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Abstract. As in the cases of freeness and monotonic independence, the notion of conditional freeness is meaningful when complex-valued states are replaced by positive conditional expectations. In this framework, the paper presents several positivity results, a version of the central limit theorem and an analogue of the conditionally free R-transform constructed by means of multilinear function series.

1. Introduction

The paper addresses a topic related to conditionally free (or, shortly, using the term from [2], c-free) probability. This notion was developed in the '90's (see [1], [2]) as an extension of freeness within the framework of ∗-algebras endowed with not one, but two states. Namely, given a family of unital algebras \( \{A_i\}_{i \in I} \), each \( A_i \) endowed with two expectations \( \phi_i, \psi_i : A_i \rightarrow \mathbb{C} \), their c-free product is the triple \((A, \varphi, \psi)\), where:

(i) \( A = \bigstar_{i \in I} A_i \) is the free product of the algebras \( A_i \).
(ii) \( \psi = \bigstar_{i \in I} \psi_i \) and \( \varphi = \bigstar_{(\psi_i)_{i \in I}} \psi_i \) are expectations given by the relations
   (a) \( \psi(a_1 \cdots a_n) = 0 \)
   (b) \( \varphi(a_1 \cdots a_n) = \varphi_{\varepsilon(1)}(a_1) \cdots \varphi_{\varepsilon(n)}(a_n) \)

for all \( a_j \in A_{\varepsilon(j)}, j = 1, \ldots, n \) such that \( \psi_{\varepsilon(j)}(a_j) = 0 \) and \( \varepsilon(1) \neq \cdots \neq \varepsilon(n) \).

A key result is that if the \( A_i \) are ∗-algebras and \( \varphi, \psi \) are positive functionals, then \( \varphi \) and \( \psi \) are also positive.

In [6], the positivity of the free product maps \( \varphi, \psi \) is proved for the case when \( \varphi_1, \varphi_2 \) are positive conditional expectations in a common \( C^* \)-subalgebra, but \( \psi_1, \psi_2 \) remain positive \( \mathbb{C} \)-valued maps. A more general situation is indeed discussed (see Theorem 3, Section 6, from [6]), but the question if \( \varphi, \psi \) are positive for \( \varphi_{1,2}, \psi_{1,2} \) arbitrary positive conditional expectations is left unanswered.

A first answer was given in [8], where we showed that for \( A \) a ∗-algebra, the analogous construction with both \( \varphi \) and \( \psi \) valued in a \( C^* \)-subalgebra \( B \) of \( A \) still retains the positivity. The present paper further develops this result (see Theorem 2.3) and also demonstrates the use of multilinear function series in c-free setting.
In [2] is constructed a c-free version of Voiculescu’s R-transform, which we will call the $^cR$-transform, with the property that $^cR_X + Y = ^cR_X + ^cR_Y$ if $X$ and $Y$ are c-free elements from the algebra $\mathcal{A}$ relative to $\varphi$ and $\psi$ (i.e. the relations (a) and (b) from the definition of the c-free product hold true for the subalgebras generated by $X$ and $Y$.)

The apparatus of multilinear function series is used in recent work of K. Dykema ([3] and [4]) to construct suitable analogues for the $R$ and $S$-transforms in the framework of freeness with amalgamation. We will show that this construction is also appropriate for the $^cR$-transform mentioned above. The techniques used differ from the ones of [3], the Fock space type construction being substituted by combinatorial techniques similar to [2] and [7]. Particularly, Theorems 3.3 and 3.6 contain new (shorter) proves of the results 6.1–6.13 from [3].

The paper is structured in four sections. In Section 2 are stated the basic definitions and are proved the main positivity results. Section 3 describes the construction and the basic property of the multilinear function series $^cR$-transform and Section 4 treats the central limit theorem and a related positivity property.

2. Definitions and positivity results

Definition 2.1. Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of algebras, all containing the subalgebra $\mathcal{B}$. Suppose $\mathcal{D}$ is a subalgebra of $\mathcal{B}$ and $\Psi_i : \mathcal{A}_i \rightarrow \mathcal{D}$ and $\Phi_i : \mathcal{A}_i \rightarrow \mathcal{B}$ are conditional expectations, $i \in I$. We say that the triple $(\mathcal{A}, \Phi, \Psi) = \bigotimes_{i \in I}(\mathcal{A}_i, \Phi_i, \Psi_i)$ is the conditionally free product with amalgamation over $(\mathcal{B}, \mathcal{D})$, or shortly, the c-free product, of the triples $(\mathcal{A}_i, \Phi_i, \Psi_i)_{i \in I}$ if

1. $\mathcal{A}$ is the free product with amalgamation over $\mathcal{B}$ of the family $(\mathcal{A}_i)_{i \in I}$
2. $\Psi = \bigotimes_{i \in I} \Psi_i$ and $\Phi = \bigotimes_{(i,j) \in I} \Phi_i$ are determined by the relations
   \[
   \Psi(a_1 a_2 \ldots a_n) = 0 \\
   \Phi(a_1 a_2 \ldots a_n) = \Phi(a_1) \Phi(a_2) \ldots \Phi(a_n),
   \]
   for all $a_i \in \mathcal{A}_{\varepsilon(i)}, \varepsilon(i) \in I$ such that $\varepsilon(1) \neq \varepsilon(2) \neq \ldots \neq \varepsilon(n)$ and $\Psi_{\varepsilon(i)}(a_i) = 0$.

When $\mathcal{D} = \mathbb{C}$, this definition reduces to the one given in [6]. When both $\mathcal{B}$ and $\mathcal{D}$ are equal to $\mathbb{C}$, this definition was given in [2].

When discussing positivity, we need a *-structure on our algebras. We will demand that $\mathcal{B}$ and $\mathcal{D}$ be $C^*$-algebras, while $\mathcal{A}_i$ and $\mathcal{A}$ are only required to be *-algebras.

The following results are slightly modified versions of Lemma 6.4 and Theorem 6.5 from [8].

Lemma 2.2. Let $\mathcal{B}$ be a $C^*$-algebra and $\mathcal{A}_1$, $\mathcal{A}_2$ be two *-algebras containing $\mathcal{B}$ as a *-subalgebra, endowed with positive conditional expectations $\Phi_j : \mathcal{A}_j \rightarrow \mathcal{B}$, $j = 1, 2$. If $a_1, \ldots, a_n \in \mathcal{A}_1$, $a_{n+1}, \ldots, a_{n+m} \in \mathcal{A}_2$ and $A = (A_{i,j})_{i,j} \in M_{n+m}(\mathcal{B})$ is the matrix with the entries

\[
A_{i,j} = \begin{cases} 
\Phi_1(a_i^* a_j) & \text{if } i, j \leq n \\
\Phi_1(a_i^*) \Phi_2(a_j) & \text{if } i \leq n, j > n \\
\Phi_2(a_i^*) \Phi_1(a_j) & \text{if } i > n, j \leq n \\
\Phi_2(a_i^* a_j) & \text{if } i, j > n 
\end{cases}
\]
then $A$ is positive.

