Darboux transformations and Fay identities for the extended bigraded Toda hierarchy

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Abstract

The extended bigraded Toda hierarchy (EBTH) is an integrable system satisfied by the Gromov–Witten total descendant potential of $\mathbb{CP}^1$ with two orbifold points. We write a bilinear equation for the tau-function of the EBTH and derive Fay identities from it. We show that the action of Darboux transformations on the tau-function is given by vertex operators. As a consequence, we obtain generalized Fay identities.

Keywords: Darboux transformation, extended bigraded Toda hierarchy, Fay identities, Lax operator, tau-function, wave function, wave operator

1. Introduction

The extended Toda hierarchy (ETH) was originally introduced by Getzler \cite{Getzler2000} and Zhang \cite{Zhang2005} in its bihamiltonian form, and later in its Lax form by Carlet et al \cite{Carlet2009}. It is obtained by adding an extra set of commuting flows to the 1D Toda hierarchy, which are given in terms of a ‘logarithm’ of the Lax operator. It was shown in \cite[corollary 1.3]{Getzler2002} that the Gromov–Witten total descendant potential of $\mathbb{CP}^1$ is a tau-function of the ETH (see also \cite{Getzler2000, Szendroi2008, Eynard2015}).

The extended bigraded Toda hierarchy (EBTH) was introduced by Carlet \cite{Carlet2011} as a generalization of the extended Toda hierarchy related to the Frobenius manifolds from \cite{Dubrovin2002}. The EBTH is defined for every pair $(k, m)$ of positive integers, and it coincides with the ETH for $k = m = 1$. The total descendant potential of $\mathbb{CP}^1$ with two orbifold points of orders $k$ and $m$ is a tau-function of the EBTH (see \cite{Getzler2002, Szendroi2008}). The EBTH contains the bigraded Toda hierarchy, which is a reduction of the 2D Toda hierarchy (see \cite{Zhang2002, Zhang2004}).

In this paper, we investigate how Darboux transformations of the EBTH affect the tau-function. Let us recall the Bäcklund–Darboux transformations and Fay identities of the KP...
hierarchy, following [2, 4, 29]. Let $\Psi$ be a wave function for the KP hierarchy and $\Psi_1 = \Psi|_{z = z_1}$ for some fixed $z_1$. Denote by $A_\Psi$ the differential operator defined by $A_\Psi, \Psi = \text{Wr}(\Psi, \Psi_1)/\Psi_1$, where $\text{Wr}$ is the Wronskian determinant. A Bäcklund–Darboux transformation maps the Lax operator $L$ to a new pseudo-differential operator $\bar{L} = A_\Psi \text{Lx}^{-1}$ and the wave function $\Psi$ to a new wave function $\bar{\Psi} = A_\Psi, \Psi$. Then we have $\bar{L}\bar{\Psi} = z\bar{\Psi}$, and $\bar{\Psi}$ is a wave function for the KP hierarchy. In this case, the tau-function $\bar{\tau}$ corresponding to the new solution is given by $\bar{\tau} = X(t, z_1)\tau$, where

$$X(t, z) = \exp\left(\sum_{j=1}^{\infty} t_j z^j\right) \exp\left(-\sum_{j=1}^{\infty} \frac{\partial}{j} z^{-j}\right)$$

is the so-called vertex operator. The proof of this theorem relies on the differential Fay identity, which is obtained by making a certain substitution in the bilinear equation for the KP hierarchy (see [2, lemma 2.1]). As a consequence of this result, one gets the generalized differential Fay identities for the KP hierarchy:

$$\text{Wr}(\Psi(t, z_1), \ldots, \Psi(t, z_N)) = \prod_{1 \leq i < j \leq N} (z_j - z_i)$$

$$\times \exp\left(\sum_{j=1}^{\infty} t_j (z_1^j + \cdots + z_N^j)\right) \frac{\tau(t - [z_1^{-1}] - \cdots - [z_N^{-1}])}{\tau(t)}.$$

where $[z^{-1}] = (z^{-1}, z^{-2}/2, z^{-3}/3, \ldots)$. Conversely, Takasaki and Takebe [38] showed that the Fay identities of the KP hierarchy imply the bilinear equation of the hierarchy. Teo proved that the same is true for the Fay identities of the 2D Toda hierarchy [39].

In this paper, we use the approach of Takasaki [37] to derive a bilinear equation for the EBTH, which is equivalent to the one from [13] after a change of variables. From this we obtain a difference Fay identity, similar to what was done in [36, 39] for the 2D Toda hierarchy. Some Fay identities for the EBTH were given in [26], but writing the Fay identity in our notation allows us to study the action of Darboux transformations on the tau-function. In [10], Carlet defined Darboux transformations on the wave functions for the ETHE, and in [27], Li and Song generalized them to the EBTH.

Our main result is that the action of Darboux transformations on the tau-function is given by applying the vertex operators

$$\Gamma_+(z) = e^{-\partial_+} \exp\left(\sum_{j=1}^{\infty} t_j z^j\right) \exp\left(-\sum_{j=1}^{\infty} \frac{\partial}{j} z^{-j}\right) = e^{-\partial_+} X(t, z)$$

and

$$\Gamma_-(z) = e^{\partial_-} \exp\left(-\sum_{j=1}^{\infty} t_j z^{-j}\right) \exp\left(\sum_{j=1}^{\infty} \frac{\partial}{j} z^j\right).$$

Thus, new tau-functions for the EBTH can be obtained from existing ones by applying a product of $\Gamma_+$ and $\Gamma_-$ evaluated at different values $z_i \in \mathbb{C}^\ast$. As an application, we derive generalized Fay identities for the EBTH.

Now let us summarize the contents of the present paper. In section 2, we start by reviewing difference operators and the extended bigraded Toda hierarchy (EBTH) following the approach of Takasaki [37]. Our version of the EBTH is related to the original definition of Carlet [11] (or to [13]) by an explicit change of variables, and we believe it is more convenient. We discuss the Lax operator $L$, the wave operators $W, \bar{W}$, wave functions $\psi, \bar{\psi}$, and tau-function $\tau$ of the EBTH.
In section 3, we give an explicit bilinear equation for the EBTH that is equivalent to the one from [13], in the notation introduced by Takasaki. We provide a shorter proof than what was done in [13]. From the bilinear equation written in this form, we get two difference Fay identities satisfied by the tau-function of the EBTH (see [36]). This is similar to what was done in [26], but we are following Takasaki’s notation.

In section 4, we review the Darboux transformations on \( L \) and \( \psi \) from [27]. We show that the action of the Darboux transformations on the tau-function is given by the vertex operators \( \Gamma_+(z) \) and \( \Gamma_-(z) \). This result is new even in the case \( k = m = 1 \) corresponding to the extended Toda hierarchy. We use it to conclude that new tau-functions can be found by acting on an existing tau-function \( \tau \) with a product of \( \Gamma_+(z) \) and \( \Gamma_-(z) \) for certain \( z \in \mathbb{C}^* \).

In section 5, we apply a sequence of \( N \) Darboux transformations and the vertex operators \( \Gamma_+(z), \Gamma_-(z) \) to derive generalized difference Fay identities for the EBTH.

Finally, section 6 contains concluding remarks and open questions.

2. Review of the extended bigraded Toda hierarchy

This section is a quick review of the EBTH following [8]. We first discuss the spaces of difference and differential-difference operators. Then we present a definition of the EBTH, its Lax operator, wave operators, wave functions, and tau-function.

2.1. Spaces of difference and differential-difference operators

Consider functions of a variable \( s \), and the shift operator \( \Lambda = e^h \) defined by \((\Lambda^f)(s) = f(s + 1)\).

The space \( \mathcal{A} \) of (formal) difference operators consists of all expressions of the form

\[
\mathcal{A} = \sum_{i \in \mathbb{Z}} a_i(s) \Lambda^i.
\]

We have \( \mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \) where \( \mathcal{A}_+ \) (respectively, \( \mathcal{A}_- \)) consists of \( A \in \mathcal{A} \) such that \( a_i = 0 \) for all \( i < 0 \) (respectively, \( i \geq 0 \)). For \( A \in \mathcal{A} \), we define its projections

\[
A_+ = \sum_{i > 0} a_i(s) \Lambda^i \in \mathcal{A}_+, \quad A_- = \sum_{i < 0} a_i(s) \Lambda^i \in \mathcal{A}_-.
\]

We let \( \mathcal{A}_{++} \) be the space of difference operators \( A \in \mathcal{A} \) such that \( a_i = 0 \) for \( i < 0 \) (i.e. the powers of \( \Lambda \) are bounded from below), and \( \mathcal{A}_{--} \) be the space of \( A \in \mathcal{A} \) such that \( a_i = 0 \) for \( i > 0 \) (i.e. the powers of \( \Lambda \) are bounded from above). Both \( \mathcal{A}_{++} \) and \( \mathcal{A}_{--} \) are associative algebras, where the product is defined by linearity and

\[
(a(s)\Lambda^i)(b(s)\Lambda^j) = a(s)b(s+i)\Lambda^{i+j}.
\]

Let \( \mathcal{A}_{\text{fin}} = \mathcal{A}_{++} \cap \mathcal{A}_{--} \). The product of a difference operator \( A \in \mathcal{A} \) by an element of \( \mathcal{A}_{\text{fin}} \) is defined, but in general, the product of an element of \( \mathcal{A}_{++} \) and an element of \( \mathcal{A}_{--} \) is not well defined.

