Growth exponents of the etching model in high dimensions

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Abstract
In this work we generalize the etching model (Mello et al 2001 Phys. Rev. E 63 041113) to $d + 1$ dimensions. The dynamic exponents of this model are compatible with those of the Kardar–Parisi–Zhang universality class. We investigate the roughness dynamics with surfaces up to $d = 6$. We show that the data from all substrate lengths and for all dimensions can be collapsed into one common curve. We determine the dynamic exponents as a function of the dimension. Moreover, our results suggest that $d = 4$ is not an upper critical dimension for the etching model, and that it fulfills the Galilean invariance.

Keywords: upper critical dimension, KPZ universality class, etching model

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(Some figures may appear in colour only in the online journal)

1. Introduction

In recent decades, surface dynamics has become an intense research topic. This interest has partially been motivated by the description of several real-world system dynamics such as stochastic processes, weather, and population dynamics. Under the statistical approach, the detailed structure of these systems is abstracted and modeled in simple ways. Those models have been grouped into universality classes of similar dynamics [2].

Of particular interest in the study of growing surfaces is the Kardar–Parisi–Zhang (KPZ) equation [3]

$$\partial_t h(\vec{x}, t) = \mu V^2 h + \frac{1}{2} \left( \vec{V} h \right)^2 + \eta(\vec{x}, t).$$ (1)
It describes the kinetics of a $d$ dimensional surface in a space of $d + 1$ dimensions. The roughness smoothing is controlled by the surface tension $\mu$, $\lambda$ quantifies the non-linear growth, and $\eta$ is the non-correlated noise. Besides its theoretical applications, the KPZ equation describes many real-world systems, such as flame front propagation [4] and deposition of thin films [5].

For the particular case of the KPZ equation in $1 + 1$ dimensions, the dynamic exponents have been known for more than 20 years [6, 7]. One exact solution was recently discussed by Sasamoto et al [8]. However, there are no such exact results for higher dimensions. In these cases, the growth exponents are obtained by two approaches: numerical simulations and approximate methods.

The non-linear character of the KPZ equation leads to behaviors similar to those found in the cellular automata models of atomistic surfaces. Some recent developments include extensions on the Eden model [9], use of the high parallelism of GPU computation [10], and numerical integration of the KPZ equation [11]. Because simulations of large substrates and higher dimensionality are memory bounded and very time consuming, numerical approaches are limited by hardware availability.

Using renormalization group techniques, it has been conjectured that the KPZ equation is the field theory of many surface growth models, such as the Eden model, ballistic deposition (BD), the restricted solid-on-solid (RSOS) model and the polynuclear growth (PNG) model [12]. A rigorous proof has been provided by Bertini and Giacomin [13] in the case of the RSOS model.

Most stochastic models of surface dynamics with relaxation present the behavior described by the Family–Vicsek (FV) relation [14]. As shown in figure 1, the roughness $w(t)$ grows initially as a power-law function of time, with exponent $\beta$. When $t \to \infty$, the roughness saturates at

$$w_s \propto L^\alpha,$$

where $L$ is the substrate size. These properties are expressed in the FV relation:

$$w(t, L) = w_s f\left(t/t_s, \beta\right) = \begin{cases} w_s t^\beta & \text{if } t \ll t_s \\ w_s & \text{if } t \gg t_s \end{cases}$$

The two regimes are separated by the saturation time $t_s$, defined as the intersection of the two preceding functions, leading to

$$t_s \propto L^z,$$

with $z = \alpha/\beta$.

Considering the attention obtained by the KPZ equation and related model, even if only numerical solutions were considered, one would expect that most of its dynamics should be fully understood by now. However, that is not the case. For example, the absence of exact solutions leads to the much debated possibility of an upper critical dimension (UCD) $d_c$ for the dynamic exponents [15]. For a concise review, see [16].

