Retrospective change-point detection and estimation in multivariate linear models

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Abstract

In this paper the problem of retrospective change-point detection and estimation in multivariate linear models is considered. The lower bounds for the error of change-point estimation are proved in different cases (one change-point: deterministic and stochastic predictors, multiple change-points). A new method for retrospective change-point detection and estimation is proposed and its main performance characteristics (type 1 and type 2 errors, the error of estimation) are studied for dependent observations in situations of deterministic and stochastic predictors and unknown change-points. We prove that this method is asymptotically optimal by the order of convergence of change-point estimates to their true values as the sample size tends to infinity. Results of a simulation study of the main performance characteristics of proposed method in comparison with other well known methods of retrospective change-point detection and estimation are presented.

Keywords: change-point; retrospective detection and estimation; performance measure; asymptotic optimality
1 Introduction

This paper deals with change-point problems for multivariate linear models. We begin with a short review of this field.

The change-point problem for regression models was first considered by Quandt (1958, 1960). Using econometric examples Quandt proposed a method for estimation of a change-point in a sequence of independent observations based upon the likelihood ratio test.

Let us describe the change-point problem for the linear regression models considered in the literature. Let $y_1, y_2, \ldots, y_n$ be independent random variables (i.r.v.'s). Under the null hypothesis $H_0$ the linear model is

$$y_i = x_i^* \beta + \epsilon_i, \quad 1 \leq i \leq n,$$

where $\beta = (\beta_1, \beta_2, \ldots, \beta_d)^*$ is an unknown vector of coefficients, $x_i^* = (1, x_{2i}, \ldots, x_{di})$ are known predictors (here and below $*$ is the transposition symbol).

The errors $\epsilon_i$ are supposed to be independent identically distributed random variables (i.i.d.r.v.'s) with $E \epsilon_i = 0$, $0 < \sigma^2 = var \epsilon_i < \infty$.

Under the alternative hypothesis $H_1$ a change at the instant $k^*$ occurs, i.e.

$$y_i = \begin{cases} x_i^* \beta + \epsilon_i, & 1 \leq i \leq k^* \\ x_i^* \gamma + \epsilon_i, & k^* < i \leq n, \end{cases}$$

where $k^*$ and $\gamma \in \mathbb{R}^d$ are unknown parameters, and $\beta \neq \gamma$.

Denote

$$\bar{y}_k = \frac{1}{k} \sum_{1 \leq i \leq k} y_i, \bar{x}_k = \frac{1}{k} \sum_{1 \leq i \leq k} x_i,$$

$$Q_n = \sum_{1 \leq i \leq n} (x_i - \bar{x}_n)(x_i - \bar{x}_n)^*$$

and $X_n = (x_1, x_2, \ldots, x_n)^*$, $Y_n = (y_1, y_2, \ldots, y_n)^*$.

The least square estimate of $\beta$ is:

$$\hat{\beta}_n = (X_n^*X_n)^{-1} X_n^* Y_n.$$

Siegmund with co-authours (James, James, Siegmund (1989)) proposed to reject $H_0$ for the large values of $\max_{1 \leq k \leq n} |U_n(k)|$, where

$$U_n(k) = \left( \frac{k}{1 - k/n} \right)^{1/2} \frac{\bar{y}_k - \bar{y}_n - \hat{\beta}_n(\bar{x}_k - \bar{x}_n)^*}{(1 - k(\bar{x}_k - \bar{x}_n)(\bar{x}_k - \bar{x}_n)^*/(Q_n(1 - k/n)))^{1/2}}.$$
Earlier, Brown, Durbin, and Evans (1975) used the cumulative sums of regression residuals

$$\sum_{1 \leq i \leq k} (y_i - \bar{y}_n - \hat{\beta}_n (x_i - \bar{x}_n)^{*}), \quad 1 \leq k \leq n.$$ 

It is easy to see that

$$U_n(k) = w_n(k) R_n(k)$$

$$R_n(k) = \left( \frac{n}{k(n-k)} \right)^{1/2} \sum_{1 \leq i \leq k} (y_i - \bar{y}_n - \hat{\beta}_n (x_i - \bar{x}_n)^{*})$$

$$w_n(k) = 1 - k(\bar{x}_k - \bar{x}_n)(\bar{x}_k - \bar{x}_n)^{*}/(Q_n(1 - k/n))^{-1/2}.$$ 

The functionals of $U_n(k)$ and $R_n(k)$ were used as the test statistics for detection of change-points in regression relationships.

Kim and Siegmund (1989) obtained the limit distribution of $\max_{1 \leq k < n} |U_n(k)|$. Alternatively, Maronna and Yohay (1978), and Worsley (1986) used the maximum likelihood method for testing $H_0$ against $H_1$ for Gaussian errors. Later Gombay and Horvath (1994) studied the limit distributions of statistics $Z_n(i,j) = \max_{i \leq k < j} |U_n(k)|$, $T_n(i,j) = \max_{i \leq k < j} |R_n(k)|$ for deterministic and stochastic regression plans. The monograph by Csorgo and Horvath (1997) puts together various results in detection of structural changes in regression models.

Besides change-point detection problems, results in change-point estimation for regressions are of especial practical importance. This theme is considered in papers by Darkhovsky (1995), Huskova (1996), Horvath, Huskova, and Serbinovska (1997). In two last papers the asymptotical characteristics of change-point estimates based upon the maximum likelihood statistics are studied. For the case of contiguous alternatives, the limit distribution of the change-point estimates is obtained and weak and strong consistency of these estimates is proved. The paper by Darkhovsky (1995) develops the nonparametric approach to retrospective change-point estimation. Here the limit characteristics of change-point estimates in the functional regression model are studied without the contiguity assumption, and the rate of convergence of these estimates to the 'true' change-point parameters is estimated. Some generalizations of these results can be found in the monograph by Brodsky and Darkhovsky (2000).

A new wave of research interest to change-point problems in regressions was formed in 2000s. Different generalizations to change-point problems for autoregressive time series (Huskova, Praskova, Steinebach (2007, 2008), Gombay (2008)), for multiple change-point estimation in non-stationary time series (Davis, Lee, Rodriguez-Yam (2006)), for
testing change-points in covariance structure of linear processes (Berkes, Gombay, Horvath (2009)) were studied.

However, as a result we see the multitude of methods proposed for solving different change-point problems in linear relationships and almost no theoretical approaches to their comparative analysis. We cannot even estimate the asymptotic efficiency of these methods. All that is empirically observed for 'structural breaks' tests in statistics and econometrics can be reduced to the following 'vague' statement: the power of these methods is rather low. Let us agree that this 'practical conclusion' requires a more serious verification.

In this paper, we pursue the following main goals:

1) To prove the prior theoretical lower bounds for the error probability in change-point estimation in multivariate models. These bounds provide the theoretical basis for the proofs of the asymptotic optimality of change-point estimates and for the comparative analysis of these estimates;

2) To propose a new nonparametric method for the problem of retrospective change-point detection and estimation in multivariate linear systems. Then we study the main performance characteristics of this method: type 1 and type 2 errors, the error of change-point estimation.

3) For the problem of multiple change-point detection and estimation, to propose a general statement in which both the number of change-points and their coordinates in the sample are unknown. For this problem statement, to propose a new asymptotically optimal method which gives consistent estimates of an unknown number of change-points and their coordinates.

The structure of this paper is as follows. In Section 2 the general change-point problem for multivariate linear systems is formulated and general assumptions are given. In Section 3 we prove the prior informational inequalities for the main performance characteristic of the retrospective change-point problem, namely, the error of change-point estimation. The lower bounds for the error of estimation are found in different situations of change-point detection (deterministic and stochastic regression plan, multiple change-points). In Section 4 we propose a new method for the retrospective change-point detection and estimation in multivariate linear models and study its main performance characteristics (type 1 and type 2 errors, the error of estimation) in different situations of change-point detection and estimation (dependent observations, deterministic and stochastic regression plan, multiple change-points). We prove that
this method is asymptotically optimal by the order of convergence of change-point estimates to their true values as the sample size tends to infinity. In Section 5 a variant of the functional limit theorem in the case of absence of change-points is given. In Section 6 a simulation study of characteristics of the proposed method for finite sample sizes is performed. The main goals of this study are as follows: to compare performance characteristics of the proposed method with characteristics of other well known methods of change-point detection in linear regression models, to consider more general multivariate linear models and performance characteristics of the proposed method in these multivariate models. Section 7 contains main conclusions. All proofs are given in the Appendix.

2 Problem statement and general assumptions

2.1 General model

The following basic specification of the multivariate system with structural changes is considered:

\[ Y(n) = \Pi X(n) + \nu_n, \quad n = 1, \ldots, N \] (1)

where \( Y(n) = (y_1n, \ldots, y_Mn)^\star \) is the vector of endogenous variables, \( X(n) = (x_1n, \ldots, x_Kn)^\star \) is the vector of pre-determined variables, \( \Pi \) is an \( M \times K \) matrix, \( \nu_n = (\nu_1n, \ldots, \nu_Mn)^\star \) is the vector of random errors.

The matrix \( \Pi = \Pi(\vartheta, n), \vartheta = (\theta_1, \ldots, \theta_k) \) can change abruptly at some unknown change-points \( m_i = \lfloor \theta_iN \rfloor, i = 1, \ldots, k \) (here and below \( [a] \) denote the integer part of number \( a \)), i.e.,

\[ \Pi(\vartheta, n) = \sum_{i=1}^{k+1} a_i \mathbb{I}([\theta_{i-1}N] < n \leq [\theta_iN]), \]

where \( \theta_i \) are unknown change-point parameters such that \( 0 \equiv \theta_0 < \theta_1 < \ldots < \theta_k < \theta_{k+1} \equiv 1 \), \( a_i \neq a_{i+1}, i = 1, \ldots, k \) are unknown matrices (here and below \( \mathbb{I}(A) \) is the indicator of the set \( A \)).

The problem is to estimate the unknown parameters \( \theta_i \) (and therefore, the change-points \( m_i \)) by observations \( Y(i), X(i), i = 1, \ldots, N \) (the case \( \theta_i \equiv 1, i = 1, \ldots, k \) corresponds to the model without change-points).

Therefore, first, we need to test an obtained dataset of observations for the presence of change-points. Second, in the case of a rejected stationarity hypothesis, we wish to
estimate all detected change-points.

Model (1) generalizes many widely used regression models, namely:

a) autoregression model (AR)

\[ y_n = c_0 + c_1 y_{n-1} + \cdots + c_m y_{n-m} + \nu_n, \]

Here \( X(n) = (1, y_{n-1}, \ldots, y_{n-m})^\prime, \Pi = (c_0, c_1, \ldots, c_m). \)

b) autoregression-moving average (ARMA) model

\[ y_n = c_1 y_{n-1} + \cdots + c_k y_{n-k} + d_1 u_{n-\Delta} + \cdots + d_m u_{n-\Delta-m} + \nu_n, \]

where \( u_n \) is the input variable, \( y_n \) is the output variable at the instant \( n \), \( \Delta \) is the delay time. Here \( X(n) = (y_{n-1}, \ldots, y_{n-m}, u_{n-\Delta}, \ldots, u_{n-\Delta-m})^\prime, \Pi = (c_1, \ldots, c_k, d_1, \ldots, d_m). \)

c) multi-factor regression model

\[ y_n = c_1 y_{n-1} + \cdots + c_k y_{n-m} + \sum_{i=1}^r \sum_{j=1}^{l_i} d_{ij} x_i(n-j) + \nu_n, \]

where \( r, m, l_i \geq 1 \). Here \( X(n) = (y_{n-1}, \ldots, y_{n-m}, x_1(n-1), \ldots, x_1(n-l_1), x_2(n-1), \ldots, x_2(n-l_2), \ldots, x_r(n-1), \ldots, x_r(n-l_r))^\prime, \Pi = (c_1, \ldots, c_k, d_{11}, \ldots, d_{rl}). \)

d) simultaneous equation systems (SES)

\[ B Y(n) + \Gamma X(n) = \epsilon_n, \]

where \( Y(n) = (y_{1n}, y_{2n}, \ldots, y_{Mn})^\prime \) is the vector of endogenous variables, \( X(n) = (x_{1n}, x_{2n}, \ldots, x_{Kn})^\prime \) is the vector of pre-determined variables (all exogenous variables plus lagged endogenous variables), \( \epsilon_n = (\epsilon_{1n}, \epsilon_{2n}, \ldots, \epsilon_{Mn})^\prime \) is the vector of random errors, \( B \) is a \( M \times M \) non-degenerate matrix (det \( B \neq 0 \)), \( \Gamma \) is a \( M \times K \) matrix.

This general structural form of the SES can be written in the following reduced form:

\[ Y(n) = -B^{-1} \Gamma X(n) + B^{-1} \epsilon_n = \Pi X(n) + \nu_n \]

This system is usually used for the analysis of change-points (structural changes) in multivariate linear models (see, e.g., Bai, Lumsdaine, Stock (1998)).

### 2.2 General assumptions

In this subsection we formulate general assumptions which will be used in our main theorems 3-5. Some specific assumptions will be formulated together with the corresponding theorems.
Let us start from the following definitions. Consider the probability space \((\Omega, \mathcal{F}, P)\). Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be two \(\sigma\)-algebras from \(\mathcal{F}\). Consider the following measure of dependence between \(\mathcal{H}_1\) and \(\mathcal{H}_2\):

\[
\psi(\mathcal{H}_1, \mathcal{H}_2) = \sup_{A \in \mathcal{H}_1, B \in \mathcal{H}_2, P(A)P(B) \neq 0} \left| \frac{P(AB)}{P(A)P(B)} - 1 \right|
\]

Suppose \((X_i, i \geq 1)\) is a sequence of random vectors defined on \((\Omega, \mathcal{F}, P)\). Denote by \(\mathcal{F}_t^s = \sigma\{X_i: s \leq i \leq t\}, 1 \leq s \leq t < \infty\) the minimal \(\sigma\)-algebra generated by random vectors \(X_i, s \leq i \leq t\). Define

\[
\psi(n) = \sup_{t \geq 1} \psi(\mathcal{F}_t^1, \mathcal{F}_{t+n}^\infty)
\]

A) Mixing condition

We say that a scalar random sequence \(\{x_n\}\) satisfies the \(\psi\)-mixing condition if the function \(\psi(n)\) (which is also called the \(\psi\)-mixing coefficient) tends to zero as \(n\) goes to infinity.

