Hawking radiation from not-extremal and extremal Reissner-Nordstrom black holes

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To dear Shmuel Agmon with great admiration

Abstract

We consider the non-extremal Reissner-Nordstrom black hole and construct a wave packet that exhibits the Hawking radiation. We find the average of the number of the created particles with respect to the $|0\rangle$ vacuum state and with respect to Unruh type vacuum state. The average of the number operator in the $|0\rangle$ vacuum state consists of two terms: one is related to the Hawking radiation and the second is not related. We use the same construction for the extremal RN black hole and get that the average of the number operator with respect to the $|0\rangle$ vacuum state is also a sum of two terms, where the one related to the Hawking radiation is equal to zero.

This result is consistent with other works on the Hawking radiation for the extremal RN black hole.

Keywords: Hawking radiation; black holes; Reissner-Nordstrom.

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1 Introduction

The phenomenon that the black hole emits quantum particles was discovered by S. Hawking in the seminal paper [7] and it is called the Hawking radiation. Any wave packet with the initial conditions having the support outside the black hole produces some particles emission. The Hawking radiation refers to the creation of particles intimately related to the black hole.

In [8] to get the Hawking radiation one takes the limit when the time tends to $-\infty$. A rigorous derivation of Hawking radiation by this way was obtained by Fredenhagen and Haag in [6] for the case of Schwartzschild black hole.

In [3] we developed a new approach to study the Hawking radiation in the case of a rotating black hole with a variable velocity. Note that the previous methods of [7], [8] and others do not work in this case. We shall use the approach of paper [3] in the present paper too.

We consider the case of two space dimensions and we construct a special wave packet of the form

$$(\rho - r_+)^\varepsilon e^{-a(\rho - r_+)} e^{iS(x_0, \rho, \varphi)} \quad \text{at} \quad x_0 = 0,$$

where $(\rho, \varphi)$ are polar coordinates, $\rho > r_+$, $r_+$ is the event horizon, $S(x_0, \rho, \varphi)$ is an approximate eikonal function that tends to the infinity when $\rho \to r_+$. Here constant $a$ plays an important role since $\text{supp} (\rho - r_+)^\varepsilon e^{-a(\rho - r_+)}$ tends to $\rho = r_+$ when $a \to \infty$.

Normalizing the wave packet, considering the average of the number of particles operator and taking the limit as $a \to \infty$, we get the the Hawking radiation.

Thus, taking the limit as $a \to \infty$ is replacing the necessity of taking the limit $T \to -\infty$ as in Fredenhagen-Haag approach.

There are two choices of selecting the vacuum state: 1) the vacuum state $|0\rangle$ that was used in [7] and [8] and others, and 2) the Unruh type vacuum state that was used in [3], which is similar to the original Unruh vacuum state [10]. Note that the limit when $a \to \infty$ of the average of the number operator with respect to $|0\rangle$ state is the sum of two terms $I_+$ and $I_-$, where $I_+$ corresponds to the Hawking radiation and $I_-$ is not (cf. §3).

Note that in the case of the average of the number of particles with respect to the Unruh type vacuum all terms correspond to the Hawking radiation (cf. §4).
In §5 we consider the case of extremal RN black hole. Our approach allows an uniform treatment of extremal and non-extremal RN black holes. It is shown that the limit when $a \to \infty$ of the average of the number operator with respect to $|0\rangle$ vacuum also contains two terms $I_+ + I_-$ and the one corresponding to the Hawking radiation is zero. This is consistent with the conclusion of other authors ([1], [2], [9]) that presents different arguments to conclude the Hawking radiation for the extremal RN black hole is zero.

The plan of the paper is the following:

In §2 we present the elements of the second quantization as in the paper [3] following mostly the lecture notes of Jacobson [8].

In §3 we study not-extremal RN black holes when the vacuum state is the $|0\rangle$ vacuum.

In §4 we consider not-extremal RN black holes when the vacuum state is the Unruh type vacuum state.

In §5 we consider the extremal RN black holes when the vacuum states are the $|0\rangle$ vacuum and the Unruh type vacuum.

2 Elements of the quantum field theory

The Reissner-Nordstom metric in the case of two space dimensions has the following form in polar coordinates $(\rho, \varphi)$ (cf. [4]):

\begin{equation}
    ds^2 = \left(1 - \frac{2m}{\rho} + \frac{e^2}{\rho^2}\right)dx_0^2 - \rho^2d\varphi^2 - 2\left(\frac{2m}{\rho} - \frac{e^2}{\rho^2}\right)dx_0d\rho - \left(\frac{2m}{\rho} - \frac{e^2}{\rho^2}\right)d\rho^2,
\end{equation}

where $m$ is the mass and $e$ is the electrical charge of the black hole. The corresponding wave equation $\Box_g u = 0$ has the form

\begin{equation}
    \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial \rho^2} - \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} - \left(- \frac{\partial}{\partial x_0} + \frac{\partial}{\partial \rho}\right)(f - 1)\left(- \frac{\partial}{\partial x_0} + \frac{\partial}{\partial \rho}\right)u = 0,
\end{equation}

where

\begin{equation}
    f = 1 - \frac{2m}{\rho} + \frac{e^2}{\rho^2}.
\end{equation}

We have

\begin{equation}
    f = \frac{(\rho - r_+)(\rho - r_-)}{\rho^2},
\end{equation}

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(2.4) \[ r_+ = m + \sqrt{m^2 - e^2}, \quad r_- = m - \sqrt{m^2 - e^2}, \]

When \( e^2 < m^2 \), \( r_+ \) and \( r_- \) are distinct real numbers and \( \rho = r_+ \) and \( \rho = r_- \) are event horizons. The RN black hole is called not-extremal in this case. If \( e^2 = m^2 \) then

(2.5) \[ r_+ = r_- = m \]

is the only event horizon and RN hole is called extremal.

We shall consider the Hawking radiation in the exterior of the black hole \( \rho = r_+ \) separately for not-extremal and extremal black holes.

First we shall briefly describe the quantum field theory for curved space-time (cf. [8]) when the background metric is not-extremal or extremal RN metric. Note that the symbol of (2.2) is

(2.6) \[ (2 - f)\eta_0^2 + 2(f - 1)\eta_0\eta_\rho - f\eta_\rho^2 - \frac{1}{\rho^2}m'^2. \]

We shall introduce, as in [3], an “orthonormal basis” of solutions of the wave equation (2.2) and we will use the same notations as in [3].

Let \( f_k^\pm(x_0, \rho, \varphi) \) be the solutions of (2.2) with the initial conditions

(2.7) \[ f_k^\pm(x_0, \rho, \varphi)|_{x_0=0} = \gamma_k e^{i k \cdot x} \]

(2.8) \[ \frac{\partial f_k^\pm(x_0, \rho, \varphi)}{\partial x_0}|_{x_0=0} = i \lambda_0^\pm(k) \gamma_k e^{i k \cdot x} \]

where \( k = (\eta_\rho, m') \), \( x = (\rho, \varphi) \), \( k \cdot x = \rho \eta_\rho + m' \varphi \), and (cf. [3])

(2.9) \[ \gamma_k = \frac{1}{\sqrt{\rho}(\eta_\rho^2 + a^2)^{\frac{1}{4}}} \frac{1}{\sqrt{2} \pi}, \]

(2.10) \[ \lambda_0^\pm(k) = \frac{-(f - 1)\eta_\rho \pm \sqrt{\eta_\rho^2 + a^2}}{2 - f}, \]

where \( a \) is arbitrary.

