VOVERTX DENSITIES AND CORRELATIONS AT PHASE TRANSITIONS *

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ABSTRACT

We present a model for the formation of relativistic vortices (strings) at a quench, and calculate their density and correlations. The significance of these to early universe and condensed-matter physics is discussed.

1. Introduction

Despite our improved understanding of the very early universe the nature of the phase transitions in the Grand Unification era is largely unknown. This would seem to be a serious drawback in our modelling of the universe since, in all but the baldest inflationary scenarios, an imprint of these transitions is visible today. In particular, it has been argued by Kibble and others that the large-scale structure of the universe can be attributed to cosmic strings (vortices in the fields) formed at that time.

However, it has been suggested that the initial conditions of any string network are largely washed out after a few expansion times, at which the network is assumed to approach a scaling regime with a few large loops and long strings per horizon volume continuing to produce smaller loops by self and mutual intersection. This has seemed to obviate the need for any detailed description of the early microscopic dynamics that set the boundary conditions for the latter classical picture.

Despite this, there are two circumstances in which this lack of detailed initial information leaves us at a loss. The first concerns the density of the defects formed. Although not of immediate importance to the cosmologists, for the reasons given above, it does relate directly to the interesting proposition that the production of vortices in superfluid $^4He$ and $^3He$ may share many attributes with the production of cosmic strings. To date, the primary data from such experiments concern defect densities. If there is a similarity between the production mechanisms in these different regimes of high $T$ (low chemical potential) and low $T$ (high chemical potential) we should be able to predict the density of vortices in both superfluid experiments and the early universe.

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Secondly, and of direct importance to the astrophysicists, the scaling solutions for the early universe mentioned above require the presence of some ‘infinite’ string, i.e., vortices that do not self-intersect. The presence of such string is, in part, determined by the initial conditions and it is important to know whether it is present for reasonable models. For example, it has been suggested that vortices produced by bubble nucleation in a strong first-order transition will only form small loops. The tendency for string to form loops should be visible in string density correlation functions. In a condensed matter context, the same correlation functions should enable us to estimate the superflow that would occur at a superfluid quench from fluctuations alone.

In this talk I shall show how these problems can be addressed in a model of vortex formation by unstable long-wavelength Gaussian fluctuations at a temperature ‘quench’. For simplicity, flat spacetime is assumed. Relying on a slow ‘rollover’, this excursionary model can only describe weak coupling systems for the short times while the domains are growing before the defects freeze out. Although this is unsatisfactory for most early universe applications and for low-temperature many-body systems, we know in principle how to include back-reaction (still within the context of a Gaussian approximation) to slow down domain growth prior to the field fluctuations spreading to the ground-state manifold. If the results of Ref.7 are a reliable guide to our model then the conclusions that we shall draw are likely to survive. Greater detail is given in recent work by myself and my collaborators, Tim Evans, Alasdair Gill and Glykeria Karra, from which this talk is drawn.

2. Vortex Distributions

Consider the theory of a complex scalar field $\phi(x,t)$. The complex order parameter of the theory is $\langle \phi \rangle = \eta e^{i\alpha}$ and the theory possesses a global $O(2)$ symmetry that we take to be broken at its (continuous) phase transition.

Initially, we take the system to be in the symmetry-unbroken (disordered) phase, in which the field is distributed about $\phi = 0$ with zero mean. We assume that, at some time $t = t_0$, the $O(2)$ symmetry of the ground-state (vacuum) is broken by a rapid change in the environment inducing an explicit time-dependence in the field parameters. Once this quench is completed the $\phi$-field potential $V(\phi) = -M^2|\phi|^2 + \lambda|\phi|^4$ is taken to have the familiar symmetry-broken form with $M^2 > 0$.

