Liouville coherent states

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Abstract – For a certain class of open quantum systems there exists a dynamical symmetry which connects different time-evolved density matrices. We show how to use this symmetry for dynamics in the Liouville space with time-dependent parameters. This allows us to introduce a concept of generalized coherent states in the Liouville space (i.e., for density matrices). The dynamics of this class of density matrices is characterized by robustness with respect to any time-dependent perturbations of the couplings. We study their dynamical context while focusing on common physical situations corresponding to compact and non-compact symmetries.

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Introduction. – The concept of coherent states plays a very important role in quantum physics. Introduced by Schrödinger for quantum harmonic oscillator [1], it received further application with the birth of quantum optics in the 1960s [2,3]. Further developments of the concept of coherent states are associated with the generalized coherent states defined for any Lie algebra, with the most important examples given by the $su(2)$ and $su(1,1)$ algebras. Coherent states have extremely wide applications in physics and mathematics, reviewed, e.g., in [4–6].

There are several definitions of the coherent states. The oscillator coherent state can be defined either as i) a state which minimizes an uncertainty relation; or ii) an eigenstate of an annihilation operator; or iii) a state obtained from the vacuum state by the action of the displacement operator. These three definitions are equivalent for the harmonic oscillators while they are not for the more general (generalized) coherent states associated to some non-Abelian algebras, other than a Heisenberg algebra. The construction of coherent states associated to a Lie algebra includes three ingredients: i) an algebra $g$ with the representation space; ii) a vacuum state of ladder operators in this representation space. This state is supposed to have an invariant subgroup defined by some subalgebra $h \subset g$; iii) a concrete representation of a group $G$. It follows from this that generalized coherent states have a profound geometrical interpretation: they are labeled by the points of a homogeneous (coset) space $G/H$ which in many important physical situations has a structure of a Kähler manifold.

Those quantum-mechanical systems which have a dynamical symmetry given by $G$ (e.g., the generators of $G$ commute with the Schrödinger operator $i\partial_t - H$, and not just with the Hamiltonian) can be described by a classical dynamical system on a coset $G/H$. One of the most important properties of the generalized coherent states exists due to their Lie group structure: if the initial state of a quantum system is a (generalized) coherent state, it will remain so at any subsequent evolution time (see, e.g., [7]). This key property will be important for the construction below.

The developments briefly outlined above concern the situation of isolated quantum systems. In this letter we propose a construction scheme of generalized coherent states for a certain class of quantum systems with dissipation. The most convenient formulation is in terms of the Liouville space on which the dynamics of the reduced density matrix is defined by the Liouville operator $\mathcal{L}$. The dynamical symmetry in this case is associated with a symmetry of the density matrix, so that there is an algebra $G$ of operators commuting with $\partial_t - \mathcal{L}$. We thus define generalized coherent states for an open system as Liouville coherent states (LCS). These states allow for a new interpretation of the Liouville dynamics and simplify the calculation of typical quantities of interest. In this paper we first introduce the very concept of LCS and then we demonstrate it on several important examples: the simplest non-trivial ones when the algebra $G$ is isomorphic to $su(2)$ algebra, which happens for spin-boson–type
models, or $su(1,1)$ algebra, which corresponds to models of harmonic-oscillator type. A defining property of these density matrices is a robustness of evolution with respect to any time-dependent driving.

**Dynamical symmetry approach to open quantum systems.** In general, the formal solution of the evolution equation for an open system is given by the time-ordered exponent $\rho(t) = T \exp(\mathcal{L}_I(t))\rho(0)$, where $\mathcal{L}_I$ is a Liouville superoperator (that is, it acts on operators) evaluated in the interaction picture. When focusing on a subsystem, the reduced density matrix is evolved in time by the Lindblad generators which are constructed using the eigenoperators of the subsystem according to the usual technique (see, e.g., [8,9]). It is possible that these superoperators fulfill some Lie-(super)-algebra identities [10]. This is the starting point of our construction. We demonstrate that a large class of physically relevant systems indeed satisfies this assumption.

