Compact Hankel operators with bounded symbols

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Abstract

We discuss the compactness of Hankel operators on Hardy, Bergman and Fock spaces with focus on the differences between the three cases, and complete the theory of compact Hankel operators with bounded symbols on the latter two spaces with standard weights. In particular, we give a new proof (using limit operator techniques) of the result that the Hankel operator $H_f$ is compact on Fock spaces if and only if $\overline{H_f}$ is compact. Our proof fully explains that this striking result is caused by the lack of nonconstant bounded analytic functions in the complex plane (unlike in the other two spaces) and extends the result from the Fock-Hilbert space to all Fock-Banach spaces. As in Hardy spaces, we also show that the compactness of Hankel operators is independent of the underlying space in the other two cases.

1 Introduction

Hankel operators $H_f$ form one of the most important classes of bounded linear operators with various applications in several areas of analysis, such as function theory, harmonic analysis, moment problems, asymptotic analysis, spectral theory, orthogonal polynomials, random matrix theory and mathematical physics. The most important settings include Hardy, Bergman and Fock spaces. In Hardy spaces, their importance is often realized through their matrix representations, which makes them suitable for many applications, see, e.g. Widom’s proof of Szegő’s strong limit theorem in [24] for the 2D Ising model. In the other two cases, their study is particularly important in connection with problems in quantum mechanics and several complex variables. Another important aspect about Hankel operators is their use in the study

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of Toeplitz operators $T_f$, which goes back to the fundamental paper [10] of Gohberg and Krein. Indeed, Widom’s identity

$$T_fT_g = T_{fg} - H_fH_g$$

makes this connection crystal clear, as explained in [24], and it naturally leads to the question of compactness (and Schatten class membership) of Hankel operators.

Hankel operators in their various forms have been extensively studied since 1957, when Nehari [17] showed that $H_f$ defined as the matrix $(f_{j+k+1})_{j,k \geq 0}$ acting on $\ell^2$ is bounded if and only if there is a function $g \in L^\infty(\mathbb{T})$ on the unit circle $\mathbb{T}$ such that $\hat{g}(n) = f_n$ for $n \in \mathbb{N}$, where $\hat{g}(n)$ is the $n$th Fourier coefficient of $g$. In particular, Nehari’s result shows that Hankel matrices with bounded symbols are bounded. It is worth noting that there is no Nehari type result for Hankel matrices on $\ell^p$ with $1 < p < \infty$ (see Open Problem 2.12 of [7]).

In order to view $H_f$ as an operator on a function space, we recall the Hardy space $H^2(\mathbb{T}) = \{ f \in L^2(\mathbb{T}) : \hat{f}(k) = 0 \text{ for } k < 0 \}$, and denote by $P$ the orthogonal (Riesz) projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. The Hankel operator $H_f$ on the Hardy space is defined by

$$H_f = PM_fQJ,$$

where $Q = I - P$ and $Jf(t) = \bar{t}f(\bar{t})$ for $t \in \mathbb{T}$. We observe that for $j, k \geq 0$,

$$\langle H_f e_k, e_j \rangle = \langle fe_{-k-1}, P e_j \rangle = \langle f, e_{j+k+1} \rangle = \hat{f}(j+k+1),$$

which is the matrix representation of the Hankel operator.

In 1958, Hartman [13] characterized compactness of Hankel matrices and showed that $H_f$ is compact if and only if there is a continuous function $g$ such that $\hat{g}(n) = f_n$ for $n \in \mathbb{N}$. In terms of the spaces of functions of bounded and vanishing mean oscillation and their decompositions

$$BMO = \left\{ f \in L^1(\mathbb{T}) : \sup_{I \subseteq \mathbb{T}} \frac{1}{|I|} \int_I |f-f_I| < \infty \right\} = \left\{ u+Pv : u,v \in L^\infty(\mathbb{T}) \right\} \quad (1)$$

and

$$VMO = \left\{ f \in BMO : \lim_{\delta \rightarrow 0} \sup_{|I| < \delta} \frac{1}{|I|} \int_I |f-f_I| = 0 \right\} = \left\{ u+Pv : u,v \in C(\mathbb{T}) \right\}, \quad (2)$$

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where $I$ is an arc of $\mathbb{T}$ and $f_I = \frac{1}{|I|} \int_I f$, the results of Nehari and Hartman say that $H_f$ is bounded if and only if $P(f)$ is in BMO, and $H_f$ is compact if and only if $P(a)$ is in VMO. The same is true for Hankel operators on Hardy spaces $H^p$ when $1 < p < \infty$, and in fact, for the essential norm of $H_f$, it is known that

$$\text{dist}_\infty(f, C + H^\infty) \leq \|H_f\|_{\text{ess}} \leq c_p \text{dist}_\infty(f, C + H^\infty),$$

(3)

where $c_p$ is the norm of $P : L^p \to H^p$; for further details, see 2.54 in Section 2.10 of [7]. We note that there is no (analogous) characterization of compact Hankel matrices on $\ell^p$ (see Open Problem 2.56 of [7]).

Hankel operators have also been studied on many other function spaces, such as Bergman and Fock spaces (see [26, 27]), where their definition differs from the one used above for Hardy spaces. We denote by $X^p_\alpha(\Omega)$ the standard Bergman or Fock spaces of a given domain $\Omega$ (see Section 2 for the precise definitions of these spaces), and let $P_\alpha$ be the orthogonal projection of $L^2_\alpha(\Omega)$ onto $X^2_\alpha(\Omega)$. Then the Hankel operator $H_f : X^p_\alpha(\Omega) \to L^p_\alpha(\Omega)$ is defined by

$$H_f g = Q_\alpha(fg),$$

where $Q_\alpha = I - P_\alpha$ is the complimentary projection of $P_\alpha$. The lack of a general definition of Hankel operators in the literature may seem odd at first, especially for those familiar with Toeplitz operators, whose definition is universal, but in fact there is no compelling reason for having one—it naturally depends on the underlying space and the problem under consideration.