Proof. The vector space $E = \mathfrak{B} \oplus \ker(\Phi_1) \oplus \ker(\Phi_2)$ has a $\mathfrak{B}$-bimodule structure given by the algebraic operations on $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Consider the $\mathfrak{B}$-sesquilinear pairing
\[
\langle \cdot, \cdot \rangle : E \times E \to \mathfrak{B}
\]
determined by the relations:
\[
\begin{align*}
\langle b_1, b_2 \rangle &= b_1^* b_2, \text{ for } b_1, b_2 \in \mathfrak{B} \\
\langle u_j, v_j \rangle &= \Phi_j(u_j^* v_j), \text{ for } u_j, v_j \in \ker(\Phi_j), j = 1, 2 \\
\langle u_1, u_2 \rangle &= \langle u_2, u_1 \rangle = 0 \text{ for } u_1 \in \ker(\Phi_1), \text{ and } u_2 \in \ker(\Phi_2). \\
\langle b, u_j \rangle &= \langle u_j, b \rangle = 0 \text{ for all } b \in \mathfrak{B}, u_j \in \mathfrak{A}_j.
\end{align*}
\]

With this notation, we have that $A_{i,j} = \langle a_i, a_j \rangle$, hence it suffices to show that $\langle a, a \rangle \geq 0$ for all $a \in E$.

Indeed, for an element $a = b + u_1 + u_2$ with $b \in \mathfrak{B}, u_j \in \ker(\Phi_j), j = 1, 2$, we have:
\[
\begin{align*}
\langle a, a \rangle &= \langle b + u_1 + u_2, b + u_1 + u_2 \rangle \\
&= \langle b, b \rangle + \langle u_1, u_1 \rangle + \langle u_2, u_2 \rangle \\
&= b^* b + \Phi_1(u_1^* u_1) + \Phi_2(u_2^* u_2) \\
&\geq 0
\end{align*}
\]

\[ \square \]

**Theorem 2.3.** Let $\mathfrak{B}$ be a $C^*$-algebra and $\mathfrak{D}$ a $C^*$-subalgebra of $\mathfrak{B}$. Suppose that $\mathfrak{A}_1, \mathfrak{A}_2$ are $*$-algebras containing $\mathfrak{B}$, each endowed with two positive conditional expectations $\Phi_j : \mathfrak{A}_j \to \mathfrak{B}$, and $\Psi_j : \mathfrak{A}_j \to \mathfrak{D}, j = 1, 2$ and consider the c-free product $(\mathfrak{A}, \Phi, \Psi) = \bigotimes_{i=1,2}(\mathfrak{A}_i, \Phi_i, \Psi_i)$.

Then the maps $\Phi$ and $\Psi$ are positive.

Proof. The positivity of $\Psi$ is by now a classical result in the theory of free probability with amalgamation over a $C^*$-algebra (for example, see [9], Theorem 3.5.6). For the positivity of $\Phi$ we have to show that $\Phi(a^* a) \geq 0$ for any $a \in \mathfrak{A}$.

Any element of $\mathfrak{A}$ can be written as
\[
a = \sum_{k=1}^N s_{1,k} \ldots s_{n(k),k},
\]
where $s_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)} \varepsilon(1, k) \neq \varepsilon(2, k) \neq \cdots \neq \varepsilon(n(k), k)$.

Writing
\[
s_{(j,k)} = s_{(j,k)} - \Psi(s_{(j,k)}) + \Psi(s_{(j,k)})
\]
and expanding the product, we can consider $a$ of the form
\[
a = d + \sum_{k=1}^N a_{1,k} \ldots a_{n(k),k}
\]
with $d \in \mathfrak{D} \subset \mathfrak{B}$ and $a_{j,k} \in \mathfrak{A}_{\varepsilon(j,k)}$ such that $\Psi_{\varepsilon(j,k)}(a_{j,k}) = 0$ and $\varepsilon(1, k) \neq \varepsilon(2, k) \neq \cdots \neq \varepsilon(n(k), k)$. 

Therefore

\[ \Phi(a^*a) = \Phi \left( d^*d + d^* \left[ \sum_{k=1}^{N} a_{1,k} \cdots a_{n(k),k} \right] + \left[ \sum_{k=1}^{N} a_{1,k} \cdots a_{n(k),k} \right]^* d + \left[ \sum_{k=1}^{N} a_{1,k} \cdots a_{n(k),k} \right]^* \left[ \sum_{k=1}^{N} a_{1,k} \cdots a_{n(k),k} \right] \right). \]

Since \( \Phi \) is a conditional expectation and \( d \in \mathcal{D} \subset \mathfrak{B} \), the above equality becomes

\[ \Phi(a^*a) = d^*d + \sum_{k=1}^{N} d^* \Phi(a_{1,k} \cdots a_{n(k),k}) + \sum_{k=1}^{N} \Phi(a_{n(k),k}^* \cdots a_{1,k}^*) d \]

\[ + \sum_{k,l=1}^{N} \Phi(a_{n(k),k}^* \cdots a_{1,k}^* a_{1,l} \cdots a_{n(l),l}). \]

Using the definition of the conditionally free product with amalgamation over \( \mathfrak{B} \) and that \( \Psi_{\epsilon(j,k)}(a_{j,k}) = 0 \) for all \( j, k \), one further has

\[ \Phi(a^*a) = d^*d + \sum_{k=1}^{N} \Phi(d^* a_{1,k}) \Phi(a_{2,k}) \cdots \Phi(a_{n(k),k}) \]

\[ + \sum_{k=1}^{N} \Phi(a_{n(k),k})^* \cdots \Phi(a_{2,k}^*) \Phi(a_{1,k}^* d) \]

\[ + \sum_{k,l=1}^{N} \left[ \Phi(a_{n(k),k})^* \cdots \Phi(a_{2,k}^*) \right] \Phi(a_{1,k}^* a_{1,l} \cdots a_{n(l),l}) \Phi(a_{2,l} \cdots \Phi(a_{n(l),l}) \right]. \]

that is

\[ \Phi(a^*a) = d^*d + \sum_{k=1}^{N} \Phi(d^* a_{1,k}) \left[ \Phi(a_{2,k}) \cdots \Phi(a_{n(k),k}) \right] \]

\[ + \sum_{k=1}^{N} \left[ \Phi(a_{2,k}) \cdots \Phi(a_{n(k),k}) \right]^* \Phi(a_{1,k}^* d) \]

\[ + \sum_{k,l=1}^{N} \left[ \Phi(a_{2,k}) \cdots \Phi(a_{n(k),k}) \right]^* \Phi(a_{1,k}^* a_{1,l}) \left[ \Phi(a_{2,l}) \cdots \Phi(a_{n(l),l}) \right]. \]

