Perfect state transfer over interacting boson networks associated with group schemes

M. A. Jafarizadeh$^{a,b,c}$ *, R. Sufiani$^{a,b}$ †, M. Azimi$^a$ and F. Eghbali Fam$^a$

$^a$Department of Theoretical Physics and Astrophysics, University of Tabriz, Tabriz 51664, Iran.

$^b$Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran.

$^c$Research Institute for Fundamental Sciences, Tabriz 51664, Iran.

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*E-mail:jafarizadeh@tabrizu.ac.ir

†E-mail:sofiani@tabrizu.ac.ir
Abstract

It is shown how to perfectly transfer an arbitrary qudit state in interacting boson networks. By defining a family of Hamiltonians related to Bose-Hubbard model, we describe a possible method for state transfer through bosonic atoms trapped in these networks with different kinds of coupling strengths between them. Particularly, by taking the underlying networks of so called group schemes as interacting boson networks, we show how choose suitable coupling strengths between the nodes, in order that an arbitrary qudit state be transferred from one node to its antipode, perfectly. In fact, by employing the group theory properties of these networks, an explicit formula for suitable coupling strengths has been given in order that perfect state transfer (PST) be achieved. Finally, as examples, PST on the underlying networks associated with cyclic group $C_{2m}$, dihedral group $D_{2n}$, Clifford group $CL(n)$, and the groups $U_{6n}$ and $V_{8n}$ has been considered in details.

Keywords: Bose-Hubbard Hamiltonian, Interacting boson networks, Perfect state transfer (PST), Qudit state, Underlying networks, Group schemes

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1 Introduction

Transfer a quantum state from one site to another one (quantum communication between two parts of a physical unit, e.g., a qubit) is a crucial ingredient for many quantum information processing protocols (QIP) [1]. However, in many other tasks, e.g., solid-state-based quantum computation, quantum state transmission is not a trivial task. Completing such a task is often needed in QIP, e.g., the quantum information exchange between two separate processors. Therefore, it is very important to find physical systems that provide robust quantum state transmission lines linking different QIP processors. There are various physical systems that can serve as quantum channels, one of them being a quantum spin system. In recent years, many results have been obtained on qubit-state transfer through spin chains with various types of neighbor couplings. The original idea of quantum state transfer through a system of interacting spin $1/2$ was introduced by Bose [2, 3]. Afterward, Christandl et al. [4, 5] and independently Nikolopoulos et al. [6] found that perfect state transfer (PST) is possible [7] in spin-1/2 networks without any additional actions from senders and receivers, including not requiring switching on and off qubit couplings. Similarly, in Ref. [8], near perfect state transfer was achieved for uniform couplings provided a spatially varying magnetic field was introduced. After the work of Bose [2], in which the potentialities of the so-called spin chains have been shown, several strategies were proposed to increase the transmission fidelity [9] and even to achieve, under appropriate conditions, perfect state transfer [4, 5, 10, 11, 12, 13, 14]. Recently, A. Bernasconi, et al. [15] have studied perfect state transfer (PST) between two particles in quantum networks modeled by a large class of cubelike graphs. Since quantum networks (and communication networks in general) are naturally associated to undirected graphs, one can use the relation between graph-theoretic properties and properties that allow PST [16].

In Ref. [17], the so called distance-regular graphs have been considered as spin networks (in the sense that with each vertex of a distance-regular graph a qubit or a spin $1/2$ particle
was associated) and PST over them has been investigated. In the paper [18], optimal state transfer (ST) of a $d$-level quantum state (qudit) over pseudo distance regular networks was investigated, where it was shown that only for pseudo distance regular networks with some certain symmetry (mirror symmetry) in the corresponding intersection numbers, PST between antipodes of the networks can be achieved. Also, in the recent paper [19], PST of a qudit state in a system of $N$ spin $j$ particles located at the nodes of an underlying network of a so-called group association scheme [20] has been studied. In paper Ref. [21], PST of any qudit state through bosonic lattices has been investigated. They have considered a model which can be implemented using the Bose-Hubbard model and proposed a protocol to perfectly transfer an unknown $n$-variable function from a processor at one end of a boson chain to another processor at the other.

In this work, we consider the underlying networks of group schemes, as interacting boson networks and impose a more general Bose-Hubbard Hamiltonian to the considered networks. More clearly, for a given vertex set (each vertex is associated with an element of a finite group $G$) we define different adjacency matrices according to different kinds of coupling strengths between the vertices (nodes). Then, we investigate perfect state transfer (PST) on the nodes of these networks and show that by choosing suitable coupling strengths between the nodes, an arbitrary qudit state can be transferred from one node to its antipode, perfectly. In fact, by employing the group theory properties of these networks, we give an explicit formula for suitable coupling strengths in terms of irreducible characters of the corresponding groups. As examples, we consider PST on the underlying networks associated with cyclic group $C_{2m}$, dihedral group $D_{2n}$, Clifford group $CL(n)$, and the groups $U_{6n}$ and $V_{8n}$, in details.

The organization of the paper is as follows: In section 2, PST over antipodes of interacting boson networks is investigated, where a method for finding suitable coupling constants in particular Bose-Hubbard Hamiltonians so that PST be possible, is given. Section 3 is devoted to PST on special networks called the underlying networks of group schemes, where a formula
for suitable coupling strengths between nodes is given in order that PST of a qudit state from an arbitrary node to its antipode be achieved. In section 4, some important examples of underlying networks of group schemes are considered and PST over them is investigated in details. The paper is ended with a brief conclusion.

