QUANTUM MULTIPLICATION OPERATORS FOR LAGRANGIAN AND ORTHOGONAL GRASSMANNIANS

DAEWOONG CHEONG

Abstract. In this article, we make a close analysis on quantum multiplication operators on the quantum cohomology rings of Lagrangian and orthogonal Grassmannians, and give an explicit description on all simultaneous eigenvectors and the corresponding eigenvalues for these operators. As a result, we show that Conjecture O of Galkin, Golyshev and Iritani holds for these manifolds.

1. INTRODUCTION

Let \( M \) be a Fano manifold. The quantum cohomology ring \( qH^*(M, \mathbb{C}) \) is a certain deformation of the classical cohomology ring \( H^*(M, \mathbb{C}) \). For \( \sigma \in qH^*(M, \mathbb{C}) \), define the quantum multiplication operator \([\sigma]\) on \( qH^*(M)\) by \([\sigma](\alpha) = \sigma \cdot \alpha\). Denote the set of eigenvalues of \([\sigma]\) by Spec([\sigma]). Suppose \( \text{Spec}(c_1(M)) = \{a_1, ..., a_m\} \), and \( T_0 = \max\{|a_1|, |a_1|, ..., |a_m|\} \), where \( c_1(M) = c_1(TM) \) is the first Chern class of the tangent bundle \( TM \) of \( M \). Then we say that \( M \) satisfies Conjecture O if

1. \( T_0 \) is an eigenvalue of \([c_1(M)]\).
2. If \( u \) is an eigenvalue of \([c_1(M)]\) such that \(|u| = T_0\), then \( u = T_0 \xi \) for some \( r \)-th root of unity, where \( r \) is the Fano index of \( M \).
3. The multiplicity of the eigenvalue \( T_0 \) is one.

Originally, while Galkin, Golyshev and Iritani studied the exponential asymptotic of solutions to the quantum differential equations, they proposed two more conjectures called Gamma Conjectures I, II which, informally, relate the quantum cohomology of \( M \) and the so-called Gamma class in terms of differential equations. We refer to \[10\] for details on these materials. The importance of Conjecture O lies in the fact that it ‘underlies’ the Gamma Conjectures. Indeed, above all, the Gamma Conjecture I was stated under the Conjecture O. And under further assumption of the semisimplicity of the quantum cohomology of \( M \), the Gamma Conjecture II, which is a refinement of a part of Dubrovin’s conjecture [12], relates eigenvalues of the operator \([c_1(M)]\) with members of a certain exceptional collection of the derived category \( D_{\text{coh}}(M) \) bijectively. Then, under the semisimplicity of the quantum cohomology ring, the Gamma Conjecture I can be viewed as a part of the Gamma conjecture II; it relates the eigenvalue \( T_0 \) to the member \( O_M \) of the aforementioned exceptional collection.

Let us mention for which manifolds Conjecture O has been proved. The Grassmannian is the first manifold for which Conjecture O has been proved. Indeed, in [10], Galkin, Golyshev and Iritani recently proved Conjecture O, and then the Gamma Conjectures I, II for the Grassmannian. In fact, we noticed that for the Grassmannian, Galkin and Golyshev gave a very short proof of Conjecture O in an earlier paper of theirs [11], and Rietsch gave an explicit description on the eigenvalues and eigenvectors of multiplication operators on the quantum cohomology ring of the Grassmannian ([19]) which contains all the ingredient necessary to prove Conjecture O for the Grassmannian. For toric
Fano manifolds, assuming Conjecture $O$, Galkin, Golyshev and Galkin proved Gamma conjectures I and II. In this article, following Rietch, we obtain simultaneous eigenvectors and corresponding eigenvalues for multiplication operators on the quantum cohomology rings (specialized at $q = 1$) of Lagrangian and orthogonal Grassmannians, which are homogeneous varieties and in particular examples of Fano manifolds. Then we use these to show that Conjecture $O$ holds for Lagrangian and orthogonal Grassmannians. Very recently, Li and the author worked out Conjecture $O$ for general homogeneous varieties by a different approach [3].

Lastly, let us explain what makes possible for these manifolds such an explicit description of the multiplication operators. In this article, we heavily use one of Peterson’s results, which states that the quantum cohomology ring of a homogeneous variety is isomorphic with the coordinate ring of so-called Peterson variety corresponding to the homogeneous variety ([16], [17], [20], [2]). Unlike general homogeneous varieties, for these manifolds together with the Grassmannian, there is a much simpler isomorphic variety that can replace the Peterson variety. Thereby we identify points of the Peterson variety (specialized at $q = 1$). On the other hand, as the Grassmannian does, they have symmetric polynomials representing the Schubert classes and the quantization of these polynomials which serve as regular functions on the Peterson variety, too. These two facts provide us with orthogonal formulas evaluated at points of the Peterson variety (cut out by $q = 1$) which play a key role in explicitly finding simultaneous eigenvectors and the associated eigenvalues. Much of material needed in this article was studied in the author’s earlier paper [2]. Acknowledgements. This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2016R1A6A3A11930321), and also by the Ministry of Science, ICT and Future Planning(NRF-2015R1A2A2A01004545).

2. SYMMETRIC FUNCTIONS

In this section, we review $Q$- and $P$-polynomials of Pragacz and Ratajski and Schur polynomials. References are [18] and [14] for the former polynomials, and [7] and [15] for the latter.

2.1. Notations and definitions. We begin with some notations concerning the labeling of symmetric polynomials. A partition $\lambda$ is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of nonnegative integers. A Young diagram is a collection of boxes, arranged in left-justified rows, with a weakly decreasing number of boxes in each row. To a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, we associate a Young diagram whose $i$-th row has $\lambda_i$ boxes. The nonzero $\lambda_i$ in $\lambda = (\lambda_1, \ldots, \lambda_m)$ are called the parts of $\lambda$. The number of the parts of $\lambda$ is called the length of $\lambda$, denoted by $l(\lambda)$; the sum of the parts of $\lambda$ is called the weight of $\lambda$, denoted by $|\lambda|$. For $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $l(\lambda) = l$, we usually write $(\lambda_1, ..., \lambda_l)$ for $(\lambda_1, \ldots, \lambda_l, 0, \ldots, 0)$ if any confusion does not arise. For positive integers $m$ and $n$, denote by $R(m, n)$ the set of all partitions whose Young diagram fits inside an $m \times n$ diagram, which is the Young diagram of the partition $(n^m)$. A partition $\lambda = (\lambda_1, \ldots, \lambda_l, \ldots, \lambda_m) \in R(m, n)$ is called strict if $\lambda_1 > \cdots > \lambda_l$ and $\lambda_{l+1} = \cdots = \lambda_m = 0$, where $l = l(\lambda)$. Denote by $D(m, n)$ the set of all strict partitions in $R(m, n)$. If $m = n$, then we write $R(n)$ and $D(n)$ for $R(n, n)$ and $D(n, n)$, respectively. If $\lambda \in D(n)$, denote by $\lambda^t$ the partition whose parts complements the parts of $\lambda$ in the set $\{1, \ldots, n\}$.

For $\lambda \in R(m, n)$, the conjugate of $\lambda$ is the partition $\lambda^t \in R(n, m)$ whose Young diagram is the transpose of that of $\lambda$.

2.2. Symmetric functions. Let $X := (x_1, \ldots, x_n)$ be the $n$-tuple of variables. For $i = 1, \ldots, n$, let $H_i(X)$ (resp. $E_i(X)$) be the $i$-th complete (resp. elementary) symmetric function. Then for any partition $\lambda$, the Schur polynomial $S_{\lambda}(X)$ is defined by

$$S_{\lambda}(X) := \text{Det}[H_{\lambda_i+j-i}(X)]_{1 \leq i, j \leq n} = \text{Det}[E_{\lambda'_i+j-i}(X)]_{1 \leq i, j \leq n},$$

where $\lambda'_i$ is the conjugate of $\lambda_i$.

References are [18] and [14] for the former polynomials, and [7] and [15] for the latter.
where \(H_0(X) = E_0(X) = 1\) and \(H_k(X) = E_k(X) = 0\) for \(k < 0\).

We define \(\tilde{Q}\) - and \(\tilde{P}\)-polynomials both of which are indexed by elements of \(\mathcal{R}(n)\). For \(i = 1, \ldots, n\), set \(\tilde{Q}_i(X) := E_i(X)\), the \(i\)-th elementary symmetric function.

