Sub-shot noise sensitivities without entanglement

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Abstract

It is commonly maintained that entanglement is necessary to beat the shot noise limit in the sensitivity with which certain parameters can be measured in interferometric experiments. Here we show that, with a fluctuating number of two-mode bosons, the shot-noise limit can be beaten by non-entangled bosonic states with all bosons in one mode. For a given finite maximum number of bosons, we calculate the optimal one- and two-mode bosonic states, and show that in the absence of losses, NOON states are the optimal two-mode bosonic states.
I. INTRODUCTION

Suppose a parameter-dependent probability distribution $\mu_\theta(\xi)$ arises from the description of a classical system consisting of $N$ independent parties. No matter what estimator one uses for estimating the parameter $\theta$ from measured values $\xi_i$ drawn from the probability distribution, a universal lower bound to the best mean square error in the determination of $\theta$ is given by the inverse of the classical Fisher information $F[\mu, \theta]$: this at best behaves as $1/N$, a scaling known as shot-noise limit or standard quantum limit [1, 2].

The field of quantum metrology concerns the use of quantum mechanical features to improve on the above classical limitation [3, 4]. In particular, using systems consisting of $N$ subsystems prepared in entangled states, one can prove that the shot-noise limit can be beaten [5]. The squared sensitivity of the determination of a parameter $\theta$ has been proved to be bounded from below by the inverse of a quantity known as quantum Fisher information [6, 7]. The quantum Fisher information can scale as fast as $N^2$ if the state $\rho_\theta$ of the system represents certain specific $N$-partite entangled states, a scaling known as the Heisenberg limit. Based on this, in the literature one often finds stated that, albeit not sufficient, entanglement is necessary for overcoming the shot-noise limit (see [8] for a recent review).

However, in experimental contexts where identical particles are used for metrological purposes, as for instance ultracold atoms trapped in double-well potentials which can be effectively described as two-mode bosons [9, 10], the very notion of entanglement, that is of quantum non-locality, has to be generalized with respect to the case when the constituent parties, say qubits, are distinguishable. Indeed, in this latter case there is a predetermined tensor product structure related to the particle aspect of first quantization: for instance, for two qubits the algebra of observables is $M_2 \otimes M_2$, where $M_2$ is the algebra of $2 \times 2$ matrices for the first and second qubit, respectively. Instead, in the case of identical particles, such a structure is no more available and one is forced to speak of entanglement always in relation to a given algebraic context specified by a suitable mode description typical of the second quantization formalism [11, 12].

In the following, we show that in the case when the number of identical bosons is not fixed, the shot-noise limit $1/\overline{N}$ represented by the inverse of the average boson number $\overline{N}$, can be beaten by non-entangled states with all bosons in one mode, without the need of partially populating the other mode. We also identify the optimal one-mode and two
mode pure states in the sense of maximum quantum Fisher information for given maximum number of bosons and show that for two modes these are NOON states.

II. QUANTUM METROLOGY WITH IDENTICAL PARTICLES

A. Basic quantum parameter estimation theory

Consider a (possibly mixed) quantum state \( \rho_\theta \) that depends on the parameter \( \theta \) whose value we want to find out as precisely as possible. \( N \) repeated generalized measurements with POVM elements (non-negative Hermitian operators \( E(\xi) \), \( \int d\xi E(\xi) = 1 \)) in the identically prepared state \( \rho_\theta \) leads to \( N \) measurement outcomes \( \xi_i \) (\( i = 1, \ldots, N \)), distributed according to \( \mu_\theta(\xi) = \text{tr}\rho_\theta E(\xi) \). One estimates the value of \( \theta \) based on these \( N \) outcomes \( \xi_i \) with an estimator function \( \theta_{\text{est}}(\xi_1, \ldots, \xi_N) \). The squared sensitivity with which \( \theta \) can be estimated from the data is defined as \( \langle (\delta\theta)^2 \rangle \), where

\[
\delta \theta = \frac{\theta_{\text{est}}}{d\theta_{\text{est}}} - \theta,
\]

and the average \( \langle \ldots \rangle \) is over \( \mu_\theta(\xi) \), \( \langle \theta_{\text{est}}(\xi_1, \ldots, \xi_N) \rangle = \int \left( \prod_{i=1}^N \mu_\theta(\xi_i) d\xi_i \right) \theta_{\text{est}}(\xi_1, \ldots, \xi_N) \).