Note that

(2.11) \[ f_k^+(x_0, \rho, \varphi) = \overline{f_k^-(x_0, \rho, \varphi)}, \]

since \(-\lambda_0^+(k) = \lambda_0^-(k)\).
Introduce the Klein-Gordon (KG) inner product of any two solutions \(u, v\) of (2.2):

\[
(2.12) \quad < u, v > = i \int_{x_0=t} \sum_{j=0}^{2} g_{0j} \left( \bar{u} \frac{\partial v}{\partial x_j} - \frac{\partial \bar{u}}{\partial x_j} v \right) dx_1 dx_2.
\]

Note that (2.12) is independent of \(t\) (cf. [8]) and in polar coordinates we have (cf. (2.2) and (2.6))

\[
(2.13) \quad < u, v > = i \int_{x_0=t} (2-f) \left( \bar{u} \frac{\partial v}{\partial x_0} - \frac{\partial \bar{u}}{\partial \rho} v \right) + (f-1) \left( \bar{u} \frac{\partial v}{\partial \rho} - \frac{\partial \bar{u}}{\partial \rho} v \right) \rho d\rho d\varphi.
\]

It follows from (2.10) that

\[
(2.14) \quad (2-f) \frac{\partial f_k^+}{\partial x_0} + (f-1) \frac{\partial f_k^+}{\partial \rho} = -i \sqrt{\eta_0^2 + a^2} f_k^+ + (f-1) \frac{1}{2\rho} f_k^+
\]

Thus, similarly to [3], we have

\[
(2.15) \quad < f_k^+, f_{k'}^+ > = \delta(k-k'), \quad < f_{-k}^+, f_{-k'}^- > = -\delta(k-k'), \quad < f_k^+, f_{-k}^- > = 0
\]

for all \(k = (\eta_0, m)\), \(k' = (\eta_0', m')\),

where

\[
(2.16) \quad \delta(k-k') = \delta(\eta_0 - \eta_0') \delta_{mm'}.
\]

Note that terms containing \(\frac{1}{2\rho}\) in (2.15) are canceled since \(\frac{1}{2\rho}\) is real. It follows from (2.13), (2.16) that

\[
\{ f_k^+(x_0, \rho, \varphi), f_{-k}(x_0, \rho, \varphi) \}
\]

form an “orthonormal” basis of solutions of (2.2).

Having the basis \(\{ f_k^+, f_{-k} \}\) we can expend any solution \(C(x_0, \rho, \varphi)\) of (2.2) as

\[
(2.17) \quad C(x_0, \rho, \varphi) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \left( C^+(k)f_k^+(x_0, \rho, \varphi) + C^-(k)f_{-k}(x_0, \rho, \varphi) \right) d\eta_0,
\]

where
where
\[ C^+(k) = \langle f^+_{k}, C \rangle, \quad C^-(k) = -\langle f^-_{-k}, C \rangle. \]

We shall call \( C(x_0, \rho, \varphi) \) a wave packet.

Analogously, when \( \Phi \) is the field operator (cf. [8]), we have
\[ \Phi = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha^+_k f^+_k + \alpha^-_{-k} f^-_{-k}) d\eta \rho, \]
where
\[ \alpha^+_k = \langle f^+_k, \Phi \rangle, \quad \alpha^-_{-k} = -\langle f^-_{-k}, \Phi \rangle. \]

Operators \( \alpha^+_k, \alpha^-_{-k} \) are called the annihilation and the creation operators, respectively. They satisfy commutation relations (cf. [8]):
\[ [\alpha^+_k, \alpha^-_{-k}] = \delta(k - k') I, \quad [\alpha^+_k, \alpha^+_k] = 0, \quad [\alpha^-_{-k}, \alpha^-_{-k}] = 0, \]
where \( I \) is the identity operator.

Let
\[ C^\pm = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} C^\pm(k) f^\pm_{\pm k}(x_0, x) d\eta \rho, \]
Thus
\[ C = C^+ + C^- . \]

It follows from (2.17), (2.19) that
\[ \langle C, \Phi \rangle = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (C^+\alpha^+_k - C^-\alpha^-_{-k}) d\eta \rho. \]

The vacuum state \( |0\rangle \) is defined (cf. [7]) by the conditions
\[ \alpha^+_k |0\rangle = 0 \text{ for all } k, \]
i.e. \( |0\rangle \) annihilates all annihilation operators.
Let \( N(C) \) be the number operator of particles created by the wave packet \( C \) (cf. [8])

\[
(2.26) \quad N(C) = \langle C, \Phi >^* \langle C, \Phi >,
\]

and let \( \langle 0 | N(C) | 0 \rangle \) be the average number of particles.

As in [8] one have the following theorem (see Theorem 2.1 in [3]).

**Theorem 2.1.** The average number of particles created by the wave packet \( C \) is given by the formula

\[
(2.27) \quad \langle 0 | N(C) | 0 \rangle = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} |C^{-}(k)|^2 d\eta, 
\]

where \( C^{-} \) is the same as in (2.22).

## 3 The case of Hawking radiation for non-extremal RN black hole in \( |0\rangle \) vacuum state

First, we find the solution of the eikonal equation

\[
(3.1) \quad (2 - f(\rho)) \hat{S}_x^2 + 2(f(\rho) - 1) \hat{S}_x \hat{S}_\rho - f(\rho) \hat{S}_\rho^2 - \frac{1}{\rho^2} \hat{S}_\varphi^2 = 0
\]

that tends to \( \infty \) when \( \rho \to r_+ \).

We assume \( \hat{S}_x = -\eta_0, \hat{S}_\varphi = m, \eta_0 > 0 \). Then

\[
(3.2) \quad \hat{S}_\rho^\pm = \frac{(f(\rho) - 1)\eta_0 \pm \sqrt{(f - 1)^2 \eta_0^2 + 2(f - 1)f \eta_0^2 - \frac{1}{\rho^2} m^2 f}}{-f} = \frac{(f(\rho) - 1)\eta_0 \pm \sqrt{\eta_0^2 - \frac{f}{\rho^2} m^2}}{-f}.
\]

Therefore

\[
(3.3) \quad \hat{S}_\rho^- = \frac{(f(\rho) - 1)\eta_0 - \sqrt{\eta_0^2 - f \frac{m^2}{\rho^2}}}{-f} = \frac{-2\eta_0}{(\rho - r_+)(r_+ - r_0)} + O(\rho - r_+),
\]
so

\begin{equation}
\hat{S}_{\rho}^- = \frac{2\eta_0 r_+^2}{(r_+ - r_-)(\rho - r_+)} + O(\rho - r_+).
\end{equation}