In practice we expect that, as the complex scalar field begins to fall from the false ground-state into the true ground-state, different points on the ground-state manifold (the circle $S^1$, labelled by the phase $\alpha$ of $\langle \phi \rangle$) will be chosen at each point in space. If this is so then continuity and single valuedness will sometimes force the field to remain in the false ground-state at $\phi = 0$. For example, the phase of the field may change by an integer multiple of $2\pi$ on going round a loop in space. This requires at least one zero of the field within the loop, each of which has topological stability and characterises a vortex (or string). As to the density of the strings, if the phase $\alpha$ is correlated over a distance $\xi$, then the density of strings passing
through any surface will be $O(\xi^{-2})$ i.e. a fraction of a string per unit correlation area. On the completion of the transition a network of strings survives whose further evolution is determined by classical considerations as the field gradients adjust to minimise the energy.

The question is, how can we infer these late-time string densities and the density correlations from the microscopic field dynamics? The answer lies in the fact, noted earlier, that the string core is a line of zeroes of the fields $\Phi_a$ ($a=1,2$). This is equally true for both relativistic and non-relativistic $O(2)$ theories. The problem is solved if we can identify those zeroes which will freeze out to define the late-time vortices. This will require careful winnowing, since it is apparent that quantum fluctuations lead to zeroes of the fields on all distance scales (even in the disordered phase). However, most of these zeroes will be transient. Two levels of screening are required before the relevant zeroes can be identified. Firstly, those zeroes whose positions vary rapidly, or which annihilate one another on time scales of $O(m^{-1})$, where $m$ provides the mass-scale, can be discounted. Secondly, since $m^{-1}$ also measures the typical vortex width, zeroes on scales smaller than this should also be ignored. For the moment we ignore this problem, and count every zero.

To see how to proceed, consider an ensemble of systems evolving from one of a set of disordered states whose relative probabilities are known, to an ordered state as indicated above. As a prologue to the problem in hand, counting the zero line densities appropriate to vortices, we consider the much simpler case of counting the zeroes of a real field $\Phi(x)$ in one space dimension. On implementing the change of state on the line, these zeroes fluctuate and annihilate, but some of them will come to define the position of 'kinks' (at which $\Phi'(x) > 0$), some the position of 'antikinks' (at which $\Phi'(x) < 0$), the one-dimensional counterparts of vortices and 'anti'-vortices.

Suppose, at a given time, the zeroes of $\Phi(x)$ occur at $x = x_1, x_2, ...$. It is useful to define two densities. The first,

$$\bar{\rho}(x) = \sum_i \delta(x - x_i),$$

is the total density of zeroes, not distinguishing between kinks and antikinks (by which we now mean zeroes at which the field has positive or negative derivative). The second is the topological density,

$$\rho(x) = \sum_i n_i \delta(x - x_i),$$

where $n_i = \text{sign}(\Phi'(x_i))$, measuring (net) topological charge, the number of kinks minus the number of antikinks.

†Because of the peculiarities of one spatial dimension, that would confuse the issue, we pretend that we are examining a one-dimensional subset of a real field in higher dimensions and ignoring other degrees of freedom.
Equivalently, in terms of the $\Phi$-field, the total density is
\[ \bar{\rho}(x) = \delta[\Phi(x)]|\Phi'(x)|, \] (3)
since $\Phi'(x)$ is the Jacobian of the transformation from zeroes to fields. Similarly, the topological density is
\[ \rho(x) = \delta[\Phi(x)]\Phi'(x). \] (4)

Analytically, it is not possible to keep track of individual transitions, but we can construct ensemble averages. If the phase change begins at time $t_0$ then, for $t > t_0$, it is possible in principle to calculate the probability $p_t[\Phi]$ that $\phi(x,t)$ (the counterpart of $\phi(x,t)$) takes the value $\Phi(x)$ at time $t$. Ensemble averaging $\langle F[\Phi] \rangle_t$ at time $t$ is understood as averaging over the field probabilities $p_t[\Phi]$. This is not thermal averaging since we are out of equilibrium.

