SDES WITH RANDOM AND IRREGULAR COEFFICIENTS

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Abstract. We consider Itô uniformly nondegenerate equations with random coefficients. When the coefficients satisfy some low regularity assumptions with respect to the spatial variables and Malliavin differentiability assumptions on the sample points, the unique solvability of singular SDEs is proved by solving backward stochastic Kolmogorov equations and utilizing a modified Zvonkin type transformation.

1. Introduction

The main purpose of this work is to study the well-posedness of stochastic differential equations (SDEs) with random and irregular coefficients. More precisely, we are concerned with the following SDE in $\mathbb{R}^d$:

\begin{equation}
X_t(\omega) = X_0(\omega) + \int_0^t \sigma_s(X_s, \omega) dW_s(\omega) + \int_0^t b_s(X_s, \omega) ds.
\end{equation}

Here $\{W_t\}_{t \in [0,1]}$ is a $d$-dimensional Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, $\mathcal{F}_t$ and $\mathcal{F}$ are generalized by $\{W_s\}_{s \in [0,1]}$ and $\{W_s\}_{s \in [0,t]}$, respectively. $\mathcal{B}$ is the Borel algebra on $\mathbb{R}^n$ and $\mathcal{P}$ is the collection of all the progressively measurable sets on $[0,1] \times \Omega$. The coefficients $\sigma : \mathbb{R}^n \times [0,1] \times \Omega \to \mathbb{R}^n \otimes \mathbb{R}^d$ and $b : \mathbb{R}^n \times [0,1] \times \Omega \to \mathbb{R}^n$ are $\mathcal{B} \times \mathcal{P}$-measurable.

In the past half century, a great deal of mathematical effort in stochastic analysis has been devoted to the study of the existence, uniqueness and regularity properties of strong solutions to Itô uniformly nondegenerate stochastic equations with deterministic and irregular drifts. When $\nabla \sigma \in L^{2d}_{\text{loc}}$ and $b$ is bounded, Veretennikov [19] proved the strong existence and uniqueness of solutions to SDE (1.1) by developing the original idea proposed by Zvonkin in [26]. In the case that $\sigma = 1$ and $b \in L^d L^p_x$ with $\frac{n}{p} + \frac{2}{q} < 1$, using Girsanov’s transformation and $L^d L^p_x$-estimate for parabolic equations, Krylov-Röckner [11] obtained the existence and uniqueness of strong solutions to (1.1). After that, a lot of works appeared to investigate properties of the strong solution to (1.1) with singular drifts. Among all, we mention that the Hölder continuity of the stochastic flow was proved.
by Fedrizzi and Flandoli in [5], provided that the coefficients meet the same condition in [11]. When \( b \) is bounded, Menoukeu et al. [15] obtained the weak differentiability of the stochastic flow and the Malliavin differentiability of \( X_t \) with respect to the sample \( \omega \) by using Malliavin’s calculus. Zhang [21] extended Veretennikov’s unique strong solvability result to the case that \( \nabla \sigma, b \in L_1^q L_2^p \) with \( \frac{p}{q} + \frac{2}{q} < 1 \). Under similar conditions, the regularities of strong solutions with respect to the initial data and sample point were also shown in [22] and [20]. For more recent results, we refer the reader to [12] and [16]. We also note that martingale problems and stochastic Lagrangian flows corresponding to (1.1) were studied by many researchers, among which we quote [2, 17, 23–25].

The well-posedness and regularities of strong solutions to SDEs with singular coefficients is not only a fundamental theoretical problem, but also has a wide range of applications in many mathematical and physical problems. For instance, in the remarkable paper [7], Flandoli, Gubinelli and Priola studied the following linear stochastic transport equation (see also [6]):

\[
\partial_t u + b \cdot \nabla u + \nabla u \odot \frac{dW_t}{dt} = 0, \quad u_0 = \varphi,
\]

where \( b : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n \) is deterministic. Using the stochastic flow of the corresponding SDE (or stochastic characteristics), they proved the existence and uniqueness for the above equation in \( L_1^\infty \)-setting, provided that the drift \( b \) is \( \alpha \)-Hölder continuous uniform in \( t \) and the divergence of \( b \) satisfies some integrability conditions. However, as mentioned in [7], one of the major obstacles to extending the regularization by noise phenomenon to the case where \( b \) is random is the fact that even when \( b \) is Hölder continuous in \( x \) the stochastic characteristics corresponding to (1.2) may not uniquely exist. Below is a simple but typical example:

**Example 1.** Let \( d = n = 1 \). Assume \( \sigma = 1 \) and

\[
b_t(x) = \sqrt{|x - W_t|} \wedge 1, \quad X_0 = 0.
\]

Denote \( Y_t := X_t - W_t \), then \( Y_t \) satisfies the following random ODE:

\[
dY_t(\omega) = b_t(Y_t(\omega) + W_t(\omega), \omega)dt = \left( \sqrt{|Y_t(\omega)|} \wedge 1 \right) dt, \quad Y_0 = 0.
\]

One can verify that \( y_t^{(1)} \equiv 0 \) and \( y_t^{(2)} = \frac{t^2}{4} \) are two solutions of the above ODE, which implies \( X_t^{(1)} = W_t \) and \( X_t^{(2)} = \frac{t^2}{4} + W_t \) are two \( \mathcal{F}_t \)-adapted solutions to equation

\[
X_t = \int_0^t b_s(X_s)ds + W_t, \quad t \in [0, 1].
\]

The above example demonstrates that the non degeneracy of the noise and the uniformly Hölder continuity of \( b_t(\cdot, \omega) \) are insufficient to guarantee the well-posedness of (1.1). To the best of our knowledge, there is little literature written to address this issue so far. The main work before this paper is [4], which is a foundational work but misses some important requirements, in particular because it asks a specific form of Malliavin derivative for the drift, and in certain situations, \( W^{1,p} \) regularity for drift with respect to
$x$, which makes the results not so competitive. This paper attempts to make some progress in this direction. Roughly speaking, our main result, Theorem 1.1, shows that if the noise is additive and nondegenerate, and $b$ is Hölder in $x$, the well-posedness of the Itô equation (1.1) is guaranteed when $Db_t(x)$, the Malliavin differentiable of $b_t(x)$, also satisfies a Hölder continuity assumption with respect to $x$.

With a little abuse of notation, in this paper $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ is abbreviated as $L^p(\Omega)$, where $m$ is an integer that may take different values in different places. Our main result is

**Theorem 1.1.** Let $\alpha \in (0, 1)$, $p > n/\alpha$, $\Lambda > 1$, $\mathcal{A} := \{(s, t) \in [0, 1]^2 : 0 \leq s \leq t \leq 1\}$ and let $D$ be the Malliavin derivative operator. Assume that $\sigma$ and $b$ are $\mathcal{B} \times \mathcal{P}$ measurable, then equation (1.1) admits a unique solution if $\sigma$ and $b$ satisfy the following assumptions:

(i) for almost surely $\omega \in \Omega$, $\sigma(\omega)$ and $b(\omega)$ are bounded, and for all $x, y \in \mathbb{R}^n, t \in [0, 1]$,

\[
|b_t(x, \omega) - b_t(y, \omega)| \leq \Lambda|x - y|^\alpha, \quad |\sigma_t(x, \omega) - \sigma_t(y, \omega)| \leq \Lambda|x - y|;
\]

(ii) for almost surely $\omega \in \Omega$ and all $(x, t) \in \mathbb{R}^n \times [0, 1]$,

\[
\Lambda^{-1}|\xi|^2 \leq \frac{1}{2} \sigma_t^{ij}(x, \omega)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^d;
\]

(iii) for each $(x, t) \in \mathbb{R}^n \times [0, 1]$, $\sigma_t(x), b_t(x)$ are Malliavin differentiable and the random fields $D_s\sigma_t(x)$ and $D_s b_t(x)$ have continuous versions as maps from $\mathbb{R}^n \times \mathcal{A}$ to $L^2(\Omega)$ such that

\[
\sup_{(s, t) \in \mathcal{A}} \left( \|D_s \sigma_t\|_{C^0(\mathbb{R}^n; L^2_p(\Omega))} + \|D_s b_t\|_{C^0(\mathbb{R}^n; L^2_p(\Omega))} \right) \leq \Lambda.
\]

We give an example of $b$ meeting the conditions in Theorem 1.1.

**Example 2.** Let $n = d = 1$, $\alpha \in (0, 1)$, $p > 1/\alpha$. Assume $\bar{b} : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is bounded function satisfying

\[
|\bar{b}_t(x, y) - \bar{b}_t(x', y)| + |\partial_x \bar{b}_t(x, y) - \partial_x \bar{b}_t(x', y)| \leq C|x - x'|^\alpha, \quad \forall x, x', y \in \mathbb{R}^n, t \in [0, 1]
\]

and

\[
b_t(x, \omega) := \bar{b}_t \left( x, \int_0^t h_r(\omega) \, dW_r(\omega) \right).
\]

Here $h$ is an adapted process satisfying

\[
\sup_{s \in [0, 1]} \mathbb{E} \left( |h_s|^2 + \int_0^1 |D_s h_r|^2 \, dr \right) < \infty.
\]

Noting that

\[
D_s b_t(x) = \partial_x \bar{b}_t \left( x, \int_0^t h_r \, dW_r \right) \left( \int_s^t D_s h_r \, dW_r + h_s \right) \mathbf{1}_\mathcal{A}(s, t),
\]
by Burkholder-Davis-Gundy’s inequality, one sees that
\[
\sup_{t \in [0,1], \omega \in \Omega} ||b_t(\cdot, \omega)||_{C^\alpha(\mathbb{R})} + \sup_{(s,t) \in J} ||D_s b_t||_{C^\alpha(\mathbb{R}^d; L^2p(\Omega))} \\
\leq C \left[ 1 + \sup_{s \in [0,1]} \mathbb{E} \left( |h_s|^{2p} + \int_0^1 |D_s h_r|^{2p}\,dr \right)^{1/2p} \right] < \infty,
\]
so \( b \) satisfies the conditions in (i) and (iii) in Theorem 1.1.

Our approach of studying the well-posedness of (1.1) is using a modified Zvonkin transformation. Such kind of trick was first proposed in [26] for solving SDEs with deterministic and bounded coefficients. To explain our main idea, let us first give a brief introduction to Zvonkin’s idea. Denote
\[
a = \frac{1}{2} \sigma \sigma^*, \quad L_t u = a^{ij}_t \partial_{ij} u + b^i_t \partial_i u.
\]
When \( a \) and \( b \) are deterministic, \( a, b \in L^\infty_t C^\alpha_x \) and \( a \) is uniformly elliptic, by Schauder’s estimate, the following backward equation
\[
\partial_t u + L_t u = -b, \quad u_T(x) = 0
\]
admits a unique solution \( u \in L^\infty_t C^{2+\alpha}_x \) with \( \partial_t u \in L^\infty_t C^\alpha_x \). Moreover, if \( T \) is sufficiently small, the map \( x \mapsto \phi_t(x) := x + u_t(x) \) is a \( C^2 \)-homeomorphism. Assuming that \( X_t \) solves (1.1), by Itô’s formula, \( Y_t := \phi_t(X_t) \) satisfies a new SDE with Lipschitz continuous coefficients. Thus, the strong uniqueness of the solution to the original equation is given by the one of the new equation. Coming to the case that \( \sigma, b \) are progressive measurable and
\[
\text{ess sup}_{w \in \Omega} (||\sigma(\omega)||_{L^\infty_t C^\alpha_x} + ||b(\omega)||_{L^\infty_t C^\alpha_x}) < \infty.
\]
Thanks to the classic Schauder’s estimate, pointwisely, one can solve the backward equation
\[
(1.4) \quad \partial_t w + L_t w + f = 0, \quad u_T(x) = 0.
\]
Moreover, \( w \) satisfies
\[
\text{ess sup}_{\omega \in \Omega} \left( ||w(\omega)||_{L^\infty_t C^{2+\alpha}_x} + ||\partial_t w(\omega)||_{L^\infty_t C^\alpha_x} \right) \leq \text{Cess sup}_{\omega \in \Omega} ||f(\omega)||_{L^\infty_t C^\alpha_x}.
\]
However, in this case for each \( x \in \mathbb{R}^d \), the process \( w(\cdot, x) : (t, \omega) \mapsto w_t(x, \omega) \) is non-adapted, so one cannot apply the Itô-Wentzell formula as in the deterministic case. A very natural way to overcome this difficulty is to consider the function \( u_t := \mathbb{E}(w_t|\mathcal{F}_t) \) instead of \( w_t \). Formally, \( u_t \) satisfies the following backward stochastic Kolmogorov equation (see Lemma 3.1):
\[
(1.5) \quad du_t + (L_t u_t + f_t)dt = v_t \cdot dW_t, \quad u_T(x) = 0.
\]
On this point, a more general class of semi-linear equations including (1.5) were already studied by Du-Qiu-Tang [3] in \( L^p \)-spaces and also by Tang-Wei [18] in Hölder spaces.
However, the main obstacle of applying their result for our purpose is that one can only expect that the vector field \( u_1 \) is in some \( L^p \) (or \( C^\alpha \)) space, which is far from enough to apply the Itô-Wentzell formula (see Lemma 5.7). On the other hand, Duboscq-Réveillac [4], studied the stochastic regularization effects of diffusions with random drift coefficients on random functions. After adding some Malliavin differentiability conditions on \( b \) and \( f \), they extended the boundedness of time average of a deterministic function \( f \) depending on a diffusion process \( X \) with deterministic drift coefficient \( b \) to random mappings \( f \) and \( b \) by investigate the backward stochastic Kolmogorov equation (1.5) (\( \alpha \equiv 1 \)) in some \( L^p \)-type space. Inspired by [4] and [26], in this paper we prove a \( C^{2+\alpha} \) type estimate (Theorem 3.4) for \((u,v)\), provided that the coefficients satisfy some Malliavin differentiability conditions. To achieve this purpose, we first extend the classic Schauder estimate to random PDEs with Banach variables. Such kind of extension gives the \( C^{2+\alpha} \) estimate for \( u \), as well as \( C^\alpha \) estimate for \( v \) (see Lemma 3.1). The main ingredient of this paper is Theorem 3.4, where we give the \( C^{2+\alpha} \) estimate for \( v \), provided that the Malliavin derivatives of the coefficients satisfy (1.3). To us, such kind of result is new and intriguing. With such regularity estimate in hand, we then use a modified Itô-Wentzell’s formula and Zvonkin type transformation to prove the well-posedness of (1.1). We believe our results have the potential to be applied to stochastic transport equations with random coefficients and some other nonlinear stochastic PDEs.

This paper is organized as follows: In Section 2, we investigate a random Banach-valued non-adapted Kolmogorov equation and prove its well-posedness in some Hölder type spaces. In Section 3, we study the solvability of backward stochastic Kolmogorov equation (1.5) in some \( C^{2+\alpha} \) space. Our main result was proved in Section 4. A Itô-Wentzell type formula and some auxiliary lemmas used in our main proofs were presented in Appendix.

2. Schauder Estimates for Random Banach-valued PDEs

In this section, we give a self-contain proof of Schauder type estimate for random Banach-valued parabolic PDEs by using Littlewood-Paley decomposition.

