The BF Theory Explanation of the Entropy for Non-rotating Isolated Horizons

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We consider the non-rotating isolated horizon as an inner boundary of a 4-dimensional asymptotically flat spacetime region. Due to the symmetry of the isolated horizon, it turns out that the boundary degrees of freedom can be described by a SO(1,1) BF theory with punctures. This provides a new approach alternative to the usual one by Chern-Simons theory to study the black hole entropy. To count the microscopical degrees of freedom with the boundary BF theory, the entropy of the isolated horizon can also be calculated in the framework of loop quantum gravity. The leading order contribution to the entropy coincides with the Bekenstein-Hawking area law only for a particular choice of the Barbero-Immirzi parameter, which is different from its value in the usual approach by Chern-Simons theory. Moreover, the quantum correction to the entropy formula is a constant term rather than a logarithmic term.

I. INTRODUCTION

While the notion of the event horizon of a black hole is based on the global structure of the spacetime[1], the notion of isolated horizon is defined quasi-locally as a portion of the event horizon which is in equilibrium[2]. As expected, the laws of black hole mechanics can be generalized to those of isolated horizon[2, 3]. The advantage of the quasi-local notation of isolated horizon admits one to explore the statistical mechanical origin of its entropy by some local quantum gravity theory. In fact, various attempts have been made in the framework of loop quantum gravity (LQG)[4, 5, 6] to account for the entropy of the isolated horizon[8, 9]. In the usual treatment the degrees of freedom of the isolated horizon are described by Chern-Simons theory with SU(2)[10, 11] (or U(1)[9]) gauge group. The relation between the approaches with the two different gauge groups are discussed in Ref.[12]. For recent review on the entropy of the isolated horizon in LQG, we refer to[13, 14, 15].

Although it is feasible to account for the entropy of isolated horizon by the boundary Chern-Simons theory, this approach can not be valid for arbitrary dimensions of the horizon since Chern-Simons theories are only well defined on odd-dimensional manifold. The aim of this paper is to develop a new approach to account for the entropy of the isolated horizon in the framework of LQG, which admits the possibility of the application to arbitrary dimensional horizon. As the first step, we consider the non-rotating isolated horizon in 4-dimensional spacetime. We will show that, with the boundary condition for the isolated horizon, the horizon degrees of freedom can be described by a SO(1,1) BF theory, which is well defined on arbitrary dimensional manifold. Hence, in this case, the entropy of the isolated horizon can also be counted by the boundary BF theory in the framework of LQG.
This alternative approach gives an entropy formula different from the approach of the Chern-Simons theory. Thus the two approach indicate different values of the Barbero-Immirzi parameter.

The paper is organized as follows. In section 2, following the covariant phase space method, we derive the symplectic structure for the spacetime with an non-rotating isolated horizon as an inner boundary. The presymplectic form can be split into the bulk term and the boundary term. In section 3, we identify the boundary degrees of freedom with those of the BF theory. We quantize the punctured BF theory and give the corresponding Hilbert space. In section 4, we set up the boundary condition to relate boundary fields to the bulk fields and calculate the entropy of the isolated horizon. In the light of LQG, the Bekenstein-Hawking area law of black hole entropy is obtained. Our results are discussed in section 5.

II. THE SYMPLECTIC STRUCTURE

Let us first consider an 4-dimensional spacetime region $\mathcal{M}$ with an isolated horizon $\Delta$ as an inner boundary. As in the usual treatment in LQG, we are going to employ the covariant phase space method [16, 17] to derive the symplectic structure of the system.

The Palatini action of general relativity on $\mathcal{M}$ reads:

$$ S[e, A] = -\frac{1}{4\kappa} \int_\mathcal{M} \varepsilon_{IJKL} e^I \wedge e^J \wedge F(A)^{KL} + \frac{1}{4\kappa} \int_{\tau_\infty} \varepsilon_{IJKL} e^I \wedge e^J \wedge A^{KL}, $$

(1)

where we let $\kappa \equiv 8\pi G$, $e^I$ is the cotetrad, $A^{IJ}$ is the so(3,1) connection 1-form and $F(A)^{KL} \equiv dA^{KL} + [A, A]^{KL}$ is the curvature of the connection $A^{KL}$. For convenience, we define the solder form $\Sigma^{IJ} \equiv e^I \wedge e^J$ and its dual $(\ast \Sigma)_{KL} = 1/2 \varepsilon_{IJKL} \Sigma^{IJ}$. All fields will be assumed to be smooth and satisfy the standard asymptotic boundary condition at infinity $\tau_\infty$.

