The comparison principle for viscosity solutions of fully nonlinear subelliptic equations in Carnot groups

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Abstract. For any Carnot group $G$ and a bounded domain $\Omega \subset G$, we prove that viscosity solutions in $C(\overline{\Omega})$ of the fully nonlinear subelliptic equation $F(u, \nabla_h u, \nabla^2_h u) = 0$ are unique when $F \in C(R \times R^m \times S(m))$ satisfies (i) $F$ is degenerate subelliptic and decreasing in $u$ or (ii) $F$ is uniformly subelliptic and nonincreasing in $u$. This extends Jensen’s uniqueness theorem from the Euclidean space to the sub-Riemannian setting of the Carnot group.

§1. Introduction

The notion of viscosity solutions of fully nonlinear 2nd order degenerate elliptic equation:

$$F(x, u(x), \nabla u(x), \nabla^2 u(x)) = 0,$$

was developed by Crandall-Lions [CL] and Evans [E1,2] in 1980’s. This idea, together with Jensen’s celebrated uniqueness theorem [J1], provides a very satisfactory theory on existence, uniqueness, and compactness theorem of weak solutions of (1.1). The theory of viscosity solutions has been very powerful in many applications, and we refer to the user’s guide [CIL] by Crandall-Ishii-Lions for many such applications.

In recent years there has been an explosion of interest in the study of analysis on sub-Riemannian, or Carnot-Carathéodory spaces. The corresponding developments in the theory of partial differential equations of subelliptic type have prompted people to consider fully nonlinear equations in Carnot groups. For examples, motivated by the very important work of Jensen [J2] on absolute minimizing Lipschitz extensions (or ALMEs, a notion first introduced by Aronsson [A]) and viscosity solutions to the $\infty$-laplacian equation in the Euclidean space, Bieske [B], Bieske-Capogna [BC], and Wang [W1] have studied absolute minimizing horizontal Lipschitz extensions and viscosity solutions to the $\infty$-sublaplacian equation on Carnot groups. In particular, the notion of viscosity solutions has been extended to fully nonlinear subelliptic equation (see [B]) and the uniqueness of viscosity solution of $\infty$-sublaplacian equation on any Carnot group was established by Wang [W1].

It is well-known (cf. the monographs [CC] by Caffarelli-Cabré and [G] by Gutierrez) that both convexity and the Monge-Ampère equation:

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$$\det(\nabla^2 u) = f,$$
have played crucial roles in the theory of fully nonlinear elliptic equation. Inspired by this, Lu-Manfredi-Stroffolini [LMS] and Danielli-Garofalo-Nhieu [DGN] have introduced and studied various notions of convexity, such as v-convexity and h-convexity, on Carnot groups (see also [BR], [W2], [JM] for some further related results). Moreover, Garofalo-Tournier [GN] and Gutierrez-Montanari [GM] have initiated the study of Monge-Ampère measures and maximum principle of convex functions on Heisenberg groups.

In this paper, we are interested in the comparison principle for viscosity solutions to 2nd order subelliptic equation which is either uniformly subelliptic, nonincreasing or degenerate subelliptic, decreasing in the sub-Riemannian setting of the Carnot group. In this aspect, we are able to extend Jensen’s uniqueness theorem from the Eucliean space to any Carnot group.

In order to describe our result, we first recall the basic properties of Carnot groups. A simply connected Lie group $G$ is called a Carnot group of step $r \geq 1$, if its Lie algebra $\mathfrak{g}$ admits a vector space decomposition in $r$ layers $\mathfrak{g} = \mathfrak{v}_1 + \mathfrak{v}_2 + \cdots + \mathfrak{v}_r$ such that (i) $\mathfrak{g}$ is stratified, i.e., $[\mathfrak{v}_1, \mathfrak{v}_j] = \mathfrak{v}_{j+1}, j = 1, \cdots, r - 1$, and (ii) $\mathfrak{g}$ is $r$-nilpotent, i.e. $[\mathfrak{v}_j, \mathfrak{v}_r] = 0, j = 1, \cdots, r$. We call $\mathfrak{v}_1$ the horizontal layer and $\mathfrak{v}_j, j = 2, \cdots, r$ the vertical layers. We choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that $\mathfrak{v}_j$’s are mutually orthogonal for $1 \leq j \leq r$. Let \( \{X_{j,1}, \cdots, X_{j,m_j}\} \) denote a fixed orthonormal basis of $\mathfrak{v}_j$ for $1 \leq j \leq r$, where $m_j = \dim(\mathfrak{v}_j)$ is the dimension of $\mathfrak{v}_j$. From now on, we also denote $m = \dim(\mathfrak{v}_1)$ as the dimension of the horizontal layer and set $X_i = X_{1,i}$ for $1 \leq i \leq m$. It is well-known (see [FS]) that the exponential map $\exp: \mathfrak{g} \cong \mathbb{R}^m \to G$ is a global diffeomorphism and yields an exponential coordinate system on $G$, with $n = \sum_{i=1}^r m_i$ the topological dimension of $G$. More precisely, any $p \in G$ has a coordinate $(p_1, \cdots, p_m, p_{2,1}, \cdots, p_{2,m_2}, \cdots, p_{r,1}, \cdots, p_{r,m_r})$ such that \[ p = \exp(\xi_1(p) + \cdots \xi_r(p)), \quad \text{with} \quad \xi_1(p) = \sum_{i=1}^m p_i X_i, \ \xi_i(p) = \sum_{j=1}^{m_i} p_{i,j} X_{i,j}, 2 \leq i \leq r. \]