We say that a vector random sequence \(\{X(n)\}, X(n) = (x_1(n), \ldots, x_k(n))^*\) satisfies the uniform \(\psi\)-mixing condition if \(\max_{i,j} \psi_{ij}(n)\) tends to zero as \(n\) goes to infinity, where \(\psi_{ij}(n)\) is the \(\psi\)-mixing coefficient for the sequence \(\{x_i(n)x_j(n)\}\).

The \(\psi\)-mixing condition is satisfied in most practical situations of change-point detection. In particular, for a Markov chain (not necessarily stationary), if \(\psi(n) < 1\) for a certain \(n\), then \(\psi(k)\) goes to zero at least exponentially as \(k \to \infty\) (see Bradley, 2005, theorem 3.3).

B) Cramer condition

Let \(\{\zeta(n)\}, \zeta(n) = (\zeta_1(n), \ldots, \zeta_k(n))^*\) be a vector random sequence. We say that the uniform Cramer condition is satisfied if there exists a constant \(L > 0\) such that

\[
\sup_n \mathbb{E} \exp(t\zeta_i(n)\zeta_j(n)) < \infty
\]

for every \(i, j = 1, \ldots, k\) and \(|t| < L\).

For a centered random sequence \(\xi_n\) this condition is equivalent to the following: there exist constants \(g > 0, T > 0\) such that for each \(|t| < T\):

\[
\sup_n \mathbb{E} e^{t\xi_n} \leq \exp\left(\frac{t^2g}{2}\right).
\]
3 Preliminary results: prior inequalities

3.1 Unique change-point

On a probability space \((\Omega, \mathcal{F}, \mathbb{P}_\theta)\) consider a sequence of i.r.v.’s \(x_1, \ldots, x_N\) with the following density function (w.r.t. some \(\sigma\)-finite measure \(\mu\))

\[
f(x_n) = \begin{cases} 
  f_0(x_n, n/N), & 1 \leq n \leq [\theta N], \\
  f_1(x_n, n/N), & [\theta N] < n \leq N.
\end{cases}
\]

(2)

Here \(0 < \theta < 1\) is an unknown change-point parameter.

Define the following objects:

\[T_N(\Delta) : \mathbb{R}^N \rightarrow \Delta \subset \mathbb{R}^1\]

(3)

is the Borel function on \(\mathbb{R}^N\) with the values in the set \(\Delta\);

\[\mathcal{M}_N(\Delta) = \{T_N(\Delta)\}\]

(4)

is the collection of all Borel functions \(T_N\).

**Theorem 1.** Suppose the following assumption is satisfied:

the functions \(J_0(t) \overset{\text{def}}{=} \mathbb{E}_0 \ln \frac{f_0(x,t)}{f_1(x,t)}\) and \(J_1(t) \overset{\text{def}}{=} \mathbb{E}_1 \ln \frac{f_1(x,t)}{f_0(x,t)}\) are continuous at \([0,1]\) and such that

\[J_0(t) \geq \delta > 0, \quad J_1(t) \geq \delta > 0.\]

Then for any fixed \(0 < \theta < 1\), \(0 < \epsilon < \theta \land (1 - \theta)\) the following inequality holds:

\[
\liminf_{N \to \infty} N^{-1} \ln \inf_{\hat{\theta}_N \in \mathbb{M}_N([0,1])} \mathbb{P}_\theta \{ |\hat{\theta}_N - \theta| > \epsilon \} \geq -\min \left( \int_{\theta-\epsilon}^{\theta+\epsilon} J_0(t)dt, \int_{\theta}^{\theta} J_1(t)dt \right).
\]

The proof of this theorem is given in the Appendix A.

**Remark 1.** The lower bound in Theorem 1 can not be improved essentially. It follows from the results of Korostelev (1997). In this work the exact lower bound for the change-point estimate in continuous time model for the Wiener process was given. The exact lower bound in Korostelev (1997) differs from our bound only by a constant factor.

Consider the following particular cases of model (2).
1. A break in the trend function $\phi(t)$ of the mathematical expectation of Gaussian observations

Let

$$
\begin{align*}
f_0(x, t) &= h(x) \exp (\phi_0(t)x - \phi_0^2(t)/2), \quad t \leq \theta \\
f_1(x, t) &= h(x) \exp (\phi_1(t)x - \phi_1^2(t)/2), \quad t > \theta,
\end{align*}
$$

where $h(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, $\phi_0(\cdot) \neq \phi_1(\cdot)$.

In this case from Theorem 1 we obtain the following lower bound for the error probability:

$$
P_\theta\{|\hat{\theta}_N - \theta| > \epsilon\} \geq (1 - o(1)) \cdot \exp \left( -\frac{N}{2} \min_\theta \left( \int_{\theta-\epsilon}^{\theta+\epsilon} (\phi_0(t) - \phi_1(t))^2 dt, \int_{\theta-\epsilon}^{\theta+\epsilon} (\phi_0(t) - \phi_1(t))^2 dt \right) \right).
$$

2. Linear regression with deterministic predictors and Gaussian errors

Let

$$
y_n = c_1(n)x_{1n} + \cdots + c_k(n)x_{kn} + \xi_n, \quad n = 1, \ldots, N,
$$

where $\{\xi_n\}$ is a sequence of independent Gaussian r.v.’s with zero mean, $\xi_n \sim \mathcal{N}(0, \sigma^2)$, $c(n) \equiv (c_1(n), \ldots, c_k(n))^* = a^1(n \leq \lfloor \theta N \rfloor) + b^1(n > \lfloor \theta N \rfloor)$, $a = (a_1, \ldots, a_k)^* \neq b = (b_1, \ldots, b_k)^*$, $x_{in} = f_i(n/N)$, $n = 1, \ldots, N$, and $f_i(\cdot) \in C[0,1]$, $i = 1, \ldots, k$.

In this case from Theorem 1 applied to the sequence of observations $y_1, \ldots, y_N$ we obtain:

$$
P_\theta\{|\hat{\theta}_N - \theta| > \epsilon\} \geq (1 - o(1)) \cdot \exp \left( -\frac{N}{2\sigma^2} \min_\theta \left( \int_{\theta-\epsilon}^{\theta+\epsilon} (\sum_{i=1}^{k} f_i(t)(a_i - b_i))^2 dt, \int_{\theta-\epsilon}^{\theta+\epsilon} (\sum_{i=1}^{k} f_i(t)(a_i - b_i))^2 dt \right) \right).
$$

3. Linear stochastic regression model with Gaussian predictors

Consider model (5) with $\xi_n \equiv 0$. Suppose that there exist continuous functions $f_i(\cdot), \sigma_i(\cdot), \ i = 1, \ldots, k$ such that $x_{in}$ are Gaussian i.r.v.’s, $x_{in} \sim \mathcal{N}(f_i(n/N), \sigma_i^2(n/N))$, $n = 1, \ldots, N$. Suppose also that $x_{in}$ and $x_{jn}$ are independent for $i \neq j$ and $c(n)$ is the same as in model (5).

Then from Theorem 1 we obtain:

$$
P_\theta\{|\hat{\theta}_N - \theta| > \epsilon\} \geq (1 - o(1)) \exp \left( -\frac{N}{2} \min_\theta \left( \int_{\theta-\epsilon}^{\theta+\epsilon} J_0(t) dt, \int_{\theta-\epsilon}^{\theta+\epsilon} J_1(t) dt \right) \right),
$$
where
\[ J_0(t) = \left( \frac{\phi_0(t)}{\Delta_0(t)} - \frac{\phi_1(t)}{\Delta_1(t)} \right)^2 + 2 \frac{\phi_0(t)}{\Delta_0(t)} \frac{\phi_1(t)}{\Delta_1(t)} \left( 1 - \frac{\Delta_0(t)}{\Delta_1(t)} \right) + 2 \ln \frac{\Delta_1(t)}{\Delta_0(t)} + \left( 1 + \frac{\phi_0^2(t)}{\Delta_0^2(t)} \right) \left( \frac{\Delta_0(t)}{\Delta_1(t)} - 1 \right), \]
and
\[ \phi_0(t) = a_1 f_1(t) + \cdots + a_k f_k(t), \quad \Delta_0^2(t) = a_1^2 \sigma_1^2(t) + \cdots + a_k^2 \sigma_k^2(t), \]
\[ \phi_1(t) = b_1 f_1(t) + \cdots + b_k f_k(t), \quad \Delta_1^2(t) = b_1^2 \sigma_1^2(t) + \cdots + b_k^2 \sigma_k^2(t). \]

### 3.2 Multiple change-points

Theorem 1 can be generalized to the case of several change-points in the sequence of independent r.v.’s with the following density function:

\[ f(x_n) = f_i(x_n, n/N) I([\theta_{i-1} N] < n \leq [\theta_i N]), \quad n = 1, \ldots, N, \]
where \( i = 1, \ldots, k + 1 \) and \( 0 \equiv \theta_0 < \theta_1 < \cdots < \theta_k < \theta_{k+1} \equiv 1. \)

Suppose the following assumptions are satisfied:

i) change-points \( \theta_i \) are such that \( \min_{1 \leq i \leq k+1} (\theta_i - \theta_{i-1}) \geq \delta > 0. \)

ii) the functions \( J_i(t) = E_i \ln \frac{f_i(x,t)}{f_{i-1}(x,t)} \) and \( J_{i-1}(t) = E_{i-1} \ln \frac{f_{i-1}(x,t)}{f_i(x,t)}, i = 1, \ldots, k \)

are continuous at \([0,1]\) and such that

\[ J_i(t) \geq \Delta > 0, \quad i = 1, \ldots, k. \]

For the multiple change-point problem we estimate both the number \( k \) and the vector \( \vartheta \equiv (\theta_1, \ldots, \theta_k) \) of change-points’ coordinates. Let \( s^* \equiv [1/\delta] \) and denote \( Q = \{1, 2, \ldots, s^*\}. \)

For any \( s \in Q \) define

\[ D_s = \{ x \in \mathbb{R}^s : \delta \leq x_i \leq 1 - \delta, x_{i+1} - x_i \geq \delta, x_0 \equiv 0, x_{s+1} \equiv 1 \} \]

\[ D^* = \bigcup_{i=1}^{s^*} D_i, D^* \subset \mathbb{R}^{s^*} \equiv \mathbb{R}^* \]

By the construction, an unknown vector \( \vartheta \) is an arbitrary point of the set \( D_k \) and an unknown number of the change-points \( k \) is an arbitrary point of the set \( Q. \)

As before, it is reasonable to consider objects (3)-(4). In this notation \( \mathcal{M}_N(D^*) \) is the set of all arbitrary estimates of the parameter \( \vartheta \) and \( \mathcal{M}_N(Q) \) is the set of all arbitrary estimates of the parameter \( k \) on the basis of observations with the sample size \( N. \)
Let $\hat{k} \in M_N(Q)$ is an estimate of an unknown number of change-points $k$ and $\hat{\vartheta} \in M_N(D_k)$ is an estimate of unknown change-point coordinates on condition that the number of the coordinates was estimated correctly.

**Theorem 2.** Suppose assumptions i) and ii) are satisfied. Then for any fixed $0 < \epsilon < \delta$ the following inequality holds:

$$\lim\inf_{N \to \infty} N^{-1} \ln \inf_{\vartheta \in M_N(D_k)} \inf_{k \in M_N(Q)} \sup_{\vartheta_i + \epsilon} \sup_{\vartheta_i - \epsilon} \mathbb{P}_{\vartheta} \{ \{ \hat{k} \neq k \} \cup \{ (\hat{k} = k) \cap \max_{1 \leq i \leq k} |\hat{\theta}_i - \theta_i| > \epsilon \} \} \geq - \min_{1 \leq i \leq k} \min( \int_{\theta_i}^{\theta_i + \epsilon} J_i^{-1}(\tau) d\tau, \int_{\theta_i - \epsilon}^{\theta_i} J_i(\tau) d\tau ).$$

The proof of this theorem is given in the Appendix B.

## 4 Main results

Now consider model (1). In this Section we assume that the uniform mixing condition (A) and the uniform Cramer condition (B) (see Section 2) are satisfied, and an unknown vector of change-point parameters $\vartheta = (\theta_1, \ldots, \theta_k)$ is such that $0 < \beta \leq \theta_1 < \theta_2 < \cdots < \theta_k \leq \alpha < 1$, where $\beta$, $\alpha$ are known numbers. Everywhere below the measure $\mathbb{P}_{\vartheta}$ corresponds to a sample with the change-point $\vartheta$ ($\mathbb{P}_0$ corresponds to a sample without change-points).

### 4.1 Unique change-point

In this subsection model (1) with unique change-point $0 < \beta \leq \theta \leq \alpha < 1$ is considered.

#### 4.1.1 Deterministic predictors

Let us formulate assumptions for model (1) in the case of a unique change-point (remind that in model (1) the vector $X(n)$ has the dimension $K$ and the vector $Y(n)$ has the dimension $M$):

a) the vector random sequence $\{\nu_n\}$ satisfies conditions (A) and (B) (see section 2).

b) there exist functions $f_i(\cdot) \in C[0,1], i = 1, \ldots, K$ such that $x_{in} = f_i(n/N), n = 1, \ldots, N.$

Denote $F(t) = (f_1(t), \ldots, f_K(t))^*, t \in [0,1].$

c) for arbitrary $0 \leq t_1 < t_2 \leq 1$, the matrix

$$A(t_1, t_2) \overset{\text{def}}{=} \int_{t_1}^{t_2} F(s) F^*(s) ds.$$
is positive definite (below we denote $A(t) \overset{\text{def}}{=} A(0,t)$, $A(1) \overset{\text{def}}{=} I$).

In virtue of our assumptions, the matrix $I$ is symmetric and positive definite.