From the first variation of the action (1) we can get the symplectic potential density:

$$ \theta(\delta) = \frac{1}{2\kappa} (\ast \Sigma)_{IJ} \wedge \delta A^{IJ}. $$

(2)

Thus the second-order variation will give the presymplectic current

$$ J(\delta_1, \delta_2) = \frac{1}{\kappa} \delta_1 (\ast \Sigma)_{IJ} \wedge \delta_2 A^{IJ}. $$

(3)

The variational principle implies $dJ = 0$. Applying the Stokes theorem to the integration $\int_\mathcal{M} dJ = 0$, we can get the following equation:

$$ \frac{1}{\kappa} \int_{M_1} \delta_1 (\ast \Sigma)_{IJ} \wedge \delta_2 A^{IJ} - \int_{M_2} \delta_1 (\ast \Sigma)_{IJ} \wedge \delta_2 A^{IJ} + \int_{\Delta} \delta_1 (\ast \Sigma)_{IJ} \wedge \delta_2 A^{IJ} = 0, $$

(4)

where $M_1, M_2$ are spacelike boundary of $\mathcal{M}$. Next we will show that the horizon integral in Eq. (4) is a pure boundary contribution, i.e, the symplectic flux across the horizon can be expressed as a sum of two terms corresponding to the two-sphere $H_1 = \Delta \cap M_1$ and $H_2 = \Delta \cap M_2$.

To describe the geometry near the isolated horizon, it is convenient to employ the Newman-Penrose formulism with the null tetrad $(l, n, m, \bar{m})$ [18]. Let the real vectors $l$ and $n$ coincide with the outgoing and ingoing future directed null vectors at the horizon $\Delta$ respectively. For the non-rotating isolated horizon which we are considering, the components $\pi$ and $\bar{\pi}$ of $l^a \nabla_a n$ along the complex null vectors $\bar{m}$ and $m$ respectively are vanishing on $\Delta$ [19]. In the neighborhood of $\Delta$, we choose the Bondi coordinates given by $(v, r, x^i)$, $i = 1, 2$, where the horizon is given by $r = 0$ [14, 20]. The fields can be expanded in power series in coordinate $r$ away from the horizon. Acting on the function the null tetrad in the neighborhood can be written as

$$ \begin{cases} 
  n^a \nabla_a = -\frac{\partial}{\partial r} \\
  l^a \nabla_a = \frac{\partial}{\partial v} + U \frac{\partial}{\partial r} + X^i \frac{\partial}{\partial x^i} \\
  m^a \nabla_a = \Omega \frac{\partial}{\partial r} + \xi^i \frac{\partial}{\partial x^i}
\end{cases} $$

(5)

where the frame functions $U$ and $X^i$ are real, while $\Omega$ and $\xi^i$ are complex functions of $(v, r, x^i)$. 

Near the horizon, up to the second order of $r$, the metric components can be written as [19, 20]:

\begin{align}
g^{rr} &= 2(\tilde{k}r + \text{Re}(\Psi_2^{(0)}r^2)), \quad g^{vv} = 1, \quad (6) \\
g^{ri} &= 4\text{Re}(1/2\Psi_3^{(0)}\xi^{(0)}_r) r^2, \quad g^{ij} = \xi^i \xi^j + \bar{\xi}^i \bar{\xi}^j,
\end{align}

where $\tilde{k}$ is the surface gravity on the horizon, $\Psi_i$ are the components of Weyl tensor, and the subscript $(0)$ denotes taking values on $\Delta$. In the following, if it is not specified, the functions in the front of $r$ are all functions of $(v, x^i)$. In the coordinate neighborhood the null co-tetrad can also be written up to the second order of $r$ as [20]:

\begin{align}
\{ & n = -dv, \\
& l = dr - (\tilde{k}r + \text{Re}(\Psi_2^{(0)}r^2)) dv - \text{Re}(\Psi_3^{(0)}\xi^{(0)}_r) r^2 dx^i, \\
& m = -1/2\Psi_3^{(0)} r^2 dv + (1 - \mu^{(0)}(r))\xi^{(0)}_r dx^i \\
& \quad - (\bar{\lambda}^{(0)}(r) + 1/2\Psi_4^{(0)} r^2)\bar{\xi}^{(0)}_i dx^i \}.
\end{align}

(7)

where $\mu$ and $\lambda$ are the spin coefficients in the Newman-Penrose formulation. Note that $\xi^{(0)}_i$ are only functions of $(x^1, x^2)$ satisfying $\xi^{(0)}_i \xi^{(0)}_i = 0$ and $\xi^{(0)}_i \bar{\xi}^{(0)}_i = 1$.

