Multirate iterative scheme with multiphysics finite element method for a fluid-saturated poroelasticity

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Abstract

In this paper, we propose a multirate iterative scheme with multiphysics finite element method for a fluid-saturated poroelasticity model. Firstly, we reformulate the original model into a fluid coupled problem to apply the multiphysics finite element method for the discretization of the space variables, and we design a multirate iterative scheme on the time scale which solve a generalized Stokes problem in the coarse time size and solve the diffusion problem in the finer time size according to the characteristics of the poroelasticity problem. Secondly, we prove that the multirate iterative scheme is stable and the numerical solution satisfies some energy conservation laws, which are important to ensure the uniqueness of solution to the decoupled computing problem. Also, we analyze the error estimates to prove that the proposed numerical method doesn’t reduce the precision of numerical solution and greatly reduces the computational cost. Finally, we give the numerical tests to verify the theoretical results and draw a conclusion to summary the main results in this paper.

Keywords: Poroelasticity, multiphysics finite element methods, multirate iterative scheme.

1. Introduction

Poromechanic is a branch of continuum mechanics and acoustics, which is a fluid-solid interaction system at pore scale. If the solid is an elastic material, then the subject of the study is known as poroelasticity. In this paper, we study the behavior of a fluid-saturated
poroelasticity model as follows:

\[-\text{div}\sigma(u) + \alpha \nabla p = f \quad \text{in } \Omega_T := \Omega \times (0, T) \subset \mathbb{R}^d \times (0, T), \quad (1.1)\]

\[c_0 p + \alpha \text{div} u_t + \text{div} \mathbf{v}_f = \phi \quad \text{in } \Omega_T. \quad (1.2)\]

Here \(\sigma(u)\) is called the (effective) stress tensor, and defined by

\[\sigma(u) := \mu \varepsilon(u) + \lambda \text{div} u I, \quad (1.3)\]

where \(\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)\) and \(\mathbf{v}_f\) is the volumetric solvent flux and is the well-known Darcy’s law and given by

\[\mathbf{v}_f := -\frac{K}{\mu_f}(\nabla p - \rho_f g). \quad (1.4)\]

In addition, \(\Omega \subset \mathbb{R}^d (d = 2, 3)\) is a bounded polygonal domain with the boundary \(\partial \Omega\), \(u\) denotes the displacement vector of the solid and \(p\) denotes the pressure of the solvent, \(f\) is the body force, \(I\) denotes the \(d \times d\) identity matrix and \(\varepsilon(u)\) is known as the strain tensor. The parameters in the above model list as follows: Lamé constants \(\lambda\) and \(\mu\); the permeability tensor \(K = K(x)\) which is assumed to be symmetric and uniformly positive definite in the sense that there exist positive constants \(K_1\) and \(K_2\) such that \(K_1|\zeta|^2 \leq K(x)\zeta \cdot \zeta \leq K_2|\zeta|^2\) for a.e. \(x \in \Omega\) and any \(\zeta \in \mathbb{R}^d\); the solvent viscosity \(\mu_f\), Biot-Willis constant \(\alpha\), and the constrained specific storage coefficient \(c_0\). Also, we denote \(\hat{\sigma}(u, p) := \sigma(u) - \alpha p I\) by the total stress tensor.

To close the problem (1.1)-(1.2), we impose the following boundary and initial conditions:

\[\hat{\sigma}(u, p) n = \sigma(u) n - \alpha p I n = f_1 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0, T), \quad (1.5)\]

\[\mathbf{v}_f \cdot n = -\frac{K}{\mu_f}(\nabla p - \rho_f g) \cdot n = -\phi_1 \quad \text{on } \partial \Omega_T, \quad (1.6)\]

\[u = u_0, \quad p = p_0 \quad \text{in } \Omega \times \{t = 0\}. \quad (1.7)\]

We remark that the Lamé constant \(\mu\) is also called the shear modulus and denoted by \(G\), and \(B := \lambda + \frac{2}{3}G\) is called the bulk modulus. \(\lambda, \mu\) and \(B\) are computed from the Young’s modulus \(E\) and the Poisson ratio \(\nu\) by the following formulas:

\[\lambda = \frac{E\nu}{(1 + \nu)(1 - 2 \nu)}, \quad \mu = G = \frac{E}{2(1 + \nu)}, \quad B = \frac{E}{3(1 - 2 \nu)}.\]

The problem (1.1)-(1.7) is widely applied to many fields such as biomedical and chemical systems, environmental and reservoir engineering, for the details, one can refer to [1–10] and the references therein.
Since the problem (1.1)-(1.7) and its solution domain are very complicated, it is
difficult to find the analytical solution of the problem. Therefore, the finite element
method are usually applied to approximate the poroelastic problem. In recent years, many
scholars have proposed different finite element methods to study the poroelasticity model,
the main difficulty is “locking” phenomenon, for the details, one can see [11]. The authors
of [9,12] proposed and analyzed a semi-discrete and a fully discrete mixed finite element
method which simultaneously approximate the pressure and its gradient along with the
displacement vector field, based on the same or similar idea, some stabilized finite element
methods are designed, one can see [13-15] and the therein references. To prevent “locking”
phenomenon and reveal the multi physical processes, the authors of [16,17] proposed fully
discrete finite element method to describe the expansion dynamics of polymer gels under
mechanical constraints. Later, the authors of [18] proposed multiphysics finite element
method for poroelasticity model, the key idea of multiphysics finite element method is
to reconstruct the original poroelastic problem by introducing two pseudo-pressure fields,
which makes the original problem is decoupled into two sub-problems at each time step.
For the coupled fluid problem, the generalized Stokes problem changes slow and the
diffusion problem changes fast along with time, in order to study this phenomenon, the
authors of [19] propose a multirate iterative scheme based on a mixed formulation. Ge and
Ma in [20] proposed a multirate iterative scheme based on the multiphysics discontinuous
Galerkin method and gave the optimal convergent order estimates, which is very different
from the method of [19]. However, the multiphysics discontinuous Galerkin method has a
large amount of computation. In this paper, in order to reduce the computational while
retaining the precision, we use the multiphysics finite element method for the discretization
of the space variables and adopt a multirate iterative scheme on the time scale which
solve a generalized Stokes problem in the coarse time size and solve the diffusion problem
in the finer time size. And we prove that the multirate iterative scheme is stable and
the numerical solution satisfies some energy conservation laws. Also, we prove that it
doesn’t reduce the precision of numerical solution and compared with the the multiphysics
discontinuous Galerkin method, the multirate iterative scheme based on the multiphysics
finite element method greatly reduces computation time.

The remainder of this paper is organized as follows. In Section 2 we present the
reconstruction of poroelasticity model and the preliminary knowledge which is needed to
study the poroelasticity problem. In Section 3 we propose and analyze the multirate
iterative scheme with the multiphysics finite element method for the reformulated model,
and give the optimal error estimates. In Section 4 we give some numerical examples to
verify theoretical results. Finally, we draw a conclusion to summary the main results in
this paper.

2. Multiphysics reformulation of poroelasticity model

To reveal the multi physical processes and propose an effective numerical method, we introduce the new variable \( q = \text{div} \mathbf{u} \) and denote \( \eta = c_0 p + \alpha q, \xi = \alpha p - \lambda q \), it is easy to check that

\[
p = \kappa_1 \xi + \kappa_2 \eta, \quad q = \kappa_1 \eta - \kappa_3 \xi, \tag{2.1}
\]

where

\[
\kappa_1 = \frac{\alpha}{\alpha^2 + \lambda c_0}, \quad \kappa_2 = \frac{\lambda}{\alpha^2 + \lambda c_0}, \quad \kappa_3 = \frac{c_0}{\alpha^2 + \lambda c_0}. \tag{2.2}
\]

Using (2.1), we reformulate (1.1)-(1.4) into the following system:

\[
-\mu \text{div} \varepsilon(\mathbf{u}) + \nabla \xi = f \quad \text{in } \Omega_T, \tag{2.3}
\]

\[
\kappa_3 \xi + \text{div} \mathbf{u} = k_1 \eta \quad \text{in } \Omega_T, \tag{2.4}
\]

\[
\eta_t - \frac{1}{\mu_f} \text{div}[K(\nabla(k_1 \xi + k_2 \eta) - \rho_f \mathbf{g})] = \phi \quad \text{in } \Omega_T. \tag{2.5}
\]

The boundary and initial conditions (1.6)-(1.7) can be rewritten as

\[
\sigma(\mathbf{u}) \mathbf{n} - \alpha(\kappa_1 \xi + \kappa_2 \eta) \mathbf{I} \mathbf{n} = f_1 \quad \text{on } \partial \Omega_T := \partial \Omega \times (0, T), \tag{2.6}
\]

\[
-\frac{K}{\mu_f}(\nabla(k_1 \xi + k_2 \eta) - \rho_f \mathbf{g}) \cdot \mathbf{n} = \phi_1 \quad \text{on } \partial \Omega_T, \tag{2.7}
\]

\[
\mathbf{u} = \mathbf{u}_0, \quad p = p_0 \quad \text{in } \Omega \times \{t = 0\}. \tag{2.8}
\]

Remark 2.1. From the problem (2.3)-(2.8), we know that \( (\mathbf{u}, \xi) \) satisfies the generalized Stokes problem of the displacement vector field along with the pseudo-pressure field, and \( \eta \) satisfies the diffusion problem of the pseudo-pressure field. Thus, the problem (2.3)-(2.8) reveals the multiphysics deformation and diffusion process.