From Lemma 2.2, the matrix \( S = \left( \Phi(a_{i,j}^* a_{1,j})_{i,j=1}^{N+1} \right) \) is positive in \( M_{N+1}(\mathfrak{B}) \), therefore

\[ S = T^*T, \] for some \( T \in M_{N+1}(\mathfrak{B}). \)
Denote now \( a_{1,N+1} = d \) and \( v_k = \Phi(a_{2,k}) \ldots \Phi(a_{n(k),k}) \). The identity for \( \Phi(a^*a) \) becomes:

\[
\Phi(a^*a) = (v_1, \ldots, v_N, 1)^* T^* T(v_1, \ldots, v_N, 1) \geq 0, \quad \text{as claimed.}
\]

\[\square\]

**Theorem 2.4.** Assume that \( \mathcal{I} = \bigcup_{j \in \mathcal{J}} \mathcal{I}_j \) is a partition of \( \mathcal{I} \). Then:

\[
\ast \in \mathcal{J} \left( \ast \in \mathcal{I}_j (\mathfrak{A}_i, \Phi_i, \Psi_j) \right) = \ast \in \mathcal{I} (\mathfrak{A}_i, \Phi_i, \Psi_j)
\]

**Proof.** The proof is identical to the proofs of similar results in [6] and [2]. Consider \( a_i \in \mathfrak{A}_{\varepsilon(i)}, 1 \leq i \leq m \) such that \( \varepsilon(1) \neq \varepsilon(2) \neq \cdots \neq \varepsilon(m) \) and \( \Psi_{\varepsilon(i)}(a_i) = 0 \). Let \( 1 = i_0 < i_1 < \cdots < i_k = m \) and \( \mathfrak{J}_i = \{ \varepsilon(i), i < i_1 \} \). Since

\[
(\ast \in \mathcal{J}_i \Psi_j)((a_{i_1} \cdots a_{i_k})) = 0,
\]

it suffices to show that

\[
\Phi(a_1 \cdots a_m) = \prod_{i=1}^k \left( (\ast \in \mathcal{J}_i \psi_j)(a_{i_1} \cdots a_{i_k}) \right).
\]

But

\[
\Phi(a_1 \cdots a_m) = \Phi_{\varepsilon(1)}(a_1) \cdots \Phi_{\varepsilon(m)}(a_m)
\]

while, since \( \Psi_{\varepsilon(i)}(a_i) = 0 \),

\[
(\ast \in \mathcal{J}_i \psi_j)(a_{i_1} \cdots a_{i_k}) = \Phi_{i_1-1}(a_{i_1-1}) \cdots \Phi_{i_k}(a_{i_k})
\]

and the conclusion follows. \[\square\]

**Definition 2.5.** Let \( \mathfrak{A} \) be an algebra (respectively a \(*\)-algebra), \( \mathfrak{B} \) a subalgebra (\(*\)-subalgebra) of \( \mathfrak{A} \) and \( \mathfrak{D} \) a subalgebra (\(*\)-subalgebra) of \( \mathfrak{B} \). Suppose \( \mathfrak{A} \) is endowed with the conditional expectations \( \Psi : \mathfrak{A} \rightarrow \mathfrak{D} \) and \( \Phi : \mathfrak{A} \rightarrow \mathfrak{D} \).

(i) The subalgebras (\(*\)-subalgebras) \( (\mathfrak{A}_i)_{i \in \mathcal{I}} \) of \( \mathfrak{A} \) are said to be \( c\)-free with respect to \( (\Phi, \Psi) \) if:

(a) \( (\mathfrak{A}_i)_{i \in \mathcal{I}} \) are free with respect to \( \Psi \).

(b) if \( a_i \in \mathfrak{A}_{\varepsilon(i)}, 1 \leq i \leq m \), are such that \( \varepsilon(1) \neq \cdots \neq \varepsilon(m) \) and \( \Psi(a_i) = 0 \), then \( \Phi(a_1 \cdots a_m) = \Phi(a_1) \cdots \Phi(a_m) \).

(ii) The elements \( (X_i)_{i \in \mathcal{I}} \) of \( \mathfrak{A} \) are said to be \( c\)-free with respect to \( (\Phi, \Psi) \) if the subalgebras (\(*\)-subalgebras) generated by \( \mathfrak{B} \) and \( X_i \) are \( c\)-free with respect to \( (\Phi, \Psi) \).

We will denote by \( \mathfrak{B} \langle \xi \rangle \) the non-commutative algebra of polynomials in the symbol \( \xi \) and with coefficients from \( \mathfrak{B} \) (the coefficients do not commute with the symbol \( \xi \)). If \( \mathcal{I} \) is a family of indices, \( \mathfrak{B} \langle \{ \xi_i \}_{i \in \mathcal{I}} \rangle \) will denote the algebra of polynomials in the non-commuting variables \( \{ \xi_i \}_{i \in \mathcal{I}} \) and with coefficients from \( \mathfrak{B} \). We will identify \( \mathfrak{B} \langle \{ \xi_i \}_{i \in \mathcal{I}} \rangle \) with the free product with amalgamation over \( \mathfrak{B} \) of the family \( \{ \mathfrak{B} \langle \xi_i \rangle \}_{i \in \mathcal{I}} \).
If $\mathfrak{A}$ is a $*$-algebra and $\mathfrak{B}$ is with the $C^*$-algebra, $\mathfrak{B}(\xi)$ will also be considered with a $*$-algebra structure, by taking $\xi^* = \xi$. If $X$ is a selfadjoint element from $\mathfrak{A}$, we define the conditional expectations

$$
\Phi_X, \Psi_X : \mathfrak{B}(\xi) \longrightarrow \mathfrak{B}
$$

given by $\Phi_X(f(\xi)) = \Phi(f(X))$ and $\Psi_X(f(\xi)) = \Psi(f(X))$, for any $f(\xi) \in \mathfrak{B}(\xi)$.

**Corollary 2.6.** Suppose that $\mathfrak{A}$ is a $*$-algebra and $X$ and $Y$ are c-free selfadjoint elements of $\mathfrak{A}$ such that the maps $\Phi_X, \Psi_X$ and $\Phi_Y, \Psi_Y$ are positive. Then the maps $\Phi_{X+Y}$ and $\Psi_{X+Y}$ are also positive.

**Proof.** The positivity of $\Psi_{X+Y}$ is an immediate consequence of the fact that $X$ and $Y$ are free with amalgamation over $\mathfrak{B}$ with respect to $\Psi$. It remains to prove the positivity of $\Phi_{X+Y}$.

