R-estimators in GARCH models; asymptotics, applications and bootstrapping

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Abstract

The quasi-maximum likelihood estimation is a commonly-used method for estimating GARCH parameters. However, such estimators are sensitive to outliers and their asymptotic normality is proved under the finite fourth moment assumption on the underlying error distribution. In this paper, we propose a novel class of estimators of the GARCH parameters based on ranks, called R-estimators, with the property that they are asymptotic normal under the existence of a more than second moment of the errors and are highly efficient. We also consider the weighted bootstrap approximation of the finite sample distributions of the R-estimators. We propose fast algorithms for computing the R-estimators and their bootstrap replicates. Both real data analysis and simulations show the superior performance of the proposed estimators under the normal and heavy-tailed distributions. Our extensive simulations also reveal excellent coverage rates of the weighted bootstrap approximations. In addition, we discuss empirical and simulation results of the R-estimators for the higher order GARCH models such as the GARCH (2, 1) and asymmetric models such as the GJR model.

Keywords: R-estimation, Empirical process, Higher order GARCH models, Weighted bootstrap.

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1 Introduction

1.1 R-estimation and GARCH models

Consider observations \( \{X_t; 1 \leq t \leq n\} \) from a financial time series with the following representation

\[
X_t = \sigma_t \epsilon_t,
\]

where \( \{\epsilon_t; t \in \mathbb{Z}\} \) are unobservable i.i.d. non-degenerate error r.v.’s with mean zero and finite variance and

\[
\sigma_t = (\omega_0 + \sum_{i=1}^{p} \alpha_{0i} X_{t-i}^2 + \sum_{j=1}^{q} \beta_{0j} \sigma_{t-j}^2)^{1/2}, \quad t \in \mathbb{Z},
\]  

(1.1)

with \( \omega_0, \alpha_{0i}, \beta_{0j} > 0, \forall i, j \). In the literature, such models are known as the GARCH \((p, q)\) model and we assume that \( \{X_t; t \in \mathbb{Z}\} \) is stationary and ergodic.

Estimation of parameters based on ranks of the residuals was discussed by Koul and Ossiander (1994) for the homoscedastic autoregressive model and Mukherjee (2007) for the heterscedastic models. Andrews (2012) proposed a class of R-estimators for the GARCH model using a log-transformation of the squared observations and then minimizing a rank-based residual dispersion function. However, our R-estimators for the GARCH model are defined through the one-step approach based on an asymptotic linearity result of a rank-based central sequence and uses data directly without requiring such transformation.

Similar to the linear regression and autoregressive models, the class of the R-estimators for the GARCH model are also asymptotically normal and highly efficient. However, unlike the commonly-used quasi-maximum likelihood estimator (QMLE) which is asymptotically normal under the finite fourth moment assumption of the error distribution, the
R-estimators proposed in this paper are asymptotically normal under the assumption of only a finite \(2 + \delta\)-th moment for some \(\delta > 0\). The efficient property of the R-estimators is further confirmed based on the simulated data from the GARCH \((1,1)\) model and the higher order GARCH \((2,1)\) model to fill some void in the literature since the computation and empirical analysis for the higher order GARCH models are not considered widely. Analysis of real data shows that the numerical values of R-estimates can be different from the QMLE and the subsequent analysis of the GARCH residuals lead to conclude that such difference may be due to the assumption of the infinite fourth moment of the innovation distribution which may not hold and consequently leads to the failure of the QMLE.

Since the proposed class of the R-estimators are shown to converge to normal distributions, of which the covariance matrices do not have explicit forms, we employ a bootstrap method to approximate the distributions of the R-estimators. Chatterjee and Bose (2005) proved the consistency of the weighted bootstrap method for an estimator defined by smooth estimating equation. We consider weighted bootstrap with residual ranks that are integer-valued and non-smooth function. Our extensive simulation study provides evidence that the weighted bootstrap has good coverage rates even under a heavy-tailed distribution and for a small simple size.

Finally, we use the R-estimators for estimating parameters of the GJR \((p,q)\) model proposed by Glosten et al. (1993), which is used to estimate the asymmetry effect of financial time series. Simulation results demonstrate good performance of the R-estimators for the GJR model.

The main contributions of the paper are threefold. First, a new class of robust and efficient estimators for the GARCH model parameters is proposed. Second, the asymptotic distributions of the proposed estimators are derived based on weaker assumption on the
error moment. Third, weighted bootstrap approximations of the distribution of the R-estimators are investigated empirically through the analysis of real data and simulations. In particular, we propose algorithms for computing the R-estimators and the bootstrap replicates, which are computational friendly and easy to implement.

1.2 A motivating example

To illustrate the advantages of the class of R-estimators over the commonly-used QMLE, we consider below some simulation results corresponding to the GARCH (2, 1) model with underlying standardized innovation density (i) the normal distribution and (ii) the Student’s t-distribution with $\nu = 3$ degrees of freedom, denoted as $t(3)$. We generate $R = 1000$ samples of size $n = 1000$ with parameter values $(\omega, \alpha_1, \alpha_2, \beta)' = (0.1, 0.1, 0.1, 0.6)'$. Simulation results described below are similar to various other choices of the true parameters. Consider three types of R-estimators given in Section 2.3. The boxplots of the QMLE and the proposed R-estimates based on the normal and $t(3)$ distributions are displayed in Figure 1 and Figure 2, respectively. Here the R-estimators and QMLE are standardized by multiplying with constant matrices $A$ and $B$ defined in Section 2.5 so that all resulting estimators can estimate the parameter $\theta_0$; see Section 2 for details.

An inspection of these two plots reveals overwhelming superiority of the R-estimators over the QMLE. Under the normal error distribution, the distribution patterns of the R-estimates of each parameter are quite similar to the QMLE around the true parameter value. However, under the $t(3)$ errors, the QMLE is lot more likely to deviate from the true parameter than the three R-estimates. Therefore, different from the QMLE, the good performance of the R-estimators under the normal distribution is not at the cost of poor performance under the $t(3)$ distribution.
Figure 1: Boxplots of the QMLE and R-estimators (signs, van der Waerden, Wilcoxon) for the GARCH (2, 1) model under the normal distribution (sample size $n = 1000$; $R = 1000$ replications). The horizontal red line represents the actual parameter value.

1.3 Outline of the paper

The rest of the paper is organized as follows. Section 2 defines the R-estimators based on an asymptotic linearity result of a rank-based central sequence. The asymptotic distributions and efficiency of the R-estimators are discussed. Also, we give an algorithm for computing the R-estimators. Section 3 contains empirical and simulation results of the R-estimators. Section 4 describes the weighted bootstrap for the R-estimators and includes extensive simulation results. Section 5 considers an application to the GJR model. Conclusion is given in Section 6. The technique used to establish the asymptotic distribution is included in the supplementary material S1.
Figure 2: Boxplots of the QMLE and R-estimators (signs, van der Waerden, Wilcoxon) for the GARCH (2, 1) model under the $t(3)$ distribution (sample size $n = 1000$; $R = 1000$ replications). The horizontal red line represents the actual parameter value.

![Boxplots of the QMLE and R-estimators](image)

2 The class of R-estimators for the GARCH model

In this section, we first define a central sequence of R-criteria $\{\hat{R}_n(\theta)\}$ based on ranks of the residuals of the GARCH model. We prove the asymptotic uniform linear expansion (2.7) of this central sequence. Consequently, we define one-step R-estimator $\hat{\theta}_n$ in (2.9).

We propose a recursive algorithm for computation in Section 2.4.

Notations: Throughout the paper, for a function $g$, we use $\dot{g}$ and $\ddot{g}$ to denote its first and second derivatives whenever they exist. We use $c$, $b$, $c_1$ to denote positive constants whose values can possibly change from line to line. Let $\epsilon$ be a generic random variable (r.v.) with the same distribution as $\{\epsilon_t\}$ and let $F$ and $f$ denote the cumulative distribution function (c.d.f.) and probability density function (p.d.f.) of $\epsilon$, respectively. Let $\eta_t := \epsilon_t / \sqrt{c_\phi}$, where
$c_\varphi > 0$ satisfies (2.4), and $\eta$ be a generic r.v. with the same distribution as $\{\eta_t\}$. Let $G$ and $g$ be the c.d.f. and p.d.f. of $\eta$, respectively. A sequence of stochastic process $\{Y_n(\cdot)\}$ is said to be $u_p(1)$ (denoted by $Y_n = u_p(1)$) if for every $c > 0$, $\sup\{|Y_n(b)|; \|b\| \leq c\} = o_p(1)$, where $\|\cdot\|$ stands for the Euclidean norm.

### 2.1 Rank-based central sequence

From Lemma 2.3 and Theorem 2.1 of Berkes et al. (2003), $\sigma_t^2$ of (1.1) has the unique almost sure representation $\sigma_t^2 = c_0 + \sum_{j=1}^{\infty} c_j X_{t-j}^2$, $t \in \mathbb{Z}$, where $\{c_j; j \geq 0\}$ are defined in (2.7)-(2.9) of Berkes et al. (2003).

Let $\theta_0 = (\omega_0, \alpha_{01}, \ldots, \alpha_{0p}, \beta_{01}, \ldots, \beta_{0q})'$ denote the true parameter belonging to a compact subset $\Theta$ of $(0, \infty)^{1+p} \times (0, 1)^q$. A typical element in $\Theta$ is denoted by $\theta = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)'$. Define the variance function by

$$v_t(\theta) = c_0(\theta) + \sum_{j=1}^{\infty} c_j(\theta) X_{t-j}^2, \quad \theta \in \Theta, t \in \mathbb{Z},$$

where the coefficients $\{c_j(\theta); j \geq 0\}$ are given in (3.1) of Berkes et al. (2003) with the property $c_j(\theta_0) = c_j$, $j \geq 0$, so that the variance functions satisfy $v_t(\theta_0) = \sigma_t^2$, $t \in \mathbb{Z}$ and

$$X_t = \{v_t(\theta_0)\}^{1/2} \epsilon_t, \quad 1 \leq t \leq n.$$  

Let $\{\hat{v}_t(\theta)\}$ be observable approximation of $\{v_t(\theta)\}$, which is defined by

$$\hat{v}_t(\theta) = c_0(\theta) + I(2 \leq t) \sum_{j=1}^{t-1} c_j(\theta) X_{t-j}^2, \quad \theta \in \Theta, \quad 1 \leq t \leq n.$$
Let $H^*(x) = x\{-\dot{f}(x)/f(x)\}$. The maximum likelihood estimator (MLE) is a solution of $\Delta_{n,f}(\theta) = 0$, where

$$\Delta_{n,f}(\theta) := n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta)}{v_t(\theta)} \left\{ 1 - H^* \left[ \frac{X_t}{v_t^{1/2}(\theta)} \right] \right\}.$$  

However, $f$ in $H^*$ is usually unknown and we therefore consider an approximation to $\Delta_{n,f}(\theta)$.

Let $\varphi : (0, 1) \to \mathbb{R}$ be a score function satisfying some regularity conditions which will be discussed later. Let $R_{nt}(\theta)$ denote the rank of $X_t/v_t^{1/2}(\theta)$ among $\{X_j/v_j^{1/2}(\theta); 1 \leq j \leq n\}$. In linear regression models, the MLE has the same asymptotic efficiency as an R-estimator based on the score function $\varphi(u) = -\dot{f}(F^{-1}(u))/f(F^{-1}(u))$. For the estimation of the scale parameters, the MLE corresponds to the central sequence

$$R_n(\theta) := R_{n,\varphi}(\theta) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta)}{v_t(\theta)} \left\{ 1 - \varphi \left[ \frac{R_{nt}(\theta)}{n+1} \right] \frac{X_t}{v_t^{1/2}(\theta)} \right\}. \quad (2.2)$$

However, since $v_t(\theta)$ is unobservable, we therefore replace it by $\hat{v}_t(\theta)$. Let $\hat{R}_{nt}(\theta)$ denote the rank of $X_t/\hat{v}_t^{1/2}(\theta)$ among $\{X_j/\hat{v}_j^{1/2}(\theta); 1 \leq j \leq n\}$. We define rank-based central sequence as

$$\hat{R}_n(\theta) := \hat{R}_{n,\varphi}(\theta) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{\hat{v}}_t(\theta)}{\hat{v}_t(\theta)} \left\{ 1 - \varphi \left[ \frac{\hat{R}_{nt}(\theta)}{n+1} \right] \frac{X_t}{\hat{v}_t^{1/2}(\theta)} \right\}. \quad (2.3)$$

### 2.2 One-step R-estimators and their asymptotic distributions

To define the R-estimator in terms of the classical Le Cam’s one-step approach as in Hallin and La Vecchia (2017) and Hallin et al. (2019), we derive the asymptotic linearity of the
rank-based central sequence under the following assumptions. Let \( c_\varphi > 0 \) be defined by
\[
\sqrt{c_\varphi} = \mathbb{E} [\varphi (F (\epsilon_t)) \epsilon_t]
\]
which satisfies
\[
\mathbb{E} \left\{ \varphi [F (\epsilon_t)] \frac{\epsilon_t}{\sqrt{c_\varphi}} \right\} = 1.
\] (2.4)

Define \( \mu (x) := \int_{-\infty}^{x} t g(t) dt \). Since \( g(x) > 0 \), \( \mu (x) \) is strictly decreasing on \((-\infty, 0] \) with range \([\mu (0), 0] \) and strictly increasing on \([0, +\infty) \) with range \([\mu (0), 0] \). The functions \( y \to \mu^{-1}(y) \) on \([\mu (0), 0] \) with ranges \((-\infty, 0\] \) and \([0, +\infty) \) are well-defined when the ranges are considered separately.

The following conditions on the distribution of \( \eta_t \) are assumed for the proof of Theorem S1 in the supplementary material on the approximation of a scale-perturbed weighted mixed-empirical process by its non-perturbed version.

**Assumption (A1).** (i). The p.d.f. \( g(x), x g(x) \) and \( x^2 g(x) \) are bounded on \( x \in \mathbb{R} \); functions \( y \to \mu^{-1}(y) g(\mu^{-1}(y)) \) and \( y \to (\mu^{-1}(y))^2 g(\mu^{-1}(y)) \) are uniformly continuous on \([\mu (0), 0] \) when they are considered separately;

(ii). \[
\limsup_{\delta \to 0} \left\{ \int_0^1 |xg(x) - (x + hx\delta)g(x + hx\delta)| dh; x \in \mathbb{R} \right\} = 0;
\]

(iii). There is a \( \delta > 0 \) such that \( \mathbb{E} |\eta_t|^{2+\delta} < \infty \).

We remark that Assumption (i) entails that \( \mu (x) \) is uniformly Lipschitz continuous in scale in the sense that for some constant \( 0 < c < \infty \) and for every \( s \in \mathbb{R} \), we have
\[
\sup_{x \in \mathbb{R}} |\mu (x + xs) - \mu (x)| \leq c |s|.
\]

A more easily verifiable condition for Assumption (ii) can be obtained, for example,
when \( g \) admits the derivative \( \dot{g} \) which satisfies that for some \( \delta > 0 \),

\[
\sup\{x^2 \sup |g(y) + y\dot{g}(y)|; x(1 - \delta) < y < x(1 + \delta)\} < \infty.
\]

In particular, Assumptions (i), (ii) and (iii) in (A1) hold for a wide range of distributions, including normal, double-exponential, logistic and \( t \)-distributions with degrees of freedom more than 2.

We also need the following assumptions on the parameter space and the score function \( \varphi \).

**Assumption (A2).** Let \( \Theta_0 \) denote the set of interior points of \( \Theta \). We assume that \( \theta_0, \theta_{\varphi} \in \Theta_0 \), where

\[
\theta_{\varphi} = (c_{\varphi}\omega_0, c_{\varphi}\alpha_{01}, ..., c_{\varphi}\alpha_{0p}, \beta_{01}, ..., \beta_{0q})'
\]

is a transformation of the true parameter \( \theta_0 \).

**Assumption (A3).** The score function \( \varphi \) is non-decreasing, right-continuous with only a finite number of points of discontinuity and is bounded on \((0, 1)\).

To state the asymptotic linearity of \( \hat{R}_n(\theta) \), write \( v_t \) and \( \dot{v}_t \) for \( v_t(\theta_{\varphi}) \) and \( \dot{v}_t(\theta_{\varphi}) \), respectively. Let

\[
\gamma(\varphi) := \int_0^1 \int_0^1 G^{-1}(u)G^{-1}(v) \{\min\{u, v\} - uv\} d\varphi(u)d\varphi(v), \quad J := E(\dot{v}_t \dot{v}'_t/v_t^2)
\]

\[
\rho(\varphi) := \int_0^1 \{G^{-1}(u)\}^2 g\{G^{-1}(u)\} d\varphi(u), \quad \sigma^2(\varphi) := E\{\varphi[G(\eta_t)]\eta_t\}^2 - 1,
\]
\[ \lambda(\varphi) := \int_0^1 \int_0^1 G^{-1}(u)I(v \leq u)(1 - G^{-1}(v)\varphi(v))dv d\varphi(u). \quad (2.6) \]

Let \( Z \) be the r.v. \( Z := \int_0^1 G^{-1}(u)B(u)d\varphi(u) \), where \( B(.) \) is the standard Brownian bridge. Then \( Z \) has mean zero and variance \( \gamma(\varphi) \); see the proof in Lemma S5 for details. Let \( \tilde{G}_n(x), x \in \mathbb{R} \) be the empirical distribution function of \( \{\eta_t\} \) (which is unobservable),

\[ Q_n := \int_0^1 n^{-1/2} \sum_{t=1}^n \frac{\dot{v}_t}{v_t} \left[ \mu(G^{-1}(u)) - \mu(\tilde{G}_n^{-1}(u)) \right] d\varphi(u), \]
\[ N_n := n^{-1/2} \sum_{t=1}^n \frac{\dot{v}_t}{v_t} \{1 - \eta_t \varphi[G(\eta_t)]\}. \]

The following proposition states the asymptotic uniform linearity of \( \hat{R}_n(\theta) \).

