Exotic cluster structures on $SL_5$

Idan Eisner

Department of Mathematics, University of Haifa, Haifa, Mount Carmel 31905, Israel

E-mail: eisner@math.haifa.ac.il

Received 31 March 2014, revised 1 July 2014
Accepted for publication 21 July 2014
Published 11 November 2014

Abstract

A conjecture by Gekhtman, Shapiro and Vainshtein suggests a correspondence between the Belavin–Drinfeld (BD) classification of solutions of the classical Yang–Baxter equation and cluster structures on simple Lie groups. This paper confirms the conjecture for $SL_5$. Given a BD class, we construct the corresponding cluster structure in $\mathfrak{sl}_5^*$, and show that it satisfies all parts of the conjecture.

This article is part of a special issue of *Journal of Physics A: Mathematical and Theoretical* devoted to ‘Cluster algebras in mathematical physics’.

Keywords: Poisson–Lie group, Cluster algebra, Belavin–Drinfeld triple

Mathematics Subject Classification: 53D17, 13F60

1. Introduction

Since cluster algebras were introduced by Fomin and Zelevinsky [4], the following natural question arose: given an algebraic variety $V$—can one find a cluster structure in the coordinate ring of $V$?

Partial answers were given for the Grassmannian $G(n, k)$ (see [11]) and for double Bruhat cells in semisimple complex Lie groups [2]. In [5, example 12.10] Fomin and Zelevinsky pointed out that the coordinate ring of the Grassmannian $Gr_{2,n}$ with $n \geq 5$ has two different cluster structures. Gekhtman, Shapiro and Vainshtein [8] recovered multiple cluster structure in $\mathbb{C}[G]$, where $G = SL_n$. They conjectured that this is the case for any simple complex Lie group. They called the newly discovered structures ‘exotic’, as opposed to the one already known, which was sometimes referred to as the ‘standard’ structure. This is due to the fact that it corresponds to the standard Sklyanin bracket on $G$. The conjecture also states that these cluster structures correspond to the BD classification of solutions to the classical Yang–Baxter equation (CYBE). This paper confirms the conjecture for $G = SL_5$. 


1.1. Cluster structures and cluster algebras

Let \{z_1, \ldots, z_m\} be a set of independent variables, and let \(S\) denote the ring of Laurent polynomials generated by \(z_1, \ldots, z_m\)
\[ S = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \]
(Here and in what follows \(z^{\pm 1}\) stands for \(z, z^{-1}\).) The ambient field \(F\) is the field of rational functions in \(n\) independent variables (distinct from \(z_1, \ldots, z_m\)), with coefficients in the field of fractions of \(S\) (if \(m = 0\) this field is just \(\mathbb{Q}\)).

Let \(\tilde{B} = (b_{ij})\) be an integer \(n \times (n + m)\) matrix, whose principal part \(B\) is skew symmetric. The variables \(x_1, \ldots, x_n\) are called cluster variables, while \(x_{n+i} = z_i, 1 \leq i \leq m\) are called stable (or frozen) variables. The set \(\mathbf{x} = (x_1, \ldots, x_n)\) is called a cluster, and the set \(\tilde{\mathbf{x}} = (x_1, \ldots, x_{n+m})\) is called an extended cluster.

The adjacent cluster in direction \(k\) is \(\mathbf{x}_k = \mathbf{x} \cup \{x_k\}\), where \(x_k\) is defined by the exchange relation
\[ x_k x_k' = \prod_{b_{ij} > 0} x_j^{b_{ij}} + \prod_{b_{ij} < 0} x_j^{-b_{ij}}. \]  
A mutation \(\mu_k(B)\) of the matrix \(B = (b_{ij})\) in direction \(k\) is defined
\[ \mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2} \left( |b_{ik}| b_{kj} + |b_{jk}| b_{ik} \right) & \text{otherwise}. \end{cases} \]
A pair \((\tilde{\mathbf{x}}, \tilde{B})\) is called a seed. The adjacent seed in direction \(k\) is \((\tilde{\mathbf{x}}_k, \mu_k(\tilde{B}))\). Two seeds are said to be mutation equivalent if they can be connected by a sequence of pairwise adjacent seeds.

Given a seed \(\Sigma = (\tilde{\mathbf{x}}, \tilde{B})\), the cluster structure \(C(\Sigma)\) in \(F\) is the set of all seeds that are mutation equivalent to \(\Sigma\).

Let \(\Sigma\) be a seed as above, and \(A = \mathbb{Z}[x_{n+1}, \ldots, x_{n+m}]\). The cluster algebra \(A = A(C) = A(\tilde{B})\) associated with the seed \(\Sigma\) is the \(A\)-subalgebra of \(F\) generated by all cluster variables in all seeds in \(C(\tilde{B})\). The upper cluster algebra \(\overline{A} = \overline{A}(C) = \overline{A}(\tilde{B})\) is the intersection of the rings of Laurent polynomials over \(A\) in cluster variables taken over all seeds in \(C(\tilde{B})\). The famous Laurent phenomenon [5] claims the inclusion \(A(C) \subseteq \overline{A}(C)\).

It is convenient to describe \(C(\tilde{B})\) (or the matrix \(\tilde{B}\)) by the exchange quiver \(Q(\tilde{B})\): it has \(n + m\) vertices, each corresponds to a variable \(x_k\), and there is a directed edge \(i \rightarrow j\) with weight \(w\) if \(\tilde{B}_{ij} = w > 0\).

Let \(V\) be a quasi-affine variety over \(\mathbb{C}\), \(C(V)\) be the field of rational functions on \(V\), and \(\mathcal{O}(V)\) be the field of regular functions on \(V\). Let \(C\) be a cluster structure in \(F\) as above. Assume that \(\{f_1, \ldots, f_{n+m}\}\) is a transcendence basis of \(C(V)\). Then the map \(\varphi: x_i \rightarrow f_i, 1 \leq i \leq n + m\), can be extended to a field isomorphism \(\varphi: \mathcal{P}_C \rightarrow C(V)\), where \(\mathcal{P}_C = F \otimes \mathbb{C}\) is obtained from \(F\) by extension of scalars. The pair \((C, \varphi)\) is called a cluster structure in \(C(V)\) (or just a cluster structure on \(V\)). \(\{f_1, \ldots, f_{n+m}\}\) is called an extended cluster in \((C, \varphi)\).

Sometimes we omit direct indication of \(\varphi\) and say that \(C\) is a cluster structure on \(V\). A cluster structure \((C, \varphi)\) is called regular if \(\varphi(x)\) is a regular function for any cluster variable \(x\). The two algebras defined above have their counterparts in \(\mathcal{P}_C\) obtained by extension of scalars; they are denoted \(A_C\) and \(\overline{A}_C\). If, moreover, the field isomorphism \(\varphi\) can be restricted to an isomorphism of \(A_C\) (or \(\overline{A}_C\)) and \(\mathcal{O}(V)\), we say that \(A_C\) (or \(\overline{A}_C\)) is naturally isomorphic to \(\mathcal{O}(V)\).
The following statement is a weaker analogue of proposition 3.37 in [7].

**Proposition 1.** Let $V$ be a Zariski open subset in $\mathbb{C}^{n+m}$ and $(C = C(\tilde{B}), \varphi)$ be a cluster structure in $\mathcal{C}(V)$ with $n$ cluster and $m$ stable variables such that

1. $\text{rank } \tilde{B} = n$;
2. there exists an extended cluster $\tilde{x} = (x_1, \ldots, x_{n+m})$ in $C$ such that $\varphi(x_i)$ is regular on $V$ for $i \in [n+m]$;
3. for any cluster variable $x_k$, $k \in [n]$, obtained by applying the exchange relation (1) to $\tilde{x}$, $\varphi(x_k)$ is regular on $V$;
4. for any stable variable $x_{n+i}, i \in [m]$, the function $\varphi(x_{n+i})$ vanishes at some point of $V$;
5. each regular function on $V$ belongs to $\mathcal{A}_C(C)$.

Then $\mathcal{A}_C(C)$ is naturally isomorphic to $\mathcal{O}(V)$.