Given two nonnegative integers \(i\) and \(j\) with \(i \geq j\), define

\[
\tilde{Q}_{i,j}(X) = \tilde{Q}_i(X)\tilde{Q}_j(X) + 2\sum_{k=1}^j (-1)^k \tilde{Q}_{i+k}(X)\tilde{Q}_{j-k}(X).
\]

Finally, for any partition \(\lambda\) of length \(l = l(\lambda)\), not necessarily strict, and for \(r = 2[(l + 1)/2]\), let \(B_\lambda = \{B_{i,j}\}_{1 \leq i,j \leq r}\) be the skew symmetric matrix defined by \(B_{i,j} = \tilde{Q}_{\lambda,i\lambda,j}(X)\) for \(i < j\).

We set

\[
\tilde{Q}_\lambda(X) = \text{Pfaffian}(B_\lambda).
\]

Given \(\lambda\), not necessarily strict, \(\tilde{P}_\lambda\) is defined by

\[
\tilde{P}_\lambda(X) := 2^{-l(\lambda)}\tilde{Q}_\lambda(X).
\]

**Proposition 2.1.** ([13]) The \(\tilde{Q}\)-polynomials satisfy the following properties.

1. For \(i = 1, \ldots, n\), \(\tilde{Q}_i(X) = E_i(x_1^2, \ldots, x_n^2)\).
2. For any \(\lambda \in \mathcal{D}(n)\),

\[
\tilde{Q}_\lambda(X)\tilde{Q}_n(X) = \tilde{Q}_{(n, \lambda_1, \ldots, \lambda_l)}(X).
\]

Let \(S_n = \langle s_1, \ldots, s_{n-1} \rangle\) be the symmetric group generated by the simple transpositions \(s_1, \ldots, s_{n-1}\).

The Weyl group \(W_n\) for Lie type \(C_n\) is an extension of the symmetric group \(S_n\) by \(s_0\) such that the following relations hold

\[
s_0^2 = 1, \quad s_0 \cdot s_1 \cdot s_0 = s_1 \cdot s_0 \cdot s_1, \quad s_0 \cdot s_i = s_i \cdot s_0 \quad \text{for} \quad i \geq 2.
\]

Recall that an element \(w\) of \(S_n\) can be represented by a sequence \(w = (w_1, \ldots, w_n)\) of numbers \(1, \ldots, n\), e.g., \(s_i = (1, \ldots, i, i+1, \ldots, n)\) for \(i = 1, \ldots, n-1\). In contrast, an element of \(W_n\) can be represented by a permutation with bars \(w = (w_1, \ldots, w_n)\), e.g., \(s_0 = (1 \bar{2}, \ldots, n)\).

With this notation, the multiplication in \(W_n\) is given as follows: For \((w_1, \ldots, w_n) \in W_n\),

\[
(w_1, \ldots, w_n) \cdot s_0 = (\bar{w_1}, w_2, \ldots, w_n);
\]

\[
(w_1, \ldots, w_n) \cdot s_i = (w_1, \ldots, w_{i-1}, w_{i+1}, w_i, w_{i+2}, \ldots, w_n) \quad \text{for} \quad i = 1, \ldots, n-1.
\]

For \(w = (w_1, \ldots, w_n) \in W_n\) and \(X = (x_1, \ldots, x_n)\), let \(X^w\) be the \(n\)-tuple \((y_1, \ldots, y_n)\) of ‘signed variables’ \(\pm x_1, \ldots, \pm x_n\), where \(y_k = x_{w_k}\) (resp. \(-x_{w_k}\)) if \(w_k\) is unbarred (resp. barred). This induces an action of \(W_n\) on the ring of polynomials in \(x_1, \ldots, x_n\), which is defined as follows: For \(w \in W_n\) and a polynomial \(P(X)\) in \(x_1, \ldots, x_n\),

\[
w \cdot P(X) := P(X^w).
\]

The following will be used to derive orthogonal formulas for the Lagrangian and orthogonal Grassmannians later.

**Proposition 2.2.** ([13]) We have the following identity of polynomials

\[
\sum_{\lambda \in \mathcal{D}(n)} \tilde{P}_\lambda(X^w)\tilde{P}_\lambda(X) = \begin{cases} S_{\rho_n}(X) & \text{if} \ w \in S_n, \\ 0 & \text{if} \ w \in W_n \setminus S_n. \end{cases}
\]

Let \(\Lambda_n\) be the algebra over \(\mathbb{Z}\) of symmetric functions in \(x_1, \ldots, x_n\). Let \(\Lambda'_n\), the algebra over \(\mathbb{Z}\) of symmetric polynomials generated by \(\tilde{P}_\lambda\) with \(\lambda \in \mathcal{R}(n)\). Note that \(\Lambda_n\) is spanned by the polynomials \(\tilde{Q}_\lambda\) with \(\lambda\) with \(\lambda \in \mathcal{R}(n)\), and so \(\Lambda'_n\) is isomorphic to \(\Lambda_n\) as \(\mathbb{Z}\)-modules.
3. QUANTUM COHOMOLOGY RINGS

3.1. Lagrangian and orthogonal Grassmannians. Let \( E = \mathbb{C}^N \) be a complex vector space equipped with a nondegenerate (skew) symmetric bilinear form \( Q \). A subspace \( W \subset E \) is called \textit{isotropic} if \( Q(v, w) = 0 \) for all \( v, w \in W \). A maximal isotropic subspace of \( E \) is of (complex) dimension \( \lfloor N/2 \rfloor \). In particular, when \( N = 2n \) and \( Q \) is a skew symmetric form, such a maximal subspace is called \textit{Lagrangian}. Let \( LG(n) \) be the parameter space of Lagrangian subspaces in \( E \). Then \( LG(n) \) is a homogeneous variety \( Sp_{2n}(\mathbb{C})/P_n \) of complex dimension \( n(n + 1)/2 \), where \( P_n \) is the maximal parabolic subgroup of the symplectic group \( Sp_{2n}(\mathbb{C}) \) associated with the ‘right end root’ in the Dynkin Diagram of Lie type \( C_n \), e.g., on Page 58 of [1].

For the case when \( N = 2n + 1 \) and \( Q \) is a symmetric bilinear form, let \( OG(n) \) be the parameter space of maximal isotropic subspaces in \( E \). Then \( OG(n) \) is a homogeneous variety \( SO_{2n+1}(\mathbb{C})/P_n \) of dimension \( n(n + 1)/2 \), where \( P_n \) is the maximal parabolic subgroup of \( SO_{2n+1}(\mathbb{C}) \) associated with a ‘right end root’ of the Dynkin diagram of type \( B_n \) (on Page 58 of [11]). Traditionally, the manifold \( OG(n) \) is called an \textit{odd orthogonal Grassmannian}. There is an ‘even counterpart’, \textit{even orthogonal Grassmannian}, written as \( SO_{2n+2}(\mathbb{C})/P_{n+1} \), where \( P_{n+1} \) is the maximal parabolic subgroup of \( SO_{2n+2}(\mathbb{C}) \) associated with the ‘right end root’ of the Dynkin diagram of Lie type \( D_{n+1} \) (on Page 58 of [11]). It is well-known that they are isomorphic (projectively equivalent) to each other. Therefore, in this paper, we treat only \( OG(n) \), and we will go without the adjective ‘odd’.

3.2. Quantum cohomology of \( LG(n) \). To describe the quantum cohomology of \( LG(n) \), we begin with the definition of Schubert varieties of \( LG(n) \). Given a complex vector space \( E \) of dimension \( 2n \) with a nondegenerate skew-symmetric form, fix a complete isotropic flag \( F \) of subspaces \( F_i \) of \( E \):

\[
F_i : 0 = F_0 \subset F_1 \subset \cdots \subset F_n \subset E,
\]

where \( \dim(F_i) = i \) for each \( i \), and \( F_n \) is Lagrangian. To \( \lambda \in \mathcal{D}(n) \), we associate the Schubert variety \( X_\lambda(F) \) defined as the locus of \( \Sigma \in LG(n) \) such that

\[
\dim(\Sigma \cap F_{n+1-\lambda}) \geq i \quad \text{for} \quad i = 1, \ldots, l(\lambda).
\]

Then \( X_\lambda(F) \) is a subvariety of \( LG(n) \) of complex codimension \( |\lambda| \). The Schubert class associated with \( \lambda \) is defined to be the cohomology class, denoted by \( \sigma_\lambda \), Poincaré dual to the homology class \( [X_\lambda(F)] \), so \( \sigma_\lambda \in H^{|\lambda|}(LG(n), \mathbb{Z}) \). It is a classical result that \( \{\sigma_\lambda \mid \lambda \in \mathcal{D}(n)\} \) forms an additive basis for \( H^*(LG(n), \mathbb{Z}) \). It is conventional to write \( \sigma_i \) for \( \sigma_{(i)} \).

\[
0 \to S \to \mathcal{E} \to Q \to 0
\]

denotes the short exact sequence of tautological vector bundles on \( LG(n) \), then \( \sigma_i \) equals the \( i \)-th Chern class \( c_i(Q) \) of the tautological quotient bundle \( Q \) on \( LG(n) \), where \( \mathcal{E} \) denotes the trivial bundle \( \mathcal{E} = LG(n) \times \mathbb{C}^{2n} \). It is known that there is a surjective ring homomorphism from \( \Lambda_n \to H^*(LG(n), \mathbb{Z}) \) sending \( Q_\lambda(X) \) to \( \sigma_\lambda \) which has the kernel generated by \( \tilde{Q}_i \) with \( i = 1, \ldots, n \).

