TAIL BOUNDS FOR STOCHASTIC APPROXIMATION

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Abstract. Stochastic-approximation gradient methods are attractive for large-scale convex optimization because they offer inexpensive iterations. They are especially popular in data-fitting and machine-learning applications where the data arrives in a continuous stream, or it is necessary to minimize large sums of functions. It is known that by appropriately decreasing the variance of the error at each iteration, the expected rate of convergence matches that of the underlying deterministic gradient method. Conditions are given under which this happens with overwhelming probability.

Key words. stochastic approximation, sample-average approximation, incremental gradient, steepest descent, convex optimization

AMS subject classifications. 90C15, 90C25

1. Introduction. Stochastic-approximation methods for convex optimization are prized for their inexpensive iterations and applicability to large-scale problems. The convergence analyses for these methods typically rely on expectation-based metrics for gauging progress towards a solution. But because the solution path is itself stochastic, practitioners—especially those relying on ad-hoc applications of such algorithms for a limited number of iterations—may pause and question how far an observed solution path is from the optimal value. Our aim is to develop bounds on the probability of deviating too far from the deterministic solution path. This result complements existing expectation-based analyses, and can furnish useful guidance for practitioners.

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a differentiable function with a Lipschitz-continuous gradient, and \( g : \mathbb{R}^n \to (-\infty, +\infty] \) be a convex lower-semicontinuous \([20, \S 7]\) function. Consider the optimization problem

\[
\text{minimize } x \in \mathbb{R}^n \quad h(x) := f(x) + g(x).
\]

(1.1)

This formulation permits us to capture a wide variety of problems, including convex constraints (by letting \( g \) represent the indicator function over that set) and nonsmooth regularizers. We are interested in the probabilistic guarantees of the approximate proximal-gradient iteration

\[
x_{k+1} = \text{prox}_{\alpha_k} \{ x_k - \alpha_k(\nabla f(x_k) + e_k) \},
\]

(1.2)

where \( \alpha_k \) is a positive step length, \( e_k \) is a random variable, and

\[
\text{prox}_\alpha (z) := \arg \min_y \{ \alpha g(y) + \frac{1}{2} \| x - y \|^2 \}
\]

is the proximal operator \([7]\). The gradient residual \( e_k \) is meant to account for error that might be incurred in the computation of the gradient \( \nabla f(x_k) \). Such situations may arise, for example, if evaluating the exact gradient requires a costly simulation, or traversing a large data set. This iteration reduces to the classical steepest-descent method when \( g \equiv 0 \) and \( e_k \equiv 0 \).

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An application of this framework is to provide tail bounds for solving stochastic optimization problems, such as when

\[ f(x) = \mathbf{E} F(x, Z), \]

where \( Z \) is a random variable. Stochastic-approximation algorithms generally proceed by generating at each iteration \( k \) a random sample \( \{Z_1, \ldots, Z_{m_k}\} \) of size \( m_k \), which is used to compute the search direction

\[ \nabla f(x_k) + e_k = \frac{1}{m_k} \sum_{i=1}^{m_k} \nabla_x F(x_k, Z_i). \quad (1.3) \]

When \( Z \) takes on a finite number of values with uniform probability, then \( f \) is equivalent to the familiar case of sums of functions

\[ f(x) = \frac{1}{M} \sum_{i=1}^{M} f_i(x), \quad (1.4) \]

and then (1.3) reduces to

\[ \nabla f(x_k) + e_k = \frac{1}{m_k} \sum_{i \in S_k} \nabla f_i(x_k), \quad (1.5) \]

where the random sample \( S_k \subseteq \{1, \ldots, M\} \) is possibly chosen without replacement.

At one extreme is a fixed sample size \( m_k \) (equal to 1, say), which yields an inexpensive iteration but generally does not converge to a minimizer unless \( \alpha_k \to 0 \); at best it converges sublinearly to the solution. At the other extreme is the deterministic proximal-gradient method, which under certain conditions (described in §1.2) converges linearly, i.e., for all iterations \( k \), there exists a constant \( \rho < 1 \) such that

\[ \pi_k \leq \rho^k \pi_0, \quad (1.6) \]

where

\[ \pi_k := h(x_k) - \inf h(x) \]

is the gap between the current and optimal values of the function. As do Friedlander and Schmidt [8], Byrd, Chin, Nocedal, and Wu [5], and So [24], we consider methods that interpolate between these extremes by increasing the sample size at a linear rate. As argued by Byrd et al. [5, see Table 1], the increasing-sample size strategy has a better complexity rate, in terms of total gradients sampled, than if the sample size is held fixed.

Our aim in this paper is to bound the probability that the rate of convergence of the stochastic method deviates from linear-convergence rate, i.e., we provide tail bounds on

\[ \Pr(\pi_k - \rho^k \pi_0 \geq \epsilon). \quad (1.7) \]

It is straightforward to recast these results to obtain tail bounds on \( \Pr(\pi_k > \epsilon) \). In §3 we describe bounds that depend on the errors generically, and in §5 apply these results to obtain exponentially decaying tail bounds in the case where the errors decrease linearly. In §6 these results are further specialized to the case where \( f \) is given by (1.3) and (1.5), and exponential tail bounds are derived that depend on the sample size.
1.1. Assumptions and notation. We make the following blanket assumptions throughout. The solution set $S$ of (1.1) is nonempty. For all $x$ and $y$, there exist positive constants $L$ and $\tau \geq 1$ such that

\[
\|\nabla f(y) - \nabla f(x)\| \leq L \|y - x\|,
\]

\[
\min_{\bar{x} \in S} \|x - \bar{x}\| \leq \tau \|x - \text{prox}_{1/L}\{x - \frac{1}{L} \nabla f(x)\}\|.
\]

Except for the discussion in section 1.2, $\alpha_k \equiv 1/L$. Assumption (1.8a) asserts the Lipschitz continuity of the gradient of $f$. Assumption (1.8b) is a global error bound on the distance from $x$ to the solution set in terms of the residual in the optimality conditions. Local versions of this error bound are described by Tseng and Yun [27] and Luo and Tseng [15]; the bound that we use here is a global version described by So [24]. This assumption is less restrictive than strong convexity; in particular, if $g \equiv 0$ and $f$ is strongly convex with parameter $\mu$, (1.8b) holds with $\tau = L/\mu$. Moreover, this assumption holds whenever $g$ is polyhedral and one of the following holds for the function $f$: it is convex and quadratic; or $f(x) = q(Ex) + c^T x$ for any matrix $E$, vector $c$, and strongly convex function $q$; or $f(x) = \max_{y \in Y} \{\langle Ex, y \rangle - q(y)\}$ for any strongly convex function $q$ that has a Lipschitz-continuous gradient. This is described in Theorem 4 of Luo and Tseng [15], and Proposition 4 of So [24].

Let $R_k := \sum_{i=0}^{k-1} \rho^i$, where $\rho < 1$ is a constant specified in Lemma 2.1 in terms of $L$ and $\tau$. Let $\mathcal{F}_k = \sigma(e_1, e_2, \ldots, e_k)$ be the $\sigma$-algebra generated by the sequence of errors $e_i$. When the context is clear, $[z]_i$ denotes the $i$th component of a vector $z$.

1.2. Existing convergence analysis. In general, if $\liminf_k \|e_k\| \neq 0$, then we necessarily require $\alpha_k \to 0$ in (1.2) in order to ensure optimality of limit points. Combettes and Wajs [7, Theorem 3.4] show that the iteration (1.2) converges to a solution when $0 < \inf \alpha_k < \sup \alpha_k < 2/L$ and $\sum_{k \in \mathbb{N}} \|e_k\| < \infty$, and also consider other kinds of perturbations that enter into the iteration; no convergence rates are given. Schmidt, Roux, and Bach [22] link the convergence rate of the iterations (including accelerated variants) to $\mathbb{E} \|e_k\|^2$, which measures the variance in the error, and to error in the proximal-map computation. In the case in which $e_k = 0$ has zero mean and finite variance, it is known that the proximal-gradient method converges as $O(1/\sqrt{k})$; see, e.g., Lanford, Li, and Zhang [12]. Contrast these rates to those that can be obtained when $e_k \equiv 0$, and in that case the method has a rate of $O(1/k)$, and its accelerated variant has a rate of $O(1/k^2)$, which is an optimal rate; see, e.g., Nesterov [18] and Beck and Teboulle [2].

For the case where $g \equiv 0$ (and hence the proximal operator in (1.2) is simply the identity map), has been extensively studied. It is well known that if $f$ is strongly convex, deterministic steepest descent without error (i.e., $e_k \equiv 0$) and with a constant stepsize $\alpha_k = 1/L$ converges linearly with a rate constant $\rho < 1$ that depends on the condition number of $f$; see [13, section 8.6]. Bertsekas and Tsitsiklis [3] describe conditions for convergence of the iteration (1.2) when the steplengths $\alpha_k$ satisfy the conditions $\sum_{k \in \mathbb{N}} \alpha_k = \infty$ and $\sum_{k \in \mathbb{N}} \alpha_k^2 < \infty$. Bertsekas and Nedić [16] show that randomized incremental-gradient methods for (1.4), with constant steplength $\alpha_k \equiv \alpha$, converge as

$$
\mathbb{E} \pi_k \leq O(1)(\rho^k + \alpha)
$$

where $O(1)$ is a positive constant. This expression is telling because the first term on the right-hand side decreases at a linear rate, and depends on the condition number
through $\rho$; this term is also present for any deterministic first-order method with constant stepsize. For a constant stepsize, Friedlander and Schmidt [8] give non-asymptotic rates that directly depend on the rate at which the error goes to zero, and for the case where $f$ is given by (1.4), they further show the dependence of the convergence rate on the sample size.

