On the Noether–Fano inequalities

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Abstract
I give a survey of Noether–Fano inequalities in birational geometry, starting with the original Noether inequality and up to the modern approach of Log Minimal Model program. The paper is based on my talk at the Fano conference in Torino in October 2002. This is the revised version: an erroneous reference in my paper published in the Proceedings of Fano conference is corrected.

1. The Noether inequality

1.1. The general theory of birational maps of plane has been developed by Cremona since 1863. The factorization conjecture of such maps into quadratic ones was given by Klebsh in 1869. He proved it for the maps of degree \( n \leq 8 \). M. Noether also announced the proof of this conjecture in 1869 and showed that it follows from the inequality

\[
\nu_1 + \nu_2 + \nu_3 > n
\]

for three maximal multiplicities of base points of linear system giving the birational map. He proved this inequality in 1871.

Independently Rosanes discovered this inequality and gave his own proof by induction.

However Noether’s factorization theorem does follow from this inequality very non-trivially. If there exists a quadratic map with base points satisfying (1) then its composition with the initial birational map has a lesser degree, but the existence of a required quadratic untwisting map is a hard problem. A lot of papers of different authors were devoted to it. It is accepted that the first complete proof of Noether’s theorem was obtained by Castelnuovo in 1901 and the most clear proof was obtained by Alexander in 1916 (see [Hu]).
The classical proofs of Noether’s inequality (1) immediately follow from two basic equalities

\[
\sum \nu_i^2 = n^2 - 1 \tag{2}
\]

\[
\sum \nu_i = 3n - 3 \tag{3}
\]

where \(n\) is the degree of linear system giving Cremona transformation \(\chi : \mathbb{P}^2 \to \mathbb{P}^2\) and \(\nu_i\) are the multiplicities of base points including infinitely close points of this linear system.

1.2. The birational and geometrical sense of Noether’s inequality was understood later and it is close to the question of birational invariance of adjunction termination. By modern proof of Noether’s theorem according to Castelnuovo (see [I–R] preface to Hudson’s book and 1.5 below) it is not necessarily to find the triple of base points with maximal multiplicities satisfying (1). It is enough to find one maximal singularity for the linear systems on \(\mathbb{P}^2\) and \(\mathbb{F}_N\) only, where \(\mathbb{F}_N = \mathbb{P}(O + O(N))\) is a standard linear surface.

1.3. Noether lemma. Let us consider the main ideas of the classical proof. Let \(F\) be a surface \(\mathbb{P}^2\) or \(\mathbb{F}_N\) and \(\chi : F \to F' = \mathbb{P}^2\) be a birational map. Let \(\mathcal{H}'\) be a linear system of lines on \(\mathbb{P}^2\) and \(\mathcal{H} = \chi_*^{-1}(\mathcal{H}')\) be a proper transform on \(F\). Write \(H = \alpha s + \beta f\), where \(H \in \mathcal{H}\) is a general divisor, \(\alpha, \beta \in \mathbb{Z}\), \(\alpha \geq 1\), \(f\) is a fiber of ruled surface, \(s\) is an exceptional section of \(F = \mathbb{F}_N\) or \(s\) is a line on \(F = \mathbb{P}^2\). It is better to write it in another basis \(\{-K_F, f\}\) of the space \(\text{Pic}(F) \otimes \mathbb{Q}\) : \(H' \sim Q - aK_F + bf\), where \(a = \alpha/3\), \(b = 0\) for \(F = \mathbb{P}^2\), and \(a = \alpha/2\), \(b = \beta - \frac{N+2}{2} \alpha\) for \(F = \mathbb{F}_N\).

Then, if \(\chi\) is not isomorphism then

(1) \(\mathcal{H}\) has a base point \(P\) with a multiplicity \(\nu > a\); or

(2) \(F = \mathbb{F}_N\) and \(b < 0\).

1.4. Remark. In the notations of Log-MMP both these statements are equivalent to the following:

: if \(K_F + \frac{1}{a}H\) is canonical and \(b \geq 0\) then \(\chi\) is an isomorphism.