For an unbiased estimator (\( \langle \theta_{\text{est}} \rangle = \theta \) locally at the value of \( \theta \) we are interested in, which we take without restriction of generality as \( \theta = 0 \) in the following), a universal lower bound of \( \langle (\delta\theta)^2 \rangle \) is provided by the inverse of the quantum Fisher information \( F[\rho] \),

\[
\langle (\delta\theta)^2 \rangle \geq \frac{1}{F[\rho]}.
\]

where \( F[\rho] = \text{Tr}(\rho L^2) \) and

\[
\partial_\theta \rho |_{\theta=0} = \frac{1}{2} \left( \rho L + L \rho \right)
\]

defines the symmetric logarithmic derivative \( L \) of the quantum state. This so-called quantum Cramér-Rao bound \([6, 7]\) limits the best sensitivity achievable for a given parameter dependent state \( \rho_\theta \), regardless of the choice of measurements and the data-analysis, as it is optimized over all POVM measurements, in addition to the optimization over all possible estimators used for the derivation of the classical Cramér-Rao bound \([1]\). According to Fisher’s theorem, the bound can be saturated in the limit of \( N \to \infty \) \([13]\). Beating the shot-noise limit, that is making

\[
\langle (\delta\theta)^2 \rangle < \frac{1}{N}
\]
necessarily requires $F[\rho] > N$. Note that instead of measuring the same system $N$ times with an identical initial preparation for each measurement, one can equivalently measure once a composite system consisting of $N$ identical subsystems in an initial product state.

The quantum Fisher information can be written as $F[\rho] = 4d_{\text{Bures}}^2(\rho_\theta, \rho_{\theta+d\theta})$ in terms of the Bures distance $d_{\text{Bures}}(\rho, \sigma) = \sqrt{2(1 - f(\rho, \sigma))}$, where the fidelity $f(\rho, \sigma) = \text{tr}(\rho^{1/2} \sigma \rho^{1/2})$ \cite{7, 14}. For pure states $\rho = |\psi\rangle\langle\psi|$, $\sigma = |\phi\rangle\langle\phi|$ the fidelity $f$ reduces to the overlap $f(\rho, \sigma) = |\langle\psi|\phi\rangle|$. Therefore, one has the intuitive and information-theoretically plausible interpretation that the distinguishability of two neighboring states whose parameters $\theta$ differ by an infinitesimal amount $d\theta$ determines the best sensitivity with which $\theta$ can be obtained through measurement of whatever observables.

If $\rho = \rho_{\theta=0}$ and $\rho_{d\theta}$ is created from $\rho$ through a unitary rotation with self-adjoint generator $J = J^\dagger$ from $\rho$,

$$\rho \mapsto \rho_{d\theta} = e^{i \theta J^\dagger} \rho_0 e^{i \theta J}, \quad (5)$$

one shows that

$$F[\rho] = 4\Delta^2_{\rho} J = 4(\text{tr}\rho_0 J^2 - (\text{tr}\rho_0 J)^2) \equiv F[\rho, J], \quad (6)$$

where in the last step and from now on we make the dependence on the generator $J$ explicit \cite{7}. For a pure, fully separable state $\rho = |\psi\rangle\langle\psi|$ of $N$ distinguishable subsystems and a generator $J$ that is a sum of operators of the individual subsystems, it turns out that $\Delta^2_{\psi} J \leq N/4$ \cite{7}. It follows that in such a situation entanglement is necessary to achieve sensitivities beyond the shot-noise limit.

Several ways are known by now how this limitation can be surpassed. The first one is the use of $k$-body interactions (also known as non-linear schemes) which can offer a scaling $\Delta^2_{\psi} J \propto N^{2k-1}$ without entanglement (and $N^{2k}$ with entanglement \cite{15–22}). This requires, however, having $N$ particles all interact with each other. In some cases, such as light induced interactions in a Bose-condensate, interactions can be relatively naturally induced. They typically lead to squeezed states and resemble in this respect the earliest examples of quantum-enhanced measurements that proposed the use of squeezed light \cite{3, 4}. Another way is having $N$ distinguishable subsystems interact with a single $N + 1$st system and read out the latter \cite{23, 24}. This method has the advantage that the system needs to accommodate only $N$ interaction terms. Furthermore, the scaling with $N$ is stable under
local decoherence, and even decoherence itself can be used as a signal, if the $N+1$st system is an environment.