From the results of numerical experiments, mathematical expressions involving integers have been proposed to describe how the dynamic exponents of models belonging to the KPZ universality class depend on the dimension $d$, i.e., $\alpha = \alpha(d)$, $\beta = \beta(d)$, and $z = z(d)$. The best known are those for the RSOS model by Kim and Kosterlitz [17], for the Eden model [18] by Kertész and Wolf, for the heuristic approach to the strong-coupling regime by Stepnow [19], for a tentative method based on quantization of the exponents by Lässig [20], and for a perturbation expansion of the KPZ equation by Bouchaud and Cates [21].
Unfortunately, further numerical results have shown that these expressions are neither exact nor precise \[22–25\].

Figure 1. Roughness \(w(L, t)\) from the etching model as a function of time, for surfaces with (a) \(d = 1\), (b) \(d = 4\), and (c) \(d = 6\), showing only data points for \(t < 15t_e\) for clarity. The lines are guides for illustration.
Analytical methods such as mapping of the directed polymer (DP) [26], perturbation expansion [27], and mode-coupling techniques [28], among others [29–32], suggest the value $d_c = 4$. An asymptotically weak noise approach by Fogedby [33] suggests $d_c < 4$. On the other hand, such a limit was not found by numerical studies [18, 34, 35], or by the numerical and theoretical results obtained by Scharwartz and Perlsman [16].

In this work, we contribute to the discussion regarding dynamic exponent values by extending the etching algorithm by Mello et al. [1] to $d + 1$ spatial dimensions. The exponents obtained in that work through simulations of this model are mostly compatible with the values of the KPZ equation.

Here, we determine the exponents for $1 \leq d \leq 6$ and we compare them with other numerical results in the existing literature, concluding that if the UCD exists, it is no less than $d_c = 6$. Moreover, we show that this version of the etching model obeys the Galilean invariance (GI).

2. The etching model in $d$ dimensions

Surface roughness obtained by numerical simulation of a discrete atomistic model often presents the FV scaling. Examples are the BD [36] and Wolf–Villain (WV) [37] models.

One of these models is the etching algorithm, which is a simple atomistic model that mimics the etching of a crystalline solid by a liquid [1]. It was originally proposed for $d = 1$, in which case the scaling exponents are very close to those of the KPZ equation, namely $\alpha = 0.4961 \pm 0.0003$ and $\beta = 0.330 \pm 0.001$. For this reason, the model is believed to belong to the KPZ universality class, although this has not been formally proven.

Some properties of this model have been investigated, such as its Kramers–Moyal coefficients and Markov length scale [38], maximum and minimum height distribution [39], height and roughness distributions in thin films [40] and a variation of the model on $+2$ dimensions used to test a novel method for roughness exponent estimation [41].

In the present work, we investigate the dynamics and the exponents of the roughness of this model extended to $d + 1$ dimensions. For this model, the ‘solid’ is a square lattice exposed to a solvent, and the removal probability of each cell is proportional to its exposed area. The cellular automata with $d = 1$ is

(i) at discrete instant $T$, one horizontal site $i = 1, 2…, L$ is randomly chosen;
(ii) $h_i(T + 1) = h_i(T) + 1$;
(iii) if $h_{i+\delta}(T) < h_i(T)$, do $h_{i+\delta}(T + 1) = h_i(T)$, where $\delta = \pm 1$ are the first neighbors.

The general case $i$ and $\delta$ are vectors and $\delta$ runs over the $2^d$ first neighbors of the hypercube. If $L$ is the substrate length in each direction, the total number of sites is $L^d$. The normalized time $t$ defines the time unity as $L^d$ cellular automata iterations, i.e., $t = T/L^d$.

Note that (i) introduces randomness in time and space; this is equivalent to the noise $\eta$ in the KPZ equation. The off-diagonal condition (iii) combines the linear Laplacian term—which tends to smooth the surface reducing its curvature—with a nonlinear term, the lateral growth, equivalent to the Burgers equation. Of course, this cellular automata is not the KPZ, but we expect it to mimic the KPZ dynamics for $d + 1$ dimensions, as it does for $1 + 1$.