Most results about Hankel operators have been obtained first in the setting of the Hardy space before their treatment in Bergman and Fock spaces. Regarding compactness, while there are similarities, there are also striking differences between the properties of Hankel operators on these three function spaces. One major difference is that in the case of Hardy spaces most results are only available for functions on the circle $\mathbb{T}$, while for Hankel operators on Bergman and Fock spaces, most results are known at least for the unit ball $\mathbb{B}_n$ and $\mathbb{C}^n$, respectively.

In 1984, Axler posed the question of characterizing compact Hankel operators on the Bergman space $A^2(\mathbb{D})$ of the open unit disk $\mathbb{D}$, and two years later, in [2], he showed that when $f \in A^2(\mathbb{D})$, the Hankel operator $H_f$ (with a conjugate analytic symbol) is compact if and only if $f$ is in the little Bloch space; that is, $(1 - |z|^2)f'(z) \to 0$ as $|z| \to 1$. In the early 1990s, Stroethoff and Zheng independently for $\Omega = \mathbb{D}$ and jointly for $\Omega = \mathbb{B}_n$ (see [21] and the references therein) showed that when $f \in L^\infty(\Omega)$, the Hankel operator $H_f$ is compact on $A^2(\Omega)$ if and only if

$$\|Q(f \circ \varphi_\lambda)\|_q \to 0 \text{ as } \lambda \to \partial \Omega$$

(4)
for some (or, equivalently, for all) $1 < q < \infty$.

In the present paper, we deal with bounded symmetric domains $\Omega$, which form a natural generalization of the open unit disk, and in particular include the unit ball in $\mathbb{C}^n$ (see Section 2 for further details on these domains). However, our results are new even in the setting of the open unit disk $\mathbb{D}$.

As in the context of the Hardy space, we can define the space of functions of vanishing mean oscillation $\text{VMO}^p(\Omega)$ analogously to (2) by replacing $I$ with $B(z, r) = \{w \in \Omega : \beta(z, w) < r\}$, where $\beta$ is the Bergman metric on $\Omega$. More precisely, we say that $f \in \text{VMO}^p_r(\Omega)$ with $r > 0$ if

$$\lim_{z \to \partial \Omega} \frac{1}{|B(z, r)|} \int_{B(z, r)} |f(w) - \hat{f}_r(z)|^p \, dv(z) = 0,$$

where $\hat{f}_r(z)$ denotes the Euclidean average of $f$ over $B(z, r)$ and $|B(z, r)|$ denotes the Euclidean volume of $B(z, r)$. It is well known that $\text{VMO}^p_r(\Omega)$ is independent of $r$ (see [25]); we set $\text{VMO}^p(\Omega) = \text{VMO}^p_1(\Omega)$.

In 1987, Zhu [25] proved that $H_f$ and $H_{\bar{f}}$ are simultaneously compact on $A^p_{\alpha}(\mathbb{B}_n)$ if and only if $f \in \text{VMO}^p(\mathbb{B}_n)$ (see [5] for a generalization to bounded symmetric domains when $p = 2$). It is worth noting here that, as in the case of Hardy spaces, there are bounded symbols $f$ for which $H_f$ is compact on $A^p(\mathbb{B}_n)$ but $H_{\bar{f}}$ is not compact. An interesting example is a Blaschke product $b$ with zeros at $\alpha_k = 1 - 1/2^k$; that is,

$$b(z) = \prod_{k=1}^{\infty} \alpha_k - z \frac{|\alpha_k|}{1 - \bar{\alpha}_k z} \alpha_k$$

for $z \in \mathbb{D}$. The function $b$ is not in the little Bloch space (see [1]) and hence $H_b$ is not compact by Axler’s result above. However, $H_b = 0$ because $b$ is a bounded analytic function in $\mathbb{D}$.

We now compare the situation with Hankel operators on Fock spaces. The result on the simultaneous compactness of $H_f$ and $H_{\bar{f}}$ is the same as in Bergman spaces, which was first proved for Hankel operators on $F^2_{\alpha}(\mathbb{C}^n)$ by Bauer [3] in 2005. For an extension to $1 < p < \infty$, see [14, 19]. Here $\text{VMO}^p(\mathbb{C}^n)$ is defined analogously to $\text{VMO}^p(\Omega)$ by replacing the Bergman metric with the Euclidean metric (see [19]). Regarding (4), Stroethoff [20] showed that the same condition is both sufficient and necessary for $H_f$ to be compact on $F^2(\mathbb{C}^n)$ (but curiously only for $q = 2$ unlike in the Bergman space where he had done it for all $1 < q < \infty$).

What is very different about Hankel operators on Fock spaces is that, for bounded symbols, $H_f$ is compact if and only if $H_{\bar{f}}$ is compact. This was proved for $p = 2$ by
Berger and Coburn [6] in 1987 and by Stroethoff [20] in 1992 using more elementary methods.

As for the reason, recently Zhu [27] commented, “A partial explanation for this difference is probably the lack of bounded analytic or harmonic functions on the entire complex plane.” This naturally suggests that Berger and Coburn’s result should remain true for the other Fock spaces $F^p_\alpha$ with $1 < p < \infty$. However, all the previously used techniques seem unsuitable when $p \neq 2$.

In what follows, we will give an alternate proof of the result of Berger and Coburn using limit operator techniques, which works for all Fock spaces $F^p_\alpha(\mathbb{C}^n)$ with $1 < p < \infty$ and fully explains the difference between Bergman and Fock spaces (see Theorem 13). Namely, it will become apparent that the only ingredient missing for the same proof to work for Hankel operators on Bergman spaces is Liouville’s theorem. In Theorem 10 and Corollary 12, we also show that the compactness of $H_f : X^p_\alpha(\Omega) \to L^p_\alpha(\Omega)$ is independent of $p$, and hence generalize the results of Stroethoff and Zheng which state that (4) is sufficient and necessary for $H_f : X^2_\alpha(\Omega) \to L^2_\alpha(\Omega)$ to be compact. These results complete the theory of compact Hankel operators on standard weighted Bergman and Fock spaces with bounded symbols.

