Shuffling Quantum Field Theory*

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Abstract

We discuss shuffle identities between Feynman graphs using the Hopf algebra structure of perturbative quantum field theory. For concrete exposition, we discuss vertex function in massless Yukawa theory.

1 Introduction

The perturbative expansion of a quantum field theory is given in terms of Feynman graphs which demand integration over positions of all internal vertices. Such spacetime integrations diverge due to singularities in the integrand located in the regions where the positions of two or more such vertices coincide. The required process of renormalization amounts to a continuation of the generalized functions provided by Feynman integrands. Those integrands are well defined on the configuration space of vertices at distinct locations and have to be continued to diagonals so that the resulting analytic expressions are integrable along the diagonals. Thus, morally speaking, renormalization is a problem dual to compactification of configuration space, but generalizes such compactifications by allowing for the freedom of scale variations in this continuation, resulting in the presence of the renormalization group. From this viewpoint, it is no surprise that the common algebraic structure underlying any perturbative quantum field theory is a Hopf algebra based on rooted trees, as the latter are known to stratify the various limits to diagonals in configuration space 1.

While the Hopf algebra of rooted trees is the universal object for the Hopf algebra structure of perturbative quantum field theory 2, for any specific theory under consideration we can formulate the Hopf algebra directly on Feynman graphs 3, 4. In this latter formulation, the primitive generators of the Hopf algebra \( \mathcal{H} \) are graphs free of subdivergences. The only non-trivial part of this

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Hopf algebra is spanned by the graphs with a non-vanishing overall degree of divergence, which form a sub-Hopf algebra $\mathcal{H}_c$ whose primitive generators are graphs which are overall divergent, but have no subdivergences. Such graphs provide well-defined scheme-independent coefficients of divergence.

This algebraic set-up completely settles the renormalization procedure as a mathematical well-defined principle, and dispenses with all conceptual criticism of local point-particle quantum field theory which is based on the appearance of ultra-violet divergences. No need remains to abandon local quantum field theories for that reason: the mathematics of the perturbation series of a local quantum field theory is, as far as the UV sector is concerned, sound and, to my mind, rich and beautiful.

With the mathematics of renormalization laid out in [2, 4, 5, 6] in particular, we can ask for more structure which can be revealed thanks to the Hopf algebra structure of renormalization.

For a long time, David Broadhurst and the author observed interesting number-theoretic content in the evaluation of Feynman graphs without subdivergences, hence precisely in the primitive generators of the very Hopf algebra of graphs.

Intuitively, a close connection between the topology of such a primitive graph and its coefficient of overall divergence was found by relating the topology of the graphs to braid-positive knots in a highly empirical fashion [9]. Nevertheless, a faithful knot-to-number dictionary was established up to seven loops and more. Up to the six-loop level all coefficients of divergence were found to be multiple zeta values, while at the seven-loop level a few coefficients remained unidentified, and may well be shown, eventually, to point towards unexpected generalizations of these classes of numbers.

The algebraic structure of those numbers can be reasonably expected to be mirrored by algebraic relations between Feynman diagrams.

In this paper, we want to make first attempts to describe such relations between diagrams. We will consider quasi-shuffle identities, as they were considered for example in [10], which are known by many to form part of the algebraic structure of Euler/Zagier sums [12].

We will formulate these quasi-shuffle identities making use of the operators $B_+$ (the closed Hochschild one-cocycle which exists in any Hopf algebra of the type considered here) and $B_-$, which is its inverse on the linear basis of decorated rooted trees, but fails to be its inverse on products of rooted trees: $B_+ B_- (\prod_i t_i) \neq 0$, see below.

We can then naturally formulate the shuffle algebra on decorated rooted trees, and will turn to massless Yukawa theory for concrete exposition.

We show that the iteration of vertex corrections provides, modulo finite terms, a quasi-shuffle algebra, where letters are provided by all two-line irre-

\footnote{Our results in [9] do not reduce the number content of field theories in even dimensions to multiple zeta values and thus do not yet fully support the conjecture in [10].}
ducible four-fermion skeleton graphs.

We will finish the paper by showing that the term which determines the deviation from a proper shuffle identity obeys a pentagon identity in its finite part.

