An exponential-type integrator for the KdV equation

Martina Hofmanová · Katharina Schratz

Abstract We introduce an exponential-type time-integrator for the KdV equation and prove its first-order convergence in $H^1$ for initial data in $H^3$. Furthermore, we outline the generalization of the presented technique to a second-order method.

We consider the Korteweg-de Vries (KdV) equation

$$\partial_t u(t, x) + \partial_x^3 u(t, x) = \frac{1}{2} \partial_x (u(t, x))^2, \quad u(0, x) = u_0(x), \quad t \in \mathbb{R},$$

$$x \in \mathbb{T} = [-\pi, \pi],$$

(1)

where for practical implementation issues we impose periodic boundary conditions. For local-wellposedness results of the periodic KdV equation in low regularity spaces we refer to [1,5,19].

In the context of the numerical time integration of (non)linear partial differential equations splitting methods as well as exponential integrators contribute attractive classes of integration methods. We refer to [7,8,10,18] for an extensive overview, and in particular to [3,4,16] for the analysis of splitting methods for Schrödinger(-Poisson) equations. In recent years, splitting as well as exponential integration schemes (including Lawson type Runge–Kutta methods [15]) have also gained a lot of attention in the context of the numerical integration of the KdV equation, see for instance [9,11–
and the references therein. We also refer to [2] for a splitting approach for the 
Kadomtsev–Petviashvili equation.

In particular, a distinguished convergence result was obtained in [11, 12]. In the latter 
it was proven that the Strang splitting, where the right-hand side of the KdV equation 
is split into the linear and Burgers part, respectively, is second-order convergent in $H^r$
for initial data in $H^{r+5}$ for $r \geq 1$ assuming that the Burgers part is solved exactly.

Here we derive a first-order exponential-type time-integrator for the KdV equation (1) based on Duhamel’s formula

$$u(t) = e^{-\frac{3}{4}t}u_0 + \frac{1}{2} \int_0^t e^{-\frac{3}{4}(t-s)}\partial_x(u(s))^2 ds \quad (2)$$

looking at the “twisted variable” $v(t) = e^{\frac{3}{4}t}u(t)$. This idea of “twisting” the variable is 
widely used in the analysis of partial differential equations in low regularity spaces (see, 
for instance [1, 5, 19] for the periodic KdV equation) and also well known in the context 
of numerical analysis, see [15] for the introduction of Lawson type Runge–Kutta 
methods. However, instead of approximating the appearing integral with a Runge– 
Kutta method (see for instance [13]) we use the key relation

$$k_1^3 + k_2^3 - (k_1 + k_2)^3 = -3(k_1 + k_2)k_1k_2 \quad (3)$$

which allows us to overcome the loss of derivative by integrating the stiff parts (i.e., the 
terms involving $\partial_x^3$) exactly. The derived exponential-type integrator is unconditionally 
stable and we will in particular show its first-order convergence in $H^1$ for initial data 
in $H^3$. A key tool in our convergence analysis is a variant of [12, Lemma 3.1].

The presented technique can be generalized to higher-order methods. We outline 
the construction of a second-order exponential-type integrator in Remark 1.4.

**Notation** In the following we will denote the Fourier expansion of some function 
$f \in L^2(\mathbb{T})$ by $f(x) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ikx}$. Furthermore, we will use the notation

$$\left(\partial_x^{-1}\right)_k := \begin{cases} 
(ik)^{-1} & \text{if } k \neq 0 \\
0 & \text{if } k = 0 
\end{cases} \quad \text{i.e., } \partial_x^{-1}f(x) = \sum_{k \in \mathbb{Z}} (ik)^{-1} \hat{f}_k e^{ikx}. \quad (4)$$

1 An exponential-type integrator

To illustrate the idea we first consider initial values with zero mean. In Remark 1.3 we 
point out the generalization to general initial values.

**Assumption 1.1** Assume that the zero-mode of the initial value is zero, i.e., $\hat{u}_0(0) = (2\pi)^{-1} \int_T u(0, x) dx = 0$. Note that the conservation of mass then implies that $\hat{u}_0(t) = 0$.

We will derive a scheme for the “twisted” variable $v(t) = e^{\frac{3}{4}t}u(t)$. With this 
transformation at hand the equation in $v$ reads

Springer
An exponential-type integrator for the KdV equation

\[ v(t) = v_0 + \frac{1}{2} \int_0^t e^{s \partial_x^3} \left( e^{-\partial_x^3 v(s)} \right)^2 ds \]  

such that

\[ v(t_n + \tau) = v(t_n) + \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \left( e^{-\partial_x^3 v(t_n+s)} v(t_n+s) \right)^2 ds. \]  

For a small time-step \( \tau \) we iterate Duhamel’s formula (5) and approximate the exact solution (6) as follows

\[ v(t_n + \tau) \approx v(t_n) + \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \left( e^{-\partial_x^3 v(t_n)} v(t_n) \right)^2 ds. \]  

The key relation (3) now allows us the following integration technique (cf. [1,5,19]):

We have

\[ \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left( e^{-\partial_x^3 v(t_n)} v(t_n) \right)^2 ds \]

\[ = \sum_{k_1,k_2} \int_0^\tau e^{-i(t_n+s)((k_1+k_2)^3-k_1^3-k_2^3)} i(k_1 + k_2) \hat{v}_{k_1}(t_n) \hat{v}_{k_2}(t_n) e^{i(k_1+k_2)x} ds \]

\[ = \sum_{k_1,k_2} \frac{e^{-i(t_n+\tau)((k_1+k_2)^3-k_1^3-k_2^3)} - e^{-i(t_n)((k_1+k_2)^3-k_1^3-k_2^3)}}{-i((k_1+k_2)^3-k_1^3-k_2^3)} \cdot i(k_1 + k_2) \hat{v}_{k_1}(t_n) \hat{v}_{k_2}(t_n) e^{i(k_1+k_2)x} \]

\[ \times \frac{1}{3k_1 k_2} \hat{v}_{k_1}(t_n) \hat{v}_{k_2}(t_n) e^{i(k_1+k_2)x} \]

\[ = \frac{1}{3} e^{\partial_x^3 (t_n+\tau)} \left( e^{-\partial_x^3 v(t_n+\tau)} \partial_x^{-1} v(t_n) \right)^2 - \frac{1}{3} e^{\partial_x^3 t_n} \left( e^{-\partial_x^3 t_n} \partial_x^{-1} v(t_n) \right)^2. \]

Together with the approximation in (7) this yields that

\[ v^{n+1} = v^n + \frac{1}{6} e^{\partial_x^3 (t_n+\tau)} \left( e^{-\partial_x^3 (t_n+\tau)} \partial_x^{-1} v^n \right)^2 - \frac{1}{6} e^{\partial_x^3 t_n} \left( e^{-\partial_x^3 t_n} \partial_x^{-1} v^n \right)^2, \]  

where \( \partial_x^{-1} \) is defined in (4) and by construction \( \hat{v}_0^{n+1} = 0 \), see Remark 1.2 below.

**Remark 1.2** The zero-mode is preserved by the scheme (9) as the key relation (3) implies that

\[ \hat{v}_0^{n+1} = \hat{v}_0^n - \frac{1}{6} \sum_{k_1+k_2=0} e^{-i(t_n+\tau)(k_1+k_2)k_1 k_2} - e^{-i(t_n)(k_1+k_2)k_1 k_2} \frac{1}{k_1 k_2} \hat{v}_{k_1} \hat{v}_{k_2} = \hat{v}_0^n. \]
In order to obtain an approximation to the original solution $u(t_n)$ of the KdV equation (1) at time $t_n = n\tau$ we then “twist” the variable back again by setting $u^n = e^{-\partial^{-1}x t_n}v^n$. This yields the following exponential-type integrator for the KdV equation (1)

$$u^{n+1} = e^{-\tau \partial^3_x}u^n + \frac{1}{6} e^{-\tau \partial^3_x \partial^{-1}x u^n} - \frac{1}{6} e^{-\tau \partial^3_x \partial^{-1}x u^n}^2,$$

where $\partial^{-1}x$ is defined in (4) and $\tilde{u}_0^{n+1} = 0$ thanks to Remark 1.2. For sufficiently smooth solutions the semi-discrete scheme (10) is first-order convergent, see Corollary 2.8 below for the precise convergence result.