### 1.2. Compatible Poisson brackets

Let $\{\cdot, \cdot\}$ be a Poisson bracket on $F$. Two elements $f_1, f_2 \in F$ are log—canonical with respect to $\{\cdot, \cdot\}$ if there exists an integer $\omega$ s.t. $\omega \{f_1, f_2\} = \{f_1 f_2\}$. A set $F \subseteq F$ is log—canonical if every pair in $F$ is log—canonical.

We say that $\{\cdot, \cdot\}$ is compatible with the cluster structure $C(\tilde{B})$ if every cluster is log—canonical w.r.t. $\{\cdot, \cdot\}$, that is for every pair $x_i, x_j$ in an extended cluster $\tilde{x}$ there exists an integer $\omega_{ij}$ such that

$$\{ x_i, x_j \} = \omega_{ij} x_i x_j. \tag{2}$$

The matrix $\Omega^x = (\omega_{ij})$ is called the coefficient matrix of $\{\cdot, \cdot\}$ (in the basis $\tilde{x}$); clearly, $\Omega^x$ is skew symmetric.

A complete characterization of Poisson brackets compatible with a given cluster structure $C(\tilde{B})$ in the case $\text{rank } B = n$ is given in [6], see also [7, Ch 4]. In particular, the following statement is an immediate corollary of theorem 1.4 in [6].

**Proposition 2.** If $\text{rank } \tilde{B} = n$ then a Poisson bracket is compatible with $C(\tilde{B})$ if and only if its coefficient matrix $\Omega^x$ satisfies $B \Omega^x = D [0 \ 0^T]$, where $D$ is a diagonal matrix.

### 1.3. Poisson–Lie groups and $R$-matrices

A Lie group $G$ with a Poisson bracket $\{\cdot, \cdot\}$ is called a Poisson–Lie group if the multiplication map $\mu: G \times G \to G, \mu( x, y ) \mapsto xy$ is Poisson. That is, $G$ with a Poisson bracket $\{\cdot, \cdot\}$ is a Poisson–Lie group if

$$\{f_1, f_2\}(xy) = \{\rho_{f_1} f_2\}(x) + \{\lambda_{f_1} f_2\}(y),$$

where $\rho_x$ and $\lambda_x$ are, respectively, right and left translation operators on $G$.

Let $g$ be the Lie algebra of a Lie group $G$ with a nondegenerate invariant bilinear form $\langle \cdot, \cdot \rangle$, and let $t \in g \otimes g$ be the corresponding Casimir element. For an element $r = \sum a_i \otimes b_i \in g \otimes g$ denote

$$[r, r] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j + \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]$$

and $r^{21} = \sum_i b_i \otimes a_i$. 

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A classical $R$-matrix is an element $r \in g \otimes g$ that satisfies the CYBE
\[
[r, r] = 0
\]
together with the condition
\[
r + r^{21} = 0.
\]
Following [10], recall the construction of the Drinfeld double of a Lie algebra $g$ with the Killing form $\langle \cdot, \cdot \rangle$; define $D(g) = g \oplus g$, with an invariant nondegenerate bilinear form
\[
\langle (\xi, \eta), (\xi', \eta') \rangle = \langle \xi, \xi' \rangle - \langle \eta, \eta' \rangle.
\]
Define subalgebras $\mathfrak{d}_+$ and $\mathfrak{d}_-$ of $D(g)$ by
\[
\mathfrak{d}_+ = \{(\xi, \xi) : \xi \in g\}, \quad \mathfrak{d}_- = \{(R_+(\xi), R_-(\xi)) : \xi \in g\},
\]
where $R_\pm \in \text{End}g$ are defined for any $R$-matrix $r$ by
\[
\mathfrak{d}_+ = \{(\xi, \xi) : \xi \in g\}, \quad \mathfrak{d}_- = \{(R_+(\xi), R_-(\xi)) : \xi \in g\},
\]
where $\langle \cdot, \cdot \rangle$ is the corresponding Killing form on the tensor square of $g$.

Any classical $R$-matrix induces Poisson bracket on $\mathfrak{g}$: choose a basis $\{I_a\}$ in $g$, and let $\partial_a$ (resp., $\partial'_a$) be the right (resp., left) invariant vector field on $\mathfrak{g}$ whose value at the unit element is $I_a$. Let $r = \sum_{\alpha, \beta} r_{\alpha, \beta} I_{\alpha} \otimes I_{\beta}$. Then
\[
\{f_1, f_2\} = \sum_{\alpha, \beta} r_{\alpha, \beta} \left( \partial_{a_1} \partial_{b_1} f_2 - \partial'_{a_1} \partial'_{b_1} f_2 \right)
\]
defines a Poisson bracket on $\mathfrak{g}$. This bracket is called the Sklyanin bracket corresponding to $r$.

The classification of classical $R$-matrices for simple complex Lie groups was given by Belavin and Drinfeld in [1].

Let $\mathfrak{g}$ be a simple complex Lie algebra with a fixed nondegenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. Fix a Cartan subalgebra $\mathfrak{h}$, a root system $\Phi$ of $\mathfrak{g}$, and a set of positive roots $\Phi^+$. Let $\Delta \subseteq \Phi^+$ be a set of positive simple roots.

A BD triple is two subsets $\Gamma_1, \Gamma_2$ of $\Delta$ and an isometry $\gamma: \Gamma_1 \to \Gamma_2$ nilpotent in the following sense: for every $\alpha \in \Gamma_1$ there exists $m \in \mathbb{N}$ such that $\gamma^m(\alpha) \in \Gamma_1$ for $j = 0, \ldots, m - 1$, but $\gamma^m(\alpha) \not\in \Gamma_1$. The isometry $\gamma$ extends in a natural way to a map between root systems generated by $\Gamma_1, \Gamma_2$. This allows one to define a partial ordering on $\Phi$: $\alpha < \beta$ if $\beta = \gamma(\alpha)$ for some $j \in \mathbb{N}$.

Select root vectors $E_\alpha \in g$ satisfying $\langle E_\alpha, E_\beta \rangle = 1$. According to the BD classification, the following is true (see, e.g., [3, chapter 3]).

**Proposition 3.** (i) Every classical $R$-matrix is equivalent (up to an action of $\sigma \otimes \sigma$ where $\sigma$ is an automorphism of $\mathfrak{g}$) to
\[
r = \gamma_0 + \sum_{\alpha \in \Phi^+} E_{-\alpha} \otimes E_\alpha + \sum_{\alpha < \beta} E_{-\alpha} \otimes E_\beta
\]
(ii) $\gamma_0 \in \mathfrak{h} \otimes \mathfrak{h}$ in (7) satisfies
\[
\langle \gamma(\alpha) \otimes \text{Id} \rangle \gamma_0 + \langle \text{Id} \otimes \alpha \rangle \gamma_0 = 0
\]
for any $\alpha \in \Gamma_\mu$, and
\[
\gamma_0^{21} = \gamma_0.
\]
where \( t_0 \) is the \( h \otimes h \) component of \( t \). (iii) Solutions \( r_0 \) to (8), (9) form a linear space of dimension \( k_r = |\Delta| \setminus I_I \).

We say that two classical \( R \)-matrices that have a form (7) belong to the same BD class if they are associated with the same BD triple. The corresponding bracket defined in (6) by an \( R \)-matrix \( r \) associated with a triple \( T \) will be denoted by \( \{ \cdot, \cdot \}_T \).

**Example 4.** As a running example, let us look at the triple \( T_{12} \equiv (\alpha, \beta, \gamma) \rightarrow (\alpha \alpha \alpha) \).

