Stability of Hamiltonian relative equilibria in symmetric magnetically confined rigid bodies

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Misconceptions in the physics literature

- **Purely magnetic systems cannot exhibit stable behavior in the absence of other long-range forces.**

- Earnshaw’s Global Instability Theorem [Ear42] – one of the most profusely cited results in the physics literature concerning stability in magnetic systems [Bas06].

- Earnshaw’s theory concerns mainly point particles BUT was generalized during the last 170 years to a large variety of systems dealing with pure/combined confinement [SHG01] exhibiting both static and dynamic [Gin47, Tam79] solutions.

- Such extensions were in many occasions not rigorously proved.

- Experimental results by Meissner [MO33], Braunbeck [Bra39], Arkadiev-Kapitsa [Ark47] (levitation with Type I superconductors), Brandt [Bra89], [Bra90] (levitation with Type II superconductors), or Harrigan [Har83] (levitron) raised questions as to the universal applicability of Earnshaw’s theory.
History of the orbitron

- 1941–1947 – results by Tamm and Ginzburg: in the case of two interacting magnetic dipoles, orbital motion is impossible using both classical and quantum mechanical descriptions [Gin47].

- Nevertheless, there exist experimental prototypes where a small permanent magnet exhibited quasiorbital motion around another fixed permanent magnet for up to six minutes [Koz74]

- The model was introduced in the 1970s; the first theoretical and experimental results were presented in [Koz74, Koz81]; numerical simulations were carried out in [Zub08, Gry09, Zub10]. These works do not contain a complete mathematical proof of nonlinear stability due to the limitations of the classical Lyapunov type approach that was followed.

- We present the positive stability results obtained for dynamic solutions of the orbitron.

- Other similar configurations that have been experimentally observed to be stable can be rigorously proved to have this property despite widespread beliefs in the opposite direction.
Work flow

- Geometric description of the orbitron: phase space, equations of motion, symmetries, and associated momentum map
- Characterization of the relative equilibria of the standard and generalized orbitron: regular and singular branches of equatorial relative equilibria
- Stability study: we apply the energy–momentum method to study the stability of the solutions
- Spectral stability/instability analysis: we introduce new linear methods to assess the sharpness of the stability conditions, apply them to study the nonlinear stability/instability of the relative equilibria
The orbitron

Consider a small axisymmetric magnetized rigid body (permanent magnet or a current-carrying loop) with magnetic moment $\mu$, in the permanent magnetic field created by two fixed magnetic poles/“charges” placed at distance $h$ [Smy39] in the absence of gravity.
Phase space

- The configuration space of the orbitron is $SE(3) = SO(3) \times \mathbb{R}^3$
- The orbitron is a simple mechanical system
- Phase space is the cotangent bundle $T^*SE(3)$ of its configuration space $SE(3)$ endowed with the canonical symplectic structure $\omega$ obtained as minus the differential of the corresponding Liouville one form
- Left/right trivializations provide an identification of the bundle $T^*SE(3)$ with the product $SE(3) \times \mathfrak{se}(3)^*$. We work in body coordinates and denote by $(A, x)$ the elements of $SE(3) = SO(3) \times \mathbb{R}^3$ and by $((A, x), (\Pi, p))$ those of $T^*SE(3) \simeq SE(3) \times \mathfrak{se}(3)^*$.
The Hamiltonian of the orbitron is

\[ h((A, x), (\Pi, p)) = T(\Pi, p) + V(A, x) \]  

(1)

with

\[ T(\Pi, p) := \frac{1}{2} \Pi^T \Pi_{\text{ref}}^{-1} \Pi + \frac{1}{2M} \|p\|^2, \]  

(2)

\[ V(A, x) := -\mu \langle B(x), Ae_3 \rangle, \]  