Following the idea in Ref. [12], we choose an appropriate set of co-tetrad fields which are compatible with the metric (6) as:

\begin{align}
e^0 &= \sqrt{\frac{1}{2}}(\alpha n + \frac{1}{\alpha} l), \quad e^1 = \sqrt{\frac{1}{2}}(\alpha n - \frac{1}{\alpha} l), \\
e^2 &= \sqrt{\frac{1}{2}}(m + \bar{m}), \quad e^3 = i\sqrt{\frac{1}{2}}(m - \bar{m}),
\end{align}

(8)

where $\alpha(x)$ is an arbitrary function of the coordinates. Each choice of $\alpha(x)$ characterizes a local Lorentz frame in the plane $\mathcal{I}$ formed by $\{e^0, e^1\}$. Restricted to the horizon $\Delta$, the revelent co-tetrad fields (9) are given by:

\begin{align}
e^0 &\triangleq e^1 \triangleq \sqrt{1/2} \alpha n, \\
e^2 &\triangleq \sqrt{2} \text{Re}(\xi^{(0)}_i) dx^i, \quad e^3 \triangleq \sqrt{2} \text{Im}(\xi^{(0)}_i) dx^i.
\end{align}

(9)

Hereafter we denote equalities on $\Delta$ by the symbol $\triangleq$.

Notice that the non-vanishing solder fields $\Sigma^{IJ}$ on $\Delta$ satisfy:

\begin{align}
\Sigma^{0i} &\triangleq \Sigma^{1i}, \quad \forall i = 2, 3, \\
\Sigma^{23} &= im \wedge \bar{m} \triangleq 2\text{Im}(\xi^{(0)}_1 \xi^{(0)}_2) dx^1 \wedge dx^2.
\end{align}

(10)

By a straightforward calculation, we can get the following properties for the connection restricted to $\Delta$:

\begin{align}
A^{01} &\triangleq \tilde{k} dv + dln\alpha \equiv d\beta(x), \quad A^{0i} \triangleq A^{1i}, \forall i = 2, 3, \quad (11)
\end{align}

where $\beta(x) = \tilde{k} v + \text{ln}(\alpha(x))$.

By Eqs. (10) and (11) the horizon integral can be reduced to

\begin{align}
\frac{1}{\kappa} \int_{\Delta} \delta_{[1}\Sigma^{JJ} \wedge \delta_{2]} A^{IJ} = \frac{2}{\kappa} \int_{\Delta} \delta_{[1} \Sigma^{23} \wedge \delta_{2]} A^{01}. \quad (12)
\end{align}

In fact $(\Sigma^{01})_01 = \Sigma^{23}$ is the area element 2-form on the slicing $v = \text{const.}$ of the horizon. Since the property of isolated horizon ensures that the area of the slice is unchanged for different $v$. We can conclude that

\begin{align}
d(\Sigma^{01})_01 = d\Sigma^{23} \triangleq 0. \quad (13)
\end{align}

Thus $\Sigma^{23}$ is closed. So we can define an 1-form $\tilde{B}$ locally such that

\begin{align}
\Sigma^{23} = d\tilde{B}. \quad (14)
\end{align}

Note that the topology of the horizon $\Delta$ is non-trivial with the second cohomology group $H^2(\mathbb{R} \times S^2) \cong \mathbb{R}$. Hence the $\tilde{B}$ field can not be globally defined on $\Delta$.

This situation is similar to the monopole in electromagnetism theory. Although there is no globally defined potential for the electromagnetic field in topologically non-trivial spacetime, one can define the so-called Wu-Yang potential [21] for separated topologically trivial regions. Indeed we have the following condition for the integral over any cross section of $\Delta$:

\begin{align}
\oint_{S^2} d\tilde{B} = \oint_{S^2} \Sigma^{23} = \oint_{S^2} 2\text{Im}(\xi^{(0)}_1 \xi^{(0)}_2) dx^1 \wedge dx^2 = a_H. \quad (15)
\end{align}

where $a_H$ represents the area of the horizon.

