Correcting Quantum Errors with Entanglement

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We show how entanglement shared between encoder and decoder can simplify the theory of quantum error correction. The entanglement-assisted quantum codes we describe do not require the dual-containing constraint necessary for standard quantum error correcting codes, thus allowing us to “quantize” all of classical linear coding theory. In particular, efficient modern classical codes that attain the Shannon capacity can be made into entanglement-assisted quantum codes attaining the hashing bound (closely related to the quantum capacity). For systems without large amounts of shared entanglement, these codes can also be used as catalytic codes, in which a small amount of initial entanglement enables quantum communication.
Entanglement plays a central role in quantum information processing. It enables the teleportation of quantum states without physically sending quantum systems (1); it doubles the capacity of quantum channels for sending classical information (2); it is known to be necessary for the power of quantum computation (3, 4). We show how shared entanglement provides a simpler and more fundamental theory of quantum error correction.

The theory of quantum error correcting codes was established a decade ago as the primary tool for fighting decoherence in quantum computers and quantum communication systems. The first nine-qubit single error-correcting code was a quantum analog of the classical repetition code, which stores information redundantly by duplicating each bit several times (5). Probably the most striking development in quantum error correction theory is the use of the stabilizer formalism (6, 7, 8, 9), whereby quantum codes are subspaces (“code spaces”) in Hilbert space, and are specified by giving the generators of an abelian subgroup of the Pauli group, called the stabilizer of the code space. Essentially, all QECCs developed to date are stabilizer codes. The problem of finding QECCs was reduced to that of constructing classical dual-containing quaternary codes (6). When binary codes are viewed as quaternary, this amounts to the well known Calderbank-Shor-Steane construction (10, 11). The requirement that a code contain its dual is a consequence of the need for a commuting stabilizer group. The virtue of this approach is that we can directly construct quantum codes from classical codes with a certain property, rather than having to develop a completely new theory of quantum error correction from scratch. Unfortunately, the need for a self-orthogonal parity check matrix presents a substantial obstacle to importing the classical theory in its entirety, especially in the context of modern codes such
as low-density parity check (LDPC) codes (\textsuperscript{12}).

Assume that the encoder Alice and decoder Bob have access to shared entanglement. We will argue that in this setting every quaternary (or binary) classical linear code, not just dual-containing codes, can be transformed into a QECC, and illustrate this with a particular example. If the classical codes are not dual-containing, they correspond to a set of stabilizer generators that do not commute; however, if shared entanglement is an available resource, these generators may be embedded into larger, commuting generators, giving a well-defined code space. We call this the entanglement-assisted stabilizer formalism, and the codes constructed from it are entanglement-assisted QECCs (EAQECCs).

**Standard stabilizer formalism.** The power of the stabilizer formalism comes from the clever use of group theory. Let $\Pi$ denote the set of Pauli operators $\{I, X, Y, Z\}$, and let $\Pi^n = \{I, X, Y, Z\}^\otimes n$ denote the set of $n$-fold tensor products of single-qubit Pauli operators. Then $\Pi^n$ together with the possible overall factors $\pm 1, \pm i$ forms a group $G_n$ under multiplication, the $n$-fold Pauli group. Here are a few useful properties of the $n$-fold Pauli group: (a) every element of $G_n$ squares to $\pm I_n$ (plus or minus the identity); (b) any two elements of $G_n$ either commute or anti-commute; (c) every element of $G_n$ is unitary; and (d) elements of $G_n$ are either Hermitian or anti-Hermitian. The connection of $G_n$ to error correction is straightforward: the elements of $G_n$ can be identified as possible sets of errors that might affect a quantum register of $n$ qubits.

Suppose $S$ is an abelian subgroup of $G_n$. We define the stabilizer code $C(S)$ associated with
\( \mathcal{C}(S) = \{ |\psi\rangle : M|\psi\rangle = |\psi\rangle, \forall M \in S \} \).

The code \( \mathcal{C}(S) \) is the subspace fixed by \( S \), so \( S \) is called the stabilizer of the code. In other words, the code space is the simultaneous +1 eigenspace of all elements of \( S \). For an \([n, k]\) stabilizer code, which encodes \( k \) logical qubits into \( n \) physical qubits, \( \mathcal{C}(S) \) has dimension \( 2^k \) and \( S \) has \( 2^{n-k} \) elements. We should notice that for group \( S \) to be the stabilizer of a nontrivial subspace, it must satisfy two conditions: the elements of \( S \) commute, and \(-I_n\) is not in \( S \). (This second condition implies that all elements of \( S \) are Hermitian, and hence have eigenvalues ±1.)