We consider an evolution of the reduced density matrix in the Lindblad form,

$$\mathcal{L}(t)\rho(t) = -\frac{i}{\hbar}[H(t), \rho(t)] + \sum \gamma_j(t)(2A_j\rho(t)A_j^\dagger - A_j^\dagger A_j\rho(t) - \rho(t)A_j^\dagger A_j),$$

(1)

which is the only form compatible with the positivity of the density matrix. Here $A_j$ are eigenoperators of the subsystem. In general, $\gamma_j(t) = \int_0^\infty \text{d}\omega \omega^{-1} \rho(\omega) \exp(-i\omega x)$ is related to the spectral density $\rho(\omega)$ which contains information about sub-system-reservoir coupling. Introducing the following notation: $(A \otimes B)\rho := A\rho B$, one defines the action of the individual terms in (1) on the density matrix as $L_i \rho$ (e.g., $L_1 \rho = A\rho A^\dagger$, $L_2 \rho = A^\dagger \rho A$, $L_3 \rho = \rho A A^\dagger$, etc.).

Now we assume that the components of the Lindblad operators obey the Lie-algebraic-type relations,

$$[L_i, L_j] \rho = f_{ijk} L_k \rho,$$

(2)

where $f_{ijk}$ are some numbers (structure coefficients). Our approach is also valid if $f_{ijk}$ are proportional to some operator which commutes with all the other operators. The assumption (2) is satisfied for a surprisingly large number of systems, basically for all the standard Liouvilleans discussed in literature, and it was shown to be a rather general property of Lindblad-type dynamics [10].

Provided that the relations (2) are satisfied, the time-ordered exponent for any time-dependent coefficients $\gamma_j(t)$ is an element of a Lie group corresponding to the Lie algebra of superoperators. Using the disentangling technique (see, e.g., [9]), the time-ordered exponential can be transformed into a product of ordinary exponentials

$$\rho(t) = \prod_j \exp(f_j(t)L_j)\rho(0).$$

(3)

The relation between the functions $f_j(t)$ and the functions $\gamma_j(t)$ can be derived easily. In particular, for the simplest case of the $su(2)$ and $su(1,1)$ algebraic structure, defined by the commutation relations $[L_-, L_+] = 2\sigma L_0$, $[L_0, L_\pm] = \pm L_\pm$, where $\sigma = \pm 1$ refer to the $su(1,1)$ and $su(2)$ cases, the function $f_+$ can be shown to satisfy the following Riccati-type equation ($j \in \{+,-,\}$),

$$\dot{f}_+ - \gamma_+(t) f_+ - \sigma \gamma_-(t) f_+^2 + \dot{\gamma}_+ = 0,$$

(4)

while the remaining functions are determined as $\gamma_- = \int_0^t (\gamma_+(t) + 2\sigma \gamma_-(t) f_+(t))dt$ and $\gamma_0 = \gamma_+(t) \exp(f_+(t))dt$. This equation can be solved (either analytically or numerically) for any functional form of $\gamma_+(t), \gamma_-(t)$, in particular for the case in which they do not depend on time. The solution in such a case reads

$$f_{\pm}(t) = \frac{(\gamma_+/D) \sinh(tD)}{\cosh(tD) - (\gamma_0/2D) \sinh(tD)},$$

$$f_0(t) = [\cosh(tD) - (\gamma_0/2D) \sinh(tD)]^{-2},$$

(5)

where $D = ((\gamma_0/2)^2 - \sigma \gamma_+ \gamma_-)^{1/2}$.

The disentangled form (3) allows a direct evaluation of any dynamical quantity of interest, such as the entropy $S = -\text{Tr}(\rho \log \rho)$ or the purity of the density matrix $\text{Tr}(\rho^2)$, for arbitrary time-dependent couplings.

The use of the Lie group structure greatly simplifies computations of any correlation functions. For example, the quantity $\text{Tr}(A \rho(t))$ with $A \in \mathcal{G}$ can be reformulated as $\partial_t \text{Tr}(\exp(A )\rho(t))|_{t=0}$: the expression in the brackets can be evaluated using the group multiplication rules [5].