The situation we have in mind is one in which, for early times when the available space permits many domains,
\[ \langle \rho(x) \rangle_t = 0. \] (5)i.e. an equal likelihood of a kink or an antikink occurring in an infinitesimal length, compatible with an initially disordered state. However, the density correlation functions
\[ C(x; t) = \langle \rho(x)\rho(0) \rangle_t \] (6)
will be non-zero, as will
\[ \bar{n}(t) = \langle \tilde{\rho}(x) \rangle_t. \] (7)

Let us now turn to the complex field $\phi(x,t)$ and its more complicated vortices. As before we assume that it is possible in principle to calculate the probability $p_t[\Phi]$ that $\phi(x,t)$ takes the value $\Phi(x)$ at time $t$. To generalise our observations about zeroes on the line to vortices, we follow Halperin in defining the topological line density $\tilde{\rho}(r)$ by
\[ \tilde{\rho}(r) = \sum_n \int ds \frac{dR_n}{ds} \delta^3[r - R_n(s)]. \] (8)
In (2.1) $ds$ is the incremental length along the line of zeroes $R_n(s)$ ($n=1,2,...$) and $\frac{dR_n}{ds}$ is a unit vector pointing in the direction which corresponds to positive winding number.

It follows that, in terms of the zeroes of $\Phi(r)$, $\rho_i(r)$ can be written as
\[ \rho_i(r) = \delta^2[\Phi(r)]\epsilon_{ijk}\partial_j\Phi_1(r)\partial_k\Phi_2(r), \] (9)
where $\delta^2[\Phi(r)] = \delta[\Phi_1(r)]\delta[\Phi_2(r)]$. The coefficient of the $\delta$-function in Eq.(5) is the Jacobian of the more complicated transformation from line zeroes to field zeroes. We shall also need the total line density $\bar{\rho}(r)$, the counterpart of $\tilde{\rho}(x)$ of Eq.(3),
\[ \bar{\rho}_i(r) = \delta^2[\Phi(r)]|\epsilon_{ijk}\partial_j\Phi_1(r)\partial_k\Phi_2(r)|. \] (10)

Throughout, it will be convenient to decompose $\Phi$ into real and imaginary parts as $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$ (and $\phi$ accordingly). This is because we wish to track the field as it falls from the unstable ground-state hump at the centre of the potential to the ground-state manifold in Cartesian field space.
As before, ensemble averaging $\langle F[\Phi]\rangle_t$ at time $t$ means averaging over the field probabilities $p_t[\Phi]$. Again we assume

$$\langle \rho_j(r) \rangle_t = 0.$$  \hspace{1cm} (11)

i.e. an equal likelihood of a string or an antistring passing through an infinitesimal area. However,

$$\bar{n}(t) = \langle \bar{\rho}_i(r) \rangle_t > 0$$  \hspace{1cm} (12)

and measures the total string density in the direction $i$, without regard to string orientation. The isotropy of the initial state guarantees that $\bar{n}(t)$ is independent of the direction $i$. Further, the line density correlation functions

$$C_{ij}(r; t) = \langle \rho_i(r) \rho_j(0) \rangle_t$$  \hspace{1cm} (13)

will be non-zero, and give information on the persistence length of strings.

In practice, our ability to construct $p_t[\Phi]$ over the whole timescale $t > t_0$ from initial quantum fluctuations to late time classicality is severely limited. It is convenient to divide time into four intervals, in each of which we adopt a different approach. If $m$ sets the mass-scale, there is an initial period $t_0 < t < t_i = O(m^{-1})$, before which the field is able to respond to the quench, however rapidly it is implemented, and which we can largely ignore. Assuming weak coupling, of which more later, this is followed by an interval $t_i < t < t_f$ in which, provided the quench is sufficiently rapid, domains in field phase form, grow, and then cease growing as the field magnitude begins to approach the $S^1$ groundstates. Vortices will appear and be driven apart by the coalescence of these domains. In the third period the field magnitude relaxes to the ground state values i.e. the vortices freeze in. Finally, in the last period, the vortices behave semiclassically. A phenomenological description of this period based on time-dependent Landau-Ginzburg theory has been given by Liu and Mazenko\(^\text{11}\), in which they fill in and extend the work of Halperin, and from whose calculations we borrow. See also Bray\(^\text{12}\).

