Walsh-Hadamard Variational Inference for Bayesian Deep Learning

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Abstract

Over-parameterized models, such as DeepNets and ConvNets, form a class of models that are routinely adopted in a wide variety of applications, and for which Bayesian inference is desirable but extremely challenging. Variational inference offers the tools to tackle this challenge in a scalable way and with some degree of flexibility on the approximation, but for over-parameterized models this is challenging due to the over-regularization property of the variational objective. Inspired by the literature on kernel methods, and in particular on structured approximations of distributions of random matrices, this paper proposes Walsh-Hadamard Variational Inference (WHVI), which uses Walsh-Hadamard-based factorization strategies to reduce the parameterization and accelerate computations, thus avoiding over-regularization issues with the variational objective. Extensive theoretical and empirical analyses demonstrate that WHVI yields considerable speedups and model reductions compared to other techniques to carry out approximate inference for over-parameterized models, and ultimately show how advances in kernel methods can be translated into advances in approximate Bayesian inference.

1 Introduction and Motivation

Since its inception, Variational Inference (VI) [21] has continuously gained popularity as a scalable and flexible approximate inference scheme for a variety of models for which exact Bayesian inference is intractable. Bayesian neural networks [31, 35] represent a good example of models for which inference is intractable, and for which VI– and approximate inference in general – is challenging due to the nontrivial form of the posterior distribution and the large dimensionality of the parameter space [14, 12]. Recent advances in VI allow to effectively deal with these issues in various ways. A flexible class of posterior approximations can be constructed using, e.g., normalizing flows [42], whereas large parameter space have pushed the research in the direction of Bayesian compression [30, 33].

Employing VI is notoriously challenging for over-parameterized statistical models. In this paper, we focus in particular on Bayesian Deep Neural Networks (DNNs) and Bayesian Convolutional Neural Networks (CNNs) as typical examples of over-parameterized models. Let’s consider a supervised learning task with \( N \) input vectors and corresponding labels collected in \( X = \{x_1, \ldots, x_N\} \) and \( Y = \{y_1, \ldots, y_N\} \), respectively; furthermore, let’s consider DNNs with weight matrices \( W = \{W^{(1)}, \ldots, W^{(L)}\} \), likelihood \( p(Y|X, W) \), and prior \( p(W) \). Following standard variational arguments, after introducing an approximation \( q(W) \) to the posterior \( p(W|X, Y) \) it is possible to obtain a lower bound to the log-marginal likelihood \( \log[p(Y|X)] \) as follows:

\[
\log[p(Y|X)] \geq E_{q(W)}[\log p(Y|X, W)] - \text{KL}\{q(W)||p(W)\}.
\]

The first term acts as a model fitting term, whereas the second one acts as a regularizer, penalizing solutions where the posterior is far away from the prior. It is easy to verify that the KL term can be
the dominant one in the objective for over-parameterized models. For example, a mean field posterior approximation turns the KL term into a sum of as many KL terms as the number of model parameters, say $Q$, which can dominate the overall objective when $Q \gg N$. As a result, the optimization focuses on keeping the approximate posterior close to the prior, disregarding the other term with measures the data fit. This issue has been observed in a variety of deep models [2], where it was proposed to gradually include the KL term throughout the optimization [2] [46], or to improve the initialization of variational parameters [43]. Alternatively, other approximate inference methods for deep models with connections to VI have been proposed, notably Monte Carlo Dropout (MCD, [12]) and Noisy Natural Gradients (NNG; [57]).

In this paper, we propose a novel strategy to cope with model over-parameterization when using variational inference, which is inspired by the literature on kernel methods. Our proposal is to reparameterize the variational posterior over model parameters by means of a structured decomposition based on random matrix theory [59], which has inspired a number of fundamental contributions in the literature on approximations for kernel methods, such as FASTFOOD [27] and Orthogonal Random Features (ORF, [55]). The key operation within our proposal is the Walsh-Hadamard transform, and this is why we name our proposal Walsh-Hadamard Variational Inference (WHVI).

Without loss of generality, consider Bayesian DNNs with weight matrices $W^{(l)}$ of size $D \times D$. Compared with mean field VI, WHVI has a number of attractive properties. The number of parameters is reduced from $O(D^2)$ to $O(D)$, reducing the over-regularization effect of the KL term in the variational objective. We derive expressions for the reparameterization and the local reparameterization tricks, showing that the computational complexity is reduced from $O(D^2)$ to $O(D \log D)$. Finally, unlike mean field VI, WHVI induces a matrix-variate distribution to approximate the posterior over the weights, thus increasing flexibility at a log-linear cost in $D$ instead of linear.

We can think of our proposal as a specific factorization of the weight matrix, so we can speculate that other tensor factorizations [39] of the weight matrix could equally yield such benefits. Our comparison against various matrix factorization alternatives, however, shows that WHVI is superior to other parameterizations that have the same complexity. Furthermore, while matrix-variate posterior approximations have been proposed in the literature of VI [28], this comes at the expense of increasing the complexity, while our proposal keeps the complexity to log-linear in $D$.

Through a wide range of experiments on DNNs and CNNs, we demonstrate that our approach allows to run variational inference on complex over-parameterized models, while being competitive with state-of-the-art alternatives. Ultimately, our proposal shows how advances in kernel methods can be instrumental in improving VI, much like previous works showed how kernel methods can improve Markov chain Monte Carlo sampling [44] [47] and statistical testing [15] [16] [56].

## 2 Walsh-Hadamard Variational Inference

### 2.1 Background on Structured Approximations of Kernel Matrices

WHVI is inspired by a line of works that developed from random feature expansions for kernel machines [41], which we briefly review here. A positive-definite kernel function $\kappa(x_i, x_j)$ induces a mapping $\phi(x)$, which can be infinite dimensional depending on the choice of $\kappa(\cdot, \cdot)$. Among the large literature of scalable kernel machines, random feature expansion techniques aim at constructing a finite approximation to $\phi(\cdot)$. For many kernel functions [41] [4], this approximation is built by applying a nonlinear transformation to a random projection $X\Omega$, where $\Omega$ has entries $N(\omega_{ij} | 0, 1)$. If the matrix of training points $X$ is $N \times D$ and we are aiming to construct $D$ random features, that is $\Omega$ is $D \times D$, this requires $N$ times $O(D^2)$ time, which can be prohibitive when $D$ is large.

FASTFOOD [27] tackles the issue of large dimensional problems by replacing the matrix $\Omega$ with a random matrix for which the space complexity is reduced from $O(D^2)$ to $O(D)$ and time complexity of performing products with input vectors is reduced from $O(D^2)$ to $O(D \log D)$. In FASTFOOD, the matrix $\Omega$ is replaced by the following:

$$\Omega \approx SH\Pi HB,$$

(2)
where Π is a permutation matrix, H is the Walsh-Hadamard matrix\(^1\), whereas G and B are diagonal random matrices with standard Normal distributions and Rademacher (\{±1\}), respectively. S is also diagonal with i.i.d. entries, and it is chosen such that the elements of Ω obtained by this series of operations are approximately independent and follow a standard Normal (see Tropp [50] for more details). FASTFOOD inspired a series of other works on kernel approximations [54, 11], whereby Gaussian random matrices are approximated by a series of products between diagonal Rademacher and Walsh-Hadamard matrices, for example \(Ω ≈ HS_1HS_2HS_3\).

2.2 From FASTFOOD to Walsh-Hadamard Variational Inference

FASTFOOD and its variants yield cheap approximations to Gaussian random matrices with pseudo-independent entries, and zero mean and unit variance. The question we address in this paper is whether we can use these types of approximations as cheap approximating distributions for VI. By considering a prior for the elements of the diagonal matrix \(G = \text{diag}(g)\) and a variational posterior \(q(g) = N(\mu, \Sigma)\), we can actually obtain a class of approximate posterior with some desirable properties as discussed next. Let \(W = W^{(l)} ∈ \mathbb{R}^{D×D}\) be the weight matrix of a DNN at layer \((l)\), and consider

\[
\hat{W} \sim q(W) \quad \text{s.t.} \quad \hat{W} = S_1H\text{diag}(\hat{g})HS_2 \quad \text{with} \quad \hat{g} \sim q(g).
\]

(3)

The choice of a Gaussian \(q(g)\) and the linearity of the operations, induce a parameterization of a matrix-variate Gaussian distribution for \(W\), which is controlled by \(S_1\) and \(S_2\) if we assume that we can optimize these diagonal matrices. Note that we have dropped the permutation matrix Π and we will show later that this is not critical for performance, while it speeds up computations.

For a generic \(D_1 × D_2\) matrix-variate Gaussian distribution, we have

\[
W \sim MN(M, U, V) \quad \text{if and only if} \quad \text{vector}(W) \sim N(\text{vector}(M), V ⊗ U),
\]

(4)

where \(M ∈ \mathbb{R}^{D_1×D_2}\) is the mean matrix and \(U ∈ \mathbb{R}^{D_1×D_1}\) and \(V ∈ \mathbb{R}^{D_2×D_2}\) are two positive definite covariance matrices among rows and columns, and \(⊗\) denotes the Kronecker product. In whvi, as \(S_2\) is diagonal, \(HS_2 = [v_1, . . . , v_D]\) with \(v_i = (S_2)i,i(H)i,i\), so \(W\) can be rewritten in terms of \(A ∈ \mathbb{R}^{D^2×D}\) and \(g\) as follows

\[
\text{vector}(W) = Ag \quad \text{where} \quad A^T = \left[\left(S_1H\text{diag}(v_1)\right)^T \ldots \left(S_1H\text{diag}(v_D)\right)^T\right].
\]

(5)

This rewriting, shows that the choice of \(q(g)\) yields \(q(\text{vector}(W)) = N(A\mu, A\Sigma A^T)\), proving that whvi assumes a matrix-variate distribution \(q(W)\), as also shown in Figure 1.