2 Quasi shuffle identities

Let \( A = a_1, a_2, \ldots \) be a locally finite graded set of letters, and \( W = k\langle A \rangle \) be the \( k \)-vector space of words built from such letters, with \( e \), the empty word, being the only word of degree zero. For all letters \( x, y \in A \) and words \( w_1, w_2 \in W \) let the shuffle product \( s : W \times W \to W \) be defined by

\[
s[xw_1, yw_2] = xs[w_1, yw_2] + ys[xw_1, w_2].
\]

(1)

Then, \((W, s)\) is known to form a commutative algebra (hence, \( s \) is commutative and associative), the shuffle algebra.

If we further set \( \tilde{s}[e, w_1] = \tilde{s}[w_1, e] = w_1 \) and

\[
\tilde{s}[xw_1, yw_2] = x\tilde{s}[w_1, yw_2] + y\tilde{s}[xw_1, w_2] + C[x, y]\tilde{s}[w_1, w_2],
\]

then \((W, \tilde{s})\) forms again a commutative associative algebra provided \( C : A \times A \to A \) is commutative, associative and adds degrees. See [11] for proof and details.

Also, a proof of this claim will be given below in terms of operators \( B_+, B_- \).

We call \((W, s)\) a quasi-shuffle algebra.

Consider an algebra homomorphism \( \rho : (W, s) \to V \) where \( V \) is some algebra, \( \rho(s[w_1, w_2]) = \rho(w_1)\rho(w_2) \). A prominent example is provided by iterated integrals [8], where letters \( a_i \in A \) are represented by differential one-forms, and the iterated integral \( F^w(x) \) fulfills the shuffle identity \( F^{s[w_1, w_2]}(x) = F^{w_1}(x)F^{w_2}(x) \).

Essentially, this identity allows to rewrite any rooted tree with sidebranchings as a sum over rooted trees without sidebranchings [7].

Consider now decorated rooted trees [2] where decorations are taken from the alphabet \( A \). Let \( H(A) \) be the resulting Hopf algebra [2, 3]. As usual, we have operators \( B_-, B^\times_+ \), \( x \in A \), such that \( B_- (\prod_{i=1}^k t_i) = \sum_{i=1}^k t_1 \ldots B_-(t_i) \ldots t_k \) and we write \( B_+ B_- \) for

\[
B_+ B_- (\prod_{i=1}^k t_i) := \sum_{i=1}^k B^\times_+(t_i)(t_1 \ldots B_-(t_i) \ldots t_k),
\]

(3)

where \( r(t) \in A \) gives the decoration at the root. With these definitions the identity operator \( id : H \to H \), \( id(X) = X \) can be written as \( id = B_- B^\times_+ \) for all \( x \in A \).

We can formulate a (quasi)-shuffle product on decorated rooted trees as follows. We identify a \( k \)-letter word \( a_{i_1} a_{i_2} \ldots a_{i_k} \) with a decorated rooted tree without sidebranchings,

\[
T = B^\alpha_{i_1}_+ \circ B^\alpha_{i_2}_+ \circ \ldots \circ B^\alpha_{i_k}_+ (e),
\]

3
and write the shuffle product as

\[ s[t_1, t_2] = B^r(t_1)(s[B_-(t_1), t_2]) + B^r(t_2)(s[t_1, B_-(t_2)]). \] (4)

This is well-defined as \( B_-(t) \) is again a single rooted tree, due to the fact that \( T \) has no sidebranching. Let us prove, \( \hat{H} \) has no sidebranching. Let \( T \to H \) be the subset in \( \mathcal{H}(A) \) of single decorated rooted trees, i.e. the linear basis of \( \mathcal{H}(A) \). In general, on \( \mathcal{H}(A), B_- \) is a map \( \mathcal{H}(A) \to \mathcal{H}(A) \) while on rooted trees without sidebranchings, \( B_- \) maps \( \mathcal{H}(A) \to \mathcal{H}(A) \).

We can define the shuffle product for arbitrary decorated rooted trees \( t \in \mathcal{H}_1(A) \) using \( u : \mathcal{H}(A) \to \mathcal{H}(A) \) given in an iterative manner by \( u(\prod_{i=1}^k t_i) = s[t_1, u(\prod_{i=2}^k t_i)], \; u(t_i) = t_i; \)

\[ s[t_1, t_2] = B^r(t_1)(s[u(B_-(t_1)), t_2]) + B^r(t_2)(s[t_1, u(B_-(t_2))]). \] (5)

We obtain a quasi-shuffle algebra on decorated rooted trees if we replace \( s \) by \( \tilde{s} \) everywhere, thus

\[ \tilde{s}[t_1, t_2] = B^r(t_1)(\tilde{s}[u(B_-(t_1)), t_2]) + B^r(t_2)(\tilde{s}[t_1, \tilde{u}(B_-(t_2))]) \]

\[ + B^r(r(t_1), r(t_2))(\tilde{s}[\tilde{u}(B_-(t_1)), \tilde{u}(B_-(t_2))]), \] (6)

