Research Article

Some New Difference Sequence Spaces of Invariant Means Defined by Ideal and Modulus Function

Sudhir Kumar, Vijay Kumar, and S. S. Bhatia

1 School of Mathematics and Computer Applications, Thapar University, Patiala, Punjab 147004, India
2 Department of Mathematics, Haryana College of Technology and Management, Kaithal, Haryana 136027, India

Correspondence should be addressed to Sudhir Kumar; sudhirgd@yahoo.in

Received 15 February 2014; Accepted 9 May 2014; Published 28 May 2014

Academic Editor: Shamsul Qamar

Copyright © 2014 Sudhir Kumar et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The main objective of this paper is to introduce a new kind of sequence spaces by combining the concepts of modulus function, invariant means, difference sequences, and ideal convergence. We also examine some topological properties of the resulting sequences. Further, we introduce a new concept of $S_{\Delta^m \omega}$-convergence and obtain a condition under which this convergence coincides with above-mentioned sequence spaces.

1. Introduction and Background

Let $\ell^\infty$ and $C$ be the Banach spaces of real bounded and convergent sequences with the usual supremum norm.

Let $\sigma$ be the mapping of the set of all positive integers into itself. A continuous linear functional $\varphi$ on $\ell^\infty$ is said to be an invariant mean or $\sigma$-mean if and only if

(i) $\varphi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$, for all $n$;
(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \ldots)$;
(iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$, for all $x \in \ell^\infty$.

If $x = (x_n)$, we write $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown by Schaefer [1] that

$$V_{\sigma} = \left\{ x \in \ell^\infty : \lim_{k \to \infty} t_{k_m} (x) = L, \right. \left. \text{uniformly in } m, \; \sigma - \lim x = L \right\},$$

where $t_{k_m} (x) = (x_m + x_{\sigma(m)} + \cdots + x_{\sigma^{(m)}(m)})/(k + 1)$.

In case $\sigma$ is the translation mapping $n \to n + 1$, $\sigma$-mean is often called a Banach limit and $V_{\sigma}$, the set of bounded sequences all whose invariant means are equal, is the set of almost convergent sequences (see Lorentz [2]). Using the concept of invariant means Mursaleen et al. [3] introduced the following sequence spaces as a generalization of Das and Sahoo [4]:

$$w_{\sigma} = \left\{ x : \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} t_{k_n} (x - \ell) \to 0, \text{uniformly in } m \right\},$$

$$[w]_{\sigma} = \left\{ x : \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \|t_{k_n} (x - \ell)\| \to 0, \text{uniformly in } m \right\},$$

$$w_{\sigma}^* = \left\{ x : \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} t_{k_n} (|x - \ell|) \to 0, \text{uniformly in } m \right\}.$$
Definition 1 (see [5]). A number sequence \( x = (x_k) \) is said to be statistically convergent to a number \( L \) (denoted by \( S - \lim_{k \to \infty} x_k = L \)) provided that, for every \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| = 0,
\]
(3)
where the vertical bars denote the cardinality of the enclosed set.

By a lacunary sequence, we mean an increasing sequence \( \theta = (k_r) \) of positive integers such that \( k_0 = 0 \) and \( k_r = k_{r-1} + \delta \) for \( r \to \infty \). The intervals determined by \( \theta \) will be denoted by \( I_r = (k_{r-1}, k_r] \), where the ratio \( k_r/k_{r-1} \) is denoted by \( q_r \). The space of lacunary strongly convergent sequence \( N_0 \) was defined by Freedman et al. [10] as follows:
\[
N_0 = \left\{ x = (x_k) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0 \text{ for some } L \right\},
\]
(4)
Fridy and Orhan [11] generalized the concept of statistical convergence by using lacunary sequence which is called lacunary statistical convergence. Further, lacunary sequences have been studied by Fridy and Orhan [12], Pehlivan and Fisher [13], Et and Gökhan [14], and Tripathy and Dutta [15]. Quite recently, Karaçay [16] combined the approach of lacunary sequence with invariant means and introduced the notion of strong \( \sigma \)-lacunary statistically convergent as follows.

Definition 2 (see [16]). Let \( \theta = (k_r) \) be a lacunary sequence. A sequence \( x = (x_k) \) is said to be lacunary strong \( \sigma \)-lacunary statistically convergent if, for every \( \varepsilon > 0 \),
\[
\lim_{r \to \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |x_k - L| \geq \varepsilon \right\} \right| = 0,
\]
(5)
uniformly in \( m \),
where \( S_{\theta} \) denotes the set of all lacunary strong \( \sigma \)-lacunary statistically convergent sequences.