We report the expression for \(M, U, \) and \(V\) and leave the full derivation to the Appendix. For the mean, we have \(M = S_1H\text{diag}(μ)HS_2\), whereas for \(U\) and \(V\), we have:

\[
U^{1/2} = S_1HT_2 \quad \text{and} \quad V^{1/2} = \frac{1}{\sqrt{\text{Tr}(U)}}S_2HT_1,
\]

(6)

where each row \(i\) of \(T_1 ∈ \mathbb{R}^{D×D^2}\) is the column-wise vectorization of \(\left(Σ_{i,j}^{1/2}(HS_1)_{i,j'}\right)_{j, j' ≤ D}\), the matrix \(T_2\) is defined similarly with \(S_2\) instead of \(S_1\), and \(\text{Tr}(\cdot)\) denotes the trace operator.

\(^1\)The Walsh-Hadamard matrix is defined recursively by \(H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\) and then \(H_{2D} = \begin{bmatrix} H_D & H_D \\ H_D & -H_D \end{bmatrix}\), possibly scaled by \(D^{-1/2}\) to make it orthonormal.
The mean of the structured matrix-variate posterior assumed by WHVI can span a $D$-dimensional linear subspace within the whole $D^2$-dimensional parameter space, and the orientation is controlled by the matrices $S_1$ and $S_2$; more details on this geometric intuition can be found in the Appendix.

Matrix-variate Gaussian posteriors for variational inference have been introduced in Louizos and Welling [28]; however, assuming full covariance matrices $U$ and $V$ is memory and computationally intensive (quadratic and cubic in $D$, respectively). WHVI captures covariances across weights (see Figure 1), while keeping memory requirements linear in $D$ and complexity log-linear in $D$.

### 2.3 Reparameterizations in WHVI for Stochastic Optimization

The so-called reparameterization trick [22] is a standard way to make the variational lower bound in Equation 1 a deterministic function of the variational parameters, so as to be able to carry out gradient-based optimization despite the stochasticity in the objective. As we assume a Gaussian posterior for $g$, the expression $g = \mu + \Sigma^{1/2}\epsilon$ separates out the stochastic component ($\epsilon \sim N(0, I)$) from the deterministic one ($\mu$ and $\Sigma^{1/2}$).

Considering input vectors $h_i$ to a given layer, an improvement over this approach is to consider the distribution of the product $W h_i$. This is also known as the local reparameterization trick [23], and it reduces the variance of stochastic gradients in the optimization, thus improving convergence. The product $Wh_i$ follows the distribution $N(m, AA^\top)$ [17], with

$$m = S_1 H \text{diag}(\mu) H S_2 h_i, \quad \text{and} \quad A = S_1 H \text{diag}(HS_2 h_i) \Sigma^{1/2}.$$  

A sample from this distribution can be efficiently computed thanks to the Walsh-Hadamard transform as: $W(\mu) h_i + W(\Sigma^{1/2}\epsilon) h_i$, with $W$ a linear matrix-valued function $W(u) = S_1 H \text{diag}(u) H S_2$.

### 2.4 Alternative Structures and Comparison with Tensor Factorization

The choice of the parameterization of $W$ in WHVI leaves space to several possible alternatives, which we compare in Table 2. For all of them, $G$ is learned variationally and the remaining diagonal $S_i$ (if any) are either optimized or treated variationally (Gaussian mean-field). Figure 2 shows the behavior of these alternatives when applied to a $2 \times 64$ network with ReLU activation functions. With the exception of the simple and highly constrained alternative $GH$, all parameterizations are converging quite easily and the comparison with MCD shows that indeed the proposed WHVI performs better both in terms of ERROR RATE and MNLL.

WHVI is effectively imposing a factorization of $W$, where parameters are either optimized or treated variationally. Tensor decompositions for DNNs and CNNs have been proposed in Novikov et al. [39]; here $W$ is decomposed into $k$ small matrices (tensor cores), such that

$$W = W_1 W_2 \cdots W_k.$$  

where each $W_i$ has dimensions $r_{i-1} \times r_i$ (with $r_1 = r_k = D$). We adapt this idea to compare WHVI with another factorization approach to obtain an approximate posterior over $W$. In order to match the space and time complexity of WHVI (Table 1), assuming $\{r_i = R \forall i = 2, \ldots, k-1\}$, we set:

$$R \propto \log_2 D \quad \text{and} \quad K \propto \frac{D}{(\log_2 D)^2}.$$  

![Figure 2: Test ERROR RATE and test MNLL with different structures for the weights. Benchmarked on DRIVE with a $2 \times 64$ network.](image)

| Table 2: List of alternative structures and test performance on DRIVE dataset. |
| --- |
| MODEL | TEST ERROR | MNLL |
| MCD | 0.097 | 0.249 |
| $GH$ | 0.226 | 0.773 |
| $S_{var}HGHS_{var}H$ | 0.043 | 0.159 |
| $S_{var}HGHS_{var}H$ | 0.061 | 0.190 |
| $S_{opt}HGHS_{opt}H$ | 0.054 | 0.199 |
| $S_{opt}HGHS_{opt}H$ | 0.031 | 0.146 |
| $S_{1, opt}HGHS_{2, opt}H$ | 0.026 | 0.094 |

Colors are coded to match the ones used in the adjacent Figure.
Also, to match the number of variational parameters, all internal cores \((i = 2, \ldots, k - 1)\) are learned with fully factorized Gaussian posterior, while the remaining are optimized. Given the same asymptotic complexity, Figure 3 reports the results when fitting a 2-hidden layer network with WHVI and with tensor decomposition. Not only WHVI can reach better solutions in terms of test performance, but optimization is also faster. We speculate that this is attributed to the redundant variational parameterization induced by the tensor cores, which makes the optimization landscapes highly multi-modal, leading to slow convergence.

2.5 Extensions

Concatenating or Reshaping Parameters for WHVI For the sake of presentation, so far we have assumed \(W \in \mathbb{R}^{D \times D}\) with \(D = 2^d\), but we can easily extend WHVI to handle parameters of any shape \(W \in \mathbb{R}^{D_1 \times D_2}\). One possibility is to use WHVI with a large \(D \times D\) matrix with \(D = 2^d\), such that a subset of its elements represent \(W\). Alternatively, a suitable value of \(d\) can be chosen so that \(W\) is a concatenation by row/column of square matrices of size \(D = 2^d\), padding if necessary.

When one of the dimensions is one so that the parameter matrix is a vector \((W = w \in \mathbb{R}^D)\), this latter approach is not ideal, as WHVI would fall back on mean-field VI. WHVI can be extended to handle these cases efficiently by reshaping the parameter vector into a matrix of size \(2^d\) with suitable \(d\), again by padding if necessary. Thanks to the reshaping, WHVI uses \(\sqrt{D}\) parameters to model a posterior over \(D\), and allows for computations in \(O(\sqrt{D} \log D)\) rather than \(D\). This is possible by reshaping the vector that multiplies the weights in a similar way. In the Appendix, we explore this idea to infer parameters of Gaussian processes linearized using large numbers of random features.

Normalizing Flows To better model approximate distributions for variational inference Rezende and Mohamed [22] introduce Normalizing Flows (NF). In the general setting, consider a set of invertible, continuous and differentiable functions \(f_k : \mathbb{R}^D \rightarrow \mathbb{R}^D\) with parameters \(\lambda_k\). Given \(z_0 \sim q_0(z_0)\), \(z_0\) is transformed with a chain of \(K\) flows to \(z_K = (f_K \circ \cdots \circ f_1)(z_0)\). The variational lower bound slightly differs from Equation 1 to take into account the determinant Jacobian of the transformations, yielding a new variational objective as follows:

\[
\mathbb{E}_{q_0} \left[ \log p_\theta(Y|X, W) \right] - \mathbb{KL}\{q_0(z_0)\|p(z_K)\} + \mathbb{E}_{q_0} \left[ \sum_{k=1}^{K} \log \left| \det \frac{\partial f_k(z_k-1; \lambda_k)}{\partial z_{k-1}} \right| \right].
\]

Setting the initial distribution \(q_0\) to a fully factorized Gaussian \(\mathcal{N}(z_0|\mu, \sigma I)\) and assuming Gaussian prior on the generated \(z_K\), the KL term is analytically tractable. The transformation \(f\) is generally chosen to allow for fast computation of the determinant Jacobian. The parameters of the initial density \(q_0\) as well as the flow parameters \(\lambda\) are optimized. In our case, we consider \(q_K\) as distribution over the elements of \(g\). This approach increases the flexibility of the form of the variational posterior in WHVI, which is no longer Gaussian, while still capturing covariances across weights. This is obtained at the expense of losing the possibility of employing the local reparameterization trick.