Define $K \times M$ matrix

$$Z(n_1, n_2) = \sum_{i=n_1}^{n_2} F(i/N)Y^*(i)$$

and $K \times K$ matrix

$$P_{n_2} \overset{\text{def}}{=} \sum_{k=n_1}^{n_2} F(k/N)F^*(k/N), \quad 1 \leq n_1 < n_2 \leq N.$$ 

The following matrix statistic is used for estimation of an unknown change-point:

$$Z_N(n) = N^{-1} \left( Z(1, n) - P_{n_1}(P_{n_1}^{-1} Z(1, N)) \right).$$ (7)

An arbitrary point $\hat{n}$ of the set $\arg\max_{[\beta N] \leq n \leq [\alpha N]} \|Z_n(n)\|^2$ is assumed to be the estimate of an unknown change-point (here and below $\|C\|$ denotes the Gilbert norm of a quadratic matrix $C$, namely $\|C\| = \sqrt{\text{tr}(CC^*)}$).

We define also the value $\hat{\theta}_N = \hat{n}/N$ - the estimate of the change-point parameter $\theta$.

Denote $B \overset{\text{def}}{=} B(\theta) = (E - I^{-1}A(\theta))(a - b)^*$.

**Theorem 3.** Suppose assumptions a)–c) are satisfied and $\text{rank}(B) = M$ if $\theta \in [\beta, \alpha]$.

Then the estimate $\hat{\theta}_N$ converges to the change-point parameter $\theta$ $P_\theta$-almost surely as $N \to \infty$.

Besides, for any fixed $(\alpha - \beta) > \epsilon > 0$ the following inequality is satisfied for $N > N_0(F)$:

$$\sup_{\beta \leq \theta \leq \alpha} P_\theta\{|\hat{\theta}_N - \theta| > \epsilon\} \leq m_0(C(\epsilon, N)/R) \begin{cases} \exp \left( -\frac{N\beta^2 (C(\epsilon, N)/R)^2}{4gm_0(C(\epsilon, N)/R)} \right), & \text{if } C(\epsilon, N) \leq \mathcal{R}gT \\ \exp \left( -\frac{T\beta^2 (C(\epsilon, N)/R)^2}{4gm_0(C(\epsilon, N)/R)} \right), & \text{if } C(\epsilon, N) > \mathcal{R}gT. \end{cases}$$ (8)

where the constants $g, T, m_0(\cdot) \geq 1$ are taken from the uniform Cramer’s and $\psi$-mixing conditions, respectively, $C(\epsilon, N) = \frac{\lambda_F}{4M} \|a - b\|^2 - L_F/N$, $N_0(F)$, $\lambda_F$, $L_F$, $\mathcal{R}$ are constants which can be exactly calculated for any given family of functions $F(t)$, and the constant $M$ is given in the proof.
Remark 2. The assumption \( \text{rank} B = M \) yields \( K \geq M \), i.e., the number \( M \) of endogenous variables in (1) cannot exceed the number \( K \) of pre-determined variables. Note that for one regression equation this assumption is always satisfied.

Remark 3. For independent random errors \( m_0(\epsilon) = 1 \).

Remark 4. Comparing theorems 1 and 3, we conclude that the order of convergence of the proposed estimate of the change-point parameter to its true value is asymptotically optimal as \( N \to \infty \).

Remark 5. For any given family of functions \( F(t) \) one can calculate the function \( f(t) = \|m(t)\|^2 \), \( m(t) = \lim_{N \to \infty} E_\theta Z_N([Nt]) \) (see the proof) and investigate this function on the square \( (\theta, t) \in [\beta, \alpha] \times [\beta, \alpha] \). Such investigation gives the opportunity to calculate all constants from the formulation.

The proof of Theorem 3 is given in the Appendix C.

From the proof we obtain the following

Corollary 1. Let \( C > 0 \) be the decision threshold and \( C \overset{\text{def}}{=} C - \frac{L_F}{N} \). Then:

- for type 1 error the following inequality is satisfied:

\[
P_0\{ \max_{[\beta N] \leq n \leq [\alpha N]} \|Z_n(n)\|^2 > C \} \leq m_0(C/R) \left\{ \begin{array}{ll}
\exp\left(-\frac{TNC\beta}{4Rm_0(C/R)}\right), & \text{if } C > RgT \\
\exp\left(-\frac{N\beta C^2}{4R^2gm_0(C/R)}\right), & \text{if } C \leq RgT,
\end{array} \right.
\]

- for type 2 error the following inequality is satisfied:

\[
P_\theta\{ \max_{[\beta N] \leq n \leq [\alpha N]} \|Z_n(n)\|^2 \leq C \} \leq m_0(d) \left\{ \begin{array}{ll}
\exp\left(-\frac{TN\beta d}{4m_0(d)}\right), & d > gT \\
\exp\left(-\frac{N\beta d^2}{4gm_0(d)}\right), & d \leq gT,
\end{array} \right.
\]

where \( d = R^{-1}\left(\|m(\theta)\| - C - \frac{L_F}{N}\right) > 0 \), \( \|m(\theta)\|^2 = \text{tr}(B^*A^2(\theta)B) \).

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4.1.2 Stochastic predictors

In this subsection we suppose that predictors $x_{ji}$ in (1) are random. On the probability space $(\Omega, \mathcal{F}, P_\theta)$ consider filtration $\{\mathcal{F}_n\}$, $n = 1, \ldots, n$, where $\{\mathcal{F}_n\} \in \mathcal{F}$, $\mathcal{F}_n$ can be interpreted as all available information up to the instant $n$.

Put $X(n) \overset{\text{def}}{=} (x_{1n}, \ldots, x_{Kn})^*$.

Suppose that the following conditions are satisfied:

a) there exists a continuous symmetric matrix function $V(t)$, $t \in [0, 1]$ such that the matrix $\int_{t_1}^{t_2} V(s)ds$ is positive definite for any $0 \leq t_1 < t_2 \leq 1$, and $E_\theta X(n)X^*(n) = V(n/N)$;

b) the sequence of random vectors $\{(X(n), \nu_n)\}$ satisfies the uniform Cramer’s and $\psi$-mixing conditions;

c) the random sequence $\{
u_n\}$ is a martingale-difference sequence w.r.t. the filtration $\{\mathcal{F}_n\}$;

d) the vector of predictors $X(n) \overset{\text{def}}{=} (x_{1n}, \ldots, x_{Kn})^*$ is $\mathcal{F}_{n-1}$-measurable.

On the segment $[0, 1]$ define the $K \times M$ matrix process

$$u_N(t) \overset{\text{def}}{=} \sum_{i=1}^{[Nt]} X(i)Y^*(i),$$

and the $K \times K$ matrix process

$$T_N(t) \overset{\text{def}}{=} \sum_{k=1}^{[Nt]} X(k)X^*(k).$$

In virtue of conditions a), b), c), the matrix process $N^{-1}T_N(t)$ weakly converges (in the Skorokhod space) to a positive definite symmetric matrix function $R(t) \overset{\text{def}}{=} \int_0^t V(s)ds$, and the rate of convergence is exponential. Below we denote $R(1) \overset{\text{def}}{=} R$.

Analogously, due to conditions a)-d), the matrix process $N^{-1}\sum_{k=1}^{[Nt]} X(k)\nu^*(k)$ weakly converges to zero with the exponential rate. Both conclusions follow from the fact that the random processes

$$N^{-1}\sum_{n=1}^{[Nt]} (x_{in}x_{jm} - E_\theta x_{in}x_{jn}),$$

$$N^{-1}\sum_{n=1}^{[Nt]} (x_{in}\nu_n), \quad i, j = 1, \ldots, k$$

weakly converge to zero (as $N \to \infty$) with the exponential rate (see Brodsky, Darkhovsky (2000)).
For estimation of an unknown change-point, the following statistic is used:

\[ Z_N(n) = N^{-1} \left( u_n(n/N) - T_N(n/N)(T_N(1))^{-1} u_N(1) \right), \quad n = 1, 2, \ldots, N. \quad (10) \]

An arbitrary point \( \hat{n} \) of the set \( \text{Arg max}_{[\beta N] \leq n \leq [\alpha N]} \|Z_N(n)\|^2 \) is assumed to be the estimate of an unknown change-point. Again we define \( \hat{\theta}_N = \hat{n}/N \) as the estimate of the change-point parameter \( \theta \).

Statistic (10) generalizes statistic (7) to the situation of stochastic predictors. Assumptions a)-d) guarantee the analogous properties of this statistic. In particular, the limit value (as \( N \to \infty \)) of the mathematical expectation of the statistic \( Z_N([Nt]) \) attains its unique global maximum on the segment \([0, 1]\) at the point \( t^* = \theta \).

Assumptions a)-d) guarantee convergence in probability of an arbitrary point of \( \text{Arg max}_{[\beta N] \leq n \leq [\alpha N]} \|Z_N(n)\|^2 \) to the point \( \theta \) with the exponential rate. Hence the \( P_{\theta}\text{-a.s.} \) convergence of the proposed estimate to \( \theta \) follows.

**Theorem 4.** Suppose that the conditions a)-d) are satisfied and \( \text{rank}(B) = M \) if \( \theta \in [\beta, \alpha] \), where \( B \overset{\text{def}}{=} B(\theta) = \left( E - R^{-1}R(\theta) \right) (a - b)^* \).

Then the estimate \( \hat{\theta}_N \) of the change-point parameter \( \theta \) converges to \( \theta \) \( P_{\theta}\text{-a.s.} \) as \( N \to \infty \).

Besides, there exists the number \( N_1 = N_1(\{X(n)\}) \) such that for \( N > N_1 \) and any fixed \( \epsilon, \ (\min ((\alpha - \beta), \|R\|/2) > \epsilon > 0) \), the following inequality holds:

\[
\sup_{\beta \leq \theta \leq \alpha} P_{\theta} \{ |\hat{\theta}_N - \theta| > \epsilon \} \leq \delta_N(\epsilon) + m_0 \left( \mathbb{C}(\epsilon, N)/R \right)
\begin{cases}
\exp \left( -\frac{N\beta \left( \mathbb{C}(\epsilon, N)/R \right)^2}{4gm_0 \left( \mathbb{C}(\epsilon, N)/R \right)} \right), & \text{if } \mathbb{C}(\epsilon, N) \leq RgT \\
\exp \left( -\frac{TN\beta \left( \mathbb{C}(\epsilon, N)/R \right)}{4m_0 \left( \mathbb{C}(\epsilon, N)/R \right)} \right), & \text{if } \mathbb{C}(\epsilon, N) > RgT,
\end{cases}
\]

where \( \mathbb{C}(\epsilon, N) = \left[ \frac{\epsilon \lambda_N}{4M} \|a - b\|^2 - \frac{L_N}{N} \right], \ M = \max_{\beta \leq t \leq \alpha} \|M(t)\|, \ the \ constants \ g, T, m_0(\cdot) \ are \ taken \ from \ the \ uniform \ Cramer’s \ and \ \psi\text{-mixing} \ conditions, \ and \ M(t), \ \lambda_N, \ L_N, \ \delta_N, \ R \ are \ described \ in \ the \ proof. \)

In particular, for independent observations \( m_0(\cdot) = 1 \).

Comparing Theorems 1 and 3, we conclude that the order of convergence of the proposed estimate of the change-point parameter to its true value is asymptotically optimal as \( N \to \infty \).
The proof of Theorem 4 is given in the Appendix D.

From the proof we obtain the following Corollary 2.

Let $S > 0$ be the decision threshold and $S \overset{def}{=} S - \frac{L_v}{N}$. Then:

- for type 1 error the following inequality is satisfied:

\[
\begin{align*}
\mathbb{P}_0 \{ & \max_{[\beta_N] \leq n \leq [\alpha_N]} \|Z_n(n)\|^2 > S \} \
& \leq \delta_N(S) + m_0(S/R) \\
\begin{cases}
\exp \left( -\frac{TNS\beta}{4Rm_0(S/R)} \right), & S > RgT \\
\exp \left( -\frac{N\beta S^2}{4R^2gm_0(S/R)} \right), & S \leq RgT.
\end{cases}
\end{align*}
\]

- for type 2 error the following inequality holds:

\[
\begin{align*}
\mathbb{P}_\theta \{ & \max_{[\beta_N] \leq n \leq [\alpha_N]} \|Z_n(n)\|^2 \leq S \} \
& \leq \delta_N(S) + m_0(r) \\
\begin{cases}
\exp \left( -\frac{TN\beta r}{4Rm_0(r)} \right), & r > RgT \\
\exp \left( -\frac{N\beta r^2}{4R^2gm_0(d)} \right), & r \leq RgT,
\end{cases}
\end{align*}
\]

where $r = R^{-1} (\|M(\theta)\| - S - L_v) > 0$; $\|M(\theta)\|^2 = \text{tr}(\mathbb{B}^*\mathbb{R}^2(\theta)\mathbb{B})$.

4.2 Multiple change-points

The proposed method can be generalized to problems of detection and estimation of multiple change-points in regression models. A widespread approach to solving these problems (see, e.g., Bai, Lumsdaine, Stock (1998)) consists in decomposition of the whole obtained sample to all possible subsamples and construction of regression estimates for each of these subsamples. The decomposition for which the minimum of the general sum of regression residuals is attained, is assumed to be the estimate of a true decomposition of the whole samples of obtained observations into subsamples with different regression regimes.

These methods turn out to be rather time consuming and have a low power. For example, if there are only two regression regimes in an obtained sample but we do not know this fact and are obliged to try all possible subsamples up to the order 20, then many false structural changes will be obtained.
In this paper we propose a new method of detection and estimation of multiple change-points which is not based upon LSE of regression parameters and computation of corresponding residuals. This method is more effective and robust to possible inaccuracies in specification of regression models.

