PREDUAL OF WEAK ORLICZ SPACES

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Abstract. In this paper, we consider the predual spaces of weak Orlicz spaces. As an application, we provide the Fefferman-Stein vector-valued maximal inequality for the weak Orlicz spaces. In order to prove this statement, we introduced the Orlicz-Lorentz spaces, and showed the boundedness of the Hardy-Littlewood maximal operator on these spaces.

Keywords weak Orlicz spaces, Orlicz-Lorentz spaces, predual spaces, Hardy-Littlewood maximal operator.

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1. Introduction

The purpose of this paper is to give the predual spaces of weak Orlicz spaces. Moreover, as an application, we provide the Fefferman-Stein vector-valued maximal inequality for the weak Orlicz spaces.

For a function $\Phi : [0, \infty] \to [0, \infty]$, let

$$a(\Phi) \equiv \sup \{ t \geq 0 : \Phi(t) = 0 \}, \quad b(\Phi) \equiv \inf \{ t \geq 0 : \Phi(t) = \infty \}.$$ 

Definition 1.1 (Young function). An increasing function $\Phi : [0, \infty] \to [0, \infty]$ is called a Young function (or sometimes also called an Orlicz function) if it satisfies the following properties;

1. $0 \leq a(\Phi) < \infty$, $0 < b(\Phi) \leq \infty$,
2. $\lim_{t \to 0} \Phi(t) = \Phi(0) = 0$,
3. $\Phi$ is convex on $[0, b(\Phi))$,
4. if $b(\Phi) = \infty$, then $\lim_{t \to \infty} \Phi(t) = \Phi(\infty) = \infty$,
5. if $b(\Phi) < \infty$, then $\lim_{t \to b(\Phi)} \Phi(t) = \Phi(b(\Phi))$.

For $t > 0$ and $f \in L^0(\mathbb{R}^n)$, the distribution function $m(f, t)$ and the rearrangement function $f^*(t)$ are defined by

$$m(f, t) \equiv | \{ x \in \mathbb{R}^n : |f(x)| > t \} |, \quad f^*(t) \equiv \inf \{ \alpha > 0 : m(f, \alpha) \leq t \}.$$ 

Here it will be understood that $\inf \emptyset = \infty$.

Definition 1.2 (weak Orlicz space). For a Young function $\Phi : [0, \infty] \to [0, \infty]$, let

$$wL^\Phi(\mathbb{R}^n) \equiv \left\{ f \in L^0(\mathbb{R}^n) : \sup_{t > 0} \Phi(t)m(kf, t) < \infty \text{ for some } k > 0 \right\},$$

$$\| f \|_{wL^\Phi} \equiv \inf \left\{ \lambda > 0 : \sup_{t > 0} \Phi(t)m \left( \frac{f}{\lambda}, t \right) \leq 1 \right\}.$$
Remark 1.3. Let \( f \in wL^\Phi (\mathbb{R}^n) \). Then there exists \( k > 0 \) such that
\[
\sup_{t>0} \Phi(t)m(kf,t) \leq 1.
\]
In fact, we put
\[
M \equiv \sup_{t>0} \Phi(t)m(kf,t) < \infty,
\]
and assume that \( M > 1 \). Note that, by the convexity of \( \Phi \),
\[
\frac{1}{M} \Phi(t) \geq \Phi\left(\frac{t}{M}\right), \quad t \in [0, \infty).
\]
Then, taking \( 0 < k_0 \leq \frac{1}{M} \), we have
\[
\sup_{t>0} \Phi(t)m(kk_0f,t) = \sup_{t>0} \Phi(t)m\left(\frac{kf}{k_0}, \frac{t}{k_0}\right) = \sup_{t>0} \Phi(k_0t)m(kf,t) \leq \sup_{t>0} \Phi\left(\frac{t}{M}\right)m(kf,t) \leq \sup_{t>0} \frac{1}{M} \Phi(t)m(kf,t) = 1.
\]

A Young function \( \Phi : [0, \infty] \to [0, \infty] \) is said to satisfy the \( \Delta_2 \)-condition, denoted \( \Phi \in \Delta_2 \), if
\[
\Phi(2r) \leq k \Phi(r) \quad \text{for} \quad r > 0,
\]
for some \( k > 1 \). A Young function \( \Phi : [0, \infty] \to [0, \infty] \) is said to satisfy the \( \nabla_2 \)-condition, denoted \( \Phi \in \nabla_2 \), if
\[
\Phi(r) \leq \frac{1}{2k} \Phi(kr) \quad \text{for} \quad r \geq 0,
\]
for some \( k > 1 \).

Example 1.4. \( 1 \) \( \Phi(t) = t^p, \ 1 \leq p < \infty \), belongs to \( \Delta_2 \).
\( 2 \) \( \Phi(t) = t^p, \ 1 < p < \infty \), belongs to \( \nabla_2 \).
\( 3 \) \( \Phi(t) = t \) does not belong to \( \nabla_2 \).
\( 4 \) \( \Phi(t) = t \log(3 + t) \) belongs to \( \Delta_2 \), but does not belong to \( \nabla_2 \).
\( 5 \) \( \Phi(t) = e^t - 1 \) belongs to \( \nabla_2 \), but does not belong to \( \Delta_2 \).

Let
\[
\Phi^{-1}(u) = \begin{cases} \inf \{t \geq 0 : \Phi(t) > u\}, & u \in [0, \infty), \\ \infty, & u = \infty. \end{cases}
\]
Then \( \Phi^{-1}(u) \) is finite for all \( u \in [0, \infty) \), continuous on \( (0, \infty) \) and right continuous at \( u = 0 \). If \( \Phi \) is bijective from \( [0, \infty] \) to itself, then \( \Phi^{-1} \) is the usual inverse function of \( \Phi \). It is also known that
\[
(1.1) \quad \Phi(\Phi^{-1}(t)) \leq t \leq \Phi^{-1}(\Phi(t)), \quad t \in [0, \infty].
\]

