Degree one maps between small 3-manifolds and Heegaard genus

Michel Boileau
Shicheng Wang

Abstract We prove a rigidity theorem for degree one maps between small 3-manifolds using Heegaard genus, and provide some applications and connections to Heegaard genus and Dehn surgery problems.

AMS Classification 57M50, 57N10

Keywords Degree one map, small 3-manifold, Heegaard genus

1 Introduction

All terminology not defined in this paper is standard, see [He] and [Ja].

Let $M$ and $N$ be two closed, connected, orientable 3-manifolds. Let $H$ be a (not necessarily connected) compact 3-submanifold of $N$. We say that a degree one map $f : M \to N$ is a homeomorphism outside $H$ if $f : (M, M - \text{int} f^{-1}(H), f^{-1}(H)) \to (N, N - \text{int} H, H)$ is a map between the triples such that the restriction $f| : M - \text{int} f^{-1}(H) \to N - \text{int} H$ is a homeomorphism. We say also that $f$ is a pinch and $N$ is obtained from $M$ by pinching $W = f^{-1}(H)$ onto $H$.

Let $H$ be a compact 3-manifold (not necessarily connected). We use $g(H)$ to denote the Heegaard genus of $H$, that is the minimal number of 1-handles used to build $H$.

We define $mg(H) = \max\{g(H_i), H_i \text{ runs over components of } H\}$. It is clear that $mg(H) \leq g(H)$ and $mg(H) = g(H)$ if $H$ is connected.

A path-connected subset $X$ of a connected 3-manifold is said to carry $\pi_1 M$ if the inclusion homomorphism $\pi_1 X \to \pi_1 M$ is surjective.

In this paper, any incompressible surface in a 3-manifold is 2-sided and is not the 2-sphere. A closed 3-manifold $M$ is small if it is orientable, irreducible and if it contains no incompressible surface.
It has been observed by Kneser, Haken and Waldhausen ([Ha], [Wa], see also [RW] for a quick transversality argument) that a degree one map $M \to N$ between two closed, orientable 3-manifolds is homotopic to a map which is a homeomorphism outside a handlebody corresponding to one side of a Heegaard splitting of $N$. This fact is known as “any degree one map between 3-manifolds is homotopic to a pinch”.

A main result of this paper is the following rigidity theorem.

**Theorem 1** Let $M$ and $N$ be two closed, small 3-manifolds. If there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset N$, then either:

1. There is a component $U$ of $H$ which carries $\pi_1 N$ and such that $g(U) \geq g(N)$, or
2. $M$ and $N$ are homeomorphic.

**Remark 1** Given $M$ and $N$ two non-homeomorphic small 3-manifolds, Theorem 1 implies that $N$ cannot be obtained from $M$ by a sequence of pinchings onto submanifolds of genus smaller than $g(N)$. However Theorem 1 does not hold when $M$ is not small. Below are easy examples:

- Let $f : P \# N \to N$ be a degree one map defined by pinching $P$ to a 3-ball in $N$. Then $f$ is a homeomorphism outside the 3-ball, which is genus zero and does not carry $\pi_1 N$.
- Let $k$ be a knot in a closed, orientable 3-manifold $N$ and let $F$ be a once punctured closed surface. Let $M$ be the 3-manifold obtained by gluing the boundaries of $F \times S^1$ and of $E(k)$ in such a way that $\partial F \times \{x\}$ is matched with the meridian of $k$, $x \in S^1$. Then a degree one map $f : M \to N$ pinching $F \times S^1$ to a tubular neighborhood $N(k)$ of $k$, is a homeomorphism outside a handlebody of genus 1. If $\pi_1 N$ is not cyclic or trivial, then $g(N(k)) < g(N)$ and $N(k)$ does not carry $\pi_1 N$.

The pinched part of a degree one map between closed, orientable non-homeomorphic surfaces has incompressible boundary [12]. The following straightforward corollary of Theorem 1 gives an analogous result for small 3-manifolds:

**Corollary 1** Let $M$ and $N$ be two closed, small, non-homeomorphic 3-manifolds. Let $f : M \to N$ be a degree one map and let $V \cup H = N$ be a minimal genus Heegaard splitting for $N$. Then the map $f$ can be homotoped to be a homeomorphism outside $H$ such that $f^{-1}(H)$ is $\partial$-irreducible.
Remark 2 Corollary \ref{cor:heegaard-genus} remains true for any strongly irreducible Heegaard splitting of $N$. Then the argument, using Casson-Gordon’s result \cite{CG}, is essentially the same as \cite{Le} Theorem 3.1, even if in \cite{Le} it is only proved for the case $M = S^3$ and $N$ a homotopy 3-sphere. The proof in \cite{Le} is based on his main result \cite{Le} Theorem 1.3, but one can also use a direct argument from degree one maps.

Theorem \ref{thm:degree-one-maps} follows directly from two rather technical Propositions (Proposition \ref{prop:technical-1} and Proposition \ref{prop:technical-2}). Theorem \ref{thm:degree-one-maps} and its proof lead to some results about Heegaard genus of small 3-manifolds and Dehn surgery on null-homotopic knots.

Theorem 2 Let $M$ be a closed, small 3-manifold. Let $F \subset M$ be a closed, orientable surface (not necessarily connected) which cuts $M$ into finitely many compact, connected 3-manifolds $U_1, \ldots, U_n$. Then there is a component $U_i$ which carries $\pi_1 M$ and such that $g(U_i) \geq g(M)$.

Remark 3 In general (see \cite{La}) one has only the upper bound:

$$g(M) \leq \sum_{i=1}^{n} g(U_i) - g(F).$$

Suppose that $k$ is a null-homotopic knot in a closed orientable 3-manifold $M$. Its unknotting number $u(k)$ is defined as the minimal number of self-crossing changes needed to transform it into a trivial knot contained in a 3-ball in $M$.