Let \( T \in (0,1] \), \( D \) be a domain of \( \mathbb{R}^n \), \( D_T = D \times [0,T] \) and \( \mathcal{B} \) be a real Banach space. For any \( \alpha \in (0,1) \) and strongly continuous function \( g \) : \( D \to \mathcal{B} \), we define

\[
\|g\|_{\mathcal{B},D} := \sup_{x \in D} |g(x)|_{\mathcal{B}}, \quad [g]_{\alpha;D} := \sup_{x,y \in D} \frac{|g(x) - g(y)|_{\mathcal{B}}}{|x-y|^\alpha}.
\]

For \( k \in \mathbb{N} \),

\[
\|g\|_{C^{k,\alpha} (D;\mathcal{B})} := \sum_{i=0}^{k} \|\nabla^i g\|_{0;D} + \|\nabla^k g\|_{\alpha;D}
\]

Here and below, all the derivatives of an \( \mathcal{B} \)-valued function are defined with respect to the spatial variable in the strong sense, namely, \( \nabla g \) is the unique map from \( \mathbb{R}^d \) to \( \mathcal{L}(\mathbb{R}^n;\mathcal{B}) \) such that \( \lim_{|h| \to 0} |g(x + h) - g(x) - \nabla g(x) \cdot h|_{\mathcal{B}} = 0 \). For any \( \beta \geq 0 \), the space
Let \( C^{\beta,0}_{x,t}(D_T;\mathcal{B}) \) consists all continuous function \( f : D_T \to \mathcal{B} \) such that

\[
\|f\|_{C^{\beta,0}_{x,t}(D_T;\mathcal{B})} := \sup_{t \in [0,T]} \|f(t)\|_{C^\beta(D;\mathcal{B})} < \infty.
\]

Below we always denote \( Q_T = \mathbb{R}^n \times [0,T] \) and \( Q = Q_1 \). If there is no confusion on the time parameter \( T \) and underlying Banach space \( \mathcal{B} \), we simply write \( C^\beta \) and \( C^{\beta}_{x,t} \) instead of \( C^\beta(\mathbb{R}^n;\mathcal{B}) \) and \( C^{\beta,0}_{x,t}(Q_T;\mathcal{B}) \), respectively.

### 2.1. Littlewood-Paley decomposition

Let \( \mathcal{S}(\mathbb{R}^n) \) be the Schwartz space of all rapidly decreasing complex valued functions on \( \mathbb{R}^n \), and \( \mathcal{S}'(\mathbb{R}^n) \) be the dual space of \( \mathcal{S}(\mathbb{R}^n) \) (tempered distribution space). Given \( f \in \mathcal{S}(\mathbb{R}^n) \), the Fourier transform and inverse transform of \( f \) are defined by

\[
\mathcal{F}(f)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx,
\]

\[
\mathcal{F}^{-1} f(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi \cdot x} f(x) dx.
\]

Let \( \chi : \mathbb{R}^n \to [0, 1] \) be a smooth radial function with

\[
\chi(\xi) = 1, \ |\xi| \leq 1; \ \chi(\xi) = 0, \ |\xi| \geq 3/2.
\]

Define

\[
\varphi(\xi) := \chi(\xi) - \chi(2\xi), \quad \varphi_{-1}(\cdot) := \chi(2\cdot), \quad \varphi_j(\cdot) := \varphi(2^{-j}\cdot) \ (j = 0, 1, 2, \cdots).
\]

It is easy to see that \( \varphi \geq 0 \) and \( \text{supp} \varphi \subset B_{3/2} \setminus B_{1/2} \) and formally

\[
(2.1) \sum_{j=-1}^{k} \varphi_j(\xi) = \chi(2^{-k} \xi) \xrightarrow{k \to \infty} 1.
\]

In particular, if \( |j - j'| \geq 2 \), then

\[
\text{supp} \varphi(2^{-j}\cdot) \cap \text{supp} \varphi(2^{-j'}\cdot) = \emptyset.
\]

Let \( \tilde{\varphi} \) be another smooth radial function, \( \text{supp} \tilde{\varphi} \subset B_{3/4} \setminus B_{1/4} \) and \( \tilde{\varphi}(x) = 1 \) for all \( x \in B_{3/4} \setminus B_{1/4} \).

Denote

\[
h_j := \mathcal{F}^{-1}(\varphi_j), \quad \tilde{h}_j := \mathcal{F}^{-1}(\tilde{\varphi}_j).
\]

For any \( f \in L^1(\mathbb{R}^n;\mathcal{B}) + L^\infty(\mathbb{R}^n;\mathcal{B}) \), define

\[
\Delta_j f := \int_{\mathbb{R}^n} h_j(x - y) f(y) dy, \quad \tilde{\Delta}_j f := \int_{\mathbb{R}^n} \tilde{h}_j(x - y) f(y) dy.
\]
2.2. A basic apriori estimate

Assume \((\Omega, \mathcal{F}, P)\) is a complete probability space, \(\mathcal{H}\) is a real Hilbert spaces and \(\mathcal{B} = L^p(\Omega, \mathcal{F}, P; \mathcal{H})\) for some \(p \geq 2\). Let \(a^{ij}, b^i, c\) be real-valued measurable functions on \(Q \times \Omega\) and define

\[
L_t := a^{ij}_t \partial_{ij} + b^i_t \partial_i + c_t.
\]

Fix \(T \in (0, 1]\), we first give the precise definition of solutions to the following \(\mathcal{B}\)-valued PDE

\[
\begin{aligned}
\partial_t w + L_t w + f &= 0 \quad \text{in} \quad Q^\alpha_T \\
w_T &= 0 \quad \text{on} \quad \mathbb{R}^n.
\end{aligned}
\]

**Definition 2.1.** A function \(w : Q_T \to \mathcal{B}\) is called a solution of (2.2) if

1. For each \(t \in [0, T]\), \(w(t, \cdot)\) is a twice strongly differentiable function from \(\mathbb{R}^n\) to \(\mathcal{B}\); 
2. For each \(x \in \mathbb{R}^n\), the process \(w(\cdot, x)\) is absolutely continuous from \([0, T]\) to \(\mathcal{B}\) satisfying

\[
w_t(x) = \int_t^T (L_s w_s + f_s)(x) ds.
\]

In order to study the solvability of (2.2), we need the following

**Assumption 1.** The map \((x, t, \omega) \mapsto (a_t(x, \omega), b_t(x, \omega), c_t(x, \omega), f_t(x, \omega))\) is \(\mathcal{B}(Q) \times \mathcal{F}\) measurable and there are constants \(\alpha \in (0, 1)\) and \(\Lambda > 1\) such that for almost surely \(\omega \in \Omega\),

\[
(H_1) \quad \|a^{ij}(\omega)\|_{C_{x,t}^{\alpha,0}} + \|b^i(\omega)\|_{C_{x,t}^{\alpha,0}} + \|c(\omega)\|_{C_{x,t}^{\alpha,0}} \leq \Lambda,
\]

and

\[
(H_2) \quad \Lambda^{-1} |\xi|^2 \leq (\omega)\xi_i \xi_j \leq \Lambda |\xi|^2.
\]

Our main result in this section is

**Theorem 2.2.** Under Assumption 1, for any \(f \in C_{x,t}^\alpha\), equation (2.2) admits a unique solution \(w\) in \(C_{x,t}^{2+\alpha}\). Moreover,

\[
(2.3) \quad \|\partial_t w\|_{C_{x,t}^{\alpha}} + \|w\|_{C_{x,t}^{2+\alpha}} + T^{-1}\|w\|_{C_{x,t}^0} \leq C\|f\|_{C_{x,t}^{\alpha}},
\]

where \(C\) only depends on \(n, p, \alpha, \Lambda\).

Like the proof for the classic Schauder estimate, we first consider the case \(a_t(x, \omega) = a_t(\omega)\) and \(b = c = 0\). Define

\[
A_{t,s} := \int_t^s a(r) dr, \quad p^a_{t,s}(x) := (\det 4\pi A_{t,s})^{-1/2} \exp(-\langle x, A_{t,s}^{-1}x \rangle)
\]

and

\[
P^a_{t,s}f(x) := \int_{\mathbb{R}^n} p^a_{t,s}(x - y) f(y) dy.
\]
Lemma 2.3. Let $T \in (0, 1], \alpha \in (0, 1)$. Assume $a$ is $x$-independent and satisfies $(\text{H}_2)$. For any $f \in C^{\alpha}_{x,t} \cap C^{2\alpha}_{x,t}$, the function $w_t(x) = \int_0^t P_{1,s}^a f_s(x) ds$ is the unique function in $C^{2\alpha}_{x,t}$ satisfying

\begin{equation}
\label{equation:2.4}
w_t = \int_0^T (a_s \partial_t w_s + f_s) ds.
\end{equation}

Moreover, there is a constant $C$ only depends on $n, \alpha, p, \Lambda$ such that

\begin{equation}
\label{equation:2.5}
\|\partial_t w\|_{C^{\alpha}_{x,t}} + \|w\|_{C^{2\alpha}_{x,t}} + T^{-1}\|w\|_{C^{\alpha}_{x,t}} \leq C\|f\|_{C^{\alpha}_{x,t}}.
\end{equation}

Proof. We first prove that the map $w$ defined above satisfies (2.5) by using Littlewood-Paley decompositions. Recall that $\mathcal{B} = L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$. For any $g \in L^1(\mathbb{R}^n; \mathcal{B}) + L^\infty(\mathbb{R}^n; \mathcal{B})$, by Minkowski’s inequality, we have

\begin{equation}
\label{equation:2.6}
\|\Delta_j P_{t,s}^a g(x)\|_{\mathcal{B}} = \left(\mathbb{E}\|\Delta_j P_{t,s}^a g(x)\|_{\mathcal{H}}^p\right)^{1/p} \leq \int_\Omega \left(\int_{\mathbb{R}^n} \left|P_{1,s}^a * \tilde{h}_j(y) \cdot \Delta_j g(x - y, \omega)\right|^p \mathbb{P}(d\omega)\right)^{1/p} \leq \|\Delta_j g\|_0 \int_\Omega \left[\operatorname{ess sup}_{\omega \in \Omega} \left|P_{1,s}^a * \tilde{h}_j(y)\right|\right] dy.
\end{equation}

By $(\text{H}_2)$,

\begin{align*}
\int_{\mathbb{R}^n} \left[\operatorname{ess sup}_{\omega \in \Omega} \left|P_{t,s}^a * \tilde{h}_j(x)\right|\right] dx &\leq \sup_{1/\Lambda \leq a \leq \Lambda} \left\|P_{t,s}^a * \tilde{h}_j(x)\right\|_{L^1_x} \\
&= \int_{\mathbb{R}^n} dx \sup_{1/\Lambda \leq a \leq \Lambda} \left|\int_{\mathbb{R}^n} P_{t,s}^a(x - y) 2^{jn} \tilde{h}_0(2^j y) dy\right| \\
&= \int_{\mathbb{R}^n} dx \sup_{1/\Lambda \leq a \leq \Lambda} \left|\int_{\mathbb{R}^n} 2^{jn} P_{t,s}^{2^j a}(2^j x - z) \tilde{h}_0(z) dz\right| \\
&= \int_{\mathbb{R}^n} dx \sup_{1/\Lambda \leq a \leq \Lambda} \left|\int_{\mathbb{R}^n} P_{t,s}^{2^j a}(x - z) \tilde{h}_0(z) dz\right|.
\end{align*}

Noting that

\[\|f\|_{L^1} \leq C_{n,N} \|1 + |x|^{2N} f(x)\|_{L^\infty}, \quad \forall N > n/2\]

and

\[\mathcal{F}^{-1}(P_{t,s}^a)(\xi) = \exp(-\langle \xi, A_{t,s} \xi \rangle),\]
we obtain
\[
\int_{\mathbb{R}^n} \left[ \text{ess sup}_{\omega \in \Omega} |p_{t,s}^{a(\omega)} \ast \tilde{h}_j(x)| \right] \, dx \leq \int_{\mathbb{R}^n} dx \sup_{I/L \in \mathcal{I}} \left| \int_{\mathbb{R}^n} p_{t,s}^{2j/a}(x-z) \tilde{h}_0(z) \, dz \right|
\]
\[
\leq C \left( 1 + |I/L|^2 \right) \sup_{I/L \in \mathcal{I}} \left| \int_{\mathbb{R}^n} p_{t,s}^{2j/a}(x-z) \tilde{h}_0(z) \, dz \right|
\]
\[
= C \sup_{I/L \in \mathcal{I}} \left\| \left( 1 + |I/L|^2 \right) |p_{t,s}^{2j/a}(x-z) \tilde{h}_0(z)\right\|_{L^\infty}
\]
\[
= C \sup_{I/L \in \mathcal{I}} \left\| \left( 1 + |I/L|^2 \right) |\mathcal{F}^{-1}(p_{t,s}^{2j/a}) \cdot \mathcal{F}^{-1}(\tilde{h}_0)\right\|_{L^1_I}
\]
\[
= C \sup_{I/L \in \mathcal{I}} \left\| \left( 1 + \Delta^N \right) \left[ \mathcal{F}^{-1}(p_{t,s}^{2j/a}) \cdot \mathcal{F}^{-1}(\tilde{h}_0)\right] (\xi) \right\|_{L^1_I}
\]
Since \( \sup_{|\alpha| = k} \partial^\alpha (e^{a|\xi|^2}) \leq C \left( 1 + |\alpha| \right) (1 + |\xi|) e^{a|\xi|^2} \), we get
\[
(2.7) \quad \int_{\mathbb{R}^n} \left[ \text{ess sup}_{\omega \in \Omega} |p_{t,s}^{a(\omega)} \ast \tilde{h}_j(x)| \right] \, dx
\]
\[
\leq C \int_{\frac{1}{2} \leq |\xi| \leq 2} \left[ 1 + (\lambda^2 (s-t)) e^{-\lambda^2 (s-t)} \right] \exp \left[ -2^2 (s-t) e^{-\lambda^2 (s-t)} \right] \, d\xi.
\]
Denote \( \Lambda_j := \lambda^2 (s-t) \) and \( \lambda_j := \frac{1}{16} \lambda^{-1} \lambda^2 (s-t) \). Combining (2.6) and (2.7), we get
\[
\| \Delta_j P_{t,s}^{a \alpha} g \|_0 = \sup_{x \in \mathbb{R}^n} \| (\Delta_j P_{t,s}^{a \alpha}) (x) \|_B \leq C \left( 1 + \lambda_j^2 \right) e^{-\lambda_j^2} \| B_{\frac{1}{2}} \setminus B_{\frac{1}{4}} \| \| \Delta_j g \|_0.
\]
By Lemma 5.1 and the elementary inequality:
\[
(1 + \lambda_j^2) e^{-\lambda_j^2} \leq C_k \left( 1 + \lambda_j^2 \right) \left( s-t \right)^{-k} \quad (\forall k \in \mathbb{N}),
\]
we get
\[
\| \Delta_j P_{t,s}^{a \alpha} g \|_0 \leq C \left( 1 + \lambda_j^2 \right) e^{-\lambda_j^2} \| g \|_\alpha \leq C_k \left( 1 + \lambda_j^2 \right) \left( s-t \right)^{-k} \| g \|_\alpha.
\]
This yields
\[
\| \Delta_j w_t \|_0 = \left\| \Delta_j \int_{t}^{T} P_{t,s}^{a \alpha} f_s \, ds \right\|_0
\]
\[
\leq C 2^{-j \alpha} \| f \|_{C_{x,t}^\alpha} \int_{t}^{T} (1 + 2^{-2j k \rho^{-k}}) \, dr.
\]
If \( t \geq T - 2^{-2j} \), then
\[
\| \Delta_j w_t \|_0 \leq C 2^{-j \alpha} \| f \|_{C_{x,t}^\alpha} \cdot (T-t) \leq C 2^{-j (2+\alpha)} \| f \|_{C_{x,t}^\alpha}.
\]
if \( t < T - 2^{-2j} \), by choosing \( k = 2 \), then
\[
\| \Delta_j w_t \|_0 \leq C 2^{-j} \| f \|_{C_{x,t}^\alpha} \cdot \left( 2^{-2j} + 2^{-4j} \int_{2^{-2j}}^{T-t} s^{-2} ds \right)
\leq C 2^{-j(2+\alpha)} \| f \|_{C_{x,t}^\alpha}.
\]
Again using Lemma 5.1, one sees that
\[
\| w \|_{C_{x,t}^{2+\alpha}} \leq C \sup_{j \geq 1} \left( 2^{-j(2+\alpha)} \| \Delta_j w_t \|_0 \right) \leq C \| f \|_{C_{x,t}^\alpha}.
\]

So we complete our proof for (2.5). By basic calculations, one can verify that \( w \) satisfies (2.4). It remains to show that \( w \) defined above is the unique solution to (2.2) in \( C_{x,t}^{2+\alpha} \).