Remark 2.2. In the problem (2.3)-(2.8), the original variable \( p \) is no longer the primary variable, but it is calculated as a linear combination of \( \xi \) and \( \eta \), which can be updated by (2.1), just because of this, it is possible to design an effective numerical method without “locking” phenomenon for the pressure of \( p \).

Next, we introduce some function spaces. For any Banach space \( B \), we let \( B = [B]^d \), and denote \( B' \) by its dual space, and \( \| \cdot \|_{L^p(B)} \) is a shorthand notation for \( \| \cdot \|_{L^p((0, T); B)} \). And we denote \( \langle \cdot, \cdot \rangle \) by the standard \( L^2(\Omega) \) and \( L^2(\partial \Omega) \) inner products, respectively. Also, we need to introduce the function spaces:

\[
L^2_0(\Omega) := \{ q \in L^2(\Omega); (q, 1) = 0 \}, \quad X := H^1(\Omega).
\]
Denote \( \mathbf{RM} := \{ \mathbf{r} = \mathbf{a} + \mathbf{b} \times x; \mathbf{a}, \mathbf{b}, x \in \mathbb{R}^d \} \) by the space of infinitesimal rigid motions. From [21], it is well known that \( \mathbf{RM} \) is the kernel of the strain operator \( \varepsilon \), that is, \( \mathbf{r} \in \mathbf{RM} \) if and only if \( \varepsilon(\mathbf{r}) = 0 \). Hence, we have

\[
\varepsilon(\mathbf{r}) = 0, \quad \text{div} \mathbf{r} = 0 \quad \forall \mathbf{r} \in \mathbf{RM}.
\] (2.9)

Let \( L^2_\perp(\partial \Omega) \) and \( H^1_\perp(\Omega) \) denote respectively the subspaces of \( L^2(\partial \Omega) \) and \( H^1(\Omega) \) which are orthogonal to \( \mathbf{RM} \), that is,

\[
H^1_\perp(\Omega) := \{ \mathbf{v} \in H^1(\Omega); (\mathbf{v}, \mathbf{r}) = 0 \quad \forall \mathbf{r} \in \mathbf{RM} \},
\]

\[
L^2_\perp(\partial \Omega) := \{ \mathbf{g} \in L^2(\partial \Omega); \langle \mathbf{g}, \mathbf{r} \rangle = 0 \quad \forall \mathbf{r} \in \mathbf{RM} \}.
\]

As for the existence and uniqueness of weak solution of the problem (2.3)-(2.8), one can refer to [18], here we omit the details.

3. Multirate iterative scheme with multiphysics finite element method

In this section, we propose the finite element solution \( (\mathbf{u}_h, \xi_h, \eta_h) \) of the problem (2.3)-(2.8). We use the multiphysics finite element method for the discretization of the space variables and adopt a multirate iterative scheme on the time scale, which solve a generalized Stokes problem in the coarse time size and solve the diffusion problem in the finer time size.

3.1. Multirate iterative scheme

Let \( \mathcal{T}_h \) be a quasi-uniform triangulation or rectangular partition of \( \Omega \) with mesh size \( h \), and \( \overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K} \). The time interval \([0, T]\) is divided into \( N \) equal intervals, denoted by \([t_{n-1}, t_n], n = 1, 2, \ldots, N\), and \( \Delta t = \frac{T}{N} \), then \( t_n = n\Delta t \).

In this paper, we use the following Taylor-Hood element:

\[
X_h = \{ \mathbf{v}_h \in C^0(\overline{\Omega}); \mathbf{v}_h|_K \in P_2(K) \quad \forall K \in \mathcal{T}_h \},
\]

\[
M_h = \{ \phi_h \in C^0(\overline{\Omega}); \phi_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h \}.
\]

Finite element approximation space \( W_h \) for \( \eta \) variable can be chosen independently, any piecewise polynomial space is acceptable provided that \( W_h \supset M_h \). The most convenient choice is \( W_h = M_h \).

Recall that \( \mathbf{RM} \) denotes the space of the infinitesimal rigid motions, evidently, \( \mathbf{RM} \subset X_h \). We define

\[
V_h = \{ \mathbf{v}_h \in X_h; (\mathbf{v}_h, \mathbf{r}) = 0, \forall \mathbf{r} \in \mathbf{RM} \}.
\] (3.1)
It is easy to check that \( \mathbf{X}_h = \mathbf{V}_h \bigoplus \mathbf{R}_M \). From [16], we know that there holds the following of inf-sup condition:

\[
\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{(\text{div} \mathbf{v}_h, \varphi_h)}{\|\mathbf{v}_h\|_{H^1(\Omega)}} \geq \beta_1 \|\varphi_h\|_{L^2(\Omega)} \quad \forall \varphi_h \in M_0, \quad \beta_1 > 0. \tag{3.2}
\]

Next, we define the bilinear forms as follows:

\[
a(\mathbf{u}, \mathbf{v}) = \mu(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})), \tag{3.3}
\]

\[
b(\mathbf{v}, \xi) = -(\text{div} \mathbf{v}, \xi), \tag{3.4}
\]

\[
c(\xi, \varphi) = k_3(\xi, \varphi). \tag{3.5}
\]

Now, we propose a multirate iterative scheme with multiphysics finite element method for the problem (2.3)-(2.8) as follows:

(i) Compute \( \mathbf{u}_h^0 \in \mathbf{V}_h \) and \( q_h^0 \in M_h \) by

\[
\mathbf{u}_h^0 = \mathcal{R}_h \mathbf{u}_0, \quad p_h^0 = \mathcal{Q}_h p_0, \quad q_h^0 = \mathcal{Q}_h q_0 \quad (q_0 = \text{div} \mathbf{u}_0),
\]

\[
\eta_h^0 = c_0 p_h^0 + \alpha q_h^0, \quad \xi_h^0 = \alpha p_h^0 - \lambda q_h^0,
\]

where \( \mathcal{R}_h \) and \( \mathcal{Q}_h \) are defined by (3.25) and (3.27), respectively.

(ii) For \( n = 0, 1, 2, \ldots \), do the following three steps:

\textbf{Step 1:} Solve for \((\mathbf{u}_h^{(n+1)m}, \xi_h^{(n+1)m}) \in \mathbf{V}_h \times M_h \) such that

\[
a(\mathbf{u}_h^{(n+1)m}, \mathbf{v}_h) + b(\mathbf{v}_h, \xi_h^{(n+1)m}) = (\mathbf{f}_h, \mathbf{v}_h) + \langle \mathbf{f}_1, \mathbf{v}_h \rangle, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{3.6}
\]

\[
b(\mathbf{u}_h^{(n+1)m}, \varphi_h) + c(\xi_h^{(n+1)m}, \varphi_h) = k_1(\eta_h^{(n+1)m}, \varphi_h), \quad \forall \varphi_h \in M_h. \tag{3.7}
\]

\textbf{Step 2:} For \( k = 1, 2, \ldots, m \), solve for \( \eta_h^{nm+k} \in M_h \) such that

\[
(d_t \eta_h^{nm+k}, \psi_h) + \frac{1}{\mu_f} (K(\nabla(\xi_h^{(n+1)m} + k_2 \eta_h^{nm+k}) - \rho_f g), \nabla \psi_h) = (\phi, \psi_h) + \langle \phi_1, \psi_h \rangle. \tag{3.8}
\]

\textbf{Step 3:} Update \( p_h^{(n+1)m} \) and \( q_h^{(n+1)m} \) by

\[
p_h^{(n+1)m} = k_1 \xi_h^{(n+1)m} + k_2 \eta_h^{(n+1)m}, \quad q_h^{(n+1)m} = k_1 \eta_h^{(n+1)m} - k_3 \xi_h^{(n+1)m}, \tag{3.9}
\]

where \( \theta = 0 \) or 1, \( m \) is a positive integer, \( d_t \eta_h^{nm+k} = \frac{\eta_h^{nm+k} - \eta_h^{nm+k-1}}{\Delta t} \).

