A characterisation of edge-affine 2-arc-transitive covers
of $K_{2^n,2^n}$

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Abstract

We introduce the notion of an $n$-dimensional mixed dihedral group, a general class
of groups for which we give a graph theoretic characterisation. In particular, if $H$ is an
$n$-dimensional mixed dihedral group then we construct an edge-transitive Cayley
graph $\Gamma$ of $H$ such that the clique graph $\Sigma$ of $\Gamma$ is a 2-arc-transitive normal cover of
$K_{2^n,2^n}$, with a subgroup of $\text{Aut}(\Sigma)$ inducing a particular edge-affine action on $K_{2^n,2^n}$.
Conversely, we prove that if $\Sigma$ is a 2-arc-transitive normal cover of $K_{2^n,2^n}$, with a
subgroup of $\text{Aut}(\Sigma)$ inducing an edge-affine action on $K_{2^n,2^n}$, then the line graph $\Gamma$ of
$\Sigma$ is a Cayley graph of an $n$-dimensional mixed dihedral group.

Furthermore, we give an explicit construction of a family of $n$-dimensional mixed
dihedral groups. This family addresses a problem proposed by Li concerning normal
covers of prime power order of the ‘basic’ 2-arc-transitive graphs. In particular, we
construct, for each $n \geq 2$, a 2-arc-transitive normal cover of 2-power order of the
‘basic’ graph $K_{2^n,2^n}$.

Key words: 2-arc-transitive, normal cover, Cayley graph, edge-transitive
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1 Introduction

All graphs we consider are finite, connected, simple and undirected. We denote the vertex
set of a graph $\Gamma$ by $V(\Gamma)$ and its edge set by $E(\Gamma)$, and call $|V(\Gamma)|$ the order of $\Gamma$. For a
subgroup $G$ of the automorphism group $\text{Aut}(\Gamma)$, a graph $\Gamma$ is said to be $(G, 2)$-arc-transitive if $G$ acts transitively on the set of all 2-arcs of $\Gamma$ (that is, the set of all triples $(u, v, w)$ such that $u, v, w \in V(\Gamma)$, $u \neq w$ and $\{u, v\}, \{v, w\} \in E(\Gamma)$); and $\Gamma$ is 2-arc-transitive if such a group exists. A graph $\Gamma$ is called a normal cover of a graph $\Sigma$ if $\Gamma$ and $\Sigma$ have the same valency, and there exists a subgroup chain $N \triangleleft G \leq \text{Aut}(\Gamma)$ such that $G$ is transitive on $V(\Gamma)$ and $\Sigma$ is the quotient of $\Gamma$ by the set of $N$-orbits in $V(\Gamma)$. We also say that $\Gamma$ is an $N$-normal cover of $\Sigma$ to specify the role of $N$, and we call $\Sigma$ the $N$-normal quotient of $\Gamma$, often denoted $\Gamma_N$.

In [14], Li proved that every 2-arc-transitive graph of prime power order is a normal cover of one of the following graphs (which we regard as ‘basic graphs’): $K_{2^n, 2^n}$ (the complete bipartite graph), $K_{p^m}$ (the complete graph), $K_{2^n, 2^n} - 2^nK_2$ (the graph obtained by deleting a 1-factor from $K_{2^n, 2^n}$) or a primitive or biprimitive affine graph (by which, see [14, p.130], Li meant the graphs in the classification by Ivanov and the second author in [10, Table 1]). Li was “inclined to think that non-basic 2-arc-transitive graphs of prime power order would be rare and hard to construct”, see [14, pp.130–131], and posed the following problem.

**Problem 1** [14, Problem] Construct and characterize the normal covers of prime power order, of the basic 2-arc-transitive graphs of prime power order.

As mentioned in [14], several non-trivial normal covers of order a 2-power of primitive affine graphs appear in [11, Theorem 1.4]. These are certain girth 5 normal covers of the Armanios–Wells graph of order 32, of which only the smallest one (of order $2^{20}$) has been confirmed to exist – by a coset enumeration performed by Leonard Soicher, [10, Section 8].

In this paper, we give a partial answer to Problem 1 by characterising a certain family of 2-arc-transitive normal covers of $K_{2^n, 2^n}$, and by explicitly constructing an infinite family of such graphs, which are all non-basic and have 2-power order.

In addition, the family of graphs we construct provides an affirmative answer to Problem 2 below, which was posed by Chen et al. in [2]. A non-complete graph $\Gamma$ of diameter $d \geq 2$ is called 2-distance-transitive if $\text{Aut}(\Gamma)$ acts transitively, for each $i = 0, 1, 2$, on the set of all pairs $(u, v)$, where $u, v \in V(\Gamma)$ and $d(u, v) = i$, and distance-transitive if $\text{Aut}(\Gamma)$ acts transitively, for each $i = 0, 1, \ldots, d$, on the set of all pairs $(u, v)$, where $u, v \in V(\Gamma)$ and $d(u, v) = i$. (See [2, 3, 12] for more information about 2-distance-transitive graphs.) Also, a Cayley graph $\text{Cay}(G, S)$ is said to be normal if $G$ is normal in $\text{Aut}(\text{Cay}(G, S))$ (see Section 2.2.3). Note that Problem 2 has been answered in the affirmative in [9], where an infinite family of such graphs was constructed. The graphs we construct here are different to those in [9].

**Problem 2** [2, Question 1.2] Is there a normal Cayley graph that is 2-distance-transitive, but is neither distance-transitive nor 2-arc-transitive?

### 1.1 Statement of main results

In order to state our main results we introduce in Definition 1.1 a class of groups, and for each such group we introduce in Definition 1.2 two associated graphs. Note that the derived
subgroup $G'$ of a group $G$ is the subgroup generated by all the commutators, that is, elements $[a, b] = a^{-1}b^{-1}ab$, for $a, b \in G$.

**Definition 1.1** Let $n \geq 1$. If $H$ is a finite group with subgroups $X, Y$ such that $X \cong Y \cong C_2^n$, $H = \langle X \cup Y \rangle$ and $H/H' \cong C_2^{2n}$, then we call $H$ an $n$-dimensional mixed dihedral group relative to $X$ and $Y$.

For $H, X, Y$ as in Definition 1.1, we show in Lemma 2.1 that the map $\phi : H \to H/H'$ given by $h \mapsto hH'$ is a homomorphism such that $H/H' = \phi(X) \times \phi(Y)$. We note that, for each $n \geq 1$, there exist infinitely many $n$-dimensional mixed dihedral groups. For example, let $m_1, m_2, \ldots, m_n$ be positive even integers, and let

$$A_i = \langle x_i, y_i \mid x_i^2 = y_i^2 = (x_i y_i)^{m_i} = 1 \rangle \cong D_{2m_i} \text{ with } i = 1, 2, \ldots, n.$$  

It is easy to see that $A_1 \times A_2 \times \cdots \times A_n \cong \prod_{i=1}^{n} D_{2m_i}$ is an $n$-dimensional mixed dihedral group relative to $\langle x_1, \ldots, x_n \rangle$ and $\langle y_1, \ldots, y_n \rangle$. These natural examples suggest the name for this family of groups but, as we shall see, there are many $n$-dimensional mixed dihedral groups apart from direct products of dihedral groups.

We now introduce a pair of graphs for each mixed dihedral group. See Subsection 2.2.3 for definitions related to the Cayley graph $\text{Cay}(G, S)$.

**Definition 1.2** Let $n \geq 2$, and let $H$ be an $n$-dimensional mixed dihedral group relative to $X$ and $Y$, as in Definition 1.1. Define the graphs $C(H, X, Y)$ and $\Sigma(H, X, Y)$ as follows:

$$C(H, X, Y) = \text{Cay}(H, S(X, Y)), \quad \text{with } S(X, Y) = (X \cup Y) \setminus \{1\};$$

and for $\Sigma = \Sigma(H, X, Y)$,

$$V(\Sigma) = \{Xh : h \in H\} \cup \{Yh : h \in H\},$$

$$E(\Sigma) = \{\{Xh, Yg\} : h, g \in H \text{ and } Xh \cap Yg \neq \emptyset\}.\quad (2)$$

**Remark 1.3** For $\Sigma = \Sigma(H, X, Y)$ in Definition 1.2, the set $\Sigma(Xh)$ of vertices adjacent to the vertex $Xh$ is $\{Yxh : x \in X\}$, and similarly, $\Sigma(Yg) = \{Xyg : y \in Y\}$. We show in Lemma 2.1 (2) that $X \cap Y = 1$, and this implies that $|\Sigma(Xh)| = |\Sigma(Yg)| = 2^n$ for all $h, g \in H$, and hence that $\Sigma$ is a regular bipartite graph of valency $2^n$.

In this paper we are interested in a particular class of normal covers of $K_{2^n, 2^n}$. The graph $\Sigma = K_{2^n, 2^n}$ is said to be $H$-edge-affine if $H \leq S_{2^n} \rtimes S_2 = \text{Aut}(\Sigma)$ and $H$ has a normal subgroup $C_2^n \times C_2$ that is intransitive on vertices and regular on edges. By Proposition 3.1, a $(G, 2)$-arc-transitive $N$-normal cover $\Gamma$ of $K_{2^n, 2^n}$ comes in four ‘flavours’, and one of these is the case where the normal quotient $\Gamma_N$ is $G/N$-edge-affine. We prove the following, further motivating our main result.

**Theorem 1.4** Let $n \geq 2$ and let $N \triangleleft G \leq \text{Aut}(\Gamma)$, for a graph $\Gamma$, such that $\Gamma$ is $(G, 2)$-arc-transitive and is an $N$-normal cover of $K_{2^n, 2^n}$. Then either $\Gamma$ is a Cayley graph, or $n \geq 4$ and $\Gamma_N$ is $G/N$-edge-affine.
We now discuss our main result Theorem 1.6 which characterises \((G, 2)\)-arc-transitive \(N\)-normal covers \(\Gamma\) of \(K_{2^n, 2^n}\) such that \(\Gamma_N\) is \(G/N\)-edge-affine. In particular Theorem 1.6 answers Problem 1 for the basic graphs \(K_{2^n, 2^n}\). It does more than this since we do not require the group \(N\) to be a 2-group, see Example 1.5 for examples. The clique graph \(C(\Gamma)\) of \(\Gamma\) is the graph with vertices the maximal cliques of \(\Gamma\) such that two different maximal cliques are adjacent if and only if they share at least one common vertex. We denote the derived subgroup of a group \(H\) by \(H'\).

**Example 1.5** An infinite family of examples satisfying the conditions in Theorem 1.6, but for which \(N\) is not a 2-group, can be obtained from a construction of Potočnik and Spiga in [17]: Let \(G \leq \text{Aut}(K_{2^n, 2^n})\) be 2-arc-transitive on \(K_{2^n, 2^n}\) with \(\text{soc}(G) \cong C_n^2 \times C_n^2\) acting intransitively on the vertices and regularly on the edges of \(K_{2^n, 2^n}\). By [17, Theorem 6], for each odd prime \(p\) there exists a \(q\)-fold regular cover \(\Gamma\) of \(K_{2^n, 2^n}\), for some power \(q\) of \(p\), such that the maximal lifted group of automorphisms of \(\Gamma\) induces precisely \(G\) on \(K_{2^n, 2^n}\). The lifted group has the form \(P \rtimes G\), with \(P\) a group of order \(q\), \(P \rtimes G\) acts as a 2-arc-transitive group on \(\Gamma\), and the lift of \(\text{soc}(G)\) is \(X := P \rtimes G\) acting regularly on the edges of \(\Gamma\). For an edge \(\{u, v\}\) of \(\Gamma\), we have \(X_u \cong X_v \cong C_2^2\), \(X = \langle X_u, X_v \rangle\) and \(X/X' \cong C_{2^n}^2\).

**Theorem 1.6** Let \(n \geq 2\), and let \(\Sigma\) be a graph, \(G \leq \text{Aut}(\Sigma)\), and \(N \trianglelefteq G\). Then the following are equivalent.

(a) \(\Sigma\) is a \((G, 2)\)-arc-transitive \(N\)-normal cover of \(K_{2^n, 2^n}\) which is \(G/N\)-edge-affine;

(b) \(G\) has a normal subgroup \(H\) with \(H' = N\) such that \(H\) is an \(n\)-dimensional mixed dihedral group relative to \(X\) and \(Y\), the line graph of \(\Sigma\) is \(C(H, X, Y)\), and \(C(H, X, Y)\) is \(G\)-edge-transitive.

We now define an infinite family of mixed dihedral groups.

**Definition 1.7** Let \(n \geq 2\), and let \(\tilde{G}(n) = \langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle\) be a finite 2-group with the following defining relations, where \(1 \leq i, j, k \leq n\):

\[
x_i^2 = y_i^2 = 1, [x_i, x_j] = [y_i, y_j] = 1, [x_i, y_j]^2 = 1, [[x_i, y_j], x_k] = [[x_i, y_j], y_k] = 1,
\]

Let \(X = \langle x_1, \ldots, x_n \rangle\) and \(Y = \langle y_1, \ldots, y_n \rangle\).

In Theorem 1.8 we show that the graph constructions in Definition 1.2 using the groups in Definition 1.7 provide an infinite family of examples answering Problem 2 in the affirmative, namely the graphs \(C(\tilde{G}(n), X, Y)\) have all the properties sought in Problem 2. Also, for each \(n \geq 2\), the graph \(\Sigma(\tilde{G}(n), X, Y)\) has all the properties required in Problem 1 for the basic 2-arc-transitive graph \(K_{2^n, 2^n}\). A graph \(\Gamma\) is said to be 2-geodesic-transitive if \(\Gamma\) is vertex-transitive and \(\text{Aut}(\Gamma)\) acts transitively on the set of all triples \((u, v, w)\) where \(u, v, w \in V(\Gamma)\) with \(\{u, v\}, \{v, w\} \in E(\Gamma)\) and \(\{u, w\} \notin E(\Gamma)\). Note that, 2-geodesic-transitivity implies 2-distance-transitivity.
Figure 1: Distance diagram of the graph \(C(\tilde{G}(2), X, Y)\), as in Theorem 1.8, for the orbits of the stabiliser of a vertex inside the full automorphism group. Computations performed in GAP [6].

**Theorem 1.8** Let \(n \geq 2\), let \(\tilde{G}(n), X, Y\) be as in Definition 1.7, let \(\Gamma = C(\tilde{G}(n), X, Y)\) and \(\Sigma = \Sigma(\tilde{G}(n), X, Y)\) be as in Definition 1.2, and let \(N = \langle [x_i, y_j] \mid 1 \leq i, j \leq n \rangle\). Then

1. \(\tilde{G}(n)\) is an \(n\)-dimensional mixed dihedral group relative to \(X\) and \(Y\), with derived subgroup \(\tilde{G}(n)' \cong C_{2^n}^2\);

2. \(\Sigma\) is a 2-arc-transitive \(N\)-normal cover of \(K_{2^n, 2^n}\), of order \(2^{n^2+n-1}\), which is \(\tilde{G}(n)/N\)-edge-affine;

3. \(\Gamma\) is a 2-geodesic-transitive normal Cayley graph; moreover, \(\Gamma\) is 2-distance-transitive, but it is neither distance-transitive nor 2-arc-transitive.