Let us explain the idea of this method by the following example of a multiple regression model (1) with deterministic predictors and the row-matrix $\Pi(\vartheta, n)$. In other words, let $\vartheta = (\theta_1, \theta_2, \ldots, \theta_k)$, $k \geq 1$ is an unknown vector of change-point parameters such that $0 \equiv \theta_0 < \beta \leq \theta_1 < \cdots < \theta_k \leq \alpha < \theta_{k+1} \equiv 1$, where, as before, $\beta, \alpha$ are known numbers, and the observations has the form

$$y_n = \Pi^*(\vartheta, n)F(n/N) + \nu_n.$$  

Here

$$\Pi(\vartheta, n) = \sum_{i=1}^{k+1} a_i I([\theta_{i-1}N] < n \leq [\theta_i N]),$$

where $a_i \neq a_{i+1}, i = 1, 2, \ldots, k$ are unknown vectors, $F(t)$ is a given vector-function (all assumptions and notations see in Subsection 4.1.1).

Consider our main statistic (7). The mathematical expectation of this statistic converges as $N \to \infty$ to the function

$$m(t) = \int_0^t F(s)F^*(s)\Pi(\vartheta, s)ds - A(t)I^{-1} \int_0^1 F(s)F^*(s)\Pi(\vartheta, s)ds.$$

In the situation when there is no change-points, i.e., the vector of regression coefficients is constant on $[0, 1]$, the vector function $m(t)$ equals to zero for each $t \in [0, 1]$. This property of $m(t)$ makes it possible to effectively reject the null hypothesis about the absence of change-points when they are really present in an obtained sample.

Consider the following method of detection and estimation of multiple change-points. Fix a small parameter $\varepsilon$, $\min(\beta, 1 - \alpha) > \varepsilon > 0$. The proposed method consists of the following steps:

1. Compute statistic (7) by the data in the diapason of arguments $\mathcal{N} \overset{\text{def}}{=} ([\beta N], \ldots, [\alpha N])$. If $\max_{n \in \mathcal{N}} \|Z_\mathcal{N}(n)\|^2 > C$, where $C = C(N)$ is the decision threshold, then compute $n_{\text{max}} = \arg\max_{n \in \mathcal{N}} \|Z_\mathcal{N}(n)\|^2$, otherwise the sample is assumed to be stationary (without change-points).

2. Put $\mathcal{N}' = n_{\text{max}} - [\varepsilon N]$ and compute statistic (7) by the data in the diapason of arguments $\mathcal{N}' \overset{\text{def}}{=} ([\beta N], \ldots, \mathcal{N}')$ according to step 1. This cycle is repeated until:
1) we obtain a stationary sub-sample in the diapason of data with arguments \([\beta N, \ldots, N]\), i.e. \(\max_{n' \in N'} \|Z_{N'}(n)\|^2 \leq C(N')\). Then we put \(n(1) = N' + [\varepsilon N]\) as the estimate of the first change-point and go to step 3.

or

2) we obtain a sample of the size \(N' \leq 2\varepsilon N\). Then we put \(n(1) = N' + [\varepsilon N]\) as the estimate of the first change-point and go to step 3.

3. Put \(n' = n(1) + [\varepsilon N]\) and compute statistic (7) by the data in the diapason of arguments \((n', \ldots, [\alpha N])\) (i.e. with the relative arguments \([1, \ldots, [\alpha N] - n' + 1]\)) and do according to steps 1 and 2. The cycle is repeated until we obtain a stationary sub-sample in the diapason of data with arguments \([n', \ldots, n_{\text{max}}]\) or \(n_{\text{max}} - n' \leq 2\varepsilon N\). Then we put \(n(2) = n_{\text{max}}\) as the estimate of the next change-point. If \(N - n(2) < 2\varepsilon N\) then stop, otherwise repeat step 3 by the data in the diapason of arguments \((n(2), \ldots, [\alpha N])\).

In this way we continue to compute the estimates \(n(3), \ldots\) of change-points. As a result we obtain the series of estimates \(n(1), n(2), \ldots\) of the true change-points \([\theta_{N1}], \ldots, [\theta_{Nk}]\). The number \(\hat{k}_N\) of these estimates is determined by the quantity of stationary sub-samples

\([1, \ldots, n(1)], \ldots, [n(i), \ldots, n(i + 1)], \ldots, [n(\hat{k}_N), \ldots, N]\).

The proposed method is based upon reduction to the case of only one change-point and the properties of the matrix \(m(t)\). The crucial point of this method is the choice of the decision threshold \(C(N)\) which depends on the sample size \(N\). Below we give an explicit formula for computation of \(C(N)\).

Let \(\hat{k}_N\) be the estimate of the number of change-points in the sample and \(\hat{\theta}_N = ([\theta_{N1}], \ldots, [\theta_{Nk}])^*\) be the vector of estimated coordinates of change-point parameters. The following theorem holds for model (11).

**Theorem 5.** Suppose assumptions of Theorem 3 are satisfied. Moreover, assume that there exist \(h > 0, B > 0\) such that for all \(i = 2, \ldots, k + 1:\)

\[0 < \|A(\theta_{i-1}, \theta_i)A^{-1}(\theta_{i-2}, \theta_{i-1})\| \leq h\]

\[\|A(\theta_{i-1}, \theta_i)(a_i - a_{i-1})\| \geq B > 0,\]

Then for sufficiently small \(\delta > 0:\)

\[P\{ (\hat{k}_N \neq k) \cup \{ (\hat{k}_N = k) \cap (\max_{1 \leq i \leq k} |\hat{\theta}_{Ni} - \theta_i| > \delta) \} \} \leq C(\delta) \exp(-D(\delta)N),\]
where constants $C(\delta) > 0, D(\delta) > 0$ do not depend on $N$.

Analogous theorem can be proved also for stochastic predictors.

From theorem 5 it follows that the estimated number of change-points converges almost surely to its unknown true value, as well as estimated coordinates of unknown change-points converge exponentially to their true values as the sample size tends to infinity. Moreover, comparing results of theorem 2 and theorem 5 we conclude that the proposed method of detection and estimation of multiple change-points is asymptotically optimal by the order of convergence of estimated change-point parameters to their true values.

The proof of theorem 5 is given in the Appendix E.

4.3 A variant of the limit distribution theorem for the decision statistic under the null hypothesis

For practical applications of the proposed method and, in particular, for the rational choice of the decision threshold $C(N)$, we need to study the limit distribution of the decision statistic under the null hypothesis.

Let us formulate a variant of the limit theorem for the simple case of unique change-point, deterministic predictors, statistically independent noises $\nu_n$, and the one-dimensional dependent variable $y_n$.

Suppose there exists a continuous function $g(t), 0 \leq t \leq 1$ such that $E_\theta \nu_n^2 = g^2(n/N)$.

Put

$$\sigma_i^2 = \frac{1}{t} \int_0^t f_i^2(s)g^2(s)ds, i = 1, \ldots, K$$

$$G(t) = (\sigma_1(t), \ldots, \sigma_K(t))^\ast, \ Z(t) = G(t)W(t), \ U(t) = Z(t) - A(t)I^{-1}Z(1),$$

where $W(t)$ is the standard Wiener process, $A(t), I$ are the above defined matrices (see Subsection 4.1.1).

Consider our main statistic, the vector process $Z_N(t) = Z_N([Nt])$ (see (7)). Then for any $\theta \in [\beta, \alpha]$, the vector process $\sqrt{N}(Z_N(t) - E_\theta Z_N(t))$ weakly converges to the vector process $U(t)$ in the Skorokhod space $D^K[\beta, \alpha]$ (see Brodsky, Darkhovsky (2000)). In particular, under the null hypothesis, the weak convergence is valid at $[0, 1]$.

Therefore, we have the following
Theorem 6.

$$\lim_{N \to \infty} P_0\{\sqrt{N} \max_{t \in [0,1]} \|Z_N(t)\| > C\} = P_0\{\max_{t \in [0,1]} \|U(t)\| > C\} \quad (12)$$

(here we use the Euclidean norm for vectors).

The vector $U(t)$ is Gaussian with zero mean and the following $K \times K$ correlation matrix $D(t)$:

$$D(t) = t \left[ G(t)G^*(t) - G(t)G^*(1)I^{-1}A(t) - A(t)G(1)G^*(t) \right] + A(t)I^{-1}G(1)G^*(1)I^{-1}A(t).$$

Therefore, we have the following equality by distribution

$$U(t) = \sqrt{D(t)}\zeta \quad (13)$$

where $\zeta = (\zeta_1, \ldots, \zeta_K)^*$ is the standard Gaussian vector.

Taking (13) into account, we get

$$\max_{0 \leq t \leq 1} \|U(t)\| = \max_{0 \leq t \leq 1} \sqrt{\sum_{i=1}^{K} d_i^2(t)\zeta_i^2} \overset{\text{def}}{=} \rho(\zeta), \quad (14)$$

where $d_i^2(t)$ are eigenvalues of the matrix $D(t)$. The function $\rho(\zeta)$ can be explicitly calculated for any given family of functions $F(t), g(t)$.

Therefore, from (14) we have

$$P_0\{\max_{0 \leq t \leq 1} \|U(t)\| > C\} = \int_{\{u: \rho(u) > C\}} \varphi(u) du, \quad (15)$$

where $\varphi(u)$ is the density of the standard Gaussian distribution.

From (12) and (15) we can conclude that type 1 error goes to zero as $\exp(-\text{const} NC^2)$ for the proposed method. This fact allows us to choose the decision threshold. Note that the same asymptotical order can be obtained from corollary 2 (see Subsection 4.1.1). For independent noises we have

$$P_0\{\max_{\beta N \leq n \leq N} \|Z_n(n)\|^2 > C\} \leq \begin{cases} \exp\left(-\frac{TNC\beta}{4R}\right), & C > gT \\ \exp\left(-\frac{N\beta C^2}{4R^2g\sigma_0(C)}\right), & C \leq gT, \end{cases}$$

(the notations see in Subsection 4.1.1).
Therefore, we conclude that type 1 error $\alpha_N$ goes to zero exponentially as $N \to \infty$ for the proposed method.

So, the threshold can be calculated from the relation

$$C = C(N) = \frac{1}{\sqrt{N}} |\ln \alpha_N| \lambda,$$

where $\lambda$ is a certain calibration parameter which depends on variations of predictors, dispersions of noises and characteristics of their statistical dependence.

A more close study allows us to obtain the following practical formula for the decision threshold $C = C(N)$:

$$C(N) = \left( \max_i \sigma_i^2 \cdot \max_{0 \leq t \leq 1} f_i^2(t) \right)^{1/2} \sqrt{N} \lambda,$$

where $\sigma_i^2$ is the dispersion of $\nu_i$ and $\lambda > 0$ is the calibration parameter.

5 Experiments

In this section we present results of a simulation study of the proposed method in comparison with other well known tests. The following methods are most often used for detection of structural changes in regression models:

- The Chow test most often used in econometric packages;
- The CUSUM (cumulative sums) test based upon recursive regression residuals (Brown, Durbin, Evans, 1975);
- The CUSUM test based upon residuals of ordinary least squares method (OLS CUSUM test, Ploberger, Kramer, 1992);
- Fluctuation test (Ploberger, Kramer, Kontrus, 1989)
- Wald test (Andrews, 1993, Andrews, Ploberger, 1994)
- LM test (Lagrange Multiplier test, Andrews, 1993).

However, it is well known (see, e.g., Maddala and Kim (1998)) that the Wald test (together with the QMLE - quasi-maximum likelihood estimation test) is the best and most often used for detection of changes in regression models because it has the best characteristics of power and accuracy of change-point estimation.

The Wald test statistic is defined as follows:

$$SupW = \max_{1 \leq m \leq N} N \left[ \frac{S(N) - S_1(m) - S_2(N-m)}{S_1(m) + S_2(N-m)} \right],$$
where \( S(N) \) is the sum of regression residuals constructed by the whole sample of the size \( N \); \( S_1(m) \) is the sum of regression residuals constructed by the sub-sample of the first \( m \) observations; \( S_2(N-m) \) is the sum of residuals of the regression model constructed by the last \( N-m \) observations.

It is natural to define the estimate of the change point as \( n_0 \in \text{arg sup } W \), and the corresponding estimate of the change-point parameter \( \hat{\theta}_N = n_0/N \).

Comparison of characteristics of different methods is carried out in the following way. First, methods are 'equalized' by the value of type 1 error by means of choice of the corresponding decision thresholds. In practice, for this purpose we use experiments with stationary samples (without structural changes) in which the 95-percent quantiles of the variation series of the decision statistics are computed (see below, table 1). Second, for the chosen sample sizes and decision thresholds, experiments with non-stationary samples are performed in which we compute estimates of the type 2 error probability and instants of change-points (see tables 2 and 4). The method of change-point detection 'a' is preferable w.r.t. the method "b" if for the same values of the type 1 error, it gives lower estimates of the type 2 error and the error of change-point estimation.

5.1 Deterministic regression plan

We compared characteristics of our method with those of the Wald test using the following regression model with deterministic predictors:

\[
y_i = c_0 + c_1 x_i + \xi_i, \quad i = 1, \ldots, N
\]

where \((x_1, \ldots, x_N)^*\) is the vector of deterministic predictors; \(\{\xi_i\}\) is the Gaussian noise sequence with zero mean and unit variance; \(c_0, c_1\) are regression coefficients which change at the instant \(n_0 = \lfloor \theta N \rfloor\), \(0 < \theta < 1\).

The number of independent trials of each experiment was equal to \(k=2000\). The estimates of decision thresholds were obtained as follows. For each stationary sample, the 95-percent and 99-percent quantiles of the variation series of maximums of the decision statistic were computed in 2000 trials. These quantiles were then assumed to be estimates of the decision thresholds for 5-percent and 1-percent error level, respectively.

The values of the threshold \(C\) given in table 1, were used as decision bounds for the confidence probability 95 percent in experiments with non-stationary regression models. The following cases were considered:
- before the change-point: $c_0 = 0$, $c_1 = 1$
- after the change-point: $c_0 = \delta$, $c_1 = 1$.

In experiments the parameter $\delta$ and the sample size $N$ were changed. The following characteristics of the proposed method were estimated:

- The empirical estimate of decision threshold $C$ (more exactly, the empirical estimate of $\max_n \|Z_n(n)\|$);
- The empirical estimate of type 2 error probability $\hat{w}_N$;
- The empirical estimate of the change-point parameter $\hat{\theta}_N$.

Results obtained for the Wald test are given in the following tables.