**Liouville coherent states (LCS).** We proceed with the Liouville space formulation. A Liouville space $\mathcal{L}$ is defined as a direct product of two Hilbert spaces, $\mathcal{L} = \mathcal{H} \otimes \tilde{\mathcal{H}}$, corresponding to the left and right vectors in terms of superoperator notations. Thus, a vector $|A\rangle \in \mathcal{L}$ corresponds to an operator as follows: $|A\rangle = \sum_n A_{m,n} |m,n\rangle \rightarrow \sum_{m,n} A_{m,n} |m\rangle \langle n|$, where the sum runs over orthonormal bases $|m\rangle$ and $|n\rangle$ of the spaces $\mathcal{H}$ and $\tilde{\mathcal{H}}$ of dimensions $d$, $d'$. To define a geometry on the Liouville space we use the usual scalar product defined by $\langle A, B \rangle := \text{Tr}[A^\dagger B]$ (cf. [9]). This scalar product allows us to introduce the bra-vectors for each ket-vector in the Liouville space. The equation for the density matrix $\tilde{\rho} = -i[H, \rho]$ gets converted in this notation into $|\tilde{\rho}\rangle = -iH |\rho\rangle$ where $\tilde{H} = H - \tilde{H}$. This map is known as the Choi-Jamiolkowski map between states and operators and it may be used to extend certain concepts known for states in the Hilbert space to states in the Liouville space, which correspond to operators in the Hilbert space: the operators $A = \sum_j a_{j,k} |j\rangle \langle k|$ are mapped to states $|A\rangle = \sum_j a_{j,k} |j, k\rangle$.

**Definition of LCS:** We define a Liouville coherent state as a generalized coherent state in the Liouville space. The operators creating generalized coherent states thus act between two Hilbert spaces constituting the Liouville space. The general condition that the initial density matrix be in the class of coherent states means that some linear combination of generators of the Lie algebra.

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annihilates the state \( \sum \xi_i L_i |\rho\rangle = 0 \) and thus defines a stationary subgroup \( H \). We explicitly demonstrate this for the \( SU(2) \) and \( SU(1, 1) \) cases, in which \( H = U(1) \).

**Geometry of dissipative dynamics.**—Since generalized coherent states are labeled by the points of the cosets \( G/H \) where \( H \) is a stationary subgroup, the geometry of this coset space has profound influence on the dynamics. The main property of this type of dynamical evolution can be formulated as follows: if the Liouville operator is an element of the Lie algebra of superoperators, and if the initial density matrix is a coherent state (in the sense that the corresponding vector in the Liouville space is a generalized coherent state), irrespectively of the precise form of the time-dependent coefficients \( \gamma_j(t) \) the subsequent dynamics will remain in this subclass of the states (density matrices). This is a direct consequence of the group multiplication property. The overlap between two such LCS is given by the eigenfunctions of the Laplace-Beltrami operator on \( G/H \) [5]. The distance function on this space is a measure of Lodschmidt echo [11,12]. The area 2-form on \( G/H \) corresponds to the imaginary part of the metric and describes the geometric (Berry) curvature of the quantum evolution. Generally, the topology of the coset \( G/H \) can be non-trivial.

**Examples.**—We consider two types of examples which have great physical importance. One example refers to the case in which the Liouville operator is a combination of generators of a compact group, \( SU(2) \), while the other example is a realization of a non-compact, \( SU(1, 1) \) group. For both \( su(2) \) and \( su(1, 1) \) algebras the coherent state is built upon a basis state (which we denote by \( 0 \)) using a shift operator \( D(\xi) \), such that \( |\xi\rangle := D(\xi) |0\rangle \), and

\[
|\xi\rangle := e^{\xi L_+} e^{i\eta L_0} e^{-\xi^* L_-} |0\rangle = e^{\xi L_+} e^{-\xi^* L_-} |0\rangle,
\]