\( \tilde{u}(\prod_{i=1}^k t_i) = \tilde{s}[t_1, \tilde{u}(\prod_{i=2}^k t_i)], \; \tilde{u}(t_i) = t_i. \) The algebras so obtained are commutative algebras if \( C \) fulfills the requirements listed above. Commutativity follows immediately by induction over the number of vertices. Let us prove, again by induction, associativity in some detail to get acquainted with the use of \( B^r, B_-, \) Our proof is essentially an elaborated version of the one in \( \square \).

Let \( T_1, T_2, T_3 \in \mathcal{H}(A) \). For \( \tilde{s}[T_1, \tilde{s}[T_2, T_3]] \) we find (the action of \( \tilde{u} \) is implicitly understood in any appearance of \( B_- \))

\[ \tilde{s}[T_1, \tilde{s}[T_2, T_3]] = \tilde{s}\left( B^r(T_1)(\tilde{s}[B_-(T_1), T_2]) + B^r(T_2)(\tilde{s}[T_1, B_-(T_2)]) \right) \]

\[ + B^r(r(T_1), r(T_2))(\tilde{s}[\tilde{u}(B_-(T_1)), \tilde{u}(B_-(T_2))]), T_3] \]

\[ = B^r(T_1)(\tilde{s}[B_-(T_1), T_2], T_3] \]

\[ + B^r(T_3)(\tilde{s}[B_-(T_3)], B_-(T_3)] \] (7)

\[ + B^r(r(T_1), r(T_3))(\tilde{s}[\tilde{s}[B_-(T_1), T_2], B_-(T_3)]) \] (8)

\[ + B^r(T_2)(\tilde{s}[\tilde{s}[T_1, B_-(T_2)], T_3]) \] (9)

\[ + B^r(T_3)(\tilde{s}[\tilde{s}[T_1, B_-(T_2)], B_-(T_3)]) \] (10)

\[ + B^r(r(T_2), r(T_3))(\tilde{s}[\tilde{s}[T_1, B_-(T_2)], B_-(T_3)]) \] (11)

\[ + B^r(r(T_1), r(T_3))(\tilde{s}[\tilde{s}[B_-(T_1), B_-(T_2)], T_3]) \] (12)

\[ + B^r(r(T_2), r(T_3))(\tilde{s}[\tilde{s}[B_-(T_1), B_-(T_2)], B_-(T_3)]) \] (13)

\[ + B^r(r(T_1), r(T_3))(\tilde{s}[\tilde{s}[B_-(T_1), B_-(T_2)], B_-(T_3)]) \] (14)

\[ + B^r(C(r(T_1), r(T_2)), r(T_3))(\tilde{s}[\tilde{s}[B_-(T_1), B_-(T_2)], B_-(T_3)]) \] (15)
while for \( \tilde{s}[T_1, T_2, T_3] \) we find

\[
\tilde{s}[T_1, T_2, T_3] = \tilde{s}\left( B^r(T_1) \tilde{s}[B_-(T_1), T_2] + B^r(T_2) (\tilde{s}[T_1, B_-(T_2)]) + B^r(T_3) (\tilde{s}[T_1, T_2, B_-(T_3)]) \right)
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

\[
= B^r(T_1) \left[ \tilde{s}[B_-(T_1), B^r(T_2) (\tilde{s}[B_-(T_2), T_3])] + B^r(T_2) (\tilde{s}[T_1, \tilde{s}[B_-(T_2), T_3]]) + B^r(T_3) (\tilde{s}[T_1, \tilde{s}[T_2, B_-(T_3)]) \right]
\]

We assume associativity at \( n \) vertices to find \( \tilde{s} = \tilde{s}[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \) and \( \tilde{s} = \tilde{s}[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \). Hence, assuming associativity of \( \tilde{s} \) for products of trees with up to \( n \) vertices, we obtain associativity at \( n + 1 \) vertices provided the map \( C \) fulfills the requirements listed above, which finally ensures \( \tilde{s} = \tilde{s}[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \). The start of the induction is easily proved with such a map \( C \).

In the next section, we will consider an alphabet essentially given by an infinite set of skeleton graphs in Yukawa theory, graded by the loop number, and investigate to what extent we can formulate a quasi-shuffle identity. Hence we have to investigate if we can find an appropriate map \( C \).