A rational map of degree \( d \) to \( LG(n) \) is a morphism \( f : \mathbb{P}^1 \to LG(n) \) such that

\[
\int_{LG(n)} f_*[\mathbb{P}^1] \cdot \sigma_1 = d.
\]

Given an integer \( d \geq 0 \) and partitions \( \lambda, \mu, \nu \in \mathcal{D}(n) \), the Gromov-Witten invariant \( < \sigma_\lambda, \sigma_\mu, \sigma_\nu >_d \) is defined as the number of rational maps \( f : \mathbb{P}^1 \to LG(n) \) of degree \( d \) such that \( f(0) \in X_\lambda(F) \), \( f(1) \in X_\mu(G) \), and \( f(\infty) \in X_\nu(H) \), for given isotropic flags \( F, G, H \) in general position. We remark that \( < \sigma_\lambda, \sigma_\mu, \sigma_\nu >_d = 0 \) unless \( |\lambda| + |\mu| + |\nu| = \dim(LG(n)) + (n + 1)d \). The quantum cohomology ring \( qH^*(LG(n), \mathbb{Z}) \) is isomorphic to \( H^*(LG(n), \mathbb{Z}) \otimes \mathbb{Z}[q] \) as \( \mathbb{Z}[q] \)-modules, where \( q \)
is a formal variable of degree \((n + 1)\) and called the \textit{quantum variable}. The multiplication in \(qH^*(LG(n), \Z)\) is given by the relation
\begin{equation}
\sigma_\lambda \cdot \sigma_\mu = \sum <\sigma_\lambda, \sigma_\mu, \sigma_\nu >_d \sigma_\nu q^d,
\end{equation}
where the sum is taken over \(d \geq 0\) and partitions \(\nu\) with \(|\nu| = |\lambda| + |\mu| - (n + 1)d\).

Set \(X^+ := (x_1, ..., x_{n+1})\), and let \(\tilde{\Lambda}_{n+1}\) be the subring of \(\Lambda_{n+1}\) generated by the polynomials \(\tilde{Q}_i(X^+)\) for \(i \leq n\) together with the polynomial \(2\tilde{Q}_{n+1}(X^+)\).

Now we are ready to give a presentation of the quantum cohomology ring of \(LG(n)\) and the quantum Giambelli formula due to Kresch and Tamvakis.

**Theorem 3.1.** (\cite{12}). There is a surjective ring homomorphism from \(\tilde{\Lambda}_{n+1}\) to \(qH^*(LG(n), \Z)\) sending \(\tilde{Q}_\lambda(X^+)\) to \(\sigma_\lambda\) for all \(\lambda \in \D(n)\) and \(2\tilde{Q}_{n+1}(X^+)\) to \(q\), with the kernel generated by \(\tilde{Q}_{i,i}\) for \(1 \leq i \leq n\). The ring \(qH^*(LG(n), \Z)\) is presented as a quotient of the polynomial ring \(\Z[\sigma_1, ..., \sigma_n, q]\) by the relations
\begin{equation}
\sigma_i^2 + 2 \sum_{k=1}^{n-i} (-1)^k \sigma_{i+k}\sigma_{i-k} = (-1)^{n-i}\sigma_{2i-n-1}q
\end{equation}
for \(1 \leq i \leq n\). The Schubert class in this presentation is given by quantum Giambelli formula
\begin{equation}
\sigma_{i,j} = \sigma_i\sigma_j + 2 \sum_{k=1}^{n-i} (-1)^k \sigma_{i+k}\sigma_{j-k} + (-1)^{n+1-i}\sigma_{i+j-n-1}q
\end{equation}
for \(i > j > 0\), and
\begin{equation}
\sigma_\lambda = \text{Pfaffian}(\sigma_{\lambda, \lambda})_{1 \leq i < j \leq r},
\end{equation}
where quantum multiplication is employed throughout.

See \cite{12} for more details on the quantum cohomology ring of \(LG(n)\).

### 3.3. Quantum cohomology of Orthogonal Grassmannian.

The quantum cohomology theory of \(OG(n)\) is parallel with that of \(LG(n)\). Let \(E\) be a complex vector space of dimension \(2n + 1\) equipped with a nondegenerate symmetric form. Given \(\lambda \in \D(n)\), the Schubert variety \(X_\lambda(F)\) is defined by the same equation (3.1) as before, relative to an isotropic flag \(F\) in \(E\). The Schubert class \(\tau_\lambda\) is defined as a cohomology class Poincaré dual to \([X_\lambda(F)]\). Then \(\tau_\lambda \in H^2(\lambda)(OG(n), \Z)\), and the cohomology classes \(\tau_\lambda, \lambda \in \D(n)\), form a \(\Z\)-basis for \(H^*(OG(n), \Z)\). The cohomology ring \(H^*(OG(n), \Z)\) can be presented in terms of \(\tilde{P}\)-polynomials. More precisely, there is a surjective ring homomorphism from \(\Lambda'_n\) to \(H^*(OG(n), \Z)\) sending \(\tilde{P}_\lambda(X)\) to \(\tau_\lambda\) is a surjective ring homomorphism with the kernel generated by the polynomials \(\tilde{P}_{i,i}(X)\) for all \(i = 1, ..., n\).

For \(OG(n)\), the Gromov-Witten invariants are defined similarly. Given an integer \(d \geq 0\), and \(\lambda, \mu, \nu \in \D(n)\), the Gromov-Witten invariant \(<\tau_\lambda, \tau_\mu, \tau_\nu >_d\) is defined as the number of rational maps \(f : \P^1 \to OG(n)\) of degree \(d\) such that \(f(0) \in X_\lambda(F), f(1) \in X_\mu(G), \) and \(f(\infty) \in X_\nu(H)\), for given isotropic flags \(F, G, \) and \(H\), in general position. Note that \(<\tau_\lambda, \tau_\mu, \tau_\nu >_d = 0\) unless \(|\lambda| + |\mu| + |\nu| = \deg(OG(n)) + 2nd\). The quantum cohomology ring of \(OG(n)\) is isomorphic to \(H^*(OG(n), \Z) \otimes \Z[q]\) as \(\Z[q]\)-modules. The multiplication in \(qH^*(OG(n), \Z)\) is given by the relation
\begin{equation}
\tau_\lambda \cdot \tau_\mu = \sum <\tau_\lambda, \tau_\mu, \tau_\nu >_d \tau_\nu q^d,
\end{equation}
where the sum is taken over \(d \geq 0\) and partitions \(\nu\) with \(|\nu| = |\lambda| + |\mu| - 2nd\). We note that the degree of the quantum variable \(q\) in \(qH^*(OG(n), \Z)\) is \(2n\), whereas that of \(q\) in \(qH^*(LG(n), \Z)\) is \((n + 1)\).
Theorem 3.2 ([13]). There is a surjective ring homomorphism from $\Lambda'_n$ to $qH^*(OG(n), \mathbb{Z})$ sending $P_{\lambda}(X)$ to $\tau_\lambda$ for all $\lambda \in D(n)$ and $P_{n,n}(X)$ to $q$, with the kernel generated by $P_i$ for $1 \leq i \leq n-1$. The quantum cohomology ring $qH^*(OG(n), \mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_n, q]$ modulo the relations $\tau_{i,i} = 0$ for $i = 1, \ldots, n-1$ together with the quantum relation

$$\tau_n^2 = q,$$

where quantum multiplication is employed throughout.

The Schubert class $\tau_\lambda$ in this presentation is given by the quantum Giambelli formulas

$$\tau_{i,j} = \tau_i \tau_j + 2 \sum_{k=1}^{\min(i,j)-1} (-1)^k \tau_{i+k} \tau_{j-k} + (-1)^j \tau_{i+j},$$

for $i > j > 0$, and

$$\tau_\lambda = \text{Pfaffian}(\tau_{\lambda_i, \lambda_j})_{1 \leq i < j \leq r},$$

where $\tau_\lambda$ is the Schubert class in the quantum cohomology ring $qH^*(OG(n), \mathbb{Z})$.

