Optimal Transmit Filters for ISI Channels under Channel Shortening Detection

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Abstract—We consider channels affected by intersymbol interference with reduced-complexity, mutual information optimized, channel-shortening detection. For such settings, we optimize the transmit filter, taking into consideration the reduced receiver complexity constraint. As figure of merit, we consider the achievable information rate of the entire system and with functional analysis, we establish a general form of the optimal transmit filter, which can then be optimized by standard numerical methods. As a corollary to our main result, we obtain some insight of the behavior of the standard waterfilling algorithm for intersymbol interference channels. With only some minor changes, the general form we derive can be applied to multiple-input multiple-output channels with intersymbol interference. To illuminate the practical use of our results, we provide applications of our theoretical results by deriving the optimal shaping pulse of a linear modulation transmitted over a bandlimited additive white Gaussian noise channel which has possible applications in the faster-than-Nyquist/time packing technique.

Index Terms—ISI channels, channel shortening, waterfilling algorithms, reduced complexity detection, mismatched receivers, MIMO-ISI, faster-than-Nyquist, time packing.

I. INTRODUCTION

The intersymbol interference (ISI) channel has played a central role in communication theory for several decades. It has been heavily researched, and today most of its fundamental properties are known. The capacity of the ISI channel was for example derived by Hirt back in 1988 in [11], and it was shown that Gaussian inputs in combination with the classical waterfilling algorithm achieves capacity. In practice, Gaussian channel inputs are not very common and discrete inputs are typically preferred. In this case the ultimate communication limit was found in the early 2000s through a series of papers [2]–[6]. Further results on capacity properties of ISI channels include Kavcic’s elegant method [7] to achieve the capacity of the ISI channel with discrete inputs through a generalized version of the Arimoto-Blahut algorithm, and also Soriaga et al.’s evaluation of the low-rate Shannon limit of ISI channels [8].

However, all of the above mentioned papers study ISI channels under the assumption that the receiver can perform optimal maximum-likelihood (ML) or maximum-a-posteriori (MAP) detection. Let \( L_H + 1 \) denote the number of taps in the channel impulse response. Forney showed in 1972 [9] that optimal ML/MAP-detection can be performed by searching a trellis whose number of states is \( U^{L_H} \), where \( U \) is the cardinality of the employed constellation. The number of trellis states will be considered in the following has a measure of the receiver complexity. In many practical scenarios \( L_H \) is far too long for practical implementation of optimal ML/MAP detection. This observation spurred significant research efforts to reduce the computational complexity of the MAP/ML algorithm (e.g., see [10], [11] and references therein) or to investigate when a properly designed linear equalizer has the same diversity order of the optimal detector (e.g., see [12], [13] and references therein). An alternative promising approach was channel shortening pioneered by Falconer and Magee in 1973 [14] and further investigated by several researchers (e.g., see [15]–[24]). Traditionally, channel shortening detectors were optimized from a minimum mean-square-error (MMSE) perspective. However, minimizing the mean-square-error does not directly correspond to achieving the highest information rate (in the Shannon sense) that can be supported by a shortening detector. Recently, the achievable rate of channel-shortening detectors was optimized in [25] by utilizing the framework of mismatched mutual information [26], [27]. The result of [25] is a closed-form expression of the achievable information rate (AIR) of an ISI channel with Gaussian inputs and an optimized channel-shortening detector that considers the channel memory to be \( L < L_H \) taps long, where \( L \) is a user-defined parameter.

In this paper, we extend [25] by designing a proper transmit filter to be employed jointly with a channel-shortening detector [1] with the aim of further improving the achievable information rate. In other words, we consider to adopt, at the receiver side, a channel-shortening detector and then solve for the optimal transmit filter to be used jointly with it. When the use of the optimal full-complexity receiver is allowed, the answer to this question is the classical waterfilling processing. We are generalizing the waterfilling concept to the case of reduced-complexity channel-shortening detectors, i.e., we essentially redo Hirt’s derivations, but this time with the practical constraint of a given receiver complexity.