**Remark 1.9** In Definition 1.7 we give an explicit definition of a group \(G(n)\) in terms of mappings of vector spaces, and in Lemma 5.4 we prove that this group \(G(n)\) is isomorphic to the group \(\tilde{G}(n)\) in Definition 1.7. This concrete representation of \(G(n)\) is useful for deriving various of its properties. Moreover, in Lemma 5.6 we determine the full automorphism group \(A\) of the graph \(C(\tilde{G}(n), X, Y)\), prove that it is equal to the full automorphism group of \(\Sigma(\tilde{G}(n), X, Y)\) and prove that \(K_{2^n, 2^n}\) is \(A/N\)-edge-affine, a stronger statement than Theorem 1.8 (2). Knowing the full automorphism group of \(C(\tilde{G}(n), X, Y)\) and \(\Sigma(\tilde{G}(n), X, Y)\) turns out to be essential for the proof of Theorem 1.8. Figure 1 gives the distance diagram of \(C(\tilde{G}(2), X, Y)\).

This paper is organised as follows. In Section 2, we outline the notation used in the paper and give several preliminary results. In Section 3, we prove Theorem 1.4; in Section 4 we prove Theorem 1.6; and in Section 5 we use Theorem 1.6 to prove Theorem 1.8.

### 2 Preliminaries

In this section, we introduce the notation and concepts we require for graphs and their symmetry properties, and we prove some preliminary results.
2.1 Notation and concepts for graphs and groups

We begin with various notions we will meet for graphs.

2.2 Concepts for graphs

Let $\Gamma$ be a graph. As in Section 1, $V(\Gamma)$, $E(\Gamma)$ and $\text{Aut}(\Gamma)$ denote its vertex set, edge set, and full automorphism group, respectively. Let $d(\Gamma)$ be the diameter of $\Gamma$ (the maximum distance between vertices). For $v \in V(\Gamma)$ and $1 \leq i \leq d(\Gamma)$, let $\Gamma_i(v)$ denote the set of vertices at distance $i$ from $v$; we often write $\Gamma(v) = \Gamma_1(v)$. A graph is said to be regular if there exists an integer $k$ such that $|\Gamma(v)| = k$ for all vertices $v \in V(\Gamma)$.

A clique of a graph $\Gamma$ is a subset $U \subseteq V(\Gamma)$ such that every pair of vertices in $U$ forms an edge of $\Gamma$. A clique $U$ is maximal if no subset of $V(\Gamma)$ properly containing $U$ is a clique.

The clique graph of $\Gamma$ is defined as the graph $\Sigma(\Gamma)$ with vertices the maximal cliques of $\Gamma$ such that two distinct maximal cliques are adjacent in $\Sigma(\Gamma)$ if and only if their intersection is non-empty. Similarly the line graph of $\Gamma$ is defined as the graph $L(\Gamma)$ with vertex set $E(\Gamma)$ such that two distinct edges $e, e' \in E(\Gamma)$ are adjacent in $L(\Gamma)$ if and only if $e \cap e' \neq \emptyset$.

A graph $\Gamma$ is bipartite if $E(\Gamma) \neq \emptyset$ and $V(\Gamma)$ is of the form $\Delta \cup \Delta'$ such that each edge consists of one vertex from $\Delta$ and one vertex from $\Delta'$. If $\Gamma$ is connected then this vertex partition is uniquely determined and the two parts $\Delta, \Delta'$ are often called the biparts of $\Gamma$.

2.2.1 Symmetry concepts for graphs

Let $G \leq \text{Aut}(\Gamma)$. For $v \in V(\Gamma)$, let $G_v = \{ g \in G : v^g = v \}$, the stabiliser of $v$ in $G$. We say that $\Gamma$ is $G$-vertex-transitive or $G$-edge-transitive if $G$ is transitive on $V(\Gamma)$ or $E(\Gamma)$, respectively. When $G = \text{Aut}(\Gamma)$, a $G$-vertex-transitive or $G$-edge-transitive graph $\Gamma$ is simply called vertex-transitive or edge-transitive, respectively. A regular graph $\Gamma$ is said to be $G$-locally primitive or locally $(G, 2)$-arc-transitive if $G \leq \text{Aut}(\Gamma)$ and $G_v$ is primitive or 2-transitive on $\Gamma(v)$, respectively, for each $v \in V(\Gamma)$. Similarly, when $G = \text{Aut}(\Gamma)$, a $G$-locally primitive or locally $(G, 2)$-arc-transitive graph $\Gamma$ is simply called locally primitive or locally 2-arc-transitive, respectively.

A group $G$ of permutations of a set $V(\Gamma)$ is called regular if it is transitive, and some (and hence all) stabilisers $G_v = 1$ are trivial. More generally $G$ is called semiregular if the stabiliser $G_v = 1$ for all $v \in V(\Gamma)$. So $G$ is regular if and only if it is semiregular and transitive.

2.2.2 Normal quotients and normal covers of graphs

Let $\Gamma$ be a regular graph. Assume that $G \leq \text{Aut}(\Gamma)$ is such that $\Gamma$ is $G$-vertex-transitive or $G$-edge-transitive. Let $N$ be a normal subgroup of $G$ such that $N$ is intransitive on $V(\Gamma)$. The $N$-normal quotient graph of $\Gamma$ is defined as the graph $\Gamma_N$ with vertices the $N$-orbits in $V(\Gamma)$ and with two distinct $N$-orbits adjacent if there exists an edge in $\Gamma$ consisting of one vertex from each of these orbits. If $\Gamma_N$ and $\Gamma$ have the same valency, then we say that $\Gamma$ is an $N$-normal cover of $\Gamma_N$. 
2.2.3 Cayley graphs

Given a finite group $G$ and an inverse-closed subset $S \subseteq G \setminus \{1\}$ (that is, $s^{-1} \in S$ for all $s \in S$), the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is a graph with vertex set $G$ and edge set $\{\{g, sg\} : g \in G, s \in S\}$. For any $g \in G$ define

$$R(g) : x \mapsto xg$$

for $x \in G$ and set $R(G) = \{R(g) : g \in G\}$. Then $R(G)$ is a regular permutation group on $V(\Gamma)$ (see, for example [21, Lemma 3.7]) and is a subgroup of Aut($\text{Cay}(G, S)$) (as $R(g)$ maps each edge $\{x, sx\}$ to an edge $\{xg, sxg\}$). For briefness, we shall identify $R(G)$ with $G$ in the following. Let

$$\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) : S^\alpha = S\}.$$

It was proved by Godsil [8] that the normaliser of $G$ in Aut($\text{Cay}(G, S)$) is $G : \text{Aut}(G, S)$. A Cayley graph $\text{Cay}(G, S)$ is said to be normal if $G$ is normal in Aut($\text{Cay}(G, S)$) (see [23]); this is equivalent to the condition Aut($\text{Cay}(G, S)$) = $G : \text{Aut}(G, S)$. Following [20], we say that $\text{Cay}(G, S)$ is normal-edge-transitive if $G : \text{Aut}(G, S)$ is transitive on the edge set of $\text{Cay}(G, S)$. Note that $\text{Cay}(G, S)$ is normal-edge-transitive if and only if either Aut($G, S$) is transitive on $S$, or has two orbits in $S$ such that each is the set of inverses of elements of the other (see [20, Proposition 1(c)]).

Cayley graphs are precisely those graphs $\Gamma$ for which Aut($\Gamma$) has a subgroup that is regular on $V(\Gamma)$. For this reason we say that a graph $\Gamma$ is a bi-Cayley graph if Aut($\Gamma$) has a subgroup $H$ which is semiregular on $V(\Gamma)$ with two orbits.

2.2.4 Notation for groups

For a positive integer $n$, $C_n$ denotes a cyclic group of order $n$, and $D_{2n}$ denotes a dihedral group of order $2n$. For a group $G$, we denote by $1$, $Z(G)$, $\Phi(G)$, $G'$, soc($G$) and Aut($G$), the identity element, the centre, the Frattini subgroup, the derived subgroup, the socle and the automorphism group of $G$, respectively. For a subgroup $H$ of a group $G$, denote by $C_G(H)$ the centralizer of $H$ in $G$ and by $N_G(H)$ the normalizer of $H$ in $G$. For elements $a, b$ of $G$, the commutator of $a, b$ is defined as $[a, b] = a^{-1}b^{-1}ab$. If $X, Y \subseteq G$, then $[X, Y]$ denotes the subgroup generated by all the commutators $[x, y]$ with $x \in X$ and $y \in Y$.

2.3 Six useful lemmas

The first result proves some properties of mixed dihedral groups (see Definition 1.1) mentioned in Section 1.

**Lemma 2.1** Let $H$ be a mixed dihedral group relative to $X$ and $Y$, where $X, Y$ are subgroups of $H$ and $X \cong Y \cong C_2^n$. Then the following hold.

1. Let $\phi : H \rightarrow H/H'$ be the natural projection map given by $\phi : h \mapsto hH'$. Then $H/H' \cong \phi(X) \times \phi(Y) \cong X \times Y$. 

(2) For all \( h, g \in H \), \( |Xh \cap Yg| \leq 1 \).

**Proof** (1) Let \( X = \langle x_1, \ldots, x_n \rangle = C_2^n \) and \( Y = \langle y_1, \ldots, y_n \rangle = C_2^n \). Then \( H/H' = \phi(H) \) is generated by the \( 2n \) elements \( \phi(x_i), \phi(y_i) \), for \( 1 \leq i \leq n \), each of which has order at most 2. Since by definition \( |H/H'| = 2^{2n} \), it follows that \( \phi(X) \cong X, \phi(Y) \cong Y, H/H' \cong \phi(X) \times \phi(Y) \), and also \( X \cap Y = 1 \).

(2) Suppose that \( a, b \in Xh \cap Yg \). Then \( Xa = Xh = Xb \), and so \( ab^{-1} \in X \). Similarly, \( Ya = Yg = Yb \), and so \( ab^{-1} \in Y \). Thus, \( ab^{-1} \in X \cap Y = 1 \), that is, \( a = b \). Hence, \( |Xh \cap Yg| \leq 1 \). \( \Box \)

The second lemma concerns normal quotients of locally primitive graphs.

**Lemma 2.2** Let \( \Gamma \) be a connected regular \( G \)-locally primitive bipartite graph of valency \( k > 1 \) and with bipartition \( V(\Gamma) = O_1 \cup O_2 \), so each \( |O_i| > 1 \). Suppose that \( N \trianglelefteq G \) is such that \( N \) fixes both \( O_1 \) and \( O_2 \) setwise, and \( N \) is intransitive on \( O_1 \) and on \( O_2 \). Then

(1) \( \Gamma \) is an \( N \)-normal cover of the quotient graph \( \Gamma_N \) of \( \Gamma \).

(2) \( N \) acts semiregularly on \( V(\Gamma) \), \( N \) is the kernel of the \( G \)-action on \( V(\Gamma_N) \), and \( G/N \leq \text{Aut}(\Gamma_N) \).

(3) \( \Gamma_N \) is \( G/N \)-locally primitive. Furthermore, if \( \Gamma \) is locally \((G, 2)\)-arc-transitive, then \( \Gamma_N \) is locally \((G/N, 2)\)-arc-transitive.

(4) For \( N \leq H \leq G \), \( H \) is regular on \( E(\Gamma) \) if and only if \( H/N \) is regular on \( E(\Gamma_N) \).

**Proof** Parts (1)–(3) are proved in [7, Lemma 5.1]. Now we prove part (4). Note that, by part (2), \( N \) is semiregular so each \( N \)-orbit in \( V(\Gamma) \) has size \( |N| \), and by part (1), \( \Gamma \) is a cover of \( \Gamma_N \) so each edge \( \{x^N, y^N\} \) of \( \Gamma_N \) corresponds to exactly \( |N| \) edges of \( \Gamma \) and the subgroup \( N \) acts regularly and faithfully on them. Suppose that \( N \leq H \leq G \). Suppose first that \( H \) is regular on \( E(\Gamma) \). Then \( |H| = |E(\Gamma)| = |N| \cdot |E(\Gamma_N)| \), and \( H \) is transitive on \( E(\Gamma_N) \). Hence the stabiliser \( H_e \) in \( H \) of an edge \( e := \{x^N, y^N\} \) of \( \Gamma_N \) has order \( |H_e| = |H|/|E(\Gamma_N)| = |N| \). Since \( N \leq H \) and \( N \) fixes the edge \( e \), we have \( N \leq H_e \), and it follows that \( H_e = N \). Hence \( H \) induces a regular group \( H/N \) on \( E(\Gamma_N) \). Conversely suppose that \( H/N \) is regular on \( E(\Gamma_N) \), so the stabiliser in \( H \) of an edge \( e := \{x^N, y^N\} \) of \( \Gamma_N \) is equal to \( N \). Since \( N \) acts regularly and faithfully on the set of \( |N| \) edges with end-points in \( x^N, y^N \), it follows that \( H \) acts regularly on \( E(\Gamma) \). \( \Box \)

The next result gives a basic property of bi-Cayley graphs.

**Lemma 2.3** Let \( \Gamma \) be a connected bi-Cayley graph of a group \( H \) such that neither of the two \( H \)-orbits in \( V(\Gamma) \) contains an edge of \( \Gamma \), and let \( N = N_{\text{Aut}(\Gamma)}(H) \). Then, for each \( v \in V(\Gamma) \), the stabiliser \( N_v \) acts faithfully on \( \Gamma(v) \).
Proof By the definition of a bi-Cayley graph, $H$ acts semiregularly on $V(\Gamma)$ with two orbits, say $U$ and $W$. We may assume that $U = \{h_0 \mid h \in H\}$ and $W = \{h_1 \mid h \in H\}$, and that $H$ acts on $U$ and $W$ as follows:

$$h_i^g = (hg)_i, \quad \text{for all } h, g \in H \text{ and } i = 0, 1.$$ 

Since by assumption $U$ and $W$ contain no edges of $\Gamma$, it follows that $\Gamma$ is a regular bi-partite graph, and we may assume that $E(\Gamma)$ is the set of pairs $\{h_0, g_1\}$ with $h, g \in H$ such that $gh^{-1}$ lies in a certain subset $S$ of $H$, see [24, p. 505]. In particular $\Gamma(1) = \{s_1 \mid s \in S\} \subseteq W$, and $\Gamma(1_1) = \{(s^{-1})_0 \mid s \in S\} \subseteq U$. Further, by [24, Lemma 3.1], we may assume also that $1 \in S$, and hence $1_1 \in \Gamma(1_0)$. Since $\Gamma$ is connected it follows that $H = \langle S \rangle$.

