On soliton solutions of the time-discrete generalized lattice Heisenberg magnet model

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Abstract

Generalized lattice Heisenberg magnet model is an integrable model exhibiting soliton solutions. The model is physically important for describing the magnon bound states (or soliton excitations) with arbitrary spin, in magnetic materials. In this paper, a time-discrete generalized lattice Heisenberg magnet (GLHM) model is investigated. By writing down the Lax pair representation of the time-discrete GLHM model, we present explicitly the underlying integrable structure like, the Darboux transformation and soliton solutions.

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1 Introduction

Discrete or (lattice) integrable systems namely systems with their independent variables defined on a lattice points have received much attention by the researchers working in the field of theoretical and applied sciences. The study of discrete integrable system not only as a physical model but also in the context of numerical analysis is of particular importance in various fields ranging from pure mathematics to experimental science. Discrete integrable systems such as, Toda lattice, Volterra lattice, Ablowitz-Ladik lattice, Hirota-Miwa equation, nonlinear $\sigma$-model, sine-Gordon equation etc have been studied extensively in the literature \cite{1-11}. Soliton solutions of these discrete systems have been computed by using various systematic methods such inverse scattering transform, Bäcklund/Darboux transformation, Hirota bilinear method etc \cite{1-11, 20}.

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The lattice Heisenberg magnet model has been studied in many references such as [10]-[13]. It exhibits many aspects of integrability for instance Lax pair representation, higher symmetries, \( r \)-matrix formulism etc. The soliton solutions have been studied by Bäcklund transformation (BT), Darboux transformation (DT), inverse scattering transform (IST) and other solution generating techniques [10]-[19].

The Lax pair representation of the time-discrete GLHM model is given by [17]

\[
\begin{align*}
\Phi^{m+1}_n &= A^m_n \Phi^m_n, \quad A^m_n = I + \lambda U^m_n, \quad (1.1) \\
\Phi^{m+1}_n &= B^m_n \Phi^m_n, \quad B^m_n = I + h \frac{\lambda}{1 - \lambda^2} J^m_n + h \frac{\lambda^2}{1 - \lambda^2} J^m_n U^m_n, \quad (1.2)
\end{align*}
\]

where \( U^m_n \) is an \( N \times N \) matrix and \( \Phi^m_n \) is also an \( N \times N \) eigen-function matrix. The conditions on the matrix \( U^m_n \) i.e., \( (U_n^m)^2 = I \) and \( J^m_n A^m_n = A^m_{n-1} J^m_n \) are assumed. The latter condition implies \( J^m_n U^m_n = U^m_{n-1} J^m_n \). The compatibility condition of the Lax pair \((1.1)-(1.2)\) implies a zero-curvature condition i.e., \( A^{m+1}_n B^m_n = B^m_{n+1} A^m_n \) which is equivalent to the equation of motion

\[
\frac{1}{h} \left[ (A^{m+1}_n)^{-1} - (A^m_n)^{-1} \right] + \frac{\lambda}{1 - \lambda^2} (J^{m+1}_n - J^m_n) = O, \quad (1.3)
\]

or equivalently,

\[
\frac{1}{h} (U_{n+1}^m - U_n^m) = J_{n+1}^m - J_n^m. \quad (1.4)
\]

The relation \( J_n^m U_n^m = U_{n-1}^m J_n^m \) is satisfied if we choose \( J_n^m = 2ia_n^m U_{n-1}^m U_n^{m+1} (U_n^m + U_{n-1}^m)^{-1} + 2b_n^m (U_n^m + U_{n-1}^m)^{-1} \). Substituting this expression into equation (1.4), we obtain

\[
\frac{1}{h} (U_{n+1}^m - U_n^m) = \Delta_n^m \left[ 2ia_n^m U_{n-1}^m (U_n^m + U_{n-1}^m)^{-1} + 2b_n^m (U_n^m + U_{n-1}^m)^{-1} \right], \quad (1.5)
\]

where \( \Delta_n^m f^m_n = f^m_{n+1} - f^m_n \). For \( N = 2 \), we get a simplest \( 2 \times 2 \) case and express the equation (1.5) as

\[
\frac{1}{h} (U_{n+1}^m - U_n^m) = \Delta_n^m \left[ a_n^m U_n^m \times U_{n-1}^{m+1} + b_n^m U_n^m + U_{n-1}^{m+1} \right], \quad (1.6)
\]

It should be noted that the generalization of the lattice Heisenberg model (1.4), (1.5) was studied by Tsuchida [17]. In the case of matrices of size \( 2 \times 2 \), it reduces to the well-known vector lattice Heisenberg chain (1.6). The Bäcklund transformations for the latter are sufficiently well studied, in particular, their derivation is given in [17], with references to earlier works. However, the problem of describing Bäcklund/Darboux transformations for the general matrix case (1.4), (1.5) is left open [17]. It is this problem that is considered in the present work.
In this paper, we present a systematic approach to find the soliton solutions of the time-discrete GLHM model (1.4). We define Darboux transformation (DT) on the solution to the Lax pair and the solutions of the matrix generalization of the time-discrete GLHM model defined by equation (1.4) with respect to the $N \times N$ matrices $J_n^m$, $U_n^m$, or, after reduction, by one equation (1.5) with respect to the matrix $U_n^m$.