A group \( S \) can be specified by a set of independent generators, \( \{ M_i \} \). These are elements in \( S \) that cannot be expressed as products of each other, and such that each element of \( S \) can be written as a product of elements from the set. If an abelian subgroup \( S \) of \( G_n \) has \( 2^{n-k} \) distinct elements up to an overall phase, then there are \( n - k \) independent generators. The benefit of using generators is that it provides a compact representation of the group; and to see whether a particular vector \( |\psi\rangle \) is stabilized by a group \( S \), we need only check whether \( |\psi\rangle \) is stabilized by these generators of \( S \).

Suppose \( \mathcal{C}(S) \) is a stabilizer code, and the quantum register is subject to errors from an error set \( E = \{ E_a \} \subset G_n \). How are the error-correcting properties of \( \mathcal{C}(S) \) related to the generators of \( S \)? First, suppose that \( E_a \) anti-commutes with a particular stabilizer generator \( M_i \) of \( S \). Then

\[
M_i E_a |\psi\rangle = -E_a M_i |\psi\rangle = -E_a |\psi\rangle.
\]

\( E_a |\psi\rangle \) is an eigenvector of \( M_i \) with eigenvalue \(-1\), and hence must be orthogonal to the code.
space (all of whose vectors have eigenvalue +1). As the error operator $E_a$ takes the code space of $\mathcal{C}(S)$ to an orthogonal subspace, an occurrence of $E_a$ can be detected by measuring $M_i$. For each generator $M_i$ and error operator $E_a$, we can define a coefficient $s_{i,a} \in \{0, 1\}$ depending on whether $M_i$ and $E_a$ commute or anti-commute:

$$M_i E_a = (-1)^{s_{i,a}} E_a M_i.$$  

The vector $\mathbf{s}_a = (s_{1,a}, s_{2,a}, \cdots, s_{n-k,a})$ represents the syndrome of the error $E_a$. In the case of a nondegenerate code, the error syndrome is distinct for all $E_a \in \mathcal{E}$, so that measuring the $n - k$ stabilizer generators will diagnose the error completely. However, a uniquely identifiable error syndrome is not always required for an error to be correctable.

What if $E_a$ commutes with the generators of $S$? If $E_a \in S$, we do not need to worry, because the error does not corrupt the space at all. The real danger comes when $E_a$ commutes with all the elements of $S$ but is not itself in $S$. The set of elements in $\mathcal{G}_n$ that commute with all of $S$ is the centralizer $\mathcal{Z}(S)$ of $S$. If $E \in \mathcal{Z}(S) - S$, then $E$ changes elements of $\mathcal{C}(S)$ but does not take them out of $\mathcal{C}(S)$. Thus, if $M \in S$ and $|\psi\rangle \in \mathcal{C}(S)$, then

$$ME|\psi\rangle = EM|\psi\rangle = E|\psi\rangle.$$  

Because $E \not\in S$, there is some state of $\mathcal{C}(S)$ that is not fixed by $E$. $E$ will be an undetectable error for this code. Putting these cases together, a stabilizer code $\mathcal{C}(S)$ can correct a set of errors $\mathcal{E}$ if and only if $E_a^\dagger E_b \in S \cup (\mathcal{G}_n - \mathcal{Z}(S))$ for all $E_a, E_b \in \mathcal{E}$.
**Entanglement-assisted stabilizer codes.** We will now illustrate the idea of the entanglement-assisted stabilizer formalism by an example. We know from the previous paragraph that a stabilizer code can be constructed from a commuting set of operators in $G_n$. What if we are given a non-commuting set of operators? Can we still construct a QECC? Let $S$ be the group generated by the following non-commuting set of operators:

\[
\begin{align*}
M_1 &= Z \; X \; Z \; I \\
M_2 &= Z \; Z \; I \; Z \\
M_3 &= X \; Y \; X \; I \\
M_4 &= X \; X \; I \; X
\end{align*}
\]

(1)

It is easy to check the commutation relations of this set of generators: $M_1$ anti-commutes with the other three generators, $M_2$ commutes with $M_3$ and anti-commutes with $M_4$, and $M_3$ and $M_4$ anti-commute. We will begin by finding a different set of generators for $S$ with a particular class of commutation relations. We then relate $S$ to a group $B$ with a particularly simple form, and discuss the error-correcting conditions using $B$. Finally, we relate these results back to the group $S$.