Since $H$ has two orbits on $\Gamma$, in order to prove the lemma it is sufficient to show that $N_v$ is faithful on $\Gamma(v)$ for $v = 1_0$ and $v = 1_1$. By [24, Theorem 1.1], we have $N_{1_0} = \{\sigma_{\alpha, g} \mid \alpha \in \text{Aut}(H), g \in H, S^\alpha = g^{-1}S\}$, where $\sigma_{\alpha, g}$ is defined as follows:

$$\sigma_{\alpha, g} : h_0 \mapsto (h^\alpha)_0, \quad h_1 \mapsto (gh^\alpha)_1, \quad \text{for all } h \in H,$$

and therefore $N_{1_01_1} = \{\sigma_{\alpha, 1} \mid \alpha \in \text{Aut}(H), S^\alpha = S\}$. Note that for $\sigma_{\alpha, 1} \in N_{1_01_1}$, we have, by definition, that $\sigma_{\alpha, 1} : (s^{-1})_0 \mapsto (s^{-\alpha})_0$ and $s_1 \mapsto (s^\alpha)_1$, for all $s \in S$. Since $\Gamma(1) = \{s_1 \mid s \in S\}$ and $\Gamma(1_1) = \{(s^{-1})_0 \mid s \in S\}$, it follows that the kernels of the actions of $N_{1_0}$ on $\Gamma(1)$, and of $N_{1_1}$ on $\Gamma(1_1)$, are equal to the same subgroup $K$, namely $K$ consists of all elements $\sigma_{\alpha, 1}$ such that $\alpha$ fixes $S$ pointwise. Since $H = \langle S \rangle$ this implies that $K$ is trivial. Thus the lemma is proved.

The last three lemmas are all related to complete bipartite graphs in some way. The first is a group theoretic characterisation.

**Lemma 2.4** Let $\Gamma$ be a connected graph with $|E(\Gamma)| > 1$, and suppose that $G \leq \text{Aut}(\Gamma)$ is abelian and edge-transitive. Then either

1. $\Gamma$ is a cycle $C_n$, for some $n \geq 3$, and $G \cong C_n$ is vertex-transitive, or
2. $\Gamma = K_{m,n}$ with biparts $\Delta, \Sigma$ of sizes $m, n$ respectively, and $G = M \times N < \text{Sym}(\Delta) \times \text{Sym}(\Sigma)$ with $|M| = m, |N| = n$ and $m + n > 1$.

**Proof** Since $|E(\Gamma)| > 1$ and $\Gamma$ is connected, it follows that the group $G$ acts faithfully on $E(\Gamma)$. Choose an edge $\{u, v\} \in E(\Gamma)$. Then since $G$ is edge-transitive and abelian, it follows that $G$ is regular on $E(\Gamma)$ (see [21, Theorem 3.2]), and so $|G| = |E(\Gamma)|$. Now each vertex $x$ lies in at least one edge, say $\{x, y\}$ (since $\Gamma$ is connected), and for some $g \in G$, $\{x, y\}^g = \{u^g, v^g\}$ (since $G$ is edge-transitive), and hence $x \in u^G$ or $x \in v^G$. Hence $G$ has at most two orbits in $V(\Gamma)$.

Suppose first that $G$ is transitive on $V(\Gamma)$. Then $\Gamma$ is regular, say of valency $k$, and counting incident vertex-edge pairs we have $|V(\Gamma)| \cdot k = |E(\Gamma)| \cdot 2$. Now $G$ is faithful on $V(\Gamma)$ (by definition) and $G$ is abelian, and hence (again see [21, Theorem 3.2]) $G$ is regular on $V(\Gamma)$. Thus $|V(\Gamma)| = |G| = |E(\Gamma)|$, and this implies that $k = 2$. As $\Gamma$ is connected, this means that $\Gamma$ is a cycle $C_n$ for some $n \geq 3$, and the only abelian edge-transitive subgroup $G$ of $\text{Aut}(C_n) = D_{2n}$ is the cyclic group of rotations $C_n$, and part (1) holds.
Thus we may assume that $G$ has two vertex orbits, namely $\Delta = u^G$ of size $m$, and $\Sigma = v^G$ of size $n$. As $\Gamma$ is connected, $\Gamma$ is bipartite with biparts $\Delta$ and $\Sigma$. Since $G$ is edge-transitive, it follows that $N := G_u$ is transitive on $\Gamma(u)$ (a subset of $\Sigma$), and as $G$ is abelian, the transitive $G$-action on $\Delta$ is regular so $N$ fixes $\Delta$ pointwise. If $x \in \Gamma_2(u)$, then $x \in \Delta$ and there exists $y \in \Gamma(u) \cap \Gamma(x)$. Then $G_x = N$ (as $G$ is regular on $\Delta$) and $G_x$ is transitive on $\Gamma(x)$ (as $G$ is edge-transitive), and so $\Gamma(x)$ is the $N$-orbit containing $y$, that is, $\Gamma(x) = y^N = \Gamma(u)$. This holds for every $x \in \Gamma_2(u)$, and it follows, since $\Gamma$ is connected that $\Sigma = \Gamma(u)$ and that $N$ is transitive on $\Sigma$. The same argument with $u$ in place of $u$ proves that $\Delta = \Gamma(v)$ and that $M := G_v$ fixes $\Sigma$ pointwise and is transitive on $\Delta$. Thus $\Gamma = K_{m,n}$, and $M \times N \leq G$. Now $|G| = |E(\Gamma)| = mn = |M \times N|$, and hence $G = M \times N$. Finally $m + n > 1$ since $|E(\Gamma)| > 1$. $\square$

We next record the result of a computer investigation of the two small graphs $K_{4,4}$ and $K_{8,8}$ using Magma [1].

**Lemma 2.5** Let $n = 2$ or $3$, and let $\Gamma = K_{2^n,2^n}$. If $G \leq \text{Aut}(\Gamma)$ is such that $\Gamma$ is $G$-vertex-transitive and $G$-locally primitive, then $\Gamma$ is $(G,2)$-arc-transitive and $G$ contains a subgroup acting regularly on $V(\Gamma)$.

**Lemma 2.6** Let $\Gamma$ be a connected $(G,2)$-arc-transitive graph, and let $u \in V(\Gamma)$. Suppose that $\Gamma$ is an $N$-normal cover of $K_{2^n,2^n}$, for some normal 2-subgroup $N$ of $G$. Then $\Gamma$ is bipartite, and one of the following holds:

1. $\Gamma$ is a normal Cayley graph of a 2-group;
2. $\Gamma$ is a bi-Cayley graph of a 2-group $H$ such that $G \leq N_{\text{Aut}(\Gamma)}(H)$;
3. $N \trianglelefteq \text{Aut}(\Gamma)$.

Moreover if the stabiliser $G_u$ acts non-faithfully on $\Gamma(u)$, then part (3) holds.

**Proof** By assumption $\Gamma_N \cong K_{2^n,2^n}$, so $\Gamma$ is bipartite, and also $\Gamma$ is regular of valency $2^n$ since $\Gamma$ covers $\Gamma_N$. Now $\Gamma$ is a $(G,2)$-arc-transitive graph with order a 2-power (since $N$ is a 2-group). We apply [16, Theorem 1.1] relative to the group $\text{Aut}(\Gamma)$ and conclude that either $\Gamma$ is a normal Cayley graph of a 2-group, or $\text{Aut}(\Gamma)$ has a normal subgroup $M$ such that $\Gamma_M \cong K_{2^m,2^m}$ for some $m$ (see [16, proof of Theorem 1.1 on p. 120] and note that the graphs in case (ii) of that result do not arise as they are not bipartite). In the former case, $G_u$ acts faithfully on $\Gamma(u)$ (see, for example, [15, p. 4610]), and hence $\Gamma$ satisfies part (1) and the lemma is proved in this case. Thus we may assume that the latter holds. By Lemma 2.2 (1) and (2), $\Gamma$ is an $M$-normal cover of $\Gamma_M$, and $M$ is semiregular on $V(\Gamma)$. In particular $\Gamma$ has valency $2^m$ so that $m = n$, and $|V(\Gamma)| = |V(\Gamma_M)| \cdot |M| = 2^{n+1}|M|$. As $\Gamma$ is also an $N$-normal cover of $K_{2^n,2^n}$, again by Lemma 2.2, $N$ is semiregular on $V(\Gamma)$ and $|V(\Gamma)| = |V(\Gamma_N)| \cdot |N| = 2^{n+1}|N|$. It follows that $|N| = |M|$, and so $M$ is a 2-group. Suppose first that $N = M$. Then part (3) holds and there is nothing further to prove.

Assume therefore that $N \neq M$, and recall that $N \triangleleft G$ and $M \triangleleft \text{Aut}(\Gamma)$. Since $\Gamma_N \cong \Gamma_M \cong K_{2^n,2^n}$, both $M$ and $N$ stabilise each bipart of $\Gamma$. Let $G^+$ be the subgroup of $G$ stabilising
each of the biparts of $\Gamma$. Then $NM \leq GM$, $NM \leq G^+M$ and $NM$ is a 2-group. Since $\Gamma$ is $(G, 2)$-arc-transitive, it follows from Lemma 2.2 (3) that $GM/M$ is 2-arc-transitive on $\Gamma_M$, and in particular, $G^+M/M$ acts 2-transitively on each part of size $2^n$ of $\Gamma_M = K_{2^n, 2^n}$. Moreover, since $N \neq M$, $NM/M$ is a non-trivial normal 2-subgroup of $G^+M/M$, and so $NM/M$ induces a regular action of $C_2^n$ on each part of $\Gamma_M$. Suppose that $NM/M$ acts faithfully on each part of $\Gamma_M$. Then $\bar{\Gamma}/(N \cap M) \cong NM/M \cong C_2^n$ (since each part of $\Gamma_M$ has size $2^n$), and $NM$ acts faithfully and regularly on each part of $\bar{\Gamma}$ (since $M$ is semiregular on $V(\bar{\Gamma})$). Thus $\bar{\Gamma}$ is a bi-Cayley graph of the 2-group $H := NM$, and as $H \leq GM$, $G$ is contained in the normaliser of $H$ in $\text{Aut}(\bar{\Gamma})$, and part (2) holds. Also $G_u$ is faithful on $\bar{\Gamma}(u)$ by Lemma 2.3, and there is nothing further to prove.

Therefore we may assume that $NM/M$ is unfaithful on at least one of the parts of $\Gamma_M$. Since $NM \leq GM$ and $GM$ is transitive on $V(\bar{\Gamma})$, the action of $NM$ is unfaithful on each of the parts of $\Gamma_M$. We will derive a contradiction, and this will complete the proof (even of the last assertion) of the lemma.

We showed above that the group induced by $NM/M$ on each of the parts of $\Gamma_M$ is $C_2^n$, and hence $NM/M$ is isomorphic to a subdirect subgroup of $C_2^n \times C_2^n$. In particular $NM/M$ is elementary abelian. Let $U, W$ be the two parts of the bipartition for $\Gamma_M$, so the actions of $NM/M$ on $U$ and on $W$ are not faithful. We may assume that the vertex $u$ lies in a part $u$ of $U$. Let $K/M$ be the kernel of $NM/M$ acting on $U$, and note that $K/M$ acts faithfully on $W$. Since $NM/M$ is abelian and regular on $U$, $K/M$ is the stabiliser of $u$ in $NM/M$. So $K/M = (NM/M)_u \leq (GM/M)_u$. Since $GM/M$ is 2-arc-transitive on $\Gamma_M = K_{2^n, 2^n}$, $(GM/M)_u$ is 2-transitive on the part $W$ of $\Gamma_M$. It follows that its nontrivial normal 2-subgroup $K/M = (NM/M)_u$ is abelian and regular on $W$, and we conclude that $NM/M \cong C_2^n \times C_2^n = C_2^{2n}$, and that $NM/M$ acts regularly on the edge set of $\Gamma_M$. It follows from Lemma 2.2 (4) that $NM$ is regular on the edges of $\Gamma$.

Now $NM/M \cong C_2^{2n}$ implies that the Frattini subgroup $\Phi(NM) \leq M$, and since $NM \leq GM$, also $\Phi(NM) \leq GM$. Since $GM$ is 2-arc-transitive on $\Gamma$, by Lemma 2.2 (1)–(3), $\Gamma$ is a $\Phi(NM)$-normal cover of the quotient graph $\Gamma_{\Phi(NM)}$ with $GM/\Phi(NM)$ as a 2-arc-transitive group of automorphisms. In particular $\Gamma_{\Phi(NM)}$ has valency $2^n$. Since $NM$ is regular on the edge set of $\Gamma$, by Lemma 2.2 (4), $NM/\Phi(NM)$ is also regular on the edge set of $\Gamma_{\Phi(NM)}$, and since $NM/\Phi(NM)$ is elementary abelian (see [22, 5.2.12]), it follows from Lemma 2.4 that $\Gamma_{\Phi(NM)} \cong K_{2^n, 2^n}$, and so $NM/\Phi(NM) \cong C_2^{2n}$. This implies that $M = \Phi(NM)$. Since $NM$ is generated by $N \cup M = N \cup \Phi(NM)$ it follows that $NM$ is generated by $N$ (see [22, 5.2.12]), and hence $NM = N$. Thus $M \leq N$. However we proved above that $|M| = |N|$, so we conclude that $M = N$, which contradicts $NM/M \cong C_2^{2n}$. □

3 Proof of Theorem 1.4

The goal of this section is to prove Theorem 1.4. The proof will depend heavily on the following result, Proposition 3.1. In turn, the proof of Proposition 3.1 relies on the classification of the finite 2-transitive permutation groups of 2-power degree, and hence on the finite simple group classification.

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Proposition 3.1 Let \( n \geq 2 \) be a positive integer and let \( \Gamma = K_{2^n, 2^n} \) with biparts \( U \) and \( W \). Let \( G \leq \text{Aut}(\Gamma) \) be 2-arc-transitive on \( \Gamma \), and let \( G^+ = G_U = G_W \), the setwise stabilizer in \( G \) of each of the biparts. Then one of the following holds.

1. \( \text{soc}(G^+) \cong A_{2^n} \times A_{2^n} \) with \( n \geq 3 \);
2. \( \text{soc}(G^+) \cong \text{PSL}(2, p) \times \text{PSL}(2, p) \) with \( p = 2^n - 1 \) a prime and \( n \geq 3 \);
3. \( \text{soc}(G^+) \cong C_2^{2n} \), acting regularly on the edge set of \( \Gamma \);
4. \( n = 3 \) and \( G^+ = \text{AGL}(3, 2), G_u = \text{PSL}(2, 7), G_w = \text{SL}(3, 2) \), with \( G_u \cap G_w = \mathbb{Z}_7 : \mathbb{Z}_3 \), where \( u \in U \) and \( w \in W \).

Moreover, if \( G \) does not contain a subgroup acting regularly on \( V(\Gamma) \), then \( n \geq 4 \) and case (3) holds.

