MULTIPLICATION FORMULAS IN MOUFANG LOOPS

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Abstract. Using groups with triality we obtain some general multiplication formulas in Moufang loops, construct Moufang extensions of abelian groups, and describe the structure of minimal extensions for finite simple Moufang loops over abelian groups.

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1. Introduction

A multiplication formula for a loop \( L \) with respect to its decomposition \( L = MN \), where \( M, N \) are subloops of \( L \), is a rule of expressing the product of two elements of the form \( mn \), with \( m \in M \) and \( n \in N \), again in the same form that often assumes the multiplication within both \( M \) and \( N \) and some other information to be known. Multiplication formulas generalize the concept of semidirect product of groups and have been used to give explicit constructions of new loops. Some examples of such formulas for Moufang loops can be found in \([2, 3, 4, 6]\).

Given a Moufang loop \( L \) with a normal subloop \( N \), we use its corresponding group with triality \( G \) to derive a general multiplication formula for the decomposition \( L = MN \) where \( M \) is a subloop. The formula takes especially simple form when the subgroup of \( G \) corresponding to \( N \) is abelian. We then give two important special examples of this formula. The first one comes from modules for wreathlike triality groups \( G \times G \times G \). We obtain a criterion when these modules admit triality and write explicitly the corresponding multiplication rule. The second example may be viewed as a generalization of group action on associative algebras to Moufang loop ‘action’ on alternative algebras. Whenever a Moufang loop \( M \) is mapped to invertible elements \( A \times U \) of an alternative algebra \( A \), one can always construct a semidirect product \( M \rtimes U \) which is a Moufang loop, where \( U \) is any subgroup of the additive group of \( A \) invariant under the operators \( L_{m,n} \) and \( T_m \) for \( m, n \in M \).

The Moufang loops that are upward extensions of abelian group by simple loops are of special interest. We describe the structure of all known minimal such extensions for finite simple noncyclic Moufang loops and conjecture that there are no others.

2. Preliminaries

For elements \( x, y \) in a group, we set \([x, y] = x^{-1} y^{-1} xy\) and define the conjugation operator \( J_y : x \mapsto y^{-1} x y \). The symmetric group of permutations of a set \( X \) is denoted by \( \text{Sym}(X) \). The notation \( V^\oplus n \) means the direct sum of \( n \) copies of a module \( V \).

A loop \( M \) is called a Moufang loop if \( xy \cdot zx = (x \cdot yz)x \) for all \( x, y, z \in M \), where we use the common shorthand notation \( x \cdot yz = x(yz) \). For basic properties of Moufang loops, see [1]. Associative subloops of \( M \) will be called subgroups.
If \((M,\cdot)\) is a Moufang loop and \(x, y \in M\) then we define
\[
\begin{align*}
yL_x &= x.y, & yR_x &= y.x, & T_x &= L_x^{-1}R_x, & P_x &= L_x^{-1}R_x^{-1}, \\
L_{x,y} &= L_xL_yL_{yx}, & R_{x,y} &= R_xR_yR_{xy}.
\end{align*}
\]
We use the notation \([x, y] = x^{-1}y^{-1}x.y\) instead of \([x, y]\) to denote the commutator in \(M\). We also set \((x, y, z) = (x.yz)^{-1}(xy.z)\).

A *pseudoautomorphism* of a Moufang loop \((M,\cdot)\) is a bijection \(A : M \to M\) with the property that there exists an element \(a \in M\) such that
\[
xA.(yA.a) = (x.y)A.a \quad \text{for all } x, y \in M.
\]
This element \(a\) is called a *companion* of \(A\). The set of pairs \((A, a)\), where \(A\) is a pseudoautomorphism of \(Q\) with companion \(a\), is a group with respect to the operation
\[
(A, a)(B, b) = (AB, aB.b).
\]
This group is denoted by \(\text{PsAut}(M)\). It is known [1, Lemma VII.2.2] that \((T_x, x^{-3})\) and \((R_{x,y},[x, y])\) belong to \(\text{PsAut}(M)\).

A group \(G\) possessing automorphisms \(\rho\) and \(\sigma\) that satisfy \(\rho^3 = \sigma^2 = (\rho\sigma)^2 = 1\) is called a *group with triality* \(\langle \rho, \sigma \rangle\) if
\[
(x^{-1}x^\sigma)(x^{-1}x^\sigma)^\rho(x^{-1}x^\sigma)^{\rho^2} = 1
\]
for every \(x \in G\). In a group \(G\) with triality \(S = \langle \rho, \sigma \rangle\), the set
\[
\mathcal{M}(G) = \{x^{-1}x^\sigma \mid x \in G\}
\]
is a Moufang loop with respect to the multiplication
\[
m.n = m^{-\rho}nm^{-\rho^2}
\]
for all \(n, m \in \mathcal{M}(G)\). Conversely, every Moufang loops arises so from a suitable group with triality. For more information on the relation between groups with triality and Moufang loops, see [5, 7].

\(S\)-invariant subgroups of a group \(G\) with triality (for short, \(S\)-subgroups) are also groups triality \(S\), whose corresponding Moufang loops are subloops of \(\mathcal{M}(G)\). If \(G\) is a group with triality then, for \(g \in G\), we define
\[
\varphi(g) = g^{-\rho}g^{\rho^2}.
\]

**Lemma 1.** Let \(G\) be a group with triality \(S = \langle \rho, \sigma \rangle\). Denote \(M = \mathcal{M}(G)\) and \(H = C_G(\sigma)\). Then, for all \(m, n, l \in M\) and \(h \in H\), we have
\[
\begin{align*}
(i) & \quad m, m^\rho, m^{\rho^2} \text{ pairwise commute}; \\
(ii) & \quad m^{-\rho}nm^{-\rho^2} = n^{-\rho^2}mn^{-\rho}; \\
(iii) & \quad [n^{\rho^2}, m^{-\rho}] = [n^{-\rho}, m^{\rho^2}] \in H; \\
(iv) & \quad m.n.m = mmn; \\
v) & \quad \varphi(H) \subseteq M \quad \text{and} \quad \varphi(M) \subseteq H; \\
vii) & \quad M^h = M; \quad \text{moreover, } (J_h, \varphi(h)) \in \text{PsAut}(M); \\
viii) & \quad \text{for } k = [m^\rho, n^{-\rho^2}], \text{ we have } (J_k, \varphi(k)) = (R_{m,n}, [m, n]); \\
(ix) & \quad \text{for } k = \varphi(m), \text{ we have } (J_k, \varphi(k)) = (T_m, m^{-3}); \\
(i) & \quad (l, m, n) = [k, l]^{\rho^2}, \text{ where } k = [n^{-\rho}, m^{\rho^2}].
\]

Lemma 2. Let \( \sigma \) be a group with triality \( S = \langle \rho, \sigma \rangle \). Denote \( M = M(G) \) and \( H = C_G(\sigma) \). We have

(i) \( g^{-1}mg^\sigma \in M \) for every \( g \in G \) and every \( m \in M \).