**Proposition 2.1.** Let Assumptions (A1)-(A3) hold. Then for \( b \in \mathbb{R}^{1+p+q} \) with \( \|b\| < c \),

\[ \hat{R}_n(\theta_0 \varphi + n^{-1/2}b) - \hat{R}_n(\theta_0 \varphi) = (1/2 + \rho(\varphi)/2)Jb + u_p(1). \quad (2.7) \]

Moreover,

\[ \hat{R}_n(\theta_0 \varphi) = Q_n + N_n + u_p(1), \quad (2.8) \]

where \( Q_n \) converges in distribution to \( E(\dot{v}_1/v_1)Z \) with mean zero and covariance matrix \( E(\dot{v}_1/v_1)E(\dot{v}_1'/v_1')\gamma(\varphi) \) and \( N_n \to N(0, J\sigma^2(\varphi)) \).

The above asymptotic linearity allows us to define a class of R-estimators through the one-step approach. Let \( \{\tilde{\Upsilon}_n\} \) be a sequence of consistent estimator of \( \Upsilon_{\varphi,g}(\theta_0 \varphi) := (1/2 + \rho(\varphi)/2)J \); see Section 2.4 for a construction of \( \tilde{\Upsilon}_n \). Let \( \tilde{\theta}_n \) be a root-\( n \) consistent and asymptotically discrete estimator of \( \theta_0 \varphi \). Here asymptotically discreteness is only of
theoretical interest since in practice $\hat{\theta}_n$ always has a bounded number of digits; see Le Cam and Yang (2000, Chapter 6) and van der Vaart (1998, Section 5.7) for more details. Then the one-step R-estimator is defined as

$$\hat{\theta}_n := \theta_n - n^{-1/2} \left( \hat{Y}_n \right)^{-1} \hat{R}_n(\theta_n). \tag{2.9}$$

Note that strictly speaking, the R-estimators based on this definition are not functions of the ranks of the residuals only. However, we borrow the terminology from the regression and the homoscedastic-autoregression settings and still call them (generalized) R-estimators. When, for example, $\varphi(u) = u - 1/2$, $\hat{\theta}_n$ is an analogue of the Wilcoxon type R-estimator.

The following theorem shows that the R-estimator defined in (2.3) is $\sqrt{n}$-consistent estimator of $\theta_{0\varphi}$. The proof is given in the supplementary material S1.

**Theorem 2.1.** Let Assumptions (A1)-(A3) hold. Then, as $n \to \infty$,

$$\sqrt{n} \left( \hat{\theta}_n - \theta_{0\varphi} \right) = -(1/2 + \rho(\varphi)/2)^{-1} J^{-1} (Q_n + N_n) + o_p(1). \tag{2.10}$$

Hence as $n \to \infty$, $\sqrt{n} \left( \hat{\theta}_n - \theta_{0\varphi} \right)$ is normal with mean 0 and covariance matrix

$$J^{-1} [4\gamma(\varphi) + 8\lambda(\varphi)] E(\dot{v}_1/v_1) E(\dot{v}'_1/v_1) + 4\sigma^2(\varphi) J^{-1}.$$

We remark that according to Theorem 2.1, the asymptotic covariance matrix of $\hat{\theta}_n$ has a complicated form. Hence we consider bootstrap methods in Section 4 to approximate the limit distribution of $\sqrt{n} \left( \hat{\theta}_n - \theta_{0\varphi} \right)$.

Notice that similar to the M-estimator for the GARCH model, $\hat{\theta}_n$ turns out to be an estimator of transformed parameter $\theta_{0\varphi}$ defined in (2.5), where the factor $c_\varphi$ is unknown.
since the distribution of $\epsilon_t$ is unknown. Although $c_\varphi$ is unknown, one can still apply the R-estimator to get a consistent estimate of some important quantity in financial risk management such as the Value at Risk (VaR) when the returns are modeled as GARCH $(p,q)$. This is because $c_\varphi$ is cancelled in the estimation process. See, e.g., Iqbal and Mukherjee (2010) for how to estimate the VaR based on the GARCH model.

2.3 Examples of the score functions

Below we cite examples of three commonly-used R-scores; for similar examples of scores in other models, see Mukherjee (2007) and Hallin and La Vecchia (2017).

**Example 1** (sign score). Let $\varphi(u) = \text{sign}(u - 1/2)$. Then for symmetric innovation distribution, $c_\varphi = (E|\epsilon|)^2$, which coincides with the scale factor of the LAD estimator in Mukherjee (2008). Therefore, the sign R-estimator is expected to be close to the LAD estimator. This is demonstrated later in the real data analysis.

**Example 2** (Wilcoxon score). Let $\varphi(u) = u - 1/2$ so that the range of $\varphi(u)$ is symmetric.

**Example 3** (van der Waerden (vdW) or normal score). One might also set $\varphi(u) = \Phi^{-1}(u)$, with $\Phi(\cdot)$ denoting the c.d.f. of the standard normal distribution. Notice that unlike the sign and Wilcoxon score, the vdW score is not bounded as $u \to 0$ and $u \to 1$. It thus does not satisfy Assumption (A3). However, an approximating sequence of bounded score functions of $\varphi$ on $(0,1)$ can be constructed as in Andrews (2012). It is demonstrated later using both real data analysis and extensive simulation that the vdW has superior performance compared with the QMLE.

We now provide heuristics for the definition of the R-estimator in (2.2). When the underlying error distribution is known, one can obtain efficient R-estimator by choosing the score function as $\varphi(u) = -\hat{f}(F^{-1}(u))/f(F^{-1}(u))$. Since for large $n$, the empirical
distribution function $R_{nt}(\theta_0\varphi)/(n+1)$ of $\{\epsilon_j; 1 \leq j \leq n\}$ evaluated at $\epsilon_t$ is close to $F(\epsilon_t)$, we have
\[
\varphi \left[ \frac{R_{nt}(\theta)}{n+1} \right] \frac{X_t}{v_t^{1/2}(\theta)} \approx H^r \left[ \frac{X_t}{v_t^{1/2}(\theta)} \right].
\]
Therefore, the criteria function of the R-estimator gets close to the MLE which is efficient. This leads to the choice of the vdW, sign and Wilcoxon under the normal, double exponential (DE) and logistic distributions, respectively. This is observed later in simulation study of the R-estimator.

2.4 Computational aspects

Here we discuss some key computational aspects and propose an algorithm to compute $\hat{\theta}_n$.

First, since $c_\varphi$ depends on the unknown density $f$, it is difficult to have a $\sqrt{n}$-consistent initial estimator $\bar{\theta}_n$ of $\theta_{0\varphi}$. However, due to finite sample size in practice, the one-step procedure is usually iterated a number of times, taking $\hat{\theta}_n$ as the new initial estimate, until it stabilizes numerically. This iteration process would mitigate the impact of different initial estimates; see van der Vaart (1998, Section 5.7) and Hallin and La Vecchia (2017) for similar comments. In fact, we observed during our extensive simulation study that irrespective of the choice of the QMLE, LAD or $\theta_0$ as initial estimates, only few iterations result in the same estimates.

Second, to compute $\hat{\theta}_n$ of (2.9), we need $\hat{\Upsilon}_n$ which is a consistent estimator of $(1/2 + \rho(\varphi)/2)J$. The matrix $J$ can be consistently estimated by
\[
\hat{J}_n(\bar{\theta}_n) := n^{-1} \sum_{t=1}^n \{\hat{v}_t(\bar{\theta}_n)\hat{v}_t'(\bar{\theta}_n)/\hat{v}_t^2(\bar{\theta}_n)\}.
\]
For estimating $\rho(\varphi)$ which is a function of the density $g$, we can use the asymptotic linearity in (2.7). Here with an arbitrarily chosen $b$, we can substitute $\hat{\theta}_n$ for $\theta_{0\varphi}$ and then solve the equation for $\rho(\varphi)$ based on (2.7). A more delicate approach for estimating $\rho(\varphi)$ can be found in Cassart et al. (2010) and Hallin and La Vecchia (2017, Appendix C). Based on our extensive simulation study and real data analysis, it appears that different values of $\rho(\varphi)$ would finally lead to same estimate after some iterations. Consequently, we set $\rho(\varphi) = 1$ during the computation which is the value corresponding to the vdW score under the normal distribution.

In summary, we propose the following iterative algorithm to compute the R-estimator.

\[
\hat{\theta}_{(r+1)} = \hat{\theta}_{(r)} - \left[ \sum_{t=1}^{n} \frac{\dot{v}_t(\hat{\theta}_{(r)}) \dot{v}_t'(\hat{\theta}_{(r)})}{\tilde{v}_t^2(\hat{\theta}_{(r)})} \right]^{-1} \times \left\{ \sum_{t=1}^{n} \frac{\dot{v}_t(\hat{\theta}_{(r)})}{\tilde{v}_t(\hat{\theta}_{(r)})} \left[ 1 - \varphi \left( \frac{R_{nt}(\hat{\theta}_{(r)})}{n+1} \right) \frac{X_t}{\tilde{v}_t^{1/2}(\hat{\theta}_{(r)})} \right] \right\}, \quad \text{for} \quad r = 0, 1, \ldots,
\]

(2.11)

with $\hat{\theta}_{(0)} = \hat{\theta}_n$ being the initial estimator.

### 2.5 Asymptotic relative efficiency

In the linear regression and autoregressive models, the asymptotic relative efficiency (ARE) of the R-estimators is high with respect to (wrt) the least squares estimator for a wide array of error distributions. For the GARCH model, we compare the ARE of the R-estimator wrt the QMLE based on Theorem 2.1.
Define a diagonal matrix of order \((1 + p + q) \times (1 + p + q)\) by

\[
A := \text{diag}(c_{-1}^{-1}, \ldots, c_{-1}^{-1}, 1, \ldots, 1).
\]

Then \(A\hat{\theta}_n\) is a \(\sqrt{n}\)-consistent estimator of \(\theta_0\) for all score functions \(\varphi\). Using the forms of \(\{c_j(\theta); j \geq 0\}\) in (3.1) of Berkes et al. (2003),

\[
v_t(A^{-1}\theta) = c_{\varphi}v_t(\theta), \quad \dot{v}_t(A^{-1}\theta) = c_{\varphi}A\dot{v}_t(\theta).
\]

Since \(\theta_{0,\varphi} = A^{-1}\theta_0\) with \(J_0 := \mathbb{E}[\dot{v}_t(\theta_0)\dot{v}_t(\theta_0)/v_t^2(\theta_0)]\), we have

\[
\frac{\dot{v}_t(\theta_{0,\varphi})}{v_t(\theta_{0,\varphi})} = A\frac{\dot{v}_t(\theta_0)}{v_t(\theta_0)}, \quad J = AJ_0A.
\]

Thus Theorem 2.1 implies that as \(n \to \infty\), \(n^{-1/2}(A\hat{\theta}_n - \theta_0)\) converges to the normal distribution with mean 0 and covariance matrix

\[
C_1 := J_0^{-1}\left[4\gamma(\varphi) + 8\lambda(\varphi)\right]E(\dot{v}_1(\theta_0)/v_1(\theta_0))E(\dot{v}_1(\theta_0)/v_1(\theta_0)) + 4\sigma^2(\varphi)J_0J_0^{-1}.
\]

For the QMLE \(\hat{\theta}_{QMLE}\), we can derive a similar result as follows. Define a \((1 + p + q) \times (1 + p + q)\) diagonal matrix \(B = \text{diag}((E\epsilon)^{1+q})^{-1},\ldots,(E\epsilon)^{1+q})^{-1},1,\ldots,1\). When \(E\epsilon < \infty\),

\[
\sqrt{n}(B\hat{\theta}_{QMLE} - \theta_0) \to N(0, C_2),
\]

16
where \( C_2 = (E\epsilon^4/(E\epsilon^2)^2 - 1)J_0^{-1} \). The ARE of the R-estimator wrt the QMLE is

\[
C_1^{-1}C_2 = J_0\left\{[4\gamma(\varphi) + 8\lambda(\varphi)]E\left(\frac{\dot{v}_1(\theta_0)}{v_1(\theta_0)}\right)E\left(\frac{\dot{v}_1'(\theta_0)}{v_1'(\theta_0)}\right) + 4\sigma^2(\varphi)J_0\right\}^{-1}
\times (1 + \rho(\varphi))^2 \left(\frac{E\epsilon^4}{(E\epsilon^2)^2 - 1}\right).
\]

For the sign R-estimator, \( \gamma(\varphi) \), \( \lambda(\varphi) \) and \( \rho(\varphi) \) are all zeros. Hence \( C_1^{-1}C_2 \) reduces to

\[
(\frac{E\epsilon^4}{(E\epsilon^2)^2 - 1})/(4\sigma^2(\varphi))I_{1+p+q},
\]

where \( I_{1+p+q} \) is the \((1 + p + q) \times (1 + p + q)\) identity matrix. Consequently, the ARE of the sign R-estimator wrt the QMLE equals

\[
(\frac{E\epsilon^4}{(E\epsilon^2)^2 - 1})/(4\sigma^2(\varphi)),
\]

which is 0.876 under the normal distribution. This corresponds to the classical result of the ARE of the mean absolute deviation wrt the mean square deviation; see, e.g., Huber and Ronchetti (2011, Chapter 1).

For the vdW and Wilcoxon R-estimators, the AREs are more difficult to calculate since \( \gamma(\varphi) \) and \( \lambda(\varphi) \) are non-zero. However, in the following simulation study in Table 2, the estimated AREs reveal that the vdW R-estimator, compared with the QMLE, does not lose any efficiency even under the normal distribution.

**Scale-transformation invariance of the ARE.** The following proposition states that the ARE of the R-estimator wrt the QMLE enjoys the invariance property of the scale-transformation in terms of the innovation. Hence, with underlying distributions of the same type but with different variances, the AREs remain the same. In particular,
let $E(\epsilon)$ denote the ARE when the innovation term is $\{\epsilon_t\}$. Then we have the following proposition.

**Proposition 2.2.** Under Assumptions (A1)-(A3), $E(\tilde{\epsilon}) = E(\epsilon)$, where $\tilde{\epsilon} = c\epsilon$ for a constant $c$.

Proof. Let $\tilde{c}_\varphi$, $\tilde{\eta}_t$, $\tilde{v}_t(\theta)$, $\tilde{\sigma}^2(\varphi)$, $\tilde{\rho}(\varphi)$, $\tilde{\gamma}(\varphi)$, $\tilde{\lambda}(\varphi)$ denote the counterparts of $c_\varphi$, $\eta_t$, $v_t(\theta)$, $\sigma^2(\varphi)$, $\rho(\varphi)$, $\gamma(\varphi)$, $\lambda(\varphi)$ when the innovation term is $\{\tilde{\epsilon}_t\}$. Since $\tilde{c}_\varphi = c^2c_\varphi$, we have $\tilde{\eta}_t = \eta_t$ and this implies that

$$\tilde{\sigma}^2(\varphi) = \sigma^2(\varphi), \tilde{\rho}(\varphi) = \rho(\varphi), \tilde{\gamma}(\varphi) = \gamma(\varphi), \tilde{\lambda}(\varphi) = \lambda(\varphi), E\tilde{\epsilon}^4/(E\epsilon^2)^2 = E\epsilon^4/(E\epsilon^2)^2.$$

Thus, in view of (2.12), we obtain $e(\tilde{\epsilon}) = e(\epsilon)$. \qed

Using the similar method of proving Proposition 2.2, it is easy to show that the scale-transformation invariance of the ARE also holds in terms of the score function. Specifically, let $e(\varphi)$ denote the ARE when the score function is $\varphi$. Then $e(\tilde{\varphi}) = e(\varphi)$ for $\tilde{\varphi} = c\varphi$ for a constant $c > 0$.

### 3 Real data analysis and simulation results

This section examines the performance of the R-estimators and compare them with the QMLE by analysing three financial time series and by carrying out extensive Monte Carlo simulation.
3.1 Real data analysis

In this section we fit GARCH (1, 1) model to three financial time series and compare the proposed three R-estimators with the M-estimators QMLE and LAD discussed in Mukherjee (2008).

In an earlier work, Muler and Yohai (2008) fitted the the GARCH (1, 1) model to the Electric Fuel Corporation (EFCX) time series for the period of January 2000 to December 2001 with sample size $n = 498$. The parameters of the model are estimated by M-estimators based on various score functions. It turned out that the M-estimates of the parameter $\beta$ differ widely depending on the score functions and so it is difficult to assess which estimate should be relied on in similar situations. Here we compare various M-estimates and R-estimates of the GARCH (1, 1) parameters for the EFCX series again shedding light on which could be some possible reasons for the difference in estimates and finally which estimation methods can be relied upon. We also compare M-estimates of the GARCH (1, 1) parameters when fitted to two other dataset, namely, the S&P 500 stock index from June 2013 to May 2017 with $n = 1005$ and the GBP/USD exchange rate from June 2013 to May 2017 with $n = 998$ to illustrate that the M- and R-estimates of $\beta$ do not differ widely when the underlying theoretical assumptions hold in general.