Proof Let \( u \in U \) and \( w \in W \). Since \( G \) is 2-arc-transitive on \( \Gamma \), there exists an element \( g \in G \) such that \( g \) interchanges \( u \) and \( w \), and replacing \( g \) by an odd power of \( g \) if necessary, we may assume that \( g \) has order a 2-power. Then \( g^2 \in G_u \cap G_w \subseteq G^+ \). Also, since \( \Gamma \) is \((G, 2)\)-arc-transitive, the subgroup \( G_u \) is 2-transitive on \( \Gamma(u) = W \) and hence also \( G^+ \) is 2-transitive on \( W \). Similarly \( G^+ \) is 2-transitive on \( U = \Gamma(w) \). Thus \( G^+ \) is a subdirect subgroup of \( H \times H^q \) for some 2-transitive subgroup \( H \) of \( \text{Sym}(U) = S_{2^n} \), and \( G_u^W, G_w^W \) are 2-transitive subgroups of \( H, H^q \), respectively.

Suppose first that \( n = 2 \). Then \( H = A_4 \) or \( S_4 \), and since \( G_u^W \) is 2-transitive, \( \text{soc}(G_u^W) = C_2^2 \). Hence \( \text{soc}(G^+) = (C_2^2)^2 = C_2^{2n} \) as in part (3). Moreover, it follows from Lemma 2.5 that \( G \) contains a subgroup acting regularly on \( V(\Gamma) \). Thus we may assume that \( n \geq 3 \). Next suppose that case (4) holds for \( G \). We note that a subgroup \( G \) of \( \text{Aut}(\Gamma) \) with all the required properties exists, see [18, Theorem (b), and pp. 26–27] which presents a construction due to Chris Rowley. In particular, by Lemma 2.5, \( G \) contains a subgroup acting regularly on \( V(\Gamma) \). Thus we may assume further that \( G \) is not as in case (4). It now follows from [5, Corollary 1.2] that

\[
G^+ = (M \times M^q).P, \text{ where } M \text{ fixes } W \text{ pointwise and } M.P \text{ is faithful and } 2\text{-transitive on } U.
\]

Let \( T \) be the socle of the 2-transitive group \( M.P \) so that \( T \leq M \leq \text{Sym}(U) \) and either \( T \) is elementary abelian or \( T \) is a nonabelian simple group (by a theorem of Burnside, see [21, Theorem 3.21]), and as the degree \( |U| = 2^n \) it follows from [16, Theorem 2.2] that either \( T \cong C_2^n \) with \( M.P \leq \text{AGL}(n, 2) \), or \( T \cong A_{2^n} \) (with \( n \geq 3 \)), or \( T \cong \text{PSL}(2, p) \) with \( p = 2^n - 1 \) a prime and \( n \geq 3 \). In each of these cases \( T \) is a minimal normal subgroup of \( G^+ \) which is transitive on \( U \) and fixes \( W \) pointwise, and also \( T^q \) is a minimal normal subgroup of \( G^+ \) with \( T^q \) transitive on \( W \) and fixing \( U \) pointwise. Thus \( T \times T^q \) is contained in \( \text{soc}(G^+) \). If \( \text{soc}(G^+) \) is strictly larger than \( T \times T^q \) then \( G^+ \) has a minimal normal subgroup \( N \) such that \( N \cap (T \times T^q) = 1 \), and in this case \( N \leq C_{G^+}(T \times T^q) \). However \( C_{G^+}(T \times T^q) \leq C_{\text{Sym}(U)}(T) \times C_{\text{Sym}(W)}(T^q) \). If \( T = C_2^n \) then \( T \) is self-centralising in \( \text{Sym}(U) \) by [21, Theorem 3.6], while if \( T \cong A_{2^n} \), or \( T \cong \text{PSL}(2, p) \) then \( T \) is 2-transitive on \( U \) and so \( C_{\text{Sym}(U)}(T) = 1 \),
by [21, Theorem 3.2]. In either case it follows that \( N \leq C_{G^+}(T \times T^g) \leq T \times T^g \). This is a contradiction, and hence \( \text{soc}(G^+) = T \times T^g \). Thus one of the cases (1), (2) or (3) holds.

To complete proof of the last assertion, we need to prove that \( G \) has a subgroup \( X \) which is regular on \( V(\Gamma) \) if case (1) or (2) holds. This follows from Lemma 2.5 if \( n = 3 \), so we may assume that \( n \geq 4 \). Therefore we have \( \text{soc}(G^+) = T \times T^g \), where either \( T \cong A_{2n} \), or \( \text{PSL}(2, p) \) with \( p = 2^n - 1 \) a prime. Suppose first that there exists an involution in \( G \setminus G^+ \).

Then we may assume that \( |g| = 2 \). In either case, \( T \) has a subgroup \( H \) acting regularly on \( U \); for example, \( H = C_2^n \) or \( H = D_{p+1} \) respectively. Then \( H^g \leq T^g \) and \( H^g \) acts regularly on \( W \) and fixes \( U \) pointwise. Moreover \( H \times H^g \leq G^+ \) and so \( L := \{ hh^g \mid h \in H \} \) is a subgroup of \( G^+ \) that is semiregular on \( V(T) \) with orbits \( U \) and \( W \). Since, for each \( h \in H \), we have \( (hh^g)^g = h^g h = hh^g \), it follows that \( L \times \langle g \rangle \) is regular on \( V(T) \). Thus we may assume that \( G \setminus G^+ \) contains no involutions.

Now \( \text{soc}(G^+) = T \times T \leq G \leq W = H \wr S_2 \), with the pair \( (T, H) = (A_{2n}, S_{2n}) \) or \( (\text{PSL}(2, p), \text{PGL}(2, p)) \), and in both cases \( W/\text{soc}(G^+) \cong D_8 \) and \( G/\text{soc}(G^+) \neq 1 \). Further, in both cases there exists an involution \( y \in H \setminus T \) (with \( H \) acting on \( U \) and fixing \( W \) pointwise) and an involution \( z \in W \setminus (H \times H) \) (generating the top group of the wreath product) such that \( \langle y, z \rangle \cong D_8 \). Hence \( G = \text{soc}(G^+) \rtimes K \) for some nontrivial subgroup \( K \) of \( \langle y, z \rangle \) with \( K \not\leq \langle y \rangle \times \langle y^2 \rangle \). The condition that \( G \setminus G^+ \) contains no involutions implies that \( K \cong C_4 \).

We now construct, in both cases, explicit subgroups of \( G \) that are regular on \( V(\Gamma) \). Let us take \( U = \{1, 2, \ldots, 2^n\} \) and \( W = \{1', 2', \ldots, (2^n)'\} \), so that the generator \( z \) of the top group of \( W \) is

\[
z = (1, 1')(2, 2') \cdots (2^n, (2^n)')\.
\]

Both \( H = S_{2n} \) and \( H = \text{PGL}(2, p) \) contain a 2-cycle \( x \) acting regularly on \( U \), and an involution \( y \) that inverts \( x \). Hence, replacing \( x, y \) (and hence also \( H \) in the second case) by conjugates in \( S_{2n} \) if necessary, we may assume that

\[
x = (1, 2, \ldots, 2^n) \text{ and } y = (2, 2^n)(3, 2^n - 1) \cdots (2^n - 1, 2^n + 2).
\]

We note that \( x, y \in H \setminus T \) so the product \( xy \in T \), and \( xy \) is an involution with \( \langle x, y \rangle \cong D_{2n+1} \). Further, taking \( x, y \) to act on \( U \) and fix \( W \) pointwise, we see that \( y, z \) are as in the previous paragraph, that is to say, \( \langle y, z \rangle \cong D_8 \) and \( W = \text{soc}(G^+) \rtimes \langle y, z \rangle \). Thus, as argued above, \( G = \text{soc}(G^+) \rtimes K \) with \( K = \langle yz \rangle \cong C_4 \). Finally we note that \( xz = (xy)(yz) \) lies in \( G \) since \( xy \in T \times 1 \leq \text{soc}(G^+) \) and \( yz \in K \leq G \). Computing the product \( xz \) explicitly we find that

\[
xz = (1, 2, \ldots, 2^n) \cdot (1, 1')(2, 2') \cdots (2^n, (2^n)')(1', 1', 2', 2', \ldots (2^n)', 2^n) = (1', 1', 2', 2', \ldots (2^n)', 2^n).
\]

Thus \( \langle xz \rangle \cong C_{2n+1} \) and acts regularly on \( V(\Gamma) \). This completes the proof.

**Proof of Theorem 1.4** Let \( \Gamma, G, N \) be as in the statement of Theorem 1.4. Then, by [19, Theorem 4.1], \( G/N \) is 2-arc-transitive on \( \Gamma_N = K_{2n, 2n} \). Let \( G^+/N \) be the subgroup of \( G/N \) stabilising both parts of the bipartition of \( \Gamma_N = K_{2n, 2n} \). Suppose that \( \Gamma \) is not a Cayley graph. Then \( G \) does not contain a subgroup acting regularly on \( V(\Gamma) \), and it follows from the last assertion of Proposition 3.1 that \( n \geq 4 \), and part (3) of Proposition 3.1 holds, that is, \( \text{soc}(G^+/N) \cong C_{2n} \), acting regularly on the edge set of \( \Gamma_N \). In other words, \( \Gamma_N \) is \( G/N \)-edge-affine. □
4 Characterisation

The goal of this section is to prove Theorem 1.6.

Throughout this section, we let $H$ be an $n$-dimensional mixed dihedral group relative to $X$ and $Y$ with $|X| = |Y| = 2^n \geq 4$, and let $C(H, X, Y)$ and $\Sigma(H, X, Y)$ be the graphs defined in Definition 1.2. We say that an edge \{h, g\} of $C(H, X, Y)$ is an $X$-edge if $hg^{-1} \in X$, or a $Y$-edge if $hg^{-1} \in Y$. By Lemma 2.1 (2), $X \cap Y = 1$. Hence these concepts are well defined and each edge of $C(H, X, Y)$ is either an $X$-edge or a $Y$-edge. Note that, by definition, each $X$-edge is of the form \{g, xg\} and each $Y$-edge is \{g, yg\}, for some $g \in H, x \in X \setminus \{1\}$, and $y \in Y \setminus \{1\}$. In our first result we describe several graph theoretic links between $C(H, X, Y)$ and $\Sigma(H, X, Y)$.

**Lemma 4.1** Let $H, X, Y, n$ be as above. Then the following hold.

1. For each triangle (3-clique) \{g, h, k\} in $C(H, X, Y)$, either all three edges are $X$-edges or all three edges are $Y$-edges.
2. Each $X$-edge \{g, xg\} of $C(H, X, Y)$ lies in a unique maximal clique, namely $Xg$, and each $Y$-edge \{g, yg\} lies in a unique maximal clique, namely $Yg$.
3. $\Sigma(H, X, Y)$ is the clique graph of $C(H, X, Y)$.
4. The map $\varphi : z \rightarrow \{Xz, Yz\}$, for $z \in H$, is a bijection $\varphi : H \rightarrow E(\Sigma(H, X, Y))$, and induces a graph isomorphism from $C(H, X, Y)$ to the line graph $L(\Sigma(H, X, Y))$ of $\Sigma(H, X, Y)$.
5. Moreover, $\text{Aut}(C(H, X, Y)) = \text{Aut}(\Sigma(H, X, Y)) = \text{Aut}(L(\Sigma(H, X, Y)))$.

**Proof**

1. For convenience, we let $\Gamma := C(H, X, Y)$ and $\Sigma := \Sigma(H, X, Y)$. Suppose that \{g, h\} is an $X$-edge so $x := hg^{-1} \in X \setminus \{1\}$, and that \{h, k\} is a $Y$-edge so $y := kh^{-1} \in Y \setminus \{1\}$. Then $kg^{-1} = yx \notin (X \cup Y) \setminus \{1\}$, so \{g, k\} is not an edge. This implies part (1).

2. It follows from the definition of $\Gamma$ in Definition 1.2 that each pair \{xg, x'g\} of distinct elements of $Xg$ is an $X$-edge, since $(x'g)(xg)^{-1} = x'x^{-1} \in X \setminus \{1\}$. Hence $Xg$ is a clique and it contains \{g, xg\}. Also, by part (1), each clique of size at least 3 containing \{g, xg\} must contain only $X$-edges. The only $X$-edges incident with $g$ or $xg$ are of the form \{g, x'g\} or \{xg, x'xg\}, respectively, for some $x' \in X \setminus \{1\}$, and hence each such clique is contained in $Xg$. Thus, $Xg$ is a maximal clique of $\Gamma$, and is the unique maximal clique containing \{g, xg\}. Similarly, $Yg$ is a maximal clique and is the unique maximal clique containing the $Y$-edge \{g, yg\}. This proves part (2).

3. By part (2), the clique graph $\Sigma(\Gamma)$ of $\Gamma$ has vertex set \{Xg | g \in H\} $\cup$ \{Yg | g \in H\}, and this is equal to $V(\Sigma)$ by Definition 1.2. Moreover, two maximal cliques are adjacent in $\Sigma(\Gamma)$ if and only if they contain at least one common vertex. Since distinct cosets of $X$, or of $Y$ are disjoint, each edge in the clique graph $\Sigma(\Gamma)$ is of the form \{Xg, Yh\} such that $g, h \in H$ and $Xg \cap Yh \neq \emptyset$. Thus, the edges of the clique graph $\Sigma(\Gamma)$ are precisely the edges of $\Sigma$, and so $\Sigma = \Sigma(\Gamma)$, proving part (3).
(4) The fact that $\Gamma$ is isomorphic to the line graph of $\Sigma$ can be deduced from [4, Corollary 1.6], since the subgraph of $\Gamma$ induced on the neighbourhood $\Gamma(1) = (X \cup Y) \setminus \{1\}$ is isomorphic to $2K_{2^n-1}$, where $|X| = |Y| = 2^n$. However we require an explicit isomorphism for our later work.

By Definition 1.2, for each $z \in H$, the image $\varphi(z)$ is an edge of $\Sigma$ since $\varphi(z) = \{Xz, Yz\}$ and $Xz \cap Yz = (X \cap Y)z = \{z\} \neq \emptyset$. Thus $\varphi : H \to E(\Sigma)$ is well defined. Next, if $z, w \in H$ and $\varphi(z) = \varphi(w)$, then $Xz = Xw$ and $Yz = Yw$, and hence $zw^{-1} \in X \cap Y = \{1\}$, so $z = w$. Thus $\varphi$ is one-to-one.

Next, it follows from Lemma 2.1(2) that, for each edge $e = \{Xg, Yh\}$ of $\Sigma$ the defining property of an edge, namely $Xg \cap Yh \neq \emptyset$, is equivalent to $Xg \cap Yh = \{z\}$ for some unique $z \in H$. Hence the edge can be written as $e = \{Xz, Yz\}$. It follows that $\varphi$ is onto, and hence is a bijection.

