BOUNDDEDNESS OF $n$-COMPLEMENTS FOR GENERALIZED POLARIZED PAIRS

GUODU CHEN

Abstract. We show the existence of $n$-complements for generalized polarized pairs with additional Diophantine approximation properties when the coefficients of boundaries belong to a DCC set.

Contents

1. Introduction 1
2. Preliminaries 4
2.1. Arithmetic of sets 5
2.2. Divisors 5
2.3. Generalized polarized pairs 5
2.4. MMP for generalized polarized pairs 8
2.5. Complements 8
3. Uniform rational polytopes 9
3.1. Accumulation points of generalized lc thresholds 9
3.2. Nakamura’s (generalized) lc rational polytopes 15
3.3. Han type polytopes for $\mathbb{R}$-complementary (generalized) pairs 16
4. Boundedness of relative complements 18
5. Proof of Theorem 1.1 22
5.1. From the DCC set to a finite set 22
5.2. Proof of Theorem 1.3 25
6. Existence of $n$-complements 25
References 28

1. INTRODUCTION

We work over the field of complex numbers $\mathbb{C}$.

The theory of complements was introduced by Shokurov when he proved the existence of log flips for threefolds [Sho92]. It turns out the theory of complements plays an important role in the recent development of birational geometry. The theory is further developed in [Sho00,PS01,Bir04,PS09,Bir19,HLS19,Sho20,CH20], see also [FM18,XuY19,FMX19]. In [Bir19], Birkar
proved the boundedness of log canonical complements for Fano type varieties when the coefficients of boundaries belong to a hyperstandard sets \( \Gamma \subseteq [0,1] \cap \mathbb{Q} \) which is a breakthrough in the study of Fano varieties. The boundedness of log canonical complements plays an important role in various contexts, see [Bir16, Bir18, Liu18, Bir19, XuCh19, BLX19].

In [HLS19], Han, Liu and Shokurov proved the boundedness of log canonical complements with additional Diophantine approximation properties for Fano type varieties with any DCC set \( \Gamma \subseteq [0,1] \). The theory of complements in this case could also be applied to prove the ACC for minimal log discrepancies of exceptional singularities [HLS19] and it has other applications.

For the purpose of induction in the birational geometry, we need to study and explore the space of generalized polarized pairs. They were first introduced in [BZ16] to deal with the effectivity of Iitaka fibrations. In recent years, they have found applications in various contexts, such as the boundedness of complements for Fano varieties [Bir19], and Fujita’s spectrum conjecture [HL17], see [Fil17, HL18, HLiu19, LT19] for more works.

It is not clear that one can address the nature questions for the usual pairs in the setting of generalized polarized pairs, such as the cone theorem. On the other hand, it is known that the nonvanishing conjecture fails for genenralized polarized pairs, and we can only expect the numerical nonvanishing for genenralized polarized pairs, see [HLiu18, LPS19, LP19]. In this paper, we focus on the boundedness of complements for generalized polarized pairs with additional Diophantine approximation properties, and show the existence of \( n \)-complements for generalized polarized pairs with DCC coefficients. Recall that the rational envelope \( V \subseteq \mathbb{R}^s \) of a point \( v \in \mathbb{R}^s \) is the smallest affine subspace containing \( v \) which is defined over the rationals.

**Theorem 1.1.** Let \( d, p, r \) and \( s \) be positive integers, \( \epsilon \) a positive real number, \( \Gamma \subseteq [0,1] \) a DCC set, \( ||| \) a norm on \( \mathbb{R}^s \), \( v_0 \in \mathbb{R}^s \setminus \mathbb{Q}^s \) a point, \( V \subseteq \mathbb{R}^s \) the rational envelope of \( v_0 \), and \( e \in V \) a non-zero vector. Then there exist a positive integer \( n \) and a point \( v \in V \) depending only on \( d, p, r, s, \epsilon, \Gamma, |||, v_0 \) and \( e \) satisfying the following.

Assume that \( (X, B + M) \) is a generalized pair with data \( X' \xrightarrow{f} X \to Z \) and \( M' \) such that

- \( \dim X = d \),
- \( X \) is of Fano type over \( Z \),
- \( B \in \Gamma \), that is, the coefficients of \( B \) belong to \( \Gamma \),
- \( rM' \) is \( b \)-Cartier, and
- \( (X/Z, B + M) \) is \( \mathbb{R} \)-complementary.

Then

1. (existence of \( n \)-complements) for any \( z \in Z \), there exists an \( n \)-complement \( (X/Z \ni z, B^+ + M) \) of \( (X/Z \ni z, B + M) \), moreover, if \( \text{Span}_{\mathbb{Q}_{\geq 0}}(\Gamma \setminus \mathbb{Q}) \cap (\mathbb{Q} \setminus \{0\}) = \emptyset \), then we may pick \( B^+ \geq B \), where \( \Gamma \) is the closure of \( \Gamma \).
approximation properties (2)–(5) in Theorem 1.1, it was proved by Shokurov \[Sho00\] for the case when \( \dim X = 2 \) and \( \Gamma \) is the standard set, by Prokhorov and Shokurov \[PS09\] for the case when \( \dim X = 3 \) and \( \Gamma \subseteq \mathbb{Q} \) is a hyperstandard set, by Birkar \[Bir19, Theorem 1.7, Theorem 1.8\] for the case when \( \Gamma \subseteq \mathbb{Q} \) is a hyperstandard set. When \( M' = 0 \), it was proved by Han, Liu and Shokurov \[HLS19\]. We refer readers to \[FM18\] for the case when \( \bar{\Gamma} \) and \( \Gamma \subseteq \mathbb{Q} \) is a hyperstandard set. When \( M' = 0 \), it was proved by Han, Liu and Shokurov \[HLS19\]. We refer readers to \[FM18\] for the case when \( \bar{\Gamma} \subseteq [0,1] \) without additional Diophantine approximation properties.

In order to show Theorem 1.1, we study a new class of complements, namely \((n, \Gamma_0)\)-decomposable \( \mathbb{R} \)-complements. Note that when \( M' = 0 \), \((n, \Gamma_0)\)-decomposable \( \mathbb{R} \)-complements are the same as \[HLS19\] Definition 1.9.

**Definition 1.2.** Let \( n \) be a positive integer, \( \Gamma_0 \subseteq (0,1] \) a finite set and \((X, B + M)\) a generalized polarized pair with data \( X' \xrightarrow{f} X \to Z \) and \( M' \). We say that \((X/Z \ni z, B^+ + M)\) is an \((n, \Gamma_0)\)-decomposable \( \mathbb{R} \)-complement of \((X/Z \ni z, B + M)\) if

1. \((X/Z \ni z, B^+ + M)\) is an \( \mathbb{R} \)-complement of \((X/Z \ni z, B + M)\),
2. \( K_X + B^+ + M = \sum a_i (K_X + B_i^+ + M) \) for some boundaries \( B_i^+ \) and some \( a_i \in \Gamma_0 \) with \( \sum a_i = 1 \), and
3. \((X/Z \ni z, B_i^+ + M)\) is an \( n \)-complement of itself for any \( i \).

We will show the existence of \((n, \Gamma_0)\)-decomposable \( \mathbb{R} \)-complements for \( \mathbb{R} \)-complementary generalized polarized pairs with DCC coefficients which is an important step in the proof of Theorem 1.1.

**Theorem 1.3.** Let \( d, r \) be positive integers, and \( \Gamma \subseteq [0,1] \) a DCC set. Then there exist a positive integer \( n \) and a finite set \( \Gamma_0 \subseteq (0,1] \) depending only on \( d, r \) and \( \Gamma \) satisfying the following.

Assume that \((X, B + M)\) is a generalized polarized pair with data \( X' \xrightarrow{f} X \to Z \) and \( M' \) such that

- \((X, B + M)\) is generalized lc of dimension \( d \),
- \( X \) of is Fano type over \( Z \),
- \( B \in \Gamma \) and \( rM' \) is \( b \)-Cartier, and
- \((X/Z, B + M)\) is \( \mathbb{R} \)-complementary.

Then for any point \( z \in Z \), there is an \((n, \Gamma_0)\)-decomposable \( \mathbb{R} \)-complement \((X/Z \ni z, B^+ + M)\) of \((X/Z \ni z, B + M)\). Moreover, if \( \Gamma \subseteq \mathbb{Q} \), then we may pick \( \Gamma_0 = \{1\} \), and \((X/Z \ni z, B^+ + M)\) is a monotonic \( n \)-complement of \((X/Z \ni z, B + M)\).
As one of the main ingredients in the proof of Theorem 1.1, we will show the existence of Nakamura’s (generalized) lc rational polytopes.

**Theorem 1.4** (Nakamura’s (generalized) lc rational polytopes). Let \( d, r \) and \( m \) be positive integers, \( v_0 := (v_0^1, \ldots, v_0^m) \in \mathbb{R}^m \) a point and \( V \subseteq \mathbb{R}^m \) the rational envelope of \( v_0 \). Then there exists an open set \( U \ni v_0 \) of \( V \) depending only on \( d, r, m \) and \( v_0 \) satisfying the following.

Assume that \( (X, (\sum_{i=1}^m v_i^0 B_i) + M) \) is a generalized polarized pair with data \( X' \to X \to Z \) and \( M' \) such that

1. \( (X/Z, (\sum_{i=1}^m v_i^0 B_i) + M) \) is generalized lc of dimension \( d \),
2. \( B_1, \ldots, B_m \geq 0 \) are Weil divisors on \( X \), and
3. \( rM' \) is b-Cartier.

Then \( (X, (\sum_{i=1}^m v_i B_i) + M) \) is generalized lc for any point \((v_1, \ldots, v_m) \in U\).

To show Theorem 1.4, we generalized a result of Nakamura [Nak16, Theorem 1.6], which is about perturbation of an irrational coefficient of generalized lc pairs, see Theorem 3.15. The proof of Theorem 3.15 is a combination of [HMX14], [Nak16] and [HLQ17]. It is worthwhile to point out that, we use Birkar-Borisov-Alexeev-Borisov Theorem to simplify the proof of Nakamura. Moreover, by using Theorem 3.15, we can show the existence of Han type polytopes for \( \mathbb{R} \)-complementary generalized polarized pairs of Fano type over the base (Theorem 3.18).

**Structure of the paper.** We outline the organization of the paper. In Section 2 we introduce some notation and tools which will be used in this paper, and prove certain basic results. In Section 3 we prove Theorem 1.4. In Section 4 we show Theorem 1.4 which is a generalized of [Bir19, Theorem 1.10]. In Section 5 we prove Theorem 1.3. In Section 6 we prove Theorem 1.1.

**Acknowledgements.** This work began when the author visited Jingjun Han at Johns Hopkins University in April of 2019, and the author would like to thank Jingjun Han for suggesting him the problem and also for useful discussions. Part of this work was done while the first author visited the MIT Mathematics Department during 2018–2020 supported by China Scholarship Council (File No. 201806010039). The author would like to thank their hospitality. The author would like to thank his advisor Chenyang Xu for constant support and encouragement, and Vyacheslav V. Shokurov for teaching him the theory of complements. The author would also like to thank Jihao Liu for his comments.

**2. Preliminaries**

In this section, we will collect some definitions and preliminary results which will be used in this paper.
2.1. Arithmetic of sets. Let $\Gamma \subseteq [0, +\infty)$ be a set, we define

$$\sum \Gamma := \{0\} \cup \left\{ \sum_{p=1}^{l} i_p \mid i_p \in \Gamma \text{ for } 1 \leq p \leq l, l \in \mathbb{Z}_{>0} \right\}.$$ 

If $\Gamma \subseteq [0, 1]$, then we define $\Phi(\Gamma) := \{1 - \frac{r}{m} \mid r \in \Gamma, m \in \mathbb{Z}_{>0}\}$ to be the set of hyperstandard multiplicities associated to $\Gamma$ (c.f. [PS09, 3.2]). Note that if we add $1 - r$ to $\Gamma$ for any $r \in \Gamma$, then we get $\Gamma \subseteq \Phi(\Gamma)$.

**Definition 2.1 (DCC and ACC sets).** We say that $\Gamma \subseteq \mathbb{R}$ satisfies the descending chain condition (DCC) if any decreasing sequence $a_1 \geq a_2 \geq \cdots$ in $\Gamma$ stabilizes. We say that $\Gamma$ satisfies the ascending chain condition (ACC) if any increasing sequence in $\Gamma$ stabilizes.

Now assume that $\Gamma \subseteq [0, 1] \cap \mathbb{Q}$ is a finite set. Then $\Phi(\Gamma)$ is a DCC set of rational numbers whose only accumulation point is 1.

2.2. Divisors. We adopt the standard notation and definitions in [BZ16] and [Bir19], and will freely use them.

Let $X$ be a normal variety, $D := \sum d_i D_i$ an $\mathbb{R}$-divisor and $a$ a real number. We define $[D] := \sum[d_i] D_i$, $\{D\} := \sum\{d_i\} D_i$, $D^{\geq a} := \sum d_i \geq a D_i$, and $[D] := \sum[d_i] D_i$. For any point $v := (v_1, \ldots, v_m) \in \mathbb{R}^m$, we define $||v||_\infty := \max_{1 \leq i \leq m} \{|v_i|\}$.

We say that $D$ is $b$-Cartier if it is $\mathbb{Q}$-Cartier and $\phi^* D$ is Cartier for some birational morphism $\phi : Y \to X$.

**Definition 2.2 (b-divisors).** Let $X$ be a variety. A $b$-$\mathbb{R}$-Cartier b-divisor over $X$ is the choice of a projective birational morphism $Y \to X$ from a normal variety $Y$ and an $\mathbb{R}$-Cartier divisor $M$ on $Y$ up to the following equivalence: another projective birational morphism $Y' \to X$ from a normal variety $Y'$ and an $\mathbb{R}$-Cartier divisor $M'$ defines the same $b$-$\mathbb{R}$-Cartier b-divisor if there is a common resolution $W \to Y$ and $W \to Y'$ on which the pullbacks of $M$ and $M'$ coincide.