Theorem 3 Let $k$ be a null-homotopic knot in a closed, small 3-manifold $M$. If $u(k) < g(M)$, then every closed 3-manifolds obtained by a non-trivial Dehn surgery along $k$ is not small. In particular $k$ is determined by its complements.

This article is organized as follows.

In Section 2 we state and prove Proposition \ref{prop:technical-1} which is the first step in the proof of Theorem \ref{thm:degree-one-maps}. The second step, given by Proposition \ref{prop:technical-2} is proved in Section 3, then Theorem \ref{thm:degree-one-maps} follows from these two propositions. Section 4 is devoted to the proof of Theorem \ref{thm:heegaard-genus} and Section 5 to the proof of Theorem \ref{thm:dehn-surgery}.

Acknowledgements We would like to thank both the referee and Professor Scharlemann for their suggestions which enhance the paper. The second author is partially supported by MSTC and NSFC.

Algebraic & Geometric Topology, Volume 5 (2005)
2 Making the preimage of $H$ $\partial$-irreducible

The first step of the proof of Theorem [1] is given by the following proposition:

**Proposition 1** Let $M$ and $N$ be two closed, connected, orientable, irreducible 3-manifolds which have the same first Betti number, but are not homeomorphic.

Suppose there is a degree one map $f_0 : M \to N$ which is a homeomorphism outside a compact irreducible 3-submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$. Then there is a degree one map $f : M \to N$ which is a homeomorphism outside an irreducible submanifold $H \subset H_0$ such that:

- $\partial H \neq \emptyset$;
- $mg(H) \leq mg(H_0)$;
- Any connected component of $f^{-1}(H)$ is either $\partial$-irreducible or a 3-ball, and there is at least one component of $f^{-1}(H)$ which is $\partial$-irreducible.

**Remark 4** Since $M$ is not homeomorphic to $N$ it is clear that at least one component of $f^{-1}(H)$ is not a 3-ball.

**Proof** In the whole proof, 3-manifolds $M$ and $N$ are supposed to meet all hypotheses given in the first paragraph of Proposition [1].

By the assumption there is a degree one map $f_0 : M \to N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$.

Let $\mathcal{H}_0$ be the set of all 3-submanifolds $H \subset H_0$ such that:

1. There is a degree one map $f : M \to N$ which is a homeomorphism outside $H$;
2. $\partial H \neq \emptyset$;
3. $mg(H) \leq mg(H_0)$;
4. $H$ is irreducible.

For an element $H \in \mathcal{H}_0$, its complexity is defined as a pair

$$c(H) = (\sigma(\partial H), \pi_0(H))$$

with the lexicographic order, and where $\sigma(\partial H)$ is the sum of the squares of the genera of the components of $\partial H$, and $\pi_0(H)$ is the number of components of $H$. 
Remark on $c(H)$  The second term of $c(H)$ is not used in this section, but will be used in the next two sections.

Clearly $\mathcal{H}_0$ is not the empty set, since by assumption $H_0 \in \mathcal{H}_0$.

A compressing disk for $\partial H$ in $H$ is a properly embedded 2-disk $(D, \partial D) \subset (H, \partial H)$ such that $\partial D = D \cap \partial H$ is an essential simple closed curve on $\partial H$ (i.e. does not bound a disk on $\partial H$). In the following we shall denote by $H \setminus \mathcal{N}(D)$ the compact 3-manifold obtained from $H$ by removing an open product neighborhood of $D$. The operation of removing such neighborhood is called splitting $H$ along $D$.

**Lemma 1**  Let $H$ be a compact orientable 3-manifold and let $(D, \partial D) \subset (H, \partial H)$ be a compressing disk. Then $mg(H_*) \leq mg(H)$, where $H_* = H \setminus \mathcal{N}(D)$ is obtained by splitting $H$ along $D$. Moreover $c(H_*) < c(H)$.

**Proof**  By Haken’s lemma for boundary-compressing disk ([BO], [CG]), a minimal genus Heegaard surface for $H$ can be isotoped to meet $D$ along a single simple closed curve. It follows that $mg(H_*) \leq mg(H)$.

Since $\partial D$ is an essential simple closed curve on $\partial H$, it is easy to see that $\sigma(\partial H_*) < \sigma(\partial H)$, therefore $c(H_*) < c(H)$. \qed

The proof of Proposition 1 follows from the following:

**Lemma 2**  Let $H \in \mathcal{H}_0$ be an element which realizes the minimal complexity, then any component of $f^{-1}(H)$ which is not a 3-ball is $\partial$-irreducible.

**Proof**  Let $W_0 \subset W = f^{-1}(H)$ be a component which is not homeomorphic to a 3-ball. Such a component exists since $M$ is not homeomorphic to $N$. To prove that $W_0$ is $\partial$-irreducible, we argue by contradiction.

If $\partial W_0$ is compressible in $W$, there is a compressing disc $(D, \partial D) \rightarrow (W, \partial W)$ whose boundary is an essential simple closed curve on $\partial W$.

Since $f : M \rightarrow N$ is a homeomorphism outside the submanifold $H \subset N$ the restriction $f| : (W, \partial W) \rightarrow (H, \partial H)$ maps $\partial W$ homeomorphically onto $\partial H$. Therefore $f(\partial D)$ is an essential simple closed curve on $\partial H$ which bounds the immersed disk $f(D)$ in $H$. By Dehn’s Lemma, $f(\partial D)$ bounds an embedded disc $D^*$ in $H$.

**Lemma 3**  By a homotopy of $f$, supported on $W = f^{-1}(H)$ and constant on $\partial W$, we can achieve that:
• \( f| : W \to H \) is a homeomorphism in a collar neighborhood of \( \partial W \cup D \),
• \( f|^{-1}(D^*) = D \cup S \), where \( S \) is a closed orientable surface.

