Preservation of Trees by semidirect Products

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Abstract

We show that the semidirect product of a group $C$ by $A * D B$ is isomorphic to the free product of $A \rtimes C$ and $B \rtimes C$ amalgamated at $D \rtimes C$, where $A$, $B$ and $C$ are arbitrary groups. Moreover, we apply this theorem to prove that any group $G$ that acts without inversion on a tree $T$ that possesses a segment $\Gamma$ for its quotient graph, such that, if the stabilizers of the vertex set $\{ P, Q \}$ and edge $y$ of a lift of $\Gamma$ in $T$ are of the form $G_P \rtimes H$, $G_Q \rtimes H$ and $G_y \rtimes H$, then $G$ is isomorphic to the semidirect product of $H$ by $(G_P \ast G_y, G_Q)$.

Using our results we conclude with a non-standard verification of the isomorphism between $\text{GL}_2(\mathbb{Z})$ and the free product of the dihedral groups $D_4$ and $D_6$ amalgamated at their Klein-four group.

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1 Introduction

In this paper we provide an analysis of an interplay between semidirect products and free products with amalgamation (i.e., tree products.) That is, we show that given any groups $A$, $B$ and $C$, the semidirect product of a group $C$ by $A * D B$ is isomorphic to the free product of the $A \rtimes C$ and $B \rtimes C$ amalgamated at $D \rtimes C$, i.e.,

$$( A * B ) \rtimes C \simeq ( A \rtimes C ) *_{D \rtimes C} ( B \rtimes C ) .$$

Intuitively, in the category of groups, the semidirect product of a group distributes (or, it is preserved) on the right over free products with amalgamation. Moreover, we show that a group $G$ that acts without inversion on a tree $\Gamma$ such that if $G_P \rtimes H$, $G_Q \rtimes H$ and
$G_y \rtimes H$ are the stabilizers of the vertex set $\{P, Q\}$ and edge $y$ of a lift of its segment $\Gamma$ in the quotient graph $\tilde{\Gamma}$, then

$$G \simeq (G_P *_{G_y} G_Q) \rtimes H.$$  

We then give an example of the isomorphism $\text{GL}_2(\mathbb{Z}) \simeq D_4 * D_2 D_6$ using these results.

2 An Exact Sequence for a Tree

Lemma 1 If $A$, $B$, $C$ and $D$ are groups, then there is an exact sequence of form

$$1 \longrightarrow A * B \xleftarrow{\nu} (A \rtimes C) *_{D \rtimes C} (B \rtimes C) \xrightarrow{\mu} C \longrightarrow 1. \quad (1)$$

Proof. Let $G = (A * B)$ and $\tilde{G} = (A \rtimes C) *_{D \rtimes C} (B \rtimes C)$ be pushouts of the diagrams

$$
\begin{array}{ccc}
D & \xrightarrow{\iota_A} & A \\
\downarrow{\iota_B} & & \downarrow{\alpha} \\
B & \xrightarrow{\beta} & G
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
D \rtimes C & \xrightarrow{\iota_A} & A \rtimes C \\
\downarrow{\iota_B} & & \downarrow{\bar{\alpha}} \\
B \rtimes C & \xrightarrow{\bar{\beta}} & G,
\end{array}
$$

where its homomorphisms are injective. In particular, $\iota_A$ and $\bar{\alpha}$ are the embeddings induced from embedding $i_A$ and the natural embedding $\alpha$, i.e.,

$$\iota_A : d \cdot c \longmapsto (\nu_A(d \cdot c)) \quad \text{and} \quad \bar{\alpha} : a \cdot c \longmapsto (a \cdot c) \tilde{N},$$

where $\tilde{N} \triangleleft \tilde{G}$ is of the form $\tilde{N} := \text{Ncl} \{ \iota_A(d \cdot c) \iota_B((d \cdot c)^{-1}) \mid d \cdot c \in D \rtimes C \}$, and $a \cdot c \in A \rtimes C$. The embeddings $\bar{\beta}$ and $\iota_B$ are defined in the same manner. Now the semidirect products $A \rtimes C$ and $B \rtimes C$ are described by the split extensions

$$
\begin{array}{ccc}
A & \xrightarrow{\nu_A} & A \rtimes C \\
& & \xrightarrow{\mu_A} C \quad \text{and} \quad
B & \xrightarrow{\nu_B} & B \rtimes C \xrightarrow{\mu_B} C,
\end{array}
$$

where $\nu_A$ is the natural embedding of $A$ into $A \rtimes C$, $\mu_A$ is the projective homomorphism defined by

$$\mu_A : A \times C \longrightarrow (A \times C)/A \simeq C \quad \text{such that} \quad \mu_A : a \cdot c \longmapsto Ac \longmapsto c$$

