We provide a class of quantum evolution beyond Markovian semigroup. This class is governed by a hybrid Davies-like generator such that dissipation is controlled by a suitable memory kernel and decoherence by standard GKLS generator. These two processes commute and both of them commute with the unitary evolution controlled by the systems Hamiltonian. The corresponding memory kernel gives rise to semi-Markov evolution of the diagonal elements of the density matrix. However, the corresponding evolution needs not be completely positive. The role of decoherence generator is to restore complete positivity. Hence, to pose the dynamical problem one needs two processes generated by classical semi-Markov memory kernel and purely quantum decoherence generator. This scheme is illustrated for a qubit evolution.

A simple generalization of quantum Markovian semigroup is proposed. The generator of Markovian semigroup in the weak coupling limit has the characteristic form derived by Davies (so called Davies generator) and consists of three commuting parts: Hamiltonian part responsible for unitary evolution, dissipative part and pure decoherence part. To include possible memory effect due to he nontrivial system-environment interaction we propose to replace Markovian dissipation by non-Markovian generator represented by non-local memory kernel. In general the corresponding evolution is no longer physically admissible since it is represented by a map which is not completely positive and trace preserving. In our scheme the memory kernel gives rise to purely classical dissipative process (so called semi-Markov process) for diagonal elements of the density operator interpreted as populations of the corresponding energy levels. Hence, whenever one restricts to diagonal elements the process is physically legitimate. However, off-diagonal elements (quantum coherences) may cause the entire evolution not legitimate. Interestingly, it turns out that the additional Markovian generator responsible for pure decoherence may wash out all unphysical terms making the evolution perfectly physically admissible. This way the total process is described by a hybrid generator, that is, an appropriate pair of semi-Markovian dissipation and Markovian decoherence. A simple example of qubit evolution is provided to illustrate our approach.

INTRODUCTION

A theory of open quantum systems provides a basic tool to analyze quantum systems which are not isolated but interact with an external environment [1–4]. Any realistic system is never perfectly isolated and hence this theory plays a key role for modelling and controlling realistic quantum systems. It is, therefore, clear that open quantum systems are fundamental for potential applications in modern quantum technologies such as quantum communication, cryptography and computation [5]. The standard approach [1] considers a total “system + environment” Hamiltonian $H$ and looks for the reduced evolution of the system density operator $\rho(t)$ defined by

$$\rho_0 \longrightarrow \rho(t) = \text{Tr}_E(e^{-iHt}\rho_0 \otimes \rho_E e^{iHt}),$$

where $\rho_E$ is an initial state of the environment and $\text{Tr}_E$ denotes a partial trace over the environmental degrees of freedom. It is well known that the map $\rho_0 \longrightarrow \rho(t) = \Lambda_0[\rho_0]$ is completely positive and trace-preserving (CPTP) and satisfies $\Lambda_0 = \mathbb{I}$ (identity map). It is usually called a (quantum) dynamical map. It was shown by Nakajima and Zwanzig [6] (see also [7, 8]) that $\rho(t)$ satisfies the following generalized master equation

$$\dot{\rho}(t) = \int_0^t \mathcal{K}_{t\rightarrow\tau} \rho(\tau) d\tau,$$

in which quantum memory effects are taken into account through the introduction of the memory kernel $\mathcal{K}_t$. This means that the rate of change of the state $\rho(t)$ at time $t$ depends on its history (starting at $t = 0$). The memory kernel is fully determined by the total Hamiltonian and the initial state of the environment. It should be stressed that in general its structure is highly nontrivial since the knowledge of the memory kernel derived from the microscopic model governed by the total Hamiltonian would be in principle equivalent to the knowledge of the full “system + environment” evolution. The main problem of memory kernel master equation [2] is the issue of complete positivity. This problem was already posed by Barnett and Stenholm [9]. An interesting approach of Shabani and Lidar [10] leads to so-called post-Markovian master equation. However, neither the phenomenological kernel of Barnett-Stenholm nor the Shabani-Lidar post-Markovian kernel guarantees that the solution $\rho_0 \longrightarrow \rho(t)$ defines a complete positive map (cf. also [11]). This problem was further extensively analyzed both from mathematical and physical point of view (see e.g. [12–20]). An interesting proposal leading to legitimate memory kernels is provided by so called collision models [21–25]. Actually, the non-local memory kernel master equation is well known for classical stochastic evolution [26, 27, 29], where the dynamical map is realized by a
family of stochastic matrices. The quantum analog of semi-Markov evolution was proposed by Breuer and Vacchini [15], and then further analyzed in [19, 20] (see also recent papers [31]). Very often one replaces Nakajima-Zwanzing equation (2) by time-local Master equation

