SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN TYPE \( q \)- DIFFERENCE OPERATOR

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ABSTRACT. In this paper, we introduce and investigate a new subclass of the function class \( \Sigma \) of bi-univalent functions defined in the open unit disk, which are associated with the Sălăgean type \( q \)- difference operator and satisfy some subordination conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients \(|a_2|\) and \(|a_3|\) for functions in the new subclass introduced here. Several (known or new) consequences of the results are also pointed out. Further we obtain Fekete-Szegő inequality for the new function class.

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1. INTRODUCTION

Let \( A \) denote the class of analytic functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

normalized by the conditions \( f(0) = 0 = f'(0) - 1 \) defined in the open unit disk

\[\Delta = \{ z \in \mathbb{C} : |z| < 1 \}.\]

Let \( S \) be the subclass of \( A \) consisting of functions of the form (1.1) which are also univalent in \( \Delta \). Let \( S^\ast(\alpha) \) and \( K(\alpha) \) denote the subclasses of \( S \), consisting of starlike and convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \), respectively. An analytic function \( f \) is subordinate to an analytic function \( g \), written \( f(z) \prec g(z) \), provided there is an analytic function \( w \) defined on \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) satisfying \( f(z) = g(w(z)) \). Ma and Minda [11] unified various subclasses of starlike and convex functions for which either of the quantities

\[
\frac{z f'(z)}{f(z)} \quad \text{or} \quad 1 + \frac{z f''(z)}{f'(z)}
\]

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function \( \varphi \) with positive real part in the unit disk \( \Delta \), \( \varphi(0) = 1 \), \( \varphi'(0) > 0 \) and \( \varphi \) maps \( \Delta \) onto a region starlike with respect to 1 and symmetric with respect to the...
real axis. In the sequel, it is assumed that such a function has a series expansion of the form
\[ \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \]  
(1.2)

In particular for the class of strongly starlike functions of order \( \alpha (0 < \alpha \leq 1) \), the function \( \phi \) is given by
\[ \phi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1), \]
(1.3)

which gives \( B_1 = 2\alpha \) and \( B_2 = 2\alpha^2 \) and on the other hand, for the class of starlike functions of order \( \beta (0 \leq \beta < 1) \),
\[ \phi(z) = \frac{1 + (1 - 2\beta) z}{1 - z} = 1 + 2(1 - \beta) z + 2(1 - \beta)^2 z^2 + \cdots \quad (0 \leq \beta < 1), \]
(1.4)
we have \( B_1 = B_2 = 2(1 - \beta) \).

The Koebe one quarter theorem \[5\] ensures that the image of \( \Delta \) under every univalent function \( f \in \mathcal{A} \) contains a disk of radius \( \frac{1}{4} \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z \), \( (z \in \Delta) \) and \( f(f^{-1}(w)) = w \) \( (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}) \). A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \Delta \) if both \( f \) and \( f^{-1} \) are univalent in \( \Delta \). Let \( \Sigma \) denote the class of bi-univalent functions defined in the unit disk \( \Delta \). Since \( f \in \Sigma \) has the Maclaurian series given by \([1,1]\), a computation shows that its inverse \( g = f^{-1} \) has the expansion
\[ g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 + \cdots. \]
(1.5)

Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see \([2,3,6,10,8,16,19,21]\)).

Quantum calculus is ordinary classical calculus without the notion of limits. It defines \( q\)-calculus. Here \( h \) ostensibly stands for Planck's constant, while \( q \) stands for quantum. Recently, the area of \( q\)-calculus has attracted the serious attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of \( q\)-calculus was initiated by Jackson\([9]\). He was the first to develop \( q\)-integral and \( q\)-derivative in a systematic way. Later, geometrical interpretation of \( q\)-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and \( q\)-analysis. A comprehensive study on applications of \( q\)-calculus in operator theory may be found in \([11]\). For the convenience, we provide some basic definitions and concept details of \( q\)-calculus which are used in this paper.