In the following we explore a third option, namely the use of indistinguishable particles. Before addressing this possibility, it is necessary to stress that, for identical particles, the notion of separability (entanglement) cannot be given independently of the modes that are selected for the description of the system.

B. Separability and entanglement for identical particles

Identical bosons are best addressed within the second quantization formalism by means of the Fock representation: we shall denote by $|\text{vac}\rangle$ the vacuum state and by $a_i, a_i^\dagger$ the annihilation and creation operators relative to an orthonormal basis $\{ |i\rangle \}_{i \in I}$ in the single particle Hilbert space. They satisfy the commutation relations $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$; furthermore, states $|n_1, n_2, \ldots, n_k\rangle$ with $n_i$ bosons in the single particle states $|i\rangle$, $i = 1, 2, \ldots, k$, are generated by acting on the vacuum as follows

$$|n_1, n_2, \ldots, n_k\rangle = \frac{\prod_{i=1}^{k} (a_i^\dagger)^{n_i}}{\sqrt{\prod_{i=1}^{k} n_i!}} |\text{vac}\rangle. \quad (7)$$

In quantum optics one deals with exactly the same type of multi-mode Fock-states defined in eq.(7), even though their physical meaning is somewhat different. The quantization of the electro-magnetic field starts with the identification of a set of orthonormal mode functions which are solutions of the classical Maxwell-equations with appropriate boundary conditions, such as plane waves with a given wave-vector and polarization in the case of vacuum and periodic boundary conditions. Each of these modes corresponds to a harmonic oscillator due to the fact that the energy of the electro-magnetic field is quadratic in both the electric and magnetic fields. The multi-mode Fock state (7) therefore has the meaning of a product state of a set of physical harmonic oscillators. The index $i$ labels the harmonic oscillator (alias mode of the classical electro-magnetic field), and arises from simple first quantization. A state $(a_i^\dagger)^n |\text{vac}\rangle$ means the $i$-th oscillator being excited in the $n$-th one of its excited states, i.e. it corresponds to $n$ photons in mode $i$ (see e.g. [25] or any other text-book on quantum optics). This is in contrast to the second quantization formalism for massive identical bosons, where $i$ labels different single particle orthonormal basis vectors; in this case, $(a_i^\dagger)^n |\text{vac}\rangle$ means that $n$ identical bosons are created in the $i$-th single particle state. In
the Bose-Hubbard approximation, instances of single atom orthonormal bases for ultracold atoms trapped by a double-well potential are states corresponding to an atom being localized in either one or the other of the two wells or, the first two energy eigen-states of the single-particle Hamiltonian. Despite the different physical meaning, the formalism is exactly the same in both cases, and we will therefore not distinguish between massive identical bosons treated in second quantization and photons in quantum optics, but have both situations in mind when we speak of “indistinguishable particles”.

In the following we shall be dealing with identical bosons that can be found in two modes identified by pairs of creation and annihilation operators \( a, a^\dagger \), respectively \( b, b^\dagger \) satisfying the canonical commutation relations \([a, a^\dagger] = [b, b^\dagger] = 1\), while the remaining ones all vanish. We shall consider the Fock representation based on a vacuum state \( |\text{vac}\rangle \) so that the states

\[
|n_a, n_b\rangle = \frac{(a^\dagger)^{n_a}(b^\dagger)^{n_b}}{\sqrt{n_a!n_b!}}|\text{vac}\rangle \quad n_{a,b} \in \mathbb{N}
\]

constitute the orthonormal basis of eigenstates of the Fock number operator \( a^\dagger a + b^\dagger b \) with \( n_a \) bosons in one mode and \( n_b \) bosons in the other one.

In the second quantization formalism there is no pre-defined algebraic tensor product structure as for distinguishable particles; in the latter case, one starts out with the tensor product of the algebras of operators acting on the Hilbert spaces of the single particles: for instance, in the case of one qubit the operator algebra is the \( 2 \times 2 \) complex matrix algebra \( M_2 \) and in the case of two distinguishable qubits, it is the \( 4 \times 4 \) matrix algebra \( M_2 \otimes M_2 \).

