CAUCHY MEANS OF DIRICHLET POLYNOMIALS

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Abstract. We study Cauchy means of Dirichlet polynomials
\[ \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{s+it}} \right|^2 \frac{dt}{\pi(t^2 + 1)}. \]
These integrals were investigated when \( q = 1, \sigma = 1, s = 1/2 \) by Wilf, using integral operator theory and Widom’s eigenvalue estimates. We show the optimality of some upper bounds obtained by Wilf. We also obtain new estimates for the case \( q \geq 1, \sigma \geq 0 \) and \( s > 0 \). We complete Wilf’s approach by relating it with other approaches (having notably connection with Brownian motion), allowing simple proofs, and also prove new results.

1. Introduction and Main Results.

In a quite inspiring paper \[8\], Wilf has considered integral operators associated with homogeneous, nonnegative kernels \( K(x, y) \) and applied his results to Dirichlet series. Consider for instance the kernel \( K(x, y) = \max(x, y) - 1 \). It has Mellin transform
\[ \mathcal{F}(s) = \int_{0}^{\infty} t^{-s}K(t, 1)dt = \frac{1}{s + 1} \frac{1}{1 - s}, \]
\( s = \sigma + it \), which is invertible on the critical line. As further \( K(x, y) \) is symmetric and decreasing, it is well-known in this case that the spectral theory of \( K(x, y) \) depends on the behavior of the Mellin transform of \( K(t, 1) \) along the critical line.

If \( x_1, \ldots, x_N \) are complex numbers, then \(8\), Theorem 3)
\[ (1.1) \quad \sum_{n=1}^{N} \bar{x}_n K(n, m)x_m = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}(\frac{1}{2} + it) \left| \sum_{n=1}^{N} \frac{x_n}{t^2 + m^2} \right|^2 dt \leq \mathcal{F}(\frac{1}{2}) \sum_{j=1}^{N} |x_j|^2. \]
The last inequality follows from Widom’s eigenvalue estimate \(8\), Theorem 2). Wilf has shown that \(1.1\) holds for the class \( \mathcal{H} \) of kernels \( K(x, y) \) such that \( K(x, y) \geq 0 \) for \( x, y \) nonnegative, and is further symmetric, decreasing and homogeneous of degree \(-1\): for every \( \alpha > 0 \) we have
\[ (1.2) \quad K(\alpha x, \alpha y) = \alpha^{-1} K(x, y) \quad \forall x > 0, \forall y > 0. \]

In the case considered, \(1.1\) implies that
\[ (1.3) \quad \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{s+it}} \right|^2 \frac{dt}{t^2 + 1} \leq 8\pi \sum_{n=1}^{N} |x_n|^2. \]
Taking \( x_n = n^{-1/2} \) yields in particular the following nice bound \(8\), (17))
\[ (1.4) \quad \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{1+it}} \right|^2 \frac{dt}{t^2 + 1} \leq 8\pi \sum_{n=1}^{N} \frac{1}{n} \leq C \log N. \]

That inequality is in turn two-sided and this can be showed without appealing to Mellin transform nor Widom’s eigenvalue estimate. The purpose of this Note is to first relate Wilf’s approach with other approaches allowing simple proofs, and next, to develop some more parts and prove new results. The above integrals are Cauchy means on the real line of Dirichlet polynomials, and

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admit an exact formulation. This is in contrast with usual mean-value of Dirichlet polynomials, with respect to measures \( \chi_{[0, T]}(t)dt/T \), where an error term always occurs due to the fact that

\[
\int_0^T \left( \frac{m}{n} \right)^n dt = \begin{cases} 
T & \text{if } m = n \\
\mathcal{O}_{m,n}(1) & \text{otherwise}.
\end{cases}
\]  

(1.5)

Both means are in turn strongly related. Cauchy means of Dirichlet polynomials are part of the theory of Dirichlet polynomials for various weights and it is expected that their study will give new insight into properties of general Dirichlet polynomials. We refer for instance to the recent works of Lubinsky [3, 4], [which we discovered while this work was much advanced].

As the weight functions in turn represent a sampling of the parameter \( t \), the properties of the weighted Dirichlet approximating polynomials can be used to study the behavior of the Riemann zeta function \( \zeta(\sigma + it) \) along the critical line \( \sigma = 1/2 \). A (rather) elaborated application of this, in the case of the Cauchy density, can be found in Lifshits and Weber [2].

We begin with giving proofs of (1.2) and (1.3) without appealing to spectral theory (Widom’s eigenvalue estimate).

1.1. Proof of (1.4) using Cauchy means. We start with an elementary lemma.

**Lemma 1.1.** Let \( s \in \mathbb{R}_+ \) and \( x_1, \ldots, x_N, y_1, \ldots, y_N \) be complex numbers. We have

\[
\int \left| \sum_{\nu=1}^M y_{\nu} x_{\nu} \right|^2 \left| \sum_{n=1}^N x_n n^{ist} \right|^2 \frac{dt}{\pi (t^2 + 1)} = \sum_{\mu, \nu=1}^M \sum_{m,n=1}^N y_{\mu} x_{\mu} y_{\nu} x_{\nu} \left( \frac{m \nu / n \mu}{m \nu / n \mu} \right)^s
\]

Moreover,

\[
\lim_{s \to \infty} \int \left| \sum_{\nu=1}^M y_{\nu} x_{\nu} \right|^2 \left| \sum_{n=1}^N x_n n^{ist} \right|^2 \frac{dt}{\pi (t^2 + 1)} = \sum_{1 \leq \mu, \nu \leq M \atop 1 \leq m, n \leq N} y_{\mu} x_{\mu} y_{\nu} x_{\nu}.
\]

**Remark 1.2.** The last assertion implies that

\[
\lim_{s \to \infty} \int \left| \sum_{n=1}^N x_n n^{ist} \right|^2 \frac{dt}{\pi (t^2 + 1)} = \sum_{n=1}^N |x_n|^2
\]

\[
\lim_{s \to \infty} \int \left| \sum_{n=1}^N x_n n^{ist} \right|^4 \frac{dt}{\pi (t^2 + 1)} = \sum_{1 \leq \mu, \nu \leq N \atop 1 \leq m, n \leq N} x_{\mu} x_{\mu} x_{\nu} x_{\nu}.
\]

Taking \( x_n = n^{-\sigma} \) yields,

\[
\lim_{s \to \infty} \int \left| \sum_{n=1}^N \frac{1}{n^{\sigma + ist}} \right|^2 \frac{dt}{\pi (t^2 + 1)} = \sum_{n=1}^N \frac{1}{n^{2\sigma}}
\]

\[
\lim_{s \to \infty} \int \left| \sum_{n=1}^N \frac{1}{n^{\sigma + ist}} \right|^4 \frac{dt}{\pi (t^2 + 1)} = \sum_{1 \leq \mu, \nu \leq N \atop 1 \leq m, n \leq N} \frac{1}{(\mu \nu \mu \nu)}^{2\sigma}.
\]