Our immediate aim is limited to discussing the second period $t_i < t < t_f$ on the (as yet unproven) assumption that the distribution of relevant zeroes at time $t_f$ is left largely unchanged by their freezing in. This distribution of vortices can then be taken as initial data for the final evolution of the network. Although we have long term plans for a semi-analytical linkage of this final stage to the initial microscopic dynamics, at the very least we can compare the data to that used in numerical simulations.

3. A Gaussian Model for Vortex Distribution

We have yet to specify the nature of the quench but it is already apparent that, if we are to make further progress, additional approximations are necessary. We return to our one-dimensional example. The most important concerns the independence,
or approximate independence, of the fields and their derivatives at the same point. If that holds then $\bar{n}(t)$ of Eq.7 separates as

$$\bar{n}(t) \approx \langle \delta[\Phi(x)]_t \rangle_t \langle | \Phi'(x) |_t \rangle_t. \quad (14)$$

and we can estimate, or bound, each factor separately. In fact, we can perform explicit calculations if $p_t[\Phi]$ is Gaussian, for which the approximate equality in Eq.14 becomes exact, and this we shall assume henceforth. That this is not a frivolous exercise in solving what we can solve but a representation of reasonable dynamics will be shown, in part, later, for the 'slow-roll' dynamics that we shall adopt.

Specifically, suppose that (for the one-dimensional case) $\Phi$ is a Gaussian field for which

$$\langle \Phi(x) \rangle_t = 0 = \langle \Phi(x) \Phi'(x) \rangle_t, \quad (15)$$

and that

$$\langle \Phi(x) \Phi(y) \rangle_t = W(|x - y|; t). \quad (16)$$

All other connected correlation functions are taken to be zero. Then all ensemble averages are given in terms of $W(r; t)$ which, from the closed timepath integral formalism, can be equally understood as the equal-time Wightman function,

$$W(|x - y|; t) = \langle \phi(x, t) \phi(y, t) \rangle \quad (17)$$

with the given initial conditions. In our case, where we shall assume thermal equilibrium initially, this is the usual thermal Wightman function. It is straightforward to see that

$$\bar{n}(t) = \frac{1}{\pi} \left| \frac{W''(0; t)}{W(0; t)} \right|^{1/2}. \quad (18)$$

where primes on $W$ denote differentiation with respect to $x$. The numerator in Eq.18 is the second factor in Eq.7. The denominator is the $\delta$-function term, as follows directly on writing $\langle \delta[\Phi(x)]_t \rangle_t$, as $\int d\alpha \langle e^{i\alpha \Phi(x)} \rangle_t$. On using the same exponentiation it takes only a little manipulation to cast the correlation function $C(x; t)$ of Eq.6 in the form

$$C(x; t) = \frac{\partial h(x; t)}{\partial x}, \quad (19)$$

where

$$h(x; t) = \frac{-W'(x; t)}{2\pi \sqrt{W(0; t)^2 - W'(x; t)^2}}. \quad (20)$$

The extension to $O(2)$ line zeroes is messy, but leads to no surprises. Specifically, suppose that

$$\langle \Phi_a(r) \rangle_t = 0 = \langle \Phi_a(r) \partial_j \Phi_b(r) \rangle_t, \quad (21)$$

and, further, that

$$\langle \Phi_a(r) \Phi_b(r') \rangle_t = W_{ab}(|r - r'|; t) = \delta_{ab}W(|r - r'|; t), \quad (22)$$
is diagonal. As before, all other connected correlation functions are taken to be zero.

The density calculation proceeds as before. It follows that

$$\bar{n}(t) = \frac{1}{2\pi} \left| \frac{W''(0;t)}{W(0;t)} \right|.$$

The line density-line density correlation functions have a much more complicated realisation, albeit still in terms of \(h(r;t)\) of Eq. 20, but we shall not consider them here in any detail.

The folly of counting all zeroes is now apparent in the ultraviolet divergence of \(W(r;t)\) at \(r = 0\) in all dimensions. None of the expressions given above is well-defined. To identify which zeroes will turn into our vortex network requires coarse-graining, determined by the dynamics.

