Nonlinear Coherent States of the Fokas-Lagerstrom Potential

Mehdi Ashrafi\textsuperscript{1}, Ali Mahdifar\textsuperscript{2,3,4} and Ehsan Amooghorban\textsuperscript{1,4}

\textsuperscript{1} Department of Physics, Faculty of Basic Sciences, Shahrekord University, Shahrekord 88186-34141, Iran.
\textsuperscript{2} Department of Physics, Faculty of Science, University of Isfahan, Hezar Jerib Str., Isfahan, 81746-73441, Iran.
\textsuperscript{3} Quantum Optics Group, Department of Physics, Faculty of Science, University of Isfahan, Hezar Jerib Str., Isfahan, 81746-73441, Iran
\textsuperscript{4} Nanotechnology Research Center, Shahrekord University, Shahrekord 88186-34141, Iran.

Abstract. In this paper, we introduce an algebraic approach to construct Fokas-Lagerstrom coherent states. To do so, we define deformed creation and annihilation operators associated to this system and investigate their algebra. We show that these operators satisfy the $f$-deformed Weyl-Heisenberg algebra. Then, we propose a theoretical scheme to generate the aforementioned coherent states. The present contribution shows that the Fokas-Lagerstrom nonlinear coherent states possess some non-classical features.

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‡ a.mahdifar@sci.ui.ac.ir
1. Introduction

It is well known that an $N$-dimensional classical or quantum system is called completely integrable if there are $N$ functionally independent well defined constants of motion (including the Hamiltonian) in involution \([1, 2]\). Furthermore, if it is possible to obtain $2N - 1$ constants of motion, the system is called maximally superintegrable, provided that the Poisson brackets (or the commutators) of these constants of motion with the Hamiltonian vanish (in general, only $N$ of $2N - 1$ constants are in involution) \([2, 3, 4, 5, 6, 7, 8]\). Famous examples of the superintegrable systems are the isotropic harmonic oscillator and the Kepler-Coulomb potential. It is worthwhile to note that their higher order symmetries, play the fundamental role in solvability and other interesting properties of these physical systems.

In the last decades, coherent states of the harmonic oscillator \([9]\) and generalized coherent states associated with various algebras \([10, 11, 12]\) have been playing an important role in various branches of physics. The coherent states, which defined as the right eigenstates of the annihilation operator $\hat{a}$, are the quantum states with classical-like properties. In other words, they are the closest analogue to classical states. On the other hand, the generalized coherent states exhibit some nonclassical properties therefore, have received an ever-increasing interest during the last decades. Among the generalized coherent states, the nonlinear coherent states or $f$-deformed coherent states \([13]\) have attracted many interests in recent years due to their applications in quantum optics and quantum technology. It is shown that the statistical characteristics of these states exhibit some nonclassical features, such as photon antibunching \([14]\) or sub-Poissonian photon statistics \([15]\) and squeezing \([16]\). These states, which are associated with nonlinear algebras \([13, 20]\), could be generated in the center-of-mass motion of an appropriately laser-driven trapped ion \([17, 18]\) and in a micromaser under intensity-dependent atom-field interaction \([19]\).

Recently, one of the authors of this paper defined the nonlinear coherent states for some of the two-dimensional superintegrable systems, including the isotropic harmonic oscillator on a sphere \([21]\) and the Kepler-Coulomb problem on a sphere \([22]\). In the present contribution, we study the Fokas-Lagerstrom potential as the another example of the two-dimensional superintegrable systems \([23]\). Our approach which is based on the $f$-deformed harmonic oscillator algebra and the nonlinear coherent states, can increase the insight about the Fokas-Lagerstrom system. Of course, the Fokas-Lagerstrom potential was considered previously \([23, 24]\). The distinction between the present paper and the mentioned references is that our approach is based on the algebraic methods. Specifically, we investigate the Fokas-Lagerstrom potential by applying the nonlinear coherent states approach.