**Table 1. Estimation of the decision thresholds for the Wald test for different sample sizes**

| $N$  | 100  | 200  | 300  | 400  | 500  | 700  | 1000 | 1200 |
|------|------|------|------|------|------|------|------|------|
| $p = 0.95$ | 10.10 | 8.09 | 9.59 | 8.66 | 8.12 | 7.62 | 7.51 | 7.43 |
| $p = 0.99$ | 12.60 | 10.88 | 14.14 | 12.10 | 12.20 | 9.97 | 11.68 | 10.02 |

**Table 2. Estimation of the change-point parameter $\theta = 0.30$ by the Wald test**

| $N$  | 300  | 400  | 500  | 700  | 1000 |
|------|------|------|------|------|------|
| $\delta = 0.3$ | $C$ | 5.63 | 6.76 | 8.24 | 9.77 | 12.09 |
|      | $\hat{w}_N$ | 0.83 | 0.71 | 0.59 | 0.46 | 0.32 |
|      | $\hat{\theta}_N$ | 0.29 | 0.25 | 0.22 | 0.19 | 0.20 |
| $\delta = 0.4$ | $C$ | 9.65 | 10.20 | 11.88 | 15.27 | 19.32 |
|      | $\hat{w}_N$ | 0.56 | 0.47 | 0.34 | 0.23 | 0.18 |
|      | $\hat{\theta}_N$ | 0.28 | 0.25 | 0.22 | 0.20 | 0.23 |

The same model was studied with the help of the method proposed in this paper.

1) Decision thresholds
In the first series of experiments, model (16) with constant coefficients $c_0 = 0$, $c_1 = 1$ was used. The following results were obtained.

**Table 3. Estimation of the decision thresholds**

| $N$  | 100  | 200  | 300  | 400  | 500  | 700  | 1000 | 1200 |
|------|------|------|------|------|------|------|------|------|
| $p = 0.95$ | 0.401 | 0.257 | 0.202 | 0.182 | 0.150 | 0.125 | 0.103 | 0.081 |
| $p = 0.99$ | 0.450 | 0.300 | 0.247 | 0.211 | 0.187 | 0.162 | 0.138 | 0.102 |

2) The estimates of the change-point parameter
Comparing results from tables 2 and 4, we conclude that type 2 error estimates for our method are lower than for the Wald test, and the error of estimation for our method is much lower than for the Wald test. Therefore, we conclude that our method is essentially better by the main performance characteristics of change-point detection than the Wald test, and so, we conclude that the proposed method is one of the most effective among all known tests for detection and estimation of structural changes in regression models.

Comparing results from table 4 and 5, we can conclude that the quality of estimation of the change-point parameter \( \theta \) depends on its location on the segment \([0, 1]\): estimation of \( \theta \) which is closer to the bounds of the segment \([0, 1]\) is more difficult.

In next two subsections we investigate our methods.

### 5.2 Stochastic regression plan

In this series of experiments the following model of observations was used:

\[ y_i = c_0 + c_1 x_i + \xi_i, \quad i = 1, \ldots, N \]
where \((x_1, \ldots, x_N)^*\) is a stationary random sequence of the following type:

\[ x_i = \rho x_{i-1} + \eta_i, \quad i = 1, \ldots, N, \quad x_0 \equiv 0, \]

\(\{\xi_i, \eta_i\}\) is the sequence of independent Gaussian r.v.’s with zero mean and unit dispersion; \(c_0, c_1\) are regression coefficients which change at the instant \(n_0 = \lfloor \theta N \rfloor, 0 < \theta < 1; |\rho| < 1.

1) Estimation of decision thresholds

In the first series of tests decision thresholds were estimated. For this purpose, stationary sequences (without change-points) were used: \(c_0 = 0, c_1 = 1, \rho = 0.3\). The following results were obtained.

Table 6. Estimation of decision thresholds (the case of stochastic predictors)

|     | \(N\) | 100 | 200 | 300 | 400 | 500 | 700 | 1000 | 1200 |
|-----|------|-----|-----|-----|-----|-----|-----|------|------|
| \(p = 0.95\) | 0.355 | 0.291 | 0.230 | 0.188 | 0.150 | 0.132 | 0.103 | 0.082 |
| \(p = 0.99\) | 0.401 | 0.332 | 0.273 | 0.218 | 0.192 | 0.171 | 0.141 | 0.100 |

2) Estimation of the change-point parameter

In the following series of experiments a model with a structural change in the regression coefficients was used:

- before the change-point: \(c_0 = 0, c_1 = 1\)
- after the change-point: \(c_0 = 0, c_1 = 1.3\).

Results obtained are presented in table 7.

Table 7. Estimation of change-point parameters (the case of stochastic predictors)

|     | \(N\) | 500 | 700 | 1000 | 1200 |
|-----|------|-----|-----|------|------|
| \(\theta = 0.5\) | \(C\) | 0.167 | 0.157 | 0.152 | 0.152 |
|     | \(\hat{w}_N\) | 0.32 | 0.21 | 0.02 | 0 |
|     | \(\hat{\theta}_N\) | 0.481 | 0.495 | 0.498 | 0.499 |
| \(\theta = 0.3\) | \(C\) | 0.156 | 0.148 | 0.142 | 0.140 |
|     | \(\hat{w}_N\) | 0.45 | 0.30 | 0.03 | 0 |
|     | \(\hat{\theta}_N\) | 0.312 | 0.310 | 0.308 | 0.301 |
5.3 Multiple structural changes in multivariate systems

The following multivariate system was used:

\[
\begin{align*}
y_i &= c_0 + c_1 y_{i-1} + c_2 z_{i-1} + c_3 x_i + \epsilon_i \\
z_i &= d_0 + d_1 y_i + d_2 x_i + \xi_i \\
x_i &= 0.5x_{i-1} + \nu_i \\
\epsilon_i &= 0.3\epsilon_{i-1} + \eta_i,
\end{align*}
\]

where \(\xi_i, \nu_i, \eta_i, i = 1, 2, \ldots\) are independent standard Gaussian random variables.

Here \((y_i, z_i)^*\) is the vector of endogenous variables, \(x_i\) is the vector of exogenous variables, \((y_{i-1}, z_{i-1}, x_i)^*\) - the vector of pre-determined variables of the considered system.

Dynamics of this system is characterized by the following vector of coefficients:

\[
u = [c_0, c_1, c_2, d_0, d_1, d_2].\]

The initial vector of coefficients is \([0.1, 0.5, 0.3, 0.7, 0.2, 0.4, 0.6]\). The first structural change occurs at the instant \(\theta_1 = 0.3\). The vector of coefficients \(\nu\) changes into \([0.1, 0.5, 0.7, 0.2, 0.4, 0.6]\). The second structural change occurs at the instant \(\theta_2 = 0.7\). Then the vector \(\nu\) changes into \([0.1, 0.5, 0.7, 0.2, 0.4, 0.9]\).

In the first series of tests the decision threshold \(C\) was estimated. For this purpose, the model with the initial vector of coefficients \(\nu\) and without change-points was used. In 2000 independent trials the maximums of the decision statistic were computed and the variation series of these maximum was constructed. Then the 95-percent and the 99-percent quantiles of this series were computed. These values are presented in table 8.

**Table 8. Estimation of decision thresholds (the case of a multivariate system)**

| \(N\) | 200 | 400 | 500 | 700 | 900 | 1000 | 1200 | 1500 |
|------|-----|-----|-----|-----|-----|------|------|------|
| \(p = 0.95\) | 0.28 | 0.20 | 0.19 | 0.18 | 0.16 | 0.15 | 0.145 | 0.14 |
| \(p = 0.99\) | 0.36 | 0.33 | 0.28 | 0.24 | 0.23 | 0.21 | 0.19 | 0.17 |

The computed 95-percent quantiles were assumed to be the decision thresholds for the corresponding sample volumes.

In the next series of tests non-stationary samples with multiple change-points were used. The true number of change-points was equal to \(p = 2\), the coordinates of these change-points were \(\theta_1 = 0.3\) and \(\theta_2 = 0.7\). In table 9 the following performance characteristics are given:
- $w$ is the estimate of the probability $P_\theta(\hat{p}_N \neq p)$ in 2000 independent trials, where $\hat{p}_N$ is the estimate of the number of change-points in the data.

- $\Delta$ is the estimation error on condition that $\hat{p}_N = p$, i.e. $\Delta = \sqrt{\sum_{i=1}^{p} (\hat{\theta}_i - \theta_i)^2}$.

Table 9. Estimation of change-point parameters (the case of a multivariate system)

| $N$  | 200 | 400 | 500 | 700 | 900 | 1000 | 1200 | 1500 |
|------|-----|-----|-----|-----|-----|------|------|------|
| $w$  | 0.96| 0.54| 0.39| 0.21| 0.04| 0.03 | 0.02 | 0.01 |
| $\Delta$ | 0.02| 0.05| 0.04| 0.02| 0.03| 0.02 | 0.01 | 0.005|

6 Conclusions

In this paper the following main results were obtained:

1. The general statement of the retrospective change-point detection and estimation problem in multivariate linear systems is given (both one change-point and multiple change-point problems, both independent and dependent sequences of observations).

2. The prior lower bounds are proved for the main performance characteristic in retrospective change-point detection and estimation: the probability of the error of change-point estimation, in different contexts of change-point estimation: from one change-point in multi-factor linear regressions with deterministic and stochastic regression plans, to multiple change-point problems in multivariate linear models.

3. A new method is proposed for the problem of retrospective change-point detection and estimation in multivariate linear systems. The main performance characteristics of this method: type 1 and type 2 errors, the error of change-point estimation, are studied theoretically. We prove that the proposed method is asymptotically optimal by the order of convergence of the change-point estimate to its true value as the sample size tends to infinity.

4. For the problem of multiple change-point detection and estimation, we propose a general setup in which both the number of change-points and their coordinates in the sample are unknown. For this problem statement, a new method is proposed which gives consistent estimates of an unknown number of change-points and their coordinates. This method is also asymptotically optimal by the order of convergence of these estimates to true change-point parameters.

5. A simulation study of characteristics of the proposed method for finite sample sizes is performed. The main goals of this study are as follows: to compare performance
characteristics of the proposed method with characteristics of other well known methods of change-point detection in linear regression models: the Wald test, the Chow test, the CUSUM tests with ordinary and recursive regression residuals, the fluctuation test; to consider more general multivariate linear models and performance characteristics of the proposed method in these multivariate models. The main conclusion: performance characteristics of the proposed method are no worse but often even better than those of well known change-point tests.

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Appendix. Proofs of theorems

A Proof of Theorem 1

Using notations (3)-(4), put

\[ \mathcal{M}(\Delta) = \{ T(\Delta) : T(\Delta) = \{ T_n(\Delta) \}_{n=1}^\infty \} \]

This is the set of all sequences of the elements \( T_n(\Delta) \in \mathcal{M}_N(\Delta) \). Consider also the collection of all consistent estimates of the parameter \( \theta \in \Delta \), i.e.,

\[ \hat{\mathcal{M}}(\Delta) = \{ T(\Delta) \in \mathcal{M}(\Delta) : \lim_{N \to \infty} \mathbb{P}_\theta(|T_n(\Delta) - \theta| > \epsilon) = 0, \forall \theta \in \Delta, \forall \epsilon > 0 \} \]

Under the assumption of Theorem 1, the set \( \hat{\mathcal{M}}([a,b]) \) is non-empty for any \( 0 < a < b < 1 \). Indeed, consider the sequence \( y_n = \ln \frac{f_0(x_n, n/N)}{f_1(x_n, n/N)} \). Due to the assumption, \( \mathbb{E}_\theta y_n \geq \delta > 0 \) before the change-point \( \theta \), \( a \leq \theta \leq b \), and less than \( (-\delta) \) after the change-point. Now, using the same idea as in Brodsky and Darkhovsky (2000), it is easy to construct the consistent estimate of the change-point.

Further, without loss of generality we can consider only consistent estimates of the change-point parameter \( \theta \), because for non-consistent estimates the probability of the error of estimation does not converge to zero and the considered inequality is satisfied trivially.

Let \( \hat{\theta}_N \) be some consistent estimate of the change-point parameter \( \theta \) constructed by the sample \( X^N = \{ x_1, \ldots, x_N \} \). Consider the random variable \( \lambda_N = \lambda_N(x_1, \ldots, x_N) = \mathbb{I}\{ |\hat{\theta}_N - \theta| > \epsilon \} \).
Under the change-point parameter $\theta$, the likelihood function for the sample $X^N$ can be written as follows:

$$f(X^N, \theta) = \prod_{i=1}^{\lfloor \theta N \rfloor} f_0(x_i, i/N) \cdot \prod_{i=\lfloor \theta N \rfloor + 1}^{N} f_1(x_i, i/N).$$

We have for any $d > 0$ and $0 < \epsilon < \epsilon'$:

$$P_\theta \{ |\hat{\theta}_N - \theta| > \epsilon \} = E_\theta \lambda_N \geq E_\theta (\lambda \Pi(f(X^N, \theta + \epsilon')/f(X^N, \theta) < e^d)) \geq$$

$$\geq e^{-d} (E_{\theta + \epsilon}(\lambda \Pi(f(X^N, \theta + \epsilon)/f(X^N, \theta) < e^d)) \geq$$

$$e^{-d} (P_{\theta + \epsilon} \{ |\theta_N - \theta| > \epsilon \} - P_{\theta + \epsilon} \{ f(X^N, \theta + \epsilon)/f(X^N, \theta) \geq e^d \})$$

(here we used the elementary inequality $P(AB) \geq P(A) - P(\Omega \setminus B)$).

