Exact S-Matrix for
2D String Theory

Gregory Moore, M. Ronen Plesser, and Sanjaye Ramgoolam
Department of Physics
Yale University
New Haven, CT 06511-8167

We formulate simple graphical rules which allow explicit calculation of nonperturbative $c = 1$ $S$-matrices. This allows us to investigate the constraint of nonperturbative unitarity, which indeed rules out some theories. Nevertheless, we show that there is an infinite parameter family of nonperturbatively unitary $c = 1$ $S$-matrices. We investigate the dependence of the $S$-matrix on one of these nonperturbative parameters. In particular, we study the analytic structure, background dependence, and high-energy behavior of some nonperturbative $c = 1$ $S$-matrices. The scattering amplitudes display interesting resonant behavior both at high energies and in the complex energy plane.
1. Introduction

The definition of the double scaling limit was a significant breakthrough in physics because it provided a framework in which nonperturbative aspects of string theory could be discussed in a rigorous and precise way [1-3]. Although the nonperturbative formulation given by the matrix models is not necessarily the “correct” one, it is the only one available and hence deserves close scrutiny.

Unfortunately, the original hope that a Painlevé I transcendent would provide the nonperturbative definition of the free energy of pure 2D quantum gravity evaporated with the discovery that the reality of the free energy inevitably clashes with “physical” constraints arising from matrix-model/topological field theory Ward identities [4]. The PI transcendent was subsequently demoted to the status of a generating functional for the string perturbation series. This fate has been shared by all solutions to the string equations for gravity coupled to unitary $c<1$ conformal matter. Many attempts to circumvent these difficulties have been made. Those which are physically well-motivated have proven to be mathematically inconsistent [5]. Others, while mathematically consistent lack any cogent physical rationale. We are thus left with the unsatisfying situation of having no nonperturbative unitary $c<1$ theory of gravity.

There is a two-fold origin of the above dilemma. First, there is no simple spacetime interpretation of the $c<1$ models. Hence questions of, e.g., whether the specific heat should or should not have an imaginary part are difficult to resolve. Second and more importantly, there is no physical principle which isolates a reasonable parametrization of physically acceptable possibilities.

The first difficulty is not essential thanks to the $c=1$ model of 2D gravity [6]. In the $c=1$ model there is a simple spacetime interpretation [10]: strings move in two target space dimensions. The nongauge field theoretic degrees of freedom of the string are described by a single massless boson field - the “massless tachyon” which is related to the eigenvalue density field of collective field theory [11,12]. The vertex operator calculations of the $c=1$ matrix model are Euclidean continuations of the scattering amplitudes of the massless boson in a half-space, i.e., the “wall” $S$-matrix amplitudes in the terminology of Polchinski [13].

With this clear physical interpretation there is an equally clear physical constraint on the theory: the $S$-matrix must be unitary. That this is true perturbatively might be

---

1 For reviews see [1-3].
expected to follow automatically either from the point of view of collective field theory or from string perturbation theory. However the question is somewhat more subtle for nonperturbative definitions of the theory.

In the present paper we study nonperturbative unitarity of the $c = 1$ $S$-matrix. We will show that the situation at $c = 1$ is the reverse of that at $c < 1$: while nonperturbative unitarity may be used to rule out some theories there is a plethora of nonperturbatively unitary $c = 1$ theories. Thus, the second and more important difficulty alluded to above will be exacerbated. Indeed, our construction applies to a wide class of matrix model potentials $V(\lambda)$. These may be divided into two classes by the asymptotic properties of $V(\lambda)$ as the eigenvalue variable $\lambda \to -\infty$. In theories of type I $V(\lambda) \to +\infty$ rapidly for $\lambda \to -\infty$ and $V(\lambda) \to -\lambda^2$ for $\lambda \to +\infty$. Thus the theory is effectively defined on a semi-infinite line. The canonical example of such theories is defined by $V(\lambda) \propto -\lambda^2$ on the interval $[A, \infty)$ together with an infinite wall at $\lambda = A$. We will sometimes specialize to the case $A = 0$ where formulas simplify. Although the potential is not analytic at $\lambda = A$ we expect that for a smoothed out version of the wall the main results will be unchanged. Theories of type II are defined by a smooth potential $V(\lambda) \to -\lambda^2$ on both ends of the real $\lambda$ axis. Thus, in perturbation theory there appear to be two disconnected worlds.

In outline, the paper is organized as follows. In section two we give a definition of the $S$-matrix in terms of large spacetime asymptotics of correlators of the eigenvalue density operator of the matrix model. We begin with an integral representation for the correlators derived in [14] and continue the resulting asymptotics to Minkowski space to obtain the $S$-matrix. The procedure of this section is an important technical advance over previous calculations. For example, in [14], the two-, three-, and four-tachyon scattering amplitudes were obtained to all orders of perturbation theory using the small length asymptotics of macroscopic loops. However, at the time [14] was written, extension of the results to more general amplitudes appeared hopeless. Moreover, while the results of [14] are valid to all orders of perturbation theory the nonperturbative foundations of these formulae are shaky. Another advantage of the procedure of section two is that the nonperturbative formulation of the theory is unambiguous.

In section three we formulate the result of applying the procedure of section two in terms of some simple graphical rules. These rules lead to a relatively simple and compact form for the $n \to m$ amplitudes (eq. (3.6) below). Some particularly simple cases, for example the nonperturbative $1 \to n$ amplitudes are written out explicitly (eq. (3.11)).

2
The graphical rules also provide considerable insight into the fundamental nature of $c = 1$ scattering, making clear how particle production is possible in a theory of free fermions.

In section four we apply the lessons learned from the graphical formalism of section three to the problem of nonperturbative unitarity. Both theories of type I and theory II are unitary to all orders of perturbation theory. In section (4.1) we prove that theories of type I are nonperturbatively unitary. In section (4.2) we remark that theory II is not unitary, essentially because the theory does not take into account fermionic soliton sectors. Our remark will strike many readers as trivial. Nevertheless, we think it is important and deserves emphasis. The unitarity proof for theory I suggests various interesting generalizations of $c = 1$ scattering and leads to a characterization of a large class of acceptable nonperturbative definitions of the $c = 1$ $S$-matrix.

The exact formulae for $c = 1$ scattering amplitudes allow us to investigate in some detail the analytic structure of the $S$-matrix in section five. We find several interesting singularities of the analytically-continued $S$-matrix elements and interpret these in terms of metastable bound states of matrix model fermions. We emphasize the dependence on the parameter $A$. This parameter does not appear in perturbative amplitudes but has an important influence on the nonperturbative answers.

In section six we make some remarks on the worldsheet/Liouville interpretation of our results. Most importantly, we show that the vertex-operator-motivated prescription used in [14] to extract correlators from small loop-length asymptotics of macroscopic loop amplitudes is equivalent (to all orders of perturbation theory) to the procedure of section two.

In section seven we make a few remarks on the background-dependence of the $S$-matrix, focusing on the background dependence of the singularity structure uncovered in section five. The dependence on tachyonic perturbations of the background will be addressed in a separate publication. In section eight we study some aspects of the high energy behavior of our scattering amplitudes. The amplitudes exhibit many nontrivial features. While this is hardly surprising from the close relation of the present $S$-matrix to that of potential scattering, the interpretation of these features might prove instructive for string theory. In the conclusion we present some remarks on directions for future research, and we summarize some technical details in several appendices.
2. Definition of the $S$-matrix

2.1. Eigenvalue Density Correlation Functions

The Euclidean Green’s functions of the eigenvalue density $\rho = \psi^\dagger \psi(\lambda, x)$, where $x$ is the “time” dimension of the $c = 1$ matrix model are defined by

$$G_{\text{Euclidean}}(x_1, \lambda_1, \ldots, x_n, \lambda_n) \equiv \langle \mu | T \left( \hat{\psi}^\dagger(x_1, \lambda_1) \cdots \hat{\psi}^\dagger(x_n, \lambda_n) \right) | \mu \rangle_c$$

(2.1)

Since the fermions are noninteracting these Green’s functions may be written in terms of the the Euclidean fermion propagator:

$$S^E(x_1, \lambda_1; x_2, \lambda_2) = e^{-\mu \Delta x} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-ip\Delta x} I(p, \lambda_1, \lambda_2)$$

(2.2)

where $I$ is the resolvent for the upside down oscillator Hamiltonian $H = \frac{1}{2} p^2 - \frac{1}{8} \lambda^2$. In particular, for $q > 0$

$$I(q, \lambda_1, \lambda_2) = (I(-q, \lambda_1, \lambda_2))^* = \langle \lambda_1 | \frac{1}{H - \mu - iq} | \lambda_2 \rangle$$

(2.3)

Thus we obtain the integral representation for the eigenvalue correlators [14]:

$$G_{\text{Euclidean}}(q, \lambda_i) = \frac{1}{n^2} \int \prod_{i=1}^n dq_i e^{-iq_i x_i} \sum_{\sigma \in \Sigma_n} \prod_{\sigma} S^E(x_{\sigma(i)}, \lambda_{\sigma(i)}; x_{\sigma(i+1)}, \lambda_{\sigma(i+1)})$$

(2.4)

$$= \frac{1}{n} \delta(\sum q_i) \int_{-\infty}^{\infty} dq \sum_{\sigma} \prod_{k=1}^n I(Q_{k}^\sigma, \lambda_{\sigma(k)}; \lambda_{\sigma(k+1)})$$

where $Q_{k}^\sigma \equiv q + q_{\sigma(1)} + \cdots q_{\sigma(k)}$.

2.2. Relation to Collective Field Theory

We now relate the exact eigenvalue correlators (2.4) to the correlation functions of the Das-Jevicki collective field formulation, as explained in [12,13,15,16] .

\[\text{Of course, the connection had also been extensively discussed in [12,16,13] before [15]. Nevertheless, the method of appendix A in [17] leads to the simple calculation of this paper, which we believe is new.}\]
First, let us find the large \( \lambda \) behavior of \( I \). We work on half of the \( \lambda \) axis, change variables to \( \lambda = 2\sqrt{\mu} \cosh \tau \) and identify \( \delta \rho \equiv \rho - \langle \rho \rangle \equiv \partial \lambda \chi \) as usual. If we hold \( \tau \) fixed then asymptotically at large \( \mu \) the integral \( I \) has the form of a direct and reflected propagator \([15]\):

\[
I(q, \tau_1, \tau_2) \xrightarrow{\mu \to \infty} \frac{i}{\sqrt{4\mu \sinh \tau_1 \sinh \tau_2}} \left[ e^{-q|\tau_1 - \tau_2|} e^{i\mu |G(\tau_1) - G(\tau_2)|} D(q, \tau_1, \tau_2) \right.
\]

\[
+ ie^{-q(\tau_1 + \tau_2)} e^{i\mu (G(\tau_1) + G(\tau_2))} R(q, \tau_1, \tau_2) \right].
\]

(2.5)

Here \( G(\tau) = \tau - \frac{1}{2} \sinh 2\tau \) is the WKB phase factor of a fermion wavefunction, and \( R = 1 + \frac{i}{\mu} C^{(1)} + \frac{1}{\mu^2} C^{(2)} + \cdots \). The \( C^{(i)} \) are real, polynomials in the \( q_i \), and rational functions of \( w_i = e^{\tau_i} \). Finally, \( D(q, \tau_1, \tau_2) = R(q, \tau_1, -\tau_2) \) for \( \tau_1 > \tau_2 \).

The correlation functions of the Das-Jevicki collective field theory are obtained by inserting (2.3) into (2.4), expanding the product, and keeping only those terms for which the nonperturbatively oscillating factors \( e^{\pm i\mu G(\tau)} \) cancel. This defines a new set of Green’s functions \( \tilde{G}(q_i, \tau_i; \mu) \) (at least as asymptotic expansions in \( 1/\mu \)) which are the Euclidean continuation of the Green’s functions of the Das-Jevicki theory to all orders of perturbation theory:

\[
\tilde{G}_{\text{Euclidean}}(\tau_i, q_i; \mu) = \prod_i \frac{1}{2\sqrt{\mu \sinh \tau_i}} \langle 0 | \prod_i \partial \tau \chi_i | 0 \rangle_c .
\]

(2.6)

Although in principle this procedure gives a straightforward method for calculating the off-shell Green’s functions of \( \partial \tau \chi \), in practice it becomes cumbersome already at one-loop \([17]\). Nevertheless, as we will see, it is an extremely useful observation for extracting the \( S \)-matrix.

2.3. Large \( \lambda \) Asymptotics

Our strategy for extracting \( S \)-matrix elements (in Minkowski space) and (microscopic) vertex operator correlators (in Euclidean space) will be to extract the coefficients of appropriate terms in the large \( \lambda \) asymptotic expression of \( \tilde{G} \). From the collective-field-theory we may invoke a coordinate space version of the LSZ reduction procedure to identify the \( S \)-matrix as the coefficient of the term multiplying appropriate incoming and outgoing wavefunctions of \( \tau \) \([18,19]\).

In the \( \tau \to \infty \) limit the complicated rational expressions in (2.3) simplify drastically. Since (2.3) is just the Green’s function for \( H \) it may itself be expressed in terms of parabolic
cylinder functions. From this representation one obtains the asymptotics for \( \lambda_i \to \pm \infty \). The calculation is outlined in appendix A. For \( \lambda_i \to + \infty \) we obtain

\[
I(q, \lambda_1, \lambda_2) \sim -i \sqrt{\lambda_1 \lambda_2} \left[ e^{-q|\tau_1 - \tau_2|} e^{i\mu |G(\tau_1) - G(\tau_2)|} \right] (1 + \mathcal{O}(e^{-\tau_i})) \quad q > 0
\]

\[
I(q, \lambda_1, \lambda_2) \sim i \sqrt{\lambda_1 \lambda_2} \left[ e^{q|\tau_1 - \tau_2|} e^{-i\mu |G(\tau_1) - G(\tau_2)|} \right] (1 + \mathcal{O}(e^{-\tau_i})) \quad q < 0 .
\] (2.7)

The “Euclidean fermion reflection matrix” \( R_q \) will play a key role in what follows. In theories of type I, defined on \( \lambda \in [A, \infty) \) we find

\[
R_q = i e^{i \mu \log \mu} \left[ \frac{1 + i e^{-\pi(\mu + |q|)}}{1 - i e^{-\pi(\mu + |q|)}} \right] \left( \frac{\Gamma(\frac{1}{2} - i \mu + |q|)}{\Gamma(\frac{1}{2} + i \mu - |q|)} \right) .
\] (2.8)

for \( A = 0 \). For nonzero \( A \) the result is given in equation (A.7).

