Abstract

We compare deformations of algebras to deformations of schemes in the setting of invariant theory. Our results generalize comparison theorems of Schlessinger and the second author for projective schemes. We consider deformations (abstract and embedded) of a scheme $X$ which is a good quotient of a quasi-affine scheme $X'$ by a linearly reductive group $G$ and compare them to invariant deformations of an affine $G$-scheme containing $X'$ as an open invariant subset. The main theorems give conditions for when the comparison morphisms are smooth or isomorphisms.

Introduction

Given a projective scheme $X$ defined by equations $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$, perturbing the equations in a flat manner so that they remain homogeneous induce deformations of $X$. In practice this is often the only way to construct examples of deformations. In more stringent terms we have a map between the degree 0 embedded deformations of the affine cone $C(X)$ and deformations of $X$ in $\mathbb{P}^n$. If we take into account trivial deformations we get a map to the deformations of $X$ as scheme. The question is when do we get all deformations this way.

To see precisely what is going on we should compare deformation functors on Artin rings. If $R = k[x_0, \ldots, x_n]$ and $S = R/(f_1, \ldots, f_m)$ then the above describes maps $\text{Def}^0_{S/R} \to \text{Hilb}_{X/\mathbb{P}^n}$ where $\text{Def}^0_{S/R}$ is the functor of degree 0 deformations of $S$ as $R$-algebra and $\text{Def}^0_S \to \text{Def}_X$ where $\text{Def}^0_S$ is the functor of degree 0 deformations of $S$ as $k$-algebra. Generalizing comparisons theorems of Schlessinger ([Sch71], [Sch73]) the second author gave in [Kle79] exact conditions for when these maps are isomorphisms. The object of this paper is to further generalize these to other situations where one can compare deformations of algebras to deformations of schemes.

The comparison map for projective schemes factors through deformations of the open subset of $C(X)$ where the vertex $\{0\}$ is removed. Thereafter one compares deformations to $X = (C(X) \setminus \{0\})/k^*$ via the quotient map. A natural question is if this can be generalized
to closed subschemes of toric varieties corresponding to ideals in the Cox ring. It turns out one can go even further, i.e. the quotient need not be by a quasi-torus.

In this paper we consider schemes $X$ that are good quotients of a quasi-affine scheme $X'$ by a linearly reductive group $G$. We assume that $X' \subseteq \text{Spec} \, S$, where $S$ is a finitely generated $k$-algebra and that $G$ acts on $S$ inducing the action on $X'$. We can then compare $\text{Def}^G_S$ to $\text{Def}_X$ where $\text{Def}^G_S$ is the functor of invariant deformations of $S$. The precise definitions of these settings are formulated with what we call $G$-quadruples - see Definition 2.1. Given such a situation we define in Definition 2.10 a $G$-subquadruple induced by a $G$-invariant ideal in $S$. This gives us a setting to compare local Hilbert functors with the deformations functor of an invariant surjection of $k$-algebras.

Linearly reductive groups have many properties coming from the Reynolds operator which make it possible to prove things, e.g. taking invariants is exact. Another reason to work with them is that the functor of invariant deformations is well defined and has the usual nice properties of a good deformation theory. This was proven by Rim.

Our main result on the local Hilbert functor is Theorem 3.3. The conditions are depth conditions along the complement of $X'$ in $\text{Spec} \, S$ and along the locus where the quotient map fails to be geometric and smooth. We state also corollaries for subschemes of toric varieties and weighted projective space.

For the abstract deformation functor $\text{Def}_X$ the results are not so exact due to the presence of infinitesimal automorphisms. It is not clear what the correct assumptions should be but we found it useful to use results of Altmann regarding rigidity of $\mathbb{Q}$-Gorenstein toric singularities as a guide. We get depth conditions as above but also along the locus where the isotropy groups are not finite. The main results for $\text{Def}_X$ are Theorem 4.19, Theorem 4.20 and Theorem 4.20. If $S$ is a regular ring then we get a rigidity statement as corollary.

An important ingredient is what we call a set of Euler derivations coming from the Lie algebra of $G$. This was done in discussions with Dmitry Timashev and made it possible to work with general groups and not just tori which we had originally studied. As a by-product we are able to give criteria for when there exists a generalized Euler sequence for the scheme $X$. See Definition 4.9 and Theorem 4.12.

We conclude with examples of how our results can be used to study deformations of toric varieties and subschemes of these. In particular we consider Calabi-Yau hypersurfaces in simplicial toric Fano varieties, first order deformations of toric singularities and reprove rigidity results of Altmann and of Totaro.

Throughout this paper $k$ is an algebraically closed field, in Section 4 we assume characteristic 0 and in Section 5 that $k = \mathbb{C}$.

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1 Preliminaries

1.1 Cotangent cohomology

To fix notation we give a short description of the cotangent modules and sheaves. Given a ring \( R \) and an \( R \)-algebra \( S \) there is a complex of free \( S \) modules; the cotangent complex \( L_{S/R}^\bullet \). See e.g. [And74, p. 34] for a definition. For an \( S \) module \( M \) we get the cotangent cohomology modules \( T^i(S/R; M) = H^i(\text{Hom}_S(L_{S/R}^i, M)) \). If \( R \) is the ground field we abbreviate \( T^i(S/R; M) = T^i_R(M) \) and \( T^i_S(S) = T^i_S = T^i_X \) if \( X = \text{Spec} \, S \).

If \( X \) is a scheme we may globalise these as follows. If \( \mathcal{R} \) is a sheaf of rings on \( X \) and \( S \) an \( \mathcal{R} \) algebra we set \( L^\bullet_{S/R} \) to be the complex of sheaves associated with the presheaves \( U \mapsto L^{S(U)/R(U)} \). Let \( \mathcal{F} \) be an \( S \) module. We get the cotangent cohomology sheaves \( T^i(S/R; \mathcal{F}) \) as the cohomology sheaves of \( \text{Hom}_S(L^i_{S/R}, \mathcal{F}) \) and the cotangent cohomology groups \( T^i(S/R; \mathcal{F}) \) as the cohomology of \( \text{Hom}_S(L^i_{S/R}, \mathcal{F}) \).

Because of the functoriality of these constructions we have \( T^i(S/R; \mathcal{F}) \) as the sheaf associated to the presheaf \( U \mapsto T^i(S(U)/\mathcal{R}(U); \mathcal{F}(U)) \) and \( T^i(S/R; \mathcal{F}) \) as the hypercohomology of \( \text{Hom}_S(L^i_{S/R}, \mathcal{F}) \). In particular there is a “local-global” spectral sequence

\[
H^p(X, T^q(S/R; \mathcal{F})) \Rightarrow T^n(S/R; \mathcal{F}).
\]  

(1.1)

If \( S \) is the structure sheaf \( O_X \) and \( \mathcal{R} \) corresponds to the ground field, then we abbreviate as above to \( T^i_X(\mathcal{F}) \).

The properties of the cotangent cohomology we use, e.g. the Zariski-Jacobi sequence and flat base change results, may be found in [And74]. We include here one result that does not seem to be well known. Let \( Z \subseteq X \) be a closed subscheme, then following Laudal [Lau79, 3.2.10] one may define cotangent cohomology with support in \( Z \) denoted \( T^i_Z(O_X/\mathcal{R}; \mathcal{F}) \). If \( Z \subseteq X = \text{Spec} \, S \) is given by \( V(I) \), we write \( T^i_I(S/R; S) = T^i_Z(O_X/\mathcal{R}; O_X) \).

**Theorem 1.1.** [Lau79, Theorem 3.2.11] There is a long exact sequence

\[
\cdots \rightarrow T^i_Z(O_X/\mathcal{R}; \mathcal{F}) \rightarrow T^i(O_X/\mathcal{R}; \mathcal{F}) \rightarrow T^i(O_X/Z/\mathcal{R}; \mathcal{F}) \rightarrow T^{i+1}_Z(O_X/\mathcal{R}; \mathcal{F}) \rightarrow \cdots
\]

and a spectral sequence yielding

\[
T^p(O_X/\mathcal{R}; H^q_Z(\mathcal{F})) \Rightarrow T^p_Z(O_X/\mathcal{R}; \mathcal{F}).
\]

1.2 Deformation functors

The deformation theory we use in this paper is described in various degrees of generality and readability in [LS67], [Sch68], [Ill71], [Lau79], [Ser06] and [Har10]. For a slightly different, but applicable, newer approach see e.g. [FM98]. For the functor of invariant deformations see [Rim80].

Let \( \mathcal{C} \) be the category of Artin local \( k \)-algebras with residue field \( k \). We list here the deformation functors on \( \mathcal{C} \) of interest to us. We denote the functor of deformations of
a scheme $X$ by $\text{Def}_X$. The functor of embedded deformations of a subscheme $X \subset Y$ is denoted $\text{Hilb}_{X/Y}$ and called the local Hilbert functor. The deformations of an $R$-algebra $S$ is denoted $\text{Def}_{S/R}$ and if $R = k$ we simply write $\text{Def}_S$.

If $G$ is an algebraic group acting on a scheme $X$ then it also acts on the set of deformations over $A$. If $\sigma \in G$ then it acts by $X \rightarrow X^{\sigma}$.

If $G$ is linearly reductive then there is a well behaved sub-deformation functor of invariant deformations $\text{Def}^G_{X}$. Similarly if $R \rightarrow S$ is equivariant for a linearly reductive group $G$ then there is a subfunctor $\text{Def}^G_{S/R}$ of invariant deformations. If $G$ is a quasi-torus so that the action corresponds to a grading by an abelian group $C$ then we often write $\text{Def}_0^{S/R}$ for the degree 0 $\in C$ deformations instead of $\text{Def}^G_{S/R}$.

The tangent and obstruction spaces for $\text{Def}_X$ are $T^i X = T^i(O_X/k; O_X)$ for $i = 1$ and 2. If $f : X \rightarrow Y$ is a closed embedding then the tangent and obstruction spaces for $\text{Hilb}_{X/Y}$ are $T^i(O_X/f^{-1}O_Y; O_X)$ for $i = 1$ and 2. The tangent and obstruction spaces for $\text{Def}_{S/R}$ are $T^i(S/R; S)$ for $i = 1$ and 2 and finally the tangent and obstruction spaces for $\text{Def}^G_{S/R}$ are $T^i(S/R; S)^G$ for $i = 1$ and 2. If $D$ is one of these deformation functors let $T^i_D$, $i = 1, 2$, denote the corresponding tangent and obstruction space.

Recall that a morphism $F \rightarrow G$ of functors is smooth if for any surjection $B \rightarrow A$ in $\mathcal{C}$, the morphism

$$F(B) \rightarrow F(A) \times_{G(A)} G(B)$$

is surjective. A functorial map of deformation functors $D \rightarrow D'$ induces maps $T^i_D \rightarrow T^i_{D'}$. We use throughout the standard criteria for smoothness, namely that $T^i_D \rightarrow T^i_{D'}$ is surjective and $T^2_D \rightarrow T^2_{D'}$ is injective.

If $T^1_D \rightarrow T^1_{D'}$ is an isomorphism and $T^2_D \rightarrow T^2_{D'}$ is injective it is not necessarily true that $D \rightarrow D'$ is an isomorphism. This is true if $D$ and $D'$ satisfy Schlessinger’s condition $H_4$, see [Sch68, 2.11 and 2.15]. If $X \subset Y$ is a closed subscheme then $\text{Hilb}_{X/Y}$ satisfies $H_4$.

In general $\text{Def}_X$ does not, but in our case we will not only have an isomorphism at the tangent level but also surjectivity of infinitesimal automorphisms allowing us to state that the functors we compare are isomorphic (Lemma [L1]).

## 2 $G$-quadruples

We consider now a standard situation in invariant theory. Let $G$ be a linearly reductive algebraic group acting on a finitely generated $k$-algebra $S$. Let $J \subset S$ be a $G$-invariant radical ideal such that if $X' = \text{Spec } S \setminus V(J)$ then there exists a good quotient $\pi : X' \rightarrow X = X'/G$. We will also need that there exists a closed (possibly empty) $Z \subset X$ such
that $\pi : \pi^{-1}(X \setminus Z) \to X \setminus Z$ is a geometric quotient and smooth of relative dimension $\dim G$. Since all of our results are stated with conditions on such $X, Z, S$ and $J$ it is convenient to give this situation a name for the sake of future reference.

**Definition 2.1.** Let $G$ be a linearly reductive algebraic group. A $G$-quadruple is $(X, Z, S, J)$ where $X$ is a noetherian $k$-scheme, $Z$ is a (possibly empty) closed subset, $S$ is a finitely generated $k$-algebra on which $G$ acts and $J \subseteq S$ is a $G$ invariant radical ideal with the following properties. If $X' = \text{Spec} S \setminus V(J)$ and $U = X \setminus Z$ then there is a commutative diagram

$$
\begin{array}{ccc}
U' & \longrightarrow & X' \\
\pi_{|U'} & \downarrow & \pi \\
U & \longleftarrow & X
\end{array}
$$

where $U' = \pi^{-1}(U)$ and

- $\pi$ is a good quotient of $X'$ by the action $G$
- $\pi_{|U'}$ is a geometric quotient and smooth of relative dimension $\dim G$
- $\text{depth}_J S \geq 1$.

The sheaf of algebras $S = \pi_* \mathcal{O}_{X'}$ is called the ferry of the $G$-quadruple.

**Remark.** A $G$-quadruple may be constructed using Geometric Invariant Theory. (See [MFK94], all though the presentation in [Muk03] applies more directly to our situation.) Starting with the action of $G$ on a reduced and irreducible affine scheme $Y = \text{Spec} S$ and a character $\chi$ of $G$, let $Y^{ss}$ and $Y^s$ be the semistable and stable points of $Y$ with respect to $\chi$. This gives a $G$-quadruple by letting $X' = Y^{ss}$, $J = I(Y \setminus Y^{ss})$, $X = Y^{ss} // G$, $U'$ equals the open subset of $Y^s$ where the quotient map is smooth, $U = U' / G$ and $Z = X \setminus U$.

Here are some examples.

**Example 2.2.** The Spec construction. If $X = \text{Spec} S$ and $G$ is trivial then $(X, \emptyset, S, (1))$ is a $G$-quadruple.

**Example 2.3.** The usual Proj construction. If $X = \text{Proj} S$ for $S$ finitely generated $\mathbb{Z}$-graded $k$-algebra generated in degree 1 with irrelevant ideal $m$ and $G = k^*$ then $(X, \emptyset, S, m)$ is a $G$-quadruple.