The paper is organized as follows. In Sec. 2, we briefly review the superintegrable systems. By using the nonlinear oscillator approach in Sec. 3, we investigate the Fokas-Lagerstrom system, as an example of two-dimensional superintegrable systems and show that the algebra of the this system can be considered as an $f$-deformed Weyl-Heisenberg...
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algebra. We define nonlinear coherent states for the Fokas-Lagerstrom potential and examine their resolution of identity in Sec. 4. In Sec. 5 we propose a scheme to generate the aforementioned coherent states. Sec. 6 is devoted to the study of the quantum statistical properties of the constructed nonlinear coherent states, including mean number of photons, Mandel parameter and quadrature squeezing. Finally, the summary and concluding remarks are given in Sec. 7.

2. Superintegrable systems

In classical mechanics, an \( N \)-dimensional system is called superintegrable, if it has more than \( N \) independent constants of motion. Furthermore, if the system has exactly \( (2N-1) \) independent constants of motion (the maximum number) and also if all of these constants are single valued and globally defined, then the system is called maximally superintegrable system \[1, 2\].

To be specific, for a classical system in two dimensions, described by the following Hamiltonian:

\[
H = H(x, y, p_x, p_y),
\]

if there exist two independent constants of motion \( I \) and \( C \), so that we have,

\[
\{ H, I \}_{PB} = \{ H, C \}_{PB} = 0, \\
\{ I, C \}_{PB} \neq 0,
\]

then this system is called a superintegrable system. Here, \( \{ \cdot, \cdot \}_{PB} \) denotes the Poisson bracket \[2\].

In quantum mechanics, a two-dimensional system described by a Hamiltonian \( \hat{H} \) is called integrable, if is possible to find an operator \( \hat{I} \) commuting with the \( \hat{H} \) \[2\]:

\[
[\hat{H}, \hat{I}] = 0.
\]

This system is also called superintegrable, if there exist another operator \( \hat{C} \), linearly independent of \( \hat{H} \) and \( \hat{I} \), that commute with and \( \hat{H} \) but not commute with \( \hat{I} \), so that,

\[
[\hat{H}, \hat{C}] = 0, \ [\hat{I}, \hat{C}] \neq 0.
\]

3. Fokas- Lagerstrom system

We consider the quantum Fokas-Lagerstrom system, described by the Hamiltonian \[23\]:

\[
\hat{H} = \frac{1}{2}(\hat{P}_x^2 + \hat{P}_y^2) + \frac{\hat{x}^2}{2} + \frac{\hat{y}^2}{18},
\]

(in this paper we put \( h = m = \omega = 1 \). By introducing the following operators:

\[
\hat{J} = \hat{P}_x^2 + \hat{x}^2, \\
\hat{B} = \frac{1}{2}\{\hat{X} \hat{P}_y - \hat{Y} \hat{P}_x, \hat{P}_y\} + \frac{\hat{Y}^3 \hat{P}_x}{27} - \frac{\{\hat{X} \hat{Y}^2, \hat{P}_y\}}{6},
\]

\[
\hat{H} = \hat{J} + \hat{B}.
\]
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where \{,\} is the usual anticommutator, and making use of Eq.\([5]\) it can be easily shown that the following commutation relations hold [24],

\[
\begin{align*}
[H, \hat{B}] &= [\hat{H}, \hat{J}] = 0, \\
[\hat{J}, \hat{R}] &= 4\hat{B}, \\
[\hat{J}, \hat{B}] &= \hat{R}.
\end{align*}
\] (8)

With these in mind, and considering the definition of the superintegrable systems, it is clear that the Fokas-Lagerstrom system is a quantum superintegrable system. Now, by defining the following operators:

\[
\begin{align*}
\hat{n} &= \frac{\hat{J}}{2} - u\hat{I}, \\
\hat{A}^\dagger &= \hat{B} + \frac{\hat{R}}{2}, \\
\hat{A} &= \hat{B} - \frac{\hat{R}}{2},
\end{align*}
\] (9)

where \(u\) is a constant to be determined, it is seen that the operators \(\hat{A}, \hat{A}^\dagger\) and \(\hat{n}\) satisfy the following closed algebra [24]:

\[
\begin{align*}
[\hat{n}, \hat{A}^\dagger] &= \hat{A}^\dagger, \\
[\hat{n}, \hat{A}] &= -\hat{A}, \\
[\hat{A}, \hat{A}^\dagger] &= \Phi(\hat{H}, \hat{n} + 1) - \Phi(\hat{H}, \hat{n}).
\end{align*}
\] (10)

where the structure function \(\phi\) is given by

\[
\Phi(E, x) = \frac{1}{9}(2x + 2u - 1)(2E + 1 - 2x - 2u) \times (6E - 6u - 6x)(6E - 6u + 5 - 6x).
\] (11)