In Table 1, we report the QMLE computed using the fGarch package in R program, the M-estimates QMLE and LAD and the R-estimates proposed in Examples 1-3 of Subsection 2.3. For the EFCX data, the R-estimates of $\beta$ for all score functions are quite close to the LAD estimate, but they are very different than the QMLE. On the contrary, for the S&P 500 and GBP/USD data, all these estimates of $\beta$ are close to each other. We can also find that the LAD estimates of all parameters are quite close to the sign estimates and this is consistent with the discussion in Example 1 of Subsection 2.3. Note also that for $\omega$ and
Table 1: The QMLE, LAD and R-estimates (sign, Wilcoxon and vdW) of the GARCH (1, 1) parameter for the EFCX, S&P 500 and GBP/USD data.

|       | fGarch | QMLE  | LAD   | sign  | Wilcoxon | vdW   |
|-------|--------|-------|-------|-------|----------|-------|
| **EFCX** |        |       |       |       |          |       |
| \(\omega\) | 1.89×10^{-4} | 6.28×10^{-4} | 6.43×10^{-4} | 6.27×10^{-4} | 8.44×10^{-5} | 1.12×10^{-3} |
| \(\alpha\) | 4.54×10^{-2} | 7.20×10^{-2} | 8.87×10^{-2} | 8.81×10^{-2} | 1.09×10^{-2} | 0.12 |
| \(\beta\) | 0.92 | 0.84 | 0.66 | 0.65 | 0.67 | 0.69 |
| **S&P 500** |        |       |       |       |          |       |
| \(\omega\) | 6.50×10^{-6} | 7.02×10^{-6} | 3.02×10^{-6} | 3.02×10^{-6} | 4.02×10^{-7} | 5.97×10^{-6} |
| \(\alpha\) | 0.18 | 0.18 | 0.11 | 0.11 | 1.41×10^{-2} | 0.18 |
| \(\beta\) | 0.72 | 0.70 | 0.73 | 0.73 | 0.73 | 0.72 |
| **GBP/USD** |        |       |       |       |          |       |
| \(\omega\) | 5.32×10^{-7} | 1.02×10^{-6} | 3.64×10^{-7} | 3.64×10^{-7} | 5.23×10^{-8} | 8.73×10^{-7} |
| \(\alpha\) | 0.11 | 0.13 | 3.74×10^{-2} | 3.74×10^{-2} | 5.18×10^{-3} | 8.53×10^{-2} |
| \(\beta\) | 0.88 | 0.85 | 0.91 | 0.91 | 0.91 | 0.89 |

\(\alpha\), the R-estimates are quite different since \(c_{\nu}\)'s of these scores have different values.

To investigate why the QMLE of \(\beta\) is different from the other R-estimates and LAD for the EFCX data, we check the assumption \(E\epsilon^4 < \infty\) for this data by using the QQ-plots of the residuals based on the QMLE and the R-estimates corresponding to the vdW score against \(t\) distributions. We consider the vdW score only since the R-estimates based on two other score functions and the LAD are close to the vdW estimates. For comparison, we have also provided QQ-plots for the S&P 500 data. The main idea behind the QQ-plots of the residuals against the \(t(d)\) distribution is simple. Since if \(\epsilon \sim t(d)\) distribution then \(E|\epsilon|^\nu < \infty\) if and only if \(\nu < d\), residuals with heavier tail than the \(t(d)\) distribution correspond to the errors with the infinite \(d\)-th moment while those with thinner tail than the \(t(d)\) distribution have the finite \(d\)-th error moment.
The top-left panel of Figure 1 shows the QQ-plot of the residuals against the $t(4.01)$ distribution for the EFCX data. The residuals have heavier right tail than the $t(4.01)$ distribution which implies that the fourth moment of the error term may not exist. On the other hand, the QQ-plot against the $t(3.01)$ distribution reveals lighter tail as shown at the bottom-left panel of Figure 1 and this implies that $E|\epsilon|^3 < \infty$.

For the S&P 500 data, the QQ-plot against $t(4.01)$ distribution at the top-right panel of Figure 1 shows that the residuals have lighter tails than $t(4.01)$ distribution. For the QQ-plot against $t(6.01)$ distribution, as shown at the bottom-right panel of Figure 1, the residuals fit the distribution better. Therefore, we may conclude that $E|\epsilon|^4 < \infty$ holds for the S&P 500 data.

3.2 Simulation study of the R-estimators

We now evaluate the performance of the R-estimators based on simulated data from various error distributions. Apart from the GARCH (1, 1) model we consider the GARCH (2, 1) model also as the computation for higher order models are not considered frequently in the literature. Let $R$ denote the number of replications and $\hat{\theta}_{ni} = (\hat{\omega}_i, \hat{\alpha}_{i1}, ..., \hat{\alpha}_{ip}, \hat{\beta}_{i1}, ..., \hat{\beta}_{iq})'$ denote the R-estimator computed from the $i$-th data, $1 \leq i \leq n$. Note that $\hat{\theta}_{ni}$ is an estimator of $\theta_{0\varphi}$, which depends on the score function used in the estimation. To compare R-estimates based on different score functions fairly, we consider the standardized bias defined by

$$\frac{1}{R} \sum_{i=1}^{R} \left( \frac{\hat{\omega}_i}{\sigma_\varphi} - \omega_0, \frac{\hat{\alpha}_{i1}}{\sigma_\varphi} - \alpha_{01}, ..., \frac{\hat{\alpha}_{ip}}{\sigma_\varphi} - \alpha_{0p}, \frac{\hat{\beta}_{i1}}{\sigma_\varphi} - \beta_{01}, ..., \frac{\hat{\beta}_{iq}}{\sigma_\varphi} - \beta_{0q} \right)'$$
Figure 3: QQ-plots of the residuals against $t$-distributions for the EFCX (left column) and S&P 500 data (right column); the residuals are obtained by using the vdW R-estimator.
and the standardized MSE defined by

\begin{equation}
\frac{1}{R} \sum_{i=1}^{R} \left( \left( \frac{\hat{\omega}_i}{c} - \omega_0 \right)^2, \left( \frac{\hat{\alpha}_{i1}}{c} - \alpha_{01} \right)^2, \ldots, \left( \frac{\hat{\alpha}_{ip}}{c} - \alpha_{0p} \right)^2, \left( \hat{\beta}_{i1} - \beta_{01} \right)^2, \ldots, \left( \hat{\beta}_{iq} - \beta_{0q} \right)^2 \right)'.
\end{equation}

We also compare the relative efficiency of the R-estimators wrt the QMLE under a finite sample size, as an estimate of the ARE, by using the formula

\begin{equation}
\overline{\text{ARE}}_{R/QMLE} = \frac{\overline{\text{MSE}}_{QMLE}}{\overline{\text{MSE}}_{R}}.
\end{equation}

**Simulation for the GARCH (1,1) model.** Here we run simulation with $R = 500$, $n = 1000$ and $\theta_0 = (6.50 \times 10^{-6}, 0.177, 0.716)'$, where our choice of $\theta_0$ is motivated by the estimate given by the fGarch for the S&P 500 data in Table 1. The estimates of the standaradized bias and MSE of the R-estimators and QMLE under various error distributions are reported in Table 2. The estimates of the ARE are shown in the parentheses. Notice that under $t(3)$ distribution, the QMLE does not converge for many replications, while the R-estimators always converge. Therefore, the bias and MSE are obtained using the replications where the QMLE converges.

It is worth noting that the vdW achieves almost the same efficiency as the QMLE under the normal distribution, and the vdW is more efficient under heavier-tailed distributions. In general, the sign score is most efficient under the DE and $t(3)$ distributions, while the Wilcoxon score is optimal under the logistic distribution. Under the $t(3)$ error distribution with infinite fourth moment, the R-estimators outperform the QMLE in terms of both bias and MSE.

To strengthen the point that the R-estimators behave better than the QMLE under a heavy-tailed distribution, we have reported simulation results for larger sample sizes.
Table 2: The estimates of the standardized bias, MSE and ARE of the R-estimators (sign, Wilcoxon and vdW) and the QMLE for the GARCH (1, 1) model under various error distributions with sample size $n = 1000$ based on $N = 500$ replications.

| Error Distribution | $\omega$ | $\alpha$ | $\beta$ | $\omega$ | $\alpha$ | $\beta$ |
|--------------------|----------|----------|----------|----------|----------|----------|
| Normal             | QMLE     | $8.96 \times 10^{-7}$ | $-4.42 \times 10^{-4}$ | $-1.54 \times 10^{-2}$ | $6.45 \times 10^{-12}$ | $1.41 \times 10^{-3}$ | $4.14 \times 10^{-3}$ |
|                    | Sign     | $9.30 \times 10^{-7}$ | $1.74 \times 10^{-3}$ | $-1.54 \times 10^{-2}$ | $8.39 \times 10^{-12}$ (0.77) | $1.62 \times 10^{-3}$ (0.87) | $5.16 \times 10^{-3}$ (0.80) |
|                    | Wilcoxon | $1.02 \times 10^{-6}$ | $3.09 \times 10^{-3}$ | $-1.61 \times 10^{-2}$ | $8.52 \times 10^{-12}$ (0.76) | $1.54 \times 10^{-3}$ (0.91) | $4.93 \times 10^{-3}$ (0.84) |
|                    | vdW      | $9.05 \times 10^{-7}$ | $4.55 \times 10^{-4}$ | $-1.55 \times 10^{-2}$ | $6.44 \times 10^{-12}$ (1.00) | $1.43 \times 10^{-3}$ (0.98) | $4.15 \times 10^{-3}$ (1.00) |
| Logistic           | QMLE     | $1.02 \times 10^{-6}$ | $3.56 \times 10^{-3}$ | $-2.26 \times 10^{-2}$ | $8.60 \times 10^{-12}$ | $2.37 \times 10^{-3}$ | $6.29 \times 10^{-3}$ |
|                    | Sign     | $5.82 \times 10^{-7}$ | $-3.42 \times 10^{-3}$ | $-1.69 \times 10^{-2}$ | $6.22 \times 10^{-12}$ (1.38) | $1.74 \times 10^{-3}$ (1.36) | $5.15 \times 10^{-3}$ (1.22) |
|                    | Wilcoxon | $6.24 \times 10^{-7}$ | $-2.93 \times 10^{-3}$ | $-1.74 \times 10^{-2}$ | $6.34 \times 10^{-12}$ (1.36) | $1.76 \times 10^{-3}$ (1.35) | $5.12 \times 10^{-3}$ (1.23) |
|                    | vdW      | $6.22 \times 10^{-7}$ | $-4.13 \times 10^{-3}$ | $-1.96 \times 10^{-2}$ | $6.51 \times 10^{-12}$ (1.32) | $1.88 \times 10^{-3}$ (1.26) | $5.45 \times 10^{-3}$ (1.15) |
| $t(3)$             | QMLE     | $1.05 \times 10^{-6}$ | $2.51 \times 10^{-3}$ | $-1.51 \times 10^{-2}$ | $7.44 \times 10^{-12}$ | $1.63 \times 10^{-3}$ | $4.28 \times 10^{-3}$ |
|                    | Sign     | $6.85 \times 10^{-7}$ | $-2.65 \times 10^{-3}$ | $-1.17 \times 10^{-2}$ | $5.40 \times 10^{-12}$ (1.38) | $1.42 \times 10^{-3}$ (1.15) | $3.66 \times 10^{-3}$ (1.17) |
|                    | Wilcoxon | $6.82 \times 10^{-7}$ | $-2.91 \times 10^{-3}$ | $-1.19 \times 10^{-2}$ | $5.24 \times 10^{-12}$ (1.42) | $1.38 \times 10^{-3}$ (1.18) | $3.56 \times 10^{-3}$ (1.20) |
|                    | vdW      | $7.06 \times 10^{-7}$ | $-3.80 \times 10^{-3}$ | $-1.34 \times 10^{-2}$ | $5.66 \times 10^{-12}$ (1.31) | $1.42 \times 10^{-3}$ (1.14) | $3.83 \times 10^{-3}$ (1.12) |

$n = 3000$ and $n = 5000$ under $t(3)$ distribution in Table 3. The QMLE failed to converge for large sample size; for example, with $n = 5000$ around 8% replications do not converge. From Table 3, when $n$ increases, the performance of the R-estimators becomes even better in terms of both the bias and MSE.

Overall, the vdW dominates the QMLE and other R-estimators sacrifice small efficiency under the normal error distribution while they achieve much higher efficiency when tails become much heavier. This provides a strong support for using the R-estimators.

Simulation for the GARCH (2, 1) model. It was reported in Francq and Zakoïan (2009) that higher order GARCH models may fit some financial time series better than the GARCH (1, 1) model. Therefore, here we examine the performance of the R-estimators
Table 3: The estimates of the standardized bias, MSE and ARE of the R-estimators (sign, Wilcoxon and vdW) and the QMLE for the GARCH (1,1) model under the $t(3)$ error distribution with larger sample sizes $n = 3000, 5000$ based on $R = 500$ replications.

|               | $n = 3000$       | $n = 5000$       |
|---------------|------------------|------------------|
|               | $\omega$ | $\alpha$ | $\beta$ | $\omega$ | $\alpha$ | $\beta$ | $\omega$ | $\alpha$ | $\beta$ |
| QMLE          | 6.34 x 10^{-7} | 1.80 x 10^{-2} | -3.48 x 10^{-2} | 1.14 x 10^{-11} | 1.61 x 10^{-2} | 1.25 x 10^{-2} | 3.66 x 10^{-7} | 1.20 x 10^{-2} | -2.07 x 10^{-2} | 8.21 x 10^{-12} | 1.20 x 10^{-2} | 8.22 x 10^{-3} |
| Sign          | 1.52 x 10^{-7} | 1.46 x 10^{-3} | -9.99 x 10^{-3} | 1.65 x 10^{-12} | 1.29 x 10^{-3} | (6.89) | 2.10 x 10^{-3} | (5.93) |
| Wilcoxon      | 1.61 x 10^{-7} | 1.47 x 10^{-3} | -1.03 x 10^{-2} | 1.76 x 10^{-12} | 1.35 x 10^{-3} | (11.95) | 2.22 x 10^{-3} | (5.63) |
| vdW           | 1.58 x 10^{-7} | 1.01 x 10^{-3} | -1.39 x 10^{-2} | 2.46 x 10^{-12} | 1.89 x 10^{-3} | (8.49) | 3.15 x 10^{-3} | (3.96) |

under the GARCH (2, 1) model by running simulations with $R = 500, n = 1000$. For choosing the true model parameters for the simulations, we fitted the FTSE 100 data from January 2007 to December 2009 to the by GARCH (2, 1) model using the fGarch package. It turned out that $\alpha_2$ is significant with $p$-value = 0.019 and the Akaike information criterion (AIC) of the GARCH (2, 1) is smaller than that of the GARCH (1, 1). Since the fGarch estimate of the true parameter is $\theta_0 = (4.46 \times 10^{-6}, 0.0525, 0.108, 0.832)'$, we choose this $\theta_0$ to generate sample from the GARCH (2, 1) model with various error distributions. The R-estimators and QMLE are compared through the standardized bias and MSE and the corresponding estimates are reported in Table 4. Similar to the GARCH (1, 1) case, the advantage of the R-estimators over the QMLE becomes prominent under heavy-tailed distributions, especially under the $t(3)$ distribution, where the bias and MSE of the R-estimators have smaller order of magnitude than the those of the QMLE.
Table 4: The estimates of the standardized bias and MSE of the R-estimators (sign, Wilcoxon and vdW scores) and the QMLE for the GARCH (2,1) model under various error distributions (sample size \( n = 1000; \ R = 500 \) replications).

|        | Standardized bias |             | Standardized MSE |             |
|--------|-------------------|-------------|------------------|-------------|
|        | \( \omega \)     | \( \alpha_1 \) | \( \alpha_2 \)   | \( \beta \) | \( \omega \) | \( \alpha_1 \) | \( \alpha_2 \) | \( \beta \) |
| Normal | QMLE              | 3.80\times10^{-6} | 8.85\times10^{-3} | -3.16\times10^{-3} | -2.01\times10^{-2} | 2.50\times10^{-11} | 1.71\times10^{-3} | 1.93\times10^{-3} | 1.35\times10^{-3} |
|        | Sign              | 3.79\times10^{-6} | 1.05\times10^{-2} | -5.76\times10^{-3} | -1.84\times10^{-2} | 2.65\times10^{-11} | 1.90\times10^{-3} | 2.19\times10^{-3} | 1.30\times10^{-3} |
|        | Wilcoxon          | 3.68\times10^{-6} | 9.91\times10^{-3} | -6.51\times10^{-3} | -1.81\times10^{-2} | 2.42\times10^{-11} | 1.74\times10^{-3} | 2.01\times10^{-3} | 1.20\times10^{-3} |
|        | vdW               | 3.95\times10^{-6} | 1.03\times10^{-2} | -7.49\times10^{-3} | -1.96\times10^{-2} | 2.67\times10^{-11} | 1.74\times10^{-3} | 1.94\times10^{-3} | 1.25\times10^{-3} |
| DE     | QMLE              | 2.61\times10^{-6} | 4.43\times10^{-3} | 2.14\times10^{-3}  | -1.99\times10^{-2} | 3.11\times10^{-11} | 2.53\times10^{-3} | 4.01\times10^{-3} | 2.33\times10^{-3} |
|        | Sign              | 1.96\times10^{-6} | 5.02\times10^{-3} | 2.57\times10^{-4}  | -1.61\times10^{-2} | 9.96\times10^{-12} | 1.85\times10^{-3} | 2.88\times10^{-3} | 1.66\times10^{-3} |
|        | Wilcoxon          | 1.85\times10^{-6} | 3.03\times10^{-3} | -2.03\times10^{-3} | -1.65\times10^{-2} | 9.57\times10^{-12} | 1.79\times10^{-3} | 2.85\times10^{-3} | 1.73\times10^{-3} |
|        | vdW               | 1.95\times10^{-6} | 1.80\times10^{-3} | -1.96\times10^{-3} | -1.81\times10^{-2} | 1.10\times10^{-11} | 1.92\times10^{-3} | 3.14\times10^{-3} | 1.97\times10^{-3} |
| Logistic | QMLE              | 4.72\times10^{-6} | 5.44\times10^{-3} | 8.41\times10^{-4}  | -1.98\times10^{-2} | 5.24\times10^{-11} | 3.75\times10^{-3} | 4.49\times10^{-3} | 2.06\times10^{-3} |
|        | Sign              | 3.17\times10^{-6} | 3.23\times10^{-3} | -2.32\times10^{-3} | -1.49\times10^{-2} | 2.09\times10^{-11} | 1.75\times10^{-3} | 2.50\times10^{-3} | 1.39\times10^{-3} |
|        | Wilcoxon          | 3.24\times10^{-6} | 2.93\times10^{-3} | -1.97\times10^{-3} | -1.51\times10^{-2} | 2.20\times10^{-11} | 1.73\times10^{-3} | 2.48\times10^{-3} | 1.42\times10^{-3} |
|        | vdW               | 3.62\times10^{-6} | 2.49\times10^{-3} | -1.97\times10^{-3} | -1.72\times10^{-2} | 2.76\times10^{-11} | 1.91\times10^{-3} | 2.67\times10^{-3} | 1.72\times10^{-3} |
| \( t(3) \) | QMLE              | 1.78\times10^{-6} | 3.06\times10^{-2} | -2.07\times10^{-2} | -3.12\times10^{-2} | 2.85\times10^{-11} | 7.88\times10^{-2} | 7.65\times10^{-2} | 1.08\times10^{-2} |
|        | Sign              | 9.92\times10^{-7} | 3.18\times10^{-3} | -3.92\times10^{-3} | -1.29\times10^{-2} | 5.67\times10^{-12} | 3.25\times10^{-3} | 5.25\times10^{-3} | 2.42\times10^{-3} |
|        | Wilcoxon          | 9.78\times10^{-7} | 3.69\times10^{-3} | -4.87\times10^{-3} | -1.28\times10^{-2} | 5.70\times10^{-12} | 3.51\times10^{-3} | 5.58\times10^{-3} | 2.50\times10^{-3} |
|        | vdW               | 9.86\times10^{-7} | 5.10\times10^{-3} | -9.49\times10^{-3} | -1.56\times10^{-2} | 7.59\times10^{-12} | 5.66\times10^{-3} | 8.08\times10^{-3} | 3.57\times10^{-3} |
4 Bootstrapping the R-estimators

Since the asymptotic covariance matrix of the R-estimators are of complicated forms, in this section we employ the weighted bootstrap technique discussed by Chatterjee and Bose (2005) in the context of M-estimators to approximate the distributions of the R-estimators and we compute corresponding coverage probabilities to exhibit the effectiveness of such bootstrap approximations. The weighted bootstrap in this context is attractive for its computational simplicity since at each bootstrap replication, only the weights need to be generated instead of resampling the data components to compute the replicates of the bootstrapped R-estimate.

