Data-driven modeling of the chaotic thermal convection in an annular thermosyphon

Jean-Christophe Loiseau

Received: date / Accepted: date

Abstract Identifying accurate and yet interpretable low-order models from data has gained renewed interest over the past decade. In the present work, we illustrate how the combined use of dimensionality reduction and sparse system identification techniques allows us to obtain an accurate model of the chaotic thermal convection in a two-dimensional annular thermosyphon. Taking as guidelines the derivation of the Lorenz system, the chaotic thermal convection dynamics simulated using a high-fidelity computational fluid dynamics solver are first embedded into a low-dimensional space using dynamic mode decomposition. After having reviewed the physical properties the reduced-order model should exhibit, the latter is identified using SINDy, an increasingly popular and flexible framework for the identification of nonlinear continuous-time dynamical systems from data. The identified model closely resembles the canonical Lorenz system, having a similar structure and exhibiting the same physical properties. Finally, extensions to other flow configurations with or without control are discussed.

Keywords Reduced-order model · System Identification · Chaos · Natural convection

1 Introduction

Fluid flows are characterized by high-dimensional nonlinear dynamics that gives rise to rich structures. Despite this apparent complexity, the dynamics often evolves on a low-dimensional attractor defined a few dominant coherent structures that contain significant energy or are useful for control [1]. Given this property, one might aim to derive or identify reduced-order models that reproduce qualitatively and quantitatively the dynamics of the full system. Over the past decades, identifying robust, accurate and efficient reduced-order models has thus become a central challenge in fluid dynamics and closed-loop flow control [2][3][4][5][6]. Amidst the numerous flows exhibiting such properties, the buoyancy-driven flow in a thermosyphon is of particular interest both from a theoretical and practical point of view. From the engineering perspective, natural convection in closed loops can play

J.-Ch. Loiseau
Laboratoire DynFluid — Arts et Métiers Institute of Technology
151 boulevard de l’hôpital, 75013 Paris, France.
E-mail: loiseau.jc@gmail.com
an important role in the design of thermal energy systems such as solar heating systems and nuclear reactors. Natural convection is also of crucial importance in numerous geophysical situations such as the mesoscale convective thunderstorms, land and sea breezes resulting from a differential heating between landmass and an adjacent body of water, or Hadley cells in the Earth’s atmosphere. From a theoretical point of view finally, natural convection has attracted a lot of attention ever since the seminal work of Edward Lorenz \cite{7} on the derivation of a low-order model for the Rayleigh-Bénard convection.

As in \cite{7}, many traditional model reduction techniques are analytical. They rely on prior knowledge of the Navier-Stokes equations and project them onto the span of an orthonormal basis of modes, resulting in a dynamical system in terms of the coefficients of this expansion basis. These modes may come from a classical expansion, such as Fourier modes or Tchebyshev polynomials, or they may be data-driven as in the proper orthogonal decomposition (POD) \cite{8,9}. Although such approaches have proven to be quite successful for linear systems \cite{5}, they have been applied only with limited success to obtain low-order approximations of nonlinear systems, mostly on flow oscillators \cite{10}. Lately, data-driven approaches have been becoming increasingly popular. They encompass a wide variety of different approaches such the eigenrealization algorithm \cite{11}, the dynamic mode decomposition \cite{12,13,14} and its variants, or NARMAX \cite{15}. Advances in machine learning are also greatly expanding our ability to extract governing dynamics purely from data. One can cite from instance the genetic programming-based system identification proposed by Bongard & Lipson \cite{16} and Schmidt & Lipson \cite{17} or the more recent framework based on sparsity-promoting regression techniques, SINDy, proposed by Brunton, Proctor & Kutz \cite{18}.

Considering a two-dimensional annular thermosyphon, the aim of the present work is to illustrate how these recent advances in dimensionality reduction and sparsity-promoting regression techniques can be harvested to enable the identification of a low-order Lorenz-like system able to accurately reproduce qualitatively and quantitatively the key features of chaotic natural convection in an annular thermosyphon purely from data. The present paper is organized as follow. First, the exact flow configuration considered is presented in §\ref{sec:flow_config} along with a brief overview of its chaotic dynamics. Then, §\ref{sec:dim_red} briefly introduces the reader to the dimensionality reduction and system identification techniques used, namely dynamic mode decomposition \cite{12,13} and the sparse identification of nonlinear dynamics \cite{18}. Finally, the key results of this study are presented in §\ref{sec:results} while §\ref{sec:discussion} discusses the new perspectives opened by the present work.