This structure function, for \(n \geq 0\) is a real definite positive function and also \(\phi(E, 0) = 0\). Now, for each energy eigenvalue it is possible to define the corresponding Fock space as below,

\[
\begin{align*}
\hat{H} |E, n\rangle &= E |E, n\rangle, \\
\hat{N} |E, n\rangle &= n |E, n\rangle, \\
\hat{A} |E, 0\rangle &= 0, \\
|E, n\rangle &= \left(\frac{1}{\sqrt{[\phi(E, n)]!}}\right) (\hat{A}^\dagger)^n |E, 0\rangle.
\end{align*}
\] (12)

For a discrete energy eigenvalue \(E\), there is the \(N + 1\)-dimensional degeneracy. So, we deal with an \(N\)-dimensional Fock space corresponding to that eigenvalue energy as,

\[
\begin{align*}
\hat{H} |E, n\rangle &= E_N |N, n,\rangle, \\
\hat{n} |E, n\rangle &= n |N, n,\rangle
\end{align*}
\] (13) (14)
The combination of this restriction along with \( \phi(E,0) = 0 \) and \( \phi(E,N+1) = 0 \), imply that \( u = \frac{1}{2} \). Therefore, the possible energy eigenvalues can be obtained as:

\[
E_N = N + 1, \\
E_N = N + \frac{2}{3}, \\
E_N = N + \frac{4}{3},
\]

and the allowable structure functions are given by,

\[
\Phi(E,N,x) = 16x(N + 1 - x)(N + A - x)(N + B - x).
\]

Here, the constants \( A \) and \( B \) are corresponding to one of the pairs \((2/3, 4/3)\), \((2/3, 1/3)\) or \((5/3, 4/3)\).

As is well known, the \( f \)-deformed annihilation and creation operators associated with an \( f \)-deformed harmonic oscillator can be defined as \cite{20},

\[
\hat{A} \hat{a} f(\hat{n}) = f(\hat{n} + 1) \hat{a}, \\
\hat{A}^\dagger \hat{a}^\dagger = f^\dagger(\hat{n}) \hat{a}^\dagger = \hat{a}^\dagger f^\dagger(\hat{n} + 1),
\]

where \( \hat{a} \), \( \hat{a}^\dagger \) and \( \hat{n} \) are the bosonic annihilation, creation and number operators, respectively, and \( f(\hat{n}) \) is a real nonnegative deformation function. These deformed operators satisfy the commutation relation

\[
[\hat{A}, \hat{A}^\dagger] = (\hat{n} + 1)f(\hat{n} + 1)\hat{f}(\hat{n} + 1) - \hat{n} f(\hat{n}) f^\dagger(\hat{n}).
\]

On the other hand, with respect to the algebraic structure of Fokas-Lagerstrom superintegrable systems, we have

\[
[\hat{A}, \hat{A}^\dagger] = \Phi(\hat{H}, N + 1) - \Phi(\hat{H}, N).
\]

Now, if we deal with a constant energy, \( E_N \), then \( \Phi(\hat{H}, \hat{n}) \) depends only on \( \hat{n} \) and if we now compare Eq. (21) with Eq. (22), we find that:

\[
n f^2(n) = \Phi(E_N, n).
\]

Therefore, we can consider the algebra of Fokas-Lagerstrom system as a deformed Weyl-Heisenberg algebra with the following deformation function:

\[
f(n) = \sqrt{16(N + 1 - n)(N + A - n)(N + B - n)}.
\]

Now, by applying the operator \( \hat{A}^\dagger \) on the state \( |N\rangle \) and making use of Eq. (20), we arrive at

\[
\hat{A}^\dagger |N\rangle = \hat{a}^\dagger f(\hat{n} + 1) |N\rangle = f(N + 1)\sqrt{N + 1} |N + 1\rangle = 0.
\]

Thus, we conclude that for any constant \( N \), corresponding to the constant value of energy \( E_N \), there is a Hilbert space with finite dimension.