Since the maps $\Phi_X : \mathfrak{B}(\xi_1) \longrightarrow \mathfrak{B}$ and $\Phi_Y : \mathfrak{B}(\xi_2) \longrightarrow \mathfrak{B}$ are positive, from Theorem 2.3 so is

$$
\Phi_X \ast (\Psi_X, \Psi_Y) \Phi_Y : \mathfrak{B}(\xi_1) \ast_{\mathfrak{B}} \mathfrak{B}(\xi_2) = \mathfrak{B}(\xi_1, \xi_2) \longrightarrow \mathfrak{B}
$$

Remark also that

$$
i_Z : \mathfrak{B}(\xi) \ni f(\xi) \mapsto f(X + Y) \in \mathfrak{B}(\xi_1) \ast_{\mathfrak{B}} \mathfrak{B}(\xi_2)
$$

is a positive $\mathfrak{B}$-functional.

The conclusion follows from the fact that the c-freeness of $X$ and $Y$ is equivalent to

$$
\Phi_{X+Y} = (\Phi_X \ast (\Psi_X, \Psi_Y) \Phi_Y) \circ i_{X+Y}.
$$

$\square$

### 3. Multilinear function series and the $cR$-transform

Let $\mathfrak{A}$ be a $*$-algebra containing the $C^*$-algebra $\mathfrak{B}$, endowed with a conditional expectation $\Psi : \mathfrak{A} \longrightarrow \mathfrak{B}$. If $X$ is a selfadjoint element of $\mathfrak{A}$, then by the moment of order $n$ of $X$ we will understand the map

$$
m_X^{(n)} : \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{n-1 \text{ times}} \longrightarrow \mathfrak{B}
$$

$$
m_X^{(n)}(b_1, \ldots, b_{n-1}) = \Psi(Xb_1X \cdots Xb_{n-1}X)
$$

If $\mathfrak{B} = \mathbb{C}$, then the moment-generating series of $X$

$$
m_X(z) = \sum_{n=0}^{\infty} \Psi(X^n)z^n
$$

encodes all the information about the moments of $X$. For $\mathfrak{B} \neq \mathbb{C}$, the straightforward generalization

$$
\overline{m}_X(z) = \sum_{n=0}^{\infty} \Psi(X^n)z^n
$$

generally fails to keep track of all the possible moments of $X$. A solution to this inconvenience was proposed in [3], namely the moment-generating multilinear function series of $X$. Before defining this notion, we will briefly recall the construction and several results on multilinear function series.
Let \( \mathfrak{B} \) be an algebra over a field \( K \). We set \( \widetilde{\mathfrak{B}} \) equal to \( \mathfrak{B} \) if \( \mathfrak{B} \) is unital and to the unitization of \( \mathfrak{B} \) otherwise. For \( n \geq 1 \), we denote by \( \mathcal{L}_n(\mathfrak{B}) \) the set of all \( K \)-multilinear maps

\[
\omega_n : \underbrace{\mathfrak{B} \times \cdots \times \mathfrak{B}}_{\text{n times}} \longrightarrow \mathfrak{B}
\]

A formal multilinear function series over \( \mathfrak{B} \) is a sequence \( \omega = (\omega_0, \omega_1, \ldots) \), where \( \omega_0 \in \widetilde{\mathfrak{B}} \) and \( \omega_n \in \mathcal{L}_n(\mathfrak{B}) \) for \( n \geq 1 \). According to [3], the set of all multilinear function series over \( \mathfrak{B} \) will be denoted by \( Mul[[\mathfrak{B}]] \).

For \( \alpha, \beta \in Mul[[\mathfrak{B}]] \), the formal sum \( \alpha + \beta \) and the formal product \( \alpha \beta \) are the elements from \( Mul[[\mathfrak{B}]] \) defined by:

\[
(\alpha + \beta)_n(b_1, \ldots, b_n) = \alpha_n(b_1, \ldots, b_n) + \beta_n(b_1, \ldots, b_n)
\]

\[
(\alpha \beta)_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} \alpha_k(b_1, \ldots, b_k) \beta_{n-k}(b_{k+1}, \ldots, b_n)
\]

for any \( b_1, \ldots, b_n \in \mathfrak{B} \).

If \( \beta_0 = 0 \), then the formal composition \( \alpha \circ \beta \in Mul[[\mathfrak{B}]] \) is defined by

\[
(\alpha \circ \beta)_0 = \alpha_0
\]

and, for \( n \geq 1 \), by

\[
(\alpha \circ \beta)_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \sum_{p_1 + \cdots + p_k = n, q_j = p_1 + \cdots + p_j - 1} \alpha_k \left( \beta_{p_1}(b_1, \ldots, b_{p_1}), \ldots, \beta_{p_k}(b_{q_k+1}, \ldots, b_{q_k+p_k}) \right)
\]

where the second summation is done over all \( k \)-tuples \( p_1, \ldots, p_k \geq 1 \) such that \( p_1 + \cdots + p_k = n \) and \( q_j = p_1 + \cdots + p_{j-1} \).

One can work with elements of \( Mul[[\mathfrak{B}]] \) as if they were formal power series. The relevant properties are described in [3], Proposition 2.3 and Proposition 2.6. As in [3], we use 1, respectively \( \nu \), to denote the identity elements of \( Mul[[\mathfrak{B}]] \) relative to multiplication, respectively composition. In other words, \( 1 = (1, 0, 0, \ldots) \) and \( \nu = (0, \nu \mathfrak{B}, 0, 0, \ldots) \). We will also use the fact that an element \( \alpha \in Mul[[\mathfrak{B}]] \) has an inverse with respect to formal composition, denoted \( \alpha^{(-1)} \), if and only if \( \alpha \) has the form \( (0, \alpha_1, \alpha_2, \ldots) \) with \( \alpha_1 \) an invertible element of \( \mathcal{L}_1(\mathfrak{B}) \).

**Definition 3.1.** With the above notation, the moment-generating multilinear function series \( \mathcal{M}_X \) of \( X \) is the element of \( Mul[[\mathfrak{B}]] \) such that:

\[
\mathcal{M}_{X,0} = \Psi(X)
\]

\[
\mathcal{M}_{X,n}(b_1, \ldots, b_n) = \Psi(Xb_1X \cdots Xb_nX).
\]

Given an element \( \alpha \in Mul[[\mathfrak{B}]] \), the multilinear function series \( R_\alpha \) is defined by the following equation (see [3], Def 6.1):

\[
R_\alpha = \left( (1 + \alpha \nu)^{-1} \right) \circ (I + \alpha \nu)^{(-1)}.
\]

(3.1)

A key property of \( R \) is that for any \( X, Y \in \mathfrak{A} \) free over \( \mathfrak{B} \), we have

\[
R_{\mathcal{M}_{X+Y}} = R_{\mathcal{M}_X} + R_{\mathcal{M}_Y}.
\]

(3.2)
These relations were proved earlier in the particular case $B = C$. One can also describe $R_\alpha$ by combinatorial means, via the recurrence relation

$$
\alpha_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} \sum R_{\alpha,k} \left( b_1 \alpha_{p(1)}(b_3, \ldots, b_{i_1-2})b_{i_1-1}, \ldots \right.
$$

$$
\ldots \left. b_{i(k-1)} \alpha_{p(k)}(b_{i(k-1)+1}, \ldots, b_{i(k)-2})b_{i(k)-1} \right) b_{i(k)} \alpha_{n-i_k}(b_{i_{k+1}}, \ldots, b_n)
$$

where the second summation is done over all $1 = i(0) < i(1) < \cdots < i(k) \leq n$ and $p(k) = i(k) - i(k-1) - 2$.