In theory II we have

\[
R_q = -\frac{i}{\sqrt{2\pi}} e^{i \mu \log \mu} \left[ e^{-\pi(\mu + |q|)} \Gamma(\frac{1}{2} - i \mu + |q|) \right] .
\] (2.9)

The factors \( R \) in (2.8) and (2.9) agree to all orders of perturbation theory

\[
R_q^{II} = \frac{1}{1 + i e^{-\pi(\mu + |q|)}} R_q^{I}
\] (2.10)

and have a perturbative expansion of the form \( R_q \sim 1 + \mathcal{O}(1/\mu) \). In theory II there are two worlds and we must also calculate the “transmission” probabilities obtained from the asymptotics of \( I \) for \( \lambda_1 = 2\sqrt{\mu} \cosh \tau_1 \to +\infty \) and \( \lambda_2 = -2\sqrt{\mu} \cosh \tau_2 \to -\infty \). In this case the “direct propagator” in (2.7) does not appear and the transmitted amplitude is

\[
T_q = -i e^{-\pi \mu - i \pi |q|} R_q .
\] (2.11)

We may now derive the large \( \lambda \) asymptotics of the collective field theory Green’s function \( \tilde{G} \). Substitute (2.7) into (2.4) and expand, keeping only terms for which factors of \( e^{i\mu G(\tau)} \) cancel. For large \( \tau \) the two-point function behaves like:

\[
\tilde{G}_{\text{Euclidean}} = \delta(q_1 + q_2) \frac{1}{4\mu \sinh \tau_1 \sinh \tau_2} \left[ |q| e^{-|q||\tau_1 - \tau_2|} + \mathcal{R}_2(q_1, -q_1; \mu) e^{-|q|(\tau_1 + \tau_2)} \right] .
\] (2.12)
giving a direct and reflected propagator for the string-theoretic massless tachyon. Specifically, for \( q > 0 \)

\[
\mathcal{R}_2(q, -q; \mu) = \int_0^q dx R_x R^*_q x
\]

(2.13)

For three or more operators the resulting expression for \( \tilde{G} \) will have large \( \tau_i \) asymptotics:

\[
\tilde{G} \sim \delta(\sum q_i) \prod_{i=1}^n e^{-|q_i|\tau_i} \mathcal{R}_n(q_1, \ldots, q_n; \mu) \left[ 1 + \mathcal{O}(e^{-\tau_i}) \right].
\]

(2.14)

Equation (2.14) may be taken as the definition of the functions \( \mathcal{R}_n \). In theory II we define in an analogous way the functions \( \mathcal{R}_n(\epsilon_i, q_i) \) where \( \epsilon_i = \pm \) for \( \lambda_i \to \pm \infty \).

2.4. Continuation to Minkowski Space

The analytic continuation to Minkowski space is most clearly understood by starting with the Minkowski space formulation of the Das-Jevicki theory, continuing that theory to Euclidean space and comparing with the Euclidean free fermion amplitudes. By studying the behavior of propagators we can obtain the following analytic continuation prescription. Consider the function \( \mathcal{R}_n(q_1, \ldots, q_k; q'_1, \ldots q'_l) \) in some kinematic region where \( q_i > 0 \) and \( q'_i < 0 \). We replace \( q_i \to -i\omega_i \) and \( q'_i \to i\omega'_i \), or, briefly, \( |q| \to -i\omega \). Here and hereafter \( \omega \) represents a real positive number, the energy of a massless tachyon.

Explicitly, in theory I with \( A = 0 \) the reflection factor becomes a pure (energy-dependent) phase:

\[
R_q \to e^{i\mu \log \mu} e^{i\omega} \sqrt{\frac{1 + i e^{-\pi(\mu + \omega)}}{1 - i e^{-\pi(\mu + \omega)}}} \sqrt{\frac{\Gamma(\frac{1}{2} - i(\mu + \omega))}{\Gamma(\frac{1}{2} + i(\mu + \omega))}} \equiv ie^{i\mu \log \mu} e^{i\omega} e^{i\Theta(\mu + \omega)} \quad q > 0
\]

\[
R_q \to ie^{i\mu \log \mu} e^{i\omega} e^{-i\Theta(\mu - \omega)} \quad q < 0
\]

(2.15)

More generally the analytic continuation will define a phase \( e^{i\Theta(\mu + \omega, A)} \). The factors of \( ie^{i\mu \log \mu} \) and \( \mu e^{i\omega} \) do not affect final amplitudes. The former cancels out of eq. (3.7) below while the latter may be absorbed as a phase redefinition of the states.

In theory II \( R \) does not become a pure phase but has absolute magnitude \((1 + e^{-2\pi(\mu + \omega)})^{-1/2}\). Similarly \( T \) has magnitude \((1 + e^{2\pi(\mu + \omega)})^{-1/2}\). For \( \mu \) large and positive any \( S \)-matrix element involving \( k \) bosons traversing the barrier is exponentially suppressed and of order \( e^{-2\pi k\mu} \).
2.5. Definition of the S-Matrix

In theory I there is an incoming and outgoing Fock space of bosonic states defined by oscillators \( \alpha(\omega) \) and normalized by

\[
[\alpha(\omega), \alpha^\dagger(\omega')] = \omega \delta(\omega - \omega') .
\]  

(2.16)

In the space of incoming states \( \mathcal{H}^\text{in} \), \(|\omega_1, \ldots, \omega_k\rangle \) represents a set of left-moving bosons approaching the wall from the left with wavefunction

\[
\psi_{\omega_1, \ldots, \omega_k}^L(t_i, \tau_i) = \frac{1}{\sqrt{k!}} \sum_\sigma \prod e^{-i\omega_\sigma(i)(t_i + \tau_i)} .
\]  

(2.17)

In the outgoing space \( \langle \omega | \) represents a set of outgoing right-moving bosons with wavefunction

\[
\psi_{\omega_1', \ldots, \omega_l'}^R(t'_i, \tau'_i) = \frac{1}{\sqrt{l!}} \sum_\sigma \prod e^{-i\omega'_\sigma(i)(t'_i - \tau'_i)} .
\]  

(2.18)

The wavefunctions \( \psi^R, \psi^L \) are continuations of the Euclidean wavefunctions \( \prod e^{-|q_i|\tau_i + iq_ix_i} \). Thus, the analytic continuation of the previous subsection shows that negative and positive \( q \) correspond to incoming and outgoing particles respectively.

Putting together \( (2.14) \) with \( (2.17)(2.18) \) we may write the large \( \tau \) asymptotics of the Minkowski space Green’s function as:

\[
\tilde{G}_{\text{Minkowski}}(\omega, \tau) \sim \delta(\sum_{i=1}^k \omega_i - \sum_{i=1}^l \omega_i') \prod_{i=1}^k (\psi_{\omega_i}^L(t_i, \tau_i))^* \prod_{i=1}^l (\psi_{\omega_i'}^R(t'_i, \tau'_i)) \sqrt{k!l!} S_c(\omega_i | \omega_i')
\]  

(2.19)

where \( S_c \) is obtained from \( \frac{k+l}{\sqrt{k!l!}} \mathcal{R}_n \) by analytic continuation.

Equation \( (2.19) \) defines the connected S-matrix element:

\[
\langle \omega_1 | S | \omega_1' \rangle = \delta(\sum \omega_i - \sum \omega_i') S_c(\omega_i | \omega_i') + \text{disconnected terms} .
\]  

(2.20)

3. Calculation of the S-matrix

3.1. Diagrammatic Formalism

The procedure of extracting terms for which \( e^{\pm i\mu G(\tau)} \) cancel from the product of the functions \( I \) can be given a diagrammatic interpretation which facilitates calculation and which will be crucial to our proof that theory I is unitary.\(^3\)

\(^3\) The formalism is closely related to old-fashioned non-covariant perturbation theory for the fermions. Several special features of our problem lead to \textit{ad hoc} rules which necessitate our detailed discussion below.
As in a Feynman diagram there is a vertex in the \((x, \tau)\) half-space corresponding to each operator \(\psi^\dagger \psi(x, \tau)\). While the final result will of course be independent of the order in which the \(\tau_i\) are increased to infinity, in intermediate steps we will choose some order and locate the vertices accordingly. Points are connected by line segments, representing the integral \(I\), to form a one-loop graph. Since the expression for \(I\) in (2.7) has two terms and we have both direct and reflected propagators as in fig. 1. Each line segment carries a momentum and an arrow. Note that in fig. 1 the reflected propagator, which we call simply a “bounce,” is composed of two segments with opposite arrows and momenta. These line segments are joined to form a one-loop graph according to the following rules:

**RH1.** Lines with positive (negative) momenta slope upwards to the right (left).

**RH2.** At any vertex arrows are conserved and momentum is conserved as time flows upwards. In particular momentum \(q_i\) is inserted at the vertex as in fig. 2.

**RH3.** Outgoing vertices at \((x_{\text{out}}, \tau_{\text{out}})\) all have later times than incoming vertices \((x_{\text{in}}, \tau_{\text{in}})\): \(x_{\text{out}} > x_{\text{in}}\).

Since diagrams drawn according to these rules correspond to real processes taking place in spacetime they will be called real histories. Some examples are shown in fig. 3 and fig. 4. There is a finite number (less than \(n!\)) of ways of connecting the dots. Since \(\sum q_i = 0\) there is an overall undetermined loop momentum \(q\), however the constraints RH1-RH3 restrict \(q\) to lie in a finite interval.

To each real history we associate an amplitude, easily derived from (2.7) and illustrated in fig. 5 and fig. 6. The total amplitude is simply obtained by summing over real histories to produce a formula which reads, schematically,

\[
\mathcal{R} = i^n \sum_{RH} \pm \int dq \prod_{\text{bounces}} R_Q(-R_Q)^* \quad (3.1)
\]

In theory II we take two copies of \((x, \tau)\) space. Transmission amplitudes are computed by connecting vertices in one half space to another. Each transmission line is weighted with a factor of \(T_q\). Note in particular that since two fermions make a boson every transmission of a boson has a nonperturbative suppression of \(e^{-2\pi\mu}\).
3.2. Explicit Formula

We now write the equation (3.1) more precisely. Each real history has the structure of a series of “brackets” fig. 7 separated by bounces since (by considering momentum conservation) direct propagators can only connect incoming to incoming and outgoing to outgoing vertices. Each bracket must contain at least one insertion of momentum, so in an \( n \to m \) amplitude there can be up to \( \text{Min}\{n,m\} \) incoming and outgoing brackets. The number of incoming brackets equals the number of outgoing brackets. As we traverse the loop, we encounter an increasing set of momentum insertions. The factors for each bounce only depend on the net momenta flowing through the bracket, but there is a combinatoric factor counting the number of ways of forming a bracket out of a given set \( T \) of momenta.

Consider for example the case where \( T \) is a set of positive momenta. The only constraints on the formation of a bracket as in fig. 7 is that, if \( q^\ast \) corresponds to the vertex \( \tau^\ast \) with the largest \( \tau \) in \( T \) then the direct propagators immediately before and after the insertion of \( q^\ast \) must have positive and negative momenta, respectively. Let \( T_1 \) be the momenta inserted before the insertion of \( q^\ast \) and \( T_2 \) the momenta inserted after. The sum over real histories involves a sum with weights:

\[
\sum_{T_1 \uplus T_2 \uplus \{q^\ast\} = T} \theta(q(T_2) + q^\ast - Q)\theta(Q - q(T_2))(-1)^{|T_2|}
\]

\[
= \sum_{T_1 \uplus T_2 \uplus \{q^\ast\} = T} (\theta(Q - q(T_2)) - \theta(Q - q(T_2) - q^\ast))(-1)^{|T_2|}
\]

\[
= \sum_{T_1 \uplus T_2 = T} (-1)^{|T_2|}\theta(Q - q(T_2))
\]

\[
= - \sum_{T_1 \uplus T_2 = T} (-1)^{|T_2|}\theta(q(T_2) - Q)
\]

\[
\equiv f_+(T,Q)
\]  

(3.2)

where the notation \( q(S) \) means the sum of momenta in a set \( S \) and in going from the second to the third expression we have noticed that the two terms in the sum correspond to subsets which do and do not include \( q^\ast \). Note in particular that the final weighting factor makes no reference to \( q^\ast \) and is therefore independent of \( \tau \)-ordering. Similarly, for a set \( T \) of negative momenta as in fig. 7 we have a weighting factor

\[
f_-(T,Q) = \sum_{T_1 \uplus T_2 = T} \theta(q(T_1) - Q)(-1)^{|T_2|}
\]

\[
= - \sum_{T_1 \uplus T_2 = T} \theta(Q - q(T_1))(-1)^{|T_2|}.
\]  

(3.3)
Note that the form of \( f \) differs for positive and negative momenta. For convenience we set \( f_\pm(0, Q) = 0 \).