### 2 Flow configuration

The flow configuration considered is that of the two-dimensional thermosyphon depicted in figure 1. The flow within this annular enclosure is assumed to be incompressible and to be governed by the Navier-Stokes equations

\[
\frac{\partial \textbf{u}}{\partial t} + \nabla \cdot (\textbf{u} \otimes \textbf{u}) = -\nabla p + \Pr \nabla^2 \textbf{u} + \text{Ra} \Pr \theta \textbf{e}_z
\]

\[
\frac{\partial \theta}{\partial t} + (\textbf{u} \cdot \nabla) \theta = \nabla^2 \theta
\]

\[
\nabla \cdot \textbf{u} = 0
\]

where \(\textbf{u}(x,t)\) is the velocity field, \(p(x,t)\) is the pressure field and \(\theta(x,t)\) is the temperature one. The coupling between the momentum equation and the temperature equation is modeled from the Boussinesq approximation. The Prandtl number \(\Pr\) and the Rayleigh number
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Fig. 1 Left: Sketch of the geometry considered. The temperature of the upper walls of the thermosyphon (dashed lines) is set to $\theta = 0$ while it is set to $\theta = 1$ on the lower walls (solid lines). The gravity is acting along $-e_y$. Right: Definition of the various non-dimensional parameters defining the problem. Throughout this manuscript, the Rayleigh number is set to 17 000 while the Prandtl one is set to 5.

Ra are defined as classicaly and set to $Pr = 5$ and $Ra = 17000$ throughout the rest of this manuscript. Finally, the velocity field $u(x,t)$ satisfies no-slip boundary conditions at both walls while the temperature field obeys the following boundary condition

$$
\theta = \begin{cases} 
1 & \text{for } y \leq 0, \\
0 & \text{for } y > 0.
\end{cases}
$$

The fluid is thus heated from below and cooled from the top. The Navier-Stokes equations are solved using the spectral element code Nek5000 [19]. The mesh consists in 32 spectral elements uniformly distributed in the azimuthal direction and 8 elements uniformly distributed in the radial one. Within each element, Lagrange interpolants of order 7 based on the Gauss-Lobatto-Legendre quadrature points are used in each direction resulting in 16 384 grid points. The temporal integration relies on a third-order accurate scheme and the timestep has been chosen as to satisfy the condition CFL $\leq 0.5$ during the whole simulation.

Figure 2 shows representative snapshots of the temperature, radial velocity and azimuthal velocity field at various instants in time. These clearly highlight that, as time moves forward, the convection cell can either rotate clockwise or anti-clockwise. The dynamics of these seemingly random switches between clockwise and anti-clockwise are better visualized in figure 3, which depicts the temporal evolution of the flow rate through a given cross-section. As already noted by other authors [20,21,22], the dynamics of the flow appear quite similar to that of the chaotic Lorenz system. Based on this observation, the rest of this paper is thus devoted to the identification of a low-dimensional Lorenz-like model, purely from data, that would reproduce these dynamics. Note that, when the ratio $R_2/R_1 \to 1$, both Yorke et al. [20] and Ehrhard & Müller [21] have proposed a low-order model derived from first principles.

3 Numerical methods

This section introduces the reader to the two techniques used to (i) obtain a low-dimensional embedding of the chaotic dynamics of the high-dimensional system (see § 3.1) and (ii) to

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1 A video of these dynamics is available online at [https://tinyurl.com/y55buvnc](https://tinyurl.com/y55buvnc)
identify, within this low-dimensional space, an accurate and interpretable low-order model of the dynamics (see §3.2). It must be noted that these two sections only provide a brief overview to these techniques. For extensive details about dynamic mode decomposition, please refer to [23] while [18] provides a good introduction to SINDy.
3.1 Dynamic Mode Decomposition

