Relaxation of nonlinear oscillations in BCS superconductivity

Razvan Teodorescu
Physics Department, Columbia University, 538 West 120th Street, Mail Code 5293,
New York, NY 10027
E-mail: rteodore@phys.columbia.edu

Abstract. The diagonal case of the $sl(2)$ Richardson-Gaudin quantum pairing model [1, 2, 3, 4, 5, 6, 7] is known to be solvable as an Abel-Jacobi inversion problem [8, 9, 10, 11, 12, 13, 14, 15]. This is an isospectral (stationary) solution to a more general integrable hierarchy, in which the full time evolution can be written as isomonodromic deformations. Physically, the more general solution is appropriate when the single-particle electronic spectrum is subject to external perturbations. The asymptotic behavior of the nonlinear oscillations in the case of elliptic solutions is derived.

1. Introduction

The integrable system described by the pairing hamiltonian introduced by Richardson and Sherman [1, 2, 3, 4, 5, 6] in the context of nuclear physics has received revived interest in recent years, after being applied to metallic superconducting grains [16] and cold fermionic systems [17, 15]. The model is intimately related [18, 19, 20] to a class of integrable systems generally referred to as Gaudin magnets [7]. These systems have been studied both at quantum and classical level [21, 22, 23, 24, 25, 26, 27, 28, 29, 30], in the elliptic case as well as trigonometric and rational degenerations, using various methods from integrable vertex models to singular limits of Chern-Simons theory.

In [17], an interesting regime of the pairing problem was considered, which may be relevant to recent experiments with cold fermionic gases exhibiting the paired BCS state [31, 32, 33, 34, 35, 36, 37, 38]. It was shown that for such systems, the time scales of the order parameter $\tau_\Delta \sim |\Delta|^{-1}$, and the quasiparticle energy relaxation time $\tau_\epsilon$ are both much larger than typical time for switching on the pairing interaction $\tau_0$, essentially given by the variation of external parameters, such as detuning from the Feshbach resonance. It was argued that in this regime, for times $t \ll \tau_\epsilon$, the dynamics of the system is given by non-linear, non-dissipative equations describing the coherent BCS fluctuations for the system out of equilibrium. In this limit, the system is integrable, and features non-perturbative behavior, such as soliton-type solutions.
In the mean-field limit, such non-trivial solutions describing the collective mode of the Anderson spins [39] were derived in [17], for a two-level effective system. This work was generalized [15] in algebro-geometrical terms.

In [40, 41, 42], the long-time behavior of the solution has been considered, under various conditions. An issue not addressed so far is the relaxation of the nonlinear oscillatory solution induced by perturbations of the spectral curve, physically justified by coupling to the environment. Several possible kinds of perturbations may be considered, which may lead to different types of relaxation.

In this paper, we consider the effect of fluctuations of single-particle energy levels, which amount to slow deformations of the Liouville tori, and can be described by hydrodynamic-type equations in phase space [43, 44]. These equations describe the evolution of the moduli for the complex curve of the system.

2. The Richardson-Gaudin Model

2.1. The quantum pairing hamiltonian

Following [21], we briefly review the Richardson pairing model. It describes a system of $n$ fermions characterized by a set of independent one-particle states of energies $\epsilon_l$, where the label $l$ takes values from a set $\Lambda$. The labels may refer, for instance, to orbital angular momentum eigenstates. Each state $l$ has a total degeneracy $d_l$, and the states within the subspace corresponding to $l$ are further labeled by an internal quantum number $s$. For instance, if the quantum number $l$ labels orbital angular momentum eigenstates, then $d_l = 2l + 1$ and $s = -l, \ldots, l$. However, the internal degrees of freedom can be defined independently of $l$. We will assume that $d_l$ is even, so for every state $(ls)$, there is another one related by time reversal symmetry $(l\bar{s})$. For simplicity, we specialize to the case $d_l = 2, s = \uparrow, \downarrow$. Let $\hat{c}_{ls}^\dagger$ represent the fermionic creation operator for the state $(ls)$. Using the Anderson pseudo-spin operators [39] (quadratic pairing operators), satisfying the $su(2)$ algebra

$$\left[ t^3_i, t^\pm_j \right] = \pm \delta_{ij} t^\pm_j, \quad \left[ t_i^+, t_j^- \right] = 2 \delta_{ij} t^3_j,$$

the Richardson pairing hamiltonian is given by

$$H_P = \sum_{l \in \Lambda} 2 \epsilon_l t^3_l - g \sum_{l,l'} t^+_l t^-_{l'} = \sum_{l \in \Lambda} 2 \epsilon_l t^3_l - gt^+ \cdot t^-,$$

where $t = \sum_l t_l$ is the total spin operator. It maps to the reduced BCS model

$$\hat{H} = \sum_{p,\sigma} \epsilon_p \hat{c}_{p,\sigma}^\dagger \hat{c}_{p,\sigma} - g \sum_{p,\sigma} \hat{c}_{p,\uparrow}^\dagger \hat{c}_{-p,\downarrow}^\dagger \hat{c}_{-k,\downarrow} \hat{c}_{k,\uparrow}$$

by replacing the translational degrees of freedom by rotational ones, where $l \in \Lambda = \{1, \ldots, n\}$ enumerates the one-particle orbital degrees of freedom, while $s = \uparrow, \downarrow$ indicates the two internal spin states per orbital ($d_l = 2$). The pairing hamiltonian
can be decomposed into the linear combination
\[
H_P = 2 \sum_{l \in \Lambda} \epsilon_l R_l + g \left[ \left( \sum_{l \in \Lambda} \epsilon_l^3 \right)^2 - \frac{1}{4} \sum_{l \in \Lambda} (d_l^2 - 1) \right].
\] (4)