Another interesting generalization of statistical convergence was introduced in [17] with the help of ideals of subsets of \( \mathbb{N} \). Let \( \mathcal{P}(\mathbb{N}) \) denote the power set of \( \mathbb{N} \). A family of sets \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \) is called an ideal in \( \mathbb{N} \) if and only if (i) \( \emptyset \in \mathcal{F} \); (ii) \( A, B \in \mathcal{F} \) imply \( A \cup B \in \mathcal{F} \); (iii) \( A \in \mathcal{F} \) and \( B \subset A \) imply \( B \in \mathcal{F} \). A nonempty family of sets \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \) is called a filter on \( \mathbb{N} \) if and only if (i) \( \emptyset \notin \mathcal{F} \); (ii) \( A, B \in \mathcal{F} \) imply \( A \cap B \in \mathcal{F} \); (iii) \( A \in \mathcal{F} \) and \( B \supset A \) imply \( B \in \mathcal{F} \). An ideal \( \mathcal{F} \) is called nontrivial if \( \mathcal{F} \neq \emptyset \) and \( \emptyset \notin \mathcal{F} \). It immediately implies that \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \) is a nontrivial ideal if and only if the class \( \mathcal{F} = \mathcal{F}(1) = \{N - A : A \in \mathcal{F} \} \) is a filter on \( \mathbb{N} \). The filter \( \mathcal{F} = \mathcal{F}(1) \) is called the filter associated with the ideal \( \mathcal{F} \). A nontrivial ideal \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \) is called an admissible ideal in \( \mathbb{N} \) if and only if it contains all singletons, that is, if it contains \( \{n\} : n \in \mathbb{N} \). Throughout the present work, \( \mathcal{F} \) denotes a nontrivial admissible ideal.

Definition 3 (see [17]). Let \( \mathcal{F} \subset \mathcal{P}(\mathbb{N}) \) be a nontrivial ideal in \( \mathbb{N} \) and let \((X, \rho)\) be a metric space. A sequence \( x = (x_k) \) in \( X \) is said to be \( \mathcal{F} \)-convergent to \( \xi \) if, for each \( \varepsilon > 0 \), the set \( A(\varepsilon) = \{ k \in \mathbb{N} : \rho(x_k, \xi) \geq \varepsilon \} \) is in \( \mathcal{F} \). In this case, we write \( \mathcal{F} - \lim_{k \to \infty} x_k = \xi \).

Recently, Das et al. [18] unified the idea of lacunary statistical convergence with ideal convergence and presented the following interesting generalization of statistical convergence.

Definition 4 (see [18]). Let \( \theta = (k_r) \) be a lacunary sequence. A sequence \( x = (x_k) \) of numbers is said to be \( \mathcal{F} \)-lacunary statistical convergent or \( S_{\theta}(\mathcal{F}) \)-convergent to \( L \), if, for every \( \varepsilon > 0 \) and \( \delta > 0 \),
\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : |x_k - L| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{F}.
\]
(6)
In this case, we write \( x_k \to L(S_{\theta}(\mathcal{F})) \) or \( S_{\theta}(\mathcal{F}) - \lim_{k \to \infty} x_k = L \). The set of all \( \mathcal{F} \)-lacunary statistically convergent sequences will be denoted by \( S_{\theta}(\mathcal{F}) \).

Definition 5 (see [18]). Let \( \theta = (k_r) \) be a lacunary sequence. A sequence \( x = (x_k) \) of numbers is said to be \( N_{\theta}(\mathcal{F}) \)-convergent to \( L \) if, for every \( \varepsilon > 0 \), we have
\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \varepsilon \right\} \in \mathcal{F}.
\]
(7)
It is denoted by \( x_k \to L(N_{\theta}(\mathcal{F})) \).

In 1981, Kizmaz [19] introduced the notion of difference sequence space as follows:
\[
\Delta(X) = \{ x = (x_k) : (\Delta x_k) \in X \},
\]
(8)
for \( X = \ell_\infty, c \), and \( c_0 \), where \( \Delta x_k = x_k - x_{k+1} \), for all \( k \in \mathbb{N} \).

