Cutoff for Mixing Times on Random Abelian Cayley Graphs

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Abstract

Consider the random Cayley graph of a finite, Abelian group \( G = \bigoplus_{j=1}^d \mathbb{Z}_{m_j} \) with respect to \( k \) generators chosen uniformly at random. We prove that the simple random walk on this graph exhibits abrupt convergence to equilibrium, known as cutoff, subject to \( k \gg 1, \log k \ll \log |G| \) and mild conditions on \( d \) and \( \min_j m_j \) in terms of \( |G| \) and \( k \).

In accordance with spirit of a conjecture of Aldous and Diaconis, the cutoff time is shown to be independent of the algebraic structure of the group; it occurs around the time that the entropy of the simple random walk on \( \mathbb{Z}^k \) is \( \log |G| \), independent of \( d \) and \( \{m_j\}_{j=1}^d \). Moreover, we prove a Gaussian profile of convergence to equilibrium inside the cutoff window.

We also prove that the order of the spectral gap is \( |G|^{-2/k} \) with high probability (as \( |G| \) and \( k \) diverge); this extends a celebrated result of Alon and Roichman.

Keywords: cutoff, mixing times, random walk, random Cayley graphs, Abelian groups

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1 Introduction

1.1 Model Setup

Consider a finite Abelian group \((G, +)\), and let \(n = |G|\). Choose \(k\) elements \(Z_1, ..., Z_k\) from \(G\) uniformly at random with replacement. We refer to \(Z_1, ..., Z_k\) as generators (even though they may fail to generate \(G\)). Throughout, we let \(k\) depend on \(n\), with \(k \to \infty\) and \(\log k / \log n \to 0\) as \(n \to \infty\).

We consider the nearest-neighbour random walk, abbreviated \(RW\) and denoted \(S = (S_t)_{t \geq 0}\), both on the undirected and on the directed Cayley multigraph generated by the multiset \(Z = [Z_1, ..., Z_k]\), which we denote by \(G(Z)\) and \(G^+(Z)\), respectively. This is the undirected, respectively directed, multigraph whose vertex set is \(G\) and whose edge multiset is given by

\[
\{(g, g + z) \mid g \in G, z \in Z\}, \quad \text{respectively} \quad \{(g, g + z) \mid g \in G, z \in Z\}.
\]

Note that if some \(z\) appears twice in \(Z\), then for every \(g \in G\) there are multiple edges between \(g\) and \(g + z\) in the undirected case and multiple directed edges from \(g\) to \(g + z\) in the directed case.

1.2 Motivation and Related Work

The cutoff phenomenon, first identified by Diaconis and Shahshahani in [9], occurs when a Markov chain on a finite state space exhibits the following abrupt convergence to its invariant distribution. In a short window, known as the cutoff window, the total variation distance of the distribution of the chain drops from a value close to one, to a value close to 0; in other words, the \(\varepsilon\) mixing time is asymptotically independent of \(\varepsilon\). See §1.3 for a precise definition.

In their seminal paper [1], where they coined the term the “cutoff phenomenon”, Aldous and Diaconis made an informal conjecture that the cutoff phenomenon should be universal for random Cayley graphs with a diverging number of generators chosen uniformly at random, and that the cutoff time should be independent of the algebraic structure of the group. There has since been much work on this conjecture. It was verified by Hildebrand [13, Theorem 3] for the cyclic group \(\mathbb{Z}_n\) with number of generators \(k = \lfloor (\log n)^a \rfloor\) with \(a > 1\), but he also showed [13, Theorem 4] that their conjecture as stated is false with \(a < 1\). Dou and Hildebrand [11, Theorem 1] showed that the same upper bound on the mixing time holds for every group for \(k = \lfloor (\log n)^a \rfloor\) with \(a > 1\). Their proof was later simplified by Roichman [18, Theorem 2]. Wilson [20, Theorem 1] established the cutoff phenomenon for the hypercube, for \(k\) random generators conditioned to generate the group.

The case \(k \lesssim \log n\) remained open for a long time with no progress. This is the regime in which Aldous and Diaconis’ conjecture was imprecise. Indeed, the cutoff time should depend on the geometry of the group: even for the class of Abelian groups, for the \(d\)-dimensional hypercube clearly \(d = \log_2 n\) generators are necessary in order to generate the group, and thus we cannot expect a general statement to hold for an arbitrary diverging \(k\) random generators. The best one can hope for is to establish universality of cutoff in this regime for a large class of groups. The only progress towards this goal was obtained recently by Hough [14, Theorem 1.7]: he considered the groups \(\mathbb{Z}_p\) with \(p\) prime and the number \(k\) of generators satisfying \(1 \ll k \leq \log p / \log \log p\).

Our main result significantly improves Hough’s result by essentially eliminating the restrictions on \(p\) and \(k\) and by applying to Abelian groups \(G = \bigoplus_{j=1}^d \mathbb{Z}_{m_j}\) with a more complex algebraic structure under mild conditions on \(d\) and \(\min_j m_j\). Also, while Hough only considers the undirected case, we are able to address both the undirected and directed cases.

Our approach has the advantage of being unified over all regimes of \(k\), i.e those satisfying \(k \gg 1\) and \(\log k \ll \log n\), and for the directed and undirected cases simultaneously. This includes the not previously covered cases where \(k \asymp \log n\), which is of particular interest since here the random Cayley graph is an expander, by Alon and Roichman [3, Corollary 1], and where \(k\) grows strictly faster than poly-logarithmically in \(n\). It also has the additional advantage of demonstrating that the convergence within the cutoff window exhibits a Gaussian profile, whereas none of the aforementioned results gave such refined information.

Hough [14, Theorem 1.7] also showed that, for every prime \(p\), the relaxation time of the RW on any Cayley graph of \(\mathbb{Z}_p\) with respect to an arbitrary set of \(k\) generators is at least \(cZ_p^{2/k} = cp^{2/k}\), for an absolute positive constant \(c\), provided that \(k \leq \log p / \log \log p\). Separately to our cutoff result, we extend Hough’s result, removing the restrictions on \(p\) and \(k\) and considering arbitrary
Abelian groups. Moreover, we also prove that a matching upper bound holds with probability at least $1 - O(2^{-k})$ when the $k$ generators are picked uniformly at random with replacement. This also extends, in the Abelian setup, a celebrated result of Alon and Roichman [3, Corollary 1], which asserts that, for any finite group, the random Cayley graph with at least $C \log n$ random generators is with high probability an expander, provided $C$ is a sufficiently large constant.

A classical result of Diaconis and Saloff-Coste [8] is that both the mixing time and relaxation time of Cayley graphs of moderate growth are proportional to the square of the diameter of the graph. Here, ‘moderate growth’ is a technical condition, similar to polynomial growth, and holds for Cayley graphs of finite Abelian groups of bounded degree. The bounds contain constants which depend implicitly on the degree. In sharp contrast to the diverging degree setup that we consider in the present work, when the degrees are bounded their result implies that the random walk does not exhibit cutoff; see eg [16, Proposition 18.4]. Moreover, we see that in this case the mixing time depends on the algebraic structure of the group: eg, for the Cayley graph of $\mathbb{Z}_n^d$ with the standard choice of generators $\{ \pm e_i : i \in \{1, ..., d\} \}$, the mixing time is of order $d n^2 \log d = d \log d |\mathbb{Z}_n^d|^{2/d}$.

In the special case of $\mathbb{Z}_n$, where $n = p$ is prime and the number $k$ of generators satisfies $k \leq \log p / \log \log p$, Hough [14, Theorem 1.4] showed that the dependence on $k$ in the aforementioned upper bound on the mixing time of Diaconis and Saloff-Coste in [8] is at most linear in $k$.

We now put our results into a broader context. A common theme in the study of mixing times is that ‘generic’ instances often exhibit the cutoff phenomenon. In this setup, a family of transition matrices chosen from a certain family of distributions is shown to, with high probability, give rise to a sequence of Markov chains which exhibits cutoff. A few notable examples include random birth and death chains [10, 19], the simple or non-backtracking random walk on various models of sparse random graphs, including random regular graphs [17], random graphs with given degrees [4, 5, 6, 7], the giant component of the Erdős-Rényi random graph [6] (where the authors consider mixing from a ‘typical’ starting point), and a large family of sparse random graphs [7].

A recurring idea in the aforementioned works establishing the cutoff phenomenon for certain families of random instances is that the cutoff time can be described in terms of entropy. More specifically, one can look at some auxiliary random process which up to the cutoff time can be coupled with, or otherwise related to, the considered Markov chain – often in the above examples this is the random walk on the corresponding Benjamini-Schramm local limit. The cutoff time is then shown to be (up to smaller order times) the time at which the entropy of the auxiliary process equals the entropy of the invariant distribution of the original Markov chain. This is the case in the present work also; in the undirected case, we use the simple random walk on $\mathbb{Z}^k$ with jump rate 1 as our auxiliary random process.

### 1.3 Mixing Definitions

Recall that $S = (S_t)_{t \geq 0}$ is the RW on the Cayley graph. Observe that, for any choice of $Z = \{Z_1, ..., Z_k\}$, the uniform distribution on $G$, which we denote $\mathcal{U}$, is invariant for $S$, by transitivity. Moreover, if $S$ is irreducible, ie if the Cayley graph generated by $Z$ is connected, then $\mathcal{U}$ is the only invariant distribution for $S$. We are interested in the mixing time of $S$ given a choice $Z = \{Z_1, ..., Z_k\}$ of generators, and so are interested in the quantity

$$d_Z(t) = \| \mathbb{P}(S(t) \in \cdot \mid Z) - \mathcal{U} \|_{\text{TV}}.$$ 

Here we are using the total variation distance, which, for probability measures $\mu$ and $\nu$, is given by

$$\| \mu - \nu \|_{\text{TV}} = \max_A |\mu(A) - \nu(A)| = \frac{1}{2} \sum_x |\mu(x) - \nu(x)|.$$ 

We define the $\varepsilon$-mixing time to be

$$t_{\text{mix}}^Z(\varepsilon) = \inf \{ t \geq 0 \mid d_Z(t) \leq \varepsilon \} \quad \text{for} \quad \varepsilon \in (0, 1);$$

note that $d_Z(t)$ and $t_{\text{mix}}^Z(\varepsilon)$ are both random variables, depending on $Z$. We shall show bounds that hold with high probability over the choice of $Z$.

Now consider a sequence of such groups $(G_N)_{N \in \mathbb{N}}$, and a sequence of integers $(k_N)_{N \in \mathbb{N}}$; now $|Z| = k_N$ depends on $N$. Denote the corresponding mixing times by $t_{\text{mix}}^{Z_N}(\cdot)$. We say an event holds
with high probability, abbreviated whp, if its probability tends to 1 as $N \to \infty$. We say that the RW exhibits cutoff whp if, for all fixed $\varepsilon \in (0,1)$, we have

$$t_{\text{mix}}^\text{Z,N}(\varepsilon)/t_{\text{mix}}^\text{Z,N}(1-\varepsilon) \to^d 1 \quad \text{as } N \to \infty,$$

where $\to^d$ denotes convergence in distribution. We also say that $(t_N)_{N \in \mathbb{N}}$ defines a cutoff window whp if $t_N \ll t_{\text{mix}}^\text{Z,N}(\frac{1}{4})$ whp and, for all $\varepsilon \in (0,1)$, there exists a constant $C_\varepsilon \in \mathbb{R}$ so that

$$t_{\text{mix}}^\text{Z,N}(\varepsilon) - t_{\text{mix}}^\text{Z,N}(1-\varepsilon) \leq C_\varepsilon t_N \quad \text{whp}.$$

### 1.4 Statement of Results

We now state a summarised form of our main theorem. A more detailed, and more general, statement is given in Theorem 1.3 in §1.6.

**Theorem A** (Cutoff). Let $d \in \mathbb{N}$. Let $G_m = \mathbb{Z}_m^d$ and $Z$ be a set of generators of size $k_m$ chosen uniformly at random from $G_m$. Suppose that $k_m \to \infty$ and $\log k_m / \log m \to 0$ as $m \to \infty$. Then the random walk on either $G_m(Z)$ or $G_m^*(Z)$ exhibits cutoff whp.

In Theorem 1.3 we give a precise description of the cutoff time and window in various regimes of $k$, in doing so precisely describing the shape of the cutoff window. In that statement, we allow more general Abelian groups $G = \bigoplus_{j=1}^d \mathbb{Z}_m^d$, with the $m_j$-s allowed to vary over $j$ and $d$ allowed to diverge with $n = |G|$, provided $d$ is not too large in terms of $k$ and $\min_j m_j$ is sufficiently large in terms of $n$ and $k$.

Perhaps surprisingly, the outline of our proof is identical for the undirected and directed cases, and for all regimes of $k$. Moreover, for both the undirected and directed cases the cutoff occurs around the same time (up to smaller order terms), with the same window and same shape, in the regimes $k \ll \log n$ and $\log n \ll k \leq n^{o(1)}$. In the regime $k \asymp \log n$, the cutoff occurs around different times in the undirected and directed cases, but both are of order $k$, with window order $\sqrt{k}$. Providing $k \lesssim \log n$, we have that the cutoff time is of order $kn^{2/k}$ with cutoff window order $\sqrt{kn^{2/k}}$. (Recall that $n$ denotes the size of the group.)

We now define the spectral gap, and the relaxation time. Let $P$ be the transition matrix of a reversible Markov chain on a finite state space $V$, with stationary distribution $\pi$. The spectral gap of $P$, denoted by $\gamma$, is defined to be 0 if $P$ is reducible, and otherwise is defined as the minimal positive eigenvalue of $I - P$, where $I$ is the identity matrix. The relaxation time is defined as $t_{\text{rel}} = 1/\gamma$. It is classical that

$$\gamma = -\lim_{t \to \infty} \frac{1}{t} \log \max_{x \in V} \left( P_t(x,x) - \pi(x) \right)$$

and

$$\frac{1}{t} e^{-\gamma t} \leq \max_{x \in V} \| P_t(x,\cdot) - \pi \|_{TV} \leq \frac{1}{2} \left( \min_{y \in V} \pi(y) \right)^{-1/2} e^{-\gamma t}, \quad (1.1)$$

where $P_t = e^{-t(I-P)}$ is the heat-kernel of the corresponding continuous-time chain; cf [16, Theorems 12.4, 12.5 and 20.5]. This gives a way of bounding the mixing time in terms of the spectral gap. The precise statement that we prove, in §5, is the following.

**Theorem B** (Spectral Gap). There exists an absolute constant $c$ with the following properties. Let $Z$ be any set of generators of size $k$. Write $t_{\text{rel}} = t_{\text{rel}}(G(Z))$. Then

$$t_{\text{rel}} \geq c|G|^{2/k}, \quad (1.2)$$

Further, suppose $G$ admits a representation $G = \bigoplus_{j=1}^d \mathbb{Z}_{m_j}$, the generators comprising $Z$ are chosen uniformly at random with replacement and $k \geq 3d$. Write $C = 1/c$. Then

$$\mathbb{P}(t_{\text{rel}} \leq C|G|^{2/k}) \geq 1 - C2^{-k}. \quad (1.3)$$
1.5 Canonical Construction of Random Walk

It will be convenient to have the following canonical construction of the RW. At rate 1, the walker chooses an element uniformly at random from the multiset $Z = \{Z_1, \ldots, Z_k\}$. Suppose the walker is at $x$ when this selection is made, and the selection is $z$. It updates its position from $x$ to $x + \xi z$, where we choose $\xi$ according to the following rules: in the directed case, we take $\xi = \alpha$; in the undirected case, we take $\xi = \pm 1$ each with probability $\frac{1}{2}$. This walk may be realised in the following (natural) way:

$$S(t) = \sum_{i=1}^{k} W_i(t)Z_i,$$

where $\{W_i(t)\}_{i=1}^{k}$ are independent random variables and $W_i = (W_i(t))_{t \geq 0}$ has the following dynamics for each $i = 1, \ldots, k$: in the directed case, $W_i$ is a rate-$1/k$ Poisson process; in the undirected case, $W_i$ is a rate-$1/k$ continuous-time simple random walk (abbreviated SRW) on $Z$. Note that in the directed case $W_i(t)$ is the number of times that generator $i$ has been picked, while in the undirected case, considering taking $\xi = -1$ as picking the inverse of $Z_i$, $W_i(t)$ is the number of times generator $i$ is picked minus the number of times its inverse is picked. By transitivity, we may take $W(0) = 0 = (0, \ldots, 0) \in \mathbb{Z}^k$.

1.6 Precise Statement of Cutoff Result

For functions $f$ and $g$, write $f \approx g$ if $f(N)/g(N) \rightarrow 1$ as $N \rightarrow \infty$; also write $f \ll g$, or $g \gg f$, if $f(N)/g(N) \rightarrow 0$ as $N \rightarrow \infty$. Write $f \preceq g$, or $g \succeq f$, if there exists a constant $C$ so that $f(N) \leq C g(N)$ for all $N$; also write $f \asymp g$ if $f \preceq g \preceq f$. Also write $f = O(g)$ if $f \preceq g$, and $f = o(g)$ if $f \ll g$. Throughout the paper, unless otherwise explicitly mentioned all limits will be as the size of the group diverges; so if a term is $o(1)$, then it tends to 0 as the group gets larger.

Write $\text{Po}(\lambda)$ for the Poisson distribution with parameter $\lambda$, $\text{Bern}(p)$ for Bernoulli with success probability $p$ and $N(\mu, \sigma^2)$ for normal with mean $\mu$ and variance $\sigma^2$.

Write $\mu_t$ for the law of $W(t)$, and write $\nu_t$ for the law of $W_1(t)$; note that $\mu_t = \nu_t^k$. Also, define

$$Q_i(t) = -\log \nu_t(W_i(t)) \quad \text{for each } i = 1, \ldots, k, \quad \text{and} \quad Q(t) = -\log \mu_t(W(t)) = \sum_{i=1}^{k} Q_i(t).$$

**Definition 1.1 (Entropic and Cutoff Times).** For $n \in \mathbb{N}$ and $k$ generators, for all $\alpha \in \mathbb{R}$, define $t_\alpha = t_\alpha(n, k)$ so that

$$\mathbb{E}(Q_1(t_\alpha)) = (\log n + \alpha \sqrt{\nu})/k \quad \text{where} \quad \nu = \text{Var}(Q_1(t_0)).$$

We call $t_0$ the entropic time and the $t_\alpha$ cutoff times.

Direct calculation with the Poisson distribution and SRW on $\mathbb{Z}$ give the following relations.

**Proposition 1.2 (Entropic and Cutoff Times).** Write $\kappa = k/\log n$. For all $\lambda > 0$, the entropic time $t_0$ satisfies

$$t_0 \approx k \cdot \begin{cases} n^{2/k}/(2\pi e) & \text{when } k \ll \log n, \\ f(\lambda) & \text{when } k \approx \lambda \log n, \\ 1/(\kappa \log \kappa) & \text{when } k \gg \log n, \end{cases}$$

and, for all $\alpha \in \mathbb{R}$, the cutoff times $t_\alpha$ satisfy $t_\alpha \approx t_0$, and furthermore satisfy

$$t_\alpha - t_0 \approx \frac{\alpha}{\sqrt{\kappa}} t_0 \cdot \begin{cases} \sqrt{2} & \text{when } k \ll \log n, \\ g(\lambda) & \text{when } k \approx \lambda \log n, \\ \sqrt{\kappa \log \kappa} & \text{when } k \gg \log n, \end{cases}$$

here $f, g : (0, \infty) \rightarrow (0, \infty)$ are continuous functions, whose value differs between the undirected and directed cases. In particular, for all $\alpha \in \mathbb{R}$, in all cases, we have $t_\alpha \approx t_0$.

We give the proof of this proposition in §A.4. Observe that, although we do not consider the general $k \approx \log n$, to prove cutoff, by passing to appropriate subsequences, it suffices to consider
only the case where \(k / \log n \to \lambda\), for arbitrary \(\lambda \in (0, \infty)\). As such, throughout the remainder of the paper, whenever \(k \sim \log n\) we assume that \(k / \log n\) actually converges to some given \(\lambda \in (0, \infty)\).