## 2 Preliminaries

Throughout this paper let $1 < p < \infty$ and $n \in \mathbb{N}$. For $\alpha > 0$, let $\mu_\alpha$ be the Gaussian measure defined on $\mathbb{C}^n$ by

$$d\mu_\alpha(z) := \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z),$$

where $v$ is the usual Lebesgue measure on $\mathbb{C}^n$, and set $L^p_\alpha := L^p(\mathbb{C}^n, d\mu_{p\alpha/2})$. For $f \in L^p_\alpha$ and $z \in \mathbb{C}^n$, we define the weighted shift $C_z$ by

$$(C_z f)(w) = f(\phi_z(w))e^{\alpha\langle w, z \rangle - \frac{\alpha}{2}|z|^2},$$

where $\phi_z(w) = w - z$ for $w \in \mathbb{C}^n$. It is easy to check that $C_z$ is a surjective isometry with $C_z^{-1} = C_{-z}$ and $C_z M_f C_z^{-1} = M_{f \circ \phi_z}$ for every multiplication operator $M_f$ with bounded symbol $f$. The Fock space $F^p_\alpha$ is the closed subspace of all analytic functions in $L^p_\alpha$. The orthogonal projection $P_\alpha$ of $L^2_\alpha$ onto $F^2_\alpha$ is given by

$$(P_\alpha f)(z) = \int_{\mathbb{C}^n} f(w)e^{\alpha\langle z, w \rangle} d\mu_\alpha(w)$$

and it extends to a bounded projection of $L^p_\alpha$ onto $F^p_\alpha$. $K^\alpha_z(w) := e^{\alpha\langle w, z \rangle}$ is called the reproducing kernel. It is easy to check that $K^\alpha_z \in F^p_\alpha$ for all $p \in (1, \infty)$, $\alpha > 0$ and
For \( f \in L^\infty(\mathbb{C}^n) \), the Berezin transform of \( f \) is defined as
\[
\tilde{f}(z) = \frac{\langle f K_z, K_z \rangle_{L^2_{\alpha}}}{\langle K_z, K_z \rangle_{L^2_{\alpha}}} = \int_{\mathbb{C}^n} (f \circ \phi_{-z})(w) \, d\mu_{\alpha}(w).
\]

Let \( \alpha > -1 \) and suppose that \( \Omega \subset \mathbb{C}^n \) is an irreducible bounded symmetric domain in its Harish-Chandra realization (see [8, 12, 23]). We define
\[
d\mu_{\alpha}(z) := c_{\alpha} h(z, z)^{\alpha} \, dv(z),
\]
where \( h \) is the so-called Jordan triple determinant (see e.g. [8, 12, 23]), and \( c_{\alpha} \) is a constant such that \( \mu_{\alpha}(\Omega) = 1 \). For the unit ball \( \Omega = B_n \) we have \( h(z, w) = 1 - \langle z, w \rangle \) and \( c_{\alpha} = \frac{\Gamma(n+\alpha+1)}{n! \Gamma(\alpha+1)} \). For \( p \in (1, \infty) \) satisfying
\[
1 + \frac{(r-1)a}{2(\alpha+1)} < p < 1 + \frac{2(\alpha+1)}{(r-1)a}
\]
we set \( L^p_{\alpha} := L^p(\Omega, d\mu_{\alpha}) \). Here, \( r \) and \( a \) are two constants depending on the domain \( \Omega \) (see [8] for a table and [12] for a discussion of (7)). For \( \Omega = B_n \), it holds \( r = 1 \) and \( a = 2 \), hence every \( p \in (1, \infty) \) is permitted.

For \( f \in L^p_{\alpha} \) and \( z \in \Omega \), we define the reflection \( C_z \) by
\[
(C_z f)(w) = f(\phi_z(w)) \frac{h(z, z)^{\alpha+g}}{h(z, w)^{2(\alpha+g)}}
\]
where \( g \) is another constant depending on \( \Omega \) (\( g = n + 1 \) for \( \Omega = B_n \)) and \( \phi_z \) is the (unique) geodesic symmetry interchanging 0 and \( z \) (Möbius transforms in case \( \Omega = B_n \)). It is not difficult to check that \( C_z \) is a surjective isometry with \( C_z^{-1} = C_z \) (observe the difference between Bergman and Fock spaces here) and \( C_z M_f C_z^{-1} = M_{f \circ \phi_z} \) for every multiplication operator \( M_f \) with bounded symbol \( f \). The Bergman space \( A^2_{\alpha} \) is the closed subspace of all analytic functions in \( L^2_{\alpha} \). The orthogonal projection \( P_{\alpha} \) of \( L^2_{\alpha} \) onto \( A^2_{\alpha} \) is given by
\[
(P_{\alpha} f)(z) = \int_{\Omega} f(w) h(z, w)^{-\alpha-g} \, d\mu_{\alpha}(w)
\]
and it extends to a bounded projection of \( L^p_{\alpha} \) onto \( A^p_{\alpha} \) (provided that \( p \) satisfies (7)). The projection \( P_{\alpha} \) is referred to as the Bergman projection and \( K^\alpha_z(w) := h(w, z)^{-\alpha-g} \) is called the reproducing kernel. It is again not difficult to check that \( K^\alpha_z \in A^p_{\alpha} \) for
all \( p \in (1, \infty), \alpha > -1 \) satisfying (7) and \( z \in \Omega \). For \( f \in L^\infty(\mathbb{C}^n) \), the Berezin transform of \( f \) is again defined as

\[
\hat{f}(z) = \frac{\langle f K_z, K_z \rangle_{L_2^\alpha}}{\langle K_z, K_z \rangle_{L_2^\alpha}} = \int_{\Omega} (f \circ \phi_z)(w) \, d\mu_\alpha(w).
\]

When the results and their proofs are similar, we denote both \( A^p_\alpha \) and \( F^p_\alpha \) by \( X^p_\alpha \) and their corresponding domains by \( \Omega \), that is,

\[ X^p_\alpha \in \{ A^p_\alpha, F^p_\alpha \}. \]

When \( X^p_\alpha = A^p_\alpha \), it is understood that \( \alpha > -1 \) and \( \Omega \) is a bounded symmetric domain in \( \mathbb{C}^n \), while for \( X^p_\alpha = F^p_\alpha, \alpha > 0 \) and \( \Omega = \mathbb{C}^n \), in which case \( \partial \Omega \) denotes the point at infinity. Finally, \( d_\Omega \) will denote the Bergman metric for bounded symmetric domains and the Euclidean metric for \( \Omega = \mathbb{C}^n \).