In the next section, we intend to construct the finite dimension coherent states corresponding to the Fokas-Lagrestrom potential.
4. Fokas-Lagrstrom nonlinear coherent states

Let us now turn to define the finite dimensional nonlinear coherent states corresponding to the Fokas-Lagrstrom potential. We follow the formalism of truncated coherent state approach introduced in [25], to define the Fokas-Lagrstrom nonlinear coherent states (FLNCSs). Therefore, we have:

$$|z\rangle_{F.L} = C^{-\frac{1}{2}} (|z|^2) \exp(z\hat{A}^\dagger)|0\rangle,$$

where $z$ is a complex number. After some calculations, the above nonlinear coherent state can be recast into the following form:

$$|z\rangle_{F.L} = C^{-\frac{1}{2}} (|z|^2) \sum_{n=0}^{N} \frac{z^n}{\sqrt{\rho(n)}} |n\rangle,$$

where $\rho(n)$ is defined as:

$$\rho(n) = \left(\frac{1}{16}\right)^n \left[ \frac{n\Gamma(n)\Gamma(N-n+1)}{N\Gamma(N)} \right] \left[ \frac{\Gamma(N+A-n)}{\Gamma(N+A)} \right] \times \left[ \frac{\Gamma(N+B-n)}{\Gamma(N+B)} \right].$$

Here, $\Gamma$ is the gamma function, and the normalization constant $C$ is given by,

$$C \left( |z|^2 \right) = \sum_{n=0}^{N} \frac{|z|^{2n}}{\rho(n)}.$$

4.1. Resolution of identity

In this section, we intend to show that the constructed FLNCSs from an overcomplete set. In other words, we would show:

$$\int d^2z |z\rangle W(|z|^2) \langle z| = \sum_{n=0}^{N} |n\rangle \langle n| = 1.$$  

The resolution of identity can be achieved by finding a measure function $W(|z|^2)$. For this purpose, by substituting $|z\rangle_{F.L}$ from the Eq. (27) into Eq. (30) we obtain,

$$\int d^2z \sum_{n,n'=0}^{\infty} \left\{ \frac{1}{\rho(n) \rho(n')} \int_{0}^{\infty} \frac{w(r^2)}{C(r^2)} r^{n+n'} d\theta \int e^{i\theta(n-n')} d\theta \right\} |n\rangle \langle n'|,$$

where we have used $z = re^{i\theta}$ and $d^2z = \frac{1}{2} dr^2 d\theta$. Now, by using the change of variable $x = r^2$ and considering $\tilde{w}(x) = \pi w(x)/C(x)$, we have:

$$\int_{0}^{\infty} \tilde{w}(x) x^n dx = \rho(n).$$
The above integral is called the moment problem and well-known mathematical methods such as Mellin transformations can be used to solve it \[8, 9\]. From definition the of Meijers $G$-function, it follows that,

$$
\int dx \ x^{k-1} G_{m,n}^{p,q} \left( \beta x \left| a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \right. \ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \right) = \frac{1}{\beta^k} \prod_{j=1}^m \Gamma(b_j + k) \prod_{j=1}^n \Gamma(1 - a_j - k) \prod_{j=m+1}^q \Gamma(1 - a_j - k). \quad (33)
$$

Therefore, by comparing Eqs. (32) and (33), we find that the weight function can be written as,

$$
\tilde{w}(x) = 16 \frac{1}{N! \Gamma(N) \Gamma(A + N) \Gamma(B + N)} \times G_{0,0}^{1,3} \left( \frac{1}{16x} \left| -N - 1, -N - A, -N - B \right. \ 0 \right). \quad (35)
$$

In this manner, it is seen that the FLNCSs satisfy the resolution of identity and consequently form an overcomplete set.