For a non-vanishing stepsize, i.e., $\liminf_k \alpha_k > 0$, Luo and Tseng [14] show that for a decreasing error sequence that satisfies $\|e_k\| = O(\|x_{k+1} - x_k\|)$, the function values converge to the optimal value at an asymptotic linear rate.

The convergence in probability of the stochastic-approximation method was first discussed by the classic Robbins [19] paper. Bertsekas and Tsitsiklis [3] give mild conditions on $e_k$ and $f$ under which $f(x_k) \rightarrow \inf f(x)$ in probability. More recently, Nemirovski, Juditsky, Lan, and Shapiro [17] show that for decreasing steplengths $\alpha_k = O(1/k)$, these methods achieve a sublinear rate according to $E\pi_k = O(1/k)$; the iteration average has similar convergence properties, and it converges sublinearly with overwhelming probability.

2. Proximal point with error. Our point of departure is the following result, which relates the progress in the objective value to the norm of the gradient residual.

**Lemma 2.1.** After $k$ iterations of algorithm (1.2),

$$\pi_k - \rho^k \pi_0 \leq \frac{1}{\vartheta} \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2,$$

where $\rho = (40\tau^2)/(1 + 40\tau^2) \in (0, 1)$ and $\vartheta = L(1 + 40\tau^2)/(40\tau^2) > 0$.

The proof of this result is laid out in Appendix A, and follows the template laid out by Luo and Tseng [14, Theorem 3.1], modified to keep the error term $e_k$ explicit. So [24] also provides a similar derivation for the case where $g \equiv 0$, in which case it seems possible to obtain tighter constants $\rho$ and $\vartheta$. If additionally $\|e_k\| = 0$, then the result reduces to the well-known fact that steepest descent decreases the objective value linearly. We note that the constants are invariant to scalings of $h$.

**Example 2.2 (Gradient descent with independent Gaussian noise, part I).** Let $e_k \sim N(0, \sigma^2 I)$. Because $\|e_k\|^2$ is a sum of $n$ independent Gaussians, it follows a chi-squared distribution with mean $E\|e_k\|^2 = n\sigma^2$. Therefore,

$$E \pi_k - \rho^k \pi_0 \leq \frac{1}{\vartheta} \sum_{i=0}^{k-1} \rho^{k-1-i} E\|e_i\|^2 = \frac{n\sigma^2}{\vartheta} \sum_{i=0}^{k-1} \rho^{k-1-i}. \quad (2.1)$$

Take the limit inferior of both sides of (2.1), and note that $\lim_{k \rightarrow \infty} \sum_{i=0}^{k-1} \rho^{k-1-i} = 1/(1 - \rho)$. Use the values of the constants in Lemma 2.1 to obtain the bound

$$E \liminf_{k \rightarrow \infty} \pi_k \leq \liminf_{k \rightarrow \infty} E \pi_k \leq (20\tau^2/L) n\sigma^2,$$

where the first inequality follows from the application of Fatou’s Lemma [21, Ch. 4]. Hence, even though $\lim_{k \rightarrow \infty} \pi_k$ may not exist, we can still provide a lower bound on the distance to optimality that is proportional to the variance of the error term.

The following result establishes sufficient conditions under which the distance to the solution $\pi_k$ exhibits a supermartingale property. The dependence on the $\sigma$-algebra $\mathcal{F}_{k-1}$ is effectively a conditioning on the history of the algorithm.
Theorem 2.3. (Supermartingale Property). Let $\bar{x}_k$ be the projection of $x_k$ onto the solution set $S$. For algorithm (1.2),

$$
E[\pi_{k+1} | F_{k-1}] \leq \pi_k \quad \text{if} \quad E[\|e_k\|^2 | F_{k-1}] \leq 1/(10\tau)\|x_k - \bar{x}_k\|^2.
$$

Proof. Lemma A.3 gives a sufficient condition for the monotonicity of the iteration. Using that as a starting point yields

$$
\pi_{k+1} \leq \pi_k + \frac{1}{L}\|e_k\|^2 - \frac{L}{4}\|x_k - x_{k+1}\|^2 \\
\quad \quad \leq \pi_k + \frac{1}{L}\|e_k\|^2 - \frac{L}{4}\left(\frac{1}{2\tau^2}\|x_k - \bar{x}_k\|^2 - \frac{5}{8L}\|e_k\|^2\right) \\
\quad \quad \leq \pi_k + \frac{27}{32L}\|e_k\|^2 - \frac{L}{8\tau^2}\|x_k - \bar{x}_k\|^2,
$$

where (i) comes from Lemma A.2b. Taking conditional expectations on both sides:

$$
E[\pi_{k+1} | F_{k-1}] \leq E \left[ \pi_k + \frac{27}{32L}\|e_k\|^2 - \frac{1}{8\tau^2}\|x_k - \bar{x}_k\|^2 \right] \leq \pi_k + \frac{27}{32L} E[\|e_k\|^2 | F_{k-1}] - \frac{1}{8\tau^2}\|x_k - \bar{x}_k\|^2 \leq \pi_k.
$$

\[ \square \]

3. Probabilistic bounds for gradient descent with random error. An immediate consequence of Lemma 2.1 is a tail bound via Markov’s inequality:

$$
\Pr(\pi_k - \rho^k\pi_0 \geq \epsilon) \leq \Pr \left( \frac{1}{\rho} \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \geq \epsilon \right) \leq \frac{1}{\rho\epsilon} \sum_{i=0}^{k-1} \rho^{k-1-i} E[\|e_i\|^2].
$$

This inequality is too weak, however, to say anything meaningful about the confidence in our solution after a finite number of iterations. We are instead interested in Chernoff-type bounds that are exponentially decreasing in $\epsilon$, and in the parameters that control the size of the error.

The first bound (section 3.1) that we develop makes no assumption on the relation of the gradient errors between iterations, i.e., the error sequence may or may not be history dependent, and we thus refer to this as a generic error sequence. The second bound (section 3.2) makes the stronger assumption about the relationship of the errors between iterations.

3.1. Generic error sequence. Our first exponential tail bounds are defined in terms of the moment-generating function

$$
\gamma_k(\theta) := E \exp(\theta\|e_k\|^2)
$$

of the error norms $\|e_k\|^2$. We make the convention that $\gamma_k(\theta) = +\infty$ for $\theta \notin \text{dom} \gamma_k$.

\begin{theorem}[Tail bound for generic errors]

For algorithm (1.2),

$$
\Pr(\pi_k - \rho^k\pi_0 \geq \epsilon) \leq \inf_{\theta > 0} \left\{ \exp(-\theta\epsilon/R_k) \frac{k-1}{\rho} \sum_{i=0}^{k-1} \rho^{k-1-i} \gamma_i(\theta) \right\}. \quad (3.1a)
$$
\end{theorem}
If $\gamma_k \equiv \gamma$ for all $k$ (i.e., the error norms $\|e_k\|^2$ are identically distributed), then the bound simplifies to

$$\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \inf_{\theta > 0} \{ \exp(-\theta \partial \epsilon / R_k) \gamma(\theta) \}.$$  \hspace{1cm} (3.1b)

**Proof.** By the definition of $R_k$, $(\sum_{i=0}^{k-1} \rho^{k-1-i}) / R_k = 1$. Thus, for $\theta > 0$,

$$\mathbb{E} \exp \left( \theta \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \right) = \mathbb{E} \exp \left( \sum_{i=0}^{k-1} \frac{\rho^{k-1-i} R_k}{\theta R_k} \|e_i\|^2 \right) \leq \mathbb{E} \sum_{i=0}^{k-1} \frac{\rho^{k-1-i}}{R_k} \exp(\theta R_k \|e_i\|^2) \leq \frac{1}{R_k} \sum_{i=0}^{k-1} \rho^{k-1-i} \gamma_i(\theta R_k),$$

where (i) follows from the convexity of $\exp(\cdot)$, and (ii) follows from the linearity of the expectation operator and the definition of $\gamma_i$. Together with Markov’s inequality, the above implies that for all $\theta > 0$,

$$\Pr \left( \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \geq \epsilon \right) = \Pr \left( \exp \left( \theta \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \right) \geq \exp(\theta \epsilon) \right) \leq \exp(\theta \epsilon) \mathbb{E} \exp \left( \theta \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \right) \leq \frac{\exp(-\theta \epsilon)}{R_k} \sum_{i=0}^{k-1} \rho^{k-1-i} \gamma_i(\theta R_k).$$  \hspace{1cm} (3.2)

This inequality, together with Lemma 2.1, implies that for all $\theta > 0$,

$$\Pr \left( \frac{\pi_k - \rho^k \pi_0}{\theta} \geq \epsilon \right) \leq \Pr \left( \frac{1}{\theta} \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \geq \theta \epsilon \right) \leq \frac{\exp(-\theta \epsilon)}{R_k} \sum_{i=0}^{k-1} \rho^{k-1-i} \gamma_i(\theta R_k),$$

where we use the elementary fact that $\Pr(X \geq \epsilon) \leq \Pr(Y \geq \epsilon)$ if $X \leq Y$ almost surely. Redefine $\theta$ as $\theta R_k$, and take the infimum of the right-hand side over $\theta > 0$, which gives the required inequality (3.1a). The simplified bound (3.1b) follows directly from the definition of $R_k$. \[\square\]

When the errors are identically distributed, there is an intriguing connection between the tail bounds described in Theorem 3.1 and the convex conjugate of the cumulant-generating function of that distribution, i.e., $(\log \circ \gamma)^*$.  

