Non-Hermitian lattices with binary-disorder

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In this paper we study for the first time the effect of non-Hermiticity on an interesting short-range correlated one dimensional disordered lattice which, in its Hermitian version, has been studied repeatedly for its unexpected delocalized states. The diagonal matrix elements of our Hamiltonian take randomly two complex values $\epsilon$ and $\epsilon^*$, each one assigned to a pair of neighboring sites. Contrary to the Hermitian case, all states in our system are localized. In addition, the eigenvalue spectrum exhibits an unexpected intricate fractal-like structure on the complex plane. Moreover, with increasing non-Hermitian disorder, the eigenvalues tend to coalesce in particular small areas of the complex plane, a feature termed "eigenvalue condensation". Despite the Anderson localization of all eigenstates, the system exhibits a novel transport by quantized jumps between states located around distant sites. The relation of our findings to recent experimental results is also discussed.

Anderson localization, which predicts the possibility of suppression of diffusion of waves in disordered media, is one of the fundamental phenomena of wave physics, and thus has been extensively studied in both quantum and classical domain. Its importance is evident by the wide impact in various fields where experiments can be implemented, such as condensed matter physics, disordered photonics and imaging, Bose-Einstein condensates and acoustic waves. However, with the exception of the random laser community, the vast majority of the studies regarding wave localization has been devoted to conservative systems, in which Hermiticity of the Hamiltonian is ensured. Whereas accurate control of the openness in many fields of wave physics is difficult or even impossible, photonics provides an ideal area where such control is possible by today's available experimental techniques.

In particular, the recent introduction of the concepts of parity-time (PT) symmetry and exceptional points in optics, which relies on the complex values of the index of refraction, has led to the development of a new research field, that of non-Hermitian photonics. In particular, the openness of these systems can be described in terms of gain (amplification based on laser materials) and/or loss (intrinsic decay mechanism) and their delicate interplay leads to unexpected novel features. The rich behavior of these structures has triggered a plethora of experimental realizations of various optical devices.

Quite recently there is a renewed interest for non-Hermitian Anderson localization problems, since it was realized that in the context of optical physics one can study experimentally realized linear random non-Hermitian Hamiltonians, away from the highly nonlinear regime of random lasers and the majority of abstract non-Hermitian random matrices. The proposed complex random discrete models can be considered the most relevant non-Hermitian analogue of the Anderson original problem. In this case, the non-Hermiticity is a direct consequence of the complex nature of the index of refraction, whereas the coupling between nearest neighbors is real and fixed. Thus the fundamental questions of whether the eigenmodes are localized or not and whether transport is possible, still remain open. Interestingly, in a recent novel experiment it was demonstrated that the non-Hermiticity of a random medium with a rectangular distribution of disorder can unexpectedly result to transport despite the strong localization of all corresponding eigenfunctions.

In this work we study for the first time the spectral and dynamic properties of one-dimensional waveguide lattices, in the context of coupled mode theory, which are characterized by non-Hermitian binary disorder with short range order. Our model, shown in Fig. 1, can be considered as the non-Hermitian extension of the extensively studied, random dimer model, which is the simplest disorder system which, in spite of being one-dimensional, still facilitates wavepacket delocalization and long-range transport, as it has been shown experimentally (see Fig. S1 in Suppl. Info.). In our non-Hermitian model though, we show, both numerically and analytically, that delocalization is impossible, for any value of the complex diagonal elements. Furthermore, we find that such binary disordered system exhibits various exotic features, such as fractal-like spectrum, as well as regions in the complex plane where many eigenvalue come arbitrarily close and form "condensates". Despite the strong Anderson localization, non-Hermitian transport with quantized jumps between eigenstates localized around distant sites is possible. We hope that our study may pave the way for future optical experiments that demonstrate the counterintuitive transport properties of non-Hermitian binary pair-correlated disordered lattices.