PROOF. Consider a resolution

\[
\begin{array}{c}
F' \to F'' = \mathbb{P}^2, \\
\sigma \\
\end{array}
\]

\[
\mathcal{H}_Z = \varphi^* \mathcal{H}' = \sigma_*^{-1} \mathcal{H}, \quad K_Z + \frac{1}{a}H_Z = \sigma^*(K_F + \frac{1}{a}H) + \sum_{i}(1 - \frac{\nu_i}{a})\sigma_i^* E_i, \quad \text{where} \quad \sigma = \sigma_r \circ \ldots \circ \sigma_1.
\]

and \(E_i\) are exceptional, \(H_Z \in \mathcal{H}_Z\), \(H \in \mathcal{H}\), \(H' \in \mathcal{H}'\) are general divisors. Applying \(\varphi_*\) we get

2
\((-3 + (1/a))H' \sim K_{\mathbb{P}^2} + \frac{1}{a}H' \sim \varphi_*\sigma^*(bf) + \sum (1 - \frac{\nu_i}{a})\varphi_*\sigma_i^*E_i.\)

If \(\nu_i \leq a\) for all \(i\) and \(b \geq 0\) then \((-3 + \frac{1}{a})H'\) is an effective divisor. It can happen only if \(a = \frac{1}{3}\), i.e. \(\alpha = 1\), \(F = \mathbb{P}^2\) and \(\mathcal{H} = \mathcal{H}'\) and hence \(\chi\) is an isomorphism. \(\square\)

1.5. Noether-Castelnuovo theorem.

**Theorem.** Any birational map \(\mathbb{P}^2 \rightarrow \mathbb{P}^2\) is the composition of the following elementary transformations (links):

- A). a blow-up of a point \(\sigma^{-1}: \mathbb{P}^2 \rightarrow \mathbb{F}_1;\)
- B). an elementary transformation \(\varepsilon: \mathbb{F}_N \rightarrow \mathbb{F}_N+1;\)
- A\(^{-1}\)). a contraction of exceptional curve \(\sigma:\mathbb{F}_1 \rightarrow \mathbb{P}^2;\)
- C). a biregular involution \(\tau:\mathbb{F}_0 \rightarrow \mathbb{F}_0.\)

**Proof.** In the previous notations, if \(\chi:\mathbb{P}^2 \rightarrow \mathbb{P}^2\) is not isomorphism then by \(\text{L.3}\) a linear system \(\mathcal{H}\) has a base point (perhaps, infinitely close one) with a multiplicity \(\nu > a\). Since the multiplicities don’t increase on the resolution then such a point exists on \(\mathbb{P}^2\).

Apply link A) \(\sigma^{-1}: \mathbb{F}_1 \rightarrow \mathbb{P}^2\) with the center in this point. Let \(\mathcal{H}_1 = \sigma^{-1}_*\mathcal{H}\). Then

\[H_1 \sim \frac{-3a - \nu}{2}K_{\mathbb{F}_1} + \frac{3(\nu - a)}{2}f_1 = -a_1K_{\mathbb{F}_1} + b_1f_1.\]

where \(H_1 \subset \mathcal{H}_1\) is a general divisor. Applying the links of type B) we can untwist all maximal singularities of multiplicity > \(a_1\) and we obtain a linear system \(\mathcal{H}_N \sim -a_1K_{\mathbb{F}_N} + b_Nf\) on the surface \(\mathbb{F}_N\) for some \(N\). By \(\text{L.3}\) \(b_N < 0\). Then there are two cases \(N = 0\) or \(N = 1\) only. Indeed, \(\mathcal{H}_N\) doesn’t have the fixed components and hence it is a nef and big divisor. By index theorem, if \(H_N \cdot C = 0\) for some curve \(C\) then \(C^2 < 0\), i.e. \(C = s_N\).

Also \(0 \leq s_N \cdot H_N < -a_1K_{\mathbb{F}_N} \cdot s_N = 2 - N\), i.e. \(N \leq 1\). A divisor \(-K_{\mathbb{F}_N}\) is ample if and only if \(N = 0, 1\).

If \(N = 1\) let us apply link \(A^{-1}) \sigma: \mathbb{F} \rightarrow \mathbb{P}^2\). Then \(\mathcal{H}' = \sigma_*\mathcal{H}_1 \sim a'(-K_{\mathbb{P}^2})\) with \(a' = a_1 + \frac{1}{3}b_N = a + \frac{\alpha - \nu}{2} + \frac{1}{3}b_N < a\), since \(\nu > a\) and \(b_N < 0\).

