PAIRS OF CONVEX BODIES IN $S^D$, $\mathbb{R}^D$ AND $H^D$, 
WITH SYMMETRIC INTERSECTIONS 
OF THEIR CONGRUENT COPIES 

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Abstract. High proved the following theorem. If the intersections of any two congruent copies of a plane convex body are centrally symmetric, then this body is a circle. In our paper we extend the theorem of High to spherical and hyperbolic planes. If in any of these planes, or in $\mathbb{R}^2$, there is a pair of closed convex sets with interior points, and the intersections of any congruent copies of these sets are centrally symmetric, then, under some mild hypotheses, our sets are congruent circles, or, for $\mathbb{R}^2$, two parallel strips. We prove the analogue of this statement, for $S^d$, $\mathbb{R}^d$, $H^d$, if we suppose $C_+^2$: again, our sets are congruent balls. In $S^2$, $\mathbb{R}^2$ and $H^2$ we investigate a variant of this question: supposing that the numbers of connected components of the boundaries of both sets are finite, we exactly describe all pairs of such closed convex sets, with interior points, whose any congruent copies have an intersection with axial symmetry (there are 1, 5 or 9 cases, respectively).

1. Introduction

By a convex body in $S^d$ (sphere), $\mathbb{R}^d$, $H^d$ (hyperbolic space) we mean a compact convex set, with non-empty interior. For convexity of $K \subset S^d$, with $K$ closed and int $K \neq \emptyset$, it suffices to suppose, that for any two non-antipodal points of $K$ the shorter great circle arc connecting them belongs to $K$; then for $\pm x \in K$, $y \in \text{int } K$ and $y \neq \pm x$, the shorter arcs $\hat{(\pm x)y}$ belong to $K$, hence some half large circle connects $\pm x$ in $K$. In $S^d$, when saying ball, or sphere, we always mean one with radius at most $\pi/2$ (thus the ball is convex). A convex body in $S^d$, $\mathbb{R}^d$, $H^d$ is strictly convex, if its boundary does not contain a non-trivial segment.

R. High proved the following theorem.

Theorem 1. ([H]) Let $K \subset \mathbb{R}^2$ be a convex body. Then the following statements are equivalent:

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(1) All intersections \((\varphi K) \cap (\psi K)\), having interior points, where \(\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}^2\) are congruences, are centrally symmetric.

(2) \(K\) is a circle. 

It seems, that his proof gives the analogous statement, when \(\varphi, \psi\) are only allowed to be orientation preserving congruences.

**Problem.** Describe the pairs of closed convex sets with interior points, in \(S^d, \mathbb{R}^d\), and \(H^d\), whose any congruent copies have a centrally symmetric intersection, provided this intersection has interior points. Evidently, two congruent balls (for \(S^d\) of radii at most \(\pi/2\)), or two parallel slabs in \(\mathbb{R}^d\), have a centrally symmetric intersection, provided it has a non-empty interior.

The authors are indebted to L. Montejano (Mexico City) and G. Weiss (Dresden) for having turned their interest to characterizations of pairs of convex bodies with all translated/congruent copies having a centrally or axially symmetric intersection or convex hull of the union, respectively, or with other symmetry properties, e.g., having some affine symmetry.

The aim of our paper will be to give partial answers to this problem. To exclude trivialities, we always suppose, that our sets are different from the whole plane, or space, and also we investigate only such cases, when the intersection has interior points. We prove the analogue of Theorem 1 of High for \(S^2\) and \(H^2\). Namely, we characterize the pairs of closed convex sets with interior points, in \(S^2, \mathbb{R}^2\), and \(H^2\), having centrally symmetric intersections of all congruent copies — provided these intersections have non-empty interiors — however, for \(H^2\) only under some mild hypothesis. For \(S^d, \mathbb{R}^d\), and \(H^d\), where \(d > 2\), we prove the analogous theorem under some regularity assumptions (weaker than \(C^2\), or \(C^2_+\), respectively). The only possibilities are for \(d = 2\) two congruent circles, or two parallel strips for \(\mathbb{R}^2\), and for \(d > 2\) two congruent balls. Moreover we investigate a variant of this question, for \(S^2, \mathbb{R}^2\), and \(H^2\), when we prescribe, rather than central, the axial symmetry of all intersections, having non-empty interiors, but we restrict ourselves to the case that the numbers of connected components of the boundaries of both sets are finite. We exactly describe all pairs of such closed convex sets with interior points: there are 1, 5 or 9 cases, respectively.

Moreover, in \(S^2, \mathbb{R}^2\), and \(H^2\), if all small intersections of congruent copies of two closed convex sets with interior points, having a non-empty interior, have some non-trivial symmetry, then all connected components of the boundaries of the two sets are cycles or straight lines. For \(S^d, \mathbb{R}^d\), and \(H^d\), under the above mentioned regularity assumptions, if all small intersections of congruent copies of two closed convex sets with interior points, having a non-empty interior, are centrally symmetric, then all connected components of the boundaries of the two sets are
congruent spheres, paraspheres, hyperspheres, or hyperplanes. (“Small” means here: of sufficiently small diameter.)

Surveys about characterizations of central symmetry, for convex bodies in \( \mathbb{R}^d \), cf. in [BF], §14, pp. 124-127, and, more recently, in [HM], §4.

In a paper under preparation [J-CM] we will give more detailed theorems about \( \mathbb{R}^d \). We will describe the pairs of closed convex sets with interior points, whose any congruent copies have 1) a centrally symmetric intersection (provided this intersection has interior points), without regularity hypotheses; 2) a centrally symmetric closed convex hull of their union, also without regularity hypotheses. These results will form additions to the results of the papers [So1], [So2].

2. New results

We mean by a non-trivial symmetry a symmetry different from the identity. Moreover, \( \text{diam}(\cdot) \) will denote the diameter of a set.

As general hypotheses in our theorems for \( d = 2 \) we give

\[
(*): \begin{cases} 
X \text{ will be } S^2, \mathbb{R}^2, \text{ or } H^2, \\
\text{and } K, L \subseteq X \text{ will be closed convex sets with interior points,} \\
\text{and } \varphi, \psi : X \to X \text{ will be orientation preserving congruences.}
\end{cases}
\]

The following Theorem 2 will be the basis of our considerations for \( d = 2 \).

**Theorem 2.** Assume \((*)\). Then we have \((1) \implies (2)\), where

1. There exists some \( \varepsilon > 0 \), such that for each \( \varphi, \psi \), for which \( \text{int}((\varphi K) \cap (\psi L)) \neq \emptyset \), and \( \text{diam}((\varphi K) \cap (\psi L)) \leq \varepsilon \), we have that \( (\varphi K) \cap (\psi L) \) has some non-trivial symmetry.

2. Each connected component of the boundaries of both \( K \) and \( L \) is a cycle (for \( X = S^2 \) a circle of radius at most \( \pi/2 \)), or a straight line.

In particular, if the symmetries in (1) are central symmetries, then in (2) the connected components of the boundaries of both \( K \) and \( L \) are congruent.

For \( X = S^2 \) and \( X = \mathbb{R}^2 \), we have \((2) \iff (1)\). Let \( X = H^2 \). If, both for \( K \) and \( L \), the infimum of the positive curvatures of the boundary components is positive, and there is at most one 0 curvature, then \((2) \iff (1)\). For \( X = H^2 \), if for, e.g., \( K \), the infimum of the positive curvatures is 0, or there are two 0 curvatures, then we have \((2) \nRightarrow (1)\). Even, we may prescribe in any way the curvatures of the connected hypercycle or straight line components of \( K \) (with multiplicity), in the above way, and then we can find an \( L \), so that (2) holds, but (1) does not hold.

**Theorem 3.** Assume \((*)\), and let \( X = S^2 \). Then the following statements are equivalent:
There exists some $\varepsilon > 0$, such that for each $\varphi, \psi$, for which $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, and $\text{diam}((\varphi K) \cap (\psi L)) \leq \varepsilon$, we have that $(\varphi K) \cap (\psi L)$ has some non-trivial symmetry.

(2) There exists some $\varepsilon > 0$, such that for each $\varphi, \psi$, for which $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, and $\text{diam}((\varphi K) \cap (\psi L)) \leq \varepsilon$, we have that $(\varphi K) \cap (\psi L)$ has an axis of symmetry.

(3) $K$ and $L$ are two circles, of radii at most $\pi/2$.

In particular, if the symmetries in (1) are central symmetries, then in (3) the two circles are congruent.

**Theorem 4.** Assume (*), and let $X = \mathbb{R}^2$. Then the following statements are equivalent:

(1) For each $\varphi, \psi$, for which $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, we have that $(\varphi K) \cap (\psi L)$ has some non-trivial symmetry.

(2) $K$ and $L$ are two circles, or one of them is a circle and the other one is a parallel strip or a half-plane, or they are two parallel strips, or they are two half-planes.

In particular, if the symmetries in (1) are central symmetries, then in (2) we have either two congruent circles, or two parallel strips. If the symmetries in (1) are axial symmetries, then in (2), for the case of two parallel strips, these strips are congruent.

The following two theorems give two different characterizations for $H^2$, under different additional hypotheses.

**Theorem 5.** Assume (*), and let $X = H^2$. If all connected components of the boundaries of both of $K$ and $L$ are straight lines, let their numbers be finite. Then the following statements are equivalent:

(1) For each $\varphi, \psi$, for which $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, we have that $(\varphi K) \cap (\psi L)$ is centrally symmetric.

(2) $K$ and $L$ are two congruent circles.

In the following theorem, the base line of a straight line is meant to be itself.

**Theorem 6.** Assume (*), and let $X = H^2$. Then we have (3) $\implies$ (2) $\implies$ (1). Supposing that all connected components of the boundaries of both of $K$ and $L$ are paracycles, hypercycles or straight lines, let their total number be finite. Then we have also (1) $\implies$ (3). Here:

(1) For each $\varphi, \psi$, for which $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, we have that $(\varphi K) \cap (\psi L)$ has some non-trivial symmetry.
(2) For each $\varphi, \psi$, for which $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, we have that $(\varphi K) \cap (\psi L)$ is axially symmetric.

(3) We have either (A), or (B), or (C), or (D), or (E), where

(A): Any of $K$ and $L$ is a circle, a paracycle, a convex domain bounded by a hypercycle, or a half-plane — however, if one of $K$ and $L$ is a convex set bounded by a hypercycle or is a half-plane, then the other one is either a circle, or a congruent copy of the first one.

(B): One of $K$ and $L$ is a circle, and the other one is bounded either by two hypercycles, whose base lines coincide, or by a hypercycle, and its base line.

(C): One of $K$ and $L$ is a circle, of radius $r$, say, and the other one is bounded by at least two hypercycles or straight lines (with all base lines different), whose mutual distances are at least $2r$.

(D): One of $K$ and $L$ is a paracycle, and the other is a parallel domain of some straight line, for some distance $l > 0$.

(E): $K$ and $L$ are congruent, and both are parallel domains of some straight lines, for some distance $l > 0$.

Now we turn to the case $d > 2$. As general hypotheses in our statements for $d > 2$, we give

\[
X \text{ will be } S^d, \mathbb{R}^d, \text{ or } H^d, \\
\text{and } K, L \subseteq X \text{ will be closed convex sets with interior points,} \\
\text{and } \varphi, \psi : X \to X \text{ will be orientation preserving congruences.}
\]

Further, we will need

\[
\varepsilon(x) > 0, \text{ or } \varepsilon(y) > 0, \text{ such that } K, \text{ or } L \text{ contains a ball of radius } \\
\varepsilon(x), \text{ or } \varepsilon(y), \text{ containing } x, \text{ or } y \text{ in its boundary, respectively.}
\]

and
Let, for each $x \in \text{bd} K$, or each $y \in \text{bd} L$, there exist an 
$\varepsilon(x) > 0$, or $\varepsilon(y) > 0$, such that the set of points of $K$, or $L$, 
lying at a distance at most $\varepsilon(x)$, or $\varepsilon(y)$, from $x$, or from $y$, 
is contained in a ball $B$ (for $X = S^d$, $\mathbb{R}^d$) 
or in a convex set $B$ bounded by a hypersphere (for $X = H^d$), 
with $\text{bd} B$ having sectional curvatures at most $\varepsilon(x)$, or $\varepsilon(y)$, 
and with $\text{bd} B$ containing $x$, or $y$, respectively.

Clearly $C^2$ implies (**), and $C^2_+$ implies (***) and (***) implies smoothness, 
and (****) implies strict convexity, respectively.

The following Theorem 7 will be the basis of our considerations for $d > 2$. 
Observe that in Theorem 7, (2), for $\mathbb{R}^d$ and $H^d$, hyperplanes cannot occur, by (****).

**Theorem 7.** Assume (** and (**)). For $X = \mathbb{R}^d$, $H^d$ assume also (**). Then 
the following statements are equivalent.

1. There exists some $\varepsilon > 0$, such that for each $\varphi, \psi$, for which 
   $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, and $\text{diam}((\varphi K) \cap (\psi L)) \leq \varepsilon$, we have that $(\varphi K) \cap (\psi L)$ is centrally symmetric.

2. The connected components of the boundaries of both $K$ and $L$ are congruent spheres (for $X = S^d$ of radius at most $\pi/2$), or paraspheres, or congruent hyperspheres.

**Theorem 8.** Assume (** and (**)). For $X = \mathbb{R}^d$, $H^d$ assume also (**). Then 
the following statements are equivalent.

1. For each $\varphi, \psi$, for which $\text{int}((\varphi K) \cap (\psi L)) \neq \emptyset$, we have that $(\varphi K) \cap (\psi L)$ is centrally symmetric.

2. $K$ and $L$ are two congruent balls, and, for $X = S^d$, their common radius is 
at most $\pi/2$.

**Remark.** Possibly Theorems 7 and 8 hold without any regularity assumption. For $\mathbb{R}^d$ (where $d \geq 2$), in [J-CM] we will give a proof, without any hypotheses, that 
(1) of Theorem 8 is equivalent to the following: $K, L$ are two congruent balls, or 
are two parallel slabs. This is a word for word generalization of Theorem 4, case 
of central symmetry. The methods of the proofs in this paper, and in [J-CM], are 
completely different.
In the proofs of our Theorems we will use some ideas of [H].

3. Preliminaries

For hyperbolic plane geometry we refer to [Ba], [Bo], [L], [P], for geometry of hyperbolic space we refer to [AVS], [C], and for elementary differential geometry we refer to [St].

We shortly recall some of the concepts to be used later. In \( S^2 \), \( H^2 \) there are the following (complete, connected, twice differentiable) curves of constant curvature (in \( S^2 \) meaning geodesic curvature). In \( S^2 \) these are the circles, of radii \( r \in (0, \pi/2] \), with (geodesic) curvature \( \cot r \in [0, \infty) \). In \( H^2 \), these are circles of radii \( r \in (0, \infty) \), with curvature \( \coth r \in (1, \infty) \), paracycles, with curvature 1, and hypercycles, i.e., distance lines, with distance \( l \) from their base lines (i.e., the straight lines that connect their points at infinity), with curvature \( \tanh l \in (0, 1) \), and straight lines, with curvature 0. Either in \( S^2 \), or in \( H^2 \) (and also in \( \mathbb{R}^2 \), where we have circles and straight lines), each sort of the above curves have different curvatures, and for one sort, with different \( r \) or \( l \), they also have different curvatures. The common name of these curves is, except for straight lines in \( \mathbb{R}^2 \) and \( H^2 \), cycles. In \( S^2 \) also a great circle is called a cycle, but when speaking about straight lines, for \( S^2 \) this will mean great circles. An elementary method for the calculation of these curvatures cf. in [V].

