A New Determination of $M_b$ Using Lattice QCD

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Final Revision — September 1994

Abstract

Recent results from lattice QCD simulations provide a realistic picture, based upon first principles, of $\Upsilon$ physics. We combine these results with the experimentally measured mass of the $\Upsilon$ meson to obtain an accurate and reliable value for the $b$-quark’s pole mass. We use two different methods, each of which yields a mass consistent with $M_b = 5.0(2)$ GeV. This corresponds to a bare mass of $M_b^0 = 4.0(1)$ GeV in our lattice theory and an $\overline{\text{MS}}$ mass of $M_{\overline{\text{MS}}}(M_b) = 4.0(1)$ GeV. We discuss the implications of this result for the $c$-quark mass.

PACS numbers: 12.38.Gc, 14.40.Gx, 14.56.Fy, 12.38.Bx

The mass $M_b$ of the $b$ quark is a fundamental parameter of the Standard Model. Various phenomenological studies have suggested masses in the range of $4.5$–$5$ GeV [1]. In this paper we use new results from lattice QCD to obtain a value for $M_b$ from the measured mass of the $\Upsilon$ meson [2]. We employ two methods for computing the quark mass; both are consistent with $M_b = 5.0(2)$ GeV, which is presently among the most accurate and reliable of determinations.

The notion of quark mass is complicated by quark confinement [3]. In a particular process, a $b$-quark’s effective, or running, mass $M_b(q)$ depends upon the typical momentum $q$ transferred to it. In perturbation theory, at least, this mass stops running when $q$ falls below $M_b(q)$. Below this point, $M_b(q)$ equals the quark’s pole mass, which is defined perturbatively, in terms of the bare mass,
by the pole in the quark propagator. It is this mass we quote for $M_b$ above. The use of perturbation theory is certainly justified when $q = M_b$. At very low $q$'s the effective mass may be modified away from our value by nonperturbative effects, but our value should hold for $\Lambda_{QCD} \ll q \ll M_b$. It is therefore the appropriate mass to use in quark potential models and other phenomenological applications that involve this momentum range. We may also use perturbation theory to relate our bare quark mass to the $\overline{\text{MS}}$ mass; we find $M_b^{\overline{\text{MS}}}(M_b) = 4.0(1)$ GeV.

Both of our mass determinations rely on numerical simulations of the $\Upsilon$ using lattice QCD. In lattice QCD spacetime is approximated by a discrete grid of finite physical volume. An action for QCD is defined in terms of quark and gluon fields on the nodes of the grid and the links joining them. Monte Carlo methods are used to evaluate the path integrals that define various correlation, or Green’s, functions. The energies and wavefunctions of the $\Upsilon$ and its excitations are extracted from these correlation functions.

Our gluon configurations were generated using the standard Wilson lagrangian for gluons. For most of our simulations we used a $16^3 \times 24$ grid with $\beta \equiv 6/g^2 = 6.0$, where $g$ is the bare coupling constant. This corresponds to an inverse lattice spacing of $a^{-1} = 2.4(1)$ GeV and a volume of approximately $1.3^3 \times 2.0$ fm$^3$. The lattice volume is more than adequate since $\Upsilon$'s have a radius of only $R_\Upsilon \approx 0.2$ fm. The finite grid spacing introduces systematic errors of order $a^2/R_\Upsilon^2$; we have estimated these to be less than 5% (for binding energies) both from quark potential models and from other simulations with smaller lattice spacings.

The gluonic configurations described above do not include effects due to light-quark vacuum polarization. There are strong theoretical arguments suggesting that $\Upsilon$ physics is largely unaffected by the light quarks, at least for states that are well below the $B\bar{B}$ threshold. To verify this we repeated a part of our analysis for gluon configurations that contain contributions from $n_f = 2$ flavors of light quarks. The mass of the light quarks in these configurations (under 100 MeV) is larger than the mass of a $u$ or $d$ quark but negligible relative to the typical momentum transfers in an $\Upsilon$ (about 1 GeV). Thus the quarks are effectively massless. The lattice spacing and volume are the same for these configurations as for the $n_f = 0$ configurations described above. The bulk of the results presented in this paper are based upon the simulations with $n_f = 0$; however, we found these essentially unchanged by the inclusion of light-quarks.

