On the norms of circulant and $r$-circulant matrices with the hyperharmonic Fibonacci numbers

Naim Tuglu and Can Kızılateş

Abstract

In this paper, we study norms of circulant and $r$-circulant matrices involving harmonic Fibonacci and hyperharmonic Fibonacci numbers. We obtain inequalities by using matrix norms.

MSC: Primary 15A60; secondary 15B05; 11B39

Keywords: hyperharmonic Fibonacci number; $r$-Circulant matrix; matrix norm

1 Introduction

The circulant and $r$-circulant matrices have a connection to signal processing, probability, numerical analysis, coding theory, and many other areas. An $n \times n$ matrix $C_r$ is called an $r$-circulant matrix defined as follows:

$$
C_r = \begin{pmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    r_0 & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
    r_0 & r_1 & c_0 & \cdots & c_{n-4} & c_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    r_{n-2} & r_{n-1} & r_{n-2} & \cdots & c & c_{n-1} \\
    c & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1}
\end{pmatrix}
$$

Since the matrix $C_r$ is determined by its row elements and $r$, we denote $C_r = \text{Circ}(c_0, c_1, c_2, \ldots, c_{n-1})$. In particular for $r = 1$

$$
C = \begin{pmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
    c & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{pmatrix}
$$

is called a circulant matrix and we denote it for brevity by $C = \text{Circ}(c_0, c_1, c_2, \ldots, c_{n-1})$. The eigenvalues of $C$ are
\[
\lambda_j = \sum_{i=0}^{n-1} c_i (w^i)^j,
\]

where \( w = e^{\frac{2\pi i}{n}} \) and \( i = \sqrt{-1} \).

Many authors have investigated the norms of circulant and \( r \)-circulant matrices. In [1], Solak studied the lower and upper bounds for the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries. In [2], Kocer et al. obtained norms of circulant and semicirculant matrices with Horadam numbers. In [3], Zhou et al. gave the spectral norms of circulant-type matrices involving binomial coefficients and harmonic numbers. In [4], Zhou calculated spectral norms for circulant matrices with binomial coefficients combined with Fibonacci and Lucas number entries. In [5], Shen and Cen have given upper and lower bounds for the spectral norms of \( r \)-circulant matrices with classical Fibonacci and Lucas number entries. In [6], Bahşi and Solak computed the spectral norms of circulant and \( r \)-circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers. In [7], Jiang and Zhou studied spectral norms of even-order \( r \)-circulant matrices.

Motivated by the above papers, we compute the spectral norms and Euclidean norms of circulant and \( r \)-circulant matrices with the harmonic and hyperharmonic Fibonacci entries. The scheme of this paper is as follows. In Section 2, we present some definitions, preliminaries, and lemmas related to our study. In Section 3, we calculate spectral norms of circulant matrix with harmonic Fibonacci entries. Moreover, we obtain the Euclidean norms of \( r \)-circulant matrices and give lower and upper bounds for the spectral norms of \( r \)-circulant matrices with harmonic and hyperharmonic Fibonacci entries.

## 2 Preliminaries

The Fibonacci numbers \( F_n \) are defined by the following recurrence relation for \( n \geq 1 \):

\[
F_{n+1} = F_n + F_{n-1},
\]

where \( F_0 = 0 \), \( F_1 = 1 \). In [8], the authors investigated the finite sum of the reciprocals of Fibonacci numbers,

\[
\mathbb{F}_n = \sum_{k=1}^{n} \frac{1}{F_k},
\]

which are called harmonic Fibonacci numbers. Then they gave a combinatoric identity related to harmonic Fibonacci numbers as follows:

\[
\sum_{k=0}^{n-1} F_{k-1}F_k = F_n \mathbb{F}_n - n.
\]

Moreover, in [8], they defined hyperharmonic Fibonacci numbers for \( n, r \geq 1 \)

\[
\mathbb{F}_n^{(r)} = \sum_{k=1}^{n} \frac{1}{F_k^{(r-1)}},
\]

where \( F_n^{(0)} = \frac{1}{F_n} \) and \( F_0 = 0 \). At this point, we give some definitions and lemmas related to our study.
**Definition 1** Let $A = (a_{ij})$ be any $m \times n$ matrix. The Euclidean norm of matrix $A$ is

$$\|A\|_E = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.$$ 

**Definition 2** Let $A = (a_{ij})$ be any $m \times n$ matrix. The spectral norm of matrix $A$ is

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ is an eigenvalue of $A^H A$ and $A^H$ is the conjugate transpose of matrix $A$.