\[ \dot{\rho}(t) = \mathcal{L}_t \rho(t) \]  

with time-dependent local generator \( \mathcal{L}_t \) [32]. An interesting discussion on intricate relation between (2) and (3) can be found in [30, 33].

Why this problem is so hard? Note, that expressing the total Hamiltonian as

\[ \mathbf{H} = H_S \otimes I_E + I_S \otimes H_E + H_{\text{int}}, \]  

with the interaction Hamiltonian given by

\[ H_{\text{int}} = \lambda \sum_{\alpha} A_\alpha \otimes B_\alpha, \]  

where \( \lambda \) denotes the coupling constant, one finds that the memory kernel \( \mathcal{K}_t \) depends on all multi-time environmental correlation functions

\[ C_{\alpha_1 \ldots \alpha_n}(t_1, \ldots, t_n) = \text{Tr}_E (B_{\alpha_1}(t_1) \ldots B_{\alpha_n}(t_n) \rho_E), \]  

Moreover, assuming the spectral representation for the system’s Hamiltonian \( H_S = \sum_k E_k P_k \)

\[ A_\alpha(\omega) = \sum_{E_i - E_j = \omega} P_k A_\alpha P_l \]  

one finds the Lamb shift correction

\[ \Delta H_{LS} = \sum_{\alpha, \beta} \sum_\omega s_{\alpha\beta}(\omega) A_\alpha^\dag(\omega) A_\beta(\omega). \]  

Finally, \( s_{\alpha\beta}(\omega) \) and \( \gamma_{\alpha\beta}(\omega) \) are controlled by 2-point correlation function via

\[ \int_0^\infty e^{i\omega t} C_{\alpha\beta}(t) dt = \frac{1}{2} \gamma_{\alpha\beta}(\omega) + is_{\alpha\beta}(\omega). \]  

The basic properties of 2-point correlation function imply that for each Bohr frequency \( \omega = E_k - E_l \) the matrix \( \gamma_{\alpha\beta}(\omega) \)

where \( B_\alpha(t) = e^{iH_E t} B_\alpha e^{-iH_E t} \) are the time evolved environmental operators. Clearly, multi-time correlation functions contribute to non-Markovian memory effects of the quantum evolution governed by (4).

For weak system-environment interaction (\( \lambda \ll 1 \)) one usually applies well known Born approximation which allows to keep only 2-point function \( C_{\alpha_1, \alpha_2}(t_1, t_2) \). If the initial environmental state \( \rho_E \) is invariant w.r.t. free evolution (\( \lambda = 0 \)), then all correlation function at time homogeneous. In particular one has \( C_{\alpha_1, \alpha_2}(t_1, t_2) = C_{\alpha_1, \alpha_2}(t_1 - t_2) \). Applying so called weak coupling approximation which assume not only Born approximation but also appropriate Markov approximation Davies [34, 35] derived Markovian Master equation

\[ \dot{\rho}(t) = \mathcal{L} \rho(t), \]  

where the time independent generator has the following well known structure

\[ \mathcal{L} \rho = -i[H_S, \rho] + \lambda^2 L \rho, \]  

with the dissipative part \( L \)

\[ L \rho = -i[\Delta H_{LS}, \rho] + \sum_{\alpha, \beta} \sum_\omega \gamma_{\alpha\beta}(\omega) \left( A_\alpha(\omega) \rho A_\beta^\dag(\omega) - \frac{1}{2} \{ A_\beta^\dag(\omega) A_\alpha(\omega), \rho \} \right), \]  

