Parabolic-like mappings

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Abstract

In this paper we introduce the notion of parabolic-like mapping, which is an object similar to a polynomial-like mapping, but with a parabolic external class, i.e. an external map with a parabolic fixed point. We prove a straightening theorem for parabolic-like maps, which states that any parabolic-like map of degree 2 is hybrid conjugate to a member of the family $\Per_1(1) = \{ P_A(z) = z + 1/z + \sqrt{A} \mid A \in \mathbb{C} \}$, and this member is unique (up to holomorphic conjugacy) if the filled Julia set of the parabolic-like map is connected.

1 Introduction

A polynomial-like map of degree $d$ is a triple $(f, U', U)$ where $U', U$ are open subsets of $\mathbb{C}$, $U', U \approx \mathbb{D}$, $U' \subsetneq U$, and $f : U' \to U$ is a proper holomorphic map of degree $d$. These were originally singled out and studied by Douady and Hubbard in the groundbreaking paper On the Dynamics of Polynomial-like Mappings, see [DH]. A polynomial-like map of degree $d$ is naturally characterized by two associated sub-dynamical systems called the internal class, which is a polynomial, and the external class of the mapping, which is a degree $d$ real-analytic orientation preserving and strongly expanding (i.e. hyperbolic) covering of the unit circle by itself. Since the external class of a polynomial-like mapping is expanding, all its periodic points must be repelling. In particular, the periodic points of an external map of a polynomial-like mapping can not be parabolic.

The aim of this paper is, in some sense, to avoid this restriction. More precisely we will define an object, a parabolic-like mapping, to describe the parabolic case. A parabolic-like mapping is thus similar to a polynomial-like mapping, but with a parabolic external class, i.e. an external map with a parabolic fixed point. This implies that the domain does not need to be compactly contained in the codomain. We will deal with the case in which the
domain is not contained in the codomain, since in the other case we would fall back into the case of polynomial-like maps.

We know that the family $\text{Per}_1(1) = \{P_A(z) = z + 1/z + \sqrt{A} \mid A \in \mathbb{C}\}$ is the family consisting of the quadratic rational maps with a parabolic fixed point of multiplier 1. In analogy with the theory of polynomial-like mappings, we prove a straightening theorem for parabolic-like maps, which states that any parabolic-like map of degree 2 is hybrid conjugate to a member of the family $\text{Per}_1(1)$, and this member is unique (up to holomorphic conjugacy) if the filled Julia set of the parabolic-like map is connected.

The maps belonging to the family $\text{Per}_1(1)$ have two simple critical points at $z = \pm 1$, and, for $A \neq 0$, a parabolic fixed point at infinity and an other fixed point at $z = -\frac{1}{\sqrt{A}}$. For $A = 0$ we obtain the map $P_0(z) = z + 1/z$, which has just one double parabolic fixed point at infinity. This map is conformally equivalent to the map $h_2 = \frac{3z^2 + 1}{3z^2 + z^2} + 3z^2 + 1$ under the Möbius transformation which sends $z = 1$ to infinity, $z = -1$ to $z = 0$ and infinity to $z = 1$. All the other maps $P_A$, with $A \neq 0$, are not globally conformally conjugate to the map $h_2$ anymore, but we will prove they are still conjugate to $h_2$ outside their filled Julia set if it is connected, or on part of the basin of infinity if not, and then that $h_2$ is the external map of $P_A$ (see Prop. 2.8).

In this paper we will first define a parabolic-like map and the filled Julia set of a parabolic-like map. Then we will construct the external class for parabolic-like maps, and give its properties. Finally, the straightening theorem will state that every parabolic-like map of degree 2 is hybrid equivalent to a member of the family $\text{Per}_1(1) = \{P_A(z) = z + 1/z + \sqrt{A} \mid A \in \mathbb{C}\}$, and moreover this member is unique (up to holomorphic conjugacy) if the filled Julia set of the parabolic-like map is connected. This theorem will be obtained by gluing outside a generic parabolic-like map of degree 2 the map $h_2 = \frac{3z^2 + 1}{3z^2 + z^2}$.

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2 Parabolic-like maps

**Definition 2.1. (Parabolic-like maps)** A parabolic-like map of degree $d$ is a 4-tuple $(f, U', U, \gamma)$ where

- $U', U$ are open subsets of $\mathbb{C}$, $U' \subset U$, $U \cup U' \approx \mathbb{D}$, $U'$ not contained into $U$
Figure 1: On a parabolic-like map \((f, U', U, \gamma)\) the arc \(\gamma\) divides \(U', U\) into \(\Omega', \Delta'\) and \(\Omega, \Delta\) respectively. These sets are such that \(\Omega'\) is compactly contained in \(U\), \(\Omega' \subset \Omega\) and \(f: \Delta' \rightarrow \Delta\) is an isomorphism.

- \(f: U' \rightarrow U\) is a proper holomorphic map of degree \(d\) with a parabolic fixed point at \(z = z_0\) of multiplier 1

- \(\gamma: [-1, 1] \rightarrow \overline{U}, \gamma(0) = z_0\) is a \(C^1\) arc forward invariant under \(f\), such that

\[
f(\gamma(t)) = \gamma(dt) \ \forall -\frac{1}{d} \leq t \leq \frac{1}{d}, \ \gamma\left(\left[\frac{1}{d}, 1\right] \cup [-1, \frac{1}{d}]\right) \subseteq U \setminus U', \ \gamma(\pm 1) \in \partial U.
\]

It lives in repelling petal(s) of \(z_0\) and it divides \(U', U\) into \(\Omega', \Delta'\) and \(\Omega, \Delta\) respectively, such that \(\Omega' \subset U\) (and \(\Omega' \subset \Omega\)), \(f: \Delta' \rightarrow \Delta\) is an isomorphism (see Fig. 1) and \(\Delta'\) contains at least one attracting fixed petal of \(z_0\).

**Remark 2.1.** We can consider \(\gamma := \gamma_+ \cup \gamma_-\), where \(\gamma_+(t) = \gamma(t), t \in [0, 1], \) and \(\gamma_-(t) = \gamma(-t), t \in [0, 1] \) \(i.e.\ \gamma_+: [0, 1] \rightarrow \overline{U}, \ \gamma_-: [0, -1] \rightarrow \overline{U}, \ \gamma_\pm(0) = z_0\). Where it will be convenient \(e.g.\ in\ the\ examples\) we will refer to \(\gamma_\pm\) instead of \(\gamma\). Therefore we will often consider a parabolic-like map as a 5-tuple \((f, U', U; \gamma_+, \gamma_-)\) instead of a 4-tuple \((f, U', U; \gamma)\). These
two notions are equivalent.

The filled Julia set and the Julia set are defined for parabolic-like maps in the same fashion as for polynomials.

**Definition 2.2.** Let $f : U' \to U$ be a parabolic-like map. We define the filled Julia set of $f$ as the set of points in $U'$ that never leave $(\Omega' \cup \gamma_{\pm}(0))$ under iteration, i.e.

$$K_f = \{ z \in U' | \forall n \geq 0 , \ f^n(z) \in \Omega' \cup \gamma_{\pm}(0) \}.$$ 

As for polynomials, we define the Julia set of $f$ as

$$J_f := \partial K_f$$

**Remarks 2.1.** An equivalent definition for the filled Julia set of $f$ is

$$K_f = \bigcap_{n \geq 0} f^{-n}(U) \cap (\Omega' \cup \gamma_{\pm}(0)).$$

The filled Julia set is a compact subset of $U \cap U'$ and, if it is connected, it is full (since it is the intersection of topological disks).

**2.0.1 Examples**

1. Consider the function $h_2 = \frac{3z^2 + 1}{3 + z^2}$. This map has a parabolic-fixed point at $z = 1$ of multiplicity 2 and a critical point (in the parabolic basin of attraction of $z = 1$) at $z = 0$. Choose $\epsilon > 0$ and define $U' = \{ z : |z| < 1 + \epsilon \}$, and $U = h_2(U')$. Since the parabolic fixed point has multiplicity 2, there are 4 petals (2 repelling petals and 2 attracting ones, alternating) whose union form a neighborhood of the parabolic fixed point $z = 1$. Let $\Xi_{\pm}$ be the repelling petals. The attracting directions of the parabolic fixed point are along the real axis, while the repelling ones are perpendicular to the real axis. Therefore the repelling petals $\Xi_{\pm}$ intersect the unit circle. Let $\phi_{\pm} : \Xi_{\pm} \to \mathbb{H}_{\pm}$ be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point $z = 1$. The image of the unit circle in the Fatou coordinate planes are periodic curves. Choose $m > 0$ such that

$$\forall z \in \phi_{+}(\mathbb{S}^1 \cap \Xi_{+}) \ \ \text{Im}(z) > -m$$
∀z ∈ φ(S¹ ∩ Ξ) \quad \text{Im}(z) < m

Then let us define

\[ \gamma_+ := φ_+^{-1}(-m + \mathbb{R}_-) : \mathbb{C} \to \Xi_+, \quad \gamma_- := φ_-^{-1}(m + \mathbb{R}_-) : \mathbb{C} \to \Xi_. \]

Reparametrizing the arcs as

\[ \gamma_+ : [0, 1] \to \Xi_+ \quad t \to \phi_+^{-1}(\log_d(t) - im), \]

and

\[ \gamma_- : [0, -1] \to \Xi_- \quad t \to \phi_-^{-1}(\log_d(-t) + im). \]

We to obtain

\[ h_2(\gamma_{±}(t)) = γ_{±}(dt) \quad ∀0 ≤ ±t ≤ \frac{1}{d} \]

and \( γ_+ : [0, 1] \to \Xi_+ \), \( γ_- : [0, -1] \to \Xi_- \).

Then \( (h_2, U', U, γ_{±}) \) is a parabolic-like map of degree 2.

2. Let \( f = z^3 - 3a^2z + 2a^3 + a \), for \( a = -2/3i \). This map has a super-attracting fixed point at \( z = a \), a parabolic fixed point at \( z = -a/2 \) and a critical point at \( z = -a \). Let \( \Xi(-a/2) \) be the parabolic basin of the parabolic fixed point \( z = -a/2 \). Then the critical point \( z = -a \) belongs to the parabolic basin \( \Xi(-a/2) \). Let \( φ : \Xi(-a/2) \to \mathbb{D} \) be the Riemann map. Then its inverse \( ψ := φ^{-1} \) extends continuously to the boundaries by the Carathéodory theorem, and we can normalize it by setting \( ψ(0) = -a \) and \( ψ(1) = -a/2 \). Then \( φ \circ f \circ ψ = h_2 \). Let \( w \) be a \( h_2 \) periodic point in the first quadrant of \( \mathbb{D} \) such that the hyperbolic geodesic \( \tilde{γ} \) connecting \( w \) and \( \overline{w} \) separates the critical value \( z = 1/3 \) from the parabolic fixed point \( z = 1 \). Let \( U \) be the Jordan domain bounded by \( \tilde{γ} = ψ(\tilde{γ}) \) union the arcs up to potential level 1 of the external rays landing at \( ψ(w) \) and \( ψ(\overline{w}) \) together with the arc of the level 1 equipotential connecting the 2 rays around \( z = a \) (see Fig. 2). Let \( U' \) be the preimage of \( U \) under \( f \) and the arcs \( γ_{±} \) be external rays landing at the parabolic fixed point \( z = 1/3i \). Then \( (f, U', U, γ_{±}) \) is a parabolic-like map of degree 2 (see Fig. 3).
Let $f = z^2 + c$, for $c = -0.125 + 0.6495i$ (fat rabbit). Let $\Xi(a)$ be the parabolic basin of the parabolic fixed point. Let $\phi : \Xi(a) \to \mathbb{D}$ be the Riemann map. Then its inverse $\psi := \phi^{-1}$ extends continuously to the boundaries by the Carathodory theorem, and we can normalize it by fixing zero and sending one to the parabolic fixed point. Then $\phi \circ f^3 \circ \psi = h_2$. Let as above $w$ be a $h_2$ periodic point in the first quadrant such that the hyperbolic geodesic $\tilde{\gamma}$ connecting $w$ and $\overline{w}$ separate the critical value $z = 1/3$ from the parabolic fixed point $z = 1$. Define $\hat{\gamma} = \psi(\tilde{\gamma})$ and $\hat{\gamma}' = f^{-1}(\hat{\gamma})$. Let $U$ be the Jordan domain bounded by $\hat{\gamma}$ union the arcs up to potential level 1 of the external rays landing at $\psi(w)$ and $\psi(\overline{w})$ union $\hat{\gamma}'$ union the arcs up to potential level 1 of the external rays landing at $f^{-1}(\psi(w))$ and $f^{-1}(\psi(\overline{w}))$, together with the arc of the level 1 equipotential connecting the 4 rays around the parabolic fixed point. Let $U'$ be the preimage of $U$ under $f^3$ and the arcs $\gamma_{\pm}$ be the external rays for the angles $1/7$ and $2/7$. Then $(f^3, U', U, \gamma_{\pm})$ is a parabolic-like map of degree 2 (see Fig. 4).