In the case \(N = 0\) we have \(H_N \sim -a_1K + b_Nf_N = 2a_1S_0 + (2a_1 + b_0)f_0 = -(2a_1 + b_0)K_{F_0} - b_0S_0\), so the statement (1) of (1.3) is true if it is considered on another structure of \(\mathbb{P}^1\)-fibration \(\mathbb{F}_0 \rightarrow \mathbb{P}^1\), since \(\mathbb{F}_0 \not\cong \mathbb{P}^2\) and \(-b_0 > 0\). So after links \(\tau\) and \(B)\) we fall into the previous situation, but with a smaller coefficient of \(-K_F\) (see also \([\text{L.R}]\) and \([\text{Isk2}]\)). \(\square\)

1.6. Remark. In fact, just these very reasonings and not very clear notes of Sarkisov have inspired M. Reid [R] to formulate the program about decomposition into elementary links of four types of birational maps between Mori-fibred spaces \(\{\phi: X \rightarrow S\}\). He
named it Sarkisov program. In dimension 2 the existing of decomposition and finiteness of this algorithm were just shown in (1.4). However it could be seen as a particular case of the general theorem of Corti, which says the existence and finiteness of the algorithm in Sarkisov-Reid program in dimension at most three (see the following paragraph).

2. Fano inequality and Sarkisov-Reid program

2.1. Studying of birational transformation of $\mathbb{P}^3$ was started in Cayley (1869-1870), Noether (1870-1871) and Cremona (1871-1872) papers (see, for example, [Hu]). However only some particular examples were considered and no general theory (as in dimension 2) was created. In the early of twenty century there was an understanding (through the birational invariant of the termination of adjunction) that for Cremona transformation of $\mathbb{P}^3$ there is an analog of the Noether inequality: there is a maximal singularity: either a basis curve of degree more than $\frac{n}{4}$, or a point with a multiplicity more than $\frac{n}{2}$, where $n$ is degree of the linear system $\mathcal{H}$ in $\mathbb{P}^3$ which define this transformation and that the joint degree of maximal curves are $\leq 15$.

I do not know whether the existing of infinitely close maximal curves for three-dimensional Cremona transformations was analyzed, but Fano in the paper ([Fa], 1915) essentially used the analog of the Noether inequality, including the infinitely closed maximal curve (only on the first blow up of a point) for his studying of birational characteristics of three-dimensional quartic $V_4 \subset \mathbb{P}^4$ and the full intersection $V_{2,3} \subset \mathbb{P}^5$.

2.2. Classical Fano inequality, 1915.

**Lemma.** Let $V$ and $V'$ be smooth Fano threefolds, $\rho(V) = \rho(V') = 1$ where $\rho$ is Picard number. Let $\chi : V \dasharrow V'$ be a birational map, $H'$ be the positive generator in Pic($V'$), $H$ be the positive generator in Pic($V$). Let $\mathcal{M} = \chi^{-1}_*(|H'|) \subset |nH|$ be proper transform of the linear system $\mathcal{H}' = |H'|$. Then, if $\chi$ is not an isomorphism, then $\mathcal{M}$ has a maximal singularity of one of the following types:

1. curve $C \subset V$, \text{mult}_C \mathcal{M} > \frac{n}{r}$, \text{deg} $C = CH < r^2 H^3$, where $r$ ia an index of $V$, i.e. $-K_V \sim rH$;
2. point $P \in V$, \text{mult}_P \mathcal{M} > \frac{2n}{r}$;
3. infinitely close curve $B^* \subset V^*$ $\sigma \longrightarrow V$, \text{mult}_B^* \mathcal{M}_{V^*} > \frac{n}{r}$, \text{mult}_{\sigma(B^*)} \mathcal{M} > \frac{n}{r}$, where $\sigma(B^*)$ is a point on $V$.

This lemma can be proved in the similar way as lemma 1.3 did. Generalizations on smooth Fano varieties and conic bundles are made in [I-M], [Isk], [I-P], where these inequalities were named Noether-Fano inequalities.

The first generalization onto singular varieties was formulated by M. Reid within the framework of the Sarkisov-Reid program for Mori-fibred spaces with $\mathbb{Q}$-factorial varieties with terminal singularities and was proved by Corti [Co] for treefolds (see also [Ma]).
2.3. Mori fibration.

**Definition.** Let $X$ be a projective $\mathbb{Q}$-factorial variety with terminal singularities. The morphism $\varphi: X \to S$ is called Mori fibration if

1. $\dim S < \dim X$, $\varphi_* \mathcal{O}_X = \mathcal{O}_S$;
2. $\rho(X/S) = 1$;
3. $-K_X$ is $\varphi$-ample.

Properties 1)-3) mean that $\varphi: X \to S$ is an extremal contraction of a fibred type, $S$ is normal $\mathbb{Q}$-factorial variety.