For \( 0 < q < 1 \) the Jackson's \( q\)-derivative of a function \( f \in \mathcal{A} \) is, by definition, given as follows \([9]\)
\[ D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1 - q)z} & \text{for} \quad z \neq 0, \\ f'(0) & \text{for} \quad z = 0, \end{cases} \]
(1.6)
and \( D_q^2 f(z) = D_q(D_q f(z)) \). From \([1,6]\), we have
\[ D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \]
(1.7)
where \( [n]_q = \frac{1 - q^n}{1 - q} \).
(1.8)
Sivasubramanian defined and discussed the Salagean q-differential operator \( z, w \) where \( n \) is sometimes called the basic number \( n \). If \( q \to 1^- \), \( [n] \to n \). For a function \( h(z) = z^n \), we obtain \( D_q h(z) = D_q z^n = \frac{1}{1-q} z^{n-1} = [n] z^{n-1} \), and \( \lim_{q \to 1^-} D_q h(z) = \lim_{q \to 1^-} ([n] z^{n-1}) = n z^{n-1} = h'(z) \), where \( h' \) is the ordinary derivative. Recently for \( f \in \mathcal{A} \), Govindaraj and Sivasubramanian [15] defined and discussed the Salagean q-differential operator as given below:

\[
D_q^0 f(z) = f(z) \\
D_q^1 f(z) = z D_q f(z) \\
D_q^k f(z) = z D_q^k (D_q^{k-1} f(z)) \\
D_q^k f(z) = z + \sum_{n=2}^{\infty} [n]_q^k a_n z^n \quad (k \in \mathbb{N}_0, z \in \Delta) \tag{1.9}
\]

We note that \( \lim_{q \to 1^-} \)

\[
D_q^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (k \in \mathbb{N}_0, z \in \Delta) \tag{1.10}
\]

the familiar Salagean derivative[15].

In this paper, making use of the Salagean q-differential operator, for functions \( g \) of the form (1.5) we define

\[
D_q^k g(w) = w - a_2 [2]_q^k w^2 + (2 a_2^2 - a_3) [3]_q^k w^3 + \cdots \tag{1.11}
\]

and introduce two new subclass of bi-univalent functions to obtain the estimates on the coefficients \( |a_2| \) and \( |a_3| \) by Ma-Minda subordination. Further by using the initial coefficient values of \( a_2 \) and \( a_3 \) we also obtain Fekete-Szegő inequalities.

2. Bi-Univalent Function Class \( \mathcal{M} \Sigma^k_q (\lambda, \phi) \)

In this section, due to Vijaya et al [18], we introduce a subclass \( \mathcal{M} \Sigma^k_q (\lambda, \phi) \) of \( \Sigma \) and find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for the functions in this new subclass, by subordination. Throughout our study, unless otherwise stated, we let

\[ 0 \leq \lambda \leq 1; \quad 0 < q < 1; \quad k \in \mathbb{N}_0 \]

**Definition 2.1.** For \( 0 \leq \lambda \leq 1 \), a function \( f \in \Sigma \) of the form (1.1) is said to be in the class \( \mathcal{M} \Sigma^k_q (\lambda, \phi) \) if the following subordination hold:

\[
(1-\lambda) \frac{D_q^{k+1} f(z)}{D_q^k f(z)} + \lambda \frac{D_q^{k+2} f(z)}{D_q^{k+1} f(z)} < \phi(z) \tag{2.1}
\]

and

\[
(1-\lambda) \frac{D_q^{k+1} g(w)}{D_q^k g(w)} + \lambda \frac{D_q^{k+2} g(w)}{D_q^{k+1} g(w)} < \phi(w), \tag{2.2}
\]

where \( z, w \in \Delta \) and \( g \) is given by (1.5).