To see how this works, we need two lemmas. (See the supporting online material for proofs.) The first lemma shows that there exists a new set of generators for $S$ such that $S$ can be decomposed into an “isotropic” subgroup $S_I$ generated by a set of commuting generators, and a “symplectic” subgroup $S_S$ generated by a set of anti-commuting generator pairs $\{(I3)\}$.

**Lemma 1.** Given any arbitrary subgroup $V$ in $G_n$ that has $2^m$ distinct elements up to overall phase, there exists a set of $m$ independent generators for $V$ of the form $\{Z_1, Z_2, \ldots, Z_{\ell}, X_1, \ldots, X_{m-\ell}\}$
where \( m/2 \leq \ell \leq m \), such that \([Z_i, Z_j] = [X_i, X_j] = 0\) for all \( i, j \); \([Z_i, X_j] = 0\) for all \( i \neq j \); and \([Z_i, X_i] = 0\) for all \( i \). Here \([A, B]\) is the commutator and \(\{A, B\}\) the anti-commutator of \(A\) with \(B\). Let \(\mathcal{V}_I = \langle Z_{m-\ell+1}, \cdots, Z_{\ell} \rangle\) denote the isotropic subgroup generated by the set of commuting generators, and let \(\mathcal{V}_S = \langle Z_1, \cdots, Z_{m-\ell}, X_1, \cdots, X_{m-\ell} \rangle\) denote the symplectic subgroup generated by the set of anti-commuting generator pairs. Then, with slight abuse of the notation, \(\mathcal{V} = \langle \mathcal{V}_I, \mathcal{V}_S \rangle\) indicates that \(\mathcal{V}\) is generated by subgroups \(\mathcal{V}_I\) and \(\mathcal{V}_S\).

For the group \(S\) that we are considering, one such set of independent generators is

\[
\begin{align*}
Z_1 &= Z \ X \ Z \ I \\
X_1 &= Z \ Z \ I \ Z \\
Z_2 &= Y \ X \ X \ Z \\
Z_3 &= Z \ Y \ Y \ X
\end{align*}
\] (2)

so that \(S_S = \langle Z_1, X_1 \rangle, S_I = \langle Z_2, Z_3 \rangle\), and \(S = \langle S_I, S_S \rangle\).

The choice of the notation \(Z_i\) and \(X_i\) is not accidental: these generators have exactly the same commutation relations as a set of Pauli operators \(Z_i\) and \(X_i\) on a set of qubits labeled by \(i\). Let \(B\) be the group generated by the following set:

\[
\begin{align*}
Z_1 &= Z \ I \ I \ I \\
X_1 &= X \ I \ I \ I \\
Z_2 &= I \ Z \ I \ I \\
Z_3 &= I \ I \ Z \ I
\end{align*}
\] (3)

From the previous lemma, \(B = \langle B_I, B_S \rangle\), where \(B_S = \langle Z_1, X_1 \rangle\) and \(B_I = \langle Z_2, Z_3 \rangle\). Therefore, groups \(B\) and \(S\) are isomorphic, which is denoted as \(B \cong S\). We can then relate \(S\) to the simpler
group $\mathcal{B}$ by the following lemma (14):

**Lemma 2.** If $\mathcal{B} \cong \mathcal{S}$, then there exists a unitary $U$ such that for all $B \in \mathcal{B}$ there exists an $S \in \mathcal{S}$ such that $B = USU^{-1}$ up to an overall phase.

As a consequence of this lemma, the error-correcting power of $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{S})$ are also related by a unitary transformation. In what follows, we will use $\mathcal{B}$ to discuss the error-correcting conditions, and then translate the results back to $\mathcal{S}$.