(3)

where \( M \) is the mass of the axisymmetric magnetic body, the reference inertia tensor \( \Pi_{\text{ref}} = \text{diag}(l_1, l_1, l_3) \), \( x = (x, y, z) \in \mathbb{R}^3 \), \( \mu \) is the magnetic moment of the axisymmetric rigid body/dipole, and \( B(x) \) is the strength of the magnetic field created by two magnetic poles/“charges” \( \pm q \) placed at the points \((0, 0, h)\) and \((0, 0, -h)\), \( h > 0 \), that is,

\[ B(x) = \frac{\mu_0 q}{4\pi} \left( \frac{x}{D(x)_+^{3/2}} - \frac{x}{D(x)_-^{3/2}}, \frac{y}{D(x)_+^{3/2}} - \frac{y}{D(x)_-^{3/2}}, \frac{z - h}{D(x)_+^{3/2}} - \frac{z + h}{D(x)_-^{3/2}} \right), \]  

(4)

with \( D(x)_+ = x^2 + y^2 + (z - h)^2 \), \( D(x)_- = x^2 + y^2 + (z + h)^2 \), and \( \mu_0 \) the magnetic permeability of vacuum.
**The standard and generalized orbitron**

**Definition**

A small axisymmetric magnetized rigid body subjected to a external magnetic field of the form specified in (4) is called a **standard orbitron**.

The external magnetic field $B$ in (4) has the following symmetry properties, namely:

(i) Equivariance with respect to rotations $R^Z_{\theta_S}$ around the $OZ$ axis:

$$B(R^Z_{\theta_S}x) = R^Z_{\theta_S}B(x) \text{ for } \theta_S \in \mathbb{R}.$$ 

(ii) Behavior with respect to the mirror transformation $(x, y, z) \mapsto (x, y, -z)$ according to the prescription:

$$B_x(x, y, z) = -B_x(x, y, -z), B_y(x, y, z) = -B_y(x, y, -z), B_z(x, y, z) = B_z(x, y, -z).$$

**Definition**

A small axisymmetric magnetized rigid body subjected to the influence of an arbitrary magnetic field in the magnetostatic approximation in a domain free of other magnetic sources that satisfies these symmetry properties is called a **generalized orbitron**.
The equations of motion of the orbitron are determined by Hamilton’s equations:

\[
\begin{align*}
\dot{A} &= A\hat{\Pi}_{\text{ref}}^{-1}\Pi, \\
\dot{x} &= \frac{1}{M}Ap, \\
\dot{\Pi} &= \Pi \times \Pi_{\text{ref}}^{-1}\Pi + A^{-1}B(x) \times e_3, \\
\dot{p} &= p \times \Pi_{\text{ref}}^{-1}\Pi + \mu A^{-1}DB(x)^T Ae_3.
\end{align*}
\]

The symbol \(\hat{\Pi}_{\text{ref}}^{-1}\Pi\) stands for the antisymmetric matrix associated to the vector \(\Pi_{\text{ref}}^{-1}\Pi \in \mathbb{R}^3\) via the Lie algebra isomorphism

\[\hat{\cdot}: (\mathbb{R}^3, \times) \longrightarrow (\mathfrak{so}(3), [\cdot, \cdot])\]

and \(D\) for the differential.
Toral symmetry and associated momentum map

The axial symmetry of the magnetic rigid body + the rotational spatial symmetry of the external magnetic field w.r.t. rotations around OZ = toral symmetry. The action on $SE(3)$:

$$\Phi : \left( T^2 = S^1 \times S^1 \right) \times SE(3) \rightarrow SE(3)$$

$$((e^{i\theta_S}, e^{i\theta_B}), (A, x)) \mapsto (R^{Z}_{\theta_S}AR^{-Z}_{-\theta_B}, R^{Z}_{\theta_S}x).$$

(9)

The cotangent lift $\Phi$ is a canonical symmetry given by

$$\Phi : \left( T^2 = S^1 \times S^1 \right) \times T^*SE(3) \rightarrow T^*SE(3)$$

$$\left( (e^{i\theta_S}, e^{i\theta_B}), ((A, x), (\Pi, p)) \right) \mapsto \left( (R^{Z}_{\theta_S}AR^{-Z}_{-\theta_B}, R^{Z}_{\theta_S}x), (R^{Z}_{\theta_B}\Pi, R^{Z}_{\theta_B}p) \right),$$

that has an invariant momentum map associated $J : T^*SE(3) \rightarrow t^*$:

$$J ((A, x), (\Pi, p)) = (\langle AP + x \times Ap, e_3 \rangle, -\langle \Pi, e_3 \rangle).$$