Consider a SO(1,1) boost on the plan spanned by $\{e^0, e^1\}$ with group element $g = \exp(\zeta)$. Under this transformation, we get $A^{01} = A^{01} - d\zeta$ and $\Sigma^{23} = \Sigma^{23}$ unchanged. Hence $A^{01}$ is a SO(1,1) connection, and $\Sigma^{23}$ is in its adjoint representation. We will see later that this is just what we need for a SO(1,1) BF theory.
Inserting Eqs. (11) and (13) into Eq. (12), we get
\[
\int_\Delta \delta_1^2 \Sigma_{23} \wedge \delta_2^2 A^01 = d \int_\Delta \delta_1^2 \Sigma_{23} \wedge \delta_2^2 \beta
\]
\[
= \oint_{H_1} \delta_1^2 \Sigma_{23} \wedge \delta_2^2 \beta - \oint_{H_2} \delta_1^2 \Sigma_{23} \wedge \delta_2^2 \beta.
\] (16)

Note that the bulk term in Eq. (4) can be rewritten as the usual form in LQG [14], with new variables \(A_\mu = \gamma A_\mu^0 - 1/2 \epsilon_{jk} A_\mu^k\), and \(\Sigma^i = \epsilon_{jk} \Sigma_{jk}\). Then the full presymplectic structure can be defined on a spatial slice \(M\) with the inner boundary \(H = M \cap \Delta\) as
\[
\Omega(\delta_1, \delta_2) = \frac{1}{2 \kappa \gamma} \int_M 2 \delta_1^2 \Sigma^i \wedge \delta_2^2 A_i + \frac{1}{\kappa} \int_H 2 \delta_2^2 \Sigma_{23} \wedge \delta_2^2 \beta
\]
\[
eq \Omega_M(\delta_1, \delta_2) + \Omega_H(\delta_1, \delta_2),
\] (17)
which is independent of the choice of the spatial surface \(M\). As we can see, the pre-symplectic form split into the bulk term and the boundary term. Hence we can handle the quantization of the bulk and boundary degrees of freedom separately. In the following section, we will show that the pre-symplectic form on the boundary is precisely that of a topological SO(1,1) BF theory with locally defined B fields on the isolated horizon.

### III. 3D SO(1,1) BF Theory

In 3-dimensional space-time \(\Sigma\) without boundary, the action of the SO(1,1) BF theory can be written as [22, 23]
\[
S[B, A] = \int_\Sigma B \wedge F(A) = \int_\Sigma dB \wedge A.
\] (18)
Since one has \(SO(1,1) \cong \mathbb{R}\), the connection field \(A\) is a real-valued 1-form, and the B field is also a real-valued 1-form. From the action (18), we can get the equation of motion easily as
\[
F = dA = 0, \quad dB = 0.
\] (19)

In the Hamiltonian formulism, the restriction of the field \(A\) and \(B\) to the spatial hypersurface gives the conjugate variables, which we still denote as \((A, B)\), satisfying the Gaussian constraint as well as the constraint: \(F = dA = 0\). The latter generates gauge transformations of the form:
\[
A \rightarrow A, \quad B \rightarrow B + d\lambda.
\] (20)

From the view point of covariant phase space, the symplectic flux can be obtained from the anti-symmetrization of the second variation of action (18) as
\[
\int_\Sigma 2 \delta_1^i (dB) \wedge \delta_2^j A,
\] (21)
from which we can get the pre-symplectic form on the covariant phase space as [22]
\[
\Omega(\delta_1, \delta_2) = \int_\Sigma 2 \delta_2^j B \wedge \delta_1^j A,
\] (22)
where \(H\) is an arbitrary 2-dimensional spatial slice in \(\Sigma\).