What is the code space $\mathcal{C}(\mathcal{B})$ described by $\mathcal{B}$? Because $\mathcal{B}$ is not a commuting group, the usual definition of $\mathcal{C}(\mathcal{B})$ does not apply, as the generators do not have a common eigenspace. However, by extending the generators, we can find a new group that is commuting, and for which the usual definition of code space can apply; the qubits of the codewords will be embedded in a larger space. Notice that we can append a $Z$ operator at the end of $Z_1$, an $X$ operator at the end of $X_1$, and an identity at the end of $Z_2$ and $Z_3$ to make $\mathcal{B}$ abelian:

$$
\begin{align*}
Z_1' &= Z I I I Z \\
X_1' &= X I I I X \\
Z_2' &= I Z I I I \\
Z_3' &= I I Z I I I 
\end{align*}
$$

We assume that the four original qubits are possessed by Alice (the sender), and the additional qubit is possessed by Bob (the receiver) and is not subject to errors. Let $\mathcal{B}_e$ be the extended group generated by $\{Z_1', X_1', Z_2', Z_3'\}$. We define the code space $\mathcal{C}(\mathcal{B})$ to be the simultaneous $+1$ eigenspace of all elements of $\mathcal{B}_e$, and we can write it down explicitly in this case:

$$
\mathcal{C}(\mathcal{B}) = \{|\Phi\rangle^{AB}|0\rangle|\psi\rangle\}
$$

8
where $|\Phi\rangle_{AB}$ is a maximally entangled state shared between Alice and Bob, and $|\psi\rangle$ is an arbitrary single-qubit pure state. Because entanglement is used, this is an EAQECC. We use the notation $[[n, k; c]]$ to denote an EAQECC that encodes $k$ qubits into $n$ qubits with the help of $c$ ebits. (Sometimes we will write $[[n, k, d; c]]$ to indicate that the “distance” of the code is $d$, meaning it can correct at least $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors.) The number of ebits $c$ needed for the encoding is equal to the number of anti-commuting pairs of generators in $B_S$. The number of ancilla bits $s$ is equal to the number of independent generators in $B_I$. The number of encoded qubits $k$ is equal to $n - c - s$, and we define the rate of the EAQECC to be $(k - c)/n$. Therefore, $C(B)$ is a $[[4, 1, 1]]$ EAQECC with zero rate: $n = 4$, $c = 1$, $s = 2$ and $k = 1$. Note that the zero rate does not mean that no qubits are transmitted by this code! Rather, it implies that a number of bits of entanglement is needed that is equal to the number of bits transmitted. In general, $k - c$ can be positive, negative, or zero.

Now we see how the error-correcting conditions are related to the generators of $B$. We saw that if an error $E_a \otimes I_B$ anti-commutes with one or more of the operators in $\{Z'_1, X'_1, Z'_2, Z'_3\}$, it can be detected by measuring these operators. This will only happen if the error $E_a$ anti-commutes with one of the operators in the original set of generators $\{Z_1, X_1, Z_2, Z_3\}$, as the entangled bit held by Bob is assumed to be error-free. Alternatively, if $E_a \otimes I_B \in B_e$, or equivalently if $E_a \in B_I$, then $E_a$ does not corrupt the encoded state. In this case we call the code degenerate. Altogether, $C(B)$ can correct a set of errors $\mathcal{E}_0$ if and only if $E_a \otimes E_b \in B_I \cup (G_4 - Z(B))$ for all $E_a, E_b \in \mathcal{E}_0$.

With this analysis of $B$, we can go back to determine the error-correcting properties of our
original stabilizer $S$. We can construct a QECC from a nonabelian group $S$ if entanglement is available, just as we did for the group $B$. We add extra operators $Z$ and $X$ to make $S$ abelian as follows:

$$
\begin{align*}
Z'_1 &= Z X Z I Z \\
X'_1 &= Z Z I Z X \\
Z'_2 &= Y X Y X Z I \\
Z'_3 &= Z Y Y X I \\
\end{align*}
$$

(6)

where the extra qubit is once again assumed to be possessed by Bob and to be error-free. Let $S_e$ be the group generated by the above operators. Since $B \cong S$, let $U$ be the unitary from Lemma 2. Define the code space $C(S)$ by $C(S) = U^{-1}(C(B))$, where the unitary $U$ is applied only on Alice’s side. This unitary $U$ can be interpreted as the encoding operation of the EAQECC defined by $S$. Observe that the code space $C(S)$ is a simultaneous eigenspace of all elements of $S_e$. As in the analysis for $C(B)$, the code $C(S)$ can correct a set of errors $E$ iff $E_a \cup E_b \in S_1 \cup (G_4 - Z(S))$ for all $E_a, E_b \in \mathcal{E}$.