A $b$-$\mathbb{R}$-Cartier b-divisor represented by some $Y \to X$ and $M$ is $b$-Cartier if $M$ is $b$-Cartier, i.e., its pullback to some resolution is Cartier.

2.3. Generalized polarized pairs.

**Definition 2.3.** We say $\pi : X \to Z$ is a contraction if $X$ and $Z$ are normal quasi-projective varieties, $\pi$ is a projective morphism, and $\pi_* \mathcal{O}_X = \mathcal{O}_Z$ ($\pi$ is not necessarily birational).

**Definition 2.4.** Let $X \to Z$ be a contraction. We say that $X$ is of Fano type over $Z$ if $(X, B)$ is klt and $-(K_X + B)$ is big and nef over $Z$ for some boundary $B$. 
**Remark 2.5.** Assume that $X$ is of Fano type over $Z$. Then we can run the MMP/$Z$ on any $\mathbb{R}$-Cartier divisor $D$ on $X$ which terminates with some model $Y$ (c.f. [PS09, Corollary 2.9], [BCHM10]).

**Definition 2.6** (Generalized polarized pairs). A generalized polarized pair consists of

- a normal variety $X$ equipped with a projective morphism $X \to Z$,
- an $\mathbb{R}$-divisor $B \geq 0$ on $X$, and
- a $b\mathbb{R}$-Cartier $b$-divisor over $X$ represented by some projective birational morphism $X' \xrightarrow{f'} X$ and $\mathbb{R}$-Cartier divisor $M'$ on $X'$,

such that $M'$ is nef $/Z$, and $K_X + B + M$ is $\mathbb{R}$-Cartier, where $M := f_* M'$.

We may say that $(X, B + M)$ is a generalized polarized pair with data $X' \xrightarrow{f'} X \to Z$ and $M'$. If $\dim Z = 0$, the generalized polarized pair is called projective, and we will omit $Z$. If $Z = X$ and $X \to Z$ is the identity map, we will omit $Z$. Since a $b\mathbb{R}$-Cartier $b$-divisor is defined birationally, we will often replace $X'$ by a higher model and replace $M'$ by its pullback.

Possibly replacing $X'$ by a higher model and $M'$ by its pullback, we may assume that $f$ is a log resolution of $(X, B)$, and write

$$K_{X'} + B' + M' = f^*(K_X + B + M)$$

for some uniquely determined $B'$. The generalized log discrepancy of a divisor $E$ on $X'$ with respect to $(X, B + M)$ is $1 - \text{mult}_E B'$ and denoted by $a(E, X, B + M)$. We define the generalized minimal log discrepancy of $(X, B + M)$ as

$$\text{mld}(X, B + M) := \min\{a(E, X, B + M) \mid E \text{ is a prime divisor over } X\}.$$

We say that $(X, B + M)$ is generalized $\epsilon$-lc (respectively generalized klt, generalized lc) for some non-negative real number $\epsilon$ if $\text{mld}(X, B + M) \geq \epsilon$ (respectively $> 0, \geq 0$). For a divisor $E$ over $X$ with $a(E, X, B + M) \leq 0$, we call $E$ a generalized nonlc place and its image on $X$ a generalized nonlc center.

A generalized lc pair $(X, B + M)$ with data $X' \xrightarrow{f'} X \to Z$ and $M'$ is generalized dlt if $(X, B)$ is a dlt pair and every generalized nonlc center of $(X, B + M)$ is a nonlc center of $(X, B)$. If in addition, the connected components of $|B|$ are irreducible, then we say that the generalized polarized pair is generalized plt.

We recall an adjunction formula for generalized polarized pairs.

**Definition 2.7** (Generalized adjunction formula). Let $(X, B + M)$ be a generalized polarized pair with data $X' \xrightarrow{f'} X \to Z$ and $M'$. Assume that $S$ is the normalization of a component of $|B|$, and that $S'$ is its strict transform on $X'$. Possibly replacing $X'$ by a higher model, we may assume that $f$ is a log resolution of $(X, B)$ and write

$$K_{X'} + B' + M' = f^*(K_X + B + M).$$
Then
\[ K_{S'} + B_{S'} + M'_{S'} = f^*(K_X + B + M)|_S, \]
where \( B_{S'} := (B' - S')|_S' \) and \( M'_{S'} := M'|_S'. \) Let \( g := f|_{S'} \) be the induced morphism, \( B_S := g_* B_{S'} \) and \( M_S := g_* M'_{S'} \). Then we get
\[ K_S + B_S + M_S = (K_X + B + M)|_S, \]
and \((S, B_S + M_S)\) is a generalized pair with data \( S' \overset{g}{\to} S \) and \( M'_{S'} \), which is referred as the generalized adjunction formula.

**Definition 2.8 (Generalized lc thresholds).** Let \((X, B + M)\) be a generalized lc pair with data \( X' \overset{f}{\to} X \to Z \) and \( M' \). Assume that \( D \geq 0 \) is an \( \mathbb{R} \)-divisor on \( X \) and \( N' \) is a nef \( \mathbb{R} \)-divisor on \( X' \), such that \( D + N \) is \( \mathbb{R} \)-Cartier, where \( N' = f_* N \). The generalized lc threshold of \( D + N \) with respect to \((X, B + M)\) is defined as
\[ \text{lct}(X, B + M; D + N) := \sup \{ t | (X, (B + tD) + (M + tN)) \text{ is generalized lc} \}. \]

We also need the following results.

**Theorem 2.9 (ACC for generalized lc thresholds, [BZ16 Theorem 1.5]).** Let \( d \) be a positive integer and \( \Gamma \) a DCC set of non-negative real numbers. Then there exists an ACC set \( g\text{LCT}(d, \Gamma) \) depending only on \( d \) and \( \Gamma \) satisfying the following.

Assume that \((X, B + M), M', N' \) and \( D \) are as in Definition 2.8 such that
\begin{enumerate}
  \item \((X, B + M)\) is generalized lc of dimension \( d \),
  \item \( B, D \in \Gamma \),
  \item \( M' = \sum \mu_j M'_j \) where \( M'_j \) are nef \( \mathbb{R} \)-divisors and \( \mu_j \in \Gamma \), and
  \item \( N' = \sum \nu_k N'_k \) where \( N'_k \) are nef \( \mathbb{R} \)-Cartier divisors and \( \nu_j \in \Gamma \).
\end{enumerate}

Then \( \text{lct}(X, B + M; D + N) \in g\text{LCT}(d, \Gamma) \).

**Theorem 2.10 (Global ACC for generalized polarized pairs, [BZ16 Theorem 1.6]).** Let \( d \) be a positive integer and \( \Gamma \) a DCC set of non-negative real numbers. Then there exists a finite subset \( \Gamma_0 \subseteq \Gamma \) depending only on \( d \) and \( \Gamma \) satisfying the following.

Assume that \((X, B + M)\) is a projective generalized pair with data \( X' \overset{f}{\to} X \) and \( M' \) such that
\begin{enumerate}
  \item \((X, B + M)\) is generalized lc of dimension \( d \),
  \item \( B \in \Gamma \),
  \item \( M' = \sum \mu_j M'_j \) where \( M'_j \) are nef Cartier divisors and \( \mu_j \in \Gamma \),
  \item \( \mu_j = 0 \) if \( M'_j \equiv 0 \), and
  \item \( K_X + B + M \equiv 0 \).
\end{enumerate}

Then \( B \in \Gamma_0 \) and \( \mu_j \in \Gamma_0 \) for any \( j \).
2.4. MMP for generalized polarized pairs. For generalized polarized pairs, one can ask whether one can run MMP and whether it terminates. However the MMP for generalized polarized pairs is not completed established, but some important cases could be derived from the standard MMP. We elaborate these results which are developed in [BZ16 §4].

Lemma 2.11 (Generalized dlt modification [HL18 Proposition 3.9]). Let \((X, B + M)\) be a generalized lc pair with data \(X' \xrightarrow{f} X \rightarrow Z\) and \(M'\). Then possibly replacing \(X'\) by a higher model, there exist a \(\mathbb{Q}\)-factorial generalized dlt pair \((Y, B_Y + M_Y)\) with data \(X' \xrightarrow{g} Y \rightarrow Z\) and \(M'\), and a contraction \(\phi : Y \to X\) such that \(K_Y + B_Y + M_Y = \phi^*(K_X + B + M)\). Moreover, each exceptional divisor of \(\phi\) is a component of \([B_Y]\).

We may use the following lemma frequently without citing it in this paper.

Lemma 2.12 ([BZ16 Lemma 4.4], [HL18 Lemma 3.5]). Let \((X, B + M)\) be a \(\mathbb{Q}\)-factorial generalized lc pair with data \(X' \xrightarrow{f} X \rightarrow Z\) and \(M'\) such that \(K_X + B + M\) is not pseudo-effective and either

1. \((X, B + M)\) is generalized klt, or
2. \((X, C)\) is klt for some boundary \(C\).

Then any generalized MMP on \(K_X + B + M\) with scaling of some ample/\(\mathbb{R}\)-Cartier divisor terminates with a Mori fiber space.

2.5. Complements.

Definition 2.13 (Complements). Let \((X, B + M)\) be a generalized polarized pair with data \(X' \xrightarrow{f} X \to Z\) and \(M'\), and \(z \in Z\) a point. We say that 
\((X/Z \ni z, B^+ + M)\) is an \(\mathbb{R}\)-complement of \((X/Z \ni z, B + M)\) if \((X, B^+ + M)\) is generalized lc, \(B^+ \geq B\) and \(K_X + B^+ + M \equiv 0\) over a neighborhood of \(z\).

Let \(n\) be a positive integer. We say that \((X/Z \ni z, B^+ + M)\) is an \(n\)-complement of \((X/Z \ni z, B + M)\), if over a neighborhood of \(z\), we have

1. \((X, B^+ + M)\) is generalized lc,
2. \(n(K_X + B^+ + M) \sim 0\), and
3. \(nB^+ \geq n[B] + \lfloor (n+1)\{B\}\rfloor\).

We say that \((X/Z \ni z, B^+ + M)\) is a monotonic \(n\)-complement of \((X/Z \ni z, B + M)\) if we additionally have \(B^+ \geq B\). We say that \((X/Z \ni z, B + M)\) is \(\mathbb{R}\)-complementary (respectively (monotonic) \(n\)-complementary) if it has an \(\mathbb{R}\)-complement (respectively (monotonic) \(n\)-complement).

If \(\dim Z = 0\), we will omit \(Z\) and \(z\). If for any \(z \in Z\), \((X/Z \ni z, B^+ + M)\) is \(\mathbb{R}\)-complementary (respectively (monotonic) \(n\)-complementary), then we say that \((X/Z, B + M)\) is \(\mathbb{R}\)-complementary (respectively (monotonic) \(n\)-complementary).

The following lemma is well-known to experts (c.f. [Bir19 6.1]). We will use the lemma frequently without citing it in this paper.
Lemma 2.14. Let \((X, B + M)\) be a generalized polarized pair with data \(X' \xrightarrow{f} X \to Z\) and \(M'\), and \(z \in Z\) a point. Assume that \(g : X \dasharrow X''/Z\) is a birational contraction and \(B'', M''\) are the strict transforms of \(B, M\).

1. If \((X/Z \ni z, B + M)\) is \(\mathbb{R}\)-complementary, then \((X''/Z \ni z, B'' + M'')\) is \(\mathbb{R}\)-complementary.

2. Let \(n\) be a positive integer. If \(g\) is \(-(K_X + B + M)\)-non-positive and \((X''/Z \ni z, B'' + M'')\) is \(\mathbb{R}\)-complementary (respectively monotonic \(n\)-complementary), then \((X/Z \ni z, B + M)\) is \(\mathbb{R}\)-complementary (respectively monotonic \(n\)-complementary).

3. Uniform rational polytopes

3.1. Accumulation points of generalized lc thresholds. The goal of this subsection is to prove Theorem N which is a generalization of [Nak16, Proposition 3.10].

Let \(X\) be a variety, \(B_i\) distinct prime divisors on \(X\) and \(d_i(t) : \mathbb{R} \to \mathbb{R}\) linear functions. We call the formal finite sum \(\sum_i d_i(t)B_i\) a linear functional divisor.

Definition 3.1 \((D_c(r, \Gamma))\). Let \(c\) be a non-negative real number, \(r\) a positive integer and \(0 \in \Gamma \subseteq [0, +\infty)\) a set. We define \(D_c(r, \Gamma)\) to be the set of linear functional divisors \(B(t) := \sum_i d_i(t)B_i\), such that

1. For each \(i\), either \(d_i(t) = 1\) or \(d_i(t) = \frac{m-1+u+kt}{m}\), where \(m \in \mathbb{Z}_{>0}\), \(u \in \sum(\Gamma \cup \{\frac{1}{r}\})\) and \(k \in \mathbb{Z}\), and
2. \(u + kt\) above can be written as \(u + kt = \sum_j (u_j + k_jt)\), where \(u_j \in \Gamma \cup \{\frac{1}{r}\}\), \(k_j \in \mathbb{Z}\), and \(u_j + k_jc \geq 0\) for each \(j\).

For convenience, we may write \(d(t) \in D_c(r, \Gamma)\) if \(d(t)\) satisfies the above conditions.

For convenience, if \(B(t)\) is a linear functional divisor, and \((X, B(c) + M)\) is generalized polarized pair with data \(X' \xrightarrow{f} X \to Z\) and \(M'\), then we say \((X, B(t) + M)\) is a \(\mathbb{Q}\)-factorial generalized polarized pair with data \(X' \xrightarrow{f} X \to Z\) and \(M'\).

The form of the coefficient \(d_i(t)\) is preserved by generalized adjunction. The proof is similar to [Nak16 Lemma 3.2] and [Bir19 Lemma 3.3], we may omit it.