Consider the probabilities in the right-hand side of the last inequality. Since $\theta_N$ is a consistent estimate of $\theta$, we have $P_{\theta + \epsilon} \{ |\theta_N - \theta| > \epsilon \} \to 1$ as $N \to \infty$. For estimation of the second probability, we take into account that

$$\ln \left( f(X^N, \theta + \epsilon)/f(X^N, \theta) \right) = \sum_{i=\lfloor \theta N \rfloor + 1}^{\lfloor \theta + \epsilon \rfloor N} \ln \left( f_0(x_i, i/N)/f_1(x_i, i/N) \right)$$

Therefore,

$$E_{\theta + \epsilon} \ln \left( f(X^N, \theta + \epsilon)/f(X^N, \theta) \right) =$$

$$= N \int_{\theta}^{\theta + \epsilon'} E_0 \ln \left( \frac{f_0(x, t)}{f_1(x, t)} \right) dt + O(1).$$

Then

$$P_{\theta + \epsilon'} \{ f(X^N, \theta + \epsilon')/f(X^N, \theta) \geq e^d \} =$$

$$= P_{\theta + \epsilon'} \left\{ \sum_{i=\lfloor \theta N \rfloor + 1}^{\lfloor \theta + \epsilon \rfloor N} (\ln(f_0(x_i, i/N)/f_1(x_i, i/N)) - E_0 \ln f_0(x_i, i/N)/f_1(x_i, i/N)) \right\}$$

$$\geq d - N \int_{\theta}^{\theta + \epsilon'} E_0 \ln \left( \frac{f_0(x, t)}{f_1(x, t)} \right) dt + O(1)$$

Put $d = d_1(N) = N(\int_{\theta}^{\theta + \epsilon'} E_0 \ln \left( \frac{f_0(x, t)}{f_1(x, t)} \right) dt + \delta)$ for some $\delta > 0$ and use the law of large numbers which holds due to existence of $E_0 \ln \left( \frac{f_0(x, t)}{f_1(x, t)} \right)$. Then we obtain

$$P_{\theta + \epsilon'} \{ f(X^N, \theta + \epsilon')/f(X^N, \theta) \geq e^{d_1(N)} \} \to 0$$
as $N \to \infty$.

The same considerations for $d = d_2(N) = N(\int_{\theta - \epsilon'}^{\theta} E_1 \ln \frac{f_1(x,t)}{f_0(x,t)} dt + \delta)$ yield

$$P_{\theta - \epsilon'} \left\{ f(X^N, \theta - \epsilon') / f(X^N, \theta) \geq e^{d_2(N)} \right\} \to 0$$

as $N \to \infty$.

Therefore,

$$P_{\theta} \left\{ |\hat{\theta}_N - \theta| > \epsilon \right\} \geq (1 - o(1)) \max(e^{-d_1(N)}, e^{-d_2(N)}).$$

It follows from here

$$\liminf_{N \to \infty} N^{-1} \ln \inf_{\theta_N \in M_N} P_{\theta} \left\{ |\hat{\theta}_N - \theta| > \epsilon \right\} \geq -\min \left( \int_{\theta}^{\theta + \epsilon'} J_0(t) dt, \int_{\theta - \epsilon'}^{\theta} J_1(t) dt \right) - \delta.$$

Note that the left-hand side of this inequality does not depend on the parameters $\delta, \epsilon'$, and the right-hand side exists for each $\delta > 0, \theta \land (1 - \theta) > \epsilon' > \epsilon > 0$. From the continuity assumption for the functions $J_0(\cdot), J_1(\cdot)$, we conclude that our result follows after taking the limits of both sides of this inequality as $\delta \to 0$ and $\epsilon' \to \epsilon$.

**B Proof of Theorem 2**

We will use notations (3)-(4) and (6). Let $x \in \mathbb{R}^p, y \in \mathbb{R}^q, u m = \max(p,q)$. Define the following natural immersions:

$$\text{im}_x : \mathbb{R}^p \to \mathbb{R}^m, \quad \tilde{x} = \text{im}_x x, \quad \text{im}_y : \mathbb{R}^q \to \mathbb{R}^m, \quad \tilde{y} = \text{im}_y y$$

(all lacking components are substituted by zeros) and put:

$$\text{dist}(x, y) = \|\tilde{x} - \tilde{y}\|^{(m)}$$

(here we use the $\| \cdot \|_{\infty}$-norm for vector $x = (x_1, \ldots, x_p)$, i.e., $\|x\|^{(p)} = \max_{1 \leq i \leq p} |x_i|$).

Consider

$$\liminf_{N \to \infty} N^{-1} \ln \inf_{\theta_N \in M_N(D^*)} \sup_{\|\vartheta_N\|^{(k)} > \epsilon} \left\{ P_{\vartheta}(\vartheta_N \in D_k; \|\vartheta_N - \vartheta\|^{(k)} > \epsilon) + P_{\vartheta}(\vartheta_N \notin D_k) \right\}$$

(B.1)
Note that for \( \epsilon < \delta \), any estimate \( \vartheta_N \in \mathcal{M}_N(D^*) \), and any \( \vartheta \in D_k \), the following relationships between events hold:

\[
\text{dist}(\vartheta_N, \vartheta) > \epsilon = (\vartheta_N \in D_k, \|\vartheta_N - \vartheta\|^{(k)} > \epsilon) \cup (\vartheta_N \notin D_k, \text{dist}(\vartheta_N, \vartheta) > \epsilon) = (\vartheta_N \in D_k, \|\vartheta_N - \vartheta\|^{(k)} > \epsilon) \cup (\vartheta_N \notin D_k).
\]

Here we used the fact that from the definition of \( \text{dist} \) and the condition \( (\vartheta_N \notin D_k) \) it follows that \( (\text{dist}(\vartheta_N, \vartheta) > \delta) \), and this condition yields \( \text{dist}(\vartheta_N, \vartheta) > \epsilon \) for \( \epsilon < \delta \).

Thus, we need to estimate the probability \( P_{\vartheta}(\text{dist}(\vartheta_N, \vartheta) > \epsilon) \).

First, note that the set \( \mathcal{M}(D_k) \) of all consistent estimates of the parameter \( \vartheta \in D_k \) is \textit{non-empty}. This fact follows from assumption ii) of the Theorem 2 and the same considerations as in proof of Theorem 1.

Second, remark that the infimum in (B.1) can be taken only on the set \( \mathcal{M}_N(D_k) \). In fact, let \( \vartheta_N \in \mathcal{M}_N(D^*) \) belongs to arg inf of the left-hand side of this inequality, i.e.,

\[
\inf \vartheta_N \in \mathcal{M}_N(D^*) \sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\vartheta_N, \vartheta) > \epsilon\} = \sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\vartheta_N, \vartheta) > \epsilon\}
\]

(without loss of generality we suppose that the infimum is attainable). Then consider the following element \( \hat{\vartheta}_N \) of the set \( \mathcal{M}_N(D_k) \):

\[
\hat{\vartheta}_N = \vartheta_N^* I(\vartheta_N^* \in D_k) + \Gamma_N I(\vartheta_N^* \notin D_k)
\]

where \( \Gamma_N \) is the element of the set \( \mathcal{M}_N(D_k) \) such that

\[
\sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\Gamma_N, \vartheta) < \epsilon/2\} \geq 1 - \kappa
\]

for some fixed \( \kappa > 0 \). Such elements exist in \( \mathcal{M}_N(D_k) \) (for large enough \( N \)), because this set contains consistent estimates.

By definition, \( \hat{\vartheta}_N \in \mathcal{M}_N(D_k) \) and for each \( \vartheta \in D_k \),

\[
P_{\vartheta}\{\text{dist}(\hat{\vartheta}_N, \vartheta) > \epsilon\} = P_{\vartheta}\{\text{dist}(\vartheta_N^*, \vartheta) > \epsilon\} + P_{\vartheta}\{\text{dist}(\Gamma_N, \vartheta) > \epsilon\}.
\]

Therefore,

\[
\sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\hat{\vartheta}_N, \vartheta) > \epsilon\} \leq \sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\vartheta_N^*, \vartheta) > \epsilon\} + \sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\Gamma_N, \vartheta) > \epsilon\}
\]

\[
= \inf \vartheta_N \in \mathcal{M}_N(D^*) \sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\vartheta_N, \vartheta) > \epsilon\} + \sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\Gamma_N, \vartheta) > \epsilon\}
\]

\[
\leq \inf \vartheta_N \in \mathcal{M}_N(D^*) \sup \vartheta \in D_k P_{\vartheta}\{\text{dist}(\vartheta_N, \vartheta) > \epsilon\} + \kappa
\]

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So,
\[ \kappa + \inf_{\vartheta_N \in \mathcal{M}_N(D^*)} \sup_{\vartheta \in D_k} \mathbb{P}_{\vartheta}\{\text{dist}(\vartheta_N, \vartheta) > \epsilon\} \geq \inf_{\vartheta_N \in \mathcal{M}_N(D_k)} \sup_{\vartheta \in D_k} \mathbb{P}_{\vartheta}\{\text{dist}(\vartheta_N, \vartheta) > \epsilon\} \geq \inf_{\vartheta_N \in \mathcal{M}_N(D^*)} \sup_{\vartheta \in D_k} \mathbb{P}_{\vartheta}\{\text{dist}(\vartheta_N, \vartheta) > \epsilon\}, \]

and this is the fact we wanted to show.

By the definition of \( \text{dist} \), we have on the set \( \mathcal{M}_N(D_k) \):
\[ \text{dist}(\vartheta_N, \vartheta) = \| \vartheta_N - \vartheta \|^{(k)}. \]

Further, for any \( i = 1, \ldots, k \) the following inclusion holds
\[ \{\| \vartheta_N - \vartheta \|^{(k)} > \epsilon, \vartheta_N \in D_k\} \supseteq \{\| \theta_i(N) - \theta_i \| > \epsilon, \vartheta_N \in D_k\}, \]

where \( \theta_i(N) \) is the \( i \)-th component of the vector \( \vartheta_N \).

Therefore,
\[ \mathbb{P}_{\vartheta}\{\| \vartheta_N - \vartheta \|^{(k)} > \epsilon, \vartheta_N \in D_k\} \geq \max_{1 \leq i \leq k} \mathbb{P}_{\vartheta}\{|\theta_i(N) - \theta_i| > \epsilon, \vartheta_N \in D_k\}. \]

But estimation of the value
\[ \liminf_{N \to \infty} N^{-1} \ln \inf_{\vartheta_N \in \mathcal{M}_N(D_k)} \sup_{\vartheta \in D_k} \mathbb{P}_{\vartheta}\{|\theta_i(N) - \vartheta_i| > \epsilon, \vartheta \in D_k\} \triangleq A_i \]

is exactly the problem already considered in the proof of Theorem 1 for the case of unique change-point. Therefore,
\[ A_i \geq -\min\left( \int_{\theta_i}^{\theta_i+\epsilon} J^{i^{-1}}(t)dt, \int_{\theta_i-\epsilon}^{\theta_i} J_i(t)dt \right). \]

So, finally we obtain
\[ \liminf_{N \to \infty} N^{-1} \ln \inf_{\vartheta_N \in \mathcal{M}_N(D^*)} \sup_{\vartheta \in D_k} \mathbb{P}_{\vartheta}\{|\vartheta_N - \vartheta (\epsilon, \vartheta_N \in D_k\} \geq \]
\[ \geq -\min \min_{1 \leq i \leq k}\left( \int_{\theta_i}^{\theta_i+\epsilon} J^{i^{-1}}(t)dt, \int_{\theta_i-\epsilon}^{\theta_i} J_i(t)dt \right). \]

This completes the proof.
C Proof of Theorem 3

Due to the assumptions, the matrix $I = \int_0^1 F(t)F^*(t)dt$ is positive definite. Therefore, there exists the matrix $[N (\mathcal{P}_1^N)^{-1}]$ for all $N > N_0(F)$. The constant $N_0(F)$ can be exactly estimated for any given family of functions $F(t)$.

Let us consider the matrix random process with continuous time $\mathcal{Z}_N(t) \overset{\text{def}}{=} \mathcal{Z}_N([Nt]), \ t \in [0, 1]$. It is easy to see that the mathematical expectation of the process $\mathcal{Z}_N(t)$ can be written as follows:

$$
\mathbb{E}_\theta \mathcal{Z}_N(t) = N^{-1} \left( \sum_{i=1}^{[Nt]} F(i/N)F^*(i/N) \Pi^*(\theta, i) - \mathcal{P}_1^{[Nt]}(\mathcal{P}_1^N)^{-1} \sum_{i=1}^N F(i/N)F^*(i/N) \Pi^*(\theta, i) \right).
$$

After simple transformations we obtain that $m(t) \overset{\text{def}}{=} \lim_{N \to \infty} \mathbb{E}_\theta \mathcal{Z}_N(t)$ has the form:

$$
m(t) = \begin{cases} 
A(t)I^{-1}(I - A(\theta))(a - b)^*, & t \leq \theta \\
(I - A(t))I^{-1}A(\theta)(a - b)^*, & t > \theta, 
\end{cases} \quad (C.0)
$$

Consider the square of the Gilbert norm of the matrix $m(t)$, i.e., the function $f(t) = \text{tr}(m^*(t)m(t))$, and show that the function $f(t)$ has a unique global maximum on the segment $[0, 1]$ at the point $t = \theta$.

First, for each $t \leq \theta$:

$$
f(\theta) - f(t) = \text{tr}(B^*(A^2(\theta) - A^2(t))B),
$$

where matrix $B$ was defined in Theorem 3. Consider the matrix

$$
A^2(\theta) - A^2(t) = A(\theta)(A(\theta) - A(t)) + (A(\theta) - A(t))A(t).
$$

Denote $L = A(\theta)(A(\theta) - A(t))$ and prove that the matrix $L$ is positive definite as $t < \theta$. In fact, since the matrix $A(\theta)$ is symmetric and positive definite, we can write

$$
x^*Lx = x^*A^{1/2}(\theta)A^{1/2}(\theta)(A(\theta) - A(t))x = y^*A^{1/2}(\theta)(A(\theta) - A(t))A^{-1/2}(\theta)y,
$$

where $y = A^{1/2}(\theta)x$.

The matrices $A(\theta) - A(t)$ and $A^{1/2}(\theta)(A(\theta) - A(t))A^{-1/2}(\theta)$ have identical characteristic polynomial and eigenvalues. Besides, $A(\theta) - A(t)$ is positive definite as $t < \theta$. 

