LOCAL REGULARITY NEAR BOUNDARY FOR THE STOKES AND NAVIER-STOKES EQUATIONS

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**Abstract.** We are concerned with local regularity of the solutions for the Stokes and Navier-Stokes equations near boundary. Firstly, we construct a bounded solution but its normal derivatives are singular in any \(L^p\) with \(1 < p\) locally near boundary. On the other hand, we present criteria of solutions of the Stokes equations near boundary to imply that the gradients of solutions are bounded (in fact, even further Hölder continuous). Finally, we provide examples of solutions whose local regularity near boundary is optimal.

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1. Introduction

We consider the non-stationary Stokes equations near flat boundary

\[
(1.1) \quad u_t - \Delta u + \nabla \pi = 0, \quad \text{div} \ u = 0 \quad \text{in} \ B_2^+ \times (0,4),
\]

where \(B_r^+ := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < r, x_n > 0\}\). Here, no-slip boundary condition is given only on the flat boundary, i.e.

\[
(1.2) \quad u = 0 \quad \text{on} \ \Sigma := (B_2 \cap \{x_n = 0\}) \times (0,4).
\]

We can also consider similar situation for the Navier-Stokes equations, i.e.

\[
(1.3) \quad u_t - \Delta u + (u \cdot \nabla) u + \nabla \pi = 0, \quad \text{div} \ u = 0 \quad \text{in} \ B_2^+ \times (0,4)
\]

with the boundary condition (1.2).

Our concern is local analysis of the solutions of the equations (1.1) or (1.3) with (1.2) in \(B_1^+ \times (0,1)\). Unlike the heat equation, non-local effect of the Stokes equations may cause limitation of local smoothing effects of solutions. In fact, the second author showed that there exists a weak solution of the Stokes equations (1.1)-(1.2) whose normal derivative is unbounded near boundary, although it is bounded and its derivatives are square integrable (see [13]), i.e.

\[
\sup_{Q_r^+} |D_{x_n} u| = \infty, \quad \sup_{Q_r^+} \|u\|_{L^4_t L^p_x(Q_r^+)} < \infty,
\]

where \(Q_r^+ := B_r^+ \times (1 - r^2, 1), \ 0 < r < 1\).

Seregin and Šverák found a simplified example as the form of shear flow in a half-space to the Stokes equations (and Navier-Stokes equations as well) such that its gradient is unbounded near boundary, although its velocity field is locally bounded (see [19]). It is noteworthy that the example in [19] is not of globally finite energy, and on the other hand, the constructed one in [13] has finite energy globally in a half-space (see [15] for the details).

Following similar constructions as in [13], even further, the authors constructed an example that \(u\) is integrable in \(L_1^t L_p^x\) but \(\nabla u\) is not square integrable (see [7]). More precisely, there exists a very weak solution of the Stokes equations (1.1) or the Navier-Stokes equations (1.3) with (1.2) such that

\[
\|\nabla u\|_{L^2(Q_r^+)} = \infty, \quad \|u\|_{L_1^t L_p^x(Q_r^+)} < \infty, \quad p < \infty.
\]

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The notions of very weak solutions are given in Section 2 (see Definition 2.3 and Definition 2.11). With the aid of this construction, the authors also proved that Caccioppoli’s inequalities of Stokes equations and Navier-Stokes equations in general may fail near boundary when only local boundary problems are considered (see [7] Theorem 1.1). It is a very important distinction in comparison to the interior case, where Caccioppoli’s (type) inequalities turn out to be true (compare to [11] and [25], and refer also [12] for generalized Navier-Stokes flow).

One may ask how bad $\nabla u$ could be, when $u$ is bounded in a local domain near boundary. One of our motivations in this paper is to answer to the question, and we obtain the following:

**Theorem 1.1.** Let $1 < p < \infty$. Then, there exists a very weak solution $u$ of Stokes equations (1.1) or Navier-Stokes equations (1.3) with the boundary condition (1.2) such that

(1.4) \[ \|u\|_{L^\infty(Q^+_1)} < \infty, \quad \|\nabla u\|_{L^p(Q^+_1)} = \infty. \]

**Remark 1.2.** If we compare to the example constructed by Seregin and Šverák, their solutions also show singular normal derivatives near boundary not only pointwise but also $L^p_{\text{loc}}$, $p > 3$, since $\partial_{x_3} u(x_3, t) \geq cx_3^{-1+2\alpha}$, $\alpha \in (0, 1/2)$ in the region near origin with $x_3^2 \geq -4t$. Theorem 1.1 is an improvement of their result, since construction of singular normal derivatives in $L^p_{\text{loc}}$ is extended up to all $p > 1$ near boundary.

**Remark 1.3.** We do not know if $p$ in (1.4) can be replaced by 1 in Theorem 1.1 and thus we leave it as an open question. In Appendix A alternative proof is given for $p = 2$ in Theorem 1.1 which seems informative.

On the other hand, the second motivation of the paper is to study optimal regularity of the local problem near boundary, in case that the pressure $\pi$ is locally integrable in $L^p$. We recall that it was shown in [18] Proposition 2] in three dimensions that for given $p, q \in (1, 2]$ the solution of the Stokes equations (1.1)·(1.2) satisfies the following a priori estimate: For any $r$ with $p \leq r < \infty$

(1.5) \[ \|u_t\|_{L^p_t L^q_x(Q^+_1)} + \|\nabla^2 u\|_{L^p_t L^q_x(Q^+_1)} + \|\nabla \pi\|_{L^p_t L^q_x(Q^+_1)} \leq C \left( \|u\|_{L^p_t W^{1,p}_x(Q^+_1)} + \|\pi\|_{L^p_t \dot{W}^{1,p}_x(Q^+_1)} \right). \]

Furthermore, due to parabolic embedding, it follows that

(1.6) \[ \|u\|_{C^\alpha_t C^\gamma_x(Q^+_1)} < \infty, \quad 0 < \alpha < 2 - \frac{2}{q}. \]

We remark that there are examples such that the Hölder continuity (1.6) is optimal. To be more precise, in case that $q < 2$, it was proved in [15] that

(1.7) \[ \|\nabla u\|_{L^\infty(Q^+_1)} = \infty. \]

**Remark 1.4.** For the case $q = 2$, we can also show that (1.7) is true, and the details of its verification will be given in Appendix [7]

The above estimate (1.5) shows that integrability of $u_t$, $\nabla^2 u$ and $\nabla \pi$ is increased for spatial variables, and it is, however, not clear if integrability in time could be improved or not.

Firstly, we extend the result of [18] Proposition 2] for the case $q > 2$. In such case, an interesting feature is that not only velocity but also the gradient of velocity fields are Hölder continuous up to boundary, contrary to the case $1 < q \leq 2$.

**Theorem 1.5.** Let $2 < q < \infty$ and $1 < p < \infty$. Suppose that $(u, \pi)$ is solution for the Stokes equations (1.1)·(1.2) satisfying $\nabla^2 u$, $u_t \in L^p_t L^q_x(Q^+_1)$ and $\pi \in L^p_t \dot{W}^{1,p}_x(Q^+_1)$. Then, for any $r$ with $p \leq r < \infty$

(1.8) \[ \|u_t\|_{L^p_t L^q_x(Q^+_1)} + \|\nabla^2 u\|_{L^p_t L^q_x(Q^+_1)} + \|\nabla \pi\|_{L^p_t L^q_x(Q^+_1)} \leq c \left( \|u\|_{L^p_t W^{1,p}_x(Q^+_1)} + \|\pi\|_{L^p_t \dot{W}^{1,p}_x(Q^+_1)} \right). \]
Furthermore, the derivative of \( u \) is Hölder continuous, i.e.
\[
\| \nabla u \|_{C^{\alpha}_{q}C^{\alpha + \frac{n}{q}}_{q}(Q^{+})} < \infty, \quad 0 < \alpha < 1 - \frac{2}{q}.
\]

In next theorem, we prove that the estimates (1.5) and (1.8) are indeed optimal. To be more precise, there is a solution of the Stokes equations (1.1)-(1.2) such that \( \nabla u \) and \( \pi \) belong to \( L^{p}(Q^{+}_1) \) but \( u_{t}, \nabla \pi \notin L^{p}(Q^{+}_2) \) for any \( \tilde{p} \) with \( \tilde{p} > p \).

**Theorem 1.6.** Let \( 1 < p, q < \infty \). Then, there exist a solution \( u \) of Stokes equations (1.1) and Navier-Stokes equations (1.3) with the boundary condition (1.2) such that
\[
\| \nabla u \|_{L^{3/2}(Q^{+}_2)} + \| D_{t}u \|_{L^{q}(Q^{+}_2)} + \| \nabla \pi \|_{L^{q}(Q^{+}_2)} < \infty,
\]
but for any \( r_{1} > q \) and \( r_{2} > \frac{3q}{2} \)
\[
\| D_{t}u \|_{L^{r_{1}}(Q^{+}_2)} = \infty, \quad \| \nabla \pi \|_{L^{r_{1}}(Q^{+}_2)} = \infty, \quad \| \nabla^{2}u \|_{L^{r_{2}}(Q^{+}_2)} = \infty.
\]

**Remark 1.7.** We do not know whether or not there exists a solution \( u \) of Stokes equations (1.1)-(1.2) such that \( \| \nabla^{2}u \|_{L^{r_{1}}(Q^{+}_{2})} = \infty \) for \( r_{1} > q \). In fact, our construction shows that \( \nabla^{2}u \in L^{\frac{3q}{2}}(Q^{+}_{2}) \).

This paper is organized as follows. In Section 2, we introduce the function spaces and we recall some known results and introduce results useful for our purpose. Section 3 is devoted to recalling Poisson kernel for Stokes equations in a half-space and two useful lemmas are proved as well. In Section 4, Section 5, and Section 6, we prove Theorem 1.1, Theorem 1.5, and Theorem 1.6 for the Stokes equations, respectively. In the case of the Navier-Stokes equations, proofs of Theorem 1.1 and Theorem 1.6 are given in Section 7. Appendix provides a simple proof of Theorem 1.1 for the case \( p = 2 \), and presents the details of Remark 1.3, Remark 1.4, and Remark 1.2 as well.

## 2. Preliminaries

For notational convention, we denote \( x = (x', x_{n}) \), where the symbol \( t \) means the coordinate up to \( n - 1 \), that is, \( x' = (x_{1}, x_{2}, \cdots, x_{n-1}) \). We write \( D_{x_{i}}u \) as the partial derivative of \( u \) with respect to \( x_{i} \), \( 1 \leq i \leq n \), i.e.,
\[
D_{x_{i}}u(x) = \frac{\partial}{\partial x_{i}}u(x).
\]
Throughout this paper we denote by \( c \) various generic positive constant and by \( c(\ast, \cdots, \ast) \) depending on the quantities in the parenthesis.

Let \( \alpha \in \mathbb{R} \) and \( 1 \leq p \leq \infty \). We define an anisotropic homogeneous Sobolev space \( \mathring{W}_{p}^{\alpha, \frac{n}{q}}(\mathbb{R}^{n+1}) \) by
\[
\mathring{W}_{p}^{\alpha, \frac{n}{q}}(\mathbb{R}^{n+1}) = \{ f \in S'(\mathbb{R}^{n+1}) \mid f = h_{\alpha} \ast g, \quad \text{for some} \quad g \in L^{p}(\mathbb{R}^{n+1}) \}
\]
with norm \( \| f \|_{\mathring{W}_{p}^{\alpha, \frac{n}{q}}(\mathbb{R}^{n+1})} := \| g \|_{L^{p}(\mathbb{R}^{n+1})} = \| h_{-\alpha} \ast f \|_{L^{p}(\mathbb{R}^{n+1})} \), where \( \ast \) is a convolution in \( \mathbb{R}^{n+1} \) and \( S'(\mathbb{R}^{n+1}) \) is the dual space of the Schwartz space \( S(\mathbb{R}^{n+1}) \). Here \( h_{\alpha} \) is a distribution whose Fourier transform in \( \mathbb{R}^{n+1} \) is defined by
\[
\hat{h}_{\alpha}(\xi, \tau) = c_{\alpha}(4\pi^{2}|\xi|^{2} + i\tau)^{-\frac{n}{2}}, \quad (\xi, \tau) \in \mathbb{R}^{n} \times \mathbb{R}.
\]
In case that \( \alpha = k \in \mathbb{N} \cup \{0\} \), we note that
\[
\| f \|_{\mathring{W}_{p}^{k, \frac{n}{q}}(\mathbb{R}^{n+1})} \approx \sum_{k_{1} + k_{2} + \cdots + k_{n} + l = k} \| D_{x_{1}}^{k_{1}}D_{x_{2}}^{k_{2}} \cdots D_{x_{n}}^{k_{n}}D_{t}^{l}f \|_{L^{p}(\mathbb{R}^{n+1})},
\]
where \( D_{t}^{1/2}f(t) = D_{t} \int_{-\infty}^{t} f(s) (t - s)^{-1/2} ds \) and \( D_{t}^{k+1/2}f = D_{t}^{k}D_{t}^{1/2}f \). In particular, when \( k = 0 \), it follows that \( \| f \|_{\mathring{W}_{p}^{0, \frac{n}{q}}(\mathbb{R}^{n+1})} = \| f \|_{L^{p}(\mathbb{R}^{n+1})} \).
Next, we recall an anisotropic homogeneous Besov space \( \dot{B}^\alpha_{pq} (\mathbb{R}^{n+1}) \). Let \( \phi \in \mathcal{S}(\mathbb{R}^{n+1}) \) such that \( \hat{\phi} \), the Fourier transform of \( \phi \), satisfies

\[
\begin{cases}
    \hat{\phi}(\xi, \tau) > 0 & \text{on } 2^{-1} < |\xi| + |\tau|^{\frac{2}{p}} < 2, \\
    \hat{\phi}(\xi, \tau) = 0 & \text{elsewhere}, \\
    \sum_{-\infty < \xi < \infty} \hat{\phi}(2^{-i} \xi, 2^{-2i} \tau) = 1 \ ((\xi, \tau) \neq (0, 0)).
\end{cases}
\]

We then introduce functions \( \phi_i \in \mathcal{S}(\mathbb{R}^{n+1}), i \in \mathbb{Z} \) in terms of \( \phi \), which are defined by

\[
(2.1) \quad \hat{\phi}_i(\xi, \tau) = \hat{\phi}(2^{-i} \xi, 2^{-2i} \tau), \quad i = 0, \pm 1, \pm 2, \ldots.
\]

Note that \( \phi_i(x, t) = 2^{(i+2)n} \hat{\phi}(2^i x, 2^{2i} t) \). For \( \alpha \in \mathbb{R} \) we define the anisotropic homogeneous Besov space \( \dot{B}^\alpha_{pq} (\mathbb{R}^{n+1}) \) by

\[
\dot{B}^\alpha_{pq} (\mathbb{R}^{n+1}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^{n+1}) \mid \| f \|_{\dot{B}^\alpha_{pq} (\mathbb{R}^{n+1})} < \infty \right\}
\]

with the norms

\[
\| f \|_{\dot{B}^\alpha_{pq} (\mathbb{R}^{n+1})} : = \left( \sum_{-\infty < i < \infty} (2^{ni} \| \phi_i \ast f \|_{L^p})^r \right)^{\frac{1}{r}}, \quad 1 \leq r < \infty,
\]

\[
\| f \|_{\dot{B}^\alpha_{pq} (\mathbb{R}^{n+1})} : = \sup_{-\infty < i < \infty} 2^{ni} \| \phi_i \ast f \|_{L^p}.
\]

Let \( I \) be a open interval. The anisotropic homogeneous Sobolev space \( \dot{W}^\alpha_{p,q} (\mathbb{R}^{n+1} \times I) \) in \( \mathbb{R}^{n+1} \times I \) is defined by

\[
\dot{W}^\alpha_{p,q} (\mathbb{R}^{n+1} \times I) = \left\{ f = F|_{\mathbb{R}^{n+1} \times I} \mid F \in \dot{W}^\alpha_{p,q} (\mathbb{R}^{n+1} \times \mathbb{R}) \right\}
\]

with norm

\[
\| f \|_{\dot{W}^\alpha_{p,q} (\mathbb{R}^{n+1} \times \mathbb{R})} = \inf \left\{ \| F \|_{\dot{W}^\alpha_{p,q} (\mathbb{R}^{n+1} \times \mathbb{R})} \mid F \in \dot{W}^\alpha_{p,q} (\mathbb{R}^{n+1} \times \mathbb{R}), \ F|_{\mathbb{R}^{n+1} \times I} = f \right\}.
\]

Similarly, we define the the anisotropic homogeneous Besov space \( \dot{B}^\alpha_{pq} (\mathbb{R}^{n+1} \times I) \). The properties of the anisotropic Besov spaces are comparable with the properties of Besov spaces. In particular, the following properties will be used later.

