Coherence length in superconductors from weak to strong coupling

L. Benfatto, A. Toschi, S. Caprara, and C. Castellani
Dipartimento di Fisica, Università di Roma “La Sapienza”,
and Istituto Nazionale per la Fisica della Materia (INFM), SMC and Unità di Roma 1,
Piazzale Aldo Moro, 2 - 00185 Roma, Italy

We study the evolution of the superconducting coherence length $\xi_0$ from weak to strong coupling, both within a $s$-wave and a $d$-wave lattice model. We show that the identification of $\xi_0$ with the Cooper-pair size $\xi_{pair}$ in the weak-coupling regime is meaningful only for a fully-gapped (e.g., $s$-wave) superconductor. Instead in the $d$-wave superconductor, where $\xi_{pair}$ diverges, we show that $\xi_0$ is properly defined as the characteristic length scale for the correlation function of the modulus of the superconducting order parameter. The strong-coupling regime is quite intriguing, since the interplay between particle-particle and particle-hole channel is no more negligible. In the case of $s$-wave pairing, which allows for an analytical treatment, we show that $\xi_0$ is of order of the lattice spacing at finite densities. In the dilute regime $\xi_0$ diverges, recovering the behavior of the coherence length of a weakly interacting effective bosonic system. Similar results are expected to hold for $d$-wave superconductors.

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I. INTRODUCTION

In the last years the analysis of the phase-fluctuation contribution to the low-temperature properties of high-$T_c$ superconducting cuprates stimulated the interest in the microscopic derivation of the phase-only action for $d$-wave superconductors \cite{1-4}. The symmetry of the order parameter has been shown to play a crucial role in determining the quantum-to-classical crossover for phase fluctuations via enhanced dissipative effects \cite{1-2}. The effective hydrodynamic phase-only action is usually cut off at large momenta. A reasonable spatial bound for the effective hydrodynamic phase-only action is usually cut off at large momenta. A reasonable spatial bound for the effective hydrodynamic phase-only action is usually cut off at large momenta. A reasonable spatial bound for the effective hydrodynamic phase-only action is usually cut off at large momenta. A reasonable spatial bound for the effective hydrodynamic phase-only action is usually cut off at large momenta. A reasonable spatial bound for the effective hydrodynamic phase-only action is usually cut off at large momenta.

Another relevant length scale for superconductors is the characteristic size of the Cooper pair

$$\xi_{pair} = \sqrt{\int d^3r |\psi(r)|^2 r^2} = \sqrt{\int dk [\partial_k \phi(k)]^2 \int dk \partial_k \phi^2(k)}.$$  \hspace{1cm} (1)

where $\psi(r)$ is the Cooper-pair wave function, $\partial_k$ is the gradient operator in $k$-space, and $\phi(k) = \Delta_0 \gamma_k / E_k$ is the Fourier transform of $\psi(r)$. Here $\Delta_0$ is the amplitude of the superconducting gap, $\gamma_k$ is the factor which controls the symmetry of the BCS wave function, $E_k = \sqrt{\xi^2 + \Delta^2}$, and $\xi_k$ is the normal-state dispersion. Throughout this paper we assume the lattice spacing $a$ as the unit length.

It is known that at weak coupling for a $s$-wave superconductor (i.e., $\gamma_k = 1$) both $\xi_0$ and $\xi_{pair}$ are, within a numerical factor, of the same order $v_F / \Delta_0 \equiv \xi_0$. Here $v_F$ is a properly defined average value of the quasiparticle velocity at the Fermi surface (see Sec. II), which reduces to $k_F / m$ in a continuum model. By analogy, one would expect the same result to hold for $d$-wave superconductors ($\gamma_k = \gamma^d_k \equiv \cos k_x - \cos k_y$). However, when evaluating Eq. (1), one finds that the mean size of the Cooper pair is infinite. This fact, which has been often overlooked (see, however, Ref. \cite{10}), can be easily seen by evaluating the dominant contribution to Eq. (1), which comes from the nodal quasiparticles. As we show in Appendix A, the nodal quasiparticles give a logarithmically divergent contribution to Eq. (1).

The question then arises whether $\xi_0$ is well defined in the case of $d$-wave pairing, and of the expected order of magnitude in the weak-coupling limit.

The strong-coupling limit is even more intriguing, as one would expect that the magnitudes of the two lengths are widely separated. For instance, in the case of $s$-wave pairing, while $\xi_{pair}$ should decrease as the pairing strength increases, $\xi_0$ should resemble the behavior of the natural length scale for the spatial variation of the condensate wave function of an effective “bosonic” system \cite{5-7}. Thus $\xi_0$ should be controlled by the inverse of the residual interaction between the bosonic Cooper pairs, and could eventually increase as the residual boson-boson interaction decreases and by approaching the diluted limit. Such a separation between the two length scales has indeed been found in Ref. \cite{10} within the continuum $s$-wave model à la Nozières and Schmitt-Rink \cite{11}.

In this paper we address the issue of the systematic analysis of the behavior of the coherence length $\xi_0$ defined through the spatial decay of the static correlation function for $|\Delta_0|, \ X_{\Delta}(r)$, which is obtained by Fourier-transforming the correlation function in momentum space $X_{\Delta}(\mathbf{q})$. We specifically consider the two-
dimensional negative-\(U\) Hubbard model for \(s\)-wave pairing and its simplest generalization for \(d\)-wave pairing.

We show that, whatever is the symmetry of the order parameter, in the weak-coupling limit \(\xi_0\) is finite and of the expected order \(v_F/\Delta_0\).

In the strong-coupling limit the modulus of the order parameter \(|\Delta|\) and the density of particles \(\rho\) are coupled. Specifically, in the case of \(s\)-wave pairing, they experience the same fluctuations: in particular at low density the two fields become proportional \[12,13\]. As a consequence, the density fluctuations contribute to \(X_\Delta(q)\) via the gap-density coupling. We derive by particle-hole symmetry). In the diluted limit \(\rho \leq 1\) (the range \(1 \leq \rho \leq 2\) being recovered by particle-hole symmetry). In the diluted limit \(\rho \approx 0\) \(X_\Delta(r)\) decays exponentially with a length scale \(\xi_0\) diverging as \(1/\sqrt{\rho}\), as it is expected for a weakly-coupled diluted Bose liquid \[12,13\]. At higher densities, and for strong coupling, \(X_\Delta(r)\) oscillates with the periodicity \(a\) of the lattice. This is due to the fact that \(X_\Delta(q)\) is dominated by momenta \(q \approx Q \equiv (\pi, \pi)\), in analogy with the density mode, which is massless for \(q = Q\) at half filling \[12\]. By means of numerical calculations we show that the length above which \(X_\Delta(r)\) is strongly suppressed with respect to its value at \(r = 0\) is of the order of the lattice constant for all densities away from \(\rho \approx 0\), even though it exhibits a long-living exponential tail governed by an increasing characteristic length scale approaching half filling.

The case of \(d\)-wave pairing in the strong-coupling limit would be much more difficult to address, as the coupling of the gap fluctuations to the particle-hole channel reflects in a Hartree-Fock-like correction to the bare band dispersion, making the analytical treatment not viable. Despite these complications, however, the physics is not expected to be different from the \(s\)-wave case, which allows for a much simpler and transparent analytical treatment. For this reason, we devote most of our strong-coupling analysis to the \(s\)-wave case.