**Remark 2.2.** Suppose \( f \in \Sigma \). If \( \lambda = 0 \), then \( \mathcal{M} \Sigma^k_q (\lambda, \phi) \equiv \mathcal{S} \Sigma^k_q (\phi) \); thus \( f \in \mathcal{S} \Sigma^k_q (\phi) \) if the following subordination holds:

\[
\frac{D_q^{k+1} f(z)}{D_q^k f(z)} < \phi(z) \quad \text{and} \quad \frac{D_q^{k+1} g(w)}{D_q^k g(w)} < \phi(w),
\]

where \( z, w \in \Delta \) and \( g \) is given by (1.5).
Remark 2.3. Suppose \( f \in \Sigma \). If \( \lambda = 1 \), then \( \mathcal{M} \Sigma^k_q(\lambda, \phi) \equiv \mathcal{K} \Sigma^k_q(\phi) \) : thus \( f \in \mathcal{K} \Sigma^k_q(\phi) \) if the following subordination holds:

\[
\frac{D_q^{k+2}f(z)}{D_q^{k+1}f(z)} \prec \phi(z) \quad \text{and} \quad \frac{D_q^{k+2}g(w)}{D_q^{k+1}g(w)} \prec \psi(w),
\]

where \( z, w \in \Delta \) and \( g \) is given by (1.5).

Remark 2.4. For \( 0 \leq \lambda \leq 1 \) and \( k = 0 \) a function \( f \in \Sigma \) of the form (1.1) is said to be in the class \( \mathcal{M} \Sigma^k_q(\lambda, \phi) \) if the following subordination hold:

\begin{align*}
(1 - \lambda) \frac{z D_q f(z)}{f(z)} &+ \lambda \frac{D_q(z D_q f(z))}{D_q f(z)} \prec \phi(z) \quad \text{(2.3)} \\
(1 - \lambda) \frac{z D_q g(w)}{g(w)} &+ \lambda \frac{D_q(w D_q g(w))}{D_q g(w)} \prec \phi(w), \quad \text{(2.4)}
\end{align*}

where \( z, w \in \Delta \) and \( g \) is given by (1.5).

It is of interest to note that \( \mathcal{M} \Sigma^0_q(0, \phi) = \mathcal{S} \Sigma^1_q(\phi), \mathcal{M} \Sigma^0_q(1, \phi) = \mathcal{K} \Sigma_q(\phi) \) new subclasses of \( \Sigma \) associated with \( q \)— difference operator not yet discussed sofar.

In order to prove our main results, we require the following Lemma:

Lemma 2.5. If a function \( p \in \mathcal{P} \) is given by

\[ p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \Delta), \]

then

\[ |p_i| \leq 2 \quad (i \in \mathbb{N}), \]

where \( \mathcal{P} \) is the family of all functions \( p \), analytic in \( \Delta \), for which

\[ p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \Delta). \]

Theorem 2.6. Let \( f \) given by (1.1) be in the class \( \mathcal{M} \Sigma^k_q(\lambda, \phi) \). Then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|2(1 + 2\lambda)[3]^k_q - (1 + 3\lambda)[2]^{2k}_q|}} \quad \text{(2.5)}
\]

and

\[
|a_3| \leq \frac{B_1}{2(1 + 2\lambda)[3]^k_q} + \left( \frac{B_1}{(1 + \lambda)[2]^{k}_q} \right)^2 \quad \text{(2.6)}
\]

where \( 0 \leq \lambda \leq 1 \).

Proof. Let \( f \in \mathcal{M} \Sigma^k_q(\lambda, \phi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : \Delta \rightarrow \Delta \), with \( u(0) = 0 = v(0) \), satisfying

\begin{align*}
(1 - \lambda) \frac{D_q^{k+1} f(z)}{D_q^{k+1} f(z)} + \lambda \frac{D_q^{k+2} f(z)}{D_q^{k+1} f(z)} &= \phi(u(z)) \quad \text{(2.7)} \\
(1 - \lambda) \frac{D_q^{k+1} g(w)}{D_q^{k+1} g(w)} + \lambda \frac{D_q^{k+2} g(w)}{D_q^{k+1} g(w)} &= \phi(v(w)). \quad \text{(2.8)}
\end{align*}
Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + q_1 z + q_2 z^2 + \cdots$$

or, equivalently,

$$u(z) := \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \cdots \right]$$

and

$$v(z) := \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[ q_1 z + \left( q_2 - \frac{q_1^2}{2} \right) z^2 + \cdots \right].$$

Then $p(z)$ and $q(z)$ are analytic in $\Delta$ with $p(0) = 1 = q(0)$. Since $u, v : \Delta \to \triangle$, the functions $p(z)$ and $q(z)$ have a positive real part in $\Delta$, $|p_1| \leq 2$ and $|q_1| \leq 2$.