Following an idea from [2], the above equation can be graphically illustrated by the picture:

$$
\text{[Diagram]} = \sum \text{[Boxes]}
$$

In the case of scalar $c$-free probability, an analogue of the Voiculescu’s $R$-transform is developed in [2]. In order to avoid confusions, we will denote it by $^cR$.

The $^cR$-transform has the property that it linearizes the $c$-free convolution of pairs of compactly supported measures. In particular, if $X$ and $Y$ are $c$-free elements from some algebra $A$, then

$$
^cRX + ^cRY = ^cRX + ^cRY.
$$

If the $*$-algebra $\mathfrak{A}$ is endowed with the $\mathbb{C}$-valued states $\varphi, \psi$ and $X$ is a selfadjoint element of $\mathcal{A}$, then (see [2]), the coefficients $\{^cR_m\}_{m \geq 0}$ of $^cRX$ are defined by the recurrence:

$$
\varphi(X^n) = \sum_{k=1}^{n} \sum_{l(1), \ldots, l(k) \geq 0} \sum_{l(1)+\cdots+l(k) = n-k} ^cR_k \cdot \psi(X^{l(1)}) \cdots \psi(X^{l(k)}) \varphi(X^{l(k)})
$$

equation that can be graphically illustrated by the picture, were the dark boxes stand for the application of $\varphi$ and the light ones for the application of $\psi$:

$$
\text{[Diagram]} = \sum \text{[Boxes]}
$$

The above considerations lead to the following definition:

**Definition 3.2.** Let $\beta, \gamma \in Mul[[\mathfrak{B}]]$. The multilinear function series $^cR_{\beta,\gamma}$ is the element of $Mul[[\mathfrak{B}]]$ defined by the recurrence relation

$$
\beta_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} \sum ^cR_{\beta,\gamma,k} \left( b_1 \gamma_{p(1)}(b_3, \ldots, b_{i_1-2})b_{i_1-1}, \ldots \right.
$$

$$
\ldots \left. b_{i(k-1)} \gamma_{p(k)}(b_{i(k-1)+1}, \ldots, b_{i(k)-2})b_{i(k)-1} \right) b_{i(k)} \beta_{n-i_k}(b_{i_{k+1}}, \ldots, b_n)
$$

where the second summation is done over all $1 = i(0) < i(1) < \cdots < i(k) \leq n$ and $p(k) = i(k) - i(k-1) - 2$. 
The following analytical description of $cR_{\beta,\gamma}$ also shows that it is unique and well-defined:

**Theorem 3.3.** For any $\beta, \gamma \in Mul[[\mathcal{B}]]$, 

$$cR_{\beta,\gamma} = [\beta(1 + I\beta)^{-1}] \circ (I + I\gamma I)^{(-1)}$$ (3.3)

Before proving the theorem, remark that the right-hand side of (3.3) is well-defined and unique, since $1 + I\gamma$ is invertible with respect to the formal multiplication, $I + I\beta I$ is invertible with respect to formal composition and its inverse has 0 as first component (see [3]). We will need the following auxiliary result:

**Lemma 3.4.** Let $\beta$ be an element of $Mul[[\mathcal{B}]]$ and $I$ the identity element with respect to formal composition, $I = (0, id_{\mathcal{B}}, 0, 0, \ldots)$.

(i) the multilinear function series $I\beta$ is given by:

$$
(I\beta)_0 = 0 \\
(I\beta)_n(b_1, \ldots, b_n) = b_1\beta_{n-1}(b_2, \ldots, b_n)
$$

(ii) the multilinear function series $I\beta I$ is given by

$$
(I\beta I)_0 = 0 \\
(I\beta I)_1(b_1) = 0 \\
(I\beta I)_n(b_1, \ldots, b_n) = b_1\beta_{n-2}(b_2, \ldots, b_{n-1})b_n
$$

**Proof.** Since $I = (0, id_{\mathcal{B}}, 0, 0, \ldots)$, one has:

$$(I\beta)_0 = I_0\beta_0 = 0.$$ 

If $n \geq 1$,

$$(I\beta)_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} I_k(b_1, \ldots, b_k)\beta_{n-k}(b_{k+1}, \ldots, b_n) \\
= I_1(b_1)\beta_{n-1}(b_{k+1}, \ldots, b_n) \\
= b_1\beta_{n-1}(b_{k+1}, \ldots, b_n).$$

For $I\beta I$, the same computations give:

$$(I\beta I)_0 = (I\beta)_0I_0 = 0$$ 

$$(I\beta I)_1 = (I\beta)_0I_1(b_1) + (I\beta)_1(b_1)I_0 \\
= 0.$$ 

If $n \geq 2$, one has:

$$(I\beta I)_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} (I\beta)_k(b_1, \ldots, b_k)I_{n-k}(b_{k+1}, \ldots, b_n) \\
= (I\beta)_{n-1}(b_1, \ldots, b_k)I_1(b_1) \\
= b_1\beta_{n-2}(b_2, \ldots, b_{n-1})b_n$$

□
Proof of the Theorem 3.3: Set $\sigma = I + I\beta I$. Then

$$\left(\circ R_{\beta,\gamma} \circ \sigma\right)_n(b_1, \ldots, b_n) = \sum_{k=1}^{n} \sum_{p_1, \ldots, p_k \geq 1 \atop p_1 + \cdots + p_k = n} \circ R_{\beta,\gamma,k}(\sigma_{p_1}(b_1, \ldots, b_{p_1}), \ldots, \sigma_{p_k}(b_{q_1+1}, \ldots, b_{q_k+p_k}))$$

where $q_i = p_1 + \cdots + p_i - 1$.

From Lemma (3.4)(ii), we have that

$$\sigma_n(b_1, \ldots, b_n) = (I + I\beta I)_n(b_1, \ldots, b_n)$$

therefore Definition 3.2 is equivalent to

$$\beta_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} \left(\circ R_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \ldots, b_k)\right) b_{k+1} \beta_{n-k-2}(b_{k+2}, \ldots, b_n)$$

Considering now Lemma 3.4(i), the above relation becomes

$$\beta_n(b_1, \ldots, b_n) = \sum_{k=0}^{n} \left(\circ R_{\beta,\gamma} \circ (I + I\beta I)_k(b_1, \ldots, b_k)\right) (I + I\beta)_{n-k}(b_{k+1}, \ldots, b_n)$$

therefore

$$\beta = \left[\circ R_{\beta,\gamma} \circ (I + I\gamma I)\right] (1 + I\beta)$$

which is equivalent to (3.3). $\square$

Remark 3.5. Up to a shift in the coefficients, equation (3.3) is similar to the result in the case $\mathcal{B} = \mathbb{C}$ from [2], Theorem 5.1.

