1. Introduction

We cover some useful techniques in computational aspects of analytic number theory, with specific emphasis on ideas relevant to the evaluation of $L$-functions. These techniques overlap considerably with basic methods from analytic number theory. On the elementary side, summation by parts, Euler-Maclaurin summation, and Mobius inversion play a prominent role. In the slightly less elementary sphere, we find tools from analysis, such as Poisson summation, generating function methods, Cauchy’s residue theorem, asymptotic methods, and the fast Fourier transform. We then describe conjectures and experiments that connect number theory and random matrix theory.

2. Basic methods

2.1. Summation by parts. Summation by parts can be viewed as a discrete form of integration by parts. Let $f$ be a function from $\mathbb{Z}^+$ to $\mathbb{R}$ or $\mathbb{C}$, and $g$ a real or complex valued function of a real variable. Then

\begin{equation}
\sum_{1 \leq n \leq x} f(n)g(n) = \left( \sum_{1 \leq n \leq x} f(n) \right) g(x) - \int_1^x \left( \sum_{1 \leq n \leq t} f(n) \right) g'(t) dt.
\end{equation}

Here we are assuming that $g'$ exists and is continuous on $[1, x]$. One verifies this identity by writing the integral as $\int_1^2 + \int_2^3 + \ldots + \int_{[x]}^x$, noticing that the sum in each integral is constant on each open interval, integrating, and telescoping. Although our integral begins at $t = 1$, it is sometimes convenient to start earlier, for example at $t = 0$. This doesn’t change the value of the integral, the sum in the integrand being empty if $t < 1$. Formula (1) can also be interpreted in terms of the Stieltjes integral.
A slightly more general form of partial summation is over a set \( \{ \lambda_1, \lambda_2, \ldots \} \) of increasing real numbers:

\[
\sum_{\lambda_n \leq x} f(n) g(\lambda_n) = \left( \sum_{\lambda_n \leq x} f(n) \right) g(x) - \int_{\lambda_1}^{x} \left( \sum_{\lambda_n \leq t} f(n) \right) g'(t) \, dt.
\]

As an application, let

\[
\pi(x) = \sum_{p \leq x} 1
\]
denote the number of primes less than or equal to \( x \), and

\[
\theta(x) = \sum_{p \leq x} \log p
\]
denote the number primes up to \( x \) with each prime weighted by its logarithm. The famous equivalence between \( \pi(x) \sim x / \log x \) and \( \theta(x) \sim x \) can be verified using partial summation. Write

\[
\pi(x) = \sum_{p \leq x} \log p \frac{1}{\log p} = \theta(x) \frac{1}{\log x} + \int_2^{x} \theta(t) \frac{dt}{t \log t},
\]

from which it follows that if \( \theta(x) \sim x \) then \( \pi(x) \sim x / \log x \). The converse follows from

\[
\theta(x) = \sum_{p \leq x} 1 \cdot \log p = \pi(x) \log x - \int_2^{x} \pi(t) \frac{dt}{t}.
\]

2.2. Euler-Maclaurin summation. A powerful application of partial summation occurs when the function \( f(n) \) is identically equal to 1 and the function \( g(t) \) is many times differentiable. In that case, summation by parts specializes to the Euler Maclaurin formula which involves one summation by parts with \( f(n) = 1 \) followed by repeated integration by parts. For \( a, b \in \mathbb{Z}, a < b \), partial summation gives

\[
\sum_{a < n \leq b} g(n) = (b-a)g(b)-\int_{a}^{b} ([t]-a)g'(t)dt = bg(b)-ag(a)-\int_{a}^{b} [t]g'(t)dt.
\]

Here, we have chosen to start the integral at \( t = a \), rather than at \( t = a+1 \). Writing \([t] = t - \{t\}\), with \( \{t\} \) the fractional of \( t \) we get

\[
\sum_{a < n \leq b} g(n) = \int_{a}^{b} g(t)dt + \int_{a}^{b} \{t\}g'(t)dt.
\]

The second term on the r.h.s. should be viewed as the necessary correction that arises from replacing the sum on the left with an integral.
The next step is to write \( \{t\} = 1/2 + (\{t\} - 1/2) \), the latter term having nicer properties than \( \{t\} \), for example being odd and also having zero constant term in its Fourier expansion. So

\[
(2) \quad \sum_{a < n \leq b} g(n) = \int_{a}^{b} g(t)dt + \frac{1}{2}(g(b) - g(a)) + \int_{a}^{b} (\{t\} - 1/2)g'(t)dt.
\]

Integrating the second integral repeatedly by parts leads naturally to the introduction of Bernoulli polynomials, named after Jacob Bernoulli (1654-1705), who discovered them in connection to the problem of studying sums of positive integer powers of consecutive integers. During the 1730’s Euler (1707-1783), who studied mathematics from Jacob’s brother Johann (1667-1748), developed the summation formula being described in connection with computing reciprocals of powers and Euler’s constant.

2.2.1. Bernoulli Polynomials. The Bernoulli polynomials are defined recursively by the following relations

\[
B_0(t) = 1
\]

\[
B'_k(t) = kB_{k-1}(t), \quad k \geq 1
\]

\[
\int_{0}^{1} B_k(t)dt = 0, \quad k \geq 1.
\]

The second equation determines \( B_k(t) \) recursively up to the constant term, and the third equation fixes the constant. The first few Bernoulli polynomials are listed in Table 1.

| \( k \) | \( B_k(t) \) |
|-------|-------------|
| 0     | 1           |
| 1     | \( t - 1/2 \) |
| 2     | \( t^2 - t + 1/6 \) |
| 3     | \( t^3 - 3/2t^2 + 1/2t \) |
| 4     | \( t^4 - 2t^3 + t^2 - 1/30 \) |
| 5     | \( t^5 - 5/2t^4 + 5/3t^3 - 1/6t \) |

Table 1. The first few Bernoulli polynomials

Let \( B_k = B_k(0) \) denote the constant term of \( B_k(t) \). \( B_k \) is called the \( k \)-th Bernoulli number. We state basic properties of the Bernoulli polynomials. Expansion in terms of Bernoulli numbers:

\[
B_k(t) = \sum_{m=0}^{k} \binom{k}{m} B_{k-m}t^m, \quad k \geq 0
\]
Generating function:
\[
\frac{ze^{zt}}{e^z - 1} = \sum_{0}^{\infty} B_k(t)z^k/k!, \quad |z| < 2\pi
\]

Fourier series:
(3) \[B_1({\{t\}}) = -\frac{1}{\pi} \sum_{1}^{\infty} \frac{\sin(2\pi mt)}{m}, \quad t \notin \mathbb{Z}\]

(4) \[B_k({\{t\}}) = -k! \sum_{m \neq 0} \frac{e^{2\pi imt}}{(2\pi im)^k}, \quad k \geq 2.\]

Functional equation:
\[B_k(t) = (-1)^kB_k(1-t), \quad k \geq 0\]

Difference equation:
(5) \[
\frac{B_{k+1}(t+1) - B_{k+1}(t)}{k+1} = t^k, \quad k \geq 0
\]

Special values:
\[B_k(1) = \begin{cases} (-1)^kB_k(0), & k \geq 0 \\
0, & k \text{ odd, } k \geq 3 \\
1/2, & k = 1 \end{cases}\]
i.e.
(6) \[B_k(1) = B_k(0), \quad \text{unless } k = 1\]

Recursion:
\[
\sum_{m=0}^{k-1} \binom{k}{m} B_m = 0, \quad k \geq 2
\]

Equation (3) can be obtained directly. The other formulae can be verified using the defining relations and induction.

Property (4) can be used to obtain a formula for \(\zeta(2m)\). Let
\[
\zeta(s) = \sum_{1}^{\infty} n^{-s}, \quad \Re s > 1.
\]

Taking \(t = 0, k = 2m\), even, in the Fourier expansion of \(B_k({\{t\}})\) gives
\[
B_{2m} = \frac{(-1)^{m+1}(2m)!}{(2\pi)^{2m}} 2\zeta(2m)
\]
so that
\[
\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!} B_{2m},
\]
a formula discovered by Euler. Because $\zeta(2m) \to 1$ as $m \to \infty$, we have

$$B_{2m} \sim \frac{(-1)^{m+1}2(2m)!}{(2\pi)^{2m}}$$

as $m \to \infty$.

2.2.2. Euler-Maclaurin continued. Returning to (2), we write

$$\int_a^b (\{t\} - 1/2)g'(t)dt = \int_a^b B_1(\{t\})g'(t)dt.$$

Breaking up the integral $\int_a^b = \int_a^{a+1} + \int_{a+1}^{a+2} + \ldots + \int_{b-1}^b$, integrating by parts, and noting that $B_2(1) = B_2(0)$, we get, assuming that $g^{(2)}$ exists and is continuous on $[a,b]$,

$$\frac{B_2}{2} (g'(b) - g'(a)) - \int_a^b \frac{B_2(\{t\})}{2} g^{(2)}(t)dt.$$

Repeating, using $B_k(1) = B_k(0)$ if $k \geq 2$, leads to the Euler-Maclaurin summation formula. Let $K$ be a positive integer. Assume that $g^{(K)}$ exists and is continuous on $[a,b]$. Then

$$\sum_{a<n \leq b} g(n) = \int_a^b g(t)dt + \sum_{k=1}^{K} \frac{(-1)^k B_k}{k!} (g^{(k-1)}(b) - g^{(k-1)}(a))$$

$$+ \frac{(-1)^{K+1}}{K!} \int_a^b B_K(\{t\})g^{(K)}(t)dt.$$

2.2.3. Application: Sums of consecutive powers. We apply Euler-Maclaurin summation to obtain Bernoulli’s formula for sums of powers of consecutive integers. Let $r \geq 0$ be an integer. Then

$$\sum_{n=1}^{N} n^r = \frac{B_{r+1}(N+1) - B_{r+1}(1)}{r+1}.$$

We can verify this directly using property (5), substituting $n = 1, 2, \ldots, N$, and telescoping. However, it is instructive to apply the Euler-Maclaurin formula, which, once begun, carries through in an automatic fashion. In this example, we have $g(t) = t^r$. Notice that $g^{(r+1)}(t) = 0$, and that

$$g^{(m)}(N) - g^{(m)}(0) = \begin{cases} 
  r(r-1) \ldots (r-m+1)N^{r-m}, & m \leq r-1 \\
  0, & m \geq r.
\end{cases}$$
If \( m = 0 \) we set \( r(r - 1) \ldots (r - m + 1) = 1 \). Then

\[
\sum_{n=1}^{N} n^r = \int_0^N t^r dt + \sum_{k=1}^{r} \frac{(-1)^k B_k}{k!} r(r - 1) \ldots (r - k + 2) N^{r-k+1}
\]

\[
= \int_0^N t^r dt + \sum_{k=1}^{r} \frac{(-1)^k B_k}{r - k + 1} \binom{r}{k} N^{r-k+1}
\]

\[
= \int_0^N \sum_{k=0}^{r} (-1)^k B_k \binom{r}{k} t^{r-k} dt = \int_0^N (-1)^r B_r(-t) dt = \int_0^N B_r(t + 1) dt
\]

\[
= (B_{r+1}(N + 1) - B_{r+1}(1))/(r + 1).
\]

If \( r \geq 1 \), the last line simplifies according to (6) and equals

\[
\frac{B_{r+1}(N + 1) - B_{r+1}}{r + 1}.
\]

2.2.4. Application: \( \zeta(s) \). The Euler-Maclaurin formula can be used to obtain the analytic continuation of \( \zeta(s) \) and also provides a useful expansion for its numeric evaluation. Consider

\[
\sum_{1}^{N} n^{-s} = 1 + \sum_{2}^{N} n^{-s}
\]

with \( \Re s > 1 \). We have started the sum at \( n = 2 \) rather than \( n = 1 \) to avoid difficulties near \( t = 0 \) below. Applying Euler-Maclaurin summation, with \( g(t) = t^{-s} \), \( g^{(m)}(t) = (-1)^m s(s + 1) \ldots (s + m - 1)t^{s-m} \), we get

\[
\sum_{1}^{N} n^{-s} = 1 + \int_1^{N} t^{-s} dt - \sum_{k=1}^{K} \frac{B_k}{k} \binom{s + k - 2}{k - 1} (N^{-s-k+1} - 1)
\]

\[
= \left( s + K - 1 \right) \int_1^{N} B_K(\{t\}) t^{-s-K} dt.
\]

Evaluating the first integral, taking the limit as \( N \to \infty \), with \( \Re s > 1 \), we get

(7)

\[
\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{2}^{K} \left( s + k - 2 \right) \frac{B_k}{k} \binom{s + K - 1}{K} \int_1^{\infty} B_K(\{t\}) t^{-s-K} dt.
\]

While we started with \( \Re s > 1 \), the r.h.s. is meromorphic for \( \Re s > -K + 1 \), so gives the meromorphic continuation of \( \zeta(s) \) in this region, with the only pole being the simple pole at \( s = 1 \).
Taking \( s = 2 - K, \ K \geq 2, \)
\[
\zeta(2 - K) = 1 - \frac{1}{K - 1} \sum_{k=0}^{K-1} (-1)^k \binom{K-1}{k} B_k = \frac{(-1)^K B_{K-1}}{K - 1}.
\]
Thus,
\[
\zeta(1 - 2m) = -B_{2m}/(2m), \quad m = 1, 2, 3, \ldots \\
\zeta(-2m) = 0, \quad m = 1, 2, 3, \ldots \\
\zeta(0) = -1/2.
\]
Applying the functional equation for \( \zeta \) (see for example Roger Heath-Brown’s notes)
\[
\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s)
\]
and
\[
\Gamma(1/2) = \pi^{1/2} = -\frac{1}{2} - \frac{3}{2} - \frac{5}{2} \ldots - \frac{(2m-1)}{2} \Gamma(1/2 - m)
\]
gives another proof of Euler’s identity
\[
\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!} B_{2m}, \quad m \geq 1.
\]

2.2.5. Computing \( \zeta(s) \) using Euler-Maclaurin summation. Next we describe how to adapt the above to obtain a practical method for numerically evaluating \( \zeta(s) \). From a computational perspective, the following works better than using (7). Let \( N \) be a large positive integer, proportional in size to \( |s| \). We will make this more explicit shortly. For \( \Re s > 1 \), write

\[
(8) \quad \zeta(s) = \sum_{1}^{\infty} n^{-s} = \sum_{1}^{N} n^{-s} + \sum_{N+1}^{\infty} n^{-s}.
\]