### 3.2. Stability analysis

**Lemma 3.1.** Let \( \{(\mathbf{u}_h^{(n+1)m}, \xi_h^{(n+1)m}, \eta_h^{nm+k})\}_{n \geq 0, 1 \leq k \leq m} \) be the solution of the multirate iterative scheme (3.6)-(3.9), then we have

\[
J_{h,\theta}^{l+1} + S_{h,\theta}^l = J_{h,\theta}^0 \quad \text{for } l \geq 0, \quad \theta = 0, 1, \tag{3.10}
\]
where
\[
J_{h, \theta}^{l+1} := \frac{1}{2} \mu \epsilon(u_h^{(l+1)m})^2_{L^2(\Omega)} + \kappa_2 \| \eta_h^{(l+\theta)m} \|^2_{L^2(\Omega)} + \kappa_3 \| \xi_h^{(l+1)m} \|^2_{L^2(\Omega)}
- 2(f, u_h^{(l+1)m}) - 2(f_1, u_h^{(l+1)m}) \]
\[
S_{h, \theta}^l := \Delta t \sum_{n=0}^{l} \sum_{k=1}^{m} \left[ \frac{K}{\mu_f} \left( \nabla p_h^{nm+k} - \rho_f g, \nabla \xi_h^{nm+k} \right) + \frac{\kappa_2 \Delta t}{2} \| d_t \eta_h^{(n-1+\theta)m+k} \|^2_{L^2(\Omega)} \right]
- (\phi, p_h^{nm+k}) - (\phi_1, p_h^{nm+k}) - (1 - \theta) \frac{\kappa_3 \Delta t}{\mu_f} M \left( K d_t \xi_h^{nm+k}, \nabla \xi_h^{nm+k} \right)
+ m \Delta t \sum_{n=0}^{l} \left( \frac{\mu \Delta t}{2} \| d_t \epsilon(u_h^{(n+1)m}) \|_{L^2(\Omega)}^2 + \frac{\kappa_3 \Delta t}{2} \| d_t \xi_h^{(n+1)m} \|_{L^2(\Omega)}^2 \right),
\]
where \( d_t^{(m)} \eta_h^{nm} \) is defined by
\[
d_t^{(m)} \eta_h^{nm} = \frac{\eta_h^{nm} - \eta_h^{nm-m}}{m \Delta t}. \tag{3.11}
\]

**Proof.** Since the proof for the case of \( \theta = 1 \) is simple, so we we only consider the case of \( \theta = 0 \). Setting \( v_h = d_t^{(m)} u_h^{(n+1)m} \) in (3.6), \( \varphi_h = \xi_h^{(n+1)m} \) in (3.7), and \( \psi_h = p_h^{nm+k} \) in (3.8) after lowering the degree from \( n \) to \( n - 1 \) and then summing over \( k \) from 1 to \( m \), we get
\[
\mu (\epsilon(u_h^{(n+1)m}), \epsilon(d_t^{(m)} u_h^{(n+1)m})) - \left( \xi_h^{(n+1)m}, \text{div} d_t^{(m)} u_h^{(n+1)m} \right)
= (f, d_t^{(m)} u_h^{(n+1)m}) + (f_1, d_t^{(m)} u_h^{(n+1)m}), \tag{3.12}
\]
\[
\kappa_3 (d_t^{(m)} \xi_h^{(n+1)m}, \xi_h^{(n+1)m}) + (\text{div} d_t^{(m)} u_h^{(n+1)m}, \xi_h^{(n+1)m})
= \kappa_1 (d_t^{(m)} \eta_h^{nm}, \xi_h^{(n+1)m}), \tag{3.13}
\]
\[
+ \sum_{k=1}^{m} \left( \| d_t \eta_h^{(n-1)m+k} \|^2_{L^2(\Omega)} + \frac{1}{2} \| d_t \xi_h^{(n-1)m+k} \|^2_{L^2(\Omega)} \right)
= \sum_{k=1}^{m} \left( \phi, p_h^{nm+k} \right). \tag{3.14}
\]

The first term on the left-hand side of (3.14) can be rewritten as
\[
\sum_{k=1}^{m} (d_t \eta_h^{(n-1)m+k}, p_h^{nm+k}) = \sum_{k=1}^{m} (d_t \eta_h^{(n-1)m+k}, \kappa_1 \xi_h^{nm+k} + \kappa_2 \eta_h^{(n-1)m+k})
= \sum_{k=1}^{m} \left( \frac{\kappa_2 \Delta t}{2} \| d_t \eta_h^{(n-1)m+k} \|^2_{L^2(\Omega)} + \frac{\kappa_2}{2} \| d_t \eta_h^{(n-1)m+k} \|^2_{L^2(\Omega)} \right)
+ \kappa_1 m (d_t^{(m)} \eta_h^{nm}, \xi_h^{(n+1)m}). \tag{3.15}
\]
Moreover, we have
\[
\sum_{k=1}^{m} \frac{K}{\mu_f} (\nabla (\kappa_1 \xi_h^{nm} + \kappa_2 \eta_h^{(n-1)m+k}) - \rho_f g, \nabla p_h^{nm+k})
\]
\[
\sum_{k=1}^{m} \left[ \frac{K}{\mu_f} (\nabla p_{h}^{\text{nm}+k} - \rho_f \mathbf{g}, \nabla p_{h}^{\text{nm}+k}) - \frac{k_1Kk_\Delta t}{\mu_f} (d_{t}(k) \nabla \xi_{h}^{\text{nm}+k}, \nabla p_{h}^{\text{nm}+k}) \right],
\]

(3.16)

\[
k_3(d_{t}(m) \xi_{h}^{(n+1)m}, \epsilon_{h}^{(n+1)m}) = \frac{k_3}{2} d_{t}(m) \| \xi_{h}^{(n+1)m} \|^2_{L^2(\Omega)} + \frac{k_3m \Delta t}{2} \| d_{t}(m) \epsilon_{h}^{(n+1)m} \|^2_{L^2(\Omega)}.
\]

(3.17)

Adding (3.12)-(3.13) (after applying the summation operator \( m\Delta t \sum_{n=0}^{l} \)) and (3.14) (after applying the summation operator \( \Delta t \sum_{n=0}^{l} \)), using (3.15)-(3.17), we see that (3.10) holds. The proof is complete.

Lemma 3.2. Let \( \{ (u_{h}^{(n+1)m}, \xi_{h}^{(n+1)m}, \eta_{h}^{\text{nm}+k}) \}_{n \geq 0, 1 \leq k \leq m} \) be the solution of the multirate iterative scheme (3.6)-(3.9), then there hold

\[
(\eta_{h}^{\text{nm}}, 1) = C_\eta(t_{nm}) \quad \text{for} \quad n = 0, 1, 2, \ldots,
\]

(3.18)

\[
(\xi_{h}^{\text{nm}}, 1) = C_\xi(t_{(n-1)+\theta}m) \quad \text{for} \quad n = 1 - \theta, 1, 2, \ldots,
\]

(3.19)

\[
(u_{h}^{\text{nm}} \cdot \mathbf{n}, 1) = C_u(t_{(n-1)+\theta}m) \quad \text{for} \quad n = 1 - \theta, 1, 2, \ldots,
\]

(3.20)

where \( C_\eta(t) = (\eta(\cdot, t), 1), C_\xi(t) = (\xi(\cdot, t), 1) = \frac{1}{d + \mu k_3}(\mu k_1 C_\eta(t) - (f, x) - \langle f_1, x \rangle), C_u(t) = (u(\cdot, t) \cdot \mathbf{n}, 1). \)

Proof. Taking \( \psi_{h} = 1 \) in (3.8) and summing over \( k \) from 1 to \( m \) and over \( n \) from 0 to \( l \), we get

\[
(\eta_{h}^{(l+1)m}, 1) = (\eta_{h}^{0}, 1) + [(\phi, 1) + \langle \phi_1, 1 \rangle]t_{(l+1)m} = C_\eta(t_{(l+1)m}), \quad l = 0, 1, 2, \ldots.
\]

(3.21)

From (3.21), we see that (3.18) holds.

