ENUMERATING PARTIAL LINEAR TRANSFORMATIONS IN A SIMILARITY CLASS

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Abstract. Let $V$ be a finite-dimensional vector space over the finite field $\mathbb{F}_q$ and suppose $W$ and $\tilde{W}$ are subspaces of $V$. Two linear transformations $T : W \to V$ and $\tilde{T} : \tilde{W} \to V$ are said to be similar if there exists a linear isomorphism $S : V \to V$ with $SW = \tilde{W}$ such that $ST = \tilde{T}S$. Given a linear map $T$ defined on a subspace $W$ of $V$, we give an explicit formula for the number of linear transformations that are similar to $T$. Our results extend a theorem of Philip Hall that settles the case $W = V$ where the above problem is equivalent to counting the number of square matrices over $\mathbb{F}_q$ in a conjugacy class.

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1. Introduction

Denote by $\mathbb{F}_q$ the finite field with $q$ elements where $q$ is a prime power. Let $\mathbb{F}_q[x]$ denote the ring of polynomials over $\mathbb{F}_q$ in the indeterminate $x$. Throughout this paper $n$ and $k$ denote nonnegative integers. A partition of a nonnegative integer $n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers with $\lambda_i \geq \lambda_{i+1}$ for $i \geq 1$ and $\sum_i \lambda_i = n$. If $\lambda_{\ell+1} = 0$ for some integer $\ell$, we also write $\lambda = (\lambda_1, \ldots, \lambda_{\ell})$. The notation $\lambda \vdash n$ or $|\lambda| = n$ will mean that $\lambda$ is a partition of the integer $n$.

Let $V$ be an $n$ dimensional vector space over $\mathbb{F}_q$ and let $W$ be a subspace of $V$. Let $L(W, V)$ denote the vector space of all $\mathbb{F}_q$-linear transformations from $W$ to $V$. Two linear transformations $T \in L(W, V)$ and $\tilde{T} \in L(\tilde{W}, V)$ defined on subspaces $W$ and $\tilde{W}$ of $V$ respectively are similar if there exists a linear

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isomorphism $S : V \rightarrow V$ such that the following diagram commutes:

$$\begin{array}{ccc}
W & \xrightarrow{T} & V \\
\downarrow{S} & & \downarrow{S} \\
\tilde{W} & \xrightarrow{\tilde{T}} & V
\end{array}$$

Let $\mathcal{L}(V)$ denote the union of the vector spaces $L(W; V)$ as $W$ varies over all possible subspaces of $V$. Given $T \in \mathcal{L}(V)$ define $\mathcal{C}(T)$, the conjugacy class of $T$, by

$$\mathcal{C}(T) := \{ \tilde{T} : \tilde{T} \in \mathcal{L}(V), \tilde{T} \text{ is similar to } T \}.$$ 

We are interested in determining the cardinality of $\mathcal{C}(T)$ for an arbitrary linear map $T$. The case where $T$ is a linear operator on $V$ is well-studied. Given such a linear operator $T$, one can view $V$ as an $\mathbb{F}_q[x]$-module where the element $x$ acts on $V$ as the linear transformation $T$. By the structure theorem for modules over a principal ideal domain [7, p. 86], $V$ is isomorphic to a direct sum

$$V \cong \frac{\mathbb{F}_q[x]}{(p_1)} \oplus \frac{\mathbb{F}_q[x]}{(p_2)} \oplus \cdots \oplus \frac{\mathbb{F}_q[x]}{(p_r)}$$

of cyclic modules where $p_1, p_2, \ldots, p_r$ are monic polynomials of degree at least one over $\mathbb{F}_q$ with $p_i$ dividing $p_{i+1}$ for $1 \leq i < r - 1$. The $p_i$ are known as the invariant factors of $T$ and uniquely determine $T$ up to similarity; two linear operators $T$ and $\tilde{T}$ on $V$ are similar if and only if they have the same invariant factors. In this case the problem of determining $|\mathcal{C}(T)|$ is equivalent to counting the number of square matrices over $\mathbb{F}_q$ in a conjugacy class. An explicit formula [13, Eq. 1.107] for the size of $\mathcal{C}(T)$ for a linear operator $T$ was given by Philip Hall based on work of Frobenius. This problem has also been studied by Kung [9] and Stong [14] who employ a generating function approach. In particular, Kung developed a vector space cycle index which is an analog of the Pólya cycle index and can be used to enumerate various types of square matrices over a finite field. We refer to the survey article of Morrison [10] for more on this topic. The invariant factors $p_i$ of a linear operator $T$ appear as the nonunit diagonal entries in the Smith Normal Form [6, p. 257] of $xI - A$ where $A$ is the matrix of $T$ with respect to some ordered basis for $V$.

In this paper we determine the size (Corollary 4.8) of the similarity class $\mathcal{C}(T)$ for an arbitrary transformation $T \in \mathcal{L}(V)$. Our methods are mostly combinatorial and we use ideas from the theory of integer partitions. The first step is to characterize the similarity invariants for a linear transformation $T$ defined only on a subspace $W$ of $V$. Accordingly, let $T \in \mathcal{L}(V)$ be a linear transformation and let $U$ denote the maximal $T$-invariant subspace. Interestingly, in this case the similarity classes are indexed by pairs $(\lambda, I)$ where $\lambda$ is an integer partition of $\dim V - \dim U$ and $I$ is an ordered set of monic polynomials corresponding to the invariant factors of the restriction of $T$ to $U$. The precise details are in Section 2. When the domain of $T$ is all of $V$, the partition $\lambda$ above is empty and
the similarity class $C(T)$ is completely determined by the invariant factors of $T$. Thus Hall’s result on matrix conjugacy class size may be viewed as the special case $\lambda = \emptyset$ of Corollary 4.7.

As a corollary of our results we give a new proof of a theorem of Lieb, Jordan and Helmke [5, Thm. 1] which is related to the problem of counting the number of zero kernel pairs of matrices or, equivalently, reachable linear systems over a finite field. This problem was initially considered by Kocięcki and Przyłuski [8]. The reader is referred to [11, 12] for the definition of zero kernel pairs and the connections with mathematical control theory.