(ii) The map \( \chi : G \to \text{Sym}(M) \), \( g \mapsto g_x \), given by \( m\chi g = g^{-1}mg^\sigma \), \( m \in M \), is a group homomorphism with \( \ker \chi = C_H(M) \).

Proof. (i) Since \( m \in M \), we have \( m = x^{-1}x^\sigma \) for some \( x \in G \). Then \( g^{-1}mg^\sigma = (xg)^{-1}(xg)^\sigma \in M \).

(ii) Clearly, \( \chi g \in \text{Sym}(M) \) and \( \chi g_1 \chi g_2 = \chi g_1 g_2 \). If \( g \in \ker \chi \) then \( 1 = 1\chi g = g^{-1}g^\sigma \), which yields \( g \in H \). Moreover, \( m = m\chi g = g^{-1}mg \) for every \( m \in M \), and so \( g \in C_H(M) \). The reverse inclusion is obvious.

The permutation action \( \chi \) of \( G \) on \( M \) defined in Lemma 2(ii) has the following implications. First, the three subsets \( M, M^\rho, M^\rho^2 \) of \( G \) are naturally permuted by \( S \) and their corresponding action \( \chi \) on \( M \) is given by the following.

Lemma 3. For every \( m \in M \), we have

\[
\chi_m = P_m; \quad \chi_{m^\rho} = L_m; \quad \chi_{m^\rho^2} = R_m.
\]

Proof. For every \( n \in M \), we have

\[
\begin{align*}
n\chi_m &= m^{-1}nm^{-1} = m^{-1}n.m^{-1} = nP_m; \\
n\chi_{m^\rho} &= m^{-\rho}nm^{-\rho^2} = m.n = nL_m; \\
n\chi_{m^\rho^2} &= m^{-\rho^2}nm^{-\rho} = n.m = nR_m
\end{align*}
\]

by Lemma 1 and the claim follows.

In particular, Lemma 3 demonstrates how the classical triality for Moufang loops

\[
P_x \mapsto P_x^\rho \mapsto L_x \mapsto R_x \mapsto P_x, \quad P_x \mapsto P_x^\sigma \mapsto P_x^{-1}, \quad L_x \mapsto L_x^{-1}, \quad R_x \mapsto R_x^{-1}.
\]

(3)
can be observed from the permutation action \( \chi \) of \( G \) on \( M \).

Also, the action \( \chi \) restricted to the subgroups \( H, H^\rho, H^\rho^2 \), where \( H = C_G(\sigma) \), is related to the induced pseudoautomorphisms \( J_h, h \in H \), of \( M \) and their companions \( \varphi(h) \), see Lemma 1(vii), in the following manner.

Lemma 4. For every \( h \in H \), we have

\[
\chi_h = J_h; \quad \chi_{h^\rho} = J_h R_{\varphi(h)}; \quad \chi_{h^\rho^2} = J_h L_{\varphi(h)}^{-1}.
\]
Proof. For every $m \in M$, we have
\begin{align*}
m\chi_h &= h^{-1} m h^\varrho = h^{-1} m h = m J_h; \\
m\chi_{h^\varrho} &= h^{-\varrho} m h^{\varrho^2} = (h^{-\varrho} h^{\varrho^2})^{-\varrho} h^{-1} m h (h^{-\varrho} h^{\varrho^2})^{-\varrho^2} = m J_h \varphi(h); \\
m\chi_{h^{\varrho^2}} &= h^{-\varrho^2} m h^{\varrho} = (h^{-\varrho} h^{\varrho^2})^{\varrho} h^{-1} m h (h^{-\varrho} h^{\varrho^2})^{-\varrho^2} = \varphi(h)^{-1} m J_h
\end{align*}

by Lemma \[\square\] and the claim follows.

\begin{lemma}
Let $M$ be a Moufang loop and let $x, y, m \in M$. Then
\begin{enumerate}[(i)]
\item $m^{-1}(mx \cdot y) = xm^{-1} \cdot my = (x \cdot ym^{-1}) \cdot m$;
\item $R_{x,y} = L_{x^{-1},y^{-1}} = R_y L_{y^{-1}} R^{-1}_{x^{-1}} L_y$;
\item $x^{-1} \cdot (xy^{-1} \cdot m) y = y^{-1} x^{-1} \cdot (xm \cdot y) = (y^{-1} \cdot mx^{-1}) \cdot xy = y^{-1} (m \cdot yx^{-1}) \cdot x$;
\item The operator $D_{x,y} : m \mapsto x^{-1} \cdot (xy^{-1} \cdot m) y$ is a pseudoautomorphism of $M$ with companion $y^{-1} xy^{-2} x^{-1}$;
\item In $\text{PsAut}(M)$, we have $(D_{x,y}, y^{-1} xy^{-2} x^{-1}) = (L_{x,y^{-1}}, [x^{-1}, y]) (T_y, y^{-3}) (L_y, [y^{-1}, x^{-1}])$.
\end{enumerate}
\end{lemma}

Proof. (i) The first equality holds, since the left Moufang identity implies
\begin{align*}
m \cdot (xm^{-1} \cdot my) &= mxm^{-1} \cdot my = mx \cdot y.
\end{align*}
The second equality is dual to the first one (i.e. follows by inversion).