Then the following inequalities hold for the Euclidean norm and the spectral norm:

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E,$$  \hspace{1cm} (3)

$$\|A\|_2 \leq \|A\|_E \leq \sqrt{n}\|A\|_2.$$  \hspace{1cm} (4)

**Lemma 1** [9] Let $A$ and $B$ be two $m \times n$ matrices. Then we have

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2,$$

where $A \circ B$ is the Hadamard product of $A$ and $B$.

**Lemma 2** [9] Let $A$ and $B$ be two $n \times m$ matrices. We have

$$\|A \circ B\|_2 \leq r_1(A)c_1(B),$$

where

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2},$$

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}.$$ 

**Definition 3** [10] The difference operator of $f(x)$ is defined as

$$\Delta f(x) = f(x + 1) - f(x).$$ 

**Definition 4** [10] A function $f(x)$ with the property that $\Delta f(x) = g(x)$ is called the antidifference operator of $g(x)$.

**Lemma 3** [10] If $\Delta f(x) = g(x)$, then

$$\sum_{x=a}^{b} g(x) \delta_x = \sum_{x=a}^{b-1} g(x) = f(b) - f(a).$$
Lemma 4 [10] We have
\[ b \sum_a u(x) \Delta v(x) \delta_x = u(x) v(x) |_{b+1} - \sum_a v(x+1) \Delta u(x) \delta_x. \] (5)

Lemma 5 [10] For \( m \neq -1 \) we have
\[ \sum_x x^m \delta_x = \frac{x^{m+1}}{m+1}, \]
where \( x^m = x(x-1)(x-2) \cdots (x-m+1) \).

3 Main results
Theorem 1 [8] Let \( C_1 = \text{Circ}(F_0, F_1, F_2, \ldots, F_{n-1}) \) be an \( n \times n \) circulant matrix. The spectral norm of \( C_1 \) is
\[ \| C_1 \|_2 = n \mathbb{F}_n - \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}}. \]

Theorem 2 [8] Let \( C^{(k)} = \text{Circ}(f^{(0)}, f^{(1)}, f^{(2)}, \ldots, f^{(k)}) \) be an \( n \times n \) circulant matrix. The spectral norm of \( C^{(k)} \) is
\[ \| C^{(k)} \|_2 = \mathbb{F}_n^{(k+1)} \]

Theorem 3 Let
\[ C = \text{Circ}(F_{-1}F_0, F_0F_1, \ldots, F_{n-2}F_{n-1}) \] (6)
be an \( n \times n \) circulant matrix. Then the spectral norm of the matrix \( C \) is
\[ \| C \|_2 = F_n \mathbb{F}_n - n. \]

Proof Since \( C \) is a circulant matrix, from (1), for all \( t = 0, 1, \ldots, s-1 \),
\[ \lambda_t(C) = \sum_{i=0}^{s-1} F_{i-1} F_i (w^t)^i. \]

Then, for \( t = 0 \),
\[ \lambda_0(C) = \sum_{i=0}^{s-1} F_{i-1} F_i \]
and from (2), \( \lambda_0(C) = F_n \mathbb{F}_n - n. \) Hence, for \( 1 \leq m \leq n - 1 \), we have
\[ |\lambda_m| = \left| \sum_{i=0}^{s-1} F_{i-1} F_i (w^t)^i \right| \leq \left| \sum_{i=0}^{s-1} F_{i-1} F_i \right| \leq \sum_{i=0}^{s-1} F_{i-1} F_i. \] (8)
Since $C$ is a normal matrix, we have

$$\|C\|_2 = \max_{0 \leq m \leq n-1} |\lambda_m|.$$  \hspace{1cm} (9)

From (7), (8), (9), and (2), we have

$$\|C\|_2 = F_n F_n - n. \quad \square$$

**Corollary 1** We have

$$\sqrt{n - \sum_{k=0}^{n-1} F_k^2 F_n - n} \leq \sqrt{n - \sum_{k=0}^{n-1} F_k^2 F_n^2}.$$