For each dimension $d$, the simulations are performed with several substrate lengths $L$. For each value of $d$ and $L$, the experiment is repeated several times, and the ensemble average is taken to reduce noise. As commonly done in surface dynamics simulations, we apply periodic boundary conditions to reduce unwanted finite length effects.
Figure 1 shows the roughness evolution for some substrate lengths of dimensions \(d = 1, 4,\) and 6 in a log–log scale. The same behavior appears for \(d = 2, 3,\) and 5 dimensions. The short-range correlations develop at very small time scales \(t \lesssim 1,\) resulting in deviations from the FV. This transient plays an important role in determining the growing exponents, mainly for higher dimensions, where computer power limitations impose small values of \(L,\) leading to small values of \(t_x.\) After that time, all substrates show the expected power-law-like behavior for \(t \ll t_x,\) saturating when \(t \gg t_x.

3. The data collapse

Later in this work, we obtain the parameters \(w_x, \beta,\) and \(t_x\) from our simulation data. Once the values of \(w_x\) and \(t_x\) are known for every \(L\) of a given \(d,\) the corresponding data may be rescaled, resulting in the collapse predicted by the FV relation (3).

The parameters \(\beta\) and \(t_x\) are responsible for the collapse at \(t \ll t_x,\) whereas the parameter \(w_x\) is responsible for the collapse at \(t \gg t_x.\) If these parameters are to be obtained from fitting the roughness as a function of the time for each value of \(d\) and \(L,\) then this is the minimal set of parameters required to collapse the data. The agreement is verified in both extreme regions of figures 2(a) and (b), indicating that these parameters were properly obtained.

4. Finding the dynamic exponents

Power-law fitting (PL) is a common method used to determine the parameters of (3). It consists of fitting the values of \(w_x, \beta,\) and \(t_x\) by using the two expressions of (3) at \(t \ll t_x\) and \(t \gg t_x.\) After this, the value of \(t_x\) can be determined by the intersections of the functions of the two regimes.

In this work, we refine the usual PL by using a numerical iterative data collapse. Our method consists of initially obtaining a set of exponents and using this initial set as a seed for the next curve. Using this simple technique, we observe a significant increase in precision for the case where \(d = 1,\) for which exact exponents are known.

The transient at \(t \lesssim 1\) is a problem for small substrates because the value of \(t_x\) may be so low that the roughness saturation mixes with the short-range correlation, affecting the evaluation of \(\beta\) and \(t_x.\) For that reason, we do not expect the parameter \(\beta\) to be independent of \(L.\) To make this clear, we use the subscript \(L\) in the parameter \(\beta_L\) obtained in such a way.

The values of \(\beta_L, w_x,\) and \(t_x\) for each value of \(d\) and \(L\) were obtained from roughness fitting and plotted in figure 3.

As prescribed by the FV, \(w_x \propto L^n\) and \(t_x \propto L^z,\) for a large enough \(L.\) When substrates are small, \(t_x \approx 1\) and the power laws are disturbed by the transient behavior occurring at this time scale. Therefore, the exponents must rely more strongly on the points with higher values of \(L.\)

When performing the fitting of figure 3, we incorporated the error obtained from the roughness fitting. The fitting error is a combination of the error due to the stochastic data fluctuation and the error due to the deviation between the fitting function and the data. Because that deviation is strong at \(t \lesssim 1,\) the use of the fitting error from the roughness data in the fittings of figure 3 reduces the weight of the points with a small \(L,\) because, in these cases, a significant part of the data has small values of \(t.\)

The PL to the points of figures 3(a) and (b) leads, respectively, to the exponents \(\alpha\) and \(z.\)

Because \(\beta_L\) is the main parameter controlling the curve shape at small values of \(t,\) it is also the parameter most strongly disturbed by the initial transient. If we define \(\beta\) as the asymptotic value of \(\beta_L,\) it can be found by using the points from figure 3(c) to do the finite-
The collapse was achieved by applying the scales indicated in the labels of the axis. Only points with \( t > 10 \) were included in the plots to exclude the transients at \( t \lesssim 1 \).