The first sum on the r.h.s. is evaluated term by term, while the second sum is evaluated using Euler-Maclaurin summation

\[
(9) \quad \sum_{N+1}^{\infty} n^{-s} = \frac{N^{1-s}}{s-1} + \sum_{1}^{K} \left( s + k - 2 \right) \frac{B_k}{k} N^{-s-k+1} - \left( s + K - 1 \right) \int_{N}^{\infty} B_K(\{t\}) t^{-s-K} dt.
\]

As before, the r.h.s. above gives the meromorphic continuation of the l.h.s. to \( \Re s > -K + 1 \). Breaking up the sum over \( n \) in this fashion allows us to throw away the integral on the r.h.s., and obtain sharp estimates for its neglected contribution. First, from property (4),
\[
|B_K(\{t\})| \leq \frac{K!}{(2\pi)^K} 2 \zeta(K).
\]
It is convenient to take $K = 2K_0$, even, in which case we have from (8)

$$|B_{2K_0}(\{t\})| \leq B_{2K_0}.$$ 

Therefore, for $s = \sigma + i\tau, \sigma > -2K_0 + 1$,

$$\left| \left( \frac{s + 2K_0 - 1}{2K_0} \right) \int_N^\infty B_{2K_0}(\{t\}) t^{-s - 2K_0} dt \right| \leq \left| \frac{(s + 2K_0 - 1)}{\sigma + 2K_0 - 1} \right| \left( \frac{(s + 2K_0 - 2)}{2K_0 - 1} \right) B_{2K_0} \frac{N^\sigma}{\sigma + 2K_0 - 1} \frac{N^{-\sigma - 2K_0 + 1}}{N^{-\sigma - 2K_0 + 1}}$$

$$= \left| \frac{(s + 2K_0 - 1)}{\sigma + 2K_0 - 1} \right| |\text{last term taken}|.$$ 

A more precise estimate follows by comparison of $B_{2K_0}$ with $\zeta(2K_0)$, and we have that the remainder is

$$\leq \frac{\zeta(2K_0)}{\pi N^\sigma} \frac{|s + 2K_0 - 1|}{\sigma + 2K_0 - 1} \prod_{j=0}^{2K_0-2} \left| \frac{s + j}{2\pi N} \right|.$$ 

We start to win when $2\pi N$ is bigger than $|s|, |s + 1|, \ldots, |s + 2K_0 - 2|$. There are two parameters which we need to choose: $K_0$ and $N$, and we also need to specify the number of digits accuracy, $\text{Digits}$, we desire. For example, with $\sigma \geq 1/2$, taking

$$2\pi N \geq 10|s + 2K_0 - 2|$$

with

$$2K_0 - 1 > \text{Digits} + \frac{1}{2} \log_{10}(|s + 2K_0 - 1|)$$

achieves the desired accuracy. The main work involves the computation of the sum $\sum_1^N n^{-s}$ consisting of $O(|s|)$ terms. Later we will examine the Riemann-Siegel formula and its smoothed variants which, for $\zeta(s)$, involves a main sum of $O(|s|^{1/2})$ terms. However, for high precision evaluation of $\zeta(s)$, especially with $s$ closer to the real axis, the Euler-Maclaurin formula remains an ideal method allowing for sharp and rigorous error estimates and reasonable efficiency.

In fact, we can turn the above scheme into a computation involving $O(|s|^{1/2})$ operations but requiring $O((\text{Digits} + \log |s|) \log |s|)$ precision due to cancellation that occurs. In (8) choose $N \sim |10s/(2\pi)|^{1/2}$, and assume that $\Re s \geq 1/2$. Expand $B_{2K_0}(\{t\})$ into its Fourier series (4). We only need $M = O(|s|^{1/2})$ terms of the Fourier expansion to assure a contribution from the neglected terms smaller than the desired
precision. Each term contributes
\[ K! \binom{s + K - 1}{K} \frac{1}{(2\pi i m)^K} \int_N^\infty e^{2\pi i m t} t^{-s-K} dt, \]
so the neglected terms contribute altogether less than
\[ \frac{1}{N^\sigma} \frac{|s + K - 1|}{\sigma + K - 1} \left( \prod_{j=0}^{K-2} \frac{|s + j|}{2\pi N} \right) \left( \sum_{M=1}^{\infty} \frac{2}{m^K} \right). \]
Here we have combined the \( \pm m \) terms together. Comparing to an integral, the sum above is \( < 2/(K-1)M^{K-1} \) and so the neglected terms contribute less than
\[ \frac{2}{(K-1)N^\sigma} \frac{|s + K - 1|}{\sigma + K - 1} \left( \prod_{j=0}^{K-2} \frac{|s + j|}{2\pi M N} \right). \]
We start to win when \( 2\pi MN \) exceeds \( |s|, \ldots, |s + K - 2| \). For \( \sigma \geq 1/2 \), choose \( K > \text{Digits} + \log_{10}(|s + K - 1|) + 1 \) and \( M = N \) with
\[ 2\pi MN \geq 10|s + K - 2|. \]
Asymptotically, we can improve the above choices so as to achieve \( M = N \sim |s|^{1/2}/(2\pi) \), the same as in the Riemann-Siegel formula. The only drawback is that extra precision as described above is needed. The individual terms summed in (10) are somewhat large in comparison to the final result, this coming form the binomial coefficients which have numerator \((s + k - 2) \ldots (s + 1)s\), and this leads to cancellation.

Finally to compute the contribution to the Fourier expansion from the terms with \(|m| \leq M\), we assume that \( 4|K| \) so that the terms \( \pm m \) together involve in (10) the integral
\[ \int_N^\infty \cos(2\pi m t) t^{-s-K} dt = (2\pi m)^{s+K-1} \int_{2\pi m N}^\infty \cos(u) u^{-s-K} du. \]
This can be expressed in terms of the incomplete \( \Gamma \) function
\[ \int_w^\infty \cos(u) u^{z-1} du = \frac{1}{2} \left( e^{-\pi iz/2} \Gamma(z, iw) + e^{\pi iz/2} \Gamma(z, -iw) \right). \]
See Section 3 which describes properties of the incomplete \( \Gamma \) function and methods for its evaluation.

The Euler-Maclaurin formula can also be used to evaluate Dirichlet \( L \)-functions. It works in that case due to the periodic nature of the corresponding Dirichlet coefficients. For general \( L \)-functions, there are smoothed Riemann-Siegel type formulae. These are described later.
2.3. Mobius inversion with an application to sums and products over primes. Computations in analytic number theory often involve evaluating sums or products over primes. For example, let $\pi_2(x)$ denote the number of twin primes $(p, p+2)$, with $p$ and $p+2$ both prime and less than or equal to $x$. The famous conjecture of Hardy and Littlewood predicts that

$$\pi_2(x) \sim 2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \frac{x}{(\log x)^2}.$$  

Generally, it is easier to deal with a sum rather than a product, so we turn this product over primes into a sum by expressing it as

$$\exp \left( \sum_{p>2} \log\left(1 - \frac{2}{p}\right) - 2 \log\left(1 - \frac{1}{p}\right) \right).$$

Letting $f(p) = \log\left(1 - \frac{2}{p}\right) - 2 \log\left(1 - \frac{1}{p}\right)$, we have

$$f(p) = -\sum_{m=1}^{\infty} \frac{2^m - 2}{mp^m}$$

hence

$$\sum_{p>2} f(p) = -\sum_{m=1}^{\infty} \frac{2^m - 2}{m} (h(m) - 1/2^m)$$

with

$$h(s) = \sum_p p^{-s}, \quad \Re s > 1.$$  

We therefore need an efficient method for computing $h(m)$. This will be dealt with below. Notice that $h(m) - 1/2^m \sim 1/3^m$ so the sum on the r.h.s. of (11) converges exponentially fast. We can achieve faster convergence by writing

$$\sum_{p>2} f(p) = \sum_{2<p\leq P} f(p) + \sum_{p>P} f(p),$$

summing the terms in the first sum, and expressing the second sum as

$$-\sum_{m=1}^{\infty} \frac{2^m - 2}{m} (h(m) - 1/2^m - \ldots - 1/P^m).$$

A second example involves the computation of constants that arise in conjectures for moments of $\zeta(s)$. The Keating-Snaith conjecture [KeS] asserts that

$$(12) \quad M_k(T) := \frac{1}{T} \int_{o}^{T} |\zeta(1/2 + it)|^{2k} dt \sim \frac{a_k g_k}{k!} (\log T)^{k^2}.$$
\( a_k = \prod_p \left( 1 - \frac{1}{p} \right)^{k^2} \sum_{m=0}^{\infty} \binom{m + k - 1}{m}^2 p^{-m} \)

\[ \prod_p \left( 1 - \frac{1}{p} \right)^{(k-1)^2} \sum_{j=0}^{k-1} \binom{k-1}{j}^2 p^{-j} \]

and

\( g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!} \).

The placement of \( k^2! \) is to ensure that \( g_k \) is an integer \([\text{CF}]\). Keating and Snaith also provide a conjecture for complex values, \( \Re k > -1/2 \), of which the above is a special case. Keating and Snaith used random matrix theory to identify the factor \( g_k \). The form of (12), without identifying \( g_k \), was conjectured by Conrey and Ghosh \([\text{CG}]\).

The above conjecture gives the leading term for the asymptotics for the moments of \( |\zeta(1/2 + it)| \). In \([\text{CFKRS}]\) a conjecture is given for the full asymptotics of \( M_k(T) \):

\[ M_k(T) \sim \sum_{r=0}^{k^2} c_r(k) (\log T)^{k^2-r} \]

where \( c_0(k) = a_k g_k / k^2! \) coincides with the Keating-Snaith leading term and where the degree \( k^2 \) polynomial is given implicitly as an elaborate multiple residue. Explicit expressions for \( c_r(k) \) are worked out in \([\text{CFKRS3}]\) and are given as \( c_0(k) \) times complicated rational functions in \( k \), generalized Euler constants, and sums over primes involving \( \log(p) \), \( \text{}_{2} F_1(k, k; 1; p^{-1}) \) and its derivatives. One method for computing the \( c_r(k) \)'s involves as part of a single step the computation of sums of the form

\[ \sum_p \frac{(\log p)^r}{p^m}, \quad m = 2, 3, 4, \ldots \quad r = 0, 1, 2, \ldots \]

We now describe how to efficiently compute \( h(s) = \sum_p p^{-s} \) and the sums in (14). Take the logarithm of

\[ \zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad \Re s > 1 \]

and apply the Taylor series for \( \log(1 - x) \) to get

\[ \log \zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m} h(ms), \quad \Re s > 1. \]
Let $\mu(n)$, the Mobius $\mu$ function, denote the Dirichlet coefficients of $1/\zeta(s)$:

$$1/\zeta(s) = \prod_p (1 - p^{-s}) = \sum_1^\infty \mu(n)n^{-s}.$$ 

We have

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by the square of an integer } > 1 \\ (-1)^\text{number of prime factors of } n & \text{if } n \text{ is squarefree} \end{cases}$$

and

$$\sum_{n|r} \mu(n) = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The last property can be proven by writing the sum of the left as $\prod_{p|r}(1 - 1)$, and it allows us to invert equation (15)

$$\sum_{m=1}^\infty \frac{\mu(m)}{m} \log \zeta(ms) = \sum_{m=1}^\infty \frac{\mu(m)}{m} \sum_{n=1}^\infty \frac{h(mns)}{n}$$

$$= \sum_{r=1}^\infty \frac{h(rs)}{r} \sum_{m|r} \mu(m) = h(s),$$

i.e.

$$\sum_p p^{-s} = \sum_{m=1}^\infty \frac{\mu(m)}{m} \log \zeta(ms).$$

This is an example of Mobius inversion, and expresses $h(s)$ as a sum involving $\zeta$. Mobius inversion can be interpreted as a form of the sieve of Eratosthenes.

Notice that $\zeta(ms) = 1 + 2^{-ms} + 3^{-ms} + \ldots$ tends to 1, and hence $\log \zeta(ms)$ tends to 0, exponentially fast as $m \to \infty$. Therefore, the number of terms needed on the r.h.s. of (16) is proportional to the desired precision.

To compute the series appearing in (14) we can differentiate $h(s)$ $r$ times, obtaining

$$\sum_p \frac{(\log p)^r}{p^s} = (-1)^r \sum_{m=1}^\infty \frac{\mu(m)}{m}(\log \zeta(ms))^{(r)}.$$ 

In both (16) and (17), we can use Euler-Maclaurin summation to compute $\zeta$ and its derivatives. The paper of Henri Cohen [C] is a good reference for computations involving sums or products of primes.
2.4. Poisson summation as a tool for numerical integration. Let \( f \in L^1(\mathbb{R}) \) and let
\[
\hat{f}(y) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i yt} dt.
\]
denote its Fourier transform. The Poisson summation formula asserts, for \( f, \hat{f} \in L^1(\mathbb{R}) \) and of bounded variation, that
\[
\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).
\]
We often encounter the Poisson summation formula as a potent theoretical tool in analytic number theory. For example, the functional equations of the Riemann \( \zeta \) function and of the Dedekind \( \eta \) function can be derived by exploiting Poisson summation. However, Poisson summation is often overlooked in the setting of numerical integration where it provides justification for carrying out certain numerical integrals in a very naive way.

Let \( \Delta > 0 \). By a change of variable
\[
\Delta \sum_{n=-\infty}^{\infty} f(n\Delta) = \sum_{n=-\infty}^{\infty} \hat{f}(n/\Delta) = \hat{f}(0) + \sum_{n\neq 0} \hat{f}(n/\Delta)
\]
so that
\[
\int_{-\infty}^{\infty} f(t) dt - \Delta \sum_{n=-\infty}^{\infty} f(n\Delta) = -\sum_{n\neq 0} \hat{f}(n/\Delta)
\]
tells us how closely the Riemann sum \( \Delta \sum_{n=-\infty}^{\infty} f(n\Delta) \) approximates the integral \( \int_{-\infty}^{\infty} f(t) dt \).

The main point is that if \( \hat{f} \) is rapidly decreasing then we get enormous accuracy from the Riemann sum, even with \( \Delta \) not too small. For example, with \( \Delta = 1/10 \), the first contribution comes from \( \hat{f}(\pm 10) \) which can be extremely small if \( \hat{f} \) decreases sufficiently fast.