Darboux transformation is one of the powerful and effective techniques used to compute solutions of a given nonlinear integrable equation in soliton theory. The main idea of this method is that, a new solution to the Lax pair (i.e., pair of linear equations associated with nonlinear integrable equation) can be obtained from the old solution by means of Darboux matrix. The covariance of the Lax pair under the Darboux transformation requires that the new solution satisfies the same Lax pair such that the relationship between the new and old solutions to the Lax pair and the solutions to the nonlinear integrable equation can be built. Hence, one can find the soliton solutions to the nonlinear integrable equation by solving a Lax pair with the given seed (or trivial) solutions. Various integrable equations have been studied successfully by means of Darboux transformation and obtained the soliton solutions have been computed. The solutions are expressed in terms of Wronskian, quasi-Grammian and quasi-determinants in the literature [8]-[9], [20]-[24].

This paper is organized as follows. Section 2, contains the derivation of the DT for the matrix generalization of the time-discrete GLHM model (1.4), (1.5). Furthermore, the solutions obtained by DT are expressed in quasideterminant form. In section 3, soliton solutions for the general $N \times N$ and simplest $N = 2$ case of the time-discrete GLHM model are obtained. Section 4, is devoted for concluding remarks.

## 2 Discrete Darboux transformation

In what follows, we apply DT on the Lax pair (1.1)-(1.2) of the time-discrete GLHM model (1.4) to obtain soliton solutions. We define a DT on the solutions to the Lax pair equations (1.1)-(1.2) by means of a $N \times N$ discrete Darboux matrix $D_n^m$. The discrete Darboux matrix $D_n^m$ acts on the solution $\Phi_n^m$ of the Lax pair (1.1)-(1.2) to give another solution $\Phi_n^m[1]$ i.e.,

$$\Phi_n^m[1] = D_n^m \Phi_n^m. \quad (2.1)$$
The covariance of the Lax pair (1.1)-(1.2) under the DT requires that the new solution Φₜᵣ[m][1] satisfies the same Lax pair equations with the new matrices Aₜᵣ[m][1], Bₜᵣ[m][1] i.e.,

\[
Aₜᵣ[m][1] = I + \lambda Uₜᵣ[m][1],
\]

\[
Bₜᵣ[m][1] = I + \frac{\lambda}{1 - \lambda²} Jₜᵣ[m][1] + \frac{\lambda²}{1 - \lambda²} Jₜᵣ[m][1] Uₜᵣ[m][1],
\]

By using (1.1)-(1.2), equations (2.2)-(2.3) imply that the discrete Darboux matrix Dₜᵣ[m][n] satisfies the following discrete Darboux-Lax equations as

\[
Aₜᵣ[m][1] Dₜᵣ[m][n] = Dₜᵣ[m][n+1] Aₜᵣ[m][1],
\]

\[
Bₜᵣ[m][1] Dₜᵣ[m][n] = Dₜᵣ[m][n+1] Bₜᵣ[m][1],
\]

We are interested in finding a DT on the matrices Uₜᵣ[m][1], Jₜᵣ[m][1]. For this, we make the ansatz for the Darboux matrix such as Dₜᵣ[m][n] = λ⁻¹I - Qₜᵣ[m][n], where Qₜᵣ[m][n] is the N × N auxiliary matrix and I is the N × N identity matrix. By substituting the latter expression of Dₜᵣ[m][n] in equation (2.4), the coefficients of λ yields the DT on the matrices Uₜᵣ[m][n], Jₜᵣ[m][n] as

\[
Uₜᵣ[m][1] = Uₜᵣ[m][1] - (Qₜᵣ[m][n+1] - Qₜᵣ[m][n]),
\]

\[
Jₜᵣ[m][1] = Jₜᵣ[m][1] - \frac{1}{h} (Qₜᵣ[m][n+1] - Qₜᵣ[m][n]),
\]

with the following conditions on the Darboux matrix Qₜᵣ[m][n]

\[
\frac{1}{h} (Qₜᵣ[m][n+1] - Qₜᵣ[m][n]) (I - (Qₜᵣ[m][n])²) = [Qₜᵣ[m][n], Jₜᵣ[m][n] (Qₜᵣ[m][n] + Uₜᵣ[m][n])].
\]