2. Similarity invariants for maps defined on a subspace

We begin by describing a complete set of similarity invariants for a linear map in $L(W, V)$. Given $T \in L(W, V)$, define a sequence of subspaces [4, sec. III.1] $W_i = W_i(T)(i \geq 0)$ by $W_0 = V, W_1 = W$ and

$$W_{i+1} = W_i \cap T^{-1}(W_i) = \{v \in W_i : Tv \in W_i\} \quad \text{for } i \geq 1.$$ 

The descending chain of subspaces $W_0 \supseteq W_1 \supseteq \cdots$ eventually stabilizes as the dimensions of the subspaces are nonnegative integers. Let $d_i = d_i(T) := \dim W_i$ for $i \geq 0$ and let

$$\ell = \ell(T) := \min\{i : W_i = W_{i+1}\}.$$ 

The subspace $W_\ell$ is clearly a $T$-invariant subspace which is evidently the maximal $T$-invariant subspace. Therefore we may consider the invariant factors of the restriction $T_{W_\ell}$ of $T$ to $W_\ell$. Denote by $\mathcal{I}_T$ the ordered set of invariant factors of $T_{W_\ell}$. Since the characteristic polynomial of $T_{W_\ell}$ equals the product of the invariant factors of $T_{W_\ell}$, it follows that

$$d_\ell = \deg \prod_{p \in \mathcal{I}_T} p.$$ 

Now define

$$\lambda_j = \lambda_j(T) := d_{j-1} - d_j \quad \text{for } 1 \leq j \leq \ell.$$ 

**Definition 2.1.** The integers $\lambda_j(T)(1 \leq j \leq \ell)$ are called the defect dimensions [4, p. 52] of $T$.

**Lemma 2.2.** For any $T \in \mathcal{L}(V)$, we have $\lambda_j(T) \geq \lambda_{j+1}(T)$ for $1 \leq j \leq \ell - 1$.

**Proof.** Let the subspaces $W_j(j \geq 1)$ be as above. Note that $T(W_j) \subseteq W_{j-1}$ for each $j$. Fix $j \geq 1$ and define a map $\varphi : W_j/W_{j+1} \to W_{j-1}/W_j$ by

$$\varphi(v + W_{j+1}) = Tv + W_j.$$ 

We claim that $\varphi$ is well defined. Suppose $v_1 + W_{j+1} = v_2 + W_{j+1}$ for some $v_1, v_2 \in W_j$. Then $v_1 - v_2 \in W_{j+1}$ and consequently $T(v_1 - v_2) \in W_j$. Therefore $Tv_1 + W_j = Tv_2 + W_j$. Thus $\varphi$ is well defined. The linearity of $\varphi$ follows easily
from the fact that $T$ is linear. In fact $\varphi$ is also injective. Suppose for some $v \in W_j$ we have
\[ \varphi(v + W_{j+1}) = Tv + W_j = 0 + W_j. \]
Then $Tv \in W_j$ and since $v$ itself lies in $W_j$, it follows that $v \in W_{j+1}$ as well. Thus $v + W_{j+1}$ is in the kernel of $\varphi$. The injectivity of $\varphi$ implies that $\dim(W_{j-1}/W_j) \geq \dim(W_j/W_{j+1})$, or equivalently, $\lambda_j \geq \lambda_{j+1}$ for $1 \leq j \leq \ell - 1$. \hfill \Box

Hereon the sequence $W_i(T)(i \geq 0)$ will be referred to as the chain of subspaces associated with $T$.

**Corollary 2.3.** For $T \in \mathcal{L}(V)$, let $\ell = \ell(T)$. The sequence $\lambda_T = (\lambda_1(T), \ldots, \lambda_\ell(T))$ is an integer partition of $n - d_\ell(T)$.

**Proof.** This follows since $\sum_{i=1}^\ell \lambda_i = d_0 - d_\ell = n - d_\ell$. \hfill \Box

We now show that the pair $(\lambda_T, I_T)$ completely determines the similarity class of a linear transformation $T$ in the sense that two linear transformations $T, \tilde{T} \in \mathcal{L}(V)$ are similar if and only if $\lambda_T = \lambda_{\tilde{T}}$ and $I_T = I_{\tilde{T}}$. We require a lemma [4, Ch. III Lem. 3.3] to prove this result. As the terminology in [4] differs considerably from that in this paper, we include a proof here for the sake of completeness.

**Lemma 2.4.** Let $W, \tilde{W}$ be subspaces of $V$. For $T \in L(W, V)$ and $\tilde{T} \in L(\tilde{W}, V)$, let $T_U$ and $\tilde{T}_{\tilde{U}}$ denote the restrictions of $T$ and $\tilde{T}$ to the subspaces
\[ U = \{v \in W : Tv \in W\} \text{ and } \tilde{U} = \{v \in \tilde{W} : \tilde{T}v \in \tilde{W}\} \]
respectively. Then $T$ is similar to $\tilde{T}$ if and only if $T_U$ is similar to $\tilde{T}_{\tilde{U}}$ and $\dim W = \dim \tilde{W}$.

**Proof.** First suppose that $T$ is similar to $\tilde{T}$. Then there exists a linear isomorphism $S : V \to V$ such that $SW = \tilde{W}$ and $ST = \tilde{T}S$. It follows that $\dim W = \dim \tilde{W}$. We claim that $T_U$ is similar to $\tilde{T}_{\tilde{U}}$ with respect to the same linear isomorphism $S$. To see this, we first show that $S$ maps $U$ onto $\tilde{U}$. Suppose $v \in U$. Then, by definition, $v \in W$ and $Tv \in W$. This implies that $Sv \in \tilde{W}$ and consequently $\tilde{T}Sv = STv \in \tilde{W}$ which further implies that $Sv \in \tilde{U}$. Thus $SU \subseteq \tilde{U}$. Now the isomorphism $S^{-1} : V \to V$ has the property that $S^{-1}\tilde{W} = W$ and $S^{-1}\tilde{T} = TS^{-1}$. By reasoning as above it follows that $S^{-1}\tilde{U} \subseteq U$. It follows that $SU = \tilde{U}$. Now since $T_U$ and $\tilde{T}_{\tilde{U}}$ are restrictions of $T$ and $\tilde{T}$ to $U$ and $\tilde{U}$ respectively, it is easy to see that $ST_U = \tilde{T}_{\tilde{U}}S$ and it follows that $T_U$ and $\tilde{T}_{\tilde{U}}$ are similar.