As a simple application, let \( f(t) = \exp(-t^2/2) \). Then \( \hat{f}(y) = \sqrt{2\pi} \exp(-2\pi^2 y^2) \), and so
\[
\sum_{n\neq 0} \hat{f}(n/\Delta) = O(\exp(-2\pi^2/\Delta^2)).
\]
Therefore
\[
\int_{-\infty}^{\infty} \exp(-t^2/2) dt - \Delta \sum_{n=-\infty}^{\infty} \exp(-(n\Delta)^2/2) = O(\exp(-2\pi^2/\Delta^2)).
\]
As everyone knows, the integral on the l.h.s. equals $\sqrt{2\pi}$. Taking $\Delta = 1/10$, we therefore get

$$\Delta \sum_{n=-\infty}^{\infty} \exp\left(-\left(n\Delta\right)^2/2\right) = \sqrt{2\pi} + \epsilon$$

with $\epsilon \approx 10^{-857}$. We can truncate the sum over $n$ roughly when

$$\frac{(n\Delta)^2}{2} > \frac{2\pi^2}{\Delta^2},$$

i.e. when $n > 2\pi/\Delta^2$. So only 628 terms (combine $\pm n$) are needed to evaluate $\sqrt{2\pi}$ to about 857 decimal place accuracy!

This method can be applied to the problem of computing certain probability distributions that arise in random matrix theory. Let $U$ be an $N \times N$ unitary matrix, with eigenvalues $\exp(i\theta_1), \ldots, \exp(i\theta_N)$, and characteristic equation

$$Z(U, \theta) = \prod_{1 \leq j < m \leq N} (\exp(i\theta_j) - \exp(i\theta_m))$$

evaluated on the unit circle at the point $\exp(i\theta)$. In making their conjecture for the moments of $|\zeta(1/2 + it)|$, Keating and Snaith [KeS] studied the analogous random matrix theory problem of evaluating the moments of $|Z(U, \theta)|$, averaged according to Haar measure on $U(N)$. The characteristic function of a matrix is a class function that only depends on the eigenvalues of the matrix. For class functions, the Weyl integration formula gives Haar measure in terms of the eigenangles, the invariant probability measure on $U(N)$ being

$$\frac{1}{(2\pi)^N N!} \prod_{1 \leq j < m \leq N} |e^{i\theta_j} - e^{i\theta_m}|^2 d\theta_1 \ldots d\theta_N.$$

Therefore, $M_N(r)$, the $r$th moment of $|Z(U, \theta)|$, is given by

$$M_N(r) = \frac{1}{(2\pi)^N N!} \int_0^{2\pi} \ldots \int_0^{2\pi} \prod_{1 \leq j < m \leq N} |e^{i\theta_j} - e^{i\theta_m}|^2 |Z(U, \theta)|^r d\theta_1 \ldots d\theta_N,$$

for $\Re r > -1$. This integral happens to be a special case of Selberg’s integral, and Keating and Snaith consequently determined that

$$M_N(r) = \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+r)}{\Gamma(j+r/2)^2}.$$

Notice that this does not depend on $\theta$.

Say we are interested in computing the probability distribution of $|Z(U, \theta)|$. One can recover the probability density function from the
moments as follows. We can express the moments of $|Z(U, \theta)|$ in terms of its probability density function. Let

$$\text{prob}(0 \leq a \leq |Z(U, \theta)| \leq b) = \int_a^b p_N(t) dt.$$  

Then

$$(18) \quad M_N(r) = \int_0^\infty p_N(t) t^r dt$$

is a Mellin transform, and taking the inverse Mellin transform we get

$$(19) \quad p_N(t) = \frac{1}{2\pi i} t^{-\nu} \int_{\nu-i\infty}^{\nu+i\infty} \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j + r)}{\Gamma(j + (r/2)^2)} t^{-r} dr$$

with $\nu$ to the right of the poles of $M_N(r)$, $\nu > -1$. There is an extra $1/t$ in front of the integral since the Mellin transform $(18)$ is evaluated at $r$ rather than at $r - 1$.

To compute $p_N(t)$ we could shift the line integral to the left picking up residues at the poles of $M_N(r)$, but as $N$ grows this becomes burdensome. Instead, we can compute the inverse Mellin transform $(19)$ as a simple Riemann sum.

Changing variables we have

$$p_N(t) = \frac{1}{2\pi i} t^{-\nu} \int_{\nu-i\infty}^{\nu+i\infty} \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j + \nu + iy)}{\Gamma(j + (\nu + iy)/2)^2} t^{-\nu-i\nu} i dy.$$  

Let

$$f_t(y) = \frac{1}{2\pi} \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j + \nu + iy)}{\Gamma(j + (\nu + iy)/2)^2} t^{-\nu-1-i\nu}.$$  

This function also depends on $\nu$ and $N$, but we do not include them explicitly on the l.h.s. so as to simplify our notation. The above integral equals

$$(20) \quad p_N(t) = \int_{-\infty}^{\infty} f_t(y) dy.$$  

To estimate the error in computing this integral as a Riemann sum using increments of size $\Delta$, we need bounds on the Fourier transform

$$(21) \quad \hat{f}_t(u) = \int_{-\infty}^{\infty} f_t(y) e^{-2\pi i uy} dy.$$  

However,

$$f_t(y) e^{-2\pi i uy} = f_{te^{2\pi u}}(y) e^{2\pi u(\nu + 1)}$$

and so

$$\hat{f}_t(u) = e^{2\pi u(\nu + 1)} p_N(te^{2\pi u}).$$
Now, $p_N(t)$ is supported in $[0, 2^N]$, because $0 \leq |Z(U, \theta)| \leq 2^N$. Hence if $u > (N \log 2 - \log t)/(2\pi)$ then $\hat{f}_t(u) = 0$. Thus, for $0 < t < 2^N$, if we evaluate (20) as a Riemann sum with step size $\Delta < 2\pi/(N \log 2 - \log t)$ the error is

$$\sum_{n \neq 0} \hat{f}_t(n/\Delta) = \sum_{n < 0} \hat{f}_t(n/\Delta)$$

since the terms with $n > 0$ are all zero. On the other hand, with $n < 0$ we get

$$\hat{f}_t(-|n|/\Delta) = e^{-2\pi(\nu+1)|n|/\Delta} p_N(t e^{-2\pi|n|/\Delta}) \leq e^{-2\pi(\nu+1)|n|/\Delta} p_{\text{max}}$$

where $p_{\text{max}}$ denotes the maximum of $p_N(t)$ (an upper bound for $p_N(t)$ can be obtained from (19)).

Therefore, choosing

$$\Delta = \frac{2\pi}{\text{Digits } \log 10 + N \log 2 - \log t}$$

and setting $\nu = 0$ we have

$$\hat{f}_t(-|n|/\Delta) < (10^{-\text{Digits}} 2^{-N} t)^{|n|} p_{\text{max}}$$

Summing over $n = -1, -2, -3, \ldots$ we get an overall bound of

$$10^{-\text{Digits}} p_{\text{max}}/(1 - 10^{-\text{Digits}}) \approx 10^{-\text{Digits}} p_{\text{max}}.$$

We could choose $\nu$ to be larger, i.e. shift our line integral (19) to the right, and thus achieve more rapid decay of $\hat{f}_t(u)$ as $u \to -\infty$. However, this leads to precision issues. As $\nu$ increases, the integrand in (19) increases in size, yet $p_N(t)$ remains constant for given $N$ and $t$. Therefore cancellation must occur when we evaluate the Riemann sum and higher precision is needed to capture this cancellation. We leave it as an exercise to determine the amount of precision needed for a given value of $\nu$.

Another application appears in [RS] where Poisson summation is used to compute, on a logarithmic scale, the probability that $\pi(x)$, the number of primes up to $x$, exceeds $\text{Li}(x) = \int_2^x dt/\log(t)$. The answer turns out to be $0.00000026\ldots$

Later in this paper, we apply this method to computing certain complicated integrals that arise in the theory of general $L$-functions.

3. Analytic aspects of $L$-function computations

3.1. Riemann-Siegel formula. The Riemann-Siegel formula expresses the Riemann $\zeta$ function as a main sum involving a truncated Dirichlet series and correction terms. The formula is often presented with $\Re s = 1/2$, but can be given for $s$ off the critical line. See [OS] for a
nice presentation of the formula for $1/2 \leq \Re s \leq 2$ and references. Here we stick to $\Re s = 1/2$.

Let

$$Z(t) = e^{it} \zeta(1/2 + it)$$

(22)

$$e^{it} = \left( \frac{\Gamma(1/4 + it/2)}{\Gamma(1/4 - it/2)} \right)^{1/2} \cos(it/2).$$

The rotation factor $e^{it}$ is chosen so that $Z(t)$ is real.

For $t > 2\pi$, let $a = (t/(2\pi))^{1/2}$, $N = \lfloor a \rfloor$, $\rho = \{ a \} = a - \lfloor a \rfloor$ the fractional part of $a$. Then

$$Z(t) = 2 \sum_{n=1}^{N} n^{-1/2} \cos(t \log(n) - \theta(t)) + R(t)$$

where

$$R(t) = \frac{(-1)^{N+1}}{a^{1/2}} \sum_{r=0}^{m} \frac{C_r(\rho)}{a^r} + R_m(t)$$

with

$$C_0(\rho) = \psi(\rho) := \cos(2\pi(\rho^2 - \rho - 1/16))/\cos(2\pi\rho)$$

$$C_1(\rho) = -\frac{1}{96\pi^2} \psi^{(3)}(\rho)$$

$$C_2(\rho) = \frac{1}{18432\pi^4} \psi^{(6)}(\rho) + \frac{1}{64\pi^2} \psi^{(2)}(\rho).$$

In general, $C_j(\rho)$ can be expressed as a linear combination of the derivatives of $\psi$. We also have

$$R_m(t) = O(t^{-(2m+3)/4}).$$

Gabcke [G] showed that

$$|R_1(t)| \leq 0.053 t^{-5/4}, \quad t \geq 200.$$

The bulk of computational time in evaluating $\zeta(s)$ using the Riemann-Siegel formula is spent on the main sum $\sum_{n=1}^{N} n^{-1/2} \cos(t \log(n) - \theta(t))$. Odlyzko and Schönhage [OS] [O] developed an algorithm to compute the main sum for $T \leq t \leq T + T^{1/2}$ in $O(t^\epsilon)$ operations providing that a precomputation involving $O(T^{1/2+\epsilon})$ operations and bits of storage are carried out beforehand. This algorithm lies behind Odlyzko’s monumental $\zeta$ computations [O] [O2]. An earlier implementation proceeded by using the Fast Fourier Transform to compute the main sum and its derivatives at equally spaced grid points to then compute the main sum in between using Taylor series. This was then improved [O 4.4]
to using just the values of the main sum at equally spaced points and
an interpolation formula from the theory of band-limited functions.

Riemann used the saddle point method to obtain $C_j$, for $j \leq 5$.
The reason that a nice formula works using a sharp cutoff, truncating
the sum over $n$ at $N$, is that all the Dirichlet coefficients are equal to
one. Riemann starts with an expression for $\zeta(s)$ which involves the
geometric series identity $1/(1-x) = \sum x^n$, the Taylor coefficients on
the right being the Dirichlet coefficients of $\zeta(s)$. For general $L$-functions
smoothing works better.

3.2. Smoothed approximate functional equations. Let

$$L(s) = \sum_{n=1}^\infty \frac{b(n)}{n^s}$$

be a Dirichlet series that converges absolutely in a half plane, $\Re(s) > \sigma_1$, and hence uniformly convergent in any half plane $\Re(s) \geq \sigma_2 > \sigma_1$
by comparison with the series for $L(\sigma_2)$.

Let

$$\Lambda(s) = \left( \prod_{j=1}^a \Gamma(k_j s + \lambda_j) \right) L(s),$$

with $Q, \kappa_j \in \mathbb{R}^+$, $\Re \lambda_j \geq 0$, and assume that:

1. $\Lambda(s)$ has a meromorphic continuation to all of $\mathbb{C}$ with simple
poles at $s_1, \ldots, s_\ell$ and corresponding residues $r_1, \ldots, r_\ell$.
2. (functional equation) $\Lambda(s) = \omega \Lambda(1-s)$ for some $\omega \in \mathbb{C}$, $\omega \neq 0$.
3. For any $\alpha \leq \beta$, $L(\sigma + it) = O(\exp t^A)$ for some $A > 0$, as
$|t| \to \infty$, $\alpha \leq \sigma \leq \beta$, with $A$ and the constant in the ‘Oh’
notation depending on $\alpha$ and $\beta$.

Remarks. a) The 3rd condition, $L(\sigma + it) = O(\exp t^A)$, is very
mild. Using the fact that $L(s)$ is bounded in $\Re s \geq \sigma_2 > \sigma_1$, the
functional equation and the estimate \((23)\), and the Phragmén-Lindelöf
Theorem \[Rud\] we can show that in any vertical strip $\alpha \leq \sigma \leq \beta$,

$$L(s) = O(t^b), \quad \text{for some } b > 0$$

where both $b$ and the constant in the ‘Oh’ notation depend on $\alpha$ and $\beta$.
b) If $b(n), \lambda_j \in \mathbb{R}$, then the second assumption reads $\Lambda(s) = \omega \Lambda(1-s)$.
c) In all known examples the $\kappa_j$’s can be taken to equal 1/2. It is useful
to know the Legendre duplication formula

$$\Gamma(s) = (2\pi)^{-1/2}2^{s-1/2}\Gamma(s/2)\Gamma((s+1)/2).$$

However, it is sometimes more convenient to work with \((24)\), and we
avoid specializing prematurely to $\kappa_j = 1/2$. 
d) The assumption that \( L(s) \) have at most simple poles is not crucial and is only made to simplify the presentation.

e) From the point of view of computing \( \Lambda(s) \) given the Dirichlet coefficients and functional equation, we do not need to assume an Euler product for \( L(s) \). Without an Euler product, however, it is unlikely that \( L(s) \) will satisfy a Riemann Hypothesis.

To obtain a smoothed approximate functional equation with desirable properties we introduce an auxiliary function. Let \( g : \mathbb{C} \rightarrow \mathbb{C} \) be an entire function that, for fixed \( s \), satisfies

\[
|\Lambda(z + s)g(z + s)z^{-1}| \rightarrow 0
\]
as \( |\Im z| \rightarrow \infty \), in vertical strips, \( -\alpha \leq \Re z \leq \alpha \). The smoothed approximate functional equation has the following form.