3 Experiments

3.1 Toy example

We generated a 1D toy regression problem with 128 inputs sampled from \(\mathcal{U}(-1, 2)\), and removed 20% inputs on a predefined interval; targets are noisy realizations of a random function (noise variance
We report the test DWHVI of the number of hidden features in a two-layer network to learn standardized realizations of the targets \([28]\) and Zhang et al. \([57]\], the network output is parameterized as a function of the input features \(x\). Data is randomly divided into 90%/10% splits for training and testing eight times. We standardize the posterior distribution using noisy natural gradients. In \(3.2\) Bayesian Neural Networks, the input domain, whereas \(\text{FFG}\) dropout rate \(0\) and two variants of \(\text{WHVI-} G\)-\(\text{WHVI}\) with Gaussian posterior and \(\text{NF-} \text{WHVI}\) with planar normalizing flows \([42]\). As Figure 4 shows, \(\text{WHVI}\) offers a sensible modeling of the uncertainty on the input domain, whereas \(\text{FFG}\) and \(\text{MCD}\) seem to be slightly over-confident.

\[ \sigma^2 = \exp(-3) \]. We model these data using a DNN with 2 hidden layers of 128 features and cosine activations. We test four models: mean-field Gaussian VI (MFG), Monte Carlo dropout (MCD) with dropout rate 0.4 and two variants of \(\text{WHVI-} G\)-\(\text{WHVI}\) with Gaussian posterior and \(\text{NF-} \text{WHVI}\) with planar normalizing flows \([42]\). As Figure 4 shows, \(\text{WHVI}\) offers a sensible modeling of the uncertainty on the input domain, whereas \(\text{FFG}\) and \(\text{MCD}\) seem to be slightly over-confident.

### 3.2 Bayesian Neural Networks

We conduct a series of comparisons with state-of-the-art VI schemes for Bayesian DNNs; see the Appendix for the list of data sets used in the experiments. We compare \(\text{WHVI}\) with MCD and NOISY-KFAC (also referred to as \(\text{NNG}\) \([57]\)). MCD draws on a formal connection between dropout and VI with Bernoulli-like posteriors, while the more recent NOISY-KFAC yields a matrix-variate Gaussian distribution using noisy natural gradients. In \(\text{WHVI}\), the last layer assumes a fully factorized Gaussian posterior.

Data is randomly divided into 90%/10% splits for training and testing eight times. We standardize the input features \(x\) while keeping the targets \(y\) unnormalized. The network has two hidden layers and 128 features with \(\text{ReLU}\) activations. Similarly to the experimental setup of Louizos and Welling \([28]\) and Zhang et al. \([57]\), the network output is parameterized as \(y = f(x) \odot \sigma_y + \mu_y\); thus forcing the network to learn standardized realizations of the targets \(y\).

We report the test RMSE and the average predictive test negative log-likelihood (MNLL). A selection of results is showed in Table 3; an extended version is available in the Appendix. On the majority of the datasets, \(\text{WHVI}\) outperforms MCD and NOISY-KFAC. Empirically, these results demonstrate the value of \(\text{WHVI}\), which offers a competitive parameterization of a matrix-variate Gaussian posterior while requiring log-linear time in \(D\).

\(\text{WHVI}\) builds its computational efficiency on the Fast Walsh-Hadamard Transform (FWHT), which allows one to cut the complexity of a \(D\)-dimensional matrix-vector multiplication from a naive \(O(D^2)\) to \(O(D \log D)\). To empirically validate this claim, we extended \textsc{PyTorch} \([40]\) with a custom C++/CUDA kernel which implements a batched-version of the FWHT. The box-plots in Figure 5 report the inference time for the entire test split on eight heterogeneously sized datasets as a function of the number of hidden features in a two-layer DNN.

### Table 3: Test RMSE and test MNLL for regression datasets

| MODEL | DATASET | MCD | NNG | TEST ERROR | MCD | NNG | TEST MNLL |
|-------|---------|-----|-----|------------|-----|-----|----------|
|       |         |     |     | \(\text{WHVI}\) |     |     | \(\text{WHVI}\) |
| BOSTON| 3.91 ± 0.86 | 3.56 ± 0.43 | \textbf{3.14 ± 0.71} | 6.90 ± 2.93 | \textbf{2.72 ± 0.09} | 4.33 ± 1.80 |
| CONCRETE | 5.12 ± 0.79 | 8.21 ± 0.55 | \textbf{4.70 ± 0.72} | 3.20 ± 0.36 | \textbf{3.56 ± 0.08} | 3.17 ± 0.37 |
| ENERGY | 2.07 ± 0.11 | 1.90 ± 0.28 | \textbf{0.58 ± 0.07} | 4.15 ± 0.15 | \textbf{2.11 ± 0.12} | 2.00 ± 0.60 |
| KINSM | 0.09 ± 0.00 | 0.07 ± 0.00 | 0.08 ± 0.00 | -0.87 ± 0.02 | -1.19 ± 0.04 | -1.19 ± 0.04 |
| NAVAL | 0.30 ± 0.30 | 0.00 ± 0.00 | 0.01 ± 0.00 | -1.00 ± 2.27 | -6.52 ± 0.09 | -6.25 ± 0.01 |
| POWERPLANT | 31.65 ± 0.07 | 4.23 ± 0.09 | \textbf{4.00 ± 0.12} | 49.78 ± 0.17 | 2.86 ± 0.02 | \textbf{2.71 ± 0.03} |
| PROTEIN | \textbf{4.23 ± 0.10} | 4.57 ± 0.47 | 4.36 ± 0.11 | \textbf{2.76 ± 0.02} | 2.95 ± 0.12 | 2.79 ± 0.01 |
| YACHT | 1.90 ± 0.54 | 5.16 ± 1.48 | \textbf{0.69 ± 0.16} | 2.95 ± 1.27 | 3.06 ± 0.27 | \textbf{1.80 ± 1.01} |
Figure 5: Inference time on the test split with a batch size of 128, Monte Carlo samples of 64 and averaged over 1000 repetitions. The workstation used is equipped with two 16C/32T Intel Xeon CPUs, four NVIDIA Tesla P100 (16 GB) and 512 GB of RAM. Each experiment is performed on an GPU fully dedicated to it.

3.3 Bayesian Convolutional Neural Networks

We continue the experimental evaluation of WHVI by analyzing its performance on CNNs. For this experiment, we replace all fully-connected layers in the CNN with the WHVI parameterization, while the convolutional filters are treated variationally using MCD. In this setup, we fit ALEXNET [25], VGG16 [49] and RESNET-18 [18] on CIFAR10. Using WHVI, we can reduce the number of parameters in the linear layers without affecting neither test performance nor calibration properties of the resulting model, as shown in Figure 6 and Table 4. As a reference, ALEXNET with its original ~23.3M parameters is reduced to just ~2.3M (9.9%) when using G-WHVI and to ~2.4M (10.2%) with WHVI and 3 planar flows. Even though we lose the benefits of the local reparameterization, the higher flexibility of normalizing flows allows to reach better test performance with respect to the Gaussian posterior. This might be improved even further using more complex families of normalizing flows [42, 57, 24, 29].

We also tested WHVI for the convolutional filters, but the resulting models had too few parameters to obtain any interesting results. For ALEXNET, we obtained a model with just to 189K, which corresponds to a sparsity of 99.2% with respect of the original model. As a reference, Wen et al. [52] was able to reach sparsity only up to 60% in the convolutional layers without impacting performance. We are currently investigating ways to adapt WHVI to be an effective method to reduce convolutional parameters.

Additional results and insights We refer the reader to the Appendix for an extended version of the results, including new applications of WHVI to Gaussian processes.

Table 4: Test performance of different Bayesian CNNs on CIFAR10

| CIFAR10     | TEST ERROR | TEST MNLL |
|-------------|------------|-----------|
| VGG16 NOisy K-FAC | 15.06%     | 0.7327    |
| VGG16 FPG    | 16.82%     | 0.6443    |
| VGG16 MCD    | 21.47%     | 0.8213    |
| VGG16 (W/ BN) NOisy K-FAC | 11.78% | N/A       |
| ALEXNET WHVI | 13.56%     | 0.6164    |
| ALEXNET NF-WHVI | 12.72%   | 0.6596    |
| RESNET18 WHVI | 10.71%     | 0.6222    |
| RESNET18 (W/ BN) WHVI | 11.46% | 0.5513    |
| RESNET18 (W/ BN) NF-WHVI | 11.42% | 0.4908    |
| VGG16 (W/ BN) WHVI | 12.85%     | 0.6995    |

Figure 6: Reliability diagram and expected calibration error (ECE) of VGG16, ALEXNET and RESNET with WHVI [18, 34].
Related Work

In the early sections of the paper, we have already briefly reviewed some of the literature on VI and Bayesian DNNs and CNNs; here we complement the literature by including other relevant works that have connections with WHVI for Bayesian deep learning.

Our work takes inspiration from the works on random features for kernel approximation [41] and FASTFOOD [27] in particular. Random feature expansions have had a wide impact on the literature on kernel methods. Such approximations have been successfully used to scale a variety of models, such as Support Vector Machines [41], Gaussian processes [26] and Deep Gaussian processes [5, 12]. This has contributed to bridging the gap between (Deep) Gaussian processes and Bayesian DNNs and CNNs [35, 10, 5, 11], which is an active area of research which aims to gain a better understanding of deep learning models through the use of kernel methods [6, 9, 13].