3.4. Quantum Euler class. The quantum Euler class $e_q(M)$ of a projective manifold $M$ is a deformation of (ordinary) Euler class $e(M)$. Originally, it is defined in the context of the so-called Frobenius algebra ([1]). Restricting ourselves to the cases $OG(n)$ and $LG(n)$ for simplicity, the quantum Euler class $e_q(M)$ can be defined as follows.

Definition. For $M = OG(n)$ or $LG(n)$, the quantum Euler classes $e_q(M)$ are respectively defined as

$$e_q(OG(n)) := \sum_{\lambda \in D(n)} \tau_\lambda \cdot \tau_\lambda,$$

and

$$e_q(LG(n)) := \sum_{\lambda \in D(n)} \sigma_\lambda \cdot \sigma_\lambda.$$

Note that if we replace the quantum product in the above definitions by the ordinary product in $H^*(M)$, we get the Euler class $e(M)$ of $M$. The object $e_q(M)$ encodes information on the semisimplicity of the quantum cohomology ring as follows.

Proposition 3.3. ([1], Theorem 3.4) For a projective manifold $M$, the quantum cohomology ring $qH^*(M)$ with the quantum parameters specialized to nonzero complex numbers is semisimple if and only if the quantum Euler class (after the specialization) is invertible in that ring.

Proposition 3.4. ([1]) If $M = G/P$ is a minuscule or cominuscule homogeneous variety, then the quantum cohomology ring $qH^*(M)$, which contains a single quantum variable $q$, is semisimple after specializing at $q = 1$.

We refer to §2 of [5] for the notion of minuscule or cominuscule homogeneous varieties of Proposition 3.4. Note that $OG(n)$ is minuscule and $LG(n)$ is cominuscule (see §2 of [5]). Thus, the rings $qH^*(OG(n))_{q=1}$ and $qH^*(LG(n))_{q=1}$ are semisimple.
4. Peterson’s result

4.1. Peterson’s result. Informally, one of Peterson’s (unpublished) results on the quantum cohomology can be stated as follows (10): Let $G$ be a semisimple algebraic group and $B$ a Borel subgroup of $G$. Let $G^\vee$ be the Langlands dual of $G$, and $B^\vee$ a Borel subgroup of $G^\vee$. For a parabolic subgroup $P$ of $G$ containing $B$, the quantum cohomology ring of the homogeneous variety $G/P$ is isomorphic with the coordinate ring $\mathcal{O}(Y_P)$ of a (an affine) subvariety $Y_P$, which is a stratum of so-called Peterson’s variety $Y \subset G^\vee/B^\vee$, i.e.,

$$Y = \bigcup_Q Y_Q,$$

where $Q$ ranges over parabolic subgroups containing $B$. For convenience, we will simply call the subvariety $Y_P$ a Peterson variety (corresponding to $P$), too, if there is no confusion. When $P$ is a minuscule parabolic subgroup of $G$, this Peterson’s result goes further (17). More precisely, in this case, the variety $Y_P$ can be replaced by a simpler isomorphic variety $V_P \subset U^\vee$, where $U^\vee$ is the unipotent radical of $B^\vee$. This Peterson’s (unpublished) result was verified for homogeneous varieties $G/P$ of Lie type $A$ ([10], [20]), and for even and odd orthogonal Grassmannians (2). On the other hand, $LG(n) = SP_{2n}(\mathbb{C})/P_n$ is not minuscule but cominuscule, but still we can find a variety $V_{P_n} \subset SO_{2n+1}(\mathbb{C})$, defined in the same way as in the minuscule case, of which the coordinate ring $\mathcal{O}(V_{P_n})$ turns out to be isomorphic with the quantum cohomology ring of $LG(n)$ (2).

4.2. Varieties $V_n$ and $W_n$. Note that the Peterson variety $V_{P_n}$ was defined Lie theoretically, and so we cannot see how the coordinate ring of $V_{P_n}$ looks like directly from the definition of $V_{P_n}$. To avoid some complexity, here we will not give the definition of $V_{P_n}$. Instead, we will give a ‘unraveled’ version of $V_{P_n}$ for our cases which serves our purpose better.

For $V_1, ..., V_n \in \mathbb{C}$, let $\tilde{v}(V_1, ..., V_n)$ be the matrix in $SL_{2n}(\mathbb{C})$

$$\tilde{v}(V_1, ..., V_n) := \begin{pmatrix}
1 & V_1 & V_2 & \cdots & V_n & 0 & \cdots & 0 \\
1 & V_1 & V_2 & \cdots & V_n & \vdots & \vdots & \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & V_1 & & & & & & \\
& 1 & V_1 & & & & & \\
& & \ddots & 1 & V_2 & & & \\
& & & \vdots & \vdots & 1 & V_1 & \\
& & & & & 1 & V_1 & 1
\end{pmatrix}.
$$

(4.7)

For $i = 1, ..., n - 1$, put

$$V_{i,i} := V_i^2 + 2 \sum_{k=1}^{i} (-1)^k V_{i+k} V_{i-k},$$

(4.8)

where $V_0 = 1$, and $V_r = 0$ if $r \geq n + 1$.

**Lemma 4.1.** ([2]) If $\tilde{v}(V_1, ..., V_n)$ is an element of $SL_{2n}(\mathbb{C})$ such that $V_{i,i} = 0$ for $i = 1, ..., n - 1$, then the matrix $\tilde{v}(V_1, ..., V_n)$ in fact belongs to $SP_{2n}(\mathbb{C})$.

Lemma 4.1 makes the following definition well-defined.

**Definition.** For $OG(n) = SO_{2n+1}(\mathbb{C})/P_n$, we define $V_n = V_{P_n}$ to be the subvariety of $SP_{2n}(\mathbb{C})$ consisting of matrices of the form $\tilde{v}(V_1, ..., V_n)$ satisfying the relations $V_{i,i} = 0$ for $i = 1, ..., n - 1$. 

Note that if we view $V_i$ as coordinate functions of $\mathcal{V}_n$, then the coordinate ring $\mathcal{O}(\mathcal{V}_n)$ of $\mathcal{V}_n$ is $\mathbb{C}[V_1, ..., V_n]/\mathcal{I}$, where $\mathcal{I}$ is generated by $V_{i,i}$ for all $i = 1, ..., n - 1$.

For $W_1, ..., W_{n+1} \in \mathbb{C}$, let $\tilde{w}(W_1, ..., W_{n+1})$ be the matrix in $SL_{2n+1}(\mathbb{C})$

\[
\begin{pmatrix}
1 & W_1 & W_2 & \cdots & W_{n+1} & 0 & \cdots & 0 \\
1 & W_1 & W_2 & \cdots & W_{n+1} & \ddots & \vdots \\
& & & & & \ddots & 0 \\
& & & & & & \ddots & \vdots \\
& & & & & & & 1 & W_1 & W_2 \\
& & & & & & & & 1 & W_1 \\
& & & & & & & & & 1
\end{pmatrix}
\]

(4.9) $\tilde{w}(W_1, ..., W_{n+1})$ := $W_{i,i} := W_i^2 + 2 \sum_{k=1}^{i} (-1)^k W_{i+k} W_{i-k},$

where $W_0 = 1$, and $W_r = 0$ if $r \geq n + 2$.

**Lemma 4.2.** \([\text{2}]\) If $\tilde{w}(W_1, ..., W_{n+1})$ is an element of $SL_{2n+1}(\mathbb{C})$ such that $W_{i,i} = 0$ for $i = 1, ..., n$, then the matrix $\tilde{w}(W_1, ..., W_{n+1})$ in fact belongs to $SO_{2n+1}(\mathbb{C})$.

By Lemma 4.2, the following definition makes sense.

**Definition.** For $LG(n) = Sp_{2n}(\mathbb{C})/P_n$, we define $\mathcal{W}_n = \mathcal{V}_n$ to be the subvariety of $SO_{2n+1}(\mathbb{C})$ consisting of matrices of the form $\tilde{w}(W_1, ..., W_{n+1})$ satisfying the relations $W_{i,i} = 0$ for $i = 1, ..., n$.

Note that if we view $W_i$ as coordinate functions of $\mathcal{W}_n$, then the coordinate ring $\mathcal{O}(\mathcal{W}_n)$ is $\mathbb{C}[W_1, ..., W_{n+1}]/\mathcal{J}$, where $\mathcal{J}$ is generated by $W_{i,i}$ for all $i = 1, ..., n$.

