Wigner’s Spins, Feynman’s Partons, and Their Common Ground

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Abstract

The connection between spin and symmetry was established by Wigner in his 1939 paper on the Poincaré group. For a massive particle at rest, the little group is $O(3)$ from which the concept of spin emerges. The little group for a massless particle is isomorphic to the two-dimensional Euclidean group with one rotational and two translational degrees of freedom. The rotational degree corresponds to the helicity, and the translational degrees to the gauge degree of freedom. The question then is whether these two different symmetries can be united. Another hard-pressing problem is Feynman’s parton picture which is valid only for hadrons moving with speed close to that of light. While the hadron at rest is believed to be a bound state of quarks, the question arises whether the parton picture is a Lorentz-boosted bound state of quarks. We study these problems within Einstein’s framework in which the energy-momentum relations for slow particles and fast particles are two different manifestations one covariant entity.

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1 Introduction

Let us start with Einstein’s energy-momentum relation. If the momentum of a particle is much smaller than its mass, the energy-momentum relation is \( E = p^2/2m + mc^2 \). If the momentum is much larger than the mass, the relation is \( E = cp \). These two different relations can be combined into one covariant formula \( E = \sqrt{m^2 + p^2} \).

There are other problems of similar nature. Particles have internal space-time variables. Massive particles have spins while massless particles have their helicities and gauge degrees of freedom. Can these symmetries be unified like Einstein’s energy-momentum relation? These is another problem in modern physics. Hadrons are bound states of quarks. However, fast-moving hadrons are incoherent collections of partons, as Feynman correctly observed in 1969 [1]. Can the quark model and the parton model be combined into a single covariant entity?

We shall discuss these problems in this report. First, let us review Wigner’s work. In 1939 [2], Eugene Wigner constructed the maximal subgroups of the Lorentz group whose transformations leave the four-momentum of a given particle invariant. These groups are known as Wigner’s little groups. Thus, the transformations of the little groups change the internal space-time variables of the particle, while leaving its four-momentum invariant. He observed that the little groups for massive and massless particles are isomorphic to the three-dimensional rotation group and the two-dimensional Euclidean group respectively.

If a massive particle is at rest, the rotation subgroup of the Lorentz group will leave its four-momentum invariant. This rotation group will however change the direction of the particle spin. We are quite familiar with what the Pauli spin matrices do. They are the generators of the \( SU(2) \) group which is locally isomorphic to the three-dimensional rotation group. In addition, the Pauli matrices correspond the physical quantity known as the spin. Indeed, Wigner’s work placed the spin of physics into Einstein’s relativistic world.

It is not possible to bring a massless particle into its rest frame. For this particle, we can fix the coordinate system in which the particle momentum is along the \( z \) direction. Wigner showed that the little group for this particle is generated by the rotation around the \( z \) axis and two other generators consisting of linear combinations of rotation and boost generators of the Lorentz group [2]. He showed further that these three generators satisfy a closed set of commutation relations, which is the same as the two-dimensional Euclidean group consisting of one rotation and two translations. It is not difficult to associate this rotation with the helicity of the massless particle, but Wigner did not give a physical meaning to those translation-like degrees of freedom. It was not until 1971 that Janner and Janssen observed that they are gauge transformations [3].

There was another issue unresolved until the 1980s. In 1953 [4], Inomu and Wigner observed that the two-dimensional Euclidean group can be obtained from a limiting case of the three-dimensional rotation group. This limiting case is known as the group contraction. The simplest way to grasp the picture is to consider a sphere with a large radius. It is then possible to make a flat surface approximation on the north pole. It is therefore likely that the \( E(2) \)-like little group for a massless particle is a limiting case of the \( O(3) \)-like little group. It is also likely that the limiting procedure is the infinite-momentum or zero-mass limit. But this problem was was not until 1990 [5].

Let us next see how this aspect of Wigner’s work is relevant to Feynman’s parton picture. Wigner’s original paper [2] tells us that the internal space-time symmetry groups for
massive and massless particles are different. It took fifty years for the physics community to establish that they are two different manifestation of one covariant entity, as in the case of Einstein’s energy-momentum relation. Likewise, we can raise the question of whether the quark model and the parton model are two different manifestation of one single covariant entity.