Moreover, assuming the spectral representation for the system’s Hamiltonian \( H_S = \sum_k E_k P_k \)

\[ A_\alpha(\omega) = \sum_{E_i - E_j = \omega} P_k A_\alpha P_l \]  

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CLASSICAL SEMI-MARKOV EVOLUTION

The classical analog of Markovian Master equation is a Pauli rate equation for the probability vector \( \mathbf{p}(t) \)

\[
\dot{\mathbf{p}}(t) = \mathbf{Lp}(t),
\]

where the classical generator \( \mathbf{L} \) is represented by a real square matrix \( \mathbf{L}_{kl} \) satisfying well known Kolmogorov conditions \[44\]

\[
\mathbf{L}_{kl} \geq 0, \quad \sum_k \mathbf{L}_{kl} = 1,
\]

and hence can be represented as follows

\[
\mathbf{L}_{kl} = \mathbf{W}_{kl} - \delta_{kl} \sum_l \mathbf{W}_{ld},
\]

with \( \mathbf{W}_{kl} \) satisfying \( \mathbf{W}_{kl} \geq 0 \) (note, that \( \mathbf{W}_{kk} \) does not affect \( \mathbf{L}_{kl} \)). In terms of \( \mathbf{W}_{kl} \) Pauli equation reads as follows

\[
\dot{p}_k(t) = \sum_l \left( W_{kl}p_l(t) - W_{lk}p_k(t) \right).
\]

Going beyond Markovian scenario one replaces (13) by the corresponding memory kernel rate equation

\[
\dot{\mathbf{p}}(t) = \int_0^t K_{t-s} \mathbf{p}(s)ds,
\]

or equivalently

\[
\dot{p}_k(t) = \int_0^t \left( W_{kl}(t-s)p_l(s) - W_{lk}(t-s)p_k(s) \right)ds,
\]

with time dependent rates \( W_{kl}(\tau) \). Now, contrary to the Markovian case, the condition for the rates \( W_{kl}(\tau) \) which guarantee that the solution to (13) provides the stochastic matrix \( T(t) \)

\[
\mathbf{p}(t) = T(t)\mathbf{p}(0),
\]

are not known. There is, however, an interesting class of classical memory kernels which can be completely characterized and corresponds to so called semi-Markov classical processes \[15, 26–29\]: one defines a semi-Markov matrix \( q_{ij}(\tau) \) for \( \tau \geq 0 \) such that \( \int_0^\tau q_{ij}(\tau)d\tau \) denotes the probability of jump from state “\( i \)” to state “\( j \)” no later than \( \tau = t \) provided that at time \( \tau = 0 \) the system stays at the state “\( i \)”.

One introduces waiting time distribution \( f_j(\tau) = \sum_i q_{ij}(\tau) \) and survival probability

\[
g_j(\tau) = 1 - \int_0^\tau f_j(\tau)d\tau,
\]

that is the probability that the system stays in the state “\( j \)” up to \( \tau = t \). Clearly \( \sum_i \int_0^\tau q_{ij}(\tau)d\tau \leq 1 \) and hence \( g_j(\tau) \in [0,1] \).

Note, that the following diagonal matrix

\[
n_{ij}(\tau) = g_j(\tau)\delta_{ij},
\]

satisfies

\[
n_{ij}(\tau) \geq 0, \quad n_{ij}(0) = \delta_{ij},
\]

but it is not a stochastic matrix since \( \sum_j n_{ij}(t) \leq 1 \). Interestingly, one may ‘normalize’ \( n_{ij}(t) \) using the semi-Markov matrix \( q_{ij}(t) \). One proves that the following matrix in the Laplace transform domain

\[
\widetilde{T}(s) = \widetilde{n}(s) \frac{1}{1 - \overline{q}(s)} = \widetilde{n}(s) \sum_{n=0}^\infty \overline{q}^n(s),
\]

defines the stochastic matrix in the time domain

\[
T_{ij}(t) = n_{ij}(t) + (n * q)_{ij}(t) + (n * q * q)_{ij}(t) + \ldots,
\]

provided that the above infinite series converges. In the above formula we use the following definitions of the Laplace transform and convolution

\[
\widetilde{f}(s) := \int_0^\infty e^{-st} f(t)dt, \quad (f * g)(t) := \int_0^t f(t - \tau)g(\tau)d\tau.
\]