Thus,

\begin{equation}
\hat{S}^- = -\eta_0 x_0 + \frac{2\eta_0 r_+^2}{r_+ - r_-} \ln |\rho - r_+| + m\varphi + O(\rho - r_+) \quad \text{for} \quad \rho > r_+.
\end{equation}

As in [3] we define the wave packet \( \hat{C}(x_0, \rho, \varphi) \) as the exact solution of the wave equation (2.2) having the following initial data at \( x_0 = 0 \):

\begin{equation}
\left| \hat{C} \right|_{x_0 = 0} = \theta(\rho - r_+) \frac{1}{\sqrt{\rho}} (\rho - r_+)^{\varepsilon} e^{-a(\rho - r_+)} \exp i\hat{\xi}_0 \ln |\rho - r_+| + im\varphi,
\end{equation}

\begin{equation}
\frac{\partial \hat{C}}{\partial x_0} \bigg|_{x_0 = 0} = i\hat{\beta} \hat{C} \bigg|_{x_0 = 0},
\end{equation}

where \( a > 0, \varepsilon > 0 \),

\begin{equation}
\hat{\xi}_0 = \frac{2\eta_0 r_+^2}{r_+ - r_-},
\end{equation}

\begin{equation}
\hat{\beta} = -\frac{f \hat{\xi}_0}{\rho - r_+}.
\end{equation}

For the convenience we take \( a > 0 \) in (3.6) equal to \( a > 0 \) in (2.9). Note that

\begin{equation}
\hat{\beta} = \frac{-f \hat{\xi}_0}{\rho - r_+} = \frac{-(\rho - r_+)\rho}{r_+^2} \cdot \frac{2\eta_0 r_+^2}{(r_+ - r_-)(\rho - r_+)} + O(\rho - r_+) = -\eta_0 + O(\rho - r_+).
\end{equation}

Calculating the KG norm of \( \hat{C} \), we get, as in [3],

\begin{equation}
< \hat{C}, \hat{C} > = \frac{4\pi \hat{\xi}_0 \Gamma(2\varepsilon)}{(2a)^{\varepsilon}}.
\end{equation}
Note that the terms containing the derivative of \( \frac{1}{\sqrt{\rho}}(\rho - r)^\varepsilon e^{\alpha(\rho - r)} \) are canceled since they are the derivatives of a real-valued function.

As in (3.4) in [5], we have

\[
\hat{C}^- (k) = - < f^-_k, \hat{C} > = \hat{C}_1^- (k) + \hat{C}_2^- (k),
\]

where

\[
\hat{C}_1^- (k) = -i \int_0^\infty \int_0^{2\pi} \left( (2 - f) \frac{\partial f_k^+}{\partial x_0} + (f - 1) \frac{\partial f_k^+}{\partial \rho} \right) \hat{C} \rho d\rho d\varphi
\]

and

\[
\hat{C}_2^- (k) = i \int_0^\infty \int_0^{2\pi} f_k^+ \left( (2 - f) \frac{\partial \hat{C}}{\partial x_0} + (f - 1) \frac{\partial \hat{C}}{\partial \rho} \right) \rho d\rho d\varphi
\]

Note that for \( x_0 = 0 \)

\[
(2 - f) \frac{\partial \hat{C}}{\partial x_0} + (f - 1) \frac{\partial \hat{C}}{\partial \rho} = \left[ - \frac{i \hat{\xi}_0}{\rho - r_+} + (f - 1) \left( \frac{\varepsilon}{\rho - r_+} - a - \frac{1}{2\rho} \right) \right] \hat{C} \bigg|_{x_0=0},
\]

where we used that

\[
(2 - f) i \hat{\beta} + (f - 1) \frac{i \hat{\xi}_0}{\rho - r_+} = - \frac{i \hat{\xi}_0}{\rho - r_+}.
\]

Using (3.13), (3.14) we have (cf. (3.5), (3.6) in [5]):

\[
\hat{C}_2^- (k) = \int_0^\infty \int_0^{2\pi} e^{i\rho \eta_0 + i\rho^* \varphi} \frac{\theta(\rho - r_+)}{\sqrt{\rho(\eta_0^2 + a^2)^{\frac{3}{2}}} \sqrt{2\pi}} \frac{(\rho - r_+)^\varepsilon}{\sqrt{\rho}} \left( \frac{\hat{\xi}_0}{\rho - r_+} - (f - 1) \left( \frac{-i \varepsilon}{\rho - r_+} + i \alpha \right) \right) e^{-a(\rho - r_+ + i \hat{\xi}_0 \ln(\rho - r_+)} e^{im\varphi} \rho d\rho d\varphi,
\]

\[
\hat{C}_1^- (k) = - \int_0^\infty \int_0^{2\pi} e^{i\rho \eta_0 + i\rho^* \varphi} \frac{\theta(\rho - r_+)(\rho - r_+)^\varepsilon}{\sqrt{\rho \sqrt{2\pi}}} \frac{\theta(\rho - r_+)^\varepsilon}{\sqrt{\rho}} \rho d\rho d\varphi,
\]

\( k = (\eta_0, m') \).
Note that the terms in $\hat{C}_1^- + \hat{C}_2^-$ containing $\frac{1}{2\pi}$ are cancelled (cf. (2.15)).

Integrating in $\varphi$ and using the formula (cf. [5])

\[
(3.19) \quad \int_0^\infty e^{it\eta d}e^{-at}dt = \frac{e^{it\eta}(\lambda+1)}{(\eta + ia)^{\lambda+1}},
\]

we get

\[
(3.20) \quad \hat{C}_2^-(k) = \frac{\delta_{m', -m} e^{ir_\eta} e^{i\frac{\pi}{2}(i\xi_0 + \varepsilon)}}{(\eta_0 + a^2)^{\frac{1}{2}}(\eta + ia)^{\frac{1}{2}}(\xi + i\lambda)(\lambda + 1)} \Gamma(i\xi_0) \Gamma(i\xi_0 + \varepsilon + 1) + \delta_{m', -m} O\left(\frac{1}{|\eta_0 + ia|^{1+\varepsilon}}\right),
\]

\[
(3.21) \quad \hat{C}_1^-(k) = -\frac{\delta_{m', -m} e^{ir_\eta} e^{i\frac{\pi}{2}(i\xi_0 + \varepsilon + 1)}}{(\eta_0 + a^2)^{\frac{1}{2}}(\eta + ia)^{\frac{1}{2}}(\xi + i\lambda)(\lambda + 1)} \Gamma(i\xi_0) \Gamma(i\xi_0 + \varepsilon + 1) + \delta_{m', -m} O\left(\frac{1}{|\eta_0 + ia|^{1+\varepsilon}}\right),
\]

where $\delta_{m_1, m_2} = 1$ when $m_1 = m_2$ and $\delta_{m_1, m_2} = 0$ when $m_1 \neq m_2$. Note that 

\[
\left| e^{i\frac{\pi}{2}(i\xi_0 + \varepsilon)} \right| = e^{-\frac{\pi}{2}i\xi_0}, \quad \Gamma(i\xi_0 + \varepsilon + 1) = (i\xi_0 + \varepsilon)e^{\frac{\pi}{2}i\xi_0} \Gamma(1) (i\xi_0 + \varepsilon),
\]