### 4.3. Comparing two presentation of the quantum cohomology ring

A Peterson's result for our cases can be stated as follows. See [2] for an elementary proof.

**Theorem 4.3** (Peterson). We have isomorphisms of two rings.

1. The map $qH^*(OG(n), \mathbb{C}) \cong \mathcal{O}(\mathcal{V}_n)$ sending $\tau_i$ to $\frac{1}{2} V_i$ for all $i \leq n$, and $q$ to $\frac{1}{4} V_n^2$ is an isomorphism.
2. The map $qH^*(LG(n), \mathbb{C}) \cong \mathcal{O}(\mathcal{W}_n)$ sending $\sigma_i$ to $W_i$ for all $i \leq n$, and $q$ to $2W_{n+1}$ is an isomorphism.

**Notation.** (1) By the isomorphism $qH^*(OG(n), \mathbb{C}) \cong \mathcal{O}(\mathcal{V}_n)$, each $\tau \in qH^*(OG(n))$ defines a function on $\mathcal{V}_n$. We denote this function by $\check{\tau} = \check{\tau}(V_1, ..., V_n)$.

(2) Similarly, for $\sigma \in qH^*(LG(n))$, $\sigma = \check{\sigma}(W_1, ..., W_{n+1})$ denotes the function on $\mathcal{W}_n$ corresponding to $\sigma$ under the isomorphism $qH^*(LG(n)) \cong \mathcal{O}(\mathcal{W}_n)$.

**Example.** For nonnegative integers $i \geq j$, let $V_{i,j}$ be the function on $\mathcal{V}_n$ defined by

\[V_{i,j} := V_i V_j + 2 \sum_{k=1}^{i} (-1)^k V_{i+k} V_{j-k},\]
where $V_0 = 1$ and $V_l = 0$ if $l < 0$ or $l > n$. Then we have $V_{i,j} = 4\tau_{i,j}$ if $j \neq 0$, and if $j = 0$ and $i \neq 0$, then $V_{i,j} = V_i = 2\tau_i$. More generally, for $\lambda \in \mathcal{D}(n)$, $\tau_\lambda$ is the function on $V_n$ defined by

$$\tau_\lambda = 2^{-r}\text{Pfaffian}(V_{\lambda,\lambda})_{1 \leq i,j \leq r},$$

where $r = 2[(l + 1)/2]$ for $l = l(\lambda)$.

5. Analysis on points of Peterson’s variety

In this section, we record an explicit description of elements of $V_n$ and $W_n$ from Section 4 of [2].

5.1. Definitions and Notations. Let $\zeta = \zeta_n$ be the primitive $2n$-th root of unity, i.e., $\zeta_n = e^{\frac{2\pi i}{2n}}$.

Let $T_n$ be the set of all $n$-tuples $J = (j_1,...,j_n)$, $-\frac{n-1}{2} \leq j_1 < \cdots < j_n \leq \frac{n-1}{2}$, such that $\zeta^J := (\zeta^{j_1},...,\zeta^{j_n})$ is an $n$-tuple of distinct $2n$-th roots of $(-1)^{n+1}$. Let us call $I = (i_1,...,i_n) \in T_n$ exclusive if $\zeta^{i_k} \neq \zeta^{i_l}$ for all $k,l = 1,...,n$.

Define subsets $I_n$, $I_n^e$ and $I_n^o$ of $T_n$ as

$$I_n := \{I \in T_n \mid I \text{ exclusive}\},$$

$$I_n^e := \{I \in T_n \mid E_n(\zeta^I) = 1\},$$

$$I_n^o := \{I \in T_n \mid E_n(\zeta^I) = -1\}.$$

Remark. We can easily check that $|I_n| = 2^n = |\mathcal{D}(n)|$. Note that $I_n = I_n^e \cup I_n^o$ since $E_n^2(\zeta^I) = 1$ for $I \in I_n$ by Lemma 5.1 below. Since $|I_n^e| = |I_n^o|$, it follows that $|I_n^e| = |I_n^o| = 2^{n-1}$.

Now we characterize exclusive $n$-tuples in terms of elementary symmetric functions.

Lemma 5.1. If $I$ is exclusive, then $E_i(\zeta^{2l}) = 0$ for $i = 1,2,...,n-1$ and $E_n^2(\zeta^I) = 1$.

Proof. Note that $I_0 := (-\frac{n-1}{2},-\frac{n-1}{2}+1,...,\frac{n-1}{2})$ is exclusive, $E_i(\zeta^{2l}) = 0$ for $i = 1,...,n-1$ and $E_n(\zeta^{2l}) = 1$. If $I$ is exclusive, we can easily check that $\zeta^{2l} = \zeta^{2l_0}$. Thus $E_i(\zeta^{2l}) = 0$ for $i = 1,...,n-1$, and $E_n^2(\zeta^I) = E_n(\zeta^{2l}) = 1$ by (1) of Proposition 2.1.

We now characterize the elements of $V_n$ and $W_n$ more explicitly. To do this, we introduce the following notations.

Notation. (1) For $a_1,...,a_n \in \mathbb{C}$, let $v(a_1,...,a_n)$ be the matrix in $SL_{2n}(\mathbb{C})$ defined by

$$v(a_1,...,a_n) = \tilde{v}(E_1(a_1,...,a_n),...,E_n(a_1,...,a_n)),$$

where $\tilde{v}$ was given in (4.7).

(2) For $b_1,...,b_{n+1} \in \mathbb{C}$, let $w(b_1,...,b_{n+1})$ be the matrix in $SL_{2n+1}(\mathbb{C})$ defined by

$$w(b_1,...,b_{n+1}) = \tilde{w}(E_1(b_1,...,b_{n+1}),...,E_{n+1}(b_1,...,b_{n+1})).$$

where $\tilde{w}$ was given in (4.9).

Proposition 5.2. ([2], Lemmas 5.2, 5.3) Elements of $V_n$ and $W_n$ are characterized as follows.

(1) All elements of $V_n$ are exactly of the form $v(t\zeta^I)$ with $t \in \mathbb{C}$ and $I \in I_n$.

(2) All elements of $W_n$ are exactly of the form $w(t\zeta^I)$ with $t \in \mathbb{C}$ and $I \in I_{n+1}$.

Remark. Note that for $\lambda \in \mathcal{D}(n)$, the function $\lambda_\lambda$ evaluates on $v(t\zeta^I) \in V_n$ to $\overline{P}(t\zeta^I)$, and $\delta_\lambda$ evaluates on $w(t\zeta^I) \in W_n$ to $\overline{Q}(t\zeta^I)$.

For later use, here we record the evaluations on $V_n$ and $W_n$ of $\dot{q}$ for the quantum variable $q$ for $OG(n)$ and $LG(n)$.

Proposition 5.3. ([2], Lemma 5.5) The functions $\dot{q}$ on $V_n$ and $W_n$ evaluate as follows.
(1) For $v = v(t\zeta^I) \in \mathcal{V}_n$, we have
\[ \dot{q}(v) = \frac{1}{4} t^{2n}. \]

(2) For $w = w(t\zeta^I) \in \mathcal{W}_n$, we have
\[ \dot{q}(w) = 2^{n+1}E_{n+1}(\zeta^I), \quad \text{and} \quad \dot{q}^2(w) = 4t^{2n+2}. \]

Definition. Let $\mathcal{V}_n'$ (resp. $\mathcal{W}_n'$) be the subvariety of $\mathcal{V}_n$ (resp. $\mathcal{W}_n$) defined by the function $\dot{q} = 1$.

Corollary 5.4. Let $\epsilon = \epsilon_n = (4)^{\frac{n}{2}}$. Then $v' \in \mathcal{V}_n'$ if and only if there exists a unique $I \in \mathcal{I}_n$ such that $v' = v(\epsilon\zeta^I)$.

Proof. The direction $(\Leftarrow)$ is obvious. For the converse, let $v' \in \mathcal{V}_n'$. Then, by Proposition 5.2, $v'$ can be written as $v' = v(t\zeta^I)$ for some $t \in \mathbb{C}$ and $I \in \mathcal{I}_n$. Note that $\dot{q}(v') = \frac{1}{4} t^{2n} = 1$ since $v' \in \mathcal{V}_n'$. Now, since $\frac{1}{4}$ is a $2n$-th root of unity, and $\zeta^J$ is an $n$-tuple $\zeta^J$ rotated by $\arg\left(\frac{1}{4}\right)$ in each entry, there is a unique $I \in \mathcal{I}_n$ such that $\zeta^I = \epsilon\zeta^J$, i.e., $\epsilon\zeta^I = t\zeta^J$. This proves the corollary. \(\Box\)

Corollary 5.5. Let $\delta = \delta_n := \left(\frac{1}{4}\right)^{\frac{n}{2}}$. Then $w' \in \mathcal{W}_n'$ if and only if there exists a unique $I \in \mathcal{T}_n^e$ such that $w' = v(\delta\zeta^I)$.