Lemma 3.2. Let \(r\) be a positive integer, \(c\) a non-negative real number and \(0 \in \Gamma \subseteq [0, +\infty)\) a set. Suppose that \((X, B(t) + M)\) is a \(\mathbb{Q}\)-factorial generalized polarized pair with data \(X' \xrightarrow{f} X \to Z\) and \(M'\) such that

1. \((X, B(c) + M)\) is generalized lc,
2. \(B(t) = \sum_i d_i(t)B_i \in D_c(r, \Gamma)\),
3. \(d_0(t) = 1\), and \(d_i(c) > 0\) for each \(i\), and
4. \(rM'\) is \(b\)-Cartier.
Let $S$ be the normalization of $B_0$, and
$$K_S + B_S(t) + M_S = (K_X + B(t) + M)|_S$$
the generalized adjunction. Then $B_S(t) \in \mathcal{D}_c(r, \Gamma)$.

We define $\mathfrak{L}_d(r, \Gamma)$, the set of generalized lc thresholds derived from a positive integer $r$ and a set of non-negative real numbers $\Gamma$, and $\mathfrak{R}_d(r, \Gamma)$, the set of generalized numerical trivial thresholds derived from $r, \Gamma$.

**Definition 3.3** ($\mathfrak{L}_d(r, \Gamma)$). Let $d$ and $r$ be positive integers, and $0 \in \Gamma \subseteq [0, +\infty)$ a set. We write $c \in \mathfrak{L}_d(r, \Gamma)$ if there exists a $\mathbb{Q}$-factorial generalized polarized pair $(X, B(t) + M)$ with data $X' \xrightarrow{f} X \rightarrow Z$ and $M'$ such that

1. $(X, B(c) + M)$ is a generalized lc pair of dimension $\leq d$,
2. $B(t) \in \mathcal{D}_c(r, \Gamma)$,
3. $rM'$ is $b$-Cartier, and
4. either $(X, B(c + \epsilon) + M)$ is not generalized lc for any $\epsilon > 0$, or $(X, B(c - \epsilon) + M)$ is not generalized lc for any $\epsilon > 0$.

**Definition 3.4** ($\mathfrak{R}_d(r, \Gamma)$). Let $d$ and $r$ be positive integers, and $0 \in \Gamma \subseteq [0, +\infty)$ a set. We write $c \in \mathfrak{R}_d(r, \Gamma)$ if there exists a $\mathbb{Q}$-factorial projective generalized polarized pair $(X, B(t) + M)$ with data $X' \xrightarrow{f} X$ and $M'$ such that

1. $(X, B(c) + M)$ is a generalized lc pair of dimension $\leq d$,
2. $B(t) \in \mathcal{D}_c(r, \Gamma)$,
3. $rM'$ is $b$-Cartier, and
4. $K_X + B(c) + M \equiv 0$, and
5. $K_X + B(c') + M \not\equiv 0$ for some $c'$.

**Lemma 3.5.** Let $d \geq 2$ and $r$ be positive integers, and $0 \in \Gamma \subseteq [0, +\infty)$ a set. Then $\mathfrak{L}_d(r, \Gamma) \subseteq \mathfrak{R}_{d-1}(r, \Gamma)$.

**Proof.** The is similar to [Nak16, Theorem 3.6].

**Theorem 3.6.** Let $d$ and $r$ be positive integers, and $\Gamma \subseteq [0, 1]$ a finite set. Then the accumulation points of $\mathfrak{R}_d(r, \Gamma)$ belong to $\text{Span}_\mathbb{Q}(\Gamma \cup \{1\})$. In particular, the accumulation points of $\mathfrak{L}_d(r, \Gamma)$ belong to $\text{Span}_\mathbb{Q}(\Gamma \cup \{1\})$.

Theorem 3.6 immediately follows from the following theorem.

**Theorem N.** Let $d$ and $r$ be positive integers, $c$ a non-negative real number and $0 \in \Gamma \subseteq [0, +\infty)$ a finite set. Suppose that for each $i \in \mathbb{Z}_{>0}$, there exist a positive real number $c_i$ and a $\mathbb{Q}$-factorial projective generalized polarized pair $(X_i, (A_i + B_i(t_i)) + M_i)$ with data $X_i' \xrightarrow{f_i} X_i$ and $M_i'$, such that

1. $(X_i, (A_i + B_i(c_i)) + M_i)$ is a generalized lc pair of dimension $\leq d$,
2. the coefficients of $A_i$ are approaching 1,
3. $B_i(t_i) \in \mathcal{D}_c(r, \Gamma)$, $rM_i'$ is $b$-Cartier,
4. $\lim c_i = c$,
5. $K_{X_i} + A_i + B_i(c_i) + M_i \equiv 0$, and
(6) $K_{X_i} + A_i + B_i(c'_i) + M_i \not\equiv 0$ for some $c'_i$.

Then $c \in \text{Span}_Q(\Gamma \cup \{1\})$.

**Theorem P.** Let $d$ and $r$ be positive integers, $c$ a non-negative real number and $0 \in \Gamma \subseteq [0, +\infty)$ a finite set. Suppose that for each $i \in \mathbb{Z}_{\geq 0}$, there exist a positive real number $c_i$ and a $\mathbb{Q}$-factorial projective generalized polarized pair $(X_i, (A_i + B_i(t)) + M_i)$ with data $X_i \overset{f}{\to} X_i$ and $M_i'$, such that

1. $(X_i, (A_i + B_i(c_i)) + M_i)$ is a generalized klt pair of dimension $\leq d$,
2. $X_i$ is a Fano variety with Picard number 1,
3. the coefficients of $A_i$ are approaching 1,
4. $B_i(t) \in D_{c_i}(r, \Gamma)$, $rM_i'$ is $b$-Cartier,
5. $\lim c_i = c$, 
6. $K_{X_i} + A_i + B_i(c_i) + M_i \equiv 0$, and
7. $K_{X_i} + A_i + B_i(c'_i) + M_i \not\equiv 0$ for some $c'_i$.

Then $c \in \text{Span}_Q(\Gamma \cup \{1\})$.

We will prove Theorem N and Theorem P inductively. Before that, we put some additional conditions.

**Remark 3.7.** In Theorem N and Theorem P, we may write $B_i(t) := \sum j d_{ij}(t)B_{ij}$ by definition. Possibly replacing $A_i$ and $B_i(t)$, we may assume that $d_{ij}(t)$ is not identically one. By [Nak16] Lemma 3.7, we may assume that

$$(\Gamma \cup \{\frac{1}{r}\}) \cap c\mathbb{Z}_{>0} = \emptyset.$$

By [Nak16] Claim 3.13, possibly passing to a subsequence, we may assume that

- if $d_{ij}(t) = \frac{m_{ij}^{-1}+u_{ij}+k_{ij}t}{m_{ij}}$, where $m_{ij} \in \mathbb{Z}_{>0}, u_{ij} \in \sum(\Gamma \cup \frac{1}{r})$ and $k_{ij} \in \mathbb{Z}$, then $u_{ij}, k_{ij}$ have only finitely many possibilities,
- $d_{ij}(c_i)$ are bounded from zero, and $d_{ij}(c_i) < 1$ for any $i, j$,
- $d_{ij}(c) > 0$ for any $i, j$, and
- the set $\{d_{ij}(c) \mid i, j\}$ satisfies the DCC.

**Lemma 3.8.** Notation as in Theorem N. Then possibly passing to a subsequence, we may assume that $(X_i, ([A_i] + B_i(c)) + M_i)$ is generalized lc.

**Proof.** We may write $B_i(t) = B_{i0} + t(B_i^+ - B_i^-)$, where $B_i^+ \geq 0$ and $B_i^- \geq 0$ have no common components. Note that $A_i + B_{i0} + cB_i^+ - cB_i^- + M_i \geq 0$. Hence $(X_i, (A_i + B_{i0} + cB_i^+ - cB_i^- + M_i)$ is generalized lc. As the coefficients of $B_{i0} + cB_i^+ - cB_i^-$ belong to a DCC set, and the coefficients of $A_i$ are increasing, by Theorem 3.11 possibly passing to a subsequence, we may assume that $(X_i, ([A_i] + B_i(c)) + M_i)$ is generalized lc for any $i$. \(\Box\)

In the following, by “(Theorem N)$_d$” (respectively “(Theorem P)$_d$”), we mean Theorem N (respectively Theorem P) with dimension $\leq d$. 
Proposition 3.9. Let $d \geq 2$ be an integer. Then (Theorem $N_{d-1}$) implies (Theorem $N_d$) if there exists a component $S_i$ of $[A_i]$ such that $(K_{X_i} + A_i + B_i(c_i') + M_i)|_{S_i} \neq 0$ for any $i$.

Proof. The proposition follows by generalized adjunction. \qed

Proposition 3.10. Let $d \geq 2$ be an integer. Then (Theorem $N_{d-1}$) implies (Theorem $N_d$) if there exists a Mori fiber space $X_i \to Z_i$ with $\dim Z_i > 0$ and $\text{Supp} A_i$ dominates $Z_i$ for any $i$.

Proof. Let $F_i$ be a general fiber of $g_i : X_i \to Z_i$. Then restricting to $F_i$ gives $K_{F_i} + A_{F_i} + B_{F_i}(t) + M_{F_i} := (K_{X_i} + A_i + B_i(t) + M_i)|_{F_i}$, and $(F_i, (A_{F_i} + B_{F_i}(c_i)) + M_{F_i})$ is a generalized lc pair of dimension $\leq d - 1$.

By Lemma 3.8, we may assume that $(F_i, (A_{F_i} + B_{F_i}(c_i)) + M_{F_i})$ is generalized lc.

If $K_{F_i} + A_{F_i} + B_{F_i}(c_i') + M_{F_i} \neq 0$, then we are done by assumption. Suppose that $K_{F_i} + A_{F_i} + B_{F_i}(c_i') + M_{F_i} \equiv 0$, then $K_{F_i} + A_{F_i} + B_{F_i}(c_i') + M_{F_i} \equiv 0$. By Theorem 2.10 possibly passing to a subsequence, we have $|A_i| = A_i$ for any $i$. In particular, there exists a component $S_i$ of $A_i$ such that $g_i(S_i) = Z_i$.

It follows that $(K_{X_i} + A_i + B_i(c) + M_i)|_{S_i} \neq 0$. Hence we are done by Proposition 3.9. \qed

Proposition 3.11. Let $d \geq 2$ be an integer. Then (Theorem $P_d$) and (Theorem $N_{d-1}$) imply (Theorem $N_d$).

Proof. Possibly passing to a subsequence, we may assume that $\dim X_i = d$. Suppose that $(X_i, A_i + B_i(c_i) + M_i)$ is not generalized klt for any $i$. Possibly replacing $(X_i, (A_i + B_i(c_i)) + M_i)$ by a generalized dlt model, we may assume that $(X_i, (A_i + B_i(c_i)) + M_i)$ is Q-factorial generalized dlt, and $|A_i| = |A_i + B_i(c_i)| \neq 0$. Then we may run a generalized MMP on $(K_{X_i} + A_i + B_i(c_i) + M_i - |A_i|)$ with scaling of an ample divisor,

$$X_i := X_i^{(0)} \to X_i^{(1)} \to \cdots \to X_i^{(m_i)} \to Z_i$$

which terminates with a Mori fiber space $X_i^{(m_i)} \to Z_i$, as $K_{X_i} + A_i + B_i(c_i) + M_i - |A_i| \equiv -|A_i|$ is not pseudo-effective. Let $A_i^{(j)}, B_i^{(j)}(t)$ and $M_i^{(j)}$ be the strict transforms of $A_i, B_i(t)$ and $M_i$ on $X_i^{(j)}$ respectively, and $D_i^{(j)} := K_{X_i^{(j)}} + A_i^{(j)} + B_i^{(j)}(c_i') + M_i^{(j)}$ for any $1 \leq j \leq m_i$.

Assume that there exists an integer $1 \leq l_i \leq m_i - 1$, such that $D_i^{(j)} \neq 0$ for any $1 \leq j \leq l_i$, and $D_i^{(l_i+1)} \equiv 0$. Then $f_i^{(l_i)} : X_i^{(l_i)} \to X_i^{(l_i+1)}$ is a divisorial contraction. Let $E_i$ be the $f_i^{(l_i)}$-exceptional divisor. Since $D_i^{(l_i)} \neq 0$, $D_i^{(l_i)} - (f_i^{(l_i)})^*D_i^{(l_i+1)} = \alpha_i E_i$ for some non-zero real number $\alpha_i$. Since $f_i^{(l_i)}$ is $[A_i^{(l_i)}]$-positive, there exists a component $S_i$ of $\text{Supp}[A_i^{(l_i)}]$ such that $E_i|_{S_i} \neq 0$ and we are done by Proposition 3.9.

We may assume that $D_i^{(m_i)} \neq 0$. Possibly replacing $X_i$ by $X_i^{(m_i)}$, and $A_i, B_i(t), M_i$ by its strict transforms respectively, we may assume that $X_i$
admits a Mori fiber space $g_i : X_i \to Z_i$. Note that $\text{Supp}A_i$ dominates $Z_i$ as $g_i$ is $[A_i]$-positive. If $\dim Z_i > 0$, we are done by Proposition 3.10. If $\dim Z_i = 0$, the we are done since then $X_i$ has Picard number one and therefore $(K_{X_i} + A_i + B_i(c'_i) + M_i)|_{T_i} \neq 0$ for any component $T_i$ of $\text{Supp}|A_i|$.

Now assume that $(X_i, (A_i + B_i(c'_i)) + M_i)$ is generalized klt for any $i$. It follows that $(X_i, A_i + B_i(c''_i) + M_i)$ is generalized klt and $K_{X_i} + A_i + B_i(c''_i) + M_i$ is not pseudo-effective for some positive real number $c''_i$. We may run a generalized MMP on $(K_{X_i} + A_i + B_i(c''_i) + M_i)$ with scaling of an ample divisor which terminates with a Mori fiber space $h_i : X_i \to V_i$. Since every step of the generalized MMP is $(K_{X_i} + A_i + B_i(c''_i) + M_i)$-negative and $K_{X_i} + A_i + B_i(c''_i) + M_i$ is not pseudo-effective, $(K_{X_i} + A_i + B_i(c''_i) + M_i) \neq 0$. We may replacing $X_i$ by $\tilde{X}_i$, and $A_i, B_i(t), M_i$ by its strict transforms respectively, and therefore assume that $X_i$ admits a Mori fiber space $h_i : X_i \to V_i$.

