Newtonian Gravity on an N-Sphere

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Abstract

We consider some elementary features of Newtonian gravity, or electrostatics, as defined on an N-sphere. In particular, we present and discuss “the shell theorem” for this system.

Introduction

The analogue of Newtonian gravity on a closed Riemannian manifold has been an interesting subject for quite a long time, especially among mathematicians (e.g. see [2] and the literature cited therein). The N-sphere presents a tractable example that can be analyzed without approximations (e.g. see [1] and the literature cited therein).

The physics of Newtonian gravity on the N-sphere is emphasized here. Alternatively, one may think of the following as electrostatics on the N-sphere. Either way, perhaps some insight and intuition can be gleaned from this simple model.

Taking this point of view, we follow a path not very far from that trodden by G. Green so long ago [5]. We identify the point-particle gravitational or electrostatic potentials with a Green function of the system, and we define the force on such particles in terms of the gradient of that potential, as is usual in physics. We then consider the analogue of Newton’s shell theorem for this simple system. We find a generalization of the shell theorem which is simple to state as well as pleasing to our taste.

N-Sphere Laplacian Green Function

On an N-sphere of radius \(R\), acting on functions \(f(\theta)\) with only polar angle dependence:

\[
\nabla^2 = \frac{1}{R^2} \left( \frac{1}{(\sin \theta)^{N-1}} \frac{\partial}{\partial \theta} \left( (\sin \theta)^{N-1} \frac{\partial}{\partial \theta} \right) \right)
\]

For small angles, \(\nabla^2 \approx \frac{1}{R^2} \left( \frac{1}{(\sin \theta)^{N-1}} \frac{\partial}{\partial \theta} \left( (\sin \theta)^{N-1} \frac{\partial}{\partial \theta} \right) \right) = \frac{1}{R^2} \left( \frac{\partial^2}{\partial \theta^2} + \frac{N-1}{\theta} \frac{\partial}{\partial \theta} \right)\). Also recall the “surface area” of the unit N-sphere \(S_N\) is \(A_N = 2\pi^{(N+1)/2}/\Gamma\left(\frac{N+1}{2}\right)\). This is the “total solid angle” for Euclidean space \(\mathbb{E}_{N+1}\).
A singular solution of the homogeneous equation

$$0 = \frac{1}{\sin^{N-1} \theta} \frac{\partial}{\partial \theta} \left( (\sin^{N-1} \theta) \frac{\partial}{\partial \theta} I(\theta, N) \right)$$  \hspace{1cm} (2)$$

is obviously given by the integral

$$I(\theta, N) = \int_{\pi/2}^{\theta} \frac{1}{(\sin \phi)^{N-1}} d\phi$$  \hspace{1cm} (3)$$

almost everywhere, except at the singularities \( \theta = 0 \) or \( \theta = \pi \). All \( I(N) \) for even \( N \) involve logarithms, but all \( I(N) \) for odd \( N \) do not.

Similarly, a singular solution of the inhomogeneous equation

$$1 = \frac{1}{\sin^{N-1} \theta} \frac{\partial}{\partial \theta} \left( (\sin^{N-1} \theta) \frac{\partial}{\partial \theta} J(\theta, N) \right)$$  \hspace{1cm} (4)$$

is just as obviously given almost everywhere by the double integral

$$J(\theta, N) = \int_{\pi/2}^{\theta} \frac{1}{(\sin \phi)^{N-1}} \left( \int_{\pi/2}^{\phi} (\sin \varphi)^{N-1} \right) d\phi \right) d\theta$$  \hspace{1cm} (5)$$

By combining these two solutions we arrive at explicit expressions for a Laplacian Green function on the \( N \)-sphere, in a form that exhibits the physics of the model in simple terms. The result is

$$G(\hat{r}, \hat{s}, N) = \frac{1}{2A_{N-1}} I(\arccos (\hat{r} \cdot \hat{s}) , N) - \frac{1}{A_N} J(\arccos (\hat{r} \cdot \hat{s}) , N)$$  \hspace{1cm} (6)$$

$$R^2 \nabla^2 G(\hat{r}, \hat{s}, N) = \delta^N (\hat{r} - \hat{s}) - \frac{1}{A_N}$$  \hspace{1cm} (7)$$

As usual in physics, \( G(\hat{r}, \hat{s}, N) \) is interpreted here as the potential at position \( \hat{r} \) produced by a unit point charge at position \( \hat{s} \).