We will also consider the space \( \mathcal{A}[\partial_s] \) of (formal) differential-difference operators, where \( \Lambda \partial_s = \partial_s \Lambda \). Note that such operators depend polynomially on \( \partial_s \). Again, there is a splitting \( \mathcal{A}[\partial_s] = \mathcal{A}_{++}[\partial_s] \oplus \mathcal{A}_{--}[\partial_s] \), and we have the associative algebras \( \mathcal{A}_{++}[\partial_s] \) and \( \mathcal{A}_{--}[\partial_s] \), where the product is defined by linearity and

\[
(a(s)\Lambda^i\partial_s^k)(b(s)\Lambda^j\partial_s^m) = \sum_{k=0}^{n} \binom{n}{k} a(s) \frac{\partial^kB}{\partial^K}(s+i)\Lambda^{i+j+k} \partial_s^{m+n-k}.
\]
Differential-difference operators can be applied to \( z' \) so that
\[
(a(s)\Lambda^n)z' = a(s)z'[(\log z)^n]z'.
\]

2.2. The extended bigraded Toda hierarchy

For fixed, positive integers \( k \) and \( m \), consider a Lax operator of the form
\[
L = \Lambda^k + u_{k-1}(s)\Lambda^{k-1} + \cdots + u_m(s)\Lambda^{-m} \in \mathcal{A}_{kn}, \quad u_{-m}(s) \neq 0.
\]

There exist wave operators (also called dressing operators):
\[
W = 1 + \sum_{i=1}^{\infty} w_i(s)\Lambda^{-i} \in 1 + \mathcal{A}_- \subset \mathcal{A}_- -\subset,
\]
\[
\bar{W} = \sum_{i=0}^{\infty} \bar{w}_i(s)\Lambda^i \in \mathcal{A}_+,
\]
\[
\bar{w}_0(s) \neq 0,
\]
\[
(L^n) = (L^n)^m = L^n, \quad n \in \mathbb{Z}_{\geq 0}.
\]

However, observe that \( L^k \neq L^n \), unless \( \frac{n}{k} = \frac{p}{q} \in \mathbb{Z}_{\geq 0} \). We define \( \log L \in \mathcal{A} \) by
\[
\log L = -\frac{1}{2} W\partial_s W^{-1} - \frac{1}{2} \bar{W}\partial_s \bar{W}^{-1} = -\frac{1}{2} \frac{\partial W}{\partial s} W^{-1} + \frac{1}{2} \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}.
\]

Then \( \log L \) commutes with all \( L^n \) for \( n \in \mathbb{Z}_{\geq 0} \), but the composition of \( \log L \) with a fractional power of \( L \) is not well defined in general.

**Definition 2.1 ([11]).** The extended bigraded Toda hierarchy (abbreviated EBTH) in Lax form is given by:
\[
\partial_s L = [(L^n)_{+}, L], \quad n \geq 1,
\]
\[
\partial_s L = [(L^n)_{-}, L], \quad n \geq 1,
\]
\[
\partial_{s}^2 L = [(2L^n \log L)_{+}, L], \quad n \geq 0.
\]

The first two equations in (2.4) describe the bigraded Toda hierarchy, which is a reduction of the 2D Toda hierarchy (see [37, 40]). For \( k = m = 1 \), the EBTH is equivalent to the extended Toda hierarchy (ETH) [12, 37].

The flows of the EBTH induce flows on the dressing operators:
\[
\partial_s W = -(L^n)_{-} W, \quad \partial_{s} W = (L^n)_{+} W,
\]
\[
\partial_s \bar{W} = -(L^n)_{-} \bar{W}, \quad \partial_{s} \bar{W} = (L^n)_{+} \bar{W},
\]
\[
\partial_{s}^2 W = -(2L^n \log L)_{-} W, \quad \partial_{s}^2 \bar{W} = (2L^n \log L)_{+} \bar{W}.
\]

(2.5)
Remark 2.2. Since $\partial_s = 0$ and $\partial_{\bar{t}} = 0$, it follows that $L$, $W$, and $\bar{W}$ are trivially depend on $x_0 + s$ and $t_{nk} + \bar{t}_{nm}$ for $n \geq 1$. Without loss of generality, we can assume $x_0 = s$ and $t_{nk} = \bar{t}_{nm}$.

Remark 2.3. To compare our version of the EBTH to the one from [13], we need to change there $\epsilon \mapsto -\epsilon$, which leads to $\Lambda \mapsto \Lambda - 1$ and $\zeta \mapsto \zeta^{-1}$ ($z$ here), and then apply the following change of variables:

$x = \epsilon s,$

$q_{n}^{k-\alpha} = \epsilon k \left( n + \frac{\alpha}{k} \right) \tau_{nk}, \quad \alpha = 1, 2, \ldots, k - 1,$

$q_{n}^{k+\beta} = \epsilon m \left( n + \frac{\beta}{m} \right) \tau_{nm}, \quad \beta = 1, 2, \ldots, m - 1,$

$q_{n}^{k+m} = \epsilon (n + 1)! \left( \tau_{(n+1)k} + \tau_{(n+1)nm} + c_{n+1} \left( \frac{1}{k} + \frac{1}{m} \right) x_{k+1} \right),$

$q_{n}^{k} = \epsilon n! x_{n}, \quad n \geq 0.$

Here $c_{n}$ are the harmonic numbers

$c_{0} = 0, \quad c_{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k},$

and $(p)_{n}$ denotes the Pochhammer symbol,

$(p)_{0} = 1,$

$(p)_{n} = \prod_{i=1}^{n} (p - i + 1), \quad n \geq 1,$

$(p)_{-n} = \prod_{i=-n+1}^{0} (p - i + 1)^{-1} = \frac{1}{(p + n)!}.$

Due to remark 2.2, from now on we will always assume $x_0 = s$ and $t_{nk} = \bar{t}_{nm}$ for $n \geq 1$.

Introduce the notation

$t = (t_1, t_2, \ldots), \quad \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots), \quad x = (x_1, x_2, \ldots),$

and

$\xi(t, z) = \sum_{i=1}^{\infty} t_i z^i, \quad \xi_{k}(t, z) = \sum_{n=1}^{\infty} t_{nk} z^{nk} = \sum_{n=1}^{\infty} \tau_{nk} z^{nk}.$

Then

$\xi_{m}(\bar{t}, z^{-1}) = \sum_{n=1}^{\infty} \tau_{nm} z^{-nm} = \sum_{n=1}^{\infty} t_{nm} z^{-nm}.$

We let

$\chi = z^{x} \xi(x, z) e^{\xi(t, z) - \frac{1}{2} \xi_{k}(t, z)}$

$\bar{\chi} = z^{x+n} \xi(x, z^{-1}) e^{-\xi(t, z^{-1}) + \frac{1}{2} \xi_{m}(t, z^{-1})}.$
Observe that, by definition,
\[ \begin{align*}
\partial_t \chi &= z^l \chi, \\
\partial_x \chi &= z^{-l} \bar{\chi}, \\
\partial_{x^k} \chi &= \partial_{x_m} \bar{\chi} = \frac{1}{2} z^{nk} \chi, \\
\partial_{x^m} \bar{\chi} &= \partial_{x_m} \chi = -\frac{1}{2} z^{-nm} \bar{\chi}.
\end{align*} \tag{2.7} \]

The wave functions \( \psi \) and \( \bar{\psi} \) of the EBTH are defined by:
\[ \begin{align*}
\psi &= \psi(s, t, t, x, z) = W \chi = w \chi, \\
\bar{\psi} &= \bar{\psi}(s, t, t, x, z) = \bar{W} \bar{\chi} = \bar{w} \bar{\chi},
\end{align*} \tag{2.8} \]

where
\[ w = 1 + \sum_{i=1}^{\infty} w_i(s) z^{-i}, \quad \bar{w} = \sum_{j=0}^{\infty} \bar{w}_j(s) z^{j} \tag{2.9} \]
are the (left) symbols of \( W \) and \( \bar{W} \) respectively. Here we view \( w \) and \( \bar{w} \) as formal power series of \( z^{-1} \) and \( z \); however, in section 4 below we will assume that \( w(z) \) is convergent for \( z \) in some domain \( U \subset \mathbb{C} \).

The wave functions satisfy
\[ \begin{align*}
L \psi &= z^l \psi, \\
L \bar{\psi} &= z^{-m} \bar{\psi}. \tag{2.10}
\end{align*} \]

We have:
\[ \begin{align*}
\partial_t \psi &= (L^L)^{+}_+ \psi, & n &\in \mathbb{Z}_{\geq 0} \setminus kz, \\
\partial_t \psi &= -(L^L)^{-}_- \psi, & n &\in \mathbb{Z}_{\geq 0} \setminus mZ, \\
\partial_{x^k} \psi &= \partial_{x_m} \bar{\psi} = A_n \psi, & n &\in \mathbb{Z}_{\geq 1}, \\
\partial_{x^m} \psi &= (L^n \partial_t + P_n) \psi, & n &\in \mathbb{Z}_{\geq 0},
\end{align*} \tag{2.11} \]
and exactly the same equations hold for \( \bar{\psi} \), where
\[ A_n = \frac{1}{2} (L^n)^+ - \frac{1}{2} (L^n)^- = (L^n)^+ - \frac{1}{2} L^n = \frac{1}{2} L^n - (L^n)^- \tag{2.12} \]
and
\[ P_n = -(L^n \frac{\partial W}{\partial s} W^{-1})^+ - (L^n \frac{\partial W}{\partial s} W^{-1})^- = L^n W \partial_t W^{-1} - (2L^n \log L)^- - L^n \partial_t \]
\[ = L^n W \partial_t W^{-1} + (2L^n \log L)^+ - L^n \partial_t. \tag{2.13} \]

