Unique Deformation of Local Quantum Field Theory Resulting in Divergence-free Amplitudes

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Abstract. An essentially unique deformation of the product of quantum fields at the same spacetime point is obtained. It is proposed to replace local quantum field theory with another structure which uses a $*$-product. The resulting theory contains a fundamental length and is free from divergences. This provides the third deformation suggested by Faddeev.

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Quantum Field Theory (QFT) is plagued with the problem of divergences. The problem is with the product of local fields at the same spacetime point. In Bogoliubov’s program [2] the ambiguity in the time-ordered product at coincident spacetime point is made use of to get finite S-matrix elements consistent with Lorentz covariance, locality, causality and unitarity. In the class of so called renormalizable theories the finite S-matrix elements depend only on a finite number of independent parameters. Unfortunately Einstein gravity does not belong to this class. There is a widely held belief that the problems of divergences in local QFT and quantized gravity are inter-related. A correct quantization of gravity may automatically remove divergences of QFT. This is all the more probable because Newton’s gravitational constant provides for fundamental length scale that is the Planck length.

Faddeev [3] (see also [4]) has observed that the special theory of relativity and quantum mechanics which replaced Galilean relativity at classical mechanics can be interpreted as deformations [1], [5], [6] of the earlier structures. In the process two fundamental parameters \(c\) the velocity of light and \(\hbar\) the Planck constant enter physics. \(c\) provides a cutoff for velocities and \(\hbar\) for phase-space. Faddeev has proposed that another deformation bringing in a fundamental length is perhaps required. This will provide a system of fundamental or natural units for all dimensionful parameters involving mass, length and time.

In this article it is shown that there is an essentially unique deformation
of the product of fields at the same spacetime point preserving associativity. Surprisingly it turns out that the deformation parameter provides an in-built momentum cutoff. The proposal here provides a natural deformation of local QFT thus implementing Faddeev’s program.

There have been many attempts to quantize spacetime and include a fundamental length by making coordinates non-commutative [5]. However this procedure is not unique and has many ad hoc assumptions and has many difficulties. In contrast the approach here is to consider deformation of the ring of functions on spacetime. This is the dual object in the mathematical sense. It is well known that deformation is natural in this dual object. For example, in the case of quantum mechanics the deformation of Poisson bracket to Moyal bracket [1], [8] works with functions on phase-space.

As mentioned earlier the problem of divergences in QFT is related to the product of local fields at the same spacetime point. Consider now a possible deformation preserving the associativity of such products. As is conventional the deformed product is denoted by $\ast$. For functions of one variable the solution is

$$f(x) \ast g(x) = \sum_{n=0}^{\infty} \frac{L^{2n}}{n!} \frac{d^n f}{dx^n} \frac{d^n g}{dx^n}.$$

$$= f(x) e^{L^2 \frac{d}{dx} \frac{d}{dx}} g(x). \quad (1)$$

In order to prove the uniqueness and associativity consider first

$$e^{ikx} \ast e^{ik'x} = f(k, k') e^{i(k+k')x}. \quad (2)$$
The requirement of associativity is
\[ e^{ikx} * (e^{ik'x} * e^{ik''x}) = (e^{ikx} * e^{ik'x}) * e^{ik''x}. \] (3)

This implies that
\[ f(k, k' + k'')f(k', k'') = f(k, k')f(k + k', k''). \] (4)

This functional equation has unique nontrivial solution
\[ f(k, k') = e^{L^2k^2} \] (5)

where \( L^2 \) is some real or complex constant. By Fourier transforming \( f(x) \) and \( g(x) \) and using equation (5), equation (1) follows.

Now the results are applied to local QFT. For the present consider Euclidean fields. It is well known that once the Euclidean field theory is well defined the corresponding Minkowski theory can be got uniquely by analytic continuation. The product of local fields is now replaced by the \(*\)-product which has a straight-forward generalization to four dimensions.

\[ f(x) * g(x) = f(x)e^{L^2\eta_{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}g(x)}. \] (6)

Using the Fourier transforms,
\[ f(x) * g(x) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} \tilde{f}(k) \tilde{g}(k') e^{iL^2k.k'} e^{i(k+k').x}. \] (7)

In particular
\[ \int d^4x f(x) * g(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) \tilde{g}(-k) e^{-L^2k^2}. \] (8)
If \( L \) is chosen to be a real positive constant it follows that high frequency components of the quantum fields are cutoff through the deformation parameter \( L \). This is exactly in analogy with \( c \) which provides a cutoff for velocity in relativity and \( \hbar \) which provides a cutoff in quantum mechanics for phase-space. In this sense this proposal provides an exact realization of Faddeev’s program for obtaining a third deformation introducing a fundamental length into physical theories.

It is proposed that all local products of fields be replaced by \( \star \)-products. Thus the functional integral for a Euclidean scalar field theory is

\[
\int \! \mathcal{D} \phi e^{-\frac{1}{2} \int d^4x \partial_\mu \phi(x) \star \partial_\mu \phi(x) + m^2 \phi(x) \star \phi(x) + \lambda \phi(x) \star \phi(x) \star \phi(x) \star \phi(x)}
\]

(9)

Now all computations becomes straight-forward going to momentum space in using formula (8). In particular the free propagator is \( \frac{e^{-k^2 L^2}}{k^2 + m^2} \). This shows that an automatic cutoff is provided by \( L \). An added advantage of this scheme is that calculations can be made in the conventional way after the introduction of the \( \star \)-product.

A physical interpretation of the \( \star \)-product is now given. Note the exact analogy of the \( \star \)-product with Wick’s theorem.

\[
: e^{ik\phi} \star e^{ik'\phi} : = e^{-kk'\Delta} : e^{i(k+k')\phi} :
\]

(10)

where \( \Delta = \langle \phi \phi \rangle \) which is the covariance for the random fields \( \phi \). Thus the deformation appears to make \( x \) a random variable with covariance \( L^2 \).
In other words, $(\Delta x)^2 \geq L^2$. It is this graininess of spacetime at length scale $L$ which is providing an ultra-violet regularization. To illustrate this relationship between ordinary product and $\ast$-product note that

$$x_\mu \ast x_\nu = x_\mu x_\nu + \eta_{\mu\nu} L^2.$$  \hspace{1cm} (11)

Thus commutativity of the coordinates is not changed in contrast to other proposals but the light cone is smoothened out.

The fundamental length $L$ also enters quantum gravity and makes it meaningful. The invariance under general coordinate transformation plays a crucial role in the general theory of relativity. Note that this principle has to be drastically modified once ordinary products are replaced by $\ast$-products. There is an easy and systematic way of guessing what this new principle is to be. In the field theory approach pioneered by Feynman, Schwinger, Gupta, Thirring, Weinberg and Deser (see discussions in [7]) flat spacetime and Lorentz invariance continue to remain. It is merely required that there is a symmetric rank two tensor which couples to the energy-momentum tensor of all fields including itself. It is known that this gives the Einstein - Hilbert action of the interaction of these spin two fields. Thus general coordinate invariance and Einstein gravity are a consequence of QFT with spin two gravitons coupling to the energy momentum tensor in flat spacetime. The same strategy can be adopted in the present approach, however the local products should be replaced by $\ast$-products. A unique theory with a geometric interpretation will follow. This will be pursued elsewhere.
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