For the converse, suppose $\dim W = \dim \tilde{W}$ and $T_U$ is similar to $\tilde{T}_{\tilde{U}}$. This implies that there exists a linear isomorphism $S' \in GL(V)$ such that $S'U = \tilde{U}$ and $S'T_U = \tilde{T}_{\tilde{U}}S'$. First construct a linear isomorphism $S'' \in GL(V)$ such that $S''W = \tilde{W}$ and $S''T_U = \tilde{T}_{\tilde{U}}S''$. Note that $T(U) \subseteq W$. We simply set $S''v = S'v$ for all $v$ lying in the subspace $U + TU$ of $W$. Since $S'(u_1 + Tu_2) = S'u_1 + S'Tvu_2 = A$
\(S'v_1 + \tilde{T}_U S'v_2 \in \tilde{U} + \tilde{T}\tilde{U},\) it is clear that \(S'' : U + TU \to \tilde{U} + \tilde{T}\tilde{U}\) is an isomorphism. Since \(\dim W = \dim \tilde{W},\) we may extend the definition of \(S''\) to all of \(W\) to obtain a linear isomorphism \(S'' : W \to \tilde{W}\) which may be further extended to a linear isomorphism \(S'' : V \to V.\)

Now we use \(S''\) to construct another linear isomorphism \(S : V \to V\) such that \(SW = \tilde{W}\) and \(ST = \tilde{T}\tilde{S}\) which will imply that the linear transformations \(T\) and \(\tilde{T}\) are similar. Let \(Sv = S''v\) for any \(v \in W\) and let \(Sv' = \tilde{T}S''v\) for any \(v' = T(v) \in T(W).\) We assert that \(S : W + TW \to \tilde{W} + \tilde{T}\tilde{W}\) is well defined and a linear isomorphism. If \(v' = Tv\) lies in \(W,\) then \(v \in U\) and hence \(S''v' = S''Tv = S''Tv = \tilde{T}_U S''v = \tilde{T}\tilde{S}''v.\) Therefore \(Sv'\) is uniquely defined. If \(Tv = Tu\) for some \(v, u \in W,\) then \(T(v - u) = 0\) lies in \(W.\) Thus, \(ST(v - u) = S''T(v - u) = 0\) which further implies \(\tilde{T}S'v - \tilde{T}S'u = \tilde{T}\tilde{S}'(v - u) = 0,\) and hence \(STv = STu.\) This implies that \(S\) is well defined. To prove that \(S\) is injective, let \(v' = Tv\) for some \(v \in W\) and \(Sv' = 0.\) This implies \(Sv' = \tilde{T}\tilde{S}''v = 0\) which further implies \(S''Tv = 0\) and since \(S''\) is invertible, it follows \(v' = 0.\) It is easy to check that \(S\) is surjective and \(ST = \tilde{T}\tilde{S}.\) Furthermore, it can be extended to a linear isomorphism \(S : V \to V.\) This completes the proof. \(\square\)

**Proposition 2.5.** The linear transformations \(T \in L(W, V)\) and \(\tilde{T} \in L(\tilde{W}, V)\) are similar if and only if \(\lambda_T = \lambda_{\tilde{T}}\) and \(\mathcal{I}_T = \mathcal{I}_{\tilde{T}}.\)

**Proof.** For \(T \in L(W, V),\) consider the sequence of subspaces \(W_i\) such that \(W_0 = V, W_1 = W\) and \(W_{i+1} = \{v \in W_i : Tv \in W_i\}.\) Let \(\ell = \min\{i : W_i = W_{i+1}\}\) and denote by \(T_i\) the restriction of \(T\) to \(W_i\) for \(1 \leq i \leq \ell.\) Similarly, define \(\tilde{W}_i, \tilde{T}_i, \ell\) for \(\tilde{T} \in L(\tilde{W}, V).\) By Lemma 2.4, it follows that \(T_1\) is similar to \(\tilde{T}_1\) if and only if \(T_2\) is similar to \(\tilde{T}_2\) and \(\dim W_1 = \dim \tilde{W}_1.\) Using the lemma again, it is clear that \(T_1\) is similar to \(\tilde{T}_1\) if and only if \(T_3\) is similar to \(\tilde{T}_3,\) \(\dim W_2 = \dim \tilde{W}_2\) and \(\dim W_1 = \dim \tilde{W}_1.\) By repeated application of the lemma, it is evident that \(T\) is similar to \(\tilde{T}\) if and only if \(T_i\) is similar to \(\tilde{T}_i\) with \(\ell = \ell\) and \(\dim W_i = \dim \tilde{W}_i\) for \(1 \leq i \leq \ell.\) The linear operators \(T_\ell : W_\ell \to W_\ell\) and \(\tilde{T}_\ell : \tilde{W}_\ell \to \tilde{W}_\ell\) are similar if and only if \(\mathcal{I}_T = \mathcal{I}_{\tilde{T}}.\) Thus, it follows that \(T\) and \(\tilde{T}\) are similar if and only if \(\lambda_T = \lambda_{\tilde{T}}\) and \(\mathcal{I}_T = \mathcal{I}_{\tilde{T}}.\) \(\square\)

**Definition 2.6.** For any ordered set of invariant factors \(\mathcal{I},\) define

\[
\deg \mathcal{I} = \deg \prod_{p \in \mathcal{I}} p.
\]