If $\dim V_i > 0$, let $F_k$ be a general fiber of $h_i$. As $h_i$ is $(K_{X_i} + A_i + B_i(c'_i) + M_i)$-negative, $(K_{X_i} + A_i + B_i(c'_i) + M_i)|_{F_i} \neq 0$. By assumption, we are done by restricting to $F_i$. If $\dim V_i = 0$, then $X_i$ has Picard number one. We finish the proof, since Theorem P holds in dimension $d$. □

**Proposition 3.12.** Let $\epsilon$ be a positive real number and $d$ a positive integer. Then (Theorem P)$_d$ holds if we additionally assume that $X_i$ is $\epsilon$-lc for every $i$. In particular, (Theorem N)$_1$ and (Theorem P)$_1$ hold.

**Proof.** Possibly passing to a subsequence, we may assume that and $\dim X_i = d$ and $A_i = 0$. By [Bir16] Theorem 1.1], $X_i$ belongs to a bounded family. In particular, there is a very ample divisor $H_i$ on $X_i$, such that $H_i^d$ and $-K_{X_i} \cdot H_i^{d-1}$ are bounded from above. Since $X_i$ has Picard number one, $M_i$ is nef and $rM$ is integral. By the assumption, we have

$$(K_{X_i} + B_i(c_i) + M_i) \cdot H_i^{d-1} = 0 \text{ and } B(c_i) \cdot H_i^{d-1} \neq B(c'_i) \cdot H_i^{d-1}$$

which implies that $c_i = c$ for $i$ sufficiently large, as there are only finitely many possibilities for $u_{ij}, k_{ij}$ that appearing in $d_{ij}(t)$.

If $d = 1$, then $X_i$ is $\mathbb{P}^1$ for any $i$. Thus (Theorem N)$_1$ and (Theorem P)$_1$ hold. □

**Proposition 3.13.** Let $d \geq 2$ be an integer. Then (Theorem N)$_{d-1}$ implies (Theorem P)$_d$.

**Proof.** Possibly passing to a subsequence, we may assume that $c_i$ is decreasing and $\dim X_i = d$. Let

$$a_i := \text{mld}(X_i, A_i + B_i(c_i) + M_i) = a(E_i, X_i, A_i + B_i(c_i) + M_i)$$

for some prime divisor $E_i$ over $X_i$. By Proposition 3.12 we may assume that $a_i$ is decreasing, $\lim_{i \to +\infty} a_i = 0$ and $a_i < 1$ for any $i$.

We first reduce to the case when $A_i \neq 0$. Possibly replacing $X_i'$ by a higher model, there exists a morphism $\phi_i : \tilde{X}_i \to X_i$ contracting $E_i$. We may write

$$K_{\tilde{X}_i} + (1 - a_i)E_i + \tilde{A}_i + \tilde{B}_i(c_i) + \tilde{M}_i = \phi_i^*(K_{X_i} + A_i + B_i(c_i) + M_i),$$

where \(\tilde{A}_i, \tilde{B}_i, \tilde{M}_i\) are the pullbacks of \(A_i, B_i, M_i\). By assumption, we have $\tilde{A}_i = 0$. We obtain

$$K_{\tilde{X}_i} + (1 - a_i)E_i + \tilde{B}_i(c_i) + \tilde{M}_i = \phi_i^*(K_{X_i} + B_i(c_i) + M_i).$$

Since $\dim V_i = 0$, by Proposition 3.10, we are done.
where $\tilde{A}_i, \tilde{B}_i(t)$ and $\tilde{M}_i$ are the strict transforms of $A_i, B_i(c_i)$ and $M_i$ on $X_i$ respectively. Possibly replacing $X_i, A_i, B_i(t)$ and $M_i$ by $X_i, (1 - a_i)E_i + A_i, B_i(t)$ and $\tilde{M}_i$ respectively, we may assume that $A_i \neq 0$. Note that $X_i$ may has Picard number $\geq 2$. Since $K_{X_i} + B_i(c_i) + M_i \equiv -A_i$ is not pseudoeffective, we may run a generalized MMP on $(K_{X_i} + B_i(c_i) + M_i)$ which terminates with a Mori fiber space $X''_i \to Z_i$. Let $A''_i, B''_i(t)'', M''_i$ be the strict transforms of $A_i, B_i(t)$ and $M_i$ respectively. Since each step of the generalized MMP is $A_i$-positive, $A''_i \neq 0$. We claim that $K_{X_i} + A_i + B_i(c'_i) + M_i \equiv 0$. Otherwise $K_{X_i} + A_i + B_i(c) + M_i \equiv 0$, which contradicts Theorem 2.10 as $(X_i, (A_i + B_i(c)) + M_i)$ is generalized lc (Lemma 3.8) and the coefficients of $A_i$ are approaching 1.

We may replace $X_i$ by $X''_i$, and assume that $X_i$ admits a Mori fiber space $g_i: X_i \to Z_i$. Note that we have $A_i \neq 0$ and Supp $A_i$ dominates $Z_i$ since $g_i$ is $A_i$-positive. If $\dim Z_i > 0$, then we are done by Proposition 3.10. Hence we may assume that $\dim Z_i = 0$ and $X_i$ has Picard number one.

**Claim 3.14.** Possibly passing to a subsequence, we may assume that $K_{X_i} + A_i + B_i(c) + M_i$ is not ample for any $i$.

**Proof of Claim 3.14.** By Lemma 3.8 possibly passing to a subsequence, we may assume that $(X_i, (A_i + B_i(c_i)) + M_i)$ is generalized lc for any $i$. We may write $B_i(t) := B_i(0) + t (B_i^+ - B_i^-)$, where $B_i^+ \geq 0$ and $B_i^- \geq 0$ have no common components.

Suppose on the contrary that $K_{X_i} + A_i + B_i(c) + M_i$ is ample. Then $(c - c_i)(B_i^+ - B_i^-)$ is ample, since $K_{X_i} + A_i + B_i(c_i) + M_i \equiv 0$. In particular, $B_i^+ - B_i^-$ is antiample, as $c_i > c$ by assumption. Then $B_i^- \equiv \gamma_i B_i^+$ for some real number $\gamma_i > 1$. Let $t_i := c_i - \frac{c - c_i}{\gamma_i}$. Then $c < t_i < c_i$ and $(X_i, A_i + B_i(0) + cB_i^+ - t_i B_i^- + M_i)$ is generalized lc. Moreover, we have

$$K_{X_i} + A_i + B_i(0) + cB_i^+ - t_i B_i^- + M_i \equiv 0,$$

which contradicts Theorem 2.10. $\square$

Now we can finish the proof. If $(X_i, ([A_i] + B_i(c_i)) + M_i)$ is not generalized lc, then we set

$$e_i := \sup \{ t \mid (X_i, ([A_i] + B_i(t)) + M_i) \text{ is generalized lc} \}.$$

Then $e_i \in L_d(r, \Gamma) \subseteq \mathfrak{m}_{d-1}(r, \Gamma)$, and $\lim e_i = \lim c_i = c$, we are done. Thus we may assume that $(X_i, ([A_i] + B_i(c_i)) + M_i)$ is generalized lc.

We define $\alpha_i$ and $\beta_i$ as

$$K_{X_i} + [A_i] + B_i(\alpha_i) + M_i \equiv 0, K_{X_i} + \beta_i [A_i] + B_i(c) + M_i \equiv 0.$$

We have that $\alpha_i < c_i$, by Claim 3.14 and the assumption that $K_{X_i} + A_i + B_i(c_i) + M_i \equiv 0$. If $\alpha_i < c < c_i$, then $K_{X_i} + [A_i] + B_i(c) + M_i$ is ample and thus $\beta_i < 1$. As the coefficients of $A_i$ are approaching one, $\lim \beta_i = 1$. In particular, we may assume that the set of coefficients of $\beta_i [A_i] + B_i(c) + M_i$ satisfies the DCC, which contradicts Theorem 2.10. Thus $c \leq \alpha_i < c_i$, and $\lim \alpha_i = c$. Note that then $(X_i, ([A_i] + B_i(\alpha_i)) + M_i)$ is generalized lc and not
generalized klt, as both \((X_i, ([A_i] + B_i(c)) + M_i)\) and \((X_i, ([A_i] + B_i(c_1)) + M_i)\) are generalized lc. We may replace \(A_i\) by \([A_i]\), and we are done by Proposition 3.9.

**Proof of Theorem N and Theorem P.** The theorems follow from Proposition 3.9, Proposition 3.10, Proposition 3.11, Proposition 3.12, and Proposition 3.13.

3.2. NAKAMURA’S (GENERALIZED) LC RATIONAL POLYTOPES. In this subsection, we show Theorem 1.3.

The proof of Theorem 3.15 is similar to that of [Nak16, Theorem 1.6]. Note that \(K_X + B(t) + M\) is \(\mathbb{R}\)-Cartier for any \(t \in \mathbb{R}\), by [HLS19, Lemma 5.4].

**Theorem 3.15.** Let \(d, c, r, m\) be positive integers, \(r_0 := 1, r_1, \ldots, r_c\) real numbers which are linearly independent over \(\mathbb{Q}\), and \(s_1, \ldots, s_m : \mathbb{R}^{c+1} \to \mathbb{R}\) \(\mathbb{Q}\)-linear functions. Then there exists a positive real number \(\epsilon\) depending only on \(d, r, r_1, \ldots, r_c\) and \(s_1, \ldots, s_m\) satisfying the following.

Assume that \((X, B(r_c) + M)\) is a generalized polarized pair with data \(X' \xrightarrow{f} X \to Z\) and \(M'\) such that

1. \(\dim X = d\),
2. \(B(t) := \sum_{i=1}^{m} s_i(r_0, \ldots, r_{c-1}, t)B_i\), where \(B_i\) are Weil divisors on \(X\),
3. \(rM'\) is \(b\)-Cartier, and
4. \((X, B(r_c) + M)\) is generalized lc.

Then \((X, B(t) + M)\) is generalized lc for any \(t\) satisfying \(|t - r_c| \leq \epsilon\).

**Proof.** We may write \(s_i(x_0, \ldots, x_c) := \sum_{j=0}^{c} q_{ij}x_j\), for any \(1 \leq i \leq m\), where \(q_{ij}\) is a rational number for any \(i, j\). Let \(n\) be a positive integer such that \(nq_{ic} \in \mathbb{Z}\) for any \(1 \leq i \leq m\). Since \(s_i(r_0, \ldots, r_c) \geq 0\) and \(r_0, \ldots, r_c\) are linearly independent over \(\mathbb{Q}\), there exist two rational numbers \(t^-< t^\ast < t^+\) such that \(t^- \leq r_c \leq t^+\), and \(s_i(r_0, \ldots, r_{c-1}, t) \geq 0\) for any \(t\) satisfying \(t^- \leq t \leq t^+\).

Suppose on the contrary that there exist generalized polarized pairs \((X_i, B_i(t) + M_i)\) with data \(X'_i \xrightarrow{g_i} X_i \to Z_i\) and \(M'_i\), and prime divisors \(B_{ij}\) (\(1 \leq j \leq m\)) satisfying the conditions and that either \(\lim_{t \to \infty} h_i^+ = r_c\) or \(\lim_{t \to \infty} h_i^- = r_c\), where \(h_i^+\) and \(h_i^-\) are defined as

\[
\begin{align*}
  h_i^+ &:= \sup\{t \geq r_c \mid (X_i, B_i(t) + M_i) \text{ is generalized lc}\}, \\
  h_i^- &:= \sup\{t \leq r_c \mid (X_i, B_i(t) + M_i) \text{ is generalized lc}\}.
\end{align*}
\]

For each \(i\), possibly replacing \(X'_i\) by a higher model, there exists a generalized dlt modification \(g_i : Y_i \to X_i\) of \((X_i, B_i(r_c) + M_i)\), such that

\[
K_{Y_i} + B_{Y_i}(r_c) + E_i + M_{Y_i} = g_i^*(K_{X_i} + B_i(r_c) + M_i),
\]

where \(M_{Y_i}, B_{Y_i}(t)\) are the strict transforms of \(M'_i, B_i(t)\) on \(Y_i\), and \(E_i\) is the sum of reduced \(g_i\)-exceptional divisors.

Since \(r_0, \ldots, r_c\) are linearly independent over \(\mathbb{Q}\), we have

\[
K_{Y_i} + B_{Y_i}(t) + E_i + M_{Y_i} = g_i^*(K_{X_i} + B_i(t) + M_i)
\]
for any \( t \in \mathbb{R} \). Possibly replacing \( (X_i, B_i(t) + M_i) \) by \( (Y_i, (B_Y(t) + E_i) + M_Y) \), we may assume that \( X_i \) is \( \mathbb{Q} \)-factorial for any \( i \).

Without loss of generality, we may assume that \( \lim_{i \to \infty} h_i^- = r_c \) and \( t^- \leq h_i^- \leq r_c \). Let \( \Gamma := \{ s_i(r_0, \ldots, r_{c-1}, t^-) \mid 1 \leq i \leq m \} \) be a finite set. Since

\[
B_i(t) = B_i(t^-) + \sum_{j=1}^{m} \frac{t^- - t}{n}(nq_{ij})B_{ij},
\]

we have that \( \frac{h_i^- - t^-}{n} \in \mathcal{L}_d(r, \Gamma) \). Hence

\[
\frac{r_c - t^-}{n} \in \text{Span}_Q(\{ \Gamma \cup \{ 1 \} \}) \subseteq \text{Span}_Q(\{ r_0, \ldots, r_{c-1} \})
\]

by Theorem 3.16, a contradiction. \( \square \)

Following the same arguments as in \cite[Theorem 5.6]{HLS19}, we have the following result.