The exponential map can induce a homogeneous pseudo-norm $N_G$ on $G$ in the following way (see [FS]). \[
N_G(p) := \left( \sum_{i=1}^r |\xi_i(p)|^{2i/(2i-1)} \right)^{1/(2i-1)}, \quad \text{if} \quad p = \exp(\xi_1(p) + \cdots \xi_r(p)), \] (1.3)

where $|\xi_1(p)| = \left( \sum_{i=1}^m p_i^2 \right)^{1/2}$, and $|\xi_i(p)| = \left( \sum_{j=1}^{m_i} p_{i,j}^2 \right)^{1/2} (2 \leq i \leq r)$. Moreover, $N_G$ yields a pseudo-distance on $G$ as follows. \[
d_G(p, q) := N_G(p^{-1} \cdot q), \quad \forall p, q \in G, \] (1.4)

where $\cdot$ is the group multiplication of $G$ and $p^{-1}$ is the inverse of $p$. It is easy to see that $d_G$ satisfies the invariance property \[
d_G(z \cdot x, z \cdot y) = d_G(x, y), \quad \forall x, y, z \in G, \] (1.5)
and is of homogeneous of degree one, i.e.,

\[ d_G(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d_G(p,q), \quad \forall \lambda > 0, \quad \forall p, q \in G \]

(1.6)

where \( \delta_\lambda(p) = \lambda \xi_1(p) + \sum_{i=2}^{r} \lambda^i \xi_i(p) \) is the non-isotropic dilations on \( G \).

Throughout this paper, we fix some notations. For \( l \geq 1 \), denote \( S(l) \) as the set of \( l \times l \) symmetric matrices. For \( M, N \in S(m) \), we say \( M \geq N \) if \( M - N \in S(m) \) is a positive semidefinite matrix, and let \( \text{trace}(M) \) denote the trace of \( M \) for \( M \in S(m) \). For \( u : G \to \mathbb{R} \), let \( \nabla u, \nabla^2 u \) denote the Euclidean gradient, hessian of \( u \) respectively, and \( \nabla_h u := (X_1 u, \ldots, X_m u) \), \( \nabla^2_h u := (\frac{X_i X_i + X_j X_j}{2})_{1 \leq i, j \leq m} \) denote the horizontal gradient, horizontal hessian of \( u \) respectively. For a given domain \( \Omega \subset G \), denote \( C(\Omega) \) as the set of continuous functions on \( \Omega \), \( C^2(\Omega) = \{ u \in C(\Omega) : \nabla u, \nabla^2 u \in C(\Omega) \} \), and \( \Gamma^2(\Omega) = \{ u \in C(\Omega) : \nabla_h u, \nabla^2_h u \in C(\Omega) \} \). A fully nonlinear partial horizontal-differential operator \( \mathcal{F}[:] \) on \( \Omega \) is defined by

\[ \mathcal{F}[\phi](x) = F(\phi(x), \nabla_h \phi(x), \nabla^2_h \phi(x)), \quad \forall x \in \Omega, \quad \forall \phi \in \Gamma^2(\Omega), \]

(1.7)

where \( F \in C(R \times R^m \times S(m)) \). We now give the definition of subellipticity and nondecreasing property of \( \mathcal{F} \).

**Definition 1.1.** The operator \( \mathcal{F}[:] \) is degenerate subelliptic if

\[ F(r, p, M) \leq F(r, p, N), \quad \text{for all } M, N \in S(m) \text{ with } M \leq N \text{ and } (r, p) \in R \times R^m. \]

(1.8)

The operator \( \mathcal{F}[:] \) is uniformly subelliptic if there exist constants \( \alpha_1, \alpha_2 > 0 \) such that

\[ F(r, p, M) - F(r, q, N) \geq \alpha_1 \text{trace}(M - N) - \alpha_2 |p - q| \]

(1.9)

for all \( M, N \in S(m) \) with \( M \geq N \) and \( (r, p, q) \in R \times R^m \times R^m \).

**Definition 1.2.** The operator \( \mathcal{F}[:] \) is nonincreasing if

\[ F(r, p, M) \leq F(s, p, M), \quad \text{for all } r \geq s, \text{ and } (p, M) \in R^m \times S(m). \]

(1.10)

The operator \( \mathcal{F}[:] \) is decreasing if there is a constant \( \alpha_3 > 0 \) such that

\[ F(r, p, M) - F(s, p, M) \leq \alpha_3 (s - t) \text{ for all } r \geq s, \text{ and } (p, M) \in R^m \times S(m). \]

(1.11)

We shall now recall the definition of viscosity solution of fully nonlinear degenerate subelliptic equation (1.7), which was introduced by Crandall-Lions (see [CL] and the user’s guides [CIL]) for fully nonlinear elliptic equations.
**Definition 1.3.** Assume that $F$ is a degenerate subelliptic operator. $w \in C(\Omega)$ is a viscosity subsolution of (1.7) if for any $(x_0, \phi) \in \Omega \times C^2(\Omega)$ such that

$$0 = \phi(x_0) - w(x_0) \geq \phi(x) - w(x), \forall x \in \Omega$$

we have

$$F(\phi(x_0), \nabla_h \phi(x_0), \nabla^2_h \phi(x_0)) \geq 0.$$  \hfill (1.12)

$w \in C(\Omega)$ is a viscosity supersolution of (1.7) if $-w$ is a viscosity subsolution of (1.7). $w \in C(\Omega)$ is a viscosity solution of (1.7) if it is both a viscosity subsolution and a viscosity supersolution.

**Remark 1.4.** (i) It is well-known that there is an equivalent formulation of viscosity solution of (1.7) using elliptic jets (see [CIL] or [J1]§2). (ii) From the sub-Riemannian point of views, it also seems natural to define an intrinsic version of viscosity solution of (1.7) by allowing the test functions $\phi \in \Gamma^2(\Omega)$ in Definition 1.3. However, since $C^2(\Omega) \subset \Gamma^2(\Omega)$, the intrinsic version of viscosity solution of (1.7) is stronger than the version given by Definition 1.3. (iii) This intrinsic version of viscosity solution of (1.7) has been previously formulated by Bieske [B] (see also Manfredi [M]) in connections with $\infty$-sublaplacian equations on Heisenberg groups, where the notion of subelliptic jets was also introduced.