We shall require the Abelian groups under consideration satisfy certain hypotheses.

**Hypotheses.** We say that a representation \(G = \oplus_{j=1}^{d} \mathbb{Z}_{m_j}\) satisfies hypotheses \(H(n, k, \eta)\) if \(|G| = n\) and the following conditions hold:

- \(m_j > n^{1/k} (\log k)^2\) for all \(j = 1, \ldots, d\);
- \(k \leq \eta \log n / \log \log n\) then \(d \leq (1 - 2\eta)k\);
- \(k > \eta \log n / \log \log n\) then \(d \leq \frac{1}{2\eta} \log n / \log \log n\).

For ease of notation, when \(k = k_N\) and \(G = G_N\), we write

\[
d_N^Z(t) = \|P(S(t) \in \cdot \mid Z) - U\|_{TV} \quad \text{with} \quad Z = [Z_1, \ldots, Z_{k_N}] \quad \text{and} \quad U \sim \text{Unif}(G_N).
\]

Also, for all \(\alpha \in \mathbb{R}\), define \(t_\alpha = t_\alpha(|G_N|, k_N)\), suppressing the \(N\)-dependence.

In summary, we prove that whp we have cutoff at the entropic time \(t_0\) given by (1.4) with window given by (1.5) and Gaussian shape given by (1.6) below. We now state our full result in generality. Here and from now on, we write \(\Psi\) for the tail distribution function of \(N(0, 1)\), i.e.

\[
\Psi(\alpha) = P(N(0, 1) \geq \alpha) = (2\pi)^{-1/2} \int_{\alpha}^{\infty} e^{-y^2/2} dy \quad \text{for} \quad \alpha \in \mathbb{R}.
\]

**Theorem 1.3 (Cutoff and Shape of TV).** Let \((k_N)_{N \in \mathbb{N}}\) be a sequence of integers and \((G_N)_{N \in \mathbb{N}}\) be a sequence of finite, Abelian groups such that, for some \(\eta\) (independent of \(N\)), \(G_N\) admits a representation satisfying hypotheses \(H(|G_N|, k_N, \eta)\) for each \(N \in \mathbb{N}\); also require \(k_N \to \infty\) and \(\log k_N / \log |G_N| \to 0\) as \(N \to \infty\).

Then, for both the directed and the directed cases, for all \(\alpha \in \mathbb{R}\), we have \(t_\alpha \approx t_0\) and

\[
d_N^Z(t_\alpha) \to^d \Psi(\alpha) \quad \text{as} \quad N \to \infty,
\]

with the convergence in distribution over the randomness in \(Z = [Z_1, \ldots, Z_{k_N}]\).

We prove this by showing, separately, a matching upper and lower bound on the limit (in distribution) of \(d_N^Z(t_\alpha)\); we show the lower bound in §3 and the upper bound in §4.

Throughout the paper, we shall always be assuming the conditions of this theorem. Only in Proposition 4.6 will the conditions on \(d\) be required, and they will be restated there.

Observe that, since the \(W_i\) are iid, and hence so are the \(Q_i\), it is the case that \(Q\) is a sum of \(k\) iid random variables. It will also turn out that

\[
\forall \text{Var}(Q(t)) \approx \text{Var}(Q(t_0)) \gg 1 \quad \text{when} \quad t \approx t_0;
\]

see Proposition A.6. Proposition 1.2, proved in §A.3, shows that \(t_\alpha \approx t_0\) for all \(\alpha \in \mathbb{R}\). Hence we are in a CLT regime; we are able to prove the following CLT application, which will be of great importance. The proof is deferred until the appendix (§A.1).

**Proposition 1.4 (CLT).** For \(k\) with \(k \gg 1\) and \(\log k \ll \log n\), for each \(\alpha \in \mathbb{R}\), we have

\[
\mathbb{P}(Q(t_\alpha) \leq \log n \pm \omega) \to \Psi(\alpha) \quad \text{for} \quad \omega = \text{Var}(Q(t_0))^{1/4} = (\kappa k)^{1/4}.
\]

We now make six remarks regarding our main theorem, Theorem 1.3.

**Remarks 1.5.** (i) In accordance with the spirit of the Aldous-Diaconis conjecture, our results are independent of the algebraic structure of the representation \(G = \oplus_{j=1}^{d} \mathbb{Z}_{m_j}\): they depend only on the number of generators \(k\) and the size of the group \(n\). This is true even for the cutoff window and the profile. Contrast this with the mixing time of the lazy RW on \([0, 1]^d\), which is \(\frac{1}{2} d \log d\), or the lazy RW on the torus \(\mathbb{Z}_m^d\) which is \(m^2 d \log d = n^{2d} / d \log d\) where \(n = m^d\) is the volume.
(ii) The lower bound in the limit (1.6) actually holds in more generality and as a deterministic statement: for all $\alpha \in \mathbb{R}$, all (finite, Abelian) groups $G$ and all $Z$, we have

$$d_Z(t_\alpha) = \|\mathbb{P}(S(t_\alpha) \in \cdot | Z) - \mathcal{U}\|_{TV} \geq \Psi(\alpha) - o(1). \tag{1.8}$$

(The $o(1)$ term decays to 0 as the size of the group grows.)

This 'holding for all realisations of $Z$' property for the lower bound appears elsewhere in the literature: Hildebrand [13, Theorem 3] and Wilson [20, Theorem 1] both have a similar result.

(iii) It turns out that the CLT (1.7) gives the dominating term in the TV distance (1.6):

- on the event $\{Q(t_\alpha) \leq \log n - \omega\}$, we lower bound the TV distance by $1 - o(1)$.
- on the event $\{Q(t_\alpha) \geq \log n + \omega\}$, we upper bound the expected TV distance by $o(1)$.

Combined with (1.7), we deduce that the TV distance is approximately $\Psi(\alpha)$ at time $t_\alpha$.

(iv) One can also quantify the error terms in the TV limit (1.6), using Berry-Esseen and an inspection of our proofs. We explore this in §A.2, and there show that the error is at most $2 \log(k/\log k)/\sqrt{k}$ when $k \leq \frac{1}{2} \log n/\log \log n$, for $n$ sufficiently large.

(v) Our proof outputs a semi-explicit form for the functions $f$ and $g$ in (1.4b) and (1.5b), respectively. Write $H(s)$ for the entropy of $W_1(sk)$, $\mu$ of the rate-1 SRW on $Z$ at time $s$ in the undirected case and of Po($s$) in the directed case. (Note that the function $H$ is independent of both $n$ and $k$.) Then $f$ is such that $h(f(\lambda)) = 1/\lambda$; since the entropy is strictly increasing, the inverse function $H^{-1}$ exists, and so

$$f(\lambda) = H^{-1}(1/\lambda).$$

In particular, this shows that $f$ is a decreasing bijection. Also, we show that

$$g(\lambda) = \sqrt{v(\lambda)}/(f(\lambda)H'(f(\lambda))) \quad \text{where} \quad v(\lambda) = \text{Var}(Q_1(f(\lambda)k));$$

in particular, since each coordinate runs at rate $1/k$, $v : (0, \infty) \to (0, \infty)$ is a continuous function (independent of $n$ and $k$), whose value differs between the undirected and directed cases.

(vi) Observe that our Cayley graph is simple if and only if no generator is picked twice, i.e. $Z_i \neq Z_j$ for all $i \neq j$, and, in the undirected case, additionally no generator is the inverse of another, i.e. $Z_i \neq -Z_j$ for all $i$ and $j$. Since $k/\sqrt{n} \to 0$ as $n \to \infty$, the probability of this event tends to 1 as $n \to \infty$. Hence our results all also hold when the generators are chosen uniformly at random from $G$ but without replacement. \(\triangle\)

2 Outline of Proof

We now give a high-level description of our approach, introducing notations and concepts along the way. Recall the definitions from the previous section.

In all cases we show that cutoff occurs around the entropic time. As $Q(t)$ is a sum of iid random variables, we expected it to be concentrated around its mean. Loosely speaking, we show that the shape of the cutoff, i.e. the profile of the convergence to equilibrium, is determined by the fluctuations of $Q(t)$ around its mean, which in turn, by the CLT (1.7), are determined by $\text{Var}(Q(t))$, for $t$ 'close' to $t_0$; note, since the $Q_i$ are iid, that $\text{Var}(Q(t)) = k \text{Var}(Q_1(t))$. We now explain this in more detail.

We start by discussing the lower bound. We show, for any $\omega$ with $1 \ll \omega \ll \log n$ and all $t$ and all $Z = [Z_1, \ldots, Z_k]$, that

$$d_Z(t) \geq \mathbb{P}(Q(t) \leq \log n - \omega) - e^{-\omega}.$$ 

Observe that this probability on the right-hand side is independent of $Z$. Thus we are naturally interested in the fluctuations of $Q(t)$ for $t$ close to $t_0$. Using the CLT application (1.7) above, with $\omega = \mathbb{V}(Q(t_0))^{1/4}$, we deduce the lower bound (1.8):

$$d_Z(t_\alpha) \geq \Psi(\alpha) - o(1) - e^{-\omega} = \Psi(\alpha) - o(1).$$
We now turn to discussing the upper bound. Our aim is to show that, for all \( \alpha \in \mathbb{R} \), we have

\[
d_Z(t, \alpha) \leq \Psi(\alpha) + o(1) \quad \text{whp over } Z.
\]

As opposed to the lower bound, here we exploit the uniform randomness of \( Z \). For clarity of presentation, we concentrate here on \( G = \mathbb{Z}_n \); we consider more general Abelian groups later.

Let \( W'(t) \) be an independent copy of \( W(t) \), and let \( V(t) = W(t) - W'(t) \). Observe that in both the undirected and directed case, the law of \( V(t) \) is that of the SRW in \( \mathbb{Z}_n \) with jump rate \( 1/k \) in each coordinate, evaluated at time \( 2t \). It is standard that the TV distance \( \| \zeta - \mathcal{U} \|_{TV} \) can be upper bounded by half the \( L_2 \) distance, i.e.

\[
2 \| \zeta - \mathcal{U} \|_{TV} \leq \| \zeta - \mathcal{U} \|_{L_2} = \sqrt{n \sum_{x \in G} (\zeta(x) - \frac{1}{n})^2},
\]

recalling that \( \mathcal{U}(x) = 1/n \) for all \( x \in G \). A standard elementary calculation shows that

\[
\| P(S(t) \in \cdot \mid Z) - \mathcal{U} \|_{L_2} = \sqrt{n \sum_{x \in G} (\zeta(x) - \frac{1}{n})^2}.
\]

where \( \zeta \equiv \text{‘equivalent modulo } n \text{’} \), noting that \( V(t) \) may take any value in \( Z \). Unfortunately,

\[
P(V(t) \cdot Z \equiv 0 \mid Z) \geq P(V(t) = (0, \ldots, 0) \in Z^k) = P(X(2t/k) = 0)^k,
\]

where \( X = (X(s))_{s \geq 0} \) is a rate-1 SRW on \( Z \), and a simple calculation shows that

\[
P(V(t_0) \cdot Z \equiv 0 \mid Z) \geq 1/n.
\]

(This calculation differs between the regimes of \( k \).) Hence the \( L_2 \)-mixing time is larger than the TV-mixing time by at least a constant factor; this is insufficiently precise for showing cutoff.

This motivates the following type of modified \( L_2 \) calculation: for any \( W(t) \subseteq Z^k \), we have

\[
\mathbb{E}(d_Z(t) - P(W(t) \not\subseteq W(t))) \leq \mathbb{E}(\| P(S(t) \in \cdot \mid Z, W(t) \in W(t)) - \mathcal{U} \|_{TV})
\]

\[
\leq \mathbb{E}\left(\sqrt{n \sum_{x \in G} (\zeta(x) - \frac{1}{n})^2} \mid W(t), W'(t) \in W(t)\right) - 1
\]

\[
\leq \sqrt{n \sum_{x \in G} (\zeta(x) - \frac{1}{n})^2} \mid W(t), W'(t) \in W(t)\right) - 1; \tag{2.1}
\]

see the start of §4.2. We think of \( W(t) \subseteq Z^k \) as a set of ‘typical values’ for \( V(t) \). To have \( w \in W(t) \), we shall impose local and global typicality requirements. The global ones say that

\[-\log \mu(w) \geq \log n + \omega \quad \text{for all } w \in W(t),\]

where \( \omega = (vk)^{1/4} \) as above; the local ones will come later. This has the advantage that now

\[
P(V(t) = (0, \ldots, 0) \mid W'(t) \in W(t)) = P(W(t) = W'(t) \mid W'(t) \in W(t)) \leq n^{-1} e^{-\omega},
\]

since \( -\log \mu \geq \log n + \omega \) if and only if \( \mu \leq n^{-1} e^{-\omega} \). Call this the ‘empty’ case.

Of course, there are other scenarios in which we may have \( V(t) \cdot Z \equiv 0 \). (We drop the \( t \)-dependence from the notation from now on.) To deal with these, we observe that, conditional on \( \{V_i\}_{i \leq k} \) and \( V \not\equiv 0 \), we have \( V \cdot Z \sim gU \) where \( g = \gcd(V_1, \ldots, V_k, n) \) and \( U \sim \text{Unif}[1, \ldots, n/g] \); see Lemma 4.8. We then deduce that

\[
P(V \cdot Z \equiv 0 \mid V \not\equiv 0) = \mathbb{E}(g/n).
\]

We then need to bound the expectation of the gcd; see Lemma 4.9. We also decompose according to the size of \( I \), writing

\[
D_\alpha = n P(V(t_\alpha) \cdot Z \equiv 0 \mid \typ) - 1 = n \sum_{|I| < L} P(V \cdot Z \equiv 0 \mid I = I, \typ) P(I = I \mid \typ)
\]

\[
+ n \sum_{|I| \geq L} P(V \cdot Z \equiv 0 \mid I = I, \typ) P(I = I \mid \typ) + n P(I = \emptyset \mid \typ)
\]

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where $\text{typ}_\alpha = \{W(t_\alpha), W'(t_\alpha) \in W(t_\alpha)\}$. The $I = \emptyset$ case has just been considered above. We also impose a local typicality requirement, namely that each coordinate of $V$ has absolute value at most $r_s = \frac{1}{2} n^{1/4} (\log k)^2$; for now, we implicitly assume this.

Recall that, in both the undirected and directed cases, each coordinate $V_i(\cdot)$ has the distribution of a SRW on $\mathbb{Z}$ run at rate $2/k$. We then use the fact that the distribution of the SRW is unimodal to write $|V_1|$ conditioned to be non-zero as a mixture of uniforms, say $\text{Unif}\{1, ..., Y_1\}$, where $Y_1$ has some distribution on $\mathbb{N}$. Then, for any $\gamma \in \mathbb{N}$, we have

$$\mathbb{P}(\gamma \text{ divides } V_1 | Y_1, V_1 \neq 0) = (Y_1/\gamma)/Y_1 \leq 1/\gamma.$$  

Using the independence of coordinates, writing $I = \{i | V_i \neq 0\}$, for any $I \subseteq \{1, ..., k\}$, we have

$$\mathbb{P}(g = \gamma | I = I) \leq \mathbb{P}(\gamma \text{ divides } V_i \forall i \in I | I = I) \leq \gamma^{-|I|}.$$  

Hence we see that for large $|I|$ even $\gamma = 2$ gives a very small contribution. So, for large $|I|$, we have $\mathbb{E}(g | I = I) = 1 + o(1)$. Summing over all $I$ with $|I| \geq L$, with $L \to \infty$, we obtain $\mathbb{E}(g | |Z| \geq L) = 1 + o(1)$; see Corollary 4.10, where, in the general case, the requirement is that $L - d \to \infty$. (This is not completely precise, since we are implicitly conditioning on typicality; the precise calculation is carried out in §4.2.)

Asking that $I = I$ for a small set $I$ imposes restrictions on a large number of coordinates, and will have low probability; see Lemma 4.12. Using independence, we have

$$\mathbb{P}(I = I | W) \leq \mathbb{P}(W' \equiv W | W) \cdot \prod_{i \in I} \frac{1}{\mathbb{P}(W'_i \equiv W_i | W_i)}.$$  

The local typicality requirements will turn the ‘≡’ into ‘=’. They will also allow us to lower bound the probability in the product: given typical $W$, we shall have $\mathbb{P}(W'_i = W_i | W) \geq p_s$ where $p_s = n^{-1/4} k^{-2}$. Using this (and the $I = \emptyset$ case), we obtain

$$\mathbb{P}(I = I; W, W' \equiv W) \leq \mathbb{P}(W(t_\alpha) \notin W(t_\alpha)) + o(1).$$

The typicality requirements have precisely the property that $\mathbb{P}(W(t_\alpha) \notin W(t_\alpha)) = \mathbb{P}(\alpha) + o(1)$; see Lemma 4.4. So from this the upper bound in (1.6) follows. For a precise statement of the typicality requirements, see Definitions 4.1 and 4.3.

3 Lower Bound on TV

In this section we prove the lower bound (1.8), which holds for every choice of $Z$.

**Proof of Lower Bound (1.8).** For this proof, we assume that $Z$ is given, and suppress it.

We convert (1.7) from a statement about $Q$ into one about $W$. Let $\alpha \in \mathbb{R}$ and write

$$\mathcal{E}_\alpha = \{\mu(W(t_\alpha)) \geq n^{-1} e^\omega\} = \{Q(t_\alpha) \leq \log n - \omega\};$$

recall that $\omega \gg 1$. From the CLT application (1.7), we have $\mathbb{P}(\mathcal{E}_\alpha) \to \mathbb{P}(\alpha)$. Consider the set

$$A_\alpha = \{x \in G | \exists w \in \mathbb{Z}^d \text{ s.t. } \mu_\alpha(w) \geq n^{-1}e^\omega \text{ and } x = w \cdot Z\}.$$  

Since we can use $W$ to generate $S$, we have $\mathbb{P}(S(t_\alpha) \in A_\alpha | \mathcal{E}_\alpha) = 1$. Since every element $x \in A_\alpha$ can be realised as $x = w_\alpha \cdot Z$ for some $w_\alpha \in \mathbb{Z}^k$ with $\mu_\alpha(w_\alpha) \geq n^{-1}e^\omega$, for all $x \in A_\alpha$, we have

$$\mathbb{P}(S(t_\alpha) = x) \geq \mathbb{P}(W(t_\alpha) = w_\alpha) = \mu_\alpha(w_\alpha) \geq n^{-1} e^\omega.$$  

From this we deduce that

$$1 \geq \sum_{x \in A_\alpha} \mathbb{P}(S(t_\alpha) = x) \geq |A_\alpha| \cdot n^{-1} e^\omega,$$

and hence $|A_\alpha|/n \leq e^{-\omega} = o(1)$.

Finally we deduce the lower bound (1.8) from the definition of TV distance:

$$\|\mathbb{P}(S(t_\alpha) \in \cdot | Z) - \mathcal{U}\|_{\text{TV}} \geq \mathbb{P}(S(t_\alpha) \in A_\alpha) - \mathcal{U}(A_\alpha) \geq \mathbb{P}(\mathcal{E}_\alpha) - \frac{1}{n}|A_\alpha| \geq \mathbb{P}(\alpha) - o(1).$$
4 Upper Bound on TV

We define a set \( W_\alpha \) in which the auxiliary walk \( W \) will ‘typically’ lie in at time \( t_\alpha \), in the sense that \( \mathbb{P}(W(t_\alpha) \notin W_\alpha) \rightarrow \Psi(\alpha) \) as \( n \rightarrow \infty \). Given that \( W(t_\alpha) \in W(t_\alpha) \), we show that the TV distance has expectation \( o(1) \). Using the upper bound

\[
\| \mathbb{P}(S(t_\alpha) \in \cdot | Z) - \mathbb{U} \|_{TV} \leq \| \mathbb{P}(S(t_\alpha) \in \cdot | Z, W(t_\alpha) \in W_\alpha) - \mathbb{U} \|_{TV} + \mathbb{P}(W(t_\alpha) \notin W_\alpha),
\]

this shows an upper bound of \( \Psi(\alpha) \) in the limit in probability. As mentioned in the outline, in order to control the TV distance given typicality, we actually upper bound it first by the \( \ell_2 \) distance.