Finally, we comment on the consequences of such an analysis on the problem of a proper definition of the momentum cutoff for phase fluctuations in the effective phase-only model.

The plan of the paper is the following. In Sec. II we discuss the weak-coupling regime for both \(s\)- and \(d\)-wave superconductors. In Sec. III we devote our analysis to the strong-coupling regime for \(s\)-wave superconductors. In particular, Sec. III A addresses the low-density regime, Sec. III B deals with the intermediate- and high-density regime, while in Sec. III C we discuss the properties of the density-density correlation function. Concluding remarks are found in Sec. IV. Detailed calculations of the coherence length and of the Cooper-pair size for both \(s\)- and \(d\)-wave superconductors at weak coupling are reported in Appendix A. In Appendix B we discuss how the strong-coupling results discussed in Sec. III are affected by the inclusion of a next-to-nearest-neighbor hopping term.

II. THE WEAK-COUPLING REGIME

We start with the BCS action on a two-dimensional lattice, at a temperature \(T = \beta^{-1}\):

\[
S = \int_0^\beta d\tau \left\{ \sum_{k,\sigma} c_{k\sigma}^+(\tau)(\partial_\tau + \xi_k)c_{k\sigma}(\tau) + H_I(\tau) \right\},
\]

\[
H_I = -\frac{U}{N_s} \sum_{k,k'q} \gamma_k \gamma_{k'} \left( \begin{array}{cc} c_{k+q,\uparrow}^+ c_{-k+q,\downarrow}^+ & c_{-k'q,\downarrow}^+ c_{k'q,\uparrow}^+ \end{array} \right). \tag{2}
\]

Here \(N_s\) is the number of lattice sites, \(U > 0\) is the pairing interaction strength, \(\xi_k = -2t(\cos k_x + \cos k_y) - \mu\) is the band dispersion associated with a nearest-neighbor hopping \(t\), \(\mu\) is the chemical potential, and the factor \(\gamma_k\) controls the symmetry of the gap. To derive the correlation function for \(|\Delta|\), we first perform the standard Hubbard-Stratonovich decoupling of \(H_I\) and then make explicit the dependence on the phase \(\theta\) and on the modulus \(|\Delta|\) of the complex order parameter \(\Delta = |\Delta|e^{i\theta}\) by means of the gauge transformation \(c_\sigma \to c_\sigma e^{i\theta/2}\) \[13\]. As a consequence, the action depends on \(|\Delta|\) through the Hubbard-Stratonovich Gaussian term and the interaction of the field with the fermions, which arises from \(H_I\), i.e.

\[
S(|\Delta|) = \frac{1}{q} \left| \Delta_n \right| \Delta_{-q} - \frac{T}{N_s} \sum_{k,k',q} \gamma_{k,k'} \Psi_{k'}^+ \left( \begin{array}{cc} 0 & |\Delta|_{k'k} \end{array} \right) \Psi_k,
\]

where \(q = (q, \Omega_m), k = (k, \omega_n), \Omega_m\) and \(\omega_n\) are the bosonic and fermionic Matsubara frequencies respectively, and we introduced the Nambu spinor \(\Psi_k = (c_{k,\uparrow}, c_{k,\downarrow})\). We assume, as usual, small fluctuations of \(|\Delta|_q\) around its saddle-point value \(\Delta_0\), i.e., \(|\Delta|_q = \Delta_0 + \delta|\Delta|_q\). After integrating out the fermionic degrees of freedom around the BCS superconducting saddle-point solution, we expand the resulting effective action \(S_{\text{eff}}\) for \(\delta|\Delta|_q\) up to the second order, obtaining

\[
S_{\text{eff}}(\delta|\Delta|) = \sum_q \left[ \frac{1}{U} - \frac{1}{2}D(q) \right] \delta|\Delta|_q\delta|\Delta|_{-q}, \tag{3}
\]

\[
D(q, \Omega_m) = \frac{T}{N_s} \sum_{k,\omega_n} \gamma^2_{k,\omega_n} \text{Tr} \left[ G_0 \left( k + \frac{q}{2}, \omega_n + \Omega_m \right) \right] \times \tau_\uparrow \tau_{\downarrow},
\]

where \(G_0\) is the Nambu Green function evaluated at the BCS level, \(\tau_\uparrow\) is the Pauli matrix and the trace is taken in the Nambu space.

2
We define

\[ X_\Delta(q, \Omega_m) = 2 < \delta|\Delta|q|\Delta|_q >, \]

i.e., the inverse of the coefficient of the Gaussian term in Eq. (3). Since we are interested in the spatial variation of the static correlation function for \( \delta|\Delta|q \) at zero temperature, we evaluate the \( T = 0 \) limit of \( X_\Delta(q, \Omega_m = 0) = X_\Delta(q) \).

\[ X_\Delta(q) = \left[ \frac{1}{U} - \frac{1}{2} D(q) \right]^{-1}, \]

\[ D(q) = \frac{1}{N_s} \sum_k \frac{\gamma_k^2}{E_+ + E_-} \left( 1 + \frac{\xi_+ \xi_- - \Delta_+ \Delta_-}{E_+ E_-} \right), \]

where, with a standard notation, \( \Delta_k = \Delta_0 \gamma_k \) [14], \( E_k = \sqrt{\xi_k^2 + \Delta_k^2} \), and \( \xi_\pm, \Delta_\pm, E_\pm \) are calculated at momenta \( k \pm q/2 \) respectively.

![Figure 1](image)

**FIG. 1.** \( X_\Delta^{-1}(q) \) at intermediate coupling \( U/t = 1 \) in the case of \( d \)-wave pairing for some values of the density \( \rho \). The wave vector \( q \) varies along the diagonal of the Brillouin zone, and an analogous behavior is observed by varying \( q \) along different directions. As it is expected in the weak-coupling regime, \( X_\Delta^{-1}(q) \) exhibits a minimum at \( q = 0 \) [which corresponds to a maximum for the correlation function \( X_\Delta(q) \) in Eq. (3)].