Observe that, due to (2.1) and (2.3), we have
\[ \begin{align*}
(L^L)^+ \ , \ (L^L)^- \ , \ A_n , \ P_n \in \mathcal{A}_{\text{fin}}, \quad P_0 = 0.
\end{align*} \]

Finally, by [13], there exists a tau-function \( \tau \) such that
\[ \begin{align*}
\psi(s, t, t, x, z) &= \frac{\tau(s, t - [z^{-1}], x)}{\tau(s, t, t, x)} \chi, \tag{2.14} \\
\bar{\psi}(s, t, t, x, z) &= \frac{\tau(s + 1, t, t + [z], x)}{\tau(s, t, t, x)} \bar{\chi}. \tag{2.15}
\end{align*} \]
where
\[ [z^{-1}] = \left( z^{-1}, \frac{z^{-2}}{2}, \frac{z^{-3}}{3}, \ldots \right), \quad [z] = \left( z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots \right). \]

**Remark 2.4.** Since \( t_{ak} = t_{km}, \) we need to specify how to do the shifts \( t - [z^{-1}] \) in (2.14) and \( t + [z] \) in (2.15). Here and further, our convention is that in (2.14), \( t - [z^{-1}] \) includes all variables \( t_1, t_2, \ldots, \) while \( t \) only includes \( i \) such that \( m \nmid i. \) Similarly, in (2.15), all \( i, t_2, \ldots \) are shifted, while \( t \) only includes \( i \) such that \( k \nmid i. \)

Given a tau-function \( \tau \) of the EBTH, one finds the wave function \( \psi \) by (2.14). The expansion of \( \psi \) as in (2.8) and (2.9) determines the wave operator \( W \) from (2.1), which in turn determines the Lax operator \( L \) by (2.2). Note that the tau-function \( \tau \) is determined only up to multiplication by a nonzero function of \( x, \) since both \( \tau \) and \( f(x)\tau \) give the same wave functions \( \psi, \bar{\psi} \) via (2.14) and (2.15).

### 3. Bilinear equation for the EBTH

In this section, we derive a bilinear equation for the EBTH using Takasaki’s approach from [37]. Our equation is equivalent to the bilinear equation from [13], and when \( k = m = 1 \) it reduces to the bilinear equation for the ETH from [37]. As a consequence, we obtain two difference Fay identities satisfied by tau-functions of the EBTH.

#### 3.1. Dual wave functions

Recall that the formal adjoint of a difference operator \( A = \sum_{i \in \mathbb{Z}} a_i(s) \Lambda^i \in \mathcal{A} \) is defined by
\[ A^* = \sum_{i \in \mathbb{Z}} \Lambda^{-i} \circ a_i(s) = \sum_{i \in \mathbb{Z}} a_i(s - i) \Lambda^{-i}. \]

It has the properties:
\[ (AB)^* = B^*A^*, \quad (A^*)^* = A, \quad (A^{-1})^* = (A^*)^{-1}. \]

For given wave operators \( W \) and \( W', \) we define the dual wave functions \( \psi^* \) and \( \bar{\psi}^* \) by:
\begin{align*}
\psi^* &= (W^*)^{-1}\chi^{-1} = (W^*)^{-1}z^{-x-\xi(xz^2)}e^{-\xi(txz+\frac{1}{2}x^2t^2)}, \\
\bar{\psi}^* &= (W^*)^{-1}\chi^{-1} = (W^*)^{-1}z^{-x-\xi(xz^{-m})}e^{\xi(txz^{-1})-\frac{1}{2}x^2t^2}.
\end{align*}
\ \ (3.1)

If \( W \) and \( W' \) satisfy (2.2), then it is easy to derive equations satisfied by \( \psi^* \) and \( \bar{\psi}^*. \) For example, we have (see (2.10)):
\[ L^* \psi^* = z^k \psi^*, \quad L^* \bar{\psi}^* = z^{-m} \bar{\psi}^*. \ \ (3.2)\]

We will not list all the other equations, which are similar to (2.11), but we will need the following lemma.

**Lemma 3.1.** For every solution of the EBTH, the dual wave functions satisfy
\begin{align*}
(\partial_n - z^k \partial_{n}) \psi^* &= -P^n_{k} \psi^*, \\
(\partial_{n} - z^{-m} \partial_{n}) \bar{\psi}^* &= -P^n_{m} \bar{\psi}^*,
\end{align*}

for all \( n \in \mathbb{Z}_{\geq 1}, \) where \( P_n \) is given by (2.13).
Proof. First, since \((\partial_{\alpha} - z^{nk} \partial_k) \chi = 0\), we have
\[
(\partial_{\alpha} - z^{nk} \partial_k) \psi^* = \left( (\partial_{\alpha} - z^{nk} \partial_k) (W^*)^{-1} \right) \chi^{-1}.
\]
Using (2.5), we find
\[
\partial_{\alpha}^{-1} = -W^{-1}(\partial_{n} W) W^{-1} = W^{-1}(2L^n \log L)_-, \quad \partial_{\alpha}^{-1} = \partial_{n} W^{-1} = W^{-1}(2 \log L)_-.
\]
Note that taking formal adjoint commutes with taking derivative with respect to \(x_n\), because the latter is done coefficient by coefficient. Hence,
\[
(\partial_{\alpha} - z^{nk} \partial_k) (W^*)^{-1} = (2L^n \log L)_- - z^{nk} (2 \log L)_- (W^*)^{-1}.
\]
Then using (3.2), (2.13) and the fact that \(P_0 = 0\), we obtain
\[
(\partial_{\alpha} - z^{nk} \partial_k) \psi^* = (2L^n \log L)_- - L^n (2 \log L)_- (W^*)^{-1} \psi^* = -P_n^a \psi^*.
\]
The second equation of the lemma is proved in the same way. \(\square\)

3.2. Bilinear equation for the wave functions

The next result provides bilinear equations satisfied by the wave functions and dual wave functions of the EBTH.

**Theorem 3.2.** The wave functions \(\psi = W \chi\) and \(\bar{\psi} = W \bar{\chi}\) solve the EBTH if and only if they satisfy the bilinear equation
\[
\oint DS \frac{dz}{2\pi i} z^{nk} \psi(s' - \xi(a, z'), t', \bar{t}', x + a, z) \bar{\psi}^*(s - \xi(b, z), t, \bar{t}, x + b, z)
\]
\[
= \oint DS \frac{dz}{2\pi i} z^{-nm} \bar{\psi}(s' - \xi(a, z^{-m}), t', \bar{t}', x + a, z)
\]
\[
\times \psi^*(s - \xi(b, z^{-m}), t, \bar{t}, x + b, z)
\]
for all \(a = (a_1, a_2, \ldots)\), \(b = (b_1, b_2, \ldots)\), \(n \in \mathbb{Z}_{\geq 0}\) and \(s - s' \in \mathbb{Z}\).

**Remark 3.3.** By Taylor expansions of \(\psi\) and \(\bar{\psi}\) about \(t' = t, \bar{t}' = \bar{t}\), the bilinear equation (3.3) is equivalent to:
\[
\oint DS \frac{dz}{2\pi i} (\partial_{\alpha} \partial_{\beta}^3 \psi(s' - \xi(a, z'), t, \bar{t}, x + a, z)) \bar{\psi}^*(s - \xi(b, z), t, \bar{t}, x + b, z)
\]
\[
= \oint DS \frac{dz}{2\pi i} (\partial_{\alpha} \partial_{\beta}^3 \bar{\psi}(s' - \xi(a, z^{-m}), t, \bar{t}, x + a, z))
\]
\[
\times \psi^*(s - \xi(b, z^{-m}), t, \bar{t}, x + b, z)
\]
for all multi-indices \(\alpha, \beta\), where \(\partial_{\alpha} = \partial_{t_1}^{\alpha_1} \partial_{t_2}^{\alpha_2} \cdots\) and \(\partial_{\beta}^3 = \partial_{t_1}^{\beta_1} \partial_{t_2}^{\beta_2} \cdots\).
Remark 3.4. By taking a linear combination of equation (3.3) for different $n \in \mathbb{Z}_{\geq 0}$, we can replace $z^{nk}$ by $f(z^k)$ on the left side of (3.3) and $z^{-nm}$ by $f(z^{-m})$ on the right side, for any formal power series $f(z) \in \mathbb{C}[[z]]$.

The following lemma from [34] will be useful in the proof of the above theorem. In this lemma and below, we will use the notation $(A)_j = a_j(s)$ for the coefficient of $\Lambda^j$ in a difference operator $A = \sum_{j \in \mathbb{Z}} a_j(s) \Lambda^j$.

Lemma 3.5. Let $A$ and $B$ be difference operators such that the product $BA^*$ is well defined. Then

\[
(BA^*)_j = \oint \frac{dz}{2\pi i} (\Lambda^j \Lambda z^j)(Bz^{-j}), \quad j \in \mathbb{Z}.
\]

In particular, suppose that $\Lambda, \tilde{B}$ are two other difference operators such that $\tilde{B} \Lambda^*$ is well defined. Then

\[
\oint \frac{dz}{2\pi i} (\Lambda^j \Lambda z^j)(Bz^{-j}) = \oint \frac{dz}{2\pi i} (\Lambda^j \Lambda z^j)(Bz^{-j})
\]

for all $j \in \mathbb{Z}$, if and only if $BA^* = \tilde{B} \Lambda^*$.