Dynamic mode decomposition (DMD) is an increasingly popular dimensionality reduction that has originated from the field of fluid dynamics \[12\,13\]. Note that the derivation that follow differs from the ones in \[12\,13\] which actually provide only a suboptimal solution to the DMD problem. Considering a zero-mean sequence of evenly sampled snapshots \{\mathbf{q}(\mathbf{x}, t_k)\}_{k=1}^{n}\) of the state vector of the system under scrutiny, DMD aims at finding the low-rank linear operator \( \mathbf{A} = \mathbf{PQ}^H \), with \( \mathbf{P} \) and \( \mathbf{Q} \in \mathbb{R}^{m \times r} \) solution to the following minimization problem

\[
\minimize_{\mathbf{P}, \mathbf{Q}} \sum_{k=1}^{n-1} \| \mathbf{q}_{k+1} - \mathbf{PQ}^H \mathbf{q}_k \|_2^2
\]

subject to \( \text{rank} \mathbf{P} = r \)

\( \text{rank} \mathbf{Q} = r \),

where \( \mathbf{q}_k = \mathbf{q}(\mathbf{x}, t_k) \). Introducing the data matrix

\[
\mathbf{X} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_{n-1} \end{bmatrix}
\]

and its time-shifted counterpart

\[
\mathbf{Y} = \begin{bmatrix} \mathbf{q}_2 & \mathbf{q}_3 & \cdots & \mathbf{q}_n \end{bmatrix},
\]

the above minimization problem can be rewritten as

\[
\minimize_{\mathbf{P}, \mathbf{Q}} \| \mathbf{Y} - \mathbf{PQ}^H \mathbf{X} \|_F^2
\]

subject to \( \text{rank} \mathbf{P} = r \)

\( \text{rank} \mathbf{Q} = r \).

Although such a rank-constrained minimization problem is non-convex, it is a special case of a wider class of problems known as Reduced Rank Regression (RRR) whose properties have been studied as early as the mid 1970’s by Izenman \[24\]. It should be emphasized moreover that principal component analysis and canonical correlation analysis also fall within this class of reduced rank regression problems. Fore mode details about RRR problems, please refer to the review by De la Torre \[25\].

Defining the sample variance-covariance matrices

\[
\mathbf{C}_{xx} = \mathbf{XX}^H, \quad \mathbf{C}_{xy} = \mathbf{XY}^H \quad \text{and} \quad \mathbf{C}_{yx} = \mathbf{YX}^H,
\]
it can be shown that the low-rank matrix $P$ is solution to the following trace-maximization problem

$$\text{maximize } \text{Tr} \left( P^H C_x C_x^{-1} C_y P \right)$$

subject to rank $P = r$

$$P^H P = I.$$ 

The $P$ matrix is thus formed by the first $r$ eigenvectors of the symmetric positive definite matrix $C_x C_x^{-1} C_y$. Note that, if $X$ is full-rank, these reduce to the first $r$ left singular vectors of the output matrix $Y$. Once the $P$ matrix has been identified, the low-rank $Q$ matrix is constructed as follows

$$Q^H = P^H C_x C_x^{-1},$$

where $C_x C_x^{-1}$ is the least-squares solution to the unconstrained DMD problem. Given the low-rank structure of the DMD operator $A = PQ^H$, one can then exploit this property to easily compute its eigendecomposition, thus obtaining its right (resp. left) eigenvectors $\Phi$ (resp. $\Psi$) as well as its eigenvalues $\mu$. Note finally that various extensions of DMD have been proposed over the years, see for instance [12,13,14,26,27,28,29,30,31]. For more details about DMD and its connections to the Koopman operator [32], interested readers are referred to the recent book by Kutz et al. [23].

### 3.2 Sparse identification of nonlinear dynamics

Identifying dynamical systems from data has been a central challenge in mathematical physics. The form of the dynamics is typically either constrained via prior knowledge, as in Galerkin projection, or a particular model structure is chosen heuristically and parameters are optimized to match the data. Simultaneous identification of the model structure and parameters is considerably more challenging as there are combinatorially many possible model structures. The **sparse identification of nonlinear dynamics (SINDy)** approach [18] bypasses the intractable brute force search by leveraging the fact that many systems may be modeled as

$$\frac{da}{dt} = f(a)$$

where $a \in \mathbb{R}^n$ is the state vector of our system and the unknown $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is sparse in the space of possible right-hand side functions.