Assume \( \tilde{w} \in C_{x,t}^{2+\alpha} \) is another function satisfy (2.4). Let \( 0 \leq \varrho \in C^\infty_c(\mathbb{R}^n) \) satisfying \( \int \varrho = 1 \) and \( \varrho_e(x) = \varrho(x/\varepsilon) \). Define \( v := w - \tilde{w} \) and \( v^e := v * \varrho_e \). For any \( k > n/p, N > 1 \) and \( \varepsilon \in (0, 1) \), by Sobolev embedding and Hölder’s inequality,
\[
\mathbb{E} \| v^e_{t_1} - v^e_{t_2} \|_{L^p(B_N; \mathcal{H})}^p = \mathbb{E} \sup_{\| h \|_H = 1} \| \langle v^e_{t_1} - v^e_{t_2}, h \rangle \|_{L^\infty(B_N)}^p
\leq CN^{k-p-n} \mathbb{E} \sup_{\| h \|_H = 1} \| \langle v^e_{t_1} - v^e_{t_2}, h \rangle \|_{W^{k,p}(B_N)}^p
\leq CN^{k-p-n} \sum_{i=0}^k \int_{B_N} \| \nabla^i \int_{t_1}^{t_2} \left( a_{ij} \partial_{ij} v^e(x) \right) ds \|_{\mathcal{H}}^p dx
\leq CN^{k-p-n} |t_2 - t_1|^{p-1} \sum_{i=2}^{k+2} \int_{B_N} \int_{t_1}^{t_2} \mathbb{E} \left| \int_{B_{N+1}} v_s(y) \nabla^i \varrho_e(x-y) dy \right|_{\mathcal{H}}^p ds dx
\leq CN^{k-p-n} |t_2 - t_1|^{p-1} \int_{t_1}^{t_2} \int_{B_{N+1}} \mathbb{E} |v_s(y)|_{\mathcal{H}}^p dy
\leq C_e N^{(k+n)p} |t_2 - t_1|^{p} \| v \|_{C_{x,t}^\alpha}^p.
\]
Due to Kolmogorov’s criterion, for almost surely \( \omega \in \Omega \) and all \( \varepsilon \in (0, 1) \), \( (x, t) \in Q_T \),
\[
\| \langle v^e_t(x, \omega) \|_{\mathcal{H}} \leq C_e(1 + |x|)^{k+n},
\]
which means \( v^e_t(\cdot, \omega) \) satisfies a certain growth condition at infinity. On the other hand, by definition, for almost surely \( \omega \in \Omega \) and each \( h \in \mathcal{H} \), the real valued function \( \langle v^e_t(\omega), h \rangle \) satisfies
\[
\partial_t \langle v^e_t(\omega), h \rangle + a_{ij}^{e} \partial_{ij} \langle v^e_t(\omega), h \rangle = 0, \quad \langle v^e_t(\omega), h \rangle = 0.
\]
Thus, we have \( \langle v^e_t(\omega), h \rangle \equiv 0 \) (see [8, Chapter 7, p176]) i.e. \( w * \varrho_e = \tilde{w} * \varrho_e \) a.s.. So
\[
\| w_t(x) - \tilde{w}_t(x) \|_2 \leq \lim_{\varepsilon \to 0} \| w_{t_1}(x) - (w * \varrho_e)(x) \|_2 + \lim_{\varepsilon \to 0} \| \tilde{w}_{t_1}(x) - (\tilde{w} * \varrho_e)(x) \|_2 = 0.
\]
So we complete our proof.

**Proof of Theorem 2.2.** Thanks to Lemma 2.3 and the method of continuity, we only need to prove the aprior estimate (2.3). Assume \( w \in C_{x,t}^{2+\alpha} \) is a solution to (2.2). Let \( \chi \in C^\infty_c(\mathbb{R}^d) \)
so that $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Fix a number $\delta > 0$, which will be determined later. Define $\chi_{\delta} = \chi((x-z)/\delta)$, then

$$
\partial_t(w\chi_{\delta}^z) + L_t^z(w\chi_{\delta}^z) + (f\chi_{\delta}^z) + [\chi_{\delta}^z L_t w - L_t^z(w\chi_{\delta}^z)] = 0,
$$

where $L_t^i w_t(x) := a_{ij}(z)\partial_{ij} w_t(x)$. Using (H$_1$) and noting that

$$
\chi_{\delta}^z L_t w - L_t^z(w\chi_{\delta}^z) = \chi_{\delta}^z(a^{ij} - a_{ij}^z)\partial_{ij} w + (b^i\chi_{\delta}^z - 2a_{ij}^z \partial_j \chi_{\delta}^z)\partial_i w + (c\chi_{\delta}^z - a_{ij}^z \partial_{ij} \chi_{\delta}^z)w,
$$

we have

$$
\left\| [\chi_{\delta}^z L_t w - L_t^z(w\chi_{\delta}^z)] \right\|_{C_{\alpha,t}^0} 
\leq C\delta^\alpha \left\| \nabla^2 w \right\|_{C_{\alpha,t}^0(B_{2\delta}(z) \times [0, T]; \mathcal{B})} + C \left( \delta^{-\alpha} \left\| \nabla^2 w \right\|_{C_{\alpha,t}^0} + \delta^{-1-\alpha} \left\| \nabla w \right\|_{C_{\alpha,t}^0} \right).
$$

(2.8)

Combining Lemma 2.3 and equation (2.8), we obtain that for any $\delta > 0$,

$$
\sup_{z \in \mathbb{R}^n} \left\| w \right\|_{C_{\alpha,t}^{2+\alpha,0}(B_{2\delta}(z) \times [0, T]; \mathcal{B})} \leq C_n \sup_{z \in \mathbb{R}^n} \left\| w \right\|_{C_{\alpha,t}^{2+\alpha,0}(B_\delta(z) \times [0, T]; \mathcal{B})} 
\leq C \sup_{z \in \mathbb{R}^n} \left\| w \right\|_{C_{\alpha,t}^{2+\alpha,0}} + C \sup_{z \in \mathbb{R}^n} \left\| f \chi_{\delta}^z + [\chi_{\delta}^z L_t w - L_t^z(w\chi_{\delta}^z)] \right\|_{C_{\alpha,t}^0} 
\leq C\delta^\alpha \sup_{z \in \mathbb{R}^n} \left\| w \right\|_{C_{\alpha,t}^{2+\alpha,0}} + C \left( \delta^{-\alpha} \left\| \nabla^2 w \right\|_{C_{\alpha,t}^0} + \delta^{-1-\alpha} \left\| \nabla w \right\|_{C_{\alpha,t}^0} \right)
$$

+ $\delta^{-2-\alpha} \left\| w \right\|_{C_{\alpha,t}^0} + \delta^{-\alpha} \left\| f \right\|_{C_{\alpha,t}^0}$. 

By choosing $\delta \in (0, 1)$ sufficiently small such that $C\delta^\alpha < 1/2$, we obtain

$$
\sup_{z \in \mathbb{R}^n} \left\| w \right\|_{C_{\alpha,t}^{2+\alpha,0}(B_{2\delta}(z) \times [0, T]; \mathcal{B})} \leq C_\delta \left( \left\| w \right\|_{C_{\alpha,t}^0} + \left\| f \right\|_{C_{\alpha,t}^0} \right).
$$

Using interpolation, we get

$$
\left\| w \right\|_{C_{\alpha,t}^{2+\alpha}} \leq C_\delta \sup_{z \in \mathbb{R}^n} \left\| w \right\|_{C_{\alpha,t}^{2+\alpha,0}(B_\delta(z) \times [0, T]; \mathcal{B})} 
\leq \varepsilon C_\delta \left\| w \right\|_{C_{\alpha,t}^{2+\alpha}} + C_{\delta, \varepsilon} \left( \left\| w \right\|_{C_{\alpha,t}^0} + \left\| f \right\|_{C_{\alpha,t}^0} \right), \quad \forall \varepsilon \in (0, 1).
$$

By choosing $\varepsilon$ small such that $\varepsilon C_\delta \leq 1/2$, we get

(2.9)

$$
\left\| w \right\|_{C_{\alpha,t}^{2+\alpha}} \leq C \left( \left\| w \right\|_{C_{\alpha,t}^0} + \left\| f \right\|_{C_{\alpha,t}^0} \right).
$$
It remains to show that \( \|w\|_{C_{\alpha, t}^0} \) can be controlled by \( \|f\|_{C_{\alpha, t}^2} \). By Minkowski’s inequality, for any \( t \in [0, T] \),

\[
(2.10) \quad \left( E \int_{B_r(x)} |w_t(y)|^p dy \right)^{1/p} = \left( E \int_{B_r(x)} \left| \int_{t}^{T} \partial_s w_s(y) ds \right|^p dy \right)^{1/p} \\
= \left( E \int_{B_r(x)} \left| \int_{t}^{T} (L_s w_s + f_s)(y) ds \right|^p dy \right)^{1/p} \\
\leq C \int_{t}^{T} \left( E \int_{B_r(x)} |L_s w_s + f_s|^p dy \right)^{1/p} ds \\
\leq C T r^{n/p} (\|w\|_{C_{\alpha, t}^2} + \|f\|_{C_{\alpha, t}^0}).
\]

One the other hand, by Hölder’s inequality,

\[
(2.11) \quad |w_t(x)|_B \leq \int_{B_r(x)} |w_t(x) - w_t(y)|_B dy + \int_{B_r(x)} |w_t(y)|_B dy \\
\leq \|\nabla w\|_{C_{\alpha, t}^0} \int_{B_r(x)} |x - y| dy + \int_{B_r(x)} \left( E \int_{B_r(x)} |w_t(y)|^p dy \right)^{1/p} \\
\leq r \|\nabla w\|_{C_{\alpha, t}^0} + r^{-n/p} \left( E \int_{B_r(x)} |w_t(y)|^p dy \right)^{1/p} .
\]

Combining (2.10) and (2.11), we obtain

\[
\|w\|_{C_{\alpha, t}^0} \leq r \|\nabla w\|_{C_{\alpha, t}^0} + C T (\|w\|_{C_{\alpha, t}^2} + \|f\|_{C_{\alpha, t}^0}).
\]

Due to (2.9),

\[
\|w\|_{C_{\alpha, t}^2} \leq C (\|f\|_{C_{\alpha, t}^0} + \|w\|_{C_{\alpha, t}^0}).
\]

Combining the above two inequalities and letting \( r \to 0 \), we get

\[
\|w\|_{C_{\alpha, t}^0} \leq C T (\|w\|_{C_{\alpha, t}^0} + \|f\|_{C_{\alpha, t}^0}).
\]

By choosing \( T \) sufficiently small such that \( C T \leq 1/2 \), we get

\[
\|w\|_{C_{\alpha, t}^0} \leq C T \|f\|_{C_{\alpha, t}^0}.
\]

This together with (2.9) implies that (2.3) holds for some small \( T > 0 \). The same estimate for arbitrary \( T \in (0, 1] \) can be obtained by induction.

\[\square\]

**Remark 2.4.** If \( f \) satisfies

\[
\text{ess sup}_{\omega \in \Omega} \|f(\omega)\|_{C_{\alpha, t}^{0, 0}(Q_T; \mathbb{R})} < \infty,
\]

then (2.2) can be solved pointwisely and by the classic Schauder estimate, it holds that

\[
(2.12) \quad \text{ess sup}_{\omega \in \Omega} \left( \|\partial_t w(\omega)\|_{C_{\alpha, t}^{0, 0}(Q_T; \mathbb{R})} + \|w(\omega)\|_{C_{\alpha, t}^{2, 0, 0}(Q_T; \mathbb{R})} \right. \\
\left. + T^{-1} \|w(\omega)\|_{C_{\alpha, t}^{0, 0}(Q_T; \mathbb{R})} \right) \leq \text{ess sup}_{\omega \in \Omega} \|f(\omega)\|_{C_{\alpha, t}^{0, 0}(Q_T; \mathbb{R})}.
\]
3. Schauder estimate for Backward SPDE

In this section, we prove the solvability of (1.5) in $C^{2+\alpha}_{x,t} \times C^{2+\alpha}_{x,t}$ space. Recall that $W_t$ is a $d$-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t = \sigma\{W_s : s \leq t\} \vee \mathcal{N}$ and $\mathcal{F} = \mathcal{F}_t$. For any $t \in [0, 1]$ and $X \in \mathcal{F}$, we denote $\mathbb{E}^X := \mathbb{E}(X|\mathcal{F}_t)$. Throughout this section, we always assume $T \in (0, 1]$, $\mathcal{H}$ is a real Hilbert space, $\mathcal{B} = L^p(\Omega; \mathcal{H})$ for some $p \geq 2$ and $H = L^2([0, 1]; \mathbb{R}^d)$. With a little abuse of notation, $L^p(\Omega) = L^p(\Omega; \mathbb{R}^m)$ for some integer $m \geq 1$ that can be changed in different places.

**Lemma 3.1.** Let $\mathcal{H} = \mathbb{R}$. Assume that $a, b, c$ are $\mathcal{B} \times \mathcal{P}$ measurable and satisfy Assumption 1, then the following BSPDE

$$u_t(x) = \int_t^T (L_s w_s + f_s)(x) ds - \int_t^T v_s(x) \cdot dW_s$$

has an $\mathcal{F}_t$-adapted solution $(u, v)$ in $C^{2+\alpha}_{x,t} \times C^\alpha(\mathbb{R}^n; L^p(\Omega; \mathcal{H})$ and $u_t = \mathbb{E}^t w_t$, where $w$ is the solution to (2.2). Moreover,

$$\|u\|_{C^{2+\alpha}_{x,t}} + T^{-1}\|u\|_{C^0_{x,t}} + \|v\|_{C^\alpha(\mathbb{R}^n; L^p(\Omega; H))} \leq C\|f\|_{C^\alpha_{x,t}},$$

where $C$ only depends on $n, d, p, \alpha, \Lambda$.