Our results are not as conclusive as in the unconstrained

\[ \text{As in [25], with the term “channel-shortening detector” we mean a detector based on a proper linear filter (the channel shortener) plus a suboptimal reduced-complexity trellis-based detector with proper branch metrics designed for a target channel response of length } L < L_H. \]
receiver complexity case. With functional analysis, we can prove that, for real channels, the optimal transmit filter has a frequency response described by $L + 1$ real-scalar values. In general, for complex channels, the optimal transmit filter is described by $L + 1$ complex scalar values. The transmit filter optimization thereby becomes a problem of finite dimensionality, and a numerical optimization provides the optimal spectrum. Note that, in practice, $L$ is limited to rather small values and $L = 1$ is an appealing choice from a complexity perspective. This essentially leads to very effective numerical optimizations.

The rest of the paper is organized as follows. In Section II we lay down the system model and formulates the problem that we intend to solve. In Section III we derive a general form of the frequency response of the optimal transmit filter. In Section V-B we derive, by using the same framework, the optimal transmit filter for multiple-input multiple-output channel (MIMO) affected by ISI (MIMO-ISI), and the optimal shaping pulse for a transmission over a bandlimited additive white Gaussian noise (AWGN) channel. Numerical examples and properties of the numerical optimization are given in Section VI. Finally, Section VII concludes the paper.

II. Preliminaries

In this section we give the system model, lay down the fundamentals of channel shortening receivers and their optimization, and formulates the problem that will be solved.

A. System Model

Let us consider the transmission of the sequence of symbols $a = \{a_k\}$ over a discrete-time channel with model\(^2\)

$$y_k = \sum_{\ell=0}^{L} a_k - \ell h_\ell + w_k, \quad (1)$$

where $h = \{h_k\}_{k=0}^{L}$ is the channel impulse response, assumed time-invariant and of finite length, and $w = \{w_k\}$ are independent and identically distributed complex Gaussian random variables, with mean zero and variance $N_0$—note that bold letters are used for vectors. This system is studied under the assumption of ideal channel state information (CSI) at both transmit and receive side, that is, perfect knowledge of the coefficients and the noise variance. The symbol vector $a$ is a precoded version of the information symbols $u = \{u_k\}$,

$$a = u \ast p, \quad (2)$$

where “$\ast$” denotes convolution and $p$ is a transmit filter subject to the power constraint $\sum_k |p_k|^2 = 1$ and with continuous spectrum $|P(\omega)|^2$, where $P(\omega)$ is the discrete time Fourier transform (DTFT) of the vector $p$. Taken together, the received signal can be expressed as

$$y = v \ast u + w, \quad (3)$$

where $v = h \ast p$. It is convenient to assemble the presentation on matrix notation, so that (3) becomes

$$y = V u + w,$$

where $V$ is a convolutional matrix formed from the vector $v$, and $y$, $u$ and $w$ are now column vectors of appropriate sizes. Assume that the combined channel-precoder response $v$ has $K + 1$ non-zero taps. The complexity of MAP sequence (implemented through the Viterbi algorithm) and symbol detection (implemented through the BCJR algorithm) is $O(U^K)$ per symbol, where $U$ is the cardinality of the employed alphabet. Falconer and Magee’s idea was to reduce this complexity by a linear filtering

$$r = y \ast q = (v \ast q) \ast u + (w \ast q).$$

Then, a Viterbi/BCJR algorithm follows assuming a target response $t$ of $L + 1$ taps ($L \leq K$), and working on a trellis with $U^L$ states. Presumably, the target response $t$ roughly equals the $L + 1$ strongest taps of $(v \ast q)$, but there must not be an exact match if it turns out that it is not optimal to do so. In matrix notation, this procedure can be viewed as if the receiver decodes on the basis of a mismatched conditional probability distribution (pdf)\(^3\)

$$p(y|u) \propto \exp \left(-\frac{\|Qy - Tu\|^2}{N_0}\right) \quad (4)$$

instead of the actual conditional pdf

$$p(y|u) \propto \exp \left(-\frac{\|y - Vu\|^2}{N_0}\right).$$

Two questions now emerge: (1) For a given target response $t$, how should the linear filter $q$ be selected? And (2) how should the target response $t$ be selected? These two questions kept researchers busy for several decades, see [14]-[23]. However, in all of those papers, the optimizations of $t$ and $q$ was done with an MMSE cost function, which does not directly correspond to the achievable information rate of the overall system\(^4\).