While random feature expansions improve scalability of kernel methods with respect to the size of the data set, FASTFOOD was designed to lower the complexity with respect to the dimensionality of the input [27]. FASTFOOD was later extended to improve its efficiency in the works on Structured Orthogonal Random Features (SORF) [54] and random spinners [1]. FASTFOOD and SORF have been applied to the problem of handling large dimensional convolutional features in the works on Deep Fried CNNs [53] and Deep Convolutional Gaussian Processes [49], respectively.

This paper focuses in particular on DNNs and CNNs as working examples to present WHVI. Recent advances in understanding DNNs have investigated the effect of over-parameterization and how model compression can be used during or after training [20, 30, 58]. Neyshabur et al. [36] empirically shows that bigger and wider neural networks can converge better and they are more resilient to overfit while Neyshabur et al. [37] derives a capacity bound that relates model performance (error rate) to network size. As we discussed in the introduction, over-parameterization is reflected on over-regularization of the variational objective, leading the optimization to converge to solutions where the posterior falls back to the prior. Several works have dealt with such issue by either rescaling the KL divergence in the variational objective [19, 3], or by gradually including it during optimization [2, 46, 43].

4 Discussion and Conclusions

Inspired by the literature on scalable kernel methods, this paper proposed Walsh-Hadamard Variational Inference (WHVI). WHVI offers a novel parameterization of the variational posterior, which is particularly attractive for over-parameterized models, such as modern DNNs and CNNs. WHVI assumes a matrix-variate posterior distribution, which therefore captures covariances across weights. Crucially, unlike previous work on matrix-variate posteriors for VI, this is achieved with a low parameterization and fast computations, bypassing the over-regularization issues of VI for over-parameterized models. The large experimental campaign, demonstrates that WHVI is a strong competitor of other variational approaches for such models, while offering considerable speedups.

One limitation of the parameterization induced by WHVI is that its mean cannot span the whole $D^2$-dimensional space. A remedy could be to modify the parameterization from $S_1HGHS_2 + M$ with $\mathbb{E}[G] = 0$ so that the mean can span the whole space thanks to $M$, while the rest would allow to model covariances across weights in a cheap way. However, we carry out a numerical study on the RMSE between the weights induced by WHVI and arbitrary weight matrices can be found in the Appendix, showing constant behavior w.r.t. $D$.

The key operation that contributes to accelerate computations in WHVI is the Walsh-Hadamard transform. This has obvious connections with other matrix/vector operations, such as the Discrete Fourier Transform, so we are currently investigating whether it is possible to generalize WHVI to other kinds of transforms to increase flexibility.

We are currently investigating other extensions where we capture the covariance between weights across layers, by either sharing the matrix $G$ across, or by concatenating all weights into a single matrix which is then treated using WHVI, with the necessary adaptations to handle the sequential nature of computations. Finally, we are looking into employing WHVI for other models, such as deep generative models.
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References

[1] M. Bojarski, A. Choromanska, K. Choromanski, F. Fagan, C. Gouy-Pailler, A. Morvan, N. Sakr, T. Sarlos, and J. Atif. Structured Adaptive and Random Spinners for Fast Machine Learning Computations. In A. Singh and J. Zhu, editors, Proceedings of the 20th International Conference on Artificial Intelligence and Statistics, volume 54 of Proceedings of Machine Learning Research, pages 1020–1029, Fort Lauderdale, FL, USA, 20–22 Apr 2017. PMLR.

[2] S. R. Bowman, L. Vilnis, O. Vinyals, A. Dai, R. Jozefowicz, and S. Bengio. Generating Sentences from a Continuous Space. In Proceedings of The 20th SIGNLL Conference on Computational Natural Language Learning, pages 10–21. Association for Computational Linguistics, 2016.

[3] C. P. Burgess, I. Higgins, A. Pal, L. Matthey, N. Watters, G. Desjardins, and A. Lerchner. Understanding disentangling in β-VAE. CoRR, abs/1804.03599, 2018.

[4] Y. Cho and L. K. Saul. Kernel Methods for Deep Learning. In Y. Bengio, D. Schuurmans, J. D. Lafferty, C. K. I. Williams, and A. Culotta, editors, Advances in Neural Information Processing Systems 22, pages 342–350. Curran Associates, Inc., 2009.

[5] K. Cutajar, E. V. Bonilla, P. Michiardi, and M. Filippone. Random feature expansions for deep Gaussian processes. In D. Precup and Y. W. Teh, editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 884–893, International Convention Centre, Sydney, Australia, Aug. 2017. PMLR.

[6] A. G. de G. Matthews, J. Hron, M. Rowland, R. E. Turner, and Z. Ghahramani. Gaussian Process Behaviour in Wide Deep Neural Networks. In International Conference on Learning Representations, 2018.

[7] M. H. DeGroot and S. E. Fienberg. The comparison and evaluation of forecasters. Journal of the Royal Statistical Society. Series D (The Statistician), 32(1/2):12–22, 1983. ISSN 00390526, 14679884.

[8] S. Ding and D. Cook. Dimension folding PCA and PFC for matrix-valued predictors. Statistica Sinica, 24(1):463–492, 2014.

[9] M. M. Dunlop, M. A. Girolami, A. M. Stuart, and A. L. Teckentrup. How Deep Are Deep Gaussian Processes? Journal of Machine Learning Research, 19(1):2100–2145, Jan. 2018. ISSN 1532-4435.

[10] D. K. Duvenaud, O. Rippel, R. P. Adams, and Z. Ghahramani. Avoiding pathologies in very deep networks. In Proceedings of the Seventeenth International Conference on Artificial Intelligence and Statistics, AISTATS 2014, Reykjavik, Iceland, April 22-25, 2014, volume 33 of JMLR Workshop and Conference Proceedings, pages 202–210. JMLR.org, 2014.

[11] Y. Gal and Z. Ghahramani. Bayesian Convolutional Neural Networks with Bernoulli Approximate Variational Inference. CoRR, abs/1506.02158, 2015.

[12] Y. Gal and Z. Ghahramani. Dropout As a Bayesian Approximation: Representing Model Uncertainty in Deep Learning. In Proceedings of the 33rd International Conference on International Conference on Machine Learning - Volume 48, ICML’16, pages 1050–1059. JMLR.org, 2016.

[13] A. Garriga-Alonso, C. E. Rasmussen, and L. Aitchison. Deep Convolutional Networks as shallow Gaussian Processes. In International Conference on Learning Representations, 2019.

[14] A. Graves. Practical Variational Inference for Neural Networks. In J. Shawe-Taylor, R. S. Zemel, P. L. Bartlett, F. Pereira, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems 24, pages 2348–2356. Curran Associates, Inc., 2011.

[15] A. Gretton, K. Fukumizu, C. H. Teo, L. Song, B. Schölkopf, and A. J. Smola. A Kernel Statistical Test of Independence. In J. C. Platt, D. Koller, Y. Singer, and S. T. Roweis, editors, Advances in Neural Information Processing Systems 20, pages 585–592. Curran Associates, Inc., 2008.

[16] A. Gretton, K. M. Borgwardt, M. J. Rasch, B. Schölkopf, and A. Smola. A Kernel Two-sample Test. Journal of Machine Learning Research, 13:723–773, Mar. 2012. ISSN 1532-4435.

[17] A. K. Gupta and D. K. Nagar. Matrix variate distributions. Chapman and Hall/CRC, 1999.
[18] K. He, X. Zhang, S. Ren, and J. Sun. Deep Residual Learning for Image Recognition. In 2016 IEEE Conference on Computer Vision and Pattern Recognition, CVPR 2016, Las Vegas, NV, USA, June 27-30, 2016, pages 770–778, 2016.

[19] I. Higgins, L. Matthey, A. Pal, C. Burgess, X. Glorot, M. Botvinick, S. Mohamed, and A. Lerchner. beta-VAE: Learning Basic Visual Concepts with a Constrained Variational Framework. In International Conference on Learning Representations, 2017.

[20] I. Hubara, M. Courbariaux, D. Soudry, R. El-Yaniv, and Y. Bengio. Binarized neural networks. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 4107–4115. Curran Associates, Inc., 2016.

[21] M. I. Jordan, Z. Ghahramani, T. S. Jaakkola, and L. K. Saul. An Introduction to Variational Methods for Graphical Models. Machine Learning, 37(2):183–233, Nov. 1999.

[22] D. P. Kingma and M. Welling. Auto-Encoding Variational Bayes. In Proceedings of the Second International Conference on Learning Representations (ICLR 2014), Apr. 2014.

[23] D. P. Kingma, T. Salimans, and M. Welling. Variational Dropout and the Local Reparameterization Trick. In Advances in Neural Information Processing Systems 28, pages 2575–2583. Curran Associates, Inc., 2015.

[24] D. P. Kingma, T. Salimans, R. Jozefowicz, X. Chen, I. Sutskever, and M. Welling. Improved Variational Inference with Inverse Autoregressive Flow. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 4743–4751. Curran Associates, Inc., 2016.

[25] A. Krizhevsky, I. Sutskever, and G. E. Hinton. ImageNet Classification with Deep Convolutional Neural Networks. In F. Pereira, C. J. C. Burges, L. Bottou, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems 25, pages 1097–1105. Curran Associates, Inc., 2012.