**Proof.** Let $w$ be the solution of (2.2). Define $u_t(x) = \mathbb{E}^t w_t(x)$. By Theorem 2.2 and Lemma 5.4,

$$\|u\|_{C^{2+\alpha}_{x,t}} + T^{-1}\|u\|_{C^0_{x,t}} \leq C\|f\|_{C^\alpha_{x,t}}.$$

Since $a_t(x), b_t(x) \in \mathcal{F}_t$, by the definitions of $u$, we have

$$u_t(x) = \mathbb{E}^t \left\{ \int_t^T [(L_s w_s + f_s)(x)] ds \right\} = \int_t^T \mathbb{E}^t[(L_s w_s + f_s)(x)] ds$$

$$+ \left\{ \int_t^T \mathbb{E}^t[(L_s w_s + f_s)(x)] ds - \int_t^T \mathbb{E}^t[(L_s w_s + f_s)(x)] ds \right\}$$

$$= \int_t^T (L_s u_s + f_s)(x) ds + m_t(x) - m_T(x).$$

Here

$$m_t(x) = \int_t^T \mathbb{E}^t[(L_s w_s + f_s)(x)] ds$$

$$+ \int_0^t \mathbb{E}^s[(L_s w_s + f_s)(x)] ds \in \mathcal{F}_t.$$  

For any $t \in [0, T]$, noting that

$$\mathbb{E}' m_T(x) = \mathbb{E}' \int_0^T \mathbb{E}^s[(L_s w_s + f_s)(x)] ds$$

$$= \mathbb{E}' \int_0^t \mathbb{E}^s[(L_s w_s + f_s)(x)] ds + \mathbb{E}' \int_t^T \mathbb{E}^s[(L_s w_s + f_s)(x)] ds$$

$$= \int_0^t \mathbb{E}^s[(L_s w_s + f_s)(x)] ds + \int_t^T \mathbb{E}^s[(L_s w_s + f_s)(x)] ds = m_t(x),$$
By Theorem 2.2, (3.1) and Lemma 5.4, one can see that \( m \in C_x^\alpha \). Thanks to martingale representation, there is an \( \mathcal{F}_t \)-adapted process \( v(x) \) such that

\[
m_t(x) - m_0(x) = \int_0^t v_s(x) \cdot dW_s.
\]

Hence, we have

\[
u_t(x) = \int_t^T (Ls u_s + f_s)(x) ds - \int_t^T v_\cdot(x) \cdot dW_s,
\]
i.e.

\[
u_t(x) = u_0(x) - \int_0^t (Lu_s + f_s)(x) ds + \int_0^t v_\cdot(x) \cdot dW_s.
\]

By (3.1) and Burkholder-Davis-Gundy inequality, we obtain

\[
E \left( \int_0^T |v_t(x) - v_t(y)|^2 dr \right)^{\frac{p}{2}} = E (m(x) - m(y))_T^p \lesssim CE |m_T(x) - m_T(y)|^p
\]
\[
\lesssim CE \int_0^T E \left[ |(L_s w_s + f_s)(x) - (L_s w_s + f_s)(y)| \right] ds
\]
\[
\lesssim C \int_0^T E |(L_s w_s + f_s)(x) - (L_s w_s + f_s)(y)|^p ds
\]
\[
\lesssim C \int_0^T E |x - y|^{|\alpha p} \left( \|w\|_{C^{2+\alpha}_{x,t}}^p + \|f\|_{C^{1+\alpha}_{x,t}}^p \right)
\]
\[
\lesssim C |x - y|^{|\alpha p} \left( \|w\|_{C^{2+\alpha}_{x,t}}^p + \|f\|_{C^{1+\alpha}_{x,t}}^p \right)
\]
which yields

\[
\|v\|_{C^\alpha([0,T];L^p(\Omega;H))} \lesssim C \|f\|_{C^{1+\alpha}_{x,t}}.
\]

So we complete our proof.

As we mentioned in the introduction, the Zvonkin type transform is an effective way to prove the well-posedness of SDEs with singular coefficients. However, the \( C^\alpha \)-regularity of \( v \) in the spatial variable is not enough to apply this trick. So we need to get better regularity estimate for \( v \) under some mild conditions. To achieve this goal, we start with some definitions and lemmas. Let \( S_b \) be the set of random variables of the form

\[
F = f(\langle h_1, W \rangle, \cdots, \langle h_m, W \rangle),
\]

where \( f \in C_b^\infty(\mathbb{R}^m), h_i \in H \) and \( \langle h_i, W \rangle := \int_0^1 h_i dW_s \). We define the operator \( D \) on \( S_b \) with values in the set of \( H \)-valued random variables, by

\[
DF = \sum_{i=1}^m \partial_i f(\langle h_1, W \rangle, \cdots, \langle h_m, W \rangle) h_i.
\]
For any $p \in [1, \infty)$, $\mathbb{D}^{1,p}$ is the closure of the set $S_b$ with respect to the norm $\|F\|_{\mathbb{D}^{1,p}} := \|F\|_p + \|DF\|_{L^p(\Omega; H)}$.

**Lemma 3.2.** Suppose $\{y_t\}_{t \in [0, 1]}$ is a process (may not be adapted) on $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$y_t = y_0 + \int_0^t \dot{y}_r \, dr,$$

with $y_0 \in \mathbb{D}^{1,2}$ and $\dot{y} \in L^2([0, 1]; \mathbb{D}^{1,2})$. Then there exists a random field $\{y_{s,t}\}_{(s,t) \in [0, 1]^2}$ such that for each $t \in [0, 1]$, $y_{.,t} = D_y t$ is in $L^2(\Omega; H)$; for each $s \in [0, 1]$, the map $[0, 1] \ni t \mapsto y_{s,t} \in L^2(\Omega; \mathbb{R}^d)$ is absolutely continuous and

$$E' y_t = E y_0 + \int_0^t E \dot{y}_s \, ds + \int_0^t E y_{s,s} \, dW_s.$$  

**Proof.** By our condition that $y_0 \in \mathbb{D}^{1,2}$ and $\dot{y} \in L^2([0, 1]; \mathbb{D}^{1,2})$, we have $Dy_0 \in L^2([0, 1] \times \Omega; \mathbb{R}^d)$ and the map $(s, t, \omega) \mapsto D_s \dot{y}_t(\omega)$ is an element in $L^2([0, 1] \times \Omega; \mathbb{R}^d)$. By Fubini’s theorem, there is a Lebesgue null set $\mathcal{N} \subseteq [0, 1]$ such that for each $s \notin \mathcal{N}$, the map $t \mapsto D_s \dot{y}_t$ is an element in $L^2([0, 1]; L^2(\Omega))$ and $D_s y_0 \in L^2(\Omega)$. For any $s \in [0, 1]$, define

$$y_{s,t} = \begin{cases} D_s y_0 + \int_0^t D_s \dot{y}_r \, dr & s \notin \mathcal{N}, \ t \in [0, 1] \\ 0 & s \in \mathcal{N}, \ t \in [0, 1]. \end{cases}$$

Obviously, for each $s \in [0, 1]$, the map $[0, 1] \ni t \mapsto y_{t,s} \in L^2(\Omega)$ is absolutely continuous. By our assumption

$$\int_0^1 \|\dot{y}_r\|_{\mathbb{D}^{1,2}} \, dr \leq \left( \int_0^1 \|\dot{y}_r\|_{\mathbb{D}^{1,2}}^2 \, dr \right)^{1/2} < \infty,$$

i.e. $\dot{y} : [0, 1] \to \mathbb{D}^{1,2}$ is Bochner integrable. Since $D$ is a continuous operator from $\mathbb{D}^{1,2}$ to $L^2(\Omega)$, we get

$$Dy_t = Dy_0 + D \int_0^t \dot{y}_r \, dr = Dy_0 + \int_0^t D \dot{y}_r \, dr.$$ 

Combining this with the definition of $y_{s,t}$, we get $y_{.,t} = D_y t$ in $L^2(\Omega; H)$ for all $t \in [0, 1]$. Moreover, by our assumption,

$$E \int_0^1 |y_{s,s}|^2 \, ds \leq E \int_0^1 |D_{s}s y_0|^2 \, ds + E \int_0^1 \left( \int_0^s \|D_s \dot{y}_r\|_2^2 \, dr \right) ds \leq \|Dy_0\|_2^2 + \int_0^T \|D \dot{y}_r\|_2^2 \, dr < \infty,$$

which means $y_{s,s}$ is an element of $L^2([0, 1] \times \Omega; \mathbb{R}^d)$. By Lemma 5.6, we have

$$E' y_t = E y_t + \int_0^t E D_s y_r \cdot dW_s = E y_t + \int_0^t E y_{s,t} \cdot dW_s = E y_t + \int_0^t E(y_{s,t} - y_{s,s}) \cdot dW_s.$$ 

(3.4)
Note that for any \( s \notin \mathcal{N}, t \in [0, 1], \)
\[
y_{s,t} - y_{s,s} = \int_s^t D_s \dot{y}_r \, dr
\]
by stochastic Fubini theorem,
\[
\int_0^t \mathbb{E}^s (y_{s,t} - y_{s,s}) \cdot dW_s = \int_0^t \mathbb{E}^s \left( \int_s^t D_s \dot{y}_r \, dr \right) \cdot dW_s = \int_0^t \int_s^t \mathbb{E}^s D_s \dot{y}_r \, dr \cdot dW_s = \int_0^t \left( \int_s^t \mathbb{E}^s D_s \dot{y}_r \, dr \right) \cdot dW_s = \int_0^t (\mathbb{E}^r \dot{y}_r - \mathbb{E}^s \dot{y}_r) \, dr = \int_0^t \mathbb{E}^r \dot{y}_r \, dr + \mathbb{E}^0 y_0 - \mathbb{E}^t y_t.
\]
Plugging this into (3.4), we obtain (3.3).

For any \( F \in \mathcal{F} \) and \( h \in H \), denote
\[
\tau_{eh} F(\omega) := F\left( \omega + \varepsilon \int_0^t h_s \, ds \right), \quad D^h_F := \frac{(\tau_{eh} F - F)}{\varepsilon}.
\]

The next lemma is taken from [14], which gives a characterization of the space \( \mathcal{D}^{1,p} \) in terms of differentiability properties.

**Lemma 3.3.** Let \( p \in (1, \infty) \) and \( F \in L^p(\Omega) \). The following properties are equivalent

1. \( F \in \mathcal{D}^{1,p} \).
2. There is \( DF \in L^p(\Omega; H) \) such that for any \( h \in H \) and \( q \in [1, p) \)
\[
\lim_{\varepsilon \to 0} \mathbb{E}|D^h_F - \langle DF, h \rangle_H|^q = 0.
\]
3. There is \( DF \in L^p(\Omega; H) \) and some \( q \in [1, p) \) such that for any \( h \in H \)
\[
\lim_{\varepsilon \to 0} \mathbb{E}|D^h_F - \langle DF, h \rangle_H|^q = 0.
\]

Moreover, in that case, \( DF = \mathcal{D}F \).

Denote \( \mathcal{A}T = \{(s, t) : 0 \leq s \leq t \leq T\}, A = A_1 \). We need the following

**Assumption 2.** For each \( (x, t) \in Q, a_t(x), b_t(x), c_t(x) \) are Malliavin differentiable and each of the random fields \( D_s a_t(x), D_s b_t(x), D_s c_t(x) \) has a continuous version as a map from \( \mathbb{R}^n \times A \) to \( L^2(\Omega) \) such that
\[
\sup_{(s, t) \in A} \left( \|D_s a_t\|_{C^\alpha(\mathbb{R}^n; L^2(\Omega))} + \|D_s b_t\|_{C^\alpha(\mathbb{R}^n; L^2(\Omega))} + \|D_s c_t\|_{C^\alpha(\mathbb{R}^n; L^2(\Omega))} \right) \leq \Lambda' < \infty.
\]

The next Theorem is the key to the main purpose of this paper.
Theorem 3.4. Let $T \in (0, 1]$, $q > 2p \geq 4$ and $C_{x,t}^\beta = C_{x,t}^{\beta,0}(Q_T; L^p(\Omega))$. Under Assumption 1 and 2, the following BSPDE

\begin{equation}
(3.6)
\quad u_t(x) = \int_t^T (L_s u_s + f_s(x))ds - \int_t^T v_s(x) \cdot dW_s
\end{equation}

has an $\mathcal{F}_t$-adapted solution $(u, v) \in C_{x,t}^{2+\alpha} \times C_{x,t}^{2+\alpha}$ provided that $f \in C_{x,t}^{\alpha,0}(Q_T; L^q(\Omega))$ and $Df \in C_{x,t}^{\alpha,0}(Q_T; L^{2p}(\Omega; H))$. Moreover, there is a constant $C$ only depends on $n, d, p, q, \alpha, \Lambda, N'$ such that

$$
\|u\|_{C_{x,t}^{2+\alpha}} + \|v\|_{C_{x,t}^{2+\alpha}} \leq C \left( \|f\|_{C_{x,t}^{\alpha,0}(Q_T; L^q(\Omega))} + \sup_{(s,t) \in \Delta T} \|D_s f_t\|_{C_{\mathbb{R}^n;L^{2p}(\Omega; \mathbb{R}^d))} \right).
$$

Proof. We divide the proof into four steps.

Step 1. Let

$$
\Lambda_f := \|f\|_{C_{x,t}^{\alpha,0}(Q_T; L^q(\Omega))} + \sup_{(s,t) \in \Delta T} \|D_s f_t\|_{C_{\mathbb{R}^n;L^{2p}(\Omega; \mathbb{R}^d))}
$$

and $w$ be the unique solution to equation (2.2) in $C_{x,t}^{2+\alpha,0}(Q_T; L^q(\Omega))$. Below we show that for each $(x, t)$, $w_t(x)$ is Malliavin differentiable, and $Dw$ satisfies the following $L^p(\Omega; H)$-valued equation:

\begin{equation}
(3.7)
\quad Dw_t = \int_t^T (L_r Dw_r + G_r)dr,
\end{equation}

where $G_r = Df_r + (\partial_{ij} w_r Da_r^{ij} + \partial_{ij} w_r Db_r^{ij} + w_r \cdot Dc_r)$. To do this, we consider the following $L^p(\Omega; H)$-valued PDE,

\begin{equation}
(3.8)
\quad D^2 w_t = \int_t^T L_r(Dw_r)dr + \int_t^T G_r dr = 0.
\end{equation}

By Assumption 1, 2 and Theorem 2.2, we get

$$
\|w\|_{C_{x,t}^{2+\alpha,0}(Q_T; L^q(\Omega))} \leq \|f\|_{C_{x,t}^{\alpha,0}(Q_T; L^q(\Omega))} \leq C \Lambda_f,
$$

$$
\sum_{i,j} \|Da_r^{ij}\|_{C_{x,t}^{\alpha,0}(Q_T; L^{2p}(\Omega; H))} + \sum_i \|Db_r^{ij}\|_{C_{x,t}^{\alpha,0}(Q_T; L^{2p}(\Omega; H))} + \|Dc\|_{C_{x,t}^{\alpha,0}(Q_T; L^{2p}(\Omega; H))} < \infty.
$$

Recalling that $q > 2p \geq 4$, Hölder’s inequality yields,

$$
\|G\|_{C_{x,t}^{\alpha,0}(Q_T; L^p(\Omega; H))} \leq C \Lambda_f.
$$

Due to Theorem 2.2 (with $\mathcal{H} = H$ therein), there is a unique solution $Dw \in C_{x,t}^{2+\alpha,0}(Q_T; L^p(\Omega; H))$ sloves (3.8). Thus, for any $h \in H$, $D^h w_t := \langle Dw_t, h \rangle$ satisfies

\begin{equation}
(3.9)
\quad D^h w_t = \int_t^T L_r(D^h w_r)dr = \int_t^T \langle G_r, h \rangle dr
\end{equation}
and

\[
\| \mathcal{D}^h w \|_{C^{2+\alpha}_{x,t}} + \| \mathcal{D}^h \partial_t w \|_{C^{\alpha}_{x,t}} \lesssim C |h| \Lambda f.
\]