It turns out that \( T(t) \) satisfies

\[
T_{ij}(t) = \int_0^t \sum_k \left( W_{ik}(t-\tau)T_{kj}(\tau) - W_{ki}(t-\tau)T_{ij}(\tau) \right)d\tau,
\]

where the rates \( W_{ij}(t) \) are defined by the following relation in the Laplace transform domain

\[
\widetilde{W}_{ij}(s) = \frac{\overline{q}_{ij}(s)}{\overline{g}_j(s)},
\]

that is, \( W_{ij}(t) \) is uniquely defined in terms of the semi-Markov matrix \( q_{ij}(t) \).
A HYBRID GENERATOR

A generic Davies generator has the following structure

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{diss}} + \mathcal{L}_{\text{dec}},$$

with $\mathcal{L}_0 \rho = -i[H, \rho]$, where $H = \sum_k E_k |k\rangle \langle k|$, and

$$\mathcal{L}_{\text{diss}} = \sum_{k \neq l} W_{kl} \left( \langle k| \langle l| \rho \langle l| - \frac{1}{2} \{\langle l| \langle l|, \rho\} \right)$$

and

$$\mathcal{L}_{\text{dec}} \rho = \sum_{k, l} D_{kl} |k\rangle \langle k| \rho \langle l| + \frac{1}{2} \sum_k D_{kk} |k\rangle \langle k| \rho,$$

Moreover, $W_{kl} \geq 0$, and the matrix $D_{kl}$ is positive definite. Note, that

$$[\mathcal{L}_0, \mathcal{L}_{\text{diss}}] = [\mathcal{L}_0, \mathcal{L}_{\text{dec}}] = [\mathcal{L}_{\text{dec}}, \mathcal{L}_{\text{diss}}] = 0,$$

and hence the corresponding dynamical map reads

$$\Lambda_t = e^{t\mathcal{L}} = \mathcal{U}_t \circ \Phi_{t}^{\text{diss}} \circ \Phi_{t}^{\text{dec}},$$

with

$$\mathcal{U}_t \rho = e^{-iHt} \rho e^{iHt}, \quad \Phi_{t}^{\text{diss}} = e^{t\mathcal{L}_{\text{diss}}}, \quad \Phi_{t}^{\text{dec}} = e^{t\mathcal{L}_{\text{dec}}}.$$  

(32)

Each map $\mathcal{U}_t$, $\Phi_{t}^{\text{diss}}$, and $\Phi_{t}^{\text{dec}}$ is CPTP and so is their concatenation $\Lambda_t$. Moreover

$$\Phi_{t}^{\text{diss}} \rho = \sum_{k, l} T_{kl}(t) |k\rangle \langle l| \rho \langle l| + \sum_{k \neq l} \lambda_{kl}(t) |k\rangle \langle k| \rho \langle l|,$$

where

$$\lambda_{kl}(t) = \exp \left( -\frac{1}{2} (w_k + w_l) t \right), \quad w_k := \sum_i W_{ik}$$

(34)

and $T_{kl}(t)$ is a stochastic matrix being a solution to the classical Pauli rate equation \[[16]\). Complete positivity of $\Phi_{t}^{\text{diss}}$ is guaranteed by the positivity of the following matrix

$$C_{kl}(t) := T_{kk}(t), \quad C_{kl}(t) := \lambda_{kl}(t), \quad (k \neq l).$$

(35)

Similarly,

$$\Phi_{t}^{\text{dec}} \rho = \sum_k |k\rangle \langle k| \rho \langle k| + \sum_{k \neq l} \mu_{kl}(t) |k\rangle \langle k| \rho \langle l|,$$

with

$$\mu_{kl}(t) = e^{-tD_{kl}}, \quad (k \neq l).$$

(37)