(ii) The first equality is known [\[\square\] Lemma VII.5.4]. Applying (i), we have
\begin{align*}
m R_{x,y} &= (mx \cdot y) \cdot y^{-1} x^{-1} = y \cdot (y^{-1} \cdot mx) x^{-1} = m R_y L_{y^{-1}} R^{-1}_{x^{-1}} L_y
\end{align*}

(iii) In order to show the first equality, it suffices to prove that $xy^{-1} \cdot m = (y^{-1} x^{-1} \cdot (xm \cdot y)) \cdot y^{-1}$. Using the Moufang identities, the right-hand side can be rewritten as
\begin{align*}
x(y^{-1} x^{-1} \cdot (x \cdot x^{-1} (xm \cdot y))) \cdot y^{-1} &= (xy^{-1} \cdot x^{-1} (xm \cdot y)) \cdot y^{-1} \\
&= x(y^{-1} x^{-1} \cdot (xm \cdot y) y^{-1}) = xy^{-1} x^{-1} x \cdot m = xy^{-1} \cdot m.
\end{align*}
The third equality is dual to the first one. In order to show the second one, it suffices to prove that $xm \cdot y = xy \cdot (y^{-1} \cdot mx^{-1}) \cdot y$. The right-hand side equals
\begin{align*}
x(mx^{-1} \cdot xy) &= xmx^{-1} x \cdot y = xm \cdot y.
\end{align*}

(iv) follows from (v).

(v) We first show that $D_{x,y} = L_{x,y^{-1}} T_y L_{y,x}$. By the second expression in (iii), we have
\begin{align*}
L_{x,y^{-1}} T_y L_{y,x} &= (R_x L_{y^{-1}} R^{-1}_y R_y L_{y^{-1}} L^{-1}_x) = R_x^{-1} L_y R_y L_{y} L_{y^{-1}} L^{-1}_x.
\end{align*}

Hence, it remains to show that $R_x^{-1} L_y R_y L_{y} L_{y} = L_x R_y$. This follows from $R_y L_{y} L_{y} = L_y R_x L_x R_y$, which in turn follows by
\begin{align*}
x(y \cdot mx \cdot y) &= x(ym \cdot xy) = (x \cdot ym \cdot x)y
\end{align*}
for every $m$ due the Moufang identities.

For the companion, we have
\begin{align*}
([y^{-1}, y] T_y y^{-3}) L_{y,x} [y^{-1}, x^{-1}] &= (y^{-1} \cdot xy^{-1} x^{-1} y \cdot y) y^{-3} \cdot yxy^{-1} x^{-1} = y^{-1} xy^{-2} x^{-1},
\end{align*}
where we have used the fact that $L_{x,y}$ acts as the identity on the subgroup $\langle x, y \rangle$ due to diassociativity.

\[\square\]
4. A multiplication formula

**Lemma 6.** Let $G$ be a group with triality $S$ and let $m,n,u,w \in \mathcal{M}(G)$. Then

$$(m.u).(n.w) = (m.n).x,$$  \hspace{1cm} (4)

where

$$x = u^{-\rho n^{-\rho} m^{\rho^2}} w^{[n^{\rho^2},m^{-\rho}]} u^{-\rho^2 n^{\rho^2} m^{-\rho}} \in \mathcal{M}(G);$$  \hspace{1cm} (5)

and

$$(m.u)^{-1} = m^{-1}.y, \quad \text{where} \quad y = u^{\rho m^{-1}} w^{\rho^2}. \; (6)$$

**Proof.** The left-hand side of (4) can be expanded using (2) and Lemma 1(iii) as follows

$$m^{-\rho} u m^{-\rho^2} n^{-\rho} w n^{-\rho^2} = (mu^{-\rho} m^{\rho^2}) n^{-\rho} w n^{-\rho^2} (m^{\rho} u^{-\rho^2} m) = (mn^{-\rho} m^{\rho^2}) \times$$

$$\times m^{-\rho^2} n\rho u^{\rho^2} m \rho^3 u^{-\rho^2} [n^{\rho^2},m^{-\rho}] w^{[n^{\rho^2},m^{-\rho}]} m^{\rho} n^{-\rho^2} u^{-\rho^2} n^{\rho^2} m^{-\rho} (m^{\rho^2} n^{-\rho^2} m)$$

$$= (m^{-\rho} n m^{-\rho^2})^{-\rho} u^{-\rho^2} m \rho^3 u^{\rho^2} w^{[n^{\rho^2},m^{-\rho}]} u^{-\rho^2} n^{\rho^2} m^{-\rho} (m^{-\rho} n m^{-\rho^2})^{-\rho^2} = (m.n).x$$

Observe that $x = g^{-1} w^h g^\sigma$, where $h = [n^{\rho^2},m^{-\rho}] \in C_G(\sigma)$ and $g = u^{\rho n^{-\rho} m^{\rho^2}}$. Hence, $x \in \mathcal{M}(G)$ by Lemma 2.

The inversion formula (6) follows from (5) by setting $n = m^{-1}$ and using Lemma 1(i). \hfill \Box

This result can be applied in the following situation.

**Lemma 7.** Let $K,V$ be $S$-subgroups of a group $G$ with triality. Suppose that $V$ is normal in $G$ and $G = KV$. Then (4) is a multiplication formula in the corresponding Moufang loop $M = \mathcal{M}(G)$ with respect to its decomposition $M = \mathcal{M}(K)\mathcal{M}(V)$, where $\mathcal{M}(V)$ is a normal subloop of $M$ and $m,n \in \mathcal{M}(K)$, $u,w \in \mathcal{M}(V)$.

**Proof.** We only need to show that the element $x$ from (5) lies in $\mathcal{M}(V)$. This follows from applying the natural homomorphism $M \rightarrow M/\mathcal{M}(V)$ to both sides of (4). \hfill \Box

Lemma 7 can be used to obtain a multiplication formula for any Moufang loop $L$ with respect to any decomposition $L = MN$, where $M,N$ are subloops of $L$ with $N$ normal, because there always exists a group with triality $G$ as stated in the lemma such that $\mathcal{M}(G) = L$, $\mathcal{M}(K) = M$ and $\mathcal{M}(N) = V$. In fact, any group $G$ with triality such that $[G,S] = G$ and $\mathcal{M}(G) = L$ will have this property, see [7].

The case where $V$ is abelian deserves special attention.

**Lemma 8.** If, under the assumptions of Lemma 7, $V$ is an abelian subgroup of $G$ then the multiplication and inversion formulas in $\mathcal{M}(G)$ take the form

$$(m.u).(n.w) = (m.n).x, \quad x = uD_{m,n} + wL_{n,m};$$

$$(m.u)^{-1} = m^{-1}.y, \quad y = -uT^{-1}_{m} \; (7)$$

for all $m,n \in \mathcal{M}(K)$, $u,w \in \mathcal{M}(V)$.