The following result shows that that the transition from a shuffle identity to a quasi-shuffle identity essentially measures the non-vanishing of the commutator \( [B_+, B_-] = B_+ B_- - B_- B_+ \). We have

\[
u([B_+, B_-](X)) = 0.
\]

Proof: Proceed by induction on the number of vertices, using the definitions of \( s \) and \( B_+ B_- \).

Figure (3) gives some \( \tilde{s} \)-products.

### 3 Yukawa Theory

In this first approach towards quasi-shuffle identities in quantum field theories, we will study vertex corrections at zero momentum transfer in massless
Figure 1: The first row gives the $\bar{s}$-product $\bar{s}[t_1(a), t_1(b)]$ of two decorated roots. The second row gives the product $\bar{s}[T_{3_2}(c, a, b), t_1(d)]$ of a rooted tree, 'the claw' $t_{3_2}$, with three vertices, both connected to the root, which is decorated by the letter $c$ while the other two vertices are decorated by letters $a, b$, with a root decorated by the letter $d$.

Yukawa theory. Hence, let us consider two-line irreducible subdivergence-free four-fermion skeleton graphs $K^{[i]}$ in massless Yukawa theory. Upon closure, we obtain an overall divergent subdivergence-free vertex correction corresponding to a graph $\Gamma^{[w]}$, cf. Fig. (2), at zero momentum transfer. At each loop number, we have a finite number of such primitive graphs, and the graphs $K^{[i]}$ provide an example of an alphabet $A$ in the sense above.

To each graph $K^{[i]}$ belongs an analytic expression $U(K^{[i]})(k, q)$ and the closure to a vertex correction is defined by the integral

$$U(\Gamma^{[i]})(q) = g \int d^D k \frac{1}{k^2} U(K^{[i]})(k, q),$$

where $g$ is the renormalized coupling constant. These graphs $K^{[i]}$ provide an alphabet, and the vertex correction corresponding to the word $w = i_1 i_2 \ldots i_k$ is

$$U(\Gamma^{[w]})(q) = g \int \ldots \int d^D k_1 \frac{d^D k_1}{k_1^2} \ldots \frac{d^D k_k}{k_k^2} \times U(K^{[i_1]})(k_1, k_2) \ldots U(K^{[i_k]})(k_k, q).$$

(27)

We can regard $U$ as a character from the Hopf algebra $\mathcal{H}(A)$ to the ring of meromorphic functions $A$ as in [4, 8] and the minimal subtracted counterterm is given by the character $U_-$ of the unique Birkhoff decomposition $U = U_1^{-1} \ast U_+$, where $U_+$ is the character which assigns the renormalized Feynman graph $U_+(\Gamma^{[w]})$ to the bare unrenormalized $U(\Gamma^{[w]})$ [4, 8].

\(^2\)Obviously, the kernels $U(K^{[i]})$ are themselves of order $[g^2]^{1+grad(K^{[i]})}$, where $grad$ gives the loop number.
Consider a different character $\bar{U}$ defined by

$$\bar{U}(\Gamma[i])(q) = g \int \cdots \int d^Dk_1 d^Dk_2 \cdots d^Dk_k \times \bar{U}(K[i_1])(k_1 - k_2) \cdots \bar{U}(K[i_k])(k_k - q),$$

(29)

where $\bar{U}(K[i])(r) = U(K[i])(r, 0)$.

Using the convolution of characters and the antipode $S$ of $\mathcal{H}(A)$, we have $U = \bar{U} * \bar{U} * S * U$ as a trivial identity. The character $\bar{U} * S * U$ maps to functions holomorphic at $D = 4$. Indeed, it delivers finite renormalized Green functions by construction: all divergent forests in $\bar{U}(\Gamma[w])$ are compensated by appropriate forests in $\bar{U} * S * U$, as such ratios $\bar{U} * S * U$ reproduce the forest formula [2, 4, 5].