In order to identify $f(a)$, time-series data is first collected and formed into the data matrix

$$A = [a(t_1) \ a(t_2) \ \cdots \ a(t_m)]^T.$$ 

A similar matrix $\dot{A}$ of time derivatives is formed as well. As a second step, a possibly over-complete library (or dictionary) $\Theta$ of candidate nonlinear functions is constructed, e.g.

$$\Theta (A) = [1 \ A \ P_2(A) \ \cdots \ P_d(A)].$$

Here $P_d(A)$ denotes a matrix with columns vectors given by all possible time-series of $d$th degree monomials in the state $a$, e.g.

$$P_2(a_1, a_2, a_3) = [a_1^2 \ a_1 a_2 \ a_1 a_3 \ a_2^2 \ a_2 a_3 \ a_3^2].$$
Note that any basis function may be included in the library $\Theta$, albeit polynomials have proven to work well for fluid problems. The unknown dynamical system may now be represented in terms of the data matrices as

$$\dot{A} = \Theta(A)\Xi.$$ 

Each column $\Xi_k$ is a vector of coefficients determining the active terms in the equation governing the dynamics of $a_k(t)$. A parsimonious model will provide an accurate model fit with as few non-zero terms as possible in $\Xi$. Identifying this sparsity pattern may be formulated as the following optimization problem

$$\minimize_{\Xi_k} \text{card}(\Xi_k)$$
$$\text{subject to } \|\dot{A} - \Theta(A)\Xi_k\|_2^2 \leq \sigma$$
$$C\Xi_k = d$$
$$h(\Xi_k) \leq 0,$$

where $\text{card}(\Xi_k) = \|\Xi_k\|_0$ is the cardinality (or $\ell_0$ norm) of $\Xi_k$, i.e. its number of non-zero entries, and the constraint $\|\dot{A} - \Theta(A)\Xi_k\|_2^2 \leq \sigma$ quantifies the fidelity of the model with respect to the data. The linear equality constraint $C\Xi_k = d$ and the convex inequality constraint $h(\Xi_k) \leq 0$ may be used to enforce additional constraints on the identified model based on prior physical knowledge such an energy-preserving quadratic nonlinearity in [33] for instance. The above minimization problem remains however a combinatorially complex optimization problem. Various convex relaxations to this problem have been proposed over the years, most of them replacing the $\ell_0$ norm with an $\ell_1$ norm such as in LASSO regression [34]. In the rest of this work, a greedy approach is used solve directly this minimization problem rather than any of its convex relation. The algorithm used is the forward regression orthogonal least-squares (FROLS), a standard algorithm used notably for the identification of discrete-time nonlinear models within the NARMAX framework [15]. The model structures obtained by FROLS are then cross-validated using the recently introduced SR3 framework [35].

4 Results

The following two sections present the key results of the present study. First, the data generation process and its embedding into a low-dimensional subspace by DMD are discussed in §4.1. The identification of the low-order model governing the dynamics of the system within this low-dimensional subspace is discussed in §4.2. Note that it is preceded by a discussion about the properties of the classical Lorenz system which we expect our identified model to verify as well.

4.1 Low-dimensional embedding

Given a random initial condition, the Navier-Stokes equations are integrated forward in time until a statistical steady state is reached. Once reached, a sequence of 20 000 snapshots of the complete state vector of the fluctuation is collected with a sampling period $\Delta T = 2.5 \times 10^{-3}$ diffusive time units. Such a sampling period enables us to properly sample each high frequency oscillation observed in figure 3 using 20 to 30 snapshots. It must be noted
moreover that, as shown in figure 2, the flow exhibits some symmetry with respect to the vertical axis. One can thus make use of this property to virtually increase the number of snapshots by 2 while enforcing this symmetry in the training dataset itself.

When performing the DMD analysis, each state variable has been treated independently for the sake of simplicity. Following the derivation of the Lorenz system, only a rank-1 DMD expansion is used for the azimuthal velocity while the fluctuation’s temperature field is approximated using a rank-2 DMD expansion. The radial velocity component is discarded from the analysis as it plays little to no role in the dynamics. Figure 4 depicts the aforementioned DMD modes. It can be observed that the leading DMD mode for the velocity expansion captures the azimuthally invariant component of the velocity field which strongly correlates with the flow rate. Regarding the DMD modes associated to the temperature field, the first mode of this rank-2 expansion captures the temperature difference between the right and left parts of the thermosyphon while the second mode captures the temperature difference between the upper and lower parts. It is interesting to note that the spatial structures of these leading DMD modes are actually in excellent agreements with the set of measurements that were deemed necessary by Ehrhard & Müller to characterize the state of the system in their experiments, namely the flow rate, the temperature difference measured by a first set of probes at 3 and 9 o’clock and a second set at 6 and 12 o’clock.