**Proof.** First, observe that $J_h = L_{n,m}$ for $h = [n^{\rho^2},m^{-\rho}]$ by Lemmas 1(iii), (vii) and 5(ii). In particular, $w^h = wL_{n,m}$. Also, if we set $w = 1$ in (4), we have $x = (mn)^{-1}(m.u)n = uD_{m,n}$. Hence, with $w = 1$ the expression (5) gives

$$uD_{m,n} = u^{-\rho n^{-\rho} m^{\rho^2}} u^{-\rho^2 n^{\rho^2} m^{-\rho}} \; (8)$$

Since $(m.u)^{-1} = u^{-1}.m^{-1} = m^{-1}.u^{-1}T^{-1}_{m}$, we have the expression for $y$. 

\hfill \Box
Assume now that $V$ is abelian. Since $V$ is a normal $S$-subgroup, all three factors on the right-hand side of (5) belong to $V$. Permuting the last two factors and using (8), we obtain (7) in the additive notation. □

**Lemma 9.** Suppose in a Moufang loop $L = MU$ with a subloop $M$ and a normal abelian subgroup $U$, the multiplication is given by (7) for all $m, n \in M$, $u, w \in U$. Then $(l, u, w) = 1$ for all $l \in L$, $u, w \in U$.

**Proof.** We use multiplicative notation in $U$. Using (7), we see that $mu \cdot w = m \cdot uw$ for every $m \in M$. Hence, if $l = mv$ for $m \in M$, $v \in U$, we have $lu \cdot w = (m \cdot vu)w = m(vu \cdot w)$ and $l \cdot uw = m(v \cdot uw)$. The claim follows, since $U$ is associative. □

Lemma 9 shows that not every Moufang loop $L$ with decomposition $L = MU$, where $U$ is normal abelian, has multiplication (7). For example, the abelian-by-cyclic Moufang loops constructed in [6] have this decomposition, but do not satisfy the conclusion of the lemma. The reason is that for those loops, in the corresponding group with triality, the normal $S$-subgroup $V$ such that $M(V) = U$ is not abelian. The following generalization of Lemma 9 holds.

**Lemma 10.** Let $G$ be a group with triality $S$ and let $V$ be an abelian $S$-subgroup of $G$. Denote $U = M(V)$, which is an abelian subgroup of $L = M(G)$. Then we have $(L, U, U) = 1$.

**Proof.** Let $l \in L$ and $u, v \in U$. Recall that $U \subseteq V$. By Lemma 1(ix), we have $(l, u, v) = [k, l]_{\rho, v^2}$, where $k = [v^{-\rho}, w^2]$. Since $V$ is $S$-invariant, we have $v^{-\rho}, w^2 \in V$. Since $V$ is abelian, we have $k = 1$. Therefore, $(l, u, v) = 1$ and the claim follows. □

In the next sections, we will show how the action of operators $D_{m,n}, L_{n,m},$ and $T_n$ that appear in the multiplication formula (7) can be restored from the group action on modules or linked with the inner multiplication of an alternative algebra on its additive group.

## 5. Modules for wreathlike triality groups

Let $G$ be a group. It it known that $T = G \times G \times G$ is a group with triality $S = \langle \rho, \sigma \rangle$ (which we call wreathlike following [7]) with respect to the natural action $(g_1, g_2, g_3)^\rho = (g_3, g_1, g_2)$, $(g_1, g_2, g_3)^\sigma = (g_2, g_1, g_3)$, whose corresponding Moufang loop is isomorphic to $G$.

Let $R$ be a commutative associative unital ring and let $V$ be an $RG$-module which is free of rank $n$ as an $R$-module. The outer tensor product $W = V \# V \# V$ is an $RT$-module [9, Definition VII.43.1] which admits the natural action of $S$ by $(v_1 \# v_2 \# v_3)^\rho = v_3 \# v_1 \# v_2$, $(v_1 \# v_2 \# v_3)^\sigma = v_2 \# v_1 \# v_3$ that is compatible with the action of $T$ on $W$ in the sense that $(w^t)^\tau = (w^\tau)^{(t^\tau)}$ for all $w \in W$, $t \in T$, $\tau \in S$, and hence is extended to $A = T \times W$. Our aim is to determine when $A$ is a group with triality $S$.

**Lemma 11.** The group $A$ constructed above has triality $S$ if and only if $n \leq 2$. If $n = 1$ then $M(A) \cong G$.

**Proof.** Let $e_1, \ldots, e_n$ be an $R$-basis of $V$. For $t = (g_1, g_2, g_3) \in T$ we have $t^{-1} t^\sigma = (g_1^{-1} g_2, g_2^{-1} g_1, 1)$. Hence, $M(T) = \{(g^{-1}, g, 1) \mid g \in G\}$. By [9] Lemma 4, $A$ has triality $S$ if and only if $W$ has triality $S_{(m)}$ for all $m \in M(T)$, where $S_{(m)} = \{\rho^2 m \rho^2, \sigma\}$. Let $m = (g^{-1}, g, 1)$. Then $\rho^2 m \rho^2 = \rho(g, 1, g^{-1})$ and $\rho m^{-1} \rho = \rho^2 (1, g, g^{-1})$. Hence, $A$
has triality if and only if, for every basis element \( e_{ijk} = e_i \# e_j \# e_k \) of \( W \) and every \( g \in G \),
\[
e_{ijk}^{(1-\sigma)(1+\rho(g,1,g^{-1})+\rho^2(1,g,g^{-1}))} = 0.
\]
Expanding and acting on both sides by \((1,1,g)\), we have
\[
(e_{ijk} - e_{jik})^{(1,1,g)} + (e_{kij} - e_{jki})^{(g,1,1)} + (e_{jki} - e_{ikj})^{(1,g,1)} = 0.
\]
Let \( e_k g = \sum_s g_{ks} e_s \), where \( g_{ks} \in R \). Then the condition is rewritten as
\[
\sum_{s=1}^n g_{ks} (e_{ijs} - e_{jis} + e_{sij} - e_{sji} + e_{jsi} - e_{isj}) = 0
\]
which must hold for all \( g \in G \), \( 1 \leq i,j,k \leq n \). In particular, setting \( g = 1 \) gives
\[
e_{ijk} - e_{jik} + e_{kij} - e_{jki} + e_{ikj} = 0.
\]
This does not hold, say, for \((i,j,k) = (1,2,3)\), if \( n \geq 3 \), since the basis elements are linearly independent. However, if \( n \leq 2 \) then (9) is satisfied, since at least two of \( i,j,s \) will coincide.