where \([fₜᵣ[m][n], gₜᵣ[m][n]]^+ := fₜᵣ[m][n] gₜᵣ[m][n] - gₜᵣ[m][n] fₜᵣ[m][n] \) in what follows, we show that for Qₜᵣ[m][n] = Θₜᵣ[m][n] Λ⁻¹(Θₜᵣ[m][n])⁻¹, where Θₜᵣ[m][n] is the N × N particular matrix solution to the Lax pair (1.1)-(1.2) which is constructed by N wave function Φₜᵣ[m][n] for different values of λ, whereas the matrix Λ is an N × N diagonal matrix with N distinct eigenvalues λᵢ (λᵢ ≠ λⱼ), i = 1, ..., N, j = i+1, ..., i+N. Take N constant column basis vectors \(|e₁⟩\), ..., \(|eₙ⟩\), so that the invertible N × N matrix Θₜᵣ[m][n] can be defined as Θₜᵣ[m][n] = (Φₜᵣ[m][n]λᵢ|e₁⟩, ..., Φₜᵣ[m][n]λᵢ|eₙ⟩) = (|θ₁⟩ₙ, ..., |θₙ⟩ₙ), such that each |θᵢ⟩ₙ = Φₜᵣ[m][n]λᵢ|eᵢ⟩ in the matrix Θₜᵣ[m][n] is a column solution to the Lax pair (1.1)-(1.2). For λ = λᵢ (i = 1, ..., N), we have

\[
|θᵢ⟩ₙ+₁ = |θᵢ⟩ₙ + λ Uₜᵣ[m][n] |θᵢ⟩ₙ,
\]

\[
|θᵢ⟩ₙ+₁ = |θᵢ⟩ₙ + h \frac{λᵢ}{1 - λᵢ²} Jₜᵣ[m][n] |θᵢ⟩ₙ + h \frac{λᵢ²}{1 - λᵢ²} Jₜᵣ[m][n] Uₜᵣ[m][n] |θᵢ⟩ₙ,
\]

For Λ = diag(λ₁, ..., λₙ), the Lax pair (2.9)-(2.10) reduce to the generalized matrix form as

\[
Θₜᵣ[m][n+1] = Θₜᵣ[m][n] + Uₜᵣ[m][n] Θₜᵣ[m][n] Λ,
\]

\[
Θₜᵣ[m][n+1] = Θₜᵣ[m][n] + h Jₜᵣ[m][n] Θₜᵣ[m][n] Λ (I - Λ²)⁻¹ + h Jₜᵣ[m][n] Θₜᵣ[m][n] Λ² (I - Λ²)⁻¹,
\]
where $\Theta^m_n$ is a particular matrix solution to the Lax pair (1.1)-(1.2). Let us define a matrix $Q^m_n$ in terms of an invertible matrix $\Theta^m_n$, i.e. $Q^m_n = \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1}$. Now, we show that the latter expression of the matrix $Q^m_n$ satisfies the set of equations (2.7)-(2.8). To do this, let us check the first condition as

$$(Q^m_{n+1} - Q^m_n) \Theta^m_n = \left( \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} - \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1} \right) \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1},$$

which is equation (2.7). Similarly for the second condition, we have

$$\frac{1}{h} \left( Q^m_{n+1} - Q^m_n \right) \left( I - (Q^m_n)^2 \right) = \frac{1}{h} \left( \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} - \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1} + \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} - \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} \right) \times \Theta^m_n \left( I - \Lambda^{-2} \right) (\Theta^m_n)^{-1},$$

$$= \frac{1}{h} \left( \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} - \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1} + \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} - \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} \right) \left( J^m_n \Theta^m_n \Lambda (I - \Lambda^{-2})^{-1} + J^m_n U^m_n \Theta^m_n \Lambda^2 (I - \Lambda^{-2})^{-1} \right) \left( I - \Lambda^{-2} \right) (\Theta^m_n)^{-1},$$

$$= \left[ Q^m_n, J^m_n (Q^m_n + S^m_n) \right]^+. \quad (2.14)$$

which is equation (2.8). Therefore, we have established that the DT on the matrix solutions $\Phi^m_n$, $U^m_n$ is given by

$$\Phi^m_n \left[ 1 \right] = (\Lambda^{-1} I - \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1}) \Phi^m_n, \quad (2.15)$$

$$U^m_n \left[ 1 \right] = U^m_n - \left( \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} - \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1} \right),$$

$$= \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_n)^{-1} \left[ U^m_n \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1} \right]. \quad (2.16)$$