**Proposition 2.1.** Let \( \Omega \) be \( \mathbb{R}^n \) or \( \mathbb{R}^n_+ \).

1. For \( \alpha > 0 \)
   \[
   \dot{B}^\alpha_{pq} (\Omega \times I) = L^p \left( I; \dot{B}^\alpha_{pq}(\Omega) \right) \cap L^p \left( \Omega; \dot{B}^\alpha_{pq}(I) \right).
   \]

2. Suppose that \( 1 \leq p_0 \leq p_1 \leq \infty, \ 1 \leq q_0 \leq q_1 \leq \infty \) and \( \alpha_0 \geq \alpha_1 \) with \( \alpha_0 - \frac{n+2}{p_0} = \alpha_1 - \frac{n+2}{p_1} \). Then, the following inclusion holds
   \[
   \dot{B}^{\alpha_0}_{p_0 q_0} (\Omega \times I) \subset \dot{B}^{\alpha_1}_{p_1 q_1} (\Omega \times I).
   \]

3. Suppose that \( f \in \dot{W}^{\alpha_0}_{p,q} (\Omega \times I) \) and \( f \in \dot{B}^{\alpha_0}_{pq} (\Omega \times I) \) with \( \alpha > \frac{1}{p} \). Then, \( f|_{x_n=0} \in \dot{B}^{\alpha - \frac{1}{p} - \frac{1}{2p}}_{pq} (\mathbb{R}^{n-1} \times I) \) and following estimates are satisfied.
   \[
   \| f \|_{\dot{B}^{\alpha - \frac{1}{p} - \frac{1}{2p}}_{pq} (\mathbb{R}^{n-1} \times I)} \leq c \| f \|_{\dot{W}^{\alpha_0}_{p,q} (\Omega \times I)},
   \]
   \[
   \| f \|_{\dot{B}^{\alpha - \frac{1}{p} - \frac{1}{2p}}_{pq} (\mathbb{R}^{n-1} \times I)} \leq c \| f \|_{\dot{B}^{\alpha_0}_{pq} (\Omega \times I)}.
   \]

**Remark 2.2.** For the proof of (1) in Proposition 2.1, one can refer to [8, Theorem 3]. We also consult Theorem 6.5.1 and Theorem 6.6.1 in [2] for (2) and (3) in Proposition 2.1, respectively.
We next remind the non-homogeneous anisotropic Sobolev space and Besov space defined by

\[
\|f\|_{W^{\alpha, q}_p(\Omega \times I)} = \|f\|_{L^p(\Omega \times I)} + \|f\|_{W^{\alpha, q}_p(\Omega \times I)}, \\
\|f\|_{B^{\alpha, q}_p(\Omega \times I)} = \|f\|_{L^p(\Omega \times I)} + \|f\|_{B^{\alpha, q}_p(\Omega \times I)},
\]

where \(\alpha > 0\) and \(1 \leq p \leq \infty\). Then, Proposition 2.1 holds for non-homogeneous anisotropic Sobolev spaces and Besov spaces (see [24,25]).

Let \(\Omega\) be a domain in \(\mathbb{R}^n\) and \(f\) be an open interval. Let \(X(\Omega)\) be a function space defined in \(\Omega\). We denote by \(L^q_tX_x(\Omega \times I)\), \(1 \leq q \leq \infty\) with norm

\[
\|f\|_{L^q_tX_x(\Omega \times I)} = \left( \int_I \|f(t)\|_{X_x(\Omega)}^q dt \right)^{\frac{1}{q}}.
\]

In case that \(X(\Omega) = L^q(\Omega \times I)\), we denote \(L^q(\Omega \times I) = L^q_tL^q_x(\Omega \times I)\).

We introduce notions of very weak solutions for the Stokes equations and the Navier-Stokes equations with non-zero boundary values in a half-space \(\mathbb{R}^n_+\). To be more precise, we consider first the following Stokes equations in \(\mathbb{R}^n_+\):

\[
w_t - \Delta w + \nabla \pi = f, \quad \text{div } w = 0, \quad \text{in } \mathbb{R}^n_+ \times (0, \infty)
\]

with zero initial data and non-zero boundary value

\[
w|_{t=0} = 0, \quad w|_{x_n=0} = g.
\]

We now define very weak solutions of the Stokes equations (2.2)-(2.3).

**Definition 2.3.** Let \(g \in L^1_{loc}(\mathbb{R}^{n-1} \times (0, \infty))\) and \(f \in L^1_{loc}(\mathbb{R}^n_+ \times (0, \infty))\). A vector field \(w \in L^1_{loc}(\mathbb{R}^n_+ \times (0, \infty))\) is called a very weak solution of the Stokes equations (2.2)-(2.3), if the following equality is satisfied:

\[
-\int_0^\infty \int_{\mathbb{R}^n_+} w \cdot \Delta \Phi dx dt = \int_0^\infty \int_{\mathbb{R}^n_+} (w \cdot \Phi_t + f \cdot \Phi) dx dt - \int_0^\infty \int_{\mathbb{R}^{n-1}} g \cdot D x_n \Phi dx' dt
\]

for each \(\Phi \in C^2_c(\mathbb{R}^{n-1} \times (0, \infty))\) with

\[
\text{div } \Phi = 0, \quad \Phi|_{\mathbb{R}^{n-1} \times (0, \infty)} = 0.
\]

In addition, for each \(\Psi \in C^1_c(\mathbb{R}^n_+)\) with \(\Psi|_{\mathbb{R}^{n-1}} = 0\)

\[
\int_{\mathbb{R}^n_+} w(x,t) \cdot \nabla \Psi(x) dx = 0 \quad \text{for all } \quad 0 < t < \infty.
\]

We recall some estimates which are useful to our purpose. In the sequel, we denote by \(N\) and \(\Gamma\) the fundamental solutions of the Laplace and the heat equation, respectively, i.e.

\[
N(x) = \begin{cases} 
-\frac{1}{(n-2)\omega_n |x|^{n-2}}, & n \geq 3 \\
\frac{1}{2\pi} \ln |x|, & n = 2
\end{cases}, \quad \Gamma(x,t) = \begin{cases} 
\frac{1}{(4\pi t)^{\frac{n}{2}} e^{-|x|^2/4t}}, & t > 0, \\
0, & t < 0,
\end{cases}
\]

where \(\omega_n\) is the measure of the unit sphere in \(\mathbb{R}^n\).

**Proposition 2.4.** Let \(w\) be solution of (2.2)-(2.3) with \(g = 0\). Then,

\[
\|w(t)\|_{L^p(\mathbb{R}^n_+)} \leq c(\|\Gamma * Pf(t)\|_{L^p(\mathbb{R}^n_+)} + \|\Gamma^* * Pf(t)\|_{L^p(\mathbb{R}^n_+)}), \quad 0 < t < \infty,
\]
where \( P \) is Helmholtz decomposition in \( \mathbb{R}^n_+ \) and
\[
\Gamma * P f(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x-y,t-s)Pf(y,s)dyds,
\]
\[
\Gamma^* * P f(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x'-y',x_n+y_n,t-s)Pf(y,s)dyds.
\]

**Proof.** See the proof of Lemma 3.3 in [5]. \( \square \)

**Proposition 2.5.** Let \( u \) solution of (2.2) satisfies
\[
\left\| \nabla u \right\|_{L^q(\mathbb{R}^n_+; \mathbb{R}^n)} \leq c \left( \left\| F \right\|_{L^q(\mathbb{R}^n_+; \mathbb{R}^n)} + \left\| F \right\|_{L^q(\mathbb{R}^n_+; \mathbb{R}^n)} \right).
\]

**Proposition 2.6.** Let \( f = \text{div } F \) with \( F_{\text{in}}|_{x_n=0} = 0 \) and \( g = 0 \). Then, for \( 1 < p_0 \leq p < \infty \),
\[
\left\| \Gamma * P f(t) \right\|_{L^p(\mathbb{R}^n_+)} + \left\| \Gamma^* * P f(t) \right\|_{L^p(\mathbb{R}^n_+)} \leq c \int_0^t (t-s)^{-\frac{n}{2b}\left(\frac{1}{p_0}-\frac{1}{p}\right)} \left\| F(s) \right\|_{L^p_0(\mathbb{R}^n_+)} ds.
\]

**Proof.** See the proof of Lemma 3.7 in [4]. \( \square \)

With aid of Proposition 2.4 and Proposition 2.5, we obtain the following proposition.

**Proposition 2.7.** (\([7\) Proposition 2.3]) Let \( f = \text{div } F \) with \( F \in L^q(0, \infty; L^p(\mathbb{R}^n_+)) \), \( F|_{x_n=0} \in L^q(0, \infty; \dot{B}^{-\frac{1}{2}}_{pp}(\mathbb{R}^{n-1})) \) and \( g = 0 \). Then, for \( 1 < p, q < \infty \) the solution \( w \) of (2.2)-(2.3) satisfies
\[
\left\| \nabla w \right\|_{L^q(0, \infty; L^p(\mathbb{R}^n_+))} \leq c \left( \left\| F \right\|_{L^q(0, \infty; L^p(\mathbb{R}^n_+))} + \left\| F \right\|_{L^q(0, \infty; \dot{B}^{-\frac{1}{2}}_{pp}(\mathbb{R}^{n-1}))} \right).
\]

**Proposition 2.8.** ([3 Theorem 1.2]) Let \( g = 0 \) and \( f = \text{div } F \) with \( F \in L^q(\mathbb{R}^n \times \mathbb{R}^+), 1 < q < \infty \). Then there is a unique very weak solution \( w \in L^q(\mathbb{R}^n_+ \times (0, T)) \) of the Stokes equations (2.2) satisfying the following inequality
\[
\left\| w \right\|_{L^q(\mathbb{R}^n_+ \times (0, T))} \leq c T^{\omega \left( 1 - \frac{2q}{np} \right)} \max(1, T^{-\omega \left( 2q - np \right)}) \left\| F \right\|_{L^p(\mathbb{R}^n_+ \times (0, T))},
\]
where \( 0 < \omega_1 \leq 1 - \frac{n+2}{p} \).

**Proposition 2.9.** ([6 Theorem 2.2]) Let \( n+2 < p < \infty \). Let \( f = \text{div } F \) with \( F \in L^p(\mathbb{R}^n_+ \times (0, T)) \) and \( g \equiv 0 \). Then, there exists unique very weak solution, \( w \in L^\infty(\mathbb{R}^n_+ \times (0, T)) \), of the Stokes equations (2.2) such that
\[
\left\| w \right\|_{L^\infty(\mathbb{R}^n_+ \times (0, T))} \leq c T^{\omega \left( 1 - \frac{2q}{np} \right)} \max(1, T^{-\omega \left( 2q - np \right)}) \left\| F \right\|_{L^p(\mathbb{R}^n_+ \times (0, T))},
\]
where \( 0 < \omega_1 \leq 1 - \frac{n+2}{p} \).

**Proposition 2.10.** Let \( \frac{n+2}{2} < p < \infty \). Let \( f \in L^p(\mathbb{R}^n_+ \times (0, T)) \) and \( g = 0 \). Then, there exists unique very weak solution, \( w \in L^\infty(\mathbb{R}^n_+ \times (0, T)) \), of the Stokes equations (2.2) such that
\[
\left\| w \right\|_{L^\infty(\mathbb{R}^n_+ \times (0, T))} \leq c T^{\omega \left( 1 - \frac{2q}{np} \right)} \left\| F \right\|_{L^p(\mathbb{R}^n_+ \times (0, T))}.
\]

The proof of Proposition 2.10 is rather straightforward, and thus we skip its details.

We similarly consider the Navier-Stokes equations in a half-space, namely
\[
u_t - \Delta u + \nabla p = -\text{div}(u \otimes u), \quad \text{div } u = 0 \quad \text{in } Q^+ := \mathbb{R}^n_+ \times (0, \infty)
\]
with zero initial data and non-zero boundary values
\[
u|_{t=0} = 0, \quad u|_{x_n=0} = g.
\]

We mean by very weak solutions of the Navier-Stokes equations (2.7)-(2.8) distribution solutions, which are defined as follows:
Definition 2.11. Let $g \in L_{loc}^1(\partial \mathbb{R}^n_+ \times (0, \infty))$. A vector field $u \in L_{loc}^2(\mathbb{R}^n_+ \times (0, \infty))$ is called a very weak solution of the non-stationary Navier-Stokes equations (2.7)-(2.8), if the following equality is satisfied:

$$-\int_0^\infty \int_{\mathbb{R}^n_+} u \cdot \Delta \Phi dxdt = \int_0^\infty \int_{\mathbb{R}^n_+} (u \cdot \Phi_t + (u \otimes u) : \nabla \Phi) dxdt - \int_0^\infty \int_{\partial \mathbb{R}^n_+} g \cdot D_{x_n} \Phi dx'dt$$

for each $\Phi \in C^2_0(\mathbb{R}^n_+ \times [0, \infty))$ satisfying (2.4). In addition, for each $\Psi \in C^1(\mathbb{R}^n_+)$ with $\Psi|_{\partial \mathbb{R}^n_+} = 0$, $u$ satisfies (2.5).

3. Stokes equations with boundary data in a half-space

For convenience, we introduce a tensor $L_{ij}$ defined by

$$L_{ij}(x, t) = D_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} D_{x_i} \Gamma(z, t) D_{x_n} N(x - z) dz, \quad i, j = 1, 2, \ldots, n.$$ (3.1)

We recall the following relations on $L_{ij}$ (see (20)):

$$\sum_{i=1}^n L_{ii} = \frac{1}{2} D_{x_n} \Gamma, \quad L_{in} = L_{ni} - B_{in} \quad \text{if} \quad i \neq n,$$ (3.2)

where

$$B_{in}(x, t) = \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - z', x_n, t) D_{z_n} N(z', 0) dz'.$$ (3.3)

Furthermore, we remind an estimate of $L_{ij}$ defined in (5.1) (see (20))

$$|D_{x_n}^{\nu_0} x_n^{\nu_0} D_{x_n}^\nu L_{ij}(x, t)| \leq \frac{c}{\nu^{m_0 + \frac{1}{2}} (|x|^2 + t)^{n_0 + \frac{1}{2}(n_k^0 + n) t^{\frac{1}{2}}}},$$ (3.4)

where $1 \leq i \leq n$ and $1 \leq j \leq n - 1$.

It is known that the Poisson kernel $K$ of the Stokes equations is given as follows (see (20)):

$$K_{ij}(x' - y', x_n, t) = -2\delta_{ij} D_{x_n} \Gamma(x' - y', x_n, t) + 4L_{ij}(x' - y', x_n, t)$$

$$+ 2\delta_{jn} \delta(t) D_{x_n} N(x' - y', x_n),$$ (3.5)

where $\delta(t)$ is the Dirac delta function and $\delta_{ij}$ is the Kronecker delta function, and thus, the solution $w$ of the Stokes equations (2.2)-(2.3) with $f = 0$ is expressed by

$$w_i(x, t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^{n-1}} K_{ij}(x' - y', x_n, t - s) g_j(y', s) dy'ds.$$ (3.6)

Next, we will construct a solution $w$ of Stokes equations via (3.6) for a certain $g$ such that $w \in L^\infty$ but $L^p$-norm of $\nabla w$ is not bounded near boundary. For convenience, we denote

$$A = \{y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} | 3 < |y'| < 4\sqrt{n}, -4\sqrt{n} < y_i < -3, \quad 1 \leq i \leq n - 1\}.$$ (3.7)

We introduce a non-zero boundary data $g : \mathbb{R}^{n-1} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ with only $n$-th component defined as follows:

$$g(y', s) = (0, \cdots, 0, g_n(y', s)) = (0, \cdots, 0, a g_n^S(y') g_n^T(s)),$$

where $g_n^S \geq 0$ and $g_n^T \geq 0$ satisfy

$$g_n^S \in C^\infty_c(A), \quad \text{supp} \ g_n^T \subset \left(\frac{3}{4}, \frac{7}{8}\right), \quad g_n^T \in L^\infty(\mathbb{R}).$$ (3.8)

Here $\alpha > 0$ is a constant, which is specified later. In this section, we assume $\alpha = 1$, without loss of generality (in section 7 the parameter $\alpha$ will be taken sufficiently small).