In this context, the weighted bootstrap version of the rank-based central sequence is

\[
\hat{R}_{n,\varphi}^*(\theta) := \hat{R}_n^*(\theta) = n^{-1/2} \sum_{t=1}^n w_{nt} \frac{\hat{v}_t(\theta)}{\hat{v}_t(\theta)} \left( 1 - \varphi \left[ \frac{\hat{R}_{nt}(\theta)}{n+1} \right] \frac{X_t}{\hat{v}_t^{1/2}(\theta)} \right),
\]

where \( \{w_{nt}; 1 \leq t \leq n; n \geq 1\} \) is a triangular array of r.v.’s which satisfies the following conditions:

(i) The weights \( \{w_{nt}; 1 \leq t \leq n\} \) are exchangeable and independent of the data \( \{X_t; 1 \leq t \leq n\} \) and errors \( \{\epsilon_t; 1 \leq t \leq n\} \);

(ii) For all \( t \geq 1, w_{nt} \geq 0; E(w_{nt}) = 1; \text{Corr}(w_{nt}; w_{n2}) = O(1/n); \text{Var}(w_{nt}) = \sigma^2_n, \) where \( 0 < c_1 < \sigma_n^2 = o(n) \), with \( c_1 > 0 \) being a constant.

Among various schemes of the weights satisfying the above conditions, we compare the following three types of weights:

(i) Scheme M: \( \{w_{n1}, \ldots, w_{nn}\} \) have a multinomial \( (n, 1/n, \ldots, 1/n) \) distribution, which is essentially the classical paired bootstrap.

(ii) Scheme E: \( w_{nt} = (nE_t)/\sum_{i=1}^n E_i \), where \( \{E_t\} \) are i.i.d. exponential r.v.’s with mean 1.
(iii) Scheme U: \( w_{nt} = (nU_t) / \sum_{i=1}^{n} U_i \), where \( \{U_i\} \) are i.i.d. uniform r.v.'s on \((0.5, 1.5)\).

We use the weighted version of (2.11) to compute the bootstrap estimator \( \hat{\theta}_n^* \):

\[
\hat{\theta}_{s(r+1)} = \hat{\theta}_{s(r)} - \left[ \sum_{t=1}^{n} w_{nt} \frac{\hat{v}_t(\hat{\theta}_{s(r)})}{\hat{v}_t(\theta_{s(r)})} \right]^{-1} \times \left\{ \sum_{t=1}^{n} w_{nt} \frac{\hat{v}_t(\hat{\theta}_{s(r)})}{\hat{v}_t(\theta_{s(r)})} \left[ 1 - \varphi \left( \frac{\hat{R}_{nt}(\hat{\theta}_{s(r)})}{n + 1} \right) \frac{X_t}{\hat{v}_t^{1/2}(\theta_{s(r)})} \right] \right\} , \quad \text{for } r = 0, 1, \ldots ,
\]

with \( \hat{\theta}_{s(0)} = \hat{\theta}_n \) being the initial estimator.

### 4.1 Bootstrap coverage probabilities

Chatterjee and Bose (2005) proved the consistency of the bootstrap for an estimator defined by smooth estimating equation. Since ranks are integer-valued discontinuous functions, the proof of the asymptotic validity of the bootstrapped R-estimator is a mathematically challenging problem which is beyond the scope of this paper. Instead, we resort to simulations to evaluate the performance of the bootstrap approximation of the R-estimators by comparing the distribution of \( \sigma_n^{-1} \sqrt{n} (\hat{\theta}_n^* - \hat{\theta}_n) \) with that of \( \sqrt{n} (\hat{\theta}_n - \theta_0\varphi) \) in terms of coverage rates.

In particular, with the choice of the true parameter \( \theta_0 = (6.50 \times 10^{-6}, 0.177, 0.716)' \) as in the simulation study of the GARCH (1, 1) model of the previous section, we generate \( R = 1000 \) data each with sample size \( n = 1000 \) based on different error distributions. For each data, the exchangeable weights \( \{w_{nt}; 1 \leq t \leq n\} \) are generated \( B = 2000 \) times. We consider cases where the error distributions are normal, DE, logistic and \( t(3) \). The bootstrap weights are based on Schemes M, E and U. The bootstrap coverage rates (in percentage) for 95%, 90% nominal levels are reported in Table 5. Notice that all bootstrap
schemes provide reasonable coverage rates under these error distributions. Scheme U is slightly better than the scheme M and E under the DE and \(t(3)\) distributions.

To check the performance of the bootstrap under different sample sizes, we run simulation with \(n = 200, 300, ..., 1000\) for the sign, Wilcoxon and vdW scores. There are \(R = 1000\) replications being generated under the normal error distribution, and each replication is bootstrapped \(B = 2000\) times with the scheme U. Figure 1 shows the bootstrap coverage rates for \(\omega\) (first row), \(\alpha\) (second row), \(\beta\) (third row) under 95\% nominal level (left column) and 90\% nominal level (right column). We notice that as the sample size increases, the coverage rates get close to the nominal levels for all parameters and all R-estimators, with only few exceptions. This tends to imply the consistency of the bootstrap approximation. With the sample size \(n \geq 500\), the bootstrap coverage rates are generally close to the nominal levels.

5 Application of the R-estimator to the GJR model

The GJR model, proposed by Glosten et al. (1993), is used frequently for the asymmetric financial data. See, e.g., Iqbal and Mukherjee (2010) where a class of M-estimators are used to estimate model parameters. In a similar fashion, we have used the new class of R-estimators to analyze the GJR model and the relevant simulation results are available in the supplementary material S2.
Table 5: The bootstrap coverage rates (in percentage) for the R-estimators (sign, Wilcoxon and vdW) under various error distributions

|            | Normal | Wilcoxon | vdW | DE | sign | Wilcoxon | vdW |
|------------|--------|----------|-----|----|------|----------|-----|
|            |        |          |     |    |      |          |     |
| Error      |        |          |     |    |      |          |     |
|            | Scheme M | Scheme E | Scheme U | Scheme M | Scheme E | Scheme U | Scheme M | Scheme E | Scheme U | Scheme M | Scheme E | Scheme U | Scheme M | Scheme E | Scheme U | Scheme M | Scheme E | Scheme U | Scheme M | Scheme E | Scheme U | Scheme M | Scheme E | Scheme U |
| Normal     |        |          |     |    |      |          |     |
| sign       | 94.1   | 93.6     | 92.8 | 90.8 | 88.5 | 89.2     |     |
|            | 93.5   | 93.3     | 92.8 | 90.3 | 88.1 | 88.8     |     |
|            | 94.8   | 94.7     | 93.7 | 91.7 | 90.2 | 89.6     |     |
| Wilcoxon   | 96.4   | 96.3     | 93.7 | 93.8 | 90.3 | 89.7     |     |
|            | 96.5   | 96.0     | 93.2 | 93.2 | 90.0 | 89.3     |     |
|            | 96.5   | 95.6     | 94.3 | 93.5 | 91.1 | 90.1     |     |
| vdW        | 94.3   | 92.3     | 93.6 | 91.3 | 89.1 | 89.0     |     |
|            | 94.2   | 92.2     | 92.7 | 90.5 | 88.5 | 88.7     |     |
|            | 95.3   | 94.1     | 93.7 | 91.4 | 90.5 | 89.2     |     |
| DE         |        |          |     |    |      |          |     |
| sign       | 90.8   | 90.5     | 91.6 | 87.8 | 86.0 | 86.8     |     |
|            | 90.4   | 89.6     | 90.4 | 87.2 | 85.3 | 86.4     |     |
|            | 91.9   | 92.7     | 92.7 | 88.8 | 88.9 | 87.8     |     |
| Wilcoxon   | 91.0   | 91.0     | 91.5 | 87.6 | 86.9 | 87.6     |     |
|            | 90.7   | 90.2     | 90.4 | 87.2 | 86.2 | 86.6     |     |
|            | 92.4   | 93.6     | 92.8 | 88.7 | 89.1 | 87.7     |     |
| vdW        | 90.9   | 87.6     | 89.7 | 87.5 | 83.9 | 85.3     |     |
|            | 90.4   | 86.9     | 88.9 | 86.9 | 83.1 | 84.8     |     |
|            | 92.4   | 90.2     | 91.0 | 89.7 | 85.6 | 86.0     |     |
| Logistic   |        |          |     |    |      |          |     |
| sign       | 93.0   | 91.1     | 92.1 | 89.0 | 87.6 | 88.6     |     |
|            | 93.4   | 92.3     | 92.5 | 89.8 | 86.3 | 88.4     |     |
|            | 93.0   | 92.3     | 91.9 | 88.7 | 87.5 | 87.1     |     |
| Wilcoxon   | 93.5   | 91.3     | 92.5 | 89.9 | 87.7 | 89.2     |     |
|            | 93.7   | 89.4     | 91.7 | 90.0 | 85.8 | 87.1     |     |
|            | 94.1   | 92.2     | 92.8 | 88.9 | 88.1 | 86.4     |     |
| vdW        | 93.1   | 91.2     | 92.3 | 89.3 | 88.0 | 87.0     |     |
|            | 92.4   | 91.1     | 91.7 | 88.5 | 87.6 | 86.4     |     |
|            | 94.4   | 93.6     | 92.2 | 90.4 | 90.8 | 86.8     |     |
| t(3)       |        |          |     |    |      |          |     |
| sign       | 88.3   | 85.3     | 88.3 | 86.0 | 82.6 | 83.5     |     |
|            | 88.3   | 85.0     | 87.6 | 84.9 | 82.4 | 82.0     |     |
|            | 91.8   | 89.0     | 90.6 | 87.5 | 85.6 | 86.4     |     |
| Wilcoxon   | 88.1   | 84.7     | 88.5 | 85.7 | 81.4 | 83.7     |     |
|            | 88.0   | 84.5     | 87.7 | 85.0 | 80.7 | 82.8     |     |
|            | 91.8   | 88.7     | 90.0 | 87.6 | 85.6 | 85.5     |     |
| vdW        | 85.6   | 82.3     | 86.1 | 81.9 | 79.7 | 81.0     |     |
|            | 84.4   | 82.1     | 86.0 | 80.9 | 78.7 | 80.2     |     |
|            | 90.3   | 85.4     | 88.9 | 86.6 | 81.0 | 83.3     |     |
Figure 4: Plot of the bootstrap coverage rates for the R-estimators (sign, Wilcoxon and vdW) at different sample sizes. The first, second and third rows are for \( \omega, \alpha \) and \( \beta \) respectively. The nominal levels are 95\% (left column) and 90\% (right column). Scheme U is employed and the errors have normal distribution.
6 Conclusion

We propose a new class of R-estimators for the GARCH model and derive the asymptotic normality of these estimators under mild moment and smoothness conditions on the error distribution. We exhibit the robustness and efficiency of R-estimators with respect to the QMLE through simulation and real data analysis. We also consider a general type of weighted bootstrap for the R-estimators which is computational-friendly and easy-to-implement. The theoretical analysis such as the asymptotic validity of the weighted bootstrap is an interesting but challenging problem that can be explored in the future.

SUPPLEMENTARY MATERIALS

Proofs of Proposition 2.1 and Theorem 2.1: The supplementary material S1 contains the proofs of Proposition 2.1 and Theorem 2.1.

Applications of the R-estimators to the GJR model: The supplementary material S2 includes simulation results of the R-estimators for the GJR model.

R-package for computing the R-estimators and their bootstrap counterparts: In RankGarch.cpp file, functions Resti and Boot_Resti are used for computing the R-estimators based on the vdW, sign and Wilcoxon scores and the corresponding bootstrap estimators for the GARCH (1, 1) model; functions Resti_21 and Boot_Resti_21 are used for computing the R-estimators for the GARCH (2, 1) model. Other .R files include codes for running simulation and computing bootstrap coverage rates.

Financial data sets: Data sets used in the illustration of the R-estimation method in Section . (.csv files)
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Supplemental Materials: R-estimators in GARCH models; asymptotics, applications and bootstrapping

S1 Proofs of Proposition 2.1 and Theorem 2.1

We will use the following facts from Berkes et al. (2003) for the proofs:

Fact 1. For any \( \nu > 0 \),
\[
\mathbb{E} \left\{ \sup_{\theta \in \Theta_0} \left| \frac{\dot{v}_1(\theta)}{v_1(\theta)} \right|^\nu \right\} < \infty. \tag{S1}
\]
and
\[
\mathbb{E} \left\{ \sup_{\theta \in \Theta_0} \left| \frac{\ddot{v}_1(\theta)}{v_1(\theta)} \right|^\nu \right\} < \infty.
\]

Fact 2. There exist random variables \( Z_0, Z_1 \), and \( Z_2 \), all independent of \( \{\epsilon_t; 1 \leq t \leq n\} \) and a number \( 0 < \rho < 1 \), such that
\[
0 < v_t(\theta) - \hat{v}_t(\theta) \leq \rho^t Z_0, \tag{S2}
\]
\[
|\dot{v}_t(\theta) - \hat{\dot{v}}_t(\theta)| \leq \rho^t Z_1, \tag{S3}
\]
\[
|\ddot{v}_t(\theta) - \ddot{\hat{v}}_t(\theta)| \leq \rho^t Z_2.
\]

Fact 3. Let \( \{(A_t, B_t, C_t); t \geq 0\} \) be a sequence of identically distributed random variables. If \( \mathbb{E} \log^+ A_0 + \mathbb{E} \log^+ B_0 + \mathbb{E} \log^+ C_0 < \infty \), then for any \( |r| < 1 \),
\[
\sum_{t=0}^{\infty} (A_t + B_t C_t) r^t \text{ converges with probability } 1. \tag{S4}
\]
Idea of the proof of Theorem 2.1. We first derive the following Theorem S1, Corollary S1.1 and Theorem S1 on empirical processes where a scale-perturbed weighted mixed-empirical process is approximated by its non-perturbed version. With \( \theta_n = \theta_0 + n^{-1/2} b \), we derive asymptotic expansion of the difference between two quantities \( T_{1n}(\theta_n) \) and \( T_{2n}(\theta_n) \) which are defined later. We then show that \( T_{1n}(\theta_n) \) can be approximated by a r.v., which is asymptotic normal, plus a term linear in \( b \). Also, we use \( T_{2n}(\theta_n) \) to approximate \( R_n(\theta_n) \) and show that asymptotically their difference is a r.v. with mean zero. Finally, we prove that the difference of \( R_n(\theta_n) \) and \( \hat{R}_n(\theta_n) \) converges in probability to zero. Using these results, we are able to derive the asymptotic linearity of \( \hat{\theta}_n \) as shown in Proposition 2.1. Finally, using the definition of the one-step R-estimator in (2.9), we are able to derive the asymptotic distribution of \( \hat{\theta}_n \).

The following results state the uniform approximation of a scale-perturbed weighted mixed-empirical process by its non-perturbed version which is used for the derivation of the asymptotic distributions of the R-estimators.