At this stage, we can say that DT (2.15)-(2.16) preserves the system i.e., if $\Phi^m_n$, $U^m_n$ are the solutions to the Lax pair (1.1)-(1.2) and GLHM model (1.3) respectively, then $\Phi^m_n[K]$, $U^m_n[K]$ (that correspond to multi-soliton solutions) are also the solutions of the same equations. The DT (2.16) is also consistent with the reduction (1.5). For $Q^m_n = \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1}$, it seems appropriate here to express the solutions $\Phi^m_n[K]$, $U^m_n[K]$ in terms
of quasideterminants. The matrix solution $\Psi^m_n[1]$ to the Lax pair (1.1)-(1.2) with the particular matrix solution $\Theta^m_n$ in terms of quasideterminant can be expressed as

$$\Psi^m_n[1] \equiv D_n\Psi^m_n = (\lambda^{-1} I - \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1}) \Phi^m_n,$$

$$= \begin{vmatrix} \Theta^m_n & \Phi^m_n \\ \Theta^m_n \Lambda^{-1} & \lambda^{-1} \Phi^m_n \end{vmatrix}.$$  \hfill (2.18)

And the one-fold Darboux transformation on the matrix field $U^m_n$ of the time-discrete GLHM is

$$U^m_n[1] = Q^m_{n+1} U^m_n (Q^m_n)^{-1} = \Theta^m_{n+1} \Lambda^{-1} (\Theta^m_{n+1})^{-1} U^m_n \left( \Theta^m_n \Lambda^{-1} (\Theta^m_n)^{-1} \right)^{-1},$$

$$= \begin{vmatrix} \Theta^m_{n+1} & I \\ \Theta^m_{n+1} \Lambda^{-1} & \Theta^m_n \Lambda^{-1} \end{vmatrix} U^m_n \begin{vmatrix} I \\ O \end{vmatrix}.$$  \hfill (2.19)

where $O$ is the $N \times N$ null matrix and $I$ is the $N \times N$ identity matrix. The results obtained in (2.18)-(2.19) can be extended and generalized to $K$-fold DT. For the matrix solutions $\Theta_k$ at $\Lambda = \Lambda_k$ ($k = 1, 2, \ldots, K$) to the Lax pair (1.1)-(1.2), the $K$-times repeated DT $\Phi^m_n[K]$ in terms of quasideterminant is written as

$$\Phi^m_n[K] = \prod_{k=1}^{K} (\lambda I - Q^m_n[K - k]) \Phi^m_n,$$

$$= \prod_{k=1}^{K} (\lambda I - \Theta^m_n[K - k] \Lambda^{-1}_{K-k} (\Theta^m_n[K - k])^{-1}) \Phi^m_n,$$

$$= \begin{vmatrix} \Theta^m_{n,1} & \Theta^m_{n,2} & \cdots & \Theta^m_{n,K} & \Phi^m_n \\ \Theta^m_{n,1} \Lambda^{-1}_1 & \Theta^m_{n,2} \Lambda^{-1}_2 & \cdots & \Theta^m_{n,K} \Lambda^{-1}_K & \lambda^{-1} \Phi^m_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta^m_{n,1} \Lambda^{-K+1}_1 & \Theta^m_{n,2} \Lambda^{-K+1}_2 & \cdots & \Theta^m_{n,K} \Lambda^{-K+1}_K & \lambda^{-K+1} \Phi^m_n \\ \Theta^m_{n,1} \Lambda^{-K}_1 & \Theta^m_{n,2} \Lambda^{-K}_2 & \cdots & \Theta^m_{n,K} \Lambda^{-K}_K & \lambda^{-K} \Phi^m_n \end{vmatrix}.$$  \hfill (2.20)

\footnote{In this paper, we will use quasideterminants that are expanded about \( n \times n \) matrix. The quasideterminant expression of \( N \times N \) expanded about \( n \times n \) matrix is given as

$$\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} = M_{22} - M_{21} M_{11}^{-1} M_{12}.\quad (2.17)$$

For further details see \cite{22, 25}.}
Similarly the quasideterminant expression for $U_n^m[K]$ is