The algebraic description is somewhat abstract, so let us translate this into a physical picture. Alice wishes to encode a single ($k = 1$) qubit state $|\psi\rangle$ into four ($n = 4$) qubits, and transmit them through a noisy channel to Bob. Initially, Alice and Bob share a single ($c = 1$) maximally entangled pair of qubits—one ebit. Alice performs the encoding operation $U$ on her bit $|\psi\rangle$, her half of the entangled pair, and two ($s = 2$) ancilla bits. She then sends the four qubits through the channel to Bob. Bob measures the extended generators $Z'_1, X'_1, Z'_2, Z'_3$ on the four received qubits plus his half of the entangled pair. The outcome of these four measurements
gives the error syndrome; as long as the error set satisfies the above requirement, Bob can correct the error and decode the transmitted qubit $|\psi\rangle$.

We have worked out the procedure for a particular example, but any EAQECC will function in the same way. The particular parameters $n, k, c, s$ will vary depending on the code. It should be noted that the first example of entanglement-assisted error correction produced a $[[3, 1, 3; 2]]$ EAQECC based on the $[[5, 1, 3]]$ standard QECC (15). Our construction differs in that it is completely general and, more important, eschews the need for commuting stabilizers.

**Construction of EAQECCs from classical quaternary codes.** We will now examine the $[[4, 1; 1]]$ EAQECC given above, and show that it can be derived from a classical non-dual-containing quaternary $[4, 2]$ code. This is a generalization of the well-known construction for standard QECCs (16).

First, note that this $[[4, 1; 1]]$ code is non-degenerate, and can correct an arbitrary one-qubit error. (Therefore the distance $d$ of the code $C(S)$ is 3.) This is because the 12 errors $X_i, Y_i$ and $Z_i, i = 1, \ldots, 4$, have distinct non-zero error syndromes. $X_i$ denotes the bit flip error on $i$-th qubit, $Z_i$ denotes the phase error on the $i$-th qubit, and $Y_i$ means that both a bit flip and phase flip error occur on the $i$-th qubit. It suffices to consider only these three standard one-qubit errors, because any other one-qubit error can be written as a linear combination of these three errors and the identity.

Next, we define the following map between the Pauli operators and elements of GF(4), the
field with four elements:

|   | Π | I | X | Y | Z |
|---|---|---|---|---|---|
| GF(4) | 0 | 1 | ω | 1 | ω |

Note that under this map, addition in GF(4) corresponds to multiplication of the Pauli operators, up to an overall phase. So multiplication of two elements of $\mathcal{G}_n$ corresponds to addition of two $n$-vectors over GF(4), up to an overall phase.

The set of generators $\{M_i\}$ given in Eq. (1) is mapped to the matrix $\tilde{H}_4$:

$$\tilde{H}_4 = \begin{pmatrix} \omega & \overline{\omega} & \omega & 0 \\ \omega & \omega & 0 & \omega \\ \overline{\omega} & 1 & \overline{\omega} & 0 \\ \overline{\omega} & \overline{\omega} & 0 & \overline{\omega} \end{pmatrix}.$$ \hspace{1cm} (7)

Examining the matrix $\tilde{H}_4$, we see that it can be written

$$\tilde{H}_4 = \begin{pmatrix} \omega H_4 \\ \overline{\omega} H_4 \end{pmatrix},$$ \hspace{1cm} (8)

where $H_4$ is the parity-check matrix of a classical $[4, 2, 3]$ quaternary code whose rows are not orthogonal, and 3 is the minimum distance between codewords:

$$H_4 = \begin{pmatrix} 1 & \omega & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$ \hspace{1cm} (9)

We get a $[[4, 1, 3; 1]]$ EAQECC from a classical $[4, 2, 3]$ quaternary code. This outperforms the best 4-bit self-dual QECC currently known, which is $[[4, 0, 2]]$. This connection between EAQECCs and quaternary classical codes is quite general. Given an arbitrary classical $[n, k, d]$ quaternary code, we can use Eq. (8) to construct a non-degenerate
The rate becomes \((2k - n)/n\) because the \(n - k\) classical parity checks give rise to \(2(n - k)\) quantum stabilizer generators. (The complete details of this construction, along with rigorous proofs of its performance can be found in the supporting online materials.)