How can we approach this problem? In their 1971 paper [6], Feynman et al. used harmonic oscillators to work out hadronic mass spectra. Even though they used relativistic oscillators, they did not address the question of whether their formalism constitute a representation of Wigner’s little group. On the other hand, Wigner did not use harmonic oscillators too often in his papers, and Feynman did not pay enough attention to Lorentz covariance. Thus, in order to combine Feynman’s initiative with Wigner’s formalism, we can construct representations of the little group using harmonic oscillators. In so doing, we construct harmonic oscillator wave functions which can be Lorentz-boosted.

In Sec. 2, we present a brief history of applications of the little groups to internal space-time symmetries of relativistic particles. It is emphasized that massive and massless particles have different internal space-time symmetries, namely the $O(3)$-like and $E(2)$-like little groups respectively. It is pointed out that the translation-like transformations of the $E(2)$-like little group corresponds to gauge transformations. In Sec. 3, it is shown that the $E(2)$-like little group can be obtained as an infinite-momentum or zero-mass limit of the $O(3)$-like little group for a massive particle. This procedure is like the the original Inomu-Wigner contraction in which $O(3)$ becomes contracted into $E(2)$.

In Sec. 4, it is noted that R. P. Feynman was quite fond of harmonic oscillators, especially trying new physical ideas. It is pointed out that harmonic oscillators embrace many useful properties of the Lorentz group. We show in Sec. 5 that this $O(1,1)$ formalism enables to construct a covariant model of relativistic extended particles. As a consequence, we show that the quark and parton model are two different aspects of one covariant object. It is shown also that this parton picture exhibits the decoherence effect.

2 Wigner’s Spins

The group of Lorentz transformations consists of three boosts and three rotations. The rotations therefore constitute a subgroup of the Lorentz group. If a massive particle is at rest, its four-momentum is invariant under rotations. Thus the little group for a massive particle at rest is the three-dimensional rotation group. Then what is affected by the rotation? The answer to this question is very simple. The particle in general has its spin. The spin orientation is going to be affected by the rotation!

If the rest-particle is boosted along the $z$ direction, it will pick up a non-zero momentum component. The generators of the $O(3)$ group will then be boosted. The boost will take the form of conjugation by the boost operator. This boost will not change the Lie algebra of the rotation group, and the boosted little group will still leave the boosted four-momentum invariant. We call this the $O(3)$-like little group.

We use in this report the metric convention $(x, y, z, t)$. Then the Lorentz group is generated by the three rotation generators $J_i$ and three boost generators $K_i$, whose third
components take the form

\[ J_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix}, \]  

respectively. The expressions for the first and second components are well known.

It is important to note that the rotation generators satisfy the closed set of commutation relations:

\[ [J_i, J_j] = i\varepsilon_{ijk} J_k, \]  

but the commutation relations among the \( K_i \) operators do not. They lead to

\[ [K_i, K_j] = -i\varepsilon_{ijk} J_k. \]  

However, we can still consider the following combinations of the generators.

\[ N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1, \]  

Then, together with \( J_3 \), they form the following closed set of commutation relations.

\[ [N_1, N_2] = 0, \quad [J_3, N_1] = iN_2, \quad [J_3, N_2] = -iN_1. \]  

It is not difficult to associate the rotation generator \( J_3 \) with the helicity degree of freedom of the massless particle. Then what physical variable is associated with the \( N_1 \) and \( N_2 \) generators? Indeed, Wigner was the one who discovered the existence of these generators, but did not give any physical interpretation to these translation-like generators. For this reason, for many years, only those representations with the zero-eigenvalues of the \( N \) operators were thought to be physically meaningful representations \[8\].

It is possible to get a hint that the \( N \) operators generate gauge transformations from Weinberg’s 1964 papers \[8, 10\]. But it was not until 1971 when Janner and Janssen explicitly demonstrated that they generate gauge transformations \[3, 9\]. In order to fully appreciate their work, let us compute the transformation matrix

\[ \exp (-i(uN_1 + vN_2)) \]  

generated by \( N_1 \) and \( N_2 \). Then the four-by-four matrix takes the form

\[ \begin{pmatrix} 1 & 0 & -u & u \\ 0 & 1 & -v & v \\ u & v & 1 - (u^2 + v^2)/2 & (u^2 + v^2)/2 \\ u & 0 & -(u^2 + v^2)/2 & 1 + (u^2 + v^2)/2 \end{pmatrix}. \]  