Figure 5(a) depicts the evolution of the amplitudes $a_i(t)$ corresponding to the projection of the snapshots of the high-dimensional state vector onto the span of these DMD modes. It appears quite striking that these time-series are very similar to those generated by a chaotic Lorenz system. This is further confirmed by looking at the attractor within this low-dimensional subspace depicted in figure 5(b). In particular, the projection of this strange attractor onto the plane $(a_1, a_3)$ (rightmost subplot of figure 5b) exhibits the characteristic double-winged structure of the Lorenz attractor. It should be noted furthermore that this low-dimensional attractor appears to satisfy the same $(a_1, a_2, a_3) \rightarrow (-a_1, -a_2, a_3)$ symmetry as the Lorenz attractor. This property is of paramount importance and we expect it to be verified by the low-order model to be identified in the next section.

**Fig. 4** (a) Leading DMD for the azimuthal velocity field. (b) and (c) depict the first and second DMD modes for the temperature.
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Fig. 5 (a) Temporal evolution of the chaotic dynamics once each snapshot of the high-dimensional system is projected onto the span of the DMD modes presented in figure 4. (b) Corresponding low-dimensional embedding. Each subfigure shows a two-dimensional projection of the attractor on various planes.

4.2 Low-order modeling

The previous section has highlighted that, within the low-dimensional subspace spanned by the leading DMD modes, the chaotic attractor onto which the dynamics of the thermosyphon flow concentrate appears quite similar to the canonical Lorenz system. Hence, prior to the identification of the low-order model in §4.2.2, let us take a step back in §4.2.1 and discuss the properties of the Lorenz system.

4.2.1 A quick detour by the Lorenz system

The now-called Lorenz system has been derived by Edward Lorenz in the early 1960’s [7], originally as a simplified model of atmospheric convection. Starting from the governing equations for the two-dimensional Rayleigh-Bénard convection between two infinite flat plates, it had been shown by Saltzmann in 1962 that, if one considers free-slip boundary conditions, then the solution could be expressed as an infinite Fourier series [36]. The Lorenz system can be obtained from it by considering only a finite number of terms in this infinite Fourier series, namely only one mode for the velocity and two for the temperature. After
some simplifications, the Lorenz system eventually reads
\[
\begin{align*}
\dot{a}_1 &= \sigma (a_2 - a_1) \\
\dot{a}_2 &= a_1 (\rho - a_3) - a_2 \\
\dot{a}_3 &= a_1 a_2 - \beta a_3,
\end{align*}
\]
where the parameters \(\sigma\), \(\rho\) and \(\beta\) (all positive) are associated to the Prandtl number, the Rayleigh number and the aspect ratio of the convection cell, respectively. It is now well known that for certain sets of parameters, e.g. \((\sigma, \rho, \beta) = (10, 28, 8/3)\), the system evolves onto a \textit{strange attractor} characterized by a fractal dimension.

A very large number of studies have been dedicated to the Lorenz system over the past 50 years. The discussion that comes thus does not aim to delve extensively into all of the properties of the Lorenz system, but rather at highlighting the key properties we expect the to-be-identified low-order model to exhibit if the dynamics of thermosyphon are indeed Lorenz-like. The first of this properties, already mentionned in the previous section, is that the system is invariant with respect to the transformation
\[
(a_1, a_2, a_3) \rightarrow (-a_1, -a_2, a_3).
\]
As will be discussed in §4.2.2, this property actually limits the pool of admissible functions for the system identification problem. The second property of interest to us is that, within the chaotic regime, the system possesses three unstable fixed points given by
\[
\begin{align*}
a^{(1)} &= (0, 0, 0) \\
a^{(2)} &= (\sqrt{\beta (\rho - 1)}, \sqrt{\beta (\rho - 1)}, \rho - 1) \\
a^{(3)} &= (-\sqrt{\beta (\rho - 1)}, -\sqrt{\beta (\rho - 1)}, \rho - 1).
\end{align*}
\]
From a physical point of view, the fixed point at the origin corresponds to the pure conducting state in the Rayleigh-Bénard problem (i.e. only temperature diffusion but no flow). The other two fixed points on the other hand correspond to the clockwise or anti-clockwise steady rotation of the convection cells. The last property of interest to us is the fact that, if \(a_1 = a_2 = 0\), then the equation for \(a_3\) reduces to
\[
\dot{a}_3 = -\beta a_3
\]
whose solution is simply given by \(a_3(t) = e^{-\beta t} a_3(0)\). The \(a_3\) axis is thus always part of the stable manifold belonging to the fixed point at the origin (the pure conduction state), no matter the parameters of the system (provided \(\beta > 0\)). It should be emphasized that the Lorenz system exhibits a number of other interesting properties. The three properties just listed are however expected to be the most important ones that an accurate and interpretable low-order model of the thermosyphon should exhibit if the latter indeed is characterized by Lorenz-like dynamics as suggested by the low-dimensional embedding.