**Theorem 1.** For \( s \notin \{s_1, \ldots, s_\ell\} \), and \( L(s), g(s) \) as above,

\[
\Lambda(s)g(s) = \sum_{k=1}^{\ell} \frac{r_k g(s_k)}{s - s_k} + Q^s \sum_{n=1}^{\infty} \frac{b(n)}{n^s} f_1(s, n)
\]

\[
+ \omega Q^{1-s} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-s}} f_2(1 - s, n)
\]

where

\[
f_1(s, n) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \prod_{j=1}^{a} \Gamma(\kappa_j(z + s) + \lambda_j)z^{-1} g(s + z)(Q/n)^z dz
\]

(26)

\[
f_2(1 - s, n) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \prod_{j=1}^{a} \Gamma(\kappa_j(z + 1 - s) + \lambda_j)z^{-1} g(s - z)(Q/n)^z dz
\]

(26)

with \( \nu > \max \{0, -\Re(\lambda_1/\kappa_1 + s), \ldots, -\Re(\lambda_a/\kappa_a + s)\} \).

**Proof.** Let \( C \) be the rectangle with vertices \( (-\alpha, -iT), (\alpha, -iT), (\alpha, iT), (-\alpha, iT) \), let \( s \in \mathbb{C} - \{s_1, \ldots, s_\ell\} \), and consider

\[
\frac{1}{2\pi i} \int_{C} \Lambda(z + s)g(z + s)z^{-1} dz
\]

(integrated counter-clockwise). \( \alpha \) and \( T \) are chosen big enough so that all the poles of the integrand are contained within the rectangle. We will also require, soon, that \( \alpha > \sigma_1 - \Re s \). On the one hand (27) equals

\[
\Lambda(s)g(s) + \sum_{k=1}^{\ell} \frac{r_k g(s_k)}{s_k - s}
\]

(28)
since the poles of the integrand are included in the set \{0, s_1 - s, \ldots, s_\ell - s\}, and are all simple. Typically, the set of poles will coincide with this set. However, if \( \Lambda(s)g(s) = 0 \), then \( z = 0 \) is no longer a pole of the integrand. But then \( \Lambda(s)g(s) \) contributes nothing to (28) and the equality remains valid. And if \( g(s_k) = 0 \), then there is no pole at \( z = s_k - s \) but also no contribution from \( r_k g(s_k)/(s_k - s) \).

On the other hand, we may break the integral over \( C \) into four integrals:

\[
\int_C = \int_{\alpha - iT}^{\alpha + iT} + \int_{\alpha + iT}^{-\alpha - iT} + \int_{-\alpha + iT}^{-\alpha - iT} + \int_{-\alpha - iT}^{\alpha - iT}
\]

The integral over \( C_1 \), assuming that \( \alpha \) is big enough to write \( L(s + z) \) in terms of its Dirichlet series i.e. \( \alpha > \sigma_1 - \Re s \), is

\[
Q^s \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \prod_{j=1}^{a} \Gamma(\kappa_j(z + s) + \lambda_j)z^{-1}g(s + z)(Q/n)^z \, dz.
\]

We are justified in rearranging summation and integration since the series for \( L(z+s) \) converges uniformly on \( C_1 \). Further, by the functional equation, the integral over \( C_3 \) equals

\[
\frac{\omega}{2\pi i} \int_{-\alpha - iT}^{-\alpha - iT} \Lambda(1 - z + s)g(z + s)z^{-1} \, dz
\]

\[
= \omega Q^{1-s} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-s}} \frac{1}{2\pi i} \int_{-\alpha - iT}^{\alpha + iT} \prod_{j=1}^{a} \Gamma(\kappa_j(1 - s - z) + \lambda_j)z^{-1}g(s + z)(Q/n)^z \, dz
\]

\[
= \omega Q^{1-s} \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-s}} \frac{1}{2\pi i} \int_{-\alpha + iT}^{\alpha + iT} \prod_{j=1}^{a} \Gamma(\kappa_j(1 - s + z) + \lambda_j)z^{-1}g(s - z)(Q/n)^z \, dz.
\]

Letting \( T \to \infty \), the integrals over \( C_2 \) and \( C_4 \) tend to zero by our assumption on the rate of growth of \( g(s) \), and we obtain (25). The integrals in (26) are, by Cauchy’s Theorem, independent of the choice of \( \nu \), so long as \( \nu > \max \{0, -\Re(\lambda_1/\kappa_1 + s), \ldots, -\Re(\lambda_a/\kappa_a + s)\} \).

\[
\square
\]

3.3. Choice of \( g(z) \). Formulae of the form (25) are well known \[L\] \[Fr\]. Usually, one finds it in the literature with \( g(s) = 1 \). For example, for the Riemann zeta function this leads to Riemann’s formula \[R\ pg 179\] \[T\].
\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) = -\frac{1}{s} - \frac{1}{1-s} + \pi^{-s/2} \sum_{n=1}^{\infty} \frac{1}{n^s} \Gamma(s/2, \pi n^2) + \pi^{(s-1)/2} \sum_{n=1}^{\infty} \frac{1}{n^{1-s}} \Gamma((1-s)/2, \pi n^2) \]

where \( \Gamma(s, w) \) is the incomplete gamma function (see Section 3.4).

However, the choice \( g(s) = 1 \) is not well suited for computing \( \Lambda(s) \) as \( |\Im(s)| \) grows. By Stirling’s formula \[\text{[Ol, pg 294]}\]

\[ |\Gamma(s)| \sim (2\pi)^{1/2} |s|^{\sigma-1/2} e^{-|t|\pi/2}, \quad s = \sigma + it \]

as \( |t| \to \infty \), and so decreases very quickly as \( |t| \) increases. Hence, with \( g(s) = 1 \), the l.h.s. of (25) is extremely small for large \( |t| \) and fixed \( \sigma \). On the other hand, we can show that the terms on the r.h.s., though decreasing as \( n \to \infty \), start off relatively large compared to the l.h.s.. Hence a tremendous amount of cancellation must occur on the r.h.s. and we would need an unreasonable amount of precision. This problem is analogous to what happens if we try to sum \( \exp(-x) = \sum (-x)^n/n! \) in a naive way. If \( x \) is positive and large, the l.h.s. is exponentially small, yet the terms on the r.h.s. are large before they become small and high precision is needed to capture the ensuing cancellation.

One way to control this cancellation is to choose \( g(s) \) equal to \( \delta^{-s} \) with \( |\delta| = 1 \) and chosen to cancel out most of the exponentially small size of the \( \Gamma \) factors. This idea appears in the work of Lavrik \[\text{[L]}\], and was also suggested by Lagarias and Odlyzko \[\text{[LO]}\] who did not implement it since it led to complications regarding the computation of (26). This method was successfully applied in the author’s PhD thesis \[\text{[Ru]}\] to compute Dirichlet \( L \)-functions and \( L \)-functions associated to cusp forms and is used extensively in the author’s \( L \)-function package \[\text{[Ru3]}\].

More recently, this approach was used in the computation of Akiyama and Tanigawa \[\text{[AT]}\] to compute several elliptic curve \( L \)-functions.

In fact when there are multiple \( \Gamma \) factors it is better to choose a different \( \delta \) for each \( \Gamma \) and multiply these together. For a given \( s \) let

\[
\begin{align*}
t_j &= \Im(\kappa_j s + \lambda_j) \\
\theta_j &= \begin{cases} 
\pi/2, & \text{if } |t_j| \leq 2c/(a\pi) \\
\c/(a|t_j|), & \text{if } |t_j| > 2c/(a\pi)
\end{cases} \\
\delta_j &= \exp(i \text{ sgn}(t_j)(\pi/2 - \theta_j)).
\end{align*}
\]
Here $c > 0$ is a free parameter. Larger $c$ means faster convergence of the sums in (25), but also more cancellation and loss of precision.

Next, we set
\begin{equation}
  g(z) := \prod_{j=1}^{a} \delta_{j}^{-\kappa_{j}z-\Im\lambda_{j}} = \beta \delta^{-z}.
\end{equation}

Because $\delta_{j}$ depends on $s$, the constants $\delta$ and $\beta$ depends on $s$. We can either use a fresh $\delta$ for each new $s$ value, or else modify the above choice of $t_{j}$ so as to use the same $t_{j}$ for other nearby $s$’s. The latter is preferred if we wish to carry out precomputations that can be recycled as we vary $s$. For simplicity, here we assume that a fresh $\delta$ is chosen as above for each new $s$.

The choice of $g$ controls the exponentially small size of the $\Gamma$ factors. Notice that the constant factor $\beta = \prod_{j=1}^{a} \delta_{j}^{-\Im\lambda_{j}}$ in (31) appears in every term in (25), and hence can be dropped from $g(z)$ without any effect on cancellation or the needed precision. However, to analyze the size of the the l.h.s. of (25) and the terms on the r.h.s. this factor is helpful and we leave it in for now, but with the understanding that it can be omitted.

To see the effect of the function $g(z)$ on the l.h.s of (25) we have, by (29) and (30)
\begin{equation}
  |\Lambda(s)g(s)| \sim \ast \cdot |L(s)| \prod_{|t_{j}| \leq 2c/\pi} \exp (-|t_{j}| \pi/2) \prod_{|t_{j}| > 2c/\pi} \exp (-c/a)
\end{equation}

where
\begin{equation}
  \ast = Q^{\sigma_{0}}(2\pi)^{a/2} \prod_{j=1}^{a} |\kappa_{j}s_{0} + \lambda_{j}|^{\kappa_{j}\sigma_{0} + \Im\lambda_{j} - 1/2}.
\end{equation}

We have thus managed to control the exponentially small size of $\Lambda(s)$ up to a factor of $\exp(-c)$ which we can regulate via the choice of $c$. We can also show that this choice of $g(z)$ leads to well balanced terms on the r.h.s. of (25).

3.4. Approximate functional equation in the case of one $\Gamma$-factor. We first treat the case $a = 1$ separately because it is the simplest, the greatest number of tools have been developed to handle this case, and many popular $L$-functions have $a = 1$.

Here we are assuming that
\begin{equation}
  \Lambda(s) = Q^{s}\Gamma(\gamma s + \lambda)L(s).
\end{equation}
According to (31) we should set
\[ g(s) = \delta^{-s} \]
(we omit the factor \( \beta \) as described following (31)) with
\[ \delta = \delta_1 \gamma \]
and
\[ t_1 = \Im(\gamma s + \lambda) \]
\[ \theta_1 = \begin{cases} \pi/2, & \text{if } |t_1| \leq 2c/\pi \\ c/|t_1|, & \text{if } |t_1| > 2c/\pi \end{cases} \]
\[ \delta_1 = \exp(i \text{ sgn}(t_1)(\pi/2 - \theta_1)). \]

In that case, the function \( f_1(s, n) \) that appears in Theorem \( \text{II} \) equals
\[
f_1(s, n) = \frac{\delta^{-s}}{2\pi i} \int_{\gamma+i\infty}^{\gamma-i\infty} \Gamma(\gamma(z + s) + \lambda)z^{-1} \left( Q/(n\delta) \right)^z dz
= \frac{\delta^{-s}}{2\pi i} \int_{\gamma+i\infty}^{\gamma+i\infty} \Gamma(u + \gamma s + \lambda)u^{-1} \left( Q/(n\delta) \right)^u du.
\]

Now
\[ \Gamma(v + u)u^{-1} = \int_0^\infty \Gamma(v, t)t^{u-1}dt, \quad \Re u > 0, \quad \Re(v + u) > 0 \]
where
\[ \Gamma(z, w) = \int_w^\infty e^{-x}x^{z-1}dx \quad |\arg w| < \pi \]
\[ = w^z \int_1^\infty e^{-wx}x^{z-1}dx, \quad \Re(w) > 0. \]
\( \Gamma(z, w) \) is known as the incomplete gamma function. By Mellin inversion
\[ f_1(s, n) = \delta^{-s}\Gamma \left( \gamma s + \lambda, \left( n\delta/Q \right)^{1/\gamma} \right). \]

Similarly
\[ f_2(1 - s, n) = \delta^{-s}\Gamma \left( \gamma(1 - s) + \lambda, \left( n/(\delta Q) \right)^{1/\gamma} \right). \]
We may thus express, when $a = 1$ and $g(s) = \delta^{-s}$, as

$$Q^a \Gamma(\gamma s + \lambda)L(s)\delta^{-s} = \sum_{k=1}^{\ell} \frac{r_k \delta^{-s_k}}{s - s_k} + (\delta/Q)^{\lambda/\gamma} \sum_{n=1}^{\infty} b(n) n^{\lambda/\gamma} G \left( \gamma s + \lambda, (n\delta/Q)^{1/\gamma} \right)$$

$$+ \frac{\omega}{\delta} (Q\delta)^{-\bar{\lambda}/\gamma} \sum_{n=1}^{\infty} \overline{b}(n) n^{\bar{\lambda}/\gamma} G \left( \gamma (1 - s) + \bar{\lambda}, (n/(\delta Q))^{1/\gamma} \right)$$

(33)

where

$$G(z, w) = w^{-z} \Gamma(z, w) = \int_1^{\infty} e^{-wx} x^{z-1} dx, \quad \Re(w) > 0.$$ (34)

Note, from (31) with $a = 1$, we have $\Re(\delta) > 0$, so both $(n\delta/Q)^{1/\gamma}$ and $(n/(\delta Q))^{1/\gamma}$ have positive $\Re$ part.

3.4.1. Examples.

1) Riemann zeta function, $\zeta(s)$: the necessary background can be found in [1]. Formula (33), for $\zeta(s)$, is

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) \delta^{-s} = -\frac{1}{s} - \frac{\delta^{-1}}{1 - s} + \sum_{n=1}^{\infty} G \left( s/2, \pi n^2 \delta^2 \right)$$

$$+ \delta^{-1} \sum_{n=1}^{\infty} G \left( (1 - s)/2, \pi n^2 / \delta^2 \right)$$ (35)

2) Dirichlet $L$-functions, $L(s, \chi)$: (see [1] chapter 9). When $\chi$ is primitive and even, $\chi(-1) = 1$, we get

$$\left( \frac{q}{\pi} \right)^{s/2} \Gamma(s/2) L(s, \chi) \delta^{-s} = \sum_{n=1}^{\infty} \chi(n) G \left( s/2, \pi n^2 \delta^2 / q \right)$$

$$+ \frac{\tau(\chi)}{\delta q^{1/2}} \sum_{n=1}^{\infty} \overline{\chi}(n) G \left( (1 - s)/2, \pi n^2 / (\delta^2 q) \right)$$

and when $\chi$ is primitive and odd, $\chi(-1) = -1$, we get

$$\left( \frac{q}{\pi} \right)^{s/2} \Gamma(s/2 + 1/2) L(s, \chi) \delta^{-s} = \delta \left( \frac{\pi}{q} \right)^{1/2} \sum_{n=1}^{\infty} \chi(n) n G \left( s/2 + 1/2, \pi n^2 \delta^2 / q \right)$$

$$+ \frac{\tau(\chi) \pi^{1/2}}{iq \delta^2} \sum_{n=1}^{\infty} \overline{\chi}(n) n G \left( (1 - s)/2 + 1/2, \pi n^2 / (\delta^2 q) \right)$$
Here, \( \tau(\chi) \) is the Gauss sum
\[
\tau(\chi) = \sum_{m=1}^{q} \chi(m)e^{2\pi im/q}.
\]

3) Cusp form \( L \)-functions: (see [Og]). Let \( f(z) \) be a cusp form of weight \( k \) for \( \text{SL}_2(\mathbb{Z}) \), \( k \) a positive even integer:

(1) \( f(z) \) is entire on \( \mathbb{H} \), the upper half plane.