2 Perfect state transfer in interacting boson networks

Consider the Bose-Hubbard model in which dynamics of bosons, in a system with $N$ sites, governed by a linearly coupled bosonic Hamiltonian as

$$H = \sum_{k=1}^{N-1} J_k (b_k^\dagger b_{k+1} + b_{k+1}^\dagger b_k) + \sum_{k=1}^N \epsilon_k n_k$$  \hspace{1cm} (2-1)

where, $n_k = b_k^\dagger b_k$ is the number operator for the bosons located at the $k$th site, $b_k^\dagger$ ($b_k$) is the bosonic creation (annihilation) operator. For simplicity, $\hbar = 1$ has considered.

We generalize the above model to more general finite graphs not only for a finite path, in a way that the nodes of graph have different kinds of coupling strengths between themselves. Suppose $\Gamma$ is a connected graph. For each coupling strength $J_k$, $k = 0, 1, \ldots, d$, we can form a graph $\Gamma_k$ in which vertices are adjacent if their coupling in $\Gamma$ equals $k$. Let $A_k$ be the adjacency matrix of $\Gamma_k$. For instance, $A_1$ is the adjacency matrix $A$ of $\Gamma$. Also, let $A_0 = I$, the identity matrix. This gives us $d + 1$ matrices $A_0, A_1, \ldots, A_d$, called the adjacency matrices of $\Gamma$. Their sum is the matrix $J$ in which every entry is 1. In the other words, we assume that the dynamics of bosons, in a system with $N$ sites (associated with the nodes of a finite group), is governed by the following Bose-Hubbard Hamiltonian

$$H = \sum_{i,j=1}^N \sum_{k=0}^d J_k (A_k)_{ij} b_i^\dagger b_j. $$  \hspace{1cm} (2-2)

Now, we assume that the adjacency matrices $A_k$ for $k = 0, 1, \ldots, d$ commute with each other such that, one can diagonalize them by a unitary matrix $U$, simultaneously. That is, we
have
\[ UA_k U^\dagger = D_k, \]
where \( D_k = \text{diag}(\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_N^{(k)}) \) is a diagonal matrix with eigenvalues of \( A_k \) as its diagonal entries. Therefore, the Hamiltonian (2-2) can be rewritten as
\[
H = \sum_{i,j} \sum_k J_k (U^\dagger D_k U)_{i,j} b_i^\dagger b_j = \sum_{k=0}^d \sum_{l=1}^N J_k \lambda_l^{(k)} \sum_i U_{il}^\dagger b_i^\dagger \sum_j U_{lj} b_j \tag{2-3}
\]
By considering the Bogoliubov transformations
\[
\tilde{b}_l = \sum_j U_{lj} b_j, \quad b_l = \sum_j U_{jl}^* \tilde{b}_j, \quad l = 1, 2, \ldots, N \tag{2-4}
\]
and defining \( \tilde{n}_l = \tilde{b}_l^\dagger \tilde{b}_l \), the Hamiltonian (2-3) is written as
\[
H = \sum_{l=1}^N \tilde{J}_l \tilde{n}_l, \tag{2-5}
\]
where,
\[
\tilde{J}_l = \sum_{k=0}^d J_k \lambda_l^{(k)}. \tag{2-6}
\]
Now, assume that we are able to prepare the initial state \( |\phi_0\rangle = \alpha |0\rangle + \beta b_1^\dagger |0\rangle \) (with \( |\alpha|^2 + |\beta|^2 = 1 \) and \( |0\rangle := |00\ldots00\rangle \) as the vacuum state). That is, we consider that only the first site is occupied. Then, the network couplings are switched on and the system is allowed to evolve under \( U(t) = e^{-iHt} \) for a fixed time interval, say \( t_0 \). The final state becomes
\[
|\phi(t_0)\rangle = e^{-iHt_0} |\phi_0\rangle = \alpha |0\rangle + \beta \sum_{j=1}^N f_{j1}(t_0) b_j^\dagger |0\rangle \tag{2-7}
\]
where, \( f_{j1}(t_0) := \langle 0 | b_j e^{-it_0} \sum_{l=1}^N \tilde{J}_l \tilde{n}_l |0\rangle \). We consider perfect state transfer, i.e., we want to transfer informations coded in the first site (starting site) to the site \( m \) (target site), perfectly. This means that, we impose the condition
\[
|f_{m1}(t_0)| = 1 \quad \text{for some} \quad 0 < t_0 < \infty \tag{2-8}
\]
which can be interpreted as the signature of perfect communication (or PST) between sites 1 and \( m \) in time \( t_0 \). In order to achieve this condition, we use the identity

\[
b_1^\dagger = \sum_i U_{i1} \tilde{b}_i^\dagger = \sum_i \tilde{b}_i^\dagger
\]

to rewrite the evolved state (2-7) as

\[
|\phi(t_0)\rangle = \alpha|0\rangle + \beta \sum_{l=1}^N e^{-it_0 \tilde{J}^l_1} \tilde{b}_l^\dagger |0\rangle = \alpha|0\rangle + \beta \sum_{j,l=1}^N e^{-it_0 \tilde{J}^l_j} U_{lj}^* b_j^\dagger |0\rangle
\]

where, we have used the Eq.(2-4). By defining column matrix \( \tilde{J} \) as

\[
\tilde{J} = \begin{pmatrix}
e^{-it_0 \tilde{J}^1_1} \\
e^{-it_0 \tilde{J}^2_2} \\
\vdots \\
e^{-it_0 \tilde{J}^N_N}
\end{pmatrix}
\]