**Remark 1.3** If $\tilde{u}_0(0) = \alpha \neq 0$ we set $\tilde{u} := u - \alpha$ and look at the modified KdV equation in $\tilde{u}$, i.e.,

$$\partial_t \tilde{u} + \partial^3_x \tilde{u} = \alpha \partial_x \tilde{u} + \frac{1}{2} \partial_x (\tilde{u})^2.$$

Note that the solution $\tilde{u}(t)$ of the modified KdV equation (11) satisfies $\tilde{u}_0(t) = \alpha$ for all $t$ as by the conservation of mass we have that $\tilde{u}_0(t) = \alpha$. Thus, we can proceed as above: We look at the twisted variable $\tilde{v}(t) = e^{(\partial^3_x - \alpha \partial_x)} \tilde{u}(t)$ and carry out an approximation as above, i.e.,

$$\tilde{v}(t_n + \tau) \approx \tilde{v}(t_n) + \frac{1}{2} \int_0^\tau e^{(t_n + s)(\partial^3_x - \alpha \partial_x) \partial_x} \left( e^{-(t_n + s)(\partial^3_x - \alpha \partial_x) \tilde{v}(t_n)} \right)^2 \, ds.$$

The relation

$$-(k_1 + k_2)^3 + \alpha (k_1 + k_2) + k_1^3 + k_2^3 - \alpha k_1 - \alpha k_2 = -(k_1 + k_2)^3 + k_1^3 + k_2^3$$

then allows us to derive similarly to above an exponential-type integration scheme

$$\tilde{v}^{n+1} = \tilde{v}^n + \frac{1}{6} e^{(\partial^3_x - \alpha \partial_x) t_n + \tau} \left( e^{-(\partial^3_x - \alpha \partial_x) (t_n + \tau) \partial_x^{-1} \tilde{v}^n} \right)^2 - \frac{1}{6} e^{(\partial^3_x - \alpha \partial_x) t_n} \left( e^{-(\partial^3_x - \alpha \partial_x) t_n \partial_x^{-1} \tilde{v}^n} \right)^2,$$

where $\partial^{-1}x$ is defined in (4) and $\tilde{u}_0^{n+1} = 0$ cf. Remark 1.2. Finally, by setting $u^n = e^{-(\partial^3_x - \alpha \partial_x) t_n} \tilde{v}^n + \alpha$ we then obtain an approximation to the exact solution $u(t_n)$ of the KdV equation (1) (with non-zero zero-mode) at time $t_n = n\tau$.

Note that higher-order approximations to the solution of the KdV equation (1) can be obtained by truncating the expansion in (6) later. In Remark 1.4 below we explain the construction of a second-order scheme in more detail.

**Remark 1.4** (A second-order exponential-type integrator) In order to derive a second-order approximation in the “twisted” variable $v(t)$ we need to include the second-order
term in the Taylor-series expansion of $v(t_n + s)$ in (6). More precisely, plugging the formal expansion
\[ v(t_n + s) = v(t_n) + s v'(t_n) + O\left(s^2 v''\right) \]
into Duhamel’s formula (6) yields that
\[ v(t_n + \tau) = v(t_n) + \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left( e^{-\partial_x^3(t_n+s)} (v(t_n) + s v'(t_n)) \right)^2 ds \]
\[ + \mathcal{R}_1(\tau, t_n, v) \]
\[ = v(t_n) + \frac{1}{2} \int_0^\tau e^{(t_n+s)\partial_x^3} \partial_x \left[ \left( e^{-\partial_x^3(t_n+s)} v(t_n) \right)^2 + 2s \left( e^{-\partial_x^3(t_n+s)} v(t_n) \right) \left( e^{-\partial_x^3(t_n+s)} v'(t_n) \right) \right] ds \]
\[ + \mathcal{R}_1(\tau, t_n, v) + \mathcal{R}_2(\tau, t_n, v), \]
where the remainders satisfy for $r > 1/2$ and some constant $c > 0$ that
\[ \| \mathcal{R}_1(\tau, t_n, v) \|_r + \| \mathcal{R}_2(\tau, t_n, v) \|_r \]
\[ \leq c \tau^3 \sup_{t_n \leq t \leq t_{n+1}} \left( \| \partial_x (v v') (t) \|_r + \| \partial_x (v')^2 (t) \|_r \right). \]

In order to construct a numerical scheme based on the expansion (13) we need to solve the appearing integral. The first term involving the product $v^2$ can be easily determined thanks to (8). Note that $\hat{v}_0(t) = \hat{v}'_0(t) = 0$. Thus, similarly, we obtain for the $v v'$ term with the aid of the key-relation (3) and integration by parts that
\[
\int_0^\tau s \times e^{(t_n+s)\partial_x^3} \partial_x \left( e^{-\partial_x^3(t_n+s)} v(t_n) \right) \left( e^{-\partial_x^3(t_n+s)} v'(t_n) \right) ds \\
= \sum_{k_1, k_2} \int_0^\tau s \times e^{-i(t_n+s)(k_1+k_2)^3-k_1^3-k_2^3)} i(k_1 + k_2) \hat{v}_{k_1}(t_n) \hat{v}'_{k_2}(t_n) e^{i(k_1+k_2)x} ds \\
= \sum_{k_1 \neq 0, k_2 \neq 0} \int_0^\tau s \times e^{-i(t_n+s)(k_1+k_2)^3-k_1^3-k_2^3)} i(k_1 + k_2) \hat{v}_{k_1}(t_n) \hat{v}'_{k_2}(t_n) e^{i(k_1+k_2)x} ds \\
= \tau \sum_{k_1 \neq 0, k_2 \neq 0} e^{-i(t_n+\tau)(k_1+k_2)^3-k_1^3-k_2^3)} \frac{1}{-3k_1k_2} \hat{v}_{k_1}(t_n) \hat{v}'_{k_2}(t_n) e^{i(k_1+k_2)x} \\
- \sum_{k_1 \neq 0, k_2 \neq 0} \int_0^\tau e^{-i(t_n+s)(k_1+k_2)^3-k_1^3-k_2^3)} \frac{1}{-3k_1k_2} \hat{v}_{k_1}(t_n) \hat{v}'_{k_2}(t_n) e^{i(k_1+k_2)x} ds \\
= \tau \sum_{k_1 \neq 0, k_2 \neq 0} e^{-i(t_n+\tau)(k_1+k_2)^3-k_1^3-k_2^3)} \frac{1}{-3k_1k_2} \hat{v}_{k_1}(t_n) \hat{v}'_{k_2}(t_n) e^{i(k_1+k_2)x} \\
- \sum_{k_1 \neq 0, k_2 \neq 0} \int_0^\tau e^{-i(t_n+s)(k_1+k_2)^3-k_1^3-k_2^3)} \frac{1}{-3k_1k_2} \hat{v}_{k_1}(t_n) \hat{v}'_{k_2}(t_n) e^{i(k_1+k_2)x} ds.
\[- \sum_{k_1 \neq 0, k_2 \neq 0 \atop k_1 + k_2 \neq 0} \frac{e^{-i(t_n + \tau)\left(\left((k_1 + k_2)^3 - k_1^3 - k_2^3\right)\right)}}{9i k_1^2 k_2^2 (k_1 + k_2)} \]

\[ \times \hat{\vartheta}_1(t_n) \hat{\vartheta}_2(t_n) e^{i(k_1 + k_2)x} \, ds \]

\[ = \frac{\tau}{3} e^{(t_n + \tau) \partial_x^3} \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v(t_n) \right) \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v'(t_n) \right) \]

\[ - \frac{1}{9} e^{(t_n + \tau) \partial_x^3} \partial_x^{-1} \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v(t_n) \right) \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v'(t_n) \right) \]

\[ + \frac{1}{9} e^{(t_n + \tau) \partial_x^3} \partial_x^{-1} \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v(t_n) \right) \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v'(t_n) \right) \] (15)

with \( \partial_x^{-1} \) defined in (4). Plugging the relations given in (8) and (15) together with the definition

\[ v^n := \frac{1}{2} e^{t_n \partial_x^3} \left( e^{-t_n \partial_x^3} v^n \right)^2 \] (16)

[see (5)] into the expansion (13) builds the basis of our numerical scheme: As a second-order approximation to the solution \( v(t_n + \tau) \) of (5) we take the exponential-type integration scheme

\[ v^{n+1} = v^n + \frac{1}{6} e^{(t_n + \tau) \partial_x^3} \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v^n \right)^2 - \frac{1}{6} e^{t_n \partial_x^3} \left( e^{-\partial_x^3 t_n} \partial_x^{-1} v^n \right)^2 \]

\[ + \frac{\tau}{3} e^{(t_n + \tau) \partial_x^3} \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v^n \right) \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v^n \right) \]

\[ - \frac{1}{9} e^{(t_n + \tau) \partial_x^3} \partial_x^{-1} \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v^n \right) \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v^n \right) \]

\[ + \frac{1}{9} e^{(t_n + \tau) \partial_x^3} \partial_x^{-1} \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v^n \right) \left( e^{-\partial_x^3 (t_n + \tau)} \partial_x^{-1} v^n \right) \]

with \( v^n \) given in (16), \( \partial_x^{-1} \) defined in (4) and by construction \( \hat{\vartheta}_0^{n+1} = 0 \) (cf. Remark 1.2).

