Buffon’s problem with a star of needles and a lattice of parallelograms

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Abstract

A star of \( n \geq 2 \) line segments (needles) of equal length with common endpoint and constant angular spacing is randomly placed onto a lattice which is the union of two families of equidistant lines in the plane with angle \( \alpha \) between the nonparallel lines. For odd \( n \), we calculate the probabilities of exactly \( i \) intersections between the star and the lattice (for even \( n \), see [3]). Using a geometrical method, we derive the limit distribution function of the relative number of intersections as \( n \to \infty \). This function is independent of \( \alpha \). We show that the relative numbers for each of the two families are asymptotically independent random variables.

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1 Introduction

We consider the random throw of a star \( S_{n,\ell} \) of line segments onto a plane ruled with two families \( R_a \) and \( R_b \) of parallel lines,

\[
R_a := \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha = ka, k \in \mathbb{Z}\}, \\
R_b := \{(x, y) \in \mathbb{R}^2 \mid y = mb, m \in \mathbb{Z}\},
\]

where \( a \) and \( b \) are positive real constants, \( \alpha \in \mathbb{R}, 0 < \alpha \leq \pi/2 \), and put \( R_{a,b,\alpha} := R_a \cup R_b \). We denote the parallelogram

\[
\mathcal{F} := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq b, \ y \cot \alpha \leq x \leq a \csc \alpha + y \cot \alpha \}
\]

shown in Fig. 1 the fundamental cell of \( R_{a,b,\alpha} \). The star \( S_{n,\ell} \) consists of \( n \) \((2 \leq n < \infty)\) line segments (needles) of equal length \( \ell \) with common endpoint and constant angular spacing \( 2\pi/n \) between neighbouring needles. (The convex hull of \( S_{n,\ell} \) is the regular \( n \)-gon with circumscribed circle of radius \( \ell \).)

The random throw \( S_{n,\ell} \) onto \( R_{a,b,\alpha} \) is defined as follows: The coordinates \( x \) and \( y \) of the centre point of \( S_{n,\ell} \) are random variables uniformly
Figure 1: Star $S_{n,\ell}$ (Example $n = 9$) and lattice and lattice $R_{a,b,\alpha}$ distributed in $[y \cot \alpha, a \csc \alpha + y \cot \alpha]$ and $[0, b]$ resp.; the angle $\phi$ between the direction perpendicular to the lines of $R_a$ and a certain needle of $S_{n,\ell}$ is a random variable uniformly distributed in $[0, 2\pi]$. All 3 random variables are stochastically independent. We assume $2\ell \sin(\pi n \lfloor n/2 \rfloor) \leq \min(a, b)$; in this case the probability that $S_{n,\ell}$ intersects two lines of $R_a$ (or $R_b$) at the same time is equal to zero. The maximum number $M$ of intersections with $R_a$ (or $R_b$) is then given by

$$M = \begin{cases} 
  n/2, & \text{if } n \text{ is even}, \\
  (n + 1)/2, & \text{if } n \text{ is odd}.
\end{cases}$$

In [8], Buffon published the solution of his famous needle problem. It is the calculation of the probability of the event that $S_{2,\ell}$ intersects $R_a$. ($S_{2,\ell}$ can be considered as single needle of length $2\ell$.) Laplace [10] pp. 359-362 calculated the intersection probability for $S_{2,\ell}$ and $R_{a,b,\pi/2}$. Santaló [12] generalized this result for $R_{a,b,\alpha}$, $0 < \alpha \leq \pi/2$, and derived the probabilities of 0, 1 or 2 intersection points (see also [13, p. 139]). Duma and Stoka [9] solved the problem for ellipses and $R_{a,b,\pi/2}$. Ren and Zhang [11] and Aleman et al. [1] calculated the intersection probability for an arbitrary convex body $K$ and $R_{a,b,\alpha}$, and proved that for $K$ there is an nonvanishing value of $\alpha$ for which the events $K$ intersects $R_a$ and $K$ intersects $R_b$ are independent; explicit results for regular $n$-gons ($n \geq 2$) and $R_{a,b,\alpha}$ were obtained by Bäsel [4]. In [8], Bäsel calculated the probabilities of exactly $i$ intersections for $S_{n,\ell}$ with even $n \geq 2$ and $R_{a,b,\alpha}$. Bonanzinga [7] found the intersection probabilities for $S_{3,\ell}$ and $R_{a,b,\alpha}$, $\pi/3 \leq \alpha \leq \pi/2$.

In the Sections 2 and 3 we calculate the probabilities of exactly $i$ intersections for $S_{n,\ell}$ with odd $n$, $3 \leq n < \infty$, and $R_{a,b,\alpha}$, $0 < \alpha \leq \pi/2$. In Section 4 we investigate the distribution functions of the relative number of intersections for $n \in \mathbb{N}$, $n \geq 2$. Using a geometrical method, we derive the limit distribution as $n \to \infty$. For abbreviation we put $\lambda = \ell/a$, $\mu = \ell/b$, and $\lfloor \cdot \rfloor$ for the integer part of $\cdot$. 

2
2 Intersection probabilities

Theorem 1. The probabilities \( p(i) \) of exactly \( i \) intersections between \( S_{n, \ell} \) and \( R_{a, b, \alpha} \) are for odd \( n \geq 3 \), \( 2 \max(\lambda, \mu) \sin(\frac{\pi}{n} \lfloor \frac{n}{2} \rfloor) \leq 1 \) and \( 0 < \alpha \leq \frac{\pi}{n} \) given by

\[
p(i) = \begin{cases} 
1 - \left[ \frac{2n(\lambda + \mu)}{\pi} \sin \frac{\pi}{n} \frac{\pi}{n} f_0(\alpha) \right], & \text{if } i = 0, \\
\frac{8n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{2n} \sin \frac{\pi}{n} \frac{\pi}{n} - \frac{4n\mu}{\pi} \left[ f_1(\alpha) \sin \frac{\pi}{n} \frac{\pi}{n} \right] \\
- f_4(\alpha) \left( \cot \frac{\pi}{n} \frac{\pi}{n} - i \cos \frac{\pi}{n} \frac{\pi}{n} \right), & \text{if } 1 \leq i \leq M - 2, \\
\frac{4n(\lambda + \mu)}{\pi} \left( \cos \frac{\pi}{2n} - \cos^2 \frac{3\pi}{4n} \right) - \frac{2n\mu}{\pi} \left[ f_2(\alpha) \right] \\
- 2f_4(\alpha) \left( \cot \frac{\pi}{n} \frac{\pi}{n} - i \cos \frac{\pi}{n} \frac{\pi}{n} \right), & \text{if } i = M - 1, \\
\frac{4n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{2n} \frac{\pi}{n} - \frac{2n\mu}{\pi} \left[ 4f_3(\alpha) - f_7(\alpha) \right], & \text{if } i = M \text{ and } n = 3, \\
\frac{4n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{2n} \frac{\pi}{n} - \frac{2n\mu}{\pi} \left[ f_3(\alpha) - 4f_5(\alpha) \sin \frac{\pi}{n} \frac{\pi}{n} \right] \\
- f_4(\alpha) \left\{ (n-5) \sin \frac{\pi}{2n} \frac{\pi}{n} + 2 \csc \frac{\pi}{n} \cos \frac{\pi}{2n} \right\}, & \text{if } i = M \text{ and } n \geq 5, \\
\frac{4n\mu}{\pi} \left[ 2f_6(\alpha) \sin \frac{(i-M)\pi}{n} + 2f_5(\alpha) \sin \frac{(i+1-M)\pi}{n} \right] \\
- f_4(\alpha) \left\{ (2M - i - 3) \cos \frac{i\pi}{n} \frac{\pi}{n} \right\}, & \text{if } M + 1 \leq i \leq 2M - 3, \\
\frac{n\mu}{2n} \left[ 16f_6(\alpha) \cos \frac{3\pi}{2n} \frac{\pi}{n} + f_7(\alpha) \right], & \text{if } i = 2M - 2 \text{ and } n \geq 5, \\
\frac{n\mu}{2n} f_8(\alpha), & \text{if } i = 2M - 1, \\
\frac{n\mu}{2n} f_9(\alpha), & \text{if } i = 2M, 
\end{cases}
\]