Now we are ready to state our comparison theorem.

**Theorem A.** Let $G$ be a Carnot group and $\Omega \subset G$ be a bounded domain. Suppose that $u \in C(\bar{\Omega})$ is a viscosity subsolution of (1.7) and $v \in C(\bar{\Omega})$ is a viscosity supersolution of (1.7). If $F$ satisfies either

(i) $F[\cdot]$ is degenerate subelliptic and decreasing,

or

(ii) $F[\cdot]$ is uniformly subelliptic and nonincreasing,

then

$$\sup_{\Omega} (u - v)^+ \leq \sup_{\partial \Omega} (u - v)^+.$$  \hfill (1.13)

We would like to remark that the operator $\bar{F}[\cdot]$ induced by the degenerate subelliptic operator $F[\cdot]$:

$$\bar{F}[w] = \bar{F}(x, w, \nabla w, \nabla^2 w) := F(w, \nabla_h w, \nabla^2_h w)$$  \hfill (1.14)

may not be degenerate elliptic (i.e $\bar{F}$ may not be monotone in its third variable, see [CIL]), and may be dependent of the spatial variable $x$ in an essential way so that the uniqueness theorems by Jensen [J1] and Ishii [I] on viscosity solutions to 2nd order elliptic equations are not applicable here. Therefore theorem A not only provides a comparison principle in
the subelliptic setting of the Carnot group but also makes the comparison principle of [J1]
(see also [I] or [CIL]) available for a considerably larger class of equations in the Euclidean
setting.

We believe that theorem A shall play an important role in the existence of viscosity
of (1.7) by the Perron’s method (see [I]) and plan to study it in a future article. We
would like to mention that Manfredi [M] proved, among other things, theorem A for any
uniformly elliptic, linear subelliptic operator \( F(w) = \sum_{i,j=1}^m a_{ij}(x)X_iX_jw \), with \( (a_{ij}(x)) \in C(G, S(m)) \) a uniformly elliptic matrix.

A direct consequence of Theorem A is the uniqueness theorem of viscosity solutions
of (1.7).

**Corollary B.** Under the same assumptions as Theorem A. There exists at most one
viscosity solution \( u \in C(\Omega) \) of (1.7).

The basic point to prove theorem A is that we can always compare between a classical
subsolution \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \) and a classical strict supersolution \( v \in C^2(\Omega) \cap C(\bar{\Omega}) \) of
(1.7), under the assumption that \( F[\cdot] \) is degenerate subelliptic.

In order to extend this idea to viscosity sub (or super) solutions of (1.7), we first
establish, in Lemma 2.1 below, that under the same conditions of theorem A, any viscosity
supersolution \( v \in C(\bar{\Omega}) \) can be perturbed into a viscosity strict supersolution of (1.7). We
would like to point out that, even in the Euclidean setting of \( \mathbb{R}^n \), Lemma 2.1 seems to be
new and can be used to simplify the proof of Jensen [J1]. Moreover, it seems necessary in
the subelliptic setting, since the counterpart of [J1] Lemma 3.20 is to estimate \( \frac{\text{trace}(\nabla^2 h w)}{|\nabla h w|} \)
from below for a semiconvex function \( w \) on \( G \) and may not be available.

The second ingredient is to approximate viscosity sub (or super) solutions by semi-
convex (or semiconcave) sub (or super) solutions of (1.7). This idea was first introduced
by Jensen in his very important paper [J1] on uniqueness of Lipschitz continuous viscosity
solutions to 2nd order elliptic equations, and Jensen’s original approximation scheme
was further simplified by the sup/inf convolution construction by Jensen-Lions-Souganidis
[JLS] in the Euclidean setting. In the subelliptic setting of the Carnot group, we succeeded,
in an earlier paper [W1] where we proved the uniqueness of viscosity solution to the subel-
liptic \( \infty \)-laplacian equation on any Carnot group \( G \), in extending the sup/inf convolution
construction of [JLS] by employing the smooth gauge pseudo-norm function \( d_G \) and get
the desired approximations. For the reader’s convenience, we review the sup/inf convolu-
tion construction of [W1] in §3 below. We would like to point out that we have used in
a very crucial way that (1.7) is invariant under the group multiplication from left, i.e. if
\( u \in C(G) \) is a viscosity solution to (1.7) then \( u_a(x) = u(a \cdot x), \ x \in G \), is also a viscosity
solution to (1.7) for any \( a \in G \). Once we have semiconvex (or semiconcave) sub (or super)
solutions to (1.7), we can apply both the regularity properties (see Evans-Gariepy [EG])
and Jensen’s maximal principle for semiconvex functions (see [J1]) in our setting.

The paper is written as follows. In §2, we show that any viscosity supersolution of (1.7) given by theorem A can be perturbed into a strict supersolution of (1.7). In §3, we recall the sup/inf convolution construction on a Carnot group $G$, which was carried out in an earlier paper [W1]. In §4, we give a proof of theorem A.