The length scale which controls the long-distance behavior of \( X_\Delta(r) \) may be extracted from the dominant part of its Fourier transform \( X_\Delta(q) \). It is generally expected that the main contribution to \( X_\Delta(q) \) comes from the region \( q \approx 0 \). This is confirmed by our numerical calculations. In Fig. 3 we show the \( q \) dependence of \( X_\Delta^{-1}(q) \) at various densities \( \rho \), for \( U/t = 1 \), in the case of \( d \)-wave pairing. Similar results are found in the \( s \)-wave case. Expanding Eq. (7) for small momenta, we get

\[ X_\Delta(q) \approx \frac{1}{m^2 + cq^2}, \]

where the “mass term” is given by

\[ m^2 = \frac{1}{N_s} \sum_k \frac{\gamma_k^2 \Delta_k^2}{2 E_k^3}, \]

and

\[ c = \frac{1}{16 N_s} \sum_k \left\{ \frac{2 \gamma_k^2}{E_k} \xi k \Delta_k (2 \Delta_k^2 - 3 \xi_k^2) (\partial k \xi_k) \cdot (\partial k \Delta_k) + \frac{\gamma_k^2}{E_k} \left[ \xi_k (\xi_k - 2 \Delta_k^2) \partial k \xi_k + 3 \Delta_k \xi_k \Delta_k^2 \right] + \frac{\gamma_k^2}{E_k} \left[ (\xi_k^4 - 4 \xi_k^2 \Delta_k^2)(\partial k \Delta_k)^2 + 5 \Delta_k^2 \xi_k (\partial k \xi_k)^2 \right] \right\}. \]

When the relevant fluctuations are those near \( q = 0 \), \( X_\Delta(q) \) is maximum at \( q = 0 \), and \( c > 0 \). In such a case the approximate expression (7) can be used to determine the long-distance behavior of \( X_\Delta(r) \). Indeed the Fourier transform of Eq. (6) has an exponential decay in real space with a characteristic length scale

\[ \xi_0 \approx \frac{c}{m^2}. \]

As we show in Appendix A, differently from \( \xi_{\text{pair}} \) given by Eq. (6), such a length scale, evaluated from Eqs. (6) and (7), is finite also in the presence of nodal quasiparticles, making it a natural candidate for the characteristic length of \( d \)-wave superconductors. In the weak-coupling regime \( \langle U/t \ll 1 \rangle \), \( \xi_0 \), Eq. (11), can be expressed both in the \( s \)-wave and in the \( d \)-wave case by means of the average values of the Fermi velocity \( v_F \) and of the gap \( \langle \Delta_F \rangle \) on the Fermi surface,

\[ \xi_0 \approx \frac{v_F}{\sqrt{12 \Delta_F}}. \]

Here we define the average value of a function \( h_k \) on the Fermi surface as

\[ h_F \equiv \sqrt{\langle h_k^2 \rangle_F} = \sqrt{\frac{\sum_k h_k^2 \delta(\xi_k)}{\sum_k \delta(\xi_k)}}. \]

In the \( s \)-wave case \( (\gamma_k = 1) \) \( \Delta_F = \Delta_0 \), and \( \xi_0 \) coincides within a numerical factor of order one with the Cooper-pair size (see Appendix A). In the \( d \)-wave case \( \Delta_F = \Delta_0 \gamma_d \), so that the effective gap \( \Delta_F \) which appears in Eq. (11) is smaller than the maximum value of the gap at the Fermi surface. In both cases, however, in the weak-coupling regime, the correlation function for \( \delta|\Delta|q \) exhibits an exponential decay over a length scale of order \( v_F/\Delta_0 \), while the different gap symmetries only introduce a numerical factor \( \gamma_F \). It is then natural to assume such a length scale as the spatial cutoff for phase fluctuations in the effective phase-only action, both in \( s \)- and in \( d \)-wave superconductors.
III. THE STRONG-COUPLING REGIME

The extension of the above results to the strong-coupling regime \( U/t \gg 1 \) is quite intriguing. Indeed, as the pairing increases the coefficient \( c \) of Eq. \( \text{(3)} \) decreases, and becomes \textit{negative}, making the definition \( \text{(14)} \) meaningless. On the other hand, it is commonly expected that in the diluted regime the fermionic system maps into a (weakly interacting) bosonic system, where the Cooper pairs act as boson particles, with a weak residual repulsion between them \( [4] \). It is well known \( [8] \) that in a Bogoljubov liquid of weakly interacting bosons, the coheren
t length \( \xi_{\text{bos}} \) which controls the correlations of the superfluid order parameter diverges as the density decreases. According to Ref. \( [8] \)

\[
\xi_{\text{bos}} = \frac{1}{\sqrt{2g m_{BB}}}, \tag{13}
\]

where \( g \) is the local repulsion, \( m_B \) and \( \rho_B \) are the mass and the particle density of bosons respectively. One would like to find a similar behavior for \( \xi_0 \) within the Hubbard model at strong coupling and in the low-density limit. As we anticipated in Sec. I, in the following we devote our attention to the case of \( s \)-wave pairing, which allows for an analytical treatment. However, one expects similar results also in the case of \( d \)-wave pairing.

At strong coupling the definition \( \text{(10)} \) must be generalized to include two effects: (i) the fluctuations of \( |\Delta| \) are strictly tied to the fluctuations of \( \rho \), in such a way that both contribute to the correlation function for \( |\Delta| \); when this interplay is considered, we find that (ii) the dominant contribution to \( X_\Delta(q) \) arises from \( q \neq 0 \) (except for the small-density regime), and the small-momentum approximation \( \text{(10)} \) is no longer appropriate.

We first address the point (i). In the \( s \)-wave Hubbard model the interaction \( H_I \) can be decoupled both in the particle-particle and in the particle-hole channel. When the density fluctuations are taken into account on the same footing as the Cooper-pair fluctuations, the effective action \( \text{(8)} \) gets modified, and reads

\[
S_{\text{eff}} = \sum_q \sum_{\mu=1,2} \sum_{\nu=1,2} \Phi_\mu^q X_{\mu\nu}^{-1}(q) \Phi_\nu^q, \tag{14}
\]

where \( \Phi_\mu^q = \delta \rho_\mu \Phi_\mu^q, \Phi_\mu^{q=2} = \delta |\Delta| q, X_{11}^{-1}(q) = 1/U - \frac{i}{2} \chi_\rho(q), X_{12}^{-1}(q) = X_{21}^{-1}(q) = \chi_\alpha(q), \) and \( X_{22}^{-1}(q) = 1/U - \frac{i}{2} D(q) \). Below, we only need the expressions in the static limit,

\[
\chi_\rho(q) = \frac{1}{N_s} \sum_k \frac{1}{E_+ + E_-} \left( 1 + \frac{\Delta_0^2 - \xi_+ \xi_-}{E_+ E_-} \right), \tag{15}
\]

\[
\chi_\rho \Delta(q) = -\frac{\Delta_0}{2N_s} \sum_k \frac{1}{E_+ + E_-} \frac{\xi_+ + \xi_-}{E_+ E_-}. \tag{16}
\]

To take into account the effect of density fluctuations on \( X_\Delta \) we integrate out the density-fluctuation field \( \delta \rho_q \) in Eq. \( \text{(14)} \), and recover the action for \( \delta |\Delta| \) only. The correlation function \( X_\Delta \) now reads

\[
X_\Delta(q) = \left[ \frac{1}{U} - \frac{1}{2} D(q) - \frac{\chi_\Delta^2(q)}{\chi_{\rho}(q)} \right]^{-1}. \tag{17}
\]