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Therefore, the matrix $A^{1/2}(A(\theta) - A(t))A^{-1/2}(\theta)$ is also positive definite as $t < \theta$ and therefore, the matrix $L$ is positive definite.

In analogy, the matrix $(A(\theta) - A(t))A(t)$ is positive definite as $t < \theta$. Therefore, the matrix $A^2(\theta) - A^2(t)$ is positive definite as $t < \theta$.

Now consider the matrix $D = B(A^2(\theta) - A^2(t))B^*$. The matrix $D$ is positive definite if rank$(B) = M$, but this is our assumption.

So, we obtain $\text{tr}(B(A^2(\theta) - A^2(t))B^*) > 0$ for $t < \theta$ and therefore, the function $f(t)$ has a unique global maximum on the segment $[0, \theta]$ at the point $t = \theta$.

The same considerations for $t < \theta$ yield that $f(t)$ monotonically decreases on the segment $[\theta, 1]$. As a result, we obtain that $f(t)$ has a unique global maximum on the segment $[0, 1]$ at the point $t = \theta$.

Further, we are going to show the following: there exists a positive constant $c$ such that $f(\theta) - f(t) \geq c \cdot |\theta - t|$. This estimate can be obtained as follows. Taking into account the continuity of the functions $f_j(t)$, we obtain

$$A(\theta) - A(t) = \int_0^\theta F(\tau)F^*(\tau) \, d\tau = (\theta - t)U(t, \theta) > 0, \quad (C.1)$$

where the matrix $U(t, \theta)$ is positive definite for $0 \leq t < \theta$ and negative definite for $t > \theta$. Due to the continuity, we can write

$$U(t, \theta) = U(\theta, \theta) + \kappa(t, \theta), \quad (C.2)$$

where $\kappa(t, \theta) \to 0$ as $t \to \theta$.

Then

$$f(\theta) - f(t) = \text{tr} \left( B^* (A^2(\theta) - A^2(t))B \right) =$$

$$= \text{tr} \left( BB^* \left( A(\theta)(A(\theta) - A(t)) \right) \right) + \text{tr} \left( BB^* (A(\theta) - A(t))A(t) \right) = \quad (C.3)$$

$$= (\theta - t) \text{tr} \left( (a - b)^*(a - b)V(t, \theta) \right),$$

where $V(t, \theta) = (E - A(\theta)I^{-1})(A(\theta)U(t, \theta) + U(t, \theta)A(t)) (E - I^{-1}A(\theta))$.

Taking into account (C.1) and (C.2), we have

$$V(t, \theta) = (E - A(\theta)I^{-1})(A(\theta)U(t, \theta) + U(t, \theta)A(t)) (E - I^{-1}A(\theta)) =$$

$$= (E - A(\theta)I^{-1}) (A(\theta)U(\theta, \theta) + U(\theta, \theta)A(\theta)) (E - I^{-1}A(\theta)) +$$

$$+ (E - A(\theta)I^{-1}) (A(\theta)\kappa(t, \theta) + \kappa(t, \theta)A(\theta)) (E - I^{-1}A(\theta)) +$$

$$(t - \theta) (E - A(\theta)I^{-1}) U(t, \theta)U(t, \theta) (E - I^{-1}A(\theta)). \quad (C.4)$$
Denote
\[
G(\theta) = (E - A(\theta)I^{-1}) (A(\theta)U(\theta, \theta) + U(\theta, \theta)A(\theta)) (E - I^{-1}A(\theta))
\]
\[
R(t, \theta) = (E - A(\theta)I^{-1}) (A(\theta)\kappa(t, \theta) + \kappa(t, \theta)A(\theta)) (E - I^{-1}A(\theta))
\]
\[
H(t, \theta) = (E - A(\theta)I^{-1}) U(t, \theta)U(t, \theta) (E - I^{-1}A(\theta)) \tag{C.5}
\]
and put
\[
\tilde{G}(\theta) = \begin{cases} 
G(\theta), & \theta > t \\
-G(\theta), & \theta \leq t.
\end{cases} \tag{C.6}
\]

Then from (C.3), (C.4), (C.5) and (C.6) we get
\[
f(\theta) - f(t) = |\theta - t| \, \text{tr} \left( (a - b)^*(a - b) \tilde{G}(\theta) \right) + \\
+ (\theta - t) \, \text{tr} \left( (a - b)^*(a - b) R(t, \theta) \right) - \\
- (\theta - t)^2 \, \text{tr} \left( (a - b)^*(a - b) H(t, \theta) \right) \tag{C.7}
\]

Since \(R(t, \theta) \to 0\) as \(t \to \theta\) and \(H(t, \theta)\) is positive definite, we conclude that
\[
f(\theta) - f(t) \geq |\theta - t| \, \text{tr} \left( (a - b)^*(a - b) \tilde{G}(\theta) \right) + o(|t - \theta|),
\]

i.e., there exists a positive definite matrix \(W(\theta)\) such that
\[
\|m(\theta)\|^2 - \|m(t)\|^2 = f(\theta) - f(t) \geq |\theta - t| \, \text{tr} \left( (a - b)^*(a - b) W(\theta) \right)
\]

for some neighborhood of \(\theta\). Therefore, we have got the estimate of sharpness of the maximum for the function \(f(t)\):
\[
f(\theta) - f(t) \geq |\theta - t| \lambda_F \, \text{tr} \left( (a - b)^*(a - b) \right), \tag{C.8}
\]

where
\[
\lambda_F \overset{\text{def}}{=} \min_{\beta \leq \theta \leq \alpha} \frac{\text{tr} \left[ (a - b)^*(a - b) W(\theta) \right]}{\text{tr} \left[ (a - b)^*(a - b) \right]}
\]

Let us describe how to calculate \(\lambda_F\). For given family of functions \(F(t)\) we can calculate the function \(f(t) = \text{tr} \left[ m^*(t)m(t) \right]\). Then it is possible to calculate
\[
\lambda_F = \min_{\beta \leq t \leq \alpha, \beta \leq \theta \leq \alpha} \frac{f(\theta) - f(t)}{|\theta - t| \, \text{tr} \left[ (a - b)^*(a - b) \right]}.
\]

Due to the condition \(0 < \beta \leq \theta \leq \alpha < 1\), we get \(\lambda_F > 0\) (see (C.5)). Note that from (C.8) and definition of \(f(t)\) we have for any \(t \in [\beta, \alpha]\):
\[
\|m(\theta)\|^2 - \|m(t)\|^2 \geq \frac{\lambda_F}{2\|m(\theta)\|} |\theta - t| \|a - b\|^2 \tag{C.9}
\]

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The process $Z_N(t)$ can be decomposed into deterministic and stochastic terms:

$$Z_N(t) = m(t) + \gamma_N(t) + \eta_N(t), \quad (C.10)$$

where the norm of the deterministic function $\gamma_N(t)$ converges to zero with the rate $L_F/N$ (this term estimates the difference between corresponding integral sum and the integral; the constant $L_F$ depends on the function family $F(t)$ and can be estimated explicitly for any given family), and the stochastic term is equal to

$$\eta_N(t) = N^{-1} \left( \sum_{i=1}^{[N]} F(i/N) \nu_i^* - \mathcal{P}_1^{[N]}(\mathcal{P}_1^{N})^{-1} \sum_{i=1}^{N} F(i/N) \nu_i^* \right).$$

The norm of the process $\eta_N(t)$ can be estimated as follows:

$$\sup_{\beta \leq \alpha} \|\eta_N(t)\| \leq R \left[ \sqrt{K} + \|I\| \cdot |I^{-1}| + \frac{L_F}{N} \left( \|I\| + \|I^{-1}\| + L_F/N \right) \right] \times \left( \max_{1 \leq i \leq K} \max_{1 \leq l \leq M} \max_{|\beta|n \leq N} N^{-1} \left| \sum_{j=1}^{N} f_i(j/N) \nu_{lj} \right| \right) \overset{\text{def}}{=} \mathcal{R} \left( \max_{1 \leq i \leq K} \max_{1 \leq l \leq M} \max_{|\beta|n \leq N} N^{-1} \left| \sum_{j=1}^{N} f_i(j/N) \nu_{lj} \right| \right), \quad (C.11)$$

where $\mathcal{R} = \mathcal{R}(F, N)$. Here we used the following relations

$$\max_{t \in [0,1]} \|N^{-1} \mathcal{P}_1^{[N]} - A(t)\| \leq \frac{L_F}{N}, \quad \max_{t \in [0,1]} \|A(t)\| \leq |I|$$

and took into account that for any matrix $M$ we have the relation $\|M\| = \sqrt{\text{tr}(M^*M)} \leq R \max_{i,j} |m_{ij}|$, where constant $R$ depends only on the dimensionality.

Denote $\tilde{S}_n = \sum_{j=1}^{n} f_i(j/N) \nu_{lj}$, $\tilde{\xi}(j) = f_i(j/N) \nu_{lj}$ and put $\sigma^2 = \sup_{i} \sup_{1 \leq n \leq N} \sup_{1 \leq l \leq M} \mathbb{E}_\theta(f_i(n/N) \nu_{ln})^2$. Choose the number $\epsilon(x)$ from the following condition

$$\ln(1 + \epsilon(x)) = \left\{ \begin{array}{ll} x^2/4g, & x \leq gT, \\ xT/4, & x > gT, \end{array} \right.$$

where the constant $T$ is taken from the uniform Cramer condition and $g > \sigma^2$.

For the chosen $\epsilon(x) = \epsilon$, we choose the number $m_0(x) \geq 1$ from the uniform $\psi$-mixing condition such that $\psi(m) \leq \epsilon$ for $m \geq m_0(x)$.

Decompose the sum $\tilde{S}_n$ into groups of weakly dependent terms:

$$\tilde{S}_n = \tilde{S}_n^1 + \tilde{S}_n^2 + \cdots + \tilde{S}_n^{m_0(x)},$$
\[ \tilde{S}_n^i = \tilde{\xi}(i) + \tilde{\xi}(i + m_0(x)) + \cdots + \tilde{\xi}\left(i + m_0(x)\left[\frac{n - i}{m_0(x)}\right]\right), \]

and \( i = 1, 2, \ldots, m_0(x). \)

The number of summands \( k(i) \) in each group is no less than \( \lfloor n/m_0(x) \rfloor \) and no more than \( \lceil n/m_0(x) \rceil \). The \( \psi \)-mixing coefficient between summands within each group is no larger than \( \epsilon \). Therefore,

\[
P_\theta\{|\tilde{S}_n|/n \geq x\} \leq m_0(x) \max_{1 \leq i \leq m_0(x)} P_\theta\{|\tilde{S}_n^i| \geq (k(i) - 1)x\}. \tag{C.12}
\]

From Chebyshev’s inequality we have:

\[
P_\theta\left\{\tilde{S}_k = \sum_{j=0}^{k} \tilde{\xi}(i + m_0j) \geq x\right\} \leq e^{-tx}E_\theta e^{t\tilde{S}_k}, \quad \forall t > 0. \tag{C.13}
\]

Further, from \( \psi \)-mixing condition it follows that (see Ibragimov, Linnik (1971)):

\[
E_\theta e^{t\tilde{\xi}(i)} \leq (1 + \epsilon)^k E_\theta \exp(t\tilde{\xi}(i)) E_\theta \exp(t\tilde{\xi}(i + m_0)) \cdots E_\theta \exp(t\tilde{\xi}(i + m_0k)). \tag{C.14}
\]

Consider the term \( E_\theta \exp(t\tilde{\xi}(i)) \). From the uniform Cramer’s condition it follows that for each \( 0 < t < T \):

\[
E_\theta e^{t\tilde{\xi}(i)} \leq \exp(t^2g/2).
\]

Then from (C.13) and (C.14) we obtain

\[
P_\theta\{\tilde{S}_k \geq x\} \leq (1 + \epsilon)^k \exp\left(kgt^2/2 - tx\right).
\]

Taking the minimum of \( kgt^2/2 - tx \) w.r.t. \( t \), write

\[
P_\theta\{\tilde{S}_k \geq x\} \leq \begin{cases} (1 + \epsilon)^k \exp(-x^2/2kg), & x \leq kgT, \\ (1 + \epsilon)^k \exp(-xT/2), & x > kgT. \end{cases}
\]

From the definition of \( \epsilon \) we obtain

\[
P_\theta\{|S_k/k| \geq x\} \leq \begin{cases} \exp(-kx^2/4g), & x \leq gT, \\ \exp(-kxT/4), & x > gT. \end{cases} \tag{C.15}
\]

Now, using (C.12) and (C.15), we obtain

\[
P_\theta\{|\tilde{S}_n/n| \geq x\} \leq \begin{cases} m_0(x) \exp(-x^2n/4gm_0(x)), & x \leq gT, \\ m_0(x) \exp(-Tnx/4m_0(x)), & x > gT. \end{cases} \tag{C.16}
\]
Remark 6. In case of only one regression relationship and independent noises \( \nu_i \), we obtain from here

\[
\sup_{\beta \leq \theta \leq \alpha} P_\theta \{ |\hat{\theta}_N - \theta| > \epsilon \} \leq \begin{cases} 
\exp \left( \frac{-N \beta (C(\epsilon, N)/R)^2}{4g m_0 (C(\epsilon, N)/R)} \right), & \text{if } C(\epsilon, N) \leq R g T \\
\exp \left( \frac{-TN \beta (C(\epsilon, N)/R)^2}{4m_0 (C(\epsilon, N)/R)} \right), & \text{if } C(\epsilon, N) > R g T.
\end{cases}
\]

From (C.11) and (C.16) we get

\[
P_\theta \{ \sup_{\beta \leq t \leq \alpha} \| \eta_N(t) \| > \epsilon \} \leq m_0(\epsilon/R) \left\{ \begin{array}{ll}
\exp \left( -\epsilon^2 N \beta / 4 g m_0(\epsilon/R) \right), & \epsilon \leq R g T \\
\exp \left( -T(\epsilon/R) N \beta / 4 m_0(\epsilon/R) \right), & \epsilon > R g T.
\end{array} \right.
\] (C.17)

In particular, for the case of independent observations, \( m_0(\epsilon) = 1 \).