**Proof** The proof is trivial from Definition 1 and the relation between the Euclidean norm and the spectral norm in (3). \quad \square

**Theorem 4** Let

$$C_r^{(k)} = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \ldots, F_{n-1}^{(k)})$$  \hspace{1cm} (10)

be an $n \times n$ $r$-circulant matrix. The Euclidean norm of $C_r^{(k)}$ is

$$\|C_r^{(k)}\|_E = \left[ \frac{n}{2} \left( n + 1 + (n - 1) |r|^2 \right) \left( \sum_{s=0}^{n-1} s |r|^2 \right)^2 \right]^{\frac{1}{2}}.$$

**Proof** From the definition of the Euclidean norm we have

$$\|C_r^{(k)}\|_E = \left[ \sum_{s=0}^{n-1} \left( n - s \right) \left( F_s^{(k)} \right)^2 + \sum_{s=0}^{n-1} s |r|^2 \left( F_s^{(k)} \right)^2 \right]^{\frac{1}{2}}.$$

Now we will use the property of the difference operator in Lemma 4. Let $u(s) = (\sum_{k=0}^{n-1} F_k^{(k)})^2$ and $\Delta v(s) = n + s(|r|^2 - 1)$. Then using the definition of the hyperharmonic Fibonacci numbers we obtain $\Delta u(s) = \sum_{s=0}^{n-1} (n - s) \left[ \sum_{s=0}^{n-1} s |r|^2 \right]$ and $v(s) = ns + \frac{n^2}{2}(|r|^2 - 1)$. By using (5), we have

$$\|C_r^{(k)}\|_E = \left[ \frac{n}{2} \left( n + 1 + (n - 1) |r|^2 \right) \left( \sum_{s=0}^{n-1} s |r|^2 \right)^2 \right]^{\frac{1}{2}}.$$

\quad \square
Corollary 2 Let $C_r = \text{Circ}(F_0, F_1, \ldots, F_{n-1})$ be an $n \times n$ $r$-circulant matrix. The Euclidean norm of $C_r$ is
\[
\|C_r\|_E = \left[ \left( n^2 + \frac{n^2}{2} |r|^2 - 1 \right) F_n^2 \right]^{\frac{1}{2}} - \sum_{s=0}^{n-1} \left( n(s+1) + \frac{(s+1)^2}{2} |r|^2 - 1 \right) \left( 2F_{s+1} + \frac{1}{F_{s+1}} \right) \frac{1}{F_{s+1}} \right]^{\frac{1}{2}}.
\]

Proof It is clear that the proof can be completed if we take $k = 1$ in Theorem 4. \hfill \square

Corollary 3 [8] Let $C_1 = \text{Circ}(F_0, F_1, \ldots, F_{n-1})$ be an $n \times n$ matrix. The Euclidean norm is
\[
\|C_1\|_E = \left[ n^2 F_n^2 - n \sum_{k=0}^{n-1} \frac{k+1}{F_{k+1}} \left( 2F_k + \frac{1}{F_{k+1}} \right) \right]^{\frac{1}{2}}.
\]

Proof It is easily seen that the proof can be completed if we take $k = r = 1$ in Theorem 4. \hfill \square

Now we give upper and lower bounds for the spectral norms of $r$-circulant matrices.

Theorem 5 Let $C^{(k)}_r = \text{Circ}(F^{(k)}_0, F^{(k)}_1, \ldots, F^{(k)}_{n-1})$ be an $n \times n$ $r$-circulant matrix.

(i) If $|r| \geq 1$, then
\[
\frac{1}{\sqrt{n}} F^{(k+1)}_{n-1} \leq \|C^{(k)}_r\|_2 \leq |r| \sqrt{n - n^{(k+1)}_{n-1}}.
\]

(ii) If $|r| < 1$, then
\[
\frac{|r|}{\sqrt{n}} F^{(k+1)}_{n-1} \leq \|C^{(k)}_r\|_2 \leq \sqrt{n - n^{(k+1)}_{n-1}}.
\]