And in particular, by using Ayyad, Cochrane and Zheng estimate [1], Theorem 3,

\[
\lim_{s \to \infty} \int \left| \sum_{n=1}^N \frac{1}{n^{ist}} \right|^4 \frac{dt}{\pi (t^2 + 1)} = \# \left\{ 1 \leq \mu, \nu \leq N, 1 \leq m, n \leq N : n \nu = m \mu \right\} = \frac{12}{\pi^2} N^2 \log N + CN^2 + O(N^{19/13} \log^{7/13} N),
\]

where \( C = \frac{2}{\pi^2} (12 \gamma - \frac{36}{\pi^2} \zeta'(2) - 3 - 2, \gamma \) is Euler’s constant and \( \zeta'(2) = \sum_{n=1}^\infty \frac{\log n}{n^2} \).

**Proof.** From the relation \( e^{-|\vartheta|} = \int e^{it} \frac{dt}{\pi (t^2 + 1)} \), it follows that

\[
\left( \frac{n}{m} \right)^s = \int \frac{1}{n^{ist} m^{-ist} \pi (t^2 + 1)} dt
\]

(1.6)
Thus
\[
\int_{\mathbb{R}} \left| \sum_{\nu=1}^{M} y_{\nu} \nu^{ist} \right|^{2} \frac{1}{n^{1+ist}} \frac{\pi}{m^{1+ist}} \frac{dt}{(t^2 + 1)} = \sum_{\mu, \nu=1}^{M} y_{\mu} y_{\nu} \frac{1}{(\nu \nu')^{1+ist}} \frac{\pi}{m^{1+ist}} \frac{dt}{(t^2 + 1)} = \sum_{\mu, \nu=1}^{M} \frac{1}{\nu \nu'} \frac{(\nu \nu' \wedge m \mu)^{s}}{(\nu \nu' \vee m \mu)^{s}}.
\]

Consequently
\[
\int_{\mathbb{R}} \left| \sum_{\nu=1}^{M} y_{\nu} \nu^{ist} \right|^{2} \left| \sum_{n=1}^{N} x_{n} n^{ist} \right|^{2} \frac{dt}{\pi(t^2 + 1)} = \sum_{\mu, \nu=1}^{M} \sum_{m, n=1}^{N} \frac{1}{\nu \nu'} \frac{1}{(\nu \nu')^{1+ist}} \frac{\pi}{m^{1+ist}} \frac{dt}{(t^2 + 1)} = \sum_{\mu, \nu=1}^{M} \sum_{m, n=1}^{N} \frac{1}{\nu \nu'} \frac{1}{(\nu \nu' \wedge m \mu)^{s}} \frac{1}{(\nu \nu' \vee m \mu)^{s}}.
\]

The second assertion follows easily. Let
\[
\delta = \max_{1 \leq \mu, \nu \leq M, \nu \neq m \mu, n \neq m \mu} \left( \frac{(\nu \nu \wedge m \mu)^{s}}{\nu \nu \vee m \mu} \right).
\]

Then $0 < \delta < 1$. And the conclusion follows from
\[
\int_{\mathbb{R}} \left| \sum_{\nu=1}^{M} y_{\nu} \nu^{ist} \right|^{2} \left| \sum_{n=1}^{N} x_{n} n^{ist} \right|^{2} \frac{dt}{\pi(t^2 + 1)} = \sum_{\mu, \nu=1}^{M} \frac{1}{\nu \nu'} \frac{1}{(\nu \nu')^{1+ist}} \frac{\pi}{m^{1+ist}} \frac{dt}{(t^2 + 1)} = \sum_{\mu, \nu=1}^{M} \frac{1}{\nu \nu'} \frac{1}{(\nu \nu' \wedge m \mu)^{s}} \frac{1}{(\nu \nu' \vee m \mu)^{s}} = (M N)^2 \mathcal{O}(\delta^s).
\]

To recover (1.4) and also to prove the corresponding lower bound, take $x_n = n^{-1}$, $M = 1 = y_1$ and $s = 1/2$. We get
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{1+ist}} \frac{1}{n^{1+ist}} \frac{dt}{\frac{1}{4} + g^2} \right|^{2} \frac{dt}{\frac{1}{4} + g^2} = \sum_{m, n=1}^{N} \frac{1}{(m \vee n)\frac{1}{2}} \frac{1}{m \vee n} ^{3/2} \frac{dt}{(m \vee n)\frac{1}{2}} ^{3/2} = \sum_{n=1}^{N} \frac{1}{n^2} + 2 \sum_{n=1}^{N} \frac{1}{n^{1/2}} \sum_{m=1}^{N} \frac{1}{m^{3/2}} \leq C \left(1 + \sum_{n=1}^{N} \frac{1}{n}\right) \leq C \log N,
\]
which is (1.4). And obviously,
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{1+ist}} \frac{1}{n^{1+ist}} \frac{dt}{\frac{1}{4} + g^2} \right|^{2} \frac{dt}{\frac{1}{4} + g^2} \geq \sum_{m, n=1}^{N} \frac{1}{(m \vee n)\frac{1}{2}} \frac{1}{m \vee n} ^{3/2} \frac{dt}{(m \vee n)\frac{1}{2}} ^{3/2} \geq C \sum_{n=1}^{N} \frac{1}{n} \geq C \log N.
\]

1.2. Proof of (1.3) using Brownian motion. Let $W = \{W(t), t \geq 0\}$ be standard one-dimensional Brownian motion issued from 0 at time $t = 0$ and with underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then
\[
(1.7) \quad K(s, t) = \frac{(s \wedge t)}{st} = \mathbb{E} \left( \frac{W(s) W(t)}{s} \right) \quad \text{and} \quad \mathbb{E} \left( \frac{W(s) W(t)}{\sqrt{s} \sqrt{t}} \right) = \left( \frac{s \wedge t}{s \vee t} \right)^{1/2}.
\]

This allows to interpret these integrals as Brownian sums, and by using the independence of the increments of $W$, to find another convenient reformulation.