Now $\varphi$ induces a natural bijection from the set of unordered pairs from $H$ to the set of unordered pairs of edges of $\Sigma$, namely $\varphi : \{h, g\} \to \{\varphi(g), \varphi(h)\}$. For an $X$-edge $\{g, xg\}$, the images $\varphi(g), \varphi(xg)$ share a common $X$-coset, namely $Xg = Xg$. Similarly, for a $Y$-edge $\{h, yh\}$, the images $\varphi(h), \varphi(yh)$ share a common $Y$-coset, namely $Yh = Yh$. Thus, for arbitrary $g, h \in H$, each of $\varphi(\{g, xg\})$ and $\varphi(\{h, yh\})$ is an edge-pair from $\Sigma$ (hence a vertex-pair from $L(\Sigma)$) which intersects nontrivially, and hence forms an edge of $L(\Sigma)$. Thus the restriction of this induced map $\varphi$ to $E(\Gamma)$ is a one-to-one map into $E(L(\Sigma))$. We claim that this restriction is onto. Since $\varphi$ on edges is one-to-one, we have $|\varphi(E(\Gamma))| = |E(\Gamma)| = |V(\Gamma)| \cdot |\Gamma(1)|/2 = |H| \cdot (2^n - 1)$. Also, since $\varphi$ (acting on $H$) is bijective, $|V(L(\Sigma))| = |E(\Sigma)| = |\varphi(H)| = |H|$. Now by Remark 1.3, $\Sigma$ is regular of valency $2^n$, and hence $L(\Sigma)$ is regular of valency $2(2^n - 1)$. It follows that $|E(L(\Sigma))| = |H| \cdot 2(2^n - 1)/2 = |\varphi(E(\Gamma))|$. Thus the induced map $\varphi$ on edges is onto, proving the claim. Therefore $\varphi$ induces a bijection on the edge sets, and hence $\varphi$ is a graph isomorphism from $\Gamma$ to the line graph $L(\Sigma)$, completing the proof of (4).

(5) Considering the induced action of $\text{Aut}(\Gamma)$ on the set of maximal cliques of $\Gamma$, and noting that adjacent cliques $Xg, Yh$ intersect in a single vertex of $\Gamma$ (by Lemma 2.1) it follows $\text{Aut}(\Gamma)$ induces a faithful action as a subgroup of automorphisms of the clique graph of $\Gamma$. Hence by part (3), $\text{Aut}(\Gamma) \leq \text{Aut}(\Sigma)$. Also, considering the induced action of $\text{Aut}(\Sigma)$ on the set of edges of $\Sigma$, and noting that adjacent edges in the line graph of $\Sigma$ intersect in a unique vertex of $\Sigma$, it follows $\text{Aut}(\Sigma)$ induces a faithful action as a subgroup of automorphisms of $L(\Sigma)$. Hence $\text{Aut}(\Sigma) \leq \text{Aut}(L(\Sigma))$, so we have $\text{Aut}(\Gamma) \leq \text{Aut}(\Sigma) \leq \text{Aut}(L(\Sigma))$. It follows from part (4) that equality holds, proving part (5).

Now we consider the symmetry of these graphs in more detail. Recall that $H'$ denotes the derived subgroup of $H$.

**Lemma 4.2** Let $H, X, Y, n$ be as above, let $\Sigma = \Sigma(H, X, Y)$, and let $G = H : A(H, X, Y)$, where $A(H, X, Y)$ is the setwise stabiliser in $\text{Aut}(H)$ of $X \cup Y$. Then the following hold.

(1) The group $G$ acts as a subgroup of automorphisms on $\Sigma$ as follows, for $h, z \in H, \sigma \in A(H, X, Y)$, and $\varphi : H \to E(\Sigma)$ as in Lemma 4.1(4):

**Vertex action:** $h : Xz \to Xzh, Yz \to Yzh$

**Edge action:** $\sigma : Xz \to X^\sigma z^\sigma, Yz \to Y^\sigma z^\sigma$

$\varphi : \varphi(z) \to \varphi(z^\sigma)$

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The subgroup \( H \) acts regularly on \( E(\Sigma) \) and has two orbits on \( V(\Sigma) \). In particular, this \( G \)-action is edge-transitive.

(2) The \( H' \)-normal quotient graph \( \Sigma_{H'} \) of \( \Sigma \) is isomorphic to \( K_{2n,2n} \) and admits \( G/H' \) acting faithfully as an edge-transitive group of automorphisms. Moreover, \( \Sigma \) is an \( H' \)-normal cover of \( K_{2n,2n} \).

(3) \( A(H, X, Y) \leq (\text{Aut}(X) \times \text{Aut}(Y)) : C_2 \cong (\text{GL}(n, 2) \times \text{GL}(n, 2)) : C_2 \).

(4) Let \( H \leq L \leq G \). If \( C(H, X, Y) \) is \( L \)-edge-transitive, then \( \Sigma(H, X, Y) \) is \( (L, 2) \)-arc-transitive.

**Proof**

(1) By Lemma 4.1(3), \( \Sigma \) is the clique graph of \( \Gamma := C(H, X, Y) \) and hence \( \text{Aut}(\Gamma) \) induces a subgroup of automorphisms of \( \Sigma \) via its induced action on subsets of \( V(\Gamma) \) (namely on the maximal cliques of \( \Gamma \)). In particular, since by definition \( \Gamma \) is a Cayley graph for \( H \) we have \( G \leq \text{Aut}(\Gamma) \), and this gives a \( G \)-action as a subgroup of automorphisms of \( \Sigma \). Since \( H \) acts by right multiplication, and \( A(H, X, Y) \) acts naturally, on \( V(\Gamma) = H \), the vertex-actions of \( h, \sigma \) are as in the statement, and in particular \( H \) has two vertex-orbits, namely \([H : X] = \{ Xz \mid z \in H \} \) and \([H : Y] = \{ Yz : z \in H \} \). (Note that \( \sigma \) fixes the set \( \{ X, Y \} \) setwise.) Also, by Lemma 4.1(4), \( \varphi : z \mapsto \{ Xz, Yz \} \) is a bijection \( H \to E(\Sigma) \), and it follows from the definition of the \( G \)-action that an edge \( \{ Xz, Yz \} \) is mapped by \( h, \sigma \) to \( \{ Xzh, Yzh \}, \{ X^\sigma z^\sigma, Y^\sigma z^\sigma \} \), respectively, and hence \( \varphi(z)^h = \varphi(zh) \) and \( \varphi(z)^\sigma = \varphi(z^\sigma) \). Thus the edge-action is as asserted. In particular, as \( H \) acts regularly by right multiplication on \( H \), it follows that \( H \) acts regularly on \( E(\Sigma) \), and hence also \( G \) is transitive on \( E(\Sigma) \). This proves part (1).

(2) Since \( H' \) is normal in \( H \), elements of \( H \) permute the \( H' \)-orbits in \( V(\Sigma) \) by right multiplication. Let \( \Delta^X = \{ Xh : h \in H' \} \) and \( \Delta^Y = \{ Yh : h \in H' \} \). Then \( \Delta^X, \Delta^Y \) are the \( H' \)-orbits in \( V(\Sigma) \) containing \( X \) and \( Y \), and they lie in the \( H \)-orbits \([H : X], [H : Y]\) on vertices, respectively. Further, it follows from Lemma 2.1 (1) that each \( g \in H \) is of the form \( g = zxy \) for some \( z \in H' \), \( x \in X, y \in Y \).

We claim that \( \Delta^X g = \Delta^X y \), and that the number of \( H' \)-orbits in \([H : X]\) is \( 2^n \). By definition, \( \Delta^X z = \Delta^X \). Also for each \( Xh \in \Delta^X \) (where \( h \in H' \)), \( Xhx = X(xhx^{-1})h = X[x, h^{-1}]h \), and since \([x, h^{-1}] \in H' \), it follows that \( Xhx = Xz' \) with \( z' = [x, h^{-1}]h \in H' \), and hence \( Xhx \in \Delta^X \). Thus also \( \Delta^X x = \Delta^X \), and so \( \Delta^X g = \Delta^X y \). Therefore each of the \( H' \)-orbits in \([H : X]\) is of the form \( \Delta^X y \) for some \( y \in Y \). Since \([Y] = 2^n \), in order to prove that the number of \( H' \)-orbits in \([H : X]\) is \( 2^n \), it is sufficient to prove that \( \Delta^X y \neq \Delta^X y' \) for distinct \( y, y' \in Y \), or equivalently, to prove that \( \Delta^X y = \Delta^X \) (with \( y \in Y \)) implies that \( y = 1 \). So suppose that \( \Delta^X y = \Delta^X \) for some \( y \in Y \). Then \( Xy \in \Delta^X \), so \( Xy = Xz \) for some \( z \in H' \), and hence \( z = xy \) for some \( x \in X \). Using the map \( \phi \) from Lemma 2.1 (1), this implies that \( \phi(x) = \phi(y^{-1}) = \phi(y^{-1}) \in \phi(X) \cap \phi(Y) = 1 \), and hence that \( x = y = 1 \). Thus the claim is proved.

An analogous proof shows that \( H' \) has exactly \( 2^n \) orbits in \([H : Y]\), and that these orbits are \( \Delta^Y x \) for \( x \in X \). Thus, the \( H' \)-normal quotient of \( \Sigma \) has \( 2 \times 2^n \) vertices and, for each \( x \in X \), by the definition of \( \Sigma \), the orbit \( \Delta^X \) is adjacent to \( \Delta^Y x \) since by Lemma 4.1 (1) the cliques \( X \) and \( Yx \) both contain \( x \). Similarly \( \Delta^Y \) is adjacent to \( \Delta^X y \) for all \( y \in Y \). Since the right multiplication action of \( H \) induces a subgroup of automorphisms of the \( H' \)-normal
quotient graph $\Sigma_{H'}$, and in this action $H$ is transitive on each of the two sets $[H : X]$ and $[H : Y]$, it follows that $\Sigma_{H'}$ is isomorphic to $K_{2n,2n}$. Since $\Sigma$ and $K_{2n,2n}$ both have valency $2^n$, it follows that $\Sigma$ is an $H'$-normal cover of $K_{2n,2n}$. Further, since $\Sigma$ is connected and a cover of $\Sigma_{H'} \cong K_{2n,2n}$, the kernel of the $G$-action on $V(\Sigma_{H'})$ is semiregular on $V(\Sigma)$, and hence is equal to $H'$. Thus $G/H'$ acts faithfully as an edge-transitive group of automorphisms of $\Sigma_{H'}$. This proves part (2).

(3) Now $G \leq \text{Aut}(\Gamma)$, and its subgroup $A(H, X, Y)$ is the stabiliser in $G$ of the vertex 1 of $\Gamma$. Since $X \cup Y$ generates $H$, $A(H, X, Y)$ acts faithfully on $\Gamma(1) = (X \cup Y) \setminus \{1\}$. It then follows from Lemma 4.1 (2) that $A(H, X, Y)$ acts faithfully on the set of maximal cliques of $\Gamma$, and from Lemma 4.1 parts (1) and (2) that each element of $A(H, X, Y)$ either fixes setwise or interchanges the subsets $X \setminus \{1\}$ and $Y \setminus \{1\}$ of $\Gamma(1)$. The subgroup, of $A(H, X, Y)$, of index 1 or 2, fixing setwise each of these subsets, induces automorphisms of $X$ and $Y$, and part (3) follows.

(4) Suppose that $H \leq L \leq G$ such that $L$ is transitive on $E(\Gamma)$, and let $S = (X \cup Y) \setminus \{1\}$, and $s \in S$. Since $H$ is transitive on $V(\Gamma)$, and $L$ is transitive on $E(\Gamma)$, it follows from [20, Proposition 1] that the arc set of $\Gamma$ is $(1, s)^L \cup (s, 1)^L$. Moreover, since $s$ is an involution, right multiplication $R(s) \in H \leq L$ by $s$ maps $(1, s)$ to $(s, s^2) = (s, 1)$, and hence $(1, s)^L = (s, 1)^L$. Thus $\Gamma$ is $L$-arc-transitive. In particular, the stabiliser $L_1$ of $1 \in H$ acts transitively on $S$, and since $L_1 \leq A(H, X, Y)$, it follows from part (3) that $L_1$ acts imprimitively on $S$ and each element of $L_1$ either fixes setwise each of $X \setminus \{1\}$ and $Y \setminus \{1\}$ or interchanges these two sets. Let $L_1^+$ denote the index 2 subgroup of $L_1$ fixing each of $X$ and $Y$ setwise. Then $L_1^+$ is transitive on each of the sets $X \setminus \{1\}$ and $Y \setminus \{1\}$.

Now $L$ is transitive on $V(\Sigma)$ by part (1) (since some element of $L_1$ interchanges $X \setminus \{1\}$ and $Y \setminus \{1\}$). The stabiliser in $L$ of the clique $X$ (a vertex of $\Sigma$), contains the subgroup $X$, which acts transitively on the set of cliques $Yx \ (x \in X)$ adjacent to $X$. Hence $\Sigma$ is $L$-arc-transitive. Also the stabiliser in $L$ of the arc $(X, Y)$ of $\Sigma$ contains $L_1^+$, which acts transitively on the set of 2-arcs $(X, Y, Xy) \ (y \in Y \setminus \{1\})$ extending this arc. It follows that $\Sigma$ is $(L, 2)$-arc-transitive, proving part (4).

**Proof of Theorem 1.6** Suppose that $n$ is an integer with $n \geq 2$, that $\Sigma$ is a graph, and $N \trianglelefteq G \leq \text{Aut}(\Sigma)$. First, we prove that part (b) implies part (a). Suppose, as in part (b), that $G$ has a normal subgroup $H$, where $H$ is an $n$-dimensional mixed dihedral group relative to $X$ and $Y$ with derived subgroup $H' = N$, and that the line graph $L(\Sigma)$ of $\Sigma$ is $C(H, X, Y)$, and $C(H, X, Y)$ is $G$-edge-transitive. Lemma 4.1 (4) gives an explicit isomorphism $\varphi$ from $C(H, X, Y) = L(\Sigma)$ to the line graph $L(\hat{\Sigma})$ of $\hat{\Sigma} = \Sigma(H, X, Y)$. Since the vertex sets of $L(\Sigma)$ and $L(\hat{\Sigma})$ are $E(\Sigma)$ and $E(\hat{\Sigma})$, respectively, $\varphi$ is a bijection $E(\Sigma) \rightarrow E(\hat{\Sigma})$, and similarly $\varphi$ yields a bijection $V(\Sigma) \rightarrow V(\hat{\Sigma})$, which preserves adjacency. Thus $\varphi$ induces an explicit isomorphism from $\Sigma$ to $\hat{\Sigma}$. By Lemma 4.2 (1), $\varphi$ also induces a permutational isomorphism between the $G$-actions on $\Sigma$ and $\hat{\Sigma}$. Thus it suffices to prove that part (a) holds for $\Sigma$.

Recall that $N = H'$. By Lemma 4.2 (2), $\hat{\Sigma}$ is an $N$-normal cover of $\hat{\Sigma}_N \cong K_{2^n,2^n}$. Also, since $C(H, X, Y)$ is $G$-edge-transitive, it follows from Lemma 4.2 (4) that $\hat{\Sigma}$ is $(G, 2)$-arc-transitive. It remains to prove that $\hat{\Sigma}_N \cong K_{2^n,2^n}$ is $G/N$-edge-affine. By Lemma 4.2 (1), $H$ acts regularly on the edge set of $\hat{\Sigma}$, and hence by Lemma 2.2 (4), $H/N$ acts regularly on the edge set of $\hat{\Sigma}_N \cong K_{2^n,2^n}$. Since $H$ is an $n$-dimensional mixed dihedral group, we have
$H/N \cong C_{2^n}^2$ (Definition 1.1) and $H/N \leq G/N$ (since $H \leq G$), and hence $\tilde{\Sigma}_N$ is $G$-edge-affine. Thus part (b) implies part (a).