(2-10)

the final state \( |\phi(t_0)\rangle \) can be written as

\[
|\phi(t_0)\rangle = \alpha|0\rangle + \beta \sum_j (\sum_i U_{ji}^\dagger \tilde{J}^l_i) b_j^\dagger |0\rangle = \alpha|0\rangle + \beta \sum_{j=1}^N (U_{ji}^\dagger \tilde{J}^l_j) b_j^\dagger |0\rangle
\]

(2-11)

By comparing the above equation with (2-7) we see that

\[
f_{j1}(t_0) = (U^\dagger \tilde{J})_j = \sum_{l=1}^N U_{jl}^\dagger e^{-it_0 \tilde{J}^l_j}.
\]

(2-12)

Now, in order to achieve PST to the \( m \)-th site, it is sufficient to have

\[
(U^\dagger \tilde{J})_i = e^{i\theta} \delta_{im}.
\]

(2-13)

In order to obtain some informations about the matrix \( U \), we project the Hamiltonian \( H \) to the single particle subspace, in which the sites are empty or occupied by only one boson. Then, by defining the kets \( |i_1, i_2, ..., i_N\rangle \) with \( i_1, ..., i_N \in \{0, 1\} \) as an orthonormal basis for Hilbert space, one can easily see that

\[
\begin{align*}
    b_j^\dagger b_j |0_{j} \ldots 0_{i} \ldots \rangle &= b_j^\dagger b_j |1_{j} \ldots 1_{i} \ldots \rangle = 0, \\
    b_j^\dagger b_j |0_{i} \ldots 0_{j} \ldots \rangle &= |0_{i} \ldots 1_{j} \ldots \rangle.
\end{align*}
\]

(2-14)
where, we have used the facts that $b|1\rangle = |0\rangle$, $b|0\rangle = 0$, $b^\dagger|0\rangle = |1\rangle$ and $b^\dagger|1\rangle = 0$. Now, let $|k\rangle$ denotes the vector state which its all components are 0 except for $k$, i.e., $|k\rangle = b^\dagger_k|0\rangle = |0...0\rangle_k$. Then, it can be easily seen that
\[ b^\dagger_i b_j |k\rangle = \delta_{jk} |i\rangle \rightarrow b^\dagger_i b_j = E_{ij}, \] (2-15)
where, $E_{ij}$ is an $n \times n$ matrix all of whose elements are zero except the $(i,j)$ element which is unity, i.e., $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Then, one can easily deduce that
\[ \sum_{i \sim_k j} b^\dagger_i b_j = \sum_{i \sim_k j} E_{ij} = A_k, \] (2-16)
where, $i \sim_k j$ means $i$ and $j$ are adjacent in the graph $\Gamma_k$ or equivalently, the coupling strength between $i$ and $j$ is $J_k$. Then, by using (2-16), the hamiltonian in (2-2) can be written in terms of the adjacency matrices $A_k$, $k = 0, 1, ..., d$ as follows
\[ H = \sum_{k=0}^{d} J_k \sum_{i \sim_k j} E_{ij} = \sum_{k=0}^{d} J_k A_k. \] (2-17)
Assume that we are able to obtain the spectrum of $A_k$ so that we can write $A_k = \sum_l \lambda_l^{(k)} E_l$ ($E_l$ is the projection operator to the subspace associated with the eigenvalue $\lambda_l^{(k)}$). Then, we have
\[ f_{j1}(t_0) = \langle 0|b_j e^{-iH_0 t_0} b_1^\dagger |0\rangle = \sum_{l} \langle 0|b_j e^{-i\lambda_l t_0} \sum_k J_k \lambda_l^{(k)} E_l b_1^\dagger |0\rangle = \sum_{l} e^{-i\lambda_l t_0} \langle 0|b_j E_l b_1^\dagger |0\rangle. \]
By comparing with (2-12), we obtain
\[ U_{jl}^\dagger = \langle 0|b_j E_l b_1^\dagger |0\rangle = \langle j|E_l|1\rangle. \] (2-18)

### 2.1 Generalization to the PST of a qudit state

Assume that we can prepare a $d$-level quantum state (qudit) as
\[ |\phi_0\rangle = \alpha_0|0\rangle + \alpha_1 b_1^\dagger |0\rangle + \alpha_2 (b_1^\dagger)^2 |0\rangle + \ldots + \alpha_d (b_1^\dagger)^d |0\rangle. \] (2-19)
Now, using the equations (2-4) and (2-5), one can write

\[ e^{-iHt}(b_i^\dagger)^i|0\rangle = e^{-iHt} \sum_{l_1,\ldots,l_i} \tilde{b}_{l_1}^\dagger \tilde{b}_{l_2}^\dagger \ldots \tilde{b}_{l_i}^\dagger |0\rangle = \sum_{l_1,\ldots,l_i} e^{-it} \tilde{J}_k \tilde{b}_{l_1}^\dagger \tilde{b}_{l_2}^\dagger \ldots \tilde{b}_{l_i}^\dagger |0\rangle = \sum_{l_1,\ldots,l_i} \left( \sum_{k_1} e^{-it\tilde{J}_k} U_{l_1,k_1}^\dagger \right) \left( \sum_{k_2} e^{-it\tilde{J}_k} U_{l_2,k_2}^\dagger \right) \ldots \left( \sum_{k_i} e^{-it\tilde{J}_k} U_{l_i,k_i}^\dagger \right) \tilde{b}_{l_1}^\dagger \tilde{b}_{l_2}^\dagger \ldots \tilde{b}_{l_i}^\dagger |0\rangle \]