$\Gamma_1$ is bounded, $\Gamma_1 = \int_0^\infty e^{i(\xi_0 + \varepsilon - 1) \ln y + i(\varepsilon - 1) \frac{\pi}{2} - iy} dy$. We have

\[
(3.22) \quad |\hat{C}^-|^2 = |\hat{C}_1^- + \hat{C}_2^-|^2
\]

\[
= \frac{\delta_{m', -m}}{2} \left| \frac{\xi_0 - i\varepsilon}{(\eta_0 + a^2)^{\frac{1}{2}}} + \frac{ia(-\xi_0 + i\varepsilon)}{(\eta_0 + a^2)^{\frac{1}{2}}(\eta + ia)} - \frac{i(\xi_0 + \varepsilon)(\eta_0 + a^2)^{\frac{1}{2}}\xi_0}{\eta_0 + ia} \right|^2 e^{-2\pi\xi_0 \Gamma_1(i\xi_0 + \varepsilon) \frac{2}{|\xi_0 + \varepsilon|}} e^{2\xi_0 \arg(\eta_0 + ia) \frac{1}{2}} e^{2\xi_0 \arg(\eta_0 + ia) \frac{1}{2}}
\]

\[
\times e^{-2\pi\xi_0 \Gamma_1(i\xi_0 + \varepsilon) \frac{2}{|\xi_0 + \varepsilon|}} e^{2\xi_0 \arg(\eta_0 + ia) \frac{1}{2}} e^{2\xi_0 \arg(\eta_0 + ia) \frac{1}{2}}
\]

\[
\times e^{-2\pi\xi_0 \Gamma_1(i\xi_0 + \varepsilon) \frac{2}{|\xi_0 + \varepsilon|}} e^{2\xi_0 \arg(\eta_0 + ia) \frac{1}{2}} e^{2\xi_0 \arg(\eta_0 + ia) \frac{1}{2}}
\]

Therefore integrating in $\eta_0$, summing in $m'$ and changing $\eta_0 = a\eta_0'$ we get

\[
(3.23) \quad \langle 0 | N(\hat{C}) | 0 \rangle = a^{-2\varepsilon} \int_{-\infty}^{\infty} \hat{C}_3(\eta_0') d\eta_0' + O(a^{-2\varepsilon - 1}),
\]

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where

\[
(3.24) \quad \hat{C}_3(\eta_\rho) = \frac{1}{2} e^{-2\pi \hat{\xi}_0} |\Gamma_1(i\hat{\xi}_0 + \varepsilon)|^2 |\hat{\xi}_0 + i\varepsilon|^2 \left| \frac{\eta_\rho}{(\eta_\rho^2 + 1)^{\frac{1}{4}}} + (\eta_\rho^2 + 1)^{\frac{1}{4}} \right|^2 \\
\cdot e^{2\hat{\xi}_0 \arg(\eta_\rho + i)} (\eta_\rho^2 + 1)^{-\varepsilon - 1}.
\]

Let \( \hat{C}_n = \frac{\hat{C}}{<\hat{C}, \hat{C}>} \). Thus \(<\hat{C}_n, \hat{C}_n> = 1 \). Therefore, taking into account (3.11), we get:

\[
(3.25) \quad \lim_{a \to \infty} \langle 0 | N(\hat{C}_n) | 0 \rangle = \frac{2^\varepsilon}{4\pi \Gamma(2\varepsilon)} \int_{-\infty}^{\infty} \frac{1}{\hat{\xi}_0} \hat{C}_3(\eta_\rho) d\eta_\rho.
\]

Note that \( \hat{C}_3(\eta_\rho) \) in (3.25) is identical to the \( C_3(\eta_\rho) \) in (3.13) in [5] after we replace \( \hat{\xi}_0 |A\) by \( \hat{\xi}_0 \).

Let

\[
I_+ = \frac{2^\varepsilon}{4\pi \Gamma(\varepsilon)} \int_{0}^{\hat{\xi}_0} \frac{1}{\hat{\xi}_0} \hat{C}_3(\eta_\rho) d\eta_\rho, \quad I_- = \frac{2^\varepsilon}{4\pi \Gamma(\varepsilon)} \int_{-\infty}^{0} \frac{1}{\hat{\xi}_0} \hat{C}_3(\eta_\rho) d\eta_\rho,
\]

i.e.

\[
(3.26) \quad \lim_{a \to \infty} \langle 0 | N(\hat{C}_n) | 0 \rangle = I_+ + I_-.
\]

As it is shown in the end of §3 in [5], \( I_+ = O(e^{-\pi \hat{\xi}_0}) \) and \( I_- = O(\hat{\xi}_0^{1-\delta(1+2\varepsilon)}) \), \( 0 < \delta < 1 \).

The integral \( I_+ \) gives the contribution to the Hawking radiation and the contribution of \( I_- \) is unrelated to the Hawking radiation.

4 The Hawking radiation for not-extremal RN black hole in the Unruh type vacuum

Consider the equation (2.2) with \( r_+ > r_- \), i.e. the case of not-extremal RN equation.

When using the vacuum state \( |0\rangle \) we get that the average number of created particles consists of two parts: the part related to the Hawking radiation and the part that is not related. In this section we introduce another
vacuum state that is called the Unruh type vacuum state. The average number of particles in this vacuum state consists only of particles related to the Hawking radiation.

To introduce the Unruh type vacuum state, we will split, as in [3]:

\begin{align}
  f^{++}_k &= f^+_k \theta(\eta_\rho), \quad f^{+-}_k = f^+_k (1 - \theta(\eta_\rho)), \\
  f^{-+}_k &= f^-_k \theta(\eta_\rho), \quad f^{--}_k = f^-_k (1 - \theta(\eta_\rho)),
\end{align}

where \( \theta(\eta_\rho) = 1 \) when \( \eta_\rho > 0 \), \( \theta(\eta_\rho) = 0 \) when \( \eta_\rho < 0 \). Analogously, we define \( \alpha^{++}_k, \alpha^{+-}_k, \alpha^{-+}_k, \alpha^{--}_k \). Therefore

\begin{equation}
  \Phi = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha^{++}_k f^{++}_k + \alpha^{+-}_k f^{+-}_k + \alpha^{-+}_k f^{-+}_k + \alpha^{--}_k f^{--}_k) d\eta_\rho,
\end{equation}

and

\begin{equation}
  C = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} C^{++}(k) f^{++}_k + C^{+-}(k) f^{+-}_k + C^{-+}(k) f^{-+}_k + C^{--}(k) f^{--}_k) d\eta_\rho,
\end{equation}

where (cf. (4.1))

\begin{equation}
  C^{++}(k) = C^{+}(\eta_\rho), \quad C^{+-}(k) = C^{+}(1 - \theta(\eta_\rho)), \\
  C^{-+}(k) = C^{-}(\eta_\rho), \quad C^{--} = C^{-}(1 - \theta(\eta_\rho)).
\end{equation}