Remarks 2.2. As we can see from the examples, there are many differ-
Figure 3: A parabolic-like restriction of the map $f = z^3 - 3a^2z + 2a^3 + a$, for $a = -2/3i$.

ent equivalent choices of the domain and codomain of a parabolic-like map. This is because the notion of parabolic-like map (as well as the notion of polynomial-like map) is local.

Moreover, the forward invariant arcs $\gamma_\pm$ (and therefore $\gamma := \gamma_+ \cup \gamma_-$) are defined up to isotopy. Here by isotopy we mean a map

$$H : [0,1] \times [-1,1] \to U$$

continuous and such that for all $s \in [0,1]$

1. $\gamma_s = H(s, \cdot) : [-1,1] \to U$ is injective,
2. $f(\gamma_s(t)) = \gamma_s(dt) \ \forall -\frac{1}{2} \leq t \leq \frac{1}{2}$
3. $\gamma_s([\frac{1}{2}, 1] \cup [-1, -\frac{1}{2}]) \subseteq U \setminus U'$,
4. $\gamma_s(\pm 1) \in \partial U$,
5. $\text{and } \gamma_s(0) = z_0$.

**Definition 2.3.** Let $(f, U', U, \gamma)$ be a parabolic-like map of degree $d$. Let $A$ be an open subset of $U$ such that $A \approx \mathbb{D}$ and the critical points and the
Figure 4: A parabolic-like restriction of the map \( f = z^2 + c \), for \( c = -0.125 + 0.6495i \).

filled Julia set of \( f \) are compactly contained in \( A \cap A' \) (where \( A' = f^{-1}(A) \)) and \( A' \cap \Omega' \subset A \cap \Omega \). Then we call \((f, A', A, \gamma)\) a \textit{parabolic-like restriction} of \((f, U', U, \gamma)\).

A parabolic-like restriction has the same degree, Julia set and filled Julia set as the original parabolic-like map. A parabolic-like restriction \((f, A', A, \gamma)\) has the same arcs of the original parabolic-like map \((f, U', U, \gamma)\), but rescaled in order to obtain \( \gamma((\frac{1}{d}, 1] \cup [-1, -\frac{1}{d}]) \subseteq A \setminus A', \quad \gamma((\frac{1}{d}, 1] \cup [-1, -\frac{1}{d}]) \subseteq A \setminus A', \quad \gamma(\pm 1) \in \partial A \).

2.1 The external class of \( f \)

In analogy with the polynomial-like setting, we want to associate to any parabolic-like map \((f, U', U, \gamma)\) of degree \( d \) a real-analytic map \( h_f : S^1 \to S^1 \) of the same degree \( d \) and with a parabolic fixed point, unique up to conjugacy by a real-analytic diffeomorphism. We will call \( h_f \) an \textit{external map} of \( f \), and we will call \([h_f]\) (its conjugacy class under analytic diffeomorphism) the external class of \( f \).
The construction of an external map of a parabolic-like map follows the construction of an external map in [DH], up to the differences given by the geometry of our setting.

Let \((f, U', U, \gamma)\) be a parabolic-like map of degree \(d\), and let us suppose first that the filled Julia set \(K_f\) is connected. Then \(K_f\) contains all the critical points of \(f\) and hence \(f : U' \setminus K_f \to U \setminus K_f\) is an holomorphic degree \(d\) covering map.

Let \(\alpha : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \{0\}\) be the Riemann map, with \(|\alpha(z)| \to 1\) as \(z \to K_f\).

Write \(W' = \alpha(U' \setminus K_f)\) and \(W = \alpha(U \setminus K_f)\) (see Fig. 5) and define the map:

\[
\begin{align*}
\alpha & : \mathbb{C} \setminus K_f \rightarrow \mathbb{C} \setminus \{0\} \\
\alpha(\gamma_+) & : W' \rightarrow W,
\end{align*}
\]

Then the map \(\alpha(\gamma_+)\) is an holomorphic degree \(d\) covering.

![Figure 5: W' = α(U' \setminus K_f), W = α(U \setminus K_f) and h⁺: W' → W.](image)

Let \(\tau(z) = 1/\bar{z}\) denote the reflection with respect to the unit circle, and define \(W_- = \tau(W), W'_- = \tau(W'_-)\), \(\overline{W} = W \cup S^1 \cup W_-\) and \(\overline{W'} = W' \cup S^1 \cup W'_-\). Applying the strong reflection principle with respect to \(S^1\) we can extend analytically the map \(h^+ : W' \to W\) to \(h : \overline{W'} \to \overline{W}\). Let \(h_f\) be the restriction of \(h\) to the unit circle, then the map \(h_f : S^1 \to S^1\) is an external map of \(f\). This map is by construction symmetric with respect to the unit circle, and it has then on both inside and outside the unit circle the dynamics that \(f\) has outside of the filled Julia set. Therefore \(h_f\) has a parabolic fixed point \(z_1\) with multiplicity 2\(n\), where \(n\) is the number of petals of \(z_0\) (the parabolic point of \(f\)) outside \(K_f\). In particular the parabolic fixed point of \(h_f\) has an even number of petals.
Remarks 2.3. 1. As we have seen, we can construct a canonical external map of $f$ when $K_f$ is connected. Therefore in the case $K_f$ connected we could speak about 'the' external map of $f$, instead of 'an' external map.

However we prefer to refer to this map as 'an' external map of $f$ and to consider more generically the external 'class' of $f$, which is its conjugacy class under real analytic diffeomorphism, in order to allow more flexibility to our setting.

2. By the remark 2.1, $\gamma = \gamma_+ \cup \gamma_-$. Let us define $\tilde{\gamma}_+ := \alpha(\gamma_+ \setminus \{z_0\})$, $\tilde{\gamma}_- := \alpha(\gamma_- \setminus \{z_0\})$ and $\tilde{\gamma} := \tilde{\gamma}_+ \cup \tilde{\gamma}_- \cup \{z_1\}$. Since the arcs $\gamma_{\pm}$ are forward invariant under $f$, the arcs $\tilde{\gamma}_{\pm}$ are forward invariant under $h$.

The conformal conjugacy $\alpha$ extends to a topological conjugacy between the arcs $\gamma$ and $\tilde{\gamma}$.

3. The arc $\tilde{\gamma}$ inherits from $\alpha$ the properties of the arc $\gamma$. Therefore it lives in repelling petals of $z_1$ and it divides $W'$, $W$ into $\Omega_W, \Delta'_W$ and $\Omega'_W, \Delta_W$ respectively, such that $h : \Delta'_W \to \Delta_W$ is an isomorphism and $\Delta'_W$ contains at least an attracting fixed petal of $z_1$.

4. Since there is at least an attracting fixed petal of $z_1$ in $\Delta_W$, it separates the arcs $\tilde{\gamma}_{\pm}$ by an angle greater then zero and then the arcs $\tilde{\gamma}_{\pm}$ cannot form a cusp.

We can see that the arcs $\tilde{\gamma}_{\pm}$ are actually tangent to $S^1$ at $z_1$.

Since the arcs $\tilde{\gamma}_{\pm}$ live in repelling petals $\Xi_{\pm}$ of $z_1$, their images under Fatou coordinates $\phi_{\pm}(\tilde{\gamma}_{\pm})$ (where $\phi_{\pm} : \Xi_{\pm} \to \mathbb{H}_{\pm}$ are Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point) are periodic curves. Since the arcs $\tilde{\gamma}_{\pm}$ are forward invariant under $h$ and $\phi_{\pm} \circ h(z) = 1 + \phi_{\pm}(z)$, the curves $\phi_{\pm}(\tilde{\gamma}_{\pm})$ are invariant under the map $T(z) = z + 1$. This implies that the curves $\phi_{\pm}(\tilde{\gamma}_{\pm})$ are periodic and bounded from both above and below, and then that they are tangent to $S^1$ at $z_1$.

2.1.1 The general case

Let $(f, U', U, \gamma_{\pm})$ be a parabolic-like map of degree $d$. To deal with the filled Julia set non connected, we need to construct annular Riemann surfaces $T$ and $T'$ that would play the role of $U \setminus K_f$ and $U' \setminus K_f$ respectively, and an analytic map $F : T' \to T$ that would play the role of $f$. 

10
The construction takes inspiration from the one in [DH]. Let $V \approx \mathbb{D}$ be a full compact connect subset of $U$ containing $\overline{\Omega}$, the critical values of the parabolic-like map and such that $f : f^{-1}(V) \rightarrow V$ is a parabolic-like restriction of $(f, U', U, \gamma_\pm)$.

Let us call $L = f^{-1}(V) \cap \Omega'$ and $M = f^{-1}(V) \cap \Delta'$. Define $X'_0 = (U \cup U') \setminus L$, $U_0 = U \setminus \overline{V}$, $A_0 = U \cap U' \setminus L$, $X_0 = U \setminus L$, $A'_0 = U' \setminus L$ and $A''_0 = U' \setminus f^{-1}(V)$.

Let $\rho_0 : X_1 \rightarrow X_0$ be a degree $d$ covering map for some Riemann surface $X_1$, and define $V_1 = \rho_0^{-1}(V \setminus L)$. Define $X''_1 = X_1 \setminus \overline{V_1}$. The map $f : A''_0 \rightarrow U_0$ is proper holomorphic of degree $d$, and $\rho_0 : X''_1 \rightarrow U_0$ is a proper holomorphic map of degree $d$. Therefore we can choose $\pi_0 : A''_0 \rightarrow X''_1$ a lift of $f : A''_0 \rightarrow U_0$ to $\rho_0 : X''_1 \rightarrow U_0$. Therefore $\pi_0$ is an isomorphism. The subset $\Delta$ has $d$ preimages under the map $\rho_0$. Let us call $\Delta_1$ the preimage of $\Delta$ under $\rho_0$ such that $\Delta_1 \cap \pi_0(A''_0 \cap \Delta') \neq \emptyset$. Since $f : \Delta' \rightarrow \Delta$ is an isomorphism, we can extend the map $\pi_0$ to $\Delta'$. Let us call $B'_1 = X''_1 \cup \Delta_1$.