It is worth noting that integrating out the density at the Gaussian level corresponds to performing the RPA resummation for the correlation function \( X_\Delta \) in the particle-hole channel. At weak coupling the bubble \( \chi_\rho \Delta \) which couples the two channels is negligible, and the result \( \text{(10)} \) is recovered. In order to estimate Eq. \( \text{(17)} \) at strong coupling, we evaluate the \( q \)-dependent leading order in \( t/U \) of the bubbles \( \chi_0, \chi_\Delta \), and \( \chi_\alpha \) in Eqs. \( \text{(18)-(20)} \) where the function \( w(q) \equiv 4(\sin^2 \frac{x}{2} + \sin^2 \frac{x}{2}) \) reduces to \( q^2 \) at small momenta, while respecting the lattice periodicity at higher momenta. In Eqs. \( \text{(18)-(20)} \) the \( q = 0 \) values are of order \( 1/U \), while the coefficients \( c_\Delta, c_\rho \Delta \) and \( c_\rho \) are of order \( t^2/U^3 \). All these coefficients are functions of the gap amplitude \( \Delta_0 \) and of the chemical potential \( \mu \), whose dependence on the density can be determined by using the self-consistent saddle-point equations

\[
\frac{1}{2} = \frac{U}{2 N_s} \sum_k \frac{1}{E_k}, \quad \delta = \frac{1}{N_s} \sum_k \frac{\xi_k}{E_k}, \tag{18}
\]

where we introduced the “doping” \( \delta \equiv 1 - \rho \) to simplify the notation \( \text{(17)} \). The above equations can be explicitly solved at strong coupling giving

\[
\Delta_0 = \frac{U}{2} \left[ (1 - 2\alpha^2) \sqrt{1 - \delta^2} + \mathcal{O}(\alpha^4) \right], \tag{19}
\]

\[
\mu = -\frac{U}{2} \left[ \delta (1 + 4\alpha^2) + \mathcal{O}(\alpha^4) \right], \tag{20}
\]

where we introduced the small parameter \( \alpha \equiv 2t/U \).
As anticipated above, we shall investigate the region $0 \leq \rho \leq 1$, i.e. $0 \leq \delta \leq 1$, while the range $1 \leq \rho \leq 2$ can be recovered by particle-hole symmetry ($\delta \to -\delta$).

With long but straightforward calculations it can be seen that at leading order in $\alpha$

\begin{align}
  c_\Delta &= \frac{\alpha^2}{U} (5\delta^4 - 5\delta^2 + 1), \\
  c_{\rho\Delta} &= \frac{\alpha^2}{2U} \delta (3 - 5\delta^2) \sqrt{1 - \delta^2}, \\
  c_\rho &= \frac{\alpha^2}{U} (1 - \delta^2)(5\delta^2 - 1),
\end{align}

whose behavior is reported in Fig. 3.

By means of Eqs. (18), (21) and Eqs. (21)-(23) we obtain the $q$-dependent strong-coupling expression of $X_\Delta^{-1}$ at leading order in $\alpha$,

\begin{align}
  X_\Delta^{-1}(q) &= \frac{1}{U} - \frac{1}{2} \left[ D(0) - c_\Delta w(q) \right] \\
  &\quad - \frac{\chi_{\rho\Delta}(0)}{U} - \frac{1}{2} \left[ \chi_\rho(0) - c_\rho w(q) \right] \\
  &\quad + \frac{2\chi_{\rho\Delta}(0)c_\rho w(q)}{U} - \frac{1}{2} \left[ \chi_\rho(0) - c_\rho w(q) \right],
\end{align}

where we neglect the subleading $c_{\rho\Delta}^2$ term.

### A. Low-density regime

In the limit of low density ($\delta \approx 1$) it is possible to rewrite Eq. (24) in a simpler form. Indeed, at the leading order in $\alpha$ the denominator of the third and fourth terms in Eq. (24) is given by

\[
\frac{1}{U} - \frac{1}{2} \left[ \chi_\rho(0) - c_\rho w(q) \right] \approx \frac{1}{U} \left[ \delta^2 + 4\alpha^2 - \frac{\alpha^2}{2} w(q) \right].
\]

As a consequence, for $\alpha \ll \delta$ we can put

\[
\frac{1}{U} - \frac{1}{2} \left[ \chi_\rho(0) - c_\rho w(q) \right] \approx \frac{1}{U} \left[ 1 - \frac{\alpha^2}{2} c_\rho w(q) \right].
\]

Substituting Eq. (23) into Eq. (24) we have

\[
X_\Delta(q) = \frac{1}{M^2 + C w(q)},
\]

where the $q = 0$ value $M^2 \equiv X_\Delta^{-1}(0)$ is given by

\[
M^2 = \frac{1 - \delta^2}{U} \left[ 1 - 6\delta^2 + 30\delta^2 \alpha^2 \right] - \frac{\delta^2(1 - 36\delta^2 + 60\alpha^2 \delta^2)}{\delta^2 + 4\alpha^2(1 - \frac{\delta^2}{2} + \frac{\delta^2}{2} \delta^4)} \frac{4\alpha^2}{U} \frac{1 - \delta^2}{\delta^2}.
\]

and

\[
C = \frac{\alpha^2}{4U} \left[ 1 - 5\delta^2 + 5\delta^4 + \frac{2\delta^2(1 - \delta^2)(5\delta^2 - 3)}{\delta^2 + 4\alpha^2} \right] + \frac{\delta^2(5\delta^2 - 1)(1 - \delta^2)^2}{\delta^2 + 4\alpha^2} \frac{2\alpha^2}{U} \frac{2\delta^2 - 1}{\delta^2}.
\]

Therefore, $C > 0$ in the diluted limit ($\delta \approx 1$), and $X_\Delta(q)$ is dominated by the small-momentum region, where $w(q) \approx q^2$ and the coherence length is given by the generalization of Eq. (4), with $c$ and $m^2$ substituted by $C$ and $M^2$ respectively,

\[
\xi_0 = \sqrt{\frac{C}{M^2}} = \sqrt{\frac{2\delta^2 - 1}{8(1 - \delta^2)}} \frac{m^2}{\mu^2} \frac{1}{4\sqrt{\rho}}.
\]

The result (29) for the coherence length $\xi_0$ shows, at low particle density, the same divergence of the bosonic coherence length $\xi_{bos}$ given by Eq. (13). Notice that a more strict comparison between Eq. (29) and Eq. (13) requires a dependence of $\xi_0$ on the effective mass of the electron pair $m_P$ and on the pair-pair residual repulsion $g_P$ in the bosonic limit, of the form $\xi_0^2 \approx 1/(g_P m_P)$.