As in [3] (cf. [10]), we define the Unruh type vacuum state \( |\Psi\rangle \) by the conditions

\begin{equation}
  \alpha^{++}_k |\Psi\rangle = 0, \quad \alpha^{--}_k |\Psi\rangle = 0, \quad \forall k.
\end{equation}

It follows from (4.3), (4.4) that

\begin{equation}
  \langle C, \Phi \rangle = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} (C^{++}(k) \alpha^{++}_k + C^{+-}(k) \alpha^{+-}_k + C^{-+}(k) \alpha^{-+}_k + C^{--}(k) \alpha^{--}_k) d\eta_\rho,
\end{equation}

The operator of the number of particles created by the wave packet \( C \) is equal to (cf. [8])

\begin{equation}
  N(C) = \langle C, \Phi \rangle^* < C, \Phi >.
\end{equation}
The average number of created particles is

$$\langle \Psi | N(C) | \Psi \rangle.$$ 

Exactly as in [3] we get (see Theorem 2.1 in [3])

$$\langle \Psi | N(C) | \Psi \rangle = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \left( - |C^{+\prime}(k)|^2 + |C^{-\prime}(k)|^2 \right) d\eta \rho.$$ 

Let \( \hat{C}(x_0, \rho, \varphi) \) be the same wave packet as in §3 (see (3.6), (3.7), (3.8), (3.9)).

To compute the average number of particles created by the wave packet \( \hat{C} \) we need to compute \( \hat{C}^{+\prime}(k) \) and \( \hat{C}^{-\prime}(k) \). \( \hat{C}^{-\prime}(k) \) was computed in §3. Thus it remains to compute \( \hat{C}^{+\prime}(k) \).

We have

$$\hat{C}^{+\prime}(k) = \hat{C}^{+\prime}_1(k) + \hat{C}^{+\prime}_2(k),$$

where \( \hat{C}^{+\prime}(k) = < f_k^{+\prime}, \hat{C} > \).

Analogously to (3.13), (3.14), we have

$$\hat{C}^{+\prime}_1(k) = i \int_0^{2\pi} \int_0^\infty \left( (2 - f) \frac{\partial f_k^{+\prime}}{\partial x_0} + (f - 1) \frac{\partial f_k^{+\prime}}{\partial \rho} \right) \hat{C} \rho d\rho d\rho d\varphi,$$

$$\hat{C}^{+\prime}_2(k) = -i \int_0^{2\pi} \int_0^\infty \left( (2 - f) \frac{\partial \hat{C}}{\partial x_0} + (f - 1) \frac{\partial \hat{C}}{\partial \rho} \right) \rho d\rho d\rho d\varphi.$$ 

Therefore \( \hat{C}^{+\prime}_1 \) and \( \hat{C}^{+\prime}_2 \) have the form:

$$\hat{C}^{+\prime}_1(k) = - \int_0^{2\pi} \int_0^\infty \left( \eta^2 + a^2 \right)^{\frac{1}{4}} e^{-i\rho \eta - i m \varphi}$$

$$\cdot \theta(\rho - r_+) \left( \rho - r_+ \right)^{\frac{\varepsilon}{\sqrt{\rho}}} e^{-a(\rho-r_+) + i \xi_0 \ln(\rho-r_+) + i m \varphi} \rho d\rho d\varphi,$$
we get
\(2 \cdot \phi \eta \)
Therefore, for
\(|\hat{C}_0^{(\rho - r_+)} - (f - 1) \left( \frac{i \xi}{\rho - r_+} - ia \right) | \rho d \rho d \varphi.\)

Integrating in \(\varphi\) and using the formula (cf. [3])

\[
\int_0^\infty e^{-i \rho t} t^\lambda e^{-a t} dt = \frac{\Gamma(\lambda + 1)e^{-\frac{a^2}{4}}}{(\eta - ia)^{\lambda + 1}},
\]

we get

\[
\hat{C}_1^{(\rho - r_+)}(k) = -\frac{\delta_{m,-m'}(\eta^2 + a^2)^k e^{-ir_{m'}} \Gamma(i\hat{\xi}_0 + \varepsilon + 1)}{(\eta^2 + a^2)^k (\eta - ia)^{i\hat{\xi}_0 + \varepsilon + 1}} e^{-\frac{ia^2}{4}}(i\hat{\xi}_0 + \varepsilon + 1),
\]

(4.15)

(4.16)

Note that the terms \(-\frac{1}{2\rho} \hat{C}_k^{(\rho - r_+)} \) in (2.12) are canceled.

Therefore, for \(\eta_0 < 0\) we have

\[
|\hat{C}_1^{(\rho - r_+)} + \hat{C}_2^{(\rho - r_+)}|^2 = \frac{\delta_{m,-m'} |\Gamma_1(i\hat{\xi}_0 + \varepsilon)|^2 e^{\frac{i\hat{\xi}_0}{\sqrt{\eta^2 + a^2}}} |\eta - ia|^{2\varepsilon}}{2 e^{2\varepsilon} (\eta^2 + a^2)^k} \left( \frac{\hat{\xi}_0 + \varepsilon}{\eta^2 + a^2} \right)^2 + O\left( \frac{1}{(\eta^2 + a^2)^{2\varepsilon + 2}} \right)
\]

\[\cdot \left( \frac{\hat{\xi}_0}{\eta - ia} - \frac{\hat{\xi}_0}{\eta^2 + a^2} \right) \left( 1 + \frac{ia}{\eta - ia} \right)^2 + O\left( \frac{1}{(\eta^2 + a^2)^{2\varepsilon + 2}} \right)
\]

(4.18)

\[
= \frac{\delta_{m,-m'}}{2} \left( \frac{\hat{\xi}_0}{\eta - ia} - \frac{\hat{\xi}_0}{\eta^2 + a^2} \right) \left( 1 + \frac{ia}{\eta - ia} \right)^2 + O\left( \frac{1}{(\eta^2 + a^2)^{2\varepsilon + 2}} \right)
\]

Note that

\[
\left( \frac{\eta}{\eta^2 + a^2} \right)^{1/2} + \frac{\eta}{\left( \eta^2 + a^2 \right)^{1/2}} \right|^2 - \left( \frac{\eta}{\eta^2 + a^2} \right)^{1/2} - \frac{\eta}{\left( \eta^2 + a^2 \right)^{1/2}} \right|^2 = 4\eta
\]

(4.19)
when \(\eta_\rho > 0\). Therefore

\[
\int_0^{\infty} |C^{-+}|^2 d\eta_\rho - \int_{-\infty}^0 |C^{+-}|^2 d\eta_\rho = 2 \delta_{m,-m'} |\hat{\xi}_0 - i\varepsilon|^2 e^{-2\pi \hat{\xi}_0 |\Gamma_1|^2} \int_0^{\infty} \frac{\eta_\rho}{(\eta_\rho^2 + a^2)^{\epsilon+1}} e^{\frac{2\hat{\xi}_0 \sin^{-1} \hat{\xi}_0}{\sqrt{\eta_\rho^2 + a^2}}} d\eta_\rho + O(a^{-2\epsilon-1}).
\]