Lemma 1.3. For any real $s \geq 0$,
\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} x_{n} n^{ist} \right|^{2} \frac{dt}{\pi(t^2 + 1)} = \mathbb{E} \left| \sum_{n=1}^{N} x_{n} W(n^{2s}) \right|^{2} \]
\[ \int_{\mathbb{R}} \left| \sum_{n=1}^{N} x_n n^{ist} \right|^4 \frac{dt}{t^2 + 1} = \sum_{j=1}^{N} (j^{2s} - (j - 1)^{2s}) \left| \sum_{\mu=j}^{N} \frac{x_\mu}{\mu^s} \right|^2 \]

As to the second one, write \( W((n\nu)^s) = \sum_{j=0}^{n\nu} g_j \), where \( g_j = W(j^{2s}) - W((j-1)^{2s}) \), \( j \geq 1 \), we also have

\[ \int_{\mathbb{R}} \left| \sum_{n=1}^{N} x_n n^{ist} \right|^4 \frac{dt}{t^2 + 1} = \sum_{n,\nu=1}^{N} N \sum_{m,\mu=1}^{N} x_\mu x_\nu \left( \frac{n\nu m\mu}{n\nu m\mu} \right)^s \]

\[ = \sum_{j=1}^{N} (j^{2s} - (j - 1)^{2s}) \left( \sum_{n,\nu \geq j}^{N} \frac{x_\nu x_n}{(n\nu)^s} \right)^2 \]

Proof. The first equality follows from Lemma 1.1 and 1.7. As to the second one, write \( W((n\nu)^s) = \sum_{j=0}^{n\nu} g_j \), where \( g_j = W(j^{2s}) - W((j-1)^{2s}) \), \( j \geq 1 \), we also have

We now need a technical lemma.

Lemma 1.4. For any \( s > 0 \) and complex numbers \( x_j, j = 1, \ldots, N \),

\[ \sum_{j=1}^{N} (j^{2s} - (j - 1)^{2s}) \left| \sum_{\mu=j}^{N} \frac{x_\mu}{\mu^s} \right|^2 \leq \begin{cases} C_s \sum_{\mu=1}^{N} |x_\mu|^2 \mu^{3/2 - 2s} & \text{if } 0 < s < 1/4, \\ C_s \sum_{\mu=1}^{N} |x_\mu|^2 \mu \log \mu & \text{if } s = 1/4, \\ C_s \sum_{\mu=1}^{N} |x_\mu|^2 \mu & \text{if } s > 1/4. \end{cases} \]

Proof. Let \( y_\mu = x_\mu / \mu^{s-1} \). By Hölder’s inequality,

\[ \sum_{j=1}^{N} (j^{2s} - (j - 1)^{2s}) \left| \sum_{\mu=j}^{N} \frac{x_\mu}{\mu^s} \right|^2 \leq \sum_{j=1}^{N} (j^{2s} - (j - 1)^{2s}) \left( \sum_{\mu=j}^{N} \frac{1}{\mu^{3/2}} \right) \left( \sum_{\mu=j}^{N} |y_\mu|^2 \right) \]

(writing \( \mu = \mu^{3/4}, \mu^{1/4} \))

\[ \leq C_s \sum_{j=1}^{N} \frac{(j^{2s} - (j - 1)^{2s}) (\sum_{\mu=j}^{N} \frac{1}{\mu^{3/2}}) (\sum_{\mu=j}^{N} |y_\mu|^2 \mu^{1/2})}{\mu^{1/2} \sum_{j \leq \mu}^{N} j^{2s-3/2}} \]

If \( 0 < s < 1/4 \), it follows that

\[ \sum_{j=1}^{N} (j^{2s} - (j - 1)^{2s}) \left| \sum_{\mu=j}^{N} \frac{x_\mu}{\mu^s} \right|^2 \leq C_s \sum_{\mu=1}^{N} \frac{|y_\mu|^2 \mu^{1/2}}{\mu^{1/2}} = C_s \sum_{\mu=1}^{N} |y_\mu|^2 \mu^{3/2 - 2s}. \]
If $s > 1/4$, 
\[
\sum_{j=1}^{N} (j^{2s} - (j-1)^{2s}) \left| \sum_{\mu=j}^{N} x_{\mu} \right|^2 \leq C_s \sum_{\mu=1}^{N} \left| \frac{y_{\mu}}{\mu^{1/2}} \right|^2 \sum_{j \leq \mu} j^{2s-3/2} \leq C_s \sum_{\mu=1}^{N} \left| \frac{y_{\mu}}{\mu^{1/2}} \right|^2 \mu^{2s-1/2} 
\]
\[
= C_s \sum_{\mu=1}^{N} \left| x_{\mu} \right|^2 \mu^{3/2-2s} \mu^{2s-1/2} = C_s \sum_{\mu=1}^{N} \left| x_{\mu} \right|^2 \mu. 
\]
And if $s = 1/4$, 
\[
\sum_{j=1}^{N} (j^{2s} - (j-1)^{2s}) \left| \sum_{\mu=j}^{N} x_{\mu} \right|^2 \leq C \sum_{\mu=1}^{N} \left| \frac{y_{\mu}}{\mu^{1/2}} \right|^2 \sum_{j \leq \mu} j^{-1} \leq C \sum_{\mu=1}^{N} \left| \frac{y_{\mu}}{\mu^{1/2}} \right|^2 \mu^{1/2} 
\]
\[
= C \sum_{\mu=1}^{N} \left| x_{\mu} \right|^2 \mu \log \mu. 
\]

\[\square\]

Indicate now how to deduce (1.3). By taking $s = 1/2$, $x_j = z_j/j^{1/2}$ we get in particular
\[
\sum_{j=1}^{N} \left| \sum_{k=1}^{N} \frac{z_j}{j} \right|^2 \leq C \sum_{j=1}^{N} \left| z_j \right|^2, 
\]
hence by Lemma 1.3
\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{z_n}{n^{1/2} + it} \right|^2 \frac{dt}{\pi(t^2 + 1)} = \sum_{k=1}^{N} \left| \sum_{j=k}^{N} \frac{z_j}{j} \right|^2 \leq C \sum_{j=1}^{N} \left| z_j \right|^2. 
\]
Making the variable change $t = 2\theta$, gives
\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{z_n}{n^{1/2} + it} \right|^2 \frac{d\theta}{\pi(\theta^2 + 1)} \leq 2C \sum_{j=1}^{N} \left| z_j \right|^2, 
\]
which is (1.3) up to the value of the constant.