First, construct \( r_0 \) following [3, chapter 3] use the coefficient matrix \( r_{ab} \) defined by

\[
\alpha \beta \alpha \beta \alpha \beta = \otimes \in \alpha \beta \alpha \beta \alpha \beta (\alpha, \beta) \in \Delta.
\]

Then conditions (8) and (9) are equivalent to the following equations:

\[
\begin{align*}
\alpha \beta \alpha \beta \alpha \beta &= \in \alpha \beta \beta \alpha \beta \alpha \beta (\alpha, \beta) \in \Delta, \\
\alpha \Gamma \beta \Delta &= \in \gamma \alpha \beta \beta \alpha \beta \alpha \beta (\alpha, \beta) \in \Delta.
\end{align*}
\]

In the \( T_{12} \) case equation (11) takes the form

\[
r_{2\beta} + r_{\beta 1} = 0 \quad \forall \beta \in \Delta,
\]

so we can set

\[
r_{ab} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 
\end{bmatrix}
\]

and take \( n_0 = \sum_{ab} r_{ab} \alpha^n \otimes \beta^a \). Now we get the CYBE solution from (7). Next we define \( X = X \otimes X \), so (6) takes the form

\[
\{ x_{ij}, x_{kl} \}_{12} = (RX - XR)_{5(i-1)+k,5(j-1)+\ell},
\]

and now for every pair of functions \( f, g \) we compute

\[
\{ f, g \}_{12} = \sum_{i,j,k,l} \frac{\partial f}{\partial x_{ij}} \frac{\partial g}{\partial x_{kl}} \{ x_{ij}, x_{kl} \}.
\]

Given a BD triple \( T \) for \( \mathcal{G} \), define the torus \( \mathcal{H}_T = \exp h_T \subset \mathcal{G} \), where

\[
h_T = \{ h \in h : \alpha(h) = \beta(h) \text{ if } \alpha<\beta \}
\]

In [8] Gekhtman, Shapiro and Vainshtein give the following conjecture:

**Conjecture 5.** Let \( \mathcal{G} \) be a simple complex Lie group. For any BD triple \( T = (\Gamma_1, \Gamma_2, \gamma) \) there exists a cluster structure \( \mathcal{C}_T \) on \( \mathcal{G} \) such that

(1) the number of stable variables is \( 2k_T \), and the corresponding extended exchange matrix has a full rank;
(2) \( \mathcal{C}_T \) is regular, and the corresponding upper cluster algebra \( \mathcal{A}_e(C_T) \) is naturally isomorphic to \( \mathcal{O}(\mathcal{G}) \);
(3) the global toric action of \( (\mathbb{C}^*)^{2k_T} \) on \( \mathcal{C}(\mathcal{G}) \) is generated by the action of \( \mathcal{H}_T \otimes \mathcal{H}_T \) on \( \mathcal{G} \) given by \( \{ H_1, H_2 \}(X) = H_1XH_2 \).
(4) for any solution of CYBE that belongs to the BD class specified by $T$, the corresponding Sklyanin bracket is compatible with $C_T$;

(5) a Poisson–Lie bracket on $\mathcal{G}$ is compatible with $C_T$ only if it is a scalar multiple of the Sklyanin bracket associated with a solution of CYBE that belongs to the Belavin-Drinfeld class specified by $T$.

The conjecture was proved for the BD class $I_1 = I_2 = \emptyset$. This trivial triple corresponds to the standard Poisson–Lie bracket. We call the cluster structures associated with the non-trivial BD data exotic.

In the Cremmer–Gervais case $I_1^\alpha = \cdots \rightarrow \{\alpha_1, \alpha_2\}$, $I_2^\alpha = \{\alpha_3, \alpha_4\}$ and $\gamma^\alpha: \alpha_1 \rightarrow \alpha_2 + 1$. This case is, in some sense, ‘the furthest’ from the standard case, because here $|I_1|$ is maximal. Conjecture 5 was proved for the Cremmer–Gervais case in [9]. The conjecture is also true for all exotic cluster structures on $SL_n$ with $n \leq 4$ (see [8]). This paper deals with $SL_5$.

2. Exotic cluster structures

Consider the two following isomorphisms of the BD data on $SL_5$: the first one transposes $I_1$ and $I_2$ and reverses the direction of $\gamma$, while the second one takes each root $\alpha_i$ to $\alpha_{\omega_0(i)}$, where $\omega_0$ is the longest element in the Weyl group (which in our case is naturally identified with the symmetric group $S_4$). These two isomorphisms correspond to the automorphisms of $SL_5$ given by $X \mapsto -X^t$ and $X \mapsto \omega_0 X \omega_0$, respectively. Since we consider $R$-matrices up to an action of $\sigma^\otimes \sigma$, in what follows we do not distinguish between BD triples obtained one from the other via these isomorphisms.

$SL_5$ has four simple roots, and the Dynkin diagram is given in figure 1.

Therefore, the BD triples are (up to the above isomorphisms):

1. $I_1 = \{\emptyset\}, I_2 = \{\emptyset\}$
2. $I_1 = \{\alpha_1\}, I_2 = \{\alpha_2\}, \gamma: \alpha_1 \rightarrow \alpha_2$
3. $I_1 = \{\alpha_1\}, I_2 = \{\alpha_3\}, \gamma: \alpha_1 \rightarrow \alpha_3$
4. $I_1 = \{\alpha_1\}, I_2 = \{\alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_4$
5. $I_1 = \{\alpha_2\}, I_2 = \{\alpha_3\}, \gamma: \alpha_2 \rightarrow \alpha_3$
6. $I_1 = \{\alpha_1, \alpha_2\}, I_2 = \{\alpha_3, \alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_1 + 1$
7. $I_1 = \{\alpha_1, \alpha_2\}, I_2 = \{\alpha_3, \alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_2 + 1$
8. $I_1 = \{\alpha_1, \alpha_3\}, I_2 = \{\alpha_2, \alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_1 + 1$
9. $I_1 = \{\alpha_1, \alpha_3\}, I_2 = \{\alpha_2, \alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_2 + 1$
10. $I_1 = \{\alpha_1, \alpha_3\}, I_2 = \{\alpha_1, \alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_1 + 1$ (mod 5)
11. $I_1 = \{\alpha_1, \alpha_2, \alpha_3\}, \gamma: \alpha_1 \rightarrow \alpha_1 + 1$ (mod 5)
12. $I_1 = \{\alpha_1, \alpha_2\}, I_2 = \{\alpha_3, \alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_5 + 1$
13. $I_1 = \{\alpha_1, \alpha_2\}, I_2 = \{\alpha_3, \alpha_4\}, \gamma: \alpha_1 \rightarrow \alpha_1 + 2$ (mod 5)

Slightly abusing the notation, we sometime refer to a root $\alpha_i \in \Delta$ just as $i$, and write $\gamma^i: i \mapsto j$ instead of $\gamma^\alpha: \alpha_i \rightarrow \alpha_j$. We say that a BD triple $T = (I_1, I_2, \gamma)$ is orientable unless there is a pair of adjacent roots $i, i + 1 \in I_1$ such that $\gamma(i + 1) = 1 = \gamma(i)$. 

Figure 1. The Dynkin diagram of $SL_5$. 

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Case 1 (the trivial case) corresponds to the standard cluster structure on $SL_5$ described in [2] (see also [8]). Case 11 is the Cremmer–Gervais case, and it is covered in [9]. Case 12 is the only non-orientable triple. It will be treated separately in section 3. The following discussion concerns all orientable BD triples.

The main result of this paper is the following theorem:

**Theorem 6.** Conjecture 5 is true for $G = SL_5$ and any BD triple 1–12 above.

This theorem will be proved by constructing a structure $C_T$ on $G$ that satisfy statements 1–5 of the conjecture.

**Remark 7.** The last case 13 was excluded since some parts of it were not computed. This is also why some of the following propositions regard only the triples numbered 1–11 in the list above (case 12 is treated separately in section 3). See section 5 for more details.

2.1. Constructing a log canonical set for the orientable cases

Let $T = (I_1, I_2, \gamma)$ be an orientable BD triple for $SL_5$. Denote by $\{\cdot, \cdot\}_T$ the corresponding Poisson–Lie bracket. We are going to construct a set of matrices $M$, so that the set of all leading principal minors of these matrices will be log canonical with respect to $\{\cdot, \cdot\}_T$. This set will be the basis of the corresponding exotic cluster structure.