where, \( \tilde{J} \) has defined by (2-10). Therefore, the final state of the system is given by

\[ |\phi_0(t)\rangle = e^{-iHt}|\phi_0\rangle = \alpha_0 |0\rangle + \alpha_1 \sum_{k_1} (U^\dagger \tilde{J})_{k_1} \tilde{b}_{k_1}^\dagger |0\rangle + \alpha_2 \sum_{k_1,k_2} (U^\dagger \tilde{J})_{k_1} (U^\dagger \tilde{J})_{k_2} \tilde{b}_{k_1}^\dagger \tilde{b}_{k_2}^\dagger |0\rangle + \ldots + \alpha_d \sum_{k_1,\ldots,k_d} (U^\dagger \tilde{J})_{k_1} \ldots (U^\dagger \tilde{J})_{k_d} \tilde{b}_{k_1}^\dagger \ldots \tilde{b}_{k_d}^\dagger |0\rangle \]  \hspace{1cm} (2-20)

Now, in order that PST from the first site to the \( m \)-th one be achieved, i.e., we can obtain the evolved state \( |\phi_0(t)\rangle \) as

\[ |\phi_0(t)\rangle = \alpha_0 |0\rangle + \alpha_1 \tilde{b}_m^\dagger |0\rangle + \alpha_2 (\tilde{b}_m^\dagger)^2 |0\rangle + \ldots + \alpha_d (\tilde{b}_m^\dagger)^d |0\rangle, \]  \hspace{1cm} (2-21)

we should have the constraint

\[ (U^\dagger \tilde{J})_i = e^{it\delta_{im}} \]  \hspace{1cm} (2-22)

which is the same condition obtained for the purpose of PST of a qubit \( (d = 2) \). This indicates that, by choosing suitable coupling strengths \( J_l, l = 0, 1, \ldots, d \), one can transfer a qubit, a qudit, and in general a qudit, simultaneously.

3 Underlying networks of group schemes

Now, we consider some special graphs which are defined via a finite group \( G \). These graphs have the preference that, the adjacency matrices \( A_k \) are simultaneously diagonalizable and the needed information about the matrix \( U \) can be obtained via the group characters. In order to define these graphs, first we recall the notion of an association scheme. For more details about association schemes and their underlying networks, refer to [19], [20], [22] and [23].
Assume that $V$ and $E$ are vertex and edge sets of a regular graph, respectively. Then, the matrices $A_i$ for $i = 0, 1, \ldots, d$ form a commutative association scheme with diameter $d$ if

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k, \quad (3-23)$$

$A_0 = I$, and the sum of $A_i$ is the all-one matrix $J$. From (3-23), it is seen that the adjacency matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative algebra $A$ known as the Bose-Mesner algebra of the association scheme. This algebra has a second basis $E_0, \ldots, E_d$ (known as primitive idempotents) so that

$$E_0 = \frac{1}{N} J, \quad E_i E_j = \delta_{ij} E_i, \quad \sum_{i=0}^{d} E_i = I. \quad (3-24)$$

Let $P$ and $Q$ be the matrices relating the two bases for $A$:

$$A_i = \sum_{j=0}^{d} P_{ij} E_j, \quad 0 \leq i \leq d,$$

$$E_i = \frac{1}{N} \sum_{j=0}^{d} Q_{ij} A_j, \quad 0 \leq i \leq d. \quad (3-25)$$

Then, clearly we have

$$A_i E_j = P_{ij} E_j,$$

$$PQ = QP = NI. \quad (3-26)$$

which shows that the $P_{ij}$ is the $j$-th eigenvalue of $A_i$ and that the columns of $E_j$ are the corresponding eigenvectors. Thus, $m_i = \text{rank}(E_i)$ is the multiplicity of the eigenvalue $P_{ij}$ of $A_i$ (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$).

### 3.1 Group association schemes

Group schemes are particular association schemes for which the vertex set contains elements of a finite group $G$ and the $i$-th adjacency matrix $A_i$ is defined as:

$$A_i = \bar{C}_i := \sum_{g \in \bar{C}_i} g,$$
where $C_0 = \{e\}, C_1, ..., C_d$ are the conjugacy classes of $G$ and $g$ is considered in the regular representation of the group. The corresponding idempotents $E_0, ..., E_d$ are the projection operators as

$$E_k = \frac{\chi_k(1)}{|G|} \sum_{\alpha \in G} \chi_k(\alpha^{-1}) \alpha$$

(3-27)

where, $\chi_k$ is the $k$th irreducible character of $G$. Thus eigenvalues of adjacency matrices $A_k$ and idempotents $E_k$ are given by

$$P_{ik} = \frac{\kappa_i \chi_k(\alpha_i)}{d_k}, \quad Q_{ik} = d_i \chi_i(\alpha_k)$$

(3-28)

respectively, where $d_j = \chi_j(1)$ is the dimension of the irreducible character $\chi_j$ and $\kappa_k \equiv |C_k|$ is the $k$th valency of the graph.