Discussion. Our entanglement-assisted stabilizer formalism enables us to construct QECCs from arbitrary classical quaternary codes without the dual-containing constraint. The simplification and unification that occurs when entanglement assistance is allowed is an effect well known in the context of quantum Shannon theory \(\cite{18,19}\).

The better the classical quaternary code is, the better the corresponding EAQECC will be. Searching for good quantum codes now becomes the problem of searching for good classical codes, which has been extensively studied and is well understood. Efficient modern codes, such as Turbo codes \(\cite{20}\) or LDPC codes \(\cite{21}\) whose performance approaches the classical Shannon limit, can now be used to construct corresponding quantum codes.

There are two interesting properties of EAQECCs constructed from Eq. (8). A classical quaternary code that saturates the Singleton bound will give rise to a quantum code saturating the quantum Singleton bound. To see this, assume that the \([n, k, d]\) classical quaternary code saturates the classical Singleton bound; that is, \(n - k \geq d - 1\). The corresponding \([n, 2k - n + c, d; c]\) quantum code then saturates

\[
n - (2k - n) = 2(n - k) \geq 2(d - 1),
\]

which is the quantum Singleton bound \(\cite{22}\).
Another feature of EAQECC is that a classical quaternary code that achieves the Shannon bound will give rise to a quantum code that achieves the “hashing” limit on a depolarizing channel \cite{23}. Let the rate (in base 4) $R_C$ of a $[n, k, d]$ quaternary code meeting the Shannon bound of the quaternary symmetric channel be

$$R_C = C_4(f) = 1 - (H_4(f) + f \log_4 3),$$

where $f$ is the error probability and $H_b(f) = -f \log_b f - (1 - f) \log_b (1 - f)$ is the entropy in base $b$. Then the rate (in base 2) $R_Q$ of the corresponding $[[n, 2k - n + c, d; c]]$ EAQECC is

$$R_Q = 2R_C - 1 = 1 - (H_2(f) + f \log_2 3),$$

which is exactly the hashing bound on a depolarizing channel. The hashing bound is a lower bound on the closely related quantum channel capacity. It was previously achieved only by inefficient random coding techniques \cite{23}.

The use of an EAQECC requires an adequate supply of entanglement. However, these codes can be useful even if there is not a large amount of pre-existing entanglement, by turning an EAQECC into a catalytic QECC (CQECC). The idea here is simple. Suppose the EAQECC has parameters $n, k, c$. Using $c$ bits of pre-existing entanglement, Alice encodes some of the qubits she wishes to transmit, plus one bit each from $c$ maximally entangled pairs that she prepares locally. After her $n$ bits have been transmitted to Bob, corrected and decoded, Bob has received $k - c$ qubits, plus $c$ new bits of entanglement have been created. These can then be used to send another $k - c$ bits, and so on. The idea is that the perfect qubit channel that is simulated by the code is a stronger resource than pre-existing entanglement \cite{19}. It is this catalytic mode
of performance that makes the rate $(k - c)/n$ a reasonable figure of merit for an EAQECC as described above. Clearly the $[[4, 1, 3; 1]]$ code described in this paper is useless as a catalytic code, though it is perfectly useful for an entanglement-assisted channel. To be a useful catalytic code, an EAQECC must have a positive value of $k - c$.

We have presented EAQECCs in a communication context up to now, but catalytic codes open the possibility of application to error correction in quantum computing, where we can think of decoherence as a channel into the future. In this case, the “seed” resource is not pre-existing entanglement, but rather a small number of qubits that are error-free, either because they are physically isolated, or because they are protected by a decoherence-free subspace or standard QECC.

CQECCs provide great flexibility in designing quantum communication schemes. For example, in periods of low usage we can use an EAQECC in the catalytic mode to build up shared entanglement between Alice and Bob. Then in periods of peak demand, we can draw on that entanglement to increase the capacity. Quantum networks of the future can use schemes like this to optimize performance. In any case, the existence of practical EAQECCs will greatly enhance the power of quantum communications, as well as providing a beautiful connection to the theory of classical error correction codes.

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