Next we show that \( D^h w_t(x) \) (see (3.5) for the definition) convergence to \( D^h w_t(x) \) in \( L^p(\Omega) \), and as a consequence, we have \( D w_t(x) = D w_t(x) \). By the definition of \( D^h w_t \), one sees

\[
D^h w_t - \int_0^T \left[ \tau_{\varepsilon h} D^h w_t + \tau_{\varepsilon h} b_i \partial_i \right. \\
\left. + \tau_{\varepsilon h} c_i \partial_i + \tau_{\varepsilon h} \partial_i w_t \right] dr,
\]

\[
= \int_0^T \left[ D^h f_t + D^h D^h b_i \partial_i w_t + D^h c_i w_t \right] dr.
\]

Noting that for any \( F \in \mathbb{D}^{1,p} \) and \( h \in H \),

\[
D^h F = \frac{\left( \tau_{\varepsilon h} F - F \right)}{\varepsilon} = \varepsilon^{-1} \int_0^E \tau_{\varepsilon h} D^h F \, d\theta,
\]

we get that for any \( q' \in [p, 2p) \),

\[
\mathbb{E}[D^h f_t(x) - D^h f_t(y)]^{q'} = \left\| \varepsilon^{-1} \int_0^E \tau_{\varepsilon h} [D^h f_t(x) - D^h f_t(y)] d\theta \right\|_{L^{q'}(\Omega)}^{q'}
\]

\[
\leq \sup_{0 \leq \theta \leq E} \| \tau_{\varepsilon h} (D^h f_t(x) - D^h f_t(y)) \|^q_{L^{q'}(\Omega)}.
\]

Due to Girsanov theorem,

\[
\frac{dP \circ \tau_{\varepsilon h}^{-1}}{dP} = \mathcal{E}(\theta h) := \exp \left( \theta \int_0^T h_t dW_t - \frac{\theta^2}{2} \int_0^T |h_t|^2 dr \right).
\]

Hence,

\[
\mathbb{E}[D^h f_t(x) - D^h f_t(y)]^{q'} \leq \sup_{0 \leq \theta \leq E} \mathbb{E}[\|D^h f_t(x) - D^h f_t(y)\|^{q'} \mathcal{E}(\theta h)]
\]

\[
\leq \sup_{0 \leq \theta \leq E} \mathbb{E}[\|D^h f_t(x) - D^h f_t(y)\|^{2p}]^{\frac{q'}{2p}} \cdot \mathbb{E}[\mathcal{E}^{-\frac{q}{2p}}(\theta h)]^{\frac{q}{2p}}
\]

\[
\leq C \|D f_t\|_{C^\alpha(\mathbb{R}^d; L^2p(\Omega; H))} |h|^q H_x |x - y|^{\alpha q'},
\]

where we use the following fact in the last inequality:

\[
\mathbb{E}\mathcal{E}^\kappa(\theta h) = \mathbb{E}\mathcal{E}(\kappa \theta h) \exp \left( \frac{\kappa^2 - \kappa}{2} |h|^2_H \right) \leq C_k.
\]

Thus,

\[
\sup_{\varepsilon \in (0, 1)} \|D^h f\|_{C^\alpha(\mathbb{R}^d; L^2p(\Omega; H))} \leq C |h| H \|D f\|_{C^\alpha(\mathbb{R}^d; L^2p(\Omega; H))}.
\]
Similarly, for any \( q'' \in (1, 2p) \),
\[
\sup_{\varepsilon \in (0, 1)} \left[ \| D^h e a \|_{C^{\alpha,0}_{x,t} (Q_T : L^{q''}(\Omega))} + \| D^h e b \|_{C^{\alpha,0}_{x,t} (Q_T : L^{q''}(\Omega))} \right. \\
\left. + \| D^h c_r \|_{C^{\alpha,0}_{x,t} (Q_T : L^{q''}(\Omega))} \right] \leq C.
\]

Choosing \( q' = p \) and \( q'' = \frac{p q}{q - p} \in (p, 2p) \), and noticing that \( \| w \|_{C^{2+\alpha,0}_{x,t} (Q_T : L^{q}(\Omega))} \leq C \Lambda f \), by Hölder’s inequality, we get
\[
(3.13) \quad \sup_{\varepsilon \in (0, 1)} \| D^h e f + D^h e a^{ij} \partial_{ij} w + D^h e b^i \partial_i w + D^h e c_w \|_{C^{\alpha,0}_{x,t} (Q_T : L^p(\Omega))} \leq C |h|_H \Lambda f.
\]

Since \( \tau_{eh} a, \tau_{eh} b, \tau_{eh} c \) satisfy (H1) and (H2), by (3.11), (3.13) and Theorem 2.2, we have
\[
(3.14) \quad \sup_{\varepsilon \in (0, 1)} \left( \| D^h e w \|_{C^{2+\alpha}_{x,t}} + \| D^h \partial_t w \|_{C^{\alpha}_{x,t}} \right) \leq C |h|_H \Lambda f.
\]

Let \( \delta^h w := D^h e w - D^h w \). Next we want to prove \( \delta^h w \to 0 \) in \( L^p(\Omega) \), for each \((x, t) \in Q_T\). By definition,
\[
\delta^h w + L_t \delta^h w = -(D^h e f - D^h f) \\
- \left[ (D^h e a^{ij} - D^h e a^{ij}) \partial_{ij} w + (D^h e b^i - D^h b^i) \partial_i w + (D^h e c - D^h c) w \right] \\
- \varepsilon \left( D^h e a^{ij} \partial_{ij} D^h e w + D^h e b^i \partial_i D^h e w + D^h e c D^h e w \right) =: - \sum_{i=1}^3 F^e, i
\]
e.g. \( \delta^h w \) is a \( L^p(\Omega) \)-valued solution to (2.2) with \( f \) replaced by \( F^e := \sum_{i=1}^3 F^e, i \). Estimates (3.10) and (3.14) yield
\[
(3.16) \quad \sup_{\varepsilon \in (0, 1)} \left( \| \delta^h w \|_{C^{2+\alpha}_{x,t}} + \| \partial_t \delta^h w \|_{C^{\alpha}_{x,t}} \right) \leq C |h|_H \Lambda f.
\]

By (3.15), for each \( R > 0 \), we have
\[
\partial_t (\delta^h w \chi_R) + L_t (\delta^h w \chi_R) + F^e \chi_R \\
- (2a^{ij} \partial_i \delta^h w \partial_j \chi_R + \delta^h w a_i^{ij} \partial_i \chi_R + \delta^h w b_i^{ij} \partial_i \chi_R) = 0,
\]
where \( \chi_R(x) = \chi(x/R) \). Due to our assumptions and (3.16),
\[
\left\| (2a^{ij} \partial_i \delta^h w \partial_j \chi_R + \delta^h w a_i^{ij} \partial_i \chi_R + \delta^h w b_i^{ij} \partial_i \chi_R) \right\|_{C^{\alpha}_{x,t}} \leq C |h|_H \Lambda f / R.
\]

So by Theorem 2.2, for any \( \alpha' \in (0, \alpha) \),
\[
(3.17) \quad \| \delta^h w \chi_R \|_{C^{2+\alpha'}_{x,t}} \leq C \| F^e \chi_R \|_{C^{\alpha'}_{x,t}} + C |h|_H \Lambda f / R.
\]

Thanks to Lemma 3.3, for each \((x, t) \in Q_T, F_t^{e, 1}(x) = D^h e f_t(x) - D^h f_t(x) \xrightarrow{L^{2p}(\Omega)} 0 \). By (3.12) and the continuity of \( D f : Q_T \to L^{2p}(\Omega; H) \), one can verify that the map \( Q_T \ni
(x, t) \mapsto D^h_{e, f_t}(x) \in L^p(\Omega) is equivalent continuous. So by Arzela-Ascoli theorem, for any sequence \( \varepsilon_n \to 0(n \to \infty) \), there exists a subsequence \( \varepsilon_{n_k} \to 0(k \to \infty) \) such that for all \( R > 0 \), \( F^{\varepsilon_{n_k}}\chi_R \to 0 \) in \( C^\alpha_{x,t} \), with some \( \alpha' \in (0, \alpha) \). Similarly, we have \( F^{\varepsilon_{n_k}}\chi_R \to 0 \) and \( F^{\varepsilon_{n_k}}\chi_R \to 0 \) in \( C^\alpha_{x,t} \) as \( k \to \infty \). Thus, \( \limsup_{\varepsilon \to 0} \| F^{\varepsilon} \chi_R \|_{C^\alpha_{x,t}} = 0 \). So by (3.17), for any \( R_0 > 0 \),

\[
\limsup_{\varepsilon \to 0} \| \delta^h_{e, w(x)} \|_{C^\alpha_{x,t}} \leq \lim_{R \to \infty} \limsup_{\varepsilon \to 0} \| \delta^h_{e, w(x)} \|_{C^\alpha_{x,t}} \leq \lim_{R \to \infty} C/R = 0,
\]

which of course implies \( D^h_{e, w(x)} - D^h_{w(x)} \to 0 \) in \( L^p(\Omega) \). Again by Lemma 3.3, for each \( (x, t) \in Q_T \), we have \( w_t(x) \in \mathbb{D}^{1,p} \) and \( D w_t(x) = D w_t(x) \in C^{2+\alpha,0}_{x,t}(Q_T; L^p(\Omega; H)) \). Estimate (3.7) follows by the definition of \( Dw \).

**Step 2.** For any \( (s, t) \in A_T \), let \( w^s_t(x) \) be the solution to the following equation

\[
(3.18) \quad w^s_t = \int_t^T (L_r w^s_r + g^s_r) \mathrm{d}r,
\]

where \( g^s := (D s \delta^i_r) \partial_i w_r + (D s b^i_r) \partial_i w_r + (D s c_r) w_r + D s f_r \). By Hölder’s inequality,

\[
\| g^s \|_{C^{\alpha}_{x,t}} \leq \| D s f \|_{C^{\alpha}_{x,t},L^p(\Omega)} + \| w \|_{C^{2+\alpha,0}_{x,t},(Q_T; L^2(\Omega))} \left( \sum_j \| D s a^{ij} \|_{C^{\alpha,0}_{x,t},(Q_T; L^2(\Omega))} \right)
\]

\[
+ \sum_j \| D s b^i \|_{C^{\alpha}_{x,t},L^2(\Omega)} + \| D s c \|_{C^{\alpha,0}_{x,t},(Q_T; L^2(\Omega))} \right)
\]

\[
\leq C \| f \|_{C^{\alpha}_{x,t},(Q_T; L^p(\Omega))} + C \sup_{(s, t) \in A_T} \| D s f_t \|_{C^{\alpha}(\mathbb{R}^d; L^p(\Omega))} \leq C \Lambda_f.
\]

Theorem 2.2 yields,

\[
(3.19) \quad \sup_{s \in [0, T]} \left( \| \partial_t w^s \|_{C^{\alpha}_{x,t}} + \| w^s \|_{C^{2+\alpha}_{x,t}} \right) \leq C \| g^s \|_{C^{\alpha}_{x,t}} \leq C \Lambda_f.
\]

**Step 3.** In this step, we prove that \( w^s_t(x) \) constructed in Step 2 is a version of \( D s w_t(x) \).

Let \( \mathcal{A}^\alpha = \{ w : w \in C^{2+\alpha}_{x,t}, \partial_t w \in C^\alpha_{x,t} \} \), \( \| w \|_{\mathcal{A}^\alpha} := \| w \|_{C^{2+\alpha}_{x,t}} + \| \partial_t w \|_{C^\alpha_{x,t}} \).

By linearity and Theorem 2.2, the solution map of (2.2)

\[
\mathcal{T} : C^\alpha_{x,t} \ni f \mapsto w \in \mathcal{A}^\alpha
\]

is Lipschitz continuous. Since \( [0, T] \ni s \mapsto g^s \in C^\alpha_{x,t} \) is measurable, \( s \mapsto w^s \) is measurable from \( [0, T] \) to \( \mathcal{A}^\alpha \). For any \( \varphi \in C^\infty_c((0, T); \mathbb{R}^d) \), define

\[
w^\varphi = \int_0^T \varphi(s) \cdot w^s \mathrm{d}s, \quad g^\varphi = \int_0^T \varphi(s) \cdot g^s \mathrm{d}s.
\]

Then, one sees that \( w^\varphi \) satisfies

\[
w^\varphi_t = \int_t^T (L_r w^\varphi_r + g^\varphi_r) \mathrm{d}r.
\]
On the other hand, notice that $Dw$ is the unique solution to (3.7), we have

$$
\langle \varphi, Dw_t \rangle_H = \int_t^T (L_r \langle \varphi, Dw_r \rangle_H + \langle \varphi, g_r \rangle_H)dr = \int_t^T (L_r \langle \varphi, Dw_r \rangle_H + g_r^\varphi)dr.
$$

So $w^\varphi = \langle \varphi, Dw \rangle$, which implies $s \mapsto w^s$ is a version of $Dw$.

**Step 4.** In this step, we prove the $C^{2+\alpha}$ regularity estimate for $v$. Define $u_t(x) = E^t w_t(x)$. Theorem 2.2 and Lemma 5.4 yield

$$
\|u\|_{C^{2+\alpha}_{x,t}} \leq \|w\|_{C^{2+\alpha}_{x,t}} \leq C\|f\|_{C^\alpha_{x,t}} \leq CA_f.
$$

Let $\dot{w}_t(x) := -[L_t w_t(x) + f_t(x)]$, by Step 1, $\dot{w} \in C^{\alpha,0}_{x,t}(Q_T; D^{1,p})$. Note that

$$
w_t(x) = w_0(x) + \int_0^t \dot{w}_s(x)ds.
$$

Thanks to Lemma 3.2, for each $(x, t) \in Q_T$,

$$
u_t(x) = E^t w_t(x) = Ew_0(x) + \int_0^t E^s \dot{w}_s(x)ds + \int_0^t E^s W_{s,s}(x) \cdot dW_s,
$$

where $W_{s,s}(x) = D_x w_0(x) + \int_0^s D_x \dot{w}_r(x)dr$ for all $(x, t) \in Q_T$ and $s \in [0, T]$ a.e.. Since

$$
W_{x,s}(x) = D_x w_0(x) + \int_0^s D_x \dot{w}_r(x)dr = \int_0^s D_x [L_r w_r + f_r](x)dr = \int_0^s [L_r w^s_r + g^s_r](x) = w^s(x), s \in [0, T] a.e.,
$$

we get

$$
u_t(x) = u_0(x) - \int_0^t E^s (L_s w_s + f_s)(x)ds + \int_0^t E^s w^s_s(x) \cdot dW_s
$$

Note $u_T(x) = 0$, we have

$$
u_0(x) = \int_0^T (L_s u_s + f_s)(x)ds - \int_0^T E^s w^s_s(x) \cdot dW_s.
$$

Combining the above two equations, we obtain

$$
u_t(x) = \int_t^T (L_s u_s + f_s)(x)ds - \int_t^T E^s w^s_s(x)dW_s
$$

Let $v_s(x) = w^s(x)$, then the above identity implies $(u_t, v_t) = (E^t w_t, E^t w^t_s)$ is a solution to (3.6). Moreover,

$$
\|v\|_{C^{2+\alpha}_{x,t}} = \sup_{0 \leq t \leq T} \|E^t w^t_s\|_{C^{2+\alpha}_{x,t;L^p}(\Omega)} \leq \sup_{s \in [0, T]} \|w^s\|_{C^{2+\alpha}_{x,t}} \leq C A_f < \infty.
$$

So we complete our proof.
Let \( \varrho \in C^\omega_c(\mathbb{R}^n) \) satisfying \( \int \varrho = 1 \), and \( \varrho_m(x) := m^n \varrho(mx) \). For any function \( g : \mathbb{R}^n \to \mathbb{R}^m \), set \( g^m := g \ast \varrho_m \).