By using (3-25) and (3-28), the result (2-18) is written as

$$U^*_{lm} = \langle m|E_l|\phi_0 \rangle = \frac{1}{|G|} Q_{lm} = \frac{1}{|G|} d_i \bar{\chi}_l(\alpha_m)$$

(3-29)

Then, the PST condition (2-13) takes the form

$$\sum_{l=0}^{d} U^*_{lm} \tilde{J}_l = \frac{1}{|G|} \sum_{l=0}^{d} d_i \bar{\chi}_l(\alpha_m) e^{-i\theta_0 \tilde{J}_l} = e^{i\theta}.$$  

(3-30)

Or equivalently,

$$\frac{1}{|G|} \sum_{l=0}^{d} d_i \bar{\chi}_l(\alpha_m) e^{-i(\theta_0 \tilde{J}_l + \theta)} = 1.$$  

(3-31)

Using the fact that $\sum_{l=0}^{d} d_l^2 = |G|$, the Eq.(3-31) gives

$$e^{-i(\theta_0 \tilde{J}_l + \theta)} = \frac{d_l}{\bar{\chi}_l(\alpha_m)}.$$  

(3-32)

It should be noticed that, $\alpha_m \in C_m$ belongs to the center of the group (and so commutes with all elements of the group); Then, the well known Schur’s lemma in the group representation theory implies that $\alpha_m$ is represented by $\mu 1$ for some $\mu \in \mathcal{C}$. Assume that $\alpha_m$ has order $r$ ($r$ is the smallest positive integer for which, we have $\alpha^r_m = 1$). Then, clearly $\mu$ must be the $r$-th root of unity, i.e., $\mu = e^{2\pi i/r}$ and so $|\mu| = 1$. consequently, we have $|\chi_k(\alpha_m)| = |\mu|d_k = d_k$ and
from the fact that \( \chi_k(\alpha_m) \) is real, then we have \( \frac{d}{\chi_l(\alpha_m)} = \pm 1 = e^{-i\pi n_l}, \ n_l \in \mathbb{Z} \). Then, the result (3-32) gives

\[
\tilde{J}_l = \frac{\pi n_l - \theta}{t_0}, \ \ l = 0, 1, \ldots, d; \ n_l \in \mathbb{Z}.
\] (3-33)

From the fact that the adjacency matrices \( A_k \) possess \( d + 1 \) distinct eigenvalues (see Eq.(3-26)) given by \( \lambda_l^{(k)} = P_{kl} = \frac{\kappa_k \chi_l(\alpha_k)}{d_l} \) for \( l = 0, 1, \ldots, d \), we have only \( d + 1 \) distinct \( \tilde{J}_l \) given by

\[
\tilde{J}_l = \sum_{k=0}^{d} J_k P_{kl}.
\] (3-34)

The above equation implies that

\[
\begin{pmatrix}
J_0 \\
\tilde{J}_1 \\
\vdots \\
\tilde{J}_d
\end{pmatrix}
= P^t
\begin{pmatrix}
J_0 \\
J_1 \\
\vdots \\
J_d
\end{pmatrix}.
\] (3-35)

By using the invertibility of the eigenvalue matrix \( P \) (see Eq. (3-26)), the above equation leads to the following relation for coupling strengths \( J_k \):

\[
J_l = \sum_{k=0}^{d} (P^{-1})_{lk} \tilde{J}_k = \frac{1}{|G|} \sum_{k=0}^{d} (Q^t)_{lk} \tilde{J}_k = \frac{1}{|G| t_0} \sum_{k=0}^{d} d_k \bar{\chi}_k(\alpha_l) \tilde{J}_k, \ l = 0, 1, \ldots, d; \ n_l \in \mathbb{Z}.
\] (3-36)

where, we have used the equations (3-26) and (3-28). Now, using the equation (3-33) we obtain an explicit formula for suitable coupling strengths \( J_l \) as

\[
J_l = \frac{1}{|G| t_0} \sum_{k=0}^{d} (\pi n_k - \theta) d_k \bar{\chi}_k(\alpha_l), \ l = 0, 1, \ldots, d; \ n_k \in \mathbb{Z}.
\] (3-37)

4 Examples

4.1 Cyclic graph \( C_{2m} \)

The undirected cyclic graph \( C_{2m} \) with \( 2m \) nodes have the following adjacency matrices

\[
A_0 = I, \ A_i = S^i + S^{-i}, \ i = 1, 2, \ldots, m - 1 ; \ A_m = S^m,
\]
where $S$ is the circulant matrix of order $n = 2m$, i.e., $S^{2m} = I$. Then, it is well known that the finite Fourier transform $F_n$ with matrix entries $(F_n)_{kl} = \frac{1}{\sqrt{n}} e^{2\pi i kl/n}$ diagonalizes the above adjacency matrices, simultaneously. Therefore, the Bose-Hubbard Hamiltonian (2-2) is given by

$$H = \sum_{i,j=1}^{2m} (J_0 I + \sum_{k=1}^{m-1} J_k (S^k + S^{-k}) + J_m S^m) b_i \dagger b_j = \sum_{i,j=1}^{2m} (J_0 I + \sum_{k=1}^{m-1} J_k (\omega^k + \omega^{-k}) + J_m \omega^m) b_i \dagger b_j = \sum_{i=0}^{2m-1} \sum_{k=1}^{m-1} (J_0 + 2 \sum_{k=1}^{m-1} J_k \cos \frac{\pi ki}{m} + J_m (-1)^i) b_i \dagger b_i = \sum_{i=0}^{2m-1} \tilde{J}_i \tilde{n}_i.$$ 