Hence

$$\mathcal{U}_t \Phi_{t}^{\text{dec}} \rho = \sum_{k, l} T_{kl}(t) |k\rangle \langle l| \rho \langle l| + \sum_{k \neq l} e^{-t(E_k - E_l)} \lambda_{kl}(t) \mu_{kl}(t) |k\rangle \langle k| \rho \langle l|.$$

(38)

Now, we propose the following generalization: we replace $\mathcal{L}_{\text{diss}}$ by the corresponding memory kernel

$$\mathcal{K}_t \rho = \sum_{k, l} W_{kl}(t) \left( |k\rangle \langle l| \rho \langle l| - \frac{1}{2} \{\langle l| \langle l|, \rho\} \right)$$

(39)

where $W_{kl}(t)$ is constructed out of semi-Markov matrix $q_{kl}(t)$ via \[[27]\). Note, that

$$[\mathcal{L}_0, \mathcal{K}_t] = [\mathcal{L}_0, \mathcal{L}_{\text{dec}}] = [\mathcal{L}_{\text{dec}}, \mathcal{K}_t] = 0,$$

and hence the corresponding dynamical map has a form \[[31]\), where now the stochastic matrix $T_{kl}(t)$ is a solution of \[[26]\) and $\lambda_{kl}(t)$ solve the following equation

$$\dot{\lambda}_{kl}(t) = -\frac{1}{2} \int_0^t (w_k(t - \tau) + w_l(t - \tau)) \lambda_{kl}(\tau) d\tau,$$

(40)

with $\lambda_{kl}(0) = 1$, which reduces to \[[34]\) in the Markovian case. The main difference between Markovian generator $\mathcal{L}_{\text{diss}}$ and semi-Markov memory kernel $\mathcal{K}_t$ is that $\mathcal{L}_{\text{diss}}$ always generates a CPTP dynamical map $\Lambda_t$, whereas it is not always true for the semi-Markov generator. The main result of this paper consists in the following observation

**Theorem 1** Given a semi-Markov matrix $q_{kl}(t)$ one can always define a positive matrix $D_{kl}$ such that

$$\mathcal{K}_t \rho = -i[H, \rho(t)] + \int_0^t \mathcal{K}_{\tau} \mathcal{L}_{\text{diss}} \rho(t) d\tau + \mathcal{L}_{\text{dec}} \rho(t),$$

(41)

gives rise to CPTP dynamical map $\Lambda_t$.

Indeed, the role of pure decoherence generator is to make the matrix $C(t)$ defined in \[[35]\) positive, that is, the corresponding dynamical map is completely positive iff

$$\Phi_{t}^{\text{dec}} |C(t)\rangle \geq 0,$$

(42)

for all $t \geq 0$. Hence, if the matrix $C(t)$ is already positive it remains positive under the action of decoherence map $\Phi_{t}^{\text{dec}}$ for completely arbitrary decoherence matrix $D_{kl}$. However, if $C(t)$ is not positive (for some $t$), then the role of $\Phi_{t}^{\text{dec}}$ is to suppress the off-diagonal elements of $C(t)$ in order to make it positive.
STOCHASTIC HAMILTONIAN AND SEMI-MARKOV DISSIPATION

Actually, the same effect of suppressing off-diagonal elements of $C(t)$ may be achieved by a family of stochastic Hamiltonians $H(t) = \sum_k E_k(t) |k\rangle \langle k|$, with

$$E_k(t) = E_k + \xi_k(t),$$

where $\xi_k(t)$ represents a collection of white noise satisfying

$$\langle \langle \xi_k(t) \rangle \rangle = 0, \quad \langle \langle \xi_k(t) \xi_l(s) \rangle \rangle = \delta_{kl} \gamma \delta(t - s).$$

One finds for the dynamical map

$$\Lambda_i(\rho) = \langle \langle U^*_k(t) \Phi^\text{diss}_k(\rho) U_k^i(t) \rangle \rangle,$$

where

$$U_k(t) = e^{-i \int_0^t H(\tau) d\tau} = \sum_k e^{-i E_k \int_0^t \xi_k(\tau) d\tau} |k\rangle \langle k|.$$