In the next lemma, we estimate spatial derivatives of the convolution of $L_{ni}$ and $g_n$ up to the second order. For convenience, we denote

$$w_i^T(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} L_{ni}(x' - y', x_n, t - s) g_n(y', s) dy'ds, \quad 1 \leq i \leq n.$$ (3.9)
Lemma 3.1. Let \( g_n \) be given in (3.7) and (3.8), and \( w_i^c \) defined in (3.9). Then, for \( |x'| < 2 \) and \( t > 0 \), \( w_i^c \) satisfies the following estimates.

\[
\begin{align*}
|w_i^c(x, t)| & \leq c\|g\|_{L^\infty}, \quad 1 \leq i \leq n, \\
|\nabla w_i^c(x, t)| & \leq c\|g\|_{L^\infty}, \quad 1 \leq i \leq n-1, \\
|\nabla w_n^c(x, t)| & \leq c(1 + (\ln \frac{x_n^2}{t})_+)\|g\|_{L^\infty}, \\
|\nabla^2 w_i^c(x, t)| & \leq c(1 + (\ln \frac{x_n^2}{t})_+)\|g\|_{L^\infty}, \quad 1 \leq i \leq n-1.
\end{align*}
\]

Proof. Let \( 1 \leq i \leq n-1 \). Noting that for \( |x'| < 2 \) and \( y' \in A \), we have \( |x' - y'| > 1 \). Hence, due to the estimate (3.4), we have that

\[
|w_i^c(x, t)| + |D_{x_i} w_i^c(x, t)| \leq c\|g\|_{L^\infty}, \quad 1 \leq k \leq n-1, \\
|D_{x_k} D_{x_i} w_i^c(x, t)| \leq c\|g\|_{L^\infty}, \quad 1 \leq k, l \leq n-1.
\]

From (3.14), we get (3.10) for \( 1 \leq i \leq n-1 \).

Set \( f(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - z', x_n, t - s) g_n(z', s) dz' ds \). Then, we note that \( -B_{n}(x, t) = R_i^t f(x, t) \), where \( R_i^t \) is \( n-1 \) dimensional Riesz transform. From the second identity of (3.2), we observe

\[
w_i^c(x, t) = D_{x_n} D_{x_i} \int_0^t \int_{\mathbb{R}^{n-1}} N(x - y) f(y, t) dy - R_i^t f(x, t).
\]

On the other hand, we note that

\[
\Delta_x \int_0^t \int_{\mathbb{R}^{n-1}} N(x - y) f(y, t) dy = \frac{1}{2} f(x, t) + I(D_{x_n} f(\cdot, x_n, t))(x'),
\]

where \( I f(\cdot, x_n, t) = \int_{\mathbb{R}^{n-1}} N(x' - y', 0) f(y', x_n, t) dy' \). Since \( D_{x_n} I = R_i^t \), we have from (3.16)

\[
D_{x_n} w_i^c(x, t) = -\Delta_x D_{x_i} \int_0^t \int_{\mathbb{R}^{n-1}} N(x - y) f(y, t) dy \\
+ D_{x_i} \left( \frac{1}{2} f(x, t) + I(D_{x_n} f(x', x_n, t))(x') \right) - D_{x_n} R_i^t f(x, t)
\]

\[
= -\sum_{k=1}^{n-1} D_{x_k} w_{ik}^c(x, t) + \frac{1}{2} D_{x_i} f(x, t),
\]

where

\[
w_{ik}^c(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} L_{ik}(x' - y', x_n, t - s) g_n(y', s) dy' ds, \quad k = 1, 2, \ldots, n-1.
\]

It follows from (3.4) that, for \( |x'| < 2 \),

\[
|D_{x_i} w_{ik}^c(x, t)| + |D_{x_i} f(x, t)| \leq \|g_n\|_{L^\infty}.
\]

Summing up (3.14), (3.17) and (3.18), we obtain (3.11).

Similarly, for \( 1 \leq l \leq n \), we have from (3.17)

\[
D_{x_n} D_{x_i} w_i^c(x, t) = -\sum_{k=1}^{n-1} D_{x_i} w_{ik}^c(x, t) + \frac{1}{2} D_{x_i} D_{x_i} f(x, t).
\]

Here, we note that

\[
|D_{x_i} D_{x_i} f(x, t)| \leq c\|g\|_{L^\infty}.
\]

Again, using the estimate (3.4), for \( 1 \leq l \leq n \) and \( 1 \leq k \leq n-1 \), we get

\[
|D_{x_i} D_{x_k} w_{ik}^c(x, t)| \leq c \int_0^t (t - s)^{-\frac{1}{2}} (x_n^2 + t - s)^{-\frac{1}{2}} ds \|g\|_{L^\infty} \leq c(1 + (\ln \frac{x_n^2}{t})_+)\|g\|_{L^\infty}.
\]
Hence, combining estimates (3.15), (3.19), (3.20) and (3.21), we obtain (3.13).

Now, it remains to estimate \( w_n \). The first equality of (3.2) implies

\[
(3.22) \quad w_n^L(x, t) = -\sum_{k=1}^{n-1} w_{n,k}^L(x, t) - 2f(x, t).
\]

Hence, we observe that, for \( 1 \leq k \leq n \),

\[
(3.23) \quad |w_n^L(x, t)| \leq c ||g||_{L^\infty}, \quad |D_{x_k} w_n^L(x, t)| \leq c(1 + (\log t)^{\frac{n^2}{2}}) ||g||_{L^\infty}.
\]

Therefore, the estimates (3.10) and (3.12) are consequences of (3.23). We complete the proof.

Next lemma shows a pointwise control for \( n - 1 \) dimensional Riesz transform of \( e^{-\frac{|x'|^2}{4t}} \), which is one of crucial estimates in our analysis.

**Lemma 3.2.** Let \( 1 \leq |X'| \leq 5 \). Then,

\[
\int_{\mathbb{R}^{n-1}} e^{-\frac{|X'-z'|^2}{4t}} \frac{z_1}{|X'|^n} dz = c_{n-1} t^{\frac{n-1}{2}} \frac{|X_1|}{|X'|^n} + J(X', t),
\]

where \( |J(X', t)| \leq ct^{\frac{n}{2}} \) and \( c_{n-1} = (4\pi)^{\frac{n-1}{2}} \).

**Proof.** We divide \( \mathbb{R}^{n-1} \) by three disjoint sets \( D_1, D_2 \) and \( D_3 \), which are defined by

\[
D_1 = \left\{ z' \in \mathbb{R}^{n-1} : |X' - z'| \leq \frac{1}{10} |X'| \right\}, \\
D_2 = \left\{ z' \in \mathbb{R}^{n-1} : |z'| \leq \frac{1}{10} |X'| \right\}, \quad D_3 = \mathbb{R}^{n-1} \setminus (D_1 \cup D_2).
\]

We then split the following integral into three terms as follows:

\[
(3.24) \quad \int_{\mathbb{R}^{n-1}} e^{-\frac{|X'-z'|^2}{4t}} \frac{z_1}{|X'|^n} dz' = \int_{D_1} \cdots + \int_{D_2} \cdots + \int_{D_3} \cdots := J_1 + J_2 + J_3.
\]

Since \( \int_{D_2} \frac{z_1}{|z'|^n} dz' = 0 \), we have \( \int_{D_2} \frac{z_1}{|z'|^n} e^{-\frac{|X'-z'|^2}{4t}} dz' = \int_{D_2} \frac{z_1}{|z'|^n} (e^{-\frac{|X'-z'|^2}{4t}} - e^{-\frac{|X'|^2}{4t}}) dz' \). Thus, using the mean-value Theorem, we have

\[
|J_2| = \left| \int_{D_2} \frac{z_1}{|z'|^n} (e^{-\frac{|X'-z'|^2}{4t}} - e^{-\frac{|X'|^2}{4t}}) dz' \right| \leq ct^{-1} |X'| e^{-\frac{|X'|^2}{4t}} \int_{D_2} \frac{1}{|z'|^{n-2}} dz' \leq ct^{-1} |X'|^2 e^{-\frac{|X'|^2}{4t}} \leq ce^{-\frac{|X'|^2}{4t}}.
\]

Since \( \int_{|z'| > a} e^{-\frac{|z'|^2}{4t}} dz' \leq c_1 e^{-c_2a^2}, a > 0 \), we have

\[
(3.26) \quad |J_3| \leq \frac{c}{|X'|^n} \int_{\left\{ |z' - X'| \geq \frac{1}{10} |X'| \right\}} e^{-\frac{|z' - X'|^2}{4t}} dz' \leq \frac{ct^{\frac{n-1}{2}}}{|X'|^{n-2}} e^{-\frac{|X'|^2}{4t}} \leq ce^{-\frac{|X'|^2}{4t}}.
\]

Due to \( 1 \leq |X'| \leq 5 \), it follows from (3.25) and (3.26) that

\[
(3.27) \quad |J_2(X', t)| + |J_3(X', t)| \leq ce^{-\frac{|X'|}{4t}}.
\]
Now, we estimate \( J_1 \). Firstly, we decompose \( J_1 \) in the following way:

\[
J_1 = \int_{D_1} e^{-\frac{|x'-z'|^2}{4}} \frac{z_1}{|z'|^n} dz' = (4t)^{n-\frac{1}{2}} \int_{\{|z'| \leq \frac{1}{10} \frac{|X'|}{|z'|} \}} e^{-|z'|^2} \frac{X_1 - 2t \frac{z_1}{z'} x_1}{|X' - 2t \frac{z_1}{z'} z'|^n} dz' \\
= (4t)^{n-\frac{1}{2}} \int_{\{|z'| \leq \frac{1}{10} \frac{|X'|}{|z'|} \}} e^{-|z'|^2} \left( \frac{X_1 - 2t \frac{z_1}{z'} x_1}{|X' - 2t \frac{z_1}{z'} z'|^n} - \frac{X_1}{|X'|^n} \right) dz' \\
- (4t)^{n-\frac{1}{2}} \frac{X_1}{|X'|^n} \int_{\{|z'| \geq \frac{1}{10} \frac{|X'|}{|z'|} \}} e^{-|z'|^2} dz' + (4t)^{n-\frac{1}{2}} \frac{X_1}{|X'|^n} \int_{\mathbb{R}^{n-1}} e^{-|z'|^2} dz' \\
= J_{11} + J_{12} + J_{13}.
\]

We observe that

\[
|J_{11}(X', t)| \leq ct^2, \quad |J_{12}(X', t)| \leq e^{-\frac{c|X'|^2}{t}} \leq ct^2.
\]

Here we set \( c_{n-1} := \int_{\mathbb{R}^{n-1}} e^{-|z'|^2} dz' = \pi^{\frac{n-1}{2}} \) and take \( J := J_2 + J_3 + J_{11} + J_{12} \). Then, combining (3.24), (3.27), (3.28) and (3.29), we complete the proof of Lemma 3.2.

\[\Box\]

4. PROOF OF THEOREM 1.1 FOR STOKES EQUATIONS

The case for the Stokes system in Theorem 1.1 can be verified by the following proposition, where a class of boundary data for temporal variable is specified to show that velocity is bounded but its gradient is not integrable in \( L^q \) near boundary. The case of the Navier-Stokes equations will be treated in subsection 7.1.

**Proposition 4.1.** Let \( 1 < p < \infty \) and \( q \) satisfy (3.7) and (3.8). Assume further that \( g_n^T \in L^\infty(\mathbb{R}) \setminus \dot{B}^{\frac{1}{2}}_{pp} \dot{B}^{\frac{1}{2p}}_{pp}(\mathbb{R}) \). Suppose that \( w \) is a solution of the Stokes equations (2.2), (2.3) defined by (3.5) with \( f = 0 \) and the boundary data \( g \). Then, \( w \) is bounded in \( B_2^+ \times (0, 4) \) but the normal derivative of tangential components for \( w \) are unbounded in \( L^p(Q_2^+) \), i.e. \( w \) satisfies

\[
\|w\|_{L^\infty(B_2^+ \times (0,4))} < \infty,
\]

\[
\int_0^1 \int_{B_2^+} |D_{x_n} w_i(x, t)|^p dx dt = \infty, \quad i = 1, \ldots, n - 1.
\]

**Remark 4.2.** In Appendix C we give an example, for a clearer understanding, of a function \( g_n^T \in L^\infty(\mathbb{R}) \setminus \dot{B}^{\frac{1}{2}}_{pp} \dot{B}^{\frac{1}{2p}}_{pp}(\mathbb{R}) \), \( 1 < p < \infty \).

**Proof.** We prove only the case that \( i = 1 \), since the arguments are similar for other cases. From (3.2) and (3.5), we have

\[
w_1(x, t) = 4 \int_0^t \int_A L_{n1}(x' - y', x_n, t - s) g_n(y', s) dy' ds \\
+ 4 \int_0^t \int_A B_{n1}(x' - y', x_n, t - s) g_n(y', s) dy' ds \\
- 2 \int_A D_{x_1} N(x' - y', x_n) g_n(y', t) dy' \\
:= w_1^T(x, t) + w_1^B(x, t) + w_1^N(x, t).
\]

We note that \( 1 \leq |x' - y'| \leq 4 \sqrt{n} \) for \( |x'| \leq 2 \) and \( y' \in A \), and thus, for \( (x, t) \in B_2^+ \times (0, 4) \) it is direct that

\[
\|w_1^N(x, t)\|_{L^p(B_2^+ \times (0,4))} + \|\nabla w_1^N(x, t)\|_{L^p(B_2^+ \times (0,4))} \leq c \|g_n\|_{L^\infty} \quad 1 < p < \infty.
\]
From Lemma \[3.1\] it is straightforward that for \(1 < r < \infty\),
\[
\|w^F_r\|_{L^\infty(B_2^+ \times (0,4))} + \|\nabla w^F_r\|_{L^r(B_2^+ \times (0,4))} \leq c\|g_n\|_{L^\infty}.
\]

Next, we estimate \(w_r^B\). Since \(1 \leq |x' - y'| \leq 4\sqrt{n}\), reminding \((3.3)\) and Lemma \[3.2\] we note that
\[
w_r^B(x,t) = c_n \int_0^t \int_{\mathbb{R}^{n-1}} g_n(y',s) \frac{x_n}{(t-s)^{\frac{1}{2}}} e^{-\frac{x_n^2}{4(t-s)}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y'-z'|^2}{4(t-s)}} \frac{z_1}{|z'|^n} dz' dy' ds
\]
\[
\quad := w_r^{B,1}(x,t) + w_r^{B,2}(x,t),
\]
where
\[
w_r^{B,1}(x,t) = c_n \int_0^t \int_{\mathbb{R}^{n-1}} g_n^T(s) \frac{x_n}{(t-s)^{\frac{1}{2}}} e^{-\frac{x_n^2}{4(t-s)}} \int_{\mathbb{R}^{n-1}} g_n^S(y') \frac{x_1 - y_1}{|x' - y'|^{n-1}} dy' ds,
\]
\[
w_r^{B,2}(x,t) = c_n \int_0^t \int_{\mathbb{R}^{n-1}} g_n^T(s) \frac{x_n}{(t-s)^{\frac{1}{2}}} e^{-\frac{x_n^2}{4(t-s)}} \int_{\mathbb{R}^{n-1}} g_n^S(y') J(x' - y', t - s) dy' ds.
\]

Since \(1 \leq |x' - y'| \leq 5\), it follows from Lemma \[3.2\] that
\[
|w_r^{B,1}(x,t)| \leq c_n \int_0^t e^{-\frac{x_n^2}{4(t-s)}} \left( \frac{x_n}{(t-s)^{\frac{1}{2}}} + \frac{x_n}{t-s} \right) ds \|g_n\|_{L^\infty} \leq c_n \|g_n\|_{L^\infty}.
\]

Thus, due to \((3.3)\), \((4.4)\) and \((4.9)\), we obtain \((4.1)\). Noting that
\[
D_{x'} w_r^{B}(x,t) = c_n \int_0^t \int_{\mathbb{R}^{n-1}} D_y g_n(y',s) \frac{x_n}{(t-s)^{\frac{1}{2}}} e^{-\frac{x_n^2}{4(t-s)}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y|-z'|^2}{4(t-s)}} \frac{z_1}{|z'|^n} dz' dy' ds,
\]
we similarly have
\[
|D_{x'} w_r^{B}(x,t)| \leq c_n \|D_y g_n\|_{L^\infty}.
\]

Using Lemma \[3.2\] we have
\[
D_{x_n} w_r^{B,2}(x,t) = c_n \int_0^t \int_{\mathbb{R}^{n-1}} g_n(y',s) \frac{1}{(t-s)^{\frac{1}{2}}} e^{-\frac{x_n^2}{4(t-s)}} (1 - \frac{x_n^2}{t-s}) J(x' - y', t - s) ds
\]
\[
\quad \leq c \int_0^t \frac{1}{t-s} e^{-\frac{x_n^2}{4(t-s)}} (1 - \frac{x_n^2}{t-s}) ds \|g_n\|_{L^\infty}
\]
\[
\quad \leq c(1 + |\ln \frac{x_n^2}{t}|) \|g_n\|_{L^\infty}.
\]