4.1 Typicality

We define two parameters \( r \) and \( p \) which will be used in our definition of typicality.

**Definition 4.1.** For all \( \alpha \in \mathbb{R} \), define \( r_\alpha(n, k) \) to be the minimal integer \( r \) such that

\[
\mathbb{P}\left( |W_1(t_\alpha) - \mathbb{E}(W_1(t_\alpha))| > r \right) \leq 1/k^{3/2},
\]

and define \( p_\alpha(n, k) \) as

\[
p_\alpha(n, k) = \min_{|j| \leq r_\alpha(n, k)} \mathbb{P}\left( W_1(t_\alpha) - \mathbb{E}(W_1(t_\alpha)) = j \right).
\]

Let us also define

\[
r_*(n, k) = \frac{1}{\alpha}n^{1/k}(\log k)^2 \quad \text{and} \quad p_*(n, k) = n^{-1/k}k^{-2}.
\]

**Proposition 4.2.** For all \( \alpha \in \mathbb{R} \) and all \( n \) sufficiently large, we have

\[
r_\alpha(n, k) \leq r_*(n, k) = \frac{1}{\alpha}n^{1/k}(\log k)^2 \quad \text{and} \quad p_\alpha(n, k) \geq p_*(n, k) = n^{-1/k}k^{-2}.
\]

We prove this proposition in §A.5. The typicality conditions will be a combination of ‘local’ (i.e coordinate-wise) and ‘global’ conditions.

**Definition 4.3** (Typicality: Local and Global). For all \( \alpha \in \mathbb{R} \), we make the following definitions.

(i) Local conditions. Define \( W_{\alpha, \ell} \) to be the set of all \( w \in \mathbb{Z}^k \) satisfying

\[
|w_i - \mathbb{E}(W_1(t_\alpha))| \leq r_\alpha(n, k) \quad \text{for each} \quad i = 1, \ldots, k.
\]

(ii) Global conditions. Define \( W_{\alpha, g} \) to be the set of all \( w \in \mathbb{Z}^k \) satisfying

\[
\mu_{\alpha}(w) = \mathbb{P}(W(t_\alpha) = w) \leq n^{-1}e^{-\omega}.
\]

Write \( W_\alpha = W_{\alpha, \ell} \cap W_{\alpha, g} \), and say that \( w \in \mathbb{Z}^d \) is \( (\alpha-) \) typical if \( w \in W_\alpha \).

An immediate consequence of the local conditions and the definitions of \( r \) and \( p \) is that

\[
\text{if} \quad |w_1 - \mathbb{E}(W_1(t_\alpha))| \leq r_\alpha(n, k) \quad \text{then} \quad \mathbb{P}(W_1(t_\alpha) = w_1) \geq p_\alpha(n, k) \quad \text{for all} \quad \alpha \in \mathbb{R}.
\]

**Lemma 4.4** (Probability of Typicality). For each \( \alpha \in \mathbb{R} \), we have

\[
\mathbb{P}(W(t_\alpha) \notin W_\alpha) \rightarrow \Psi(\alpha).
\]

**Proof.** By our application of the CLT (1.7), the probability of the global conditions’ holding converges to \( 1 - \Psi(\alpha) \). By the union bound, the probability of the local conditions’ failing to hold is at most \( k^{-1/2} = o(1) \). The claim follows. \( \square \)
4.2 Calculations

Throughout this section, we fix $\alpha \in \mathbb{R}$ and set $t = t_\alpha$; we drop this from the notation. First we condition that $W$ is typical:

$$\|P(S \in \cdot \mid Z) - U\|_{TV} \leq \|P(S \in \cdot \mid Z, W \in W) - U\|_{TV} + P(W \notin W).$$

The second term is determined in Lemma 4.4. For the first term, we use a modified $\ell_2$-calculation, as mentioned in the introduction. To do this, let $W'$ be an independent copy of $W$, and let $S' = W' \cdot Z$; then $S'$ is an independent copy of $S$. Also let $V = W - W'$. Write

$$D_\alpha = nP(V(t_\alpha) \cdot Z = 0 \mid \text{typ}_\alpha) - 1 \quad \text{where} \quad \text{typ}_\alpha = \{W(t_\alpha), W'(t_\alpha) \in W_\alpha\}.$$

We shall also sometimes drop the subscript $\alpha$ from $D_\alpha$ and $\text{typ}_\alpha$. We also decompose the typicality requirements into the local and global parts, as defined in Definition 4.3:

$$\text{typ}_t = \{W, W' \in W_t\} \quad \text{and} \quad \text{typ}_g = \{W, W' \in W_g\} ; \quad \text{note that} \quad \text{typ} = \text{typ}_t \cap \text{typ}_g.$$

Lemma 4.5 (TV-$\ell_2$ Relation). For all $\alpha \in \mathbb{R}$, we have

$$\mathbb{E}(\|P(S(t_\alpha) \in \cdot \mid Z, W(t_\alpha) \in W_\alpha) - U\|_{TV}) \leq \frac{1}{2\sqrt{D_\alpha}}.$$

Proof. Using Cauchy-Schwarz, we upper bound the TV distance by $\ell_2$ distance:

$$4 \|P(S \in \cdot \mid Z, W \in W) - U\|^2 \leq n \sum_x \{P(S = x \mid Z, W \in W) - \frac{1}{2}\}^2$$

$$= n \sum_x P(S = x \mid Z, W \in W)^2 - 1. \quad (4.5)$$

Now let $S'$ be an independent copy of $S$, with auxiliary $W'$ independent of $W$. We then have

$$\sum_x P(S = x \mid Z, W \in W) = \sum_x P(S = S' = x \mid Z, W \in W) = P(S = S' \mid Z, W \in W).$$

We then take expectation with respect to $Z$ apply (4.5) to deduce the claim:

$$4 \mathbb{E}(\|P(S \in \cdot \mid Z, W \in W) - U\|^2) \leq 4 \mathbb{E}(\|P(S \in \cdot \mid Z, W \in W) - U\|^2_{TV})$$

$$\leq n \mathbb{E}(P(S = S' \mid W, W' \in W) - 1 = D_\alpha \quad \Box$$

The equalities for $S$ and $Z$ are all in our group $G$; eg $V \cdot Z = 0$ means ‘equal to the identity of $G$’. Recall that we are considering groups of the form $G = \bigoplus_{j=1}^d \mathbb{Z}_{m_j}$. In the same way as when embedding a torus $T \subseteq \mathbb{Z}^d$ in $\mathbb{Z}^d$, we wish to consider elements of $G$ as the torus $\prod_{j=1}^d \mathbb{Z}_{m_j}$, and embed it in $G$. Under this embedding, we identify each $x \in \mathbb{Z}^d$ with the unique $y \in G$ so that $x_j \equiv y_j \mod m_j$ for all $j$, and write $x \equiv y$. Accordingly, we often treat $V \cdot Z$ as an element of $\mathbb{Z}^d$.

We prove the following upper bounds on $D_\alpha$. The proposition has two parts.

Proposition 4.6a ($\ell_2$ Given Typicality). Let $\eta \in (0,1)$, and suppose that $k \leq \eta \log n / \log \log n$ and $d \leq \frac{1}{2} \eta \log n / \log k$. For all $\alpha \in \mathbb{R}$, we have $D_\alpha = o(1)$.

Proposition 4.6b ($\ell_2$ Given Typicality). Let $\eta \in (0,1)$, and suppose that $k \geq \eta \log n / \log \log n$ and $d \leq \frac{1}{2} \eta \log n / \log k$. For all $\alpha \in \mathbb{R}$, we have $D_\alpha = o(1)$.

It is simple to check in each case that the conditions in the above two theorems imply that $d \leq k$, a necessary condition for $Z$ to generate $G$. Write $[k] = \{1, \ldots, k\}$. For $v \in \mathbb{Z}^k$, write

$$I(v) = \{i \in [k] \mid v_i \neq 0 \mod m_j \ \forall j = 1, \ldots, d\}.$$

We shall always be considering $V$ conditioning on typicality, and note that the local requirements say that $|V_i| \leq 2r < m_j$ for all $i$ and $j$. Thus, conditioned on local typicality, ie $\text{typ}_t$, we have

$$I(V) = \{i \in [k] \mid V_i \neq 0\}.$$
Also, write $I = I(V)$ for ease of notation. Thus we may write

$$D + 1 = n \sum_{I \subseteq [k]} P(V \cdot Z = 0, I = I \mid \text{typ}).$$

We now split the sum into 'large $I$, 'small $I$' and 'empty $I$. In the sums below, we always have $I \subseteq [k]$. Let $L$ be a number greater than 1, allowed to depend on $n$. We then have

$$D + 1 \leq n \sum_{I \subseteq [k]} P(V \cdot Z = 0 | I = I, \text{typ})P(I = I | \text{typ})$$

$$+ n \sum_{|I| \geq L} P(V \cdot Z = 0 | I = I, \text{typ})P(I = I | \text{typ}) + n P(I = \emptyset | \text{typ}), \quad (4.6)$$

noting that if $I = \emptyset$ then $V \equiv 0$ (as a vector), and hence $V \cdot Z = 0$.

We first consider when $I = \emptyset$.

**Lemma 4.7** (Empty $I$). We have

$$n P(I = \emptyset | \text{typ}) \leq e^{-\omega}/P(\text{typ}). \quad (4.7)$$

**Proof.** By direct calculation, we have

$$P(I = \emptyset, \text{typ}) = P(V = 0, \text{typ}) = P(W = W', W \in W)$$

$$= \sum_{w \in W} P(W = w)P(W' = w) = \sum_{w \in W} P(W = w)^2,$$

since $W$ and $W'$ are iid copies. Recall the global typicality conditions. From these we obtain

$$n P(I = \emptyset | \text{typ}) \leq n \sum_{w \in W} P(W = w)^2 / P(\text{typ}) \leq e^{-\omega}/P(\text{typ}).$$

We now consider $I \neq \emptyset$. For $j = 1, \ldots, d$, define

$$g_j = \gcd(V_1, \ldots, V_k, m_j) = \gcd(\{V_i \mid i \in I(V) \cup \{m_j\}).$$

Also define

$$g = \gcd(V_1, \ldots, V_k, n) = \gcd(\{V_i \mid i \in I(V) \cup \{n\}).$$

Observe then that $g_j \leq g$ since $m_j$ divides $n$, for all $j = 1, \ldots, d$. We now state a technical lemma, which we shall use immediately and prove (independently) at the end.

**Lemma 4.8** (Technical Lemma). Conditional on $\{V_i\}_{i=1}^d$ and $I(V) \neq \emptyset$, we have

$$V \cdot Z \sim \text{Unif}(\prod_{j=1}^d g_j Z / g_j) \sim \text{Unif}(\prod_{j=1}^d \{g_j, 2g_j, \ldots, m_j\}).$$

The importance of this lemma is that we deduce that

$$n P(V \cdot Z = 0 | I = I, \text{typ}) = n E(\prod_{j=1}^d (g_j/m_j) | I = I, \text{typ}) \leq E(g^d | I = I, \text{typ}),$$

since by local typicality we have $|V_i| < m_j$ for all $i$ and $j$ and observing that the conditioning affects $V$, but not $Z$. We now consider the expectation of this gcd.

**Lemma 4.9** (gcd Calculation). For any $I \subseteq [k]$ with $I \neq \emptyset$, we have

$$n P(V \cdot Z = 0 | I = I, \text{typ}) \leq \begin{cases} 1 + 3 \cdot 2^{d-|I|}/P(\text{typ}) & \text{when } |I| \geq d + 2, \quad (4.8) \\
C(2 r_u)^{d-|I|+2}/P(\text{typ}) & \text{when } |I| \leq d + 1. \quad (4.9) \end{cases}$$

Furthermore, recalling the definition of $r_u$ from (4.3), we also have

$$n P(V \cdot Z = 0 | I = I, \text{typ}) \leq (2 r_u)^d = n^{d/k} (\log k)^{2d}. \quad (4.10)$$

**Corollary 4.10** (Sum Over $|I| \geq d + 2$). For any $L$ with $L \geq d + 2$, we have

$$n \sum_{|I| \geq L} P(V \cdot Z = 0, I = I \mid \text{typ}) \leq 1 + 3 \cdot 2^{d-L}/P(\text{typ}). \quad (4.11)$$
We deduce that
\[ n \sum_{|I| \geq L} P(V \cdot Z \equiv 0, I = I | \text{typ}) = n \sum_{|I| \geq L} P(V \cdot Z \equiv 0 | I = I, \text{typ}) P(I = I | \text{typ}) \]
\[ \leq \sum_{|I| \geq L} (P(I = I | \text{typ}) + \frac{1}{2} \cdot 2^{-|I|} P(I = I) / P(\text{typ})) \]
\[ \leq P(|I| \geq L | \text{typ}) + \frac{3 \cdot 2^{-|I|} P(|I| \geq L) / P(\text{typ})}{1 + \frac{3 \cdot 2^{-|I|}}{P(\text{typ})}}. \]

In order to prove Lemma 4.9, we use the following divisibility property of the coordinates of \( V \), which we recall are independent. We state this property now, as Lemma 4.11, and prove it in the appendix, in \( \S \text{A.1} \). Below, we write \( \alpha \mid \beta \) if \( \alpha \) divides \( \beta \).

**Lemma 4.11** (Divisibility Lemma). For any \( I \subseteq [k] \) with \( I \neq \emptyset \), for any \( \gamma \in \mathbb{N} \), we have
\[ P(\gamma \mid V_i \forall i \in I | I = I, \text{typ}) \leq (1/\gamma)^{|I|}. \]

**Proof of Lemma 4.9.** As stated previously, from Lemma 4.8 we deduce that
\[ n P(V \cdot Z \equiv 0 | I = I, \text{typ}) = n E\left( \prod_{j=1}^{d} (g_j/m_j) \mid I = I, \text{typ} \right) \leq E(g^d | I = I, \text{typ}). \]
From this, we immediately deduce the final claim (4.10). Let us write \( \tilde{P} \) and \( \tilde{E} \) to denote probability and expectation, respectively, conditioned on \( I = I \) and \( \text{typ} \). Continuing the above, we obtain
\[ n P(V \cdot Z \equiv 0 | I = I, \text{typ}) \leq 1 + E(g^d - 1 | I = I, \text{typ}) \leq 1 + \tilde{E}(g^d - 1) / P(\text{typ}). \]
(4.12)
Hence, to prove (4.8, 4.9), we need to bound \( \tilde{E}(g^d) = E(g^d | I = I, \text{typ}) \). To do this, observe that
\[ \tilde{E}(g^d) = \sum_{\gamma=1}^{2r} \gamma^d \tilde{P}(g = \gamma) \leq \sum_{\gamma=1}^{2r} \gamma^d \tilde{P}(\gamma \mid V_i \forall i \in I). \]
Applying the Lemma 4.11, we obtain
\[ \tilde{E}(g^d) \leq \sum_{\gamma=1}^{2r} \gamma^{d-|I|}. \]
To bound this sum, we now consider separate cases, according to the value of \( d - |I| \).

*Suppose that \( d - |I| \leq -2 \). Then the series converges, and it is easy to see that
\[ \tilde{E}(g^d) \leq \sum_{\gamma=1}^{\infty} \gamma^{d-|I|} \leq 1 + 2^{d-|I|}. \]

*Suppose that \( d - |I| = -1 \). Then we have a (diverging) harmonic series, and so
\[ \tilde{E}(g^d) \leq 2 \log r. \]

*Suppose that \( d - |I| = 0 \). Then the summand is always 1, and so
\[ \tilde{E}(g^d) \leq 2r. \]

*Suppose that \( d - |I| \geq 1 \). Then the sum is growing polynomially, and so
\[ \tilde{E}(g^d) \leq (2r)^{d-|I|+1} / (d - |I| + 2). \]

In particular, we can summarise all these cases in the following upper bound:
\[ \tilde{E}(g^d) \leq \begin{cases} 1 + 3 \cdot 2^{d-|I|} & \text{when } d - |I| \leq -2, \\ C(2r)^{d-|I|+2} & \text{when } d - |I| \geq -1, \end{cases} \]
where \( C \) is the implicit constant in the previous equation. We thus deduce (4.8, 4.9). \( \Box \)
We now consider the probability of a given realisation of $\mathcal{I}$.

**Lemma 4.12 (Set Calculations).** We have

$$\mathbb{P}(\mathcal{I} = I, \text{typ}) \leq n^{-1} e^{-\omega / p_*^{|I|}} = e^{-\omega n^{-1+|I|/k} k^{2|I|}}.$$  \hfill (4.13)

Moreover, for $k \ll \log n$, we have

$$\mathbb{P}(\mathcal{I} = I, \text{typ}_f) \leq 2^{k-|I|} n^{-1+|I|/k}$$  \hfill (4.14)

(This expression on the left-hand side only includes local typicality conditions, not global ones.)

**Proof.** Requiring $\mathcal{I} = I$ places restrictions on the coordinates in $I^c$, but not on the coordinates of $I$ other than that they are non-zero; we ignore the latter to get an upper bound.

We prove (4.14) first. Recall that for $k \ll \log n$, for all $\alpha \in \mathbb{R}$, we have $t_\alpha \approx t_0 \approx kn^{2/k}/(2\pi e)$, and in particular $t_\alpha/k \gg 1$; see Proposition 1.2. By the local CLT, which we state precisely in Theorem A.1 and (A.1), letting $X = (X_\alpha)_{\alpha \geq 0}$ be a SRW on $\mathbb{Z}$, we have

$$\mathbb{P}(V_1(t) = 0) = \mathbb{P}(X_{2t/k} = 0) = (2\pi \cdot 2t/k)^{-1/2} \left(1 + o(1)\right) \leq 2n^{-1/k},$$

noting that $(e/2)^{1/2} \leq 2$. From this we deduce (4.14), as follows:

$$\mathbb{P}(\mathcal{I} = I, \text{typ}_f) \leq \mathbb{P}(\mathcal{I} = I, \text{typ}_f) \leq 2^{k-|I|} n^{(k-|I|)/k} = 2^{k-|I|} n^{-1+|I|/k}.$$

We now move on to consider (4.13). For a vector $w$, let us write

$$\mathcal{W}_I(w) = \{w' | \mathcal{I}(w - w') = I\}.$$

We then have, using the independence of $W$ and $W'$, that

$$\mathbb{P}(\mathcal{I} = I, W \in \mathcal{W}) = \sum_{w \in \mathcal{W}} \mathbb{P}(W = w) \mathbb{P}(W' \in \mathcal{W}_I(w)).$$

Hence, using the independence of the coordinates of $W'$, given $w \in \mathcal{W}$ we have

$$\mathbb{P}(W' \in \mathcal{W}_I(w)) = \mathbb{P}(W' = w) \cdot \prod_{i \in I} \frac{\mathbb{P}(W'_i \neq w_i)}{\mathbb{P}(W'_i = w_i)} \leq \mathbb{P}(W' = w) \cdot \prod_{i \in I} \frac{1}{\mathbb{P}(W'_i = w_i)}.$$

From the typicality requirements (and their immediate consequence), for $w \in \mathcal{W}$, we then obtain

$$\mathbb{P}(W' \in \mathcal{W}_I(w)) \leq \mathbb{P}(W' = w) / p_*^{|I|} \leq n^{-1} e^{-\omega / p_*^{|I|}}.$$

Hence, by summing over all $w \in \mathcal{W}$, we obtain (4.13), as follows:

$$\mathbb{P}(\mathcal{I} = I, \text{typ}) \leq \mathbb{P}(\mathcal{I} = I, W \in \mathcal{W}) \leq n^{-1} e^{-\omega} \sum_{w \in \mathcal{W}} \mathbb{P}(W = w) \leq n^{-1} e^{-\omega} p_*^{-|I|};$$

finally we substitute the definition $p_* = n^{-1/k} k^{-2}$ from (4.3). \hfill $\square$

We now combine these two lemmas to prove Propositions 4.6a and 4.6b.