**Corollary 3.2** (Tail bound for identically-distributed errors). Suppose that
the error norms $\|e_k\|^2$ are identically distributed. Then for algorithm (1.2),

$$\log \Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq -[\log \gamma(\cdot)]^* (\vartheta \epsilon/R_k).$$

**Proof.** Take the log of both sides of (3.1b) to get

$$\log \Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \log \inf_{\theta > 0} \{\exp(-\theta \vartheta \epsilon/R_k) \gamma(\theta)\},$$

which we recognize as the negative of the conjugate of $\log \circ \gamma$ evaluated at $\vartheta \epsilon/R_k$. □

Note that these bounds are invariant with regard to scaling, in the sense that if the objective function $f$ is scaled by some $\alpha > 0$, then the bounds hold for $\alpha \epsilon$.

The following example illustrates an application of this tail bound to the case in which the errors follow a simple distribution with a known moment-generating function.

**Example 3.3** (Gradient descent with independent Gaussian noise, part II). As in Example 2.2, let $e_k \sim N(0, \sigma^2 I)$. Then $\|e_k\|^2$ is a scaled chi-squared distribution with moment-generating function

$$\gamma_k(\theta) = (1 + 2\sigma^2 \theta)^{-n/2}, \ \theta \in [0, \frac{1}{2\sigma^2}],$$

Note that

$$[\log \gamma(\cdot)]^*(\mu) = \frac{\mu - n\sigma^2}{2\sigma^2} + \frac{n}{2} \log(n\sigma^2/\mu) \text{ for } \mu > n\sigma^2.$$

We can then apply Corollary 3.2 to this case to deduce the bound

$$\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \left(\frac{\exp(1)}{n} \cdot \frac{\vartheta \epsilon}{\sigma^2 R_k}\right)^{n/2} \exp\left(-\frac{\vartheta \epsilon}{2\sigma^2 R_k}\right) \text{ for } \epsilon > \frac{n\sigma^2 R_k}{\vartheta}.$$

The bound can be further simplified by introducing an additional perturbation $\delta > 0$ that increases the base of the exponent:

$$\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) = \mathcal{O}\left[\exp\left(-\delta \vartheta \frac{\epsilon}{2\sigma^2 R_k}\right)\right] \text{ for all } \delta \in [0,1), \quad (3.3)$$

which highlights the exponential decrease of the bound in terms of $\epsilon$.

**3.2. Unconditionally bounded error sequence.** In contrast to the previous section, we now assume that there exists a deterministic bound on the conditional expectation $\mathbb{E}[\exp(\theta \|e_k\|^2) | \mathcal{F}_{k-1}]$. We say that this bound holds unconditionally because it holds irrespective of the history of the error sequence.

**Assumption 3.4.** Assume that $\mathbb{E}[\exp(\theta \|e_k\|^2) | \mathcal{F}_{k-1}]$ is finite over $[0, \sigma)$, for some $\sigma > 0$. Therefore there exists, for each $k$, a deterministic function $\bar{\gamma}_k : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ such that

$$\bar{\gamma}_k(0) = 1 \text{ and } \mathbb{E}[\exp(\theta \|e_k\|^2) | \mathcal{F}_{k-1}] \leq \bar{\gamma}_k(\theta).$$
(Thus, the bound is tight at $\theta = 0$.)

The existence of such a function in fact implies a bound on the moment-generating function of $\|e_k\|^2$. In particular,

$$
\gamma_k(\theta) := \mathbf{E} \exp(\theta \|e_k\|^2) = \mathbf{E} \left[ \mathbf{E} \left[ \exp(\theta \|e_k\|^2) \mid F_{k-1} \right] \right] \leq \mathbf{E} \tilde{\gamma}_k(\theta) = \tilde{\gamma}_k(\theta). \tag{3.4}
$$

The converse, however, is not necessarily true. To see this, consider the case in which the errors $e_1, \ldots, e_{k-1}$ are independent Bernoulli-distributed random variables, and $e_k$ is a deterministic function of all the previous errors, e.g., $\Pr(e_i = 0) = \Pr(e_i = 1) = 1/2$ for $i = 1, \ldots, k - 1$, and the error on the last iteration is completely determined by the previous errors:

$$
e_k = \begin{cases} 
1 & \text{if } e_1 = e_2 = \cdots = e_{k-1}, \\
0 & \text{otherwise.}
\end{cases}
$$

Therefore, $\Pr(e_k = 1) = (1/2)^{k-1}$ and $\Pr(e_k = 0) = 1 - (1/2)^{k-1}$, and the moment-generating function of $e_k$ is $\gamma_k(\theta) = 1 - 2^{1-k}(1 + \exp \theta)$. Then,

$$
\mathbf{E} \left[ \exp(\theta e_k^2) \mid e_1, \ldots, e_{k-1} \right] = \begin{cases} 
\exp \theta & \text{if } e_1 = e_2 = \cdots = e_{k-1}, \\
1 & \text{otherwise,}
\end{cases}
$$

whose tightest deterministic upper bound is $\tilde{\gamma}_k(\theta) = \exp \theta$. However, $\tilde{\gamma}_k(\theta) \geq \gamma_k(\theta)$ for all $\theta \geq 0$.

The following result is analogous to Theorem 3.1.

**Theorem 3.5** (Tail bounds for unconditionally bounded errors). Suppose that Assumption 3.4 holds. Then for algorithm (1.2),

$$
\Pr(\pi_k - \rho k \pi_0 \geq \epsilon) \leq \inf_{\theta > 0} \left\{ \exp(-\theta \epsilon) \prod_{i=0}^{k-1} \tilde{\gamma}_i(\theta \rho^{k-i-1}) \right\}.
$$

**Proof.** The proof follows the same outline as many martingale-type inequalities [1, 6]. We obtain the following relationships:

$$
\mathbf{E} \exp \left[ \theta \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \right]^{(i)} \leq \mathbf{E} \left[ \mathbf{E} \left[ \exp \left( \theta \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \right) \mid F_{k-2} \right] \right]
$$

$$
= \mathbf{E} \left[ \mathbf{E} \left[ \exp \left( \theta \rho \|e_{k-1}\|^2 + \theta \sum_{i=0}^{k-2} \rho^{k-1-i} \|e_i\|^2 \right) \mid F_{k-2} \right] \right]
$$

$$
= \mathbf{E} \left[ \exp \left( \theta \sum_{i=0}^{k-2} \rho^{k-1-i} \|e_i\|^2 \right) \mathbf{E} \left[ \exp \left( \theta \|e_{k-1}\|^2 \right) \mid F_{k-2} \right] \right]
$$

$$
\leq \mathbf{E} \left[ \exp \left( \theta \sum_{i=0}^{k-2} \rho^{k-1-i} \|e_i\|^2 \right) \tilde{\gamma}_{k-1}(\theta) \right]
$$

$$
\leq \prod_{i=0}^{k-1} \tilde{\gamma}_i(\theta \rho^{k-i-1}),
$$
where (i) follows from the law of total expectations, i.e., \( \mathbb{E}_Y [\mathbb{E}[X|Y]] = \mathbb{E}[X] \); (ii) follows from the observation that the random variable \( \exp(\theta \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2) \) is a deterministic function of \( e_0, \ldots, e_{k-2} \), and hence is measurable with respect to \( \mathcal{F}_{k-1} \) and can be factored out of the expectation; (iii) uses Assumption 3.4; and to obtain (iv) we simply repeat the process recursively.

Thus, we now have a bound on the moment-generating function of the discounted sum of errors \( \theta \sum_{i=0}^{k-1} \rho^{k-1-i} \|e_i\|^2 \), and we can continue by using the same approach used to derive (3.2). The remainder of the proof follows that of Theorem 3.1, except that the sums over \( i = 0, \ldots, k \) are replaced by products over that same range.