Proof of theorem 3.2. First, following the approach of [37], we will prove that the equations of the EBTH imply the bilinear equation (3.3). By (2.8), (3.1) and lemma 3.5, we have

\[
\oint \frac{dz}{2\pi i} (\Lambda^j \psi(s, t, \bar{t}, x, z)) \psi^*(s, t, \bar{t}, x, z)
= \oint \frac{dz}{2\pi i} (\Lambda^j \psi(s, t, \bar{t}, x, z)) \psi^*(s, t, \bar{t}, x, z)
\]

for all $j \in \mathbb{Z}$. Therefore,

\[
\oint \frac{dz}{2\pi i} \psi(s', t, \bar{t}, x, z) \psi^*(s, t, \bar{t}, x, z)
= \oint \frac{dz}{2\pi i} \psi(s', t, \bar{t}, x, z) \psi^*(s, t, \bar{t}, x, z)
\]

for all $s', s$ with $s - s' \in \mathbb{Z}$.

Now applying $L^2$ as a difference operator with respect to $s'$ to both sides of (3.5) and using (2.10), we obtain

\[
\oint \frac{dz}{2\pi i} z^{nk} \psi(s', t, \bar{t}, x, z) \psi^*(s, t, \bar{t}, x, z)
= \oint \frac{dz}{2\pi i} z^{-mn} \psi(s', t, \bar{t}, x, z) \psi^*(s, t, \bar{t}, x, z)
\]

for all $n \in \mathbb{Z}_{\geq 0}$ and $s - s' \in \mathbb{Z}$. Recall that the action of the derivatives with respect to $t$ and $\bar{t}$ on the wave functions is given by difference operators (see (2.11)). We can apply the generating function $\exp(\sum_{n=1}^{\infty} c_n \partial_n)$ to $\psi$ and $\tilde{\psi}$ in the above equation, thus shifting $t$ by a constant $c$. Let us denote $t + c$ by $t'$. Doing the same for $\bar{t}$, we get
\[
\int \frac{dz}{2\pi i} z^{nk} \psi(s', t', \tilde{t}', x, z) \psi^*(s, t, \tilde{t}, x, z)
= \int \frac{dz}{2\pi i} z^{-nm} \tilde{\psi}(s', t', \tilde{t}', x, z) \tilde{\psi}^*(s, t, \tilde{t}, x, z).
\] (3.7)

Notice that, by (2.10) and (2.11),
\[
(\partial_\xi - z^k \partial_\xi) \psi = Q_\xi \psi,
(\partial_\xi - z^{-m} \partial_\xi) \tilde{\psi} = Q_\xi \tilde{\psi}, \quad Q_\xi = P_\xi - \partial(\xi^*) \frac{\partial}{\partial s},
\]
where \(P_\xi\) is given by (2.13). We can apply the difference operator \(Q_\xi\) to the variable \(s'\) on both sides of (3.7) to obtain
\[
\int \frac{dz}{2\pi i} z^{nk} \left((\partial_\xi - z^k \partial_\xi) \psi(s', t', \tilde{t}', x, z)\right) \psi^*(s, t, \tilde{t}, x, z)
= \int \frac{dz}{2\pi i} z^{-nm} \left((\partial_\xi - z^{-m} \partial_\xi) \tilde{\psi}(s', t', \tilde{t}', x, z)\right) \tilde{\psi}^*(s, t, \tilde{t}, x, z)
\]
for all \(n \geq 0, \ell \geq 1\). Using the generating function
\[
\exp\left(\sum_{\ell=1}^{\infty} a_\ell (\partial_\xi - z^k \partial_\xi)\right) \psi(s', t', \tilde{t}', x, z)
= \psi(s' - \xi(a, z^k), t', \tilde{t}', x + a, z),
\]
we get
\[
\int \frac{dz}{2\pi i} z^{nk} \psi(s' - \xi(a, z^k), t', \tilde{t}', x + a, z) \psi^*(s, t, \tilde{t}, x, z)
= \int \frac{dz}{2\pi i} z^{-nm} \tilde{\psi}(s' - \xi(a, z^{-m}), t', \tilde{t}', x + a, z) \tilde{\psi}^*(s, t, \tilde{t}, x, z).
\]
Similarly, by acting with \(-P_\xi^*\) on \(s\) in both sides of this equation and using lemma 3.1, we obtain the bilinear equation (3.3).

Conversely, we have to prove that if \(\psi\) and \(\tilde{\psi}\) satisfy the bilinear equation (3.3), then they obey the equations of the EBTH. More precisely, suppose that the functions
\[
\psi = W\chi, \quad \psi^* = T\chi^{-1}, \quad \tilde{\psi} = W\tilde{\chi}, \quad \tilde{\psi}^* = T\tilde{\chi}^{-1}
\]
satisfy (3.3), where \(W, W, T, T\) are difference operators such that
\[
W, T^* \in 1 + \mathcal{A}_{-}, \quad W, T^* \in \mathcal{A}_{+}
\]
(see (2.1), (2.6), (2.8) and (3.1)). Then we will prove that \(\psi, \tilde{\psi}\) are the wave functions and \(\psi^*, \tilde{\psi}^*\) are the dual wave functions of a solution of the EBTH.

First, setting \(a = b = 0, t = t', \tilde{t} = \tilde{t}'\) in (3.3), we obtain (3.6) as a special case. Then putting \(n = 0\) gives (3.5), and equivalently, (3.4). By lemma 3.5, equation (3.4) implies that \(TW^* = TW^*\). Since \((TW^*)^* = WT^* \in 1 + \mathcal{A}_{-}\) and \((TW^*)^* = WT^* \in \mathcal{A}_{+}\), we conclude that
\[ T = (W^*)^{-1}, \quad T = (W^*)^{-1}, \]

and (3.1) holds.

Second, we define \( L = W^a W^{-1} \) and want to prove (2.2). Notice that \( L\psi = W^a W^{-1} W \chi = z^1 \psi \). Applying \( L \) with respect to \( s' \) to both sides of (3.5) and using (3.6) for \( n = 1 \), we get

\[
\oint \frac{dz}{2\pi i} \left( L\bar{\psi}(s', t, \iota, x, z) \right) \bar{\psi}^*(s, t, \iota, x, z) = \oint \frac{dz}{2\pi i} z^{-m} \bar{\psi}(s', t, \iota, x, z) \bar{\psi}^*(s, t, \iota, x, z).
\]

For \( s' = s + j \) with \( j \in \mathbb{Z} \), we have:

\[
L\bar{\psi}(s', t, \iota, x, z) = \Lambda L \bar{\chi},
\]

\[
z^{-m} \bar{\psi}(s', t, \iota, x, z) = \Lambda W^a \Lambda^{-m} \bar{\chi}.
\]

From lemma 3.5, it follows that

\[
(W^*)^{-1} (L\bar{\psi})^* = (W^*)^{-1} (\Lambda W^a \Lambda^{-m} \bar{\chi})^*.
\]

This simplifies to \( L = \Lambda W^a \Lambda^{-m} W^{-1} \), thus proving (2.2) and (2.10).

Next, we will show that we can identify \( t_{nk} \) with \( t_{nm} \) in \( L \), \( W \) and \( \bar{W} \) for \( n \in \mathbb{Z}_{\geq 1} \) (see remark 2.2). Observe that, by (2.7) and (2.8),

\[
\frac{\partial \psi}{\partial t_{nk}} = \frac{\partial W}{\partial t_{nk}} \chi + \frac{1}{2} z^k W \chi, \quad \frac{\partial \bar{\psi}}{\partial t_{nk}} = \frac{\partial W}{\partial t_{nk}} \bar{\chi} - \frac{1}{2} z^{-n} W \bar{\chi},
\]

\[
\frac{\partial \psi}{\partial t_{nm}} = \frac{\partial W}{\partial t_{nm}} \chi + \frac{1}{2} z^k W \chi, \quad \frac{\partial \bar{\psi}}{\partial t_{nm}} = \frac{\partial W}{\partial t_{nm}} \bar{\chi} - \frac{1}{2} z^{-n} W \bar{\chi},
\]

hence,

\[
\left( \frac{\partial}{\partial t_{nk}} - \frac{\partial}{\partial t_{nm}} \right) \psi = \left( \frac{\partial W}{\partial t_{nk}} - \frac{\partial W}{\partial t_{nm}} \right) \chi,
\]

\[
\left( \frac{\partial}{\partial t_{nk}} - \frac{\partial}{\partial t_{nm}} \right) \bar{\psi} = \left( \frac{\partial W}{\partial t_{nk}} - \frac{\partial W}{\partial t_{nm}} \right) \bar{\chi}.
\]

By remark 3.3, we can apply \( \partial_{ts} - \partial_{sw} \) to \( \psi \) and \( \bar{\psi} \) in the bilinear equation (3.5) to obtain

\[
\oint \frac{dz}{2\pi i} \left( \left( \frac{\partial W}{\partial t_{ts}} - \frac{\partial W}{\partial t_{sw}} \right) \chi(s') \right) (W^*)^{-1} \chi^{-1}(s) = \oint \frac{dz}{2\pi i} \left( \left( \frac{\partial W}{\partial t_{ts}} - \frac{\partial W}{\partial t_{sw}} \right) \bar{\chi}(s') \right) (W^*)^{-1} \bar{\chi}^{-1}(s)
\]

for \( s - s' \in \mathbb{Z} \). Using lemma 3.5 as before, we get

\[
(W^*)^{-1} \left( \frac{\partial W}{\partial t_{ts}} - \frac{\partial W}{\partial t_{sw}} \right) = (W^*)^{-1} \left( \frac{\partial W}{\partial t_{ts}} - \frac{\partial W}{\partial t_{sw}} \right)^*.
\]
or equivalently,
\[
\left( \frac{\partial W}{\partial t_{nk}} - \frac{\partial W}{\partial t_{nm}} \right) W^{-1} = \left( \frac{\partial W}{\partial l_{nk}} - \frac{\partial W}{\partial l_{nm}} \right) \tilde{W}^{-1}.
\]

By (2.1), the left-hand side of this equation lies in \( A_\pm \), while the right-hand side in \( A_\mp \). Therefore, both sides vanish.