2.4. Sarkisov-Reid program. This is a program (which is still hypothetical in dimension more then three) of decomposition of every birational map between Mori fibred spaces

$X \xrightarrow{\chi} X'$

\[ \varphi \quad \downarrow \quad \varphi' \]
\[ S \quad \downarrow \quad S' \]

(which is not necessarily preserve fibration structures) into the finite composition of elementary links – commutative diagrams of one of the following four types

**Type I:**

\[
\begin{array}{c}
\sigma \\
\downarrow \sigma_1 \\
X_1 \\
\varphi_1 \\
\downarrow \\
S_1 \\
\leftarrow \alpha \\
\downarrow \\
S_2 \\
\end{array}
\]

$Z \xleftarrow{\psi} X_2$

where $\varphi_1$, $\varphi_2$ are Mori fibrations, $\sigma$ is a divisorial extremal contraction, $\psi$ is a sequence of Mori flips, anti-flips and flops, $\alpha$ is a surjective map with connected fibres. (This is actually a high dimensional analog of the link of the type A) in 1.4);

**Type II:**

\[
\begin{array}{c}
\sigma_1 \\
\downarrow \\
Z_1 \\
\varphi_1 \\
\downarrow \\
S_1 \\
\end{array}
\]

\[
\begin{array}{c}
\sigma_2 \\
\downarrow \\
Z_2 \\
\varphi_2 \\
\downarrow \\
S_2 \\
\end{array}
\]

$X_1 \xrightarrow{\psi} X_2$

\[
\begin{array}{c}
\sigma_1 \\
\downarrow \\
Z_1 \\
\varphi_1 \\
\downarrow \\
S_1 \\
\end{array}
\]

\[
\begin{array}{c}
\sigma_2 \\
\downarrow \\
Z_2 \\
\varphi_2 \\
\downarrow \\
S_2 \\
\end{array}
\]

$X_1 \xrightarrow{\psi} X_2$

where $\varphi_1$, $\varphi_2$ are extremal divisorial contractions Mori, $\psi$ is a sequence of Mori flips, anti-flips and flops. (This is actually a high dimensional analog of the link of the type B) in 1.4);
This link is opposite to the one of type (I) (a higher dimensional analog of link of the type $A^{-1}$);

where $\varphi$ is a sequence of logflips, $\alpha_1$, $\alpha_2$ – surjective morphisms with connected fibres (this is actually a higher dimensional analog of link of the type C).

2.5. To construct the algorithm of decomposition we need a numerically ordered characteristic, which is decreasing during the decomposition. We can use $\deg(\chi, H')$ as such a characteristic. It is define by the map $\chi$ and previously fixed on the whole process of decomposition very ample linear system $H' = | - \mu'K_{V'} + \varphi'^*A' |$ in the following diagram

(2.1) \[ \mathcal{H} \sim - \mu K_X + \varphi^* A, \quad X - \xrightarrow{\chi} X' \quad \mathcal{H}' = | - \mu' K_{X'} + \varphi'^* A' | \]

where $\mathcal{H} = \mathcal{H}_X$ – proper transform of the linear system $\mathcal{H}'$.

By definition we have $\deg(\chi, H') = (\mu, \lambda, e)$, $\mu \in \mathbb{Q}_{>0}$, $\lambda \in \mathbb{Q}_{\geq 0}$, $e \in \mathbb{Z}_{\geq 0}$, is lexicographically ordered triple, where $\mu$ – as in the diagram 2.1.

$\lambda = \frac{1}{c}$, where $c := \max\{ t \in \mathbb{Q}_{>0} | K_X + tH_X \text{ is canonical} \}$, $H_X \in \mathcal{H}$ – is a general divisor, in other words, $c$ is canonical threshold for log-pair $(X, H_X)$.

If

\[ X \xrightarrow{\chi} X' \]
is a common resolution for $\chi$, such that $p$ is log-resolution for pair $(X, H_X)$ and

$$K_Y = p^* K_X + \sum a_k E_k,$$
$$q^* H_{X'} = p^* H_{X'} - \sum b_k E_k,$$

where $E_k$ are all $p$-exceptional divisors, then

$$\lambda = \max \left\{ \frac{b_k}{a_k} \right\}$$

The property to be canonical for the pair $(X, cH_X)$ means that inequalities for discrepancies $a(E, X, cH_X) \geq 0$ for all exceptional divisors $E$ over $X$, i.e. $(a_k - cb_k) \geq 0$ for every $k$, and $c = \min \{ \frac{a_k}{b_k} \}$, are true.