(2) \( f(\sigma z) = (cz + d)^k f(z) \), \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \ z \in \mathbb{H} \).

(3) \( \lim_{t \to \infty} f(it) = 0 \).

Assume further that \( f \) is a Hecke eigenform, i.e. an eigenfunction of the Hecke operators. We may expand \( f \) in a Fourier series
\[
f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}, \quad \Im(z) > 0
\]
and associate to \( f(z) \) the Dirichlet series
\[
L_f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^{(k-1)/2}} n^{-s}.
\]

We normalize \( f \) so that \( a_1 = 1 \). This series converges absolutely when \( \Re(s) > 1 \) because, as proven by Deligne [Del],
\[
|a_n| \leq \sigma_0(n)n^{(k-1)/2},
\]
where \( \sigma_0(n) := \sum_{d|n} 1 = O(n^{\epsilon}) \) for any \( \epsilon > 0 \).

\( L_f(s) \) admits an analytic continuation to all of \( \mathbb{C} \) and satisfies the functional equation
\[
\Lambda_f(s) := (2\pi)^{-s} \Gamma(s + (k - 1)/2)L_f(s) = (-1)^{k/2} \Lambda_f(1 - s).
\]

With our normalization, \( a_1 = 1 \), the \( a_n \)'s are real since they are eigenvalues of self adjoint operators, the Hecke operators with respect to the Petersson inner product (see [Og, III-12]). Furthermore, the required rate of growth on \( L_f(s) \), condition 3 on page 18, follows from the modularity of \( f \).

Hence, in this example, formula (33) is
\[
(2\pi)^{-s} \Gamma(s + (k - 1)/2)L_f(s)\delta^{-s} = (\delta 2\pi)^{(k-1)/2} \sum_{n=1}^{\infty} a_n G(s + (k - 1)/2, 2\pi n\delta) + \frac{(-1)^{k/2}}{\delta} \left( \frac{2\pi}{\delta} \right)^{(k-1)/2} \sum_{n=1}^{\infty} a_n G(1 - s + (k - 1)/2, 2\pi n/\delta)
\]
4) Twists of cusp forms: $L_f(s, \chi)$, $\chi$ primitive, $f(z)$ as in the previous example. $L_f(s, \chi)$ is given by the Dirichlet series

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{a_n\chi(n)}{n^{(k-1)/2}} n^{-s}.$$ 

$L_f(s, \chi)$ extends to an entire function and satisfies the functional equation

$$\Lambda_f(s, \chi) := \left( \frac{q}{2\pi} \right)^{s} \Gamma(s + (k - 1)/2) L_f(s, \chi)$$

$$= (-1)^{k/2} \chi(-1) \frac{\tau(\chi)}{\tau(\chi)} \Lambda_f(1 - s, \chi).$$

In this example, formula (33) is

$$\left( \frac{q}{2\pi} \right)^{s} \Gamma(s + (k - 1)/2) L_f(s, \chi) \delta^{-s} =$$

$$\left( 2\pi \delta \right)^{(k-1)/2} \sum_{n=1}^{\infty} a_n\chi(n) G(s + (k - 1)/2, 2\pi n \delta / q)$$

$$+ \frac{(-1)^{k/2}}{\delta} \chi(-1) \frac{\tau(\chi)}{\tau(\chi)} \left( \frac{2\pi}{q \delta} \right)^{(k-1)/2} \sum_{n=1}^{\infty} a_n\overline{\chi}(n) G(1 - s + (k - 1)/2, 2\pi n / (\delta q)).$$

5) Elliptic curve $L$-functions: (see [Kn, especially chapters X,XII]). Let $E$ be an elliptic curve over $\mathbb{Q}$, which we write in global minimal Weierstrass form

$$y^2 + c_1 xy + c_3 y = x^3 + c_2 x^2 + c_4 x + c_6$$

where the $c_j$'s are integers and the discriminant $\Delta$ is minimal.

To the elliptic curve $E$ we may associate an Euler product

$$L_E(s) := \prod_{p \nmid \Delta} \left( 1 - a_p p^{-1/2-s} \right)^{-1} \prod_{p | \Delta} \left( 1 - a_p p^{-1/2-s} + p^{-2s} \right)^{-1}$$

where, for $p \nmid \Delta$, $a_p = p + 1 - \#E_p(\mathbb{Z}_p)$, with $\#E_p(\mathbb{Z}_p)$ being the number of points $(x, y)$ in $\mathbb{Z}_p \times \mathbb{Z}_p$ on the curve $E$ considered modulo $p$, together with the point at infinity. When $p | \Delta$, $a_p$ is either 1, $-1$, or 0. If $p \nmid \Delta$, a theorem of Hasse states that $|a_p| < 2p^{1/2}$. Hence, (36) converges when $\Re(s) > 1$, and for these values of $s$ we may expand $L_E(s)$ in an absolutely convergent Dirichlet series

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{1/2}} n^{-s}.$$
The Hasse-Weil conjecture asserts that $L_E(s)$ extends to an entire function and has the functional equation

$$
\Lambda_E(s) := \left( \frac{N^{1/2}}{2\pi} \right)^s \Gamma(s + 1/2) L_E(s) = -\varepsilon \Lambda_E(1 - s).
$$

where $N$ is the conductor of $E$, and $\varepsilon$, which depends on $E$, is either $\pm 1$. The Hasse-Weil conjecture and also the required rate of growth on $L_E(s)$ follows from the Shimura-Taniyama-Weil conjecture, which has been proven by Wiles and Taylor [TW] [Wi] for elliptic curves with square free conductor and has been extended, by Breuil, Conrad, Diamond and Taylor to all elliptic curves over $\mathbb{Q}$ [BCDT].

Hence we have

$$
\left( \frac{N^{1/2}}{2\pi} \right)^s \Gamma(s + 1/2) L_E(s) \delta^{-s} = \left( \frac{2\pi \delta}{N^{1/2}} \right)^{1/2} \sum_{n=1}^{\infty} a_n G \left( s + 1/2, 2\pi n \delta / N^{1/2} \right)
- \frac{\varepsilon}{\delta} \left( \frac{2\pi}{N^{1/2}\delta} \right)^{1/2} \sum_{n=1}^{\infty} a_n G \left( 1 - s + 1/2, 2\pi n / (\delta N^{1/2}) \right).
$$

6) Twists of elliptic curve $L$-functions: $L_E(s, \chi)$, $\chi$ a primitive character of conductor $q$, $(q, N) = 1$. Here $L_E(s, \chi)$ is given by the Dirichlet series

$$
L_E(s, \chi) = \sum_{n=1}^{\infty} \frac{a_n}{n^{1/2}} \chi(n)n^{-s}.
$$

The Weil conjecture asserts, here, that $L_E(s, \chi)$ extends to an entire function and satisfies

$$
\Lambda_E(s, \chi) := \left( \frac{q N^{1/2}}{2\pi} \right)^s \Gamma(s + 1/2) L_E(s, \chi) = -\varepsilon \chi(-N) \frac{\tau(\chi)}{\bar{\tau}(\chi)} \Lambda_E(1 - s, \bar{\chi}).
$$

Here $N$ and $\varepsilon$ are the same as for $E$. In this example the conjectured formula is

$$
\left( \frac{q N^{1/2}}{2\pi} \right)^s \Gamma(s + 1/2) L_E(s, \chi) \delta^{-s} = \left( \frac{2\pi \delta}{q N^{1/2}} \right)^{1/2} \sum_{n=1}^{\infty} a_n \chi(n) G \left( s + 1/2, 2\pi n \delta / (q N^{1/2}) \right)
- \frac{\varepsilon}{\delta} \chi(-N) \frac{\tau(\chi)}{\bar{\tau}(\chi)} \left( \frac{2\pi}{q N^{1/2}\delta} \right)^{1/2} \sum_{n=1}^{\infty} a_n \bar{\chi}(n) G \left( 1 - s + 1/2, 2\pi n / (\delta q N^{1/2}) \right).
$$

We have reduced in the case $a = 1$ the computation of $\Lambda(s)$ to one of evaluating two sums of incomplete gamma functions. The $\Gamma(\gamma s + \lambda) \delta^{-s}$ factor on the left of [33] and elsewhere is easily evaluated using several terms of Stirling’s asymptotic formula and also the recurrence $\Gamma(z + 1) = z\Gamma(z)$ applied a few times. The second step is needed for
small $z$. Some care needs to be taken to absorb the $e^{-\pi|\Im(\gamma s + \lambda)|/2}$ factor of $\Gamma(\gamma s + \lambda)$ into the $e^{\pi|\Im(\gamma s + \lambda)|/2}$ factor of $\delta^{-s}$. Otherwise our effort to control the size of $\Gamma(\gamma s + \lambda)$ will have been in vain, and lack of precision will wreak havoc.

To see how many terms in (33) are needed we can use the rough bound

$$|G(z, w)| < e^{-\Re(w)} \int_0^\infty e^{-(\Re(w) - \Re(z) + 1)t} dt = \frac{e^{-\Re(w)}}{\Re(w) - \Re(z) + 1},$$

valid for $\Re(w) > \Re(z) - 1 > 0$. We have put $t = x - 1$ in (34) and have used $t + 1 \leq e^t$. Also, for $\Re(w) > 0$ and $\Re(z) \leq 1$,

$$|G(z, w)| < \frac{e^{-\Re(w)}}{\Re(w)}.$$

These inequalities tells us that the terms in (33) decrease exponentially fast once $n$ is sufficiently large.

For example, in equation (35) for $\zeta(s)$ we get exponential drop off roughly when

$$\Re(\pi n^2 \delta^2) >> 1.$$  

But

$$\Re(\pi n^2 \delta^2) = \pi n^2 \Re(\delta^2) \sim 2\pi n^2 c/t$$

so the number of terms needed is roughly

$$>> (t/c)^{1/2}.$$

3.4.2. Computing $\Gamma(z, w)$. Recall the definitions

$$\Gamma(z, w) = \int_w^\infty e^{-t} t^{z-1} dt, \quad |\arg w| < \pi$$

$$G(z, w) = w^{-z} \Gamma(z, w).$$

Let

$$\gamma(z, w) := \Gamma(z) - \Gamma(z, w) = \int_0^w e^{-x} x^{z-1} dx, \quad \Re z > 0, \quad |\arg w| < \pi$$

be the complimentary incomplete gamma function, and set

(38)  

$$g(z, w) = w^{-z} \gamma(z, w) = \int_0^1 e^{-wt} t^{z-1} dt$$

so that $G(z, w) + g(z, w) = w^{-z} \Gamma(z)$. The function $g(z, w)/\Gamma(z)$ is entire in $z$ and $w$.

The incomplete $\Gamma$ function undergoes a transition when $|w|$ is close to $|z|$. This will be described using Temme’s uniform asymptotics for $\Gamma(z, w)$. The transition explains the difficulty in computing $\Gamma(z, w)$
without resorting to several different expressions or using uniform asymptotics.

A combination of series, asymptotics, and continued fractions are useful when $|z|$ is somewhat bigger than or smaller than $|w|$. When the two parameters are close in size to one another, we can employ Temme’s more involved uniform asymptotics. We can also apply the Poisson summation method described in Section 2, or an expansion due to Nielsen. Below we look at a few useful approaches.

Integrating by parts we get

$$g(z, w) = e^{-w} \sum_{j=0}^{\infty} \frac{w^j}{(z)_{j+1}}$$

where

$$(z)_j = \begin{cases} z(z+1)\ldots(z+j-1) & \text{if } j > 0; \\ 1 & \text{if } j = 0. \end{cases}$$

(The case $j = 0$ occurs below in an expression for $G(z, w)$). While this series converges for $z \neq 0, -1, -2, \ldots$ and all $w$, it is well suited, say if $\Re z > 0$ and $|w| < \alpha |z|$ with $0 < \alpha < 1$. Otherwise, not only does the series take too long to converge, but precision issues arise.

The following continued fraction converges for $\Re z > 0$

$$g(z, w) = e^{-w} \frac{zw}{z + 1 + \frac{w}{z + 2} \frac{(z+1)w}{z + 3 + \frac{2w}{z + 4} \frac{(z+2)w}{z + 5 + \cdots}}}$$

The paper of Akiyama and Tanigawa [AT] contains an analysis of the truncation error for this continued fraction, as well as the continued fraction in (39) below, and show that the above is most useful when $|w| < |z|$, with poorer performance as $|w|$ approaches $|z|$.

Another series, useful when $|w| << 1$, is

$$g(z, w) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{w^j}{z + j}.$$ 

This is obtained from (38) by expanding $e^{-wt}$ in a Taylor series and integrating termwise. As $|w|$ grows, cancellation and precision become an issue in the same way it does for the sum $e^{-w} = \sum (-w)^j / j!$. 