To finish the proof of the theorem, it is left to show that if \( \psi \) and \( \bar{\psi} \) satisfy the bilinear equation (3.3), then they satisfy (2.11). First, consider the derivatives with respect to \( t_{nk} \) and \( \bar{t}_{nm} \) for \( n \in \mathbb{Z} \geq 1 \). As above, we have
\[
\frac{\partial \psi}{\partial t_{nk}} = \frac{\partial W}{\partial t_{nk}} \chi + \frac{1}{2} \tilde{\varepsilon}^{nk} W \chi = \frac{\partial W}{\partial t_{nk}} \chi + \frac{1}{2} L^n \tilde{W} \chi,
\]
which implies
\[
\left( \frac{\partial}{\partial t_{nk}} - A_n \right) \psi = \left( \frac{\partial W}{\partial t_{nk}} + (L^n)_- W \right) \chi,
\]
where \( A_n \) is given by (2.12). Similarly,
\[
\left( \frac{\partial}{\partial t_{nk}} - A_n \right) \bar{\psi} = \left( \frac{\partial \bar{W}}{\partial t_{nk}} - (L^n)_+ \bar{W} \right) \bar{\chi}.
\]

We can apply the operator \( \partial_{t_{nk}} - A_n \) to \( \psi \) and \( \bar{\psi} \) in the bilinear equation (3.5). By lemma 3.5 again, we obtain
\[
\frac{\partial W}{\partial t_{nk}} = \frac{\partial \tilde{W}}{\partial t_{mk}} = -(L^n)_- W, \quad \frac{\partial W}{\partial t_{nk}} = \frac{\partial \bar{W}}{\partial t_{mn}} = (L^n)_+ \tilde{W},
\]
as claimed.

Next, let \( n \) be such that \( k \) does not divide \( n \). Using (2.10), we get
\[
\left( \frac{\partial}{\partial n} - (L^n)_+ \right) \psi = \frac{\partial W}{\partial n} \chi + \tilde{\varepsilon}^n W \chi - (L^n)_+ W \chi = \frac{\partial W}{\partial n} + (L^n)_- \tilde{W} \chi,
\]
and similarly,
\[
\left( \frac{\partial}{\partial n} - (L^n)_+ \right) \bar{\psi} = \frac{\partial \bar{W}}{\partial n} - (L^n)_+ \tilde{W} \bar{\chi}.
\]
Applying the operator \( \partial_{l_{nk}} - (L^n)_+ \) to \( \psi \) and \( \bar{\psi} \) in (3.5) and using lemma 3.5 gives
\[
\frac{\partial W}{\partial n} + (L^n)_- W = \frac{\partial \bar{W}}{\partial n} - (L^n)_+ \tilde{W} = 0.
\]

Finally, consider the derivatives with respect to the logarithmic variables \( x_n \). By (2.13) and \( \psi = W \chi \), we see that
\[
\left( \frac{\partial}{\partial x_n} - (L^n \partial_t + P_n) \right) \psi \\
= \frac{\partial W}{\partial x_n} \chi + W \frac{\partial \chi}{\partial x_n} - \left( L^n W \partial_s W^{-1} - (2L^n \log L) - \right) \psi \\
= \frac{\partial W}{\partial x_n} \chi + \zeta^n \log(z) W \chi - \zeta^n \log(z) W \chi + (2L^n \log L) - W \chi \\
= \left( \frac{\partial W}{\partial x_n} + (2L^n \log L) - W \right) \chi.
\]

Similarly,
\[
\left( \frac{\partial}{\partial x_n} - (L^n \partial_t + P_n) \right) \bar{\psi} = \left( \frac{\partial \bar{W}}{\partial x_n} - (2L^n \log L) + \bar{W} \right) \bar{\chi}.
\]

Applying the operator \( \frac{\partial}{\partial x_n} - (L^n \partial_t + P_n) \) to \( \psi \) and \( \bar{\psi} \) in (3.5) gives
\[
\oint \frac{dz}{2\pi i} \left( \left( \frac{\partial W}{\partial x_n} + (2L^n \log L) - W \right) \chi(s') \right) (W^*)^{-1} \chi^{-1}(s) \\
= \oint \frac{dz}{2\pi i} \left( \left( \frac{\partial \bar{W}}{\partial x_n} - (2L^n \log L) + \bar{W} \right) \bar{\chi}(s') \right) (\bar{W}^*)^{-1} \bar{\chi}^{-1}(s).
\]

By lemma 3.5, this implies
\[
\left( \frac{\partial W}{\partial x_n} + (2L^n \log L) - W \right) W^{-1} = \left( \frac{\partial \bar{W}}{\partial x_n} - (2L^n \log L) + \bar{W} \right) \bar{W}^{-1}.
\]

Since the left side is in \( A_- \) and the right side is in \( A_+ \), both sides must vanish. This completes the proof of theorem 3.2.

3.3. Bilinear equation for the tau-function

In this subsection, we will derive a bilinear equation satisfied by the tau-function \( \tau \) of the EBTH. Recall that the wave functions \( \psi \) and \( \bar{\psi} \) can be expressed in terms of \( \tau \) by (2.14) and (2.15). Next, we do it for the dual wave functions defined by (3.1).

**Proposition 3.6.** The dual wave functions \( \psi^* \) and \( \bar{\psi}^* \) of the EBTH can be expressed in terms of the tau-function \( \tau \) as follows:
\[
\psi^*(s, t, \hat{t}, x, z) = \frac{\tau(s, t + [z^{-1} \hat{t}], x)}{\tau(s, t, \hat{t}, x)} \chi^{-1}, \\
(3.8)
\]
\[
\bar{\psi}^*(s, t, \hat{t}, x, z) = \frac{\tau(s - 1, t, \hat{t} - [z], x)}{\tau(s, t, \hat{t}, x)} \bar{\chi}^{-1}, \\
(3.9)
\]

where we use the convention of remark 2.4.

**Proof.** Let us write
\[
\psi = w \chi, \quad \bar{\psi} = \bar{w} \bar{\chi}, \quad \psi^* = w^* \chi^{-1}, \quad \bar{\psi}^* = \bar{w}^* \bar{\chi}^{-1}.
\]
for some functions $w, \bar{w}, w^*, \bar{w}^*$ (see (2.8) and (3.1)). Setting $s' = s, a = b = 0$ in the bilinear equation (3.3), we get

$$
\int \frac{dz}{2\pi i} z^{nk} e^{\xi'(t'-t_0) - \frac{1}{2} \xi_1 (t'-t_0)} w(s', t', t, x, z) w^*(s, t, t, x, z) = \int \frac{dz}{2\pi i} z^{-nm} e^{-\xi(t'-t_0) - \frac{1}{2} \xi_1 (t'-t_0)} \bar{w}(s', t', t, x, z) \bar{w}^*(s, t, t, x, z).
$$

According to remark 3.4, we can replace $z^{nk}$ in the left-hand side by $f(z^k)$, and $z^{-nm}$ in the right-hand side by $f(z^{-m})$, for any $f(z) \in \mathbb{C}[[z]]$. If we do it for

$$
f(z^k) = e^{\frac{1}{2} \xi_1 (t'-t_0)} = \exp \left( \frac{1}{2} \sum_{n=1}^{\infty} (t'_{nk} - t_{nk}) z^{nk} \right) = \exp \left( \frac{1}{2} \sum_{n=1}^{\infty} (\bar{t}_{nm} - \bar{t}_{mn}) z^{nk} \right),
$$

then $f(z^{-m}) = e^{\frac{1}{2} \xi_1 (t'-t_0)^{-1}}$, and we obtain

$$
\int \frac{dz}{2\pi i} e^{\xi'(t'-t_0)} w(s', t', t, x, z) w^*(s, t, t, x, z) = \int \frac{dz}{2\pi i} \exp \left( - \sum_{m|i} (\bar{t}'_m - \bar{t}_m) z^{-1} \right) \bar{w}(s', t', t, x, z) \bar{w}^*(s, t, t, x, z).
$$

Now setting $t'_m = \bar{t}_m$ for $m \nmid i, t' = t + [u^{-1}]$ and using

$$
\xi([u^{-1}], z) = \sum_{i=1}^{\infty} \frac{u^{-i}}{i} z^i = - \log \left( 1 - \frac{z}{u} \right), \quad (3.10)
$$

we get

$$
\int \frac{dz}{2\pi i} \left( 1 - \frac{z}{u} \right)^{-1} w(s, t + [u^{-1}], t, x, z) w^*(s, t, t, x, z) = \int \frac{dz}{2\pi i} \bar{w}(s, t + [u^{-1}], t, x, z) \bar{w}^*(s, t, t, x, z).
$$

Notice that $\bar{w}$ and $\bar{w}^*$ are formal power series of $z$, while $w - 1$ and $w^* - 1$ are formal power series of $z^{-1}$ (see (2.9)). Hence, the right-hand side of this equation vanishes, and by Cauchy’s formula, the left-hand side is

$$
u \left( w(s, t + [u^{-1}], t, x, u) w^*(s, t, t, x, u) - 1 \right) = 0.
$$

From this and (2.14), we can derive (3.8). Equation (3.9) is proved similarly. \hfill \Box

**Theorem 3.7.** A function $\tau$ is a tau-function of the EBTH if and only if it satisfies the following bilinear equation:
\[
\int \frac{dz}{2\pi i} e^{\mu t - \mu' t'} \tau \left( \frac{z}{z^{-1}} \right) \times \tau \left( s' - \xi(a,z^k), t' + [z^{-1}], x' + a \right) \\
\times \tau \left( s - \xi(b,z^{k'}), t + [z^{-1}], x + b \right)
\]
\[
= \int \frac{dz}{2\pi i} e^{\mu t - \mu' t'} \tau \left( \frac{z}{z^{-1}} \right) \times \tau \left( s' + 1 - \xi(a,z^{-m}), t' + [z], x + a \right) \\
\times \tau \left( s - 1 - \xi(b,z^{-m}), t, x + b \right)
\]
(3.11)

for all \( a = (a_1, a_2, \ldots) \), \( b = (b_1, b_2, \ldots) \), \( n \in \mathbb{Z}_{\geq 0} \) and \( s - s' \in \mathbb{Z} \).