Taking \( \mathbf{v}_{h} = x \) in (3.6) and \( \varphi_{h} = 1 \) in (3.7), we have

\[
\mu(\text{div} u_{h}^{(n+1)m}, 1) - d(\xi_{h}^{(n+1)m}, 1) = (f, x) + \langle f_1, x \rangle,
\]

(3.22)

\[
\kappa_3(\xi_{h}^{(n+1)m}, 1) + (\text{div} u_{h}^{(n+1)m}, 1) = \kappa_1 C_\eta(t_{(n+1)\theta}m).
\]

(3.23)

Using (3.22) and (3.23), we obtain

\[
(d + \mu k_3)(\xi_{h}^{(n+1)m}, 1) = \mu k_1 C_\eta(t_{(n+1)\theta}m) - (f, x) - \langle f_1, x \rangle.
\]

(3.24)

From the definition of \( C_\xi(t) \) and (3.24), we conclude that (3.19) holds for all \( n \geq 1 - \theta \).

Using (3.18), (3.19), (3.23) and the Gauss divergence theorem, we see that (3.20) holds. The proof is complete.

Using Lemma 3.1 and Lemma 3.2, taking the similar argument to one of [18] or [22], we can get the following result and omit the detail of its proof here.

Theorem 3.3. Let \( u_0 \in H^1(\Omega), f \in L^2(\Omega), f_1 \in L_2(\partial\Omega), p_0 \in L^2(\Omega), \phi \in L^2(\Omega), \) and \( \phi_1 \in L^2(\partial\Omega). \) Then there exists a unique numerical solution to the problem (3.6)-(3.9).
3.3. Error estimates

Next, we give the error estimate of the multirate iterative scheme. To do that, we firstly introduce some projection operators. Firstly, for any $V \in H^1_\perp(\Omega)$, we define its elliptic projection $R_h : H^1_\perp(\Omega) \to V_h$ by

$$
(\varepsilon(R_h V - v), \varepsilon(w_h)) = 0 \quad \forall w_h \in V_h.
$$

(3.25)

Secondly, for any $\varphi \in H^1(\Omega)$ we define the projection operator $S_h : H^1(\Omega) \to M_h$ by

$$
(\nabla S_h \varphi, \nabla \varphi_h) = (\nabla \varphi, \nabla \varphi_h) \quad \forall \psi_h \in M_h.
$$

(3.26)

Finally, for any $\varphi \in L^2(\Omega)$, we define the $L^2$-projection $Q_h : L^2(\Omega) \to M_h$ as

$$
(Q_h \varphi - \varphi, \psi_h)_K = 0, \forall \psi_h \in P_{r_2}(K).
$$

(3.27)

From [21], we know that the following estimates hold:

**Lemma 3.4.** The projection operators of $R_h, S_h, Q_h$ satisfy

$$
\|R_h v - v\|_{L^2(\Omega)} + h^{s} \|\nabla(R_h v - v)\|_{L^2(\Omega)} \leq C h^{s+1} \|v\|_{H^{s+1}(\Omega)}, 0 \leq s \leq k;
$$

(3.28)

$$
\|S_h \varphi - \varphi\|_{L^2(\Omega)} + h^{s} \|\nabla(S_h \varphi - \varphi)\|_{L^2(\Omega)} \leq C h^{s+1} \|\varphi\|_{H^{s+1}(\Omega)}, 0 \leq s \leq k \varepsilon,
$$

(3.29)

$$
\|Q_h \varphi - \varphi\|_{L^2(\Omega)} + h^{s} \|\nabla(Q_h \varphi - \varphi)\|_{L^2(\Omega)} \leq C h^{s+1} \|\varphi\|_{H^{s+1}(\Omega)}, 0 \leq s \leq k,
$$

(3.30)

where $k$ is the degree of piecewise polynomial of finite element space.

To derive the error estimates for the numerical solution, we split the errors into two parts by the following forms:

$$
E^n_u := u(t_n) - u^n_h = u(t_n) - R_h(u(t_n)) + R_h(u(t_n)) - u_h^n := \Lambda^n_u + \Theta^n_u,
$$

$$
E^n_\zeta := \zeta(t_n) - \zeta_h^n = \zeta(t_n) - Q_h(\zeta(t_n)) + Q_h(\zeta(t_n)) - \zeta^n_h := \Lambda^n_\zeta + \Theta^n_\zeta,
$$

$$
E^n_\eta := \eta(t_n) - \eta_h^n = \eta(t_n) - Q_h(\eta(t_n)) + Q_h(\eta(t_n)) - \eta^n_h := \Lambda^n_\eta + \Theta^n_\eta,
$$

$$
E^n_p := p(t_n) - p_h^n = p(t_n) - Q_h(p(t_n)) + Q_h(p(t_n)) - p_h^n := \Lambda^n_p + \Theta^n_p,
$$

$$
E^n_\xi := p(t_n) - p_h^n = p(t_n) - S_h(p(t_n)) + S_h(p(t_n)) - p_h^n := \Psi^n_p + \Phi^n_\xi.
$$

Trivially, we have $\Phi^n_p = \Lambda^n_p - \Psi^n_p + \Theta^n_p$.

To convenience, we introduce the notations as follows:

$$
\zeta_h^{l+1} = \frac{1}{2} \left[ \mu \|\varepsilon(\Theta_u^{l+1})\|_{L^2(\Omega)}^2 + k_2 \|\Theta^{l+1}\|_{L^2(\Omega)}^2 \right],
$$

$$
R^{(n+1)m}_h = \int_{t_{nm}}^{t_{nm+1}} (s - t_{nm}) \eta_{tt}(s) ds.
$$

(3.31)

(3.32)
Lemma 3.5. Let \( \{u^{(n+1)m}_h, \xi^{(n+1)m}_h, \eta^{(n+\theta)m}_h\} \) be the solution of the multirate iterative scheme \((3.6)-(3.9)\), then we have

\[
\zeta^{t+1}_h + m\Delta t \sum_{n=0}^{l} \left[ \frac{K}{\mu_f} (\nabla \Phi_p^{(n+1)m} - K \rho_f g, \nabla \Phi_p^{(n+1)m}) + \frac{\kappa_2 \Delta t}{2} \|d_t^{(m)} \Theta^{(n+\theta)m}\|_{L^2(\Omega)}^2 \right] \\
+ m\Delta t \sum_{n=0}^{l} \frac{\Delta t}{2} \left( \mu \|d_t^{(m)} \varepsilon(\Theta_u^{(n+1)m})\|_{L^2(\Omega)}^2 + \kappa_3 \|d_t^{(m)} \Theta^{(n+1)m}\|_{L^2(\Omega)}^2 \right) \\
= \zeta^0_h + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7, \tag{3.33}
\]

where

\[
E_1 = m\Delta t \sum_{n=0}^{l} \left[ (\Lambda^{(n+1)m}_\xi, \text{div}(d_t^{(m)} \Theta_u^{(n+1)m})) - (\text{div}(d_t^{(m)} \Lambda^{(n+1)m}_u), \Theta^{(n+1)m}_\xi) \right],
\]

\[
E_2 = -k_3 m\Delta t \sum_{n=0}^{l} (d_t^{(m)} \Lambda^{(n+1)m}_\xi, \Theta^{(n+1)m}_\xi),
\]

\[
E_3 = m\Delta t \sum_{n=0}^{l} (R^{(n+\theta)m}_h, \Phi_p^{(n+1)m}),
\]

\[
E_4 = (1 - \theta) k_1 (m \Delta t)^2 \sum_{n=0}^{l} (d_t^{(m)} \eta(t^{(n+1)m}), \Theta^{(n+1)m}_\xi),
\]

\[
E_5 = m\Delta t \sum_{n=0}^{l} (d_t^{(m)} \Theta^{(n+\theta)m}_\eta, \Psi_p^{(n+1)m} - \Lambda^{(n+1)m}_p).
\]

\[
E_6 = (1 - \theta) (m \Delta t)^2 \sum_{n=0}^{l} \frac{Kk_1}{\mu_f} (d_t^{(m)} \nabla \Lambda^{(n+1)m}_\xi, \nabla \Phi_p^{(n+1)m}).
\]

\[
E_7 = (1 - \theta) (m \Delta t)^2 \sum_{n=0}^{l} \frac{Kk_1}{\mu_f} (d_t^{(m)} \nabla \Theta^{(n+1)m}_\xi, \nabla \Phi_p^{(n+1)m}).
\]