Size scaling

\[
\beta_L = \beta \left( 1 + \frac{A_0}{L^\gamma} \right),
\]

(5)

where \( \gamma \approx 1 \) and \( A_0 \) are parameters to be adjusted. From now on we will use the size independent value \( \beta \).
5. Universality

In the previous section, we have seen that, for a given dimension $d$, when we scale the axes as $w \to w/w_x$ and $t \to t/t_x$, all curves collapse in a single one. Because we know that the exponents depend on the dimension, we conjecture as to how those curves would behave if now we scale the axes as $(w/w_x)^{1/d}$ and $t/t_x$. The results are exhibited in figure 4, where we plot all curves for all dimensions. We use the data from the previous results as input and a method of minimization of the distance between two sets of data (curves). From that, we can obtain
the collapse of all data, i.e., for all dimensions $d$ and for all lengths $L$. This data collapse seems to be a universal behavior. Moreover, the exponents obtained by this method are much more precise than when we have the collapse for just one single dimension $d$. It is notable that, even considering that the $y$ scale should amplify statistical noise, the data from all curves collapses very well around the saturation time $t_X$, as well as the extremes for $t/\beta \rightarrow \infty$ and $t/\beta \rightarrow 0$.

These results suggest that $(w/w_0)^\beta = g(t/\beta)$ where the function $g(x)$ satisfies

$$w(t, L) = w_0 g(t/\beta)^\beta = \begin{cases} \frac{w_0 t}{t_X} & \text{if } t \ll t_X, \\ w_0 & \text{if } t \gg t_X, \end{cases}$$

(6)

i.e., the FV relation.

### 6. UCD and GI

We present, in tables 1 and 2, the values of $\alpha$ and $\beta$ from other authors. Ko and Seno [42] simulated BD with overhangs. Perlsman [43] studied the DP in general dimensions. Ala-Nissila [25] performed simulations of the RSOS growth model for dimensions $d \geq 4$. Marinari et al [34] and Kim and Kosterlitz [49] used a RSOS discretization of the surface, finding values for surfaces of up to $4 + 1$ dimensions. Katzav and Schartz [44] obtained these

![Figure 4. Collapse of the data from all values of $L$ for all dimensions $d$. The collapse was achieved by applying the scales indicated in the labels of the axis. The inset contains a zoom around the saturation time $t_X$.](image)

| Ko and Seno [42] | Ala-Nissila [25] | Marinari et al [34] | Katzav and Schartz [44] | Odor et al [47] | Canet et al [48] | Pagnani and Odor et al [47] |
|------------------|------------------|---------------------|------------------------|-----------------|-----------------|-----------------------------|
| $d$              | $\alpha$         | $\beta$             | $\alpha$              | $\beta$         | $\alpha$        | $\beta$          |
| 1                | 0.50             | 0.50                | 0.45                   | 0.50            | 0.50            | 0.50            |
| 2                | 0.39(3)          | 0.26                | 0.395(5)               | 0.33            | 0.395(5)        | 0.33            |
| 3                | 0.3135(15)       | 0.12                | 0.29(1)                | 0.17            | 0.29(1)         | 0.17            |
| 4                | 0.255(3)         | 0.245(5)            | 0.075                  | 0.2537(8)       | 0.075           | 0.2537(8)       |
| 5                | 0.141(1)         | 0.22(1)             | 0.12                   | 0.22(1)         | 0.22(1)         | 0.22(1)         |
| 6                | 0.22(1)          | 0.075               | 0.075                  | 0.075           | 0.075           | 0.075           |

Table 1. Growth exponent $\alpha$ as obtained from several authors.
exponents for BD. Ghaisas [45] introduced a solid-on-solid lattice model with conditional evaporation. Pagnani and Parisi [46] used multi-surface coding on a four-dimensional RSOS model. Ódor et al. [47] mapped growth models onto driven lattice gases of \(d\)-mers. Canet et al. [48] developed a simple approximation of the non-perturbative renormalization group for the KPZ equation. Tang et al. [23] proposed a hypercube-stacking model.