[26] M. Lázaro-Gredilla, J. Quinonero-Candela, C. E. Rasmussen, and A. R. Figueiras-Vidal. Sparse Spectrum Gaussian Process Regression. Journal of Machine Learning Research, 11:1865–1881, 2010.

[27] Q. Le, T. Sarlos, and A. Smola. Fastfood - Approximating Kernel Expansions in Loglinear Time. In 30th International Conference on Machine Learning (ICML), 2013.

[28] C. Louizos and M. Welling. Structured and Efficient Variational Deep Learning with Matrix Gaussian Posteriors. In M. F. Balcan and K. Q. Weinberger, editors, Proceedings of The 33rd International Conference on Machine Learning, volume 48 of Proceedings of Machine Learning Research, pages 1708–1716, New York, New York, USA, 20–22 Jun 2016. PMLR.

[29] C. Louizos and M. Welling. Multiplicative Normalizing Flows for Variational Bayesian Neural Networks. In D. Precup and Y. W. Teh, editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 2218–2227, International Convention Centre, Sydney, Australia, 06–11 Aug 2017. PMLR.

[30] C. Louizos, K. Ullrich, and M. Welling. Bayesian Compression for Deep Learning. In I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett, editors, Advances in Neural Information Processing Systems 30, pages 3288–3298. Curran Associates, Inc., 2017.

[31] D. J. C. Mackay. Bayesian methods for backpropagation networks. In E. Domany, J. L. van Hemmen, and K. Schulten, editors, Models of Neural Networks III, chapter 6, pages 211–254. Springer, 1994.

[32] A. G. d. G. Matthews, M. van der Wilk, T. Nickson, K. Fujii, A. Boukouvalas, P. León-Villagrá, Z. Ghahramani, and J. Hensman. GPflow: A Gaussian process library using TensorFlow. Journal of Machine Learning Research, 18(40):1–6, apr 2017.

[33] D. Molchanov, A. Ashukha, and D. Vetrov. Variational Dropout Spursifies Deep Neural Networks. In D. Precup and Y. W. Teh, editors, Proceedings of the 34th International Conference on Machine Learning, volume 70 of Proceedings of Machine Learning Research, pages 2498–2507, International Convention Centre, Sydney, Australia, 06–11 Aug 2017. PMLR.

[34] M. P. Naeini, G. F. Cooper, and M. Hauskrecht. Obtaining well calibrated probabilities using Bayesian binning. In AAAI, pages 2901–2907. AAAI Press, 2015.

[35] R. M. Neal. Bayesian Learning for Neural Networks. Springer-Verlag, Berlin, Heidelberg, 1996. ISBN 0387947248.
[36] B. Neyshabur, R. Tomioka, and N. Srebro. In Search of the Real Inductive Bias: On the Role of Implicit Regularization in Deep Learning. In ICLR (Workshop), 2015.

[37] B. Neyshabur, Z. Li, S. Bhojanapalli, Y. LeCun, and N. Srebro. The role of over-parametrization in generalization of neural networks. In International Conference on Learning Representations, 2019.

[38] A. Niculescu-Mizil and R. Caruana. Predicting Good Probabilities with Supervised Learning. In Proceedings of the 22Nd International Conference on Machine Learning, ICML ’05, pages 625–632, New York, NY, USA, 2005. ACM.

[39] A. Novikov, D. Podoprikhin, A. Osokin, and D. P. Vetrov. Tensorizing Neural Networks. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, Advances in Neural Information Processing Systems 28, pages 442–450. Curran Associates, Inc., 2015.

[40] A. Paszke, S. Gross, S. Chintala, G. Chanan, E. Yang, Z. DeVito, Z. Lin, A. Desmaison, L. Antiga, and A. Lerer. Automatic differentiation in PyTorch. In NIPS-W, 2017.

[41] A. Rahimi and B. Recht. Random Features for Large-Scale Kernel Machines. In J. C. Platt, D. Koller, Y. Singer, and S. T. Roweis, editors, Advances in Neural Information Processing Systems 20, pages 1177–1184. Curran Associates, Inc., 2008.

[42] D. Rezende and S. Mohamed. Variational inference with normalizing flows. In F. Bach and D. Blei, editors, Proceedings of the 32nd International Conference on Machine Learning, volume 37 of Proceedings of Machine Learning Research, pages 1530–1538, Lille, France, 07–09 Jul 2015. PMLR.

[43] S. Rossi, P. Michiardi, and M. Filippone. Good Initializations of Variational Bayes for Deep Models. In K. Chaudhuri and R. Salakhutdinov, editors, Proceedings of the 36th International Conference on Machine Learning, volume 97 of Proceedings of Machine Learning Research, pages 5487–5497, Long Beach, California, USA, 09–15 Jun 2019. PMLR.

[44] D. Sejdinovic, H. Strathmann, M. L. Garcia, C. Andrieu, and A. Gretton. Kernel Adaptive Metropolis-Hastings. In E. P. Xing and T. Jebara, editors, Proceedings of the 31st International Conference on Machine Learning, volume 32 of Proceedings of Machine Learning Research, pages 1665–1673, Beijing, China, 22–24 Jun 2014. PMLR.

[45] K. Simonyan and A. Zisserman. Very Deep Convolutional Networks for Large-Scale Image Recognition. CoRR, abs/1409.1556, 2014.

[46] C. K. Sønderby, T. Raiko, L. Maaløe, S. K. Sønderby, and O. Winther. Ladder Variational Autoencoders. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 3738–3746. Curran Associates, Inc., 2016.

[47] H. Strathmann, D. Sejdinovic, S. Livingstone, Z. Szabo, and A. Gretton. Gradient-free Hamiltonian Monte Carlo with Efficient Kernel Exponential Families. In C. Cortes, N. D. Lawrence, D. D. Lee, M. Sugiyama, and R. Garnett, editors, Advances in Neural Information Processing Systems 28, pages 955–963. Curran Associates, Inc., 2015.

[48] S. Surjanovic and D. Bingham. Virtual library of simulation experiments: Test functions and datasets. Retrieved May 22, 2019, from http://www.sfu.ca/~ssurjano

[49] G.-L. Tran, E. V. Bonilla, J. Cunningham, P. Michiardi, and M. Filippone. Calibrating Deep Convolutional Gaussian Processes. In K. Chaudhuri and M. Sugiyama, editors, Proceedings of Machine Learning Research, volume 89 of Proceedings of Machine Learning Research, pages 1554–1563. PMLR, 16–18 Apr 2019.

[50] J. A. Tropp. Improved Analysis of the subsampled Randomized Hadamard Transform. Advances in Adaptive Data Analysis, 3(1-2):115–126, 2011.

[51] R. Van den Berg, L. Hasenclever, J. M. Tomczak, and M. Welling. Sylvester Normalizing Flows for Variational Inference. In UAI ’18: Proceedings of the Thirty-Fourth Conference on Uncertainty in Artificial Intelligence, 2018.

[52] W. Wen, C. Wu, Y. Wang, Y. Chen, and H. Li. Learning Structured Sparsity in Deep Neural Networks. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 2074–2082. Curran Associates, Inc., 2016.

[53] Z. Yang, M. Moczulski, M. Denil, N. d. Freitas, A. Smola, L. Song, and Z. Wang. Deep fried convnets. In 2015 IEEE International Conference on Computer Vision (ICCV), pages 1476–1483, Dec 2015.
[54] F. X. Yu, A. T. Suresh, K. M. Choromanski, D. N. Holtmann-Rice, and S. Kumar. Orthogonal Random Features. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 1975–1983. Curran Associates, Inc., 2016.

[55] F. X. Xu, A. T. Suresh, K. M. Choromanski, D. N. Holtmann-Rice, and S. Kumar. Orthogonal Random Features. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 1975–1983. Curran Associates, Inc., 2016.

[56] W. Zaremba, A. Gretton, and M. Blaschko. B-test: A Non-parametric, Low Variance Kernel Two-sample Test. In C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems 26, pages 755–763. Curran Associates, Inc., 2013.

[57] G. Zhang, S. Sun, D. Duvenaud, and R. Grosse. Noisy Natural Gradient as Variational Inference. In J. Dy and A. Krause, editors, Proceedings of the 35th International Conference on Machine Learning, volume 80 of Proceedings of Machine Learning Research, pages 5852–5861, Stockholmsmässan, Stockholm Sweden, 10–15 Jul 2018. PMLR.