**Proof of Theorem 1.4.** The result follows from Theorem 3.15. \( \square \)

**Remark 3.16.** When \( v \in \mathbb{Q}^m \), then \( U = V = \{ v \} \). We also note that \( U \) does not depend on \( X \).

### 3.3. Han type polytopes for \( \mathbb{R} \)-complementary (generalized) pairs.

**Theorem 3.17.** Let \( d, r, c \) and \( m \) be positive integers, \( r_0 := 1, r_1, \ldots, r_c \) real numbers which are linearly independent over \( \mathbb{Q} \), and \( s_1, \ldots, s_m : \mathbb{R}^{c+1} \to \mathbb{R} \) \( \mathbb{Q} \)-linear functions. Then there exists a positive real number \( \epsilon \) depending only on \( d, r, r_1, \ldots, r_c \) and \( s_1, \ldots, s_m \) satisfying the following.

Assume that \( (X, B(t) + M) \) is a generalized polarized pair with data \( X' \xrightarrow{\varphi} X \to Z \) and \( M' \) such that

1. \( \dim X = d, X \) is of Fano type over \( Z \),
2. \( B(t) := \sum_{i=1}^{m} s_i(r_0, \ldots, r_{c-1}, t)B_i \), where \( B_i \) are Weil divisors on \( X \),
3. \( rM' \) is b-Cartier, and
4. \( (X/Z, B(r_c) + M) \) is \( \mathbb{R} \)-complementary.

Then \( (X/Z, B(t) + M) \) is \( \mathbb{R} \)-complementary for any \( t \) satisfying \( |t - r_c| \leq \epsilon \).

The proof is very similar to that of \cite[Theorem 5.16]{HLS19}.

**Proof.** We claim that there exists a positive real number \( \epsilon \), such that \( -(K_X + B(t) + M) \) is pseudo-effective for any \( t \) satisfying \( |t - r_c| \leq \epsilon \). Suppose that the claim does not hold. By Theorem 3.15, there exist generalized polarized pairs \( (X_i, B_i(t) + M_i) \) with data \( X'_i \xrightarrow{\varphi} X_i \to Z_i \) and \( M'_i \) satisfying the conditions and either \( \lim h_i^+ = r_c \) or \( \lim h_i^- = r_c \), where

\[
\begin{align*}
    h_i^+ &:= \sup \{ t \geq r_c \mid -(K_{X_i} + B_i(t) + M_i) \text{ is pseudo-effective}/Z \}, \\
    h_i^- &:= \sup \{ t \leq r_c \mid -(K_{X_i} + B_i(t) + M_i) \text{ is pseudo-effective}/Z \}.
\end{align*}
\]
Without loss of generality, we may assume that \( \lim h_i^+ = r_c \). Possibly passing to a subsequence, we may assume that \( h_i^+ \) is strictly decreasing. Possibly passing to a subsequence again, by Theorem 3.15, we may assume that there exists a sequence of real numbers \( t_i \), such that \( h_i^{+} < t_i \leq h_{i-1}^{+} \). \((X_i, B_i(t_i) + M_i)\) is generalized dlt, and \(- (K_{X_i} + B_i(t_i) + M_i)\) is not pseudo-effective over \( Z \).

We may run an MMP/\( Z_t \) on \(- (K_{X_t} + B_t(t_t) + M_t)\) which terminates with a model \( Y_t \to Z_t \) over \( Z_t \), such that \(- (K_{Y_t} + B_t(t_t) + M_{Y_t})\) is semiample over \( Z_t \), where \( B_t(t_t), M_{Y_t} \) are the strict transforms of \( B_t(t_t), M_t \) on \( Y_t \). Since \(- (K_{Y_t} + B_t(h_i^{+}) + M_{Y_t})\) is pseudo-effective over \( Z_t \), \(- (K_{Y_t} + B_t(h_i^{+}) + M_{Y_t})\) is nef over \( Z_t \). Thus there exists a real number \( \eta_t \) such that \( h_i^{+} \leq \eta_t < t_t \) and

\[
(K_{Y_t} + B_t(\eta_t) + M_{Y_t})|_{F_t} \equiv 0,
\]

where \( F_t \) is a general fiber of \( Y_t \to Z_t \). Since \( (X_t/Z_t, B_t(r_c) + M_t) \) is \( \mathbb{R} \)-complementary, \((Y_t, B_t(r_c) + M_{Y_t})\) is generalized lc. By Theorem 3.15, we may assume that both \((Y_t, B_t(t_t) + M_{Y_t})\) and \((Y_t, B_t(h_i^{+}) + M_{Y_t})\) are generalized lc. Thus \((K_{Y_t} + B_t(\eta_t) + M_{Y_t})|_{F_t} \) is generalized lc. Since \( \lim_{t \to +\infty} \eta_t = r_c \), by Theorem 3.16, \( r_c \in \text{Span}_\mathbb{Q}(\{r_0, r_1, \ldots, r_{c-1}\}) \), a contradiction.

For any \( t \) satisfying \( |t - r_c| \leq \epsilon \), we may run an MMP/\( Z \) on \(- (K_X + B(t) + M)\) which terminates with a model \( Y_t \), such that \(- (K_{Y_t} + B_Y(t) + M_{Y_t})\) is semiample over \( Z \), where \( B_{Y_t}(t), M_{Y_t} \) are the strict transforms of \( B(t), M \) on \( Y_t \). Since \((Y_t, B_Y(r_c) + M_{Y_t})\) is generalized lc, by the claim again, \((Y_t, B_Y(t) + M_{Y_t})\) is generalized lc. Thus \((Y_t/Z, B_Y(t) + M_{Y_t})\) is \( \mathbb{R} \)-complementary and so is \((X/Z, B(t) + M)\).

By Theorem 3.17 and [HLS19, Theorem 5.17], we can show the existence of Han type polytopes for \( \mathbb{R} \)-complementary (generalized) polarized pairs.

**Theorem 3.18** (Han type polytopes for \( \mathbb{R} \)-complementary (generalized) pairs). Let \( d, r \) and \( m \) be positive integers, \( v_0 := (v_1^0, \ldots, v_m^0) \in \mathbb{R}^m \) a point and \( V \subseteq \mathbb{R}^m \) the rational envelope of \( v_0 \). Then there exists an open set \( U \ni v_0 \) of \( V \) depending only on \( d, r, m \) and \( v_0 \) satisfying the following.

Assume that \((X, (\sum_{i=1}^m v_i^0 B_i) + M)\) is a generalized polarized pair with data \( X \to X \to Z \) and \( M' \) such that

1. \( \dim X = d, X \) is of Fano type over \( Z \),
2. \( B_1, \ldots, B_m \geq 0 \) are Weil divisors on \( X \),
3. \( rM' \) is \( b \)-Cartier, and
4. \((X/Z, (\sum_{i=1}^m v_i B_i) + M)\) is \( \mathbb{R} \)-complementary.

Then \((X/Z, (\sum_{i=1}^m v_i B_i) + M)\) is \( \mathbb{R} \)-complementary for any \((v_1, \ldots, v_m) \in U \).
4. Boundedness of relative complements

In this section, we show the following result which is the relative version of [Bir19, Theorem 1.10]. Proofs are very similar to those in [Bir19, Section 8].

**Theorem 4.1.** Let $d$ and $r$ be positive integers and $\Gamma \subseteq [0,1] \cap \mathbb{Q}$ a finite set. Then there exists a positive integer $n$ depending only on $d$, $r$ and $\Gamma$ satisfying the following.

Assume that $(X, B + M)$ is a generalized polarized pair with data $X' \xrightarrow{f} X \rightarrow Z$ and $M'$ such that

- $(X, B + M)$ is a generalized lc pair of dimension $d$,
- $B \in \Phi(\Gamma)$ and $rM'$ is $b$-Cartier,
- $X$ is of Fano type over $Z$, and
- $-(K_X + B + M)$ is nef over $Z$.

Then for any point $z \in Z$, $(X/Z \ni z, B + M)$ has a monotonic $n$-complement.

**Proposition 4.2.** Let $d \geq 2$ be a positive integer. Then (Theorem 4.1) implies (Theorem 4.1) for those $(X, B + M)$ such that

1. $B \in \Gamma$,
2. $(X, \Sigma + \alpha M)$ is $\mathbb{Q}$-factorial generalized plt for some boundary $\Sigma$ and $\alpha \in (0,1)$,
3. $-(K_X + \Sigma + \alpha M)$ is ample over $Z$,
4. $S = |\Sigma| \leq |B|$ is irreducible, and
5. $S$ intersects $\pi^{-1}(z)$, where $\pi$ is the morphism $X \rightarrow Z$.

**Proof.** We may find a boundary $\Sigma_1$ such that $(X, \Sigma_1)$ is plt, $-(K_X + \Sigma_1)$ is ample over $Z$, and $|\Sigma_1| = S$. By the same arguments as in [Bir19, Proposition 8.1], $S \rightarrow \pi(S)$ is a contraction.

Possibly replacing $X'$ by a higher model, we may assume that $f : X' \rightarrow X$ is a log resolution of $(X, B + \Sigma)$, $rM'$ is Cartier, and the induced map $\phi : S' \rightarrow S$ is a morphism, where $S'$ is the strict transform of $S$ on $X'$. Let $N' := K_{X'} + B' + M' = f^*(K_X + B + M)$, and $K_{X'} + \Sigma' + \alpha M' := f^*(K_X + \Sigma + \alpha M)$. Then we have the generalized adjunction

$$K_S + B_S + M_S \sim_Q (K_X + B + M)|_S$$

such that $rM_S$ is Cartier. Moreover, by [Bir19, 3.1(2)], we may assume that $r(K_S + B_S + M_S) \sim r(K_X + B + M)|_S$.

By [Bir19] Lemma 3.3, $B_S \in \Phi(\Gamma_1)$ for some finite set $\Gamma_1 \subseteq [0,1] \cap \mathbb{Q}$ depending only on $r$ and $\Gamma$. Restricting $K_X + \Sigma + \alpha M$ to $S$ shows that $S$ is a of Fano type over $\pi(S)$ by [Bir19, 2.13(6)]. Thus there exists a positive integer $n$ divisible by $r$ depending only on $d - 1$, $r$ and $\Gamma_1$ such that $(S, B_S + M_S)$ has a monotonic $n$-complement $(S, B_S^+ + M_S)$ over $z$, where
\[B^+_S := B_S + R_S \text{ for some } \mathbb{R}\text{-divisor } R_S \geq 0. \] Possibly replacing \( n \) by a larger number, we may assume that \( n \Gamma \subseteq \mathbb{Z} \). Let \( R_{S'} := \phi^* R_S \), then we have
\[ nN'|_{S'} \sim -n\phi^*(K_S + B_S + M_S) \sim nR_{S'} \geq 0. \]

In the following, we want to lift \( R_{S'} \) from \( S' \) to \( X' \).
Let \( T' := [B'^{\geq 0}] \) and \( \Delta' := B' - T' \). Define
\[ L' := -nK_{X'} - nT' - [(n + 1)\Delta'] - nM' = n\Delta' - [(n + 1)\Delta'] + nN', \]
which is an integral divisor. Possibly replacing \( \Sigma' \) by \((1 - a)\Sigma' + aB'\), and \( \alpha M' \) by \(((1 - a)\alpha + a)M' \) for some \( a \in (0, 1) \) sufficiently close to 1, we may assume that \( \alpha \) is sufficiently close to 1 and \( B' - \Sigma' \) has sufficiently small coefficients.

We claim that there exists a divisor \( P' \) on \( X' \), such that \( P' \) is exceptional over \( X \), \((X', N')\) is plt and \( |\Delta'| = S' \), where \( \Delta' := \Sigma' + n\Delta' - [(n + 1)\Delta'] + P' \). Indeed let \( \text{mult}_{S'} P' = 0 \), and for each prime divisor \( D' \neq S' \), let \( \text{mult}_{D'} P' = -\text{mult}_{D'} |\Sigma'| = -\text{mult}_{D'} |\Sigma' - \Delta' + ((n + 1)\Delta')| \).

This implies that \( 0 \leq \text{mult}_{D'} P' \leq 1 \) for any prime divisor \( D' \). In fact if \( D' \neq S' \) is a component of \( T' \), then \( D' \) is not a component of \( \Delta' \) and \( \text{mult}_{D'} \Sigma' \neq 0 \), hence \( \text{mult}_{D'} P' = 0 \). If \( D' \) is not a component of \( T' \), then \( \text{mult}_{D'} (\Sigma' - \Delta') = \text{mult}_{D'} (\Sigma' - B') \) is sufficiently small and \( \text{mult}_{D'} P' = 0 \).

It suffices to show that \( P' \) is exceptional over \( X \). Assume that \( D' \) is a component of \( P' \) that is not exceptional over \( X \). Then \( D' \neq S' \), and since \( nB \) is integral, \( \text{mult}_{D'} n\Delta' \) is integral. Hence \( \text{mult}_{D'} [(n + 1)\Delta'] = \text{mult}_{D'} [n\Delta'] \) which implies that \( \text{mult}_{D'} P' = -\text{mult}_{D'} |\Sigma'| = 0 \), a contradiction.

By construction,
\[ (L' + P')|_{S'} = (n\Delta' - [(n + 1)\Delta'] + nN' + P')|_{S'}, \]
\[ \sim_Z nR_{S'} + n\Delta' - [(n + 1)\Delta'] + P_{S'} =: G_{S'}, \]
where \( \Delta_{S'} := \Delta'|_{S'} \) and \( P_{S'} := P'|_{S'} \). Note that \( G_{S'} \) is integral by the choice of \( R_{S'} \). Moreover, as the coefficients of \( n\Delta' - [(n + 1)\Delta'] \) belong to \((-1, 1)\), \( G_{S'} \geq 0 \).