Proof. Using \((3.6)-(3.8)\) and the definition of \(Q_h, R_h, S_h\), we get

\[
\mu(\varepsilon(\Theta_u^{(n+1)m}), \varepsilon(\mathbf{v}_h)) - (\Lambda^{(n+1)m}_\xi + \Theta^{(n+1)m}_\xi, \text{div} \mathbf{v}_h) = 0, \tag{3.34}
\]

\[
k_3 (\Lambda^{(n+1)m}_\xi + \Theta^{(n+1)m}_\xi, \varphi_h) + (\text{div} \Lambda^{(n+1)m}_u + \text{div} \Theta^{(n+1)m}_u, \varphi_h) \\
= k_1 (\Lambda^{(n+\theta)m}_\eta + \Theta^{(n+\theta)m}_\eta, \varphi_h) + (1 - \theta) k_1 m \Delta t (d_t^{(m)} \eta(t^{(n+1)m}), \varphi_h), \forall \varphi_h \in M_h \tag{3.35}
\]

\[
(d_t \Lambda^{(n+\theta)m}_\eta + d_t \Theta^{(n+\theta)m}_\eta, \psi_h) + \frac{K}{\mu_f} (\nabla \Phi_p^{(n+1)m} - \rho_f g, \nabla \psi_h) \\
- (1 - \theta) m \Delta t \frac{K k_1}{\mu_f} (d_t^{(m)} \nabla \Theta^{(n+1)m}_\xi, \nabla \psi_h) = (R^{(n+\theta)m}_h, \psi_h), \forall \psi_h \in M_h. \tag{3.36}
\]

Setting \(\mathbf{v}_h = d_t^{(m)} \Theta^{(n+1)m}_u\) in \((3.34)\), \(\varphi_h = \Theta^{(n+1)m}_\xi\) after applying the difference operator \(d_t^{(m)}\) to the equation in \((3.35)\), applying the summation operator \(m \Delta t \sum_{n=0}^{l}\) to both
sides, and taking \( \psi_h = \Phi_p^{(n+1)m} = \Delta_p^{(n+1)m} - \Psi_p^{(n+1)m} + \kappa_1 \Theta_\xi^{(n+1)m} + \kappa_2 \Theta_\eta^{(n+\theta)m} \) in (3.36), after applying the summation operator \( m\Delta t \sum_{n=0} \) to both sides and adding the resulting equations, we see that (3.33) holds. The proof is complete.

**Theorem 3.6.** Suppose that \( \mathbf{u} \in L^\infty(0,T; \mathbf{H}_x^1(\Omega)), \xi \in L^\infty(0,T; L^2(\Omega)), \eta \in L^\infty(0,T; L^2(\Omega)) \cap H^1(0,T; H^1(\Omega)'), p \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \) are the solution of the problem (2.3)-(2.8), and let \( \{ (\mathbf{u}_h^{(n+1)m}, \xi_h^{(n+1)m}, \eta_h^{(n+\theta)m}) \}_{n \geq 0, \theta = 0, 1} \) be the solution of the multirate iterative scheme (3.6)-(3.9), then we have

\[
\max_{0 \leq n \leq l} \| \varepsilon(\Theta^{(n+1)m}) \|_{L^2(\Omega)} + \sqrt{k_2} \| \Theta^{(n+\theta)m} \|_{L^2(\Omega)} + \sqrt{k_3} \| \Theta^{(n+1)m} \|_{L^2(\Omega)}
\]

\[
+ m\Delta t \sum_{n=0}^l \frac{K}{\mu_f} \| \nabla \Phi_p^{(n+1)m} \|_{L^2(\Omega)}^2 \leq C_1(T) m\Delta t + C_2(T) m h^2
\]

provided that \( \Delta t = O(h^2) \) when \( \theta = 0 \) and \( \Delta t > 0 \) when \( \theta = 1 \), where \( C_1(T) = C \| \eta \|_{L^2((0,T); L^2(\Omega))} + C \| \eta u \|_{L^2((0,T); H^{-1}(\Omega))} \) and \( C_2(T) = C \| \xi \|_{L^2((0,T); H^2(\Omega))} + C \| \xi \|_{L^\infty((0,T); H^2(\Omega))} + C \| \text{div } \mathbf{u} \|_{L^2((0,T); H^2(\Omega))} \).

**Proof.** Using Lemma [3.5] and the fact of \( \Theta_\xi^0 = 0, \Theta_\eta^0 = 0 \), we have

\[
\zeta_h^{l+1} + m \Delta t \sum_{n=0}^l \left[ \frac{K}{\mu_f} (\nabla \Phi_p^{(n+1)m} - \rho_f \mathbf{g}, \nabla \Phi_p^{(n+1)m}) + \frac{\kappa_2 \Delta t}{2} \| d_t \Theta_\eta^{(n+\theta)m} \|_{L^2(\Omega)}^2 \right]
\]

\[
+ m \Delta t \sum_{n=0}^l \frac{m \Delta t}{2} (\mu \| d_t^{(m)} \varepsilon(\Theta^{(n+1)m}) \|_{L^2(\Omega)}^2 + \kappa_3 \| d_t^{(m)} \Theta_\xi^{(n+1)m} \|_{L^2(\Omega)}^2) = E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7.
\]

Next, we estimate each term on the right-hand side of (3.38). To bound \( E_1 \), we recall the Korn’s inequality:

\[
\| \text{div } \Theta^{(n+1)m} \|_{L^2(\Omega)}^2 \leq C \| \varepsilon(\Theta^{(n+1)m}) \|_{L^2(\Omega)}^2.
\]

Using the Cauchy-Schwarz inequality and (3.39), we obtain

\[
E_1 \leq \frac{1}{2} \| \Lambda_\xi^{(l+1)m} \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \text{div } \Lambda_\xi^{(l+1)m} \|_{L^2(\Omega)}^2 + \frac{1}{2} \Delta t \sum_{n=1}^l \| d_t \Lambda_\xi^{(n+1)m} \|_{L^2(\Omega)}^2 + C \| \varepsilon(\Theta^{(n+1)m}) \|_{L^2(\Omega)}^2 + \| \text{div } d_t^{(m)} \Lambda_\xi^{(n+1)m} \|_{L^2(\Omega)}^2 + \| \Theta_\xi^{(n+1)m} \|_{L^2(\Omega)}^2.
\]

Using the Cauchy-Schwarz inequality and Young inequality, we get

\[
E_2 \leq \frac{k_3}{2} \Delta t \sum_{n=0}^l \left( \| d_t^{(m)} \Lambda_\xi^{(n+1)m} \|_{L^2(\Omega)}^2 + \| \Theta_\xi^{(n+1)m} \|_{L^2(\Omega)}^2 \right).
\]
Using the fact that
\[ \| R_h^{(n+\theta)m} \|_{H^{-1}(\Omega)}^2 \leq \frac{m \Delta t}{3} \int_{t_{nm}}^{t_{(n+\theta)m}} \| \eta t \|_{H^{-1}(\Omega)} dt, \]
we can bound \( E_3 \) as follows:
\[
E_3 \leq m \Delta t \sum_{n=0}^{l} \| R_h^{(n+\theta)m} \|_{H^{-1}(\Omega)} \| \nabla \Phi_p^{(n+1)m} \|_{L^2(\Omega)}
\leq m \Delta t \sum_{n=0}^{l} \left( \frac{K}{4m} \| \nabla \Phi_p^{(n+1)m} \|_{L^2(\Omega)}^2 + \frac{\mu f_m \Delta t}{3K_1} \| \eta t \|_{L^2((t_{nm}, t_{(n+1)m}); H^{-1}(\Omega)))} \right). \tag{3.42}
\]