It is straightforward to express the results (6) in terms of hypergeometric functions (4), which Mathematica will do without much coaxing, but the hypergeometric functions in question always reduce to combinations of elementary functions. (See the Appendix for \( N \leq 10 \).) Alternatively, the Green function can be written as a sum of bilinears in a complete set of hyperspherical harmonics (1) divided by their Laplacian eigenvalues, but excluding the zero mode solution (2), hence the \(-1/A_N \) term in (7)

The Dirac delta produced by acting with \( \nabla^2 \) at the “north pole” of the \( N \)-sphere is most easily exhibited by small angle expansions. For example:

$$G(\hat{r}, \hat{s} = \hat{z}, N = 2) = \frac{1}{4\pi} \ln (1 - \cos \theta) = \frac{\ln \theta}{2\pi} - \frac{1}{4\pi} \ln 2 - \frac{\theta^2}{48\pi} + O(\theta^4)$$  \hspace{1cm} (8)$$

$$G(\hat{r}, \hat{s} = \hat{z}, N = 3) = \frac{1}{4\pi^2} (\theta - \pi) \cot \theta = -\frac{1}{4\pi \theta} + \frac{1}{4\pi^2} + \frac{\theta}{12\pi} + O(\theta^2)$$  \hspace{1cm} (9)$$

etc. For \( N > 2 \) the leading term is always \(-1/((N - 2)A_N\theta^{N-2})\), thereby revealing the Dirac delta when the Laplacian acts as \( \frac{1}{R^2} \left( \frac{\partial^2}{\partial \sigma^2} + \frac{N-1}{\theta} \frac{\partial}{\partial \theta} \right) \) for small \( \theta \).
With $G(\widehat{r}, \widehat{s}, N)$ interpreted as a potential, then the gradient of $G$ is to be interpreted as a repulsive/attractive force exerted on a unit point charge/mass at position $\widehat{r}$ produced by an identical point charge/mass at position $\widehat{s}$. Placing a source point charge at the north pole of the N-sphere will produce a repulsive force on another, same sign point charge at position $(\theta, \cdots)$. In this particular situation, the repulsive force depends only on the polar angle $\theta$, is independent of all the other “azimuthal” angles, and has only a $\theta$ component as given by

$$F(\theta, N) = \frac{d}{d\theta} G(\theta, N) = \frac{1}{4\pi^{N/2}} \left( \Gamma\left(\frac{N}{2}\right) - \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{N + 1}{2}\right) K(\theta, N) \right) \frac{1}{(\sin\theta)^{N-1}}$$

(10)

where $K(\theta, N) = \int_{\pi/2}^{\theta} (\sin\phi)^{N-1} d\phi = (\sin\theta)^{N-1} \frac{d}{d\theta} J(\theta, N)$. For example:

$$F(\theta, 2) = \frac{1 + \cos\theta}{4\pi \sin\theta}$$

$$F(\theta, 3) = \frac{\pi - \theta + \cos\theta \sin\theta}{4\pi^3 (\sin^3 \theta)}$$

The result for $F(\theta, N)$ evinces Gauss’ law on the N-sphere, as Faraday would surely have realized. The field line flux through any $S_{N-1}$ subsphere at polar angle $\theta$, of radius $R \sin \theta$, is $\Phi = \int F(\theta, N) R^{N-1} \sin^{N-1} \theta d^{N-1}\Omega = F(\theta, N) R^{N-1} A_{N-1} \sin^{N-1} \theta$. This flux
is proportional to the total charge on the hyperspherical cap $S_{N|<\theta}$. Thus $\Phi$ decreases monotonically as $\theta$ increases for $0 < \theta < \pi$, and goes to zero as $\theta \to \pi$ corresponding to zero total charge on the entire hypersphere. For example:

$$\int F(\theta, 2) \sin \theta d\phi = \frac{1}{2} (1 + \cos \theta) = 1 - \int_0^\theta \frac{1}{4\pi} \sin \vartheta d\vartheta \int_0^{2\pi} d\phi$$

$$\int F(\theta, 3) \sin^2 \theta d^2\Omega = \frac{1}{\pi} \left( \pi - \theta + \cos \theta \sin \theta \right) = 1 - \int_0^\theta \frac{1}{2\pi^2} \sin^2 \vartheta d\vartheta \int d^2\Omega$$

etc. The “1” on the RHS is the Dirac delta contribution, while the integral is due to the uniform negative charge density on the hypersphere.