Let us consider the paraxial wave propagation in a one-dimensional disordered medium.
We do not refer to the case where $\epsilon_{1,2}$ are complex as it gives us the same qualitative results. Since the only quantity of interest is the fraction $\frac{\delta \epsilon}{c}$, we will set $c = 1$ and $\epsilon_1 = -\epsilon_2 = i\alpha$ for simplicity, with $\alpha$ being our single parameter from now on:

$$\epsilon_{2m} = \epsilon_{2m+1} = \begin{cases} i\alpha, & \text{with } P_1 = \frac{1}{2}, \\ -i\alpha, & \text{with } P_2 = \frac{1}{2} \end{cases} \quad (2)$$

with $P_{1,2}$ being the associated probabilities.

Now we focus on the study of the spectrum of the system, for various values of the parameter $\alpha$. We present our results in Fig. 2. The spectrum for small values $\alpha$ is concentrated near the real axis, except for the edges whose imaginary part extends through $(-\alpha, \alpha)$ (Fig. 2(a)). These results are reasonable, and can be intuitively associated to the density of states of the corresponding Hermitian problem. However, for $\alpha = 0.5$ the picture is quite different. The whole spectrum now tends to move away from the real axis and to form an intricate fractal-like structure in the complex plane (Fig. 2(b)), which resembles the spectrum of the quasi-periodic Harper model\textsuperscript{61}, in exhibiting a similar regularity in spite of its randomness. In addition, a gap opens around the imaginary axis Real($\omega$) = 0. It must be pointed out that these features are associated with the binary pair-correlated character of our model; they disappear in the absence of pair-correlation and in the case of a rectangular distribution of the random variable. If we further increase the value of $\alpha$ then the eigenvalues do not extend over the whole complex plane, as one might expect, but they rather tend to “collapse” into specific points of the complex plane (Fig. 2(c)) leading to rather sparse spectrum. We term this behavior as “eigenvalue condensation”. Such eigenvalue coalescence can be directly shown if one plots the integral density of states $R(\omega)$ as a function of the real part Real($\omega$) of the eigenvalues ($R(\omega)$ counts all states with the real part of the corresponding eigenvalue less than $\omega$). All these results correspond to $N = 3000$.

Figure 2. (a)-(c) Eigenvalue spectrum on the complex plane for the non-Hermitian random binary pair-correlated model and for (a) $\alpha = 0.1$, (b) $\alpha = 0.5$ and (c) $\alpha = 1$. (d) Integral density of states $R(\omega)$ as a function of the real part Real($\omega$) of the eigenvalues ($R(\omega)$ counts all states with the real part of the corresponding eigenvalue less than $\omega$). All these results correspond to $N = 3000$.

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One may notice that some kind of ”transition” occurs for $\alpha$ between 0.1 and 0.5. This change of behavior is captured by the (normalized) level spacing distribution $P(s)$, where $s$ is the minimum distance between two eigenvalues in the complex plane. For $\alpha = 0.1$ we get the expected Wigner-Dyson distribution: $P_{WD}(s) = \frac{3s}{2\pi}e^{-\frac{s^2}{2}}$ and the eigenvalues show level repulsion. If we set $\alpha = 0.5$ though, the level spacing distribution rather acquires a Poissonian form. However, in contrast with the classical Poisson distribution: $P_P(s) = e^{-s}$, where $P(s = 0) = 1$, here we...
get an even stronger peak for \( s = 0 \). The obtained distribution can be approximated by a sum of two exponential functions:

\[
P(s) \approx Ae^{-\lambda_1 s} + Be^{-\lambda_2 s}, \quad s \neq 0
\]

where the first exponent \( \lambda_1 > 1 \) and dominates for small values of \( s \), describing the abrupt drop of \( P(s) \) in this region, while the second exponent \( \lambda_2 < 1 \) and describes the slow drop of \( P(s) \) for \( s \geq \frac{1}{2} \). We must point here that this expression is accurate for \( s \neq 0 \), while for \( s = 0 \) there may be a singularity in the distribution which is difficult to capture numerically. We think that this behavior is in agreement with the aforementioned eigenvalue condensation, since a large number of eigenvalues tend to coalesce.

