Radiative corrections to the quark masses in the ferromagnetic Ising and Potts field theories

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Abstract
We consider the Ising Field Theory (IFT), and the 3-state Potts Field Theory (PFT), which describe the scaling limits of the two-dimensional lattice $q$-state Potts model with $q = 2$, and $q = 3$, respectively. At zero magnetic field $h = 0$, both field theories are integrable away from the critical point, have $q$ degenerate vacua in the ferromagnetic phase, and $q(q - 1)$ particles of the same mass - the kinks interpolating between two different vacua. Application of a weak magnetic field induces confinement of kinks into bound states - the "mesons" (for $q = 2, 3$) consisting predominantly of two kinks, and "baryons" (for $q = 3$), which are essentially the three-kink excitations. The kinks in the confinement regime are also called "the quarks". We review and refine the Form Factor Perturbation Theory (FFPT), adapting it to the analysis of the confinement problem in the limit of small $h$, and apply it to calculate the corrections to the kink (quark) masses induced by the multi-kink fluctuations caused by the weak magnetic field. It is shown that the subleading third-order $\sim h^3$ correction to the kink mass vanishes in the IFT. The leading second order $\sim h^2$ correction to the kink mass in the 3-state PFT is estimated by truncation the infinite form factor expansion at the first term representing contribution of the two-kink fluctuations into the kink self energy.

Keywords: Potts model, form factors, confinement

1. Introduction
Integrable models of statistical mechanics and field theory [1, 2] provide us with a very important source of information about the critical behavior of condensed matter systems. Any progress in analytical solutions of such models is highly desirable, since it does not only yield exact information about the model itself but also about the whole universality class it represents. On the other hand, integrable models can serve as zeroth-order approximations in the perturbative analysis of their non-integrable deformations, providing a useful insight.
into a rich set of physical phenomena that never occur in integrable models: confinement of topological excitations, particle decay and inelastic scattering, false-vacuum decay, etc.

The Ising Field Theory (IFT) is the Euclidean quantum field theory that describes the scaling limit of the two-dimensional lattice Ising model near its phase transition point. Upon making a Wick rotation, the IFT can be also viewed as a Lorentz-covariant field theory describing the dynamics of a one-dimensional quantum ferromagnet at zero temperature near its quantum phase transition point \( T_c \). The IFT is integrable at all temperatures for zero magnetic field \( h = 0 \). Directly at the critical point \( T = T_c, \ h = 0 \) it reduces \[4\] to the minimal conformal field theory \( \mathcal{M}_3 \), which describes free massless Majorana fermions. These fermions acquire a nonzero mass \( m \sim |T - T_c| \) at non-critical temperatures, but remain free at \( h = 0 \). In the ordered phase \( T < T_c \), the fermions are ordinary particles, while in the ferromagnetic phase \( T < T_c \) they become topological excitations - the kinks interpolating between two degenerate ferromagnetic vacua. Application of the magnetic field \( h > 0 \) induces interactions between fermions and breaks the integrability of the IFT at \( T \neq T_c \). In the ordered phase \( T < T_c \), it explicitly breaks also the degeneracy between ferromagnetic vacua. This induces an attractive long-range linear potential between the kinks, which leads to their confinement into two-kink bound states. Due to the analogy with quantum chromodynamics, such bound states are often called "mesons", while the kink topological excitations in such a confinement regime are also called "quarks". In what follows, we shall synonymously use the terms "kinks" and "quarks".

This mechanism of confinement known as the McCoy - Wu scenario was first described for the IFT by these authors \[5\] in 1978, and attracted much interest in the last two decades. Recently it was experimentally observed and studied in one-dimensional quantum ferro- and anti-ferromagnets \[6, 7, 8, 9, 10\]. Since the IFT is not integrable at \( h > 0, \ m > 0 \), different approximate techniques have been used for the theoretical understanding of the kink confinement in this model, such as analytical perturbative expansions \[11, 12, 13, 14\] in the weak confinement regime near the integrable direction \( h = 0 \), and numerical methods \[12, 15\].

The idea to use the magnetic field as a perturbative parameter characterizing a small deformation of an integrable massive field theory was first realized in the Form Factor Perturbation Theory (FFPT) introduced by Delfino, Musardo, and Simonetti \[16\]. It turns out, however, that their original FFPT cannot be applied directly to the kink confinement problem and requires considerable modification. The reason is that even an arbitrarily weak long-ranged confining interaction leads to qualitative changes of the particle content at the confinement-deconfinement transition: isolated kinks cannot exist any more in the presence of the magnetic field, and the mass spectrum \( M_n(m, h), \ n = 1, 2, \ldots \) of their bound states (the mesons), become dense in the interval \( 2m < M_n < \infty \) in the limit \( h \rightarrow +0 \). This in turn makes straightforward perturbation theory based on the adiabatic hypothesis unsuitable. A different, non-perturbative technique to study the IFT meson mass spectrum was developed by Fonseca and
Zamolodchikov [11]. This technique is based on the Bethe-Salpeter equation, which was derived for the IFT in [11] in the two-quark approximation. The latter approximation implies that at small magnetic fields $h \to +0$, the meson eigenstate

$$|\Psi_P\rangle = |\Psi^{(2)}_P\rangle + |\Psi^{(4)}_P\rangle + |\Psi^{(6)}_P\rangle + \ldots \quad (1)$$

of the IFT Hamiltonian, with $P$ being the meson momentum, is approximated by the two-quark component

$$|\Psi^{(2)}_P\rangle = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \delta(p_1 + p_2 - P) \Psi_P(p_1, p_2) |p_1 p_2\rangle, \quad (2)$$

neglecting the multi-quark contributions represented by further terms in the right-hand side of (1). Here $p_1, p_2$ denote the momenta of two quarks coupled into a meson.

It was shown in [14], that the FFPT can be modified to adapt it to the confinement problem, if one takes into account the long-range attractive potential already at zeroth order and applies a certain $h$-dependent unitary transform in the Fock space of the free IFT. Such a modified FFPT incorporates the Bethe-Salpeter equation in its leading order. This perturbative technique can be effectively used in the weak confinement regime $h \to +0$ despite the break of the adiabatic hypothesis at the confinement-deconfinement transition at $h = 0$.

Two kinds of asymptotic expansions for the meson masses $M_n(m, h)$ have been obtained for the IFT in the weak confinement regime $h \to +0$. The low energy expansion [5, 11, 12, 14] in fractional powers of $h$ describes the initial part of the meson mass spectrum, while the semiclassical expansion [12, 13, 14] in integer powers of $h$ describes the meson masses $M_n(m, h)$ with $n \gg 1$. High accuracy of both expansions has been established [12, 15] by comparison with the IFT meson mass spectra calculated by direct numerical methods based on the Truncated Conformal Spaced Approach [17, 18].

The leading terms in the low energy and semiclassical expansions can be gained from the Bethe-Salpeter equation. This indicates [12], that the two-quark approximation is asymptotically exact to the leading order in $h \to 0$. It was shown [11, 12], however, that starting from the second order in $h$ in both low energy and semiclassical expansions, one must take into account the mixture of four-quark, six-quark, etc. configurations in the meson state (1).

The leading multi-quark correction to the meson masses in the IFT was obtained by Fonseca and Zamolodchikov [12]. This correction is of order $h^2$, and originates from the renormalization of the quark mass. The third-order $\sim h^3$ multi-quark corrections to the IFT meson masses have so far only partly been known. These corrections arise from contributions of three effects.

- Renormalization of the long-range attractive force between the neighboring kinks (the 'string-tension') of order $h^3$ which was determined in [12].

- Multi-quark fluctuations modify the regular part of the Bethe-Salpeter kernel, which is responsible for the pair interaction between quarks at short
The corresponding contribution $\sim h^3$ to the meson masses was found in [14].

- The radiative corrections of the quark mass of the third-order in $h$, which was unknown.

The first aim of this paper is to complete the calculation of the meson mass spectrum in the IFT in the weak confinement regime $h \rightarrow +0$ to third order in $h$. To this end, we review and further modify the form factor perturbative technique developed for the confinement problem in [14]. The FFPT contains a well known problem caused by the so-called kinematic singularities in the matrix elements of the spin operator. Merging of such singularities in the integrals arising in the FFPT leads to ill-defined quantities like $\delta(0)$, or $\delta(p)/p$. We propose a consistent regularization procedure that allows one to perform high-order FFPT calculations in a controlled fashion avoiding ill-defined quantities in intermediate expressions. The key idea is to replace the uniform magnetic field in the Hamiltonian of the infinite system by its nonuniform counterpart switched on in a finite interval of the length $R$, to perform all calculations at a large but finite $R$, and to proceed to the limit $R \rightarrow \infty$ afterwards. To verify the efficiency of this regularization procedure, we use it to reproduce several well-known results and to obtain some new ones for the scaling limit of the Ising model. Then we apply the same procedure to calculate the third-order radiative correction to the quark mass in the ferromagnetic IFT showing that it vanishes.

The mechanism of confinement outlined above is quite common in two-dimensional quantum field theories, that are invariant under some discrete symmetry group and display a continuous order-disorder phase transition. If such a model has several degenerate vacua in the ordered phase, the application of an external field typically leads to confinement of kinks interpolating between different vacua. Realizations of this scenario in different two-dimensional models have been the subject of considerable interest in recent years [19, 20, 21, 22, 23]. In this paper we shall address to some aspects of the confinement problem in the three-state Potts Field Theory (PFT).

The three-state PFT represents the scaling limit of the two-dimensional lattice three-state Potts model [1, 24]. At zero magnetic field, it is invariant under the permutation group $S_3$ and displays the continuous order-disorder phase transition. It was shown by Dotsenko [25], that the conformal field theory corresponding to the critical point of the three-state Potts model can be identified as the minimal unitary model $\mathcal{M}_5$. In the ordered phase at zero magnetic field, the three-state PFT has three degenerate vacua and six kinds of massive particles of the same mass - the kinks (‘quarks’) $K_{\mu\nu}$ interpolating between vacua $|0\rangle_\mu$ and $|0\rangle_\nu$, where $\mu, \nu \in \mathbb{Z} \mod 3$. The three-state PFT is integrable at zero magnetic field [26], and the quark scattering matrix is exactly known [27]. This scattering matrix is non-trivial, which indicates that the quarks in the three-state PFT are not free at zero magnetic field, but strongly interact with each other at small distances, in contrast to the IFT. The form factors of the physically
Application of the magnetic field \( \hbar \neq 0 \) breaks integrability of the PFT and leads to confinement of quarks. The quark bound states in the \( q \)-state PFT in the confinement regime were classified by Delfino and Grinza [20], who also showed that besides the mesonic (two-quark) bound states, the baryonic (three-quark) bound states are allowed at \( q = 3 \). First numerical calculations of the meson and baryon mass spectra in the \( q \)-state PFT were described in [20, 29]. The meson masses in the \( q \)-state PFT in the weak confinement regime were analytically calculated to leading order in \( \hbar \) in [30], where the generalization of the IFT Bethe-Salpeter equation to the PFT was also described. The masses of several lightest baryons in the three-state PFT in the leading order in \( \hbar \) have been calculated in [31]. Analytical predictions of [30, 31] for the meson and baryon masses in the three-state PFT were confirmed in direct numerical calculations performed by Lencsés and Takács [15].

The second subject of the present paper is to estimate the second-order radiative correction to the quark masses in the 3-state PFT in the weak confinement regime. This correction to the quark mass gives rise to the multi-quark corrections to the meson and baryon masses in second order in \( \hbar \). Starting from the Lehmann expansion for the quark mass radiative correction, we calculate its first term representing the quark self-energy diagram with two virtual quarks in the intermediate state.

The remainder of this paper is organized as follows. In the next section we start with recalling some well-known properties of the \( q \)-state Potts model on the square lattice, and then describe briefly its scaling limit in the case \( q = 3 \), and zero magnetic field. In Section 3 we review the FFPT adapted in [14] to the confinement problem in the IFT. We further improve this FFPT technique in order to regularize the products of singular matrix elements of the spin operator which arise in this method. We then apply the improved version of the FFPT to recover some well-known results and to obtain several new ones for the IFT. In Section 4 we describe the form factors of the disorder spin operators in the three-state PFT at zero magnetic field in the paramagnetic phase, which were found by Kirillov and Smirnov [28]. Applying the duality transform to these form factors, we obtain the matrix elements of the order spin operators in the ferromagnetic three-state PFT between the one- and two-quark states. These matrix elements are used in Section 5 to estimate the second-order correction to the quark mass in the latter model in the presence of a weak magnetic field. Concluding remarks are given in Section 6. Finally, there are four appendixes describing technical details of some of the required calculations.

2. Potts Field Theory

In this section we following [20] review some well known properties of the \( q \)-state Potts model on the square lattice, and then proceed to its scaling limit. Consider the two-dimensional square lattice \( \mathbb{Z}^2 \) and associate with each lattice site \( x \in \mathbb{Z}^2 \) a discrete spin variable \( s(x) = 1, 2, \ldots, q \). The model Hamilton-
nian is defined as
\[ E = -\frac{1}{T} \sum_{\langle x, y \rangle} \delta(s(x), s(y)) - H \sum_x \delta(s(x), q). \] (3)

Here the first summation is over nearest neighbour pairs, \( T \) is the temperature, \( \delta(s(x), q) \) is the Kronecker symbol. At \( H = 0 \), the Hamiltonian (3) is invariant under the permutation group \( S_q \); at \( H \neq 0 \) the symmetry group reduces to \( S_{q-1} \). At \( q = 2 \), model (3) reduces to the Ising model.

The order parameters \( \langle \sigma_\alpha \rangle \) can be associated with the variables
\[ \sigma_\alpha(x) = \delta(s(x), \alpha) - \frac{1}{q}, \quad \alpha = 1, \ldots, q. \]

The parameters \( \langle \sigma_\alpha \rangle \) are not independent, since
\[ \sum_{\alpha=1}^q \sigma_\alpha(x) = 0. \] (4)

Two complex spin variables \( \sigma(x) \) and \( \bar{\sigma}(x) \) defined by the relations
\[ \sigma(x) = \exp[2\pi i s(x)/q] = \sum_{\alpha=1}^q \exp(2\pi i \alpha/q) \sigma_\alpha(x), \] (5)
\[ \bar{\sigma}(x) = \exp[-2\pi i s(x)/q] = \sum_{\alpha=1}^q \exp(-2\pi i \alpha/q) \sigma_\alpha(x), \] (6)
are useful in proceeding to the continuous limit.

At zero magnetic field, the model undergoes a ferromagnetic phase transition at the critical temperature
\[ T_c = \frac{1}{\log(1 + \sqrt{q})}. \] (7)

This transition is continuous for \( 2 \leq q \leq 4 \). The ferromagnetic low-temperature phase at zero field is \( q \)-times degenerated. The Potts model (3) at \( H = 0 \) possesses the dual symmetry, which generalizes the Kramers-Wannier duality of the Ising model. This symmetry connects the properties of the model in the ordered and disordered phases. By duality, the partition functions of the zero-field Potts model coincide at the temperatures \( T \) and \( \tilde{T} \), provided
\[ \left( e^{1/T} - 1 \right) \left( e^{1/\tilde{T}} - 1 \right) = q. \]
For a review of many other known properties of the Potts model see [24, 1].

The scaling limit of the model (3) at \( H \to 0 \), \( T \to T_c \), and \( q \in [2, 4] \) is described by the Euclidean action [20]
\[ \mathcal{A}^{(q)} = \mathcal{A}^{(q)}_{CFT} - \tau \int d^2 x \varepsilon(x) - h \int d^2 x \sigma_q(x), \] (8)
Here $x$ denotes the points of the plane $\mathbb{R}^2$ having the cartesian coordinates $(x, y)$. The first term $A_{CFT}^{(q)}$ corresponds to the conformal field theory, which is associated with the critical point. Its central charge $c(q)$ takes the value

$$c(q) = 1 - \frac{6}{t(t+1)}, \quad \text{where} \quad \sqrt{q} = 2 \sin \frac{\pi(t-1)}{2(t+1)}. \quad (9)$$

The fields $\epsilon(x)$ (energy density) and $\sigma_q(x)$ (spin density) are characterized by the scaling dimensions

$$X^{(q)}_{\epsilon} = \frac{1}{2} \left( 1 + \frac{3}{t} \right), \quad X^{(q)}_{\sigma} = \frac{(t-1)(t+3)}{8t(t+1)}.$$ 

The parameters $\tau \sim (T - T_c)$ and $h \sim H$ are proportional to the deviations of the temperature and the magnetic field from their critical point values. At $h = 0$ and $\tau \neq 0$ the field theory is integrable, i.e. it has infinite number of integrals of motion and a factorizable scattering matrix [26].

In the rest of this section we shall concentrate on the $q = 3$ Potts field theory. The simpler and better studied Ising case corresponding to $q = 2$ will be discussed in Section 3.

2.1. Disordered phase at $h = 0$

The model has a unique ground state $|0\rangle_{\text{par}}$ in the disordered phase, at $\tau > 0$ and $h = 0$. The particle content of the model consists of a massive scalar particle and its antiparticle. Their momentum $p$ and energy

$$\omega(p) = \sqrt{p^2 + m^2} \quad (10)$$

can be conveniently parametrized by the rapidity $\beta$,

$$p(\beta) = m \sinh \beta, \quad \omega(\beta) = m \cosh \beta. \quad (11)$$

Here $m \sim \tau^{5/6}$ is the particle mass.

The space of states is generated by the Faddeev-Zamolodchikov creation/annihilation operators $Z^*_\varepsilon(\beta)$, $Z_\varepsilon(\beta)$, where the index $\varepsilon = \pm 1$ distinguishes particles ($\varepsilon = 1$) and antiparticles ($\varepsilon = -1$). These operators satisfy the following equations

$$Z_{\epsilon_1}(\beta_1) Z_{\epsilon_2}(\beta_2) = S_{\epsilon_1, \epsilon_2}(\beta_1 - \beta_2) Z_{\epsilon_2}(\beta_2) Z_{\epsilon_1}(\beta_1), \quad (12)$$

$$Z^*_{\epsilon_1}(\beta_1) Z^*_{\epsilon_2}(\beta_2) = S_{\epsilon_1, \epsilon_2}(\beta_1 - \beta_2) Z^*_{\epsilon_2}(\beta_2) Z^*_{\epsilon_1}(\beta_1), \quad (13)$$

$$Z_{\epsilon_1}(\beta_1) Z^*_{\epsilon_2}(\beta_2) = S_{\epsilon_2, \epsilon_1}(\beta_2 - \beta_1) Z^*_{\epsilon_2}(\beta_2) Z_{\epsilon_1}(\beta_1) + \delta_{\epsilon_1, \epsilon_2} \delta(\beta_1 - \beta_2), \quad (14)$$

where

$$S_{-1,-1}(\beta) = S_{1,1}(\beta) = \frac{\sinh[\beta + 2\pi i/3]/2}{\sinh[(\beta - 2\pi i/3)/2]}, \quad (15)$$

$$S_{1,-1}(\beta) = S_{-1,1}(\beta) = S_{1,1}(i\pi - \beta).$$

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Equation (14) implies that the one-particle states are normalized as

$$\langle 0 | Z_{\varepsilon_1}^{\beta_1} Z_{\varepsilon_2}^{\beta_2} | 0 \rangle_{\text{par}} = \delta_{\varepsilon_1, \varepsilon_2} \delta(\beta_1 - \beta_2).$$

(16)

The two-particle scattering amplitudes (15) were found by Köberle and Swieca [27]. The generators of the permutation group $S_3 \approx Z_3 \times Z_2$ act on the paramagnetic vacuum and particles as follows

$$\Omega | 0 \rangle_{\text{par}} = | 0 \rangle_{\text{par}}, \quad C | 0 \rangle_{\text{par}} = | 0 \rangle_{\text{par}},$$

(17)

$$\Omega Z_{\varepsilon}^{\beta} \Omega^{-1} = \nu^\varepsilon Z_{\varepsilon}^{\beta}, \quad C Z_{\varepsilon}^{\beta} C^{-1} = Z_{-\varepsilon}^{\beta}.$$

(18)

Here $\nu = \exp(2\pi i/3)$, $\Omega$ is the generator of the cyclic permutation group $Z_3$, $\Omega^3 = 1$, $C$ is the charge conjugation, $C^2 = 1$.