**The Shell Theorem**

An interesting physical feature of this N-sphere model is embodied in the “shell theorem” — or perhaps the “new shell theorem” would be a more appropriate description.

In Euclidean space the well-known shell theorem is the statement that an ideal uniform spherical shell of charge or mass will produce no force on a test particle placed within the shell. Actually, the theorem is true for general closed, charged, equipotential shells (somewhat obviously from the point of view of the uniqueness theorem) as was first proven for spheroidal shells by Isaac Newton, in the *Principia*, and much later for general ellipsoidal shells by James Ivory, remarkably before potential theory was invented.

However, a uniformly charged $S_{N-1}$ sub-sphere located on the N-sphere will exert force on a test particle located at almost all points on $S_N$. The only points on $S_N$ where the force on the test particle will be zero are the “two antipodal Faraday points” of the system, as well as all points on the uniformly charged $S_{N-1}$ itself. Note that the antipodal Faraday points (or “Far points” for brevity) may be defined as the two points on $S_N$ for which all points on the embedded $S_{N-1}$ are equidistant, with each Far point as far as possible from the sub-sphere as well as from each other. These two Faraday points maximize, locally, their respective distances from the $S_{N-1}$ sub-sphere. (Of course, to specify these Far points for the N-sphere, distances may be computed either intrinsically or else extrinsically, if in the latter case both the $S_{N-1}$ and the $S_N$ manifolds are canonically embedded in $\mathbb{E}_{N+1}$.)

For such a uniformly charged or massive $S_{N-1} \subset S_N$, the shell theorem is supplanted by the following (*new shell theorem*):

**Proposition 1** The force on any test particle placed on $S_N \setminus S_{N-1}$ is the same as would result if all the uniformly distributed charge or mass on $S_{N-1}$ were moved away from the test particle and relocated entirely at the opposing Faraday point.

To clarify the last point, note that the sub-sphere partitions $S_N \setminus S_{N-1}$ into two disjoint regions, with one region containing the test particle and the other containing the “opposing”

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1More generally, on a generically curved, closed Riemannian manifold, the interior of a charged, closed, equipotential submanifold is not necessarily at a constant potential, despite what one might naively expect from harmonic function theory. Recall the potential, identified as a Green function, is not necessarily harmonic in the interior of the submanifold due to terms analogous to the $-1/A_N$ in [7].
Faraday point. Thus the position of the test particle determines which of the Far points is
the opposing one.

So stated, the spherical shell theorem in Euclidean space becomes a special case of the
Proposition. If a test particle is placed outside a uniformly charged \( S_{N-1} \) spherical shell
embedded in \( \mathbb{E}_N \), the force on that test particle is the same as though all the charge were
concentrated at the center of the shell, as is well-known. On the other hand, if the test
particle is placed inside the shell, there is no force, but that is exactly the same null result
that would be obtained if all the charge on the shell were moved out to the “point at infinity”.
In this case, the center of the charged shell and the point at infinity play the roles of the
antipodal Faraday points for the system. That is to say, the Euclidean space shell theorem
for \( \mathbb{E}_N \) results from taking the radius of \( S_N \) to infinity while keeping fixed the radius of the
charged \( S_{N-1} \) sub-sphere.

All this is most easily visualized and verified mathematically for the 2-sphere.

The Simplicity of \( N = 2 \)

For a 2-sphere of radius \( R \), let \( \vec{r} \) and \( \vec{s} \) be two points on the sphere, so \( r = R = s \). Then
the standard Green function for the Helmholtz equation is

\[
G_{\text{Helmholtz}} (\vec{r}, \vec{s}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm} (\hat{r}) Y_{lm}^* (\hat{s})}{k^2 - l (l + 1) / R^2} = \sum_{l=0}^{\infty} \frac{2l + 1}{4\pi} \frac{P_l (\hat{r} \cdot \hat{s})}{k^2 - l (l + 1) / R^2}
\]