In the case \( n = 1 \), the normal subgroup \( W \) of \( A \) is spanned by \( e_{111} \) on which \( S \) acts trivially. Hence \( M(A) \cong M(A/W) \cong M(T) \cong G \).

The case \( n = 2 \) is of main interest and we consider it now in more detail.

**Theorem 12.** Let \( G \) be a subgroup of \( GL_2(R) \). Let \( V \) be the free \( R \)-module of rank 2 with the natural action \( \circ \) of \( G \). Denote by \( G \ltimes V \) the set of pairs \((g,u)\) for \( g \in G \), \( u \in V \). Then with respect to the operation
\[
(g,u) \cdot (h,w) = (gh, u \circ (\det h)gh^{-2}g^{-1} + w \circ [h^{-1},g^{-1}])
\]
\( G \ltimes V \) becomes a Moufang loop with identity \((1,0)\) and inversion
\[
(g,u)^{-1} = (g^{-1}, -u \circ (\det g)^{-1}g^2)
\]
Moreover, this Moufang loop is isomorphic to \( M(A) \), where \( A = T \ltimes W \) is the triality group constructed above with respect to the \( RG \)-module \( V \).

**Proof.** Let \( e_1, e_2 \) be an \( R \)-basis of \( V \). Then \( e_{ijk} = e_i \# e_j \# e_k \), \( i,j,k = 1,2 \), is a basis of \( W = V \# V \# V \). As above, let \( T = G \times G \times G \) with \( A = T \ltimes W \) admitting the natural action of \( S \). By Lemma 11, \( A \) is a group with triality. We see that \( M(W) \) is spanned by \( f_1 = e_{121} - e_{211} \) and \( f_2 = e_{122} - e_{212} \). Let \( m,n \in M(T) \) and \( u,w \in M(W) \). Lemma 8 implies that in the loop \( M(A) \) we have \((m.u).(n.w) = (m.n).x \), where
\[
x = u^{\rho n \rho m \rho^2 - \rho^2 n^2 m - \rho^2} + w^{n^2 m^2 - \rho^2}.
\]
Let \( m = (g^{-1},g,1) \) and \( n = (h^{-1},h,1) \) for suitable \( g,h \in G \). Then \( n^2 = (h,1,h^{-1}) \) and \( m^{-\rho} = (1,g,1) \). Therefore,
\[
[n^2, m^{-\rho}] = (h^{-1},1,h)(1,g^{-1},g)(h,1,h^{-1})(1,g,g^{-1}) = (1,1,[h^{-1},g^{-1}]).
\]
Observe that \( f_k = (e_1 \# e_2 - e_2 \# e_1) \# e_k \), \( k = 1,2 \). Hence, the matrix of action of \((1,1,[h^{-1},g^{-1}])\) in \( \{f_1, f_2\} \) coincides with the matrix of action of \([h^{-1},g^{-1}]\) in \( \{e_1, e_2\} \). Similarly, we have
\[
-\rho n \rho m \rho^2 - \rho^2 n^2 m - \rho^2 = -n^2 m \rho - n m^{-\rho} \rho^2 = -(h,h^{-1},1)(1,g^{-1},g)\rho
\]
\[
-(h^{-1},h,1)(1,g^{-1},g)\rho = -(h,h^{-1}g^{-1},g)\rho - (h^{-1}g^{-1},h,g)\rho^2,
\]
and this operator send the basis element \( e_{ijk} \) to
\[
-e_{kg}\# e_ih\# e_jh^{-1}g^{-1} - e_jh\# e_{kg}\# e_ih^{-1}g^{-1}.
\]
Therefore, \( f_k = (e_1\# e_2 - e_2\# e_1)\# e_k \) is sent to
\[
(e_{kg}\# e_2h - e_2h\# e_{kg})\# e_1h^{-1}g^{-1} - (e_{kg}\# e_1h - e_1h\# e_{kg})\# e_2h^{-1}g^{-1}.
\] (12)

Observe that, for any \( a, b \in M_2(R) \) with \( a = (a_{ij}), b = (b_{ij}) \) and any \( k, i = 1, 2 \), we have
\[
e_{ka}\# e_ib - e_i\# e_k a = (a_{k1}e_1 + a_{k2}e_2)\# (b_{1i}e_1 + b_{2i}e_2) - (b_{1i}e_1 + b_{2i}e_2)\# (a_{k1}e_1 + a_{k2}e_2) = (a_{k1}b_{1i} - a_{k2}b_{2i})(e_1\# e_2 - e_2\# e_1).
\]

Denoting \( C_{ki}^{ab} = a_{k1}b_{i2} - a_{k2}b_{i1} \), we check that
\[
\left( \begin{array}{cc}
C_{12}^{ab} & -C_{11}^{ab} \\
-C_{22}^{ab} & C_{21}^{ab}
\end{array} \right) = ab^*,
\] (13)
where \( b^* \) is the adjoint matrix of \( b \), i.e. \( bb^* = (\det b)I \) with \( I \) the identity matrix. Using this notation, (12) can be rewritten as
\[
C_{k2}^{g,h}(e_1\# e_2 - e_2\# e_1)\# (e_1h^{-1}g^{-1}) - C_{k1}^{g,h}(e_1\# e_2 - e_2\# e_1)\# (e_2h^{-1}g^{-1})
\]
\[
= (C_{k2}^{g,h}(h^{-1}g^{-1})_{11} - C_{k1}^{g,h}(h^{-1}g^{-1})_{21})f_1 + (C_{k2}^{g,h}(h^{-1}g^{-1})_{12} - C_{k1}^{g,h}(h^{-1}g^{-1})_{21})f_2.
\]

Therefore, (13) implies that the matrix of the transformation (11) in the basis \( \{ f_1, f_2 \} \) equals
\[
\left( \begin{array}{cc}
C_{12}^{g,h}(h^{-1}g^{-1})_{11} & C_{11}^{g,h}(h^{-1}g^{-1})_{21} \\
C_{22}^{g,h}(h^{-1}g^{-1})_{11} & C_{21}^{g,h}(h^{-1}g^{-1})_{21}
\end{array} \right) = gh^*h^{-1}g^{-1} = (\det h)gh^{-2}g^{-1}.
\]

It is clear from this discussions that under the map \( m.u \mapsto (g, u) \), where \( m = (g^{-1}, g, 1) \), the multiplication formula from Lemma 8 for the loop \( M(A) \) takes the required form (10) thus giving the claimed isomorphism. The assertions about the identity and inversion readily follow. \( \square \)

The loop \( G \bowtie V \) from Theorem 12 has a naturally embedded subgroup isomorphic to \( G \) with elements of the form \((g, 0)\) and a normal subgroup isomorphic to \( V \) with elements \((1, v)\). We will identify \( G \) and \( V \) with these corresponding subgroups and call \( G \bowtie V \) their (outer) Moufang semidirect product.