(10)

The invariance of the Hamiltonian w.r.t. the cotangent toral action

$$h \circ \Phi_{(e^{i\theta_S}, e^{i\theta_B})} = h, \quad \text{for any} \quad (e^{i\theta_S}, e^{i\theta_B}) \in T^2.$$

By Noether’s Theorem [AM78], the level sets of $J$ are preserved by the associated Hamiltonian dynamics, i.e., if $F_t$ is the flow of the vector field $X_h$ then $J \circ F_t = J$ for any $t$. 

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Stability of Hamiltonian relative equilibria in symmetric magnetic fields
Relative equilibria: setup and background

Consider a vector field $X \in \mathfrak{X}(M)$ on a manifold $M$ that is equivariant with respect to action of a Lie group $G$ on it. We say that the point $m \in M$ is a relative equilibrium with velocity $\xi \in \mathfrak{g}$ and the infinitesimal generator $\xi_M$ associated to it if

$$X_h(m) = \xi_M(m).$$  \hfill (11)

Or, equivalently,

$$F_t(m) = \exp t\xi \cdot m,$$  \hfill (12)

where $F_t$ is the flow associated to $X$ at $m \in M$, $\exp$ is the Lie group exponential map $\exp : \mathfrak{g} \to G$ (the one-parameter Lie subgroup of $G$ generated by $\xi \in \mathfrak{g}$). Consider now a symmetric Hamiltonian system $(M, \omega, h, G, J : M \to \mathfrak{g}^*)$ and assume that the momentum map $J$ is coadjoint equivariant; it can be shown [AM78] that the point $m \in M$ is a relative equilibrium of the Hamiltonian vector field $X_h$ with velocity $\xi \in \mathfrak{g}$ if and only if

$$d \left( h - J^\xi \right)(m) = 0,$$  \hfill (13)

where $J^\xi := \langle J, \xi \rangle$. The combination $h - J^\xi$ is usually referred to as the augmented Hamiltonian.
If the relative equilibrium $m \in M$ is such that $J(m) = \mu \in \mathfrak{g}^*$ and we denote its isotropy subgroup with respect to the $G$ action by $G_m$, the law of conservation of the isotropy [OR04] and Noether’s Theorem imply [OR99, Theorem 2.8] that $\xi \in \text{Lie} \left(N_{G_\mu} (G_m)\right)$, where $G_\mu$ is the coadjoint isotropy of $\mu \in \mathfrak{g}^*$ and $N_{G_\mu} (G_m)$ is the normalizer group of $G_m$ in $G_\mu$ (note that $G_m \subset G_\mu$ necessarily due to the equivariance of the momentum map). Finally, notice that the velocity of a relative equilibrium with nontrivial isotropy is not uniquely defined; indeed, it is clear in (12) that if $\xi \in \mathfrak{g}$ is a velocity for the relative equilibrium $m$, then so is $\xi + \eta$ for any $\eta \in \text{Lie} \left(G_m\right)$. 
Proposition

Consider the orbitron system whose Hamiltonian function is given by (1) and let
\( z = ((A, x), (\Pi, p)) \in T^*SE(3) \). Then:

(i) The point \( z \) is a relative equilibrium of the orbitron with velocity \( (\xi_1, \xi_2) \in \mathbb{R}^2 \) with respect to the introduced toral symmetry if and only if the following identities are satisfied:

\[
\mu [B(x) \times Ae_3] + \xi_1 [Ap \times (x \times e_3) - A\Pi \times e_3] = 0, \tag{14}
\]

\[
- \mu DB(x)^T(Ae_3) - \xi_1 (Ap \times e_3) = 0, \tag{15}
\]

\[
I^{-1}_{ref} \Pi + \xi_2 e_3 - \xi_1 A^{-1} e_3 = 0, \tag{16}
\]

\[
\frac{1}{M} p - \xi_1 A^{-1} (e_3 \times x) = 0. \tag{17}
\]
Proposition (Continued)

