QUANTITATIVE WEIGHTED ESTIMATES FOR ROUGH SINGULAR INTEGRALS ON HOMOGENEOUS GROUPS

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Abstract. In this paper, we study weighted $L^p(w)$ boundedness ($1 < p < \infty$ and $w$ a Muckenhoupt $A_p$ weight) of singular integrals with homogeneous convolution kernel $K(x)$ on an arbitrary homogeneous group $H$ of dimension $Q$, under the assumption that $K_0$, the restriction of $K$ to the unit annulus, is mean zero and $L^q$ integrable for some $0 < q < \infty$, where $q_0$ is a fixed constant depending on $w$. We obtain a quantitative weighted bound, which is consistent with the one obtained by Hytönen–Roncal–Tapiola in the Euclidean setting, for this operator on $L^p(w)$. Comparing to the previous results in the Euclidean setting, our assumptions on the kernel and on the underlying space are weaker. Moreover, we investigate the quantitative weighted bound for the bi-parameter rough singular integrals on product homogeneous Lie groups.

1. Introduction

Let $H = \mathbb{R}^n$ be a homogeneous group (see [24, 50]), which is a nilpotent Liegroup with multiplication, inverse, dilation, and norm structures $(x, y) \mapsto xy$, $x \mapsto x^{-1}$, $(t, x) \mapsto t \circ x$, $x \mapsto \rho(x)$ for $x, y \in H$, $t > 0$. The multiplication and inverse operations are polynomials and form a group with identity $0$, the dilation structure preserves the group operations and is given in coordinates by

$$t \circ (x_1, \ldots, x_n) = (t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n)$$

for some constants $0 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. Besides, $\rho(x) := \max\{|x_j|^{1/a_j}\}$ is a norm associated to the dilation structure. We call $n$ the Euclidean dimension of $H$, and the quantity $Q = \sum_{j=1}^n \alpha_j$ the homogeneous dimension of $H$, respectively.

We now recall the notion of homogeneous singular integrals on homogeneous group. Let $\Sigma := \{x \in H : \rho(x) = 1\}$ and $K$ be a homogeneous convolution kernel on $H$, so that

$$K(x) = \frac{\Omega(\rho(x)^{-1} \circ x)}{\rho(x)^Q} \quad \text{with} \quad \int_{\Sigma} \Omega(\theta) dS(\theta) = 0.$$ 

The homogeneous singular integral operator $T$ is defined initially for $f \in C_0^\infty(H)$ as follows

$$T(f)(x) := \text{p.v.} \int_H K(y) f(y^{-1} x) dy = \lim_{R \to \infty} \int_{|y| < R} K(y) f(y^{-1} x) dy.$$ 

The study of rough singular integral operators dates back to Calderón and Zygmund’s work [5, 6]. It is well-known that when $H$ is an isotropic Euclidean space, Calderón and Zygmund [6] used the method of rotations to show that if $Q \in L\log L(S^{n-1})$, then $T$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Later it was shown by Christ [9, 11], Hofmann [26] and Seeger [48] that such

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operators are of weak-type \((1, 1)\) and by Tao [51] that the underlying space can be generalized to homogeneous group \(\mathbb{H}\). There are also many other important progress on rough singular integral operators (see for example \([8, 14, 15, 19, 21, 25, 42]\)).

Furthermore, there has been numerous work on weighted inequalities of singular integral with rough kernels, see for example \([18, 22, 38, 41]\) and the references therein for its development and applications. Recently, the sharp weight inequalities for standard Calderón–Zygmund operators was proved by Hytönen [27] via constructing the representation theorem, which gives the following weighted \(L^p\) bound with sharp dependence on \([w]_{A_p}\).

\[
\|Tf\|_{L^p(w)} \leq C_{p,T}[w]_{A_p}^{\max\{1,1/(p-1)\}}\|f\|_{L^p(w)}, \quad 1 < p < \infty.
\]

Besides, Lerner [35, 36] and Lacey [34] gave alternative approaches to this result. Then a natural question arises: can we also obtain a sharp weight bound for rough homogeneous singular integral operators? We point out that this topic has been studied intensively especially in the last three years (the pointwise version originated in [34]) with the key tool sparse domination, see for example \([7, 13, 30, 32, 33, 37, 39, 45]\). Among these results, we would like to highlight that Hytönen–Roncal–Tapiola [30] first quantitatively proved that if \(\Omega \in L^\infty(\mathbb{S}^{n-1})\), then

\[
(1.1) \quad \|T\|_{L^p(w) \to L^p(w)} \leq C_{n,p}\|\Omega\|_{L^\infty}(w)_{A_p}.
\]

In particular,

\[
\|T\|_{L^2(w) \to L^2(w)} \leq C_n\|\Omega\|_{L^\infty}[w]_{A_2}^2,
\]

(For the definitions of \([w]_{A_p}\) and \((w)_{A_p}\), we refer the readers to Section 2). Different proofs of the quantitative bound of \(T\) (as in (1.1)) via a sparse domination principle were obtained by Conde-Alonso, Culiuc, Di Plinio and Ou [13], and by Lerner [37].

Inspired by Tao’s work [51], Sato [46] extended part of the classical results related to singular integral to homogeneous group. He obtained the \(L^p(w)\) boundedness for rough homogeneous singular integral operators under the assumption that \(w \in A_p\) for some \(1 < p < \infty\) and \(\Omega \in L^\infty(\Sigma)\). However, it is still unclear that whether a quantitative weight bound can be obtained in this setting and that whether the condition \(\Omega \in L^\infty(\Sigma)\) can be weakened.

The purpose of this paper is to address these points. As in [51], let \(K_0\) be the restriction of \(K\) to the annulus \(\Sigma_0 = \{x \in \mathbb{H} : 1 \leq \rho(x) \leq 2\}\), then it is clear that \(\Omega \in L^q(\Sigma)\) implies \(K_0 \in L^q(\Sigma_0)\) for the same \(q \in (1, \infty)\). Our main result is the following theorem.

**Theorem 1.1.** Let \(1 < p < \infty\), \(w \in A_p\). Suppose that \(K_0\) has mean zero and there exists a constant \(c_{Q,p} > 0\) such that \(K_0 \in L^q\) for some \(q > c_{Q,p}(w)_{A_p}\), then

\[
\|T\|_{L^p(w) \to L^p(w)} \leq C_{Q,p,q}[w]_{A_p}(w)_{A_p}.
\]

In particular,

\[
\|T\|_{L^2(w) \to L^2(w)} \leq C_{Q,q}[w]_{A_2}^2
\]

for some constants \(C_{Q,p,q}\) and \(C_{Q,q}\) independent of \(w\).
Comparing with the previous closely related results, we point out that the weight bound \([w]_{A_p}(w)_{A_p}\) we obtained is consistent with that obtained in \([30]\). It is still unknown that whether this is sharp, but it is the best known quantitative result for this class of operators.

To show Theorem 1.1, we borrow the idea from \([30]\) to divide the proof into two steps:

- In the first step, noting that the Fourier transform is not applicable in general homogeneous groups, we provide a new decomposition of the operator \(T\) into a summation of Dini-type Calderón-Zygmund operators \(\tilde{T}_{j_1,j_2}^N\) defined by (4.3), (4.4) and (4.5). Then we combine Cotlar-Knapp-Stein Lemma with a key \(L^p\) estimate originated from \([51]\) (later extended by \([46]\)) to show the unweighted \(L^2\) estimate for \(\tilde{T}_{j_1,j_2}^N\), that is, for any \(j_1, j_2 \geq 1\),
  \[
  \|\tilde{T}_{0,0}^N(f)\|_2 \leq \|K_0\|_{q}\|f\|_2, \\
  \|\tilde{T}_{j_1,0}^N(f)\|_2 \leq C_{q,p}2^{-\alpha N(j_1-1)}\|K_0\|_{q}\|f\|_2, \\
  \|\tilde{T}_{0,j_2}^N(f)\|_2 \leq C_{q,p}2^{-\alpha N(j_2-1)}\|K_0\|_{q}\|f\|_2, \\
  \|\tilde{T}_{j_1,j_2}^N(f)\|_2 \leq C_{q,p}2^{-\alpha N(j_1-1)}2^{-\alpha N(j_2-1)}\|K_0\|_{q}\|f\|_2.
  \]

- In the second step, we prove a quantitative weighted \(L^p\) inequality and a quantitative good unweighted \(L^p\) estimate for \(\tilde{T}_{j_1,j_2}^N\), both of which contain an extra bad factor \(2^{-\frac{N(j_1)}{q}}\), that is,
  \[
  \|\tilde{T}_{j_1,j_2}^N(f)\|_{L^p(w)} \leq 2^{-\frac{N(j_1)}{q}}(1 + N(j_2))\|K_0\|_{q}\|w\|_{A_p}\|f\|_{L^p(w)}, \\
  \|\tilde{T}_{j_1,j_2}^N(f)\|_{L^p} \leq 2^{-\beta_pN(j_1-1)2^{-\alpha N(j_2-1)}}2^{-\frac{N(j_1)}{q}}(1 + N(j_2))\|K_0\|_{q}\|f\|_{L^p}.
  \]

Finally, Theorem 1.1 follows from choosing appropriate \(N(j)\) and repeating a standard argument of interpolation theorem with change of measures.

As a direct application, we obtain the quantitative weighted bound for the rough singular integrals studied by Sato \([46]\). To state our result, we first recall some notations introduced in \([46]\).

For \(q \geq 1\), let \(d_q\) denote the collection of measurable functions \(h\) on \(\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}\) satisfying \(\|h\|_{d_q} = \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^{q} \frac{dt}{t}\right)^{1/q} < \infty\). We define \(\|h\|_{d_{\infty}} := \|h\|_{L^q(\mathbb{R}_+)}\). Besides, for \(t \in (0, 1]\),

let \(u(h, t) := \sup_{|s| < t/2} \int_{R} \frac{h(r - s)}{|s|^{\eta}} \frac{dt}{r}\), where the supremum is taken over all \(s\) and \(R\) such that \(|s| < tR/2\). For \(\eta > 0\), let \(\Lambda^\eta\) denote the family of functions \(h\) such that \(\|h\|_{\Lambda^\eta} := \sup_{t \in (0, 1]} t^{-\eta}u(h, t) < \infty\). Define \(\Lambda^\eta_q := d_q \cap \Lambda^\eta\) and set \(\|h\|_{\Lambda^\eta_q} := \|h\|_{d_q} + \|h\|_{\Lambda^\eta}\) for \(h \in \Lambda^\eta_q\).

Consider
\[
T(f)(x) = \text{p.v. } f \ast L(x) = \text{p.v. } \int_{H} f(y)L(y^{-1} x)dy,
\]
where \(L(x) := h(\rho(x))K(x)\) and \(K\) is defined in Section 1. Then we have

**Theorem 1.2.** Let \(1 < p < \infty\), \(w \in A_p\). Suppose that \(K_0\) has mean zero and there exists a constant \(c_{q,p} > 0\) such that \(K_0 \in L^q\) for some \(q > c_{q,p}(w)_{A_p}\). Suppose that \(h \in \Lambda^\eta_q\) for some \(\eta > 0\), then
\[
\|T\|_{L^p(w)\rightarrow L^p(w)} \leq C_{q,p,q}\|K_0\|_{q}\|h\|_{\Lambda^\eta_q}\|w\|_{A_p}(w)_{A_p},
\]
for some constant \(C_{q,p,q}\) independent of \(w\).
We also have an investigation on the quantitative weighted estimate for bi-parameter rough singular integrals on the product homogeneous groups $\mathbb{H}_1 \times \mathbb{H}_2$. Recall that the Euclidean version was introduced by R. Fefferman [23, page 198] where $\Omega$ is Lipschitz, and later studied by Duoandikoetxea [17]. See also [1, 2, 16] for previous works about rough singular integrals on product of Euclidean spaces. Consider the singular integral

$$Tf(x, y) = p.v. f \ast K(x, y) = p.v. \int_{\mathbb{H}_1 \times \mathbb{H}_2} f(xu^{-1}, yv^{-1})K(u, v)dudv,$$

where $K(u, v)$ satisfies

$$K(t_1 \circ_1 u, t_2 \circ_2 v) = t_1^{-Q_1}t_2^{-Q_2}K(u, v),$$

for all $t_i \in \mathbb{R}_+$ and $(u, v) \in \mathbb{H}_1 \times \mathbb{H}_2$. For $i = 1, 2$, let $D_0^{(i)} = \{x_i \in \mathbb{H}_i : 1 \leq \rho_i(x_i) \leq 2\}$ and $D_0 = D_0^{(1)} \times D_0^{(2)}$. Denote $x = (x_1, x_2) \in \mathbb{H}_1 \times \mathbb{H}_2$ and $K^0(x) = K(x)_X D_0(x)$. In the bi-parameter setting, we also abuse the notation $w \in A_p$ to denote that $w$ is a product $A_p$ weight. We now state our result in the bi-parameter setting. For the sake of simplicity, we refer the readers to Section 6 for all details of definitions and notations.

**Theorem 1.3.** Let $w \in A_2$. Suppose that $K^0$ satisfies

$$\int_{D_0^{(i)}} K(u, v)du = \int_{D_0^{(i)}} K(u, v)dv = 0, \text{ for all } (u, v) \in D_0,$$

and there exists a constant $c_{Q_1, Q_2} > 0$ such that $K^0 \in L^q(D_0)$ for some $q > c_{Q_1, Q_2}(w)_{A_2}$, then

$$\|T\|_{L^2(w) \to L^2(w)} \leq C_{Q_1, Q_2, q} \max(\|K^0\|_q, \|K_0\|_q^2)(w)^{12}_A[w]^2_{A_\infty},$$

for some constant $C_{Q_1, Q_2, q}$ independent of $w$.

Based on the framework of the proof of Theorem 1.1, the key idea to prove Theorem 1.3 is to decompose the bi-parameter rough singular integral $T$ into a summation of bi-parameter Dini-type Calderón–Zygmund operators $\tilde{T}^N_{j_1, j_2, j_3, j_4}$ with the modified Dini-1 condition as in [3, Section 5] and with the cancellation on the kernel $K_{j_1, j_2, j_3, j_4}^N$.

Then we have the standard bi-parameter representation theorem for each $\tilde{T}^N_{j_1, j_2, j_3, j_4}$, which, together with [4, Corollary 3.2] and the sparse domination for the Shifted Square Function [4, Section 5], gives Theorem 1.3.

It is not clear whether the quantitative estimate appearing in Theorem 1.3 can be pushed down further using our methods.

This paper is organized as follows. In Section 2 we provide the preliminaries, including the fundamental properties of the Muckenhoupt $A_p$ weights, a system of dyadic cubes on $\mathbb{H}$ and the definition of Calderón–Zygmund operators with Dini-continuous kernel. In section 3, we show a sparse domination principle for $\omega$-Calderón-Zygmund operator with $\omega$ satisfying the Dini condition. In Section 4, we prove our main result Theorem 1.1. In Section 5, we prove Theorem 1.2, the quantitative weighted bound of the rough singular integrals studied by Sato [46]. In the last section we investigate the quantitative weighted bound in the bi-parameter setting.
2. Preliminaries

2.1. Muckenhoupt $A_p$ weights. We denote the average of a function $f$ over a ball $B$ by

$$\langle f \rangle_B = \frac{1}{|B|} \int_B f \, dx,$$

where $|B|$ denotes the Lebesgue measure of $B$.