For a Young function \( \Phi : [0, \infty] \to [0, \infty] \), the complementary function is defined by
\[
\bar{\Phi}(r) \equiv \begin{cases} \sup \{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty), \\ \infty, & r = \infty. \end{cases}
\]
Then \( \bar{\Phi} \) is also a Young function and \( \bar{\Phi} = \Phi \). Note that \( \Phi \in \nabla_2 \) if and only if \( \bar{\Phi} \in \Delta_2 \). It is known that
\[
(1.2) \quad r \leq \Phi^{-1}(r)\bar{\Phi}^{-1}(r) \leq 2r \quad \text{for} \quad r \geq 0.
\]
Theorem 1.5. Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function such that \( \Phi \in \Delta_2 \). Then

\[
L^{\Phi,1}(\mathbb{R}^n)^* = wL^{\tilde{\Phi}}(\mathbb{R}^n)
\]

with equivalence of quasi-norms, where the space \( L^{\Phi,1}(\mathbb{R}^n) \) is defined by the set of all measurable functions with the finite quasi-norm

\[
\|f\|_{L^{\Phi,1}} \equiv \int_0^\infty \Phi^{-1}\left(\frac{1}{t}\right)^{-1} f^*(t) \frac{dt}{t}.
\]

Definition 1.6 (Hardy-Littlewood maximal operator). For a measurable function \( f \) defined on \( \mathbb{R}^n \), define a function \( Mf \) by

\[
Mf(x) \equiv \sup_{Q \in Q(\mathbb{R}^n)} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)| dy, \quad x \in \mathbb{R}^n,
\]

where \( Q(\mathbb{R}^n) \) denotes the family of all cubes with parallel to coordinate axis in \( \mathbb{R}^n \).

Theorem 1.7. Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function, and let \( 1 < q < \infty \).

1. If \( \Phi \in \Delta_2 \), then

\[
\left\| \sup_{j \in \mathbb{N}} M f_j \right\|_{wL^q} \lesssim \left\| \sup_{j \in \mathbb{N}} |f_j| \right\|_{wL^q}
\]

for any sequence \( \{f_j\}_{j=1}^\infty \) of measurable functions.

2. If \( \Phi \in \Delta_2 \cap \nabla_2 \), then

\[
\left\| \left( \sum_{j=1}^\infty (M f_j)^q \right)^{\frac{1}{q}} \right\|_{wL^q} \lesssim \left\| \left( \sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_{wL^q}
\]

for any sequence \( \{f_j\}_{j=1}^\infty \) of measurable functions.

We organize the remaining part of the paper as follows: We prepare the statements for the proof of Theorem 1.5 in Section 2 and show Theorem 1.5 in Section 3. Next, we provide the boundedness of the Hardy-Littlewood maximal operator on generalized Lorentz spaces in Section 4. Finally, we prove Theorem 1.7 in Section 5.

2. Preliminaries

2.1. Statements of inverse Young function for \( \Delta_2 \) and \( \nabla_2 \) conditions.

Lemma 2.1. Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function. Then the followings are obtained:

1. \( \Phi \in \Delta_2 \) if and only if there exists \( k > 1 \) such that

\[
\Phi^{-1}(ku) \geq 2\Phi^{-1}(u).
\]

2. \( \Phi \in \nabla_2 \) if and only if there exists \( k > 1 \) such that

\[
\Phi^{-1}(2ku) \leq k\Phi^{-1}(u).
\]

Proof. First we show (1). We assume that \( \Phi \in \Delta_2 \). Then

\[
\Phi(2\Phi^{-1}(u)) \leq k\Phi(\Phi^{-1}u) \leq ku
\]

and therefore we have

\[
2\Phi^{-1}(u) \leq \Phi^{-1}(ku).
\]
Conversely, we assume that there exists \( k > 1 \) such that
\[ \Phi^{-1}(ku) \geq 2 \Phi^{-1}(u). \]
For \( u \geq 0 \), taking \( t = \Phi(u) \), we have \( u \leq \Phi^{-1}(\Phi(u)) = \Phi^{-1}(t) \). Hence
\[ \Phi(2u) \leq \Phi(2\Phi^{-1}(t)) \leq \Phi(\Phi^{-1}(kt)) \leq kt \leq k\Phi(u). \]

Next we show (2). We assume that \( \Phi \in \nabla_2 \). Then, for any \( v \geq 0 \),
\[ \Phi\left(\frac{1}{k} \Phi^{-1}(v)\right) \leq \frac{1}{2k} \Phi(\Phi^{-1}(v)) \leq \frac{v}{2k}, \]
and hence
\[ \frac{1}{k} \Phi^{-1}(v) \leq \Phi^{-1}\left(\frac{v}{2k}\right). \]
Putting \( v \mapsto 2ku \), we have
\[ \Phi^{-1}(2ku) \leq k\Phi^{-1}(u). \]
Conversely, we assume that there exists \( k > 1 \) such that
\[ \Phi^{-1}(2ku) \leq k\Phi^{-1}(u). \]
For \( u \geq 0 \), taking \( t = \Phi(2ku) \), we have \( ku \leq \Phi^{-1}(\Phi(2ku)) = \Phi^{-1}(t) \). Hence
\[ \Phi(u) \leq \Phi\left(\frac{1}{k} \Phi^{-1}(t)\right) \leq \Phi\left(\Phi^{-1}\left(\frac{t}{2k}\right)\right) \leq \frac{t}{2k} = \frac{1}{2k} \Phi(2ku). \]

\[ \square \]

**Lemma 2.2.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function. If \( \Phi \in \Delta_2 \), then there exists \( q \in (0, \infty) \) and \( C \in [1, \infty) \) such that
\[ \frac{1}{t} \Phi^{-1}\left(\frac{1}{t}\right)^{-q} \leq C \frac{1}{s} \Phi^{-1}\left(\frac{1}{s}\right)^{-q} \quad \text{for} \quad t \leq s. \]

Conversely, if there exists \( q \in (0, \infty) \) and \( C \in [1, \infty) \) such that
\[ \frac{1}{t} \Phi^{-1}\left(\frac{1}{t}\right)^{-q} \leq C \frac{1}{s} \Phi^{-1}\left(\frac{1}{s}\right)^{-q} \quad \text{for} \quad t \leq s, \]
then \( \Phi \in \Delta_2 \).

**Proof.** If \( \Phi \in \Delta_2 \), putting \( q = \log_2 k \) and \( C = k \), we calculate
\[ \Phi^{-1}\left(\frac{1}{t}\right) = \Phi^{-1}\left(k^{\log_2 \frac{1}{t}} \frac{1}{t}\right) \geq \Phi^{-1}\left(k^{\log_2 \frac{1}{t}} \frac{1}{s}\right) \geq 2^{\log_2 \frac{1}{s}} \Phi^{-1}\left(\frac{1}{s}\right) \geq 2^{\log_2 \frac{1}{s}} \Phi^{-1}\left(\frac{1}{s}\right), \]
where \([\cdot]\) stands for the Gauss symbol. This is the desired result.