This is indeed the case, since in the bosonic limit of the fermionic model $\rho_B = \rho/2$, $m_P \approx U/\chi t^2$, and the residual repulsion of the bosonic model corresponds to the inverse of the compressibility of the fermionic system $g_P \sim \chi^{-1}$. Following the analysis of Ref. [4], it can be seen that at strong coupling the compressibility of the Hubbard model is $\chi = U/8t^2$ [18], leading to the explicit cancellation of $t$ and $U$ in Eq. (13), in agreement with Eq. (29). This supports the reliability of the mapping of the diluted large-$U$ Hubbard model into an effective boson model [2].
B. High-density regime

As $\delta$ decreases, according to Eq. (28), the coefficient $C$ decreases and becomes negative for $\delta < 1/\sqrt{2}$. This change of sign is not due to a failure of our approximation, since at strong coupling $\alpha \ll 1$, so that at $\delta = 1/\sqrt{2}$ the simplified expression (26) is still valid. The vanishing of the leading coefficient in the small-$q$ expansion is instead a signal of the fact that the maximum of the correlation function $X_\Delta(q)$ is moving to a finite $q$. Since the $q$-dependence of $X_\Delta(q)$ comes entirely from the function $w(q)$, the only candidate alternative to $q = (0,0)$ is $q = Q = (\pi, \pi)$. Indeed, evaluating $M^2_\pi = X_\Delta^{-1}(Q)$ according to Eq. (24) we find that

$$M^2_\pi = \frac{4\alpha^2}{U}$$

(30)

at all doping, so that $M^2_\pi < M^2$ as far as $\delta < 1/\sqrt{2}$, as depicted in Fig. 3. In such a case, the dominant $|\Delta|^q$ modes are located near the wave vector $Q$. This is an effect of the coupling of the Cooper to the density, which is a massless mode at exactly half filling ($\delta = 0$) for $q = Q$. Notice that this behavior is peculiar of the case $\rho = 1$ for a band dispersion arising from a nearest-neighbor hopping. In the presence of an attractive on-site interaction the system exhibits an enlarged symmetry with respect to the instability in the particle-particle channel (at $q = 0$) or in the particle-hole channel (at $q = Q$). Once the symmetry has been explicitly broken in the Cooper channel, the density becomes a Goldstone mode, similarly to the phase, reflecting such a degeneracy. Since at strong coupling $\delta \rho$ tends to fluctuate coherently with $\delta |\Delta|$ [33], at small doping and strong coupling both $X_\Delta(q)$ and $X_\rho(q)$ have a maximum at the wave vector which controls the instability of the density mode approaching half filling, i.e., $q = Q$.

As a result, for $M^2_\pi < M^2$ the long-distance behavior of $X_\Delta(r)$ should be controlled by the characteristic length $\xi_\pi$ obtained by considering the expansion of $X_\Delta(q)$ around $Q$,

$$X_\Delta(r) \simeq \int dq \frac{e^{iqr}}{M^2_\pi + A(q - Q)^2}$$

$$= \frac{1}{4} \left. \frac{\partial^2 X_\Delta^{-1}(q)}{\partial q^2} \right|_{q = Q} = \frac{\alpha^2 (1 - 2\delta^2)}{2U \delta^2},$$

(31)

so that the resulting $X_\Delta(r)$ is a staggered function with an exponential envelope controlled by the stiffness $A$ of the $q$-modes near $Q$ and by the mass $M^2_\pi$. The parameter $A$ is obtained by evaluating, at $q = Q$, the second-order derivative of $X_\Delta^{-1}(q)$, as given by the expression (24),

$$A = \frac{1}{4} \left. \frac{\partial^2 X_\Delta^{-1}(q)}{\partial q^2} \right|_{q = Q} = \frac{\alpha^2 (1 - 2\delta^2)}{2U \delta^2}.$$ 

(32)

According to Eq. (24), the long-distance decay of $X_\Delta(r)$ is exponential, with a characteristic length scale

$$\xi_\pi = \sqrt{\frac{A}{M^2_\pi}} = \sqrt{\frac{1 - 2\delta^2}{8\delta^2}},$$

(33)

which matches continuously at $\delta = 1/\sqrt{2}$ with the $\xi_0$ previously defined in Eq. (28).

![FIG. 3](image_url)

**FIG. 3.** Doping dependence of $M^2$ and $M^2_\pi$ given by Eqs. (27) and (28) respectively, for $t = 1$ and $U = 20.$

![FIG. 4](image_url)

**FIG. 4.** Absolute value of $X_\Delta(r)$ normalized to its $r = 0$ value for $r = n\xi$ at various $\delta$. Left panel: the slope of the long-distance (exponential) decay of $|X_\Delta(r)|$ in the diluted limit is given by $\frac{1}{\xi_0}$, with $\xi_0$ from Eq. (23). Right panel: $|X_\Delta(r)|$ near half filling. Although the characteristic length $\xi_\pi$ of the exponential tail increases as the doping decreases, $|X_\Delta(r)|$ is strongly suppressed at much shorter distances.
According to our strong-coupling expression (24) both $\xi_0$ and $\xi_\kappa$ vanish at $\delta = 1/\sqrt{2}$. This is clearly an artifact of retaining only the order $\mathcal{O}(\alpha^2)$ in the $q$ dependence of $X^{-1}_\Delta(q)$. By including in Eqs. (18)-(21) the next terms, which are of the form $\sim \alpha^4 w^2(q) + \text{const} \times [\sin^4(q_x/2) + \sin^4(q_y/2)]$, the ratio between the coefficient of $q^4$ and the mass term in Eq. (24) gives a coherence length of order $\xi_0 \sim \xi_\kappa \sim \mathcal{O}(\alpha^3)$.

One usually defines the correlation length $\xi_0$ as

$$\xi_0^{-1} = - \lim_{r \to \infty} \frac{\ln X_\Delta(r)}{r}.$$ 

According to this definition, the identification between $\xi_0$ the and $\xi_\kappa$ for $\delta < 1/\sqrt{2}$ would imply a divergent $\xi_0$ at half filling. Such a divergence is surprising, since in this regime one would naively expect that the correlation length for the amplitude fluctuations attains a value of the order of the lattice constant. Indeed this problem is only apparent, as we can see by evaluating numerically $X_\Delta(r)$ near half filling from Eqs. (13)-(17). In Fig. 4 we report $X_\Delta(r)$ normalized to its $r = 0$ value at various doping, both in the diluted (left panel) and nearly half-filled (right panel) limit. In particular, near half filling, it can be seen that, although the long-distance behavior of $X_\Delta(r)$ is controlled by the large length scale $\xi_\kappa$, Eq. (23), which effectively diverges as $\delta \to 0$, $X_\Delta(r)$ is severely suppressed with respect to its $r = 0$ value at much shorter distances, of the order of the lattice spacing. A simple criterion to estimate the coherence length consists in determining the distance $\xi_\kappa$ at which $X_\Delta(r)$ has reached a fixed percent $\kappa$ of its $r = 0$ value, say $\kappa = 0.1$, in Fig. 4. One can identify $\xi_0$ as the minimum value between $\xi_\kappa$ and the lattice spacing, which is the lower bound for $\xi_0$ in a lattice model. From Fig. 4 it follows that the maximum value of $\xi_0$ at $U/t = 30$ is about 5 lattice spacings, and is reached at $\delta = 0.05$. At lower doping the decay of $X_\Delta(r)$ is much more rapid, even though the long-distance tail extends over a greater distance. On the other hand, in the low-density regime the coherence length is really diverging. In the left panel of Fig. 4 we report $|X_\Delta(r)|/|X_\Delta(0)|$ at high doping. $X_\Delta(r)$ is exponentially decreasing in this regime, with the characteristic length given by Eq. (25). Indeed, as $\delta$ increases, the slope of the curve, i.e. $-\xi_0^{-1}$, decreases, and the overall decay of $X_\Delta(r)$ extends up to larger distances.