4. Gaussian Dynamics and its Coarsening

It is not difficult to justify our adoption of Gaussian field fluctuations for the early period of vortex production. We have already assumed that the initial conditions correspond to a disordered state. In the absence of any compelling evidence to the contrary we achieve this by adopting thermal equilibrium at a temperature \(T\) higher than the critical temperature \(T_c\) for \(t < t_0\). The potential describing this state behaves, near the origin in field space, as

$$V(\phi) = \frac{1}{2} m^2(T) \phi^2,$$

for an effective mass \(m(T)\), with \(m^2(T) > 0\). For the sake of argument we take \(V(\phi)\) to be exactly as in Eq. 24. The resulting Gaussian field distribution will not be seriously disturbed by weak coupling. Further, as we shall see later, initial conditions generally give slowly varying behaviour in the correlation function, in contrast to the rapid variation due to the subsequent dynamics, and calculations are insensitive to them.

In order to have as simple a change of phase as possible, we assume an idealised quench, described by giving a time-dependence to the effective mass \(m(T) \equiv m(t)\) so that, at \(t = t_0\), \(m^2(t)\) changes sign everywhere. Given that the resulting theory displays symmetry-breaking, this change in sign in \(m^2(t)\) can be interpreted as due to a reduction in temperature. Even more, we assume that, for \(t > t_0\), \(m^2(t)\) takes the negative value \(m^2(t) = -M^2 < 0\) immediately, where \(-M^2\) is the mass parameter of the (cold) relativistic Lagrangian. That is, the potential at the origin has been instantaneously inverted, breaking the global \(O(2)\) symmetry, interpreted as a temperature quench from high-\(T\) to, effectively, zero temperature. If, as we shall further assume, the \(\lambda |\phi|^4\) field coupling is very weak then, for times \(M(t-t_0) < \ln(1/\lambda)\), the \(\phi\)-field, falling down the hill away from the metastable vacuum, will not yet have experienced the upturn of the potential, before the point of inflection.
Thus, for these small time intervals, \( p_t[\Phi] \) is Gaussian, as required. Henceforth, we take \( t_0 = 0 \).

For such a weakly coupled field the onset of the phase transition at time \( t = 0 \) is characterised by the instabilities of long wavelength fluctuations permitting the growth of correlations. Although the initial value of \( \langle \phi \rangle \) over any volume is zero, we anticipate that the resulting evolution will lead to domains of constant \( \langle \phi \rangle \) phase, whose boundaries will trap vortices.

Consider small amplitude fluctuations of \( \phi_a \), at the top of the parabolic potential hill. Long wavelength fluctuations, for which \( |k|^2 < M^2 \), begin to grow exponentially. If their growth rate \( \Omega(k) = \sqrt{M^2 - |k|^2} \) is much slower than the rate of change of the environment which is causing the quench, then those long wavelength modes are unable to track the quench. Unsurprisingly, the time-scale at which domains appear in this instantaneous quench is \( t_i = O(M^{-1}) \). As long as the time taken to implement the quench is comparable to \( t_i \) and much less than \( t_f = O(M^{-1}\ln(1/\lambda)) \) the approximation is relevant.

This picture essentially tells us how to coarsegrain the ultraviolet zeroes to identify the vortices that will form the network. Firstly, we note that if \( \Phi \) is Gaussian, then so is the coarse-grained field on scale \( L \),

\[
\Phi_L(r) = \int d^3r' I(|r - r'|)\Phi(r'),
\]

where \( I(r) \) is an indicator (window) function, normalised to unity, which falls off rapidly for \( r > L \). The only change is that Eq.22 is now replaced by

\[
\langle \Phi_{L,a}(r)\Phi_{L,b}(r') \rangle_t = W_{L,ab}(|r - r'|; t) = \delta_{ab}W_L(|r - r'|; t),
\]

where \( W_L = \iint I W I \) is now cut off at distance scale \( L \). \( W_L(0; t) \), its derivatives, and all relevant quantities constructed from \( W_L \) are ultraviolet finite. The distribution of zeroes, or line zeroes, of \( \Phi_L \) is given in terms of \( W_L \) as in the previous section.