The following corollary of Theorem 3.4 is standard.

**Corollary 3.5** (Stability). Assume \( a, b, c \) satisfy Assumption 1 and 2. Let \( w_t^m \) (respectively \((u^m, v^m)\)) be the solution to (2.2) (respectively (3.6)) in \( C^{2+\alpha}_{x,t} \) (respectively \( C^{2+\alpha}_{x,t} \times C^{2+\alpha}_{x,t} \)) with \( a, b, c, f \) replaced by \( a^m, b^m, c^m, f^m \). Then for any \( \beta \in (0, \alpha) \), it holds that

\[
\| \partial_t (w - w^m) \|_{C^{2+\beta}_{x,t}} + \| w - w^m \|_{C^{2+\alpha}_{x,t}} + T^{-1} \| w - w^m \|_{C^{0}_{x,t}} \to 0 \quad (n \to \infty),
\]

\[
\| u - u^m \|_{C^{2+\beta}_{x,t}} + \| v - v^m \|_{C^{2+\beta}_{x,t}} \to 0 \quad (n \to \infty).
\]

4. SDEs with random singular coefficients

In this section, we give the proof for our main result.

**Proof of Theorem 1.1.** We first point out that it is enough to prove the well-posedness of (1.1) for \( t \in [0, T/2] \), where \( T \) is a universal constant depending only on \( n, \alpha, \Lambda, p \).

**Pathwise uniqueness:** Assume \( X_t \) is a solution to (1.1). We prove the uniqueness by Zvonkin type transformation. With a little abuse of notation, we denote \( \Lambda \) where \( \Lambda \) is an integer that can be changed in different places. Recalling that \( L_t = a^{ij}_{t} \partial_{ij} + b^i_t \partial_i \). We consider the following BSPDE:

\[
(4.1) \quad d u_t + (L_t u_t + b_t) dt = v_t \cdot d W_t, \quad u_T(x) = 0.
\]

By our assumptions and Theorem 3.4, (4.1) has an \( \mathcal{F}_t \)-adapted solution \((u_t, v_t)\) and

\[
||u||_{C^{2+\alpha}_{x,t}} + ||v||_{C^{2+\alpha}_{x,t}} < \infty.
\]

Since \( u_t = \mathbf{E}^t w_t \), \( w_t \) solves

\[
\partial_t w + L_t w + b = 0, \quad w_T(x) = 0
\]

and

\[
\text{ess sup}_{\omega \in \Omega} \left( \sup_{t \in [0, T]} ||b_t(\cdot, \omega)||_{C^\alpha} + \sup_{(s,t) \in \Delta_T} ||D_s b_t(\cdot, \omega)||_{C^\alpha} \right) < \infty.
\]

By Remark 2.4, we have

\[
\text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T]} \left( ||w_t(\cdot, \omega)||_{C^{2+\alpha}} + T^{-1} ||w_t(\cdot, \omega)||_{C^\alpha} \right)
\]

\[
\leq \text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T]} ||b_t(\cdot, \omega)||_{C^\alpha}.
\]

Interpolation inequality and above estimate yield

\[
\text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T]} ||u_t(\cdot, \omega)||_{C^1} \leq \text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T]} ||w_t(\cdot, \omega)||_{C^1} \leq C_T,
\]
where \( C_T \to 0 \) as \( T \to 0 \). Below we fix \( T = T(n, \alpha, \Lambda, \rho) > 0 \) so that

\[
\text{ess sup}_{\omega \in \Omega} \sup_{t \in [0, T]} \| u_t(\cdot, \omega) \|_{C^1} \leq \frac{1}{2}.
\]

Let \( \phi_t(x) = x + u_t(x) \), then

\[
\frac{1}{2} \leq \text{ess sup}_{\omega \in \Omega} \sup_{0 \leq t \leq T} \| \nabla \phi_t(x, \omega) \|_{L^\infty} \leq \frac{3}{2}.
\]

So for almost surely \( \omega \in \Omega \), \( \phi_t(\cdot, \omega) \) is a stochastic \( C^{2+\alpha} \)-differential homeomorphism from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). By the definition of \( \phi \),

\[
d\phi_t(x) = -(L_t u_t(x) + b_t(x))dt + v_t(x) \cdot dW_t = du_t(x) = dg_t(x) + dm_t(x),
\]

where

\[
g_t(x) = - \int_0^t (L_s u_s(x) + b_s(x))ds,
\]

\[
m_t(x) := \int_0^t v_s(x)dW_s.
\]

We want to show that \( \phi, g, u, v \) and \( X \) are regular enough to apply the Itô-Wenzell formula (see Lemma 5.7). Since \( \| v \|_{C^{2+\alpha}} < \infty \), we have

\[
\sup_{t \in [0, T]: x \neq y} \frac{E|\nabla^2 v_t(x) - \nabla^2 v_t(y)|^p}{|x - y|^{\alpha p}} < \infty.
\]

Note that \( p > n/\alpha \), so for any \( \beta \in (n/p, \alpha) \) and \( N > 0 \), by Garsia-Rademich-Rumsey’s inequality,

\[
\sup_{t \in [0, T]} E \left( \sup_{x, y \in B_N} \frac{|\nabla^2 v_t(x) - \nabla^2 v_t(y)|}{|x - y|^{\beta - n/p}} \right)^p 
\leq C_N \sup_{t \in [0, T]} E \left( \int_{B_N} \int_{B_N} \frac{|\nabla^2 v_t(x) - \nabla^2 v_t(y)|^p}{|x - y|^{d+\beta p}} dx dy \right) 
\leq C_N \int_{B_N} \int_{B_N} |x - y|^{-d+(\alpha - \beta) p} \leq C_N.
\]

Combining this and the fact that \( \sup_{t \in [0, T]} E|\nabla^2 v_t(0)|^p < \infty \), we get

\[
\sup_{t \in [0, T]} E \left( \sup_{x \in B_N} |\nabla^2 v_t(x)|^p \right) < \infty, \quad \forall N > 0.
\]

Moreover, one can also prove

\[
\sup_{t \in [0, T]} E\| v_t \|_{C^2(B_N)}^p < \infty, \quad \forall N > 0.
\]

Recalling that \( g_t(x) \) and \( m_t(x) \) are defined in (4.4), let

\[
\eta_t(x) := \int_0^t g_s(x)ds \overset{(4.1)}{=} u_t(x) - u_0(x) - m_t(x).
\]
By Burkholder-Davis-Gundy’s inequality, for each \( k = 0, 1, 2 \)
\[
E \left| \nabla^k m_t(x) - \nabla^k m_t(y) \right|^p \leq CE \left[ \int_0^t \left| \nabla^k v_s(x) - \nabla^k v_s(y) \right|^k ds \right]^{\frac{p}{k}}
\]
\[
\leq CE \int_0^t \left| \nabla^k v_s(x) - \nabla^k v_s(y) \right|^p ds \leq C|x - y|^\alpha p \left\| \nabla^k v \right\|_{L^p}^p C_{x,t}^{2+\alpha},
\]
which together with (4.2) implies
\[
\left\| \eta \right\|_{C^{2+\alpha}} \leq C \left( \left\| u \right\|_{C_{x,t}^{2+\alpha}} + \left\| v \right\|_{C_{x,t}^{2+\alpha}} \right).
\]
By the definition of \( \eta \),
\[
\left\| \partial_t \eta \right\|_{C_{x,t}^{\alpha}} \leq \left\| g \right\|_{C_{x,t}^{\alpha}} \leq \left\| L_t u + b \right\|_{C_{x,t}^{\alpha}} \leq C \left( \left\| u \right\|_{C_{x,t}^{2+\alpha}} + \left\| b \right\|_{C_{x,t}^{2+\alpha}} \right).
\]
Thanks to Lemma 5.3, for any \( \beta \in (n/p, \alpha) \) and \( \theta = \frac{1}{2} + \frac{\alpha - \beta}{2} \in (\frac{1}{2}, 1) \), we have
\[
\left\| \eta \right\|_{C^{\phi}} \leq C \left\| \partial_t \eta \right\|_{C_{x,t}^{\alpha}}^{\theta} \left\| \eta \right\|_{C_{x,t}^{2+\alpha}}^{1-\theta}.
\]
By the same procedure of proving (4.5), we have
\[
\left[ E \left\| \int_{t_1}^{t_2} g_s ds \right\|_{C^1(B_N)}^p \right]^{1/p} \leq C \left| t_1 - t_2 \right|^\theta, \quad \theta \in (1/2, 1).
\]
On the other hand, \( E|X_{t_1} - X_{t_2}|^{p'} \leq C\left| t_1 - t_2 \right|^{p'/p} \) (\( p' = p/(p-1) \)). So \( \phi, g, v, X \) satisfy all the conditions in Lemma 5.7. Using (5.10), we get
\[
d\phi_t(X_t) = -L_t u_t(X_t) - b_t(X_t) dt + v_t^k(X_t) dW_t^k
\]
\[
+ [b_t^i(X_t) \partial_i \phi_t(X_t) + a_t^{ij}(X_t) \partial_{ij} \phi_t(X_t) + \partial_i v_t^k(X_t) \sigma_t^{ik}(X_t)] dt
\]
\[
+ \partial_i \phi_t(X_t) \sigma_t^{ik}(X_t) dW_t^k
\]
\[
= \partial_i v_t^k(X_t) \sigma_t^{ik}(X_t) dt + \partial_i \phi_t(X_t) \sigma_t^{ik}(X_t) dW_t^k + v_t^k(X_t) dW_t^k.
\]
Set
\[
Y_t = \phi_t(X_t), \quad \tilde{b}_t(y) = \partial_i v_t^k \sigma_t^{ik} \circ \phi_t^{-1}(y) \text{ and } \tilde{\sigma}_t(y) = [\nabla \phi_t \sigma_t + v_t] \circ \phi_t^{-1}(y).
\]
By the above calculations, one sees that
\[
Y_t = Y_0 + \int_0^t \tilde{b}_s(Y_s) ds + \int_0^t \tilde{\sigma}_s(Y_s) dW_s.
\]
Thanks to Lemma 5.2, \( \tilde{b} \) and \( \tilde{\sigma} \) are \( \mathcal{B} \times \mathcal{P} \)-measurable. For any \( x, y \in B_N \) and \( t \in [0, T] \), by the definitions of \( \tilde{b} \) and \( \tilde{\sigma} \), we have
\[
\left| \tilde{b}_t(0) \right| + \left| \tilde{\sigma}_t(0) \right| \leq C K_t^N,
\]
[\tilde{b}_t(x) - \tilde{b}_t(y)] + |\tilde{\sigma}_t(x) - \tilde{\sigma}_t(y)| \leq CK^N_t|x - y|,

where \(K^N_t := \|u_t\|_{C^2(B_N)} + \|v_t\|_{C^2(B_N)}\). It is not hard to see that \(K^N_t\) is progressive measurable and satisfies

\[
\mathbb{E} \int_0^T K^N_t \, dt \leq T \sup_{t \in [0,T]} \mathbb{E} K^N_t < \infty.
\]

Thanks to Theorem 1.2 of [9], equation (4.6) admits a unique solution, which implies \(X_t\) is unique up to indistinguishability.

**Existence:** Let \(b^m_t = b_t \circ \varphi_m\) and \(X^m\) be the solution to

\[
(4.7) \quad X^m_t = X_0 + \int_0^t b^m_s(X^m_s) \, ds + \int_0^t \sigma_s(X^m_s) \, dW_s, \quad t \in [0, T].
\]

We claim that \(X^m_t\) uniform convergence on compacts in probability (ucp convergence in short) to a process \(X_t\). Let \((u^m, \nu^m)\) be the pair of functions constructed in Theorem 3.4 satisfying

\[
du^m_t + \left[ a_t^{ij} \partial_j u^m_t + (b^m_t)^{ij} \partial_i u^m_t + b^m_t \right] \, dt = \nu^m_t \, dW_t.
\]

Like before, we can find a uniform constant \(T = T(n, \alpha, \Lambda, p) > 0\) such that \(\|\nabla u^m\|_{L^\infty} \leq 1/2\). Define \(\phi^m_t(x) := x + u^m_t(x), Y^m_t := \phi^m_t(X^m_t)\) and \(Z_t^{m,m'} := Y^m_t - Y^{m'}_t\). Again by Itô-Wentzell’s formula, we have

\[
\begin{align*}
Z_{s}^{m,m'} = & \left( Y^{m'}_{s} - Y^m_{s} \right) = u^m_0(X_0) - u^{m'}_0(X_0) + \int_0^s \left[ \tilde{b}^m_s(X^m_s) - \tilde{b}^{m'}_s(X^{m'}_s) \right] \, ds \\
& + \int_0^s \left[ \tilde{\sigma}^m_s(X^m_s) - \tilde{\sigma}^{m'}_s(X^{m'}_s) \right] \, dW_s,
\end{align*}
\]

where

\[
\tilde{b}^m_t := [\partial_i u^{m,k}_t \sigma^{ij}_t] \circ \phi^{m^{-1}}_t, \quad \tilde{\sigma}^m_t := \left[ (\nabla \phi^m_t) \sigma_t + \nu^m_t \right] \circ \phi_t^{-1}.
\]

By Itô’s formula, for any stopping time \(\tau \leq T\),

\[
\left| Z_{t \wedge \tau}^{m,m'} \right|^2 = |u^m_0(X_0) - u^{m'}_0(X_0)|^2 + 2 \int_0^{t \wedge \tau} Z_{s}^{m,m'} \cdot \left[ \tilde{b}^m_s(Y^m_s) - \tilde{b}^{m'}_s(Y^{m'}_s) \right] \, ds \\
+ \int_0^{t \wedge \tau} \text{tr} \left[ \tilde{\sigma}^m_s(Y^m_s) - \tilde{\sigma}^{m'}_s(Y^{m'}_s) \right] \left[ \tilde{\sigma}^m_s(Y^m_s) - \tilde{\sigma}^{m'}_s(Y^{m'}_s) \right]^* \, ds + m_{t \wedge \tau},
\]

where

\[
m_t = 2 \int_0^t Z_{s}^{m,m'} \cdot \left[ \tilde{\sigma}^m_s(Y^m_s) - \tilde{\sigma}^{m'}_s(Y^{m'}_s) \right] \, dW_s.
\]

For any \(N, k \in \mathbb{N}\), let \(K_t^{m,N} := \|u^m_t\|_{C^2(B_N)} + \|v^m_t\|_{C^2(B_N)}\),

\[
\tau^{N,k} = \inf_m \inf \left\{ t \geq 0 : \int_0^t (K_s^{m,N})^2 \, ds \geq k \right\} \wedge T,
\]

and

\[
\sigma^N = \inf_m \inf \left\{ t \geq 0 : |Y^m_t| > N/2 \right\} \wedge T, \quad \sigma^{N,k} := \sigma^N \wedge \tau^{N,k}.
\]
For all \( x, y \in B_{N/2} \) and \( t \in [0, \sigma_{N,k}] \), we have

\[
\sup_{m \in \mathbb{N}} \left( |\tilde{b}_1^m(x) - \tilde{b}_0^m(y)| + |\tilde{\sigma}_1^m(x) - \tilde{\sigma}_0^m(y)| \right) \leq C_k |x - y|.
\]