Next, integrate \( G(z, w) \) by parts to obtain the asymptotic series

\[
G(z, w) = e^{-w} \sum_{j=0}^{M-1} \frac{(1-z)_j}{(-w)^j} + \epsilon_M(z, w)
\]

with

\[
\epsilon_M(z, w) = \frac{(1-z)_M}{(-w)^M} G(z - M, w).
\]

This asymptotic expansion works well if \(|w| > \beta |z|\) with \(\beta > 1\) and \(|z|\) large. In that region the following continued fraction also works well

\[
(39) \quad G(z, w) = e^{-w} \frac{1}{w + \frac{1}{1 + \frac{2}{w + \frac{3}{1 + \cdots}}}}
\]

Temme’s uniform asymptotics for \( \Gamma(z, w) \) provide a powerful tool for computing the function in its transition zone and elsewhere. Following the notation in [11], let

\[
Q(z, w) = \frac{\Gamma(z, w)}{\Gamma(z)}
\]

\[
\lambda = \frac{w}{z}
\]

\[
\eta^2/2 = \lambda - 1 - \log \lambda
\]

where the sign of \( \eta \) is chosen to be positive for \( \lambda > 1 \). Then

\[
Q(z, w) = \frac{1}{2} \text{erfc}(\eta(z/2)^{1/2}) + R_z(\eta)
\]

where

\[
\text{erfc} = \frac{2}{\pi^{1/2}} \int_{z}^{\infty} e^{-t^2} dt,
\]

and \( R_z \) is given by the asymptotic series, as \( z \to \infty \),

\[
(40) \quad R_z(\eta) = \frac{e^{-z\eta^2/2}}{(2\pi z)^{1/2}} \sum_{n=0}^{\infty} \frac{c_n(\eta)}{z^n}.
\]
Here

\[
\begin{align*}
c_0(\eta) & = \frac{1}{\lambda - 1} - \frac{1}{\eta} \\
c_1(\eta) & = \frac{1}{\eta^3} - \frac{1}{(\lambda - 1)^3} - \frac{1}{(\lambda - 1)^2} - \frac{1}{12(\lambda - 1)} \\
\eta c_n(\eta) & = \frac{d}{d\eta} c_{n-1}(\eta) + \frac{\eta}{\lambda - 1} \gamma_n, \quad n \geq 1
\end{align*}
\]

with

\[
\Gamma^*(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{z^n}
\]

being the asymptotic expansion of

\[
\Gamma^*(z) = (z/(2\pi))^{1/2}(e/z)^{z}\Gamma(z).
\]

The first few terms are \(\gamma_0 = 1, \gamma_1 = -1/12, \gamma_2 = 1/288, \gamma_3 = 139/51840\). The singularities at \(\eta = 0\), i.e. \(\lambda = 1, z = w\), are removable. Unfortunately, explicit estimates for the remainder in truncating \((40)\) when the parameters are complex have not been worked out, but in practice the expansion seems to work very well.

To handle the intermediate region \(|z| \approx |w|\) we could also use the following expansion of Nielsen to step through the troublesome region \((41)\)

\[
\gamma(z, w + d) = \gamma(z, w) + w^{z-1} e^{-w} \sum_{j=0}^{\infty} \frac{(1 - z)^j}{(-w)^j} (1 - e^{-d} e_j(d)), \quad |d| < |w|
\]

where

\[
e_j(d) = \sum_{m=0}^{j} \frac{d^m}{m!}
\]

A proof can be found in \[EMOT\]. This expansion is very well suited, for example, for \(L\)-functions associated to modular forms, since in that case we increment \(w\) in equal steps from term to term in \((33)\) and precomputations can be arranged to recycle data. Numerically, this expansion is unstable if \(|d|\) is big. This can be overcome by taking many smaller steps, but this then makes Nielsen’s expansion an inefficient choice for \(\zeta(s)\) or Dirichlet \(L\)-functions.

In computing \((41)\) some care needs to be taken to avoid numerical pitfalls. One pitfall is that, as \(j\) grows, \(e^{-d} e_j(d) \to 1\). So once \(|1 - e^{-d} e_j(d)| < 10^{-\text{Digits}}\), the error in computation of \(1 - e^{-d} e_j(d)\) is bigger than its value, and this gets magnified when we multiply by \((1 - z)j/(-w)^j\). So in computing \((1 - z)j/(-w)^j (1 - e^{-d} e_j(d))\) one
must avoid the temptation to view this as a product of \((1 - z)_j / (-w)^j\) and \(1 - e^{-d} e_j(d)\). Instead, we let
\[
a_j(z, w, d) = \frac{(1 - z)_j}{(-w)^j} (1 - e^{-d} e_j(d)).
\]

Now, \(1 - e^{-d} e_j(d) = e^{-d} (e^d - e_j(d))\), and we get
\[
a_{j+1}(z, w, d) = a_j(z, w, d) \frac{z - (j + 1)}{w} \left( \sum_{m=0}^{\infty} \frac{d^m}{m!} \right) / \left( \sum_{m=0}^{\infty} \frac{d^m}{m!} \right)
\]
\[
= a_j(z, w, d) \frac{z - (j + 1)}{w} (1 - 1/\beta_j(d)), \quad j = 1, 2, 3, \ldots
\]
where
\[
\beta_j(d) = \sum_{m=0}^{\infty} \frac{d^m}{(j + 2)_m}.
\]
Furthermore
\[
\beta_j(d) - 1 \sim d/(j + 2), \quad \text{as } |d|/j \to 0.
\]
Hence, for \(|w| \approx |z|\), we approximately have (as \(|d|/j \to 0\))
\[
\left| \frac{z - (j + 1)}{w} (1 - 1/\beta_j(d)) \right| \leq \left( 1 + \frac{j + 1}{|w|} \right) \frac{|d|}{j + 2} \leq \frac{|d|}{j + 2} + \frac{|d|}{|w|}.
\]
Thus, because \(|d/w| < 1\), we have, for \(j\) big enough, that the above is < 1, and so the sum in (41) converges geometrically fast, and hence only a handful of terms are required.

One might be tempted to compute the \(\beta_j(d)\)'s using the recursion
\[
\beta_{j+1}(d) = (\beta_j(d) - 1)(j + 2)/d
\]
but this leads to numerical instability. The \(\beta_j(d)\)'s are all equal to \(1 + O_d(1/(j + 2))\) and are thus all roughly of comparable size. Hence, a small error due to roundoff in \(\beta_j(d)\) is turned into a much larger error in \(\beta_{j+1}(d), (j + 2)/|d|\) times larger, and this quickly destroys the numerics.

There seems to be some potential in an asymptotic expression due to Ramanujan [3, pg 193, entry 6]
\[
G(z, w) \sim w^{-z} \Gamma(z)/2 + e^{-w} \sum_{k=0}^{M} p_k(w - z + 1)/w^{k+1}, \quad \text{as } |z| \to \infty,
\]
for \(|w - z|\) relatively small, where \(p_k(v)\) is a polynomial in \(v\) of degree \(2k + 1\), though this potential has not been investigated substantially.
We list the first few $p_k(v)$'s here:

\[ p_0(v) = -v + 2/3 \]
\[ p_1(v) = -\frac{v^3}{3} + \frac{v^2}{3} - \frac{4}{135} \]
\[ p_2(v) = -\frac{v^5}{15} + \frac{v^3}{9} - \frac{2v^2}{135} - \frac{4v}{135} + \frac{8}{2835} \]
\[ p_3(v) = -\frac{v^7}{105} - \frac{v^6}{45} + \frac{v^5}{45} - \frac{7v^4}{135} - \frac{8v^3}{405} - \frac{16v^2}{567} + \frac{16v}{2835} + \frac{8}{8505} \]
\[ p_4(v) = -\frac{v^9}{945} - \frac{2v^8}{315} - \frac{2v^7}{315} + \frac{8v^6}{405} + \frac{11v^5}{405} - \frac{62v^4}{2835} - \frac{32v^3}{1215} + \frac{16v^2}{1701} + \frac{16v}{2835} - \frac{8992}{12629925} \]
\[ p_5(v) = -\frac{v^{11}}{10395} - \frac{v^{10}}{945} - \frac{2v^9}{567} - \frac{2v^8}{2835} + \frac{43v^7}{2835} + \frac{41v^6}{2835} - \frac{968v^5}{42525} - \frac{68v^4}{2835} + \frac{368v^3}{25515} \]
\[ + \frac{138064v^2}{12629925} - \frac{35968v}{12629925} - \frac{334144}{492567075} \]

It is worth noting that when many evaluations of $\Lambda(s)$ are required, we can reduce through precomputations the bulk of the work to that of computing a main sum. This comes from the identity

\[ G(z, w) = w^{-z}\Gamma(z) - g(z, w). \]

The above discussion indicates that, in \(33\), we should use $g(z, w)$ and this identity to compute $G(z, w)$ roughly when $|w|$ is smaller than $|z|$. For example, with $\zeta(1/2 + it)$, the region $|w| < |z|$ corresponds in \(35\) to $|\pi n^2 \delta^2| < |1/4 + it/2|$ and $|\pi n^2 / \delta^2| < |1/4 - it/2|$. Because $|\delta| = 1$ this leads to a main sum consisting of approximately $|t/(2\pi)|^{1/2}$ terms, the same as in the Riemann-Siegel formula.

3.5. The approximate functional equation when there is more than one $\Gamma$-factor, and $\kappa_j = 1/2$. In this case, the function $f_1(s, n)$ that appears in Theorem 1 is

\[ f_1(s, n) = \frac{\delta^{-s}}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \prod_{j=1}^{a} \Gamma((z + s)/2 + \lambda_j)z^{-1}(Q/(\delta n))^z \, dz. \]

This is a special case of the Meijer $G$ function and we develop some of its properties.

Let $M(\phi(t); z)$ denote the Mellin transform of $\phi$

\[ M(\phi(t); z) = \int_{0}^{\infty} \phi(t)t^{z-1}. \]

We will express $\prod_{j=1}^{a} \Gamma((z + s)/2 + \lambda_j)z^{-1}$ as a Mellin transform analogous to \(32\).
Letting $\phi_1 * \phi_2$ denote the convolution of two functions

$$(\phi_1 * \phi_2)(v) = \int_0^\infty \phi_1(v/t)\phi_2(t)\frac{dt}{t}$$

we have (under certain conditions on $\phi_1, \phi_2$)

$$M(\phi_1 * \phi_2; z) = M(\phi_1; z) \cdot M(\phi_2; z).$$

Thus

$$\prod_{j=1}^a M(\phi_j; z) = \int_0^\infty (\phi_1 * \cdots * \phi_a)(t) t^{z-1} dt,$$

with

$$(\phi_1 * \cdots * \phi_a)(v) = \int_0^\infty \cdots \int_0^\infty \phi_1(v/t_1)\phi_2(t_1/t_2) \cdots \phi_{a-1}(t_{a-2}/t_{a-1})\phi_a(t_{a-1})\frac{dt_1}{t_1} \cdots \frac{dt_{a-1}}{t_{a-1}}.$$ 

Now

$$\prod_{j=1}^a \Gamma((z+s)/2 + \lambda_j) z^{-1} = \left( \prod_{j=1}^{a-1} \Gamma((z+s)/2 + \lambda_j) \right) (\Gamma((z+s)/2 + \lambda_0) z^{-1}).$$

But

$$\Gamma((z+s)/2 + \lambda) = M \left( 2e^{-t^2} t^{2\lambda + s}; z \right),$$

and (32) gives

$$\Gamma((z+s)/2 + \lambda) z^{-1} = M(\Gamma(s/2 + \lambda, t^2); z).$$

So letting

$$\phi_j(t) = \begin{cases} 2e^{-t^2} t^{2\lambda_j + s} & j = 1, \ldots, a-1; \\ \Gamma(s/2 + \lambda_a, t^2) & j = a, \end{cases}$$

and applying Mellin inversion, we find that (42) equals

$$f_1(s, n) = \delta^{-s}(\phi_1 * \cdots * \phi_a)(n\delta/Q),$$

where

$$(\phi_1 * \cdots * \phi_a)(v) = v^{2\lambda_1+s} \int_0^\infty \cdots \int_0^\infty 2^{a-1} \prod_{j=1}^{a-1} e^{-\left( \frac{t_1^2 + t_2^2 + \cdots + t_{a-1}^2}{t_{a-1}} \right)} \left( \int_1^\infty e^{-t_{a-1}^2 x^{s/2+\lambda_a-1}} \frac{dt_{a-1}}{t_{a-1}} \right) \frac{dt_1}{t_1} \cdots \frac{dt_a}{t_a}.$$ 

Substituting $u_j = \frac{(v^2 x)/a}{v^2} t_j^2$ and rearranging order of integration this becomes

$$v^{2\mu+s} \int_1^\infty E_{\lambda} \left( xv^2 \right) x^{s/2+\mu-1} dx,$$
where

\[
\mu = \frac{1}{a} \sum_{l=1}^{a} \lambda_j,
\]

\[
E_\lambda(w) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{j=1}^{a-1} u_j^{\lambda_{j+1} - \lambda_j} e^{-w^{1/a} \left( \frac{u_1}{u_1 + u_2 + \cdots + u_{a-1}} \right)} \frac{du_1}{u_1} \cdots \frac{du_{a-1}}{u_{a-1}}.
\]

So, returning to (44), we find that

\[
f_1(s, n) = \left( n\delta/Q \right)^{2\mu} \left( n/Q \right)^{s} \int_{1}^{\infty} E_\lambda \left( x \left( n\delta/Q \right)^{2} \right) x^{s/2 + \mu - 1} dx.
\]

Note that because (42) is symmetric in the \( \lambda_j \)'s, so is \( E_\lambda \).

Similarly

\[
f_2(1-s, n) = \delta^{-1} \left( n/(\delta Q) \right)^{2\mu} \left( n/Q \right)^{1-s} \int_{1}^{\infty} E_{\overline{\lambda}} \left( x \left( n/(\delta Q) \right)^{2} \right) x^{(1-s)/2 + \overline{\mu} - 1} dx.
\]

Hence,

\[
Q^s \prod_{j=1}^{a} \Gamma \left( s/2 + \lambda_j \right) L(s) \delta^{-s} = \sum_{k=1}^{\ell} \frac{r_k \delta^{-s_k}}{s - s_k} + (\delta/Q)^{2\mu} \sum_{n=1}^{\infty} b(n)n^{2\mu} G_{\lambda} \left( s/2 + \mu, (n\delta/Q)^{2} \right) + \frac{\omega}{\delta} (\delta Q)^{-2\mu} \sum_{n=1}^{\infty} \overline{b}(n)n^{2\mu} G_{\overline{\lambda}} \left( (1-s)/2 + \overline{\mu}, (n/(\delta Q))^2 \right)
\]

with

\[
G_{\lambda}(z,w) = \int_{1}^{\infty} E_\lambda(xw) x^{z-1} dx
\]

(\( \mu \) and \( E_\lambda \) are given by (45), (46)).

3.5.1. Examples. When \( a = 2 \)

\[
E_\lambda(xw) = \int_{0}^{\infty} t^{\lambda_2 - \lambda_1} e^{-(wx)^{1/2}(1/t+t)} \frac{dt}{t}
\]

(48)

\[
= 2K_{\lambda_2 - \lambda_1} \left( 2(wx)^{1/2} \right) = 2K_{\lambda_1 - \lambda_2} \left( 2(wx)^{1/2} \right),
\]

\( K \) being the \( K \)-Bessel function, so that \( G_{\lambda} \) is an incomplete integral of the \( K \)-Bessel function.