**Definition 2.1.** Let $w(x)$ be a nonnegative locally integrable function on $\mathbb{H}$. For $1 < p < \infty$, we say that $w$ is an $A_p$ weight, written $w \in A_p$, if

$$[w]_{A_p} := \sup_B \left( \frac{1}{|B|} \int_B w \, dx \right)^{1/(p-1)} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{H}$. The quantity $[w]_{A_p}$ is called the $A_p$ constant of $w$. For $p = 1$, if $M(w)(x) \leq w(x)$ for a.e. $x \in \mathbb{H}$, then we say that $w$ is an $A_1$ weight, written $w \in A_1$, where $M$ denotes the Hardy-Littlewood maximal function. Besides, let $A_\infty := \cup_{1 \leq p < \infty} A_p$ and we have

$$[w]_{A_\infty} := \sup_B \left( \frac{1}{|B|} \int_B w \, dx \right) \left( \int_B \log \frac{1}{w} \, dx \right) < \infty.$$

In order to state our weighted estimates much more efficiently, we recall the following variants of the weight characteristic (see for example [30]):

$$[w]_{A_p} := [w]_{A_p}^{1/p} \max \{[w]_{A_\infty}^{1/p}, [w^{1/p}]_{A_\infty}^{1/p} \}, \quad (w)_{A_p} := \max \{[w]_{A_\infty}, [w^{1/p}]_{A_\infty} \}.$$

**Lemma 2.2.** Let $1 < p < \infty$, and $w \in A_p$. Then there exists a constant $c_\mathcal{Q}$ small enough such that for every $0 < \delta \leq c_\mathcal{Q}/(w)_{A_p}$, we have that $w^{1+\delta/2} \in A_p$ and

$$(w^{1+\delta/2})_{A_p} \leq C_\mathcal{Q}(w)_{A_p}^{1+\delta/2}, \quad [w^{1+\delta/2}]_{A_p} \leq C_\mathcal{Q}[w]_{A_p}^{1+\delta/2}.$$

**Proof.** In the setting of Euclidean space, the proof was given in [30, Corollary 3.18]. For the case in homogeneous groups, it suffices to note that a similar sharp reverse Hölder inequality also holds (see [29]).

2.2. A System of Dyadic Cubes. To begin with, we define a left-invariant quasi-distance $d$ on $\mathbb{H}$ by $d(x, y) = \rho(x^{-1}y)$, which means that there exists a constant $A_0 \geq 1$ such that for any $x, y, z \in \mathbb{H}$,

$$d(x, y) \leq A_0[d(x, z) + d(z, y)].$$

Next, let $B(x, r) := \{y \in \mathbb{H} : d(x, y) < r \}$ be the open ball with center $x \in \mathbb{H}$ and radius $r > 0$.

Let $\mathcal{A}_\alpha$ be $k$-th countable set of Index. A countable family $\mathcal{D} := \cup_{k \in \mathbb{Z}} \mathcal{D}_k$, $\mathcal{D}_k := \{Q^k_\alpha : \alpha \in \mathcal{A}_\alpha \}$, of Borel sets $Q^k_\alpha \subset \mathbb{H}$ is called a system of dyadic cubes with parameters $\delta \in (0, 1)$ and $0 < a_1 \leq A_1 < \infty$ if it has the following properties:

1. $\mathbb{H} = \bigcup_{\alpha \in \mathcal{A}} Q^k_\alpha$ (disjoint union) for all $k \in \mathbb{Z}$;
2. If $\ell \geq k$, then either $Q^\ell_\beta \subset Q^k_\alpha$ or $Q^k_\alpha \cap Q^\ell_\beta = \emptyset$;
3. For each $(k, \alpha)$ and each $\ell \leq k$, there exists a unique $\beta$ such that $Q^k_\alpha \subset Q^\ell_\beta$;
(4) for each \((k, \alpha)\) there exists at most \(M\) (a fixed geometric constant) \(\beta\) such that 
\[Q^{k+1}_\beta \subseteq Q^k_\alpha, \text{ and } Q^k_\alpha = \bigcup_{Q \in D_{k+1}, Q \subseteq Q^k_\alpha} Q;\]

(5) \(B(x^k_\alpha, a_1 \delta^k) \subseteq Q^k_\alpha \subseteq B(x^k_\alpha, A_1 \delta^k) =: B(Q^k_\alpha);\)

(6) if \(\ell \geq k\) and \(Q^k_\beta \subseteq Q^\ell_\alpha\), then \(B(Q^k_\beta) \subseteq B(Q^\ell_\alpha).\)

The set \(Q^k_\alpha\) is called a dyadic cube of generation \(k\) with centre \(x^k_\alpha \in Q^k_\alpha\) and sidelength \(\ell(Q^k_\alpha) = \delta^k.\)

From the properties of the dyadic system above, we see that there exists a constant \(\tilde{A}_0 > 0\) such that for any \(Q^k_\alpha\) and \(Q^{k+1}_\beta\) satisfying \(Q^{k+1}_\beta \subseteq Q^k_\alpha\), the following inequalities holds:
\[(2.1) \quad |Q^{k+1}_\beta| \leq |Q^k_\beta| \leq \tilde{A}_0 |Q^{k+1}_\beta|.
\]

We now recall from [28] the following lemma, which provides a construction of a system of dyadic cubes (see also M. Christ [10] and Sawyer–Wheeden [47]).

**Lemma 2.3.** There exists a system of dyadic cubes with parameters \(0 < \delta \leq (12A_0^3)^{-1}\) and \(a_1 := (3A_0^2)^{-1}, A_1 := 2A_0.\) The construction only depends on some fixed set of countably many centre points \(x^k_\alpha,\) satisfying that \(d(x^k_\alpha, x^k_\beta) \geq \delta^k\) with \(\alpha \neq \beta, \min_\alpha d(x, x^k_\alpha) < \delta^k\) for all \(x \in \mathbb{H},\) and a certain partial order “\(\leq\)” among their index pairs \((k, \alpha)\). Indeed, this system can be constructed as follows.

\[
\overline{Q}^k_\alpha = \{x^k_\beta : (\ell, \beta) \leq (k, \alpha)\}, \quad \overline{Q}^k_\alpha := \text{int} \overline{Q}^k_\alpha = \left(\bigcup_{\beta \neq \alpha} \overline{Q}^k_\beta\right)^c, \quad Q^k_\alpha \subseteq Q^k_\alpha \subseteq \overline{Q}^k_\alpha,
\]

where \(Q^k_\alpha\) are obtained from the closed sets \(\overline{Q}^k_\alpha\) and the open sets \(\overline{Q}^k_\alpha\) by finitely many set operations.

2.3. **Adjacent Systems of Dyadic Cubes.** A finite collection \(\mathcal{D}^t: t = 1, 2, \ldots, t\) of the dyadic families is called a collection of adjacent systems of dyadic cubes with parameters \(\delta \in (0, 1), 0 < a_1 \leq A_1 < \infty\) and \(1 \leq \text{C}_{\text{adj}} < \infty\) if it has the following two properties:

1. For any \(t \in \{1, 2, \ldots, t\},\) \(\mathcal{D}^t\) is a system of dyadic cubes with parameters \(\delta \in (0, 1)\) and \(0 < a_1 \leq A_1 < \infty;\)

2. For any ball \(B(x, r) \subseteq \mathbb{H}\) with \(\delta^{k+3} < r \leq \delta^{k+2}, k \in \mathbb{Z},\) there exist \(t \in \{1, 2, \ldots, t\}\) and \(Q \in \mathcal{D}^t\) of generation \(k\) and with centre \(x^k_\alpha\) such that \(d(x, x^k_\alpha) < 2A_0 \delta^k\) and
\[(2.2) \quad B(x, r) \subseteq Q \subseteq B(x, C_{\text{adj}} r).
\]

We recall from [28] the following construction.

**Theorem 2.4.** There exists a collection \(\{\mathcal{D}^t: t = 1, 2, \ldots, t\}\) of adjacent systems of dyadic cubes with parameters \(\delta \in (0, (96A_0^6)^{-1}), a_1 := (12A_0^4)^{-1}, A_1 := 4A_0^2\) and \(C_{\text{adj}} := 8A_0^3\delta^{-3}.\) For each \(t \in \{1, 2, \ldots, t\},\) the centres \(x^k_\alpha\) of the cubes \(Q \in \mathcal{D}^t_k\) satisfy the following two properties
\[d(x^k_\alpha, x^k_\beta \geq (4A_0^{5/2})^{-1}\delta^k \quad (\alpha \neq \beta), \quad \min_\alpha d(x, x^k_\alpha) < 2A_0 \delta^k \quad \text{for all } x \in \mathbb{H}.
\]

We recall from [31, Remark 2.8] that the number \(t\) of the adjacent systems of dyadic cubes as in the theorem above satisfies the estimate
\[t = t(A_0, A_1, \delta) \leq A_0^8 (A_0^{4/3})^\log_2 A_1,
\]
where \(A_1\) is the geometrically doubling constant, see [31, Section 2].
2.4. Calderón-Zygmund operators with Dini-continuous kernel. Let $T$ be a bounded linear operator on $L^2(\mathbb{H})$ represented as

$$
T(f)(x) = \int_{\mathbb{H}} K(x, y) f(y) dy, \quad \forall x \notin \text{supp } f.
$$

A function $\omega : [0, 1] \to [0, \infty)$ is a modulus of continuity if it satisfies the following three properties:

1. $\omega(0) = 0$;
2. $\omega(s)$ is an increasing function;
3. For any $s_1, s_2 > 0$, $\omega(s_1 + s_2) \leq \omega(s_1) + \omega(s_2)$.

**Definition 2.5.** We say that the operator $T$ is an $\omega$-Calderón-Zygmund operator if the kernel $K$ satisfies the following two conditions:

1. (size condition):
   $$
   |K(x, y)| \leq \frac{C_T}{d(x, y)^q},
   $$
   for some constant $C_T > 0$;

2. (smoothness condition):
   $$
   |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \omega \left( \frac{d(x, x')}{d(x, y)^q} \right) \frac{1}{d(x, y)^Q}
   $$
   for $d(x, y) \geq 2A_0 d(x, x') > 0$.

Moreover, $K$ is said to be a Dini-continuous kernel if $\omega$ satisfies the Dini condition:

$$
\| \omega \|_{\text{Dini}} := \int_0^1 \frac{\omega(s)}{s} ds < \infty.
$$

2.5. Notation of the paper. For $1 \leq p \leq +\infty$, we denote the norm of a function $f \in L^p(\mathbb{H})$ by $\|f\|_p$. If $T$ is a bounded linear operator from $L^p(\mathbb{H})$ to $L^q(\mathbb{H})$, $1 \leq p, q \leq +\infty$, we write $\|T\|_{p \to q}$ for the operator norm of $T$. The indicator function of a subset $E \subseteq X$ is denoted by $\chi_E$. We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some constant $C > 0$.

3. Domination of Dini-type Calderón–Zygmund operator by sparse operators

To begin with, we recall the definition of sparse family given in [20] on general spaces of homogeneous type in the sense of Coifman and Weiss [12], which can be applied to our setting of homogeneous groups.

**Definition 3.1.** Let $0 < \eta < 1$, a collection $S \subset \mathcal{D}$ of dyadic cubes is said to be $\eta$-sparse if for every $Q \in S$, there exists a measurable subset $E_Q \subset Q$ such that $\mu(E_Q) \geq \eta \mu(Q)$ and the sets $\{E_Q\}_{Q \in S}$ have only finite overlap.

**Definition 3.2.** Let $\Lambda > 1$, a collection $S \subset \mathcal{D}$ of dyadic cubes is said to be $\Lambda$-Carleson if for every cube $Q \in \mathcal{D}$,

$$
\sum_{P \in S, P \subseteq Q} \mu(P) \leq \Lambda \mu(Q).
$$
It was shown in [20] that the above two definitions are equivalent in homogeneous group. We now recall the definition of the sparse operator in this setting.

**Definition 3.3.** Given a sparse family, we define a sparse operator \( \mathcal{A}_S \) by

\[
\mathcal{A}_S(f)(x) = \sum_{Q \in S} (f)_{Q} \chi_{Q}(x).
\]

In this subsection, the main task is to show the following quantitative version of Lacey’s pointwise domination inequality.

**Proposition 3.4.** Let \( T \) be an \( \omega \)-Calderón-Zygmund operator with \( \omega \) satisfying the Dini condition. Then for any compactly supported function \( f \in L^1(\mathbb{H}) \), there exists a sparse family \( S \) such that for a.e. \( x \in \mathbb{H} \),

\[
|T(f)(x)| \leq C_Q(\|T\|_{L^2 \rightarrow L^2} + C_T + \|\omega\|_{\text{Dini}})\mathcal{A}_S(|f|)(x).
\]

To show Proposition 3.4, we need some auxiliary maximal operators. To begin with, we define the maximal truncated operator given by

\[
T^*(f)(x) = \sup_{e>0} \left| \int_{d(x,y)>e} K(x,y)f(y)dy \right|.
\]

Next, let \( j_0 \) be the smallest integer such that

\[
2^{j_0} > \max\{3A_0, 2A_0 \cdot C_{adj}\}
\]

and let \( C_{j_0} := 2^{j_0+2}A_0 \). We now define the grand maximal truncated operator \( \mathcal{M}_T \) as follows.

\[
\mathcal{M}_T f(x) = \sup_{B \ni x} \operatorname{ess sup}_{\xi \in B} |T(f\chi_{\mathbb{H} \setminus C_{j_0}})(\xi)|,
\]

where the first supremum is taken over all balls \( B \subset \mathbb{H} \) containing \( x \). We can see later that this operator plays a crucial role in the proof. Given a ball \( B_0 \subset \mathbb{H} \), for \( x \in B_0 \) we also define a local version of \( \mathcal{M}_T \) by

\[
\mathcal{M}_{T,B_0} f(x) = \sup_{B_0 \supset B_0} \operatorname{ess sup}_{\xi \in B} |T(f\chi_{C_{j_0}B_0 \setminus C_{j_0}B_0})(\xi)|.
\]

**Lemma 3.5.** There exists a constant \( C_Q > 0 \) such that for a.e. \( x \in B_0 \),

\[
|T(f\chi_{C_{j_0}B_0})(x)| \leq C_Q\|T\|_{L^1 \rightarrow L^1} |f(x)| + \mathcal{M}_{T,B_0} f(x).
\]

**Proof.** The result in Euclidean setting was proven in [37]. Here we adapt the proof in [37] to our setting of homogeneous group.

Recall that almost every \( x \in B_0 \) is a interior point and Lebesgue point of \( B_0 \), then \( x \) is also a point of approximate continuity of \( T(f\chi_{C_{j_0}B_0}) \). Then for every \( \varepsilon > 0 \), the sets

\[
E_{\varepsilon}(x) = \{y \in B(x,r) : |T(f\chi_{C_{j_0}B_0})(y) - T(f\chi_{C_{j_0}B_0})(x)| < \varepsilon\}
\]

satisfy

\[
\lim_{r \to 0} \frac{|E_{\varepsilon}(x)|}{|B(x,r)|} = 1.
\]
Let \( r \) sufficient close to 0 such that \( B(x, r) \subset B_0 \). Then for a.e. \( y \in E_r(x) \),
\[
|T(f \chi_{C_{j_0}B_0})(x)| < |T(f \chi_{C_{j_0}B_0})(y)| + \epsilon \leq |T(f \chi_{C_{j_0}B(x,r)})(y)| + M_{T,B_0}f(x) + \epsilon.
\]
Therefore, the weak type \((1, 1)\) boundedness of \( T \) yields
\[
|T(f \chi_{C_{j_0}B_0})(x)| \leq \inf_{y \in E_r(x)} |T(f \chi_{C_{j_0}B(x,r)})(y)| + M_{T,B_0}f(x) + \epsilon
\]
\[
\leq \|T\|_{L^1 \to L^{1,\infty}} \frac{1}{|E_r(x)|} \int_{C_{j_0}B(x,r)} |f(y)|dy + M_{T,B_0}f(x) + \epsilon.
\]
Finally, we let \( r \to 0 \) and \( \epsilon \to 0 \) to obtain the estimate \((3.2)\).

**Lemma 3.6.** There exists a constant \( C_Q > 0 \) such that for a.e. \( x \in \mathbb{H} \),
\[
(3.3) \quad M_{T}f(x) \leq C_Q(\|\omega\|_{\text{Dini}} + C_T)Mf(x) + T^*f(x),
\]
where \( M \) denotes the Hardy-Littlewood maximal function.