We used the NRQCD lagrangian for the $b$ quarks. This is a nonrelativistic lagrangian that is tuned to efficiently reproduce results from continuum relativistic QCD order-by-order in the lattice spacing $a$ and quark velocity $v$. We included the leading relativistic and finite-lattice-spacing corrections; further corrections are almost certainly negligible for our purposes. The only parameter in this lagrangian, other than the QCD charge $g$, is the bare quark mass $M_b^0$. This is tuned so that the simulation gives the correct dispersion relation for the $\Upsilon$. The energy of a low-momentum $\Upsilon$ can be computed in the simulation,
and has the form

$$E_{\Upsilon}(p) \approx E_{\text{NR}}(\Upsilon) + \frac{p^2}{2M_{\text{kin}}(\Upsilon)} ,$$  

(1)

where $E_{\text{NR}}(\Upsilon)$ is related to the nonrelativistic binding energy of the meson, and $M_{\text{kin}}(\Upsilon)$ is the kinetic mass of the meson. We tuned the bare quark mass $M_0^b$ until $M_{\text{kin}}(\Upsilon)$ was equal to the measured mass of the $\Upsilon$. We found that the correct bare mass is given by $aM_0^b = 1.7(1)$.

Our simulations very successfully reproduce the general features of $\Upsilon$ physics. This is illustrated by Fig. 1, which compares simulation results for the spectrum of the lowest-lying states with experiment. In Fig. 2, simulation results for the fine structure of the lowest lying $P$ state are compared with experiment. The excellent agreement supports the reliability of these simulations. We emphasize that these are calculations from first principles; our approximations can be systematically improved. The only inputs are the lagrangians describing gluons and quarks, and the only parameters are the bare coupling constant and quark mass. In particular, these simulations are not based on a phenomenological quark potential model.

Our first procedure for computing the $b$-quark mass uses simulation results for the nonrelativistic energy $E_{\text{NR}}(\Upsilon)$ of the $\Upsilon$ (Eq. (1)). The quark mass is

$$M_b = (1/2) \left( M_\Upsilon - (E_{\text{NR}} - 2E_0) \right),$$

(2)

where $M_\Upsilon = 9.46$ GeV is the experimentally measured mass of the $\Upsilon$, and $E_0$ is the nonrelativistic energy of a $b$ quark with $p = 0$ in NRQCD. The quantity $E_{\text{NR}}(\Upsilon) - 2E_0$ is the effective binding energy of the meson. The quark energy $E_0$ is ultraviolet divergent as the lattice spacing $a$ vanishes, and so at small $a$, it depends primarily on physics at momenta of order $1/a$. Thus it can be computed using weak-coupling perturbation theory [10], with

$$E_0 = \left( \frac{b_0}{a} \right) \alpha_V(q_0) \left( 1 + \mathcal{O}(\alpha_V) \right).$$

(3)

Here $\alpha_V(q_0)$ is the strong coupling constant defined in [11].

In Table 1 we present our simulation results for $E_{\text{NR}}$, and the corresponding $E_0$’s from perturbation theory. We list results for several values of the bare quark mass; for the $b$-quark, $aM_0^b = 1.7(1)$. From these results we conclude that the pole mass of the $b$ quark is $M_b = 5.0(2)$ GeV. The major sources of uncertainty in this determination are:

1. In a purely nonrelativistic theory, Galilean invariance implies that the kinetic mass of a bound state equals the sum of the masses of its constituents. In a relativistic theory the binding energy enters as well, resulting in a 10% shift for $\Upsilon$’s. Thus it is important to include relativistic corrections in the quark action when tuning the quark mass using $M_{\text{kin}}$.

2. All perturbative results in this paper are obtained using the techniques and conventions of [11]. In particular, the definition of the running coupling constant and the procedure for setting the coupling scale are from this reference. The value of $\alpha_V$ is determined from the measured expectation value of the plaquette operator, as discussed in [11]. This coupling constant is related to the MS coupling by: $\alpha_{\text{MS}}(Q) = \alpha_V(\exp(5/6) Q) \left( 1 + 2 \alpha_V/\pi \right)$. 

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Figure 1: NRQCD simulation results for the spectrum of the \( \Upsilon \) system including radial excitations. Experimental values (dashed lines) are indicated for the triplet \( S \)-states, and for the spin-average of the triplet \( P \)-states. The energy zero from simulation results is adjusted to give the correct mass to the \( \Upsilon(1^{3}S_{1}) \). These data are for \( n_f = 0 \).
Figure 2: Simulation results for the spin splittings between the lowest lying $P$-wave states in the $\Upsilon$ family. The dashed lines are the experimental values for the triplet states. Energies are measured relative to the center of mass of the triplet states. These data are for $n_f = 0$. 


Table 1: Simulation and perturbative results used in the first method for determining $M_b$. Values for $M_b$ are in GeV, and are determined using Eq. (6). Results are presented for $n_f = 0$ and 2 light-quark flavors, and for a range of bare quark masses; the $b$-quark has $aM_0^b = 1.7(1)$. Results are also given for just the leading term in the NRQCD lagrangian ($\delta H = 0$).