Conversely, we suppose that there exists \( q \in (0, \infty) \) and \( C \in [1, \infty) \) such that
\[ \frac{1}{t} \Phi^{-1}\left(\frac{1}{t}\right)^{-q} \leq C \frac{1}{s} \Phi^{-1}\left(\frac{1}{s}\right)^{-q} \quad \text{for} \quad t \leq s. \]

Then, by the change of variables \( u = \frac{1}{t} \) and \( v = \frac{1}{s} \), we compute
\[ u \Phi^{-1}(u)^{-q} \leq v \Phi^{-1}(v)^{-q}, \]
or equivalently
\[ \Phi^{-1}(v) \leq \left(\frac{v}{u}\right)^{\frac{1}{q}} \Phi^{-1}(u). \]
Consequently, choosing \( u = kv \) and \( k = 2^iC > 1 \), we obtain

\[
\Phi^{-1} (u) \leq \left( \frac{1}{k} \right)^\frac{1}{i} \Phi^{-1} (kv) \leq \frac{1}{2} \Phi^{-1} (kv)
\]
as desired by Lemma 2.1 (1).

\[\square\]

2.2. Orlicz-Lorentz spaces and weak-type Orlicz spaces.

**Definition 2.3** (Orlicz-Lorentz space). For a parameter \( 0 < q \leq \infty \) and a Young function \( \Phi : [0, \infty] \to [0, \infty] \), let

\[
L^{\Phi, q}(\mathbb{R}^n) \equiv \left\{ f \in L^0(\mathbb{R}^n) : \| f \|_{L^{\Phi, q}} < \infty \right\},
\]

endowed with the quasi-norm

\[
\| f \|_{L^{\Phi, q}} \equiv \begin{cases} 
\left( \int_0^\infty \left[ \Phi^{-1} \left( \frac{1}{t} \right) f^*(t) \right]^{\frac{q}{\Phi'}} \frac{dt}{t} \right)^\frac{1}{q}, & 0 < q < \infty, \\
\sup_{t > 0} \Phi^{-1} \left( \frac{1}{t} \right) f^*(t), & q = \infty.
\end{cases}
\]

**Definition 2.4** (Orlicz space). For a Young function \( \Phi : [0, \infty] \to [0, \infty] \), let

\[
L^\Phi(\mathbb{R}^n) \equiv \left\{ f \in L^0(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(|f(x)|) \, dx < \infty \text{ for some } k > 0 \right\},
\]

\[
\| f \|_{L^\Phi} \equiv \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
\]

**Definition 2.5.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function. The weak-type Orlicz space \( WL^\Phi(\mathbb{R}^n) \) is defined by

\[
WL^\Phi(\mathbb{R}^n) \equiv \left\{ f \in L^0(\mathbb{R}^n) : \| f \|_{WL^\Phi} \equiv \sup_{t > 0} t \| \chi_{\{x \in \mathbb{R}^n : |f(x)| > t\}} \|_{L^\Phi} \right\}.
\]

**Proposition 2.6.** Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function. Then we have \( wL^\Phi(\mathbb{R}^n) = WL^\Phi(\mathbb{R}^n) = L^{\Phi, \infty}(\mathbb{R}^n) \).

**Proof.** See [3] Proposition 1.4.5 (16) for the equality \( WL^\Phi(\mathbb{R}^n) = L^{\Phi, \infty}(\mathbb{R}^n) \) in detail.

When we choose \( f \in wL^\Phi(\mathbb{R}^n) \), we prove \( \| f \|_{WL^\Phi} \leq \| f \|_{wL^\Phi} \). Fix a sufficiently small number \( \varepsilon > 0 \). Then

\[
\sup_{t > 0} \Phi(t)m \left( \frac{t}{\| f \|_{wL^\Phi} + \varepsilon} \right) \leq 1.
\]

Changing \( t \mapsto \frac{t}{\| f \|_{wL^\Phi} + \varepsilon} \), we have

\[
\sup_{t > 0} \Phi \left( \frac{t}{\| f \|_{wL^\Phi} + \varepsilon} \right) m(f, t) \leq 1.
\]

By the equation (1.1), for all \( t > 0 \),

\[
\frac{t}{\| f \|_{wL^\Phi} + \varepsilon} \leq \Phi^{-1} \left( \frac{1}{m(f, t)} \right),
\]
or equivalently,

\[
t\Phi^{-1} \left( \frac{1}{m(f, t)} \right)^{-1} \leq \| f \|_{wL^\Phi} + \varepsilon.
\]

Thus we obtain

\[
\| f \|_{WL^\Phi} \leq \| f \|_{wL^\Phi} + \varepsilon.
\]
Let \( f \in W^\Phi_L(R^n) \). We verify \( \|f\|_{W^\Phi_L} \geq \|f\|_{w^\Phi_L} \). Remark that for all \( t > 0 \),
\[
t \Phi^{-1} \left( \frac{1}{m(f,t)} \right)^{-1} \leq \|f\|_{W^\Phi_L}.
\]
We calculate
\[
\Phi \left( \frac{t}{\|f\|_{W^\Phi_L}} \right) \leq \Phi \left( \Phi^{-1} \left( \frac{1}{m(f,t)} \right) \right) \leq \frac{1}{m(f,t)},
\]
and then
\[
\Phi \left( \frac{t}{\|f\|_{W^\Phi_L}} \right) m(f,t) \leq 1.
\]
Hence
\[
\sup_{t > 0} \Phi \left( \frac{t}{\|f\|_{W^\Phi_L}} \right) m(f,t) \leq 1.
\]
Therefore \( \|f\|_{w^\Phi_L} \leq \|f\|_{W^\Phi_L} \). \( \square \)

**Proposition 2.7.** Let \( 0 < q < \infty \), and let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function. Then for any measurable set \( E \subset R^n \), the followings are hold:

1. \( \|\chi_E\|_{L^\Phi_q} \geq \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1} \) and \( \|\chi_E\|_{L^\Phi_\infty} = \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1} \).
2. \( \Phi \in \Delta_2 \) implies \( \|\chi_E\|_{L^\Phi_q} \sim \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1} \).