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The homomorphisms $\nu_B$ and $\mu_B$ and defined in the similar manner. Then, putting the data of diagrams (2) and (3), we get the commutative diagram

\[
\begin{array}{ccc}
 D & \xrightarrow{\iota_A} & A \\
 \downarrow{\alpha} & & \downarrow{\tilde{\alpha}} \\
 G & \xrightarrow{\iota_A(d)} & G \\
 \downarrow{\beta} & & \downarrow{\tilde{\beta}} \\
 B & \xrightarrow{\iota_B} & B \\
\end{array}
\]

where $d$, $\iota_A(d) = \iota_B(d) = \iota_A(d) \cdot 1_C = \iota_B(d) \cdot 1_C$, i.e., the diagram commutes. In addition, since $G$ is a pushout through $D$ by $\iota_A$ and $\iota_B$, there is a unique $\nu: G \rightarrow \tilde{G}$ such that the diagram

\[
\begin{array}{ccc}
 D & \xrightarrow{\iota_A} & A \\
 \downarrow{\alpha} & & \downarrow{\tilde{\alpha}} \\
 G & \xrightarrow{\nu} & \tilde{G} \\
 \downarrow{\beta} & & \downarrow{\tilde{\beta}} \\
 B & \xrightarrow{\iota_B} & B \\
\end{array}
\]

commutes. In particular, let $g$ be a word in $G$. Then $g$ has a unique normal form

\[ g = a_1 b_1 \cdots a_n b_n N \]

where each $a_i \in A$, $b_i \in B$ and $N := \text{Ncl} \{ \iota_A(d) \iota_B(d^{-1}) \mid d \in D \}$. Then defining

\[ \nu(g) := a_1 \tilde{\alpha} \circ \iota_A A_1 \tilde{\beta} \circ \iota_B \cdots a_n \tilde{\alpha} \circ \iota_A A_n \tilde{\beta} \circ \iota_B N = (a_1 \cdot 1_C)(b_1 \cdot 1_C) \cdots (a_n \cdot 1_C)(b_n \cdot 1_C)N = a_1 b_1 \cdots a_n b_n N, \]

give us a well-defined embedding, by definition and the uniqueness of normal forms for free products with amalgamation. Moreover, given $a \in A$, we can naturally check that

\[ \nu \circ \alpha(a) := a \tilde{N} = \tilde{\alpha} \circ \iota_A (a \cdot c) \quad \text{and} \quad \nu \circ \beta(b) := b \tilde{N} = \tilde{\alpha} \circ \iota_A (b \cdot c), \]

i.e., $\nu \circ \alpha = \tilde{\alpha} \circ \iota_A$ and $\nu \circ \beta = \tilde{\beta} \circ \iota_B$. Therefore $\nu$ is a monomorphism its image is of the form

\[ \text{Im} \nu = \{ a_1 b_1 \cdots a_n b_n N \mid a_i \in A \wedge b_i \in B \} = G. \]

Now we would like to extend this sequence to an exact sequence. To do this we use the
diagrams (2) and (3) again to get the commutative diagram

\[ D \times C \quad \xymatrix{ A \times C \ar[r]^{\mu_A} \ar[d]_{\alpha} & A \times C/A \ar[d]^{l} & \mu_A \ar@{|->}[d] \ar[d]^{l} \\
G \ar[r]_{\iota} & C \ar[r]_{\iota} & \iota_A(d) \cdot c \ar[r]^{\mu_A} & Ac \ar[d]^{l} \\
B \times C \ar[r]_{\mu_B} \ar[u]_{\beta} & B \times C/B \ar[u]_{l} & \iota_B(d) \cdot c \ar[r]_{\mu_B} \ar[u]_{\beta} & Bc \ar[u]_{l} 
} \]

where \( d \cdot c = \iota_B(d) \cdot c = c = c \).