(13)

\[
(\nabla^2 + k^2) G_{\text{Helmholtz}} (\vec{r}, \vec{s}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm} (\hat{r}) Y_{lm}^* (\hat{s}) = \delta^2 (\hat{r} - \hat{s})
\]

(14)

where \( \hat{r} \) has direction \((\theta, \phi)\), \( \hat{s} \) has direction \((\vartheta, \varphi)\), and \( \hat{r} \cdot \hat{s} = \cos \Theta \), with

\[
\cos \Theta = \cos \theta \cos \vartheta + \sin \theta \cos \phi \sin \vartheta \cos \varphi + \sin \theta \sin \phi \sin \vartheta \sin \varphi
\]

(15)

and \( \delta^2 (\hat{r} - \hat{s}) = \frac{1}{|\sin \Theta|} \delta (\Theta) \delta (\Phi) \). Note that \( \sqrt{1 - \cos \Theta} \) is proportional to the length
of the chord (not the arc length) that connects the two points on the sphere. Unfortunately,
\textit{Mathematica} cannot readily carry out the sum over \( l \) to obtain a simple closed form for
\( G_{\text{Helmholtz}} \), even after regularizing the sum through use of an \( i\epsilon \) prescription.

This is in contrast to the Laplacian Green function on the 2-sphere. In this case \textit{Mathematica}
can perform the relevant sum.

\[
G_{\text{Laplace}} (\vec{r}, \vec{s}) = \text{constant} + \sum_{NB \ l=1}^{\infty} \sum_{m=-l}^{l} \frac{Y_{lm} (\hat{r}) Y_{lm}^* (\hat{s})}{-l (l + 1) / R^2} = \text{constant} + \sum_{l=1}^{\infty} \frac{2l + 1}{4\pi} \frac{P_l (\cos \Theta)}{-l (l + 1) / R^2} = \text{constant} + \frac{R^2}{4\pi} \left( \ln \left( 1 - \cos \Theta \right) + 1 - \ln 2 \right)
\]

(16)

Note the issue with the \( l = 0 \) mode (for example, see \textit{https://en.wikipedia.org/wiki/Laplace-Beltrami_operator}).
It is convenient to choose the constant to be \( \frac{R^2}{4\pi} (-1 + \ln 2) \) such that

\[
G_{\text{Laplace}} (\hat{r}, \hat{s}) = \frac{R^2}{4\pi} \ln (1 - \cos \Theta)
\] (17)

Also note that interchange of the Laplacian with the sum gives

\[
\nabla^2 G_{\text{Laplace}} (\hat{r}, \hat{s}) = \sum_{\text{NB} \ l=1}^{\infty} \sum_{m=-l}^{l} Y_{lm} (\hat{r}) Y_{lm}^* (\hat{s}) = \delta^2 (\hat{r} - \hat{s}) - \frac{1}{4\pi}
\] (18)

That is to say, \( \int_{S_2} \nabla^2 G_{\text{Laplace}} (\hat{r}, \hat{s}) d^2r = 0 \), no matter what the location of \( \hat{s} \in S_2 \). With the interpretation that \( \nabla^2 G_{\text{Laplace}} \propto \sigma \), the surface charge (or mass) density, the fact that \( \nabla^2 G_{\text{Laplace}} \) integrates to zero means the total charge (or mass) on the sphere is always zero (cf. Faraday’s field line interpretation). This fact is quite curious and may be surprising when first encountered. And while it is certainly not a big deal for electric charge, it is rather more provocative for mass. It remains to be seen if this fact requires modification of some long-established ways of thinking about cosmology in a Newtonian framework [3].

In any case, we will define the position dependent Newtonian gravitational potential energy on the sphere, between point charges at \( \hat{r}_1 \) and \( \hat{r}_2 \), with masses \( m_1 \) and \( m_2 \), to be

\[
V (\hat{r}_1, \hat{r}_2) = \kappa m_1 m_2 \ln (1 - \hat{r}_1 \cdot \hat{r}_2)
\] (19)

where \( \kappa = G/R \) is the “Newtonian gravitational constant on the sphere” with units for \( G \) chosen to agree with those in 3D Euclidean space.

2-Sphere Shell theorem

Now it should be more or less obvious that the proof of the 2D Euclidean space shell theorem, as given by Newton, does not work on the 2-sphere.