**Example 2.4.** The Proj construction in general. Let $S$ be a finitely generated $\mathbb{Z}_+$ graded $k$-algebra with $S_0 = k$ and set $X = \text{Proj} S$. Assume $S = R/I$ where $R = k[x_0, \ldots, x_n]$ is graded with $\deg x_i = q_i \in \mathbb{N}$. Let $m = (x_0, \ldots, x_n)$ and suppose $\text{depth}_m S \geq 1$. This defines an embedding of $X$ into the weighted projective space $\mathbb{P}(q) = \mathbb{P}(q_0, \ldots, q_n)$. 5
Following Miles Reid we say that $P(q)$ is well formed if no $n$ of the $q_0, \ldots, q_n$ have a common factor. Similarly we will say that the corresponding grading on $S$ is well formed. We may assume that all gradings are well formed, see e.g. [Del75, Proposition 1.3] or [Dol82 1.3.1].) Let $J'_k$ be the ideal of $R$ generated by $\{x_i : k \nmid q_i\}$ and set

$$J' = \bigcap_{k \geq 2} J'_k = \bigcap_{p \text{ prime}} J'_p.$$ 

The singular locus of the well formed $P(q)$ is $Z = V(J') \subseteq P(q)$.

Then $(X, X \cap Z, S, m)$ is a $G$-quadruple.

**Example 2.5.** The Cox construction for complex toric varieties. This is our main example and it includes the previous ones, see e.g. [Cox95] and [CLS11 Chapter 5]. To fix notation for the rest of this paper we recall the construction. Using the standard notation for toric geometry let $X = X_\Sigma$ be an $n$-dimensional toric variety given by a fan $\Sigma$ in $N_R$. We assume here for simplicity that $X_\Sigma$ has no torus factors, but this is not necessary for applying our results (see [CLS11 5.1.11]). Let $\Sigma(1) = \{\rho_1, \ldots, \rho_N\}$ be the set of rays of $\Sigma$ and let $v_i$ be the primitive generator of $\rho_i \cap N$. The divisor class group of $X$ is given by the exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{b} \mathbb{Z}^N \xrightarrow{\text{Cl}(X)} 0$$

where $b(u) = (\langle u, v_1 \rangle, \ldots, \langle u, v_N \rangle)$. The ring $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$ is naturally graded by the abelian group $\text{Cl}(X)$ and with this grading it is called the Cox ring or total homogeneous coordinate ring of $X$.

For each cone $\sigma$ in $\Sigma$ there is a monomial

$$\prod_{\rho \in \sigma} x_i \in S$$

and define $B(\Sigma)$ to be the ideal of $S$ generated by these. It is called the irrelevant ideal of the Cox ring. Let $Z(\Sigma) = V(B(\Sigma)) \subseteq \text{Spec } S$. Let $G$ be the quasi-torus $\text{Hom}_\mathbb{Z}(\text{Cl}(X_\Sigma), \mathbb{C}^*)$. The theorem of Cox states that $X$ is an almost geometric quotient for the action of $G$ on $\text{Spec } S \setminus Z(\Sigma)$.

Let $\text{Sing}(X_\Sigma)$ be the singular locus, i.e.

$$\text{Sing}(X_\Sigma) = \bigcup_{\sigma \notin \text{smooth}} V(\sigma)$$

where $V(\sigma)$ is the closure of the torus orbit corresponding to $\sigma$. The smooth locus is given by the subfan consisting of smooth cones in $\Sigma$. Then it follows from the construction (see e.g. [CLS11 Exercise 5.1.10]) that $(X_\Sigma, \text{Sing}(X_\Sigma), S, B(\Sigma))$ is a $G$-quadruple. Note that a $G$ invariant $S$-module $M$ is the same thing as a $\text{Cl}(X)$ graded $S$-module and that $M^G = M_0$, the degree $0 \in \text{Cl}(X)$ part of $M$. Moreover if $\alpha = [D] \in \text{Cl}(X)$ then $S_\alpha = \Gamma(X, \mathcal{O}_X(D))$. In particular the Ferry in this case is

$$\mathcal{S} = \bigoplus_{[D] \in \text{Cl}(X)} \mathcal{O}_X(D).$$
Example 2.6. Multigraded sheaves of algebras. All the examples above and also the Cox ring construction for general varieties fit into the following set up. Let $\Gamma$ be a finitely generated abelian group. Consider a reduced noetherian $k$-scheme $X$ with a sheaf of $\Gamma$-graded algebras $S$ which is locally of finite type. Assume $S_0 = \mathcal{O}_X$ and that $S = H^0(X, S)$ is a finitely generated $k$-algebra. Then $X' = \text{Spec}_X S \subseteq \text{Spec} S$ is quasi-affine. Set $J$ to be the radical ideal defining $\text{Spec} S \setminus X'$.

Let $G = \text{Spec} k[\Gamma]$ be the corresponding quasi-torus and assume $k$ and $G$ are so that $G$ is linearly reductive. Choose $Z \subset X$ so that $G$ acts freely on $\pi^{-1}(X \setminus Z)$. (In the Cox ring construction one could choose $Z = \text{Sing}(X)$.). Then $(X, Z, S, J)$ is a $G$-quadruple.

Example 2.7. Quotient singularities. Let $G \subseteq \text{GL}_n(\mathbb{C})$ be a finite group without pseudo-reflections. Then $(\mathbb{C}^n/G, \text{Sing}(\mathbb{C}^n/G)), \mathbb{C}[x_1, \ldots, x_n], (1)$ is a $G$-quadruple.

Example 2.8. Grassmannians. Consider the Grassmannian $G(d, n)$ and let $S$ be the polynomial ring on variables $x_{ij}$ for $i = 1, \ldots, d$ and $j = 1, \ldots, n$. Thus $G = \text{GL}_d$ acts on $S$ by viewing the variables as entries in a $d \times n$ matrix. Let $J$ be the ideal generated by the $\binom{n}{d}$ maximal minors in such a matrix. Then $(G(d, n), \emptyset, S, J)$ is a $G$-quadruple.

Example 2.9. Moduli spaces. Compactifications of moduli spaces constructed using GIT on affine schemes give $G$-quadruples as explained above. Among these are the moduli of smooth hypersurfaces in $\mathbb{P}^n$, vector bundles and quiver representations.

We may also define $G$-subquadruples.

Definition 2.10. Given a $G$-quadruple $(Y, W, R, J)$ and a $G$-invariant ideal $I \subseteq R$ we may construct a new $G$-quadruple as follows. Let $S = R/I$, set $\bar{J} = (I + J)/I$ and assume $\text{depth}_J S \geq 1$. Let $\pi : Y' \to Y$ be the good quotient as in Definition 2.1. The quasi-affine scheme $X' = \text{Spec} S \setminus V(\bar{J})$ is a closed $G$-invariant subset of $Y'$. Thus $\pi : X' \to X = \pi(X')$ is a good quotient where $X$ has structure sheaf $(\pi_* \mathcal{O}_{X'})^G$. It follows that $(X, X \cap W, S, J)$ is also a $G$-quadruple. We call it the $G$-subquadruple induced by $I$.

A $G$-subquadruple determines a diagram as in Figure 1 where $Z = X \cap W$, $U_X = X \cap U_Y$ and by definition $U'_X = \pi^{-1}(U_X)$. Let $S = \pi_* \mathcal{O}_{X'}$ be the ferry of $(X, Z, S, J)$. Let $f : X \to Y$ be the closed embedding. Let $I_W$ be the radical ideal of $\text{Spec} R \setminus U_Y$ and set $I_Z = (I_W + I)/I$. We will use this notation and the notation in the diagram throughout.

Now $\pi|_{U'_Y}$ is a geometric quotient, so in particular inverse images of points are $G$-orbits. Thus for any $G$-invariant subset $V \subseteq U'_Y$ we have $\pi^{-1}(\pi(V)) = V$. Thus $X' \cap U'_Y = \pi^{-1}(\pi(X' \cap U'_Y)) = \pi^{-1}(\pi(X') \cap \pi(U'_Y))$ since $x, y \in U'_Y$ and $\pi(x) = \pi(y)$ implies $x$ and $y$ are in the same $G$-orbit. This shows that $U'_X = X' \cap U'_Y = \text{Spec} S \setminus V(I_Z)$, so the above diagram is commutative. Note that the front vertical square is Cartesian but the back one needs not be.

3 Deformations of the embedded scheme - $\text{Hilb}^b_{X/Y}$

We begin with a general lemma.

7
Lemma 3.1. If in a Cartesian square of schemes

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \pi \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

the morphism \(\pi\) is flat and affine and \(f\) is a closed immersion then

\[
T^i(\mathcal{O}_X/f^{-1}\mathcal{O}_Y; \pi_*\mathcal{O}_{X'}) \cong T^i(\mathcal{O}_{X'}/f'^{-1}\mathcal{O}_{Y'}; \mathcal{O}_{X'})
\]

for all \(i \geq 0\).

Proof. We first compare the corresponding sheafs. Since all morphisms are affine and \(\pi\) is flat the diagram locally corresponds to a cocartesian diagram of rings

\[
\begin{array}{ccc}
A' & \leftarrow & B' \\
\uparrow & & \uparrow \pi^\# \\
A & \leftarrow & B
\end{array}
\]

with \(\pi^\#\) flat, so \(T^i(A/B; A') \cong T^i(A'/B'; A')\). By comparing values on open affine sets we get therefore

\[
T^q(\mathcal{O}_X/f^{-1}\mathcal{O}_Y; \pi_*\mathcal{O}_{X'}) \cong \pi_*T^q(\mathcal{O}_{X'}/f'^{-1}\mathcal{O}_{Y'}; \mathcal{O}_{X'})
\]

for all \(q \geq 0\). Since \(\pi\) is affine and being affine is preserved under base change, \(H^p(X, \pi_*\mathcal{F}) \cong H^p(X', \mathcal{F})\) for any quasi-coherent \(\mathcal{F}\). Thus all terms in the two spectral sequences (1.1) for the two cohomology groups are isomorphic and the result follows.

Lemma 3.2. Let \((X, Z, S, \bar{J})\) be a \(G\)-subquadruple of \((Y, W, R, J)\) induced by \(I \subseteq R\) with ferry \(S\). If \(\text{depth}_Z S \geq 2\) then
(i) \( \text{depth}_Z \mathcal{O}_X \geq 2 \)

(ii) \( H^0_\varnothing(S) = H^0_{I_Z}(S) = 0 \)

(iii) there is an isomorphism \( H^1_\varnothing(S) \cong H^1_{I_Z}(S) \).

**Proof.** Since taking invariants is exact the first statement follows from \( \mathcal{H}^i_Z(S)^G = \mathcal{H}^i_Z(\mathcal{O}_X) \).

We have \( \text{depth}_J S \geq 1 \) and \( I_Z \subseteq \bar{J} \) by the assumption that \((X, Z, S, \bar{J})\) is a \( G \)-subquadruple, so we get (ii) and (iii) by considering the commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & S & \rightarrow & H^0(X, S) & \rightarrow & H^1_\varnothing(S) & \rightarrow & 0 \\
0 & \rightarrow & H^0_{I_Z}(S) & \rightarrow & S & \rightarrow & H^0(U_X, S|_{U_X}) & \rightarrow & H^1_{I_Z}(S) & \rightarrow & 0
\end{array}
\]

where the middle vertical map is an isomorphism by the depth condition.

**Remark.** Note that in fact for any positive integer \( d \)

\[
\text{depth}_Z S \geq d \quad \text{and} \quad \text{depth}_J S \geq d \iff \text{depth}_{I_Z} S \geq d.
\]

We will use this fact throughout.

**Theorem 3.3.** If \((X, Z, S, \bar{J})\) is a \( G \)-subquadruple of \((Y, W, R, J)\) induced by \( I \subseteq R \) with ferry \( S \) and

(i) \( \text{depth}_Z S \geq 2 \)

(ii) \( \text{Hom}_R(I, H^1_\varnothing(S))^G = 0 \)

then \( \text{Def}^G_{S/R} \) and \( \text{Hilb}_{X/Y} \) are isomorphic deformation functors.

**Proof.** We will prove that both deformation functors are isomorphic to \( \text{Hilb}_{U_X/U_Y} \). Note first that if \( f : X \rightarrow Y \) is any closed embedding then \( T^0(\mathcal{O}_X/f^{-1}\mathcal{O}_Y; \mathcal{F}) = 0 \) for any \( \mathcal{O}_X \)-module \( \mathcal{F} \). Thus the spectral sequence in Theorem 1.1 yields \( T^1_{I_Z}(S/R; S) \cong \text{Hom}_R(I, H^1_\varnothing(S)) = 0 \) by Lemma 3.2. Furthermore this and Lemma 3.2 (iii) show that \( T^2_{I_Z}(S/R; S) \cong \text{Hom}_R(I, H^1_{I_Z}(S)) \cong \text{Hom}_R(I, H^1_\varnothing(S)) \).

We start with applying the long exact sequence in Theorem 1.1 and get

\[
0 \rightarrow T^1(S/R; S) \rightarrow T^1(\mathcal{O}_{U_X}/R; \mathcal{O}_{U_X}) \rightarrow T^2_{I_Z}(S/R; S) \rightarrow T^2(S/R; S) \rightarrow T^2(\mathcal{O}_{U_X}/R; \mathcal{O}_{U_X})
\]

when we write \( R \) for \((f')^{-1}i_Y^{-1}\mathcal{O}_{\text{Spec} R} \) where \( i_Y : U'_Y \hookrightarrow \text{Spec} R \). Now \( i_Y \) is flat so

\[
T^1(\mathcal{O}_{U_X}/R; \mathcal{O}_{U_X}) \cong T^1(\mathcal{O}_{U_X}/(f')^{-1}\mathcal{O}_{U'_Y}; \mathcal{O}_{U_X})
\]

and by Lemma 3.1

\[
T^1(\mathcal{O}_{U_X}/(f')^{-1}\mathcal{O}_{U'_Y}; \mathcal{O}_{U_X}) \cong T^1(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y}; \pi_*\mathcal{O}_{U_X}).
\]
Therefore after taking invariants and using the conditions we see that

\[ T^i(S/R; S)^G \to T^i(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y}; \mathcal{O}_{U_X}) \]

is an isomorphism for \( i = 1 \) and injective for \( i = 2 \). Thus \( \text{Def}^G_{S/R} \cong \text{Hilb}_{U_X/U_Y} \).