It should be noted that if \(H\) is topologically non-trivial and the B field is not globally defined as the case in the last section, the definition of the integration in the pre-symplectic form (22) is a delicate issue. However, as shown in the Appendix, in the case of a two-sphere \(H = S^2\), the integration can be well defined as the sum of integrals over two topological trivial patches and one of their boundaries. Then the boundary pre-symplectic form \(\Omega_H\) in Eq. (17) can be regarded as that of SO(1,1) BF theory by making the identification:
\[
B \leftrightarrow \frac{1}{\kappa} A \leftrightarrow A^{01}.
\] (23)

Hence on the non-rotating isolated horizon, the boundary degrees of freedom of general relativity can be described effectively by a SO(1,1) BF theory. Since the fundamental group of the manifold \(\Delta\) is trivial, i.e., \(\pi_1(\mathbb{R} \times S^2)\) is trivial, the quantum BF theory has trivial Hilbert space [25, 26].

Recall that in canonical LQG, the kinematical Hilbert space is spanned by spin network states \(|\Gamma, \{j_e\}, \{i_v\} > 0\)
\[\mathcal{H},\] where \(\Gamma\) denotes some graph in the spatial manifold \(M\), each edge \(e\) of \(\Gamma\) is labeled by a half-integer \(j_e\) and each vertex \(v\) is labeled by an intertwiner \(i_v\). In the case when \(M\) has a boundary \(H\), some edges of \(\Gamma\) may intersect \(H\) and endow it a quantum area at each puncture [10].
Thus, to account for the isolated horizon degrees of freedom, we need to consider the quantum BF theory with punctures. Eqs. (11) and (14) imply the equation of motion of our punctured BF theory as

$$F = dA = 0, \quad dB = \frac{\Sigma^1}{2\kappa}. \quad (24)$$

Compared with Eq. (19), we can see that those punctures will only affect the $B$ field rather than the $A$ field.

Let’s assume that the graph $\Gamma$ underling a spin network state intersects $H$ by $n$ punctures denoted by $\mathcal{P} = \{p_i|i = 1, \cdots, n\}$. For every puncture $p_i$ we associated a bounded neighborhood $s_i$ which contains it and doesn’t intersect with each other. We denote the boundary of $s_i$ by $\eta_i$. Since $H$ is homeomorphism to a two-sphere, the holonomy of flat connections is trivial. Taking account of the gauge transformations (20), the physical degrees of freedom of our punctured BF theory are encoded in the flux functions

$$f_i = \int_{s_i} dB = \oint_{\eta_i} B, \quad (25)$$

which are gauge-invariant functions of $B$. Since to each puncture $p_i$ we can associate a real valued variable $f_i$ to it, the configuration space of the BF theory with $n$ punctures is $\mathbb{R}^n$. Therefore, we can employ the well-known Lebesque measure to define the quantum Hilbert space $\mathcal{H}_H^n$ as the space of $L^2$ functions on $\mathbb{R}^n$. Note that, as configuration operators, $\hat{f}_i$ act on any wave function by multiplications. The common eigenstates of all these $\hat{f}_i$ are the Dirac distributions $\{a_p, \mathcal{P}\| \equiv \{a_1, a_2, \cdots, a_n\}$ characterized by $n$ real numbers $\{a_i|i = 1, \cdots, n\}$. As unbounded self-adjoint operators, the collection $\{\hat{f}_i|i = 1, \cdots, n\}$ comprises of a complete set of observables in $\mathcal{H}_H^n = L^2(\mathbb{R}^n)$. There is a spectral decomposition of $\mathcal{H}_H^n$ with respect to each $\hat{f}_i$, i.e.,

$$\langle \{a_p\}, \mathcal{P}\| \hat{f}_i \langle \{a_p\}, \mathcal{P}\| a_i = (\{a_p\}, \mathcal{P}\|a_i. \quad (26)$$

IV. BOUNDARY CONDITION AND STATE COUNTING

The form of the pre-symplectic form (17) motivates us to handle the quantization of the bulk and horizon degrees of freedom separately. As in the standard LQG one first consider the bulk kinematical Hilbert space $\mathcal{H}_M^n$ defined on a graph $\Gamma \subset M$ with the $n$ punctures $\mathcal{P}$ as the end points on $H$. This Hilbert space can be spanned by the spin network states $|\mathcal{P}, \{j_p, m_p\}; \cdots >$, where $j_p$ and $m_p$ are respectively the spin labels and magnetic numbers of the edge $e_p$ with endpoint $p \in \mathcal{P}$. Note that the integral $\Sigma^1(H) = \int_H \Sigma^1$ can be promoted as an operator $\hat{\Sigma}^1(H)$ in $\mathcal{H}_{H}^n$, and $|\mathcal{P}, \{j_p, m_p\}; \cdots >$ are common eigenstates of $\hat{\Sigma}^1(H)$ and the horizon area operator $\hat{a}_H$ from the viewpoint of bulk LQG. Thus we have $[0, 27]$ and $[10]$