We start our construction with an element $(X, Y)$ in the double $D(sl_5)$. The building blocks of our matrices are submatrices of the matrices $X$ and $Y$ of the following form: set $k \in \{0, \ldots, 4\}$. A building block is obtained either by deleting the first $k$ rows and last $k$ columns of $X$, or by deleting the first $k$ columns and the last $k$ rows of $Y$. Using the notation $X_R^C$ for the submatrix of $X$ with set of rows indices $R$ and set of column indices $C$, we have two kinds of these blocks: $U_i = X^{[1,5]}_{[i,5]} - I$ and $V_j = Y^{[1,5]}_{[j,6]-j}$.

For a matrix $A$ with lower right-hand entry $x_{ij}$, let $\xi(A) = \min (i, j)$. Define

$$\sigma(A) = \begin{cases} 1, & \text{if } A \text{ is } U_i, \\ -1, & \text{if } A \text{ is } V_j, \end{cases}$$

and $\sigma(A) = \frac{1}{2}(3 - \sigma(A))$, so

$$\sigma(A) = \begin{cases} 1 & \text{if } \sigma(A) = 1 \\ 2 & \text{if } \sigma(A) = -1. \end{cases}$$

Let $A = W_{ij}^k$, where $W$ is a matrix $X$ or $Y$, and $P$, $Q$ are subsets of $\{1, \ldots, 5\}$ of the form $\{1, \ldots, k\}$ or $\{\ell, \ldots, 5\}$. An extension of a submatrix $A$ by a number $t$ is adding rows or columns of the matrix $W$ to $A$ according to the following rule: if $t > 0$, the set $\{1, \ldots, k\}$ becomes $\{1, \ldots, k + t\}$. If $t < 0$ the set $\{\ell, \ldots, 5\}$ becomes $\{\ell - |t|, \ldots, 5\}$. Denote the extended matrix by $A(t)$. Thus for a positive integer $t$, a matrix of type $U_i$ can be extended by $t$ to $U_i(t) = X^{[1,5]}_{[i,5]} - I$ or by $-t$ to $U_i(-t) = X^{[1,5]}_{[i,5]} + I$. Similarly, extending $V_j$ by $t$ gives $V_j(t) = Y^{[1,5]}_{[j,6]-j} + I$ and extending it by $-t$ gives $V_j(-t) = Y^{[1,5]}_{[j,6]-j} - I$. Clearly, for a given submatrix $A$ of type $V$ or $U$ the two possible (positive and negative) directions of extension of $A$ are determined by $\sigma(A)$.

An extended block diagonal matrix is a block diagonal matrix whose blocks are submatrices of $X$ and $Y$ that have been extended as above, with two restrictions: the first block...
was not extended in the negative direction and the last block was not extended in the positive direction. These two conditions assure the matrix is still a square matrix. An example is given in figure 2: the striped rectangles are extensions of the blocks $B_k$. A block of an extended block diagonal matrix is an extended submatrix of $X$ or $Y$. Such a block is not necessarily square, but may still be extended.

Let $A$ be an extended block diagonal matrix with two blocks or more, $B_1, \ldots, B_k$. We define $\hat{\xi}(A) = \hat{\xi}(B_k)$, $\sigma(A) = \sigma(B_k)$ and $\hat{\sigma}(A) = \hat{\sigma}(B_k)$. Extending such a matrix $A$ is simply extending the last (lower) block $B_k$.

The direct sum of a matrix $A_1$ and a matrix $A_2$ is a new block diagonal matrix $A$ with blocks $A_1$ and $A_2$. That is, $A = A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$.

Last, we need to define the extension number: for a submatrix $A$, we look at the root $\alpha = \hat{\xi}(A)$. We look at the subgraph obtained by intersecting the Dynkin diagram of $\mathrm{SL}_5$ and $\Gamma_{\sigma(A)}$. The connected component of $\alpha$ in this intersection is a path $\left( p_1 < \cdots < \alpha < \cdots < p_k \right)$. Let $t^r(A)$ be the number of vertices in the path $\left( p_1, \ldots, p_k \right)$, i.e., $t^r(A) = p_k - \alpha + 1$, and let $t^\gamma(A)$ be the number of vertices in the path $\left( p_1, \ldots, \alpha \right)$, that is $t^\gamma(A) = \alpha - p_1 + 1$. Note that $t^r(A)$ and $t^\gamma(A)$ are set to 1 if $\alpha$ is the maximal or minimal root in this subgraph, respectively.

To define the set $\mathcal{M}$, start with the set of all $V_j$ with $j - 1 \notin I_j$ and all $U_i$ with $i - 1 \notin I_j$.

To every matrix $A$ in this set, apply the following steps.

(1) Put $A_0 = A$.
(2) If $\hat{\xi}(A_i) \notin \Gamma_{\sigma(A)}$, then stop the process and add $A_i$ to $\mathcal{M}$.
(3) Otherwise, if $\sigma(A_i) = 1$, define $A_i^+ = V_{t^r(\hat{\xi}(A_i)) + 1}$. If $\sigma(A_i) = -1$, define $A_i^+ = U_{t^\gamma(\hat{\xi}(A_i)) + 1}$.
(4) Take the direct sum $\begin{bmatrix} A_i & 0 \\ 0 & A_i^+ \end{bmatrix}$.
(5) Extend $A_i$ by $t^r(A_i)$, and extend $A_i^+$ by $t^\gamma(A_i)$. Note that this matrix is now extended block diagonal.
(6) Let $A_{i+1}$ be the matrix obtained in step 5. Go back to step 2 with $i = i + 1$.

Eventually, we get a set of matrices which are all either just submatrices of $X$ and $Y$ or extended block diagonal matrices. We will denote this set by $\mathcal{M}$, and use it to define our log canonical functions.

**Example 8.** Construct the set $\mathcal{M}$ for our running example $T_{12}$: start with submatrices

$$M_1 = X, \quad M_2 = Y, \quad M_3 = \begin{bmatrix} y_{12} & y_{13} & y_{14} & y_{15} \\ y_{22} & y_{23} & y_{24} & y_{25} \\ y_{32} & y_{33} & y_{34} & y_{35} \\ y_{42} & y_{43} & y_{44} & y_{45} \end{bmatrix}.$$ 

$$M_4 = \begin{bmatrix} y_{14} & y_{15} \\ y_{24} & y_{25} \end{bmatrix}.$$ 

$$M_5 = \begin{bmatrix} y_{15} \end{bmatrix}.$$ 

$$M_6 = \begin{bmatrix} x_{31} & x_{32} & x_{33} \\ x_{41} & x_{42} & x_{43} \\ x_{51} & x_{52} & x_{53} \end{bmatrix}.$$ 

$$M_7 = \begin{bmatrix} x_{41} & x_{42} \\ x_{51} & x_{52} \end{bmatrix}.$$ 

$$M_8 = \begin{bmatrix} x_{51} \end{bmatrix}.$$ 

Apply steps 1–6 above to every matrix in $\mathcal{M}$: for $M_4$ and $M_2$ we stop at step 2 (this is true for all BD triples, as $\xi(X) = \xi(Y) = 5 \notin \mathcal{F}_1, \mathcal{F}_2$). Similarly, matrices $M_5, M_6, M_7$ stop the algorithm at step 2, because for these matrices $\xi(M_k) \notin \mathcal{F}_{\mathcal{P}(M_k)}$. This is not the case with the other two matrices: for $M_4$ we have $\xi(M_4) = 2 \in \mathcal{F}_2$. Therefore we set $A^+ = U_{r^{-1}(2)+1} = U_5$, or

$$A^+ = \begin{bmatrix} x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \\ x_{51} & x_{52} & x_{53} & x_{54} \end{bmatrix}.$$ 

The direct sum is

$$M_4 \oplus A^+ = \begin{bmatrix} y_{14} & y_{15} & 0 & 0 & 0 & 0 \\ y_{24} & y_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{21} & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{31} & x_{32} & x_{33} & x_{34} \\ 0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} \\ 0 & 0 & x_{51} & x_{52} & x_{53} & x_{54} \end{bmatrix}.$$
and the extension numbers are \( t^* = t^- = 1 \). The result is the extended block matrix

\[
\begin{bmatrix}
  y_{14} & y_{15} & 0 & 0 & 0 \\
  y_{24} & y_{25} & x_{11} & x_{12} & x_{13} & x_{14} \\
  y_{34} & y_{35} & x_{21} & x_{22} & x_{23} & x_{24} \\
  0 & 0 & x_{31} & x_{32} & x_{33} & x_{34} \\
  0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} \\
  0 & 0 & x_{51} & x_{52} & x_{53} & x_{54}
\end{bmatrix}.
\]

Returning to step 2 with this matrix terminates the algorithm, since \( 4 \notin I_1 \).