(ii) Consider now \( A_0 = R^Z_{\theta_0} \), \( x_0 = (x, y, 0) \), \( \Pi_0 = l_3 (\xi_1 - \xi_2) e_3 \) and \( p_0 = M \xi_1 A_0^{-1} (-y, x, 0) \). The point \( z_0 = ((A_0, x_0), (\Pi_0, p_0)) \) is a relative equilibrium of the standard orbitron with velocity \((\xi_1, \xi_2)\), where \( \xi_2 \) is an arbitrary real number and \( \xi_1 \) is either arbitrary when \( x_0 = 0 \) or

\[
\xi_1 = \pm \left( -\frac{3h \mu q \mu_0}{2\pi MD(x_0)^{5/2}} \right)^{1/2},
\]

(18) when \( x_0 \neq 0 \) (the existence is only guaranteed when \( \mu q < 0 \)).

(iii) In the case of the generalized orbitron: \( B_z(x, y, z) = f(x^2 + y^2, z) \) for some \( f \in C^\infty(\mathbb{R}^2) \), and the spatial velocity \( \xi_1 \) of the relative equilibria with \( x_0 \neq 0 \) is

\[
\xi_1 = \pm \left( -\frac{2M \mu f_1'}{f_1} \right)^{1/2},
\]

(19) where \( f_1' = \frac{\partial f(v, z)}{\partial v} \bigg|_{v=x^2+y^2,z=0} \) (exists only when \( \mu f_1' < 0 \)).

The relative equilibria for which \( x_0 \neq 0 \) (resp. \( x_0 = 0 \)) have trivial (resp. nontrivial \( H \)) isotropy and hence belong to the orbit type \((T^*SE(3))_{\{(e)\}}\) (resp. \((T^*SE(3))_{\{H\}}\)); we refer to them as regular relative equilibria (resp. singular relative equilibria).
Figure: Regular and singular relative equilibria of the standard orbitron. $r_{\text{min}}$ and $r_{\text{max}}$ represent the stability region in config. space determined by the conditions below.
**Definition**

Let $X \in \mathfrak{X}(M)$ be a $G$–equivariant vector field on the $G$–manifold $M$ and let $G'$ be a subgroup of $G$. A relative equilibrium $m \in M$ of $X$, is called $G'$–**stable**, or **stable modulo** $G'$, if for any $G'$–invariant open neighborhood $V$ of the orbit $G' \cdot m$, there is an open neighborhood $U \subset V$ of $m$, such that if $F_t$ is the flow of the vector field $X$ and $u \in U$, then $F_t(u) \in V$ for all $t \geq 0$. 
The energy–momentum method provides a sufficient condition for the $G_{\mu}$–stability of a given relative equilibrium [Pat92, OR99, PRW04, MRO11].

**Theorem (Energy-momentum method)**

Let $(M, \omega, h)$ be a symplectic Hamiltonian system with a symmetry given by the Lie group $G$ acting properly on $M$ with an associated coadjoint equivariant momentum map $J : M \to g^*$. Let $m \in M$ be a relative equilibrium such that $J(m) = \mu \in g^*$ and assume that the coadjoint isotropy subgroup $G_{\mu}$ is compact. Let $\xi \in \text{Lie} N_{G_{\mu}}(G_m)$ be a velocity of the relative equilibrium. If the quadratic form

$$d^2(h - J^\xi)(m)|_{W \times W}$$

is definite for some (and hence for any) subspace $W$ such that

$$\ker T_m J = W \oplus T_m(G_{\mu} \cdot m),$$

then $m$ is a $G_{\mu}$–stable relative equilibrium. If $\dim W = 0$, then $m$ is always a $G_{\mu}$–stable relative equilibrium. The quadratic form $d^2(h - J^\xi)(m)|_{W \times W}$, will be called the stability form of the relative equilibrium $m$ and $W$ a stability space.
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Stability of Hamiltonian relative equilibria in symmetric magnetic confinement systems

\[ \left( R_{A}a \right)^{1/2} > \frac{1}{\sqrt{2}} \]

\[ R_{A}^{-1} < \frac{1}{\sqrt{3}} \]