If we apply this matrix to the four-vector to the four-momentum vector

\[ p = (0, 0, \omega, \omega) \]  

of a massless particle, the momentum remains invariant. It therefore satisfies the condition for the little group. If we apply this matrix to the electromagnetic four-potential

\[ A = (A_1, A_2, A_3, A_0) \exp (i(kz - \omega t)), \]  

with \( A_3 = A_0 \) which is the Lorentz condition, the result is a gauge transformation. This is what Janner and Janssen discovered in their 1971 and 1972 papers \[3\]. Thus the matrices \( N_1 \) and \( N_2 \) generate gauge transformations.
The E(2)-like Little as a Contraction of the O(3)-like Little Group

We know how to boost the the four-momentum of the rest particle along the $z$ direction. The boost matrix takes the form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cosh \eta & \sinh \eta \\
0 & 0 & \sinh \eta & \cosh \eta
\end{pmatrix},
\]

(10)

Then the the generators of the $O(3)$-like little group can also be boosted. Under this boost, $J_3$ will remain invariant, but $J_1$ and $J_2$ will become

\[
J'_1 = B J_1 B^{-1}, \quad J'_2 = B J_2 B^{-1},
\]

(11)

Because the above transformations are similarity transformations, these new generators will still satisfy the commutation relations for the three-dimensional rotation group.

On the other hand, we can take the following limit in the spirit of the Inonu-Wigner group contraction.

\[
N_1 = \frac{1}{R} B^{-1} J_2 B, \quad N_2 = -\frac{1}{R} B^{-1} J_1 B,
\]

(12)

where $N_1$ and $N_2$ are given in Eq. (9). The generators $N_1$ and $N_2$ are the contracted $J_2$ and $J_1$ respectively in the infinite-momentum and/or zero-mass limit. It was noted that $N_1$ and $N_2$ generate gauge transformations on massless particles. Thus the contraction of the transverse rotations leads to gauge transformations.

We have seen in this section that Wigner’s $O(3)$-like little group can be contracted into the $E(2)$-like little group for massless particles. Here, we worked out explicitly for the spin-1 case, but this mechanism should be applicable to all other spins. Of particular interest is spin-1/2 particles. This has been studied by Han, Kim and Son. They noted that there are also gauge transformations for spin-1/2 particles, and the polarization of neutrinos is a consequence of gauge invariance. It has also been shown that the gauge dependence of spin-1 particles can be traced to the gauge variable of the spin-1/2 particle. It would be very interesting to see how the present formalism is applicable to higher-spin particles.

Another case of interest is the space-time symmetry of relativistic extended particles. In 1973, Kim and Noz constructed a ground-state harmonic oscillator wave function which can be Lorentz-boosted. It was later found that this oscillator formalism can be extended to represent the $O(3)$-like little group. This oscillator formalism has a stormy history because it ultimately plays a pivotal role in combining quantum mechanics and special relativity.

With these wave functions, we propose to solve the following problem in high-energy physics. The quark model works well when hadrons are at rest or move slowly. However, when they move with speed close to that of light, they appear as a collection of an infinite-number of partons. The question then is whether the parton model is a Lorentz-boosted quark model. This question has been addressed before, but it can generate more interesting problems. The present situation is presented in the Table 1.

We are now ready to consider the third row of Table 1. In the this table, we would like to say that the quark model and the parton model are two different manifestation
Table 1: Massive and massless particles in one package. Wigner’s little group unifies the internal space-time symmetries for massive and massless particles. It is a great challenge for us to find another unification: the unification of the quark and parton pictures in high-energy physics.

| Massive, Slow | COVARIANCE | Massless, Fast |
|---------------|-------------|----------------|
| Energy-Momentum | \( E = \frac{p^2}{2m} \) | Einstein’s \( E = \sqrt{p^2 + m^2} \) | \( E = p \) |
| Internal Space-time Symmetry | \( S_3 \) | Wigner’s Little Group | \( S_3 \) |
| Relativistic Extended Particles | Quark Model | One Covariant Theory | Parton Model |

of one covariant entity. In order to appreciate fully this covariant aspect, let us examine Feynman’s style of doing physics.

4 Feynman’s Harmonic Oscillators

Quantum field theory has been quite successful in terms of perturbation techniques in quantum electrodynamics. However, this formalism is basically based on the S matrix for scattering problems and useful only for physically processes where a set of free particles becomes another set of free particles after interaction. Quantum field theory does not address the question of localized probability distributions and their covariance under Lorentz transformations. The Schrödinger quantum mechanics of the hydrogen atom deals with localized probability distribution. Indeed, the localization condition leads to the discrete energy spectrum. Here, the uncertainty relation is stated in terms of the spatial separation between the proton and the electron. If we believe in Lorentz covariance, there must also be a time separation between the two constituent particles.