The vector space $L_{\text{par}}$ of paramagnetic states is spanned by the paramagnetic vacuum $| 0 \rangle_{\text{par}}$, and the $n$-particle vectors

$$| \beta_n, \ldots, \beta_2, \beta_1 \rangle_{\varepsilon_n, \ldots, \varepsilon_2, \varepsilon_1} = Z_{\varepsilon_n}^{\beta_n} \ldots Z_{\varepsilon_2}^{\beta_2} Z_{\varepsilon_1}^{\beta_1} | 0 \rangle_{\text{par}},$$

(20)

with $n = 1, 2, \ldots$. Corresponding to (20) bra-vector is denoted as

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n(\beta_1, \beta_2, \ldots, \beta_n) \equiv \langle 0 | Z_{\varepsilon_1}^{\beta_1} Z_{\varepsilon_2}^{\beta_2} \ldots Z_{\varepsilon_n}^{\beta_n}.$$

Let us denote by $L_{\text{sym}}$ the subspace of $L_{\text{par}}$ spanned by the vacuum $| 0 \rangle$ and vectors (20), for which $\sum_{j=1}^{n} \varepsilon_j = 0 \mod 3$. Operator $\Omega$ acts as the identity operator on the subspace $L_{\text{sym}}$.

The $n$-particle vectors (20) are not linearly independent, but satisfy a number of linear relations, which are imposed on them by the commutation relations (13). For example,

$$| \beta_1, \beta_2 \rangle_{\varepsilon_1, \varepsilon_2} = S_{\varepsilon_1, \varepsilon_2}(\beta_1 - \beta_2) | \beta_2, \beta_1 \rangle_{\varepsilon_2, \varepsilon_1}.$$  

(21)

The "in"-basis in the $n$-particle subspace $L_{\text{par}}^{(n)}$ of $L_{\text{par}}$ is formed by the vectors of the form (20) with $\beta_n > \beta_{n-1} > \ldots > \beta_1$, and the "out"-basis in the same subspace $L_{\text{par}}^{(n)}$ is formed by the vectors (20) with $\beta_n < \beta_{n-1} < \ldots < \beta_1$.

Reconstruction of the matrix elements of local operators between such basis states in integrable models is the main subject of the form factor bootstrap program [32]. For the three-state PFT, this program was realized by Kirillov and Smirnov in [28], where the explicit representations for the form factors of the main operators naturally arising in this model were obtained. We postpone the discussion of these results to Section 4.

2.2. Ordered phase at $h = 0$

In the low temperature phase $\tau < 0$, the ground state $| 0 \rangle_{\mu}$, $\mu = 0, 1, 2 \mod 3$ is three-fold degenerate at $h = 0$. The elementary excitations are topologically charged being represented by six kinks $| K_{\mu,\nu}(\beta) \rangle$, $\mu, \nu \in Z \mod 3$ interpolating
between two different vacua $|0\rangle_\mu$ and $|0\rangle_\nu$. These kinks are massive relativistic particles with the mass $m \sim (-\tau)^{5/6}$.

The generators of the symmetry group $S_3$ act on the vacua and one-kink states as follows,

\[
\tilde{\Omega} |0\rangle_\mu = |0\rangle_{\mu+1}, \tag{22}
\]
\[
\tilde{C} |0\rangle_\mu = |0\rangle_{-\mu}, \tag{23}
\]
\[
\tilde{\Omega} |K_{\mu_0, \mu_{n-1}}(\beta_n) \ldots K_{\mu_2, \mu_1}(\beta_2) K_{\mu_1, \mu_0}(\beta_1)\rangle, \tag{24}
\]
\[
\tilde{C} |K_{\mu_0, \mu_{n-1}}(\beta_n) \ldots K_{\mu_2, \mu_1}(\beta_2) K_{\mu_1, \mu_0}(\beta_1)\rangle. \tag{25}
\]

The subspace $L^{(n)}_{fer}$ of the $n$-kink states in the ferromagnetic space $L_{fer}$ is spanned by the vectors

\[
|K_{\mu_0, \mu_{n-1}}(\beta_n) \ldots K_{\mu_2, \mu_1}(\beta_2) K_{\mu_1, \mu_0}(\beta_1)\rangle. \tag{26}
\]

Corresponding bra-vector is denoted as

\[
\langle K_{\mu_0, \mu_{n-1}}(\beta_n) \ldots K_{\mu_2, \mu_1}(\beta_2) K_{\mu_1, \mu_0}(\beta_1) |. \tag{27}
\]

The $n$-kink states (26) are called topologically neutral, if $\mu_n = \mu_0$, and topologically charged otherwise. We denote by $L_0$ the topologically neutral subspace of $L_{fer}$ spanned by the ferromagnetic vacuum $|0\rangle_0$, and vectors (26) with $\mu_n = \mu_0 = 0$.

The Kramers-Wannier duality of the PFT requires that the mapping $D$ must be unitary, i.e. the inverse mapping $\{D^{-1}; L_{sym} \to L_0\}$ must exist, and
$D^{-1} = D^\dagger$. These requirements lead to a number of linear relations between the $n$-kink states (26). For example, acting on the equality

$$|\beta_1, \beta_2\rangle_{1,-1} = S_{1,-1}(\beta_1 - \beta_2)|\beta_2, \beta_1\rangle_{-1,1}$$

[following from (21)] by the mapping $D^{-1}$, one obtains,

$$|K_{02}(\beta_1)K_{20}(\beta_2)\rangle = S_{1,-1}(\beta_1 - \beta_2)|K_{01}(\beta_2)K_{10}(\beta_1)\rangle.$$

Application of the same procedure to the $n$-particle states (26) leads to the Faddeev-Zamolodchikov commutation relations

$$K_{\mu\nu}(\beta_1)K_{\nu\gamma}(\beta_2) = S_{1,1}(\beta_1 - \beta_2)K_{\mu\rho}(\beta_2)K_{\rho\mu}(\beta_1), \quad (30a)$$

$$K_{\mu\nu}(\beta_1)K_{\nu\mu}(\beta_2) = S_{1,-1}(\beta_1 - \beta_2)K_{\mu\rho}(\beta_2)K_{\rho\mu}(\beta_1), \quad (30b)$$

where $\rho \neq \nu$. According to the conventional agreement [34], notations $K_{\alpha\alpha'}(\beta_1)$ in the above relations can be understood as the formal non-commutative symbols representing the kinks in the $n$-kink states (26).

Relations (30) describe the two-kink scattering processes in the ferromagnetic phase. Due to the PFT dual symmetry, they are characterized by the same scattering amplitudes, as the two-particle scattering in the paramagnetic phase. Furthermore, the scattering theories in the high- and low-temperature phases are equivalent. Such duality arguments can be also extended to the matrix elements of physical operators. In particular, the matrix elements of the order spin operators in the ferromagnetic phase can be expressed in terms of the form factors of the disorder spin operators [35] in the paramagnetic phase. We shall return to this issue in Section 4.

3. Quark mass in the ferromagnetic IFT

The IFT action $A_{IFT} = A^{(2)}$ is defined by equation (8) with $q = 2$. The conformal field theory $A_{C,IFT}^{(2)}$ associated with the critical point is the minimal model $M_3$, which contains free massless Majorana fermions [4]. These fermions acquire a mass $m \sim |\tau|$, as the temperature deviates from the critical point.

They remain free at $h = 0$. However, application of a magnetic field $h > 0$ induces interaction between the fermions. The Hamiltonian corresponding to the action $A_{IFT}$ can be written as

$$H = H_0 + hV, \quad (31)$$

where

$$H_0 = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \omega(p)a^\dagger(p)a(p), \quad (32)$$

$$V = -\int_{-\infty}^{\infty} dx \sigma(x), \quad (33)$$

and $\omega(p)$ is the spectrum [10] of free fermions. These fermions are ordinary spinless particles in the disordered phase $\tau > 0$, and topologically-charged kinks interpolating between two degenerate vacua in the ordered phase $\tau < 0$. Fermionic
operators $\mathbf{a}^\dagger(p')$, $\mathbf{a}(p)$ obey the canonical anticommutational relations

$$\{\mathbf{a}(p), \mathbf{a}^\dagger(p')\} = 2\pi \delta(p-p'), \quad \{\mathbf{a}(p), \mathbf{a}(p')\} = \{\mathbf{a}^\dagger(p), \mathbf{a}^\dagger(p')\} = 0.$$  

Commonly used are also fermionic operators $a(\beta)$, $a^\dagger(\beta)$, corresponding to the rapidity variable $\beta = \arcsinh(p/m)$:

$$a(\beta) = \omega(p)^{1/2} \mathbf{a}(p), \quad a^\dagger(\beta) = \omega(p)^{1/2} \mathbf{a}^\dagger(p). \quad (34)$$

The notations

$$\langle p_1, \ldots, p_N | = \mathbf{a}^\dagger(p_1) \ldots \mathbf{a}^\dagger(p_N)|0\rangle, \quad \langle p_1, \ldots, p_N | = \langle 0| \mathbf{a}(p_1) \ldots \mathbf{a}(p_N),$$

for the fermionic basis states with definite momenta will be used.

The order spin operator $\sigma(x) = \sigma(x,y)|_{y=0}$ in the ordered phase $\tau < 0$ is completely characterized by the matrix elements $\langle \beta_1, \ldots, \beta_K | \sigma(0)| \beta'_1, \ldots, \beta'_{K'} \rangle$, whose explicit expressions are well known [36, 11], see equation (2.14) in [11]. These matrix elements are different from zero only if $K + N = 0 \pmod{2}$. The matrix elements with $K + N = 2$ read as

$$\langle p|\sigma(x)|k\rangle = \frac{i \bar{\sigma} \exp[i x (k-p)] \omega(p) + \omega(k)}{p - k \sqrt{\omega(p)\omega(k)}}, \quad (35)$$

$$\langle 0|\sigma(x)k_1k_2\rangle = \frac{i \bar{\sigma} \exp[i x (k_1 + k_2)] \omega(k_1) - \omega(k_2)}{k_1 + k_2 \sqrt{\omega(k_1)\omega(k_2)}}, \quad (36)$$

$$\langle k_1k_2|\sigma(x)|0\rangle = \frac{i \bar{\sigma} \exp[-i x (k_1 + k_2)] \omega(k_1) - \omega(k_2)}{k_1 + k_2 \sqrt{\omega(k_1)\omega(k_2)}}, \quad (37)$$

where $\bar{\sigma} = \bar{s}|m|^{1/8}$ is the zero-field vacuum expectation value of the order field (spontaneous magnetization), and

$$\bar{s} = 2^{1/12}e^{-1/8}A^{3/2} = 1.35783834..., \quad (38)$$

where $A = 1.28243...$ stands for the Glaisher’s constant. The matrix elements of the order spin operator with $K + N > 2$ can be determined from (35)-(37) by means of the Wick expansion. For real $p$ and $k$, the "kinematic" pole at $p = k$ in (35) is understood in the sense of the Cauchy principal value

$$\frac{1}{p - k} \rightarrow \mathcal{P} \frac{1}{p - k} = \frac{1}{2} \left( \frac{1}{p - k + i0} + \frac{1}{p - k - i0} \right). \quad (39)$$

The field theory defined by the Hamiltonian (31)-(33) is not integrable for generic $m > 0$ and $h > 0$, but admits exact solutions along the lines $h = 0$ and $m = 0$. The line $h = 0$ corresponds to the Onsager’s solution [37], whose scaling limit describes free massive fermions. Integrability of the IFT along the line $m = 0, h \neq 0$ was established by Zamolodchikov [38].

Close to integrable directions, it is natural to treat the non-integrable quantum field theories as deformations of integrable ones. As it was mentioned in
the Introduction, realization of this idea leads to the FFPT, whose original ver-

sion [16], however, cannot be applied directly to the confinement problem since
the magnetic field changes the particle content of the theory at arbitrary small
$h > 0$. The problem manifests itself already in the naive first-order correction
formula for the kink mass [16]

$$\delta^{(1)} m = - \lim_{p \to 0} \lim_{k \to p} h \langle p|\sigma(0)|k \rangle,$$

(40)

which is infinite due to the kinematic pole in the matrix element (35) of the spin
operator. To avoid this problem, a modified version of the FFPT was de-
volved in [14]. Since it is substantially used in this section, it will be helpful to rec-
ald here its main issues.

The kea idea of the modified FFPT is to absorb a part of the interaction
into the unitary operator $U(h)$, for which the formal expansion in powers of $h$
is postulated,

$$U(h) = 1 + \sum_{n=1}^{\infty} h^n F_n.$$

(41)

This operator has been used to define creator and annihilator operators for the
"dressed" fermions,

$$\mathbf{a}(p) = U(h)^{-1} \mathbf{a}(p) U(h), \quad \mathbf{a}^\dagger(p) = U(h)^{-1} \mathbf{a}^\dagger(p) U(h),$$

(42)

which are underlined to distinguish them from the "bare" ones. Similarly, the

dressing unitary transform is defined for arbitrary operators and states,

$$\mathbf{A} = U(h)^{-1} \mathbf{A} U(h), \quad |\Phi\rangle = U(h)^{-1} |\Phi\rangle.$$

It was required in [14] that the number of dressed fermions conserves in the
evolution defined by the Hamiltonian (31)-(33), i.e.

$$[\mathcal{N}, \mathcal{H}] = 0,$$

(43)

where

$$\mathcal{N} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \mathbf{a}^\dagger(p) \mathbf{a}(p).$$

It was required, further, that operators $F_n$ change the number of dressed fermions, i.e.

$$\langle p|F_n|k \rangle = 0 \quad \text{for} \quad n(p) = n(k).$$

(44)

Here the shortcut notations $|k\rangle = |k_1, \ldots, k_{n(k)}\rangle$, $\langle p\rangle = \langle p_1, \ldots, p_{n(p)}\rangle$, have been
used.

Conditions (43), (44) together with the unitarity requirement

$$U(h)U(h)^{-1} = 1,$$

(45)

allow one to determine the coefficients $F_n$ in the expansion (41). In particular,
the matrix elements of the first one read as

$$\langle p|F_1|k \rangle = \frac{\langle p|V|k \rangle}{\omega(p) - \omega(k)}, \quad \text{for} \quad n(p) \neq n(k),$$

(46)
where we again use the abbreviation $\omega(q) \equiv \omega(q_1) + \ldots + \omega(q_n(q))$.

Note that the matrix element (46) diverges at the hyper-surface determined by the "resonance relation"

$$\omega(p_1) + \ldots + \omega(p_n(p)) = \omega(k_1) + \ldots + \omega(k_n(k)).$$

This indicates that, strictly speaking, the unitary operator $U(h)$ satisfying requirements (41)-(43) does not exist. However, in calculations of the small-$h$ asymptotic expansions of certain quantities [e.g. the ground state energy $E_{vac}(m, h)$] the resonance terms do not appear, and the modified FFPT can be effectively used and leads to unambiguous results. This situation is similar to the perturbation theory for nonlinear systems in classical mechanics [39]. The Birkhoff’s theorem states that, if the classical nonlinear system is close to some linear one, and the characteristic frequencies of the latter do not satisfy the resonance relations, the dynamics of the nonlinear system can be well approximated by the integrable system which Hamiltonian has the Birkhoff normal form, see page 387 in [39]. The unitary operator $U(h)$ can be viewed as the quantum analogue of the canonical transform, which maps the original Hamiltonian of a non-integrable classical system to the integrable Birkhoff normal form.

The second difficulty, which is inherent to the FFPT, comes from the kinematic singularities in the matrix elements of the spin order operator between the states with nonzero numbers of kinks. Such singularities contributing in the leading and higher-orders of the FFPT lead to infinite and ill-defined quantities like $\delta(0)$, which require regularization. This problem has been widely discussed in the literature, mostly in the context of finite-temperature correlation function calculations [40, 41, 42, 43, 44]. Several regularization procedures have been proposed, such as finite volume regularization [44, 45], and appropriate infinitesimal shiftings of the kinematic poles into the complex plane [40, 43, 14]. Here we apply a different regularization scheme, which seems to be more convenient for the problem considered.

Keeping the length of the system infinite, we replace the uniform magnetic field $h > 0$ by the non-uniform field $h_R(x)$, which is switched on only in the large, but finite interval $[-R/2, R/2]$, $R \gg m^{-1}$,

$$h_R(x) = h \chi(x; -R/2, R/2),$$

where $\chi(x; -R/2, R/2) = \begin{cases} 1, & \text{if } x \in [-R/2, R/2], \\ 0, & \text{if } x \notin [-R/2, R/2]. \end{cases}$

After performing all calculations, we proceed to the limit $R \to \infty$.

Accordingly, instead of the IFT Hamiltonian (31), we get a set of Hamiltonians $\mathcal{H}_R$ parametrized by the length $R$,

$$\mathcal{H}_R = \mathcal{H}_0 + h V_R,$$

$$V_R = -\int_{-R/2}^{R/2} dx \sigma(x).$$
After diagonalization of the Hamiltonian $\mathcal{H}_R$ in the fermionic number along the lines described in Section 5 of [14], we arrive to equations (35)-(39) of [14], modified by the following replacements:

\[ V \rightarrow V_R, \quad \mathcal{H} \rightarrow \mathcal{H}_R, \quad U \rightarrow U_R, \quad \Lambda \rightarrow \Lambda_R. \quad (51) \]

In the rest of this Section, the efficiency of the described version of the FFPT will be demonstrated by the recovery of some well-known features of the IFT in the weak confinement regime and the derivation of several new results.

### 3.1. Vacuum sector

To warm-up, let us consider the small-$\hbar$ expansion of the ferromagnetic ground state energy in the IFT. The results will be used in the subsequent subsection in calculations of the radiative corrections to the kink dispersion law and string tension.