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\[ \xi = 2aR^{-n} \]

\[ L_{22} = \mu_{0}a^{2} \left( \frac{1 + \frac{3}{4} \xi^{2} - \frac{15}{32} \xi^{4} + \frac{35}{64} \xi^{6} + \cdots}{\left( 1 + \frac{1}{4} \xi^{2} + \frac{31}{128} \xi^{4} - \frac{247}{1536} \xi^{6} + \cdots \right)^{2}} \right) \]

\[ S = 3R^{-1}U_{r} + U_{rr} > 0 \]

\[ 0 < \xi < 1 \]

\[ \xi > 0.85 \]
размером орбиты $r_0$ и размером магнита $2l$ происходит «переход» через закон $U \sim -\frac{1}{r^2}$, разделяющий взаимодействия с устойчивыми и неустойчивыми траекториями. Для одинаковых длинных цилиндрических магнитов это соответствует условию (5.4.7).

Если размеры одного магнита намного больше размеров другого, вместо (5.4.7) используем условие

$$l_2 r_0^{-1} > 0.5.$$ (5.4.8)

Если импульсы собственного вращения магнитов намного больше импульса орбитального вращения

$$M_i \gg M_0 > 0, \quad i = 1, 2,$$

что реализуем выбором, например, скорости собственного вращения магнитов, то коэффициенты $U_{0,0_i}$ и определитель (5.4.5) будут положительными при любых $\beta$ и $\gamma$.

Таким образом, планетарная конфигурация двух цилиндрических магнитов, совершающих быстрое собственное вращение вокруг магнитной оси, перпендикулярной к плоскости орбиты, является устойчивой, если только размер невозмущенной орбиты не превосходит размера, сравнимого с размером магнита.

Если размеры орбиты большие по сравнению с размером магнита, устойчивость невозможна (нарушается условие (5.4.6)), и мы приходим к известному результату неустойчивости движения пары магнитных моментов. [96]
Theorem

Consider the relative equilibria introduced in Proposition 3. Then:

(i) The regular relative equilibria of the standard orbitron in part (ii) of Proposition 3, that is, those for which \( x_0 \neq 0 \), are \( T^2 \)-stable whenever the following three inequalities are satisfied:

\[
\frac{2}{3} < \frac{r^2}{h^2} < 4,
\]

\[
\text{sign}(\xi^0_1)l_3 \xi_2 < -|\xi^0_1| \left(l_1 - l_3 + \frac{2}{3}M \left(\frac{r^2 + h^2}{3r^2 - 2h^2}\right)\right),
\]

where \( r^2 = \| x_0 \|^2 \), \( \xi^0_1 = \pm \left(-\frac{3h \mu q \mu_0}{2\pi MD(x_0)^{5/2}}\right)^{1/2} \), and \( \mu q < 0 \). The singular relative equilibria (\( x_0 = 0 \)) are always formally unstable, in the sense that the stability form (20) exhibits a nontrivial signature.
The regular relative equilibria of the generalized orbitron in part (iii) of Proposition 3 are $\mathbb{T}^2$–stable whenever the following conditions hold:

\begin{align*}
\mu f_1' &< 0, \\
\mu \left(2 f_1' + r^2 f_1''\right) &< 0, \\
\mu f_2'' &< 0, \\
\text{sign}(\xi_1^0) l_3 \xi_2 &< -|\xi_1^0| \left((l_1 - l_3) + \frac{1}{2} M \left(\frac{f_0}{f_1'} + 4r^2 \frac{f_1'}{f_2''}\right)\right),
\end{align*}

where $r^2 = \|x_0\|^2$, $f \in C^\infty(\mathbb{R}^2)$ is the function such that $B_z(x, y, z) = f(r^2, z)$, $f_0 = f(r^2, 0)$, $f_1' = \left.\frac{\partial f(v, z)}{\partial v}\right|_{v=r^2, z=0}$, $f_1'' = \left.\frac{\partial^2 f(v, z)}{\partial v^2}\right|_{v=r^2, z=0}$, $f_2'' = \left.\frac{\partial^2 f(v, z)}{\partial z^2}\right|_{v=r^2, z=0}$, and $\xi_1^0 = \pm \left(-\frac{2}{M} \mu f_1'\right)^{1/2}$. 
Theorem (Continued)