§2 Viscosity strict supersolutions

In this section, we show that any viscosity supersolution given by theorem A can be perturbed into a viscosity strict supersolution by a suitable small perturbation. More precisely, we have

**Lemma 2.1.** Suppose that $F \in C(R \times R^m \times S(m))$ and $v \in C(\Omega)$ is a viscosity supersolution to

$$F[w] := F(w, \nabla_h w, \nabla^2_h w) = 0, \text{ in } \Omega,$$

under either (i) $F$ is degenerate subelliptic and decreasing or (ii) $F$ is uniformly subelliptic, nonincreasing. Then for any $\delta \in (0, \delta_0)$ there are $c_\delta > 0$ and $v^\delta \in C(\Omega)$ so that

$$v(x) \leq v^\delta(x) \leq v(x) + \delta$$

and $v^\delta$ is a viscosity supersolution to

$$F(w, \nabla_h w, \nabla^2_h w) + c_\delta = 0, \text{ in } \Omega.$$

**Proof.** We first recall that for $x \in G$ if $(x_1, \ldots, x_m)$ denotes the horizontal component of its coordinate and $((x_{2,1}, \ldots, x_{2,m_2}), \ldots, (x_{r,1}, \ldots, x_{r,m_r}))$ denotes the vertical component of its coordinate, then the horizontal vector fields $X_l, 1 \leq l \leq m$, can be expressed as (see [FS])

$$X_l = \frac{\partial}{\partial x_l} + \sum_{i=2}^{r} \sum_{j=1}^{m_i} a_{ij}(x) \frac{\partial}{\partial x_{i,j}}, 1 \leq l \leq m,$$

where $\{a_{ij}\}$ are smooth on $G$ for $2 \leq i \leq r, 1 \leq j \leq m_i$. For $k > 0$, denote $c_1 = \inf_{x \in \Omega} x_1 \in R$ and define

$$\alpha_k(x) = 1 - \frac{1}{k} e^{-k(x_1 + 1 - c_1)}, \forall x \in \Omega.$$

Then (2.4) implies that, for any $x \in \Omega$, we have

$$X_1 \alpha_k(x) = e^{-k(x_1 + 1 - c_1)}, X_2 \alpha_k(x) = \cdots = X_m \alpha(x) = 0,$$

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$$X_{ij\alpha k}(x) = -ke^{-k(x_1+1-c_1)}, \text{ if } i = j = 1,$$
$$= 0, \quad \text{ otherwise.} \tag{2.6}$$

For any $\delta > 0$ and $k \geq 2$ to be chosen later, we consider $v^\delta(x) := v(x) + \delta \alpha_k(x) : \Omega \to R$. Since $0 \leq \alpha_k \leq 1$, it is easy to see that $v^\delta$ satisfies (2.2). We want to show that $v^\delta$ is also a viscosity supersolution to eqn. (2.4). To do this, let $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ touch $v^\delta$ from below at $x = x_0$, i.e.

$$0 = (v + \delta \alpha_k - \phi)(x_0) \geq (v + \delta \alpha_k - \phi)(x), \quad \forall x \in \Omega.$$

This implies that $\phi - \delta \alpha_k \in C^2(\Omega)$ touches $v$ from below at $x_0$. Since $v$ is a viscosity supersolution to eqn.(2.1), we have

$$F(\phi - \delta \alpha_k, \nabla_h \phi - \delta \nabla_h \alpha_k, \nabla^2_h \phi - \delta \nabla^2_h \alpha_k)(x_0) \leq 0. \tag{2.7}$$

Now we need to show that there exists a $k_0 > 0$ such that for any $k \geq k_0$ (2.3) is true. We proceed it as follows.

**Case 1.** $\mathcal{F}$ is degenerate subelliptic and decreasing:

It follows from (2.6) that $\nabla^2_h (-\alpha_k)$ is positive semidefinite. Therefore, the degenerate subellipticity (1.8) and decreasing property (1.11) of $\mathcal{F}$ imply

$$F(\phi - \delta \alpha_k, \nabla_h \phi - \delta \nabla_h \alpha_k, \nabla^2_h \phi - \delta \nabla^2_h \alpha_k)(x) \geq F(\phi, \nabla_h \phi - \delta \nabla_h \alpha_k, \nabla^2_h \phi)(x)$$
$$\geq F(\phi, \nabla_h \phi, \nabla^2_h \phi)(x) + \alpha_3 \delta \alpha_k(x)$$
$$\geq F(\phi, \nabla_h \phi, \nabla^2_h \phi)(x) + \alpha_3 \delta \alpha_k(x) - \omega_2(\delta |\nabla_h \alpha_k|)(x), \tag{2.8}$$

where $\omega_2$ is the modular of continuity of $F$ with respect to it second variable. Since $\alpha_k(x) \geq \frac{1}{2}$ and $|\nabla_h \alpha_k|(x) \leq \frac{1}{2k}$, (2.8) implies

$$F(\phi, \nabla_h \phi, \nabla^2_h \phi)(x_0) \leq \omega_2(\delta k^{-1}) - \frac{k_0 \delta}{2} \equiv c_\delta < 0, \tag{2.9}$$

if we choose $k$ so large that $\omega_2(\delta k^{-1}) \leq \frac{k_0 \delta}{4}$. This verifies (2.4) under the condition (i) of Lemma 2.1.

**Case 2.** $\mathcal{F}$ is uniformly subelliptic, nonincreasing:

Since $\nabla^2_h (-\alpha_k)$ is positive semidefinite, the uniform ellipticity (1.9) and nonincreasing property (1.10) of $\mathcal{F}$ imply

$$F(\phi - \delta \alpha_k, \nabla_h \phi - \delta \nabla_h \alpha_k, \nabla^2_h \phi - \delta \nabla^2_h \alpha_k)(x) \geq F(\phi, \nabla_h \phi - \delta \nabla_h \alpha_k, \nabla^2_h \phi)(x) + \alpha_1 \text{trace}(\nabla^2_h (-\alpha_k))(x)$$
$$\geq F(\phi, \nabla_h \phi - \delta \nabla_h \alpha_k, \nabla^2_h \phi)(x) + \alpha_1 k e^{-k(x_1+1-c_1)}$$
$$\geq F(\phi, D_h \phi, D^2_h \phi)(x) + \alpha_1 k e^{-k(x_1+1-c_1)} - \alpha_2 \delta |\nabla_h \alpha_k|(x)$$
$$\geq F(\phi, D_h \phi, D^2_h \phi)(x) + e^{-k(x_1+1-c_1)}(\alpha_1 k - \alpha_2 \delta). \tag{2.10}$$
Therefore if we choose \( k \geq \frac{2\alpha_2\delta}{\kappa_1} \), then we have
\[
F(\phi, \nabla_h \phi, \nabla_h^2 \phi)(x_0) \leq -\frac{\alpha_2\delta}{2} e^{-k(x_1+1-c_1)},
\] (2.11)

this, combined with \( \inf_{x \in \Omega} e^{-k(x_1+1-c_1)} = 2c_2(k) > 0 \), implies that (2.4) holds with \( c_\delta = c_2(k)\alpha_2\delta > 0 \). Therefore, the proof of Lemma 2.1 is complete.