We have \( \mathcal{H}_2(\mathcal{O}_X) = 0 \) for \( i \leq 1 \) by Lemma 3.2 so again the spectral sequence and vanishing of \( T^0(\mathcal{O}_X/f^{-1}\mathcal{O}_Y; \mathcal{O}_X) \) implies \( T^2(\mathcal{O}_X/f^{-1}\mathcal{O}_Y; \mathcal{O}_X) = 0 \) for \( i \leq 2 \). We apply the long exact sequence in Theorem 1.1 again to get \( T^1(\mathcal{O}_X/f^{-1}\mathcal{O}_Y; \mathcal{O}_X) \cong T^1(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y}; \mathcal{O}_{U_X}) \)

and \( T^2(\mathcal{O}_X/f^{-1}\mathcal{O}_Y; \mathcal{O}_X) \to T^2(\mathcal{O}_{U_X}/f^{-1}\mathcal{O}_{U_Y}; \mathcal{O}_{U_X}) \) injective. Thus \( \text{Hilb}_{X/Y} \cong \text{Hilb}_{U_X/U_Y} \).

**Remark.** (i) From the proof we see that the slightly weaker assumptions depth_{T^2} S \geq 1, \( \text{Hom}_{\mathcal{O}_Y}(T, \mathcal{H}_2^G(\mathcal{O}_X)) = 0 \) also imply the result in the theorem.

(ii) If depth_{T^2} S \geq 2 then both assumptions in the theorem are satisfied.

(iii) If the theorem applies and \( T^2(S/R; S)^G = 0 \) then \( \text{Hilb}_{X/Y} \) is unobstructed even though \( H^1(X, N_X/Y) \) or \( H^2(X, T^2_X/Y) \) do not vanish. See Example 3.7.

In the case of the Cox construction for toric varieties (Example 2.5) we get a corollary which is a generalization of the comparison theorem as stated in [PS85], cf. [Kle79, Theorem 3.6 and Remark 3.7]. Proofs and full statements of the results we use here may be found in [CLS11, Chapter 5 and 6].

Let \( Y \) be a toric variety with Cox ring \( R \). Every closed subscheme \( X \) of \( Y \) corresponds to a homogeneous, with respect to the \( \text{Cl}(Y) \) grading, ideal \( I \subseteq R \). Moreover there is a sheafification construction taking any graded \( R \)-module \( M \) to a sheaf \( \widetilde{M} \) on \( Y \). In particular if \( S = R/I \) then \( \widetilde{S} = \bigoplus_{\alpha \in \text{Cl}(Y)} \mathcal{O}_X(\alpha) \). Also, as in the case of projective space, one may compute sheaf cohomology from local cohomology at the irrelevant ideal ([EMS00, Proposition 2.3]). In particular there is an exact sequence

\[ 0 \to H^0_B(S) \to S \to \bigoplus_{\alpha \in \text{Cl}(Y)} H^0(X, \mathcal{O}_X(\alpha)) \to H^1_B(S) \to 0. \]

We therefore get the following result for subschemes of toric varieties.

**Corollary 3.4.** Let \( X \) be a subscheme of a toric variety \( Y \) corresponding to a homogeneous ideal \( I \) in the Cox ring \( R \) of \( Y \). Set \( S = R/I \) and let \( Z \) be the intersection of the singular locus of \( Y \) with \( X \). Assume \( I \) is generated by homogeneous polynomials of degrees \( \alpha_1, \ldots, \alpha_m \in \text{Cl}(Y) \). If

- (i) \( \text{depth}_S \bigoplus_{\alpha \in \text{Cl}(Y)} \mathcal{O}_X(\alpha) \geq 2 \)
- (ii) \( S_{\alpha_i} \cong H^0(X, \mathcal{O}_X(\alpha_i)) \) for all \( i = 1, \ldots m \)

then \( \text{Def}^0_{S/R} \) and \( \text{Hilb}_{X/Y} \) are isomorphic deformation functors.

**Corollary 3.5.** Let \( X = \text{Proj} S \) be a subscheme of weighted projective space \( \mathbb{P}(q) = \text{Proj} R \) defined by the homogeneous ideal \( I \). Let \( m \) be the irrelevant maximal ideal of \( R \) and \( Z \) the intersection of the singular locus of \( \mathbb{P}(q) \) with \( X \). If
(i) $\text{depth}_Z \bigoplus_{m \in \mathbb{Z}} O_X(m) \geq 2$

(ii) $\text{Hom}_R(I, H^1_m(S))_0 = 0$

then $\text{Def}_{S/R}^0$ and $\text{Hilb}_{X/Y}$ are isomorphic deformation functors.

We give some examples to illustrate the conditions in Theorem 3.3 and Corollary 3.4. Note that in the (multi)-graded case we may think of $\text{Def}_{S/R}^0$ as deformations that preserve the Hilbert function and that they correspond to Hilbert function strata of the Hilbert scheme, see [Kle98, Theorem 1.1] and [HS04].

**Example 3.6. Points.** If $X = \text{Proj} S$ is $s$ points in general enough position in $\mathbb{P}^n$, then the Hilbert function of $X$ is

$$h_X(\nu) = \dim S_\nu = \inf \{s, \binom{\nu + n}{n} \}$$

by e.g. [GMR83]. Let $\nu_0$ be the smallest integer with

$$s \leq \binom{\nu_0 + n}{n}.$$

The exact sequence

$$0 \to S_\nu \to H^0(X, O_X(\nu)) \to H^1(\mathcal{I}_X(\nu)) \to 0$$

and the fact that $h^0(O_X(\nu)) = s$ yield $I_\nu = 0$ for $\nu < \nu_0$ and $H^1(\mathcal{I}_X(\nu)) \cong H^1_m(S)_\nu = 0$ for $\nu \geq \nu_0$ so $\text{Def}_{S/R}^0 \cong \text{Hilb}_{X/\mathbb{P}^n}$ in this case.

Six general points in $\mathbb{P}^2$ will have Hilbert function $(1, 3, 5, 6, \ldots)$. On the other hand the complete intersection of a quadric and a cubic will have $h_X = (1, 3, 5, 6, \ldots)$. Thus $\dim I_2 = h^1(\mathcal{I}_X(2)) = 1$, so $\text{Hom}(I, H^1_m(S))_0 \cong k$ and the functors need not be isomorphic. Indeed since both obstruction spaces vanish ($S$ is a complete intersection) the map $\text{Def}_{S/R}^0 \to \text{Hilb}_{X/\mathbb{P}^n}$ corresponds dually to a surjection of formally smooth complete $k$-algebras. The above two cases show that the Hilbert function stratum given by $(1, 3, 5, 6, \ldots)$ is an open subscheme of $\text{Hilb}^6(\mathbb{P}^2)$ while $(1, 3, 5, 6, \ldots)$ gives a smooth codimension 1 stratum.

**Example 3.7. Curves.** Consider a smooth curve $C$ sitting on a hypersurface $V$ of degree $s < \sum_{i=0}^3 q_i$ in $\mathbb{P}(q_0, q_1, q_2, q_3)$. Let $C = \text{Proj} S \subset \mathbb{P}(\mathfrak{q}) = \text{Proj} R$ with $S = R/I$ and suppose that $\text{Spec} R/I_V \setminus \{0\}$ is smooth, i.e. $V$ quasi-smooth. There is an exact sequence of normal bundles

$$0 \to \mathcal{N}_{C/V} \to \mathcal{N}_{C/\mathbb{P}(\mathfrak{q})} \to O_C(s) \to 0$$

and $\mathcal{N}_{C/V} \cong (\omega_C^{-1} \otimes \omega_V)^{-1}$. Thus by Serre duality $H^1(\mathcal{N}_{C/V}) = 0$ and $H^1(\mathcal{N}_{C/\mathbb{P}(\mathfrak{q})}) \cong H^1(O_C(s))$ while $H^0(\mathcal{N}_{C/\mathbb{P}(\mathfrak{q})}) \to H^0(O_C(s))$ is surjective.

Applying this to the long exact sequence in Theorem 1.1 we get

$$0 \to \text{Hom}_R(I, S)_0 \to H^0(\mathcal{N}_{C/\mathbb{P}(\mathfrak{q})}) \to \text{Hom}_R(I, H^1_m(S))_0 \to T^2(S/R; S)_0 \to 0,$$
because \( T^2_m(S/R; S)_0 \cong \text{Hom}_R(I, H^1_m(S))_0 \) and the composition

\[
H^1(\mathcal{N}_{C/P(q)}) \to T^2_m(S/R; S)_0 \to \text{Hom}_R(I, H^2_m(S))_0 \to H^1(\mathcal{O}_C(s))
\]

is injective. Similarly there is a commutative diagram

\[
\begin{array}{ccc}
H^0(\mathcal{N}_{C/P(q)}) & \longrightarrow & \text{Hom}_R(I, H^1_m(S)) \\
& \downarrow & \\
H^0(\mathcal{O}_C(s)) & \longrightarrow & H^1(\mathcal{I}_C(s))
\end{array}
\]

so if \( \text{Hom}_R(I, H^1_m(S))_0 \cong H^1(\mathcal{I}_C(s)) \) then \( T^2(S/R, S)_0 = 0 \) and \( \text{Def}^{0}_{S/R} \) is unobstructed. If \( s = d_0, d_1, \ldots, d_m \) are the degrees of the generators of \( I \), then it is easy to see that there is such an isomorphism if \( H^1(\mathcal{I}_C(d_i)) = 0 \) for all \( i = 1, \ldots, m \).

If \( \mathbb{P}(q) = \mathbb{P}^3 \) and \( s = 2 \) then \( V \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and we can use the Künneth formula to compute these cohomology groups. The outcome is that the above vanishing holds and we get \( \text{Hom}_R(I, H^1_m(S))_0 \cong H^1(\mathcal{I}_C(2)) \), so \( T^2(S/R, S)_0 = 0 \). Moreover if \( C \) has bidegree \((p, q)\) with \( 2 \leq p \leq q \) then

\[
h^1(\mathcal{I}_C(2)) = \begin{cases} 
0 & \text{if } p > 2 \\
\max\{0, q - 3\} & \text{if } p = 2
\end{cases}.
\]

Thus \( \text{Def}^{0}_{S/R} \cong \text{Hilb}_{C/P^3} \) are smooth if \( p > 2 \) or \( q = 3 \) while \( \text{Def}^{0}_{S/R} \) corresponds to a smooth stratum of codimension \( q - 3 \) in \( \text{Hilb}^{d,q}(\mathbb{P}^3) \) otherwise, see [Tan80].

If \( s = 3 \) then among curves on a cubic in \( \mathbb{P}^3 \) we find the curve that gives rise to Mumford’s example of an irreducible component of \( \text{Hilb}^{14,24}(\mathbb{P}^3) \) which is not reduced at its generic point \([\text{Mum}62]\). Let \( H \) be a hyperplane section of \( V \) and \( E \) a line on \( V \). Take \( C \) to be a generic element of the complete linear system \([4H + 2E]\) on \( V \). Such a curve may be constructed as the linked curve to the curve consisting of two disjoint conics in a general \((3, 6)\) complete intersection, and there is a resolution

\[
0 \to R(-9) \to R(-8)^2 \oplus R(-7)^2 \to R(-6)^3 \oplus R(-3) \to I \to 0.
\]

One computes that \( h^1(\mathcal{I}_C(d)) = 0 \) when \( d \geq 6 \) and \( h^1(\mathcal{I}_C(3)) = h^1(\mathcal{O}_C(3)) = 1 \), so \( \text{Hom}_R(I, H^1_m(S)) \cong H^1(\mathcal{I}_C(3)) \cong k \) and \( T^2(S/R; S)_0 = 0 \). Since \( \text{Def}^{0}_{S/R} \) is unobstructed it corresponds to the reduced subscheme of this component.

### 4 Deformations of the scheme - \( \text{Def}_X \)

In this section we must assume that the characteristic of the ground field is 0. Fix for the whole of this section a \( G \)-quadruple \((X, Z, S, J)\) with ferry \( S \). Moreover fix \( M \), a finitely generated \( SG \)-module, and set \( \mathcal{F} = \pi_*(\mathcal{M}_X) \) to be the corresponding sheaf of \( SG \) module on \( X \). This section is mostly concerned with computing cotangent groups with values in \( S \) or \( \mathcal{O}_X \), but because of future applications it is better to be a little more general and consider values in \( M, \mathcal{F} \) and \( \mathcal{F}^G \).
4.1 Conditions for Def$^G_S \to$ Def$_X$ to be smooth or an isomorphism

Lemma 4.1. If $T^2(S/O_X; S)^G = 0$ and $T^2_j(S/k; S)^G = 0$ then Def$^G_S \to$ Def$_X$ is smooth. If moreover $T^1(S/O_X; S)^G = 0$ and $T^1_j(S/k; S)^G = 0$ then Def$^G_S \to$ Def$_X$ is an isomorphism.

Proof. First note that since taking $G$ invariants is exact we have $T^1_X(S)^G \simeq T^1_X$. Secondly [And74, Appendix, Proposition 56] and the fact that $\pi$ is affine imply $T^1_X(S) \simeq T^1(\pi^{-1}O_X/k; O_X)$. The Zariski-Jacobi sequence for $k \to \pi^{-1}O_X \to O_X'$ reads

$$\to T^i(O_X; \pi^{-1}O_X; O_X') \to T^i(\pi^{-1}O_X/k; O_X) \to T^{i+1}(O_X/\pi^{-1}O_X; O_X') \to$$

so the first condition yields $(T^i_X)^G \to T^i_X$ surjective for $i = 1$ and injective for $i = 2$. The exact sequence in Theorem 1.1 for $X' \subseteq \text{Spec} S$ is

$$\cdots \to T^i_j(S/k; S) \to T^i_S \to T^i_X \to T^{i+1}_j(S/k; S) \to \cdots$$

so the second condition implies the same for $(T^i_S)^G \to (T^i_X)^G$ and we have proven the first statement.

To prove the second statement we need to show that the conditions imply Def$^G_S(A) \to$ Def$_X(A)$ injective for all objects $A$ of C. More explicitly we must show that if $S^1_A$ and $S^2_A$ are invariant deformations over $A$ mapping to $X^1_A$ and $X^2_A$, then for every isomorphism of deformations $\varphi_{X_A} : X^1_A \to X^2_A$ there exists an equivariant isomorphism $\varphi_{S_A} : S^1_A \to S^2_A$ inducing $\varphi_{X_A}$.