Note further that if \( \lambda_1 = \lambda/2, \lambda_2 = (\lambda + 1)/2 \) then (48) is

\[
2K_{1/2} \left( 2(wx)^{1/2} \right) = \left( \pi^{1/2}/(wx)^{1/4} \right) e^{-2(wx)^{1/2}}
\]
(see [EMOT]), so $G_{(\lambda/2,(\lambda+1)/2)}(z, w) = 2(2\pi)^{1/2}(4w)^{-z}\Gamma(2z-1/2, 2w^{1/2})$, i.e. the incomplete gamma function. This is what we expect since, using (24), we can write the gamma factor $\Gamma((s + \lambda)/2)\Gamma((s + \lambda + 1)/2)$ in terms of $\Gamma(s + \lambda)$, for which the $a = 1$ expansion, (33), applies.

Maass cusp form $L$-functions: (background material can be found in [Bu]). Let $f$ be a Maass cusp form with eigenvalue $\lambda = 1/4 - v^2$, i.e. $\Delta f = \lambda f$, where $\Delta = -y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, and Fourier expansion

$$f(z) = \sum_{n \neq 0} a_n y^{1/2} K_v(2\pi |n| y) e^{2\pi i n x},$$

with $a_{-n} = a_n$ for all $n$, or $a_{-n} = -a_n$ for all $n$. Let

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Re s > 1$$

(absolute convergence in this half plane can be proven via the Rankin-Selberg method), and let $\varepsilon = 0$ or 1 according to whether $a_{-n} = a_n$ or $a_{-n} = -a_n$. We have that

$$\Lambda_f(s) := \pi^{-s}\Gamma((s + \varepsilon + v)/2)\Gamma((s + \varepsilon - v)/2)L_f(s)$$

extends to an entire function and satisfies

$$\Lambda_f(s) = (-1)^\varepsilon \Lambda_f(1 - s).$$

Hence, formula (17), for $L_f(s)$, is

$$\pi^{-s}\Gamma((s + \varepsilon + v)/2)\Gamma((s + \varepsilon - v)/2)L_f(s)\delta^{-s} =$$

$$(\delta \pi)^\varepsilon \sum_{n=1}^{\infty} a_n n^\varepsilon G_{((\varepsilon + v)/2, (\varepsilon - v)/2)} \left( s/2 + \varepsilon/2, (n\delta \pi)^2 \right)$$

$$+ \frac{(-1)^\varepsilon}{\delta} (\pi/\delta)^\varepsilon \sum_{n=1}^{\infty} a_n n^\varepsilon G_{((\varepsilon + \pi)/2, (\varepsilon - \pi)/2)} \left( (1 - s)/2 + \varepsilon/2, (n\pi/\delta)^2 \right)$$

where, by (18),

$$G_{((\varepsilon + v)/2, (\varepsilon - v)/2)} \left( s/2 + \varepsilon/2, (n\delta \pi)^2 \right) = 4 \int_1^{\infty} K_v(2n\delta \pi t)t^{s+\varepsilon-1} dt$$

$$G_{((\varepsilon + \pi)/2, (\varepsilon - \pi)/2)} \left( (1 - s)/2 + \varepsilon/2, (n\pi/\delta)^2 \right) = 4 \int_1^{\infty} K_{\pi}(2n\pi t/\delta) t^{-s+\varepsilon} dt.$$
Next, let
\[ \Gamma_\lambda(z, w) = w^z G_\lambda(z, w) = \int_w^\infty E_\lambda(t) t^{-1} dt, \]
(49)
\[ \Gamma_\lambda(z) = \int_0^\infty E_\lambda(t) t^{-1} dt, \]
\[ \gamma_\lambda(z, w) = \int_0^w E_\lambda(t) t^{-1} dt, \]
with \( E_\lambda \) given by (46).

**Lemma 1.**
\[ \Gamma_\lambda(z) = \prod_{j=1}^a \Gamma(z - \mu + \lambda_j) \]
where \( \mu = \frac{1}{a} \sum_{j=1}^a \lambda_j \).

**Proof.** Let \( \psi_j(t) = e^{-t \lambda_j}, j = 1, \ldots, a \), and consider
\[
(\psi_1 \ast \ldots \ast \psi_a)(v) = v^\lambda \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{a-1} t_j^{\lambda_j+1-\lambda_j} e^{-\left(\frac{v}{1+t_1+t_2+\ldots+t_{a-1}}\right) t_1 \ldots t_{a-1}} dt_1 \ldots dt_{a-1}
\]
\[
= v^\mu \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{a-1} x_j^{\lambda_j+1-\lambda_j} e^{-v^{1/a}\left(\frac{1}{x_1+\frac{x_2}{x_1}+\ldots+\frac{x_{a-1}}{x_{a-2}}+x_{a-1}}\right) x_1 \ldots x_{a-1}} dx_1 \ldots dx_{a-1}
\]
(we have put \( t_j = v^{1-j/a} x_j \)). Thus, from (46)
\[ E_\lambda(v) = v^{-\mu}(\psi_1 \ast \ldots \ast \psi_a)(v), \]
and hence (49) equals
\[ \int_0^\infty (\psi_1 \ast \ldots \ast \psi_a)(t) t^{z-\mu-1} dt \]
which, by (49) is \( \prod_{j=1}^a \Gamma(z - \mu + \lambda_j) \). \( \square \)

Inverting, we get
\[ E_\lambda(t) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \Gamma_\lambda(z) t^{-z} dz \]
with \( \nu \) to the right of the poles of \( \Gamma_\lambda(z) \). Shifting the line integral to the left, we can express \( E_\lambda(t) \) as a sum of residues, and hence obtain through termwise integration a series expansion for \( \gamma_\lambda(z, w) \). An algorithm for doing so is detailed in [Do], though with different notation. Such an expansion is useful for \( |w| << 1 \). That paper also describes how to obtain an asymptotic expansion for \( E_\lambda(t) \) and hence,
by termwise integration, for $\Gamma_\lambda(z, w)$, useful for $|w|$ large in comparison to $|z|$. The paper has, implicitly, $g(z) = 1$ and does not control for cancellation. Consequently, it does not provide a means to compute $L$-functions away from the real axis other than increasing precision.

If one wishes to use the methods of this paper to control for cancellation, then one will have $w$ varying over a wide range of values for which the series expansion in [15] is not adequate. We thus need an alternative method to compute $G_\lambda(z, w)$ especially in the transition zone $|z| \approx |w|$. It would be useful to have Temme’s uniform asymptotics generalized to handle $G_\lambda(z, w)$. Alternatively, we can apply the naive but powerful Riemann sum technique described in section 2.

### 3.6. The functions $f_1(s, n), f_2(1 - s, n)$ as Riemann sums.

Substituting $z = v + iu$ into (26) we have

$$f_1(s, n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{j=1}^{a} \Gamma(\kappa_j(s + v + iu) + \lambda_j) \frac{g(s + v + iu)}{v + iu} (Q/n)^{v + iu} du.$$ 

Let

$$h(u) = \frac{1}{2\pi} \prod_{j=1}^{a} \Gamma(\kappa_j(s + v + iu) + \lambda_j) \frac{g(s + v + iu)}{v + iu} (Q/n)^{v + iu}.$$ 

With the choice of $g(z)$ as in (31), an analysis similar to that following (21) shows that $\hat{h}(y)$ decays exponentially fast as $y \to -\infty$, and doubly exponentially fast as $y \to \infty$. Hence, we can successfully evaluate $f_1(s, n)$, and similarly $f_2(1 - s, n)$ as simple Riemann sums, with step size inversely proportional to the number of digits of precision required.

The Riemann sum approach gives us tremendous flexibility. We are no longer bound in our choice of $g(z)$ to functions for which (26) has nice series or asymptotic expansions. For example, we can, with $A > 0$, set

$$g(z) = \exp(A(z - s)^2) \prod_{j=1}^{a} \delta_j^{-\kappa_j z}.$$ 

The extra factor $\exp(A(z - s)^2)$ is chosen so as to cut down on the domain of integration. Recall that in $f_1(s, n)$ and $f_2(1 - s, n)$, $g$ appears as $g(s \pm (v + iu))$, hence $\exp(A(z - s)^2)$ decays in the integral like $\exp(-Au^2)$. Ideally, we would like to have $A$ large. However, this would cause the Fourier transform $\hat{h}(y)$ to decay too slowly. The Fourier transform of a product is a convolution of Fourier transforms, and the
Fourier transform of \( \exp(A(v + iu)^2) \) equals
\[
(\pi/A)^{1/2} \exp(\pi y (2Av - \pi y)/A).
\]
A large value of \( A \) leads to a small \( 1/A \) and this results in poor performance of \( \hat{h}(y) \). We also need to specify \( v \), for the line of integration. Larger \( v \) means more rapid decay of \( \hat{h}(y) \) but more cancellation in the Riemann sum and hence loss of precision.

Another advantage to the Riemann sum approach is that we can rearrange sums, putting the Riemann sum on the outside and the sum over \( n \) on the inside. Both sums are finite since we truncate them once the tails are within the desired precision. This then expresses, to within an error that we can control by our choice of stepsize and truncation, \( \Lambda(s) \) as a sum of finite Dirichlet series evaluated at equally spaced points and hence gives a sort of interpolation formula for \( \Lambda(s) \).

Details related to this approach will appear in a future paper.

3.7. Looking for zeros. To look for zeros of an \( L \)-function, we can rotate it so that it is real on the critical line, for example working with \( Z(t) \), see (22), rather than \( \zeta(1/2 + it) \).

We can then advance in small steps, say one quarter the average gap size between consecutive zeros, looking for sign changes of this real valued function, zooming in each time a sign change occurs. Along the way, we need to determine if any zeros have been missed, and, if so, go back and look for them, using more refined step sizes. We can also use more sophisticated interpolation techniques to make the search for zeros more efficient [O]. If this search fails to turn up the missing zeros, then presumably a bug has crept into one’s code, or else one should look for zeros of the \( L \)-function nearby but off the critical line in violation of the Riemann hypothesis.

To check for missing zeros, we could use the argument principle and numerically integrate the logarithmic derivative of the \( L \)-function along a rectangle, rounding to the closest integer. However, this is inefficient and difficult to make numerically rigorous.

It is better to use a test devised by Alan Turing [Tu] for \( \zeta(s) \) but which seems to work well in general. Let \( N(T) \) denote the number of zeros of \( \zeta(s) \) in the critical strip above the real axis and up to height \( T \):

\[
N(T) = |\{\rho = \beta + i\gamma | \zeta(\rho) = 0, 0 \leq \beta \leq 1, 0 < \gamma \leq T\}|.
\]

A theorem of von Mangoldt states that

\[
N(T) = \frac{T}{2\pi} \log(T/(2\pi)) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1})
\]
with

\[ S(T) = O(\log T). \]

However, a stronger inequality due to Littlewood and with explicit constants due to Turing [Tu] and Turing [Le] is given by

\[ \left| \int_{t_1}^{t_2} S(t) dt \right| \leq 2.3 + 0.128 \log(\frac{t_2}{\pi}) \]

for all \( t_2 > t_1 > 168\pi \), i.e. \( S(T) \) is 0 on average. Therefore, if we miss one sign change (at least two zeros), we'll quickly detect the fact. To illustrate this, Table 2 contains a list of the imaginary parts of the zeros of \( \zeta(s) \) found naively by searching for sign changes of \( Z(t) \) taking step sizes equal to two. We notice that near the ninth zero on our list a missing pair is detected, and similarly near the twenty fifth zero. A more refined search reveals the pairs of zeros with imaginary parts equal to 48.0051508812, 49.7738324777, and 94.6513440405, 95.8706342282 respectively.

It would be useful to have a general form of the explicit inequality (51) worked out for any \( L \)-function. The papers of Rumely [Rum] and Tollis [To] generalize this inequality to Dirichlet \( L \)-functions and Dedekind zeta functions respectively.

The main term, analogous to (50), for a general \( L \)-function is easy to derive. Let \( L(s) \) be an \( L \)-function with functional equation as described in [23]. Let \( N_L(T) \) denote the number of zeros of \( L(s) \) lying within the rectangle \( |\Im s| \leq T, 0 < \Re s < 1 \). Notice here we are considering zeros lying both above and below the real axis since the zeros of \( L(s) \) will not be located symmetrically about the real axis if its Dirichlet coefficients \( b(n) \) are non-real.

Assume for simplicity that \( L(s) \) is entire. The argument principle and the functional equation for \( L(s) \) suggests a main term for \( N_L(T) \) equal to

\[ N_L(T) \sim \frac{2T}{\pi} \log(Q) + \frac{1}{\pi} \sum_{j=1}^{a} \Im \left( \log \left( \frac{\Gamma((1/2 + iT)\kappa_j + \lambda_j)}{\Gamma((1/2 - iT)\kappa_j + \lambda_j)} \right) \right). \]

If we assume further that the \( \lambda_j \)'s are all real, then the above is, by Stirling’s formula, asymptotically equal to

\[ N_L(T) \sim \frac{2T}{\pi} \log(Q) + \sum_{j=1}^{a} \left( \frac{2T\kappa_j}{\pi} \log(T\kappa_j/e) + (\kappa_j/2 + \lambda_j - 1/2) \right). \]

A slight modification of the above is needed if \( L(s) \) has poles, as in the case of \( \zeta(s) \). See Davenport [13, chapters 15,16] where rigorous proofs
are presented for $\zeta(s)$ and Dirichlet $L$-functions (the original proof is due to von Mangoldt).

| $j$ | $t_j$       | $N((t_j + t_{j-1})/2) - j + 1$ |
|-----|-------------|--------------------------------|
| 1   | 14.1347251417 | 0.11752                        |
| 2   | 21.0220396388 | 0.04445                        |
| 3   | 25.0108575801 | 0.03216                        |
| 4   | 30.4248761259 | 0.01102                        |
| 5   | 32.9350615877 | 0.01000                        |
| 6   | 37.5861781588 | 0.05699                        |
| 7   | 40.9187190121 | 0.07354                        |
| 8   | 43.3270732809 | 0.07314                        |
| 9   | 52.9703214777 | 0.81717                        |
| 10  | 56.4462476971 | 2.01126                        |
| 11  | 59.3470440026 | 2.12394                        |
| 12  | 60.8317785246 | 1.90550                        |
| 13  | 65.1125440481 | 1.95229                        |
| 14  | 67.0798105295 | 2.11039                        |
| 15  | 69.5464017112 | 1.94654                        |
| 16  | 72.0671576745 | 1.90075                        |
| 17  | 75.7046906991 | 2.09822                        |
| 18  | 77.1448400689 | 2.10097                        |
| 19  | 79.3373750202 | 1.82662                        |
| 20  | 82.9103808541 | 1.99205                        |
| 21  | 84.7354929805 | 2.09800                        |
| 22  | 87.4252746131 | 2.03363                        |
| 23  | 88.8091112076 | 1.88592                        |
| 24  | 92.4918992706 | 1.95640                        |
| 25  | 98.8311942182 | 3.10677                        |
| 26  | 101.3178510057| 4.03517                        |
| 27  | 103.7255380405| 4.11799                        |

**Table 2.** Checking for missing zeros. The second column lists the imaginary parts of the zeros of $\zeta(s)$ found by looking for sign changes of $Z(t)$, advancing in step sizes equal to two. The third column compares the number of zeros found to the main term of $N(T)$, namely to $\tilde{N}(T) := \left(T/(2\pi)\right) \log(T/(2\pi e)) + 7/8$, evaluated at the midpoint between consecutive zeros, with $t_0$ taken to be 0. This detects a pair of missing zeros near the ninth and twenty fifth zeros on our list.
4. Experiments involving $L$-functions

Here we describe some of the experiments that reflect the random matrix theory philosophy, namely that the zeros and values of $L$-functions behave like the zeros and values of characteristic functions from the classical compact groups [KS2]. Consequently, we are interested in questions concerning the distribution of zeros, horizontal and vertical, and the value distribution of $L$-functions.