**Remark 2.8.** Let \( \Phi \) be a Young function, and let \( 0 < q < \infty \). By the convexity of \( \Phi \), \( L^\Phi_q(R^n) \neq \{0\} \) if and only if
\[
\int_0^1 \Phi^{-1} \left( \frac{1}{t} \right)^{-q} \frac{dt}{t} < \infty,
\]
generally.

1. If \( b(\Phi) < \infty \), then, we have \( L^\Phi_q(R^n) = \{0\} \). Indeed, it is suffices to show that for all measurable sets \( E \subset R^n \) with \( |E| \neq 0 \),
\[
\|\chi_E\|_{L^\Phi_1} = \infty.
\]
By definition, for \( u \geq 0 \),
\[
\Phi^{-1}(u) = \inf \{ 0 \leq t \leq b(\Phi) : \Phi(t) > u \} \leq b(\Phi)
\]
Then,
\[
\|\chi_E\|_{L^\Phi_1} = \int_0^{|E|} \Phi^{-1} \left( \frac{1}{u} \right)^{-1} \frac{du}{u} \geq \frac{1}{b(\Phi)} \int_0^{|E|} \frac{du}{u} = \infty.
\]
2. Let
\[
\Phi(t) = \begin{cases} 
0, & t \leq 1, \\
1, & t > 1.
\end{cases}
\]
Then, \( a(\Phi) > 0 \) and
\[
\Phi^{-1}(u) = \inf \{ t > 1 : t - 1 > u \} = u + 1.
\]
Additionally, for each measurable set \( E \subset R^n \),
\[
\|\chi_E\|_{L^\Phi_1} = \int_0^{|E|} \left( \frac{1}{u + 1} \right)^{-1} \frac{du}{u} = \int_0^{|E|} \frac{1}{u + 1} \frac{du}{u} = \log(1 + |E|) \neq \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1}.
\]
(3) Taking $\Phi(t) = e^t - 1$, $t \geq 0$, we have $a(\Phi) = 0$ and $b(\Phi) = \infty$. Especially, $L^{\Phi,1}(\mathbb{R}^n) = \{0\}$. Indeed,

$$\Phi^{-1}(u) = \inf\{t \geq 0 : e^t - 1 > u\} = \log(1 + u),$$

and then, for any measurable set $E \subset \mathbb{R}^n$ with $|E| \leq \frac{1}{2}$,

$$\|\chi_E\|_{L^{\Phi,1}} = \int_0^{|E|} \frac{1}{\log \left(1 + \frac{1}{t} \right)} \frac{dt}{u} = \int_0^{|E|} \frac{1}{\log(u + 1) + \log \frac{1}{u}} \frac{du}{u} \geq \int_0^{|E|} \frac{1}{2 \log \frac{1}{u}} \frac{du}{u} = \infty.$$

Meanwhile, $q > 1$ implies $L^{\Phi,q}(\mathbb{R}^n) \neq \{0\}$.

**Proof of Proposition 2.7**

(1) $\|\chi_E\|_{L^{\Phi,\infty}} = \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1}$ is obtained by Proposition 2.6 and the following calculation:

$$\|\chi_E\|_{L^{\Phi,\infty}} = \|\chi_E\|_{W^L} = \|\chi_E\|_{L^{\Phi}} = \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1}.$$

In addition, since $\frac{1}{t} \Phi^{-1} \left( \frac{1}{t} \right)^{-1}$ is decreasing, we compute

$$\|\chi_E\|_{L^{\Phi,q}} = \left( \int_0^{|E|} \Phi^{-1} \left( \frac{1}{t} \right)^{-q} \frac{dt}{t} \right)^\frac{1}{q} \geq \frac{1}{|E|} \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1} \left( \int_0^{|E|} t^q \frac{dt}{t} \right)^\frac{1}{q}$$

$$= \frac{1}{q^\frac{1}{q}} \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1}$$

as desired.

(2) We assume that $\Phi \in \Delta_2$. By Lemma 2.2 there exists $q \in (0, \infty)$ and $C \in [1, \infty)$ such that

$$\frac{1}{t} \Phi^{-1} \left( \frac{1}{t} \right)^{-q} \leq C \frac{1}{s^q} \Phi^{-1} \left( \frac{1}{s^q} \right)^{-q} \quad \text{for} \quad t \leq s,$$

and then,

$$\|\chi_E\|_{L^{\Phi,q}} = \int_0^{|E|} \Phi^{-1} \left( \frac{1}{t} \right)^{-q} \frac{dt}{t} \leq C \frac{1}{|E|} \Phi^{-1} \left( \frac{1}{|E|} \right)^{-q} \int_0^{|E|} \frac{dt}{t} = C \Phi^{-1} \left( \frac{1}{|E|} \right)^{-q}.$$

**3. Proof of Theorem 1.5**

Given $T \in L^{\Phi,1}(\mathbb{R}^n)^*$, we consider the measure $\mu(E) = T \chi_E$. Since $\mu$ satisfies

$$|\mu(E)| \leq \|T\|_{(L^{\Phi,1})^*} \|\chi_E\|_{L^{\Phi,1}} \sim \|T\|_{(L^{\Phi,1})^*} \Phi^{-1} \left( \frac{1}{|E|} \right)^{-1},$$

it follows that $\mu$ is absolutely continuous with respect to the Lebesgue measure $|\cdot|$. By the Radon-Nykodym theorem, there exists a unique measurable function $g \in L^1(\mathbb{R}^n)$ such that

$$\mu(E) = \int_E g(x) \, dx$$
for all measurable sets $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$. Therefore, we obtain

$$T \chi_E = \int_{\mathbb{R}^n} \chi_E(x) g(x) \, dx, \quad 0 < |E| < \infty, \ E \subset \mathbb{R}^n.$$ 

In addition, we obtain

$$\left| \int_{\mathbb{R}^n} f(x) g(x) \, dx \right| \leq \|T\|_{(L^{\Phi,1})^*} \|f\|_{L^{\Phi,1}}. \tag{3.1}$$

In fact, given $f \in L^{\Phi,1}(\mathbb{R}^n)$, we take a sequence of non-negative simple functions $\{f_j\}_{j \geq 1}$ such that

$$f_j \uparrow |f|, \quad \text{a.e.}$$

Then, by the Fatou lemma, we have

$$\int_{\mathbb{R}^n} |f(x) g(x)| \, dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} |f_j(x) g(x)| \, dx, \quad g \in L^1(\mathbb{R}^n).$$