At the end, we have $e$ is number $\sharp \{ E_k \mid \lambda a_k - b_k = 0 \}$ of ”maximal singularities”.

**2.6. Remark.** In the case of smooth surfaces, as in 1.4, $\lambda$ is multiplicity of maximal singularity of $H_X$. In the dimension three and higher this connection with maximal multiplicity is not quite straightforward. It is connected with the fact that for smooth threefold discrepancies depend not only on incidences in the graph of the exceptional divisors, but also on the weight 2 or 1 which appear with exceptional divisor when we blow up point or curve.

As to singular (terminal) points, the situation is much more complicated, because $a_k, b_k \in \mathbb{Q}_{\geq 0}$ are rational numbers and decreasing of the degree $\deg(\chi, H')$, in particular the process may be infinite. However, for the algorithm to be finite one needs the break of descending sequens of lexicographically ordered triples $(\mu, \lambda, e)$.

The similar way as we have in dimension 2, the main ingredient in the process of decomposition is Noether-Fano inequality – the criterion of stopping of the decomposition.

**2.7. Noether-Fano criterion.** In the previous notations

$$\chi$$ is an isomorphism between two Mori-fibred spaces with $\mathbb{Q}$-factorial terminal singularities if $\lambda \leq \mu$ (i.e the linear system $H_X$ has no maximal singularities) and divisor $K_X + \frac{1}{\mu} H_X$ is nef.

The proof of the theorem is similar to the proof of the classical case (see [Co], [Ma]). We consider a common resolution $X \xrightarrow{p} Y \xrightarrow{q} X'$ and study intersections of divisors $(K_X + \frac{1}{\mu} H_X)$ and $(K_{X'} + \frac{1}{\mu} H_{X'})$ on $Y$ with generic curves in fibres of morphisms $\varphi' p$ and $\varphi q$ using Negativity lemma. In contrast to the previous reasoning with effective divisors (geometrical case), here we use an intersection theory (nef case) which is easier and more
convenient to work within log-MMP. In geometrical formulation the divisor $K_X + \frac{1}{\mu}H_X$ 
(when $\lambda \leq \mu$) has to be effective (or quasi-effective).

3. Generalizations

3.1. In the paper [Br-Ma] (see also [Ma]) log-variant of Sarkisov-Reid program in 
the category of $\mathbb{Q}$-factorial Kawamata log-terminal pairs $(X, B)$ with log-MMP relation 
is studied.
3.2. Log-MMP relation.

Definition. A finite number of projective log-pairs $(X_i, B_i), i = 1, \ldots, k$ with only $\mathbb{Q}$-factorial and klt singularities are said to be log-MMP related iff there exists a log-pair $(Y, B_Y)$ with nonsingular projective $Y$ and a boundary $\mathbb{Q}$-divisor $B_Y$ with only normal crossings such that all log-pairs $(X_i, B_i)$ are obtained from $(Y, B_Y)$ via log-MMP.

For klt pairs of the fibred type $(X, B) \xrightarrow{\phi} S$ and $(X', B_{X'}) \xrightarrow{\phi'} S'$ which are log-MMP related and for a birational map between them

$$ (3.1) \quad H \sim -\mu(K_X + B) + \varphi^*A, \quad (X, B) \xrightarrow{\varphi} (X', B_{X'}) \quad (X', B_{X'}) \xrightarrow{\varphi'} \quad S \quad S' $$

we can define a degree $(\mu, \lambda, e)$, where $\lambda$ is a maximal multiplicity of extremal ray for $K_X + B_X + cH_X$ on a good common log-resolution $(Y, B_Y) \rightarrow (X, B_X), (Y, B_Y) \rightarrow (X', B_{X'})$.

3.3. Noether-Fano criterion for the log Sarkisov-Reid program with klt singularities.

Theorem. In the diagram (3.1) $\chi$ is an isomorphism if $\lambda \leq \mu$ (i.e. $K_X + B_X + \frac{1}{\mu}H_X$ is canonical) and $K_X + B_X + \frac{1}{\mu}H_X$ is nef. In the geometrical variant the condition to be nef is replaced to be effective.

The most general variant of Noether-Fano inequality was proposed by Shokurov and Cheltsov which is reduced to the statement about the uniqueness of the canonical model for log-pair $(X, B)$ and birational invariance of log-kodaira dimension.