(1) *Tuning errors in the bare quark mass.* The binding energy, $E_{NR} - 2E_0$, like the radial and orbital level splittings, should be almost independent of the quark's mass for masses of order $M_b$. Thus $M_b$, as determined from Eq. (2), should also be independent of the quark's mass. This is confirmed by our results in Table 1. Errors in $M_b$ due to uncertainty in the bare quark mass are less than 1%.

(2) *Two-loop corrections to $E_0$.‖ Quantitative studies \[11\] of the reliability of perturbation theory at scales of order $q_0$ suggest that two-loop corrections to $E_0$ could range from 1–30% of $E_0$. A 20% contribution would shift $M_b$ by about 3%. Two-loop corrections could not be much larger than this withoutruining the agreement we find between results obtained with widely different quark masses. Nonperturbative effects are most likely smaller.

(3) *Simulation errors in $E_{NR}(\Upsilon)$.‖ The dominant sources of error in $E_{NR}$ are the \[O(a^2)\] errors in the gluon action, and our use of zero and two flavors of light quarks rather than three. Based upon the results in the Table 1, together with results from simulations with smaller lattice spacings \[6\], we expect these effects to shift the quark mass by less than 100 MeV. Statistical errors are negligible here.

(4) *Simulation errors in $a^{-1}$.‖ The lattice spacing is determined by matching simulation results for the $\Upsilon$ spectrum to experiment. It is easily measured to within $\pm 5\%$. Since $E_{NR}$ and $2E_0$ are each only about 10% of $M_\Upsilon$, this amounts to an uncertainty of less than 0.5% in $M_b$.

The relativistic and finite-lattice-spacing corrections in the quark lagrangian should have only a small effect on the total mass of the $\Upsilon$. This is confirmed by our simulation, as illustrated in Table 1. The last entry in the table was obtained from a simulation without these corrections. It agrees well with results from the full simulation, confirming that further corrections are probably unimportant.
Our second procedure for determining $M_b$ is to tune the bare quark mass $M_b^0$ until the kinetic mass of the $\Upsilon$, as computed in the simulation (Eq. (1)), agrees with the $\Upsilon$’s true mass. Then the pole mass is $Z_m M_b^0$, where the renormalization constant $Z_m$ is computed using perturbation theory [10], with

$$Z_m = 1 + b_m \alpha V(q_m) + \mathcal{O}(\alpha^2 V).$$

(4)

To reduce the sensitivity of the result to the values of the lattice spacing and bare quark mass, we rewrite the expression for the pole mass as

$$M_b = Z_m M_{\Upsilon} a M_{\text{kin}(\Upsilon)}. \quad (5)$$

Here $M_{\Upsilon} = 9.46$ GeV is the $\Upsilon$’s experimental mass, while $M_{\text{kin}(\Upsilon)}$ is its mass as determined from a simulation using the bare mass $M_b^0$. Our results are summarized in Table 2. For $n_f = 0$, the kinetic mass is closest to the $\Upsilon$ mass when $a M_b^0 = 1.71$, and therefore the renormalized quark mass is $M_b = 4.9(2)$ GeV. The corresponding bare quark mass is $M_b^0 = 4.1(1)$ GeV (from Eq. (3) with $Z_m \to 1$). The $n_f = 2$ data must be extrapolated slightly in $a M_b^0$ to obtain the correct $\Upsilon$ mass. Using the $n_f = 0$ data as a guide for this extrapolation, we obtain $M_b = 5.0(2)$ GeV and $M_b^0 = 4.0(1)$ GeV for $n_f = 2$.

The main sources of uncertainty in these determinations are:

(1) **Two-loop corrections to $Z_m$.** The one-loop corrections to $Z_m$ shift $M_b$ by 15–20%. If two-loop corrections are 15–20% of the one-loop corrections, they could shift $M_b$ by 2–4%. The bare mass is unaffected by these corrections.

(2) **Simulation errors in $a M_{\text{kin}(\Upsilon)}$.** These are mainly due to the omission of $\mathcal{O}(v^4)$ relativistic corrections in the quark action. Such effects should shift $M_b$ by less than 1%. (The $\mathcal{O}(v^2)$ corrections, which we include, shift $M_b$ by only 7%.) Statistical errors in $a M_{\text{kin}}$ are of order 1%.

(3) **Tuning errors in the bare quark mass.** Since $a^{-1}$ is known only to ±4%, the $\Upsilon$’s kinetic mass can only be tuned with an accuracy of ±4%. However our determination of $M_b$ (Eq. (5)) minimizes sensitivity to such tuning, as is evident from Table 2. Thus tuning errors have a negligible effect on $M_b$.