In absence of a definite tensor product structure, a new approach to locality (of observables) and separability (of states) must be developed \[11,12\]: observe that the main property of local observables \( A \otimes 1 \) and \( 1 \otimes B \) for a bi-partite system consisting of distinguishable particles is that they commute. In the case of two bosonic modes, the tensor product structure can thus be replaced by pairs of commuting sub-algebras \( (A, B) \) generated by \( \{a, a^\dagger\} \), respectively \( \{b, b^\dagger\} \) whereby operators of the form \( AB \) with \( A \in A \) and \( B \in B \) can be termed local with respect to the pair \( (A, B) \), or \( (A, B) \)-local. Furthermore, one can extend the notion of separability as follows: states \( \omega \) on the Bose algebra \( \mathcal{M} \) of the two-mode system are generic expectations (linear positive and normalized functionals) \( \mathcal{M} \ni X \mapsto \omega(X) \in \mathbb{C} \). Then, a state \( \omega \) will be called separable with respect to the pair \( (A, B) \), or \( (A, B) \)-separable, if, on local observables, it splits into a convex combinations of products of expectations with
respect to other states, namely if
\[ \omega(AB) = \sum_i \lambda_i \omega_i^{(1)}(A) \omega_i^{(2)}(B), \]
for all \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \).

The simplest examples of \((\mathcal{A}, \mathcal{B})\)-separable states are the Fock states in (8); indeed, expectations of \((\mathcal{A}, \mathcal{B})\)-local observables \(AB, A \in \mathcal{A}, B \in \mathcal{B}\) factorize,
\[ \langle n_a, n_b | AB | n_a, n_b \rangle = \langle n_a, n_b | A | n_a, n_b \rangle \langle n_a, n_b | B | n_a, n_b \rangle, \]
and show no correlation among these commuting observables.

The definitions of locality and separability given above reduce to the standard ones in the case of two distinguishable particles; in this case one identifies the sub-algebras \( \mathcal{A} \) and \( \mathcal{B} \) with the single particle algebras of operators, \( \mathcal{M} \) with their tensor product and the states \( \omega(X) \) with the expectations \( \text{Tr}(\rho X) \) corresponding to density matrices \( \rho \).

What should be remarked is that, while in the case of distinguishable particles the tensor product structure is somewhat taken for granted and one need not specify that locality and separability always refer to it, it is not so in the case of identical bosons: in such a case it must always be specified with respect to which pair \((\mathcal{A}, \mathcal{B})\) a state is separable. Indeed, it is easy to see that Bogolubov transformations, as those implemented by beam-splitters either in quantum optics or in cold atom interferometry, transform the mode operators \( \{a, a^\dagger\}, \{b, b^\dagger\} \) into new mode operators \( \{c, c^\dagger\}, \{d, d^\dagger\} \) such that the \((\mathcal{A}, \mathcal{B})\)-separable state in (8) turns out not to be separable with respect to the new pair of commuting sub-algebras generated by \( \{c, c^\dagger\} \) and \( \{d, d^\dagger\} \). Other definitions of entanglement have been discussed in the literature, e.g. in the context of “generalized entanglement” (see [26–28] and references therein).

C. Fock states

In order to illustrate the physical consequences of the mode-dependent formulation of locality and entanglement, we show that entanglement is not necessary for beating the shot noise limit when one deals with identical particles. To this end, consider an \((\mathcal{A}, \mathcal{B})\)-separable Fock state as in (8) with \( k \) bosons in mode \( a \) and \( N - k \) bosons in mode \( b \),
\[ |k, N - k\rangle = \frac{(a^\dagger)^k (b^\dagger)^{N-k} |\text{vac}\rangle}{\sqrt{k!(N-k)!}}. \]
We call a state with \( k = N \) or \( k = 0 \) a one-mode state, as all bosons are in the first or second mode, respectively.

Let us now consider the Schwinger-representation that associates to the two-mode bosons the angular-momentum-like operators

\[
J_x = \frac{a^\dagger b + ab^\dagger}{2}, \quad J_y = \frac{a^\dagger b - ab^\dagger}{2i}, \quad J_z = \frac{a^\dagger a - b^\dagger b}{2}.
\]  

A pseudo-rotation \( U(\theta) = e^{i \theta J_x} \) is \((\mathcal{A}, \mathcal{B})\)-non-local for \( U(\theta) \) as it cannot be split into the product of some \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \). Indeed, the action of \( U(\theta) \) on the \((\mathcal{A}, \mathcal{B})\)-separable state \( |k, N-k\rangle \) makes the state \((\mathcal{A}, \mathcal{B})\)-entangled. Furthermore, unlike for distinguishable qubits, the corresponding Fisher information,

\[
F[|k, N-k\rangle, J_x] = 4 \Delta^2 |k, N-k\rangle J_x = 4 \langle k, N-k | J_x^2 |k, N-k \rangle
\]

\[
= N (2k + 1) - 2k^2,
\]

exceeds \( N \) for all \( k \neq 0 \) and \( k \neq N \).