where

\[
f_0(\alpha) = 2\left[ \frac{\pi}{n} \cos \alpha + g(\frac{\pi}{n} - \alpha) + h(\alpha) \right] \cos^2 \frac{\pi}{2n} \frac{\pi}{n}, \\
f_1(\alpha) = \left[ \frac{\pi}{n} \cos \alpha + g(\frac{\pi}{n} - \alpha) + h(\alpha) \right] \sin \frac{\pi}{n} \frac{\pi}{n}, \\
f_2(\alpha) = \left[ \frac{2\pi}{n} \cos \frac{\pi}{n} \frac{\pi}{n} \cos \alpha + g(\frac{3\pi}{2n} - \alpha) - g(\frac{\pi}{2n} - \alpha) + h(\frac{\pi}{2n} + \alpha) \right] \sin \frac{\pi}{n} \frac{\pi}{n}, \\
f_3(\alpha) = g(\frac{\pi}{2n} - \alpha) \sin \frac{\pi}{n} \frac{\pi}{n}, \\
f_4(\alpha) = \left[ \frac{\pi}{n} \cos \alpha + g(\frac{\pi}{n} - \alpha) + h(\alpha) \right] \sin^2 \frac{\pi}{2n} \frac{\pi}{n}, \\
f_5(\alpha) = \left[ \frac{2\pi}{n} \cos \frac{\pi}{n} \frac{\pi}{n} \cos \alpha + g(\frac{3\pi}{2n} - \alpha) - g(\frac{\pi}{2n} - \alpha) + h(\frac{\pi}{2n} + \alpha) \right] \sin^2 \frac{\pi}{2n} \frac{\pi}{n}, \\
f_6(\alpha) = g(\frac{\pi}{2n} - \alpha) \sin^2 \frac{\pi}{2n} \frac{\pi}{n}, \\
f_7(\alpha) = \frac{\pi}{n} (3 - 2 \cos \frac{2\pi}{n} \frac{\pi}{n} \cos \alpha - g(\frac{3\pi}{2n} - \alpha) + 3g(\frac{2\pi}{n} - \alpha) - g(\frac{\pi}{n} - \alpha) \\
+ 7h(\alpha) - h(\frac{\pi}{n} + \alpha) - h(\frac{2\pi}{n} + \alpha) 
\]
\[ f_8(\alpha) = -\frac{\pi}{n} \cos \alpha - g\left(\frac{2\pi}{n} - \alpha\right) + 2g\left(\frac{\pi}{n} - \alpha\right) - 4h(\alpha) + h\left(\frac{\pi}{n} + \alpha\right), \]
\[ f_9(\alpha) = \frac{\pi}{n} \cos \alpha - g\left(\frac{\pi}{n} - \alpha\right) + 3h(\alpha) \]

with
\[ g(x) = \sin x + \alpha \cos x \quad \text{and} \quad h(x) = \sin x - \alpha \cos x. \]

**Proof.** We denote by \( w(\phi) \) the width of \( S_{n, \ell} \) (with angle \( \phi \)) perpendicular to the lines of \( R_a \) (see Fig. 2 (left side) and Fig. 3), and by \( s(k, \phi) \) the breadth functions of exactly \( k \), \( k \in \{1, 2, \ldots, M\} \), intersections between \( S_{n, \ell} \) (with angle \( \phi \)) and \( R_a \). \( s(k, \phi) \) is the breadth of one stripe or the sum of the breadths of two stripes. An example of \( s(3, \phi) \) for \( S_{9, \ell} \) is shown on the
right side of Fig. 2. Here, \( s(3, \phi) \) is the sum of the breadths \( b_1 = b_1(\phi) \) and \( b_2 = b_2(\phi) \). From the symmetry of \( S_{n, \ell} \), it follows that \( w \) and \( s(k, \cdot) \) are \( \pi/n \)-periodic functions. In the following, we have to consider these functions in the half-open intervals:

\[
I_1 := \left[ 0, \frac{\pi}{2n} \right), \quad I_2 := \left[ \frac{\pi}{2n}, \frac{\pi}{n} \right) \quad \text{and} \quad I_3 := \left[ \frac{\pi}{n}, \frac{3\pi}{2n} \right).
\]

The required restrictions of the function \( w \) are given by

\[
w_{12}(\phi) := w|_{I_1 \cup I_2}(\phi) = 2\ell \cos \frac{\pi}{2n} \cos \left( \phi - \frac{\pi}{2n} \right),
\]

\[
w_{3}(\phi) := w|_{I_3}(\phi) = 2\ell \cos \frac{\pi}{2n} \cos \left( \phi - \frac{3\pi}{2n} \right).
\]

For \( s(k, \cdot) \) and \( 1 \leq k \leq M - 2 \), one finds

\[
s_{12}(k, \phi) := s|_{I_1 \cup I_2}(k, \phi) = 4\ell \sin \frac{k\pi}{n} \sin \frac{\pi}{2n} \cos \left( \phi - \frac{\pi}{2n} \right),
\]

\[
s_{3}(k, \phi) := s|_{I_3}(k, \phi) = 4\ell \sin \frac{k\pi}{n} \sin \frac{\pi}{2n} \cos \left( \phi - \frac{3\pi}{2n} \right),
\]

for \( k = M - 1 \),

\[
s_1(k, \phi) := s|_{I_1}(k, \phi) = \ell \left[ 2 \cos \frac{\pi}{2n} \sin \phi - \sin \left( \phi - \frac{3\pi}{2n} \right) \right],
\]

\[
s_2(k, \phi) := s|_{I_2}(k, \phi) = \ell \left[ -2 \cos \frac{\pi}{2n} \sin \left( \phi - \frac{\pi}{n} \right) + \sin \left( \phi + \frac{\pi}{2n} \right) \right],
\]

\[
s_3(k, \phi) := s|_{I_3}(k, \phi) = \ell \left[ 2 \cos \frac{\pi}{2n} \sin \left( \phi - \frac{\pi}{n} \right) - \sin \left( \phi - \frac{3\pi}{2n} \right) \right],
\]

and for \( k = M \),

\[
s_1(k, \phi) = s|_{I_1}(k, \phi) = -\ell \sin \left( \phi - \frac{\pi}{2n} \right),
\]

\[
s_2(k, \phi) = s|_{I_2}(k, \phi) = \ell \sin \left( \phi - \frac{3\pi}{2n} \right),
\]

\[
s_3(k, \phi) = s|_{I_3}(k, \phi) = -\ell \sin \left( \phi - \frac{\pi}{2n} \right).
\]