The approximation to the original solution \( u(t_n) \) of the KdV equation (1) at time \( t_n = n \tau \) is then obtained by “twisting” the variable back again, i.e., by setting \( u^n = e^{-\partial_t^n} v^n \). This yields that

\[ u^{n+1} = e^{-\partial_t \tau} u^n + \frac{1}{6} \left( e^{-\partial_t \partial_x^{-1} u^n} \right)^2 - \frac{1}{6} e^{-\partial_t \partial_x^{-1} u^n} \]

\[ + \frac{\tau}{3} \left( e^{-\partial_t \partial_x^{-1} u^n} \right)^\frac{1}{2} (u^n)^2 \]

\[ - \frac{1}{9} \partial_x^{-1} \left( e^{-\partial_t \partial_x^{-1} u^n} \right)^\frac{1}{2} (u^n)^2 \]

\[ + \frac{1}{9} \partial_x^{-1} e^{-\partial_t \partial_x^{-1} u^n} \left( \partial_x^{-1} \frac{1}{2} (u^n)^2 \right) \] (17)
where $\partial_x^{-1}$ is defined in (4) and by construction $\hat{u}_0^{n+1} = 0$ (cf. Remark 1.2). The semi-discret exponential-type integration scheme (17) allows formally second-order convergence in $H^r$ for sufficiently smooth solutions $u(t) \in H^{r+5}$ with $r > 1/2$ thanks to the local error bound (14) together with the observation that

$$\|v''(t)\|_r \leq c \left( \|\partial_x^2 v(t)\|_r^3 + \|\partial_x^4 v(t)\|_r^2 \right) = c \left( \|\partial_x^2 u(t)\|_r^3 + \|\partial_x^4 u(t)\|_r^2 \right).$$

We do not pursue this here and only underline the second-order convergence rate numerically in Sect. 3.

In the following section we give a detailed convergence analysis of the first-order exponential-type integration scheme (10).

2 Error analysis

For simplicity we carry out the error analysis for initial values satisfying Assumption 1.1. Furthermore, in the following we denote by $(\cdot, \cdot)$ the $L^2$ scalar product, i.e., $(f, g) = \int_T fg dx$ and by $\|\cdot\|_{L^2}$ the corresponding $L^2$ norm.

In order to obtain a convergence result in $H^1$ we follow the strategy presented in [12,16]: We first prove convergence order of one half of the numerical scheme (9) in $H^2$ for solutions in $H^3$, see Sect. 2.1, Theorem 2.6. This yields essential a priori bounds on the numerical solution in $H^2$ and allows us to prove first-order convergence globally in $H^1$, see Theorem 2.7 in Sect. 2.2. The latter in particular implies first-order convergence of the exponential-type integration scheme (10) towards the KdV solution (1), see Corollary 2.8 below for the precise convergence result.

2.1 Error analysis in $H^2$

We commence with the error analysis of the numerical scheme (9) in $H^2$. In Sect. 2.1.1 we carry out the stability analysis in $H^2$. In Sect. 2.1.2 we show that the method is consistent of order one half in $H^2$ for solutions in $H^3$.

2.1.1 Stability analysis

Set

$$\Phi^\tau_t(v) := v + \frac{1}{6} e^{\partial_x^3 (t+\tau)} \left( e^{-\partial_x^3 (t+\tau)} \partial_x^{-1} v \right)^2 - \frac{1}{6} e^{\partial_x^3 t} \left( e^{-\partial_x^3 t} \partial_x^{-1} v \right)^2 \quad (18)$$

such that for all $k$ we have $v^{k+1} = \Phi^\tau_t(v^k)$. The following stability result holds for the numerical flow $\Phi^\tau_t$:

**Lemma 2.1** Let $f \in H^2$ and $g \in H^3$. Then, for all $t \in \mathbb{R}$ we have

$$\|\partial_x^2 (\Phi^\tau_t(f) - \Phi^\tau_t(g))\|_{L^2} \leq \exp(\tau L) \|\partial_x^2 (f - g)\|_{L^2},$$

where $L$ depends on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^3 g\|_{L^2}$. 

\[ \square \] Springer
Proof Note that

\[
\| \partial_x^2 (\Phi_1^\tau f) - \Phi_1^\tau g \|_{L^2}^2 = \| \partial_x^2 (f - g) \|_{L^2}^2 \\
+ \frac{1}{3} \left\{ \partial_x^2 e^{\partial_x^3 (t+\tau)} \left[ \left( e^{-\partial_x^3 (t+\tau)} \partial_x f \right)^2 - \left( e^{-\partial_x^3 (t+\tau)} \partial_x g \right)^2 \right] \cdot \partial_x^2 (f - g) \right\} \\
- \frac{1}{3} \left\{ \partial_x^2 e^{\partial_x^3 (t+\tau)} \left[ \left( e^{-\partial_x^3 (t+\tau)} \partial_x f \right)^2 - \left( e^{-\partial_x^3 (t+\tau)} \partial_x g \right)^2 \right] \cdot \partial_x^2 (f - g) \right\} \\
+ \frac{1}{6} \left\| \partial_x^2 e^{\partial_x^3 (t+\tau)} \left[ \left( e^{-\partial_x^3 (t+\tau)} \partial_x f \right)^2 - \left( e^{-\partial_x^3 (t+\tau)} \partial_x g \right)^2 \right] \right\|_{L^2}^2 \\
- \partial_x^2 e^{\partial_x^3 (t+\tau)} \left[ \left( e^{-\partial_x^3 (t+\tau)} \partial_x f \right)^2 - \left( e^{-\partial_x^3 (t+\tau)} \partial_x g \right)^2 \right] \right\|_{L^2}^2 \\
=: \| \partial_x^2 (f - g) \|_{L^2}^2 + \frac{1}{3} I_1 + \frac{1}{6^2} I_2.
\]

Lemma 2.3 and 2.4 below allow us the following bounds on $I_1$ and $I_2$: We have

\[
|I_1 + I_2| \leq \tau L \left\| \partial_x^2 (f - g) \right\|_{L^2}^2,
\]

where $L$ depends on $\| \partial_x^2 f \|_{L^2}$ and $\| \partial_x^3 g \|_{L^2}$. Hence,

\[
\left\| \partial_x^2 (\Phi_1^\tau f) - \Phi_1^\tau g \right\|_{L^2}^2 \leq (1 + \tau L) \left\| \partial_x^2 (f - g) \right\|_{L^2}^2
\]

which yields the assertion. \qed

In the rest of Sect. 2.1.1 we will show the essential bound (19). We start with a useful Lemma.

**Lemma 2.2** The following estimates hold for $u, v, w \in H^2$

\[
\left\| \partial_x^2 u, v w - \partial_x^3 \left[ \left( e^{-\partial_x^3 \tau} u \right) \left( e^{-\partial_x^3 \tau} w \right) \right] \right\|_{L^2} \leq c \tau \left\| \partial_x^2 u \right\|_{L^2} \left\| \partial_x^2 v \right\|_{L^2} \left\| \partial_x^2 w \right\|_{L^2}
\]

\[
\left\| u, \left( \partial_x v \right)^2 - \partial_x^3 \left( e^{-\partial_x^3 \tau} \partial_x v \right) \right\|_{L^2} \leq c \tau \left\| \partial_x^2 u \right\|_{L^2} \left\| \partial_x^2 v \right\|_{L^2}^2
\]

for some constant $c > 0$.