Repeat the process for \( M_8 = \begin{bmatrix} x_{51} \end{bmatrix} \); here \( \xi(\{M_8\}) = 1 \in I_1 \), and so \( A^+ = Y_{[\{1 \ldots 5\}]}, \) and the extension numbers are again \( t^* = t^- = 1 \). We end up with the matrix

\[
\begin{bmatrix}
  x_{51} & x_{52} & 0 & 0 \\
  y_{12} & y_{13} & y_{14} & y_{15} \\
  y_{22} & y_{23} & y_{24} & y_{25} \\
  y_{32} & y_{33} & y_{34} & y_{35}
\end{bmatrix}.
\]

This matrix has \( y_{35} \) at the lower right-hand corner, and after returning to step 2 the algorithm stops since \( 3 \notin I_2 \).

The set \( \mathcal{M} \) is now defined. It has eight matrices: \( X, Y \), four submatrices of \( X \) and \( Y \), and two extended block diagonal matrices consisted of blocks that are submatrices of \( X \) and \( Y \).

Note that \( U_i = X \) and \( V_i = Y \) are always in the set \( \mathcal{M} \). The number of elements in \( \mathcal{M} \) is just the number of submatrices we start with. For each root \( i \in \mathcal{I} \) we have such a submatrix, as well as for each \( f_i \notin I_2 \). Adding one for \( U_i = X \) and \( V_i = Y \), we end up with \( |\mathcal{M}| = 2 \left| I_1 \right| + 2 = 2k_I + 2 \) matrices.

We can now define our log canonical functions: for every \( M \in \mathcal{M} \) take all the leading principal minors of \( M \). If we denote the number of rows (and columns) of a matrix \( M \) by \( s(M) \), these are all the functions \( \det_{[1 \ldots r]} M \) with \( 1 \leq r \leq s(M) \). Thus we have a set of functions \( F(X, Y) \) on \( \text{Diag} \). The projection of \( F(X, Y) \) on the diagonal subgroup can be viewed as a function \( f(X) = F(X, X) \) on \( \text{SL}_5 \). Note that after this projection all the minors of \( U_i = X \) coincide with those of \( V_i = Y \).

For any of these functions \( f(X) \) consider the lower right matrix entry of the submatrix associated with \( f \). This entry is \( x_{ij} \) for some \( i, j \in [5] \). This defines a map \( \rho \) from the set of functions to \( [5] \times [5] \).

**Proposition 9.** The map \( \rho \) is bijective.

**Proof.** Consider the set \( \{ U_i, V_j \} \) of building blocks of the matrices of \( \mathcal{M} \). Each block was used once: either as a starting block if \( i \in \mathcal{I} \) or \( j \notin I_2 \), or as a direct summand block if \( i \notin \mathcal{I} \) or \( j \in I_2 \). Since the main diagonals of these blocks are exactly all the diagonals of the matrix \( X \), it follows that every \( x_{ij} \) occurs exactly once on the main diagonal of one matrix \( M \in \mathcal{M} \).

This allows us to write \( f_{ij} = \rho^{-1}(i, j) \). Removing the function \( f_{55} = \det X \) (which is constant on \( \text{SL}_5 \)), we get a set of 24 regular functions on \( \text{SL}_5 \). Denote this set by \( B = \{ f_{ij} \}_{i,j=1}^5 \setminus \{ f_{55} \} \).
Proposition 10. For every orientable triple $T$ the set $\mathcal{B}$ is algebraically independent.

**Proof.** This can be verified by checking that the gradients of $\{f_i\}_{i,j=1}^5$ form a linearly independent set. It is sufficient to evaluate the gradients at a random point and show that they are linearly independent. This was done with Maple. □

Let $T$ be an orientable BD triple, and let $\{\cdot,\cdot\}_T$ denote the Sklyanin bracket associated with the triple $T$.

Proposition 11. For every triple $T$ numbered 1–11 in the list above, the set $\mathcal{B}$ is log canonical with respect to $\{\cdot,\cdot\}_T$.

**Proof.** To show that we just compute $\frac{\partial f_i}{\partial f_j}$. Using Maple, we see that this term is constant for every pair of functions $f_i, f_j \in \mathcal{B}$. □

For every pair of functions $\phi_i, \phi_j \in B$ we used Maple to compute the coefficient $\omega_{ij} = \frac{\phi_i \phi_j}{\phi_j \phi_i}$. Let $\Omega = (\omega_{ij})$ be the Poisson coefficient matrix. According to the proposition 2, a cluster structure in $\mathcal{O}(SL_5)$ with a seed $(B, B)$, is compatible with the Poisson structure $\{\cdot,\cdot\}$ if $B$ satisfies $\Omega B = [D \ 0]$ where $D$ is a diagonal matrix. In all the cases described in this paper, $\Omega$ is a $24 \times 24$ matrix of rank 24, so we simply compute $B = \Omega^{-1}$.

**Example 12.** In the case $T_{i=2}$ write $\phi_i = \phi_{5(i-1)+j}$. The Poisson coefficient matrix is then $5\Omega^{(i)} = \begin{bmatrix} ... \end{bmatrix}$

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2.2. Recovering the cluster structure for the orientable case

Motivated by [9], we look at the functions that are determinants of the matrices \( M \in \mathcal{M} \): let \( S = \{ f_{m,k} \mid 1 \leq m, k \leq 4, \text{ and } m \notin I_1, k \notin I_2 \} \), and take the corresponding set of functions on the double \( \tilde{S} = \{ \det M \mid M \in \mathcal{M} \setminus \{X, Y\} \} \).

**Proposition 13.** (1) The functions \( F_{XY}(,) \) of \( \tilde{S} \) are semi-invariant of the left and right action of \( D_+ = \exp \mathfrak{d}_+ \). (2) The functions \( f_X() \) of \( S \) are log canonical with \( x_{ij} \) for every \( i, j \in [1...5] \).

**Proof.** (1) We want to verify that the determinants of \( M \in \mathcal{M} \) are semi-invariant for every BD triple. Every triple in the list at the beginning of this section will be treated in a similar way. The case \( T_{12} \) is given below, and similar considerations can be made for all other cases. First look at the subgroup \( D_- \) of \( SL_2 \) that corresponds to the subalgebra \( \mathfrak{d}_- \) of \( \mathfrak{d}(sl_n) \) as defined in (5). In the example \( T_{12} \) we have

\[
D_- = \left\{ \begin{bmatrix} A & * & * & * \\ 0 & a_1 & * & * \\ 0 & 0 & a_2 & * \\ 0 & 0 & 0 & a_3 \end{bmatrix}, \begin{bmatrix} a_3 & * & * \\ 0 & A & * \\ 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \end{bmatrix} \right\},
\]

where \( A \in GL_2 \) and the * stand for terms that are not necessary for the computation. Denote

\[
U_1 = \begin{bmatrix} A & * & * & * \\ 0 & a_1 & * & * \\ 0 & 0 & a_2 & * \\ 0 & 0 & 0 & a_3 \end{bmatrix}, \quad U_2 = \begin{bmatrix} A' & * & * & * \\ 0 & a_1' & * & * \\ 0 & 0 & a_2' & * \\ 0 & 0 & 0 & a_3' \end{bmatrix},
\]

and

\[
L_1 = \begin{bmatrix} a_3 & 0 & 0 & 0 \\ * & A & 0 & 0 \\ * & * & a_1 & 0 \\ * & * & * & a_2 \end{bmatrix}, \quad L_2 = \begin{bmatrix} a_3' & 0 & 0 & 0 \\ * & A' & 0 & 0 \\ * & * & a_1' & 0 \\ * & * & * & a_2' \end{bmatrix}.
\]

Then the two-sided action of \( D_- \) on \( (X, Y) \in D(SL_2) \) in this case is

\( (X, Y) \mapsto (U_1XU_2, L_1YL_2) \).