Proof Since we have the matrix
\[
C^{(k)}_r = \begin{pmatrix}
F^{(k)}_0 & F^{(k)}_1 & F^{(k)}_2 & \ldots & F^{(k)}_{n-2} & F^{(k)}_{n-1} \\
F^{(k)}_0 & F^{(k)}_1 & F^{(k)}_2 & \ldots & F^{(k)}_{n-3} & F^{(k)}_{n-2} \\
& \vdots & \vdots & \ddots & \vdots & \vdots \\
F^{(k)}_0 & F^{(k)}_1 & F^{(k)}_2 & \ldots & F^{(k)}_{n-3} & F^{(k)}_{n-2} \\
F^{(k)}_0 & F^{(k)}_1 & F^{(k)}_2 & \ldots & F^{(k)}_{n-3} & F^{(k)}_{n-2} \\
F^{(k)}_0 & F^{(k)}_1 & F^{(k)}_2 & \ldots & F^{(k)}_{n-3} & F^{(k)}_{n-2}
\end{pmatrix},
\]
we have
\[
\|C^{(k)}_r\|_2 = \sqrt{\sum_{s=0}^{n-1} (n-s)(F^{(k)}_s)^2 + \sum_{s=0}^{n-1} s|r|^2 (F^{(k)}_s)^2}.
\]

(i) In [8], for the sum of the squares of hyperharmonic Fibonacci numbers, we have
\[
\frac{1}{\sqrt{n}} F^{(r+1)}_{n-1} \leq \sqrt{\sum_{k=0}^{n-1} (F^{(r)}_k)^2} \leq F^{(r+1)}_{n-1}.
\]
Since $|r| \geq 1$ and by (11), we have

$$\|C^k_r\|_2 \geq \sqrt{\sum_{s=0}^{n-1} (n-s)(F^k_s)^2} \geq \sqrt{n \sum_{s=0}^{n-1} (F^k_s)^2} \geq \|X^{(k+1)}\|_{n-1}.$$  

From (3)

$$\frac{1}{\sqrt{n}} X^{(k+1)} \leq \|C^k_r\|_2.$$  

On the other hand, let the matrices $A$ and $B$ be defined by

$$A = \begin{pmatrix} F^k_0 & 1 & 1 & \cdots & 1 & 1 \\ r & F^k_0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & r & \cdots & F^k_0 & 1 \\ r & r & r & \cdots & r & F^k_0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} F^k_0 & F^k_1 & F^k_2 & \cdots & F^k_{n-2} & F^k_{n-1} \\ F^k_1 & F^k_0 & F^k_2 & \cdots & F^k_{n-3} & F^k_{n-2} \\ F^k_2 & F^k_3 & F^k_4 & \cdots & F^k_{n-4} & F^k_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F^k_{n-1} & F^k_{n-2} & F^k_{n-3} & \cdots & F^k_0 & F^k_{n-2} \\ F^k_{n-2} & F^k_{n-3} & F^k_{n-4} & \cdots & F^k_1 & F^k_{n-3} \end{pmatrix}.$$  

That is, $C^k_r = A \circ B$. Then we obtain

$$r_1(A) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{(n-1)|r|^2}$$

and

$$c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} (F^k_i)^2}.$$  

Hence, from (11) and Lemma 1, we have

$$\|C^k_r\|_2 \leq |r| \sqrt{n-1} X^{(k+1)}_{n-1}.$$  

Thus, we have

$$\frac{1}{\sqrt{n}} X^{(k+1)}_{n-1} \leq \|C^k_r\|_2 \leq |r| \sqrt{n-1} X^{(k+1)}_{n-1}.$$
(ii) From $|r| < 1$ and from (11), we have

$$
\| C^{(k)}_r \|_2 = \sqrt{\sum_{s=0}^{n-1} \left( (n-s)(F_0^{(k)})^2 + \sum_{s=0}^{n-1} s|r|^2(F_s^{(k)})^2 \right)}
$$

$$
\geq \sqrt{\sum_{s=0}^{n-1} (n-s)|r|^2(F_s^{(k)})^2 + \sum_{s=0}^{n-1} s|r|^2(F_s^{(k)})^2}
$$