It is easily checked that \( G \bowtie V \) is nonassociative if and only if \( G \) is nonabelian. Let \( G_0 \) be the scalar subgroup of \( G \). Then \( G_0 \) is a central (hence, normal) subloop of \( G \bowtie V \). The factor loop \((G \bowtie V)/G_0 \), which we denote by \( \overline{G} \bowtie V \) with \( \overline{G} = G/G_0 \), has a ‘projective’ analog of the formulas (10) with \( g, h \) replaced by \( \overline{g}, \overline{h} \), because the operators \((\det h)gh^{-2}g^{-1} \) and \([h^{-1}, g^{-1}] \) are constant on the cosets \( G : G_0 \). We note that \( \overline{G} \bowtie V \) can be nonassociative even when \( \overline{G} \) is abelian.

The existence of the following Moufang semidirect product of finite simple groups follows from the above construction along with the subgroup structure [12] of \( \text{GL}_2(q) \):

- \( \text{PSL}_2(q) \bowtie (\mathbb{F}_q \oplus \mathbb{F}_q) \), where \( q \geq 4 \) is a prime power;
- \( A_5 \bowtie (\mathbb{F}_p \oplus \mathbb{F}_p) \), where \( p \equiv \pm 1 \) (mod 10) is prime;
- \( A_5 \bowtie (\mathbb{F}_{p^2} \oplus \mathbb{F}_{p^2}) \), where \( p \equiv \pm 3 \) (mod 10) is prime.
6. Embedding in a Cayley algebra

A loop of shape $GL_2(R) \ltimes (R \oplus R)$ also appears as a parabolic subloop of the invertible elements of the split Cayley $R$-algebra $O = O(R)$. We recall that $O$ can be defined as the set of all Zorn matrices

$$\begin{pmatrix} a & v \\ w & b \end{pmatrix}, \quad a, b \in R, \quad v, w \in R^3$$

with the natural structure of a free $R$-module and multiplication given by the rule

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix}, \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + v_1 \cdot w_2 & a_1v_2 + b_2v_1 \\ a_2w_1 + b_1w_2 & w_1 \cdot v_2 + b_1b_2 \end{pmatrix} + \begin{pmatrix} 0 & -w_1 \times w_2 \\ v_1 \times v_2 & 0 \end{pmatrix}, \quad (14)$$

where, for $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ in $R^3$, we denoted

$$v \cdot w = v_1w_1 + v_2w_2 + v_3w_3 \in R, \quad v \times w = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \in R^3.$$

It is well-known that $O$ is an alternative algebra and the set of its invertible elements $O^\times$ is a Moufang loop. The parabolic subloop of $O^\times$ can be identified with the set of elements of the form

$$\begin{pmatrix} a_{11} \\ (r_2, a_{21}, 0) \end{pmatrix}, \quad (15)$$

where $(a_{ij}) \in GL_2(R)$ and $r_1, r_2 \in R$.

**Lemma 13.** Let $V$ be the free $R$-module of rank 2 with the natural action “$\circ$” of $GL_2(R)$. Then the outer Moufang semidirect product $GL_2(R) \ltimes V$ is isomorphic to a parabolic subloop of the Cayley algebra $O(R)$.

**Proof.** Identifying the element $(15)$ with the pair $(a, r \circ a^{-1})$, where $a = (a_{ij})$, $r = (r_1, r_2)$, one easily checks using $(14)$ that such pairs are multiplied in $O^\times$ as follows

$$(a, r \circ a^{-1}) \cdot (b, s \circ b^{-1}) = (ab, (r \circ b^* + s \circ a) \circ (ab)^{-1}),$$

where $b^*$ is the adjoint matrix of $b$. Setting $u = r \circ a^{-1}$ and $w = s \circ b^{-1}$, we have

$$(a, u) \cdot (b, w) = (ab, u \circ ab^*b^{-1}a^{-1} + w \circ bab^{-1}a^{-1}),$$

which coincides with the formula $(10)$. \hfill \Box

As a corollary we see that the Moufang semidirect product $PSL_2(q) \ltimes (F_q \oplus F_q)$ is isomorphic to a maximal parabolic subloop of the finite simple Moufang loop $M(q)$, see \cite{10}.

7. The action of Moufang loops on abelian groups

Let $R$ be a commutative associative unital ring and let $A$ be an alternative $R$-algebra with unit $1$. It is well known \cite[Lemma 2.3.7]{8} that the Moufang identities hold in $A$. In particular, the set of invertible elements $A^\times$ of $A$ is a Moufang loop. For every $x \in A^\times$, the maps $L_x : a \mapsto xa, \quad R_x : a \mapsto ax$ are invertible linear operators of $A$ and we may also define the following operators

$$T_x = L_x^{-1}R_x, \quad L_{x,y} = L_xL_yL_{y,x}^{-1}, \quad D_{x,y} = L_xR_yL_{y,x}^{-1}. \quad (16)$$
Clearly, when restricted to $A^\times$ these operators coincide with those defined in (I) and Lemma 5(iii) for Moufang loops.

**Lemma 14.** Let $A$ be an alternative $R$-algebra. Then, for all $m, n, k \in A^\times$, we have

(i) $D_{m,n} = L_{m,n^{-1}}T_{n}L_{n,m}$;

(ii) $L_{n,m}D_{mn,km} = D_{n,k}L_{nk,m}D_{m-nk,m}$;

(iii) $D_{k,n}L_{km, mn} = L_{k,n}L_{nk,m}D_{m-nk,m}$;

(iv) $D_{m,n}D_{mn, km} + L_{m,k}L_{km, mn} = D_{m,nk}D_{m-nk,m} + L_{m,mn}$.

**Proof.** (i) This can be proved as Lemma 5(v), where we only used the Moufang identities which also hold in alternative algebras.