Clearly, the determinant \( \det X \) (and \( \det Y \)) is semi-invariant. The case \( T_{12} \) has six other matrices in \( \mathcal{M} \):

\[
(1) \begin{bmatrix} x_{3,1} & x_{3,2} & x_{3,3} \\ x_{4,1} & x_{4,2} & x_{4,3} \\ x_{5,1} & x_{5,2} & x_{5,3} \end{bmatrix},
(2) \begin{bmatrix} x_{4,1} & x_{4,2} \\ x_{5,1} & x_{5,2} \end{bmatrix}.
\]
Now we can verify that the determinants of these matrices are indeed semi-invariants of the left and right action of $D$. As mentioned above, this is done in a similar manner for every triple $T$. (2) Here we simply compute $\omega = \{f_{ij, x_0} \} / f_{ij, x_0}$ with Maple and see that for every $f \in S$ and $i, j \in [1...5]$ we have $\omega \in \mathbb{C}$.

Setting the set $S$ as the set of stable variables, and $\tilde{B}$ as the suitable submatrix of $B$ produces a seed $(B, \tilde{B})$. By its definition, the set $S$ has $2 |(A \setminus T)| = 2k_T$ functions. One can also see that the set $\tilde{S}$ has the determinants of all matrices in $M$ except $X$ and $Y$. Since the set $M$ has $|M| = 2k_T + 2$ matrices, this means $|\tilde{S}| = 2k_T$. Both ways fit assertion 1 of conjecture 5 about the number of stable variables. For reasons that will be explained below, we will sometimes extend $(B, \tilde{B})$ from a cluster structure in $\mathcal{O}(SL_5)$ to one in $\mathcal{O}(\text{Mat}_3)$, adding $\det X$ as a stable variable, and the appropriate column to $\tilde{B}$. We will use this form when describing the quivers of the exotic cluster structures.

Recall that the standard cluster structure on $SL_5$ is the one that corresponds to the trivial BD triple $T = T_1 = T_2 = \emptyset$. The quiver of the standard cluster structure is shown in figure 3 (circles represent mutable variables and squares represent stable variables). The vertex in the $i$th row and $j$th column corresponds to the cluster variable $f_{ij}$. Note that in the standard case all functions of the form $f_{ij}$ or $f_{ji}$ are stable variables.

All the exotic quivers have similar form, with minor changes. For a triple $T = (T_1, T_2, T)$, let $Q_T$ be the quiver obtained from the standard one through the following operations:

- For every $i \in T_1$ the stable variable $f_{ij}$ becomes a cluster variable. Similarly, for every $j \in T_2$ the stable variable $f_{ji}$ becomes a cluster variable.
- For every new cluster variable $f_{ij}$ (that was stable in the standard quiver), arrows are added from $f_{ij}$ to $f_{i,j+1}$, from $f_{ij}$ to $f_{i,j+1}$, and from $f_{i,j+1}$ to $f_{ij}$. 


For every new cluster variable $f_j$ (that was stable in the standard quiver), arrows are added from $f_j$ to $f_{\gamma^{-1}(j),1}$, from $f_j$ to $f_{j+1,5}$ and from $f_{\gamma^{-1}(j)+1,1}$ to $f_j$. As an example the quiver $Q_{1\rightarrow 2}$ is shown in figure 4. The ‘non standard’ arrows are dashed.

Remark 14. If $T$ is an orientable triple with $|I_1| > 1$, the operations above can be applied independently for each $i \in I_1$ and $j \in I_2$. For example, the quiver of $T = (\{1, 2\}, \{2, 3\}, \gamma: i \mapsto i + 1)$ can be viewed as a ‘superposition’ of the quiver of $T = (\{1\}, \{2\}, \gamma: i \mapsto i + 1)$ and the quiver of $T = (\{2\}, \{3\}, \gamma: i \mapsto i + 1)$: both quivers have the same the set of vertices, and the set of edges of the new quiver is the union of sets of edges of the two.

Proposition 15. For every triple $T$ numbered $1\rightarrow 11$ in the list above, the quiver $Q_T$ describes a cluster structure on $SL_5$ that is compatible with the Sklyanin bracket $\{\cdot, \cdot\}_T$ associated with the triple $T$.

Proof. Recall that the quiver $Q_T$ was obtained from the matrix $B = \Omega^{-1}$. Thus, proposition 2 assures that the cluster structure $C(B)$ is compatible with the bracket $\{\cdot, \cdot\}_T$. \qed

Note that the quiver of the standard structure is planar. The quivers of all the exotic structures are non-planar, since there are edges connecting $f_{5,j}$ and $f_{j,1}$ or $f_{3,j}$ and $f_{j,1}$. However, any orientable exotic quiver can be embedded on the torus such that there are no crossing edges. Figure 5 illustrates such an embedding for the quiver of the cluster structure $\{1\} \rightarrow \{2\}$. We identify opposite edges of the dashed square oriented as indicated by the arrows.

Starting with the seed $(B, \tilde{B})$, we can mutate in direction $k$ using the exchange relation (1).

Proposition 16. Let $T$ be a triple numbered $1\rightarrow 11$ in the list above. Then for every mutable cluster variable $x_k = f_{ij}$, the variable $x'_k = f'_{ij}$ defined by (1) is a regular function on $SL_5$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{quiver.png}
\caption{The quiver of the standard cluster structure.}
\end{figure}
Proof. This can be verified directly by computing all the adjacent clusters by the exchange relation (1).

Proposition 17. Let $T$ be a triple numbered 1–11 in the list above. The upper cluster algebra $\mathcal{A}(B_T, \bar{B}_T)$ is naturally isomorphic to $\mathcal{O}(SL_3)$.

Proof. We use proposition 1. Since $SL_3$ is not a Zariski open subset of $\mathcal{E}^{25}$, we extend it to $\text{Mat}_3$ and extend the cluster structure $C(B, \bar{B})$ to a cluster structure in $\text{Mat}_3$ by adding the function $\det X$ as a stable variable. The extra column on the right of $\bar{B}$ can be obtained using the homogeneity of the exchange relations: wherever a line of $\bar{B}$ encodes an inhomogeneous polynomial it can be made homogeneous by multiplying one of the summands by $\det X$. This determines a ±1 in the corresponding entry of the extra column.

Conditions 1, 2 and 4 of proposition 1 are clearly true, and condition 3 holds by proposition 16. The ring of regular functions on $\text{Mat}_3$ is generated by the matrix entries $x_{ij}$. By theorem 3.21 in [7], condition 1 implies that the upper cluster algebra coincides with the intersection of rings of Laurent polynomials in cluster variables taken over the initial cluster and all its adjacent clusters. So it suffices to check that every matrix entry can be expressed as a Laurent polynomial in the variables of each of these clusters. This is verified by direct computation with Maple: solving a system of equations one can express any $x_{ij}$ in terms of cluster variables. These all turn out to be Laurent polynomials.

3. The non-orientable case

As mentioned in section 2, there is one non-orientable case. This is the BD triple $I_1 = \{a_1, a_2\}, I_2 = \{a_3, a_4\}, I_3: a_1 \to a_{5-i}$ which appears to be somewhat different. The main difference is the structure of the functions $f_{ij}$: start with the same construction described in section 2. This gives the set of matrices $\mathcal{M} = \{M_{I_i}^0\}_{i=1}^6$ where

![Figure 4. The quiver of the cluster structure $\{1\} \mapsto \{2\}$.

