A 2–Cocycle on a Group of Symplectomorphisms

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A 2-COCYCLE ON A GROUP OF SYMPLECTOMORPHISMS

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Abstract. For a symplectic manifold \((M, \omega)\) with exact symplectic form we construct a 2-cocycle on the group of symplectomorphisms and indicate cases when this cocycle is not trivial.

1. Introduction

For a symplectic manifold \((M, \omega)\) such that \(H^1(M, \mathbb{R}) = 0\) and the symplectic form \(\omega\) is exact we indicate a formula defining a 2-cocycle on the group \(\text{Diff}(M, \omega)\) of symplectomorphisms with values in the trivial \(\text{Diff}(M, \omega)\)-module \(\mathbb{R}\). Let \(G\) be a connected real simple Lie group and \(K\) a maximal compact subgroup. For the symmetric Hermitian space \(M = G/K\) endowed with the induced symplectic structure, we prove that the restriction of this cocycle to the group \(G\) is non-trivial. Thus this cocycle is non-trivial on the whole group \(\text{Diff}(M, \omega)\), too. In particular, this implies that the cocycle is non-trivial for the symplectic manifold \((\mathbb{R}^2 \times M, \omega_0 + \omega_M)\), where \((M, \omega_M)\) is a non-compact symplectic manifold with exact symplectic form \(\omega_M\) such that \(H^1(M, \mathbb{R}) = 0\) and \(\omega_0\) is the standard symplectic form on \(\mathbb{R}^2\).

For the convenience of the reader, in an appendix we consider the corresponding 2-cocycle on the Lie algebra of locally Hamiltonian and Hamiltonian vector fields and indicate when this cocycle is non-trivial.

Note that in [7] a similar 2-cocycle was constructed for the group of volume preserving diffeomorphisms on a compact \(n\)-dimensional manifold \(M\). This cocycle takes its values in the space \(H^{n-2}(M, \mathbb{R})\). Neretin in [10] constructed a 2-cocycle on the group of symplectomorphisms with compact supports.

Throughout the paper \(M\) is a connected \(C^\infty\)-manifold.

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2. Preliminaries

We recall some standard facts on central extensions of groups and two-dimensional cohomology of groups (see, for example, [8], ch. 4).

Consider a group $G$ and the field $\mathbb{R}$ as a trivial $G$-module. Let $C^p(G, \mathbb{R})$ be the set of maps from $G^p$ to $\mathbb{R}$ for $p > 0$ and let $C^0(G, \mathbb{R}) = \mathbb{R}$. Define a map $D^p : C^p(G, \mathbb{R}) \to C^{p+1}(G, \mathbb{R})$ as follows: for $f \in C^p(G, \mathbb{R})$ and $g_1, \ldots, g_{p+1} \in G$

\[(D^p f)(g_1, \ldots, g_{p+1}) = f(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i f(g_1, \ldots, g_ig_{i+1}, \ldots, g_{p+1}) + (-1)^{p+1} f(g_1, \ldots, g_p).\]

By definition $C^*(G, \mathbb{R}) = (C^p(G, \mathbb{R}), D^p)_{p \geq 0}$ is the standard complex of nonhomogeneous cochains of the group $G$ with values in the $G$-module $\mathbb{R}$ and its cohomology $H^*(G, \mathbb{R}) = (H^p(G, \mathbb{R}))_{p \geq 0}$ is the cohomology of the group $G$ with values in the trivial $G$-module $\mathbb{R}$. Recall that a cochain $f \in C(G, \mathbb{R})$ is called normalized if $f(g_1, \ldots, g_p) = 0$ whenever at least one of the $g_1, \ldots, g_p \in G$ equals the identity $e$ of $G$. It is known that the inclusion of the subcomplex of normalized cochains into $C^*(G, \mathbb{R})$ induces an isomorphism in cohomology.