**Proof** We define a homotopy \( F : W \times [0,1] \to H \) by the following steps:

1. \( F(x, 0) = f(x) \) for every \( x \in W \);
2. \( F(x, t) = F(x, 0) \) for every \( x \in \partial f^{-1}(H) = \partial W \) and for every \( t \in [0,1] \);
3. Then we extend \( F(x, 1) : D \times \{1\} \to D^* \) by a homeomorphism.

We have defined \( F \) on \( D \times \{0\} \cup \partial D \times [0,1] \cup D \times \{1\} \) which is homeomorphic to a 2-sphere \( S^2 \). Since \( H \) is irreducible, by the Sphere theorem \( \pi_2(H) = \{0\} \). Hence:

4. We can extend \( F \) to \( D \times [0,1] \);

Now \( F \) has been defined on \( W \times \{0\} \cup \partial W \times [0,1] \cup D \times [0,1] \), which is a deformation retract of \( W \times [0,1] \), therefore:

5. We can finally extend \( F \) on \( W \times [0,1] \).

After this homotopy we may assume that \( f(x) = F(x, 1) \), for every \( x \in W \). Then by construction this new \( f \) sends \( \partial W \cup D \) homeomorphically to \( \partial H \cup D^* \).

By transversality, we may further assume that \( f| : W \to H \) is a homeomorphism in a collar neighborhood of \( \partial W \cup D \) and that \( f|^{-1}(D^*) = D \cup S \), where \( S \) is a closed surface.

The following lemma will be useful:

**Lemma 4** Suppose \( f : M \to N \) is a degree one map between two closed orientable 3-manifolds with the same first Betti number \( \beta_1(M) = \beta_1(N) \). Then \( f_* : H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z}) \) is an isomorphism.

**Proof** Since \( f : M \to N \) is a degree one map, by [Br, Theorem I.2.5], there is a homomorphism \( \mu : H_2(N;\mathbb{Z}) \to H_2(M;\mathbb{Z}) \) such that \( f_* \circ \mu : H_2(N;\mathbb{Z}) \to H_2(N;\mathbb{Z}) \) is the identity, where \( f_* : H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z}) \) is the homomorphism induced by \( f \).

In particular \( f_* : H_2(M;\mathbb{Z}) \to H_2(N;\mathbb{Z}) \) is surjective. Then the injectivity follows from the fact that \( H_2(M;\mathbb{Z}) \) and \( H_2(N;\mathbb{Z}) \) are torsion free abelian groups with the same finite rank \( \beta_2(M) = \beta_1(M) = \beta_1(N) = \beta_2(M) \).
Since the degree one map \( f : M \to N \) is a homeomorphism outside \( H \), the Mayer-Vietoris sequence and Lemma 1 imply that \( f_* : H_2(W; \mathbb{Z}) \to H_2(H; \mathbb{Z}) \) is an isomorphism.

Let \( S' \) be a connected component of \( S \). Since \( f(S') \subset D^* \), the homology class \( [f(S')] = f_*([S']) \) is zero in \( H_2(H, \mathbb{Z}) \). Hence the homology class \( [S'] \) is zero in \( H_2(W, \mathbb{Z}) \), because \( f_* : H_2(W, \mathbb{Z}) \to H_2(H, \mathbb{Z}) \) is an isomorphism. It follows that \( S' \) is the boundary of a compact submanifold of \( W \). Therefore \( S' \) divides \( W \) into two parts \( W_1 \) and \( W_2 \) such that \( \partial W_2 = S' \) and \( W_1 \) contains \( \partial W \cup D \).

We can define a map \( g : W \to H \) such that:

(a) \( g|_{W_1} = f|_{W_1} \) and \( g(W_2) \subset D^* \).

Then by slightly pushing the image \( g(W_2) \) to the correct side of \( D^* \), we can improve the map \( g : W \to H \) such that:

(b) \( g|_{\partial W} = f|_{\partial W} \),

(c) \( g^{-1}(D^*) = D \cup (S \setminus S') \) and \( g : \mathcal{N}(D) \to \mathcal{N}(D^*) \) is a homeomorphism.

After finitely many such steps we get a map \( h : W \to H \) such that:

(b) \( h|_{\partial W} = f|_{\partial W} \),

d) \( h^{-1}(D^*) = D \) and \( h : \mathcal{N}(D) \to \mathcal{N}(D^*) \) is a homeomorphism.

Let \( H_* = H \setminus \mathcal{N}(D) \) obtained by splitting \( H \) along \( D \). Then \( H_* \) is still an irreducible 3-submanifold of \( N \) with \( \partial H_* \neq \emptyset \).

Now \( f|_{M - \text{int}W} \) and \( h|_{W} \) together provide a degree one map \( h : M \to N \). The transformation from \( f \) to \( h \) is supported in \( W \), hence \( h \) is a homeomorphism outside the irreducible submanifold \( H_* \) of \( N \).

Since \( H_* \) is obtained by splitting \( H \) along a compressing disk, we have \( H_* \subset H_0 \) and \( H_* \) belongs to \( \mathcal{H}_0 \). Moreover \( mg(\mathit{H}_*) \leq mg(H) \) and \( c(\mathit{H}_*) < c(H) \) by Lemma 1.

This contradiction finishes the proof of Lemma 2 and thus the proof of Proposition 1.

3 Finding a closed incompressible surface in the domain

Since closed, orientable, small 3-manifolds are irreducible and have first Betti number equal to zero, Theorem 1 is a direct corollary of the following proposition:

\[ \text{Algebraic \& Geometric Topology, Volume 5 (2005)} \]
Proposition 2 Let $M$ and $N$ be two closed, connected, orientable, irreducible 3-manifolds with the same first Betti number. Suppose that there is a degree one map $f : M \rightarrow N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ such that for each connected component $U$ of $H_0$ either $g(U) < g(N)$ or $U$ does not carry $\pi_1 N$. Then either $M$ contains an incompressible orientable surface or $M$ is homeomorphic to $N$.