\( E_{k, m}, 0 \leq k, m < M \), denotes the event that \( S_{n, \ell} \) has exactly \( k \) intersections with \( R_a \) and (at the same time) exactly \( m \) intersections with \( R_b \). For fixed value of \( \phi \), this event occurs if the centre point of \( S_{n, \ell} \) is in one, two or four disjunct parallelograms that are subsets of \( F \). (An example is shown in Fig. 3.) For the given angle \( \phi \), the event \( E_{k, 2} \) occurs if the centre point of \( S_{n, \ell} \) is in one of the four hatched parallelograms.) \( s(k, \phi) s(m, \phi + \alpha) / \sin \alpha \) is the area of the one parallelogram or the sum of the areas of the two or four parallelograms if \( 1 \leq k, m < M \). Therefore, the conditional probability of the event \( E_{k, m} \) for fixed angle \( \phi \) is given by

\[
P(E_{k, m} \mid \phi) = \frac{s(k, \phi) s(m, \phi + \alpha) / \sin \alpha}{\text{Area } F} = \frac{1}{ab} s(k, \phi) s(m, \phi + \alpha).
\]

For \( 0 \leq k, m < M \), we have

\[
P(E_{0, 0} \mid \phi) = \frac{1}{ab} \left[ a - w(\phi) \right] \left[ b - w(\phi + \alpha) \right],
\]

For 0 ≤ k, m < M, we have

\[
P(E_{0, 0} \mid \phi) = \frac{1}{ab} \left[ a - w(\phi) \right] \left[ b - w(\phi + \alpha) \right],
\]
\[ P(E_{0,m} | \phi) = \frac{1}{ab} [a - w(\phi)] s(m, \phi + \alpha), \]
\[ P(E_{k,0} | \phi) = \frac{1}{ab} s(k, \phi) [b - w(\phi + \alpha)]. \]

The density function of the random variable \( \phi \) is given by
\[
f(\phi) = \begin{cases} 
\frac{n}{\pi} & \text{if } \phi \in I_1 \cup I_2, \\
0 & \text{if } \phi \in \mathbb{R} \setminus I_1 \cup I_2.
\end{cases}
\]

Therefore, the (total) probability of the event \( E_{k,m} \) is given by
\[
P(E_{k,m}) = \int_{\pi/n}^{\pi/n} P(E_{k,m} | \phi) f(\phi) d\phi = \frac{n}{\pi} \int_{\pi/n}^{\pi/n} P(E_{k,m} | \phi) d\phi.
\]

From the piecewise definition of the functions \( w \) and \( s (k, \cdot) \), it follows that we have to distinguish (in general) the cases
\[ 0 \leq \phi < \frac{\pi}{2n} - \alpha, \quad \frac{\pi}{2n} - \alpha \leq \phi < \frac{\pi}{2n}, \quad \frac{\pi}{2n} < \phi < \frac{\pi}{n} - \alpha, \quad \frac{\pi}{n} - \alpha \leq \phi < \frac{\pi}{n}. \]

We calculate the probabilities \( P(E_{k,m}) \) in some examples. For \( k = 0 \) and \( 1 \leq m \leq M - 2 \), we get
\[
P(E_{0,m}) = \frac{n}{\pi ab} \left( \int_{0}^{\pi/n-\alpha} + \int_{\pi/n-\alpha}^{\pi/n} \right) [a - w(\phi)] s(m, \phi + \alpha) d\phi
\[
= \frac{n}{\pi ab} \left( \int_{0}^{\pi/n-\alpha} [a - w_{12}(\phi)] s_{12}(m, \phi + \alpha) d\phi
\[
+ \int_{\pi/n-\alpha}^{\pi/n} [a - w_{12}(\phi)] s_{3}(m, \phi + \alpha) d\phi \right)
\[
= \frac{8n\mu}{\pi} \sin^{2} \frac{\pi}{2n} \sin \frac{m\pi}{n} - \frac{2n\lambda\mu}{\pi} \sin \frac{m\pi}{n} f_{1}(\alpha).
\]

Due to symmetry, for \( 1 \leq k \leq M - 2 \) and \( m = 0 \), we get
\[
P(E_{k,0}) = \frac{8n\lambda}{\pi} \sin^{2} \frac{\pi}{2n} \sin \frac{k\pi}{n} - \frac{2n\lambda\mu}{\pi} \sin \frac{k\pi}{n} f_{1}(\alpha).
\]

For \( 1 \leq k, m \leq M - 2 \), we find
\[
P(E_{k,m}) = \frac{n}{\pi ab} \left( \int_{0}^{\pi/n-\alpha} + \int_{\pi/n-\alpha}^{\pi/n} \right) s(k, \phi) s(m, \phi + \alpha) d\phi
\[
= \frac{8n\lambda\mu}{\pi} \sin \frac{k\pi}{n} \sin \frac{m\pi}{n} f_{4}(\alpha),
\]

\[ \text{6} \]
for $k = M - 1$ and $m = M$,

$$P(E_{M-1,M}) = \frac{n}{\pi a b} \left( \int_0^{\frac{n}{\pi} - \alpha} + \int_{\frac{n}{\pi} - \alpha}^{\frac{n}{2\pi} + \alpha} + \int_{\frac{n}{2\pi} - \alpha}^{\frac{n}{\pi} + \alpha} + \int_{\frac{n}{\pi} + \alpha}^\infty \right) s(M - 1, \phi)$$

$$\times s(M, \phi + \alpha) \, d\phi = \frac{n\lambda \mu}{2\pi} f_8(\alpha),$$

and due to symmetry, $P(E_{M,M-1}) = P(E_{M-1,M})$.

The remaining calculations deliver the results

$$P(E_{k,m}) = 1 - \left( \frac{2n(\lambda + \mu)}{\pi} \sin \frac{\pi}{n} - \frac{n\lambda \mu}{\pi} f_0(\alpha) \right), \quad (k = 0 = m),$$

$$P(E_{k,m}) = \frac{4n\lambda}{\pi} \left( \cos \frac{\pi}{2n} - \cos^2 \frac{3\pi}{4n} \right) - \frac{n\lambda \mu}{\pi} f_2(\alpha), \quad (k = 0, m = M - 1),$$

$$P(E_{k,m}) = 4n\lambda \mu \sin \frac{k\pi}{n} f_3(\alpha), \quad (k = 0, m = M),$$

$$P(E_{k,m}) = 4n\lambda \mu \sin \frac{m\pi}{n} f_5(\alpha), \quad (1 \leq k \leq M - 2, m = M - 1),$$

$$P(E_{k,m}) = 4n\lambda \mu \sin \frac{m\pi}{n} f_5(\alpha), \quad (k = M - 1, 1 \leq m \leq M - 2),$$

$$P(E_{k,m}) = 4n\lambda \mu \sin \frac{k\pi}{n} f_6(\alpha), \quad (1 \leq k \leq M - 2, m = M),$$

$$P(E_{k,m}) = 4n\lambda \mu \sin \frac{m\pi}{n} f_0(\alpha), \quad (k = M, 1 \leq m \leq M - 2),$$

$$P(E_{k,m}) = \frac{n\lambda \mu}{2\pi} f_7(\alpha), \quad (k = M - 1 = m),$$

$$P(E_{k,m}) = \frac{n\lambda \mu}{2\pi} f_9(\alpha), \quad (k = M = m).$$

The probabilities $p(i)$ of exactly $i$ intersections between $S_{n,\ell}$ und $R_{a,b,\alpha}$ are given by

$$p(i) = \begin{cases} 
\sum_{k=0}^{i} P(E_{k,i-k}) & \text{for } 0 \leq i \leq M, \\
\sum_{k=i-M}^M P(E_{k,i-k}) & \text{for } M + 1 \leq i \leq 2M.
\end{cases}$$