Proof The key relation (3) together with the Cauchy–Schwarz inequality allows us the following bound

\[\text{Springer}\]
\[
\left\| \partial_x^2 u, \, v w - e^{\partial_x^3 \tau} \left[ \left( e^{-\partial_x^3 \tau} v \right) \left( e^{-\partial_x^3 \tau} w \right) \right] \right\|
= \left| \sum_{k_1, k_2} (k_1 + k_2)^2 \hat{u}_{-(k_1+k_2)} \left( 1 - e^{-i \tau \left( (k_1+k_2)^3 - k_1^3 - k_2^3 \right)} \right) \hat{v}_{k_1} \hat{w}_{k_2} \right|
= \left| \sum_{k_1, k_2} (k_1 + k_2)^2 \hat{u}_{-(k_1+k_2)} \left( 1 - e^{-i \tau 3k_1k_2(k_1+k_2)} \right) \hat{v}_{k_1} \hat{w}_{k_2} \right|
\leq 3\tau \sum_{k_1, k_2} |(k_1 + k_2)^2 \hat{u}_{-(k_1+k_2)}||(k_1 + k_2)k_1k_2 \hat{v}_{k_1} \hat{w}_{k_2}|
= 3\tau \sum_{l, k} l^2 |\hat{u}_{-l}| |l(k - k)||\hat{v}_{l} \hat{w}_{l-k}|
\leq 3\tau \sum_{l, k} l^2 |\hat{u}_{-l}| \left( |l(k - k)^2||\hat{v}_{l} \hat{w}_{l-k}| + |k|^2 |l - k||\hat{v}_{l} \hat{w}_{l-k}| \right)
\leq 3\tau \left( \sum_{l} l^4 |\hat{u}_{l}|^2 \right)^{1/2} \left( \sum_{k} \left( \sum_{l} |l||\hat{v}_{l}||l - k||\hat{w}_{l-k}| \right)^2 \right)^{1/2}
+ 3\tau \left( \sum_{l} l^4 |\hat{u}_{l}|^2 \right)^{1/2} \left( \sum_{k} \left( \sum_{l} |k|^2 |\hat{v}_{k}||l - k||\hat{w}_{l-k}| \right)^2 \right)^{1/2}
\leq 3\tau \| \partial_x^2 u \|_{L^2} \left( \| v^{(1)} \ast w^{(2)} \|_{l^2} + \| v^{(2)} \ast w^{(1)} \|_{l^2} \right),
\]
where \( v^{(j)}(k) := |k|^j |\hat{v}_k| \) and \( w^{(j)}(k) := |k|^j |\hat{w}_k| \). By the Young and Cauchy–Schwarz inequality we furthermore obtain that
\[
\| v^{(1)} \ast w^{(2)} \|_{l^2} + \| v^{(2)} \ast w^{(1)} \|_{l^2} \leq \| v^{(1)} \|_{l^1} \| w^{(2)} \|_{l^2} + \| w^{(1)} \|_{l^1} \| v^{(2)} \|_{l^2}
\leq c \| v^{(2)} \|_{l^2} \| w^{(2)} \|_{l^2}
\leq c \| \partial_x^2 v \|_{L^2} \| \partial_x^2 w \|_{L^2}
\]
for some constant \( c > 0 \). Plugging (22) into (21) yields the first assertion.

Similarly we have that
\[
\left| u, (\partial_x v)^2 - e^{\partial_x^3 \tau} \left( e^{-\partial_x^3 \tau \partial_x v} \right)^2 \right|
\leq 3\tau \sum_{k_1, k_2} |(k_1 + k_2) \hat{u}_{-(k_1+k_2)}| |k_1k_2|^2 |\hat{v}_{k_1} \hat{v}_{k_2}|
= 3\tau \sum_{k, l} |l| |\hat{u}_{-l}| |k|^2 |l - k|^2 |\hat{v}_{l} \hat{w}_{l-k}|
\leq 3\tau \left( \sum_{k} k^4 |\hat{v}_{k}|^2 \right)^{1/2} \| u^{(1)} \ast v^{(2)} \|_{l^2}
\leq c\tau \| \partial_x^2 v \|_{L^2} \| u^{(1)} \|_{l^1} \| v^{(2)} \|_{l^2}
\leq c\tau \| \partial_x^2 v \|_{L^2}^2 \| \partial_x^2 u \|_{L^2}
\]
which yields the second assertion.
Lemma 2.3 (Bound on $I_1$) We have

$$|I_1| \leq c \tau \left( \| \partial_x^2 f \|_{L^2} + \| \partial_x^3 g \|_{L^2} \right) \| \partial_x^2 (f - g) \|_{L^2}^2$$

for some constant $c > 0$.

Proof Note that for all $t \in \mathbb{R}$ the following relation holds

$$\langle e^{t \partial_x^3 f}, g \rangle = \langle f, e^{-t \partial_x^3 g} \rangle.$$

Thus, by setting $\tilde{f} = e^{-t \partial_x^3 f}$ and $\tilde{g} = e^{-t \partial_x^3 g}$ we obtain that

$$I_1 = \left\{ \partial_x^2 \left( e^{-\partial_x^3 \partial_x^{-1} (\tilde{f} - \tilde{g})} \right)^2 - \partial_x^2 \left( e^{-\partial_x^3 \partial_x^{-1} \tilde{g}} \right)^2 \right\} \left\{ e^{-\partial_x^3 \partial_x^{-1} (\tilde{f} - \tilde{g})} \right\} \left\{ e^{-\partial_x^3 \partial_x^{-1} \tilde{g}} \right\}
- \left\{ \partial_x^2 \left( \partial_x^{-1} (\tilde{f} - \tilde{g}) \right)^2 + 2 \partial_x^2 \left( \partial_x^{-1} (\tilde{f} - \tilde{g}) \right) \left( \partial_x^{-1} \tilde{g} \right) \right\} \left\{ \partial_x (\tilde{f} - \tilde{g}) \right\}
= 2 \left\{ \left( e^{-\partial_x^3 \partial_x^{-1} (\tilde{f} - \tilde{g})} \right)^2 \left( e^{-\partial_x^3 \partial_x^{-1} \tilde{g}} \right) \left( e^{-\partial_x^3 \partial_x^{-1} \tilde{g}} \right) \right\}
+ 2 \left\{ \left( e^{-\partial_x^3 \partial_x^{-1} (\tilde{f} - \tilde{g})} \right) \left( e^{-\partial_x^3 \partial_x^{-1} \tilde{g}} \right) + 2 \left( e^{-\partial_x^3 \partial_x^{-1} (\tilde{f} - \tilde{g})} \right) \left( e^{-\partial_x^3 \partial_x^{-1} \tilde{g}} \right) \right\}
+ 2 \left\{ \left( e^{-\partial_x^3 \partial_x^{-1} (\tilde{f} - \tilde{g})} \right) \left( e^{-\partial_x^3 \partial_x^{-1} \tilde{g}} \right) \right\}
- 2 \left\{ \left( \partial_x (\tilde{f} - \tilde{g}) \right) \left( \partial_x^{-1} (\tilde{f} - \tilde{g}) \right) + (\tilde{f} - \tilde{g})^2, \partial_x^2 (\tilde{f} - \tilde{g}) \right\}
- 2 \left\{ \left( \partial_x^{-1} (\tilde{f} - \tilde{g}) \right) \partial_x \tilde{g} + 2(\tilde{f} - \tilde{g}) \tilde{g} + \left( \partial_x (\tilde{f} - \tilde{g}) \right) \left( \partial_x^{-1} \tilde{g} \right) \right\} \partial_x^2 (\tilde{f} - \tilde{g}) \right\}.
$$

Next we use another key fact namely that

$$\langle vu, \partial_x u \rangle = \frac{1}{2} \langle v, \partial_x (u)^2 \rangle = -\frac{1}{2} \langle \partial_x v, u^2 \rangle.$$
as well as that

\[\langle u - v, (\partial_x(u - v))^2\rangle = -\frac{1}{2}((u - v)^2, \partial_x^2(u - v)).\]