When \( \theta = 0 \), using the summation by parts formula and \( d_t^{(m)} \eta_h(t_0) = 0 \) to bound \( E_4 \), we have
\[
E_4 = k_1 m \Delta t (d_t^{(m)} \eta(t_{(l+1)m}), \Theta^{(l+1)m}) - k_1 (m \Delta t)^2 \sum_{n=1}^{l} (d_t^{(m)} \eta(t_{nm}), d_t^{(m)} \Theta^{(n+1)m}). \tag{3.43}
\]
Using \( (3.43) \), the Cauchy-Schwarz inequality and the Young inequality, we get
\[
\begin{align*}
\kappa_1 m \Delta t (d_t^{(m)} \eta(t_{(l+1)m}), \Theta^{(l+1)m}) & \leq k_1 m \Delta t \| d_t^{(m)} \eta(t_{(l+1)m}) \|_{L^2(\Omega)} \| \Theta^{(l+1)m} \|_{L^2(\Omega)} \\
& \leq \frac{C \kappa_1 \mu}{\beta_1^2} m(\Delta t)^2 \| \eta t \|_{L^2((t_{nm}, t_{(n+1)m}); L^2(\Omega)))} + \frac{C \kappa_1}{\beta_1^2} m(\Delta t)^2 \| \Lambda^{(l+1)m} \|_{L^2(\Omega)} \\
& + \frac{\kappa_1 m \mu}{4} \| \varepsilon (\Theta^{(l+1)m}) \|_{L^2(\Omega)}^2, \tag{3.44}
\end{align*}
\]
\[
\begin{align*}
\sum_{n=1}^{l} (d_t^{(m)} \eta(t_{nm}), d_t^{(m)} \Theta^{(n+1)m}) & \leq \sum_{n=1}^{l} \| d_t^{(m)} \eta(t_{nm}) \|_{L^2(\Omega)} \| d_t^{(m)} \Theta^{(n+1)m} \|_{L^2(\Omega)} \\
& \leq \sum_{n=1}^{l} \left( \frac{\mu}{4} \| d_t^{(m)} \varepsilon (\Theta^{(n+1)m}) \|_{L^2(\Omega)}^2 + \frac{C}{\beta_1^2} \| d_t^{(m)} \Lambda^{(n+1)m} \|_{L^2(\Omega)}^2 \right) \\
& + \frac{C \mu}{\beta_1^2} \| \eta t \|_{L^2((0, T); L^2(\Omega)))}^2. \tag{3.45}
\end{align*}
\]

We can bound \( E_5 \) as follows:
\[
E_5 = m \Delta t \sum_{n=0}^{l} (d_t^{(m)} \Theta^{(n+\theta)m}, \Psi^{(n+1)m} - \Lambda^{(n+1)m}) \tag{3.46}
\leq m \Delta t \sum_{n=1}^{l} \sum_{k=1}^{m} \| d_t^{(m)} \Theta^{(n+\theta)m} \|_{L^2(\Omega)}^2 + \frac{m \Delta t}{2} \sum_{n=0}^{l} (\| \Psi^{(n+1)m} \|_{L^2(\Omega)}^2 + \| \Lambda^{(n+1)m} \|_{L^2(\Omega)}^2).
\]

We can bound \( E_6 \) as follows:
\[
E_6 = (m \Delta t)^2 \sum_{n=0}^{l} \frac{K k_1}{\mu f} (d_t^{(m)} \Lambda^{(n+1)m}, \nabla \Phi_p^{(n+1)m}) \tag{3.47}
\]
\[ \leq \frac{1}{2} (m \Delta t)^2 \sum_{n=0}^{l} \frac{K \kappa_1}{\mu_f} \| d_t^{(m)} \nabla \Lambda_{\xi}^{(n+1)m} \|_{L^2(\Omega)}^2 + \frac{1}{2} (m \Delta t)^2 \sum_{n=0}^{l} \frac{K \kappa_1}{\mu_f} (\| \nabla \Phi_{p}^{(n+1)m} \|_{L^2(\Omega)}^2). \]

When \( \theta = 0 \), we can bound \( E_7 \) as follows:

\[ \sum_{n=0}^{l} (d_t^{(m)} \nabla \Theta_{\xi}^{(n+1)m}, \nabla \Phi_{p}^{(n+1)m}) \]

\[ \leq c h^{-1} \sum_{n=0}^{l} \| d_t^{(m)} \Theta_{\xi}^{(n+1)m} \|_{L^2(\Omega)} \| \nabla \Phi_{p}^{(n+1)m} \|_{L^2(\Omega)} \]

\[ \leq \sum_{n=0}^{l} \left( \frac{\mu^2 \kappa_1 \Delta t}{\beta_1^2 h^2} \| d_t^{(m)} \varepsilon(\Theta_u^{(n+1)m}) \|_{L^2(\Omega)}^2 + \frac{\kappa_1 \Delta t}{h^2 \beta_1^2} \| d_t^{(m)} \Lambda_{\xi}^{(n+1)m} \|_{L^2(\Omega)}^2 \right) \]

\[ + \sum_{n=0}^{l} \frac{1}{4 \Delta t \kappa_1} \| \nabla \Phi_{p}^{(n+1)m} \|_{L^2(\Omega)}^2. \]  \hspace{1cm} (3.48)

Substituting (3.37)-(3.48) into (3.38), we obtain

\[ \mu \| \varepsilon(\Theta_u^{(l+1)m}) \|_{L^2(\Omega)}^2 + \kappa_2 \| \Theta_{\eta}^{(l+\theta)m} \|_{L^2(\Omega)}^2 + \kappa_3 \| \Theta_{\xi}^{(l+1)m} \|_{L^2(\Omega)}^2 \]

\[ + m \Delta t \sum_{n=0}^{l} \frac{K}{\mu_f} \| \nabla \Phi_{p}^{(n+1)m} \|_{L^2(\Omega)}^2 \]

\[ \leq m \left( \frac{1}{2} + \frac{C \kappa_1}{\beta_1^2} \right) \| \Lambda_{\xi}^{(l+1)m} \|_{L^2(\Omega)}^2 + \frac{m^2}{\mu} \Delta t \sum_{n=0}^{l} \| d_t^{(m)} \Lambda_{\xi}^{(n+1)m} \|_{L^2(\Omega)}^2 \]

\[ + \frac{1}{2} m \Delta t \sum_{n=0}^{l} \| \text{div} d_t^{(m)} \Lambda_{\eta}^{(n+1)m} \|_{L^2(\Omega)}^2 + (1 - \theta) \frac{m^2 \mu^2 \kappa_2^2 K \Delta t}{\mu_f \beta_1^2 h^2} \| \varepsilon(\Theta_u^{(l+1)m}) \|_{L^2(\Omega)}^2 \]

\[ + (\Delta t)^2 \left( \frac{\mu m^2}{\beta_1} + \frac{\kappa_1 m^2}{\beta_1^2} \right) \| \eta_t \|_{L^2((0,T);L^2(\Omega))}^2 + \frac{\mu m \Delta t^2}{3 K_1} \| \eta_t \|_{L^2((0,T);H^1(\Omega))}^2 \]

\[ + \frac{2}{2} \Delta t \sum_{n=1}^{l} \| \Psi_{(n+1)m} \|_{L^2(\Omega)}^2 + \frac{m}{2} \Delta t \sum_{n=1}^{l} \| \Lambda_{p}^{(n+1)m} \|_{L^2(\Omega)}^2 \]

\[ \leq C m (\Delta t)^2 \left( \| \eta_t \|_{L^2((0,T);L^2(\Omega))}^2 + \| \eta_t \|_{L^2((0,T);H^1(\Omega))}^2 \right) \]

\[ + C m h^4 \left[ \| \xi_t \|_{L^2((0,T);H^2(\Omega))}^2 + \| \xi_t \|_{L^2((0,T);H^2(\Omega))}^2 + \| \text{div} \mathbf{u}_t \|_{L^2((0,T);H^2(\Omega))}^2 \right]. \]  \hspace{1cm} (3.49)