The next important issue we would like to address, is whether or not delocalization in this non-Hermitian model is possible. A direct and elegant way to see this is by considering a periodic lattice with a single dimer defect. In this case, one can analytically obtain an expression for the reflection probability-\( R \) from the impurity (see Suppl. Info). For the non-Hermitian dimer, the expression reads:

\[
R = \frac{\alpha^2[\cos^2(k) + \alpha^2]}{C(\alpha, k)}
\]

where \( C \) is a real, strictly positive function of \( \alpha \) and \( k \). The above relation clearly shows that the equation \( R = 0 \) does not admit acceptable solutions as long as the on-site energies become imaginary. This indicates that, contrary to the Hermitian case, all the eigenstates are localized. In order to verify this statement, we calculate the localization length \( \xi \) using the transfer matrix method (see for more details) and plot \( \xi \) which corresponds to every eigenvalue on the complex plane, as is shown in Fig. 4. We can clearly see that our assessment was correct, since \( \xi \) does not exceed the value of 30. Moreover, we find that \( \xi \) remains finite for every value of \( \alpha \neq 0 \). The fact that all eigenstates of the spectrum are localized, in contrast to the Hermitian case, is one of the most direct consequences of the non-Hermiticity on our system.

So far we have investigated the spectral properties of our model. Unexpected dynamic phenomena however also occur. In order to systematically study the wave dynamics, we examine the evolution pattern and the variance \( M(z) \) of an initial single-channel excitation as a function of \( z \). Here we assume that the total site number is odd \( N = 2\kappa + 1 \), with \( \kappa \) an integer number. In particular, we examine the case of a single-channel excitation in the middle of our lattice: \( \psi_n(z = 0) = \delta_{n,n_0} \), where \( n_0 = \kappa + 1 \). More specifically and study the variance of the intensity pattern as a function of \( z \), which is defined by the following relation:

\[
M(z) = \sum_n (n - n_0)^2|\psi_n(z)|^2
\]

For \( \alpha = 0 \) the lattice is periodic and \( M \sim z^2 \), as expected, which indicates ballistic transport. If we set \( \alpha \neq 0 \) though, all the states become exponentially localized and \( M(z) \) exhibits a very interesting behavior to be described below. Our results are depicted in Fig. 5. At this point we note that we have plotted the normalized field amplitude:

\[
|\phi_n| = \frac{|\psi_n|}{\sum_n |\psi_n|}, \quad \text{since the field amplitude } |\psi_n| \text{ diverges exponentially due to the presence of gain.}
\]

The wave evolution pattern is rather surprising. Even though all the eigenstates are localized, the wave exhibits “non-Hermitian jumps” between distant sites and thus we obtain short range transport, which most probably will increase as the system becomes larger. This behavior is also captured by the plot of the variance over \( z \) in Fig. 5(b), for the same realization of disorder, which also exhibits a number of finite jumps. This behavior occurs
only in non-Hermitian systems and has not a Hermitian analogue. In order to understand the physical mechanism behind this unexpected feature, we calculate the field at $z = z_{\text{max}} = 10^4$ for the same realization of disorder as in Fig. 5(a), and compare it with the field profile of the most gainy mode, namely the mode which corresponds to the eigenvalue with the largest value of the real part of $i\omega$ (Suppl. Info Fig. S3(d)).

Our result can be physically explained as follows. We excite a single-channel, which in turn is a superposition of various eigenstates. The mode with the largest value of the real part of $i\omega$ will dominate among them after a finite propagation distance due to its amplification, no matter how small is the overlap with the initial condition, as long as it is not zero. Thus, we get the first jump. Now, however, we get a new linear combination of some other modes, some of which probably correspond to eigenvalues with even larger value of the real part of $i\omega$ and will eventually be the ones that are dominant. This procedure carries on until the wave reaches the most gainy mode. When this happens, the jumps stop and we no longer get transport. This termination of the process after a finite number of jumps for a finite system is reflected by the absence of further jumps in Fig. 5(a) and by the final flat behavior in Fig. 5(b). However, if the system is sufficiently spatially increased, it is almost certain that a new, more gainy mode will be realized beyond the old boundaries and the initial excitation will find its way to this mode and hence will further diffuse away from the initial system.