**Proof.** Let \( x, \xi \in B := B(x_0, r) \). Let \( B_x \) be the closed ball centered at \( x \) with radius \( 4(A_0 + C_{j_0})r \). Then \( C_{j_0}B \subset B_x \), and we obtain
\[
|T(f \chi_{\mathbb{H}\setminus (C_{j_0}B)}(\xi))| \leq |T(f \chi_{\mathbb{H}\setminus B_x}(\xi)) - T(f \chi_{\mathbb{H}\setminus B_0}(\xi))| + |T(f \chi_{B_0\setminus C_{j_0}B}(\xi))| + |T(f \chi_{B_x}(\xi))|.
\]
It follows from the smooth condition of \( T \) that
\[
|T(f \chi_{\mathbb{H}\setminus B_x}(\xi)) - T(f \chi_{\mathbb{H}\setminus B_0}(\xi))| \leq \int_{d(x,y) > 4A_0r} |f(y)|\omega \left( \frac{2r}{d(x,y)} \right) \frac{1}{d(x,y)^Q}dy
\]
\[
\leq \sum_{k=1}^{\infty} \left( \frac{1}{(2^k r)^Q} \int_{B(x, 2^{k+2}r)} f(y)dy \right) \omega(2^{-k}) \leq C_Q\|\omega\|_{\text{Dini}}Mf(x).
\]
Next, by the size condition,
\[
|T(f \chi_{B_0\setminus C_{j_0}B}(\xi))| \leq C_QC_T \frac{1}{\|B_x\|} \int_{B_x} |f(y)|dy \leq C_QC_T Mf(x).
\]
Finally, we also have \( |T(f \chi_{\mathbb{H}\setminus B_x}(\xi))| \leq T^*f(x) \). Combining these estimates together, we prove \((3.3)\).

Next, we give the proof of Proposition 3.4.

**Proof of Proposition 3.4.** We follow the ideas in [37] for this domination, and adapt it to our setting of homogeneous group.

We first suppose that \( f \) is supported in a ball \( B_0 := B(x_0, r) \subset \mathbb{H} \), and then decompose \( \mathbb{H} \) with respect to this ball \( B_0 \). To this end, we define the annuli \( U_j := 2^{j+1}B_0 \setminus 2^jB_0 \), \( j \geq 0 \) and we choose \( j_0 \) to be the smallest integer such that
\[
(3.4) \quad j_0 > \bar{j}_0 \quad \text{and} \quad 2^{j_0} > 4A_0.
\]
Next, for each \( U_j \), we choose the balls
\[
(3.5) \quad \{\bar{B}_{j,\ell}\}_{\ell=1}^{L_j}
\]
centred in $U_j$ and with radius $2^{j-j_0}r$ to cover $U_j$. It follows from the doubling property (see for example [12]) that

$$
\sup_j L_j \leq C_{A_0,j_0},
$$

where $C_{A_0,j_0}$ is an absolute constant depending only on $A_0$ and $j_0$.

We now recall the properties of these $B_{j,t}$. Denote $\tilde{B}_{j,t} := B(x_{j,t}, 2^{j-j_0}r)$, where $j_0$ is defined as in (3.1). Then we have $C_{adj}\tilde{B}_{j,t} := B(x_{j,t}, C_{adj}2^{j-j_0}r)$. It was shown in the proof of [20, Theorem 3.7] that

$$
C_{adj}\tilde{B}_{j,t} \cap U_{j+j_0} = \emptyset, \quad \forall j \geq 0 \quad \text{and} \quad \forall \ell = 1, 2, \ldots, L_j,
$$

and that

$$
C_{adj}\tilde{B}_{j,t} \cap U_{j-j_0} = \emptyset, \quad \forall j \geq j_0 \quad \text{and} \quad \forall \ell = 1, 2, \ldots, L_j.
$$

Now combining the properties (3.7) and (3.8), we see that each $C_{adj}\tilde{B}_{j,t}$ only intersects with at most $2j_0 + 1$ annuli $U_j$’s. Moreover, for every $j$ and $\ell$, $C_{\tilde{j_0}}\tilde{B}_{j,t}$ covers $B_0$.

Next observe that by (2.2), there exists an integer $t_0 \in \{1, 2, \ldots, t\}$ and $Q_0 \in \mathcal{D}^{t_0}$ such that $B_0 \subseteq Q_0 \subseteq C_{adj}B_0$. Moreover, for this $Q_0$, as in section 2.2 we use $B(Q_0)$ to denote the ball that contains $Q_0$ and has measure comparable to $Q_0$. Therefore, $B(Q_0)$ covers $B_0$ and $|B(Q_0)| \lesssim |B_0|$, where the implicit constant depends only on $C_{adj}$ and $A_1$.

Next we claim that there exists a $1/2$-sparse family $\mathcal{F}^{t_0} \subset \mathcal{D}^{t_0}(Q_0)$, the set of all dyadic cubes in $t_0$-th dyadic system that are contained in $Q_0$, such that for a.e. $x \in B_0$,

$$
|T(f\chi_{C_{j_0}B(Q_0)}(x))| \leq C_Q(|\mathcal{T}|_2 + 2 + \|\omega\|_{\text{Dini}}) \sum_{Q \in \mathcal{F}^{t_0}} \langle |f| \rangle_{C_{j_0}B(Q)}\chi_Q(x).
$$

To prove the claim it suffices to show the following recursive estimate: there exist pairwise disjoint cubes $P_j \in \mathcal{D}^{t_0}(Q_0)$ such that $\sum_j |P_j| \leq 1/2|Q_0|$ and for a.e. $x \in B_0$,

$$
|T(f\chi_{C_{j_0}B(Q_0)}(x))| \chi_{Q_0}(x) \leq C_Q(|\mathcal{T}|_2 + 2 + \|\omega\|_{\text{Dini}})\langle |f| \rangle_{C_{j_0}B_0}\chi_{Q_0}(x)
$$

$$
+ \sum_j |T(f\chi_{C_{j_0}B(P_j)}(x))| \chi_{P_j}(x).
$$

Indeed, iterating this estimate, we directly get (3.9) with $\mathcal{F}^{t_0}$ being the union of all the families $\{P_j^k\}$ where $\{P_j^0\} = \{Q_0\}$, $\{P_j^1\} = \{P_j\}$ as mentioned above, and $\{P_j^k\}$ are the cubes obtained at the $k$-th stage of the iterative process. It is not difficult to see that $\mathcal{F}^{t_0}$ is a $1/2$-sparse family.

We now give the proof of the recursive estimate. For any arbitrary family of disjoint cubes $\{P_j\} \subset \mathcal{D}^{t_0}(Q_0)$, we see that

$$
|T(f\chi_{C_{j_0}B(Q_0)}(x))| \chi_{Q_0}(x)
$$

$$
\leq |T(f\chi_{C_{j_0}B(Q_0)}(x))| \chi_{Q_0 \cup \cup_j P_j}(x) + \sum_j |T(f\chi_{C_{j_0}B(Q_0)}(x))| \chi_{P_j}(x)
$$

$$
\leq |T(f\chi_{C_{j_0}B(Q_0)}(x))| \chi_{Q_0 \cup \cup_j P_j}(x) + \sum_j |T(f\chi_{C_{j_0}B(Q_0)\cap C_{j_0}B(P_j)}(x))| \chi_{P_j}(x)
$$
+ \sum_j |T(f \mathcal{X}_{B(P_j)})| |-\mathcal{X}_{P_j}|

So it suffices to show that we can choose a family of pairwise disjoint cubes \( \{P_j\} \subset \mathcal{D}^0(Q_0) \) such that \( \sum_j |P_j| \leq \frac{1}{2}|Q_0| \) and that for a.e. \( x \in B_0 \),

\[
|T(f \mathcal{X}_{B(P_0)}(x)| |-\mathcal{X}_{P_0}|
\]

(3.10)

\[
\leq CQ(||T||_{L^2} + CT + ||\omega||_{\text{dim}})(|f|)_{C^0_{B(Q_0)}}.
\]

To begin with, an examination of standard proofs ([50]) shows that

\[
\max(||T||_{L^2}, ||T||_{L^1}) \leq CQ(||T||_{L^2} + CT + ||\omega||_{\text{dim}}).
\]

This, in combination with Lemma 3.6, implies that \( ||M_T||_{L^1} \leq CQ(||T||_{L^2} + CT + ||\omega||_{\text{dim}}) \).

Therefore, one can choose \( \alpha \) sufficient large (depending on \( C_{\tilde{\lambda}}, C_{\tilde{\lambda}} \) and \( A_1 \)) such that the set

\[
E = \{ x \in B_0 : |f(x)| > \alpha(|f|)_{C^0_{B(Q_0)}} \}
\]

\[
\cup \{ x \in B_0 : M_{T,B_0}f(x) > \alpha(||T||_{L^2} + CT + ||\omega||_{\text{dim}})(|f|)_{C^0_{B(Q_0)}} \}
\]

satisfy

\[
|E| \leq \frac{1}{4A_0}|B_0|,
\]

where \( A_0 \) is defined in Section 2.2. We now apply the Calderón–Zygmund decomposition to the function \( \mathcal{X}_E \) on \( B_0 \) at the height \( \lambda := \frac{1}{2A_0} \), to obtain pairwise disjoint cubes \( \{P_j\} \subset \mathcal{D}^0(Q_0) \) such that

\[
\frac{1}{2A_0}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|
\]

and \( |E \cup_j P_j| = 0 \). This implies that

\[
\sum_j |P_j| \leq \frac{1}{2}|B_0| \quad \text{and} \quad P_j \cap E^c \neq \emptyset.
\]

Therefore,

\[
\text{ess sup}_{\xi \in \xi_j} |T(f \mathcal{X}_{\mathcal{C}_{\tilde{\lambda}}B_0})_{\mathcal{C}_{\tilde{\lambda}}}(\xi)| \leq C(||T||_{L^2} + CT + ||\omega||_{\text{dim}})(|f|)_{C^0_{B(Q_0)}}.
\]

Next it follows from Lemma 3.5 that for a.e. \( x \in B_0 \setminus \cup_j P_j \),

\[
|T(f \mathcal{X}_{\mathcal{C}_{\tilde{\lambda}}B_0})(x)| \leq CQ(||T||_{L^2} + CT + ||\omega||_{\text{dim}})(|f|)_{C^0_{B(Q_0)}}.
\]

This, in combination with the estimate (3.11), proves (3.10) and so (3.9).

To extend this result to almost every \( x \in \mathbb{H} \), we consider the partition of the homogeneous group \( \mathbb{H} \) as follows.

\[
\mathbb{H} = \bigcup_{j=0} \sum_{j=0} 2^j B_0.
\]

We next study the annuli \( U_j := 2^{j+1}B_0 \setminus 2^j B_0 \) for \( j \geq 0 \) and the covering \( \{\tilde{B}_{j,\ell}\}_{\ell=1} \) of \( U_j \) as in (3.5). Note that for each \( \tilde{B}_{j,\ell} \), there exist \( t_{j,\ell} \in \{1, 2, \ldots, t\} \) and \( \tilde{Q}_{j,\ell} \in \mathcal{D}^{t_{j,\ell}} \) such that \( \tilde{B}_{j,\ell} \subseteq \tilde{Q}_{j,\ell} \subseteq C_{ad_j} \tilde{B}_{j,\ell} \).

Furthermore, we also observe that since \( \mathcal{C}_{\tilde{\lambda}} \tilde{B}_{j,\ell} \) covers \( B_0 \), for each such \( \tilde{B}_{j,\ell} \), the enlargement \( \mathcal{C}_{\tilde{\lambda}} \tilde{Q}_{j,\ell} \) covers \( B_0 \).
We then apply (3.9) to each \(\tilde{B}_{j,\ell}\) and then obtain a \(\frac{1}{2}\)-sparse family \(\tilde{F}_{j,\ell} \subset D^j(\tilde{Q}_{j,\ell})\) such that (3.9) holds for a.e. \(x \in \tilde{B}_{j,\ell}\).

Set \(\mathcal{F} \coloneqq \bigcup_{j,\ell} \tilde{F}_{j,\ell}\). Note that the balls \(C_{ad j} \tilde{B}_{j,\ell}\) overlap at most \(C_{Ad, j_0}(2j_0 + 1)\) times, where \(C_{Ad, j_0}\) is the constant in (3.6). Then we conclude that \(\mathcal{F}\) is a \(\frac{1}{2}C_{Ad, j_0}(2j_0 + 1)\)-sparse family and for a.e. \(x \in \mathbb{H}\),

\[
|T(f)(x)| \leq C_Q(\|T\|_{L^2 \to L^2} + \|\omega\|_{Dini}) \sum_{Q \in \mathcal{F}} \langle |f| \rangle_{C_{j_0, B(Q)}(\tilde{Q})} \chi_{\tilde{Q}}(x).
\]

We further set \(S \coloneqq \{C_{j_0} B(Q) : Q \in \mathcal{F}\}\), then \(S\) is a \(\frac{1}{2}C_{Ad, j_0}(2j_0 + 1)\)-sparse family, where \(\tilde{c}\) is a constant depending on \(C_{j_0}\). For this sparse family, we have

\[
|T(f)(x)| \leq C_Q(\|T\|_{L^2 \to L^2} + \|\omega\|_{Dini}) \mathcal{A}_S(|f|)(x).
\]

This finishes the proof of Proposition 3.4.

4. PROOF OF THEOREM 1.1

In this section, we will combine the ideas from [30] and [51] to show Theorem 1.1. Throughout this section, we assume that \(T\) is a rough homogeneous singular integral operator with \(K_0\) satisfying the conditions in Theorem 1.1. To show Theorem 1.1, the main difficulties are the lack of suitable Fourier transforms and the convolution on homogeneous groups are not commutative in general. We combine the ideas from [30] and [51] via using Littlewood–Paley decompositions and Cotlar–Knapp–Stein lemma to overcome the difficulties.

To begin with, we recall that for appropriate functions \(f\) and \(g\) on \(\mathbb{H}\), the convolution \(f \ast g\) is defined by

\[
f \ast g(x) = \int_{\mathbb{H}} f(y)g(y^{-1}x)dy.
\]

4.1. Kernel truncation and frequency localization. To begin with, we first partion the kernel \(K\) dyadically. Note that

\[
K = \frac{1}{\ln 2} \int_0^\infty \Delta[t]K_0 \frac{dt}{t},
\]

where for each \(t\), we define the scaling map \(\Delta[t]\) by

\[
\Delta[t]f(x) := t^{-Q} f(t^{-1} \circ x).
\]

Therefore we have the decomposition

\[
K = \sum_{j \in \mathbb{Z}} A_j K_0,
\]

where \(A_j\) is the operator

\[
A_j F = 2^{-j} \int_0^\infty \varphi(2^{-j}t)\Delta[t]F dt
\]
and \( \phi \) is a bump function localized in \( \{ t \sim 1 \} \) such that \( \sum_{j \in \mathbb{Z}} 2^{-j} \varphi(2^{-j} t) = \frac{1}{\ln 2} \). Hence,

\[
T(f) = \sum_{j \in \mathbb{Z}} T_j(f),
\]

where we denote \( T_j(f) = f \ast A_j K_0 \). Besides, since \( \text{supp} K_0 \subset \{ x \in \mathbb{H} : 1 \leq \rho(x) \leq 2 \} \), we have for any \( q > 1 \),

\[
(4.2) \quad \| A_j K_0 \|_1 \leq C \| K_0 \|_1 \leq C \| K_0 \|_q.
\]

The next step is to introduce a form of Littlewood-Paley theory, but we avoid any explicit use of the Fourier transform. To this end, let \( \phi \in C_c^\infty(\mathbb{H}) \) be a smooth cut-off function such that

1. \( \text{supp} \phi \subset \{ x \in \mathbb{H} : \frac{1}{200} \leq \rho(x) \leq \frac{1}{100} \} \);
2. \( \int_\mathbb{H} \phi(x) dx = 1 \);
3. \( \phi \geq 0 \);
4. \( \phi = \hat{\phi} \). Here \( \hat{\phi} \) denotes the function \( \hat{\phi}(x) = F(x^{-1}) \). For each integer \( j \), write

\[
\Psi_j = \Delta[2^{-j-1}] \phi - \Delta[2^{j}] \phi.
\]

Then \( \Psi_j \) is supported on the ball of radius \( C2^j \), has mean zero, and \( \hat{\Psi}_j = \Psi_j \).