Finally, normalize \(\hat{C}\), i.e. replace \(\hat{C}\) by

\[
\hat{C}_n = \frac{\hat{C}}{<\hat{C},\hat{C}>^{1/2}},
\]

make change of variables \(\eta_\rho = a\eta'_\rho\) and use (3.11). Then we will get, as in Theorem 3.1 in [3], that

\[
\lim_{a \to \infty} \frac{\langle \Psi | N(\hat{C}_n^2) | \Psi \rangle}{2\pi \Gamma(\epsilon)} = \frac{2\varepsilon e^{-2\pi \hat{\xi}_0 |\Gamma_1|^2}}{\hat{\xi}_0} \int_{-\infty}^0 \frac{|\eta_\rho|}{(\eta_\rho^2 + 1)^{\epsilon+1}} e^{\frac{2\hat{\xi}_0 \sin^{-1} \hat{\xi}_0}{\sqrt{\eta_\rho^2 + 1}}} d\eta_\rho.
\]

Therefore we proved that the Hawking radiation for non-extremal RN black hole has the same form as the Hawking radiation for the rotating acoustic black hole with \(\hat{\xi}_0 = \frac{2m_0^2}{r_+ - r_-}\) replacing \(\xi_0 |A| = (\eta_0 - \frac{Bm}{|A|^2}) |A|\).

The proof given here is slightly different from the proof in [3] since we wanted to use the fact that \(\int_0^{\infty} |\hat{C}^{-+}|^2 d\eta_\rho\) is already computed in \(\S 3\).

5 The Hawking radiation for extremal RN black hole

In the case of two space dimensions one have the wave equation (3.3) when \(e^2 = m^2\). Thus

\[
r_+ = r_- = m.
\]
In particular,

\[(5.2)\quad f = 1 - \frac{2m}{\rho} + \frac{\rho^2}{c^2} = \left(1 - \frac{m}{\rho}\right)^2.\]

We shall treat the extremal RN case following the prescription of §3. We start with the eikonal that tends to $\infty$ when $\rho \to m$ (cf. (3.3)). We have

\[(5.3)\quad \tilde{S}_\rho^- = \frac{(f(\rho) - 1)\eta_0 - \sqrt{\eta_0^2 - \frac{f m^2}{\rho^2}}}{-f} = \frac{-2\eta_0}{-(\rho - m)^2} + O(\rho - m) = \frac{2\eta_0 m^2}{(\rho - m)^2} + O(\rho - m),\]

where $\eta_0 > 0$, $m' \in \mathbb{Z}$. Therefore the eikonal $\tilde{S}^-$ has the form

\[(5.4)\quad \tilde{S}^- = -\eta_0 x_0 - \frac{2\eta_0 m^2}{\rho - m} + O(\rho - m) + m' \varphi.\]

As in §3 we use the approximation of $\tilde{S}^-$ to construct the wave packet $\tilde{C}(x_0, \rho, \varphi)$ as the exact solution of (2.2) with the following initial data

\[(5.5)\quad \tilde{C}\big|_{x_0=0} = \theta(\rho - m) \frac{1}{\sqrt{\rho}}(\rho - m)^{\frac{\epsilon}{2}} e^{-a(\rho - m)} e^{-\frac{\xi_0}{\rho - m} + im' \varphi},\]

\[(5.6)\quad \frac{\partial \tilde{C}}{\partial x_0}\big|_{x_0=0} = i \tilde{\beta} \tilde{C}\big|_{x_0=0},\]

where

\[(5.7)\quad \tilde{\xi}_0 = 2\eta_0 m^2,\]

\[(5.8)\quad \tilde{\beta} = \frac{-f \tilde{\xi}_0}{2 - f}.\]

Note that

\[(5.9)\quad (2 - f) \tilde{\beta} + (f - 1) \frac{\tilde{\xi}_0}{(\rho - m)^2} = \frac{-\tilde{\xi}_0}{(\rho - m)^2}.\]
Note also that
\[ (2 - f) \frac{\partial \tilde{C}}{\partial x_0} \bigg|_{x_0=0} + (f - 1) \frac{\partial \tilde{C}}{\partial \rho} = \left[ (2 - f)i\tilde{\beta} + (f - 1)\left( \frac{i\tilde{\xi}_0}{(\rho - m)^2} + \frac{\varepsilon}{\rho - m} - a - \frac{1}{2\rho} \right) \right] \tilde{C} \bigg|_{x_0=0}. \]

Using (3.9), (3.10) we shall compute, as in [3], the KG norm of \( \tilde{C} \):
\[ < \tilde{C}, \tilde{C} > = \int_0^{2\pi} \int_0^\infty (\rho - m)^{2\varepsilon} e^{-2a(\rho - m)} \frac{2\tilde{\xi}_0}{(\rho - m)^2} d\rho d\varphi = 4\pi \tilde{\xi}_0 \int_0^\infty t^{2\varepsilon - 2} e^{-2at} dt = \frac{4\pi \tilde{\xi}_0}{(2a)^{2\varepsilon - 1}} \Gamma(2\varepsilon - 1), \]
where we assume that \( \varepsilon > \frac{1}{2} \).

Now we compute
\[ \langle 0 | N(\tilde{C}) | 0 \rangle = \sum_{m'=\infty}^{\infty} \int_0^{\infty} |C^-(k)|^2 d\eta. \]
As in (3.10) of [3], we have
\[ \tilde{C}^- = < f_k^+, \tilde{C} >= \tilde{C}_1^- + \tilde{C}_2^- (k). \]
Analogously to (3.13)–(3.16), we get
\[ \tilde{C}_1^-(k) = - \int_0^{2\pi} \int_0^{\infty} \left( \frac{\eta_\rho^2 + a^2}{\sqrt{2\pi} \sqrt{p}} \right)^{1/2} e^{i m \rho' + i m' \varphi} \theta(\rho - m) \frac{(\rho - m)^\varepsilon}{\sqrt{\rho}} e^{-a(\rho - m)} \cdot e^{-i \rho' m' + i m'' \varphi} \rho d\rho d\varphi, \]
where \( m', m'' \) are integers. We used in (5.14) (cf. 2.15) that
\[ (2 - f) \frac{\partial f_k^+}{\partial x_0} + (f - 1) \frac{\partial f_k^+}{\partial \rho} = -i \sqrt{\eta_\rho^2 + a^2} f_k^+ - (f - 1) \left( -\frac{1}{2\rho} \right) f_k^+. \]
Analogously,

\begin{equation}
\tilde{C}_2^{-} = \int_{0}^{\infty} \int_{0}^{2\pi} \frac{1}{\sqrt{2} 2\pi (\eta_\rho^2 + a^2)^{\frac{1}{4}}} e^{i\rho \eta \varphi + i\eta \tilde{\xi}_0} \theta(\rho - m) (\rho - m)^{\varepsilon} \nonumber \\
\cdot e^{-a(\rho - m) - i\eta \tilde{\xi}_0 + i\varphi} \left[ \frac{\tilde{\xi}_0}{(\rho - m)^2} + i(f - 1) \left( \frac{2}{\rho - m} - a - \frac{1}{2\rho} \right) \right] d\rho d\varphi.
\end{equation}