This yields that

\[I_1 = 2 \left\{ \left( e^{-\partial_x^2 \tau} (\tilde{f} - \tilde{g}) \right)^2 \cdot e^{-\partial_x \tau} \partial_x^2 (\tilde{f} - \tilde{g}) \right\}
+ \frac{1}{2} \left\{ \left( e^{-\partial_x^2 \tau} (\tilde{f} - \tilde{g}) \right)^2 \cdot \partial_x^2 e^{-\partial_x \tau} (\tilde{f} - \tilde{g}) \right\}
- \left\{ e^{-\partial_x \tau} \tilde{g}, \left( e^{-\partial_x \tau} \partial_x(\tilde{f} - \tilde{g}) \right)^2 \right\}
+ 4 \left\{ e^{-\partial_x \tau} (\tilde{f} - \tilde{g}) \left( e^{-\partial_x \tau} \tilde{g}, e^{-\partial_x^2 \tau} (\tilde{f} - \tilde{g}) \right) \right\}
+ 2 \left\{ e^{-\partial_x \tau} \partial_x^{-1}(\tilde{f} - \tilde{g}) \left( e^{-\partial_x \tau} \partial_x \tilde{g}, e^{-\partial_x^2 \tau} (\tilde{f} - \tilde{g}) \right) \right\}
- \frac{1}{2} \left\{ (\tilde{f} - \tilde{g})^2, \partial_x^2 (\tilde{f} - \tilde{g}) \right\}
- 2 \left\{ (\tilde{f} - \tilde{g})^2, \partial_x^2 (\tilde{f} - \tilde{g}) \right\}
- 2 \left\{ (\partial_x^{-1}(\tilde{f} - \tilde{g}), \partial_x \tilde{g}, \partial_x^2 (\tilde{f} - \tilde{g}) \right\}
- 4 \left\{ (\tilde{f} - \tilde{g}) \partial_x (\tilde{f} - \tilde{g}) \right\}
+ \left\{ \tilde{g}, \partial_x(\tilde{f} - \tilde{g}) \right\}.\]

Thus, rearranging the terms leads to

\[I_1 = \frac{5}{2} \left\{ e^{\partial_x^2 \tau} \left( e^{-\partial_x \tau} (\tilde{f} - \tilde{g}) \right)^2 - (\tilde{f} - \tilde{g})^2, \partial_x^2 (\tilde{f} - \tilde{g}) \right\}
- \left\{ e^{\partial_x^2 \tau} \left( e^{-\partial_x \tau} \partial_x(\tilde{f} - \tilde{g}) \right)^2 - \left( \partial_x(\tilde{f} - \tilde{g}) \right)^2, \tilde{g} \right\}
+ 4 \left\{ e^{\partial_x^2 \tau} \left( e^{-\partial_x \tau} (\tilde{f} - \tilde{g}) \right) \left( e^{-\partial_x \tau} \tilde{g}, \partial_x^2 (\tilde{f} - \tilde{g}) \right) \right\}
+ 2 \left\{ e^{\partial_x^2 \tau} \left( e^{-\partial_x \tau} \partial_x^{-1}(\tilde{f} - \tilde{g}) \right) \left( e^{-\partial_x \tau} \partial_x \tilde{g}, \partial_x^{-1}(\tilde{f} - \tilde{g}) \right) \right\} \partial_x \tilde{g}, \partial_x^2 (\tilde{f} - \tilde{g}) \right\}.\]

With the aid of Lemma 2.2 we thus obtain that

\[|I_1| \leq c \left( \|\partial_x^2(f - g)\|_{L^2} + \|\partial_x^3 g\|_{L^2} \right) \|\partial_x^2(f - g)\|_{L^2}^2\]

for some constant \(c > 0\). \(\square\)
Lemma 2.4 (Bound on $I_2$) We have

$$|I_2| \leq \tau M \|\partial_x^2(f - g)\|^2_{L^2},$$

where $M$ depends on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^2 g\|_{L^2}$.

Proof In the following let $M$ denote a constant depending on $\|\partial_x^2 f\|_{L^2}$ and $\|\partial_x^2 g\|_{L^2}$. Setting $\tilde{f} = e^{-t\partial_x^3} f$ and $\tilde{g} = e^{-t\partial_x^3} g$ yields that

$$I_2 = \left\{ \partial_x^2 \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{g} \right)^2 \right\} - 2 \left\{ \partial_x^2 e^{\partial_x^3} \left[ \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{f} \right)^2 - \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{g} \right)^2 \right] \right\} + \left\{ \partial_x^2 \left( \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left( \partial_x^{-1} \tilde{g} \right)^2 \right\}$$

$$= I^a_2 + I^b_2$$

with

$$I^a_2 = \left\{ \partial_x^2 \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{g} \right)^2 \right\} - \partial_x^2 \left[ \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{f} \right)^2 - \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{g} \right)^2 \right],$$

$$I^b_2 = -\partial_x^2 \left[ \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{f} \right)^2 - \left( e^{-\partial_x^3} \partial_x^{-1} \tilde{g} \right)^2 \right] - \partial_x^2 \left( \partial_x^{-1} \tilde{f} \right)^2 - \partial_x^2 \left( \partial_x^{-1} \tilde{g} \right)^2.$$
Plugging the bounds (25) and (26) into (24) yields the assertion.

\[ \leq 3\tau \sum_{k_1, k_2} \left| \hat{F}_{-(k_1+k_2)}(k_1 + k_2) \right| (k_1 + k_2)^2 \left( \hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \]
\[ \leq 3\tau \sum_{k, l} \left| \hat{F}_{-l} \right| ((l - k)^2 + 2|l - k|k) \left( \hat{f}_l - \hat{g}_l \right) \hat{f}_{l-k} + \hat{g}_k (\hat{f}_{l-k} - \hat{g}_{l-k}) \]
\[ \leq 6\tau \| \partial_x F \|_{L^2} \left( \sum_{j=0,1} \| (\hat{f} - \hat{g})^{(2j)} * \hat{g}^{(2j)} \|_{l^2} + \| (\hat{f} - \hat{g})^{(2j)} * \hat{f}^{(2-2j)} \|_{l^2} \right) \]
\[ + 6\tau \| \partial_x F \|_{L^2} \left( \| (\hat{f} - \hat{g})^{(1)} \|_{l^2} + \| (\hat{f} - \hat{g})^{(1)} * \hat{g}^{(1)} \|_{l^2} \right) \leq c\tau \| \partial_x F \|_{L^2} \| \partial_x^2 (f - g) \|_{L^2} (\| \partial_x^2 \hat{f} \|_{L^2} + \| \partial_x^2 \hat{g} \|_{L^2}) \]
\[ \leq \tau M \| \partial_x^2 (f - g) \|_{L^2}^2. \tag{25} \]

where again we used the notation \( \Phi^{(j)}(k) := |k|^j |\hat{\Phi}_k|. \) Similarly, we obtain for \( I_2^a \) with \( F := \partial_x^2 \left( e^{-\partial_x^3 \tau \partial_x^{-1} f} \right)^2 - \partial_x^2 \left( e^{-\partial_x^3 \tau \partial_x^{-1} g} \right)^2 \) that
\[ |I_2^a| = \left| \sum_{k_1, k_2} \hat{F}_{-(k_1+k_2)} \frac{(k_1 + k_2)^2}{k_1 k_2} \left( e^{i\tau(k_1^3 + k_2^3)} - e^{i\tau(k_1 + k_2)^3} \right) \left( \hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \]
\[ \leq \sum_{k_1, k_2} \left| \hat{F}_{-(k_1+k_2)} \frac{(k_1 + k_2)^2}{k_1 k_2} \right| \left| 1 - e^{-i\tau 3k_1 k_2} (k_1 + k_2) \right| \left| \left( \hat{f}_{k_1} \hat{f}_{k_2} - \hat{g}_{k_1} \hat{g}_{k_2} \right) \right| \]
\[ \leq \tau M \| \partial_x^2 (f - g) \|_{L^2}^2. \tag{26} \]

Plugging the bounds (25) and (26) into (24) yields the assertion. \( \square \)

2.1.2 Local error analysis

Let \( \phi^t \) denote the exact flow associated to the reformulated KdV equation (5), i.e., \( v(t) = \phi^t(v(0)). \) The following local error bound holds for the exponential-type integrator \( \Phi^T \) defined in (18) with \( v^{k+1} = \Phi^T_{\tau} (v^k). \)

**Lemma 2.5** Let \( v(t_k + \tau) = \phi^t(v(t_k)) \in H^3 \) for \( 0 \leq t \leq \tau. \) Then
\[ \| \partial_x^2 (\phi^T(v(t_k)) - \Phi^T_{\tau}(v(t_k))) \|_{L^2} \leq c\tau^{3/2}, \]

where \( c \) depends on \( \sup_{0 \leq t \leq \tau} \| \partial_x^2 (\phi^t(v(t_k))) \|_{H^3}. \)