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Figure 5. Embedding the quiver $1 \to 2$ on the torus.

\[ M_1 = \begin{bmatrix} X \end{bmatrix}, \quad M_2 = \begin{bmatrix} Y \end{bmatrix}, \]

\[ M_3 = \begin{bmatrix} y_{12} & y_{13} & y_{14} & y_{15} & 0 & 0 & 0 & 0 \\ y_{22} & y_{23} & y_{24} & y_{25} & 0 & 0 & 0 & 0 \\ y_{32} & y_{33} & y_{34} & y_{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{31} & x_{32} & x_{33} & x_{34} \\ 0 & 0 & 0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} \\ 0 & 0 & 0 & 0 & x_{51} & x_{52} & x_{53} & x_{54} \end{bmatrix}, \]

\[ M_4 = \begin{bmatrix} y_{13} & y_{14} & y_{15} & 0 & 0 & 0 \\ y_{23} & y_{24} & y_{25} & 0 & 0 & 0 \\ y_{33} & y_{34} & y_{35} & x_{21} & x_{22} & x_{23} \\ y_{43} & y_{44} & y_{45} & x_{31} & x_{32} & x_{33} \\ 0 & 0 & 0 & x_{41} & x_{42} & x_{43} \\ 0 & 0 & 0 & x_{51} & x_{52} & x_{53} \end{bmatrix}, \]

\[ M_5 = \begin{bmatrix} x_{41} & x_{42} & x_{43} & 0 \\ x_{51} & x_{52} & x_{53} & 0 \\ 0 & y_{13} & y_{14} & y_{15} \\ 0 & y_{23} & y_{24} & y_{25} \end{bmatrix}, \quad M_6 = \begin{bmatrix} x_{51} & x_{52} \\ y_{14} & y_{15} \end{bmatrix}. \]
Define now six other matrices as follows:

\[ M'_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & 0 & 0 & 0 & x_{21} & x_{22} & x_{23} & x_{24} \\ y_{12} & y_{13} & y_{14} & y_{15} & 0 & 0 & 0 & 0 \\ y_{22} & y_{23} & y_{24} & y_{25} & 0 & 0 & 0 & 0 \\ y_{42} & y_{43} & y_{44} & y_{45} & 0 & 0 & 0 & 0 \\ y_{52} & y_{53} & y_{54} & y_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} \\ 0 & 0 & 0 & 0 & x_{51} & x_{52} & x_{53} & x_{54} \end{bmatrix}, \]

\[ M'_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & x_{11} & x_{12} & x_{13} \\ y_{13} & y_{14} & y_{15} & 0 & 0 & 0 \\ y_{23} & y_{24} & y_{25} & 0 & 0 & 0 \\ y_{53} & y_{54} & y_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{41} & x_{42} & x_{43} \\ 0 & 0 & 0 & x_{51} & x_{52} & x_{53} \end{bmatrix}, \]

\[ M'_3 = \begin{bmatrix} 0 & 0 & x_{42} & x_{43} \\ 0 & 0 & x_{52} & x_{53} \\ y_{13} & y_{14} & 0 & 0 \\ y_{23} & y_{24} & 0 & 0 \end{bmatrix}, \]

\[ M'_6 = \begin{bmatrix} 0 & x_{53} \\ y_{13} & 0 \end{bmatrix}. \]

For every \( k \in [6] \), take all the functions \( \det(M_k)_{1 \ldots r} + (-1)^k \det(M'_k)_{1 \ldots r} \). Each of these function is labeled \( f_{ij} \) when \( (M_k)_{i,j} = x_{ij} \) (i.e., the map \( \rho \) is defined in the same way, regarding only the matrix \( M_k \) and ignoring \( M'_k \)). This also guarantees that proposition 9 still holds, as it uses the matrices as constructed for the orientable cases).

The set \( B = \{ f_{ij} \} \) is then log canonical with respect to the Sklyanin bracket associated with the triple \( I_1 = \{ \alpha_1, \alpha_2 \}, I_2 = \{ \alpha_3, \alpha_4 \}, \gamma : \alpha_i \rightarrow \alpha_{5-i} \). Proceeding as described in the orientable cases one can verify that propositions 10, 11, 13, 16, 17 hold here as well.

The quiver of this cluster structure is almost the same as in the orientable case. The only difference is that there are no edges between the vertices \( f_{35} \) and \( f_{45} \) and between the vertices \( f_{51} \) and \( f_{52} \)—see figure 6. It is not hard to check that proposition 15 also holds in this case, and this quiver is compatible with the bracket \( \{ , \} \). Note that unlike all the quivers of orientable triples, this quiver can not be embedded on the torus with no crossing edges. However, it can be embedded on the projective plane. Figure 7 shows the quiver on the universal cover of the projective plane (identifying opposite edges of the dashed square oriented as indicated by the arrows). This justifies the terminology: the quivers of the orientable triples can be embedded on the torus, which is an orientable surface, while this quiver is embedded on a non orientable surface.

The removal of the two edges \((f_{35}, f_{45})\) and \((f_{51}, f_{52})\) can be explained as an attempt to preserve the structure of the quiver on the projective plane: when we identify the opposite edges of the dashed square in figure 7, and look at the vertices \( f_{51} \) and \( f_{52} \) we expect an edge \( f_{51} \rightarrow f_{52} \), as there is an edge directed to the right between any two horizontally adjacent vertices. But if we would put their copies \( f'_{51} \) above \( f_{14} \) and \( f'_{52} \) above \( f_{13} \) the edge should be
directed from \( f_5^{\prime} \) to \( f_4^{\prime} \), because now \( f_5^{\prime} \) is on the right side of \( f_2^{\prime} \). The same holds for the edge between \( f_{35} \) and \( f_{45} \). This contradiction is settled by removing these two edges.

4. Toric action

Assertion 3 of conjecture 5 can be interpreted as follows: for any \( H \in \mathcal{H} \) and any weight \( \omega \in \mathfrak{h}^* \) put \( H^w = e^{\omega(h)} \) where \( H = \exp h \). Let \( (\mathbf{x}, \mathbf{B}) \) be a seed in \( C_T \), and \( y_i = \varphi(x_i) \) for \( i \in [n + m] \). Then 3 is equivalent to the following:

(1) for any \( H_1, H_2 \in \mathcal{H}_T \) and any \( x \in G \),

\[
y_i(H_1 X H_2) = H_1^\eta H_2^{\zeta_i} y_i(X)
\]

for some weights \( \eta_i, \zeta_i \in \mathfrak{h}_T^* \) (\( i \in [n + m] \));

(2) \( \text{span} \{ \eta_i \}_{i=1}^{\dim G} = \text{span} \{ \zeta_i \}_{i=1}^{\dim G} = \mathfrak{h}_T^* \);

(3) for every \( i \in [\dim G - 2k_T] \),

\[
\sum_{j=1}^{\dim G} b_{ij} \eta_j = \sum_{j=1}^{\dim G} b_{ij} \zeta_j = 0.
\]

To verify assertion 3 of conjecture 5, we parametrize the left and the right action of \( \mathcal{H}_T \) for every triple \( T \) by diagonal matrices as shown in table 1.

As an example we look at case 2. \([1] \mapsto [2] \): the torus \( \mathcal{H}_{1\mapsto 2} \) has dimension 3. We parametrize the left and the right action of \( \mathcal{H}_{1\mapsto 2} \) by \( \text{diag}(r, s, s^2 t^{-1}, t s^{-3}, t^{-1}) \) and \( \text{diag}(u, v, v^2 u^{-1}, w v^{-3}, w^{-1}) \), respectively. Then condition 1 above holds with three-dimensional vectors \( \eta_i, \zeta_i \) given by
The above conditions 2 and 3 can now be verified via direct computation.

5. The case $T_{1234 \rightarrow 341}$

The case $T = ([1, 2, 4], \{1, 3, 4\}, \gamma: i \mapsto i + 2 \pmod{5})$ is not significantly different than the described above, but some of the functions in the initial cluster (constructed as explained previously) were too ‘big’ in terms of computer memory. For this reason some of the computations could not be completed.
Let us review the process from 2.1: The construction of the set \( \mathcal{M} \) starts with submatrices

\[
M_1 = X, \quad M_2 = Y, \\
M_3 = \begin{bmatrix}
y_{13} & y_{14} & y_{15} \\
y_{23} & y_{24} & y_{25} \\
y_{33} & y_{34} & y_{35}
\end{bmatrix}, \\
M_4 = \begin{bmatrix}
x_{41} & x_{42} \\
x_{51} & x_{52}
\end{bmatrix}.
\]

For \( A_0 = X \) and \( A_0 = Y \) the process stops at step 2, because in both cases \( \xi (A_0) = 5 \), and \( \xi (A_0) \not\in \mathcal{I}_{\tau(A_0)} \) (for either \( \tau(A_0) = 1 \) or \( \tau(A_0) = 2 \)), therefore \( X, Y \in \mathcal{M} \).