At a fixed value of the component \( t^3 \) of the total angular momentum, the last term becomes a constant and is dropped from the hamiltonian. The operators \( R_l \) (generalized Gaudin magnets [7]) are given by
\[
R_l = t_l^3 - \frac{g}{2} \sum_{l' \neq l} \frac{\mathbf{t}_l \cdot \mathbf{t}_{l'}}{\epsilon_l - \epsilon_{l'}}.
\] (5)

These operators solve the Richardson pairing hamiltonian because [18] they are independent, commute with each other, and span all the degrees of freedom of the system. Richardson showed [1, 2] that the exact \( N \)–pair wavefunction of his hamiltonian is given by application of operators \( b_k^+ = \sum_l \frac{d_l^k}{\sqrt{2} \epsilon_l - \epsilon_k} \) to vacuum (zero pairs state). The unnormalized \( N \)–pair wavefunction reads
\[
\Psi_R(\epsilon_i) = \prod_{k=1}^N b_k^+ |0\rangle.
\]

The eigenvalues \( e_k \) satisfy the self-consistent algebraic equations
\[
\frac{1}{g} = \sum_{p \neq k} \frac{2}{e_k - e_p} + \sum_l \frac{1}{2\epsilon_l - e_k},
\] (6)

which can be given a 2D electrostatic interpretation [21] with energy
\[
U(\epsilon_i, e_k) = \frac{2}{g} \left[ \sum_{k=1}^N \mathcal{R}e(e_k) - \sum_{l=1}^n \mathcal{R}e(\epsilon_l) \right] + 
\]
\[
2 \sum_{i=1}^n \sum_{k=1}^N \log |e_k - 2\epsilon_i| - 4 \sum_{k<p} \log |e_k - e_p| - \sum_{i<j} \log |2\epsilon_i - 2\epsilon_j|
\] (8)

Equations (6) appear as equilibrium conditions for a set of charges of strength \( q = 2 \) placed at points \( e_k \), in the presence of fixed charges of strength \( q = -1 \) at points \( 2\epsilon_l \), and uniform electric field of strength \( \frac{1}{g} \), pointing along the real axis. This interpretation proves to be very useful for the conformal field theory (CFT) description of the Richardson problem. The electrostatic energy (7) is minimized for values \( \{e_k\} \) corresponding to pair energies. In (7), \( n, N \) represent the number of single-particle levels and the number of pair energies, respectively. For large interaction constant \( g \), the equilibrium positions \( \{e_k\} \) form a set of complex conjugated pairs defining a curve \( \gamma \) in the complex plane of energies. We note that the eigenvalues \( r_i \) of Gaudin hamiltonia are proportional in this language to the values of the electric field at positions \( 2\epsilon_i \),
\[
2r_i = g \frac{\partial U}{\partial \epsilon_i}.
\]

For a set of single-particle energies \( \{\epsilon_i\} \), the BCS ground state is obtained by minimizing the electrostatic energy (7) with respect to positions of the free charges at \( \{e_k\} \). Once found, they also determine exactly the values of the electric field at positions \( \{2\epsilon_i\} \) on the real axis, which are proportional to the ground-state eigenvalues \( \{r_i^{GS}\} \). For any other values of \( \{r_i\} \), the electrostatic energy (7) is not minimized. This indicates that for arbitrary values \( r_i \neq r_i^{GS} \), the system is not in equilibrium.
2.2. The mean field limit of Richardson-Gaudin models

General description of the classical model  In the mean-field limit, the spin operators \( t_l \) are replaced by their quantum mechanical averages. Written in terms of the classical vectors \( S_l = 2 \langle t_l \rangle \), the semiclassical approximation for the pairing hamiltonian becomes

\[
H_{MF} = \sum_{l \in \Lambda} \epsilon_l S_l^3 - \frac{g}{4} |J^-|^2,
\]

where \( J = \sum_{l \in \Lambda} S_l \) and the BCS gap function is given by \( \Delta = gJ^-/2 \). Replacing commutators by canonical Poisson brackets,

\[
\{ S^\alpha_i, S^\beta_j \} = 2 \epsilon^{\alpha\beta\gamma} S_\gamma^i \delta_{ij},
\]

variables \( S_i^\alpha \) become smooth functions of time. In this limit, the problem can analyzed with tools of classical integrable systems, and the solution is known to be exact as \( n \to \infty \).

The Poisson brackets (10) and hamiltonian (9) lead to the equations of motion

\[
\dot{S}_i = 2(-\Delta + \epsilon_i \hat{z}) \times S_i,
\]

where \( 2\Delta = (gJ_x, gJ_y, 0) \) and \( \vec{J} \) is the total spin. The semiclassical limit of Gaudin hamiltonia are independent constants of motion,

\[
r_i = \frac{1}{2} \left[ S_i^z - \frac{g}{2} \sum_{j \neq i} \frac{S_i^z \cdot S_j^z}{\epsilon_i - \epsilon_j} \right], \quad \dot{r}_i = 0.
\]

Equations (11) describe a set of strongly interacting spins and have generic non-linear oscillatory solutions. The exact solution may be obtained through the Abel-Jacobi inverse map \([9, 45, 15]\).