Therefore, except when all identical bosons are in one mode and none in the other, despite the separability of the Fock state, one can beat the shot-noise limit profiting from the non-locality of the rotation generated by \( J_x \).

In the following section we show that state-entanglement is not necessary to beat the shot-noise limit even when, differently from the case just discussed, all identical qubits are in one of the two modes.

**III. PHASE ESTIMATION WITH A MACH-ZEHNDER INTERFEROMETER**

In this section, we identify the identical bosons with photons in a Mach-Zehnder interferometer (MZ); notice, however, that the same formalism applies to matter interferometry based on trapped ultracold atoms, which is also currently pursued (see for instance [29]).

Let us consider a MZ consisting of two equal beam-splitters (BS); the first BS generates a Bogoliubov rotation of the two modes \( a \) and \( b \) associated with the two arms of the interferometer,

\[
U_{BS}(\alpha) a U_{BS}(-\alpha) = a \cos |\alpha| - b e^{i \text{arg}(\alpha)} \sin |\alpha|
\]

\[
U_{BS}(\alpha) b U_{BS}(-\alpha) = b \cos |\alpha| - a e^{i \text{arg}(\alpha)} \sin |\alpha|,
\]
with $\alpha$ a complex parameter characteristic of the $BS$, via the unitary operator

$$U_{BS}(\alpha) = e^{\alpha a^\dagger b - \alpha^* a b^\dagger}.$$  \hfill (13)

After the first BS, along the arm of the MZ described by $a, a^\dagger$, a unitary rotation by an angle $\phi$ is generated by the number operator $a^\dagger a$. This is in turn followed by a second BS which recombine the beams by means of $U_{BS}(-\alpha)$.

The total effect on an incoming state $\rho$ is described by the unitary operator

$$U_{MZ}(\alpha, \phi) = U_{BS}(-\alpha) e^{i\phi a^\dagger a} U_{BS}(\alpha) = e^{i\phi J}$$

$$J = \left( \cos^2 |\alpha| a^\dagger a + \sin^2 |\alpha| b^\dagger b \right.$$

$$- \cos |\alpha| \sin |\alpha| (e^{i \text{arg}(\alpha)} a^\dagger b - e^{-i \text{arg}(\alpha)} b^\dagger a) \right).$$  \hfill (14)

A. One mode photon states

Suppose a system of identical bosons is prepared as input to the interferometer in a state of the form

$$|\Psi\rangle = \sum_k c_k |k, 0\rangle, \quad |k, 0\rangle = \frac{(a^\dagger)^k}{\sqrt{k!}} |\text{vac}\rangle,$$  \hfill (15)

with all bosons in mode $a$, that is $a^\dagger a |k, 0\rangle = k |k, 0\rangle, b^\dagger b |k, 0\rangle = 0$. We shall only demand that the mean boson number

$$\overline{N} = \langle \Psi | (a^\dagger a + b^\dagger b) |\Psi\rangle = \sum_k p_k k$$  \hfill (16)

be finite, where $p_k = |c_k|^2, \sum_k p_k = 1$. Using (6), the quantum Fisher information corresponding to such a state and the generator $J$ in (14) is readily computed:

$$F[\Psi, J] = 4 \left[ \cos^4 |\alpha| \left( \sum_k p_k k^2 - \left( \sum_k p_k k \right)^2 \right) \right.$$  \hfill (17)

$$+ \cos^2 |\alpha| \sin^2 |\alpha| \sum_k p_k k \right].$$

The term in parenthesis in the first line is the mean square error of a stochastic variable $X$ taking values on the natural numbers $X \in \mathbb{N}$ distributed according to the probabilities $p(X = k) = p_k$. The variance is always non-negative; whence, choosing $\alpha = \pi/4$ yields $F[\Psi, J] \geq \overline{N}$. This already indicates the possibility of beating the shot-noise limit, that is the bound (4) with the mean photon number $\overline{N}$ in the place of $N$.  