We have

$$p(0) = P(E_{0,0}) = 1 - \left( \frac{2n(\lambda + \mu)}{\pi} \sin \frac{\pi}{n} - \frac{n\lambda \mu}{\pi} f_0(\alpha) \right).$$
For $1 \leq i \leq M - 2$, one finds

$$p(i) = P(E_{0,i}) + P(E_{i,0}) + \sum_{k=1}^{i-1} P(E_{k,i-k})$$

\[
= 8n(\lambda + \mu) \frac{\pi}{2} \frac{\sin^2 \frac{\pi}{2n} i \frac{\pi}{n} - 4n\lambda\mu f_1(\alpha) \sin \frac{i \pi}{n}}{\pi} + \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=1}^{i-1} \sin \frac{k\pi}{n} \frac{(i-k)\pi}{n}
\]

with

\[
\sum_{k=1}^{i-1} \sin \frac{k\pi}{n} \frac{(i-k)\pi}{n} = \frac{1}{2} \left( \cot \frac{\pi}{n} \sin \frac{i \pi}{n} - i \cos \frac{i \pi}{n} \right).
\]

For $i = M - 1$, we get

$$p(i) = P(E_{0,M-1}) + P(E_{M-1,0}) + \sum_{k=1}^{M-2} P(E_{k,i-k})$$

\[
= 4n(\lambda + \mu) \left( \cos \frac{\pi}{2n} - \cos^2 \frac{3\pi}{4n} \right) - \frac{2n\lambda\mu}{\pi} f_2(\alpha) + \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=1}^{i-1} \sin \frac{k\pi}{n} \frac{(i-k)\pi}{n}
\]

with the sum as above. For $i = M = 2$ and $n = 3$, we find

$$p(M) = P(E_{0,M}) + P(E_{M,0}) + P(E_{M-1,M-1})$$

\[
= \frac{4n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{4n} - \frac{2n\lambda\mu}{\pi} f_3(\alpha) + \frac{8n\lambda\mu}{\pi} f_5(\alpha) \sin \frac{\pi}{n},
\]

and for $i = M$ and $n \geq 5$,

$$p(M) = P(E_{0,M}) + P(E_{M,0}) + P(E_{1,M-1}) + P(E_{M-1,1}) + \sum_{k=2}^{M-2} P(E_{k,i-k})$$

\[
= \frac{4n(\lambda + \mu)}{\pi} \sin^2 \frac{\pi}{4n} - \frac{2n\lambda\mu}{\pi} f_3(\alpha) + \frac{8n\lambda\mu}{\pi} f_5(\alpha) \sin \frac{\pi}{n} + \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=2}^{M-2} \sin \frac{k\pi}{n} \frac{(M-k)\pi}{n}
\]

with

\[
\sum_{k=2}^{M-2} \sin \frac{k\pi}{n} \frac{(M-k)\pi}{n} = \frac{1}{2} \left( -(M-3) \cos \frac{M\pi}{n} + \csc \frac{\pi}{n} (M-3) \frac{\pi}{n} \right)
\]

\[
= \frac{1}{4} \left( (n-5) \sin \frac{\pi}{2n} + 2 \csc \frac{\pi}{n} \cos \frac{5\pi}{2n} \right).
\]
For the case $M + 1 \leq i \leq 2M - 3$, we put $i = 2M - \nu$. So we have to consider all $\nu$ with $3 \leq \nu \leq M - 1$. One finds

$$p(2M - \nu) = \sum_{k=(2M-\nu)-M}^{M} P(E_{k, 2M-\nu-k}) = \sum_{k=M-\nu}^{M} P(E_{k, 2M-\nu-k})$$

$$= P(E_{M-\nu, M}) + P(E_{M, M-\nu}) + P(E_{M-(\nu-1), M-1}) + \sum_{k=M-(\nu-2)}^{M-2} P(E_{k, 2M-\nu-k})$$

$$= \frac{8n\lambda\mu}{\pi} f_6(\alpha) \frac{\sin (M-\nu)\pi}{n} + \frac{8n\lambda\mu}{\pi} f_5(\alpha) \frac{\sin [M - (\nu - 1)]\pi}{n}$$

$$+ \frac{8n\lambda\mu}{\pi} f_4(\alpha) \sum_{k=M-(\nu-2)}^{M-2} \frac{\sin k\pi}{n} \frac{\cos (2M - \nu - k)\pi}{n}$$

with

$$\sum_{k=M-(\nu-2)}^{M-2} \frac{\sin k\pi}{n} \frac{\cos (2M - \nu - k)\pi}{n}$$

$$= \frac{1}{2} \left( -(\nu - 3) \cos \frac{(2M - \nu)\pi}{n} + \csc \frac{\pi}{n} \sin \frac{(\nu - 3)\pi}{n} \right),$$

and therefore, with $\nu = 2M - i$,

$$p(i) = \frac{4n\lambda\mu}{\pi} \left[ 2f_6(\alpha) \frac{i-M)\pi}{n} + 2f_5(\alpha) \frac{(i+1-M)\pi}{n}$$

$$- f_4(\alpha) \left( 2M - i - 3 \cos \frac{i\pi}{n} - \csc \frac{\pi}{n} \sin \frac{(2M - i - 3)\pi}{n} \right) \right].$$

For $i = 2M - 2$ and $n \geq 5$, we get

$$p(2M - 2) = \sum_{k=M-2}^{M} P(E_{k, (2M-2)-k})$$

$$= P(E_{M-2, M}) + P(E_{M, M-2}) + P(E_{M-1, M-1})$$

$$= \frac{8n\lambda\mu}{\pi} f_6(\alpha) \frac{(M - 2)\pi}{n} + \frac{n\lambda\mu}{2\pi} f_7(\alpha)$$

$$= \frac{n\lambda\mu}{2\pi} \left( 16f_6(\alpha) \cos \frac{3\pi}{2n} + f_7(\alpha) \right).$$

Furthermore, we find

$$p(2M - 1) = \sum_{k=M-1}^{M} P(E_{k, (2M-1)-k}) = P(E_{M-1, M}) + P(E_{M, M-1})$$

$$= \frac{n\lambda\mu}{\pi} f_8(\alpha)$$

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and finally
\[ p(2M) = P(E_{M,M}) = \frac{n\lambda \mu}{2\pi} f_\theta(\alpha). \]
So, the proof is complete. \(\square\)

In the following, we write \(p(i, \alpha)\) instead of \(p(i)\) and \(P_\alpha(E_{k,m})\) instead of \(P(E_{k,m})\) to emphasize the dependence on \(\alpha\).

**Theorem 2.** For fixed values of odd \(n \geq 3\), \(a, b\) and \(\ell\), the function
\[ p(i, \cdot) : [0, \pi/2] \to [0, 1], \quad \alpha \mapsto p(i, \alpha) \]
is \(\pi/n\)-periodic. The restriction \(p|_{[0, \pi/n]}\) is symmetric in relation to the line \(\alpha = \pi/(2n)\).