If $T^1(S/O_X; S)^G = 0$ and $T^1_j(S/k; S)^G = 0$ then the same argument as above shows that $(T^1_S)^G \to T^1_X$ is surjective for $i = 0$ and injective for $i = 1$. We will prove the statement by induction on the length of $A$. Let $\rho : B \to A$ be a small extension and assume the statement is true for $A$. Let $S^2_B$ be invariant deformations of $S$ over $B$ mapping to $X^1_B$ and $X^2_B$. Assume $\varphi_{X_B} : X^1_B \to X^2_B$ is an isomorphism of deformations. Let $\varphi_{X_A} : X^1_A \to X^2_A$ be the induced isomorphism where $X^1_A = X^1_B \times_{\text{Spec} B} \text{Spec} A$. By induction there exists an isomorphism $\varphi_{S_A} : S^1_A \to S^2_A$ inducing $\varphi_{X_A}$ and where $S^2_A = S^2_B \otimes_B A$.

The obstruction to lifting $\varphi_{S_A}$ to an isomorphism $S^1_A \to S^2_B$ is in $(T^1)^G \otimes \ker \rho$ and maps to 0 in $T^1_X \otimes \ker \rho$ because a lifting of $\varphi_{X_A}$ exists (namely $\varphi_{X_B}$). Thus by injectivity at the $T^1$ level the obstruction vanishes and there exists a lifting $\varphi'_{S_B} : S^1_B \to S^2_B$. It maps to an isomorphism $\varphi'_{X_B}$ which may differ from our given $\varphi_{X_B}$. However $\varphi_{X_B} - \varphi'_{X_B}$ defines an element of $T^0_X \otimes \ker \rho$ and the surjectivity at the $T^0$ level yields an element $D_{S,B} \in (T^0)^G \otimes \ker \rho$ mapping to $\varphi_{X_B} - \varphi'_{X_B}$. Thus $\varphi_{S,B} := \varphi'_{S,B} + D_{S,B}$ is an isomorphism inducing $\varphi_{X_B}$. \hfill \square

We will later need a statement with more general values. The proof is the same as the first part above.

Lemma 4.2. If $T^2(S/O_X; F)^G = 0$ and $T^2_j(S/k; M)^G = 0$ then there is a surjective morphism $T^1_S(M)^G \to T^1_X(F^G)$.

The modules $T^1_j(S/k; M)$ are easily described in terms of local cohomology.

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Lemma 4.3. If \( \text{depth}_J M \geq 1 \) then there is an isomorphism \( T_J^1(S/k; M) \simeq \text{Der}_k(S, H_J^1(M)) \) and an exact sequence

\[
0 \to T^1(S/k; H_J^1(M)) \to T_J^2(S/k; M) \to T^0(S/k; H_J^2(M)) \to T^2(S/k; H_J^1(M)).
\]

In particular if \( \text{depth}_J M \geq 2 \) or \( S \) is regular then \( T^2_J(S/k; M) \simeq \text{Der}_k(S, H_J^2(M)) \) as \( SG \)-modules.

Proof. Since \( H_J^0(M) = 0 \) the exact sequence comes from the edge exact sequence for the spectral sequence in Theorem 1.1. \( \Box \)

4.2 Euler derivations

We show that if \( \text{depth}_Z S \geq 2 \) then \( \Theta_{X/Z} \) is globally free of rank equal to the dimension of \( G \). This will allow us to give finer criteria for the equivalence of deformation functors. The result follows from general considerations and we start with some lemmas.

Let \( G \) be a reductive group acting on an affine scheme \( X = \text{Spec} A \) over a field \( k \) of characteristic 0. Let \( g = T_e \) be the Lie algebra of \( G \) (where \( e \) is the identity element in \( G \)). Consider the corresponding representation of Lie algebras \( \phi : A \otimes g \to \text{Der}_{AC}(A, A) \).

Recall that if \( \xi \in g = \text{Der}_k(k[G], k(e)) \) then \( \phi(1 \otimes \xi) \) is the composition

\[
A \xrightarrow{\mu} A \otimes k[G] \xrightarrow{1 \otimes \xi} A \otimes k(e) \simeq A \tag{4.1}
\]

where \( \mu \) is the comultiplication of the group action. Note that if \( f \) is invariant then \( \mu(f) = f \otimes 1 \) so the composition is in \( \text{Der}_{AC}(A, A) \).

If \( D \in \text{Der}_{AC}(A, A) \) and \( x \) is a closed point of \( \text{Spec} A \), let \( D_x \) be the value in \( \text{Der}_{AC}(A, k(x)) \).

For a fixed closed point \( x \) let \( p_x : G \to Gx \) be the orbit map. Then \( \phi(1 \otimes \xi)_x(f) = \xi(p_x^*(f)) \) so \( \phi(1 \otimes \xi)_x \) equals the image of \( \xi \) via the map of Zariski tangent spaces \( dp_x : T_e \to T_{Gx,x} \).

Note that \( T_{Gx,x} = \text{Der}_k(\mathcal{O}_{Gx}, k(x)) \subseteq \text{Der}_{AC}(A, k(x)) \subseteq \text{Der}_k(A, k(x)) = T_{X,x} \).

Lemma 4.4. If there is an open dense subset of \( \text{Spec} A \) where all isotropy groups are finite, then \( \phi : A \otimes_k g \to \text{Der}_{AC}(A, A) \) is injective.

Proof. Choose a basis \( \xi_1, \ldots, \xi_r \) for \( g \) and assume \( \phi(\sum a_i \otimes \xi_i) = 0 \). Then \( \sum a_i dp_x(\xi_i) = 0 \) in \( T_{Gx,x} \) for all \( x \). But by assumption \( dp_x \) is an isomorphism on an open dense subset \( U \).

Thus for all \( i = 1, \ldots, r \) we have \( a_i = 0 \) on \( U \) so \( a_i = 0 \) in \( A \). \( \Box \)

Lemma 4.5. If the quotient map \( \text{Spec} A \to \text{Spec} A^G \) is smooth of relative dimension \( \dim G \) and a geometric quotient then \( \phi : A \otimes_k g \to \text{Der}_{AC}(A, A) \) is an isomorphism.

Proof. Injectivity follows from Lemma 4.4. We will show surjectivity for the maps of stalks \( A_x \otimes g \to \text{Der}_{AC}(A_x, A)_x = \text{Der}_{AC}(A_x, A_x) \) where \( x \mapsto y \) via the quotient map. By assumption \( \mathcal{O}_{Gx} \simeq A \otimes_{AC} k(y) \) and so \( \text{Der}_{AC}(A_x, k(x)) \simeq \text{Der}_k(\mathcal{O}_{Gx}, k(x)) = T_{Gx,x} \simeq g \). On the other hand since \( A^G \to A \) is smooth we have \( \text{Der}_{AC}(A_x, k(x)) \simeq \text{Der}_{AC}(A_x, A_x) \otimes_k k(x) \) and so the surjectivity follows from Nakayama’s Lemma. \( \Box \)
We write \( M^\vee = \text{Hom}_A(M, A) \) for the dual module. The above construction yields also a map \( \psi : \Omega_{A/G^G} \to \text{Hom}_k(g, A) \) namely the composition

\[
\Omega_{A/G^G} \to \Omega^\vee_{A/G^G} = \text{Hom}_A(\text{Der}_G(A, A), A) \xrightarrow{\phi^\vee} \text{Hom}_k(g, A).
\]

Thus \( \psi(da)(\xi) = \phi(1 \otimes \xi)(a) \)

**Lemma 4.6.** If \( x \) is a closed point in \( \text{Spec} \, A \) mapping to \( y \in \text{Spec} \, A^G \) with finite isotropy group and \( Gx \) is closed in \( \text{Spec} \, A \) then \( \psi_x : \Omega_{A,x/G^G_y} \to \text{Hom}_k(g, A_x) \) is surjective.

**Proof.** By Nakayama’s Lemma it is enough to prove \( \Omega_{A,x/G^G_y} \otimes_{A_x} k(x) \to \text{Hom}_k(g, k(x)) \) is surjective, which we can do by showing that the \( k \)-dual map is injective. But this is the map \( g \otimes k(x) \to \text{Der}_{A^G_y}(A_x, k(x)) \) induced by \( \phi \). Thus it factors in the following way

\[
g \otimes k(x) \simeq T_{G_x,x} \subseteq \text{Der}_{A^G_y}(A_x, k(x)).
\]

□

We now apply these local lemmas to our situation. By \( F \otimes_k g \) we mean the \( SG \)-module that as \( S \)-module is isomorphic to the sum of \( \dim g \) copies of \( F \) and the \( G \)-action is given by \( g(f \otimes \xi) = gf \otimes \text{Ad}_g(\xi) \) where \( \text{Ad}_g : g \to g \) is the adjoint action. Similarly we can define \( F \otimes_k g^* \) for the dual of the adjoint action.

**Lemma 4.7.** If \( \text{depth}_Z \, S \geq 2 \) then there is an \( SG \) isomorphism

\[
\mathcal{T}^0(S/O_X; S) \cong S \otimes_k g.
\]

More generally if \( \text{depth}_Z \, F \geq 2 \) then \( \mathcal{T}^0(S/O_X; F) \cong F \otimes_k g \) as \( SG \)-modules.

**Proof.** By definition \( \pi_U \) satisfies locally the conditions in Lemma 4.5. This globalizes to an isomorphism \( \Theta_{U'/U} \simeq O_{U'} \otimes_k g \). Since \( \pi \) is smooth on \( U' \) we get \( \Omega_{U'/U}^1 \simeq O_{U'} \otimes g^* \) so. Since \( \mathcal{T}^0(S/O_X; F) = \text{Hom}(\Omega^1_{S/O_X}, F) \) it also has depth \( \geq 2 \) in \( Z \). It is therefore isomorphic to \( j_* \text{Hom}(\pi_* \Omega^1_{U'/U}, F) \mid_{U'} \simeq F \otimes_k g \) where \( j \) is the inclusion \( U \subseteq X \).

□

For example let \( G = \mathbb{G}_m \) and \( A \) a finitely generated algebra, or a localization of such, with algebra generators \( x_1, \ldots, x_n \). Then \( \mu : A \to A \otimes k[t, t^{-1}] \) is of the form \( f(x_1, \ldots, x_n) \mapsto f(t^{a_1}x_1, \ldots, t^{a_n}x_n) \) for integers \( a_1, \ldots, a_n \). We get

\[
\phi(1 \otimes \frac{d}{dt(t=1)})(f) = \frac{d}{dt} f(t^{a_1}x_1, \ldots, t^{a_n}x_n)_{t=1} = \sum_{i=1}^n a_i x_i \frac{\partial f}{\partial x_i}
\]

by the chain rule. Thus the generator of \( g \) maps to the Euler derivation of the \( \mathbb{Z} \)-graded algebra \( A \). This motivates the name in the following definition.

**Definition 4.8.** Assuming \( \text{depth}_Z \, S \geq 2 \), let \( E_1, \ldots, E_r \) be a set of global sections that generate the free sheaf \( \mathcal{T}^0(S/O_X; S) \). We call such a set a set of Euler derivations for the \( G \)-quadruple. A set of Euler derivations defines an \( SG \) map \( E : \Omega^1_{S/O_X} \to S \otimes g^* \). Let \( Q = \text{coker}(E) \) and \( Z(Q) \) be the support of \( Q \).
The morphism $E$ is just the natural map $\Omega^1_{S/\mathcal{O}_X} \to (\Omega^1_{S/\mathcal{O}_X})^{\vee\vee}$ after choosing a basis for the free sheaf $(\Omega^1_{S/\mathcal{O}_X})^{\vee\vee}$ thus we have an exact sequence

$$0 \to \text{Tor}(\Omega^1_{S/\mathcal{O}_X}) \to \Omega^1_{S/\mathcal{O}_X} \xrightarrow{E} S \otimes g^* \to Q \to 0 \quad (4.2)$$

where $\text{Tor}(\Omega^1_{S/\mathcal{O}_X})$ is the torsion submodule of $\Omega^1_{S/\mathcal{O}_X}$.

If $\text{depth}_Z S \geq 2$ and the sheaf $\mathcal{T}^1(S/\mathcal{O}_X; S)^G = 0$ then the Zariski-Jacobi sequence for $k \to \mathcal{O}_X \to S$ leads to a short exact sequence

$$0 \to \mathcal{T}^0(S/\mathcal{O}_X; S)^G \to (\mathcal{T}^0_S)^G \to \mathcal{T}^0(\mathcal{O}_X/k; S)^G \to 0.$$

Therefore by Lemma 4.7 we can make the following definition.

**Definition 4.9.** If $\text{depth}_Z S \geq 2$ and the sheaf $\mathcal{T}^1(S/\mathcal{O}_X; S)^G = 0$ then we call the short exact sequence

$$0 \to (S \otimes g)^G \to \Theta_S^G \to \Theta_X \to 0$$

the Euler sequence associated to the $G$-quadruple. Here $\Theta_X = \mathcal{T}^0(\mathcal{O}_X/k; \mathcal{O}_X)$.

**Remark.** If $G \to \text{GL}(V)$ is a representation and $S = \text{Sym}(V^*)$ then $\text{Der}_k(S) \simeq S \otimes V$ as $SG$-module. From the above we have an $SG$-map $S \otimes g \to S \otimes V$. In this case it is induced from the natural map $g \to \text{Hom}(V, V) \simeq \text{Sym}^1(V^*) \otimes V$ given by the representation.

**Lemma 4.10.** If $\text{depth}_Z F \geq 2$ then

$$\mathcal{T}^i(S/\mathcal{O}_X; F) \simeq \mathcal{E}xt^i_S(\Omega^1_{S/\mathcal{O}_X}, F) \text{ for } i = 0, 1, 2.$$

**Proof.** Consider the spectral sequence $\mathcal{E}xt^p_S(\mathcal{T}_q(S/\mathcal{O}_X), F) \Rightarrow \mathcal{T}^{p+q}(S/\mathcal{O}_X; F)$. Since the $\mathcal{T}_q(S/\mathcal{O}_X; S)$ are supported on $Z$ when $q \geq 1$, the depth condition yields the result. \(\square\)

Assume $S$ is a sheaf of $\mathcal{O}_X$-algebras on a scheme $X$ and $Z \subseteq X$ is locally closed. If $F$ and $G$ are $S$-modules, denote by $\text{Ext}^p_{S, Z}(F, G)$, respectively $\mathcal{E}xt^p_{S, Z}(F, G)$, the higher derived functors of $G \mapsto H^0_Z(X, \mathcal{H}om_S(F, G))$, respectively $G \mapsto H^0_Z(X, \mathcal{H}om_S(F, G))$. We refer to SGA 2 Exposé VI ([Gro85]) for details and the results we will use.

**Lemma 4.11.** Assume $\text{depth}_Z F \geq 2$.

(i) There is an isomorphism $\mathcal{E}xt^1_S(\Omega^1_{S/\mathcal{O}_X}, F) \simeq \mathcal{H}om_S(Q, \mathcal{H}^2_Z(F))$.