We compute

$$\int_{\mathbb{R}^n} |f_j(x) g(x)| \, dx = T[f_j \text{ sgn } g] \leq \|T\|_{(L^{\Phi,1})^*} \|f_j \text{ sgn } g\|_{L^{\Phi,1}} \leq \|T\|_{(L^{\Phi,1})^*} \|f\|_{L^{\Phi,1}}.$$ 

Note that, for each $t > 0$,

$$\left\| \frac{g}{|g|} \chi_{\{x \in \mathbb{R}^n : |g(x)| > t\}} \right\|_{L^{\Phi,1}} \sim \Phi^{-1} \left( \frac{1}{\{x \in \mathbb{R}^n : |g(x)| > t\}} \right)^{-1} \leq \Phi^{-1} \left( \frac{t}{\|g\|_{L^1}} \right)^{-1} < \infty.$$ 

Taking $f = \frac{g}{|g|} \chi_{\{x \in \mathbb{R}^n : |g(x)| > t\}} \in L^{\Phi,1}(\mathbb{R}^n)$ for each $t > 0$ in (3.1), we have

$$t |\{x \in \mathbb{R}^n : |g(x)| > t\}| \leq \|T\|_{(L^{\Phi,1})^*} \Phi^{-1} \left( \frac{1}{\{x \in \mathbb{R}^n : |g(x)| > t\}} \right)^{-1},$$

and, by (1.2), then

$$t \Phi^{-1} \left( \frac{1}{\{x \in \mathbb{R}^n : |g(x)| > t\}} \right)^{-1} \leq \|T\|_{(L^{\Phi,1})^*}.$$

Consequently, take the supremum over $t > 0$ to obtain that

$$\|g\|_{wL^\Phi} \leq \|T\|_{(L^{\Phi,1})^*}.$$ 

Conversely, using Exercise 1.4.1 (b) in [3] and equation (1.2), we see that if $f \in L^{\Phi,1}(\mathbb{R}^n)$ and $h \in wL^{\hat{\Phi}}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |f(x) h(x)| \, dx \leq \int_0^\infty f^*(t) h^*(t) \, dt \leq 2 \int_0^\infty \Phi^{-1} \left( \frac{1}{t} \right)^{-1} f^*(t) \cdot \Phi^{-1} \left( \frac{1}{t} \right)^{-1} h^*(t) \frac{dt}{t}$$

$$\leq 2 \|f\|_{L^{\Phi,1}} \|h\|_{wL^{\hat{\Phi}}}.$$

Thus every $h \in wL^{\hat{\Phi}}(\mathbb{R}^n)$ gives rise to a bounded linear functional $f \mapsto \int h f \, dx$ on $L^{\Phi,1}(\mathbb{R}^n)$ with norm at most $\|h\|_{wL^{\hat{\Phi}}}.$
4. Generalized Lorentz spaces

Definition 4.1. Let $0 < q \leq \infty$, and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. We define the generalized Lorentz space $\Lambda^{p,q}(\mathbb{R}^n)$ by the set of all measurable functions $f$ with the finite quasi-norm

$$
\|f\|_{\Lambda^{p,q}} \equiv \begin{cases} 
\left( \int_0^\infty [\varphi(t)f^*(t)]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & q < \infty, \\
\text{ess sup}_{t>0} \varphi(t)f^*(t), & q = \infty.
\end{cases}
$$

Remark 4.2. If $\varphi$ is a non-decreasing function satisfying the doubling condition; there exists $C \geq 1$ such that for all $s, t > 0$,

$$
\frac{1}{C} \leq \frac{\varphi(t)}{\varphi(s)} \leq C, \quad \text{if} \quad \frac{1}{2} \leq \frac{t}{s} \leq 2,
$$

then, the generalized Lorentz space $L^{p,q}(\mathbb{R}^n)$ is vector space.

Remark 4.3. (1) If $\varphi$ is a non-increasing function and $q < \infty$, then, $\Lambda^{p,q}(\mathbb{R}^n) = \{0\}$.

(2) If $\varphi(t) = t^p$, then, $\Lambda^{p,q}(\mathbb{R}^n)$ is the classical Lorentz space $L^{p,q}(\mathbb{R}^n)$.

(3) If $\varphi(t) = \Phi^{-1}\left(\frac{1}{t}\right)^{-1}$, then, $\Lambda^{p,q}(\mathbb{R}^n)$ is the Orlicz-Lorentz space $L^{p,q}(\mathbb{R}^n)$.

4.1. Boundedness on Hardy-Littlewood maximal operator on generalized Lorentz spaces.

Theorem 4.4 ([2 Corollary 1.9]). Let $0 < q < \infty$, and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. Then the Hardy-Littlewood maximal operator $M$ is bounded on $\Lambda^{p,q}(\mathbb{R}^n)$ if and if for every $r > 0$,

$$
(4.1) \quad \int_r^\infty \frac{\varphi(t)}{t} \left( \frac{q}{t} \right)^{\frac{q}{r}} dt \lesssim \frac{1}{r^q} \int_0^r \varphi(t) \frac{dt}{t}.
$$

Lemma 4.5. Let $0 < q < \infty$, and $\Phi$ be a Young function. If $\Phi \in \nabla_2$, then $\varphi(t) \equiv \Phi^{-1}\left(\frac{1}{t}\right)^{-1}$ satisfies the equation $(4.1)$.

Proof. Let $q < \infty$.