Hence, using the following property of the Gaussian noise

$$\langle \langle e^{-i \int_0^t [\xi_k(\tau) - \tilde{\xi}_k(\tau)] d\tau} \rangle \rangle = e^{-\frac{1}{2} \langle \langle \xi_k(\tau) \xi_k(\tau) \rangle \rangle} = e^{-\frac{1}{2} \delta_{kl}},$$

one finds that $\Lambda_i$ is represented by (38) with

$$\mu_{kl}(t) = e^{-\gamma(\kappa + \gamma)t}.$$

Again, one can always find $\gamma$ such that $\Lambda_i$ is completely positive. One can in principle consider also time dependent rates $\gamma(t)$ which lead to $\mu_{kl}(t) = \exp(-\int_0^t \gamma(\tau) + \gamma(t) d\tau)$. Note, that due to the assumption that the noise is Gaussian the decoherence is fully controlled by 2-point correlation only. One can also consider more general scenario, i.e. a noise which is not Gaussian, giving rise to decoherence controlled by higher order correlation functions.

EXAMPLE: QUBIT EVOLUTION

To illustrate our approach let us analyze a qubit evolution. One has $H = \frac{\omega}{2} \sigma_z$, with $\omega = E_1 - E_0$,

$$\mathcal{L}_{\text{diss}} \rho = k_+(t) \left( \sigma_+ \rho \sigma_- - \frac{1}{2} [\sigma_- \sigma_+, \rho] \right)$$

$$+ k_-(t) \left( \sigma_- \rho \sigma_+ - \frac{1}{2} [\sigma_+ \sigma_-, \rho] \right),$$

and finally the pure dephasing generator

$$\mathcal{L}_{\text{dec}} \rho = \frac{\gamma}{2} (\sigma_+ \rho \sigma_- - \rho).$$

Actually, authors of [16] analyzed memory kernel master equation

$$\rho(t) = \int_0^t \mathcal{L}_{\text{diss}} \rho(t) d\tau,$$

taking a semi-Markov matrix

$$q(t) = \begin{pmatrix} 0 & k_+(t) \\ k_-(t) & 0 \end{pmatrix}, \quad k_\pm(t) = \kappa_\pm e^{-\gamma t}.$$

It was found that indeed one finds legitimate stochastic matrix $T_{ij}(t)$, however, in general the solution $\Phi^\text{diss}_k$ violates complete positivity. Now, the role of purely dephasing generator (50) is to restore complete positivity for sufficiently large dephasing rate $\gamma$. Following [16] consider $k_\pm(t) = \kappa_\pm e^{-\gamma t}$, with the additional condition $\gamma^2 \geq \max\{4 \kappa_+, 4 \kappa_-\}$. This choice provides legitimate classical semi-Markov evolution and one finds for the diagonal elements of the stochastic matrix

$$T_{00}(t) = \frac{\kappa_+}{\kappa_+ + \kappa_-} + \frac{\kappa_-}{\kappa_+ + \kappa_-} e^{-\gamma t/2} \left[ \cosh(dt/2) + \frac{\gamma}{d} \sinh(dt/2) \right],$$

and

$$T_{11}(t) = \frac{\kappa_-}{\kappa_+ + \kappa_-} + \frac{\kappa_+}{\kappa_+ + \kappa_-} e^{-\gamma t/2} \left[ \cosh(dt/2) + \frac{\gamma}{d} \sinh(dt/2) \right],$$

with

$$d = \sqrt{\gamma^2 - 4 (\kappa_+ + \kappa_-)}.$$

Now, the decoherence factor $\lambda_{01}(t)$ reads

$$\lambda_{01}(t) = e^{-\gamma t/2} \left[ \cosh(dt/2) + \frac{\gamma}{d} \sinh(dt/2) \right],$$

with

$$d = \sqrt{\gamma^2 - 4 (\kappa_+ + \kappa_-)}.$$

Complete positivity is controlled by positivity of

$$C(t) = \begin{pmatrix} T_{00}(t) & \lambda_{01}(t) \\ \lambda_{01}(t) & T_{11}(t) \end{pmatrix}.$$