1.3. Example. One can deduce similar estimates for integrals of power four.
\[
C_1 (\log N)^3 \leq \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{1+it/2}} \right|^4 \frac{dt}{\pi(t^2 + 1)} \leq C_2 (\log N)^3. 
\]
Take $s = 1/2$, $x_n = 1/n$. Then
\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{1+it/2}} \right|^4 \frac{dt}{\pi(t^2 + 1)} = \sum_{j=1}^{N^2} \left| \sum_{1 \leq n, \nu \leq N \atop n\nu \geq j} \frac{1}{(n\nu)^{1/2}} \right|^2. 
\]
Next
\[
\sum_{1 \leq n, \nu \leq N \atop n\nu \geq j} \frac{1}{(n\nu)^{1/2}} \leq \sum_{j \leq \nu \leq N} \frac{1}{n^{1/2}} + \sum_{1 \leq n \leq \nu} \frac{1}{n^{1/2}} \sum_{1 \leq \nu \leq N} \frac{1}{n^{1/2}} \leq C \frac{\log j}{j^{3/2}} + C \frac{\log j}{j^{3/2}} \leq C \frac{\log j}{j}. 
\]
Thus
\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{1+it/2}} \right|^4 \frac{dt}{\pi(t^2 + 1)} \leq C \sum_{j=1}^{N^2} \frac{\log^2 j}{j} \leq C(\log N)^3. 
\]
Further, for \( j \leq N/2 \),
\[
\sum_{1 \leq n \leq j} \frac{1}{n^{3/2}} \geq \sum_{1 \leq n \leq j} \frac{1}{n^2} \sum_{\nu \geq j/n} \frac{1}{\nu^2} \geq C \sum_{1 \leq n \leq j} \frac{1}{n^2} \left( \frac{n}{j} \right)^{1/2} = C \frac{\log j}{j^{1/2}},
\]
and
\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{1+2it}} \right|^4 \frac{dt}{\pi(t^2+1)} \geq C (\log N)^3.
\]

1.4. Lubinsky's space \( \mathcal{L} \). It is natural to consider the (Hilbert) space \( \mathcal{L} \) consisting with all Borel-measurable functions \( f : \mathbb{R} \to \mathbb{C} \) such that
\[
\|f\|^2 = \int_{\mathbb{R}} |f(t)|^2 \frac{dt}{\pi(t^2+1)} < \infty.
\]
That question was recently investigated by Lubinsky in \cite{3}. Let \( \lambda_0 = 0 \) and \( 1 = \lambda_1 < \lambda_2 < \ldots \) with \( \lim_{k \to \infty} \lambda_k = \infty \). Applying the Gram-Schmidt process to \( \{\lambda_n^{-it}, n \geq 1\} \), produces the sequence of orthonormal Dirichlet polynomials
\[
\phi_n(t) = \frac{\lambda_n^{1-it} - \lambda_n^{-1-it}}{\sqrt{\lambda_n^2 - \lambda_n^{-2}}}, \quad n = 1, 2, \ldots
\]
Let \( F(t) = \sum_{n=1}^{\infty} a_n \lambda_n^{-it} \) where \( \{a_n, n \geq 1\} \subset \mathbb{C} \) and let \( s > 0 \). Recall Th. 1.1 in \cite{3}. Assume that the series
\[
\sum_{n=1}^{\infty} |\lambda_n^{2s} - \lambda_n^{-2s}| \left| \sum_{n=k}^{\infty} a_n \right|^2
\]
converges. Then \( F(s) \in \mathcal{L} \) and
\[
\int_{\mathbb{R}} |F(st)|^2 \frac{dt}{\pi(1+t^2)} = \sum_{n=1}^{\infty} |\lambda_n^{2s} - \lambda_n^{-2s}| \left| \sum_{n=k}^{\infty} a_n \right|^2.
\]
Further, \( F(s) \) is the limit in \( \mathcal{L} \) of some (explicated) subsequence of its partial sums.
Consequently, in Lemma \cite{13} we also have that
\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{\infty} x_n n^{ist} \right|^2 \frac{dt}{\pi(t^2+1)} = \sum_{j=1}^{\infty} \left| j^{2s} - (j-1)^{2s} \right| \left| \sum_{\mu=j}^{\infty} x_\mu \right|^2
\]
(1.9)
\[
= \mathbb{E} \left| \sum_{n=1}^{\infty} x_n W(n^{2s}/n^s) \right|^2
\]
powered that the Brownian series \( \sum_{n=1}^{\infty} x_n W(n^{2s}) \) converges in \( L^2(\mathbb{P}) \).

New sufficient conditions for \( F \) to belong to \( \mathcal{L} \) can further easily be derived from Lemma \cite{14}. More precisely,

**Corollary 1.5.** Let \( F(t) = \sum_{n=1}^{\infty} x_n n^{-it} \) where \( x_n \geq 0 \) and let \( s > 0 \). A sufficient condition for \( F(s) \in \mathcal{L} \) is
\[
\begin{align*}
\sum_{n=1}^{\infty} x_n^2 \mu^{3/2-2s} &< \infty \quad \text{if } 0 < s < 1/4, \\
\sum_{\mu=1}^{\infty} x_n^3 \mu \log \mu &< \infty \quad \text{if } s = 1/4, \\
\sum_{\mu=1}^{\infty} x_n^3 \mu &< \infty \quad \text{if } s > 1/4.
\end{align*}
\]

**Proof.** Under either of these conditions, the corresponding series
\[
\sum_{j=1}^{\infty} \left| j^{2s} - (j-1)^{2s} \right| \left( \sum_{\mu=j}^{\infty} x_\mu \right)^2
\]
is convergent, since for instance if \( s > 1/4 \), by Lemma 1.3 for all \( N_0 \geq 1 \), for all \( N \geq N_0 \),

\[
\sum_{j=1}^{N} (j^{2s} - (j - 1)^{2s}) \left( \sum_{\mu=1}^{N} x_\mu \right)^2 \leq C_s \sum_{\mu=1}^{N} x_\mu^2 \mu.
\]

The conclusion thus follows from the aforementioned Lubinsky’s result.

1.5. Higher moments. Let \( s \geq 0, r > 0 \). Consider the more general integrals

\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{ist}} \right|^{2k} \frac{dt}{\pi(t^2 + 1)},
\]

and in particular, for any positive integer \( k \),

\[
I_k(N, \sigma, s) = \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma+ist}} \right|^{2k} \frac{dt}{\pi(t^2 + 1)}.
\]

corresponding to Dirichlet approximating polynomials. By simple iteration, Lemmas 1.1 and 1.3 extend to general integer moments.