Let $f$ be a normalized 2-cocycle of $G$ with values in a trivial $G$-module $\mathbb{R}$. Let $E(G, \mathbb{R}) = G \times \mathbb{R}$, with multiplication $(g_1, a_1)(g_2, a_2) = (g_1g_2, a_1a_2 + a_2 + f(g_1, g_2))$ for $a_1, a_2 \in \mathbb{R}$ and $g_1, g_2 \in G$. Then $E(G, \mathbb{R})$ is a group, and the natural projection $E(G, \mathbb{R}) = G \times \mathbb{R} \to G$ is a central extension of the group $G$ by $\mathbb{R}$. The extension $E(G, \mathbb{R})$ is non-split iff the cocycle $f$ is non-trivial.

If $G$ is a topological group (finite-dimensional or infinite-dimensional Lie group) one can define a subcomplex $C^*_\text{cont}(G, \mathbb{R})$ ($C^*_\text{diff}(G, \mathbb{R})$) of the complex $C^*(G, \mathbb{R})$ (see [4], ch. 3) consisting of cochains which are continuous (smooth) functions. The cohomologies of the complexes $C^*_\text{cont}(G, \mathbb{R})$ and $C^*_\text{diff}(G, \mathbb{R})$ are isomorphic whenever $G$ is a finite-dimensional Lie group (see [4], ch. 3 and [9]). Note that if the 2-cocycle $f$ is continuous (differentiable), the extension $E(G, \mathbb{R})$ is a topological group (Lie group).

3. A 2-COCYCLE ON THE GROUP OF SYMPLECTOMORPHISMS

Let $(M, \omega)$ be a non-compact symplectic manifold such that $H^1(M, \mathbb{R}) = 0$ and the symplectic form $\omega$ is exact. Let $\omega_1$ be a 1-form on $M$ such that $d\omega_1 = \omega$. Denote by $\text{Diff}(M, \omega)$ the group of symplectomorphisms of $M$. We define a 2-cocycle on the group $G = \text{Diff}(M, \omega)$ with values in the trivial $G$-module $\mathbb{R}$ as follows. Fix a point $x_0 \in M$. Then for $g_1, g_2 \in G$ we put

\[C_{x_0}(g_1, g_2) = \int_{x_0}^{g_2x_0} (g_1^*\omega_1 - \omega_1),\]
where the integral is taken along a smooth curve connecting the point $x_0$ with the point $g_2x_0$. Since $H^1(M, \mathbb{R}) = 0$ the 1-form $g^*_1\omega_1 - \omega_1$ is exact and the value of this integral does not depend on the choice of such a curve.

**Theorem 3.1.** The function $C_{x_0} : G^2 \to \mathbb{R}$ defined by (2) is a normalized 2-cocycle on the group $G$ with values in the trivial $G$-module $\mathbb{R}$. The cohomology class of $C_{x_0}$ is independent of the choice of the point $x_0$ and the form $\omega_1$.

**Proof.** By (1) it is easy to check that $D^2C_{x_0} = 0$. Moreover, the 2-cocycle $C_{x_0}$ is normalized. Since for each $g \in G$ the 1-form $g^*\omega_1 - \omega_1$ is exact, for any points $x_1, x_2 \in M$ we have $C_{x_1} - C_{x_2} = Da$, where $a$ is a 1-cochain on $G$ defined by $a(g) = \int_{x_1}^{x_2} (g^*\omega_1 - \omega_1)$.

By definition, the cocycle $C_{x_0}$ is a continuous function on $G \times G$.

**Remark 3.2.** Let $M$ be a manifold such that $H^1(M, \mathbb{R}) = 0$ and let $\omega$ be an exact 2-form on $M$. Let $\text{Diff}(M, \omega)$ be the group of diffeomorphisms of $M$ preserving the form $\omega$. Then the formula (2) for $g_1, g_2 \in \text{Diff}(M, \omega)$ gives a 2-cocycle on the group $\text{Diff}(M, \omega)$ and all statements of theorem (3.1) are true for this cocycle.