The different role played by $\xi_0$, Eq. (24), and $\xi_\kappa$, Eq. (23), in determining the decay of $X_\Delta(r)$ can be better understood by analyzing in greater detail the $q$ dependence of $X_\Delta(q)$ approaching half filling. $X^{-1}_\Delta(q)$ is reported in Fig. 5. Let us consider the curve for $\delta = 0.005$. Even though the minimum of $X^{-1}_\Delta(q)$ is at $q = Q$, the width of this minimum decreases with $\delta$. This reflects the fact that, according to Eqs. (32) and (30), the divergence of $\xi_\kappa$, Eq. (23), at half filling arises from an increasing stiffness $A$ of the $q$ modes near $Q$ with a fixed mass. As a consequence near half filling $X_\Delta(q)$ is almost constant (equal to its $q = 0$ value) except in a narrow region near $Q$, which then contributes to the integral (31) only at very large distances and with a small weight.

In the inset of Fig. 5 it is shown that at $\delta = 0.005$ a local minimum of $X^{-1}_\Delta(q)$ at $q = 0$ exists besides the global minimum at $q = Q$. Indeed, the second-order derivative of $X^{-1}_\Delta(q)$ at $q = 0$, given by $C$ of Eq. (28) is again positive at $\delta \leq 4\alpha^2$. At exactly $\delta = 0$ only the $q = 0$ minimum survives, since, according to Eqs. (10) and (24), $\chi_\rho = 0$ so that the amplitude fluctuations decouple from the density fluctuations and $X_\Delta(q)$ reduces to the form (6). As a consequence, the maximum of $X_\Delta(q)$ is again at zero momentum and the coherence length is

$$\xi_0(\delta = 0) = \sqrt{\frac{c_\Delta}{U - D(0)}} = \sqrt{\frac{\alpha^2}{2}}.$$ 

This small value matches continuously with the characteristic length scale which controls the short-distance decay of $X_\Delta(r)$ near half filling, and can be extracted form the data reported in the right panel of Fig. 3.

C. The density-density correlation function

In the last part of this section, to gain some more insight into the physics of the interplay between the particle-particle and particle-hole channels at strong coupling, we compare the above results for the pairing correlation function with the outcomes of the same calculations for the density-density correlation function. In analogy with the case of $X_\Delta(q)$, starting from the same Eq. (14), we find that the density-density correlation
function, defined as $X_{\rho}(q, \Omega_m) = 2 < \delta \rho_q \delta \rho_{-q} >$, is given by
\[ X_{\rho}(q) = \left[ \frac{1}{U} - \frac{1}{2} \chi_{\rho}(q) - \frac{\chi_{\rho \Delta}^2(q)}{U - \frac{1}{2} D(q)} \right]^{-1}, \tag{34} \]
i.e. the RPA resummation for $X_{\rho}$ in the particle-particle channel. Evaluating Eq. (34) in the strong-coupling regime by means of Eqs. (15)-(23) and Eqs. (21)-(23), we find that at leading order in $\alpha X_{\rho}^{-1}(q) = M^2 + C w(q)$, where $M^2 \equiv X_{\rho}^{-1}(0) = 4\alpha^2 / U$, and $C = (\alpha^2 / 2U)(2\delta^2 - 1)(1 - \delta^2)$. It follows that $C > 0$ at $\delta > 1/\sqrt{2}$, so that the maximum of $X_{\rho}(\mathbf{q})$ is at $\mathbf{q} = 0$ and the coherence length $\xi_{\rho}$, which controls the exponential decay of $X_{\rho}(\mathbf{r})$, does coincide with $\xi_0$, Eq. (29),
\[ \xi_{\rho} = \sqrt{\frac{C}{M^2}} = \sqrt{\frac{2\delta^2 - 1}{8(1 - \delta^2)}}. \tag{35} \]
Instead, for $\delta < 1/\sqrt{2}$, $C < 0$ and the maximum of $X_{\rho}(\mathbf{q})$ is at $\mathbf{q} = \mathbf{Q}$. Indeed, the mass term for the density-density correlation function at $\mathbf{q} = \mathbf{Q}$ is
\[ M_\pi^2 \equiv X_{\rho}^{-1}(\mathbf{Q}) = M^2 + 8C = \frac{4\alpha^2}{U} \frac{\delta^2}{1 - \delta^2}, \tag{36} \]
and the second-order derivative at $\mathbf{q} = \mathbf{Q}$ is
\[ A \equiv \frac{1}{4} \partial^2_{\mathbf{q}} X_{\rho}^{-1}(\mathbf{q}) \bigg|_{\mathbf{q} = \mathbf{Q}} = -C. \]
As a consequence, for $\delta < 1/\sqrt{2}$, $M_\pi^2 < M^2$, and the function $X_{\rho}(\mathbf{r})$ oscillates with an exponential envelope controlled by the characteristic length
\[ \xi_{\rho} = \sqrt{\frac{A}{M_\pi^2}} = \sqrt{\frac{1 - 2\delta^2}{8\delta^2}}, \tag{37} \]
which coincides with $\xi_\pi$, Eq. (33). However, contrary to the case of the $|\Delta|$ mode, which is suppressed at much shorter distances due to the presence of a second small length scale associated with the secondary minimum of $X_{\Delta}^{-1}(\mathbf{q})$ at $\mathbf{q} = 0$, the exponential decay of $|X_{\rho}(\mathbf{r})|$ is controlled by a single length scale given by Eq. (37), which diverges at $\delta = 0$. This divergence is due to the fact that the density mode becomes a Goldstone mode for the enlarged symmetry of the Hubbard model at half filling, so that the mass $M_\pi^2$, Eq. (23), vanishes, while the stiffness $A$ is finite. As a consequence, while $\xi_0$ for pairing fluctuations is ultimately cut off by the lattice constant, the divergence of $\xi_{\rho}$ as $\xi_\pi$ according to Eq. (37) is the signal of the degenerate instability at $\rho = 1$. On the other hand, since in the diluted limit the pairing and density fluctuations become proportional, as shown in Ref. (13), the characteristic length of the spatial decay of both pairing [Eq. (29)] and density mode [Eq. (23)] diverges according to the mapping with the bosonic model. We like to point out that the degenerate instability at $\rho = 1$ is specific of the Hubbard model with nearest-neighbor hopping. Indeed, as discussed in Appendix B, the inclusion of a next-to-nearest-neighbor hopping term $t'$ spoils this degeneracy and the charge sector becomes massive even at $\rho = 1$. However, by approaching $\rho = 1$, the charge mode at $\mathbf{q} = \mathbf{Q}$ still affects the pairing correlation function as discussed in this section, provided $t'/t$ is not too large.

The extension of the above results to the strong-coupling regime for the $d$-wave symmetry is not straightforward. Indeed, the decoupling of the interaction $H_I$, Eq. (3), in the particle-hole channel does not simply reduce to a density-density interaction, but rather renormalizes the bare band dispersion, limiting the possibility for an analytical treatment. Nevertheless, supported also by the analysis of the weak-coupling case, we expect that in the strong-coupling limit results similar to those of the $s$-wave superconductor hold. In particular, one would find a coherence length of the order of the lattice parameter for all doping except in the low-density regime, where the mapping to the bosonic system should be recovered. Anyway, further numerical work is required to address this issue in more detail.