**Proof.** First, we plug in (3.3) the expressions for \( \psi, \tilde{\psi}, \psi^*, \tilde{\psi}^* \) in terms of \( \tau \) (see (2.14), (2.15), (3.8), (3.9)). Then, by remark 3.4, we can replace \( z^m \) on the left-hand side of (3.3) by

\[
z^m \tau \left( s' - \xi(a,z^k), t', x + a \right) \tau \left( s - \xi(b,z^{k'}), t, x + b \right),
\]

and \( z^{-m} \) on the right-hand side by

\[
z^{-m} \tau \left( s' - \xi(a,z^{-m}), t', x + a \right) \tau \left( s - \xi(b,z^{-m}), t, x + b \right).
\]

Therefore, (3.3) is equivalent to (3.11).

If we apply the change of variables from remark 2.3, we get the bilinear equation from [13] (see (85)–(87) there) as a special case of (3.11) after setting \( x = 0, a = x', b = x'' \) in (3.11). Conversely, we can obtain (3.11) from the bilinear equation of [13] by observing that if \( \tau(s,t,x) \) is a tau-function for the EBTH, then so is \( \tau(s,t,x + c) \) for any constant \( c \).

### 3.4. Two difference Fay identities for the EBTH

From theorem 3.7, we can derive the following difference Fay identities for the EBTH (see [36]). We will again use the shift convention of remark 2.4.

**Theorem 3.8.** If \( \tau \) is a tau-function of the EBTH, then for any \( \lambda, \mu \in \mathbb{C}^* \), we have

\[
(\lambda - \mu) \tau(s,t,x) \tau(s - 1, t - [\lambda^{-1}] - [\mu^{-1}], x) \\
= \lambda \tau(s,t - [\lambda^{-1}], x) \tau(s - 1, t - [\mu^{-1}], x) \\
- \mu \tau(s,t - [\mu^{-1}], x) \tau(s - 1, t - [\lambda^{-1}], x)
\]
(3.12)

and

\[
(\lambda - \mu) \tau(s + 1, t, [\lambda] + [\mu], x) \tau(s, t, x) \\
= \lambda \tau(s + 1, t + [\lambda], x) \tau(s, t, x) \\
- \mu \tau(s + 1, t + [\mu], x) \tau(s, t, x)
\]
(3.13)

**Proof.** Using the same trick as in the proof of proposition 3.6, we can rewrite the bilinear equation (3.11) as
\[
\int \frac{dz}{2\pi i} z^{n_k s} e^{z(t - \mu z)} \tau(s' - \zeta(a, z^k), t' - [z^{-1}], \bar{t}, x + a) \\
\times \tau(s - \zeta(b, z^k), t + [z^{-1}], \bar{t}, x + b) = \int \frac{dz}{2\pi i} e^{-m n s - k s'} \exp(-\sum_{m \neq i} (\bar{t}' - \bar{t}) z^{-i}) \\
\times \tau(s' + 1 - \zeta(a, z^{-m}), t', \bar{t}' + [z], x + a) \\
\times \tau(s - 1 - \zeta(b, z^{-m}), t, \bar{t} - [z], x + b).
\]

Then setting
\[n = 0, \quad s' - s = 1, \quad a = b = 0, \quad t' = \tau + [\lambda^{-1}] + [\mu^{-1}], \quad \bar{t}' = \bar{t},\]
for \(m \nmid i\) gives
\[
\int \frac{dz}{2\pi i} \frac{z}{(1 - z\lambda^{-1})(1 - z\mu^{-1})} \\
\times \tau(s + 1, \tau + [\lambda^{-1}] + [\mu^{-1}] - [z^{-1}], \bar{t}, x) \tau(s, \tau + [z^{-1}], \bar{t}, x) = \int \frac{dz}{2\pi i} z \tau(s + 2, \tau + [\lambda^{-1}] + [\mu^{-1}] + [z], \bar{t}, x) \tau(s - 1, \tau - [z], \bar{t}, x) = 0.
\]

To compute the residue in the left side, we use
\[
\frac{z}{(1 - z\lambda^{-1})(1 - z\mu^{-1})} = \frac{1}{\lambda^{-1} - \mu^{-1}} \left( \frac{1}{1 - z\lambda^{-1}} - \frac{1}{1 - z\mu^{-1}} \right)
\]
and
\[
\int \frac{dz}{2\pi i} \frac{f(z)}{1 - z\lambda^{-1}} = \lambda(f(\lambda) - f_0), \quad \text{if} \quad f(z) = \sum_{i=0}^{\infty} f_i z^{-i}.
\]

We obtain
\[
(\lambda - \mu) \tau(s + 1, \tau + [\lambda^{-1}] + [\mu^{-1}], \bar{t}, x) \tau(s, \tau + [\lambda^{-1}], \bar{t}, x) \\
- \lambda \tau(s + 1, \tau + [\mu^{-1}], \bar{t}, x) \tau(s, \tau + [\lambda^{-1}], \bar{t}, x) \\
+ \mu \tau(s + 1, \tau + [\lambda^{-1}], \bar{t}, x) \tau(s, \tau + [\mu^{-1}], \bar{t}, x) = 0,
\]
which gives (3.12) after the shift \(s \mapsto s - 1, \tau \mapsto \tau - [\lambda^{-1}] - [\mu^{-1}]\). Equation (3.13) can be proved similarly, by making the substitution \(s' = s - 1, \bar{t}' = \bar{t} - [\lambda] - [\mu] \) and \(t' = t_i \) for \(k \nmid i\) in (3.11).

3.5. Fay identities for the ETH

In this subsection, we discuss Fay identities for the ETH (when \(k = m = 1\)). First, we introduce the notation
\[
F_1(s', t', x'; s, t, x) = \tau(s', t', x')\tau(s - 1, t - [\lambda^{-1}] - [\mu^{-1}], x),
\]
\[
F_2(s', t', x'; s, t, x) = \tau(s', t' - [\lambda^{-1}], x')\tau(s - 1, t - [\mu^{-1}], x),
\]
\[
F_3(s', t', x'; s, t, x) = \tau(s', t' - [\mu^{-1}], x')\tau(s - 1, t - [\lambda^{-1}], x).
\]

Then for the ETH the Fay identity (3.12) reduces to
\[
((\lambda - \mu)F_1 - \lambda F_2 + \mu F_3) \bigg|_{s'=s, t'=t, x'=x} = 0.
\]

To write a Fay identity involving shifts in the logarithmic variables, we let \(k = m = 1, n = 0, s' - s = 1, t' = t + [\lambda^{-1}] + [\mu^{-1}],\) and \(a = (a_1, 0, 0, \ldots)\) and \(b = 0\) in (3.11). Then through a similar calculation, we obtain
\[
\left((\lambda - \mu)(\lambda + \mu)\partial_{\nu} - \partial_{t_1} + \partial_{\nu}(\partial_{t_1} - \partial_{t_2})\right)F_1 + \lambda(\partial_{t_1} - \lambda\partial_{\nu})F_2 - \mu(\partial_{t_1} - \mu\partial_{\nu})F_3 \bigg|_{s'=s, t'=t, x'=x} = 0.
\]

This process can be generalized to obtain Fay identities involving derivatives with respect to the other variables \(x_r\). It is possible to do this for the EBTH as well, but the formulas become too complicated to list them here.

4. Darboux transformations on the tau-function

In this section, we first review the action of Darboux transformations on the Lax operator and wave function of the EBTH. Then, using the Fay identity (3.12), we determine the action of Darboux transformations on the tau-function.

4.1. Darboux transformations

We first review the notion of a Darboux transformation of an ordinary differential or difference operator \(L\) (see, e.g. [2, 4, 20, 22, 29, 30]). Let \(\psi\) be an eigenfunction of \(L\) and \(P_\psi\) be an ordinary differential or difference operator of order 1 such that \(P_\psi \psi = 0\); in the differential case
\[
P_\psi = \partial \frac{\partial (\psi)}{\psi},
\]
and in the difference case we let
\[
P_\psi = I - \frac{\psi}{\Lambda^{-1}(\psi)} \Lambda^{-1},
\]
where \(I\) denotes the identity operator (see [10]). Then a Darboux transformation of \(L\) sends it to the new operator \(\tilde{L} = P_\psi LP_\psi^{-1}\). If \(L\psi = z\psi\), then the operator \(L - zI\) factors as \(QP_\psi\) for some differential or difference operator \(Q\) (see [23, section 5.4] and [9, theorem 2.3]). Hence,
\[
\tilde{L} = P_\psi LP_\psi^{-1} = zI + P_\psi Q.
\]

Darboux transformations are useful because they send all eigenfunctions of \(L\) to eigenfunctions of \(\tilde{L}\). Indeed, if \(L\phi = \lambda\phi\), then \(\tilde{L}\phi = P_\psi \phi\) satisfies \(\tilde{L}\phi = \lambda\phi\).