This solution can be described exactly in the language of hyperelliptic Riemann surfaces (see Appendix A for details). At this point, it is useful to make use of the intuitively clear features of this construction (Figure 1). For a given set of initial conditions for the spins \( \{ \vec{S}_i \} \), i.e. also of the constants of motion \( \{ r_i \} \), a polynomial \( Q(u) \) of degree \( 2n \) and with \( n \) pairs of complex conjugated roots \( E_{2k+2} = \overline{E}_{2k+1}, k = 0, \ldots, n-1 \), is constructed. A schematic representation of these roots is given in Figure 1. Between each pair of roots, we place a simple cut \( C_k = [E_{2k+1}, E_{2k+2}] \) on the complex plane of energies. The surface thus obtained is a representation of a torus of smooth genus \( g = n - 1 \).

Variables \( u_k \) are introduced for \( n - 1 \) of these cuts, with respect to which the equations of motion separate. The variables \( u_k \) evolve in time in a complicated fashion, solving a system of nonlinear coupled differential equations (A.4). Up to a constant, the time dependence of the gap parameter amplitude is given by

\[
\log |\Delta(t)| = \Im \int u(t) dt, \quad u(t) = \sum_k u_k(t).
\]

The widths of the cuts \( C_k \) and the periods of the non-linear oscillators \( u_k \) are determined by the values of constants of motion \( \{ r_i \} \). For the particular choice \( r_i = r_i^{GS} \), all
the cuts $C_k$, $k = 1, \ldots, n - 1$ vanish, and the width of the remaining cut $C_n$ equals the equilibrium value of the gap function: $|E_{2n-1}^{GS} - E_{2n}^{GS}| = 2|\Delta|^{GS}$. In that case, the oscillators $u_k = E_{2k-1} = E_{2k}$ are at rest, and the only time dependence left in the system is the uniform precession of the parallel planar spins $S_i^-$, with frequency $\omega = 2 \sum_{k=1}^n \epsilon_k - 2 \sum_{p=1}^{n-1} E_{2p-1} - \sum_{i=1}^n S_i^z$. In the case of particle-hole symmetry, $\omega$ vanishes as well.

2.3. The onset of pairing interaction and long-time behavior of oscillations

Equations (11) do not impose any particular constraints on the integrals of motion. These constants are known for the metallic state $t < 0$, but change abruptly during the short interval when the electron-electron interaction is turned on. The onset of pairing interaction is a delicate problem in itself [40, 41, 42], which deserves further investigation. There are several reasons which make the issue non-trivial. We review here several of them.

a) In the case of KdV and KP2 hierarchies, it is known that finite-gap solutions (for which only a few cuts $C_k$ are non-degenerate) are dense in the space of all periodic solutions. A conjecture of Krichever [44] extends this fact to arbitrary algebraic curves. Therefore, solutions where an infinite number of independent frequencies contribute to the mean-field approximation require very special initial conditions. The system is most likely to be described either by (i) a finite-gap solution, or (ii) a non-oscillatory function. The case (ii) has not been investigated in previous studies.

b) Another issue related to solutions described by only a few frequencies is that the quantum integrability of the system may impose additional constraints and provide a selection criterion, non-existent in the mean-field approximation. The exact solution of quantum XXX Gaudin system under a sudden variation of the spectral
Relaxation of nonlinear oscillations in BCS superconductivity is a subject of active research and will be addressed elsewhere [46].

c) Concerning the finite-gap solutions of the mean-field approximation, we note that the onset of attractive interaction is realized through some mechanism which effectively changes the parameters of the spectral curve. It is reasonable to assume that this variation is not instantaneous and in fact may continue as a perturbation for the rest of the evolution of the system. This mechanism describes coupling of the system to the environment (the external magnetic field and optical trap, for degenerate Fermi gases) and therefore may induce fluctuations in the parameters describing the spectral curve. We therefore consider the effect of such perturbations which do not change the monodromy of the solution. The concept of isomonodromic deformations of integrable systems was introduced in [47, 48, 49] and used extensively in the theory of Painlevé transcendents. We give here a brief review of the method.

We consider the unperturbed problem given by a nonlinear scalar differential equation

\[ R(u, \dot{u}, \ddot{u}, \ldots) = 0, \]

where \( R \) is a rational function of the solution \( u(t) \) and its derivatives \( \dot{u}, \ddot{u}, \ldots \). Specializing to the cases described by hyperelliptic spectral curves, we express it through the Lax pair of \( 2 \times 2 \) operators \( L, A \), solving the linear vector problem

\[
L \Psi = \mu \Psi, \quad \partial_t \Psi = A \Psi, \quad \partial_t L = [A, L],
\]

where \( \lambda, \mu \) are two auxiliary complex variables, \( A(\lambda, u) \) a \( 2 \times 2 \) matrix, and \( \Psi(\lambda, \mu, t) \) is a column vector. The operators are chosen such that elements of the identity \( \dot{L} = [A, L] \) are equivalent with \( R(u, \dot{u}, \ddot{u}, \ldots) = 0 \). A trivial calculation gives the isospectral property \( \dot{\mu} = 0 \).