When we take \( A_0 = M_3 \) we have \( \tau(A_0) = 2 \) and \( \xi (A_0) = 3 \), therefore \( \xi (A_0) \in \mathcal{I}_{\tau(A_0)} \), and we proceed to step 2. Here, since \( \sigma (A_0) = -1 \) we take the direct sum of \( A_0 \) and the submatrix \( A_0^\sigma = U_{\tau^{-1}(\xi (A_0))} = U_2 \) and get the matrix

\[
\begin{bmatrix}
y_{13} & y_{14} & y_{15} & 0 & 0 & 0 & 0 \\
y_{23} & y_{24} & y_{25} & 0 & 0 & 0 & 0 \\
y_{33} & y_{34} & y_{35} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{21} & x_{22} & x_{23} & x_{24} \\
0 & 0 & 0 & x_{31} & x_{32} & x_{33} & x_{34} \\
0 & 0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} \\
0 & 0 & 0 & x_{51} & x_{52} & x_{53} & x_{54}
\end{bmatrix}.
\]

To determine the extension numbers we look at the intersection of the set \( \mathcal{I}_{\tau(A_0)} \) and the Dynkin diagram (see figure 8): the connected component of the root 3 in this intersection has two vertices—3 and 4 connected by an edge. Therefore \( t^+(A_0) = 2 \) and \( t^-(A_0^\sigma) = 1 \). Hence, after extending in step 5, we get the matrix

\[
\begin{bmatrix}
y_{13} & y_{14} & y_{15} & 0 & 0 & 0 & 0 \\
y_{23} & y_{24} & y_{25} & 0 & 0 & 0 & 0 \\
y_{33} & y_{34} & y_{35} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{21} & x_{22} & x_{23} & x_{24} \\
0 & 0 & 0 & x_{31} & x_{32} & x_{33} & x_{34} \\
0 & 0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} \\
0 & 0 & 0 & x_{51} & x_{52} & x_{53} & x_{54}
\end{bmatrix}.
\]
We return to step 1. with the new matrix $A_1$. Now $\xi(A_1) = 4$ and $\tilde{\sigma}(A_1) = 1$, so $\xi(A_1) \in I_{\tilde{\sigma}(A_1)}$ again, and we proceed to step 2. Since $\sigma(A_1) = 1$, we add (as a direct summand) the submatrix $A_1^+ = V_{r(\xi(A_1))} = V_2$ to get the matrix

$$A_1 = \begin{bmatrix}
y_{13} & y_{14} & y_{15} & 0 & 0 & 0 \\
y_{23} & y_{24} & y_{25} & 0 & 0 & 0 \\
y_{33} & y_{34} & y_{35} & x_{11} & x_{12} & x_{13} \\
y_{43} & y_{44} & y_{45} & x_{21} & x_{22} & x_{23} \\
y_{53} & y_{54} & y_{55} & x_{31} & x_{32} & x_{33} \\
0 & 0 & 0 & x_{41} & x_{42} & x_{43} \\
0 & 0 & 0 & x_{51} & x_{52} & x_{53}
\end{bmatrix}.$$

Now the connected component of 4 in the intersection of the set and $I_{\tilde{\sigma}(A_1)}$ with the Dynkin diagram has only one vertex 4. So $r(\xi(A_1)) = r(1) = 1$, and after extending in step 5 we have

$$A_2 = \begin{bmatrix}
y_{13} & y_{14} & y_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{23} & y_{24} & y_{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{33} & y_{34} & y_{35} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 0 & 0 & 0 \\
y_{43} & y_{44} & y_{45} & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & 0 & 0 & 0 \\
y_{53} & y_{54} & y_{55} & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & 0 \\
0 & 0 & 0 & 0 & 0 & y_{21} & y_{22} & y_{23} & y_{24} & y_{25} & 0 \\
0 & 0 & 0 & 0 & 0 & y_{31} & y_{32} & y_{33} & y_{34} & y_{35} & 0 \\
0 & 0 & 0 & 0 & 0 & y_{41} & y_{42} & y_{43} & y_{44} & y_{45} & 0
\end{bmatrix}.$$

We now go back to step 2. Again, we look at $\xi(A_2) = 4$ and $\sigma(A_2) = 2$. That means $\xi(A_2) \in I_{\tilde{\sigma}(A_2)}$ and we move on. Here $\sigma(A_2) = -1$, and we add (as a direct summand) the submatrix $A_2^{++} = U_{r^{-1}(\xi(A_2))} = U_3$.

The connected component of 4 in the intersection of the set $I_3$ with the Dynkin diagram has two vertices 3 and 4. This gives the extension numbers $r(\xi(A_2)) = 1$ and $r(\sigma(A_2)) = 2$, and the new matrix is
Finally, returning to step 2 with this matrix we have $\xi(A_3) = 3$ and $\sigma(A_3) = 1$. Now $\xi(A_3) \notin \mathcal{I}_\Gamma(A_3)$, which stops the process. We add the matrix $A_3$ to the set $\mathcal{M}$ and turn to the last matrix $M_4$.

Starting the algorithm with $M_4$ leads to the matrix

$$
A_3 = \begin{bmatrix}
    y_{13} & y_{14} & y_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    y_{23} & y_{24} & y_{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    y_{33} & y_{34} & y_{35} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    y_{43} & y_{44} & y_{45} & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    y_{53} & y_{54} & y_{55} & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & y_{21} & y_{22} & y_{23} & y_{24} & y_{25} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & y_{31} & y_{32} & y_{33} & y_{34} & y_{35} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\
    0 & 0 & 0 & 0 & 0 & y_{41} & y_{42} & y_{43} & y_{44} & y_{45} & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\
    0 & 0 & 0 & 0 & 0 & y_{51} & y_{52} & y_{53} & y_{54} & y_{55} & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

which is the last matrix in the set $\mathcal{M}$.

The matrix $A_3$ is of size $14 \times 14$. Its determinant is a polynomial of $187,814$ summands. The leading principal minors of sizes $13 \times 13$ and $12 \times 12$ form polynomials of $79,628$ and $42,468$ summands, respectively. All the computations were done with Maple, using a computer of $128$ Gb RAM. Apparently, this was not enough, and the terms $\{f, g\}$ could not be computed for $28$ pairs $f$ and $g$. (presumably, $\{f, g\}$ is proportional to the product $f \cdot g$). Therefore, Proposition 11 could not be rigorously proved in this case.

However, for each of these pairs the term

$$
\omega_{fg} = \frac{\{f, g\}}{f \cdot g}
$$

was evaluated at a random point of $SL_5$. Plugging these values in the coefficient matrix $\Omega$ leads to a quiver (and a cluster structure) that matches exactly the construction described in 2.2. This does give a strong evidence supporting the conjecture: if $\omega_{fg}$ is indeed constant for

Figure 8. The intersection of $\mathcal{I} = \{1, 3, 4\}$ and the Dynkin diagram.
every such pair \( f, g \), then the quiver must be the one predicted. And naturally, if proposition 11 is true here, then proposition 15 is also true.

For the same reason, it could not be verified that all the neighboring clusters are regular, and so propositions 16 and 17 were not verified as well.

Acknowledgments

The author was supported by ISF grant #162/12. Parts of this paper were written during my stay at MSRI (Cluster Algebras program, August–December 2012). I would like to thank this institution for the warm hospitality and excellent working conditions. I am grateful to Michael Gekhtman for his enlightening comments at certain points. Special thanks to Alek Vainshtein for his support and encouragement, as well as his mathematical, technical and editorial advice.

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