(ii) By definition, we have

\[
L_{n,m}D_{mn, km} = L_{n,m}L_{m}^{-1}L_{mn}R_{km}L_{mn, km}^{-1} = L_{n,m}R_{km}L_{mn, km}^{-1},
\]

\[
D_{n,k}L_{nk,m}D_{m-nk,m} = L_{n,k}L_{nk,m}^{-1}L_{nk}L_{mn}L_{m-nk,m}^{-1}L_{mn}^{-1} = L_{n,k}L_{nk,m}^{-1}L_{mn}^{-1}.
\]

Since $A^\times$ is a Moufang loop, it remains to show that $L_{m}R_{km} = R_{k}L_{m}R_{m}$. This follows from $ma \cdot km = (m \cdot ak) \cdot m$ for every $a \in A$, since the Moufang identities hold in $A$.

(iii) We have

\[
D_{k,n}L_{km, mn} = L_{k,n}L_{km}^{-1}L_{km}L_{mn}L_{mn, km}^{-1} = L_{k,n}L_{km}L_{mn, km}^{-1},
\]

\[
L_{k,n}L_{nk,m}D_{m-nk,m} = L_{k,n}L_{nk,m}^{-1}L_{nk}L_{mn}L_{m-nk,m}^{-1}L_{mn}^{-1} = L_{k,n}L_{nk,m}^{-1}L_{mn}^{-1}.
\]

Again, it suffices to show that $R_{m}L_{mn} = L_{m}R_{m}$. This follows from the Moufang identity $mn \cdot am = (m \cdot na) \cdot m$.

(iv) We have

\[
D_{m,n}D_{mn, km} + L_{m,k}L_{km, mn} = L_{m}R_{n}L_{mn}^{-1}L_{mn}R_{km}L_{mn, km}^{-1} + L_{m}L_{k}L_{km}^{-1}L_{km}L_{mn}L_{mn, km}^{-1} = L_{m}R_{n}R_{km} + L_{k}L_{mn}L_{mn, km}^{-1},
\]

\[
D_{m,nk}D_{m-nk,m} + L_{m,m-nk} = L_{m}R_{n}L_{m-nk,m}^{-1}L_{m-nk,m}R_{m}L_{m-nk,m}^{-1} + L_{m}L_{m-nk}L_{mn}L_{m-nk,m}^{-1} = L_{m}R_{n}R_{m} + L_{m-nk}L_{m-nk,m}^{-1}.
\]

The equality $R_{n}R_{km} + L_{k}R_{mn} = R_{nk}R_{m} + L_{m-nk}$ follows from

\[
an \cdot km + mn \cdot ka = (a \cdot nk) \cdot m + (m \cdot nk)a
\]

for every $a \in A$, which in turn is the linearization of the Moufang identity $xn \cdot kx = (x \cdot nk)x$ with $x = m + a$. \qed

**Theorem 15.** Let $A$ be an alternative $R$-algebra and let $M$ be a subloop of $A^\times$. Let $U$ a subgroup of the additive group of $A$ that is invariant under the operators $T_{m}$ and $L_{m,n}$ for all $m, n \in M$. Denote by $M \ltimes U$ the set of pairs $(m, u)$ for $m \in M$, $u \in U$. Then with respect to the operation

\[
(m, u) \cdot (n, w) = (mn, uD_{m,n} + wL_{n,m})
\]

$M \ltimes U$ becomes a Moufang loop with identity $(1, 0)$ and inversion

\[
(m, u)^{-1} = (m^{-1}, -uT_{m}^{-1})
\]
Proof. First observe that $U$ is also invariant under $D_{m,n}$ due to Lemma \ref{lem:mufl}(i); in particular, $uD_{m,n} + wL_{n,m}$ lies in $U$.

We show that the Moufang identity $ab \cdot ca = (a \cdot bc)a$ holds for arbitrary $a = (m, u)$, $b = (n, w)$, $c = (k, v)$ in $M \not\triangleleft U$. We have
\[
ab \cdot ca = (mn, uD_{m,n} + wL_{n,m}) \cdot (km, vD_{k,m} + uL_{m,k}) \\
= (mn \cdot km, (uD_{m,n} + wL_{n,m})D_{mn,km} + (vD_{k,m} + uL_{m,k})L_{km,mn}).
\]
Also,
\[
(a \cdot bc)a = (a \cdot (nk, wD_{n,k} + vL_{k,n}))a = (m \cdot nk, uD_{mn,nk} + (wD_{n,k} + vL_{k,n})L_{nk,m})a \\
= ((m \cdot nk) \cdot m, (uD_{mn,nk} + (wD_{n,k} + vL_{k,n})L_{nk,m})D_{mn,km} + uL_{m,mnk}).
\]
The first components are equal, since $M$ is a Moufang loop. The second components are equal, since the operators acting on $v, w, u$ coincide due to (ii), (iii), (iv) of Lemma \ref{lem:mufl} respectively.

The assertions about the identity and inversion are readily verified, once we note that $D_{x,x^{-1}} = T_{x^{-1}}$, $D_{1,x} = T_x$, and $D_{x,1} = L_{x,x^{-1}} = L_{1,x} = L_{x,1}$ is the identity operator.

\[\square\]

Remark 1. Instead of considering $M$ as a subloop of $A^\times$ in Theorem \ref{thm:mufl} we may clearly generalize this to an arbitrary loop homomorphism $M \rightarrow A^\times$. In this situation we will say that the loop $M$ acts on the abelian subgroup $U$ of $A$ and call $L = M \not\triangleleft U$ an outer semidirect product of $M$ and $U$. We will identify $M$ and $U$ with their isomorphic images in $L$ consisting of elements $(m, 0)$ and $(1, u)$, respectively.

Remark 2. Once we make this identification, the multiplication and inversion formulas for $L$ take the inner form
\[
mu \cdot nw = mn \cdot x, \quad \text{with} \quad x = uD_{m,n} + wL_{n,m}, \\
(mu)^{-1} = m^{-1} \cdot y, \quad \text{with} \quad y = -uT_{m^{-1}},
\]
which means that the operators $T_m, D_{m,n}, L_{n,m}$ can be viewed as inner loop operators in $L$ without reference to the alternative algebra $A$, thus agreeing with the formulas \ref{eq:mufl}. Indeed, for example, $(1, u)$ is sent by the loop operator $L_{(m,0),(n,0)}$ to
\[
((nm)^{-1}, 0)(n, 0) \cdot (m, 0)(1, u)) = ((nm)^{-1}, 0) \cdot (n, 0)(m, u) \\
= ((nm)^{-1}, 0) \cdot (nm, uL_{m,n}) = (1, uL_{m,n}),
\]
and similarly for $D_{m,n}$ and $T_m$.