Specializing the operator \( L \) to have the form [50] (the \( u \) functional dependence is implicit throughout)

\[
L(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda) \end{bmatrix},
\]

the eigenvalue equation for \( \Psi \) becomes

\[
\mu^2 + \det L = 0, \quad \mu(\lambda) = \pm i \sqrt{\det L(\lambda)}.
\]

When functions \( a, b, c \) in (15) are rational, equation (16) defines the hyperelliptic Riemann surface \( \mu(\lambda) \) called spectral curve of the system (14). As noted before, the time evolution leaves the spectral curve invariant, which is why the problem is sometimes called isospectral.

The isomonodromic deformation [44] is introduced through

\[
L \Psi = \mu \Psi + \epsilon \partial_\lambda \Psi, \quad [\epsilon \partial_\lambda - L, \partial_t - A] = 0,
\]

where the second equation is the isomonodromy requirement, and \( 0 \leq \epsilon \ll 1 \). We recover the unperturbed problem by setting \( \epsilon = 0 \). The first equation is easily integrated and gives the formal solutions

\[
\log \Psi_\pm(\lambda, \mu, t) = \frac{1}{\epsilon} \left[ -\mu \lambda \pm i \int^\lambda \sqrt{\det L(\sigma)} d\sigma \right],
\]

(18)
Relaxation of nonlinear oscillations in BCS superconductivity

up to constants in $\lambda$. Let us now impose the saddle-point (or turning-point) condition $\partial_\lambda \log \Psi = 0$. This will give $\mu = \pm i \sqrt{\det L}$, i.e. the spectral curve. Therefore, we may see the isospectral problem as a saddle-point (turning point) approximation. The compatibility equations

$$[\epsilon \partial_\lambda - L, \partial_\tau - A] = 0$$

can be recast in the form

$$\epsilon (\partial_\nu L - \partial_\lambda A) + \partial_\tau L - [A, L] = 0,$$

where we have split the time dependence $\partial_t$ into a fast time scale $\partial_\tau$ and a slow one $\epsilon \partial_\nu$. At zero order in $\epsilon$, (20) is simply the unperturbed problem. The first order correction gives the slow-time scale dependence of the modulated solution $u_\epsilon(\tau, \nu)$. Moreover, from the matrix elements of (20), we get after averaging over the fast motion in $\tau$ [51],

$$\partial_\nu \det L = -(2aA_{11} + bA_{21} + cA_{12}),$$

where the bar signifies $\tau$-averaging. Since for the unperturbed problem, $\partial_\nu \det L = 0$, the averaging is justified.

Equation (21) is simply the result of Bogoliubov-Whitham [52] averaging for the problem (17). It tells us how the previously invariant spectral curve now changes slowly in time over the large time scale $\nu$. It also gives us the form of the deformed solution $u_\epsilon(\tau, k_1(\nu))$, as a modulation of the original solution.

Note The fact that such perturbations generalize the autonomous Garnier system of [15] to a non-autonomous system of Schlesinger type was indicated in [51]. Extensions of these systems to include a constant matrix were discussed in [53].

3. The effect of weak perturbations

3.1. Topological classification of finite-gap solutions

In the presence of spectrum symmetry $\epsilon_k = -\epsilon_{-k}$, $S^z_k = -S^z_{-k}$, the distribution of cuts $C_k$ obeys the same symmetry. The number of non-degenerate cuts is therefore even. This analysis uses the fact that at $t = 0$ all Anderson pseudo-spins are aligned along the $z$ axis. In the simplest non-trivial case, there are only two non-degenerate cuts as shown in Figure 2 and $g = 1$, while the corresponding variable $u_1$ is given by elliptic functions. Other non-trivial cuts $C_k$ may exist in general, associated with dynamics of variables $u_k$. In the limit of “small” cuts [54], their contributions separate are given by trigonometric functions and the behavior of the gap parameter takes the simplified form

$$\log \left| \frac{\Delta(t)}{\Delta(0)} \right| = \Im \int u(t) dt = U_{\text{ell}}(t) + \sum_k U^k_{\text{trig}}.$$  

As we shall see, the most relevant contribution is due to the elliptic part $U_{\text{ell}}$, studied in the next section.
Relaxation of nonlinear oscillations in BCS superconductivity

3.2. Modulations of elliptic solutions

For the root distribution shown in Figure 2 there is only one variable $u_k$, taking imaginary values \[15\]. It solves an equation of the type \[15\]

$$ (\dot{u})^2 + (u^2 + m^2)(u^2 + M^2) = 0. \quad (23) $$

For simplicity, we therefore make the transformation $u \rightarrow iu$, keeping the time variable real. The new function is a real quartic oscillator which satisfies

$$ (\dot{u})^2 = (u^2 - M^2)(u^2 - m^2). \quad (24) $$

The solutions corresponding to this distribution of roots are

$$ u_1(t) = m \cdot sn(Mt + \phi_1, k_1), \quad k_1 = m/M, \quad (25) $$

$$ u_2(t) = M \cdot sn(mt + \phi_2, k_2), \quad k_2 = M/m, \quad (26) $$

where $sn$ is the Jacobi sine function, and $\phi_{1,2}$ are arbitrary phases. In the degenerate case $m = M$, the solutions become hyperbolic functions. Solution $u_2$ is non-physical in our case.