In this paper, we have presented two different determinations of the $b$-quark mass, each consistent with a pole mass of $M_b = 5.0(2)$ GeV. The systematic errors are quite different in each case, and have been probed by computing with a variety of quark masses and quark lagrangians, with and without light-quark vacuum polarization. The complete agreement between the two methods is a strong indication of the validity of each, and more generally of the lattice QCD techniques on which they rely. In both cases the dominant uncertainties are in the perturbative calculations. A two-loop evaluation of either the quark energy $E_0$ or the mass renormalization constant $Z_m$ would probably reduce the errors in $M_b$ by a factor of two or more. We are currently examining the feasibility of such calculations.
Table 2: Simulation and perturbative results used in the second method for determining \( M_b \). Values for \( M_{\text{kin}}(\Upsilon) \), \( M_b \) and \( M^0_b \) (last column) are in GeV. Values for \( M_b \) are determined using Eq. (9). Values for \( M^0_b \) are obtained using Eq. (9) with \( Z_m \to 1 \). Results are given for \( n_f = 0 \) and 2 light-quark flavors, and for a range of bare quark masses; the \( b \)-quark has \( a M^0_b = 1.7(1) \).

| \( \beta \) | \( n_f \) | \( a M^0_b \) | \( a M_{\text{kin}}(\Upsilon) \) | \( b_m \) | \( a q_m \) | \( Z_m - 1 \) | \( M_{\text{kin}}(\Upsilon) \) | \( M_b \) | \( M^0_b \) |
|------|-----|---------|------------|------|------|---------|------------|------|------|
| 6.0  | 0   | 1.71    | 3.94(3)   | 0.48 | 0.49 | 0.20    | 9.5(4)     | 4.92 | 4.11 |
| 1.80 | 0   | 4.09(3) | 0.46      | 0.51 | 0.18 | 9.8(4)  | 4.93       | 4.16 |
| 2.00 | 0   | 4.49(4) | 0.42      | 0.54 | 0.16 | 10.7(4) | 4.88       | 4.21 |
| 3.00 | 0   | 6.57(7) | 0.24      | 0.52 | 0.09 | 15.8(6) | 4.73       | 4.32 |
| 5.6  | 2   | 1.80    | 4.17(2)   | 0.46 | 0.51 | 0.23    | 10.0(4)    | 5.02 | 4.08 |

Combining the standard result for the pole mass in terms of the \( \overline{\text{MS}} \) mass \cite{12},

\[
M_b = M^\overline{\text{MS}}_b(M_b) \left\{ 1 + 0.424 \alpha_V(0.22 M_b) + 0.164 \alpha^2_V + \cdots \right\},
\]

with our result in terms of the bare quark mass, \( M_b = Z_m M^0_b \), we obtain a relation between the bare quark mass on the lattice and the \( \overline{\text{MS}} \) mass:

\[
M^\overline{\text{MS}}_b(M_b) = M^0_b \left\{ 1 + 0.06 \alpha_V(q_{\overline{\text{MS}}}) + \cdots \right\},
\]

where \( q_{\overline{\text{MS}}} \approx 0.8/a \) for our \( M^0_b \)'s. Using this relation, our data implies that \( M^\overline{\text{MS}}_b(M_b) = 4.0(1) \) GeV. \( O(\alpha^2_V) \) corrections should be smaller than in Eq. (5) for the pole mass. This is because the scale for \( \alpha_V \) is significantly larger here and therefore \( \alpha_V \) is smaller. The larger scale is sensible since the \( \overline{\text{MS}} \) mass is a bare mass and so should be less infrared sensitive than the on-shell pole mass.

Our \( b \)-quark mass can be used to estimate the \( c \)-quark’s pole mass. Because of heavy-quark symmetry, the difference between the spin-averaged mass of the \( B \) and \( B^* \) mesons, \( M_{B/B^*} \), and \( M_b \) is the same as the difference between the spin-averaged \( D/D^* \) mass and \( M_c \) up to corrections of order \( \Lambda_{\text{QCD}}^2/M_c \). Thus we expect

\[
M_c \approx M_{D/D^*} + (M_b - M_{B/B^*}) = 1.6(2) \text{ GeV}.
\]

A different estimate is possible if the binding energies of the \( \Upsilon \) and \( \psi \) are roughly equal. This is probably the case since radial and orbital excitation energies are roughly equal in the two systems. Also our simulation shows that the binding energy is almost mass independent for masses larger than \( M_b \). Thus we expect

\[
M_c \approx (1/2) (M_{\psi} + (2M_b - M_{\Upsilon})) = 1.8(2) \text{ GeV}.
\]

We are currently extending our lattice analysis to allow a direct determination of \( M_c \).
We thank Greg Anderson and Paul Mackenzie for discussions concerning this analysis. The simulations described here were carried out at the Ohio Supercomputer Center. This work was supported by grants from the NSF, the SERC, and the DOE (DE-FG02-91ER40690, DE-FC05-85ER250000, DE-FG05-92ER40742, DE-FG05-92ER40722).

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