Let \( A := -(K_X + \Sigma + \alpha M) \), and \( A' := f^* A \). Then
\[ L' + P' = n\Delta' - [(n + 1)\Delta'] + nN' + P' \]
\[ = K_{X'} + \Sigma' + \alpha M' + A' + n\Delta' - [(n + 1)\Delta'] + nN' + P' \]
\[ = K_{X'} + \Lambda' + A' + \alpha M' + nN'. \]
Possibly shrinking \( Z \) near \( z \), we may assume that \( Z \) is affine. Since \( A' + \alpha M' + nN' \) is nef and big over \( Z \), and \((X', N' - S') \) is klt, \( h^1(L' + P' - S) = 0 \) by the relative Kawamata-Viehweg vanishing theorem \([KMM87 \text{ Theorem } 1-2-5] \), and hence
\[ H^0(L' + P') \rightarrow H^0((L' + P')|_{S'}) \]
is surjective. Therefore, there exists \( G' \geq 0 \) on \( X' \) such that \( L' + P' \sim G' \) and \( G'|_{S'} = G_{S'} \).
As $P'$ is exceptional over $X$, we have
$$f_*(L' + P') = L = -nK_X - nT - [(n + 1)\Delta] - nM \sim_Z G \geq 0,$$
where $T, \Delta, L, G$ are the strict transforms of $T', \Delta', L', G'$ respectively. Since $nB$ is integral, $[(n + 1)\Delta] = n\Delta$, and
$$-n(K_X + B + M) = -nK_X - nT - n\Delta - nM = L \sim_Z G =: nR \geq 0.$$

Let $B^+ := B + R$. Then we have $n(K_X + B^+ + M) \sim_Z 0$.

It is enough to show that $(X, B^+ + M)$ is generalized lc over some neighborhood of $z$ since then $(X/Z \ni z, B^+ + M)$ is a monotonic $n$-complement of $(X/Z \ni z, B + M)$.

We first show that $R|_S = R_S$. Since
$$nR' := G' - P' + [(n + 1)\Delta'] - n\Delta' \sim L' + [(n + 1)\Delta'] - n\Delta' = nN' \sim_{Q, X} 0,$$
and $[(n + 1)\Delta] = n\Delta$, we have that $f_*(nR') = G = nR$ and $R' = f^*R$. Thus
$$nR_{S'} = G_{S'} - P_{S'} + [(n + 1)\Delta_{S'}] - n\Delta_{S'} = (G' - P' + [(n + 1)\Delta'] - n\Delta')|_{S'} = nR'|_{S'},$$
which means that $R_{S'} = R'|_{S'}$. In particular, $R_S = R|_S$ and
$$K_S + B^+_S + M_S = K_S + B_S + R_S + M_S = (K_X + B^+ + M)|_S.$$

By the inversion of generalized adjunction [Bir19, Lemma 3.2], $(X, B^+ + M)$ is generalized lc near $S$.

Suppose that $(X, B^+ + M)$ is not generalized lc. Then there exists a real number $a \in (0, 1)$ which is sufficiently close to 1, such that $(X, (aB^+ + (1 - a)\Sigma) + ((1 - a)\alpha + a)M)$ is not generalized lc and is generalized lc near $S$. In particular, the generalized non-klt locus of $(X, (aB^+ + (1 - a)\Sigma) + ((1 - a)\alpha + a)M)$ is not connected. Since $-(K_X + (aB^+ + (1 - a)\Sigma) + ((1 - a)\alpha + a)M) = -a(K_X + B^+ + M) - (1 - a)(K_X + \Sigma + aM)$ is ample over $Z$, it contracts the connectedness principle [Bir19, Lemma 2.14]. Therefore $(X, B^+ + M)$ is generalized lc over a neighborhood of $z$. □

**Proof of Theorem 4.4.1.** We show the statement by induction on the dimension. Assume that Theorem 4.4.1 holds in dimension $d - 1$.

We may assume that $1 \in \Gamma$. According to Theorem 5.1.1 we may assume that $B \in \Gamma$ and $(X, B + M)$ is $\mathbb{R}$-complementary.

Let $N \geq 0$ be a Cartier divisor on $Z$ passing through $z$, and $t$ the generalized lc threshold of $\pi^*N$ with respect to $(X, B + M)$ over a neighborhood of $z$, where $\pi$ is the morphism $X \to Z$. Let $\Omega_0 := B + t\pi^*N$. Possibly shrinking $Z$ near $z$, we may assume that $(X, \Omega_0 + M)$ is generalized lc. Let $(X'', \Omega''_0 + M'')$ be a generalized dlt modification of $(X, \Omega_0 + M)$. Then $X''$ is of Fano type over $Z$. Moreover, there exists a boundary $\Omega''_1 \leq \Omega''_0$ such that $\Omega''_1 \in \Gamma$, some component of $|\Omega''_1|$ intersecting $\pi^{-1}(z)$, and $B \leq \Omega_1$, where $\Omega_1$ is the strict transform of $\Omega''_1$ on $\Omega_1$. We may run an MMP $Z$ on $-(K_{X''} + \Omega''_1 + M'')$ which terminates with a model $X'''$ such that $-(K_{X'''} + \Omega''_1 + M''')$ is nef over
Moreover, possibly shrinking $\Omega''_Z$ of $Z$ and $X$ so is $(X'', \Omega''_Z + M'')$. Moreover, no component of $[\Omega''_Z]$ is contracted by the MMP, as $a(S'', X'', \Omega''_Z + M'') < 0$ for any contracted divisor $S''$. Possibly replacing $(X, B + M)$ by $(X'', \Omega''_Z + M'')$, we may assume that $X$ is $\mathbb{Q}$-factorial, $-(K_X + B + M)$ is nef over $Z$, and $\text{Supp}[B]$ intersecting $\pi^{-1}(z)$.

We claim that there exist boundaries $\tilde{\Delta} \leq \Delta$ such that $-(K_X + \Delta + \alpha M)$ and $-(K_X + \tilde{\Delta} + \alpha M)$ are nef and big over $Z$, some component of $[\Delta]$ intersects $\pi^{-1}(z)$, $(X, \Delta + \alpha M)$ is generalized dlt, and $(X, \tilde{\Delta} + \alpha M)$ is generalized klt for some $\alpha \in (0, 1)$.

Since $-K_X$ is big over $Z$ and $-(K_X + B + M)$ is nef over $Z$,

$-(K_X + \alpha B + \alpha M) = -(K_X + B + M) - (1 - \alpha)K_X$

is big over $Z$ for any $\alpha \in (0, 1)$. Assume that $\alpha$ is sufficiently close to 1, we define $\Delta$ as follows. For any prime divisor $D$ which is vertical over $Z$, we let $\text{mult}_D \Delta = \text{mult}_D B$, otherwise let $\text{mult}_D \Delta = \text{mult}_D \alpha B$. Then $(X, \Delta + \alpha M)$ is generalized lc, $\alpha B \leq \Delta \leq B$, $\text{Supp}[\Delta]$ intersects $\pi^{-1}(z)$, and $-(K_X + \Delta + \alpha M)$ is big over $Z$ as $\Delta = \alpha B$ near the generic fiber.

Let $X \to V/Z$ be the contraction defined by $-(K_X + B + M)$. We may run an MMP/V on $-(K_X + \Delta + \alpha M)$ which terminates with a model $X''_\beta$ such that $-(K_{X''_{\beta}} + \Delta''_{\beta} + \alpha M''_{\beta})$ is nef over $V$, where $\Delta''_{\beta}, M''_{\beta}$ are the strict transforms of $\Delta, M$ respectively. Possibly replacing $\Delta$ by $\alpha B + (1 - a)\Delta$ for some $\alpha \in (0, 1)$ sufficiently close to 1, we may assume that $-(K_{X''_{\beta}} + \Delta''_{\beta} + \alpha M''_{\beta})$ is nef and big over $Z$. Possibly replacing $(X, \Delta + \alpha M)$ by $(X'', \Delta''_{\beta} + \alpha M''_{\beta})$ and $B$ by its strict transform, we may assume that $-(K_X + \Delta + \alpha M)$ is nef and big over $Z$.

Let $X \to T/Z$ be the morphism define by $-(K_X + \Delta + \alpha M)$, $\tilde{\Delta} := \beta \Delta$ for some $\beta \in (0, 1)$. We may run an MMP/T on $-(K_X + \tilde{\Delta} + \alpha M)$ and terminates with $X'_\beta$. Note that the MMP is $(K_X + B + M)$-trivial, as it is $(K_X + \Delta + \alpha M)$-trivial and $\Delta \leq B$. Possibly replacing $X$ by $X'_\beta$, and also the corresponding divisors, and pick $\beta$ sufficiently close to 1, we may assume that $-(K_X + \tilde{\Delta} + \alpha M)$ is nef and big over $Z$. Possibly replacing $(X, B + M)$ by a generalized dlt modification, increasing $\alpha, \beta$, and replacing $(X, \Delta + \alpha M)$ and $(X, \tilde{\Delta} + \alpha M)$ by its crepant pullbacks, the claim holds. Moreover, possibly shrinking $Z$ near $z$, we may assume that every component of $[\Delta]$ intersects $\pi^{-1}(z)$.

We may write $-(K_X + \Delta + \alpha M) \sim_{\mathbb{R}, Z} A + G$, where $A \geq 0$ is ample/Z and $G \geq 0$. Suppose that $\text{Supp}G$ does not contain any generalized non-klt center of $(X, \Delta + \alpha M)$. Then $(X, (\Delta + \delta G) + \alpha M)$ is generalized dlt for some sufficiently small positive real number $\delta$. Moreover,

$-(K_X + \Delta + \delta G + \alpha M) \sim_{\mathbb{R}, Z} \delta A + (1 - \delta)(A + G)$

is ample over $Z$. There exists a boundary $\Sigma$ such that $(X, \Sigma + \alpha M)$ is generalized plt, $S := [\Sigma] \subseteq [B]$ is irreducible and intersects $\pi^{-1}(z)$, and $-(K_X + \Sigma + \alpha M)$ is ample over $Z$. By Proposition 4.2, the theorem holds.
In the following, we may assume that \(\text{Supp} G\) contains some non-klt center of \((X, \Delta + \alpha M)\).

Possibly replacing \(\Delta\), we may assume that \(\Delta - \tilde{\Delta}\) has sufficiently small coefficients, and the generalized lc threshold \(t\) of \(G + \Delta - \tilde{\Delta}\) with respect to \((X, \tilde{\Delta} + \alpha M)\) over a neighborhood of \(z\) is sufficiently small such that any generalized non-klt center of \((X, \Omega + \alpha M)\) is a generalized non-klt center of \((X, \Delta + \alpha M)\), where \(\Omega := \tilde{\Delta} + t(G + \Delta - \tilde{\Delta})\). Moreover,

\[-(K_X + \Omega + \alpha M) = -(K_X + \tilde{\Delta} + t(G + \Delta - \tilde{\Delta}) + \alpha M)\]

\[= -(K_X + \Delta + \alpha M) + \Delta - \tilde{\Delta} - t(G + \Delta - \tilde{\Delta})\]

\[\sim_{\mathbb{R}, Z} A + G - tG + (1 - t)(\Delta - \tilde{\Delta}) = tA + (1 - t)(A + G + \Delta - \tilde{\Delta})\]

is ample over \(Z\).

If \(|\Omega| \neq 0\), then there exist a component \(S\) of \([\Omega] \subseteq [\Delta] \subseteq [B]\) and a boundary \(\Sigma_0\) such that \((X, \Sigma_0 + \alpha M)\) is generalized plt, \(S = [\Sigma_0]\) intersects \(\pi^{-1}(z)\), and \(-(K_X + \Sigma_0 + \alpha M)\) is ample over \(Z\). We are done by Proposition 4.2.

Suppose that \(|\Omega| = 0\). Let \((X''_3, \Omega''_3 + \alpha M''_3)\) be a generalized plt modification of \((X, \Omega + \alpha M)\). Possibly shrinking \(Z\) near \(z\), we may assume that every component of \([\Omega''_3]\) intersects \(\pi^{-1}(z)\). We may run a generalized MMP/Z on \(K_{X''_3} + [\Omega''_3] + \alpha M''_3\) which terminates with \(X\) since \([\Omega''_3]\) is the reduced exceptional divisor of \(X''_3 \to X\) and \((X, \alpha M)\) is generalized klt. Let \(X'''_3 \to X\) be the last step of the generalized MMP which is a divisorial contraction contracting a prime divisor \(S'''_3\). Let \((X'''_3, B'''_3 + M'''_3)\) be the crepant pullback of \((X, B + M)\), then \(S'''_3\) is a component of \([B'''_3]\). We finish the proof by Proposition 4.2.

5. Proof of Theorem 1.1

5.1. From the DCC set to a finite set.

**Theorem 5.1.** Let \(d\) and \(r\) be positive integers, \(\alpha\) a positive real number, and \(\Gamma \subseteq [0, 1]\) a DCC set. Then there exist a finite set \(\Gamma' \subseteq \Gamma\) and a projection \(g : \Gamma \to \Gamma'\) depending only on \(d, r, \alpha\) and \(\Gamma\) satisfying the following.

Assume that \((X, \sum_{i=1}^{s} b_i B_i + M)\) is a generalized polarized pair with data \(X' \xrightarrow{f} X \to Z\) and \(M'\) such that

- \((X, \sum_{i=1}^{s} b_i B_i + M)\) is generalized lc of dimension \(d\),
- \(B_i \geq 0\) is a \(Q\)-Cartier Weil divisor for any \(i\),
- \(b_i \in \Gamma\) for any \(1 \leq i \leq s\), and
- \(rM'\) is \(b\)-Cartier.

Then

1. \(\gamma + \alpha \geq g(\gamma) \geq \gamma\) for any \(\gamma \in \Gamma\),
2. \(g(\gamma') \geq g(\gamma)\) for any \(\gamma' \geq \gamma\), \(\gamma, \gamma' \in \Gamma\), and
3. \((X, \sum_{i=1}^{s} \gamma(b_i) B_i + M)\) is generalized lc.
The proof of Theorem 5.1 is very similar to that of [HLS19, Theorem 5.18].