Let us use the simplest hadron consisting of two quarks bound together with an attractive force, and consider their space-time positions \( x_a \) and \( x_b \), and use the variables

\[
X = \frac{(x_a + x_b)}{2}, \quad x = \frac{(x_a - x_b)}{2\sqrt{2}}.
\]

(13)

The four-vector \( X \) specifies where the hadron is located in space and time, while the variable \( x \) measures the space-time separation between the quarks. According to Einstein, this space-time separation contains a time-like component which actively participates in
the Lorentz transformation, as can be seen from

\[
\begin{pmatrix}
  z' \\
  t'
\end{pmatrix} =
\begin{pmatrix}
  \cosh \eta & \sinh \eta \\
  \sinh \eta & \cosh \eta
\end{pmatrix}
\begin{pmatrix}
  z \\
  t
\end{pmatrix},
\]

(14)

when the hadron is boosted along the \(z\) direction. In terms of the light-cone variables defined as \[21\]

\[
u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}.
\]

(15)

The boost transformation of Eq.(14) takes the form

\[
u' = e^{\eta}u, \quad v' = e^{-\eta}v.
\]

(16)

The \(u\) variable becomes expanded while the \(v\) variable becomes contracted.

Does this time-separation variable exist when the hadron is at rest? Yes, according to Einstein. In the present form of quantum mechanics, we pretend not to know anything about this variable. There is also an uncertainty relation between the time and energy variables. However, there are no known time-like excitations. Unlike Heisenberg’s uncertainty relation applicable to position and momentum, the time and energy separation variables are c-numbers, and we are not allowed to write down the commutation relation between them. Indeed, the time-energy uncertainty relation is a c-number uncertainty relation \[22\], as is illustrated in Fig. 1.

How does this space-time asymmetry fit into the world of Lorentz covariance \[12\]? The answer is that Wigner’s \(O(3)\)-like little group is not a Lorentz-invariant symmetry, but is a covariant symmetry \[2\]. It has been shown that the time-energy uncertainty applicable to the time-separation variable fits perfectly into the \(O(3)\)-like symmetry of massive relativistic particles \[14\].

The c-number time-energy uncertainty relation allows us to write down a time distribution function without excitations \[14\]. If we use Gaussian forms for both space and time distributions, we can start with the expression

\[
\left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2} (z^2 + t^2)\right\}
\]

(17)
for the ground-state wave function. In their classic 1971 paper [6], Feynman et al. start with the following Lorentz-invariant differential equation.

$$\frac{1}{2} \left\{ x_\mu^2 - \frac{\partial^2}{\partial x_\mu^2} \right\} \psi(x) = \lambda \psi(x). \quad (18)$$

This partial differential equation has many different solutions depending on the choice of separable variables and boundary conditions. Feynman et al. insist on Lorentz-invariant solutions which are not normalizable. On the other hand, if we insist on normalization, the ground-state wave function takes the form of Eq.(17). It is then possible to construct a representation of the Poincaré group from the solutions of the above differential equation [14]. If the system is boosted, the wave function becomes

$$\psi_\eta(z,t) = \left( \frac{1}{\pi} \right)^{1/2} \exp \left\{ -\frac{1}{2} \left( e^{-2\eta u^2} + e^{2\eta v^2} \right) \right\}. \quad (19)$$

This wave function becomes Eq.(17) if $\eta$ becomes zero. The transition from Eq.(17) to Eq.(19) is a squeeze transformation. The wave function of Eq.(17) is distributed within a circular region in the $uv$ plane, and thus in the $zt$ plane. On the other hand, the wave function of Eq.(19) is distributed in an elliptic region with the light-cone axes as the major and minor axes respectively, as illustrated in Fig. 2. If $\eta$ becomes very large, the wave function becomes concentrated along one of the light-cone axes. Indeed, the form given in Eq.(19) is a Lorentz-squeezed wave function.