On the other hand, we note that
\[
w_r^{B,1}(x,t) = c_n \int_0^t D_{x_n} \Gamma_1(x_n, t - s) g_n^T(s) ds \psi(x'),
\]
where \(\Gamma_1\) is one dimensional Gaussian kernel and \(\psi(x') = \int_{\mathbb{R}^{n-1}} \frac{x_n}{|x'-y'|^{n-1}} g_n^S(y') dy'\), which is smooth in \(|x'| \leq 2\). Since \(\text{supp} g_n^T \subset (\frac{2}{3}, \frac{7}{8})\), we have for \(t > 1\)
\[
\left| \int_0^t D_{x_n}^2 \Gamma_1(x_n, t - s) g_n^T(s) ds \right| \leq t^{-\frac{3}{2}} e^{-\frac{x_n^2}{t}} \|g_n^T\|_{L^\infty}
\]
and for \(x_n > 1\), we have
\[
\left| \int_0^t D_{x_n}^2 \Gamma_1(x_n, t - s) g_n^T(s) ds \right| \leq c\|g_n^T\|_{L^\infty} \begin{cases} x_n^{-1}, & t < 1, \\ \frac{t^{-\frac{3}{2}} e^{-\frac{x_n^2}{t}}}{t}, & t > 1. \end{cases}
\]
This implies that
$$
\int_1^\infty \int_0^\infty | \int_0^t D_x^2 \Gamma_1(x_n, t-s) g_n^T(s) ds |^p dx_n dt \leq c \| g_n^T \|_{L^\infty(\mathbb{R})},
$$
(4.13)
$$
\int_1^\infty \int_0^\infty | \int_0^t D_x^2 \Gamma_1(x_n, t-s) g_n^T(s) ds |^p dt dx_n \leq c \| g_n^T \|_{L^\infty(\mathbb{R})}.
$$

By the trace theorem of anisotropic space (see (3) of Proposition 2.1), we have
$$
\| g_n^T \|_{B^\beta_p(\mathbb{R}^n)_x} \leq c \| \int_0^t D_x \Gamma_1(x_n, t-s) g_n^T(s) ds \|_{L^p(\mathbb{R}^n)} \| \psi \|_{L^p(B^1_x)}
$$
(4.14)
$$
\leq c \| D_x \int_0^t D_x \Gamma_1(x_n, t-s) g_n^T(s) ds \|_{L^p(\mathbb{R}^n \times \mathbb{R})}.
$$

With the aid of (4.12), (4.13) and (4.14), we conclude that
$$
\| D_x w^\Gamma_x \|_{L^p(Q^+_\alpha)} \geq c \| D_x \int_0^t D_x \Gamma_1(x_n, t-s) g_n^T(s) ds \|_{L^p(\mathbb{R}^n \times \mathbb{R})} \| \psi \|_{L^p(B^1_x)}
$$
$$
- c \| g_n \|_{L^\infty(\mathbb{R}^n \times (0,\infty))}
$$
$$
\geq c \| g_n^T \|_{B^\beta_p(\mathbb{R}^n)_x} \| \psi \|_{L^p(B^1_x)} - c \| g_n \|_{L^\infty(\mathbb{R}^n \times (0,\infty))}.
$$

Since the righthand side is unbounded, we obtain (4.2). Hence, we complete the proof Proposition 4.1. □

5. PROOF OF THEOREM 1.5

Before we prove Theorem 1.5 for the Stokes system, we first show an elementary estimate, which is useful for our purpose.

Lemma 5.1. For 1 < p < \infty, 1 \leq r \leq \infty and 0 < \beta < 1,
$$
\| \nabla \Gamma * f(t) \|_{B^\beta_p(\mathbb{R}^n)_y} + \| \nabla \Gamma^* * f(t) \|_{B^\beta_p(\mathbb{R}^n)_x} \leq c \int_0^t (t-s)^{\frac{1}{2} - \frac{\beta}{p}} \| f(s) \|_{L^p(\mathbb{R}^n)} ds,
$$
where
$$
\Gamma * f(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x - y, t-s) f(y,s) dy ds,
$$
$$
\Gamma^* * f(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x' - y', x_n + y_n, t-s) f(y,s) dy ds.
$$

Proof. We prove only the case \Gamma * f, since the other case can be treated similarly. For h \in L^p(\mathbb{R}^n), we have
$$
\| \nabla \Gamma_t \* h \|_{L^p(\mathbb{R}^n)} \leq c t^{-\frac{1}{2}} \| h \|_{L^p(\mathbb{R}^n)} \quad 0 < t < \infty,
$$
where k is a non-negative integer and
$$
\Gamma_t \* h(x) = \int_{\mathbb{R}^n} \Gamma(x - y, t) h(y) dy.
$$
Using the property of real interpolation between k = 1 and k = 2, we have
$$
\| \nabla \Gamma_t \* h \|_{B^\beta_p(\mathbb{R}^n)_y} \leq c t^{-\frac{1}{2} - \frac{\beta}{p}} \| h \|_{L^p(\mathbb{R}^n)}.
$$
Let $\tilde{f}(t)$ be a zero extension of $f(t)$ over $\mathbb{R}^n$. Then, we have

$$
\|\nabla \Gamma * f(t)\|_{B^0_{p\prime}(\mathbb{R}^n_+)} \leq \|\nabla \Gamma * \tilde{f}(t)\|_{B^0_{p\prime}(\mathbb{R}^n_+)} \leq c \int_0^t \|\nabla \Gamma_{t-s} * f(s)\|_{B^0_{p\prime}(\mathbb{R}^n)} ds
$$

$$
\leq c \int_0^t (t-s)^{-\frac{n}{2} - \frac{3}{q}} \|f(s)\|_{L^p(\mathbb{R}^n)} ds
$$

$$
\leq c \int_0^t (t-s)^{-\frac{n}{2} - \frac{3}{q}} \|f(s)\|_{L^p(\mathbb{R}^n)} ds.
$$

Therefore, we complete the proof of Lemma 5.1.

To prove Theorem 1.5, we change the local problem into a problem in a half-space by multiplying a test function. Using Bogoski’s formula to control non-divergence free term caused localization, we appropriately decompose the solution to compute estimates of Hölder continuity.

Firstly, let $\phi_1 \in C_c^\infty (\mathbb{R}^n)$ be a cut-off function satisfying $\phi_1 \geq 0$, supp $\phi_1 \subset B_{\frac{1}{2}}$ and $\phi_1 \equiv 1$ in $B_{\frac{3}{4}}$. Also, let and $\phi_2 \in C_c^\infty (-\infty, \infty)$ be a cut-off function satisfying $\phi_2 \geq 0$, supp $\phi_2 \subset (\frac{1}{2}, 2)$ and $\phi_2 \equiv 1$ in $(\frac{3}{4}, 1)$. Let $\phi(x, t) = \phi_1(x)\phi_2(t)$. Let $U = u\phi$ and $\Pi = \pi\phi$ such that $U|_{Q_t^+} = u$ and $\Pi|_{Q_t^+} = \pi$. Then, $(U, \Pi)$ satisfies the following equations:

$$
U_t - \Delta U + \nabla \Pi = \tilde{f}, \quad \text{div } U = h \quad \text{in } \mathbb{R}^n_+ \times (0, 1),
$$

$$
U|_{t=0} = 0, \quad U|_{x_n=0} = 0,
$$

where

$$
\tilde{f} = -2(\nabla u)\nabla \phi - \Delta \phi u + \phi_t u + \pi \nabla \phi, \quad h = \nabla \phi \cdot u.
$$

We note that $\tilde{f}$, $h$, $\nabla h$, $h_t \in L^p_t L^q_x (\mathbb{R}^n_+ \times (0, 1))$, $1 < r < \infty$ with

$$
\|\tilde{f}\|_{L^p_t L^q_x (\mathbb{R}^n_+ \times (0, 1))} \leq c(\|\nabla u\|_{L^q_t L^q_x (Q_t^+)} + \|u\|_{L^q_t L^q_x (Q_t^+)} + \|\pi\|_{L^q_t L^q_x (Q_t^+)}),
$$

$$
\|h\|_{L^p_t L^q_x (\mathbb{R}^n_+ \times (0, 1))} \leq c\|u\|_{L^q_t L^q_x (Q_t^+)},
$$

$$
\|\nabla h\|_{L^p_t L^q_x (\mathbb{R}^n_+ \times (0, 1))} \leq c(\|u\|_{L^q_t L^q_x (Q_t^+)} + \|\nabla u\|_{L^q_t L^q_x (Q_t^+)})
$$

$$
\|h_t\|_{L^p_t L^q_x (\mathbb{R}^n_+ \times (0, 1))} \leq c(\|u\|_{L^q_t L^q_x (Q_t^+)} + \|u_t\|_{L^q_t L^q_x (Q_t^+)})
$$

(5.1)

For the fifth inequality, we used Proposition 1 in [18].

Let $H(x, t) = \int_{\mathbb{R}^n} E(x, y)h(y, t)dy$ be a Bogoski’s formula (see [9]) such that

$$
\text{div } H(\cdot, t) = h(\cdot, t), \quad \text{in } \mathbb{R}^n_+ \quad \text{and } \quad H(\cdot, t)|_{x_n=0} = 0 \quad \text{on } \ (x_n = 0).
$$

Since $h(t) \in W^{1, p}_0 (\mathbb{R}^n_+)$ for all $0 < t < \infty$, $H(t)$ satisfies

$$
\|\nabla^k H(t)\|_{L^r(\mathbb{R}^n)} \leq c\|\nabla^{k-1} h(t)\|_{L^r(\mathbb{R}^n)} \leq c\|\nabla^{k-1} h(t)\|_{L^r(Q_t^+)} , \quad k = 1, 2 \quad 1 < r < \infty,
$$

(5.2)

$$
\|H_t(t)\|_{L^{r*}(\mathbb{R}^n)} \leq c\|\nabla H_t(t)\|_{L^r(\mathbb{R}^n)} \leq c\|H_t(t)\|_{L^r(\mathbb{R}^n)} \leq c\|H_t(t)\|_{L^r(Q_t^+)} ,
$$

where $r^* = \frac{nr}{n-r}$ for $r < n$ (See Chapter 3 in [9]). Take $1 < r < \infty$ and $0 < \epsilon$ such that $\alpha + \frac{r}{r} + \epsilon < 1$. Using the Besov imbedding and the property of real interpolation in (5.2), we obtain

$$
\|\nabla H\|_{L^{q\prime}_t C^\epsilon_q (\mathbb{R}^n_+ \times (0, 1))} \leq c(\|\nabla H\|_{L^q_t B^{\alpha+\frac{2}{p}}_{q,r} (\mathbb{R}^n_+ \times (0, 1))} \leq c\|h\|_{L^q_t C^{\alpha+\frac{2}{p}+\epsilon}_q (Q_t^+)})
$$

(5.3)
Note that supp \( h \subset B_{\frac{1}{\sqrt{2}}} \setminus B_{\frac{3}{8}} \). Since \( |E(x, y)| \leq c \) for \( |x| < \frac{1}{4} \) and \( |y| > \frac{3}{4} \), in case that \( |x| < \frac{1}{4} \), we observe that

\[
\begin{aligned}
|\nabla H(x, t) - \nabla H(x, s)| & \leq \int_{B_{\frac{1}{\sqrt{2}}} \setminus B_{\frac{3}{8}}} |E(x, y)(h(y, t) - h(y, s))| dy \\
& \leq \|h\|_{C_t^q L_x^p(Q^+_\frac{1}{\sqrt{2}})} |t - s|^\frac{q}{4}.
\end{aligned}
\]

Hence, we have

\[
\|\nabla H\|_{C_t^q L_x^p(Q^+_\frac{1}{4})} \leq c\|h\|_{C_t^q L_x^p(Q^+_\frac{1}{\sqrt{2}})}.
\]

Since \( \alpha + \frac{q}{r} + \epsilon < 1 \), from Proposition 2 and Lemma 1 in [18], there are \( 1 < q_0 < 2 \) and \( 1 < p_0 < p \) such that

\[
\|h\|_{L^\infty_t C_{x}^{\alpha + \frac{q}{r} + \epsilon}(Q^+_\frac{1}{4})} \leq c\|h\|_{L^\infty_t C_{x}^{\alpha + \frac{q}{r} + \epsilon}(Q^+_\frac{1}{\sqrt{2}})}
\]

\[
\leq c\left(\|\nabla u\|_{L^{q_0}_t L^{p_0}_x(Q^+_\frac{1}{4})} + \|u\|_{L^{q_0}_t L^{p_0}_x(Q^+_\frac{1}{4})} + \|\nabla \Pi\|_{L^{q_0}_t L^{p_0}_x(Q^+_\frac{1}{4})}\right)
\]

\[
\leq c\left(\|\nabla u\|_{L^q_t L^p_x(Q^+_1)} + \|u\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla \Pi\|_{L^q_t L^p_x(Q^+_1)}\right).
\]

Summing up all estimates, we have

\[
\|\nabla H\|_{C_t^q C_x^q(Q^+_1)} \leq c\left(\|\nabla u\|_{L^q_t L^p_x(Q^+_1)} + \|u\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla \Pi\|_{L^q_t L^p_x(Q^+_1)}\right).
\]

We then decompose \( U = H + W \) in \( Q^+_\frac{1}{4} \) such that \( W \) solves the following equations:

\[
\begin{aligned}
W_t - \Delta W + \nabla \Pi = \tilde{f}_1, & \quad \text{in} \quad Q^+_1, \\
\text{div} W = 0, & \quad \text{in} \quad Q^+_1, \\
W|_{t=0} = 0, & \quad W|x_n=0 = 0,
\end{aligned}
\]

where \( Q^+_1 := \mathbb{R}^n_+ \times (0, 1) \) and \( \tilde{f}_1 = \tilde{f} - H_t + \Delta H \). If \( n \leq p < r \), then choose \( p_1 < n \) such that \( r = \frac{np_1}{n-p_1} \).

Then, from (5.1), (5.2), Sobolev imbedding and Proposition 1 in [18], we have

\[
\begin{aligned}
\|\tilde{f}_1\|_{L^q_t L^p_x(Q^+_1)} & \leq c\left(\|\nabla u\|_{L^q_t L^p_x(Q^+_1)} + \|u\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla \Pi\|_{L^q_t L^p_x(Q^+_1)}\right)
\leq c\left(\|\nabla^2 u\|_{L^{q_0}_t L^{p_0}_x(Q^+_\frac{1}{4})} + \|\nabla u\|_{L^{q_0}_t L^{p_0}_x(Q^+_\frac{1}{4})} + \|\nabla \Pi\|_{L^{q_0}_t L^{p_0}_x(Q^+_\frac{1}{4})}\right)
\leq c\left(\|\nabla u\|_{L^q_t L^p_x(Q^+_1)} + \|u\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla \Pi\|_{L^q_t L^p_x(Q^+_1)}\right).
\end{aligned}
\]

(5.5)

From well-known result (see Theorem 1.1 in [21]), we have

\[
\|\nabla^2 W\|_{L^q_t L^p_x(Q^+_1)} + \|D_t W\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla \Pi\|_{L^q_t L^p_x(Q^+_1)} \leq c\|\tilde{f}_1\|_{L^q_t L^p_x(Q^+_\frac{1}{4})}.
\]

Then, due to (5.1), (5.2) and (5.5), we have

\[
\begin{aligned}
\|\nabla^2 u\|_{L^q_t L^p_x(Q^+_1)} + \|D_t u\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla \Pi\|_{L^q_t L^p_x(Q^+_1)}
\leq c\|\tilde{f}_1\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla^2 H\|_{L^q_t L^p_x(Q^+_\frac{1}{4})} + \|H_t\|_{L^q_t L^p_x(Q^+_\frac{1}{4})}
\leq c\left(\|\tilde{f}_1\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla h\|_{L^q_t L^p_x(Q^+_\frac{1}{4})} + \|h_t\|_{L^q_t L^p_x(Q^+_\frac{1}{4})}\right)
\leq c\left(\|\nabla u\|_{L^q_t L^p_x(Q^+_1)} + \|\nabla \Pi\|_{L^q_t L^p_x(Q^+_1)}\right).
\end{aligned}
\]

Hence, we obtain (1.8) for \( p \geq n \).
In case that \( p < n \), we set \( p^* = \frac{np}{n-p} \). If \( p^* > n \) then for the same reason as the case \( p \geq n \), we have

\[
\|\nabla^2 u\|_{L^p_tL^n\nu (Q^+_4)} + \|D_1 u\|_{L^p_tL^n\nu (Q^+_4)} + \|\nabla \pi\|_{L^p_tL^n\nu (Q^+_4)} \\
\leq c\left(\|\nabla u\|_{L^p_tL^n\nu (Q^+_4)} + \|\pi\|_{L^p_tL^n\nu (Q^+_4)} + \|u\|_{L^p_tL^n\nu (Q^+_4)}\right).
\]

If \( p^* < n \), then iterating the process up to \( p^* > n \), we obtain (1.8).