Continuing on this way, the notion was further generalized by Et and Çolak [20] by introducing the sequence spaces as follows:
\[
\Delta^m(X) = \{ x = (x_k) : \Delta^m x_k \in X \},
\]
(9)
for \( X = \ell_\infty, c \), and \( c_0 \), where \( m \in \mathbb{N} \) and \( \Delta^m x_k = \Delta^{m-1} x_{k+1} \), so that \( \Delta^m x_k = \sum_{r=0}^{m-1}(-1)^r(m/r)x_{k+r+1} \). For extensive view in this area, we refer to the series of papers ([21–27]).

The notion of modulus function was introduced by Nakano [28] as follows: by a modulus function, we mean a function \( f \) from \([0, \infty)\) to \([0, \infty)\) such that (i) \( f(x) = 0 \) if and only if \( x = 0 \), (ii) \( f(x + y) \leq f(x) + f(y) \), for all \( x, y \geq 0 \), (iii) \( f \) is increasing, and (iv) \( f \) is continuous from right at 0. It follows that \( f \) must be continuous everywhere on \([0, \infty)\).

A modulus function may be bounded or unbounded. In the recent past the notion of modulus function was investigated from different aspects and sequence spaces have been studied by Buckley [29], Maddox [30], Et [31], Pehlivan and Fisher [13], Savas [32], Et and Gökhan [14], Kumar et al. [33], and many others.

The following well-known lemma is required for establishing a very important result in our paper.

**Lemma 6.** Let \( f \) be a modulus function and let \( 0 < \delta < 1 \). Then, for each \( x > \delta \), we have \( f(x) \leq (2 \cdot f(1x)) / \delta \).
The following inequality will be used throughout the paper. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup_k p_k = H$ and $C = \max(1, 2^{H-1})$. Then, for all $a_k, b_k \in C$, for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^p \leq C \left| |a_k|^p + |b_k|^p \right|.$$ (10)

Inspired by the above works, we presently introduce some new kind of sequence spaces by using ideal convergence, modulus function, and invariant mean. Further, we also obtain some relevant connections of these spaces with $S_0^m(\mathcal{F})$-convergence.

### 2. Main Results

Throughout the paper, $\mathcal{F} \subset \mathcal{F}(\mathbb{N})$ is considered a nontrivial admissible ideal and $w(X)$ denotes the space of all sequences $x = (x_k) \in X$. 

**Definition 7.** Let $\mathcal{F}$ be an admissible ideal, let $f$ be a modulus function, and let $p = (p_k)$ be any sequence of strictly positive real numbers. Then, for each $\epsilon > 0$, we define the following sequence spaces:

$$w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_{\mathrm{lo}} = \left\{ x \in w(X) : \right. $$

$$ \left\{ r \in \mathbb{N} : \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m x) \right| \right) \right|^p \geq \epsilon \right\} \in \mathcal{F} \right\},$$

$$w^f_\sigma[\Delta^m, \mathcal{F}, \theta] = \left\{ x \in w(X) : \exists \epsilon > 0, $$

$$ \left\{ \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m x - \epsilon) \right| \right) \right|^p \geq \epsilon \right\} \in \mathcal{F} \right\},$$

$$w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_{\mathrm{co}} = \left\{ x \in w(X) : \exists \epsilon > 0, $$

$$ \left\{ \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m x) \right| \right) \right|^p \geq \epsilon \right\} \in \mathcal{F} \right\},$$

uniformly in $n$.

**Remark 8.** By taking some particular cases, we obtain the following.

(i) If we take $m = 0$, for all $k \in \mathbb{N}$, then the above spaces reduce to $w^f_\sigma[I, p, \theta]_0$, $w^f_\sigma[I, p, \theta]$, and $w^f_\sigma[I, p, \theta]_{\mathrm{co}}$, respectively.

(ii) If we choose $m = 0$, for all $k \in \mathbb{N}$ and $\sigma(n) = n + 1$, then we obtain $\tilde{w}^f_\sigma[I, p, \theta]_0$, $\tilde{w}^f_\sigma[I, p, \theta]$, and $\tilde{w}^f_\sigma[I, p, \theta]_{\mathrm{co}}$ instead of $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_0$, $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]$, and $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_{\mathrm{co}}$. 