**Proof.** The functions \(w\) and \(s(k, \cdot), 1 \leq k \leq M\), are \(\pi/n\)-periodic. For \(1 \leq k, m \leq M\) and \(\nu \in \mathbb{Z}\) we get
\[
\begin{align*}
P_{\alpha + \nu \pi/n}(E_{k,m}) &= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi) s(m, \phi + \nu \pi/n) \, d\phi \\
&= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi) s(m, \phi + \alpha) \, d\phi = P_\alpha(E_{k,m}).
\end{align*}
\]
This result holds for all values of \(k\) and \(m\), \(0 \leq k, m \leq M\). Since \(p(i, \alpha)\) is a sum of \(\pi/n\)-periodic functions, it is \(\pi/n\)-periodic.
\(s(k, \phi) s(m, \phi + \alpha)\) are \(\pi/n\)-periodic functions. Hence
\[
\begin{align*}
P_\alpha(E_{k,m}) &= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi) s(m, \phi + \alpha) \, d\phi \\
&= \frac{n}{\pi ab} \int_{-\alpha}^{\pi/n-\alpha} s(k, \phi) s(m, \phi + \alpha) \, d\phi \\
&= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi - \alpha) s(m, \phi) \, d\phi,
\end{align*}
\]
and therefore, with \(\nu \in \mathbb{Z}\),
\[
\begin{align*}
P_{\nu \pi/n-\alpha}(E_{k,m}) &= \frac{n}{\pi ab} \int_0^{\pi/n} s \left( k, \phi - \left( \nu \frac{\pi}{n} - \alpha \right) \right) s(m, \phi) \, d\phi \\
&= \frac{n}{\pi ab} \int_0^{\pi/n} s(k, \phi + \alpha) s(m, \phi) \, d\phi = P_\alpha(E_{m,k}).
\end{align*}
\]
For \(1 \leq k, m \leq M\), it follows that
\[
\begin{align*}
P_{\nu \pi/n-\alpha}(E_{k,k}) &= P_\alpha(E_{k,k}), \\
P_{\nu \pi/n-\alpha}(E_{k,m}) + P_{\nu \pi/n-\alpha}(E_{m,k}) &= P_\alpha(E_{k,m}) + P_\alpha(E_{m,k}).
\end{align*}
\]
Analogously, one gets
\[ P_{\nu \pi/n - \alpha}(E_0, 0) = P_{\alpha}(E_0, 0), \]
\[ P_{\nu \pi/n - \alpha}(E_0, m) + P_{\nu \pi/n - \alpha}(E_m, 0) = P_{\alpha}(E_0, m) + P_{\alpha}(E_m, 0). \]

With \( \nu = 1 \), we have
\[ p(0, \pi/n - \alpha) = P_{\pi/n - \alpha}(E_0, 0) = P_{\alpha}(E_0, 0) = p(0, \alpha), \]
\[ p(2M, \pi/n - \alpha) = P_{\pi/n - \alpha}(E_M, M) = P_{\alpha}(E_M, M) = p(2M, \alpha). \]

For \( 1 \leq i \leq 2M - 1 \), we find: If \( i \) is odd, \( p(i, \alpha) \) is the sum of terms \( P_{\alpha}(E_k, i - k) + P_{\alpha}(E_i - k, k) \). If \( i \) is even, \( p(i, \alpha) \) is the sum of terms \( P_{\alpha}(E_k, i - k) + P_{\alpha}(E_i - k, k) \) and one term \( P(E_{i/2}, i/2) \).

So, for every \( i \), \( 0 \leq i \leq 2M \), we have \( p(i, \pi/n - \alpha) = p(i, \alpha) \); therefore, the restriction \( p(i, \alpha) \mid [0, \pi/n] \) is symmetric in relation to the line \( \alpha = \pi/(2n) \). \( \square \)

From Theorem 2 one easily gets the following corollary:

**Corollary 1.** The probabilities \( p(i, \alpha) \) for \( 0 < \alpha \leq \pi/2 \) are given by

\[
p(i, \alpha) = \begin{cases} 
  p(i, \alpha - \delta(\alpha)) & \text{if } \alpha - \delta(\alpha) \leq \frac{\pi}{2n}, \\
  p\left( i, \frac{\pi}{n} - [\alpha - \delta(\alpha)] \right) & \text{if } \alpha - \delta(\alpha) > \frac{\pi}{2n}
\end{cases}
\]

with
\[
\delta(\alpha) = \left\lfloor \frac{n\alpha}{\pi} \right\rfloor \frac{\pi}{n}.
\]

\( p(0, \alpha) \) is strictly decreasing for \( 0 < \alpha < \frac{\pi}{2n} \), which can be seen as follows:

We denote by \( f_0^*(\alpha) \) the restriction of \( f_0(\alpha) \) to the interval \([0, \pi/n]\). It may be written as
\[
f_0^*(\alpha) = 2 \cos^2 \frac{\pi}{2n} \left[ \sin \alpha + \sin \left( \frac{\pi}{n} - \alpha \right) + \alpha \cos \left( \frac{\pi}{n} - \alpha \right) + \left( \frac{\pi}{n} - \alpha \right) \cos \alpha \right].
\]

One finds
\[
f_0^*'(\alpha) = \frac{d}{d\alpha} f_0^*(\alpha) = 2 \cos^2 \frac{\pi}{2n} \left[ \alpha \sin \left( \frac{\pi}{n} - \alpha \right) - \left( \frac{\pi}{n} - \alpha \right) \sin \alpha \right]
\]
and hence
\[
\frac{f_0^*'(\alpha)}{\sin \alpha \sin \left( \frac{\pi}{n} - \alpha \right)} = 2 \cos^2 \frac{\pi}{2n} \left( \frac{\alpha}{\sin \alpha} - \frac{\frac{\pi}{n} - \alpha}{\sin \left( \frac{\pi}{n} - \alpha \right)} \right).
\]

For \( 0 < \alpha \leq \frac{\pi}{2n} \), we have \( \alpha \leq \frac{\pi}{n} - \alpha \), and therefore,
\[
\frac{\alpha}{\sin \alpha} - \frac{\frac{\pi}{n} - \alpha}{\sin \left( \frac{\pi}{n} - \alpha \right)} \leq 0.
\]
It follows that $f_0^*(\alpha) \leq 0$ and hence $p(0, \alpha) \leq 0$ in $0 < \alpha \leq \frac{\pi}{2n}$, where the equality signs hold only if $\alpha = \frac{\pi}{2n}$. Due to the symmetry of $p(0, \alpha)$ (see Theorem 2), $p(0, \alpha)$ is strictly increasing in $\frac{\pi}{2n} < \alpha < \frac{\pi}{n}$. Therefore, the probability of at least one intersection is strictly increasing in $0 < \alpha < \frac{\pi}{2n}$ and strictly decreasing in $\frac{\pi}{2n} < \alpha < \frac{\pi}{n}$ (cp. [4]).

Due to its additivity, the expectation $\sum_{i=0}^{2M} i p(i, \alpha)$ of the number of intersections is always given by $2n(\lambda + \mu)/\pi$.

3 Special cases

The probability of at least one intersection is given by

$$\frac{2n(\lambda + \mu)}{\pi} \sin \frac{\pi}{n} - \frac{n\lambda\mu}{\pi} f_0(\alpha).$$

This is one result of Theorem 2.1 in [4].