According to (2-4), we have $\tilde{b}_i \dagger = \frac{1}{\sqrt{2m}} \sum_k \omega^{-ik} b_k \dagger$. Then, for the purpose of PST, it suffices that the probability amplitude

$$\langle m | \frac{1}{2m} \sum_{i,k} \omega^{-ik} e^{-ita} \tilde{J}_i | k \rangle = \frac{1}{2m} \sum_{l=0}^{2m-1} \omega^{-lm} e^{-ita} \tilde{n}_l.$$ 

be an arbitrary phase $e^{i\theta}$. This leads to

$$\tilde{J}_l = \frac{-\pi l - \theta}{t_0},$$

and so, we obtain

$$J_l = \frac{1}{2mt_0} \{ -\theta + 2 \sum_{k=1}^{m-1} (-\pi k - \theta) \cos \frac{\pi kl}{m} - (-1)^l (\pi m + \theta) \}.$$ 

### 4.2 Dihedral group $D_{2n}$

The dihedral group $G = D_{2n}$ is generated by two generators $a$ and $b$ with the following relations:

$$D_{2n} = \langle a, b : a^n = 1, b^2 = 1, b^{-1} ab = a^{-1} \rangle$$  \hspace{1cm} (4-38)
We consider the case of even \( n = 2m \), the case of odd \( n \) can be considered similarly. The Dihedral group \( D_{2n} \) with even \( n = 2m \) has \( m + 3 \) conjugacy class so that \( C_m = \{a^m\} \).

\[
C_0 = \{e\}, \quad C_i = \{a^i, a^{-i}\}, \quad i = 1, \ldots, m - 1, \quad C_m = \{a^m\}, \quad C_{m+1} = \{a^{2j}b, 0 \leq j \leq m - 1\}, \quad C_{m+2} = \{a^{2j+1}b, 0 \leq j \leq m - 1\}
\]

Then, the group scheme \( D_{2n} \) with \( n = 2m \) have the following adjacency matrices

\[
A_0 = I_2 \otimes I_n, \quad A_i = I_2 \otimes (S^i + S^{-i}), \quad i = 1, 2, \ldots, m - 1; \quad A_m = I_2 \otimes S^m,
\]

\[
A_{m+1} = \sigma_x \otimes (I_n + S^2 + \ldots + S^{2(m-1)}), \quad A_{m+2} = \sigma_x \otimes (S + S^3 + \ldots + S^{2m-1}).
\]

where \( S \) is the circulant matrix of order \( 2m \), i.e., \( S^{2m} = I \). The character table of \( D_{2n} \) with \( n = 2m \) is given by [24]

| \( D_{2n} \) | \( e \) | \( a^m \) | \( a^r \) \((1 \leq r \leq m - 1)\) | \( b \) | \( ab \) |
| --- | --- | --- | --- | --- | --- |
| \( \chi_0 \) | 1 | 1 | 1 | 1 | 1 |
| \( \chi_1 \) | 1 | 1 | 1 | -1 | -1 |
| \( \chi_2 \) | 1 | \((-1)^m\) | \((-1)^r\) | 1 | -1 |
| \( \chi_3 \) | 1 | \((-1)^m\) | \((-1)^r\) | -1 | 1 |
| \( \psi_j \) \((1 \leq j \leq m - 1)\) | 2 | \((-1)^j\) | \(2\cos(2\pi j r/n)\) | 0 | 0 |

Then, by using the result (3-37), we obtain

\[
J_0 = -\frac{(m + 3)(\theta + \pi m/2)}{2nt_0},
\]

\[
J_1 = \frac{1}{2nt_0}\{- (m + 1)\theta - \pi (m - 2)(m - 3 + 6)\},
\]

\[
J_2 = J_3 = \frac{1}{2nt_0}\{- \theta((-1)^m + \sum_{r=1}^{m-1}(-1)^r) + \pi (1 - \sum_{j=1}^{m-3}(-1)^j j)\}
\]

\[
J_{l+3} = \frac{2}{nt_0}\{- \theta(1 + (-1)^l + \sum_{r=1}^{m-1}\cos \frac{2\pi lr}{n}) - \pi \sum_{r=1}^{m-1} r \cos \frac{2\pi lr}{n}\},
\]

\[
l = 1, 2, \ldots, m - 1.
\]

\[(4-39)\]
4.3 Clifford group

The Clifford algebra with \( n \) generator matrices \( \gamma_1, \gamma_2, \ldots, \gamma_n \), obeys the following relations \([25]\)

\[
\gamma_i \gamma_j + \gamma_j \gamma_i = 2 \delta_{ij} I
\]

(4-40)

Thus, the \( \gamma \)'s have square 1 and anti-commute. The Clifford group denoted by \( CL(n) \) has \( 2^{n+1} \) elements as

\[
CL(n) = \{ \pm 1, \pm \gamma_{i_1} \ldots \gamma_{i_j}; \quad i_1 < \ldots < i_j, j = 1, \ldots, n \},
\]

where, \( i_r \in \{1, 2, \ldots, n\} \). We suppose \( n > 2 \) throughout. It is well known that \([25]\), the center of \( CL(n) \) denoted by \( Z(CL(n)) \), consists of \( \{ \pm 1 \} \) if \( n \) is even and \( \{ \pm 1, \pm \gamma_{1} \ldots \gamma_{n} \} \) if \( n \) is odd. \( CL(n) \) has \( 2^n \) one-dimensional representations, each real. In each such representation, \( U(-1) = I \); Any irreducible representation with dimension greater than 1 has \( U(-1) = -I \).