In an application where both \( \gamma_k \) and \( \tilde{\gamma}_k \) are available, it is not true in general that either of the bounds obtained in Theorems 3.1 and 3.5 are tighter than the other. When only a bound \( \tilde{\gamma}_k \) that satisfies Assumption 3.4 is available, however, (which is the case in the sampling application of section 6) we could leverage (3.4) and apply Theorem 3.1 to obtain a valid bound in terms of \( \tilde{\gamma}_k \) by simply substituting it for \( \gamma_k \). However, as shown below, in this case it is better to apply Theorem 3.5 because it yields a uniformly better bound:

\[
\Pr \left( \pi_k - \rho^k \pi_0 \geq \epsilon \right) \leq \inf_{\theta > 0} \left\{ \exp \left( -\theta \epsilon + \sum_{i=0}^{k-1} \log \tilde{\gamma}_i (\theta \rho^{k-1-i}) \right) \right\},
\]  

(3.5)

while Theorem 3.1 (with \( \gamma_k \) replaced by \( \tilde{\gamma}_k \)) gives us

\[
\Pr \left( \pi_k - \rho^k \pi_0 \geq \epsilon \right) \leq \inf_{\theta > 0} \left\{ \exp \left( -\theta \epsilon + \log \left[ \frac{1}{\rho_k} \sum_{i=0}^{k-1} \rho^{k-1-i} \tilde{\gamma}_i (\theta R_k) \right] \right) \right\},
\]  

(3.6)

where we rescale \( \theta \) by \( R_k \). A direct comparison of the two bounds show that they only differ by one term:

\[
\log \left[ \frac{1}{\rho_k} \sum_{i=0}^{k-1} \rho^{k-1-i} \tilde{\gamma}_i (\theta R_k) \right] \quad \text{vs.} \quad \sum_{i=0}^{k-1} \log \tilde{\gamma}_i (\theta \rho^{k-1-i}).
\]

Because \( R_k = \sum_{i=0}^{k-1} \rho^{k-1-i} \), the term in the log on the left is a convex combination of the functions \( \tilde{\gamma}_i \). Therefore,

\[
\log \left[ \frac{1}{\rho_k} \sum_{i=0}^{k-1} \rho^{k-1-i} \tilde{\gamma}_i (\theta R_k) \right] \overset{(i)}{\geq} \sum_{i=0}^{k-1} \rho^{k-1-i} \log \tilde{\gamma}_i (\theta R_k) \overset{(ii)}{\geq} \sum_{i=0}^{k-1} \log \tilde{\gamma}_i (\theta R_k \rho^{k-1-i}/R_k) = \sum_{i=0}^{k-1} \log \tilde{\gamma}_i (\theta \rho^{k-1-i}),
\]

where (i) is an application of Jensen’s inequality and the concavity of \( \log \), and (ii) follows from the convexity of the cumulant generating function. It is then evident that (3.5) implies (3.6).

As with Corollary 3.2, by taking logs of both sides above, a connection can be made between our bound and the infimal convolution when \( \tilde{\gamma} \) is log-concave:

\[
\log \Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \left[ \bigotimes_{i=0}^{k-1} \left[ \log \tilde{\gamma}_i (\cdot \rho^{k-1-i}) \right]^* \right] (\epsilon/R_k),
\]
where $\otimes$ denotes the infimal convolution operator.

**Example 3.6** (Gradient descent with independent Gaussian noise, part III). As in Example 3.3, let $e_k \sim \mathcal{N}(0, \sigma^2 I)$. Because the errors $e_k$ are independent, $\mathbb{E} \left[ \exp(\theta \|e_k\|^2) \mid \mathcal{F}_{k-1} \right] = \mathbb{E}(\exp(\theta \|e_k\|^2)) = \gamma_k(\theta)$, which satisfies Assumption 3.4 with $\bar{\gamma}_k(\theta) := \gamma_k(\theta)$. Apply Theorem 3.5 to obtain the bound

$$\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \inf_{\theta > 0} \left\{ \exp(-\theta \epsilon) \cdot \prod_{i=0}^{k-1} \left( 1 - 2\sigma^2 \theta \rho^{k-1-i} - \epsilon^2 / 2 \right) \right\}. \quad (3.7)$$

Apply Lemma B.2 to obtain

$$\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \left( \exp(1) \cdot \frac{n \alpha}{\sigma^2} \right) \exp \left( -\frac{\theta \epsilon}{\sigma^2} \right) \quad \text{for} \quad \epsilon > \frac{n \alpha \sigma^2}{\theta},$$

where $\alpha = 1 - (\log \rho)^{-1}$. We simplify the bound to obtain

$$\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) = O \left[ \exp \left( -\delta \cdot \frac{\theta \epsilon}{\sigma^2} \right) \right] \quad \text{for all} \quad \delta \in (0, 1); \quad (3.8)$$

cf. (3.3).

As an aside, we note that we can easily accommodate correlated noise, i.e., $e_k \sim \mathcal{N}(0, \Sigma^2)$ where $\Sigma$ is an $n \times n$ positive definite matrix. The error $\|e_k\|^2$ then has the distribution of a sum of chi-squared random variables that are weighted according to the eigenvalues $\sigma_j$ of $\Sigma$ [10]:

$$\|e_k\|^2 \sim n \sum_{j=1}^{n} \sigma_j^2 \chi_1^2,$$

and so the above tail bounds hold with $\sigma = \sigma_{\text{max}}$.

The bounds obtained in Examples 3.3 and 3.6 illustrate the relative strengths of Theorems 3.1 and 3.5. Comparing (3.3) and (3.8), we see that the asymptotic bounds only differ by a factor of $1/R_k$. Hence, for large $\epsilon$, the bound in Example 3.3 is uniformly weaker than the bound in Example 3.6. Note that this holds despite the simplification (i.e., Lemma B.2) used to simply (3.7).

**4. From tail bounds to moment-generating bounds.** Let $\mathcal{G}$ be a $\sigma$-algebra. Consider the exponential bound on the conditional probability [11, Definition 8.11] of a sequence of univariate random variables $X_i$:

$$\Pr(X_i \geq \epsilon \mid \mathcal{G}) := \mathbb{E} \left[ \mathbb{I}_{X_i \geq \epsilon} \mid \mathcal{G} \right] \leq \exp\left( -\epsilon^2 / \nu \right) \quad \text{for some} \quad \nu > 0. \quad (4.1)$$

In this section we show that this bound translates into a deterministic bound on the conditional moment-generating function

$$\mathbb{E} \left[ \exp(\theta \|X\|^2) \mid \mathcal{G} \right],$$

where $X = (X_1, X_2, \ldots, X_n)$ is an $n$-vector. The subsidiary lemmas follow standard arguments (e.g., see [4, Chapter 2]), except for the requirement to condition on $\mathcal{G}$; hence, we rederive the required results.
**Lemma 4.1** (Bounds on moments). If (4.1) holds for some $\nu > 0$, then

$$E[X_t^{2v} | G] \leq v!\nu^v$$

for all $v = 0, 1, 2, \ldots$.

**Proof.** We follow a similar argument to [4, Theorem 2.1]). Use the substitution $\epsilon^{2v} = \tau$ to obtain

$$\Pr(Y^{2v} \geq \tau | G) \leq \exp\left(-\frac{\tau}{v\nu}\right).$$

Integrate to get

$$E[Y^{2v} | G] = \int_0^\infty E[1_{Y^{2v} \geq \tau} | G] d\tau \leq \int_0^\infty \exp(-\tau/\nu) d\tau = \Gamma(1 + v)\nu^v = v!\nu^v,$$

where the first equality comes from the conditional layer-cake representation of positive random variables [25].

With this result, we can translate the bound (4.1) into a bound on the moment-generating function of $Y^2$.

**Lemma 4.2** (Bound on conditional MGF). If (4.1) holds for some $\nu > 0$, then

$$E[\exp(\theta Y^2) | G] \leq \frac{1}{1 - \theta\nu}$$

for $\theta \in [0, 1/\nu]$.

**Proof.** Using the Taylor expansion of $E[\exp(\theta Y^2) | G]$,

$$E[\exp(\theta Y^2) | G] = E\left[\sum_{i=0}^\infty \frac{\theta^i Y^{2i}}{i!} | G\right]$$

$$\leq \sum_{i=0}^\infty \frac{\theta^i E[Y^{2i} | G]}{i!}$$

$$\leq \sum_{i=0}^\infty \frac{\theta^i i!\nu^i}{i!} = \frac{1}{1 - \theta\nu}.$$

Equality ($i$) is obtained via the conditional monotone convergence theorem [28, Theorem 9.7c], which allows us to exchange limits and conditional expectations; inequality ($ii$) is obtained using Lemma 4.1.

We now generalize this last result to the case in which $X$ is a random $n$-vector.

**Theorem 4.3** (From tail bounds to moment-generating bounds). Let $X$ be a random $n$-vector for which the tail bound (4.1) holds for each $i$ for some $\nu > 0$. Then

$$E[\exp(\theta \|X\|^2) | G] \leq \frac{1}{1 - \theta\nu n}$$

for $\theta \in [0, 1/\nu n]$.

**Proof.** From Lemma 4.2,

$$E\left[\exp(\theta n [X]^2_i) | G\right] \leq \frac{1}{1 - \theta n\nu}.$$
The following inequalities hold:

$$E \left[ \exp (\theta \|X\|^2) \bigg| \mathcal{G} \right] = E \left[ \exp \left( \theta n \sum_{i=1}^{n} \frac{1}{n} [X]_i^2 \right) \bigg| \mathcal{G} \right]$$

$$\leq E \left[ \sum_{i=1}^{n} \frac{1}{n} \exp \left( \theta n [X]_i^2 \right) \bigg| \mathcal{G} \right]$$

$$= \sum_{i=1}^{n} \frac{1}{n} E \left[ \exp \left( \theta n [X]_i^2 \right) \bigg| \mathcal{G} \right]^{(ii)} \leq \frac{1}{1 - \theta n \nu},$$

where (i) follows from Jensen’s inequality and (ii) follows from (4.2).