**Theorem S1, Corollary S1.1 and Theorem S1.**

Let \( \{ (\eta_t, \gamma_{nt}, \delta_{nt}), 1 \leq t \leq n \} \) be an array of 3-tuple r.v.'s defined on a probability space such that \{ \( \eta_t \), \( 1 \leq t \leq n \) \} are i.i.d. with c.d.f. \( G \) and \( \eta_t \) is independent of (\( \gamma_{nt}, \delta_{nt} \)) for each \( 1 \leq t \leq n \). Let \( \{ \mathcal{A}_t; 1 \leq t \leq n \} \) be an array of increasing sub-\( \sigma \)-fields in both \( n \) and \( t \) so that \( \mathcal{A}_t \subset \mathcal{A}_{n+1}, \mathcal{A}_t \subset \mathcal{A}_{(n+1)\sigma}, 1 \leq t \leq n-1, n \geq 2 \). Assume also that \( (\gamma_{n1}, \delta_{n1}) \) is \( \mathcal{A}_{n1} \) measurable, and \( \{ (\gamma_{nt}, \delta_{nt}); 1 \leq t \leq j \}, \eta_1, \eta_2, \ldots, \eta_{j-1} \} \) are \( \mathcal{A}_{nj} \) measurable, \( 2 \leq j \leq n \). For \( x \in \mathbb{R} \), recall that \( \mu(x) = E[\eta I(\eta < x)] = \int_{-\infty}^{x} t g(t) dt \) and consider the following
weighted mixed-empirical processes

\[ \tilde{V}_n(x) := n^{-1/2} \sum_{t=1}^{n} \gamma_{nt} \eta_t I(\eta_t < x + x \delta_{nt}), \]  
(S5)

\[ \tilde{J}_n(x) := n^{-1/2} \sum_{t=1}^{n} \gamma_{nt} \mu(x + x \delta_{nt}), \]  

\[ V^*_n(x) := n^{-1/2} \sum_{t=1}^{n} \gamma_{nt} \eta_t I(\eta_t \leq x), \quad J^*_n(x) := n^{-1/2} \sum_{t=1}^{n} \gamma_{nt} \mu(x), \]  

\[ \tilde{U}_n(x) := \tilde{V}_n(x) - \tilde{J}_n(x), \quad U^*_n(x) := V^*_n(x) - J^*_n(x). \]

Assume the following conditions on the weights \( \{\gamma_{nt}\} \) and perturbations \( \{\delta_{nt}\} \).

Let \( C_n := \sum_{t=1}^{n} E|\gamma_{nt}|^q \) for some \( q > 2 \). Let \( a \) with \( 0 < a < q/2 \) be such that

\[ C_n/n^{q/2-a} = o(1). \]  
(S6)

\[ \left( n^{-1} \sum_{t=1}^{n} \gamma_{nt}^2 \right)^{1/2} = \gamma + o_p(1) \text{ for a positive r.v.} \gamma. \]  
(S7)

\[ E \left( n^{-1} \sum_{t=1}^{n} \gamma_{nt}^2 \right)^{q/2} = O(1). \]  
(S8)

\[ \max_{1 \leq t \leq n} n^{-1/2} |\gamma_{nt}| = o_p(1). \]  
(S9)

\[ \max_{1 \leq t \leq n} |\delta_{nt}| = o_p(1). \]  
(S10)

\[ \frac{n^{q/2-\epsilon}}{C_n} E \left[ n^{-1} \sum_{t=1}^{n} \{\gamma_{nt}^2 \delta_{nt}\} \right]^{q/2} = o(1). \]  
(S11)

\[ n^{-1/2} \sum_{t=1}^{n} |\gamma_{nt} \delta_{nt}| = O_p(1). \]  
(S12)

The following theorem shows that uniformly over the entire real line, the perturbed process
\( \tilde{U}_n \) can be approximated by \( U_n^* \).

**Theorem S1.** Under the above set-up and Assumptions (S6)-(S13) and (A1),

\[
\sup_{x \in \mathbb{R}} |\tilde{U}_n(x) - U_n^*(x)| = o_p(1). \tag{S13}
\]

**Proof.** The proof is similar to the proof in Mukherjee (2007, Theorem 6.1). In particular, we show point-wise convergence for each \( x \) and then invoke the monotone structure of the mean processes to achieve the uniform convergence. For weighted empirical, the monotonically increasing mean process is given by the distribution function. Although \( \mu \) in the present case is not a monotone function on \((-\infty, \infty)\), we use its monotone property separately on \((-\infty, 0] \) and \([0, \infty)\).

We remark that this theorem is different from Koul and Ossiander (1994, Theorem 1.1) and Mukherjee (2007, Theorem 6.1) where weighted empirical processes were considered for the estimation of the mean parameters. For the estimation of the scale parameters, in this paper we consider weighted mixed-empirical process which is a weighted sum of the mixture of error and its indicator process.

The following corollary describes a Taylor-type expansion of the weighted sum of indicator functions \( \tilde{V}_n(x) \).

**Corollary S1.1.** Under the above setup and under the Assumptions (S6)-(S13) and (A1),

\[
\sup_{x \in \mathbb{R}} |\tilde{J}_n(x) - J_n^*(x) - x^2 g(x) n^{-1/2} \sum_{t=1}^n \gamma_{nt} \delta_{nt}| = o_p(1). \tag{S14}
\]
Hence,

$$\sup_{x \in \mathbb{R}} |\tilde{V}_n(x) - V^*_n(x) - x^2 g(x)n^{-1/2} \sum_{t=1}^{n} \gamma_{nt} \delta_{nt}| = o_p(1).$$ \hspace{1cm} (S15)

**Proof.** Here (S15) follows from (S14) and (S13). Therefore, it remains to prove (S14). Notice that the LHS of (S14) equals

$$\sup_{x \in \mathbb{R}} \left| n^{-1/2} \sum_{t=1}^{n} \gamma_{nt} \left[ x \int_{x}^{x+\delta_{nt}} t g(t) dt - x^2 g(x) \delta_{nt} \right] \right|$$

$$= \sup_{x \in \mathbb{R}} \left| n^{-1/2} \sum_{t=1}^{n} \gamma_{nt} \delta_{nt} \left[ x \int_{0}^{1} (x + hx\delta_{nt}) g(x + hx\delta_{nt}) dh - x^2 g(x) \right] \right|$$

$$= o_p(1)$$

due to (S12) and Assumption (A1).

The next theorem provides an extended version of (S13) when the weights are functions on appropriately scaled parameter space. We define the following processes of two arguments as follows.

**Probabilistic framework:** Let \( \{\eta_t, 1 \leq t \leq n\} \) be i.i.d. with the c.d.f. \( G \), \( \{l_{nt}; 1 \leq t \leq n\} \) be an array of measurable functions from \( \mathbb{R}^m \) to \( \mathbb{R} \) such that for every \( b \in \mathbb{R}^m \) and
$1 \leq t \leq n$, $(l_{nt}(b), u_{nt}(b))$ are independent of $\eta_t$. For $x \in \mathbb{R}$ and $b \in \mathbb{R}^m$, let

$$
\tilde{V}(x, b) := n^{-1/2} \sum_{t=1}^{n} l_{nt}(b) \eta_t I \left( \eta_t < x + xu_{nt}(b) \right),
$$

$$
\tilde{J}(x, b) := n^{-1/2} \sum_{t=1}^{n} l_{nt}(b) \mu \left( x + xu_{nt}(b) \right),
$$

$$
\tilde{U}(x, b) := \tilde{V}(x, b) - \tilde{J}(x, b),
$$

$$
\hat{V}^*(x, b) := n^{-1/2} \sum_{t=1}^{n} l_{nt}(b) \eta_t I \left( \eta_t < x \right),
\hat{J}^*(x, b) := n^{-1/2} \sum_{t=1}^{n} l_{nt}(b) \mu(x),
$$

$$
\hat{U}^*(x, b) := \hat{V}^*(x, b) - \hat{J}^*(x, b) = n^{-1/2} \sum_{t=1}^{n} l_{nt}(b) \left[ \eta_t I \left( \eta_t < x \right) - \mu(x) \right].
$$

Here $\hat{U}^*(\cdot, \cdot)$ is a sequence of ordinary non-perturbed weighted mixed-empirical processes with weights $\{l_{nt}(\cdot)\}$ and $\tilde{U}(\cdot, \cdot)$ is a sequence of perturbed weighted mixed-empirical processes with scale perturbations $\{u_{nt}(\cdot)\}$. In Theorem S1 below it is shown that $\tilde{U}$ can be uniformly approximated by $\hat{U}$ under the following conditions (S16)-(S24) for $\{l_{nt}(\cdot)\}$ and $\{u_{nt}(\cdot)\}$. Note that the statements on assumptions and convergence hold point-wise for each fixed $b \in \mathbb{R}^m$.

There exist numbers $q > 2$ and $a$ (both free from $b$) satisfying $0 < a < q/2$ such that with $C_n(b) := \sum_{i=1}^{n} E|l_{ni}(b)|^q$,

$$
C_n(b)/n^{q/2-a} = o(1), \text{ for each } b \in \mathbb{R}^m. \quad \text{(S16)}
$$
For some positive random process $\ell(b)$,

$$
\left( n^{-1} \sum_{t=1}^{n} l_{nt}^2(b) \right)^{1/2} = \ell(b) + o_p(1), \quad b \in \mathbb{R}^m. \quad (S17)
$$

$$
\mathbb{E} \left( n^{-1} \sum_{t=1}^{n} l_{ni}^2(b) \right)^{q/2} = O(1), \quad b \in \mathbb{R}^m. \quad (S18)
$$

$$
\max_{1 \leq t \leq n} n^{-1/2} |l_{nt}(b)| = o_p(1), \quad b \in \mathbb{R}^m. \quad (S19)
$$

$$
\max_{1 \leq t \leq n} \{|u_{nt}(b)|\} = o_p(1), \quad b \in \mathbb{R}^m. \quad (S20)
$$

$$
\frac{n^{q/2-a}}{C_n(b)} \mathbb{E} \left[ n^{-1} \sum_{t=1}^{n} l_{nt}^2(b)|u_{nt}(b)| \right]^{q/2} = o(1), \quad b \in \mathbb{R}^m. \quad (S21)
$$

$$
n^{-1/2} \sum_{t=1}^{n} l_{nt}(b)u_{nt}(b) = O_p(1), \quad b \in \mathbb{R}^m. \quad (S22)
$$

$$
\forall b \text{ and } \varepsilon > 0, \exists \delta > 0, \text{ and } n_1 \in \mathbb{N} \text{ whenever } \|s\| \leq b, \text{ and } n > n_1, \quad (S23)
$$

$$
P \left( n^{-1/2} \sum_{t=1}^{n} |l_{nt}(s)| \left\{ \sup_{\|t-s\|\leq \delta} |u_{nt}(t) - u_{nt}(s)| \right\} \leq \varepsilon \right) > 1 - \varepsilon.
$$

$$
\forall b \text{ and } \varepsilon > 0, \exists \delta > 0, \text{ and } n_2 \in \mathbb{N} \text{ whenever } \|s\| \leq b, \text{ and } n > n_2, \quad (S24)
$$

$$
P \left( \sup_{\|t-s\|\leq \delta} n^{-1/2} \sum_{t=1}^{n} |l_{nt}(t) - l_{nt}(s)| \leq \varepsilon \right) > 1 - \varepsilon.
$$

Conditions (S16)-(S24) are regularity conditions on the weights and perturbations of the two-parameters empirical processes. Conditions (S23)-(S24) are smoothness conditions on the weights and perturbations. Under stationarity and ergodicity, many of these conditions reduce to much simpler conditions based on existence of the moments.

The following theorem generalizes (S13) when the weights are functions of $b$.

**Theorem S2.** Under the above framework, suppose that conditions (S16)-(S24) and As-
sumption (A1) hold. Then for every $0 < b < \infty$,

$$\sup_{x \in \mathbb{R},\|b\| \leq b} |\tilde{U}(x, b) - \mathcal{U}^*(x, b)| = o_p(1). \quad (S25)$$

**Proof.** Clearly, under conditions (S16)-(S22), Theorem S1 entails that for each fixed $b$,

$$\sup_{x \in \mathbb{R}} |\tilde{U}(x, b) - \mathcal{U}^*(x, b)| = o_p(1).$$

The uniform convergence with respect to $b$ over compact sets can be proved as in Mukherjee (2007, Lemma 3.2) using conditions (S23) and (S24). \qed

The following facts are useful in the proofs of various results of this paper. Let $m = 1 + p + q$ be the total number of parameters and fix $b \in \mathbb{R}^m$. Let $\theta_n = \theta_{0\varphi} + n^{-1/2}b$,

$$u_{nt}(b) = \frac{v_{t}^{1/2}(\theta_n)}{v_{t}^{1/2}(\theta_{0\varphi})} - 1, \quad v_{nt}(b) = \frac{v_{t}^{1/2}(\theta_{0\varphi})}{v_{t}^{1/2}(\theta_n)} - 1. \quad (S26)$$

Then $\{u_{nt}(b)\}$ satisfies (S20) since

$$u_{nt}(b) = \frac{v_{t}(\theta_n) - v_{t}(\theta_{0\varphi})}{v_{t}^{1/2}(\theta_{0\varphi})\{v_{t}^{1/2}(\theta_n) + v_{t}^{1/2}(\theta_{0\varphi})\}} = \frac{n^{-1/2}v_{t}'(\theta^*)b}{v_{t}^{1/2}(\theta_{0\varphi})\{v_{t}^{1/2}(\theta_n) + v_{t}^{1/2}(\theta_{0\varphi})\}}, \quad (S27)$$

for some $\theta^* = \theta^*(n, t, b)$ in the neighbourhood of $\theta_{0\varphi}$ for large $n$. The $n^{-1/2}$-factor is used later for deriving convergence of some sequence of random vectors. Similarly, for some $\theta^*$,

$$v_{nt}(b) = \frac{v_{t}(\theta_{0\varphi}) - v_{t}(\theta_n)}{v_{t}^{1/2}(\theta_n)\{v_{t}^{1/2}(\theta_n) + v_{t}^{1/2}(\theta_{0\varphi})\}} = \frac{-n^{-1/2}v_{t}'(\theta^*)b}{v_{t}^{1/2}(\theta_{0\varphi})\{v_{t}^{1/2}(\theta_n) + v_{t}^{1/2}(\theta_{0\varphi})\}} = n^{-1/2}v_{nt}, \quad (S28)$$

8
say. Let \( a_{nt}(b) = v_t^{1/2}(\theta_{0\varphi})/v_t^{1/2}(\theta_n) = 1 + v_{nt}(b) = 1 + n^{-1/2}\xi_{nt} \). Then

\[
\frac{X_t}{v_t^{1/2}(\theta_n)} = a_{nt}(b)\eta_t = \eta_t + n^{-1/2}\eta_t\xi_{nt} = \eta_t + n^{-1/2}z_{nt},
\]

where

\[
z_{nt} = \eta_t\xi_{nt} = \eta_t \times \frac{-\dot{v}'(\theta^*)b}{v_t^{1/2}(\theta_n)\{v_t^{1/2}(\theta_n) + v_t^{1/2}(\theta_{0\varphi})\}}.
\]

For \( \delta > 0 \) in Assumption (A1) and any \( c > 0 \),

\[
P \left[ n^{-1/2} \max_{1 \leq t \leq n} |z_{nt}| > c \right] \leq \sum_{t=1}^{n} P \left[ n^{-1/2}|z_{nt}| > c \right] \leq n \frac{E \left[ n^{-1-\delta/2}|\eta_t|^{2+\delta}|\xi_{nt}|^{2+\delta} \right]}{c^{2+\delta}} = o(1)
\]
since all moments of \( \{ |\xi_{nt}| \} \) are finite and \( \eta_t \) and \( \xi_{nt} \) are independent for all \( t \). Therefore

\[
\max_{1 \leq t \leq n} \left| \frac{X_t}{v_t^{1/2}(\theta_n)} - \eta_t \right| = o_p(1). \tag{S29}
\]

If \( \dot{v}_t(\theta_n)/v_t(\theta_n) \) appears as the coefficients, we replace it by \( \dot{v}_t(\theta_{0\varphi})/v_t(\theta_{0\varphi}) \) and the difference is controlled as follows. Notice that all derivatives below exist with bounded moments and so

\[
\frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} - \frac{\dot{v}_t(\theta_{0\varphi})}{v_t(\theta_{0\varphi})} = n^{-1/2} A_t(\theta_{0\varphi}) b + n^{-1} A^*_{nt} \tag{S30}
\]

where \( A_t(\theta_{0\varphi}) = \dot{v}_t(\theta_{0\varphi})/v_t(\theta_{0\varphi}) - \dot{v}_t(\theta_{0\varphi})/\{v_t(\theta_{0\varphi})\}^2 \). Only the term \( n^{-1/2} A_t(\theta_{0\varphi}) b \) is of our interest since others are of higher order than \( n \).

Take \( l_{nt}(b) \) to be equal to the \( j \)-th coordinate \((1 \leq j \leq m = 1 + p + q)\) of

\[
L_{nt}(b) = \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \times \frac{v_t^{1/2}(\theta_{0\varphi})}{v_t^{1/2}(\theta_n)} \tag{S31}
\]

9
and \( u_{nt}(b) \) as in (S26). We now show that (S17)-(S24) hold with such choice.

For each \( t \) with \( 1 \leq t \leq n \), \( \{ L_{nt}(b), u_{nt}(b) \} \) are independent of \( \eta_t \). Using a Taylor expansion of \( \ln(b) \) at \( \theta_0 \phi \) for each \( 1 \leq t \leq n \) and noting the existence of all moments of \( v_t(\theta_0 \phi) \) and its derivatives of all higher orders, (S17) and (S18) hold. Existence of all higher moments of \( \{ l_{nt}(b), u_{nt}(b) \} \) ensure conditions (S19)-(S21).

To verify (S22), we use (S27) and that for each \( t \), \( v_t(\cdot) \) is a smooth function with derivatives of all order to conclude that

\[
\frac{1}{n} \sum_{t=1}^{n} \frac{\Delta v_t(\theta_n)}{v_t(\theta_n)} \frac{1}{v_t^{1/2}(\theta_n)} \sigma_2(\theta, \eta) = J b/2 + o_p(1).
\]

Conditions (S23) and (S24) can be verified using the mean value theorem.