The optimization for achievable information rate was completely solved in [25] under the assumption of Gaussian input symbols and by using a slightly more general model for channel shortening. This generalization is now described. By expansion of the exponent in (4) we get

$$p(y|u) \propto \exp \left(-\frac{\|Qy - Tu\|^2}{N_0}\right) \propto \exp \left(-\frac{2R\{u^T Q y - u^T T u\}}{N_0}\right), \quad (5)$$

where all terms independent of $u$ have been left out. A MAP sequence detector based on [5] was proposed by Ungerboeck in 1974 [28] and an algorithm for MAP symbol detection in 2005 by Colavolpe and Barbieri [29]. In [25], a reduced complexity channel shortening detector is obtained by substituting

\(^2\)For simplicity of exposition, we refer here to this discrete-time model of a channel with finite ISI. We will discuss later the case of a continuous-time, bandlimited AWGN channel.

\(^3\)By $T$ and $Q$ we mean the convolutional matrices formed from the vectors $t$ and $q$, respectively.

\(^4\)With “overall system”, we mean the chain: prefilter-channel-reduced complexity receiver.
in \( T^\dagger Q \) with \((H^r)^\dagger\) and \(T^\dagger T\) with \(G^r\). In addition, the noise density \(N_0\) is also absorbed into \(H^r\) and \(G^r\). This results in a mismatched conditional pdf of the form
\[
\hat{p}(y|\omega) = \exp\left(2R\{u^\dagger(H^r)^\dagger y\} - u^\dagger G^r u\right).
\]

While the front-end \(H^r\) is unconstrained, the matrix \(G^r\) must satisfy
\[
G^r_{\ell k} = 0, \quad |\ell - k| > L
\]
in order to satisfy the reduced-complexity constraint. The matrix \(T^\dagger T\) in (5) must be positive semi-definite, while no such constraint applies to the matrix \(G^r\). Hence, a more general model than (4) for channel shortening is obtained. The AIR of a general mismatched receiver is derived in [26], [27] and equals
\[
I_{\text{AIR}} = \lim_{N \to \infty} \frac{1}{N} \left[-E_y \log_2(\hat{p}(y)) + E_{y,u} \log_2(\hat{p}(y|u))\right],
\]
where \(N\) is the number of input symbols (i.e., the length of the vector \(u\)), \(E_y\) denotes the expectation operator with respect to the random variable \(y\) and
\[
\hat{p}(y) = \sum_u \hat{p}(y|u)p_u(u).
\]
The rate \(I_{\text{AIR}}\) is directly impacted by the choices of \(G^r\) and \(H^r\). The optimization problem reads
\[
I_{\text{OPT}} = \max_{G^r,H^r} I_{\text{AIR}},
\]
under the constraints specified in (6). Problem (7) for a discrete alphabet is a hard task. On the other hand, it can be solved in closed form under the assumption that transmitted symbols are independent Gaussian random variables [25]. In this case of Gaussian inputs, closed-form expressions for \(G^r\) and \(H^r\) can be found with the following algorithms:
- Compute the sequence \(\{b_k\}_{-L}^{L}\) as
  \[
b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{N_0}{|V(\omega)|^2 + N_0} e^{j\omega k} d\omega
  = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{N_0}{|H(\omega)|^2|P(\omega)|^2 + N_0} e^{j\omega k} d\omega.
\]
where \(H(\omega)\) and \(V(\omega)\) are the DTFT of \(h\) and \(v\).
- Compute the real-valued scalar
  \[
c = b_0 - bB^{-1}b\]
  where \(b = [b_1, b_2, \ldots, b_L]\), and \(B\) is \(L \times L\) Toeplitz with entries \(B_{ij} = b_{j-i}\).
- Define the vector \(u = \frac{1}{\sqrt{c}}[0, -bB^{-1}]\) and find the optimal target response as
  \[
  G^r(\omega) = |U(\omega)|^2 - 1.
  \]
- Finally, the optimal channel shortener is found as
  \[
  H^r(\omega) = \frac{V(\omega)}{|V(\omega)|^2 + N_0}(G^r(\omega) + 1).
  \]
By using the optimal channel shortener and the target response \(I_{\text{OPT}}\) results to be
\[
I_{\text{OPT}} = -\log_2(c).
\]

### B. Problem Formulation

The problem we aim at solving is to maximize \(I_{\text{OPT}}\) over the transmit filter \(P(\omega)\), i.e., the DTFT of \(p\). Thus, we have the following optimization problem at hand
\[
\min_{P(\omega)} c[P(\omega)]
\]
such that
\[
\int_{-\pi}^{\pi} |P(\omega)|^2 d\omega = 2\pi
\]
In [25], we have made explicit the dependency of \(c\) on \(P(\omega)\), but not on \(N_0\) and \(H(\omega)\), since these are not subject to optimization. Since the starting point is the expression of the AIR when the optimal channel-shortening detector is employed, we are thus jointly optimizing the channel shortening filter, the target response, and the transmit filter, although for Gaussian inputs only. However, as shown in the numerical results, when a low-cardinality discrete alphabet is employed, a significant performance improvement is still observed (see also [25]).