For \( f \in L^\infty(\Omega) \), we define the Hankel operator \( H_f : X^p_\alpha \to L^p_\alpha \) by

\[ H_f g = (I - P_\alpha)(fg) \]

and the Toeplitz operator \( T_f : X^p_\alpha \to X^p_\alpha \) by

\[ T_f g = P_\alpha(fg). \]

The following Banach algebra plays an important role in our analysis:

\[ \mathcal{A} := \{ f \in L^\infty(\Omega) : H_f \text{ is compact} \}. \]

Our first auxiliary result is the following lemma, which demonstrates the role of \( C_z \) in connection with compactness and plays an important role in the proofs of our main results.

**Lemma 1.** If \( H : X^p_\alpha \to L^p_\alpha \) is a compact operator, then \( \| C_z H C_z^{-1} g \| \to 0 \) for every \( g \in X^p_\alpha \) as \( z \to \partial \Omega \).

**Proof.** Let \( g \in X^p_\alpha \) and \( B(0, r) := \{ z \in \mathbb{C}^n : d_\Omega(0, z) < r \} \) for \( r > 0 \). Then

\[
\| C_z H C_z^{-1} g \| \leq \| C_z H P_\alpha M_{\chi_{B(0, r)}} C_z^{-1} g \| + \| C_z H P_\alpha M_{1-\chi_{B(0, r)}} C_z^{-1} g \|
\]

\[
\leq \| H P_\alpha \| \| M_{\chi_{B(0, r)}} C_z^{-1} g \| + \| H P_\alpha M_{1-\chi_{B(0, r)}} \| \| g \|
\]

\[
= \| H P_\alpha \| \| M_{\chi_{B(0, r)}} g \| + \| H P_\alpha M_{1-\chi_{B(0, r)}} \| \| g \|.
\]
where we used $C_z M_{\chi_{B(0,r)}} C_z^{-1} = M_{\chi_{B(x,r)}}$ and the fact that $C_z$ and $C_z^{-1}$ are both isometries. Since $\chi_{B(z,r)} \to 0$ pointwise as $z \to \partial \Omega$, the first term tends to 0 by dominated convergence. Similarly, $M_{1-\chi_{D(0,r)}}$ tends strongly to 0 as $r \to \infty$. As $H$ is assumed to be compact, this implies $\|HP_\alpha M_{1-\chi_{D(0,r)}}\| \to 0$ as $r \to \infty$. Hence, choosing $r$ sufficiently large, we can assume that $\|HP_\alpha M_{1-\chi_{D(0,r)}}\|$ is arbitrarily small. We conclude that $\|C_z HC_z^{-1}g\| \to 0$ as $z \to \partial \Omega$.

Let $\beta \Omega$ denote the Stone-Čech compactification of $\Omega$. By its universal property, any continuous map $f$ from $\Omega$ to a compact Hausdorff space $K$ can be uniquely extended to a continuous map $f: \beta \Omega \to K$. Here, we do not distinguish between $f$ and its extension to $\beta \Omega$. Note that $\beta \Omega$ can be realized as the maximal ideal space of bounded continuous functions defined on $\Omega$. Every maximal ideal corresponds to a point in $\beta \Omega$ via evaluation.

**Proposition 2.** Let $f \in A$ and $x \in \beta \Omega \setminus \Omega$. Then there is a bounded analytic function $h_x$ such that for all nets $(z_\gamma)$ in $\Omega$ converging to $x$:

1. $\|f \circ \phi_{z_\gamma} - h_x\|_{L^p_\alpha} \to 0$ as $z_\gamma \to x$,
2. $C_{z_\gamma} M_f C_{z_\gamma}^{-1} = M_{f \circ \phi_{z_\gamma}}$ converges strongly to $M_{h_x}$,
3. $C_{z_\gamma} M_f C_{z_\gamma}^{-1} = M_{f \circ \phi_{z_\gamma}}$ converges strongly to $M_{h_x}$.

**Remark 3.** In the literature two different compactifications are used to achieve more or less the same thing, namely the Stone-Čech compactification e.g. in [12, 16, 20] and the maximal ideal space of bounded uniformly continuous functions e.g. in [4, 11, 15]. Usually, this is just a matter of labeling limit operators. More precisely, if there are two compactifications of $\Omega$, say $\hat{\Omega}$ and $\bar{\Omega}$, and a net $(z_\gamma)$ in $\Omega$ converging to some $x \in \hat{\Omega}$, then by compactness there is a subnet, again denoted by $(z_\gamma)$, such that $(z_\gamma)$ also converges in $\bar{\Omega}$. For an arbitrary operator $A$ the convergence of the corresponding net $(C_{z_\gamma} AC_{z_\gamma}^{-1})$, by definition, does not depend on the chosen compactification. Hence the set of all limits of nets of the form $(C_{z_\gamma} AC_{z_\gamma}^{-1})$ is the same for either compactification, i.e. exactly the closure of $\{z \in \Omega : C_z AC_z^{-1}\}$ in the strong operator topology. In fact, since bounded sets are metrizable in the strong operator topology, one may even take sequences instead of nets. However, it is convenient to label the limits in terms of boundary elements of the compactification. For this to make sense, for every net $(z_\gamma)$ converging to the same $x \in \hat{\Omega} \setminus \Omega$, the limit of $(C_{z_\gamma} AC_{z_\gamma}^{-1})$ needs to be the same. For the Stone-Čech compactification, this is rather easy. One only needs to show that $z \mapsto C_z AC_z^{-1}$ is weakly continuous as it may then be continuously extended to $\beta \Omega$, which implies the uniqueness.
We will need the following lemma, which is a corollary of [12, Proposition 14] and [4, Proposition 5.3]:

**Lemma 4.** Let \( f \in L^\infty(\Omega) \) and \( g \in X^p_\alpha \). Then the map \( z \mapsto C_z T_f C^{-1}_z g \) extends continuously to \( \beta \Omega \).