In order to set up the isomonodromic deformation method, consider the Lax pair \[55\]:

$$ L = -\left[2\dot{u}\sigma_1 + 4u\lambda\sigma_2 + (4\lambda^2 + \xi + 2u^2)i\sigma_3\right], \quad (27) $$

$$ A = i\lambda\sigma_3 - u\sigma_2, \quad \epsilon\partial_\lambda \Psi = L\Psi, \quad \partial_t \Psi = A\Psi, \quad (28) $$

where $\sigma_\alpha$, $\alpha = 1, 2, 3$ are the Pauli matrices, $\epsilon$ is a small real number, and $\Psi = (\psi_1, \psi_2)^t$ is the Baker-Akhiezer function. Choosing the initial condition $\psi_1 = 0, \psi_2 = 1$, we can extract the amplitude of the ratio $\Delta(t)/\Delta(0)$ from $\Psi(t)$ as

$$ \left| \frac{\Delta(t)}{\Delta(0)} \right| = \left[ \psi_1(t) + \overline{\psi}_1(t) \right] \overline{2}. \quad (29) $$

The compatibility (zero-curvature) conditions

$$ [\partial_t - A, \epsilon\partial_\lambda - L] = 0 \quad (30) $$
yield the system of equations
\[ \partial_t \xi = \epsilon, \quad \partial_t^2 u = 2u^3 - \xi u. \]  
(31)

We note that
\[ \det L = -4\{(\dot{u})^2 - [u^4 - \xi u^2 + (2\lambda^2 - \xi/2)^2]\}. \]  
(32)

Setting \( \epsilon = 0 \) gives the equation \( (\dot{u})^2 - u^4 + \xi u^2 = \text{constant} \), while the limit \( \epsilon = 1 \) yields the Painlevé II equation. In fact, the unperturbed case \( \epsilon = 0 \) allows to retrieve the full elliptic solution, from the equation
\[ L\Psi = 0, \quad \det(L) = 0, \]  
(33)

which gives the elliptical function \( u \) satisfying
\[ (\dot{u})^2 - [u^4 - \xi u^2 + (2\lambda^2 - \xi/2)^2] = 0. \]  
(34)

The physical solution \( u_1 \) is obtained identifying
\[ \xi = m^2 + M^2, \quad 2\lambda = M + m. \]  
(35)

Now restore \( \epsilon \neq 0 \), write \( \partial_t = \partial_\tau + \epsilon \partial_\xi \), and retain terms of order \( \epsilon \) from (30). Averaging over the fast variable \( \tau \) gives
\[ \partial_\xi \det L = L_{22} \partial_\lambda A_{11} + L_{11} \partial_\lambda A_{22}, \]  
(36)

or equivalently,
\[ \partial_\xi \det L = -(4u^2 - 2\xi + 8\lambda^2). \]  
(37)

Performing the computations, we obtain
\[ \partial_\xi [u^4 - \xi u^2 - (\dot{u})^2] = -u^2. \]  
(38)

Writing the elliptic equation as
\[ (\dot{u})^2 = u^4 - \xi u^2 + \mu^2, \]  
(39)

the physical solution takes the form
\[ u_1(t) = \sqrt{\xi - \sqrt{\xi^2 - 4\mu^2}} \text{sn} \left[ \sqrt{\xi + \sqrt{\xi^2 - 4\mu^2}} t \right], \]

up to an arbitrary initial phase \( \phi \), and elliptic modulus
\[ k^2 = \frac{1 - \sqrt{1 - (\frac{2\mu}{\xi})^2}}{1 + \sqrt{1 - (\frac{2\mu}{\xi})^2}}. \]  
(40)

The Whitham averaging equation has the form
\[ 4 \frac{\partial \mu^2}{\partial \xi^2} = 1 - \frac{2 \lambda \mathcal{E}(k^2)}{2 - k^2 \mathcal{K}(k^2)}, \]  
(41)

where \( \mathcal{E}, \mathcal{K} \) are the complete elliptic integrals of the first and second kind, respectively. Equation (41) has a fixed point at \( k = 0, \frac{\mu}{\xi} \rightarrow 0 \). This shows that, on the slow
time scale, the parameter $\mu/\xi = m/M + O(m^3/M^3)$ goes to zero, as $\xi = m^2 + M^2$ increases. Expanding the solution in this limit, and integrating under the separation of time scales assumption, we obtain for the elliptic contribution to the gap parameter, the approximation

$$\Delta_{\text{ell}}(t) = \Delta(0)e^{k^2\int \text{sn}(\tau,k^2)d\tau},$$

and $k^2 = \frac{\mu^2}{\xi^2} + O(\mu^4/\xi^4) \to 0$ as $t \to \infty$.

**Asymptotic behavior of modulated elliptic solutions** Starting from the Lax pair, we can obtain the asymptotic behavior of $\Psi(\lambda, t)$ as $t \to \infty$, in the whole complex $\lambda$ plane. The analysis is simpler when working in the variable $z = \frac{\lambda}{\sqrt{\xi}}$. There are six Stokes sectors at $z \to \infty$, with canonical asymptotes for $\Psi$, but the region of interest is $z = 1/2$, where

$$\frac{4\lambda^2}{\xi} \to 1, \quad \frac{\mu^2}{\xi^2} \to 0.$$ (43)

Using the Whitham equation, separation of scales, and identifying the strength of the fluctuations $\epsilon$ with an effective temperature $T$, we obtain for the elliptic contribution at large times

$$U_{\text{ell}}(t) = \frac{\mathcal{F}(t)}{tT},$$ (44)

where $\mathcal{F}$ is a bounded oscillatory function $|\mathcal{F}| = O(1)$.