We have thus isolated the UV-divergences of such vertex corrections in $\bar{U}(\Gamma[w])$, and hence in the minimal subtracted counterterms $\bar{U}_-(\Gamma[w])$. Obviously, a finite renormalization by $U_- * \bar{U}_- * S * \bar{U}_-$:

$$U_- = U_- * \bar{U}_- * S * \bar{U}_-.$$

(30)

The character $\bar{U}$ can be calculated with considerable ease, as it relies solely on knowledge of $\bar{U}(K[i])(q)$, which, being a function of the single parameter $q^2$, has the form, with $z := (D - 4)/2$,

$$\bar{U}(K[i])(q) = [q^2]^{-1+zgrad(K[i])} F[i](z),$$

(31)
where \( F^{[i]}(z) \) is holomorphic at \( D = 4 \). Hence,

\[
\tilde{U}(\Gamma^{[i]})(q) = G_q^2(1, 1 + (\text{grad}(K^{[i]}) + 1)z)F^{[i]}(z),
\]

where we use the \( G \)-function

\[
G_q^2(a, b) := \int d^Dk \frac{1}{[k^2]^a((k + q)^2)^b} \tag{33}
\]

which evaluates to

\[
G_q^2(a, b) = [q^2]^{2-z-a-b} \frac{\Gamma(a + b - 2 + z)\Gamma(2 - z - a)\Gamma(2 - z - b)}{\Gamma(a)\Gamma(b)\Gamma(D - a - b)}. \tag{34}
\]

The evaluation of words \( w \) via \( \tilde{U}(\Gamma^{[w]})(q) \) then delivers products of \( G \)- and \( F^{[i]} \)-functions. The leading pole term of Feynman diagrams fulfills a shuffle identity \([4]\), and indeed, here we find, for example,

\[
\tilde{U}(\Gamma^{[\bar{s}[i_1, i_2]]}) - \tilde{U}(\Gamma^{[i_1]})\tilde{U}(\Gamma^{[i_2]}) \sim \frac{1}{z}, \tag{35}
\]

where \( s[i_1, i_2] = i_1i_2 + i_2i_1 \), in this notation. We want to investigate if the term \( \sim 1/z \) can be understood as resulting from a quasi-shuffle identity. This is possible as a quasi shuffle identity would precisely demand that this term is of the form \( \tilde{U}(\Gamma^{[C(i_1, i_2)]}) \), where \( C(i_1, i_2) \) must be regarded as a single letter, hence be given by an expression without subdivergences, hence \( \sim 1/z \). More precisely, we want to know if, for arbitrary letters represented as \( \tilde{U}(K^{[i_1]}), \tilde{U}(K^{[i_2]}) \), we can define an analytic expression \( \tilde{U}(K^{[C(i_1, i_2)]}) \) such that

\[
\tilde{U}(\Gamma^{[\bar{s}[i_1, i_2]]}) - \tilde{U}(\Gamma^{[i_1]})\tilde{U}(\Gamma^{[i_2]}) \tag{36}
\]
is finite. Here,

\[
\tilde{U}(\Gamma^{[\bar{s}[i_1, i_2]]}) = \tilde{U}(\Gamma^{[i_1i_2]}) + \tilde{U}(\Gamma^{[i_2i_1]}) + \tilde{U}(\Gamma^{[C(i_1, i_2)]}). \tag{37}
\]

Let us now define such a map which assigns to any two letters a new one. We define

\[
\tilde{U}(K^{[C_2(i_1, i_2)]})(p) := \int d^Dr\tilde{U}(K^{[i_1]})(r) \frac{1}{r^2} \tilde{U}(K^{[i_2]})(r + p) \frac{(r + p) \cdot r}{(r + p)^2}. \tag{38}
\]

This map \( C_2 \) crosses kernels and is best explained in Fig.\([\bar{8}]\).

It evaluates to

\[
\tilde{U}(K^{[C_2(i_1, i_2)]})(p) = [p^2]^{-1+\bar{s}(1+n_1+n_2)}F^{[i_1]}(z)F^{[i_2]}(z) \times [G_1(2+n_1z, 2+n_2z) - G_1(1+n_1z, 2+n_2z) - G_1(2+n_1z, 1+n_2z)], \tag{39}
\]

\[\]
where $n_j = \text{deg}(K^{[i_j]})$, $j = 1, 2$. Closure to a vertex correction delivers

$$\hat{U}(\Gamma^{[C_2(i_1, i_2)]})(q) = G[q^2(1 + z(2 + n_1 + n_2))]F^{[i_1]}(z)F^{[i_2]}(z)$$

$$\times [G_1(2 + n_1 z, 2 + n_2 z)$$

$$- G_1(1 + n_1 z, 2 + n_2 z) - G_1(2 + n_1 z, 1 + n_2 z)].$$

(40)

Explicit evaluation, i.e. expansion in $z$, shows that the limit

$$\lim_{z \rightarrow 0} [G_1(2 + n_1 z, 2 + n_2 z) - G_1(1 + n_1 z, 2 + n_2 z) - G_1(2 + n_1 z, 1 + n_2 z)]$$

(41)

exists and that Eq. (36)) is fulfilled, solely due to the remarkable properties of $G$-functions. The $F[x]$-functions play no role here as our definition of $\hat{U}$ reduces their role to mere coefficients. Also, one immediately confirms that $C_2(i_1, i_2) = C_2(i_2, i_1)$.