Using Proposition 2.4 and the property of real interpolation, for \( 1 < r_1 < \infty, \ 1 \leq r_2 \leq \infty \) and \( \alpha > 0 \), \( W \) holds the following estimate

\[
\|W(t)\|_{L^{r_1}_{x,t}(\mathbb{R}^n_+)} \leq c\left(\|\Gamma * \mathbb{P}\tilde{f}_1(t)\|_{L^{r_1}_{x,t}(\mathbb{R}^n_+)} + \|\Gamma * \mathbb{P}\tilde{f}_1(t)\|_{L^{r_1}_{x,t}(\mathbb{R}^n_+)}\right).
\]

From the Besov embedding, (5.6) and Lemma 5.1 we have

\[
\|\nabla W(t)\|_{C^\alpha(\mathbb{R}^n_+)} \leq c\|\nabla W(t)\|_{B^{\alpha+\frac{n}{p}}_{r_1} (\mathbb{R}^n_+)} \leq c\int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{q} - \frac{n}{p}} \|\tilde{f}_1(s)\|_{L^r(\mathbb{R}^n_+)} ds,
\]

\[
\|\nabla W(t)\|_{L^{\infty}(\mathbb{R}^n_+)} \leq \|\nabla W(t)\|_{B^{\alpha+\frac{n}{p}}_{r_1} (\mathbb{R}^n_+)} \leq c\int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{q} - \frac{n}{p}} \|\tilde{f}_1(s)\|_{L^r(\mathbb{R}^n_+)} ds
\]

\[(5.8)\]

Combining (5.5), (5.6), (5.7) and (5.8) and H"{o}lder inequality, and taking \( r < \infty \) satisfying \( \frac{n}{r} + \frac{n}{q} < 1 - \alpha \), we obtain

\[
\|\nabla W\|_{L^{\infty}C^\alpha\nu (Q^+_n)} \leq c\|\tilde{f}_1\|_{L^p_tL^n\nu (Q^+_4)} \leq c\left(\|\nabla u\|_{L^p_tL^n\nu (Q^+_4)} + \|\pi\|_{L^p_tL^n\nu (Q^+_4)} + \|u\|_{L^p_tL^n\nu (Q^+_4)}\right).
\]

Next, we compute H"{o}lder continuous estimate with respect to \( t \). From (21), there is the kernel \( K^P \) such that \( W \) is represented by

\[
W(x,t) = \int_0^t \int_{\mathbb{R}^n_+} K^P(x,y,t-s)\mathbb{P}\tilde{f}_1(y,s) dyds,
\]

where

\[
K^P_{ij}(x,y,t) = \delta_{ij}(\Gamma(x-y,t) - \Gamma(x-y^*,t)) + 4(1 - \delta_{jn}) D_{xj} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \Gamma(x-y^* - z, t) D_{x_i} N(z) dz.
\]

Furthermore, it is known that \( K^P \) satisfies that for all \( k \in \mathbb{N} \cup \{0\}, l = (l', l_n) \in (\mathbb{N} \cup \{0\})^n \), (see Proposition 2.5 in [21])

\[
|D^l D^{l_n}_{x_n} D^{l'_n}_{x'_n} K^P(x,y,t)| \leq \frac{ce^{-\xi_1 \frac{y^*}{t}}}{t^k(t + x_n^2)^{\frac{n}{2}}(|x - y^*|^2 + t)^{\frac{n}{2} + l'}}.
\]

We remark that one can also refer [14] for the representation formula via unrestricted Green tensor. Since we use \( L^p \)-type estimate of \( f \), not the pointwise estimate of \( f \), the formula (5.10) with the restricted Green tensor \( K^P \) is enough for our purpose. Continuing computations, it follows that

\[
\nabla_x W(x,t) - \nabla_x W(x,s) = \int_s^t \int_{\mathbb{R}^n_+} (\nabla_x K^P(x,y,t - \tau) - \nabla_x K^P(x,y,s - \tau))\mathbb{P}\tilde{f}_1(y,\tau) dyd\tau
\]

\[
+ \int_s^t \int_{\mathbb{R}^n_+} \nabla_x K^P(x,y,t - \tau)\mathbb{P}\tilde{f}_1(y,\tau) dyd\tau
\]

\[(5.12)\]

:= I_1 + I_2.
Using Hölder inequality, we first estimate $I_2$

$$|I_2| \leq c \int_s^t \left( \int_{\mathbb{R}^n_+} |\nabla_x K^P(x, y, t - \tau)|' dy \right) \frac{1}{\tau'} \left\| \mathbb{P} f_1(\tau) \right\|_{L^q(\mathbb{R}^n_+)} d\tau.$$  

It follows from (5.11) that

$$\int_{\mathbb{R}^n_+} |\nabla_x K^P(x, y, t - \tau)|' dy \leq c \int_{\mathbb{R}^n_+} \left( |x - y|^2 + (t - \tau) \right)^{-\frac{n'}{2} - \frac{n}{2} + \frac{n}{2}} e^{-c_1 \frac{y^2}{2\tau}} dy$$

$$\leq c \int_0^{\infty} \left( x_n^2 + t - \tau \right)^{-\frac{n'}{2} - \frac{n}{2} + \frac{n}{2}} e^{-c_1 \frac{y^2}{2\tau}} dy_n$$

$$= (t - r)^{-\frac{n'}{2} - \frac{n}{2} + \frac{n}{2}}.$$  

Hence, for $1 < r < \infty$ satisfying $1 > \frac{n}{r} + \frac{2}{q}$, we have

$$|I_2| \leq c \int_s^t (t - \tau)^{-\frac{1}{2} - \frac{2n}{q}r} \left\| f_1(\tau) \right\|_{L^r(\mathbb{R}^n_+)} d\tau$$

$$\leq c \left( \int_s^t (t - \tau)^{-\frac{1}{2} - \frac{2n}{q}r} d\tau \right)^{\frac{1}{r}} \left\| f_1(\tau) \right\|_{L^r(\mathbb{R}^n_+)}$$

$$= c (t - s)^{-\frac{1}{2} - \frac{2n}{q}r} \left\| f_1(\tau) \right\|_{L^r(\mathbb{R}^n_+)}.$$  

(5.13)

Next, we estimate $I_1$. Using Hölder inequality, it follows that

$$I_1 = c \int_0^{t-s} \int_0^t \int_{\mathbb{R}^n_+} D_t \nabla_x K^P(x, y, \eta + s - \tau) \mathbb{P} f_1(y, \tau) dy dy d\tau$$

$$\leq c \int_0^{t-s} \left( \int_{\mathbb{R}^n_+} |D_t \nabla_x K^P(x, y, \eta + s - \tau)|' dy \right)^{\frac{1}{r'}} \left\| f_1(\tau) \right\|_{L^r(\mathbb{R}^n_+)} d\tau.$$  

Due to (5.11), we have

$$\int_{\mathbb{R}^n_+} |D_t \nabla_x K^P(x, y, \eta + s - \tau)|' dy$$

$$\leq c \int_{\mathbb{R}^n_+} \left( \eta + s - \tau \right)^{\frac{n'}{2} + \frac{n}{2}} e^{-c_1 \frac{y^2}{2\tau}} dy$$

$$\leq c \int_0^{\infty} \left( \eta + s - \tau \right)^{\frac{n}{2} + \frac{n}{2}} e^{-c_1 \frac{y^2}{2\tau}} dy_n$$

$$\leq \frac{c}{(\eta + s - \tau)^{\frac{n}{2} + \frac{n}{2}}}.$$  

Using change of variable, we have

$$\int_0^{t-s} (\eta + s - \tau)^{-\frac{n}{2} - \frac{n}{2r}} d\eta = (s - \tau)^{-\frac{n}{2} - \frac{n}{2r}} \int_0^{\tau} (s + 1)^{-\frac{n}{2} - \frac{n}{2r}} d\eta$$

$$\leq \begin{cases} c(t - s)(s - \tau)^{-\frac{n}{2} - \frac{n}{2r}} & \text{if } \tau < 2s - t, \\ c(s - \tau)^{-\frac{n}{2} - \frac{n}{2r}} & \text{if } \tau > 2s - t. \end{cases}$$  

We note that for $1 > \frac{n}{r} + \frac{2}{q}$

$$(t - s) \left( \int_0^{2s-t} (s - \tau)^{-\frac{1}{2} - \frac{n}{2r}} d\tau \right)^{\frac{1}{r'}} \leq c(t - s)^{\frac{3}{2} - \frac{n}{2r} - \frac{1}{q}},$$

$$\left( \int_0^{s} (s - \tau)^{-\frac{1}{2} - \frac{n}{2r}} d\tau \right)^{\frac{1}{r'}} \leq c(t - s)^{-\frac{1}{2} - \frac{n}{2r} - \frac{1}{q}}.$$
Hence, we have
\begin{equation}
|I_1| \leq c(t-s)^{\frac{3}{2}-\frac{n}{2p} - \frac{2}{q}} \|\tilde{f}_1\|_{L^p_t L^q_x(Q_+)}.
\end{equation}
Due to (5.12), (5.13) and (5.14), for \( \alpha \leq \frac{3}{2} - \frac{n}{2p} - \frac{2}{q} \), it follows that
\begin{equation}
\|W\|_{L^{\infty}_t C^{\alpha}_x(Q^+_1)} \leq \|\tilde{f}_1\|_{L^p_t L^q_x(Q_+)}.
\end{equation}
Combining (5.9) and (5.15), we obtain
\begin{equation}
\|W\|_{C^{\alpha}_t C^{\alpha}_x(Q^+_1)} \leq \|\tilde{f}_1\|_{L^p_t L^q_x(Q_+)} + \|\tilde{f}_1\|_{L^p_t L^q_x(Q_+)} + \|u\|_{L^p_t L^q_x(Q_+)}.
\end{equation}
Since \( U = W + H \) and \( u = U \) in \( Q^+_1 \), we have via (5.4) and (5.16)
\begin{equation}
\|\nabla u\|_{C^{\alpha}_t C^{\alpha}_x(Q^+_1)} = \|\nabla U\|_{C^{\alpha}_t C^{\alpha}_x(Q^+_1)}
\leq c(\|\tilde{f}_1\|_{L^p_t L^q_x(Q_+)} + \|h\|_{L^\infty_t C^{\alpha}_x(Q^+_1)} + \|h\|_{L^\infty_t L^\infty_x(Q^+_1)})
\leq c(\|\nabla u\|_{L^p_t L^q_x(Q_1)} + \|\pi\|_{L^p_t L^q_x(Q^+_1)} + \|u\|_{L^p_t L^q_x(Q^+_1)} + \|u\|_{L^p_t C^{\alpha}_x(Q^+_1)} + \|u\|_{L^p_t L^q_x(Q^+_1)}).
\end{equation}
Due to Proposition 2 and Lemma 1 in [18], there are \( 1 < q_0 < 2 \) and \( 1 < p_0 < p \) such that
\begin{equation}
\|u\|_{L^\infty_t C^{\alpha}_x(Q^+_1)} + \|u\|_{L^\infty_t L^{\infty}_x(Q^+_1)} \leq c(\|\nabla u\|_{L^{q_0}_t L^{p_0}_x(Q^+_1)} + \|\pi\|_{L^{q_0}_t L^{p_0}_x(Q^+_1)} + \|u\|_{L^{q_0}_t L^{p_0}_x(Q^+_1)})
\leq c(\|\nabla u\|_{L^{q_0}_t L^{p_0}_x(Q^+_1)} + \|\pi\|_{L^{q_0}_t L^{p_0}_x(Q^+_1)} + \|u\|_{L^{q_0}_t L^{p_0}_x(Q^+_1)}).
\end{equation}
Hence, we obtain (1.9). This completes the proof of Theorem 1.5.

6. PROOF OF THEOREM 1.6 FOR STOKES EQUATIONS

Let \( g \) be a vector field defined in (3.7) such that
\begin{equation}
g_n^S \in C^\infty_c(B_2 \setminus B_{\frac{1}{2}}, g_n^T \in C(0,1) \quad \text{with} \quad g_n^T(0) = 0, \quad D_t g_n^T \in L^q(0,1),
\end{equation}
\begin{equation}
D_t g_n^T \notin L^r \left( \frac{3}{4}, 1 \right) \quad \text{for all} \quad r > q, \quad g_n^T \notin B^1_{rr} \left( \frac{3}{4}, \frac{7}{8} \right) \quad \text{for all} \quad r > \frac{3q}{2}.
\end{equation}

**Remark 6.1.** An example of \( g_n^T \) satisfying (6.1) is the following:
\begin{equation}
g_n^T(t) = \begin{cases}
0, & 0 < t \leq \frac{3}{4}, \\
(t - \frac{3}{4})^{-\frac{1}{2}} \ln(t - \frac{3}{4})^{-1} & \frac{3}{4} < t < \frac{7}{8}.
\end{cases}
\end{equation}

In Appendix C for clarity, we give its details.

Let \( \phi(x,t) = c_n \int_{\mathbb{R}^{n-1}} \frac{1}{|x - y|^{n-2}} g_n(y',t) dy' \). We define \( w^H = \nabla \phi \) and \( p^H(x,t) = -D_t \phi(x,t) \). Then, we note that \( (w^H, p^H) \) is the solution of the Stokes equations in \( \mathbb{R}^n_+ \times (0, \infty) \)
\begin{equation}
\begin{cases}
w^H_t - \Delta w^H + \nabla p^H = 0, & \text{div } w^H = 0, \\
w^H |_{t=0} = 0, & w^H|_{x_n=0} = (R'_1 g_n, \cdots, R'_{n-1} g_n, g_n).
\end{cases}
\end{equation}
We set \( G = g - w^H |_{x_n=0} = (-R'_1 g_n, \cdots, -R'_{n-1} g_n, 0) \). Let \( (w^S, p^S) \) be a solution of the following equations in \( \mathbb{R}^n_+ \times (0, \infty) \)
\begin{equation}
\begin{cases}
w^S_t - \Delta w^S + \nabla p^S = 0, & \text{div } w^S = 0, \\
w^S |_{t=0} = 0, & w^S|_{x_n=0} = G.
\end{cases}
\end{equation}
Then, \( (w^H + w^S, p^H + p^S) \) is solution of (2.2) with external force \( f = 0 \).
Since for $x' \in B_1'$ and $y' \in B_2' \setminus B_1'$, $|x' - y'| \geq \frac{1}{2}$, we have
\[
\left| D^k_x \int_{\mathbb{R}^{n-1}} \frac{1}{|x - y'|^n} g_n^S(y') dy' \right| \leq c\|g_n^S\|_{L^\infty(B_2' \setminus B_1')} \quad (6.4)
\]
for $x' \in B_1'$ and $k \geq 0$. We note that for $1 \leq \theta \leq \infty$ and $0 < r < 1$,
\[
\|D^2_x w^H\|_{L^\theta(Q_+^1)} = \|g_n^T\|_{L^\theta(1-r, 1)} \|D^2_x \int_{\mathbb{R}^{n-1}} \frac{1}{|x - y'|^n} g_n^S(y') dy'\|_{L^\theta(B_1')} = \|D_t g_n^T\|_{L^\theta(1-r, 1)} \|D^2_x \int_{\mathbb{R}^{n-1}} \frac{1}{|x - y'|^n} g_n^S(y') dy'\|_{L^\theta(B_1')} = \|D^2_x p^H\|_{L^\theta(Q_+^1)} = \|D_t g_n^T\|_{L^\theta(1-r, 1)} \|D^1_x \int_{\mathbb{R}^{n-1}} \frac{1}{|x - y'|^n} g_n^S(y') dy'\|_{L^\theta(B_1')}.
\]
Since $\|g_n^T\|_{B_{3q/2}^{q/2}(\mathbb{R})} \leq \|g_n^T\|_{W_2^1(\mathbb{R})} < \infty$, we have
\[
\|G\|_{B_{3q/2}^{q/2}(\mathbb{R}^{n-1} \times \mathbb{R})} = c\|g_n\|_{B_{3q/2}^{q/2}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|g_n^G\|_{B_{3q/2}^{q/2}(\mathbb{R}^{n-1} \times \mathbb{R})} \leq \|g_n^S\|_{L_2^{3q/2}(\mathbb{R}^{n-1})} \|g_n^T\|_{B_1^{1-q/2}(\mathbb{R})} + \|g_n^S\|_{B_{3q/2}^{q/2}(\mathbb{R}^{n-1})} \|g_n^T\|_{L_2^{3q/2}(\mathbb{R})} < \infty.
\]
From (17), we obtain
\[
\|D^2_x w^S\|_{L_2^{3q/2}((Q_+^1)_1)} + \|D^2_x w^S\|_{L_2^{3q/2}((Q_+^1)_{1/2})} + \|D^2_x p^S\|_{L_2^{3q/2}((Q_+^1)_{1/2})} \leq c\|G\|_{B_1^{3q/2}((Q_+^1)_{1/2})} < \infty.
\]
Combining (6.4) and (6.5), we obtain
\[
\|\nabla p\|_{L^\theta(Q_+^1)} + \|D_t w\|_{L^\theta(Q_+^1)} + \|D^2_x w\|_{L_2^{3q/2}(Q_+^1_{1/2})} < \infty,
\]
\[
\|\nabla p\|_{L^\theta(1, \infty)} = \infty, \quad \|D_t w\|_{L^\theta(Q_+^1_{1/2})} = \infty \quad r_1 > q.
\]
For the last result in (11.11), we use the decomposition of $w_1 = w_1^B + w_1^S + w_1^N$ defined in Proposition 4.1. We note that for $(x, t) \in B(0, 1) \times (0, 1)$ it is direct that
\[
|\nabla^2 w_1^N(x, t)| \leq c\|g_n\|_{L^\infty}.
\]
From (6.6) and (3.13), for $1 < r < \infty$, we have
\[
\|\nabla^2 (w_1 - w_1^B)\|_{L^r(Q_+^1)} \leq \|g\|_{L^\infty}.
\]
We note that
\[
D_{x_n} D_{x_n} w_1^B(x, t) = c_n \int_0^t \int_{\mathbb{R}^{n-1}} D_s g_n(y', s) \frac{x_n}{(t - s)^{n+2}} \int_{\mathbb{R}^{n-1}} e^{-|x' - x'|^2/t - s} e^{-|y' - y'|^2/t - s} z_1 \frac{|y'|^2}{|z'|^n} d y' d s
\]
\[
- c_n \int_0^t \int_{\mathbb{R}^{n-1}} \Delta g_n(y', s) \frac{x_n}{(t - s)^{n+2}} \int_{\mathbb{R}^{n-1}} e^{-|x' - x'|^2/t - s} e^{-|y' - y'|^2/t - s} z_1 \frac{|y'|^2}{|z'|^n} d y' d s
\]
\[
:= I_1 + I_2.
\]
Firstly, it follows from (4.9) that, for $|x'| \leq 1$,
\[
|I_2(x, t)| \leq c\|\nabla^2 g_n\|_{L^\infty} < \infty.
\]
Using Lemma [3.2] we divide $I_1 = I_{11} + I_{12}$, where

$$I_{11}(x,t) = c_n \int_0^t D_x \Gamma_1(x_n,t-s)D_{sgn}(s)ds\psi(x'),$$

$$I_{12}(x,t) = c_n \int_0^t D_x \Gamma_1(x_n,t-s)D_{sgn}(s) \int_{\mathbb{R}^n-1} g_s^S(y') J(x'-y', t-s)dy'sds,$$

where $\psi(x') = \int_{\mathbb{R}^n-1} \frac{x_n-y_n}{|x'-y'|} g_s^S(y')dy'$. We consider first $I_{12}$. Noting that

$$|I_{12}(x,t)| \leq c \int_0^t \frac{x_n}{t-s} e^{-\frac{r^2}{t-s}} |D_s g_n^{T}(s)|ds \|g_n^S\|_{L^\infty}.$$