Sometimes we will include straight lines among the hypercycles. Then the base line of a straight line is meant to be itself.

The space \( H^d \) has two usual models, in the interior of the unit ball in \( \mathbb{R}^d \), namely the collinear (Caley-Klein) model, and the conformal (Poincaré) model. In analogy, we will speak about collinear and conformal models of \( S^d \) in \( \mathbb{R}^d \), meaning the ones obtained by central projection (from the centre), or by stereographic projection (from the north pole), to the tangent hyperplane of \( S^d \), at the south pole, in \( \mathbb{R}^{d+1} \). These exist of course only on the open southern half-sphere, or on \( S^d \) minus the north pole, respectively. Their images are \( \mathbb{R}^d \).

A paraball in \( H^d \) is a closed convex set bounded by a parabola.

The congruences of \( S^2 \), \( \mathbb{R}^2 \) and \( H^2 \) can be given as follows. The orientation preserving ones are rotations in \( S^2 \), rotations and translations in \( \mathbb{R}^2 \), and rotations, “rotations about an infinite point”, and translations along a straight line (preserving this line) in \( H^2 \). The orientation reversing ones are glide reflections in each of \( S^2 \), \( \mathbb{R}^2 \), and \( H^2 \).
If a non-empty closed convex set $K$ in $\mathbb{R}^2$ or $H^2$ admits a non-trivial translation, or a glide reflection that is not a reflection, as a congruence to itself, then $K$ contains the closed convex hull of the orbit of some point, w.r.t. the subgroup generated by this congruence. Thus, $K$ contains a straight line. If a non-empty closed convex set $K$ in $H^2$ admits a non-trivial rotation about an infinite point, then, by the analogous reasoning, $K$ contains a paracycle. In most cases in our proofs, these containments are impossible.

4. Proofs of our theorems

In the proofs of our theorems by the boundary components of a set we will mean the connected components of the boundary of that set. For $x_1, x_2 \in X$, we write $x_1 x_2$ for the distance of $x_1$ and $x_2$, and $[x_1, x_2]$ for the segment with these endpoints (supposing that $x_1, x_2$ are not antipodal points of $X = S^d$), and $\overline{x_1 x_2}$ for an arc of the boundary of a closed convex set with interior points, with these end-points, which convex set will be always specified.

Proof of Theorem 2. 1. We begin with the proof of the implication $(1) \implies (2)$.

2. We begin with showing that $(1)$ implies that both $K$ and $L$ are smooth. Then, this will imply, by convexity, that both of them are $C^1$.

In fact, suppose, e.g., that $K$ is not smooth. Let $x \in \text{bd} K$ be a point of non-smoothness. Let $\alpha \in (0, \pi)$ denote the angle of the positively oriented half-tangents of $K$ at $x$.

Let $y \in \text{bd} L$ arbitrary. Let $y'$ be a point of $\text{bd} L$ very close to $y$, that follows $y$ on $\text{bd} L$ in the positive sense. Consider the shorter, i.e., counterclockwise arc $\overline{yy'}$ of $\text{bd} L$. (If $\text{bd} L$ is homeomorphic to $S^1$; if the connected component of the boundary of $L$, containing $y$, is homeomorphic to $\mathbb{R}$, then there is just one such arc. Here, and also later, when writing shorter arc, we mean the shorter one in the first case, and the unique one in the second case.) This arc is almost like an arc in the Euclidean plane. In particular, its map in the conformal model of $S^2$ or $H^2$ is a very short arc (when $y$ is mapped to 0 in the model), which therefore has a total (geodesic) curvature almost 0. So, for each point of the relative interior of this arc the angle of the positively oriented half-tangents (in the conformal model, but then also in $S^2$ or $H^2$) is very small. The same statement holds for $\mathbb{R}^2$ as well.

Let $x', x'' \in \text{bd} K$ be points very close to $x$, such that the smaller, say, counterclockwise open arc $\overline{x'x''}$ contains $x$. Furthermore, we choose the points $x', x''$ so, that, additionally, for the ratio of the distances we have $xx' : xx'' = b : c$, where $b, c \in (0, \infty)$ satisfy, that a Euclidean triangle $T$ with one angle $\pi - \alpha$ and adjacent sides $b, c$ is not isosceles. Close to $x$ we have, that $\text{bd} K$ behaves almost like two (geodesic) segments. In particular, for $x', x''$ close enough to $x$
the circles with centre $x$ and with radii at most $xx' + xx''$ intersect $\text{bd} K$ just only in two points, and there transversally, so, that $x$ is on the smaller arc of $\text{bd} K$ determined by these two points. In fact, for points of $\text{bd} K$ in a certain neighbourhood of $x$ this follows by the differential geometric behaviour of $\text{bd} K$, while for the points of $\text{bd} K$, not in a certain neighbourhood of $x$ this follows by compactness of an anyhow long arc, with two end-points, of $\text{bd} K$ (that is embedded homeomorphically in $S^2$, $\mathbb{R}^2$, or $H^2$), or by the fact that, outside a very long arc, with two end-points, of $\text{bd} K$, the points of $\text{bd} K$ are approaching infinity (only for $\mathbb{R}^2$ and $H^2$, and then only if the connected component of $\text{bd} K$ in question is homeomorphic to $\mathbb{R}$).

The analogous statement holds also for $\text{bd} L$, with centre of circle $y$, and radius of circle at most $yy'$, for $y'$ sufficiently close to $y$.

Choosing $x'x'' = yy'$, there exist orientation preserving congruences $\varphi$ and $\psi$, such that $\varphi(x') = \psi(y')$, and $\varphi(x'') = \psi(y)$, and $(\varphi K) \cap (\psi L)$ is bounded by the shorter arcs $\varphi(x')\psi(x'')$ of $\text{bd} (\varphi K)$ and $\psi(y)\psi(y')$ of $\text{bd} (\psi L)$. Thus, this intersection is almost like a Euclidean triangle, with inner angle at $\varphi(x)$ equal to $\pi - \alpha$, hence at the other two vertices $\varphi(x') = \psi(y')$ and $\varphi(x'') = \psi(y)$ the angles between the positively oriented half-tangents are at least about $\pi - \alpha$. In particular, this intersection has a non-empty interior, hence has a non-trivial symmetry. There are just three points on the boundary of this intersection, where the angles of the positively oriented half-tangents are at least about $\min\{\pi - \alpha, \alpha\}$, at all other points these angles are about $0$.

Then, the set $(\varphi K) \cap (\psi L)$ is almost like a Euclidean triangle, that is not isosceles. So, for a sufficiently small distance $x'x'' = yy'$, this set has an arbitrarily small diameter, and cannot have any non-trivial symmetry. This is a contradiction, showing, that both $K$ and $L$ are $C^1$.

3. Now, we begin the proof of the fact, that all boundary components both of $K$ and $L$ are either cycles, or straight lines.

By compactness, and $C^1$, on any compact arc of the boundary of $K$, or $L$, the italicized statement from 2 holds uniformly at the points of the compact arc (i.e., $x$ lying in the compact arc), for the values of the radius at most some $\varepsilon > 0$. Let us consider two connected boundary components $K'$, or $L'$, of $K$, or $L$, respectively. Let $K''$, or $L''$ be some compact arc of $K'$, or $L'$, respectively, provided $K'$, or $L'$ is homeomorphic to $\mathbb{R}$ (and then necessarily tends to infinity in both directions, for $\mathbb{R}^2$ and $H^2$). For $K'$ or $L'$ compact, i.e., when it is homeomorphic to $S^1$, and when necessarily $K' = \text{bd} K$, or $L' = \text{bd} L$, we choose $K''$, or $L''$ equal to $\text{bd} K$, or $\text{bd} L$, respectively.
Let $\varepsilon > 0$ be sufficiently small. Then, we may assume that the italicized statement from 2 holds uniformly on $K''$, or $L''$ (i.e., for $x$ in $K''$, or $L''$), respectively, for all radius values at most $\varepsilon$. Let $[x_1, x_2]$, or $[y_1, y_2]$ be a chord of $K$ or $L$, respectively, of length $\varepsilon$, where $x_2$ follows $x_1$ on $\text{bd} K$ in the positive sense, and $y_2$ follows $y_1$ on $\text{bd} L$ in the negative sense. Let $x_1, x_2 \in K'$, and $y_1, y_2 \in L'$, with at least one of $x_1, x_2$ belonging to the relative interior (w.r.t. $K'$) of $K''$, and at least one of $y_1, y_2$ belonging to the relative interior (w.r.t. $L'$) of $L''$. Let us choose $\varphi$ and $\psi$ so, that $\varphi(x_i) = \psi(y_i)$ ($i = 1, 2$).

First suppose, that not both shorter arcs $\overline{x_1x_2}$ and $\overline{y_1y_2}$ are equal to the corresponding chord. Then, $(\varphi K) \cap (\psi L)$ is bounded by the shorter arcs $\varphi(\overline{x_1x_2})$ and $\psi(\overline{y_1y_2})$. By the hypothesis about the arcs, this intersection has a non-empty interior, hence has a non-trivial symmetry. Observe, that this intersection has just two points of non-smoothness, namely $\varphi(x_1) = \psi(y_1)$ and $\varphi(x_2) = \psi(y_2)$. Thus, any non-trivial symmetry of $(\varphi K) \cap (\psi L)$ is a central symmetry, with centre the midpoint of the segment joining these two non-smooth points, or is an axial symmetry, either with axis passing through these two points, or with axis the perpendicular bisector of the segment with endpoints these two non-smooth points.

Now, consider the case, that both above arcs are equal to the corresponding chord, that has length $\varepsilon$. Then, $(\varphi K) \cap (\psi L)$ may strictly contain this chord, thus, in particular, its diameter may be not small. In this case, therefore, we will consider, rather than this intersection, this common chord, as a degenerate closed convex set (i.e., with empty interior). Observe, that this common chord (in general not equal to $(\varphi K) \cap (\psi L)$) has an arbitrarily small diameter, and has all three above mentioned non-trivial symmetries.

In both cases, the intersection (in the first case above), or the chord (in the second case above), has an arbitrarily small diameter, and has (at least) one of the above mentioned non-trivial symmetries. We will say, that the direction of the straight line joining the two points $\varphi(x_i) = \psi(y_i)$ (for $i = 1, 2$) is vertical, and their perpendicular bisector is horizontal.

4. We begin with the case, when for some sequence $\varepsilon_n \to 0$, where each $\varepsilon_n$ is sufficiently small, we have the following. Either $K'$, or $L'$ has a chord $[x_1, x_2]$, or $[y_1, y_2]$, with $x_2$ following $x_1$ in the positive sense, or $y_2$ following $y_1$ in the negative sense, and with at least one endpoint in the relative interior (w.r.t. $K'$, or $L'$) of $K''$, or $L''$, such that the following holds. The chord $[x_1, x_2]$, or $[y_1, y_2]$ is of length $\varepsilon_n$, and the smaller arc determined by this chord, either on $K'$, or on $L'$, is not symmetrical to the perpendicular halving straight line of the chord (in particular, the respective smaller arc is different from the chord).
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Then, for these $\varepsilon_n$’s, we have the following. Let $[x_1, x_2]$, or $[y_1, y_2]$ be a chord of $K'$, or of $L'$, with at least one endpoint in the relative interior (w.r.t. $K'$, or $L'$) of $K''$, or $L''$, and of length $\varepsilon_n$, with $x_2$ following $x_1$ on $\partial K$ in the positive sense, or $y_2$ following $y_1$ on $\partial L$ in the negative sense, respectively. Let $\varphi$ and $\psi$ be chosen so, that $\varphi(x_i) = \psi(y_i)$ (for $i = 1, 2$), and $(\varphi K) \cap (\psi L)$ is bounded by the shorter arcs $\varphi(x_1)\varphi(x_2)$ and $\psi(y_1)\psi(y_2)$. (Observe that, since at least one of the arcs $\widetilde{x_1x_2}$ and $\widetilde{y_1y_2}$ is different from the respective chord, the case that $(\varphi K) \cap (\psi L)$ strictly contains this chord, and thus is degenerate, cannot occur.) Then, the intersection $(\varphi K) \cap (\psi L)$ has a non-empty interior, and has an arbitrarily small diameter. Hence it has some non-trivial symmetry, which cannot be a symmetry w.r.t. the horizontal axis. That is, this symmetry is a central symmetry, or is an axial symmetry with respect to the vertical axis.

Observe, that both central symmetry, and axial symmetry with respect to the vertical axis, cannot occur. Namely, then we would have also an axial symmetry w.r.t. the horizontal axis, that has already been excluded.

In the case of central symmetry the two (smaller) arcs $\widetilde{x_1x_2}$ of $\partial K$ and $\widetilde{y_1y_2}$ of $\partial L$, respectively, are congruent, with $x_1$ corresponding to $y_2$, and $x_2$ corresponding to $y_1$. In case of axial symmetry w.r.t. the vertical axis, once more the above arcs are congruent, but now with $x_1$ corresponding to $y_1$, and $x_2$ corresponding to $y_2$.

We will consider the one-sided curvatures, provided they exist, of $K''$ at $x_i$, in the sense towards $x_{2-i}$, and similarly, of $L''$ at $y_j$, in the sense towards $y_{2-j}$, where $x_i$ is in the relative interior of $K''$ (w.r.t. $K'$), and $y_j$ is in the relative interior of $L''$ (w.r.t. $L'$). For both considered symmetries, the above considered two one-sided curvatures exist and are equal at the corresponding points, or they both do not exist at the corresponding points.

Now recall, that any of $x_1, x_2,$ or $y_1, y_2$ could be any relative interior point of $K''$, or $L''$, respectively.

First suppose the case that, for all choices of $x_1, x_2, y_1, y_2$, we have central symmetry. Then $\varphi(x_1)$ corresponds by the symmetry to $\psi(y_2)$. Recall that $x_1, y_2$ could be any relative interior points of $K''$ and $L''$. Then, for all relative interior points of $K''$ and $L''$, the considered one-sided curvatures exist and are equal, or they do not exist for any points. However, convex curves — and surfaces — are almost everywhere twice differentiable (more exactly, the functions having, locally, in a suitable coordinate system, these graphs, have Taylor series expansions, of second degree, with error term $o(\|x\|^2)$; cf. [Sch], pp. 31-32, for $\mathbb{R}^d$, that extends to $S^d$ and $H^d$ by using the collinear models; observe, that we already know, that both $\partial K$ and $\partial L$ are $C^1$, that simplifies the condition in [Sch]).
This rules out the second case. Now, replacing $x_1, y_2$ by $x_2, y_1$, we obtain the same for one-sided curvatures, but now in the opposite sense. Therefore, at all relative interior points of $K''$ and $L''$, the curvatures exist and are equal.