**Remark 2.7.** In view of the above proposition similarity classes in \(\mathcal{L}(V)\) are indexed by pairs \((\lambda, \mathcal{I})\) where \(\lambda\) is an integer partition (possibly the empty partition) and \(\mathcal{I} \subseteq \mathbb{F}_q[x]\) is an ordered set of invariant factors satisfying

\[|\lambda| + \deg \mathcal{I} = \dim V.\]
Denote the similarity class in $\mathcal{L}(V)$ corresponding to the pair $(\lambda, \mathcal{I})$ by $\mathcal{C}(\lambda, \mathcal{I})$. For a given subspace $W$ of $V$ and an integer partition $\lambda$ with largest part $\dim V - \dim W$, denote by $\mathcal{C}_{W,V}(\lambda, \mathcal{I})$ the set of all linear transformations in $L(W, V)$ corresponding to the pair $(\lambda, \mathcal{I})$, i.e.,

$$\mathcal{C}_{W,V}(\lambda, \mathcal{I}) := L(W, V) \cap \mathcal{C}(\lambda, \mathcal{I}).$$

In the case $W = V$, the similarity class $\mathcal{C}_{V,V}(\lambda, \mathcal{I})$ is defined only when $\lambda$ is the empty partition and it and depends only on the invariant factors $\mathcal{I}$. In this case $\mathcal{C}_{V,V}(\emptyset, \mathcal{I})$ is abbreviated to $\mathcal{C}(\mathcal{I})$. A closed formula for the size of $\mathcal{C}(\mathcal{I})$ can be found in Stanley [13, Eq. 1.107] or [12, Eq. 1].

3. Simple linear transformations

**Definition 3.1.** A linear transformation $T \in \mathcal{L}(V)$ is simple if, for each $T$-invariant subspace $U$, either $U = \{0\}$ or $U = V$.

It follows from the definition that simple maps are injective. If $T \in \mathcal{L}(V)$ is simple with domain a proper subspace of $V$, then the maximal $T$-invariant subspace is necessarily the zero subspace and therefore $T \in \mathcal{C}(\lambda, \emptyset)$ for some integer partition $\lambda$ of $\dim V$ with largest part $\dim V - \dim W$ where $W$ is the domain of $T$. In this section we determine the size of $\mathcal{C}(\lambda, \emptyset)$ for an arbitrary partition $\lambda$ of $\dim V$. We begin with some combinatorial lemmas.

The number of $k$-dimensional subspaces of an $n$-dimensional vector space over $\mathbb{F}_q$ is given by the $q$-binomial coefficient [15, p. 292]

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}.$$

**Lemma 3.2.** Let $U \subseteq W$ be subspaces of an $n$-dimensional vector space $V$ over $\mathbb{F}_q$ with $\dim U = d$ and $\dim W = k$. The number of $k$-dimensional subspaces of $V$ whose intersection with $W$ is $U$ equals

$$\begin{bmatrix} n - k \\ k - d \end{bmatrix}_q q^{(k-d)^2}.$$

**Proof.** We count the number of $k$-dimensional subspaces $W'$ for which $W \cap W' = U$. Given any ordered basis of $U$, there are $\prod_{i=k}^{2k-d-1} (q^n - q^i)$ ways to extend it to an ordered basis of $W'$. Counting in this manner, the same subspace $W'$ arises in precisely $\prod_{i=d}^{i=k-1} (q^k - q^i)$ ways. Thus the total number of such subspaces $W'$ is given by

$$\frac{\prod_{i=k}^{2k-d-1} (q^n - q^i)}{\prod_{i=d}^{i=k-1} (q^k - q^i)} = \begin{bmatrix} n - k \\ k - d \end{bmatrix}_q q^{(k-d)^2}.$$

$\square$
**Definition 3.3.** A flag \([3, \text{ p. 95}]\) of length \(r\) in a vector space \(V\) is an increasing sequence of subspaces \(W_i(0 \leq i \leq r)\) such that
\[
\{0\} = W_0 \subset W_1 \subset \cdots W_{r-1} \subset W_r = V.
\]

The following lemma \([10, \text{ Sec. 1.5}]\) counts the number of flags of length \(r\) with subspaces of given dimensions.

**Lemma 3.4.** Let \(n_1, \ldots, n_r\) be positive integers with \(n_1 + \cdots + n_r = n\). The number of flags \(W_0 \subset \cdots \subset W_r\) of length \(r\) in an \(n\)-dimensional vector space \(V\) over \(\mathbb{F}_q\) with \(\dim W_i = n_1 + n_2 + \cdots + n_i\) is given by the \(q\)-multinomial coefficient
\[
\left[ \begin{array}{c} n \\ n_1, n_2, \ldots, n_r \end{array} \right]_q := \frac{[n]_q!}{[n_1]_q! [n_2]_q! \cdots [n_r]_q!},
\]
where \([n]_q := \frac{q^n - 1}{q - 1}\) and \([n]_q! := [n]_q [n-1]_q \cdots [1]_q\).

In what follows the number of nonsingular \(k \times k\) matrices over \(\mathbb{F}_q\) is denoted by \(\gamma_q(k) = \prod_{i=0}^{k-1} (q^k - q^i)\) \([10, \text{ Sec. 1.2}]\).

**Theorem 3.5.** Let \(\lambda\) be a partition of \(n\). Then
\[
|\mathcal{C}(\lambda, \emptyset)| = q^{\sum_{j \geq 2} \lambda_j^2} \left[ \begin{array}{c} n \\ n - \lambda_1, \lambda_1 - \lambda_2, \ldots, \lambda_\ell \end{array} \right]_q \gamma_q(n - \lambda_1).
\]

**Proof.** We count the number of simple linear transformations \(T \in \mathcal{L}(V)\) having defect dimensions \((\lambda_1, \lambda_2, \ldots, \lambda_\ell)\) defined on some subspace of \(V\) of dimension \(n - \lambda_1\). Fix a subspace \(W\) of \(V\) of dimension \(n - \lambda_1\). We first determine the cardinality of \(\mathcal{C}_{W,V}(\lambda, \emptyset)\). For \(T \in \mathcal{C}_{W,V}(\lambda, \emptyset)\) consider the chain of subspaces \(\{W_i = W_i(T)\}_{i=0}^{\ell}\) associated with \(T\). Define a sequence \(\{W'_i\}_{i=1}^{\ell}\) by \(W'_i = T(W_i)\).