For example, put a uniform ring of total mass \( M \) but negligible thickness on a circle of latitude specified by \( \vartheta \). The Faraday points of this system are obviously the north and south poles of the sphere. Next, consider the force on an ideal point test particle, of mass \( m \), located at \((\theta, \phi)\). The uniform mass density of the ring is \( \lambda = M / (2\pi R \sin \vartheta) \) since its radius is \( R \sin \vartheta \), thus \( dM = \frac{1}{2\pi} M d\phi \) for azimuthal angular segments of the ring. Therefore the net force on the test particle is

\[
\vec{F} (\theta, \phi) = F_{\theta} \hat{\theta} + F_{\phi} \hat{\phi} = \frac{\kappa m M}{2\pi R} \int_{0}^{2\pi} \left( -\frac{\partial}{\partial \theta} \ln (1 - \hat{r}_1 \cdot \hat{r}_2) - \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \ln (1 - \hat{r}_1 \cdot \hat{r}_2) \right) d\phi
\] (20)

Clearly the \( F_{\phi} \) component will integrate to zero, by symmetry, but perhaps less obviously, for \( 0 \leq \theta \leq \pi \),

\[
f (\theta, \vartheta) \equiv 2\pi R F_{\theta} / (\kappa m M)
\]

\[
= 2\pi \left( \frac{1}{1 + \cos \theta} \text{Heaviside} (\vartheta - \theta) - \frac{1}{1 - \cos \theta} \text{Heaviside} (\theta - \vartheta) \right) \sin \theta
\] (21)
In other words, there is in general an attractive force on the test particle, due to this ring of total mass $M$, and that force is the same as the force which would be produced by a point charge of mass $M$, located at the Faraday point in the spherical region on the other side of the ring, opposite the test particle. That is to say,

$$f(\theta, \vartheta) = f(\theta, \pi) \text{Heaviside}(\vartheta - \theta) - f(\theta, 0) \text{Heaviside}(\theta - \vartheta)$$  \hspace{1cm} (22)$$

To help visualize this new shell theorem, we plot $f(\theta, \vartheta)$ for various ring latitudes, along with the force curves produced by point particles at either north (i.e. $f(\theta, 0)$) or south (i.e. $f(\theta, \pi)$) poles of the sphere. First, an equatorial ring.

$f(\theta, \pi/2)$ in red, compared to $f(\theta, 0) = 2\pi \frac{\sin \theta}{\cos \theta - 1}$ in green & $f(\theta, \pi) = 2\pi \frac{\sin \theta}{\cos \theta + 1}$ in blue.

Next, a ring placed at the Tropic of Cancer, latitude $23^\circ 26' 10.8'' = 23.43632^\circ$ N.

$f(\theta, \theta_{\text{Cancer}})$ in red, compared to $f(\theta, 0) = 2\pi \frac{\sin \theta}{\cos \theta - 1}$ in green & $f(\theta, \pi) = 2\pi \frac{\sin \theta}{\cos \theta + 1}$ in blue.

Note the discontinuity in $f$ at the ring’s latitude is $4\pi/\sin \vartheta$. 

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Generalizing all this to higher N-spheres is now too obvious to warrant further discussion here. But again, it remains to be seen if this requires modification of some long-established ways of thinking about cosmology in a Newtonian framework [3].

Miscellaneous Additional Notes for the 2-Sphere

The perspicacious reader will have noticed that there is an obvious solution to the inhomogeneous equation

\[
\frac{1}{\sin \theta} \partial_{\theta} \left( (\sin \theta) \frac{\partial}{\partial \theta} h(\theta) \right) = -\frac{1}{4\pi}
\]

Namely,

\[
h(\theta) = \frac{1}{4\pi} \ln (\sin \theta) + \text{const} \tag{24}
\]

Combining with the previous result,

\[
\frac{1}{4\pi} \ln (1 - \cos \theta) - \frac{1}{4\pi} \ln (\sin \theta) = \frac{1}{4\pi} \ln \left( \tan \frac{\theta}{2} \right) \tag{25}
\]

Check for \(\theta \neq 0\):

\[
\frac{\partial}{\partial \theta} \left( (\sin \theta) \frac{\partial}{\partial \theta} \left( \frac{1}{4\pi} (\ln (\tan (\theta/2))) \right) \right) = 0, \text{ or at least this is true so long as } \theta \neq 0.
\]