Proof. The proof is similar to the proof of Corollary 5.4 \(\Box\)

5.2. Orthogonality formulas.

Lemma 5.6. For $I, J \in \mathcal{I}_n$ with $I \neq J$, there is $w \in W_n \setminus S_n$ such that $(\zeta^I)^w = \zeta^J$.

Proof. Write $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$. Let us determine (entries $w_i$ of) $w = (w_1, \ldots, w_n) \in W_n \setminus S_n$. First note that since $I$ is exclusive, the union of two sets $\{\zeta^{i_1}, \ldots, \zeta^{i_n}\} \cup \{-\zeta^{i_1}, \ldots, -\zeta^{i_n}\}$ equals the set of all $2n$-th roots of $(-1)^{n+1}$. Therefore for each $k = 1, \ldots, n$, the entry $\zeta^{j_k}$ of $\zeta^J$ is $\zeta^{i_m}$ or $-\zeta^{i_m}$ for some $1 \leq m \leq n$. Then put $w_m = k$ (resp. $-k$) if $\zeta^{j_k} = \zeta^{i_m}$ (resp. $\zeta^{j_k} = -\zeta^{i_m}$). Then for $w = (w_1, \ldots, w_n)$ thus obtained, it is obvious that $(\zeta^I)^w = \zeta^J$, and $w \in W_n \setminus S_n$ since $\{\zeta^{i_1}, \ldots, \zeta^{i_n}\} \neq \{\zeta^{j_1}, \ldots, \zeta^{j_n}\}$. \(\Box\)

Proposition 5.7. (1) For $I, J \in \mathcal{I}_n$ and $t \in \mathbb{C}$, we have
\[ \sum_{\lambda \in D(n)} \tilde{P}_\lambda(t\zeta^I)\tilde{P}_\lambda(t\zeta^J) = \delta_{I,J}S_{\rho_n}(t\zeta^I), \]

(2) For $I, J \in \mathcal{T}_n^e$ and $t \in \mathbb{C}$, we have
\[ (5.11) \sum_{\lambda \in D(n)} t^{n+1}\tilde{Q}_\lambda(t\zeta^I)\tilde{Q}_\lambda(t\zeta^J) = \delta_{I,J}2^n S_{\rho_{n+1}}(t\zeta^I), \]

where we denote $\rho_n := (n, n-1, \ldots, 1)$.

Proof. By Lemma 5.6, for $I, J \in \mathcal{I}_n$ there is $w \in W_n \setminus S_n$ such that $(\zeta^{i_1}, \ldots, \zeta^{i_n})^w = (\zeta^{j_1}, \ldots, \zeta^{j_n})$. Therefore (1) is immediate from Proposition 2.2. The equality (2) follows from the equalities
\[ 2^{n+1}S_{\rho_{n+1}}(t\zeta^I) = \sum_{\lambda \in D(n+1)} \tilde{Q}_\lambda(t\zeta^I)\tilde{Q}_\lambda(t\zeta^I) \]
\[ = \sum_{\mu \in D(n)} 2\tilde{Q}_{\mu+1}(t\zeta^I)\tilde{Q}_{\mu}(t\zeta^I) = \sum_{\mu \in D(n)} 2t^{n+1}E_{n+1}(\zeta^I)\tilde{Q}_{\mu}(t\zeta^I)\tilde{Q}_{\mu}(t\zeta^I). \]

Here the first equality follows from (1) of Proposition 5.7 and the second equality follows from the fact that for each pair $(\mu, \tilde{\mu}) \in D(n) \times D(n)$, there are exactly two pairs $(\lambda, \tilde{\lambda}) \in D(n+1) \times D(n+1)$,
Then we have
\[ k \text{ distinct nonzero vectors in the complex plane satisfying} \]
Prelemma 6.3.

Obvious from the definitions of the length and summation of vectors in the complex plane.

Let Prelemma 6.2.

and the strict inequalities hold if there is a

upper half plane of the complex plane satisfying

Then we have
\[ S \text{ Proposition 5.7, we get} \]

\[ qH \text{ (1) For} \]

\[ qH \text{ which proves (1) of the lemma. Similarly, using the semisimplicity of} \]

\[ S \text{ Thus it follows from Proposition 3.3 that} \]

\[ \text{Proof. Denote by} \]

\[ \hat{\varepsilon}_q \text{ the function on} \]

\[ \text{Proposition 5.7, we have} \]

\[ \varepsilon_q(e^{\varepsilon_l'}) = \sum_{\lambda \in D(n)} \tilde{P}_\lambda(e^{\varepsilon_l'}) \tilde{P}_\lambda(e^{\varepsilon_l'}) = S_{\rho_n}(e^{\varepsilon_l'}). \]

Recall that \( OG(n) \) is minuscule and hence the ring \( qH^*(OG(n))_{q=1} \) is semisimple (Proposition 5.4). Thus it follows from Proposition 5.3 that \( S_{\rho_n}(e^{\varepsilon_l'}) \neq 0 \), equivalently, \( S_{\rho_n}(\varepsilon_l') \neq 0 \) for \( I \in I_n \), which proves (1) of the lemma. Similarly, using the semisimplicity of \( qH^*(LG(n))_{q=1} \) and (2) of Proposition 5.7, we get \( S_{\rho_n}(\varepsilon_l') \neq 0 \) for \( J \in I_{n+1} \). This proves the lemma.

Prelemma 6.2.

Let \( \{v_0, v_1, ..., v_m\} \) and \( \{v'_0, v'_1, ..., v'_m\} \) be the sets of nonzero vectors in the (closed) upper half plane of the complex plane satisfying

(i) \( v_0 = v'_0 \) and \( v_0 \in \mathbb{R}_{>0} \),
(ii) \( |v_k| = |v'_k| \) for \( k = 1, ..., n \),
(iii) \( 0 < \arg(v_1) < \cdots < \arg(v_m) \leq \pi; \) and \( 0 < \arg(v'_1) < \cdots < \arg(v'_m) \leq \pi \),
(iv) \( \arg(v'_{k+1}) - \arg(v'_k) \leq \arg(v_{k+1}) - \arg(v_k) \) for all \( k = 0, ..., m - 1 \).

Then we have
\[ \sum_{i=1}^{m} v'_i \geq \sum_{i=1}^{m} v_i, \quad \arg(\sum_{i=1}^{m} v_i) \geq \arg(\sum_{i=1}^{m} v'_i), \]
and the strict inequalities hold if there is a \( k \) with \( 1 \leq k \leq m \) such that \( \arg(v_k) > \arg(v'_k) \).

Proof. Obvious from the definitions of the length and summation of vectors in the complex plane.

The following is more general than Prelemma 6.2.

Prelemma 6.3.

Let \( \{v_{-l}, ..., v_{-1}, v_0, v_1, ..., v_m\} \) and \( \{v'_{-l}, ..., v'_{-1}, v'_0, v'_1, ..., v'_m\} \) be the sets of distinct nonzero vectors in the complex plane satisfying

(i) \( v_k \text{ and } v'_k \text{ lie on the (closed) upper half plane if } k = 0, 1, ..., m, \text{ and on the (open) lower half plane if } k = -1, ..., -l \).
(ii) \( \text{The vectors } v_k \text{, and } v'_k \text{ in the (closed) upper half plane satisfy the conditions in Prelemma 6.2} \).
(iii) \( \text{The vectors } v_k \text{, and } v'_k \text{ in the lower half plane satisfy the similar conditions; } \)
(a) \( |v_k| = |v'_k| \text{ for } k = -1, ..., -l \),
(b) \(-\pi < \arg(v_{-l}) < \cdots < \arg(v_{-1}) < 0; \) and \(-\pi < \arg(v'_{-l}) < \cdots < \arg(v'_{-1}) \leq 0 \),
(c) \( \arg(v_k) - \arg(v_{k-1}) \geq \arg(v'_k) - \arg(v'_{k-1}) \) for all \( k = 0, ..., -1 + 1 \).