5. The Physical Generation of FLNCSs

To generate an arbitrary but finite superposition of Fock states, a physical scheme has been proposed in Ref. [26]. In this method, $N$ two-level atoms which are prepared in a specific superposition of the excited state $|a\rangle$ and the ground state $|b\rangle$ interact with a resonant mode of radiation field in a cavity via the Jaynes-Cummings Hamiltonian. The cavity field is initially in the vacuum state. By choosing atoms with appropriate initial states, it is possible to produce the desired state. In the resonator the measurement of the internal state of the atom after it has passed through the cavity leaves the quantum states of field in a pure state. After the interaction of $(K-1)\text{th}$ atom with cavity field, we make a measurement on the atomic state. If we find the atom in the ground state $|b\rangle$, the state of the radiation field reads:

$$
|\varphi^{(k-1)}\rangle = \sum_n \varphi_n^{(k-1)} |n\rangle. \quad (36)
$$

If the atom found in the excited state, our attempt to create the desired field state fails and we go back to vacuum state and start the procedure again. Therefore, the state of the atom-field system after the $K\text{th}$ atom in the atomic state $|a\rangle + i\varepsilon_k |b\rangle$ exit the cavity is given by

$$
|\Phi^{(k)}\rangle = \sum_n \varphi_n^{(k-1)} C_n^{(k)} |n, a\rangle - iS_n^{(k)} |n + 1, b\rangle + i\varepsilon_k C_{n-1}^{(k)} |n, b\rangle + \varepsilon_k S_{n-1}^{(k)} |n - 1, a\rangle, \quad (37)
$$
where
\[ C^k_n = \cos \left( g \tau_k \sqrt{n + 1} \right), \quad \text{(38)} \]
\[ S^k_n = \sin \left( g \tau_k \sqrt{n + 1} \right). \quad \text{(39)} \]

Besides, the new coefficients \( \varphi^{(k)}_n \) are given in terms of the \( \varphi^{(k-1)}_n \) as
\[ \varphi^{(k)}_n = S^{(k)}_{n-1} \varphi^{(k-1)}_{n-1} - \varepsilon_k C^{(k)}_{n-1} \varphi^{(k-1)}_n. \quad \text{(40)} \]

We are now in a position to generate the finite dimensional FLNCSs \( |z\rangle_{F.L} \) based on this approach. For this purpose, we need to obtain that field combination state,
\[ |\varphi^{(N-1)}\rangle = \sum_{n=0}^{N-1} \varphi^{(N-1)}_n |n\rangle, \quad \text{(41)} \]
of \( N \) number states such that after the \( N' \)th atom with the atomic superposition \( |a\rangle + i \varepsilon_N |b\rangle \) passed through the cavity and has been detected in the ground state, the field in the cavity become,
\[ |z\rangle_{F.L} = \sum_{n=0}^{N} d_n |n\rangle, \quad \text{(42)} \]
where
\[ d_n = C^{-\frac{1}{2}} \left( |z|^2 \right) \frac{z^n}{\sqrt{\rho(n)}}. \quad \text{(43)} \]

By using Eq. (41), we get a set of \( N+1 \) equations which \( N \) coefficients \( \varphi^{(N-1)}_n \) and the parameter \( \varepsilon_N \) can be obtained from it. The unknown coefficient \( \varphi^{(N-1)}_n \) is given by \[ \varphi^{(N-1)}_n = \sum_{\nu=1}^{N-n} \left[ \prod_{\mu=0}^{n+\mu-2} \varphi^{(N)}_{\mu} \right] \frac{d_{n+\nu}}{S^{(N)}_{n+\nu-1}} \varepsilon^{\nu} = 0. \quad \text{(44)} \]

We also have the characteristic equation for \( \varepsilon_N \) as
\[ d_0 + \sum_{\nu=1}^{N} \left[ \prod_{\mu=0}^{\nu-2} \varphi^{(N)}_{\mu} \right] \frac{d_{\nu}}{S^{(N)}_{\nu-1}} \varepsilon^{\nu} = 0. \quad \text{(45)} \]

We choose \( \varepsilon_N \) as one of the \( N \) roots of the Eq. (38) which is a polynomial of degree \( N \). In order to obtain other parameters \( \varepsilon_{N-1}, ..., \varepsilon_2 \) and \( \varepsilon_1 \) by a recurrence relation, we take \( |\varphi^{(N-1)}\rangle \) as a new desired state which can be generated by sending \( N - 1 \) atoms through the cavity. By following the same procedure, \( N - 1 \) coefficient \( \varphi^{(N-2)} \) and the parameter \( \varepsilon_{N-1} \) are obtained. Repeating the calculations yields a sequence of complex numbers \( \varepsilon_1, \varepsilon_2, ..., \varepsilon_N \) that defines the internal state of the \( N \) injected atoms into the cavity in order to generate the desired state \( |z\rangle_{F.L} \).