It was already observed in [16] that $C(t)$ needs not be positive and hence the evolution in general is not CPTP. Figure 1 shows the plot of $\det C(t)$. The blue curve shows that the evolution is evidently not CPTP. Now, we turn on decoherence generator $\mathcal{L}_{\text{dec}}$ controlled by $\gamma$. Figure 1 shows that for small $\gamma$ the evolution still violates CP (yellow curve), however for $\gamma = 1$ (green curve) the evolution is perfectly CP.
CONCLUSIONS

We provided a class of quantum evolution beyond Markovian semigroup. This class is governed by a hybrid Davies like generator: dissipation is controlled by a memory kernel and decoherence by standard GKLS generator. These two processes commute and both of them commute with the unitary evolution controlled by the Hamiltonian $H$. The corresponding memory kernel gives rise to semi-Markov evolution of the diagonal elements of the density matrix. However, the corresponding map $\Phi_{t}^{\text{diss}}$ needs not be completely positive (actually, in general it is not even positive). The role of decoherence generator $\mathcal{L}^{\text{dec}}$ is to restore complete positivity. Hence, to pose the dynamical problem one needs that two processes generated by $\mathcal{K}^{\text{diss}}$ and $\mathcal{L}^{\text{dec}}$ are properly adjust. They are no longer independent as one has for a semigroup evolution. However, given $\mathcal{K}^{\text{diss}}$ one can always find $\mathcal{L}^{\text{dec}}$ which properly suppresses all terms violating complete positivity. This simple scheme is illustrated for a qubit evolution.

Note, that presented approach may be easily generalized for time-dependent decoherence matrix $D_{kl}(t)$ and time-dependent Hamiltonian $H(t) = \sum_{k} E_k(t) |k\rangle \langle k|$. In particular modeling a decoherence process such that

$$T_{kk}(t) T_{ll}(t) = |\lambda_{kl}(t)| \exp\left(\int_{0}^{t} D_{kl}(\tau) d\tau\right)^{2} \quad (59)$$

one realizes an evolution which maximally preserves the coherence compatible with the dissipation process encoded in $T_{kl}(t)$ (cf. [45]). It would be interesting to further analyze this hybrid approach in connection to quantum non-Markovianity (cf. recent reviews [46–48]). Clearly, a simple qubit example serves as a simple illustration of the general method. It would be interesting to analyze a hybrid approach to many body quantum systems. We postpone this for further research.

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Data availability

The data that supports the findings of this study are available within the article.