### III. General Form of the Optimal Transmit Filter

The optimization problem (9) is an instance of calculus of variations. We have not been able to solve it in closed form, but we can reduce the optimization problem into an \(L + 1\) dimensional problem, which can then efficiently be solved by standard numerical methods. The main result of the paper is the following theorem.

**Theorem 1:** The optimal transmit filter with continuous spectrum for the channel \(H(\omega)\) with a memory \(L\) channel-shortening detector satisfies
\[
|P(\omega)|^2 = \max \left(0, \frac{N_0}{\sqrt{|H(\omega)|^2}} \sum_{\ell=-L}^{L} A_\ell e^{j\omega \ell} - \frac{N_0}{|H(\omega)|^2} \right),
\]
where \(\{A_\ell\}\) are complex-valued scalar constants with Hermitian symmetry, i.e., \(A_\ell = A_{-\ell}^*\).

For a proof see the Appendix A.

### IV. Interlude: Full Complexity Detectors

Theorem 1 gives a general form of the optimal transmit filter to be used for a memory \(L\) channel shortening detector. By definition, it becomes the classical waterfilling filter when \(L = K\). Hence, it also provides an insight to the behavior of the transmit filter for the classical waterfilling algorithm. We remind the reader that \(L_H + 1\) denotes the duration of the channel impulse response and \(K + 1\) denotes the duration of the combined transmit filter and channel response. We summarize our finding in the following

**Theorem 2:** Let \(P(\omega)\) be the transmit filter found through the waterfilling algorithm. Then,
\[
K \geq L_H.
\]
For a proof, see the Appendix B.

Whereas the statement is trivial when the transmit filter and the channel have a finite impulse response (FIR), the theorem proves that this fact holds also when they have infinite impulse responses (IIR). Thus, for a FIR channel response,
the waterfilling solution cannot contain any pole that cancels a zero of the channel, while, for IIR channels, the waterfilling solution cannot contain any zero that cancels a pole. Thus, the overall channel cannot be with memory shorter than the original one.

Theorem 2 reveals the interesting fact that the waterfilling algorithm trades a rate gain for detection complexity. By using the optimal transmit filter, a capacity gain is achieved, but the associated decoding complexity (of a full complexity detector) must inherently increase. Thus, with waterfilling, it is not possible to achieve both a rate gain and a decoding complexity reduction at the same time.

V. OTHER PRACTICAL APPLICATIONS OF THE OPTIMAL TRANSMIT FILTER

Although we restricted our attention on the discrete-time ISI channel [1], the same framework can be used to derive the optimal precoder for other channels.

A. MIMO-ISI Channels with perfect CSI

Consider the MIMO-ISI channel

\[ y_k = \sum_{\ell=0}^{L} H_\ell a_k - \ell + w_k. \]

Without loss of generality, we assume that the channel is \( N \times N \), i.e., matrices \( \{ H_\ell \}_{\ell=0}^{L} \) have dimension \( N \times N \) and \( \{ y_k \}, \{ a_k \}, \{ w_k \} \) are column vectors \( N \times 1 \). In case \( N \times M \) channels, they can be converted in an equivalent \( N \times N \) channel by means of the QR decomposition [25]. Channel shortening receivers for MIMO-ISI channels have been studied before, e.g., in [19], but here we optimize the receiver with respect to mutual information rather than an MMSE cost function as in [19].

The DTFT of \( \{ H_\ell \} \), defined as \( H(\omega) = \sum_{\ell=0}^{L} H_\ell e^{-j\ell\omega} \), can be factorized by means of singular value decomposition (SVD) as

\[ H(\omega) = U_H(\omega) \Sigma(\omega) V_H^\dagger(\omega), \]

where \( U_H(\omega) \) and \( V_H(\omega) \) are unitary matrices and \( \Sigma(\omega) \) is a diagonal matrix with elements \( \Sigma_n(\omega) \). By adopting the MIMO filter \( V_H(\omega) \) at the transmitter and the filter \( U_H^\dagger(\omega) \) at the receiver, without any information we optimize the receiver with respect to mutual information rather than an MMSE cost function as in [19].