In (3.49), if \( \frac{m \mu \kappa_2^2 K \Delta t}{\mu_f \beta_1^2 h^2} < 1 \), we can bound the term of \( \| \varepsilon(\Theta_u^{(l+1)m}) \|_{L^2(\Omega)}^2 \); so we need the condition: \( \Delta t \leq \frac{\mu_f \beta_1^2 h^2}{m \mu \kappa_2^2 K} \). Thus, using (3.28), (3.29), (3.30), and (3.49), if \( \theta = 0 \), \( \Delta t \leq \frac{\mu_f \beta_1^2 h^2}{m \mu \kappa_2^2 K} \); when \( \theta = 1 \), \( \Delta t > 0 \), we see that (3.37) holds. The proof is finished. \( \square \)

**Theorem 3.7.** Suppose that \( (\mathbf{u}, \xi, \eta) \) and \( (\mathbf{u}_h^{(n+1)m}, \xi_h^{(n+1)m}, \eta_h^{(n+\theta)m}) \) are the solutions of the problem (2.3)-(2.8) and of the problem (3.6)-(3.9), respectively, then we have the following error estimates:

\[ \max_{0 \leq n \leq N} \| \varepsilon(\mathbf{u}(t_{nm}) - \mathbf{u}_h^{nm}) \|_{L^2(\Omega)} + \sqrt{\kappa_2} \| \eta(t_{nm}) - \eta_h^{nm} \|_{L^2(\Omega)} \]
provided that $\Delta t = O(h^2)$ when $\theta = 0$; $\Delta t > 0$ when $\theta = 1$. $C_1(T) : = C_1(T)$, $C_2(T) : = C_2(T) + \|\eta\|_{L^\infty((0,T);H^2(\Omega))} + \|\nabla u\|_{L^\infty((0,T);H^2(\Omega))} + \|\xi\|_{L^\infty((0,T);H^2(\Omega))}$.

\textit{Proof.} The assertions follow easily from a straightforward use of the triangle inequality on

$$u(t_{nm}) - u_h(t_{nm}) = \Lambda_u^{nm} + \Theta_u^{nm}, \quad \xi(t_{nm}) - \xi_h = \Lambda_\xi^{nm} + \Theta_\xi^{nm},$$

$$\eta(t_{nm}) - \eta_h^{nm} = \Lambda_\eta^{nm} + \Theta_\eta^{nm}, \quad p(t_{nm+k}) - p_h^{nm+k} = \Lambda_p^{nm+k} + \Theta_p^{nm+k},$$

and applying \((3.29)-(3.30)\) and Theorem \(3.6\) we imply that \((3.50)\) and \((3.51)\) hold. The proof is complete. \hfill \Box

4. Numerical tests

In this section, we will present three two-dimensional numerical experiments to validate theoretical results for the proposed numerical methods, to numerically examine the performances of the approach and methods as well as to compare them with existing methods in the literature on two benchmark problems. The numerical examples show that our approach and numerical methods have a build-in mechanism to prevent the "locking" phenomenon. Also, we denote CR by the shorthand notation of convergence rates.

**Test 1.** Let $\Omega = [0,1] \times [0,1], \Gamma_1 = \{(x,0); 0 \leq x \leq 1\}, \Gamma_2 = \{(1,y); 0 \leq y \leq 1\}, \Gamma_3 = \{(x,1); 0 \leq x \leq 1\}, \Gamma_4 = \{(0,y); 0 \leq y \leq 1\},$ and $T = 1, \Delta t = 1e - 6$. We consider problem \((1.1)-(1.7)\) with the following source functions:

$$\mathbf{f} = -(\lambda + \mu)T + \alpha \cos(x+y)e^t(1,1)^T,$$

$$\phi = \left(c_0 + \frac{2K}{\mu_f}\right) \sin(x+y)e^t + \alpha(x+y),$$

and the following boundary and initial conditions:

$$p = \sin(x+y)e^t \quad \text{on} \quad \partial \Omega_T,$$

$$u_1 = \frac{1}{2}x^2t \quad \text{on} \quad \Gamma_j \times (0,T), \quad j = 2, 4,$$

$$u_2 = \frac{1}{2}y^2t \quad \text{on} \quad \Gamma_j \times (0,T), \quad j = 1, 3,$$

$$\sigma(u)n - \alpha p l n = f_1 \quad \text{on} \quad \partial \Omega_T,$$
\[ u(x, 0) = 0, \quad p(x, 0) = \sin(x + y) \quad \text{in} \quad \Omega, \]

where

\[ f_1 = \mu(x n_1, y n_2)^T t + \lambda(x + y)(n_1, n_2)^T t - \alpha \sin(x + y)(n_1, n_2)^T e^t, \]

It is easy to check that the exact solution for this problem is

\[ u = \frac{t}{2} (x^2, y^2)^T, \quad p = \sin(x + y)e^t. \]

Table 1: Physical parameters

| Parameter | Description                     | Value     |
|-----------|---------------------------------|-----------|
| \(\lambda\) | Lamé constant                  | 1.43e-4   |
| \(\mu\)   | Lamé constant                  | 3.57e-5   |
| \(c_0\)   | Constrained specific storage coefficient | 1e-5     |
| \(\alpha\) | Biot-Willis constant          | 0.83      |
| \(K\)     | Permeability tensor            | \((1e - 5)I\) |
| \(E\)     | Young’s modulus               | 1e-4      |
| \(\nu\)   | Poisson ratio                 | 0.4       |

Table 2: The errors and convergence rates of \(u_h^n\) when \(m = 5\)

| \(h\)  | \(\|e_u\|_{L^\infty(L^2)}\) | CR | \(\|e_u\|_{L^\infty(H^1)}\) | CR |
|--------|-----------------------------|----|-----------------------------|----|
| 0.18   | 0.00106336                  |    | 0.0666679                   |    |
| 0.09   | 9.00707e-5                  | 3.5614 | 0.0116539 | 2.5162 |
| 0.045  | 7.79098e-6                  | 3.5312 | 0.00204438 | 2.5111 |
| 0.0225 | 6.87406e-7                  | 3.5026 | 0.000359823 | 2.5063 |

Table 3: The errors and convergence rates of \(u_h^n\) when \(m = 1\)

| \(h\)  | \(\|e_u\|_{L^\infty(L^2)}\) | CR | \(\|e_u\|_{L^\infty(H^1)}\) | CR |
|--------|-----------------------------|----|-----------------------------|----|
| 0.18   | 0.00106336                  |    | 0.0666679                   |    |
| 0.09   | 9.00707e-5                  | 3.5614 | 0.0116539 | 2.5162 |
| 0.045  | 7.79083e-6                  | 3.5312 | 0.00204438 | 2.5111 |
| 0.0225 | 6.85585e-7                  | 3.5064 | 0.000359823 | 2.5063 |
Table 2 and Table 3 display the errors of displacement $u^n_h$ in $L^\infty(0, T; L^2(\Omega))$-norm and $L^\infty(0, T; H^1(\Omega))$-norm and the convergence rates with respect to $h$ at the terminal time $T$ when $m = 5$ and $m = 1$, respectively, one can see that the convergence rates is almost identical. However, the multirate iterative scheme greatly reduces the computational cost, for example, when $m = 5$, $h = 0.18$, the execution time of multirate iterative scheme with multiphysics finite element method is $t = 6.333$ s; when $m = 1$, $h = 0.18$, the execution time is $t = 17.846$ s.

If the parameters are same, when $m = 5$, $h = 0.18$, the execution time of multirate iterative scheme based on multiphysics discontinuous Galerkin method is $t = 13.168$ s; when $m = 1$, $h = 0.18$, the execution time is $t = 30.302$ s. So, we can conclude that the multirate iterative scheme with multiphysics finite element method save a huge computation cost.

Figure 1 display the computed pressure $p^n_h$ at $T = 1$ when $m = 5$, from the above two figures, we see that our numerical method has no "locking" phenomenon.