$$\begin{align*}
\hat{a}_H|\mathcal{P}, \{j_p, m_p\}; \cdots > = 8\pi \gamma l_p^2\sum_{p=1}^{n} \sqrt{j_p(j_p + 1)}|\mathcal{P}, \{j_p, m_p\}; \cdots >, \quad (27) \\
\hat{\Sigma}^1(H)|\mathcal{P}, \{j_p, m_p\}; \cdots > = 16\pi \gamma l_p^2\sum_{p \in \Gamma \cap H} m_p|\mathcal{P}, \{j_p, m_p\}; \cdots >. \quad (28)
\end{align*}$$

Classically, the restriction of Eq. (24) to the spatial slice $H = \Delta \cap M$ imply the following boundary condition to relate the boundary and bulk degrees of freedom,

$$dB = \frac{\Sigma^1}{2\kappa}, \quad (29)$$

where $\hat{=} = \text{means equal on } H$. Eq. (29) motive us to input the quantum version of the horizon boundary condition as

$$\left(\text{Id} \otimes \hat{f}_i(s_i) - \frac{\hat{\Sigma}^1(s_i)}{2\kappa} \otimes \text{Id}\right)(\Psi_v \otimes \Psi_h) = 0, \quad (30)$$

where $s_i$ is the neighborhood of an arbitrary puncture $p_i \in \mathcal{P}$, $\Psi_v \in \mathcal{H}_M^n$ and $\Psi_h \in \mathcal{H}_H^n$. For a given bulk spin network state $|\mathcal{P}, \{j_p, m_p\}; \cdots >$, the solutions of Eq. (30) restrict the generalized eigenstates of $\hat{f}_p$ to be $\{m_p\}; \mathcal{P}$ with eigenvalues
\[ a_p = \gamma m_p. \]  
(31)

This means that by applying the quantum boundary condition, the eigenvalues \( a_p \) of \( \hat{f}_p \) for all \( p \in \mathcal{P} \) can take values only in the subset of the real number consisting of the integers times a constant. Thus the quantum boundary condition not only relate the bulk and the boundary theories, but also reduce the dimension of the boundary Hilbert space.

The space of kinematical states on a fixed graph \( \Gamma \), satisfying the boundary condition, can be written as

\[ \mathcal{H}_\Gamma = \bigoplus_{(j_p, m_p) \in \Gamma \cap \mathcal{H}} \mathcal{H}_\mathcal{P}^\Gamma(\mathcal{P}^\mathcal{M}(\{j_p, m_p\})) \otimes \mathcal{H}_H^\Gamma(\{m_p\}), \]  
(32)

where \( \mathcal{H}_\mathcal{P}^\Gamma(\{m_p\}) \) denotes the subspace corresponds to the spectrum \( \{m_p\} \) in the spectral decomposition of the punctured BF Hilbert spaces \( \mathcal{H}_H^\Gamma \) with respect to the operators \( \hat{f}_p \) on the boundary.

It should be noted that the imposition of the diffeomorphism constraint implies that one only needs to consider the diffeomorphism equivalence class of quantum states. Hence, in the following states counting, we will only take account of the number of punctures on \( H \), while the possible position of punctures are irrelevant.

To calculate the entropy of the isolated horizon that we are considering, we will follow the viewpoint of LQG to trace out the degrees of freedom corresponding to the bulk but take account of the horizon degrees of freedom. Then the entropy will be

\[ S = \log(\mathcal{N}), \]  
(33)

where \( \mathcal{N} \) is the dimension of horizon Hilbert space compatible with the given macroscopic horizon area \( a_H \) and satisfying the horizon boundary constraint (31).