**5. Convergence rates for linearly decreasing errors.** Section 3 describes tail bounds for (1.7) in terms of any available bound on the moment-generating function of the error $e_k$. A goal of this section is to show that an exponential tail bound on the error translates to an exponential tail bound on (1.7). Thus we consider the case where the tails on each component of $e_k$ are exponentially decreasing (cf. Hypothesis 5.1.B below). We also consider two additional conditions on the error sequence, which illustrate the exponential tail bound’s relative strength in the following hierarchy of assumptions. In section 6 we show how various sampling strategies satisfy these conditions.

**Hypothesis 5.1 (Uniform bounds).** Suppose that for each $\beta \in (0, 1)$,

$$U_k \leq \lambda \beta^k \quad (5.1)$$

for some constant $\lambda > 0$ and for all $k$. Consider the following hypotheses:

A. [Variance] $E \|e_k\|^2 \leq U_k$;

B. [Exponential Tail] $\Pr (\|e_k\| \geq \epsilon | \mathcal{F}_{k-1}) \leq \exp \left( -\epsilon^2 / U_k \right)$;

C. [Norm] $\|e_k\|^2 \leq U_k$.

These conditions are ordered in increasing strength: if (C) holds, then (B) holds by Hoeffding’s inequality (Theorem B.5), and if (B) holds, then (A) holds because the exponential bound implies a bound on the second moment, i.e.,

$$E \left[ \|e_k\|^2 | \mathcal{F}_{k-1} \right] = \int_0^\infty \Pr (\|e_k\|^2 \geq \epsilon | \mathcal{F}_{k-1}) \, d\epsilon \leq \int_0^\infty \exp \left( -\epsilon^2 / U_k \right) \, d\epsilon < \infty.$$

**5.1. Expectation-based and deterministic bounds.** Although our main goal is to derive tail bounds, it is useful to compare these against the expectation-based and deterministic bounds derived in Friedlander and Schmidt [8, Theorem 3.3]. We give here a reformulation of these results, which rely on parts A and C of Hypothesis 5.1.
Theorem 5.2 (Bound in expectation). If Hypothesis 5.1.A holds, then
\[ \mathbb{E} \pi_k - \rho^k \pi_0 = O(\max\{\beta, \rho\} + \zeta k) \text{ for all } \zeta > 0, \]
and if \( \rho \neq \beta \), then the bound holds with \( \zeta = 0 \). If Hypothesis 5.1.C holds, than this result holds verbatim, except without the expectation operator.

Proof. For \( \beta \leq \rho \), it follows from Lemma 2.1 and Hypothesis 5.1.A that
\[ \mathbb{E} \pi_k - \rho^k \pi_0 \leq \frac{1}{\vartheta} \sum_{i=0}^{k-1} \rho^{k-i-1} \mathbb{E} \|e_i\|^2 \leq \frac{\lambda \rho^{k-1}}{\vartheta} \sum_{i=0}^{k-1} (\beta/\rho)^i \leq \frac{\lambda}{\vartheta} \rho^{k-1} k. \] (5.2)
Similarly, for \( \beta > \rho \),
\[ \mathbb{E} \pi_k - \rho^k \pi_0 \leq \frac{\lambda \beta^{k-1}}{\vartheta} \sum_{i=0}^{k-1} (\rho/\beta)^i \leq \frac{\lambda}{\vartheta} \beta^{k-1} k. \] (5.3)
We summarize these last two bounds in the single expression
\[ \mathbb{E} \pi_k - \rho^k \pi_0 \leq \frac{\lambda}{\vartheta} \max\{\beta, \rho\}^{k-1} k = O(\max\{\beta, \rho\} + \zeta k) \]
for all \( \zeta > 0 \).

If \( \beta \neq \rho \), then it follows from the second inequality in (5.2) and the first inequality in (5.3), and the summation formula for geometric series, that
\[ \mathbb{E} \pi_k - \rho^k \pi_0 \leq \frac{\lambda}{\vartheta} \max\{\beta, \rho\}^{k-1} \frac{1}{|\beta - \rho|} = O(\max\{\beta, \rho\}^k). \] (5.4)

If Hypothesis 5.1.C holds, than the proof above proceeds verbatim, except that the expectation operator above can be removed. \( \square \)

5.2. Tail bounds. The next result gives exponential tail bounds in terms the iteration \( k \), and the deviation \( \epsilon \) from the linear rate of deterministic steepest descent.

Theorem 5.3 (Tail bounds). If Hypothesis 5.1.B holds, then
\[ \Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) = O\left(\exp\left[-\frac{\epsilon}{\max\{\beta, \rho\}^k} \cdot \zeta\right]\right) \text{ for some } \zeta > 0. \] (5.5)

Proof. From Theorem 4.3 the conditioned moment-generating function of \( \|e_k\|^2 \) is bounded:
\[ \mathbb{E}[\exp(\theta \|e_k\|^2) \mid \mathcal{F}_{k-1}] \leq \frac{1}{1 - \theta \vartheta_k} \text{ for } \theta \in \left[0, \frac{1}{nU_k}\right]. \] (5.6)
Define
\[ \alpha_1 = \max_k \rho^{k-i-1} nU_k \text{ and } \alpha_2 = \max\{\beta, \rho\}. \]
We can now use Theorem 3.5, where we identify \( \bar{\gamma} \) with the bound in (5.6) (and define \( \bar{\gamma}(\theta) = \infty \) outside of the required interval), to obtain the tail bound

\[
\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \inf_{\theta \in [0,1/\alpha_1]} \left\{ \frac{\epsilon}{\alpha_3} \cdot \frac{1}{\alpha_2} \cdot \exp \left( -\frac{\epsilon}{\alpha_2} \right) \right\},
\]

where (i) follows from the definition of \( \alpha_1 \), (ii) follows from (5.1), and (iii) follows from the definition of \( \alpha_2 \).

Define \( \alpha_3 = 1 + 1/\log(1/\min\{\beta/\rho, \rho/\beta\}) = 1 + 1/|\log \beta - \log \rho| \), and apply Lemma B.4 to (5.7) to obtain, for all \( \epsilon \geq \alpha_3 \alpha_2^{-1} n \lambda \theta \),

\[
\Pr \left( \pi_k - \rho^k \pi_0 \geq \epsilon \right) \leq \frac{\epsilon}{\alpha_3} \cdot \frac{1}{\alpha_2} \cdot \exp \left( -\frac{\epsilon}{\alpha_2} \right).
\]

Next, note that \( \min\{\beta/\rho, \rho/\beta\} \leq 1 \), and so from (5.7), for all \( \epsilon \geq k \alpha_2^{-1} n \lambda \theta \),

\[
\Pr \left( \pi_k - \rho^k \pi_0 \geq \epsilon \right) \leq \inf_{\theta \in [0,1/\alpha_1]} \left\{ \frac{\epsilon}{\alpha_3} \cdot \frac{1}{\alpha_2} \cdot \exp \left( -\frac{\epsilon}{\alpha_2} \right) \right\},
\]

where (i) follows from Lemma B.4. Let \( \bar{\alpha}_k := \min\{\alpha_3, k\} \). Inequalities (5.8) and (5.9) can be expressed together, for all \( \epsilon \geq \bar{\alpha}_k \alpha_2^{-1} n \lambda \theta \), as

\[
\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \left( \frac{\epsilon}{\alpha_3} \cdot \frac{1}{\alpha_2} \cdot \exp \left( -\frac{\epsilon}{\alpha_2} \right) \right). \tag{5.10}
\]

Consider the case in which \( \epsilon \to \infty \). Then

\[
\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq O \left[ \exp \left( -\delta \cdot \frac{\epsilon}{\alpha_2} \right) \right],
\]

for some positive \( \delta \) independent of \( \theta \) and \( \alpha_2 \).