**Proof** As \( e^{\partial_x^3 t} \) is a linear isometry in \( H^r \) for all \( t \in \mathbb{R} \) the iteration of Duhamel’s formula (5) yields that
\[ \| \phi^T(v(t_k)) - \Phi^T_{\tau}(v(t_k)) \|_{H^2} \]
\[ \leq \int_0^\tau \left\| \left( e^{-\partial_x^3 (t+s)} \phi^s(v(t_k)) \right)^2 - \left( e^{-\partial_x^3 (t+s)} v(t_k) \right)^2 \right\|_{H^3} \ ds \]
\[ \leq \tau c \sup_{0 \leq t \leq \tau} \| \phi^t(v(t_k)) - v(t_k) \|_{H^3}, \tag{27} \]

\( \square \)
where $c_1$ depends on $\sup_{0 \leq t \leq \tau} \| \phi^t (v(t)) \|_{H^3}$. Duhamel’s formula (5) and integration by parts furthermore yields that

\[
\| \phi^t (v(t_j)) - v(t_j) \|_{H^3} \leq \left[ \int_0^t e^{(t_j+s)\partial_x} \left( e^{-\partial_x^3 (t_j+s)} v(t_j + s) \right)^2 \, ds \right]_{H^3} \\
\leq \left[ \sum_{k_1,k_2} \frac{1}{k_1 k_2} e^{-3i\tau k_1 k_2 (k_1 + k_2)} \left( e^{-3i\tau k_1 k_2 (k_1 + k_2)} - 1 \right) \times \hat{v}_{k_1} (t_j + t) \hat{v}_{k_2} (t_j + t) e^{i (k_1 + k_2) x} \right]_{H^3} \\
+ \left[ \int_0^t \sum_{k_1,k_2} e^{-3i\tau (t_j+s) k_1 k_2 (k_1 + k_2)} \frac{1}{k_1 k_2} \frac{d}{ds} (\hat{v}_{k_1} (t_j + s) \hat{v}_{k_2} (t_j + s)) d e^{i (k_1 + k_2) x} \right]_{H^3} \\
\leq c t^{1/2} \sup_{k_1,k_2 \in \mathbb{Z} \setminus \{0\}} \frac{|k_1 + k_2|^{1/2}}{|k_1 k_2|^{1/2}} \| v(t_j + t) \|_{H^3}^2 \\
+ c t \sup_{k_1,k_2 \in \mathbb{Z} \setminus \{0\}} \frac{|k_1| + |k_2|}{|k_1 k_2|} \sup_{0 \leq s \leq t} \| v(t_j + s) \|_{H^3}^3. \tag{28}
\]

Plugging (28) into (27) yields the assertion. \hfill \square

### 2.1.3 Global error bound

The stability analysis in Sect. 2.1.1 and local error analysis in Sect. 2.1.2 allows us the following global error bound in $H^2$.

**Theorem 2.6** Let the solution of (5) satisfy $v(t) \in H^3$ for $t \leq T$. Then there exists a $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ and $t_n \leq T$ we have

\[
\| v(t_n) - v^n \|_{H^2} \leq c \tau^{1/2},
\]

where $c$ depends on $\sup_{0 \leq t \leq t_n} \| v(t) \|_{H^3}$ and $t_n$, but can be chosen independently of $\tau$.

**Proof** The triangular inequality yields that

\[
\| v(t_{k+1}) - v^{k+1} \|_{H^2} = \| \phi^{t_{k+1}} (v(t_k)) - \Phi^{t_{k+1}} (v^k) \|_{H^2} \\
\leq \| \phi^{t_{k+1}} (v(t_k)) - \Phi^{t_{k}} (v(t_k)) \|_{H^2} \\
+ \| \Phi^{t_{k}} (v(t_k)) - \Phi^{t_{k}} (v^k) \|_{H^2}. \tag{29}
\]

Thus, iterating the estimate (29) we obtain with the aid of Lemma 2.1 (with $g = v(t_k) \in H^3$) and Lemma 2.5 that as long as $v^k \in H^2$ (for $0 \leq k \leq n$) we have that
\[ \| v(t_{n+1}) - v^n \|_{L^2} \leq c \tau^{3/2} + \exp(\tau L) \| v(t_n) - v^n \|_{L^2} \]
\[ \leq c \tau^{3/2} + \exp(\tau L) \left( c \tau^{3/2} + c \tau L \| v(t_{n-1}) - v^{n-1} \|_{L^2} \right) \]
\[ \leq c \tau^{3/2} \sum_{k=0}^{n} e^{\tau L} \leq c \tau^{1/2} \tau e^{\tau L}, \]

where \( c \) depends on \( \sup_{0 \leq t \leq t_{n+1}} \| v(t) \|_{H^3} \), \( L \) depends on \( \sup_{0 \leq k \leq n} \| v(t_k) \|_{H^3} \) as well as on \( \sup_{0 \leq k \leq n} \| v^k \|_{L^2} \) and we have used the fact that \( \hat{v}_0(t_n) \equiv \hat{v}_0^n \). The assertion then follows by a bootstrap, respectively, “Lady Windermere’s fan” argument, see, for example \([3, 6, 12, 16]\).

\[ \square \]

### 2.2 Error analysis in \( H^1 \)

The error analysis in \( H^2 \) of the numerical scheme \((9)\) given in Sect. 2.1 yields a priori bounds on the numerical solution in \( H^2 \) for solutions in \( H^3 \). This allows us to derive the following first-order convergence bound in \( H^1 \).

**Theorem 2.7** Let the solution of \((5)\) satisfy \( v(t) \in H^3 \) for \( t \leq T \). Then there exists a \( \tau_0 > 0 \) such that for all \( \tau \leq \tau_0 \) and \( t_n \leq T \) we have

\[ \| v(t_n) - v^n \|_{L^2} \leq c \tau, \]

where \( c \) depends on \( \sup_{0 \leq t \leq t_n} \| v(t) \|_{H^3} \) and \( t_n \), but can be chosen independently of \( \tau \).

**Proof** Note that Duhamel’s formula \((5)\) implies the first-order consistency bound

\[ \| \partial_x (\phi^t(v(t_k)) - \Phi^t_k(v(t_k))) \|_{L^2} \]
\[ \leq \int_0^\tau \| \partial_x^2 \left[ \left( e^{-\partial_x^2 (t_k+s)} \phi^s(v(t_k)) \right)^2 - \left( e^{-\partial_x^2 (t_k+s)} v(t_k) \right)^2 \right] \|_{L^2} ds \]
\[ \leq \tau c_1 \sup_{0 \leq t \leq \tau} \| \phi^t(v(t_k)) - v(t_k) \|_{H^2} \]
\[ \leq \tau^2 c_1 \sup_{0 \leq t \leq \tau} \| \phi^t(v(t_k)) \|_{H^3}, \]

where \( c_1 \) depends on \( \sup_{0 \leq t \leq \tau} \| \phi^t(v(t_k)) \|_{H^2} \).

Furthermore, as \( v(t) \in H^3 \) for \( t \leq T \) we have the boundedness of the numerical solution in \( H^2 \) a priori thanks to Theorem 2.6, i.e., there exists a \( \tau_0 > 0 \) such that for all \( \tau \leq \tau_0 \) \( v^n \in H^2 \) as long as \( t_n \leq T \). In particular a stability estimate of type

\[ \| \partial_x (\Phi^t_f - \Phi^t_g) \|_{L^2} \leq \exp(\tau L) \| \partial_x (f - g) \|_{L^2}, \quad L = L(\| \partial_x^2 f \|_{L^2}, \| \partial_x^3 g \|_{L^2}) \]

is therefore sufficient for our bootstrapping argument in \( H^1 \) by choosing \( f = v^n \in H^2 \) and \( g = v(t_n) \in H^3 \). The stability bound \((31)\) follows similarly to Lemma 2.1: Note that
Similarly to the proof of Lemma 2.3 we can rewrite $I_1$ as

$$I_1 = \left\{ \tilde{f} - \tilde{g}, (\tilde{f} - \tilde{g})^2 - e^{\frac{\partial_1 \tau}{2}} \left( e^{-\frac{\partial_1 \tau}{2}} (\tilde{f} - \tilde{g}) \right)^2 \right\}$$

$$+ \left\{ \tilde{g}, (\tilde{f} - \tilde{g})^2 - e^{\frac{\partial_1 \tau}{2}} \left( e^{-\frac{\partial_1 \tau}{2}} (\tilde{f} - \tilde{g}) \right)^2 \right\}$$

$$- 2 \left\{ \tilde{f} - \tilde{g}, \tilde{g}(\tilde{f} - \tilde{g}) - e^{\frac{\partial_1 \tau}{2}} \left[ (e^{-\frac{\partial_1 \tau}{2}} \tilde{g}) \left( e^{-\frac{\partial_1 \tau}{2}} (\tilde{f} - \tilde{g}) \right) \right] \right\}$$

$$- 2 \left\{ \tilde{f} - \tilde{g}, (\partial_1 \tilde{g}) \left( \partial_1^{-1} (\tilde{f} - \tilde{g}) \right) \right\}$$

$$- e^{\frac{\partial_1 \tau}{2}} \left[ (e^{-\frac{\partial_1 \tau}{2}} \partial_1 \tilde{g}) \left( e^{-\frac{\partial_1 \tau}{2}} \partial_1^{-1} (\tilde{f} - \tilde{g}) \right) \right].$$