(ii) If $\text{depth}_Z(F \otimes g)^G \geq 3$ or $Q = 0$ then

$$\mathcal{T}^1(S/\mathcal{O}_X; F)^G = \mathcal{E}xt^1_S(\Omega^1_{S/\mathcal{O}_X}, F)^G = \mathcal{H}om_S(Q, \mathcal{H}^2_Z(F))^G = 0.$$

**Proof.** By Lemma 4.7, $\mathcal{H}om(\Omega^1_{S/\mathcal{O}_X}, F) \simeq F \otimes g$ so we get an exact sequence

$$\cdots \to \mathcal{E}xt^1_{S, Z}(\Omega^1_{S/\mathcal{O}_X}, F) \to \mathcal{E}xt^1_S(\Omega^1_{S/\mathcal{O}_X}, F) \to \mathcal{R}^1 j_*(F \otimes g)|_U \xrightarrow{\sim} \mathcal{E}xt^2_{S, Z}(\Omega^1_{S/\mathcal{O}_X}, F) \to \cdots$$
from the long exact sequence for $\mathcal{E}xt^1_{S,Z}(\Omega^1_{S/\mathcal{O}_X}, \mathcal{F})$. Here $j$ is the inclusion of $U$ in $X$. Now since depth$_Z\mathcal{F} \geq 2$ the left sheaf vanishes. The sheaf $R^1j_*(\mathcal{F} \otimes g)_{|U}$ is isomorphic to $\mathcal{H}^2_Z(\mathcal{F} \otimes g) \cong \mathcal{H}^2_Z(\mathcal{F}) \otimes g$. Using the spectral sequence $\mathcal{E}xt^p_S(\Omega^1_{S/\mathcal{O}_X}, \mathcal{H}^q_Z(\mathcal{F})) \Rightarrow \mathcal{E}xt^{p+q}_{S,Z}(\Omega^1_{S/\mathcal{O}_X}, \mathcal{F})$ we get

$$\mathcal{E}xt^2_{S,Z}(\Omega^1_{S/\mathcal{O}_X}, \mathcal{F}) \cong \text{Hom}_S(\Omega^1_{S/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F})).$$

Thus our exact sequence becomes

$$0 \to \mathcal{E}xt^1_{S}(\Omega^1_{S/\mathcal{O}_X}, \mathcal{F}) \to \mathcal{H}^2_Z(\mathcal{F}) \otimes g \xrightarrow{\varepsilon} \text{Hom}_S(\Omega^1_{S/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F})) \to \cdots$$

where $\varepsilon$ is the map induced by $E$ in the sequence (4.2). It follows that

$$\mathcal{E}xt^1_{S}(\Omega^1_{S/\mathcal{O}_X}, \mathcal{F}) = \ker \varepsilon \cong \text{Hom}_S(Q, \mathcal{H}^2_Z(\mathcal{F}))$$

proving the first statement.

Clearly if $Q = 0$ then $\mathcal{E}xt^1_{S}(\Omega^1_{S/\mathcal{O}_X}, \mathcal{F}) = 0$ and the inclusion

$$\mathcal{E}xt^1_{S}(\Omega^1_{S/\mathcal{O}_X}, \mathcal{F})^G \hookrightarrow \mathcal{H}^2_Z(\mathcal{F} \otimes g)^G \cong \mathcal{H}^2_Z((\mathcal{F} \otimes g)^G)$$

yields the second statement.

We now have conditions for the existence of an Euler sequence.

**Theorem 4.12.** If depth$_Z\mathcal{S} \geq 2$ and $\text{Hom}_S(Q, \mathcal{H}^2_Z(\mathcal{S}))^G = 0$ then there is an Euler sequence

$$0 \to (\mathcal{S} \otimes g)^G \to \Theta_{\mathcal{S}}^G \to \Theta_X \to 0.$$ 

**Proposition 4.13.** The closed subset $Z(Q) \subset X$ is contained in the set of points $x \in X$ where $\pi^{-1}(x)$ contains a point with positive dimensional isotropy group.

**Proof.** We may prove the result locally on an affine chart, i.e. for $\pi : \text{Spec} A \to \text{Spec} A^G$. A standard result in invariant theory says that the stable points are the complement of $\pi^{-1}(\pi(L))$ where $L$ is the locus of points with positive dimensional isotropy. The result follows now from Lemma 4.6. 

**Example 4.14.** For toric varieties Proposition 4.13 says that $Z(Q)$ is contained in the non-simplicial locus. In fact one can prove directly using Euler derivations that these loci are equal. In particular for simplicial toric varieties $Q = 0$ and we recover the Euler sequence of [BC94, Theorem 12.1]

**Example 4.15.** For Grassmannians (Example 2.8) even $Z$ is empty so Proposition 4.13 applies. If $\mathcal{L}$ is the universal subbundle and $\mathcal{P}$ is the universal quotient bundle then the tangent sheaf is $\mathcal{P} \otimes \mathcal{L}^\vee$. Our Euler sequence is just the usual sequence induced by the tautological exact sequence.
4.3 The groups $T^i(S/\mathcal{O}_X; F)$

Lemma 4.16. If $\text{depth}_Z F \geq 2$ and $\text{Hom}_S(Q, H^2_Z(F))^G = 0$ then

$$T^i(S/\mathcal{O}_X; F)^G \simeq \text{Ext}^i_S(\Omega^i_{S/\mathcal{O}_X}, F)^G$$

for $i = 0, 1, 2$.

Proof. Lemma 4.11 says that under the conditions $T^i(S/\mathcal{O}_X; F)^G = \text{Ext}^i_S(\Omega^i_{S/\mathcal{O}_X}, F)^G = 0$. We have local global spectral sequences which at the $E_2$ level are

$$H^p(X, T^q(S/\mathcal{O}_X; F)) \Rightarrow T^{p+q}(S/\mathcal{O}_X; F)$$

$$H^p(X, \text{Ext}^q_S(\Omega^i_{S/\mathcal{O}_X}, F)) \Rightarrow \text{Ext}^{p+q}_S(\Omega^i_{S/\mathcal{O}_X}, F).$$

Write (just for this proof) $T^i = T^i(S/\mathcal{O}_X; F)$ and $\mathcal{E}^i = \text{Ext}^i_S(\Omega^i_{S/\mathcal{O}_X}, F)$. The edge homomorphisms for the spectral sequence $\text{Ext}^p_S(T_q(S/\mathcal{O}_X; S), \mathcal{E}) \Rightarrow T^{p+q}(S/\mathcal{O}_X; F)$ yield natural maps $\mathcal{E}^p \rightarrow T^i$. Lemma 4.10 implies $T^i(S/\mathcal{O}_X; F)^G \simeq \text{Ext}^i_S(\Omega^i_{S/\mathcal{O}_X}, F)^G$ for $i = 0, 1$. Since $\text{Ext}^1_S(\Omega^1_{S/\mathcal{O}_X}, F)^G = 0$ the spectral sequences yield a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^2(X, \mathcal{E}^0)^G & \rightarrow & \text{Ext}^2_S(\Omega^1_{S/\mathcal{O}_X}, F)^G & \rightarrow & H^0(X, \mathcal{E}^2)^G & \rightarrow & H^3(X, \mathcal{E}^0)^G \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H^2(X, T^0)^G & \rightarrow & T^2(S/\mathcal{O}_X; F)^G & \rightarrow & H^0(X, T^2)^G & \rightarrow & H^3(X, T^0)^G
\end{array}
\]

so the isomorphism for $i = 2$ follows from Lemma 4.10 and the Five Lemma. \qed

Proposition 4.17. Assume $\text{depth}_Z F \geq 2$

\(\text{(i)}\) If $\text{Hom}_S(Q, H^2_Z(F))^G = 0$ then $T^1(S/\mathcal{O}_X; F)^G \simeq H^1(X, (F \otimes g)^G)$ and there is an exact sequence

$$0 \rightarrow H^2(X, (F \otimes g)^G) \rightarrow T^2(S/\mathcal{O}_X; F)^G \rightarrow H^0(X, \text{Ext}^2_S(\Omega^1_{S/\mathcal{O}_X}, F)^G)$$

$$\rightarrow H^3(X, (F \otimes g)^G)$$

\(\text{(ii)}\) If $\text{depth}_Z (F \otimes g)^G \geq 3$ then there is an exact sequence

$$0 \rightarrow \text{Hom}_S(\Omega^1_{S/\mathcal{O}_X}, H^2_Z(F))^G \rightarrow H^0(X, \text{Ext}^2_S(\Omega^1_{S/\mathcal{O}_X}, F)^G) \rightarrow \text{Hom}_S(Q, H^2_Z(F))^G.$$

\(\text{(iii)}\) If $Q = 0$ then $\text{Ext}^2_S(\Omega^1_{S/\mathcal{O}_X}, F)^G \simeq \text{Hom}_S(Tor(\Omega^1_{S/\mathcal{O}_X}), H^2_Z(F)).$

Proof. Since the condition in (i) implies $T^1(S/\mathcal{O}_X; F)^G = 0$ by Lemma 4.10 and Lemma 4.11 (i) follows from the local-global spectral sequence and Lemma 4.7 and Lemma 4.16.

Let $j$ be the inclusion of $U$ in $X$. To prove (ii) we consider the long exact sequence for $\text{Ext}^2_S(Z, \Omega^1_{S/\mathcal{O}_X}, F)$ as in the proof of Lemma 4.11 and get

$$H^2_Z(F \otimes g) \rightarrow \text{Ext}^2_S(Z, \Omega^1_{S/\mathcal{O}_X}, F) \rightarrow \text{Ext}^2_S(\Omega^1_{S/\mathcal{O}_X}, F) \rightarrow H^2_Z(F \otimes g)$$

$$\rightarrow \text{Ext}^2_S(Z, \Omega^1_{S/\mathcal{O}_X}, F). \quad (4.3)$$
Taking invariants and using the depth assumption this yields
\[ 0 \to \mathcal{E}xt^2_{S,Z}(\Omega^1_{S/O_X}, F)^G \to \mathcal{E}xt^2_{S}(\Omega^1_{S/O_X}, F)^G \to \mathcal{H}^3_Z(F \otimes g)^G \to \mathcal{E}xt^3_{S,Z}(\Omega^1_{S/O_X}, F)^G. \]

Using the spectral sequence \( \mathcal{E}xt^2_{S}(\Omega^1_{S/O_X}, \mathcal{H}^2_Z(F)) \Rightarrow \mathcal{E}xt^3_{S,Z}(\Omega^1_{S/O_X}, F) \) we get
\[ \mathcal{E}xt^2_{S,Z}(\Omega^1_{S/O_X}, F)^G \simeq \mathcal{E}xt^3_{S}(\Omega^1_{S/O_X}, \mathcal{H}^2_Z(F))^G. \]

Moreover there is a composite map \( \varepsilon = v \circ u \)
\[ \mathcal{H}^3_Z(F \otimes g) \to \mathcal{E}xt^3_{S,Z}(\Omega^1_{S/O_X}, F)^G \to \mathcal{H}om_S(\Omega^1_{S/O_X}, \mathcal{H}^2_Z(F)) \]
where \( v \) is the map to \( E^{0,3} \) in the above spectral sequence. Thus \( \ker u \subseteq \ker \varepsilon \). On the other hand \( \varepsilon \) is the map induced by \( E \) in the exact sequence \((4.2)\). Therefore \( \ker \varepsilon = \mathcal{H}om_S(Q, \mathcal{H}^2_Z(F)) \) and we get the exact sequence in the statement.

To prove (iii) note that if \( Q = 0 \) then even without the depth assumption the above argument shows that \( \ker \varepsilon = 0 \). Thus \( u \) is injective. Moreover by Lemma \( 4.11 \) \( \mathcal{E}xt^1_{S}(\Omega^1_{S/O_X}, F) \) vanishes so \((4.3)\) becomes
\[ 0 \to \mathcal{H}^2_Z(F \otimes g) \to \mathcal{H}om_S(\Omega^1_{S/O_X}, \mathcal{H}^2_Z(F)) \to \mathcal{E}xt^2_{S}(\Omega^1_{S/O_X}, F) \to 0. \]
The sequence \((4.2)\) is now short exact and we can apply \( \mathcal{H}om_S(\cdot, \mathcal{H}^2_Z(F)) \) to get an exact sequence
\[ 0 \to \mathcal{H}^2_Z(F \otimes g) \to \mathcal{H}om_S(\Omega^1_{S/O_X}, \mathcal{H}^2_Z(F)) \to \mathcal{H}om_S(\mathcal{F}or(\Omega^1_{S/O_X}), \mathcal{H}^2_Z(F)) \to 0. \]
This proves (iii).

\[ \square \]

**Remark.** If \( S \) is regular then \( H^0(X, \mathcal{E}xt^2_{S}(\Omega^1_{S/O_X}, F)^G) \to \mathcal{H}om_S(Q, \mathcal{H}^3_Z(F))^G \) is surjective. See the proof of Theorem \( 5.10 \) below.

### 4.4 Results

**Example 4.18.** *Finite G.* We can apply the above to the situation in Example \( 2.7 \) i.e. \( G \subset \text{GL}_n(\mathbb{C}) \) is finite without pseudo-reflections and
\[ (X, Z, S, J) = (\mathbb{C}^n/G, \text{Sing}(\mathbb{C}^n/G), \mathbb{C}[x_1, \ldots, x_n], (1)). \]
Assume that \( \text{codim}_Z X \geq 3 \) (the singularity need not be isolated). Then since \( X \) is Cohen-Macaulay, \( \text{depth}_Z O_X \geq 3 \) and since \( G \) is finite, \( \text{depth}_{I_Z} S \geq 3 \) and \( Q = 0 \). It follows from Proposition \( 4.17 \) Lemma \( 4.10 \) and Lemma \( 4.1 \) that \( X \) is rigid. This was first proven by Schlessinger, see [Sch71] and [Sch73].

