Abstract

An 'isomorphism' between the 'moduli space' of star products on $\mathbb{R}^2$ and 'moduli space' of all formal Poisson structures on $\mathbb{R}^2$ is established.
Quantization of Poisson Structures on \( \mathbb{R}^2 \)

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This is a preliminary version of the paper!

The problem of quantization of Poisson structures originates from [1]. It is well known that any Poisson structure on a two-dimensional manifold is quantizable. In this paper we establish an 'isomorphism' between the 'moduli space' of star products on \( \mathbb{R}^2 \) and 'moduli space' of all formal Poisson structures on \( \mathbb{R}^2 \) by construction of a map from Poisson structures to star products. Certainly, this isomorphism follows from the Kontsevich formality conjecture [2]. Most likely, our map can be used as a first step in constructing the \( L_{\infty} \) quasiisomorphism in the formality conjecture for \( \mathbb{R}^2 \). The author would like to thank Boris Tsygan and Paul Bressler for the attention and helpful suggestions.

The set of all star-products \( S \) is acted upon by the group \( \mathcal{D} + \text{Diffeo}\mathbb{R}^2 \), where \( \mathcal{D} \) is the group of operators of the form \( 1 + hD_1 + h^2D_2 + \ldots \) with \( D_k \) to be arbitrary differential operators. The set of all formal Poisson structures \( \mathcal{P} \) consists of formal series in \( h \) with bivector fields as the coefficients. Formal Poisson structures are acted upon by the group \( \text{Diffeo}\mathbb{R}^2 + \exp(h\text{Vect}([h])) \), where \( \text{Vect} \) is the Lie algebra of vector fields on \( \mathbb{R}^2 \). These actions define equivalence relations. We want to have a pair of maps \( f_1 : S \to \mathcal{P} \) and \( f_2 : \mathcal{P} \to S \) such that

\[
\begin{align*}
  f_1 \circ f_2(x) &\equiv x & f_2 \circ f_1(x) &\equiv x, \\
  x \equiv y &\Rightarrow f_{1,2}x \equiv f_{1,2}y.
\end{align*}
\]

By a map from \( S \) we mean a differential expression in terms of the coefficients of the bidifferential operators corresponding to the star products. Maps from \( \mathcal{P} \) are defined similarly.

We can replace \( S \) by a subspace. Let \( P, Q \) be a non degenerate pair of (real) polarizations of \( \mathbb{R}^2 \). Define a subset \( S_{P,Q} \) of \( S \) in the following...
way \( m \in \mathcal{S}_{P,Q} \) iff \( m(f, g) = fg \) if \( f \) is constant along \( P \) or \( g \) is constant along \( Q \).

**PROPOSITION 1** Let \( x, y \) be a nondegenerate coordinate system on \( \mathbb{R}^2 \) such that \( x \) is constant along \( Q \) and \( y \) is constant along \( P \). Then there exists a unique map \( \mathcal{S} \to \mathcal{D} : m \mapsto U(m) = 1 + hV(m) \) such that

1) \( m_{P,Q}(m) = U^{-1}(m(Uf, Ug)) \in \mathcal{S}_{P,Q} \)

2) \( Ux = x, \ Uy = y, \ U1 = 1. \)

\( U \) is uniquely defined by the condition \( U(x^m \ast y^n) = x^m y^n \) (where star denotes the star product \( m \)).

We denote by \( m_{P,Q} : \mathcal{S} \to \mathcal{S}_{P,Q} \) the map which sends \( m \) to \( m_{P,Q}(m) \). Further, \( x, y \) will mean the same as in Proposition 1. Thus, it is enough to find maps \( p_1 : \mathcal{S}_{P,Q} \to \mathcal{P} \) and \( p_2 : \mathcal{P} \to \mathcal{S}_{P,Q} \) with the same properties as \( f_1, f_2 \) have. Indeed, put \( f_2 = i \circ p_2 \) and \( f_1 = p_1 \circ m_{P,Q} \) (here \( i : \mathcal{S}_{P,Q} \to \mathcal{S} \) is the injection).