Since $\pi_0(\Delta' \setminus A''_0) \cap X''_1 = \emptyset$, the extension $\pi_0 : A''_0 \rightarrow B'_1$ is an isomorphism (see Fig. 6).

Let us call $B_1 = \pi_0(A''_0)$.

Define $A'_1 = \rho_0^{-1}(A_0)$ and $f_1 = \pi_0 \circ \rho_0 : A'_1 \rightarrow B_1$. It is a proper holomorphic map of degree $d$ (see Fig. 7). Indeed $\rho_0 : A'_1 \rightarrow A_0$ is a degree $d$ covering by definition and $\pi_0 : A_0 \rightarrow B_1$ is an isomorphism because it is a restriction of an isomorphism. Define $X'_1 = X_1 \setminus \pi_0(A''_0 \setminus A_0)$, then $B_1 \subset X'_1$.

Let $\rho_1 : X'_2 \rightarrow X'_1$ be a degree $d$ covering map for some Riemann surface $X_2$, and call $B''_1 = \rho_0^{-1}(B_1)$. Define $\pi_1 : A'_1 \rightarrow B''_1$ as a lift of $f_1$ to $\rho_1$. Then $\pi_1$ is an isomorphism, since $f_1 : A'_1 \rightarrow B_1$ is a degree $d$ covering and $\rho_1 : B''_1 \rightarrow B_1$.

Figure 6: On the left: in yellow $U_0 = U \setminus \overline{V}$, in green plus purple $A'_0 = U' \setminus L$. On the right: in green plus purple $B'_1 = X''_1 \cup \Delta_1$. The map $\pi_0 : A''_0 \rightarrow B'_1$ is an isomorphism.
Figure 7: Let us define $f_1 = \pi_0 \circ \rho_0 : A'_1 \to B_1$. It is a proper holomorphic map of degree $d$. Indeed $\rho_0 : A'_1 \to A_0$ is a degree $d$ covering and $\pi_0 : A_0 \to B_1$ is an isomorphism because it is a restriction of an isomorphism.

is a degree $d$ covering as well. Define $A_1 = A'_1 \cap X'_1$, and $B_2 = \pi_1(A_1)$.

Define $A'_2 = \rho_1^{-1}(A_1)$ and $f_2 = \pi_1 \circ \rho_1 : A'_2 \to B_2$. It is a proper holomorphic map of degree $d$, indeed $\rho_1 : A'_2 \to A_1$ is a degree $d$ covering and $\pi_1 : A_1 \to B_2$ is an isomorphism. Define $X'_2 = X_2 \setminus \pi_1(A'_1 \setminus A_1)$, then $B_2 \subset X'_2$.

Figure 8: Let $\rho_1 : X_2 \to X'_1$ be a degree $d$ covering map for some Riemann surface $X_2$, and call $B'_2 = \rho_1^{-1}(B_1)$. Define $\pi_1 : A'_1 \to B'_2$ as a lift of $f_1$ to $\rho_1$. Then $\pi_1$ is an isomorphism. Define $A_1 = A'_1 \cap X'_1$, and $B_2 = \pi_1(A_1)$, $A'_2 = \rho_1^{-1}(A_1)$ and $f_2 = \pi_1 \circ \rho_1 : A'_2 \to B_2$.

Define recursively $\rho_{n-1} : X_n \to X'_{n-1}$ for $n > 1$ as holomorphic degree
covering for some Riemann surface $X_n$ and call $B'_n = \rho_{n-1}^{-1}(B_{n-1})$. Define recursively $\pi_{n-1} : A'_{n-1} \to B'_n \subset X_n$ as a lift of $f_{n-1}$ to $\rho_{n-1}$. Then $\pi_{n-1}$ is an isomorphisms.

Define $A_{n-1} = A'_{n-1} \cap X'_{n-1}$, and $B_n = \pi_{n-1}(A_{n-1})$. Define $A'_n = \rho_{n-1}^{-1}(A_{n-1})$ and $f_n = \pi_{n-1} \circ \rho_{n-1} : A'_n \to B_n$. Then all the $f_n$ are proper holomorphic maps of degree $d$, indeed $\rho_{n-1} : A'_n \to A_{n-1}$ are degree $d$ coverings and $\pi_{n-1} : A_{n-1} \to B_n$ are isomorphisms. Define $X'_n = X_n \setminus \pi_{n-1}(A'_{n-1} \setminus A_{n-1})$, then $B_n \subset X'_n$. Let us define $X = \bigsqcup_{n \geq 0} X'_n$ and $X' = \bigsqcup_{n \geq 1} X_n$ (disjoint union). Let $T$ be the quotient of $X$ by the equivalence relation identifying $x \in A'_n$ with $x' = \pi_n(x) \in X_{n+1}$, and $T'$ be the quotient of $X$ by the same equivalence relation. Therefore $T$ is an annulus, since it is constructed by identifying at each level an inner annulus $A_i \subset X'_i$ with an outer annulus $B_{i+1} \subset X'_{i+1}$ in the next level. Similarly $T'$ is an annulus, since it is constructed by identifying at each level an inner annulus $A'_i \subset X_i$ with an outer annulus $B'_{i+1} \subset X'_{i+1}$ in the next level. Hence (since $\forall i > 1 \ : \ X'_i \subset X_i$) $T \cup T' = T' \cup X'_0/\sim$ is an annulus, since $X'_0$ is an annulus and $\pi_0$ identifies an inner annulus of $X_0$ (which is $A'_0$) with an outer annulus of $X'_1$ ($B'_1$), and $T'$ is an annulus. The covering maps $\rho_n$ induce a degree $d$ holomorphic covering map $F : T' \to T$. Indeed, $F$ is well defined, since at each level $f_n = \pi_{n-1} \circ \rho_{n-1}$ by definition and $\pi_n$ is as a lift of $f_n$ to $\rho_n$. Therefore $\rho_n \circ \pi_n = f_n = \pi_{n-1} \circ \rho_{n-1}$, and the following diagram commutes

$$
\begin{array}{ccc}
A'_n & \xrightarrow{\pi} & B'_{n+1} \\
\downarrow{\rho_{n-1}} & & \downarrow{\rho_n} \\
A_n & \xrightarrow{\pi_{n-1}} & B_n
\end{array}
$$

Finally, the map $F$ is proper since by definition $F|_{X_n} = \rho_{n-1} : X_n \to X'_{n-1}$ is a proper map (and $F|_{X_1} = \rho_0 : X_1 \to X'_0$ is proper into its domain, which is $X_0$).

Now, let us construct an external map for $f$. Let us say that the modulus of the annulus $T \cup T'$ is $m$. Let $A \subseteq \mathbb{C}$ be any annulus with inner boundary $\mathbb{S}^1$ and modulus $m$. Then there exists an isomorphism

$$
\alpha : T \cup T' \longrightarrow A
$$

with $|\alpha(z)| \to 1$ when $z \to K_f$. Then we just have to repeat the construction done for the case $K_f$ connected.
2.1.2 Singly parabolic external maps

So far we have constructed external maps from parabolic-like maps. We have then considered external maps in relation to parabolic-like maps.

We now want to separate these two concepts, and then consider external maps as maps from the unit circle to itself with some specific properties, without referring to a particular parabolic-like map. In order to do that we need to give a definition of external map which endow it with all the properties it would have if it would have been constructed from a parabolic-like map.

An external map $h_f$ constructed from a parabolic-like map $f$ of degree $d$ is an orientation preserving real-analytic map $h_f : S^1 \to S^1$ of the same degree $d$ with a parabolic fixed point $z = z_1$. The repelling petals of $z_1$ intersect the unit circle, since the repelling petal(s) of the parabolic fixed point of $f$ intersect the Julia set. When we consider a generic orientation preserving real-analytic degree $d > 1$ map $h : S^1 \to S^1$ with a parabolic fixed point $z = z_*$ of multiplier 1, we cannot insure that the repelling petals of the parabolic fixed point $z_*$ intersect the unit circle. In order to assure this it is sufficient to require $h$ to be weakly expanding. Here weakly expanding means that $|h'(z)| = 1$, if $h(z) = z_*$, and $|h'(z)| > 1$, if $h(z) \neq z_*$.

Definition 2.4. (Singly parabolic external map) Let $h : S^1 \to S^1$ be an orientation preserving weakly expanding real-analytic degree $d > 1$ map. We say that $h : S^1 \to S^1$ is a singly parabolic external map, if $h$ has one parabolic fixed point $z = z_*$ of multiplier 1 and all its other periodic points are repelling.

The multiplicity of $z_*$ as parabolic fixed point of $h$ is even and in particular greater than 1 (exactly as for the fixed point of an external map constructed from a parabolic-like map), since the map $h$ is symmetric with respect to the unit circle and it is weakly expanding.

Remarks 2.4.  
- A parabolic-like map as defined in 2.1 is called singly parabolic, because its external class is singly parabolic. We can generalize this concept to parabolic-like maps with external map with several parabolic fixed points. A generic parabolic-like map has as many pairs of invariant arcs $\gamma_\pm$ (which divide $U$ and $U'$ in $\Omega, \Delta_1, \Delta_2, ... , \Delta_n$ and $\Omega', \Delta'_1, \Delta'_2, ..., \Delta'_n$, respectively) as the number of parabolic fixed points.
We gave the definition of singly parabolic-like map, instead of a generic one, in order to simplify the notation.

- We are considering maps with a parabolic fixed point, rather than a parabolic periodic orbit, in order to simplify the notation. We can easily generalize to the case of parabolic periodic orbits.
2.2 Conjugacy between parabolic-like maps

The aim of this section is to prove that, given a parabolic-like map of degree \(d\) and chosen a singly parabolic external map of the same degree \(d\), we can construct a parabolic-like map which has as external map the chosen one and which is hybrid conjugate to the given parabolic-like map. We start by defining the notion of conjugacies between parabolic-like maps.

**Remark 2.2.** Remember that if \((f, U', U, \gamma)\) is a parabolic like map, we can consider \(\gamma : [-1, 1] \to U\) as \(\gamma := \gamma_+ \cup \gamma_-\), where \(\gamma_+ = \gamma[0, 1]\) and \(\gamma_+(t) = \gamma(t), t \in [0, 1]\), and \(\gamma_-(t) = \gamma(-t), t \in [0, 1]\).

**Definition 2.5.** (Conjugacy for parabolic-like mappings)

Let \((f, U', U, \gamma_+ f, \gamma_f)\) and \((g, V', V, \gamma_+ g, \gamma_g)\) be two parabolic-like mappings.

We say that \(f, g\) are topologically conjugate if there exist parabolic like restrictions \((f, A', A, \gamma_+ f, \gamma_f)\) and \((g, B', B, \gamma_+ g, \gamma_g)\) of \((f, U', U, \gamma_+ f, \gamma_f)\) and \((g, V', V, \gamma_+ g, \gamma_g)\) respectively, and a homeomorphism \(\phi : (A \cup A') \to (B \cup B')\) such that \(\phi(\gamma_\pm f) = \gamma_\pm g\) and

\[\phi(f(z)) = g(\phi(z)) \quad \text{on } \Omega_{A_f} \cup \gamma_\pm f\]

If moreover \(\phi\) is quasiconformal \(K_f\) and \(\bar{\partial}\phi = 0\) a.e. on \(K_f\), we say that \(f, g\) are hybrid conjugate.

**Definition 2.6.** (External equivalence)

Let \((f, U', U, \gamma_+ f, \gamma_f)\) and \((g, V', V, \gamma_+ g, \gamma_g)\) be two parabolic-like mappings.