4.2.2 System identification

Now that a good understanding of the expected properties the low-order model should exhibit has been obtained, let us turn our attention to its identification. Given the time-series of the low-dimensional state vector \(a(t)\) obtained from projection of the high-dimensional
Given our choice of library $\Theta(a)$ of candidate nonlinear functions is constructed

$$
\Theta(a_1, a_2, a_3) = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_1^2 & a_1 a_2 & a_1 a_3 & a_2 a_3 & a_3^2 \end{bmatrix}.
$$

It must be emphasized that, prior to the identification step, these time-series have been standardized, i.e. transformed to have zero-mean and unit variance. As discussed in the previous section, the system appears to be invariant with respect to the transformation

$$(a_1, a_2, a_3) \rightarrow (-a_1, -a_2, a_3).$$

Rewriting the unknown system as

$$
a_1 = \Theta(a_1, a_2, a_3) \Xi_1,
$$

$$
a_2 = \Theta(a_1, a_2, a_3) \Xi_2,
$$

$$
a_3 = \Theta(a_1, a_2, a_3) \Xi_3,
$$

this invariant property implies that

$$
(\Theta(a_1, a_2, a_3) + \Theta(-a_1, -a_2, a_3)) \Xi_1 = 0,
$$

$$
(\Theta(a_1, a_2, a_3) + \Theta(-a_1, -a_2, a_3)) \Xi_2 = 0,
$$

$$
(\Theta(a_1, a_2, a_3) - \Theta(-a_1, -a_2, a_3)) \Xi_3 = 0.
$$

Given our choice of library $\Theta(a_1, a_2, a_3)$, these constraints impose a certain sparsity pattern on the vectors of coefficients $\Xi_k$, namely

$$
\Xi_1 = \begin{bmatrix} 0 & \Xi_1^{(x)} & \Xi_1^{(y)} & 0 & 0 & 0 & \Xi_1^{(xz)} & 0 & \Xi_1^{(yz)} & 0 \end{bmatrix},
$$

$$
\Xi_2 = \begin{bmatrix} 0 & \Xi_2^{(x)} & \Xi_2^{(y)} & 0 & 0 & 0 & \Xi_2^{(xz)} & 0 & \Xi_2^{(yz)} & 0 \end{bmatrix},
$$

$$
\Xi_3 = \begin{bmatrix} \Xi_3^{(x)} & 0 & \Xi_3^{(z)} & \Xi_3^{(yz)} & \Xi_3^{(y)} & 0 & \Xi_3^{(yz)} & 0 & \Xi_3^{(z)} & 0 \end{bmatrix}.
$$

It is worthy to note that these are equality constraints of the form $C_k \Xi_k = 0$ which can easily be enforced in a convex minimization problem. Similarly, the constraint that the $a_1$-axis has to belong to the stable manifold of the pure conduction state can easily be enforced by imposing the inequality constraint

$$
\Xi_3^{(c)} < 0.
$$

Combining everything together, a first low-order model is identified by simple constrained least-squares

$$
a_1 = -92.26 a_1 + 93.67 a_2 + 8.33 a_1 a_3
$$

$$
a_2 = 13 a_1 - 30 a_1 a_3 - 6.84 a_2 a_3
$$

$$
a_3 = -42.92 - 15.62 a_3 + 41.25 a_1 a_2 + 5.07 a_2^2.
$$

A few observations need to be made. First of all, a few terms appear to have comparatively small coefficients compared to the others, suggesting that a simpler model could be identified by replacing the constrained least-squares procedure with a constrained sparsity-promoting procedure as discussed in §3.2. Additionally, the first two coefficients in the equation governing the dynamics of $a_1$ seem to be of the same magnitude, albeit of opposite signs, as is observed for the first equation of the Lorenz system. In order once again to further simplify
the model, these two coefficients can be forced to be of equal magnitude and opposite sign by supplementing the convex minimization problem with the following equality constraint

\[
\Xi^{(1)}_1 + \Xi^{(2)}_1 = 0.
\]