Now assume that part (a) of Theorem 1.6 holds, that is, $\Sigma$ is a $(G, 2)$-arc-transitive $N$-normal cover of $K_{2^n, 2^n}$, and $K_{2^n, 2^n}$ is $G/N$-edge-affine. Thus in particular $\Sigma$ and $K_{2^n, 2^n}$ have the same valency, namely $2^n$. By Lemma 2.2 (2), $N$ is the kernel of the $G$-action on $\Sigma_N \cong K_{2^n, 2^n}$, and hence $G/N \leq \text{Aut}(\Sigma_N)$. Since $K_{2^n, 2^n}$ is $G/N$-edge-affine, there exists $N < H \leq G$ such that $H/N \cong C_{2^n}^2$ and $H/N$ is regular on the edge set of $\Sigma_N$. By Lemma 2.2 (4), $H$ is also regular on the edge set of $\Sigma$. Since $H'$ is characteristic in $H$ and $H \leq G$, we have $H' \leq G$. Consider the $H'$-normal quotient $\Sigma'$. Note that $\Sigma$ is bipartite, say with bipartition $V(\Sigma) = O_1 \cup O_2$, and $O_1, O_2$ are the two $H$-orbits in $V(\Sigma)$. Since $H/N \cong C_{2^n}^2$ the derived subgroup $H' \leq N$, and since $N$ is intransitive on both $O_1$ and $O_2$, it follows that also $H'$ is intransitive on both $O_1$ and $O_2$. Then since $G$ is $2$-arc-transitive on $\Sigma$, it follows from Lemma 2.2 parts (1) and (4) that $\Sigma$ is an $H'$-normal cover of $\Sigma'$ (so $\Sigma'$ has valency $2^n$), and $H/H'$ is regular on the edge set of $\Sigma'$.

The group $H/H'$ is abelian and edge transitive on $\Sigma'$, and so by Lemma 2.4, $\Sigma' \cong K_{2^n, 2^n}$ (since $\Sigma'$ has valency $2^n \geq 4$). In particular, $|H/H'| = |E(\Sigma')| = 2^{2n}$. Since $H' \leq N < H$, it follows that $H' = N$ and $H/H' = H/N \cong C_{2^n}^2$. Since $H$ is regular on $E(\Sigma)$, the line graph $L(\Sigma)$ of $\Sigma$ is a Cayley graph of $H$, say $L(\Sigma) = \text{Cay}(H, S)$. Let $\{u, v\}$ be an edge of $\Sigma$. Then
\[
S = \{h \in H : \{u, v\}^h \text{ is incident with } \{u, v\}\}.
\]
Now the set of edges of $\Sigma$ incident with $\{u, v\}$ is
\[
F = \{\{u, x\}, \{v, y\} : v \neq x \in \Sigma(u), u \neq y \in \Sigma(v)\}.
\]
Let $X = H_u$ and $Y = H_v$. Then $XH'$ is the subgroup of $H$ stabilising the $H'$-orbit in $V(\Sigma)$ containing $u$. Since $H/H' \cong C_{2^n}^2$ is regular on the set of edges of $\Sigma' \cong K_{2^n, 2^n}$, we have $XH'/H' \cong C_{2^n}^2$. Similarly, $YH'/H' \cong C_{2^n}^2$. By Lemma 2.2 (2), $H'$ is semiregular on $V(\Sigma)$, so $X \cong XH'/H' \cong C_{2^n}^2$ and $Y \cong YH'/H' \cong C_{2^n}^2$. Again, since $H$ is regular on the edge set of $\Sigma$, it follows that $X$ and $Y$ act regularly on $\Sigma(u)$ and $\Sigma(v)$, respectively. Thus, $S = (X \cup Y) \setminus \{1\}$. Since $\Sigma$ is connected, $L(\Sigma)$ is also connected, and so $H = \langle S \rangle = \langle X, Y \rangle$. It follows from Definition 1.1 that $H$ is an $n$-dimensional mixed dihedral group relative to $X$ and $Y$, and from Definition 1.2 that $L(\Sigma) = C(H, X, Y)$. Since $G$ is $2$-arc-transitive on $\Sigma$, it follows that $G_{\{u, v\}}$ is transitive on $F$, and so $L(\Sigma)$ is $G$-edge-transitive. Thus part (b) holds. This completes the proof of Theorem 1.6.

\section{Construction of 2-arc-transitive covers of $K_{2^n, 2^n}$}

In this section, we apply Theorem 1.6 to construct 2-arc-transitive normal covers, of 2-power order, of $K_{2^n, 2^n}$ and prove Theorem 1.8. First we define in Definition 5.1 the group $G(n)$ we will use in the construction. While it is not obvious, the group $G(n)$ is isomorphic to the group $\tilde{G}(n)$ in Definition 1.7, a fact we prove in Lemma 5.4. The more explicit definition of multiplication for $G(n)$ will help in our graph construction and analyses.
Definition 5.1 Let $n \geq 2$, let $X$ and $Y$ be $n$-dimensional vector spaces over $\mathbb{F}_2$, and let $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $A_1, A_2 \in X \otimes Y$. Define $G(n)$ to be the set $X \oplus Y \oplus (X \otimes Y)$ with multiplication defined as follows: for $g_1 = x_1 + y_1 + A_1$ and $g_2 = x_2 + y_2 + A_2$ in $G(n)$, given by

$$g_1 g_2 = g_1 + g_2 + x_2 \otimes y_1,$$  

(3)

where each addition occurs in $X \oplus Y \oplus (X \otimes Y)$, considered as a vector space over $\mathbb{F}_2$.

More explicitly, $g_1 g_2$ is the element $(x_1 + x_2) + (y_1 + y_2) + (A_1 + A_2 + x_2 \otimes y_1)$ of $G(n)$. In particular we denote by 0 the element $x + y + A \in G(n)$ with each of $x, y, A$ equal to the zero vector of the corresponding space. It turns out that $G(n)$ with this multiplication is a group.

Lemma 5.2 Let $X$, $Y$ and $G(n)$ be as in Definition 5.1. Then $G(n)$ is a group of order $2^{n^2 + 2n}$ with identity 0. Furthermore, for $g = x + y + A \in G(n)$, with $x \in X$, $y \in Y$, $A \in X \otimes Y$, the inverse is

$$g^{-1} = g + x \otimes y.$$  

(4)

Proof By definition $|G(n)| = 2^{n^2 + 2n}$. First, we show that the multiplication is associative. For $i = 1, 2, 3$, let $g_i = x_i + y_i + A_i \in G(n)$, where $x_i \in X$, $y_i \in Y$, $A_i \in X \otimes Y$. Then, applying the multiplication defined in Equation (3),

$$(g_1 g_2) g_3 = ((x_1 + x_2) + (y_1 + y_2) + (A_1 + A_2 + x_2 \otimes y_1)) g_3$$

$$= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) + (A_1 + A_2 + A_3 + x_2 \otimes y_1 + x_3 \otimes (y_1 + y_2))$$

$$= g_1 ((x_2 + x_3) + (y_2 + y_3) + (A_2 + A_3 + x_3 \otimes y_2))$$

$$= g_1 (g_2 g_3).$$

A direct computation using (3) yields $g 0 = 0 = g 0$, so 0 is the identity of $G(n)$. Finally, for $g = x + y + A \in G(n)$, using (3) we have

$$g (g + x \otimes y) = g + (g + x \otimes y) + x \otimes y = 2g + 2(x \otimes y) = 0$$

and similarly $(g + x \otimes y) g = 0$. Hence $g^{-1} = g + x \otimes y$. In particular $G(n)$ is a group. \[\square\]

Next we derive some relevant properties of $G(n)$.

Lemma 5.3 Let $X$, $Y$ and $G(n)$ be as in Definition 5.1. Then the following hold.

1. As subgroups of $G(n)$, we have $X \cong Y \cong C_2^n$ and $X \otimes Y \cong C_2^{2n}$.
2. $G(n) = \langle X, Y \rangle$.
3. The derived subgroup $G(n)' = X \otimes Y$, and $G(n)'$ is equal to the centre $Z(G(n))$.
4. $G(n)/G(n)' \cong C_2^{2n}$ and $|G(n)| = 2^{n^2 + 2n}$.

In particular, $G(n)$ is an $n$-dimensional mixed dihedral group with respect to $X$ and $Y$.  

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Proof. Throughout the proof, for each $i = 1, 2$, we let $x_i \in X$, $y_i \in Y$, $A_i \in X \otimes Y$ and $g_i \in \mathcal{G}(n)$ such that $g_i = x_i + y_i + A_i$.

(1) By Equation (3), $x_1x_2 = x_1 + x_2 - x_3x_1$ and $x_2^2 = 0$. Since $|X| = 2^n$, we have $X \cong C_{2^n}$. By a similar argument, $Y \cong C_{2^n}$. Also, $A_1A_2 = A_1 + A_2 = A_2A_1$, and the product $A_1A_1 = 2A_1 = 0$ and $|X \otimes Y| = 2^{n^2}$. Hence $X \otimes Y \cong C_{2^n}^2$ and part (1) is proved.

(2) A typical element of $\mathcal{G}(n)$ has the form $g_1 = x_1 + y_1 + A_1$. By Equation (3), the product $x_1y_1A_1 = x_1 + y_1 + A_1 = g_1$ and so, in order to prove that $\mathcal{G}(n) = \langle X, Y \rangle$ it suffices to prove that $A_1$ lies in the subgroup $\langle X, Y \rangle$. As $\mathbb{F}_2$-vector spaces, let $\{e_1, \ldots, e_n\}$ be a basis for $X$ and $\{f_1, \ldots, f_n\}$ be a basis for $Y$ so that $\{e_i \otimes f_j \mid 1 \leq i, j \leq n\}$ is a basis for $X \otimes Y$. Since $A_1 \in X \otimes Y$, there exist $c_{ij} \in \{0, 1\}$, for $1 \leq i, j \leq n$, such that

$$A_1 = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} (e_i \otimes f_j).$$

By part (1), $X \otimes Y$ is an abelian subgroup of $\mathcal{G}(n)$, and hence, by the definition of multiplication (3), we can rewrite the above summation as

$$A_1 = \prod_{i=1}^{n} \prod_{j=1}^{n} (e_i \otimes f_j)^{c_{ij}}.$$

Again using (3), $(e_i f_j)^2 = (e_i + f_j)^2 = e_i \otimes f_j$, and hence $e_i \otimes f_j \in \langle X, Y \rangle$, for each $i, j$. It then follows from the product expression for $A_1$ displayed above that $A_1 \in \langle X, Y \rangle$. Therefore $\mathcal{G}(n) = \langle X, Y \rangle$, and part (2) is proved.

(3) Recalling the form for the inverse of an element of $\mathcal{G}(n)$ in Equation (4), we have

$$[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2 = (g_1 + x_1 \otimes y_1)(g_2 + x_2 \otimes y_2)g_1 g_2$$

$$= (g_1 + g_2 + x_1 \otimes y_1 + x_2 \otimes y_2 + x_2 \otimes y_1)(g_1 + g_2 + x_2 \otimes y_1)$$

$$= x_1 \otimes y_1 + x_2 \otimes y_2 + (x_1 + x_2) \otimes (y_1 + y_2)$$

$$= x_1 \otimes y_2 + x_2 \otimes y_1.$$ 

Thus $\mathcal{G}(n)' \leq X \otimes Y$. Moreover, in terms of the basis for $X \otimes Y$ introduced in (2) above, we have $[e_i, f_j] = (e_i + f_j)^2 = e_i \otimes f_j$ for each $i, j$. Hence $X \otimes Y$ is generated by the commutators $[e_i, f_j]$, for $1 \leq i, j \leq n$, so $\mathcal{G}(n)' = X \otimes Y$. In the displayed equation above, choosing $g_2 = A_2 \in X \otimes Y = \mathcal{G}(n)'$ leads to $[g_1, g_2] = 0$ for all $g_1 \in \mathcal{G}(n)$, and hence $\mathcal{G}(n)' \leq Z(\mathcal{G}(n))$. On the other hand, if $g_2 \in \mathcal{G}(n) \setminus \mathcal{G}(n)'$, then at least one of $x_2, y_2 \neq 0$. Without loss of generality, $x_2 = \sum_{i=1}^{n} a_i e_i$ with, say, $a_\ell = 1$. Then choosing $g_1 = y_1 = f_\ell$, we have $[g_1, g_2] = x_2 \times y_1 = \sum_{i=1}^{n} a_i e_i \otimes f_\ell$, which is nonzero since $a_\ell \neq 0$. Thus $g_2 \not\in Z(\mathcal{G}(n))$, and hence $\mathcal{G}(n)' = Z(\mathcal{G}(n))$, proving part (3).

(4) Finally, working in the quotient $\mathcal{G}(n)/\mathcal{G}(n)'$, $g_1(X \otimes Y)g_2(X \otimes Y) = (x_1 + x_2 + y_1 + y_2)(X \otimes Y) = g_2(X \otimes Y)g_1(X \otimes Y)$, and $(g_1(X \otimes Y))^2 = X \otimes Y$, so $\mathcal{G}(n)/\mathcal{G}(n)'$ is an elementary abelian 2-group. By Lemma 5.2, $|\mathcal{G}(n)| = 2^{n^2 + 2n}$, and by part (3), $|\mathcal{G}(n)'| = 2^{n^2}$, so part (4) follows. The last assertion follows from Definition 1.1, completing the proof. □

Now we prove the isomorphism of the groups $\mathcal{G}(n)$ and $\tilde{\mathcal{G}}(n)$, and we note that the assertions of Theorem 1.8 (1) then follow from Lemmas 5.3 and 5.4.
Lemma 5.4 For each $n \geq 2$, the group $G(n)$ in Definition 5.1 is isomorphic to the group $	ilde{G}(n)$ given by the presentation in Definition 1.7.

Proof We claim that $G(n)$, as defined in Definition 5.1, satisfies all the relations in Definition 1.7. Let $e_1, \ldots, e_n$ be a basis for $X$ and $f_1, \ldots, f_n$ be a basis for $Y$. By Lemma 5.3 (2) we have that $G(n) = \langle e_1, \ldots, e_n, f_1, \ldots, f_n \rangle$. Since the multiplication defined in Equation (3) reduces to vector addition when restricted to elements of $X$, or, respectively, when restricted to elements of $Y$, we have $e_i^2 = 0$, $f_i^2 = 0$, $[e_i, e_j] = 0$ and $[f_i, f_j] = 0$ for all $i, j \in \{1, \ldots, n\}$. Also, for each $i, j \in \{1, \ldots, n\}$ we have $[e_i, f_j] = (e_i + f_j)^2 = e_i \otimes f_j$. It then follows from Equation (3) that $[e_i, f_j]^2 = 0$ and that $[e_i, f_j]$ commutes with $e_k$ and $f_k$ for all $k \in \{1, \ldots, n\}$. This proves the claim.