In general consider an affine \( G \)-quadruple
\[ (X, Z, S, J) = (\text{Spec} S^G, Z, S, (1)) \]
with finite \( G \) and \( Z \) the locus where \( \pi : \text{Spec} S \to X \) is not étale. Then by the above results we have \( \text{Def}^G_S \simeq \text{Def}_X \) if \( \text{depth}_{I_Z} S \geq 3 \). In particular if \( S \) is Cohen-Macaulay and equidimensional then it is enough to assume \( \text{codim} Z \geq 3 \). See also [Ste88 Section 7].
Our main result in its most general form is

**Theorem 4.19.** Let \((X, Z, S, J)\) be a \(G\)-quadruple with ferry \(S\) and assume \(\text{depth}_Z S \geq 2\). If

(i) \(\text{Hom}_S(Q, \mathcal{H}_Z^2(S))^G = 0\)

(ii) \(H^0(X, \mathcal{E}xt^2_S(\Omega^1_{S/\mathcal{O}_X}, S))^G = 0\)

(iii) \(H^2(X, (S \otimes g)^G) = 0\)

(iv) \(T^1(S/k; H^1_J(S))^G = 0\) and \(\text{Der}_k(S, H^2_J(S))^G = 0\)

then \(\text{Def}^G_S \rightarrow \text{Def}_X\) is smooth. If moreover

(v) \(H^1(X, (S \otimes g)^G) = 0\)

(vi) \(\text{Der}_k(S, H^1_J(S))^G = 0\)

then \(\text{Def}^G_S \rightarrow \text{Def}_X\) is an isomorphism.

**Proof.** The result follows directly from Lemma 4.1 and Proposition 4.17. 

Using the same proposition and lemma we get a slight improvement if \(Q = 0\).

**Theorem 4.20.** Let \((X, Z, S, J)\) be a \(G\)-quadruple with ferry \(S\) and assume \(\text{depth}_Z S \geq 2\). If

(i) \(Q = 0\)

(ii) \(\text{Hom}_S(\mathcal{T}or(\Omega^1_{S/\mathcal{O}_X}), \mathcal{H}_Z^2(S))^G = 0\)

and (iii) and (iv) of Theorem 4.19 hold, then \(\text{Def}^G_S \rightarrow \text{Def}_X\) is smooth. If moreover (v) and (vi) of Theorem 4.19 hold then \(\text{Def}^G_S \rightarrow \text{Def}_X\) is an isomorphism.

We give two examples to illustrate how the statement fails if the assumptions are not met.

**Example 4.21.** For smooth curves in \(\mathbb{P}^3\) we only need to verify (iv) of Theorem 4.20 to conclude that \(\text{Def}^0_S \rightarrow \text{Def}_X\) is smooth since \(Z\) is empty and \((S \otimes g)^G = \mathcal{O}_X\). We claim that a general space curve \(X = \text{Proj} S\) in \(\text{Hilb}^{d,g}(\mathbb{P}^3)\) satisfying \(g < d + 3\) also satisfy (iv). Indeed in this case \(J\) is the irrelevant maximal ideal and \(H^2_J(S) \simeq \bigoplus_{\nu} H^1(X, \mathcal{O}_X(\nu))\). We get \(\text{Der}_k(S, H^2_J(S))_0 = 0\) provided \(H^1(X, \mathcal{O}_X(1)) = 0\). It is well known that a general curve with \(g < d + 3\) satisfies this property.

We need to show \(T^1(S/k, H^1_J(S))_0 = 0\). Let \(I\) be the homogeneous ideal of \(X\) in \(\mathbb{P}^3 = \text{Proj} R\). Since \(\text{Hom}_R(I, H^1_J(S))_0 = T^1(S/R, H^1_J(S)) \rightarrow T^1(S/k, H^1_J(S))_0\) is surjective, it suffices to show \(\text{Hom}(I, H^1_J(S))_0 = 0\). Space curves of maximal rank (i.e. \(H^1(\mathcal{I}_X(\nu)) = 0\) provided \(H^0(\mathcal{I}_X(\nu)) \neq 0\) necessarily satisfy this property. The main theorem of Ellia and Ballico in [BE85] implies that the general curve in the range \(g < d - 3\) has maximal rank. Thus the first four assumptions of the theorem are satisfied.

It follows that \(\text{Def}^0_S \rightarrow \text{Def}_X\) is smooth in this range. But if \(g > 0\) then \(H^1(X, \mathcal{O}_X) \neq 0\), i.e. (v) does not hold and we do not expect \(\text{Def}^0_S \rightarrow \text{Def}_X\) to be an isomorphism.
Example 4.22. Let $X = \text{Spec } A$ be the affine cone over $\mathbb{G}(2, 4)$ in the Plücker embedding. It is a node in $\mathbb{A}^6$ so dim $T^1_X = 1$. On the other hand $A = S^G$ where $S = k[x_{1,1}, \ldots, x_{2,4}]$ and $G = \text{SL}_2$ corresponding to an affine $G$-quadruple $(X, \{0\}, S, (1))$. Of course $T^1_S = 0$. If $m \subset A$ is the ideal of $\{0\}$ in $X$, then the ideal $I_Z = mS \subset S$ is generated by the $2 \times 2$ minors of a general $2 \times 4$ matrix so depth$_Z S = 3$. Thus conditions (i), (iii) and (iv) of Theorem 4.19 are satisfied.

We compute $\text{Ext}_S^2(\Omega^1_{S/A}, S)$. Let $f_{ij}$ be the minor with columns $i$ and $j$. If $1 \leq i < j \leq 4$ write $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. Then one checks that for each pair $(\alpha, \beta)$ with $1 \leq \alpha < \beta \leq 4$

$$
\sum_{1 \leq i < j \leq 4} \epsilon_{ij} \frac{\partial f_{ij}}{x_{\alpha\beta}} \cdot f_{kl} = 0
$$

for suitable signs $\epsilon_{ij}$. We may use this to construct a free $SG$-resolution

$$
0 \to S \xrightarrow{M} S^6 \to S \otimes_k V^* \to \Omega^1_{S/A} \to 0
$$

where the entries of $M$ generate $I_Z$ and $V$ is the vector space of $2 \times 4$ matrices. In particular

$$
\text{Ext}_S^2(\Omega^1_{S/A}, S)^G \simeq (S/I_Z)^G = A/m = k
$$

and indeed condition (ii) of Theorem 4.19 fails.

We assume now that $G$ is a quasitorus. In particular the adjoint action is trivial so $F \otimes g \simeq F^r$ as $SG$-modules.

Lemma 4.23. If $G$ is a quasitorus and $G$ is an $SG$-module then there are inclusions

$$
\text{Hom}_S(Q, G)^G \subseteq \text{Hom}_{\mathcal{O}_X}(Q^G, G^G) \subseteq rG^G.
$$

Proof. Since now $Q$ is a quotient of $S^r$ it is generated by $G$ invariants. Thus

$$
\text{Hom}_S(Q, G)^G \to \text{Hom}_{\mathcal{O}_X}(Q^G, G^G)
$$

is injective. The second inclusion follows from first taking invariants of (4.2) and then applying $\text{Hom}_{\mathcal{O}_X}(-, G^G)$. □

The results become better if the codimension of $Z(Q)$ is sufficiently large.

Proposition 4.24. If $G$ is a quasitorus and

$$
depth_Z F \geq 2, \quad depth_Z F^G \geq 3 \text{ and } depth_Z(Q), F^G \geq 4
$$

then there is an exact sequence

$$
0 \to rH^2(X, F^G) \to T^2(S/\mathcal{O}_X; F)^G \to \text{Hom}_S(\Omega^1_{S/\mathcal{O}_X}, \mathcal{H}^2_F)^G \to rH^3(X, F^G)
$$

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Proof. From Proposition 4.17 and Lemma 4.23 it is enough to prove that
\[ \text{Hom}_{\mathcal{O}_X}(Q^G, \mathcal{H}^3_Z(\mathcal{F}^G)) = 0 \]
but this follows from the assumptions and the following lemma. \qed

Lemma 4.25. Let \( A \) be a noetherian ring, \( M \) a finitely generated \( A \)-module, \( I \subseteq J \) two ideals of \( A \) and suppose \( \text{depth}_I M \geq d \) and \( \text{depth}_J M \geq d + 1 \). If \( Q \) is an \( A \)-module with support contained in \( V(J) \) then \( \text{Hom}_A(Q, H^d_I(M)) = 0 \).

Proof. Consider the two spectral sequences
\[ E_2^{p,q} = H^p_I(\text{Ext}^q_A(Q, M)) \quad \text{and} \quad E_2'^{p,q} = \text{Ext}^p_A(Q, H^q_I(M)) \]
which both converge to \( \text{Ext}_{A,I}^{p+q}(Q, M) \). We have \( H^q_I(M) = 0 \) for \( q < d \) so the second spectral sequence yields \( \text{Hom}_A(Q, H^d_I(M)) \simeq \text{Ext}^d_A(Q, M) \). On the other hand since \( \text{depth}_J M \geq d + 1 \) we have \( \text{Ext}^q_A(Q, M) = 0 \) for \( q < d + 1 \). Thus by the first spectral sequence \( \text{Ext}^d_A(Q, M) = 0 \). \qed

Theorem 4.26. Assume \( G \) is a quasitorus and let \((X, Z, S, J)\) be a \( G \)-quadruple with ferry \( S \) and assume \( \text{depth}_Z S \geq 2 \). If

(i) \( \text{depth}_Z \mathcal{O}_X \geq 3 \) and \( \text{depth}_{Z(Q)} \mathcal{O}_X \geq 4 \)
(ii) \( \text{Hom}_S(\mathcal{O}_S, \mathcal{H}^2_Z(S))^G = 0 \)
(iii) \( G \) is finite or \( H^2(X, \mathcal{O}_X) = 0 \)

and (iv) of Theorem 4.19 hold, then \( \text{Def}^G_S \to \text{Def}_X \) is smooth. If moreover

(v) \( G \) is finite or \( H^1(X, \mathcal{O}_X) = 0 \)

and (vi) of Theorem 4.19 hold then \( \text{Def}^G_S \to \text{Def}_X \) is an isomorphism.

Proof. Apply Lemma 4.1, Proposition 4.17 and Proposition 4.24 \qed

If \( S \) is regular then we get a statement about the rigidity of \( X \) and more generally about rigidity of \( X \) along a sheaf \( \mathcal{F} \), i.e. when \( T^1_X(\mathcal{F}) = 0 \). Note that if \((X, Z, S, J)\) is a \( G \)-subquadruple of \((Y, W, R, J)\) as defined in Definition 2.10 then the vanishing of \( T^1_X(f_*\mathcal{O}_X) \) implies that the forgetful map \( \text{Hilb}_{X/Y} \to \text{Def}_X \) is smooth. This is often useful for proving unobstructedness.

Corollary 4.27. Assume \( G \) is a quasitorus and let \((X, Z, S, J)\) be a \( G \)-quadruple with \( S \) a regular ring, \( M \) a finitely generated \( SG \)-module and \( \mathcal{F} = \pi_*(\widetilde{M}_{X'}) \). If

(i) \( \text{depth}_Z \mathcal{F} \geq 2 \), \( \text{depth}_Z \mathcal{F}^G \geq 3 \) and \( \text{depth}_{Z(Q)} \mathcal{F}^G \geq 4 \)
(ii) \( G \) is finite or \( H^2(X, \mathcal{F}^G) = 0 \)

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(iii) $\text{Hom}_S(\Omega^1_{S/\mathcal{O}_X}, \mathcal{H}^2_Z(\mathcal{F}))^G = 0$

(iv) $\text{Der}_k(S, \mathcal{H}^2_Z(M))^G = 0$

then $T^1_X(\mathcal{F}) = 0$. In particular if the conditions hold for $M = S$ then $X$ is rigid.

Proof. This follows from Lemma 4.2, Lemma 4.3 and Proposition 4.24.

Remark. By the Hochster-Roberts theorem $X$ will be Cohen-Macaulay and equidimensional if $S$ is a regular ring so we may exchange depth with codimension if $M = S$.

Example 4.28. Weighted projective space. Consider now $X = \mathbb{P}(q) = \mathbb{P}(q_0, \ldots, q_n)$ as described in Example 4.22. We use Corollary 4.27 to prove that if no $n - 1$ of the $q_0, \ldots, q_n$ have a common factor then $\mathbb{P}(q)$ is rigid.

The subscheme $Z$ is the singular locus of $X$ and the condition means that codim $Z \geq 3$. Let $S = k[x_0, \ldots, x_n]$ with $n \geq 2$. We have

$$J' = \bigcap_{p \text{ prime}} \{ x_i : p \mid q_i \}$$

so in fact codim $J' \geq 3$. The sheaf $Q$ is trivial since the isotropy is finite everywhere. This takes care of condition (i).

The cohomology $H^2(X, \mathcal{O}_X) = 0$ and since $J = (x_0, \ldots, x_n)$ clearly $H^2_Z(S) = 0$. We are left with (iii) but as in Example 4.18 since locally the quotient is by a finite group on a smooth space, depth $Z S = \text{codim} Z$ and $\mathcal{H}^2_Z(S) = 0$.

5 Applications to toric varieties

We collect here some results for toric varieties $X_\Sigma$ over $\mathbb{C}$ that illustrate the various aspects of our comparison theorems. That is the $G$-quadruple $(X_\Sigma, \text{Sing}(X_\Sigma), S, B(\Sigma))$ described in Example 2.5 and we use the notation from there. Note that toric varieties are Cohen-Macaulay and $S$ is a polynomial ring. Moreover the condition depth $Z S \geq 2$ is always satisfied since $U' = \text{Spec } S \setminus Z(\Sigma')$ where $\Sigma'$ is the fan of smooth cones in $\Sigma$ and codim $Z(\Sigma') \geq 2$.

5.1 Subschemes of simplicial toric varieties

We will look at the $G$-subquadruple induced by a homogeneous $I$ in the Cox ring $R$ of a toric variety $Y = Y_\Sigma$. Let $X \subset Y$ be the corresponding subscheme. We have $Z = \text{Sing}(Y) \cap X$, $S = R/I$ and $J = (I + B(\Sigma))/I$.

We assume for simplicity that $Y$ is simplicial and all maximal cones have dimension $d$ and that $X$ is Cohen-Macaulay and equidimensional. This means that the quotient $\pi : Y' = \text{Spec } (R \setminus V(B(\Sigma))) \rightarrow Y$ is a geometric quotient with all isotropy finite. In
particular $Q = 0$. Moreover locally on a chart $U_\sigma \subset Y$, corresponding to a maximal cone in $\Sigma$, the quotient map sits in a commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}^d & \longrightarrow & \mathbb{C}^d \times (\mathbb{C}^*)^r \\
\downarrow & & \downarrow \\
\mathbb{C}^d / G_\sigma & \cong & U_\sigma
\end{array}
$$

where $G_\sigma$ is finite abelian. See e.g. the proof of [BC94, Theorem 1.9]. If $Z$ is closed in $X \subseteq Y$ then codim $\pi^{-1}(Z) = \text{codim } Z$.