For even \( n \), the conjugacy classes are given by

\[
C_0 = \{1\}, \quad C_1 = \{-1\}, \quad C_2 = \{\gamma_1, -\gamma_1\}, \ldots, \quad C_j = \{\gamma_{i_1} \ldots \gamma_{i_j}, -\gamma_{i_1} \ldots \gamma_{i_j}\},
\]

\[
C_{2n+1} = \{\gamma_{1} \ldots \gamma_{n}, -\gamma_{1} \ldots \gamma_{1}\},
\]

whereas for odd \( n \), we have

\[
C_0 = \{1\}, \quad C_1 = \{-1\}, \quad C_2 = \{\gamma_1, -\gamma_1\}, \ldots, \quad C_j = \{\gamma_{i_1} \ldots \gamma_{i_j}, -\gamma_{i_1} \ldots \gamma_{i_j}\},
\]

\[
C_{2n+1} = \{\gamma_{1} \ldots \gamma_{n}\}, \quad C_{2n+2} = \{-\gamma_{1} \ldots \gamma_{1}\}.
\]

The characters of the \( 2^n \) one dimensional representations are given by

\[
\chi_k(1) = \chi_k(-1) = 1, \quad 0 \leq k \leq 2^n - 1,
\]

\[
\chi_{2^n}(\pm \gamma_A) = \pm \delta_{A0} 2^{n/2} \quad \Rightarrow \quad \chi_{2^n}(1) = 2^{n/2}, \quad \chi_{2^n}(-1) = -2^{n/2}.
\]

Then, by using the result (3-36), we obtain the suitable coupling strengths as

\[
J_0 = -\frac{1}{2^{n+1} \ell_0} \{ (2^n + 1) \theta + \pi l \},
\]
\[ J_1 = J_2 = \ldots = J_{2^n-1} = \frac{1}{2^{n+1}t_0} (-\theta + \pi l), \]
\[ J_{2^n-1+1} = \ldots = J_{2^n} = -\frac{1}{2nt_0} (\theta + \pi l), \]
\[ J_{2^n} = 0, \quad l \in \mathbb{Z}. \]

### 4.4 The group \( U_{6n} \)

The group \( U_{6n} \) is a group of order \( 6n \) which is defined as
\[ U_{6n} = \langle a, b : a^{2n} = b^3 = 1, \quad a^{-1}ba = b^{-1} \rangle. \]

The \( 3n \) conjugacy classes of \( U_{6n} \) are, for \( 0 \leq r \leq n - 1, \)
\[ \{a^{2r}\}, \quad \{a^{2r}b, a^{2r}b^2\}, \quad \{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2\}. \]

The character table of \( U_{6n} \) is given by [24]

| \( U_{6n} \) | \( a^{2r} \) | \( a^{2r}b \) | \( a^{2r+1} \) |
|-------------|-------------|-------------|-------------|
| \( \chi_j \) (\( 0 \leq j \leq 2n - 1 \)) | \( \omega^{2jr} \) | \( \omega^{2jr} \) | \( \omega^{j(2r+1)} \) |
| \( \psi_k \) (\( 0 \leq k \leq n - 1 \)) | \( 2\omega^{2kr} \) | \( -\omega^{2kr} \) | \( 0 \) |

with \( \omega := e^{2\pi i/2n} \). Then by using the character table and the result (3-36), we have
\[ J_l = \frac{1}{6nt_0} \left( \sum_{k=0}^{2n-1} \chi_k(a_l)(\frac{2\pi k}{n} - \theta) + 2 \sum_{k=0}^{n-1} \psi_k(a_l)(\frac{2\pi k}{n} - \theta) \right), \quad (4-41) \]

where, we have chosen \( \alpha_m = a^2 \) and substituted \( \frac{d_k}{\chi_k(a^2)} = \frac{d_k}{\psi_k(a^2)} = \omega^{-2k} = e^{-\frac{2\pi ik}{n}} \). By using (4-41), one can obtain
\[ J_0 = \frac{\pi(5n-3) - 6n\theta}{6nt_0}, \]
\[ J_l = \frac{\pi}{3nt_0} \left( \sum_{k=0}^{2n-1} k\omega^{-2kl} + 4 \sum_{k=0}^{n-1} k\omega^{-2kl} \right), \quad l = 1, 2, \ldots, n - 1, \]
\[ J_l = \frac{\pi}{3nt_0} \left( \sum_{k=0}^{2n-1} k\omega^{-2kl} - 2 \sum_{k=0}^{n-1} k\omega^{-2kl} \right), \quad l = n, n + 1, \ldots, 2n - 1, \]
\[ J_l = \frac{\pi}{3nt_0} \sum_{k=0}^{2n-1} k\omega^{-k(2l+1)}, \quad l = 2n, 2n+1, \ldots, 3n-1. \]

For instance, for the case \( n = 2 \) (the group \( U_{12} \)), the coupling strengths \( J_l \) are given by

\[ J_0 = \frac{7\pi - 12\theta}{12t_0}, \quad J_1 = -\frac{\pi}{2t_0}, \quad J_2 = \frac{\pi}{3t_0}, \quad J_3 = 0, \quad J_4 = J_5^* = \frac{\pi(i-1)}{6t_0}. \]