Also, since \( \tilde{G} \) is a pushout through \( D \times C \) by \( \iota_A \) and \( \iota_B \), then there exists a unique \( \mu : \tilde{G} \longrightarrow C \) such that the diagram

\[ D \times C \quad \xymatrix{ A \times C \ar[r]^{\mu_A} \ar[d]_{\alpha} & A \times C/A \ar[d]^{l} \\
G \ar[r]_{\iota} & C \ar[r]_{\iota} & \iota_A(d) \cdot c \ar[r]^{\mu_A} & Ac \ar[d]^{l} \\
B \times C \ar[r]_{\mu_B} \ar[u]_{\beta} & B \times C/B \ar[u]_{l} & \iota_B(d) \cdot c \ar[r]_{\mu_B} \ar[u]_{\beta} & Bc \ar[u]_{l} 
} \]

commutes. In particular, if \( g \) is a word in \( G \), then \( \tilde{g} \) has a unique normal form

\[
\tilde{g} := (a_1 \cdot c_1)(b_1' \cdot c_1') \cdots (a_n \cdot c_n)(b_n' \cdot c_n') \tilde{N}
= a_1 b_1' c_1 a_2 b_2' c_1 c_2 \cdots a_n b_n' c_1 c_2 \cdots c_{n-1} c_n' \cdot (c_1 c_1' \cdots c_n c_n') \tilde{N}
= a_1 b_1 \cdots a_n b_n \cdot c \tilde{N}
\]

where \( a_i \cdot c_i \in A \times C \), \( b_i' \cdot c_i' \in B \times C \) and \( b_i \in B \). Hence

\[
\mu(g) := a_1^{\mu_A} b_1^{\mu_B} a_2^{\mu_A} b_2^{\mu_B} \cdots a_n^{\mu_A} b_n^{\mu_B} \cdot c
\]

which is clearly a well-defined epimorphism (again, by definition and the uniqueness of normal forms for free products with amalgamation.) The kernel of \( \mu \) has the form

\[
\text{Ker} \mu = \{ a_1 b_1 \cdots a_n b_n N \mid a_i \in A \land b_i \in B \} = G.
\]

Therefore, \( \text{Ker} \mu = \text{Im} \nu \). Moreover, given \( a \in A \), we can check that \( \mu \circ \tilde{\alpha} = \alpha \circ \mu_A \) and \( \mu \circ \tilde{\beta} = \alpha \circ \mu_B \). Therefore we get the diagram

\[
1 \longrightarrow A \ast B \xrightarrow{\nu} (A \times C) \ast (B \times C) \xrightarrow{\mu} C \longrightarrow 1,
\]

which is an exact sequence. \( \square \)
3 A Preservation of a Tree by a Semidirect Product

Proposition 1 Let $\text{Grp}$ be the category of groups, let $C$ be a group and let

$$\text{Grp}_{\times C} : \text{Grp} \to \text{Grp}$$

be the assignment defined as follows:

- $\text{Grp}_{\times C} : G \mapsto G \times C$ for any group $G$.
- $\text{Grp}_{\times C} : \psi \mapsto \psi \times 1_C$ for any $\psi \in \text{Hom}(G, H)$ such that $1_C$ is the identity automorphism of $C$ and

$$\psi \times 1_C : G \times C \to H \times C$$

is defined by $\psi \times 1_C : g \cdot c \mapsto g^\psi \cdot c$,

where $g \in G$ and $c \in C$.

Then $\text{Grp}_{\times C}$ is a functor.

Proof. By definition, the map is well-defined on the class of groups and homomorphisms. Let $G$ and $H$ be groups, and let $\psi : G \mapsto H$ be a homomorphism. Suppose $g \in G$ and $c \in C$, then $\text{Grp}_{\times C}(\psi)(g \cdot c) := \psi \times 1_C(g \cdot c) = g^\psi \cdot c$. Therefore

$$G \xrightarrow{\psi} H$$

$$(G \times C) \xrightarrow{\psi \times 1_C} (H \times C).$$

In particular, if $\psi$ is the identity automorphims $1_C : G \mapsto G$, then $\text{Grp}_{\times C}(1_G)$ is equal to $1_G \times 1_C$. Now $1_G \times 1_C(g \cdot c) = g \cdot c = 1_{G \times C}(g \cdot c)$ and therefore