Let $(M, N)$ be a pair of closed orientable 3-manifolds such that there is a degree one map from $M$ to $N$. We say that condition $(\ast)$ holds for the pair $(M, N)$ if:

\[(\ast) \quad \pi_1 N = \{1\} \text{ implies } M = S^3.\]

For the proof we first assume that condition $(\ast)$ holds for the pair $(M, N)$.

Proof of Proposition 2 under condition $(\ast)$

By the assumptions, there is a degree one map $f : M \rightarrow N$ which is a homeomorphism outside an irreducible submanifold $H_0 \subset N$ with $\partial H_0 \neq \emptyset$ and such that for each connected component $U$ of $H_0$ either $g(U) < g(N)$ or $U$ does not carry $\pi_1 N$. We assume moreover that $M$ is not homeomorphic to $N$. Our goal is to show that $M$ contains an incompressible surface.

Similar to Section 2 let $\mathcal{H}$ be the set of all 3-submanifolds $H \subset N$ such that:

1. There is a degree one map $f : M \rightarrow N$ which is a homeomorphism outside $H$.
2. $\partial H$ is not empty.
3. For each component $U$ of $H$, either $g(U) < g(N)$ or $U$ does not carry $\pi_1 N$.
4. $H$ is irreducible.

The set $\mathcal{H}$ is not empty by our assumptions.

The complexity $c(H) = (\sigma(\partial H), \pi_0(H))$ for the elements of $\mathcal{H}$ is defined like in Section 2.

Lemma 5 Assume that there is a degree one map $f : M \rightarrow N$ which is a homeomorphism outside a submanifold $H \subset N$. If $H$ contains 3-ball component $B^3$, then $f$ can be homotoped to be a homeomorphism outside $H_*$, where $H_* = H - B^3$. Moreover if $H$ is irreducible, then $H_*$ is also irreducible.
Proof By our assumption, there is a degree one map $f : M \to N$ which is a homeomorphism outside a submanifold $H \subset N$ and $H$ contains a $B^3$ component. Since $f| : f^{-1}(\partial H) \to \partial H$ is a homeomorphism, then $f^{-1}(\partial B^3)$ is a 2-sphere $S^2_\ast \subset M$. Since $M$ is irreducible, $S^2_\ast$ bounds a 3-ball $B^3_\ast$ in $M$. Then either

(a) $M - \text{int } f^{-1}(B^3) = B^3_\ast$, or
(b) $f^{-1}(B^3) = B^3_\ast$.

In case (a), $N = f(B^3_\ast) \cup B^3$ is a union of two homotopy 3-balls with their boundaries identified homeomorphically, and clearly $\pi_1N = \{1\}$. So $M = S^3$ by assumption (*). Hence (b) holds in either case.

In case (b), by a homotopy of $f$ supported in $f^{-1}(B^3)$, we can achieve that $f| : f^{-1}(B^3) \to B^3$ is a homeomorphism. Then $f$ becomes a homeomorphism outside the irreducible 3-submanifold $H_\ast \subset N$, obtained from $H$ by deleting the 3-ball $B^3$.

The last sentence in Lemma 5 is obviously true.

Let $H \in \mathcal{H}$ be an element which realizes the minimal complexity. By Lemma 5 no component of $H$ is a 3-ball, hence no component of $\partial H$ is a 2-sphere since $H$ is irreducible. Therefore no component of $f^{-1}(H)$ is a 3-ball and $\partial f^{-1}(H)$ is incompressible in $f^{-1}(H)$ by the proof of Lemma 2.

Since $f : M - \text{int } f^{-1}(H) \to N - \text{int } H$ is a homeomorphism, $\partial f^{-1}(H)$ is incompressible in $M - \text{int } f^{-1}(H)$ if and only if $\partial H$ is incompressible in $N - \text{int } H$. For simplicity we will set $V = N - \text{int } H$, then $N = V \cup H$.

Then the proof of Proposition 2 under condition (*) follows from:

Lemma 6 If $\partial H$ is compressible in $V$, then there is $H_\ast \in \mathcal{H}$ such that $c(H_\ast) < c(H)$.

Proof Suppose $\partial H$ is compressible in $V$. Let $(D, \partial D) \subset (V, \partial V)$ be a compressing disc. By surgery along $D$, we get two submanifolds $H_1$ and $V_1$ as follows:

$$H_1 = H \cup \mathcal{N}(D), \quad V_1 = V \setminus \mathcal{N}(D).$$

Since $H_1$ is obtained from $H$ by adding a 2-handle, for each component $U'$ of $H_1$ there is a component $U$ of $H_0$ such that $g(U') \leq g(U)$ and $\pi_1U'$ is a quotient of $\pi_1U$, hence $H_1$ verifies the defining condition (3) of $\mathcal{H}$. Moreover $f$ is still a homeomorphism outside $H_1$ because $H_1$ contains $H$ as a subset.
Clearly $\partial H_1 \neq \emptyset$. Hence $H_1$ satisfies also the defining conditions (1) and (2) of $\mathcal{H}$. We notice that $c(H_1) < c(H)$ because $\sigma(\partial H_1) < \sigma(\partial H)$.

We will modify $H_1$ to become $H_* \in \mathcal{H}$ with $c(H_*) \leq c(H_1)$. The modification will be divided into two steps carried by Lemma 8 and Lemma 9 below. First the following standard lemma will be useful:

**Lemma 7** Suppose $U$ is a connected 3-submanifold in $N$ and let $B^3 \subset N$ be a 3-ball with $\partial B^3 = S^2$.

(i) Suppose $S^2 \subset \partial U$. If $\text{int} \, U \cap B^3 \neq \emptyset$, then $U \subset B^3$. Otherwise $U \cap B^3 = S^2$.