If \(K_f\) and \(K_g\) are connected, we say that \(f\) and \(g\) are externally equivalent if there exist parabolic like restrictions \((f, A', A, \gamma_+ f, \gamma_f)\) and \((g, B', B, \gamma_+ g, \gamma_g)\) of \((f, U', U, \gamma_+ f, \gamma_f)\) and \((g, V', V, \gamma_+ g, \gamma_g)\) respectively, and a biholomorphic map

\[\psi : (A \cup A') \setminus K_f \to (B \cup B') \setminus K_g\]

such that \(\psi(\gamma_\pm f)\) is isotopic to \(\gamma_\pm g\) and \(\psi \circ f = g \circ \psi\).

**Remarks 2.5.** Two parabolic-like maps \(f\) and \(g\) with connected filled Julia sets are externally equivalent if and only if their external maps are real-analytic conjugate.

In the other case (filled Julia set not connected), we say that \(f\) and \(g\) are externally equivalent if their external maps are real-analytically conjugate.
Remarks 2.6. Note that by continuity $\phi(\gamma_f) = \gamma_g$ implies $\phi(\Omega_{A_f}) = \Omega_{B_g}$ and $\phi(\Delta_{A_f}) = \Delta_{B_g}$. In the same way, $\psi(\gamma_f) = \gamma_g$ implies $\psi(\Omega_{A_f}) = \Omega_{B_g}$ and $\psi(\Delta_{A_f}) = \Delta_{B_g}$.

Lemma 2.1. Let $f : U' \to U$ and $g : V' \to V$ be two parabolic-like mappings with disconnected Julia sets. Let $W_f \approx \mathbb{D}$ be a full compact connected subset of $U$ containing $\Omega_f'$ and the critical values of $f$, and such that $f^{-1}(W_f) \subset U \cap U'$. Define $L := f^{-1}(W_f) \cap \Omega_f'$. Let $W_g \approx \mathbb{D}$ be a full compact connected subset of $V$ containing $\Omega_g'$ and the critical values of $g$, and such that $g^{-1}(W_g) \subset V \cap V'$. Define $M := f^{-1}(W_g) \cap \Omega_g'$. Let $\phi : (U \cup U') \setminus L \to (V \cup V') \setminus M$ be a biholomorphic map such that

\[
\begin{align*}
U' \setminus L & \quad \xrightarrow{f} \quad U \setminus \Omega_f' \\
\downarrow \phi & \quad \downarrow \phi \\
V' \setminus M & \quad \xrightarrow{g} \quad V \setminus \Omega_g'
\end{align*}
\]

Then $h_f$ and $h_g$ are analytically conjugate, and we say that $\overline{\phi} : (U \cup U') \setminus L \to (V \cup V') \setminus M$ is an external conjugacy between the parabolic-like maps.

Proof. Let $(X_{nf}, \rho_{(n-1)f}, \pi_{(n-1)f}, f_n)_{n \geq 1}$ and $(X_{ng}, \rho_{(n-1)g}, \pi_{(n-1)g}, g_n)_{n \geq 1}$ be as in 2.1.1. Let us set $\phi_0 = \phi$ and define recursively $\phi_n = \rho_{(n-1)g}^{-1} \circ \phi_{n-1} \circ \rho_{(n-1)f} : X_{nf} \to X_{ng}$.

\[
\begin{align*}
X_{nf} & \quad \xrightarrow{\phi_n} \quad X_{ng} \\
\downarrow \rho_{nf} & \quad \downarrow \rho_{ng} \\
X_{n-1f} & \quad \xrightarrow{\phi_{n-1}} \quad X_{n-1g}
\end{align*}
\]

Then every $\phi_n : X_{nf} \to X_{ng}$ thus defined is an isomorphism and a conjugacy between $f_n$ and $g_n$.

\[
\begin{align*}
A_{nf} & \quad \xrightarrow{f_n} \quad X_{nf} \\
\downarrow \phi_n & \quad \downarrow \phi_n \\
A_{ng} & \quad \xrightarrow{g_n} \quad X_{ng}
\end{align*}
\]

Thus the family of isomorphisms $(\phi_n : X_{nf} \to X_{ng})$ induce an isomorphism $\Phi : T_f \to T_g$ compatible with dynamics, and then the external maps $h_f$ and $h_g$ are real analytically conjugated.

We can now prove the main statement of this section:
Proposition 2.7. Let \((f, U, U', \gamma_f)\) be a parabolic-like mapping of some degree \(d > 1\), and \(h : S^1 \to S^1\) be a singly parabolic external map of the same degree \(d\). Then there exists a parabolic-like mapping \((g, V, V', \gamma_g)\) which is hybrid equivalent to \(f\) and whose external class is \(h\).

Throughout this proof we assume, in order to simplify the notation, \(U\) and \(U'\) with \(C^1\) boundaries (if \(U\) and \(U'\) do not have \(C^1\) boundaries we would consider a parabolic-like restriction of \((f, U, U', \gamma_f)\) with \(C^1\) boundaries).

Let \(h : S^1 \to S^1\) be a singly parabolic external map of degree \(d > 1\), and \(z_\ast\) be its parabolic fixed point.

Since the map \(h : S^1 \to S^1\) is real-analytic, then it extends analytically to \(h : W' \to W\), (where \(W'\) and \(W\) are neighborhoods of \(S^1\) in \(\mathbb{C}\)). Let \(\partial B\) and \(\partial B'\) be the outer boundaries of \(W\) and \(W'\) respectively and let \(B\) and \(B'\) be the discs bounded by \(\partial B\) and \(\partial B'\) respectively.

We are going to construct now arcs \(\tilde{\gamma}_+ : [0, 1] \to B \setminus \mathbb{D}\) and \(\tilde{\gamma}_- : [0, -1] \to B \setminus \mathbb{D}\) forward invariant under \(h\), conjugate to \(\gamma_\pm\) (where \(\gamma_+ \cup \gamma_- = \gamma_f\)), tangent to \(S^1\) at \(z_\ast\) and such that \(\tilde{\gamma}_\pm(0) = z_\ast\).

Let \(h_f\) be the external map of \(f\), and \(z_1\) be its parabolic fixed point. Then by construction \(h_f\) extends to a map \(h_f : W'_f \to W'_f\), where \(W'_f\), \(W'_f\) are neighborhoods of \(S^1\), and it is conformally conjugate to \(f\) under the isomorphism \(\alpha\) on \(W'_f \setminus \mathbb{D}\) (see 2.1). By the remark 2.3 the arcs \(\gamma_{h_f\pm}\) are tangent to \(S^1\) at \(z_1\) and such that \(\tilde{\gamma}_\pm(0) = z_1\).

Let \(\Xi_{h_f\pm}\) be repelling petals of the parabolic fixed point \(z_1\), which intersect the unit circle and \(\tilde{\phi}_\pm : \Xi_{h_f\pm} \to \mathbb{H}_-\) be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point \(z_1\). Let \(\Xi_h\) be repelling petals of the parabolic fixed point \(z_\ast\) which intersect the unit circle and \(\tilde{\phi}_\pm : \Xi_h \to \mathbb{H}_-\) be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point \(z_\ast\).

Define \(\tilde{\gamma}_+ = \tilde{\phi}_+^{-1}(\tilde{\phi}_{h_f\pm}(\gamma_{h_f\pm}))\) and \(\tilde{\gamma}_- = \tilde{\phi}_-^{-1}(\tilde{\phi}_{h_f\pm}(\gamma_{h_f\pm}))\). Since \(\gamma_{h_f\pm}\) are tangent to the unit circle at \(z_1\) (see Remark 2.3), then \(\tilde{\gamma}_+\), \(\tilde{\gamma}_-\) are tangent to the unit circle at \(z_\ast\).

The arcs \(\tilde{\gamma}_\pm\) divide the set \(B\) into \(\Omega_B, \Delta_B\) (with \(\mathbb{D} \in \Omega_B\)) and the set \(B'\) into \(\Omega'_B, \Delta'_B\) (with \(\mathbb{D} \in \Omega'_B\)).

Define \(Q_f = \Omega \setminus \Omega'\), \(Q_h = \Omega_B \setminus \Omega'_B\) (see Fig. 9).

Let \(z = z_0\) be the parabolic fixed point of \(f\). Define orientation preserving diffeomorphisms that conjugate dynamics \(\psi_+ : \gamma_+ \to \tilde{\gamma}_+\) and \(\psi_- : \gamma_- \to \tilde{\gamma}_-\).
Figure 9: The arcs $\tilde{\gamma}_\pm$ devide the set $B$ into $\Omega_B$, $\Delta_B$ (with $D \in \Omega_B$) and the set $B'$ into $\Omega'_B$, $\Delta'_B$ (with $D \in \Omega'_B$). Define $Q_f = \Omega \setminus \Omega'$ and $Q_h = \Omega_B \setminus \Omega'_B$.

as follows:

$$
\psi_+(z) = \begin{cases} 
\tilde{\phi}_+^{-1} \circ \phi_{h_f+} \circ \alpha & \text{on } \gamma_+ \setminus \{z_0\} \\
 z & \text{on } z_0 
\end{cases}
$$

$$
\psi_-(z) = \begin{cases} 
\tilde{\phi}_-^{-1} \circ \phi_{h_f-} \circ \alpha & \text{on } \gamma_- \setminus \{z_0\} \\
 z & \text{on } z_0 
\end{cases}
$$

Let $\psi_0 : \partial U \to \partial B$ be an orientation preserving diffeomorphism coinciding with $\psi_\pm$ on $\gamma_\pm \cap \partial U$ (it exists because both $U$ and $B$ have smooth boundary), and let $\psi_1 : \partial U' \to \partial B'$ be the lift of $\psi_0$ coinciding with $\psi_\pm$ on $\gamma_\pm \cap \partial U'$ (see Fig. 10).

Let us construct a quasiconformal map $\Psi_0$ between the topological rectangles $Q_f$ and $Q_h$ which agrees with $\psi_\pm$ on $\gamma_\pm$, with $\psi_0$ on $\partial U$ and with $\psi_1$ on $\partial U'$. Let us call $a = Q_f \cap \gamma_+$, $b = Q_f \cap \partial U$, $c = Q_f \cap \gamma_-$ and $d = Q_f \cap \partial U'$ (see Fig. 10).

Let $\Psi_f, \Psi_h$ be the unique conformal maps sending $Q_f$ and $Q_h$ respectively onto straight rectangles. Let $\tilde{\Psi}_0 : \Psi_f(\partial Q_f) \to \Psi_h(\partial Q_h)$ be the quasisymmetric map given by:

$$
\tilde{\Psi}_0(z) = \begin{cases} 
\Psi_h \circ \psi_+ \circ \Psi_f^{-1} & \text{on } \Psi_f(a) \\
\Psi_h \circ \psi_0 \circ \Psi_f^{-1} & \text{on } \Psi_f(b) \\
\Psi_h \circ \psi_- \circ \Psi_f^{-1} & \text{on } \Psi_f(c) \\
\Psi_h \circ \psi_1 \circ \Psi_f^{-1} & \text{on } \Psi_f(d) 
\end{cases}
$$

and let $\tilde{\psi}_0 : \Psi_f(Q_f) \to \Psi_h(Q_h)$ be a quasiconformal extension (cfr. BF pg.48). Then $\Psi_R := \Psi_h^{-1} \circ \tilde{\psi}_0 \circ \Psi_f$ is a quasiconformal map between $Q_f$ and
Figure 10: Construction of the quasiconformal map $\Psi_0$ between the topological rectangles $Q_f$ and $Q_h$.