Since \(T\) is injective, we have \(\dim W_i' = \dim W_i = d_i\) for \(i \geq 1\). By the choice of \(T\) we have \(d_{i-1} - d_i = \lambda_i\). Note that \(W_i \cap W'_i = W'_{i+1}\) for \(1 \leq i \leq \ell - 1\). The sequence \(\{W_i\}_{i=0}^{\ell}\) is a flag in \(W\) of length \(\ell - 1\):
\[
\{0\} = W_\ell \subset \cdots W_2 \subset W_1 = W
\]
such that \(\dim W_i = d_i = \lambda_\ell + \lambda_{\ell-1} + \cdots + \lambda_{i+1}\). By Lemma 3.4, the number of such flags is
\[
\left[ \begin{array}{c} n - \lambda_1 \\ \lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_2 \end{array} \right]_q.
\]

For a given choice of \(\{W_i\}_{i=0}^{\ell}\) the total number of choices for the sequence \(\{W'_i\}_{i=0}^{\ell}\) equals the total number of flags
\[
\{0\} = W'_\ell \subset \cdots W'_2 \subset W'_1 = T(W)
\]
of length \(\ell - 1\) where \(\dim W'_i = d_i\) and \(W_i \cap W'_i = W'_{i+1}\) for \(1 \leq i \leq \ell - 1\). Thus \(W'_{i-1}\) is a subspace of \(W_{i-2}\) of dimension \(d_{i-1}\) that intersects \(W_{i-1}\) trivially. It
follows by Lemma 3.2 that $W'_{\ell-1}$ can be chosen in
\[
\left[ \frac{d_{\ell-2} - d_{\ell-1}}{d_{\ell-1} - d_{\ell}} \right] q^{(d_{\ell-1} - d_{\ell})^2} \left[ \begin{array}{c} \lambda_{\ell-1} \\ \lambda_{\ell} \end{array} \right] \left[ \begin{array}{c} \lambda_{\ell-2} \\ \lambda_{\ell-1} \end{array} \right] = q^{\lambda_{\ell}^2}
\]
ways. Similarly, the conditions $W'_{\ell-2} \subseteq W_{\ell-3}$ and $W_{\ell-2} \cap W'_{\ell-2} = W'_{\ell-1}$ imply that $W'_{\ell-2}$ can be chosen in
\[
\left[ \frac{d_{\ell-3} - d_{\ell-2}}{d_{\ell-2} - d_{\ell-1}} \right] q^{(d_{\ell-2} - d_{\ell-1})^2} \left[ \begin{array}{c} \lambda_{\ell-2} \\ \lambda_{\ell-1} \end{array} \right] = q^{\lambda_{\ell-1}^2}
\]
ways. Proceeding in this manner, it is seen that the total number of choices for the sequence $\{W'_i\}_{i=1}^\ell$ is equal to
\[
\left[ \begin{array}{c} \lambda_{\ell-1} \\ \lambda_{\ell} \end{array} \right] \left[ \begin{array}{c} \lambda_{\ell-2} \\ \lambda_{\ell-1} \end{array} \right] \cdots \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] = q^{\sum_{i=2}^\ell \lambda_i^2} \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] \cdots \left[ \begin{array}{c} \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{\ell-1} \end{array} \right] \lambda_{\ell-1} \gamma_q(\lambda_{\ell-1}).
\]

For each choice of the flags $\{W_i\}_{i=0}^\ell$ and $\{W'_i\}_{i=0}^\ell$, we count the number of possibilities for $T$. Note that $T$ is injective and $TW_i = W'_i$ for $1 \leq i \leq \ell$. Thus the number of ways to map $W_{\ell-1}$ onto $W'_{\ell-1}$ is equal to the number of invertible $\lambda_\ell \times \lambda_\ell$ matrices over $\mathbb{F}_q$, i.e., $\gamma_q(\lambda_\ell)$. The number of ways to extend $T$ to $W_{\ell-2}$ such that $TW_{\ell-2} = W'_{\ell-2}$ is evidently
\[
d_{\ell-1} + \lambda_{\ell-1} - 1 \prod_{i=d_{\ell-1}}^{d_{\ell-2}} (q^{d_{\ell-2} - d_{\ell-1}}) = q^{d_{\ell-1} \lambda_{\ell-1} \gamma_q(\lambda_{\ell-1})}.
\]

Following this line of reasoning, the total number of choices for the map $T$ for a given choice of $\{W_i\}_{i=0}^\ell$ and $\{W'_i\}_{i=0}^\ell$ equals
\[
q^{\sum_{i=2}^\ell \lambda_i} \prod_{i=2}^\ell \gamma_q(\lambda_i).
\]

It follows that
\[
|C_{W,V}(\lambda, \emptyset)| = q^{\sum_{i=2}^\ell \lambda_i^2} \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] \cdots \left[ \begin{array}{c} \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_\ell \end{array} \right] \gamma_q(\lambda_1) \gamma_q(\lambda_2) \cdots \gamma_q(\lambda_\ell).
\]

We expand the values of $\gamma_q(\lambda_i)$ and simplify the above expression.
\[
|C_{W,V}(\lambda, \emptyset)| = q^{\sum_{i=2}^\ell \lambda_i^2} \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] \cdots \left[ \begin{array}{c} \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_\ell \end{array} \right] \prod_{i=2}^\ell \gamma_q(\lambda_i)
\]
\[
\times (q - 1)^{\lambda_1} q^{\lambda_2} \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \end{array} \right] \gamma_q(\lambda_1) \gamma_q(\lambda_2) \cdots \gamma_q(\lambda_\ell).
\]
\[\begin{align*}
&= q^{\sum_{i=2}^{t} \lambda_i^2} \left[ \lambda_1 - \lambda_2 - \lambda_3, \ldots, \lambda_t \right] q^{n - \lambda_1} \frac{q^{\sum_{i=2}^{t} d_i (q - 1)^2 + \cdots + (q - 1)}}{q^{\sum_{i=2}^{t} \lambda_i}} \\
&= q^{\sum_{i=2}^{t} \lambda_i^2} \left[ \lambda_1 - \lambda_2 - \lambda_3, \ldots, \lambda_t \right] q^{n - \lambda_1} \left( q - 1 \right)^{n - \lambda_1} q^{\gamma_q(n - \lambda_1)}.
\end{align*}\]