Proof. Possibly replacing $\Gamma$ by $\bar{\Gamma}$, we may assume that $\Gamma = \bar{\Gamma}$. Let $\Gamma' := g \mathcal{LCT}(d, \Gamma \cup \{Z \geq 0\})$ which is an ACC set by Theorem 2.9. By [HLS19, Lemma 5.17], there exist a finite set $\Gamma' \subseteq \Gamma$ a projection $g : \Gamma \to \Gamma'$ such that

1. $\gamma + \alpha \geq g(\gamma) \geq \gamma$ for any $\gamma \in \Gamma$,
2. $g(\gamma') \geq g(\gamma)$ for any $\gamma' \geq \gamma$, $\gamma, \gamma' \in \Gamma$, and
3. for any $\beta \in \Gamma'$ and $\gamma \in \Gamma$, if $\beta \geq \gamma$, then $\beta \geq g(\gamma)$.

It suffices to show that $(X, (\sum_{i=1}^{s} g(b_i)B_i) + M)$ is generalized lc. Otherwise, there exists some $1 \leq j \leq s$, such that $(X, (\sum_{i=1}^{j} g(b_i)B_i + \sum_{i=j+1}^{s} b_iB_i) + M)$ is generalized lc, and $(X, (\sum_{i=1}^{j+1} g(b_i)B_i + \sum_{i=j+2}^{s} b_iB_i) + M)$ is not generalized lc. Let

$$\beta := \text{lct}(X, \sum_{i=1}^{j} g(b_i)B_i + \sum_{i=j+2}^{s} b_iB_i + M; B_{j+1} + 0).$$

Then $g(b_{j+1}) > \beta \geq b_{j+1}$. Since $g(b_i), b_i \in \Gamma$ for any $i$, $\beta \in \Gamma''$ and $\beta \geq g(b_{j+1})$, a contradiction.

Theorem 5.2. Let $d$ and $r$ be positive integers, and $\Gamma \subseteq [0, 1]$ a DCC set. Then there exist a finite set $\Gamma' \subseteq \Gamma$, and a projection $g : \Gamma \to \Gamma'$ depending only on $d$, $r$ and $\Gamma$ satisfying the following.

Assume that $(X, (\sum b_iB_i) + M)$ is a generalized polarized pair with data $X' \xrightarrow{f} X \to Z$, and $M'$ such that

1. $\dim X = d$,
2. $X$ is of Fano type over $Z$,
3. $B_i \geq 0$ is a $\mathbb{Q}$-Cartier Weil divisor and $b_i \in \Gamma$ for any $i$,
4. $rM'$ is $b$-Cartier, and
5. $(X/Z, (\sum b_iB_i) + M)$ is $\mathbb{R}$-complementary.

Then

1. $g(\gamma) \geq \gamma$ for any $\gamma \in \Gamma$,
2. $g(\gamma') \geq g(\gamma)$ for any $\gamma' \geq \gamma$, $\gamma, \gamma' \in \Gamma$, and
3. $(X/Z, (\sum g(b_iB_i) + M)$ is $\mathbb{R}$-complementary.

The proof of Theorem 5.2 is very similar to that of [HLS19, Theorem 5.20].

Proof. We first claim that there exist a finite set $\Gamma' \subseteq \bar{\Gamma}$, and a projection $g : \Gamma \to \Gamma'$ depending only on $d$, $r$, and $\Gamma$ such that for any generalized polarized pair $(X, (\sum b_iB_i) + M)$ with data $X' \xrightarrow{f} X \to Z$ and $M'$ satisfying the conditions, then

1. $g(\gamma) \geq \gamma$ for any $\gamma \in \Gamma$,
2. $g(\gamma') \geq g(\gamma)$ for any $\gamma' \geq \gamma$, $\gamma, \gamma' \in \Gamma$, and
• \((X, \sum g(b_i)B_i + M)\) is generalized lc, and
• \(-(K_X + \sum g(b_i)B_i + M)\) is pseudo-effective over \(Z\).

We may assume that \(1 \in \Gamma\). Suppose that the claim does not hold. By Theorem 5.1 there exist a sequence of \(d\)-dimensional generalized polarized pairs \((X_k, B_{(k)} + M_k)\) with data \(X'_k \xrightarrow{\phi_k} X_k \rightarrow Z_k\) and \(M'_k\), where \(B_{(k)} := \sum_{i} b_{k,i} B_{k,i}\), and a sequence of projections \(g_k : \bar{\Gamma} \rightarrow \Gamma\), such that for any \(k, i,\) \ we have

- \(b_{k,i} \in \Gamma, b_{k,i} + \frac{1}{k} \geq g_k(b_{k,i}) \geq b_{k,i}\),
- \((X_k/Z_k, B_{(k)} + M_k)\) is \(\mathbb{R}\)-complementary
- \((X_k, B''_{(k)} + M_k)\) is generalized lc, where \(B''_{(k)} := \sum_{i} g_k(b_{k,i})B_{k,i}\), and
- \(-(K_{X_k} + B''_{(k)} + M_k)\) is not pseudo-effective over \(Z_k\).

Possibly replacing \(X_k\) by a dlt modification of \((X_k, B_{(k)} + M_k)\), we may assume that \(X_k\) is \(\mathbb{Q}\)-factorial for any \(k\). We may run an MMP over \(- (K_{X_k} + B''_{(k)} + M_k)\) with scaling of an ample/Z divisor which terminates with a Mori fiber space \(Y_k \rightarrow Z'_k\) over \(Z_k\), such that \(- (K_{Y_k} + B''_{(Y_k)} + M_{Y_k})\) is antiample over \(Z'_k\), where \(B''_{(Y_k)}, M_{Y_k}\) are the strict transforms of \(B''_{(k)}, M_k\) on \(Y_k\). Since \(- (K_{X_k} + B_{(k)} + M_k)\) is pseudo-effective over \(Z_k\), \(- (K_{Y_k} + B_{(Y_k)} + M_{Y_k})\) is nef over \(Z'_k\), where \(B_{(Y_k)}\) is the strict transform of \(B_{(k)}\) on \(Y_k\).

For each \(k\), there exist a positive integer \(k_j\) and a positive real number \(0 \leq b_k^+ \leq 1\), such that \(b_{k,k_j} \leq b_k^+ < g_k(b_{k,k_j})\), and \(K_{F_k} + B_{F_k}^+ + M_{F_k} \equiv 0\), where

\[
K_{F_k} + B_{F_k}^+ + M_{F_k} := (K_{Y_k} + (\sum_{i < k_j} g_k(b_{k,i})B_{Y_k,i}) + b_k^+B_{Y_k,k_j} + (\sum_{i > k_j} b_{k,i}B_{Y_k,i}) + M_{Y_k})|_{F_k},
\]

\(B_{Y_k,i}\) is the strict transform of \(B_{k,i}\) on \(Y_k\) for any \(i\), and \(F_k\) is a general fiber of \(Y_k \rightarrow Z'_k\). Since \((X_k, B_{(k)} + M_k)\) is \(\mathbb{R}\)-complementary, \((Y_k, B_{(Y_k)} + M_k)\) is generalized lc. Thus \((Y_k, B''_{(Y_k)} + M_{Y_k})\) and \((F_k, B_{F_k}^+ + M_{F_k})\) are generalized lc.

Since \(g_k(b_{k,k_j})\) belongs to the DCC set ~\(\Gamma\) for any \(k, k_j\), possibly passing to a subsequence, we may assume that \(g_k(b_{k,k_j})\) is increasing. Since \(g_k(b_{k,k_j}) - b_k^+ > 0\) and \(\lim_{k \to +\infty} (g_k(b_{k,k_j}) - b_k^+) = 0\), by [HLS19] Lemma 5.21, possibly passing to a subsequence, we may assume that \(b_k^+\) is strictly increasing.

Now \(K_{F_k} + B_{F_k}^+ + M_{F_k} \equiv 0\), the coefficients of \(B_{F_k}^+\) belong to the DCC set ~\(\bar{\Gamma} \cup \{b_k^+\}_{k=1}^\infty\), and \(b_k^+\) is strictly increasing. This contradicts Theorem 2.10.

Since \(- (K_X + \sum g(b_i)B_i + M)\) is pseudo-effective over \(Z\), we may run an MMP over \(- (K_X + \sum g(b_i)B_i + M)\) which terminates with a good minimal model \(X''\), such that \(- (K_{X''} + \sum g(b_i)B''_i + M'')\) is semiample over \(Z\), where \(B''_i, M''\) are the strict transforms of \(B_i, M\) on \(X''\). Since \((X/Z, \sum b_i B''_i + M'')\) is \(\mathbb{R}\)-complementary, \((X''/Z, \sum b_i B''_i + M'')\) is \(\mathbb{R}\)-complementary. Hence
(X'', \sum b_i B''_i + M'') is generalized lc, and (X'', \sum g(b_i) B''_i + M'') is generalized lc. Thus (X'/Z, (\sum g(b_i) B''_i) + M'') is \mathbb{R}-complementary, and (X/Z, (\sum g(b_i) B_i) + M) is also \mathbb{R}-complementary. \square

5.2. Proof of Theorem 1.3

Proof of Theorem 1.3. We may assume that 1 ∈ Γ. Possibly replacing (X, B + M) by a generalized dlt modification, we may assume that X is \mathbb{Q}-factorial. By Theorem 5.2 we may assume that Γ is a finite set.

By Theorem 3.18 there exist two finite sets Γ₀ ⊆ (0, 1] and Γ₁ ⊆ [0, 1] \cap \mathbb{Q} depending only on d, r and Γ such that possibly shrinking Z near z, we have (X, B + M) is generalized lc, (K_X + B + M) is pseudo-effective over Z, and \sum a_i = 1,

K_X + B + M = \sum a_i(K_X + B_i + M)

for some a_i ∈ Γ₀ and B_i ∈ Γ₁. Moreover, if Γ ⊆ \mathbb{Q}, then we may pick Γ₀ = \{1\}, and B₁ = B.

For each i, we may run an MMP/Z on (K_X + B_i + M) and terminates with a model Y_i, such that −(K_{Y_i} + B_{Y_i,i} + M_i) is nef over Z, where B_{Y_i,i} and M_i are the strict transforms of B_i and M' on Y_i respectively. Since (X/Z, B + M) is \mathbb{R}-complementary, (Y_i, B_{Y_i,i} + M_i) is generalized lc, where B_{Y_i,i} is the strict transform of B on Y_i. According to the construction of Γ₁, (Y_i, B_{Y_i,i} + M) is generalized lc. By Theorem 4.1 there exists a positive integer n depending only on d and Γ₁, such that (Y_i/Z, B_{Y_i,i} + M_i) has a monotonic n-complement.

By Lemma 2.14 (X/Z, B_i + M) has a monotonic n-complement (X/Z, (B_i + G_i) + M) for some \mathbb{Q}-Cartier divisor G_i ≥ 0. Let B⁺ := \sum a_i(B_i + G_i). Hence (X/Z, B⁺ + M) is an (n, Γ₀)-decomposable \mathbb{R}-complement of (X/Z, B + M). \square

6. Existence of n-complements

The following lemma is similar to [HLS19, Lemma 6.6] and [CH20, Lemma 6.1], but in a slightly different form.

Lemma 6.1. Let n₀, s be two positive integers, ε₀ a positive real number, \nu_0 ∈ \mathbb{R}^s \setminus \mathbb{Q}^s a point, V ⊆ \mathbb{R}^s the rational envelope of \nu_0, \|\cdot\| a norm on V and \epsilon ∈ V a non-zero vector. Let Γ ⊆ [0, 1] and Γ₀ ⊆ (0, 1] be finite sets. Then there exist a positive integer n₀|n and a vector \nu depending only on n₀, s, \epsilon₀, \nu_0, \|\cdot\|, \epsilon, Γ and Γ₀ satisfying the following.

Assume that a_i ∈ Γ₁ and b_{ij} ∈ \frac{1}{n₀}Z \cap [0, 1] (1 ≤ i ≤ k, 1 ≤ j ≤ m) such that \sum_{i=1}^k a_i = 1 and \sum_{i=1}^k a_i b_{ij} ∈ Γ. Then there exists a point \alpha' := (a'_1, \ldots, a'_k) ∈ \mathbb{R}^k_{>0} such that

1. \sum_{i=1}^k a'_i = 1,
2. n(\alpha', \nu) ∈ n₀Z^s + k,
3. \|\nu - \nu_0\| < \frac{\epsilon₀}{n},
(4) \[ \| \frac{v - v_0}{\|v - v_0\|} - \frac{e}{\|e\|} \| < \epsilon_0, \quad \text{and} \]
(5) \[ \sum_{i=1}^{k} a_i b_{ij} = \sum_{i=1}^{k} a_i b_{ij} + (n+1) \{ \sum_{i=1}^{k} a_i b_{ij} \} \] for any \( 1 \leq j \leq s \).

Moreover, if there exist real numbers \( r_0 := 1, r_1, \ldots, r_c \) which are linearly independent over \( \mathbb{Q} \) such that \( \Gamma \subseteq \text{Span}_{\mathbb{Q}}(\{r_0, \ldots, r_c\}) \), then we can additionally require that \( \sum_{i=1}^{k} a_i b_{ij} \geq \sum_{i=1}^{k} a_i b_{ij} \) for any \( 1 \leq j \leq m \).

Proof. Let \( v_1, \ldots, v_c \in \mathbb{Z}^s \) be a basis of \( V \) such that \( e \in \text{Span}_{\mathbb{R}_{>0}}(\{v_1, \ldots, v_c\}) \).

Then there exist \( c', r_1, \ldots, r_{c'} \) and \( e_1, \ldots, e_c \geq 0 \) such that \( r_0 := 1, r_1, \ldots, r_{c'} \) are linearly independent over \( \mathbb{Q} \), \( \Gamma_0 \subseteq \text{Span}(\{r_0, \ldots, r_{c'}\}) \), and

\[
v = \sum_{i=1}^{c} r_i v_i, \quad e = \sum_{i=1}^{c} e_i v_i.
\]

There exist positive integers \( l, M \) and \( \mathbb{Q} \)-linear functions \( a_i(r) : \mathbb{R}^{c'} \rightarrow \mathbb{R} \) depending only on \( \Gamma_0 \) such that \( a_i(r_0) = a_i \) and \( a_i(r) \) is a \( \mathbb{Z} \)-linear function and

\[
|a_i(r) - a_i(0)| \leq M \|r\|_\infty
\]

for any \( r \in \mathbb{R}^{c'} \) and \( i \), where \( r_0 := (r_1, \ldots, r_{c'}) \) and \( 0 := (0, \ldots, 0) \in \mathbb{R}^{c'} \).