Denote by $E(\text{Diff}(M, \omega))$ the central extension of the group $\text{Diff}(M, \omega)$ by $\mathbb{R}$ defined by the cocycle $C_{x_0}$. Now we give a geometric interpretation of the extension $E(\text{Diff}(M, \omega))$. We choose a form $\omega_1$ with $d\omega_1 = \omega$ and put $\omega_2(g) = \int_{x_0}^{x} (\omega_1 - g^*\omega_1)$. Consider the trivial $\mathbb{R}$-bundle $M \times \mathbb{R}$. Clearly, the form $dt + \omega_1$ is a connection with curvature $\omega$ of this bundle. Denote by $\text{Aut}(M \times \mathbb{R}, \omega)$ the group of those bundle automorphisms which respect the connection $dt + \omega_1$ and which are projectable to diffeomorphisms in $\text{Diff}(M, \omega)$. It is easy to check that the group $\text{Aut}(M \times \mathbb{R}, \omega)$ is isomorphic to the group $E(\text{Diff}(M, \omega)) = \text{Diff}(M, \omega) \times \mathbb{R}$ which acts as follows on $M \times \mathbb{R}$: $(x, t) \to (g(x), \omega_2(g)(x) + t + a)$, where $(x, t) \in M \times \mathbb{R}$ and $(g, a) \in G \times \mathbb{R}$. This gives an equivalent definition of the extension $E(\text{Diff}(M, \omega))$ as a group of automorphisms of the trivial principal $\mathbb{R}$-bundle $M \times \mathbb{R}$ with connection $dt + \omega_1$.

If we replace the form $\omega_1$ by the form $\omega_1 + df$, where $f$ is a smooth function on $M$, we get an action of $G$ on $M \times \mathbb{R}$ which is related to the initial one by the gauge transformation $(x, t) \to (x, t - f(x))$ of the bundle $M \times \mathbb{R} \to M$.

4. **Examples of non-trivial 2-cocycles**

The authors are not able to prove that the cocycle $C_{x_0}$ is non-trivial for any symplectic manifold $M$ with an exact symplectic 2-form $\omega$. In this section we prove that for some symplectic manifolds the restrictions of this cocycle to some subgroups of $G \subset \text{Diff}(M, \omega)$ turn out to be non-trivial.

4.1. **The linear symplectic space $\mathbb{R}^{2n}$ and the Heisenberg group.** Consider the space $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = \sum_{k=1}^{n} dx_k \wedge dx_{k+n}$ and the group $G = \mathbb{R}^{2n}$ acting on the space $\mathbb{R}^{2n}$ by translations. Applying (2) to the form
\[ \omega_0, \text{ the 1-form } \omega_1 = \frac{1}{2} \sum_{k=1}^{n} (x_{n+k} dx_k - x_k dx_{n+k}), \] and the point \( x_0 = 0 \in \mathbb{R}^{2n} \) we get a 2-cocycle on the group \( G \) given by

\[ C_0(x, y) = \frac{1}{2} \sum_{k=1}^{n} (x_k y_{n+k} - y_k x_{n+k}), \]

where \( x = (x_1, \ldots, x_{2n}) \) and \( y = (y_1, \ldots, y_{2n}) \). The central extension of the group \( \mathbb{R}^{2n} \) by \( \mathbb{R} \) defined by this cocycle is the Heisenberg group. This extension is non-split since the Heisenberg group is noncommutative and thus the cocycle \( C_0(x, y) \) is non-trivial.

### 4.2. Symmetric Hermitian spaces and the Guichardet-Wigner cocycle.

Consider a non-compact symmetric space \( M = G/K \), where \( G \) is a connected real simple Lie group and where \( K \) is a maximal compact subgroup. Then \( M \) is diffeomorphic to \( \mathbb{R}^n \), where \( n = \dim M \). We suppose that \( M \) admits a \( G \)-invariant complex structure, i.e., \( M \) is a symmetric Hermitian space. This condition is satisfied (up to finite covering) for the following groups: \( SU(p, q) \) \((p, q \geq 1)\), \( SO_0(2, q) \) \((q = 1 \text{ or } q \geq 3)\), \( Sp(n, \mathbb{R}) \) \((n \geq 1)\), \( SO^*(2n) \) \((n \geq 2)\), and certain real forms of \( E_6 \) and \( E_7 \).