The expansion of the ground state energy $E_{\text{vac}}(m, h, R)$ can be read from Subsection 5.1 of Reference [14], with substitutions (51):

\[
E_{\text{vac}}(m, h, R) \equiv \langle 0 | \mathcal{H}_R | 0 \rangle = \langle 0 | U_R(h) (\mathcal{H}_0 + hV_R) U_R(h)^{-1} | 0 \rangle = \sum_{j=1}^{\infty} \delta_j E_{\text{vac}}(m, h, R), \quad (52)
\]

where $\delta_j E_{\text{vac}}(m, h, R) \sim h^j$, and

\[
\delta_1 E_{\text{vac}}(m, h, R) = h \langle 0 | V_R | 0 \rangle = -\hbar \bar{\sigma} R, \quad (53)
\]

\[
\delta_2 E_{\text{vac}}(m, h, R) = -h^2 \sum_{n(q) \neq 0} \frac{\langle 0 | V_R | q \rangle \langle q | V_R | 0 \rangle}{\omega(q)}, \quad (54)
\]

\[
\delta_3 E_{\text{vac}}(m, h, R) = -h^3 \langle 0 | V_R | 0 \rangle \sum_{n(q) \neq 0} \frac{\langle 0 | V_R | q \rangle \langle q | V_R | 0 \rangle}{|\omega(q)|^2} + h^3 \sum_{q, q' \neq 0, n(q') \neq n(q)} \frac{\langle 0 | V_R | q \rangle \langle q | V_R | q' \rangle \langle q' | V_R | 0 \rangle}{\omega(q) \omega(q')} \quad (55)
\]

The same abbreviation as in equation (46) have been used, $n(q)$ denotes the number of fermions in the intermediate state $|q\rangle \equiv |q_1, q_2, \ldots, q_n(q)\rangle$.

Four comments on equations (52)-(55) are in order.

1. There are no resonance poles [like in equation (46)] in expansion (52), while the kinematic singularities are present in its third and higher order terms.

2. Equation (52) is nothing else but the Rayleigh-Schrödinger expansion (see, for example §38 in [46]) for the ground state energy of the Hamiltonian (49). This expansion in $\hbar$ is asymptotic. In the limit $R \rightarrow \infty$, its convergence radius goes to zero due to the weak essential droplet singularity.
The second-order term $\delta_2 E_{\text{vac}}(m, h, R)$ is defined by means of the Lehmann expansion (54), whose explicit form reads as

$$\delta_2 E_{\text{vac}}(m, h, R) = \sum_{l=1}^\infty \delta_{2,2l} E_{\text{vac}}(m, h, R),$$

(64)
where
\[ \delta_{2,E_{vac}}(m, h, R) = -\frac{h^2}{\nu!} \int_{-R/2}^{R/2} dx_1 \, dx_2 \int_{-\infty}^{\infty} \frac{dq_1 \ldots dq_\nu}{(2\pi)^\nu}. \] (65)

\[ \exp[i(q_1 + \ldots + q_\nu)(x_1 - x_2)] \langle 0|\sigma(0)|q_1, \ldots, q_\nu\rangle \langle q_\nu, \ldots, q_1|\sigma(0)|0\rangle. \]

Straightforward summation of \((64)\) yields,
\[ \delta_{2,E_{vac}}(m, h, R) = -\int_{-R/2}^{R/2} dx_1 \, dx_2 \int_{-\infty}^{\infty} dy_1 \langle 0|\Delta\sigma(x_1, y_1) \Delta\sigma(x_2, 0)|0\rangle, \] (66)

where
\[ \Delta\sigma(x, y) = \exp(H_0 y)\sigma(x) \exp(-H_0 y) - \bar{\sigma}. \]

Since the matrix element in the integrand does not depend on \((x_1 + x_2)/2\) and vanishes exponentially for \(|x_1 - x_2| m \gg 1\), we can easily proceed to the limit \(R \to \infty\), arriving at the well-known representation of the magnetic susceptibility in terms of the spin-spin correlation function,
\[ \delta_{2,\rho}(m, h) = \lim_{R \to \infty} \frac{\delta_{2,E_{vac}}(m, h, R)}{R} = -h^2 \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \langle 0|\Delta\sigma(x, y) \Delta\sigma(0, 0)|0\rangle. \] (67)

Let us return now to the Lehmann expansion \((64)\) for the ground state energy, perform the elementary integration over \(x_1, x_2\) in \((65)\), and proceed to the limit \(R \to \infty\), exploiting the equality
\[ \lim_{R \to \infty} \frac{4\sin^2(qR/2)}{R^2} = 2\pi \delta(q). \] (68)

As a result, we arrive at the familiar spectral expansion \([51]\) for the ground state energy density
\[ \delta_{2,\rho}(m, h) = \sum_{l=1}^{\infty} \delta_{2,2l}(m, h), \] (69)
\[ \delta_{2,2l}(m, h) = -h^2 \frac{1}{(2l)^2} \int_{-\infty}^{\infty} dq_1 \ldots dq_{2l} \int_{-\infty}^{\infty} dq_{2l-1} \frac{\delta(q_1 + \ldots + q_{2l})}{(2\pi)^{2l-1}} \langle 0|\sigma(0)|q_1, \ldots, q_{2l}\rangle \langle q_{2l}, \ldots, q_1|\sigma(0)|0\rangle. \] (70)

The first term in expansion \((69)\) can be easily calculated using the explicit expressions \([36], [37]\) for the form factors, giving
\[ \delta_{2,2}(m, h) = -\frac{h^2 \bar{\sigma}^2}{12\pi m}. \] (71)
The corresponding two-fermion contribution \( \tilde{G}_{2,2} \) to the universal amplitude \( \tilde{G}_2 \)

\[
\tilde{G}_{2,2} = -\frac{s^2}{12\pi} = -0.0489063 \ldots
\]  

(72)

reproduces the well-known result of Tracy and McCoy [52], which is rather close to the exact value [51, 11, 49] \( \tilde{G}_2 = -0.0489532897203 \ldots \).

Now let us turn to the third order term [55] in the expansion [52] for the ground state energy \( E_{\text{vac}}(m, h, R) \). Unlike the previous case of the second-order correction, kinematic singularities do contribute to \( \delta E_{\text{vac}}(m, h, R) \) through the matrix element \( \langle q|V_R|q' \rangle \) in the second line of [55]. Nevertheless, the right-hand side of [55] is well defined due to the chosen regularization [51].

After summation of the Lehmann expansion in [55] one arrives in the limit \( R \to \infty \) at the well-known integral representation [5] for \( \delta_3 \rho(m, h) \) in terms of the three-point correlation function,

\[
\delta_3 \rho(m, h) = -h^3 \int \int \int dx_1 dx_3 \int_0^\infty dy_3 \langle 0|\Delta \sigma(x_1, y_1)\Delta \sigma(0, 0)\Delta \sigma(x_3, y_3)|0 \rangle.
\]

(73)

Alternatively, one can truncate the spectral series [55] which defines \( \delta E_{\text{vac}}(m, h, R) \) at the level of the two-kink intermediate states \( n(q) = n(q') = 2 \). Denoting the result by \( \delta_{3,2} E_{\text{vac}}(m, h, R) \), we get explicitly

\[
\delta_{3,2} E_{\text{vac}}(m, h, R) = A_{3,2}(m, h, R) + B_{3,2}(m, h, R),
\]

(74)

where

\[
A_{3,2}(m, h, R) = \frac{h^3 \bar{\sigma} R}{2} \int \int \int dq_1 dq_2 \frac{1}{(2\pi)^2} \frac{1}{\omega(q_1) + \omega(q_2)^2}.
\]

(75)

\[
B_{3,2}(m, h, R) = -\frac{h^3}{4} \int \int \int dq_1 dq_2 \frac{1}{(2\pi)^2} \frac{1}{\omega(q_1) + \omega(q_2)} \int \int \int dq'_1 dq'_2 \frac{1}{\omega(q'_1) + \omega(q'_2)} \int \int \int dx_1 dx_2 dx_3 e^{i(x_1 - x_2)(q_1 + q_2)} e^{i(x_2 - x_3)(q'_1 + q'_3)}.
\]

(76)

Here the two-kink matrix elements of the spin operator are determined by equations [55]-[57], while the four-kink matrix element in the last line can be expressed in terms of the latter by means of the Wick expansion:

\[
\langle q_2, q_1|\sigma(0)|q'_1, q'_2 \rangle = (\langle q_2, q_1|\sigma(0)|0 \rangle \langle 0|\sigma(0)|q'_1, q'_2 \rangle + \langle q_1|\sigma(0)|q'_2 \rangle \langle q_2|\sigma(0)|q'_1 \rangle - \langle q_1|\sigma(0)|q'_2 \rangle \langle q_2|\sigma(0)|q'_1 \rangle) \bar{\sigma}^{-1}.
\]

(77)
Since the two last terms in the square brackets in the right-hand side provide equal contributions to the integral (76), we can replace the four-kink matrix element in its integrand as follows

\[ \langle q_2, q_1 | \sigma(0) | q'_1, q'_2 \rangle \sim [\langle q_2, q_1 | \sigma(0) | 0 | \sigma(0) | q'_1, q'_2 \rangle + 2 \langle q_1 | \sigma(0) | q'_1 \rangle \langle q_2 | \sigma(0) | q'_2 \rangle] \delta^{-1}. \]  

(78)

The second term in the bracket containing the product of two kinematic singularities can be modified to the form

\[ 2 \langle q_1 | \sigma(0) | q'_1 \rangle \langle q_2 | \sigma(0) | q'_2 \rangle = 
-2 \tilde{\sigma}^2 \omega(q_1) + \omega(q'_1) \omega(q_2) + \omega(q'_2) \]  

\[ \frac{1}{\sqrt{\omega(q_1)\omega(q'_1)}} \frac{1}{\sqrt{\omega(q_2)\omega(q'_2)}} P \frac{1}{q_1 - q'_1} P \frac{1}{q_2 - q'_2} = 
8\pi^2 \tilde{\sigma}^2 \delta(q_1 - q'_1) \delta(q_2 - q'_2) - \tilde{\sigma}^2 \frac{\omega(q_1) + \omega(q'_1) \omega(q_2) + \omega(q'_2)}{\sqrt{\omega(q_1)\omega(q'_1)}} \frac{1}{\sqrt{\omega(q_2)\omega(q'_2)}} \]  

\[ \left( \frac{1}{q_1 - q'_1 + i0} \frac{1}{q_2 - q'_2 - i0} \frac{1}{q_1 - q'_1 - i0} \frac{1}{q_2 - q'_2 + i0} \right). \]

In deriving (79) we have used (35), (39), together with the equality

\[ P \frac{1}{q_1 - q'_1} P \frac{1}{q_2 - q'_2} = -\pi^2 \delta(q_1 - q'_1) \delta(q_2 - q'_2) + \frac{1}{2} \left( \frac{1}{q_1 - q'_1 + i0} \frac{1}{q_2 - q'_2 - i0} \frac{1}{q_1 - q'_1 - i0} \frac{1}{q_2 - q'_2 + i0} \right). \]

(80)

After substitution of (79) into (78), (76), the term \( 8\pi^2 \tilde{\sigma}^2 \delta(q_1 - q'_1) \delta(q_2 - q'_2) \) in the right-hand side of (78) gives rise to the contribution in \( B_{3,2}(m, h, R) \), which cancels exactly with the term \( A_{3,2}(m, h, R) \) in (74). Performing the integration over \( x_1, x_2, x_3 \) over the cube \((-R/2, R/2)^3\) in the remaining part and dividing the result by \( R \), we obtain

\[ \frac{\delta_{3,2} E_{vac}(m, h, R)}{R} = -\frac{h^3 \tilde{\sigma}^3}{4} \int_{-\infty}^{\infty} dq_1 dq_2 dq'_1 dq'_2 (2\pi)^4 \Delta_3(q_1 + q_2, q'_1 + q'_2, R) \cdot \]  

\[ \mathcal{G}(q_1, q_2, q'_1, q'_2), \]

(81)

where

\[ \Delta_3(p, k, R) = \frac{8 \sin(pR/2) \sin(kR/2) \sin((k-p)R/2]}{R \sin(kR) \sin[(k-p)R]/(k-p)}, \]

(82)
and
\[ G(q_1, q_2, q_1', q_2') = \frac{\omega(q_1) - \omega(q_2)}{\sqrt{\omega(q_1) \omega(q_2)}} \frac{\omega(q_2') - \omega(q_1')}{\sqrt{\omega(q_2') \omega(q_1')}} \frac{1}{(q_1 + q_2)(q_1' + q_2')} \tag{83} \]
\[ \frac{1}{\omega(q_1) + \omega(q_2) \omega(q_1') + \omega(q_2')} \left\{ \frac{\omega(q_1) - \omega(q_2')}{\sqrt{\omega(q_1) \omega(q_2')}} \frac{\omega(q_2) - \omega(q_1)}{\sqrt{\omega(q_1) \omega(q_2)}} \right\} + \frac{1}{(q_1 + q_2)(q_1' + q_2')} \]
\[ \left( \frac{1}{q_1 - q_1' + i0} q_2 - q_2' - i0 \right) + \frac{1}{q_1 - q_1' - i0} q_2 - q_2' + i0 \right) \}

It is possible to show that the weak large-\(R\) limit of the function \(\Delta_3(p, k, R)\) is proportional to the two-dimensional \(\delta\)-function,
\[ \lim_{R \to \infty} \Delta_3(p, k, R) = 4\pi^2 \delta(p) \delta(k). \tag{84} \]

The simplest way to prove this equality is to integrate \(\Delta_3(p, k, R)\) multiplied with the plane-wave test function. The result reads as
\[ \int_{-\infty}^{\infty} dp \, dk \, \Delta_3(p, k, R) \exp[i(px + ky)] = 4\pi^2 \left[ 1 - \frac{\max(|x|, |y|, |x + y|)}{R} \right], \tag{85} \]
if \(\max(|x|, |y|, |x + y|) < R\). Taking the limit \(R \to \infty\) in (85), we arrive at (84).

Exploiting (84), one can proceed to the limit \(R \to \infty\) in (81), yielding
\[ \delta_{3,2} \rho(m, h) \equiv \lim_{R \to \infty} \frac{\delta_{3,2} E_{\text{vac}}(m, h, R)}{R} = \frac{\hbar^3 \bar{\sigma}^3}{4} \int_{-\infty}^{\infty} dq \, dq' \frac{1}{(2\pi)^2} G(q_1, -q_1, q_1', -q_1') = \frac{\hbar^3 \bar{\sigma}^3}{16\pi^2 m^4} (C_1 + C_2), \tag{86} \]
where
\[ C_1 = -\frac{m^4}{4} \left( \int_{-\infty}^{\infty} dq \, \frac{q^2}{|\omega(q)|^3} \right)^2 = -\frac{1}{9}, \tag{87} \]

and
\[ C_2 = -\frac{m^4}{4} \int_{-\infty}^{\infty} dq \, dq' q q' \frac{|\omega(q) + \omega(q')|^2}{|\omega(q)\omega(q')|^4} \cdot \frac{1}{(q - q' + i0)^2} + \frac{1}{(q - q' - i0)^2} = \frac{4}{3} + \frac{\pi^2}{8}. \tag{88} \]

Calculation of the integral in equation (87) is straightforward. The calculation of the double integral \(C_2\) is harder and described in Appendix A.
Combining (80)-(88), we obtain finally
\[ \delta_{3,2} \rho(m, h) = \frac{\hbar^3 \bar{\sigma}^3}{16m^4} \left( \frac{11}{9\pi^2} + \frac{1}{8} \right). \tag{89} \]
For the two-kink contribution $\tilde{G}_{3,2}$ to the amplitude $\tilde{G}_3$, this yields

$$\tilde{G}_{3,2} = \frac{s^3}{16} \left( \frac{11}{9\pi^2} + \frac{1}{8} \right) = 0.0389349 \ldots$$

(90)

The exact value of the universal amplitude $\tilde{G}_3$ is unknown. In 1978, McCoy and Wu [50] performed a thorough analysis of the three- and four-point spin correlation functions in the zero-field Ising model on the square lattice, from which they obtained the approximate value for this amplitude,

$$\tilde{G}_3 \approx \frac{11s^3}{72} = 0.0387529 \ldots$$

(91)

Recently, at least six digits of the exact amplitude $\tilde{G}_3$ have become available

$$\tilde{G}_3 = 0.0388639 \ldots$$

(92)

due to the very accurate numerical calculations carried out by Mangazeev et al. [53, 49] for the square and triangular lattice Ising models.

Comparison of (90) and (91) with (92) indicates, that (i) the two-kink contribution (90) approximates the ”exact” amplitude (92) somewhat better than (91), (ii) the two-kink configurations provide the dominant contribution to the universal amplitude $\tilde{G}_3$. The configurations with four and more kinks in intermediate states contribute less than 0.2% in the spectral sum (55).

3.2. One-fermion sector

In this subsection we address the modified FFPT in the one-fermion sector $n(p) = n(k) = 1$, and extend it to the third order in $h$.

The matrix element of the Hamiltonian (49) between the dressed one-fermion states $\langle p |$ and $| k \rangle$ can be written as

$$\langle p | \mathcal{H}_R | k \rangle = \langle p | U_R(h) \mathcal{H}_R U_R(h)^{-1} | k \rangle = 2\pi \delta(p - k) \omega(p) + \delta \langle p | \mathcal{H}_R | k \rangle.$$  

(93)

Expanding here the unitary operator $U_R(h)$ and its inverse in powers of $h$, one arrives at the perturbation expansion

$$\delta \langle p | \mathcal{H}_R | k \rangle = \sum_{j=1}^{\infty} \delta_j \langle p | \mathcal{H}_R | k \rangle.$$  

(94)

Three initial terms in this expansion can be obtained from equation (37)-(39).

1The values of the amplitude $\tilde{G}_3$ reported in [49] for the square and triangular lattices are 0.038863932(3) and 0.0388639290(1), respectively.
be analytically continued to larger momenta, the latter conditions guarantee that the resonance poles do not appear in the expansion (94) for large enough momenta. The resonance poles contribute to the second and higher orders of R braces in (96), (97). Nevertheless, at finite j for symmetry relations:

\[
\delta_1 \langle p | H_R | k \rangle = h \langle p | V_R | k \rangle, \\
\delta_2 \langle p | H_R | k \rangle = -\frac{h^2}{2} \sum_{n(q) \neq n(p)} \langle p | V_R | q \rangle \langle q | V_R | k \rangle \left\{ \frac{1}{\omega(q) - \omega(p)} + \frac{1}{\omega(q) - \omega(k)} \right\}, \\
\delta_3 \langle p | H_R | k \rangle = \frac{h^3}{2} \sum_{q, q'} \langle p | V_R | q \rangle \langle q | V_R | q' \rangle \langle q' | V_R | k \rangle \left\{ [1 - \delta_{n(q), n(p)}][1 - \delta_{n(q'), n(p)}] - \frac{1}{\omega(q') - \omega(p)} \frac{1}{\omega(q) - \omega(k)} \right\},
\]

where n(p) = n(k) = 1.