The transceiver block diagram is as shown in Fig. 1b, for the case \( N = 2 \). The objective function to be maximized is

\[ I_{\text{OPT}} = \sum_{n=1}^{N} - \log_2(c_n) \]

under the constraint

\[ \sum_{n=1}^{N} \int |P_n(\omega)|^2 d\omega = 2\pi N \]

where \( c_n \) is given in (8) and \( P_n(\omega) \) is the precoder for the channel \( \Sigma_n(\omega) \). By solving the Euler-Lagrange equation, the optimal precoders have spectra of the form (10).
Thus the optimization problem is still given by (9) where the optimal shaping pulse is such that

\[ |P(f)|^2 = T |P(2\pi Tf)|^2 \]

with \(|P(\omega)|^2\) given in (10).

Clearly, when \(2WT \geq 1\), the optimal solution is trivial and \(|P(\omega)|^2\) is flat. Thus, for \(2WT = 1\) the \(\tilde{p}(t)\) is a sinc function, whereas for \(2WT > 1\) the \(\tilde{p}(t)\) can be a pulse whose spectrum has vestigial symmetry (e.g., pulses with a root raised cosine (RRC) spectrum). For \(2WT < 1\), the symbol time is such that the Nyquist condition for the absence of ISI cannot be satisfied. Thus, we are working in the domain of the faster-than-Nyquist (FTN) paradigm [30]–[32] or its extension represented by time packing [33], [34]. Note that, as said before, the discrete-time channel model, will depend on the values of \(W\) and \(T\). When changing the values of \(W\) and/or \(T\), the corresponding optimal pulse will change and so the maximum value of the AIR for the given allowed complexity. In general, when reducing the value of \(WT\), the maximum AIR value will decrease. However, the spectral efficiency, defined as the ratio between the AIR and the product \(WT\), can be a pulse whose spectrum has vestigial symmetry (e.g., pulses with a root raised cosine (RRC) spectrum). For \(2WT < 1\), the symbol time is such that the Nyquist condition for the absence of ISI cannot be satisfied. Thus, we are working in the domain of the faster-than-Nyquist (FTN) paradigm [30]–[32] or its extension represented by time packing [33], [34]. Note that, as said before, the discrete-time channel model, will depend on the values of \(W\) and \(T\). When changing the values of \(W\) and/or \(T\), the corresponding optimal pulse will change and so the maximum value of the AIR for the given allowed complexity. In general, when reducing the value of \(WT\), the maximum AIR value will decrease. However, the spectral efficiency, defined as the ratio between the AIR and the product \(WT\), can be a pulse whose spectrum has vestigial symmetry (e.g., pulses with a root raised cosine (RRC) spectrum). For \(2WT < 1\), the symbol time is such that the Nyquist condition for the absence of ISI cannot be satisfied. Thus, we are working in the domain of the faster-than-Nyquist (FTN) paradigm [30]–[32] or its extension represented by time packing [33], [34]. Note that, as said before, the discrete-time channel model, will depend on the values of \(W\) and \(T\). When changing the values of \(W\) and/or \(T\), the corresponding optimal pulse will change and so the maximum value of the AIR for the given allowed complexity. In general, when reducing the value of \(WT\), the maximum AIR value will decrease. However, the spectral efficiency, defined as the ratio between the AIR and the product \(WT\), can be a pulse whose spectrum has vestigial symmetry (e.g., pulses with a root raised cosine (RRC) spectrum).

Theorem 1 provides a general form of the optimal transmit filter for channel shortening detection of ISI channels. What remains to be optimized is the \(L + 1\) complex-valued constants \(\{A_k\}\). A closed form optimization seems out of reach since \(|P(\omega)|^2\) has no simple analytical form in \(\{A_k\}\).

We have applied a straightforward numerical optimization of the variables \(\{A_k\}\) under the constraints in (9). With a standard workstation and any randomly generated channel impulse response, the optimization is stable, converges to the same solution no matter the starting position as long as the signal-to-noise-ratio (SNR) is not very high or very low, and is altogether a matter of fractions of a second.