Next we define the partial sum operators \( S_j \) by

\[
S_j(f) = f \ast \Delta[2^{j-1}] \phi.
\]

Their differences are given by

\[
S_j(f) - S_{j+1}(f) = f \ast \Psi_j.
\]

Since \( S_j(f) \rightarrow f \) as \( j \rightarrow -\infty \), for any sequence of integer numbers \( \{ N(j) \}_{j=0}^\infty \), with \( 0 = N(0) < \cdots < N(j) \rightarrow +\infty \), we have the following identity

\[
T_k = \left \{ S_k + \sum_{j_1=1}^\infty (S_{k-N(j_1)} - S_{k-N(j_1-1)}) \right \} T_k \left \{ S_k + \sum_{j_2=1}^\infty (S_{k-N(j_2)} - S_{k-N(j_2-1)}) \right \}
\]

\[
= S_k T_k S_k + \sum_{j_1=1}^\infty (S_{k-N(j_1)} - S_{k-N(j_1-1)}) T_k S_k + \sum_{j_2=1}^\infty S_k T_k (S_{k-N(j_2)} - S_{k-N(j_2-1)})
\]

\[
+ \sum_{j_1=1}^\infty \sum_{j_2=1}^\infty (S_{k-N(j_1)} - S_{k-N(j_1-1)}) T_k (S_{k-N(j_2)} - S_{k-N(j_2-1)}).
\]

In this way, \( T = \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \tilde{T}_{j_1,j_2} = \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \tilde{T}_{j_1,j_2}^N \), where

\[
\tilde{T}_{0,0} := \tilde{T}_{0,0}^N := \sum_{k \in \mathbb{Z}} S_k T_k S_k
\]

and, for \( j_1, j_2 \geq 1 \),

\[
\tilde{T}_{j_1,0} := \sum_{k \in \mathbb{Z}} (S_{k-j_1} - S_{k-(j_1-1)}) T_k S_k,
\]

\[
\tilde{T}_{0,j_2} := \sum_{k \in \mathbb{Z}} S_k T_k (S_{k-j_2} - S_{k-(j_2-1)}),
\]

\[
\tilde{T}_{j_1,j_2} := \sum_{k \in \mathbb{Z}} (S_{k-j_1} - S_{k-(j_1-1)}) T_k (S_{k-j_2} - S_{k-(j_2-1)}).
\]
(4.3) \[
\tilde{T}_{j_0,0}^N := \sum_{k \in \mathbb{Z}} (S_{k-N(j_1)} - S_{k-N(j_1-1)}) T_k S_k = \sum_{i_1=N(j_1-1)+1}^{N(j_1)} \tilde{T}_{i_1,0}.
\]

(4.4) \[
\tilde{T}_{0,j_2}^N := \sum_{k \in \mathbb{Z}} S_k T_k (S_{k-N(j_2)} - S_{k-N(j_2-1)}) = \sum_{j_2=N(j_2-1)+1}^{N(j_2)} \tilde{T}_{0,j_2}.
\]

(4.5) \[
\tilde{T}_{j_1,j_2}^N := \sum_{k \in \mathbb{Z}} (S_{k-N(j_1)} - S_{k-N(j_1-1)}) T_k (S_{k-N(j_2)} - S_{k-N(j_2-1)}) = \sum_{i_1=N(j_1-1)+1}^{N(j_1)} \sum_{i_2=N(j_2-1)+1}^{N(j_2)} \tilde{T}_{i_1,i_2}.
\]

For \(j_1, j_2 \in \mathbb{Z}\), we also consider the operator \(G_{j_1,j_2}\) defined by

\[
G_{j_1,j_2} = \sum_{k \in \mathbb{Z}} (S_{k-j_1} - S_{k-(j_1-1)}) T_k (S_{k-j_2} - S_{k-(j_2-1)}).
\]

Then we can further decompose \(\tilde{T}_{0,0}, \tilde{T}_{j_1,0}\) and \(\tilde{T}_{0,j_2}\) in the following way.

(4.7)
\[
\tilde{T}_{0,0} = \sum_{j_1=-\infty}^{0} \sum_{j_2=-\infty}^{0} G_{j_1,j_2}, \tilde{T}_{j_1,0} = \sum_{j_2=-\infty}^{0} G_{j_1,j_2}, \tilde{T}_{0,j_2} = \sum_{j_1=-\infty}^{0} G_{j_1,j_2}.
\]

4.2. \(L^2\) estimate for \(\tilde{T}_{j_1,j_2}^N\). In this subsection, we will give the \(L^2\)-estimate of the operators \(\tilde{T}_{j_1,j_2}^N\), which plays a crucial role in the proof of Theorem 1.1.

**Proposition 4.1.** Let \(q > 1\). Then there exist constants \(C_{q,q} > 0\) and \(\alpha > 0\) such that for any \(j_1, j_2 \geq 0\),

\[
\|\tilde{T}_{j_1,j_2}(f)\|_2 \leq C_{q,q} 2^{-\alpha j_1} 2^{-\alpha j_2} \|K_0\|_q \|f\|_2
\]

and for any \(j_1, j_2 \geq 1\),

\[
\|\tilde{T}_{j_1,0}^N(f)\|_2 \leq C_{q,q} 2^{-\alpha N(j_1-1)} \|K_0\|_q \|f\|_2;
\]

\[
\|\tilde{T}_{0,j_2}^N(f)\|_2 \leq C_{q,q} 2^{-\alpha N(j_2-1)} \|K_0\|_q \|f\|_2;
\]

\[
\|\tilde{T}_{j_1,j_2}^N(f)\|_2 \leq C_{q,q} 2^{-\alpha N(j_1-1)} 2^{-\alpha N(j_2-1)} \|K_0\|_q \|f\|_2.
\]

4.2. \(L^2\) estimate for \(G_{j_1,j_2}\) and then \(\tilde{T}_{j_1,j_2}^N\).

**Lemma 4.2.** Let \(q > 1\). Then there exist constants \(C_{q,q} > 0\) and \(\alpha > 0\) such that for any \(j_1, j_2 \in \mathbb{Z}\),

\[
\|G_{j_1,j_2}(f)\|_2 \leq C_{q,q} 2^{-\alpha j_1} 2^{-\alpha j_2} \|K_0\|_q \|f\|_2.
\]

**Proof.** For simplicity, we set

\[
G_{j_1,j_2,k}(f) := (S_{k-j_1} - S_{k-(j_1-1)}) T_k (S_{k-j_2} - S_{k-(j_2-1)})(f),
\]

then \(G_{j_1,j_2}(f) = \sum_{k \in \mathbb{Z}} G_{j_1,j_2,k}(f)\). By Cotlar-Knapp-Stein Lemma (see [50]), it suffices to show that:

\[
\|G_{j_1,j_2,k} G_{j_1,j_2,k'} \|_{2 \to 2} + \|G_{j_1,j_2,k}^* G_{j_1,j_2,k'}^* \|_{2 \to 2} \leq C 2^{-2\alpha j_1} 2^{-2\alpha j_2} 2^{-c|k-k'|} \|K_0\|_q^2,
\]

for some \(C, c > 0\) and \(\alpha > 0\). We only estimate the first term, since the second term is similar. A direct calculation yields

\[
G_{j_1,j_2,k} G_{j_1,j_2,k'}(f) = f \ast \Psi_{k-j_2}^* A_k K_0 \ast \Psi_{k-j_1}^* \Psi_{k-j_1} \ast A_k \tilde{K}_0 \ast \Psi_{k-j_2}.
\]
On the one hand, we first recall that Tao [51] applied iterated \((TT')^N\) method to obtain the following inequality with \(q = \infty\) and then Sato [46] extended it to general \(q > 1\): there exist constants \(C_{q, \alpha} > 0\) and \(\alpha > 0\) such that for any integers \(j, k\), and any \(L^q\) function \(F\) on the annulus with mean zero,

\[
\|f \ast A_jF \ast \Psi_k\|_2 \leq C_{q, \alpha} 2^{-\alpha j - k} \|f\|_2 \|F\|_q.
\]

(4.11)

It follows from the above fact and its duality version that

\[
\|G_{j_1, j_2}^* G_{j_1, j_2}^* (f)\|_2 \leq \|(f \ast \Psi_{\alpha - j_2} \ast A_k \tilde{K}_0) \ast (\Psi_{\alpha - j_2} \ast \Psi_{k-1}) \ast (A_k \tilde{K}_0 \ast \Psi_{k-1})\|_2
\]

\[
\leq C 2^{-\beta j_2} \|\tilde{K}_0\|_q \|(f \ast \Psi_{\alpha - j_2} \ast A_k \tilde{K}_0) \ast (\Psi_{\alpha - j_2} \ast \Psi_{k-1})\|_2
\]

\[
\leq C 2^{-\beta j_2} \|\tilde{K}_0\|_q \|f \ast \Psi_{\alpha - j_2} \ast A_k \tilde{K}_0\|_2 \|\Psi_{k-1}\|_1
\]

\[
\leq C 2^{-\beta j_2} \|\tilde{K}_0\|_q 2^{-\epsilon(k-\alpha)} \|f \ast \Psi_{\alpha - j_2} \ast A_k \tilde{K}_0\|_2
\]

(4.12)

for some constants \(C, c > 0\) and \(\beta > 0\), where in the next to the last inequality we used the cancellation and the smoothness properties of \(\Psi_{\alpha - j_2}\) and \(\Psi_{k-1}\).

On the other hand,

\[
\|G_{j_1, j_2}^* G_{j_1, j_2}^* (f)\|_2 \leq \|(f \ast \Psi_{\alpha - j_2} \ast (A_k \tilde{K}_0 \ast \Psi_{k-1}) \ast (A_k \tilde{K}_0 \ast \Psi_{k-1})\|_2
\]

\[
\leq C \|(f \ast \Psi_{\alpha - j_2} \ast (A_k \tilde{K}_0 \ast \Psi_{k-1}) \ast (A_k \tilde{K}_0 \ast \Psi_{k-1})\|_2
\]

\[
\leq C 2^{-\beta j_1} \|\tilde{K}_0\|_q \|f \ast \Psi_{\alpha - j_2}\|_2 \|A_k \tilde{K}_0\|_2 \|\Psi_{k-1}\|_2
\]

\[
\leq C 2^{-\beta j_1} \|\tilde{K}_0\|_q \|f \ast \Psi_{\alpha - j_2}\|_2
\]

(4.13)

Taking geometric mean of (4.12) with (4.13), we obtain the estimate (4.10). This finishes the proof of Lemma 4.2. \(\square\)

Next, we give the proof of Proposition 4.1.

**Proof of Proposition 4.1.** It suffices to show the first inequality, since the remaining three inequalities can be obtained by simply summing the geometric series \(\sum_{i=N(j-1)+1}^{N(j)} 2^{-\alpha i}\). It follows from the fact \(G_{j_1, j_2} = \tilde{T}_{j_1, j_2}\) for \(j_1, j_2 \geq 1\) and Lemma 4.2 that the estimate (4.8) holds for \(j_1, j_2 \geq 1\). For the case \(j_1 = j_2 = 0\), By applying Lemma 4.2 to the equality (4.7), we obtain

\[
\|\tilde{T}_{0,0}(f)\|_2 \leq \sum_{j_1=0}^{0} \sum_{j_2=0}^{0} \|G_{j_1, j_2}(f)\|_2 \leq C \sum_{j_1=0}^{0} \sum_{j_2=0}^{0} 2^{\alpha j_1} 2^{\alpha j_2} \|K_0\|_q \|f\|_2 \leq C \|K_0\|_q \|f\|_2.
\]

Hence, the proof of Proposition 4.1 is finished. \(\square\)

**4.3. Calderón–Zygmund theory of \(\tilde{T}_{N,j_1,j_2}\).**

**Lemma 4.3.** The operator \(\tilde{T}_{N,j_1,j_2}\) is a Calderón–Zygmund operator satisfying for any \(q > 1\),

\[
C_{j_1,j_2}^N := C_{j_1,j_2}^N \leq C_{q, \alpha} 2^{N(j_1 + j_2)} \|K_0\|_q, \quad \omega_{j_1,j_2}^N (t) := \omega_{j_1,j_2}^N (t) \leq C_{q, \alpha} 2^{N(j_1 + j_2)} \min\{1, 2^{N(j_1,j_2)}\} \|K_0\|_q,
\]

for any \(q > 1\).
which satisfies
\[
\int_0^1 \omega_{j_1,j_2}^N(t) \frac{dt}{t} \leq C_{Q,q} 2^{\frac{N(j_2)q}{q}} (1+N(j_2)) \| K_0 \|_q.
\]

**Proof.** From Proposition 4.1 we can see that \( \tilde{T}^N_{j_1,j_2} \) is a bounded operator in \( L^2 \). In order to obtain the required estimates for the kernel of \( \tilde{T}^N_{j_1,j_2} \), we first study the kernel of each \( S_{\mathbb{Z}^{j_1}} T_k S_{\mathbb{Z}^{j_2}} \). Note that
\[
|A_k K_0(x)| = 2^{-k} \left| \int_{-\infty}^{+\infty} \phi(2^{-k} t) \rho(z)^{-Q} K((2^k t)^{-1} \circ x) \chi_{\{1 \leq: \rho \circ (r^{-1} \circ x) \leq 2 \}}(x) dt \right|
= \left| \int_{-\infty}^{+\infty} t \phi(t) \rho(z)^{-Q} K((2^k t)^{-1} \circ x) \chi_{\{1 \leq: \rho \circ (r^{-1} \circ x) \leq 2 \}}(x) dt \right|
\leq C \rho(x)^{-Q} |K((2^k t)^{-1} \circ x) \chi_{\{1 \leq: \rho \circ (r^{-1} \circ x) \leq 2 \}}(x)|.
\]
Since \( \text{supp} \phi \subset \{ x \in \mathbb{H} : \rho(x) \leq \frac{1}{100} \} \),
\[
|\Delta [2^{k-N(j_2)-1}] \phi \ast A_k K_0(x)|
\leq C \int_{\mathbb{H}} 2^{-[k-N(j_2)-1]} |\phi(2^{-[k-N(j_2)-1]} \circ x y^{-1})| \rho(y)^{-Q} \chi_{\{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2} \}}(y) K((\rho(y)^{-1} \circ y)|y|) dy
\leq C 2^{-[k-N(j_2)-1]} \rho(x)^{-Q} \chi_{\{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2} \}}(x) \int_{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2}} |\phi(2^{-[k-N(j_2)-1]} \circ x y^{-1})| \rho(y)^{-Q} \chi_{\{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2} \}}(y) dy.
\]
By Hölder’s inequality, for \( 1/q + 1/q' = 1 \),
\[
\int_{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2}} |\phi(2^{-[k-N(j_2)-1]} \circ x y^{-1})| \rho(y)^{-Q} \chi_{\{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2} \}}(y) dy
\leq \left( \int_{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2}} |\phi(2^{-[k-N(j_2)-1]} \circ x y^{-1})| |y|^{q} dy \right)^{1/q} \left( \int_{\mathbb{H}} |\phi(2^{-[k-N(j_2)-1]} \circ x y^{-1})| |y|^{q'} dy \right)^{1/q'}
\leq C 2^{\frac{N(j_2)q}{q}} 2^{\frac{N(j_2)q}{q}} \| K_0 \|_q.
\]
Combining the above two inequalities, we get that
\[
|\Delta [2^{k-N(j_2)-1}] \phi \ast A_k K_0(x)| \leq C 2^{\frac{N(j_2)q}{q}} \rho(x)^{-Q} \| K_0 \|_q \chi_{\{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2} \}}(x).
\]
Therefore,
\[
|\Delta [2^{k-N(j_2)-1}] \phi \ast A_k K_0 \ast \Delta [2^{k-N(j_1)-1}] \phi(x)|
\leq C 2^{\frac{N(j_2)q}{q}} \| K_0 \|_q \int_{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2}} \rho(z)^{-Q} |\Delta [2^{k-N(j_2)-1}] \phi(z^{-1} x)| dz
\leq C 2^{\frac{N(j_2)q}{q}} \| K_0 \|_q \chi_{\{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2} \}}(x) \rho(x)^{-Q}.
\]
Similarly, we obtain the gradient estimate as follows.
\[
|\nabla \Delta [2^{k-N(j_2)-1}] \phi \ast A_k K_0 \ast \Delta [2^{k-N(j_1)-1}] \phi(x)| \leq C 2^{N(j_2)\left(1+\frac{q}{q'}\right)} \rho(x)^{-Q} \chi_{\{2^{k-2} \leq: \rho \circ (r^{-1} \circ y) \leq 2^{k+2} \}}(x) \| K_0 \|_q,
\]
where \( \nabla \) is the gradient on homogeneous groups (see for example [24]). Hence,
\[
\sum_{k \in \mathbb{Z}} |\nabla \Delta [2^{k-N(j_2)-1}] \phi \ast A_k K_0 \ast \Delta [2^{k-N(j_1)-1}] \phi(x)| \leq C 2^{N(j_2)\left(1+\frac{q}{q'}\right)} \rho(x)^{-Q-1} \| K_0 \|_q.
\]
From the triangle inequality and $N(j - 1) < N(j)$ we can see that the kernel
\[ K^N_{j_1, j_2} := \sum_{k \in \mathbb{Z}} (\Delta[2^{k-N(j_2)} - 1] \phi - \Delta[2^{k-N(j_1)} - 1] \phi) * A_k K_0 * (\Delta[2^{k-N(j_1)} - 1] \phi - \Delta[2^{k-N(j_1-1)} - 1] \phi) \]
of $\tilde{T}^N_{j_1, j_2}$ satisfies the same estimates (4.15) and (4.16). That is,
\[ |K^N_{j_1, j_2}(x, y)| = |K^N_{j_1, j_2}(y^{-1} x)| \leq C2^{\frac{N(j_1)Q}{4}} d(x, y)^{-Q} \|K_0\|_q, \]
\[ |\nabla K^N_{j_1, j_2}(x, y)| \leq C2^{N(j_2)(1+\frac{Q}{2})} d(x, y)^{-\frac{Q}{2}} \|K_0\|_q. \]
Note that for $j_1 = 0$ or $j_2 = 0$, the subtraction is not even needed. The first bound above is already the required estimate for $C^N_{j_1, j_2}$. Besides, for $d(x, y) \geq 2A_0 d(x, x')$, by mean value theorem on homogeneous groups (see for example [24]),
\[ |K^N_{j_1, j_2}(x, y) - K^N_{j_1, j_2}(x', y)| = |K^N_{j_1, j_2}(y^{-1} x) - K^N_{j_1, j_2}(y^{-1} x')| \]
\[ \leq C2^{N(j_2)(1+\frac{Q}{2})} d(x, y)^{-\frac{Q}{2}} d(x, x') \|K_0\|_q. \]
By the triangle inequality, we also have
\[ |K^N_{j_1, j_2}(x, y) - K^N_{j_1, j_2}(x', y)| \leq C2^{\frac{N(j_1)Q}{4}} d(x, y)^{-Q} \|K_0\|_q. \]
Combining the two estimates we obtain above and by symmetry, we conclude that
\[ |K^N_{j_1, j_2}(x, y) - K^N_{j_1, j_2}(x', y)| + |K^N_{j_1, j_2}(y, x) - K^N_{j_1, j_2}(y, x')| \leq C\omega^N_{j_1, j_2} \left( \frac{d(x, x')}{d(x, y)} \right) d(x, y)^{-Q}, \]
where
\[ \omega^N_{j_1, j_2}(t) \leq C2^{\frac{N(j_1)Q}{4}} \min\{1, 2^{N(j_2)}t\} ||K_0||_q. \]
A direct calculation yields that
\[ \int_0^1 \omega^N_{j_1, j_2}(t) \frac{dt}{t} \leq C2^{\frac{N(j_1)Q}{4}} (1 + N(j_2)) ||K_0||_q. \]
This ends the proof of Lemma 4.3. \qed