**Modulations of trigonometric solutions** In order to analyze the slow dynamics of the small cuts, let $\xi \to \infty$ in and write the solution as

$$u = \frac{\mu}{\sqrt{\xi}} \cos(\sqrt{\xi}t + \phi), \quad \mu^2/\xi \to 0.$$ (45)

This assumption is consistent with. Formally, this is simply the trigonometric approximation of the general elliptic solution, in the limit of small modulus $k^2$. However, the parameter $\xi$ in must be sent to $\infty$ much faster than the physical $\xi = M^2 + m^2$, in order to give the correct spherical limit. The contributions from the small cuts therefore vanish faster than the elliptic component.

**Higher-genus contributions** In the case where there are several non-degenerate cuts at $t = 0$ corresponding to a high genus $g > 1$, the isomodromic deformation method can be applied in the same way as for $g = 1$, leading to modulated hyper-elliptic solutions of Painlevé equations degree higher than 2. There are few systematic results in this field, and a complete classification does not exist at this time. Specific high-genus solutions may have interesting topological properties with physical interpretations in terms of the collective excitations of the Gaudin magnet.
Appendix A. Isospectral case and the Abel-Jacobi inversion problem

In [9, 15], the system with fixed spectral curve was solved through inverse Abel-Jacobi mapping, by using Sklyanin separation of variables techniques [8, 56]. Interesting connections to generalized Neumann systems and Hitchin systems were discovered in [24]. The solution starts from the Lax operator

$$L(\lambda) = 2g^{\sigma_3} + \sum_{i=0}^{n} \vec{S}_i \cdot \vec{\sigma}_\lambda - \epsilon_i,$$

(A.1)

where $\sigma_\alpha$, $\alpha = 1, 2, 3$ are the Pauli matrices, and $\lambda$ is an additional complex variable, the spectral parameter. Let $u_k$, $k = 1, \ldots, n - 1$ be the roots of the coefficient $c(\lambda)$. Poisson brackets for variables $S^\alpha_i$ read

$$\{S^\alpha_j, S^\beta_k\} = 2\epsilon_{\alpha\beta\gamma} S^\gamma_k \delta_{jk}.$$

(A.2)

The Lax operator (A.1) defines a Riemann surface (the spectral curve) $\Gamma(y, \lambda)$ of genus $g = n - 1$, through

$$y^2 = Q(\lambda) = \det L(\lambda) \left[ gP(\lambda) - 2 \right]^2,$$

(A.3)

where $P(\lambda) = \prod_{i=1}^{n}(\lambda - \epsilon_i)$. The equations of motion for the Hamiltonian (9) become

$$\dot{u}_i = \frac{2iy(u_i)}{\prod_{j \neq i}(u_i - u_j)}, \quad iJ^- = J^- \left[ gj^3 + 2 \sum_{k=1}^{n} \epsilon_k - u \right].$$

(A.4)

In (A.4), $u = -2\sum_{i=1}^{n-1} u_i$, $b(u_i) = 0$.

From the equations of motion, it is clear that knowledge of the initial amplitude of $J^-$ and of the roots $\{u_i\}$ is enough to specify the $n$ unit vectors $\{S_i\}$, for a given set of constants of motion $\{R_t\}$ given by the classical limit of Gaudin Hamiltonian. The Dubrovin equations (A.4) are solved by the inverse of the Abel-Jacobi map, as we explain in the following. We begin by noting that the polynomial $Q(\lambda)$ has degree $2n$, and is positively defined on the real $\lambda$ axis. Therefore, the curve $\Gamma(y, \lambda)$ has $n$ cuts between the pairs of complex roots $[E_{2i-1}, E_{2i}]$, $i = 1, 2, \ldots, n$, perpendicular to the real $\lambda$ axis. The points $u_i$ belong to $n - 1$ of these cuts, $u_i \in [E_{2i-1}, E_{2i}]$, $i = 1, \ldots, n - 1$. These $g = n - 1$ cuts allow to define a canonical homology basis of $\Gamma$, consisting of cycles $\{\alpha_i, \beta_i\}$, $i = 1, \ldots, g$. With respect to these cycles, a basis of normalized holomorphic differentials $\{\omega_i\}$ can be defined, through

$$\mu_i = \lambda^{g-i} \frac{d\lambda}{y}, \quad M_{ij} = \int_{\alpha_j} \mu_i, \quad \omega = M^{-1} \mu.$$  

(A.5)

The period matrix $B_{ij} = \int_{\beta_j} \omega_i$ is symmetric and has positively defined imaginary part. The Riemann $\theta$ function is defined with the help of the period matrix as

$$\theta(z|B) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i(n^Tz + \frac{1}{2}n^TBn)}.$$

(A.6)
The $g$ vectors $B_k$ consisting of columns of $B$ and the basic vectors $e_k$ define a lattice in $\mathbb{C}^g$. The Jacobian variety of the curve $\Gamma$, is then the $g-$dimensional torus defined as the quotient $J(\Gamma) = \mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g)$. The Abel-Jacobi map associates to any point $P$ on $\Gamma$, a point ($g-$dimensional complex vector) on the Jacobian variety, through $A(P) = \int_{x}^{P} \omega$. Considering now a $g-$dimensional complex vector of points $\{P_k\}, k = 1, \ldots, g$ on $\Gamma$, defined up to a permutation, we can associate to it the point on the Jacobian
\[ z = a(P) = \sum_{k=1}^{g} A(P_k) + K,\] (A.7) where $K$ is the Riemann characteristic vector for $\Gamma$.