Second suppose the case that, for all choices of $x_1, x_2, y_1, y_2$, we have axial symmetry, with respect to the vertical axis. Then $\varphi(x_1)$ corresponds by the symmetry to $\psi(y_1)$. Now, $x_1, y_1$ could be any relative interior points of $K''$ and $L''$. Then, with this notational change, we repeat the arguments of the preceding paragraph, and gain that, at all relative interior points of $K''$ and $L''$, the curvatures exist and are equal.

As third case, there remains the case that, for some choice of $x_1, x_2, y_1, y_2$ we have central symmetry, and for some other choice of these points we have axial symmetry, with respect to the vertical axis. Now, take into consideration, that an arc, or $S^1$, is a connected topological space, and thus products of arcs, or $S^1$'s, are connected topological spaces as well. Clearly, the configurations of the points $x_1, x_2, y_1, y_2$ in $K' \times K' \times L' \times L'$ (with $x_2$ following $x_1$ in the positive sense, and $y_2$ following $y_1$ in the negative sense), where still we suppose, that one of $x_1, x_2$ belongs to the relative interior of $K''$, and one of $y_1, y_2$ belongs to the relative interior of $L''$ (and, of course, still $x_1 x_2 = y_1 y_2 = \varepsilon_n$), is a connected topological space as well. Moreover, the set of configurations of the points $x_1, x_2, y_1, y_2$, for which one of the considered symmetry properties holds, is a closed subset. Further, the union of these two closed subsets is the entire space of all above configurations of the points $x_1, x_2, y_1, y_2$. By connectedness, these two closed subsets must intersect. That is, we must have a configuration, that simultaneously possesses both the central symmetry, and the axial symmetry w.r.t. the vertical axis. This, however, contradicts the second paragraph of 4.

So, the third case cannot occur. Therefore, we must have the first, or second case. Both had the conclusion that, at all relative interior points of $K''$ and $L''$, the curvatures exist and are equal. In other words, both $K''$ and $L''$ have equal constant curvatures, i.e., both are arcs of congruent cycles (including entire compact cycles, i.e., circles), or are segments.

Since $K''$, or $L''$ were arbitrary compact subarcs of $K'$, or $L'$, if $K'$, or $L'$ were homeomorphic to $\mathbb{R}$ (and they were equal to $K' = \text{bd } K$, or $L' = \text{bd } L$, if $K'$, or $L'$ was homeomorphic to $S^1$), we have that, in both cases, $K'$ and $L'$ are congruent cycles, or are straight lines.

Recall, that at the beginning of 4 we have considered the case, that the chord $[x_1, x_2]$, or $[y_1, y_2]$, respectively, is of length $\varepsilon_n$, and the smaller arc determined by this chord, either on $K'$, or on $L'$, is not symmetrical to the perpendicular halving straight line of the chord.
However, this contradicts the fact, that $K'$ and $L'$ are congruent cycles, or are straight lines. Hence, we have obtained a contradiction. Therefore, the case considered at the beginning of 4 cannot occur.

5. Thus, there remains the case that, for each sufficiently small $\varepsilon > 0$, both for $K''$ and $L''$, we have that all smaller arcs of $K''$ and $L''$, having corresponding chords of length $\varepsilon$, hence having arbitrarily small diameters, are symmetrical to the perpendicular halving straight line of the chord. Observe, that this axis of symmetry halves the smaller arc, and is perpendicular to it at its midpoint.

Now, let $x', x''$ belong to the relative interior (w.r.t. $K'$) of $K''$. Then, there exist $x' = x_1, \ldots, x_n = x''$ in the relative interior (with respect to $K'$) of $K''$, following each other in the same sense, and such, that the distance of $x_i$ and $x_{i+1}$ is less than $\varepsilon$ (for $i = 1, \ldots, n-1$). Then, $x_i$ and $x_{i+1}$ are symmetrical to each other, with respect to the perpendicular bisector of the chord $[x_i, x_{i+1}]$. Then $x_i$ and $x_{i+1}$ are symmetrical also w.r.t. the perpendicular bisector of some other chord, for which the corresponding shorter arc $I'$ contains the closed shorter arc $I = \widehat{x_i x_{i+1}}$ in its relative interior, $I'$ being only slightly larger than $I$ (and the two arcs have the same midpoint). In particular, either $K''$ has equal curvatures at $x_i, x_{i+1}$, or does not have a curvature at these points. Hence, for $x', x''$, we have that $K''$ has equal curvatures at $x', x''$, or does not have a curvature at these points. However, convex curves have a curvature at almost all of their points ([Sch], pp. 31-32, cited in detail in 4 of this proof). Hence the second alternative cannot hold, i.e., $K''$ has a constant curvature at each of its relative interior points.

Since $K''$ was any compact arc of $K'$ (and was equal to $K' = \text{bd} K$, if $K'$ was homeomorphic to $S^d$), we have that $K'$ is a $C^2$ curve of constant curvature, i.e., a cycle, or a straight line. A similar conclusion holds for $L'$. This proves the implication $(1) \implies (2)$, that is the first statement of our theorem.

6. The particular case of (1), with central symmetry, follows easily. Let $\varphi K$ and $\psi L$ touch each other, and push them slightly towards each other. Then central symmetry of the new intersection implies equality of the curvatures of the two originally touching boundary curves.

7. We turn to the third statement, i.e., to the investigation of the implication $(2) \implies (1)$.

For $X = S^2$, $(2)$ clearly implies $(1)$.

For $X = \mathbb{R}^2$, we take in consideration the following. The closed convex sets in $\mathbb{R}^2$, whose boundaries are disconnected, are just the parallel strips. Furthermore, the closed convex sets in $\mathbb{R}^2$, with connected boundaries, whose boundaries are cycles or straight lines, are just circles or half-planes, respectively. Thus, any of $K$ and $L$ can be a circle, a parallel strip, or a half-plane.
If, e.g., $K$ is a circle, or both $K$ and $L$ are halfplanes, then $(\varphi K) \cap (\psi L)$ is axially symmetric. If both $K$ and $L$ are parallel strips, then $(\varphi K) \cap (\psi L)$ is centrally symmetric. If, e.g., $K$ is a half-plane and $L$ is a parallel strip, then, if $(\varphi K) \cap (\psi L)$ has a non-empty interior, then it is unbounded. Thus (1) is satisfied in each case.

Let $X = H^2$. Both $K$ and $L$ is either a circle, or a paracycle, or has boundary components which are hypercycles or straight lines. The infimum of the positive curvatures of the boundary components is of course the same infimum, taken only for the hypercycle components (if there is one). Let us first suppose that the infimum of the positive curvatures of the hypercycle boundary components (if there is one) of both of $K$ and $L$ is positive, and both for $K$ and $L$ there is at most one 0 curvature. That is, the distances, for which these hypercycles are distance lines, have an infimum $c > 0$, say, and there may be still at most one straight line component, both for $K$ and $L$.

Let, e.g., $K_1$ and $K_2$ be two boundary components of $K$, and let $x_1 \in K_1$, and $x_2 \in K_2$. Let $K'$ and $L'$ be defined, as the non-empty closed convex sets (possibly with empty interiors), bounded by all the straight lines for which the boundary components are distance lines, and by the at most one straight line component. In particular, $K_1$ and $K_2$ are distance lines for $K'_1$ and $K'_2$, with a non-negative distance. Then the segment $[x_1, x_2]$ intersects both $K'_1$ and $K'_2$, at points $x'_1, x'_2$, and for the distances we have $x_1 x_2 \geq x_1 x'_1 + x'_2 x_2 \geq c$. This means that the distances of the different boundary components both of $K$, and of $L$, are bounded from below by $c$. The same holds vacuously for circles and paracycles. Hence, if $\text{diam} \left[(\varphi K) \cap (\psi L)\right] < c$, then $(\varphi K) \cap (\psi L)$ is compact, and is bounded by portions of only one boundary component of $\varphi K$, and of $\psi L$.

Thus $(\varphi K) \cap (\psi L)$ is the intersection of two sets, both being a circle, a paracycle, or a convex domain bounded by a hypercycle, including a half-plane. Observe that a circle, and a paracycle are axially symmetric w.r.t. any straight line passing through their centres. Thus, if both above sets are a circle or a paracycle, then their intersection is axially symmetric. There remain the cases when one set is a convex set bounded by a hypercycle, and the other one is a circle, a paracycle, or a convex set bounded by a hypercycle. In the first case an axis of symmetry of the intersection is a straight line passing through the centre of the circle, and orthogonal to the base line of the hypercycle. In the second case, by compactness of the intersection, the centre of the paracycle cannot lie at an endpoint of the base line. Therefore an axis of symmetry of the intersection is a straight line passing through the centre of the paracycle, and orthogonal to the base line of the hypercycle.
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In the third case, again by compactness of the intersection, the base lines of the hypercycles are not intersecting and not parallel. Therefore, the unique straight line orthogonal to both of them is an axis of symmetry.

Now we turn to the case when the infimum of the positive curvatures of the boundary components of $K$ is 0, or there are at least two 0 curvatures (and the set of these curvatures, with multiplicity, is prescribed).

We begin with an example, where both $\text{bd} K$ and $\text{bd} L$ consists of two straight lines. We consider the collinear model. Let $l_1, l_2 \subset H^2$ be parallel (but distinct) straight lines, with axis of symmetry $l$. Let $x_i, y_i \in l_i$ be points symmetric w.r.t. $l$, with all six pairwise distances at most $\varepsilon$. Then $x_1x_2y_2y_1$ is a symmetrical quadrangle of arbitrarily small diameter, and is the intersection of the convex sets $K$, bounded by $l_1, l_2$, and $L$, bounded by the straight lines $x_1x_2, y_1y_2$. A small generic perturbation of this quadrangle, preserving the relations $x_i, y_i \in l_i$, will have no non-trivial symmetry, will preserve $K$, and will perturb $L$ to a convex set bounded by two non-intersecting and non-parallel straight lines.

If the set (with multiplicity) of the positive curvatures of the connected hypercycle components $K_i$ of $K$ is prescribed, and has infimum $c = 0$, or there are at least two 0 curvatures, then we make the following modification of the above example. These hypercycles are distance lines, for distances $c_i$. We consider a closed convex set $K'$, bounded in the collinear model by at most countably infinitely many chords of the model circle, one for each $i$, so that with at most one exception, these chords occur in pairs having one common endpoint (possibly the set of these pairs is empty). Then we replace these chords by the corresponding distance lines, outwards from $K'$. If there are two 0 curvatures, then the corresponding chords should occur in a pair, and if $c = 0$, then there should be pairs for which both distances $c_i$ are arbitrarily small. In the first case just copy the above construction. In the second case we have that the hypercycles are arbitrarily close to their base lines, and then we have two points on both of these hypercycles, which form a convex quadrangle, with arbitrarily small diameter, which has generically no non-trivial symmetry.

Proof of Theorem 3. The implications $(3) \implies (2) \implies (1)$ are evident. Last, $(1) \implies (3)$ follows from Theorem 2. The particular case with central symmetries in $(1)$ follows immediately.

Proof of Theorem 4. The implication $(2) \implies (1)$ is evident.

For the implication $(1) \implies (2)$ we apply Theorem 2, taking in consideration the following. In 7 of the proof of Theorem 2 we have seen that any of $K$ and $L$ can be a circle, a parallel strip, or a half-plane, and with the exception
of the case that, e.g., $K$ is a half-plane and $L$ is a parallel strip, that \((\varphi K) \cap (\psi L)\) has some non-trivial symmetry. However, the case that, e.g., $K$ is a parallel strip and $L$ is a half-plane, contradicts (1). Thus (1) \(\implies\) (2) holds.

The two particular cases, with central, or axial symmetries in (1), follow by easy discussions. ■

Proof of Theorem 5. 1. The implication (2) \(\implies\) (1) is evident, so we turn to the proof of (1) \(\implies\) (2).

2. Observe that (1) of Theorem 5 implies (1) of Theorem 2, and (1) of Theorem 2 implies, by Theorem 2, that the connected components of the boundaries both of $K$ and $L$ are congruent cycles or straight lines.

3. From 2 we have, that $K$ and $L$ are two congruent circles, two paracycles, or all their boundary components are either congruent hypercycles, or straight lines. However, in the case of straight lines, their total number is finite, by the hypothesis of the theorem.

The case, that $K$ and $L$ are paracycles is clearly impossible. Namely, we may choose $\varphi$ and $\psi$ so, that $\varphi K = \psi L$, and then their intersection is a paracycle. However, this has exactly one point at infinity, hence is not centrally symmetric.

We are going to show, that also the case of (finitely many) straight lines, and the case of hypercycles is impossible.

4. First we deal with the case, when each boundary component both of $K$ and $L$ are straight lines, when, by hypothesis, their total number is finite.

Now, it will be convenient to use the collinear model for $H^2$. Then, in this model, both $K$ and $L$ will be bounded by finitely many non-intersecting chords of the boundary circle of the model. Possibly we have chords with common endpoints. Let $K_1$, or $L_1$ be some connected component of $\text{bd } K$, or $\text{bd } L$, respectively. We may choose $\varphi$ and $\psi$ so, that $\varphi K_1 = \psi L_1 = (\varphi K) \cap (\psi L)$, and this set contains the centre of the model. Thus, $\varphi K$ and $\psi L$ lie on opposite sides of this straight line. Let us change $\varphi$ and $\psi$ a bit, so that in the model $\varphi K$ and $\psi L$ rotate a very little bit about the centre of the model. We will not use new notations for the new orientation preserving congruences, but will retain the old ones $\varphi$ and $\psi$. Let the closure — taken in the model with its boundary circle — of the intersection $C$ of the half-circles, bounded by $\varphi K_1$, or $\psi L_1$, and containing $\varphi K$, or $\psi L$, in their new positions, respectively, satisfy the following. It does not contain any end-point of any chord, which in the model represents some boundary component of $\varphi K$ or $\psi L$, except of course one end-point of $\varphi K_1$, and one end-point of $\psi L_1$. 
This can be attained, and implies the following. The set $C$ does not intersect any other boundary components of $\varphi K$, or $\psi L$, than those, which satisfy the following properties 1) and 2):

1) They are in the collinear model chords of the model circle with one common end-point with the chords $\varphi K_1$, or $\psi L_1$, respectively, and moreover this/these common end-point/s lie in $C$ (i.e., is/are end-point/s of the circular arc corresponding to $C$).

2) From this/these connected component/s of the boundaries only a half-line is in $C$.

Then, $(\varphi K) \cap (\psi L)$ is, in the collinear model, a sector of the model circle, a triangle, with two sides parallel in $H^2$, and having two finite vertices, or a quadrangle, with opposite sides parallel in $H^2$. The first case gives a set having exactly one non-smooth boundary point. If it were centrally symmetric, this boundary point would be the center of symmetry, which is a contradiction. In the second case we have a set having exactly one point at infinity, hence it is not centrally symmetric, which is a contradiction. In the third case, if there would be a centre of symmetry, that would be an inner point of our set. Then one side and its centrally symmetric image would span straight lines, which are not intersecting, and not parallel. However, any two sides of this quadrangle are either intersecting, or parallel. So we have a contradiction in each of the three cases.