**Remark 4.4.** From the above proof we can see that if we apply Hölder’s inequality to the expression $A_k K_0 * \Delta[2^{k-N(j_1)} - 1] \phi$ instead of $\Delta[2^{k-N(j_2)} - 1] \phi * A_k K_0$, then we can also obtain similar upper bounds for $C^N_{j_1, j_2}$ and for $\omega^N_{j_1, j_2}(t)$ with $N(j_1)$ replaced by $N(j_2)$. Then taking geometric mean of these two estimates involving $N(j_1)$ and $N(j_2)$, we can obtain an estimate with a symmetry form:
\[ C^N_{j_1, j_2} \leq C_2 Q_d 2^{\frac{N(j_1)Q}{2m}} 2^{\frac{N(j_2)Q}{2}} \|K_0\|_q. \]
\[ \int_0^1 \omega^N_{j_1, j_2}(t) \frac{dt}{t} \leq C2^{\frac{N(j_1)Q}{4}} 2^{\frac{N(j_2)Q}{2}} (1 + N(j_1))^{1/2} (1 + N(j_2))^{1/2} \|K_0\|_q. \]
But in the remaining steps we can see that these factors can be absorbed via interpolation.

With the help of Propositions 3.4, 4.1 and Lemma 4.3, we can easily follow a similar procedure in [30] to show a bad quantitative $L^p$ weighted inequality and a good quantitative unweighted $L^p$ estimate for the operators $\tilde{T}^N_{j_1, j_2}$. 
Lemma 4.5. Let $1 < p < \infty$ and $q > 1$, then for any $w \in A_p$, there exists a constant $C_{Q,p,q} > 0$ such that
\[
\|\hat{T}_{j_1,j_2}^N(f)\|_{L^p(w)} \leq C_{Q,p,q}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q\|w\|_{A_p}\|f\|_{L^p(w)}.
\]

Proof. By Propositions 3.4, 4.1 and Lemma 4.3 as well as the $L^p(w)$ boundedness of the sparse operators (See for example [20], [43, Theorem 3.1]),
\[
\|\hat{T}_{j_1,j_2}^N\|_{L^p(w)} \leq C_{Q,p,q}(\|\hat{T}_{j_1,j_2}^N\|_{L^2 \rightarrow L^2} + C_{j_1,j_2}^N + \|\omega_{j_1,j_2}^N\|_{Dini})(w)A_p\|f\|_{L^p(w)}
\]
\[
\leq C_{Q,p,q}(2^{-\alpha N(j_1-1)2^{-\alpha N(j_2-1)}}\|K_0\|_q + 2^{-\alpha N(j_1-1)/q}\|K_0\|_q + 2^{-\alpha N(j_2-1)/(q+1)}(1 + N(j_2))\|K_0\|_q)\|w\|_{A_p}\|f\|_{L^p(w)}
\]
\[
\leq C_{Q,p,q}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q\|w\|_{A_p}\|f\|_{L^p(w)}.
\]
This finishes the proof of Lemma 4.5. \hfill \Box

Lemma 4.6. Let $1 < p < \infty$ and $q > 1$, then there exist constants $C_{Q,p,q}$ and $\beta_p > 0$ such that
\[
\|\hat{T}_{j_1,j_2}^N(f)\|_{L^p} \leq C_{Q,p,q}2^{-\beta_p N(j_1-1)2^{-\beta_p N(j_2-1)}}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q\|f\|_{L^p}.
\]

Proof. We first consider the case $p > 2$ and let $s = 2p$ so that $2 < p < s$. This implies that $\frac{1}{p} = \frac{1-q}{2} + \frac{q}{s}$, for $0 < \theta := \frac{p-2}{p-1} < 1$. Then, it follows from Proposition 4.1, Lemma 4.5 with $w(x) \equiv 1$ and complex interpolation that
\[
\|\hat{T}_{j_1,j_2}^N\|_{L^p \rightarrow L^p} \leq \|\hat{T}_{j_1,j_2}^N\|_{L^2 \rightarrow L^2}\|\hat{T}_{j_1,j_2}^N\|_{L^s \rightarrow L^p}
\]
\[
\leq (C_{Q,q}2^{-\alpha N(j_1-1)2^{-\alpha N(j_2-1)}}\|K_0\|_q)^{1-\theta}(C_{Q,2,p,q}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q)^{\theta}
\]
\[
\leq C_{Q,p,q}2^{-\beta_p N(j_1-1)2^{-\beta_p N(j_2-1)}}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q,
\]
where $\beta_p = \alpha(1-\theta) = \alpha/(p-1)$.

For the case $p < 2$, let $s := \frac{2p}{1+p}$ so that $1 < s < p < 2$. In this case, $\frac{1}{p} = \frac{1-q}{2} + \frac{q}{s}$, for $0 < \theta := 2 - p < 1$. Applying the interpolation theorem between $L^2$ and $L^s$, we obtain a similar $L^p$ estimate. \hfill \Box

4.4. Proof of Theorem 1.1. Let us denote $\varepsilon := \frac{1}{2}C_q/(w)\lambda_p$. It follows from Lemma 4.5 and 2.2 that for this choice of $\varepsilon$,
\[
\|\hat{T}_{j_1,j_2}^N\|_{L^p(w^{1+\varepsilon}) \rightarrow L^p(w^{1+\varepsilon})} \leq C_{Q,p,q}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q\|w^{1+\varepsilon}\|_{A_p}
\]
\[
\leq C_{Q,p,q}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q\|w^{1+\varepsilon}\|_{A_p}.
\]
Besides, by Lemma 4.6, we also have
\[
\|\hat{T}_{j_1,j_2}^N\|_{L^p \rightarrow L^p} \leq C_{Q,p,q}2^{-\beta_p N(j_1-1)2^{-\beta_p N(j_2-1)}}2^{N(j_2)q\theta/(q+1)}(1 + N(j_2))\|K_0\|_q.
\]

We now apply the interpolation theorem with change of measures ([49, Theorem 2.11]) to $T = \hat{T}_{j_1,j_2}^N$ with $p_0 = p_1 = p$, $w_0 = 1$ and $w_1 = w^{1+\varepsilon}$ so that $\theta = \varepsilon/(1+\varepsilon)$ and
\[
\|\hat{T}_{j_1,j_2}^N\|_{L^p(w) \rightarrow L^p(w)} \leq \|\hat{T}_{j_1,j_2}^N\|_{L^p \rightarrow L^p}^{1/(1+\varepsilon)\|\hat{T}_{j_1,j_2}^N\|_{L^p(w^{1+\varepsilon})}^{1/(1+\varepsilon)}}
\]
\[
\leq C_{Q,p,q}\|K_0\|_q2^{N(j_2)q\theta/(q+1)}2^{-\beta_p N(j_1-1)\varepsilon/(1+\varepsilon)}2^{-\beta_p N(j_2-1)\varepsilon/(1+\varepsilon)}(1 + N(j_2))\|w\|_{A_p}
\]
\[
\leq C_{Q,p,q}\|K_0\|_q2^{N(j_2)q\theta/(q+1)}2^{-\beta_p N(j_1-1)\varepsilon/(1+\varepsilon)}2^{-\beta_p N(j_2-1)\varepsilon/(1+\varepsilon)}(1 + N(j_2))\|w\|_{A_p},
\]
for some constants $\beta_p, \beta_{\mathbb{Q}, p} > 0$. 

Thus,

$$
\|T\|_{L^p(w) \to L^p(w)} \leq \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \|\tilde{T}_{j_1,j_2}^N\|_{L^p(w) \to L^p(w)} 
$$

(4.17) 

$$
\leq C_{\mathbb{Q}, p,q} \|K_0\|_{w_A} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} 2^{\frac{N(j_1)Q}{q}} 2^{-\beta_{\mathbb{Q}, p} N(j_1-1)/(w_A)} 2^{-\beta_{\mathbb{Q}, p} N(j_2-1)/(w_A)} (1 + N(j_2)).
$$

Note that if we choose $N(j) = 2^j$ for $j \geq 1$ and $q > \frac{2\mathbb{Q}(w_A)}{\beta_{\mathbb{Q}, p}} := c_{\mathbb{Q}, p}(w_A)$, then

$$
\sum_{j_2=0}^{\infty} 2^{\frac{N(j_2)Q}{q}} 2^{-\beta_{\mathbb{Q}, p} N(j_2-1)/(w_A)} (1 + N(j_2)) \leq \sum_{j_2=0}^{\infty} 2^{j_2} 2^{-\beta_{\mathbb{Q}, p} 2^{j_2} (w_A)^{-1}}
$$

$$
\leq C_{\mathbb{Q}, p,q} \left( \sum_{j_2:2^{j_2} \leq (w_A)_{A_p}} 2^{j_2} + \sum_{j_2:2^{j_2} > (w_A)_{A_p}} 2^{j_2} \left( \frac{(w_A)_{A_p}}{2^{j_2}} \right)^2 \right)
$$

$$
\leq C_{\mathbb{Q}, p,q}(w_A),
$$

for some constant $\beta_{\mathbb{Q}, p,q} > 0$. Besides, the summation with respect to $j_1$ can be estimated much more easily. These, in combination with the estimate (4.17), complete the proof of Theorem 1.1. □

5. Application: Quantitative Estimate of Singular Integrals Studied by Sato

In the previous section, we proved quantitative weighted estimates for classical rough homogeneous singular integrals on homogeneous group. Indeed, our argument can also be applied to draw a parallel conclusion for a larger class of singular integrals considered by [46].

Proof of Theorem 1.2. The proof is a minor modification of Theorem 1.1. We list the differences of the proof here. To begin with, we decompose the kernel $L$ into the summation of $B_j K_0$, where $B_j$ is the operator

$$
B_j F = 2^{-j} h(\rho(x)) \int_0^{\infty} \varphi(2^{-j} t) \Delta [t] F dt.
$$

Then the operators $T_j, \tilde{T}_{j_1,j_2}, \tilde{T}_{j_1,j_2}^N$ and $G_{j_1,j_2}, G_{j_1,j_2,k}$ can be constructed with $A_j$ (defined by (4.1)) replaced by $B_j$. By Lemma 1 in [46] and a similar almost orthogonal argument, we obtain that

$$
\|G_{j_1,j_2,k} G_{j_1,j_2,k'}(f)\|_2 \leq C 2^{-\beta_{j_1} j_1} 2^{-\beta_{j_2} j_2} 2^{-\beta_{k,k'} j_2} \|K_0\|_{q}^{2} \|h\|_{A_{q'} q'}^{2} \|f\|_2.
$$

Similar to the proof of Proposition 4.1, we obtain that for any $j_1, j_2 \geq 1$,

(5.1) 

$$
\|\tilde{T}_{j_1,j_2}(f)\|_2 \leq C_{\mathbb{Q}, q} 2^{-\alpha N(j_1-1)} 2^{-\alpha N(j_2-1)} \|K_0\|_{q}^{2} \|h\|_{A_{q'} q'}^{2} \|f\|_2.
$$

In the next step, note that for $2/q + 1/q_0 = 1,$

$$
\int_{2^{k_1-1} \leq \rho(y) \leq 2^{k_2+1}} |\phi(2^{-[k-N(j_2)-1]} \circ y^{-1})| K(\rho(y)^{-1} \circ y) |h(\rho(y))| dy
$$

$$
\leq \left( \int_{2^{k_1-1} \rho(y) \leq 2^{k_2+1}} |K(\rho(y)^{-1} \circ y)| dy \right)^{1/q} \left( \int_{2^{k_1-1} \rho(y) \leq 2^{k_2+1}} |h(\rho(y))|^{q_0} dy \right)^{1/q_0} \left( \int_{\mathbb{H}} |\phi(2^{-[k-N(j_2)-1]} \circ y)|^{q_0} dy \right)^{1/q_0}.
$$
\[\leq C2^{\frac{Q}{2}}2^{-\frac{N(j_i)Q}{2}}\|K_0\|_q\|h\|_{d_i}.\]

Then a simple modification of Lemma 4.3 yields that the operator \(\tilde{T}^N_{j_1,j_2}\) is a Calderón–Zygmund operator satisfying for any \(q > 2\),
\[
C_{j_1,j_2}^N \leq C_{Q,q}2^{-\frac{2N(j_2)Q}{q}}\|K_0\|_q\|h\|_{d_i}, \quad \omega^{N}_{j_1,j_2}(t) \leq C_{Q,q}2^{-\frac{2N(j_2)Q}{q}}\min\{1, 2^{N(j_2)t}\}\|K_0\|_q\|h\|_{d_i},
\]
which satisfies
\[
\int_0^1 \omega^{N}_{j_1,j_2}(t) \frac{dt}{t} \leq C_{Q,q}2^{-\frac{2N(j_2)Q}{q}}(1 + N(j_2))\|K_0\|_q\|h\|_{d_i}.
\]

Next we obtain simple variants of Lemmata 4.5 and 4.6, and then the proof of Theorem 1.2 is complete.

\[\square\]

6. An investigation in the bi-parameter setting: proof of Theorem 1.3

In this section, we show that our argument can be also applied to obtain a parallel result in the bi-parameter setting. To begin with, let \(\mathbb{H}_i = \mathbb{R}^{n_i}, i = 1, 2\), be homogeneous groups with dilations \(\varnothing_1, \varnothing_2\), and norm functions \(\rho_1, \rho_2\), respectively. Each \(\varnothing_i\) is an automorphism of the group structure and is of the form
\[
t \circ_i (x_1, \ldots, x_{n_i}) = (t^{a_1^i}x_1, \ldots, t^{a_{n_i}^i}x_{n_i}), \quad \forall (x_1, \ldots, x_{n_i}) \in \mathbb{H}_i,
\]
for some constants \(0 < a_1^i \leq a_2^i \leq \ldots \leq a_{n_i}^i\). We call the quantity \(Q_i = \sum_{j=1}^{n_i} a_j^i\) the homogeneous dimension of \(\mathbb{H}_i\). We define a left-invariant quasi-distance \(d_i\) on \(\mathbb{H}_i\) by \(d_i(x, y) = \rho_i(x^{-1}y)\), which means that there exists a constant \(A_0^{(i)} \geq 1\) such that for any \(x, y, z \in \mathbb{H}_i\),
\[
d_i(x, y) \leq A_0^{(i)}[d_i(x, z) + d_i(z, y)].
\]
Let \(B_i(x_i, r)\) be the ball in with center \(x_i \in \mathbb{H}_i\) and radius \(r \in \mathbb{R}_+\) defined by \(B_i(x_i, r) = \{y_i \in \mathbb{H}_i : d_i(x_i, y_i) < r\}\).