We are now going to construct a quasiconformal map $\Phi_{\Delta} : \Delta \to \Delta_B$ which extends to $\psi_\pm$ on $\gamma_{f\pm}$ and to $\psi_0$ on $\partial U \cap \partial \Delta$.

Consider $h_f : W'_f \to W_f$ the external map of $f$ extended to a neighborhood of the unit circle and its parabolic fixed point $z_1$. The arcs $\gamma_{h_f\pm}$ devide $W_f, W'_f$ in $\Delta_W, \Omega_W$ and $\Delta'_W, \Omega'_W$ respectively.

Since the map $\alpha$ is an analytic conjugacy between $f$ and $h_f$ outside the filled Julia set of $f$ (if it is connected, or outside a compact connected subset of $\overline{\Omega}$ which contain the filled Julia set if this is not connected), in order to obtain a quasiconformal map $\Phi_{\Delta} : \Delta \to \Delta_B$ which extends to $\psi_\pm$ on $\gamma_{f\pm}$ and to $\psi_0$ on $\partial U \cap \partial \Delta$ it is sufficient to construct a quasiconformal map $\Phi_{\Delta_W} : \Delta_W \to \Delta_B$ which extends to $\psi_\pm \circ \alpha^{-1}$ on $\gamma_{h_f\pm}$ (we extend $\psi_\pm \circ \alpha^{-1}$ to $z_1$ by setting $\alpha^{-1}(z_1) = z_0$, where $z_0$ is the parabolic fixed point of $f$) and to $\psi_0 \circ \alpha^{-1}$ on $\alpha(\partial U \cap \partial \Delta)$. Then we set $\Phi_{\Delta} = \Phi_{\Delta_W} \circ \alpha$.

Let define $\beta = \partial W_f \cap \partial \Delta_W$ (see picture 12). Since

$$\psi_+(z) = \begin{cases} \tilde{\phi}_+^{-1} \circ \phi_{h_f+} \circ \alpha & \text{on } \gamma_+ \setminus \{z_0\} \\ z_0 & \text{on } z_0 \end{cases}$$

$$\psi_-(z) = \begin{cases} \tilde{\phi}_-^{-1} \circ \phi_{h_f-} \circ \alpha & \text{on } \gamma_- \setminus \{z_0\} \\ z_0 & \text{on } z_0 \end{cases}$$

19
then

\[
\psi_+ \circ \alpha^{-1}(z) = \begin{cases} 
\tilde{\phi}_+^{-1} \circ \phi_{h_f^+} & \text{on } \gamma_{h_f^+} \setminus \{z_1\} \\
\psi_+ & \text{on } z_1
\end{cases}
\]

\[
\psi_- \circ \alpha^{-1}(z) = \begin{cases} 
\tilde{\phi}_-^{-1} \circ \phi_{h_f^-} & \text{on } \gamma_{h_f^-} \setminus \{z_1\} \\
\psi_- & \text{on } z_1
\end{cases}
\]

Figure 11: The map \( h_f \) is the external map of \( f \). Since the map \( \alpha \) is a conformal conjugacy between \( f \) and \( h_f \) on a neighborhood of \( \Delta \), in order to obtain a quasiconformal map \( \Phi_{\Delta} : \Delta \to \Delta_B \) which extends to \( \psi_\pm \) on \( \gamma_{h_f^\pm} \) and to \( \psi_0 \) on \( \partial U \cap \partial \Delta \) it is sufficient to construct a quasiconformal map \( \Phi_{\Delta_W} : \Delta_W \to \Delta_B \) which extends to \( \psi_\pm \circ \alpha^{-1} \) on \( \gamma_{h_f^\pm} \) and to \( \psi_0 \circ \alpha^{-1} \) on \( \beta \).

Define an orientation preserving homeomorphism \( \tilde{\Phi}_{\partial \Delta_W} : \partial \Delta_W \to \partial \Delta_B \) by:

\[
\tilde{\Phi}_{\partial \Delta_W}(z) = \begin{cases} 
\psi_+ \circ \alpha^{-1} & \text{on } \gamma_{h_f^+} \\
\psi_- \circ \alpha^{-1} & \text{on } \gamma_{h_f^-} \\
\psi_0 \circ \alpha^{-1} & \text{on } \beta
\end{cases}
\]
We can assume the angles between them (we can choose $\beta$ larger). By taking restrictions of $W'_i$ and $W''_i$, we can choose $\gamma_{h+}$ and $\gamma_{h-}$. By the remark 2.3 the angle $\pi$ separates $\gamma_{h+}$ and $\gamma_{h-}$. Therefore $\partial\Delta_W$ is a quasicircle.

Now we have to prove that the map $\tilde{\Phi}_{\partial W}$ is quasisymmetric. Since we can choose both $\psi_0$ and $\beta$ we can suppose $\tilde{\Phi}_{\partial W}$ quasisymmetric on both $\gamma_{h+} \cup \beta$ and $\gamma_{h-} \cup \beta$. Therefore, in order to prove that the map $\tilde{\Phi}_{\partial W}$ is quasisymmetric on all its domain, it is enough to prove that it is a quasisymmetric conjugacy between the arcs $\gamma_{h+} \cup \gamma_{h-}$ and $\tilde{\gamma}_+ \cup \tilde{\gamma}_-$. Therefore, in order to prove that the map $\tilde{\Phi}_{\partial W}$ is quasisymmetric between the arcs $\gamma_{h+} \cup \gamma_{h-}$ and $\tilde{\gamma}_+ \cup \tilde{\gamma}_-$ it is enough to prove that the parametrization of the arcs $\tilde{\gamma} = \tilde{\gamma}_+ \cup \tilde{\gamma}_-$ and $\gamma_{h} = \gamma_{h+} \cup \gamma_{h-}$ is quasisymmetric on a neighborhood of the parabolic fixed point.

**Lemma 2.2.** Let $h_i : S^1 \to S^1$, $i = 1, 2$ be singly parabolic external maps of the same degree $d > 1$, and let $z_i$ be their parabolic fixed points. Let $h_i : W'_i \to W_i$ be complex analytic extensions of $h_i$. Then there exist arcs $\tilde{\gamma}_{i+} : [0, 1] \to W'_i \setminus \mathbb{D}$ and $\tilde{\gamma}_{i-} : [0, -1] \to W'_i \setminus \mathbb{D}$, which are forward invariant under $h_i$, satisfies $\tilde{\gamma}_{i\pm}(0) = z_i$ and which are tangent to $S^1$ at $z_i$. The arc $\tilde{\gamma}_i = \tilde{\gamma}_{i+} \cup \tilde{\gamma}_{i-}$ is a quasisymmetric conjugacy between $h_i$ and the parametrization $t \to dt$.

**Proof.** Let $\Xi_{i\pm}$, $i = 1, 2$ be the repelling petals of $z_i$ which intersect the unit circle and $\phi_{i\pm} : \Xi_{i\pm} \to \mathbb{H}_-$ be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point $z_i$.

The images of the unit circle in the Fatou coordinate planes are periodic curves. Choose $m > 0$ such that

$$\forall z \in \phi_{i+}(S^1 \cap \Xi_{i+}) \cup \phi_{i-}(S^1 \cap \Xi_{i-}) \quad \text{Im}(z) > -m$$

$$\forall z \in \phi_{i+}(S^1 \cap \Xi_{i-}) \cup \phi_{i-}(S^1 \cap \Xi_{i+}) \quad \text{Im}(z) < m$$

Then let us define $\tilde{\gamma}_{i\pm} = (\phi_{i\pm}^{-1})(-mi + \mathbb{R}_-)$.
\[ \tilde{\gamma}_{i-} = (\phi_i^{-1})_- (+mi + \mathbb{R}_-) \]

Reparametrizing the arcs as

\[ \tilde{\gamma}_{i+}(t) = \phi_i^{-1}(\log_d(t) - im) : [0, 1] \to \Xi_{+i} \]
\[ \tilde{\gamma}_{i-}(t) = \phi_i^{-1}(\log_d(-t) + im) : [0, -1] \to \Xi_{-i} \]

we obtain

\[ h_i(\tilde{\gamma}_{i\pm}(t)) = \tilde{\gamma}_{i\pm}(dt), \quad \forall t : -\frac{1}{d} \leq \pm t \leq \frac{1}{d}. \]

Hence the arc \( \tilde{\gamma}_i = \tilde{\gamma}_{i+} \cup \tilde{\gamma}_{i-} \) is a conjugacy between \( h_i \) and the map \( t \to dt \).

Let us show now that it is quasisymmetric.

Define \( \hat{\gamma}_i : [-1, 1] \to \tilde{\gamma}_{i+} \cup \tilde{\gamma}_{i-} \). Let us see that the parametrized arc \( \hat{\gamma}_i \) is a quasisymmetric conjugacy between \( h_i \) and the map \( t \to dt \).

That is, we want to prove that \( \exists M > 0 \) such that \( \forall t, t + h, t - h \in [-1, 1] \)

\[ \frac{1}{M} \leq \frac{|\hat{\gamma}_i(t + h) - \hat{\gamma}_i(t)|}{|\hat{\gamma}_i(t) - \hat{\gamma}_i(t - h)|} \leq M. \]

Note that for \( t, t + h, t - h \in [0, 1] \) the condition is satisfied because the arc is preimage of straight lines under Fatou coordinates, and the same holds for \( t, t + h, t - h \in [0, -1] \). Therefore it is enough to prove that \( \exists M > 0 \) such that \( \forall t \in (-\epsilon, \epsilon) \) the following expression holds:

\[ \frac{1}{M} \leq \frac{|\hat{\gamma}_i(t) - \hat{\gamma}_i(0)|}{|\hat{\gamma}_i(0) - \hat{\gamma}_i(-t)|} = \frac{|\hat{\gamma}_i(t)|}{|\hat{\gamma}_i(-t)|} \leq M. \]

Since \( \hat{\gamma}_{i+}(t) = \phi_i^{-1}_+ \circ (\log_d(t) - im) \) and \( \hat{\gamma}_{i-} = \phi_i^{-1}_- \circ (\log_d(-t) + im) \), then \( \phi_i^+ \circ \hat{\gamma}_{i+}(t) = \log_d(t) - im \) and \( \phi_i^- \circ \hat{\gamma}_{i-}(-t) = \log_d(t) + im \). Hence we can write

\[ \phi_i^+ \circ \hat{\gamma}_{i+}(t) + im = \log_d(t) \]
and
\[ \phi_i \circ \tilde{\gamma}_i (-t) - im = \log_d(t) \]

Therefore
\[ \frac{\phi_{i+} \circ \tilde{\gamma}_{i+}(t) + im}{\phi_{i-} \circ \tilde{\gamma}_{i-}(-t) - im} = \frac{\log_d(t)}{\log_d(t)} = 1. \]

Let us compute this expression. Since it is the same expression for both \( \tilde{\gamma}_i \), \( i = 1, 2 \) in order to fix the notation let us consider just \( \tilde{\gamma}_1 \) (for \( \tilde{\gamma}_2 \) the computation is analogous.) On \( \Xi_{\pm 1} \) (and then on the arc) the map \( \tilde{h}_1 \) is conformally conjugate (let us say by \( \gamma \)) to the map \( \tilde{\gamma}_1 \) (for \( \tilde{\gamma}_2 \) the parabolic fixed point at \( z \)).