Also check the Dirac delta:

\[
\int_{\text{cap at north pole}} \nabla^2 \ln \left( \tan \frac{\theta}{2} \right) d^2 \Omega = \frac{1}{R^2} \int_{\partial \text{cap}} \left( \frac{\partial}{\partial \theta} \ln \left( \tan \frac{\theta}{2} \right) \right) \sin \theta d\phi = \frac{2\pi}{R^2} \tag{26}
\]

upon using \(\frac{\partial}{\partial \theta} \ln \left( \tan \frac{\theta}{2} \right) \sin \theta = 1\). Therefore, \(\nabla^2 \ln \left( \tan \frac{\theta}{2} \right) = \frac{2\pi}{R^2} \delta^2 (\hat{z})\), or at least that is the case near the north pole of the sphere. What about the south pole?

\[
\int_{\text{cap at south pole}} \nabla^2 \ln \left( \tan \frac{\theta}{2} \right) d^2 \Omega = -\frac{1}{R^2} \int_{\partial \text{cap}} \left( \frac{\partial}{\partial \theta} \ln \left( \tan \frac{\theta}{2} \right) \right) \sin \theta d\phi = -\frac{2\pi}{R^2} \tag{27}
\]

So there is another point source at the south pole. That is to say,

\[
\nabla^2 \ln \left( \tan \frac{\theta}{2} \right) = \frac{2\pi}{R^2} \left( \delta^2 (\hat{z}) - \delta^2 (-\hat{z}) \right) = \frac{4\pi}{R^2} \delta^2 (\hat{z}) - \frac{2\pi}{R^2} \left( \delta^2 (\hat{z}) + \delta^2 (-\hat{z}) \right) \tag{28}
\]

Subtracting the \(\ln (\sin \theta)\) term cancelled the uniform density on the sphere due to the \(\ln (1 - \cos \theta)\) term, but in addition, the \(\ln (\sin \theta)\) term contributed equal strength Dirac deltas at both the north and the south poles, such that the overall contribution of \(\ln (\sin \theta)\) to the total charge was still zero. Hence, the modification

\[
G (\vec{r}, \vec{s}) = \frac{R^2}{4\pi} \ln \left( \tan \left( \frac{1}{2} \arccos (\vec{r} \cdot \vec{s}) \right) \right) \tag{29}
\]

yields a Green function whose source is only Dirac deltas, but again with total charge zero.

\[
\nabla^2 G (\vec{r}, \vec{s}) = \frac{1}{2} \delta^2 (\vec{r} - \vec{s}) - \frac{1}{2} \delta^2 (\vec{r} + \vec{s}) \propto \sigma_{\text{total}} \quad \text{with} \quad \int_{S^2} \sigma_{\text{total}} d^2 r = 0 \tag{30}
\]
Yet another way to look at the \( \ln (\sin \theta) \) term is to write
\[
\ln (\sin \theta) = \frac{1}{2} \ln (\sin^2 \theta) = \frac{1}{2} \ln (1 - \cos^2 \theta) = \frac{1}{2} (\ln (1 - \cos \theta) + \ln (1 + \cos \theta))
\]  \hspace{1cm} (31)
which reveals the two Dirac deltas more immediately.

Alternatively, one could just as well subtract from the \( \ln (1 - \hat{r} \cdot \hat{s}) \) term a contribution due to a single, equal strength Dirac delta at any other location. For example, if the original point source is at the north pole of the sphere, put an opposite sign point source at the south pole, to obtain
\[
\ln \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) = \ln \left( \frac{\sin^2 \theta / 2}{\cos^2 \theta / 2} \right) = 2 \ln \left( \tan \frac{\theta}{2} \right)
\]  \hspace{1cm} (32)
Thus, if \( \theta \neq 0 \) and \( \theta \neq \pi \), once again check: \[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( (\sin \theta) \frac{\partial}{\partial \theta} \ln \left( \tan \frac{\theta}{2} \right) \right) = 0.
\]
Moreover,
\[
G (\hat{r}, \hat{s}) = \frac{R^2}{2\pi} \ln \left( \tan \left( \frac{1}{2} \arccos (\hat{r} \cdot \hat{s}) \right) \right), \quad \nabla^2 G (\hat{r}, \hat{s}) = \delta^2 (\hat{r} - \hat{s}) - \delta^2 (\hat{r} + \hat{s})
\]  \hspace{1cm} (33)
In fact, an arbitrary number of Dirac deltas, representing point particles with various strengths at various locations on \( S_2 \), will eliminate the constant charge density provided all the Dirac delta charges sum to zero. Similar remarks apply for other \( S_N \).