We are now in a position to write equation \((3.1)\) precisely. Let \( S = \{q_1, \ldots, q_n\} \) be the set of momenta, \( S^\pm \) the set of positive and negative momenta, and define an admissible filtration (AF) of order \( k \) to be a tower of subsets of momenta:

\[
\emptyset = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{2k} = S
\]

such that

\[
F_j - F_{j-1} \subset S^+ \quad j = 0 \pmod{2}
\]

\[
F_j - F_{j-1} \subset S^- \quad j = 1 \pmod{2}
\]

Then we have the (Euclidean space) “filtration formula”

\[
\mathcal{R}_n = i^n \sum_{k=1}^{m} \frac{1}{k} \sum_{AF_k} \int dQ \prod_{j=1}^{2k} f_{(-1)^j}(F_j - F_{j-1}, Q + q(F_j)) \prod_{j=1}^{k} R_{Q+q(F_{2j-1})} R^*_{Q+q(F_{2j})} \quad (3.6)
\]

where \( m = \text{Min}\{|S^+|, |S^-|\} \). The combinatoric factors \( f_\pm \) impose kinematic constraints which render the \( Q \)-integral finite.

Examining the reflection factors of \((2.8),(2.9)\) we see that the \( \mu \) dependence of \( R_n \) is only through the combination \( \mu + i|q| \), except for the (noncancelling) prefactor \( \mu^{-|q|} \). This observation allows us to compute the derivative \( \partial_\mu \left( \mu^{\frac{1}{2}} \sum |q_i| \mathcal{R} \right) \) by converting it to a \( Q \) derivative. This has the effect of killing the loop integration, extracting a boundary term wherever the argument of one of the theta functions vanishes. Thus we can write a formula

\[
\frac{\partial}{\partial \mu} \left( \mu^{\frac{1}{2}} \sum |q_i| \mathcal{R}_n \right) = i^{n+1} \mu^{\frac{1}{2}} \sum |q_i| \sum_{RH} \prod_{\text{bounces}} R_Q R^*_Q
\]

where the real histories allowed in \((3.7)\) have the added condition

RH4. Exactly one direct or reflected propagator carries zero-momentum. Direct propagators with zero momentum have a factor +1.

The following useful properties are immediate consequences of these expressions. The functions \( \mathcal{R}_n(q_1, \ldots, q_n; \mu) \) are real and totally symmetric functions of the \( q_i \) and are defined on the plane \( \sum q_i = 0 \) in \( \mathbb{R}^n \). If \( S \) is any subset of momenta then the functions are continuous but have discontinuous derivatives across the hyperplanes \( H_S \) defined by \( q(S) \equiv \sum_{q \in S} q = 0 \). On the complements of these hyperplanes the \( \mathcal{R}_n \) are analytic. Indeed, we can Taylor-expand for small \( q_i \) and the Taylor-coefficients are polynomials in polygamma functions (see e.g. appendix C), although the coefficients in the Taylor expansion change across \( H_S \). Similarly, in the asymptotic expansion of \( \mathcal{R}_n \) for \( \mu \to +\infty \) the coefficients of \( 1/\mu^n \) are polynomials in the \( q_i \), although the polynomial changes across the hypersurfaces \( H_S \). The analytic structure of \( \mathcal{R}_n \) will be discussed further in section 5 below.
3.3. A Low-Energy Theorem

Suppose any momentum $p_*$ is taken to zero. Physically the vertex operator inserting $p_*$ becomes proportional to the cosmological constant and the resulting limit should be expressed as a derivative with respect to $\mu$ of an $n - 1$ point function. This can be proven by noting that if $p_* \in F_j - F_{j-1}$ then

$$f_\pm(F_j - F_{j-1}, Q) \to -p_* \frac{\partial}{\partial Q} f_\pm(F'_j - F_{j-1}, Q)$$  (3.8)

where $F'_j = F_j - \{p_*\}$. Hence combining this remark with (3.7)

$$\mu^{\frac{1}{2}} \sum |q_1| \mathcal{R}_n \to p_* \frac{\partial}{\partial \mu} \left( \mu^{\frac{1}{2}} \sum |q_1| \mathcal{R}_{n-1} \right)$$  (3.9)

as any momentum $p_*$ goes to zero.

3.4. Special Cases

The general formula (3.6) is somewhat awkward to work with. We give the explicit formula in some special cases.

1. If $S^+ = \{q_1, \ldots q_{n-1}\}, q_n < 0$. Then we have:

$$\frac{\partial}{\partial \mu} \left( \mu^{\frac{1}{2}} |q_n| \mathcal{R}_n \right) = i^{n+1} \sum_{S_1 \cup S_2 = S^+} (-1)^{|S_2|} R_{q(S_1)} R^*_{q(S_2)} .$$  (3.10)

Correspondingly, in theory I the $S$-matrix element for an incoming state $|\omega\rangle$ to produce $\langle \omega_1, \ldots, \omega_{n-1}|$ is given by

$$S(\omega|\omega_i) = \frac{i^{n+1}}{\sqrt{(n-1)!}} \sum_{S_1 \cup S_2 = S^+} (-1)^{|S_2|} \int_{0}^{\omega(S_2)} dx e^{i\Theta(\mu + \omega - x) + i\Theta(x - \mu)} .$$  (3.11)

This allows us to compute the probabilities for particle production when an incoming tachyon impinges upon the wall. A discussion of this effect follows in section 8.

We also note that the content of (3.10) and (3.11) can be expressed in terms of linear Ward identities on the amplitudes, which could be interesting in Liouville theory.

2. We can use our rules to write a relatively simple result for the $S$-matrix for scattering $\omega_1 + \omega_2 \to \sum \omega'_i$. This consists of two terms with either 2 or 4 bounces. Let
\[ S^+ = \{q_1, \ldots q_{n-2}\}, \text{ and } S^- = \{q_{n-1}, q_n\}, \text{ then:} \]

\[ R_n = \sum_{T_1 \cup T_2 = S^+} (-1)^{|T_2|} \int_0^{q(T_2)} (1 - \theta(Q + q_{n-1}) - \theta(Q + q_n)) R_Q^* R_{Q+q(S^-)} dQ \]

\[ + \frac{1}{2} \sum_{S_1 \cup S_2 = S^+} \int_0^{q(S_2)} f_+(S_2, Q) R_{q(S_2)-Q} R_Q^* \left[ f_+(S_1, Q + q_n + q(S_1)) R_{q_n+Q} R_{Q+q_n+q(S_1)} \right. \]

\[ + f_+(S_1, Q + q_{n-1} + q(S_1)) R_{q_{n-1}+Q} R_{Q+q_{n-1}+q(S_1)} \left. \right] . \]  

(3.12)

Specializing further, consider \(2 \rightarrow 2\) scattering. Let \(n = 4\) above, and take \(q_1 + q_3 > 0, q_1 + q_4 > 0\). We find:

\[ \frac{\partial}{\partial \mu} \left( \frac{1}{2} \sum |q_i| R_A \right) = -4\mu^2 \sum |q_i| \text{Im} \left[ R_{q_1+q_2} R_{q_0}^* - R_{q_1} R_{q_2}^* \right. \]

\[ \left. - R_{q_2} R_{q_2+q_3} R_{q_4} R_{q_0}^* - R_{q_2} R_{q_2+q_4} R_{q_3} R_{q_0}^* \right] . \]  

(3.13)

The previously published result for \(2 \rightarrow 2\) scattering in [14] involved an infinite sum of gamma functions and was rather unwieldy. This illustrates nicely that the formulae of this paper are a substantial improvement upon those given in [14]. (And also that there are some remarkable identities on gamma functions, this particular case is written out in appendix B.)

4. Unitarity of the \(S\)-matrix

4.1. Theories of Type I are Unitary

The essential idea of the unitarity proof is to regard the time evolution of the real histories of the previous section as a composition of three maps: fermionization, free fermion scattering, and bosonization: \(i_{f \rightarrow b} \circ S_{ff} \circ i_{b \rightarrow f}\) as in fig. 8.

The most complicated map is the bosonization map, although this is well-known. We will let \(a^\dagger(\nu), a(\nu)\) be delta function normalized fermion creation operators acting on the Fermi sea:

\[ a(\xi, +)|\mu\rangle \equiv a^\dagger(\mu + \xi)|\mu\rangle = 0 \quad \xi > 0 \]

\[ a(\xi, -)|\mu\rangle \equiv a(\mu - \xi)|\mu\rangle = 0 \quad \xi > 0 . \]  

(4.1)
An operator in the fermionic theory is normal ordered if it is a sum of terms of the form $\prod_i a(\xi_i, \epsilon_i)$ with $\epsilon_i \xi_i < 0$. Under bosonization a one-particle incoming state is mapped according to

$$i_{b \rightarrow f} : |\omega\rangle \rightarrow \int_{-\infty}^{\infty} d\xi a(\mu + \xi) a^\dagger(\mu + \xi - \omega) |\mu\rangle = \int_{0}^{\omega'} d\xi a(\mu + \xi) a^\dagger(\mu + \xi - \omega) |\mu\rangle. \tag{4.2}$$

Note that if incoming states are normalized according to $\langle \omega | \omega' \rangle = \omega \delta(\omega - \omega')$ then the map $i_{b \rightarrow f}$ is an isometry. This isometry may be extended to the entire Fock space by

$$i_{b \rightarrow f} : |\omega_1 \rangle \otimes \cdots \otimes |\omega_n\rangle \rightarrow \int_{-\infty}^{\infty} \prod_i d\xi_i \prod_{i} a(\mu + \xi_i) a^\dagger(\mu + \xi_i - \omega_i) |\mu\rangle. \tag{4.3}$$

Note that the operator acting on $|\mu\rangle$ on the RHS is not normal-ordered and hence the state does not have definite particle-hole number. We may now imagine normal-ordering the operators in (4.3). To each normal-ordered fermionic monomial we can associate a diagram of the type drawn in our real histories. In particular, the state $\prod_i a(\xi_i, \epsilon_i) |\mu\rangle$ created by a normal-ordered operator is associated with a set of incoming bounce-lines carrying negative momentum $\epsilon_i \xi_i$ and having upward(downward) - pointing arrows for $\epsilon = \pm$, respectively. By an inductive argument one can show that the normal-ordering process corresponds exactly to forming all possible incoming bracket structures discussed in subsection 3.2 above. The details of this argument appear in appendix D.

Thus the first third of the set of real histories corresponds in a precise sense to the fermionization of the incoming bosons. The second stage corresponds to free fermion scattering. In theory I this scattering is diagonal in the momentum basis $\prod_i a(\xi_i, \epsilon_i) |\mu\rangle$ and merely reverses the signs of all $\epsilon_i$. In particular,

$$\langle \mu | \prod_{i=1}^{n} a(\xi_i, \epsilon_i) S_{ff} \prod_{j=1}^{m} a(\xi'_j, \epsilon'_j) |\mu\rangle = \delta_{n,m} \sum_{\sigma \in \Sigma_n} \prod_{j=1}^{n} \delta(\xi_j + \xi'_j) \delta_{\epsilon_j, \epsilon'_j} e^{-i \epsilon_j \Theta(\mu + \xi_j)}. \tag{4.4}$$

The final third of the map is just the inverse of the isometry $i_{b \rightarrow f}$, since our graphical rules implement this inverse (simply consider reading the diagram from top to bottom).

We have written the bosonic S-matrix as a composition of two isometries and a unitary free-fermion S-matrix; it is therefore nonperturbatively unitary.

Remarks:
1. The above discussion allows us to characterize a large set of nonperturbatively unitary $c = 1$ $S$-matrices. Note that at no point were any special properties of the function $\Theta(x)$ employed: the unitarity equations may be checked without resort to any identities on the Bernoulli numbers. The specific form of $\Theta$ only enters in the comparison with string perturbation theory. Thus we may construct a nonperturbatively unitary string $S$-matrix (in two dimensions!) using any real-valued function $\Theta(x)$ which has an asymptotic expansion

$$
\Theta(x) \sim \text{arg}\Gamma\left(\frac{1}{2} - ix\right)
\sim -x\log x + x + \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{2n(2n - 1)}(1 - 2^{-2n-1})x^{-2n-1}
$$

(4.5)

for $x \to +\infty$. For example, putting the infinite wall at a position $\lambda = A$ leads to the reflection coefficient given in eq. (A.7) below. For any $A$ this coefficient defines a distinct $S$-matrix, corresponding to a $\Theta(x, A)$ satisfying (4.5). Indeed, by “inverse scattering” we expect one can assign a matrix model potential $V(\lambda; \Theta)$ to any function $\Theta$ satisfying (4.5). It would be interesting to describe the set of potentials which are characterized by the requirement (4.5), since, according to the philosophy of section 7 below, this ambiguity corresponds to the existence of differences of background geometries detectable only nonperturbatively.

2. The fermionic formulation partially clarifies the question of the $W_\infty$ symmetry which the system is known to respect \cite{[20],[30],[13]]. This is not manifest as a symmetry of the $S$-matrix we have computed. From the point of view of the free fermion system, the Cartan subalgebra of $W_\infty$ generated by $O^{2s,0}$ is a manifest symmetry of (4.4), corresponding to the infinite set of conserved charges $Q_{2s} = \sum p^{2s}$ constructed from the fermion momenta (note that due to the reflection off the wall, only even powers are conserved). The realization of the rest of the symmetry algebra is not obvious. We now see why the symmetry is so deeply hidden in the bosonic formalism – the conserved charges do not commute with tachyon number.

3. The above formalism also makes clear how it is that one can have particle production in a “free” theory. The essential point is that the fermions scatter with an energy-dependent phase $e^{i\Theta(E; V)}$. Thus a particle-hole pair corresponding to a single boson scatters into a “dispersed” particle-hole pair with overlap on states of arbitrary boson number. This may be regarded as a nonperturbative generalization of the dispersion of tachyon wavepackets described in \cite{[13]}. 

15
4. The above discussion suggests an obvious yet interesting generalization of the present S-matrix. We may imagine adding “flavor” quantum numbers to our bosons and fermions and may further replace the crossings of fermion lines “in the bulk” by any factorizable S-matrix compatible with the bounce factors. The compatibility conditions have been investigated in [31,32]. The above remarks also provide a general mechanism by which one could use factorizable S-matrices to write nontrivial S-matrices with particle production.