This ends the proof of impossibility of the case, when all (finitely many) boundary components are straight lines.

5. There remained the case, when all connected components of the boundaries of both $K$ and $L$ are congruent hypercycles. Both for $K$ and $L$, there is at least one such component, since $K, L \subseteq H^2$. Denote by $l$ the common value of the distance, for which these hypercycles are distance lines of their base lines.

Now, it will be convenient to consider the collinear model. Replace the image by $\varphi$, or by $\psi$, of each above hypercycle, for $K$, or $L$, by its base line, respectively. These will bound closed convex sets $K_0$ and $L_0$ (possibly without interior points), not containing any of the image hypercycles. The parallel domain of $K_0$, or $L_0$, with distance $l$, contains $\varphi K$, or $\psi L$, respectively. However, also these parallel domains are contained in $\varphi K$, or $\psi L$, respectively. Namely, if $x \in K_0$, and the distance of a point $z \notin K_0$ from $x$ is at most $l$, then the segment $[x, z]$ intersects some boundary component of $K_0$, say, in a point $x'$. Then, the distance of $z$ from $x'$ is at most $l$, hence $z$ lies in $\varphi K$. 
Let $K_{0,1}$ or $L_{0,1}$ denote some boundary component of $K_0$, or $L_0$, respectively. Let us suppose, that $\varphi K$ and $\psi L$ are in such a position, that one end-point of $K_{0,1}$ and one end-point of $L_{0,1}$ coincide, their other endpoints are different, and the interiors of $K_0$ or $L_0$ (if not empty), lie on the opposite side of $K_{0,1}$ or $L_{0,1}$, as $L_{0,1}$ or $K_{0,1}$, respectively. (This can be attained by applying some orientation preserving congruences.) Let $K_1$, or $L_1$ denote the boundary component of $\varphi K$, or $\psi L$, whose base line is $K_{0,1}$, or $L_{0,1}$, respectively (if there are two such ones, the one that lies on the same side of $K_{0,1}$ or $L_{0,1}$, as $L_{0,1}$ or $K_{0,1}$, respectively). Let us consider the intersection $M$ of the closed convex sets bounded by $K_1$ and $L_1$ (which evidently contain $\varphi K$, or $\psi L$, respectively). This is bounded by some arcs of $K_1$ and $L_1$, having one common infinite endpoint. We have $(\varphi K) \cap (\psi L) \subset M$. We are going to show, that also $M \subset (\varphi K) \cap (\psi L)$.

It will suffice to show $M \subset \varphi K$, or, in other words, that $M$ lies in the parallel domain of $K_0$ with distance $l$ (the other inclusion is proved analogously). The straight line $K_{0,1}$ cuts $H^2$ in two half-planes. In the half-plane containing $L_{0,1}$, a point belonging to $M$ clearly belongs to $\varphi K$. In the other half-plane, a point $p$ belonging to $M$ satisfies that the distance of $p$ and $L_{0,1}$ is at most $l$, hence the distance of $p$ and $K_{0,1}$ is at most $l$, hence the distance of $p$ and $K_0$ is at most $l$, as well, as was to be shown.

Thus, we have $(\varphi K) \cap (\psi L) = M$. The set $M$ has just one point at infinity, which implies that it cannot be centrally symmetric.

This ends the proof of impossibility of the case, when all boundary components are hypercycles.

Before passing to the proof of Theorem 6, we introduce some terminology. If we have a topological space, $Y$, say, then we say that some property of a point $y \in Y$ holds generically, if it holds outside a nowhere dense closed subset.

If $Y$ happens to be a connected (real) analytic manifold, and $f, g : Y \rightarrow \mathbb{R}$ are analytic functions, then either $f$ and $g$ coincide, or else they cannot coincide on any non-empty open subset (this is the principle of analytic continuation). Otherwise said, in the second case, generically, for $y \in Y$, we have $f(y) \neq g(y)$.

Observe that a finite union of nowhere dense closed subsets is itself nowhere dense and closed. In 6 A of the proof of Theorem 6 we will have the following situation. On a connected (real) analytic manifold (in fact, on $H^2$) there are finitely many, pairwise different analytic functions, $f_1, \ldots, f_n : Y \rightarrow \mathbb{R}$, say. Then generically, for $y \in Y$, we have that $f_1(y), \ldots, f_n(y)$ are all different.
Before the proof we show a trigonometrical type formula in $H^2$. It is in a sense an analogue of the law of cosines for an angle of a triangle in $H^2$. Namely, the law of cosines allows us, for two circles, or radii $r, R$, and distance of centres $c$, to determine the half central angle of the arc of the circle of radius $r$, lying in the circle of radius $R$. We will need an analogous formula, for a circle of radius $r$, and a hypercycle, with distance $l$ from its base line, for the half central angle of the arc of the circle of radius $r$, lying in the convex domain bounded by the hypercycle, when the distance $c$ of the centre of the circle and the base line of the hypercycle is given. We consider $c$ and $l$ as signed distances. We admit degeneration to a straight line, i.e., we admit $l = 0$: then we choose one of the half-planes bounded by this straight line (cf. below). This formula is surely known, but we could not find an explicit reference. Therefore we sketch its simple proof.

So, let us consider a hypercycle, with distance $l$ from its base line. Moreover, let us consider a circle of radius $r$, whose centre $O$ lies at a distance $c \geq 0$ from the base line of the hypercycle. Correspondingly, later we will consider $l$ as a signed distance, with positive sign determined so that we should have $c \geq 0$ (for $c = 0$ we choose the sign some way). We want to determine the half-angle of the arc of the circle, lying in the convex domain bounded by the hypercycle (if the hypercycle degenerates to a straight line, then we take the half-plane bounded by it that consists of the points with non-positive signed distance to this straight line). For $l$ the signed distance, we will mean our question as the determination of the half-angle of the arc of the circle, lying in the set given by \( \{ x \in H^2 \mid \text{dist}(x, B) \leq l \} \), where dist is signed distance, and $B$ is the base line of the hypercycle.

Clearly, the intersection is non-empty if and only if \(|c - l| \leq r\). At deriving our formula (** we assume \(|c - l| \leq r\).

The conformal model shows that the circle and the hypercycle have either two common points, or they are tangent to each other, or they are disjoint (their images are a circle, and a circular arc or segment that cuts the model into two connected parts).

Let $C$ be one of the common points of the circle and the hypercycle, and let $A$ and $B$ be the orthogonal projections of $O$ and $C$ to the base line of the hypercycle (thus $BC = l$). We let $d := AC$. So we have to determine the angle $\omega = \angle COA$ (for $O$ lying on the base-line we define $\omega$ by the evident limit procedure).
By the law of cosines we have
\[ \cosh d = \cosh r \cdot \cosh c - \sinh r \cdot \sinh c \cdot \cos \omega. \]

Now we calculate the angle \( \alpha := \angle OAC \) (for \( O \) lying on the base-line defined as a limit). Preliminarily let us suppose \( l \neq 0 \), that implies \( d \neq 0 \). By the law of sines we have
\[ \sin^2 \alpha = \sin^2 \omega \cdot \sinh^2 r / \sinh^2 d. \]

Last, from the right triangle \( ACB \) we have
\[ \sinh^2 (BC) = \sin^2 (\pi / 2 - \alpha) \cdot \sinh^2 d. \]

So, fixing \( r, c \), and supposing \( \cos \omega \) as given, we determine, by substitutions, successively, first \( \cosh d \), then \( \sin^2 \alpha \), then \( \sinh^2 (BC) \). This last expression should equal \( \sinh^2 l \). Solving this last equation for \( \cos \omega \) (which is a quadratic equation), we obtain, by rearranging,
\[
(*) \quad \begin{cases} \pm \sinh l = \cosh r \cdot \sinh c - \sinh r \cdot \cosh c \cdot \cos \omega \\ = \cosh r \cdot \cosh c \cdot (\tanh c - \tanh r \cdot \cos \omega). \end{cases}
\]

We will show that here in fact we have
\[
(**) \quad \sinh l = \cosh r \cdot \sinh c - \sinh r \cdot \cosh c \cdot \cos \omega.
\]

Recall that \( |c - l| \leq r \) is assumed.

In (*) the expression in the middle lies in \([\sinh (c - r), \sinh (c + r)]\). So, for \( 0 \leq r \leq c \), it is non-negative, and, since the signed distance \( l \) was taken to be positive, so that we should have \( c \geq 0 \), therefore here the first expression must be \( \sinh l \), i.e., we have (**)\). Now let \( r > c \geq 0 \). Then the boundary of our circle intersects the base line in two points. The case \( l = 0 \) corresponds to a well known formula for a right triangle in \( H^2 \): it is equivalent to \( \tanh c = \tanh r \cdot \cos \omega \). In particular, (*) and (**) are valid for \( l = 0 \) as well. Let us increase \( \omega \), and thus the middle expression of (*). Then the signed distance of the end-point of the radius of our circle, enclosing an angle \( \omega \) with the radius of our circle orthogonally intersecting the base line (for \( O \) on the base line this is meant as a limit), increases. This corresponds to the fact that we have \( \sinh l \) in the first expression in (*), i.e., we have (**). Last we extend the validity of (**) to \( c < 0 \). Let us apply (**) to \(-c, -l, \pi - \omega\) rather than \( c, l, \omega \). Then the validity of (**) for these values implies its validity for \( c, l, \omega \).
Later, in the proof of Theorem 6, we will consider the case when \( l \geq 0 \); then, of course, \( c \) varies in \( \mathbb{R} \).

Observe that (***) implies the necessary and sufficient condition for the existence of a point of intersection, namely \(|c - l| \leq r\). In fact, the right hand side of (***), \( \sinh(c - r), \sinh(c + r) \).

From this there follows the converse implication. Namely: if (***) is satisfied, then \(|c - l| \leq r\), and there exist 1) a hypercycle, having a signed distance \( l \) from its base line \( B \), and 2) a circle of radius \( r \), that has a centre at a distance \( c \) from the base line of the hypercycle, such that 3) the circle intersects \( \{x \in H^2 \mid \text{dist}(x, B) \leq l\} \) in a circular arc of half central angle \( \omega \).

**Proof of Theorem 6.**

1. The implication (2) \( \Rightarrow \) (1) is trivial.
2. We continue with the proof of (3) \( \Rightarrow \) (2).

We begin with case (A).

A circle is axially symmetric w.r.t. a straight line spanned by any of its diameters. A paracycle is axially symmetric with respect to any straight line passing through its centre (its point at infinity), and a convex domain bounded by a hypercycle, or a half-plane is axially symmetric w.r.t. any straight line, that intersects its base line, or its boundary, orthogonally, respectively. These imply, that if any of \( \varphi K \) and \( \psi L \) is either a circle or a paracycle, then their intersection is axially symmetric w.r.t. (any) straight line joining their centres. If one of \( \varphi K \) and \( \psi L \) is a circle, and the other one is a convex set bounded by a hypercycle, or is a half-plane, then the straight line passing through the centre of the circle, and orthogonal to the base line of the hypercycle, or to the boundary of the half-plane, is an axis of symmetry of the intersection.

Last, let \( \varphi K \) and \( \psi L \) be congruent convex sets, both bounded by hypercycles, or let them be two half-planes. Consider the base lines of these hypercycles, or the boundaries of these half-planes, respectively. There are four cases. These lines

a) may coincide; or
b) may intersect; or

Case a) is evident. In case b), \( \varphi K \neq \psi L \), and \( \text{bd}(\varphi K) \) and \( \text{bd}(\psi L) \) intersect transversally at some point \( p \) (for this use the conformal model). Then, \( (\varphi K) \cap (\psi L) \) has an inner angle at \( p \), of measure less than \( \pi \), and the halving straight line of this angle is an axis of symmetry of \( (\varphi K) \cap (\psi L) \). In case c), if one of \( \varphi K \) and \( \psi L \) contains the other, the intersection is evidently axially symmetric.
Otherwise, the symmetry axis of the base lines is an axis of symmetry of the intersection. In case d), we consider the pair of points on the base lines, realizing the distance of these lines. The straight line connecting these points is orthogonal to both lines, and is an axis of symmetry of \((\varphi K) \cap (\psi L)\).

We continue with case (B). If \(K\) is a circle, and \(\psi L\) is bounded by two hypercycles, whose base lines coincide (one of them possibly degenerating to a straight line), then the straight line passing through the centre of \(\varphi K\), and orthogonal to the above base line, is an axis of symmetry of \((\varphi K) \cap (\psi L)\).

We continue with case (C). If \(K\) is a circle of radius \(r\), and the boundary hypercycle or straight line components of \(L\) have pairwise distances at least \(2r\), then \(\text{int}\ (\varphi K)\) can intersect at most one boundary component of \(\psi L\).

If \(\text{int}\ (\varphi K)\) does not intersect any boundary component of \(\psi L\) (and, by hypothesis, \(\text{int}\ [((\varphi K) \cap (\psi L))] \neq \emptyset\)), then \((\varphi K) \cap (\psi L) = \varphi K\) is a circle, hence is axially symmetric.

If \(\text{int}\ (\varphi K)\) intersects exactly one boundary component \(L_1\) of \(\psi L\), then \((\varphi K) \cap (\psi L)\) is the same as the intersection of \(\varphi K\) and of the closed convex set, bounded by \(L_1\), and containing \(\psi L\). This has an axis of symmetry, cf. case (A).

We continue with case (D). Let \(K\) be a paracycle, and \(L\) a parallel domain of some straight line, with some distance \(l > 0\). Consider the common base line of the two hypercycles, bounding \(\varphi K\). If the infinite point of the paracycle \(\varphi K\) lies on this common base line, then this straight line is an axis of symmetry of \((\varphi K) \cap (\psi L)\). If the infinite point of \(\varphi K\) does not lie on this common base line, then there is a unique straight line that passes through the infinite point of \(\varphi K\), and is orthogonal to the common base line. Then this unique straight line is an axis of symmetry.

Last we turn to case (E). Consider the common base lines of the two hypercycles bounding \(\varphi K\), and of the two hypercycles bounding \(\psi L\). These two straight lines can coincide, or can intersect, or can be parallel (but distinct), or can be neither intersecting nor parallel. In any case there is an axial symmetry interchanging these two straight lines. This axial symmetry interchanges the parallel domains of these straight lines, with distance \(l\), as well. Hence it is a symmetry of the intersection of these parallel domains, i.e., of \((\varphi K) \cap (\psi L)\).

3. Last we turn to the proof of \((1) \implies (3)\). By Theorem 2 we know, that each boundary component of both \(K\) and \(L\) is either a cycle, or a straight line. Thus, for each of \(K\) and \(L\), we have the following possibilities: it is a circle, or a paracycle, or its boundary components are hypercycles and straight lines.

We make a case distinction. Either both \(\text{bd}\ K\) and \(\text{bd}\ L\) are connected, or one of them has several connected components.
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4. We begin with the case when both $\text{bd } K$ and $\text{bd } L$ are connected, and will show that then we have (A) of (3) of the theorem.

We have to investigate the cases when

a) $\varphi K$ and $\psi L$ are one paracycle and one convex set bounded by a hypercycle or a straight line, or
b) $\varphi K$ and $\psi L$ are two incongruent convex sets, both bounded by a hypercycle or a straight line,

and in both cases we have to find a contradiction.