\hfill 9
Indeed, consider a balanced BS ($\alpha = \pi/4$) and choose an input state with a finite fixed maximum number $K$ of bosons,

$$|\Psi\rangle = \sum_{k=0}^{K} c_k |k,0\rangle.$$  \hfill (18)

By isolating in (17) the term $k = K$ and using $K p_K = \overline{N} - \sum_{k=1}^{K-1} p_k k$, one gets

$$F[\Psi, J] = \overline{N} \left( 1 + K - \overline{N} \right) + \sum_{k=1}^{K-1} p_k k(k - K).$$

Each term in the last sum is negative. The quantum Fisher information is thus optimized by choosing $p_k = 0$ for all $k \neq 0, K$, which in turn implies $p_K = \overline{N}/K$ and

$$|\Psi\rangle = \sqrt{1 - \frac{\overline{N}}{K}} |00\rangle + e^{i\chi} \sqrt{\frac{\overline{N}}{K}} |K,0\rangle.$$  \hfill (19)

Then,

$$F[\Psi, J] = \overline{N} \left( K - \overline{N} + 1 \right).$$  \hfill (20)

Since the Fisher information is larger than the mean particle number, it thus follows that the shot-noise limit can be beaten by choosing a suitable $\overline{N}$. A similar conclusion has been reached in [30], where, differently from here, the authors considered a superposition of the vacuum state with a squeezed state acted upon by a rotation generated by the number operator. Earlier work on optimizing states for minimal phase uncertainty, notably in the context of squeezed states, can be found in [31–35]. Most of this earlier work used the notion of an approximate phase operator. In [36] it was shown that optimal sensitivity of a Mach-Zehnder interferometer using multimode Gaussian states can be reached without entanglement by appropriate mode-engineering.

One might wonder about the importance of the scaling with the average boson number $\overline{N}$ instead of $N$. Of course, $N$ is only well defined for a Fock state, whereas for all other states one has to live with fluctuating $N$. For laser light in a coherent state with $\overline{N} \gg 1$ or even the most squeezed states currently available [37, 38], the fluctuations of $N$ still satisfy $\sigma(N)/N \ll 1$, and the average photon number is therefore representative of the photon number in any realization. A state of the form (19) with $\overline{N} \approx K/2$ maximizes the photon-number fluctuations, however, and obviously the average value of $N$ is never realized (only $N = 0$ or $N = K$ are). Nevertheless, the scaling with $\overline{N}$ is highly relevant practically,
as it corresponds to the mean energy in the state, which is indeed what makes producing
the state costly. In [31] a state was proposed that leads to a very sharp maximum in the
distribution of measured rotation angles, suggesting even exponential scaling of the phase
uncertainty with $N$.

From a physical perspective it makes sense that the optimal state leads to maximum
uncertainty in $N$, as, for a Heisenberg-uncertainty limited state, this corresponds to min-
imal uncertainty in $\theta$. Since there is no entirely satisfying definition of a phase operator,
“Heisenberg-uncertainty limited” means here a state that satisfies the Cramér-Rao bound.
Indeed, inequality (2) has been understood from the beginning as a generalization of Heisen-
berg’s uncertainty relation [7]. Eq.(6) shows that the relevant “complementary observable”
is the generator $J$ of eq.(5). Furthermore, for the one-mode states considered here, the fluc-
tuations of $J$ are given by the fluctuations of $N$. Therefore the optimal one mode state must be
indeed the state that maximizes the photon number fluctuations in that mode. The single
mode “ON” state ($|0\rangle + |N\rangle)/\sqrt{N}$ was also identified as the optimal state for obtaining the
best possible sensitivity of mass measurements with nano-mechanical harmonic oscillators [39]. It has a Wigner function with $N$ lobes in azimuthal direction.

### B. Optimal state

One might wonder what state is optimal for a given maximum photon number. In [40]
umerical and some analytical evidence was shown that the so-called NOON-state [41] is the
optimal state when losses are neglected. It has meanwhile become clear that NOON states
are not very useful in practice, as the slightest chance of photon loss leads, in the limit of
large $N$, back to the standard quantum limit [42–44]. The analysis in [40] was ba-
sed on the classical Fisher information and photon counting measurements at the two output ports
of the MZ. Here we give a simple demonstration using the quantum Cramér-Rao bound that the
NOON state in absence of losses and dephasing is optimal for a balanced MZ-interferometer
no matter what measurement is performed in the end.