The following lemmas and their proofs represent the intermediate steps in the proofs of Proposition 2.1 and Theorem 2.1.

**Lemma S3, Lemma S4, Lemma S5 and Lemma S6.**

Let

\[
T_{n1}(\theta_n) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \left( 1 - \frac{X_t}{v_t^{1/2}(\theta_n)} \phi[G(\eta)] \right),
\]

\[
T_{n2}(\theta_n) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \left( 1 - \frac{X_t}{v_t^{1/2}(\theta_n)} \phi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right),
\]

and note that the difference in the definitions of these two quantities lies only in the argument of \( \phi(G(.)) \). We show in Lemma S3 below that \( \int_0^1 [\hat{V}(u, b) - V^*(u, b)] d\phi(u) = T_{n1}(\theta_n) - T_{n2}(\theta_n) \). Using results on empirical processes in Theorem , \( \int_0^1 [\hat{V}(u, b) - V^*(u, b)] d\phi(u) \) is linear in \( b \). Consequently, we obtain the following uniform approximations of \( T_{n1}(\theta_n) - T_{n2}(\theta_n) \) over \( ||b|| \leq c \) where \( c > 0 \).
Lemma S3. Let Assumptions (A1)-(A3) hold. Then, as $n \to \infty$,

$$T_{n2}(\theta_n) - T_{n1}(\theta_n) = M b + u_p(1), \quad \text{(S32)}$$

where $M = J \rho(\varphi)/2$.

Proof. To use Theorem S1 in the proof, let $b = n^{1/2}(\theta_n - \theta_0 \varphi)$ and $x = G^{-1}(u)$ for some $0 < u < 1$. For simplicity, we use the notation $\tilde{V}(u, b)$ to denote $\tilde{V}(G^{-1}(u), b)$ which is defined in the probabilistic framework above. Accordingly

$$\tilde{V}(u, b) := n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} I \left[ \eta_t < G^{-1}(u) \frac{v_t^{1/2}(\theta_n)}{v_t^{1/2}(\theta_0 \varphi)} \right]$$

and

$$V^*(u, b) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} I (\eta_t < G^{-1}(u)).$$

With the choice based on (S31) and (S26) and using

$$\frac{v_t^{1/2}(\theta_0 \varphi)}{v_t^{1/2}(\theta_n)} \eta_t = \frac{X_t}{v_t^{1/2}(\theta_n)},$$

$$\tilde{V}(u, b) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} I \left[ \frac{X_t}{v_t^{1/2}(\theta_n)} < G^{-1}(u) \right]$$

$$= n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} I \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) < u \right].$$

11
Similarly,

\[ \mathcal{V}^*(u, b) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} I(G(\eta_t) < u). \]

Since

\[ \int_0^1 I \left\{ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) < u \right\} d\varphi(u) = \varphi(1) - \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right], \]

we get

\[ \int_0^1 \tilde{\mathcal{V}}(u, b) d\varphi(u) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} \left\{ \varphi(1) - \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right\} \]

and

\[ \int_0^1 \mathcal{V}^*(u, b) d\varphi(u) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} \left( \varphi(1) - \varphi(G(\eta_t)) \right). \]

Cancelling \( \varphi(1) \), \( \int_0^1 [\tilde{\mathcal{V}}(u, b) - \mathcal{V}^*(u, b)] d\varphi(u) \) equals

\[ n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} \left\{ -\varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] + \varphi(G(\eta_t)) \right\} \]

\[ = T_{n2}(\theta_n) - T_{n1}(\theta_n). \]

Using (S14) and (S27) with

\[ \tilde{\mathcal{J}}(u, b) := n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{v_t^{1/2}(\theta_0\varphi)}{v_t^{1/2}(\theta_n)} \mu \left[ G^{-1}(u) \frac{v_t^{1/2}(\theta_n)}{v_t^{1/2}(\theta_0\varphi)} \right], \]

\[ \mathcal{J}^*(u, b) := n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{v_t^{1/2}(\theta_0\varphi)}{v_t^{1/2}(\theta_n)} \mu \left( G^{-1}(u) \right), \]
we have

$$\sup_{u \in (0,1)} |\tilde{J}(u, b) - J^*(u, b) - [G^{-1}(u)]^2 g(G^{-1}(u)) n^{-1/2} \sum_{t=1}^n \dot{v}_t(\theta_n) v_t^{1/2}(\theta_0) u_{nt}(b)| = u_p(1).$$

Also,

$$|n^{-1/2} \sum_{t=1}^n \dot{v}_t(\theta_n) v_t^{1/2}(\theta_0) u_{nt}(b) - J| = u_p(1).$$

Hence,

$$\int_0^1 [\tilde{J}(u, b) - J^*(u, b)] d\varphi(u) = \int_0^1 [G^{-1}(u)]^2 g(G^{-1}(u)) d\varphi(u) Jb/2 + u_p(1)$$

$$= Mb + u_p(1) \tag{S33}$$

by recalling that $M = J \rho(\varphi)/2$. Finally, (S32) follows from Theorem [S1]. \hfill \square

The following lemma states that the difference between $T_{n1}(\theta_n)$ and $N_n$ is asymptotically linear in $b$.

**Lemma S4.** Let Assumptions (A1)-(A3) hold. Then, as $n \to \infty$,

$$T_{n1}(\theta_n) - N_n = Jb/2 + u_p(1), \tag{S34}$$

where

$$N_n \to \mathcal{N}(0, J \sigma^2(\varphi)), \tag{S35}$$

with $\sigma^2(\varphi) = \text{Var}\{\eta_1 \varphi[G(\eta_1)]\}$.

**Proof.** The difference between $T_{n1}$ and $N_n$ lies in comparing $X_t/v_t^{1/2}(\theta_n) = \eta_t + n^{-1/2}z_{nt}$ and $\eta_t$ and involves smooth function of $b$. So the proof follows easily with the details below.
Notice that

\[
T_{n1}(\theta_n) - N_n = n^{-1/2} \sum_{t=1}^{n} \left[ \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} - \frac{\dot{v}_t(\theta_0\varphi)}{v_t(\theta_0\varphi)} \right] \{ 1 - a_n(t(b)\eta_t\varphi[G(\eta_t)]) \}
- n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_0\varphi)}{v_t(\theta_0\varphi)} v_{nt}(b)\eta_t\varphi[G(\eta_t)] = F_{n1} - F_{n2}.
\]

Using (S30),

\[
F_{n1} = n^{-1} \sum_{t=1}^{n} A_t(\theta_0\varphi) b \{ 1 - a_n(t(b)\eta_t\varphi[G(\eta_t)]) \} + u_p(1)
= n^{-1} \sum_{t=1}^{n} A_t(\theta_0\varphi) b \{ 1 - \eta_t\varphi[G(\eta_t)] \} - n^{-1} \sum_{t=1}^{n} A_t(\theta_0\varphi) b v_{nt}(b)\eta_t\varphi[G(\eta_t)] + u_p(1).
\]

Using the LLN, the first term in the above decomposition of \( F_{n1} \) is \( u_p(1) \) since

\[
E \{ A_t(\theta_0\varphi) b \{ 1 - \eta_t\varphi[G(\eta_t)] \} \} = E[A_t(\theta_0\varphi) b] E \{ 1 - \eta_t\varphi[G(\eta_t)] \} = 0.
\]

For the second term, using (S28) we have \( n^{-1/2} \) factor of \( v_{nt}(b) \) and consequently it is \( u_p(1) \).

For \( F_{n2} \), we approximate \( v_{nt}(b) \) by \( -n^{-1/2} \dot{v}_t'(\theta_0\varphi) b / \{ 2v_t(\theta_0\varphi) \} \) and use \( E\{\eta_t\varphi[G(\eta_t)]\} = 1 \) to obtain \( F_{n2} = -Jb/2 + u_p(1) \). Hence (S34) is proved.

Using the independence of \( v_t \) and \( \eta_t \) for each \( t \), \( N_n \) is a sum of the vectors of martingale differences and so (S35) follows from the martingale CLT.

Now consider the rank-based counterpart of \( T_{n2}(\theta_n) \)

\[
R_n(\theta_n) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \left\{ 1 - \frac{X_t}{v_t^{1/2}(\theta_n)} \varphi \left( \frac{R_{nt}(\theta_n)}{n + 1} \right) \right\}.
\]
The following lemma provides the difference between \( T_{n2}(\theta_n) \) and \( R_n(\theta_n) \). It shows that the effect of replacing observations in \( T_{2n}(\theta_n) \) by ranks is asymptotically a r.v. with mean zero.

**Lemma S5.** Let Assumptions (A1)-(A3) hold. Then, as \( n \to \infty \),

\[
R_n(\theta_n) - T_{n2}(\theta_n) = Q_n + u_p(1). \tag{S36}
\]

Also, \( Q_n \) converges in distribution to \( E(\dot{v}_1(\theta_0)/v_1(\theta_0))Z \), where \( Z \) has mean zero and variance \( \gamma(\varphi) \).

**Proof.** Consider the following decomposition

\[
R_n(\theta_n) - T_{n2}(\theta_n) = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} \left\{ \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right. - \left. \varphi \left[ \frac{R_{nt}(\theta_n)}{n+1} \right] \right\}
\]

\[
= n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} \eta_t \left\{ \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right. - \left. \varphi \left[ \frac{R_{nt}(\theta_n)}{n+1} \right] \right\}
\]

\[
+ n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} v_{nt}(b) \eta_t \left\{ \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right. - \left. \varphi \left[ \frac{R_{nt}(\theta_n)}{n+1} \right] \right\}
\]

\[
+ n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_0 \varphi)}{v_t(\theta_0 \varphi)} v_{nt}(b) \eta_t \left\{ \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right. - \left. \varphi \left[ \frac{R_{nt}(\theta_n)}{n+1} \right] \right\}
\]

\[
+ n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_0 \varphi)}{v_t(\theta_0 \varphi)} v_{nt}(b) \eta_t \left\{ \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right. - \left. \varphi \left[ \frac{R_{nt}(\theta_n)}{n+1} \right] \right\}
\]

\[
= D_{n1} + D_{n2} + D_{n3} + D_{n4}.
\]

Using the \( n^{-1/2} \)-factor in \( \text{[S30]} \) and \( \text{[S28]} \), \( D_{n1}, D_{n2} \) and \( D_{n4} \) are \( u_p(1) \). We next prove that \( D_{n3} = Q_n + u_p(1) \) in detail. Recall that \( \tilde{G}_n(x) \) is the empirical distribution function
Let \( G_n(x) \), \( x \in \mathbb{R} \) be the empirical distribution function of \( \{X_t/v_t^{1/2}(\theta_n)\} \). Then

\[
D_n^3 = n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_0 \varphi)}{v_t(\theta_0 \varphi)} \eta_t \left\{ \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] - \varphi \left[ G_n \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] \right\} \\
= n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_0 \varphi)}{v_t(\theta_0 \varphi)} \eta_t \left\{ \varphi \left[ G \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] - \varphi \left[ G(\eta_t) \right] \right\} \\
- n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_0 \varphi)}{v_t(\theta_0 \varphi)} \eta_t \left\{ \varphi \left[ G_n \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) \right] - \varphi \left[ \tilde{G}_n(\eta_t) \right] \right\} \\
+ n^{-1/2} \sum_{t=1}^{n} \frac{\dot{v}_t(\theta_0 \varphi)}{v_t(\theta_0 \varphi)} \eta_t \left\{ \varphi \left[ G(\eta_t) \right] - \varphi \left[ \tilde{G}_n(\eta_t) \right] \right\} \\
= D_{n1}^* - D_{n2}^* + D_{n3}^*.
\]

Since \( D_{n1}^* \) is the weighted sum of the difference of a c.d.f. evaluated at two different r.v.'s and integrated wrt \( \varphi \), using the same technique for proving \( S33 \), \( D_{n1}^* = Mb + u_p(1) \).

Write \( w_t = \dot{v}_t(\theta_0 \varphi)/v_t(\theta_0 \varphi) \). Since \( D_{n2}^* \) is the weighted sum of the difference of two different c.d.f.'s evaluated at two different r.v.'s and integrated wrt \( \varphi \),

\[
D_{n2}^* = \int_0^1 n^{-1/2} \sum_{t=1}^{n} w_t \eta_t I \left[ G_n \left( \frac{X_t}{v_t^{1/2}(\theta_n)} \right) < u \right] - I \left[ \tilde{G}_n(\eta_t) < u \right] d\varphi(u) \\
= \int_0^1 n^{-1/2} \sum_{t=1}^{n} w_t \eta_t I \left[ \frac{X_t}{v_t^{1/2}(\theta_n)} < G_n^{-1}(u) \right] - I \left[ \eta_t < \tilde{G}_n^{-1}(u) \right] d\varphi(u) \\
= \int_0^1 n^{-1/2} \sum_{t=1}^{n} w_t \eta_t I \left[ \eta_t < G_n^{-1}(u) \frac{1}{1 + v_n(b)} \right] - I \left[ \eta_t < \tilde{G}_n^{-1}(u) \right] d\varphi(u).
\]
Using (S29), sup \( \{ G_n^{-1}(u) - \tilde{G}_n^{-1}(u) \}; u \in (0, 1) \} = u_p(1). \) Hence, by Theorem S1,

\[
D_{n^2}^* = \int_0^1 n^{-1/2} \sum_{t=1}^n w_t \left[ \mu \left( \tilde{G}_n^{-1}(u) \frac{1}{1 + v_{nt}(b)} \right) - \mu \left( \tilde{G}_n^{-1}(u) \right) \right] d\varphi(u) + u_p(1)
\]

\[
= \int_0^1 n^{-1/2} \sum_{t=1}^n w_t \mu \left( \tilde{G}_n^{-1}(u) \right) \left( 1 + v_{nt}(b) \right) d\varphi(u) + u_p(1)
\]

\[
= \int_0^1 n^{-1/2} \sum_{t=1}^n w_t \tilde{G}_n^{-1}(u) \left( 1 + v_{nt}(b) \right) d\varphi(u) + u_p(1)
\]

\[
= \int_0^1 n^{-1/2} \sum_{t=1}^n w_t \tilde{G}_n^{-1}(u) \left( 1 + v_{nt}(b) \right) d\varphi(u) + u_p(1)
\]

\[
= \int_0^1 n^{-1/2} \sum_{t=1}^n \tilde{G}_n^{-1}(u) \left( 1 + v_{nt}(b) \right) d\varphi(u) + u_p(1)
\]

Finally consider \( D_{n^3}^* \) written as

\[
D_{n^3}^* = \int_0^1 n^{-1/2} \sum_{t=1}^n w_t \eta_t \left\{ I \left[ \eta_t \leq \tilde{G}_n^{-1}(u) \right] - I \left[ \eta_t \leq G_n^{-1}(u) \right] \right\} d\varphi(u)
\]

\[
= \int_0^1 n^{-1/2} \sum_{t=1}^n w_t \eta_t \left\{ I \left[ \eta_t \leq \tilde{G}_n^{-1}(u) \right] - I \left[ \eta_t \leq G_n^{-1}(G(\tilde{G}_n^{-1}(u))) \right] \right\} d\varphi(u)
\]

\[
= \int_0^1 \left[ M_n(u) - M_n(G(\tilde{G}_n^{-1}(u))) \right] d\varphi(u)
\]

\[
+ \int_0^1 n^{-1/2} \sum_{t=1}^n w_t \left[ \mu(G_n^{-1}(u)) - \mu(\tilde{G}_n^{-1}(u)) \right] d\varphi(u),
\]

where \( M_n(u) := n^{-1/2} \sum_{t=1}^n w_t \left\{ \eta_t I \left[ \eta_t \leq G_n^{-1}(u) \right] - \mu(G_n^{-1}(u)) \right\} \). We show that

\[
\left| \int_0^1 \left[ M_n(u) - M_n(G(\tilde{G}_n^{-1}(u))) \right] d\varphi(u) \right| = o_p(1), \quad (S37)
\]
\[ Q_n = \int_0^1 n^{-1/2} \sum_{i=1}^n w_i \left[ \mu(G^{-1}(u)) - \mu(\tilde{G}_n^{-1}(u)) \right] d\varphi(u) \to \mathbb{E}(\dot{\psi}_1/v_1)Z. \quad (S38) \]

For (S37), note that \( \{M_n(.)\} \) converges weakly to a Brownian Bridge on \((0,1)\) since for each fixed \( u \), \( M_n(u) \) converges to a normal distribution using the martingale CLT and it is tight using the bound on the moment of the difference process in Billingsley (1968, Theorem 12.3).