The map (A.7) suggests that we now have a way to describe the dynamics on $\Gamma$ by following the image point on the Jacobian. Given a point on the $g-$dimensional Jacobian $z = (\zeta_1, \ldots, \zeta_{n-1})$, we can find an unique set of points $\{\lambda_k\}, k = 1, \ldots, g$ on $\Gamma$, such that $z = a(\lambda)$, and $\theta(a(P) - z|B) = 0$. The system evolves in time according to the point $z(t)$
\[ \zeta_i = ic_k, \ 1 \leq i \leq g - 1, \ \zeta_{n-1} = i(c_{n-1} + t),\] (A.8) where $\{c_k\}$ is a set of initial conditions, such that $z_0 = z(t = 0) = a(c)$, and $c$ is the set of initial conditions for positions of $\lambda$ on $\Gamma$. Together with the initial condition which determines the initial amplitude of $J^-$, this set will determine entirely the evolution of the functions $u_i(t), J^-(t)$.

Acknowledgments

The author is grateful to I Aleiner, I Gruzberg, I Krichever and P Wiegmann for suggestions and contributions. Useful discussions with A G Abanov, B Altshuler, E Bettelheim, and E Yuzbashyan are acknowledged. The author also thanks I Aleiner and A Millis for support.

References

[1] R.W. Richardson. Phys. Lett., 3:277, 1963.
[2] R.W. Richardson. Phys. Lett., 5:82, 1963.
[3] R. W. Richardson and N. Sherman. Exact eigenstates of the pairing-force Hamiltonian. Nuclear Phys., 52:221–238, 1964.
[4] R. W. Richardson. Exact eigenstates of the pairing-force Hamiltonian. II. J. Mathematical Phys., 6:1034–1051, 1965.
[5] R.W. Richardson. Phys. Rev., 141:949, 1966.
[6] R.W. Richardson. J. Math. Phys., 9:1327, 1968.
[7] M. Gaudin. Diagonalisation d’une classe d’Hamiltoniens de spin. J. Physique, 37(10):1089–1098, 1976.
[8] E. K. Sklyanin. Separation of variables in the Gaudin model. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 164(Differentialnaya Geom. Gruppy Li i Mekh. IX):151–169, 1987.
[9] A. N. W. Hone, V. B. Kuznetsov, and O. Ragnisco. Bäcklund transformations for the sl(2) Gaudin magnet. J. Phys. A, 34(11):2477–2490, 2001.
Relaxation of nonlinear oscillations in BCS superconductivity

[10] V. B. Kuznetsov. Quadrics on Riemannian spaces of constant curvature. Separation of variables and a connection with the Gaudin magnet. *Teoret. Mat. Fiz.*, 91(1):83–111, 1992.

[11] V. Kuznetsov and P. Vanhaecke. Bäcklund transformations for finite-dimensional integrable systems: a geometric approach. *J. Geom. Phys.*, 44(1):1–40, 2002.

[12] E. G. Kalnins, V. B. Kuznetsov, and Willard Miller, Jr. Quadrics on complex Riemannian spaces of constant curvature, separation of variables, and the Gaudin magnet. *J. Math. Phys.*, 35(4):1710–1731, 1994.

[13] V. B. Kuznetsov. Isomorphism of an n-dimensional Neumann system and an n-site Gaudin magnet. *Funktsional. Anal. i Prilozhen.*, 26(4):88–90, 1992.

[14] V. B. Kuznetsov. Equivalence of two graphical calculi. *J. Phys. A*, 25(22):6005–6026, 1992.

[15] E. A. Yuzbashyan, B. L. Altshuler, V. B. Kuznetsov, and V. Z. Enolskii. Nonequilibrium Cooper Pairing in the Non-adiabatic Regime. *ArXiv Condensed Matter e-prints*, May 2005.

[16] G. Sierra. Integrability and conformal symmetry in the BCS model. In *Statistical field theories (Como, 2001)*, volume 73 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 317–328. Kluwer Acad. Publ., Dordrecht, 2002.

[17] R. A. Barankov, L. S. Levitov, and B. Z. Spivak. Collective Rabi Oscillations and Solitons in a Time-Dependent BCS Pairing Problem. *Phys. Rev. Lett.*, 93(16):160401, October 2004.

[18] M. C. Cambiaggi, A. M. F. Rivas, and M. Saraceno. Integrability of the pairing hamiltonian. *Nuclear Physics A*, 624:157–167, February 1997.

[19] L. Amico, A. di Lorenzo, and A. Osterloh. Integrable Model for Interacting Electrons in Metallic Grains. *Phys. Rev. Lett.*, 86:5759–5762, June 2001.

[20] J. Dukelsky, C. Esebbag, and P. Schuck. Class of Exactly Solvable Pairing Models. *Phys. Rev. Lett.*, 87(6):066403, August 2001.

[21] J. Dukelsky, S. Pittel, and G. Sierra. Colloquium: Exactly solvable Richardson-Gaudin models for many-body quantum systems. *Rev. Modern Phys.*, 76(3):643–662, 2004.