As before, we use the Schwinger-representation [9]. For a balanced MZ-interferometer
($\alpha = \pi/4$) eq. (14) gives

\[
J = (a^\dagger a + b^\dagger b + a^\dagger b + ab^\dagger)/2 = \frac{\hat{N}}{2} + J_x ,
\]

(21)
where \( \hat{N} = a^\dagger a + b^\dagger b \) is the total photon number operator. It proves convenient to introduce a new class of orthogonal states adapted to the Schwinger representation: these are the pseudo-angular momentum states \( |jm\rangle_\ell, \ell = x, y, z \), such that

\[
J_\ell|jm\rangle_\ell = m|jm\rangle_\ell, \quad J^2|jm\rangle_\ell = j(j+1)|jm\rangle_\ell.
\]

(22)

One easily shows that \( J^2 = (\hat{N}/2)(\hat{N}/2 + 1) \), which shows that the usual pseudo-angular momentum states \( |jm\rangle_\ell \) are also eigenstates of \( \hat{N}, \hat{N}|jm\rangle_\ell = 2j|jm\rangle_\ell \). Furthermore, if \( \ell = z \), using the expression of \( J_z \) in eq. (9), the Fock states \( |k,N-k\rangle \) can be recast as common eigenstates \( |jm\rangle_z \) of \( J^2 \) and \( J_z \) with

\[
j = \frac{N}{2}, \quad -j \leq m = k - \frac{N}{2} \leq j.
\]

(23)

Consider now first a state \( |\psi\rangle \) with fixed \( j \). It is useful to write \( |\psi\rangle \) in the \( J_x \) eigenbasis,

\[
|\psi\rangle = \sum_{m=-j}^j c_m|jm\rangle_x,
\]

with \( J_x|jm\rangle_x = m|jm\rangle_x \). There are no fluctuations from \( \hat{N}/2 \) in \( J \), and from eq. (21) we have thus \( \Delta^2_\psi J = \langle J_x^2 \rangle - \langle J_x \rangle^2 \). Inserting the expansion of \( |\psi\rangle \) in the \( J_x \) eigenbasis, we are led to

\[
\Delta^2_\psi J = \sum_{m=-j}^j p_m m^2 - \left( \sum_{m=-j}^j p_m m \right)^2,
\]

(24)

with \( p_m = |c_m|^2 \).

Let \( \Delta_\pi \) denote the right hand side of eq. (24) depending on the distribution \( \pi = \{p_m\}_{m=-j}^j \); because of the apparent symmetry of the expression, the distribution \( \pi' = \{p_{-m}\}_{m=-j}^j \) obtained by exchanging \( p_m \) with \( p_{-m} \) leads to \( \Delta_{\pi'} = \Delta_\pi \). Let us then consider the symmetrized distribution \( \pi_{\text{sym}} = \left\{ \frac{p_m + p_{-m}}{2} \right\}_{m=-j}^j \); convexity yields

\[
\left( \frac{\sum_{m=-j}^j \frac{p_m + p_{-m}}{2} m}{\sum_{m=-j}^j \frac{p_m + p_{-m}}{2}} \right)^2 \leq \frac{1}{2} \left( \sum_{m=-j}^j p_m m \right)^2 + \frac{1}{2} \left( \sum_{m=-j}^j p_{-m} m \right)^2,
\]

whence \( \Delta_{\pi_{\text{sym}}} \geq \Delta_\pi \), i.e. a generic distribution \( \pi \) cannot provide a \( \Delta_\pi \) larger than the \( \Delta_{\pi_{\text{sym}}} \) obtained by symmetrizing it. The maximum \( \Delta_\pi \) must then be attained at distributions with the property that \( p_m = p_{-m} \); this means maximizing \( \Delta^2_\psi J = 2 \sum_{m=1}^j p_m m^2 \) under the constraints \( 0 \leq p_m \leq 1/2 \) for all \( m \) and \( \sum_{m=-j}^j p_m = 1 \). Since \( \langle \psi|J^2_x|\psi\rangle \leq j^2 \), for fixed \( j \), the maximum of the variance is attained at \( p_j = 1/2 = p_{-j} \).
Next, consider a general state with at most $j_{\text{max}}$ excitations,

$$\ket{\psi} = \sum_{j=0}^{j_{\text{max}}} \sum_{j=-m}^{m} c_{jm} \ket{j m}_x .$$