We will perform iterated Darboux transformations in the following setting. Suppose that \(\psi(z)\) is a family of eigenfunctions of \(L\), so that \(L\psi(z) = z\psi(z)\), where the spectral parameter \(z\)
belongs in some open set $\mathcal{U} \subset \mathbb{C}$. Then for a fixed $z_1 \in \mathcal{U}$, we have an eigenfunction $\psi_1 = \psi(z_1)$ and the corresponding operator $P_1^{[i]} = P_{\psi_1}$ given by (4.1) or (4.2) with $\psi_1$ in place of $\psi$. As above, we have $L\psi_1 = z_1\psi_1$ and $P_1^{[i]}\psi_1 = 0$, which give

$$L - z_1I = Q_1^{[i]}P_1^{[i]}$$

for some operator $Q_1^{[i]}$. Then the Darboux transformation is

$$L \rightarrow L_1^{[i]} = P_1^{[i]}L(P_1^{[i]})^{-1} = z_1I + P_1^{[i]}Q_1^{[i]}.$$

The new operator $L_1^{[i]}$ has a family of eigenfunctions $\psi_1^{[i]}(z) = P_1^{[i]}\psi(z)$ such that $L^{[i]}\psi_1^{[i]}(z) = z_1\psi_1^{[i]}(z)$ for $z \in \mathcal{U}$. After picking $z_2 \in \mathcal{U}$, we can make another Darboux transformation by starting from $L^{[i]}$, $\psi_1^{[i]}(z)$ and $z_1$ in place of $L$, $\psi_1(z)$ and $z_1$, respectively. In this way, we obtain $L^{[2]}$ and $\psi_1^{[2]}(z)$ satisfying $L^{[2]}\psi_1^{[2]}(z) = z_2\psi_1^{[2]}(z)$, and so on. In many important cases, the eigenfunction $\psi^{[N]}(z)$ obtained after $N$ Darboux transformations can be given in terms of a (discrete) Wronskian of the initial eigenfunction $\psi(z)$; see [22, 30] and the next subsection.

### 4.2. Darboux transformations of the Lax operator of the EBTH

Darboux transformations for the ETH were first considered by Carlet in [10], and a generalization to the EBTH was given in [27]. In order to state the result, we need to introduce some notation. We define the discrete Wronskian (or Casoratian) of functions $f_i = f_i(s)$ by

$$\text{Wr}_{\Lambda}(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \Lambda^{-1}(f_1) & \Lambda^{-1}(f_2) & \cdots & \Lambda^{-1}(f_n) \\ \vdots & \vdots & \cdots & \vdots \\ \Lambda^{-n+1}(f_1) & \Lambda^{-n+1}(f_2) & \cdots & \Lambda^{-n+1}(f_n) \end{vmatrix}.$$

Let $L$ be a Lax operator for the EBTH with a wave function $\psi$. We will suppose that $\mathcal{U} \subset \mathbb{C}$ is an open set such that the wave function $\psi(s, t, i, x, z)$ is defined for $z \in \mathcal{U}$, i.e. the formal power series $w(z)$ from (2.9) is convergent for $z \in \mathcal{U}$. Fix values $z_1, \ldots, z_N \in \mathcal{U}$. Recall that $L\psi(z) = \frac{\partial^2}{\partial s^2}\psi$ for all $z \in \mathcal{U}$ (see (2.10)). Starting from $L$, $\psi$ and $z_1$, we use the eigenfunction $\psi_1 = \psi(z_1)$, of $L$ to get a Darboux transformation $L \rightarrow L^{[i]} = P_1^{[i]}L(P_1^{[i]})^{-1}$ as in the previous subsection, where $P_1^{[i]}$ is given by (4.2):

$$P_1^{[i]} = I - \frac{\psi_1}{\Lambda^{-1}(\psi_1)}\Lambda^{-1}.$$

Then

$$L - z_1I = Q_1^{[i]}P_1^{[i]}, \quad L^{[i]} - z_1I = P_1^{[i]}Q_1^{[i]}$$

for some difference operator $Q_1^{[i]}$.

We have $L^{[i]}\psi_1^{[i]}(z) = z_1\psi_1^{[i]}(z)$ where

$$\psi_1^{[i]} = P_1^{[i]}\psi_1$$

and

$$\psi_1^{[i]} = \text{Wr}_{\Lambda}(\psi_1, \psi) \Lambda^{-1}(\psi_1) = \text{Wr}_{\Lambda}(\psi_1, \psi) \Lambda^{-1}(\psi_1)).$$

Using the eigenfunction $\psi_1^{[i]}(z)_{z=z_2} = P_1^{[i]}\psi_1(z)_{z=z_2}$ of $L^{[i]}$, we make another Darboux transformation to get a new operator $L^{[2]}$ with an eigenfunction $\psi^{[2]}$, and so on. The following theorem is
equivalent to theorem 3.4 from [27] and gives a formula for the wave function, $\psi^{[N]}$, and Lax operator, $L^{[N]}$, after $N$ iterations of the Darboux transformation.

**Theorem 4.1 ([27]).** Let $\psi$ be a wave function for the EBTH and $L$ its corresponding Lax operator. For fixed $N \geq 1$ and $z_1, \ldots, z_N \in \mathbb{U}$, consider the difference operator $P$ defined by

$$ Pf = (-1)^N \frac{W_{\Lambda}(\psi_1, \ldots, \psi_N, f)}{W_{\Lambda}(\Lambda^{-1}(\psi_1), \ldots, \Lambda^{-1}(\psi_N))}, $$

where $\psi_i = \psi|_{z=z_i}$. Then

$$ L^{[N]} = P \Lambda^{-1} P^{-1}, \quad \psi^{[N]} = P \psi $$

are a Lax operator and wave function for the EBTH, which are obtained from $L$ and $\psi$ after $N$ Darboux transformations.

The fact that $L^{[N]}$ and $\psi^{[N]}$ are again solutions of the EBTH is one of the claims of theorem 4.1 (see also [41]). We refer to [41] for more details and for a proof of theorem 4.1 different from that of [27].

### 4.3. Action of Darboux transformations on $\tau$

Using theorem 4.1 and the Fay identity (3.12), we will prove that the action of a Darboux transformation on the tau-function is given by the vertex operator

$$ \Gamma_+(z) = e^{-\theta_1} e^{\xi(z)} \exp \left( - \sum_{n=1}^{\infty} \frac{\theta_n}{n} z^{-n} \right). \quad (4.4) $$

Note that $\exp \left( - \sum_{n=1}^{\infty} \frac{\theta_n}{n} z^{-n} \right)$ acts as the shift operator $t \mapsto t - [z^{-1}]$, while $e^{-\theta_1} = \Lambda^{-1}$ acts as the shift $s \mapsto s - 1$.

**Theorem 4.2.** Let $\psi$ be a wave function for the EBTH, and $\psi^{[1]}$ be the wave function after one Darboux transformation on $\psi$ (see (4.3)). Let $\tau$ and $\tau^{[1]}$ be their corresponding tau-functions. Then $\tau^{[1]} = \Gamma_+(z_1) \tau$, i.e.

$$ \tau^{[1]}(s, t, \bar{t}, x) = e^{\xi(z_1)} \tau(s - 1, t - [z_1^{-1}], \bar{t}, x). \quad (4.5) $$

**Proof.** Using (4.3) and (2.14) and $\Lambda^{-1}(\chi) = z^{-1} \chi$, we express $\psi^{[1]}$ in terms of $\tau$ as follows:

$$ \psi^{[1]} = \frac{\chi}{\tau(s, t, \bar{t}, x) \tau(s - 1, t - [z_1^{-1}], \bar{t}, x)} \times \left( \tau(s, t - [z^{-1}], \bar{t}, x) \tau(s - 1, t - [z_1^{-1}], \bar{t}, x) \right. $$

$$ - \left. z^{-1} z_1 \tau(s, t - [z_1^{-1}], \bar{t}, x) \tau(s - 1, t - [z^{-1}], \bar{t}, x) \right). \quad (4.6) $$

On the other hand, again by (2.14),

$$ \psi^{[1]} = \frac{\tau^{[1]}(s, t - [z_1^{-1}], \bar{t}, x)}{\tau^{[1]}(s, t, \bar{t}, x)} \chi. $$
Substituting $\tau^{[1]} = \Gamma_+(z_1) \tau$ into the right side of this equation gives
\begin{equation}
\frac{(1 - z^{-1} z_1) \tau(s - 1, t - [z_1^{-1}])}{\tau(s - 1, t - [z_1^{-1}], \bar{t}, x)} \chi,
\end{equation}
where we used that, by (3.10),
\begin{equation}
e^{\xi(t-[z^{-1}],z_1)} = e^{\xi(t,z_1)} e^{-\xi([z^{-1}],z_1)} = e^{\xi(t,z_1)}(1 - z^{-1} z_1).
\end{equation}
If we set $\lambda = z$, $\mu = z_1$ in the Fay identity (3.12), we see that the above two expressions (4.6) and (4.7) are equal. Therefore, $\tau^{[1]} = \Gamma_+(z_1) \tau$.