Therefore we can write
\[
\begin{align*}
\Xi_{\pm 1} & \xrightarrow{\tilde{h}_1} \Xi_{\pm 1} \\
\mathbb{H}_{-} & \xrightarrow{\tilde{h}_1^*} \mathbb{H}_{-}
\end{align*}
\]

Therefore we can write
\[
\frac{\phi_{1+} \circ \tilde{\gamma}_{1+}(t) + im}{\phi_{1-} \circ \tilde{\gamma}_{1-}(-t) - im} = 1
\]
as
\[
\frac{-\left(\frac{1}{(2n\tilde{\gamma}_{1+}(t))^{2n}} + \log\left(-\frac{1}{(2n\tilde{\gamma}_{1+}(t))^{2n}}\right)\right) + c_{+} + o(1)}{-\left(\frac{1}{(2n\tilde{\gamma}_{1-}(-t))^{2n}} + \log\left(-\frac{1}{(2n\tilde{\gamma}_{1-}(-t))^{2n}}\right)\right) + c_{-} + o(1)} = 1
\]

where \( c_{+} = c_{+} + im \) and \( c_{-} = c_{-} - im \). That is:
\[
\begin{align*}
&\frac{-\left(\frac{1}{(\gamma_{1+}+t)^{2n}} + \log\left(-\frac{1}{(2n\tilde{\gamma}_{1+}(t))^{2n}}\right)\right) + c_{+} + o(1)}{-\left(\frac{1}{(\gamma_{1-}(-t))^{2n}} + \log\left(-\frac{1}{(2n\tilde{\gamma}_{1-}(-t))^{2n}}\right)\right) + c_{-} + o(1)} = 1 \\
&\frac{-c_{+}(\tilde{\gamma}_{1+}(t))^{2n}(2n)^{2n} - o(1)(\tilde{\gamma}_{1+}(t))^{2n}(2n)^{2n}}{-c_{-}(\tilde{\gamma}_{1-}(-t))^{2n}(2n)^{2n} - o(1)(\tilde{\gamma}_{1-}(-t))^{2n}(2n)^{2n}} = 1
\end{align*}
\]
Therefore
\[
\frac{\frac{-1}{(\gamma_1(0))^2} \log \left( \frac{1}{(\gamma_1(t))^2} \right)(\gamma_1(t) + \gamma_1(0))^2 + o(1)(\gamma_1(t))^2(2n)^2}{\frac{-1}{(\gamma_1(-t))^2} \log \left( \frac{1}{(\gamma_1(-t))^2} \right)(\gamma_1(-t) + \gamma_1(0))^2 + o(1)(\gamma_1(-t))^2(2n)^2} = 1.
\]

where \( \hat{b} = b(2n)^2 \), \( \bar{c} = \hat{b} \log \left( \frac{-1}{(2n)^2} \right) - c'(2n) \) and \( \hat{d} = d(2n)^2 \), \( \bar{c} = \hat{d} \log \left( \frac{-1}{(2n)^2} \right) - c'(2n)^2 \). Finally
\[
\frac{[1 - \hat{b} \log \left( \frac{-1}{(\gamma_1(t))^2} \right)(\gamma_1(t) + \gamma_1(0))) + o(1)(\gamma_1(t))^2(2n)^2]}{[1 - \hat{d} \log \left( \frac{-1}{(\gamma_1(-t))^2} \right)(\gamma_1(-t) + \gamma_1(0))) + o(1)(\gamma_1(-t))^2(2n)^2] = \frac{(\gamma_1(-t))^2}{(\gamma_1(t))^2}.
\]

This implies that
\[
\frac{\gamma_1(-t)}{\gamma_1(t)} \to 1,
\]
and finally that \( \frac{\gamma_1(-t)}{\gamma_1(t)} \) tends to one approaching the parabolic fixed point.

Therefore the map \( \gamma_1 \) (and similarly the map \( \gamma_2 \)) is quasisymmetric on a neighborhood of 0. Hence the arc \( \gamma_i = \gamma_{i+1} \cup \gamma_{i-1} \) is a quasisymmetric conjugacy between \( h_i \) and the parametrization \( t \to dt \).

Let \( \tilde{\psi} : \overline{Q_f} \cup \Delta \to \overline{Q_h} \cup \Delta_B \) be a quasiconformal homeomorphism with the following properties:
\[
\tilde{\psi}(z) = \begin{cases} 
\psi_{\pm} & \text{on } \gamma_{\pm} \\
\Psi_{0} & \text{on } Q_f \\
\Phi_{\Delta} & \text{on } \Delta 
\end{cases}
\]

Consider
\[
\tilde{f}(z) = \begin{cases} 
\tilde{\psi}^{-1} \circ \tilde{h} \circ \tilde{\psi} & \text{on } \Delta' = \tilde{\psi}^{-1}(\Delta_B \cap \Delta_B') \\
\tilde{f} & \text{on } \Omega'
\end{cases}
\]

Define on \( U \) an almost complex structure \( \sigma \) as (see Fig. 2.2):
\[
\sigma(z) = \begin{cases} 
\sigma_0 & \text{on } K_f \\
\psi^*(\sigma_0) & \text{on } Q_f \text{ and on } \Delta \\
(f^*)^n \sigma_1 & \text{on } f^{-n}(Q_f \cup \Delta)
\end{cases}
\]

Remarks 2.7. The almost complex structure \( \sigma \) is \( f \)-invariant on \( \Delta \).
Figure 12: Define on $U$ an almost complex structure $\sigma$ as $\sigma_0$ on $K_f$, $\sigma_1 = \tilde{\psi}^* (\sigma_0)$ on $Q_f \cup \Delta$ and $(\tilde{f}^n)^* \sigma_1$ on $\tilde{f}^{-n}(Q_f \cup \Delta)$.

Let us define $\tilde{\Delta} = \tilde{\psi}^{-1}(\tilde{h}(\Delta_B \cap \Delta'_B))$, $\tilde{U} = (\Omega \cup \tilde{\Delta}) \cap U$, $\tilde{U} \subset U$ and $\tilde{U}' = \tilde{f}^{-1}(\tilde{U})$, $\tilde{U}' \subset U'$. The map $\tilde{f} : \tilde{U}' \to \tilde{U}$ is quasiregular of degree $d$.

The almost complex structure $\sigma$ is bounded and $\tilde{f}$-invariant. Therefore, by the Integrability theorem, there exists $\varphi : U \to \mathbb{D}$ such that $\varphi^* \sigma_0 = \sigma$.

Let $g := \varphi \circ \tilde{f} \circ \varphi^{-1} : \varphi(U \cap \tilde{U}') \to \varphi(U)$. Let us call $V' = \varphi(U \cap \tilde{U}')$, $V = g(U \cap \tilde{U}')$, $\gamma_{g+} = \varphi(\gamma_+)$ and $\gamma_{g-} = \varphi(\gamma_-)$.

Then $(g, V, V', \gamma_{g+}, \gamma_{g-})$ is a parabolic-like map of the same degree as $f$, quasiconformally conjugate to $\tilde{f}$ and therefore quasiconformally conjugate to $f$ on $\Omega'_g = \varphi(\Omega')$. Moreover, by construction, $\varphi^* \sigma_0 = \sigma_0$ on $K_f$, therefore $g$ is hybrid conjugate to $f$.

If $K_f$ is connected, define

$$\hat{\psi}(z) = \begin{cases} \tilde{\psi} & \text{on } Q_f \cup \Delta \\ \tilde{h}^{-n} \circ \psi \circ \tilde{f}^n & \text{on } \tilde{f}^{-n}(Q_f \cup \Delta) \end{cases}$$

This is a quasiconformal map $\hat{\psi} : \tilde{U} \setminus K_f \to B \setminus \mathbb{D}$. Then the quasiconformal map $\overline{\psi} = \hat{\psi} \circ \varphi^{-1} : (V \cup V') \setminus K_g \to B \setminus \mathbb{D}$ is a conformal conjugacy between $g$ and $\tilde{h}$ on $(V \cup V') \setminus K_g$, since it is holomorphic (indeed $\hat{\psi}^* \sigma_0 = \sigma$ and $\varphi^* \sigma_0 = \sigma$, then $(\hat{\psi} \circ \varphi^{-1})^* \sigma_0 = \sigma_0$) and $\overline{\psi} \circ g = \tilde{h} \circ \overline{\psi}$ on $(V \cup V') \setminus K_g$ (see Fig. 12). Hence the external map of $g$ is $h$. 

25
Figure 13: Since $\hat{\psi}^*\sigma_0 = \sigma$ and $\varphi^*\sigma_0 = \sigma_0$, the map $\hat{\psi} = \hat{\psi} \circ \varphi^{-1}$ is holomorphic, and since $\hat{\psi} \circ g = \tilde{h} \circ \hat{\psi}$ it is a conformal conjugacy between $g$ and $\tilde{h}$ on $(V \cup V') \setminus K_g$, and thus the external map of $g$ is $h$.

If $K_f$ is not connected, let us consider $V_f \approx \mathbb{D}$, a full compact connected subset of $\overline{U}$ containing $\overline{\Omega}$, the critical values of $f$ and such that $V_f' = f^{-1}(V_f) \subset \subset \overline{U} \cap \overline{U'}$, and $V_g \approx \mathbb{D}$, a full compact connected subset of $V$ containing $\overline{\Omega}_g$, the critical values of $g$ and such that $V_g' = f^{-1}(V_g) \subset \subset V \cap V'$. Let us call $L = V_f' \cap \Omega', M = V_g' \cap \Omega'_g$, and consider the restriction $\varphi : (\overline{U} \cup \overline{U'}) \setminus L \to (V \cup V') \setminus M$.

Define the map

$$\hat{\psi}(z) = \begin{cases} \hat{\overline{\psi}} & \text{on } Q_f \cup \Delta \\ \tilde{h}^{-n} \circ \hat{\overline{\psi}} \circ \tilde{f}^n & \text{on } (\overline{U} \cup \overline{U'}) \setminus L \end{cases}$$

Then the map $\hat{\psi} = \hat{\psi} \circ \varphi^{-1} : (V \cup V') \setminus M \to B \setminus \overline{D}$ is a biholomorphic map (since it is quasiconformal and $(\hat{\psi} \circ \varphi^{-1})^*\sigma_0 = \sigma_0$) and $\hat{\psi} \circ g = \tilde{h} \circ \hat{\psi}$ on $(V \cup V') \setminus M$. Then $g$ and $\tilde{h}$ are externally conjugate by the Lemma 2.1.
2.2.1 The Straightening Theorem

The filled Julia set is defined in the literature only for polynomials and polynomial-like maps. So far we have defined as filled Julia set of a parabolic-like map \((f,U',U,\gamma_+,\gamma_-)\) the set of points in \(U'\) that never leave the set \(\Omega' \cup \gamma_\pm(0)\) under iteration. Now we are going to work with rational maps defined on the whole Riemann sphere and their filled Julia sets, hence we have to define what do we mean as filled Julia set of a rational function. Let \(f: \hat{C} \to \hat{C}\) be a rational map. The map \(f\) has a parabolic-like restriction if there exist open connected sets \(U, U' \approx D\) and forward invariant arcs \(\gamma_\pm\) such that \((f,U',U,\gamma_+,\gamma_-)\) is a parabolic-like map of some degree \(d\), and if the degree of \(f\) is equal to the degree of its parabolic-like restriction (i.e. \(U\) contains all the critical points of \(f\) but one), the parabolic-like restriction is maximal. Then we consider as filled Julia set of \(f: \hat{C} \to \hat{C}\) the filled Julia set of its maximal parabolic-like restriction. In the remainder of the paper we are considering maximal parabolic-like restrictions without further reference.