**Proof.** For bounded symmetric domains this is shown in [12, Proposition 14]. In the case \( \Omega = \mathbb{C}^n \), [4, Proposition 5.3] proves the result for the maximal ideal space of bounded uniformly continuous functions instead of \( \beta \mathbb{C}^n \). Hence the result is obtained via Remark 3. To illustrate the argument, we provide some more details here. Let \( \tilde{\Omega} \) denote the maximal ideal space of bounded uniformly continuous functions on \( \mathbb{C}^n \). In [4, Proposition 5.3] it is shown that \( z \mapsto C_z T_f C^{-1}_z \) is strongly continuous on \( \Omega \). In particular, it is weakly continuous. Moreover, it is bounded by \( \|P_\alpha\| \|f\|_\infty \). As bounded sets are relatively compact in the weak operator topology, \( z \mapsto C_z T_f C^{-1}_z \) can be extended to a weakly continuous map on \( \beta \Omega \). It remains to show that \( z \mapsto C_z T_f C^{-1}_z \) is also strongly continuous on \( \beta \Omega \). Indeed, choose a net \((z_\gamma)\) in \( \Omega \) that converges to some \( x \in \beta \Omega \). As \( \tilde{\Omega} \) is compact, every subnet of \((z_\gamma)\) has a subnet, again denoted by \((z_\gamma)\) converging in \( \tilde{\Omega} \). For each of these subnets the corresponding subnet \((C_{z_\gamma} T_f C^{-1}_{z_\gamma})\) converges strongly by [4, Proposition 5.3]. The weak continuity implies that all these limits are the same. Hence the whole net converges strongly and thus \( z \mapsto C_z T_f C^{-1}_z \) is strongly continuous on \( \beta \Omega \).

**Proof of Proposition 2.** For \( g \in X^p_\alpha \), we have
\[
C_z M_f C^{-1}_z g = C_z T_f C^{-1}_z g + C_z H_f C^{-1}_z g. \tag{8} \]

Let \( x \in \beta \Omega \) and choose a net \((z_\gamma)\) in \( \Omega \) that converges to \( x \). By Lemma 1 and Lemma 4, we get that \( C_{z_\gamma} M_f C^{-1}_{z_\gamma} = M_{f \circ \phi_{z_\gamma}} \) converges strongly on \( X^p_\alpha \) to some operator \( T_x \) as \( z_\gamma \to x \) and \( T_x \) only depends on \( x \) (i.e. not on the chosen net \((z_\gamma)\)). Define \( h_x := T_x \mathbb{1} \), where \( \mathbb{1} \) is the constant function 1. Then
\[
\|f \circ \phi_{z_\gamma} - h_x\|_{L^p_\alpha} = \|M_{f \circ \phi_{z_\gamma}} \mathbb{1} - T_x \mathbb{1}\|_{L^p_\alpha} \to 0.
\]

As the functions \( f \circ \phi_{z_\gamma} \) are uniformly bounded, \( h_x \) is also bounded. It follows that \( M_{f \circ \phi_{z_\gamma}} \to M_{h_x} \) and \( M_{f \circ \phi_{z_\gamma}} \to M_{h_x} \) strongly on \( L^p_\alpha \). In particular, \( T_x = M_{h_x} \). By (8) and Lemma 1, this implies \( C_{z_\gamma} T_f C^{-1}_{z_\gamma} \to M_{h_x} \) strongly on \( X^p_\alpha \). As \( X^p_\alpha \) is closed, \( h_x \) has to be a bounded analytic function.

For the essential spectrum of Toeplitz operators with symbols in \( \mathcal{A} \), we have the following corollary.
Corollary 5. Let $f \in A$. Then
\[ \text{sp}_{\text{ess}}(T_f) = \bigcup_{y \in \beta\Omega \setminus \Omega} \bar{f}(y) = \bar{f}(\beta\Omega \setminus \Omega), \]
where $\bar{f}$ denotes the extension of the Berezin transform of $f$ to $\beta\Omega$.

Proof. Let $x \in \beta\Omega \setminus \Omega$ and $(z_\gamma)$ be a net in $\Omega$ converging to $x$. By Lemma 1 and Proposition 2, $C_{z_\gamma}T_fC_{z_\gamma}^{-1}$ converges strongly to $T_{h_x}$ with $h_x$ analytic. The spectrum of $T_{h_x}$ is equal to $\overline{\text{clos}(h_x(\Omega))}$. Indeed, $T(h_x - \lambda)^{-1}$ is an inverse of $T_{h_x - \lambda}$ if $\lambda \notin \overline{\text{clos}(h_x(\Omega))}$ and conversely
\[ T_{h_x}^* K_w = P_\alpha(h_x K_w) = h_x(w)K_w, \]
i.e. $h_x(w)$ is an eigenvalue of $T_{h_x}^*$ for every $w \in \Omega$. By [12, Corollary 31] and [9, Corollary 29] (cf. Remark 3), we therefore obtain
\[ \text{sp}_{\text{ess}}(T_f) = \bigcup_{x \in \beta\Omega \setminus \Omega} \overline{\text{clos}(h_x(\Omega))}. \]
Since $\tilde{f} \circ \phi_x = \tilde{f} \circ \phi_x$ for all $z \in \Omega$ and $\tilde{g} = g$ for analytic functions $g$, we get
\[ \lim_{z_\gamma \to x} \tilde{f}(\phi_{z_\gamma}(w)) = \lim_{z_\gamma \to x} \tilde{f} \circ \phi_{z_\gamma}(w) = \tilde{h}_x(w) = h_x(w) \tag{9} \]
for every $w \in \Omega$. Since $\phi_{z_\gamma}(w) \to \partial\Omega$ as $z_\gamma \to x$, $\tilde{f}(\phi_{z_\gamma}(w))$ converges to $\tilde{f}(y)$ for some $y \in \beta\Omega \setminus \Omega$. Hence
\[ \text{sp}_{\text{ess}}(T_f) = \bigcup_{x \in \beta\Omega \setminus \Omega} \overline{\text{clos}(h_x(\Omega))} \subseteq \bigcup_{y \in \beta\Omega \setminus \Omega} \bar{f}(y). \]
The other inclusion follows directly from (9) with $w = 0$. \qed