### 4.5 The group \( V_{8n} \)

Let \( n \) be an odd positive integer. The group \( V_{8n} \) is a group of order \( 8n \) which is defined as

\[ V_{8n} = \langle a, b : a^{2n} = b^4 = 1, \ ba = a^{-1}b^{-1}, \ b^{-1}a = a^{-1}b \rangle. \]

The \( 2n + 3 \) conjugacy classes of \( V_{8n} \) are

\[ \{1\}, \ \{b^2\}, \ \{a^{2r+1}, a^{-(2r+1)}b^2\} \ (0 \leq r \leq n - 1), \]

\[ \{a^{2s}, a^{-2s}\}, \ \{a^{2s}b^2, a^{-2s}b^2\} \ (1 \leq s \leq \frac{n-1}{2}), \]

\[ \{a^jb^k : j \ \text{even}, \ k = 1 \ or \ 3\}, \ \text{and} \ \{a^jb^k : j \ \text{odd}, \ k = 1 \ or \ 3\}. \]

The character table of \( V_{8n} \) is given by [24]

| \( V_{8n} \) | \( e \) | \( b^2 \) | \( a^{2r+1}(0 \leq r \leq n - 1) \) | \( a^{2s}(1 \leq s \leq (n-1)/2) \) | \( a^{2s}b^2 \) | \( b \) | \( ab \) |
|---|---|---|---|---|---|---|---|
| \( \chi_0 \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \( \chi_1 \) | 1 | 1 | 1 | 1 | 1 | -1 | -1 |
| \( \chi_2 \) | 1 | 1 | -1 | 1 | 1 | 1 | -1 |
| \( \chi_3 \) | 1 | 1 | -1 | 1 | 1 | -1 | 1 |
| \( \psi_j \) (\( 0 \leq j \leq n - 1 \)) | 2 | -2 | 2i \sin(2\pi j(2r + 1)/n) | 2 \cos(4\pi js/n) | -2 \cos(4\pi js/n) | 0 | 0 |
| \( \phi_j \) (\( 1 \leq j \leq n - 1 \)) | 2 | 2 | 2 \cos(2\pi j(2r + 1)/n) | 2 \cos(2\pi js/n) | 2 \cos(2\pi js/n) | 0 | 0 |

where, \( \omega := e^{2\pi i/2n} \). Again, by using the character table and the result (3-36), we obtain

\[ J_l = \frac{1}{4nt_0}\{-\theta[2 + \sum_{k=0}^{n-1} \bar{\psi}_k(\alpha_l) + \sum_{k=1}^{n-1} \bar{\phi}_k(\alpha_l)] + \pi \sum_{k=0}^{n-1} \bar{\psi}_k(\alpha_l)\}, \quad (4-42) \]
where, we have chosen $\alpha_m = b$ and substituted $\frac{d_k}{\chi_k(b)} = \frac{d_k}{\phi_k(b)} = -\frac{d_k}{\psi_k(b)} = 1$. By using (4-42), one can obtain

$$J_0 = \frac{\pi - 2\theta}{2t_0}, \quad J_1 = -\frac{\pi}{2t_0},$$

$$J_l = \frac{1}{2nt_0}(\theta - \pi i \sum_{k=1}^{n-1} \sin \frac{2\pi k(2l - 3)}{n}), \quad l = 2, \ldots, n + 1,$$

$$J_l = \frac{1}{2nt_0}\{\theta \sum_{k=0}^{n-1} \cos \frac{4\pi k(l - n - 1)}{n} + \cos \frac{2\pi k(l - n - 1)}{n}\} + \pi \sum_{k=0}^{n-1} \cos \frac{4\pi k(l - n - 1)}{n}, \quad l = n + 2, \ldots, \frac{3n + 1}{2},$$

$$J_l = \frac{1}{2nt_0}\{\theta \sum_{k=0}^{n-1} \cos \frac{4\pi k(l - 2n + 1)}{n} - \cos \frac{2\pi k(l - 2n + 1)}{n}\} - \pi \sum_{k=0}^{n-1} \cos \frac{4\pi k(l - 2n + 1)}{n}, \quad l = \frac{3(n + 1)}{2}, \ldots, 2n,$$

$$J_{2n+1} = J_{2n+2} = 0.$$

In the case $n = 3$ (the group $V_{24}$), the coupling strengths $J_l$ are given by

$$J_0 = \frac{\pi - 2\theta}{2t_0}, \quad J_1 = -\frac{\pi}{2t_0}, \quad J_2 = J_3 = J_4 = \frac{\theta}{6t_0}, \quad J_5 = J_6 = J_7 = J_8 = 0.$$

## 5 Conclusion

Perfect state transfer of a qudit in boson networks was investigated where, a family of Hamiltonians related to the Bose-Hubbard model is defined which enable PST of an arbitrary qudit state. By choosing the underlying networks of finite group schemes as boson networks (i.e., with each vertex (site) of the network, a bosonic number operator for the bosons located at that site, is associated), we showed how to perfectly transfer, arbitrary qudit states in interacting boson lattices. In fact, by employing the group theory properties of these networks, an explicit analytical formula for coupling constants in the Hamiltonians was given, so that the state of a particular qudit initially encoded on one site can be perfectly evolved to the opposite site without any dynamical control. Finally, PST on underlying networks associated with some finite groups was considered in details.
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