To be specific, we illustrate step by step generation of the Fokas-Lagerstrom coherent states with \( N = 2 \) by using the space distribution Q-function \[ Q(\alpha) = \left| \langle \alpha | \psi \rangle \right|^2 / \pi, \quad \text{(46)} \]
where $|\alpha\rangle$ is the standard coherent state.

Fig. 1 displays the Q-function $Q_k(\alpha) = |\langle \alpha | \varphi^{(k)} \rangle|^2 / \pi$ and the counter lines of the Q-function for the cavity state $|\varphi^{(k)}\rangle$ after the $k$'th atom has passed through the resonator and has been detected in ground state for a fix interaction parameter $g\tau = \frac{\pi}{5}$ and $k = 0, 1, 2$.

Figure 1: The Q-function $Q_k(\alpha) = |\langle \alpha | \varphi^{(k)} \rangle|^2 / \pi$ and the counter lines of the Q-function for the cavity state $|\varphi^{(k)}\rangle$ after the $k$'th atom has passed through the resonator and has been detected in ground state for a fix interaction parameter $g\tau = \frac{\pi}{5}$ and (a) $k = 0$, (b) $k = 1$ and (c) $k = 2$ (the three steps for generation of the FLCSs with $N = 2$).

6. Quantum Statistical Properties of the FLNCSs

In this section, we shall proceed to study some quantum statistical properties of the FLNCSs, including probability of finding $n$ quanta, mean number of photons, Mandel parameter and quadrature squeezing.
6.1. Photons-number distribution

By using Eqs. (27) and (28), the probability \( P(n) \) of finding \( n \) photon in the FLNCSs is given by

\[
P(n) = \frac{(4z)^{2n} \frac{N!}{n!(N-n)!} \frac{(N+A+1)!}{(N+A-1-n)!} \frac{(N+B+1)!}{(N+B-1-n)!}}{1 + \sum_{n=1}^{N} (4z)^{2n} \left( \frac{N!}{n!(N-n)!} \right) \left( \frac{(N+A+1)!}{(N+A-1-n)!} \right) \left( \frac{(N+B+1)!}{(N+B-1-n)!} \right)}.
\] (47)

As it is clear from the above complex equation, it is difficult to predict the results analytically. Therefore, in Fig. 2 we show the effect of the parameter \( z \) on the probability of finding \( n \) photons in FLNCSs with \( A = 2/3 \) and \( B = 1/3 \). To get further insight, let us consider the limiting case \( z \to \infty \). Form Eq. (47), we obtain that the probabilities \( P(n) \) tends to \( \delta_{n,N} \). In other words, by increasing \( z \), the FLNCS \(|z\rangle\) approaches to the number state \(|N\rangle\).

![Figure 2: Probability of finding \( n \) photons in FLNCSs, versus \( z \), for \( N=3, A = 2/3 \) and \( B = 1/3 \); \( P(0) \), solid red line; \( P(1) \), dashed blue line; \( P(2) \), dashed-dotted orange line; \( P(3) \), dotted green line.](image)

The mean number of photons in the FLNCSs is calculated as follows:

\[
\langle \hat{n} \rangle = \langle z | \hat{a}^\dagger \hat{a} | z \rangle = \sum_{n=0}^{N} np(n) = \sum_{n=1}^{N} (4z)^{2n} \frac{N!}{n!(N-n)!} \frac{(N+A+1)!}{(N+A-1-n)!} \frac{(N+B+1)!}{(N+B-1-n)!}.
\] (48)