A similar relation holds for the essential norm:

Corollary 6. Let $f \in A$. Then
\[ \max_{y \in \beta\Omega \setminus \Omega} |\bar{f}(y)| \leq \|T_f\|_{\text{ess}} \leq \|P_\alpha\| \max_{y \in \beta\Omega \setminus \Omega} |\bar{f}(y)|. \tag{10} \]

In particular, $\|T_f\|_{\text{ess}} = \max_{y \in \beta\Omega \setminus \Omega} \bar{f}(y)$ for $p = 2$. 

10
Note that \( \max_{y \in \beta \Omega} |\tilde{f}(y)| = \limsup_{z \to \partial \Omega} |\tilde{f}(z)|. \)

**Proof.** Let \( x \in \beta \Omega \) and choose a net \((z_\gamma)\) in \( \Omega \) that converges to \( x \). As in the proof of Corollary 5, \( C_z T f C_z^{-1} \) converges strongly to \( T_{h_x} \) with \( h_x \) analytic. By [12, Theorem 22] and [9, Theorem 31], we have

\[
\sup_{x \in \beta \Omega \setminus \Omega} \| T_{h_x} \| \leq \| T_f \|_{\text{ess}} \leq \| P_\alpha \| \sup_{x \in \beta \Omega \setminus \Omega} \| T_{h_x} \|.
\]

As \( h_x \) is bounded and analytic, we have \( \| T_{h_x} \| \leq \| h_x \|_{\infty} \). On the other hand, we observed that \( \text{sp}(T_{h_x}) = \text{clos}(h_x(\Omega)) \) in the proof of Corollary 5. As the spectral radius can never exceed the norm, we get \( \| T_{h_x} \| = \| h_x \|_{\infty} \). Moreover, Equation (9) implies

\[
\sup_{x \in \beta \Omega \setminus \Omega} \| h_x \|_{\infty} = \max_{y \in \beta \Omega \setminus \Omega} |\tilde{f}(y)|.
\]

Combining these estimates, we obtain (10). \( \square \)

**Definition 7.** ([9, Definition 6], [11, Definition 9])

An operator \( A \in \mathcal{L}(L^p_\alpha) \) is called a band operator if there exists a positive real number \( \omega \) such that \( M_f A M_g = 0 \) for all \( f, g \in L^\infty(\Omega) \) with \( \text{dist}_{d_0}(\text{supp } f, \text{supp } g) > \omega \). An operator \( A \in \mathcal{L}(L^p_\alpha) \) is called band-dominated if it is the norm limit of a sequence of band operators.

With a slight abuse of notation, we will call an operator \( A : X^p_\alpha \to L^p_\alpha \) or \( A : X^p_\alpha \to X^q_\alpha \) band-dominated if \( AP_\alpha : L^p_\alpha \to L^p_\alpha \) is band-dominated. As \( P_\alpha \) is itself band-dominated (see [9, Proof of Theorem 7], [11, Proof of Theorem 15]) and every product of band-dominated operators is again a band-dominated operator (see [9, Proposition 13], [11, Proposition 13]), all Toeplitz and Hankel operators with bounded symbols are also band-dominated.

**Proposition 8.** Let \( H : X^p_\alpha \to L^p_\alpha \) be a band-dominated operator and suppose that \( C_z H C_z^{-1} \to 0 \) strongly as \( z \to \partial \Omega \). Then \( H \) is compact.

**Proof.** The main difficulty is that for bounded symmetric domains the operators \( P_\alpha \) and \( C_z \) do not commute unless \( p = 2 \). We therefore focus on the case of bounded symmetric domains. Recall that \( C_z^{-1} = C_{\bar{z}} \) here.

Let \( \frac{1}{p} + \frac{1}{q} = 1 \) and let \( \tilde{C}_z : L^q_\alpha \to L^q_\alpha \) be the reflection operator on \( L^q_\alpha \) corresponding to \( z \in \bar{\Omega} \). We define \( C^*_z : L^p_\alpha \to L^p_\alpha \) by

\[
\langle C^*_z f, g \rangle_{L^p_\alpha} = \langle f, \tilde{C}_z g \rangle_{L^q_\alpha}
\]

for all \( f \in L^p_\alpha, \ g \in L^q_\alpha \).
i.e. the adjoint of $\tilde{C}_z$. Let $(z_γ)$ be a net in $Ω$ that converges to some $x ∈ βΩ$. The product $C_z P_α C^*_z$ is strongly convergent on $X^p_ξ$, see [12, Proposition 17]. As $C_z$ is a surjective isometry, $C^*_z$ is again a (surjective) isometry. Moreover, since $C_z M_f C_z = M_{f_0 φ_x}$ and $P_α C_z P_α = C_z P_α$, we have $C^*_z M_f C^*_z = M_{f_0 φ_x}$ and $P_α C^*_z P_α = P_α C^*_z$.