where \( \eta = -\sigma \log (1 - \sigma |\xi|^2) \) and the relation between \( \xi \) and \( \zeta = \text{tanh} |\xi|^2 \) \( \zeta \) with \( \zeta = \frac{1}{2} |\xi|^2 \) for \( su(1, 1) \) and \( \zeta = -\text{tanh} (\Theta/2) e^{-i\sigma} \) for \( su(2) \). Modifying the arguments of [5] for the present case, we may write down the resolution of identity on the Liouville space in the following form:

\[
1 = \sum_k \int d\mu_k(\zeta) |k, \zeta\rangle \langle k, \zeta|,
\]

where \( k \) runs over all representations of the underlying dynamical symmetry group and \( \mu_k(\zeta) \) is the group measure [5].

The action of the evolution superoperator on a coherent state can be easily calculated starting from its disentangled form (3). We introduce the operator

\[
\mathcal{O} := e^{f_+ L_+ + f_0 L_0} e^{-f_- L_-} e^{f_+ L_+ - \xi L_-},
\]

which can be rewritten in a product form,

\[
\mathcal{O} = e^{g_+ L_+} e^{g_0 L_0} e^{g_- L_-} = D(g_+) \cdot e^{g_0 L_0} \cdot \exp \left( g_- e^{g_0 - \eta} L_- \right),
\]

in which the functions \( g_i \) are governed by equations

\[
\begin{align*}
g_+ &= f_+ + \xi e^{f_0} \Lambda, \\
e^{g_0} &= (1 - \sigma |\zeta|^2)^{-1} e^{f_0} \Lambda^{-1}, \\
g_- &= -\xi^* (1 - \sigma |\zeta|^2)^{-1} \sigma^* e^{-f_0} \Lambda^{-1},
\end{align*}
\]

where \( \Lambda = 1 - \sigma e^{-f_0} \). As expected, the evolution of an initial coherent state \( |\zeta\rangle \) leads to a coherent state \( |g_+\rangle \), described by the equation

\[
T \exp(\mathcal{L}(t) |\zeta\rangle = e^{\langle t, \gamma_0, \gamma_+ \rangle |g_+\rangle},
\]

with the prefactor \( e^{\langle t, \gamma_0, \gamma_+ \rangle} = \bar{c}(t) \cdot e^{\langle \gamma_0 - \eta \rangle k} \), where \( k \) is a representation-dependent constant, defined by \( L_0 |0\rangle = k |0\rangle \), and the factor \( \bar{c}(t) \) is determined solely by the dynamics within the \( U(1) \) sector. Note that the relation between \( g_+ \) and \( \zeta \) is a Möbius transform, under which circles in the complex plane of \( \zeta \) are mapped onto circles in the plane of \( g_+ \); a circle \( |\zeta| e^{i\varphi} \) is mapped to a circle of radius \( R \) and origin \( z \) given by

\[
R = \frac{|\zeta||f_0|}{1 - |f_0|^2 |z|^2}, \quad z = f_+ + \sigma f_0 \bar{z} = \frac{|\zeta|^2 |\varphi|}{1 - |f_0|^2 |z|^2}.
\]

Consideration of a mapping of circles is a convenient way to think about the evolution of coherent-state parameters. Namely, one can see that while the radius of a circle remains approximately proportional to the radius of the initial circle, the origin shifts to some finite value; therefore the “average” value of a coherent-state parameter at a later time is shifted from zero.

**Spin-boson-type models: SU(2) LCS.**—For the case of spin-boson–like models within the RWA the system is described by a Lindblad equation of the form (1), in which we set \( H(t) = \sigma^+ \cdot \Omega(t)/2 \), \( A_0 = \sigma^+ \), \( A_+ = \sigma_+ \), \( A_- = \gamma_- = \sigma_- \) and \( \gamma_+ = \gamma(t) n \). As expected, the evolution of an initially coherent state \( |\zeta\rangle \) leads to a coherent state \( |g_+\rangle \), described by the equation

\[
\mathcal{O} := e^{f_+ L_+ + f_0 L_0} e^{-f_- L_-} e^{f_0 L_0} \exp \left( f_- e^{f_0 - \eta} L_- \right),
\]