Now it will be convenient to use the conformal model. In case a), let the centre of the paracycle be one endpoint of the base line of the hypercycle, or one endpoint of the straight line. Then $(\varphi K) \cap (\psi L)$ has a smooth boundary, except at one point $p$, that is the intersection of $\text{bd } (\varphi K)$ and $\text{bd } (\psi L)$. In case b), let the base lines of the two hypercycles, or the base line of the hypercycle and the straight line intersect, respectively (two straight lines cannot occur). Then, also $\text{bd } (\varphi K)$ and $\text{bd } (\psi L)$ intersect, at a single point, and this point $p$ is the only non-smooth point of $(\varphi K) \cap (\psi L)$.

Both in case a) and b), any non-trivial symmetry of $(\varphi K) \cap (\psi L)$ would have $p$ as a fixed point. Thus, it would be an axial symmetry, w.r.t. the angle bisector of the inner angle of $(\varphi K) \cap (\psi L)$ at $p$. Thus, this symmetry should interchange the portions of the boundaries of $\varphi K$ and $\psi L$, bounding $(\varphi K) \cap (\psi L)$. However, these portions of boundaries have different curvatures, which is a contradiction.

Thus, in the case investigated in 4, we have shown (A) of (3) of the theorem.

5. There remained the case when one of $K$ and $L$ has at least two boundary components. Observe that this rules out the cases when $K, L$ are two circles, or two paracycles, or one circle and one paracycle. There remain the cases when one of $K$ and $L$ is bounded by hypercycles and straight lines, and the other one is a circle, of some radius $r$, or when one of $K$ and $L$ is bounded by finitely many hypercycles and straight lines, and the other one is either a paracycle, or also is bounded by finitely many hypercycles and straight lines. We will investigate these three cases separately.

If one of $K$ and $L$ is bounded by hypercycles and straight lines, then the boundary components $K_i$ of $\varphi K$, or the boundary components $L_i$ of $\psi L$ have a natural cyclic order, in the positive sense, on $\text{bd } (\varphi K)$, or $\text{bd } (\psi L)$, respectively. We associate to $\varphi K$, or to $\psi L$ a graph, whose vertices are the infinite points of the $K_i$'s, or $L_i$'s, and between two such points there is an edge, if they are the two infinite points of some $K_i$, or $L_i$, respectively. We say that this edge is $K_i$, or $L_i$, respectively. This graph can be a union of vertex-disjoint paths, or can be a cycle.
Here we admit a cycle of length 2, shortly a 2-cycle, when the graph consists of two vertices, and two edges between these two vertices, which are two $K_i$’s ($L_i$’s) with both infinite points common.

If we have two edges in these graphs with a common vertex, and they are e.g., $K_1$ and $K_2$, then by this notation we will mean that $K_2$ follows $K_1$ on $bd(\varphi K)$ in the positive sense. If $K_1$ and $K_2$ form a 2-cycle, then the notation is fixed some way. (A similar convention holds for $L$).

When one of $K$ and $L$ is bounded by hypercycles and straight lines, then, later in the proof, for brevity we will write hypercycle for a hypercycle, or for a straight line; i.e., the curvature is allowed to be 0 as well. The base line of a straight line is considered to be the straight line itself.

6. Let $K$ be a circle of radius $r$ and centre $O$, and $L$ be bounded by hypercycles and straight lines. We have to show that either $L$ is bounded by two hypercycles with common base line (i.e., (B) of (3) of the theorem holds), or $L$ has at least two boundary components (which holds by the assumption in the beginning of 5) and these boundary components have pairwise distances at least $2r$ (i.e., (C) of (3) of the theorem holds). Let us suppose the contrary, i.e., that we have both that $L$ is not bounded by two hypercycles with common base line, and that $dist(L_1, L_2) < 2r$, for some different boundary components $L_1$ and $L_2$ of $\psi L$. By $dist(L_1, L_2) < 2r$, we have, for some $\varphi$, that $int(\varphi K)$ intersects both $L_1$ and $L_2$.

In this case, $int(\varphi K)$ intersects $L_i$ for some $i$’s, and $\varphi K$ touches $L_i$ for some other $i$’s. We will show that by an arbitrarily small perturbation of the centre $O$ of $\varphi K$ we can attain that $(\varphi K) \cap (\psi L)$ has no non-trivial symmetry. Clearly then we need not care those $L_i$’s, for which $(\varphi K) \cap L_i = \emptyset$. (Observe that any compact set in $H^2$ intersects only finitely many $L_i$’s.)

A. Then $(\varphi K) \cap (\psi L)$ is a convex body, bounded, alternately, by (at least two) non-trivial arcs of $bd(\varphi K)$, and (at least two) non-trivial arcs of some $L_i$’s, for different $L_i$’s. The curvatures of these arcs are greater than 1, or smaller than 1, respectively, so each congruence of $(\varphi K) \cap (\psi L)$ preserves both above types of arcs, separately. Now consider conv $[bd(\varphi K) \cap (\psi L)]$, that is preserved by each congruence of $(\varphi K) \cup (\psi L)$. It is obtained from the circle $\varphi K$, by cutting off disjoint circular segments by several, but at least two disjoint, non-trivial chords, having endpoints the points of intersection of the single $L_i$’s with $bd(\varphi K)$, for all $L_i$’s intersecting $int(\varphi K)$. We want to attain that

\begin{equation}
(1) \quad \text{all these chords are of different lengths.}
\end{equation}
We are going to show that this can be attained by a small, generic motion of the centre $O$ of $\varphi K$.

The lengths of the above chords of $\varphi K$ are uniquely determined by the half central angles corresponding to the chords. We use formula (***) before the proof of this theorem. We have several such equations, corresponding to several $L_i$’s, with respective values $l_i$ and $c_i$ (but with $r > 0$ fixed). We have $l_i \geq 0$ for each $i$, and then $c_i$ will vary in $\mathbb{R}$. We will use arbitrarily small generic perturbations of the centre $O$ of our circle $\varphi K$. Then the set of hypercycles that intersect the perturbed int ($\varphi K$) is a subset of the set of all those hypercycles $L_i$, that intersect a fixed concentric closed circle (concentric meant before perturbation) of some radius $r' > r$. This second set is finite (cf. the second paragraph of 6); let it be $\{L_i \mid i \in I\}$. Hence, it suffices to exclude all pairwise equalities of finitely many expressions for $\cos \omega$ — obtained from solving the equations (***) before the proof of this theorem, for all $i \in I$ — namely those of the form

$$(\cosh r \cdot \sinh c_i - \sinh l_i) / (\sinh r \cdot \cosh c_i).$$

Observe that all these expressions are analytic in $O$, since the $c_i$’s are analytic in $O$ (and $r$ and the $l_i$’s are fixed).

Moreover, none of these equations is an identity. Namely, we can consider a circle $\varphi K$ outside of the convex set bounded by the boundary component $L_i$ of $\psi L$, containing $\psi L$, where $i \in I$. By a certain motion we may attain that $\varphi K$ just touches this $L_i$, and is otherwise outside of the convex set in the last sentence. Then the $i$’th expression for $\cos \omega$ has value 1, but all other $j$’th expressions, where $j \in I$, have values for $\cos \omega$ not in $[-1, 1]$. Hence the $i$’th expression and any other $j$’th expression, for $i, j \in I$, are not identical.

Therefore all our finitely many analytic equations are not identities. Hence each of them holds only for $O$ belonging to a nowhere dense closed subset. Therefore, except for $O$ belonging to a nowhere dense closed subset, none of our equations hold. That is, we have proved what was claimed in (1).

**B.** From now on we will suppose (1). There are two possibilities. Either we can have at least three such chords — as in (1) — or we always have exactly two such chords.
Any congruence of \((\varphi K) \cap (\psi L)\) to itself preserves \(\varphi K\), and conv [(bd \((\varphi K)\)) \cap (\psi L)], and also the above mentioned at least three, or exactly two disjoint chords, since their lengths are different. However, a non-trivial congruence preserving a single chord is an axial symmetry w.r.t. the orthogonal bisector straight line of the chord. However, there are no three disjoint circular segments, cut off by chords orthogonal to a single straight line, containing the centre of the circle (namely, orthogonal to their common orthogonal bisector).

There remains the case when we can have only exactly two disjoint circular segments cut off by chords of \(\varphi K\). These must correspond to the above considered hypercycles \(L_1\) and \(L_2\). Then \(\varphi K\) is not even touched by any other \(L_i\), since then by a small motion of \(\psi L\) we could attain that int \((\varphi K)\) intersects at least three \(L_i\)'s, which case was above settled.

The above reasoning gives that, in this case, the orthogonal bisecting straight lines of the two chords coincide, furthermore, contain \(O\); moreover, these remain true after an arbitrary small motion of the centre \(O\) of \(\varphi K\), except those into a nowhere dense closed subset, cf. \(A\).

However, the orthogonal bisecting straight lines of these chords are orthogonal to the base lines of \(L_1\) and \(L_2\). We have that these base lines are different, since their coincidence was excluded in the first paragraph of \(6\). Then they have no finite point in common, and they have either one, or no infinite point in common.

If they have one infinite point in common, then they admit no common orthogonal straight line.

If they have no infinite points in common, then they have exactly one common orthogonal straight line, that should contain the centre \(O\) of \(\varphi K\), for all small motions of \(O\), except those into a nowhere dense closed subset, cf. \(A\). This is clearly impossible.

Thus we have proved, what was promised in the beginning of \(6\): namely that, in the case investigated in \(6\), we have cases (B) or (C) of (3) of our theorem.

7. Now let \(K\) be a paracycle, and let \(L\) be bounded by finitely many, but at least two hypercycles and straight lines.

If the graph of \(\psi L\) consists of paths, then, using one end-point of one path, and the adjacent edge of the graph, we choose \(\varphi K\) in the conformal model as a circle of small radius, that is thus far from all other boundary components of \(\psi L\). Then we can repeat the consideration from 4, case a), and we obtain a contradiction.

There remains the case when the graph of \(\psi L\) is a cycle. Hence there are two edges \(L_1, L_2\), with a common vertex, in the graph of \(\psi L\).
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We consider the conformal model. Fixing the position of $L_1$ and $L_2$, we consider their common vertex (or one of their common vertices) at infinity, 1, say. Let $L_2$ follow $L_1$ at 1 in the positive sense. We choose $\varphi K$ so that it touches the boundary of the model either 1) at 1, or 2) very close to 1 but not at 1, and its interior intersects both $L_1$ and $L_2$, and its image in the model is a circle of very small radius. Then $\varphi K$ is far from all other boundary components of $\psi L$. Hence, $(\varphi K) \cap (\psi L)$ is either 1) an arc triangle, bounded by an arc of $L_2$, an arc of $\text{bd} (\varphi K)$, and an arc of $L_1$, or 2) an arc quadrangle, bounded by an arc of $\varphi K$, an arc of $L_2$, another arc of $\varphi K$, and an arc of $L_1$, in this cyclic order, in the positive sense.

In case 1) $(\varphi K) \cap (\psi L)$ has exactly one infinite point, hence its only non-trivial symmetry is an axial symmetry w.r.t. an axis passing through this infinite point, and interchanging $L_1$ and $L_2$ (observe that a rotation about this infinite point is impossible, by the last paragraph of §3). Hence $L_1$ and $L_2$ are congruent.

In case 2) neither of the diagonals can be an axis of symmetry, and there is no symmetry that would be combinatorially a 4-fold rotation. So, a non-trivial symmetry of $(\varphi K) \cap (\psi L)$ can be a central symmetry, or an axial symmetry w.r.t. the common orthogonal bisector straight lines of the opposite arc-sides.

We begin with the case of central symmetry. Then the opposite arc-sides of $(\varphi K) \cap (\psi L)$ on the paracycle $\varphi K$ are centrally symmetric images of each other. Let us suppose that the centre of symmetry is the centre of the (conformal) model. Then the hyperbolic central symmetry coincides with the Euclidean central symmetry. The paracycle in the model is a Euclidean circle touching the boundary of the model in one point. The centrally symmetric image of the paracycle w.r.t. the centre of the model intersects the paracycle only in at most two points, thus in no arc.

We continue with the case of axial symmetry w.r.t. the common orthogonal bisector straight lines of the opposite arc-sides, lying on $\text{bd} (\varphi K)$. However, a common orthogonal bisector straight line to the two opposite (thus disjoint) arc-sides of $(\varphi K) \cap (\psi L)$, lying on $\text{bd} (\varphi K)$, cannot exist. Namely, such bisectors are different and parallel, thus have no finite point in common, but have only an infinite point in common, namely the infinite point of $\varphi K$.

We continue with the case of axial symmetry w.r.t. the common orthogonal bisector straight lines of the opposite arc-sides, lying on $L_1$ and $L_2$. However, a common orthogonal straight line to $L_1$ and $L_2$ is a common orthogonal to their base lines as well. These base lines are parallel, hence admit no common orthogonal, unless they coincide. If they coincide, then $\psi L$ is bounded just by $L_1$ and $L_2$.

The considerations in 1) and 2) yield (D) of (3) of the theorem.
8. Last, let both $K$ and $L$ be bounded by finitely many hypercycles and straight lines. Then, by the first paragraph of 5, either $K$, or $L$ has at least two boundary components.

We will show that the graphs of $\varphi K$ and $\psi L$ must have only a few edges, and we will clarify the structure of these graphs, till we will obtain that we must have case (E) of (3) of our theorem.

We will make the following case distinction, for the graphs of $\varphi K$ and $\psi L$.

1) Both graphs contain a pair of edges with at least one common end-point.
2) One graph contains a pair of edges with at least one common endpoint, but the other graph does not contain such a pair, i.e., it consists of vertex disjoint edges, and the number of these edges is at least 2.
3) One graph contains a pair of edges with at least one common endpoint, but the other graph does not contain such a pair, i.e., it consists of vertex disjoint edges, and the number of these edges is 1.
4) None of the graphs contains a pair of edges with at least one common endpoint, i.e., both of them consist of vertex disjoint edges. Here, by 5, at least one of the graphs contains at least 2 edges.

These cases are exhaustive, and mutually exclusive.

9. We begin with the proof of case 1).

Then each of the graphs of $\varphi K$ and $\psi L$ contains a path of length 2 or a 2-cycle. The corresponding boundary components of $\varphi K$, $\psi L$ are denoted by $K_1, K_2$, and $L_1, L_2$, with $K_2$ following $K_1$ on $\text{bd} (\varphi K)$, and $L_2$ following $L_1$ on $\text{bd} (\psi L)$, according to the positive orientation. (If one of the graphs is a 2-cycle, this does not determine $K_1$ etc.; then we fix some notation.)

We are going to show that $K_1 \cup K_2$ and $L_1 \cup L_2$ are images of each other by an orientation reversing congruence, moreover, that the graphs of $\varphi K$ and $\psi L$ both are 2-cycles. Then these will imply that $\text{bd} (\varphi K) = K_1 \cup K_2$, and $\text{bd} (\psi L) = L_1 \cup L_2$.