To check associativity of $C_2$ let us define

$$J(n_1, n_2) := G_1(2 + n_1 z, 2 + n_2 z) - G_1(1 + n_1 z, 2 + n_2 z) - G_1(2 + n_1 z, 1 + n_2 z).$$

(42)

Then, one finds

$$J(n_1, n_2)J(n_1 + n_2, n_3) - J(n_2, n_3)J(n_1, n_2 + n_3) \sim z,$$

(43)

which suffices to show that

$$\hat{U}(\Gamma^{[C_2(C_2(i_1, i_2), i_3)]})(q) - \hat{U}(\Gamma^{[C_2(i_1, C_2(i_2, i_3))]})(q)$$

(44)

is finite.
Figure 4: $\tilde{s}[\tilde{s}[a, b], c]$ given explicitly. From the three terms in the rectangle we read off the contributions of $C_3$. Those three terms provide decorations, in which $C_2$ terms (abbreviated by $C$ in the figure) are nested. $C_3$ terms contributes to $\tilde{s}[\tilde{s}[a, b], c]$ in the cyclic sum $C_3(a, b, c) + C_3(b, c, a) + C_3(c, a, b)$. As $C_3$ is commutative in the last two arguments, $C_3(a, b, c) = C_3(a, c, b)$, no violations of associativity arise from $C_3$, as $\tilde{s}[a, \tilde{s}[b, c]]$ provides the same terms.

So far we worked up to finite parts. The results show that we identify a shuffle algebra in the UV-divergences of iterated vertex corrections, and we realized the beginning of a shuffle identity in Eq.(36). The finite parts which we disregarded so far will be essential in the full quasi-shuffle algebra. They will appear inside further divergent loop integrations, and hence modify the latter in a non-trivial manner.

Let us finish this paper with an investigation to what extent these finite parts obstruct a quasi-shuffle identity.

The violation of the quasi-shuffle identity in the finite part contributes, when integrated against a further letter $a$, by

$$C_3(a, i_1, i_2) = \int \frac{d^Dk}{k^2} \left[ \tilde{U}(\Gamma^{[B_+, B_-]}(i_1, i_2))(k) - \tilde{U}(\Gamma^{C_2}(i_1, i_2))(k) \right] K^{[a]}(k - q, 0).$$

Here,

$$\Gamma^{[B_+, B_-]}(i_1, i_2) = \Gamma^{[i_1 i_2]} + \Gamma^{[i_2 i_1]} - \Gamma^{[i_1]} \Gamma^{[i_2]}.$$

This has no contribution to the violation of associativity, as it is demonstrated in Fig.(4). It can be incorporated in a redefinition of $C_2$ respecting the quasi-shuffle algebra.

More interesting is the finite violation of associativity generated by $C_2$. Explicitly, one finds

$$J(n_1, n_2)J(n_1 + n_2, n_3) - J(n_2, n_3)J(n_1, n_2 + n_3) = -4z(n_1 - n_3) + O(z^2) \ (45)$$
so that the violation of associativity is proportional to the difference between the loop numbers, the degree, of the first and last letter. Hence, the finite terms violate associativity in a well-defined manner: the pentagon equation

\[(n_1-n_3)+(n_1-n_4)+(n_2-n_4) = 2n_1+n_2-n_3-2n_4 = (n_1+n_2-n_4)+(n_1-n_3-n_4)\]

is satisfied, cf. Fig.(5).

### 4 Conclusions

We have shown how quasi-shuffle algebras can be naturally formulated using the operators $B_+, B_-$. We then investigated representations of this algebra by Feynman diagrams, and found that, in the case of Yukawa theory considered here, such representations could be obtained modulo finite violations of associativity, and that these representations deviate by finite parts from a quasi-shuffle identity. These violations obey a pentagon identity in the finite part, and it will be an interesting exercise to work out the coherence laws of higher orders in $z$ in the future. Also, the generalization to other theories and Green functions is possible using appropriate representation of $B_\pm$ on Feynman graphs and will be presented in future work. We regard the results in this paper as a first attempt towards an algebraic understanding of the number-theoretic content of Feynman diagrams.

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