The following theorem gives an explicit construction for \( p_2 \) which appears to be a bijective map so that we can put \( p_1 = p_2^{-1} \). Denote by \( C_P \) (resp. \( C_Q \)) the space of functions, constant along \( Q \) (respectively \( P \)). Denote by \( V_P \) (resp. \( V_Q \)) the space of vector fields preserving the polarizations and tangent to \( P \) (resp. \( Q \)). Denote by \( \mathcal{D}_P \) the subalgebra of the algebra of the differential operators consisting of the operators \( D \) such that \( D(C_Q) \subseteq C_Q \) and \( D(fg) = fD(g) \) if \( f \in C_P \). Denote by \( \mathcal{D}_Q \) the same algebra, where \( P \) and \( Q \) are interchanged. In the coordinates \( x, y \) we have \( C_P = \{ f(x) \} \), \( V_P = \{ f(x)\partial_x \} \), \( \mathcal{D}_P = \sum f_i(x)\partial_x^i \) and the same things with \( P \) replaced by \( Q \) and \( x \) replaced by \( y \). Denote by \( \overline{\mathcal{D}_P} \) (respectively \( \overline{\mathcal{D}_Q} \)) the subring of \( \mathcal{D}_P \) (respectively \( \mathcal{D}_Q \)) consisting of the operators which annihilate constant functions.

Note that the space of bivector fields is isomorphic to \( V_P \otimes \mathbb{R} V_Q \). Let \( \mathcal{D}_{P,k} \) be the space of maps \( V_P^k \to \overline{\mathcal{D}_P} \) (which are differential operators in terms of the coefficients).

**THEOREM 1** There exists a unique sequence \( c_k \in \mathcal{D}_{P,k} \otimes \mathcal{D}_{Q,k} \), \( k = 0, 1, 2, \ldots \) \( c_k = \sum a_k \otimes b_k \), \( c_0(X, Y) = 1 \otimes 1 \) such that for any bivector field \( \Psi = \sum X_i \wedge Y_i, \ X_i \in V_P, \ Y_i \in V_Q \), the formula

\[
m(\Psi, P, Q, f, g) = fg + \sum_{k, i_1, \ldots, i_{k+1}} h^{k+1} L_{X_{i_1}} \{ a_k^n(X_{i_2}, X_{i_3}, \ldots, X_{i_{k+1}})f \} L_{Y_{i_1}} \{ b_k^n(Y_{i_2}, Y_{i_3}, \ldots, Y_{i_{k+1}})g \}
= \sum_k m_k(f, g).
\]
gives a star-product.

**Remark 1.** The ansatz for the formula originates from the following observation. Given a product \( m \) from \( S_{P,Q} \), consider the ‘set of zeros’ of the \( m(x, y) \). One can easily show that this set (up to 'biregular isomorphisms') is an invariant of the star product. Therefore, it is natural to require that \( m(\Psi, x, y) \) would be divisible by \( \Psi \).

To prove this theorem we need some preparation. Let us pass to the coordinates \( x, y \). Then
\[
X_i = \xi_i \partial_x, \quad Y_i = \eta_i \partial_y,
\]
where \( A_n^k(\xi_1, \ldots, \xi_k, f) \) are polydifferential operators depending on the derivatives of \( \xi \) and \( f \) with respect to \( x \). Similarly, \( b_n^k(Y_1, \ldots, Y_k, g) = B_n^k(\eta_1, \ldots, \eta_k, g) \). Our task is to solve the recurrence equation
\[
b m_k = -\frac{1}{2} \sum_{i=1}^{k-1} [m_i, m_{k-i}], \tag{4}
\]

such that all \( m_k \) are of the form in (3). Here \( b \) is the Hochschild differential. Let us specify the meaning of the conditions imposed by (3). First, note that all \( m_i \) as well as \([m_i, m_j]\) belong to a subcomplex \( K^* \) of the Hochshild complex \( C^* (\tilde{A}, A) \) (\( A = C^\infty (R^2) \) and \( \tilde{A} = A/R \)) such that \( K^0 = K^1 = 0; K^i = \) polydifferential operators \( D(f_1, \ldots, f_i) \), such that \( D(\phi(y)f_1, \ldots, f_i\psi(x)) = \phi(y)\psi(x)D(f_1, \ldots, f_i) \). That is, we take the cochains that only depend on \( \partial^m_x f_1, \partial^k_y f_i \) and derivatives of \( f_2, \ldots, f_{i-1} \).