Now consider the case in which \( k \to \infty \). Take the logarithm of both sides of (5.10):

\[
\log \Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq \bar{\alpha}_k \log \left( \frac{\delta}{\alpha_3 \lambda \alpha_2} \right) + \bar{\alpha}_k - \frac{\epsilon}{\alpha_2} = O \left( -\frac{\epsilon}{\alpha_2} \right).
\]

This implies (5.5). \( \square \)

**Corollary 5.4 (Overwhelming tail bounds).** Suppose that Hypothesis 5.1.B holds. Take \( k \) fixed. There exists for all \( A > 0 \) a constant \( C_A > 0 \) such that

\[
\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq C_A \epsilon^{-A}.
\]
Take $\epsilon$ fixed. There exists a constant $C_A > 0$ such that for all $A > 0$,

$$\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) \leq C_A A^{-k}.$$  

**Proof.** Because the required result follows from Theorem 5.3, we can pick up from the proof of that result. In particular, the right-hand side of (5.10) can be equivalently expressed in two ways as

$$
\left( \frac{\exp(1)}{\bar{\alpha}_k} \cdot \frac{\vartheta \epsilon}{n\lambda \alpha_2^k} \right)^{\bar{\alpha}_k} \exp \left( - \frac{\vartheta \epsilon}{n\lambda \alpha_2^k} \right) = \begin{cases} \mathcal{O}(1) \cdot \epsilon^{\bar{\alpha}_k} \exp(-\epsilon \cdot \mathcal{O}(1)) \\
\exp(\phi_1(k)) \exp(-\exp(\phi_2(k))) \end{cases},
$$

where

$$\phi_1(k) := \bar{\alpha}_k \log(\vartheta \epsilon \alpha_2/\bar{\alpha}_k \lambda) + \bar{\alpha}_k - k \bar{\alpha}_k \log \alpha_2$$

and

$$\phi_2(k) := \log(\vartheta \epsilon \alpha_2/\lambda) - k \log \alpha_2,$$

and the notation $\mathcal{O}(1)$ stands for positive constants. The result then follows from Lemma B.1. $\Box$

6. **Stochastic and sample average approximations.** The results of section 5 are agnostic to the source of the gradient errors that are made at each iteration. We translate these generic results into a sampling policies that yields a linear convergence rate, both in expectation and with overwhelming probability.

---

**Fig. 6.1.** An illustration of the bounds derived in Theorem 6.2; this figure plots the non-asymptotic bound shown in (5.10). The thick black line (bottom left) shows the bound in expectation (see Part 1 of Theorem 6.2). For comparison, the thick red line (top right) shows the deterministic bound on the distance to the solution (see [8, Theorem 3.1]). The thin lines in between give the bounds on $\pi_k - \rho^k \pi_0$ that correspond to probabilities $10^{-i}$ for $i = 10, 20, \ldots, 100$. Assume $M = 300$, $\beta = 0.9$, and $\rho = 0.9$. 

---
**Theorem 6.1** (Stochastic-approximation convergence rates). Consider the stochastic-approximation algorithm described by (1.2) and (1.3) where

\[
\frac{1}{m_k} \leq \lambda \beta^k
\]

for all \( k \) for some \( \beta \in (0,1) \) and \( \lambda > 0 \). Then the following hold.

1. **[Expectation bound]** If the variance of the error is bounded, i.e.,

   \[
   \sup_x \mathbb{E} \left\| \nabla f(x) - \nabla F(x, Z) \right\|^2 < \infty,
   \]

   then

   \[
   \mathbb{E} \pi_k - \rho^k \pi_0 = \mathcal{O}(\max\{\beta, \rho\}^k \zeta^k) \quad \text{for all} \quad \zeta > 0.
   \]

   If \( \rho \neq \beta \), then the bound holds with \( \zeta = 0 \).

2. **[Tail bound]** If the diameter of the error is bounded, i.e.,

   \[
   \sup_x \left\{ \sup_{z \in \Omega} [\nabla F(x, z)]_i - \inf_{z \in \Omega} [\nabla F(x, z)]_i \right\} < \infty,
   \]

   for all \( i = 1, \ldots, n \), and \( \Omega \) is the sample space, then

   \[
   \Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) = \mathcal{O}\left( \exp\left[ -\epsilon \max\{\beta, \rho\}^k \zeta \right] \right) \quad \text{for some} \quad \zeta > 0.
   \]

**Proof.**

**Part 1 (Expectation Bound).** Because the random variables \( Z_1, \ldots, Z_{m_k} \) are independent copies of \( Z \), the expected sample error is equal to the sample average. Thus,

\[
\mathbb{E} \left\| e_k \right\|^2 = \frac{1}{m_k^2} \mathbb{E} \left\| \sum_{i=1}^{m_k} [\nabla f(x_k) - \nabla F(x_k, Z_i)] \right\|^2 = \mathbb{E} \left\| \nabla f(x_k) - \nabla F(x_k, Z) \right\|^2 / m_k
\]

\[
\leq \sup_x \mathbb{E} \left\| \nabla f(x) - \nabla F(x, Z) \right\|^2 / m_k \leq \lambda \beta^k,
\]

therefore satisfying Hypothesis 5.1.A and thus the hypothesis of Theorem 5.2.

**Part 2 (Tail Bound).** This follows from Hoeffding’s Inequality; see Theorem B.5. Thus we satisfy Hypothesis 5.1.B and therefore the hypothesis of Theorem 5.3. \( \square \)

**Theorem 6.2** (Sample average gradient convergence rates). Consider the algorithm described by (1.2) and (1.5) where

\[
\frac{1}{m_k} \left( 1 - \frac{m_k - 1}{M} \right) \leq \lambda \beta^k
\]

(6.1)

for all \( k \) for some \( \beta \in (0,1) \) and \( \lambda > 0 \). Then the following hold.
1. [Expectation bound] If the population variance is bounded, i.e.,

\[
\sup_x \frac{1}{M} \left( \sum_{i=1}^{M} \| f(x) - \nabla f_i(x) \|^2 \right) < \infty,
\]

then

\[
\mathbb{E} \pi_k - \rho^k \pi_0 = O \left( [\max\{\beta, \rho\} + \zeta]^k \right) \text{ for all } \zeta > 0.
\]

If \( \rho \neq \beta \), then the bound holds with \( \zeta = 0 \).

2. [Tail bound] If the population diameter is bounded, i.e.,

\[
\sup_x \left\{ \max_j \left[ \nabla f_j(x) \right]_i - \min_j \left[ \nabla f_j(x) \right]_i \right\} < \infty,
\]

for all \( i = 1, \ldots, n \), then

\[
\Pr(\pi_k - \rho^k \pi_0 \geq \epsilon) = O \left( \exp \left[ -\frac{\epsilon}{\max\{\beta, \rho\}^k} \cdot \zeta \right] \right) \text{ for some } \zeta > 0.
\]

3. [Deterministic bound] If the diameter of the error is bounded, i.e.,

\[
\sup_x \| f_i(x) \|^2 < \infty
\]

for all \( i = 1, \ldots, n \), then

\[
\pi_k - \rho^k \pi_0 = O \left( [\max\{\beta, \rho\} + \zeta]^k \right) \text{ for all } \zeta > 0.
\]

If \( \rho \neq \beta \), then the bound holds with \( \zeta = 0 \).

**Proof.**

**Part 1 (Expectation Bound).** Let

\[
S(x) := \frac{1}{M} \sum_{i=1}^{M} \| f(x) - \nabla f_i(x) \|^2.
\]

Then from Friedlander and Schmidt \([8, \S 3.2]\),

\[
\mathbb{E} \| e_k \|^2 = \left( 1 - \frac{m_k}{M} \right) \frac{S(x_k)}{m_k} \leq \left( 1 - \frac{m_k - 1}{M} \right) \frac{\sup_x S(x)}{m_k} \leq \lambda \beta^k,
\]

therefore satisfying Hypothesis 5.1.A and thus the hypotheses of Theorem 5.2.

**Part 2 (Tail Bound).** This follows from Serfling’s Inequality; see Theorem B.6. Thus we satisfy Hypothesis 5.1.B and therefore the hypothesis of Theorem 5.3.

**Part 3 (Deterministic Bound).** Refer to Friedlander and Schmidt \([8, \S 3.1]\). □

The asymptotic notation in the theorem statements helps us simplify the results, however non asymptotic bounds are available explicitly within the proofs. Figure 6.1 illustrates the non asymptotic bounds (5.4) and (5.10) that correspond to parts 1 and 2 of Theorem 6.2: the deterministic bounds follow from Friedlander and Schmidt \([8, \text{ Theorem 3.1}]\).
7. Numerical experiments. Figure 7.1 shows the results of a Monte Carlo simulation on a logistic regression problem, where

\[ f_i(x) = \log(1 + \exp[-b_i(a_i, x)]) , \]

where \( a_i \in \mathbb{R}^n \) is a vector of input features, and \( b_i \in \{-1, 1\} \) is the corresponding observation. For this problem, we generate a dataset with \( M = 100 \) pairs \((a_i, b_i)\) of random points.
Algorithm (1.2) and (1.5), where the sample size satisfies (6.1) with $\beta \approx .91$, is run 10K times on this fixed dataset. The starting point between runs is the same, and the only difference is the randomness of the sampling. Figure 7.1 summarizes the results of this experiment. As expected, the sample paths are concentrated tightly around the mean. Furthermore, the probability of deviating from the mean decays doubly-exponentially (cf. 6.2), as evidenced by the linear tail shown in the bottom panel.

**A. Proof of Lemma 2.1.** In all results of this section, we assume that the sequence $x_k$ is generated by (1.2). For this section only, we use the abbreviation $[\cdot]^+ := \text{prox}_{1/(Lg)} \{ \cdot \}$. The following result is a simple modification of the “three-point property” frequently used in the literature [26], in order to make $e_k$ explicit.

**Lemma A.1 (Three-point property with error).** For all $y \in \text{dom } g$,

$$g(y) \geq g(x_k) + \langle \nabla f(x_k) + e_k, x_k - y \rangle + \frac{L}{2} \| y - x_k \|^2.$$

**Proof.** Let $\psi_k(x) := g(x) + \langle \nabla f(x_k) + e_k, x - x_k \rangle + \frac{L}{2} \| x - x_k \|^2$. Because $\psi_k$ is strongly convex,

$$\psi_k(y) \geq \psi_k(x) + \langle q, y - x \rangle + \frac{L}{2} \| y - x \|^2$$

for all $x, y$ and all $q \in \partial \psi_k(x)$.