4.2. Theory II is not Unitary

The nonunitarity of theory II is a simple consequence of the fact that an incoming particle-hole pair can dissociate as in fig. 8 into a leftmoving and rightmoving fermion. One fermion can be reflected back into its original half space while the other can tunnel through into the adjoining half space. The Hilbert spaces for left- and right- movers are then the one-soliton sectors, which have no analogue in the Das-Jevicki theory.

More formally, we may consider the unitarity equations for $1 \rightarrow 2$ scattering on the same side of the potential. The contributions of transmitted amplitudes to the equation must be of order $e^{-4\pi\mu}$. However, the reflected amplitudes fail to satisfy the unitarity equations at order $e^{-2\pi\mu}$, as we now show. Expanding in small energies $\omega_i$ we have the S-matrix elements

$$S(\omega, +|\omega, +) = \frac{1}{1 + e^{-2\pi\mu}\omega} + \cdots$$

$$S(\omega, +|\omega_1, +; \omega_2, +) = -\frac{\pi}{2} \frac{1}{\cosh^2 \frac{\pi}{\mu}} \omega_1 \omega_2$$
$$+ i\omega_1 \omega_2 (\omega_1 + \omega_2) \left( \frac{1}{1 + e^{-2\pi\mu}} \psi_1 + \frac{\pi}{2} \frac{1}{\cosh^2 \frac{\pi}{\mu}} \psi_0 \right) + \cdots \quad (4.6)$$

$$S(\omega_1, +; \omega_2, +|\omega, +) = -\frac{\pi}{2} \frac{1}{\cosh^2 \frac{\pi}{\mu}} \omega_1 \omega_2$$
$$+ i\omega_1 \omega_2 (\omega_1 + \omega_2) \left( \frac{1}{1 + e^{-2\pi\mu}} \psi_1 + \frac{\pi}{2} \frac{1}{\cosh^2 \frac{\pi}{\mu}} \psi_0 \right) + \cdots$$

where $\psi_0 = \text{Re}(\psi(\frac{1}{2} - i\mu))$ and $\psi_1 = \frac{d\psi}{d\mu}$. The first term is order $O(e^{-2\pi\mu})$ while the second is order 1. Checking the unitarity equation $\langle \omega, +SS^\dagger |\omega_1, +; \omega_2, + \rangle = 0$ in a small

---

4 The Das-Jevicki theory does have soliton sectors corresponding to one-eigenvalue instantons [33]. It is possible that taking these into account will restore unitarity within the context of the collective field theory.
energy expansion we see that the $O(\omega^3)$ terms, which are perturbative, cancel but the nonperturbative $O(\omega^2)$ terms cannot cancel. Moreover the contributions of the transmitted amplitudes to the unitarity equations are $O(e^{-4\pi \mu})$. Hence unitarity is violated at order $e^{-2\pi \mu}$ as expected from the simple physical picture.

The violation of unitarity in the massless tachyon theory is extremely suggestive in view of the conjectured connection of the $c = 1$ matrix model to black hole physics [34-37]. Unfortunately, that relation must be further clarified before we can confidently apply the above results to the black hole problem.

5. Analytic Structure of the $S$-matrix

5.1. General Remarks

In standard field theory the analyticity properties of an $S$-matrix are related to important physical properties of the theory. For example, causality implies analyticity in suitable domains, and the existence of unstable particles and thresholds imply the existence of poles and cuts in analytically-continued $S$-matrix elements. Therefore, in this section we examine the analytic properties of the $c = 1$ $S$-matrix hoping to understand better the nature of the physics of 2D strings.

Many aspects of our problem are different from the more familiar examples of relativistic field theories in Minkowskian spacetimes. A first difference is that in standard $S$-matrix theory [38] $S$-matrix elements are considered as analytic functions of $s_{ijk\ldots} = (p_i \pm p_j \pm p_k + \cdots)^2$. Here we will analytically continue in the energies $\omega_i$, which are, roughly speaking, $\sqrt{s_{ijk\ldots}}$. A second (related) difference is that in our problem we only have time translation and time reversal invariance so many restrictions of Lorentz invariance are lost. A third difference is that the particles in our case are massless, whereas in more realistic examples massless particles like photons and gravitons, which interact with one another in asymptotic regions of spacetime, are usually excluded from discussion.

The third point above leads to a rather different analytic structure of $S$-matrix amplitudes than that normally encountered in massive relativistic field theory. Consider, for example, the $2 \rightarrow 2$ scattering amplitude in a scalar field theory with particles of mass $m$. In the complex $s = (p_1 + p_2)^2$ plane the amplitude has the structure indicated in fig. 10.

---

5 We would like to thank R. Shankar and A. Zamolodchikov for very useful discussions relevant to this section.
There are elastic threshold branch points at $0, 4m^2$. On the physical sheet there are no other singularities, but if we analytically continue to the second sheet we may discover resonance poles at some complex mass-squared $s = \mu^2$ as indicated in fig. 10. Moreover, the presence of these resonance poles implies the existence of further cuts, for example the one beginning at $s = (m + \mu)^2$ indicated in fig. 10. Consider the limit of fig. 10 as $m \to 0$. In this limit the physical sheet separates into two disjoint half-spaces. There should be no singularities on the physical sheet and the physical $S$ matrix is defined by two different analytic functions $S^\pm$ defined on these two regions. If we analytically continue, say, $S^+$ from the upper half plane into the lower half plane we will encounter resonance branch cuts located at the position formerly occupied by the resonance poles.

In the $c = 1$ $S$-matrix, the cut structure analogous to the separation of $S^+$ from $S^-$ above has already been investigated at tree level in [39]. We will see below that nonperturbatively some new singularities arise corresponding to the presence of resonances.

5.2. Analytic Properties of Bounce Factors

The analytic properties of the $S$-matrix follow from those of the bounce factors. For theories of type I with $A = 0$ the bounce factor is

$$e^{i\Theta(x)} = \sqrt{\frac{1 + ie^{-\pi x}}{1 - ie^{-\pi x}}} \sqrt{\frac{\Gamma(\frac{1}{2} - ix)}{\Gamma(\frac{1}{2} + ix)}}$$

$$= \sqrt{\frac{2}{\pi}} e^{i\pi/4} \cos\left(\frac{\pi}{2}(1 + ix)\right) \frac{1}{\Gamma\left(\frac{1}{2} - ix\right)} .$$

(5.1)

From the first equality it is obviously a phase for $x$ real. From the second we see that it can be continued into the complex $x$-plane, where it has simple poles at $x = z_\ell \equiv -i(2\ell + \frac{3}{2})$ for $\ell = 0, 1, 2 \ldots$, with residues $\rho_\ell = \sqrt{\frac{2}{\pi}} e^{i\pi/4}(-1)^\ell \frac{1}{(2\ell+1)!}$. When $A \neq 0$ we have instead

$$e^{i\Theta(x,A)} = \frac{1 + (e^{ix} - i)X(x,A)}{1 + (e^{ix} + i)X(x,A)} e^{i\Theta(x)}$$

(5.2)

6 In [39] D. Kutasov and Ph. Di Francesco observed an interesting analytic property of the tree-level $S$-matrix. Namely, by restricting attention to “one-particle irreducible” elements one could work with analytic expressions (polynomials in the $\omega_i$) valid in all kinematic regimes. Unfortunately this property does not persist at higher loops (as one might guess from physical grounds). Using the explicit formulae above it is easy to check that the “one-particle irreducible” amplitudes defined in [39] are not analytic in the momenta at one loop order.
where
\[
X(x, A) \equiv \frac{e^{-\pi x^2 / 2}}{\sqrt{2\pi}} \Gamma \left( \frac{3}{4} + i \frac{x}{2} \right) \Gamma \left( \frac{3}{4} - i \frac{x}{2} \right) A^{\frac{1}{2}} F_1 \left( \frac{\frac{3}{4} - i \frac{x}{2}}{2}, \frac{\frac{3}{4} + i \frac{x}{2}}{2}; i A^2 \right).
\] (5.3)

The bounce factor in (5.2) defines a meromorphic function of \( x \) which has a sequence of poles at \( x = z_\ell(A) \), smoothly evolving from the poles at \( A = 0 \). We denote the residue at the pole by \( \rho_\ell(A) \).

The poles \( z_\ell(A) \) in the bounce factors have a simple interpretation in terms of the free fermions. The denominator of (5.2) vanishes for \( z \) such that the wavefunction (see appendix A for a definition) \( \chi^+_{R}(z, A) = 0 \). For \( \text{Im} z < 0 \) the wavefunction \( \chi^+_{R}(z, \lambda) \) decays as \( \lambda \rightarrow +\infty \) so the condition \( \chi^+_{R}(z, \lambda = A) = 0 \) is exactly the condition to have a bound state energy \( z \) in the upside down oscillator potential with an infinite wall at \( A \). The “energy” \( z \) is complex because this bound state is unstable. As \( A \rightarrow -\infty \) there is a deeper and deeper well on the far side of the parabola and the resonant state with energy \( z_\ell(A) \) becomes more and more long-lived, i.e., the imaginary parts of \( z_\ell(A) \) go to zero. Indeed, the poles \( z_\ell(A) \) approach the positive real axis and become dense there, eventually becoming a massless free fermion cut. (It is possible to obtain some analytic results on \( z_\ell(A) \) which confirm the above picture deduced on physical grounds.)

5.3. Theory II and the Limit of Theories of Type I

One may take the limit \( A \rightarrow -\infty \) to obtain the bounce factor of theory II provided one works at \( \text{Im} z > 0 \):
\[
\lim_{A \rightarrow -\infty} R^I_q(z, A) = R^{II}_q(z) .
\] (5.4)

At \( \text{Im} z < 0 \) this is not true and we have instead
\[
\lim_{A \rightarrow -\infty} R^I_q(z, A) = (1 + e^{-2\pi z}) R^{II}_q(z) .
\] (5.5)

The reason for the discrepancy is that as \( A \rightarrow -\infty \) a new cut for the fermion, the coalescence of the poles \( z_\ell(A) \), appears along the real axis. Indeed, for \( \text{Im} z = 0 \) the \( A \rightarrow -\infty \) limit of \( R^I \) is ill-defined.

The analytic structure of \( R^{II} \) may be studied using the formula (2.9). One finds a series of poles in the lower half plane similar to that found for \( R^I \), leading to a similar analytic structure for the scattering amplitudes in the type-II theory.
5.4. General Properties of “Massless Tachyon” Amplitudes

We may now continue the amplitudes in (3.6) by choosing all $\omega_i$ but one to be independent and continuing in those. Upon analytic continuation the integrals in (3.6) become contour integrals, and singularities arise when the endpoints of the integrals hit poles of the integrand or when poles of the integrand pinch the contour. Since we have written the integral as a sum of terms there will in general be several choices for how to continue the contours in the separate integrals. One can define a notion of a “physical sheet” by starting in a given kinematic region and analytically continuing into an appropriate half-space. Note that if a bounce line carries positive energy $\omega$ then the corresponding factor $e^{\pm i\Theta(\mu \pm \omega)}$ has poles at $\omega = \mp \mu - i(2\ell + \frac{3}{2})$. Therefore, from rule RH1, if we define the physical sheet to be included in the half-space $\text{Im}(\omega) > 0$, there can be no singularities in the integral. On the other hand, it can happen that when we continue out of this half-space a pole in the integrand is forced to hit the endpoint of the contour integral. This results in a branch cut singularity. Moreover special configurations of $\omega_i$ can produce more complicated singularities, and continuation to further sheets can produce new singularities. Such singularities are indicative of new states and degrees of freedom in the system not directly seen in the physical $S$-matrix. These points are best illustrated by explicit examples.

5.5. Analytic Structure of $S(\omega|\omega)$

In the $1 \rightarrow 1$ amplitude

$$S(\omega|\omega; A) = \int_0^\omega dx e^{i\Theta(\omega-x,A)+i\Theta(\mu+x,A)}$$

(5.6)

the integrand has a set of fixed poles at $x = -\mu + z_\ell(A)$ and set of moving (with $\omega$) poles at $x = \omega - \mu - z_\ell(A)$. The contour integral is therefore forced to hit these poles when $\omega = \pm \mu + z_\ell(A)$ and at these points the integral has a logarithmic singularity. We define the second sheet by choosing the branch cuts as in fig. [11]. From the discussion of section 5.2 we see that the branch cut singularities are naturally described by a simple physical picture: one fermion gets trapped in a metastable state. The other fermion bounces off the potential and can radiate and reabsorb massless tachyons, thus producing a cut singularity.

It is interesting to proceed and continue to the third sheet by passing through one of the cuts of fig. [11]. For simplicity we consider continuing through a cut at $\omega = -\mu + z_\ell(A)$,
although a similar story holds for the cuts at \( \omega = \mu + z_\ell(A) \). Comparing the contour integrals defined in the \( x \) plane in fig. 12 we see these differ by

\[
S^{(3)}(\omega|\omega; A) = S^{(2)}(\omega|\omega; A) + \rho_\ell(A)e^{i\Theta(\omega-z_\ell(A),A)}
\]

(5.7)

where the superscript refers to sheet number. The extra factor produces a new set of poles on the third sheet at \( \omega = z_\ell(A) + z_{\ell'}(A) \). These poles may be interpreted as a metastable state obtained when a particle-hole pair resonates. Since the fermions are free the “energies” \( z_\ell \) simply add. Since both are trapped, neither fermion can shake off massless tachyons so the singularity produces a pole and not a cut. Note that when \( A = 0 \) these poles occur at \( \omega = -in \) for \( n = 3, 5, \ldots \).