As in Lemma 2.2 we obtain by the key relation (3) that

$$I(u, v, w) := \left\| u, vw - e^{\frac{\partial_1 \tau}{2}} \left[ \left( e^{-\frac{\partial_1 \tau}{2}} v \right) \left( e^{-\frac{\partial_1 \tau}{2}} w \right) \right] \right\| \leq 3\tau \sum_{k_1, k_2} \| (k_1 + k_2)\hat{u}_{-(k_1 + k_2)} \| k_1 k_2 \| \hat{v}_{k_1} \hat{w}_{k_2} \|.$$

The Cauchy–Schwarz and Young inequality furthermore yield that

$$I(u, v, w) \leq 3\tau \sum_{k, l} \| l \| \| \hat{u}_{-l} \| (l - k) \| \hat{v}_{k} \hat{w}_{l-k} \| \leq 3\tau \left( \sum_{l} \| l \| \| \hat{u}_{l} \| \right)^{1/2} \| v^{(1)} \ast w^{(1)} \|_{l^2}$$

$$\leq c \tau \| \partial_1 u \|_{L^2} \min \left( \| v^{(1)} \|_{l^2}, \| w^{(1)} \|_{l^2}, \| v^{(1)} \|_{l^2} \| w^{(1)} \|_{l^2} \right)$$

$$\leq c \tau \| \partial_1 u \|_{L^2} \min \left( \| \partial_1^2 v \|_{L^2} \| \partial_1 w \|_{L^2}, \| \partial_1^2 w \|_{L^2} \| \partial_1 v \|_{L^2} \right).$$

The above bound allows us to control the first and last two terms in (33) as long as $f - g, g \in H^2$. Furthermore,
\[ I(u, v, v) \leq 3\tau \sum_{l,l'} |l||\tilde{u}_{-l}||l-k||\tilde{v}_{-l-k}|| \leq 3\tau \left( \sum_k |k|^2 |\tilde{v}_k|^2 \right)^{1/2} ||u^{(1)} \ast v^{(1)}||_{L^2} \]
\[ \leq 3\tau ||\partial_x v||_{L^2} ||u^{(1)}||_{L^1} ||v^{(1)}||_{L^2} \leq c\tau ||\partial_x v||_{L^2}^2 ||\partial_x^2 u||_{L^2}, \]

which allows us to control the second term in (33) as long as \( f - g \in H^1 \) and \( g \in H^2 \). Using the bounds (34) and (35) in (33) yields that

\[ |I_1| \leq \tau L ||\partial_x (f - g)||_{L^2}^2, \quad L = L(||\partial_x^2 (f - g)||_{L^2}, ||\partial_x^2 g||_{L^2}). \]

Next we write \( I_2 = I_2^a + I_2^b \) with

\[
I_2^a = \left( \partial_x \left( e^{-\partial_x^3 t} \partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left( e^{-\partial_x^3 t} \partial_x^{-1} \tilde{g} \right)^2 - \partial_x e^{-\partial_x^3 t} \left[ \left( \partial_x^{-1} \tilde{f} \right)^2 - \left( \partial_x^{-1} \tilde{g} \right)^2 \right] \right)
\]

\[
I_2^b = -\left( \partial_x e^{-\partial_x^3 t} \left[ \left( e^{-\partial_x^3 t} \partial_x^{-1} \tilde{f} \right)^2 - \left( e^{-\partial_x^3 t} \partial_x^{-1} \tilde{g} \right)^2 \right] \right).
\]

Note that by the Cauchy–Schwarz and Young inequality we have with \( F := \partial_x \left( \partial_x^{-1} \tilde{f} \right)^2 - \partial_x \left( \partial_x^{-1} \tilde{g} \right)^2 \) that

\[
|I_2^a| = \left| \sum_{k_1, k_2} \tilde{F}_{-(k_1+k_2)} (k_1 + k_2) \frac{(k_1 + k_2)}{k_1 k_2} \left( 1 - e^{-i\tau \left( k_1 + k_2 \right)^3 \partial_x^{-3} \partial_x^{-1}} \right) \left( \tilde{f}_{k_1} \tilde{f}_{k_2} - \tilde{g}_{k_1} \tilde{g}_{k_2} \right) \right|
\]

\[
\leq 3\tau \sum_{k_1, k_2} |\tilde{F}_{-(k_1+k_2)}(k_1 + k_2)||(k_1 + k_2)|| \left( \tilde{f}_{k_1} \tilde{f}_{k_2} - \tilde{g}_{k_1} \tilde{g}_{k_2} \right) \right|
\]

\[
\leq 3\tau \left( \sum_l |\tilde{F}_l|^2 \right)^{1/2} \left( ||\tilde{f} - \tilde{g}||_{L^2} (1) \ast \tilde{f}^{(0)}||_{L^2} + ||\tilde{f} - \tilde{g}||_{L^2} (0) \ast \tilde{f}^{(1)}||_{L^2} \right)
\]

\[
+ 3\tau \left( \sum_l |\tilde{F}_l|^2 \right)^{1/2} \left( ||\tilde{f} - \tilde{g}||_{L^2} (1) \ast \tilde{g}^{(0)}||_{L^2} + ||\tilde{f} - \tilde{g}||_{L^2} (0) \ast \tilde{g}^{(1)}||_{L^2} \right)
\]

\[
\leq c\tau ||\partial_x F||_{L^2} \left( ||\tilde{f} - \tilde{g}||_{L^2} (0) \ast \tilde{f}^{(1)}||_{L^2} + ||\tilde{g}^{(0)}||_{L^1} \right)
\]

\[
+ ||\tilde{f} - \tilde{g}||_{L^2} (0) \ast \tilde{g}^{(1)}||_{L^2} \leq M \tau ||\partial_x (\tilde{f} - \tilde{g})||_{L^2}^2,
\]
where \( M \) depends on \( \| \partial_x f \|_{L^2} \) and \( \| \partial_x g \|_{L^2} \). A similar bound holds for \( I_2^a \) which implies that

\[
|I_2| \leq M \tau \| \partial_x (f - g) \|_{L^2}^2, \quad M = M(\| \partial_x f \|_{L^2}, \| \partial_x g \|_{L^2}). \tag{38}
\]

Plugging the bounds (36) as well as (38) into (32) yields the stability estimate (31).

With the aid of the stability estimate (31) and the local error bound (30) the proof then follows the line of argumentation to the proof of Theorem 2.6. □

**Corollary 2.8** Let the solution of the KdV equation (1) satisfy \( u(t) \in H^3 \) for \( t \leq T \). Then there exists a \( \tau_0 > 0 \) such that for all \( \tau \leq \tau_0 \) and \( t_n \leq T \) the exponential-type integration scheme (10) is first-order convergent in \( H^1 \), i.e.,

\[
\| u(t_n) - u^n \|_{H^1} \leq c \tau,
\]

where \( c \) depends on \( \sup_{0 \leq t \leq t_n} \| u(t) \|_{H^3} \) and \( t_n \), but can be chosen independently of \( \tau \).

**Proof** The assertion follows from Theorem 2.7 as \( e^{i \partial_x^3} \) is a linear isometry in \( H^1 \) for all \( t \in \mathbb{R} \). □

### 3 Numerical experiments

In this section, we numerically underline the first- and second-order convergence rates of the exponential-type integration schemes (10) and (17), respectively, towards the exact solution of the KdV equation (1). For the space discretization we use a Fourier pseudo spectral method, see [17], where we choose the largest Fourier mode \( K = 2^{12} \).