$$
= |r| \sqrt{n \sum_{s=0}^{n-1} (F_s^{(k)})^2}
$$

$$
\geq |r|^{\frac{(k+1)}{n-1}}.
$$

From (3),

$$
\frac{|r|}{\sqrt{n}} \frac{(k+1)}{n-1} \leq \| C^{(k)}_r \|_2.
$$

On the other hand, let the matrices $A$ and $B$ be defined by

$$
A = \begin{pmatrix}
F_0^{(k)} & 1 & 1 & \cdots & 1 & 1 \\
1 & F_0^{(k)} & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & r & r & r \\
r & r & \cdots & r & r & \cdots & r \\
r & r & \cdots & r & r & \cdots & r
\end{pmatrix}
$$

and

$$
B = \begin{pmatrix}
F_0^{(k)} & F_1^{(k)} & F_2^{(k)} & \cdots & F_{n-2}^{(k)} & F_{n-1}^{(k)} \\
F_1^{(k)} & F_0^{(k)} & F_1^{(k)} & \cdots & F_{n-3}^{(k)} & F_{n-2}^{(k)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
F_{n-2}^{(k)} & F_{n-3}^{(k)} & F_{n-4}^{(k)} & \cdots & F_0^{(k)} & F_1^{(k)} \\
F_{n-1}^{(k)} & F_{n-2}^{(k)} & F_{n-3}^{(k)} & \cdots & F_0^{(k)} & F_1^{(k)} \\
F_{n-2}^{(k)} & F_{n-3}^{(k)} & F_{n-4}^{(k)} & \cdots & F_0^{(k)} & F_1^{(k)}
\end{pmatrix}.
$$

Thus $C^{(k)}_r = A \circ B$. Then we obtain

$$
r_1(A) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{(F_0^{(k)})^2} + n - 1 = \sqrt{n - 1}
$$

and

$$
c_1(B) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{s=0}^{n-1} (F_s^{(k)})^2}.
$$

Therefore, from (11) and Lemma 1, we have

$$
\| C^{(k)}_r \|_2 \leq \sqrt{n - 1 \varepsilon^{(k+1)}_{n-1}}.
$$
Thus, we have
\[
\frac{|r|}{\sqrt{n}} F_{n-1}^{(k+1)} \leq \|C_r^{(k)}\|_2 \leq \sqrt{n - 1} F_{n-1}^{(k+1)}.
\]

Corollary 4 Let \( C_r = \text{Circ}(F_0, F_1, \ldots, F_{n-1}) \) be an \( n \times n \) \( r \)-circulant matrix.
(i) If \(|r| \geq 1\), then
\[
\frac{1}{\sqrt{n}} F_{n-1}^{(2)} \leq \|C_r\|_2 \leq |r| \sqrt{n - 1} F_{n-1}^{(2)}.
\]
(ii) If \(|r| < 1\), then
\[
\frac{|r|}{\sqrt{n}} F_{n-1}^{(2)} \leq \|C_r\|_2 \leq \sqrt{n - 1} F_{n-1}^{(2)}.
\]

Proof It is easily seen that the proof can be completed if we take \( k = 1 \) in Theorem 5.

4 Numerical examples
In this section, we present some numerical examples by using Maple 11.

Example 1 Let \( C = \text{Circ}(F_{-1}F_0, F_1F_0, \ldots, F_{n-2}F_{n-1}) \) be as in (6). We obtain the spectral norms of some \( n \times n \) \( C \) matrices, with the aid of Theorem 3 (see Table 1).

Example 2 Let \( C_r^{(k)} = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \ldots, F_4^{(k)}) \) be \( 5 \times 5 \) \( r \)-circulant matrix as in (10). We obtain Euclidean norms of \( C_r^{(k)} \) for some values of \( r \) and \( k \), with the aid of Theorem 4 (see Table 2).

Example 3 Let \( C_r^{(k)} = \text{Circ}(F_0^{(k)}, F_1^{(k)}, \ldots, F_4^{(k)}) \) be \( 5 \times 5 \) \( r \)-circulant matrix as in (10). We obtain some lower and upper bounds for the spectral norms of \( C_r^{(k)} \) for some values of \( r \) and \( k \), with the aid of Theorem 5 (see Tables 3 and 4).