5.6. Analytic Structure of the Four-point Function

Proceeding along similar lines one can investigate in detail, e.g., the four-point function \( S(\omega_1, \omega_2|\omega_3, \omega_4; A) \) for \( \omega_2 < \omega_3, \omega_4 < \omega_1 \). We take \( \omega_1 \) to be dependent and continue from a real subspace of \( \mathbb{R}^3 \). One finds a rather complicated singularity structure. The most interesting singularities appear to be double-poles arising from the four-bounce terms. These correspond to processes where an incoming tachyon (say, \( |\omega_2\rangle \)) becomes trapped in a resonant state \( \omega_2 = z_\ell(A) + z_{\ell'}(A) \) which later decays to \( \langle \omega_4 \)\. More precisely, letting \( \omega_2 = z_\ell + z_{\ell'} + \epsilon_1 \) and \( \omega_4 = z_\ell + z_{\ell'} + \epsilon_2 \) with the \( \epsilon \)'s small we get double poles

\[
\rho_\ell\rho_{\ell'}^2 \left[ \frac{1}{\epsilon_1\epsilon_2}e^{i\Theta(\omega_3-z_\ell)} + \frac{1}{\epsilon_1(\epsilon_1-\epsilon_2)}e^{i\Theta(\omega_3-z_{\ell'})} \right]
\]

(5.8)

The residue of this double pole, which is essentially a bounce factor may be interpreted as an amplitude for scattering off an “excited background” corresponding to adding a single resonant tachyon to the fermi sea.

5.7. Lessons from the Analytic Structure of \( S \)

The main lesson we may draw from these remarks is that it is not possible to neglect the fermionic degrees of freedom beyond perturbation theory. Even in theories of type I, where we have a nonperturbatively unitary \( S \)-matrix without including soliton sectors or matrix model fermions, the existence of the fermions can be detected by studying the analytic properties of the \( S \)-matrix to find resonance poles and cuts. Presumably the nature of the residues and discontinuities at these singularities would indicate the fermionic nature of the particles. It must be emphasized that these analytic singularities are not
mathematical artifacts but have real effects. If \( A \) is large and negative the resonant states we have discussed have long lifetimes. If, for example, these lifetimes exceed those of experimentalists measuring the \( S \)-matrix at \( \lambda = +\infty \) they will find themselves puzzling over an apparent loss of unitarity.

A second lesson is that the rich singularity structure of the higher point amplitudes indicates the existence of a correspondingly rich spectroscopy of excited, but unstable, backgrounds. These time-dependent backgrounds, resulting from changes in the Fermi sea, are the nonperturbative analogues of the tachyonic backgrounds studied by Polchinski in [13].

6. Worldsheet Interpretation

6.1. Connection to Liouville Theory

So far our discussion has emphasized the free-fermion formulation of the matrix model. Since the double-scaled matrix model is a sum over continuum surfaces we expect that the continuum amplitudes can also be described by the conformal field theory of a massless scalar \( X \) coupled to a \( c = 25 \) Liouville theory \( \phi \) with worldsheet action

\[
A = \int \frac{1}{2} \partial X \bar{\partial} X + \partial \phi \bar{\partial} \phi + \sqrt{2} R^{(2)} \phi + \mu e^{\sqrt{2} \phi}
\]  

(6.1)

As explained in [15] the Liouville and \( \tau \) coordinates are not the same. Recall that the objects calculated in a sum over continuous geometries on 2-surfaces with boundary are “macroscopic loop amplitudes” defined by fixing the boundary values of the two-metric \( e^{\sqrt{2} \phi} \) so that the bounding circles \( C \) have lengths \( \ell = \oint_C e^{\phi/\sqrt{2}} \) [10–14].

In [12,14] it was proposed that one extract the \( c = 1 \) correlators of tachyons by extracting the terms proportional to nonanalytic powers \( \ell^{|p|} \) in the small \( \ell \) expansion of the macroscopic loop amplitudes. From the continuum 2D path integral point of view \( W(\ell,p) \) corresponds to an expansion of a sum of local operators. As discussed in [13] that sum of operators can be written as

\[
W_{in}(\ell, p) = -\mathcal{T}_p \frac{\pi}{\sin \pi |p|} \mu^{-|p|/2} I_{|p|}(2\sqrt{\mu} \ell) - \sum_{r=1}^{\infty} \hat{B}_{r,p} \frac{2(-1)^r r}{r^2 - p^2} \mu^{-r/2} I_r(2\sqrt{\mu} \ell)
\]  

(6.2)

where \( \mathcal{T}_p \) is proportional to the tachyon vertex operator and \( \hat{B}_{r,p} \) are redundant operators for \( p \notin \mathbb{Z} \).
We now relate the prescription of this paper to that of [14,44]. From the matrix-model representation of macroscopic loop operators \( W(\ell, q) \) we see that the amplitudes are related by an integral transform:

\[
\langle \prod W(\ell_i, q_i) \rangle_c \equiv \int_A \prod d\lambda_i e^{-\ell_i \lambda_i} \langle \prod \rho(\lambda_i, q_i) \rangle_c \quad (6.3)
\]

in theories of type I. The \( \ell \to 0 \) asymptotic behavior of the integral (6.3) is dominated by the \( \lambda \to \infty \) behavior. Changing variables to \( \tau \) and using the asymptotic expression (2.14) we see that

\[
\langle \prod w(\ell_i, q_i) \rangle \sim \int_\infty d\tau e^{-\ell_i 2 \sqrt{\mu} \cosh \tau_i} \prod e^{-|q_i| \tau_i} \left[ \mathcal{R}_n + \mathcal{O}(e^{-\tau_i}) \right] \quad (6.4)
\]

For small \( z \) and \( q \notin \mathbb{Z} \) we have the asymptotics:

\[
\int_\infty d\tau e^{-\ell 2 \sqrt{\mu} \cosh \tau} e^{-|q| \tau} \sim (\ell \sqrt{\mu})^{|q|} \Gamma(-|q|) \quad (6.5)
\]

plus terms regular in \( \ell \). Hence \( \langle \prod T_{q_i} \rangle = \mathcal{R}(q_1, \ldots, q_n; \mu) \). To make a connection to standard vertex operator normalizations we compare with the vertex operator calculations found, e.g., in [39] which show that

\[
T_q = \frac{\Gamma(|q|)}{\Gamma(-|q|)} \int_\Sigma e^{i q X / \sqrt{2} e^{1/2} (1-1/2|q|) \phi} \equiv \frac{\Gamma(|q|)}{\Gamma(-|q|)} V_q \quad (6.6)
\]

Thus we finally have a prediction for all integrated vertex operator correlators in the \( c = 1 \) theory:

\[
\prod_i \frac{\Gamma(-|q_i|)}{\Gamma(|q_i|)} \mathcal{R}_n(q_1, \ldots, q_n; \mu) \overset{\mu \to \infty}{\sim} \sum_{g \geq 0} \left( \frac{1}{\mu} \right)^{2g-2+n} \int_{\mathcal{M}_{g,n}} \langle \prod_{i=1}^n \bar{b} \bar{b} \prod_i c \bar{c} V_{q_i} \rangle \bigg|_{\mu = 1} \quad (6.7)
\]

where \( \mathcal{M}_{g,n} \) is the moduli space of curves of genus \( g \) with \( n \) punctures.

(Let us note parenthetically that the “wavefunction normalization” factors \( \frac{\Gamma(-|q_i|)}{\Gamma(|q_i|)} \) in (6.7) have been the subject of much discussion [45-48,39]. These factors are singular when

7 In theories of type II we must be more careful since the eigenvalue distribution grows on both sides of the axis. The procedure advocated in [14] is to take a Fourier transform with \( \ell = iz, \) \( z \) real, and analytically continue positive and negative frequency components separately. This procedure is correct to all orders of perturbation theory, but its status as a nonperturbative definition is unclear. Hence we limit our discussion to theories of type I.
$q \in \mathbb{Z}$ and it has been proposed that the external line divergences should be reinterpreted as the contributions of “intermediate” on-shell states (in some space-time field theory). Such an interpretation requires inclusion of the “special state” vertex operators obtained from dressing nontrivial Virasoro primaries in Fock spaces with charge $p \in \mathbb{Z}$ with highest Virasoro weight $\frac{p^2}{4}$. An alternative interpretation of these factors was proposed in [15] based on the smoothness of macroscopic loops for $q \in \mathbb{Z}$. The apparent singularity arises since the extraction of nonanalytic powers of $\ell$ is not well-defined for $q \in \mathbb{Z}$. Indeed the integral (6.5) can be written more precisely as

$$
\int_{A}^\infty d\tau e^{-2\mu \cosh \tau} e^{-|q|\tau} = -\frac{\pi}{\sin \pi |q|} I_{|q|}(2\sqrt{\mu \ell}) \sum_{n \geq 0} (-1)^n (\ell \sqrt{\mu})^n \sum_{m=0}^n \frac{1}{m!(n-m)!} \frac{e^{A(m-n-|q|)}}{m-n-|q|} \quad (6.8)
$$

Clearly the left hand side is smooth as $q$ becomes integral, but the decomposition of the right hand side breaks down. This was interpreted in [15] as the fact that $B_{s,q}$ while in general a redundant boundary operator becomes a bulk operator at special values of $q$. It thus appears to be consistent to use the vertex operator normalization $T_q$, bearing in mind that for $q \in \mathbb{Z}$ the operator is a linear combination of special state and tachyon vertex operators.

6.2. Continuations to Minkowski Space

The $c = 1 \times$ Liouville system is a conformal field theory with two Euclidean signature bosons. The Minkowskian spacetime physics deduced from this Euclidean theory depends strongly on how we analytically continue.

The analytic continuation described in section 2 above identifies $X$ as Euclidean time. Thus we continue $X \rightarrow it$ and $|k| \rightarrow -i\omega$. If we formally continue the expression in (6.7) we obtain correlators of macroscopic state operators [49]

$$
\int e^{\pm i\omega t(z,\bar{z})} e^{(\sqrt{\tau}+i\omega)} \phi(z,\bar{z}) \quad (6.9)
$$

These operators have positive Liouville energy and, at least semiclassically, create surfaces with large holes and singular metrics. Indeed the difficulties associated with the so-called “$c = 1$ barrier” were ascribed by N. Seiberg to the destruction of a smooth world sheet arising from inclusion of such operators in the action [15] [8]. It would therefore be very

---

8 See [50] for a different opinion.
interesting to see if the formal analytic continuation of the RHS of (6.7) can be given a more rigorous justification.

The analytic continuation of $X$ to Minkowskian time also leads to a simple spacetime interpretation of the Seiberg bound [49,50] which states that Liouville exponentials $e^{\alpha \phi}$ must satisfy $\alpha < \frac{1}{2}Q$. The fact that only one root of the KPZ equation is allowed is simply the fact that for scattering on the right half-line, incoming particles must be leftmovers and outgoing particles must be rightmovers. Note in this connection that if we work in theory II, with another asymptotic universe then, nonperturbatively, there can exist "wrong branch" states, i.e., incoming rightmovers from the point of view of the righthand universe.\footnote{A related remark was made by N. Seiberg. He noted that the wavefunctions of [44] associated to the "wrong branch" could be obtained from continuations appropriate to the far side of the parabola.}

Finally the issue of correct normalization of tachyon vertex operators discussed above becomes academic in this continuation [8]. The $S$-matrix obtained by continuing $\mathcal{R}_n$ is simply related by the phase redefinition of statevectors

$$|\omega_1 \ldots \omega_n\rangle \rightarrow \prod \frac{\Gamma(i\omega_i)}{\Gamma(-i\omega_i)} |\omega_1 \ldots \omega_n\rangle$$

$$\langle \omega_1 \ldots \omega_n | \rightarrow \prod \frac{\Gamma(i\omega_i)}{\Gamma(-i\omega_i)} \langle \omega_1 \ldots \omega_n |$$

(6.10)

to the $S$-matrix obtained from vertex operator calculations. Hence, no probability amplitudes are changed and the theories are physically indistinguishable.

Another choice of continuation which is sometimes adopted is to regard the Liouville coordinate as Euclidean time. This prescription "$\phi \rightarrow i\phi$" is rather less well-defined. Indeed serious objections to it have been raised in [50]. Roughly speaking time becomes effectively semi-infinite, the Liouville wall corresponding to something like a big bang or a big crunch. In the former case the amplitudes $\mathcal{R}_n(q_1, \ldots, q_n)$ correspond to "amplitudes" for producing right-movers and leftmovers (corresponding to positive and negative $q_i$) from the finite past. Since time is semi-infinite there is no obvious analog to the unitarity constraint.
6.3. Mathematical Applications

The equality of the asymptotic expansions in (6.7) identifies the coefficients of the large \( \mu \) asymptotics of \( R_n(p_1, \ldots, p_n; \mu) \) with integrals over \( \mathcal{M}_{g,n} \) of some natural densities provided by the Liouville theory. From our construction of \( \mathcal{R} \) the answers are simply expressed in terms of Bernoulli numbers, thus giving formulae reminiscent of [51, 52]. These observations add more evidence to the oft-quoted conjecture that there is a topological field theory version of the \( c = 1 \) model. To make the connection precise one should (1) generalize our formulae to the self-dual radius and (2) understand the appropriate nontrivial \( k \to \infty \) limit of the twisted \( N = 2 \) minimal model. Step (1) might be accomplished using the results of [53, 54], but step (2) appears to be more difficult.

7. Background Dependence of the S-matrix

The most interesting questions in 2D string theory center on the nature of strings in different backgrounds. It should be possible to perturb the conformal field theory (6.1) to obtain more general nonlinear sigma models with two target space coordinates \( (X, \phi) \). In general, nearby theories are expected to have an action of the schematic form

\[
A + \delta A = \int \Sigma \left[ \frac{1}{2} \partial X \bar{\partial} X + \partial \phi \bar{\partial} \phi + \sqrt{2} R^{(2)} \phi + \mu e^{\sqrt{2} \phi} \right] + \int dp \epsilon(p) V_p + \sum t_{r,s} V_{r,s} \quad (7.1)
\]

where \( V_{r,s} \) denotes some basis of “special state operators.”