Since the domain of \(T\) is an arbitrary \(n - \lambda_1\) dimensional subspace of \(V\), we sum over all \((n - \lambda_1)\) dimensional subspaces of \(V\) to obtain

\[|C(\lambda, \emptyset)| = \sum_{W: \dim W = n - \lambda_1} |C_{W,V}(\lambda, \emptyset)| = \left[ \frac{n}{n - \lambda_1} \right] q^{\gamma_q(n - \lambda_1)}.\]

Substituting the expression for \(|C_{W,V}(\lambda, \emptyset)|\) obtained earlier, we obtain

\[|C(\lambda, \emptyset)| = q^{\sum_{i=2}^{t} \lambda_i^2} \left[ \lambda_1 - \lambda_2 - \lambda_3, \ldots, \lambda_t \right] \gamma_q(n - \lambda_1).
\]

\([1]\)

Corollary 3.6. Let \(W\) be a proper subspace of an \(n\)-dimensional vector space \(V\) over \(\mathbb{F}_q\). Let \(\lambda \vdash n\) with \(\lambda_1 = \dim V - \dim W\). Then the number of simple linear transformations defined on \(W\) with defect dimensions \(\lambda\) is given by

\[\sigma(\lambda) := |C_{W,V}(\lambda, \emptyset)| = q^{\sum_{i=2}^{t} \lambda_i^2} \gamma_q(n - \lambda_1) \prod_{i=1} \left[ \frac{\lambda_i}{\lambda_{i+1}} \right] q_{\lambda_i+1}^q.\]

\[\text{Proof.} \quad \text{Follows from Equation (1) in the proof of the above theorem.}\]

The above corollary may be used to deduce the number of simple linear transformations with a fixed domain. We first collate some results on partitions. A useful graphic representation of an integer partition is the corresponding Young diagram. Given a partition \(\lambda = (\lambda_1, \lambda_2, \ldots)\), put \(\lambda_i\) (unit) cells in row \(i\) to obtain its Young diagram. For instance, the Young diagram of the partition \((6, 3, 2)\) is shown in Figure 1.

![Figure 1. The Young diagram of (6, 3, 2).](image)
Definition 3.7. For integers \( m, r, s \) denote by \( p(m, r, s) \) the number of partitions of \( m \) with at most \( r \) parts in which each part is at most \( s \).

The geometric interpretation of \( p(m, r, s) \) is that it counts the number of partitions of \( m \) whose Young diagrams fit in a rectangle of size \( r \times s \). The following lemma \([1, \text{Prop. 1.1}]\) shows that the generating function for \( p(m, r, s) \) for fixed values of \( r \) and \( s \) is a \( q \)-binomial coefficient.

Lemma 3.8. We have

\[
\binom{r+s}{s}_q = \sum_{i \geq 0} p(i, r, s) q^i.
\]

The rank of a partition \( \lambda \) is the largest integer \( i \) for which \( \lambda_i \geq i \). Geometrically the rank of a partition corresponds to side length of the largest square, called the Durfee square, contained in the Young diagram of \( \lambda \). The Durfee square of the partition \( \lambda = (6, 4, 3, 2) \) is indicated by the shaded cells in Figure 2.

Proposition 3.9. For positive integers \( m \leq n \), we have

\[
\sum_{\lambda \vdash n} q^{\sum \lambda_i} \prod_{i \geq 1} \left[ \frac{\lambda_i}{\lambda_{i+1}} \right]_q = q^{m^2+n-m} \sum_{i=m}^{n-1} \left[ \frac{i-1}{m-1} \right]_q.
\]

Proof. Let \( S \) denote the set of all partitions \( \mu \) of rank \( m \) and largest part \( n \) with precisely \( m \) parts. Visually \( S \) consists of partitions whose Young diagrams fit inside an \( m \times n \) rectangle \( R \) and have at least \( m \) cells in each row with precisely \( n \) cells in the first row. We compute the sum

\[
\sum_{\mu \in S} q^{\mu}
\]

in two different ways. Note that each \( \mu \in S \) is uniquely determined by the partition \( \mu' = (\mu_2 - m, \mu_3 - m, \ldots) \). This is because the first row and first \( m \) columns of the Young diagram of \( \mu \) are fixed and hence \( \mu \) is uniquely determined by the corresponding partition \( \mu' \) that fits in the \( m-1 \times n-m \) rectangle at the bottom right corner of \( R \). By Lemma 3.8 we have

\[
\sum_{\mu \in S} q^{\mu} = q^{m^2+n-m} \sum_{\mu' \in S} q^{\mu'}
\]
Figure 3. The partition $\varphi(\mu) = (4, 2, 1, 1)$ corresponding to $\mu = (8, 7, 6, 5)$.

\[
= q^{m^2 + n - m - 1} \left[ \begin{array}{c} n - 1 \\ m - 1 \end{array} \right]_q,
\]

which accounts for the expression on the right hand side of the proposition. Now for any $\mu \in S$ consider the partition $\varphi(\mu) = \lambda \vdash n$ defined as follows: $\lambda_1$ is the rank of $\mu$. The part $\lambda_2$ is the rank of the partition whose diagram is to the right of the Durfee square of $\mu$ etc. For example, when $\mu = (8, 7, 6, 5)$, we have $\varphi(\mu) = (4, 2, 1, 1)$ as shown in Figure 3. As $\mu$ varies over $S$, the partition $\varphi(\mu)$ varies over all partitions of $n$ with largest part $m$. Therefore

\[
\sum_{\mu \in S} q^{\vert \mu \vert} = \sum_{\lambda \vdash n, \lambda_1 = m} \sum_{\varphi(\mu) = \lambda} q^{\vert \mu \vert}.
\]

Consider the inner sum on the right hand side. If $\varphi(\mu) = \lambda$, then $\lambda$ defines a sequence of squares (corresponding to the shaded cells in Figure 3) which accounts for $\sum_i \lambda_i^2$ cells in the diagram of $\mu$. The cells of $\mu$ that do not lie in any square in the sequence (the unshaded cells in Figure 3) correspond to a sequence of partitions: the first is a partition that fits in a rectangle of size $(\lambda_1 - \lambda_2) \times \lambda_2$, the second is a partition that fits in a rectangle of size $(\lambda_2 - \lambda_3) \times \lambda_3$ etc. Putting these observations together and applying Lemma 3.8, it is clear that

\[
\sum_{\mu \in S, \varphi(\mu) = \lambda} q^{\vert \mu \vert} = q^{\sum \lambda_i^2} \left[ \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \vdots \end{array} \right]_q \left[ \begin{array}{c} \lambda_2 \\ \lambda_3 \\ \vdots \end{array} \right]_q \ldots
\]

and the proposition follows. \hfill \Box

We now deduce the theorem of Lieb, Jordan and Helmke \cite[Thm. 1]{5} alluded to in the introduction.