(ii) if $\partial U \subset B^3$, then either $U \subset B^3$, or $N - \text{int} \, U \subset B^3$.

**Proof** For (i): Suppose first $\text{int} \, U \cap B^3 \neq \emptyset$. Let $x \in \text{int} \, U \cap B^3$. Since $U$ is connected, then for any $y \in U$, there is a path $\alpha \subset U$ connecting $x$ and $y$. Since $S^2$ is a component of $\partial U$, $\alpha$ does not cross $S^2$. Hence $x \in B^3$ and $y \in B^3$, therefore $U \subset B^3$.

Now suppose $\text{int} \, U \cap B^3 = \emptyset$. Let $x \in \partial U \cap B^3$. If $x \in \text{int} \, B^3$, then there is $y \in \text{int} \, U \cap B^3$. It contradicts the assumption. So $x \in \partial B^3 = S^2$.

For (ii): Suppose that $U$ is not a subset of $B^3$, then there is a point $x \in U \cap (N - \text{int} \, B^3)$. Let $y \in N - \text{int} \, U$. If $y \in N - \text{int} \, B^3$, there is a path $\alpha$ in $N - \text{int} \, B^3$ connecting $x$ and $y$, since $N - \text{int} \, B^3$ is connected. This path $\alpha$ does not meet $\partial U$, because $\partial U \subset B^3$. This would contradict that $x \in U$ and $y \in N - \text{int} \, U$. Hence we must have $y \in B^3$, and therefore $N - \text{int} \, U \subset B^3$.  

**Lemma 8** Suppose $H_1$ meets the defining conditions (1), (2) and (3) of the set $\mathcal{H}$. Then $H_1$ can be modified to be a 3-submanifold $H_* \subset N$ such that:

(i) $\partial H_*$ contains no 2-sphere;

(ii) $c(H_*) \leq c(H_1)$;

(iii) $H_*$ still meets the the defining conditions (1) (2) (3) of $\mathcal{H}$.

**Proof** We suppose that $\partial H_1$ contains a 2-sphere component $S^2$, otherwise Lemma 8 is proved. Then $S^2$ bounds a 3-ball $B^3$ in $N$ since $N$ is irreducible. We consider two cases:

**Case (a)** $S^2$ does bound a 3-ball $B^3$ in $H_1$.

In this case $B^3$ is a component of $H_1$. By Lemma 8, $f$ can be homotoped to be a homeomorphism outside $H_2 = H_1 - B^3$. 

*Algebraic & Geometric Topology, Volume 5 (2005)*
Case (b) $S^2$ does not bound a 3-ball $B^3$ in $H_1$.

Let $H'_1$ be the component of $H_1$ such that $S^2 \subset \partial H'_1$. By Lemma 7 (i), either:

(b') $H'_1 \subset B^3$, or
(b'') $H'_1 \cap B^3 = S^2$.

In case (b'), let $H_2 = H_1 - B^3$. By Lemma 5, $f$ can be homotoped to be a homeomorphism outside $H_2$. Note $H_2 \neq \emptyset$, otherwise $M$ and $N$ are homeomorphic, which contradicts our assumption.

In case (b''), let $H_2 = H_1 \cup B^3$, then $\partial H_2$ has at least one component less than $\partial H_1$. Since we are enlarging $H_1$, $f$ is a homeomorphism outside $H_2$.

It is easy to check that in each case (a), (b'), (b'') the components of $H_2$ verify the defining condition (3) of $\mathcal{H}$ and $c(H_2) \leq c(H_1) < c(H)$. Moreover $H_2$ is not empty because $M$ and $N$ are not homeomorphic, and $\partial H_2 \neq \emptyset$ since $g(H_2) \leq g(H_1) < g(N)$. Hence each of the transformations (a), (b') and (b'') preserves properties (ii) and (iii) in the conclusion of Lemma 8. Since each one strictly reduces the number of components of $H_1$ or of $\partial H_1$, after a finite number of such transformations we reach a 3-submanifold $H_*$ of $N$ such that $H_*$ meets the properties (ii) and (iii) of Lemma 8 and $\partial H_*$ contains no 2-sphere components. This proves Lemma 8.

\textbf{Lemma 9} Suppose that $H_1$ meets conditions (i), (ii) and (iii) in the conclusion of Lemma 8. Then $H_1$ can be modified to be a 3-submanifold $H_*$ of $N$ such that:

(a) $H_*$ is irreducible;

(b) $c(H_*) \leq c(H_1)$ is not increasing;

(c) $H_*$ still meets the the defining conditions (1), (2), (3) of $\mathcal{H}$.

In particular $H_*$ belongs to $\mathcal{H}$.

\textbf{Proof} If there is an essential 2-sphere $S^2$ in $H_1$, it must separate $N$ since $N$ is irreducible. Let $H'_1$ be the component of $H_1$ containing $S^2$. The 2-sphere $S^2$ induces a connected sum decomposition of $H'_1$: it separates $H'_1$ into two connected parts $K_\circ$ and $K'_\circ$, such that:

$$H'_1 = K \#_{S^2} K' = K_\circ \cup_{S^2} K'_\circ.$$

$K_\circ \subset H_1$ (resp. $K'_\circ \subset H_1$) is homeomorphic to a once punctured $K$ (resp. a once punctured $K'$).
By Haken’s Lemma, we have:

\[ g(H'_1) = g(K) + g(K'). \]

Neither \( K_\circ \) nor \( K'_\circ \) is a \( n \)-punctured 3-sphere, \( n \geq 0 \), because \( \partial H_1 \) contains no 2-sphere component, hence:

\[ g(K) < g(H'_1) \quad \text{and} \quad g(K') < g(H'_1) \]

Since \( N \) is irreducible, \( S^2 \) bounds a 3-ball \( B^3 \) in \( N \). We may assume that \( \text{int}K_\circ \cap B^3 = \emptyset \) and \( \text{int}K'_\circ \cap B^3 \neq \emptyset \). By Lemma 7 (i), we have \( K_\circ \cap B^3 = S^2 \) and \( K'_\circ \subset B^3 \).