One can easily see, that the matrix elements \( \delta_j \langle p | H_R | k \rangle \) obey the following symmetry relations:

\[
\delta_j \langle p | H_R | k \rangle = (-1)^j \delta_j \langle k | H_R | p \rangle, \\
\delta_j \langle p | H_R | k \rangle^* = (-1)^j [\delta_j \langle p | H_R | k \rangle]^*,
\]

for \( j = 1, 2, \ldots \). The kinematic singularity is present already in the first order term (95). The resonance poles contribute to the second and higher orders of expansion (94) for large enough momenta \( p \) and \( k \), due to the terms, like those in braces in (96), (97). Nevertheless, at finite \( R \), the right-hand sides of equations (95), (97) determine well defined generalized functions, if the absolute values of momenta \( p \) and \( k \) are small enough,

\[
\omega(p) < 3m, \quad \omega(k) < 3m.
\]

The latter conditions guarantee that the resonance poles do not appear in expansion (94). The constraints (100) will be imposed in the subsequent FFPT calculations at finite \( R \). After proceeding to the limit \( R \to \infty \), the results will be analytically continued to larger momenta, \( |p| > \sqrt{2} m \).

We postulate the following definition of the renormalized quark dispersion law \( \epsilon(p, m, h) \),

\[
\lim_{R \to \infty} \{ \langle p | H_R | k \rangle - \pi \delta(p - k) [E_{vac}(m, h, R) + E_{vac}(m, -h, R)] \} = 2 \pi \epsilon(p, m, h) \delta(p - k) + 2 \pi i f(m, h) \delta'(p - k).
\]

Just as in the case of definition (60), both sides in the above equation must be understood as formal power series in \( h \). Equating the coefficients in these power series and taking into account (98) and (62), one finds

\[
\lim_{R \to \infty} \{ \delta_j \langle p | H_R | k \rangle - 2 \pi \delta(p - k) \delta_j \langle 1 | H_R | 1 \rangle \} = 2 \pi \delta_j \epsilon(p, m, h) \delta(p - k),
\]

\[
\lim_{R \to \infty} \{ \delta_j \langle p | H_R | k \rangle - 2 \pi \delta(p - k) \delta_j \langle 1 | H_R | 1 \rangle \} = 2 \pi \delta_j \epsilon(p, m, h) \delta(p - k),
\]

(102)
for even $j = 2, 4, \ldots$, and
\[ \lim_{R \to \infty} \left\{ \delta_j \langle p | H_R | k \rangle + 4\pi i \delta' (p - k) R^{-1} \delta_j \langle 0 | H_R | 0 \rangle \right\} = 0, \quad (103) \]
\[ \delta_j \epsilon(p, m, h) = 0, \quad (104) \]
for odd $j = 1, 3, \ldots$. So, we can argue on the basis of the above heuristic analysis, that the Taylor expansion of the quark dispersion law $\epsilon(p, m, h)$ contains only even powers of $h$, which are determined by equation (102).

It was shown in [54] that the renormalized quark dispersion law $\epsilon(p, h)$, does not have the Lorentz covariant form in the confinement regime. Nevertheless, the 'dressed quark mass' $m_q(m, h)$ can be extracted from large-$p$ asymptotics of $\epsilon(p, h)$ in the following way [54, 14],
\[ [m_q(m, h)]^2 = \lim_{p \to \infty} \{2p \epsilon(p, m, h) - p\}. \quad (105) \]
This relation is understood, of course, in the sense of a power series in $h$, or, equivalently, in the parameter $\lambda = 2h\bar{\sigma}/m^2$. It follows from (104), that this expansion contains only even powers,
\[ m_q^2 = m^2 + m^2 \sum_{l=1}^{\infty} a_l \lambda^{2l}. \quad (106) \]

In order to validate the latter statement, it remains to show that the large-$R$ limits in the left-hand sides of equations (102) and (103) exist, and to prove equalities (103). In what follows, we shall do it for the three initial values $j = 1, 2, 3$.

The case $j = 1$ is quite simple. The term [95] linear in $h$ in expansion [94] reads as
\[ \delta_1 \langle p | H_R | k \rangle = -h \int_{-R/2}^{R/2} dx \langle p | \sigma(x) | k \rangle = \left( i h\bar{\sigma} \omega(p) + \omega(k) \right) \frac{2 \sin[R(k - p)/2]}{[\omega(p)\omega(k)]^{1/2}} \frac{1}{k - p}. \quad (107) \]

Even though the right-hand side contains the kinematic singularity, it describes a well defined generalized function at arbitrary finite $R$. Furthermore, exploiting the equality
\[ \lim_{R \to \infty} \frac{2 \sin(qR/2)}{q} \frac{1}{q} = -2\pi \delta'(q), \quad (108) \]
we can proceed to the limit $R \to \infty$ in equation (107), obtaining
\[ \lim_{R \to \infty} \delta_1 \langle p | H_R | k \rangle = 4\pi i \delta'(p - k) h\bar{\sigma}. \quad (109) \]
This proves (103) for $j = 1$, since $\delta_1 \langle 0 | H_R | 0 \rangle = -h\bar{\sigma}R$. 

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Turning to the term (96) quadratic in $h$, we first perform the summation over the number $n(q)$ of the fermions in the intermediate state $|q\rangle$, subject to the requirement (100). The result can be written in the compact form

$$\delta_2 \langle p|H_R|k\rangle = -\frac{h^2}{2} \int_0^\infty dy \int_{-R/2}^{R/2} dx_1 \, dx_2 \left( 1 + e^{y[\omega(k) - \omega(p)]} \right) \cdot \langle p|\sigma(x_1 - x_2, 0 - y)(1 - P_1)\sigma(0, 0)|k\rangle e^{i\pi z_k(k - p)} ,$$

where $P_1$ denotes the orthogonal projection operator onto the one-fermion subspace of the Fock space. The matrix element in the right-hand side can be represented as

$$\langle p|\sigma(x, y)(1 - P_1)\sigma(0, 0)|k\rangle = 2\pi \delta(p - k) \left[ \langle 0|\sigma(x, y)\sigma(0, 0)|0 \rangle - \sigma^2 \right] + \langle p|\sigma(x, y)(1 - P_1)\sigma(0, 0)|k\rangle_{reg} ,$$

where $x = x_1 - x_2$. The first singular term in the right-hand side represents the 'direct propagation part' [54], while the second term is a regular function of momenta at $k \to p$.

After substitution of (111) into (110) and subtraction the singular term we get

$$\delta_2 \langle p|H_R|k\rangle - 2\pi \delta(p - k) \delta_2 E_{vac}(h, R) =$$

$$-\frac{h^2}{2} \int_0^\infty dy \int_{-R/2}^{R/2} dx_1 \, dx_2 \left( 1 + e^{y[\omega(k) - \omega(p)]} \right) e^{i\pi z_k(k - p)} .$$

In this equation we can safely proceed to the limit $R \to \infty$. Comparing the result with (101), one finds the second order correction to the kink dispersion law

$$\delta_2 \epsilon(p, m, h) = -h^2 \int_0^\infty dy \int_{-\infty}^{\infty} dx \cdot \lim_{k \to p} \left[ \langle p|\sigma(x, y)\sigma(0, 0)|k\rangle - \langle p|\sigma(x, y)P_1\sigma(0, 0)|k\rangle \right] .$$

Even though the above relation was derived for small $|p|$ satisfying the first inequality in (100), we shall extend it to all real momenta $p$ by analytic continuation.

The second order correction to the squared quark mass can be read from (105) and (113),

$$\delta_2 [m_q(m, h)]^2 = -2h^2 \lim_{\beta \to \infty} \int_0^\infty dy \int_{-\infty}^{\infty} dx \cdot \lim_{\beta' \to \beta} \left[ \langle \beta'|\sigma(x, y)\sigma(0, 0)|\beta \rangle - \langle \beta'|\sigma(x, y)P_1\sigma(0, 0)|\beta \rangle \right] .$$

This integral representation for the second order correction to the quark mass [written in a slightly different form (B.25)] was first derived by Fonseca
and Zamolodchikov [54]. Exploiting the Ward identities, they managed to express the matrix element in the right-hand side in terms of solutions of certain differential equations, and obtained the value

$$a_q = s^2 \cdot 0.142021619(1) \ldots$$

for the parameter $a_q$,

$$a_q = 2s^2 a_2$$

by numerical integration of the double integral in (114) over the half-plane in polar coordinates $r, \theta$.

It turns out, that the integral in the polar angle can be evaluated analytically. The details of this calculation are relegated to Appendix B. The results read as,

$$U(r) \equiv \int_0^{\pi} \frac{d\theta}{\pi} \lim_{\beta' \to \beta} \langle \beta' | \sigma(r \cos \theta, r \sin \theta) \sigma(0, 0) | \beta \rangle =$$

and

$$W(r) \equiv \lim_{\beta \to \infty} \int_0^{\pi} \frac{d\theta}{\pi} \lim_{\beta' \to \beta} \langle \beta' | \sigma(r \cos \theta, r \sin \theta) P_1 \sigma(0, 0) | \beta \rangle =$$

where $b_0(r)$ stands for the solution of the second order differential equation

$$b''_0(r) + r^{-1} b_0(r) = \cosh[2 \varphi(r)] b_0(r),$$

which vanishes at $r \to \infty$, and behaves at small $r \to 0$ as

$$b_0(r) = \frac{1}{\Omega(r)} + O(r^4).$$

The auxiliary functions $\varphi(r)$, $\chi(r)$, and $\Omega(r)$ were defined in [54], $I_j(r)$ and $K_j(r)$ are the Bessel function of the imaginary argument and the McDonald’s function, respectively. In order to harmonize notations with Appendix B and reference [54], we have chosen the units of mass in equations (117) and (118) so that $m = 1$.

Though the integrals (117) and (118) both increase linearly at large $r$, their difference vanishes exponentially at $r \to \infty$. The remaining radial integration in (110) leads to the explicit representation for the coefficient $a_2$ in expansion (106),

$$a_2 = \frac{\pi}{2s^2} \int_0^\infty dr \left[ W(r) - U(r) \right].$$
Numerical evaluation of this integral yields
\[ a_2 = 0.0710108 \ldots \] (122)
in agreement with (115).

The described calculation procedure is based both on the summation of the
infinite form factor series (96), and on the explicit representations for the matrix
elements of the product of two spin operators between the one-fermion states,
derived by Fonseca and Zamolodchikov in [54]. Unfortunately, it is problematic
to extend this approach to other integrable models, since it essentially exploits
some rather specific features of the IFT, see the ‘Discussion’ Section in [54]. On
the other hand, a very good approximation for the constant \( a_2 \) can be obtained
by truncating the form factor series (96) at its first term accounting for the
three-kink intermediate states, \( n(q) = 3 \). We shall describe this technique in
some details here, and apply it in Section 5 to estimate the leading quark-mass
perturbative correction in the three-state PFT.

The first term \( \delta_{2,3} \langle p|H_R|k \rangle \) in the form factor series (96), which describes
contribution of the three-kink intermediate states has the following explicit form,
\[ \delta_{2,3} \langle p|H_R|k \rangle = -\frac{1}{2} \hbar^2 \int_{-\infty}^{\infty} dq_1 dq_2 dq_3 \frac{1}{(2\pi)^3} \Delta(Q - p, R) \Delta(Q - k, R) \cdot (123) \]

\[ \left\{ \frac{1}{\omega(q_1) + \omega(q_2) + \omega(q_3) - \omega(p)} + \frac{1}{\omega(q_1) + \omega(q_2) + \omega(q_3) - \omega(k)} \right\} \cdot \langle p|\sigma(0,0)|q_1, q_2, q_3 \rangle \langle q_3, q_2, q_1|\sigma(0,0)\rangle|k\rangle, \]
where \( Q = q_1 + q_2 + q_3 \), and
\[ \Delta(z, R) = \frac{2\sin(zR/2)}{z}. \] (124)

Note, that the two \( R \)-dependent factors in the integrand in (123) give rise to
the momentum conservation law in the large-\( R \) limit,
\[ \lim_{R \to \infty} \Delta(Q - p, R) \Delta(Q - k, R) = 4\pi^2 \delta(Q - p) \delta(k - p). \] (125)

The right-hand side of (123) is a well-defined generalized function for all
finite \( R \) under the conditions (100). Exploiting the Wick expansion, the product
of two matrix elements in the third line of (123) can be represented as the sum
of nine terms. Taking into account the symmetry of the integrand in (123) with
respect to permutations of momenta of three virtual kinks, one can leave only
two terms in this expansion multiplied by appropriate combinatoric factors. As
the result, the substitution
\[ \langle p|\sigma(0,0)|q_1, q_2, q_3 \rangle \langle q_3, q_2, q_1|\sigma(0,0)\rangle|k\rangle \sim 
6 \bar{\sigma}^{-2} \langle 0|\sigma(0,0)|q_2, q_3 \rangle \langle q_2, q_1|\sigma(0,0)\rangle|0\rangle \langle p|\sigma(0,0)|q_1 \rangle \langle q_3|\sigma(0,0)\rangle|k\rangle + 
3 \bar{\sigma}^{-2} \langle 0|\sigma(0,0)|q_2, q_3 \rangle \langle q_3, q_2|\sigma(0,0)\rangle|0\rangle \langle p|\sigma(0,0)|q_1 \rangle \langle q_1|\sigma(0,0)\rangle|k\rangle \] (126)
in the integrand in (123) leaves the integral unchanged.

One cannot proceed directly to the limit $R \to \infty$ in equation (123) exploiting equality (125). The problem comes from the product of two kinematic singularities in the form factors in the right-hand side of (126),

$$\langle p | \sigma(0,0) | q_1 \rangle \langle q_1 | \sigma(0,0) | k \rangle =$$

$$= \frac{2}{\sqrt{\omega(p)\omega(q_1)}} \frac{\omega(q_1) + \omega(k)}{\sqrt{\omega(q_1)\omega(k)}} \frac{1}{p - q_1} \frac{1}{q_1 - k},$$

where

$$\left[ \langle p | \sigma(0,0) | q_1 \rangle \langle q_1 | \sigma(0,0) | k \rangle \right]_{reg} =$$

$$= \frac{\omega(p) + \omega(q_1) + \omega(k)}{2 \sqrt{\omega(p)\omega(q_1)\omega(k)}} \left[ \frac{1}{(p - q_1)(k - q_1)} + \frac{1}{(p - q_1 + i0)(k - q_1 + i0)} \right].$$

Multiplication of the first term in the right-hand side of (127) representing the direct propagation part by the right-hand side of equations (125) leads to the familiar meaningful factor $[\delta(p - k)]^2$. This is not surprising, since the vacuum energy $E_{vac}(m,h,R) \sim R$ contributing to $\langle p | H_R | k \rangle$ diverges in the limit $R \to \infty$.

One can easily see, that the direct propagation part of the form factors (127), upon substitution into (126) and (123), gives rise to the term

$$2\pi \delta(p - k) \delta_{2,2} E(m,h,R),$$

where $\delta_{2,2} E(m,h,R)$ was defined in (65). After subtraction of (140) from (128), we obtain a generalized function that has a well defined limit at $R \to \infty$. According to (101), this limit must be identified with the three-kink contribution to the second order correction to the kink dispersion law,

$$2\pi \delta(p - k) \delta_{2,3} \epsilon(m,h,p) = \lim_{R \to \infty} \left[ \delta_{2,3} \langle p | H_R | k \rangle - 2\pi \delta(p - k) \delta_{2,2} E(m,h,R) \right].$$

After analytical continuation to all real $p$ and proceeding to the limit $p \to \infty$, one obtains from (130) and (102), the corresponding correction to the squared kink mass

$$\delta_{2,3} \left[ m_q(m,h) \right]^2 = m^2 \lambda^2 a_{2,3},$$

where

$$a_{2,3} = \frac{1}{16\pi^2} \lim_{p \to \infty} \left[ 2I_2(p) - I_1(p) \right]$$

is the three-kink contribution to the amplitude $a_2$. The explicit form of the
integers $I_j(p)$ reads as

$$I_j(p) = \frac{\delta(q_1 + q_2 + q_3 - p)}{\omega(q_1) \omega(q_2) \omega(q_3) \omega(q_1) + \omega(q_2) + \omega(q_3) - \omega(p)},$$

(133)

$$J_1(q_1, q_2, q_3) = \frac{[\omega(q_2) - \omega(q_3)]^2}{(q_2 + q_3)^2} \cdot \frac{[\omega(p) + \omega(q_1)]^2}{\mathcal{P} \left( \frac{1}{p - q_1} \right)^2},$$

(134)

$$J_2(q_1, q_2, q_3) = \frac{\omega(q_2) - \omega(q_1)}{q_2 + q_1} \cdot \frac{\omega(q_2) - \omega(q_3)}{q_2 + q_3} \cdot \frac{[\omega(p) + \omega(q_1)] \cdot [\omega(p) + \omega(q_3)]}{\mathcal{P} \left( \frac{1}{p - q_1} \right) \mathcal{P} \left( \frac{1}{p - q_3} \right)},$$

(135)

where

$$\mathcal{P} \left( \frac{1}{p - q} \right)^2 = \frac{1}{2} \left[ \frac{1}{(p - k - i0)^2} + \frac{1}{(p - k + i0)^2} \right].$$

(136)

The constant (132) was first numerically estimated by Fonseca and Zamolodchikov [11], $a_{2,3} \approx 0.07$. Its exact value

$$a_{2,3} = \frac{1}{16} + \frac{1}{12\pi^2} = 0.0709434 \ldots,$$

(137)

which is remarkably close to the total amplitude $a_2$ [see (122)], was announced later without derivation in [14]. To fill this gap, we present the rather involved derivation of (137) in Appendix C.

4. Form factors in the three-state PFT

The form factors of physically relevant operators in the three-state PFT were found in 1988 by Kirillov and Smirnov in the preprint [28] of the Kiev Institute for Theoretical Physics. In this section we briefly recall their results with
emphasis on the form factors of the disorder spin operator in the paramagnetic phase. Exploiting the duality of the PFT, one can simply relate them to the form factors of the spin order operators in the ferromagnetic phase, which will be used in the next section.

The set of nine operators operators $O_{ij}(x)$, $i, j = 0, 1, 2$ and their descendants were considered in [28]. The operators $O_{ij}$ transform in the following way under the action of the generator of the cyclic permutation $\Omega$ and charge conjugation $C$,

$$\Omega^{-1}O_{ij}\Omega = \upsilon^{i}O_{ij}, \quad C^{-1}O_{ij}C = O_{ij},$$

where $\upsilon = \exp(2\pi i/3)$, and $j = 3 - j \mod 3, 0 \leq j \leq 2$. The operators $O_{ij}(x)$ were identified in [28] as the main ones arising naturally in the three-state PFT. In particular, the operators $O_{j0}$ with $j = 1, 2$ are proportional to the disorder spin operators $\mu$ and $\bar{\mu}$,

$$O_{01}(x) = \mu(x), \quad O_{02}(x) = \bar{\mu}(x),$$

where $\langle \mu \rangle = \langle 0 | \mu(0) | 0 \rangle_{\text{par}} = \langle 0 | \bar{\mu}(0) | 0 \rangle_{\text{par}}$, and $| 0 \rangle_{\text{par}}$ is the (non-degenerate) paramagnetic vacuum. The operators $O_{j0}(x)$ transform as scalars under rotations. The operators $O_{j\sigma}$, $(j = 1, 2)$ are proportional to the order spin operators $\sigma$ and $\bar{\sigma}$, respectively. The operators $O_{jj}$, $(j = 1, 2)$ correspond to parafermions $\psi_{j}$ (regularized $\sigma\mu$ and $\bar{\sigma}\bar{\mu}$), while $O_{j\bar{j}}$ $(j = 1, 2)$ are parafermions $\bar{\psi}_{j}$ (regularized $\sigma\bar{\mu}$ and $\bar{\sigma}\mu$). Finally, the descendants of the operator $O_{00}(x)$ correspond to the components of the energy-momentum density tensor and to other local conserved fields. The conformal limit of these fields is described in [55].