**Definition 6.1.** Let \(w(x_1, x_2)\) be a nonnegative locally integrable function on \(\mathbb{H}_1 \times \mathbb{H}_2\). For \(1 < p < \infty\), we say that \(w\) is a product \(A_p\) weight, written \(w \in A_p\), if
\[
[w]_{A_p} := \sup_R \left(\int_R w dx_1 dx_2\right) \left(\int_R \left(\frac{1}{w}\right)^{\frac{1}{p-1}} dx_1 dx_2\right)^{p-1} < \infty,
\]
where the supremum is taken over all rectangles \(R \subset \mathbb{H}_1 \times \mathbb{H}_2\). The quantity \([w]_{A_p}\) is called the \(A_p\) constant of \(w\). For \(p = 1\), if \(M_s(w)(x_1, x_2) \leq w(x_1, x_2)\) for a.e. \((x_1, x_2) \in \mathbb{H}_1 \times \mathbb{H}_2\), then we say that \(w\) is a product \(A_1\) weight, written \(w \in A_1\), where \(M_s\) denotes the strong maximal function on \(\mathbb{H}_1 \times \mathbb{H}_2\). Besides, let \(A_\infty := \cup_{1 \leq p < \infty} A_p\) and we have
\[
[w]_{A_\infty} := \sup_R \left(\int_R w dx_1 dx_2\right) \exp \left(\int_R \log \left(\frac{1}{w}\right) dx_1 dx_2\right) < \infty.
\]

Throughout this section, for appropriate functions \(f\) and \(g\) on \(\mathbb{H}_1 \times \mathbb{H}_2\), the convolution \(f \ast g\) is defined by
\[
(f \ast g)(x, y) = \int_{\mathbb{H}_1 \times \mathbb{H}_2} f(xu^{-1}, yv^{-1})g(u, v)dudv.
\]

We now provide the proof of Theorem 1.3.
Proof of Theorem 1.3.} It suffices for us to provide the decomposition of $T$ into a suitable collection \( \{ T_{j_1,j_2,j_3,j_4}^N \} \) and then to verify that each \( T_{j_1,j_2,j_3,j_4}^N \) is a paraproduct-free operator in the class of bi-parameter singular integral operators with the modulus of continuity \( \omega_{j_1,j_2,j_3,j_4}^{N,k} \), \( k = 1, 2 \), satisfying the modified Dini condition \[3, \text{ Section 5}]:

\[
(6.1) \quad \| \omega_{j_1,j_2,j_3,j_4}^{N,k} \|_{\text{Dini}} := \int_0^1 \omega_{j_1,j_2,j_3,j_4}^{N,k}(t)(1 + \log \frac{1}{t}) dt \leq C 2^{\frac{N(j_1)q_1}{9} + \frac{N(j_2)q_2}{9}} \| K^0 \|_q (1 + N(j_{k+2}))^2.
\]

We first partition the kernel $K$ dyadically.

\[
K(x,y) = \frac{1}{(\ln 2)^2} \int_0^\infty \int_0^\infty \Delta[t_1, t_2] K^0(x,y) dt_1 dt_2,
\]

where for each $t_1, t_2 \in \mathbb{R}_+$, we define the product scaling map $\Delta[t_1, t_2]$ by $\Delta[t_1, t_2] = \Delta^{(1)}[t_1] \otimes \Delta^{(2)}[t_2]$, and $\Delta^{(0)}[t_i]f(x_i) := t_i^{-1} f(t_i^{-1} x_i), i = 1, 2$. Therefore we have the decomposition

\[
T(f)(x,y) = \sum_{(j_1,j_2) \in \mathbb{Z}^2} f * A_{j_1,j_2} K^0(x,y) =: \sum_{(j_1,j_2) \in \mathbb{Z}^2} T_{j_1,j_2}(f)(x,y),
\]

where $A_{j_1,j_2}$ is the operator

\[
(6.2) \quad A_{j_1,j_2} F(x,y) = 2^{-j_1-2j_2} \int_0^\infty \int_0^\infty \varphi(2^{-j_1} t_1) \varphi(2^{-j_2} t_2) \Delta[t_1,t_2] F(x,y) dt_1 dt_2.
\]

Let $\phi^{(i)} \in C_c(\mathbb{H}_t)$ be a smooth cut-off function supported in $B_0(\frac{1}{100}) \setminus B_0(\frac{1}{200})$ such that $\int_{\mathbb{H}_t} \phi^{(i)} dx_i = 1$, $\phi^{(i)} = \overline{\phi^{(i)}}$, $\phi^{(i)}(x_i) \geq 0$ for all $x_i \in \mathbb{H}_t$. Denote

\[
\Psi^{(i)}_j = \Delta^{(i)}[2^{-j_1}] \phi^{(i)} - \Delta^{(i)}[2^j] \phi^{(i)},
\]

then $\Psi^{(i)}_j$ satisfies $\text{supp} \Psi^{(i)}_j \subseteq B_0(\frac{1}{200} C^2)$, has mean zero, and $\overline{\Psi^{(i)}_j} = \Psi^{(i)}_j$.

We define the partial sum operators $S_{j_1,j_2}$ by

\[
S_{j_1,j_2}(f) = f * (\Delta^{(1)}[2^{-j_1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{-j_2}] \phi^{(2)}).
\]

Next, we define the difference operators $\Delta_{k_1,k_2}$ and $\Delta_{N(j_1),N(j_2)}^{(N(j_1),N(j_2))}$ by

\[
\Delta_{k_1,k_2} f = f * \Psi_{k_1,k_2}, \quad \Delta_{N(j_1),N(j_2)}^{(N(j_1),N(j_2))} f = f * \Psi_{N(j_1),N(j_2)},
\]

where we denote

\[
\Psi_{k_1,k_2} = \Psi_{k_1}\otimes \Psi_{k_2}, \quad \Psi_{N(j_1),N(j_2)} = \sum_{\ell_1=N(j_1)-1}^{N(j_1)} \sum_{\ell_2=N(j_2)-1}^{N(j_2)} \Psi_{k_1-\ell_1,k_2-\ell_2}.
\]

We also define the mixed difference operators $(S \Delta)^{N(j_2)}_{k_1,k_2}$ and $(\Delta S)^{N(j_1)}_{k_1,k_2}$ by

\[
(S \Delta)^{N(j_2)}_{k_1,k_2} f := f * (\Delta^{(1)}[2^{-k_1}] \phi^{(1)} \otimes (\Delta^{(2)}[2^{-k_2-N(j_2)}] \phi^{(2)} - \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)})),
\]

\[
(\Delta S)^{N(j_1)}_{k_1,k_2} f := f * ((\Delta^{(1)}[2^{-k_1-N(j_1)}] \phi^{(1)} - \Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes (\Delta^{(2)}[2^{k_2-1}] \phi^{(2)}).
\]

Then we have the following inequality

\[
T_{k_1,k_2} = \left( S_{k_1,k_2} + \sum_{j_1=1}^{\infty} (\Delta S)^{N(j_1)}_{k_1,k_2} + \sum_{j_2=1}^{\infty} (S \Delta)^{N(j_2)}_{k_1,k_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \Delta_{N(j_1),N(j_2)} f \right) \left( S_{k_1,k_2} + \sum_{j_1=1}^{\infty} (\Delta S)^{N(j_1)}_{k_1,k_2} \right).
\]
\[ + \sum_{j_4=1}^{\infty} (S \Delta)^{N(j_4)}_{k_1,k_2} + \sum_{j_3=1}^{\infty} \sum_{j_4=1}^{\infty} \Delta^{N(j_3),N(j_4)}_{k_1,k_2} \].

In this way, we get
\[ T = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} \hat{T}_{j_1,j_2,j_3,j_4} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} \hat{T}^N_{j_1,j_2,j_3,j_4}, \]
where
\[ \hat{T}_{0,0,0,0} := \hat{T}_{0,0,0,0} := \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} S_{k_1,k_2} T_{k_1,k_2} S_{k_1,k_2}, \]
and for \( j_1, j_2, j_3, j_4 \geq 1, \)
\[ \hat{T}_{j_1,j_2,j_3,j_4} := \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \Delta_{k_1} T_{k_1,k_2} \Delta_{k_2} T_{k_1,k_2}, \]
\[ \hat{T}^N_{j_1,j_2,j_3,j_4} := \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \Delta^{N(j_1),N(j_2)}_{k_1,k_2} T_{k_1,k_2} \Delta^{N(j_3),N(j_4)}_{k_1,k_2} = \sum_{\ell_1=N(j_1)-1}^{\infty} \sum_{\ell_2=N(j_2)-1}^{\infty} \sum_{\ell_3=N(j_3)-1}^{\infty} \sum_{\ell_4=N(j_4)-1}^{\infty} \hat{T}_{\ell_1,\ell_2,\ell_3,\ell_4}, \]
and when there is at least one \( j_i = 0, \) we can also define \( \hat{T}_{j_1,j_2,j_3,j_4} \) and \( \hat{T}^N_{j_1,j_2,j_3,j_4} \) in an obvious way. For example,
\[ \hat{T}_{j_1,0,0,j_4} := \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} (\Delta S)^{j_1}_{k_1,k_2} T_{k_1,k_2} (S \Delta)^{j_4}_{k_1,k_2}. \]
\[ \hat{T}^N_{j_1,0,0,j_4} := \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} (\Delta S)^{N(j_1)}_{k_1,k_2} T_{k_1,k_2} (S \Delta)^{N(j_4)}_{k_1,k_2}. \]

Next, we show that for each \( j_1, j_2, j_3, j_4, \) the operator \( \hat{T}^N_{j_1,j_2,j_3,j_4} \) is bounded on \( L^2 \) with the operator norm dominated by \( C_{Q_1,Q_2,q} \sum_{\ell=1}^{4} 2^{-\alpha N(j-1)} ||K^0||_q. \)

**Proposition 6.2.** Let \( q > 1. \) Then there exist constants \( C_{Q_1,Q_2,q} > 0 \) and \( \alpha > 0 \) such that for any \( j_1, j_2 \geq 0, \)
\[ ||\hat{T}_{j_1,j_2,j_3,j_4}(f)||_2 \leq C_{Q_1,Q_2,q} \sum_{\ell=1}^{4} 2^{-\alpha j} ||K^0||_q ||f||_2 \]
and for any \( j_1, j_2, j_3, j_4 \geq 1, \)
\[ ||\hat{T}^N_{j_1,j_2,j_3,j_4}(f)||_2 \leq C_{Q_1,Q_2,q} \sum_{\ell=1}^{4} 2^{-\alpha N(j-1)} ||K^0||_q ||f||_2. \]

**Proof.** It suffices to show the first estimate when \( j_1, j_2, j_3, j_4 \geq 1. \) For other cases we can repeat the argument in the setting of one-parameter. For simplicity, we set
\[ T^{k_1,k_2}_{j_1,j_2,j_3,j_4}(f) := \Delta_{k_1} T_{k_1,k_2} \Delta_{k_2} T_{k_1,k_2} (f), \]
then \( T_{j_1,j_2,j_3,j_4}(f) = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} T^{k_1,k_2}_{j_1,j_2,j_3,j_4}(f). \) By Cotlar-Knapp-Stein Lemma, it suffices to show that:
\[ ||(T^{k_1,k_2}_{j_1,j_2,j_3,j_4})^* T^{k_1,k_2}_{j_1,j_2,j_3,j_4}||_{2 \to 2} + ||T^{k_1,k_2}_{j_1,j_2,j_3,j_4}(T^{k_1,k_2}_{j_1,j_2,j_3,j_4})^*||_{2 \to 2} \leq C \sum_{\ell=1}^{4} 2^{-\alpha j} 2^{-\ell k_1} 2^{-\ell k_2} ||K^0||_q^2, \]
for some constants \( C, c > 0 \) and \( \alpha > 0 \). We only estimate the first term, since the second term is similar. Note that
\[
(T^{k_1,k_2}_{j_1,j_2,j_3,j_4})^* T^{k_1',k_2'}_{j_1,j_2,j_3,j_4}(f) = f \ast \Psi_{k_1',j_1,k_2'-j_2} \ast A_{k_1,k_2}K^0 \ast \Psi_{j_1,j_3,k_2-j_4} \ast A_{k_1,k_2}K^0 \ast \Psi_{k_1,j_3,k_2-j_4}.
\]

On the one hand, by Lemma 1 in [16] and its duality version,
\[
\|(T^{k_1,k_2}_{j_1,j_2,j_3,j_4})^* T^{k_1',k_2'}_{j_1,j_2,j_3,j_4}(f)\|_2 \\
= \|f \ast \Psi_{k_1',j_1,k_2'-j_2} \ast A_{k_1,k_2}K^0 \ast (\Psi_{k_1-j_1,k_2-j_2} \ast \Psi_{j_1,j_3,k_2-j_4} \ast A_{k_1,k_2}K^0) \ast \Psi_{k_1-j_3,k_2-j_4}\|_2 \\
\leq C 2^{-\beta|j|} 2^{-\beta|j'|} \|K^0\|_q \|f \ast \Psi_{k_1',j_1,k_2'-j_2} \ast A_{k_1,k_2}K^0 \ast \Psi_{j_1,j_3,k_2-j_4} \ast A_{k_1,k_2}K^0\|_2 \|f\|_2 \\
\leq C 2^{-\beta|j|} 2^{-\beta|j'|} \|K^0\|_q \|f \ast \Psi_{k_1',j_1,k_2'-j_2} \ast A_{k_1,k_2}K^0\|_2 \|f\|_2 \\
(6.4)
\]
for some constants \( C, c > 0 \) and \( \beta > 0 \), where in the next to the last inequality we used the cancellation and the smoothness properties of \( \Psi^{(1)}_{k_1'-j_1}, \Psi^{(1)}_{k_1-j_3}, \Psi^{(2)}_{k_2'-j_2} \) and \( \Psi^{(2)}_{k_2-j_4} \).

On the other hand,
\[
\|(T^{k_1,k_2}_{j_1,j_2,j_3,j_4})^* T^{k_1',k_2'}_{j_1,j_2,j_3,j_4}(f)\|_2 \\
= \|f \ast \Psi_{k_1',j_1,k_2'-j_2} \ast (A_{k_1,k_2}K^0 \ast \Psi_{k_1-j_1,k_2-j_2}) \ast (\Psi_{j_1,j_3,k_2-j_4} \ast A_{k_1,k_2}K^0) \ast \Psi_{k_1-j_3,k_2-j_4}\|_2 \\
\leq C \|f \ast \Psi_{k_1',j_1,k_2'-j_2} \ast (A_{k_1,k_2}K^0 \ast \Psi_{k_1-j_1,k_2-j_2}) \ast (\Psi_{j_1,j_3,k_2-j_4} \ast A_{k_1,k_2}K^0)\|_2 \|f\|_2 \\
\leq C 2^{-\beta|j|} 2^{-\beta|j'|} \|K^0\|_q \|f \ast \Psi_{k_1',j_1,k_2'-j_2} \ast (A_{k_1,k_2}K^0) \ast \Psi_{j_1,j_3,k_2-j_4}\|_2 \|f\|_2 \\
(6.5)
\]
Taking geometric mean of (6.4) and (6.5), we obtain the estimate (6.3). This ends the proof of Proposition 6.2.