**Lemma 1** The cohomology of \( K^* \) is generated over \( C^\infty (R^2) \) by the class of \( \partial_x \otimes \partial_y \).

Now, let us specify exactly the space in which all \( m_k \) should be. Note that
\[
m_k = \sum_{n, i_1, \ldots, i_k} \xi_{i_1} \partial_x \circ A_{k-1}^n(\xi_i, \ldots, \xi_{i_k}) \otimes \eta_{i_1} \partial_y \circ B_{k-1}^n(\eta_{i_2}, \ldots, \eta_{i_k}).
\]

Denote by \( \mathcal{E}_P \in D_{P,k} \) the space of operators of the form \( D = \partial_x \circ D_1 \) with \( D_1 \in D_{P,k} \). Certainly, \( \mathcal{E}_P \) depends on a choice of the coordinate \( x \). Define \( \mathcal{E}_Q \) in the same fashion.

Recall that \( \Psi = \sum_i \xi_i \eta_i \partial_x \wedge \partial_y \). Put \( \phi = \sum_i \xi_i \eta_i \). Then our theorem means exactly that
\[
m_k = \phi K, \tag{5}
\]

where \( K \in \mathcal{E}_P \otimes \mathcal{E}_Q \).
Let us investigate how $\mathcal{E}_P$ and $\mathcal{E}_Q$ interact with the Hochshild differential. Let $A$ be the subalgebra of functions depending on the derivatives of $\xi_1, \ldots, \xi_k$. Put $L^i_P = D^i \otimes A^k$ and the similar for $L^i_Q$. Then we have the Hochshild differential $b: L^i_P \to L^{i+1}_P$.

**Lemma 2** The sequence

$$0 \to \mathcal{E}_P \xrightarrow{b} L^2_P \xrightarrow{b} L^3_P$$

is exact. This is also true if we replace $P$ by $Q$.

**Proof.** Since the one-dimensional Hochshild complex is acyclic for dimensions bigger than 1, it suffices to show that $L^1_P = D^1 \otimes A^k$ is equal to $\mathcal{E}_P \oplus D^1_P$, where $D^1_P$ is the subset of operators of order 1 from $D^1_P$. But this splitting is given by the Euler-Lagrange operator $E: D_P \to D^1_P$, $E(\sum a_i \partial_i \xi) = \left( \sum (-1)^{i-1} \partial_i^{i-1} a_i \right) \partial_i$, since the kernel of $E$ is exactly $\mathcal{E}_P$.

**Proof of the Theorem 1.** Suppose we have found $m_1, \ldots, m_1$. Show that we can solve (4) for $m_k$ so that it is of the form in (5).

Denote by $\mathcal{A}$ the right hand side of (4). Note that $bA = 0$ and $A \in K^3$. This means that $A = bS$, where $S \in K^2$. Note that

$$bK^2 \subset L^2_P \otimes L^1_Q \oplus L^1_P \otimes L^2_Q.$$  \hspace{1cm} (6)

Let us define a projector $p_P : K^3 \to L^2_P \otimes L^1_Q$. For this let us notice that the right hand side can be interpreted as a differential map from $\mathcal{C}_P \otimes \mathcal{C}_P \otimes \mathcal{C}_Q$ to $C^\infty \mathbb{R}^2$, (depending on $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k$), whereas the left hand side is a differential map from $\mathcal{C}_P \otimes C^\infty \mathbb{R}^2 \otimes \mathcal{C}_Q$ to $C^\infty \mathbb{R}^2$. Thus, $p$ can be defined as a restriction from $\mathcal{C}_P \otimes C^\infty \mathbb{R}^2 \otimes \mathcal{C}_Q$ to $\mathcal{C}_P \otimes \mathcal{C}_P \otimes \mathcal{C}_Q$. In the same way we can define the projector $p_Q$ onto the second summand in (6). Thus,