For $\mu = 0$ one gets the result for one lattice $\mathcal{R}_a$ of parallel lines in [2, pp. 17-18].

Fig. 4 . . . 9 show diagrams with the intersection probabilities $p(i, \alpha)$, $0 \leq \alpha \leq \pi/n$, for $\lambda = 1/3$, $\mu = 1/4$ and $n = 5$.

Using the formulas in Theorem 1, we get the following approximate expressions in the case $n = 5$:

$$p(0, \alpha) \approx 1 - 0.87098(\lambda + \mu) + c_0\lambda\mu$$
$$p(1, \alpha) \approx 0.71465(\lambda + \mu) - c_1\lambda\mu$$
$$p(2, \alpha) \approx 1.00054(\lambda + \mu) - c_2\lambda\mu$$
$$p(3, \alpha) \approx 0.155792(\lambda + \mu) + c_3\lambda\mu$$
$$p(i, \alpha) \approx c_i\lambda\mu, \quad i = 4, 5, 6,$$

with

$c_0 = 3.50133$, $c_1 = 2.67478$, $c_2 = 3.23888$, $c_3 = 0.854102$, $c_4 = 1.23316$, $c_5 = 0.292814$, $c_6 = 0.032254$

if $\alpha = 0, \pi/5, 2\pi/5$, and

$c_0 = 3.49988$, $c_1 = 2.67367$, $c_2 = 3.22768$, $c_3 = 0.840122$, $c_4 = 1.21437$, $c_5 = 0.330696$, $c_6 = 0.0162876$

if $\alpha = \pi/10, 3\pi/10, \pi/2$.

From the calculation of many special cases, we conjecture that $p(i, \alpha)$ is strictly increasing in $0 < \alpha < \frac{\pi}{2n}$ if $i \in \{1, \ldots, M-1\}$ or $i = 2M-1$, and strictly decreasing in this interval if $i \in \{M, \ldots, 2M-2\}$ or $i = 2M$. 

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Figure 4: $p(1, \alpha)$

Figure 5: $p(2, \alpha)$

Figure 6: $p(3, \alpha)$

Figure 7: $p(4, \alpha)$

Figure 8: $p(5, \alpha)$

Figure 9: $p(6, \alpha)$
4 Distribution functions

In the following, let \( X_{n, \alpha} \) denote the ratio
\[
\frac{\text{number of intersections between } S_{n, \ell} \text{ and } R_{a, b, \alpha}}{n}
\]
(short: relative number of intersections) and \( F_{n, \alpha} : \mathbb{R} \to [0, 1] \) the distribution function of \( X_{n, \alpha} \).

\[
F_{n, \alpha}(\xi) = P(X_{n, \alpha} \leq \xi) = \begin{cases} 
0 & \text{for } -\infty < \xi < 0, \\
\sum_{i=0}^{\lfloor n\xi \rfloor} p(i, \alpha) & \text{for } 0 \leq \xi < \frac{2M}{n}, \\
1 & \text{for } \frac{2M}{n} \leq \xi < \infty.
\end{cases}
\]

We put
\[
X_{n, \lambda} := \frac{\text{number of intersections between } S_{n, \ell} \text{ and } R_{a}}{n} \quad \text{and} \quad X_{n, \mu} := \frac{\text{number of intersections between } S_{n, \ell} \text{ and } R_{b}}{n}.
\]

In the case of the independence of \( X_{n, \lambda} \) and \( X_{n, \mu} \), the distribution function \( F_n \) of \( X_n := X_{n, \lambda} + X_{n, \mu} \) is given by

\[
F_n(\xi) = P(X_n \leq \xi) = \begin{cases} 
0 & \text{for } -\infty < \xi < 0, \\
\sum_{i=0}^{\lfloor n\xi \rfloor} \sum_{k=0}^{i} p_{\lambda}(k) p_{\mu}(i-k) & \text{for } 0 \leq \xi < \frac{2M}{n}, \\
1 & \text{for } \frac{2M}{n} \leq \xi < \infty,
\end{cases}
\]

where
\[
p_{\lambda}(i) := \begin{cases} 
p(i, \alpha), & \text{if } 0 \leq i \leq M, \\
0, & \text{if } M+1 \leq i \leq 2M,
\end{cases}
\]
if \( \mu = 0 \) and \( \lambda \neq 0 \), and
\[
p_{\mu}(i) := \begin{cases} 
p(i, \alpha), & \text{if } 0 \leq i \leq M, \\
0, & \text{if } M+1 \leq i \leq 2M,
\end{cases}
\]
if \( \lambda = 0 \) and \( \mu \neq 0 \).

The horizontal lines in the diagrams in Fig. 4 . . . 9 show the values of the probabilities
\[
p^*(i) = \sum_{k=0}^{i} p_{\lambda}(k) p_{\mu}(i-k).
\]
The question arises if an angle $\alpha$ exists such that $F_n \equiv F_{n, \alpha}$. The calculation of many examples shows that it is (in general) not possible to find such a value of $\alpha$ for finite $n$. Therefore, $X_{n, \lambda}$ and $X_{n, \mu}$ are (in general) dependent random variables.

The random variables $X_{n, \lambda}$ and $X_{n, \mu}$ converge uniformly to the random variables $X_\lambda$ with distribution function

$$F_\lambda(\xi) = \begin{cases} 
0 & \text{for } -\infty < \xi < 0, \\
1 - 2\lambda \cos \pi \xi & \text{for } 0 \leq \xi < \frac{1}{2}, \\
1 & \text{for } \frac{1}{2} \leq \xi < \infty,
\end{cases}$$

and $X_\mu$ with distribution function

$$F_\mu(\xi) = \begin{cases} 
0 & \text{for } -\infty < \xi < 0, \\
1 - 2\mu \cos \pi \xi & \text{for } 0 \leq \xi < \frac{1}{2}, \\
1 & \text{for } \frac{1}{2} \leq \xi < \infty,
\end{cases}$$

respectively (see [2, p. 24]). If $X_\lambda$ and $X_\mu$ are independent, the distribution of $X_\lambda + X_\mu$ can be calculated with the convolution

$$F(\xi) = P(X_\lambda + X_\mu \leq \xi) = \int_{-\infty}^{\xi} F_\lambda(\xi - \eta) \, dF_\mu(\eta) \quad \text{see [6, p. 90]},$$

which yields

$$F(\xi) = \begin{cases} 
0 & \text{for } -\infty < \xi < 0, \\
1 - 2(\lambda + \mu) \cos \pi \xi & \text{for } 0 \leq \xi < \frac{1}{2}, \\
-2\lambda\mu(\pi \xi \sin \pi \xi - 2 \cos \pi \xi) & \text{for } 0 \leq \xi < \frac{1}{2}, \\
1 - 2\lambda\mu(1 - \xi) \sin \pi \xi & \text{for } \frac{1}{2} \leq \xi < 1, \\
1 & \text{for } 1 \leq \xi < \infty,
\end{cases}$$

(1)

(cf. [5]). The following theorem shows that $F$ is not only the distribution function of the sum $X_\lambda + X_\mu$ but also of the random variable $X := \lim_{n \to \infty} X_{n, \alpha}$. Therefore, $X_{n, \lambda}$ and $X_{n, \mu}$ are asymptotically independent.

**Theorem 3.** As $n \to \infty$, the random variables $X_{n, \alpha}$ converge to the random variable $X$ whose distribution function is given by formula (1).