Consider the symplectic manifold \((M, \omega)\), where the symplectic form \( \omega \) is defined by the Hermitian metric on \( M \). It is known that on each of the Lie groups mentioned above, in the complex \( C^\text{diff}(G, \mathbb{R}) \) there is a non-trivial Guichardet-Wigner 2-cocycle (see [5] and [4]). By [2] this cocycle is given as follows, up to a nonzero factor:

\[ (g_1, g_2) \mapsto \int_{(x_0, g_1 x_0, g_1 g_2 x_0)} \omega, \]

where \( g_1, g_2 \in G, \ x_0 = K \in G/K \), and the integral is taken over the oriented geodesic cone with vertex \( x_0 \) and the segment of a geodesic from \( g_1 x_0 \) to \( g_1 g_2 x_0 \) as base.

We prove that the restriction of the cocycle \( C_{x_0} \) to the group \( G \) is cohomologous to the cocycle given by (3).

For the base point \( x_0 \) we define a 1-cochain \( \gamma_{x_0} \) on the group \( G \) as follows:

\[ \gamma_{x_0}(g) = \int_{x_0}^{g x_0} \omega_1, \]

where \( g \in G \) and the integral is taken along the geodesic segment from \( x_0 \) to \( g x_0 \). Consider \( C_{x_0} \) on \( G \) given by formula (2), where we choose for the curve between the points \( x_0 \) and \( g_2 x_0 \) a geodesic segment from \( x_0 \) to \( g_2 x_0 \). It is easy to check that on the group \( G \) the cocycle \( C_{x_0} + D\gamma_{x_0} \) equals the cocycle given by (3). Thus the cocycle \( C_{x_0} \) on the group \( G \) is non-trivial in the complex \( C^\ast_{\text{diff}}(G, \mathbb{R}) \).

In particular, for the group \( G = \text{SL}(2, \mathbb{R}) = \text{SU}(1, 1) \) the symmetric space \( M = G/K \) is the hyperbolic plane \( H^2 \) and \( \omega \) is the area form on \( H^2 \). Instead of the group \( \text{SL}(2, \mathbb{R}) \) we will later consider the group \( \text{PSL}(2, \mathbb{R}) \) which acts effectively on \( H^2 \).

Since \( \text{SL}(2, \mathbb{R}) \) is a two-sheet cover of \( \text{PSL}(2, \mathbb{R}) \), the cohomologies of these groups
with values in $\mathbb{R}$ are the same. It is easy to check that the corresponding symplectic manifold $(M, \omega)$ is isomorphic to the symplectic manifold $(\mathbb{R}^2, \omega_0)$, where $\omega_0$ is the standard symplectic form on $\mathbb{R}^2$. Unfortunately, for the groups $G \neq \text{SU}(1,1)$ mentioned above we do not know whether the symplectic manifolds $(M, \omega)$ and $(\mathbb{R}^{2n}, \omega_0)$, where $\dim M = 2n$, are isomorphic or not.

The following proposition is known. We do not know a good reference for this; then we give a short proof communicated to us by Yu.A. Neretin.

**Proposition 4.3.** For each symmetric Hermitian space $M = G/K$, where $G$ is a connected simple Lie group and $K$ is its maximal compact subgroup, the corresponding Guichardet-Wigner cocycle is non-trivial in the complex $C^*(G, \mathbb{R})$.

**Proof.** Let $p : \tilde{G} \to G$ be the universal cover and let $a = C_{x_0}$ be the Guichardet-Wigner cocycle for the group $G$. Consider the corresponding to $a$ 2-cocycle $\tilde{a}$ on $\tilde{G}$ induced by $p$. By construction, the cocycle $\tilde{a}$ is trivial, i.e., there is a smooth function $b$ defined on $\tilde{G}$ such that for any $g, h \in \tilde{G}$ we have $\tilde{a}(g, h) = b(h) - b(gh) + b(g)$.

Assume that the cocycle $a$ is trivial in the complex $C^*(G, \mathbb{R})$, i.e., there exists a function $f : G \to \mathbb{R}$ such that for $g, h \in \Gamma$ we have $a(g, h) = f(h) - f(gh) + f(g)$.