Fix $r > 0$. Using Lemma 2.1 (2), we calculate

$$
\int_r^\infty \left[ \frac{1}{t} \Phi^{-1}\left(\frac{1}{t}\right)^{-1} \right]^q \frac{dt}{t} = \int_r^\infty \left[ \frac{1}{rt} \Phi^{-1}\left(\frac{1}{rt}\right)^{-1} \right]^q \frac{dt}{t} \leq \sum_{j=1}^\infty \int_{(2k)^{j-1}}^{(2k)^j} \left[ \frac{1}{rt} \Phi^{-1}\left(\frac{1}{r(2k)^j}\right)^{-1} \right]^q \frac{dt}{t},
$$

and

$$
\int_0^r \Phi^{-1}\left(\frac{1}{t}\right)^{-q} dt = \int_0^1 \Phi^{-1}\left(\frac{1}{rt}\right)^{-q} dt \geq \sum_{j=1}^\infty \int_{(2k)^{j-1}}^{(2k)^j} \Phi^{-1}\left(\frac{1}{r(2k)^j}\right)^{-q} \frac{dt}{t}
$$

$$
\geq \sum_{j=1}^\infty \frac{1}{k(q-1)} \Phi^{-1}\left(\frac{1}{r}\right)^{-q} \left( \log \frac{1}{k^{j-1}} - \log \frac{1}{k^j} \right) \sim \Phi^{-1}\left(\frac{1}{r}\right)^{-q}.
$$
Then we obtain
\[
\int_r^\infty \left[ \frac{1}{t} \Phi^{-1} \left( \frac{1}{t} \right)^{-q} \right] \frac{dt}{t} / \int_r^\infty \Phi^{-1} \left( \frac{1}{t} \right)^{-q} \frac{dt}{t} \lesssim 1.
\]

\[\Box\]

**Theorem 4.6.** Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a measurable function. Then the Hardy-Littlewood maximal operator \( M \) is bounded on \( \Lambda^\infty_\varphi (\mathbb{R}^n) \) if and if for every \( r > 0 \),

\[
(4.2) \quad \text{ess sup}_{t>0} \frac{\varphi(t)}{t} \int_0^t \frac{ds}{\text{ess sup}_{0<\tau<s} \varphi(\tau)} < \infty.
\]

To prove this theorem, we may use the following lemmas.

**Lemma 4.7** \((\text{[5, Theorem 4.7]})\). Let \( v, w : \mathbb{R}_+ \to \mathbb{R}_+ \) be measurable functions. Then the inequality

\[
\text{ess sup}_{t>0} w(t) \int_0^t g(s) \frac{ds}{t} \lesssim \text{ess sup}_{t>0} v(t) g(t)
\]

holds for all non-negative and non-increasing \( g \) on \((0, \infty)\) if and only if

\[
\text{ess sup}_{t>0} w(t) \int_0^t \frac{ds}{\text{ess sup}_{0<\tau<s} v(\tau)} < \infty.
\]

**Lemma 4.8.** Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a measurable function. Then the Hardy-Littlewood maximal operator \( M \) is bounded on \( \Lambda^\infty_\varphi (\mathbb{R}^n) \) if and if for all non-negative and non-increasing \( g \) on \( \mathbb{R}_+ \),

\[
(4.3) \quad \text{ess sup}_{t>0} \frac{\varphi(t)}{t} \int_0^t g(s) \frac{ds}{t} \lesssim \text{ess sup}_{t>0} \varphi(t) g(t).
\]

To prove this lemma, we use the following lemma:

**Lemma 4.9.** Let \( g : (0, \infty) \to (0, \infty) \) be a right-continuous non-increasing function. Then, taking \( f(x) \equiv g(\nu_n |x|^n) \) for \( x \in \mathbb{R}^n \), where \( \nu_n \) be a volume of the \( n \)-dimensional unit ball, we have \( f^*(t) = g(t) \) for \( t > 0 \).

**Proof.** For any \( \lambda > 0 \),

\[
m(f, \lambda) = |\{ x \in \mathbb{R}^n : g(\nu_n |x|^n) > \lambda \}| = \nu_n \sup \{ s > 0 : g(\nu_n s^n) > \lambda \}^n = \sup \{ s > 0 : g(s) > \lambda \}.
\]

Then, we note that

\[
f^*(t) = \inf \{ \lambda > 0 : \sup \{ s > 0 : g(s) > \lambda \} \leq t \}.
\]

Fix \( t > 0 \). By the non-increasingly of \( g \),

\[
\sup \{ s > 0 : g(s) > g(t) \} \leq t,
\]

and then, \( f^*(t) \leq g(t) \). Meanwhile, by the non-increasingly and right-continuity of \( g \), for all sufficiently small number \( \varepsilon > 0 \), there exists \( t_0 > 0 \) such that \( g(t_0) \geq g(t) - \varepsilon \). It follows that

\[
\sup \{ s > 0 : g(s) > g(t) - \varepsilon \} \geq t_0 > t,
\]

and therefore, \( f^*(t) \geq g(t) - \varepsilon \). This is the desired result. \( \Box \)
Lemma 5.2. Let \( \Phi : [0, \infty) \to [0, \infty] \) be a Young function. Then by Lemma 4.9, for measurable function \( f \) (see for example \( [3] \) p. 306). We assume that the Hardy-Littlewood maximal operator \( M \) is bounded on \( \Lambda^{\Phi, \infty}(\mathbb{R}^n) \). Fix a non-negative non-increasing function \( g \) on \( \mathbb{R}_+ \) and define \( f \in L^0(\mathbb{R}^n) \) by \( f(x) = g(\nu_n|x|^n) \). Then by Lemma 4.9

\[
\frac{1}{t} \int_0^t f^*(s) \, ds, \quad t > 0,
\]

for measurable function \( f \) (see for example \( [3] \) p. 306).

We assume that the Hardy-Littlewood maximal operator \( M \) is bounded on \( \Lambda^{\Phi, \infty}(\mathbb{R}^n) \). Fix a non-negative non-increasing function \( g \) on \( \mathbb{R}_+ \) and define \( f \in L^0(\mathbb{R}^n) \) by \( f(x) = g(\nu_n|x|^n) \). Then by Lemma 4.9

\[
f^*(t) = g(t),
\]

and then we can verify the equation \( \mathcal{L}_3 \), immediately.

Conversely, taking non-increasing \( g \) in \( \mathcal{L}_3 \) by \( f^* \), we have

\[
\sup_{t>0} \varphi(t)f^*(t) \geq \frac{\varphi(t)}{t} \int_0^t f^*(s) \, ds \sim \sup_{t>0} \varphi(t)(Mf)^*(t)
\]

for all \( f \in \Lambda^{\Phi, \infty}(\mathbb{R}^n) \). This is the desired result. \( \square \)

Lemma 4.10. Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function in \( \nabla_2 \). Then \( \varphi(t) = \Phi \left( \frac{1}{t} \right)^{-1} \), \( t > 0 \), satisfies the equation \( \mathcal{L}_3 \).