**Proof.** For fixed coordinates $(x, y)$ of the centre point of $S_{n, \ell}$, the relative number of intersections tends to $\xi = (\sigma + \tau)/(2\pi)$ as $n \to \infty$ (see Fig. 11). The outer parallelogram is the fundamental cell $F$. $\sigma = \sigma(x, y)$ is the angle of possible intersections with $R_a$, and $\tau = \tau(x, y)$ the angle of possible intersections with $R_b$. 

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At first we consider the situation for fixed value of $\xi$ with $0 \leq \xi < 1/2$. The relative number of intersections is equal to $\xi$ if the centre point of $S_{\alpha, \ell}$ with $n \to \infty$ lies on the boundary curve of the set $F^* \subset F$; it is $< \xi$ if the centre point lies inside $F^*$. ($F^*$ is the inner parallelogram without the four grey coloured sets in its corners.) We denote by $A_1$ and $A_2$ the areas of $F_1$ and $F_2$ respectively. Therefore, the limit distribution is given by

$$F(\xi) = \frac{\text{Area} \ F^*(\xi)}{\text{Area} \ F} = \frac{(a - 2 \ell \cos \pi \xi)(b - 2 \ell \cos \pi \xi)/\sin \alpha - 2(A_1 + A_2)}{ab/\sin \alpha}$$

$$= \frac{[ab - 2(a + b) \cos \pi \xi - 4 \ell^2 \cos^2 \pi \xi] - 2(A_1 + A_2) \sin \alpha}{ab}$$

$$= 1 - 2(\lambda + \mu) \cos \pi \xi + 4 \lambda \mu \cos^2 \pi \xi - \frac{2(A_1 + A_2) \sin \alpha}{ab}.$$

In the following, we need the equations of the lines $G_1, \ldots, G_4$. They are respectively defined in Hesse normal form by

$$G_1 = \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha = 0\},$$

$$G_2 = \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha - \ell \cos \pi \xi = 0\},$$

$$G_3 = \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha - (a - \ell \cos \pi \xi) = 0\},$$

$$G_4 = \{(x, y) \in \mathbb{R}^2 \mid x \sin \alpha - y \cos \alpha - a = 0\}.$$

The subset $F_1 \subset F$ is given by

$$F_1 = \{(x, y) \in \mathbb{R}^2 \mid \ell \cos \pi \xi \leq y \leq \ell, \ g_2(y) \leq x \leq f_1(y)\},$$

where

$$g_2(y) = \frac{1}{\sin \alpha} \left(y \cos \alpha + \ell \cos \pi \xi\right)$$
is the equation of \( G_2 \), and \( f_1(y) \) the equation of the curve \( C_1 \). We get the equation of \( C_1 \) from

\[
\xi = \frac{\sigma + \tau}{2\pi} = \frac{1}{\pi} \left( \arccos \frac{x \sin \alpha - y \cos \alpha}{\ell} + \arccos \frac{y}{\ell} \right)
\]

which yields

\[
f_1(y) = \frac{1}{\sin \alpha} \left[ \ell \cos \left( \pi \xi - \arccos \frac{y}{\ell} \right) + y \cos \alpha \right]; \tag{2}
\]

therefore,

\[
f_1(y) - g_2(y) = \frac{\ell}{\sin \alpha} \left[ \cos \left( \pi \xi - \arccos \frac{y}{\ell} \right) - \cos \pi \xi \right].
\]

So the area of \( F_1 \) is given by

\[
A_1 = \int_{\ell \cos \pi \xi}^{\ell} \left[ f_1(y) - g_2(y) \right] dy
= \frac{\ell}{\sin \alpha} \int_{\ell \cos \pi \xi}^{\ell} \cos \left( \pi \xi - \arccos \frac{y}{\ell} \right) dy - \frac{\ell \cos \pi \xi}{\sin \alpha} \int_{\ell \cos \pi \xi}^{\ell} dy =: I
\]

We calculate the integral \( I \). With the substitution \( u = y/\ell \), one finds

\[
I = \ell \int_{\cos \pi \xi}^{1} \cos(\pi \xi - \arccos u) du
= \ell \left[ \cos \pi \xi \int_{\cos \pi \xi}^{1} \cos(\arccos u) du + \sin \pi \xi \int_{\cos \pi \xi}^{1} \sin(\arccos u) du \right]
= \ell \left[ \cos \pi \xi \int_{\cos \pi \xi}^{1} u du + \sin \pi \xi \int_{\cos \pi \xi}^{1} \sqrt{1 - u^2} du \right]
= \ell \left[ \frac{u^2 \cos \pi \xi}{2} + \left( u \sqrt{1 - u^2} + \arcsin u \right) \sin \pi \xi \right]_{\cos \pi \xi}^{1} = \ell \frac{2}{\pi \xi} \pi \xi \sin \pi \xi
\]

and hence

\[
A_1 = \frac{\ell^2}{2 \sin \alpha} \left( \pi \xi \sin \pi \xi - 2 \cos \pi \xi + 2 \cos^2 \pi \xi \right).
\]

Now we calculate the area \( A_2 \) of

\[
\mathcal{F}_2 = \{(x, y) \in \mathbb{R}^2 \mid \ell \cos \pi \xi \leq y \leq \ell, \ f_2(y) \leq x \leq g_3(y)\},
\]

where

\[
g_3(y) = \frac{1}{\sin \alpha} (a + y \cos \alpha - \ell \cos \pi \xi)
\]
is the equation of the line $G_3$, and $f_2(y)$ the equation of the curve $C_2$. One gets the equation of $C_2$ from

$$
\xi = \frac{\sigma + \tau}{2\pi} = \frac{1}{\pi} \left( \arccos \frac{-x \sin \alpha - y \cos \alpha - a}{\ell} + \arccos \frac{y}{\ell} \right)
$$

which gives

$$
f_2(y) = \frac{1}{\sin \alpha} \left[ a + y \cos \alpha - \ell \cos \left( \pi \xi - \arccos \frac{y}{\ell} \right) \right] , \quad (3)
$$

and hence

$$
g_3(y) - f_2(y) = \ell \frac{\cos \left( \pi \xi - \arccos \frac{y}{\ell} \right)}{\sin \alpha} - \cos \pi \xi = f_1(y) - g_2(y) .
$$

Due to Cavallieri’s principle, we have found that $A_2 = A_1$; therefore,

$$
\frac{2(A_1 + A_2) \sin \alpha}{ab} = \frac{2\ell^2(\pi \xi \sin \pi \xi - 2 \cos \pi \xi + 2 \cos^2 \pi \xi)}{ab} = 2\lambda \mu (\pi \xi \sin \pi \xi - 2 \cos \pi \xi + 2 \cos^2 \pi \xi)
$$

and

$$
F(\xi) = 1 - 2(\lambda + \mu) \cos \pi \xi + 4\lambda \mu \cos^2 \pi \xi - 2\lambda \mu \left( \pi \xi \sin \pi \xi - 2 \cos \pi \xi + 2 \cos^2 \pi \xi \right)
$$

$$
= 1 - 2(\lambda + \mu) \cos \pi \xi - 2\lambda \mu \left( \pi \xi \sin \pi \xi - 2 \cos \pi \xi \right).
$$