Since $Q = 0$ we may apply Theorem 4.20, but we still need to ensure that $\text{codim } \pi^{-1}(Z) = \text{codim } Z$. This will be the case if $\text{codim } Z \geq 3$.

Theorem 5.1. If $X$ is an equidimensional Cohen-Macaulay subscheme of a simplicial toric variety $Y$ and

(i) $\text{codim } Z \geq 3$

(ii) $H^2(X, \mathcal{O}_X) = 0$

(iii) $T^i(S/k; H^j_J(S))_0 = 0$ and $\text{Der}_k(S, H^2_J(S))_0 = 0$

then $\text{Def}^0_S \to \text{Def}_X$ is smooth. If moreover

(i) $H^1(X, \mathcal{O}_X) = 0$

(ii) $\text{Der}_k(S, H^1_J(S))_0 = 0$

then $\text{Def}^G_S \to \text{Def}_X$ is an isomorphism.

We give now two often studied situations where the theorem applies. First, let $X = \text{Proj} S$ where $S$ is a finitely generated $\mathbb{Z}_+$ graded algebra as in Example 2.4.

Corollary 5.2. Let $X = \text{Proj} S$ be an equidimensional Cohen-Macaulay subscheme of weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$ defined by the homogeneous ideal $I$. Let $\mathfrak{m}$ be the irrelevant maximal ideal of $S$ and $Z$ the intersection of the singular locus of $\mathbb{P}(q)$ with $X$. Assume $\text{codim } Z \geq 3$ and $\text{depth}_S \mathfrak{m} \geq 2$. If $H^2(X, \mathcal{O}_X) = 0$ and $H^1(X, \mathcal{O}_X(q_i)) = 0$ for all $i = 0, \ldots, n$ then $\text{Def}^0_S \to \text{Def}_X$ is smooth. If moreover $H^1(X, \mathcal{O}_X) = 0$ then $\text{Def}^G_S \to \text{Def}_X$ is an isomorphism.

Secondly let $X$ be a Calabi-Yau hypersurface in a simplicial Gorenstein Fano toric variety $Y$. (See the survey book [CK99] or the original paper by Batyrev [Bat94] for details.) Thus $Y$ is given by the normal fan of a simple reflexive polytope and $X$ is a divisor in the class of $-K_Y$ and therefore ample and Cartier. In particular $\omega_X = \mathcal{O}_X$ and
\[ H^i(X, \mathcal{O}_X) = 0 \] for \( i \geq 1 \). We refer to it as a Calabi-Yau hypersurface even though it may be highly singular.

Let \( D_i \) be the divisors corresponding to the rays \( \rho_i \). The hypersurface \( X \) is defined by some \( f \in R_\beta \) where \( \beta = \sum D_i \). For our results it is not necessary to assume any generality for \( f \). We will need the following lemma and are grateful to Benjamin Nill for supplying the proof. (We use standard toric geometry notation as found in e.g. [CLS11].)

**Lemma 5.3.** Let \( P \) be a simple reflexive lattice polytope with inward normal fan \( \Sigma \) and let \( Y \) be the corresponding simplicial Gorenstein Fano toric variety. For every ray \( \rho_i \) in \( \Sigma \) the \( \mathbb{Q} \)-Cartier divisor \( E_i = \sum_{j \neq i} D_j = -K_Y - D_i \) is nef and big.

**Proof.** Let \( P^* \) be the dual reflexive polytope and \( v_i \) the primitive lattice points on \( \rho_i \) (i.e., the vertices of \( P^* \)). Let \( h \) be the piecewise linear function on \( N_\mathbb{R} \) such that \( h(v_i) = 0 \) and \( h(v_j) = -1 \) for \( j \neq i \). Then \( E_i \) is the divisor associated to \( h \) and \( P_h = \{ m \in M_\mathbb{R} : \langle m, v_i \rangle \geq 0 \text{ and } \langle m, v_j \rangle \geq -1 \text{ for } i \neq j \} \) is the corresponding polytope. Geometrically, \( P_h \) is \( P \) after moving the facet \( F_i = (v_i)^* \) of \( P \) one step inwards (so that the origin now lies on the boundary of \( P_h \)). Thus \( \dim P_h = \dim P \) and \( E_i \) is big.

Let \( \sigma \) be a maximal cone of \( \Sigma \), and \( \sigma^* \) the corresponding vertex of \( P \). Since \( \Sigma \) is simplicial, \( E_i \) is \( \mathbb{Q} \)-Cartier so there exists \( l_\sigma \in M_\mathbb{Q} \) with \( \langle l_\sigma, v \rangle = h(v) \) for \( v \in \sigma \). It is well known that if \( l_\sigma \) is contained in \( P_h \) for all \( \sigma \) then \( E_i \) is nef. There are two cases, \( v_i \notin \sigma \) or \( v_i \in \sigma \). If \( v_i \notin \sigma \), then \( l_\sigma \) evaluates to \(-1 \) on any vertex of \( \sigma \). In other words, \( l_\sigma = \sigma^* \in P \). Thus \( l_\sigma \) evaluates \(-1 \) on any \( v_j \). Moreover, since by duality \( l_\sigma = \sigma^* \notin F_i \), we get that \( l_\sigma \) evaluates \( > -1 \) on \( v_i \). However, \( P \) is a lattice polytope, hence \( l_\sigma \) as a vertex of \( P \) is a lattice point, so \( l_\sigma \) evaluates \( \geq 0 \) on \( v_i \).

Assume \( v_i \in \sigma \). Since \( \Sigma \) is simplicial, the facet of \( P^* \) corresponding to the maximal cone \( \sigma \) is the convex hull of \( v_i \) and a \((d-2)\)-dimensional face \( G \) of \( P^* \). Note that by duality \( G^* \) is an edge of \( P \), and \( \sigma^* \) is contained in \( G^* \) and the facet \( F_i = (v_i)^* \). In particular, the intersection of the affine hull of \( G^* \) with the hyperplane \( H \) orthogonal to \( v_i \) (which is parallel to \( F_i \)) is precisely the point \( l_\sigma \). Since \( G^* \) is an edge of \( P^* \), it has a vertex \( w \) different from \( \sigma^* \). Since \( w \) is not in \( F_i \) (otherwise, \( G^* \) would be contained in \( F_i \), hence \( v_i \) would be contained in \( G \)), we have that \( v_i \) evaluates \( > -1 \) on \( w \). Because \( P \) is a lattice polytope, \( w \) is a lattice point, so \( v_i \) evaluates \( \geq 0 \) on \( w \). In particular, \( l_\sigma \) (as it lies on the affine hull of \( G^* \) and evaluates to \( 0 \) with \( v_i \)) lies between \( \sigma^* \) and \( w \). Therefore, \( l_\sigma \in P \). This implies that not only \( l_\sigma \) evaluates to \( 0 \) on \( v_i \), but it evaluates \( \geq -1 \) with all other \( v_j \)'s, as desired. \( \square \)

**Theorem 5.4.** Let \( X \) be a Calabi-Yau hypersurface in a simplicial Gorenstein Fano toric variety \( Y \) defined by \( f \in R_\beta \) where \( R = \mathbb{C}[x_1, \ldots, x_N] \) is the Cox ring of \( Y \) and \( \beta \) is the anti-canonical class. If \( \dim Y \geq 3 \) and \( \text{codim}(X \cap \text{Sing}(Y)) \geq 3 \) in \( X \) then \( \text{Def}_X \) is smooth and its tangent space is isomorphic to the degree \( \beta \) part of the Jacobian algebra of \( f \). That is

\[ T_X^\beta \cong \left( \mathbb{C}[x_1, \ldots, x_N]/ \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N} \right) \right)_\beta. \]

**Proof.** We try to apply Theorem 5.1. Since \( H^i(X, \mathcal{O}_X) = 0 \) for \( i = 1, 2 \) by the Calabi-Yau property, it suffices to show that \( H^1_\beta(S), H^2_\beta(S)_{D_i}, \) and \( H^1_\beta(S)_{D_i} \) all vanish. From the exact
sequence
\[ 0 \to R(-\beta) \to R \to S \to 0 \]
we see that this would follow from

(i) \( H^1_j(R)_{\beta} = 0 \) and \( H^1_j(R)_{D_i} = 0 \)

(ii) \( H^2_j(R)_0 \simeq H^1(Y, \mathcal{O}_Y) = 0 \)

(iii) \( H^2_j(R)_{D_i} \simeq H^1(Y, \mathcal{O}_Y(D_i)) = 0 \)

(iv) \( H^{j+1}_j(R)_{D_i-\beta} \simeq H^j(Y, \mathcal{O}_Y(K_Y + D_i)) = 0 \) for \( j = 1, 2 \).

Now (i) is true because \( \text{depth}_{R} \geq 2 \) and (ii) is true for all complete toric varieties. To prove (iii) we have that \(-K_Y\) is ample so we may e.g. apply a vanishing theorem of Mustată as stated in [CLS11, Theorem 9.3.7]. The last vanishing follows from the assumption that \( \dim Y \geq 3 \), Lemma 5.3 and the \( \mathbb{Q}\)-Cartier version of a vanishing result of Batyrev-Borisov as stated in [CLS11, Theorem 9.3.5].

### 5.2 Local cohomology computations

In the following it will be important to be able to compute with the modules \( H^i_B(S) \) where \( S \) is the Cox ring of a toric variety and \( B \) the irrelevant ideal for some fan. There is a combinatorial method due to Mustatǎ and we will make some simplifications in the case \( i = 2 \).

Let \( \{m_1, \ldots, m_s\} \) be monomial generators for any squarefree monomial ideal \( B \subseteq S \). For \( I \subseteq \{1, \ldots, s\} \) let \( T_I \) be the simplicial complex on the vertex set \( \{1, \ldots, s\} \) where \( \{j_1, \ldots, j_k\} \) is a face if \( x_i \nmid \text{lcm}(m_{j_1}, \ldots, m_{j_k}) \) for some \( i \in I \). If \( p \in \mathbb{Z}^m \) define \( \text{neg}(p) \subseteq \{1, \ldots, m\} \) to be the set \( \{i \mid p_i \leq -1\} \). Let \( \{e_1, \ldots, e_m\} \) be the standard generators of \( \mathbb{Z}^m \).

**Theorem 5.5.** ([Mus00, Theorem 2.1]) If \( p \in \mathbb{Z}^m \) then there are isomorphisms \( H^i_B(S)_p \simeq \widetilde{H}^{i-2}(T_I; k) \) when \( I = \text{neg}(p) \). Moreover the map

\[ H^i_B(S)_p \xrightarrow{x} H^i_B(S)_{p+e_i} \]

corresponds to the map \( H^{i-2}(T_{\text{neg}(p)}; k) \to H^{i-2}(T_{\text{neg}(p+e_i)}; k) \) induced in cohomology by the inclusion \( T_{\text{neg}(p+e_i)} \subseteq T_{\text{neg}(p)} \). In particular, if \( p_i \neq -1 \), then \( x_i \) is an isomorphism.

From now on assume \( S \) is the Cox ring and \( B \) the irrelevant ideal for a fan \( \Sigma \).

**Lemma 5.6.** The codimension 2 prime ideals of \( B \) are the \((x_i, x_j)\) with \( \rho_i \) and \( \rho_j \) not in the same cone in \( \Sigma \).

**Proof.** This follows directly from the description of the prime components of \( B \). See e.g. [CLS11, Proposition 5.1.6].

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Let $B_2$ be the intersection of the codimension 2 primes of $B$. Let $K$ be the simplicial complex which has $B_2$ as Stanley-Reisner ideal. Let $\Gamma$ be the graph with vertices $\{0, \ldots, m\}$ and edges $\{i, j\}$ when $\rho_i$ and $\rho_j$ are in the same cone in $\Sigma$. Define $C(\Gamma)$ to be the clique complex of $\Gamma$. Let $\Gamma_I$ be the induced subgraph with vertices in $I$.

**Lemma 5.7.** The complex $K$ is the Alexander dual of $C(\Gamma)$. In particular there is a one to one correspondence between facets of $C(\Gamma)$ and a minimal generating set for $B_2$ given by $F \mapsto x_{Fc}$. This correspondence identifies $T_I$ with the complex with vertex set equal the set of facets of $C(\Gamma)$ containing an element of $I$ and $\{F_{i_1}, \ldots, F_{i_k}\}$ is a face if $i \in \bigcap F_{i_j}$ for some $i \in I$.

**Proof.** The facets of $K$ are the complements $\{i, j\}^c$ where $\{i, j\}$ is a non-edge of $\Gamma$. We have $f \in C(\Gamma)^c \iff f^c \notin C(\Gamma) \iff f^c$ contains a non-edge of $\Gamma \iff f \subseteq \{i, j\}^c$ for a non-edge $\{i, j\} \iff f \in K$. A set $\{x_{F_{i_1}}, \ldots, x_{F_{i_k}}\}$ is a face of $T_I$ if there is an $i \in I$ with $i \notin \bigcup F_{i_k} = (\bigcap F_{i_k})^c$, that is if $i \notin \bigcap F_{i_k}$.

**Lemma 5.8.** If $i \in I$ let $f_i$ be the face of $T_I$ given by $\{F : i \in F\}$. The map $H_0(\Gamma_I) \to H_0(T_I)$ defined by $[i] \mapsto [F]$ where $F$ is any vertex in $f_i$ is an isomorphism.

**Proof.** First note that $\{i, j\}$ is an edge in $\Gamma_I$ iff there is a maximal clique $F$ of $\Gamma$ containing $\{i, j\}$ which is iff $f_i \cap f_j \neq \emptyset$. To show that the map is well defined assume $[i] = [j]$. Then there is an edge path through the vertices $i = i_0, i_1, \ldots, i_r = j$ in $\Gamma_I$. Then since the $f_{i_k} \cap f_{i_{k+1}} \neq \emptyset$ we may find a path from any vertex of $f_i$ to any vertex in $f_j$.

The map is clearly surjective. To show injectivity assume $F \in f_i$ and $G \in f_j$ are connected by an edge path $F = F_0, F_1, \ldots, F_r = G$. This means that the intersection of cliques $F_k \cap F_{k+1}$ contains an element $i_k \in I$. This implies that there are edges $\{i_k, i_{k+1}\} \in \Gamma$ and therefore in the induced subgraph $\Gamma_I$. Thus $[i] = [j]$.

**Proposition 5.9.** There is an isomorphism $H^2_B(S)_p \simeq \bar{H}^0(\Gamma_I, k)$ where $I = \neg(p)$.

**Proof.** This follows now directly from Theorem 5.5, Lemma 5.7 and 5.8.

### 5.3 $T^1$ for toric singularities

There has been much interest in computing deformations of affine toric varieties. In particular there are several combinatorial descriptions of these due to Altmann, see e.g. [Alt00] and the references there in. We give here an alternative description using the Cox ring and compute $T^1$ in some special cases. To simplify matters we will assume codim $\Sing(X) \geq 3$.