Since \( \sup\{|u - G(\tilde{G}_n^{-1}(u))|; u \in (0,1)\} = \sup\{|G(x) - \tilde{G}_n(x)|; x \in \mathbb{R}\} = o_p(1) \), by the Arzela-Ascoli theorem,

\[
\sup \left\{ \left| M_n(u) - M_n(G(\tilde{G}_n^{-1}(u))) \right| ; u \in (0,1) \right\} = o_p(1),
\]

and consequently, (S37) is proved. For (S38), we use the Bahadur representation; see Bahadur (1966) and Ghosh (1971) for details. Since \( \dot{g}(x) \) is bounded and \( g \) is positive on \( \mathbb{R} \),

\[
n^{1/2} \left( G^{-1}(u) - \tilde{G}_n^{-1}(u) \right) - n^{-1/2} \sum_{i=1}^n \frac{I\{\eta_i \leq G^{-1}(u)\} - u}{g(G^{-1}(u))} = o(1) \quad \text{a.s.}
\]

Applying the mean value theorem,

\[
n^{1/2} \left[ \mu(G^{-1}(u)) - \mu(\tilde{G}_n^{-1}(u)) \right] - \dot{\mu}(G^{-1}(u))n^{-1/2} \sum_{i=1}^n \frac{I\{\eta_i \leq G^{-1}(u)\} - u}{g(G^{-1}(u))} = o(1) \quad \text{a.s.}
\]

(S39)
Using $\dot{\mu}(x) = xg(x)$,

$$Q_n = n^{-1} \sum_{t=1}^{n} w_t \int_{0}^{1} \left[ \dot{\mu}(G^{-1}(u)) n^{-1/2} \sum_{i=1}^{n} I\{\eta_i \leq G^{-1}(u)\} - u \right] d\varphi(u) + o_p(1)$$

$$= n^{-1} \sum_{t=1}^{n} w_t \int_{0}^{1} \left[ G^{-1}(u) n^{1/2} \left( \tilde{G}_n(G^{-1}(u)) - u \right) \right] d\varphi(u) + o_p(1).$$

Since $n^{-1} \sum_{t=1}^{n} w_t \to E(\dot{v}_1(\theta_0\phi)/v_1(\theta_0\phi))$, and using van der Vaart (1998, Theorem 19.3),

$$n^{1/2} \left( \tilde{G}_n(G^{-1}(u)) - u \right) \to B(u),$$

we obtain (S38) with the r.v. $Z$ having mean zero. The variance of $Z$ is given by

$$E(Z^2) = E \left[ \int_{0}^{1} \int_{0}^{1} G^{-1}(u) G^{-1}(v) B(u) B(v) d\varphi(u) d\varphi(v) \right]$$

$$= \int_{0}^{1} \int_{0}^{1} G^{-1}(u) G^{-1}(v) E[B(u) B(v)] d\varphi(u) d\varphi(v)$$

$$= \int_{0}^{1} \int_{0}^{1} G^{-1}(u) G^{-1}(v) [\min\{u,v\} - uv] d\varphi(u) d\varphi(v)$$

$$= \gamma(\varphi).$$

Now recall the rank-based central sequence

$$\hat{R}_n(\theta_n) = n^{-1/2} \sum_{t=1}^{n} \frac{\hat{v}_t(\theta_n)}{v_t(\theta_n)} \left\{ 1 - \frac{X_t}{\hat{v}_t^{1/2}(\theta_n)} \varphi \left( \frac{\hat{R}_{nt}(\theta_n)}{n+1} \right) \right\},$$

which is an approximation to $R_n(\theta_n)$. We have the following lemma dealing with the difference between $R_n(\theta_n)$ and $\hat{R}_n(\theta_n)$.
Lemma S6. Let Assumptions (A1)-(A3) hold. Then, as 

$$n \to \infty,$$

$$R_n(\theta_n) - \hat{R}_n(\theta_n) = u_p(1).$$  \hfill (S40)

Proof. Note that $R_n(\theta_n) - \hat{R}_n(\theta_n)$ equals

$$n^{-1/2} \sum_{t=1}^{n} \left[ \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} - \frac{\dot{\hat{v}}_t(\theta_n)}{\hat{v}_t(\theta_n)} \right]$$

$$+ n^{-1/2} \sum_{t=1}^{n} \left[ \frac{\dot{\hat{v}}_t(\theta_n)}{\hat{v}_t(\theta_n)} \frac{X_t}{\hat{v}_t^{1/2}(\theta_n)} - \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \frac{X_t}{v_t^{1/2}(\theta_n)} \right] \varphi \left( \frac{R_{nt}(\theta_n)}{n+1} \right)$$

$$- n^{-1/2} \sum_{t=1}^{n} \dot{v}_t(\theta_n) \frac{X_t}{v_t(\theta_n)} \frac{\varphi (R_{nt}(\theta_n)) - \varphi (\hat{R}_{nt}(\theta_n))}{n+1}.$$  \hfill (S41)

Due to (S2), (S3) and $\hat{v}_t(\theta) \geq c_0(\theta) > 0$, we have

$$\left| \frac{\dot{\hat{v}}_t(\theta_n)}{\hat{v}_t(\theta_n)} - \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \right| = \left| \frac{\dot{v}_t(\theta_n) - \dot{\hat{v}}_t(\theta_n)}{\hat{v}_t(\theta_n)} + \frac{\dot{v}_t(\theta_n) - \dot{v}_t(\theta_n)}{\hat{v}_t(\theta_n)} \right|$$

$$\leq \frac{\left| \dot{v}_t(\theta_n) - \dot{\hat{v}}_t(\theta_n) \right|}{\hat{v}_t(\theta_n)} + \frac{\left| \dot{v}_t(\theta_n) - \dot{\hat{v}}_t(\theta_n) \right|}{\hat{v}_t(\theta_n)} \frac{|\dot{v}_t(\theta_n)|}{v_t(\theta_n)}$$

$$\leq C \rho \left[ Z_1 + Z_0 \frac{|\dot{v}_t(\theta_n)|}{v_t(\theta_n)} \right].$$  \hfill (S42)

Hence, in view of (S1) and (S4), for every $0 < b < \infty$,

$$\sup_{\|b\|<b} \sum_{t=1}^{n} \left| \frac{\dot{\hat{v}}_t(\theta_n)}{\hat{v}_t(\theta_n)} - \frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} \right| = O_p(1),$$

which implies that (S41) is $u_p(1)$. Since $\varphi$ is bounded, (S42) is $u_p(1)$. For (S43), since there
is a $n^{-1/2}$ factor from
\[
\frac{\dot{v}_t(\theta_n)}{v_t(\theta_n)} X_t v_t^{1/2}(\theta_n) - \frac{\dot{v}_t(\theta_0\phi)}{v_t(\theta_0\phi)} \eta_t,
\]
it suffices to prove that
\[
K_n := n^{-1/2} \sum_{t=1}^n \frac{\dot{v}_t(\theta_0\phi)}{v_t(\theta_0\phi)} \eta_t \left[ \varphi \left( \frac{R_{nt}(\theta_n)}{n+1} \right) - \varphi \left( \frac{\hat{R}_{nt}(\theta_n)}{n+1} \right) \right] = u_p(1).
\]
Let $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. We split the sum in $K_n$ into two parts: in the first part, $t$ runs till $\lfloor n^k \rfloor - 1$ where $0 < k < 1/2$. We show that this part is $u_p(1)$ by noting that its expectation is of the form $n^k n^{-1/2} = o(1)$ multiplied by expectation of $\dot{v}_t(\theta_0\phi)/v_t(\theta_0\phi)\eta_t$ and a bounded quantity because $\varphi$ is bounded. The number of summands in the second term is $n - n^k$ which is large but there we bound expectation of the sum of by a quantity of the form $n\rho \lfloor n^k \rfloor$ with $0 < k < 1/2$ and $0 < \rho < 1$ and this is $o(1)$. Accordingly
\[
K_n = n^{-1/2} \sum_{t=1}^{\lfloor n^k \rfloor - 1} \frac{\dot{v}_t(\theta_0\phi)}{v_t(\theta_0\phi)} \eta_t \left[ \varphi \left( \frac{R_{nt}(\theta_n)}{n+1} \right) - \varphi \left( \frac{\hat{R}_{nt}(\theta_n)}{n+1} \right) \right] + n^{-1/2} \sum_{t=\lfloor n^k \rfloor}^n \frac{\dot{v}_t(\theta_0\phi)}{v_t(\theta_0\phi)} \eta_t \left[ \varphi \left( \frac{R_{nt}(\theta_n)}{n+1} \right) - \varphi \left( \frac{\hat{R}_{nt}(\theta_n)}{n+1} \right) \right]. \tag{S45}
\]
To show (S45) is $u_p(1)$, we prove that for every $0 < b < \infty$,
\[
\sup_{\|\tilde{b}\|<b} \left\| \varphi \left( \frac{R_{nt}(\theta_n)}{n+1} \right) - \varphi \left( \frac{\hat{R}_{nt}(\theta_n)}{n+1} \right) \right\| = O_p(n^{k-1}). \tag{S46}
\]
Since sequences $\{R_{nt}(\theta_n)\}$ and $\{\hat{R}_{nt}(\theta_n)\}$ are permutations of $\{1,\ldots,n\}$, with the proba-
bility tending to one as \( n \to \infty \), both \( \{ R_{nt}(\theta_n) \} \) and \( \{ \hat{R}_{nt}(\theta_n) \} \) are at points of continuity of \( \varphi \) that has a finite number of the points of discontinuity. Therefore, to prove (S46), it suffices to prove

\[
\sup_{\| \theta \| < b} \left| \frac{R_{nt}(\theta_n)}{n+1} - \frac{\hat{R}_{nt}(\theta_n)}{n+1} \right| = O_p(n^{k-1}). \tag{S47}
\]

For \( \lfloor n^k \rfloor \leq t \leq n \), we decompose ranks as

\[
\frac{R_{nt}(\theta_n)}{n+1} - \frac{\hat{R}_{nt}(\theta_n)}{n+1} = \frac{1}{n+1} \sum_{i=1}^{\lfloor n^k \rfloor - 1} \left\{ I \left[ \frac{X_i}{v_i^{1/2}(\theta_n)} < \frac{X_t}{v_t^{1/2}(\theta_n)} \right] - I \left[ \frac{X_i}{\hat{v}_i^{1/2}(\theta_n)} < \frac{X_t}{\hat{v}_t^{1/2}(\theta_n)} \right] \right\} + \frac{1}{n+1} \sum_{i=\lfloor n^k \rfloor}^{n} \left\{ I \left[ \frac{X_i}{v_i^{1/2}(\theta_n)} < \frac{X_t}{v_t^{1/2}(\theta_n)} \right] - I \left[ \frac{X_i}{\hat{v}_i^{1/2}(\theta_n)} < \frac{X_t}{\hat{v}_t^{1/2}(\theta_n)} \right] \right\}, \tag{S48}
\]

where the first sum is \( O_p(n^{k-1}) \). For the second sum, writing

\[
I \left[ \frac{X_i}{\hat{v}_i^{1/2}(\theta_n)} < \frac{X_t}{\hat{v}_t^{1/2}(\theta_n)} \right] = I \left[ \frac{X_i}{v_i^{1/2}(\theta_n)\hat{v}_i^{1/2}(\theta_n)} < \frac{X_t}{v_t^{1/2}(\theta_n)\hat{v}_t^{1/2}(\theta_n)} \right],
\]

the modulus of (S48) is bounded above by

\[
\sup_{x \in \mathbb{R}} \frac{1}{n+1} \sum_{i=\lfloor n^k \rfloor}^{n} I \left[ \frac{X_i}{v_i^{1/2}(\theta_n)} < x \right] - I \left[ \frac{X_i}{v_i^{1/2}(\theta_n)\hat{v}_i^{1/2}(\theta_n)} < x \right].
\]
Using $|I(A) - I(B)| \leq I(A \cap B^c) + I(A^c \cap B)$, this is bounded above by

$$\sup_{x \in \mathbb{R}} \frac{1}{n+1} \sum_{i=\lceil n^k \rceil}^{n} I(A_{i,x}, \theta),$$

where the set $A_{i,x}, \theta$ is defined as

$$A_{i,x}, \theta := \begin{cases} \frac{X_i}{v_i^{1/2}(\theta)} < x, \frac{X_i}{v_i^{1/2}(\theta)} \frac{\hat{v}_i^{1/2}(\theta)}{v_i^{1/2}(\theta)} \geq x \\ \frac{X_i}{v_i^{1/2}(\theta)} \geq x, \frac{X_i}{v_i^{1/2}(\theta)} \frac{\hat{v}_i^{1/2}(\theta)}{v_i^{1/2}(\theta)} < x \end{cases} \cup \begin{cases} \frac{X_i}{v_i^{1/2}(\theta)} < x, \frac{X_i}{v_i^{1/2}(\theta)} \frac{\hat{v}_i^{1/2}(\theta)}{v_i^{1/2}(\theta)} \geq x \\ \frac{X_i}{v_i^{1/2}(\theta)} \geq x, \frac{X_i}{v_i^{1/2}(\theta)} \frac{\hat{v}_i^{1/2}(\theta)}{v_i^{1/2}(\theta)} < x \end{cases}.$$

Therefore, it suffices to prove that $\sum_{i=\lceil n^k \rceil}^{n} I(A_{i,x}, \theta) = o_p(1)$ uniformly with respect to both $x$ and $\theta$. We show this with sets containing $A_{i,x}, \theta$.

Recall that using (S2), $\hat{v}_i(\theta) \geq c_0(\theta) > c$ for a positive constant $c$ and so

$$0 < v_i^{1/2}(\theta) - \hat{v}_i^{1/2}(\theta) \leq \frac{\rho^j Z_0}{v_i^{1/2}(\theta) + \hat{v}_i^{1/2}(\theta)} \leq \frac{\rho^j Z_0}{2c_0^{1/2}(\theta)}.$$

Now using the triangular inequality, (S49)
Therefore (S49) is bounded above by
\[
\frac{\rho^i Z_0}{2c_0^{1/2}(\theta)} + \frac{\rho^i Z_0}{2c_0^{1/2}(\theta)}.
\]

In view of (S49), we get
\[
\left| \frac{X_i}{v_i^{1/2}(\theta)} - \frac{X_i}{v_i^{1/2}(\theta)} \right| \leq (\rho^i + \rho^t) Z_4 \left| \frac{X_i}{v_i^{1/2}(\theta)} \right|,
\]
where \( Z_4 = Z_0/(2C^{1/2}) \).

Therefore, \( A_{i,x,\theta} \) is a subset of
\[
\mathcal{B}_{i,x,\theta} := \left\{ \frac{X_i}{v_i^{1/2}(\theta)} < x, \frac{X_i}{v_i^{1/2}(\theta)} + (\rho^i + \rho^t) Z_4 \left| \frac{X_i}{v_i^{1/2}(\theta)} \right| \geq x \right\}
\]
\[
\cup \left\{ \frac{X_i}{v_i^{1/2}(\theta)} \geq x, \frac{X_i}{v_i^{1/2}(\theta)} - (\rho^i + \rho^t) Z_4 \left| \frac{X_i}{v_i^{1/2}(\theta)} \right| < x \right\}
\]
\[
= \left\{ \eta_i < x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)}, \eta_i + (\rho^i + \rho^t) Z_4 |\eta_i| \geq x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)} \right\}
\]
\[
\cup \left\{ \eta_i \geq x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)}, \eta_i - (\rho^i + \rho^t) Z_4 |\eta_i| < x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)} \right\}
\]
\[
= \left\{ x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)} - (\rho^i + \rho^t) Z_4 |\eta_i| \leq \eta_i < x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)} \right\}
\]
\[
\cup \left\{ x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)} \leq \eta_i < x \frac{v_i^{1/2}(\theta)}{v_i^{1/2}(\theta_0\varphi)} + (\rho^i + \rho^t) Z_4 |\eta_i| \right\}.
\]

Consider r.v.s \( X \) and \( L \geq 0 \) with \( X \) independent of \( \eta \). Then \( P[X < \eta < X + L] \leq \)
sup\{g(y)\}E(L) where the p.d.f. \( g \) is bounded. Consequently,

\[
sup_{x \in \mathbb{R}} E \left[ \sum_{i=\lfloor n^k \rfloor}^{n} I(A_{i,x}, \theta) \right] \leq sup_{y \in \mathbb{R}} g(y) \sum_{i=\lfloor n^k \rfloor}^{n} E \{(\rho^i + \rho^t)Z_4|\eta_i|\}.
\]

Notice that since \( \lfloor n^k \rfloor \leq t \leq n \),

\[
\sum_{i=\lfloor n^k \rfloor}^{n} E \{\rho^t Z_4 | \eta_i|\} \leq n \rho^{\lfloor n^k \rfloor} E(Z_4)E|\eta| = o(1)
\]
due to \( 0 < \rho < 1 \) and \( E|\eta| < \infty \). Hence, \( \sum_{i=\lfloor n^k \rfloor}^{n} I(A_{i,x}, \theta) \) converges in mean to zero uniformly with respect to both \( x \) and \( \theta \), which entails \( \sum_{i=\lfloor n^k \rfloor}^{n} I(A_{i,x}, \theta) = o_p(1) \) uniformly.

With all the results above, we can easily prove Proposition 2.1 and the asymptotic result for the R-estimator as follows.

**Proof of Proposition 2.1.**

Proof. Combining (S34), (S36), (S32) and (S40), we get

\[
\hat{R}_n(\theta_n) - Mb - Q_n - N_n - Jb/2 = u_p(1),
\]

which, by letting \( b = 0 \), entails

\[
\hat{R}_n(\theta_0) = Q_n + N_n + u_p(1).
\]
Hence, (2.7) follows by recalling that $M = J \rho(\varphi)/2$.

The proof of (2.8) follows directly from (S35) and (S38).