We now describe some illuminating examples. In all cases, the transmit power is the same both in the absence and presence of the optimal transmit filter. We first consider the complex channel \(h = [0.5, 0.5, -0.5, -0.5, 0.5]\) with memory \(L_H = 3\). As stated by Theorem 2.

![Fig. 2. AIRs for Gaussian inputs when different values of the memory \(L\) are considered at receiver.](image)

The figure also gives \(I_{OPT}\) for a flat transmit power spectrum (i.e., no transmit filter at all) and the channel capacity (i.e., when using the spectrum obtained by means of the waterfilling algorithm and assuming a receiver with unconstrained complexity). It can be seen that using an optimized transmit filter for each \(L\), significant gains are achieved w.r.t. the flat power spectrum at all SNRs. The flat spectrum reaches its maximum information rate when \(L = L_H\) but suffers a loss to the channel capacity. On the other hand, we can see that the optimized transmit filter when \(L = L_H\) achieves an achievable rate which is close to the channel capacity. However, there is not an exact match. This loss is due to the fact that \(L_H\) must be lower than the combined channel-precoder memory \(K\) as stated by Theorem 2.

This behavior is clearly illustrated by Fig. 3 which plots the information rate when the transmit filter is found through the waterfilling algorithm and the receiver complexity is constrained with values of the memory \(L\). It can be seen that when the memory \(L\) is increased more and more, even above \(L_H\), the information rate becomes closer and closer to the channel capacity. Moreover, it is important to notice that if, naively, a transmit filter found through the waterfilling algorithm is used when the receiver complexity is constrained, a loss w.r.t. the optimized case occurs and it may even be better to not have any transmit filter at all for high SNR values.

Although the results of this paper were so far presented only for Gaussian symbols, we now show that when the optimized transmit filter and detector for Gaussian inputs are used for low-cardinality discrete alphabets, the ensuing \(I_{AIR}\) is still excellent. Fig. 4 shows the AIR for a binary phase shift keying (BPSK) modulation. It can be noticed that the behavior among the curves for BPSK reflects the behavior for Gaussian symbols. The AIR can be approached in practice with proper modulation and coding formats. Fig. 5 shows the bit error rate (BER) of a BPSK-based system using the DVB-S2 low-density parity-check code with rate 1/2. In all cases, 10 internal iterations within the LDPC decoder and 10 global iterations
were carried out. It can be noticed that the performance are in accordance with the AIR results. All simulations that we have presented were also carried out for other channels (e.g., EPR4, Proakis B and C). However, we have not presented any result for these channels since our findings for those channels are in principle identical to those for the channel presented in the paper.

A. MIMO-ISI Channels with perfect CSI

We now considered a $2 \times 2$ MIMO-ISI channel, with $L_H = 3$. Fig. 5 shows the AIR $I_{OPT}$ for Gaussian inputs as a function of $E_H/N_0$, being $E_H = \sum\text{tr}(H_iH_i^\dagger)$. The transmit filters are optimized for the equivalent channels $\Sigma_1(\omega)$ and $\Sigma_2(\omega)$ for different values of the memory $L$ considered by the receiver. For comparison, the figure also gives $I_{OPT}$ for flat transmit power spectra (i.e., $E\{a_ka_{k+m}\} = I_0$, where $I$ is the identity matrix and $\delta_m$ is the Kronecker delta) and the channel capacity (i.e., when using the spectra obtained by means of the waterfilling algorithm and assuming a receiver with unconstrained complexity). It can be seen that conclusions for scalar ISI channels also hold for MIMO-ISI. However, we found that, for MIMO-ISI channel, the objective function seems to have some local maxima, and thus the optimization can depend on the starting position. This problem can be easily solved by running the optimization more times (three times were always enough in all our tests) and keeping the maximum value.

B. Bandlimited AWGN channels

We computed the optimal shaping pulse on a bandlimited AWGN channel with $2WT = 0.48$. Hence, we are in the realm of FTN/time packing and the considered ISI is only due to the adoption of such a technique. Fig. 6 shows the achievable spectral efficiency (ASE) $\eta = I_{AR}/WT$ for a BPSK modulation on the continuous-time AWGN channel as a function of the ratio $E_b/N_0$, $E_b$ being the received signal energy per information bit. Two values of the memory, namely $L = 1$ and $L = 2$ are considered at the detector. For comparison, the figure also gives the ASE for pulses with RRC spectrum and roll-off $\alpha = 0.1$ or $\alpha = 0.2$, and the unconstrained capacity for the AWGN channel. It can be seen that the optimized pulse outperforms the other pulses.