Let $X$ be a selfadjoint element of $\mathfrak{A}$. If $\mathfrak{A}$ is endowed with two $\mathcal{B}$-valued conditional expectations $\Phi, \Psi$, the element $X$ will have two moment-generating multilinear function series, one with respect to $\Psi$, that we will denote by $\mathcal{M}_X$, and one with respect to $\Phi$, denoted $\mathfrak{M}_X$. For brevity, we will use the notation $\circ R_X$ for the multilinear function series $\circ R_{\mathcal{M}_X, \mathfrak{M}_X}$.

**Theorem 3.6.** Let $X$ and $Y$ be two elements of $\mathfrak{A}$ that are $\circ$-free with respect to the pair of conditional expectations $(\Phi, \Psi)$. Then

$$\circ R_{X+Y} = \circ R_X + \circ R_Y$$

**Proof.** Let $\mathcal{A}$ be an algebra containing $\mathfrak{B}$ as a subalgebra and endowed with the conditional expectations $\Phi, \Psi : \mathcal{A} \rightarrow \mathfrak{B}$. Consider the set $\mathcal{A}_0 = \mathcal{A} \setminus \mathfrak{B}$ (set difference). For $n \geq 1$ define the maps

$$\circ r : \mathcal{A}_0 \times \cdots \times \mathcal{A}_0 \rightarrow \mathfrak{B}$$

given by the recurrence formula:

$$\Phi(a_1 \cdots a_n) = \sum_{k=1}^{n} \sum_{(1) < \cdots < (k) \atop 1 < (1), (k) \leq n} \circ r_k(a_1[\Psi(a_2 \cdots a_{(1)-1})], \ldots, a_{(k-1)}[\Psi(a_{(k-1)+1} \cdots a_{(k)})], a_{(k)}[\Phi(a_{(k)+1} \cdots a_n)])$$
Note that \( c_r \) is well defined, and that, for any \( b_1, \ldots, b_n \in \mathcal{B} \),
\[
c_{r+1}(X, b_1 X, \ldots, b_n X) = c_{R_{X,n}}(b_1, \ldots, b_n). \tag{3.4}
\]

As in Section 2, consider \( \mathcal{B}(\xi_i) \), the noncommutative algebras of polynomials in the symbols \( \xi_i, i = 1, 2 \) and with coefficients from \( \mathcal{B} \) and the conditional expectations
\[
\Phi_X, \Psi_X : \mathcal{B}(\xi_1) \to \mathcal{B}
\]
given by
\[
\Phi_X(f(\xi_1)) = \Phi(f(X))
\]
\[
\Psi_X(f(\xi_1)) = \Psi(f(X))
\]
and their analogues \( \Phi_Y, \Psi_Y \) for \( \mathcal{B}(\xi_2) \).

On \( \mathcal{B}(\xi_1, \xi_2) \), identified to \( \mathcal{B}(\xi_1) \ast_{\mathcal{B}} \mathcal{B}(\xi_2) \), consider the conditional expectations \( \Psi_0, \Phi_0, \varphi \) given by:
\[
\Psi_0 = \Psi_X \ast \Psi_Y
\]
\[
\Phi_0(f(\xi_1, \xi_2)) = \Phi(f(X, Y))
\]
\[
\varphi(a_1 a_2 \ldots a_n) = \sum_{k=1}^{n} \sum_{l(1)< \ldots < l(k)} \rho_k(a_l \varphi(a_{l(1)} \ldots a_{l(k-1)})),
\]
\[
\ldots, a_{l(k-1)}(\Psi_0(a_{l(k-1)+1} \ldots a_{l(k)})), a_{l(k)}(\varphi(a_{l(k)+1} \ldots a_n))
\]
where \( a_1, \ldots, a_n \) are elements of the set \( \mathcal{B}(\xi_1, \xi_2)_0 = \mathcal{B}(\xi_1) \cup \mathcal{B}(\xi_2) \setminus \mathcal{B} \), and the maps
\[
\rho_n : \mathcal{B}(\xi_1, \xi_2)_0 \times \ldots \mathcal{B}(\xi_1, \xi_2)_0 \to \mathcal{B}
\]
are given by:
\[
\rho_n(a_1, \ldots, a_n) = \begin{cases} 
  c_{\rho}(a_1, \ldots, a_n) & \text{if all } a_1, \ldots, a_n \in \mathcal{B}(\xi_1) \\
  c_{\rho}(a_1, \ldots, a_n) & \text{if all } a_1, \ldots, a_n \in \mathcal{B}(\xi_2) \\
  0 & \text{otherwise}
\end{cases}
\]

We will show that \( \varphi = \Phi_0 \), in particular \( \varphi \) is also well-defined. Consider the element \( a \in \mathcal{B}(\xi_1, \xi_2) \) of the form \( a = a_1 \cdots a_n \) with \( a_j \in \mathcal{B}(\xi_{r(j)}) \), such that \( \varepsilon(1) \neq \varepsilon(2) \neq \cdots \neq \varepsilon(n) \) and \( \Psi_0(a_j) = 0 \). The computation of \( \varphi(a_1 \cdots a_n) \) is done via the recurrence relation above. Because of the definition of \( \rho \) and the fact that \( \Psi_0 = \Psi_X \ast \Psi_Y \), only the term with \( k = 1 \) contribute at the sum, i.e.
\[
\varphi(a_1 \cdots a_n) = \varphi(a_1 \varphi(a_2 \cdots a_n))
\]
\[
= \varphi_{\varepsilon(1)}(a_1 \varphi(a_2 \cdots a_n))
\]
\[
= \varphi_{\varepsilon(1)}(a_1 \varphi(a_2 \cdots a_n))
\]
and the identity between \( \varphi \) and \( \Phi_0 \) follows by induction over \( n \).

Since \( \varphi = \Phi_0 \), the maps \( \rho_n \) and \( \rho_n \) are satisfying the same recurrence relation, hence
\[
\rho_n(a_1, \ldots, a_n) = c_r(a_1, \ldots, a_n).
\]
In particular
\[ c_{X+Y,n}(b_1, \ldots, b_n) = c_{n+1}(X + Y)b_1(X + Y) \cdots (X + Y)b_n(X + Y) \]
\[ = \rho_{n+1}(X + Y)b_1(X + Y) \cdots (X + Y)b_n(X + Y) \]
\[ = \rho_{n+1}(X)b_1(X) \cdots (X)b_n(X) + \rho_{n+1}(Y)b_1(Y) \cdots (Y)b_n(Y) \]
\[ = c_{X,n}(b_1, \ldots, b_n) + c_{Y,n}(b_1, \ldots, b_n). \]

\[ \square \]

4. Central limit theorem

Consider the ordered set \( \langle n \rangle = \{1, 2, \ldots, n\} \) and \( \pi \) a partition of \( \langle n \rangle \) with blocks \( B_1, \ldots, B_m: \)
\[ \langle n \rangle = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m. \]

The blocks \( B_p \) and \( B_q \) of \( \pi \) are said to be crossing if there exist \( i < j < k < l \) in \( \langle n \rangle \) such that \( i, k \in B_p \) and \( j, l \in B_q \).