Then the difference $b - f \circ p$ is a homomorphism $\tilde{G} \to \mathbb{R}$. This homomorphism vanishes near the identity element of $\tilde{G}$ since the group $\tilde{G}$ is simple, and thus it vanishes on the whole of $\tilde{G}$ since $\tilde{G}$ is connected. Then the function $f$ is smooth and the cocycle $a$ is trivial in the complex $C^*_\text{diff}(G, \mathbb{R})$. This contradiction proves our statement. $\square$

5. **Cases of nontriviality of the cocycle $C_{x_0}$ for groups of symplectomorphisms**

Let $(M, \omega_M)$ be a non-compact symplectic manifold such that $H^1(M, \mathbb{R}) = 0$ with an exact symplectic form $\omega_M$.

By formula (2), the form $\omega_M$ defines a 2-cocycle $C_{x_0}$ for the group $\text{Diff}(M, \omega_M)$ with values in the trivial $\text{Diff}(M, \omega_M)$-module $\mathbb{R}$. The aim of this section is to indicate cases when this cocycle is non-trivial and thus the corresponding central extension of the group $\text{Diff}(M, \omega_M)$ by $\mathbb{R}$ is non-split.

Let $M = G/K$ be an Hermitian symmetric space $M$ and let $(M, \omega)$ be the corresponding symplectic manifold which we considered in subsection 4.2.

**Theorem 5.1.** For the Hermitian symmetric space $M = G/K$ and for the corresponding symplectic manifold $(M, \omega)$ the cocycle $C_0$ on the group $\text{Diff}(M, \omega)$ is non-trivial.

**Proof.** Since the group $G$ is a subgroup of the group $\text{Diff}(M, \omega)$ the statement follows from proposition 4.3. $\square$

Recall that the symplectic manifold $(H^2, \omega)$ where $\omega$ is the area form is symplectomorphic to $(\mathbb{R}^2, \omega_0)$ where $\omega_0$ is the standard symplectic form.
Theorem 5.2. Let \((M, \omega)\) be a non-compact symplectic manifold such that the symplectic form \(\omega_M\) is exact and let \(H^1(M, \mathbb{R}) = 0\). Consider the product \(\mathbb{R}^2 \times M\) of the manifold \(\mathbb{R}^2\) and \(M\) as a symplectic manifold with the symplectic form \(\omega = \omega_0 + \omega_M\). Then for each point \(x_0 \in \mathbb{R}^2 \times M\) the cocycle \(C_{x_0}\) on the group \(\text{Diff}(\mathbb{R}^2 \times M, \omega)\) is non-trivial.

Proof. Choose \(\omega_{M,1} \in \Omega^1(M)\) with \(d\omega_{M,1} = \omega_M\) and let \(\omega_1 = x \, dy + \omega_{M,1}\). The group \(\text{Diff}(\mathbb{R}^2, \omega_0)\) acting on the first factor \(\mathbb{R}^2\) of \(\mathbb{R}^2 \times M\) is naturally included as a subgroup into the group \(\text{Diff}(\mathbb{R}^2 \times M, \omega)\). Thus \(g^*\omega_1 - \omega_1 = g^*(x \, dy) - x \, dy\) for all \(g\) in the subgroup \(\text{Diff}(\mathbb{R}^2, \omega_0)\). Thus the cocycle \(C_{x_0}\) constructed from the form \(dx \wedge dy + \omega_M\) on \(\mathbb{R}^2 \times M\) restricts to a nontrivial cocycle on the subgroup of \(\text{Diff}(\mathbb{R}^2, \omega_0)\) by proposition 4.3 applied to the group \(\text{PSL}(2, \mathbb{R})\). \(\square\)

We leave to the reader to formulate the corresponding results for other symmetric Hermitian spaces \(G/K\) instead of \(H^2\).

5.3. Problem. Consider an open disk \(M\) in the Euclidean plane equipped with the standard area 2-form \(\omega\). Is the 2-cocycle \(C_{x_0}\) defined by the form \(\omega\) non-trivial?