**Proof of Proposition 4.6a.** Suppose $k \leq \eta \log n / \log \log n$; note then that $k \ll \log n$. Fix an $\eta \in (0, 1)$ and suppose that $d \leq (1 - 2\eta)k$.

Consider first $I \subseteq [k]$ with $|I| \leq d + 1$. We have

$$n \mathbb{P}(V \cdot Z \equiv 0, \mathcal{I} = I, \text{typ}) = n \mathbb{P}(V \cdot Z \equiv 0 | \mathcal{I} = I, \text{typ}) \mathbb{P}(\mathcal{I} = I, \text{typ})$$

\hfill (4.9)

$$\leq C(2r_\alpha)^{-|I|+2} \mathbb{P}(\mathcal{I} = I, \text{typ}_f) / \mathbb{P}(\text{typ}_f | \mathcal{I} = I, \text{typ}_f)$$

$$= C(2r_\alpha)^{-|I|+2} \mathbb{P}(\mathcal{I} = I, \text{typ}_f)$$

\hfill (4.14)

$$\leq C(2r_\alpha)^{-|I|+2} \cdot 2^{k-|I|} n^{-1+|I|/k}$$

\hfill (4.4)

$$\leq C2^{k-|I|} (\log k)^2 (d-|I|+2) n^{-1+(d+2)/k}$$

\hfill (4.4)

$$\leq (\log k)^{7k} n^{-1+(d+2)/k}$$

\hfill (4.4)
We now sum over the $I$ with $1 \leq |I| \leq d + 1$:

$$n \sum_{1 \leq |I| \leq d + 1} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq 2dk^d (log k)^{7k} n^{-1 + (d+2)/k} = n^{-1 + d/k + o(1)}, \quad (4.15)$$

with the final relation holding because $d < k \leq \eta \log n / \log log n$ and so $k^d \leq n^\eta$ and $(log k)^7 = n^{o(1)}$. Since $d \leq (1 - 2\eta)k$, we can upper bound this last term by $n^{-\eta + o(1)}$.

Consider now $I \subseteq [k]$ with $|I| \geq d + 2$. As above, we have

$$n \sum_{d+2 \leq |I| \leq (1 - \eta)k} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq 3Ck^k 2^{k} n^{-1 + (1 - \eta)} = n^{-\eta + o(1)}, \quad (4.16)$$

with the final relation holding because $k \ll \log n$ and so $2^k = n^{o(1)}$.

Finally we consider $I \subseteq [k]$ with $(1 - \eta)k \leq |I| \leq k$. Setting $L = (1 - \eta)k$, (4.11) says that

$$n \sum_{(1 - \eta)k \leq |I| \leq k} P(V \cdot Z \equiv 0, I = I | \text{typ}) \leq 1 + 3 \cdot 2^d - L / P(\text{typ}) \leq 1 + 3 \cdot 2^{-\eta k} / P(\text{typ}). \quad (4.17)$$

Plugging (4.7, 4.15–4.17) into (4.6), we obtain

$$D = n \sum I P(V \cdot Z \equiv 0, I = I | \text{typ}) - 1 \leq o(1)/P(\text{typ}).$$

The result follows since $P(\text{typ}) \approx (1 - \Psi(\alpha))^2$, a constant depending only on the fixed $\alpha \in \mathbb{R}$. □

**Proof of Proposition 4.6b.** Suppose $k \geq \eta \log n / \log log n$. Set $L = \frac{1}{10} \eta \log n / \log k$. We require $d \leq \frac{1}{2}L$, and hence $L - d \gg 1$. (Recall that $\log k \ll \log n$, so $L \gg 1$.)

Consider first $I \subseteq [k]$ with $1 \leq |I| \leq L$. We have

$$n \sum_{1 \leq |I| \leq L} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq 2 L k^d n^{-1 + (d+2)/k} = n^{-1 + d/k + o(1)}, \quad (4.10, 4.13)$$

(4.4) \leq n^d / (\log k)^d \cdot n^{-1 - \omega} \cdot n^{-|I|/k^2} \cdot n^{-|I|/k^2} \cdot 2^{|I| + d \log log k / \log k}.

We now sum over the $I$ with $1 \leq |I| \leq L$:

$$n \sum_{1 \leq |I| \leq L} P(V \cdot Z \equiv 0, I = I | \text{typ}) \leq L k^d n^{-1 + (d+2)/k} \cdot 2^{|I| + d \log log k / \log k}.$$

We now use the fact that $d + |I| \leq 2L = \frac{1}{2} \eta \log n / \log k$ and $k \geq \eta \log n / \log log n$:

$$(d + |I|)/k \leq \frac{1}{2} \eta \log n / (k \log k) \quad \text{and} \quad k \log k \geq \frac{1}{2} \eta \log n;$$

hence we have $(d + |I|)/k \leq \frac{1}{2} \eta$. Since $|I| \leq L$ and $d \leq \frac{1}{2} L$, we have

$$k^2 |I| + 2d \leq 3k \log k = e^{3 \eta \log n} n^{1/10} = n^{3\eta/10},$$

by definition of $L$. Hence, using the fact that $\eta \leq 1$, we have

$$n \sum_{1 \leq |I| \leq L} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq n^{-1 + 2/5 + 3\eta/10} \leq n^{-3/10}. \quad (4.18)$$

Finally we consider $I \subseteq [k]$ with $L \leq |I| \leq k$. (4.11) says that

$$n \sum_{L \leq |I| \leq k} P(V \cdot Z \equiv 0, I = I | \text{typ}) \leq 1 + 3 \cdot 2^d - L / P(\text{typ}). \quad (4.19)$$

Recall that $d \leq \frac{1}{2}L$, giving an upper bound of $1 + 3 \cdot 2^{-L/2} / P(\text{typ}) = 1 + o(1) / P(\text{typ})$.

Plugging (4.7, 4.18, 4.19) into (4.6), we obtain

$$D = n \sum I P(V \cdot Z \equiv 0, I = I | \text{typ}) - 1 \leq o(1) / P(\text{typ}).$$

The result follows since $P(\text{typ}) \approx (1 - \Psi(\alpha))^2$, a constant depending only on the fixed $\alpha \in \mathbb{R}$. □
It remains to give the two deferred proofs, of Lemmas 4.8 and 4.11.

**Proof of Lemma 4.8.** For this whole proof, we do not consider $V$ as a random variable, but rather a given vector of $\mathbb{Z}^k$ satisfying $V_i \not\equiv 0 \mod m_i$ for all $i$ and $j$.

First recall that $Z_t \sim \text{Unif}(G)$ where $G = \prod_{i=1}^d \mathbb{Z}_{m_i}$. Hence, for each $i = 1, \ldots, k$, we may write $Z_i = (\zeta_{i,1}, \ldots, \zeta_{i,d})$ with $\zeta_{i,j} \sim \text{Unif}(\mathbb{Z}_{m_i})$ with all the $\zeta_{i,j}$ independent. We then have

$$(V \cdot Z)_j = \sum_{i=1}^k V_i \zeta_{i,j},$$

where $(V \cdot Z)_j$ is the $j$-th component of $V \cdot Z \in \mathbb{Z}^d$. Assuming the $d = 1$ case, the above then shows that $(V \cdot Z)_j \sim \text{Unif}(g_j \mathbb{Z}_{m_j})$ for each $j$. Hence it is sufficient to prove the $d = 1$ case and show that coordinates of $V \cdot Z$ are independent. But since the $\zeta_{i,j}$ are all independent (over $i$ and $j$), the coordinates of $(V \cdot Z)_j$ must be independent. Hence it suffices to prove the $d = 1$ case.

We now prove the $d = 1$ case. We prove this by induction on $|I|$. First consider $U \sim \text{Unif}\{1, \ldots, n\}$ and set $R = mU$ where $m \in \{1, \ldots, n\}$. Write

$$R = mU = g \cdot (rU) \quad \text{where} \quad g = \gcd(m, n) \text{ and } r = m/g.$$

We then have $\gcd(r, n) = 1$, and so $rU \sim \text{Unif}\{1, \ldots, n\}$: indeed, for any $x \in \{1, \ldots, n\}$, we have

$$\mathbb{P}(rU = x) = \mathbb{P}(U = x^{-1}) = \frac{1}{n} \quad \text{where} \quad r^{-1} \text{ is the inverse of } r \text{ mod } n.$$

Thus we have $R = g \cdot (rU) \sim \text{Unif}\{g, 2g, \ldots, ng\}$, since $g \nmid n$. This proves the base case $|I| = 1$.

Now consider independent $X, Y \sim \text{Unif}\{1, \ldots, n\}$ and set $R = aX + bY$. By pulling out a constant as above, we may assume that $a, b \mid n$. Write $c = \gcd(a, b, n)$. Then there exist $r, s \in \{1, \ldots, n\}$ with

$$ar + bs \equiv c \mod n, \quad \text{and hence} \quad a(mr) + b(ms) \equiv cm \mod n \text{ for any } m \in \{1, \ldots, n\}.$$

Thus $(c, 2c, \ldots, n) \subseteq \text{supp}(R)$. By writing $R = c(ac^{-1}X + bc^{-1}Y)$, with $c^{-1}$ the inverse mod $n$, we see that in fact $\text{supp}(R) = \{c, 2c, \ldots, n\}$. It remains to show that $R$ is uniform on its support.

By pulling out the factor $c$, it is enough to consider $\gcd(a, b) = 1$. For $m \in \{0, 1, \ldots, n\}$, set

$$\Omega_m = \{(x, y) \in [n]^2 \mid ax + by = m\}.$$

We show that $|\Omega_m|$ is the same for all $m$, and hence deduce that $R$ is uniform on $\{c, 2c, \ldots, n\}$. Indeed, for every $m$ there exists a pair $(x_m, y_m) \in [n]^2$ so that $ax_m + by_m = m$. If also $(x, y) \in \Omega_m$, then letting $x' = x - x_m$ and $y' = y - y_m$, we see that $(x', y') \in \Omega_0$. This proves that $R \sim \text{Unif}\{c, 2c, \ldots, n\}$. As in the above case, we may then write $R = cU$ where $U \sim \text{Unif}\{1, \ldots, n\}$. This proves the case $|I| = 2$.

Now suppose that $X_1, \ldots, X_{\ell+1} \sim \text{Unif}\{1, \ldots, n\}$ and $a_1, \ldots, a_{\ell+1} \in \{1, \ldots, n\}$. By the inductive hypothesis, we have

$$\sum_{i=1}^{\ell} a_i X_i \sim c_0 U \quad \text{where} \quad U \sim \text{Unif}\{1, \ldots, n\} \quad \text{and} \quad c_0 = \gcd(a_1, \ldots, a_{\ell}, n).$$

Now, $X_{\ell+1}$ is independent of this sum, and so the previous case applies to say that

$$\sum_{i=1}^{\ell+1} a_i X_i \sim cU \quad \text{where} \quad U \sim \text{Unif}\{1, \ldots, n\} \quad \text{and} \quad c = \gcd(c_0, a_{\ell+1}, n) = \gcd(a_1, \ldots, a_{\ell+1}, n).$$

This completes the induction, and hence proves the claim. \qed

**Proof of Lemma 4.11.** The main ideas of this proof have been sketched in the outline.

Let $X = (X_s)_{s \geq 0}$ be a SRW on $\mathbb{Z}$. To calculate the expectation, we use that $V = W - W'$ has the distribution of a SRW run at twice the speed; in particular, $V_t(t) \sim X_{2t/k}$, and coordinates of $V$ are independent. (This is the case for both the directed and undirected cases.) Clearly the distribution of $X$ is symmetric (about 0). We show at the end of this proof that $m \mapsto \mathbb{P}(|X_s| = m) : \mathbb{N} \to [0, 1]$ is decreasing for any $s \geq 0$. We may then write

$$|V_s| \sim \text{Unif}\{1, \ldots, Y_s\} \quad \text{where} \quad Y_s \text{ has some distribution.}$$
Since the local typicality conditions are symmetric about 0 and \(|V_i|\) is a mixture of uniforms, the same is true for \(|V_i|\) conditioned on \(\text{typ}_e\). Recall also that the local typicality conditions imply that \(|V_i| \leq 2r < m_j\) for all \(i\) and \(j\). Hence we have
\[
P(\gamma \mid V_i \mid i \in I, \text{typ}_e, Y_i) = \frac{|Y_i/\gamma|}{Y_i} \leq 1/\gamma.
\]
The lemma follows from the independence of the coordinates of \(V\).

It remains to verify that \(m \mapsto P(|X_m| = m) : \mathbb{N} \rightarrow [0, 1]\) is decreasing. First note that \(Z\) is vertex-transitive and the walk \(X\) is reversible. Fix \(s \geq 0\). Write \(P_t(x, y) = P(X_t = y \mid X_0 = x)\). It is then well-known – see, for example, [2, Lemma 3.20, (3.60)] – that, for any \(z\), we have
\[
\max_{x, y} P_t(x, y) \leq \max_x P_t(x, x) = P_t(z, z).
\]
Since \(X\) starts from 0, for each \(m \in \mathbb{N}_0\) this says that \(P(X_t = 0) \geq P(X_t = m)\).

Let \(\tau = \inf\{t \geq 0 \mid X_t = m\}\), and let \(f\) be its density function. Then, for each \(m \in \mathbb{N}_0\), we have
\[
P_t(0, m) = \int_0^t f(s) P_{t-s}(0, 0) \, ds \geq \int_0^t f(s) P_{t-s}(0, 0) \, ds = P_t(0, m + 1).
\]
Since \(X\) has the same law as \(-X\), this completes the proof. \(\square\)

5 Spectral Gap

In this section we calculate the spectral gap. The precise statement that we prove was given in Theorem B in §1.4; the reader should recall that statement now.

**Proof of Theorem B.** Write \(n = |G|\). We first prove (1.2). We may assume that \(k \leq \log_2(n/2)\), as otherwise (1.2) indeed holds for some \(c > 0\). Let \(L = \lfloor (1/2)(n^{1/k} - 1) \rfloor\). By our assumption on \(k\), we have \(L \geq 1\). Consider the set
\[
A = \{ w \cdot Z \mid w \in \mathbb{Z}^k \text{ and } |w_i| \leq L \forall i = 1, ..., k \} \subseteq G.
\]
Clearly \(|A| \leq (2L + 1)^k \leq \frac{1}{2}n\). Let \(t \geq 0\), and let \((Y_s)_{s \geq 0}\) be a continuous-time rate-1 SRW on \(Z\). Writing \(\tau_{A^c} = \inf\{s \geq 0 \mid S_s \notin A\}\) for the exit time of \(A\) by the SRW \(S\), observe that
\[
P_0(\tau_{A^c} > t) = P_0(\max_{s \in [0, t/k]} Y_s \leq L)^k, \tag{5.1}
\]
where \(0 = (0, ..., 0) \in A\) is the zero element of the group. It follows from Lemma 5.2 below that
\[
P_0(\max_{s \in [0, t/k]} Y_s \leq L) \geq \exp\left(-\frac{1}{8} \pi^2 (t/k)/(L + 1)^2\right).
\]
Substituting this into (5.1) we get
\[
P_0(\tau_{A^c} > t) \geq \exp\left(-\frac{1}{8} \pi^2 (t/k)/(L + 1)^2\right). \tag{5.2}
\]

The minimal Dirichlet eigenvalue of a set \(A\) is defined to be the minimal eigenvalue of minus the generator of the walk killed upon exiting \(A\); we denote it by \(\lambda_A\). For connected \(A\), we show in Lemma 5.4 below that, for all \(a \in A\), we have
\[
\frac{1}{t} \log P_a(\tau_{A^c} > t) \to \lambda_A \text{ as } t \to \infty.
\]
From this and (5.2), it then follows that \(\lambda_A \geq \lambda\) where
\[
\lambda = \frac{1}{8} \pi^2/(L + 1)^2 \geq \frac{\pi^2}{(1/2)(n^{1/k} - 1)^2}.
\]
Since \(|A| \leq 1/2n\), applying [2, Corollary 3.34], we get
\[
t_{\text{rel}} \geq (1 - 1/2n)|A|/\lambda \geq 1/(2\lambda).
\]
This concludes the proof of (1.2).
We now prove (1.3). Recall that here we consider a torus $G = \oplus_{j=1}^d \mathbb{Z}_{m_j}$. An orthogonal basis of eigenvectors for $P$, the transition matrix of the corresponding discrete-time walk, is given by

$$(f_x \mid x \in G) \quad \text{where} \quad f_x(y) = \cos(2\pi \sum_{i=1}^d x_i y_i/m_i),$$

with corresponding eigenvalues are given by

$$(\lambda_x \mid x \in G) \quad \text{where} \quad \lambda_x = \frac{1}{k} \sum_{i=1}^k \cos(2\pi \langle \bar{x}, Z_i \rangle),$$

where $\bar{x}_j = x_j/m_j$ for all $j = 1, \ldots, d$ and $\bar{x} \cdot Z_i = \sum_{j=1}^d x_j Z_{ij}/m_j$ is the standard inner-product on $\mathbb{R}^d$, where $Z_{ij}$ is the $j$-th coordinate of the $i$-th generator $Z_i$; here we identify $\bar{x}$ and $Z_i$ with elements of $\mathbb{R}^d$ in a natural manner.

Observe that $\lambda_0 = 1$. Our goal is to bound $\min_{x \in G \setminus \{0\}} \{1 - \lambda_x\}$ from below. For $a \in \mathbb{R}$, let $\{a\}$ be the unique number in $(-\frac{1}{2}, \frac{1}{2}]$ such that $a - \{a\} \in \mathbb{Z}$. It follows from Lemma 5.3 below that

$$1 - \lambda_x \geq \frac{2\pi^2}{m^2} \sum_{i=1}^d \{\bar{x} \cdot Z_i\}^2. \quad (5.3)$$

For each $x \in G$, we make the following definitions:

$$g_j = g_j(x) = \gcd(x_j, m_j) \quad \text{for each} \quad j = 1, \ldots, d,$$

$$s_* = s_*(x) = \max\{m_j/g_j \mid j \in \{1, \ldots, d\}\},$$

$$A(s) = \{x \in G \mid s_*(x) = s\} \quad \text{for each} \quad s \geq 0,$$

$$\phi(j) = \left|\{j' \in \{1, \ldots, j\} \mid \gcd(j, j') = 1\}\right| \quad \text{for each} \quad j = 1, \ldots, d.$$

From this, we claim that we are able to deduce, for $s \geq 2$, that

$$|A(s)| \leq \left(\sum_{j=1}^s \phi(j)\right)^d \leq (1 + \sum_{j=2}^s (j-1))^d \leq \left(\frac{s}{2}\right)^d. \quad (5.4)$$

Indeed, $\phi(j) \leq j - 1$ for $j \geq 2$, and observe that

if $r$ divides $m$, \quad then \quad $|\{a \in \{1, \ldots, m\} \mid \gcd(a, m) = r\}| = \phi(m/r)$;

hence, summing over the set of possible values for $m_j/g_j$, which by definition of $A(s)$ is $\{1, \ldots, s\}$, we have $|A(s)|^{1/d} \leq \sum_{j=1}^s \phi(j)$. We are then able to deduce (1.3) from the following proposition.