For example, let us consider the map 
\[
h_2 = z^2 + \frac{1}{3 + z^2}.
\]
It has a parabolic fixed point at \(z = 1\), simple critical points at \(z = 0\) and at \(z = \infty\), and it is symmetric with respect to the unit circle, which is an invariant set. This map has (cfr. [2.0.1]) a parabolic-like restriction, defining \(U' = \{z : |z| < 1 + \varepsilon\}\) (for some \(\varepsilon > 0\)) and \(U = h_2(U')\). Since its filled Julia set as parabolic-like map is \(\overline{D}\), we consider \(\overline{D}\) the filled Julia set of the whole function.

So far we defined the concepts of external map and external conjugacy just for parabolic-like maps. We consider as the external class of \(f: \hat{C} \to \hat{C}\) the external class of its parabolic-like restriction, and we say that two holomorphic maps \(f, g: \hat{C} \to \hat{C}\) are externally conjugate if their parabolic-like restrictions are externally conjugate.

Remarks 2.8. We can take parabolic-like restrictions of parabolic-like maps without changing the filled Julia set (cfr. 2.4), and thus there exist many different equivalent parabolic-like restrictions of a map. This will be really useful in the proof of the following proposition.

Let us consider the family 
\[
P_A = z + \frac{1}{z} + \sqrt{A}, \text{ for } A \in \mathbb{C}.
\]
Then

Proposition 2.8. For every \(A \in \mathbb{C}\) the external map of \(P_A\) is \(h_2 = \frac{z^2 + \frac{1}{z}}{1 + \frac{1}{z^2}}\).

Proof. The map \(\phi(z) = \frac{z + 1}{z - 1}\) is a conformal conjugacy between the maps \(P_0(z) = z + 1/z\) and \(h_2 = \frac{3z^2 + 1}{3 + z^2}\). Therefore, in order to prove that \(h_2\) is the external map of \(P_A\), it is sufficient to prove that \(P_0\) with filled Julia set \(\overline{H}^- = \phi(\overline{D})\) is externally equivalent to \(P_A\), for \(A \in \mathbb{C}\).
Let us contract an external equivalence first in the case $K_{P_A}$ is connected. Replacing $A$ by $-A$ if necessary, we can assume the critical point $z = 1$ is attracted by $\infty$. Let $\Xi_0$ be an attracting petal of $P_0$ containing the critical point $z = 1$, and let $\Phi_0 : \Xi_0 \to \mathbb{H}_+$ be the incoming Fatou coordinates of $P_0$ normalized by sending the critical value $z = 2$ to $1$. Let $\Xi_A$ be an attracting petal of $P_A$ and let $\Phi_A$ be the incoming Fatou coordinate of $P_A$, normalized by sending the critical value $z = 2 + A$ to $1$. Define $\eta = \Phi_A^{-1} \circ \Phi_0 : \Xi_0 \to \Xi_A$.

The map $\eta(z)$ is a conformal conjugacy between $P_0$ and $P_A$ on $\Xi_0$. Defining $\Xi_0^n, n > 0$ as the connected component of $P_0^{-n}(\Xi_0)$ containing $\Xi_0$, and $\Xi_A^n, n > 0$ as the connected component of $P_A^{-n}(\Xi_A)$ containing $\Xi_A$, we can lift the map $\eta$ to $\eta_n : \Xi_0^n \to \Xi_A^n$ such that it extends to a conformal conjugacy between $P_0$ and $P_A$ on $\Xi_0^n$. Since $K_A$ is connected by iterated lifting of $\eta$ we obtain an external conjugacy $\hat{\eta} : \hat{\mathbb{C}} \setminus K_0 \to \hat{\mathbb{C}} \setminus K_A$ between $P_0$ and $P_A$. Thus the external map $h_A$ of $P_A$ is $h_2 = \frac{3z^2 + 1}{3 + z^2}$.

In the case $K_{P_A}$ is not connected we construct parabolic-like restrictions $(P_0, U_0, U_0', \gamma_{+0}, \gamma_{-0})$ and $(P_A, U_A, U_A', \gamma_{+A}, \gamma_{-A})$ of the maps $P_0$ and $P_A$ respectively and an external conjugacy between them. We are assuming, replacing $A$ by $-A$ if necessary, the critical point $z = 1$ is the first attracted by $\infty$ under the map $P_A$.

As above let $\Phi_0 : \Xi_0 \to \mathbb{H}_+$ be the incoming Fatou coordinate of $P_0$ normalized by sending the critical value $z = 2$ to $1$, and let $\Phi_A : \Xi_A \to \mathbb{H}_+$ be the incoming Fatou coordinates of $P_A$, normalized by sending the critical value $z = 2 + A$ to $1$. Define $\eta = \Phi_A^{-1} \circ \Phi_0 : \Xi_0 \to \Xi_A$. The map $\eta(z)$ is a conformal conjugacy between $P_0$ and $P_A$ on the region delimited by the Fatou equipotential passing through $z = 1$.

Let us construct now parabolic-like restrictions $(P_0, U_0, U_0', \gamma_{+0}, \gamma_{-0})$ and $(P_A, U_A, U_A', \gamma_{+A}, \gamma_{-A})$ of the maps $P_0$ and $P_A$ respectively which allow us to extend the map $\eta(z)$ to an external conjugacy between them. Since the critical point $z = 1$ is the first attracted by infinity for both the maps $P_0$ and $P_A$, it cannot belong to the domains $U_0', U_A'$ of their parabolic-like restrictions, but it will belong to the codomains $U_0, U_A$, to allow us to extend the map $\eta(z)$ enough. On the other hand the critical point $z = -1$ needs to belong to the domains of the parabolic-like restrictions of $P_0$ and $P_A$, and in particular it needs to belong to $\Omega_0'$ and $\Omega_A'$.

Let us denote as $\Phi_A$ the Fatou coordinate of $P_A$ extended to the whole basin of attraction of $\infty$ by iterated lifting. Choose $r > \max\{\text{Im}(\Phi_A(A - 2)) + 1, 2\}$ and $z_0, r < z_0 < r + 1$ such that $A - 2 \notin \Phi_A^{-1}(\mathbb{D}(z_0, r))$. Then
Figure 14: The construction of the parabolic-like restrictions of $P_0$ and $P_A$ which allow us to extend the map $\eta(z)$ to an external conjugacy between them. In the picture we are assuming the critical value $z = -2 + A$ in $\Omega_A \setminus \Omega'_A$. In this case the critical value $z = -2 + A$ belongs to the attracting petal $\Xi_A$, and then on $-2 + A$ we have $\Phi_A = \Phi$. Note that it is almost the worst setting possible (the worst one is when $\text{Re}\Phi_A(2 + A) = \text{Re}\Phi_A(-2 + A)$, which happens when $A$ is purely imaginary).

for $r < r' < z_0$ with $r'$ sufficiently close to $r$ we have $A - 2 \notin \Phi_A^{-1}(\mathbb{D}(z_0, r))$.

Let $\tilde{\gamma}_+, \tilde{\gamma}_-$ be horizontal lines starting at $-\infty$ and landing at $\partial \mathbb{D}(z_0, r)$ such that the point $\Phi_A(A - 2)$ is contained between them (see Fig. 13) and they do not leave the disk $T^{-1}(\mathbb{D}(z_0, r))$ (i.e. the disk of radius $r$ and center $z_1 = z_0 - 1$) after have entered in it.

Define $U_0 = (\Phi_0^{-1}(\mathbb{D}(z_0, r)))^c$, $U'_0 = P_0^{-1}(U_0)$, $\gamma_+ = \Phi_0^{-1}(\tilde{\gamma}_+)$, and $\gamma_- = \Phi_0^{-1}(\tilde{\gamma}_-)$. In the same way define $U_A = (\Phi_A^{-1}(\mathbb{D}(z_0, r)))^c$, $U'_A = P_A^{-1}(U_A)$, $\gamma_+ = \Phi_A^{-1}(\tilde{\gamma}_+)$, and $\gamma_- = \Phi_A^{-1}(\tilde{\gamma}_-)$.
Then the parabolic-like restriction of \( P_0 \) we consider is \((P_0, U_0, U_0', \gamma_+, \gamma_-)\), and the parabolic-like restriction of \( P_A \) we consider is \((P_A, U_A, U_A', \gamma_+, \gamma_-)\).

The arc \( \gamma_- \cup \gamma_+ \) divides \( U_0', U_A \) into \( \Omega_A', \Delta_A', \Omega_A, \Delta_A \) respectively (with \( \Omega_A' \subset \subset U_A, \Omega_A \subset \Omega_A \) and \( \Delta_A' \cap \Delta_A \neq \emptyset \)). By construction, the critical point \( z = -1 \) belongs to \( \Omega_A' \). In the same way the arc \( \gamma_- \cup \gamma_+ \) divides \( U_0', U_0 \) into \( \Omega_0', \Delta_0', \Omega_0, \Delta_0 \) respectively (with \( \Omega_0' \subset \subset U_0, \Omega_0 \subset \Omega_0 \) and \( \Delta_0' \cap \Delta_0 \neq \emptyset \)), and the critical point \( z = -1 \) belongs to \( \Omega_0' \) since \( K_{P_0} \) is connected. Hence (since \( \Delta_0', \Delta_A' \) do not contain any critical point) we can lift \( \eta \) such that it extends to a conformal conjugacy between \( P_0 \) and \( P_A \) on \( \Delta_0 \).

In order to obtain an external conjugacy we need \( \eta \) to be defined on an annulus, then we need to extend \( \eta \) to some annulus at the boundary of \( U_0' \).

Define \( D_0 = \Phi_0^{-1}(\mathbb{D}(z_0, r')) \), \( D_A = \Phi_A^{-1}(\mathbb{D}(z_0, r')) \), and let \( D' = T^{-1}(\mathbb{D}(z_0, r')) \). Hence \( 2 \notin D_0, -2 + A \notin D_A \), and \( T^{-1}(\mathbb{D}(z_0, r')) \subset D' \).

Figure 15: The construction of the external conjugacy \( \eta \) between the parabolic-like restriction of \( P_0 \) and the parabolic-like restriction of \( P_A \). In the picture we are assuming the critical value \( z = -2 + A \) in \( \Omega_A \setminus \Omega_A' \).
Define $D_0' = P_0^{-1}(D_0)$ and $D_A' = P_A^{-1}(D_A)$. Then $(U_0')^c \subset D_0', (U_0)^c \subset D_0$, $(U_A')^c \subset D_A'$ and $(U_A)^c \subset D_A$ (see Fig. 14). Since the annulus $D_0' \setminus (U_0')^c$ does not contain any of the critical point, the restriction $P_0 : D_0' \setminus (U_0')^c \to D_0 \setminus (U_0)^c$ is a degree 2 covering, and for the same reason the restriction $P_A : D_A' \setminus (U_A')^c \to D_A \setminus (U_A)^c$ is a degree 2 covering as well. On the other hand, the map $\eta : D_0 \setminus (U_0)^c \to D_A \setminus (U_A)^c$ is a conformal conjugacy between the parabolic-like restrictions (since the boundary of $D_0$ do not cross the equipotential passing through $z = 1$ by construction). Then we can lift $\eta$ such that it extends to a conformal conjugacy between $P_0$ and $P_A$ on $D_0' \setminus (U_0')^c$. Therefore the conjugacy $\eta$ is now defined on the annulus $D_0' \setminus (U_0')^c \cup \Delta_0'$.

Let us define the sets $V_0 = P_0((D_0')^c)$ and $V_A = P_A((D_A')^c)$, hence $L = \overline{(D_A')^c \cap \Omega_A}$ and $M = \overline{(D_A')^c \cap \Omega_A'}$. Then the result follows applying the lemma 2.1.