6. Appendix

In this appendix, for a symplectic manifold \((M, \omega)\) we define a 2-cocycle on the Lie algebra \(\text{Vect}(M, \omega)\) of locally Hamiltonian or Hamiltonian vector fields, corresponding to the 2-cocycle \(C_{x_0}\) on the group \(\text{Diff}(M, \omega)\), and study conditions of its nontriviality.

Let \(g\) be a Lie algebra over \(\mathbb{R}\) and let \(\mathbb{R}\) be the trivial \(g\)-module. Denote by \(C^p(g, \mathbb{R})\) the space of skew-symmetric \(p\)-forms on \(g\) with values in \(\mathbb{R}\). For \(c \in C^p(g, \mathbb{R})\) and \(x_1, \ldots, x_{p+1}\) put

\[
(\delta^p c)(x_1, \ldots, x_{p+1}) = \sum_{i<j} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \hat{x_i}, \ldots, \hat{x_j} \ldots, x_{p+1}),
\]

where, as usual, \(\hat{x}\) means that \(x\) is omitted. Then \(C^*(g, \mathbb{R}) = (C^p(g, \mathbb{R}), \delta^p)_{p\geq 0}\) is the complex of standard cochains of the Lie algebra \(g\) with values in the trivial \(g\)-module \(\mathbb{R}\) and the cohomology \(H^*(g, \mathbb{R})\) of this complex is the cohomology of the Lie algebra \(g\) with values in the trivial \(g\)-module \(\mathbb{R}\).

In particular, there is a bijective correspondence between \(H^2(g, \mathbb{R})\) and the set of isomorphism classes of central extensions of the Lie algebra \(g\) by \(\mathbb{R}\).

Let \((M, \omega)\) be a symplectic manifold. Denote by \(\text{Vect}(M, \omega)\) the Lie algebra of locally Hamiltonian vector fields and by \(\text{Vect}_0(M, \omega)\) the Lie algebra of Hamiltonian vector fields on \(M\). For a point \(x_0 \in M\) and \(X, Y \in \text{Vect}(M, \omega)\) put \(c_{x_0}(X, Y) = \omega(X, Y)(x_0)\).

Proposition 6.1. The function \(c_{x_0} : g^2 \to \mathbb{R}\) is a 2-cocycle on the Lie algebra \(g\) with values in the trivial \(g\)-module \(\mathbb{R}\). The cohomology class of \(c_{x_0}\) is independent of the choice of the point \(x_0\).
Proof. The proof is given by direct calculations and is based on the standard formulas \([\mathcal{L}_X, i_Y] = i_{[X,Y]}\) and \(\mathcal{L}_X = i_X d + di_X\), where \(i_X\) is the operator of the inner product by \(X\) and \(\mathcal{L}_X\) is the Lie derivative with respect to a vector field \(X\). (see, for example, [6], ch. 4). In particular, we have for any \(x \in M\) and \(X, Y \in \text{Vect}(M, \omega)\) the following equality

\[
(5) \quad c_x(X, Y) - c_{x_0}(X, Y) = -\int_{x_0}^{x} i_{[X,Y]} \omega.
\]

\(\square\)

Let \(G\) be a Lie group and let \(\mathfrak{g}\) be its Lie algebra. We have a natural homomorphism of complexes \(C^*_\text{diff}(G, \mathbb{R}) \to C^*(\mathfrak{g}, \mathbb{R})\) (see, for example, [4], ch. 3). In particular, if \(c \in C^2_\text{diff}(G, \mathbb{R})\), the corresponding cochain \(\tilde{c} \in C^2(\mathfrak{g}, \mathbb{R})\) is defined as follows:

\[
\tilde{c}(X, Y) = \frac{\partial^2}{\partial t \partial s} (c(\exp tX, \exp sY) - c(\exp sY, \exp tX)|_{t=0, s=0}
\]

where \(X, Y \in \mathfrak{g}\).