$$U_n^m[K] = \begin{vmatrix}
\Theta_{m+1,1}^n & \Theta_{m+1,2}^n & \ldots & \Theta_{m+1,K}^n \\
\Theta_{m+1,1}^n \Lambda_{1}^{-1} & \Theta_{m+1,2}^n \Lambda_{2}^{-1} & \ldots & \Theta_{m+1,K}^n \Lambda_{K}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
\Theta_{m+1,1}^n \Lambda_{1}^{-K+1} & \Theta_{m+1,2}^n \Lambda_{2}^{-K+1} & \ldots & \Theta_{m+1,K}^n \Lambda_{K}^{-K+1} \\
\Theta_{m+1,1}^n \Lambda_{1}^{-K} & \Theta_{m+1,2}^n \Lambda_{2}^{-K} & \ldots & \Theta_{m+1,K}^n \Lambda_{K}^{-K}
\end{vmatrix}^{-1} \Theta_{n+1,1}^m & \Theta_{n+1,2}^m & \ldots & \Theta_{n+1,K}^m \\
\Theta_{n+1,1}^m \Lambda_{1}^{-1} & \Theta_{n+1,2}^m \Lambda_{2}^{-1} & \ldots & \Theta_{n+1,K}^m \Lambda_{K}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
\Theta_{n+1,1}^m \Lambda_{1}^{-K+1} & \Theta_{n+1,2}^m \Lambda_{2}^{-K+1} & \ldots & \Theta_{n+1,K}^m \Lambda_{K}^{-K+1} \\
\Theta_{n+1,1}^m \Lambda_{1}^{-K} & \Theta_{n+1,2}^m \Lambda_{2}^{-K} & \ldots & \Theta_{n+1,K}^m \Lambda_{K}^{-K}
\end{vmatrix} \times \begin{vmatrix}
I \\
O \\
O \\
O
\end{vmatrix}. \quad (2.21)$$

Equations (2.20) and (2.21) represent respectively, the required $K$th quasideterminant solutions $\Phi_n^m[K]$ to the Lax pair and $U_n^m[K]$ of the time-discrete GLHM model. These results can be easily proved by induction.

### 3 Soliton solutions

In this section, we obtain the soliton solutions from a seed (trivial) solution by solving the Lax pair of the time-discrete GLHM model. For this, we re-write the matrix $(Q_n^m)^{(K)}$ from (2.21) in a more convenient form as follows

$$Q_n^m(K) = \begin{vmatrix}
\tilde{G}_n^m & T^{(K)}
\end{vmatrix}, \quad (3.1)$$

where $T^{(K)}$, $\tilde{G}_n^m$ and $G_n^m$ are $NK \times N$, $N \times NK$ and $NK \times NK$ matrices respectively. These matrices are given by

$$T^{(K)} = \begin{pmatrix} I & O & \ldots & O \end{pmatrix}^T,
\tilde{G}_n^m = \begin{pmatrix} \Theta_{n,1}^m \Lambda_{1}^{-K} & \Theta_{n,2}^m \Lambda_{2}^{-K} & \ldots & \Theta_{n,K}^m \Lambda_{K}^{-K} \end{pmatrix},
G_n^m = \begin{pmatrix}
\Theta_{n,1}^m & \Theta_{n,2}^m & \ldots & \Theta_{n,K}^m \\
\Theta_{n,1}^m \Lambda_{1}^{-1} & \Theta_{n,2}^m \Lambda_{2}^{-1} & \ldots & \Theta_{n,K}^m \Lambda_{K}^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
\Theta_{n,1}^m \Lambda_{1}^{-K+1} & \Theta_{n,2}^m \Lambda_{2}^{-K+1} & \ldots & \Theta_{n,K}^m \Lambda_{K}^{-K+1}
\end{pmatrix} \quad (3.2)$$
Likewise, with the elements given by

\[ Q_{n, ij}^{m(K)} = \begin{vmatrix} G_n^m & T_j^{(K)} \end{vmatrix}_{ij} = \begin{vmatrix} G_n^m & T_j^{(K)} \end{vmatrix}_{ij} = -\frac{\det (G_n^m)_{ij}}{\det (G_n^m)}, \quad i, j = 1, 2, \ldots, K. \tag{3.3} \]

decomposed into the \( \tilde{G}_n^m \) and \( T_j^{(K)} \) respectively.

For the simplest matrix of size 2 \( \times \) 2, the matrix \( Q_{n, ij}^{m(K)} \) can be expressed as

\[ Q_{n}^{m(K)} = \begin{pmatrix} Q_{n, 11}^{m(K)} & Q_{n, 12}^{m(K)} \\ Q_{n, 21}^{m(K)} & Q_{n, 22}^{m(K)} \end{pmatrix} = \begin{vmatrix} G_n^m & T_j^{(K)} \end{vmatrix}_{ij} = \begin{vmatrix} G_n^m & T_j^{(K)} \end{vmatrix}_{ij} = -\frac{\det (G_n^m)_{ij}}{\det (G_n^m)}, \quad i, j = 1, 2. \tag{3.4} \]

with the elements given by

\[ Q_{n, ij}^{m(K)} = \begin{vmatrix} G_n^m & T_j^{(K)} \end{vmatrix}_{ij} = -\frac{\det (G_n^m)_{ij}}{\det (G_n^m)}, \quad i, j = 1, 2. \tag{3.5} \]