Using the integral Minkowski’s inequality, we have

$$\left(\int_{B_1} |I_{12}(x,t)|^{p_0} dx\right)^{\frac{1}{p_0}} \leq c \int_0^t \frac{1}{t-s} |D_s g_n^{T}(s)| \left(\int_0^1 x_n^{p_0} e^{-\frac{r^2}{t-s}} dx_n\right)^{\frac{1}{p_0}} ds \|g_n^S\|_{L^\infty} \leq c \int_0^t (t-s)^{-\frac{1}{q}+\frac{1}{p_0}} |D_s g_n^{T}(s)|ds \|g_n^S\|_{L^\infty}.$$

Due to Sobolev-Littlewood-Hardy inequality and Holder inequality, we obtain

$$\|I_{12}\|_{L^{p_0}(Q^+_t)} \leq c \|D_s g_n^{T}\|_{L^q} \|g_n^S\|_{L^\infty}, \quad 1 < q < 2, \quad p_0 = \frac{3q}{2-q},$$

$$\|I_{12}\|_{L^{q}(Q^+_t)} \leq c \|D_s g_n^{T}\|_{L^q} \|g_n^S\|_{L^\infty}, \quad 2 \leq q < \infty, \quad p_0 > q.$$

On the other hand, since $\int_0^t D_x \Gamma_1(x_n,t-s)g_n^{T}(s)ds\big|_{x_n=0} = g_n^{T}(t)$, from the result of boundary value problem of heat equation and trace theorem in half space ((3) of Proposition [2.1]), we have that for $1 < r < \infty,$

$$2\|D_t D_x \Gamma_1 * g_n^{T}\|_{L^r(\mathbb{R} \times (0,\infty))} = \|D_t D_x \Gamma_1 * g_n^{T}\|_{L^r(\mathbb{R} \times (0,\infty))} + \|D_x^{3/2} \Gamma_1 * g_n^{T}\|_{L^r(\mathbb{R} \times (0,\infty))}$$

$$\geq c \|g_n^{T}\|_{B_r^{1,\frac{1}{2}}(\mathbb{R})}.$$  

Since $\|g_n^{T}\|_{B_r^{1,\frac{1}{2}}(\mathbb{R})} = \infty$ for $\frac{3q}{2} < r_2$, it follows from (4.6), (4.11), (4.12), (4.13) and (6.8) that

$$\|I_{11}\|_{L^r(Q^+_t)} \geq c \|g_n^{T}\|_{B_r^{1,\frac{1}{2}}(\mathbb{R})} \|\psi\|_{L^r(B'_r)} = c = \infty.$$

Thus, we complete the proof of Theorem [1.6].

Following similar computations in the proof of Theorem [1.6], we obtain the following corollary. Although its verification is not difficult, for clarity, we present its details.

**Corollary 6.2.** Let $1 < q < \infty$ and $g_n^S$ and $g_n^{T}$ satisfy the assumption [6.1]. Let $p_0 = \frac{3q}{2-q}$ if $1 < q < 2$ and $p_0$ be a any number larger than $q$ if $q \geq 2$. Then, the constructed solution $w$ of the Stokes equations (1.1) satisfies

$$\|D_x w\|_{L^{p_0}(Q^+_t)} < \infty.$$  

In case that $1 < q < 2$, if $g_n^{T}$ is taken as the function in Remark [6.1] and if $r > p_0$, then

$$\|D_x w\|_{L^r(Q^+_t)} = \infty.$$  

**Proof.** We recall the decomposition of $w = w^C + w^B + w^N$ defined in Proposition [4.1]. Flowing similarly the estimate (6.7), for $1 < r < \infty$, we have

$$\|\nabla(w - w^B)\|_{L^r(Q^+_t)} \leq \|g\|_{L^\infty}.$$
Here we compute only the normal derivative, since we can show that tangential derivatives are rather easily controlled. Thus, taking the normal derivative to \( w_1^R \), we compute
\[
D_{x_n} w_1^R(x, t) = c_n \int_0^t \int_{\mathbb{R}^{n-1}} D_s g_n(y', s) \frac{1}{(t-s)^\frac{1}{2}} e^{-\frac{|x-y'-s|^2}{2(t-s)}} \frac{z_i}{|z'|^n} dy'ds \\
+ c_n \int_0^t \int_{\mathbb{R}^{n-1}} \Delta' g_n(y', s) \frac{1}{(t-s)^\frac{1}{2}} e^{-\frac{|x-y'-s|^2}{2(t-s)}} \frac{z_i}{|z'|^n} dy'ds \\
:= I_1 + I_2.
\]
Due to (4.9), we have
\[
\|I_2\|_{L^\infty(Q^+_1)} \leq c\|\nabla_x^2 g_n\|_{L^\infty} < \infty.
\]
Using Lemma 3.2, we divide \( I_1 = I_{11} + I_{12} \), where
\[
I_{11}(x, t) = c_n \int_0^t \Gamma_1(x_n, t-s) D_s g_n^T(s) ds \psi(x'), \\
I_{12}(x, t) = c_n \int_0^t \Gamma_1(x_n, t-s) D_s g_n^T(s) \int_{\mathbb{R}^{n-1}} g_n^S(y') J(x' - y', t-s) dy'ds,
\]
where \( \psi(x') = \int_{\mathbb{R}^{n-1}} \frac{x - y}{|x-y|^n} g_n^S(y') dy' \). It follows from Lemma 3.2 that for \( 1 < r < \infty \),
\[
\|I_{12}(t)\|_{L^r(B^+_1)} \leq c \int_0^t \|D_s g_n^T(s)\| \left( \int_0^1 e^{-\frac{s^2}{2(t-s)}} ds \right)^\frac{1}{r} \|g_n^S\|_{L^\infty} \\
\leq c \int_0^t (t-s)^{\frac{1}{2r}} \|D_s g_n^T(s)\| \|g_n^S\|_{L^\infty} \\
\leq c \|D_t g_n^T\|_{L^p(0,1)} \|D_y' g_n^S\|_{L^\infty}.
\]
Continuing computations for \( I_{11} \),
\[
\|I_{11}(t)\|_{L^p_0(B^+_1)} \leq c \int_0^t (t-s)^{-\frac{1}{2}} \|D_s g_n^T(s)\| \left( \int_0^1 e^{-\frac{s^2}{2(t-s)}} ds \right)^\frac{1}{p_0} \|g_n^S\|_{L^\infty} \\
\leq c \int_0^t (t-s)^{-\frac{1}{2r} + \frac{1}{2p_0}} \|D_s g_n^T(s)\| \|g_n^S\|_{L^\infty}.
\]
Due to Young’s inequality, we obtain
\[
\|I_{11}\|_{L^{\rho_0}(Q^+_1)} \leq c \|D_t g_n^T\|_{L^p(0,1)} \|D_y' g_n^S\|_{L^{\rho_\infty}}.
\]
Hence, summing up the estimates, for \( 1 \leq i \leq n-1 \), we have
\[
(6.13) \quad \|\nabla w_i\|_{L^{\rho_0}(Q^+_1)} < \infty.
\]
From the first equality of (6.2), we obtain (6.13) for \( i = n \). Hence, we complete the proof of (6.10). It remains to show (6.11). It follows from the above estimate that
\[
(6.14) \quad \|\nabla w - I_{11}\|_{L^r(Q^+_1)} \leq \infty \quad 1 < r < \infty.
\]
Noting that for \( \frac{3}{4} < t < 1 \),
\[
|D_t g_n^T(t)| = (t - \frac{3}{4})^{-\frac{1}{2}} \left| \ln^{-1} \left( t - \frac{3}{4} \right) \right| \left( (1 - \frac{1}{q}) - \ln^{-1} (t - \frac{3}{4}) \right) \\
\geq (1 - \frac{1}{q}) (t - \frac{3}{4})^{-\frac{1}{2}} \left| \ln^{-1} (t - \frac{3}{4}) \right|.
\]
Hence, for $\frac{3}{4} < t < 1$, we have
\[
\left| \int_0^t \Gamma_1(x_n, t - s) D_s g_n^T(s) ds \right| \geq c \int_0^{\frac{3}{4} (t + \frac{3}{4})} (t - s)^{-\frac{1}{4}} e^{-\frac{s^2}{2(t-s)}} (s - \frac{3}{4})^{-\frac{1}{4}} \ln^{-1}(s - \frac{3}{4}) ds
\]
\[
\geq c(t - \frac{3}{4})^{-\frac{1}{4}} e^{-\frac{s^2}{2(t-s)}} \int_0^{\frac{3}{4} (t + \frac{3}{4})} (s - \frac{3}{4})^{-\frac{1}{4}} \ln^{-1}(s - \frac{3}{4}) ds
\]
\[
= c(t - \frac{3}{4})^{-\frac{1}{4}} e^{-\frac{s^2}{2(t-s)}} \int_0^1 s^{-\frac{1}{4}} \ln^{-1}(s) ds.
\]

By L’Hôpital’s Theorem, we have \( \lim_{a \to 0} \frac{\int_0^a s^{-\frac{1}{4}} \ln^{-1}(s) ds}{a^{1-\frac{1}{q}}} = \frac{q-1}{q} \), we have
\[
\left| \int_0^t \Gamma_1(x_n, t - s) D_s g_n^T(s) ds \right| \geq c(t - \frac{3}{4})^{\frac{1}{2}} e^{-\frac{s^2}{2(t-s)}} \int_0^1 s^{-\frac{1}{4}} \ln^{-1}(t - \frac{3}{4}) ds.
\]
Hence, for \( r > p_0 \), we have
\[
\| I_{11} \|_{L^r(B_1^n)} \geq c \int_0^1 (t - \frac{3}{4})^{\frac{1}{2}} e^{-\frac{s^2}{2(t-s)}} \ln^{-1}(t - \frac{3}{4}) dt \| \psi \|_{L^r(B_1^n)}^r = \infty.
\]
Thus, combining the estimate (6.14), we have (6.11). We complete the proof of Corollary 6.2 \( \square \)

7. PROOF OF MAIN RESULTS FOR NAVIER-STOKES EQUATIONS

Previously, we presented the proofs of Theorem 1.1 and Theorem 1.6 for the Stokes equations. In this section we complete the proofs of those results by providing the details for the case of Navier-Stokes equations. We start with the case of Theorem 1.1.

7.1. Proof of Theorem 1.1 for Navier-Stokes equations. Let \( g \) be a boundary data defined in (3.7) and (3.8), and \( w \) be a solution of the Stokes equations (2.2)-(2.3) with \( f = 0 \) defined by (3.6). Let \( \phi_1 \in C_c^\infty(\mathbb{R}^n) \) be a cut-off function satisfying \( \phi_1 \geq 0 \), \( \text{supp} \phi_1 \subset B_2 \) and \( \phi_1 \equiv 1 \) in \( B_1 \). Also, let \( \phi_2 \in C_c^\infty(-\infty, \infty) \) be a cut-off function satisfying \( \phi_2 \geq 0 \), \( \text{supp} \phi_2 \subset (-2, 2) \) and \( \phi_2 \equiv 1 \) in \( (-1, 1) \). We set \( \phi(x, t) = \phi_1(x)\phi_2(t) \).

We define \( W = \phi w \). Then, it is direct that \( W = w \) in \( Q^+_1 \). Furthermore, since supports of \( g_n \) and \( \phi \) are disjoint, we note that \( W|_{x_n=0} = 0 \) and \( W|_{t=0} = 0 \). By the result of Proposition 4.1, we observe that
\[
\| W \|_{L^r(Q^+_1)} \leq c \| w \|_{L^r(B^+_1 \times (0,4))} \leq c \| g \|_{L^r(\mathbb{R}^{n-1} \times (0,\infty))} \leq ca
\]
for all \( 1 \leq r \leq \infty \), where \( a > 0 \) is defined in (3.7).

We consider the following perturbed Navier-Stokes equations in \( \mathbb{R}^n_+ \times (0,1) \):
\[
v_t - \Delta v + \nabla g + \text{div} (v \otimes v + v \otimes v + W) = -\text{div} (W \otimes W), \quad \text{div} v = 0
\]
with homogeneous initial and boundary data, i.e. \( v(x,0) = 0 \) and \( v(x,t) = 0 \) on \( x_n = 0 \). Our aim is to establish the existence of solution \( v \) for (7.2) satisfying \( v \in L^r(\mathbb{R}^n_+ \times (0,1)) \cap L^\infty(\mathbb{R}^n_+ \times (0,1)) \) and \( \nabla v \in L^r(\mathbb{R}^n_+ \times (0,1)) \) for all \( n + 2 < r < \infty \). In order to do that, we consider the iterative scheme for (7.2), which is given as follows: For a positive integer \( m \geq 1 \)
\[
v_t^{m+1} - \Delta v^{m+1} + \nabla q^{m+1} = -\text{div} (v^m \otimes v^m + v^m \otimes W + W \otimes v^m + W \otimes W), \quad \text{div} v^{m+1} = 0
\]
with homogeneous initial and boundary data, i.e. \( v^{m+1}(x,0) = 0 \) and \( v^{m+1}(x,t) = 0 \) on \( \{x_n = 0\} \). We set \( v^1 = 0 \). By Proposition 2.6, we have
\[
\| v^2(t) \|_{L^r(\mathbb{R}^n_+)} \leq c \int_0^t (t - s)^{-\frac{1}{2}} \| W(s) \otimes W(s) \|_{L^r(\mathbb{R}^n_+)} ds.
\]
From now on, we denote $Q_+ := \mathbb{R}_+^n \times (0, 1)$, unless any confusion is to be expected. By Young’s inequality, we have
\begin{equation}
\|v^2\|_{L^r(Q_+)} \leq c\|W \otimes W\|_{L^r(Q_+)} \leq c\|W\|_{L^r(Q_+)}\|W\|_{L^\infty(Q_+)}. \tag{7.3}
\end{equation}
Since $r > n + 2$, according to Proposition 2.9, we have
\begin{equation}
\|v^2\|_{L^\infty(Q_+)} \leq c\|W \otimes W\|_{L^r(Q_+)} \leq c\|W\|_{L^r(Q_+)}\|W\|_{L^\infty(Q_+)}.
\end{equation}
According to Proposition 2.7, we have
\begin{equation}
\|\nabla v^2\|_{L^r(Q_+)} \leq c\|W \otimes W\|_{L^r(Q_+)} \leq c\|W\|_{L^r(Q_+)}\|W\|_{L^\infty(Q_+)}.
\end{equation}
By (7.1), we have $A := \|W\|_{L^r(Q_+)} + \|W\|_{L^\infty(Q_+)} \leq 2cA < 1$, where $A$ is defined in (3.7). Taking $\alpha > 0$ small such that $A < \frac{1}{\alpha}$, where $c$ is the constant in (7.3)-(7.5) such that
\begin{equation}
\|v^2\|_{L^r(Q_+)} + \|v^2\|_{L^\infty(Q_+)} + \|\nabla v^2\|_{L^r(Q_+)} < \frac{3c}{2}A^2 < A.
\end{equation}
Suppose that for $m \geq 2$,
\begin{equation}
\|\nabla v^m\|_{L^r(Q_+)} + \|v^m\|_{L^r(Q_+)} + \|v^m\|_{L^\infty(Q_+)} < A. \tag{7.6}
\end{equation}
Then, iterative arguments show that
\begin{align*}
\|\nabla v^{m+1}\|_{L^r(Q_+)} + \|v^{m+1}\|_{L^r(Q_+)} + \|v^{m+1}\|_{L^\infty(Q_+)} &\leq c\left(\|v^m\|^2 + \|v^mW\| + \|W\|^2\|\nabla v^m\|_{L^r(Q_+)}\right) \\
&\leq c\left(\|v^m\|_{L^r(Q_+)} + \|W\|_{L^r(Q_+)}\right)\left(\|v^m\|_{L^\infty(Q_+)} + \|W\|_{L^\infty(Q_+)}\right) \\
&\leq 4cA^2 < A.
\end{align*}
Hence, by mathematical induction, (7.6) holds for all $m \geq 2$.