(iii) By taking $\mathcal{F} = \mathcal{F}_f = \{ E \subset \mathbb{N} : E \text{ is a finite subset} \}$, $m = 0$, $f(x) = x$, $\theta = (2^f)$, and $p_k = 1$, for all $k \in \mathbb{N}$, then we obtain $w_\sigma$ defined by Mursaleen et al. [3] instead of $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]$ and $\tilde{w}^f_\sigma[I, p, \theta]_0$ reduces to $[\tilde{w}]$ defined in Das and Sahoo [4].

**Theorem 9.** Let $f$ be a modulus function and $p = (p_k)$ is a bounded sequence of strictly positive real numbers; then, $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_0$, $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]$, and $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_{\mathrm{co}}$ are linear spaces over $\mathbb{C}$.

**Proof.** We will prove the assertion only for $w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_0$ and others can be treated similarly. Suppose that $x = (x_k)$, $y = (y_k) \in w^f_\sigma[\Delta^m, \mathcal{F}, \theta]_0$. Then, for every $\epsilon > 0$ and uniformly in $n$, the sets

$$A_\theta \left( \frac{\epsilon}{2} \right) = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m x) \right| \right) \right|^p \geq \frac{\epsilon}{2} \right\},$$

$$B_\theta \left( \frac{\epsilon}{2} \right) = \left\{ r \in \mathbb{N} : \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m y) \right| \right) \right|^p \geq \frac{\epsilon}{2} \right\}$$

belong to $\mathcal{F}$.

Let $\alpha, \beta \in \mathbb{C}$ and $\Delta^m$ is linear; then,

$$\frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m (\alpha \cdot x + \beta \cdot y)) \right| \right) \right|^p$$

$$= \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m x + \beta \cdot \Delta^m y) \right| \right) \right|^p$$

$$\leq C \cdot (K_\alpha)^H \cdot \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m x) \right| \right) \right|^p$$

$$+ C \cdot (K_\beta)^H \cdot \frac{1}{H_r} \sum_{k \in \mathbb{N}} \left| f \left( \left| t_{k_n} (\Delta^m y) \right| \right) \right|^p,$$

by (10),

where $K_\alpha$, $K_\beta$ are two positive numbers such that $|\alpha| \leq K_\alpha$ and $|\beta| \leq K_\beta$. 

$$\left( \frac{\epsilon}{2} \right),$$
Then, for given $\epsilon > 0$, we have the following containment:

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^m \left( \alpha \cdot x + \beta \cdot y \right) \right) \right| \right]^p \geq \epsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^m x \right) \right| \right)^p \geq \frac{\epsilon}{2C \cdot (K_0)^H} \right\}$$

$$\cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^m y \right) \right| \right)^p \geq \frac{\epsilon}{2C \cdot (K_0)^H} \right\}$$

uniformly in $n$.

Since $x, y \in w_0^f[\Delta^m_n, \mathcal{I}, \theta]_{0, L, \infty}$, it follows that the later sets belong to $\mathcal{I}$. By using the property of an ideal set on the left hand side in the above expression also belongs to $\mathcal{I}$. This completes the proof.

**Theorem 10.** For $m \geq 1$, then the inclusion

$$w_0^f[\Delta^{m-1}_p, \mathcal{I}, \theta]_{0, L, \infty} \subset w_0^f[\Delta^m_p, \mathcal{I}, \theta]_{0, L, \infty}$$

is strict.

**Proof.** We will prove the result for $w_0^f[\Delta^{m-1}_p, \mathcal{I}, \theta]_{0}$ only. Suppose that $x = (x_k) \in w_0^f[\Delta^{m-1}_p, \mathcal{I}, \theta]_{0}$; by definition, for each $\epsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^{m-1} x_k \right) \right| \right]^p \geq \epsilon \right\} \in \mathcal{I}$$

uniformly in $n$.

Since $f$ is a modulus function, therefore we have the following inequality:

$$\frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^m x_k \right) \right| \right]^p \leq \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^{m-1} x_k \right) \right| \right) + f \left( \left| t_k \left( \Delta^{m-1} x_{k+1} \right) \right| \right)^p \leq C \cdot \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^{m-1} x_k \right) \right| \right)^p + C \cdot \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^{m-1} x_{k+1} \right) \right| \right)^p \right.$$}

uniformly in $n$.

Now, for given $\epsilon > 0$, we have the following containment:

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^m x_k \right) \right| \right)^p \geq \epsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^{m-1} x_k \right) \right| \right)^p \geq \frac{\epsilon}{2C \cdot (K_0)^H} \right\}$$

$$\cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f \left( \left| t_k \left( \Delta^{m-1} x_{k+1} \right) \right| \right)^p \geq \frac{\epsilon}{2C \cdot (K_0)^H} \right\}$$

uniformly in $n$.