We use the conformal model. Recall that any three different points on the boundary of the model can be taken by a congruence to any other three different boundary points of the model. Therefore we may suppose the following. The considered common vertex of $K_1$ and $K_2$ is 1, and their other vertices are very close to $-1$ — hence all other boundary components of $\varphi K$ are very close to $-1$, as well — and the considered common vertex of $L_1$ and $L_2$ is $i$, and their other vertices are very close to $-i$ — hence all other boundary components of $\psi L$ are very close to $-i$, as well (and $K_2$ follows $K_1$ on $\text{bd} (\varphi K)$ at 1 in the positive sense, and similarly for $L_2, L_1$ at $i$).
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Then we have that the distance of 0 to $(\varphi K) \cap (\psi L)$ is small (possibly 0), and $(\varphi K) \cap (\psi L)$ is bounded by arcs of $K_1, L_1, K_2, L_2$, in this cyclic order, in the positive sense. In fact, all other boundary components, both of $\varphi K$, and of $\psi L$, are in the model very close to the boundary of the model, hence cannot cut off parts of this arc-quadrangle, which is not close to the boundary.

Thus $(\varphi K) \cap (\psi L)$ is a compact arc-quadrangle. Its possible non-trivial symmetries are, combinatorially, the following: two four-fold rotations, one central symmetry, and axial symmetries w.r.t. diagonals or common orthogonal bisector straight lines of two opposite edges. If we have a symmetry that is combinatorially a four-fold rotation, then we also have a symmetry that is a combinatorial central symmetry. Hence we need not exclude the case of a combinatorial four-fold rotation, exclusion of a combinatorial central symmetry will suffice. Observe that a non-trivial symmetry of $(\varphi K) \cap (\psi L)$ extends, by analycity, to a non-trivial symmetry of $K_1 \cup K_2 \cup L_1 \cup L_2$.

A. We begin with the case when $(\varphi K) \cap (\psi L)$ has a central symmetry. Then $K_1$ and $K_2$ have two common infinite points (images of each other by this symmetry), and the same statement holds for $L_1$ and $L_2$, hence the graphs of both $\varphi K$ and $\psi L$ are 2-cycles. Clearly then the curvatures of $K_1$ and $K_2$, as well as those of $L_1$ and $L_2$, are equal, and are positive.

We are going to show that also the curvatures of $K_i$ and $L_i$ coincide, i.e., we have case (E) of (3) of our theorem. Let, e.g., the curvature of $K_i$ be less than the curvature of $L_i$. Let us choose a new position for $\psi L$, in such a way that the the infinite points of each of $K_1, K_2, L_1, L_2$ are $\pm 1$, and $K_1, L_1$ are in the lower half-plane, and $K_2, L_2$ are in the upper half-plane. Let us rotate $\psi L$ about the infinite point 1, counterclockwise, by a rotation of a small measure. Then, in the (conformal) model, the Euclidean tangents of $L_1$ and $L_2$ at 1 do not change during this rotation. Therefore, in the new position, $(\varphi K) \cap (\psi L)$ is an arc-triangle, bounded by $K_2, L_2, K_1$, in this order. This has a unique infinite point, hence a non-trivial symmetry must be an axial symmetry w.r.t. an axis passing through this infinite point, i.e., the common base line for $K_1$ and $K_2$, i.e., the real line. Then this axial symmetry should preserve $L_2$, and therefore this axis should be the orthogonal side bisector of the arc-side of $(\varphi K) \cap (\psi L)$ on $L_2$. However, one of the angles of this axis and $L_2$ equals one of their angles at their other common point 1 in the model (with its boundary circle), that is acute. Hence, in the new position, $(\varphi K) \cap (\psi L)$ has no non-trivial symmetry, a contradiction.
That is, we have obtained case (E) of (3) of our theorem.

**B.** We continue with the case when \((\varphi K) \cap (\psi L)\) has an axial symmetry w.r.t. a common orthogonal bisector straight line of two opposite edges.

Observe that a common orthogonal straight line to opposite sides, say, on \(K_1\) and \(K_2\), is a common orthogonal to the respective base lines, which are parallel, or coinciding. In the case when these base lines do not coincide, this is impossible.

If these base lines coincide, then the graph of \(\varphi K\) is a 2-cycle, and \(\varphi K\) is bounded just by \(K_1\) and \(K_2\). Let the other common infinite point of \(K_1\) and \(K_2\) be \(-1\) (and let \(\psi L\) be as described in the fourth paragraph of 9). The axis of our symmetry is a straight line orthogonal to this common base line, i.e., to the real axis. This axis of symmetry must contain the common infinite point \(i\) of \(L_2\) and \(L_2\). Thus this axis of symmetry is the imaginary axis. Now a small rotation of \(\psi L\) about \(i\) will make \(\psi L\) not symmetric w.r.t. the imaginary axis. From now on we will consider this small perturbation of the arc-quadrangle \((\varphi K) \cap (\psi L)\), rather than the original arc-quadrangle. Clearly, in the new position, this axial symmetry becomes destroyed. Moreover, for all sufficiently small (depending on the measure of the above small rotation) new perturbations of \((\varphi K) \cap (\psi L)\), in its new position, this axial symmetry will not exist. (Else the new position itself would have this axial symmetry.)

**C.** There remained the case, when the first perturbed \((\varphi K) \cap (\psi L)\) has an axial symmetry w.r.t. some diagonal. Since this symmetry of \((\varphi K) \cap (\psi L)\) yields a congruence between \(K_1 \cup K_2\) and \(L_1 \cup L_2\), these last sets are also axially symmetric images of each other, hence are images of each other by an orientation reversing congruence, as promised.

Thus the graphs of \(\varphi K\) and \(\psi L\) either

a) simultaneously contain paths of length 2, namely \(K_1 K_2\) and \(L_1 L_2\), or

b) are simultaneously 2-cycles, with edges \(K_1, K_2, L_1, L_2\), respectively.

In case a), recall that at the beginning of 9, \(\varphi K\) and \(\psi L\) were chosen as follows: the considered common infinite point of \(K_1\) and \(K_2\) is 1, the considered common infinite point of \(L_1\) and \(L_2\) is \(i\), and the other end-points of \(K_1\) and \(K_2\) are close to \(-1\), and the other end-points of \(L_1\) and \(L_2\) are close to \(-i\). Observe that the small rotation applied in B preserves these properties.

Then \((\varphi K) \cap (\psi L)\) is a compact arc-quadrangle, such that the distance of 0 to it is small (possibly is 0). If one of the diagonals is the axis of symmetry, then it is an angle bisector of the angles at the vertices that it connects. Choose the arc-sides with one endpoint at one of these vertices, say, the arc-sides on \(K_2\) and \(L_1\). These determine this diagonal uniquely. Then choose a third arc-side, on \(K_1\), say.
By symmetry, this third arc-side already uniquely determines the fourth arc-side. However, fixing the hypercycles containing these three arc-sides (i.e., $K_1, K_2, L_1$), by a small rotation of $L_2$ about $i$, that extends to an orientation preserving congruence of $\psi L$, preserving $L_1$, we attain that this symmetry is destroyed. By sufficiently smallness of the second perturbation, the second perturbed $(\varphi K) \cap (\psi L)$ does not have the symmetry investigated in $B$, and, by construction, does not have the symmetry investigated in $C$ either. This is a contradiction, hence case a) (i.e., when $K_1K_2$ and $L_1L_2$ are paths of length 2) cannot exist. (Recall that case $A$ was settled above: then we obtained case (E) of (3) of our theorem.)

10. We investigate further the situation described at the end of 9. By a suitable notation, we have that the curvatures of $K_1$ and $L_1$ are equal, and also that the curvatures of $K_2$ and $L_2$ are equal. We are going to show that $K_1$ and $K_2$, and then also $L_1$ and $L_2$, each has the same curvature, i.e., that (E) of (3) of our theorem holds. Let us suppose that the curvature of $K_2$ is greater than the curvature of $K_1$.

Observe that $\varphi K$, or $\psi L$, is symmetrical w.r.t. any straight line orthogonal to the common base line of the $K_i$’s, or of the $L_i$’s, respectively. Hence any two congruent copies of $K$ (and of $L$) are simultaneously directly and indirectly congruent. Since $K$ and $L$ are indirectly congruent by 9, they are also directly congruent.

Let us fix $\varphi K$ so that its points at infinity are $\pm 1$, and $K_1$ lies in the closed lower half-plane, and $K_2$ lies in the closed upper half-plane (with at least one of them lying in the respective open half-plane). Let us obtain $\psi L$ by rotating $\varphi K$ about 1 in positive sense a bit, with the image of $K_i$ being $L_i$. Then $(\varphi K) \cap (\psi L)$ is bounded by two arcs (in the model, with its boundary circle), one lying on $L_2$, the other one lying on $K_1$. Then $(\varphi K) \cap (\psi L)$ has one non-smooth point, at $L_2 \cap K_1$. Hence a non-trivial symmetry of $(\varphi K) \cap (\psi L)$ is an axial symmetry, w.r.t. the angle bisector of the angle of $(\varphi K) \cap (\psi L)$ at this point. However, this symmetry interchanges the two portions of the boundary, lying on $L_2$ and $K_1$. This contradicts the fact that the curvatures of $L_2$ and $K_1$ are different.

That is, we have obtained case (E) of (3) of our theorem.

11. Now we turn to the proof of case 2) from 8.

So let, e.g., the graph of $\varphi K$ consist of vertex disjoint edges, whose number is at least 2. Let us choose two vertex-disjoint edges of this graph, $K_1, K_2$, say.
Further, let the graph of \( \psi L \) contain a path of length 2 or a 2-cycle, consisting of \( L_1 \) and \( L_2 \), where \( L_2 \) follows \( L_1 \) in the positive orientation (if we have a 2-cycle, then their numeration is done in some way). We are going to show that this case cannot occur.

We fix \( \varphi K \) and thus \( K_1 \) and \( K_2 \), and will choose \( \psi L \) in the following way. The set \( \varphi K \) lies in the convex set bounded by \( K_1 \) and \( K_2 \). Then we have relatively open arcs \( I_1 \) and \( I_2 \) of the boundary of the (conformal) model, bounded by the infinite points of \( K_1 \) and \( K_2 \), and lying outside of the above mentioned convex set. We choose the (considered) common infinite point of \( L_1 \) and \( L_2 \) at the midpoint of \( I_1 \), and the other infinite points of \( L_1 \) and \( L_2 \) (possibly coinciding) very close to the centre of \( I_2 \).

Then \( (\varphi K) \cap (\psi L) \) is contained in a compact arc-quadrangle \( Q \), bounded by arcs lying on \( K_1, L_2, K_2, L_1 \), in this order, say.

Observe that all boundary components of \( \psi L \), other than \( L_1 \) and \( L_2 \), are in the model very close to the boundary of the model, hence cannot cut off parts of this arc-quadrangle \( Q \), which arc-quadrangle is not close to the boundary. So these boundary components have no arcs on \( \partial (\varphi K) \cap (\psi L) \). So we need not deal with these boundary components.

However, there may exist several boundary components \( K_i \) of \( \varphi K \), with \( i \neq 1, 2 \), which cut off parts of this arc-quadrangle, hence have non-trivial arcs on \( \partial (\varphi K) \cap (\psi L) \).

Since we investigate case 2), we have that the \( K_i \)'s have no common endpoints. Of course, \( L_1 \) and \( L_2 \) have at least one common endpoint. However, by construction, neither \( L_1 \) and any \( K_i \) (including \( K_1 \) and \( K_2 \)), nor \( L_2 \) and any \( K_i \) (including \( K_1 \) and \( K_2 \)), have any common endpoint.

We are going to show that any non-trivial congruence of \( (\varphi K) \cap (\psi L) \) is a congruence of \( Q \) as well; moreover, it is a congruence preserving the opposite pairs of arc-edges of \( Q \).

We have that \( \partial (\varphi K) \cap (\psi L) \) consists of arcs, following each other, in the positive sense, lying on \( K_1, L_2, K_{i(1)}, L_2, K_{i(2)}, ... K_{i(j)}, L_2, K_2, L_1, K_{i(j+1)}, L_1, K_{i(j+2)}, ..., K_{i(k)}, L_1 \), say. From all of these arcs only those lying on \( L_1 \) and \( L_2 \) lie on different hypercycles, which have at least one point in common.

Let us introduce a symmetric relation \( R \) on the arc-sides of \( (\varphi K) \cap (\psi L) \). For two such arc-sides \( S_1, S_2 \) we have \( S_1 R S_2 \), if the hypercycles spanned by these sides are different, and have at least one common end-point. Clearly any non-trivial congruence of \( (\varphi K) \cap (\psi L) \) preserves this relation \( R \), hence also the set of arc-sides \( S := \{ S_1 \mid \exists S_2 \text{ such that } S_1 R S_2 \} \).
Observe that the relation $R$ induces a complete bipartite graph on the vertex set $S$, with classes $L_i$, for $i = 1, 2$, where $L_i$ is the set of arc-sides of $(\varphi K) \cap (\psi L)$, lying on $L_i$, for $i = 1, 2$.

Therefore, each non-trivial congruence of $(\varphi K) \cap (\psi L)$ preserves the two-element set $\{L_1, L_2\}$. Of course, also the cyclic order of the arc-sides of $(\varphi K) \cap (\psi L)$ is preserved, up to inversion. Let the first end-point of the first arc-side and last endpoint of the last arc-side in $L_i$ (i.e., lying on $L_i$), be $v_{i,1}$ and $v_{i,2}$. Then the set $\{\{v_{1,1}, v_{1,2}\}, \{v_{2,1}, v_{2,2}\}\}$ is preserved by each non-trivial congruence of $(\varphi K) \cap (\psi L)$, as well. So, $Q$ is preserved by each non-trivial congruence of $(\varphi K) \cap (\psi L)$ as well, even in such a way, that separately the opposite pairs of sides are preserved.

12. By 11 we need to discuss only the congruences of the arc-quadrangle $Q$, more exactly only those of them, that preserve the opposite pairs of sides. Therefore, combinatorially, the possible non-trivial congruences, to be investigated, and to be excluded, are central symmetry, and axial symmetries w.r.t. common orthogonal side-bisector straight lines of opposite sides.

**A.** We begin with the case of central symmetry of $(\varphi K) \cap (\psi L)$. The central symmetry interchanges the arc-sides lying on $K_1$ and $K_2$, hence also $K_1$ and $K_2$, hence also the infinite points of $K_1$ and the infinite points of $K_2$. Thus its centre must be the intersection of the straight lines connecting the interchanged end-points. (This determines the interchanged pairs of end-points uniquely.) We may assume that this centre of symmetry is 0. Also, by central symmetry, the graph of $\psi L$ must be a 2-cycle, with $L_1$ and $L_2$ having the same curvatures. Then the centre of symmetry must lie on the common base line of $L_1$ and $L_2$.

**B.** We continue with the case of axial symmetry of $(\varphi K) \cap (\psi L)$ w.r.t. the common orthogonal bisector of the arc-sides lying on $L_1$ and $L_2$. Such a common orthogonal straight line is orthogonal to the base lines of $L_1$ and $L_2$ as well, hence it exists only if the base lines of $L_1$ and $L_2$ coincide (they cannot be parallel but different), i.e., the graph of $\psi L$ is a 2-cycle. Then the symmetry interchanges $K_1$ and $K_2$, hence its axis is the unique axis of symmetry interchanging $K_1$ and $K_2$. We may suppose that this axis is the imaginary axis. Hence the axis of this symmetry is orthogonal to the common base line of $L_1$ and $L_2$.