Interestingly, these “non-Hermitian jumps” are a feature that occurs not only in the pair-correlated binary randomness but in all the non-Hermitian lattices. In a recent experimental work, $^{56}$ it has been shown that “Anderson” transport via quantized jumps can also occur in a lattice with a rectangular distribution of non-Hermitian disorder. Moreover, a similar kind of behavior has also been mentioned in the context of open quantum systems $^{63,64}$, whose only feature in common with our model is the non-Hermiticity. Thus, taking into account our own results and experimental $^{56}$ and other $^{63,64}$ work, we suppose that the phenomenon of unusual jumpy transport in Anderson-localized random system is a general characteristic of all the non-Hermitian disordered systems, which appears due to the presence of gainy modes. Finally, we note that our system possesses no exceptional points, even if the randomness is chosen to be spatially antisymmetric, i.e. in the case of $\mathcal{PT}$ symmetry.

In conclusion, we have systematically studied for the first time, the spectral and wave dynamic characteristics of non-Hermitian one-dimensional lattices with binary pair-correlated disorder. The underlying physical system is that of evanescently coupled waveguide channels with randomly distributed gain or loss per site, described by a random non-Hermitian tridiagonal matrix. The short range correlation of the randomness parameter leads to novel features that have no analogue in the corresponding Hermitian model. The spectrum appears to have a fractal-like intricate structure, and for higher values of the imaginary randomness parameter $\alpha$ many eigenvalues are concentrated in very small areas of the complex plane, a feature termed here “eigenvalue condensation”; this is a direct outcome of non-Hermiticity, binary distribution and pair-correlated short range order. Even more surprising is the fact that even though all eigenfunctions are localized, transport by spatial jumps is possible due to the partial overlap of the excited eigenmodes. These “Anderson jumps” are only possible in non-Hermitian systems, and challenge our notion of forbidden transport due to localization. We believe that this systematic study will open the way for the direct experimental realization of these phenomena in integrated photonic waveguide structures.

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1. P. W. Anderson, Phys. Rev. 109, 1492 (1958).
2. E. Akkermans and G. Montambaux, Mesoscopic Physics of Electrons and Photons (Cambridge University Press, Cambridge, 2007).
3. I. M. Lifshitz, S. A. Gredeskul, and L. A. Pastur, Introduction to the theory of disordered systems (WileyInterscience, 1988).
4. N. F. Mott and E. A. Davis, Electronic processes in non-cristalline materials (Oxford university press, 2012).
5. P. W. Anderson, Philos. Mag. B, 52, 505-509 (1985).
Supplemental information: Non-Hermitian lattices with binary-disorder

I. HERMITIAN BINARY DISORDER

The Hermitian one-dimensional model with binary disorder with or without short range correlations has been studied both theoretically and experimentally\cite{S1-S4}. Nevertheless, we redo here some of these calculations in order to be able to compare the resulting figures with new results associated with the non-Hermitian generalization.

Let us consider a waveguide array that exhibits a binary distribution $\epsilon_1$ or $\epsilon_2$ of its propagation constants (which here play the role of the on-site energies in the context of condensed matter physics), with the same probability $P = \frac{1}{2}$ and without any correlations:

Uncorrelated binary disorder : $\epsilon_n = \begin{cases} \epsilon_1, & \text{with } P_1 = \frac{1}{2} \\ \epsilon_2, & \text{with } P_2 = \frac{1}{2} \end{cases}$ (S1)

where $\epsilon_1, \epsilon_2 \in \mathbb{R}$ (Hermitian case). We also assume that the coupling coefficient between neighboring channels is constant and equal to $c$.