Then we have \( |\sum_{k=-l}^{-1} v_k| \leq |\sum_{k=-l}^{-1} v'_k| \), and the strict inequality holds if there is a \( k \) with \( -l \leq k \leq m \) such that \( |\arg(v_k)| > |\arg(v'_k)| \).
Proof. When \( l \leq 1 \) and \( m \leq 1 \), it is trivial to check the prelemma. Furthermore, when \( m = 1 \) and \( l = 1 \), i.e., when we work with the two sets \( \{v_1, v_0, v_1\} \) and \( \{v_1', v_0, v_1'\} \), the prelemma is true even though we relax the condition \( |v_k| = |v_k'| \) into the condition \( |v_k| \leq |v_k'| \) for \( k = -1, 1 \). For general case, let \( V_1 := \sum_{k=1}^m v_k \) (resp. \( V_1' := \sum_{k=1}^m v_k' \)) and \( V_{-1} := \sum_{k=1}^l v_{-k} \) (resp. \( V_{-1}' := \sum_{k=1}^l v_{-k}' \)). Then, by Prelemma 6.2, the vectors in the sets \( \{V_{-1}, v_0, V_1\} \) and \( \{V_{-1}', v_0, V_1'\} \) satisfy all the conditions of the prelemma (for \( m = 1 \) and \( l = 1 \)) except for the condition \( |V_k| = |V_k'| \) for \( k = -1, 1 \). Instead, they satisfy the condition \( |V_k| \leq |V_k'| \) for \( k = -1, 1 \). Then, by the above special case, we have \( |V_{-1} + v_0 + V_1| \leq |V_{-1}' + v_0 + V_1'| \), equivalently, \( |\sum_{k=-1}^m v_k| \leq |\sum_{k=-1}^m v_k'| \). \( \square \)

Recall that the entries \( \zeta^{i_1}, \ldots, \zeta^{i_n} \) of \( \zeta \) lie on the unit circle. Therefore one can rotate the points \( \zeta, \ldots, \zeta \) simultaneously by a certain angle \( \theta \) so that the set of rotated points \( \{e^{2\pi \theta \zeta}, \ldots, e^{2\pi \theta \zeta} \} \) equals the set \( \{\zeta^{j_1}, \ldots, \zeta^{j_n} \} \) for some \( J = (j_1, \ldots, j_n) \in \mathbb{I}_n \). In this situation, we simply say that \( \zeta \) is obtained by rotating \( \zeta ^J \) (by \( \theta \)). Similarly, one can flip the points \( \zeta^{i_1}, \ldots, \zeta^{i_n} \) simultaneously with respect to a line \( L \) passing through the origin, so that the set of flipped points \( \{ (\zeta^{i_1})', \ldots, (\zeta^{i_n})' \} \) equals the set \( \{\zeta^{i_1}, \ldots, \zeta^{i_n} \} \) for some \( J = (j_1, \ldots, j_n) \in \mathbb{I}_n \). In this situation, we simply say that \( \zeta \) is obtained by flipping \( \zeta ^J \) (with respect to \( L \)).

Definition. (1) We say that for \( I, J \in \mathbb{I}_n \), \( \zeta ^J \) has the same configuration as \( \zeta ^J \) if \( \zeta ^J \) is obtained by rotating and (or) flipping \( \zeta ^J \).

(2) For \( J \in \mathbb{I}_n \), \( \zeta ^J \) is called closed if \( j_{k+1} = j_k + 1 \) for \( k = 1, \ldots, n - 1 \). For example, \( \zeta ^{lo} \) is closed.

Lemma 6.4. For the n-tuples \( \zeta ^J \) with \( I \in \mathbb{I}_n \), we have the following properties.

(1) If \( \zeta ^J \) and \( \zeta ^J \) have the same configuration, then \( |E_1(\zeta ^J)| = |E_1(\zeta ^J)| \).

(2) \( E_1(\zeta ^{lo}) \) is a positive real number which is equal to \( E_1(\zeta ^{lo}) = \frac{1}{\sin(\pi/2n)} \).

(3) \( \zeta ^J \) is a closed n-tuple if and only if \( |E_1(\zeta ^J)| \) is maximal among \( |E_1(\zeta ^J)| \) with \( I \in \mathbb{I}_n \). In particular, \( |E_1(\zeta ^{lo})| = E_1(\zeta ^{lo}) \) is maximal among \( |E_1(\zeta ^J)| \) with \( I \in \mathbb{I}_n \).

(4) If \( \zeta ^J \) and \( \zeta ^J \) are closed n-tuples with \( I \neq J \in \mathbb{I}_n \), then we have \( E_1(\zeta ^J) = \eta E_1(\zeta ^J) \) for some \( 2n \)-th root \( \eta \) of unity with \( \eta \neq 1 \), and, in particular, \( E_1(\zeta ^J) \neq E_1(\zeta ^J) \).

Proof. (1) is obvious. For (2), see Page 542 of [19]. For (3), first note that for any \( \theta \in \mathbb{R} \), \( |E_1(\zeta ^J)| \) is a maximal element of the set \( \{|E_1(\zeta ^J)| \mid I \in \mathbb{I}_n \} \) if and only if \( |E_1(e^{2\pi \theta \zeta})| \) is a maximal element of the set \( \{|E_1(e^{2\pi \theta \zeta})| \mid I \in \mathbb{I}_n \} \). Let \( \zeta ^J \) be a closed n-tuple with \( J = (j_1, \ldots, j_n) \). Then, fix a component \( \zeta ^J \) of the n-tuple \( \zeta ^J \) and take \( \theta \in \mathbb{R} \) so that the vector \( v_0 := e^{2\pi \theta \zeta ^J} \) lies on the positive real axis. Now we consider the set \( \{|E_1(e^{2\pi \theta \zeta ^J})| \mid I \in \mathbb{I}_n \} \). Then by Prelemma 6.3, \( |E_1(e^{2\pi \theta \zeta ^J})| \) is a maximal element of the set \( \{|E_1(e^{2\pi \theta \zeta ^J})| \mid I \in \mathbb{I}_n \} \), and hence \( |E_1(\zeta ^J)| \) is a maximal element of the set \( \{|E_1(\zeta ^J)| \mid I \in \mathbb{I}_n \} \). The converse is immediate from Prelemma 6.3. For (4), note that \( \zeta ^J \) and \( \zeta ^J \) are closed n-tuples with \( I \neq J \), then there is a \( 2n \)-th root \( \eta \) of unity such that \( \zeta ^J = \eta \zeta ^J \), and hence \( E_1(\zeta ^J) = \eta E_1(\zeta ^J) \). \( \square \)

Remark. Note that the converse of (1) is not true in general. Indeed, it is not difficult to find \( I, J \in \mathbb{I}_n \) such that \( \zeta ^J \) and \( \zeta ^J \) do not have the same configuration, and \( |E_1(\zeta ^J)| \neq |E_1(\zeta ^J)| \). However, (3) implies that with the maximality condition on the modulus, this can not happen.

We remark that the ring \( qH^*(OG(n), \mathbb{C})_{q=1} \) (resp. \( qH^*(LG(n), \mathbb{C})_{q=1} \)) is a \( 2^n \) dimensional complex vector space with the Schubert basis \( \{ r_\lambda \mid \lambda \in \mathbb{D}(n) \} \) (resp. \( \{ s_\lambda \mid \lambda \in \mathbb{D}(n) \} \)). Now we use the orthogonality formulas in Proposition 5.7 to find another basis for each of these vector spaces which, in fact, turns out to be a simultaneous eigenbasis for all multiplication operators \( [F] \). The original idea for finding this eigenbasis by using the orthogonality formula is due to Rietsch. Indeed, Rietsch obtained an eigenbasis of the quantum cohomology ring of the Grassmannian which Theorems 6.5 and 6.6 below are modeled on (Page 551 of [19]).
Theorem 6.5. For each \( I \in \mathcal{I}_n \), let \( \tau_I = \sum_{\nu \in \mathcal{D}(n)} \tilde{P}_\nu (\epsilon \zeta^I) \tau_\nu \). Then for each \( \lambda \in \mathcal{D}(n) \), the quantum multiplication operator \([\tau_I]\) on \( qH^*(OG(n), \mathbb{C})_{q=1} \) has eigenvectors \( \tau_I \) with eigenvalues \( \tilde{P}_\lambda (\epsilon \zeta^I) \). In particular, \( \{ \tau_I \mid I \in \mathcal{I}_n \} \) forms a simultaneous eigenbasis of the vector space \( qH^*(OG(n), \mathbb{C})_{q=1} \) for the operators \([F]\) with \( F \in qH^*(OG(n), \mathbb{C})_{q=1} \). Here \( \epsilon = (4/3)^{1/2} \) as before.