**Lemma 1.6.** For any positive integer \( q \), we have

\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n^{n^{-ist}}}{n^{ist}} \right|^{2q} \frac{dt}{\pi(t^2 + 1)} = \sum_{1 \leq \mu_1, \ldots, \mu_q \leq N} x_{\mu_1} \cdots x_{\mu_q} \frac{W((\nu_1, \ldots, \nu_q)^{2s})}{(\nu_1, \ldots, \nu_q)^s}. \]

And

\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} x_n^{n^{-ist}} \right|^{2q} \frac{dt}{\pi(t^2 + 1)} = E \left| \sum_{1 \leq \nu_1, \ldots, \nu_q \leq \sigma N} x_{\nu_1} \cdots x_{\nu_q} \frac{W((\nu_1, \ldots, \nu_q)^{2s})}{(\nu_1, \ldots, \nu_q)^s} \right|^2 = \sum_{j=1}^{N^q} (j^{2s} - (j - 1)^{2s}) \left| \sum_{1 \leq \nu_1, \ldots, \nu_q \leq j, \nu_1, \ldots, \nu_q \in \mathbb{N}} x_{\nu_1} \cdots x_{\nu_q} \frac{W((\nu_1, \ldots, \nu_q)^{2s})}{(\nu_1, \ldots, \nu_q)^s} \right|^2.
\]

We omit the proof. By 1.3 and the considerations made after, it also follows that

**Corollary 1.7.** Let \( q \) be a positive integer, \( s > 0 \) and let \( F^q(t) = (\sum_{n=1}^{\infty} x_n^{n^{-ist}})^q \).

\[
F^q(s, t) \in \mathcal{L} \quad \text{and thus} \quad \int_{\mathbb{R}} \left| \sum_{n=1}^{\infty} x_n^{n^{-ist}} \right|^{2q} \frac{dt}{\pi(t^2 + 1)} < \infty
\]

if the Brownian sum

\[
\sum_{1 \leq \nu_1, \ldots, \nu_q \leq \sigma N} x_{\nu_1} \cdots x_{\nu_q} \frac{W((\nu_1, \ldots, \nu_q)^{2s})}{(\nu_1, \ldots, \nu_q)^s}
\]

converges in \( L^2(\mathbb{P}) \).

1.6. Connection with mean-values of Dirichlet polynomials. Of first importance in the previous formulas is the role played by the parameter \( s \), and more precisely the behavior of the Cauchy means when \( s \to \infty \).

Lubinsky has established a clarifying link with mean-values of general Dirichlet polynomials. We state it under slightly weaker assumptions than in 29 p. 428.

**Lemma 1.8.** Let \( g : \mathbb{R} \to \mathbb{C} \) and define formally for any \( s \geq 0 \), \( \mathcal{M}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(t)| dt \). Then,

\[
\int_{0}^{\infty} |g(t)| \log \frac{1}{t} dt < \infty \iff \int_{0}^{\infty} \mathcal{M}(s) ds < \infty.
\]

(1.10)

\[
\left( \int_{\mathbb{R}} \frac{|g(t)|}{1 + t^2} dt < \infty \text{ and } \int_{0}^{\infty} |g(t)| \log \frac{1}{t} dt < \infty \right) \iff \int_{\mathbb{R}} \frac{\mathcal{M}(s)}{1 + s^2} ds < \infty.
\]
Under any of the previous properties, we further have
\begin{equation}
(1.12) \quad \int_{\mathbb{R}} |g(st)| \frac{dt}{\pi(1+t^2)} = 4 \int_{0}^{\infty} \mathcal{M}(su) \frac{u^2}{\pi(1+u^2)^2} du.
\end{equation}
And if moreover, \( \mathcal{M}(s) \) is locally bounded, then
\begin{equation}
(1.13) \quad \lim_{s \to \infty} \int_{\mathbb{R}} |g(st)| \frac{dt}{\pi(1+t^2)} = \lim_{s \to \infty} \mathcal{M}(s),
\end{equation}
if the preceding limit exists and is finite.

Proof. Assertion (1.10) follows by integration by part. Further, if \( \eta > 0 \),
\[ \int_{\eta \leq |t| < \infty} \frac{|g(t)|}{1+t^2} dt < \infty \iff \int_{\eta \leq s < \infty} \frac{\mathcal{M}(s)}{1+s^2} ds < \infty. \]
Hence (1.11) follows. An integration by part gives (1.12). Since \( \mathcal{M}(s) \to \lambda \), say, and \( |\lambda| < \infty \), there is a real \( A > 0 \) and a real \( Y > 0 \) such that we have \( |\mathcal{M}(y)| \leq A \) if \( y \geq Y \). By assumption, \( \mathcal{M}(s) \) is locally bounded, we also have \( \mathcal{M}(y) \leq B \) if \( 0 \leq y \leq Y \). Thus \( \mathcal{M}(y) \leq A \lor B \) on \( \mathbb{R}^+ \).

Therefore
\[ \frac{\mathcal{M}(su)u^2}{(1+u^2)^2} \leq \frac{(A \lor B)u^2}{(1+u^2)^2} \in L^1(\mathbb{R}^+). \]

And (1.13) follows from the dominated convergence theorem. \( \square \)

Letting \( g = \sum_{n=1}^{\infty} x_n n^{-it} 2^q \), where \( q \) is a positive integer yields
\[ \lim_{s \to \infty} \int_{\mathbb{R}} \left| \sum_{n=1}^{\infty} x_n n^{-ist} 2^q \right|^2 \frac{dt}{\pi(t^2+1)} = \lim_{s \to \infty} \frac{1}{2s} \int_{-s}^{s} \left| \sum_{n=1}^{\infty} x_n n^{-it} 2^q \right|^2 dt, \]
provided that the second limit exists.

Another link with standard mean-values of Dirichlet sums is provided with the next lemma.

Lemma 1.9. Let \( q, S, T \) be positive reals. Then
\[ \int_{S}^{\sqrt{S^2+T^2}} \left( \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 \frac{d\theta}{\pi(t^2+1)} \right) ds = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 \log \left( 1 + \frac{T^2}{\theta^2 + S^2} \right) d\theta. \]
Moreover,
\[ \frac{1}{S} \int_{0}^{S} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 d\theta \leq \left( \frac{2\pi}{\log 2} \right) \sup_{S \leq s \leq 2S} \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 \frac{dt}{\pi(t^2+1)}. \]
This provides a partial converse to Lubinsky’s observation. Indeed, assume that
\[ \lim_{s \to \infty} \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 \frac{dt}{\pi(t^2+1)} = \lambda. \]
Then it follows from the second part of the Lemma that
\[ \lim_{S \to \infty} \frac{1}{S} \int_{0}^{S} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 d\theta \leq \left( \frac{2\pi}{\log 2} \right) \lambda. \]

Proof. By using the variable change \( t = \theta/s \), we get
\[ \int_{S}^{\sqrt{S^2+T^2}} \left( \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 \frac{dt}{\pi(t^2+1)} \right) ds = \int_{S}^{\sqrt{S^2+T^2}} \left( \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{x_n}{n^{i\theta}} \right|^2 \frac{s d\theta}{\pi(\theta^2 + s^2)} \right) ds = \int_{S}^{\sqrt{S^2+T^2}} \left( \int_{S}^{\sqrt{S^2+T^2}} \frac{s ds}{\pi(\theta^2 + s^2)} \right) d\theta. \]
where we set

\[ \sum_{n=1}^{\infty} \frac{x_n}{n^{1+\sigma+i\theta}} \]

exist. The series \( \sum_{n=1}^{\infty} \frac{x_n}{n^{1+\sigma+i\theta}} \) is thus convergent. For the values \( \alpha = 2k, k = 1, 2, \ldots \), we recall that

\[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} n^{-it} \right|^\alpha \ dt \]

exist. The series \( \sum_{j=1}^{\infty} \frac{M_j}{j^2} \) is thus convergent. For the values \( \alpha = 2k, k = 1, 2, \ldots \), we recall that

\[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} n^{-it} \right|^{2k} \ dt = \sum_{1 \leq m \leq N} \frac{d_{k,N}^2(m)}{m^{2\sigma}}, \]

where \( d_{k,N}(m) \) denotes the number of representations of \( m \) as a product of \( k \) factors less or equal to \( N \).