Using (2.9) and (2.10) in (2.7) and (2.8) respectively, we have

$$(1 - \lambda) \frac{D_q^{k+1} f(z)}{D_q^k f(z)} + \lambda \frac{D_q^{k+2} f(z)}{D_q^{k+1} f(z)} = \phi \left( \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \cdots \right] \right)$$

and

$$(1 - \lambda) \frac{D_q^{k+1} g(w)}{D_q^k g(w)} + \lambda \frac{D_q^{k+2} g(w)}{D_q^{k+1} g(w)} = \phi \left( \frac{1}{2} \left[ q_1 w + \left( q_2 - \frac{q_1^2}{2} \right) w^2 + \cdots \right] \right).$$

In light of (1.1) - (1.11), and from (2.11) and (2.12), we have

$$1 + (1 + \lambda)[2]_q^k a_2 z + [2(1 + 2\lambda)[3]_q^k a_3 - (1 + 3\lambda)[2]_q^{2k} a_2^2] z^2 + \cdots$$

$$= 1 + \frac{1}{2} B_1 p_1 z + \left[ \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2 p_1^2 \right] z^2 + \cdots$$

and

$$1 - (1 + \lambda)[2]_q^k a_2 w + \{(8\lambda + 4)[3]_q^k - (3\lambda + 1)[2]_q^{2k} a_2^2 - 2(1 + 2\lambda)[3]_q^k a_3\} w^2 + \cdots$$

$$= 1 + \frac{1}{2} B_1 q_1 w + \left[ \frac{1}{2} B_1 (q_2 - \frac{q_1^2}{2}) + \frac{1}{4} B_2 q_1^2 \right] w^2 + \cdots$$

which yields the following relations:

$$(1 + \lambda)[2]_q^k a_2 = \frac{1}{2} B_1 p_1$$

$$-(1 + 3\lambda)[2]_q^{2k} a_2^2 + 2(1 + 2\lambda)[3]_q^k a_3 = \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2})$$

$$+ \frac{1}{4} B_2 p_1^2$$

$$(1 + \lambda)[2]_q^k a_2 = \frac{1}{2} B_1 q_1$$
and

\[(4(1 + 2 \lambda)[3]_q^k - (1 + 3 \lambda)[2]_q^{2k})a_3^2 - 2(1 + 2 \lambda)[3]_q^k a_3 = \frac{1}{2}B_1(q_2 - q_1^2) + \frac{1}{4}B_2q_1^2.\]  

(2.16)

From (2.13) and (2.15) it follows that

\[p_1 = -q_1\]  

(2.17)

and

\[8(1 + \lambda)^2[2]_q^{2k}a_2^2 = B_1^2(p_1^2 + q_1^2).\]  

(2.18)

From (2.14), (2.16) and (2.18), we obtain

\[a_2^2 = \frac{B_1^2(p_2 + q_2)}{4\{2(1 + 2 \lambda)[3]_q^k - (1 + 3 \lambda)[2]_q^{2k}\}B_1^2 + (1 + \lambda)^2(B_1 - B_2)[2]_q^{2k}}.\]  

Applying Lemma 2.5 to the coefficients \(p_2\) and \(q_2\), we have

\[|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{\{2(1 + 2 \lambda)[3]_q^k - (1 + 3 \lambda)[2]_q^{2k}\}B_1^2 + (1 + \lambda)^2(B_1 - B_2)[2]_q^{2k}}}.\]  

(2.20)

By subtracting (2.16) from (2.14) and using (2.17) and (2.18), we get

\[a_3 = \frac{B_1^2(p_1^2 + q_1^2)}{8(1 + \lambda)^2[2]_q^{2k}} + \frac{B_1(p_2 - q_2)}{8(1 + \lambda)[2]_q^{2k}}.\]  

(2.21)

Applying Lemma 2.5 once again to the coefficients \(p_1, p_2, q_1\) and \(q_2\), we get

\[|a_3| \leq \frac{B_1}{2(1 + 2 \lambda)[3]_q^k} + \left(\frac{B_1}{(1 + \lambda)[2]_q^k}\right)^2.\]  