§3. The construction of sup/inf convolutions on \( G \)

For the convenience of readers, we recall the construction of sup/inf convolution on any Carnot group \( G \), which was carried out earlier by Wang [W1]. The key observation is that the equation (1.7) is invariant under group multiplication from left on \( G \). We would like to point out that this construction is an extension of that by Jensen-Lions-Souganidis [JLS] in the Euclidean space.

Let \( \Omega \subset G \) be a bounded domain and \( d_G(\cdot, \cdot) \) be the smooth gauge distance defined by (1.3). For any \( \epsilon > 0 \), define
\[
\Omega_\epsilon = \{ x \in \Omega : \inf_{y \in G \setminus \Omega} d_G(x^{-1}, y^{-1}) \geq \epsilon \}.
\]

**Definition 3.1.** For any \( u \in C(\bar{\Omega}) \) and \( \epsilon > 0 \), the sup involution \( u_\epsilon \) of \( u \) is defined by
\[
u_\epsilon(x) = \sup_{y \in \Omega} (u(y) - \frac{1}{2\epsilon} d_G(x^{-1}, y^{-1})^{2r!}), \quad \forall x \in \Omega.
\] (3.1)

Similarly, the inf involution \( v_\epsilon \) of \( v \in C(\bar{\Omega}) \) is defined by
\[
v_\epsilon(x) = \inf_{y \in \Omega} (v(y) + \frac{1}{2\epsilon} d_G(x^{-1}, y^{-1})^{2r!}), \quad \forall x \in \Omega.
\] (3.2)

For \( p \in G \), let \( \|p\|_E := (\sum_{i=1}^r |\xi_i(p)|^2)^{1/2} \) be the euclidean norm of \( p \). We recall

**Definition 3.2.** A function \( u \in C(\bar{\Omega}) \) is called semiconvex, if there is a constant \( C > 0 \) such that \( u(p) + C\|p\|_E^2 \) is convex in the Euclidean sense; and \( u \) is called semiconcave if \( -u \) is semiconvex. Note that, for \( u \in C^2(\Omega) \), if \( \nabla^2 u(p) + C \) is positive semidefinite for any \( p \in \Omega \), then \( u \) is semiconvex.

Now we have

**Proposition 3.3.** For \( u, v \in C(\bar{\Omega}) \), denote \( R_0 = \max\{\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\} \). Then, for any \( \epsilon > 0 \), \( u_\epsilon, v_\epsilon \in W^{1,\infty}_{cc}(\Omega) \) satisfy

1. \( u_\epsilon \) is semiconvex and \( v_\epsilon \) is semiconcave.
(2) $u^\epsilon$ is monotonically nondecreasing w.r.t. $\epsilon$ and converges uniformly to $u$ on $\Omega$; and $v^\epsilon$ is monotonically nonincreasing w.r.t. $\epsilon$ and converges uniformly to $v$ on $\Omega$.

(3) if $u$ (or $v$ respectively) is a viscosity subsolution (or supersolution respectively) to a degenerate subelliptic equation:

$$F(u, \nabla_h u, \nabla^2_h u) = 0 \text{ in } \Omega,$$

where $F \in C(R \times R^m \times S(m))$. Then $u^\epsilon$ (or $v^\epsilon$) is a viscosity subsolution (or supersolution respectively) to eqn. (3.3) in $\Omega_{2R_0}$.

**Proof.** Since the proof of $v^\epsilon$ can be done by the same way as that of $u^\epsilon$, it suffices to consider $u^\epsilon$. For $\Omega \subset G$ is bounded, the formula (1.3) of $d_G$ implies

$$C(\Omega, d_G) \equiv \|\nabla^2_x (d_G(x^{-1}, y^{-1})^{2r!})\|_{L^\infty(\Omega \times \Omega)} < \infty.$$ 

Therefore, for any $y \in \bar{\Omega}$, the full hessian of

$$\tilde{u}^\epsilon(x, y) := u(y) - \frac{1}{2\epsilon} d_G(x^{-1}, y^{-1})^{2r!} + \frac{C(\Omega, d_G)}{2\epsilon} \|x\|^2_E, \forall x \in \Omega,$$

is positive semidefinite so that $\tilde{u}^\epsilon$ is convex. Note that the superum for a family of convex functions is still convex, this implies that

$$u^\epsilon(x) + \frac{C(\Omega, d_G)}{2\epsilon} \|x\|^2_E = \sup_{y \in \Omega} \tilde{u}^\epsilon(x, y), \forall x \in \Omega$$

is convex so that $u^\epsilon$ is semiconvex. It is well-known that semiconvex functions are Lipschitz continuous with respect to the euclidean metric (cf. Evans-Gariepy [EG]). Therefore $u^\epsilon$ is Lipschitz continuous in $\Omega$ with respect to $d_G$. This gives (1).