**Proposition 5.1.** There exist absolute constants $c_1 \in (0, 1)$ and $C_2$ such that

$$\mathbb{P}\left(\frac{1}{k} \sum_{i=1}^k \{\bar{x} \cdot Z_i\}^2 \leq c_1 n^{-2/k}\right) \leq \begin{cases} (s_*(x))^{-9k/10} & \text{when} \quad s_*(x) \leq C_2 n^{1/k}, \\ 2^{-k/n} & \text{when} \quad s_*(x) > C_2 n^{1/k}. \end{cases} \quad (5.5a)$$

Indeed, assuming Proposition 5.1 and Lemma 5.3 and writing

$$p(s) = \max_{x \mid s_*(x) = s} \mathbb{P}(1 - \lambda_x \leq c_1 n^{-2/k}),$$

by (5.3) and (5.5) combined, letting $c_1' = c_1 \cdot (3/2\pi^2)$, we have

$$\sum_{x \in G \setminus \{0\}} \mathbb{P}(1 - \lambda_x \leq c_1' n^{-2/k}) \leq n \max_{s > C_2 n^{1/k}} p(s) + \sum_{2 \leq s \leq C_2 n^{1/k}} |A(s)|p(s) \leq 2^{-k} + 2^{-d} \sum_{s \geq 2} s^{2d(2s)^{-9k/10}} \lesssim 2^{-k},$$

where we have used $k \geq 3d$ and the fact that $s_*(x) > 1$ for all $x \neq 0$. \qed

It remains to prove the quoted results, namely Proposition 5.1 and Lemmas 5.2 to 5.4.
Proof of Proposition 5.1. Fix some $x \in G$. We first consider the case that $s = s_*(x) > C_2 n^{1/k}$, ie (5.5b). Let $j = j(x)$ be a coordinate satisfying $s = m_j / g_j$. Denote $m = m_j(x)$ and $g = g_j(x)$. Observe, by Lemma 4.8, that $x_j Z_{t_j}^{i_j} \sim \text{Unif}\{g, 2g, ..., n\}$ for each $i$. Hence, for each $i$, we have

$$U_i = x_j Z_{t_j}^{i_j} \sim \text{Unif}\{1/s, 2/s, ..., 1\}. \quad (5.6)$$

By averaging over $(a_i)^k_{i=1}$, where $a_i = \{\sum_{\ell \in \{1, ..., d\} \setminus\{j\}} x_j Z_{t_j}^{i_j} / m_j\}$, it suffices to show that

$$\max_{b_1, ..., b_k \in [-1/2, 1/2]} \mathbb{P}\left(\frac{1}{k} \sum_{i=1}^k (U_i + b_i)^2 \leq c_1 n^{-2/k}\right) \leq 2^{-k}/n. \quad (5.7)$$

Replacing $c_1$ with $4c_1$ we may assume that $b_1 = 1/2$. Indeed, if $|b_j - \ell/s| \leq 1/(2s)$, ie $|b_j - \ell/s| = \min\{|b_j - a| \mid a \in \mathbb{Z}/2\}$, then $\{U_j + \ell/s\}^2 \leq 4\{U_j + b_j\}^2$, and so if $\frac{1}{k} \sum_{j=1}^k \{U_j + b_j\}^2 \leq c_1 n^{-2/k}$, then $\frac{1}{k} \sum_{j=1}^k (U_j + \ell/s)^2 \leq 4c_1 n^{-2/k}$. In this case, $\{U_j + b_j\}$ has the same law as $\{U_j\}$. Hence it suffices to prove (5.7) for $b_1 = \cdots = b_k = 0$.

We now split $[0, 1]$ into $M = \lceil 4n^{1/k} \rceil$ consecutive intervals of equal length $J_1, \ldots, J_M$, where $J_1 = [0, \frac{1}{2M}]$ and $J_j = \left(\frac{j - 1}{2M}, \frac{j}{2M}\right]$ for $j > 1$. Let $Y_i = \ell - 1$ if $\{U_i\} \in J_\ell$. Clearly, $\frac{1}{2} Y_i / M^2 \leq \frac{1}{4} Y_i^2 / M^2 \leq \{U_i\}^2$. It thus suffices to show that

$$\mathbb{P}\left(\frac{1}{k} \sum_{i=1}^k Y_i \leq \frac{1}{10}\right) \leq 2^{-k}/n.$$  

This final claim follows by a simple counting argument: there are $M^k$ total assignments of the $Y_i$-s, but at most $L(k) = \left(\frac{11k^{10}}{k-1}\right)$ $2^k$ assignments satisfy $\frac{1}{k} \sum_{i=1}^k Y_i \leq \frac{1}{10}$, since $L(k)/M^k \leq 2^{-k} n^{-1}$.

We now prove the case $s = s_*(x) \leq C_2 n^{1/k}$, ie (5.5a). By the same reasoning as for (5.7), it suffices to show that

$$\max_{b_1, ..., b_k \in [-1/2, 1/2]} \mathbb{P}\left(\frac{1}{k} \sum_{i=1}^k (U_i + b_i)^2 \leq c_1 n^{-2/k}\right) \leq s^{-9k/10}. \quad (5.8)$$

Regardless of $b_i$, there is at most one $a = a(b_i) \in \{1/s, 2/s, ..., 1\}$ such that $\{a + b_i\}^2 < (2s)^{-2}$, and hence by (5.6), for all $i$, we have

$$\mathbb{P}\left(\{U_i + b_i\}^2 < (2s)^{-2}\right) \leq 1/s.$$

If there is no such value $a(b_i)$, then set $a(b_i) = -1$.

If $\{U_i + b_i\}^2 \geq (2s)^{-2}$ for at least $q = k \cdot 4c_1 s^2 n^{-2/k}$ of the $i$-s, ie if

$$\left|\{i \in \{1, ..., k\} \mid U_i \neq a(b_i)\}\right| \geq q,$$

then $\frac{1}{k} \sum_{i=1}^k \{U_i + b_i\}^2 \geq c_1 n^{-2/k}$, as desired. As $s \leq C_2 n^{1/k}$, by taking $c_1$ sufficiently small in terms of $C_2$, we can make $q/k$ sufficiently small so that the following holds:

$$\mathbb{P}\left(\left|\{i \in \{1, ..., k\} \mid U_i \neq a(b_i)\}\right| < q\right) \leq \left(\frac{q}{k}\right) s^{q-k} \leq s^{-9k/10}. \quad \square$$

It remains to state and prove Lemmas 5.2 to 5.4.

**Lemma 5.2.** Let $\ell \in \mathbb{N}$ and $\tau = \inf\{s \geq 0 \mid Y_s = \ell\}$, where $(Y_s)_{s \geq 0}$ is a continuous-time rate-1 SRW on $\mathbb{Z}$. Let $\lambda = 1 - \cos \theta$ and $\theta = 4\pi/\ell$. Then, for all $s \geq 0$, we have

$$\mathbb{P}_0(\tau > s) \geq e^{-\lambda s} \geq \exp\left(-\left(\frac{1}{\theta^2} \exp\left(-\frac{\pi^2}{4\pi^2}\right) \pi/\ell\right)^2 \right) .$$

**Proof.** The second inequality follows from Lemma 5.3 below.

For the first inequality, we first note that

$$\mu = x \mapsto \cos(\theta x) / \sum_{j=-\ell}^{\ell} \cos(\theta j) : [-\ell, ..., \ell] \rightarrow [0, 1]$$

is a distribution satisfying $\mu(\pm \ell) = 0$ and

$$(\mu \hat{P})(x) = \mu(x) \cos \theta \text{ for } x \in J = \{-\ell + 1, ..., \ell - 1\},$$

where $\hat{P}$ is a probability distribution on $\mathbb{Z}$ such that

$$\hat{P}(x) = \frac{1}{2\ell} \left(1 - \frac{2}{\ell} \cos(\theta x) \right).$$
where \( \hat{P} \) is the transition matrix of discrete-time SRW on \( \{-\ell, \ldots, \ell\} \) with absorption at the boundary. Indeed, using \( \mu(\pm \ell) = 0 \) we have \( \mu \hat{P}(x) = \frac{1}{2} \mu(x+1) + \mu(x-1) = \mu(x) \cos(\pi/(2\ell)) \), where we have used \( \cos(a+b) + \cos(a-b) = 2 \cos a \cos b \). If follows that starting from initial distribution \( \mu \) we have \( \mu \hat{P}^i(J) = (1-\lambda)^i \), where \( \hat{P}^i \) is the matrix \( \hat{P} \) raised to the power \( i \), and so \( \mu \hat{P}^i(J) \) is the probability of not getting absorbed at the boundary by the \( i \)-th step when the initial distribution is \( \mu \). It follows that

\[
P_{\mu}(\tau > s) = \sum_{i=0}^{\infty} \mu \hat{P}^i(J) \mathbb{P}(\text{Po}(t) = i) = e^{-\lambda s}.
\]

To conclude the proof, by considering the chain \((Y_i)_{i \geq 0}\), we obtain, for all \( s \geq 0 \), that

\[
P_0(\tau > s) = \max_{j \in J} \mathbb{P}_{\mu}(\tau > s);
\]

this can be seen by a simple coupling argument, like that in [16, Example 5.1].

**Lemma 5.3.** For \( \theta \in [-\frac{1}{2}, \frac{1}{2}] \), we have

\[
2(\pi \theta)^2 \geq 1 - \cos(2\pi \theta) \geq 2 \exp\left(-\frac{7\pi^2 \theta^2}{18}\right)(\pi \theta)^2 \geq \frac{2}{3}(\pi \theta)^2.
\]

**Proof.** Let \( \theta \in [-\frac{1}{2}, \frac{1}{2}] \). Then, using the fact that \( \log(1 - x) \geq -x - x^2 \) for \( |x| < 1 \), that \( \sum_{i} 1/i^2 = \frac{\pi^2}{6} \), that \( \sum_{i} 1/i^4 = \frac{\pi^4}{90} \) and that \( \theta \in [-\frac{1}{2}, \frac{1}{2}] \), we can calculate directly:

\[
1 \geq \frac{1 - \cos(2\pi \theta)}{2(\pi \theta)^2} = \left(\frac{\sin(\pi \theta)}{\pi \theta}\right)^2 = \prod_{\ell=1}^{\infty} \left(1 - \theta^2/\ell^2\right)
\geq \exp\left(-2 \sum_{\ell=1}^{\infty} (\theta^2/\ell^2 + \theta^4/\ell^4)\right) \geq \exp\left(-\frac{7}{18} \pi^2 \theta^2\right) \geq 0.383 \geq \frac{1}{3}.
\]

For a transition matrix \( P \) and a set \( A \), let \( \lambda_A \) be the minimal Dirichlet eigenvalue, defined to be the minimal eigenvalue of minus the generator of the chain killed upon exiting \( A \), ie of

\[
I_A - P_A \quad \text{where} \quad (I_A - P_A)(x, y) = \mathbb{1}(x, y \in A)(1(x = y) - P(x, y)).
\]

Also, for a set \( A \), write \( \tau_{A^c} \) for the (first) exit time of this set by the chain.

**Lemma 5.4.** Consider a rate-1, continuous-time, reversible Markov chain with transition matrix \( P \). Let \( A \) be a connected set, and let \( \lambda_A \) and \( \tau_{A^c} \) be as above. Then, for all \( a \in A \), we have

\[
-\frac{1}{t} \log \mathbb{P}_{a}(\tau_{A^c} > t) \to \lambda_A \quad \text{as} \ t \to \infty.
\]

**Proof.** For connected \( A \), by the Perron-Frobenius theorem, the quasi-stationary distribution of \( A \), which we denote by \( \mu = (\mu_a)_{a \in A} \), is positive everywhere on \( A \). (See [2, §3.6.5] for the definition of quasi-stationarity.) Since \( \mathbb{P}_{\mu}(\tau_{A^c} > t) = \sum_{a \in A} \mu_a \mathbb{P}_a(\tau_{A^c} > t) \), we have

\[
\mathbb{P}_a(\tau_{A^c} > t) \leq \mu_a^{-1} \mathbb{P}_{\mu}(\tau_{A^c} > t) = \mu_a^{-1} \exp(-\lambda_A t),
\]

since the exit time starting from the quasi-stationary distribution is exponential with rate \( \lambda_A \), as shown in the equation proceeding (3.83) in [2]. This proves the upper bound, taking \( t \to \infty \).

For the other direction, we claim that there exists a constant \( c \), independent of \( a \) and \( t \), so that

\[
\min_{a \in A} \mathbb{P}_a(\tau_{A^c} > t) \geq c \max_{a \in A} \mathbb{P}_a(\tau_{A^c} > t + 1).
\]

Indeed, let \( a' \) be an element of \( A \) attaining the maximum at time \( t + 1 \). Using the connectedness of \( A \), for any other \( a \in A \) there exists a path from \( a' \) to \( a \) consisting of states belonging to \( A \). The probability that the walk traverses this path, and does so in time less than 1, is at least \( c \), for some \( c \) independent of \( t \). From this we deduce that

\[
\min_{a \in A} \mathbb{P}_a(\tau_{A^c} > t) \geq c \mathbb{P}_a(\tau_{A^c} > t + 1) = c \exp(-\lambda_A(t + 1)).
\]

This proves the lower bound, taking \( t \to \infty \), and hence proves the lemma.

Now that we have completed the proof, we make a few remarks.
Remarks. (i) Our proof gives an explicit form for $C_1$ in (1.2). If $k \ll \log n$, then we get

$$t_{rel} \geq 2\pi^{-2} |G|^{2/k} \cdot (1 + o(1)),$$

as in this case, we can take in the proof above $L = \lfloor \frac{1}{2}(\varepsilon n)^{1/k} \rfloor$ for an arbitrary small $\varepsilon > 0$, making $|A|/|G|$ arbitrary small. One can improve the constant by replacing $A$ with

$$\{w : Z | w \in \mathbb{Z}^k \text{ and } \sum_{i=1}^k |w_i|^2 \leq L(k, n)\},$$

where $L(k, n)$ is the maximal integer satisfying $|\{w \in \mathbb{Z}^k \mid \sum_{i=1}^k |w_i|^2 \leq L(k, n)\}| \leq \frac{1}{2}n$.

(ii) To complement (1.1), we note that the same holds with $P^*$ in the role of $P$, if we replace $e^{-\gamma t}$ with $(1 - \gamma)^t$, where

$$\gamma_* = 1 - \lim_{t \to \infty} (\max_{x \in V} (P^{2t}(x, x) - \pi(x)))^{1/2t}$$

is the absolute spectral gap, defined to be 0 if $P$ has more than one unit eigenvalue, and otherwise as min$_\lambda |1 - |\lambda||$, where the minimum is taken over all non-unit eigenvalues of $P$.

The argument in the proof of (1.3) can be used to show, for a positive constant $C$, that

$$\mathbb{P}(1/\gamma_* \leq C|G|^{2/k}) \geq 1 - C2^{-k},$$

where $\gamma_*$ is the absolute spectral gap of the transition matrix of the SRW.

(iii) The condition $k \geq 3d$ can be relaxed to $k \geq (2 + \delta)d$, for $\delta > 0$; in this case, $C_2$ will depend on $\delta$. In fact, if $v_{ij} = p$ for all $j$ for a prime $p$, then one can relax this further to $k \geq (1 + \delta)d$, and even allow $\delta$ to tend to 0, provided $p$ diverges. (In this case, the term $2^{-k}$ has to be replaced by another term which tends to zero at a slower rate as $k \to \infty$.) This follows from the fact that in this case, we only need to consider (5.4) above with $s = p$ and we can replace (5.4) with $|A(p)| = p^d - 1$.

To close this section, we related the spectral gap to the Cheeger constant and make a conjecture. First, the Cheeger constant of a finite $d$-regular graph $G = (V, E)$ is defined as

$$\Phi_* = \frac{1}{2} \min_{1 \leq |A| \leq |V|} \{\{\{a, b\} \in E \mid a \in A, b \in A^c\}\} / |A|.$$

By the well-known discrete analogue of Cheeger’s inequality, discovered independently by multiple authors – see, for example, [16, Theorem 13.10] – we have $\frac{1}{2}\gamma \leq \Phi_* \leq \sqrt{2\gamma}$. Determining the correct order of $\Phi_*$ in our model remains an open problem. We conjecture that the correct order is given by $\sqrt{\gamma}$, ie order $|G|^{-1/k}$, using Theorem B for the order of the spectral gap.

Conjecture. There exists an absolute constant $c$ so that, for all $\varepsilon \in (0, 1)$, there exist constants $N(\varepsilon)$ and $M(\varepsilon)$ so that, for every finite group $G$ of size at least $N(\varepsilon)$, the probability that the Cheeger constant of a random Cayley graph with $k$ generators, with $k \geq M(\varepsilon)$, is less than $c|G|^{-1/k}$ is at most $\varepsilon$.

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References

[1] D. Aldous and P. Diaconis (1986). Shuffling Cards and Stopping Times. Amer. Math. Monthly. 93.5, (333–348) MR841111 DOI

[2] D. Aldous and J. A. Fill (2002). Reversible Markov Chains and Random Walks on Graphs. Unfinished Monograph. Available at www.stat.berkeley.edu/~aldous/RWG/book.html
A Appendix

The main purpose of this appendix is to derive properties of \( W_1(t) \), i.e. of the SRW on \( \mathbb{Z} \) evaluated at \( t/k \) or of \( \text{Po}(t/k) \), for \( t \) around the entropic time \( t_0 \). First though, we justify the CLT application, Proposition 1.4, before using this, and an inspection of the proof of Proposition 4.6, to quantify the \( o(1) \) error terms in the TV, as mentioned in Remarks 1.5(iv).

Of repeated use will be a local CLT for Poisson and simple random walk distributions. We state it here precisely; the particular version is a combination of [15, Lawler and Limic, Proposition 2.5.5 and Theorem 2.5.6].
Theorem A.1 (Local CLT). Let $s \geq 0$. Let $X_s$ be a random variable with one of the following distributions: $\text{Po}(s) - s$; the location of a rate-1 continuous-time, one-dimensional, symmetric, simple random walk started from 0 and run for time $s$. If $|x| \leq \frac{1}{2}s$, then

$$
\mathbb{P}(X_s = x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \exp\left(O\left(\frac{|x|^3}{s}\right)\right).
$$

In particular, if $|x| \leq s^{7/12}$, then

$$
\mathbb{P}(X_s = x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) \exp\left(O(s^{-1/4})\right).
$$

(A.1)

A.1 Justification of CLT Application

We first justify our CLT application in (1.7). The distribution of $Q_i(t_n)$ depends on $k$ (and $n$), and so we cannot apply the standard CLT. Instead, we apply a CLT for ‘triangular arrays’; specifically, we now state a special case of the Lindeberg-Feller theorem.

Theorem A.2 (CLT for Triangular Arrays). For each $k \in \mathbb{N}$, let $\{Y_{i,k}\}_{i=1}^k$ be an iid sequence of centralised, normalised random variables, and suppose that $\mathbb{E}(Y_{i,k}^4) \ll k$. Then

$$
\sum_{i=1}^k Y_{i,k}/\sqrt{k} \rightarrow^d N(0,1) \quad \text{as} \ k \rightarrow \infty,
$$

where $N(0,1)$ is a standard normal.

This version can be deduced easily using Markov’s (or Chebyshev’s) inequality from, for example, the version given in [12, Theorem 3.4.5].