VII. Conclusion

We have studied ISI channels with channel shortening detection. The channel shortening detector that we used is optimized from a mutual information perspective and allows for the highest possible data rate. We then optimized the
transmit filter for a given receiver complexity and ISI channel. This is an optimization problem of infinite dimensionality, but we managed to reduce it through functional analysis into an optimization problem of a dimension that equals the memory of the receiver plus one. A standard numerical optimization procedure then follows. Since the memory \( L \) of the receiver is in practice typically set to a small value, such as \( L = 1 \), the numerical optimization can be easily carried out.

As a side result, we also show that the classical waterfilling algorithm for ISI channels can never result in a shorter channel response at the receiver than the length of the channel response itself. From our numerical experiments, we have found that it is crucial to take the receiver complexity into account when designing the transmit filter, since if the transmit filter found through the waterfilling algorithm is used, then a loss can occur compared with a flat transmit filter.

We have finally shown that the same framework can be used to derive the optimal shaping pulse on a bandlimited AWGN channel.

**APPENDIX A: PROOF OF THEOREM 1**

We first note that \( P(\omega) \) only enters the optimization through its square magnitude, and we therefore make the variable substitution \( S_p(\omega) = |P(\omega)|^2 \) and optimize over \( S_p(\omega) \) instead.

The proof will consist of three steps

- A formula for stationary points.
- The observation that some of these do not have strictly positive spectrum.
- Fixing the problem identified in the previous bullet.

Let us now start with the first bullet.

From Cramer’s rule [38], we get that

\[
\text{B}^{-1} = \frac{1}{\det(\text{B})} [C_{ij}],
\]

where \( C_{ij} \) is the cofactor of entry \((i, j)\) in \( \text{B} \). This implies that we can express \( \text{bB}^{-1}\text{b}^\dagger \) as

\[
\sum_{m=1}^{M} \alpha_m \phi_m \phi_{m-1} \phi_{m-2} \cdots \phi_{m-L} b_{m-L}^{-1} (b_m^{-1})_m \phi_{m,2L} - \sum_{n=1}^{N} \beta_n b_n b_{n+1} b_{n+2} \cdots b_{n-L-1}^{-1} (b_n^{-1})_n \phi_{n,2L}.
\]

where \( M \) and \( N \) are finite constants that depend on \( L \), \( \alpha_m, \beta_m \in \{\pm 1\} \), and both \( \phi_{m,t} \) and \( \psi_{n,t} \) are non-negative integers which satisfy

\[
\sum_{t=0}^{2L} \phi_{m,t} = L + 1 \quad \text{and} \quad \sum_{t=0}^{2L-2} \psi_{n,t} = L.
\]

We next introduce the variable substitution

\[
y(\omega) = \frac{N_0}{|H(\omega)|^2 S_p(\omega) + N_0}, \quad S_p(\omega) = \frac{N_0}{|H(\omega)|^2} \left[ \frac{1}{y(\omega)} - 1 \right].
\]

The constraint \( \int S_p(\omega) \text{d}\omega = 2\pi \) translates into

\[
e[y(\omega)] = \int_{-\pi}^{\pi} \frac{1}{y(\omega)|H(\omega)|^2} \text{d}\omega = \int_{-\pi}^{\pi} \frac{1}{|H(\omega)|^2} \text{d}\omega + \frac{2\pi}{N_0}.
\]

Furthermore, we have

\[
b_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(\omega) e^{j\omega} \text{d}\omega.
\]

The constrained Euler-Lagrange equation [39] becomes

\[
\frac{\delta c}{\delta y} = \frac{\delta e}{\delta y} = -\frac{\lambda}{|H(\omega)|^2 y^2(\omega)}.
\]

The functional derivative \( \delta b_k / \delta y \) equals

\[
\frac{\delta b_k}{\delta y} = \delta \left[ \int_{-\pi}^{\pi} y(\omega) e^{j\omega} \text{d}\omega \right] = s \left[ \int_{-\pi}^{\pi} y(\omega) e^{j\omega} \text{d}\omega \right] e^{j\omega} = sb_k e^{j\omega}.
\]