\begin{align*}
A &= (p_P + p_Q)A \\
&= \sum \phi \partial_x \circ a^n_p(\xi_{i_2}, \ldots, \xi_{i_n}) \circ \xi_{j_1} \partial_x \circ a^m_q(\xi_{j_2}, \ldots, \xi_{j_q}) \\
&\quad \otimes \eta_{i_1} \partial_y b^n_q(\eta_{j_2}, \ldots, \eta_{j_n}) \otimes \partial_y b^m_q(\eta_{i_1}, \ldots, \eta_{i_n}) \\
&\quad - \phi \partial_x a^n_q(\xi_{i_2}, \ldots, \xi_{i_n}) \circ \xi_{j_1} \partial_x a^m_q(\xi_{j_2}, \ldots, \xi_{j_q}) \\
&\quad \otimes \eta_{j_1} \partial_y b^n_q(\eta_{i_2}, \ldots, \eta_{i_n}) \otimes \eta_{j_1} \partial_y b^m_q(\eta_{j_2}, \ldots, \eta_{j_q}).
\end{align*}
From this we deduce that $A = \phi T$, where $T \in L^2_P \otimes \mathcal{E}_Q \oplus \mathcal{E}_P \otimes L^2_Q$. Set $T = t_1 + t_2$, where $t_1 \in L^2_P \otimes \mathcal{E}_Q$ and $t_2 \in \mathcal{E}_P \otimes L^2_Q$. Since $bA = 0$, we see that $(b \otimes 1)t_2 = 0$. By Lemma 2, $t_1 = (bl_i) \otimes m_i$, where $l_i \in \mathcal{E}_P$, $m_i \in \mathcal{E}_Q$. Similarly, $t_2 = u_i \otimes bv_i$, where $u_i \in \mathcal{E}_P$, $v_i \in \mathcal{E}_Q$. Using the closeness of $A$, we have $-bl_i \otimes bm_i + bu_i \otimes bv_i = 0$. Since (by Lemma 2) $b : \mathcal{E}_P \to L^2$ is an inclusion, we have $c = l_i \otimes m_i = u_i \otimes v_i$ and $\phi bc = A$. Thus, we can put $m_k = \phi c$. This proves the existence. Let us prove the uniqueness. By Lemma 1, the ambiguity in the choice of $m_k$ is of the form $f\partial_x \otimes \partial_y$. This should be of the form $\phi K$ with $K \in \mathcal{E}_P \otimes \mathcal{E}_Q$, which is impossible, since all non-zero operators in $\mathcal{E}_P, \mathcal{E}_Q$ have order greater than 1. This proves the theorem.

The following corollary follows immediately from the uniqueness of the constructed star-product-product.

**COROLLARY 1** 1) The map $(\Psi, P, Q) \mapsto m(\Psi, P, Q)$ is equivariant with respect to $\text{Diff}(\mathbb{R}^2)$. 2) All operators $a_k^i, b_k^i$ are invariant with respect to the natural action of the group $\text{Diff}(\mathbb{R}^1)$.

It looks very plausible that all the operators $a_k(X_1, X_2, \ldots, X_k)$ are just linear combinations of $L_{X_{i_1}} \ldots L_{X_{i_k}}$, where $(i_1, \ldots, i_k)$ is a permutation, and the same for $b_k$. This is correct at least for $k \leq 4$.

**PROPOSITION 1** The constructed map $p_1 : P \to S_{P, Q}$ is a bijection. The inverse map $p_2 : S_{P, Q} \to P$ is a well defined map.

*Proof.* Consider the map $p_3 : S_{P, Q} \to P : m \mapsto \frac{1}{h}(m(x, y) - m(y, x))\partial_x \wedge \partial_y$. This is a well defined map, and it is not hard to see that $p_3$ is injective. Note that $p_3 \circ p_1(\psi) = \psi + O(h)$, hence, $p_3 \circ p_1$ is invertible, and we can put $p_2 = (p_3 \circ p_1)^{-1} \circ p_3$.