Proof of Proposition 1.4. For our application, for each $\alpha \in \mathbb{R}$, we take

$$
Y_{i,k} = \frac{Q_i(t_\alpha) - \mathbb{E}(Q_i(t_\alpha))}{\sqrt{\mathbb{V}(Q_i(t_\alpha))}}.
$$

(A.2)

Observe that $\mathbb{E}(Y_{i,k}) = 0$ and $\mathbb{V}(Y_{i,k}) = \mathbb{V}(Y_{i,k}^2) = 1$. Assuming that $\mathbb{E}(Y_{i,k}^4) \ll k$, we deduce the following result: for any sequence $(\zeta_n)_{n \geq 1}$ which converges to $\zeta$, we deduce that

$$
\mathbb{P}\left(Q(t) - \mathbb{E}(Q(t)) \geq \zeta_n \sqrt{\mathbb{V}(Q(t))}\right) \rightarrow \Psi(\zeta).
$$

(A.3)

(We are also using Slutsky’s theorem to allow $\zeta_n$ to depend on $n$, and, of course, the fact that $k \rightarrow \infty$ as $n \rightarrow \infty$.) We also rely on the followingClaim: if $t \approx t_0$, then $\mathbb{V}(Q(t_0)) \approx \mathbb{V}(Q(t))$; also $\mathbb{V}(Q(t)) \gg 1$. (A.4)

We prove these two statements in this claim (independently of the application of the CLT) in Proposition A.6 in §A.3. Now recall (1.5), which says, for all $\alpha \in \mathbb{R}$, that $t_n \approx t_0$. Taking

$$
\zeta_n = -\alpha \sqrt{\mathbb{V}(Q(t_0))} / \sqrt{\mathbb{V}(Q(t_\alpha))} \pm \omega / \sqrt{\mathbb{V}(Q(t_\alpha))} \quad \text{with} \quad \omega = \mathbb{V}(Q(t_0))^{1/4} \gg 1,
$$

(A.5)

applying (A.3, A.4) along with the above recollections we obtain the desired result:

$$
\mathbb{P}(Q(t_\alpha) \leq \log n \pm \omega) \rightarrow \Psi(\alpha).
$$

(1.7)

It remains to verify that $\mathbb{E}(Y_{i,k}^4) \ll k$. Roughly, we have that

$$
|W_1(t)| \quad \text{is ‘well approximated’ by} \quad \begin{cases} 
|N(\mathbb{E}(W_1(t), t/k)| & \text{when } k \ll \log n \\
\text{Bern}(t/k) & \text{when } k \gg \log n.
\end{cases}
$$

In the interim regime $k \approx \log n$, we have that $W_1$ behaves like an ‘order 1’ random variable, in the sense that its mean and variance are order 1 in $n$ (ie do not converge to 0 or diverge to $\infty$). It will
actually turn out that the normal approximation is sufficient in the \( k = \log n \) regime also. Below, we abbreviate \( Q_1(t_o) \) by \( Q_1 \), \( W_1(t_o) \) by \( W_1 \) and \( t_o \) by \( t \).

Write \( s = t/k \). We shall consider separately the cases \( s \gtrsim 1 \) and \( s \ll 1 \). When \( s \gtrsim 1 \), we have \( t \gtrsim k \gg 1 \); when considering \( s \ll 1 \), however, we shall only consider \( t \) with \( 1 \ll t \ll k \). We shall be interested in \( t = t_o \approx t_o \), and Proposition 1.2 says that \( t_o \gg 1 \) in all regimes; hence we need only consider \( t \gg 1 \).

Consider first \( s = t/k \) with \( s \gtrsim 1 \). In this regime, we approximate \( W_1(t) \) by a \( N(\mathbb{E}(W_1), s) \) distribution, where \( s = t/k \). Let \( Z \sim N(\mathbb{E}(W_1), s) \), and write \( f \) for its density function:

\[
  f(x) = (2\pi s)^{-1/2} \exp(-\frac{1}{2s}(x - \mathbb{E}(W_1))^2) \quad \text{for} \quad x \in \mathbb{R}. \tag{A.6}
\]

Let \( R_1 \) be a real valued random variable defined so that

\[
  R_1 = -\log f(x) \quad \text{when} \quad W_1 = x. \tag{A.7}
\]

Also write \( G = W_1 + U \), where \( U \sim \text{Unif}[\frac{1}{2}, \frac{1}{2}] \) is independent of \( W_1 \); then \( G \) has density function

\[
  g(x) = \mathbb{P}(W_1 = [x]) \quad \text{for} \quad x \in \mathbb{R}, \tag{A.8}
\]

where \( [x] \in \mathbb{Z} \) is \( x \in \mathbb{R} \) rounded to the nearest integer (rounding up when \( x \in \mathbb{Z} + \frac{1}{2} \)).

Using convexity of the 4-norm, we have

\[
  (a - b)^4 \leq 2^8 ((a - a')^4 + (a' - b')^4 + (b' - b)^4) \quad \text{for all} \quad a, a', b, b' \in \mathbb{R}.
\]

Applying this inequality with \( a = Q_1 \), \( a' = R_1 \), \( b = \mathbb{E}(Q_1) \) and \( b' = \mathbb{E}(R_1) \), we obtain

\[
  2^{-8} \mathbb{E}((Q_1 - \mathbb{E}(Q_1))^4) \leq \mathbb{E}((Q_1 - R_1)^4) + \mathbb{E}((R_1 - \mathbb{E}(R_1))^4) + \mathbb{E}(R_1 - Q_1)^4
  \leq \mathbb{E}((R_1 - \mathbb{E}(R_1))^4) + 2 \mathbb{E}((Q_1 - R_1)^4), \tag{A.9}
\]

with the second inequality following from Jensen (or Cauchy-Schwarz twice). We study these terms separately. Approximately, the local CLT will say that the second term is small; up to an error term which we control with the local CLT, the first term we can calculate directly using properties of the normal distribution.

We consider first the first term of (A.9). In terms of an integral, it is given by

\[
  \mathbb{E}((R_1 - \mathbb{E}(R_1))^4) = \int_{\mathbb{R}} g(x)(-\log f(x) - \mathbb{E}(R_1))^4 dx.
\]

The local CLT suggests that we can approximately replace the \( g(x) \) factor by \( f(x) \), at least for a large range of \( x \). So let us first study

\[
  \int_{\mathbb{R}} f(x)(-\log f(x) - \mathbb{E}(R_1))^4 dx = \int_{\mathbb{R}} f(x + \mathbb{E}(W_1))(-\log f(x + \mathbb{E}(W_1)) - \mathbb{E}(R_1))^4 dx.
\]

This expression depends only on properties of the normal distribution, and direct calculation, via expanding the fourth power and using moments of \( N(0, 1) \), finds that this equals \( \frac{15}{4} \).

Now, by the local CLT (A.1), we have

\[
  \int_{s/\sqrt{12}}^{s\sqrt{12}} g(x)(-\log f(x) - \mathbb{E}(R_1))^4 dx \leq (1 + O(s^{-1/4})) \int_{s/\sqrt{12}}^{s\sqrt{12}} f(x)(-\log f(x) - \mathbb{E}(R_1))^4 dx
  \leq (1 + O(s^{-1/4})) \cdot \frac{15}{4}.
\]

Using bounds on the tail of the SRW and Poisson distribution, as given in Propositions A.14 and A.15, it is straightforward to see, in both the undirected and directed cases, that

\[
  \int_{\mathbb{R}[(-s^{7/12}, s^{7/12})]} f(x)(-\log f(x) - \mathbb{E}(R_1))^4 dx = o(s^{-10}). \tag{A.10}
\]

(In fact, it is easy to see that it is \( O(\exp(-cs^{1/6})) \) for some sufficiently small constant \( c \).) Hence

\[
  \mathbb{E}((R_1 - \mathbb{E}(R_1))^4) = \frac{15}{4}(1 + O(s^{-1/4})) = \frac{15}{4}(1 + o(1)) \tag{A.11}
\]
We now turn to the second term of (A.9). In terms of an integral, it is given by
\[ \mathbb{E}((Q_1 - R_1)^4) = \int_{\mathbb{R}} g(x) \log(f(x)/g(x))^4 \, dx. \]
Again by the local CLT (A.1), we have
\[ \int_{-s^{7/12}}^{s^{7/12}} g(x) \log(f(x)/g(x))^4 \, dx = O(s^{-1/4}) \int_{-s^{7/12}}^{s^{7/12}} g(x) \, dx \leq O(s^{-1/4}), \]
and a similar application of the tail bounds in Propositions A.14 and A.15 shows that
\[ \int_{\mathbb{R} \setminus [-s^{7/12}, s^{7/12}]} g(x) \log(f(x)/g(x))^4 \, dx = o(s^{-10}) = O(s^{-1/4}). \quad (A.12) \]
Hence, combining (A.11, A.12) into (A.9), we obtain
\[ \mathbb{E}((Q_1 - E(Q_1))^4) \leq \frac{15}{4} \cdot 2^8 + o(1) \leq 1000. \]

We must now consider \( \mathbb{V} \text{ar}(Q_1) \). We do this in Proposition A.6 in §A.3. Recall also that
\( t_0 \gtrless k \) when \( k \lesssim \log n \); this follows from the continuity of the function \( f \) in (1.4). Using then the
continuity of the function \( g \) in (A.19), we see that if there exists a constant \( c \) so that \( s = t/k \geq c \), then there
exists a constant \( C \) (depending on \( c \)) so that \( \mathbb{E}(Y_{1,k}^4) \leq C \); in particular we certainly have \( \mathbb{E}(Y_{1,k}^4) \ll k \). The completes the proof for the regime \( k \lesssim \log n \).

Consider now \( s = t/k \) with \( s \ll 1 \) but \( t \gg 1 \). In this regime, we shall approximate the number
of steps taken by \( \text{Bern}(t/k) \). Indeed, we have
\[ \mathbb{E}(W_1 = 0) = 1 - s + O(s) \quad \text{and} \quad \mathbb{E}(|W_1| = 1) = s + O(s^2). \]
We also use the fact that, for both the undirected and directed cases, for \( x \geq 0 \) we have
\[ \mathbb{P}(W_1 = x) \geq \mathbb{P} \left( \text{Pois}(s) = x \right) \cdot 2^{-x} = 2^{-x} e^{-s} s^x/x! \geq (s^2/x)^x; \quad (A.13) \]
from this one deduces that \( -\log \mathbb{P}(W_1 = x) \leq x \log(x/s^2) = x(x + 2 \log(1/s)). \) We use this to show that the terms with \( |x| \geq 2 \) contribute sub-leading order to the expectation
\[ \mathbb{E}(Q_1) = \sum_x \mathbb{P}(W_1 = x) \log 1/\mathbb{P}(W_1 = x) = s \log(1/s) + O(s). \]
Similarly, we can use (A.13) to ignore the terms with \( |x| \geq 2 \) in
\[ \mathbb{E}((Q_1 - E(Q_1))^r) = \sum_x \mathbb{P}(W_1 = x) \left| \log \mathbb{P}(W_1 = x) - s \log(1/s) + O(s) \right|^r \]
\[ = s \log(1/s)^r (1 + O(s)), \quad (A.14) \]
for any fixed \( r \in \mathbb{N} \) with \( r \geq 2 \), say \( r \in \{2, 3, 4\} \).
In particular, this says that \( \mathbb{V} \text{ar}(Q_1) \approx s \log(1/s)^2 \), and so
\[ \mathbb{E}(Y_{1,k}^4) \approx (s \log(1/s)^4) / (s \log(1/s)^2)^2 = 1/s = k/t \ll k, \]
with the final relation holding since \( s \ll 1 \) we do have \( k \gg 1 \).

We now have all that we need to get on and calculate the entropic time \( t_0 \) in the three regimes
of \( k \). However, in order to find the cutoff times \( t_o \), we need to know what the variance of the terms
in the sum \( Q(t) \), ie \( \mathbb{V} \text{ar}(Q_1(t)) \), is for \( t \approx t_0 \). First we digress to consider the rate of convergence
of the CLT, as mentioned in Remarks 1.5(iv).

### A.2 Quantification of \( o(1) \) Errors

In this section, our aim is to quantify the errors in the CLT application (1.7) and in the main
total variation distance claim (1.6) from Theorem 1.3; this was outlined in Remarks 1.5(iv).
**Theorem A.3** (Error in TV). In the setup of Theorem 1.3, for \( k \leq \frac{1}{2} \log n / \log \log n \), with \( \omega = \log(k/\log k) \), for each \( \alpha \in \mathbb{R} \), we have

\[
\mathbb{E}(|d_Z(t_\alpha) - \Psi(\alpha)|) \leq 2\omega / \sqrt{k} = 2 \log(k/\log k) / \sqrt{k}.
\]

Moreover, the corresponding lower bound on \( d_Z(t_\alpha) \) holds deterministically:

\[
d_Z(t_\alpha) \geq \Psi(\alpha) - \varepsilon_\alpha \quad \text{where} \quad \varepsilon_\alpha = 2\omega / \sqrt{k} = 2 \log(k/\log k) / \sqrt{k}.
\]

We do not try to prove the strongest result possible, but rather just look at the case \( k \leq \frac{1}{2} \log n / \log \log n \), because it is the simplest to present. Other cases \( k \ll \log n \) and \( k \gg \log n \) can be handled similarly, but more care is required when \( k \approx \log n \): in this regime, more precise knowledge of \( \text{Var}(Q_1(t_\alpha)) \) is required.

Instead of using Lindeberg-Feller to simply say that the normalised version of \( Q(t) \) converges to a normal distribution, we can use Berry-Esséen to quantify the rate of convergence.

**Theorem A.4** (Berry-Esséen). Let \( Z_1, Z_2, \ldots \) be iid centralised, normalised random variables with \( \mathbb{E}(|Z_1|^3) = \rho < \infty \). Let \( S_k = \sum_{i=1}^k Z_i / \sqrt{k} \). Then, for each \( \xi \in \mathbb{R} \), we have

\[
|\mathbb{P}(S_k \geq \xi) - \Psi(\xi)| \leq 3\rho / \sqrt{k}.
\]

This particular version is taken from [12, Theorem 3.4.9]. Recall that \( \Psi(\xi) = \mathbb{P}(N(0,1) \geq \xi) \). We apply Berry-Esséen to obtain the following result.

**Proposition A.5** (Error in CLT). For \( k \) with \( k \gg 1 \) and \( k \leq \frac{1}{2} \log n / \log \log n \), we have

\[
|\mathbb{P}(Q(t_\alpha) \leq \log n \pm \omega) - \Psi(\alpha)| = \mathcal{O}(\omega / \sqrt{k}).
\]

**Proof.** We apply Berry-Esséen with \( \zeta = \zeta_n \), defined in (A.5), and \( \zeta \) as the limit of \( \zeta_n \) as \( n \to \infty \). Before we used Slutsky’s theorem to allow \( \zeta_n \) to depend on \( n \). Now we quantify this with the triangle inequality. Define \( Y_{i,k} \) as in (A.2) and let \( S_k = \sum_{i=1}^k Y_{i,k} / \sqrt{k} \). Then

\[
|\mathbb{P}(S_k \geq \zeta_n) - \Psi(\zeta)| \leq |\mathbb{P}(S_k \geq \zeta_n) - \Psi(\zeta_n)| + |\Psi(\zeta_n) - \Psi(\zeta)|.
\]  

(A.15)

The first term we control will Berry-Esséen; we have already bounded the fourth moment \( \mathbb{E}(Y_{i,k}^4) \leq 1000 \), and so \( \mathbb{E}(|Y_{i,k}^3|) \leq 1000^{3/4} \leq 1000 \), by Hölder.

For the second term, note that \( \Psi \) is a smooth function with derivative bounded by 1, and so

\[
|\Psi(\zeta_n) - \Psi(\zeta)| \leq |\zeta_n - \zeta|.
\]  

(A.16)

We now bound \( |\zeta_n - \zeta| \). Using the definition (A.5), we have

\[
|\zeta_n - \zeta| \leq |\alpha| \cdot |1 - \sqrt{\text{Var}(Q_1(t_\alpha))/\text{Var}(Q(t_\alpha))}| + \omega / \sqrt{\text{Var}(Q_1(t_\alpha))}.
\]  

(A.17)

Using techniques similar to those used in the previous section, in the next section we show, in the proof of Proposition A.6, the following: writing \( s = t/k \), we have

\[
\text{Var}(Q_1(t)) \approx \begin{cases} 
\frac{1}{2}(1 + \mathcal{O}(s^{-1/4} \log s)) & \text{when } s \gg 1, \\
\log(1/s)^2(1 + \mathcal{O}(s)) & \text{when } s \ll 1.
\end{cases}
\]

Using the fact that \( t_\alpha \approx t_0 \), by Proposition 1.2, and the fact that

\[
|1 - \frac{t_\alpha + s}{t_0 + s}| \approx |\varepsilon_0 - \varepsilon_1| \leq |\varepsilon_0| + |\varepsilon_1| \quad \text{when} \quad \varepsilon_0, \varepsilon_1 \ll 1,
\]

this controls the variance-ratio in (A.17). We plug in the expression (1.4a) for \( t_0 \), and get that

\[
s^{-1/4} \log s \leq 1 / \sqrt{k} \quad \text{if} \quad k \leq \frac{1}{2} \log n / \log \log n.
\]

Since \( \omega \gg 1 \), this shows that (A.17) is dominated by its second term. Moreover, in this regime the third moment in Berry-Esséen was bounded, the error from Berry-Esséen is order \( 1 / \sqrt{k} \) also. Hence the whole error is dominated by the second term of (A.17), which is order \( \omega / \sqrt{k} \), since the variance is approximately \( \frac{1}{2} \) in this regime. 

\[\square\]
We now use this, along with an inspection of the proofs of the TV distance results, to find the overall error. We only consider $k \leq \frac{1}{2} \log n / \log \log n$, but other cases can be handled similarly.

**Proof of Theorem A.3.** An inspection of the proofs of Propositions 4.6a and 4.6b shows that the error in calculating the upper bound on the TV distance on the event $\{Q(t) \geq \log n + \omega\}$ is $o(1/\sqrt{k})$, in the regime $k \ll \log n$. For the lower bound, from the proof of (1.8) in §3, we have an error of $e^{-\omega}$ on the event $\{Q(t) \leq \log n - \omega\}$. Combining this with the CLT error from Proposition A.5, we see that the overall error is

$$e^{-\omega} + \omega / \sqrt{k} + O(1/\sqrt{k}) + o(1/\sqrt{k}).$$

In our choice of $\omega$, for the CLT we only needed that $\omega \ll \sqrt{\text{Var}(Q(t))} \approx \sqrt{k/2}$. Instead of picking $\omega = \text{Var}(Q(t))^{1/4}$, we take $\omega = \log(k / \log k)$, which satisfies the required condition, and has

$$e^{-\omega} = \log(k / \log k) \ll \log(k / \log k) / \sqrt{k} = \omega / \sqrt{k}.$$

Hence the error is dominated by $\omega / \sqrt{k}$, which completes the proof. □

### A.3 Variance of $Q_1(t)$

Recall that, for all $t \geq 0$, we have

$$Q(t) = -\log \mu(t) = -\sum_{i=1}^{k} \log \nu_i(W_i(t)) = \sum_{i=1}^{k} Q_i(t),$$

and that the $Q_i(t)$-s are iid (for fixed $t$). We now determine what its variance is at the entropic time $t_0$, and how the variance changes around this time. Note that $\text{Var}(Q(t)) = k \text{Var}(Q_1(t))$.

**Proposition A.6.** For all regimes of $k$, in both the undirected and directed cases, we have that

$$\text{if } t \approx t_0, \text{ then } \text{Var}(Q_1(t)) \approx \text{Var}(Q_1(t_0)) \gg 1/k. \quad (A.18)$$

Moreover, for all $\lambda > 0$, we have

$$\text{Var}(Q_1(t_0)) \approx \begin{cases} 1/2 & \text{when } k \ll \log n, \\ \nu(\lambda) & \text{when } k \approx \lambda \log n, \\ \log n \log(k / \log n)/k & \text{when } k \gg \log n, \end{cases} \quad (A.19a)$$

where $\nu : (0, \infty) \to (0, \infty)$ is a continuous function whose value differs between the undirected and directed cases.

All that we shall use from the previous section is the form of $t_0$, as given by (1.4), namely

$$t_0 \approx \begin{cases} kn^{2/k}/(2\pi e) & \text{when } k \ll \log n, \\ f(\lambda)k & \text{when } k \approx \lambda \log n, \\ \log n / \log(k / \log n) & \text{when } k \gg \log n. \end{cases} \quad (A.19b)$$

As mentioned at the end of the previous section, control of the variance is only needed to calculating the cutoff times $t_0$; however, due to the similarity between the way we calculate the entropic time $t_0$ and cutoff times $t_0$, it is natural to have these proofs together, rather the separate.