Both the sets on the right hand side in the above containment belong to $\mathcal{I}$ by (16). It follows that $x \in w_0^f[\Delta^m_p, \mathcal{I}, \theta]_{0}$.

Since $\mathcal{I}$ is an admissible ideal and the inclusion is strict as the sequence $x = (x_k) = (k^{m-1})$ belongs to $x \in w_0^f[\Delta^m_p, \mathcal{I}, \theta]_{0}$, it does not belong to $w_0^f[\Delta^{m-1}_p, \mathcal{I}, \theta]_{0}$, for $f(x) = x, \delta(x) = (x_n), \theta = (2')$, and $p_k = 1$, for all $k \in \mathbb{N}$.

**Theorem 11.** Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers and $f'$, $f''$ are modulus functions. If

$$\limsup_{t \rightarrow \infty} f'(t) = M > 0,$$

then $w_0^f[\Delta^m_p, \mathcal{I}, \theta]_{0} \subset w_0^{f''}[\Delta^m_p, \mathcal{I}, \theta]_{0}$.

**Proof.** Assume that $\limsup_{t \rightarrow \infty} (f'(t)/f''(t)) = M$; there exists a positive number $K > 0$ such that $f'(t) \geq K \cdot f''(t)$, for all $t \geq 0$, which implies that

$$\frac{1}{h_r} \sum_{k \in I_r} \left[ f' \left( \left| t_k \left( \Delta^m x_k \right) \right| \right)^p \geq (K)^H \frac{1}{h_r} \sum_{k \in I_r} \left[ f'' \left( \left| t_k \left( \Delta^m x_k \right) \right| \right)^p \right.$$}

uniformly in $n$.

Thus, for any $\epsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f'' \left( \left| t_k \left( \Delta^m x_k \right) \right| \right)^p \geq \epsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left[ f' \left( \left| t_k \left( \Delta^m x_k \right) \right| \right)^p \geq \epsilon \cdot (K)^H \right\}$$

uniformly in $n$. Therefore, the above containment gives the result.
Theorem 12. If \( f, f', \) and \( f'' \) are modulus functions, then
\[
\begin{align*}
(i) & \quad \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \subset \omega'_\sigma[\Delta^m_p, \mathcal{F}, \theta], \\
(ii) & \quad \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \cap \omega'_\sigma[\Delta^m_p, \mathcal{F}, \theta] \subset \omega'_\sigma[\Delta^m_p, \mathcal{F}, \theta].
\end{align*}
\]

Proof. (i) Let \( x = (x_k) \in \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \) and let \( \varepsilon > 0 \) be given. Choose \( \delta \in (0, 1) \) such that \( f(t) < \varepsilon \), for all \( 0 < t < \delta \), since \( x \in \omega'_\sigma[\Delta^m_p, \mathcal{F}, \theta] \) such that
\[
A_\delta = \left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f'(t_{k_n}(\Delta^m x - \ell)) \right|^p \geq \delta \right\} \in \mathcal{F}
\]
uniformly in \( n \).

On the other hand, we have
\[
\begin{align*}
\frac{1}{h_r} \sum_{k \in I_r} \left| f \circ f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p & = \frac{1}{h_r} \sum_{k \in I_r, \theta \in [f'(t_{k_n}(\Delta^m x - \ell))]^{p < \delta}} \left| f \circ f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \\
& \quad + \frac{1}{h_r} \sum_{k \in I_r, \theta \in [f'(t_{k_n}(\Delta^m x - \ell))]^{p \geq \delta}} \left| f \circ f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \\
& \leq (\varepsilon H + \max \left( 1, \left( 2 \cdot \frac{f(1)}{\delta} \right)^H \right) \\
& \quad \frac{1}{h_r} \sum_{k \in I_r} \left| f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \text{ by Lemma 6},
\end{align*}
\]
uniformly in \( n \).

Then, for any \( \eta > 0 \),
\[
\left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f \circ f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \geq \eta \right\} \\
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \geq \frac{(\eta - \varepsilon)}{K} \right\},
\]
where \( K = \max(1, (2, f(1)/\delta)^H) \).