Let us check the properties (7). Let $D_t = \exp(tX)$ be a one-parameter local Lie group of diffeomorphisms corresponding to a vector field $X$. Then $D_t$ acts naturally on $S$. Using Corollary from the Theorem 1 and Proposition, we can write

$$p_1(D_t\Psi) = D_t(m(\psi, D_t^{-1}P, D_t^{-1}Q)) = m_{P, Q}(m(\psi, D_t^{-1}P, D_t^{-1}Q)) = m(\chi_t, P, Q),$$

where $\chi_t = p_2m(\psi, D_t^{-1}P, D_t^{-1}Q)$. Put $\delta_X(\Psi) = \frac{d}{dt}\chi_t|_{t=0}$. This is a well defined map, linear in $X$. Furthermore, $\delta_X = O(h)$. It is enough to prove the following.
**Theorem 2.** There exists a linear differential operator $A : \text{Vect}[[h]] \rightarrow \text{Vect}[[h]]$, depending on $\Psi \in P$, such that

$$\delta_X(\Psi) = L_{A(X)} \Psi.$$

Let us explain why it is enough. First, disregarding, if needed, some terms, we can make $A(X)$ to be $O(h)$. Then (\ref{eq:1}) can be rewritten as $p_1(\Psi + tL_X \Psi) \equiv p_1(\Psi + tL_{A(X)} \Psi) \equiv p_1(\Psi) + o(t)$. Since $A(X) = O(h)$, the equation $Y = X - A(X)$ is solvable for all $Y$ and we deduce that $p_1(D_t \Psi) \equiv p_1(\Psi)$. Since $p_1$ is automatically equivariant with respect to the reflection $(x, y) \rightarrow (-x, y)$, this would mean that $p_1$ is equivariant with respect to the whole group of diffeomorphisms. Thus, $p_1(x) \equiv p_1(y)$. Suppose that $m_1 \equiv m_2$. Then $p_2(m_{1,2}) = p_2(\mu_{1,2})$, where $\mu_{1,2} = m_{P,Q}(m_{1,2})$. Then $\mu_2(f, g) = UD\mu_1((UD)^{-1}f, (UD)^{-1}g)$, where $D$ is a formal diffeomorphism (that is, an element of Diffo + exp hVect[[h]]) and $U$ is a differential operator $U = 1 + hV$, satisfying the condition 1 of Proposition 1. Thus, $p_2(\mu_2) = p_2(D^{-1}\mu_1)$. If $D^{-1} = \exp(tX)$, and $p_2(\mu_1) = \Psi$, then $p_2\mu_2 = \chi_t \equiv \Psi$. We only need to check the equivariance with respect to the reflection $(x, y) \rightarrow (-x, y)$, which immediately follows from the Corollary 1. Thus, we only need to prove Theorem 2. We need to make a reduction. Denote by $\mathcal{V}$ the space of all linear over $\mathbb{R}[[h]]$ differential operators $\text{Vect}[[h]] \rightarrow \text{Vect}[[h]]$, depending on $\Psi \in P$. Let $\Psi = \phi \partial_x \wedge \partial_y$. Denote $\mathcal{V}_r = \overline{\mathcal{V}[\phi^{-1}]}$, where the bar means the completion in the $h$-adic topology.

**Proposition 2** It is enough to find $A$ in $\mathcal{V}_r$.

*Proof.* Suppose we have found such an $A$. Let $l$ be the least degree in $h$, where $A$ has singularity. Let $N$ be the least positive integer such that $B = \phi^N A$ does not have singularities up to $h^{l+1}$. Further we will write $a \equiv b$ if $\phi^{N-1}(a - b)$ does not have singularities up to $h^{l+1}$. Put $B = U\partial_x + V\partial_y$. Then

$$0 = L_A \Psi \equiv (1 + n)U\phi_x + V\phi_y(\partial_x \wedge \partial_y).$$

Hence,

$$B \equiv \frac{W(\phi_x \partial_y - \phi_y \partial_x)}{\phi^n} = \frac{1}{N} \left\{ \frac{W}{\phi^N} : \Psi \right\},$$

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where \{,\} is the Schouten bracket. Put \( A_1 = A - \frac{1}{N} \{ \frac{W}{\phi}, \Psi \} \). Then \( L_{A_1(X)} \Psi = L_{A(X)} \Psi \) and \( A_1 \) has no singularities of order \( N \) up to \( h^{l+1} \).