which has a structure of the direct product \( su(2) \times u(1) \) and consequently the supergroup operator can be disentangled using the procedure outlined above. Therefore the formal operator solution of the evolution equation \( \hat{\rho}(t) = \mathcal{L}(t) \rho(t) \), with

\[
\mathcal{L} = -i \bar{\Omega}(t) R + \gamma(t) n L_+ + \gamma(t) (n + 1) L_- - \gamma(t) L_0 \gamma(t)(n + 1),
\]

reads

\[
\rho(t) = e^{f_+ L_+ + f_0 L_0} e^{f_- L_-} e^{-iR} e^{-i \gamma(t)(n + 1/2) f_0 \rho(0)} e^{i \gamma(t) 0},
\]

where \( f_{\pm}(t) \) satisfy the Ricatti equation (4).
evolution is plotted for various values of detuning $\Delta$. Both $R(t)$ and $z(t)$ are periodic functions of time.

Any matrix, namely the density matrix can be decomposed into irreducible parts classified by the Casimir operator $\vec{L}^2 = \frac{1}{2} (L^+ L^- + L^- L^+) + L^z L^z$, which can be explicitly expressed as $\vec{L}^2 = \frac{3}{8} (\rho + \sigma_z \rho \sigma_z)$. In fact, using the basis of $2 \times 2$ matrices $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_{\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we can write $\rho = \rho_+ + \rho_- = \rho_0 + \rho_+ + \rho_0 + \rho_-$ and $\rho_{j=1/2} = \rho_1 (1 + \sigma_z) + \rho_1 (1 - \sigma_z)$. The most general density matrix $\rho$ for a $\text{su}(2)$ system with spin $1/2$ can be written as a linear combination of spin coherent states $\rho = c_0 |0, + \rangle + c_0^* |0, - \rangle + c_1 (\zeta |1/2; \zeta \rangle + \zeta^* | -1/2, \zeta \rangle$, where $|0, \pm \rangle$ corresponds to $\sigma_{\pm}$ (the subspace with $j=0$) and $|1/2, \zeta \rangle$ is a coherent state with parameter $\zeta$ built upon the basis state $|j=1/2, m=-1/2 \rangle$ corresponding to $(1 - \sigma_z)/2$. The evolution operator acts independently on the two subspaces. Obviously, the coherent-state structure on the subspace $j=0$ is irrelevant, the only non-trivial evolution there being the evolution of the coefficients $c$. From the condition $\text{Tr} \rho = 1$ we derive $c_1 (\zeta) = \sqrt{1 + |\zeta|^2}/(1 + \zeta)$.

For the sake of a particular example, we follow [13] and consider a two-level system with level splitting $\omega$, interacting with a one-mode bath of frequency $\gamma = \omega + \Delta$. The non-Markovian evolution can be treated by a Lindblad equation, however with time-dependent terms. A time-dependent energy splitting $\Omega(t) = 3 \mathcal{A}(t)$ and a time-dependent dissipation $\gamma(t) = 3 \mathcal{X}(t)$ are given by

$$
\mathcal{X}(t) = i \left( \omega - \frac{g'}{2} \right) \frac{r s + e^{i \Delta' t}}{s + e^{i \Delta' t}},
$$

where $\Delta' = \sqrt{\Delta^2 + g^2}$, $g' = \Delta' - \Delta$, $s = g'/(\Delta + \Delta')$ and $r = [\omega + (\Delta + \Delta')/2]/[\omega + (\Delta - \Delta')/2]$; we also set $\bar{n} = 0$. As $\gamma_+$ is absent, we have $f_+ = 0$, and the solution to the corresponding Riccati equations simplifies greatly: $f_-(t) = -f_0(t) = \int_0^t \gamma(t) dt := \Gamma(t)$. The time evolution of a coherent state can be deduced from eq. (11).