To begin with we examine the evolution pattern assuming a single-channel excitation in the middle of our lattice, namely: $\psi_n(z=0) = \delta_{n,n_0}$, where $n_0 = \kappa + 1$ (here we assume that the total site number is odd $N = 2\kappa + 1$). More specifically, we are interested to consider the averaged variance of the intensity pattern as a function of the propagation distance $z$:

$$M(z) = \langle \sum_n (n-n_0)^2|\psi_n(z)|^2 \rangle$$ (S2)

where $\langle .. \rangle$ denotes averaging over many realizations of disorder.

Our numerical calculations for this case are shown in Fig. S1(a),(b) and are in agreement with the corresponding experimental results\cite{S4}. For $\epsilon_1 = \epsilon_2 \Rightarrow \delta\epsilon = \epsilon_1 - \epsilon_2 = 0$ the lattice is periodic and $M \sim z^2$, which indicates ballistic transport. If we set $\epsilon_1 \neq \epsilon_2$ though, all the states become exponentially localized and $M(z)$ saturates for large values of $z$; we get localization in this case since $M \sim z^0$. The single-channel excitation remains localized near its initial position.

However, one gets completely different physical results if short-range order is introduced in this model. For that purpose, we now consider a “dimer” waveguide array, where each dimer consists of two subsequent channels with the same propagation constant; this is a model originally introduced by Dunlap. et. al.\cite{S1}:

Dimer array : $\epsilon_{2m} = \epsilon_{2m+1} = \begin{cases} \epsilon_1, & \text{with } P_1 = \frac{1}{2} \\ \epsilon_2, & \text{with } P_2 = \frac{1}{2} \end{cases}$ (S3)

Repeating the same calculations as in\cite{S1}, we can see that now, $M \sim z^\gamma$, in all the cases, where $\gamma \simeq 2, \frac{3}{2}, 1$ and 0, which correspond to ballistic, superdiffusive, diffusive and localized motion, accordingly (computed numbers are: 1.99, 1.56, 0.98, 0.15) for $\frac{\delta\epsilon}{c} = 0, \frac{3}{2}, 1$ and 0 respectively. These results indicate that the spectrum now possesses delocalized eigenvectors. Indeed, one can prove that eigenstates with eigenvalue $\omega = \epsilon_{1,2}$ are extended, as long as $|\delta\epsilon| \ll 2c$. All the relevant results are presented in Fig. S1 and are again in perfect agreement with the experimental ones\cite{S4}.
A direct way to obtain a physical insight of these results is to consider a periodic lattice with a single dimer defect. One can show that the reflection probability from the impurity is given by the following expression:

$$ R = \frac{\delta\epsilon^2[\delta\epsilon + 2c\cos(k)]^2}{\delta\epsilon^2[\delta\epsilon + 2c\cos(k)]^2 + 4c^4\sin^2(k)} \quad (S4) $$

where $k = \cos^{-1}(\frac{\omega}{2c})$ is the Bloch wavenumber. From the expression above we can see that waves with $\omega = \delta\epsilon$ are perfectly transmitted, provided that $|\delta\epsilon| \leq 2$. Furthermore, it was found that the total number of states with localization length greater than the system’s size is of measure $\sqrt{N}^{S1}$. Thus, in Fig. S1(c), the two propagating peaks correspond to these $\sim \sqrt{N}$ delocalized states, leading to transport, while the central peak indicating no propagation is associated with the vast majority of localized states.

The criterion for localization originally proposed by Anderson $^{S5}$ was the asymptotic behavior of the amplitude of the wavefunction around its initial site, in the sense that absence of diffusion is associated with the limit: $\lim_{z \to \infty} |\psi_{n_0}(z)|$ being non-zero. Thus, it is reasonable to examine the probability $P(z)$ for the wave to be located in its initial position as a function of the propagation distance for the two cases of disorder discussed here.
Figure S2. Probability $P(z)$ for the wave to being found in its initial position as a function of the propagation distance $z$ for (a) the uncorrelated binary disorder and (b) the dimer case and for different values of $\frac{\delta\epsilon}{c}$. An averaging over 50 realizations of disorder has been performed for each plot. Inset: a zoom in the plot of $P - z$ for $\delta\epsilon = c$ (green line) and the least square fitting of the curve (black line). The slope of the line is shown in the title of the graph.