**Conclusion**

We identified Newtonian and Coulombic potentials, and forces, with Green functions on \( S_N \), and their gradients. We then considered the analogue of Newton’s shell theorem for this simple system. We found a tasteful generalization of the shell theorem which is simple to state. We believe a similar shell theorem can be established for spheroidal and ellipsoidal if not more general closed manifolds, as we intend to show in subsequent work.

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**Appendix**

The general integral expressions for the Green functions on \( S_N \), as given by (6) in the text, are not appropriate for the circle, \( S_1 \), when \( \theta \) is allowed to be negative with \( -\pi \leq \theta \leq \pi \). In that case \( G \) is best obtained from first principles, with the result
\[
G (\hat{r}, \hat{s}) = \frac{R^2}{2\pi} \ln \left( \tan \left( \frac{1}{2} \arccos (\hat{r} \cdot \hat{s}) \right) \right), \quad \nabla^2 G (\hat{r}, \hat{s}) = \delta^2 (\hat{r} - \hat{s}) - \delta^2 (\hat{r} + \hat{s})
\]  \hspace{1cm} (33)
The \( \theta^2 \) term eliminates a Dirac delta at \( \theta = \pi \), implicit in the second derivative of the \( |\theta| \) term, by ensuring the slope of \( G (\theta) \) is continuous at the south pole of the circle, i.e. \( \lim_{\theta \to \pm \pi} dG (\theta) / d\theta = 0 \). Thus
\[
\frac{d^2}{d\theta^2} G (\theta) = \delta (\theta) - \frac{1}{2\pi}
\]
Otherwise, the general integral expressions in the text are easily evaluated for specific integer \( N \geq 2 \). Here we list \( G(\hat{r}, \hat{s} = \hat{z}, N) \) for \( 2 \leq N \leq 10 \), as computed by Maple directly from those integral expressions.

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 2) = \frac{1}{4\pi} \ln (1 - \cos \theta)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 3) = \frac{\cos \theta}{4\pi^2 \sin \theta} (\theta - \pi)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 4) = \frac{1}{8\pi^2} \left( \ln (1 - \cos \theta) - \frac{\cos \theta}{1 - \cos \theta} \right)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 5) = \frac{\cos \theta}{8\pi^3 \sin^3 \theta} \left( 3 (\theta - \pi) + 2 (\pi - \theta) \cos^2 \theta - \sin \theta \cos \theta \right)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 6) = \frac{1}{16\pi^3} \left( 3 \ln (1 - \cos \theta) + \frac{(4 \cos \theta - 5) \cos \theta}{(1 - \cos \theta)^2} \right)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 7) = \frac{\cos \theta}{16\pi^4 \sin^4 \theta} \left( (\theta - \pi) \left( 3 + 4 \sin^2 \theta + 8 \sin^4 \theta \right) - 3 \cos \theta \sin \theta \left( 1 + 2 \sin^2 \theta \right) \right)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 8) = \frac{1}{32\pi^4} \left( 15 \ln (1 - \cos \theta) - \frac{\cos \theta}{(1 - \cos \theta)^3} \left( 33 - 54 \cos \theta + 23 \cos^2 \theta \right) \right)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 9) = \frac{\cos \theta}{32\pi^5 \sin^5 \theta} \left( 3 (\theta - \pi) \left( 5 + 6 \sin^2 \theta + 8 \sin^4 \theta + 16 \sin^6 \theta \right) - \sin \theta \cos \theta \left( 15 + 28 \sin^2 \theta + 44 \sin^4 \theta \right) \right)
\]

\[
G(\hat{r}, \hat{s} = \hat{z}, N = 10) = \frac{1}{64\pi^5} \left( 105 \ln (1 - \cos \theta) + \frac{\cos \theta}{(1 - \cos \theta)^4} \left( 176 \cos^3 \theta - 599 \cos^2 \theta + 696 \cos \theta - 279 \right) \right)
\]

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