The partition \( \pi \) is said to be non-crossing if all pairs of distinct blocks of \( \pi \) are not crossing. We will denote by \( NC_2(n) \) the set of all non-crossing partitions of \( \langle n \rangle \) whose blocks contain exactly 2 elements and by \( NC_{\leq s}(n) \) the set of all non-crossing partitions of \( \langle n \rangle \) whose blocks contain at most \( s \) elements.

Let now \( \gamma \) be a non-crossing partition of \( \langle n \rangle \) and \( B \) and \( C \) be two blocks of \( \pi \). We say that \( B \) is interior to \( C \) if there exist two indices \( i < j \) in \( \langle n \rangle \) such that \( i, j \in C \) and \( B \subset \{i+1, \ldots, j-1\} \). The block \( B \) is said to be outer if it is not interior to any other block of \( \gamma \). In a non-crossing partition of \( \langle n \rangle \), the block containing 1 is always outer.

Consider now an element \( X \) of \( \mathfrak{A} \). Let \( \pi \) be a partition from \( NC_2(n+1) \) (\( n = \) odd) and \( B_1 = (1, k) \) be the block of \( \pi \) containing 1. We define, by recurrence, the following expressions:

\[ V_\pi(X, b_1, \ldots, b_n) = \Psi(Xb_1V_{\pi\{2, \ldots, j-1\}}(X, b_2, \ldots, b_{k-2})b_{k-1}X)b_k \]
\[ V_{\pi\{k+1, \ldots, n+1\}}(X, b_{k+1}, \ldots, b_n) \]
\[ W_\pi(X, b_1, \ldots, b_n) = \Phi(Xb_1V_{\pi\{2, \ldots, j-1\}}(X, b_2, \ldots, b_{k-2})b_{k-1}X)b_k \]
\[ W_{\pi\{k+1, \ldots, n+1\}}(X, b_{k+1}, \ldots, b_n) \]

Theorem 4.1. (Central Limit Theorem) Let \( (X_n)_{n \geq 1} \) be a sequence of \( c \)-free elements of \( \mathfrak{A} \) such that:

1. all \( X_n \) have the same moment-generating multilinear function series, \( \mathfrak{M} \) with respect to \( \Phi \) and \( M \) with respect to \( \Psi \).
2. \( \Psi(X_n) = \Phi(X_n) = 0. \)

Set
\[ S_N = \frac{X_1 + \cdots + X_N}{\sqrt{N}}, \]

Then:

(i) \( \lim_{N \to \infty} c_{R_{S_N}} = (0, \mathfrak{M}_1(\cdot), 0, \ldots) \)
(ii) \( \lim_{N \to \infty} R_{\mathcal{S}N} = (0, M_1(\cdot), 0, \ldots) \)

(iii) there exist two conditional expectations \( \nu : \mathcal{B}(\xi) \to \mathcal{B} \), depending only on \( M_1(\cdot) \), and \( \mu : \mathcal{B}(\xi) \to \mathcal{B} \), depending only on \( M_1(\cdot) \) and \( \mathcal{M}_1(\cdot) \), such that

\[
\lim_{N \to \infty} \Psi_{\mathcal{S}N} = \nu \\
\lim_{N \to \infty} \Phi_{\mathcal{S}N} = \mu
\]

in the weak sense; in particular,

\[
\nu(\xi b_1 \xi \ldots b_n \xi) = \sum_{\pi \in NC_2(n)} V_{\pi}(X_1, b_1, \ldots, b_n) \\
\mu(\xi b_1 \xi \ldots b_n \xi) = \sum_{\pi \in NC_2(n)} W_{\pi}(X_1, b_1, \ldots, b_n).
\]

Proof. Let \( X \) be an element of \( \mathfrak{A} \) with the same moment generating series as \( X_j, j \geq 1 \). As shown in [3],

\[
R_{\mathcal{S}N} = \sum_{k=1}^N R X_k \sqrt{N} = N R X \sqrt{N}.
\]

Also, from Theorem 2.4 and Theorem 3.6, it follows that

\[
^c R_{\mathcal{S}N} = \sum_{k=1}^N c R X_k \sqrt{N} = N^c c R X \sqrt{N}.
\]

Since \( R \) and \( c R \) are multilinear and \( M_0 = \mathcal{M}_0 = 0 \), we have that

\[
\lim_{N \to \infty} c R_{\mathcal{S}N,n} = \lim_{N \to \infty} N c R X_n = \begin{cases} 
0 & \text{if } n \neq 1 \\
M_1(\cdot) & \text{if } n = 1
\end{cases}
\]

and the similar relations for \( R_{\mathcal{S}N,n} \), hence (i) and (ii) are proved.

For (iii) it suffices to check the relations for \( \nu(\xi b_1 \xi \ldots b_n \xi) \) and \( \mu(\xi b_1 \xi \ldots b_n \xi) \), which are a trivial corollary of (i), (ii), and the recurrence formulas that define \( R \) and \( c R \). \( \square \)

Remark 4.2. For \( \mathcal{B} = \mathbb{C} \), the theorem is a weaker version of Theorem 4.3 from [2]. If \( \Psi \) is \( \mathbb{C} \)-valued, then the result is similar to Corollary 5.1 from [6]. Also, under the assumptions that for some \( a, b \in \mathcal{B} \) we have that:

\[
\lim_{N \to \infty} N \Psi(X_1 \cdots X_N) = a \\
\lim_{N \to \infty} N \Psi(X_1 \cdots X_N) = b
\]

the same techniques lead to a Poisson-type limit Theorem, similar to Corollary 2, Section 5 of [6].
In the following remaining pages we will describe the positivity of the limit functionals \( \mu \) and \( \nu \) in terms of \( \Phi \) and \( \Psi \). The central result is Corollary 4.4.

For simplicity, suppose that \( \mathcal{B} \) is a unital \(*\)-algebra (otherwise, we can replace \( \mathcal{B} \) by its unitisation). Consider the symbol \( \xi \), the \(*\)-algebra \( \mathcal{B}(\xi) \) of polynomials in \( \xi \) with coefficients from \( \mathcal{B} \), as defined before, and consider also the linear space \( \mathcal{B}(\mathcal{B}) \) generated by the set \( \{b_1\xi b_2; \ b_1, b_2 \in \mathcal{B}\} \) with the \( \mathcal{B} \)-bimodule structure given by

\[
a_1b_1\xi b_2a_2 = (a_1b_1)\xi(b_2a_2)
\]

for all \( a_1, a_2, b_1, b_2 \in \mathcal{B} \).