In Fig. 3 the mean number of photons in FLNCSs is plotted in terms of \( z \) for \( (A,B) = (2/3, 1/3) \) and for different values of \( N \). It is seen that for a constant \( N \), by increasing \( z \), the mean number of photons increases, and in the limit of \( z \to \infty \) we get: \( \langle z | \hat{n} | z \rangle \to N \). In addition, for a fixed value of \( z \), the mean number of photons increase by the increasing the dimension of Hilbert space \( N \). It is worth noting that these results are consistent with the results in Fig. 2.
6.2. Mandel parameter

In this subsection, we investigate deviation from the Poisson distribution for the FLNCSs by using the Mandel parameter. This parameter is defined as \[ M = \frac{(\Delta \langle n \rangle)^2 - \langle n \rangle}{\langle n \rangle}, \] \[ (49) \]

where the positive, zero and negative values represent super-Poissonian, Poissonian and sub-Poissonian distribution, respectively. Due to the complexity of the final form of the Mandel parameter for the FLNCSs, we do not attempt to obtain its analytic form. Instead, we numerically study the Mandel parameter for these states. In Fig. 4, we have plotted the Mandel parameter of the FLNCSs with respect to \( z \) for \((A, B) = (2/3, 1/3)\) and different values of \( N \). The results show that, for a fixed value of \( N \), the photon counting statistic of the FLNCSs becomes more sub-Poissonian with increasing \( z \) and then tends to \(-1\) at very large values of \( z \). We can justify this result by the fact that the quantum number states \( |N\rangle \) have the Mandel parameter \( M = -1 \). Since, the FLNCSs approach to the state \( |N\rangle \) with increasing \( z \), as is seen in Fig. 2.

6.3. Quadrature squeezing

In this subsection, we consider the quadrature operators \( \hat{X}_1 \) and \( \hat{X}_2 \) defined in terms of creation and annihilation operators \( \hat{a} \) and \( \hat{a}^\dagger \) as follows:

\[ \hat{X}_1 = \frac{1}{2} (\hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi}), \]
\[ \hat{X}_2 = \frac{1}{2} (\hat{a}e^{i\phi} - \hat{a}^\dagger e^{-i\phi}). \] \[ (50) \]

By using the commutation relation of \( \hat{a} \) and \( \hat{a}^\dagger \), the following uncertainty relation is obtained

\[ (\Delta X_1)^2(\Delta X_2)^2 \geq \frac{1}{16} |\langle \left[ \hat{X}_1, \hat{X}_2 \right] \rangle|^2 = \frac{1}{16}. \] \[ (51) \]
Figure 4: Mandel parameter of the FLNCSs versus $z$ for $A = 2/3$ and $B = 1/3$, the dotted green corresponds to $N = 2$, the dashed blue to $N = 4$ and the solid red curve to $N = 6$.

As is known, the quadrature squeezing occurs if we have $(\Delta X_i)^2 < 1/4(i = 1\text{ or } 2)$ or equally $S_i \equiv 4(\Delta X_i)^2 - 1 < 0$. Figs. 5(a) and 5(b) display the squeezing parameters $S_1$ and $S_2$, respectively, for the FLNCSs with $(A, B) = (2/3, 1/3)$ as a function of $\varphi$ for $z = 0.02$ and different values of $N$.

Figure 5: The squeezing parameters (a) $S_1$ and (b) $S_2$ versus $\varphi$ for $z = 0.02$ and $(A, B) = (2/3, 1/3)$. Here, the solid-red, dashed-blue and dotted-green lines correspond to $N = 20$, $N = 17$ and $N = 15$, respectively.

It is seen that the range of the parameter $\varphi$, in which the squeezing occurs, decreases by increasing $N$. In this range, the effect of increasing the dimension $N$ on the squeezing is dependant on $\varphi$. Besides, in the areas where the maximum squeezing occurs, an increase in $N$ leads to increase squeezing, while for other $\varphi$s available in this range, $N$ increasing causes a reduction in squeezing.
7. Summary and Concluding Remarks

In this paper, we have introduced an algebraic approach to the Fokas-Lagerstrom system. We have found that the two-dimensional Fokas-Lagerstrom algebra can be considered as a deformed one-dimensional harmonic oscillator algebra. In this manner, we have succeeded to construct the nonlinear coherent states for this potential and studied their quantum statistical properties. Finally, we have proposed a physical scheme to generate the FLNCSs.

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