For any $\epsilon_1 < \epsilon_2$, it is easy to see that $u^{\epsilon_1}(x) \leq u^{\epsilon_2}(x)$ so that $\{u^\epsilon\}$ is monotonically nondecreasing with respect to $\epsilon$. Observe that for any $x \in \Omega$ there exists a $x_\epsilon \in \bar{\Omega}$ such that

$$u(x) \leq u^\epsilon(x) = u(x_\epsilon) - \frac{1}{2\epsilon} d_G(x^{-1}, x_\epsilon^{-1}) \leq R_0.$$  

(3.4)

This implies

$$u(x_\epsilon) - u^\epsilon(x) = \frac{1}{2\epsilon} d_G(x^{-1}, x_\epsilon^{-1}) \leq u(x_\epsilon) - u(x) = \omega_u(\|x_\epsilon - x\|_E), \forall x \in \Omega,$$

(3.5)

where $\omega_u$ denotes the modular of continuity of $u$. On the other hand, the monotonicity of $u^\epsilon$ with respect to $\epsilon$ implies

$$u^\epsilon(x) \geq u(x_\epsilon) - \frac{1}{\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!} = u_\epsilon(x) - \frac{1}{2\epsilon} d(x^{-1}, x_\epsilon^{-1})^{2r!}.$$
so that
\[ \lim_{\epsilon \to 0} \frac{1}{2e} d(x^{-1}, x^{-1}_\epsilon)^{2r!} = 0, \forall x \in \Omega. \]  
(3.6)

This implies that \( \lim_{\epsilon \to 0} x_\epsilon = x \) and \( \lim_{\epsilon \to 0} u^{\epsilon}(x) = u(x) \) for any \( x \in \Omega \). Moreover, (3.5) implies
\[ d_G(x^{-1}, x^{-1}_\epsilon) \leq 2\epsilon \omega_u(\|x_\epsilon - x\|_E) \leq 2R_0\epsilon \]  
(3.7)

so that \( \|x_\epsilon - x\|_E \leq C(R_0\epsilon)^{\frac{1}{r}} \), where \( r \) is the step of \( G \). This, combined with (3.5) again, implies
\[ \max_{x \in \Omega} |u^{\epsilon}(x) - u(x)| \leq \omega_u(C(R_0\epsilon)^{\frac{1}{r}}) \to 0, \quad \text{as } \epsilon \to 0 \]
so that \( u^{\epsilon} \) converges to \( u \) uniformly. Therefore (2) is proved.

For (3), we first observe that (3.7) implies that for \( x^0 \in \Omega_{2R_0\epsilon} \), \( u^{\epsilon}(x^0) \) is attained by a \( x^0_\epsilon \in \Omega \). Now we let \( \phi \in C^2(\Omega) \) be such that
\[ u^{\epsilon}(x^0) - \phi(x^0) \geq u^{\epsilon}(x) - \phi(x), \quad \forall x \in \Omega_{2R_0\epsilon}. \]

Then we have, for any \( x, y \in \Omega_{2R_0\epsilon} \),
\[ u(x^0_\epsilon) - \frac{1}{2e} d_G((x^0)^{-1}, (x^0_\epsilon)^{-1})^{2r!} - \phi(x^0) \geq u(y) - \frac{1}{2e} d_G(x^{-1}, y^{-1})^{2r!} - \phi(x). \]  
(3.8)

For \( y \) near \( x^0_\epsilon \), since \( x = x^0 \cdot (x^0_\epsilon)^{-1} \cdot y \in \Omega_{2R_0\epsilon} \), we can substitute \( x \) into (3.8) to get
\[ u(x^0_\epsilon) - \phi(x^0) \geq u(y) - \phi(x^0 \cdot (x^0_\epsilon)^{-1} \cdot y). \]

Set \( \bar{\phi}(y) = \phi(x^0 \cdot (x^0_\epsilon)^{-1} \cdot y) \) for \( y \in \Omega_{2R_0\epsilon} \) close to \( y_0 \). Then \( \bar{\phi} \) touches \( u \) from above at \( y = x^0_\epsilon \) so that \( u \) being a viscosity subsolution of eqn. (3.3) implies
\[ F(u(x^0_\epsilon), \nabla_h \bar{\phi}(x^0_\epsilon), \nabla^2_h \bar{\phi}(x^0_\epsilon)) \leq 0. \]  
(3.9)

Note that the left-invariance of \( X_i \), we know
\[ \nabla_h \bar{\phi}(y) = \nabla_h \phi(x^0 \cdot (x^0_\epsilon)^{-1} \cdot y), \quad \nabla^2_h \bar{\phi}(y) = \nabla^2_h \phi(x^0 \cdot (x^0_\epsilon)^{-1} \cdot y). \]

Hence we have
\[ F(u(x^0_\epsilon), \nabla_h \phi(x_0), \nabla^2_h \phi(x_0)) \leq 0. \]  
(3.10)

Taking \( \epsilon \) into zero, (3.10) implies that \( u^\epsilon \) is a viscosity subsolution of eqn.(3.3) on \( \Omega_{2R_0\epsilon} \). The proof is complete.

§4. Proof of Theorem A
This section is devoted to the proof of the comparison Theorem. The idea is to prove the comparison property between the strict supersolution obtained by Lemma 2.1 and the subsolution by comparing their sup/inf convolutions. The almost everywhere twice differentiability ([EG]) and Jensen’s maximum principle ([J1, 2]) for semiconvex functions play very important roles in this aspect.

Through this section, we express the horizontal vector fields \( X_i \) by the formula (2.4).

**Proof of Theorem A.**

Suppose that (1.13) were false. Then

\[
\delta_0 = \sup_{\Omega} (u - v)^+ - \sup_{\partial \Omega} (u - v)^+ > 0. \tag{4.1}
\]

Denote \( c^+ = \sup_{\partial \Omega} (u - v)^+ \geq 0 \). Note that \( v + c^+ \) is also a viscosity supersolution to eqn. (1.7), and (4.1) implies

\[
\delta_0 = \sup_{\Omega} (u - (v + c^+))^+ - \sup_{\partial \Omega} (u - (v + c^+))^+ > 0.
\]

Hence we may assume \( c^+ = 0 \) (i.e. \( u(x) \leq v(x) \) for any \( x \in \partial \Omega \)) so that (4.1) implies \( \delta_0 = \sup_{\Omega} (u - v) > 0 \).