We now further prove that the kernel \( K^{N}_{j_1,j_2,j_3,j_4}(x,y) \) of the operator \( \tilde{T}^{N}_{j_1,j_2,j_3,j_4} \) satisfies the Dini-type Calderón–Zygmund kernel condition with the Dini condition [3, Section 5].

**Lemma 6.3.** For any \( q > 1 \), there exists a constant \( C_{Q_1,Q_2,q} > 0 \) such that the kernel \( K^{N}_{j_1,j_2,j_3,j_4}(x,y) \) satisfies the size estimate
\[
|K^{N}_{j_1,j_2,j_3,j_4}(x,y)| \leq C_{Q_1,Q_2,q} 2^{-\frac{Nj_1|Q_1|}{q}} \frac{1}{d_1(x_1,y_1)^{Q_1}} \frac{1}{d_2(x_2,y_2)^{Q_2}} \|K^0\|_q,
\]
the Hölder estimate
\[
|K^{N}_{j_1,j_2,j_3,j_4}(x,y) - K^{N}_{j_1,j_2,j_3,j_4}(x',y)| \leq C_{Q_1,Q_2,q} \omega^{N,1}_{j_1,j_2,j_3,j_4} \frac{1}{d_1(x_1,y_1)^{Q_1}} \frac{1}{d_1(x_1,y_1)^{Q_1}} \omega^{N,2}_{j_1,j_2,j_3,j_4} \frac{1}{d_2(x_2,y_2)^{Q_2}} \frac{1}{d_2(x_2,y_2)^{Q_2}},
\]
where \( \omega^{N,1}_{j_1,j_2,j_3,j_4} \) and \( \omega^{N,2}_{j_1,j_2,j_3,j_4} \) are the size estimates for \( K^{N}_{j_1,j_2,j_3,j_4}(x,y) \) and \( K^{N}_{j_1,j_2,j_3,j_4}(x',y) \), respectively.
whenever \( d_1(x_1, y_1) \geq 2A_0^{(1)}d_1(x_1, x_1') \) and \( d_2(x_2, y_2) \geq 2A_0^{(2)}d_2(x_2, x_2') \), and the mixed Hölder and size estimates

\[
|K_{j_1,j_2,j_3,j_4}^N(x, y) - K_{j_1,j_2,j_3,j_4}^N((x_1', x_2'), y)| \leq C_{Q_1, Q_2, \phi} \omega_{j_1,j_2,j_3,j_4}^{N,1} \left( \frac{d_1(x_1, x_1')}{d_1(x_1, y_1)} \right) \frac{1}{d_1(x_1, y_1)^{Q_1}} \frac{1}{d_2(x_2, y_2)^{Q_2}},
\]

whenever \( d_1(x_1, y_1) \geq 2A_0^{(1)}d_1(x_1, x_1') \) and

\[
|K_{j_1,j_2,j_3,j_4}^N(x, y) - K_{j_1,j_2,j_3,j_4}^N((x_1', x_2'), y)| \leq C_{Q_1, Q_2, \phi} \frac{1}{d_1(x_1, y_1)^{Q_1}} \omega_{j_1,j_2,j_3,j_4}^{N,2} \left( \frac{d_2(x_2, x_2')}{d_2(x_2, y_2)} \right) \frac{1}{d_2(x_2, y_2)^{Q_2}},
\]

whenever \( d_2(x_2, y_2) \geq 2A_0^{(2)}d_2(x_2, x_2') \).

Moreover, for each \( k = 1, 2 \), \( \omega_{j_1,j_2,j_3,j_4}^{N,k} \) satisfies the modified Dini \( 1 \) condition:

\[
(6.6) \quad \|\omega_{j_1,j_2,j_3,j_4}^{N,k} \|_{\text{Dini}_1} \leq C 2^{N(j_1)Q_1} 2^{N(j_2)Q_2} \|K^0\|_q (1 + N(j_{k+2}))^2.
\]

**Proof.** In order to obtain these estimates for \( K_{j_1,j_2,j_3,j_4}^N \), we first study the kernel of

\[
S_{k_1-N(j_1), k_2-N(j_2)} T_{k_1,k_2} S_{k_1-N(j_3), k_2-N(j_4)}.
\]

A direct calculation implies

\[
|A_{k_1,k_2}^0(x_1, x_2)| = 2^{-k_1} 2^{-k_2} \left| \int_0^\infty \int_0^\infty \varphi(2^{-k_1} t_1) \varphi(2^{-k_2} t_2) \Delta[t_1, t_2] K_0^0(x_1, x_2) dt_1 dt_2 \right|
\]

\[
\leq C \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} |K(\rho_1(x_1)^{-1} \circ_1 x_1, \rho_2(x_2)^{-1} \circ_2 x_2)| \chi_{2^{k_1-1} \leq \rho_1(x_1) \leq 2^{k_1+2}}(x_1) \chi_{2^{k_2-1} \leq \rho_2(x_2) \leq 2^{k_2+2}}(x_2).
\]

This, in combination with the observation that for \( i = 1, 2 \), \( \text{supp } \phi^{(i)} \subset \{ x_i \in \mathbb{R}^n : \rho_i(x_i) \leq \frac{1}{100} \} \), indicates

\[
|\Delta^{(1)}[2^{-k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{-k_2-N(j_2)-1}] \phi^{(2)} \ast A_{k_1,k_2}^0(x_1, x_2)|
\]

\[
\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 2^{-[k_1-N(j_1)-1]Q_1} |\phi^{(1)}(2^{-k_1-N(j_1)-1} \circ_1 x_1, u)| \rho_1(u)^{-Q_1} \chi_{2^{k_1-1} \leq \rho_1(u) \leq 2^{k_1+2}}(u)
\]

\[
\times 2^{-[k_2-N(j_2)-1]Q_2} |\phi^{(2)}(2^{-k_2-N(j_2)-1} \circ_2 x_2, v)| \rho_2(v)^{-Q_2} \chi_{2^{k_2-1} \leq \rho_2(v) \leq 2^{k_2+2}}(v)
\]

\[
\times |K(\rho_1(u)^{-1} \circ_1 u, \rho_2(v)^{-1} \circ_2 v)| \mathrm{d}u \mathrm{d}v
\]

\[
\leq C 2^{-[k_1-N(j_1)-1]Q_1} 2^{-[k_2-N(j_2)-1]Q_2} \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} \chi_{2^{k_1-1} \leq \rho_1(x_1) \leq 2^{k_1+2}}(x_1) \chi_{2^{k_2-1} \leq \rho_2(x_2) \leq 2^{k_2+2}}(x_2)
\]

\[
\times \int_{2^{k_1-1} \leq \rho_1(u) \leq 2^{k_1+2}} \int_{2^{k_2-1} \leq \rho_2(v) \leq 2^{k_2+2}} |\phi^{(1)}(2^{-[k_1-N(j_1)-1]} \circ_1 x_1, u)| |\phi^{(2)}(2^{-[k_2-N(j_2)-1]} \circ_2 x_2, v)|
\]

\[
\times |K(\rho_1(u)^{-1} \circ_1 u, \rho_2(v)^{-1} \circ_2 v)| \mathrm{d}u \mathrm{d}v.
\]

By Hölder’s inequality, for \( 1/q + 1/q' = 1 \), we have

\[
\int_{2^{k_1-1} \leq \rho_1(u) \leq 2^{k_1+2}} \int_{2^{k_2-1} \leq \rho_2(v) \leq 2^{k_2+2}} |\phi^{(1)}(2^{-[k_1-N(j_1)-1]} \circ_1 x_1, u)| |\phi^{(2)}(2^{-[k_2-N(j_2)-1]} \circ_2 x_2, v)|
\]

\[
\times |K(\rho_1(u)^{-1} \circ_1 u, \rho_2(v)^{-1} \circ_2 v)| \mathrm{d}u \mathrm{d}v
\]

\[
\leq \left( \int_{2^{k_1-1} \leq \rho_1(u) \leq 2^{k_1+2}} \int_{2^{k_2-1} \leq \rho_2(v) \leq 2^{k_2+2}} |K(\rho_1(u)^{-1} \circ_1 u, \rho_2(v)^{-1} \circ_2 v)|^q \mathrm{d}u \mathrm{d}v \right)^{1/q}
\]
This implies that
\[
\times \left( \int_{\mathbb{H}_1 \times \mathbb{H}_2} |\phi^{(1)}(2^{-[k_1-N(j_1)-1]} o_1 x_1 u^{-1})|^{q'} |\phi^{(2)}(2^{-[k_2-N(j_2)-1]} o_2 x_2 v^{-1})|^{q'} du dv \right)^{1/q'} \leq C \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \frac{1}{2^{q(1)+\frac{q_1}{2}}} ||K^0||_q.
\]

Combining the two estimates we obtain above, we see that
\[
||\big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big) \ast A_{k_1,k_2} K^0(x_1, x_2)|| \leq C \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \frac{1}{2^{q(1)+\frac{q_1}{2}}} \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} ||K^0||_q \chi_{x_1^2 \leq 2^{k_1-1} \xi^2 \leq 2^{k_1+1}}(x_1) \chi_{x_2^2 \leq 2^{k_2-1} \xi^2 \leq 2^{k_2+1}}(x_2).
\]

Hence,
\[
||\big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big) \ast A_{k_1,k_2} K^0 \ast \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big)(x_1, x_2)|| \leq C \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \frac{1}{2^{q(1)+\frac{q_1}{2}}} \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} ||K^0||_q \chi_{x_1^2 \leq 2^{k_1-1} \xi^2 \leq 2^{k_1+1}}(x_1) \chi_{x_2^2 \leq 2^{k_2-1} \xi^2 \leq 2^{k_2+1}}(x_2).
\]

This implies that
\[
\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} ||\big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big) \ast A_{k_1,k_2} K^0 \ast \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big)(x_1, x_2)|| \leq C \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \frac{1}{2^{q(1)+\frac{q_1}{2}}} \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} ||K^0||_q.
\]

Similarly, we obtain the gradient estimates as follows.
\[
\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} ||\nabla x_1 \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big) \ast A_{k_1,k_2} K^0 \ast \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big)(x_1, x_2)|| \leq C \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \frac{1}{2^{q(1)+\frac{q_1}{2}}} \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} ||K^0||_q,
\]

and
\[
\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} ||\nabla x_2 \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big) \ast A_{k_1,k_2} K^0 \ast \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big)(x_1, x_2)|| \leq C \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \frac{1}{2^{q(1)+\frac{q_1}{2}}} \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} ||K^0||_q.
\]

We also have mixed gradient estimate
\[
\sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} ||\nabla x_1 \nabla x_2 \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big) \ast A_{k_1,k_2} K^0 \ast \big(\Delta^{(1)}[2^{k_1-N(j_1)-1}] \phi^{(1)} \otimes \Delta^{(2)}[2^{k_2-N(j_2)-1}] \phi^{(2)}\big)(x_1, x_2)|| \leq C \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \frac{1}{2^{q(1)+\frac{q_1}{2}}} \rho_1(x_1)^{-Q_1} \rho_2(x_2)^{-Q_2} ||K^0||_q.
\]
From the triangle inequality and $N(j - 1) < N(j)$ we can see that the kernel $K_{j_1,j_2,j_3,j_4}^N$ satisfies the same estimates (6.8), (6.9) and (6.10). That is,

$$\left| K_{j_1,j_2,j_3,j_4}^N(x_1, x_2, y_1, y_2) \right| = \left| K_{j_1,j_2,j_3,j_4}^N(y_1^{-1} x_1, y_2^{-1} x_2) \right| \leq C 2^{\frac{N(j)Q_1}{q}} 2^{\frac{N(j)Q_2}{q}} d_1(x_1, y_1)^{-Q_1} d_2(x_2, y_2)^{-Q_2} \| K^0 \|_q.$$ 

$$\left| \nabla_{(x_1,y_1)} K_{j_1,j_2,j_3,j_4}^N(x_1, x_2, y_1, y_2) \right| \leq C 2^{N(j_1)} \left( 1 + \frac{Q_1}{q} \right) 2^{\frac{N(j)Q_1}{q}} d_1(x_1, y_1)^{-Q_1} d_2(x_2, y_2)^{-Q_2} \| K^0 \|_q.$$ 

$$\left| \nabla_{(x_2,y_2)} K_{j_1,j_2,j_3,j_4}^N(x_1, x_2, y_1, y_2) \right| \leq C 2^{N(j_2)} \left( 1 + \frac{Q_2}{q} \right) 2^{\frac{N(j)Q_2}{q}} d_1(x_1, y_1)^{-Q_1} d_2(x_2, y_2)^{-Q_2} \| K^0 \|_q.$$ 

$$(\text{For } j_1 = 0 \text{ or } j_2 = 0, \text{ the subtraction is not even needed.) Besides, for } d_1(x_1, y_1) \geq 2A_0^{(1)} d(x_1, x')_1 \text{, by mean value theorem on homogeneous groups,}$$

$$\left| K_{j_1,j_2,j_3,j_4}^N(x, y) - K_{j_1,j_2,j_3,j_4}^N((x', x_2), y) \right| = \left| K_{j_1,j_2,j_3,j_4}^N(y_1^{-1} x_1, y_2^{-1} x_2) - K_{j_1,j_2,j_3,j_4}^N(y_1^{-1} x_1', y_2^{-1} x_2') \right| \leq C 2^{N(j)} \left( 1 + \frac{Q_1}{q} \right) 2^{\frac{N(j)Q_2}{q}} d_1(x_1, y_1)^{-Q_1} d_1(x, x') d_2(x_2, y_2)^{-Q_2} \| K^0 \|_q.$$ 

By triangle inequality, we also have

$$\left| K_{j_1,j_2,j_3,j_4}^N(x, y) - K_{j_1,j_2,j_3,j_4}^N((x', x_2), y) \right| \leq C 2^{N(j_1)Q_1} 2^{N(j_2)Q_2} d_1(x_1, y_1)^{-Q_1} d_2(x_2, y_2)^{-Q_2} \| K^0 \|_q.$$ 

Combining the two estimates we obtain above and by symmetry, we conclude that

$$\left| K_{j_1,j_2,j_3,j_4}^N(x, y) - K_{j_1,j_2,j_3,j_4}^N((x', x_2), y) \right| \leq C \left( \frac{d_1(x_1, x')}{d_1(x_1, y_1)} \right) \left( \frac{1}{d_1(x_1, y_1)} \right) \left( \frac{1}{d_2(x_2, y_2)} \right),$$

where

$$(6.11) \quad \omega_{j_1,j_2,j_3,j_4}^{N_1}(t) \leq C 2^{\frac{N(j_1)Q_1}{q}} 2^{\frac{N(j_2)Q_2}{q}} \| K^0 \|_q \min(1, 2^{N(j_3)} t).$$

Similarly, whenever $d(x_2, y_2) \geq 2A_0^{(2)} d(x_2, x'_2)$, we have

$$\left| K_{j_1,j_2,j_3,j_4}^N(x, y) - K_{j_1,j_2,j_3,j_4}^N((x_1, x'_2), y) \right| \leq C \left( \frac{d_1(x_1, x')}{d_1(x_1, y_1)} \right) \left( \frac{1}{d_1(x_1, y_1)} \right) \left( \frac{1}{d_2(x_2, y_2)} \right),$$

where

$$(6.12) \quad \omega_{j_1,j_2,j_3,j_4}^{N_2}(t) \leq C 2^{\frac{N(j_1)Q_1}{q}} 2^{\frac{N(j_2)Q_2}{q}} \| K^0 \|_q \min(1, 2^{N(j_3)} t).$$

Furthermore,

$$\left| K_{j_1,j_2,j_3,j_4}^N(x, y) - K_{j_1,j_2,j_3,j_4}^N((x_1, x'_2), y) - K_{j_1,j_2,j_3,j_4}^N((x'_1, x_2), y) + K_{j_1,j_2,j_3,j_4}^N(x', y) \right| \leq C \omega_{j_1,j_2,j_3,j_4}^{N_1} \left( \frac{d_1(x_1, x'_1)}{d_1(x_1, y_1)} \right) \left( \frac{1}{d_1(x_1, y_1)} \right) \omega_{j_1,j_2,j_3,j_4}^{N_2} \left( \frac{d_2(x_2, x'_2)}{d_2(x_2, y_2)} \right) \left( \frac{1}{d_2(x_2, y_2)} \right),$$

whenever $d_1(x_1, y_1) \geq 2A_0^{(1)} d_1(x_1, x'_1)$ and $d_2(x_2, y_2) \geq 2A_0^{(2)} d_2(x_2, x'_2)$.
Now we verified that for each \( k = 1, 2 \), the estimate \((6.6)\) holds. Indeed, it follows from the pointwise estimates \((6.11)\) and \((6.12)\) that
\[
\int_0^1 \omega_{j_1, j_2, j_3, j_4}^{N, k} (1 + \log \frac{1}{t}) \frac{dt}{t} \leq C 2^{\frac{N_1}{2}} 2^{\frac{N_2}{2}} \|K_0\|_q \left( \int_0^1 \min \{1, 2^{N_{j_1+2}}t \} \frac{dt}{t} + \int_0^1 \min \{1, 2^{N_{j_2+2}}t \} \log \frac{1}{t} \frac{dt}{t} \right).
\]
(6.13)
Note that
\[
\int_0^1 \min \{1, 2^{N_{j_2+2}}t \} \frac{dt}{t} \leq C \left( \int_0^{2^{-N_{j_2+2}}} 2^{N_{j_2+2}} t dt + \int_0^1 \log t dt \right) \leq C(1 + N(j_{k+2})).
\]
(6.14)
Besides, integration by parts yields
\[
\int_0^1 \min \{1, 2^{N_{j_2+2}}t \} \log \frac{1}{t} \frac{dt}{t} = \left( - \int_0^{2^{-N_{j_2+2}}} 2^{N_{j_2+2}} \log t dt - \int_0^1 \log t dt \right) \leq C(N(j_{k+2}) + 1) + C(N(j_{k+2}))^2 \leq C(1 + N(j_{k+2}))^2.
\]
(6.15)
Combining the estimates \((6.13), (6.14)\) and \((6.15)\), we verify the Dini condition \((6.6)\) and then the proof of Lemma 6.3 is complete.