**Remark 3.1** We employ the following fully discrete Fourier pseudo spectral version of (10): Set \( B^K = \{-K/2, \ldots, K/2 - 1\} \) and let \( \mathcal{F}_K : B^K \rightarrow B^K \) denote the discrete Fourier transform and \( \mathcal{F}_K^{-1} \) its inverse. Denote by \( u^{K,0} \) the discretized initial value vector on the grid \( x_a = \frac{2\pi}{K} a, a \in B^K \) and set

\[
\xi^{K,0} = \mathcal{F}_K (u^{K,0}) = \left[ (\mathcal{F}_K u^{K,0})_{-\frac{K}{2}}, (\mathcal{F}_K u^{K,0})_{-\frac{K}{2} + 1}, \ldots, (\mathcal{F}_K u^{K,0})_{\frac{K}{2} - 1} \right].
\]

With this notation at hand a fully discrete Fourier pseudo spectral version of (10) reads

\[
\xi^{K,n+1} = e^{-\tau \partial_x^3} \xi^{K,n} + \frac{1}{6} \mathcal{F}_K \left( (\mathcal{F}_K^{-1} \left( e^{-\tau \partial_x^3} \mathcal{F}_K \xi^{K,n} \right))^2 \right)
\]

\[
- \frac{1}{6} e^{-\tau \partial_x^3} \mathcal{F}_K \left( (\mathcal{F}_K^{-1} \left( \partial_x \mathcal{F}_K \xi^{K,n} \right))^2 \right)
\]

with \( u^{K,n+1} = \mathcal{F}_K^{-1} (\xi^{K,n+1}) \). Thereby, the multiplication of two vectors is taken point-wise, i.e.,

\[
[x_1, \ldots, x_K] [y_1, \ldots, y_K] = [x_1 y_1, \ldots, x_K y_K]
\]
Fig. 1 (Initial value (39)) Numerical simulation of the first- and second-order exponential-type integration schemes (10) and (17). a Orderplot (double logarithmic). Convergence rates of the first-order scheme (10) (blue, circle) and the second-order scheme (17) (red, star). The slopes of the dashed and dashed-dotted lines are one and two, respectively. b Time evolution of the reference solution $u_{ref}^n$ (orange, dotted) and the first-order approximate solution $u^n$ with $\tau \approx 10^{-2}$ (blue, continuous) (color figure online)

and the discrete differential operators acting in Fourier space are defined through

$$\hat{\partial}_{x, K} := i \left[ -\frac{K}{2}, \ldots, \frac{K}{2} - 1 \right], \quad \hat{\partial}_{x, K}^{-1} := \frac{1}{T} \left[ \frac{1}{K/2}, \ldots, 1, 0, 1, \ldots \frac{1}{K/2 - 1} \right].$$

Example 3.2 In the first numerical experiment we choose the initial value

$$u(0, x) = 2 \text{sech}^2 \left( \frac{x}{2} \right) \sin(x) \quad \text{with} \quad \text{sech}(x) = \frac{1}{\cosh(x)} \quad (39)$$

and integrate the exponential-type integration schemes (10) and (17) up to $T = 2$. As the exact solution is unknown we take as a reference solution the second-order scheme itself with a very small time-step size $\tau = 10^{-7}$. The error between the numerical solutions and the reference solution at time $T = 2$ as well as a graph of the initial value, the reference solution and the first-order approximate solution is given in Fig. 1.

Example 3.3 (Solitary waves) The KdV equation

$$\partial_t \phi + \partial_x^3 \phi + \frac{1}{2} \partial_x (\phi^2) = 0, \quad x \in \mathbb{R}$$

allows solitary wave solutions of type

$$\phi(t, x) = 3c \text{sech}^2 \left( \frac{\sqrt{c}}{2} (x - ct - a) \right) \quad \text{with} \quad a \in \mathbb{R}, \ c > 0. \quad (40)$$

In order to test the resolution of solitary waves under the schemes (10) and (17) we choose a “large torus” $\mathbb{T}_L = \left[-\frac{\pi}{L}, \frac{\pi}{L} \right]$ with $L = 0.1$ such that boundary errors are
Fig. 2  (Solitary wave)
Orderplot (double logarithmic). Convergence rates of the first-order scheme (10) (blue, circle) and the second-order scheme (17) (red, star) measured in a discrete $H^1$ norm. The slopes of the dashed and dashed-dotted lines are one and two, respectively (color figure online).

Fig. 3  Time evolution of the solitary wave (40) (yellow, dotted), the first-order approximate solution (12) (blue, dashed-dotted) and second-order approximate solution (17) (red, continuous) for two different time-step sizes $\tau$ (color figure online).
	negligible. Furthermore, we fix $c = 1$ and $a = 0$. The $H^1$-error between the first- and second-order exponential-type integration schemes (10) and (17), respectively, and the exact solution (40) at time $T = 2$ is illustrated in Fig. 2.

A graph of the time evolution of the solitary wave solution (40) (with $c = 1.2$, $a = -5\pi$) and the corresponding first- and second-order approximate solutions (12) and (17), respectively, for two different time step sizes is illustrated in Fig. 3.

Acknowledgements  K. Schratz gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

References

1. Bourgain, J.: Fourier transform restriction phenomena forcertain lattice subsets and applications to nonlinear evolventequations. Part II: The KdV-equation. Geom. Funct. Anal. 3, 209–262 (1993)
2. Einkemmer, L., Ostermann, A.: A splitting approach for the Kadomtsev–Petviashvili equation. J. Comput. Phys. 299, 716–730 (2015)
3. Faou, E.: Geometric Numerical Integration and Schrödinger Equations. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich (2012)
4. Gauckler, L.: Convergence of a split-step Hermite method for the Gross–Pitaevskii equation. IMA J. Numer. Anal. 31, 1082–1106 (2011)
5. Gubinelli, M.: Rough solutions for the periodic Korteweg-de Vries equation. Comm. Pure Appl. Anal. 11, 709–733 (2012)
6. Hairer, E., Nørsett, S.P., Wanner, G.: Solving Ordinary Differential Equations I. Nonstiff Problems, 2nd edn. Springer, Berlin (1993)
7. Hairer, E., Lubich, C., Wanner, G.: Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations, 2nd edn. Springer, Berlin (2006)
8. Hochbruck, M., Ostermann, A.: Exponential integrators. Acta Numer. 19, 209–286 (2010)
9. Holden, H., Karlsen, K.H., Risebro, N.H.: Operator splitting methods for generalized Korteweg-de Vries equations. J. Comput. Phys. 153, 203–222 (1999)
10. Holden, H., Karlsen, K.H., Lie, K.-A., Risebro, N.H.: Splitting for Partial Differential Equations with Rough Solutions. European Math. Soc. Publishing House, Zürich (2010)
11. Holden, H., Karlsen, K.H., Risebro, N.H., Tao, T.: Operator splitting methods for the Korteweg-de Vries equation. Math. Comp. 80, 821–846 (2011)
12. Holden, H., Lubich, C., Risebro, N.H.: Operator splitting for partial differential equations with Burgers nonlinearity. Math. Comp. 82, 173–185 (2012)
13. Kassam, A.-K., Trefethen, L.N.: Fourth-order time-stepping for stiff PDEs. SIAM J. Sci. Comput. 26, 1214–1233 (2005)
14. Klein, C.: Fourth order time-stepping for low dispersion Korteweg-de Vries and nonlinear Schrödinger equation. ETNA 29, 116–135 (2008)
15. Lawson, J.D.: Generalized Runge–Kutta processes for stable systems with large Lipschitz constants. SIAM J. Numer. Anal. 4, 372–380 (1967)
16. Lubich, C.: On splitting methods for Schrödinger–Poisson and cubic nonlinear Schrödinger equations. Math. Comp. 77, 2141–2153 (2008)
17. Maday, Y., Quarteroni, A.: Error analysis for spectral approximation of the Korteweg-de Vries equation. RAIRO -Modélisation mathématique et analyse numérique 22, 821–846 (1988)
18. McLachlan, R.I., Quispel, G.R.W.: Splitting methods. Acta Numer. 11, 341–434 (2002)
19. Tao, T.: Nonlinear Dispersive Equations. Local and Global Analysis. American Mathematical Society, Providence (2006)
20. Tappert, F.: Numerical solutions of the Korteweg-de Vries equation and its generalizations by the split-step Fourier method. In: Newell, A.C. (ed.) Nonlinear Wave Motion, pp. 215–216. American Mathematical Society, Providence (1974)