this notion...
together with setting various $y_{\alpha,\beta}$ to 0; we indicate the latter by marking in bold the corresponding edge of $\Gamma$.

As a first step towards showing that special points are nonsingular, we consider the equations in Lemma 7.2 and their meaning in the context of Figure 6.

**Lemma 7.4.** Suppose that $\Gamma$ represents a point in $Z$. Then any two related triangular subgraphs $T$ and $T^*$ have the same shape and they must appear as one of the five possibilities shown in Figure 7.

![Figure 7. Possible related triangles.](image)

**Proof.** The quadric relations in Lemma 7.2 imply that related triangles will have the same shape. Using this fact and the linear relations of Lemma 7.2, it is then easy to verify that the only combinations of zero values for such a pair are those shown in Figure 7.

**Proposition 7.5.** The subvariety $Z_{sp}$ consists of 66 isolated points and 4 subvarieties of positive dimension. Modulo the action of the symmetric group and the duality exchanging lines and 3-spaces, there are five types of points of $Z_{sp}$. In terms of $\Gamma$, these types appear in Figures 8–12. Figure 12 represents a point in the positive-dimensional locus.

**Proof.** Let $\tilde{p} \in \tilde{U}$ be minimally split, and let $p$ be its image in $U$. Then $n_k(\tilde{p}) = 2$ for $k \leq 3$, which implies $\text{Card}\{\pi_I(\tilde{p})\} = 6$. Up to symmetry a minimally split point must have its subspaces partitioned as follows:

1. The lines must collapse together as $(3, 1)$ or $(2, 2)$. (The notation $(p, q)$ means that the two distinct lines are the image of $p$ and $q$ lines, where $p + q = 4$.)
2. The 2-planes must collapse together as $(5, 1)$, $(4, 2)$, or $(3, 3)$.
3. The 3-planes, like the lines, must collapse together as $(3, 1)$ or $(2, 2)$.

With these facts and Figures 8 and 7 in hand, computing $Z_{sp}$ becomes a combinatorial exercise. We leave the pleasure of this computation to the reader.

Footnote: The labels of the figures refer to certain divisors in $Z$, cf. 2. The numbers in parentheses indicate how many of each type of component appear, without modding out by the action of $S_4$. The notation $(a + a)$ indicates that there are $2a$ components of this type; we have only depicted one of each dual pair.
Theorem 7.6. The core $Z$, and thus $\tilde{X}$, is nonsingular.

Proof. We apply the Jacobian condition for nonsingularity in an affine neighborhood of each point of $Z_{sp}$. Let $z = \eta(\tilde{p}) \in Z_{sp}$. Fix an affine neighborhood of $z$ in $\prod_{\beta \in H} \mathbb{P}_{\beta}$.
as follows. For each $\beta \in \mathcal{H}$, choose one $\alpha(\beta) \in \mathcal{E}(\beta)$ with $y_{\alpha(\beta),\beta} \neq 0$, and set this coordinate equal to 1. The remaining variables $\{y_{\alpha,\beta} \mid \alpha \neq \alpha(\beta)\}$ form a system of local parameters at $z$ in $\prod_{\beta} \mathbb{P}_{\beta}$.

Let $\Omega_z^1$ be the $\mathbb{C}$-vector space of differentials of $Z$ at the point $z$. Since $Z$ is a three-fold, we have $\dim \Omega_z^1 \geq 3$, with equality only if $Z$ is nonsingular at $z$. Furthermore,

$$\Omega_z^1 = \left( \bigoplus_{(\alpha,\beta)} \mathbb{C} \cdot dy_{\alpha,\beta} \right) / J,$$

where $J$ is the subspace generated by the differentials (evaluated at $z$) of all functions vanishing on $Z$. To study this quotient, we will use the following combinatorial rules for computing with differentials in $\Omega_z^1$. These follow immediately from the equations in Lemma 7.2; we omit the simple proof.

**Lemma 7.7.** Suppose that $y_{\alpha_1,\beta_1}y_{\alpha_2,\beta_2} - y_{\alpha_2,\beta_1}y_{\alpha_1,\beta_2}$ vanishes on $Z$.