In (3.15) and (3.16) we will drop terms containing \(-\frac{1}{2\rho}\), since they will cancel each other when we will take the sum \(\tilde{C}_1^{-}\) and \(\tilde{C}_2^{-}\). Integrate in \(\varphi\) in (5.14) and (5.16) and change \(\rho = m + t\). We get

\begin{equation}
\tilde{C}_1^{-} = \frac{-\delta_{m', m''}}{\sqrt{2}} \left( \eta_\rho^2 + a^2 \right)^{\frac{1}{4}} e^{i m \eta \varphi} A_1(\eta_\rho, \tilde{\xi}_0, a),
\end{equation}

\begin{equation}
\tilde{C}_2^{-} = \frac{\delta_{m', m''}}{\sqrt{2}} \left( \eta_\rho^2 + a^2 \right)^{-\frac{1}{4}} e^{i m \eta \varphi} A_2(\eta_\rho, \tilde{\xi}_0, a),
\end{equation}

where

\begin{equation}
A_1(\eta_\rho, \tilde{\xi}_0, a) = \int_{0}^{\infty} e^{it\eta_\rho \tau} e^{-at - i\tilde{\xi}_0 t} dt,
\end{equation}

\begin{equation}
A_2(\eta_\rho, \tilde{\xi}_0, a) = \int_{0}^{\infty} e^{it\eta_\rho \tau} e^{-at - i\tilde{\xi}_0 t} \left[ \frac{\tilde{\xi}_0}{\tau^2} + i(f - 1) \left( \frac{\varepsilon}{\tau} - a \right) \right] dt,
\end{equation}

When \(\eta_\rho > 0\) make in (5.19), (5.20) change of variables

\((-i\eta_\rho + a)t = \tau\).

We get

\begin{equation}
A_1 = (-i\eta_\rho + a)^{-\varepsilon - 1} \int_{\Gamma_{\eta_\rho + a}} \tau^\varepsilon e^{-\tau} e^{i\eta_\rho \eta_\rho - i\tilde{\xi}_0 \tau} d\tau,
\end{equation}

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where $\Gamma_{i\eta^0 + a}$ is the ray $\{( -i\eta + a)t, t \geq 0 \}$. Using the Cauchy integral theorem we transform (5.21) to the integral over positive semi axis:

\begin{equation}
A_1 = \left( -i\eta + a \right)^{-\varepsilon - 1} \int_0^\infty \tau^\varepsilon e^{-\tau} e^{-\frac{i\eta^0 - i\eta^0}{\tau}} d\tau,
\end{equation}

Note that $-\tilde{\xi}_0 \eta^0 < 0$ since $\eta^0 > 0$. Changing coordinates as in (5.21) and using the Cauchy theorem, we get from (5.20)

\begin{equation}
A_2 = \tilde{\xi}_0 \left( -i\eta + a \right)^{-\varepsilon + 1} \int_0^\infty \tau^{-\varepsilon - 2} e^{-\tau} e^{-\frac{i\eta^0 - i\eta^0}{\tau}} \cdot \left[ 1 + O(-i\eta + a)^{-1} \tau + O(-i\eta + a)^{-2} \tau^2 a \right] d\tau,
\end{equation}

where $\varepsilon > 1$. If $\tilde{C}_n = \frac{\tilde{C}}{\langle \tilde{C} \tilde{C} \rangle^{1/2}}$ is the normalized wave packet then

\begin{equation}
\langle 0 | N(C_n) | 0 \rangle = \sum_{m' = -\infty}^\infty \langle \tilde{C} \tilde{C} \rangle^{-1} \int_{-\infty}^\infty |\tilde{C}_1^- + \tilde{C}_2^-|^2 d\eta^0,
\end{equation}

We compute first $\sum_{m' = -\infty}^\infty \langle \tilde{C} \tilde{C} \rangle^{-1} \int_{0}^\infty |\tilde{C}_1^-|^2 d\eta^0$. It follows from (5.11) and (5.22) that

\begin{equation}
\sum_{m' = -\infty}^\infty \langle \tilde{C} \tilde{C} \rangle^{-1} \int_{0}^\infty |\tilde{C}_1^-|^2 d\eta^0 \leq C \int_{0}^\infty \frac{d\eta^0}{| - i\eta^0 + a |^{2\varepsilon + 2}} \leq C \int_{0}^\infty \frac{d\eta^0}{| - i\eta^0 + a |^2} \to 0 \text{ when } a \to \infty.
\end{equation}

To compute the term containing $|A_2|^2$ we need the integration by parts. We have

\begin{equation}
\frac{d}{d\tau} e^{-\frac{i\eta^0}{\tau} (-i\eta + a)} = \frac{i\tilde{\xi}_0 (-i\eta + a)}{\tau^2} e^{-\frac{i\eta^0}{\tau} (-i\eta + a)}.
\end{equation}
Substituting (5.26) in (5.23) and integrating by parts in $\tau$, we get

(5.27) \[ A_2 = \tilde{\xi}_0 (-i\eta_\rho + a)^{-\varepsilon + 1} \cdot O\left( \frac{1}{-i\eta_\rho + a} \right). \]

Therefore

(5.28) \[ \sum_{m'=\infty}^{\infty} \int_0^{\infty} |\tilde{C}^-_2| d\eta_\rho \leq C \int_0^{\infty} \frac{a^{2\varepsilon - 1} d\eta_\rho}{| -i\eta_\rho + a|^{2\varepsilon} (\eta_\rho^2 + a^2)^{\frac{1}{2}}} \]

\[ \leq C \int_0^{\infty} \frac{d\eta_\rho}{| -i\eta_\rho + a|^2} \to 0 \text{ when } a \to \infty. \]

Consider now $< \tilde{C}, \tilde{C} >^{-1} \sum_{m'=\infty}^{\infty} \int_0^{\infty} |\tilde{C}^-_1 + \tilde{C}^-_2|^2 d\eta_\rho$.

Making the changes of variables

(5.29) \[ \tau = at, \quad \eta_\rho'' = \frac{\eta_\rho}{a^2}, \]

we get from (5.19)

(5.30) \[ A_1(a^2 \eta_\rho'', \tilde{\xi}_0, a) = a^{-\varepsilon - 1} \int_0^{\infty} e^{i\eta_\rho'' \tau - \eta_\rho'' \tilde{\xi}_0} \frac{\tau^\varepsilon}{\tau} d\tau. \]

Apply the stationary phase method to the integral

(5.31) \[ \int_0^{\infty} e^{i a \gamma(\tau, \eta_\rho'')} - \tau \tau^\varepsilon d\tau, \]

where

(5.32) \[ \gamma(\tau, \eta_\rho) = \eta_\rho'' \tau - \frac{\tilde{\xi}_0}{\tau}. \]

The critical point $\gamma_\tau(\tau_0, \eta_\rho'') = 0$ is $\eta_\rho'' + \frac{\tilde{\xi}_0}{\tau_0} = 0$, i.e. $\tau_0 = \frac{\sqrt{\tilde{\xi}_0}}{\sqrt{-\eta_\rho''}}$.