As long as the background perturbations preserve conformal invariance and the asymptotically flat nature of the target space geometry, the string \( S \)-matrix should make sense, and we should therefore discuss the functions \( S(\{\omega_i\} \to \{\omega'_i\}; \text{backgrounds}) \) on the infinite-dimensional space of background geometries. These can be parametrized, in an infinitesimal neighborhood of (6.1) by a subspace spanned by the directions \( \epsilon(p), t_{r,s} \). Ultimately, we would like to know what principle determines the allowed backgrounds and what equations govern the background dependence of \( S \).

The background dependence on \( \epsilon(p) \) is an interesting question which we will address elsewhere. For the present we describe how \( S \) changes in an infinitesimal neighborhood around (6.1) by perturbing the parameters \( t_{r,s} \). The matrix-model identification

\[
V_{r,s} \leftrightarrow \hat{B}_{s,q=r} = \int \lambda^s e^{i r x} \psi^\dagger \psi(\lambda, x) d\lambda dx + \cdots \quad (7.2)
\]
for $r \in \{s, s - 2, \ldots, -s\}$ suggests, by exponentiation, that perturbations of the $\sigma$-model action by the parameters $t_{r,s}$ correspond to perturbations of the matrix model potential $V(\lambda) = -\frac{1}{2} \lambda^2 + \sum t_{r,s} \int \lambda^s e^{irx}$. Moreover, this reasoning suggests that changes in $V$ which affect the nonperturbative S-matrix but not its perturbation expansion could be interpreted as “small” perturbations in the background geometry which only affect non-perturbative physics. The dependence on $A$ discussed in section five above is an example of such a perturbation.

The change in the S matrix due to a change in background corresponding to $\delta V(\lambda)$ is easily computed (in principle) since the fermions remain free, and one need only compute the change in the fermion two-point function. In particular, for time-independent perturbations we need only compute the change in the resolvent [11]:

$$I \rightarrow \langle \lambda_1 | \frac{1}{H + \delta V - z} | \lambda_2 \rangle$$

(7.3)

To be more explicit we must realize that the operators $\int \lambda^s$ are ill-defined due to ultraviolet worldsheet divergences [14,15] so we consider matrix-model potentials $\delta V(\lambda)$ with rapid falloff for $\lambda \to \infty$. These may be viewed as linear combinations of special state operators. The new S-matrix is obtained from the old by a simple change of the bounce factor which is given to lowest order by (we put $A = 0$ here):

$$\delta \Theta(z) = -\pi \int_0^\infty (\psi^{-}(z, \lambda))^2 \delta V(\lambda) d\lambda + \cdots$$

(7.4)

The background dependence of the scattering matrix leads to background dependence of the location of the singularities described in section five. The variation with background can be computed in the $A = 0$ theory using a trick. The poles $z_\ell$ are bound state poles, and indeed at $z_\ell = -i(2\ell + \frac{3}{2})$ the bound state wavefunctions are just given by the continuation of harmonic oscillator wavefunctions:

$$\psi_\ell(\lambda) = H_{2\ell+1}(2^{1/2} e^{i\pi/4} \lambda) e^{-i\lambda^2/4}$$

(7.5)

which satisfy the identity:

$$\int_0^\infty \psi_\ell(\lambda) \psi_{\ell'}(\lambda) d\lambda = 2^{2\ell+1/2} \sqrt{\pi} (2\ell + 1)! e^{-i\pi/4} \delta_\ell,\ell'$$

(7.6)

The meaning of the ellipsis in (7.2) and the hat notation is explained in [14,15].
and are thus orthogonal without complex conjugation. It follows that if \( \delta V(\lambda) \) has an expansion in terms of even powers of \( \lambda \) (these correspond to the nonredundant operators \([26,15]\)) then the variation of the bound-state pole with \( \delta V \) is given by the expectation value in the normal harmonic oscillator:

\[
\delta z_{\ell} = \langle 2\ell + 1 | \delta V(\sqrt{2}e^{-i\pi/4}x) | 2\ell + 1 \rangle
\]  

(7.7)

where \(|n\rangle\) is an orthonormal basis for the standard harmonic oscillator. Thus the locations of the singularities \(z_{\ell}\) change as we “turn on” the special state operators displaying the background dependence of these singularities.

8. 2D String Physics at High Energies

The importance of studying string theory in the limit of ultrahigh energies has been emphasized by several authors \([55-59]\). In this section we will throw some light – literally – on the nature of the tachyon condensate wall by studying the reaction of the wall long after a single boson of energy \(\omega\) has been thrown at it. It was pointed out in \([14]\) that at large energies \(\omega \gg \mu\), with fixed string coupling, the asymptotic expansions of string perturbation theory cease to make sense and there must be new physics in this regime. We emphasize that this large order behavior of perturbation theory is in principle derivable from the continuum Liouville approach and therefore a true string-theoretic effect, independent of the matrix model regularization. On the other hand, the particular kind of new physics that arises is probably sensitive to the nonperturbative definition of the model. In what follows we will attempt to study this new physics concentrating on the nonperturbative definition we have labeled theory I with \(A = 0\).

8.1. Particle Production

The unitarity equations for \(1 \rightarrow 1\) scattering read:

\[
1 = \frac{1}{\omega^2} |S(\omega|\omega)|^2 + \frac{1}{\omega} \sum_{n \geq 2} \int_0^\infty \prod_{i=1}^n \frac{d\omega_i}{\omega_i} \delta(\sum_{i=1}^n \omega_i - \omega)|S(\omega|\omega_i)|^2
\]  

(8.1)

Thus, the probability that a single incoming boson of energy \(\omega\) produce two or more outgoing particles is simply expressed as

\[
P(\omega) = 1 - \left| \int_0^1 dt e^{i\Theta(t\omega + \mu) + i\Theta((1-t)\omega - \mu)} \right|^2
\]  

(8.2)

At large \(\omega\) this behaves like \(P(\omega) \sim 1 - \frac{\pi}{2\omega}\). Nevertheless there is an intricate resonant structure in \(P(\omega)\) for finite \(\omega\). We have plotted this function in fig. 13.
8.2. Energy Distributions

We can also look explicitly at the probability that two particles of energies $\omega_1 + \omega_2 = \omega$ are created. The result can be described in terms of a normalized energy distribution for $\omega_2$

$$D(\omega_2, \omega; \mu) = \frac{1}{2} \frac{1}{(\omega - \omega_2)\omega_2} |A|^2$$  \hspace{1cm} (8.3)

where

$$A = T(\omega_2; \omega, -\mu) - T(\omega_2; \omega, \mu)$$

$$= \int_{-\mu}^{\omega_2-\mu} dx e^{i\Theta(\omega-x)+i\Theta(x)} - \int_{\mu}^{\omega_2+\mu} dx e^{i\Theta(\omega-x)+i\Theta(x)}$$  \hspace{1cm} (8.4)

Again this distribution has very interesting large-energy behavior. One finds that the phase integral defining $T(x; \omega, \mu)$ has no stationary phase point for $x < \frac{1}{2} \omega - \mu$, the integral being of order $1/\log \omega$. At $x \equiv \frac{1}{2} \omega - \mu$ the integral turns on rapidly and is of order $(\pi \omega/2)^{1/2}$. Thus we may approximate $T$ by a step function. For $\frac{1}{2} \omega - \mu < \omega_2 < \frac{1}{2} \omega + \mu$, $A$ is flat and of order $(\pi \omega/2)^{1/2}$.

Outside of this range one may approximate (here we take $\mu \ll \omega_2 \ll \omega$):

$$A \sim \sin(\Theta(\mu) + \mu \log \omega) \left[ 1 - e^{i\omega_2 \log \omega} e^{i\Theta(\omega_2)} \right]$$  \hspace{1cm} (8.5)

up to some irrelevant phases. The rapidly oscillating second term leads to resonant behavior. A typical energy distribution is shown in fig. 14.

The above analysis leads to a simple approximate expression for the matrix element for particle creation $\omega \rightarrow \omega_1 + \cdots + \omega_n$ in the case where no $\omega_i$ is small:

$$S \approx e^{2i\Theta(\frac{1}{2}\omega)} \sqrt{\frac{i\pi \omega}{2}} \sum_{S \subseteq S^+} \theta(\omega(S) - \frac{1}{2} \omega + \mu)(-1)^{|S|}$$  \hspace{1cm} (8.6)

Note that this is much softer than the behavior obtained at any order of perturbation theory.

8.3. Particle Number Distributions

At low energies and weak string coupling it is easy to estimate the particle number distribution defined by the $n^{th}$ term in the series (8.1):

$$P(n, \omega) \sim \mu^2 \left( \frac{\omega}{\mu} \right)^{2n} \frac{((n-2)!)^2}{n!(2n-1)!} \sim \mu^2 \left( \frac{\omega}{2\mu} \right)^{2n} \frac{1}{n!n^{5/2}}$$  \hspace{1cm} (8.7)
This is almost a Poisson distribution: at low energies the fermions bounce with small phase shifts, and the subsequent bosonization proceeds via independent processes. The high-energy distribution function would be extremely interesting to work out. It is rather difficult since the approximation (8.6) breaks down near the boundaries of the simplex \( \sum \omega_i = \omega \) and these boundaries contribute significantly to the answer. We must leave the analysis of this point for the future.

9. Conclusions

Finally, we discuss some points of possible relevance to future work.

The nonperturbative violation of unitarity in theory II deserves to be understood better. From the discussion in sec. 4.2 it is clear that theory II can be made unitary by the inclusion of all the soliton sectors. This suggests the following interesting open question. Let us begin with theory II and consider the most general way of adding degrees of freedom to restore unitarity. Will we be uniquely led to the free fermions of the matrix model? It is natural to wonder if a similar situation will some day occur in higher dimensional string theories, and that nonperturbative unitarity will unveil the correct degrees of freedom with which we should describe the theory.

We have made some preliminary remarks on backgrounds, the main point being that all background dependence on \( V(\lambda) \) can be summarized in the bounce factor \( e^{i\Theta(x;V)} \), which is essentially the scattering matrix of a free fermion in the potential \( V \). It is thus clear that inverse scattering theory is well-suited to this problem, and we hope to investigate this point more thoroughly in the future.\[1\]

Several other points must be clarified before we can achieve a complete understanding of background dependence. In general, the relation between the Liouville coordinate and the matrix model eigenvalue should be better understood. Also, the important underlying \( W_\infty \) symmetry of the \( c = 1 \) theory has played a minor role in this paper. We expect that a complete understanding of the background dependence will require some nontrivial use of this symmetry. That in turn will require a better understanding of how \( W_\infty \) acts on \( S \) than was obtained in section seven.

\[1\] In this connection C. Vafa has made the interesting suggestion that the bounce factor be regarded as the \( KP \) tau function and that it should satisfy some difference equations analogous to the string equation. One could then guess that background dependence would essentially be KP flow on the bounce factor.
We have discovered nonperturbative dependence on parameters like the position of the infinite wall at $\lambda = A$. Recall that in double-scaling the theory the wall is at a distance of order $1/N$ from the quadratic maximum so we are modifying the potential in the scaling region and cannot really expect physics to be independent of $A$. Thus, we should not pessimistically call the $A$-dependence a nonperturbative violation of universality, rather, we should optimistically term it the discovery of nonperturbative parameters analogous to the theta parameter of QCD. The existence of such parameters, and indeed the existence of different “consistent” backgrounds raises once again the persistent spectre of the unpredictability of string theory. There are two possible ways this situation can be remedied. First, there might be further consistency conditions not yet considered which might rule out some possibilities. The absence-of-real-poles constraint on solutions to Painlevé I is a case in point. In our case we have seen that the $S$-matrix has a very intricate analytic structure. Accordingly there might be physically motivated constraints on this singularity structure which exclude some backgrounds from consideration. Second, some dynamical principle might favor some backgrounds and exclude others. Of course, this is an ancient hope, but with the advent of solvable string models and a deeper understanding of backgrounds an answer might be forthcoming.

Acknowledgements

We would like to thank L. Susskind for asking a good question. We would also like to thank T. Banks, I. Cherednik, L. Crane, D. Kutasov, H. Saleur, N. Seiberg, R. Shankar, S. Shenker, C. Vafa, A. Zamolodchikov, and G. Zuckerman for very helpful discussions and correspondence. We are also grateful to N. Seiberg and S. Shenker for comments on a previous draft of the manuscript. This work is supported by DOE grant DE-AC02-76ER03075 and by a Presidential Young Investigator Award.

Appendix A. The Resolvent of the Upside-down Oscillator

We list here some relevant formulae for working out the large $\lambda$ asymptotics of $I(q, \lambda_1, \lambda_2) = (I(-q, \lambda_1, \lambda_2))^* = i\langle \lambda_1 | \frac{1}{\Pi_{-\mu-iq}} | \lambda_2 \rangle$ defined in (2.3).