**Proof of Lemma 1.10.** Let

\[ u_k = \int_{-k-1}^{k} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma+i\theta}} \right|^{2k} \ dt, \quad k = 1, 2, \ldots \]

and note that

\[ \sum_{k=0}^{\infty} \frac{u_k}{\pi(k^2 + 1)} \leq \int_{0}^{\infty} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma+i\theta}} \right|^{2k} \frac{dt}{\pi(t^2 + 1)} \leq \sum_{k=0}^{\infty} \frac{u_k}{\pi((k-1)^2 + 1)}, \]

Let \( D_j = \sum_{k=1}^{j} u_k, j \geq 1 \). By applying Abel summation

\[ \sum_{k=1}^{r} u_k y_k = D_r y_{r+1} + \sum_{j=1}^{r} D_j (y_j - y_{j+1}), \]
Lemma 2.1. Let

\[ P \]

We obtain the following very precise uniform estimate.

In the next subsection, we investigate the behavior of Cauchy integrals when the parameter \( s \) is small and the moments are high.

1.7. Behavior of \( I_k(N, \sigma, s) \) for \( s = s(k) \) small and \( k \) large. We now consider the behavior of these integrals when \( s \) and \( k \) are simultaneously varying. More precisely, we will study the case when \( s = 1/\sqrt{c_{\sigma, N}} k \) where \( c_{\sigma, N} \sim c \) as \( k \to \infty \) (\( c = c(\sigma) \) will be an explicit positive constant).

We obtain the following very precise uniform estimate.

Theorem 1.12. There exist two positive numerical constants \( c_0, C \) such that for all positive integers \( N, k \) and \( 0 < \sigma < 1 \),

\[
\left| \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma + it}} \right|^{2k} \frac{dt}{\pi(t^2 + 1)} - c_0 \left( \sum_{n=1}^{N} \frac{1}{n^\sigma} \right)^{2k} \right| \leq C \frac{(1 - \sigma) \log N}{k^{1/2}} \left( \sum_{n=1}^{N} \frac{1}{n^\sigma} \right)^{2k}
\]

where

\[
c_{\sigma, N} = \frac{2}{(1 - \sigma)^2} + O(N^{\sigma - 1} (\log N)^2).
\]

2. Proof of Theorem 1.12

Our proof is probabilistic. We introduce a random model and first establish an interesting property (Lemma 2.3) of this one. We don’t know whether this model has been investigated somewhere.

2.1. A random model. Let \( \sigma \geq 0 \). Let \( N \) be some positive integer and note \( L_N = \sum_{n=1}^{N} \frac{1}{n^\sigma} \).

Let \( Y_1, \ldots, Y_k \) be independent copies of \( Y \) and note \( S_k = Y_1 + \ldots + Y_k \).

Lemma 2.1. Let \( \tilde{S}_k \) denote a symmetrization of \( S_k \). Then,

\[
\left| \sum_{n=1}^{N} \frac{1}{n^{\sigma + it}} \right|^{2k} = \left( \sum_{n=1}^{N} \frac{1}{n^\sigma} \right)^{2k} \mathbb{E} e^{it \tilde{S}_k}.
\]

Proof. We indeed have

\[
\mathbb{P}\{S_k = \log m\} = \sum_{1 \leq \ell_1, \ldots, \ell_k \leq N} \mathbb{P}\{Y_1 = \log n_1, \ldots, Y_k = \log n_k\} = \frac{\delta_k(N(m))}{m^{\sigma} L_N^k}
\]

where we set \( \delta_k(N(m)) = \# \{(n_1, \ldots, n_k) \in \{1, N\}^k : m = n_1 \ldots n_k \} \).

Further

\[
\left| \sum_{n=1}^{N} \frac{1}{n^{\sigma + ist}} \right|^{2k} = \left( \sum_{m=1}^{N} \frac{1}{m^{\sigma + ist}} \right)^N \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma - ist}} \right)^N = \left( \sum_{\mu=1}^{N} \frac{\delta_k(N(\mu))}{\mu^{\sigma + ist}} \right)^N \left( \sum_{\nu=1}^{N} \frac{\delta_k(N(\nu))}{\nu^{\sigma - ist}} \right)^N.
\]
Proof. Let \( L_N^{2k} \left( \sum_{\mu=1}^{N} \frac{\mathbb{P}\{S_\mu = \log \mu \}}{\mu_{\text{fst}}} \right) \left( \sum_{\nu=1}^{N} \frac{\mathbb{P}\{S_\nu = \log \nu \}}{\nu_{\text{ist}}} \right) \)

\[ = L_N^{2k} \left( \sum_{\mu=1}^{N} \frac{\mathbb{P}\{S_\mu = \log \mu \}}{\mu_{\text{ist}}} \right)^2 = L_N^{2k} \mathbb{E} e^{-i\pi S_k} = L_N^{2k} \mathbb{E} e^{i\pi S_k}. \]

Hence,

\[ \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma + i\pi}} \right|^{2k} = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \mathbb{E} e^{i\pi S_k}. \]

Lemma 2.2. We have the relations

\[
\frac{1}{T} \int_{-T}^{T} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma + it}} \right|^{2k} \, dt = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \mathbb{E} \int_{-T}^{T} e^{i\pi S_k} \, dt = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \mathbb{E} \frac{\sin T S_k}{T S_k}.
\]

Proof. By Fubini’s theorem,

\[
\frac{1}{T} \int_{-T}^{T} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma + it}} \right|^{2k} \, dt = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \frac{1}{T} \mathbb{E} \int_{-T}^{T} e^{i\pi S_k} \, dt = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \mathbb{E} \frac{\sin T S_k}{T S_k}.
\]

It also follows by integrating that

\[
\int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma + it}} \right|^{2k} \frac{dt}{\pi(t^2 + 1)} = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \int_{\mathbb{R}} \mathbb{E} e^{i\pi S_k} \frac{dt}{\pi(t^2 + 1)}
\]

\[ = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \mathbb{E} \int_{\mathbb{R}} e^{i\pi S_k} \frac{dt}{\pi(t^2 + 1)}
\]

\[ = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \mathbb{E} e^{-\pi S_k}.\]

A interesting fact of this model is that the variance of \( \tilde{Y} \) is small (almost constant). This is made precise in the lemma below.