**Corollary 3.10.** Let $W$ be a proper $k$-dimensional subspace of a vector space $V$ of dimension $n$ over $F_q$. The number of simple linear transformations with domain $W$ equals $\prod_{i=1}^k (q^n - q^i)$.

**Proof.** The number of simple linear transformations with domain $W$ is equal to

\[
\sum_{\lambda \vdash n, \lambda_1 = n - k} \sigma(\lambda) = \gamma_q(k) \sum_{\lambda \vdash n, \lambda_1 = n - k} q^{\sum_{i \geq 2} \lambda_i^2} \prod_{i \geq 1} \left[ \begin{array}{c} \lambda_i \\ \lambda_{i+1} \end{array} \right]_q
\]
by Corollary 3.6. Setting \( m = n - k \) in Proposition 3.9 the sum on the right hand side above equals
\[
q^k \binom{n - 1}{k}_q \gamma_q(k) = q^k \frac{(q^{n-1} - 1) \cdots (q^{n-k-1} - 1)}{(q^k - 1) \cdots (q^k - q^{k-1})} \prod_{i=0}^{k-1} (q^k - q^i)
\]
\[
= \prod_{i=1}^{k} (q^n - q^i).
\]
\( \square \)

The corollary above can also be obtained [2, Cor. 2.12] by counting certain unimodular matrices over a finite field.

4. Arbitrary Linear Transformations Defined on a Subspace

In this section we extend the results obtained on the conjugacy class size of simple linear transformations to arbitrary maps in \( \mathcal{L}(V) \).

Let \( T \in \mathcal{L}(V) \) be a fixed but arbitrary linear transformation with domain \( W \) and let \( \overline{W} \) denote the maximal invariant subspace of \( T \). Define a map \( \hat{T} \) from the quotient space \( W/\overline{W} \) into \( V/\overline{W} \) by
\[
\hat{T}(v + \overline{W}) = T v + \overline{W}.
\]

Then \( \hat{T} \) is well defined. If \( v_1 + \overline{W} = v_2 + \overline{W} \) for some \( v_1, v_2 \in W \) then \( v_1 - v_2 \in \overline{W} \) and consequently \( T(v_1 - v_2) \in \overline{W} \) since \( \overline{W} \) is \( T \)-invariant. It follows that \( T v_1 + \overline{W} = T v_2 + \overline{W} \) and thus \( \hat{T} \) is well defined. The linearity of \( \hat{T} \) is an easy consequence of the fact that \( T \) is linear.

**Lemma 4.1.** Let \( U = \{ v \in W : T v \in W \} \) and \( \overline{U} = \{ \alpha \in W/\overline{W} : \hat{T}(\alpha) \in W/\overline{W} \} \). Then \( \overline{U} = U/\overline{W} \).

**Proof.** Note that \( \overline{W} \subseteq U \). We have
\[
v + \overline{W} \in \overline{U} \iff v + \overline{W} \in W/\overline{W} \text{ and } T v + \overline{W} \in W/\overline{W}
\]
\[
\iff v \in W \text{ and } T v \in W
\]
\[
\iff v \in U. \quad \square
\]

**Lemma 4.2.** Let \( W \) be a proper subspace of an \( n \)-dimensional vector space \( V \) over \( \mathbb{F}_q \) and let \( T \in \mathcal{L}(W,V) \). Let \( \overline{W} \) denote the maximal \( T \)-invariant subspace and suppose \( \dim \overline{W} = d \). Suppose \( T \in \mathcal{C}(\lambda, \mathcal{I}) \) for some integer partition \( \lambda \vdash n - d \). Then the linear transformation \( \hat{T} : W/\overline{W} \to V/\overline{W} \) defined by \( \hat{T}(v + \overline{W}) = T v + \overline{W} \) is simple and \( \hat{T} \in \mathcal{C}(\lambda, \emptyset) \).

**Proof.** To show that \( \hat{T} \) is simple, it suffices to show that the maximal invariant subspace of \( \hat{T} \) is the zero subspace. Consider the chain of subspaces \( \{W_i\}_{i=0}^\ell \) associated with \( T \). Note that \( W_0 = \overline{W} \). Similarly, there is a chain of subspaces \( \{\overline{W}_i\}_{i=0}^\ell \) associated with \( \hat{T} \). It follows by Lemma 4.1 that \( \overline{W}_2 = \overline{W}_2/\overline{W} \). By applying the lemma again to the restriction of \( \hat{T} \) to \( W_2/\overline{W} \), we obtain \( \overline{W}_3 = \ldots \).
By repeated application of the lemma it is clear that $\hat{W}_i = W_i/W$ for $0 \leq i \leq \ell$. This implies that $\ell' = \ell$ and that the maximal invariant subspace $\hat{W}_\ell$ of $\hat{T}$ is the zero subspace. Thus $\hat{T}$ is simple. Since
\[
\text{dim } W_{j-1}/W - \text{dim } W_j/W = \text{dim } W_{j-1} - \text{dim } W_j = \lambda_j
\]
for $1 \leq j \leq \ell$, the sequence of defect dimensions of $\hat{T}$ is $\lambda$. \hfill \Box

**Definition 4.3.** For $T \in \mathcal{L}(V)$, the map $\hat{T}$ defined above is called the *simple part* of $T$.

**Definition 4.4.** For $T \in \mathcal{L}(V)$, the *operator part* of $T$ denotes the linear operator obtained by restricting $T$ to its maximal invariant subspace.