For one soliton \( K = 1 \), we have

\[ T^{(1)} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_n^m = \Theta_{n, 1}^m = \begin{pmatrix} \theta_{m, 11}^{(1)} & \theta_{m, 12}^{(2)} \\ \theta_{m, 21}^{(1)} & \theta_{m, 22}^{(2)} \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \bar{\lambda}_1 \end{pmatrix}, \]

\[ \tilde{G}_n^m = \Theta_{n, 1}^m \Lambda_1^{-1} = \begin{pmatrix} \lambda_1^{-1} \theta_{m, 11}^{(1)} & \bar{\lambda}_1^{-1} \theta_{m, 12}^{(2)} \\ \lambda_1^{-1} \theta_{m, 21}^{(1)} & \bar{\lambda}_1^{-1} \theta_{m, 22}^{(2)} \end{pmatrix}. \tag{3.6} \]

By using equation (3.6) in (3.5), the matrix element \( Q_{n, 12}^{m(1)} \) of the matrix \( Q_n^m \) can be computed as

\[ Q_{n, 12}^{m(1)} = \begin{vmatrix} G_n^m & T^{(1)}_2 \end{vmatrix}_{12} = \begin{vmatrix} G_n^m & T^{(1)}_2 \end{vmatrix}_{12} = -\frac{\det (G_n^m)}{\det (G_n^m)} \begin{vmatrix} \theta_{m, 11}^{(1)} & \theta_{m, 12}^{(2)} \\ \theta_{m, 21}^{(1)} & \theta_{m, 22}^{(2)} \end{vmatrix} \begin{vmatrix} \lambda_1^{-1} & 0 \\ 0 & \bar{\lambda}_1 \end{vmatrix} \begin{vmatrix} \theta_{m, 11}^{(1)} & \theta_{m, 12}^{(2)} \\ \lambda_1^{-1} \theta_{m, 11}^{(1)} & \bar{\lambda}_1^{-1} \theta_{m, 12}^{(2)} \end{vmatrix} \begin{vmatrix} 0 & 1 \\ \theta_{m, 21}^{(1)} & \theta_{m, 22}^{(2)} \end{vmatrix} = \frac{\lambda_1^{-1} - \bar{\lambda}_1^{-1}}{\theta_{m, 11}^{(1)} \theta_{m, 22}^{(2)} - \theta_{m, 12}^{(2)} \theta_{m, 21}^{(1)}}. \tag{3.7} \]

Likewise,

\[ Q_{n, 21}^{m(1)} = -\frac{\lambda_1^{-1} - \bar{\lambda}_1^{-1}}{\theta_{m, 11}^{(1)} \theta_{m, 22}^{(2)} - \theta_{m, 12}^{(2)} \theta_{m, 21}^{(1)}}. \tag{3.8} \]
Similarly, we have

\[
\begin{align*}
Q_{n,11}^{(1)} &= -\lambda_1^{-1}\theta_{n,11}^{(1)} + \theta_{n,22}^{(2)} - \lambda_1^{-1}\theta_{n,12}^{(2)}\theta_{n,21}^{(1)}, \\
Q_{n,22}^{(1)} &= -\lambda_1^{-1}\theta_{n,11}^{(1)} + \theta_{n,22}^{(2)} - \lambda_1^{-1}\theta_{n,12}^{(2)}\theta_{n,21}^{(1)}. \\
\end{align*}
\]

(3.9)

To obtain an explicit form of the soliton solution for the general \( N \times N \) case, let us take \( U_0 \equiv U_n = i\begin{pmatrix} c_1 & \cdots & c_N \end{pmatrix} \), \( a_n^m = 0 \), \( b_n^m = 1 \) as a seed solution, where \( c_i \) are real constants and \( \text{Tr}(U_n) = 0 \), so that solution to the Lax pair (1.1)-(1.2) is given by

\[
\Phi_n = \begin{pmatrix} Z_{n,p}^m & O \\ O & Z_{n,N-p}^m \end{pmatrix},
\]

(3.10)

where \( Z_{n,p}^m = \text{diag}(\zeta_1(\lambda), \ldots, \zeta_p(\lambda)) \), \( Z_{n,N-p}^m = \text{diag}(\zeta_{p+1}(\lambda), \ldots, \zeta_N(\lambda)) \) are \( p \times p \) and \( (N-p) \times (N-p) \) matrices respectively, whereas \( n \) and \( m \) appearing in the latter expressions, denote the discrete indices. And

\[
\zeta(\lambda) = (1 + i c_i \lambda)^n \left( 1 - ihc_i^{-1} \frac{\lambda}{1 - \lambda^2} + h \frac{\lambda^2}{1 - \lambda^2} \right)^m.
\]

(3.11)