Iterating this procedure, we will get rid of all the singularities.

Let us make the following reductions. Similarly to the differential operators on vector fields put \( F = \text{Fun}(P)[\phi^{-1}] \), where \( \text{Fun}(P) \) is the space of functions on \( P \). For \( F \in F \) put \( \delta_X F(\Psi) = \frac{d}{dt} F(\Psi + t\delta_X \Psi) |_{t=0} \). Put \( \omega = \frac{1}{\phi} dx \wedge dy \). Then we only need to prove that

\[
\delta_X \omega = d\theta(\Psi, X).
\]

for some 1-form \( \theta \). It is clear that it is enough to prove this for some form

\[
\Omega = \omega + d\alpha. \quad (8)
\]

Let us find a suitable form \( \Omega \). Also, we can assume that out vector field \( X \) is tangent to \( Q \) (since any vector field is a sum of a vector field tangent to \( P \) and a vector field tangent to \( Q \)).

**PROPOSITION 3** Let \( z \) be some function on \( \mathbb{R}^2 \) such that \((x, z)\) form a nondegenerate coordinate system. Put \( \Psi = \phi \partial_x \wedge \partial_z \).

For any \( m = m(\Psi, P, Q) \) the differential operator \( \frac{1}{h} \text{ad} x \) can be represented as \( \phi \partial_z (1+S(m)\circ \partial_z) \) for some well defined differential operator \( S = S(\Psi, P, Q) = O(h) \).

2) There exists a unique \( f = f(x, z, P, Q, \Psi) \in F \) such that

\[
\phi(1 + \partial_z \circ S)f = 1. \quad (9)
\]

3) Put \( \Omega = \Omega(x, z, P, Q, \Psi) = f dx \wedge dz \). Then (8) holds.

**Proof.** 1) This immediately follows from (8). The second statement holds because \( S = O(h) \). The last statement is true because (9) can be rewritten as

\[
f = 1/\phi + \partial_z \tau \quad (10)
\]

for some \( \tau \).

**Remark.** If we had an antiderivative \( F \) of \( f \), such that \( F_y = f \), then it would be \([x, F] = h \) and \( \Omega = dx \wedge dF \). Thus, \( \Omega \) is nothing else but the Berezin curvature (8). Also, it is not hard to prove that our construction does not depend on a choice of \( z \).
Now we are ready to prove the invariance. Formula (7) can be rewritten as

\[ m_{P,Q_t} m(\chi_t, P, Q) = m(\Psi, P, Q_t), \]

where \( Q_t = D_t^{-1}Q \). Let \( A_t \circ \partial_z = 1/h \text{ad}_{m(\chi_t, P, Q)} x \), \( B_t \circ \partial_z = 1/h \text{ad}_{m(\Psi, P, Q_t)} x \).

Recall that \( m_{P,Q_t} \) is a conjugation with respect to some operator \( U_t = 1 + hV_t \), and \( U_t(x^n) = x^n \) (see (3)).

Therefore, \( U_t = 1 + hW_t \partial_z \). Also, \( B_t \partial_z = U_t A_t(\partial_z) U_t^{-1} \) and \( A_t f(\chi_t, P, Q) = B_t(f(\Psi, x, z, P, Q_t)) = 1 \).

One can check that \( f(\Psi, x, z, P, Q_t) = f(\chi_t, P, Q) + \partial_z \circ hW_t f(\chi_t, P, Q) \).

Using (10), we immediately get \( \omega = \chi_t^{-1} + \partial_z \tau_t dx \wedge dz \), where \( \chi_t^{-1} = (\chi_t, dx \wedge dy)^{-1} dx \wedge dy \). Therefore, \( \delta \chi \omega = d\alpha \), where

\[ \alpha = \frac{d}{dt} d(\tau_t dx)|_{t=0}. \]

Which proves theorem 2.

References

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