Note that $\eta_\rho'' < 0$. Since

(5.33) \[ \gamma_\tau''(\tau, \eta_\rho'') = \frac{-2 \tilde{\xi}_0}{\tau_0^3} = -2 \frac{\tilde{\xi}_0}{\tau_0^3} |\eta_\rho''|^{\frac{1}{2}}, \]

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we have

\[ \int_0^\infty e^{ia\gamma(\tau,\eta')} e^{-\tau\varepsilon} d\tau = \sqrt{\frac{\pi}{a}} \frac{e^{ia\gamma(\tau_0,\eta'')}}{\sqrt{2\hat{\xi}_0 |\eta'\rangle}} e^{-\frac{1}{2} - \frac{1}{2}} \left( \frac{\hat{\xi}_0}{|\eta'\rangle} \right)^\frac{\varepsilon}{2} + O\left( \frac{1}{a} \right). \]

Therefore,

\[ \lim_{a \to \infty} \left< \hat{C}, \hat{C}^{-1} \right> \sum_{m'=-\infty}^0 \int_0^\infty |\hat{C}_m|\, d\eta' = \lim_{a \to \infty} C' \int_{-\infty}^0 a^2 a^{2\varepsilon-1} (a^4 \eta''^2 + a^2) a^{-2\varepsilon-2} \left[ \frac{1}{2} |\hat{\eta}'\rangle - \frac{1}{2} e^{-2(\frac{\hat{\xi}_0}{|\eta'\rangle})} \left( \frac{\hat{\xi}_0}{|\eta'\rangle} \right)^{\varepsilon} + O\left( \frac{1}{a} \right) \right] d\eta''
\]

where \( C' \) in \((5.35)\) and below means various constants independent of \( \hat{\xi}_0 \).

Changing in \((5.35)\) \( \eta'' = \hat{\xi}_0 \hat{\eta} \) we get that the right hand side of \((5.35)\) is equal to \( C' \hat{\xi}_0 \). Analogously consider \((5.20)\). Note that

\[ f(\rho) = f(m + t) = -\frac{2m}{m + t} + \frac{m^2}{(m + t)^2}. \]

Thus \( A_2(\eta, \hat{\xi}_0, a) \) behaves similarly to \( A_1(\eta, \hat{\xi}_0, a) \).

Make changes of variables \((5.29)\) in \((5.20)\) and compute the stationary phase integral

\[ \int_0^\infty e^{ia\gamma(\tau,\eta')} e^{-\tau\varepsilon-2\left( \hat{\xi}_0 + i(f - 1) \left( \frac{\varepsilon}{a} - \frac{\tau^2}{a} \right) \right)} d\tau. \]
Thus

\begin{equation}
(5.38) \quad < \tilde{C}, \tilde{C} > ^{-1} \sum_{m'=-\infty}^{\infty} \int_{-\infty}^{0} |C_2^-|^2 d\eta_{\rho} = \int_{-\infty}^{0} \frac{a^{2\varepsilon - 1}}{(a^2 \eta_{\rho}^2 + a^2)^{\varepsilon - 1}} \cdot \frac{\xi_0^2}{a^{2(\varepsilon - 1)}} \cdot \frac{1}{(\sqrt{a})^2} \xi_0^2 |\eta_{\rho}'|^{-\frac{3}{2}} \cdot e^{-2} \left( \frac{\xi_0}{|\eta_{\rho}'|} \right)^{\varepsilon - 2} \left( 1 + O \left( \frac{1}{a} \right) \right) a^2 d\eta_{\rho}''
\end{equation}

\begin{align*}
&= C' \int_{-\infty}^{0} \frac{\xi_0^2}{|\eta_{\rho}'|^2} \cdot \left( \frac{\xi_0}{|\eta_{\rho}'|} \right)^{\varepsilon - 2} \left( 1 + O \left( \frac{1}{a} \right) \right) a^2 d\eta_{\rho}'' \\
&= C' \int_{-\infty}^{0} \frac{\xi_0^2}{|\eta_{\rho}'|^2} \cdot \left( \frac{\xi_0}{|\eta_{\rho}'|} \right)^{\varepsilon - 2} \left( 1 + O \left( \frac{1}{a} \right) \right) a^2 d\eta_{\rho}''
\end{align*}

Taking the limit when \( a \to \infty \) and changing \( \eta_{\rho}' = \tilde{\xi}_0 \tilde{\eta}_{\rho} \), we get

\begin{equation}
(5.39) \quad \lim_{a \to \infty} \sum_{m'=-\infty}^{\infty} < \tilde{C}, \tilde{C} > ^{-1} \int_{-\infty}^{0} |\tilde{C}_2^-|^2 d\tilde{\eta}_{\rho} = C' \tilde{\xi}_0.
\end{equation}

Analogously, \( \lim_{a \to \infty} < \tilde{C}, \tilde{C} > ^{-1} \int_{-\infty}^{0} 2\Re(C_1^- \overline{C_2^-}) d\eta_{\rho} = C' \tilde{\xi}_0 \). Therefore

\begin{equation}
(5.40) \quad \lim_{a \to \infty} < \tilde{C}, \tilde{C} > ^{-1} \int_{-\infty}^{0} |\tilde{C}_1^- + \tilde{C}_2^-|^2 d\eta_{\rho} = C' \tilde{\xi}_0.
\end{equation}

Thus we have

\begin{equation}
\lim_{a \to \infty} \langle 0 | N(\tilde{C}_n) | 0 \rangle = I_1 + I_2,
\end{equation}

where \( I_1 \) is the integral over \( \int_{-\infty}^{0} \) and \( I_2 \) is the integral over \( \int_{-\infty}^{0} \).

We have proved that \( I_1 = 0 \) and \( I_2 = C' \tilde{\xi}_0 \neq 0 \). Note that \( I_1 \) is the portion of the average number of particles that related to the Hawking radiation.

Remark 5.1. Consider the Hawking radiation for the extremal RN black hole when the vacuum state is the Unruh type vacuum state as in §4.
Then formula (4.9) holds with $C^{-+}(k)$ and $C^{+-}(k)$ replaced by $\tilde{C}^{-+}(k)$ and $\tilde{C}^{+-}(k)$, respectively.

Normalizing $\tilde{C}(k)$ and taking the limit as $a \to \infty$ we get (see (5.25) and (5.28)) that the contribution of $\tilde{C}^{-+}(k)$ to the Hawking radiation is zero. Similarly computations show that the contribution of $\tilde{C}^{+-}(k)$ is also zero. Thus

\begin{equation}
\lim_{a \to \infty} \langle \Psi | N(\tilde{C}_n) | \Psi \rangle = 0.
\end{equation}

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