**Theorem 2.9.** Every parabolic-like mapping $f : U' \to U$ of degree 2 is hybrid equivalent to a member of the family $\text{Per}_1(1)$.

Moreover, if $K_f$ is connected, the member is unique up to conjugacy by a holomorphic map.

**Proof.** Let $g : V' \to V$ be the map obtained by proposition 2.7 where the parabolic-like map is of degree 2 and the external map is $h_2 = \frac{z^2 + \frac{1}{2}}{4 + z^4}$. Let $\psi$ be an external conjugacy between the maps $g$ and $h_2$. Let $S$ be the Riemann surface obtained by gluing $V \cup V'$ and $\hat{\mathbb{C}} \setminus \overline{D}$, by the equivalence relation identifying $z$ to $\psi(z)$, i.e.

$$S = (V \cup V') \bigsqcup \hat{\mathbb{C}} \setminus \overline{D} / z \sim \psi(z)$$

By the Uniformization theorem, $S$ is isomorphic to the Riemann sphere. Consider the map

$$\tilde{g}(z) = \begin{cases} g & \text{on } V' \\ h_2 & \text{on } \hat{\mathbb{C}} \setminus \overline{D} \end{cases}$$

Since the map $h_2$ is the external map of $g$, the map $\tilde{g}$ is continuous and then holomorphic. Let $\hat{\phi} : S \to \hat{\mathbb{C}}$ be an isomorphism that sends the parabolic fixed point of $\tilde{g}$ to infinity, the critical point of $\tilde{g}$ to $z = -1$, and the preimage of the parabolic fixed point of $\tilde{g}$ to $z = 0$. Define $P_2 = \hat{\phi} \circ \tilde{g} \circ \hat{\phi}^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The map $P_2$ is a holomorphic function hybrid conjugate to the map $f$. Let us show that $P_2$ is a member of the family $\text{Per}_1(1)$. We know that

$$\text{Per}_1(1) = \{ P_A = z + 1/z + \sqrt{A} \} = \{ \text{quadratic rational function with a parabolic fixed point of multiplier 1} \}.$$
The map $P_2$ is holomorphic on the Riemann sphere and with degree 2, so it is a quadratic rational function. Moreover, by construction, it has a parabolic fixed point of multiplier 1 at $z = \infty$ with preimage $z = 0$, and it has a critical point at $z = -1$. Therefore $P_2 = P_A$ for some $A$.

Let us prove now that $P_A$ is unique if $K_f$ is connected. If $K_f$ is connected, the map $\tilde{\psi} : (V \cup V') \setminus K_g \to (B'_{h_2} \cup B_{h_2}) \setminus \mathbb{D}$ is a conformal conjugacy between $g$ and $h_2$ on $(V \cup V') \setminus K_g$.

Let us suppose that $K_f$ is connected but $P_A$ is not unique. Then let $g_1 : V_1' \to V_1$ and $g_2 : V_2' \to V_2$ be two maps obtained by proposition 2.7 where the parabolic-like map is of degree 2 and the external map is $h_2 = \frac{z^2 + 1}{1 + \frac{z}{3}}$.

Let $(A', A, f, \gamma_+, \gamma_-), (g_1, B'_{g_1}, B_{g_1}, \gamma_{g_1+}, \gamma_{g_1-})$ and $(g_2, B'_{g_2}, B_{g_2}, \gamma_{g_2+}, \gamma_{g_2-})$ be parabolic-like restriction of $(U', U, f, \gamma_+, \gamma_-), (g_1, V_1', V_1, \gamma_{g_1+}, \gamma_{g_1-})$ and $(g_2, V_2', V_2, \gamma_{g_2+}, \gamma_{g_2-})$ respectively, and let $\varphi_1 : (A' \cup A) \to (B'_{g_1} \cup B_{g_1}), \varphi_2 : (A' \cup A) \to (B'_{g_2} \cup B_{g_2})$ be hybrid conjugacies between $f$ and $g_1, g_2$ respectively (see Fig.15). Then the map $\varphi_2 \circ \varphi_1^{-1}$ is the harmonic conjugacy between $g_1$ and $g_2$ on $B'_{g_1} \cup B_{g_1}$. Let $\psi_1 : (B'_{g_1} \cup B_{g_1}) \setminus K_{g_1} \to (B'_{h_2} \cup B_{h_2}) \setminus \overline{\mathbb{D}}$, $\psi_2 : (B'_{g_2} \cup B_{g_2}) \setminus K_{g_2} \to (B'_{h_2} \cup B_{h_2}) \setminus \overline{\mathbb{D}}$ be conformal conjugacies between $g_1, g_2$ respectively and $h_2$ on $(B'_{g_1} \cup B_{g_1}) \setminus K_{g_1}$ and $(B'_{g_2} \cup B_{g_2}) \setminus K_{g_2}$ respectively. Then the map $\psi_2^{-1} \circ \psi_1$ is a conformal conjugacy between $g_1$ and $g_2$ on $(B'_{g_1} \cup B_{g_1}) \setminus K_{g_1}$.

Define the map $\Phi : (B'_{g_1} \cup B_{g_1}) \to (B'_{g_2} \cup B_{g_2})$ as

$$
\Phi(z) = \begin{cases} 
\varphi_2 \circ \varphi_1^{-1} & \text{on } K_{g_1} \\
\psi_2^{-1} \circ \psi_1 & \text{on } (B'_{g_1} \cup B_{g_1}) \setminus K_{g_1} 
\end{cases}
$$

By construction the map $\Phi : (B'_{g_1} \cup B_{g_1}) \to (B'_{g_2} \cup B_{g_2})$ conjugates the maps $g_1$ and $g_2$ conformally on $(B'_{g_1} \cup B_{g_1}) \setminus K_{g_1}$ and quasiconformally with $\partial \Phi = 0$ on $K_{g_1}$. We want to prove that the map $\Phi$ is holomorphic. Let us first prove that the map $\Phi$ is continuous.

The map $\psi_2 \circ \varphi_2 \circ \varphi_1^{-1} \circ \psi_1^{-1} : (B'_{h_2} \cup B_{h_2}) \setminus \overline{\mathbb{D}} \to (B'_{h_2} \cup B_{h_2}) \setminus \overline{\mathbb{D}}$ is a quasi-conformal conjugacy between $h_2$ and itself on $\Omega'_{h_2} \cup \gamma_{h_2}, \gamma_{h_2}(0)$, that we can extend to $\Omega'_{h_2} \cup \gamma_{h_2}$ by setting $\psi_2 \circ \varphi_2 \circ \varphi_1^{-1} \circ \psi_1^{-1}(1) = 1$. Applying the strong reflection principle with respect to the unit circle, and then restricting to $\mathbb{S}^1$, we obtain a quasi-symmetric conjugacy between $h_2$ and itself on the unit circle. Since the preimages of the parabolic fixed point $z = 1$ are dense in $\mathbb{S}^1$, a continuous conjugacy on the unit circle between $h_2$ and itself has to be the identity. Then we obtain $\psi_2 \circ \varphi_2 \circ \varphi_1^{-1} \circ \psi_1^{-1} |_{z_0} = Id$.

Since the map $\psi_2 \circ \varphi_2 \circ \varphi_1^{-1} \circ \psi_1^{-1}$ is a homeomorphism from the annulus $(B'_{h_2} \cup B_{h_2}) \setminus \overline{\mathbb{D}}$ to itself, and coincide to the identity on $\mathbb{S}^1$, then the hyperbolic
Figure 16: The maps $\varphi_1 : (A' \cup A) \rightarrow B'_{g_1} \cup B_{g_1}$, $\varphi_2 : (A' \cup A) \rightarrow B'_{g_2} \cup B_{g_2}$ (defined in the sets with the stars) are a hybrid conjugacies between $f$ and $g_1$, $g_2$ respectively, and the maps $\psi_1 : (B'_{g_1} \cup B_{g_1}) \setminus K_{g_1} \rightarrow (B'_{g_2} \cup B_{g_2}) \setminus K_{g_2}$, $\psi_2 : (B'_{g_2} \cup B_{g_2}) \setminus K_{g_2} \rightarrow (B'_{h_2} \cup B_{h_2}) \setminus D$ (defined in the sets in yellow) are conformal conjugacies between $g_1$, $g_2$ respectively and $h_2$ on $(B'_{g_1} \cup B_{g_1}) \setminus K_{g_1}$ and $(B'_{g_2} \cup B_{g_2}) \setminus K_{g_2}$ respectively. Then the map $\varphi_2 \circ \varphi_1^{-1}$ is a hybrid conjugacy between $g_1$ and $g_2$ on $B'_{g_1}$ and the map $\psi_2^{-1} \circ \psi_1$ is a conformal conjugacy between $g_1$ and $g_2$ on $(B'_{g_1} \cup B_{g_1}) \setminus K_{g_1}$.

The distance between a point tending to $S^1$ and its image is bounded:

$$
d_{D(B'_{h_2} \cup B_{h_2}) \setminus D}(z, \psi_2 \circ \varphi_2 \circ \varphi_1^{-1} \circ \psi_1^{-1}(z)) \leq M \quad \text{as } z \rightarrow S^1
$$

for some $M > 0$. Since $\psi_1$ and $\psi_2$ are isometries, we obtain

$$
d_{D(B'_{g_1} \cup B_{g_1}) \setminus K_{g_1}}(\psi_2^{-1} \circ \psi_1(z), \varphi_2 \circ \varphi_1^{-1}(z)) \leq M \quad \text{as } z \rightarrow K_{g_1}.
$$

Then $\psi_2^{-1} \circ \psi_1(z)$ and $\varphi_2 \circ \varphi_1^{-1}(z)$ converge to the same value as $z$ converges to $J_{g_1}$, and therefore the map

$$
\Phi(z) = \begin{cases} 
\varphi_2 \circ \varphi_1^{-1} & \text{on } K_{g_1} \\
\psi_2^{-1} \circ \psi_1 & \text{on } (B'_{g_1} \cup B_{g_1}) \setminus K_{g_1}
\end{cases}
$$
is continuous. To see that $\Phi$ is holomorphic we apply now the Rickmann lemma (for the proof of Rickmann lemma we refer to [DH]):

**Lemma 2.3. Rickmann** Let $U \subset \mathbb{C}$ be open, $K \subset U$ be compact, $\phi : U \to \mathbb{C}$ and $\Phi : U \to \mathbb{C}$ be two maps which are homeomorphisms onto their images. Suppose that $\phi$ is quasi-conformal, that $\Phi$ is quasi-conformal on $U \setminus K$ and that $\Phi = \phi$ on $K$. Then $\Phi$ is quasiconformal and $D\Phi = D\phi$ almost everywhere on $K$.

Hence the map $\Phi$ is holomorphic, and this implies that the member of $\text{Per}_1(1)$ to which a parabolic-like map $f$ with connected fille Julia set is hybrid conjugate is unique up to conjugacy by a holomorphic map.  

**References**

[BF] B. Branner & N. Fagella, *Quasiconformal Surgery in Holomorphic dynamics*, preprint.

[DH] A. Douady & J. H. Hubbard, On the Dynamics of Polynomial-like Mappings, *Ann. Sci. École Norm. Sup.*, (4), Vol.18 (1985), 287-343.

[Sh] M. Shishikura, Bifurcation of parabolic fixed points, *The Mandelbrot set, Theme and Variations*, (325-363), *London Math. Soc. Lecture Note Ser.*, 274 Cambridge Univ. Press, (2000).