(25)

Since both $\hat{N}$ and $J_x$ conserve $j$, we get

$$\langle J \rangle = \sum_{m,n} c^*_n c_{jm} \langle j n | J | j m \rangle_x ,$$

(26)

and similarly for $J^2$. It is useful to introduce the notation $\langle X \rangle_j = j \langle \psi | X | \psi \rangle_j$, where $|\psi\rangle_j$ is the wave-function in the $j$ sector, that is $|\psi\rangle_j = \sum_m c_{jm} \ket{j m}_x / \sqrt{p_j}$, with $p_j = \sum_m |c_{jm}|^2$ assuring the correct normalization. This gives

$$\Delta^2_{\psi J} = \sum_j p_j \langle J^2 \rangle_j - \left( \sum_j p_j \langle J \rangle_j \right)^2 = \sum_j p_j \langle (J - \langle J \rangle_j)^2 \rangle_j .$$

(27)

Since $\max_{\{c_{jm}\}} \langle (J - \langle J \rangle_j)^2 \rangle_j = j^2$ grows monotonically with $j$, we obtain the maximum of $\Delta^2_{\psi J}$ over all $c_{jm}$ by choosing $p_{j_{\text{max}}} = 1$ (and correspondingly all other $p_j = 0$). Thus, the state that maximizes the quantum Fisher information is

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left( |j_{\text{max}} j_{\text{max}}\rangle_x + e^{i\chi} |j_{\text{max}} - j_{\text{max}}\rangle_x \right) ,$$

(28)

where $\chi$ is an arbitrary phase.

In order to give a physical interpretation to such states, let us consider the Bogolubov transformation

$$a = \frac{c + d}{\sqrt{2}} , \quad b = \frac{c - d}{\sqrt{2}}$$

(29)

to new modes described by creation (annihilation) operators $c$, $d$ ($c^\dagger$, $d^\dagger$). With reference to the new modes, the pseudo angular momentum operator $J_x$ becomes

$$J_x = \frac{c^\dagger c - d^\dagger d}{2} .$$

(30)

In terms of occupation number states of these modes, it follows that the state (28) has the form of a NOON state (see (23), namely a superposition of all photons in mode $c$ and all photons in mode $d$:

$$|\psi\rangle = \frac{(c^\dagger)^{2j_{\text{max}}} + e^{i\chi} (d^\dagger)^{2j_{\text{max}}}}{\sqrt{2}} |0\rangle .$$

(31)

Besides to photons, the above analysis also applies to interferometric setups based on ultracold atom gases trapped by double well potentials: the modes $a, b$ describe atoms confined in the left and right well, whereas the modes $c, d$ are related to the first two tunneling split single particle energy eigenstates (in the limit of high barrier).
IV. CONCLUSIONS

Entanglement is not necessary for beating the shot-noise limit when using identical bosons. In the case of fixed boson number $N$, states $|k, N - k\rangle$ with $k$ bosons in mode $a$ and $N - k$ in mode $b$ are separable with respect to these modes in the sense that there are no correlations between observables of the first and the second mode. Nevertheless, unless when $k = 0, N$ (the single mode case), these states can achieve squared sensitivities scaling faster than $1/N$ if subjected to beam splitting transformations generated by pseudo angular momentum operators like $J_x = (a^{\dagger} b + a b^{\dagger})/2$ that are non-local with respect to the given modes. Superpositions of Fock-states can beat the shot-noise limit even for single mode states. An “ON” state in one mode, i.e. a superposition of 0 and $N$ photons in one mode, leads to a scaling of the squared sensitivity of a Mach-Zehnder interferometer as $\propto 1/N^2$, which corresponds to the Heisenberg limit. The latter scaling is also obtained, at least in principle under ideal unitary evolution, for the NOON state. Using the quantum Fisher information we showed that the NOON state is — in such a highly idealized situation — the optimal two mode state, in the sense that it saturates the quantum Cramér-Rao bound for pure states with the same maximum number of excitations fed into a Mach-Zehnder interferometer.

Acknowledgments

DB would like to thank Wolfgang Schleich for interesting discussions.

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