If we do $N$ Darboux transformations of $\tau$, we can apply theorem 4.2 repeatedly to obtain the tau-function
\begin{equation}
\tau^{[N]} = \Gamma_+(z_N) \cdots \Gamma_+(z_2) \Gamma_+(z_1) \tau,
\end{equation}
which corresponds to the Lax operator $L^{[N]}$ and wave function $\psi^{[N]}$ from theorem 4.1. The product of vertex operators in (4.9) is well known (see, e.g. [24, chapter 14]) and easy to compute using (4.5) and (4.8). It follows that
\begin{equation}
\tau^{[N]}(s, t, \bar{t}, x) = V_N e^{\sum_{i=1}^N \xi(t, z_i)} \tau(s - N, t - [z_1^{-1}] - \cdots - [z_N^{-1}], \bar{t}, x),
\end{equation}
where
\begin{equation}
V_N = \prod_{1 \leq i < j \leq N} \left(1 - \frac{z_i}{z_j}\right).
\end{equation}
One can verify directly that, for any tau-function $\tau$ of the EBTH, the function $\tau^{[N]}$ given by (4.10) satisfies the bilinear equation (3.11) and hence is a tau-function of the EBTH as well.

**Remark 4.3.** The authors of [27] also give Darboux transformations on the second wave function, which is denoted $\bar{\psi}$ here. In this case, the action of the Darboux transformation on the tau-function is given by the vertex operator
\begin{equation}
\Gamma_-(z) = e^\theta e^{-\xi(t, z^{-1})} \exp\left(\sum_{n=1}^\infty \frac{\partial_n}{n} z^n\right).
\end{equation}
The proof of this claim is very similar to the proof of theorem 4.2 and uses (3.13) instead of (3.12); see [41].

As above, one can also use the bilinear equation (3.11) to show directly that if $\tau$ is a tau-function for the EBTH, then $\Gamma_-(z_1) \tau$ is as well. We conclude that
\begin{equation}
\Gamma_\epsilon(z_N) \cdots \Gamma_\epsilon(z_1) \tau
\end{equation}
is a tau-function for the EBTH for any choice of signs $\epsilon_j = \pm$ (see [24, chapter 14]). It would be interesting to understand whether suitable linear combinations of the vertex operators $\Gamma_+(z_1)$ and $\Gamma_-(z_1)$ also produce a new tau-function when applied to $\tau$; however, so far we have not been able to do this.
5. Generalized Fay identities

In this section, as an application of theorems 4.1 and 4.2, we derive generalized difference Fay identities for the EBTH (see [2] for the case of KP hierarchy). We will continue to use the notation of section 4.

**Theorem 5.1.** Let ψ be a wave function for the EBTH with a corresponding tau-function τ, and let ψ1 = ψ|z=z1. Then

\[
Wr_A(ψ1, \ldots, ψ_N) = \chi_1 \cdots \chi_N \prod_{1 \leq i < j \leq N} (z_j^{-1} - z_i^{-1})
\]

\[
\times \frac{\tau(s - N + 1, t - [z_1^{-1}] - \cdots - [z_N^{-1}], \bar{\iota}, x)}{\tau(s - N + 1, t, t, x)},
\]

(5.1)

where \( χ_i = χ|z=z_i \).

In this theorem, \( z_1, \ldots, z_N \) are complex numbers in a certain domain \( \mathcal{D} \subset \mathbb{C} \), in which \( ψ \) is defined. Alternatively, equation (5.1) makes sense as an identity of formal power series in \( z_1^{-1}, \ldots, z_N^{-1} \), if we write \( ψ = wz \) for a formal power series \( w \) in \( z^{-1} \) (see (2.9)), while the exponents in \( χ \) are not expanded.

**Proof of theorem 5.1.** We will prove the claim by induction on \( N \). The case \( N = 1 \) reduces to (2.14) for \( z = z_1 \), since \( Wr_A(ψ_1) = ψ_1 \). Now suppose that (5.1) holds for some \( N \geq 1 \).

By theorem 4.1, we have

\[
ψ^{[N]} = (-1)^N \frac{Wr_A(ψ_1, \ldots, ψ_N, ψ)}{Wr_A(\Lambda^{-1}(ψ_1), \ldots, \Lambda^{-1}(ψ_N))}.
\]

After setting \( z = z_{N+1} \), we obtain

\[
ψ^{[N]}|_{z=z_{N+1}} = (-1)^N \frac{Wr_A(ψ_1, \ldots, ψ_N, ψ_{N+1})}{Wr_A(\Lambda^{-1}(ψ_1), \ldots, \Lambda^{-1}(ψ_{N+1}))}.
\]

By the inductive assumption, the denominator is given by (5.1) after shifting \( s \mapsto s - 1 \):

\[
Wr_A(\Lambda^{-1}(ψ_1), \ldots, \Lambda^{-1}(ψ_{N+1})) = z_1^{-1} \cdots z_N^{-1} \chi_1 \cdots \chi_N \prod_{1 \leq i < j \leq N} (z_j^{-1} - z_i^{-1})
\]

\[
\times \frac{\tau(s - N + 1, t - [z_1^{-1}] - \cdots - [z_N^{-1}], \bar{\iota}, x)}{\tau(s - N + 1, t, t, x)}.
\]

On the other hand, again by (2.14),

\[
ψ^{[N]}(s, t, t, x, z) = \frac{τ^{[N]}(s, t - [z^{-1}], \bar{\iota}, x)}{τ^{[N]}(s, t, t, x)} χ.
\]

Let us plug here the formula (4.10) for \( τ^{[N]} \) and set \( z = z_{N+1} \). Using (4.8) as before, we see that

\[
τ^{[N]}(s, t - [z_{N+1}^{-1}], \bar{\iota}, x) = V_N \prod_{i=1}^N (1 - z_{N+1}^{-1}) \frac{\xi(t_{z_i})}{\xi(t_{z_i})}
\]

\[
\times \frac{\tau(s - N + 1, t - [z_1^{-1}] - \cdots - [z_{N+1}^{-1}], \bar{\iota}, x)}{\tau(s - N + 1, t, t, x)}.
\]
Hence,
\[
\psi^{[N]}_{|z=\tau_{N+1}} = \chi_{N+1} \prod_{i=1}^{N} (1 - z_i z_{N+1}^{-1})^{-1} \times \frac{\tau(s-N, t - [z_{i}^{-1}] - \cdots - [z_{N+1}^{-1}], \mathbf{i}, \mathbf{x})}{\tau(s-N, t - [z_{i}^{-1}] - \cdots - [z_{N}^{-1}], \mathbf{i}, \mathbf{x})}.
\]

Comparing the above two expressions for \(\psi^{[N]}_{|z=\tau_{N+1}}\), we obtain (5.1) with \(N+1\) in place of \(N\). This completes the proof of the theorem. \(\square\)

Similarly, using remark 4.3 and [27, theorem 5.3], we can obtain Fay identities with respect to \(\bar{\psi}\) given by (see [41]):
\[
\text{Wr}_{\Lambda}^{\bar{\psi}}(\bar{\psi}_1, \ldots, \bar{\psi}_N) = \bar{\chi}_1 \cdots \bar{\chi}_N \prod_{1 \leq i < j \leq N} (z_j - z_i) \times \frac{\tau(s+N, t, \bar{t} + [z_1] + \cdots + [z_N], \mathbf{x})}{\tau(s, t, \bar{t}, \mathbf{x})},
\]
where \(\bar{\psi}_i = \psi|_{z=\tau_i}, \bar{\chi}_i = \chi|_{z=\tau}\), and
\[
\text{Wr}_{\Lambda}^{\bar{\psi}}(f_1, f_2, \ldots, f_n) = \left| \begin{array}{cccc}
    f_1 & f_2 & \cdots & f_n \\
    \Lambda(f_1) & \Lambda(f_2) & \cdots & \Lambda(f_n) \\
    \vdots & \vdots & \cdots & \vdots \\
    \Lambda^{n-1}(f_1) & \Lambda^{n-1}(f_2) & \cdots & \Lambda^{n-1}(f_n) 
\end{array} \right|.
\]

6. Conclusion

In this paper, we proved a bilinear equation for the extended bigraded Toda hierarchy (EBTH), which is equivalent to the bilinear equation of Carlet and van de Leur [13] after a change of variables but uses Takasaki’s more convenient notation from [37]. From the bilinear equation, we derived difference Fay identities for the EBTH and showed that the action of the Darboux transformations on the wave functions \(\psi, \bar{\psi}\) corresponds to acting on the tau-function by certain vertex operators \(\Gamma^+\) and \(\Gamma^−\). As an application, we obtained generalized Fay identities for the EBTH.

A natural question is to determine explicitly the initial tau-function corresponding to the trivial Lax operator \(L = \Lambda^k + \Lambda^{-m}\), from which we can generate other solutions of the EBTH with Darboux transformations. Wave functions for this Lax operator were given in [10, 27] in the cases \(k = m = 1\) and \(k = m = 2\), but they correspond to a wave function \(\psi\) satisfying \(L \psi = z^k \psi\). We would like to determine the initial tau-function for the version of the EBTH presented here.

Another interesting question is whether one can generate a \(\mathcal{W}\)-algebra from the vertex operators \(\Gamma^+\) and \(\Gamma^−\), as was done for the KP hierarchy in [1, 2, 14]. One can construct a Virasoro algebra based on [8, 16], but it would be interesting to try to construct a more general \(\mathcal{W}\)-algebra of symmetries by modifying the vertex operators \(\Gamma^+\) and \(\Gamma^−\) so that they depend explicitly on \(x\) (see [3, 7, 32]).

We would also like to use our results about Darboux transformations to find solutions to the bispectral problem [17] for the EBTH (see [4–6]). The bispectral problem was first extended
to difference operators in the case of the discrete KP hierarchy in [21] and then expanded upon in [19].

Recently, a fermionic version of the EBTH was introduced in [25], while Darboux transformations in the super case were studied in [28]. It would be interesting to see whether our results generalize to the fermionic EBTH.

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