We shall use notations (20) for the 3-state PFT rapidity basis states as well as the normalization convention (16) in order to harmonize the notations with [28]. The form factors of the operator $O_{ij}(0)$ are defined as the matrix elements of the form

$$f_{ij}(\beta_{1}, \ldots, \beta_{n})_{\epsilon_{1}, \ldots, \epsilon_{n}} = \text{par} \langle 0 | O_{ij}(0) | \beta_{n}, \ldots, \beta_{1} \rangle_{\epsilon_{n}, \ldots, \epsilon_{1}}.$$  

(142)

Due to their $\mathbb{Z}_{3}$-transformation properties, the form factors differ from zero only if $\sum_{k=1}^{n} \epsilon_{k} = i \mod 3$.

The following axioms [32, 28] are postulated for the form factors.

1. The symmetry property:

$$f_{ij}(\beta_{1}, \ldots, \beta_{n})_{\epsilon_{1}, \ldots, \epsilon_{n}} = \text{par} \langle 0 | O_{ij}(0) | \beta_{n}, \ldots, \beta_{1} \rangle_{\epsilon_{n}, \ldots, \epsilon_{1}}.$$  

(143)

2. The analytical continuation axiom:

$$f_{ij}(\beta_{1}, \ldots, \beta_{n} + 2\pi i)_{\epsilon_{1}, \ldots, \epsilon_{n}} =$$

$$\upsilon^{-j \epsilon_{n}} f_{ij}(\beta_{n}, \beta_{1}, \ldots, \beta_{n-1})_{\epsilon_{n}, \epsilon_{1}, \ldots, \epsilon_{n-1}}.$$  

(144)
3. The function \( f_{ij}(\beta_1, \ldots, \beta_n) \) analytically depends on the complex variables \( \beta_n \) and has only simple poles in the strip \( 0 \leq \text{Im} \beta_n \leq \pi \) located at the points \( \beta_n = \beta_k + \frac{2 \pi i}{3} \), and \( \beta_n = \beta_k + \pi i \). The residues at these points are:

\[
(2\pi)^{1/2} 3^{-1/4} \text{Res}_{\beta_n = \beta_k + 2\pi i/3} f_{ij}(\beta_1, \ldots, \beta_n) = \delta_{\epsilon_n, \epsilon_k} f_{ij}(\beta_1, \ldots, \beta_n + \frac{\pi i}{3}, \ldots, \beta_n - 1)_{\epsilon_1, \ldots, \epsilon_n - \epsilon_k, \ldots, \epsilon_n} \prod_{l > k} S_{\epsilon_l, \epsilon_k} (\beta_k - \beta_l + \frac{2\pi i}{3}),
\]

\[
2\pi i \text{Res}_{\beta_n = \beta_k + \pi i} f_{ij}(\beta_1, \ldots, \beta_n) = \delta_{\epsilon_n, \epsilon_k} f_{ij}(\beta_1, \ldots, \beta_k, \ldots, \beta_n - 1)_{\epsilon_1, \ldots, \epsilon_k, \ldots, \epsilon_n - \epsilon_k, \ldots, \epsilon_n} \prod_{l > k} S_{\epsilon_l, \epsilon_k} (\beta_k - \beta_l - \pi i \epsilon_k) \prod_{l < k} S_{\epsilon_k, \epsilon_l} (\beta_k - \beta_l).
\]

The calculation of the form factors \( f_{ij}(\beta_1, \ldots, \beta_n) \) determined by the above axioms was performed by Kirillov and Smirnov in [28]. Here we describe their results for the case \( i = 0 \), and \( j = 1, 2 \). It follows from (145), that the form factor \( f_{0j}(\beta_1, \ldots, \beta_n) \) can be expressed in terms of \( f_{0j}(\beta_1, \ldots, \beta_3n-1, \ldots, 1) \). The latter form factor will be denoted as \( f_{0j}(\beta_1, \ldots, \beta_{3n}) \). Its explicit representation reads as

\[
f_{0j}(\beta_1, \ldots, \beta_{3n}) = c^{-3n} g_{0j}(\beta_1, \ldots, \beta_{3n}) \exp \left( -\frac{1}{3} \sum_{q=1}^{3n} \beta_q \right) \prod_{1 \leq i < k \leq 3n} \zeta_{11}(\beta_i - \beta_k).
\]

Here

\[
c = -i \sqrt{2\pi} 3^{-1/12} \exp \left[ \frac{\psi^{(1)}(1/3) - \psi^{(1)}(2/3)}{12\sqrt{3} \pi} \right] = -i \cdot 2.5474074563745797..., \tag{148}
\]

where \( \psi^{(1)}(z) = \frac{d}{dz} \ln \Gamma(z) \) is the polygamma function. The function \( \zeta_{11}(\beta) \) is defined by the integral representation

\[
\zeta_{11}(\beta) = i 2^{-2/3} \sinh(\beta/2) \left\{ 2 \int_0^\infty dk \frac{\sin^2\left[\frac{\pi}{2}(\beta + i\pi)k\right] + \frac{\pi}{2} \sin^2\left(\frac{\pi k}{3}\right)}{k \sin^2(\pi k)} \sinh\left(\frac{\pi k}{3}\right) \right\},
\]

which converges in the strip \(-8\pi/3 < \text{Im} \beta < 2\pi/3 \). This function can be analytically continued into the whole complex \( \beta \)-plane, where it is meromorphic.
and satisfies the equalities,

\[ \zeta_{11}(\beta) S_{11}(\beta) = \zeta_{11}(-\beta), \quad \zeta_{11}(\beta - 2\pi i) = \zeta_{11}(-\beta), \quad (150) \]

\[ \zeta_{11} \left( \beta - \frac{2\pi i}{3} \right) \zeta_{11}(\beta) \zeta_{11} \left( \beta + \frac{2\pi i}{3} \right) = \frac{1}{4 \sinh \left( \frac{\beta}{2} - \frac{\pi i}{3} \right) \sinh \left( \frac{\beta}{2} + \frac{\pi i}{3} \right)}. \quad (151) \]

The function \( \zeta_{11}(\beta) \) has a simple pole at \( \beta = -2\pi i/3 \) with the residue

\[ \text{Res}_{\beta = -2\pi i/3} \zeta_{11}(\beta) = 3^{1/6} i \exp \left[ \frac{\psi(1)(1/3) - \psi(1)(2/3)}{12\sqrt{3} \pi} \right] = -\frac{3^{1/4} c}{\sqrt{2\pi}}. \quad (152) \]

Note that in equations (149) and (151) we have corrected some misprints which were present in [28].

The functions \( g_{0j}(\beta_1, \ldots, \beta_{3n}) \) have the following representation,

\[ g_{0j}(\beta_1, \ldots, \beta_{3n}) = P_{0j,n} \left( e^{\beta_1}, \ldots, e^{\beta_{3n}} \right) \exp \left[ -(n - 1) \sum_{q=1}^{3n} \beta_q \right], \quad (153) \]

where \( P_{0j,n}(x_1, \ldots, x_{3n}) \) is the uniform symmetric polynomial of the degree \( \deg(P_{0j,n}) = 3n^2 - nj \). The polynomial \( P_{0j,n}(x_1, \ldots, x_{3n}) \) can be represented as the determinant of the matrix \( M_{0j,n} \) of the order \( (2n-1) \times (2n-1) \), which has the matrix elements

\[ (M_{0j,n})_{pq} = \sigma_{3p-q-[q-1+j]/2}(x_1, \ldots, x_{3n}), \quad (154) \]

where \([a]\) denotes the integer part of \( a \), and \( \sigma_k \) is the elementary symmetric polynomial of the variables \( x_1, \ldots, x_{3n} \) of the degree \( k \), and \( \sigma_k = 0 \) for \( k < 0 \), and for \( k > 3n \).

The first polynomials \( P_{0j,n}(x_1, \ldots, x_{3n}) \) have the form,

\[ P_{01,1}(x_1, x_2, x_3) = \sigma_1 = x_1 + x_2 + x_3, \quad (155a) \]
\[ P_{02,1}(x_1, x_2, x_3) = \sigma_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad (155b) \]
\[ P_{01,2}(x_1, \ldots, x_6) = \sigma_1\sigma_3\sigma_4 - \sigma_4^2 - \sigma_2^2\sigma_6, \quad (155c) \]
\[ P_{02,2}(x_1, \ldots, x_6) = \sigma_2\sigma_3\sigma_5 - \sigma_2^2 - \sigma_3^2\sigma_6. \quad (155d) \]

Accordingly, the form factors (147) with \( n = 0, 1 \) read as,

\[ f_{01}(\varnothing) = 1, \quad f_{01}(\beta_1, \beta_2, \beta_3) = c^{-3} \left[ e^{(-\beta_1 - \beta_2 + 2\beta_3)/3} + e^{(-\beta_1 + \beta_2 + 2\beta_3)/3} + e^{(-\beta_2 - \beta_3 + 2\beta_1)/3} \right] \prod_{1 \leq i < k \leq 3} \zeta_{11}(\beta_i - \beta_k), \quad (156a) \]
\[ e^{(-\beta_2 - \beta_3 + 2\beta_1)/3} \prod_{1 \leq i < k \leq 3} \zeta_{11}(\beta_i - \beta_k), \quad (156b) \]

\[ f_{02}(\beta_1, \beta_2, \beta_3) = c^{-3} \left[ e^{(\beta_1 + \beta_2 - 2\beta_3)/3} + e^{(\beta_1 + \beta_3 - 2\beta_2)/3} + e^{(\beta_2 + \beta_3 - 2\beta_1)/3} \right] \prod_{1 \leq i < k \leq 3} \zeta_{11}(\beta_i - \beta_k). \quad (156c) \]
The matrix elements of general form can be constructed from the form factor by means of the crossing relations 32, 28. In particular,

\[-1 \langle \beta | \mu(0) | \beta_2, \beta_1 \rangle_{11} = \text{par} \langle 0 | \mu(0) | \beta_2, \beta_1, \beta - i\pi \rangle_{11} = (\mu)_{f01}(\beta - i\pi, \beta_1, \beta_2), \tag{157} \]

\[-1 \langle \beta | \bar{\mu}(0) | \beta_2, \beta_1 \rangle_{11} = \text{par} \langle 0 | \bar{\mu}(0) | \beta_2, \beta_1, \beta - i\pi \rangle_{11} = (\mu)_{f02}(\beta - i\pi, \beta_1, \beta_2). \tag{158} \]

The above matrix elements of the disorder operators relate to the paramagnetic phase. Let us connect them with the matrix elements of the order spin operators in the ferromagnetic phase. This can be easily done by means of the duality relations

\[
\mu(x)D = D\sigma(x), \tag{159a} \]

\[
\bar{\mu}(x)D = D\bar{\sigma}(x), \tag{159b} \]

which connect the order and disorder spin operators. It is implied in (159) that the order spin operators \(\sigma(x), \bar{\sigma}(x)\) act in the subspace \(L_0\) of the ferromagnetic space \(L_{\text{fer}}\), while the disorder spin operators \(\mu(x), \bar{\mu}(x)\) act in the subspace \(L_{\text{sym}}\) of the paramagnetic space \(L_{\text{par}}\). All these vector spaces were described in Section 2. Since \(|\beta_2, \beta_1, \beta - i\pi\rangle_{111} \in L_{\text{sym}}\), we can represent this vector as

\[
|\beta_2, \beta_1, \beta - i\pi\rangle_{111} = D|K_{02}(\beta_2)K_{21}(\beta_1)K_{10}(\beta - i\pi)). \]

After substitution of this equality into (157) and straightforward manipulations exploiting (159) and unitarity of the mapping \(D\), one obtains

\[
\text{par} \langle 0 | \mu(0) | \beta_2, \beta_1, \beta - i\pi \rangle_{111} = \langle 0 | \sigma(0) | K_{02}(\beta_2)K_{21}(\beta_1)K_{10}(\beta - i\pi). \]

Application of the crossing relation\(^2\) to the right-hand side yields

\[
\langle 0 | \sigma(0) | K_{02}(\beta_2)K_{21}(\beta_1)K_{10}(\beta - i\pi) = \langle K_{10}(\beta) | \sigma(0) | K_{02}(\beta_2)K_{21}(\beta_1). \]

The right-hand side can be further transformed to the form

\[
\langle K_{10}(\beta) | \sigma(0) | K_{02}(\beta_2)K_{21}(\beta_1) = v \langle K_{02}(\beta) | \sigma(0) | K_{21}(\beta_2)K_{10}(\beta_1), \]

exploiting the transformation rule \(\sigma(0) = v \tilde{\Omega} \sigma(0) \tilde{\Omega}^{-1}\), and 24. Thus, we obtain finally from the above analysis,

\[
-1 \langle \beta | \mu(0) | \beta_2, \beta_1 \rangle_{11} |_{\text{par}} = v \langle K_{02}(\beta) | \sigma(0) | K_{21}(\beta_2)K_{10}(\beta_1) |_{\text{fer}}. \tag{160} \]

Similarly, one can connect the matrix elements of the operators \(\bar{\mu}(0)\) and \(\bar{\sigma}(0)\),

\[
-1 \langle \beta | \bar{\mu}(0) | \beta_2, \beta_1 \rangle_{11} |_{\text{par}} = v^{-1} \langle K_{02}(\beta) | \bar{\sigma}(0) | K_{21}(\beta_2)K_{10}(\beta_1) |_{\text{fer}}. \tag{161} \]

\(^2\)The crossing relations in the ferromagnetic PFT was discussed by Delfino and Cardy in the Appendix A of reference 56.
Combining (160), (161) with (157), (156) we find the three-kink matrix element of the order operator \( \sigma_3(0) = (\sigma(0) + \bar{\sigma}(0))/3 \) in the ferromagnetic phase, which will be used in the next Section,

\[
\langle K_{02}(\beta)|\sigma_3(0)|K_{21}(\beta_2)K_{10}(\beta_1)\rangle_{\text{fer}} =
\]

\[
\frac{1}{3\epsilon^3} \zeta_{11}(\beta - \beta_1 - i\pi) \zeta_{11}(\beta - \beta_2 - i\pi) \zeta_{11}(\beta_1 - \beta_2)
\times \left\{ (e^{\beta_1} + e^{\beta_2} - e^{\beta}) \exp \left[ -\frac{\beta_1 + \beta_2 + \beta_3 + \pi i}{3} \right] +
(e^{-\beta_1} + e^{-\beta_2} - e^{-\beta}) \exp \left[ \frac{\beta_1 + \beta_2 + \beta_3 + \pi i}{3} \right] \right\}.
\]

Note that the function \( \zeta_{11}(\beta) \) defined by equation (149) admits the following explicit representation in terms of the dilogarithm function \( \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \),

\[
\zeta_{11}(\beta - i\pi) = -e^{-\beta/3} \left( 1 + e^{-\beta} \right) \left( 1 - e^{-\beta} + e^{-2\beta} \right)^{-5/6}
\times \left\{ \left. \frac{e^\beta - e^{i\pi/3}}{e^\beta - e^{-i\pi/3}} \right| \frac{i}{2\pi} \left[ \text{Li}_2 \left( e^{-\beta - i\pi/3} \right) - \text{Li}_2 \left( e^{-\beta + i\pi/3} \right) \right] \right\}.
\]

The function in the right-hand side is even and real at real \( \beta \). At \( \text{Re} \beta \to +\infty \) it behaves as

\[
\zeta_{11}(\beta - i\pi) = -e^{-\beta/3} \left[ 1 + \frac{11\pi + 3\sqrt{3}(1 + \beta)}{6\pi} e^{-\beta} + \right.
\]

\[
\left. \frac{55\pi^2 + 27(1 + \beta)^2 + 3\sqrt{3}\pi(25 + 28\beta)}{72\pi^2} e^{-2\beta} \right] + O(\beta^3 e^{-3\beta}).
\]

To conclude this section, let us present a useful formula for the dilogarithm function \( \text{Li}_2(e^{i\pi p/q}) \), with \( p < q \) for \( p, q \in \mathbb{N} \):

\[
\text{Li}_2(e^{i\pi p/q}) = \sum_{j=1}^{q} e^{i\pi jp/q}s(p, q, j),
\]

where

\[
s(p, q, j) = \sum_{l=0}^{\infty} \frac{e^{i\pi lp}}{(ql + j)^2} = \begin{cases} 
\frac{\psi^{(1)}(j/q)}{4q^2}, & \text{for even } p, \\
\frac{\psi^{(1)}(j/2q) - \psi^{(1)}((j+q)/(2q))}{4q^2}, & \text{for odd } p.
\end{cases}
\]

In particular,

\[
\text{Li}_2(e^{2i\pi/3}) = -\frac{\pi^2}{18} + i \frac{\psi^{(1)}(1/3) - \psi^{(1)}(2/3)}{6\sqrt{3}}.
\]

This equality has been used to derive from (163) the expression (152) for the residue of the function \( \zeta_{11}(\beta) \) at \( \beta = -2\pi i/3 \).
5. Second-order quark mass correction in the ferromagnetic three-state PFT

In this section we estimate the second-order radiative correction to the kink mass in the ferromagnetic 3-state PFT in the presence of a weak magnetic field \( h > 0 \) coupled to the spin component \( \sigma_3 \). Since very similar calculations for the case of the IFT were described in great details in Subsection 3.2 and Appendix C we can be brief.