Re-identifying the model using this additional constraint and a sparsity-promoting technique finally leads to the following low-order model

\[
\begin{align*}
\dot{a}_1 &= 96.65(a_2 - a_1) + 9.73a_1a_3 \\
\dot{a}_2 &= 14.62a_1 - 35.7a_1a_3 \\
\dot{a}_3 &= -17a_3 + 47.12a_1a_2 - 43.25.
\end{align*}
\]

Clearly, the identified model shares strong similarities with the classical Lorenz model, with the exception of having two additional terms. While the constant term in the equation governing the dynamics of \(a_3\) simply corresponds to a shift of the origin of the phase space, the extra \(a_1a_3\) term in the equation governing the dynamics of \(a_1\) suggests a weak coupling between the flow rate and the temperature difference between the upper and lower parts of the thermosyphon due to the onset of convection. A precise investigation of the properties and physical significance of this model is however beyond the scope of the present paper.

As to cut this long story short, figure 6 depicts a visual comparison of the time evolution of the state vector \(\mathbf{a}(t)\) obtained by projection of the high-dimensional onto the span of the DMD modes and that predicted by the identified low-order model. Because the system is chaotic, matching exactly the true dynamics is impossible. Nonetheless, it is quite clear from this figure that an excellent agreement is obtained regarding the statistical and dynamical properties of the two systems as suggested notably by the overlap of the true and identified strange attractors in figure 6(b). Finally, note that one could easily show that the identified model possesses three unstable fixed points corresponding to the pure conduction state as well as the clockwise and anti-clockwise rotation of the convection cell.

5 Conclusion and perspectives

Identifying accurate yet interpretable low-order model from data has been an overarching goal in mathematical physics which has gained a renewed interest over the past decade with the advances made in dimensionality reduction and sparsity-promoting regression techniques. In this short paper, we have illustrated how such techniques, namely dynamic mode decomposition [13] and SINDy [18], could be used to identify a low-order model of the chaotic thermal convection taking place in an annular thermosyphon at sufficiently high Rayleigh number. A particular emphasis has been put on the necessity for the low-dimensional embedding and the identified system to comply with prior physical knowledge. To do so, the construction of the low-order model proposed herein closely follows the derivation of the canonical Lorenz system from the Oberbeck-Boussinesq equations modeling the Rayleigh-Bénard convection between two infinite flat plates. Given the simplicity of the identified model compared to the numerical simulation of the original high-dimensional system, excellent qualitative and quantitative agreements have been obtained as assessed by the similar structure of the strange attractor onto which the dynamics evolve in both situations.

Our ability to identify low-order models from data stemming from a high-dimensional system with various practical applications opens new possibilities. In particular, both dynamic mode decomposition and SINDy can easily be extended to include external input [38, 39], enabling us to identify accurate low-order models for control purposes. These models
Fig. 6 (a) Comparison of the true evolution of $u(t)$ (dark gray) obtained by projection of the high-dimensional snapshots onto the span of the leading DMD modes and the one predicted by the identified low-order model (light green). (b) Corresponding low-dimensional embedding for the true data and strange attractor of the identified model.

could moreover be used in conjunction with optimal sensor placement \[40,41\] to enable real-time monitoring and estimation of the system. Finally, using tool from model predictive control, such low-order models could be used to strengthen or decrease the chaotic dynamics depending on the applications. A first step toward this goal has been achieved by Kaiser et al. \[42\].

Acknowledgements This work is adapted from a contribution to the long program Machine Learning for Physics and the Physics of learning organized by the Institute for Pure and Applied Mathematics at UCLA (Los Angeles, USA) in 2019 and made possible thanks to the financial support of IPAM. The authors would also like to gratefully thank Steven Brunton, Kathleen Champion, Onofrio Semeraro, Alessandro Bucci and many others for the fruitful discussions on this topic.

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