[58] M. Zhu and S. Gupta. To Prune, or Not to Prune: Exploring the Efficacy of Pruning for Model Compression. In ICLR (Workshop). OpenReview.net, 2018.
A Matrix-variate posterior distribution induced by WHVI

We derive the parameters of the matrix-variate distribution \( q(W) = MN(M, U, V) \) of the weight matrix \( \hat{W} \in \mathbb{R}^{D \times D} \) given by WHVI,

\[
\hat{W} = S_1H \text{diag}(g)HS_2 \quad \text{with} \quad g \sim N(\mu, \Sigma). \tag{11}
\]

The mean \( M = S_1H \text{diag}(\mu)HS_2 \) is directly obtained using the linearity of the expectation. The covariance matrices \( U \) and \( V \) are non-identifiable in the sense that for any scale factor \( s > 0 \), we have \( MN(M, sU, \frac{1}{s}V) \). Therefore, we constrain the parameters such that \( \text{Tr}(V) = 1 \). The covariance matrices verify the following second-order expectations (see e.g. Section 1 in the Appendix of [8])

\[
U = \mathbb{E} \left[ (W - M)(W - M)^\top \right],
\]

\[
V = \frac{1}{\text{Tr}(U)} \mathbb{E} \left[ (W - M)^\top (W - M) \right].
\]

Recall that the Walsh-Hadamard matrix \( H \) is symmetric. Denoting by \( \Sigma^{1/2} \) a root of \( \Sigma \) and considering \( \epsilon \sim N(0, I) \), we have

\[
U = \mathbb{E} \left[ S_1H \text{diag}(\Sigma^{1/2} \epsilon)HS_2^2H \text{diag}(\Sigma^{1/2} \epsilon)HS_1 \right]. \tag{12}
\]

If we define the matrix \( T_2 \in \mathbb{R}^{D \times D^2} \) where the \( i \)th row is the column-wise vectorization of the matrix \( (\Sigma^{1/2}_{1,j'}(HS_2)_{i,j'})_{j,j' \leq D} \). We have

\[
(T_2^\top T_2)^{1/2}_{i,i'} = \sum_{j,j' = 1}^{D} \Sigma^{1/2}_{1,j}(HS_2)_{i,j'}(HS_2)_{i',j'}
\]

\[
= \sum_{j,j',j'' = 1}^{D} \Sigma^{1/2}_{1,j}(HS_2)_{i,j'}\mathbb{E}[\epsilon_j \epsilon_{j''}]\Sigma^{1/2}_{1,j''}(HS_2)_{i',j''}
\]

\[
= \sum_{j' = 1}^{D} \mathbb{E} \left[ \left( \sum_{j = 1}^{D} \epsilon_j \Sigma^{1/2}_{1,j}(HS_2)_{i,j'} \right) \left( \sum_{j'' = 1}^{D} \epsilon_{j''} \Sigma^{1/2}_{1,j''}(HS_2)_{i',j''} \right) \right]
\]

\[
= \mathbb{E} \left[ \text{diag}(\Sigma^{1/2} \epsilon)HS_2^2H \text{diag}(\Sigma^{1/2} \epsilon) \right]_{i,i'}.
\]

Using [12], we can then write a formula for a root of \( U = U^{1/2}U^{1/2 \top} \):

\[
U^{1/2} = S_1HT_2. \tag{13}
\]

Similarly for \( V \), we have

\[
V^{1/2} = \frac{1}{\sqrt{\text{Tr}(U)}}S_2HT_1, \quad \text{with} \quad T_1 = \left[ \begin{array}{c} \text{vect} \left( \Sigma_{1,1} (HS_1)_{1,1}^\top \right) \\
\vdots \\
\text{vect} \left( \Sigma_{D,1} (HS_1)_{D,1}^\top \right) \end{array} \right]. \tag{14}
\]

B A geometric interpretation of WHVI

The matrix \( A \) in Section 2.2 expresses the linear relationship between the weights \( W = S_1HGH \) and the variational random vector \( g \), i.e. \( \text{vect}(W) = Ag \). Recall the definition of

\[
A = \begin{bmatrix} S_1H \text{diag}(v_1) \\
\vdots \\
S_1H \text{diag}(v_D) \end{bmatrix}, \quad \text{with} \quad v_i = (S_2)_{i,i}(H)_{i,i}. \tag{15}
\]

We show that a \( LQ \)-decomposition of \( A \) can be explicitly formulated.
Figure 8: Distribution of the minimum RMSE between $S_1 H G H S_2$ and a sample matrix with i.i.d. $U(-1, 1)$ entries. For each dimension, the orange dots represent 20 repetitions. The median distance is displayed in black. Few outliers (with distance greater than 3.0) appeared, possibly due to imperfect numerical optimization. They were kept for the calculation of the median but not displayed.

**Proposition.** Let $A$ be a $D^2 \times D$ matrix such that $\text{vect}(W) = A g$, where $W$ is given by $W = S_1 H \text{diag}(g) H S_2$. Then a $LQ$-decomposition of $A$ can be formulated as

$$
\text{vect}(W) = [s^{(2)}_i S_1 H \text{diag}(h_i)]_{i=1,...,D} g = LQg,
$$

where $h_i$ is the $i$th column of $H$, $L = \text{diag}([s^{(2)}_i s^{(1)}]_{i=1,...,D})$, $\text{diag}(s^{(1)}) = S_1$, $\text{diag}(s^{(2)}) = S_2$, and $Q = [H \text{diag}(h_i)]_{i=1,...,D}$.

**Proof.** Equation (16) derives directly from block matrix and vector operations. As $L$ is clearly lower triangular (even diagonal), let us proof that $Q$ has orthogonal columns. Defining the $d \times d$ matrix $Q^{(i)} = H \text{diag}(h_i)$, we have:

$$
Q^{\top} Q = \sum_{i=1}^{D} Q^{(i)} \top Q^{(i)}
= \sum_{i=1}^{D} \text{diag}(h_i) H \top H \text{diag}(h_i)
= \sum_{i=1}^{D} \text{diag}(h^2_i) = \sum_{i=1}^{D} 1_D I = I.
$$

This decomposition gives direct insight on the role of the Walsh-Hadamard transforms: with complexity $D \log(D)$, they perform fast rotations $Q$ of vectors living in a space of dimension $D$ (the plane in Figure 7) into a space of dimension $D^2$ (the cube in Figure 7). Treated as parameters gathered in $L$, $S_1$, and $S_2$ control the orientation of the subspace by distortion of the canonical axes.

We empirically evaluate the minimum RMSE, as a proxy for some measure of average distance, between $W$ and any given point $\Gamma$. More precisely, we compute for $\Gamma \in \mathbb{R}^{D \times D}$,

$$
\min_{s_1,s_2,g \in \mathbb{R}^D} \sqrt{\sum_{i=1}^{D} \sum_{j=1}^{D} (\gamma_{ij} - (\text{diag}(s_1) H \text{diag}(g) H \text{diag}(s_2))_{ij})^2 / D^2}.
$$

Figure 8 shows this quantity evaluated for $\Gamma$ sampled with i.i.d $U(-1, 1)$ with increasing value of $D$. The bounded behavior suggests that WHV1 can approximate any given matrices with a precision that does not increase with the dimension.
C Additional details on Normalizing Flows

In the general setting, given a probabilistic model with observations \( x \), latent variables \( z \) and model parameters \( \theta \), the variational lower bound to the log-marginal likelihood is defined as

\[
\log p_\theta(x) = \log \int p_\theta(x|z)p(z)dz \\
= \log \int p_\theta(x|z) \frac{p(z)}{q_\phi(z|x)} q_\phi(z|x)dz \\
\geq -KL(q_\phi(z|x)||p(z)) + E_{q_\phi(z|x)} \log p_\theta(x|z) ,
\]

where \( p_\theta(x|z) \) is the likelihood function with \( \theta \) model parameters and \( p(z) \) is the prior on the latents. The objective is then to minimize the negative variational bound (NELBO):

\[
\mathcal{L}(\theta, \phi) = -E_{q_\phi(z|x)} \log p_\theta(x|z) + KL(q_\phi(z|x)||p(z)) .
\]

Consider an invertible, continuous and differentiable function \( f : \mathbb{R}^D \rightarrow \mathbb{R}^D \). Given \( z_0 \sim q(z_0) \), then \( z_1 = f(z_0) \) follows \( q(z_1) \) defined as

\[
q(z_1) = q(z_0) \left| \det \frac{\partial f}{\partial z_0} \right|^{-1} .
\]

As a consequence, after \( K \) transformations the log-density of the final distribution is

\[
\log q(z_K) = \log q(z_0) - \sum_{k=1}^K \log \left| \det \frac{\partial f_k}{\partial z_k} \right| .
\]

We shall define \( f_k(z_{k-1}; \lambda_k) \) the \( k \)th transformation which takes input from the previous flow \( z_{k-1} \) and has parameters \( \lambda_k \). The final variational objective is

\[
\mathcal{L}(\theta, \phi) = -E_{q_\phi(z|x)} \{ \log p_\theta(x|z) \} + KL(q_\phi(z|x)||p(z)) \\
= E_{q_\phi(z|x)} \{ -\log p_\theta(x|z) - \log p(z) + \log q_\phi(z|x) \} \\
= E_{q_\phi(z_0)} \{ -\log p_\theta(x|z_K) - \log p(z_K) + \log q_\phi(z_K|x) \} \\
= E_{q_\phi(z_0)} \left\{ -\log p_\theta(x|z_K) - \log p(z_K) + \log q_\phi(z_0|x) - \sum_{k=1}^K \log \left| \det \frac{\partial f_k(z_{k-1}; \lambda_k)}{\partial z_{k-1}} \right| \right\} \\
= -E_{q_\phi(z_0)} \log p_\theta(x|z) + KL(q_\phi(z_0|x)||p(z_K)) - E_{q_\phi(z_0)} \sum_{k=1}^K \log \left| \det \frac{\partial f_k(z_{k-1}; \lambda_k)}{\partial z_{k-1}} \right| .
\]

Setting the initial distribution \( q_0 \) to a fully factorized Gaussian \( \mathcal{N}(z_0|\mu, \sigma^2) \) and assuming a Gaussian prior on the generated \( z_K \), the \( k \)th term is analytically tractable. A possible family of transformation is the planar flow [42]. For the planar flow, \( f \) is defined as

\[
f(z) = z + uh(w^Tw + b) ,
\]

where \( \lambda = \{ u \in \mathbb{R}^D, w \in \mathbb{R}^D, b \in \mathbb{R} \} \) and \( h(\cdot) = \tanh(\cdot) \). This is equivalent to a residual layer with single neuron MLP – as argued by Kingma et al. [24]. The log-determinant of the Jacobian of \( f \) is

\[
\log \left| \det \frac{\partial f}{\partial z} \right| = \log \left| \det (I + \frac{\partial h}{\partial z}) \right| \\
= 1 + \frac{\partial h}{\partial z}w^T \frac{\partial h}{\partial z} + b^T .
\]

Alternatively can be found in [42][51][24][29].