Combining the Proposition 6.2 and Lemma 6.3, we see that by applying Theorem 5.12 of [3] to our \( \tilde{T}^N_{j_1, j_2, j_3, j_4} \) (with Dini condition \((6.6)\)) for \( \omega_{j_1, j_2, j_3, j_4}^{N, 1} \) and \( \omega_{j_1, j_2, j_3, j_4}^{N, 2} \), we get the representation theorem
\[
\langle \tilde{T}^N_{j_1, j_2, j_3, j_4} f, g \rangle = C_{\mathbb{E}_\sigma} \sum_{k=1}^{k_1} \sum_{k_2} \omega_{j_1, j_2, j_3, j_4}^{N, 1} (2^{-k_1}) \omega_{j_1, j_2, j_3, j_4}^{N, 2} (2^{-k_2}) \langle V_{k, \sigma} f, g \rangle,
\]
(6.16)
where \( V_{k, \sigma} \) is the standard bi-parameter dyadic Haar shifts since \( \tilde{T}^N_{j_1, j_2, j_3, j_4} \) is paraproduct-free [3, Lemma 5.11]. Here the only concern is that we are working on \( \mathbb{H}_1 \times \mathbb{H}_2 \) while the setting in [3] is \( \mathbb{R}^n \times \mathbb{R}^n \). In fact, one can obtain this result parallel to the Euclidean setting by using the Haar basis on space of homogeneous type constructed in [31] and the probability space and expectation in [44]. To be more specific, \( V_{k, \sigma} \) is given by
\[
\langle V_{k, \sigma} f, g \rangle = \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} \langle \mathcal{S}_{k_1, i_1, \sigma}^{k_2, i_2} f, g \rangle
\]
with
\[
\langle \mathcal{S}_{k_1, i_1, \sigma}^{k_2, i_2} f, g \rangle = \sum_{K_1, K_2} \sum_{f_1^{(i_1)} = K_1} \sum_{f_2^{(i_2)} = K_2} a_{i_1, i_2, k_1, k_2, f_1^{(i_1)} = K_1, f_2^{(i_2)} = K_2} \langle f, h_{i_1} h_2 \rangle \langle g, h_{j_1} h_2 \rangle,
\]
where \( h_{i_1}, h_{j_1} \) are the Haar basis in \( \mathbb{H}_1 \) (for the explicit definition, we refer to [31]). From [4, Theorem 2], we get that
\[
|\langle \mathcal{S}_{k_1, i_1, \sigma}^{k_2, i_2} f, g \rangle| \leq 2^{-k_1 - k_2 - i_1 - i_2} \sum_{R \in \Lambda_{k_1, k_2, j_1, j_2}} |R(\mathcal{S}_{\sigma}^{k_1, k_2, i_1, i_2} f) R(\mathcal{S}_{\sigma}^{k_1, k_2, i_1, i_2} g) R|,
\]
where $\Lambda_{k_1,k_2,i_1,i_2}$ is a sparse collection of dyadic rectangles depending on $f, g$, and $S^{k_1,k_2,i_1,i_2}_f$ is the shifted square function \cite[equation (12)]{4}.

Hence, we have

\begin{equation}
\begin{aligned}
|\langle \tilde{T}^N_{j_1,j_2,j_3,j_4}, f, g \rangle| &\lesssim \mathbb{E}_\sigma \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} \omega^{N,1}_{j_1,j_2,j_3,j_4}(2^{-k_1}) \omega^{N,2}_{j_1,j_2,j_3,j_4}(2^{-k_2}) \\
&\quad \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} 2^{-i_1-i_2} \sum_{R \in \Lambda_{k_1,k_2,i_1,i_2}} |R|(S^{k_1,k_2,i_1,i_2}_f g)(S^{k_1,k_2,i_1,i_2}_g)R.
\end{aligned}
\end{equation}

By noting that $\|S^{k_1,k_2,i_1,i_2}_f f\|_{L^2(w)} \leq [w]^{4}_{2}[w]_{\infty} \|f\|_{L^2(w)}$ \cite[Section 5]{4} and that $\|M_s f\|_{L^2(w)} \leq [w]^{2}_{2} \|f\|_{L^2(w)}$, we get that

\begin{equation}
\begin{aligned}
\sum_{R \in \Lambda_{k_1,k_2,i_1,i_2}} |R|(S^{k_1,k_2,i_1,i_2}_f g)(S^{k_1,k_2,i_1,i_2}_g)R &\leq \|M_s S^{k_1,k_2,i_1,i_2}_f f\|_{L^2(w)} \|M_s S^{k_1,k_2,i_1,i_2}_g g\|_{L^2(w^{-1})} \\
&\leq [w]^{4}_{2} \|S^{k_1,k_2,i_1,i_2}_f f\|_{L^2(w)} \|S^{k_1,k_2,i_1,i_2}_g g\|_{L^2(w^{-1})} \\
&\leq [w]^{12}_{2} [w]_{\infty} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}.
\end{aligned}
\end{equation}

As a consequence, we get that

\begin{equation}
\begin{aligned}
\|\tilde{T}^N_{j_1,j_2,j_3,j_4} f\|_{L^2(w)} &\leq C_{Q_1,Q_2} [w]^{12}_{A_2}[w]_{A_2}^{2} \|K^0\|_{q}^{2} \left( \prod_{k=1}^{2} 2^{2N(j_{k+2})/q} \right) (1 + N(j_{k+2})^2) \|f\|_{L^2(w)}.
\end{aligned}
\end{equation}

Finally, similar to the proof in the one-parameter setting, by applying the interpolation theorem with change of measure to the estimates (6.2) and (6.18), we see that there exists a constant $c_{Q_1,Q_2} > 0$ such that if $K^0 \in L^q(D_0)$ for some $q > c_{Q_1,Q_2}(w)A_2$ then

\begin{equation}
\begin{aligned}
\|T\|_{L^2(w) \rightarrow L^2(w)} &\leq C_{Q_1,Q_2,q} \max(\|K^0\|_{q}, \|K^0\|_{q}^{2}) [w]_{A_2}^{12} [w]_{A_2}^{2},
\end{aligned}
\end{equation}

for some constant $C_{Q_1,Q_2,q}$ independent of $w$.

The proof of Theorem 1.3 is complete.

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\begin{thebibliography}{10}
\bibitem{1} H. Al-Qassem and Y. Pan, $L^p$ boundedness for singular integrals with rough kernels on product domains, \textit{Hokkaido Math. J.} \textbf{31} (2002), 555–613.
\bibitem{2} A. Al-Salman, H. Al-Qassem and Y. Pan, Singular integrals on product domains, \textit{Indiana Univ. Math. J.} \textbf{55} (2006), 369–387.
\bibitem{3} E. Aitta, H. Martikainen and E. Vuorinen, Modern singular integral theory with mild kernel regularity, Available at arXiv: 2006.05807 (2020).
\bibitem{4} A. Barron and J. Pipher, Sparse domination for bi-parameter operators using square functions, Available at arXiv: 1709.05009 (2017).
\end{thebibliography}
[5] A.P. Calderón and A. Zygmund, On the existence of certain singular integrals, *Acta Math.* **88** (1952), 85–139.

[6] A.P. Calderón and A. Zygmund, On singular integrals, *Amer. J. Math.* **78** (1956), 289–309.

[7] J. Canto, K. Li, L. Roncal, O. Tapiola, $C^p$ estimates for rough homogeneous singular integrals and sparse forms, Available at arXiv: 1909.08344 (2019).

[8] Y. Chen and Y. Ding, $L^p$ bounds for the commutators of singular integrals and maximal singular integrals with rough kernels, *Trans. Amer. Math. Soc.* **367** (2015), 1585–1608.

[9] M. Christ, Weak type $(1,1)$ bounds for rough operators, *Ann. of Math.* **128** (1988), 19–42.

[10] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, *Colloq. Math.* **60/61** (1990), 601–628.

[11] M. Christ and J.L. Rubio de Francia, Weak type $(1,1)$ bounds for rough operators. II, *Invent. Math.* **93** (1988), 225–237.

[12] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes. Étude de certaines intégrales singulières*, Lecture Notes in Math. **242**, Springer-Verlag, Berlin, (1971). 7, 10

[13] J.M. Conde-Alonso, A. Culiuc, F. Di Plinio and Y. Ou, A sparse domination principle for rough singular integrals, *Anal. PDE.* **10** (2017), 1255–1284.

[14] Y. Ding and X.D. Lai, Weak type $(1,1)$ bound criterion for singular integrals with rough kernel and its applications, *Trans. Amer. Math. Soc.* **371** (2019), 1649–1675.

[15] Y. Ding and S. Lu, Weighted norm inequalities for fractional integral operators with rough kernel, *Canad. J. Math.* **50** (1998), 29–39.

[16] Y. Ding and S. Sato, Singular integrals on product homogeneous groups, *Integr. Equ. Oper. Theory.* **76** (2013), 55–79.

[17] J. Duoandikoetxea, Multiple singular integrals and maximal functions along hypersurfaces, *Ann. Inst. Fourier (Grenoble).* **36** (1986), 185–206.

[18] J. Duoandikoetxea, weighted norm inequalities for homogeneous singular integrals, *Trans. Amer. Math. Soc.* **336** (1993), 869–880.

[19] J. Duoandikoetxea and J.L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, *Invent. Math.* **84** (1986), 541–561.

[20] X.T. Duong, R. Gong, M.S. Ku, L. Grafakos and A. Stefanov, $L^p$ bounds for singular integral operators on spaces of homogeneous type and applications. Available at arXiv: 1809.07942 (2018).

[21] D. Fan and Y. Pan, Singular integrals operators with rough kernels supported by subvarieties, *Amer. J. Math.* **119** (1997), 799–839.

[22] D. Fan, Y. Pan and D. Yang, A weighted norm inequality for rough singular integrals, *Tohoku Math. J.* **51** (1999), 141–161.

[23] R. Fefferman, Singular integral on product domains, *Bull. Amer. Math. Soc.* **4** (1981), 195–201.

[24] G.B. Folland and E.M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, Princeton, N. J. (1982).

[25] L. Grafakos and A. Stefanov, $L^p$ bounds for singular integral and maximal singular integrals with rough kernels, *Indiana Univ. Math. J.* **47** (1998), 455–469.

[26] S. Hofmann, Weak $(1,1)$ boundedness of singular integrals with nonsmooth kernel, *Proc. Amer. Math. Soc.* **103** (1988), 260–264.

[27] T. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators, *Ann. of Math.* **175** (2012), 1473–1506.

[28] T. Hytönen and A. Kairema, Systems of dyadic cubes in a doubling metric space, *Colloq. Math.* **126** (2012), 1–33.

[29] T. Hytönen, C. Pérez and E. Rela, Sharp reverse Hölder property for $A_{\infty}$ weights on spaces of homogeneous type, *J. Funct. Anal.* **263** (2012), 3883–3899.

[30] T. Hytönen, L. Roncal and O. Tapiola, Quantitative weighted estimates for rough homogeneous singular integrals, *Israel J. Math.* **218** (2017), 133–164.

[31] A. Kairema, J. Li, C. Pereyra and L. A. Ward, Haar bases on quasi-metric measure spaces, and dyadic structure theorems for function spaces on product spaces of homogeneous type, *J. Funct. Anal.* **271** (2016), 1793–1843.

[32] G. Karagulyan and M.T. Lacey, On logarithmic bounds of maximal sparse operators, *Math. Z.* **294** (2020), no. 3-4, 1271–1281.
[33] R. Kesler, M. Lacey and D. Mena, Sparse bounds for the discrete spherical maximal functions, *Pure Appl. Anal.* 2 (2020), no. 1, 75–92.

[34] M.T. Lacey, An elementary proof of the $A_2$ bound, *Israel J. Math.* 217 (2017).

[35] A.K. Lerner, On an estimate of Calderón-Zygmund operators by dyadic positive operators, *J. Anal. Math.* 121 (2013), 141–161.

[36] A.K. Lerner, A simple proof of the $A_2$ conjecture, *Int. Math. Res. Not. IMRN.* (2013), 3159–3170.

[37] A.K. Lerner, On pointwise estimates involving sparse operators, *New York J. Math.* 22 (2016), 341–349.

[38] A.K. Lerner, A note on weighted bounds for rough singular integrals, *C. R. Acad. Sci. Paris, Ser. I* 356 (2018), 77–80.

[39] A.K. Lerner, A weak type estimate for rough singular integrals, *Rev. Mat. Iberoam.* 35 (2019), 1583–1602.

[40] A.K. Lerner and F. Nazarov, Intuitive dyadic calculus: the basics, *Exp. Math.* 37 (2019), 225–265.

[41] K. Li, C. Pérez, I.P. Rivera-Ríos and L. Roncal, Weighted Norm Inequalities for Rough Singular Integral Operators, *J. Geom. Anal.* 47 (2019), 2526–2564.

[42] F. Liu, S. Mao and H. Wu, On rough singular integrals related to homogeneous mappings, *Collect. Math.* 67 (2016), 113–132.

[43] K. Moen, Sharp weighted bounds without testing or extrapolation, *Archiv der Mathematik.* 99 (2012), 457–466.

[44] F. Nazarov, A. Reznikov, and A. Volberg, The proof of $A_2$ conjecture in a geometrically doubling metric space, *Indiana Univ. Math. J.* 62 (2013), 1503–1533.

[45] C. Pérez, I.P. Rivera-Ríos, L. Roncal, $A_1$ theory of weights for rough homogeneous singular integrals and commutators, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 19 (2019), 169–190.

[46] S. Sato, Estimates for singular integrals on homogeneous groups, *J. Math. Anal. Appl.* 400 (2013), 311–330.

[47] E. Sawyer and R. L. Wheeden, Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces, *Amer. J. Math.* 114 (1992), 813–874.

[48] A. Seeger, Singular integral operators with rough convolution kernels, *J. Amer. Math. Soc.* 9 (1996), 95–105.

[49] E.M. Stein and G. Weiss, interpolation of operators with change of measures, *Trans. Amer. Math. Soc.* 87 (1958), 159–172.

[50] E. M. Stein, Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals. *Princeton, NJ: Princeton University Press.* (1993).

[51] T. Tao, The Weak-type (1,1) of $L^{logL}$ Homogeneous Convolution Operator, *Indiana Univ. Math. J.* 48 (1999), 1547–1584.

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