[22] M. Asorey, F. Falceto, and G. Sierra. Chern-Simons theory and BCS superconductivity. *Nuclear Phys. B*, 622(3):593–614, 2002.

[23] N. Manojlović and H. Samtleben. Schlesinger transformations and quantum R-matrices. *Comm. Math. Phys.*, 230(3):517–537, 2002.

[24] K. Hikami. Separation of variables in the BC-type Gaudin magnet. *J. Phys. A*, 28(14):4053–4061, 1995.

[25] F. Falceto and K. Gawędzki. Unitarity of the Knizhnik-Zamolodchikov-Bernard connection and the Bethe ansatz for the elliptic Hitchin systems. *Comm. Math. Phys.*, 183(2):267–290, 1997.

[26] F. Falceto and K. Gawędzki. Chern-Simons states at genus one. *Comm. Math. Phys.*, 159(3):549–579, 1994.

[27] G. Ortiz, R. Somma, J. Dukelsky, and S. Rombouts. Exactly-solvable models derived from a generalized Gaudin algebra. *Nuclear Phys. B*, 707(3):421–457, 2005.

[28] D. Korotkin and H. Samtleben. On the quantization of isomonodromic deformations on the torus. *Internat. J. Modern Phys. A*, 12(11):2013–2029, 1997.

[29] J. Harnad. Quantum isomonodromic deformations and the Knizhnik-Zamolodchikov equations. In *Symmetries and integrability of difference equations (Estérel, PQ, 1994)*, volume 9 of *CRM Proc. Lecture Notes*, pages 155–161. Amer. Math. Soc., Providence, RI, 1996.

[30] A. Di Lorenzo, L. Amico, K. Hikami, A. Osterloh, and G. Giaquinta. Quasi-classical descendants of disordered vertex models with boundaries. *Nuclear Phys. B*, 644(3):409–432, 2002.
Relaxation of nonlinear oscillations in BCS superconductivity

[35] A. Truscott, K. Strecker, G. Partridge, R. Hulet, and R. Hulet. Science, 291:2570, 2001.

[36] T. Loftus, C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin. Resonant Control of Elastic Collisions in an Optically Trapped Fermi Gas of Atoms. Phys. Rev. Lett., 88(17):173201, April 2002.

[37] K. M. O’Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas. Observation of a Strongly Interacting Degenerate Fermi Gas of Atoms. Science, 298:2179–2182, December 2002.

[38] K. M. O’Hara, S. L. Hemmer, S. R. Granade, M. E. Gehm, J. E. Thomas, V. Venturi, E. Tiesinga, and C. J. Williams. Measurement of the zero crossing in a Feshbach resonance of fermionic Li. Phys. Rev. A, 66(4):041401, October 2002.

[39] P. W. Anderson. Random-Phase Approximation in the Theory of Superconductivity. Physical Review, 112:1900–1916, December 1958.

[40] A. V. Andreev, V. Gurarie, and L. Radzihovsky. Nonequilibrium Dynamics and Thermodynamics of a Degenerate Fermi Gas Across a Feshbach Resonance. Phys. Rev. Lett., 88(17):173201, April 2002.

[41] G. L. Warner and A. J. Leggett. Quench dynamics of a superfluid Fermi gas. Phys. Rev. B, 71(13):134514, April 2005.

[42] E. A. Yuzbashyan, O. Tsyplyatyev, and B. L. Altshuler. Relaxation and persistent oscillations of the order parameter in the non-stationary BCS theory. ArXiv Condensed Matter e-prints, November 2005.

[43] I. Krichever. Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations. Mosc. Math. J., 2(4):717–752, 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[44] I. Krichever. Vector bundles and Lax equations on algebraic curves. Comm. Math. Phys., 229(2):229–269, 2002.

[45] L. A. Dickey. Soliton equations and Hamiltonian systems, volume 26 of Advanced Series in Mathematical Physics. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[46] I. Aleiner and R. Teodorescu. Unpublished.

[47] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III. Phys. D, 4(1):26–46, 1981/82.

[48] M. Jimbo, T. Miwa, and K. Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function. Phys. D, 2(2):306–352, 1981.

[49] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. Phys. D, 2(3):407–448, 1981.

[50] D. Mumford. Tata lectures on theta. II, volume 43 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1984.

[51] K. Takasaki. Spectral curves and Whitham equations in isomonodromic problems of Schlesinger type. Asian J. Math., 2(4):1049–1078, 1998.

[52] G. B. Whitham. Linear and nonlinear waves. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, 1999.

[53] A. Beauville. Jacobienne des courbes spectrales et systèmes hamiltoniens complètement intégrables. Acta Math., 164(3-4):211–235, 1990.

[54] E.D. Belokolos, A.I. Bobenko, V.Z. Enol’skii, A.R. Its, and V.B. Matveev. Algebro-Geometric Approach to Nonlinear Integrable Equations, volume 11 of Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, 1994.

[55] A. R. Its. The Painlevé transcendentals as nonlinear special functions. In Painlevé transcendentals (Sainte-Adèle, PQ, 1990), volume 278 of NATO Adv. Sci. Inst. Ser. B Phys., pages 49–59. Plenum, New York, 1992.

[56] A. Nakayashiki and F. A. Smirnov. Cohomologies of affine hyperelliptic Jacobi varieties and integrable systems. Comm. Math. Phys., 217(3):623–652, 2001.