Lemma 2.3. Let \( 0 < \sigma < 1 \). We have

\[
\mathbb{E} \tilde{Y}^2 = 2 \left\{ \sum_{m=1}^{N} \frac{(\log m)^2}{m^\sigma L_N} - \left( \sum_{m=1}^{N} \frac{(\log m)}{m^\sigma L_N} \right)^2 \right\} = \frac{2}{(1 - \sigma)^2} + \mathcal{O}(N^{\sigma-1}(\log N)^2).
\]

It will follow from the proof that the almost constant behavior of the variance arises from cancellation of auxiliary sums.

Proof. We use Euler-Maclaurin formula. Let \( h : [1, N] \to \mathbb{R} \) be a twice differentiable function. Then

\[ \sum_{k=1}^{N} h(k) = \int_1^{N} h(t) \, dt + \frac{1}{2} (h(1) + h(N)) + \sum_{k=1}^{N-1} \int_{t=0}^{1} \frac{t - t^2}{2} h''(k + t) \, dt. \]
Applying this to \( h(t) = t^\alpha \), \(-1 < \alpha < 0\), we get

\[
\sum_{k=1}^{N} k^\alpha = \frac{N^{\alpha+1}}{\alpha + 1} + O((N^\alpha)) + \left( \frac{1}{2} - \frac{1}{\alpha + 1} \right) + \alpha(\alpha - 1) \sum_{k=1}^{\infty} \int_{0}^{1} \frac{t-t^2}{2} (k + t)^{\alpha-2} \, dt
\]

\[= N^{\alpha+1} / (\alpha + 1) + C_\alpha + O(N^\alpha),\]

where

\[C_\alpha = \frac{1}{2} - \frac{1}{\alpha + 1} + \alpha(\alpha - 1) \sum_{k=1}^{\infty} \int_{0}^{1} \frac{t-t^2}{2} (k + t)^{\alpha-2} \, dt .\]

Thus

\[L_N = \frac{N^{1-\sigma}}{1-\sigma} + C_\sigma + O(N^{-\sigma}).\]

Apply it now to \( h(t) = (\log t)t^{-\sigma} \). We get

\[
\sum_{k=1}^{N} \frac{\log k}{k^\sigma} = \frac{N^{1-\sigma} \log N}{1-\sigma} - \frac{N^{1-\sigma} - 1}{(1-\sigma)^2} + \frac{N^{-\sigma} \log N}{2} + C'_\sigma.
\]

Next

\[
\sum_{k=1}^{N} \frac{(\log k)^2}{k^\sigma} = \frac{N^{1-\sigma} \log N}{1-\sigma} - \frac{2N^{1-\sigma} (\log N)}{(1-\sigma)^2} + \frac{2(N^{1-\sigma} - 1)}{(1-\sigma)^3} + \frac{N^{-\sigma} (\log N)^2}{2} + C'_\sigma.
\]

We moreover have

\[
\frac{1}{L_N} = \frac{1}{\frac{N^{1-\sigma}}{1-\sigma} + C_\sigma} + O(N^{-2+\sigma}).
\]

Therefore,

\[
\sum_{k=1}^{N} \frac{(\log k)^2}{L_Nk^\sigma} = \left( \frac{1}{\frac{N^{1-\sigma}}{1-\sigma} + C_\sigma} \right) \sum_{k=1}^{N} \frac{(\log k)^2}{k^\sigma} + O(N^{-1}(\log N)^2).
\]

Now

\[
\left( \frac{1}{\frac{N^{1-\sigma}}{1-\sigma} + C_\sigma} \right) \sum_{k=1}^{N} \frac{(\log k)^2}{k^\sigma} = \left( \frac{1}{\frac{N^{1-\sigma}}{1-\sigma} + C_\sigma} \right) \left\{ \frac{N^{1-\sigma} (\log N)^2}{1-\sigma} - \frac{2N^{1-\sigma} (\log N)}{(1-\sigma)^2} + \frac{2(N^{1-\sigma} - 1)}{(1-\sigma)^3} + \frac{N^{-\sigma} (\log N)^2}{2} + C'_\sigma \right\}
\]

\[= \left( \frac{1}{1 + C_\sigma N^{\sigma-1}} \right) \left\{ (\log N)^2 - \frac{2 \log N}{1-\sigma} + \frac{2}{(1-\sigma)^2} + \frac{N^{-1}(\log N)^2}{2} + (1-\sigma)C'_\sigma N^{\sigma-1} \right\}
\]

\[= \left( \frac{1}{1 + C_\sigma N^{\sigma-1}} \right) \left\{ (\log N)^2 - \frac{2 \log N}{1-\sigma} + \frac{2}{(1-\sigma)^2} + O(N^{\sigma-1}) \right\}.
\]

We have

\[1 - \frac{1}{1 + C_\sigma N^{\sigma-1}} = O(N^{\sigma-1}).\]

Therefore

\[
\left( \frac{1}{\frac{N^{1-\sigma}}{1-\sigma} + C_\sigma} \right) \sum_{k=1}^{N} \frac{(\log k)^2}{k^\sigma} = \left( 1 + O(N^{\sigma-1}) \right) \left\{ (\log N)^2 - \frac{2 \log N}{1-\sigma} + \frac{2}{(1-\sigma)^2} + O(N^{\sigma-1}) \right\}.
\]
Choose Consequently, By reporting we get

\[ \sum_{k=1}^{N} \frac{(log k)^2}{L_N k^\sigma} = (log N)^2 - \frac{2 log N}{1 - \sigma} + \frac{2}{(1 - \sigma)^2} + O(N^{\sigma-1}(log N)^2) + O(N^{2(\sigma-1)}) \]

Similarly,

\[ \sum_{k=1}^{N} \frac{log k}{L_N k^\sigma} = (log N)^2 - \frac{2 log N}{1 - \sigma} + \frac{2}{(1 - \sigma)^2} + O(N^{\sigma-1}(log N)^2). \]

Further

\[ \sum_{k=1}^{N} \frac{log k}{L_N k^\sigma} = \left( \frac{1}{N^{1-\sigma} + C_\sigma} \right) \sum_{k=1}^{N} \frac{log k}{k^\sigma} + O(N^{-1} log N) \]

\[ = \left( \frac{1}{1 + C_\sigma(1 - \sigma)^N} \right) \left\{ \log N - \frac{1}{1 - \sigma} + N^{\sigma-1} \frac{1}{1 - \sigma} + \frac{N^{\sigma-1} log N}{2} \right\} \]

\[ = \left( 1 + O(N^{\sigma-1}) \right) \left\{ \log N - \frac{1}{1 - \sigma} + O(N^{\sigma-1}) \right\} \]

\[ = log N - \frac{1}{1 - \sigma} + O(N^{\sigma-1} log N). \]