In the presence of the magnetic field, the Hamiltonian of the PFT associated with the action (8) with \( q = 3 \) takes the form

\[
H = H_0 - h \int_{-\infty}^{\infty} dx \sigma_3(x),
\]

where the Hamiltonian \( H_0 \) corresponds to the integrable ferromagnetic 3-state PFT at zero magnetic field. The kinks \( K_{\mu \nu}(p) \) with the dispersion law

\[
\omega(p,m) = \sqrt{p^2 + m^2}
\]

are elementary excitations of the model at \( h = 0 \). For \( h > 0 \), they form mesonic and baryonic bound states at \( h > 0 \) in the confinement regime. Nevertheless, one can determine the kink dispersion law \( \delta_2 \epsilon(p,m,h) \) perturbatively in \( h \), as described in Subsection 3.2. For the leading second order radiative correction \( \delta_2 \epsilon(p,m,h) \sim h^2 \) to the dispersion law of the kink \( K_{20}(p) \), one can write down the form factor expansion

\[
\delta_2 \epsilon(p,m,h) = \sum_{n=2}^{\infty} \delta_{2,n} \epsilon(p,m,h),
\]

\[
\delta_{2,n} \epsilon(p,m,h) = -\frac{1}{n!} \frac{(2\pi)^2 h^2}{\omega(p)} \sum_{\mu_1,\ldots,\mu_{n-1}=0}^{2} \prod_{i=1}^{n-1} d\beta_i \frac{\delta(p_1 + \ldots + p_n - p)}{\omega_1 + \cdots + \omega_n - \omega} \times |\langle K_{02}(\beta)|\sigma_3(0)|K_{21}(\beta_2)K_{10}(\beta_1)\rangle|_{\text{reg}}^2,
\]

which is analogous to (106). Here \( p_j = m \sinh \beta_j, \omega_j = m \cosh \beta_j, p = m \sinh \beta, \omega = m \cosh \beta \). Of course, the same result holds for the kinks \( K_{10}(p) \), \( K_{02}(p) \) and \( K_{01}(p) \). The matrix elements in the right-hand side of (170) may contain kinematic singularities at \( \beta_j = \beta \), which must be regularized as done for the IFT in Subsection 3.2. The second-order radiative correction to the squared kink mass can be gained from \( \delta_2 \epsilon(p,m,h) \) by taking the ultra-relativistic limit. Using (105) gives

\[
\delta_2 m_q(m,h) = \lim_{p \to \infty} [\omega(p,m) \delta_2 \epsilon(p,m,h)].
\]

Let us truncate the form factor expansion (169) at its first term with \( n = 2 \),

\[
\delta_{2,2} \epsilon(p,m,h) = -\frac{1}{2} \frac{(2\pi)^2 h^2}{\omega(p)} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \frac{\delta(p_1 + p_2 - p)}{\omega_1 + \omega_2 - \omega} \times |\langle K_{02}(\beta)|\sigma_3(0)|K_{21}(\beta_2)K_{10}(\beta_1)\rangle|_{\text{reg}}^2.
\]
The matrix element in the right-hand side was calculated in the previous section, see equation (102). Since it is regular at all real $\beta, \beta_1, \beta_2$, it does not require regularization, in contrast to the subsequent terms in the expansion (169) with $n = 3, 4, \ldots$

Let us represent it in the form analogous to (106),

$$
\delta_{2,2} [m_q(m, h)]^2 = -(2\pi h)^2 \lim_{\beta \to \infty} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \frac{\delta(p_1 + p_2 - p)}{\omega_1 + \omega_2 - \omega} \times |\langle K_{02}(\beta) | \sigma_3(0) | K_{21}(\beta) K_{10}(\beta_1) \rangle|^2.
$$

(173)

Let us represent it in the form analogous to (106),

$$
\delta_{2,2} [m_q(m, h)]^2 = \lambda^2 a_{2,2} m^2,
$$

(174)

where $\lambda = f_0/m^2$ is the familiar dimensionless parameter proportional to the magnetic field $h$, and

$$
f_0 = h[0|0|\sigma_3(0)|0]\omega_0 - 2|0|\sigma_3(0)|0\omega_0] = \frac{3}{2} h[0|0|\sigma_3(0)|0]\omega_0 = h(0|\mu(0)|0)_{\text{par}}
$$

(175)

is the "bare" string tension in the weak confinement regime. For the dimensionless amplitude $a_{2,2}$, we obtain from (173) and (102),

$$
a_{2,2} = -\frac{16\pi^2}{9|c|^6} \lim_{\beta \to \infty} \int_{-\infty}^{\infty} d\beta_1 d\beta_2 \delta(\sinh \beta_1 + \sinh \beta_2 - \sinh \beta) \times (\cosh \beta_1 + \cosh \beta_2 - \cosh \beta) \left| \cosh \left( \frac{\beta_1 + \beta_2 + \beta + \pi i}{3} \right) \right|^2 \times |\zeta_{11}(\beta - \beta_1 - i\pi)\zeta_{11}(\beta - \beta_2 - i\pi)\zeta_{11}(\beta_1 - \beta_2)|^2.
$$

(176)

After changing the integration variables to $x_j = \sinh(\beta_j)/\sinh(\beta)$, $j = 1, 2$, and integrating over $x_2$ exploiting the $\delta$-function, one obtains

$$
a_{2,2} = \lim_{\beta \to \infty} \int_{-\infty}^{\infty} dx_1 \mathcal{M}(x_1, \sinh \beta).
$$

(177)

The function $\mathcal{M}(x_1, p)$ is even with respect to the reflection $x_1 \to 1 - x_1$, and has the following asymptotic behavior at large $p \to \infty$,

$$
\mathcal{M}(x_1, p) = \begin{cases} 
\mathcal{M}(x_1, \infty) + O(p^{-1}), & \text{for } 0 < x_1 < 1, \\
O(p^{-2}), & \text{for } x_1 < 0, \text{ and for } x_1 > 1,
\end{cases}
$$

(178)

where

$$
\mathcal{M}(x_1, \infty) = -\frac{8\pi^2}{9|c|^6} \frac{1 - x_1 + x_1^2}{x_1^{4/3}(1 - x_1)^{4/3}} \times |\zeta_{11}(-\ln x_1 - i\pi)\zeta_{11}[-\ln(1 - x_1) - i\pi]\zeta_{11}[\ln x_1 - \ln(1 - x_1)]|^2.
$$

(179)
Figure 1: Plot of the function $M(x_1, p)$ defined by (176), (177) for $p = 100$ (blue solid line), and of its ultra-relativistic limit $M(x_1, \infty)$ given by (179) (red circles).

Plots of $M(x_1, p)$ versus $x_1$ at $p = 100$ and at $p = \infty$ are shown in Figure 1.

Thus, we arrive at the result

$$a_{2,2} = \int_{0}^{1} dx_1 M(x_1, \infty),$$

with $M(x_1, \infty)$ given by (179). We did not manage to evaluate the integral in the right-hand side analytically, and instead computed it numerically using (163) and (164). The resulting number

$$a_{2,2} = -\frac{4}{27} + \delta, \quad \text{with } |\delta| < 2 \times 10^{-16}$$

is remarkably close to $-\frac{4}{27}$, which we assume to be the exact value of the amplitude $a_{2,2}$.

6. Conclusions

In this paper we have investigated the effect of the multi-quark (multi-kink) fluctuation on the universal characteristics of the IFT and 3-state PFT in the weak confinement regime, which is realized in these models in the low-temperature phase in the presence of a weak magnetic field. For this purpose we refined the form factor perturbation technique which was adapted in [14] for the confinement problem in the IFT. Due to proper regularization of the merging kinematic singularities arising from the products of spin-operator matrix elements, the refined technique allowed us to perform systematic high-order form factor perturbative calculations in the weak confinement regime. After verifying the efficiency of the proposed method by recovering several well-known results for the Ising model in the ferromagnetic phase in the scaling region, we have applied it to obtain the following new results.
• The explicit expression \( \tilde{G}_{3,2} \) for the contribution \( \tilde{G}_3 \) caused by two-quark fluctuations to the universal amplitude \( \tilde{G}_3 \), which characterizes the third derivative of the free energy of the scaling ferromagnetic Ising model with respect to the magnetic field \( h \) at \( h = 0 \).

• Proof of the announced earlier \[14\] exact result \( (137) \) for the amplitude \( a_{2,3} \) describing the contribution of three-quark fluctuations to the second order correction to the quark mass in the IFT in the weak confinement regime.

• We showed that the third order \( \sim h^3 \) correction to the quark self-energy and to the quark mass vanishes in the ferromagnetic IFT. This completes also calculations of the low-energy and semiclassical expansions for the meson masses \( M_n(h, m) \) in the weak confinement regime to third order in \( h \). The final expansions for \( M_n^2(h, m) \) to third order in \( h \) are described by the representations given in \[14\], since only the terms (which are now shown to be zero) proportional to the third order quark mass corrections were missing there.

In addition, a new representation \((117)-(121)\) for the amplitude \( a_2 \) characterizing the second order radiative correction to the quark mass in the ferromagnetic IFT was obtained by performing the explicit integration over the polar angle in the double-integral representation \((B.25)\) for this amplitude obtained in \[54\].

Finally, exploiting the explicit expressions for the form factors of the spin operators in the 3-state PFT at zero magnetic field obtained in \[28\], we have estimated the second-order radiative correction to the quark mass in the ferromagnetic 3-state PFT, which is induced by application of a weak magnetic field \( h > 0 \). To this end, we have truncated the infinite form factor expansion for the second-order correction to the quark mass at its first term, which represents fluctuations with two virtual quarks in the intermediate state. Our result for the corresponding amplitude \( a_{2,2} \) defined in \((174)\) is given in equations \((179)-(181)\), \((183)\).

To conclude, let us mention two possible directions for further developments.

Though the Bethe-Salpeter for the \( q \)-state PFT was obtained in paper \[14\], it was not used there for the calculation of the meson mass spectrum. Instead, the latter was determined in \[14\] to the leading order in \( h \) exploiting solely the zero-field scattering matrix known from \[26\]. The integral kernel in the Bethe-Salpeter for the \( q \)-state PFT equation contains matrix elements of the spin operator \( \sigma_q(0) \) between the two-quark states, that are not known for general \( q \). In the case of \( q = 3 \), however, such matrix elements can be gained from the form factors found by Kirillov and Smirnov \[28\]. This opens up the possibility to use the Bethe-Salpeter equation for the 3-state PFT for analytical perturbative evaluation of the meson masses in subleading orders in small \( h \). On the other hand, one can also study the magnetic field dependence of the meson masses in the 3-state PFT at finite magnetic fields by numerical solution of the Bethe-Salpeter equation. It was shown in \[12\] that the Bethe-Salpeter
equation reproduces surprisingly well the mesons masses in the IFT not only in the limit $h \to 0$, but also at finite, and even at large values of the magnetic field $h$. It would be interesting to check, whether this situation also takes place in the case of the 3-state PFT.

Recently, a dramatic effect of the kink confinement on the dynamics following a quantum quench was reported in [57, 58] for the IFT and for its discrete analogue - the Ising chain in both transverse and longitudinal magnetic fields. It was shown, in particular, that the masses of light mesons can be extracted from the spectral analysis of the post-quench time evolution of the one-point functions. It would be interesting to extend these results to the 3-state PFT, in which both mesons and baryons are allowed.

### Acknowledgements

I am grateful to A. B. Zamolodchikov for many important and stimulating discussions on the subject, and to H. W. Diehl for interesting communications and numerous suggestions leading to improvement of the text. I would like to thank F. A. Smirnov for sending me his preprint [28]. In the initial stage, this work was supported by Deutsche Forschungsgemeinschaft (DFG) via Grant Ru 1506/1.

### Appendix A. Calculation of the integral

Let us rewrite the double integral (88) in the rapidity variables $\beta = \text{arcsinh}(q/m)$, $\beta' = \text{arcsinh}(q'/m)$,

$$C_2 = -\frac{1}{4} \int_{-\infty}^{\infty} d\beta \frac{\sinh \beta}{\cosh^3 \beta} \int_{-\infty}^{\infty} d\beta' \frac{\sinh \beta'}{\cosh^3 \beta'} \left[ \coth^2 \left( \frac{\beta - \beta' + i0}{2} \right) + \coth^2 \left( \frac{\beta - \beta' - i0}{2} \right) \right],$$

(A.1)

and consider the function

$$u(\beta) = \int_{-\infty}^{\infty} d\beta' \frac{\sinh \beta'}{\cosh^3 \beta'} \coth^2 \left( \frac{\beta - \beta'}{2} \right)$$

(A.2)

defined in the strip $\Gamma = \{ \beta \in \Gamma | 0 < \text{Im} \beta < 2\pi \}$. It is straightforward to check its following properties.

1. The function $u(\beta)$ is analytic in the strip $\Gamma$ and vanishes there as $|\beta| \to \infty$.
2. $u(\pi i/2) = \pi i/4$, $u(3\pi i/2) = -\pi i/4$, $u(\pi i) = 0$.
3. The function $u(\beta)$ can be analytically continued to the whole complex $\beta$-plane, where it is meromorphic and obeys the quasi-periodicity relation

$$u(\beta + 2\pi i) = u(\beta) - 2\pi i v(\beta),$$

(A.3)

with

$$v(\beta) = \frac{4 [2 - \cos(2\beta)]}{\cosh^4 \beta}.$$  

(A.4)
4. The poles of \( u(\beta) \) lie at \( \frac{\pi i}{2} + in \), with \( n = -1, \pm 2, \pm 3, \ldots \) 

It is easy to prove that the above properties determine the function \( u(\beta) \) uniquely, and to obtain its explicit form,

\[
    u(\beta) = \frac{1}{4 \cosh^4 \beta} \left\{ 16(\beta - \pi i)[\cosh(2\beta) - 2] - 23\pi \sinh \beta - 24 \sinh(2\beta) + \pi \sinh(3\beta) \right\}.
\]  (A.5)

On the other hand, the double integral in equation (A.1) defining the constant \( C_2 \) can be rewritten in terms of the functions \( u(\beta), v(\beta) \) as

\[
    C_2 = -\frac{1}{4} \int_{-\infty}^{\infty} d\beta \frac{\sinh \beta}{\cosh^3 \beta} [2u(\beta) + 2\pi i v(\beta)].
\]  (A.6)

After substitution of (A.4) and (A.5) into the right-hand side of (A.6) and straightforward integration, one obtains finally,

\[
    C_2 = \frac{4}{3} + \frac{\pi^2}{8}.
\]  (A.7)

**Appendix B. Integration in the polar angle in (114)**

The subject of this Appendix is twofold. First, we prove that the representation (114) for the second-order radiative correction to the quark mass in the ferromagnetic IFT, which was derived in Section 3 in the frame of the modified form factor perturbative technique, is equivalent to the double-integral representation for the same quantity, which was obtained previously by Fonseca and Zamolodchikov, see equations (5.6), (5.10) in [54]. Second, we perform analytical integration over the polar angle in the above-mentioned double-integral representation, and express the amplitude \( a_2 \) as a single integral in the radial variable \( r \).

In order to simplify our further notations, we shall set throughout Appendices B and C the units of mass, length and momentum so that

\[
    m = 1.
\]  (B.1)

The second order correction to the quark mass in the ordered phase is given by equation (114), which determines the dimensionless amplitude \( a_2 \) in expansion (110),

\[
a_2 = -\frac{1}{2s^2} \lim_{\beta \to \infty} \int_0^\infty dy \int_{-\infty}^\infty dx \lim_{\beta' \to \beta} \langle \beta'|\sigma(x,y)(1 - P_1)\sigma(0,0)|\beta \rangle = \frac{-\pi}{2s^2} \lim_{\beta \to \infty} \int_0^\infty dr \int_0^{\pi} \frac{d\theta}{\pi} [G(r,\theta;\beta|\beta) - S_1(r,\theta;\beta|\beta)].
\]  (B.2)

In the second line we have proceeded to the polar coordinates \( r, \theta \) in the Euclidean half-plane, and used notations of [54] for the matrix elements of the spin.
operators,

\[ G(r, \theta; \beta | \beta) = \lim_{\beta' \to \beta} \langle \beta'|\sigma(x, y)\sigma(0, 0)|\beta \rangle, \]  

(B.3)

\[ S_1(r, \theta; \beta | \beta) = \lim_{\beta' \to \beta} \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \langle \beta'|\sigma(x, y)|\eta \rangle \langle \eta | \sigma(0, 0)|\beta \rangle, \]  

(B.4)

where \( x = r \cos \theta, y = r \sin \theta \).

Two further functions

\[ S_{-1}(r, \theta; \beta | \beta) = \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \langle 0 | \sigma(x, y)|\eta \rangle \langle \eta | \sigma(0, 0)|0 \rangle, \]  

(B.5)

\[ S_0(r, \theta; \beta | \beta) = S_{+1}(r, \theta; \beta | \beta) + S_{-1}(r, \theta; \beta | \beta) \]  

(B.6)

will be used in the sequel. Functions \( S_{+1}(r, \theta; \beta | \beta) \), describe the contributions of two different one-particle reducible components in the matrix element \( \langle \beta | \sigma(x, y)\sigma(0, 0)|\beta \rangle \), which were illustrated by two diagrams Fig. 3 in [54]. Their explicit expressions read as

\[ S_1(r, \theta; \beta | \beta) = 2rs^2 \cosh(\beta + i\theta) + \]  

(B.7a)

\[ s^2 \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \coth^2 \left( \frac{\beta + i\theta - \eta}{2} \right) e^{ir[\sinh \eta - \sinh(\beta + i\theta)]}, \]  

\[ S_{-1}(r, \theta; \beta | \beta) = s^2 \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \tanh^2 \left( \frac{\beta + i\theta - \eta}{2} \right) e^{ir[\sinh \eta + \sinh(\beta + i\theta)]}. \]  

(B.7b)

The above representations hold for real \( r, \beta, \theta \) lying in the region \( r > 0, -\infty < \beta < \infty \), and \( 0 < \theta < \pi \), and can be extended to complex values of these variables by analytical continuation. Note that the notation \( S(r, \theta; \beta | \beta) \) was used in [54] for the function \( S_0(r, \theta; \beta | \beta) \).

It follows from (B.6) and (B.7) that functions \( S_j(r, \theta; \beta | \beta) \), with \( j = 0, \pm 1 \), depend in fact on \( r \) and the combination \( \vartheta = \theta - i\beta \), being entire functions of the complex variable \( \vartheta \). These functions satisfy the following monodromy relations

\[ S_j(r, \varpi; \beta | \beta) = S_{-j}(r, 0; \beta | \beta) - 2jr s^2 \sinh \beta, \]  

(B.8)

One can easily see from (B.7b), that the function \( S_{-1}(r, \theta; \beta | \beta) \) vanishes in the limit \( \beta \to \infty \) at fixed \( r > 0 \) and \( \vartheta \in [0, \pi] \). This allows one to replace the function \( S_1(r, \theta; \beta | \beta) \) by \( S_0(r, \theta; \beta | \beta) \) in the integrand in the second line in (B.2),

\[ a_2 = -\frac{\pi}{2s^2} \lim_{\beta \to \infty} \int_{0}^{\infty} dr \int_{0}^{\pi} \frac{d\vartheta}{\pi} [G(r, \theta; \beta | \beta) - S_0(r, \theta; \beta | \beta)]. \]  

(B.9)

The explicit expression for the function \( G(r, \theta; \beta | \beta) \) in terms of the pair correlation functions

\[ G(r) = \langle 0 | \sigma(x, y)\sigma(0, 0)|0 \rangle, \quad \tilde{G}(r) = \langle 0 | \mu(x, y)\mu(0, 0)|0 \rangle, \]  

(B.10)

and associated auxiliary functions \( \Psi_{\pm}(r, \vartheta) \)

\[ G(r, \theta; \beta | \beta) = iG(r) [\Psi_{\pm}(r, \vartheta) \partial_{\vartheta} \Psi_{\pm}(r, \vartheta) - \Psi_{\pm}(r, \vartheta) \partial_{\vartheta} \Psi_{\pm}(r, \vartheta)] + \]  

\[ G(r) \Psi_{\pm}(r, \vartheta) \Psi_{\pm}(r, \vartheta), \]  

with \( \vartheta = \theta - i\beta \)  

(B.11)
were given in [54].

The IFT correlation functions \( G(r) \), \( \tilde{G}(r) \) were found by Wu, McCoy, Tracy, and Barouch in the classical paper [51]. Properties of these and related functions \( \Psi_{\pm}(r, \vartheta) \) are described in much detail in [54]. Following this paper, we shall reproduce some of these properties for later use.