(2.22)

\[\square\]

**Remark 2.7.** If \(f \in \mathcal{M}\Sigma_q^k(\lambda, \left(\frac{1 + \beta}{1 - z}\right)\alpha)\) then, we have the following estimates for the coefficients \(|a_2|\) and \(|a_3|\) :

\[|a_2| \leq \frac{2\alpha}{\sqrt{\{2(1 + 2 \lambda)[3]_q^k - (1 + 3 \lambda)[2]_q^{2k}\} \alpha + (1 - \alpha)(1 + \lambda)^2[2]_q^{2k}}}.\]

and

\[|a_3| \leq \frac{4\alpha^2}{(1 + \lambda)^2[2]_q^{2k}} + \left(\frac{\alpha}{(1 + \lambda)[2]_q^{2k}}\right)^2.\]

For functions \(f \in \mathcal{M}\Sigma_q^k(\lambda, \left(\frac{1 + (1 - 2\beta)\alpha}{1 - z}\right)\alpha)\), the inequalities (2.5) and (2.6) yields the following estimates

\[|a_2| \leq \sqrt{\frac{2(1 - \beta)}{2(1 + 2 \lambda)[3]_q^k - (1 + 3 \lambda)[2]_q^{2k}}}.\]

and

\[|a_3| \leq \frac{4(1 - \beta)^2}{(1 + \lambda)^2[2]_q^{2k}} + \left(\frac{1 - \beta}{(1 + 2 \lambda)[3]_q^k}\right)^2.\]

**Remark 2.8.** Consequently, when \(\lambda = 0\) and \(\lambda = 1\) one has the estimates for the classes \(\mathcal{S}\Sigma_q^k(\alpha)\), \(\mathcal{S}\Sigma_q^k(\beta)\) and \(\mathcal{K}\Sigma_q^k(\alpha)\), \(\mathcal{K}\Sigma_q^k(\beta)\) respectively. We note that, for \(\lim_{\eta \to 1^-}\) and for \(k = 0\) these estimates coincides with the results stated in [21].

From Remark 2.4, Theorem 2.6 yields the following corollary.
Corollary 2.9. Let $f$ given by (1.1) be in the class $M_\Sigma(q, \lambda, \phi)$. Then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(2(1+2\lambda)[3]q-(1+3\lambda)[2]q)B_1^3+(1+\lambda)^2(B_1-B_2)[2]q|}}$$

(2.23)

and

$$|a_3| \leq \frac{B_1}{2(1+2\lambda)[3]q} + \left(\frac{B_1}{(1+\lambda)[2]q}\right)^2$$

(2.24)

In the following section due to Frasin and Aouf [4] and Panigarhi and Murugusundaramoorthy [14] we define the following new subclass involving the Sălăgean operator [15].

3. Bi-Univalent Function Class $F_\Sigma^q_k(\mu, \phi)$

**Definition 3.1.** For $0 \leq \mu \leq 1$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $F_\Sigma^q_k(\mu, \phi)$ if the following subordination hold:

$$(1-\mu)\frac{D_q^k f(z)}{z} + \mu (D_q^k f(z))' \prec \phi(z)$$

(3.1)

and

$$(1-\mu)\frac{D_q^k g(w)}{w} + \mu (D_q^k g(w))' \prec \phi(w)$$

(3.2)

where $z, w \in \Delta$, $g$ is given by (1.5) and $D_q^k f(z)$ is given by (1.9).

**Remark 3.2.** Suppose $f(z) \in \Sigma$. If $\mu = 0$, then $F_\Sigma^q(0, \phi) \equiv H_\Sigma^q(\phi)$: thus, $f \in H_\Sigma^q(\phi)$ if the following subordination holds:

$$\frac{D_q f(z)}{z} \prec \phi(z) \quad \text{and} \quad \frac{D_q g(w)}{w} \prec \phi(w)$$

where $z, w \in \Delta$ and $g$ is given by (1.5).