**C.** We continue with the case of axial symmetry of $(\varphi K) \cap (\psi L)$ w.r.t. the common orthogonal bisector straight line of the arc-sides lying on $K_1$ and $K_2$. This axis is the unique straight line orthogonal to $K_1$ and $K_2$ (and hence also to their base lines). Since the centre of symmetry considered in **A** was 0, and the axis of symmetry considered in **B** was the imaginary axis, this axis is the real axis.
Then the axis of the unique axial symmetry of $L_1 \cup L_2$, interchanging $L_1$ and $L_2$, is the real axis.

Considering all three possible cases A, B, C, we have the following. By a small generic perturbation of $\psi L$ we can attain that the axis of the axial symmetry of $L_1 \cup L_2$, interchanging $L_1$ and $L_2$, intersects the imaginary axis at a point different from 0 (thus does not contain 0), and the angle enclosed by this axis of symmetry and the imaginary axis is different from $\pi/2$. Then we also have that this axis of symmetry is different from the real axis. Thus this perturbation simultaneously destroys all three possible non-trivial symmetries of $(\varphi K) \cap (\psi L)$, discussed in A, B, and C. This is a contradiction. Hence, the case investigated in 11 cannot occur, as promised in the first paragraph of 11.

13. We turn to the proof of case 3) from 8.

Let, e.g., the graph of $\varphi K$ consist of a single edge $K_1$, i.e., this is the unique boundary component, and let the graph of $\psi L$ contain a path of length 2 or a 2-cycle. We are going to show that this is impossible.

Let the graph of $\psi L$ contain two edges $L_1$ and $L_2$ with a common vertex, following each other in the positive orientation, at the considered common vertex. We consider the conformal model. We consider $L_1$ and $L_2$ as fixed, and $\varphi K$ being in a small Euclidean neighbourhood of the/some common infinite point 1 of $L_1$ and $L_2$. Then $\varphi K$ does not intersect any other $L_i$. However, we may suppose that $K_1$ intersects both $L_1$ and $L_2$, and that $\varphi K$ lies on that side of $K_1$, as the considered common infinite endpoint 1 of $L_1$ and $L_2$.

Then $(\varphi K) \cap (\psi L)$ is an arc-triangle, with one infinite vertex. Hence any of its non-trivial symmetries is an axial symmetry, w.r.t. an axis passing through 1 (it cannot be a rotation about the infinite point 1), and such that $L_1$ and $L_2$, as well as the base lines of $L_1$ and $L_2$, are images of each other by this symmetry. This unique axis of symmetry must intersect the arc-side of $(\varphi K) \cap (\psi L)$ on $K_1$ orthogonally. However, a small rotation of $K_1$, about the intersection point of the above axis of symmetry and $K_1$, destroys this unique symmetry.

14. Last we turn to the proof of case 4) from 8.

That is, both graphs consist of vertex-disjoint edges. We are going to show that this is impossible. By 5, e.g., $L$ has at least two boundary components.

Let $L_1$ and $L_2$ denote two neighbourly boundary components of $\psi L$, with $L_2$ following $L_1$ in the positive orientation.
(That is, passing on the boundary of $\psi L$, taken in one of the models, together with its boundary circle, from $L_1$ to $L_2$, in the positive sense, there are no other connected components of $\text{bd} (\psi L)$, taken in $H^2$, between them.) Then, denoting by $l_1$ the second infinite point of $L_1$, and by $l_2$ the first infinite point $l_2$ of $L_2$ (both taken in the positive orientation), the counterclockwise arc $l_1l_2$ contains no infinite point of any boundary component of $\psi L$. We may suppose that the base lines of $L_1$ and $L_2$ are symmetric images of each other w.r.t. the real axis, with the base line of $L_1$ being in the open lower half-plane, and the base line of $L_2$ being in the open upper half-plane. Let $K_1$ be a boundary component of $\varphi K$. Let its infinite end-points be $k'_1$ and $k''_1$, following each other in this order in the positive sense, on the boundary of $\varphi K$, taken in the model together with its boundary circle. Let us begin with the position, when $k'_1 = l_2$ and $k''_1 = l_1$, and $\varphi K$ lies on the same side of $K_1$, as 1.

Now let us translate $\varphi K$, and thus also $K_1$, along the real axis a bit, to the left. For the new congruent copy of $K$ we will not apply a new notation, but will preserve the old notation $\varphi K$. Then $k'_1$ and $k''_1$ move a bit (in the conformal model, taken with its boundary circle). We want to determine the intersection $(\varphi K) \cap (\psi L)$.

Let the boundary components of $\varphi K$, or of $\psi L$, be, in the positive sense, $K_1, K_2, ..., K_n$, or $L_1, ..., L_m$, respectively. Using the collinear model, we see that any of $K$ and $L$ can be obtained from a convex polygon, with all vertices at infinity, whose number of vertices is even, by putting hypercycles, outwards, on each second side, and replacing the remaining sides with the corresponding arcs of the boundary of the model. (Including the case when this convex polygon is a 2-gon, i.e., a segment.)

Then we may suppose that all boundary components of $\varphi K$, except $K_1$ (if any), lie strictly on the right hand side of the straight line $l_1l_2$. All these will be boundary components of $(\varphi K) \cap (\psi L)$ as well. There is still one boundary component of $(\varphi K) \cap (\psi L)$. This begins at $l_2$, then passes on $L_2$, then passes on $K_1$, then on some $L_{i(1)}$, then once more on $K_1$, then on some $L_{i(2)}$, ..., then on some $L_{i(k)}$, then once more on $K_1$, then on $L_1$, and ends at $l_1$. (One has to observe only that $K_1$ must cross $L_2$, transversally — observe that both $K_1$ and $L_2$ are circular arcs in the conformal model — and then some small arc of it still remains in $\text{cl conv} (L_1 \cup L_2)$ a bit, so that this small arc cannot be "cut off" by any other $L_j$. A similar reasoning is valid for $L_1$.)

Then any non-trivial symmetry of \((\varphi K) \cap (\psi L)\) preserves this unique non-smooth boundary component of \((\varphi K) \cap (\psi L)\). Thus this symmetry is an axial symmetry, which maps \(L_1\) to \(L_2\), and hence the base line of \(L_1\) to that of \(L_2\) — hence has axis the real axis — and maps the first and last arcs of \(K_1\) on \(\text{bd}[(\varphi K) \cap (\psi L)]\) to each other (if there is only one such arc, then it is mapped to itself). In both cases, the symmetry maps the whole \(K_1\) to the whole \(K_1\), hence its axis is orthogonal to \(K_1\).

This is no contradiction, since, by construction, \(K_1\) is symmetric w.r.t. the real axis. Now let us consider the point of intersection of \(K_1\) with the real axis. Let us rotate a bit \(\varphi K\) about this point. Then the combinatorial structure of \((\varphi K) \cap (\psi L)\) remains of the same type (only possibly the set of indices \(\{i(1), ..., i(k)\}\) will change, but this does not invalidate the above considerations). So the unique non-trivial symmetry has as axis the real axis, that should be orthogonal to \(K_1\). However, this is already a contradiction, since, by the above rotation, the rotated image of \(K_1\) becomes not orthogonal to the real axis. ■

Proof of Theorem 7. 1. We begin with the proof of \((1) \implies (2)\).

2. We will make a case distinction. Either both \(K\) and \(L\) are strictly convex, or one of \(K\) and \(L\) is not strictly convex.

We begin with the proof of the case when both \(K\) and \(L\) are strictly convex. (Observe that, for \(X = \mathbb{R}^d\) and \(X = H^d\), this follows from the hypothesis (****) of the theorem.)

3. First, we are going to show that, for any \(x \in \text{bd} K\) and any \(y \in \text{bd} L\), all sectional curvatures exist, and are equal to some non-negative constant, and, in case of \(\mathbb{R}^d\) and \(H^d\), even to some positive constant.

4. Let \(n, m\) denote the outer unit normals of \(K\), or \(L\), at \(x \in \text{bd} K\), or \(y \in \text{bd} L\), respectively. (Recall that (****) implies smoothness.) Let us choose an \(O \in X\), and let \(e, f\) be opposite unit vectors in the tangent space of \(X\) at \(O\). Let us choose \(\varphi_0, \psi_0\), such that \(\varphi_0 x = \psi_0 y = O\), and the images (in the tangent bundle) of \(n\) or \(m\) (by the maps induced by \(\varphi_0\) or \(\psi_0\) on the tangent bundle) should be \(e\) or \(f\), respectively. Then \((\varphi_0 K) \cap (\psi_0 L) \supset \{O\}\). Let \(l\) be the geodesic from \(O\) in the direction of \(e\) (equivalently, of \(f\)). Let us move \(\varphi_0 K, \psi_0 L\) toward each other, so that their points originally coinciding with \(O\) should move on the straight line \(l\), to the respective new positions \(O_K\) and \(O_L\), while we allow any rotations of them, about the axis \(l\). We denote these new images by \(\varphi K, \psi L\).
Let the amount of the moving of the points originally coinciding with \( O \), both for \( \varphi_0 K \) and \( \psi_0 L \), be a common distance \( O O_K = O O_L = \varepsilon > 0 \). We may assume that \( O_K = \varphi x \in \text{int} (\psi L) \) and \( O_L = \psi y \in \text{int} (\varphi K) \).

Then, \( C := (\varphi K) \cap (\psi L) \) has a non-empty interior, and, by strict convexity of \( K \) and \( L \), has an arbitrarily small diameter. Hence it has a centre of symmetry, \( c \), say. We are going to show that, for \( \varepsilon > 0 \) sufficiently small, \( c \) coincides with \( O \). First observe that, for \( \varepsilon > 0 \) sufficiently small, we have, by hypothesis (***) of the theorem, that the ball of centre \( O \) and radius \( \varepsilon \) is contained in \( C \).

5. First we deal with the case of \( S^d \). Let \( \varphi K' \), or \( \psi L' \) denote the half-\( S^d \) containing \( \varphi K \), or \( \psi L \), and containing \( O_K \), or \( O_L \) in its boundary, and thus being there tangent to \( \text{bd} (\varphi K) \), or \( \text{bd} (\psi L) \), respectively. By \( \varphi K \subset \varphi K' \) and \( \psi L \subset \psi L' \), we have also \( C \subset (\varphi K') \cap (\psi L') \). However, \( (\varphi K') \cap (\psi L') \) contains a unique ball of maximal radius, namely that with centre \( O \), and radius \( \varepsilon \). Then the same statement holds for \( C \) as well. Thus, the centre of symmetry \( c \) of \( C \) must coincide with \( O \).

Now, we turn to the case of \( \mathbb{R}^d \) and \( H^d \). Then, by hypothesis (****) of the theorem, we have that, in an open \( \varepsilon \)-neighbourhood of \( x \), or \( y \) there holds the following implication. If a point belongs to \( K \setminus \{x\} \), or \( L \setminus \{y\} \), then it belongs to \( \text{int} K'' \) or \( \text{int} L'' \), where \( K'' \) or \( L'' \) are closed balls (for \( \mathbb{R}^d \)), or closed convex sets bounded by some hyperspheres (for \( H^d \)), respectively, with \( x \in \text{bd} K'' \) and \( y \in \text{bd} L'' \), and with \( \text{bd} K'' \) and \( \text{bd} L'' \) having sectional curvatures at most \( \varepsilon \). Moreover, the images of \( K'' \), or \( L'' \), by \( \varphi \), or \( \psi \), contain \( \varphi (x) = O_K \), or \( \psi (y) = O_L \), and are there tangent to \( \text{bd} (\varphi K) \), or \( \text{bd} (\psi L) \), and then necessarily have there their concave sides towards \( \text{int} (\varphi K) \), or \( \text{int} (\psi L) \), respectively. (Observe, that we may have to decrease \( \varepsilon (x) > 0 \), or \( \varepsilon (y) > 0 \), from (***) and (****) before Theorem 7, to obtain this.)

Now we make a case distinction. First we deal with the case \( X = H^d \), and second we will deal with the case \( X = \mathbb{R}^d \).

So, let \( X = H^d \). Without loss of generality, we may assume, that \( K'' \) and \( L'' \) are distance surfaces, with equal distances \( \varepsilon'(x) = \varepsilon'(y) > 0 \) from their base hyperplanes. Further, we may suppose \( 0 < \varepsilon < \varepsilon'(x) = \varepsilon'(y) \). We may suppose, that \( (\varphi K'') \cap (\psi L'') \) lies in the intersection of the images by \( \varphi \), or \( \psi \), of the neighbourhoods of \( x \), or \( y \), mentioned in the beginning of the last but one paragraph, respectively. Then, locally, \( (\varphi K) \cap (\psi L) \) is contained in \( [\text{int} ((\varphi K'') \cap (\psi L''))] \cup \{O_K, O_L\} \). However, then also globally we have the inclusion \( (\varphi K) \cap (\psi L) \subset [\text{int} ((\varphi K'') \cap (\psi L''))] \cup \{O_K, O_L\} \).
We show, that the unique ball of maximal radius, contained in $(\varphi K'') \cap (\psi L'')$, is the one with centre $O$, and radius $\varepsilon$. Then, the same statement holds for $(\varphi K) \cap (\psi L)$ as well, hence the coincidence of $c$ and $O$ will be proved.

Observe, that $(\varphi K'') \cap (\psi L'')$ is rotationally symmetric about the axis $O_K O_L$, and is symmetric to the orthogonal halving plane $H$ of the segment $O_K O_L$. We will show that, if we have a ball, included in $(\varphi K'') \cap (\psi L'')$, with centre different from $O$, and with the centre on the (closed) side of the orthogonal halving plane of $O_K O_L$, on which $O_K$ lies, then its radius is less than $\varepsilon$. (The case of $O_L$ is analogous.) Clearly, we may restrict ourselves to the case $d = 2$.

Let $K'''$ be the base line of $K''$ (i.e., $K''$ is a distance line for $K'''$). Clearly, the straight line containing $O, O_K, O_L$ is orthogonal to $\varphi K'''$, $H$ and $\varphi K''$ (these last three curves being distinct, and their intersections with the straight line $O_K O_L$ follow each other in the given order, by $0 < \varepsilon < \varepsilon'(x)$). Let the intersection of this straight line with $\varphi K'''$ be $O'$. The straight lines $\varphi K'''$ and $H$ have no common finite or infinite point. The minimal distance of these two straight lines is attained in the position when we take the point $O'$ on $\varphi K'''$, and the point $O$ on $H$.

Now, let us draw straight lines orthogonal to $\varphi K'''$ at each point $O^* \in \varphi K'''$. Then, for the constant value $\varepsilon'(x) > 0$, we have to pass from any point $O^* \in \varphi K'''$, a segment of length $\varepsilon'(x)$ on the respective orthogonal straight line, towards $\varphi K''$, till we reach a point, say, $O^{**}$, on $\varphi K''$. During this motion, we may cross $H$, at some point $O^{**}$. We suppose that this point $O^{**}$ exists, and, moreover, it lies in $(\varphi K'') \cap (\psi L'')$.