Let $X$ be an affine toric variety with cone $\sigma$ and Cox ring $S$. In the grading defined by the abelian group $\Cl(X)$ we have $X = \Spec S_0$. Set $Z = \Sing(X)$. Thus $(X, Z, S, (1))$ is an affine $G$-quadruple where $G = \Hom_{\Bbbk}(\Cl(X), \Bbbk^*)$. The ferry is just $S$ itself. If $\Sigma$ is the fan consisting of smooth faces of $\sigma$ and $B = B(\Sigma)$ the corresponding irrelevant ideal then the sheafs $\mathcal{H}_Z^2(S)$ correspond to the modules $H^2_B(S)$.

Let $n = \dim X = \rk N$, $m = |\Sigma(1)|$ and $r = \rk \Cl(X) = m - n$. Let $C \subseteq \Cl(X)$ be the free part of the abelian group. Fix an isomorphism $C \simeq \mathbb{Z}^r$ so that the map $\mathbb{Z}^m \to C$ has
matrix $A = (a_{ij})$. The $\mathbb{Z}^r$ part of the $\text{Cl}(X)$ grading on $S = \mathbb{C}[x_1, \ldots, x_m]$ is given by the columns of $A$, i.e. $\deg x_j = (a_{1j}, \ldots, a_{rj})$. A set of Euler derivations as in Definition 4.8 is

$$\{E_i = \sum_j a_{ij} x_j \frac{\partial}{\partial x_j} : i = 1, \ldots, r\}.$$ 

Thus the module $Q$ corresponding to the sheaf $\mathcal{Q}$ in Definition 4.8 has a graded presentation

$$\bigoplus_{j=1}^m S(-\deg x_j) \overset{\tilde{A}}{\rightarrow} S^r \rightarrow Q \rightarrow 0 \quad (5.1)$$

where $\tilde{A} = (a_{ij}x_j)$. The support of $Q$ on $X$ is the non-simplicial locus corresponding to the irrelevant ideal $B(\mathcal{Q}) \subset S$ for the fan consisting of simplicial faces of $\sigma$.

**Theorem 5.10.** Let $X$ be an affine toric variety.

(i) If $X$ is simplicial then $\text{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 \simeq T^1_X$.

(ii) If $\text{codim} \text{Sing}(X) \geq 3$, then there is an exact sequence

$$0 \rightarrow \text{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 \rightarrow T^1_X \rightarrow \text{Hom}_S(Q, H^3_B(S))_0 \rightarrow 0$$

and inclusions $\text{Hom}_S(Q, H^3_B(S))_0 \subseteq \text{Hom}_{S_0}(Q_0, H^3_B(S)) \subseteq H^3_{B_0}(S_0)^r$.

**Proof.** As in the proof of Lemma 3.1 $T^1_X = T^1_{S_0} = T^1_{S_0}(S)_0$. The Zariski-Jacobi sequence for $\mathbb{C} \rightarrow S_0 \rightarrow S$ and the fact that $S$ is regular yields $T^1_{S_0}(S) \simeq T^2_{S/S_0}$. Furthermore by Lemma 4.10 we have $T^2_{S/S_0} \simeq \text{Ext}^2_S(\Omega^1_{S/S_0}, S)$. The result, except for the 0 on the right in the exact sequence in (ii), now follows by applying the proof of Proposition 4.17 to the affine case.

Recall from the proof of Proposition 4.17 that the cokernel of $\text{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S)) \rightarrow \text{Ext}^2_S(\Omega^1_{S/S_0}, S)$ is the kernel $K$ of $H^3_B(S)^r \rightarrow \text{Ext}^3_{S,B}(\Omega^1_{S/S_0}, S)$ in the long exact sequence for $\text{Ext}^1_{S,B}(\Omega^1_{S/S_0}, S)$. We have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \rightarrow & K & \rightarrow & H^3_B(S)^r & \rightarrow & \text{Ext}^3_{S,B}(\Omega^1_{S/S_0}, S) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Hom}_S(Q, H^3_B(S)) & \rightarrow & H^3_B(S)^r & \rightarrow & \text{Hom}_S(\Omega^1_{S/S_0}, H^3_B(S)) & \rightarrow & 0
\end{array}$$

where the right column comes from the spectral sequence for $\text{Ext}^1_{S,B}(\Omega^1_{S/S_0}, S)$. By assumption $H^3_B(S)_0 = 0$ so by the below Lemma 5.11 $\text{Ext}^1_S(\Omega^1_{S/S_0}, H^3_B(S))_0 = 0$ and $K \simeq \text{Hom}_S(Q, H^3_B(S))$. \qed
Lemma 5.11. Let $G$ be a linearly reductive group acting on an $n$-dimensional $k$ vector space $V$. Let $A = \text{Sym}(V)$ and let $N$ be an $AG$-module with $N^G = 0$. Then $\text{Ext}_A^1(\Omega^1_{A/AG}, N)^G = 0$.

Proof. If $A^G = k[\varphi_1, \ldots, \varphi_m] \subseteq A = k[x_1, \ldots, x_n]$ then the beginning of an $AG$ projective resolution of $\Omega^1_{A/AG}$ looks like

$$\ldots \rightarrow A^m \xrightarrow{(\partial \varphi_i / \partial x_j)} A \otimes_k V^* \rightarrow \Omega^1_{A/AG} \rightarrow 0.$$ 

If $M$ is the image of $A^m$ in $A \otimes_k V^*$ then $M$ is generated by invariants, so $\text{Hom}_A(M, N)^G = 0$. Thus $\text{Ext}_A^1(\Omega^1_{A/AG}, N)^G = 0$ by the long exact sequence for $\text{Ext}$.

We can ask when the contribution from $\text{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 = \text{Der}_{S_0}(S, H^2_B(S))_0$ vanishes. This would happen if $H^2_B(S)_{\alpha_i} = 0$ for $i = 1, \ldots, m$ where $\alpha_i$ is the degree of $x_i$. Lifting this to the $\mathbb{Z}^m$ grading we need $H^2_B(S)_p = 0$ when $p = q + e_i$ and $q = (\ldots, \langle u, v_i \rangle, \ldots)$ for some $u \in M$. If we assume codim $\text{Sing}(X) \geq 3$ then $H^2_B(S)_0 = H^2_{\text{Sing}(X)}(S_0) = 0$. In the $\mathbb{Z}^m$ grading this means $H^2_B(S)_q = 0$ when $q$ is as above. Thus by Theorem 5.5 it is enough to check when the coordinate $\langle u, v_i \rangle$ of $q$ equals $-1$.

Let $\Gamma^f$ be the graph with vertices $\{0, \ldots, m\}$ and edges $\{i, j\}$ when $\{\rho_i, \rho_j\}$ span a 2-dimensional face of $\sigma$ and $\Gamma^f$ the induced subgraph as above. If codim $\text{Sing}(X) \geq 3$ then $\Gamma^f$ is a subgraph of $\Gamma_I(\Sigma)$ with the same vertex set so the map $\tilde{H}_0(\Gamma^f_I) \rightarrow \tilde{H}_0(\Gamma_I)$ is surjective. If $u \in M$ and $\langle u, v_i \rangle = -1$ let

$$I_i(u) = \{j \in \{1, \ldots, m\} : \langle u, v_j \rangle \leq -1\} \setminus \{i\}$$

and set $\Gamma_i(u) = \Gamma^f_{I_i(u)}$. (These graphs also appear in the study of deformations of smooth complete toric varieties in [Ilt11].) Thus we have

Proposition 5.12. Let $X$ be an affine toric variety that is smooth in codimension 2 and $\Gamma_i(u)$ is connected for all $u \in M$. Then $\text{Hom}_S(\Omega^1_{S/S_0}, H^2_B(S))_0 = 0$.

The connectivity of these graphs can be proven if there is a polytope we can use. For this we need the following lemma.

Lemma 5.13. Let $P$ be a polytope, $H^*$ a closed half-space with bounding hyperplane $H$ and $v$ a vertex of $P$ with $v \in H$. If $\Gamma_v(H)$ is the induced subgraph of the edge graph of $P$ on the vertices in

$$\{w \in \text{vert } P : w \in H^*\} \setminus \{v\}$$

then $\Gamma_v(H)$ is connected.
Proof. If $H \cap \text{int} \ P \neq \emptyset$ let $H'$ be the supporting hyperplane in $H^*$ that is parallel to $H$. By standard arguments every vertex in $\Gamma_i(H)$ is either on $H'$ or has a neighbor vertex in $H^*$ that is nearer to $H'$. Thus $\Gamma_v(H)$ is connected in this case.

If $H \cap \text{int} \ P = \emptyset$ then $H$ is a supporting hyperplane. If $P \subset H^*$ then $\Gamma_v(H)$ is the edge graph of $P$ with one vertex removed. Since edge graphs of $d$-polytopes are $d$-connected $\Gamma_v(H)$ is connected (trivially also for $d = 0, 1$). If $P \cap H^* = F$ is a face then the same argument applied to $F$ yields the result. \hfill \Box

Proposition 5.14. A $\mathbb{Q}$-Gorenstein affine toric variety that is smooth in codimension 2 has $\text{Hom}(\Omega^i_S/S_0, H_B^2(S))_0 = 0$.

Proof. An affine toric variety is $\mathbb{Q}$-Gorenstein if and only if there is a primitive $u_0 \in M$ and a positive integer $g$ such that $\langle u_0, v_i \rangle = g$ for all generators $v_i$ of the rays of $\sigma$. This means that if $P$ is the convex hull of the $v_i$ in the hyperplane $\langle u_0, - \rangle = g$, then for all $u \in M$ the graph $\Gamma_i(u)$ is a $\Gamma_v(H)$ as in Lemma 5.13. \hfill \Box

Example 5.15. We compute $T^3_3$ for a non-simplicial toric 3-dimensional Gorenstein isolated singularity. Note that it follows from Theorem 5.10 and Proposition 5.14 that a toric $d$-dimensional $\mathbb{Q}$-Gorenstein isolated singularity is rigid if $d \geq 4$. (For a complete description of $T^1$ for all toric $\mathbb{Q}$-Gorenstein singularities see [Alt95] and [Alt00].) From Theorem 5.10 and Proposition 5.14 it follows that $T^3_3 \simeq \text{Hom}(Q, H_B^3(S))_0$.

The cone $\sigma$ has a special form as it comes from a plane lattice polygon $P$ with smooth face fan (and more than 3 vertices). If $(\alpha_1, \beta_1), ..., (\alpha_m, \beta_m)$ are the vertices of such a $P$ then the primitive generators of the rays of $\sigma$ are $v_i = (\alpha_i, \beta_i, 1)$.

The fan $\Sigma$ consists of all faces of $\sigma$ except $\sigma$ itself. Using the description of the prime components in [CLS11 Proposition 5.1.6] one sees that $B = B_2$ in the notation Section 5.2. Thus we may use the methods there to show that $H_B^3(S)_p \simeq H^1(\Gamma, k)$ where $I = \text{neg}(p)$. Now it is easy to see that $H_B^3(S)_p \neq 0$ if and only if all $p_i \leq -1$. Note that the Gorenstein property can be seen through the one to one correspondence $p \mapsto -p - (1, \ldots, 1)$ between non-zero pieces of $H_B^3(S)_0$ and $S_0$.

In our situation $\text{Cl}(X) \simeq \mathbb{Z}^r$ so there is an exact sequence

$$0 \to \mathbb{Z}^3 \xrightarrow{A'} \mathbb{Z}^m \xrightarrow{A} \mathbb{Z}^r \to 0$$



where the rows of $A'$ are $[\alpha_i, \beta_i, 1]$ and $A = (a_{ij})$ gives the set of Euler derivations. Using the presentation of $Q$ in (5.1) we may describe the $\mathbb{Z}^m$ graded pieces of $\text{Hom}(Q, H_B^3(S))_0$ as follows. It is convenient to use the above one-to-one correspondence so set $q = -p - (1, \ldots, 1)$. If $\text{Hom}(Q, H_B^3(S))_{0,q} \neq 0$ then $A \cdot q = 0$ and $\text{neg}(q) = \emptyset$. If this is satisfied then $\text{Hom}(Q, H_B^3(S))_{0,p}$ is isomorphic to the kernel of the submatrix $A'_q$ consisting of the rows $A'_q$ of $A'$ where $q_i > 0$.

From the smoothness of $P$ it follows that all the $3 \times 3$ minors of $A'$ are non-zero. Thus $A$ has the special property:

All $r \times r$ minors of $A$ are non-zero. \hfill (5.2)
Let $k$ be the number of $i$ with $q_i \neq 0$. If $k \geq r$ then (5.2) implies that $A^t_q$ has trivial kernel. On the other hand if $1 \leq k < r$ let $q' = (q_{i_1}, \ldots, q_{i_k})$ be the vector of non-zero components. Since $Aq = 0$ we must have $(A^t_q)q' = 0$, but this is impossible since by (5.2) the columns of $(A^t_q)^t$ are independent.

We conclude that the only non-zero Hom$(Q,H^3_B(S))_{0,p}$ is when $p = (-1, \ldots, -1)$ and that $\dim T^1_X = \dim \text{Hom}(Q,H^3_B(S))_{0} = r = m - 3$.

### 5.4 Rigidity results

The above together with Corollary 4.27 yield rigidity results for toric singularities. First we reprove a theorem of Altmann in [Alt95].

**Corollary 5.16.** A $\mathbb{Q}$-Gorenstein affine toric variety that is smooth in codimension 2 and simplicial in codimension 3 is rigid.

**Proof.** This follows directly from Corollary 4.27 and Proposition 5.14.

With the same type of arguments we can reprove Totaro’s generalization, [Tot12, Theorem 5.1] of theorems of Bien-Brion and de Fernex-Hacon.

**Corollary 5.17.** A toric Fano variety that is smooth in codimension 2 and simplicial in codimension 3 is rigid.

**Proof.** Corollary 5.16 takes care of the local situation so what is left to prove is condition (iv) in Corollary 4.27. This follows from $H^2_B(\Sigma)(S)_{D_i} = H^1(X,O_X(D_i)) = 0$ since $X$ is Fano. Alternatively, the Fano condition implies that the fan $\Sigma$ is the face fan of a polytope. Thus the same arguments as above show that $H^2_B(\Sigma)(S)_{D_i} = 0$.

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