3.4. Canonical model.

Definition. A pair $(V, B_V)$ is called canonical model for $(X, B)$ if there is a birational map $\psi : X \dashrightarrow V$ such that $(V, B_V) = (V, \psi(B))$ is canonical and divisor $K_V + B_V$ is ample.

Proposition (Shokurov). If a canonical model for pair $(X, B)$ exists then it is unique.

3.5. Iitaka map and Kodaira dimension.

Definition. For pair $(X, B)$ we consider a birational map $\alpha : Y \dashrightarrow X$ such that log pair $(Y, \phi^{-1}(B)) = (Y, B_Y)$ is canonical. The rational map $\varphi = \varphi_n : (Y, B_Y) \dashrightarrow (Z, B_Z) = (\varphi_n(Y), \varphi_n(B_Y))$ define by the linear system $|n(K_Y + B_Y)|, n \gg 1$, is called Iitaka map. By the Kodaira dimension we mean the number $\kappa(X, B) := \dim(Z, B_Z)$, if $|n(K_Y + B_Y)| \neq \emptyset$ for some $n$, otherwise $\kappa(X, B) = -\infty$. 

Proposition (Shokurov). The map $\varphi$ and $\kappa(X,B)$ not depend on the birational map $\alpha$.

Such a generalisation allow us to study not only birational maps between Mori-fibred spaces but birational maps between so-called $K$-trivial bundles, in sense of [Ch1], i.e. those bundles the general fibre of which has the Kodaira dimension 0.

This was showed in [Ch2] for general smooth hypersurfaces of degree $N$ in $\mathbb{P}^N$, $N \geq 4$.

3.6.

Theorem (Ch2). Let $X = X_N \subset \mathbb{P}^N$ be a generic smooth hypersurfaces of degree $N \geq 4$. Then $X$ is not birational to fibrations, whose generic fiber has Kodaira dimension 0, except for fibrations induced by projections from $(N-2)$-dimensional linear subspace in $\mathbb{P}^N$.

Remark. In the first version of this paper [Isk3] the theorem on birational geometry of generic Fano hypersurfaces $X_N \subset \mathbb{P}^N$ was erroneously attributed to I.Cheltsov (Sec. 3.6 of my paper).

The fact that a generic Fano hypersurface $X_N \subset \mathbb{P}^N$ is birationally superrigid was proved by A.V.Pukhlikov and published in [P1].

The claim, formulated in my paper as part (2) of the Theorem of Sec. 3.6, is an immediate consequence of the superrigidity, stated in [P1, Sec. 2].

The claim, formulated in my paper as part (1) of the Theorem of Sec. 3.6, follows from the superrigidity in an elementary way.

Both statements 1) and 2) is proved for any smooth hypersurface $X_N \subset \mathbb{P}^N$ by Pukhlikov [Pu1] for $N \geq 6$.

The main idea of the proof of the theorem is as follows. Let $\mathcal{H}$ be our linear system with no fixed components, $\mathcal{H} \subset |-nK_X|$, $CS(X, \frac{1}{n}H)$ be the locus of canonical singularities for the pair $(X, \frac{1}{n}H)$, where $H \in \mathcal{H}$ is a general divisor. It was proved in [P1] that the pair $(X, \frac{1}{n}H)$ is canonical. Using this fact, Cheltsov shows that either $CS(X, \frac{1}{n}H) = \emptyset$ or $CS(X, \frac{1}{n}H) = (X \cap L)$ for some $L \cong \mathbb{P}^{N-2}$.

We have if $CS = \emptyset$ (i.e. $(X, \frac{1}{n}H)$ is terminal) then $(X, (\frac{1}{n} + \varepsilon)H)$ is still terminal for a small $0 < \varepsilon \ll 1$. Hence the divisor $K_X + (\frac{1}{n} + \varepsilon)H \sim -\varepsilon K_X$ ia ample. The uniqueness of the canonical model implies that $\mathcal{H}$ defines an isomorphism.

If $CS \neq \emptyset$, then the Kodaira dimension $\kappa(X,H) = 1$ and $\mathcal{H}$ defines a birational map $\varphi_{\mathcal{H}} : X \dasharrow \mathbb{P}^1$ which is birationally equivalent to a projection bundle $\varphi_L : X \dasharrow \mathbb{P}^1$ from a linear space $L$. So $\varphi_{\mathcal{H}}$ and $\varphi_L$ are birationally equivalent $K$-trivial rational fibrations.

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