12 History repeats itself here: see the conclusions to the paper of Douglas and Shenker [2].
This resolvent can itself be expressed in terms of parabolic cylinder functions at a complex energy \( z = \mu + i q \). The asymptotics of the even and odd parabolic cylinder functions, defined in terms of degenerate hypergeometric functions by:

\[
\psi^+(a,x) = \frac{1}{\sqrt{4\pi(1 + e^{2\pi a})^{1/2}}} \left( W(a,x) + W(a,-x) \right) = \frac{1}{\sqrt{4\pi(1 + e^{2\pi a})^{1/2}}} 2^{1/4} \left| \frac{\Gamma(1/4 + ia/2)}{\Gamma(3/4 + ia/2)} \right|^{1/2} e^{-iax^2/4} F_1(1/4 - ia/2; 1/2; ix^2/2) = \frac{e^{-i\pi/8}}{2\pi} e^{-a\pi/4} |\Gamma(1/4 + ia/2)| \frac{1}{\sqrt{|x|}} M_{ia/2, -1/4}(ix^2/2) \tag{A.1}
\]

\[
\psi^-(a,x) = \frac{1}{\sqrt{4\pi(1 + e^{2\pi a})^{1/2}}} \left( W(a,x) - W(a,-x) \right) = \frac{1}{\sqrt{4\pi(1 + e^{2\pi a})^{1/2}}} 2^{3/4} \left| \frac{\Gamma(3/4 + ia/2)}{\Gamma(1/4 + ia/2)} \right|^{1/2} xe^{-iax^2/4} F_1(3/4 - ia/2; 3/2; ix^2/2) = \frac{e^{-3i\pi/8}}{\pi} e^{-a\pi/4} |\Gamma(3/4 + ia/2)| \frac{x}{|x|^{3/2}} M_{ia/2, 1/4}(ix^2/2) \tag{A.2}
\]

are most simply expressed in terms of left and right-mover “plane wave” combinations:

\[
\chi^\pm_R \equiv (\sqrt{k} \mp \frac{i}{\sqrt{k}}) \psi^+ - (\sqrt{k} \pm \frac{i}{\sqrt{k}}) \psi^\mp \xrightarrow{\lambda \to \pm \infty} \frac{\mp 2i}{(2\pi \lambda \sqrt{1 + e^{2\pi z}})^{1/2}} e^{\pm i F(\lambda, z)}
\]

\[
\chi^\pm_L \equiv (\sqrt{k} \mp \frac{i}{\sqrt{k}}) \psi^+ + (\sqrt{k} \pm \frac{i}{\sqrt{k}}) \psi^\mp \xrightarrow{\lambda \to \mp \infty} \frac{\mp 2i}{(2\pi i |\lambda| \sqrt{1 + e^{2\pi z}})^{1/2}} e^{\pm i F(|\lambda|, z)} \tag{A.3}
\]

where

\[
F(\lambda, z) = \frac{1}{4} \lambda^2 - z \log \lambda + \Phi(z)
\]

\[
\Phi(z) \equiv \frac{\pi}{4} + \frac{i}{4} \log \left[ \frac{\Gamma(\frac{1}{2} - iz)}{\Gamma(\frac{1}{2} + iz)} \right] \tag{A.4}
\]

\[
k(z) = \sqrt{1 + e^{2\pi z} - e^{\pi z}}
\]

\[
k(z)^{-1} = \sqrt{1 + e^{2\pi z} + e^{\pi z}}.
\]

By demanding that the resolvent be properly normalized and vanish at infinity we see that if we act on \( L^2(\mathbb{R}) \) then we have

\[
\langle \lambda_1 | \frac{1}{H - z} | \lambda_2 \rangle = -\frac{\pi}{2} \theta(\lambda_1 - \lambda_2) \chi^+_R(z, \lambda_1) \chi^-_L(z, \lambda_2) + \lambda_1 \leftrightarrow \lambda_2 \tag{A.5}
\]
for $\text{Im}(z) > 0$. From this and the above asymptotics of $\chi$ we obtain (2.9). If instead we choose a semi-infinite space $L^2[A, \infty)$ we have for $\text{Im}(z) > 0$

$$
\langle \lambda_1 | \frac{1}{H - z} | \lambda_2 \rangle = -O(z) \theta(\lambda_1 - \lambda_2) \chi_R(z, \lambda_1) \left[ \psi^-(z, \lambda_2) - \frac{\psi^-(z, A) \chi^+_{\theta}(z, \lambda_2)}{\psi^+(z, A)} \psi^+(z, \lambda_2) \right] + \lambda_1 \leftrightarrow \lambda_2
$$

(A.6)

Normalizing $O(z)$ and using the large $\lambda$ asymptotics leads to the reflection coefficient:

$$
R_q = e^{i\mu \log \mu - |q|} i \frac{\sqrt{k - i \sqrt{k}} - \frac{\psi^-(z, A)}{\psi^+(z, A)} \sqrt{k + i \sqrt{k}}} {\sqrt{k + i \sqrt{k}} - \frac{\psi^-(z, A)}{\psi^+(z, A)} \sqrt{k - i \sqrt{k}}} \sqrt{\frac{\Gamma(\frac{1}{2} - i \mu + |q|)}{\Gamma(\frac{1}{2} + i \mu - |q|)}} \Gamma(\frac{1}{2} - i \mu + |q|)
$$

(A.7)

where $z = \mu + iq$.

In particular, for $A = 0$ we recover the simpler expression (2.8).

**Appendix B. An Identity on Gamma Functions**

In [14] the four point function was calculated as the limit at small loop lengths of the macroscopic loop amplitude. The formalism of [14] allows calculations to all orders in the genus expansion. Comparing our result (3.13) to eqn. (4.38) of [14], we find that the first two terms of each (corresponding to the two-bounce contribution) are equal to within corrections that vanish at any order. Requiring agreement of the remainder of the two formulas for the amplitude leads to the rather remarkable result

$$
\text{Im} \left\{ e^{i\pi S/2} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(-q_1 + n)}{\Gamma(-q_1)} \left( \frac{\Gamma(q_3 + n)}{\Gamma(q_3)} + \frac{\Gamma(q_4 + n)}{\Gamma(q_4)} \right) \right. 
$$

$$
\left. \times \left( \frac{\Gamma(\frac{1}{2} - i \mu + S - n)}{\Gamma(\frac{1}{2} - i \mu)} - \frac{\Gamma(\frac{1}{2} - i \mu + q_1 - n)}{\Gamma(\frac{1}{2} - i \mu - q_2)} \right) \right\} =
$$

$$
\text{Im} \left\{ e^{i\pi S/2} \left[ \frac{\Gamma(\frac{1}{2} - i \mu + q_2)}{\Gamma(\frac{1}{2} - i \mu + q_2 + q_3)} \frac{\Gamma(\frac{1}{2} - i \mu - q_3)}{\Gamma(\frac{1}{2} - i \mu - q_2 + q_4)} + \frac{\Gamma(\frac{1}{2} - i \mu + q_2)}{\Gamma(\frac{1}{2} - i \mu + q_2 + q_4)} \frac{\Gamma(\frac{1}{2} - i \mu - q_3)}{\Gamma(\frac{1}{2} - i \mu)} \right] \right\}
$$

(B.1)

where $S = \frac{1}{2} \sum |q_i|$. This equality holds to all orders in an asymptotic expansion at large $\mu$. Expanding both sides and equating the coefficients, which are expressed in terms of polygamma functions, will lead to identities on these.
Appendix C. Small Energy and Topological Expansions

We write here the small energy expansions of some scattering amplitudes to all orders in the $\frac{1}{\mu}$ expansion. These are useful for some explicit unitarity checks to all orders in the genus expansion, since higher n-point functions enter the unitarity equations at higher powers of the energy. We give here the two-point function to $\mathcal{O}(q^9)$, the three-point function to $\mathcal{O}(q^6)$, the four-point function to $\mathcal{O}(q^7)$ in two kinematic regimes and the n-point function to $\mathcal{O}(q^{n+1})$ in the $1 \to n$ regime. We also give the two-point function at genus one, two and three.

The coefficients at each order in the small energy expansion are written in terms of $\psi_n$ where

$$\psi_0 \sim \log\mu + \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{2n} (1 - 2^{-2n+1}) \mu^{-2n}$$

$$\psi_n = \left( \frac{d}{d\mu} \right)^n \psi_0$$

(C.1)

C.1. Two Point Function

(a) Small Energy Expansion

$$\mu^9 R(q; -q) = q e^{q \psi_0} \left[ 1 - \frac{q^3}{12} \psi_2 - \frac{q^4}{24} \psi_1^2 + \frac{q^5}{360} \psi_4 + \frac{q^6}{240} \psi_2^2 \
+ \frac{q^7}{180} \psi_1 \psi_3 + q^7 \left( \frac{7}{1440} \psi_1^2 \psi_2 - \frac{1}{20160} \psi_6 \right) \
+ q^8 \left( \frac{1}{1920} \psi_1^4 - \frac{23}{120960} \psi_3^2 - \frac{19}{60480} \psi_2 \psi_4 - \frac{1}{6720} \psi_1 \psi_5 \right) \
+ q^9 \left( \frac{-29}{181440} \psi_2^2 - \frac{41}{60480} \psi_1 \psi_2 \psi_3 - \frac{1}{5040} \psi_1^2 \psi_4 + \frac{1}{1814400} \psi_8 \right) + \cdots \right]$$

(C.2)

where $q$ is the euclidean momentum and $q > 0$.

(b) Genus Expansion

The genus one and two results are written here in a form which emphasizes that our results can be expanded to give integrals of $n$ vertex operators with any choice of charges over moduli spaces of riemann surfaces of arbitrarily high genus. The answer is always some polynomial in the charges $q_i$ with coefficients that are a combination of Bernoulli numbers.
\[
\int_{\mathcal{M}_{1,2}} \langle T_q T_{-q} \rangle = -\frac{1}{24} q^2 (q - 1)(q^2 - q - 1)
\] (C.3)

\[
\int_{\mathcal{M}_{2,2}} \langle T_q T_{-q} \rangle = \frac{q^2}{5760} \prod_{r=1}^{3} (q - r)(3q^4 - 10q^3 - 5q^2 + 12q + 7)
\] (C.4)

\[
\int_{\mathcal{M}_{3,2}} \langle T_q T_{-q} \rangle = -\frac{q^2}{2903040} \prod_{r=1}^{5} (q - r)(9q^6 - 63q^5 + 42q^4 + 217q^3 - 205q - 93)
\] (C.5)

### C.2. Three Point Function

\[
\mu |q_3| R(q_1, q_2; q_3) = -e^{|q_3|\psi_0} q_1 q_2 |q_3\rangle _3 \left[ \psi_2 + |q_4| \psi_1^2 - \frac{1}{12}(q_1^2 + q_1 q_2 + q_2^2)\psi_3 - \frac{1}{12} |q_3| (2q_2^2 + q_1 q_2 + q_1^2) \psi_1 \psi_2 + \cdots \right],
\] (C.6)

where \(q_1, q_2 > 0\) and \(q_3 < 0\).

### C.3. Four Point Function

The four point function in the \(1 \rightarrow 3\) and \(2 \rightarrow 2\) kinematic regimes is given up to \(O(q^7)\) where we can check that one particle irreducible amplitudes of [39] are not analytic at genus one.

\[
\mu |q_4| R(q_1, q_2, q_3; q_4) = e^{\psi} q_1 q_2 q_3 |q_4\rangle _4 \left[ \psi_2 + |q_4| \psi_1^2 - \frac{1}{24}(q_1^2 + q_2^2 + q_3^2 + q_4^2)\psi_4 - \frac{\psi_1^3}{2} \left( \frac{q_1^3}{3} + \frac{q_1^2 q_2}{2} + \frac{q_1 q_2^2}{2} + \frac{q_2^3}{3} + \frac{q_1^2 q_3}{2} + \frac{q_1 q_2 q_3}{2} + \frac{q_2^3}{3} \right) - \frac{\psi_1 \psi_3}{2} \left( \frac{q_1^3}{2} + \frac{5q_1^2 q_2}{6} + \frac{5q_1 q_2^2}{6} + \frac{q_2^3}{2} + \frac{5q_1 q_2^2}{6} + \frac{5q_1 q_3^2}{6} + \frac{5q_2 q_3^2}{6} + \frac{q_3^3}{2} \right) + \cdots \right]
\] (C.7)

In the following \(q_1 = \max |q_i| \)
\[ \mu^{q_1+q_2} R(q_1, q_2; q_3, q_4) = e^{q_4 |\psi_0 q_1 q_2|q_3 || q_4} \left[ \psi_2 - q_1 \psi_1^2 - \frac{\psi_4}{12} (q_1^2 + q_1 q_2 + q_2^2 + q_1 q_3 + q_2 q_3 + q_3^2) \right. \\
\left. - \psi_2^2 \left( \frac{q_1^3}{6} + \frac{q_1^2 q_2}{4} + \frac{q_1 q_2^2}{4} + \frac{q_2^3}{12} + \frac{q_1^2 q_3}{4} + \frac{q_2 q_3^2}{4} + \frac{q_3^3}{4} \right) \right] \right] . \tag{C.8} \]

### C.4. N-point Function

For \( n > 2 \) we have:

\[ \mu^{q_n} R(q_1, q_2, \ldots ; q_n) = q_1 q_2 \cdots q_{n-1} |q_n| \left[ \psi_{n-2} \right. \right. \\
\left. \left. + \frac{|q_n|}{2} \left( \sum_{r=0}^{n-2} \binom{n-2}{r} \psi_{n-2-r} \psi_r \right) + \cdots \right] . \tag{C.9} \]

These first two terms can easily be obtained by using the recursion relation

\[ R(q_1, q_2, \ldots ; q_n) \sim q_1 \frac{\partial}{\partial \mu} R(q_2, \ldots ; q_n) \tag{C.10} \]

for \( q_1 \to 0 \), and they are precisely what we expect from unitarity.