Let \(G\) be a Lie group of diffeomorphisms of \(M\) contained in the group \(\text{Diff}(M, \omega)\). Then for the 2-cocycle \(c = C_{x_0}\) of section 3, the cocycle \(\tilde{c}\) is cohomologous to the restriction of the cocycle \(c_{x_0}\) to the Lie algebra \(\mathfrak{g}\) of \(G\). Unfortunately, we cannot apply this procedure to the whole group \(\text{Diff}(M, \omega)\) and the Lie algebra \(\text{Vect}(M, \omega)\). Therefore, the problems of nontriviality of 2-cocycles \(C_{x_0}\) on the group \(\text{Diff}(M, \omega)\) and \(c_{x_0}\) on the Lie algebra \(\text{Vect}(M, \omega)\) should be solved independently.

For each \(X \in \text{Vect}(M, \omega)\) denote by \(\alpha_X\) the closed 1-form such that \(\alpha_X = i_X \omega\).

For all vector fields \(X, Y \in \text{Vect}(M, \omega)\) we have the following equality:

\[
(6) \quad \omega(X, Y) \omega^n = n \alpha_X \wedge \alpha_Y \wedge \omega^{n-1}
\]

which can be easily checked in Darboux coordinates.

Denote by \(X_f\) a Hamiltonian vector field defined by a function \(f \in C^\infty(M)\). Consider the Poisson algebra \(\mathcal{P}(M) = \mathcal{P}(M, \omega)\) on \((M, \omega)\), i.e., the algebra \(C^\infty(M)\) endowed with the Poisson bracket \(\{f, g\} = -\omega(X_f, X_g)\) for \(f, g \in C^\infty(M)\).

The map \(\mathcal{P}(M) \to \text{Vect}_0(M, \omega)\) given by \(f \to X_f\) is a homomorphism of Lie algebras which defines an extension of \(\text{Vect}_0(M, \omega)\) by \(\mathbb{R}\). It is easy to check that this extension is isomorphic to one given by the cocycle \(-c_{x_0}\).

**Theorem 6.2.** For a non-compact symplectic manifold \((M, \omega)\) the cocycle \(c_{x_0}\) on the Lie algebras \(\text{Vect}(M, \omega)\) and \(\text{Vect}_0(M, \omega)\) is non-trivial.

**Proof.** It suffices to prove our statement for the Lie algebra \(\text{Vect}_0(M, \omega)\).

First we prove that for each form \(\beta \in \Omega^{2n-1}(M)\) there is a unique form \(\alpha \in \Omega^1(M)\) such that \(\beta = \alpha \wedge \omega^{n-1}\). Indeed, using Darboux coordinates it is easy to check that this has a unique local solution \(\alpha\). These are compatible and we get a global solution by gluing them.
Note that for each form \( \alpha \in \Omega^1(M) \) there is a positive integer \( N \) and \( 2N \) functions \( f_k, g_k \in C^\infty(M) \) \((k = 1, \ldots, N)\) such that \( \alpha = \sum_{k=1}^N f_k \, d g_k \) which follows easily from the existence (by dimension theory) of a finite atlas for \( M \).

Since \( H^{2n}(M, \mathbb{R}) = 0 \) there is a form \( \beta \in \Omega^{2n-1}(M) \) such that \( \omega^n = d \beta \). Then we have \( \omega^n = \sum_{k=1}^N f_k \, d g_k \wedge \omega^{n-1} \). By (6) and using this equality we get

\[
\sum_{k=1}^N \{ f_k, g_k \} = -n.
\]

Assume that the extension \( P(M) \to \text{Vect}_0(M, \omega) \) is split. Then \( P(M) \) is a direct sum of the space of constant functions on \( M \) and an ideal isomorphic to \( \text{Vect}_0(M, \omega) \) by \( P(M) \to \text{Vect}_0(M, \omega) \). Equality (7) means that these summands have nonzero intersection. This contradiction proves the statement.

Now we consider a compact symplectic manifold \((M, \omega)\). It is known that the extension \( P(M) \to \text{Vect}_0(M, \omega) \) is split.

For a closed form \( \alpha \) denote by \([\alpha]\) the cohomology class of \( \alpha \). Denote by \( L \) the linear map \( H^p(M, \mathbb{R}) \to H^{p+2}(M, \mathbb{R}) \) defined by \( a \to a \wedge [\omega] \), where \( a \in H^p(M, \mathbb{R}) \).