**Lemma 4.3.** For any positive \( \mathcal{B} \)-sesquilinear pairing \( \langle \cdot, \cdot \rangle \) on \( \mathcal{B}(\xi) \) there exists a positive conditional expectation \( \varphi : \mathcal{B}(\xi) \to \mathcal{B} \) such that for any \( b_1, b_2 \in \mathcal{B} \) one has that

\[
\varphi(\xi b_1^* b_2) = \langle b_1\xi, b_2\rangle
\]

**Proof.** Without loss of generality, we can suppose that \( \mathcal{B} \) is unital (otherwise we can replace \( \mathcal{B} \) by its unitization).

Consider the Full Fock bimodule over \( \mathcal{B}(\xi) \)

\[
\mathcal{F}(\xi) = \mathcal{B} \oplus \left( \bigoplus_{n \geq 1} \mathcal{B}(\xi) \otimes \mathcal{B} \otimes \cdots \otimes \mathcal{B}(\xi) \right)
\]

with the pairing given by

\[
\langle a, b \rangle = a^* b
\]

\[
\langle a_1 \xi \otimes \cdots \otimes a_n \xi, b_1 \xi \otimes \cdots \otimes b_m \xi \rangle = \delta_{m,n} \langle a_n \xi, \langle \ldots, \langle a_1 \xi, b_1 \xi \rangle, \ldots, b_n \xi \rangle \rangle.
\]

\((a, a_j, b, b_j \in \mathcal{B}, j = 1, \ldots, n)\)

Note that the \( \mathcal{B} \)-linear operators \( A_1, A_2 : \mathcal{F}(\xi) \to \mathcal{F}(\xi) \) described by the relations

\[
A_1 b = \xi b
\]

\[
A_1(a_1 \xi \otimes \cdots \otimes a_n \xi b) = \xi \otimes a_1 \xi \otimes \cdots \otimes a_n \xi b
\]

\[
A_2 b = 0
\]

\[
A_2(a_1 \xi \otimes \cdots \otimes a_n \xi b) = \langle \xi, a_1 \xi a_2 \xi \otimes \cdots \otimes a_n \xi b
\]

are self-adjoint to each other, in the sense that

\[
\langle A_1 \tilde{\zeta}_1, \tilde{\zeta}_2 \rangle = \langle \tilde{\zeta}_1, A_2 \tilde{\zeta}_2 \rangle
\]

for any \( \tilde{\zeta}_1, \tilde{\zeta}_2 \in \mathcal{F}(\xi) \), therefore \( S = A_1 + A_2 \) is selfadjoint.

Moreover, for any \( a, b \in \mathcal{B} \),

\[
\langle 1, Sa^* b S1 \rangle = \langle a S1, b S1 \rangle
\]

\[
= \langle a(A_1 + A_2)1, b(A_1 + A_2)1 \rangle
\]

\[
= \langle a\xi, b\xi \rangle
\]

and the conclusion follows by setting \( \varphi(p(\xi)) = \langle 1, p(S)1 \rangle \) for all \( p \in \mathcal{B}(\xi) \). \( \square \)

**Corollary 4.4.** The maps \( \mu \) and \( \nu \) from Theorem 4.1 are positive if and only if for any \( b \in \mathcal{B} \) one has that \( \Phi(X b^* b X) \geq 0 \) and \( \Psi(X b^* b X) \geq 0 \).
Proof. One implication is trivial, since, if \( \nu \) and \( \mu \) are positive, then

\[
\Psi(X^*bX) = \nu(Xb^*bX) = \nu((bX)^*bX) \geq 0
\]

and

\[
\Phi(Xb^*bX) = \mu(Xb^*bX) = \mu((bX)^*bX) \geq 0.
\]

Suppose now that \( \Phi(Xb^*bX) \geq 0 \) and \( \Psi(Xb^*bX) \geq 0 \) for all \( b \in \mathcal{B} \). We will use the same argument as in [9] and [8].

Consider the set of selfadjoint symbols \( \{\xi_i\}_{i \geq 1} \). On each \( \mathcal{B}\xi_i\mathcal{B} \) we have the positive \( \mathcal{B} \)-sesquilinear pairings \( \langle \cdot , \cdot \rangle_\Phi \) and \( \langle \cdot , \cdot \rangle_\Psi \) determined by

\[
\langle a\xi_i, b\xi_i \rangle_\Phi = \Phi(Xa^*bX)
\]

\[
\langle a\xi_i, b\xi_i \rangle_\Psi = \Psi(Xa^*bX).
\]

As shown in Lemma 4.3, the above \( \mathcal{B} \)-sesquilinear pairings determine positive conditional expectations \( \varphi, \psi : \mathcal{A}_i \to \mathcal{B} \), where \( \mathcal{A}_i = \mathcal{B}\langle \xi_i \rangle \) be the \( \ast \)-algebras of polynomials in \( \xi \) with coefficients from \( \mathcal{B} \), \( i \geq 1 \).

For \( \tau : \mathcal{B}\langle \xi \rangle \to \mathcal{B} \) a conditional expectation, and \( \lambda \geq 0 \), note with \( D_{\lambda}\tau \) the dilation with \( \lambda \) of \( \tau \), i.e.

\[
D_{\lambda}\tau(\xi b_1 \xi \cdots b_n \xi) = \lambda^{n+1} \tau(\xi b_1 \xi \cdots b_n \xi)
\]

Remark that if \( \tau \) is positive, then \( D_{\lambda}\tau \) is also positive.

With the notations above, consider, as in Definition 2.1, the conditionally free product \( (\mathcal{A}, \Phi, \Psi) = \bigstar_{i \in I}(\mathcal{A}_i, \Phi_i, \Psi_i) \). The elements \( \{\xi_i\}_{i \geq 1} \) are conditionally free in \( \mathcal{A} \), so Theorem 4.1 implies that:

\[
\mu = \lim_{N \to \infty} \frac{\Phi_{\xi_1 + \cdots + \xi_N}}{\sqrt{N}} = D_{\frac{1}{\sqrt{N}}} \Phi_{\xi_1 + \cdots + \xi_N}
\]

\[
\nu = \lim_{N \to \infty} \frac{\Psi_{\xi_1 + \cdots + \xi_N}}{\sqrt{N}} = D_{\frac{1}{\sqrt{N}}} \Psi_{\xi_1 + \cdots + \xi_N}
\]

\[
= \frac{1}{\sqrt{N}} \bigstar_{i=1}^N \Psi_{\xi_i} \geq 0
\]

We have that \( \bigstar_{i=1}^N \Psi_{\xi_i} \geq 0 \) since it is the free product of states (see, for example [9]), hence the positivity of \( \nu \).

Also, Theorem 2.4 and Corollary 2.6 imply that \( \Phi_{\xi_1 + \cdots + \xi_N} \geq 0 \), therefore \( \mu \geq 0 \).

\[\square\]

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