**Test 2.** In this test, we consider so-called Barry-Mercer’s problem, and we set $\Omega = [0, 1] \times [0, 1]$ and the boundary segments $\Gamma_j, j = 1, 2, 3, 4$ are same as Test 1, and $\Delta t = 1e^{-5}, T = 1, f \equiv 0$ and $\phi \equiv 0$, and we take the following boundary conditions:

\begin{align*}
p &= 0 & \text{on} & \Gamma_j \times (0, T), & j &= 2, 3, 4, \\
p &= p_2 & \text{on} & \Gamma_j \times (0, T), & j &= 1, \\
u_1 &= 0 & \text{on} & \Gamma_j \times (0, T), & j &= 2, 4, \\
u_2 &= 0 & \text{on} & \Gamma_j \times (0, T), & j &= 1, 3, \\
\sigma n - \alpha p I n &= \vec{f}_1 := (0, \alpha p)^T & \text{in} & \partial \Omega_T,
\end{align*}
where

\[ p_2(x, t) = \begin{cases} 
\sin t, & (x, t) \in [0.2, 0.8] \times (0, T), \\
0, & \text{others.}
\end{cases} \]

Table 4: Physical parameters

| Parameter | Description                              | Value     |
|-----------|------------------------------------------|-----------|
| \( \lambda \) | Lamé constant                            | 0.0044    |
| \( \mu \)  | Lamé constant                            | 0.0158    |
| \( c_0 \)  | Constrained specific storage coefficient | 0.9       |
| \( \alpha \) | Biot-Willis constant                      | 0.31      |
| \( K \)   | Permeability tensor                       | \((3e-6)I\) |
| \( E \)   | Young’s modulus                          | 3.5e-2    |
| \( \nu \) | Poisson ratio                             | 0.11      |

Table 5: The errors and convergence rates of \( u_h^n \) when \( m = 5 \)

| \( h \) | \( \|e_u\|_{L^\infty(L^2)} \) | CR | \( \|e_u\|_{L^\infty(H^1)} \) | CR |
|---------|-------------------------------|----|-----------------------------|----|
| 0.18    | 4.7027e-8                     | 4.66882e-7 |                      |    |
| 0.09    | 1.67091e-8                    | 1.4929  | 2.49994e-7    | 0.9012 |
| 0.045   | 5.6643e-9                     | 1.5607  | 1.16283e-7    | 1.1043 |
| 0.0225  | 1.44007e-9                    | 1.9758  | 4.8909e-8     | 1.2495 |

Table 6: The errors and convergence rates of \( u_h^n \) when \( m = 1 \)

| \( h \) | \( \|e_u\|_{L^\infty(L^2)} \) | CR | \( \|e_u\|_{L^\infty(H^1)} \) | CR |
|---------|-------------------------------|----|-----------------------------|----|
| 0.18    | 1.56757e-8                    | 1.55627e-7 |                      |    |
| 0.09    | 5.66969e-9                    | 1.4929  | 8.33314e-8   | 0.9012 |
| 0.045   | 1.8881e-9                     | 1.5607  | 3.8761e-8     | 1.1043 |
| 0.0225  | 4.80022e-10                   | 1.9758  | 1.6303e-8     | 1.2495 |
From Table 5 and Table 6, we know that the errors of displacement $u_h^n$ when $m = 1$ are better than ones of $u_h^n$ when $m = 5$, and the convergence rates are identical. However, the execution times of multirate iterative scheme based on multiphysics finite element method are $t = 208.977s$ for the case of $m = 5$ and $t = 291.067s$ for the case of $m = 1$ (when $h = 0.18$). Also, if the parameters are same, the execution time of multirate iterative scheme based on multiphysics discontinuous Galerkin method is $t = 300.597s$ when $m = 5$, $h = 0.18$, and the execution time is $t = 372.01s$ when $m = 1$, $h = 0.18$. So, we can conclude that the multirate iterative scheme greatly reduces the computational cost.

Figure 2 and Figure 3 display the computed displacement $u_h^n$ and pressure $p_h^n$ at $T = 1$ when $m = 5$, respectively. From the above two figures, we see that there is no "locking" phenomenon.

**Test 3.** Again, we consider problem with $\Omega = [0, 1] \times [0, 1]$. Let $\Gamma_j$ be same as in **Test 1**, and $\Delta t = 1e - 5, T = 1$. There is no source, that is, $f \equiv 0$ and $\phi \equiv 0$. The
boundary conditions are taken as
\[-\frac{K}{\mu_f} (\nabla p - \rho_f g) \cdot n = 0 \quad \text{on} \quad \partial \Omega,\]
\[u = 0 \quad \text{on} \quad \Gamma_3 \times (0, T),\]
\[\sigma n - \alpha p I n = f_i \quad \text{on} \quad \Gamma_j \times (0, T), \quad j = 1, 2, 3,\]

where \(f_i = (f_i^1, f_i^2)^T\) and

\[f_i^1 \equiv 0 \quad \text{on} \quad \partial \Omega_T,\]

\[f_i^2 = \begin{cases} 0, & (x, t) \in \Gamma_j \times (0, T), j = 1, 2, 4, \\ -1, & (x, t) \in \Gamma_3 \times (0, T). \end{cases} \quad (4.1)\]

The zero initial conditions are assigned for both \(u\) and \(p\) in this test.

### Table 7: Physical parameters

| Parameter | Description                        | Value      |
|-----------|------------------------------------|------------|
| \(\lambda\) | Lamé constant                     | 142857.14  |
| \(\mu\)   | Lamé constant                     | 35714.29   |
| \(c_0\)   | Constrained specific storage coeff | 0.01       |
| \(\alpha\) | Biot-Willis constant               | 0.93       |
| \(K\)     | Permeability tensor                | (1e-1)I    |
| \(E\)     | Young’s modulus                    | 1e5        |
| \(\nu\)   | Poisson ratio                      | 0.4        |

### Table 8: The errors and convergence rates of \(p^h_n\) when \(m = 5\)

| \(h\)  | \(\|e_p\|_{L^2(L^2)}\) | CR    | \(\|e_p\|_{L^2(H^1)}\) | CR    |
|--------|------------------------|-------|------------------------|-------|
| 0.18   | 1.11026e-9             |       | 3.38447e-8             |       |
| 0.09   | 3.286e-10              | 1.7565| 2.17254e-8             | 0.6395|
| 0.045  | 1.04069e-10            | 1.6588| 1.51649e-8             | 0.5186|
| 0.0225 | 2.38175e-11            | 2.1274| 6.54142e-9             | 1.2131|
Table 9: The errors and convergence rates of $p^n_h$ when $m = 1$

| $h$   | $\|e_p\|_{L^2(L^2)}$ | CR  | $\|e_p\|_{L^2(H^1)}$ | CR  |
|-------|------------------------|-----|-----------------------|-----|
| 0.18  | 6.38714e-10            | 1.94327e-8 |
| 0.09  | 1.91689e-10            | 1.7364  | 9.9166e-9            | 0.9706 |
| 0.045 | 6.1088e-11             | 1.6498  | 5.00331e-9           | 0.9870 |
| 0.0225| 1.35621e-11            | 2.1713  | 3.23977e-9           | 0.6270 |

Figure 4: Computed displacement $u^n_h$ at $T = 1$ when $m = 5$.

Figure 5: Computed pressure $p^n_h$ at $T = 1$ when $m = 5$.

From Table 8 and Table 9, we know that the errors of $p^n_h$ in $L^2(0,T;L^2(\Omega))$-norm and $L^2(0,T;H^1(\Omega))$-norm when $m = 1$ are better than ones of $m = 5$. However, the convergence rates are almost equal for the cases of $m = 5$ and $m = 1$. When $m = 5$, $h = 0.18$, the execution time of multirate iterative scheme based on multiphysics finite element method is $t = 215.692s$; when $m = 1$, $h = 0.18$, the execution time is
\[ t = 282.331 \text{s}, \text{ if the parameters are same, when } m = 5, \ h = 0.18, \text{ the execution time of multirate iterative scheme based on multiphysics discontinuous Galerkin method is } t = 342.197 \text{s}; \text{ when } m = 1, \ h = 0.18, \text{ the execution time is } t = 421.611 \text{s}. \] So, we can conclude that the multirate iterative scheme greatly reduces the computational cost. Also, from Figure 4 and Figure 5, we see that the numerical method in this paper has no "locking" phenomenon.

5. Conclusion

In this paper, we propose a multirate iterative scheme with multiphysics finite element method for a poroelasticity model. And we prove that the multirate iterative scheme is stable and the numerical solution satisfies some energy conservation laws, and it doesn’t reduce the precision of numerical solution and greatly reduces the computational cost. In the future work, we will apply the proposed approaches to more complex practical problems and nonlinear poroelasticity model.

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