**Remark 3.3.** Suppose $f(z) \in \Sigma$. If $\mu = 1$, then $F_\Sigma^q(1, \phi) \equiv P_\Sigma^q(\phi)$: thus, $f \in P_\Sigma^q(\phi)$ if the following subordination holds:

$$(D_q^k f(z))' \prec \phi(z) \quad \text{and} \quad (D_q^k g(w))' \prec \phi(w)$$

where $z, w \in \Delta$ and $g$ is given by (1.5).

It is of interest to note that $F_\Sigma^q(\mu, \phi) = F_\Sigma^q(\mu, \phi)$ if the following subordination hold:

$$(1-\mu)\frac{f(z)}{z} + \mu (D_q f(z)) \prec \phi(z)$$

(3.3)

and

$$(1-\mu)\frac{g(w)}{w} + \mu (D_q g(w)) \prec \phi(w)$$

(3.4)

where $z, w \in \Delta$, $g$ is given by (1.5) and $D_q^k f(z)$ is given by (1.9).

**Theorem 3.4.** Let $f$ given by (1.1) be in the class $F_\Sigma^q_k(\mu, \phi)$. Then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(1+2\mu)[3]qB_1^2+(1+\mu)^2[2]q^2(B_1-B_2)|}}$$

(3.5)
and
\[ |a_3| \leq B_1 \left( \frac{B_1}{(1 + \mu)^2[2k]_q} + \frac{1}{(1 + 2\mu)[3k]_q^3} \right). \] (3.6)

**Proof.** Proceeding as in the proof of Theorem 2.6 we can arrive the following relations.

\[ (1 + \mu)[2k]_q a_2 = \frac{1}{2} B_1 p_1 \] (3.7)
\[ (1 + 2\mu)[3k]_q a_3 = \frac{1}{2} B_1 (p_2 - \frac{p_1^2}{2}) + \frac{1}{4} B_2 p_1^2 \] (3.8)
\[ -(1 + \mu)[2k]_q a_2 = \frac{1}{2} B_1 q_1 \] (3.9)

and
\[ 2(1 + 2\mu)[3k]_q a_2^2 - (1 + 2\mu)[3k]_q a_3 = \frac{1}{2} B_1 (q_2 - \frac{q_1^2}{2}) + \frac{1}{4} B_2 q_1^2. \] (3.10)

From (3.7) and (3.9) it follows that
\[ p_1 = -q_1 \] (3.11)

and
\[ 8(1 + \mu)^2[2k]_q a_2^2 = B_1^2 (p_1^2 + q_1^2). \] (3.12)

From (3.8), (3.10) and (3.12), we obtain
\[ a_2^2 = \frac{B_1^2 (p_2 + q_2)}{4[(1 + 2\mu)[3k]_q^3 + (B_1 - B_2)(1 + \mu)^2[2k]_q^3]}. \]

Applying Lemma 2.5 to the coefficients \( p_2 \) and \( q_2 \), we immediately get the desired estimate on \( |a_2| \) as asserted in (3.5).

By subtracting (3.10) from (3.8) and using (3.11) and (3.12), we get
\[ a_3 = \frac{B_1^2 (p_1^2 + q_1^2)}{8(1 + \mu)^2[2k]_q^3} + \frac{B_1 (p_2 - q_2)}{4(1 + 2\mu)[3k]_q^3}. \]

Applying Lemma 2.5 to the coefficients \( p_1, p_2, q_1 \) and \( q_2 \), we get the desired estimate on \( |a_3| \) as asserted in (3.6).

**Remark 3.5.** Consequently, when \( \mu = 0 \) and \( \mu = 1 \) and by taking \( \phi \) as in (1.3) and (1.4) one can deduce the estimates for the classes \( \mathcal{K} \Sigma_q^*(\alpha), \mathcal{K} \Sigma_q^*(\beta) \) easily. We note that, for \( k = 0 \) and \( \lim_q \to 1^- \) these estimates coincides with the results stated in [4, 6, 16].

4. **Fekete-Szegő inequalities**

Making use of the values of \( a_2^2 \) and \( a_3 \), and motivated by the recent work of Zaprawa [23] we prove the following Fekete-Szegő result.