### Appendix D. Proof of Equivalence to Bosonization

We wish to prove that the \( S \)-matrix we compute may be obtained from the simple scattering amplitude (4.4) for free fermions via the bosonization prescription of (4.2). We will construct the proof in Euclidean space to connect to the derivation of the ‘filtration formula’. As with that formula, all results are later continued to Minkowski space. Formally, we need to show

\[ \int_{-\infty}^{\infty} \prod_{i} d\xi_i \prod_{i=1}^{n} a(\mu + \xi_i) a^{\dagger}(\mu + \xi_i + q_i) |\mu\rangle = \\
\sum_{k=1}^{n} \sum_{T_1 \Pi \ldots \Pi T_k = S} \prod_{j=1}^{k} \int d\xi_j f_-(T_j, -\xi_j) \prod_{j=1}^{k} a^{\dagger}(\mu + \xi_j + q(T_j)) a(\mu + \xi_j) |\mu\rangle , \tag{D.1} \]
where $S^- = \{ q_1, \ldots q_n \}$. Note that since $f_-(T,Q) \propto -\theta(-Q)\theta(Q - q(T))$, the left-hand side of (D.1) is normal-ordered. The proof proceeds by induction on $n$. For $n = 1$, the claim is explicitly verified in (4.2). Assuming the assertion holds for $n - 1$ tachyons, we proceed to $n$ by multiplying the “fermionized” state corresponding to $S = \{ q_2, \ldots q_n \}$ on the left by $\int d\xi_1 a(\mu + \xi_1) a^\dagger(\mu + \xi_1 + q_1)$. To do this, we split the integral into a normal-ordered part $I_1 = \int_{-q_1}^0 d\xi_1$ and the remainder $I_2$. In $I_1$, the operators can be commuted to the right to canonically order the result, leading to

$$I_1 = \sum_{k=1}^{n-1} \sum_{T_1 \cup \ldots \cup T_k = S^-} \int \prod_j d\xi_j \prod_{j=1}^k f_-(T_j, -\xi_j) a^\dagger(\mu + \xi_j + q(T_j)) a(\mu + \xi_j) |\mu\rangle.$$  

(D.2)

In $I_2$, the operators on the left annihilate $|\mu\rangle$, so the contribution is a sum of anticommutator terms. Using the identity

$$f_-(T,Q) = f_-(T \setminus \{q^*\}, Q - q^*) - f_-(T \setminus \{q^*\}, Q)$$  

(D.3)

for $q^* \in T$, one verifies that the $j$th contribution is

$$\left(\int_{-\infty}^0 + \int_{-q_1}^\infty\right) d\xi_1 \left[ a(\mu + \xi_1) a^\dagger(\mu + \xi_1 + q_1), a^\dagger(\mu + \xi_j + q(T_j)) a(\mu + \xi_j) \right] = f_-(T_j \cup \{ q_1 \}, -\xi_j) a^\dagger(\mu + \xi_j + q(T_j) + q_1) a(\mu + \xi_j).$$  

(D.4)

Adding the two contributions proves the claim by induction.
References

[1] E. Brézin and V. Kazakov, “Exactly Solvable Field Theories of Closed Strings,” Phys. Lett. 236B (1990) 144.
[2] M.R. Douglas and S. Shenker, “Strings in Less Than One Dimension,” Nucl. Phys. B335 (1990) 635.
[3] D. Gross and A. Migdal, “Non-perturbative Two Dimensional Quantum Gravity,” Phys. Rev. Lett. 64 (1990) 127.
[4] F. David, “Loop equations and non-perturbative effects in 2D Quantum Gravity” Mod. Phys. Lett. A5 (1990) 1019.
[5] M. Douglas, N. Seiberg, and S. Shenker, “Flow and Instability in Quantum Gravity” Phys. Lett. B244 (1990).
[6] E. Brézin, C. Itzykson, G. Parisi, and J.-B. Zuber, “Planar Diagrams,” Comm. Math. Phys. 59 (1978) 35.
[7] V. Kazakov, “Bosonic strings and string field theories in one dimensional target space,” to appear in the proceedings of the Cargèse Workshop on Random Surfaces, Quantum Gravity, and String Theory, 1990.
[8] I. Klebanov, “String theory in two dimensions,” Princeton preprint PUPT-1271, lectures delivered at the ICTP Spring School on String Theory and Quantum Gravity.
[9] D. Kutasov, “Some properties of (Non) Critical Strings,” Princeton preprint PUPT-1277, lectures delivered at the ICTP Spring School on String Theory and Quantum Gravity.
[10] J. Polchinski, “Critical Behavior of Random Surfaces in One Dimension,” Nucl. Phys. B346 (1990) 253.
[11] S.R. Das and A. Jevicki, “String Field Theory and Physical Interpretation of D=1 Strings,” Mod. Phys. Lett. A5 (1990) 1639.
[12] A.M. Sengupta and S.R. Wadia, “Excitations and interactions in d = 1 string theory;” Int. Jour. Mod. Phys. A6 (1991) 1961.
[13] J. Polchinski, “Classical limit of 1+1 Dimensional String Theory,” Texas preprint UTTG-06-91.
[14] G. Moore, “Double Scaled Field Theory at c = 1,” Rutgers/Yale preprint, Nucl. Phys. B, to appear.
[15] G. Moore and N. Seiberg, “From Loops to Fields in 2D Gravity,” Rutgers/Yale preprint, to appear in the Int. Jour. of Mod. Phys.
[16] D. Gross and I. Klebanov, “Fermionic String Field Theory of c = 1 2D Quantum Gravity,” Nucl. Phys. B352 (1991) 671.
[17] The one-loop off-shell propagator was calculated in this way by G. Moore and S. Shenker; unpublished. The procedure described in the text is ad hoc, but is necessary to reproduce even the perturbative expansion. It is related to the neglect of similarly
nonperturbatively oscillating terms in \[12,16\]. While a better understanding is certainly called for, we will not address the problem in more detail here since in any case the nonperturbatively oscillating terms do not contribute to the $S$-matrix.

[18] L. Faddeev, in *Methods in Field Theory* Proc. of the Les Houches Summer School 1975, Balian et. al. eds. World Scientific

[19] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, Mc Graw-Hill, 1980.

[20] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov, and A. Orlov, “Matrix Models of 2D gravity and Toda Theory,” Lebedev preprint; “Matrix Models as Integrable Systems: From Universality to Geometrodynamical Principle of String Theory,” Lebedev preprint.

[21] I. Bakas and E. Kiritsis, in *Common Trends in Mathematics and Quantum Field Theories*, Proceedings of the 1990 Yukawa International Seminar, Prog. Theor. Phys. Supp. 102; Berkeley preprint UCB-PTH-90/32.

[22] M. Fukuma, H. Kawai, and N. Nakayama, Tokyo preprint UT-562; T. Yoneya, UT-Komaba preprint (1991).

[23] M.A. Awada and S.J. Sin, “Twisted $W_\infty$ Symmetry of the KP Hierarchy and the String Equation of $d = 1$ Matrix Models,” Florida preprint UFIFT-HEP-90-33; “The String Difference Equation of $d = 1$ Matrix Model and $W_{1+\infty}$ Symmetry of the KP Hierarchy,” Florida preprint UFIFT-HEP-91-3.

[24] H. Itoyama and Y. Matsuo, “$W_{1+\infty}$-Type Constraints in Matrix Models at Finite N,” Stony Brook preprint ITP-SB-91-10.

[25] A.M. Semikhatov, ZhETF Lett. 53 (1991) 12; “Virasoro Algebra Action on Integrable Hierarchies and Virasoro Constraints in Matrix Models,” preprint.

[26] D. Minic, J. Polchinski, and Z. Yang, “Translation-Invariant Backgrounds in 1+1 Dimensional String Theory,” Texas preprint UTTG-16-91.

[27] J. Avan and A. Jevicki, “Classical Integrability and Higher Symmetries of Collective Field Theory,” Brown preprint BROWN-HET-801; “Quantum Integrability and Exact Eigenstates of the Collective String Field Theory,” BROWN-HET-824.

[28] S.R. Das, A. Dhar, G. Mandal, S. R. Wadia, “Gauge Theory Formulation of the $c=1$ Matrix Model: Symmetries and Discrete States,” IAS preprint IASSNS-HEP-91/52.

[29] E. Witten, “Ground Ring of Two Dimensional String Theory,” IAS preprint IASSNS-HEP-91/51.

[30] I.R. Klebanov and A.M. Polyakov, “Interaction of Discrete States in Two-Dimensional String Theory,” Princeton preprint PUPT-1281.

[31] I. Cherednik, “Notes on Affine Hecke Algebras,” Bonn-HE-90-04.

[32] E. Cremmer and J.-L. Gervais, “The Quantum Strip: Liouville Theory for Open Strings,” LPTENS 90/32.

[33] A. Jevicki, “Nonperturbative collective field theory,” BROWN-HET-807

[34] E. Witten, “String Theory and Black Holes,” Phys. Rev. D44 (1991) 314.
[35] G. Mandal, A.M. Sengupta, and S.R. Wadia, “Classical Solutions of Two-Dimensional String Theory,” IAS preprint IASSNS-HEP/91/10.

[36] E. Martinec and S. Shatashvili, “Black hole physics and Liouville theory,” EFI-91-22.

[37] R. Dijkgraaf, H. Verlinde, and E. Verlinde, “String propagation in a Black Hole Geometry,” Princeton preprint PUPT-1252.

[38] R. J. Eden, P. V. Landshoff, D. Olive and J. C. Polkinghorne, The Analytic S-matrix, Cambridge University Press.

[39] D. Kutasov and Ph. DiFrancesco, “World Sheet and Space Time Physics in Two Dimensional (Super) String Theory,” Princeton preprint PUPT-1276.

[40] T. Banks, M. Douglas, N. Seiberg, and S. Shenker, “Macroscopic and Microscopic Loops in Nonperturbative Two-Dimensional Quantum Gravity,” Phys. Lett. 238B (1990) 279.

[41] J. Ambjorn, J. Jurkiewicz, and Yu. Makeenko, “Multiloop Correlators for two-dimensional Quantum Gravity,” Phys. Lett. 251B (1990) 517.

[42] I.K. Kostov, “Loop Amplitudes for Non Rational String Theories,” Phys. Lett. 266B (1991) 42,317.

[43] E. Martinec, G. Moore, and N. Seiberg, “Boundary Operators in 2D Gravity,” Phys. Lett. 263B 190.

[44] G. Moore, N. Seiberg, and M. Staudacher, “From Loops to States in Two-Dimensional Quantum Gravity,” Nucl. Phys. B362 (1991)

[45] A.M. Polyakov, “Self-Tuning Fields and Resonant Correlations in 2D-Gravity,” Mod. Phys. Lett. A6 (1991) 635.

[46] D.J. Gross, I.R. Klebanov, and M.J. Newman, “The Two-Point Correlation Functions of the One Dimensional Matrix Model,” Nucl. Phys. B350 (1991) 621 ; D.J. Gross and I.R. Klebanov, “S = 1 for c = 1,” Princeton preprint PUPT-1241 ; D.J. Gross and U.H. Danielsson, “On the Correlation Functions of the Special Operators in c = 1 Quantum Gravity,” Princeton preprint PUPT-1258.

[47] D. Minic and Z.Yang, “ Is S = 1 for c = 1?” Texas preprint UTTG-23-91.

[48] N. Sakai and Y. Tanii, “Factorisation and Topological states in c = 1 Matter Coupled to 2D Gravity” TIT/HEP-173.

[49] N. Seiberg, “Notes on Quantum liouville Theory and Quantum Gravity,” in Common Trends in Mathematics and Quantum Field Theories, Proceedings of the 1990 Yukawa International Seminar, Prog. Theor. Phys. Supp. 102.

[50] J. Polchinski, “Remarks on the Liouville Field Theory,” UTTG-19-90, to appear in Strings ’90, Texas AM.

[51] E. Witten, “On Quantum Gauge Theories in Two Dimensions,” IASSNS-HEP-91/3

[52] C. Itzykson and J. B. Zuber, “Matrix Integration and Combinatorics of Modular Groups,” Comm. Math. Phys. 134 197 (1990) ; J. Distler and C. Vafa, “The Penner Model and D = 1 String Theory,” Princeton
preprint PUPT-1212, lecture given at Cargése Workshop on Random Surfaces, Quantum Gravity and String Theory, 1990

[53] D.J. Gross and I.R. Klebanov, “One Dimensional String Theory on a Circle,” Nucl. Phys. B344 (1990) 475.

[54] I. Klebanov and D. Lowe, “Correlation functions in 2D Quantum Gravity Coupled to a Compact Scalar Field,” Princeton preprint PUPT-1256.

[55] D.J. Gross and P.F. Mende, “The High Energy Behaviour of String Scattering Amplitudes,” Phys. Lett. 197B (1987) 129; “String Theory beyond the Planck Scale,” Nucl. Phys. B303 (1988) 407.

[56] D. Amati, M. Ciafaloni, and G. Veneziano, “Superstring Collisions at Planckian Energies,” Phys. Lett. 197B (1987) 81.

[57] D. Gross, “High Energy Symmetries of String Theory,” Phys. Rev. Lett. 60 (1988) 1229.

[58] E. Witten, “Space-Time and Topological Orbifolds,” Phys. Rev. Lett. 61 (1988) 670.

[59] P.F. Mende and H. Ooguri, “Borel Summation of String Theory for Planck Scale Scattering,” Nucl. Phys. B339 (1990) 641.
Figure Captions

Fig. 1. a.) A pictorial version of the integral $I$ for positive momentum. (b) A pictorial version of the integral $I$ for negative momentum

Fig. 2. Incoming and outgoing vertices. The dotted line carrying negative (positive) momentum $q_i$ should be thought of as an incoming (outgoing) boson with energy $|q_i|$. Momentum carried by lines is always conserved as time flows upwards.

Fig. 3. The possible real histories for $1 \rightarrow 2$ scattering

Fig. 4. Some possible real histories for $2 \rightarrow n$ scattering

Fig. 5. Diagrammatic rules in Euclidean space

Fig. 6. Diagrammatic rules in Minkowski space

Fig. 7. Typical bracket configurations

Fig. 8. A real history as a composition of three maps

Fig. 9. An incoming boson dissociates

Fig. 10. Analytic structure in the complex $s$ plane

Fig. 11. A set of branch cuts for $S(\omega|\omega)$, illustrated here fore the case $A = 0$.

Fig. 12. Two paths defining the $1 \rightarrow 1$ amplitude on the second and third sheets.

Fig. 13. The amplitude $1 - P(\omega)$ for $\mu = 8$ as a function of $\omega$. The complicated structure observed here is probably not an artifact of the existence of the wall. Indeed we may also plot the amplitudes in theory II to obtain similar pictures.

Fig. 14. Energy distribution integral $|A|^2$ for $\mu = 5$ and $\omega = 6$. Again, similar results hold in theory II.