Now assume that $H$ is not compact. Then, since $P_α M_{χB(0,s)}$ is compact for every $s > 0$ (see e.g. [11, Proposition 15]), there exists an $ε > 0$ such that $∥HP_α M_{1−χB(0,s)}∥ > ε$ for all $s > 0$. By [12, Proposition 21], there is a radius $r$ such that for all $s > 0$ there is a midpoint $z_s$ such that

$$∥HP_α M_{χB(z_s,r)}∥ ≥ ∥HP_α M_{1−χB(0,s)} M_{χB(z_s,r)}∥ > \frac{ε}{2}. \quad (11)$$

Clearly, $z_s → Ω$ as $s → ∞$. Therefore $(z_α)$ has a subnet, denoted by $(z_{s_α})$, that converges to some $x ∈ βΩ \setminus Ω$. It follows

$$∥(C_{z_{s_α}} H C_{z_{s_α}})(C_{z_{s_α}} P_α C^*_z) P_α M_{χB(0,r)}∥ = ∥HP_α C^*_z M_{χB(0,r)} C^*_z∥ = ∥HP_α M_{χD(z_{s_α},r)}∥.$$ 

As $z_{s_α} → x$, the left-hand side tends to 0, a contradiction to (11). This completes the proof for bounded symmetric domains. The second part of the proof together with $C_z P_α = P_α C_z$ and the obvious modifications yields a (much simpler) proof for the Fock space.

We need one more preliminary lemma for our main results.

**Lemma 9.** Let $g ∈ L^p_α$ and $z ∈ Ω$. Then

$$∥(C_z P_α C_z^{-1} − P_α)g∥_{L^p_α} ≤ ∥P_α∥ ∥(I − P_α)g∥_{L^p_α}.$$ 

**Proof.** As $C_z$ and $C_z^{-1}$ are isometries, we have

$$∥(C_z P_α C_z^{-1} − P_α)g∥_{L^p_α} = ∥(P_α C_z^{-1} − C_z^{-1} P_α)g∥_{L^p_α} ≤ ∥P_α∥ ∥C_z^{-1} − C_z^{-1} P_α∥_{L^p_α} = ∥P_α∥ ∥(I − P_α)g∥_{L^p_α},$$

where we also used the fact that $P_α C_z^{-1} P_α = C_z^{-1} P_α$. □
3 Main results

We use the following characterization to show that compactness of Hankel operators is independent of the underlying space.

Theorem 10. Let \( f \in L^\infty(\Omega) \) and \( 1 < p < \infty \). The following are equivalent:

(i) \( H_f : X^p_\alpha(\Omega) \to L^p_\alpha \) is compact.

(ii) For every \( g \in X^p_\alpha \), we have \( \| C_z H_f C^{-1}_z g \|_{L^p_\alpha} \to 0 \) as \( z \to \partial \Omega \).

(iii) For every \( x \in \beta \Omega \setminus \Omega \) there is a bounded analytic function \( h_x \) such that for all nets \( (z_\gamma) \) in \( \Omega \) converging to \( x \) we have

\[
\| f \circ \phi_{z_\gamma} - h_x \|_{L^p_\alpha} \to 0
\]

as \( z_\gamma \to x \).

(iv) \( \| (I - P_\alpha)(f \circ \phi_z) \|_{L^p_\alpha} \to 0 \) as \( z \to \partial \Omega \).

(v) For every \( g \in X^p_\alpha \) we have \( \| H_{f \circ \phi_z} g \|_{L^p_\alpha} \to 0 \) as \( z \to \partial \Omega \).

Proof. The equivalence of (i) and (ii) follows from Lemma 1 and Proposition 8. By Proposition 2, (i) implies (iii). As \( P_\alpha h_x = h_x \) and \( P_\alpha \) is bounded, (iv) follows immediately from (iii). So assume (iv). Let \( (z_\gamma) \) be a net converging to some \( x \in \beta \Omega \setminus \Omega \). By Lemma 4, the operator \( C_{z_\gamma} T_f C^{-1}_{z_\gamma} \) converges strongly to some operator \( T_x \). Set \( h_x := T_x 1 \in X^p_\alpha \). Then

\[
\| f \circ \phi_{z_\gamma} - h_x \|_{L^p_\alpha} = \| f \circ \phi_z - C_{z_\gamma} T_f C^{-1}_{z_\gamma} 1 \|_{L^p_\alpha} + \| C_{z_\gamma} T_f C^{-1}_{z_\gamma} 1 - h_x \|_{L^p_\alpha}
\]

\[
\leq \| (I - P_\alpha)(f \circ \phi_z) \|_{L^p_\alpha} + \| P_\alpha(f \circ \phi_{z_\gamma}) - C_{z_\gamma} P_\alpha M f C^{-1}_{z_\gamma} 1 \|_{L^p_\alpha}
\]

\[
+ \| (C_{z_\gamma} T_f C^{-1}_{z_\gamma} - T_x) 1 \|_{L^p_\alpha}
\]

\[
\leq \| (I - P_\alpha)(f \circ \phi_z) \|_{L^p_\alpha} + \| (P_\alpha - C_{z_\gamma} P_\alpha C^{-1}_{z_\gamma})(f \circ \phi_{z_\gamma}) \|_{L^p_\alpha}
\]

\[
+ \| (C_{z_\gamma} T_f C^{-1}_{z_\gamma} - T_x) 1 \|_{L^p_\alpha}
\]

\[
\to 0
\]

as \( z_\gamma \to x \) by assumption and Lemma 9. Now, as in the proof of Proposition 2, this implies \( M_{f \circ \phi_{z_\gamma}} \to M_{h_x} \) strongly. As \( h_x \in X^p_\alpha \), we have \( (I - P_\alpha)M_{h_x} = 0 \) on \( X^p_\alpha \). It follows

\[
\| H_{f \circ \phi_{z_\gamma}} g \|_{L^p_\alpha} = \| (I - P_\alpha)M_{f \circ \phi_{z_\gamma}} g \|_{L^p_\alpha} \to 0
\]
for all \( g \in X^p_\alpha \) as \( z \gamma \to x \). As the net \((z_\gamma)\) was arbitrary, this implies (v). A simple algebraic computation shows

\[
C_z H_f C_z^{-1} - H_{f \circ \phi_z} = (P_\alpha - C_z P_\alpha C_z^{-1}) M_{f \circ \phi_z}
\]

and therefore (v) implies (ii) by Lemma 9.

\[\square\]

**Corollary 11.** If \( f \in \mathcal{A} \), then \( \| f \circ \phi_z - \tilde{f} \circ \phi_z \|_{L^p_\alpha} \to 0 \) as \( z \to \partial \Omega \). In particular, \( T_{f-\tilde{f}}, H_{f-\tilde{f}}, H_f, T_{f-\tilde{f}}, H_{f-\tilde{f}} \) are all compact.