Thanks to the absence of dissipation (the bath has one mode only), $f_-(t)$ is a periodic function of time and therefore the coherent-state parameter is periodic as well. As mentioned above, a circle in the plane of the initial coherent-state parameter $\zeta(0)$ is mapped onto a circle in the plane of coherent-state parameter $\zeta(t)$ at time $t$. The “pulsation” of the parameters $R$ and $z$ of eq. (12) is plotted in fig. 1, while the time evolution of the purity of the density matrix is shown in fig. 2.

**Harmonic-oscillator–type models: SU(1,1) LCS.**

In analogy with the LCS for spin-boson–type models, we can study models of a harmonic-oscillator type, which lead to a non-compact symmetry group SU(1,1). A generic

Fig. 1: (Colour on-line) Time dependence of the parameters $R$ and $z$, introduced in eq. (12), for the $\text{su}(2)$-type model defined by eq. (17) (with parameters $\omega = 2$ and $g = 1$). The initial coherent-state parameter $\zeta$ lies on a circle $|\zeta| = 1/2$. The evolution is plotted for various values of detuning $\Delta$. Both $R(t)$ and $z(t)$ are periodic functions of time.

Fig. 2: (Colour on-line) The time evolution of the purity $\text{Tr} \rho^2(t)$ of the density matrix $\rho$ for a $\text{su}(2)$-type system (upper panel) defined by eq. (17), and a $\text{su}(1,1)$-type system (lower panel). The initial state of the evolution is a LCS with initial parameter $\zeta$ (respectively, $\zeta_0$) with different initial purities. The $\text{su}(1,1)$ system is characterized by time-dependent decay rates $\gamma_\pm(t) = \gamma(1 + \cos(8 \pi t))/2$. Full lines describe the evolution when $c_m = 0$ in eq. (22) for $m > 0$ (diagonal matrix). Dashed lines involve a small admixture of a coherent state $|1/2, \zeta_1 \rangle$.

- Harmonic-oscillator–type models: SU(1,1) LCS.

  - In analogy with the LCS for spin-boson–type models, we can study models of a harmonic-oscillator type, which lead to a non-compact symmetry group SU(1,1). A generic
model of a harmonic oscillator in contact with a reservoir is described by the Lindblad equation of the form (1) (cf. [9]) with $H(t) = \omega(t)a^\dagger a$, $A_+ = a^\dagger$, $A_- = a$, $\gamma_+ = \gamma_2(t)\hat{n}$ and $\gamma_- = \gamma_1(t)(\hat{n}+1)$, where $\hat{n}$ is the equilibrium thermal occupancy of the oscillator and couplings $\gamma_{1,2}(t)$ are system specific. To proceed we denote $\hat{n} = a^\dagger a$ and introduce the following superoperators:

$$K_-\rho = a\rho a^\dagger, \quad K_+\rho = a^\dagger\rho a$$

$$K_0\rho = \frac{i}{2}(\hat{n}\rho + \rho\hat{n}) + R\rho = \hat{n}\rho - \rho\hat{n},$$

which satisfy the $su(1,1)$ commutation relations

$$[K_, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm,$$

$$[R, K_\alpha] = 0.$$  \hspace{1cm} (18)

The Casimir invariant is given by $[K_0^2 - 1/2(K_+ K_- + K_- K_+)]\rho = \frac{i}{2}(-\rho + \hat{n}^2\rho + \rho\hat{n}^2 - 2\hat{n}\rho\hat{n})$. The algebraic structure in this case is $su(1,1) \times u(1)$ and we may again disentangle the evolution operator using the method presented above. The evolution equation $\dot{\rho}(t) = \mathcal{L}\rho(t)$ with

$$\mathcal{L} = [\gamma_1(t)\hat{n} + 1 - \gamma_2(t)\hat{n}] - i\omega(t)R$$

$$+2\gamma_1(t)(\hat{n} + 1)K_- + 2\gamma_2(t)\hat{n}K_+$$

$$- 2[\gamma_1(t)(\hat{n} + 1) + \gamma_2(t)\hat{n}]K_0$$

is solved by

$$\rho(t) = e^{f_+K_+}e^{hK_0}e^{f_-K_-}e^{-\mathcal{L}t}$$

$$\times e^{f_+K_+}e^{hK_0}e^{f_-K_-}d\alpha e^{\int[\gamma_1(t)(\hat{n} + 1) - \gamma_2(t)\hat{n}]d\alpha} \rho(0),$$ \hspace{1cm} (21)

where $f_\pm, h$ satisfy eq. (4) for $\sigma = 1$.