Given a subspace $W$ of $V$ and any $T \in L(W,V)$, associate with it a unique pair $(\overline{T},\hat{T})$ where $\overline{T}$ denotes the operator part of $T$ and $\hat{T}$ denotes the simple part of $T$. The following proposition asserts that the number of linear transformations having prescribed simple and operator parts is a power of $q$.

**Proposition 4.5.** Let $U \subseteq W$ be subspaces of an $n$-dimensional vector space $V$ over $\mathbb{F}_q$ and suppose that the dimensions of $U$ and $W$ are $d$ and $k$ respectively. Let $T_o$ be a linear operator on $U$ with ordered set of invariant factors $\mathcal{I}$ and let $T_s \in L(W/U, V/U)$ be a simple linear transformation with defect dimensions $\lambda \vdash n - d$. The number of linear transformations $T \in L(W,V)$ with operator part $T_o$ and simple part $T_s$ is given by $q^{d(k-d)}$.

**Proof.** Let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_d\}$ be an ordered basis for $U$. Extend $\mathcal{B}$ to a basis $\mathcal{B}' = \{\alpha_1, \ldots, \alpha_k\}$ for $W$. Let $T_o$ and $T_s$ be as in the statement of the theorem. If a linear transformation $T \in L(W,V)$ has operator part $T_o$, then $T$ is uniquely defined at each element of $\mathcal{B}$. It remains to define $T$ on each $\alpha_i$ for $d+1 \leq i \leq k$. Suppose that $T_s(\alpha_i + U) = \beta_i + U$ for some $\beta_i \in V$ and $d + 1 \leq i \leq k$. Then $T\alpha_i + U = \beta_i + U$ for $d + 1 \leq i \leq k$. It therefore suffices to count maps $T$ satisfying
\[
T(\alpha_i) = \beta_i + \gamma_i \text{ for some } \gamma_i \in U \quad (d + 1 \leq i \leq k).
\]
The number of such maps is clearly $q^{d(k-d)}$. \hfill \Box

The function $\sigma(\lambda)$ defined in Corollary 3.6 counts the number of simple maps with defect dimensions $\lambda$ when $\lambda$ is a partition of a positive integer. As the simple part of any linear operator on $V$ is trivial, it is natural to extend the definition of $\sigma(\lambda)$ to the empty partition by declaring $\sigma(\emptyset) = 1$.

**Theorem 4.6.** Let $U \subseteq W$ be subspaces of an $n$-dimensional vector space $V$ over $\mathbb{F}_q$ and suppose $\dim U = d$ and $\dim W = k$. Let $\lambda \vdash n - d$ with $\lambda_1 = n - k$ and $\mathcal{I}$ be an ordered set of invariant factors of degree $d$. The number of maps in $\mathcal{C}_{W,V}(\lambda, \mathcal{I})$ with maximal invariant subspace $U$ equals
\[
q^{d(k-d)} |\mathcal{C}(\mathcal{I})| \sigma(\lambda).
\]
Proof. There are precisely $|C(\mathcal{I})|$ possibilities for the operator part of $T$. Setting $W' = W/U$ and $V' = V/U$, the simple part of $T$ can be chosen in $|C_{W',V'}(\lambda, \emptyset)| = \sigma(\lambda)$ ways. The result now follows from Proposition 4.5.

Corollary 4.7. Let $W$ be a $k$-dimensional subspace of an $n$-dimensional vector space $V$ over $\mathbb{F}_q$. Let $\mathcal{I}$ be an ordered set of invariant factors with $\deg \mathcal{I} = d \leq \dim W$ and let $\lambda \vdash n - d$ with $\lambda_1 = n - k$. Then

$$|C_{W,V}(\lambda, \mathcal{I})| = q^{d(k-d)} \left[ \begin{array}{c} k \\ d \end{array} \right]_q |C(\mathcal{I})| \sigma(\lambda).$$

Proof. The corollary follows from Theorem 4.6 as there are $\left[ \begin{array}{c} k \\ d \end{array} \right]_q$ possibilities for the maximal invariant subspace.

In the case $W = V$, the above expression for $|C_{W,V}(\lambda, \mathcal{I})|$ reduces to $|C(\mathcal{I})|$, the number of square matrices whose invariant factors are given by $\mathcal{I}$. The next corollary determines the size of the similarity classes in $L(V)$.

Corollary 4.8. Let $V$ be a vector space over $\mathbb{F}_q$ of dimension $n$. If $\deg \mathcal{I} = d$ and $\lambda \vdash n - d$, then

$$|C(\lambda, \mathcal{I})| = q^{d(k-d)} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \left[ \begin{array}{c} k \\ d \end{array} \right]_q |C(\mathcal{I})| \sigma(\lambda),$$

where $k = n - \lambda_1$.

Proof. Any map in $C(\lambda, \mathcal{I})$ has domain of dimension $k$. The result follows from Corollary 4.7 by summing $|C_{W,V}(\lambda, \mathcal{I})|$ over all $k$-dimensional subspaces of $V$.

The following result was proved in [12, Thm. 3.8].

Corollary 4.9. Let $W$ be a fixed $k$-dimensional subspace of an $n$-dimensional vector space $V$ over $\mathbb{F}_q$. The number of linear transformations $T \in L(W, V)$ for which the operator part of $T$ has invariant factors $\mathcal{I}$ with $\deg \mathcal{I} = d$ equals

$$\left[ \begin{array}{c} k \\ d \end{array} \right]_q |C(\mathcal{I})| \prod_{i=d+1}^k (q^n - q^i).$$

Proof. By Corollary 4.7 the desired number of linear transformations equals

$$\sum_{\lambda \vdash n-d, \lambda_1 = n-k} |C_{W,V}(\lambda, \mathcal{I})| = q^{d(k-d)} \left[ \begin{array}{c} k \\ d \end{array} \right]_q |C(\mathcal{I})| \sum_{\lambda \vdash n-d, \lambda_1 = n-k} \sigma(\lambda)$$

$$= q^{d(k-d)} \left[ \begin{array}{c} k \\ d \end{array} \right]_q |C(\mathcal{I})| \prod_{j=1}^{k-d} (q^{n-d} - q^j),$$

where the second equality follows from Corollary 3.10 and the result follows.
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