Remark 3. Let $M_0$ be the scalar multiples of 1 contained in $M$. It is easily checked that $M_0$ is a central subloop of $M \not\triangleleft U$ and we denote the factor loop $(M \not\triangleleft U)/M_0$ by $\overline{M} \not\triangleleft U$, where $\overline{M} = M/M_0$.

Remark 4. If $U_0$ is a subgroup of $U$ invariant under $T_m, L_{n,m}$ then $U_0$ is a normal subgroup of $M \not\triangleleft U$. In particular, this gives the Moufang loop $(M \not\triangleleft U)/U_0$ which we denote by $(M \not\triangleleft \overline{U})$ with $\overline{U} = U/U_0$. Combining with the previous remark, we also have a Moufang loop $\overline{M} \not\triangleleft \overline{U}$.

8. Semidirect Products for Simple Moufang Loops

We may use the previous results to construct outer semidirect product of loops that arise from simple alternative algebras.

Let $A$ be a Cayley-Dickson algebra over a field $F$ equipped with a nondegenerate quadratic form $Q$ that makes $A$ a composition algebra and an orthogonal space, see \cite[Chap. 2]{cayley-dickson}. Let $U = 1^\perp$, which is a 7-dimensional subspace of $A$. For all $m, n \in A^\times$
the operators $L_{m,n}$ and $T_m$ leave fixed the unit 1, and since they also preserve $Q$, they leave invariant the subspace $U$. If $M \leq A^\times$, by Theorem [15], there exists the Moufang loop $M \triangleleft \mathcal{U}$. Moreover, if $F$ has characteristic 2, we have $U_0 = \langle 1 \rangle \leq U$. Then $U/U_0$ is 6-dimensional and we have the Moufang loop $M \triangleleft \mathcal{U}$.

Denote by SL$(A)$ the loop of units of $A$ of norm 1 and by PSL$(A)$ its quotient by the scalars. The latter loop is not always simple, but it is such in the following two important special cases [11]. First, if $A$ is the classic division algebra of octonions over $\mathbb{R}$, then PSL$(A)$ is simple and we have the loops PSL$(A) \triangleleft \mathbb{R}^\oplus 7$. Second, if $A$ is a split Cayley algebra over any field $F$. Then PSL$(A)$ is simple and we have the loops PSL$(A) \triangleleft F^\oplus n$, where $n = 6$ or 7 according as $F$ has characteristic 2 or otherwise.

In particular, the following semidirect products for finite simple Paige–Moufang loops $M(q)$ exist:

- $M(q) \triangleleft \mathbb{F}_q^\oplus 7$, where $q$ is an odd prime power;
- $M(q) \triangleleft \mathbb{F}_q^\oplus 6$, where $q$ is a power of 2;
- $M(2) \triangleleft \mathbb{F}_p^\oplus 7$, where $p$ is an odd prime.

The extensions in the last exceptional case exist due to the embedding of $M(2)$ as a maximal subloop into $M(p)$ for $p$ an odd prime, see [10].

9. Extensions of Moufang loops with abelian kernel

Let $M$ be a Moufang loop and $U$ an abelian group. Suppose there is a short exact sequence of Moufang loops

$$1 \to U \to E \to M \to 1$$

We will identify $U$ with its image in $E$ and say that the extension [(18)] of $M$ is minimal, if $U$ contains no subgroup that is a normal subloop of $E$.

The known nontrivial (i.e. nonassociative and not of the form $U \times M$) minimal extensions of finite simple noncyclic Moufang loops are as follows:

- Moufang semidirect products $G \triangleleft V$ for a finite simple group $G$ listed in Section 5
- Outer semidirect products $M \triangleleft U$ for finite simple Moufang loops $M$ listed in Section 8
- Nonsplit central extensions $1 \to \mathbb{Z}/2\mathbb{Z} \to E \to M(q) \to 1$, where $q$ is an odd prime power or $q = 2$.

The extensions in the last case are isomorphic to SL$(A)$, where $A = \mathbb{O}(q)$ is the finite Cayley algebra if $q$ is odd, and to the exceptional double cover of $M(2)$ of order 240 if $q = 2$, see [10]. The minimality of these extensions is quite apparent from their construction. We put forward

**Conjecture 1.** Up to isomorphism, the only nontrivial minimal extensions for finite simple noncyclic Moufang loops are those given in the list above.
References

[1] R. H. Bruck, A survey of binary systems. Springer-Verlag, 1958.
[2] O. Chein, Moufang loops of small order, I, *Trans. Am. Math. Soc.* **188**, (1974), 31–51.
[3] A. Rajah, Moufang loops of odd order $pq^3$, *J. Algebra*, **235**, N 1 (2001), 66-93.
[4] S. M. Gagola III, Cyclic extensions of Moufang loops induced by semi-automorphisms, *J. Algebra Appl.*, **13** (2014), N 4, 1350128, 7 pp.
[5] A. N. Grishkov, A. V. Zavarnitsine, Groups with triality, *J. Algebra Appl.*, **5**, N 4 (2006), 441–463.
[6] A. N. Grishkov, A. V. Zavarnitsine, Abelian-by-cyclic Moufang loops, *Comm. Alg.*, **41**, N 6 (2013), 2242-2253.
[7] S. Doro, Simple Moufang loops, *Math. Proc. Camb. Phil. Soc.*, **83**, (1978), 377–392.
[8] K. A. Zhevlakov, A. M. Slin’ko, I. P. Shestakov, A. I. Shirshov, Rings that are nearly associative, Pure and Applied Mathematics, 104. Academic Press, New York-London, 1982.
[9] C. W. Curtis, I. Reiner, Representation theory of finite groups and associative algebras. Pure and Applied Mathematics, Vol. XI Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.
[10] A. N. Grishkov, A. V. Zavarnitsine, Maximal subloops of finite simple Moufang loops, *J. Algebra*, **302**, N 2 (2006), 646–677.
[11] L. J. Paige, A class of simple Moufang loops, *Proc. Amer. Math. Soc.*, **7** (1956), 471–482.
[12] J. N. Bray, D. F. Holt, C. M. Roney-Dougal, The maximal subgroups of the low-dimensional finite classical groups. London Mathematical Society Lecture Note Series, 407. Cambridge University Press, Cambridge (2013), xiv+438 pp.