The correlation functions (B.10) admit the following representations

\[
G(r) = \exp[\chi(r)] \sinh[\varphi(r)], \quad \tilde{G}(r) = \exp[\chi(r)] \cosh[\varphi(r)]
\]

in terms of the solutions of the ordinary Painlevé III differential equation,

\[
\varphi''(r) + \frac{1}{r} \varphi'(r) = \frac{1}{2} \sinh[2\varphi(r)],
\]

\[
\chi''(r) + \frac{1}{r} \chi'(r) = \frac{1}{2} \{1 - \cosh[2\varphi(r)]\}.
\]

The required solution is specified by its asymptotic behavior at \( r \to 0 \),

\[
\varphi(r) = -\ln \frac{r}{2} - \ln(-\Omega) + O(r^4 \Omega^2),
\]

\[
\chi(r) = \frac{1}{2} \ln(4r) + \ln(-\Omega) + O(r^2),
\]

where

\[
\Omega = \ln \left( \frac{e^\gamma}{8} r \right),
\]

and \( \gamma \) is the Euler’s constant. The solution \( \varphi(r) \) decays at large \( r \to \infty \) as

\[
\varphi(r) = \frac{1}{\pi} K_0(r) + O(e^{-3r}).
\]

The auxiliary functions \( \Psi_{\pm}(r, \vartheta) \) solve the system of partial differential equations

\[
\partial_r \Psi_+ = \frac{1}{4} \left( e^{\varphi+i\vartheta} - e^{-\varphi-i\vartheta} \right) \Psi_-, \quad (B.19a)
\]

\[
\partial_\vartheta \Psi_+ = \frac{i}{2} r \varphi' \Psi_+ + \frac{i}{4} \left( e^{\varphi+i\vartheta} + e^{-\varphi-i\vartheta} \right) r \Psi_-,
\]

and

\[
\partial_r \Psi_- = -\frac{1}{4} \left( e^{-\varphi+i\vartheta} - e^{\varphi-i\vartheta} \right) \Psi_+,
\]

\[
\partial_\vartheta \Psi_- = \frac{i}{2} r \varphi' \Psi_- - \frac{i}{4} \left( e^{-\varphi+i\vartheta} + e^{\varphi-i\vartheta} \right) r \Psi_+.
\]

They are entire functions of the complex variable \( \vartheta \) and satisfy the monodromy properties

\[
\Psi_+(r, \vartheta + \pi) = i \Psi_+(r, \vartheta), \quad \Psi_-(r, \vartheta + \pi) = -i \Psi_-(r, \vartheta).
\]

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The following equality

\[ \Psi_+(r, \vartheta) = \Psi_-(r, -\vartheta) \tag{B.22} \]

holds when both \( r \) and \( \vartheta \) are real. The overdetermined system of partial linear differential equation \( \text{(B.19), (B.20)} \) represents the Lax equations corresponding to the nonlinear Painlevé III equation \( \text{(B.13)} \).

The function \( G(r, \vartheta; \beta | \beta) \) depends in fact on \( r \) and the combination \( \vartheta = \theta - i\beta \), being the entire \( \pi \)-periodical function of the complex variable \( \vartheta \). The latter property leads to the equality

\[ G(r, 0; \beta | \beta) = G(r, \pi; \beta | \beta) \tag{B.23} \]

for all real \( \beta \) and \( r \).

Analyticity in \( \vartheta \) and the monodromy properties \( \text{(B.8), (B.23)} \) guaranty that the integral

\[ \int_0^\pi \frac{d\vartheta}{\pi} [G(r, \vartheta; \beta | \beta) - S_0(r, \vartheta; \beta | \beta)] \tag{B.24} \]

in the right-hand side of \( \text{(B.9)} \) does not depend on \( \beta \). This allows one to drop the \( \lim_{\beta \to \infty} \) sign in equation \( \text{(B.9)} \), yielding

\[ a_2 = -\frac{\pi}{2s^2} \int_0^\infty r \, dr \int_0^\pi \frac{d\theta}{\pi} [G(r, \vartheta; \beta | \beta) - S_0(r, \vartheta; \beta | \beta)], \tag{B.25} \]

with \( \beta \)-independent right-hand side. The integral representation \( \text{(B.24)} \) for the amplitude \( a_2 \) is equivalent to equations (5.6), (5.10) obtained for the amplitude \( a_q \) [see equation (116)] by Fonseca and Zamolodchikov in \[54\].

Now, let us proceed to the calculation of the integral \( \text{(B.24)} \). We start from the first term

\[ \mathcal{U}(r) = \int_0^\pi \frac{d\vartheta}{\pi} G(r, \vartheta; \beta | \beta). \tag{B.26} \]

Since the right-hand side does not depend on \( \beta \), we shall put \( \beta = 0 \) in it, and replace \( \vartheta \) by \( \theta \) in equations \( \text{(B.11), (B.22)} \).

Let us introduce three functions

\[ f_1(r, \theta) = [\Psi_+(r, \theta)]^2 \tag{B.27} \]
\[ f_2(r, \theta) = [\Psi_-(r, \theta)]^2, \]
\[ f_3(r, \theta) = \Psi_+(r, \theta)\Psi_-(r, \theta), \]

which provide the ‘spin-1’ Lax representation

\[ -i \partial_\theta f_i(r, \theta) = \sum_{j=1}^{3} f_j(r, \theta) U_{ji}(r, \theta), \tag{B.28} \]
\[ \partial_r f_i(r, \theta) = \sum_{j=1}^{3} f_j(r, \theta) V_{ji}(r, \theta) \]
for the Painlevé III equation \( \text{[B.13]} \). The matrices \( U_{ji}(r, \theta) \) and \( V_{ji}(r, \theta) \) are defined as

\[
U(r, \theta) = \begin{pmatrix}
-r \varphi'(r) & 0 & -\frac{1}{2} \left[ e^{\varphi(r)} + e^{-\varphi(r)} v \right] \\
0 & r \varphi'(r) & \frac{1}{2} \left[ e^{\varphi(r)} v + e^{-\varphi(r)} \right] \\
\frac{1}{2} \left[ e^{\varphi(r)} v + e^{-\varphi(r)} \right] & -\frac{1}{2} \left[ e^{\varphi(r)} - e^{-\varphi(r)} v \right] & 0
\end{pmatrix},
\]

\[
V(r, \theta) = \begin{pmatrix}
0 & 0 & \frac{1}{2} \left[ e^{\varphi(r)} v - e^{-\varphi(r)} v \right] \\
0 & 0 & \frac{1}{2} \left[ e^{\varphi(r)} v - e^{-\varphi(r)} v \right] \\
\frac{1}{2} \left[ e^{\varphi(r)} v - e^{-\varphi(r)} \right] & \frac{1}{2} \left[ e^{\varphi(r)} v - e^{-\varphi(r)} v \right] & 0
\end{pmatrix},
\]

where \( v = e^{i\theta} \). The above 'spin-1' Lax equations can be easily deduced from the 'spin-1/2' Lax equations \( \text{[B.19]} \), \( \text{[B.20]} \) for the functions \( \Psi_\pm(r, \theta) \).

Taking into account the symmetry properties \( \text{[B.21]} \), \( \text{[B.22]} \), the Fourier expansions for the functions \( f_j(r, \theta) \) can be written in the form

\[
f_1(r, \theta) = \sum_{l=-\infty}^{\infty} v^{2l+1} a_l(r), \tag{B.31}
\]

\[
f_2(r, \theta) = \sum_{l=-\infty}^{\infty} v^{-2l-1} a_l(r),
\]

\[
f_3(r, \theta) = b_0(r) + \sum_{l=1}^{\infty} (v^{2l} + v^{-2l}) b_l(r).
\]

Using equations \( \text{[B.28]} \), all Fourier coefficients \( a_l(r) \) and \( b_l(r) \) can be expressed recursively in terms of the coefficient \( b_0(r) \) and its derivative \( b_0'(r) \). The latter function solves the second order linear differential equation \( \text{[B.19]} \) which also follows from \( \text{[B.28]} \).

Asymptotical behavior of the function \( b_0(r) \) at small and large \( r \) can be gained from the known asymptotical behavior of the functions \( \Psi_\pm(r, \theta) \) described in \( \text{[54]} \). The result for small \( r \to 0 \) reads as

\[
b_0(r) = \frac{1}{\Omega} + r^4 g_4 + r^8 g_8 + ..., \tag{B.32}
\]

where

\[
g_4 = \frac{16\Omega^3 - 8\Omega^2 + 1}{2^{11}\Omega^2}, \tag{B.33}
\]

\[
g_8 = \frac{8192\Omega^6 - 12288\Omega^5 + 7296\Omega^4 - 1568\Omega^3 - 111\Omega + 64}{2^{28}\Omega^3}.
\]

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For the $r \to \infty$ asymptotics one finds,
\begin{equation}
\mathfrak{b}_0(r) = 2 I_0(r) + O(e^{-r}). \tag{B.34}
\end{equation}

Exploiting equations (B.28), the function $G(r, \theta; 0|0)$ determined by (B.11) can be represented as a linear combination of functions $f_j(r, \theta)$,
\begin{equation}
G(r, \theta; 0|0) = \tilde{G}(r) f_3(r, \theta) + G(r) \left[ f_3(r, \theta) r \phi'(r) - \frac{r}{4} \left( v e^{-\varphi(r)} + v^{-1} e^{\varphi(r)} \right) f_2(r, \theta) \right]. \tag{B.35}
\end{equation}

After substitution of the Fourier expansions (B.31) in the right-hand side, the integration over the polar angle in (B.26) becomes trivial. As the result, one represents the integral $U(r)$ as a linear combination of the Fourier coefficients $\mathfrak{b}_0(r)$, $\mathfrak{a}_0(r)$, and $\mathfrak{a}_{-1}(r)$. Expressing the latter two coefficients in terms of $\mathfrak{b}_0(r)$ and $\mathfrak{b}_0'(r)$, one arrives at the result given by equation (117).

In order to complete the evaluation of the integral (B.24), it remains to calculate the second term,
\begin{equation}
\mathcal{W}(r) = \int_0^\pi \frac{d\theta}{\pi} S_0(r, \theta; \beta|\beta). \tag{B.36}
\end{equation}

Since the right-hand side does not depend on $\beta$, we shall put $\beta = 0$ in it.

Let us define an auxiliary function of the complex variable $\tilde{\beta}$,
\begin{equation}
\mathfrak{f}(\tilde{\beta}, r) = \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \exp[i r (\sinh \eta - \sinh \tilde{\beta})] \coth \frac{\eta - \tilde{\beta}}{2}, \tag{B.37}
\end{equation}
where $0 < \text{Im} \tilde{\beta} < 2\pi$, and the radius $r > 0$ is fixed. The function $\mathfrak{f}(\tilde{\beta}, r)$, analytically continued to the whole complex $\tilde{\beta}$-plane, satisfies there the quasiperiodicity relation
\begin{equation}
\mathfrak{f}(\tilde{\beta} + 2\pi i, r) = \mathfrak{f}(\tilde{\beta}, r) + 2i. \tag{B.38}
\end{equation}

For the derivative $\partial_{\tilde{\beta}} \mathfrak{f}(\tilde{\beta}, r)$, one can easily derive the following two representations from (B.37),
\begin{equation}
\partial_{\tilde{\beta}} \mathfrak{f}(\tilde{\beta}, r) = \frac{r}{\pi} \left[ i K_0(r) \sinh \tilde{\beta} - K_1(r) \right] \exp(-ir \sinh \tilde{\beta}), \tag{B.39}
\end{equation}
for all complex $\tilde{\beta}$, and
\begin{equation}
\partial_{\tilde{\beta}} \mathfrak{f}(\tilde{\beta}, r) = -ir \mathfrak{f}(\tilde{\beta}, r) \cosh \tilde{\beta} - \frac{K_0(r) \exp(-ir \sinh \tilde{\beta})}{2\pi} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\eta}{2\pi} \exp[i r (\sinh \eta - \sinh \tilde{\beta})] \coth^2 \frac{\eta - \tilde{\beta}}{2}, \tag{B.40}
\end{equation}
for $0 < \text{Im} \tilde{\beta} < 2\pi$. 

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Comparison of (B.40) with (B.7) yields

\[
\frac{S_1(r, \theta; \beta|\beta)}{s^2} = \left\{ 2r[1 + i f(\tilde{\beta}, r)] \cosh \tilde{\beta} + \frac{K_0(r)e^{-ir \sinh \tilde{\beta}}}{\pi} + 2 \partial_{\tilde{\beta}} f(\tilde{\beta}, r) \right\}_{||\tilde{\beta} = \beta + i\theta},
\]

\[
\frac{S_{-1}(r, \theta; \beta|\beta)}{s^2} = \left\{ -2i r f(\tilde{\beta} + i\pi, r) \cosh \tilde{\beta} + \frac{K_0(r)e^{ir \sinh \tilde{\beta}}}{\pi} + 2 \partial_{\tilde{\beta}} f(\tilde{\beta} + i\pi, r) \right\}_{||\tilde{\beta} = \beta + i\theta}.
\]

Upon adding these two equalities and putting \(\beta = 0\) in the result, one finds,

\[
\frac{S_0(r, \theta; 0|0)}{s^2} = \left\{ 2r[1 + i f(\tilde{\beta}, r) - i f(\tilde{\beta} + i\pi, r)] \cos \theta + \frac{K_0(r)e^{r \sin \tilde{\theta}} + e^{-r \sin \tilde{\theta}}}{\pi} - 2i \left[ \partial_{\tilde{\beta}} f(i\theta, r) + \partial_{\tilde{\beta}} f(i\theta + i\pi, r) \right] \right\}.
\]

Subsequent straightforward integration over \(\theta\) and use of equalities (B.38) and (B.39) yields finally,

\[
\mathcal{W}(r) = \int_0^\pi \frac{d\theta}{\pi} S_0(r, \theta; 0|0) = \frac{2s^2}{\pi} \left\{ (1 - 2r^2) I_0(r) K_0(r) - 2r K_1(r) [I_0(r) + r I_1(r)] \right\},
\]

Appendix C. Calculation of \(a_{2,3}\)

In this Appendix we perform the exact calculation of the amplitude \(a_{2,3}\) given by equation (132), which characterize the three-kink contribution to the second-order radiative correction to the kink mass in the ferromagnetic IFT. To this end, we evaluate the integrals \(I_1(p)\) and \(I_2(p)\) determined by equations (133)-(136) in the limit \(p \to \infty\), and show that

\[
\lim_{p \to \infty} I_1(p) = -\frac{4}{3}, \quad \lim_{p \to \infty} I_2(p) = \frac{\pi^2}{2}.
\]

The momentum variables will be normalized throughout this Appendix to the "bare" kink mass according to the convention (B.1).

Proceeding to the calculation of the large-\(p\) asymptotics of the integral \(I_1(p)\), let us transform it to the variables \(x_j = q_j/p, j = 1, 2, 3\), and expand the
For the first term, we get
\[ p \]
we split it into two terms as
\[ \frac{1}{2p} \left( \frac{1}{|x_1|} + \frac{1}{|x_2|} + \frac{1}{|x_3|} - 1 \right) + O(p^{-3}) \]
becomes small \( \sim p^{-1} \) on the part of the hyperplane defined by
\[ x_1 + x_2 + x_3 = 1, \quad \text{with} \quad x_{1,2,3} > 0, \] (C.3)
let us assume for a while that the leading contribution to the integral in the limit \( p \to \infty \) comes from the region (C.3) \(^3\). Under this assumption, one obtains from (C.3), (C.4) at large \( p \),
\[ I_1(p) = 2 \int_0^1 dx_1 dx_2 dx_3 \frac{\delta(x_1 + x_2 + x_3 - 1)}{(x_2 + x_3)(1 - x_2)(1 - x_3)} \] (C.4)
\[ \left( \frac{x_2 - x_3}{x_2 + x_3} \right)^2 (1 + x_1)^2 \mathcal{P} \left( \frac{1}{1 - x_1} \right)^2 + O(p^{-1}). \]
After trivial integration over \( x_1 \) and proceeding to the symmetric variables \( u = x_2 + x_3, v = x_2 x_3, \) one obtains from (C.4),
\[ I_1(p) = 4 \int_0^1 du \frac{(2 - u)^2}{u^5} \int_0^{u^2/4} dv \frac{(u^2 - 4v)^{1/2}}{1 - u + v} + O(p^{-1}) = \]
\[ 4 \int_0^1 du \frac{(2 - u)^2}{u^5} [-2u + (u - 2) \ln(1 - u)] + O(p^{-1}). \] (C.5)
The last integral diverges near its lower bound \( u = 0 \). This divergence indicates that the developed procedure cannot correctly describe the contribution of small momenta \( |q_{2,3}| \ll p \) to the integral \( I_1(p) \) defined by (C.3), (C.4) in the limit \( p \to \infty \). In order to regularize the integral \( \int_0^1 du \) in the right-hand side of (C.5), we split it into two terms as \( \int_0^1 du + \int_\epsilon^1 du \), where \( \epsilon \) is an arbitrary small positive number. Thus, \( I_1(p) \) becomes
\[ I_1(p) = I_{1,>}(\epsilon) + I_{1,<}(\epsilon) + O(p^{-1}). \] (C.6)
For the first term, we get
\[ I_{1,>}(\epsilon) = 4 \int_\epsilon^1 du \frac{(2 - u)^2}{u^5} [-2u + (u - 2) \ln(1 - u)] = \frac{8}{3\epsilon} - \frac{4}{3} + O(\epsilon). \] (C.7)
We replace the second (diverging) integral \( I_{1,<}(\epsilon) \) in (C.6) by the \( p \to \infty \) limit of its converging finite-\( p \) counterpart,
\[ \lim_{p \to \infty} I_1(p) = I_{1,>}(\epsilon) + \lim_{p \to \infty} I_{1,<}(p, \epsilon p), \] (C.8)
\(^3\)This assumption is not completely correct. As is shown below, the vicinity of the point \( x_1 = 1, x_2 = 0, x_3 = 0 \) also gives a considerable contribution to the integral \( I_1(p) \) at \( p \to \infty \).
where
\[ I_{1,\langle}(p, q) = \int_{-\infty}^{\infty} dq_1 dq_2 dq_3 \frac{\delta(q_1 + q_2 + q_3 - p)}{\omega(q_1) \omega(q_2) \omega(q_3) \omega(q_1) + \omega(q_2) + \omega(q_3) - \omega(p)} \cdot \mathcal{J}_1(q_1, q_2, q_3) \eta(q - q_2 - q_3). \]  
(C.9)

Here \( \eta(z) \) stands for the unit-step function,

\[ \eta(z) = \begin{cases} 1, & \text{for } z > 1, \\ 0, & \text{for } z \leq 0, \end{cases} \]  
(C.10)

and \( q = \epsilon p \) denotes the cut-off momentum.