5 Feynman’s Parton Picture

In 1969, Feynman observed that a fast-moving hadron can be regarded as a collection of many “partons” whose properties do not appear to be quite different from those of the quarks [4]. For example, the number of quarks inside a static proton is three, while the
number of partons in a rapidly moving proton appears to be infinite. The question then is how the proton looking like a bound state of quarks to one observer can appear different to an observer in a different Lorentz frame? Feynman made the following systematic observations.

a. The picture is valid only for hadrons moving with velocity close to that of light.

b. The interaction time between the quarks becomes dilated, and partons behave as free independent particles.

c. The momentum distribution of partons becomes widespread as the hadron moves fast.

d. The number of partons seems to be infinite or much larger than that of quarks.

Because the hadron is believed to be a bound state of two or three quarks, each of the above phenomena appears as a paradox, particularly b) and c) together.

In order to resolve this paradox, let us write down the momentum-energy wave function corresponding to Eq. (19). If the quarks have the four-momenta $p_a$ and $p_b$, we can construct two independent four-momentum variables

$$
P = p_a + p_b, \quad q = \sqrt{2}(p_a - p_b),$$

(20)

where $P$ is the total four-momentum and is thus the hadronic four-momentum. $q$ measures the four-momentum separation between the quarks. Their light-cone variables are

$$
q_u = (q_0 - q_z)/\sqrt{2}, \quad q_v = (q_0 + q_z)/\sqrt{2}.
$$

(21)

The resulting momentum-energy wave function is

$$
\phi_\eta(q_z, q_0) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2} \left(e^{-\eta q_u^2} + e^{\eta q_v^2}\right)\right\}.
$$

(22)

Because we are using here the harmonic oscillator, the mathematical form of the above momentum-energy wave function is identical to that of the space-time wave function. The Lorentz squeeze properties of these wave functions are also the same. This aspect of the squeeze has been exhaustively discussed in the literature [14, 17, 18].

When the hadron is at rest with $\eta = 0$, both wave functions behave like those for the static bound state of quarks. As $\eta$ increases, the wave functions become continuously squeezed until they become concentrated along their respective positive light-cone axes. Let us look at the z-axis projection of the space-time wave function. Indeed, the width of the quark distribution increases as the hadronic speed approaches that of the speed of light. The position of each quark appears widespread to the observer in the laboratory frame, and the quarks appear like free particles.

The momentum-energy wave function is just like the space-time wave function, as is shown in Fig. 3. The longitudinal momentum distribution becomes wide-spread as the hadronic speed approaches the velocity of light. This is in contradiction with our expectation from nonrelativistic quantum mechanics that the width of the momentum distribution is inversely proportional to that of the position wave function. Our expectation is that
Figure 3: Lorentz-squeezed space-time and momentum-energy wave functions. As the hadron’s speed approaches that of light, both wave functions become concentrated along their respective positive light-cone axes. These light-cone concentrations lead to Feynman’s parton picture.
if the quarks are free, they must have their sharply defined momenta, not a wide-spread distribution.

However, according to our Lorentz-squeezed space-time and momentum-energy wave functions, the space-time width and the momentum-energy width increase in the same direction as the hadron is boosted. This is of course an effect of Lorentz covariance. This indeed is the key to the resolution of the quark-parton paradox [14, 17].

The most puzzling problem in the parton picture is that partons in the hadron appear as incoherent particles, while quarks are coherent when the hadron is at rest. Does this mean that the coherence is destroyed by the Lorentz boost? The answer is NO, and here is the resolution to this puzzle.

When the hadron is boosted, the hadronic matter becomes squeezed and becomes concentrated in the elliptic region along the positive light-cone axis. The length of the major axis becomes expanded by $e^\eta$, and the minor axis is contracted by $e^{-\eta}$.

This means that the interaction time of the quarks among themselves become dilated. Because the wave function becomes wide-spread, the distance between one end of the harmonic oscillator well and the other end increases. This effect, first noted by Feynman [1], is universally observed in high-energy hadronic experiments. The period is oscillation increases like $e^\eta$.

On the other hand, the interaction time with the external signal, since it is moving in the direction opposite to the direction of the hadron, travels along the negative light-cone axis. If the hadron contracts along the negative light-cone axis, the interaction time decreases by $e^{-\eta}$. The ratio of the interaction time to the oscillator period becomes $e^{-2\eta}$. The energy of each proton coming out of the Fermilab accelerator is 900 GeV. This leads the ratio to $10^{-6}$. This is indeed a small number. The external signal is not able to sense the interaction of the quarks among themselves inside the hadron.

This covariant picture of Feynman’s parton model can be placed on the third row of Table 1.

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