Consequently,

\[ \frac{1}{2} E(\tilde{Y})^2 = \sum_{m=1}^{N} \frac{(log m)^2}{m^\sigma L_N} - \left( \sum_{m=1}^{N} \frac{log m}{m^\sigma L_N} \right)^2 \]

\[ = (log N)^2 - \frac{2 log N}{1 - \sigma} + \frac{2}{(1 - \sigma)^2} + O(N^{\sigma-1}(log N)^2) \]

\[ - \left( \log N - \frac{1}{1 - \sigma} + O(N^{\sigma-1} log N) \right)^2 \]

\[ = \frac{1}{(1 - \sigma)^2} + O(N^{\sigma-1}(log N)^2). \]

\[ \square \]

2.2. Proof of Theorem It follows from the previous Lemma that

\[ s_k^2 = E(\tilde{S}_k)^2 = kE(\tilde{Y})^2 = \frac{2k}{(1 - \sigma)^2} + O(kN^{\sigma-1}(log N)^2). \]

Choose \( s = 1/s_k \). Let \( g \) be a Gaussian standard random variable. Then,

\[ \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma+it/s_k}} \right|^2 \frac{dt}{\pi(t^2 + 1)} = \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} \{ E e^{-|g|} + E e^{-|\tilde{S}_k|/s_k} - E e^{-|g|} \}. \]

Hence

\[ \left| \int_{\mathbb{R}} \left| \sum_{n=1}^{N} \frac{1}{n^{\sigma+it/s_k}} \right|^2 \frac{dt}{\pi(t^2 + 1)} - \left( \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \right)^{2k} E e^{-|g|} \right| \]
metrical function. Assume that the series
\[ f = s \]
σ
By the transfert formula,
\[ C = A \]
where we used Berry–Esseen theorem’s in the last inequality, \( A \) being a universal constant. Using the plain bound \( \mathbb{E} \, |Y|^3 \leq (\log N) \mathbb{E} \, |\overline{Y}|^2 \), we therefore deduce
\[
\left| \int_R \left( \sum_{n=1}^N \frac{1}{n^{\sigma+i+t/s_k}} \right)^{2k} \frac{dt}{\pi(t^2 + 1)} \right| \leq \left( \sum_{n=1}^N \frac{1}{n^{\sigma}} \right)^{2k} \mathbb{E} \, |\overline{Y}|^3 \leq C \frac{\log N}{k^{1/2}} \left( \sum_{n=1}^N \frac{1}{n^{\sigma}} \right)^{2k}.
\]
And \( C \) is a universal constant. By taking \( c_0 = \mathbb{E} \, e^{-|g|} \), this achieves the proof. \( \square \)

3. Concluding Remarks.

The questions treated in \( [8] \) are also considered in \( [7] \) in the setting of Widom’s theory of Toeplitz integral kernels and their connection with finite sections of classical inequalities, such as Carleman or Hilbert’s inequality.

We believe that these are really interesting and motivating questions, which should deserve more investigations, notably because of the connection with Dirichlet sums and the link with Carleman or Hilbert’s inequality.

We conclude with a simple remark concerning a second application of \( (1.1) \) (using the Hilbert kernel \( H(x, y) = (x + y)^{-1} \)) given in \( [8] \), where the following formula in which \( \sigma > 1/2 \) and \( \lambda(n) \) is the Liouville function is established,

\[ \sum_{m=1}^\infty \frac{\lambda(m)}{m^2} \sum_{d|m} \frac{d^{2it}}{d + (m/d)} = \int_0^\infty \frac{\zeta(2(\sigma + i)) + 2i(t + \theta)}{\zeta(\sigma + i + i(t + \theta))} \cosh \pi \theta \, d\theta. \]

In fact, the same arguments used to establish \( (3.1) \) also apply for the kernel \( K(x, y) = \max(x, y)^{-1} \), and to other arithmetical functions. More precisely, let \( f(n) \) be a completely multiplicative arithmetical function. Assume that the series
\[ \sum_{n=1}^{\infty} \frac{|f(m)|}{m^{\sigma_0}} \]
converges for some \( \sigma_0 > 1 \). Let \( F(z) = \sum_{m=1}^{\infty} \frac{f(m)}{m^z} \). Then for \( \sigma \geq \sigma_0 - \frac{1}{2} \), (recalling that \( s = \sigma + it \)),

\[ \sum_{m=1}^{\infty} \frac{f(m)}{m^s} \left( \sum_{d|m} \frac{d^{2it}}{\max(d, (m/d))} \right) = \frac{1}{2\pi} \int_{\mathbb{R}} \left| F(\sigma + \frac{1}{2} + i(t + \theta)) \right|^2 \, d\theta \]

Indeed, by \( (1.1) \),

\[ \sum_{n,m=1}^{N} \frac{f(m)f(n)}{m^{\sigma}n^{\sigma}} K(m, n) \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{n=1}^{N} \frac{f(n)}{n^{\sigma + \frac{1}{2} + i(t + \theta)}}, \]

\[ \sum_{n,m=1}^{N} \frac{f(m)f(n)}{m^{\sigma}n^{\sigma}} K(m, n) \]

\[ = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{n=1}^{N} \frac{f(n)}{n^{\sigma + \frac{1}{2} + i(t + \theta)}}, \]
Now

\begin{equation}
(3.4) \quad \sum_{n,m=1}^N f(m)f(n) \frac{K(m,n)}{m^n n^s} = \sum_{\nu=1}^{N^2} f(\nu) \left( \sum_{\substack{d | \nu \\nu \leq d \leq N}} K(d, \nu/d) d^{2it} \right)
\end{equation}

Further

\begin{align*}
\left| \sum_{\substack{d | \nu \\nu \leq d \leq N}} K(d, \nu/d) d^{2it} \right| &\leq \sum_{\substack{d | \nu \\nu \leq d \leq N}} \frac{1}{\max(d, \nu/d)} = \frac{1}{\nu} \sum_{\substack{d | \nu \\nu \leq d \leq N}} \frac{1}{\max(d/\sqrt{\nu}, \sqrt{\nu}/d)} \leq \frac{d(\nu)}{\sqrt{\nu}},
\end{align*}

where \( d(n) \) is the divisor function (counting the number of divisors of the natural \( n \)), and we recall that \( d(n) = \mathcal{O}_\varepsilon(n^{\varepsilon}) \). Hence by assumption (3.2), (3.4) and letting \( N \) tend to infinity, the result follows.

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