IV. CONCLUSIONS

In summary, we studied the evolution of the superconducting coherence length $\xi_0$ from weak to strong coupling. For both $s$-wave and $d$-wave superconductors, $\xi_0$ is defined through the spatial decay of the correlation function $X_\Delta$ for the fluctuations of the modulus of the order parameter. At weak coupling the issue arises of the comparison between $\xi_0$ and the Cooper-pair size $\xi_{\text{pair}}$. These two length scales are both of order $v_F/\Delta_0$ in $s$-wave superconductors. However, as we showed, this identification of $\xi_0$ and $\xi_{\text{pair}}$ at weak coupling does not hold for $d$-wave superconductor, where $\xi_{\text{pair}}$ is divergent, while $\xi_0$ is finite and of order $v_F/\Delta_0$.

At strong coupling the modulus and density fluctuations are coupled, leading to a contribution of the density mode to the correlation function $X_\Delta$. The case of $s$-wave pairing within the negative-$U$ Hubbard model is simpler and allows for a detailed analytical treatment. We then evaluated $\xi_0$ for $s$-wave superconductors by properly including density fluctuations. We found that the coherence length in the diluted regime diverges as $\xi_0 \sim 1/\sqrt{\rho}$, according to the mapping of the Hubbard model into a weakly-interacting effective bosonic model, in the strong-coupling and low-density regime. Away from the diluted limit the coherence length attains a value of order of few lattice spacings, as reflecting the local character of the Cooper pairing. Similar results are expected in the case.
of \(d\)-wave pairing, where however the analytical treatment cannot be carried out, and a deeper insight can only be gained by further numerical investigation.

Finally, we comment on the issue of choosing an appropriate cutoff for the phase-only action when analyzing phase-fluctuation effects in superconductors. Our results indicate that at weak coupling it is reasonable to consider momenta up to a cutoff \(|q_c| \simeq \Delta_0/v_F\) for both \(s\)-wave and \(d\)-wave superconductors. In the strong-coupling regime, as we explicitly showed in the \(s\)-wave case, one should take \(|q_c| \simeq 1/a\) over a wide doping region, with the noticeable exception of the low-density region, where \(\xi_0\) diverges and \(|q_c| \simeq \sqrt{\rho/a}\).

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APPENDIX A: COOPER-PAIR SIZE AND COHERENCE LENGTH AT WEAK COUPLING

In this appendix we report the detailed calculation of \(\xi_{\text{pair}}\) and \(\xi_0\) in the \(s\)-wave and \(d\)-wave case at weak coupling. Let us start with \(\xi_{\text{pair}}\), which is defined in Eq. (A1) as \(\xi_{\text{pair}} = \sqrt{N/D}\) where

\[
N \equiv \int d\mathbf{r} |\psi(\mathbf{r})|^2 = \frac{1}{N_s} \sum_{k} |\partial_k \phi(\mathbf{k})|^2
= \frac{1}{N_s} \sum_{k} \left\{ \xi_k^4 \left( \frac{\Delta_0^2}{E_k^0} \right)^2 + \frac{\Delta_0^2 v^2 k}{E_k^0} (\partial_k \xi_k)^2 - \frac{2 \Delta_0^2 v^2 k}{E_k^0} (\partial_k \xi_k) \cdot (\partial_k \xi_k) \right\},
\]

and

\[
D \equiv \int d\mathbf{r} |\psi(\mathbf{r})|^2 = \frac{1}{N_s} \sum_{k} \phi^2(\mathbf{k}) = \frac{1}{N_s} \sum_{k} \frac{\Delta_0^4}{E_k^0}.
\]

In the \(s\)-wave case only the second term in the right-hand side of Eq. (A1) survives, and letting \(v_k^2 \equiv (\partial_k \xi_k)^2\) we have

\[
N = \frac{1}{N_s} \sum_{k} \frac{\Delta_0^2 v^2 k^2}{E_k^0}
= \Delta_0^2 \int dx \frac{x^2}{(x^2 + \Delta_0^2)^3} \frac{1}{N_s} \sum_{k} v_k^2 \delta(x - \xi_k)
= \Delta_0^2 \int dx \frac{x^2}{(x^2 + \Delta_0^2)^3} V(x),
\]

where we define \(V(x) \equiv (1/N_s) \sum_k v_k^2 \delta(x - \xi_k)\). Since at weak coupling the main contribution to the above integral comes from \(x \simeq 0\), we take \(V(x) \simeq V(0) = (1/N_s) \sum_k v_k^2 \delta(\xi_k)\) and we extend the integral over \(x\) between \(-\infty\) and \(+\infty\) to extract the leading behavior

\[
N \simeq \Delta_0^2 V(0) \int_{-\infty}^{+\infty} dx \frac{x^2}{(x^2 + \Delta_0^2)^3} = \frac{\pi V(0)}{8 \Delta_0^4}.
\]

Similarly,

\[
D = \frac{1}{N_s} \sum_{k} \frac{\Delta_0^2}{E_k^0} = \Delta_0^2 \int dx \frac{N(x)}{x^2 + \Delta_0^2} \simeq \pi N(0) \Delta_0,
\]

where \(N(x) \equiv (1/N_s) \sum_k \delta(x - \xi_k)\) is the density of states in the metallic phase. Then we find

\[
\xi_{\text{pair}}^2 = \frac{v_F^2}{8 \Delta_0^2} \quad \text{(A2)}
\]

where, according to Eq. (A1), we introduce the mean value of the velocity at the Fermi surface

\[
v_F^2 = \langle v_k^2 \rangle_p = \frac{V(0)}{N(0)}.
\]

It is worth noting that the approximations leading to Eq. (A2) are no longer valid when the chemical potential approaches the extrema or the saddle points of the spectrum \(\xi_k\) (critical points), where a more refined evaluation is needed to get the correct result. However, by means of numerical calculations we checked that the estimate (A2) is very accurate even close to the critical points.

In the \(d\)-wave case \(\xi_{\text{pair}}\) is infinite, since the nodal quasiparticles give a logarithmically divergent contribution to \(N\) in Eq. (A1). Near the nodes the quasiparticle spectrum \(E_k\) is cone-like, \(\xi_k \simeq v_1 k_1, \Delta_k \simeq v_2 k_2\), and \(E_k \simeq \sqrt{v_1^2 k_1^2 + v_2^2 k_2^2}\). Here \(v_1\) and \(v_2\) are the Fermi velocity and the slope of the gap \(\Delta_k\) at the node, and \(k_1, k_2\) are the components of \(k\) along the directions perpendicular and parallel to the Fermi surface respectively, measured from the node. By introducing the polar coordinates \((E, \theta)\), such that when \(k\) is near a node \(\xi_k = E \cos \theta\) and \(\Delta_k = E \sin \theta\), and using the identity,

\[
\frac{1}{N_s} \sum_{k_1 k_2} = \int d\theta dE \frac{E}{v_1 v_2},
\]

we find that the first two (positive) terms in the right-hand side of Eq. (A1) are divergent, while the last term, which in principle has not a definite sign, vanishes due to the orthogonality of \(\partial_k \Delta_k\) and \(\partial_k \xi_k\) for a cone-like spectrum. For example, the first term gives

\[
\frac{1}{N_s} \sum_{k} \frac{\xi_k^4}{E_k^0} (\partial_k \Delta_k)^2
\]

\[
\simeq \frac{v_1}{v_2} \int_{0}^{2\pi} d\theta \int_{0}^{\Lambda} dE E^3 \cos^4 \theta \int_{0}^{\Lambda} dE \frac{E}{E} \rightarrow \infty,
\]

where \(\Lambda\) is a properly defined upper cutoff. Notice, instead, that the contribution of the nodal quasiparticles to \(D\) is finite,
\[ D = \frac{1}{N_s} \sum_k \Delta^2_k E_k^2 \]

\[ \simeq \frac{1}{v_1 v_2} \int_0^{2\pi} d\theta \int_0^\Lambda dE \frac{E^3 \sin^2 \theta}{E^2} \propto \int_0^\Lambda dE. \]