It follows that $G(n)$ is isomorphic to a quotient of the group $	ilde{G}(n)$ in Definition 1.7. Moreover, it follows from the relations in Definition 1.7 that every element of the group $	ilde{G}(n)$ may be written as a product

$$\left( \prod_{i=1}^{n} x_i^{a_i} \right) \left( \prod_{i=1}^{n} y_i^{b_i} \right) \left( \prod_{i=1}^{n} \prod_{j=1}^{n} [x_i, y_j]^{c_{ij}} \right),$$

for some $a_i, b_i, c_{ij} \in \{0, 1\}$, where $i, j \in \{1, \ldots, n\}$. The number of distinct tuples of parameters $a_i, b_i, c_{ij}$ is $2^{n^2+2n}$, and hence $\tilde{G}(n)$ has order at most $2^{n^2+2n}$. Since by Lemma 5.3 (4), $|G(n)| = 2^{n^2+2n}$, we conclude that $|G(n)| = |\tilde{G}(n)|$ and hence that $G(n) \cong \tilde{G}(n)$, completing the proof.

Our next task is to determine the subgroup of $\text{Aut}(G(n))$ that leaves $X \cup Y$ invariant.

Lemma 5.5 Let $X$, $Y$, $G(n)$ be as in Definition 5.1 and let $S = (X \cup Y) \setminus \{0\}$. Then the subgroup $\text{Aut}(G(n), S)$ of $\text{Aut}(G(n))$ stabilising $S$ is $(\text{Aut}(X) \times \text{Aut}(Y)) : C_2 \cong \text{GL}_n(2) \wr S_2$.

Proof First, by Lemma 5.3, $X \cong Y \cong C_2^n$, and hence $\text{Aut}(X) \cong \text{Aut}(Y) \cong \text{GL}_n(2)$. As $\mathbb{F}_2$-vector spaces, let $\{e_1, \ldots, e_n\}$ be a basis for $X$ and let $\{f_1, \ldots, f_n\}$ be a basis for $Y$ so that $\{e_i \otimes f_j \mid 1 \leq i, j \leq n\}$ is a basis for $X \otimes Y$. Let

$$g_1 = x_1 + y_1 + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(e_i \otimes f_j) \quad \text{and} \quad g_2 = x_2 + y_2 + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}(e_i \otimes f_j),$$

where $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $a_{ij}, b_{ij} \in \mathbb{F}_2$ for $1 \leq i, j \leq n$. For $\phi \in \text{Aut}(X)$ we define

$$g_1^\phi = x_1^\phi + y_1 + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(e_i^\phi \otimes f_j).$$

Since $\phi$ is an invertible linear transformation of $X$, it follows that $\phi$ defines a bijection on $X \oplus Y \oplus (X \otimes Y)$. Also,

$$g_1^\phi g_2^\phi = \left( x_1^\phi + y_1 + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(e_i^\phi \otimes f_j) \right) \left( x_2^\phi + y_2 + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}(e_i^\phi \otimes f_j) \right)$$

$$= x_1^\phi + x_2^\phi + y_1 + y_2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} + b_{ij})(e_i^\phi \otimes f_j) + x_2^\phi \otimes y_1,$$
while
\[(g_1g_2)^\phi = \left( x_1 + x_2 + y_1 + y_2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} + b_{ij})(e_i \otimes f_j) + x_2 \otimes y_1 \right)^\phi \]
\[= x_1^\phi + x_2^\phi + y_1 + y_2 + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} + b_{ij})(e_i^\phi \otimes f_j) + x_2^\phi \otimes y_1. \]

Thus \(g_1^\phi g_2^\phi = (g_1g_2)^\phi\) and so \(\phi \in \text{Aut}(G(n))\) and \(\phi\) leaves \(X \cup Y\) invariant. Similarly if \(\phi \in \text{Aut}(Y)\) then the map
\[g_1^\phi = x_1 + y_1^\phi + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(e_i \otimes f_j) \]
defines an element of \(\text{Aut}(G(n))\) and \(\phi\) leaving \(X \cup Y\) invariant. Thus \(\text{Aut}(X) \times \text{Aut}(Y) \leq \text{Aut}(G(n), S)\). Further, for
\[g = \sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{n} b_i f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}(e_i \otimes f_j) \in G(n),\]
define a map \(\delta : G(n) \to G(n)\) by
\[g^\delta := \sum_{i=1}^{n} b_i e_i + \sum_{i=1}^{n} a_i f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_j b_i + c_{ji})(e_i \otimes f_j).\]
Observe that \(\delta\) is the composition of the map that interchanges \(g\) and \(g^{-1}\) (see Equation (4)) with the map that interchanges \(e_i\) and \(f_i\), for \(1 \leq i \leq n\). It follows that \(\delta\) is a bijection, and also that \(S^\delta = S\). To see that \(\delta\) is an automorphism consider \(g\) as above and
\[g' = \sum_{i=1}^{n} a_i' e_i + \sum_{i=1}^{n} b_i' f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}'(e_i \otimes f_j),\]
and compute as follows:
\[g^\delta(g')^\delta = \left( \sum_{i=1}^{n} b_i e_i + \sum_{i=1}^{n} a_i f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_j b_i + c_{ji})(e_i \otimes f_j) \right) \]
\[\left( \sum_{i=1}^{n} b_i' e_i + \sum_{i=1}^{n} a_i' f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_j' b_i' + c_{ji}') (e_i \otimes f_j) \right) \]
\[= \sum_{i=1}^{n} (b_i + b_i') e_i + \sum_{i=1}^{n} (a_i + a_i') f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_j b_i + a_j b_i' + a_j' b_i + c_{ji} + c_{ji}') (e_i \otimes f_j).\]

and
\[(gg')^\delta = \left( \sum_{i=1}^{n} (a_i + a_i') e_i + \sum_{i=1}^{n} (b_i + b_i') f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i' b_j + c_{ij} + c_{ij}') (e_i \otimes f_j) \right) \]
\[= \sum_{i=1}^{n} (b_i + b_i') e_i + \sum_{i=1}^{n} (a_i + a_i') f_i + \sum_{i=1}^{n} \sum_{j=1}^{n} (a_j b_i + a_j b_i' + a_j' b_i + c_{ji} + c_{ji}') (e_i \otimes f_j).\]
Thus $\delta \in \text{Aut}(\mathcal{G}(n), S)$, and hence we have proved that $\text{Aut}(\mathcal{G}(n), S)$ contains $(\text{Aut}(X) \times \text{Aut}(Y)) : C_2$. However, by Lemma 4.2 (3), $\text{Aut}(\mathcal{G}(n), S)$ is a subgroup of $(\text{Aut}(X) \times \text{Aut}(Y)) : C_2$, and hence equality holds, completing the proof. \hfill $\square$

Now we investigate the graphs $\Gamma(n) := C(\mathcal{G}(n), X, Y)$ and $\Sigma(n) := \Sigma(\mathcal{G}(n), X, Y)$ for $\mathcal{G}(n)$, as defined in Definition 1.2. Figure 1 shows a distance diagram for the smallest of these graphs $\Gamma(2)$. We note that Theorem 1.8 (2) follows from Lemma 5.6 (2).

**Lemma 5.6** Let $X$, $Y$, $\mathcal{G}(n)$ be as in Definition 5.1, let $\Gamma(n) = C(\mathcal{G}(n), X, Y)$, and $\Sigma(n) = \Sigma(\mathcal{G}(n), X, Y)$, as in Definition 1.2, and let $S = (X \cup Y) \setminus \{0\}$ and $N = \mathcal{G}(n)'$. Then

1. $\text{Aut}(\Gamma(n)) = \text{Aut}(\Sigma(n)) = \mathcal{G}(n) : \text{Aut}(\mathcal{G}(n), S) \cong \mathcal{G}(n) : (\text{GL}_n(2) \wr S_2)$.
2. $\Sigma(n)$ is a 2-arc-transitive graph of order $2n^2 + n + 1$, and an $N$-normal cover of $K_{2^n, 2^n}$, which is $\text{Aut}(\Sigma(n))/N$-edge-affine, and in particular is $\mathcal{G}(n)/N$-edge-affine.

**Remark 5.7** In the following proof we make use of a Magma [1] computation in the cases $n = 2$ or 3, which we now describe. First, the group $\tilde{\mathcal{G}}(n)$, with $n = 2$ or 3, is input in the category GrpFP via the presentation given in Definition 1.7. Next, the $\text{pQuotient}$ command is used to construct the largest 2-quotient $H$ of $\tilde{\mathcal{G}}(n)$ having lower exponent-2 class at most 100 as group in the category GrpPC. Comparing the orders of these groups, we find $|\tilde{\mathcal{G}}(n)| = |H|$, so that $\tilde{\mathcal{G}}(n) \cong H$. We then construct a graph isomorphic to $\Gamma(n)$ as a Cayley graph on $H$. Computing the order of the full automorphism of this graph then shows that $\text{Aut}(\Gamma(n)) = A = \mathcal{G}(n) : \text{Aut}(\mathcal{G}(n), S)$. We have made available the Magma programs in the Appendix of this paper.

**Proof** Write $\Gamma = \Gamma(n)$ and $\Sigma = \Sigma(n)$. By Lemma 5.5, the automorphism group $\text{Aut}(\Gamma)$ contains as a subgroup

$$A := \mathcal{G}(n) : \text{Aut}(\mathcal{G}(n), S) \cong \mathcal{G}(n) : (\text{GL}_n(2) \wr S_2).$$

Also $\Gamma$ is $A$-edge-transitive, and $\mathcal{G}(n)$ acts regularly on $E(\Sigma)$ and has two orbits on $V(\Sigma)$, by Lemma 4.2 (1). By Lemma 4.1 (4), the map $\varphi : z \to \{Xz, Yz\}$ defines an isomorphism from $\Gamma$ to the line graph of $\Sigma$, and by Lemma 4.1 (5), $\text{Aut}(\Gamma) = \text{Aut}(\Sigma)$. By Lemma 5.3 (3), the 2-group $N := \mathcal{G}(n)' = Z(\mathcal{G}(n))$ is both the derived subgroup and the centre of $\mathcal{G}(n)$. Furthermore, by Lemma 4.2 (2), the $N$-normal quotient $\Sigma_N$ is isomorphic to $K_{2^n, 2^n}$. Since (as noted above) $A$ is edge-transitive on $\Gamma(n)$, it follows from Lemma 4.2 (4) that $A$ is 2-arc-transitive on $\Sigma$, and then from Lemma 2.2 (3) that $A/N$ is 2-arc-transitive on $\Sigma_N$. By Lemma 4.2 (2) $\Sigma$ is an $N$-normal cover of $K_{2^n, 2^n}$. Hence $\Sigma$ has order $2n^2 + 1$ and $|N|$. By Lemma 5.3 (4), we have $|N| = 2n^2$. Thus $\Sigma$ has order $2n^2 + n + 1$. We now apply Lemma 2.6 to $\Sigma$ with 2-arc-transitive group $A$. It follows from Lemma 5.5 that the kernel of the action of the stabiliser $A_X$ on $\{Yx : x \in X\}$ contains $\text{Aut}(Y)$ and in particular is nontrivial, and hence, the last assertion of Lemma 2.6 implies that $N \leq \text{Aut}(\Sigma)$.

For any $T$ such that $N \leq T \leq \text{Aut}(\Sigma)$, let $T^+/N$ denote the subgroup of $T/N$ stabilising both parts of the bipartition of $\Sigma_N \cong K_{2^n, 2^n}$. By Lemma 2.2 (2), the kernel of the action of $\text{Aut}(\Sigma)$ on $\Sigma_N$ is $N$ so $\mathcal{G}(n)/N \leq \text{Aut}(\Sigma)^+/N \leq \text{Aut}(\Sigma_N)$, and also, by Lemma 5.3 (4),
By Lemma 4.2 (1), $G(n)$ is regular on $E(\Sigma)$, and hence $G(n)/N$ is transitive on $E(\Sigma_N)$; moreover, since $G(n)/N$ is abelian the latter action is regular. Since $G(n)$ has two orbits on $V(\Sigma)$, it follows that $G(n)/N$ has two orbits on $V(\Sigma_N)$, and since $G(n)/N$ is edge-regular, it follows that $\Sigma_N$ is $A/N$-edge-affine. Note that $\text{Aut}(\Sigma_N) = S_{2^n} \wr S_2$. If $\text{Aut}(\Sigma)/N$ has socle $C_2^{2^n}$, then $A/N \leq \text{Aut}(\Sigma)/N \leq C_2^{2^n} : (\text{GL}(n,2) \wr S_2) \cong A/N$, and it follows that $\text{Aut}(\Sigma)/N = A/N$ and $\text{Aut}(\Sigma) = A$. In this case $\Sigma_N$ is $\text{Aut}(\Sigma)/N$-edge-affine and both parts of the lemma are proved in this case.

For the small values $n = 2$ or 3, we check, using Magma [1], that $A = \text{Aut}(\Gamma(n))$, and hence the lemma is proved for $n = 2$ or 3 (see Remark 5.7 for a description of these computations). Now assume that $n \geq 4$, and assume, for a contradiction, that the socle of $\text{Aut}(\Sigma_N)/N$ is not $C_2^{2^n}$. We will apply Proposition 3.1 to $\Sigma_N \cong K_{2^n,2^{2n}}$ and $\text{Aut}(\Sigma)/N$. Since $n \geq 4$, case (4) of Proposition 3.1 does not hold. In case (2) of Proposition 3.1, the group $\text{Aut}(\Sigma)/N$ does not contain a subgroup isomorphic to $C_2^{2^n}$, so this case does not hold either. Thus case (1) of Proposition 3.1 holds, so we have normal subgroups $M, T_1, T_2$ of $\text{Aut}(\Sigma)N$, all containing $N$, such that $M/N = \text{soc}(\text{Aut}(\Sigma)/N) = T_1/N \times T_2/N \cong A_{2^n} \times A_{2^n}$. We showed above that $G(n)/N$ is regular on $E(\Sigma_N)$ and has two vertex-Orbits. It follows that $G(n)/N = (G(n)/N)^U \times (G(n)/N)^W$, where $U, W$ are the two parts of the bipartition of $\Sigma_N$, and $(G(n)/N)^U$ and $(G(n)/N)^W$ are the permutation groups induced by $G(n)/N$ on $U$ and $W$, respectively. Moreover, $(G(n)/N)^U, (G(n)/N)^W$ is regular on $U, W$, respectively. Since $n \geq 4$, the permutation induced on $U$ by a nontrivial element of $(G(n)/N)^U$ is a product of $2^{n-1}$ cycles of length 2, and hence is an even permutation of $U$. Similarly every element of $(G(n)/N)^W$ induces an even permutation of $W$. Thus $G(n)/N \leq M/N \cong A_{2^n} \times A_{2^n}$, and the normaliser of $G(n)/N$ in $M/N$ is isomorphic to $C_2^{2^n} : (\text{GL}(n,2) \times \text{GL}(n,2))$. It follows that the normaliser of $G(n)/N$ in $M/N$ is $A^+/N = (G(n) : (\text{Aut}(X) \times \text{Aut}(Y)))/N$. We may assume that $\text{Aut}(X)/N/N \leq T_1/N$ and $\text{Aut}(Y)/N/N \leq T_2/N$. Now $N \cong C_2^{n_2}$, and so $N \leq C_M(N)$ and $M/C_M(N) \leq \text{Aut}(N) \cong \text{GL}(n^2,2)$. We saw in the proof of Lemma 5.5 that both $\text{Aut}(X)$ and $\text{Aut}(Y)$ act nontrivially and faithfully on $N$, so in particular neither $T_1/N$ nor $T_2/N$ is contained in $C_M(N)$. Hence $C_M(N)/N$ is a proper normal subgroup of $M/N \cong A_{2^n} \times A_{2^n}$, containing neither simple direct factor. It follows that $C_M(N)/N = 1$. Hence $M/C_M(N) = M/N \cong A_{2^n} \times A_{2^n}$. Since $M/C_M(N) \leq \text{GL}(n^2,2)$ and since $2^n \geq 16$, it follows from [13, Proposition 5.3.2] that $n^2 \geq 2^n - 2$, which implies that $n = 4$. However $|M/N| = |A_{16}|^2$ is divisible by $13^2$, while $13^2$ does not divide $|\text{GL}(16,2)|$ (as the least $j$ such that 13 divides $2^j - 1$ is $j = 12$), and hence $13^2$ does not divide $|M/C_M(N)|$. This contradiction completes the proof.