Choosing \( L = M^{-1} \) solves all our problems simultaneously. At wavelengths \( k^{-1} < L \ i.e. \ k > M \) the field fluctuations are oscillatory, with time scales \( O(M^{-1}) \). Only those long wavelength fluctuations with \( k^{-1} > L \) have the steady exponential growth that can lead to the field migrating on larger scales to its groundstates. Further, as the field settles to its groundstates, the typical vortex thickness is \( O(M^{-1}) \), and we only wish to attribute one zero to each vortex. By taking \( L = M^{-1} \) we are choosing not to count zeroes within a string, apart from the central core.

We are now in a position to evaluate \( p_t[\Phi] \), or rather \( W_L(r; t) \), which we now write as \( W_M(r; t) \) for \( t > 0 \), and calculate the defect density accordingly. This situation of inverted harmonic oscillators was studied many years ago by Guth and Pi\(^4\) and Weinberg and Wu\(^4\). In the context of domain formation, we refer to the recent work of Boyanovsky et al.\(^4\) and our own. For weak coupling we recover what would have been our first naive guess for the coarse-grained correlation function

\[
\langle \phi_{L,a}(r, t)\phi_{L,b}(r', t) \rangle
\]

based on the growth of the unstable modes \( \phi_a(k, t) \approx e^{\Omega(k)t} \) alone,

\[
W_M(r; t) \simeq \int_{|k|<M} \delta^3 k e^{ikr} e^{2\Omega(k)t},
\]

as in the previous section.
provided we have not quenched from too close to the transition \(^5\) and \(t > M^{-1}\) (preferably \(t \gg M^{-1}\)). The reason why \(W_M(r,t)\) is approximately independent of the initial conditions is that the integral at time \(t\) is dominated by a peak in the integrand \(k^2 e^{2\Omega(k)t}\) at \(k\) around \(k_c\), where \(tk_c^2 \approx M\). Any temperature dependence only occurs in slowly varying factors (like \(\coth\beta\omega/2\)) and can be ignored.

5. Densities and Correlations

In the first instance we understand the dominance of wavevectors at \(k_c^2 = M/t\) in the integrand as defining a length scale

\[
\xi(t) = O(\sqrt{t/M}),
\]

(28)

once \(Mt > 1\), over which the independently varying fields \(\phi_a\) are correlated in magnitude. However, in our introductory comments we anticipated that string distributions would be determined by the length over which the phase of the field is correlated. For this simple model, with only a single length, \(\xi(t)\) has to serve both purposes.

To see how this happens, we take \(\xi(t) = 2\sqrt{t/M}\). To calculate the number density of vortices at early times we insert the expression Eq.27 for \(W_M\) into the equations derived earlier, to find

\[
\bar{n}(t) = \frac{1}{\pi} \frac{1}{\xi(t)^2},
\]

(29)

permitting us, from our introductory comments, to interpret \(\xi(t)\) as a correlation length for phases. We note that the dependence on time \(t\) of both the density and density correlations is only through the correlation length \(\xi(t)\). We have a scaling solution in which, as the domains of coherent field form and expand, the interstring distance grows accordingly. Since the only way the defect density can decrease without the background space-time expanding is by defect-antidefect annihilation, we deduce that the coalescence of domains proceeds by the annihilation of small loops of string, preserving roughly one string zero per coherence area, a long held belief for whatever mechanism.