We next turn to the evaluation of the coherence length \( \xi_0 \) in the weak-coupling regime, defined in Eq. (10). It can be seen that at weak coupling the leading contribution to \( c \) comes from the last term in Eq. (3), so in Eq. (3) we must take

\[ c \simeq \frac{5}{16N_s} \sum_k \frac{\gamma^2_k \Delta^2_k \xi_k^2 v^2}{\xi_k^2 v_k^2}, \]

while \( m^2 \) is given by Eq. (8). Observe that both \( c \) and \( m^2 \) are finite whatever is the gap symmetry. In the \( d \)-wave case it can be seen that, thanks to the presence of the \( \gamma_k^2 \) factor in the previous equation, the contribution of nodal quasiparticles to \( c \) is indeed proportional to \( \int d\theta dE E^2 \sin^2 \theta \sin^2 \theta / E^2 \). Let us more generally consider the case of arbitrary gap modulation \( \gamma_k \) and band dispersion \( \xi_k \). We first evaluate the mass term

\[ m^2 = \frac{\Delta^2}{2N_s} \sum_k \frac{\gamma_k^4}{(\xi_k^2 + \Delta_k^2 \xi^2_0)^2} = \frac{\Delta^2}{2} \int dz \frac{G(z)}{(z^2 + \Delta^2_0)^2}, \]

where \( G(z) = (1/N_s) \sum_k |\gamma_k| \delta(z - \xi_k/\gamma_k) \). Again, the main contribution to the above integral comes from \( z \approx 0 \), and we obtain

\[ m^2 \simeq G(0) = \frac{1}{N_s} \sum_k \gamma_k^2 \delta(\xi_k). \]

Analogously, we have

\[ c = \frac{5\Delta^2}{16} \int dz \frac{z^2}{(z^2 + \Delta^2_0)^2} W(z) \simeq \frac{W(0)}{12 \Delta_0^2}, \]

where \( W(z) = (1/N_s) \sum_k (\gamma_k^2/|\gamma_k|) \delta(z - \xi_k/\gamma_k) \) and \( W(0) = V(0) \). As a consequence, according to the definition (12), we can write

\[ \xi_0^2 = \frac{V(0)}{12 \Delta_0^2 G(0)} = \frac{v_F^2}{12 \Delta_0^2}, \]

which represent the generalization of the standard result for a continuum model, to the lattice case with arbitrary band dispersion \( \xi_k \), and for arbitrary symmetry \( \gamma_k \) of the gap parameter.

In the \( s \)-wave case Eq. (A3) reduces to \( \xi_0^2 = v_F^2/12 \Delta_0^2 \), so that \( \xi_{\text{pair}} \), Eq. (12), and \( \xi_0 \) differ only by the numerical factor, \( \xi_{\text{pair}}/\xi_0 = \sqrt{3}/2 \). Both lengths are, in turn, proportional to the Pippard length scale \( \xi_c = v_F/\pi \Delta_0 \), which appears in the electromagnetic response function \[ [13]. \]

**APPENDIX B: THE EFFECT OF A NEXT-TO-NEXT-NEAREST-NEIGHBOR HOPPING**

Although the strong-coupling results obtained in Sec. III are specific of the negative-\( U \) Hubbard model with nearest neighbor hopping, we argue that they should not be dramatically modified by extensions of the original model. In particular, we discuss here in some detail the effect of a next-to-next-nearest-neighbor hopping term \( t' \). It is known that such a term breaks the extended symmetry of the half-filled model with \( t' = 0 \), making charge ordering unfavorable. At weak coupling this is the case, due to the spoiling of the perfect nesting of the Fermi surface as soon as \( t' \neq 0 \). Therefore, at weak coupling the superconducting state is favored at all doping down to \( \delta = 0 \), and never becomes degenerate to the charge-ordered state. Since at small \( U \) the coupling of the particle-particle and particle-hole channel is negligible, the weak-coupling results discussed in Sec. II are unaffected by the modification of the charge sector due to the \( t' \) term.

At strong coupling the extended symmetry is spoiled as well by the inclusion of \( t' \) and again superconductivity is favored with respect to charge ordering, even at \( \rho = 1 \). This can be more easily understood exploiting the spin analogue of the extended symmetry, which is achieved by mapping the negative-\( U \) Hubbard model onto a positive-\( U \) model at half filling. We point out that, contrary to the case \( t' = 0 \), in which the repulsive Hubbard model is recovered [20], the transformed \( t' \) term acquires an extra sign which depends on the spin of the hopping fermion, \( t'(-U) \rightarrow \sigma t'(U) \). Subsequently, in the limit of large \( U \), we map this positive-\( U \) model onto an effective spin model which is the usual antiferromagnetic Heisenberg model with coupling \( J \sim t'^2/|U| \) for nearest-neighbor spins, but yields the non \( SU(2) \)-invariant coupling

\[ J' \sum_{i,j>0} [S_i^x S_j^x - (S_i^z S_j^z + S_i^y S_j^y)], \]

with \( J' \sim t'^2/|U| \), between spins located on next-to-next-nearest-neighbor sites \( <i,j> \). Evidently the magnetic order along the \( z \)-axis (which corresponds to charge ordering in the negative-\( U \) Hubbard model) is frustrated due to the competition of the two antiferromagnetic couplings, whereas the magnetic order on the \( xy \)-plane (which corresponds to superconductivity) is not frustrated due to the absence of interference between an antiferromagnetic nearest-neighbor and a ferromagnetic next-to-nearest-neighbor coupling. Indeed, the charge sector acquires a mass of order \( t'^2/|U| \) at half filling. Nevertheless the results discussed in Sec. III are not dramatically modified, as long as the mass in the charge sector stays small near half filling, so that the strong peak at \( q = Q \) survives in \( X_p(q) \).
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[15] We introduce the notation $\Delta_k \equiv \Delta_0 \gamma_k$ for the $k$-dependent BCS gap. This should not be confused with $|\Delta_q|$ which is the $q \equiv (q, \Omega_m)$ component of the fluctuating amplitude $|\Delta|$ of the order parameter.
[16] This result is formally equivalent to the derivation of Ref. [10] with respect to the case of a $s$-wave model à la Nozières and Schmitt-Rink [11].
[17] As it is customary, our chemical potential $\mu$ includes the Hartree shift $\mu = \bar{\mu} + (U/2)\rho$, where $\bar{\mu}$ is the true chemical potential.
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