Now how to define the area operator of the horizon is a delicate issue. In the original treatment, one employed the standard area operator defined in the kinematical Hilbert space of LQG. However, for the bulk Hilbert space \( \mathcal{H}_M^\Gamma \) with a horizon boundary \( H \), the flux-area operator \( \hat{a}_H^{\text{flux}} \) corresponding to the classical area \( \int_H dB \) of \( H \) can also be naturally well-defined as

\[ \hat{a}_H^{\text{flux}} |\mathcal{P}, \{j_p, m_p\}; \cdots > = a^{\text{flux}}(\{m_p\}) |\mathcal{P}, \{j_p, m_p\}; \cdots > \]  
(34)

where

\[ a^{\text{flux}}(\{m_p\}) = 8\pi\gamma l_p^2 \sum_{p=1}^n |m_p|. \]  
(35)

With this choice, we have the area constraint:

\[ \sum_{p \in \mathcal{P}} |m_p| = a, \quad m_p \in \mathbb{N}/2, \]  
(36)

where \( a = \frac{a_H}{8\pi\gamma l_p^2} \). Hence, for a given horizon area \( a_H \), Eq. (31) implies that the horizon states satisfying the boundary condition can be labeled by sequences \( (m_1, \cdots, m_n) \) subject to the constraint (36), where \( 2m_i \) are integers. As in the usual treatment in LQG, we assume that for each given ordering sequence \( (m_1, \cdots, m_n) \), there exists at least one state in the bulk Hilbert space of LQG, which is annihilated by the Hamiltonian constraint. Then the dimension of the horizon Hilbert space compatible with the given macroscopic horizon area can be calculated as:

\[ \mathcal{N} = \sum_{n=2a-1}^{n=2a-1} C_{2a-1}^n 2^{n+1} = 2 \times 3^{2a-1}, \]  
(37)

where \( C_i^j \) are the binomial coefficients. So the entropy is given by

\[ S = \log \mathcal{N} = 2a \log 3 + \log \frac{2}{3} = \frac{\log 3}{\pi \gamma} \frac{a_H}{4 l_p^2} + \log \frac{2}{3}. \]  
(38)

Thus we have got the entropy for an arbitrary non-rotating isolated horizon, which is proportional to its area at leading order. If we fix the value of the Barbero-Immirizi parameter as \( \gamma = \log 3/\pi \), the Bekenstein-Hawking area law is obtained.

It should be noted that the choice of the flux-area operator (31) is necessary in order to get a consistent result for states counting. Had we chosen the area operator (27)
in full LQG to represent the horizon area \( a_H \), we would have the area constraint:
\[
8\pi \gamma \sqrt{\frac{\alpha}{|\mathcal{P}|}} \sum_{p=1}^{n} \sqrt{j_p(j_p + 1)} = a_H. \tag{39}
\]

On the other hand, there is another global constraint which follows from the quantum versions of Eqs. (14) and (31):
\[
\sum_{p \in \mathcal{P}} |a_p| = \gamma \sum_{p \in \mathcal{P}} |m_p| = a_H / \kappa. \tag{40}
\]

Since \( m_i \in \{-j_i, \cdots, j_i\} \), there is no common solution for the both constraints.

V. DISCUSSION

In the previous sections, the non-rotating isolated horizon in 4-dimensional spacetime has been studied, and its entropy has been calculated in the framework of LQG. By the gauge choice of Eqs. (3), the degrees of freedom of the horizon can be encoded in a SO(1,1) BF theory. From the view point of LQG, the spin networks of the bulk quantum geometry puncture the horizon, endowing it with quantum area. This picture not only transforms the horizon boundary condition (31) into the quantum condition (30), but also indicates the area constraint (36). Thus, for a given macroscopic horizon area, the microscopic degrees of freedom of the horizon can be calculated as in Eq. (37), which accounts for its entropy and suggest a value for the Barbero-Immiriz parameter.

By comparison with the Chern-Simons theory approach, our BF theory approach indicates a different value of the Barbero-Immiriz parameter. However our value for the parameter coincides with its value obtained in a particular case in Ref. [13] by employing the same flux-area operator as ours but in the approach of Chern-Simons theory. Whether this coincidence implies any relation between the two approaches deserves further investigating. The quantum correction to the Bekenstein-Hawking area law in our approach is a constant \( \log(2/3) \), while the Chern-Simons theory approach usually gives logarithmic correction at first order. As a delicate issue, the quantum correction to the classical area law of the isolated horizon is also discussed in Ref. [27]. Irrespective of these differences, the virtue of our BF theory approach is that it admits extension to arbitrary dimensional horizon [28], while the Chern-Simons theory can only lives on odd-dimensional manifold.