We next prove various symmetry properties of $\Gamma(n)$ which complete the proof of Theorem 1.8 (3).

**Lemma 5.8** Let $X, Y, G(n)$ be as in Definition 5.1, let $\Gamma(n) = C(G(n), X, Y)$, as in Definition 1.2, and let $S = (X \cup Y) \setminus \{0\}$. Then $\Gamma(n)$ is a 2-geodesic-transitive normal Cayley graph; moreover $\Gamma(n)$ is 2-distance-transitive, but is neither 3-distance-transitive (and in particular not distance-transitive), nor 2-arc-transitive.

**Proof** By Definition 1.2, $\Gamma := \Gamma(n)$ is a Cayley graph, and by Lemma 5.6, $\text{Aut}(\Gamma) = G(n) : \text{Aut}(G(n), S) \cong G(n) : (\text{GL}(n,2) \wr S_2)$. Thus $\Gamma$ is a normal Cayley graph. Let $G = \text{Aut}(\Gamma)$. Since $G(n)$ acts transitively on $V(\Gamma)$, to prove the result it is sufficient to consider, for the
vertex 0 ∈ G(n) = V(Γ), the action of G_0 = Aut(G(n), S) on the vertices at distance one, two and three from 0. The set of vertices that are distance one from 0 is precisely Γ(0) = S, on which G_0 acts transitively, so Γ is 1-arc-transitive.

We now consider the set Γ(2) of vertices at distance two from 0. Let X_0 = X \ {0}, let Y_0 = Y \ {0}, let x, x′ be distinct elements of X_0 and let y, y′ be distinct elements of Y_0. Then, recalling that multiplication is as in Equation (3), xx′ ∈ X, yy′ ∈ Y, xy = x + y ∈ \G(n) \ (X \cup Y) and yx = x + y + x ⊗ y ∈ \G(n) \ (X \cup Y). Hence, the set of vertices at distance two from 0 is

\[ Γ(0) = \{x + y \mid x ∈ X_0, y ∈ Y_0\} \cup \{x + y + x ⊗ y \mid x ∈ X_0, y ∈ Y_0\}. \]

Let e_1, . . . , e_n be a basis for X, let f_1, . . . , f_n be a basis for Y, and let δ be as in the proof of Lemma 5.5, that is, δ is the composition of the map interchanging g and g^{-1} (for g ∈ \G(n), see Equation (4)) with the map interchanging e_i and f_i, for i ∈ {1, . . . , n}. Then, there exist σ and σ′ in Aut(X) × Aut(Y) such that (x+y)^σ = e_1 + f_1 and (x+y + x ⊗ y)^σ′ = e_1 + f_1 + e_1 ⊗ f_1. Since (e_1 + f_1)^δ = e_1 + f_1 + e_1 ⊗ f_1, we conclude that (x+y)^σ ⊗ σ′ = x + y + x ⊗ y. Hence Γ(0) is a G_0-orbit, and thus Γ is 2-distance-transitive. Considering the right multiplication action by elements of X ⊔ Y on various edges \{0, s\}, for s ∈ S, we see that

\[ Γ(x) = \{0\} \cup \{x′ : x′ ∈ X_0, x′ ≠ x\} \cup \{yx : y ∈ Y_0\} \]

and

\[ Γ(y) = \{0\} \cup \{y′ : y′ ∈ Y_0, y′ ≠ y\} \cup \{xy : x ∈ X_0\} \]

and hence the only vertex at distance one from both 0 and xy = x + y is y. It follows that Γ is 2-geodesic-transitive.

We now determine the set Γ(3) of vertices that are at distance three from 0, each such vertex is of the form sz for some s ∈ S, z ∈ Γ(2) (note that we obtain the edge \{z, sz\} by right-multiplying the edge \{0, s\} by the element z ∈ Γ(2)). We have the following cases, where x, x′ are distinct elements of X_0, y, y′ are distinct elements of Y_0; we set x″ = xx′ and y″ = yy′, and note that x″ ∉ \{0, x, x′\} and y″ ∉ \{0, y, y′\}:

\[ x(x + y) = y ∈ S, \]
\[ x(x + y + x ⊗ y) = y + x ⊗ y ∈ \G(n) \setminus (X \cup Y \cup Γ(2)), \]
\[ y(x + y) = x + x ⊗ y ∈ \G(n) \setminus (X \cup Y \cup Γ(2)), \]
\[ y(x + y + x ⊗ y) = x ∈ S, \]
\[ x′(x + y) = x″ + y ∈ Γ(2), \]
\[ x′(x + y + x ⊗ y) = x″ + y + x ⊗ y ∈ \G(n) \setminus (X \cup Y \cup Γ(2)), \]
\[ y′(x + y) = x + y″ + x ⊗ y′ ∈ \G(n) \setminus (X \cup Y \cup Γ(2)) \]
\[ y′(x + y + x ⊗ y) = x + y″ + x ⊗ y″ ∈ Γ(2). \]

Hence, the set of vertices at distance three from 0 is

\[ Γ(3) = \{y + x ⊗ y \mid x ∈ X_0, y ∈ Y_0\} \cup \{x + y + x ⊗ y′ \mid x ∈ X_0 ; y, y′ ∈ Y_0, y ≠ y′\} \]
\[ \cup \{x + x ⊗ y \mid x ∈ X_0, y ∈ Y_0\} \cup \{x + y + x′ ⊗ y \mid x, x′ ∈ X_0, x ≠ x′ ; y ∈ Y_0\}. \]
Now, each of the sets \( \{ y + x \otimes y \mid x \in X_0, y \in Y_0 \} \) and \( \{ x + x \otimes y \mid x \in X_0, y \in Y_0 \} \) are \((\text{Aut}(X) \times \text{Aut}(Y))\)-orbits, and these sets are interchanged by \( \delta \). Thus

\[
\{ y + x \otimes y \mid x \in X_0, y \in Y_0 \} \cup \{ x + x \otimes y \mid x \in X_0, y \in Y_0 \}
\]

is a \( G_0 \)-orbit. Hence \( G_0 \) does not act transitively on \( \Gamma_3(0) \) and thus \( \Gamma \) is not 3-distance-transitive. To see that \( \Gamma \) is not 2-arc-transitive, we observe that \((x,0,x')\) and \((x,0,y)\) are 2-arcs and that \( \{x,x'\} \) is an edge, while \( \{x,y\} \) is not, and hence there is no element of \( \text{Aut}(\Gamma) \) mapping \((x,0,x')\) to \((x,0,y)\). This completes the proof. \( \square \)

Finally we formalise the proof of Theorem 1.8.

**Proof of Theorem 1.8** As noted above, Theorem 1.8 (1) follows from Lemmas 5.3 and 5.4, while Theorem 1.8 (2) follows from Lemma 5.6 (2). Finally Theorem 1.8 (3) follows from Lemma 5.8.

**Appendix: Magma programs for proving Lemma 5.6 in case \( n = 2 \) or 3.**

**Appendix 1.** A function for constructing Cayley graphs:

```magma
Cay:=function(G,S);
  V:=g:g in G;
  E:=g,s*g:g in G,s in S;
  return Graph<V|E>;
end function;
```

**Appendix 2.** The case \( n = 2 \):

Input the group \( \tilde{G}(2) \):

\[
G<x1,x2,y1,y2>:=Group< x1,x2,y1,y2 \mid x1^2,x2^2,y1^2,y2^2, (x1,x2)=(y1,y2)=1, ((x1,y1),x1)=((x1,y1),x2)=((x1,y1),y1)=((x1,y1),y2)=1, ((x1,y2),x1)=((x1,y2),x2)=((x1,y2),y1)=((x1,y2),y2)=1, ((x2,y1),x1)=((x2,y1),x2)=((x2,y1),y1)=((x2,y1),y2)=1, ((x2,y2),x1)=((x2,y2),x2)=((x2,y2),y1)=((x2,y2),y2)=1>;
\]

Construct the largest 2-quotient group of \( \tilde{G}(2) \) having lower exponent-2 class at most 100 as group in the category GrpPC:

\[
G2,q:=pQuotient(G,2,100);
\]

26
Order of $G_2$ (The result shows that $|G_2| = \tilde{G}(2)$, and so $G_2 \cong \tilde{G}(2)$):

FactoredOrder(G2);

Construct the graph $\Gamma(2) = \text{Cay}(G_2, S)$:

\[
x_1 := x_1 @ q; \quad x_2 := x_2 @ q; \quad y_1 := y_1 @ q; \quad y_2 := y_2 @ q;
\]

\[
S := x_1, x_2, x_1 \ast x_2, y_1, y_2, y_1 \ast y_2;
\]

\[
\text{Gamma2} := \text{Cay}(G_2, S);
\]

Automorphism Group of $\Gamma(2)$ (The result shows that $|\text{Aut}(\Gamma(2))| = |G(2)||\text{GL}_2(2) \wr S_2|)$:

\[
A := \text{AutomorphismGroup}(\text{Gamma2});
\]

\[
\#A \text{ eq } (\#G_2) \ast (\#(\text{GL}_2(2)) \ast (\#(\text{GL}_2(2))) \ast 2;
\]

Appendix 3. The case $n = 3$:

Input group $\tilde{G}(3)$:

\[
G<x_1, x_2, x_3, y_1, y_2, y_3> := \text{Group}<x_1, x_2, x_3, y_1, y_2, y_3 \mid x_1 \cdot 2, x_2 \cdot 2, x_3 \cdot 2, y_1 \cdot 2, y_2 \cdot 2, y_3 \cdot 2, (x_1, x_2)=(x_1, x_3)=(x_2, x_3)=(y_1, y_2)=(y_1, y_3)=(y_2, y_3)=1, (x_1, y_1) \cdot 2=(x_1, y_2) \cdot 2=(x_1, y_3) \cdot 2=(x_2, y_1) \cdot 2=(x_2, y_2) \cdot 2=(x_2, y_3) \cdot 2=(x_3, y_1) \cdot 2= 1, (x_3, y_2) \cdot 2=(x_3, y_3) \cdot 2=1, ((x_1, y_1), x_1)=(x_1, y_1), x_2)=((x_1, y_1), x_3)=((x_1, y_1), y_1)=((x_1, y_1), y_2)=((x_1, y_1), y_3)=1, ((x_1, y_2), x_1)=((x_1, y_2), x_2)=((x_1, y_2), x_3)=((x_1, y_2), y_1)=((x_1, y_2), y_2)=((x_1, y_2), y_3)=1, ((x_1, y_3), x_1)=((x_1, y_3), x_2)=((x_1, y_3), x_3)=((x_1, y_3), y_1)=((x_1, y_3), y_2)=((x_1, y_3), y_3)=1, ((x_2, y_1), x_1)=((x_2, y_1), x_2)=((x_2, y_1), x_3)=((x_2, y_1), y_1)=((x_2, y_1), y_2)=((x_2, y_1), y_3)=1, ((x_2, y_2), x_1)=((x_2, y_2), x_2)=((x_2, y_2), x_3)=((x_2, y_2), y_1)=((x_2, y_2), y_2)=((x_2, y_2), y_3)=1, ((x_2, y_3), x_1)=((x_2, y_3), x_2)=((x_2, y_3), x_3)=((x_2, y_3), y_1)=((x_2, y_3), y_2)=((x_2, y_3), y_3)=1, ((x_3, y_1), x_1)=((x_3, y_1), x_2)=((x_3, y_1), x_3)=((x_3, y_1), y_1)=((x_3, y_1), y_2)=((x_3, y_1), y_3)=1, ((x_3, y_2), x_1)=((x_3, y_2), x_2)=((x_3, y_2), x_3)=((x_3, y_2), y_1)=((x_3, y_2), y_2)=((x_3, y_2), y_3)=1, ((x_3, y_3), x_1)=((x_3, y_3), x_2)=((x_3, y_3), x_3)=((x_3, y_3), y_1)=
\]

27
Construct the largest 2-quotient group of $\tilde{G}(3)$ having lower exponent-2 class at most 100 as group in the category GrpPC:

\[ G_3, q := \text{pQuotient}(G, 2, 100); \]

Order of $G_3$ (The result shows that $|G_3| = \tilde{G}(3)$, and so $G_3 \cong \tilde{G}(3)$):

\[ \text{FactoredOrder}(G_3); \]

Construct the graph $\Gamma(3) = \text{Cay}(G_3, S)$:

\[ x_1 := x_1@q; \ x_2 := x_2@q; \ x_3 := x_3@q; \ y_1 := y_1@q; \ y_2 := y_2@q; \ y_3 := y_3@q; \]
\[ X := \text{sub}<G_3| x_1, x_2, x_3>; \ Y := \text{sub}<G_3| y_1, y_2, y_3>; \]
\[ S := x : x \text{ in } X | x \neq G_3!1 \text{ join } y : y \text{ in } Y | y \neq G_3!1; \]
\[ \text{Gamma3} := \text{Cay}(G_3, S); \]

Automorphism Group of $\Gamma(3)$ (The result shows that $|\text{Aut}(\Gamma(3))| = |G(3)||\text{GL}_3(2) \wr S_2|)$:

\[ A := \text{AutomorphismGroup}(\text{Gamma3}); \]
\[ \#A \text{ eq } (\#G_3)*\#(\text{GL}(3,2))\#(\text{GL}(3,2))*2; \]

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