**Theorem 4.1.** Let the function \( f(z) \in M \Sigma_q^k(\lambda, \phi) \) and \( \tau \in \mathbb{C} \), then
\[
|a_3 - \tau a_2^2| \leq \begin{cases} 
\frac{B_1}{2(1 + 2\lambda)[3k]_q^3}, & 0 \leq |\Theta(\tau)| < \frac{1}{8(1 + 2\lambda)[3k]_q^3}, \\
4B_1 |\Theta(\tau)|, & |\Theta(\tau)| \geq \frac{1}{8(1 + 2\lambda)[3k]_q^3}.
\end{cases}
\] (4.1)
Proof. From (2.21) we have $a_3 = a_2^2 + \frac{B_1(p_2 - q_2)}{8(1 + 2\lambda)[\frac{3k}{q}]}$. Using (2.19),

$$a_3 - \tau a_2^2 = \frac{B_1(p_2 - q_2)}{8(1 + 2\lambda)[\frac{3k}{q}]} + (1 - \tau)a_2^2$$

$$+ (1 - \tau)\left(\frac{B_3^2(p_2 + q_2)}{4[(2(1 + 2\lambda)[\frac{3k}{q}]} - (1 + 3\lambda)[\frac{1}{2}B_1^2]B_1^2 + (1 + \lambda)^2(B_1 - B_2)[\frac{2k}{q}]\right)$$

by simple calculation we get

$$a_3 - \tau a_2^2 = B_1\left[\left(\Theta(\tau) + \frac{1}{8(1 + 2\lambda)[\frac{3k}{q}]}\right)p_2 + \left(\Theta(\tau) - \frac{1}{8(1 + 2\lambda)[\frac{3k}{q}]}\right)q_2\right],$$

where

$$\Theta(\tau) = \frac{B_2^2(1 - \tau)}{4[(2(1 + 2\lambda)[\frac{3k}{q}]} - (1 + 3\lambda)[\frac{1}{2}B_1^2]B_1^2 + (1 + \lambda)^2(B_1 - B_2)[\frac{2k}{q}]}.$$

Since all $B_j$ are real and $B_1 > 0$, we have

$$|a_3 - \tau a_2^2| \leq 2B_1\left|\left(\Theta(\tau) + \frac{1}{8(1 + 2\lambda)[\frac{3k}{q}]}\right) + \left(\Theta(\tau) - \frac{1}{8(1 + 2\lambda)[\frac{3k}{q}]}\right)\right|,$$

which completes the proof. □

Proceeding as in above theorem one can easily prove the following result for $f(z) \in \mathcal{F}\Sigma^k_q(\mu, \phi)$ hence we state the following without proof.

Theorem 4.2. Let the function $f(z) \in \mathcal{F}\Sigma^k_q(\mu, \phi)$ and $\tau \in \mathbb{C}$, then

$$|a_3 - \tau a_2^2| \leq 2B_1\left|\left(\Phi(\tau) + \frac{1}{4(1 + 2\mu)[\frac{3k}{q}]}\right) + \left(\Phi(\tau) - \frac{1}{4(1 + 2\mu)[\frac{3k}{q}]}\right)\right|,$$

where

$$\Phi(\tau) = \frac{B_1^2(1 - \tau)}{4[(1 + 2\mu)[\frac{3k}{q}]}B_1^2 + (B_1 - B_2)(1 + \mu)^2[\frac{2k}{q}]}.$$

Concluding Remarks: Taking $\lambda = 0$ (and $1$) in Theorem 4.1 we can state the Fekete-Szegö inequality for the function class $\mathcal{S}\Sigma^k_q(\phi).$ (and $\mathcal{K}\Sigma^k_q(\phi)$) respectively. Putting $\mu = 0$ (and $1$) in Theorem 4.2 we can state the Fekete-Szegö inequality for the function class $\mathcal{H}\Sigma^k_q(\phi).$ (and $\mathcal{P}\Sigma^k_q(\phi)$) respectively.

Future Work: Making use of the values of $a_2$ and $a_3$, and finding $a_4$ we can calculate Hankel determinant coefficient for the function classes.

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