Table 1 Spectral norms of \( C \)

| \( n \) | \( \|C\|_2 \) |
|---|---|
| 5 | 0.1016666667 x 10^2 |
| 10 | 0.1731757972 x 10^3 |
| 50 | 0.422842484 x 10^{11} |
| 100 | 0.1190154990 x 10^{22} |
| 500 | 0.4684460937 x 10^{105} |
| 1000 | 0.1460426641 x 10^{210} |

Table 2 Euclidean norms of \( C_r^{(k)} \) for \( n = 5 \)

| \( k/r \) | \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) |
|---|---|---|---|---|
| \( r = -2 \) | 6.3035 | 6.3035 | 6.3035 | 6.3035 |
| \( r = -0.5 \) | 6.3035 | 6.3035 | 6.3035 | 6.3035 |
| \( r = 0.1 \) | 6.3035 | 6.3035 | 6.3035 | 6.3035 |
| \( r = 0.9 \) | 6.3035 | 6.3035 | 6.3035 | 6.3035 |
| \( r = 1 \) | 6.3035 | 6.3035 | 6.3035 | 6.3035 |
| \( r = 1.1 \) | 6.3035 | 6.3035 | 6.3035 | 6.3035 |
| \( r = 10 \) | 6.3035 | 6.3035 | 6.3035 | 6.3035 |
### Table 3 Some lower and upper bounds for the spectral norms of $C_r^{(k)}$ for $n = 5$ and $|r| \geq 1$

| $|r| \geq 1$ | $\frac{1}{\sqrt{n}} \sqrt{\frac{k+1}{n-1}}$ | $\|C_r^{(k)}\|_2$ | $|r| \sqrt{n-1} \sqrt{\frac{k+1}{n-1}}$ |
|---|---|---|---|
| $k = 1$ | $r = -2$ | 3.726779962 | 13.07393997 |
| | $r = 1$ | 3.726779962 | 8.333333332 |
| | $r = 1.1$ | 3.726779962 | 6.187213310 |
| $k = 2$ | $r = -2$ | 7.975309119 | 29.94984421 |
| | $r = 1$ | 7.975309119 | 17.83333333 |
| | $r = 1.1$ | 7.975309119 | 19.01179206 |
| $k = 3$ | $r = -2$ | 14.45990625 | 56.50736302 |
| | $r = 1$ | 14.45990625 | 32.33333334 |
| | $r = 1.1$ | 14.45990625 | 34.60097132 |
| $k = 4$ | $r = -2$ | 23.62778496 | 94.70523788 |
| | $r = 1$ | 23.62778496 | 52.83333332 |
| | $r = 1.1$ | 23.62778496 | 56.68238740 |

### Table 4 Some lower and upper bounds for the spectral norms of $C_r^{(k)}$ for $n = 5$ and $|r| < 1$

| $|r| < 1$ | $\frac{1}{\sqrt{n}} \sqrt{\frac{k+1}{n-1}}$ | $\|C_r^{(k)}\|_2$ | $\sqrt{n-1} \sqrt{\frac{k+1}{n-1}}$ |
|---|---|---|---|
| $k = 1$ | $r = -0.5$ | 1.863389981 | 6.051069352 |
| | $r = 0.1$ | 0.372677996 | 5.912862964 |
| | $r = 0.9$ | 3.354101966 | 7.87454252 |
| $k = 2$ | $r = -0.5$ | 3.987654558 | 13.14607243 |
| | $r = 0.1$ | 0.797530912 | 12.64953828 |
| | $r = 0.9$ | 7.17778206 | 16.74610810 |
| $k = 3$ | $r = -0.5$ | 7.229953125 | 24.39979647 |
| | $r = 0.1$ | 1.445990625 | 23.46328510 |
| | $r = 0.9$ | 13.01391563 | 30.2686870 |
| $k = 4$ | $r = -0.5$ | 11.81389248 | 40.74250279 |
| | $r = 0.1$ | 2.362778496 | 39.32798158 |
| | $r = 0.9$ | 21.26500646 | 49.37728355 |

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
Each of the authors, NT, and CK, contributed to each part of this work equally and read and approved the final version of the manuscript.

Author details
1 Department of Mathematics, Gazi University, Teknikokullar, Ankara, 06500, Turkey. 2 Department of Mathematics, Bülent Ecevit University, Zonguldak, 67100, Turkey.

Acknowledgements
The authors are grateful to two anonymous referees and the associate editor for their careful reading, helpful comments, and constructive suggestions, which improved the presentation of the results.

Received: 12 June 2015 Accepted: 5 August 2015 Published online: 20 August 2015

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