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VI. APPENDIX

We now show that the boundary pre-symplectic form in Eq. (17) is indeed that of a BF theory with locally defined \( B \) fields on the horizon.

Note that the \( B \equiv \dot{B} \) field is defined locally though Eq. (14). We can cover the two-sphere \( S^2 \) with two topologically trivial patches \( S_+ \) and \( S_- \) with boundary \( c_1 \) and \( c_2 \) respectively. The intersection region is denoted by \( S_0 = S_+ \cap S_- \) with boundary \( c_1 + c_2 \). The potentials in the regions \( S_+ \) and \( S_- \) can be well defined as \( B_+ \) and \( B_- \) separately, satisfying \( dB_+ = dB_- = \Sigma^{23} \). In the region \( S_0 \), we have both \( B_+ \) and \( B_- \) such that \( B_+ - B_- = g \), where \( g \) is a closed 1-form. So we have
\[
\oint_{S^2} \delta_1 \Sigma^{23} \wedge \delta_2 \beta = \int_{S_+} \delta_1 (dB_+) \wedge \delta_2 \beta + \int_{S_-} \delta_1 (dB_-) \wedge \delta_2 \beta - \int_{S_0} \delta_1 (dB_+) \wedge \delta_2 \beta
\]
\[
= \int_{S_+} \delta_1 B_+ \wedge \delta_2 A^{01} + \int_{c_1} \delta_1 B_+ \wedge \delta_2 \beta + \int_{S_-} \delta_1 B_- \wedge \delta_2 A^{01} + \int_{c_1 + c_2} \delta_1 B_+ \wedge \delta_2 \beta
\]
\[
= \int_{S^2 - S_-} \delta_1 B_+ \wedge \delta_2 A + \int_{S_-} \delta_1 B_- \wedge \delta_2 A - \int_{\partial(S_-)} \delta_1 g \wedge \delta_2 \beta, \tag{41}
\]
where we used the Leibniz rule, the Stokes’ theorem and the definition \( d\beta = A^0 \). Then we need to show that Eq.\((11)\) can be understood as the pre-symplectic form for BF theory with locally defined \( B \) fields such that \( dB = \sum_{23} \) and \( A(x) = d\beta(x) \). Since the \( B \) fields cannot be globally defined on \( S^2 \), the integration of the pre-symplectic form \((22)\) has to be defined carefully. An innocent definition could be

\[
\oint_{S^2} \delta_1 B \wedge \delta_2 A := \int_{S^+} \delta_1 B_+ \wedge \delta_2 A + \int_{S^-} \delta_1 B_- \wedge \delta_2 A - \int_{S^0} \delta_1 B_+ \wedge \delta_2 A.
\]

(42)

However, since both \( B_+ \) and \( B_- \) are on the same footing in the region \( S_0 \), one may also employ the \( B_- \) instead of \( B_+ \) in the last integration of Eq.\((42)\). Obviously the two formulas are not equivalent to each other. Actually we have

\[
\int_{S_0} \delta_1 (B_+ - B_-) \wedge \delta_2 A = \int_{S_0} \delta_1 g \wedge \delta_2 d\beta = - \oint_{\partial c_{1+} \cup \partial c_{1-}} \delta_1 g \wedge \delta_2 \beta,
\]

(43)

and hence

\[
\int_{S_0} \delta_1 B_+ \wedge \delta_2 A + \oint_{c_2} \delta_1 (B_+ - B_-) \wedge \delta_2 \beta = \int_{S_0} \delta_1 B_- \wedge \delta_2 A + \oint_{c_1} \delta_1 (B_+ - B_-) \wedge \delta_2 \beta.
\]

(44)

Therefore, the reasonable definition for the pre-symplectic form \((22)\) with locally defined \( B \) fields, which is independent of the choice between \( B_+ \) and \( B_- \), should be

\[
\oint_{S^2} \delta_1 B \wedge \delta_2 A := \int_{S^+} \delta_1 B_+ \wedge \delta_2 A + \int_{S-_0} \delta_1 B_- \wedge \delta_2 A - \oint_{\partial (S_-)} \delta_1 g \wedge \delta_2 \beta,
\]

(45)

which coincides with Eq.\((41)\).

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