The singular branch \((x_0 = 0)\) is \(\mathbb{T}^2\)-stable if the following conditions are satisfied:

\[ \mu f'_1 < 0, \]
\[ \mu f''_2 < 0, \]
\[ \xi_1^2 < -\frac{2}{M} \mu f'_1, \]
\[ \text{sign}(\xi_1) \Pi_0 > \frac{l_1 \xi_1^2 - \mu f_0}{|\xi_1|}, \]

where \(\Pi_0 = l_3 (\xi_1 - \xi_2)\) and we use the same notation as above for \(f_0, f'_1,\) and \(f''_2,\) replacing \(v = r^2\) by \(v = 0.\) When \(\mu f_0 < 0\) and \(\frac{f_0}{f'_1} < \frac{2}{M} l_1,\) the conditions (29) and (30) can be replaced by the following single \(\xi_1\)-independent optimal condition:

\[ |\Pi_0| > 2 \sqrt{-\mu f_0 l_1}. \]

This optimal condition is achieved by using the spatial velocities \(\xi_1 = \pm (-\mu f_0 / l_1)^{1/2};\) the positive (resp. negative) sign for the velocity corresponds to positive (resp. negative) values of \(\Pi_0.\)
Conditions (27)–(30) can be used in the design of magnetic fields capable of confining magnetic rigid bodies that do not exhibit spatial rotation. This is the working principle of devices such as magnetic contactless flywheels or levitrons. In the case of flywheels, up until now only actively controlled versions have been developed; as to the levitron, the potentials that have been considered so far [DE99, Dul04, KM06] do not allow to conclude nonlinear stability using the methods put at work in Theorem 7 and only the spectral stability of the corresponding linearized systems has been considered. We plan to explore in detail these systems in a future publication.
The use of the energy-momentum method provides sufficient but not necessary nonlinear stability conditions. The complementary spectral stability analysis of the linearized system is required.
Linear stability/instability analysis of relative equilibria

- Standard equilibria: examine the spectral stability of the linearization at the equilibrium of the vector field in question.

- Regular relative equilibria: examine the spectral stability of the linearization of the reduced Hamiltonian vector field at the equilibrium corresponding to the relative equilibrium in the symplectic Marsden–Weinstein reduced space [MW74].

- Singular case: there exist reduced spaces that generalize the Marsden–Weinstein reduced space [SL91, OR06a, OR06b], the equivalence between $G_{\mu}$-stability of a relative equilibrium and standard nonlinear stability of the corresponding reduced equilibrium does not hold anymore, which makes necessary the formulation of a criterion that provides a linear stability analysis tool for relative equilibria whose formulation does not need reduction.
Proposition

Let \( G \) be a Lie group acting canonically and properly on the symplectic manifold \((M, \omega)\) and suppose that there exists a coadjoint equivariant \( m. m. \ J : M \to \mathfrak{g}^\ast \) that can be associated to it. Let \( h \in C^\infty(M)^G \) be a \( G \)-invariant Hamiltonian and let \( m \in M \) be a relative equilibrium of the corresponding \( G \)-equivariant Hamiltonian vector field \( X_h \) with velocity \( \xi \in \mathfrak{g} \). Consider a \( G_m \)-invariant stability space \( W \) such that

\[
\ker T_m J = W \oplus T_m (G_\mu \cdot m),
\]

with \( \mu := J(m) \) and \( G_\mu \subset G \) the coadjoint isotropy of \( \mu \in \mathfrak{g}^\ast \). Then:

(i) \((W, \omega_W)\) with \( \omega_W := \omega(m)|_W \) is a symplectic vector subspace of \((T_m M, \omega(m))\).