Proof of Theorem 2.1

Proof. From the definition of $\hat{\theta}_n$ in (2.9), (2.7) and (2.8) in Proposition 2.1, consistency of $\hat{\Upsilon}_n$ and the asymptotic discreteness of $\bar{\theta}_n$, we have

$$n^{1/2}(\hat{\theta}_n - \theta_{0\varphi}) = n^{1/2} \left\{ \check{\theta}_n - n^{-1/2}(\check{\Upsilon}_n)^{-1} \check{R}_n(\check{\theta}_n) - \theta_{0\varphi} \right\} = n^{1/2} \left\{ \check{\theta}_n - n^{-1/2}(\check{\Upsilon}_n)^{-1} \left[ \check{R}_n(\theta_{0\varphi}) + (1/2 + \rho(\varphi)/2)J_n^{1/2}(\check{\theta}_n - \theta_{0\varphi}) \right] - \theta_{0\varphi} \right\} + o_p(1) = n^{1/2} \left\{ \check{\theta}_n - n^{-1/2}(1/2 + \rho(\varphi)/2)J^{-1}\check{R}_n(\theta_{0\varphi}) - (\theta_n - \theta_{0\varphi}) - \theta_{0\varphi} \right\} + o_p(1) = - (1/2 + \rho(\varphi)/2)^{-1}J^{-1}\check{R}_n(\theta_{0\varphi}) + o_p(1).$$

In view of (2.8), we have

$$n^{1/2}(\hat{\theta}_n - \theta_{0\varphi}) = - (1/2 + \rho(\varphi)/2)^{-1}J^{-1}(Q_n + N_n) + o_p(1).$$

Now, it remains to obtain the asymptotic covariance matrix of $\sqrt{n}(\hat{\theta}_n - \theta_{0\varphi})$. Recall (S35) and (S38). Since the asymptotic covariance matrices of $Q_n$ and $N_n$ have been derived, it remains to obtain the covariance matrix $\text{Cov}(Q_n, N_n)$. Note that $E[Q_nN_n']$
equals

\[
\begin{align*}
\mathbb{E}\left\{ \left[ \int_0^1 n^{-1/2} \sum_{t=1}^n \frac{\dot{v}_t(\theta_{0\varphi})}{v_t(\theta_{0\varphi})} \left[ \mu(G^{-1}(u)) - \mu(G_n^{-1}(u)) \right] d\varphi(u) \right] \times \left[ n^{-1/2} \sum_{t=1}^n \frac{\dot{v}_t'(\theta_{0\varphi})}{v_t(\theta_{0\varphi})} \left[ 1 - \eta_t \varphi [G(\eta_t)] \right] \right] \right\}.
\end{align*}
\] (S51)

Using (S39) and \( n^{-1} \sum_{t=1}^n \dot{v}_t(\theta_{0\varphi})/v_t(\theta_{0\varphi}) \to \mathbb{E}(\dot{v}_1(\theta_{0\varphi})/v_1(\theta_{0\varphi})) \), as \( n \to \infty \), (S51) has the same limit as

\[
E\left( \frac{\dot{v}_1(\theta_{0\varphi})}{v_1(\theta_{0\varphi})} \right) \times \lim_{n \to \infty} \mathbb{E}\left\{ \int_0^1 G^{-1}(u) \left\{ \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n [I\{\eta_i \leq G^{-1}(u)\} - u] \frac{\dot{v}_j'(\theta_{0\varphi})}{v_j'(\theta_{0\varphi})} [1 - \eta_j \varphi [G(\eta_j)]] \right\} d\varphi(u) \right\}
\]

\[
= E\left( \frac{\dot{v}_1(\theta_{0\varphi})}{v_1(\theta_{0\varphi})} \right) \times \lim_{n \to \infty} \mathbb{E}\left\{ \int_0^1 G^{-1}(u) \left\{ \frac{1}{n} \sum_{i=1}^n [I\{\eta_i \leq G^{-1}(u)\} - u] \frac{\dot{v}_i'(\theta_{0\varphi})}{v_i'(\theta_{0\varphi})} [1 - \eta_i \varphi [G(\eta_i)]] \right\} d\varphi(u) \right\}
\]

\[
= E\left( \frac{\dot{v}_1(\theta_{0\varphi})}{v_1(\theta_{0\varphi})} \right) \times \int_0^1 G^{-1}(u) \mathbb{E}\left\{ \left[ I\{\eta_1 \leq G^{-1}(u)\} - u \right] [1 - \eta_1 \varphi [G(\eta_1)]] \right\} d\varphi(u)
\]

\[
= E\left( \frac{\dot{v}_1(\theta_{0\varphi})}{v_1(\theta_{0\varphi})} \right) \times \int_0^1 G^{-1}(u) \mathbb{E}\left\{ I\{\eta_1 \leq G^{-1}(u)\} [1 - \eta_1 \varphi [G(\eta_1)]] \right\} d\varphi(u),
\]

where the first equality is due to independence of \( \eta_i \) and \( \eta_j \) for \( i \neq j \), independence of \( v_j \) and \( \eta_j \), and Assumption (A1). The second equality is due to independence of \( v_i \) and \( \eta_i \). The last equality is due to Assumption (A1).
Recall the definition of $\lambda(\varphi)$ in (2.6), which can also be written as

$$\lambda(\varphi) = \int_0^1 G^{-1}(u) E \{ I\{\eta_1 \leq G^{-1}(u)\} [1 - \eta_1 \varphi [G(\eta_1)]]\} \, d\varphi(u).$$

We then have

$$\lim_{n \to \infty} \text{Cov}(Q_n, N_n) = E \left( \frac{\dot{v}_1(\theta_0\varphi)}{v_1(\theta_0\varphi)} \right) E \left( \frac{\dot{v}_1'(\theta_0\varphi)}{v_1'(\theta_0\varphi)} \right) \lambda(\varphi).$$

Hence, by recalling (S38) and in view of (2.10), the asymptotic covariance matrix of $\sqrt{n}(\hat{\theta}_n - \theta_0\varphi)$ is

$$J^{-1} \left[ \frac{4\gamma(\varphi) + 8\lambda(\varphi)}{(1 + \rho(\varphi))^2} \right] E \left( \frac{\dot{v}_1(\theta_0\varphi)}{v_1(\theta_0\varphi)} \right) E \left( \frac{\dot{v}_1'(\theta_0\varphi)}{v_1'(\theta_0\varphi)} \right) \lambda(\varphi) J^{-1}$$

$$= J^{-1} \left[ \frac{4\gamma(\varphi) + 8\lambda(\varphi)}{(1 + \rho(\varphi))^2} \right] E \left( \frac{\dot{v}_1(\theta_0\varphi)}{v_1(\theta_0\varphi)} \right) E \left( \frac{\dot{v}_1'(\theta_0\varphi)}{v_1'(\theta_0\varphi)} \right) + 4\sigma^2(\varphi)J J^{-1}. \quad \square$$

### S2 R-estimators in GJR models and applications

Glosten et al. (1993) proposed the GJR $(p,q)$ model for the asymmetric volatility observed in many financial dataset exhibit asymmetry property. The GJR $(p,q)$ model is defined by

$$X_t = \sigma_t \epsilon_t$$
where
\[
\sigma_t^2 = \omega_0 + \sum_{i=1}^{p} (\alpha_0 + \gamma_0 I(X_{t-i} < 0)) X_{t-i}^2 + \sum_{j=1}^{q} \beta_{0j} \sigma_{t-j}^2, \quad t \in \mathbb{Z},
\]
with \(\omega_0, \alpha_0, \gamma_0, \beta_{0j} > 0, \forall i, j\). Since \(\sigma_t^2\) is linear in parameters, we define the R-estimators for the GJR model using the same rank-based central sequence as in (2.3). See also Iqbal and Mukherjee (2010) for the extension of M-estimators from the GARCH model to the GJR model. We do not prove any asymptotic theory for the R-estimators of the GJR model but present here empirical analysis using the same algorithm as in (2.11) to compute the R-estimators. The following extensive simulation study, similar to the GARCH case, demonstrates the superior performance of the R-estimators compared to the QMLE that is often used in this model. We also carry out simulation with increasing sample sizes to show the consistency of the R-estimators. Three types of R-estimators and the QMLE are compared below under various error distributions. We run simulations with the sample size \(n = 1000\), number of replications \(R = 500\) and true parameter
\[
\theta_0 = (3.45 \times 10^{-4}, 0.0658, 0.0843, 0.8182)',
\]
which is motivated by the estimate in Tsay (2010) for the IBM stock monthly returns from 1926 to 2003. The estimates of the standardized bias and MSE of the QMLE and R-estimators and those of the ARE of the R-estimators wrt the QMLE are reported in Table S1.

We remark that the results are consistent with those in Table 2: the vdW score still dominates the QMLE uniformly; the optimal scores under the DE and logistic distributions are also the sign and Wilcoxon, respectively. It is worth noting that under \(t(3)\) distribution, the R-estimators are much more efficient than the QMLE for the parameter \(\gamma\).
Table S1: The estimates of the standardized bias, MSE and ARE of the R-estimators (sign, Wilcoxon and vDW) and the QMLE for the GJR (1, 1) model under various error distributions (sample size \( n = 1000 \); \( R = 500 \) replications).

|                  | Standardized bias | Standardized MSE and ARE |
|------------------|-------------------|---------------------------|
|                  | \( \omega \)      | \( \alpha \)   | \( \gamma \)   | \( \beta \)   | \( \omega \) | \( \alpha \) | \( \gamma \) | \( \beta \) |
| **Normal**       |                   |                           |               |               |               |               |               |               |
| QMLE             | 8.70 \times 10^{-5} | -2.03 \times 10^{-3} | 7.47 \times 10^{-3} | -2.03 \times 10^{-2} | 4.17 \times 10^{-8} | 7.63 \times 10^{-4} | 1.53 \times 10^{-3} | 3.80 \times 10^{-3} |
| Sign             | 9.01 \times 10^{-5} | -1.20 \times 10^{-3} | 8.43 \times 10^{-3} | -2.00 \times 10^{-2} | 4.73 \times 10^{-8} | 9.40 \times 10^{-4} | 1.92 \times 10^{-3} | 4.30 \times 10^{-3} |
| Wilcoxon         | 9.35 \times 10^{-5} | -8.15 \times 10^{-4} | 8.65 \times 10^{-3} | -2.00 \times 10^{-2} | 4.96 \times 10^{-8} | 8.72 \times 10^{-4} | 1.76 \times 10^{-3} | 4.24 \times 10^{-3} |
| vDW              | 8.72 \times 10^{-5} | -1.61 \times 10^{-3} | 7.59 \times 10^{-3} | -2.01 \times 10^{-2} | 4.24 \times 10^{-8} | 7.72 \times 10^{-4} | 1.56 \times 10^{-3} | 3.82 \times 10^{-3} |
| **DE**           |                   |                           |               |               |               |               |               |               |
| QMLE             | 7.07 \times 10^{-5} | -1.28 \times 10^{-3} | 1.13 \times 10^{-2} | -1.87 \times 10^{-2} | 4.66 \times 10^{-8} | 1.23 \times 10^{-3} | 3.19 \times 10^{-3} | 4.72 \times 10^{-3} |
| Sign             | 4.91 \times 10^{-5} | -2.26 \times 10^{-3} | 6.99 \times 10^{-3} | -1.62 \times 10^{-2} | 3.44 \times 10^{-8} | 9.58 \times 10^{-4} | 2.36 \times 10^{-3} | 4.10 \times 10^{-3} |
| Wilcoxon         | 5.06 \times 10^{-5} | -2.23 \times 10^{-3} | 7.25 \times 10^{-3} | -1.61 \times 10^{-2} | 3.45 \times 10^{-8} | 9.74 \times 10^{-4} | 2.41 \times 10^{-3} | 4.06 \times 10^{-3} |
| vDW              | 4.98 \times 10^{-5} | -3.35 \times 10^{-3} | 6.73 \times 10^{-3} | -1.71 \times 10^{-2} | 3.54 \times 10^{-8} | 1.02 \times 10^{-3} | 2.50 \times 10^{-3} | 4.23 \times 10^{-3} |
| **Logistic**     |                   |                           |               |               |               |               |               |               |
| QMLE             | 8.01 \times 10^{-5} | 7.88 \times 10^{-4} | 8.85 \times 10^{-3} | -1.62 \times 10^{-2} | 3.86 \times 10^{-8} | 1.07 \times 10^{-3} | 2.40 \times 10^{-3} | 3.46 \times 10^{-3} |
| Sign             | 6.06 \times 10^{-5} | -2.59 \times 10^{-4} | 5.54 \times 10^{-3} | -1.49 \times 10^{-2} | 3.26 \times 10^{-8} | 8.97 \times 10^{-4} | 1.96 \times 10^{-3} | 3.36 \times 10^{-3} |
| Wilcoxon         | 5.97 \times 10^{-5} | -5.22 \times 10^{-4} | 5.38 \times 10^{-3} | -1.44 \times 10^{-2} | 3.07 \times 10^{-8} | 8.76 \times 10^{-4} | 1.93 \times 10^{-3} | 3.18 \times 10^{-3} |
| vDW              | 6.27 \times 10^{-5} | -1.18 \times 10^{-3} | 5.18 \times 10^{-3} | -1.56 \times 10^{-2} | 3.25 \times 10^{-8} | 9.26 \times 10^{-4} | 2.04 \times 10^{-3} | 3.35 \times 10^{-3} |
| **t(8)**         |                   |                           |               |               |               |               |               |               |
| QMLE             | 9.91 \times 10^{-5} | -1.00 \times 10^{-3} | 9.21 \times 10^{-2} | -6.45 \times 10^{-2} | 1.14 \times 10^{-7} | 4.38 \times 10^{-3} | 1.18 \times 10^{-1} | 2.31 \times 10^{-2} |
| Sign             | 5.68 \times 10^{-5} | 4.23 \times 10^{-4} | 2.71 \times 10^{-2} | -2.77 \times 10^{-2} | 3.78 \times 10^{-8} | 1.35 \times 10^{-3} | 4.69 \times 10^{-3} | 6.33 \times 10^{-3} |
| Wilcoxon         | 5.69 \times 10^{-5} | 3.94 \times 10^{-5} | 2.74 \times 10^{-2} | -2.85 \times 10^{-2} | 3.93 \times 10^{-8} | 1.40 \times 10^{-3} | 4.80 \times 10^{-3} | 6.62 \times 10^{-3} |
| vDW              | 6.12 \times 10^{-5} | -1.61 \times 10^{-3} | 3.43 \times 10^{-2} | -3.71 \times 10^{-2} | 5.13 \times 10^{-8} | 1.93 \times 10^{-3} | 7.45 \times 10^{-3} | 9.58 \times 10^{-3} |
Table S2: The standardized bias, MSE of the R-estimators (sign, Wilcoxon and vdW) for the GJR (1, 1) model under normal error distributions with different sample sizes ($R = 500$ replications).

|                | Standardized Bias | Standardized MSE |
|----------------|-------------------|------------------|
|                | $\omega$          | $\alpha$        | $\gamma$ | $\beta$ | $\omega$ | $\alpha$ | $\gamma$ | $\beta$ |
| **Sign**       |                   |                  |          |         |          |          |          |         |
| $n = 500$      | 1.78×10^{-4}      | 2.42×10^{-3}     | 1.88×10^{-2} | -4.84×10^{-2} | 1.43×10^{-7} | 1.64×10^{-3} | 3.72×10^{-3} | 1.33×10^{-2} |
| $n = 1000$     | 9.01×10^{-5}      | -1.20×10^{-3}    | 8.43×10^{-3} | -2.00×10^{-2} | 4.73×10^{-8} | 9.40×10^{-4} | 1.92×10^{-3} | 4.30×10^{-3} |
| $n = 3000$     | 2.76×10^{-5}      | -6.82×10^{-4}    | 1.97×10^{-3} | -4.47×10^{-3} | 8.75×10^{-9} | 3.05×10^{-4} | 6.11×10^{-4} | 8.73×10^{-4} |
| $n = 5000$     | 2.07×10^{-5}      | -4.43×10^{-4}    | 2.05×10^{-3} | -3.43×10^{-3} | 4.57×10^{-9} | 1.70×10^{-4} | 3.68×10^{-4} | 4.62×10^{-4} |
| **Wilcoxon**   |                   |                  |          |         |          |          |          |         |
| $n = 500$      | 1.77×10^{-4}      | 2.52×10^{-3}     | 1.88×10^{-2} | -4.75×10^{-2} | 1.37×10^{-7} | 1.52×10^{-3} | 3.45×10^{-3} | 1.26×10^{-2} |
| $n = 1000$     | 9.35×10^{-5}      | -8.15×10^{-4}    | 8.65×10^{-3} | -2.00×10^{-2} | 4.96×10^{-8} | 8.72×10^{-4} | 1.76×10^{-3} | 4.24×10^{-3} |
| $n = 3000$     | 3.01×10^{-5}      | -2.94×10^{-5}    | 2.68×10^{-3} | -4.30×10^{-3} | 8.18×10^{-9} | 2.82×10^{-4} | 5.60×10^{-4} | 7.85×10^{-4} |
| $n = 5000$     | 2.45×10^{-5}      | 1.52×10^{-4}     | 2.82×10^{-3} | -3.54×10^{-3} | 4.50×10^{-9} | 1.63×10^{-4} | 3.55×10^{-4} | 4.20×10^{-4} |
| **vdW**        |                   |                  |          |         |          |          |          |         |
| $n = 500$      | 1.67×10^{-4}      | 1.42×10^{-3}     | 1.61×10^{-2} | -4.84×10^{-2} | 1.31×10^{-7} | 1.44×10^{-3} | 3.03×10^{-3} | 1.27×10^{-2} |
| $n = 1000$     | 8.72×10^{-5}      | -1.61×10^{-3}    | 7.59×10^{-3} | -2.01×10^{-2} | 4.24×10^{-8} | 7.72×10^{-4} | 1.56×10^{-3} | 3.82×10^{-3} |
| $n = 3000$     | 2.88×10^{-5}      | -1.72×10^{-4}    | 1.60×10^{-3} | -4.59×10^{-3} | 7.42×10^{-9} | 2.61×10^{-4} | 4.90×10^{-4} | 7.19×10^{-4} |
| $n = 5000$     | 2.42×10^{-5}      | 9.50×10^{-6}     | 2.42×10^{-3} | -4.02×10^{-3} | 4.38×10^{-9} | 1.49×10^{-4} | 3.28×10^{-4} | 4.26×10^{-4} |

Simulation under different sample size. We next investigate the behaviour of R-estimators by carrying out simulations with different sample sizes. The number of replications and true parameter are the same as those used for Table S1 and the error distribution is normal. The estimates of the standardized bias and MSE of the R-estimators for the GJR (1, 1) model are shown in Table S2. In general, for all R-estimators, both the bias and MSE decrease when the sample size increases from $n = 500$ to $n = 5000$. This tends to reflect that the R-estimators are consistent estimators of $\theta_{0X}$ for the GJR (1, 1) model.