$$\text{Grp}_{\times C}(1_G) = 1_{G \times C}.$$  

Also, if $\varphi : H \mapsto K$, where $K$ is a group, then

$$\text{Grp}_{\times C}(\varphi \circ \psi)(g \cdot c) = g^{\psi \circ \varphi} \cdot c = \text{Grp}_{\times C}(\varphi)(g^\psi \cdot c) = \text{Grp}_{\times C}(\varphi)\left(\text{Grp}_{\times C}(\psi)(g \cdot c)\right)$$

i.e., $\text{Grp}_{\times C}(\varphi \circ \psi) = \text{Grp}_{\times C}(\varphi) \circ \text{Grp}_{\times C}(\psi)$. Therefore $\text{Grp}_{\times C}$ is a functor.

Theorem 1 For any group $C$ the functor $\text{Grp}_{\times C} : \text{Grp} \to \text{Grp}$ preserves free products with amalgamation, i.e.,

$$(A \ast B) \ast C \simeq (A \times C) \ast (B \times C)$$

given any group $A$, $B$ and $D$.  

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Proof. Let $G = (A * B) \rtimes C$ and let $\tilde{G} = (A \rtimes C) *_{D \rtimes C} (B \rtimes C)$. By lemma 1

$$A * B \xrightarrow{\nu} (A \rtimes C) *_{D \rtimes C} (B \rtimes C) \xrightarrow{\mu} C$$

is exact; hence, it suffices to show that this sequence is also a split extension. The colimit of the diagrams

$$\begin{array}{cc}
D & A \xrightarrow{i_A} A \\
\downarrow{i_B} & \downarrow{\alpha} \\
B & G
\end{array} \quad \text{and} \quad \begin{array}{cc}
D \rtimes C & A \rtimes C \xrightarrow{i_A} A \\
\downarrow{i_B} & \downarrow{\alpha} \\
B \rtimes C & G
\end{array}$$

are the tree products $G$ and $\tilde{G}$, respectively. These groups are described by its commutative diagrams, where $i_A$ and $\tilde{\alpha}$ are the embeddings induced from embedding $i_A$ and the natural embedding $\alpha$ (as described in the proof of lemma 1). The embeddings $\tilde{\beta}$ and $\tilde{i}_B$ are defined in the same manner. Now the semidirect products $A \rtimes C$ and $B \rtimes C$ are described by the split extensions

$$\begin{array}{ccc}
A & \xrightarrow{\nu_A} & A \rtimes C \xrightarrow{\mu_A} C \\
\downarrow{\tau_A} & & \downarrow{\tau_A} \\
B \rtimes C & \xleftarrow{\tau_B} & C
\end{array} \quad \text{and} \quad \begin{array}{ccc}
B & \xrightarrow{\nu_B} & B \rtimes C \xleftarrow{\mu_B} C \\
\downarrow{\tau_B} & & \downarrow{\tau_B} \\
A \rtimes C & \xleftarrow{\tau_A} & C
\end{array}$$

where $\nu_A$ is the natural embedding, $\mu_A$ is the projective homomorphism and $\tau_A$ is the transversal homomorphism of the split extension. In particular, for any $g \in A \rtimes C$ there are unique $a \in A$ and $c \in C$ such that $g = a \cdot c$. Then $Ag = A(a \cdot c) = Ac$ and

$$\tau_A : C \simeq (A \rtimes C)/A \longrightarrow A \rtimes C$$

is defined by $\tau_A : c \simeq Ag \longmapsto 1_A \cdot c$,

which satisfies $\mu_A \circ \tau_A = 1_A$. The homomorphisms $\nu_B$, $\mu_B$ and $\tau_B$ are defined in the similar manner, as before. Let us define a homomorphism $\tau : C \longrightarrow \tilde{G}$ by the natural embedding $\tau : c \longrightarrow c\tilde{N}$. Then, putting the data of the diagrams (2) and (3), we get the commutative diagram

$$\begin{array}{ccc}
C & \xrightarrow{\tau} & \tilde{G} \\
\downarrow{\tau_B} & & \downarrow{\tau_B} \\
B \rtimes C & \xleftarrow{\tau_A} & A \rtimes C
\end{array}$$