D Additional results

D.1 Results - Gaussian Processes with Random Feature Expansion

We test WHV1 for scalable GP inference, by focusing on GPs with random feature expansions [26]. In GP models, latent variables \( f \) are given a prior \( p(f) = \mathcal{N}(0|K) \); the assumption of zero mean can be easily relaxed. Given a random feature expansion of the kernel matrix, say \( K \approx \Phi \Phi^\top \), the latent variables can be rewritten as:

\[
f = \Phi w
\]
We report the results on five datasets (10000 which are known to lead to lower approximation error on the elements on the kernel matrix compared to random with

\[
\text{Table 6:} \quad \text{Complexity table for GPs with random feature and inducing points approximations. In the case of random features, we include both the complexity of computing random features and the complexity of treating the linear combination of the weights variationally (using VI and WHVI).}
\]

| NAME           | TASK | N. | D-IN | D-OUT |
|----------------|------|----|------|-------|
| BOSTON         | Regr.| 506| 13   | 1     |
| CONCRETE       | Regr.| 1030| 8    | 1     |
| ENERGY         | Regr.| 768| 8    | 2     |
| KINSMN         | Regr.| 8192| 8    | 1     |
| NAVAL          | Regr.| 11934| 16   | 2     |
| POWERPLANT     | Regr.| 9568| 4    | 1     |
| PROTEIN        | Regr.| 45730| 9    | 1     |
| YACHT          | Regr.| 308| 6    | 1     |
| BOREHOL        | Regr.| 200000| 8   | 1     |
| HARTMAN6       | Regr.| 30000| 6    | 1     |
| RASTRIGIN5     | Regr.| 100000| 5   | 1     |
| ROBOT          | Regr.| 150000| 8   | 1     |
| OTLCIRCUIT     | Regr.| 200000| 6   | 1     |
| EEG            | Class.| 14980| 14   | 2     |
| MAGIC          | Class.| 19020| 10   | 2     |
| MINIBOOS       | Class.| 130064| 50  | 2     |
| LETTER         | Class.| 20000| 16   | 26    |
| DRIVE          | Class.| 58509| 48   | 11    |
| MOCLASS        | Class.| 78095| 37   | 5     |
| CIFAR10        | Class.| 60000| 3 x 28 x 28 | 10  |

By reshaping the vector of parameters \( \mathbf{w} \) of the linear model into a \( D \times D \) matrix, WHVI allows for the linearized GP model to reduce the number of parameters to optimize (see Table 6). We compare WHVI with two alternatives; one is VI of the Fourier features GP expansion that uses less random features to match the number of parameters used in WHVI, and another is the sparse Gaussian process implementation of GPFLOW [32] with a number of inducing points (rounded up) to match the number of parameters used in WHVI.

We report the results on five datasets (10000 \( \leq N \leq 200000 \), \( 5 \leq D \leq 8 \), see Table 5). The data sets are generated from space-filling evaluations of well known functions in analysis of computer experiments (see e.g. [48]). Dataset splitting in training and testing points is random uniform, 20% versus 80%. The input variables are rescaled between 0 and 1. The output values are standardized for training. All GPs have the same prior (centered GP with RBF covariance), initialized with equal hyperparameter values: each of the \( D \) lengthscales to \( \sqrt{D/2} \), the GP variance to 1, the Gaussian likelihood standard deviation to 0.02 (prior observation noise). The training is performed with 12000 steps of Adam optimizer. The observation noise is fixed for the first 10000 steps. Learning rate is \( 6 \times 10^{-4} \), except for the dataset HARTMAN6 with a learning rate of \( 5 \times 10^{-3} \). Sparse GPs are run with whitened representation of the inducing points.

The results are shown in Figure 9 with diagonal covariance for the three variational posteriors, and with full covariance in Figure 10. In both mean field and full covariance settings, this variant of WHVI using the reshaping of \( \mathbf{W} \) into a column largely outperforms the direct VI of Fourier features. However, it appears that this improvement of the random feature inference for GPs is not enough to reach the performance of VI using inducing points. Inducing point approximations are based on the Nyström approximation of kernel matrices, which are known to lead to lower approximation error on the elements on the kernel matrix compared to random features approximations. This is the reason we attribute to the lower performance of WHVI compared to inducing points approximations in this experiment.

**D.2 Extended results - DNNS**

Being able to increase width and depth of a model without drastically increasing the number of variational parameters is one of the competitive advantages of WHVI. Figure 11 shows the behavior of WHVI for different network configurations. At test time, increasing the number of hidden layers and the numbers of hidden features

\[
\text{Note: } M \text{ is the number of pseudo-data/inducing points and } N_{RF} \text{ is the number of random features used in the kernel approximation.}
\]
allow the model to avoid overfitting while delivering better performance. This evidence is also supported by the analysis of the test MNLL during optimization of the ELBO, as showed in Figure 14 and Figure 12.

Thanks to WHVI structure of the weights matrices, expanding and deepening the model is beneficial not only at convergence but during the entire learning procedure as well. While the analysis of the optimization problem is not a primary focus of this work, the behavior of the training ELBO reported in Figure 13 shows that performance at convergence depends exclusively on the model size. Furthermore, the derived NELBO is still a valid lower bound of the true marginal likelihood and, therefore, a suitable objective function for model selection. Differently from the issue addressed in [43], during our experiments we didn’t experience problems regarding initialization.

**Figure 9:** Comparison of test errors with respect to the number model parameters.

**Figure 10:** Comparison of test errors with respect to the number model parameters.
of variational parameters. We claim that this is possible thanks to both the reduced number of parameters and the effect of the Walsh-Hadamard transform.

**Timing profiling of the Fast Walsh-Hadamard transform** Key to the log-linear time complexity is the Fast Walsh-Hadamard transform, which allows to perform the operation $Hx$ in $O(D \log D)$ time without requiring to generate and store $H$. For our experimental evaluation, we implemented a FWHT operation in PyTorch (v. 0.4.1) in C++ and CUDA to leverage the full computational capabilities of modern GPUs. Figure 15 presents a timing profiling of our implementation versus the naive matmul (batch size of 512 samples and profiling repeated 1000 times). The breakeven point for the CPU implementation is in the neighborhood of 512/1024 features, while on GPU we see FWHT is consistently faster.

![Figure 11: Analysis of model capacity for different features and hidden layers.](image)

![Figure 12: Selection of test performance curves versus training optimization steps with 2 hidden layers.](image)
Figure 13: Selection of training ELBO (2 hidden layers). Optimization is not an issue and the performance at convergence depends only on the model (no problems with initialization, over regularization, ...)

Figure 14: Comparison of test performance. Being able to increase features and hidden layers without worrying about overfitting/overparametrize the model is advantageous not only at convergence but during the entire learning procedure.

Figure 15: On the (left), time performance versus number of features ($D$) with batch size fixed to 512. On the (right) distribution of inference time versus batch size ($D=512$) with MATMUL and FWHT on GPU.
### Table 7: Results of Bayesian DNN on 6 classification datasets. Note: NL: number of hidden layers, NF: number of hidden features