(ii) There exists a symplectic slice \((S, \omega_S)\) at \( m \in M \) such that \((T_m S, \omega_S(m)) = (W, \omega_W)\).

(iii) The Hamiltonian vector field \( X_{h_S^\xi} \in \mathfrak{X}(S) \) in \( S \) associated to the Hamiltonian function \( h_S^\xi := (h - J^\xi)|_S \) exhibits an equilibrium at the point \( m \in S \subset M \).
Proposition (Continued)

(iv) The linearization $X'_{h_S} \in \mathfrak{X}(T_mS) = \mathfrak{X}(W)$ of $X_{h_S}$ at $m \in S$ coincides with the linear Hamiltonian vector field $X_Q$ on $(W, \omega_W)$ that has as Hamiltonian vector field the stability form

$$Q(w) := d^2 \left( h - J^\xi \right)(m)(w, w), \quad w \in W.$$ 

(v) Suppose that the two tangent spaces $T_m(G_\mu \cdot m)$ and $T_m(G \cdot m)$ coincide. Then

$$T_mM = W \oplus W^\omega.$$  

(32)

Additionally, let $h^\xi := h - J^\xi \in C^\infty(M)$ be the augmented Hamiltonian and let $X'_{h_S} \in \mathfrak{X}(T_mM)$ be the linearization of the Hamiltonian vector field $X_{h_S}$ at $m$. Then

$$X_Q = P_W X'_{h_S} i_W,$$  

(33)

where $i_W : W \hookrightarrow T_mM$ is the inclusion, $P_W : T_mM \twoheadrightarrow W$ is the projection according to (32), and $X'_{h_S}$ is the linearization of $X_{h_S}$ at $m$.

(vi) If the linear vector field $X_Q$ is spectrally unstable in the sense that it exhibits eigenvalues with a nontrivial real part, then the relative equilibrium $m \in M$ of $X_h$ is nonlinearly $K$–unstable, for any subgroup $K \subset G$. 

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Proposition

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $T^* G$ be its cotangent bundle endowed with the canonical symplectic form. Consider now the body coordinates expression $G \times \mathfrak{g}^*$ of $T^* G$ and let $h \in C^\infty(G \times \mathfrak{g}^*)$ be a Hamiltonian function whose associated Hamiltonian vector field $X_h$ exhibits an equilibrium at point $(g, \mu) \in G \times \mathfrak{g}^*$. Then:

(iii) The linearization $X_{Q^g} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ for any $(\xi, \tau) \in \mathfrak{g} \times \mathfrak{g}^*$ is given by:

$$
X_{Q^g}(\xi, \tau) = \left( \pi^*_{\mathfrak{g}}(\text{Hess}(\xi, \tau)), -\pi_{\mathfrak{g}}(\text{Hess}(\xi, \tau)) + \text{ad}^*_{\pi^*_{\mathfrak{g}}(\text{Hess}(\xi, \tau))} \mu \right),
$$

(34)

where $\pi_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}$, $\pi^*_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ are the canonical projections and $\text{Hess} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} \times \mathfrak{g}^*$ is the linear map associated to the Hessian of $h^g$ at $(e, \mu)$ by the relation