[1] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford Univ. Press, Oxford, 2007).
[2] U. Weiss, Quantum Dissipative Systems, (World Scientific, Singapore, 2000).
[3] Á. Rivas and S. F. Huelga, Open Quantum Systems. An Introduction (Springer Briefs in Physics, Springer 2011).
[4] R. Alicki and K. Lendi, Quantum Dynamical Semigroups and Applications (Springer, Berlin, 1987).
[5] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (CUP, Cambridge, 2000).
[6] S. Nakajima, Prog. Theor. Phys. 20, 948 (1958); R. Zwanzig, J. Chem. Phys. 33, 1338 (1960).
[7] S. Chaturvedi and J. Shibata, Z. Phys. B 35, 297 (1979); N. H. F. Shibata and Y. Takahashi, J. Stat. Phys. 17, 171 (1977).
[8] F. Haake, Statistical Treatment of Open Systems by Generalized Master Equations, (Springer Tracts in Modern Physics) vol. 66 (Berlin: Springer 1973).
[9] S. M. Barnett and S. Stenholm, Phys. Rev. A 64, 033808 (2001).
[10] A. Shabani and D.A. Lidar, Phys. Rev. A 71, 020101(R) (2005).
[11] S. Maniscalco, Phys. Rev. A 72, 024103 (2005); S. Maniscalco and F. Petruccione, Phys. Rev. A 73, 012111 (2006).
[12] A. A. Budini, Phys. Rev. A 69, 042107 (2004).
[13] J. Wilkies, Phys. Rev. E 62, 8808 (2000); J. Wilkies and Y.M. Wong, J. Phys. A: Math. Theor. 42, 015006 (2009).
[14] A. Kossakowski and R. Rebolledo, Open Syst. Inf. Dyn. 14, 265 (2007); ibid. 16, 259 (2009).
[15] H.-P. Breuer and B. Vacchini, Phys. Rev. Lett. 101 (2008) 140402.
[16] H.-P. Breuer and B. Vacchini, Phys. Rev. E 79, 041147 (2009).
[17] D. Chrusciński and A. Kossakowski, EPL 97, 20005 (2012).
[18] F. A. Wudarski, P. Należyty, G. Sarobicz, and D. Chrusciński, Phys. Rev. A 91, 042105 (2015).
[19] D. Chrusciński and A. Kossakowski, Phys. Rev. A 94, 020103(R) (2016).
[20] B. Vacchini, Phys. Rev. Lett. 117, 230401 (2016).
[21] V. Scarani et al. Phys. Rev. Lett. 88, 097905 (2002); M. Ziman, P. Štelmachovič, V. Bužek, M. Hillery, V. Scarani, and N. Gisin, Phys. Rev. A65, 042105, (2002); M. Ziman and V. Bužek, Phys. Rev. A 72, 022110, (2005).
[22] V. Giovannetti and G. M. Palma, Phys. Rev. Lett. 108, 040401 (2012).
[23] F. Ciccarello, G. M. Palma, V. Giovannetti, Phys. Rev. A 87, 040103(R) (2013).
[24] B. Vacchini, Phys. Rev. A 87, 030101(R) (2013); B. Vacchini, Int. J. Quantum Inform. 12, 1461011 (2014).
[25] S. Lorenzo, F. Ciccarello, and G. M. Palma, Phys. Rev. A 93, 052111 (2016).
[26] E. W. Montroll and G. H. Weiss, J. Math. Phys. 6, 167 (1965); V. M. Kenre, E. W Montroll, M. F Shlesinger, J. Stat. Phys. 9, 45 (1973).
[27] D. T. Gillespie, Phys. Lett. A. 64, 22 (1977).
[28] M. Esposito and K. Lindenberg, Phys. Rev. E 77, 051119 (2008).
[29] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, New J. Phys. 13, 093004 (2011).
[30] N. Megier, A. Smirne, and B. Vacchini, New J. Phys. 22, 083011 (2020).
[31] N. Megier, A. Smirne, and B. Vacchini, Entropy 22, 796 (2020).
[32] D. Chrusciński and A. Kossakowski, Phys. Rev. Lett. 104, 070406 (2010).
[33] K. Nestmann, V. Bruch, and M. R. Wegwijs, How quantum evolution with memory is generated in a time-local way, arXiv:2002.07232.
[34] E. B. Davies, Comm. Math. Phys. 39, 91 (1974).
[35] E. B. Davies, Quantum Theory of Open Systems, (Academic Press, London, 1976).
[36] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. 17, 821 (1976).
[37] G. Lindblad, Comm. Math. Phys. 48, 119 (1976).
[38] R. Dümcke and H. Spohn, Z. f. Phys. B 34, 419 (1979).
[39] R. Dümcke, J. Math. Phys. 24, 311 (1983).
[40] N. Megier, A. Smirne, and B. Vacchini, Rep. Math. Phys. 66, 311 (2010).
[41] M. Merkli, Ann. Phys. 412, 167996 (2020).
[42] H. J. Carmichael, Statistical Methods in Quantum Optics I: Master Equations and Fokker-Planck Equations, Springer (2008).
[43] C. Gardiner and P. Zoller, Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics, (Springer Series in Synergetics) 3rd Edition (2004).
[44] N. G. van Kampen, Stochastic Processes in Physics and Chemistry, North Holland, Amsterdam 2007.
[45] K. Korzekwa, S. Czachórski, Z. Puchała, and K. Życzkowski, New J. Phys. 20, 043028 (2018).
[46] Á. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. 77, 094001 (2014).
[47] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. 88, 021002 (2016).
[48] I. de Vega and D. Alonso, Rev. Mod. Phys. 89, 015001 (2017).