1. If $y_{\alpha_1,\beta_1} = y_{\alpha_2,\beta_1} = y_{\alpha_1,\beta_2} = 0$ and $y_{\alpha_2,\beta_2} \neq 0$, then $dy_{\alpha_1,\beta_1} = 0$.
2. If $y_{\alpha_1,\beta_1} = y_{\alpha_1,\beta_2} = 0$ and $y_{\alpha_2,\beta_2} = y_{\alpha_2,\beta_1} \neq 0$, then $dy_{\alpha_1,\beta_1} = dy_{\alpha_1,\beta_2}$.

Using these rules, we add data for $\Omega_z^1$ to $\Gamma$ as in Figure 13. The 0 means that the differential of the variable corresponding to the edge is 0, and the two das indicate that the two differentials coincide in $\Omega_z^1$.

**Figure 13.** Rules for differentials.

First we verify nonsingularity at the isolated points of $Z_{sp}$. Since the computations for the various points are all very similar, we explain the case $DDE$ in detail and will leave the others to the reader. We fix the affine neighborhood of a point of type $DDE$ by assigning the value 1 to exactly one thin edge in each of the 19 components in Figure 8. Since the differentials of the linear polynomials are in $J$, the differential
corresponding to any thin edge is a linear combination of differentials corresponding to bold edges. Thus, $\Omega_1^z$ is generated by $dy_{\alpha,\beta}$, where $\alpha$ is a bold edge of the component $\beta$.

Consider the hypersimplex connected components of Figure 8. Applying Lemma 7.7 we find three independent differentials $da$, $db$, and $dc$ in these components; the other differentials in these components are 0. Now consider the other connected components of Figure 8. Using Lemma 7.7 we see that the remaining differentials are either 0 or are equal to $da$, $db$, or $dc$. The result is summarized in Figure 14. Hence $\Omega_1^z$ is 3-dimensional, and all the points of type DDE are nonsingular points of $Z$.

Figure 14. All differentials for DDE.

Finally consider the family of special points $CC^*_opD$ in Figure 12. In contrast to the isolated case, to verify nonsingularity we have to use the cubic and quartic polynomials of Lemma 7.2. Let $z \in Z$ be a point in a subvariety of type $CC^*_opD$. As before we construct an affine neighborhood of $z$ choosing a thin edge in each connected component of Figure 12 and setting it to 1. At the point $z$, the linear relations imply that all of the thin edges except those in the four thin triangles will also have value one. To complete the graph $\Gamma$ to represent the point $z$, we apply the quadratic relations to find $u, v \in \mathbb{C}$ such that the values are as in Figure 13 (up to the choice of the edges with values 1). Hence, a priori, this is a 2-dimensional component of $Z_{sp}$.

The quartic relations from Lemma 7.2, however, imply that the two parameters $u$ and $v$ satisfy a linear relation. For the choice of parameters in Figure 13, for example, this relation is

$$1 \cdot \left(u - \frac{1}{2}\right) \cdot 1 \cdot 1 = 1 \cdot \left(v - \frac{1}{2}\right) \cdot 1 \cdot 1, \quad \text{or} \quad u = v.$$  

Thus this component of $Z_{sp}$ is in fact a curve.

We now complete the proof of the theorem. As in the isolated case, the differentials on all thin edges, except for those in the four thin triangles, can be expressed as linear combinations of the differentials on bold edges. Moreover the differentials on edges of
the thin triangles can be expressed as linear combinations of $du$ and $dv$. Thus, using Lemma 7.7, we can find 8 differentials that span $\Omega_1^z$ (Figure 16):

$$da, db, dc, da^*, db^*, dc^*, du, dv.$$

Note that the span of these is at most 5-dimensional, because of the relations $da + db + dc = da^* + db^* + dc^* = 0$ induced by the differentials of the linear relations, and the relation $du = dv$ induced by the linear relation between $u$ and $v$.

To finish, we claim that the spans of $da, db, dc$ and $da^*, db^*, dc^*$ are each 1-dimensional. Indeed, a quadric relation implies that the front face of the octahedron in Figure 16 has the same shape as the corresponding face in Figure 15, which implies

$$db - (u - \frac{1}{2}) dc = 0.$$

This relation, and a similar one involving $db^*$ and $dc^*$, shows that $\dim \Omega_1^z = 3$. This completes the proof of the main theorem.
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