Proof. Fix \( t > 0 \). Using Lemma 4.1 (2), we calculate

\[
\frac{1}{t} \Phi^{-1} \left( \frac{1}{t} \right)^{-1} \int_0^t \frac{ds}{\sup_{0<s<t} \Phi^{-1} \left( \frac{1}{s} \right)} = \frac{1}{t} \Phi^{-1} \left( \frac{1}{t} \right)^{-1} \int_0^t \Phi^{-1} \left( \frac{1}{s} \right) \, ds
\]

\[
= \Phi^{-1} \left( \frac{1}{t} \right)^{-1} \int_0^1 \Phi^{-1} \left( \frac{1}{ts} \right) \, ds
\]

\[
\leq \Phi^{-1} \left( \frac{1}{t} \right)^{-1} \sum_{j=1}^{\infty} \int_{(2j)^{1/t}}^{(2j+1)^{1/t}} \Phi^{-1} \left( \frac{(2j)^{1/t}}{t} \right) \, ds
\]

\[
\leq \Phi^{-1} \left( \frac{1}{t} \right)^{-1} \sum_{j=1}^{\infty} \left( \frac{1}{(2j)^{2/t-1}} - \frac{1}{(2j)^{2/t}} \right) k^{j-1} \Phi^{-1} \left( \frac{1}{t} \right)
\]

\[
\leq 1.
\]

\( \square \)

Theorem 4.11. Let \( 0 < q \leq \infty \), and let \( \Phi \) be a Young function in \( \nabla_2 \). Then the Hardy-Littlewood maximal operator \( M \) is bounded on \( L^{\Phi, q}(\mathbb{R}^n) \). In particular, the Hardy-Littlewood maximal operator \( M \) is bounded on \( wL^\Phi(\mathbb{R}^n) \).

5. Proof of Theorem 1.7

Definition 5.1. Let \( \Phi : [0, \infty] \to [0, \infty] \) be a Young function. We define

\[
p_+ = p_+(\Phi) := \inf \{ 1 \leq p \leq \infty \mid \Phi(\lambda r) \leq \lambda^p \Phi(r) \text{ for } r \geq 0, \lambda > 1 \}
\]

\[
p_- = p_-(\Phi) := \sup \{ 1 \leq p \leq \infty \mid \Phi(\lambda r) \leq \lambda^p \Phi(r) \text{ for } r \geq 0, 0 < \lambda < 1 \}.
\]

Lemma 5.2. Let \( \Phi \) be a Young function. Then we have

1. If \( p_+ < \infty \), then we have \( \tilde{p}_- := p_-(\Phi) \geq p'_+ \).
2. If \( p_- > 1 \), then we have \( \tilde{p}_+ := p_+(\Phi) \leq p'_- \).

[RAW TEXT]
Proof. (1) We assume that \( \Phi(\lambda r) \leq \lambda^{p+} \Phi(r) \) for any \( \lambda > 1 \) and \( r > 0 \). From the definition of \( \tilde{\Phi} \), we have

\[
\tilde{\Phi}(r) = \sup_{s>0} \{sr - \Phi(s)\} \leq \sup_{s>0} \{sr - \lambda^{-p+} \Phi(\lambda s)\} = \lambda^{-p+} \sup_{s>0} \{\lambda^{p+} sr - \Phi(\lambda s)\} = \lambda^{-p+} \tilde{\Phi}(\lambda^{p+} r).
\]

When \( \Phi(\lambda r) \leq \lambda^{p+} \Phi(r) \), \( r \geq 0 \), by the change of variables \( s = \lambda^{p+} \frac{1}{r} \) and \( \mu = \lambda^{1-p} \), we have

\[
\tilde{\Phi}(\mu s) \leq \mu^{p-} \tilde{\Phi}(s),
\]

where \( \lambda^{p-} = s \) and \( \lambda^{1-p} = \mu \). Since \( 0 < \lambda < 1 \) if and only if \( 0 < \mu < 1 \), we have the desired result.

(2) We assume that \( \Phi(\lambda r) \leq \lambda^{p-} \Phi(r) \) for any \( 0 < \lambda < 1 \) and \( r > 0 \). So far, if \( \Phi(\lambda r) \leq \lambda^{p-} \Phi(r) \), then we have

\[
\tilde{\Phi}(\mu s) \leq \mu^{p-} \tilde{\Phi}(s),
\]

where \( \lambda^{p-} = s \) and \( \lambda^{1-p} = \mu \). Since \( 0 < \lambda < 1 \) if and only if \( 0 < \mu < 1 \), we have the conclusion. \( \square \)

From Lemma 5.2, we have the following corollary.

**Lemma 5.3.** Let \( \Phi : [0, \infty) \rightarrow [0, \infty) \) be a Young function.

(1) \( p_+ < \infty \) if and only if \( \Phi \in \Delta_2 \).

(2) \( p_- > 1 \) if and only if \( \Phi \in \nabla_2 \).

**Proof.** We assume \( p_+ < \infty \). Letting \( \lambda = 2 \) in the definition of \( p_+ \), we have

\[
\Phi(2r) \leq 2^{p_+} \Phi(r).
\]

From \( p_+ < \infty \), we get \( 1 < 2^{p_+} \). Meanwhile, we assume \( \Phi \in \Delta_2 \). For \( t > 0 \), we compute \( t\Phi'(t) \leq \Phi(2t) \leq k\Phi(t) \), where \( k > 1 \) is a constant appeared in the definition of the condition \( \Delta_2 \). Which implies

\[
\frac{\Phi'(t)}{\Phi(t)} \leq \frac{k}{t}.
\]

Let \( \lambda > 1 \). Integrating both sides of the above inequality from \( r \) to \( \lambda r \), we have

\[
\int_r^{\lambda r} \frac{\Phi'(t)}{\Phi(t)} dt \leq k \int_r^{\lambda r} \frac{dt}{t}
\]

\[
\frac{\Phi(\lambda r)}{\Phi(r)} \leq k \log \frac{\lambda r}{r} \leq k \log \lambda r \leq \lambda^k \Phi(r).
\]

Thus, we get \( p_+ < k < \infty \). Now, we turn to prove (2). From Lemma 5.2 and (1), we get that the conditions \( p_- > 1 \) and \( \Phi \in \nabla_2 \) are equivalent. \( \square \)

**Lemma 5.4.** Let \( \Phi \) be a young function and \( \frac{1}{p_-(\Phi)} \leq \theta < \infty \). We define

\[
\Phi_\theta(r) = \int_0^r \frac{\Phi(t)}{t} dt.
\]