For any \( \delta \in (0, \frac{\delta_0}{4}) \), let \( v^\delta \in C(\overline{\Omega}) \) be the strict supersolution of eqn. (1.7) given by Lemma 2.1. In particular, \( v^\delta \) is a viscosity supersolution to

\[
F(w, \nabla_h w, \nabla_h^2 w) + c^\delta = 0, \text{ in } \Omega. \tag{4.2}
\]

For any \( \epsilon \in (0, \delta) \), we now let \( u^\epsilon \) (\( v^\delta \), respectively) be the sup-convolution (inf-convolution, respectively) of \( u \) (\( v^\delta \) respectively) given by Proposition 3.3. By considering a smaller domain, we may assume that \( u^\epsilon \) is a viscosity subsolution of eqn. (1.7) and \( v^\epsilon \) is a viscosity supersolution of eqn. (4.2) in \( \Omega \), and

\[
\sup_{\Omega} (u^\epsilon - v^\delta) > 0 \geq \sup_{\partial \Omega} (u^\epsilon - v^\delta)
\]

is achieved at a point \( x_0 \in \Omega \). Since Proposition 3.3 implies that \( u^\epsilon - v^\delta \) is semiconvex, we know (cf. [J2] page 67) that

\[
\nabla u^\epsilon(x_0), \nabla v^\delta(x_0) \text{ both exist and are equal,} \tag{4.3}
\]

\[
u^\epsilon(x) - u^\epsilon(x_0) - \langle \nabla u^\epsilon(x_0), x - x_0 \rangle_E = O(||x - x_0||_E^2), \tag{4.4}
\]

\[
v^\delta(x) - v^\delta(x_0) - \langle \nabla v(x_0), x - x_0 \rangle_E = O(||x - x_0||_E^2). \tag{4.5}
\]
where \( \langle \cdot, \cdot \rangle_E \) denotes the Euclidean inner product on \( \mathbf{G} \). Let \( R_0 = \text{dist}_E(x_0, \partial \Omega) = \inf_{x \in \partial \Omega} \|x - x\|_E > 0 \) be the euclidean distance from \( x_0 \) to \( \partial \Omega \) and \( R_1 > 0 \) be such that both (4.4) and (4.5) hold with \( \|x - x_0\| < R_1 \). Set \( R_2 = \min \{R_0, R_1\} > 0 \). For simplicity, we will denote \( u, v \) as \( u^\epsilon, v^\epsilon \) respectively from now on. For any small \( \rho > 0, \) define the rescaled maps \( u^\rho, v^\rho \) in the euclidean ball \( B^{E}_{R,\rho} \) by
\[
\begin{align*}
u^\rho(x) &= \frac{1}{\rho^2}(v(x_0 + \rho x) - v(x_0) - \rho \langle \nabla v(x_0), x \rangle_E), \\
u^\rho(x) &= \frac{1}{\rho^2}(u(x_0 + \rho x) - u(x_0) - \rho \langle \nabla u(x_0), x \rangle_E),
\end{align*}
\]
where we have used the Euclidean addition and scalar multiplication. Then it is easy to see
\[
(u^\rho - v^\rho)(0) > 0, \quad (u^\rho - v^\rho)(x) \geq (u^\rho - v^\rho)(x), \quad \forall x \in B^{E}_{R,\rho}.
\]
(4.6)
It follows from (4.4) and (4.5) that, for any \( R > 0 \), there exists an \( \rho_0 = \rho_0(R) > 0 \) such that (i) \( \{u^\rho\}_{0 < \rho \leq \rho_0} \) are uniformly bounded, uniformly semiconvex, and uniformly Lipschitz continuous in \( B^E_R \); and (ii) \( \{v^\rho\}_{0 < \rho \leq \rho_0} \) are uniformly bounded, uniformly semiconcave, and uniformly Lipschitz continuous in \( B^E_R \). Therefore, by the Cauchy diagonal process, we may assume that there is \( \rho_i \downarrow 0 \) such that \( u^\rho_i \to u^* \), \( v^\rho_i \to v^* \) locally uniformly in \( \mathbb{R}^n \), where \( n = \text{dim}(\mathbf{G}) \). In particular, (i) and (ii) imply that \( u^* \) is locally bounded, semiconvex in \( \mathbb{R}^n \), and \( v^* \) is locally bounded, semiconcave in \( \mathbb{R}^n \), and
\[
(u^* - v^*)(0) > 0, \quad (u^* - v^*)(x) \geq (u^* - v^*)(x), \quad \forall x \in \mathbb{R}^n.
\]
Now we have

**Claim 4.1.** \( u^* \) satisfies, in the sense of viscosity,
\[
F(u(x_0), \nabla u(x_0), \{ \sum_{k,l=1}^n (A^{ij}_{kl})_k \partial^2 u + a_{ik}(x_0) \partial a_{jl}(x_0) \partial u \partial x_l (x_0) \}_{1 \leq i,j \leq m} \} \geq 0, \quad \text{in } \mathbb{R}^n,
\]
(4.7)
and \( v^* \) satisfies, in the sense of viscosity,
\[
F(v(x_0), \nabla v(x_0), \{ \sum_{k,l=1}^n (A^{ij}_{kl})_k \partial^2 v + a_{ik}(x_0) \partial a_{jl}(x_0) \partial v \partial x_l (x_0) \}_{1 \leq i,j \leq m} \} + c_\delta \leq 0, \quad \text{in } \mathbb{R}^n,
\]
(4.8)
where \( A^{ij}_{kl} = a_{ik}(x_0)a_{jl}(x_0) \), for \( 1 \leq k,l \leq n, 1 \leq i,j \leq m \), and \( c_\delta > 0 \).