In Fig. S2 we show plots of $P(z)$ for the uncorrelated binary (Fig. S2(a)) and the dimer array (Fig. S2(b)), for $\frac{\delta\epsilon}{c} = 0, 1$ and 2. The difference between the two graphs is small but crucial. While in Fig. S2(a), and for $\delta\epsilon \neq 0 P(z)$ fluctuates around a specific, constant value, in Fig. S2(b) $P(z)$ slowly drops as $z$ increases, with a slope of $\sim 10^{-5}$, which is actually the value of the localization length for $\omega$ near $\epsilon_{1,2}$. This is shown clearly in the inset of Fig. S2(b), where a least square fit (black line) is also plotted with $P(z)$. This statement is in agreement with Anderson criterion of localization, as it should be. However, due to the many fluctuations and the very small value of the linear fitting’s slope, $P(z)$ is not a convenient numerical criterion for localization in this case.

II. NON-HERMITIAN ANDERSON TRANSPORT

In this section we discuss further new results regarding the phenomenon of Anderson transport in our non-Hermitian binary pair-correlated disordered model.

Let us consider the wave evolution of a single channel excitation (in the middle of the lattice), as a function of the propagation distance $z$. Since the spectrum contains eigenvalues that correspond to amplification, we normalize the field in every step in order to obtain a physically meaningful diffraction pattern. The result is depicted for a particular realization in Fig. S3(a). We can clearly see a finite number of jumps in the transverse direction. This dynamical behavior is also reflected in the discontinuities of the variance $M$ as is shown in the plot of the logarithm of the variance as a function of log($x$), in Fig. S3(b), as well as in the abrupt drop in the amplitude of the initial site $n_0$, in Fig. S3(c).

In order to highlight the underlying physical mechanism of these transverse jumps, we calculate the eigenstate that corresponds to the most gainy eigenvalue (black line in Fig. S3(d)). The single channel excitation at $z = 0$ excites many localized eigenmodes that have complex eigenvalues. The superposition of these modes generates a complex diffraction pattern as a result of the interference of these non-orthogonal eigenstates. No matter how small is the amplitude of the projection coefficient that corresponds to the most gainy eigenvalue, for long propagation distances it will always dominate over the other modes. Therefore, the location of the final jump is solely determined by the most gainy eigenstate. This physical explanation of the jumps is also supported by direct numerical simulations. In Fig. S3(d) we can clearly see that the field at end of the lattice ($z = z_{max}$) is almost the same with the filed profile of the most gainy eigenstate, as we expected.
Figure S3. (a) Normalized field amplitude $|\psi|$ as a function of the logarithm of the propagation distance $z$, under single channel excitation and a particular realization of the random lattice with $\alpha = 1$ (see Eq. (2) in main text). (b) Logarithm of the averaged variance $M$ as a function of the logarithm of the propagation distance $z$. (c) Probability $P$ for the wave to return to its initial position as a function of the propagation distance $z$. (d) Normalized field profile $|\psi|$ at the end of the lattice $z = z_{\text{max}}$ (blue bars). The field profile of the most gainy eigenstate is also plotted here for comparison (black line). The gain and loss distributions (imaginary part of the potential) are depicted with red, green color, respectively.

[S1] D. H. Dunlap, H. L. Wu and P. W. Phillips, Phys. Rev. Let., 65, 88 (1990).
[S2] P. Phillips and H. L. Wu, Science, 252, 1805-1812 (1991).
[S3] V. Bellani, E. Diez, R. Hey, L. Toni, L. Tarricone, G. B. Parravicini, and R. Gómez-Alcalá, Phys. Rev. Let., 82, 2159 (1999).
[S4] U. Naether, S. Stützer, R. A. Vicencio, M. I. Molina, A. Tünnermann, S. Nolte, T. Kottos, D. N. Christodoulides and A. Szameit, New Jour. of Phys., 15, 013045 (2013).
[S5] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[S6] S. Weidemann, M. Kremer, S. Longhi and A. Szameit, arXiv:2007.00294 (2020).