Then the minimum length of $O^* O^{**}$ is $O'O$, and is attained only for $O^* = O'$. Hence, the maximum length of $O^{**} O^{***}$ is $OO_K$, and is attained only for $O^* = O'$. Therefore, also the distance of any point $P$, lying on the segment $O^{**} O^{***}$, to $O^{***}$, is maximal exactly when $O^* = O'$ and $P = O^{**} = O$. As promised above, this ends the proof, that $c = O$ for the case of $H^d$.

There remained the case of $X = \mathbb{R}^d$. Then elementary geometrical considerations yield that the ball of maximal radius, contained in $(\varphi K'') \cap (\psi L'')$, has centre $c = O$ (and radius $\varepsilon$).

6. Thus, $(\varphi K) \cap (\psi L)$ has as centre of symmetry $O$, and it has a chord $[O_K, O_L]$, passing through $O$, hence $O_K = \varphi x$ and $O_L = \psi y$ are centrally symmetric images of each other w.r.t. $O$. Then the same holds for some of their neighbourhoods, relative to $\text{bd} (\varphi K)$, or $\text{bd} (\psi L)$, respectively, for $\varepsilon$ sufficiently small (recall that $O_K = \varphi x \in \text{int} (\psi L)$ and $O_L = \psi y \in \text{int} (\varphi K)$).
Now, take some 2-plane containing the straight line $O_KO_L$. Then, the intersections of some neighbourhoods of $O_K$ and $O_L$, relative to $\bd (\varphi K)$, or $\bd (\psi L)$, respectively, with this 2-plane, are centrally symmetric images of each other. Therefore, these two curves have, at $O_K$ and $O_L$, the same curvatures (sectional curvatures), if one of them exists, or they do not have curvatures there. Observe, that $\varphi$ and $\psi$ were not determined uniquely, but at their definitions there were allowed to apply any rotations about the axis $l$. Hence, either all sectional curvatures (i.e., the curvatures of all above curves), of both $K$ and $L$, at the points $x$ and $y$ are equal, or all of them do not exist. Observe, that $x$ and $y$ were arbitrary points of $\bd K$ and $\bd L$. So either

a) all sectional curvatures of both $K$ and $L$ exist, at each boundary point of $K$ and $L$, and they are equal, namely to some number $\kappa \geq 0$, or
b) they do not exist anywhere.

However, convex surfaces in $\mathbb{R}^d$ are almost everywhere twice differentiable (cf. [Sch], pp. 31-32, cited in detail in the sixth paragraph of 4 of the proof of Theorem 2). Using the collinear models for $S^d$ and $H^d$, this holds for $S^d$ and $H^d$ as well. This rules out possibility b), so possibility a) holds, as promised above. Clearly, for $\mathbb{R}^d$ and $H^d$, the hypothesis of our Theorem implies $\kappa > 0$.

7. Observe, that the above proof also gives, that locally $\bd K$, or $\bd L$ is rotationally symmetric about the normal $n$ at $x$, or $m$ at $y$, respectively. (Recall, that $\varphi$ and $\psi$ were defined only up to arbitrary rotations about the straight line $l$, respectively, and we always had symmetry about $c = O$.) Their 2-dimensional normal sections, i.e., the sections by 2-planes containing $n$, or $m$, are normal sections for all of their points close to $x$, or $y$, just by local rotational symmetry of $\bd K$, or $\bd L$, respectively. Therefore, these 2-dimensional normal sections have everywhere the same constant curvature $\kappa$. Hence, locally, these sections are congruent cycles in the respective 2-dimensional subspaces (for $\mathbb{R}^d$ and $H^d$ they cannot be straight lines, by hypothesis (****) of the theorem). Therefore, $\bd K$ and $\bd L$ are, locally, for $S^d$, congruent spheres, including half-$S^d$’s, and, for $\mathbb{R}^d$ and $H^d$, they are, locally, congruent spheres, paraspheres, or congruent hyperspheres (they cannot be hyperplanes). Thus, locally, any of $\bd K$ and $\bd L$ is an analytic surface, given up to congruence.

Now, let $x \in \bd K$ be arbitrary. For some relatively open geodesic $(d - 1)$-ball $B_x$ on $\bd K$, with centre $x$, we have that $B_x$ is a subset of an above analytic hypersurface; if the above hypersurfaces are spheres, then we assume that the $B_x$’s are at most half-spheres of these spheres.
For \( x_1, x_2 \in \text{bd} K \), with \( B_{x_1} \cap B_{x_2} \neq \emptyset \), we have that \( B_{x_1} \) and \( B_{x_2} \) are subsets of the same analytic hypersurface, i.e., they are open subsets of the same sphere, parasphere, or hypersphere. Now, let us introduce an equivalence relation on the points \( x \) of \( \text{bd} K \). Two such points \( x', x'' \) are called equivalent, if there exists a finite sequence \( x' = x_1, \ldots, x_n = x'' \in \text{bd} K \), such that \( B_{x_i} \cap B_{x_{i+1}} \neq \emptyset \), for each \( i = 1, \ldots, n - 1 \). Clearly, the union of each equivalence class is a relatively open subset of a sphere, parasphere or hypersphere, and also is relatively open in \( \text{bd} K \). Thus, they form a relatively open partition of \( \text{bd} K \), which implies, that they form a relatively open-and-closed partition of \( \text{bd} K \). Thus, the union of each equivalence class is the union of some components of \( \text{bd} K \). Clearly, no \( B_x \) can intersect different connected components of \( \text{bd} K \), since \( B_x \) is connected. Hence, the unions of the equivalence classes are subsets of some connected components of \( \text{bd} K \). Since also they are unions of some connected components of \( \text{bd} K \), they are exactly the connected components of \( \text{bd} K \).

Up to now, we know the following. The connected components of \( \text{bd} K \), and of \( \text{bd} L \), are relatively open subsets of some congruent spheres/paraspheres/hyperspheres. Since \( \text{bd} K \) is closed in \( X \), its connected components, being relatively closed in \( \text{bd} K \), are closed in \( X \) as well. Thus, the connected components of \( \text{bd} K \) are non-empty, relatively open-and-closed subsets of some congruent spheres/paraspheres/hyperspheres. However, spheres, paraspheres and hyperspheres are connected, i.e., have no non-empty, relatively open-and-closed proper subsets. Therefore, the connected components of \( \text{bd} K \), and, similarly, of \( \text{bd} L \), are congruent spheres/paraspheres/hyperspheres.

This shows the implication \((1) \implies (2)\) for the case, when both \( K \) and \( L \) are strictly convex, i.e., in the first case in 2.

8. Now suppose that one of \( K \) and \( L \) is not strictly convex, that is the second case in 2. By hypothesis (***) of the theorem, this can happen only for \( X = S^d \).

By (**) both \( K \) and \( L \) are smooth. We consider two cases for \( K \) (and analogously for \( L \)). We have either \( \text{diam} K < \pi \), or \( \text{diam} K = \pi \).

In the first case, \( K \) is contained in an open half-sphere. Let us suppose that this half-sphere is the southern half-sphere. Then the collinear model is defined in a neighbourhood of \( K \), and the image of \( K \) is a compact convex set in the model \( \mathbb{R}^d \). Such a set has an exposed point, i.e., a point \( z \) such that \( \{z\} \) is the intersection of the image of \( K \) and a hyperplane in the model \( \mathbb{R}^d \), cf. [Sch], Theorem 1.4.7 (Straszewicz’s theorem).
In the second case, $K$ contains two antipodal points of $S^d$, and we may suppose that these are $(0, \ldots, 0, \pm 1)$. Since $K$ is smooth at $(0, \ldots, 0, 1)$, therefore we may suppose that it has at $(0, \ldots, 0, 1)$ the tangent hyperplane (in $S^d$) $\{(x_1, \ldots, x_d, x_{d+1}) \in S^d \mid x_1 = 0\}$, and $K$ lies on the side $\{(x_1, \ldots, x_d, x_{d+1}) \in S^d \mid x_1 \geq 0\}$ of this hyperplane. Clearly $K$ consists of entire half-meridians, connecting $(0, \ldots, 0, \pm 1)$. By the hypothesis about the tangent hyperplane, each half-meridian, whose relative interior lies in the open half-sphere, given by $x_1 > 0$, lies entirely in $K$. Therefore, $K$ contains the closed half-sphere, given by $x_1 \geq 0$. Since, by hypothesis (***) of the theorem, we have $K \neq S^d$, we have that $K$ is a half-sphere.

Considering also $L$, we have also that either $L$ has an exposed point, or $L$ is a half-sphere. So, unless both $K$ and $L$ are half-spheres — when we are done — we have that, e.g., $K$ has an exposed point $x$. Then let $y \in \text{bd } L$. Now we can repeat the procedure described in 4. Then $(\varphi K) \cap (\psi L)$ has an arbitrarily small diameter, hence is centrally symmetric by (1). Then we have the situation described in 4 and the first paragraph of 5. Then the first two sentences of 6 are valid also here. That is, some small neighbourhoods of $O_K = \varphi x$, or $O_L = \psi y$, relative to $\text{bd } (\varphi K)$, or to $\text{bd } (\psi L)$, respectively, are centrally symmetric images of each other w.r.t. $O$, with $O_K = \varphi x$ and $O_L = \psi y$ being the centrally symmetric images of each other w.r.t. $O$.

This implies, that also $y$ is an exposed point of $L$ (observe that to be an exposed point of a closed convex set is a local property). That is, all boundary points of $L$ are exposed points of $L$. Now, changing the roles of $K$ and $L$, we obtain, that also all boundary points of $K$ are exposed points of $K$. In other words, both $K$ and $L$ are strictly convex. However, this contradicts the hypothesis in the first sentence of 8.

9. The implication $(2) \implies (1)$ of the theorem is proved by copying the respective proof from 7 of the proof of Theorem 2. Now we will have central symmetry, since in $(2)$ of the theorem we have congruent connected components.

In fact, the intersection of two congruent balls (with non-empty interior) is centrally symmetric. A compact intersection of two paraballs $\varphi K$ and $\psi L$ (with non-empty interior) is centrally symmetric. In fact, the infinite points of the two paraballs, say, $k$ and $l$, are different. We consider the straight line $kl$. Let the other points of $\text{bd } (\varphi K)$ and $\text{bd } (\psi L)$ on $kl$ be $k'$ and $l'$. 
We may suppose that the order of the points on \( kl \) is \( k, l', k', k \). Then the midpoint of the segment \( k'l' \) is the centre of symmetry of \((\varphi K) \cap (\psi L)\).

For the case when the boundary components are congruent hyperspheres, these hyperspheres are distance surfaces for some distance \( c > 0 \). Like in the proof of 7 of the proof of Theorem 2, we may restrict ourselves to the case when \((\varphi K) \cap (\psi L)\) is bounded only by one boundary component of \( \varphi K \), and of \( \psi L \). (Even the different boundary components have here distances at least \( 2c \).) That is, we have a compact intersection (with non-empty interior) of two convex sets, \( \varphi K \) and \( \psi L \), bounded by congruent hyperspheres. Then the sets of infinite points of \( \varphi K \) and \( \psi L \) are disjoint. Considering the collinear model, this implies that the base hyperplanes of \( \text{bd} (\varphi K) \) and \( \text{bd} (\psi L) \) have no finite, or infinite points in common. Let us consider the segment realizing the distance of these hyperplanes. Then its midpoint is the centre of symmetry of \((\varphi K) \cap (\psi L)\).

Proof of Theorem 8. 1. We have to prove only \((1) \implies (2)\).

Observe that \((1)\) of Theorem 8 implies \((1)\) of Theorem 7, and \((1)\) of Theorem 7 implies, by Theorem 7, that the connected components of the boundaries both of \( K \) and \( L \) are either 1) congruent spheres (for \( X = S^d \) of radius at most \( \pi/2 \)), or 2) paraspheres, or 3) congruent hyperspheres.

In case 1) \( K \) and \( L \) are congruent balls, hence \((2)\) is proved.

There remained the cases when we have \( X = H^d \), and \( K \) and \( L \) are 2) two paraballs, or 3) the boundary components both of \( K \) and \( L \) are congruent hyperspheres, and their numbers are at least 1, but at most countably infinite. We will copy the respective parts of the proof of Theorem 5, 3 and 5. We are going to show, that neither of these cases can occur.

In case 2) \( K \) and \( L \) are paraballs. We choose \( \varphi \) and \( \psi \) so, that \( \varphi K = \psi L \). Then their intersection is a paraball, that is not centrally symmetric, like at Theorem 5, 3.

We turn to case 3). It will be convenient to use the conformal model. Let all boundary components \( K_i \) of \( \varphi K \), and \( L_i \) of \( \psi L \), be congruent hyperspheres, with base hyperplanes \( K_{0,i} \) and \( L_{0,i} \). Denote by \( l \) the common value of the distance, for which these hyperspheres are distance surfaces for their base hyperplanes. (By hypothesis (****) of the theorem we have \( l > 0 \).) These base hyperplanes bound closed convex sets \( K_0 \), or \( L_0 \), possibly with empty interior, not containing any of the hyperspheres \( K_i \), or \( L_i \).
and such that the parallel domain of $K_0$, or $L_0$, with distance $l$, equals $\varphi K$, or $\psi L$, respectively. Cf. the proof of Theorem 5, 5.

We choose such positions of $\varphi K$ and $\psi L$, that $K_{0,1}$ and $L_{0,1}$ project to a copy of $H^2$ in $H^d$ (with a projection along straight lines orthogonal to the copy of $H^2$), so that their projections are like $K_{0,1}$ and $L_{0,1}$ from the proof of Theorem 5, 5. For simplicity, here we assume that this copy of $H^2$ contains the centre of the model, and the axis of symmetry of the above projections, in this copy of $H^2$, passes through the centre of the model. (This implies that, in the conformal model, these projections have the same lengths.) Then, the proof of Theorem 5, 5 gives, that $(\varphi K) \cap (\psi L)$ is the intersection of two closed convex sets, bounded by the congruent hyperspheres $K_1$ and $L_1$.

Now, following the proof of Theorem 5, 5, we will show that this intersection is not centrally symmetric. In fact, in the conformal model, the hyperspheres $K_1$ and $L_1$ are subsets of spherical surfaces (congruent, in the Euclidean sense, in the conformal model), with their centres $k$ and $l$ in the Euclidean plane spanned by the above (conformal) model circle of $H^2$. Moreover, in the conformal model, their intersection is a $(d-2)$-sphere, of (Euclidean) radius less than 1, that is (in Euclidean sense) rotationally symmetric about the straight line $kl$, and touches the boundary of the model ball at one point (namely at the intersection of the projections of $K_{0,1}$ and $L_{0,1}$), and has all other points in the model. Hence, in the conformal model, the intersection of the closed convex sets bounded by $K_1$ and $L_1$ also is (in Euclidean sense) rotationally symmetric about the axis $kl$, and also touches the boundary of the model at one point, and has all other points in the model (in fact, it is contained in the Thales $(d-1)$-sphere of the above $(d-2)$-sphere, taken in the model). Therefore this intersection cannot be centrally symmetric. ■

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