Then, \( \Phi_\theta(r) \) is a Young function and we have

\[
\theta p_-(\Phi) \leq p_-(\Phi_\theta) \leq p_+(\Phi_\theta) \leq \theta p_+(\Phi).
\]

Moreover, we have \( \| \cdot \|_{L^p} \sim \| \cdot \|_{L^q}, \) and \( \| \cdot \|_{wL^p} \sim \| \cdot \|_{wL^q}. \)
Proof. By the change of variables, we have
\[
\Phi_\theta(r) = \int_0^r \frac{\Phi(t)}{t} dt = \int_0^r \frac{\Phi(t^\theta)}{t^\theta} dt,
\]
which means that \(\theta\Phi(t^\theta)t^{-1}\) is the primitive function of \(\Phi_\theta\). Since \(\frac{1}{p_-(\Phi)} \leq \theta < \infty\) and \(\Phi(\lambda t) \leq \lambda^p \Phi(t)\) for any \(t > 0\) and any \(0 < \lambda < 1\), we get
\[
\frac{\Phi(\lambda^\theta t)}{\lambda t} \leq \lambda^{p-1} \frac{\Phi(t^\theta)}{t} \leq \frac{\Phi(t^\theta)}{t},
\]
for any \(t > 0\) and any \(0 < \lambda < 1\). Thus, we obtain the fact that the primitive function \(\theta\Phi(t^\theta)t^{-1}\) is non-decreasing, which implies \(\Phi_\theta\) is a Young function. We assume \(\Phi(\lambda t) \leq \lambda^p \Phi(t)\) for any \(t > 0\) and any \(0 < \lambda < 1\) (resp. \(\lambda > 1\)). Then, we get
\[
\Phi_\theta(\lambda r) = \int_0^{(\lambda r)^\theta} \frac{\Phi(t)}{t} dt = \int_0^r \frac{\Phi(\lambda^\theta t)}{t^\theta} dt \leq \lambda^p \int_0^r \frac{\Phi(t^\theta)}{t} dt = \lambda^p \Phi_\theta(r),
\]
for any \(r > 0\) and any \(0 < \lambda < 1\) (resp. \(\lambda > 1\)). Which concludes (5.1). From \(\frac{\Phi(t)}{t} \leq \Phi'(t) \leq \frac{\Phi(2t)}{t}\) for \(t > 0\), it is easy to show that
\[
\Phi_1(r) \leq \Phi(r) \leq \Phi_1(2r),
\]
for \(r > 0\). Thus, we have \(\|\cdot\|_{L^\Phi} \sim \|\cdot\|_{L^{\Phi_1}}\) and using Proposition 2.4, we also have \(\|\cdot\|_{wL^\Phi} \sim \|\cdot\|_{wL^{\Phi_1}}\). □

Remark that \(\Phi_\theta(r) = \Phi_1(r^\theta)\).

Here we start the Proof of Theorem 1.7.

(1) If \(q = \infty\), by the pointwise estimate
\[
Mf_k(x) \leq M \left[ \sup_{j \in \mathbb{N}} |f_j| \right](x), \quad k \in \mathbb{N},
\]
then this is an easy consequence of the boundedness of \(M\) on \(wL^\Phi(\mathbb{R}^n)\) (Proposition 1.11).

(2) Let \(q < \infty\). From \(\Phi \in \Delta_2 \cap \nabla_2\), we have \(1 < p_-(\Phi) \leq p_+ (\Phi) < \infty\). Let \(\Psi = \Phi_\frac{q}{p_-}\) for \(\eta \in (1, p_-(\Phi))\). Then, \(\Psi\) is a Young function from Lemma 5.4. We fix a measurable non-negative function \(\varphi\) in \(L^{\Psi, 1}(\mathbb{R}^n)\) such that \(\|\varphi\|_{L^{\Psi, 1}} = 1\). Thus by duality and
\[
\left\| \left( \sum_{j=1}^\infty Mf_j(x)^q \right)^{\frac{1}{q}} \right\|_{wL^{\Psi}} \sim \left\| \left( \sum_{j=1}^\infty Mf_j(x)^q \right)^{\frac{1}{q}} \right\|_{wL^{\Psi_1}} \sim \left\| \left( \sum_{j=1}^\infty Mf_j(x)^q \right)^{\frac{1}{q}} \right\|_{wL^{\Psi_1}},
\]
it suffices to show that
\[
\int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty Mf_j(x)^q \right)^{\frac{1}{q}} \varphi(x) dx \lesssim \left\| \left( \sum_{j=1}^\infty |f_j|^q \right)^{\frac{1}{q}} \right\|_{wL^{\Psi_1}}.
\]
Now, choosing \(\theta\) so that
\[
0 < \theta < 1, \quad \left( \frac{p_+(\Phi)}{\eta} \right)'/\theta > 1.
\]
We note that $\tilde{\Psi}_\theta \in \nabla_2$. In fact, from Lemma 5.2 and Lemma 5.4, we obtain
\[
p_-(\tilde{\Psi}_\theta) \geq \theta \cdot p_-(\tilde{\Psi}) \geq \theta \cdot p'_+(\Phi_\frac{1}{\eta}) \geq \theta \cdot \left( \frac{p_+(\Phi)}{\eta} \right)' > 1.
\]
Consequently, we obtain
\[
\int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty |f_j(x)|^q \right)^{\frac{\eta}{q}} \varphi(x) \, dx \leq \int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty M f_j(x)^q \right)^{\frac{\eta}{q}} M \left[ \varphi^{\frac{1}{\eta}} \right] (x)^\theta \, dx
\]
\[
\lesssim \int_{\mathbb{R}^n} \left( \sum_{j=1}^\infty |f_j(x)|^q \right)^{\frac{\eta}{q}} \left( M \left[ \varphi^{\frac{1}{\eta}} \right] \right)^\theta \, dx
\]
\[
\lesssim \left\| \left( \sum_{j=1}^\infty |f_j(x)|^q \right)^{\frac{1}{q}} \right\|_{wL^\eta} \sim \left\| \left( \sum_{j=1}^\infty |f_j(x)|^q \right)^{\frac{1}{q}} \right\|_{wL^\eta},
\]
where in the second inequality we used [1] Theorem 3.1, since $(M[\varphi^{1/\theta}])^\theta \in A_\eta$ (see [3] Theorem 7.7).

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