As well as determining the string density, we are also looking for signs of anti-correlation, which enables us to determine the persistence length of strings in the network (\(i.e.\) how bendy they are). The bendier, the more string that will occur in small loops. This is important, since the conventional string model for large scale structure formation in the universe requires a certain amount of infinite string. In fact, there may be no need to calculate the line density correlation functions \(C_{ij}\) to appreciate that there is a higher fraction of string in loops than would have been anticipated from the early simulations which laid out a regular domain structure. \(\square\) On

\(^5\)In which case initial fluctuations are so large that the field magnitude has already sampled the ground states and a slow roll is inappropriate.
inspection, the *broad* peaking in wavelength \( l = k^{-1} \) about \( k = k_c \) in the integrand of Eq.\(27\) is understood as indicating that the domains, both in field magnitude and phase, of characteristic linear dimension \( \xi(t) \), have a substantial variation \( \Delta \xi(t) \) in size. Simple calculation shows that

\[
\frac{\Delta \xi(t)}{\xi(t)} = \frac{\Delta k}{k_c} = \frac{1}{2}.
\]

independent of time.

To make use of Eq.\(30\) requires two further assumptions. The first, which can be checked using the methods of Ref.\(7\), is that the domains growing as \( t^{\frac{1}{2}} \) stop expanding almost immediately as soon as the field *magnitudes* probe the spinodal region and the instability switches off. The second is that, during this time and the time it takes for the field magnitude to freeze on the \( S^1 \) groundstates, \( \Delta \xi/\xi \) of Eq.\(30\) describes (albeit approximately) the variation in domain size over which field *phase* is correlated. In numerical simulation of string networks the inclusion of variance in the size of field phase domains shows\(9\) that, the greater the variance, the more string is in small loops. The variance of Eq.\(30\) suggests much more string (\( e.g. \) twice or more) in small loops. However, it is not easy to marry the somewhat different distributions of this simulation to ours. We can say no more than there seems to be some infinite string and, in an early universe context, some is enough.

To do a little better we need to evaluate density correlation functions. For \( W_M \) above we find that, in units of \( \bar{n}^2(t) \), the anticorrelation is large. In particular, a calculation of \( C_{33} \) for separation \( r \) in the 1-2 plane gives

\[
C_{33}(r; t) = \frac{1}{r} \frac{\partial}{\partial r} h^2(r; t)
= \bar{n}^2(t) \left[ -1 + O \left( \frac{r^2}{\xi^2(t)} \right) \right]
\]

when \( r < \xi(t) \). Although it is difficult to be precise, this again suggests a significant amount of string in small loops. For \( r \gg \xi \), \( C_{33} \) falls off exponentially as

\[
C_{33}(r; t) = O(e^{-2r^2/\xi^2(t)}).
\]

showing that \( \xi \) indeed sets the scale at which strings see one another. Unfortunately, it has not yet proved possible to turn expressions for the \( C_{ij} \) into statements about self-avoidance, fractal dimension, or whatever is required to understand the length distribution of the resulting string network.

However, we can use the correlation functions to determine the variance in vortex winding number \( N_S \) through an open surface \( S \) in the 1-2 plane, which we take to be a disc of radius \( R \). There is a complication in that, as can be seen from Eq.\(34\), \( \rho_3 \) counts zeroes weighted by the cosine of the angle with which they pierce \( S \), whereas winding number counts zeroes in \( S \) without weighting. The problem is therefore
essentially a two-dimensional problem. Using topological charge conservation across the whole plane, it is not difficult to see that

\[(\Delta N)^2_S = O(n^2 \xi^3(t)R) = O(R/\xi(t)). \quad (33)\]

All that is required is short range zero-density correlation functions. Since \((\Delta N)_S\) is \((\Delta \alpha)_S/2\pi\), where \((\Delta \alpha)_S\) is the variance in the field phase around the perimeter \(\partial S\) of \(S\), this means in turn that

\[(\Delta \alpha)^2_S = O(R/\xi(t)). \quad (34)\]

This is naturally interpreted as a random walk in phase along the perimeter with average step length \(\xi(t)\), within which the phase is correlated. Calculations will be given elsewhere.

Since only the boundary region of \(S\) gives a contribution to the variance, we would get the same result if we were to quench in a annulus. It has already been proposed that quenches in an annulus be performed for superfluid \(^4\He\), for which \((\Delta \alpha)_S\) can be identified with a supercurrent.

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