| MODEL | NL | DATASET | DRIVE | EEG | LETTER | MAGIC | MINIBOO | TEST ERROR | MOCAP |
|-------|----|---------|-------|-----|--------|-------|---------|------------|-------|
| MCD   | 2  | 64      | 0.19 ± 0.11 | 0.16 ± 0.01 | 0.45 ± 0.05 | 0.13 ± 0.02 | 0.07 ± 0.00 | 0.02 ± 0.02 | 0.52 ± 0.24 | 0.36 ± 0.02 | 1.27 ± 0.26 | 0.37 ± 0.12 | 0.18 ± 0.00 | 0.11 ± 0.10 |
| 128    | 0.17 ± 0.07 | 0.19 ± 0.11 | 0.45 ± 0.04 | 0.16 ± 0.08 | 0.15 ± 0.21 | 0.04 ± 0.07 | 0.47 ± 0.19 | 0.36 ± 0.09 | 1.39 ± 0.22 | 0.33 ± 0.04 | 0.24 ± 0.17 | 0.10 ± 0.11 |
| 256    | 0.16 ± 0.09 | 0.20 ± 0.15 | 0.45 ± 0.06 | 0.13 ± 0.01 | 0.07 ± 0.00 | 0.16 ± 0.13 | 0.50 ± 0.29 | 0.33 ± 0.08 | 1.32 ± 0.25 | 0.35 ± 0.09 | 0.17 ± 0.00 | 0.29 ± 0.21 |
| 512    | 0.18 ± 0.11 | 0.18 ± 0.15 | 0.44 ± 0.02 | 0.18 ± 0.10 | 0.07 ± 0.00 | 0.03 ± 0.06 | 0.47 ± 0.27 | 0.95 ± 1.61 | 1.41 ± 0.17 | 0.40 ± 0.06 | 0.20 ± 0.04 | 0.17 ± 0.22 |
| 3     | 64  | 0.34 ± 0.10 | 0.13 ± 0.01 | 0.50 ± 0.06 | 0.16 ± 0.07 | 0.08 ± 0.02 | 0.09 ± 0.09 | 0.88 ± 0.25 | 0.55 ± 0.61 | 1.56 ± 0.28 | 0.42 ± 0.16 | 0.20 ± 0.05 | 0.18 ± 0.15 |
| 128    | 0.32 ± 0.10 | 0.21 ± 0.14 | 0.48 ± 0.09 | 0.16 ± 0.07 | 0.23 ± 0.28 | 0.11 ± 0.19 | 0.86 ± 0.28 | 1.46 ± 2.78 | 1.40 ± 0.34 | 0.44 ± 0.13 | 0.28 ± 0.18 | 0.34 ± 0.28 |
| 256    | 0.32 ± 0.21 | 0.23 ± 0.17 | 0.43 ± 0.05 | 0.14 ± 0.00 | 0.23 ± 0.28 | 0.28 ± 0.26 | 0.87 ± 0.51 | 0.40 ± 0.99 | 1.34 ± 0.19 | 0.62 ± 0.07 | 0.31 ± 0.22 | 0.61 ± 0.48 |
| 512    | 0.36 ± 0.09 | 0.14 ± 0.11 | 0.49 ± 0.06 | 0.14 ± 0.01 | 0.23 ± 0.28 | 0.23 ± 0.12 | 0.93 ± 0.27 | 0.74 ± 0.78 | 1.92 ± 0.23 | 1.02 ± 0.15 | 0.30 ± 0.20 | 0.45 ± 0.27 |

### Table 8: Test error of Bayesian DNN with 2 hidden layers on regression datasets. NF: number of hidden features

| MODEL | DATASET | BOSTON | CONCRETE | ENERGY | KINSM | NAVAL | POWERPLANT | PROTEIN | TEST ERROR | YACHT |
|-------|---------|--------|----------|--------|-------|-------|------------|--------|------------|-------|
| MCD   | 64      | 3.80 ± 0.88 | 5.43 ± 0.69 | 2.13 ± 0.12 | 0.17 ± 0.22 | 0.07 ± 0.00 | 32.57 ± 0.10 | 4.36 ± 0.12 | 2.02 ± 0.51 |
| 128    | 3.91 ± 0.86 | 5.12 ± 0.79 | 2.07 ± 0.11 | 0.09 ± 0.00 | 0.30 ± 0.30 | 31.63 ± 0.07 | 4.23 ± 0.10 | 1.90 ± 0.54 |
| 256    | 3.62 ± 1.01 | 5.03 ± 0.74 | 2.04 ± 0.11 | 0.10 ± 0.00 | 0.07 ± 0.00 | N/A | 4.09 ± 0.11 | 2.09 ± 0.66 |
| 512    | 3.56 ± 0.85 | 4.81 ± 0.79 | 2.03 ± 0.12 | 0.09 ± 0.00 | 0.07 ± 0.00 | 31.45 ± 0.05 | 3.87 ± 0.11 | 2.09 ± 0.55 |
| NNG   | 64      | 3.20 ± 0.26 | 6.00 ± 0.59 | 1.54 ± 0.18 | 0.07 ± 0.00 | 0.00 ± 0.00 | 3.94 ± 0.05 | 3.90 ± 0.02 | 3.57 ± 0.79 |
| 128    | 3.56 ± 0.43 | 8.21 ± 0.55 | 1.96 ± 0.28 | 0.07 ± 0.00 | 0.00 ± 0.00 | 4.23 ± 0.09 | 4.57 ± 0.47 | 5.16 ± 1.48 |
| 256    | 4.87 ± 0.94 | 8.18 ± 0.57 | 3.41 ± 0.55 | 0.07 ± 0.00 | 0.00 ± 0.00 | 4.07 ± 0.00 | 4.88 ± 0.00 | 5.60 ± 0.65 |
| 512    | 5.19 ± 0.62 | 11.67 ± 2.06 | 5.12 ± 0.37 | 0.10 ± 0.00 | 0.00 ± 0.00 | 4.97 ± 0.00 | N/A | 5.91 ± 0.80 |
| WHV   | 64      | 3.33 ± 0.82 | 5.24 ± 0.77 | 0.73 ± 0.11 | 0.08 ± 0.00 | 0.01 ± 0.00 | 4.07 ± 0.11 | 4.49 ± 0.12 | 0.82 ± 0.18 |
| 128    | 3.14 ± 0.71 | 4.70 ± 0.72 | 0.56 ± 0.07 | 0.08 ± 0.00 | 0.01 ± 0.00 | 4.09 ± 0.12 | 4.36 ± 0.11 | 0.69 ± 0.16 |
| 256    | 2.99 ± 0.85 | 4.63 ± 0.78 | 0.52 ± 0.07 | 0.08 ± 0.00 | 0.01 ± 0.00 | 3.95 ± 0.12 | 4.24 ± 0.11 | 0.76 ± 0.13 |
| 512    | 2.99 ± 0.69 | 4.51 ± 0.80 | 0.51 ± 0.04 | 0.07 ± 0.00 | 0.01 ± 0.00 | 3.96 ± 0.12 | 4.14 ± 0.09 | 0.71 ± 0.16 |
Table 9: Test MNLL of Bayesian DNN with 2 hidden layers on regression datasets. NF: number of hidden features

| MODEL | DATASET  | NF | BOSTON | CONCRETE | ENERGY | KN9SM | NAVAL | POWERPLANT | PROTEIN | TEST MNLL |
|-------|----------|----|--------|----------|--------|-------|--------|-------------|---------|-----------|
| MNLL  | 64       | 5.67 ± 2.35 | 3.19 ± 0.28 | 4.19 ± 0.15 | -0.78 ± 0.69 | -2.68 ± 0.00 | 51.73 ± 0.47 | 2.79 ± 0.01 | 2.85 ± 1.02 |
|       | 128      | 6.90 ± 2.93 | 3.20 ± 0.36 | 4.15 ± 0.15 | -0.87 ± 0.02 | -1.00 ± 2.27 | 49.78 ± 0.17 | 2.76 ± 0.02 | 2.95 ± 1.27 |
|       | 256      | 6.60 ± 3.59 | 3.31 ± 0.45 | 4.13 ± 0.15 | -0.70 ± 0.05 | -2.70 ± 0.00 | N/A    | 2.72 ± 0.04 | 3.79 ± 1.88 |
| 512   | 7.28 ± 3.31 | 3.45 ± 0.59 | 4.11 ± 0.17 | -0.76 ± 0.03 | -2.71 ± 0.00 | 49.43 ± 0.17 | 2.68 ± 0.02 | 3.76 ± 1.65 |
| NNG   | 64       | 2.69 ± 0.06 | 3.40 ± 0.15 | 1.95 ± 0.08 | -1.14 ± 0.05 | -5.83 ± 1.49 | 2.80 ± 0.01 | 2.78 ± 0.01 | 2.71 ± 0.17 |
|       | 128      | 2.72 ± 0.09 | 3.56 ± 0.08 | 2.11 ± 0.12 | -1.19 ± 0.04 | -6.52 ± 0.09 | 2.86 ± 0.02 | 2.95 ± 0.12 | 3.06 ± 0.27 |
|       | 256      | 3.04 ± 0.22 | 3.52 ± 0.07 | 2.64 ± 0.17 | -1.19 ± 0.03 | -5.73 ± 0.21 | 2.84 ± 0.00 | 3.02 ± 0.01 | 3.15 ± 0.13 |
|       | 512      | 3.13 ± 0.14 | 3.91 ± 0.20 | 3.07 ± 0.07 | -0.80 ± 0.00 | -5.30 ± 0.05 | 3.51 ± 0.00 | N/A         | 3.21 ± 0.14 |
| WHVI  | 64       | 3.68 ± 1.40 | 3.19 ± 0.34 | 2.18 ± 0.37 | -1.13 ± 0.02 | -6.25 ± 0.01 | 2.73 ± 0.03 | 2.82 ± 0.01 | 2.56 ± 1.33 |
|       | 128      | 4.33 ± 1.80 | 3.17 ± 0.37 | 2.00 ± 0.60 | -1.19 ± 0.04 | -6.25 ± 0.01 | 2.73 ± 0.03 | 2.79 ± 0.01 | 1.80 ± 1.01 |
|       | 256      | 4.99 ± 2.65 | 3.35 ± 0.59 | 2.06 ± 0.72 | -1.28 ± 0.04 | -6.25 ± 0.01 | 2.70 ± 0.03 | 2.77 ± 0.01 | 1.53 ± 0.53 |
|       | 512      | 5.41 ± 2.30 | 3.33 ± 0.56 | 2.05 ± 0.46 | -1.22 ± 0.04 | -6.25 ± 0.01 | 2.70 ± 0.03 | 2.74 ± 0.01 | 1.37 ± 0.57 |