**Theorem 6.3.** Let \((M, \omega)\) be a compact symplectic manifold. The cocycle \( c_{x_0} \) on the Lie algebra \( \text{Vect}(M, \omega) \) is non-trivial iff the linear map

\[
L^{n-1} : H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})
\]

is not equal zero.

**Proof.** We may assume that \( \int_M \omega^n = 1 \). Put for brevity \( V = \text{Vect}(M, \omega) \) and \( V_0 = \text{Vect}_0(M, \omega) \). Set

\[
b(X, Y) = \int_M \omega(X, Y) \omega^n,
\]

where \( X, Y \in V \). It is easy to check that \( b \) is a 2-cocycle on \( V \).

Multiplying both sides of equality (5) by \( \omega^n \) and integrating over \( M \) we get

\[
b(X, Y) - c_{x_0}(X, Y) = \int_M \left( \int_{x_0}^{X} i_{[X,Y]_0} \omega \right) \omega^n.
\]

Since the right hand side of (8) is a coboundary of a 1-cochain in \( C^1(V, \mathbb{R}) \), the cocycles \( c_{x_0} \) and \( b \) are cohomologous. By (6) we have

\[
b(X, Y) = n \int_M \alpha_X \wedge \alpha_Y \wedge \omega^{n-1},
\]

for any \( X, Y \in V \). If \( X \in V_0 \) the form \( \alpha_X \) is exact, and \( b(X, Y) = 0 \) by (9). This proves (1).

Suppose that the cocycle \( b \) is trivial, i.e., there is a linear functional \( f \) on \( V \) such that for any \( X, Y \in V \) we have \( b(X, Y) = f([X,Y]) \). By [1] we have \([V, V] = [V_0, V_0] = V_0\). This implies \( b = 0 \). So the cocycle \( b \) is trivial iff it equals zero. By (9) and the Poincaré duality this implies that \( L^{n-1} = 0 \) on \( H^1(M, \mathbb{R}) \). This proves (2). \( \square \)
We know no example when $H^1(M, \mathbb{R}) \neq 0$ and the map $L^{n-1} : H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$ is an isomorphism (see, for example, [11], ch. 4). Thus in this case the cocycle $c_{x_0}$ is non-trivial whenever $H^1(M, \mathbb{R}) \neq 0.$

References

[1] V. I. Arnold, The one-dimensional cohomology of the Lie algebra of divergence-free vector fields, and the winding numbers of dynamical systems, Functional. Anal. Appl., 3, no. 4, (1969), 77-78 (Russian). English translation: Functional. Anal. Appl., 3, (1969), 319-321.
[2] J.-L. Dupont, A. Guichardet, A propos de l'article "Sur la cohomologie réelle des groupes de Lie simples réels," Ann. Sci. Ec. Norm. Sup., 11 (1978), 293-296.
[3] W. T. van Est, Group cohomology and Lie algebra cohomology in Lie groups. I, II, Indagationes Mathematicae, 15 (1953), 484-492; 493-504.
[4] A. Guichardet, Cohomologie des groupes topologiques et des algèbres de Lie, Cedic/Nathan, Paris, 1980.
[5] A. Guichardet, D. Wigner, Sur la cohomologie réelle des groupes de Lie simples réels, Ann. Sci. Ec. Norm. Sup., 11 (1978), 277-292.
[6] C. Godbillon, Geometrie differentielle et mecanique analytique, Hermann, Paris, 1969.
[7] R. S. Ismagilov, Infinite-dimensional groups and their representations, in: Proc. Intern. Congress Math. (Warszawa, 1983), 861-875 (Russian).
[8] S. MacLane, Homology, Grundl. der. Math. Wiss., 114, Springer-Verlag, 1963.
[9] G. D. Mostow, Cohomology on topological groups and solvmanifolds, Ann. Math., 73 (1961), 20-48.
[10] Ya. A. Neretin, Central extensions of groups of symplectomorphisms, arXiv:math.DG/0406213
[11] A. Weil, Introduction à l’étude des variétés kähleriennes, Hermann, Paris, 1958.

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