Therefore, for any \( \eta > 0 \),
\[
\left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f \circ f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \geq \eta \right\} \\
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f' \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \geq \frac{(\eta - \varepsilon)}{K} \right\},
\]
uniformly in \( n \), where \( \sup_p \beta_k = H \) and \( C = \max(1, 2^H - 1) \).

Theorem 13. If \( f \) is a modulus function and \( p = (p_k) \) is a sequence of positive real numbers, then \( \omega_{p_k}[\Delta^m_p, \mathcal{F}, \theta] \subseteq \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \).

Proof. This can be proved similarly as in Theorem 12(i).

Theorem 14. Let \( f \) be a modulus function. If \( \lim \sup_{t \to \infty} \frac{f(t)}{t} = M > 0 \), then
\[
\omega'_p[\Delta^m_p, \mathcal{F}, \theta] \subseteq \omega_{p_k}[\Delta^m_p, \mathcal{F}, \theta].
\]

Proof. Suppose that \( x = (x_k) \in \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \). It is given that \( \lim \sup_{t \to \infty} \frac{f(t)}{t} = M > 0 \); there exists a constant \( K > 0 \) such that \( f(t) \geq K \cdot t \), for all \( t \geq 0 \),

which implies that
\[
\left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \geq \frac{(\eta - \varepsilon)}{K} \right\},
\]
uniformly in \( n \).

Then, for each \( \varepsilon > 0 \),
\[
\left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \geq \varepsilon \right\} \\
\subseteq \left\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| f \left( t_{k_n}(\Delta^m x - \ell) \right) \right|^p \geq \frac{(\eta - \varepsilon)}{K} \right\},
\]
uniformly in \( n \).

Since \( x \in \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \), therefore the latter set belongs to \( \mathcal{F} \). It follows that \( x \in \omega_{p_k}[\Delta^m_p, \mathcal{F}, \theta] \).

Theorem 15. If \( 0 < p_k \leq q_k \) and \( (q_k/p_k) \) are bounded, then \( \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \subseteq \omega'_p[\Delta^m_p, \mathcal{F}, \theta] \).

Proof. The proof of this theorem is easy and so it is omitted.

3. \( S_{\sigma}^{\Delta^m_p}(\mathcal{F}) \)-Convergence

In this section, we define the notion of \( S_{\sigma}^{\Delta^m_p}(\mathcal{F}) \)-convergence with the help of ideal and invariant means and difference sequences. Further, we also establish some relations between \( S_{\sigma}^{\Delta^m_p}(\mathcal{F}) \)-convergence and \( \omega_{p_k}[\Delta^m_p, \mathcal{F}, \theta] \).

Definition 16. Let \( \mathcal{F} \subseteq \mathcal{P}(\mathbb{N}) \) be a nontrivial ideal. A sequence \( x = (x_k) \) is said to be \( \Delta^m(\mathcal{F}) \)-strong lacunary \( \sigma \)-statistically convergent or \( S_{\sigma}^{\Delta^m_p}(\mathcal{F}) \)-convergent to a number \( \ell \), provided that, for every \( \varepsilon > 0 \) and \( \delta > 0 \),
\[
\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left| k \in I_r : \left| t_{k_n}(\Delta^m x - \ell) \right| \geq \varepsilon \right| \right| \geq \delta \right\} \in \mathcal{F},
\]
uniformly in \( n \).
In this case, we write $x_k \to \ell(S_{\theta_\varphi}^m(\mathcal{F}))$ or $S_{\theta_\varphi}^m(\mathcal{F})$ denote the set of all $S_{\theta_\varphi}^m(\mathcal{F})$-convergent sequences.

**Theorem 17.** Let $f$ be a modulus function and let $p = (p_k)$ be a sequence of strictly positive real numbers. If $0 < \inf k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $w_o^f(\Delta_p^m, \mathcal{F}, \theta)_0 \subset S_{\theta_\varphi}^m(\mathcal{F})$.