From the definition of the estimate \( \hat{\theta}_N \) and (C.9) we can write

\[
P_\theta \left\{ |\hat{\theta}_N - \theta| > \epsilon, \hat{\theta}_N \in \text{Arg max}_{\beta \leq t \leq \alpha} \| Z_N(t) \| \right\} = 
\left\{ \begin{array}{ll}
P_\theta \{ \| Z_N(\hat{\theta}_N) \| \geq \| Z_N(t) \|, t \in [\beta, \alpha], |\hat{\theta}_N - \theta| > \epsilon \} & \\
\leq P_\theta \{ \| \eta_N(\hat{\theta}_N) \| - \| \eta_N(\theta) \| \geq m(\theta) - m(\hat{\theta}_N) + L_f/2, |\hat{\theta}_N - \theta| > \epsilon \} & \\
\leq P_\theta \{ \| \eta_N(t) \| \geq \left[ \frac{\epsilon L_f}{4 \| m(\theta) \|} \text{tr}(a - b)^*(a - b) - \frac{L_f}{N} \right] \} & \\
\leq P_\theta \{ \| \eta_N(t) \| \geq \left[ \frac{\epsilon L_f}{4 M} \text{tr}(a - b)^*(a - b) - \frac{L_f}{N} \right] \},
\end{array} \right.
\] (C.18)

where \( M = \max_{\beta \leq t \leq \alpha} \| m(\theta) \| \).

Denote \( C(\epsilon, N) = \left[ \frac{\epsilon L_f}{4 M} \| a - b \|^2 - \frac{L_f}{N} \right] \). Then, finally we obtain from (C.18):
Theorem 3 is proved.

Corollary 2 can be obtained (as it follows from the proof) from the estimates of \( P_\theta \{ \sup_{\beta \leq t \leq \alpha} \| \eta_N(t) \| > \epsilon \}, \theta = 0 \) or \( \theta \neq 0 \).

D Proof of Theorem 4

The proof is based on the same ideas as in Section C, and so we give the sketch of the proof.

Let us consider the matrix random process with continuous time \( Z_N(t) \equiv Z_N([Nt]), t \in [0, 1] \).

It is easy to see that the mathematical expectation of the process \( Z_N(t) \) can be written as follows:

\[
E_\theta Z_N(t) = N^{-1} \left( \sum_{n=1}^{[Nt]} V(n/N) \Pi^*(\theta, n) - T_1^{[Nt]} T_1^{-1} \sum_{n=1}^N V(n/N) \Pi^*(\theta, n) \right)
\]

Denote \( M(t) \equiv \lim_{N \to \infty} E_\theta Z_N(t) \). After simple transformation we have

\[
M(t) = \begin{cases} 
\mathbb{R} (\mathbb{R}^{-1} (\mathbb{R} - \mathbb{R}(\theta)) (a - b)^*), & t \leq \theta \\
(\mathbb{R} - \mathbb{R}(\theta)) \mathbb{R}^{-1} \mathbb{R}(\theta) (a - b)^*), & t > \theta 
\end{cases} \quad (D.1)
\]

It can be shown from (D.1) (by the analogous arguments as in Section C) that the function \( \Phi(t) \equiv \| M(t) \|^2 = \text{tr} (M(t) M^*(t)) \) has unique global maximum on the segment \([0, 1]\) at the point \( t = \theta \) and there exists \( \lambda_\nu > 0 \) such that the following inequality holds

\[
\Phi(\theta) - \Phi(t) \geq \lambda_\nu |t - \theta| \text{tr} [(a - b)(a - b)^*] 
\]

for any \( \beta \leq t \leq \alpha \). The constant \( \lambda_\nu \) depends only of \( V(t) \) and can be estimated analogously the constant \( \lambda_F \) from Section C.

Consider matrix sequence \( N^{-1} T_1^N \). Due to the assumptions, this sequence \( P_\theta \text{-a.s.} \) tends to the positive definite matrix \( \mathbb{R} = \int_0^1 V(s) ds \), and the rate of the convergence is exponential. Therefore, there exists number \( N_1 = N_1 (\{X(n)\}) \) such that as \( N > N_1 \) we get

\[
P_\theta \{ \| N^{-1} T_1^N - \mathbb{R} \| > \epsilon \} \leq L(\epsilon) \exp (-K(\epsilon) N), \quad (D.3)
\]
where functions \( L(\epsilon) \), \( K(\epsilon) \) can be exactly estimated (taking into account \( \psi \)-mixing condition and Cramer’s condition) by the scheme of Section C. The number \( N_1 \) can be estimated by the random sequence \( \{X(n)\} \).

Process \( Z_N(t) \) can be written as follows

\[
Z_N(t) = M(t) + \Gamma_N(t) + \zeta_N(t),
\]

where \( \Gamma_N(t) = E_\theta Z_N(t) - M(t) \) and \( \zeta_N = Z_N(t) - E_\theta Z_N(t) \).

Note that \( \max_{0 \leq t \leq 1} \| \Gamma_N(t) \| \leq L_N \) (because this is the difference between the sum and the integral), and constant \( L_N \) can be estimated exactly for any given function \( V(t) \).

Fix \( \epsilon \), \( 0 < \epsilon < \min ((\alpha - \beta), \|R\|/2) \) and consider the events

\[
D_N = \{ N^{-1}T_1^N - R \| \leq \|R\|/2, \\
\max_{0 \leq i \leq 1} \| N^{-1}T_1^{[N]} - R(t) \| < \epsilon, \| N(T_1^N) - R^{-1} \| < \epsilon \},
\]

\( \bar{D}_N = \Omega \setminus D_N \).

Note that matrix \( N^{-1}T_1^N \) is non-degenerate on the set \( D_N \). Then, due to (D.3),

\[
\delta_N(\epsilon) \overset{\text{def}}{=} P_\theta(\bar{D}_N) \leq 3L(\epsilon) \exp (-K(\epsilon)N). \quad (D.4)
\]

Further, analogously (C.11), we can write on the set \( D_N \)

\[
\sup_{\beta \leq t \leq \alpha} \| \zeta_N(t) \| \leq R \left[ \sqrt{K} + \|R\| : \|R^{-1}\| + \epsilon(\|R\| + \|R^{-1}\| + \epsilon) \right] \times \left( \max_{1 \leq i \leq K} \max_{1 \leq l \leq M} \max_{|\beta N| \leq n \leq N} N^{-1} \left| \sum_{j=1}^{n} x_{ij} \nu_{lj} \right| \right) \overset{\text{def}}{=} R \left( \max_{1 \leq i \leq K} \max_{1 \leq l \leq M} \max_{|\beta N| \leq n \leq N} N^{-1} \left| \sum_{j=1}^{n} x_{ij} \nu_{lj} \right| \right), \quad (D.5)
\]

where \( R = R(V, \epsilon) \).

Now we can use (C.17) and get (by the analogous reasons) from (D.5) on the set \( D_N \)

\[
P_\theta \{ \sup_{\beta \leq t \leq \alpha} \| \zeta_N(t) \| > \epsilon, \, \mathbb{I}(D_N) \} \leq m_0(\epsilon/R) \left\{ \begin{array}{ll}
\exp \left( -\left( \epsilon/R \right)^2 N \beta / 4gm_0(\epsilon/R) \right), \\
\epsilon \leq RgT \\
\exp \left( -T(\epsilon/R)N \beta / 4m_0(\epsilon/R) \right), \\
\epsilon > RgT,
\end{array} \right. \quad (D.6)
\]

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Using (D.4), (D.6), and the analogous considerations as in (C.18), we get

\[
\sup_{\beta \leq \theta \leq \alpha} \mathbb{P}_\theta \{ |\hat{\theta}_N - \theta| > \epsilon \} \leq \delta_N(\epsilon) + m_0(C(\epsilon, N)/R) \left\{ \begin{array}{ll}
\exp \left( \frac{-N\beta \left( \frac{C(\epsilon, N)}{R} \right)^2}{4g_m \left( \frac{C(\epsilon, N)}{R} \right)} \right), & \text{if } C(\epsilon, N) \leq RgT \\
\exp \left( \frac{-TN\beta \left( \frac{C(\epsilon, N)}{R} \right)}{4m_0 \left( \frac{C(\epsilon, N)}{R} \right)} \right), & \text{if } C(\epsilon, N) > RgT,
\end{array} \right.
\]

where \( C(\epsilon, N) = \left[ \frac{\epsilon \lambda_v}{4M} \|a - b\|^2 - \frac{L_v}{N} \right], \ M = \max_{\beta \leq t \leq \alpha} \|M(t)\| \).

Theorem 4 is proved.

### E Proof of Theorem 5

The proposed method of multiple change-point detection and estimation is based upon the idea of recurrent reduction to the case of one change-point.

In order to prove theorem 5 we need to prove the following two propositions:

i) in the case of a stationary sub-sample the norm of the decision statistic does not exceed the threshold with the great probability. This fact is exactly the result of Corollary 2;

ii) in the case of a non-stationary sub-sample with at least two change-points, the norm of the decision statistic exceeds the decision threshold with the great probability.

In order to illustrate ii), let us consider a sub-sample of size \( N \) with two change-points \( 0 < \theta_1 < \theta_2 < 1 \).

In this case the decision statistic can be decomposed into a deterministic and a stochastic term (see (C.10)).

We have from (C.0) for \( 0 \leq t \leq \theta_1 \):

\[
m(t) = A(t)a_1 - A(t)A^{-1}(1) \left( A(\theta_1)a_1 + A(\theta_1, \theta_2)a_2 + A(\theta_2, 1)a_3 \right) \\
= A(t)(a_1 - A^{-1}(1)u), \tag{E.1}
\]

where \( u = A(\theta_1)a_1 + A(\theta_1, \theta_2)a_2 + A(\theta_2, 1)a_3 \).

Again using (C.0), we get for \( \theta_1 \leq t \leq \theta_2 \):

\[
m(t) = A(\theta_1)a_1 + A(\theta_1, t)a_2 - A(t)A^{-1}(1)u = \\
= A(\theta_1)(a_1 - A^{-1}(1)u) + A(\theta_1, t)(a_2 - A^{-1}(1)u).
\]
If
\[ \|m(\theta_1)\| \geq \Lambda \overset{\text{def}}{=} \frac{B}{2(h+1)} > 0, \]
then \( \max_{\beta \leq t \leq \alpha} \|m(t)\| \geq \Lambda > 0. \)

Otherwise, let \( \|m(\theta_1)\| < \Lambda. \) Then
\[ \|m(\theta_2)\| \geq \|A(\theta_1, \theta_2)(a_2 - A^{-1}(1)(u))\| - \Lambda = \|A(\theta_1, \theta_2)(a_2 - a_1 + a_1 - A^{-1}(1)u)\| - \Lambda \geq \|A(\theta_1, \theta_2)(a_2 - a_1)\| - \|A(\theta_1, \theta_2)(a_1 - A^{-1}(1)u)\| - \Lambda \geq B - \|A(\theta_1, \theta_2)A^{-1}(\theta_1)\| - \Lambda \geq B - \Lambda(1 + h) > \Lambda. \]

Therefore, taking into account (E.1), we get: there exists \( \Lambda > 0 \) such that
\[ \max_{\beta \leq t \leq \alpha} \|m(t)\| \geq \Lambda \quad (E.2) \]

From (E.2) it follows that we get ii) with the great probability.

After these preliminary considerations, let us consider the probability of the event:
\[ \{ \hat{k}_N \neq k \} \cup \{ (\hat{k}_N = k) \cap (\max_{1 \leq i \leq k} |\hat{\theta}_N_i - \theta_i| > \delta) \} \quad (E.3) \]
for some fixed \( \delta, \epsilon > \delta > 0. \) Let us consider the following cases:

a) \( \{ \hat{k}_N < k \} \), b) \( \{ \hat{k}_N > k \} \), c) \( \{ (\hat{k}_N = k) \cap (\max_{1 \leq i \leq k} |\hat{\theta}_N_i - \theta_i| > \delta) \} \).

Case a)

In this case the proposed method does not detect at least one change-point, i.e., a certain sub-sample of size \( \tilde{N} \geq [2\delta N] \) containing at least one true change-point, is classified as stationary. Then
\[ P_\vartheta \{ \hat{k}_N < k \} \leq P_\vartheta \{ \max_{\beta \leq t \leq \alpha} \|Z_N(t)\| \leq C(\tilde{N}) \} \quad (E.4) \]
where \( C(\tilde{N}) \) is the decision threshold for the sub-sample.

Choose \( C(\tilde{N}) < \Lambda. \) Then due to (E.4) and (C.10) we have
\[ P_\vartheta \{ \max_{\beta \leq t \leq \alpha} \|Z_N\| \leq C(\tilde{N}) \} \leq P_\vartheta \{ \max_{\beta \leq t \leq \alpha} \|\eta_N(t)\| \geq \max_{\beta \leq t \leq \alpha} \|m(t)\| - \frac{L_F}{N} - C(\tilde{N}) \} \leq P_\vartheta \{ \max_{\beta \leq t \leq \alpha} \|\eta_N(t)\| \geq \Lambda - \frac{L_F}{N} - C(\tilde{N}) \} \]

Now we can use (C.17), changing \( \epsilon \) by \( \{ \Lambda - \frac{L_F}{N} - C(\tilde{N}) \} \), and get the exponential estimate for the event \( \{ \hat{k}_N < k \}. \)
Case b)

In this case there exists a stationary sub-sample of the size $\hat{N} \geq [\delta N]$ such that it is classified as non-stationary. Then

$$P_0\{k_N > k\} \leq P_0\{\max_{\beta \leq t \leq \alpha} \|Z_N(t)\| > C(\hat{N})\} \quad (E.5)$$

But the exponential estimate of the right-hand side (E.5) can be taken from (9).

Case c)

In this case there exists a sub-sample of the size $N^* \geq [2\delta N]$ such that the distance between a true change-point parameter $\theta_i$ and its estimate $\hat{\theta}_{Ni}$ is larger than $\delta$. This is exactly the case of Theorem 3, and we get the exponential estimate of this event from (8).

Therefore, we get the exponential estimate for the event (E.3). This completes the proof of Theorem 5.