Since for each \((x, t) \in B_{N/2} \times [0, T], (\phi_t^m)^{-1}(x) \in B_N\), we obtain that for any \( x \in B_{N/2} \) and \( t \in [0, \tau_{N,k}] \),

\[
|\tilde{b}_1^m(x) - \tilde{b}_0^m(x)| \\
\leq \left| \partial_i v_t^{m,k} \sigma_t^{ik} \circ (\phi_t^m)^{-1}(x) - \partial_i v_t^{m',k} \sigma_t^{ik} \circ (\phi_t^m)^{-1}(x) \right| \\
+ \left| \partial_i v_t^{m',k} \sigma_t^{ik} \circ (\phi_t^m)^{-1}(x) - \partial_i v_t^{m',k} \sigma_t^{ik} \circ (\phi_t^{m'})^{-1}(x) \right| \\
\leq C \| \nabla v_t^m - \nabla v_t^{m'} \|_{L^\infty(B_N)} + C \| v_t^{m'} \|_{C^2(B_N)} |(\phi_t^m)^{-1}(x) - (\phi_t^{m'})^{-1}(x)| \\
\leq C \| v_t^m - v_t^{m'} \|_{C^2(B_N)} + C \| v_t^m \|_{C^2(B_N)} \sup_{y \in B_N} |\phi_t^{m'}(y) - \phi_t^m(y)| \\
\leq C_k \left( \| u_t^m - u_t^{m'} \|_{C^2(B_N)} + \| v_t^m - v_t^{m'} \|_{C^2(B_N)} \right).
\]

Similarly, for each \( x \in B_{N/2} \) and \( t \in [0, \tau_{N,k}] \),

\[
|\tilde{\sigma}_1^m(x) - \tilde{\sigma}_0^m(x)| \leq C_k \left( \| u_t^m - u_t^{m'} \|_{C^2(B_N)} + \| v_t^m - v_t^{m'} \|_{C^2(B_N)} \right)
\]

By our Theorem 3.4, Corollary 3.5 and the same procedure of proving (4.5), we have

\[
\sup_{t \in [0, T], m \in \mathbb{N}} \mathbb{E}|K_1^{m,N}|^p = \sup_{t \in [0, T], m \in \mathbb{N}} \mathbb{E}\left( \| u_t^m \|_{C^2(B_N)} + \| v_t^m \|_{C^2(B_N)} \right)^p < \infty
\]

and

\[
\lim_{m \to \infty} \sup_{t \in [0, T]} \mathbb{E}\left( \| u_t - u_t^m \|_{C^2(B_N)} + \| v_t - v_t^m \|_{C^2(B_N)} \right)^p = 0.
\]

Thus,

\[
\lim_{k \to \infty} \tau_{N,k} = T, \quad \lim_{N \to \infty} \sigma^N = T
\]

and

\[
\lim_{m, m' \to \infty} \mathbb{E}\left( \| \tilde{\sigma}_1^m - \tilde{\sigma}_0^m \|_{L^\infty(B_{N/2})} + \| \tilde{b}_t^m - \tilde{b}_t^{m'} \|_{L^\infty(B_{N/2})} \right)^p = 0.
\]
Let $\tau = \sigma^{N,k}$ in (4.8). Using (4.9), we have

\[
\left|Z_{t,\tau}^{m,m'}\right|^2 \leq u_0^m(X_0) - u_0^{m'}(X_0) + 2C_k \int_0^t \left(\|Z_s^{m,m'}\| + ||\bar{b}_s^{m'} - \bar{b}_s^m||_{L^\infty(B_{N/2})}\right) ds
\]

\[
+ C_k \int_0^t \left(\|Z_s^{m,m'}\| + ||\bar{\sigma}_s^{m'} - \bar{\sigma}_s^m||_{L^\infty(B_{N/2})}\right)^2 ds + m_{t,\sigma^{N,k}}.
\]

By Gronwall’s inequality and (4.11), we get

\[
\mathbb{E}\left(\left|Z_{t,\tau}^{m,m'}\right|^2\right) \leq C_k \|u_0^m - u_0^{m'}\|_{L^\infty}^2
\]

(4.12)

\[
+ C_k \mathbb{E} \int_0^T \left(\|\bar{\sigma}_s^{m'} - \bar{\sigma}_s^m\|_{L^\infty(B_{N/2})}^2 + \|\bar{b}_s^{m'} - \bar{b}_s^m\|_{L^\infty(B_{N/2})}^2\right) 1_{[0,\sigma^{N,k}]}(s) ds
\]

(4.11) $\rightarrow 0$ $(m, m' \rightarrow \infty)$.

On the other hand,

\[
|X_t^m - X_t^{m'}| = |(\phi_t^m)^{-1}(\phi_t^m(X_t^m)) - (\phi_t^{m'}(X_t^{m'}))| \leq 2|\phi_t^m(X_t^m) - \phi_t^{m'}(X_t^{m'})|
\]

\[
\leq 2|\phi_t^m(X_t^m) - \phi_t^{m'}(X_t^{m'})| + 2|\phi_t^{m'}(X_t^{m'}) - \phi_t^{m}(X_t^{m'})|
\]

\[
\leq 2\|u_t^m - u_t^{m'}\|_{L^\infty} + 2|Y_t^m - Y_t^{m'}|.
\]

Combining (4.12) and (4.13), we get

\[
\lim_{m,m' \rightarrow \infty} \mathbb{E} \sup_{t \in [0,T]} |X_t^m - X_t^{m'}|_{\tau,\sigma^{N,k}} = 0
\]

\[
\leq 2 \lim_{m,m' \rightarrow \infty} \mathbb{E} \sup_{t \in [0,T]} \|u_t^m - u_t^{m'}\|_{L^\infty} + 2 \lim_{m,m' \rightarrow \infty} \mathbb{E}(Z_{t,\tau}^{m,m'})_{\tau,\sigma^{N,k}} = 0.
\]

Noting

\[
\lim_{N \rightarrow \infty} \lim_{k \rightarrow \infty} \sigma^{N,k} = \lim_{N \rightarrow \infty} \sigma^N = T,
\]

we obtain

\[
\lim_{m,m' \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0,T/2]} |X_t^m - X_t^{m'}| > \varepsilon\right) = 0, \quad \forall \varepsilon > 0.
\]

(4.14)

This implies that there is a continuous process $\{X_t\}_{t \in [0,T/2]}$ such that $X^m \rightarrow X$ in the sense of uc. Hence,

\[
\int_0^t \sigma_s(X_s^m) \cdot dW_s \xrightarrow{\mathbb{P}} \int_0^t \sigma_s(X_s) dW_s, \quad \forall t \in [0,T/2],
\]
and for each \( t \in [0, T/2] \) and \( \varepsilon > 0 \),
\[
\mathbf{P} \left( \left| \int_0^t b_s^m(X_s^m) \, ds - \int_0^t b_s(X_s) \, ds \right| > \varepsilon \right) \\
\leq \mathbf{P} \left( \sup_{t \in [0, T/2]} |b_s^m(X_s^m) - b(X_s)| > \frac{\varepsilon}{2} \right) + \mathbf{P} \left( \sup_{t \in [0, T/2]} |b_t(X_t^m) - b_t(X_t)| \, ds > \frac{\varepsilon}{2} \right) \\
\leq \mathbf{P} \left( \|b_m - b\|_{L^\infty(\Omega)} > \frac{\varepsilon}{2} \right) + \mathbf{P} \left( \sup_{t \in [0, T/2]} |X_t^m - X_t|^\alpha > \frac{\varepsilon}{2\Lambda} \right) \to 0, \quad (m \to \infty).
\]

Taking limit on both side of (4.7), one sees that \( X \) is a solution to (1.1).  

\section{5. appendix}

In this section, we give some Lemmas used in the previous sections. The following basic result is useful.

\textbf{Lemma 5.1.} Let \( f \in L^1(\mathbb{R}^n; \mathcal{B}) + L^\infty(\mathbb{R}^n; \mathcal{B}) \).

1. (Bernstein’s inequality) For any \( k = 0, 1, 2, \cdots \), there is a constant \( C = C(n, k) > 0 \) such that for all \( j = -1, 0, 1, \cdots \),
\[
\|\nabla^k \Delta_j f\|_0 \leq C 2^{kj} \|\Delta_j f\|_0; \tag{5.1}
\]

2. For any \( \alpha \in (0, 1) \), there is a constant \( C = C(\alpha, n) > 1 \) such that
\[
C^{-1} \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_0 \leq \|f\|_{C^\alpha} \leq C \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_0. \tag{5.2}
\]

One can find the proof of above lemma in [1] for \( \mathcal{B} = \mathbb{R} \). We present its Banach-valued version below for the reader’s convenience.

\textbf{Proof.} For any \( j = 0, 1, 2, \cdots \), we have \( \int_{\mathbb{R}^n} h_j(z) \, dz = \varphi_j(0) = 0 \), so
\[
\|\Delta_j f(x)\|_\mathcal{B} = \left\| \int_{\mathbb{R}^n} h_j(x - y) [f(y) - f(x)] \, dy \right\|_\mathcal{B} \\
= \left\| \int_{\mathbb{R}^n} 2^{jn} h(2^j (x - y)) [f(y) - f(x)] \, dy \right\|_\mathcal{B} \\
\leq C \|f\|_{C^\alpha} \int_{\mathbb{R}^n} 2^{jn} h(2^j z) |z|^\alpha \, dz = C 2^{-j\alpha} \|f\|_{C^\alpha},
\]

which implies
\[
\sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_0 \leq C_\alpha \|f\|_{C^\alpha}.
\]
On the other hand,
\[
\left\| f(x) - \sum_{j=1}^{k} \Delta_j f(x) \right\|_B = \left\| \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\chi(2^{-k} \cdot))(y) \left[ f(x) - f(x - y) \right] dy \right\|_B \\
= \left\| \int_{B_{2^{-k} \varepsilon}} \mathcal{F}^{-1}(\chi)(z) \left[ f(x) - f(x - 2^{-k} z) \right] dz \right\|_B \\
+ \left\| \int_{B_{2^{-k} \varepsilon}^c} \mathcal{F}^{-1}(\chi)(z) \left[ f(x) - f(x - 2^{-k} z) \right] dz \right\|_B \\
\leq \text{osc}_{B_{\varepsilon}(x)} f \cdot \int_{\mathbb{R}^n} |\mathcal{F}^{-1}(\chi)(y)| dy + 2 \| f \|_0 \int_{B_{2^{-k} \varepsilon}} |\mathcal{F}^{-1}(\chi)(x)| dy.
\]

Let \( k \to \infty \) and then \( \varepsilon \to 0 \), we obtain that for each \( f \in C_b(\mathbb{R}^n; \mathcal{B}) \) and \( x \in \mathbb{R}^n \), \( f(x) = \sum_{j=1}^{\infty} \Delta_j f(x) \). Thus, for any \( K > 0 \),
\[
|f(x) - f(y)|_B \leq \sum_{j=1}^{\infty} |\Delta_j f(x) - \Delta_j f(y)|_B \leq |x - y| \sum_{j=1}^{\infty} \| \nabla \Delta_j f \|_0 + 2 \sum_{j>K} \| \Delta_j f \|_0 \\
\leq C\alpha(|x - y|^{2(1-\alpha)K} + C2^{-\alpha K}) \sup_{j>K} 2^{\alpha j} \| \Delta_j f \|_0.
\]

For any \( |x - y| < 1 \), by choosing \( K = -\log_2(|x - y|) \), we obtain
\[
|f(x) - f(y)|_B \leq C\alpha |x - y|^{\alpha} \sup_{j>1} 2^{\alpha j} \| \Delta_j f \|_0.
\]

So we complete our proof.

Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous homeomorphism on \( \mathbb{R}^n \), its inverse map is denoted by \( f^{-1} \). Our next auxiliary lemma is used in the proof of Theorem 1.1.

**Lemma 5.2.** Suppose \((S; S)\) is a measurable space, \( F : (S \times \mathbb{R}^n; S \times \mathcal{B}) \to (\mathbb{R}^n; \mathcal{B})\).

1. Assume \( X \) is another measurable map from \((S; S)\) to \((\mathbb{R}^n; \mathcal{B})\). Then the map \( a \mapsto F(a, X(a)) \) is measurable from \((S; S)\) to \((\mathbb{R}^n; \mathcal{B})\).

2. For any \( L > 0 \) define
\[
H_L := \{ f : \mathbb{R}^n \to \mathbb{R}^n | f \text{ is a continuous homeomorphism and} \}
\]
\[
L^{-1}|x - y| \leq |f(x) - f(y)| \leq L|x - y| \}.
\]

If \( F : (S \times \mathbb{R}^n; S \times \mathcal{B}) \to (\mathbb{R}^n; \mathcal{B})\) and for each \( a \in S \), \( F(a, \cdot) \in H_L \), then the map \( F^{-1} : S \times \mathbb{R}^n \ni (a, x) \mapsto [F^{-1}(a, \cdot)](x) \in \mathbb{R}^n \) is \( S \times \mathcal{B} \) measurable.

**Proof.** (1) This conclusion is trivial since the map \( a \mapsto (a, X(a)) \) is \( S / S \times \mathcal{B} \) measurable.
(2). Define
\[ d(f, g) := \sup_{x \in \mathbb{R}^n} \frac{|f(x) - g(x)|}{1 + |x|}, \quad \forall f, g \in H_L. \]

It is easy to verify that $H_L$ is a metric space equipped with metric $d$. For any $f \in H_L$ and $\varepsilon > 0$, by the continuity of $x \mapsto F(a, x)$, we get
\[ \{ a : d(F(a, \cdot), f) < \varepsilon \} = \bigcap_{q \in \mathbb{Q}^n : r \in \mathbb{Q} \cap [0, 1)} \left\{ a : \frac{|F(a, q) - f(q)|}{1 + |q|} < \varepsilon \right\} \in S. \]

So the map $\overline{F} : (S, \mathcal{S}) \to (H_L, \mathcal{B}(H_L; d))$ is measurable. Obviously, the map
\[ \text{Inv} : H_L \ni f \mapsto f^{-1} \in H_L, \]

is well-defined. Now assume $d(f_n, f) \to 0$. Given $x \in \mathbb{R}^n$, assume $y = f^{-1}(x)$, then
\[ |f_n^{-1}(x) - f^{-1}(x)| = |f_n^{-1} \circ f(y) - f_n^{-1} \circ f_n(y)| \leq L |f(y) - f_n(y)| \leq L(1 + |y|)d(f_n, f). \]

By definition of $H_L$,
\[ |f(y) - f(0)| \geq L^{-1}|y|, \]

which implies
\[ |x| = |f(y)| \geq L^{-1}|y| - |f(0)|. \]

So
\[ |f_n^{-1}(x) - f^{-1}(x)| \leq L(1 + Lf(0) + L|x|)d(f_n, f) \leq C_f L(1 + |x|)d(f_n, f), \]

which implies $d(f_n^{-1}, f^{-1}) \leq C_f Ld(f_n, f) \to 0$. Thus, the map $\text{Inv} : H_L \to H_L$ is continuous. Hence, the map $\overline{F}^{-1} := \text{Inv} \circ \overline{F}$ from $(S, \mathcal{S})$ to $(H_L, \mathcal{B}(H_L))$ is also measurable. As a consequence, the map
\[ F^{-1} : S \times \mathbb{R}^n \ni (a, x) \mapsto [\text{Inv} \circ F(a, \cdot)](x) \in \mathbb{R}^n \]
is $S \times \mathcal{B}/\mathcal{B}$ measurable.