**Proof.** Assume that $x \in w_o^f(\Delta_p^m, \mathcal{F}, \theta)_0$ and $\epsilon > 0$ is given. Then, we have

$$\frac{1}{h_r} \sum_{k \in \ell_r} [f((t_{x_n}(\Delta_m^k x)))]^p_k$$

$$= \frac{1}{h_r} \sum_{k \in \ell_r, k \notin \Delta_m^k x \geq x} [f((t_{x_n}(\Delta_m^k x)))]^p_k$$

$$+ \frac{1}{h_r} \sum_{k \in \ell_r, k \notin \Delta_m^k x \geq x} [f((t_{x_n}(\Delta_m^k x)))]^p_k$$

$$\geq \frac{1}{h_r} \sum_{k \in \ell_r, k \notin \Delta_m^k x \geq x} [f((t_{x_n}(\Delta_m^k x)))]^p_k$$

$$\geq \frac{1}{h_r} \sum_{k \in \ell_r} \min\left(\frac{[f(e)]^h}{[f(e)]^h}\right)$$

$$\geq \frac{1}{h_r} \left|\{(k \in \mathbb{N} : [t_{x_n}(\Delta_m^k x)] \geq \epsilon)\}\right|$$

$$\cdot \min\left(\frac{[f(e)]^h}{[f(e)]^h}\right)$$

uniformly in $n$.

Then, for every $\delta > 0$, we have the following containment:

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left|\{(k \in \mathbb{N} : [t_{x_n}(\Delta_m^k x)] \geq \epsilon)\}\right| \geq \delta \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in \ell_r} [f((t_{x_n}(\Delta_m^k x)))]^p_k \geq K \cdot \delta \right\}$$

uniformly in $n$, where $K = \min\{(f(e))^h, [f(e)]^h\}$, since $x \in w_o^f(\Delta_p^m, \mathcal{F}, \theta)_0$, which implies that $x \in S_{\theta_\varphi}^m(\mathcal{F})$. □

**Theorem 18.** Let $f$ be a bounded modulus function and $0 < \inf k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$; then, $S_{\theta_\varphi}^m(\mathcal{F}) \subset w_o^f(\Delta_p^m, \mathcal{F}, \theta)_0$.

**Proof.** Using the same technique of [26, Theorem 3.3], it is easy to prove. □

**Theorem 19.** If $0 < \inf k p_k = h \leq p_k \leq \sup_k p_k = H < \infty$, then $S_{\theta_\varphi}^m(\mathcal{F}) = w_o^f(\Delta_p^m, \mathcal{F}, \theta)_0$ if and only if $f$ is bounded.

**Proof.** This part is the direct consequence of Theorems 17 and 18. □

Conversely, suppose that $f$ is unbounded defined by $f(k) = k$, for all $k \in \mathbb{N}$, and let $\theta = (2^t)$ be a lacunary sequence. We take a fixed set $A \subset \mathcal{F}$ and define $x = (x_k)$ as follows:

$$x_k = \begin{cases} k^{m+1}, & \text{if } r \notin A, \ 2^{-1} + 1 \leq k \leq 2^{-1} + \left\lfloor \sqrt{h_r} \right\rfloor, \\ 0, & \text{otherwise,} \end{cases}$$

where $I_r = (2^{-1}, 2^t)$ and $h_r = 2^t - 2^{-1}$.

For given $\epsilon > 0$, we have

$$\lim_{r \to \infty} \frac{1}{h_r} \left|\{(k \in I_r : [t_{x_n}(\Delta_m^k x - 0)] \geq \epsilon)\}\right|$$

$$= \lim_{r \to \infty} \frac{1}{h_r} \left|\{(k \in I_r : [t_{x_n}(\Delta_m^k x - 0)] \geq \epsilon)\}\right|$$

$$\leq \frac{1}{h_r} \rightarrow 0,$$

for all $r \notin A$, and uniformly in $n$.

Hence, for $\delta > 0$, there exists a positive integer $r_0$ such that

$$\frac{1}{h_r} \left|\{(k \in I_r : [t_{x_n}(\Delta_m^k x - 0)] \geq \epsilon)\}\right| < \delta$$

for $r \notin A, \ r \geq r_0$.

Now, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left|\{(k \in I_r : [t_{x_n}(\Delta_m^k x - 0)] \geq \epsilon)\}\right| \geq \delta \right\}$$

$$\subset (A \cup (1, 2, \ldots, r_0 - 1)).$$

Since $\mathcal{F}$ is an admissible ideal, it follows that $S_{\theta_\varphi}^m(\mathcal{F}) - \lim_{r \to \infty} x_k = 0$.

If we take $p_k = 1$, for all $k = 1, 2, \ldots$, then $x_k \notin w_o^f(\Delta_p^m, \mathcal{F}, \theta)_0$. This contradicts the fact that $S_{\theta_\varphi}^m(\mathcal{F}) = w_o^f(\Delta_p^m, \mathcal{F}, \theta)_0$.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**

[1] P. Schaefer, "Infinite matrices and invariant means," *Proceedings of the American Mathematical Society*, vol. 36, pp. 104–110, 1972.