Proof. First note that \( \tau_I \) is a nonzero vector for all \( I \in \mathcal{I}_n \) by Lemma 6.1 and (1) of Proposition 5.7. Evaluating the function \( \dot{\tau}_I \) on the points \( v(\epsilon \zeta^J) \) for \( J \in \mathcal{I}_n \) and using (1) of Proposition 5.7, we obtain the equality of functions on \( V_n' \)

\[
\dot{\tau}_I = \tilde{P}_\lambda (\epsilon \zeta^I) \tau_I.
\]

Then the first part of the theorem is obvious since \( qH^*(OG(n), \mathbb{C})_{q=1} \) is identified with \( O(V'_n) \). Since (6.12) holds for all Schubert basis elements \( \tau_\lambda \), the vector \( \tau_I \) is a simultaneous eigenvector for all operators \([F]\) with an eigenvalue \( F(v(\epsilon \zeta^I)) \) for \( F \in qH^*(OG(n), \mathbb{C})_{q=1} \). Now we show that \( \{ \tau_I \mid I \in \mathcal{I}_n \} \) forms a simultaneous eigenbasis for \( qH^*(OG(n), \mathbb{C})_{q=1} \). Since \( |\mathcal{I}_n| = |D| \), it suffices to show that the vectors \( \tau_I, I \in \mathcal{I}_n \), are linearly independent. So suppose that

\[
\sum_{I \in \mathcal{I}_n} a_I \tau_I = 0.
\]

Now let us evaluate the function \( \dot{\tau} := \sum_{I \in \mathcal{I}_n} a_I \dot{\tau}_I \) on the points \( v(\epsilon \zeta^J) \) with \( J \in \mathcal{I}_n \). Then, by (1) of Proposition 5.7, we have the evaluation

\[
\dot{\tau}(v(\epsilon \zeta^J)) = a_J \dot{S}_{\rho_\nu}(v(\epsilon \zeta^J)) = a_J S_{\rho_\nu}(\epsilon \zeta^J).
\]

Since \( S_{\rho_\nu}(\epsilon \zeta^J) \) is nonzero for any \( J \in \mathcal{I}_n \) by Lemma 6.1, \( a_J = 0 \) for each \( J \in \mathcal{I}_n \). Therefore the vectors \( \tau_I, I \in \mathcal{I}_n \), are linearly independent. This completes the proof. \( \square \)

Theorem 6.6. For each \( I \in \mathcal{I}_{n+1} \), let \( \sigma_I = \sum_{\nu \in \mathcal{D}(n)} \tilde{Q}_\nu (\delta \zeta^I) \sigma_\nu \). Then for each \( \lambda \in \mathcal{D}(n) \), the quantum multiplication operator \([\sigma_\lambda]\) on \( qH^*(LG(n), \mathbb{C})_{q=1} \) has eigenvectors \( \sigma_I \) with eigenvalues \( \tilde{Q}_\lambda (\delta \zeta^I) \). In particular, \( \{ \sigma_I \mid I \in \mathcal{I}_{n+1} \} \) forms a simultaneous eigenbasis of the vector space \( qH^*(LG(n), \mathbb{C})_{q=1} \) for the operators \([F]\) with \( F \in qH^*(LG(n), \mathbb{C})_{q=1} \). Here \( \delta = (4/3)^{1/2} \) as before.

Proof. From (2) of Proposition 5.7 we have

\[
\sum_{\lambda \in \mathcal{D}(n)} \tilde{Q}_\lambda (\delta \zeta^I) \tilde{Q}_\lambda (\delta \zeta^J) = \delta_{IJ} 2^n S_{\rho_{n+1}} (\delta \zeta^I) \quad \text{for} \quad I \in \mathcal{I}_{n+1},
\]

from which we get eigenvectors \( \sigma_I \) and corresponding eigenvalues \( \tilde{Q}_\lambda (\delta \zeta^I) \)

\[
(6.13) \quad \sigma_\lambda \cdot \sigma_I = \tilde{Q}_\lambda (\delta \zeta^I) \sigma_I,
\]

as in the case of \( OG(n) \). The proof of linear independence of vectors \( \sigma_I, I \in \mathcal{I}_{n+1} \), is the same as in the case of \( OG(n) \). \( \square \)

For a complex manifold, the \( i \)-th Chern class \( c_i(M) \) is defined to be the \( i \)-th Chern class \( c_i(TM) \) of the tangent bundle \( TM \).

Lemma 6.7. (\cite{S}, Lemma 3.5) The first Chern class of \( OG(n) \) and \( LG(n) \), respectively, are given by

\[
c_1(OG(n)) = 2n \tau_1 \quad \text{and} \quad c_1(LG(n)) = (n + 1) \sigma_1.
\]

For \( I \in \mathcal{I}_n \), let

\[
f(I) := n \epsilon E_i(\zeta^I) = 2 \dot{\tau}_I(v(\epsilon \zeta^I)).
\]
Then since $c_1(OG(n)) = 2n\tau_1$, by Theorems 6.3, $f(I)$ is the eigenvalue of the operator $[c_1(OG(n))]$ corresponding to the eigenbasis $\tau_I$. Therefore, we have

$$\text{Spec } ([c_1(OG(n))]) = \{ f(I) \mid I \in \mathcal{I}_n \}.$$

Similarly, for $I \in \mathcal{I}_{n+1}$, letting

$$g(I) := (n + 1)\delta_{E_1}(\xi_I) = (n + 1)\delta_1(w(\delta_{I}')),$$

we have

$$\text{Spec } ([c_1(LG(n))]) = \{ g(I) \mid I \in \mathcal{I}_{n+1} \}.$$

**Lemma 6.8.** For $I \in \mathcal{I}_n$ for $OG(n)$ (resp. $I \in \mathcal{I}_{n+1}$ for $LG(n)$), the multiplicity of the eigenvalue $f(I)$ (resp. $g(I)$) is equal to the cardinality of the set $\{ J \in \mathcal{I}_n \mid f(J) = f(I) \}$ (resp. $\{ J \in \mathcal{I}_{n+1} \mid g(J) = g(I) \}$). In particular, if an eigenvalue $f(I)$ (resp. $g(I)$) has a maximal modulus among the eigenvalues for $OG(n)$ (resp. $LG(n)$), then $f(I)$ (resp. $g(I)$) is a simple eigenvalue.

**Proof.** The first statement is obvious, and the second is immediate from (3), (4) of Lemma 6.4 □

**Theorem 6.9.** $OG(n)$ and $LG(n)$ satisfy Conjecture $\mathcal{O}$.

**Proof.** By (3) of Lemma 6.4, the operator $[c_1(OG(n))]$ has the eigenvalue $T_0 := f(I_0)$, which is a positive real number of maximal modulus among $f(I)$ with $I \in \mathcal{I}_n$. Therefore the condition (1) of Conjecture $\mathcal{O}$ is satisfied by $OG(n)$. For the same reason, the condition (1) holds for $LG(n)$. The condition (3) of Conjecture $\mathcal{O}$ for both cases follows from Lemma 6.8 and (3), (4) of Lemma 6.4.

For the condition (2) for $OG(n)$, suppose that $J \in \mathcal{I}_n$ is such that $|f(J)| = f(I_0)$. Then $|E_1(J)| = E_1(I_0)$, and so $E_1(J)$ is maximal among $|E_1(\xi_I)|$ with $I \in \mathcal{I}_n$. Then by (3), (4) of Lemma 6.4, there is a $2n$-th root $\xi$ of unity such that $\xi = \eta\xi_0$, equivalently, $E_1(\xi^d) = \eta E_1(\xi_0)$. Thus $f(J) = \eta f(I_0)$. But since the Fano index of $OG(n)$ is $2n$, the condition (2) for $OG(n)$ is immediate.

For the condition (2) for $LG(n)$, suppose that $J \in \mathcal{I}_{n+1}$ is such that $|g(J)| = g(I_0)$. Then, as above, applying (4) of Lemma 6.4 to $J \in \mathcal{I}_{n+1}$, there is a $(2n + 2)$-root $\xi$ of unity such that $E_1(\xi^d) = \xi E_1(\xi_0)$, and so $g(J) = \xi g(I_0)$. Note that, a priori, $\xi$ is not necessarily a $(n + 1)$-th root of unity. But since both $I_0$ and $J$ belong to $\mathcal{I}_{n+1}$, $\xi$ is in fact a $(n + 1)$-th root of unity. Since the Fano index of $LG(n)$ is $n + 1$, it follows that the condition (2) is satisfied by $LG(n)$. This completes the proof. □

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Department of Mathematical Sciences, KAIST, 291 Daehak--ro, Yuseong-gu, Daejeon 34141, Korea
E-mail address: daewoongc@kias.re.kr