We now note that \( b_k \), raised to any power, is a constant that depends explicitly on \( y \). Therefore, by an application on the quotient rule for the derivative and the chain rule to (8), we obtain an expression of the form

\[
\frac{\delta c}{\delta y} = 1 - \sum_{\ell=-L}^{L} A_{\ell}[y] e^{j\ell\omega} C[y],
\]

where the constants \( A_{\ell}[y] \) and \( C[y] \) explicitly depend on \( y \), e.g.,

\[
C[y] = \left[ \sum_{n=1}^{N} \beta_n b_0 \beta_{n+1} \beta_{n+2} \cdots b_{n-L-1}^{-1} \right]^2.
\]

By manipulation of the Euler-Lagrange equation and by introducing a new set of constants \( \{ B_{\ell}[y] \} \), we obtain

\[
y(\omega) = \frac{1}{\sqrt{|H(\omega)|^2 [\sum_{\ell=-L}^{L} B_{\ell}[y] e^{j\ell\omega}]}},
\]

This translates into a general form of the optimal \( S_p(\omega) \) which reads

\[
S_p^{opt}(\omega) = \frac{N_0}{\sqrt{|H(\omega)|^2}} \left[ \sum_{\ell=-L}^{L} A_{\ell} e^{j\ell\omega} - \frac{N_0}{|H(\omega)|^2} \right]
\]

where the \( A_{\ell} \) must have Hermitian symmetry.

We have now found a general form for any stationary point. Unfortunately, for a given \( H(\omega) \), this stationary point may lie outside of the domain of the optimization. The
optimal spectrum $S_p(\omega)$ must therefore lie on the boundary of the optimization domain, which in this case implies that $S_p(\omega) = 0$ for $\omega \in \mathcal{I}_0 \subset [-\pi, \pi]$. Let us define $\mathcal{I}_+$ as the subset $[-\pi, \pi]$ where $S_p(\omega) > 0$ except for the endpoints of $\mathcal{I}_+$ where $S_p(\omega) = 0$ due to the assumption of a continuous spectrum. Note that $\mathcal{I}_+$ may be the union of several disjoint sub-intervals of $[-\pi, \pi]$. We can now rewrite the constraint and the expressions of $b_k$ as

$$e[y(\omega)] = \int_{\mathcal{I}_+} \frac{1}{|H(\omega)|^2} d\omega + \frac{2\pi}{N_0}$$

and

$$b_k = \frac{1}{2\pi} \int_{\mathcal{I}_+} y(\omega)e^{jk\omega} d\omega.$$  

From the first part of the proof, i.e., identifying a necessary condition for stationary points, we have that (11) must hold within the interval $\mathcal{I}_+$, and the constants $\{A_k\}$ must be such that $S_p^{\text{opt}}(\omega) = 0$ at the end-points of each sub-interval within $\mathcal{I}_+$. Hence, no matter what $\mathcal{I}_+$ is, we can express the optimal $S_p^{\text{opt}}(\omega)$ as in (10).

**APPENDIX B: PROOF OF THEOREM 2**

The waterfilling algorithm provides a transmit filter that satisfies (11)

$$|P(\omega)|^2 = \max \left(0, \theta - \frac{N_0}{|H(\omega)|^2}\right),$$  

for some power constant $\theta$. In view of Theorem 1, $|P(\omega)|^2$ in (12) must also satisfy (10). Equating (12) and (10) yields

$$\theta - \frac{N_0}{|H(\omega)|^2} = \frac{N_0}{\sqrt{|H(\omega)|^2}} \sum_{\ell = -K}^K A_\ell e^{j\ell\omega} - \frac{N_0}{|H(\omega)|^2}.$$  

From (13), it can be seen that we must have

$$\sum_{\ell = -K}^K A_\ell e^{j\ell\omega} = \gamma |H(\omega)|^2,$$

for some constant $\gamma$. However,

$$|H(\omega)|^2 = \sum_{\ell = 0}^{L_H} h_\ell e^{-j\ell\omega} = \sum_{\ell = -L_H}^{L_H} g_\ell e^{-j\ell\omega},$$

where

$$g_\ell = \sum_k h_k b_{k-\ell}^*.$$  

Clearly, to satisfy

$$\sum_{\ell = -K}^K A_\ell e^{j\ell\omega} = \gamma \left[\sum_{\ell = -L_H}^{L_H} g_\ell e^{-j\ell\omega}\right],$$

$K$ must at least equal $L_H$.  

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