We define a norm \( \|\cdot\|_* \) on \( V \). For any \( x \in V \), there exist unique real numbers \( x_1, \ldots, x_c \) such that \( x = \sum_{i=1}^{c} x_i v_i \), we define \( \|x\|_* := \max_{1 \leq i \leq c} \{|x_i|\} \).

By [HLS19, Lemma 6.4], there exists a positive real number \( M_1 \) such that

\[
\frac{\|x - y\|}{\|x\|} \leq M_1 \frac{\|x\|}{\|y\|_*}
\]

for any non-zero vectors \( x, y \in V \). Moreover, possibly replacing \( M_1 \) by a larger number, we may assume that \( \|x\| \leq M_1 \|x\|_* \) for any vector \( x \in V \).

Let \( \epsilon' \) be a positive real number such that

\[
\epsilon' < \min_{\gamma_1 \in \Gamma, \gamma_2 \in \Gamma_0} \{ \gamma_1 > 0, 1 - \gamma_1 > 0, \gamma_2, 1 \},
\]

and \( \epsilon'' \) a positive real number such that \( \epsilon'' < \min\{ \frac{e^2}{M M_1} \} \).

By [HLS19, Lemma 6.5], there exist an integer \( n_0 |n \) and a point \( r'_0 := (r'_1, \ldots, r'_{c'}) \in \mathbb{R}^{c'} \), such that

- \( n r'_0 \in \text{ln}_0 \mathbb{Z}^c \),
- \( \|r'_0 - r_0\|_\infty < \frac{\epsilon''}{n} \), and
- \( \|\frac{c - d}{\|c - d\|_\infty} - \frac{e_0}{\|e_0\|_\infty}\|_\infty < \epsilon'' \), where \( c := (r'_1, \ldots, r'_{c}) \), \( d := (r_1, \ldots, r_c) \) and \( e_0 := (e_1, \ldots, e_c) \).

Let \( v := \sum_{i=1}^{c} r'_i v_i \), then we have \( n v = \sum_{i=1}^{c} n r'_i v_i \in n_0 \mathbb{Z}^s \),

\[
\|v - v_0\| \leq M_1 \|v - v_0\|_* \leq M_1 \|r'_0 - r_0\|_\infty < \frac{\epsilon_0}{n},
\]
and
\[
\frac{\|v - v_0\|}{\|v - v_0\|} - \frac{e}{\|e\|} \leq M_1 \frac{\|v - v_0\|}{\|v - v_0\|} - \frac{e}{\|e\|} = M_1 \frac{c - d}{\|c - d\|_{\infty}} - \frac{e_0}{\|e_0\|_{\infty}} < M_1 e'' < e_0.
\]

Let \( a'_i := a_i(r'_0) \) for any \( 1 \leq i \leq k \) and \( a' := (a'_1, \ldots, a'_k) \). Since \( \sum_{i=1}^k a_i = 1 \) and \( r_1, \ldots, r_{c'} \) are linearly independent over \( \mathbb{Q} \), \( \sum_{i=1}^k a_i(r) = 1 \) for any \( r \in \mathbb{R}^c \). In particular, \( \sum_{i=1}^k a'_i = 1 \). Moreover, we have \( na'_i = n_0 \frac{n_i}{m_0} a_i(r'_0) \in n_0 \mathbb{Z} \) for any \( 1 \leq i \leq k \), and
\[
\|a - a'\|_{\infty} \leq M\|r_0 - r'_0\|_{\infty} < M\frac{e''}{n} < \frac{e^2}{n},
\]
where \( a := (a_1, \ldots, a_k) \). In particular, \( a'_i \) are positive real numbers, since
\[
a'_i \geq a_i - |a_i - a'_i| \geq a_i - \|a - a'\|_{\infty} > a_i - \frac{e^2}{n} > 0.
\]

It suffices to show (6). If \( \sum_{i=1}^k a_i b_{ij} = 1 \) for some \( j \), then \( b_{ij} = 1 \) for any \( 1 \leq i \leq k \) as \( \sum_{i=1}^k a_i = 1 \) and \( 1 \geq b_{ij} \geq 0 \). Thus \( \sum_{i=1}^k a'_i b_{ij} = 1 \). Hence we may assume that \( 1 > \sum_{i=1}^k a_i b_{ij} > 0 \) and \( \sum_{i=1}^k a'_i b_{ij} < 1 \). Since \( n \sum_{i=1}^k a'_i b_{ij} = \sum_{i=1}^k \frac{n}{n_0} a'_i \cdot (n_0 b_{ij}) \in \mathbb{Z} \), we only need to show that
\[
n \sum_{i=1}^k a'_i b_{ij} + 1 > (n + 1) \sum_{i=1}^k a_i b_{ij} \geq n \sum_{i=1}^k a'_i b_{ij}.
\]

The above inequalities hold since \( \sum_{i=1}^k a_i b_{ij} \in \Gamma, k e' < \sum_{i=1}^k a_i = 1 \), and
\[
n \sum_{i=1}^k |a'_i - a_i| b_{ij} < nk \cdot \frac{e^2}{n} < e' < \min\{\sum_{i=1}^k a_i b_{ij}, 1 - \sum_{i=1}^k a_i b_{ij}\}.
\]

If \( \Gamma \subseteq \text{Span}_{\mathbb{Q}\geq0}\{r_0, \ldots, r_{c'}\} \), then by our choices of \( a'_i \) and \( r'_0 \),
\[
\sum_{i=1}^k a'_i b_{ij} = \sum_{i=1}^k a_i(r'_0) b_{ij} \geq \sum_{i=1}^k a_i(r_0) b_{ij} = \sum_{i=1}^k a_i b_{ij}
\]
for any \( 1 \leq j \leq s \). \( \square \)

**Proof of Theorem 1.1.** According to Theorem 1.3, there exist a positive integer \( n_0 \) and a finite set \( \Gamma_0 \subseteq [0, 1] \) depending only on \( d, r \) and \( \Gamma \), such that possibly shrinking \( Z \) near \( z_0 \), \( (X/Z) (B + M) \) has an \( (n_0, \Gamma_0) \)-decomposable \( \mathbb{R} \)-complement \( (X/Z) (B' + M) \). In particular, there exist \( a_i \in \Gamma_0 \) and boundaries \( B'_i \) such that \( (X/Z) (B'_i + M) \) is an \( n_0 \)-complement of itself for any \( i \), and
\[
K_X + B' + M = \sum a_i(K_X + B'_i + M).
\]
By Lemma 6.1, there exist a positive integer $n$ divisible by $pn_0$ and a vector $v \in V$ depending only on $\epsilon, p, n_0, v_0, e$ and $\Gamma_0$ such that there exist positive rational numbers $a_i'$ with the following properties:

- $\sum a_i' = 1$,
- $nv \in \mathbb{Z}^s$, and $n a_i' \in n_0 \mathbb{Z}$ for any $i$,
- $||v - v_0|| < \frac{\epsilon}{n}$,
- $||v - v_0|| - ||e|| < \epsilon$, and
- $n B^+ \geq n \lfloor B' \rfloor + \lfloor (n + 1) \{B'\} \rfloor$, where $B^+ := \sum a_i' B_i'$.

In particular,

$$n(K_X + B^+ + M) = n \sum a_i'(K_X + B_i' + M_i) = \sum \frac{a_i'n_0}{n_0} n_0(K_X + B_i' + M_i) \sim_\mathbb{Z} 0.$$ 

Hence $(X/Z, B^+ + M)$ is an $n$-complement of $(X/Z, B + M)$, since $B' \geq B$. Moreover, if $\text{Span}_{\mathbb{Q}^\geq 0}(\Gamma_0 \setminus \mathbb{Q}) \cap (\mathbb{Q} \setminus \{0\}) = \emptyset$, then $B^+ \geq B' \geq B$ by Lemma 6.1 and [HLS19, Lemma 6.3].

\[\Box\]

References

[Bir04] C. Birkar. Boundedness of $\epsilon$-log canonical complements on surfaces, [https://www.dpmms.cam.ac.uk/~cb496/surfcomp.pdf](https://www.dpmms.cam.ac.uk/~cb496/surfcomp.pdf), preprint, 2004.

[Bir16] C. Birkar. Singularities of linear systems and boundedness of Fano varieties. [arXiv:1609.05543](https://arxiv.org/abs/1609.05543), 2016.

[Bir18] C. Birkar. Log Calabi-Yau fibrations. [arXiv:1811.10709](https://arxiv.org/abs/1811.10709), 2018.

[Bir19] C. Birkar. Anti-pluricanonical systems on Fano varieties. Ann. of Math. (2), 190(2):345–463, 2019.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan. Existence of minimal models for varieties of log general type. J. Amer. Math. Soc., 23(2):405–468, 2010.

[BLX19] H. Blum, Y. Liu, and C. Xu. Openness of K-semistability for Fano varieties. [arXiv:1907.02246](https://arxiv.org/abs/1907.02246), 2019.

[BZ16] C. Birkar and D. Zhang. Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs. Publ. Math. Inst. Hautes Études Sci., 123:283–331, 2016.

[CH20] G. Chen and J. Han. Boundedness of $(\epsilon, n)$-complements for surfaces. [arXiv:2002.02246](https://arxiv.org/abs/2002.02246), 2020.

[Fil17] S. Filipazzi. On a generalized canonical bundle formula and generalized adjunction. to appear in Annali della Scuola Normale di Pisa - Classe di Scienze., 2018.

[FM18] S. Filipazzi and J. Moraga. Strong $(\delta, n)$-complements for semi-stable morphisms. [arXiv:1810.01990](https://arxiv.org/abs/1810.01990), 2018.

[FMX19] S. Filipazzi, J. Moraga, and Y. Xu. Log canonical 3-fold complements. [arXiv:1909.10098](https://arxiv.org/abs/1909.10098), 2019.

[HL17] J. Han and Z. Li. On Fujita’s conjecture for pseudo-effective thresholds. [arXiv:1705.08862](https://arxiv.org/abs/1705.08862), 2017.

[HL18] J. Han and Z. Li. Weak Zariski decompositions and log terminal models for generalized polarized pairs. [arXiv:1806.01274](https://arxiv.org/abs/1806.01274), 2018.

[HLiu18] J. Han and W. Liu. On numerical nonvanishing for generalized log canonical pairs. to appear in Documenta Mathematica, 2018.

[HLiu19] J. Han and W. Liu. On a generalized canonical bundle formula for generically finite morphisms. [arXiv:1905.12542v1](https://arxiv.org/abs/1905.12542v1), 2019.
[HLQ17] J. Han, Z. Li, and L. Qi. ACC for log canonical threshold polytopes. \texttt{arXiv:1706.07628}, 2017.

[HLS19] J. Han, J. Liu, and V. V. Shokurov. ACC for minimal log discrepancies of exceptional singularities. \texttt{arXiv:1903.04338v1}, 2019.

[HM06] C. D. Hacon and J. McKernan. Boundedness of pluricanonical maps of varieties of general types. \textit{Invent. Math.}, 166:1–25, 2006.

[HMX14] C. D. Hacon, J. McKernan, and C. Xu. ACC for log canonical thresholds. \textit{Ann. of Math. (2)}, 180(2):523–571, 2014.

[KMM87] Y. Kawamata, K. Matsuda, and K. Matsuki. Introduction to the minimal model problem. In \textit{Algebraic geometry, Sendai, 1985}, volume 10 of \textit{Adv. Stud. Pure Math.}, pages 283–360. North-Holland, Amsterdam, 1987.

[Liu18] J. Liu. Toward the equivalence of the ACC for $a$-log canonical thresholds and the ACC for minimal log discrepancies. \texttt{arXiv:1809.04839}, 2018.

[LT19] V. Lazic and N. Tsakanikas. On minimal models. \texttt{arXiv:1905.05576v2}, 2019.

[LP18] V. Lazic and T. Peternell. On generalised abundance, I. \textit{to appear in Publ. Res. Inst. Math. Sci.}, 2018.

[LP19] V. Lazic and T. Peternell. On generalised abundance, II. \textit{to appear in Peking Math. J.}, 2018.

[Nak16] Y. Nakamura. On minimal log discrepancies on varieties with fixed gorenstein index. \textit{Michigan Math. J.}, 65(1):165–187, 03 2016.

[PS01] Y. G. Prokhorov and V. V. Shokurov. The first fundamental theorem on complements: from global to local. \textit{Izv. Ross. Akad. Nauk Ser. Mat.}, 65(6):99–128, 2001.

[PS09] Y. G. Prokhorov and V. V. Shokurov. Towards the second main theorem on complements. \textit{J. Algebraic Geom.}, 18(1):151–199, 2009.

[Sho92] V. V. Shokurov. Three-dimensional log perestroikas. \textit{Izv. Ross. Akad. Nauk Ser. Mat.}, 56(1):105–203, 1992.

[Sho00] V. V. Shokurov. Complements on surfaces. \textit{J. Math. Sci. (New York)}, 102(2):3876–3932, 2000. Algebraic geometry, 10.

[Sho20] V. V. Shokurov. Existence and boundedness of n-complements. \texttt{preprint}, 2020.

[XuC19] C. Xu. A minimizing valuation is quasi-monomial. \texttt{arXiv:1907.01114} \textit{to appear in Annals of Mathematics}, 2019.

[XuY19] Y. Xu. Complements on log canonical Fano varieties. \texttt{arXiv:1901.03891}, 2019.

Guodu Chen, Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China

E-mail address: gdchen@pku.edu.cn