Table 2. Growth exponent \(\beta\) as obtained from several authors.

| \(d\) | Kim and Kos- | Tang et al | Ko and Seno | Perlman | Ala-Nissila | Ghaisas | Ódor et al |
|------|-------------|------------|-------------|---------|-------------|---------|------------|
| 1    | 0.33        | 0.333(1)   | --          | --      | 0.332       | 0.333(5)|            |
| 2    | 0.25        | 0.240(1)   | 0.33         | 0.242   | --          | 0.221   | 0.240(1)   |
| 3    | 0.20        | 0.180(5)   | 0.25         | 0.188   | --          | 0.168   | 0.184(5)   |
| 4    | 0.16        | --         | 0.18         | 0.153   | 0.16(1)     | --      | 0.15(1)    |
| 5    | --          | --         | --           | 0.130   | 0.11(1)     | --      | 0.115(5)   |
| 6    | --          | --         | --           | 0.114   | 0.09(1)     | --      |            |

Table 3. Dynamic exponents obtained from the fittings of figure 3. Evidence of the precision of these exponents is the value of \(\alpha + z\), which should be 2.

| \(d\) | \(\alpha\) | \(\beta\) | \(z\) | \(\alpha + z\) |
|------|------------|-----------|------|----------------|
| 1    | 0.497(5)   | 0.331(3)  | 1.50(8) | 2.00(1)       |
| 2    | 0.369(8)   | 0.244(2)  | 1.61(5) | 1.98(2)       |
| 3    | 0.280(7)   | 0.168(1)  | 1.75(9) | 2.03(2)       |
| 4    | 0.205(3)   | 0.116(3)  | 1.81(3) | 2.02(1)       |
| 5    | 0.154(2)   | 0.079(3)  | 1.88(6) | 2.04(1)       |
| 6    | 0.117(1)   | 0.054(1)  | 1.90(6) | 2.01(1)       |

The renormalization group yields a version of the GI [50] for the KPZ equation as

\[ \alpha + z = 2. \]  \hspace{1cm} (7)

This relation is considered to be true for all dimensions, making it a reliable metric of exponent values for higher dimensions. The universality of this relation has been questioned in a recent work [51], where Wio et al demonstrated that it is possible for a system to show KPZ scaling without obeying the GI.

In the fifth column of table 3, we present our values of \(\alpha + z\). It is difficult to compare our results concerning the GI with other authors because some use \(z = \alpha/\beta\) and the GI to produce \(\beta\) and \(z\). We have determined those exponents independently. The dynamic exponents obtained here are precise enough, up to five dimensions, to allow us to conclude that the GI holds for the etching model. However, it is interesting to note that previous works by Tang et al. [52, 53] analyzed the etching model on surfaces with a fractional dimension such as the Sierpinski carpet (\(d = 1.465\)), the Sierpinski arrowhead, and the crab (\(d = 1.585\)). For these works, the expected condition of \(\alpha + z = 2\) is not confirmed for such dimensions. This is worth noting; however, it is not a surprise because fractal geometries are not continuous.
7. Conclusion

We have successfully generalized the etching model for $d+1$ dimensions, which has characteristics similar to KPZ model, obtained the growth exponents for $d \leq 6$, and have shown the data collapse for several substrate lengths and dimensions (see figure 4). This was obtained through a simple scaling of the axes, resulting in a perfect collapse of the data, at least up to $d = 6$.

We studied the dependence of the dynamic exponents with the dimension $d$. Our data suggests that there is no UCD $d_c = 4$ for the etching model. The exponents obtained here are precise enough to allow us to conclude that the GI holds for the dimensions $d \leq 6$ integer. It is interesting to mention that the etching model on surfaces with a fractional dimension [52, 53] could violate the GI. If the GI is violated on fractal dimensions, such result should not be a surprise, because there is no continuum space transformations in fractal geometry.

To date, we have not presented a proof that the etching model is equivalent to the KPZ, such as the proof given by Bertini and Giacomin [13] for the RSOS model. The combination of the renormalization of probability of height distribution in a lattice [54] with recent scaling for asymptotic times [55] could yield some result to this problem. Moreover, we are working to obtain the function $g$ discussed in section 5 in subsequent publications.

More than 20 years after the KPZ seminal work, we still do not have a final solution for some important questions. Nevertheless, very good numerical methods and theoretical approaches have been suggested, which clearly indicates this is still a rich research field. We hope this work stimulates new research toward solutions to those problems.

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