$$
\langle \text{Hess}(\xi, \tau), (\eta, \rho) \rangle = d^2 h^g(e, \mu)((\xi, \tau), (\eta, \rho)), \quad (\xi, \tau), (\eta, \rho) \in \mathfrak{g} \times \mathfrak{g}^*.
$$
Let \( h \in C^\infty(T^*(SE(3))) \) be a Hamiltonian function and let \( X_h \) be the corresponding Hamiltonian vector field that we assume has an equilibrium at the point \( z_0 = ((A_0, x_0), (\Pi_0, p_0)) \), that is, \( dh(z_0) = 0 \). Let \( g = (A_0, x_0) \in SE(3) \) and let \( z = ((I, 0), (\Pi_0, p_0)) \); clear that \( z_0 = \varphi_g(z) \). Let \( \text{Hess}(z) : se(3) \times se(3)^* \to se(3) \times se(3)^* \) be the linear map associated to the Hessian of \( h \circ \varphi_g \) at \( z \), that is, for any \( v, w \in T_z(T^*SE(3)) \simeq se(3) \times se(3)^* \), \( \langle v, \text{Hess}(z)w \rangle = d^2(h \circ \varphi_g)(z)(v, w) \). Now, given \( v = (\delta A, \delta x, \delta \Pi, \delta p) \in se(3) \times se(3)^* \), define the projections:

\[
\begin{align*}
\pi_{\delta A} & : se(3) \times se(3)^* \to \mathbb{R}^3 \\
(\delta A, \delta x, \delta \Pi, \delta p) & \mapsto \delta A
\end{align*}
\]

By Proposition 12 the linearization \( X_h' \) of \( X_h \) at \( z_0 \) is given by

\[
X_h' = \Phi_g \circ X_{h^g} \circ \Phi_g^{-1},
\]

where \( X_{h^g} : se(3) \times se(3)^* \simeq \mathbb{R}^{12} \to se(3) \times se(3)^* \simeq \mathbb{R}^{12} \) is the linear map

\[
X_{h^g}' = \begin{pmatrix}
\pi_{\delta \Pi} \text{Hess}(z_0) \\
\pi_{\delta p} \text{Hess}(z_0) \\
-\pi_{\delta A} \text{Hess}(z_0) + \hat{\Pi}_0 \pi_{\delta \Pi} \text{Hess}(z_0) + \hat{p}_0 \pi_{\delta p} \text{Hess}(z_0) \\
-\pi_{\delta x} \text{Hess}(z_0) + \hat{p}_0 \pi_{\delta \Pi} \text{Hess}(z_0)
\end{pmatrix}.
\]
Theorem

Consider the relative equilibria introduced in Proposition 3. Then:

(i) In the case of the standard orbitron in part (ii):

(a) The regular relative equilibria that do not satisfy the Kozorez relation \( r^2/h^2 < 4 \) are unstable; the stability condition is sharp. The conditions in (21) and (22) are not sharp, i.e. there are regions in parameter space that do not satisfy them and where the linearized system is spectrally stable.

(b) The singular relative equilibria are nonlinearly unstable.

(ii) In the case of the generalized orbitron in part (iii):

(a) The regular relative equilibria that do not satisfy the generalized Kozorez relation (24) \( \mu (2f_1' + r^2f_2'') < 0 \), are unstable; the stability condition is sharp. The conditions (23), (25), and (26) are not sharp.

(b) The spectral stability of the singular relative equilibria is equivalent to:

\[
\mu f_1' < 0, \quad \mu f_2'' < 0, \quad \Pi_0^2 > -4\mu f_0 l_1, \tag{37}
\]

where \( \Pi_0 = l_3(\xi_1 - \xi_2) \). The conditions (27) and (28) are sharp, the remaining conditions are not.
The red bullets indicate the critical values of $r$ (m) and $\xi_2$ (rad·s$^{-1}$) determined by the stability conditions. The grey bands correspond to the stability gaps (the system is spectrally stable while the stability form exhibits a nontrivial signature).

Figure: Standard orbitron with $h = 0.05$ m, $M = 0.0068$ kg, $\mu_0 = 4\pi \cdot 10^{-7}$ N·A$^{-2}$, $\mu = -0.18375$ A·m$^2$, $q = 17.58$ A·m, $I_1 = 0.17 \cdot 10^{-6}$ kg·m$^2$, $I_3 = 0.1 \cdot 10^{-6}$ kg·m$^2$. The red bullets indicate the critical values of $r$ (m) and $\xi_2$ (rad·s$^{-1}$) determined by the stability conditions. The grey bands correspond to the stability gaps (the system is spectrally stable while the stability form exhibits a nontrivial signature).
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