For the matrix of size \( 2 \times 2 \), take \( U_0 \equiv U_n = i\begin{pmatrix} c & -c \end{pmatrix} \), \( a_n^m = 0 \), \( b_n^m = 1 \) as the seed solution, so that a trivial calculation yields a matrix solution \( \Phi_n^m \) of the Lax pair (1.1)-(1.2), given by

\[
\Phi_n^m = \begin{pmatrix} \zeta(\lambda) \\ \overline{\zeta}(\lambda) \end{pmatrix}, \quad \zeta(\lambda) = (1 + i c \lambda)^n \left( 1 - ihc^{-1} \frac{\lambda}{1 - \lambda^2} + h \frac{\lambda^2}{1 - \lambda^2} \right)^m.
\]

(3.12)

From equation (3.12), the \( 2 \times 2 \) matrix \( \Theta_n^m \), as a particular solution to the Lax pair (1.1)-(1.2) can be constructed as follows

\[
\Theta_n^m = (\Phi_n^m(\lambda_1) \mid 1), \ (\Phi_n^m(\lambda_1) \mid 2)) = \begin{pmatrix} \zeta(\lambda_1) & -\zeta(\lambda_1) \\ \overline{\zeta}(\lambda_1) & \overline{\zeta}(\lambda_1) \end{pmatrix}.
\]

(3.13)

On substituting the matrix \( \Theta_n^m \) in equation (3.4) with (3.7)-(3.9), we obtain the expression of the matrix \( Q_n^{(1)} \), given by

\[
Q_n^{(1)} = \frac{1}{\lambda_n^{(1)} \lambda_n^{(-1)}} \begin{pmatrix} \lambda_n^{(1)} \lambda_n^{(1)} & \lambda_n^{(-1)} \lambda_n^{(-1)} \\ \lambda_n^{(-1)} \lambda_n^{(1)} & \lambda_n^{(-1)} \lambda_n^{(-1)} \end{pmatrix},
\]

(3.14)

where

\[
\lambda_n^{(1)} = (1 \pm ic \lambda_1)^n \left( 1 - ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2} \right)^m, \quad (1 \pm ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2})^m.
\]

\[
\lambda_n^{(-1)} = (1 \pm ic \lambda_1)^n \left( 1 - ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2} \right)^m, \quad (1 \pm ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2})^m.
\]

\[
\lambda_n^{(1)} = (1 \pm ic \lambda_1)^n \left( 1 - ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2} \right)^m, \quad (1 \pm ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2})^m.
\]

\[
\lambda_n^{(-1)} = (1 \pm ic \lambda_1)^n \left( 1 - ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2} \right)^m, \quad (1 \pm ihc^{-1} \frac{\lambda_1}{1 - \lambda_1^2} + h \frac{\lambda_1^2}{1 - \lambda_1^2})^m.
\]
From (2.19) and (3.14), we present a one-soliton solution given by

\begin{equation}
U_n^m[1] = \begin{pmatrix} u_n^m & v_n^m(+) \\ v_n^m(-) & -u_n^m \end{pmatrix},
\end{equation}

where

\begin{equation}
u_n^m = ic + (\lambda_1^{-1} - \bar{\lambda}_1^{-1}) \frac{\chi_{n+1}^m(+)\chi_n^m(-) - \chi_{n+1}^m(-)\chi_n^m(+)\chi_{n+1}^m(+) + \chi_{n+1}^m(-)\chi_n^m(+)\chi_{n+1}^m(+) + \chi_{n+1}^m(-)}{\chi_{n+1}^m(+) + \chi_{n+1}^m(-)},
\end{equation}

\begin{equation}
v_n^{m\pm} = (\lambda_1^{-1} - \bar{\lambda}_1^{-1}) \frac{\gamma_{n+1}^m(\pm)\chi_n^m(+) + \chi_n^m(-) - \gamma_n^m(\pm)\chi_{n+1}^m(+) + \chi_{n+1}^m(-)}{\chi_{n+1}^m(+) + \chi_{n+1}^m(-)}.
\end{equation}

Figure 1: Propagation of discrete one-soliton solution (3.16) with the choice of parameters: $c = -0.3$, $\lambda_1 = 1.2i$.