Let’s assume Claim 4.1 for the moment and proceed as follows. Since \( u^* - v^* \) is semiconvex and achieves its maximum at \( x = 0 \), we can apply Jensen’s maximum principle for semiconvex functions (see [J1] [J2]) to conclude that there exists \( x_* \in \mathbb{R}^n \) such
that $\nabla^2 u^*(x_\star), \nabla^2 v^*(x_\star)$ both exist and $\nabla^2(u^* - v^*)(x_\star)$ is negative semidefinite. Denote $M_1, M_2 \in \mathcal{S}(m)$ by

$$M_{ij}^1 = \sum_{k,l=1}^n \{ A_{ikl}^{ij} \frac{\partial^2 u^*}{\partial x_k \partial x_l}(x_\star) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial u}{\partial x_l}(x_0) \}, \quad 1 \leq i, j \leq m,$$

and

$$M_{ij}^2 = \sum_{k,l=1}^n \{ A_{ikl}^{ij} \frac{\partial^2 v^*}{\partial x_k \partial x_l}(x_\star) + a_{ik}(x_0) \frac{\partial a_{jl}}{\partial x_k}(x_0) \frac{\partial v}{\partial x_l}(x_0) \}, \quad 1 \leq i, j \leq m.$$

Since (4.2) implies $\nabla u(x_0) = \nabla v(x_0)$, we have

$$\sum_{1 \leq i,j \leq m} (M_{ij}^1 - M_{ij}^2)p_ip_j = \sum_{k,l=1}^n \eta_k \eta_l \frac{\partial^2 (u - v)}{\partial x_k \partial x_l}(x_\star) \leq 0, \forall p \in \mathbb{R}^m,$$

where $\eta_k = \sum_{i=1}^n p_i a_{ik}(x_0)$, for $1 \leq k \leq n$. Hence $(M_1 - M_2)$ is negative semidefinite. Note also that $u(x_0) > v(x_0)$, $\nabla_h u(x_0) = \nabla_h v(x_0)$. Therefore the subellipticity and nonincreasing property of $\mathcal{F}$ implies

$$F(u(x_0), \nabla_h u(x_0), M_1) \leq F(v(x_0), \nabla_h v(x_0), M_2). \quad (4.9)$$

This clearly contradicts with (4.7) and (4.8), since (4.7) implies

$$F(u(x_0), \nabla_h u(x_0), M_1) \geq 0,$$

and (4.8) implies

$$F(v(x_0), \nabla_h v(x_0), M_2) \leq -c_\delta.$$

Therefore the theorem is proved.

Now we indicate the proof of claim 4.1. This claim follows from the compactness theorem (cf. [CIL]) among a family of viscosity sub/supersolutions to 2nd order PDEs. For simplicity, we only indicate how to prove (4.7). First we claim that $\mu^\rho$ satisfies, in the sense of viscosity, in $B_{R^2 \rho -1}^E$,

$$F(u(x_0 + \rho x), \nabla_h u(x_0) + \rho \mu^\rho(x), 0),$$

$$\{ \sum_{k,l=1}^n A_{ij,k l}(x) \frac{\partial^2 \mu^\rho}{\partial x_k \partial x_l}(x) + B_{ij,l}(x) \left( \frac{\partial u}{\partial x_l}(x_0 + \rho \frac{\partial u}{\partial x_l}(x)) \right) \}_{1 \leq i,j \leq m} = 0, \quad (4.10)$$

where $A_{ij,k l}(x) = a_{i k}^\rho a_{j l}^\rho(x), B_{ij,l}(x) = \sum_{k=1}^n a_{i k}^\rho (\frac{\partial a_{j l}}{\partial x_k})^\rho, \mu^\rho(x) = (\mu^\rho_1(x), \cdots, \mu^\rho_m(x)), \mu^\rho_i(x) = X_i(x_0 + \rho x), a_{i k}^\rho(x) = a_{i k}(x_0 + \rho x), \text{ and } (\frac{\partial a_{i k}}{\partial x_k})^\rho(x) = \frac{\partial a_{i k}}{\partial x_k}(x_0 + \rho x).$
To see (4.10), let \((\bar{x}, \phi) \in B_{R_2 \rho^{-1}}^E \times C^2(B_{R_2 \rho^{-1}}^E)\) be such that
\[
0 = u^\rho(\bar{x}) - \phi(\bar{x}) \geq u^\rho(x) - \phi(x), \quad \forall x \in B_{R_2 \rho^{-1}}^E.
\]
It is straightforward to see
\[
\phi_\rho(x) \equiv u(x_0) + \langle \nabla u(x_0), x - x_0 \rangle_E + \rho^2 \phi\left(\frac{x - x_0}{\rho}\right), \forall x \in B_{R_2}(x_0)
\]
satisfies
\[
0 = u(x_0 + \rho \bar{x}) - \phi_\rho(x_0 + \rho \bar{x}) \geq u(x) - \phi_\rho(x), \quad \forall x \in B_{R_2}(x_0).
\]
This, combined with the fact that \(u\) is a viscosity subsolution to eqn.(1.7), implies
\[
F(u(x_0 + \rho \bar{x}), \nabla_h \phi_\rho(x_0 + \rho \bar{x}), \nabla^2_h \phi_\rho(x_0 + \rho \bar{x})) \geq 0. \tag{4.11}
\]
Direct calculations yield
\[
\frac{\partial \phi_\rho}{\partial x_k}(x_0 + \rho \bar{x}) = \frac{\partial u}{\partial x_k}(x_0) + \rho \frac{\partial \phi}{\partial x_k}(\bar{x}), \quad \forall 1 \leq k \leq n,
\]
\[
\frac{\partial^2 \phi_\rho}{\partial x_k \partial x_l}(x_0 + \rho \bar{x}) = \frac{\partial^2 \phi}{\partial x_k \partial x_l}(\bar{x}), \quad \forall 1 \leq k, l \leq n.
\]
Substituting these into (4.11), we obtain (4.10).

It is clear that, by taking \(\rho \to 0\), (4.10) implies (4.7). This proves claim 4.1.

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