Choose $x = x_{k+1} := \arg \min \phi_k(x)$. Because $0 \in \partial \psi_k(x_{k+1})$, we have $\psi_k(y) \geq \psi_k(x_{k+1}) + \frac{L}{2} \| y - x_{k+1} \|^2$, which, after simplifying, yields the required result. $\Box$

**Lemma A.2.** Let $\bar{x}_k$ be the projection of $x_k$ onto $\mathcal{S}$. Then

$$\| x_k - \bar{x}_k \| \leq \tau \| x_k - x_{k+1} \| + \frac{\tau^2}{2} \| e_k \|; \quad (A.1a)$$

$$\| x_k - \bar{x}_k \|^2 \leq 2\tau^2 \| x_k - x_{k+1} \|^2 + \frac{3\tau^2}{2} \| e_k \|^2; \quad (A.1b)$$

$$\| x_{k+1} - \bar{x}_k \| \leq (1 + \tau) \| x_k - x_{k+1} \| + \frac{\tau}{2} \| e_k \|; \quad (A.1c)$$

$$\| x_{k+1} - \bar{x}_k \|^2 \leq \frac{1}{2}[2 + 5\tau + 3\tau^2] \| x_k - x_{k+1} \|^2 + \frac{1}{2\tau^2}[3\tau^2 + \tau] \| e_k \|^2. \quad (A.1d)$$

**Proof.**

**Part (A.1a).** For all $k$,

$$(i) \| x_k - \bar{x}_k \| \leq \tau \| x_k - x_{k+1} \| + \frac{\tau}{2} \| \nabla f(x_k) \|_+ \leq \tau \| x_k - x_{k+1} \| + \tau \| x_{k+1} - [x_k + \frac{1}{L} \nabla f(x_k)]_+ \| = \tau \| x_k - x_{k+1} \| + \tau \| x_k - [x_k - \frac{1}{L} \nabla f(x) + e_k]_+ \|_+ \leq \tau \| x_k - x_{k+1} \| + \frac{\tau}{2} \| e_k \|,$$

where $(i)$ follows from Assumption (1.8b) and $(ii)$ follows from the nonexpansiveness of the proximal operator.

**Part (A.1b).** Square both sides of (A.1a) and then apply the inequality

$$ab \leq \frac{a^2}{2\alpha} + \frac{ab^2}{2}, \quad \forall \alpha > 0,$$  

(A.2)
to bound the cross terms:

\[ \|x_k - \bar{x}_k\|^2 \leq \tau^2\|x_k - x_{k+1}\|^2 + (\tau/L)^2\|e_k\|^2 + (\tau^2/L)\|x_k - x_{k+1}\|\|e_k\| \]

\[ \leq (\tau^2 + \frac{\tau^2}{2L^2})\|x_k - x_{k+1}\|^2 + (\frac{\tau^2}{2L^2} + \frac{\tau^2}{2L^2})\|e_k\|^2 \quad (\forall \alpha > 0) \]

\[ \leq 2\tau^2\|x_k - x_{k+1}\|^2 + \frac{3}{4}(\tau^2/L^2)\|e_k\|^2. \]

**Part (A.1c).** Use the triangle inequality and (A.1a):

\[ \|x_{k+1} - \bar{x}_k\| \leq \|x_{k+1} - x_k\| + \|x_k - \bar{x}_k\| \leq (1 + \tau)\|x_k - x_{k+1}\| + (\tau/L)\|e_k\|. \]

**Part (A.1d).** Square both sides above, and use the same technique used in Part (A.1b) to bound the cross-terms:

\[ \|x_{k+1} - \bar{x}_k\|^2 \leq \frac{1}{2}(2 + 5\tau + 3\tau^2)\|x_k - x_{k+1}\|^2 + \frac{1}{2L^2}(3\tau^2 + \tau)\|e_k\|^2. \]

\[ \square \]

**Lemma A.3 (Sufficient decrease).** For all \( k \),

\[ \pi_{k+1} \leq \left( 1 - \frac{1}{1 + 4\tau^2} \right) \pi_k + \frac{1}{L} \cdot \frac{40\tau^2}{1 + 4\tau^2} \|e_k\|^2. \]

**Proof.** First, specialize Lemma A.1 with \( y = x_k \):

\[ g(x_{k+1}) \leq g(x_k) - \langle \nabla f(x_k) + e_k, x_{k+1} - x_k \rangle - L\|x_{k+1} - x_k\|^2. \quad (A.3) \]

Then,

\[ h(x_{k+1}) \overset{(i)}{\leq} f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2 + g(x_{k+1}) \]

\[ \overset{(ii)}{\leq} f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2}\|x_{k+1} - x_k\|^2 + g(x_k) \]

\[ - \langle \nabla f(x_k) + e_k, x_{k+1} - x_k \rangle - L\|x_{k+1} - x_k\|^2 \]

\[ = h(x_k) - \langle e_k, x_{k+1} - x_k \rangle - \frac{L}{2}\|x_k - x_{k+1}\|^2 \]

\[ \leq h(x_k) + \frac{1}{2\alpha}\|e_k\|^2 + (\frac{\alpha}{2} - \frac{L}{2})\|x_k - x_{k+1}\|^2, \]

where \((i)\) uses Assumption (1.8a) and \((ii)\) uses the (A.3). Choose \( \alpha = L/2 \) and rearrange terms to obtain the required result. \( \square \)

We now proceed with the proof of Lemma 2.1. Let \( \bar{x}_k \) be the projection of \( x_k \) onto the solution set \( S \). By the mean value theorem,

\[ f(x_{k+1}) - f(\bar{x}_k) = \langle \nabla f(x), x_{k+1} - \bar{x}_k \rangle. \quad (A.4) \]

From Lemma A.1, we have

\[ g(x_{k+1}) - g(\bar{x}_k) \leq - \langle \nabla f(x_k) + e_k, x_{k+1} - \bar{x}_k \rangle \]

\[ - \frac{L}{2}\|x_{k+1} - x_k\|^2 - \frac{L}{2}\|\bar{x}_k - x_{k+1}\|^2 + \frac{L}{2}\|\bar{x}_k - x_k\|^2 \]

\[ \leq - \langle \nabla f(x_k) + e_k, x_{k+1} - \bar{x}_k \rangle + \frac{L}{2}\|\bar{x}_k - x_k\|^2. \quad (A.5) \]
Also note that
\[
\langle \nabla f(x) - \nabla f(x_k), x_{k+1} - \bar{x}_k \rangle \leq \| \nabla f(x_k) - \nabla f(x) \| \| x_{k+1} - \bar{x}_k \|
\]
\[\leq L \| x - x_k \| \| x_{k+1} - \bar{x}_k \|\]
\[\leq L [ \| x_{k+1} - x_k \| + \| x_k - \bar{x}_k \| ] \cdot \| x_{k+1} - \bar{x}_k \|
\]
\[\leq [L(1 + \tau) \| x_k - \bar{x}_k \| + \| \epsilon_k \|] \\
\cdot [(1 + \tau) \| x_{k+1} \| + \frac{\tau}{L} \| \epsilon_k \|]
\]
\[= L(1 + \tau)^2 \| x_k - x_{k+1} \|^2 \\
+ 2[\tau(1 + \tau)] \| x_k - x_{k+1} \| \| \epsilon_k \| + \tau^2/L \| \epsilon_k \|^2
\]
\[\leq [L(1 + \tau)^2 + \frac{1}{\tau}(1 + \tau)] \| x_k - x_{k+1} \|^2 \\
+ \frac{\tau^2}{L} + \alpha \tau(1 + \tau)] \| \epsilon_k \|^2
\]
\[\leq L(1 + 3\tau + 2\tau^2) \| x_k - x_{k+1} \|^2 + \frac{1}{L} (2\tau^2 + \tau) \| \epsilon_k \|^2,
\]
where (i) follows from (1.8a). In the steps which follow, we apply the relevant inequalities in Lemma A.2, group terms, bound every cross term using (A.2), and repeat the process until we reach the final result:

\[
h(x_{k+1}) - h(\bar{x}_k) \leq \langle \nabla f(\xi) - \nabla f(x_k), x_{k+1} - \bar{x}_k \rangle - \langle e_k, x_{k+1} - \bar{x}_k \rangle + \frac{L}{2} \| \bar{x}_k - x_k \|^2
\]
\[\leq L(1 + 3\tau + 2\tau^2) \| x_k - x_{k+1} \|^2 + \frac{1}{L} (2\tau^2 + \tau) \| \epsilon_k \|^2 \\
+ \frac{\tau}{2} \| \epsilon_k \|^2 + \frac{\tau}{4\alpha} \| \bar{x}_k - x_k \|^2 + \frac{\tau}{2} \| \bar{x}_k - x_k \|^2
\]
\[\leq L(1 + 3\tau + 2\tau^2) \| x_k - x_{k+1} \|^2 + \frac{1}{L} (2\tau^2 + \tau) \| \epsilon_k \|^2 \\
+ \frac{\tau}{2} \| \epsilon_k \|^2 + \frac{\tau}{4\alpha} [2 + 5\tau + 3\tau^2] \| x_k - x_{k+1} \|^2
\]
\[+ \frac{\sqrt{L}}{4\alpha} \| \epsilon_k \|^2 + L\tau^2 \| x_k - x_{k+1} \|^2 + \frac{5L}{8} (\tau^2/L^2) \| \epsilon_k \|^2
\]
\[\leq \left( L(1 + 3\tau + 2\tau^2) + \frac{1}{4\alpha} [2 + 5\tau + 3\tau^2] + L\tau^2 \right) \| x_k - x_{k+1} \|^2 \\
+ \left( \frac{1}{2} (2\tau^2 + \tau) + \frac{\alpha}{2} + \frac{\sqrt{L}}{4\alpha} [3\tau^2 + \tau] + \frac{5L}{8} (\tau^2/L^2) \right) \| \epsilon_k \|^2
\]
\[\leq 10L\tau^2 \| x_k - x_{k+1} \|^2 + \frac{1}{L} 10\tau^2 \| \epsilon_k \|^2
\]
\[\leq 40\tau^2 \| h(x_k) - h(x_{k+1}) \| + (4/L^2 + \frac{1}{L} 10\tau^2) \| \epsilon_k \|^2
\]
\[\leq 40\tau^2 \| h(x_k) - h(x_{k+1}) \| + \frac{40\tau^2}{1 + 40\tau^2} \| \epsilon_k \|^2
\]
\[\leq (1 + 40\tau^2) h(x_{k+1}) - (1 + 40\tau^2) h(\bar{x}_k) \leq 40\tau^2 (h(x_k) - h(\bar{x}_k)) + \frac{1}{L} 40\tau^2 \| \epsilon_k \|^2,
\]
which is true if and only if the desired result holds:

\[
\pi_{k+1} \leq \left( 1 - \frac{1}{1 + 40\tau^2} \right) \pi_k + \frac{1}{L} \cdot \frac{40\tau^2}{1 + 40\tau^2} \| \epsilon_k \|^2.
\]
B. Auxiliary results.

**Lemma B.1.** Suppose that

\[
\phi_1(k) = O(k^{O(1)}) \exp(-k^{O(1)}),
\]

\[
\phi_2(k) = \exp(O(k^{O(1)})) \exp(- \exp(k^{O(1)})),
\]

where \(O(1)\) stands for positive constants. Then for each \(A > 0\) there exists a positive constant \(C_A\) such that

\[
\phi_1(k) \leq C_A k^{-A},
\]

\[
\phi_2(k) \leq C_A A^{-k}.
\] **Proof.** The statement follows by taking the logarithms on both sides of (B.1) and (B.2).

**Lemma B.2.** For \(y \in (0, 1)\) and \(x \in [0, 1]\),

\[
(1 - x)^{1-1/\log y} \leq \prod_{i=0}^{\infty} (1 - xy^i).
\] **Proof.** To prove the lower bound, we use the following fact:

\[
\ln(1 - x) \geq -\frac{x}{1 - x} \text{ for all } x \in [0, 1).
\]

Therefore,

\[
\prod_{i=1}^{\infty} (1 - xy^i) = \exp \left( \sum_{i=1}^{\infty} \log \left( 1 - xy^i \right) \right)
\]

\[
\geq \exp \left( \sum_{i=1}^{\infty} -\frac{y^i}{1/x - y^i} \right)
\]

\[
\geq \exp \left( - \int_{0}^{\infty} \frac{y^i}{1/x - y^i} \, di \right)
\]

\[
= \exp \left( - \frac{\log(1 - x)}{\log(y)} \right) \geq (1 - x)^{-1/\log y}.
\]

Thus,

\[
\prod_{i=0}^{\infty} (1 - xy^i) = (1 - x) \prod_{i=1}^{\infty} (1 - xy^i) \geq (1 - x)^{1-1/\log y},
\]

as required.

**Lemma B.3.** For \(y \in (0, 1)\) and \(x \in [0, 1]\),

\[
\exp \left( - \frac{\log(1 - x/y) - \log(1 - xy^{N+1})}{\log(y)} \right) \leq \prod_{i=0}^{N} (1 - xy^i).
\]
Proof. Similar to the proof of the previous inequality

\[
\prod_{i=1}^{N} (1 - xy^i) = \exp \left( \sum_{i=1}^{N} \log \left(1 - xy^i \right) \right) \\
\geq \exp \left( \sum_{i=1}^{N} - \frac{xy^i}{1 - xy^i} \right) \\
\geq \exp \left( - \int_{0}^{N} \frac{xy^i}{1 - xy^i} \, dt \right) \\
\geq \exp \left( - \frac{\log(1 - x) - \log(1 - xy^N)}{\log(y)} \right).
\]

Thus,

\[
\prod_{i=0}^{N} (1 - xy^i) = \prod_{i=1}^{N+1} (1 - (x/y)y^i) \\
\geq \exp \left( - \frac{\log(1 - x/y) - \log(1 - xy^{N+1})}{\log(y)} \right),
\]

as required. \( \Box \)

Lemma B.4. Let \( k > 0, \mu > 0, \) and \( \epsilon > 0. \) Then for \( y \in (0, 1) \) and \( x \in (0, 1], \)

\[
\inf_{\theta > 0} \left\{ \exp\left( -\theta \epsilon \nu \right) \prod_{i=0}^{N-1} (1 - \theta xy^i)^{-k} \right\} \leq \left( \frac{\exp(1)}{\alpha} \cdot \frac{\epsilon \nu}{x} \right)^{\alpha} \exp \left( - \frac{\epsilon \nu}{x} \right),
\]

where \( \alpha = \frac{1}{k} \left( \frac{1}{\log(1/y)} + 1 \right). \)

Proof. By inverting both sides of (B.3) we obtain the following inequality

\[
\prod_{i=0}^{\infty} (1 - xy^i)^{-k} \leq \exp \left( - \log(1 - x) \left[ \frac{1}{\log(1/y)} + 1 \right] \right).
\]

Therefore, for \( \epsilon \geq \alpha x/v, \)

\[
\inf_{\theta > 0} \left\{ \exp(-\theta \epsilon \nu) \prod_{i=0}^{N-1} (1 - \theta xy^i)^{-k} \right\}
\]

\[
\leq \inf_{\theta > 0} \left\{ \exp(-\theta \epsilon \nu) \prod_{i=0}^{\infty} (1 - \theta xy^i)^{-k} \right\}
\]

\[
\leq \inf_{\theta > 0} \left\{ \exp \left( - \frac{1}{k} \left[ \frac{1}{\log(1/y)} + 1 \right] \log (1 - \theta x) - \theta v \epsilon \right) \right\}
\]

\[
= \inf_{\theta > 0} \left\{ \exp(-\alpha \log(1 - \theta x) - \theta v \epsilon) \right\}
\]

\[
= \exp \left( -\alpha \log \left( 1 - \left( \frac{1}{x} - \frac{\alpha}{v} \right) x \right) - \left( \frac{1}{x} - \frac{\alpha}{v} \right) v \epsilon \right)
\]

\[
= \left( \frac{\exp(1)}{\alpha} \cdot \frac{\epsilon \nu}{x} \right)^{\alpha} \exp \left( - \frac{\epsilon \nu}{x} \right),
\]

as required.
where (i) follows from (B.4); and (ii) uses the substitution $\theta = 1/x - \alpha/\nu e$, which can be shown to be the optimal choice of $\theta$. Because $\theta > 0$, $\epsilon > \alpha x/\nu$.

For the remainder of this section, define the sample average

$$S_m := \frac{1}{m} \sum_{i=1}^{m} X_i$$

for a sequence of random variables \{X_1, \ldots, X_m\}.

**Theorem B.5** (Hoeffding [9, Theorem 2]). Consider independent random variables \{X_1, \ldots, X_m\}, $X_i : \Omega \to \mathbb{R}$. If the random variables are bounded, i.e.,

$$d := \sup_{\omega \in \Omega} X_i(\omega) - \inf_{\omega \in \Omega} X_i(\omega)$$

is finite, then

$$\Pr( S_m - \mathbf{E} S_m \geq \epsilon ) \leq \exp\left(-\epsilon^2 / \eta_m\right), \quad \text{where} \quad \eta_m = d^2/(2m).$$

**Theorem B.6** (Serfling [23, Corollary 1.1]). Let $x_1, \ldots, x_M$ be a population, \{X_1, \ldots, X_m\} be samples drawn without replacement from the population, and let $d := \max_i x_i - \min_i x_i$. Then

$$\Pr( S_m - \mathbf{E} S_m \geq \epsilon ) \leq \exp\left(-\epsilon^2 / \eta_m\right), \quad \text{where} \quad \eta_m = d^2 \left(\frac{1}{2m} \left(1 - \frac{m-1}{M}\right)\right).$$

Because $\eta_m$ is strictly decreasing in $m$, the Serfling bound is uniformly better than the Hoeffding bound. Note that the Serfling bound is not tight: in particular, when $M = m$ (i.e., $S_m = \mathbf{E} S_m$), the bound is not zero (except for degenerate population).

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