After integration over \( q_1 \) and proceeding to the limit \( p \to \infty \) at a fixed positive \( q \), one obtains,

\[ I_{1,\langle}(p, q) = 4p J_{\langle}(q) + O(p^{-1}), \]  
(C.11)

where
\[ J_{\langle}(q) = \int_{-\infty}^{\infty} dq_2 dq_3 \frac{1}{\omega(q_2) \omega(q_3) - \omega(q_2) + \omega(q_3) - q_3} \cdot \left[ \frac{\omega(q_2) - \omega(q_3)}{q_2 + q_3} \right]^2 P \left( \frac{1}{q_2 + q_3} \right)^2 \eta(q - q_2 - q_3). \]  
(C.12)

First, let us show that the integral (C.12) vanishes, if the unit-step function in the integrand is dropped,

\[ J = \int_{-\infty}^{\infty} dq_2 dq_3 \frac{1}{\omega(q_2) \omega(q_3) - \omega(q_2) + \omega(q_3) - q_3} \cdot \left[ \frac{\omega(q_2) - \omega(q_3)}{q_2 + q_3} \right]^2 P \left( \frac{1}{q_2 + q_3} \right)^2 = 0. \]  
(C.13)

Really, after a change of the integration variables to
\[ x = q_2 + \omega(q_2) + q_3 + \omega(q_3), \quad y = [q_2 + \omega(q_2)](q_3 + \omega(q_3)), \]  
(C.14)

we get
\[ J = 8 \int_0^\infty dy P \frac{y^2}{(y - 1)^2} \int_2^{\infty} dx \frac{\sqrt{x^2 - 4y}}{x^5} = \frac{\pi}{16} \int_0^\infty dy \sqrt{y} P \left( \frac{1}{y - 1} \right)^2 = 0. \]

Due to (C.13), one concludes that
\[ J_{\langle}(q) = -J_{\rangle}(q), \]  
(C.15)

where
\[ J_{\rangle}(q) = \int_{-\infty}^{\infty} dq_2 dq_3 \frac{1}{\omega(q_2) \omega(q_3) - \omega(q_2) + \omega(q_3) - q_3} \cdot \left[ \frac{\omega(q_2) - \omega(q_3)}{q_2 + q_3} \right]^2 \eta(q_2 + q_3 - q) \frac{(q_2 + q_3)}{(q_2 + q_3)^2}. \]  
(C.16)
and it remains to calculate the large-$q$ asymptotics of the integral \((C.16)\). Transforming in this integral to the variables \((C.14)\), one obtains
\[
J_>(q) = 8 \int_0^\infty dx \int_0^\infty dy \frac{y^2}{(y-1)^2} \frac{\sqrt{x^2 - 4y}}{x^5} \eta(x^2 - 4y)\eta((y-1)x - 2qy). \tag{C.17}
\]

After one more change of the integration variable \(y = x^2 w\), we get
\[
J_>(q) = 8 \int_0^{1/4} dw \frac{w^2}{4w^2} \sqrt{1 - 4w} X(w, q), \tag{C.18}
\]
where
\[
X(w, q) = \int_{x_0(w, q)}^\infty dx \frac{x^2}{(1 - x^2 w)^2}, \tag{C.19}
\]
and
\[
x_0(w, q) = q + \sqrt{q^2 + w^{-1}}.
\]
Elementary integration in \(x\) yields
\[
X(w, q) = \frac{1 + 2q\sqrt{w} \arccoth \left( q \frac{w^{1/2} + \sqrt{1+q^2 w}}{1-q^2} \right)}{4qw^2}.
\]
Substitution of the large-$q$ asymptotics of this function
\[
X(w, q) = \frac{1}{2qw^2} + O(q^{-3})
\]
into \((C.18)\) and subsequent integration over \(w\) leads finally to the asymptotics
\[
J_>(q) = \frac{2}{3q} + O(q^{-3}) \tag{C.20}
\]
at \(q \gg 1\). Combining this result with \((C.15)\) and \((C.11)\), one obtains,
\[
\lim_{p \to \infty} \mathcal{I}_{1,<}(p, \epsilon p) = -\frac{8}{3\epsilon} + O(\epsilon) \tag{C.21}
\]
at \(\epsilon \ll 1\). Adding \((C.21)\) with \((C.7)\), we arrive at the result \((C.1)\).

Now let us proceed to the proof of equality \((C.2)\). Starting from the equations \((133)\) and \((135)\), which define the integral \(\mathcal{I}_2(p)\), we first perform the integration over \(q_2\) by means of the \(\delta\)-function, then change the integration variables to \(x_j = q_j/p\), with \(j=1,3\), and formally proceed to the limit \(p \to \infty\). The result reads as
\[
\lim_{p \to \infty} \mathcal{I}_2(p) = \int_0^1 dx_1 \int_{x_0}^{1-x_1} dx_3 Y(x_1, x_3), \tag{C.22}
\]
where
\[
Y(x_1, x_3) = 2 \frac{(1+x_1)(1+x_3)(1-2x_1-x_3)(1-x_1-2x_3)}{(1-x_1)^3(1-x_3)^3}.
\]
The double integral in the right-hand side over the triangle AOB shown in Figure C.2 logarithmically diverges near the edges A and B of the triangle. In order to regularize this integral, we divide the triangle AOB into the polygon $\Gamma(\epsilon)$, which is dashed in Figure C.2 and two small rectangular triangles $\Delta_A,\Delta_B(\epsilon)$ adjacent to the edges A and B. The legs of these small triangles have the length $\epsilon$. The integral over the polygon $\Gamma(\epsilon)$ approaches in the limit $\epsilon \to 0$ a constant value,

\[
\lim_{\epsilon \to 0} \iint_{\Gamma(\epsilon)} dx_1 dx_3 Y(x_1, x_3) = \frac{\pi^2}{2}.
\]

(C.23)

The similar integrals over small triangles adjacent to the points A and B are equal to one another, but formally diverge. To prove equation (C.2), it remains to show that these integrals vanish after regularization. To this end, let us consider the integral

\[
\mathcal{I}_{2,A}(p,q) = \int_{-\infty}^{\infty} dq_1 dq_2 dq_3 \frac{\delta(q_1 + q_2 + q_3 - p)}{\omega(q_1) \omega(q_2) \omega(q_3)} \left( \omega(q_1) + \omega(q_2) + \omega(q_3) - \omega(p) \right) \cdot \mathcal{J}_2(q_1, q_2, q_3) \eta(q - q_2 - q_3),
\]

(C.24)

where $\mathcal{J}_2(q_1, q_2, q_3)$ is given by (135), and $q = \epsilon p$. Clearly, this well-defined
After integration over \( p \) regularized counterpart of the diverging integral
\[
\int_{\Delta_\epsilon} dx_1 dx_3 Y(x_1, x_3).
\] (C.25)
After integration over \( q_1 \), one finds from (C.24) at fixed \( q > 0 \) in the limit \( p \to \infty \),
\[
\lim_{p \to \infty} I_{2, \epsilon}(p, q) = 2 \int_{-\infty}^{\infty} dq_2 dq_3 \frac{1}{\omega(q_2)\omega(q_3)} \frac{\omega(q_2) - q_2 + \omega(q_3) - q_3}{q_2 + q_3} \frac{1}{\eta(q - q_2 - q_3)} \mathcal{P} \left( \frac{1}{q_2 + q_3} \right) = 0,
\] (C.26)
since the integrand in the right-hand side is odd with respect of the permutation \( q_2 \leftrightarrow q_3 \). This completes the proof of equations (C.2).

**Appendix D. Proof of equations (D.2)**

The form factor expansion (97) contains the product of three matrix elements of the spin operators
\[
\langle p|\sigma(0, 0)|q_1\rangle \langle q_1|\sigma(0, 0)|q_1'\rangle \langle q_1'|\sigma(0, 0)|k\rangle.
\] (D.1)
Here \( |q_1\rangle \) and \( |q_1'\rangle \) stand for the intermediate fermionic states with odd numbers of kinks, \( n(q_1) \), and \( n(q_1') \), respectively. Note, that \( n(q_1) + n(q_1') \geq 4 \). By means of the Wick expansion, the product (D.1) can be brought into the sum of products of \( n(q_1) + n(q_1') + 1 \) elementary form factors (35)-(37). Some terms in this Wick expansion contain the products of three elementary form factors of the form
\[
\langle p|\sigma(0, 0)|q_i\rangle \langle q_i|\sigma(0, 0)|q_j\rangle \langle q_j|\sigma(0, 0)|k\rangle, \quad 1 \leq i \leq n(q), \quad 1 \leq i \leq n(q').
\] (D.2)
where \( 1 \leq i \leq n(q) \), and \( 1 \leq i \leq n(q') \). Extracting the direct propagation part from such a product, one can represent it as
\[
\langle p|\sigma(0, 0)|q_i\rangle \langle q_i|\sigma(0, 0)|q_j'\rangle \langle q_j'|\sigma(0, 0)|k\rangle = 4\pi^2 \langle p|\sigma(0, 0)|k\rangle \delta(q_i - k) \delta(q_j' - k) + \langle p|\sigma(0, 0)|q_i\rangle \langle q_i|\sigma(0, 0)|q_j'\rangle \langle q_j'|\sigma(0, 0)|k\rangle \rangle_{reg},
\] (D.3)
The regularized product of two elementary form factors standing in the last line was defined in (128). One can also extract the direct propagation part from the product (D.2) in a different way
\[
\langle p|\sigma(0, 0)|q_i\rangle \langle q_i|\sigma(0, 0)|q_j'\rangle \langle q_j'|\sigma(0, 0)|k\rangle = 4\pi^2 \langle p|\sigma(0, 0)|k\rangle \delta(q_i - p) \delta(q_j' - p) + \langle p|\sigma(0, 0)|q_i\rangle \langle q_i|\sigma(0, 0)|q_j'\rangle \rangle_{reg}(q_j'|\sigma(0, 0)|k\rangle.
\] (D.4)
Taking the arithmetic average of equations \((D.3)\) and \((D.4)\), we get the right-hand side in the symmetrized form
\[
\langle p|\sigma(0,0)|q_i\rangle \langle q_i|\sigma(0,0)|q'_j\rangle \langle q'_j|\sigma(0,0)|k\rangle = \tag{D.5}
\]
\[
\left[\langle p|\sigma(0,0)|q_i\rangle \langle q_i|\sigma(0,0)|q'_j\rangle \langle q'_j|\sigma(0,0)|k\rangle\right]_{\text{dpp}} +
\]
\[
\left[\langle p|\sigma(0,0)|q_i\rangle \langle q_i|\sigma(0,0)|q'_j\rangle \langle q'_j|\sigma(0,0)|k\rangle\right]_{\text{reg}},
\]
where
\[
\left[\langle p|\sigma(0,0)|q_i\rangle \langle q_i|\sigma(0,0)|q'_j\rangle \langle q'_j|\sigma(0,0)|k\rangle\right]_{\text{dpp}} = \tag{D.6}
2\pi^2 \langle p|\sigma(0,0)|k\rangle \delta(q_i - k) \delta(q'_j - k) + \delta(q_i - p) \delta(q'_j - p),
\]
\[
\left[\langle p|\sigma(0,0)|q_i\rangle \langle q_i|\sigma(0,0)|q'_j\rangle \langle q'_j|\sigma(0,0)|k\rangle\right]_{\text{reg}} = \tag{D.7}
\frac{1}{2} \left[\langle q_i|\sigma(0,0)|q'_j\rangle \langle q'_j|\sigma(0,0)|k\rangle\right]_{\text{reg}} \langle p|\sigma(0,0)|q_i\rangle +
\]
\[
\frac{1}{2} \left[\langle p|\sigma(0,0)|q_i\rangle \langle q_i|\sigma(0,0)|q'_j\rangle\right]_{\text{reg}} \langle q'_j|\sigma(0,0)|k\rangle.
\]
Collecting all terms in the Wick expansion, that contain the factors of the form \((D.2)\), and leaving in those only the symmetrized direct propagation parts \((D.6)\), one obtains the direct propagation part \(\delta_3(p|H_R|k)_{\text{dpp}}\) of the matrix element \(\delta_3(p|H_R|k)\). Denoting the rest of the latter by \(\delta_3(p|H_R|k)_{\text{reg}}\), we arrive to equation \((D.6)\).

The matrix element \(\delta_3(p|H_R|k)\) defines in a standard way a distribution, which acts on a 'good enough' test function \(\phi(p, k) \in \mathcal{D}_S\) having a compact support \(\mathcal{S}\),
\[
\delta_3 H_R [\phi] = \iint dp\, dk\, \delta_3(p|H_R|k)\, \phi(p, k). \tag{D.8}
\]

To avoid the resonance poles, the support of the test function will be taken inside the square, \(\mathcal{S} \subset (-p_0, p_0)^2\) with \(p_0 = 3^{1/2}m\). Due to the symmetry relation \((D.6)\), the test functions \(\phi(p, k)\) can be chosen odd,
\[
\phi(p, k) = -\phi(k, p) \quad \text{for} \quad \phi \in \mathcal{D}_S, \tag{D.9}
\]
without loss of generality.

Similarly to \((D.3)\), one can determine the action on \(\phi \in \mathcal{D}_S\) of the distribution \(\delta_3 H_{R,\text{dpp}}\) associated with the direct propagation part of the matrix element \(\delta_3(p|H_R|k)_{\text{dpp}}\),
\[
\delta_3 H_{R,\text{dpp}}[\phi] = \iint dp\, dk\, \delta_3(p|H_R|k)_{\text{dpp}}\, \phi(p, k) = \tag{D.10}
\]
\[
2i \iint dp\, dk\, \phi(p, k) \int \frac{dQ\, dQ'}{4\pi^2} D_1(p - k, Q, Q'; R) \mathcal{Y}_{\text{dpp}}(p, k, Q, Q'; m, h),
\]
with
\[
D_1(P, Q, Q'; R) = \frac{8\sin[(P - Q)R/2]\sin[(Q - Q')R/2]\sin[Q'R/2]}{(P - Q)(Q - Q'Q')} \frac{1}{P^r}. \tag{D.11}
\]
Here $Q$ and $Q'$ denote the total momenta of the intermediate kink states in the form factor expansion. The function $\mathcal{Y}_{dpp}(p, k, Q, Q'; m, h)$ in the right-hand side of (D.10), which is analytic in its momenta variables for $\{p, k\} \in \mathcal{S}$ and all $Q, Q'$, has the following symmetry properties

$$\mathcal{Y}_{dpp}(p, k, Q, Q'; m, h) = \mathcal{Y}_{dpp}(k, p, Q', Q; m, h).$$

(D.12)

$$\mathcal{Y}_{dpp}(p, k, Q, Q'; m, h) = \mathcal{Y}_{dpp}(p, k, -Q, -Q'; m, h).$$

(D.13)

and reduces at $p = k$, and $Q = Q' = 0$ to the third correction to the vacuum energy density (73),

$$\mathcal{Y}_{dpp}(k, k, 0, 0; m, h) = \delta_3 \rho(m, h).$$

(D.14)

It remains to proceed to the large-$R$ limit in equation (D.10). To this end, let us determine how the distribution (D.11) acts on the plane wave test function,

$$J(X; R) = \int_{-\infty}^{\infty} dP dQ dQ' D_{1}(P, Q, Q'; R) \cdot \exp\{i[PX_1 + Q(X_2 - X_1) + Q'(X_3 - X_2)]\},$$

where $X = \{X_1, X_2, X_3\}$. After the change of integration variables

$$P = u_1 + u_2 + u_3, \quad Q = u_2 + u_3, \quad Q' = u_3,$

the integral representation for the function $J(X; R)$ takes a symmetric form, and can be easily calculated

$$J(X; R) = \int_{-\infty}^{\infty} du_1 du_2 du_3 \mathcal{P} \frac{1}{u_1 + u_2 + u_3} \prod_{j=1}^{3} \frac{2 e^{i u_j X_j} \sin(u_j R/2)}{u_j} =

\frac{i}{2} \int_{-\infty}^{\infty} d\lambda \text{sign}\lambda \prod_{j=1}^{3} \int_{-\infty}^{\infty} du_j \frac{2 e^{i u_j (X_j - \lambda)} \sin(u_j R/2)}{u_j} =

4\pi^3 i \eta(R - X_{\text{max}} + X_{\text{min}}) \cdot
[\text{max}(X_{\text{min}}, -R - X_{\text{min}}) + \text{min}(X_{\text{max}}, R - X_{\text{max}})],$$

(D.15)

where $\eta(z)$ is the unit-step function (C.10). In the second line in (D.15), the integral representation

$$\mathcal{P} \frac{1}{u} = \frac{i}{2} \int_{-\infty}^{\infty} d\lambda e^{-i\lambda u} \text{sign}\lambda$$

has been used. Proceeding to the limit $R \to \infty$ in (D.15), one obtains

$$J(X; \infty) = 4\pi^3 i (X_{\text{max}} + X_{\text{min}}).$$

(D.16)

This results indicates that the distribution (D.11) remains nonlocal in the limit $R \to \infty$. It turns out, however, that the large-$R$ limit of (D.11) determines the...
following local distribution

$$
\lim_{R \to \infty} \int_{-\infty}^{\infty} dP dQ dQ' \ D_1(P, Q, Q'; R) \Phi(P, Q, Q') = (D.17)
$$

$$
4\pi^2 \left( 2 \partial_P + \partial_Q + \partial_{Q'} \right) \Phi(P, Q, Q') \bigg|_{P=Q=Q'=0},
$$

when it is restricted to the space of test functions, that satisfy the symmetry relation

$$
\Phi(P, Q, Q') = -\Phi(-P, -Q', -Q). \quad (D.18)
$$

To prove equality (D.17), it is sufficient to check that it holds for the 'antisymmetrized plane-wave' test function

$$
\exp \left\{ i[PX_1 + Q(X_2 - X_1) + Q'(X_3 - X_2)] \right\} - \exp \left\{ i[-PX_1 - Q'(X_2 - X_1) - Q(X_3 - X_2)] \right\},
$$

which obeys (D.18). This can be easily done by application of (D.16). Combining (D.17) with (D.9)–(D.13), we arrive at (139a).

The proof of equation (139b) is simpler. The regular part $\delta_3\langle p|H_R|k \rangle_{\text{reg}}$ of the matrix element $\delta_3\langle p|H_R|k \rangle$ was defined according to equation (138) as

$$
\delta_3\langle p|H_R|k \rangle_{\text{reg}} = \delta_3\langle p|H_R|k \rangle - \delta_3\langle p|H_R|k \rangle_{\text{dpp}}. \quad (D.19)
$$

After integration over the variables $x_1, x_2, x_3$, it takes the form

$$
\delta_3\langle p|H_R|k \rangle_{\text{reg}} = \int \frac{dQ dQ'}{4\pi^2} D_0(p, k, Q', Q; R) Y_{\text{reg}}(p, k, Q, Q'; m, h), \quad (D.20)
$$

where

$$
D_0(p, k, Q, Q'; R) = \frac{8 \sin[(p - Q)R/2] \sin[(Q - Q')R/2] \sin[(Q' - k)R/2]}{(p - Q)(Q - Q')(Q' - k)}. \quad (D.21)
$$

The function $Y_{\text{reg}}(p, k, Q, Q'; m, h)$ is regular near the hyperplane $p = k$, and vanishes on it

$$
Y_{\text{reg}}(k, k, Q, Q'; m, h) = 0 \quad (D.22)
$$

due to the symmetry relation

$$
Y_{\text{reg}}(p, k, Q, Q'; m, h) = -Y_{\text{reg}}(k, p, Q', Q; m, h).
$$

Exploiting the equality

$$
\lim_{R \to \infty} D_0(p, k, Q, Q'; R) = 8\pi^3 \delta(p - k) \delta(Q - k) \delta(Q' - k),
$$

one can proceed to the limit $R \to \infty$ in equation (D.19), which, by virtue of (D.22), leads the result (139b).
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