For two soliton, take the matrices $\mathcal{I}^{(2)}$, $\times^m_{n,1}$, $\Theta^m_{n,2}$, $\Lambda_1$ and $\Lambda_2$ to be

\begin{equation}
\mathcal{I}^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},
\end{equation}

\begin{equation}
\Theta^m_{n,1} = \begin{pmatrix} \theta^m_{n,11}^{(1)} & \theta^m_{n,12}^{(2)} \\ \theta^m_{n,21}^{(1)} & \theta^m_{n,22}^{(2)} \end{pmatrix}, \quad \Theta^m_{n,2} = \begin{pmatrix} \theta^m_{n,11}^{(3)} & \theta^m_{n,12}^{(4)} \\ \theta^m_{n,21}^{(3)} & \theta^m_{n,22}^{(4)} \end{pmatrix},
\end{equation}

\begin{equation}
\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \bar{\lambda}_1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \bar{\lambda}_2 \end{pmatrix}.
\end{equation}
so that the matrices $G_n^m$ and $\tilde{G}_n^m$ become

$$
G_n^m = \begin{pmatrix}
\Theta_{n,1}^m \Lambda_1^{-1} & \Theta_{n,2}^m \Lambda_2^{-1} \\
\Theta_{n,1}^m \Lambda_1 & \Theta_{n,2}^m \Lambda_2
\end{pmatrix} = \begin{pmatrix}
\theta_{n,11}^m & \theta_{n,12}^m \\
\theta_{n,21}^m & \theta_{n,22}^m
\end{pmatrix},
$$

$$
\tilde{G}_n^m = \begin{pmatrix}
\Theta_{n,1}^m \Lambda_1^{-2} & \Theta_{n,2}^m \Lambda_2^{-2} \\
\Theta_{n,1}^m \Lambda_1 & \Theta_{n,2}^m \Lambda_2
\end{pmatrix} = \begin{pmatrix}
\lambda_1^{-2} \theta_{n,11}^m & \lambda_1^{-2} \theta_{n,12}^m \\
\lambda_1^{-2} \theta_{n,21}^m & \lambda_1^{-2} \theta_{n,22}^m
\end{pmatrix},
$$

The two-fold scalar solutions $u_n^m[2]$ and $v_n^m[2]$ are given by

$$
u_n^m[2] = -\left(Q_{n+1,11}^m - Q_{n,11}^m\right), \quad \text{(3.19)}
$$

$$
u_n^m[2] = -\left(Q_{n+1,12}^m - Q_{n,12}^m\right). \quad \text{(3.20)}
$$

By using (3.23), one can compute the matrix elements $Q_{n,11}^m$, $Q_{n,12}^m$ as follow

$$
Q_{n,11}^m = \begin{pmatrix}
\theta_{n,11}^m & \theta_{n,12}^m \\
\theta_{n,21}^m & \theta_{n,22}^m
\end{pmatrix}
$$

$$
= \det \begin{pmatrix}
\bar{\lambda}_1^{-2} \theta_{n,11}^m & \bar{\lambda}_1^{-2} \theta_{n,12}^m \\
\lambda_1^{-2} \theta_{n,21}^m & \lambda_1^{-2} \theta_{n,22}^m
\end{pmatrix}, \quad \text{(3.21)}
$$

Similarly

$$
Q_{n,12}^m = \begin{pmatrix}
\theta_{n,11}^m & \theta_{n,12}^m \\
\theta_{n,21}^m & \theta_{n,22}^m
\end{pmatrix}
$$

$$
= \det \begin{pmatrix}
\bar{\lambda}_1^{-2} \theta_{n,11}^m & \bar{\lambda}_1^{-2} \theta_{n,12}^m \\
\lambda_1^{-2} \theta_{n,21}^m & \lambda_1^{-2} \theta_{n,22}^m
\end{pmatrix}. \quad \text{(3.22)}
$$
The graphical representation of discrete soliton solution (3.19) of the time-discrete GLHM model has been depicted in Figure 2-3.

To obtain three-soliton solution, we take three particular matrix solutions $\Theta_{n,k}^m$ corresponding to eigenvalue matrices $\Lambda_k$, $(k = 1, 2, 3)$. Figures 2-3 describes the interactions of two discrete solitons with their own lumps of energies moving with different velocities. These independent solitons propagate in space and keeps their profiles unchanged before and after collision. After collision, they separate and travel through each other independently and retains their amplitudes and velocities invariant. It is to be noted that, the
shape of these solitons are characterized under given parametric conditions. When these conditions changes, the structure of the soliton can also be changed. Similarly, by an application of DT K-times on a seed solution, one can obtain K-soliton (or multisoliton) solutions of time-discrete GLHM model.

4 Concluding remarks

In this paper, we have studied Darboux transformation and soliton solutions of time-discrete GLHM model. We have defined Darboux transformation on the solution to the Lax pair and the solutions of time-discrete GLHM model. The solutions are expressed in terms of quasideterminants. Finally, soliton solutions of the time-discrete GLHM model are calculated for the general and simple cases. We have computed expressions of one-soliton solution by expanding quasideterminants. Further, we remarks here that

- The time-discrete GLHM model studied in the present paper can also be served as a numerical scheme for the numerical simulation of the continuous time GLHM model.

- The work can be further extended by studying Hirota bilinearization of the time-discrete GLHM model.

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