**Proof of Equation (A.19a).** This proof is similar to the $k \ll \log n$ case in justifying the CLT application. In particular, if

$$g(x) = \mathbb{P}(W_1(t) = [x]) \text{ and } f(x) = (2\pi s)^{-1/2} \exp(-\frac{1}{2s}(x - \mathbb{E}(W_1(t)))^2),$$

then the local CLT (A.1) says, for $s \geq 1$, that

$$g(x) = f(x) \left(1 + O(s^{-1/4})\right) \text{ for } x \text{ with } |x - \mathbb{E}(W_1(t))| \leq s^{7/12}.$$
Under the assumption that $W_1(t)$ is actually distributed as $N(\mathbb{E}(W_1), s)$, direct calculation as in the previous section shows that the variance is then $\frac{1}{2}$. Considering the same approximations as before, namely splitting the integration range into $|x - \mathbb{E}(W_1)| \leq s^{7/12}$ and $|x - \mathbb{E}(W_1)| > s^{7/12}$, and using the local CLT to argue that $\log(g(x)/f(x)) = O(s^{-1/4})$ for $x$ in the first range, we obtain

$$\text{Var}(Q_1(t)) = \frac{1}{2} + O(s^{-1/4} \log s) \approx \frac{1}{2} \quad \text{when} \quad s = t/k \gg 1.$$ 

Since this is valid for any $t \gg k$ and $t_0 \gg k$, we have

$$\text{Var}(Q_1(t)) \approx \text{Var}(Q_1(t_0)) \approx \frac{1}{2} \quad \text{when} \quad t \approx t_0.$$ 

**Proof of Equation (A.19b).** Recall first that the variance of a random variable is 0 if and only if it is (almost surely) constant, and observe that this is not the case for $Q_1(t) = -\log \nu_t(W_1(t))$.

By (1.4b), we have $s = t_0/k \rightarrow f(\lambda)$, and each coordinate runs at rate $1/k$. Note that the variance of $Q_1(\cdot)$ is a continuous function, and hence given $C > 0$ there exists an $M$ so that

$$\frac{1}{M} \leq \text{Var}(Q_1(t)) \leq M \quad \text{for all} \quad t \text{ with } 1/C \leq t/k \leq C.$$ 

Hence, by continuity, $\text{Var}(Q_1(t_0)) \rightarrow v$ for some constant $v \in (0, \infty)$ depending only on $\lambda$. Note that this $v$ is not (necessarily) the same in the directed and undirected cases.

**Proof of Equation (A.19c).** In the CLT justification in the regime $k \gg \log n$, we showed that

$$\mathbb{E}((Q_1(t) - \mathbb{E}(Q_1(t)))^r) = s \log(1/s)^r + O(s^2 \log(1/s)^r), \quad (A.14)$$

and in particular deduced that $\text{Var}(Q_1(t)) \approx s \log(1/s)^2$, where $s = t/k$. Set $s_0 = t_0/k$, and so

$$s_0 = \frac{t_0}{k} \approx \frac{\log n/k}{\log(k/\log n)} = \frac{1}{\kappa \log \kappa} \quad \text{where} \quad \kappa = \frac{k}{\log n} \gg 1.$$ 

We then also have

$$\log(1/s_0) = -\log \log \kappa - \log \kappa \approx -\log \kappa,$$

and hence

$$s_0 \log(1/s_0)^2 \approx (\log \kappa)^2/\kappa \log \kappa \approx \log \kappa/\kappa = \log n \log(k/\log n)/k.$$ 

Note that while this has $\text{Var}(Q_1(t_0)) \ll 1$, it does have $\text{Var}(Q(t_0)) = k \text{Var}(Q_1(t_0)) \gg 1$.

### A.4 Calculating the Entropic and Cutoff Times

In this section we calculate the entropic time $t_0$, and the cutoff times $t_o$. Write

$$h_t = h(t) = \mathbb{E}(Q_1(t));$$

note that $h_t$ is the entropy of $W_1(t)$. In particular, we prove Proposition 1.2; the reader should recall that statement now.

We first consider the regime $k \ll \log n$.

**Proposition A.7 (Entropy).** For $t \gg k$, we have

$$h(t) = \frac{1}{2} \log(2\pi e t/k) + O((t/k)^{-1/4}). \quad (A.20)$$

**Proof.** We consider both the directed and undirected cases together. Write $s = t/k$. Define $f$, $R_1$ and $g$ as in (A.6), (A.7) and (A.8), respectively. By (A.12), we have

$$|\mathbb{E}(Q_1) - \mathbb{E}(R_1)| \leq \mathbb{E}((Q_1 - R_1)^4)^{1/4} = o(s^{-5/2}) \approx O(s^{-1/4}) \quad \text{when} \quad s \gg 1.$$ 

A similar calculation as used for (A.11) gives

$$\mathbb{E}(R_1) = (1 + O(s^{-1/4})) \cdot \log(2\pi e s).$$

Hence we obtain our desired expression, (A.20).
We now calculate the derivative of this entropy.

**Proposition A.8** (Derivative of Entropy). For \( t \gg k \), we have
\[
h'(t) = (2t)^{-1}(1 + \mathcal{O}((t/k)^{-10})).
\]

**Proof.** Write \( s = t/k \). Define \( f \), \( R \) and \( g \) as in (A.6), (A.7) and (A.8), respectively. We have
\[
h(t) = h(sk) = -\sum_{x \in \mathbb{Z}} P(X_s = x) \log P(X_s = x).
\]
Differentiating this with respect to \( t \) we obtain
\[
k h'(t) = \frac{d}{dt} h(sk) = -\sum_{x \in \mathbb{Z}} \frac{d}{ds} P(X_s = x) (\log P(X_s = x) + 1).
\]
Consider first the undirected case. Let \( X \) be a rate-1 continuous-time SRW on \( \mathbb{Z} \). Then \( W_1(t) \sim X_s \). Using the Kolmogorov backward equations, we obtain
\[
\frac{d}{ds} P(X_s = x) = \frac{1}{s} P(X_s = x + 1) + \frac{1}{s} P(X_s = x - 1) - P(X_s = x).
\]
Now write \( p_s(x) = P(X_s = x) \) and \( g_s(x) = p_s([x]) \). Since \( \sum_{x \in \mathbb{Z}} p_s(x) = 1 \), we obtain
\[
k h'(t) = \sum_{x \in \mathbb{Z}} \left( p_s(x) - \frac{1}{s} (p_s(x+1) + p_s(x-1)) \right) \log p_s(x)
\]
\[
\begin{align*}
= \int_{\mathbb{R}} (g_s(x) - \frac{1}{s} (g_s(x+1) + g_s(x-1))) \log g_s(x) \, dx \\
+ \int_{\mathbb{R}} (g_s(x) - \frac{1}{s} (g_s(x+1) + g_s(x-1))) \log (g_s(x)/f_s(x)) \, dx.
\end{align*}
\]
(A.21a)

(A.21b)
The same arguments as used for (A.10) show that the integral in (A.21b) is \( o(s^{-10}) \). Now consider the integral in (A.21a). Using a simple shift, we have
\[
\int_{\mathbb{R}} g_s(x+1) \log f_s(x) \, dx = \int_{\mathbb{R}} (g_s(x) - \frac{1}{s} (g_s(x+1) + g_s(x-1))) \log g_s(x) \, dx
\]
and we consider \( \int_{\mathbb{R}} g_s(x-1) \log f_s(x) \, dx \) similarly; hence we have
\[
\int_{\mathbb{R}} (g_s(x) - \frac{1}{s} (g_s(x+1) + g_s(x-1))) \log f_s(x) \, dx
\]
\[
= \frac{1}{s} \int_{\mathbb{R}} g_s(x) \log (f_s(x+1)/f_s(x)) + \log (f_s(x+1)/f_s(x)) \, dx
\]
\[
+ \frac{1}{s} \int_{\mathbb{R}} g_s(x) \log (f_s(x-1)/f_s(x)) \, dx.
\]
Since \( f_s(x) = (2\pi s)^{-1/2} \exp(-x^2/(2s)) \), this log is precisely \( 1/s \) (independent of \( x \)). Since it is a distribution, \( g_s \) integrates to \( 1 \), so we have that the integral equals \( 1/(2s) \). Combining the bounds for (A.21) and dividing through by \( k \) proves the undirected case.

Now consider the directed case. Let \( X_s \sim \text{Po}(s) \), which has \( P(X_s = x) = e^{-s}x^s/s! \). Then \( W_1(t) \sim X_s \). Direct differentiation shows that
\[
\frac{d}{ds} P(X_s = x) = P(X_s = x - 1) - P(X_s = x) = e^{-s} s^{x-1} (x-s)/x! \text{ for } x \in \mathbb{N},
\]
and \( \frac{d}{ds} P(X_s = 0) = P(X_s = 0) = e^{-s} \). (These are the backward equations for the Markov chain which starts at 0 and jumps to the right at rate 1.) Hence, as above, we have
\[
k h'(t) = e^{-s} \sum_{x \in \mathbb{N}} (p_s(x) - p_s(x-1)) \log p_s(x)
\]
\[
= \int_{1/2}^{\infty} (g_s(x) - g_s(x-1)) \log g_s(x) \, dx
\]
\[
+ \int_{1/2}^{\infty} (g_s(x) - g_s(x-1)) \log (g_s(x)/f_s(x)) \, dx.
\]
(A.22a)

(A.22b)
As for (A.21b) above, the same arguments as used for (A.10) show that the integral in (A.22b) is \( o(s^{-10}) \). Note also that \( se^{-s} = o(s^{-10}) \). Now consider the integral in (A.22a). Using a simple shift as before, we have
\[
\int_{1/2}^{\infty} (g_s(x) - g_s(x-1)) \log f_s(x) \, dx = -\int_{1/2}^{\infty} g_s(x) \log (f_s(x+1)/f_s(x)) \, dx
\]
\[
= \int_{1/2}^{\infty} g_s(x)( (x-s)/s + 1/(2s) ) \, dx = 1/(2s),
\]
recalling that here \( f_s(x) = (2\pi s)^{-1/2} \exp(-(x-s)^2/(2s)) \), \( \mathbb{E}(X_s) = s \) and \( g_s \) integrates to 1. As above, combining the bounds for (A.22) and dividing through by \( k \) proves the directed case. \( \square \)
We wish to find the times $t_\alpha$ defined so that

$$h(t_\alpha) = (\log n + \alpha \sqrt{v})/k$$

where $v = \text{Var}(Q_1(t_\alpha))$;

in this case, we have $v \approx \frac{1}{2}$, recalling (A.19a) in the previous section.

**Proposition A.9** (Entropic and Cutoff Times). For $k \ll \log n$, we have

$$t_0 \approx kn^{2/k}/(2\pi e), \quad (1.4a)$$

and, for each $\alpha \in \mathbb{R}$, we have $t_\alpha \approx t_0$, and furthermore

$$t_\alpha/t_0 - 1 \approx \alpha \sqrt{2/k}.$$  \hspace{1cm} (1.5a)

**Proof.** We consider the directed and undirected cases simultaneously. Directly manipulating (A.20), we see that if $h(t_0) = \log n/k$ then

$$t_0 = kn^{2/k}/(2\pi e) \cdot (1 + \mathcal{O}((t_0/k)^{-1/4})) \approx kn^{2/k}/(2\pi e),$$

with the final relation holding since $k \ll \log n$.

We now turn to finding $t_\alpha$. Fix $\alpha \in \mathbb{R}$. Note that $t \mapsto h_t$ is increasing and $\alpha \sqrt{v/k} = o(1)$, and so from the form of $h_t$ we see that, for all $\varepsilon > 0$, we have $(1-\varepsilon)t_0 \leq t_\alpha \leq (1+\varepsilon)t_0$ for $n$ sufficiently large (depending on $\alpha$); hence $t_\alpha \approx t_0$ for all $\alpha \in \mathbb{R}$.

By definition of $t_\alpha$, we have

$$h(t_\alpha) - h(t_0) = \alpha \sqrt{v/k}, \quad \text{and hence} \quad \frac{d}{da} h'(t_\alpha) = \sqrt{v/k}.$$  \hspace{1cm}

Hence we have

$$t_\alpha - t_0 = \int_0^\alpha \frac{da}{h'(t_\alpha)} = \sqrt{v/k} \int_0^\alpha 1/h'(t_0) \, da.$$  \hspace{1cm}

But we may write $h'(t) = (2t)^{-1/2}(1 + o(1))$ with $o(1)$ term uniform over $t \in [\frac{1}{2}t_0, 2t_0]$, which is an interval containing the cutoff window. Hence, recalling that $v \approx \frac{1}{2}$ in this regime, (1.5a) follows:

$$t_\alpha - t_0 = 2\alpha \sqrt{v/k} t_0 (1 + o(1)) = \alpha \sqrt{2/k} t_0 (1 + o(1)).$$

We next consider the regime $k \approx \lambda \log n$ with $\lambda \in (0, \infty)$. For $s \geq 0$, write $H(s) = \mathbb{E}(Q_1(s))$, i.e. the entropy of a rate-1 SRW or Poisson process in the undirected or directed case, respectively.

**Proposition A.10** (Entropic and Cutoff Times). There exists a decreasing, continuous bijection $f : (0, \infty) \to (0, \infty)$, whose value differs between the undirected and directed cases, so that, for all $\lambda > 0$, for $k \approx \lambda \log n$, we have

$$t_0 \approx f(\lambda) k \quad \text{where} \quad f(\lambda) = H^{-1}(1/\lambda), \quad (1.4b)$$

and, for each $\alpha \in \mathbb{R}$, we have $t_\alpha \approx t_0$, and furthermore

$$t_\alpha/t_0 - 1 \approx \alpha g(\lambda)/\sqrt{k} \quad \text{where} \quad g(\lambda) = \sqrt{\text{Var}(Q_1(f(\lambda) k))/(f(\lambda) H'(f(\lambda)))}. \quad (1.5b)$$

**Proof.** Since $\log n/k \approx 1/\lambda$, we must choose $t = t_0$ so that $h(t/k) \approx 1/\lambda$. From this, and the fact that each coordinate runs at rate $1/k$, we deduce that $t_0/k$ must also converge as $n \to \infty$, and so $t_0/\log n$ converges as $n \to \infty$, with limit depending only on $\lambda$, say $f(\lambda)$. This theory holds for both the directed and undirected cases, but the limit is not (necessarily) the same in each case.

Moreover, as noted in Remarks 1.5, the increasing and continuity properties of the entropy say that $f(\lambda) = H^{-1}(1/\lambda)$ and that $f$ is a decreasing bijection from $(0, \infty)$ to $(0, \infty)$.

We wish to find times $t_\alpha$ defined so that

$$H(t_\alpha/k) = h(t_\alpha) = (\log n + \alpha \sqrt{v})/k \quad \text{where} \quad v = \text{Var}(Q_1(t_\alpha)),$$

in this case, we have $v \approx v_\epsilon$, for some constant $v_\epsilon$, whose value differs between the undirected and directed cases, recalling (A.19b) in the previous section.
We now turn to finding \( t_\alpha \). Fix \( \alpha \in \mathbb{R} \). Note that \( h \) is increasing and \( \alpha \sqrt{v/k} = o(1) \), and so from the continuity of \( s \mapsto H(s) \) and the fact that the function \( H \) is independent of \( n \), we see that, for all \( \varepsilon > 0 \), we have \( (1 - \varepsilon)t_0 \leq t_\alpha \leq (1 + \varepsilon)t_0 \) for \( n \) sufficiently large (depending on \( \alpha \)); hence \( t_\alpha \approx t_0 \) for all \( \alpha \in \mathbb{R} \).

Similarly to in the previous derivative proof, noting that \( h'(t) = k^{-1}H'(t/k) \), we have
\[
t_\alpha - t_0 = \sqrt{v/k} \int_0^\alpha \frac{da}{f(a)} = \sqrt{v/k} \int_0^\alpha 1/H'(t_\alpha/k) \, da.
\]
Continuity now of \( H' \) along with the fact that \( t_\alpha \approx t_0 \approx f(\lambda)k \) then says that
\[
t_\alpha - t_0 \approx \alpha \sqrt{v/k} H'(t_\alpha/k) \approx \alpha \sqrt{v/k} H'(f(\lambda)) \approx \alpha t_0 \sqrt{\text{Var}(Q_1(f(\lambda))/k)} / (f(\lambda)H'(f(\lambda))).
\]
Noting that \( v = \text{Var}(Q_1(t_0)) \) and \( t_0 \approx f(\lambda)k \) and using continuity proves (1.5b).

Finally, it remains to show that the expression in (1.5b) is indeed \( o(1) \). This again follows straightforwardly since \( g \) is a function independent of \( n \). Indeed, the expressions for \( Q_1 \) and \( h \) are defined by running a single coordinate at rate \( 1/k \), so \( H(s) = h(sk) \) and \( Q_1(sk) \) are independent of \( n \) if \( s \) is independent of \( n \).

Finally we consider the regime \( k \gg \log n \). We have to handle the directed and undirected cases slightly differently here. The entropic time \( t_0 \) and cutoff times \( t_\alpha \) will be the same (up to smaller order terms), but the technical details of the proofs will differ ever so slightly.

**Proposition A.11** (Entropy). For \( t \ll k \), writing \( s = t/k \), we have
\[
h_t = s \log(1/s) + O(s).
\]
**(A.23)**

**Proof.** This is an immediate consequence of (A.14) given in the justification of the CLT in the regime \( k \gg \log n \) (§A.1).

**Proposition A.12** (Derivate of Entropy). For \( t \ll k \), writing \( s = t/k \), we have
\[
h'(t) = k^{-1} (\log(1/s) + O(1)).
\]

**Proof.** We proceed as in the previous derivative proof, i.e the proof of Proposition A.8.

Consider first the undirected case. Let \( X \) be a rate-1 continuous time simple random walk on \( \mathbb{Z} \). Then \( W_1(t) \sim X_s \) where \( s = t/k \). Using the backward equations as in the proof of Proposition A.8, we have
\[
k h'(t) = \sum_{x \in \mathbb{Z}} (p_s(x) - \frac{1}{2} (p_s(x + 1) + p_s(x - 1))) \log p_s(x).
\]
Recall that we have
\[
\mathbb{P}(X_s = 0) = 1 - s + O(s^2) \quad \text{and} \quad \mathbb{P}(X_s = x) = \frac{1}{2}s + O(s^2) \quad \text{for} \quad x \in \{-1, 1\},
\]
and hence \( \mathbb{P}(X_s = x) = O(s^2) \) for \( x \notin \{0, \pm 1\} \). Also, as previously, in the above sum we may ignore the \( x \) with \( x \notin \{0, \pm 1\} \) to give an error \( O(s \log(1/s)) \). (Note that it is not \( O(s^2 \log(1/s)) \), since the \( x \)-th term of the sum contains \( p_s(x + 1) \) and \( p_s(x - 1) \)!) Direct calculation then gives
\[
k h'(t) = \log(1/s) + \log 2 + O(s) = \log(1/s) + O(1).
\]
This proves the undirected case.

We now consider the directed case. Let \( X_s \sim \text{Po}(s) \), which has \( \mathbb{P}(X_s = x) = e^{-s}x^s/s! \). Then \( W_1(t) \sim X_s \). Direct differentiation shows that
\[
\frac{d}{dx} \mathbb{P}(X_s = x) = \mathbb{P}(X_s = x - 1) - \mathbb{P}(X_s = x) = e^{-s} s^{x-1} (x - s)/x! \quad \text{for} \quad x \in \mathbb{N},
\]
and \( \frac{d}{dx} \mathbb{P}(X_s = 0) = -\mathbb{P}(X_s = 0) = -e^{-s} \), as in the previous derivative proof. As there, we have
\[
k h'(t) = - \sum_{x \in \mathbb{Z}^+} \frac{d}{dx} \mathbb{P}(X_s = x) (\log \mathbb{P}(X_s = x) + 1).
\]

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As previously, we may ignore the terms with \( x \not\in \{0, \pm 1\} \), giving an error \( O(s \log(1/s)) \). Plugging in the derivative, we obtain
\[
k'h(t) = -e^{-s} \log(e^{-s}) - e^{-s}(1-s) \log(se^{-s}) + O(s \log(1/s))
\]
\[
= s(1-s + O(s^2)) - (1-s)(1-s + O(s^2)) \log s - s + O(s \log(1/s))
\]
\[
= -\log(1/s) + O(s \log(1/s)) = \log(1/s) + O(1).
\]
This proves the directed case. \( \square \)

We wish to find the times \( t_\alpha \) defined so that
\[
h(t_\alpha) = (\log n + \alpha \sqrt{\epsilon})/k \quad \text{where} \quad \epsilon = \text{Var}(Q_1(t_0));
\]
in this case, we have \( h \approx (\log n/k) \log(k/\log n) \), recalling \( A.19c \) in the previous section.