Moreover \( \partial H'_1 \cap B^3 \neq \emptyset \), otherwise \( K'_\circ \) is homeomorphic to \( B^3 \), in contradiction with the assumption that \( S^2 \) is a 2-sphere of connected sum.

**Lemma 10** \( \partial H'_1 \) is not a subset of \( B^3 \).

**Proof** We argue by contradiction. If \( \partial H'_1 \) is a subset of \( B^3 \), we have \( N - \text{int}H'_1 \subset B^3 \) by Lemma 7 (ii), since \( H'_1 \) is not a subset of \( B^3 \). Then:

\[ N = H'_1 \cup (N - \text{int}H'_1) = H'_1 \cup B^3 = (K_\circ \#_{S^2} K'_\circ) \cup B^3 = K_\circ \cup_{S^2} B^3 = K. \]

Hence \( K \) is homeomorphic to the whole \( N \). If \( g(H'_1) < g(N) \), this contradicts the fact that \( g(K) < g(H'_1) < g(N) \). If \( H'_1 \) does not carry \( \pi_1N \) this contradicts the fact that \( K \subset H'_1 \).

By Lemma 10 \( \partial H'_1 \) (and therefore \( \partial H_1 \)) has components disjoint from \( B^3 \).

Therefore if we replace \( H_1 \) by \( H_2 = H_1 \cup B^3 \), then \( \partial H_2 \) is not empty and it has no component which is a 2-sphere. Moreover the application of Haken’s Lemma above shows that \( g(H_2) < g(H_1) \).

Since we are enlarging \( H_1 \), \( f \) is a homeomorphism outside \( H_2 \), and clearly \( H_2 \) still meets the the defining condition (3) of \( \mathcal{H} \). Moreover \( c(H_2) \leq c(H_1) \). Hence the transformation from \( H_1 \) to \( H_2 \) preserves properties (b) and (c) in the conclusion of Lemma 8. Since it strictly reduces \( g(H_1) \), after a finite number of such transformations we will reach a 3-submanifolds \( H_* \subset N \) such that \( H_* \) meets conditions (b) and (c) in the conclusion of Lemma 8 but does not contain any essential 2-sphere. This proves Lemma 8.

**Proof** Lemma 8 and Lemma 9 imply Lemma 6. Hence we have proved Proposition 2 under condition (\( \ast \)).

*Algebraic & Geometric Topology, Volume 5 (2005)*
Proof of Proposition 2 Let $M$ and $N$ be two closed, small 3-manifolds which are not homeomorphic. Suppose there is degree one map $f : M \rightarrow N$ which is a homeomorphism outside an irreducible submanifold $H \subset N$ such that: for each component $U$ of $H$, either $g(U) < g(N)$ or $U$ does not carry $\pi_1N$.

Condition $(\ast)$ in the above proof of Proposition 2 is only used in the proof of Lemma 5, when $H$ contains a 3-ball component $B^3$ and that $M - \text{int}f^{-1}(B^3) = B_3^*$ and $f^{-1}(B^3) \neq B_3^*$. Indeed we can now prove that this case cannot occur.

If this case happens then $\pi_1N = \{1\}$ and thus $mg(H) < g(N)$, since every component of $H$ carries $\pi_1N$. By replacing $f^{-1}(B^3)$ by a 3-ball $B_3^*$, we obtain a degree one map $\tilde{f} : S^3 = B_3^* \cup B_3^* \rightarrow N$ defined by $\tilde{f}|B_* = f|B_*$ and $\tilde{f}|B_3^* \rightarrow B_3$ is a homeomorphism. Then $\tilde{f} : S^3 \rightarrow N$ is a map which is a homeomorphism outside a submanifold $H' = H - B^3$. Clearly $mg(H') = mg(H) < g(N)$. Furthermore condition $(\ast)$ now holds.

Since Proposition 2 has been proved under condition $(\ast)$, we have that $N$ must be homeomorphic to $S^3$, since $S^3$ does not contain any incompressible surface. It would follow that $mg(H) < 0$, which is impossible.

The proof of Proposition 2 and hence of Theorem H is now complete.  

4 Heegaard genus of small 3-manifolds

This section is devoted to the proof of Theorem H.

Let $M$ be a closed orientable irreducible 3-manifold. Let $F \subset M$ be a closed orientable surface (not necessary connected) which splits $M$ into finitely many compact connected 3-manifolds $U_1, \ldots, U_n$.

Let $M \setminus \mathcal{N}(F)$ be the manifold $M$ split along the surface $F$. We define the complexity of the pair $(M, F)$ as

$$ c(M, F) = \{\sigma(F), \pi_0(M \setminus \mathcal{N}(F))\}, $$

where $\sigma(F)$ is the sum of the squares of the genera of the components of $F$ and $\pi_0(M \setminus \mathcal{N}(F))$ is the number of components of $M \setminus \mathcal{N}(F)$.

Let $\mathcal{F}$ be the set of all closed surfaces $F$ such that for each component $U_i$ of $M \setminus F$, either $g(U_i) < g(M)$ or $U_i$ does not carry $\pi_1M$.

Remark 5 This condition implies that the surface $F \neq \emptyset$ for every $F \in \mathcal{F}$.  

*Algebraic & Geometric Topology, Volume 5 (2005)*
With the hypothesis of Theorem 2, the set $F$ is not empty. Let $F \in F$ be a surface realizing the minimal complexity. Then the following Lemma implies Theorem 2.