We denote, for convenience, $V^{m+1} := v^{m+1} - v^m$ and $Q^{m+1} := q^{m+1} - q^m$ for $m \geq 1$. We then see that $(V^{m+1}, Q^{m+1})$ solves
\begin{align*}
V^{m+1}_t - \Delta V^{m+1} + \nabla Q^{m+1} &= -\text{div} \left( V^m \otimes v^m + v^{m-1} \otimes V^m + V^m \otimes W + W \otimes V^m \right), \\
\text{div} V^{m+1} &= 0,
\end{align*}
with homogeneous initial and boundary data, i.e. $V^{m+1}(x, 0) = 0$ and $V^{m+1}(x, t) = 0$ on $\{x_n = 0\}$. Taking sufficiently small $\alpha > 0$ such that $A < \frac{1}{6c}$, we obtain from (7.7) that
\begin{align*}
\|V^{m+1}\|_{L^r(Q_+)} + \|V^{m+1}\|_{L^\infty(Q_+)} + \|\nabla V^{m+1}\|_{L^r(Q_+)} &\leq c\left(\|V^m \otimes v^m + v^{m-1} \otimes V^m + V^m \otimes W + W \otimes V^m\|_{L^r(Q_+)}\right) \\
&\leq c\|V^m\|_{L^r(Q_+)}(\|v^m\|_{L^\infty(Q_+)} + \|v^{m-1}\|_{L^\infty(Q_+)} + \|W\|_{L^\infty(Q_+)}),
\end{align*}
\begin{equation}
\|V^m\|_{L^r(Q_+)} \leq \frac{1}{2}\|V^m\|_{L^r(Q_+)}.
\end{equation}
This implies that $(v^m, \nabla v^m)$ converges to $(v, \nabla v)$ in
\begin{align*}
(L^r(Q_+) \cap L^\infty(Q_+)) \times L^r(Q_+)
\end{align*}
such that $v$ solves (7.2) with an appropriate distribution $q$. We then set $u := v + W$ and $p = \pi + q$, which becomes a weak solution of the Navier-Stokes equations in $\mathbb{R}_+^n \times (0, 1)$, namely
\begin{align*}
u t - \Delta u + \nabla p &= -\text{div} (u \otimes u), \quad \text{div} u = 0 \quad \text{in} \ Q_+^1
\end{align*}
with boundary data $u(x, t) = 0$ on $\Sigma = (B_1 \cap \{x_n = 0\}) \times (0, 1)$ such that
\begin{equation}
\|u\|_{L^\infty(Q_+^1)} \leq c, \quad \|\nabla u\|_{L^p(Q_+^1)} = \infty.
\end{equation}
This completes the proof of Theorem 1.1 for the case of the Navier-Stokes equations. \hfill \qed
7.2. Proof of Theorem 1.6 for Navier-Stokes equations. Let \( g_0 \) be a boundary data defined in (6.1) and \( w \) be a solution of the Stokes equations (2.2)-(2.3) with \( f = 0 \) defined by (3.6). Let \( \phi \in C_c^\infty (\mathbb{R}^n \times (0, \infty)) \) be a cut-off function defined in Section 7.1. Let \( W = \phi w \) and \( P = \phi \pi \) such that \( W = w \) and \( P = \pi \) in \( Q_1^+ \). As before, we denote \( Q_+ := \mathbb{R}^n_+ \times (0, 1) \) for simplicity.

Let \( a > 0 \) be the number defined in (5.7). From Section 6 and Corollary 6.2 we have

\[
(7.10) \quad \| D_t W \|_{L^p(Q_+)} + \| D_x^2 W \|_{L^q(Q_+)} + \| \nabla P \|_{L^q(Q_+)} + \| D_x W \|_{L^p(Q_+)} < c a,
\]

where \( p_0 := \frac{3}{q-2} \) if \( q < 2 \) and \( p_0 \) is any real number satisfying \( p_0 > \frac{3q}{2q-2} \) if \( q \geq 2 \). We note that \( p_0 > \frac{3q}{2q-2} \). From (3.4) and (6.4), applying Section 6, we have

\[
(7.11) \quad \| W \|_{L^r_t L^q(Q_+)} < \| w \|_{L^r_t L^q(Q_+)} < c a, \quad \forall r < \infty.
\]

We consider the following perturbed Navier-Stokes equations in \( Q_+ \):

\[
(7.12) \quad v_t - \Delta v + \nabla q + \text{div} (v \otimes v + v \otimes W + W \otimes v) = -\text{div} (W \otimes W), \quad \text{div} v = 0
\]

with homogeneous initial and boundary data, i.e. \( v(x, 0) = 0 \) and \( v(x, t) = 0 \) on \( x_n = 0 \).

Our aim is to establish the existence of solution \( v \) for (7.12) satisfying \( v \in L^\infty(Q_+), \quad D_x v, D_x^2 v, D_t v, q \in L^{p_0}(Q_+). \)

Since the proofs are exactly the same, we only prove for the case of \( q < 2 \). We consider the iterative scheme for (7.12), which is given as follows: For a positive integer \( m \geq 1 \)

\[
v^{m+1}_t - \Delta v^{m+1} + \nabla q^{m+1} = -\text{div} (v^{m} \otimes v^{m} + v^{m} \otimes W + W \otimes v^{m} + W \otimes W), \quad \text{div} v^{m+1} = 0
\]

with homogeneous initial and boundary data, i.e. \( v^{m+1}(x, 0) = 0 \) and \( v^{m+1}(x, t) = 0 \) on \( \{x_n = 0\} \). We set \( v^1 = 0 \). From the well-known result of initial-boundary value problem for Stokes equations in half space, we have

\[
\| \nabla v^2 \|_{L^p(Q_+)} + \| D_t v^2 \|_{L^p(Q_+)} \leq c \| \nabla W \cdot W \|_{L^p(Q_+)} \\
\leq c \| \nabla W \|_{L^p(Q_+)} \| W \|_{L^\infty(Q_+)}.
\]

(7.13)

According to Proposition 2.7, Proposition 2.8 and Proposition 2.10 we have

\[
\| \nabla v^2 \|_{L^p(Q_+)} \leq c \| W \otimes W \|_{L^p(Q_+)} \leq c \| W \|_{L^\infty(Q_+)}^2,
\]

(7.14)

\[
\| v^2 \|_{L^p(Q_+)} \leq c \| W \otimes W \|_{L^p(Q_+)} \leq c \| W \|_{L^\infty(Q_+)}^2.
\]

Take \( r_0 < \infty \) satisfying \( \frac{2}{q} + \frac{3}{2q} < 2 \). From Proposition 2.4, Young’s inequality and Holder inequality, we have

\[
| v^2(x, t) | \leq c \int_0^t (t-s)^{-\frac{m}{r_0}} \| \nabla W(s) \cdot W(s) \|_{L^r_+(\mathbb{R}^n_+)} ds \\
\leq ct^{-\frac{m}{r_0}+1-\frac{2}{q}} \| \nabla W \cdot W \|_{L^r_t L^q_+(Q_+)}.
\]

Hence, we obtain

\[
(7.15) \quad \| v^2 \|_{L^\infty(Q_+)} \leq c \| \nabla W \|_{L^r_t L^q_+(Q_+)} \| W \|_{L^\infty(Q_+)}.
\]

Moreover, from Proposition 2.4, Proposition 2.5, Young’s inequality and Hardy-Littlewood-Sobolev’s inequality, we have

\[
(7.16) \quad \| v^2 \|_{L^r_t L^q_+(Q_+)} \leq c \| W \otimes W \|_{L^r_+(\mathbb{R}^n_+)} \leq c \| W \|_{L^\infty(Q_+)}^2.
\]
From (3.4) and (6.4), applying Section 6, we obtain \( \|W\|_{L^\infty(\mathbb{R}^n \times (0,1))} \leq \|w\|_{L^\infty(Q^+)} \leq ca \), where \( a > 0 \) is defined in (3.7). In Corollary 6.2, we showed that \( \|\nabla w\|_{L^p(\frac{1}{2})} < ca \). Then, from (7.11), we have
\[
A := \|\nabla W\|_{L^p(\mathbb{R}^n)} + \|\nabla W\|_{L^p(\mathbb{R}^n)} + \|W\|_{L^\infty(\mathbb{R}^n)} \leq ca.
\]
Taking \( a > 0 \) small such that \( ca < A \leq \min(\frac{1}{2\varepsilon}, \frac{1}{2}) \), where \( c \) is the constant in (7.13), (7.14), (7.15) and (7.16), we have
\[
\|\nabla^2 v^2\|_{L^p(\mathbb{R}^n)} + \|D_v v^2\|_{L^p(\mathbb{R}^n)} + \|\nabla^2 v^2\|_{L^p(\mathbb{R}^n)} + \|v^2\|_{L^p(\mathbb{R}^n)}
+ \|v^2\|_{L^\infty(\mathbb{R}^n)} + \|\nabla v^2\|_{L^q(\mathbb{R}^n)} + \|v^2\|_{L^q(\mathbb{R}^n)} < A.
\]
Suppose that for \( m \geq 2 \),
\[
\|\nabla^2 v^m\|_{L^p(\mathbb{R}^n)} + \|D_v v^m\|_{L^p(\mathbb{R}^n)} + \|\nabla v^m\|_{L^p(\mathbb{R}^n)} + \|v^m\|_{L^p(\mathbb{R}^n)}
+ \|v^m\|_{L^\infty(\mathbb{R}^n)} + \|\nabla v^m\|_{L^q(\mathbb{R}^n)} + \|v^m\|_{L^q(\mathbb{R}^n)} < A.
\]
(7.17)
Then, first we obtain
\[
\|\nabla^2 v^{m+1}\|_{L^p(\mathbb{R}^n)} + \|D_v v^{m+1}\|_{L^p(\mathbb{R}^n)}
\leq c\left(\|\nabla v^m \otimes v^m + |\nabla v^m \otimes W| + |\nabla W \otimes v^m| + |\nabla W \otimes W|\right)_{L^p(\mathbb{R}^n)}
\]
(7.18)
\[\leq 2c\left(\|\nabla v^m\|_{L^p(\mathbb{R}^n)} + \|\nabla W\|_{L^p(\mathbb{R}^n)}\right)\left(\|v^m\|_{L^\infty(\mathbb{R}^n)} + \|W\|_{L^\infty(\mathbb{R}^n)}\right) \leq 4cA^2 < \frac{1}{2}A.
\]
Similarly, we have
\[
\|v^{m+1}\|_{L^\infty(\mathbb{R}^n)} \leq c\left(\|\nabla v^m\|_{L^q(\mathbb{R}^n)} + \|\nabla v^m\|_{L^q(\mathbb{R}^n)} + \|W\|_{L^\infty(\mathbb{R}^n)} + \|W\|_{L^\infty(\mathbb{R}^n)} \right) \leq A^2 < \frac{1}{2}A.
\]
(7.19)
Continuing computations for \( L^\infty \) (from (7.15)), we have
\[
\|v^{m+1}\|_{L^\infty(\mathbb{R}^n)} \leq c\left(\|\nabla v^m\|_{L^q(\mathbb{R}^n)} + \|\nabla v^m\|_{L^q(\mathbb{R}^n)} + \|W\|_{L^\infty(\mathbb{R}^n)} + \|W\|_{L^\infty(\mathbb{R}^n)} \right) \leq A^2 < \frac{1}{2}A.
\]
For mixed norm, we compute likewise, that is,
\[
\|\nabla v^{m+1}\|_{L^q(\mathbb{R}^n)} \leq c\left(\|v^m \otimes v^m + |v^m \otimes W| + |W \otimes v^m| + |W \otimes W|\right)_{L^q(\mathbb{R}^n)}
\]
\[\leq c\left(\|v^m\|_{L^q(\mathbb{R}^n)}^2 + \|v^m\|_{L^q(\mathbb{R}^n)} + \|W\|_{L^q(\mathbb{R}^n)} + \|W\|_{L^q(\mathbb{R}^n)} \right) \leq A^2 < \frac{1}{2}A.
\]
In the same way as above, we get
\[
\|v^{m+1}\|_{L^q(\mathbb{R}^n)} \leq c\left(\|v^m \otimes v^m + |v^m \otimes W| + |W \otimes v^m| + |W \otimes W|\right)_{L^q(\mathbb{R}^n)}
\]
\[\leq c\left(\|v^m\|_{L^q(\mathbb{R}^n)}^2 + \|v^m\|_{L^q(\mathbb{R}^n)} + \|W\|_{L^q(\mathbb{R}^n)} + \|W\|_{L^q(\mathbb{R}^n)} \right) \leq A^2 < \frac{1}{2}A.
\]
Hence, by mathematical induction, (7.17) holds for all \( m \geq 2 \).
Next, we denote \( V^{m+1} := v^{m+1} - v^m \) and \( Q^{m+1} := q^{m+1} - q^m \) for \( m \geq 1 \). We then see that \( (V^{m+1}, Q^{m+1}) \) solves
\[
V^{m+1} - \Delta V^{m+1} + \nabla Q^{m+1} = -\text{div}\left(V^m \otimes v^m + v^{m+1} \otimes v^m + V^m \otimes W + W \otimes v^m\right),
\]
\[\text{div} V^{m+1} = 0,
\]
with homogeneous initial and boundary data, i.e. $V^{m+1}(x,0) = 0$ and $V^{m+1}(x,t) = 0$ on $\{x_n = 0\}$. Taking sufficiently small $\alpha > 0$ such that $A < \frac{1}{6c}$, from (7.18), we obtain

$$\|\nabla V^{m+1}\|_{L^p(Q_+)} + \|D_t V^{m+1}\|_{L^p(Q_+)} \leq 2c(\|\nabla V^m\|_{L^p(Q_+)} + \|V^m\|_{L^\infty(Q_+)} + \|W\|_{L^\infty(Q_+)} + \|\nabla W\|_{L^p(Q_+)}).$$

(7.20)

Similarly, we have

$$\|V^{m+1}\|_{L^p(Q_+)} + \|\nabla V^{m+1}\|_{L^p(Q_+)} \leq c(\|V^m \otimes v^m + v^{m-1} \otimes V^m + V^m \otimes W + W \otimes V^m\|_{L^p(Q_+)}).$$

$$\leq c\|V^m\|_{L^p(Q_+)}(\|v^m\|_{L^\infty(Q_+)} + \|v^{m-1}\|_{L^\infty(Q_+)} + \|W\|_{L^\infty(Q_+)}).$$

(7.21)

For $L^\infty$ estimate, we compute

$$\|V^{m+1}\|_{L^\infty(Q_+)} \leq c(\|\nabla V^m\|_{L^p_t L^\infty_x(Q_+)}\|v^m\|_{L^\infty(Q_+)} + \|\nabla V^m\|_{L^p_t L^\infty_x(Q_+)}\|W\|_{L^\infty(Q_+)}) \leq \frac{1}{2}\|\nabla V^m\|_{L^p_t L^\infty_x(Q_+)}.$$  

(7.22)

In the same way as above, we get

$$\|V^m\|_{L^{2q}_t L^{2r_0}_x(Q_+)} \leq \frac{1}{2}\|V^{m-1}\|_{L^{2q}_t L^{2r_0}_x(Q_+)};$$

$$\|\nabla V^m\|_{L^p_t L^r_\infty_x(Q_+)} \leq \frac{1}{2}\|\nabla V^{m-1}\|_{L^p_t L^r_\infty_x(Q_+)}.$$  

Therefore, there is $v$ satisfying $v \in L^{2q}_t L^{2r_0}_x(Q_+) \cap L^\infty(Q_+), D_x v \in L^q_t L^{r_0}_x(Q_+) \cap L^p(Q_+)$ and $D^2_x v$, $D_t v \in L^p(Q_+)$ such that

$$v^m \to v \quad \text{in} \quad L^{2q}_t L^{2r_0}_x(Q_+) \cap L^\infty(Q_+),$$

$$D_x v^m \to D_x v \quad \text{in} \quad L^q_t L^{r_0}_x(Q_+) \cap L^p(Q_+),$$

$$D^2_x v^m \to D^2_x v \quad \text{in} \quad L^p(Q_+),$$

$$D_t v^m \to D_t v \quad \text{in} \quad L^p(Q_+).$$

Moreover, $v$ solves (7.12) with appropriate pressure $q \in L^p(Q_+)$.