**Proof.** Let \( x \in \beta \Omega \) and choose a net \((z_\gamma)\) in \( \Omega \) that converges to \( x \). Theorem 10 implies

\[
\| f \circ \phi_{z_\gamma} - h_x \|_{L^p_\alpha} \to 0
\]

for some bounded analytic function \( h_x \). By Equation (9), \( \tilde{f} \circ \phi_{z_\gamma} \) converges pointwise to \( h_x \). As \( \{ \tilde{f} \circ \phi_{z_\gamma} \} \) is equicontinuous with respect to \( d_\Omega \), the convergence is uniform on compact sets. In particular,

\[
\| \tilde{f} \circ \phi_{z_\gamma} - h_x \|_{L^p_\alpha} \to 0.
\]

As \((z_\gamma)\) was arbitrary, this implies the first assertion. The compactness of the mentioned operators now follows from Proposition 8 applied to \( M_{f-\tilde{f}} : X^p_\alpha \to L^p_\alpha \) and \( C_z M_{f-\tilde{f}} C_z^{-1} = M_{f \circ \phi_z-\tilde{f} \circ \phi_z} \) (similarly for \( \bar{f} \)).

\[\square\]

**Corollary 12.** Let \( f \in L^\infty(\Omega) \). Then the compactness of \( H_f : X^p_\alpha \to L^p_\alpha \) does not depend on \( p \) or \( \alpha \).

**Proof.** As \( \| (I - P_\alpha)(f \circ \phi_z) \|_{L^p_\alpha} \leq C_{p,\alpha} \| f \|_\infty \) for some constant \( C_{p,\alpha} \), this follows from standard estimates (Hölder) and Theorem 10 (iv), see e.g. [21, Theorem 7].

In our final result we give a characterization of compact Hankel operators in terms of the classical \( \text{VMO} \)-spaces, which were defined in (5). We split it into two theorems because there is a crucial difference between the two cases. In the Fock space, \( H_f \) is compact if and only if \( H_{\bar{f}} \) is compact as a consequence of Liouville’s theorem. For the Bergman space, we cannot rely on Liouville and therefore need to additionally assume the compactness of \( H_{\bar{f}} \).

**Theorem 13.** If \( f \in L^\infty(\Omega) \) and \( \Omega = \mathbb{C}^n \), then (i) to (v) in Theorem 10 are further equivalent to

(vi) \( H_{\bar{f}} : F^p_\alpha \to L^p_\alpha \) is compact.
(vii) \( \| f \circ \phi_z - \tilde{f}(-z) \|_{L^p_\alpha} \to 0 \) as \( z \to \partial \Omega \).

(viii) \( f \in \text{VMO}^p(\mathbb{C}^n) \cap L^\infty(\mathbb{C}^n) \).

**Proof.** On \( \mathbb{C}^n \) bounded analytic functions are constant by Liouville’s theorem. Hence condition (iii) in Theorem 10 is symmetric in \( f \) and \( \tilde{f} \) and therefore \( f \in \mathcal{A} \) is equivalent to (vi).

To show that \( f \in \mathcal{A} \) implies (vii), take \( x \in \beta \Omega \) and choose a net \((z_\gamma)\) in \( \Omega \) that converges to \( x \). Equation (9) implies that \( \tilde{f}(-z_\gamma) \) converges to \( h_x(0) \). As \( h_x \) is a constant, Theorem 10 (iii) implies (vii).

Now assume (vii). As \( \tilde{f}(-z) \) is a constant and \( P_\alpha \) is bounded, we get
\[
\| (I - P_\alpha)(f \circ \phi_z) \|_{L^p_\alpha} \leq \| f \circ \phi_z - \tilde{f}(-z) \|_{L^p_\alpha} + \| P_\alpha (\tilde{f}(-z) - f \circ \phi_z) \|_{L^p_\alpha} \leq (1 + \| P_\alpha \|) \| f \circ \phi_z - \tilde{f}(-z) \|_{L^p_\alpha} \to 0
\]
as \( z \to \partial \Omega \), hence Theorem 10 (iv) holds.

The equivalence of (vii) and (viii) is standard and can be found in [19, Theorem 3] and its generalization [14].

**Theorem 14.** If \( f \in L^\infty(\Omega) \) and \( \Omega \) is a bounded symmetric domain, then the following are equivalent:

(vi) \( H_f, H_{\tilde{f}} : A^p_\alpha \to L^p_\alpha \) are both compact.

(vii) \( \| f \circ \phi_z - \tilde{f}(z) \|_{L^p_\alpha} \to 0 \) as \( z \to \partial \Omega \).

(viii) \( f \in \text{VMO}^p(\Omega) \cap L^\infty(\Omega) \).

**Proof.** Assuming the compactness of both \( H_f \) and \( H_{\tilde{f}} \) again implies that \( h_x \) in Theorem 10 (iii) is constant. Thus (vi) and (vii) are equivalent by the same argument as in Theorem 13. The equivalence of (vii) and (viii) was proven in [25] for \( \Omega = \mathbb{B}_n \). A similar proof works for bounded symmetric domains; we omit the details.

**Remark 15.** It is actually not difficult to see that all statements in Theorem 13 and Theorem 14 are independent of \( p \) and \( \alpha \) (cf. Corollary 12). Therefore it would have been sufficient to cite the corresponding results for \( p = 2 \) in [5] and [6]. However, we decided to include short proofs in order to stress this key difference between Bergman and Fock spaces caused by Liouville’s theorem.
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