The space of density matrices can be decomposed into a direct sum of irreducible subspaces of the underlying $su(1,1)$ algebra. Namely, the operators of the form $|n+m\rangle\langle n|$ or $|n\rangle\langle n+m|$ belong to an irreducible subspace, according to the action of the Casimir operator on them $\hat{K}_2^2|n+m\rangle\langle n| = k(k - 1)|n+m\rangle\langle n|$ with $k = \frac{1}{2}(1+m)$ (similarly for $|n\rangle\langle n+m|$). Thus, the space of density matrices is decomposed into a direct sum of subspaces, corresponding to discrete-series representations of the $su(1,1)$ algebra. We define $|m; n\rangle$ as $|m+n\rangle\langle n|$ for $m \geq 0$ and $|n\rangle\langle n+m|$ for $m < 0$; then $K_0|m; n\rangle = (k+n)|m; n\rangle$, with $n \geq 0$. The evolution operator acts independently on each of these subspaces.

The coherent states are introduced separately for each irreducible subspace $m$ according to the recipe presented above in this paper. Thus we parametrize each subspace $m$ by coherent states $|m; \zeta_m\rangle$ ($\zeta_m$ being the coherent-state parameter) built upon the basis state $|m; 0\rangle$. A general density matrix can be expressed in terms of the overcomplete basis of these coherent states as

$$\rho(t) = \sum_{m \geq 0} \int d\mu_m(\zeta) c_m(\zeta) |m; \zeta\rangle + \text{h.c.},$$ \hspace{1cm} (22)

where $c_m(\zeta)$ are appropriate time-dependent functions and $d\mu_m$ is the group measure (see eq. (7)). The coherent-state parameters $\zeta_m$ are all time dependent; their evolution is given by eq. (11). In particular, we can consider a density matrix being a purely coherent state within each $m$-sector. There are, however, constraints on the admissible set of parameters $c_m$, $\zeta_m$, in order for the linear combination to indeed represent a physical density matrix. The simplest one reads $1 = Tr\rho = (1 - \zeta_0^2)^{-1/2}c_0(1 - \zeta_0) + \text{c.c.}$. Other constraints, such as $0 \leq Tr\rho^2 \leq 1$, are more involved; as a result not every purely coherent state is physically allowed. Interestingly, the decomposition of an arbitrary physical density matrix may contain even these unphysical states. Because of this feature no simple parametrization of a coherent state in terms of its purity exists, contrary to the su(2) case.

In order to demonstrate the LCS technique in the $su(1,1)$ case on a particular example, we focus on a harmonic oscillator interacting with a bath in such a way
that the decay rates $\gamma_{1,2}(t)$ are both equal to a periodic function

$$\gamma_1(t) = \gamma_2(t) = \gamma(t) = \gamma(a + \cos \Gamma t), \quad (23)$$

where $a$ is a free parameter. The time evolution of the parameters $R$ and $z$ of eq. (12) can be seen in fig. 3, while the evolution of the purity $\text{Tr} \rho(t)^2$ is captured in fig. 2.

**Conclusion.** – In this paper we introduced the notion of Liouville coherent states, an analogue of generalized coherent states for the density matrices. This concept identifies a class of density matrices whose time evolution is robust with respect to any time-dependent driving: if the initial density matrix belongs to this class it will remain so for any $t > 0$. We demonstrated this on several physical examples, involving compact and non-compact dynamical symmetry groups. Many geometric and topological properties of these states deserve further study. As these states form an overcomplete basis, they can be used as a platform for investigating a more complicated Liouville dynamics analogously to the way in which the usual coherent states are used in path-integral formulation of the quantum dynamics.

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