We then set $u := v + W$ and $p = \pi + q$, which becomes a weak solution of the Navier-Stokes equations in $Q_+$, namely

$$u_t - \Delta u + \nabla p = -\text{div} (u \otimes u), \quad \text{div} u = 0 \quad Q_+^1$$

with boundary data $u(x,t) = g(x,t)$ on $\{x_n = 0\}$ such that

$$\|\nabla^2 u\|_{L^{\frac{3q}{2}}_t L^{\frac{3r}{2}}(Q_+^1)} + \|D_t u\|_{L^q(Q_+^1)} + \|\nabla p\|_{L^r(Q_+^1)} < \infty.$$  

Since integrability of spatial variables can be improved, we can have that for given $p > 1$

$$\|\nabla^2 u\|_{L^{\frac{3q}{2}}_t L^{\frac{3r}{2}}(Q_+^1)} + \|D_t u\|_{L^q_t L^r_x(Q_+^1)} + \|\nabla \pi\|_{L^q_t L^r_x(Q_+^1)} < \infty.$$  

However, it is straightforward via construction that for any $r_1 > q$ and $r_2 > \frac{3q}{2}$

$$\|D_t u\|_{L^{r_1}_t L^{r_2}_x(Q_+^1)} = \infty, \quad \|\nabla \pi\|_{L^{r_1}_t L^{r_2}_x(Q_+^1)} = \infty, \quad \|\nabla^2 u\|_{L^{r_1}_t L^{r_2}_x(Q_+^1)} = \infty.$$  

This completes the proof. \qed
The term \( \Gamma \)

Hence, we complete the proof of Remark 1.3.

Using Plancherel Theorem with respect to \( (1.1) \), we have

\[
\begin{align*}
\|w^L\|_{L^\infty(Q^+_1)} + \|w^N\|_{L^\infty(Q^+_1)} + \|w^{B,2}\|_{L^\infty(Q^+_1)} &< \infty, \\
\|\nabla w^L\|_{L^\ast(Q^+_1)} + \|\nabla w^N\|_{L^\ast(Q^+_1)} + \|\nabla w^{B,2}\|_{L^\ast(Q^+_1)} &< \infty \quad 1 < r < \infty.
\end{align*}
\]

The term \( w^{B,1}_i \) is represented by

\[
w^{B,1}_i(x, t) = c_n \int_0^t D_{x_n} \Gamma_1(x_n, t-s) g^T_n(s) ds \psi_i(x'),
\]

where \( \Gamma_1 \) is the one dimensional Gaussian kernel and \( \psi_i(x') = \int_{\mathbb{R}^{n-1}} \frac{x_i - y_i}{|x'|^2} g^n_S(y') dy' \) is smooth in \( |x'| \leq 1 \).

Using the decay of \( \Gamma_1 \) (see (4.13)), we can obtain

\[
\begin{align*}
\int_1^\infty \int_0^\infty \left| \int_0^t D_{x_n} \Gamma_1(x_n, t-s) g^T_n(s) ds \right|^2 dx_n dt &\leq c \|g^T_n\|_{L^\infty(\mathbb{R})}, \\
\int_1^\infty \int_0^\infty \left| \int_0^t D_{x_n} \Gamma_1(x_n, t-s) g^T_n(s) ds \right|^2 dt dx_n &\leq c \|g^T_n\|_{L^\infty(\mathbb{R})}.
\end{align*}
\]

(1.1)

Note that \( D^2_{x_n} \Gamma_1(x_n, t) = x_n^{-2} K_{\chi_n^2}(t) \), where \( K_{\chi_n^2}(t) = x_n^{-2} K(\frac{t}{x_n^2}) \) with

\[
K(t) = \frac{1}{\sqrt{4\pi}} \left( -\frac{4}{t^2} + \frac{16}{t^2} \right) e^{-\frac{t}{4\pi}} \chi_{t>0} = D_t \left( \frac{1}{\sqrt{4\pi}} t^{-\frac{3}{2}} e^{-\frac{t}{4\pi}} \chi_{t>0} \right).
\]

Using Plancherel Theorem with respect to \( t \) and change of variables, we have

\[
\begin{align*}
\int_0^\infty \int_0^\infty \left| \int_0^t D_{x_n} \Gamma_1(x_n, t-s) g^T_n(s) ds \right|^2 dx_n dt &= \int_0^\infty \int_0^\infty x_n^{-2} |K_{\chi_n^2} * g^T_n(t)|^2 dx_n dt \\
&= \int_0^\infty \left| \frac{\partial g^T_n}{\partial \tau} \right|^2 \int_0^\infty x_n^{-2} |\hat{K}(x_n \tau)|^2 dx_n d\tau \\
&= \int_0^\infty \left| \frac{\partial g^T_n}{\partial \tau} \right|^2 |\tau|^{-\frac{3}{2}} d\tau \int_0^\infty x_n^{-\frac{3}{2}} |\hat{K}(x_n)|^2 dx_n \\
&= \|g^T_n\|_{H^\frac{3}{2}_2(\mathbb{R})} \int_0^\infty x_n^{-\frac{3}{2}} |\hat{K}(x_n)|^2 dx_n.
\end{align*}
\]

(1.2)

Since \( \int_0^\infty K(t) dt = 0 \), it follows that \( |\hat{K}(x_n)| \leq c|x_n| \) near zero and since \( K \in L^1(\mathbb{R}) \), we have that \( \hat{K}(x_n) \) is also bounded. Hence, \( \int_0^\infty x_n^{-\frac{3}{2}} |\hat{K}(x_n)|^2 dx_n \) is well-defined. Thus, from (1.2), we have

\[
\int_0^\infty \int_0^\infty \left| \int_0^t D_{x_n} \Gamma_1(x_n, t-s) g^T_n(s) ds \right|^2 dx_n dt = c \|g^T_n\|_{H^\frac{3}{2}_2(\mathbb{R})}.
\]

(1.3)

Since \( \|g^T_n\|_{H^\frac{3}{2}_2(\mathbb{R})} = \infty \), it follows from (1.1), (1.2) and (1.3) that

\[
\int_{Q^+_1} |D_{x_n} w^{B,1}_i(x, t)|^2 dx dt \geq c \|g^T_n\|_{H^\frac{3}{2}_2(\mathbb{R})} \|\psi\|_{L^\infty(B^*_1)} - c \|g_n\|_{L^\infty(\mathbb{R}^{n-1} \times (0, \infty))} = \infty.
\]

Hence, we complete the proof of Remark 1.3.
The claim in Remark 1.4 is proved in the next lemma.

**Lemma 2.1.** Let

\[
\begin{align*}
g_n^T(t) &= \eta(t)(1-t)^{\frac{1}{2}} \ln^{-1}(1-t), \quad 0 < t < 1, \\
g_n^T(t) &= 0, \quad 1 \leq t,
\end{align*}
\]

where \(\eta \in C_c^\infty(\frac{3}{4}, 2)\) satisfying \(\eta \geq 0\) and \(\eta = 1\) in \(\frac{3}{4} \leq t \leq \frac{3}{2}\) and \(g_n^S \in C_c^\infty(\mathbb{R}^{n-1})\) satisfy the conditions of (6.1). Then,

\[\|p\|_{L^2(Q_1^+)} + \|\nabla p\|_{L^2(Q_1^+)} + \|\nabla w\|_{L^2(Q_1^+)} < \infty\]

for \(1.6\), we have

\[\|\nabla w\|_{L^\infty(Q_1^+)} = \infty.\]

**Proof.** Note that \(g_n^T\) satisfies the condition in (6.1) for \(q = 2\) without the last one. From the proof of Theorem 1.6 we have

\[\|p\|_{L^2(Q_1^+)} + \|\nabla p\|_{L^2(Q_1^+)} + \|u\|_{L^2(Q_1^+)} \leq c\left(\|g_n^T\|_{L^2(0,1)} + \|D_2g_n^T\|_{L^2(0,1)}\right) < \infty.
\]

Hence, (2.4) holds. To prove (2.5), we decompose \(w = w^C + w^R \) and \(w^N\) defined in Proposition 4.1. In the proof of Proposition 4.1 we have

\[\|\nabla (w - w^{B,1})\|_{L^\infty(Q_1^+)} \leq \|g\|_{L^\infty}.
\]

Note that for \(\frac{3}{4} < t,

\[D_{2x_n}^2 \Gamma_1 \ast g_n^T(x_n, t) = c_n \int_{\frac{3}{4}}^t \frac{1}{(t-s)^{\frac{3}{4}}} e^{-\frac{s^2}{4(t-s)}} D_2 g_n^T(s)ds
\]

\[= -c_n \int_{\frac{3}{4}}^t \frac{1}{(t-s)^{\frac{3}{4}}} e^{-\frac{s^2}{4(t-s)}} \frac{1}{2} (1-s)^{-\frac{1}{2}} \ln^{-1}(1-s) (1 - 2 \ln^{-1}(1-s)) ds.
\]

Since \((1 - 2 \ln^{-1}(1-s)) \geq - \ln^{-1}(1-s), \frac{3}{4} < s < 1,\) we have

\[|D_{2x_n}^2 \Gamma_1 \ast g_n^T(x_n, 1)| \geq c \int_{\frac{1}{4}}^1 \frac{1}{(1-s)^{\frac{3}{4}}} e^{-\frac{s^2}{4(1-s)}} (1-s)^{-\frac{1}{2}} |\ln(1-s)|^{-1} ds
\]

\[= c \int_0^1 s^{-1} e^{-\frac{s^2}{4s}} |\ln s|^{-1} ds
\]

\[= c \int_{\frac{c}{4\pi n}}^\infty s^{-1} e^{-s} |\ln \frac{2\pi}{s} n|^2 ds
\]

\[\geq c \int_{\frac{c}{4\pi n}}^1 s^{-1} ds
\]

\[= -c (\ln(4\pi n^2)) - c.
\]

Summing up the above estimates, we have \(\|\nabla w^{B,1}\|_{L^\infty(B_1 \times (0,1))} = \infty,\) depending on the sign of \(\int_{\mathbb{R}^{n-1}} g_n^S(y') \frac{x_3^1}{(x_3^2 - y')^{\frac{n-1}{2}}} dy' ds,\) unless it vanishes. It is not difficult to choose \(g_n^S\) such that the integral is not zero. Thus, we complete the proof of Lemma 2.1.
APPENDIX C. EXAMPLES IN REMARK 4.2 AND REMARK 6.1

- \((g_n^T \in L^\infty(\mathbb{R}) \setminus \dot{B}_{2p}^{\frac{1}{2} - \frac{1}{2p}}(\mathbb{R}), 1 < p < \infty \) in Remark 4.2

Let \(0 < a < 1\) be a number satisfying \(3 - \frac{2}{1+a} < p\). Define \(g_n^T(t)\) by

\[
g_n^T(t) = \sum_{k=1}^{\infty} \chi((2k+1)^{-a},(2k)^{-a})(t),
\]

where \(\chi\) is a characteristic function. It is direct that \(g_n^T \in L^\infty(0,1)\). We claim that \(g_n^T \notin \dot{B}_{2p}^{\frac{1}{2} - \frac{1}{2p}}(0,1)\). Indeed,

\[
\int_0^1 \int_0^1 \frac{|g_n^T(t) - g_n^T(s)|^p}{|t-s|^{1+p(\frac{1}{2}-\frac{1}{2p})}} \, ds \, dt \\
\geq \sum_{k \geq 1} \int (2k)^{-a} \int (2k+1)^{-a} \frac{1}{|t-s|^{\frac{1}{2}+\frac{a}{2}}} \, ds \, dt.
\]

(3.6)

For \(t \in ((2k+1)^{-a}, (2k)^{-a})\) and \(l < k\), there is \(c_1 > 0\) independent of \(k, l\) such that

\[
\int (2l-1)^{-a} \frac{1}{|t-s|^{\frac{1}{2}+\frac{a}{2}}} \, ds \geq \int (2l-2)^{-a} \frac{1}{|t-s|^{\frac{1}{2}+\frac{a}{2}}} \, ds.
\]

This implies that

\[
\sum_{1 \leq l \leq k} \int (2l-1)^{-a} \frac{1}{|t-s|^{\frac{1}{2}+\frac{a}{2}}} \, ds \geq \frac{1}{2} \int (2k)^{-a} \frac{1}{|t-s|^{\frac{1}{2}+\frac{a}{2}}} \, ds dt
\]

\[
= c \left( \frac{1}{((2k)^{-a} - t)^{\frac{1}{2}+\frac{a}{2}}} - \frac{1}{(1-t)^{\frac{1}{2}+\frac{a}{2}}} \right)
\]

\[
\geq c \left( \frac{1}{((2k)^{-a} - t)^{\frac{1}{2}+\frac{a}{2}}} \right).
\]

(3.7)

Hence, \(g_n^2 \notin \dot{B}_{2p}^{\frac{1}{2} - \frac{1}{2p}}(0,1)\) for \(p \geq 3\) and \(a > 0\). Using mean-value Theorem, there is \(\xi \in (2k,2k+1)\), we have

\[
(2k)^{-a} - (2k+1)^{-a} = \frac{(2k+1)^a - (2k)^a}{(2k)^a(2k+1)^a} = \frac{a \xi^{a-1}}{(2k)^a(2k+1)^a} \geq c k^{-1-a}.
\]

(3.8)

By (3.6), (3.7) and (3.8), for \(1 < p < 3\), we have

\[
\int_0^1 \int_0^1 \frac{|g_n^T(t) - g_n^T(s)|^p}{|t-s|^{1+p(\frac{1}{2}-\frac{1}{2p})}} \, ds \, dt \\
\geq c_4 \sum_{k \geq 1} ((2k)^{-a} - (2k+1)^{-a})^{\frac{3}{2} - \frac{a}{2}}
\]

(3.9)

Since \(-(1+a)(\frac{3}{2} - \frac{p}{2}) > -1\), the right-hand side of (3.9) is infinite. Therefore, \(g_n^T \notin \dot{B}_{2p}^{\frac{1}{2} - \frac{1}{2p}}(0,1)\). \(\square\)

- \((D_t g_n^T \notin L^q(0,1) \setminus L^r(0,1)\) for \(q < r\) and \(g_n^T \notin B_{rr}^{1-\frac{1}{2r}}(0,1)\) for \(r > \frac{3p}{2}\) in Remark 6.1
Direct calculations show
\[ \|D_t g_n^T\|_{L^q(0,1)}^q \leq c \int_0^1 \left(1 - \frac{3}{4}\right)^{-\frac{q}{r}} |\ln(t - \frac{3}{4})|^{-q} dt < \infty, \]
\[ \|D_t g_n^T\|_{L^q(0,1)}^r \geq c \int_0^1 \left(1 - \frac{3}{4}\right)^{-\frac{r}{q}} |\ln(t - \frac{3}{4})|^{-r} dt = \infty \quad r > q. \]

Reminding the definition of Besov space, we have
\[ \|g_n^T\|_{B^{r-1}_{q,r}(0,1)} = \int_0^1 \int_0^1 \frac{|g_n^T(t) - g_n^T(s)|^r}{|t-s|^{1+r(1-\frac{3}{4})}} dsdt \]
\[ \geq \int_0^1 \int_0^1 \frac{|g_n(t) - g_n(s)|^r}{|t-s|^{1+r(1-\frac{3}{4})}} dsdt. \]

Using mean-value Theorem, for \( 0 < t < 1 \) and \( \frac{3}{4} < s < t \), there is \( \xi \in (s, t) \) such that \( |g_n^T(t) - g_n^T(s)| = |D_t g_n^T(\xi)||t-s| \geq c(t - \frac{3}{4})^{-\frac{3}{2}} |\ln(t - \frac{3}{4})|^{-1}|t-s|. \) Hence, we have
\[ \|g_n^T\|_{B^{r-1}_{q,r}(0,1)} \geq c \int_0^1 \left(1 - \frac{3}{4}\right)^{-\frac{3}{2}} |\ln(t - \frac{3}{4})|^{-r} \int_0^1 \left(1 - \frac{3}{4}\right)^{-\frac{3}{2} + \frac{r}{2}} |\ln(t - \frac{3}{4})|^{-r} dt \]
\[ = \infty \quad \text{for} \quad r > \frac{3q}{2}. \]

Therefore, this completes to provides an example mentioned Remark 6.1.

[\Box]

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