**Lemma 11** A surface $F \in F$ realizing the minimal complexity contains no 2-sphere component and is incompressible.

**Proof** The arguments are analogous to those used in the proof of Proposition 2. We argue by contradiction.

Suppose that $F$ contains a 2-sphere component $S^2$. It bounds a 3-ball $B^3 \subset M$, since $M$ is irreducible. Let $U_1$ and $U_2$ be the closures of the components of $M \setminus \mathcal{N}(F)$ which contain $S^2$. Then by Lemma 7 (i), either:

- $U_2 \subset B^3$ and $U_1 \cap B^3 = S^2$, or
- $U_1 \subset B^3$ and $U_2 \cap B^3 = S^2$.  

Since those two cases are symmetric, we may assume that we are in the first case. We consider the surface $F'$ corresponding to the decomposition \{U'_1, \ldots, U'_k\} of $M$ with $U'_1 = U_1 \cup B^3$, after forgetting all $U_i \subset B^3$ and then re-indexing the remaining $U_i$’s to be $U'_2, \ldots, U'_k$. This operation does not increase the Heegaard genus of any one of the components of the new decomposition. Moreover if $U_1$ does not carry $\pi_1M$, the same holds for $U'_1$. Hence $F'$ still belongs to $F$. However, this operation strictly decreases the number of components of $F$, hence $c(F') < c(F)$, in contradiction with our choice of $F$.

Suppose that the surface $F$ is compressible. Then some essential simple closed curve $\gamma$ on $F$ bounds an embedded disk in $M$. Let $D'$ be such a compression disk with the minimum number of circles of intersection in int$D' \cap F$. Then a subdisk of $D'$ bounded by an innermost circle of intersection is contained inside one of the $U_i$, say $U_1$.

Let $(D, \partial D) \subset (U_1, F \cap \partial U_1)$ be such an innermost disk. Let $U_2$ be adjacent to $U_1$ along $F$, such that $\partial D \subset \partial U_2$. By surgery along $D$, we get a new surface $F'$ which gives a new decomposition \{U'_1, \ldots, U'_n\} of $M$ as follows:

$$U'_1 = U_1 \setminus \mathcal{N}(D), \quad U'_2 = U_2 \cup \mathcal{N}(D), \quad U'_i = U_i, \, \text{for} \, i \geq 3.$$  

Then $g(U'_i) \leq g(U_i)$, for $i = 1, \ldots, n$. Moreover if $U_i$ does not carry $\pi_1M$, the same holds for $U'_i$. Hence $F' \in F$. However, $\sigma(F') < \sigma(F)$ since $\partial D$ is an essential circle on $F$. Therefore $c(F') < c(F)$ and we reach a contradiction.  

*Algebraic & Geometric Topology, Volume 5 (2005)*
5 Null-homotopic knot with small unknotting number

In this section we prove Theorem 3.

Suppose $M$ is a closed, small 3-manifold and $k \subset M$ is a null-homotopic knot with $u(k) < g(M)$. Then clearly $M$ is not the 3-sphere.

If $k$ is a non-trivial knot in a 3-ball $B^3 \subset M$. Then the compact 3-manifold $B^3(k, \lambda)$ obtained by any non-trivial surgery of slope $\lambda$ on $k$ will not be a 3-ball by [GL]. Therefore $M(k, \lambda)$ contains an essential 2-sphere.

Hence below we assume that $k$ is not contained in a 3-ball.

Since the knot $k \subset M$ is null-homotopic with unknotting number $u(k)$, $k$ can be obtained from a trivial knot $k' \subset B^3 \subset M$ by $u(k)$ self-crossing changes. Let $D' \subset M$ be an embedded disk bounded by $k'$. If we let $D'$ move following the self-crossing changes from $k'$ to $k$, then each self-crossing change corresponds to an identification of pairs of arcs in $D'$. Hence one obtains a singular disk $\Delta$ in $M$ with $\partial \Delta = k$ and with $u(k)$ clasp singularities. Since $\Delta$ has the homotopy type of a graph, its regular neighborhood $N(\Delta)$ is a handlebody of genus $g(N(\Delta)) = u(k) < g(M)$.

First we prove the following lemma which is a particular case of a more general result about Dehn surgeries on null-homotopic knots, obtained in [BBDM].

Since this paper is not yet available, we give here a simpler proof in this particular case.

Lemma 12 With the hypothesis above, if the slope $\alpha$ is not the meridian slope of $k$, then $M(k, \alpha)$ is not homeomorphic to $M$.

Proof Since $M$ is irreducible and $k \subset M$ is not contained in a 3-ball, $M - \text{int} N(k)$ is irreducible and $\partial$-irreducible. Hence $1 \leq u(k) < g(M)$ and $M$ cannot be a lens space.

Let consider the set $W$ of compact, connected, orientable, 3-submanifolds $W \subset M$ such that:

1. $k \subset W$ is null-homotopic in $W$;
2. there is no 2-sphere component in $\partial W$;
3. $g(W) < g(M)$.
By hypothesis the set $W$ is not empty since a regular neighborhood $\mathcal{N}(\Delta)$ of a singular unknotting disk for $k$ is a handlebody of genus $\geq 1$.

**Claim 1** For a 3-submanifold $W_0 \in W$ with a minimal complexity $c(W_0) = \sigma(\partial W_0)$, the surface $\partial W_0$ is incompressible in the exterior $M - \text{int}\mathcal{N}(k)$.

**Proof** If $\partial W_0$ is compressible in $M - \text{int} W_0$, let $(D, \partial D) \hookrightarrow (M - \text{int} W_0, \partial W_0)$ be a compression disk for $\partial W_0$. The 3-manifold $W_1 = W_0 \cup \mathcal{N}(D)$, obtained by adding a 2-andle to $W_0$, is a compact, connected submanifold of $M$ containing $k$.

Any 2-sphere in $\partial W_1$ bounds a 3-ball in $M - \text{int}\mathcal{N}(k)$ since it is irreductible. Hence after gluing some 3-ball along the boundary, we may assume that $W_1$ contains no 2-sphere component. Moreover $k \subset W_1$ is null-homotopic in $W_1$ and $g(W_1) \leq g(W_0) < g(M)$. It follows that $W_1 \in W$. Since $c(W_1) < c(W_0)$ we get a contradiction.