In particular, given $c \in C$, then $\mu \circ \tau(c) := \mu(c\tilde{N}) = c$, i.e., $\mu \circ \tau = 1_c$. Therefore, $\tau$ is a transversal homomorphism of the extension represented by diagram (7). Hence,

$$1 \longrightarrow A * B \xrightarrow{\nu} (A \rtimes C) *_{D \rtimes C} (B \rtimes C) \xrightarrow{\mu} C \longrightarrow 1$$

(10)
is a split extension; consequently \((A \ast B) \times C \simeq (A \times C) \ast (B \times C)\). \(\square\)

**Theorem 2** Let \(G\) act without inversion on a tree \(\tilde{\Gamma}\) and let \(\Gamma = G \setminus \tilde{\Gamma}\) denote its factor graph. If \(\Gamma\) is a segment such that \(G_P \rtimes H, G_Q \rtimes H\) and \(G_y \rtimes H\) are the stabilizers of the vertex set \(\{P, Q\}\) and edge \(y\) of a lift of \(\Gamma\) in \(\tilde{\Gamma}\), then

\[ G \simeq (G_P \ast_{G_y} G_Q) \rtimes H. \]

*Proof.* The canonical homomorphism

\[ \varphi : (G_P \rtimes H) \ast_{G_y} (G_Q \rtimes H) \longrightarrow G \]

is an isomorphism, *a priori*. Therefore \(G \simeq (G_P \ast_{G_y} G_Q) \rtimes H\) by theorem 1. \(\square\)

### 4 A Quick Application to \(\text{GL}_2(\mathbb{Z})\)

The *general linear group* \(\text{GL}_n(R)\) over a commutative \(R\)-module \(M\) consists of the set of invertible linear operators over \(M\). We are particularly interested in the group \(\text{GL}_2(\mathbb{Z})\), which has a representation

\[ \text{GL}_2(\mathbb{Z}) := \{ A \in M_{2 \times 2} \mid \det A = \pm 1 \} \]

since any invertible matrix \(A \in M_{2 \times 2}\) is the form \(A^{-1} = \left(\frac{1}{\det A}\right) \text{adj} A\), where \(\text{adj} A\) is the *classical adjoint* of \(A\) and \(\det A\) is the *determinant* of \(A\). The *special linear group* of \(2 \times 2\) matrices, denoted by \(\text{SL}_2(\mathbb{Z})\), is a subgroup \(\text{GL}_2(\mathbb{Z})\) with representation \(\text{SL}_2(\mathbb{Z}) := \{ A \in M_{2 \times 2} \mid \det A = 1 \}\). The theory of group actions on trees tells us that

\[ \text{SL}_2(\mathbb{Z}) \simeq \mathbb{Z}_4 \ast \mathbb{Z}_2 \mathbb{Z}_6. \]

We now use the above result to relate \(\text{GL}_2(\mathbb{Z})\) with \(\text{SL}_2(\mathbb{Z})\) by way of theorem 1.

**Theorem 3** \(\text{GL}_2(\mathbb{Z}) \simeq D_4 \ast_{D_2} D_6\).

*Proof.* The sequence of groups

\[
1 \longrightarrow \text{SL}_2(\mathbb{Z}) \longleftarrow \text{GL}_2(\mathbb{Z}) \overset{\det}{\longrightarrow} \mathbb{Z}_2 \longrightarrow 1 \tag{11}
\]

is exact, since \(\text{SL}_2(\mathbb{Z}) = \text{Ker} (\det)\). The map

\[ \tau : \mathbb{Z}_2 \longrightarrow \text{GL}_2(\mathbb{Z}) \text{ defined by } \tau : -1 \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

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is clearly a well-defined homomorphism. In particular, since the \((\det \circ \tau) = 1 A\), the extension in diagram is a split extension; hence \(\text{GL}_2(\mathbb{Z}) \simeq \text{SL}_2(\mathbb{Z}) \times \mathbb{Z}_2\). Thus

\[
\text{SL}_2(\mathbb{Z}) \rtimes \mathbb{Z}_2 \simeq (\mathbb{Z}_4 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \quad — \text{a priori;}
\]

\[
\simeq (\mathbb{Z}_4 \rtimes \mathbb{Z}_2) \rtimes (\mathbb{Z}_6 \rtimes \mathbb{Z}_2) \quad — \text{by theorem and}
\]

\[
\simeq D_4 \rtimes D_6.
\]

Therefore \(\text{GL}_2(\mathbb{Z}) \simeq D_4 \rtimes D_6\).

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