Figure 11: Calculation of $F$ for $\frac{1}{2} \leq \xi < 1$

Now we consider the situation for fixed value of $\xi$ with $\frac{1}{2} \leq \xi < 1$ (Fig. 11). The parallelogram is the fundamental cell $\mathcal{F}$. $\mathcal{F}^*$ is $\mathcal{F}$ without
the four grey coloured sets in its corners. The limit distribution is given by

$$F(\xi) = \frac{\text{Area} \mathcal{F}'(\xi)}{\text{Area} \mathcal{F}} = \frac{ab/\sin \alpha - 2(A_1' + A_2')}{ab/\sin \alpha} = 1 - \frac{2(A_1' + A_2') \sin \alpha}{ab},$$

where $A_1'$ and $A_2'$ are the areas of $\mathcal{F}_1'$ and $\mathcal{F}_2'$ respectively. The subset $\mathcal{F}_1' \subset \mathcal{F}$ is defined by

$$\mathcal{F}_1' = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \ell \sin \pi \xi, \ g_1(y) \leq x \leq f_1(y)\},$$

where $g_1(y) = y \cot \alpha$ is the equation of $G_1$, and $f_1(y)$ the equation of $C_1$ (see (2)). Here the upper limit for the variable $y$ is obtained from

$$\xi = \frac{1}{2} + \frac{\tau}{2\pi} = \frac{1}{2} + \frac{1}{\pi} \arccos \frac{y}{\ell} \quad \Rightarrow \quad y = \ell \sin \pi \xi.$$

So we have

$$f_1(y) - g_1(y) = y \cot \alpha + \frac{\ell}{\sin \alpha} \cos \left(\pi \xi - \arccos \frac{y}{\ell}\right) - y \cot \alpha$$

and

$$A_1' = \int_0^{\ell \sin \pi \xi} \left[f_1(y) - g_1(y)\right] dy = \frac{\ell}{\sin \alpha} \int_0^{\ell \sin \pi \xi} \cos \left(\pi \xi - \arccos \frac{y}{\ell}\right) dy.$$

Using the substitution $u = y/\ell$, we get

$$A_1' = \frac{\ell^2}{\sin \alpha} \int_0^{\sin \pi \xi} \cos(\pi \xi - \arccos u) \, du$$

$$= \frac{\ell^2}{2 \sin \alpha} \left[ u^2 \cos \pi \xi + u \sqrt{1 - u^2} \sin \pi \xi + \arcsin u \sin \pi \xi \right]_0^{\sin \pi \xi}$$

$$= \frac{\ell^2}{2 \sin \alpha} \left[ \sin^2 \pi \xi \cos \pi \xi + \sin \pi \xi \sqrt{\cos^2 \pi \xi} \sin \pi \xi + \arcsin(\sin \pi \xi) \sin \pi \xi \right].$$

From $\frac{1}{2} \leq \xi < 1$, it follows that $\cos \pi \xi \leq 0$ and $\arcsin(\sin \pi \xi) = \pi(1 - \xi)$; therefore,

$$A_1' = \frac{\ell^2}{2 \sin \alpha} \left[ -\sin^2 \pi \xi |\cos \pi \xi| + \sin^2 \pi \xi |\cos \pi \xi| + \pi(1 - \xi) \sin \pi \xi \right]$$

$$= \frac{\ell^2}{2 \sin \alpha} \pi(1 - \xi) \sin \pi \xi.$$

The subset $\mathcal{F}_2' \subset \mathcal{F}$ is defined by

$$\mathcal{F}_2' = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq \ell \sin \pi \xi, \ f_2(y) \leq x \leq g_4(y)\},$$

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where
\[ g_4(y) = \frac{1}{\sin \alpha} (a + y \cos \alpha) \]
is the equation of \( G_4 \), and \( f_2(y) \) the equation of \( C_2 \) (see (3)). We get
\[ g_4(y) - f_2(y) = \frac{\ell}{\sin \alpha} \cos \left( \pi \xi - \arccos \frac{y}{\ell} \right) = f_1(y) - g_1(y). \]

Due to Cavallieri’s principle, we have found that \( A'_2 = A'_1 \). Therefore,
\[ F(\xi) = \frac{ab/\sin \alpha - 4A}{ab/\sin \alpha} = 1 - \frac{4A \sin \alpha}{ab} = 1 - 2\lambda \mu \pi (1 - \xi) \sin \pi \xi, \]
and the proof is complete. \( \Box \)

Note the interesting fact that the limit distribution \( F \) is independent of the angle \( \alpha \)! It is the same limit distribution as for the distribution functions of corresponding clusters of needles (with equal values of \( \lambda \) and \( \mu \), respectively) [5, p. 221, Theorem 2].

The diagrams in Fig. 12 and Fig. 13 show for \( \lambda = 1/3 \) and \( \mu = 1/4 \) examples of distribution functions and the limit distribution \( F \).

The calculation of many special cases show (as the diagrams suggest) that it is most likely that the \( F_{n, \alpha} \) converge uniformly to \( F \).
Figure 12: $F_{7, \alpha}$, $\alpha = k\pi/7$, $k = 0, 1, \ldots, 3$, and $F$

Figure 13: $F_{25, \alpha}$, $\alpha = k\pi/25$, $k = 0, 1, \ldots, 12$, and $F$
References

[1] A. Aleman, M. Stoka, T. Zamfirescu: Convex bodies instead of needles in Buffon’s experiment, *Geometriae Dedicata* 67 (1997), 301-308.

[2] U. Bäsel: *Geometrische Wahrscheinlichkeiten für nichtkonvexe Testelemente*, Dissertation, FernUniversität Hagen, Hagen 2008.

[3] U. Bäsel: Geometrische Wahrscheinlichkeiten für Nadelsterne und Parallelogrammgitter, *Fernuniversität Hagen: Seminarberichte aus der Fakultät für Mathematik und Informatik* 83 (2010), 29-48.

[4] U. Bäsel: Buffon’s problem with regular polygons, *Beitr. Algebra Geom.* 53 No. 1 (2012), 247-259.

[5] U. Bäsel: Buffon’s problem with a cluster of line segments and a lattice of parallelograms, *Math. Commun.* 16 (2011), 215-225.

[6] J. Bellach, P. Franken, E. Warmuth, W. Warmuth: Maß, Integral und bedingter Erwartungswert, Akademie-Verlag, Berlin, 1978.

[7] V. Bonanzinga: Buffon’s problem with a 3-star and a lattice of parallelograms, *1st summer school quantitative methods for economic, agricultural-food and environmental sciences*, Castiglione di Sicilia, Italy, 22-24 September 2010. (unpublished)

[8] G. L. L. Buffon: *Essai d’arithmétique morale*, Appendix to ‘Histoire naturelle générale et particulière’, Vol. 4 (1777), 139-153.

[9] A. Duma, M. Stoka: Hitting probabilities for random ellipses and ellipsoids, *J. Appl. Prob.* 30 (1993), 971-974.

[10] P.-S. Laplace: *Théorie analytique des probabilités*, Courcier, Paris, 1812.

[11] D. Ren, G. Zhang: Random convex sets in a lattice of parallelograms, *Acta Math. Sci.* 11 (1991), 317-326.

[12] L. A. Santaló: Sur quelques problèmes de probabilités géométriques, *Tôhoku Math. J.* 47 (1940), 159-171.

[13] L. A. Santaló: *Integral Geometry and Geometric Probability*, Addison-Wesley, London, 1976.

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