Roughly speaking, the above lemma shows that if $(a, x) \mapsto F(a, x)$ is measurable then $(a, x) \mapsto F^{-1}(a, \cdot)(x)$ is also measurable.

The following interpolation lemma is used several times in our paper.

**Lemma 5.3.** Let $0 < \gamma_0 < \gamma_1 < \gamma_2$ with $\gamma_1 \notin \mathbb{N}$ and $\theta := (\gamma_2 - \gamma_1)/(\gamma_2 - \gamma_0) \in (0, 1)$, $Q_T = \mathbb{R}^n \times [0, T]$ and $\mathcal{B}$ be a Banach space. Then there is a constant $C > 0$, such that for all $f \in C^\gamma_{x,t}$ with $\partial_t f \in C^\gamma_{x,t}$,
\[ \|f_{t_1} - f_{t_2}\|_{\mathcal{C}^{\gamma_1}} \leq C |t_1 - t_2|^\theta \|\partial_t f\|_{\mathcal{C}^{\gamma_0}}^{\theta} \|f\|_{\mathcal{C}^{\gamma_2}}^{1-\theta}. \]
**Proof.** First of all, for any \( t \in [0, 1] \), we have
\[
\| f_t \|_{C^{\gamma_1}} \leq C \| f_1 \|_{C^{\gamma_0}}^{\theta} \| f_t \|_{C^{\gamma_2}}^{1-\theta}.
\]

For any \( 0 \leq t_0 < t_1 < T, \beta (0, \theta) \) and \( q > 1/\theta \), by Garsia-Rademich-Rumsey’s inequality, we have
\[
\frac{\| f_t - f_{t_0} \|_{C^{\gamma_1}}^q}{|t_1 - t_0|^{\beta q - 1}} \leq C \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{\| f_t - f_s \|_{C^{\gamma_1}}^q}{|t - s|^{1+\beta q}} \, ds \, dt \leq C \int_{t_0}^{t_1} \int_{t_0}^{t_1} \| f_t - f_s \|_{C^{\gamma_0}}^{\theta q} \| f_t - f_s \|_{C^{\gamma_2}}^{1-(1-\theta)q} |t - s|^{-1-\beta q} \, ds \, dt \leq C \left( \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{|t - s|^{\theta q}}{|t - s|^{1+\beta q}} \, ds \, dt \right) \| \partial_t f \|_{C^{\gamma_0}_{\gamma, t}}^{\theta q} \| u \|_{C^{\gamma_2}_{\gamma, t}}^{1-(1-\theta)q},
\]
which gives (5.3).

**Lemma 5.4.** Suppose \( \beta \geq 0 \), \( \mathcal{H} \) is a real Hilbert space and \( C^\beta_{\gamma, x} = C^\beta (\mathbb{R}^n; L^p (\Omega; \mathcal{H})) \). Assume \( \mathcal{G} \) is a subalgebra of \( \mathcal{F} \), then
\[
\| \mathbb{E}(X|\mathcal{G}) \|_{C^\beta_{\gamma, x}} \leq \| X \|_{C^\beta_{\gamma, x}}.
\]
Moreover, for any \( k \in \mathbb{N} \) with \( k \leq \beta \),
\[
\nabla^k \mathbb{E}(X(x)|\mathcal{G}) = \mathbb{E}\left( \nabla^k X(x)|\mathcal{G} \right).
\]

**Proof.** We only prove (5.4) when \( \beta \in (0, 1) \). Denote \( \mathbb{E}\mathcal{G} X(\cdot) := \mathbb{E}(X(\cdot)|\mathcal{G}) \), by Jensen’s inequality,
\[
\mathbb{E}\left| \mathbb{E}\mathcal{G} X(x) - \mathbb{E}\mathcal{G} X(y) \right|_{\mathcal{H}}^p \leq \mathbb{E}\left[ \mathbb{E}\mathcal{G} |X(x) - X(y)| \right]_{\mathcal{H}}^p \leq \mathbb{E}\left[ \mathbb{E}\mathcal{G} |X(x) - X(y)| \right]_{\mathcal{H}}^p \leq |x - y|^{\beta p} \| X \|_{C^\beta_{\gamma, x}}^p,
\]
which yields
\[
\| \mathbb{E}\mathcal{G} X \|_{C^\beta_{\gamma, x}} = \sup_{x, y \in \mathbb{R}^d} \frac{\| \mathbb{E}\mathcal{G} X(x) - \mathbb{E}\mathcal{G} X(y) \|_{\mathcal{H}}^{1/p}}{|x - y|^{\beta}} \leq \| X \|_{C^\beta_{\gamma, x}}.
\]

For (5.5), we only give the proof for \( k = 1 \). Again by Jensen’s inequality,
\[
\left| \mathbb{E}\mathcal{G} X(x + h) - \mathbb{E}\mathcal{G} X(x) - \left[ \mathbb{E}\mathcal{G} \nabla X(x) \right] \cdot h \right|_{\mathcal{H}} \leq \mathbb{E}\mathcal{G} |X(x + h) - X(x) - \nabla X(x) \cdot h|_{\mathcal{H}}.
\]
Thus,
\[
\mathbb{E}\left| \mathbb{E}\mathcal{G} X(x + h) - \mathbb{E}\mathcal{G} X(x) - \left[ \mathbb{E}\mathcal{G} \nabla X(x) \right] \cdot h \right|_{\mathcal{H}}^p \leq \mathbb{E}\mathbb{E}\mathcal{G} \left( |X(x + h) - X(x) - \nabla X(x) \cdot h|_{\mathcal{H}}^p \right) = |X(x + h) - X(x) - \nabla X(x) \cdot h|_{\mathcal{H}}^p \to 0 \ (h \to \infty),
\]
which gives the desired result.
Lemma 5.5. Suppose \( f : \mathbb{R}^n \times \Omega \to \mathbb{R}^m \) is \( \mathcal{B} \times \mathcal{F} \) measurable and \( f \in C^1(\mathbb{R}^n; \mathbb{D}^{1,p}) \), then \( \nabla f \in C(\mathbb{R}^n; \mathbb{D}^{1,p}) \) and

\[
\nabla D f = D \nabla f
\]

Proof. We assume \( n = m = 1 \) for simple. For any \( x \in \mathbb{R} \), by definition

\[
\partial_{x, \theta} f(x) := \frac{f(x + \theta) - f(x)}{\theta} \xrightarrow{L^p(\Omega)} \partial_x f(x) \quad (\theta \to 0).
\]

On the other hand, since \( Df \in C^1(\mathbb{R}; L^p(\Omega, H)) \), we have

\[
D\partial_{x, \theta} f(x) = \frac{Df(x + \theta) - Df(x)}{\theta} \xrightarrow{L^p(\Omega; H)} \partial_x (Df)(x) \quad (\theta \to 0).
\]

By the closability of Mallivian derivative, we get \( D\partial_x f(x) = \partial_x Df(x) \in \mathbb{D}^{1,p} \) and

\[
\|\partial_x f(x)\|_{\mathbb{D}^{1,p}} = \liminf_{|\theta| \to 0} \left\| \frac{f(x + \theta) - f(x)}{\theta} \right\|_{\mathbb{D}^{1,p}} \leq \|f\|_{C^1(\mathbb{R}; \mathbb{D}^{1,p})}.
\]

For any \( F \in \mathbb{D}^{1,2} \), we have the following remarkable Clark-Ocone formula,

\[
F = \mathbb{E}(F) + \int_0^1 \mathbb{E}^s D_s F \cdot dW_s := \mathbb{E}(F) + \sum_{k=1}^d \int_0^1 \mathbb{E} \left( D_t^k F | \mathcal{F}_t \right) dW^k_t.
\]

(5.7) implies the following simple lemma.

Lemma 5.6. Suppose \( F \in \mathbb{D}^{1,2} \), then for each \( t \in [0, 1] \),

\[
\mathbb{E}^t F = \mathbb{E} F + \int_0^t \mathbb{E}^s D_s F \cdot dW_s.
\]

Proof. By Clark-Ocone’s formula,

\[
m_t = \mathbb{E} F + \int_0^t \mathbb{E}^s D_s F \cdot dW_s,
\]

is a \( \mathcal{F}_t \)-martingale with \( m_1 = F \). Thus,

\[
\mathbb{E}^t F = \mathbb{E}^t m_1 = m_t = \mathbb{E} F + \int_0^t \mathbb{E}^s D_s F \cdot dW_s.
\]

The following Lemma is a modification of Theorem 1.1 in [13], which is need in our proof of main result. Similar result on distributional valued processes can be found in [10].

Lemma 5.7 (Itô-Wentzell’s formula). Let \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) be a standard filtered probability space satisfying the common conditions, \( p, p' \in [1, \infty] \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \alpha_1, \alpha_2 \in (0, 1) \) with \( \alpha_1 + \alpha_2 > 1 \). Suppose \( X_t = (X^1_t, \cdots, X^n_t) \) be continuous semimartingales and \( \phi_t(x) \) be a random field continuous in \( (x, t) \in Q \) almost surely. Assume \( \phi \) and \( X \) satisfy
(1) for each \( t \in [0, 1], \mathbb{R}^n \ni x \mapsto \phi_t(x) \in \mathbb{R} \) is \( C^2 \) continuous a.s.,
(2) for each \( x \in \mathbb{R}^n, t \mapsto \phi_t(x) \) is a continuous \( \mathcal{F}_t \)-semimartingale represented as

\[
\phi_t(x) = \phi_0(x) + \int_0^t g_s(x)ds + \int_0^t v_s^k(x)dm_s^k,
\]

where \( m^1, \cdots, m^d \) are continuous martingales, and the random field \( g, v \) are locally bounded and

(a) for each \( x \in \mathbb{R}^n, t \mapsto g_t(x) \) and \( t \mapsto v_t(x) \) are \( \mathcal{F}_t \)-adapted processes;
(b) for each \( t \in [0, 1], x \mapsto \phi_t(x) \) is \( C^1 \) a.s.;
(c) for each \( t \in [0, 1], x \mapsto g_t(x) \) is continuous,

\[
E \sup_{x \in B_N} \left| \nabla \int_{t_1}^{t_2} g_s(x)ds \right|^p \leq p, \quad |t_1 - t_2|^{\alpha_1p},
\]

\[
E \left| X_{t_1 \vee \tau_N} - X_{t_2 \vee \tau_N} \right|^p \leq p' |t_1 - t_2|^{\alpha_2p'},
\]

where \( \tau_N = \inf \{ t > 0 : |X_t| > N \} \).

Then we have

\[
d\phi_t(X_t) = g_t(X_t)dt + v_t^k(X_t)dm_t^k + \partial_i \phi_t(X_t)dX^i_t
\]

\[
+ \frac{1}{2} \partial_{ij} \phi_t(X_t)d\langle X^i, X^j \rangle_t + \partial_i v_t^k(X_t)d\langle m^k, X^i \rangle_t
\]

(5.10)

**Proof.** The proof is similar with Theorem 1.1 of [13]. Without loss of generality, we can assume \( |X_t| \) is bounded by a constant \( N \). For any \( t > 0 \), let \( t_l = lt/n, l = 0, \cdots, n. s(n) := t[sn/t]/n \) and \( X^n_s := X_{s(n)} \). Then,

\[
\phi_t(X_t) - \phi_0(X_0) = \sum_{l=0}^{n-1} [\phi_{lt+1}(X_{lt}) - \phi_{lt}(X_{lt})] + \sum_{l=0}^{n-1} [\phi_{lt+1}(X_{lt+1}) - \phi_{lt+1}(X_{lt})]
\]

\[
= : I^n_1 + I^n_2.
\]

By (5.9) and the definition of \( X^n \)

\[
I^n_1 = \sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} g_s(X_{lt})ds + \sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} v_s^k(X_{lt})dm_s^k = \int_0^t g_s(X^n_s)ds + \int_0^t v_s^k(X^n_s)dm_s^k.
\]

Since \( g_s(X^n_s) \rightarrow g_s(X_s), v_s(X^n_s) \rightarrow v_s(X_s) \) a.s. and \( g, v \) are uniformly bounded in \([0, 1] \times B_N\), we obtain

\[
I^n_1 \xrightarrow{P} \int_0^t g_s(X_s)ds + \int_0^t v_s^k(X_s)dm_s^k, \quad (n \to \infty).
\]

By Taylor expansion,

\[
I^n_2 = \sum_{l=0}^{n-1} \partial_i \phi_{lt+1}(X_{lt}) (X_{lt+1}^i - X_{lt}^i) + \frac{1}{2} \sum_{l=0}^{n-1} \partial_{ij} \phi_{lt+1}(X_{lt}) (X_{lt+1}^i - X_{lt}^i) (X_{lt+1}^j - X_{lt}^j)
\]

\[
= : I^n_{21} + I^n_{22},
\]
where $\xi_i$ are some random variables between $X_{t_i}$ and $X_{t_{i+1}}$. It is standard to show that

$$I_{22}^n \overset{P}{\rightarrow} \frac{1}{2} \int_0^t \partial_{ij} \phi_s(X_s) d\langle X^i, X^j \rangle_s, \quad (n \to \infty).$$

For $I_{21}^n$, we rewrite it as

$$I_{21}^n = \sum_{i=0}^{n-1} \partial_i \phi_{t_i}(X_{t_i})(X_{t_{i+1}}^i - X_{t_i}^i) + \sum_{i=0}^{n-1} \left[ \partial_i \phi_{t_{i+1}}(X_{t_{i+1}}) - \partial_i \phi_{t_i}(X_{t_i}) \right] (X_{t_{i+1}}^i - X_{t_i}^i)
= : I_{211}^n + I_{212}^n.$$

Like before,

$$I_{211}^n \overset{P}{\rightarrow} \int_0^t \partial_i \phi_s(X_s) dX_s^i, \quad (n \to \infty).$$

Again by (5.9),

$$I_{212}^n = \sum_{i=0}^{n-1} \partial_i \left( \int_{t_i}^{t_{i+1}} g_s(X_{t_i}) ds \right) (X_{t_{i+1}}^i - X_{t_i}^i) + \sum_{i=0}^{n-1} \left( \int_{t_i}^{t_{i+1}} \partial_i v_s^k(X_{t_i}) dm_s^k \right) (X_{t_{i+1}}^i - X_{t_i}^i)
= : I_{2121}^n + I_{2122}^n.$$

By our assumption (c) and Hölder inequality,

$$E|I_{2121}^n|^p \lesssim \sum_{i=0}^{n-1} \left[ E \sup_{x \in B_N} \left| \nabla \int_{t_i}^{t_{i+1}} g_s(x) ds \right|^{p' \prime} \right]^{1/p} \left[ E \left| X_{t_{i+1}} - X_{t_i} \right|^{p' \prime} \right]^{1/p'}
\lesssim \sum_{i=0}^{n-1} |t_{i+1} - t_i|^{\alpha_1 + \alpha_2} \lesssim n^{-\alpha_1 - \alpha_2 + 1} \to 0, \quad (k \to \infty).$$

It is standard to show

$$I_{2122}^n \overset{P}{\rightarrow} \int_0^t \partial_i v_s^k(X_s) d\langle m_s^k, X^i \rangle_s, \quad (k \to \infty).$$

Combine all the above calculations, we obtain (5.10).

**Acknowledgements.** The author would like to thank Professor Luo Dejun for raising this problem to him and also for having many useful discussions.

**Funding.** Research of G. Zhao is supported by the German Research Foundation (DFG) through the Collaborative Research Centre(CRC) 1283 “Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications”.

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