[2] G. G. Lorentz, "A contribution to the theory of divergent sequences," *Acta Mathematica*, vol. 80, pp. 167–190, 1948.

[3] Mursaleen, A. K. Gaur, and T. A. Chishti, "On some new sequence spaces of invariant means," *Acta Mathematica Hungarica*, vol. 75, no. 3, pp. 209–214, 1997.
G. Das and S. K. Sahoo, “On some sequence spaces,” *Journal of Mathematical Analysis and Applications*, vol. 164, no. 2, pp. 381–398, 1992.

H. Fast, “Sur la convergence statistique,” *Colloquium Mathematicum*, vol. 2, pp. 241–244, 1951.

J. S. Connor, “The statistical and strong p-Cesáro convergence of sequences,” *Analysis. International Mathematical Journal of Analysis and its Applications*, vol. 8, no. 1-2, pp. 47–63, 1988.

J. A. Fridy, “On statistical convergence,” *Analysis*, vol. 5, no. 4, pp. 301–313, 1985.

I. J. Maddox, “Statistical convergence in a locally convex space,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 104, no. 1, pp. 141–145, 1988.

R. Freedman, J. J. Sember, and R. Raphael, “Some p-Cesáro type summability spaces,” *Proceedings of the London Mathematical Society*, vol. 37, no. 2, pp. 508–529, 1978.

J. A. Fridy and C. Orhan, “Lacunary statistical convergence,” *Pacific Journal of Mathematics*, vol. 160, no. 1, pp. 43–51, 1993.

J. A. Fridy and C. Orhan, “Lacunary statistical summability,” *Journal of Mathematical Analysis and Applications*, vol. 173, no. 2, pp. 497–504, 1993.

S. Pehlivan and B. Fisher, “Lacunary strong convergence with respect to a sequence of modulus functions,” *Commentationes Mathematicae Universitatis Carolinae*, vol. 36, no. 1, pp. 69–76, 1995.

M. Et and A. Gökhan, “Lacunary strongly almost summable sequences,” *Studia. Universitatis Babes-Bolyai. Mathematica*, vol. 53, no. 4, pp. 29–38, 2008.

B. C. Tripathy and H. Dutta, “On some lacunary difference sequence spaces defined by a sequence of Orlicz functions and q-lacunary $\Delta_q^n$-statistical convergence,” *Analele Stiintifice ale Universitii Ovidius Constanta*, vol. 20, no. 1, pp. 417–430, 2012.

V. Karakaya, “Some new sequence spaces defined by a sequence of Orlicz functions,” *Taiwanese Journal of Mathematics*, vol. 9, no. 4, pp. 617–627, 2005.

P. Kostyrko, T. Šalát, and W. Wilczyński, “$\mathcal{F}$-convergence,” *Real Analysis Exchange*, vol. 26, no. 2, pp. 669–686, 2000.

P. Das, E. Savas, and S. Kr. Ghosal, “On generalizations of certain summability methods using ideals,” *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1509–1514, 2011.

H. Kizmaz, “On certain sequence spaces,” *Canadian Mathematical Bulletin*, vol. 24, no. 2, pp. 169–176, 1981.

M. Et and R. Çolak, “On some generalized difference sequence spaces,” *Soochow Journal of Mathematics*, vol. 21, no. 4, pp. 377–386, 1995.

A. A. Bakery, E. A. E. Mohamed, and M. A. Ahmed, “Some generalized difference sequence spaces defined by ideal convergence and Musielak-Orlicz function,” *Abstract and Applied Analysis*, vol. 2013, Article ID 972363, 9 pages, 2013.

M. Başarır, Ş. Konca, and E. E. Kara, “Some generalized difference statistically convergent sequence spaces in 2-normed space,” *Journal of Inequalities and Applications*, vol. 2013, article 177, pp. 1–10, 2013.

V. K. Bhardwaj and S. Gupta, “Cesàro summable difference sequence space,” *Journal of Inequalities and Applications*, vol. 2013, article 315, pp. 1–9, 2013.

M. Et and M. Başarır, “On some new generalized difference sequence spaces,” *Periodica Mathematica Hungarica*, vol. 35, no. 3, pp. 169–175, 1997.
Submit your manuscripts at http://www.hindawi.com