**Proposition A.13** (Entropic and Cutoff Times). For \( k \gg \log n \), we have
\[
t_0 = \frac{\log n}{(\log k) \log n}, \tag{1.4c}
\]
and, for each \( \alpha \in \mathbb{R} \), we have \( t_\alpha \approx t_0 \), and furthermore
\[
t_\alpha/t_0 - 1 \approx \alpha \sqrt{\log(k/\log n)/\log n} = o(1). \tag{1.5c}
\]
**Proof.** By \( A.23 \), we desire \( s = t/k \) with \( t = t_0 \) to satisfy
\[
\log(1/s) \approx \log(k/\log n).
\]
Taking logs of this, we obtain
\[
\log(1/s) \approx \log(k/\log n).
\]
Hence we obtain
\[
t_0 = sk \approx \log n/(\log k) \log n).
\]

We now turn to finding \( t_\alpha \). Fix \( \alpha \in \mathbb{R} \). From the form \( A.23 \) of \( h(t) \), observe that
\[
h(t_0(1+\epsilon)) = (1+\epsilon) h(t_0) + O(s_0) = (1+\epsilon) h(t_0) \cdot (1+o(1)),
\]
where \( s_0 = t_0/k \), and \( s_0 \ll 1 \) so \( h(t_0) \approx s_0 \log(1/s_0) \approx s_0 \). Note also that
\[
\sqrt{\epsilon k} \approx \sqrt{\log n \log(k/\log n)} \ll \log n,
\]
since \( k = n^{o(1)} \), and so \( h(t_\alpha) = h(t_0) \cdot (1 + o(1)) \). Hence, for all \( \epsilon > 0 \), we have \( t_0(1-\epsilon) \leq t_\alpha \leq t_0(1+\epsilon) \) for \( n \) sufficiently large (depending on \( \alpha \)); hence \( t_\alpha \approx t_0 \) for all \( \alpha \in \mathbb{R} \).

As in the previous derivative proofs, we have
\[
t_\alpha - t_0 = \sqrt{\epsilon} k \int_0^\alpha da \Rightarrow \sqrt{\epsilon} k \int_0^\alpha 1/h(t_\alpha) da.
\]
But we may write \( h'(t) = k^{-1} \log(1/s)(1+o(1)) \) with \( o(1) \) term uniform over \( t \in [t_\alpha/2, t_\alpha] \), which is an interval containing the cutoff window. Hence, recalling the expression for \( v \), \( 1.5c \) follows:
\[
\left| t_\alpha - t_0 \right| = |\alpha| \sqrt{\epsilon} k \frac{d}{da} \log(k/t_0)(1+o(1))
\]
\[
= |\alpha| \sqrt{\log n \log(k/\log n)} / \log(k/\log n) (1+o(1))
\]
\[
= |\alpha| \sqrt{\log n \log(k/\log n)} (1+o(1))
\]
\[
= |\alpha| \sqrt{\log(k/\log n)} \log n t_0 (1+o(1)).
\]
Note that \( k = n^{o(1)} \), and so \( \log(k/\log n) / \log n \ll 1 \). (Recall that \( k \gg \log n \).) So we do indeed have \( |t_\alpha - t_0| = o(t_0) \).\( \square \)

**Remark.** In the directed case, we can actually find an explicit closed-form solution for the entropy. Direct calculation shows that
\[
E(Q_1(t)) = s \log(1/s) + s + e^{-s} \sum_{\ell=2}^\infty s^\ell \log(\ell)!/\ell!.
\]
Hence we see that
\[
h_t = E(Q(t)) = t(\log(1/s) + 1 + e^{-s} \sum_{\ell=2}^\infty s^{\ell-1} \log(\ell)!/\ell!).
\]
In the regime \( k \gg \log n \) we have \( s = t_\alpha/k \ll 1 \), and so it is easy to see that \( h_s \approx s \log(1/s) \). \( \square \)
A.5 Proving Bounds on \( r \) and \( p \)

The following propositions provide asymptotic estimates for tails of the Poisson distribution and for continuous-time SRW on \( \mathbb{Z} \), as well as for the ratio between the ‘tail’ and ‘point’ probabilities. We note that in the regime \( r \in [\sqrt{s}, s^{2/3}] \) stronger assertions can be made via the local CLT (A.1).

Below, for \( a, b \in \mathbb{R} \), we write \( a \vee b = \max\{a, b\} \) and \( a \wedge b = \min\{a, b\} \).

**Proposition A.14** (Poisson Bounds). For \( s \in (0, \infty) \), let \( X_s \sim \text{Po}(s) \). Then, uniformly in \( s \in (0, \infty) \) and in \( r \geq \sqrt{s} \) and \( s + r \in \mathbb{Z} \), we have the following relations:

\[
- \log \mathbb{P}(X_s \geq s + r) \simeq r (r/s \wedge 1) \log((r/s) \vee 1); \quad (A.24a)
\]

\[
\mathbb{P}(X_s \geq s + r) / \mathbb{P}(X_s = s + r) \simeq (s/r) \vee 1. \quad (A.25a)
\]

Moreover, uniformly in \( s \in (0, \infty) \) and in \( r \in [\sqrt{s}, s] \) with \( s - r \in \mathbb{Z} \) we have the following relations:

\[
- \log \mathbb{P}(X_s \leq s - r) \simeq r (r/s \wedge 1) \log((r/s) \vee 1); \quad (A.24b)
\]

\[
\mathbb{P}(X_s \leq s - r) / \mathbb{P}(X_s = s - r) \simeq (s/r) \vee 1. \quad (A.25b)
\]

**Proposition A.15** (SRW Bounds). Let \( X = (X_s)_{s \geq 0} \) be a rate-1 SRW on \( \mathbb{Z} \) started at 0. Then, uniformly in \( s \in (0, \infty) \) and in \( r \geq \sqrt{s} \) and \( s + r \in \mathbb{Z} \), we have the following relations:

\[
- \log \mathbb{P}(X_s \geq r) \simeq r (r/s \wedge 1) \log((r/s) \vee 1); \quad (A.26)
\]

\[
\mathbb{P}(X_s \geq r) / \mathbb{P}(X_s = r) \simeq (s/r) \vee 1. \quad (A.27)
\]

**Proof of Proposition A.14** (Poisson). For \( s \leq 10 \), all that is needed is the observation that

\[
\mathbb{P}(X_s \geq r) \simeq \mathbb{P}(X_s = r) \simeq s^r / r! \approx (es/r)^r / \sqrt{r}.
\]

We now consider the case \( s \geq 1 \). First we state that, for all \( r \geq 0 \), we have

\[
\max\{\mathbb{P}(X_s \geq s + r), \mathbb{P}(X_s \leq s - r)\} \leq \exp\left(-\frac{1}{2}r^2 / (s + r/3)\right); \quad (A.28)
\]

this follows from Bernstein’s inequality, by taking an appropriate limit.

A direct calculation involving Stirling’s approximation shows, uniformly in \( s \) and in \( r \) with \( r \geq \frac{1}{2} s \) and \( s + r \in \mathbb{Z} \), respectively \( \frac{1}{2}s \leq r \leq s \), the following relations:

\[
\mathbb{P}(X_s \geq s + r) \simeq \mathbb{P}(X_s = s + r) \simeq e^r \left(\frac{s}{s + r}\right)^{s + r} \approx \frac{e^r(s/(s + r))^{s + r}}{\sqrt{2\pi(s + r)}},
\]

\[
\mathbb{P}(X_s \leq s - r) \simeq \mathbb{P}(X_s = s - r) \simeq e^r \left(\frac{s}{s - r}\right)^{s - r} \approx \frac{e^r(s/(s - r))^{s - r}}{\sqrt{2\pi(s - r)}},
\]

from these, one can verify (A.25a, A.25b) for such \( r \).

We can obtain lower bounds on \( \mathbb{P}(X_s \geq s + r) \) and \( \mathbb{P}(X_s \leq s - r) \) for \( r \leq \frac{1}{2}s \), from which, together with (A.28), we can verify (A.25a, A.25b) for such \( r \):

\[
\mathbb{P}(X_s = s + r) \sqrt{2\pi(s + r)} \approx e^r \left(\frac{s}{s + r}\right)^{s + r} \approx \exp\left(-\frac{r^2}{2(s + r)} - \mathcal{O}\left(\frac{r^3}{(s + r)^2}\right)\right),
\]

\[
\mathbb{P}(X_s = s + r) \sqrt{2\pi(s + r)} \approx e^r \left(\frac{s}{s - r}\right)^{s - r} \approx \exp\left(-\frac{r^2}{2(s - r)} - \mathcal{O}\left(\frac{r^3}{(s - r)^2}\right)\right);
\]

these are found using Stirling’s approximation, and both hold uniformly for \( r \leq \frac{1}{2}s \).

We now prove (A.24a); the proof of (A.24b) is similar and is omitted. We consider \( s \geq 10 \), having already considered \( s \leq 10 \) initially. Observe that \( r \mapsto \mathbb{P}(X_s = s \pm r) \) is decreasing on \( r \geq 0 \) with \( s \pm r \in \mathbb{Z} \). Using the formula for \( \mathbb{P}(\text{Po}(\lambda) = k) \), we have

\[
\frac{\mathbb{P}(X_s = s + r)}{\mathbb{P}(X_s = s + r + 1)} = \frac{s + r + 1}{s},
\]

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If \( r \geq \frac{1}{4}s \), then this ratio is at least 11/9, when \( s \geq 10 \), from which one can readily see that (A.24a) holds. Now suppose that \( r \in [\sqrt{5}, \frac{1}{2} \sqrt{s}] \). To conclude the proof, we show that there exist universal constants \( c_1, c_2 \in (0, 1) \) so that, for such \( r \), we have

\[
c_1 \mathbb{P}(X_s = s + r) \leq \mathbb{P}(X_s = s + r + \lfloor s/(2r) \rfloor) \leq c_2 \mathbb{P}(X_s = s + r).
\]  

(A.29)

This, together with the decreasing statement above, can easily be seen to imply (A.24a). We now prove (A.29). If \( \sqrt{s} \leq r \leq \frac{1}{4}s \), then

\[
\mathbb{P}(X_s = s + r) = \prod_{i=1}^{j} \frac{s + r + i}{s} = \prod_{i=1}^{j} \left(1 + \frac{r + i}{s}\right)
\]

\[
\leq \exp\left(\sum_{i=1}^{j} \frac{r + i}{s}\right) = \exp\left(\frac{j}{2}(j + 2r + 1)/s\right).
\]

If in addition \( j \leq \frac{1}{2}s/r \), then the last estimate is tight up to a constant factor. Indeed, in this case we have \( \exp\left(\frac{j}{2}(j + 2r + 1)/s\right) \leq e^3 \). Conversely, using the fact that \( 1 + \theta \geq \exp(\theta - \theta^2) \) for \( \theta \in [0, \frac{1}{2}] \), we find some universal constant \( c_0 > 1 \) so that \( \exp\left(\frac{j}{2}(j + 2r + 1)/s\right) \geq c_0 \).

\[\square\]

**Proof of Proposition A.15 (SRW).** Fix an \( s \in (0, \infty) \); wlog assume \( r \geq 0 \). Recall that \( X \) has the same law as \( Y_N = \sum_{i=1}^{N} \xi_i \), where \( \xi_i \in \mathbb{N} \) is an iid sequence of random variables with \( \mathbb{P}(\xi_i = 1) = \frac{1}{3} = \mathbb{P}(\xi_i = -1) \) and \( N \sim \text{Po}(s) \), independent of \( \xi_i \in \mathbb{N} \). Then, setting \( Y_k = \sum_{i=1}^{k} \xi_i \) for \( k \in \mathbb{Z}_+ \), we have that \( (Y_k)_{k \in \mathbb{Z}_+} \) is a discrete-time SRW on \( \mathbb{Z} \) started at the origin.

We first prove (A.26). Observe that \( \mathbb{E}(e^{\lambda Y_k}) = \frac{1}{3}e^{\lambda} + \frac{1}{3}e^{-\lambda} \leq e^{\lambda^2/2} \), and so \( \mathbb{E}(e^{\lambda Y_k}) \leq e^{\lambda^2 k/2} \), and hence \( \mathbb{P}(Y_k \geq r) \leq \exp(-r^2/(2k)) \), by taking \( \lambda = r/k \). Further, an elementary calculation involving Stirling’s approximation shows, uniformly over \( r \) with \( \sqrt{k} \log k < r \leq k \), that

\[
-\log \mathbb{P}(Y_k \geq r) \leq -\log \mathbb{P}(Y_k \in \{r, r+1\}) \asymp r^2/k;
\]

for \( \sqrt{k} \leq r \leq \sqrt{k} \log k \) one can use the local CLT (A.1) to verify that

\[
-\log \mathbb{P}(Y_k \geq r) \asymp r^2/k.
\]

For \( r \leq \sqrt{2}s \), we average over \( N \) and use the above bounds on \( Y_k \). In particular, we have

\[
\mathbb{E}(e^{\lambda Y_k}) \leq \sum_{k=0}^{\infty} \mathbb{P}(N = k) e^{\lambda^2 k/2} = \mathbb{E}(e^{\lambda^2 N/2}) = \exp(s(e^{\lambda^2/2} - 1)) \leq \exp(s(\lambda^2/2 + (\lambda^2/2)^2)),
\]

with the final inequality holding when \( \lambda^2 \leq 2 \), applying the inequality \( e^\theta - 1 \leq \theta + \theta^2 \) valid for \( \theta \in [-1,1] \). We now set \( \lambda = r/s \) and use Chernoff to deduce that

\[
\mathbb{P}(X_s \geq r) \leq \exp\left(-\frac{1}{2}(r^2/s)(1 - \frac{1}{2}(r/s))\right) \leq \exp\left(-\frac{1}{2}(r^2/s)\right).
\]

For \( r \geq \sqrt{2}s \), we use the inequalities

\[
\mathbb{P}(X_s \geq r) \leq \mathbb{P}(\text{Po}(s) \geq r) \quad \text{and} \quad \mathbb{P}(X_s \geq r) \geq \mathbb{P}(N = 2r) \mathbb{P}(Y_{2r} \geq r).
\]

This case is completed by applying (A.24, A.25), i.e Proposition A.14.

We now prove (A.27). For \( r \geq \frac{1}{2}s \), this follows from the fact that \( r \mapsto \mathbb{P}(X_s = r) \) is decreasing in \( r \) and that

\[
\sup_{s, r \text{ s.t. } r \geq s/2} \mathbb{P}(X_s = r + 2)/\mathbb{P}(X_s = r) < 1,
\]

which can be verified via a direct calculation involving averaging over \( N \) and applying Stirling’s approximation; we omit the details. For \( r \leq \frac{1}{2}s \), it suffices to prove the following corresponding result for \( (Y_k)_{k \in \mathbb{Z}_+} \): uniformly in \( k > 0 \) and \( r \in [\sqrt{k}, \frac{1}{2}k] \) with \( r \in \mathbb{Z} \), we have

\[
\frac{\mathbb{P}(Y_{2k} \geq 2r)}{\mathbb{P}(Y_{2k} = 2r)} \asymp k/r \asymp \frac{\mathbb{P}(Y_{2r+1} \geq 2r+1)}{\mathbb{P}(Y_{2r+1} = 2r+1)}.
\]  

(A.30)
from this, the original claim follows by averaging over $N$. Using Stirling’s approximation, it is not hard to verify for $r \in [\sqrt{k}, \frac{1}{3}k]$ that there exist universal constants $c_1, c_2 \in (0, 1)$ such that the following hold:

$$c_1 \mathbb{P}(Y_{2k} = 2r) \leq \mathbb{P}(Y_{2k} = 2(r + [k/r])) \leq c_2 \mathbb{P}(Y_{2k} = 2r);$$
$$c_1 \mathbb{P}(Y_{2k+1} = 2r + 1) \leq \mathbb{P}(Y_{2k+1} = 2(r + [k/r]) + 1) \leq c_2 \mathbb{P}(Y_{2k+1} = 2r + 1).$$

This, together with the fact that both $r \mapsto \mathbb{P}(Y_{2k} = 2r)$ and $r \mapsto \mathbb{P}(Y_{2k+1} = 2r + 1)$ are decreasing on $[0,k]$, is easily seen to imply (A.30).

Recall the definitions of $r$, $p$, $r_*$ and $p_*$ from the start of §4.1. We prove Proposition 4.2.

**Proof of Proposition 4.2.** Here we take $s = t_0/k \approx t_0/k$. Writing $\kappa = k/\log n$, we have

$$t_0/k \lesssim n^{2/k} \quad \text{when} \quad k \lesssim \log n, \quad \text{(A.31a)}$$
$$t_0/k \approx 1/(\kappa \log \kappa) \ll 1 \quad \text{when} \quad k \gg \log n. \quad \text{(A.31b)}$$

Consider the SRW, which corresponds to the undirected case. Equations (A.24–A.27) are all “$f \asymp g$”-type statements; let $c > 0$ be a constant such that $c$ is a universal lower bound and $C = 1/c$ is a universal upper bound for these relations.

For $r$, it is enough to find an $\hat{r}$ so that

$$- \log \mathbb{P}(X_s \geq \hat{r}) \geq 2 \log k.$$  

For $p$, since we only consider $j$ with $|j| \leq r$, and $r$ is defined as a minimum, we have $\mathbb{P}(X_s \geq |j|) \geq k^{-3/2}$ for all such $j$. We split into two regimes, namely $s \leq 2C \log k$ and $s > 2C \log k$; the reason for this will become clear later.

First suppose that $s \leq 2C \log k$. Set $\tilde{r} = \sqrt{2Cs \log k}$. Then $\tilde{r} \leq s$, and so, by (A.26), we have

$$- \log \mathbb{P}(X_s \geq \tilde{r}) \geq c \tilde{r}((\tilde{r}/s) \land 1) \log((\tilde{r}/s) \lor \epsilon) = c \tilde{r}^2/s \geq 2 \log k.$$  

For $p_*$, since $\tilde{r} \leq s$, by (A.27), we have

$$\mathbb{P}(X_s = j) \gtrsim (s/r) \mathbb{P}(X_s \geq j) \gtrsim (\log k)^{1/2} n^{-1/k} \cdot k^{-3/2} \gg n^{-1/k} k^{-2}.$$  

Suppose now that $s < 2C \log k$. Set $\tilde{r} = 2C \log k$. Then $\tilde{r} \geq s$, and so, by (A.26), we have

$$- \log \mathbb{P}(X_s \geq \tilde{r}) \geq c \tilde{r}((\tilde{r}/s) \land 1) \log((\tilde{r}/s) \lor \epsilon) \geq c \tilde{r} = 2 \log k.$$  

For $p_*$, since $\tilde{r} \geq s$, by (A.27), we have

$$\mathbb{P}(X_s = j) \gtrsim \mathbb{P}(X_s \geq j) \gtrsim k^{-3/2} \gg k^{-2} \geq n^{-1/k} k^{-2}.$$  

Observe that, in either regime, we have $\tilde{r} \leq r_*$, with $r_*$ defined in (4.3). This completes the proof of (4.4) in the undirected case.

The proof of the directed case, ie using Poisson instead of SRW, is in essence the same, due to the similarity of Propositions A.14 and A.15, albeit slightly messier to write down, since one must take care that $s + r \geq 0$.  