If $\partial W_0$ is compressible in $W_0 - \text{int}\mathcal{N}(k)$, let $(D, \partial D) \hookrightarrow (W_0 - \text{int}\mathcal{N}(k), \partial W_0)$ be a compression disk for $\partial W_0$. Let $W_2$ be the component of the 3-manifold $W_0 \setminus \mathcal{N}(D)$ which contains $k$. As above, after possibly gluing some 3-ball along the boundary, we may assume that $\partial W_2$ contains no 2-sphere component. The knot $k \subset W_2$ is null-homotopic in $W_2$, since it is null-homotopic in $W_0$ and $\pi_1 W_2$ is a factor of the free product decomposition of $W_0$ induced by the $\partial$-compression disk $D$. Moreover by Lemma 11 $g(W_2) \leq g(W_0) < g(M)$. It follows that $W_2 \in W$ and $c(W_2) < c(W_0)$. As above this contradicts the minimality of $c(W_0)$.

To finish the proof of Lemma 12 we distinguish two cases:

(a) The surface $\partial W_0$ is compressible in $W_0(k, \alpha)$ Then one can apply Scharlemann’s theorem [Sch, Thm 6.1]. The fact that $k \subset W_0$ is null-homotopic rules out cases a) and b) of Scharlemann’s theorem. Moreover by [BW, Prop.3.2] there is a degree one map $g : W_0(k, \alpha) \to W_0$, and thus there is a simple closed curve on $\partial W_0$ which is a compression curve both in $W_0(k, \alpha)$ and in $W_0$. Therefore case d) of Scharlemann’s theorem cannot occur. The remaining case c) of Scharlemann’s theorem shows that $k \subset W_0$ is a non-trivial cable of a knot $k_0 \subset W_0$ and that the surgery slope $\alpha$ corresponds to the slope of the cabling annulus. But then the manifold $M(k, \alpha)$ is the connected sum of a non-trivial Lens space with a manifold obtained by Dehn surgery along $k_0$. If $M(k, \alpha)$ is homeomorphic to the small 3-manifold $M$, then $M$ and $M(k, \alpha)$ both would be homeomorphic to a Lens space, which is impossible since $1 \leq u(k) < g(M)$. 

Algebraic & Geometric Topology, Volume 5 (2005)
(b) The surface $\partial W_0$ is incompressible in $W_0(k, \alpha)$. Since $\partial W_0$ is incompressible in $M - \mathcal{N}(k)$, it is incompressible in $M(k, \alpha)$. Therefore $M(k, \alpha)$ and $M$ cannot be homeomorphic since $M$ is a small manifold.

It follows from [BW, Prop.3.2] that there is a degree one map $f : M(k, \alpha) \to M$ which is a homeomorphism outside $\mathcal{N}(\Delta)$. Since $g(\mathcal{N}(\Delta)) = u(k) < g(M)$, Theorem 3 is a consequence of Theorem 1 and Lemma 12.

References

[BBDM] M Boileau, S Boyer, M Domergue, Y Mathieu, Killing slopes, in preparation

[BW] M Boileau, S Wang, Non-zero degree maps and surface bundles over $S^1$, J. Differential Geom. 43 (1996) 789–806 [MR1412685]

[BO] F Bonahon, J-P Otal, Scindements de Heegaard des espaces lenticulaires, Ann. Sci. École Norm. Sup. (4) 16 (1983) 451–466 (1984) [MR740078]

[Br] W Browder, Surgery on simply-connected manifolds, Springer-Verlag, New York (1972) [MR0358813]

[CG] A J Casson, C M Gordon, Reducing Heegaard splittings, Topology Appl. 27 (1987) 275–283 [MR918537]

[Ed] A L Edmonds, Deformation of maps to branched coverings in dimension two, Ann. of Math. (2) 110 (1979) 113–125 [MR541331]

[GL] C McA Gordon, J Luecke, Reducible manifolds and Dehn surgery, Topology 35 (1996) 385–409 [MR1380506]

[Ha] W Haken, On homotopy 3-spheres, Illinois J. Math. 10 (1966) 159–178 [MR0219072]

[He] J Hempel, 3-Manifolds, Princeton University Press, Princeton, N. J. (1976) [MR0415619]

[Ja] W H Jaco, Lectures on three-manifold topology, CBMS Regional Conference Series in Mathematics 43, American Mathematical Society, Providence, RI (1980) [MR0565450]

[La] M Lackenby, The Heegaard genus of amalgamated 3-manifolds, Geom. Dedicata 109 (2004) 139–145 [MR2113191]

[Le] F Lei, Complete systems of surfaces in 3-manifolds, Math. Proc. Cambridge Philos. Soc. 122 (1997) 185–191 [MR1443595]

[RW] Y W Rong, S C Wang, The preimages of submanifolds, Math. Proc. Cambridge Philos. Soc. 112 (1992) 271–279 [MR1171163]
[Sch] M Scharlemann, Producing reducible 3-manifolds by surgery on a knot, Topology 29 (1990) 481–500 [MR1071370]

[Wa] F Waldhausen, On mappings of handlebodies and of Heegaard splittings, from: “Topology of Manifolds (Proc. Inst. Univ. of Georgia, Athens, Ga. 1969)”, Markham, Chicago, Ill. (1970) 205–211 [MR0271941]

Laboratoire Émile Picard, CNRS UMR 5580, Université Paul Sabatier 118 Route de Narbonne, F-31062 TOULOUSE Cedex 4, France
and
LAMA Department of Mathematics, Peking University
Beijing 100871, China

Email: boileau@picard.ups-tlse.fr, wangsc@math.pku.edu.cn

Received: 20 July 2005