4.1. Horizontal distribution of the zeros. Riemann himself computed the first few zeros of $\zeta(s)$, and detailed numerical studies were initiated almost as soon as computers were invented. See Edwards [E] for a historical survey of these computations. To date, the most impressive computations for $\zeta(s)$ have been those of Odlyzko [O] [O2] and Wedeniwski [W]. The latter adapted code of van de Lune, te Riele, and Winter [LRW] for grid computing over the internet. Several thousand computers have been used to verify that the first $8.5 \cdot 10^{11}$ nontrivial zeros of $\zeta(s)$ fall on the critical line. Odlyzko’s computations have been more concerned with examining the distribution of the spacings between neighbouring zeros, although the Riemann Hypothesis has also been checked for the intervals examined. In [O], Odlyzko computed 175 million consecutive zeros of $\zeta(s)$ lying near the $10^{20}$th zero, and more recently, billions of zeros in a higher region [O2]. The Riemann-Siegel formula has been at the heart of these computations. Odlyzko also uses FFT and interpolation algorithms to allow for many evaluations of $\zeta(s)$ at almost the same cost of a single evaluation.

Dirichlet $L$-functions were not computed on machines until 1961 when Davies and Haselgrove [DH] looked at several $L(s, \chi)$ with conductor $\leq 163$. Rumely [Rum], using summation by parts, computed the first several thousand zeros for many Dirichlet $L$-functions with small moduli. He both verified RH and looked at statistics of neighbouring zeros.

Yoshida [Y1] [Y2] has also used summation by parts, though in a different manner, to compute the first few zeros of certain higher degree, with two or more $\Gamma$-factors in the functional equation, $L$-functions.

Lagarias and Odlyzko [LO] have computed the low lying zeros of several Artin $L$-functions using expansions involving the incomplete gamma function. They noted that one could compute higher up in the critical strip by introducing the parameter $\delta$, as explained in section 3.3, but did not implement it since it led to difficulties concerning the computation of $G(z, w)$ with both $z$ and $w$ complex.

Other computations of $L$-functions include those of Berry and Keating [BK] and Paris [P] ($\zeta(s)$), Tollis [To] (Dedekind zeta functions),
Keiper [Ke] and Spira [Sp] (Ramanujan $\tau$ L-function), Fermigier [F] and Akiyama-Tanigawa [AT] (elliptic curve L-functions), Strombergsson [St] and Farmer-Krance-Lemurell [FKL] (Maass waveform L-functions), and Dokchister [Do] (general L-functions near the critical line).

The author has verified the Riemann hypothesis for various L-functions. These computations use the methods described in section 3 and are not rigorous in the sense that no attempt is made to obtain explicit bounds for truncation errors on some of the asymptotic expansions and continued fractions used, and no interval arithmetic to bound round off errors is carried out. Tables of the zeros mentioned may be obtained from the author’s homepage [Ru4]. These include the first tens of millions zeros of all $L(s, \chi)$ with the conductor of $\chi$ less than 20, the first 300000 zeros of $L_{\tau}(s)$, the Ramanujan $\tau$ L-function, the first 100000 zeros of the L-functions associated to elliptic curves of conductors 11, 14, 15, 17, 19, the first 1000 zeros for elliptic curves of conductors less than 1000, the first 100 zeros of elliptic curves with conductor less than 8000, and hundreds/millions of zeros of many other L-functions.

In all these computations, no violations of the Riemann hypothesis have been found.

4.2. Vertical distribution: correlations and spacings distributions. The random matrix philosophy predicts that various statistics of the zeros of L-functions will mimic the same statistics for the eigenvalues of matrices in the classical compact groups.

Montgomery [Mo] achieved the first result connecting zeros of $\zeta(s)$ with eigenvalues of unitary matrices. Write a typical non-trivial zero of $\zeta$ as

$$1/2 + i\gamma.$$ 

Assume the Riemann Hypothesis, so that the $\gamma$’s are real. Because the zeros of $\zeta(s)$ come in conjugate pairs, we can restrict our attention to those lying above the real axis and order them

$$0 < \gamma_1 \leq \gamma_2 \leq \gamma_3 \ldots$$

We can then ask how the spacings between consecutive zeros, $\gamma_{i+1} - \gamma_i$, are distributed, but first, we need to ‘unfold’ the zeros to compensate for the fact that the zeros on average become closer as one goes higher in the critical strip. We set

$$\tilde{\gamma}_i = \gamma_i \left( \frac{\log(\gamma_i/(2\pi e))}{2\pi} \right) \quad (52)$$
and investigate questions involving the $\tilde{\gamma}$’s. This normalization is chosen so that the mean spacing between consecutive $\tilde{\gamma}$’s equals one. Summing the consecutive differences, we get a telescoping sum

$$\sum_{\gamma_i \leq T} (\tilde{\gamma}_{i+1} - \tilde{\gamma}_i) = \tilde{\gamma}(T) + O(1) = \gamma(T) \frac{\log(\gamma(T)/(2\pi e))}{2\pi} + O(1)$$

where $\gamma(T)$ is the largest $\gamma$ less than or equal to $T$. By (50), the r.h.s above equals

$$N(\gamma(T)) + O(\log(\gamma(T))) = N(T) + O(\log(T)),$$

hence $\tilde{\gamma}_{i+1} - \tilde{\gamma}_i$ has mean spacing equal to one.

From a theoretical point of view, studying the consecutive spacing distribution is difficult since this assumes the ability to sort the zeros. The tool that is used for studying spacings questions about the zeros, namely the explicit formula, involves a sum over all zeros of $\zeta(s)$, and it is easier to consider the pair correlation, a statistic incorporating differences between all pairs of zeros. Montgomery conjectured that for $0 \leq \alpha < \beta$ and $M \to \infty$,

$$M^{-1}\left|\{1 \leq i < j \leq M : \tilde{\gamma}_j - \tilde{\gamma}_i \in [\alpha, \beta]\}\right|$$

\[\sim \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt.\] (53)

Notice that $M^{-1}$, and not, say, $(\frac{M}{2})^{-1}$, is the correct normalization. For any $j$ there, are just a handful of $i$’s with $\tilde{\gamma}_j - \tilde{\gamma}_i \in [\alpha, \beta]$.

Montgomery was able to prove that

$$M^{-1} \sum_{1 \leq i < j \leq M} f(\tilde{\gamma}_j - \tilde{\gamma}_i) \to \int_0^{\infty} f(t) \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt,$$ (54)
as $M \to \infty$, for test functions $f$ satisfying the stringent restriction that $f$ be supported in $(-1, 1)$.

An equivalent way to state the conjecture as $M \to \infty$, and one which Odlyzko uses in his numerical experiments, is to let

$$\delta_i = (\gamma_{i+1} - \gamma_i) \frac{\log(\gamma_i/(2\pi))}{2\pi},$$

and replace the condition $\tilde{\gamma}_j - \tilde{\gamma}_i \in [\alpha, \beta]$ with the condition $\delta_i + \delta_{i+1} + \cdots + \delta_{i+k} \in [\alpha, \beta]$ for $1 \leq i \leq M, k \geq 0$. The main difference is the absence of the $1/e$ in the logarithm. This is done so as to maintain a mean spacing tightly asymptotic to one. Set

$$C(T) = \sum_{\gamma_i \leq T} (\gamma_{i+1} - \gamma_i),$$
and sum by parts
\[ \sum_{\gamma_i \leq T} \delta_i = C(T) \frac{\log(T/(2\pi))}{2\pi} - \frac{1}{2\pi} \int_{\gamma_1}^{T} C(t) \frac{dt}{t} . \]

Now, \( C(t) \) telescopes, and von Mangoldt’s formula \((50)\) implies that \( C(t) = t + O(1) \), so that the r.h.s above equals \( N(T) + O(\log(T)) \), and \( \delta_i \) is on average equal to one. In carrying out numerical experiments with zeros one can either use the normalization given in \((52)\) or \((55)\). For the theoretical purpose of examining leading asymptotics of, say, the pair correlation, the factors appearing in these normalizations in the logarithm, \(1/(2\pi e)\) or \(1/(2\pi)\), are not important as they only affect lower order terms. However, for the purpose of comparing numerical data to theoretical predictions it is crucial to include them.

On a visit by Montgomery to the Institute for Advanced Study, Freeman Dyson out that large unitary matrices have the same pair correlation. Let \( e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_N} \) be the eigenvalues of a matrix in \( U(N) \), sorted so that \( 0 \leq \theta_1 \leq \theta_2 \ldots \leq \theta_N < 2\pi \).

Normalize the eigenangles
\[(56) \tilde{\theta}_i = \theta_i N/(2\pi) \]
so that \( \tilde{\theta}_{i+1} - \tilde{\theta}_i \) equals one on average. Then, a classic result in random matrix theory \[M\] asserts that
\[ N^{-1} \{1 \leq i < j \leq N, \tilde{\theta}_j - \tilde{\theta}_i \in [\alpha, \beta)\} \]
equals, when averaged according to Haar measure over \( U(N) \) and letting \( N \to \infty \),
\[ \int_{\alpha}^{\beta} \left( 1 - \left( \frac{\sin \pi t}{\pi t} \right)^2 \right) dt. \]
Odlyzko \[O\] \[O2\] has carried out numerics to verify Montgomery’s conjecture \((53)\). His most extensive data to date involves billions of zeros near the \(10^{23}\)rd zero of \( \zeta(s) \). With kind permission we reproduce \[O4\]
Odlyzko’s pair correlation picture in figure \[P\].

This picture compares the l.h.s. of \((53)\) for many bins \([a,b]\) of size \( b - a = .01 \) to the curve
\[ 1 - \left( \frac{\sin \pi t}{\pi t} \right)^2. \]
Odlyzko’s histogram fits the theoretical prediction beautifully. Bogo-
molny and Keating [K BoK], using conjectures of Hardy and Littlewood, have explained the role played by secondary terms in the pair correlation of the zeros of $\zeta(s)$ and these terms are related to $\zeta(s)$ on the one line. A nice description of these results are contained in [BK2]. Recently, Conrey and Snaith [CS] obtained the main and lower terms of the pair correlation using a conjecture for the full asymptotics of the average value of a ratio of four zeta functions rather than the Hardy-Littlewood conjectures.

Montgomery’s pair correlation theorem (54) has been generalized by Rudnick and Sarnak [RudS] to any primitive $L$-function, i.e. one which does not factor as a product of other $L$-functions, as well as to higher correlations which are defined in a way similar to the pair correlation. Again, there are severe restrictions on the fourier transform of the allowable test functions, and further, for $L$-functions of degree greater than three, Rudnick and Sarnak assume a weak form of the Ramanujan conjectures. Bogomolny and Keating provide a heuristic derivation of the higher correlations of the zeros of $\zeta(s)$ using the Hardy-Littlewood conjectures [BoK2].

The author has tested the pair correlation conjecture for a number of $L$-functions. Figure 2 depicts the same experiment as in Odlyzko’s
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figure, but for various Dirichlet $L$-functions and $L$-functions associated to cusp forms. Altogether there are eighteen graphs.

The first twelve graphs depict the pair correlation for all primitive Dirichlet $L$-functions, $L(s, \chi)$ for conductors $q = 3, 4, 5, 7, 8, 9, 11, 12, 13, 15, 16, 17$. Each graph shows the average pair correlation for each $q$, i.e. the pair correlation was computed individually for each $L(s, \chi)$, and then averaged over $\chi \mod q$.

In the case of $q = 3, 4$ there is only one primitive $L$-function for either $q$, and approximately five million zeros were used for each ($4,772,120$ and $5,003,411$ zeros respectively to be precise). In the case of $q = 5, 7, 8, 9, 11, 12, 13, 15, 16, 17$ there are $3, 5, 2, 4, 9, 1, 11, 3, 4, 15$ primitive $L$-functions respectively. For $q = 5, 7, 8, 9, 11, 12$ either $2,000,000$ zeros or $1,000,000$ zeros were computed for each $L(s, \chi)$, depending on whether $\chi$ was real or complex. In the case of $q = 16, 17$ half as many zeros were computed.

The last six graphs are for $L$-functions associated to cusp forms. The first of these six shows the pair correlation of the first $284,410$ zeros of the Ramanujan $\tau$ $L$-function, corresponding to the cusp form of level one and weight twelve. The next five depict the pair correlation of the first $100,000$ zeros of the $L$-functions associated to the elliptic curves of conductors $11, 14, 15, 17, 19$. These last six graphs use larger bins since data in these cases is more limited.

The quality of the fit is comparable to what one finds with zeros of $\zeta(s)$ up to the same height. See, for example, figures 1 and 3 in [O3]. It would be possible to extend the $L(s, \chi)$ computations and obtain data near the $10^{20}$th or higher zero, at least for reasonably sized $q$. Using the methods of section 3, the time required to compute $L(1/2 + it, \chi)$ is $O(|qt|^{1/2})$, compared to $O(|t|^{1/2})$ for $\zeta(1/2 + it)$. Adapting the Odlyzko-Schönhage algorithm would allow for many evaluations of these $L$-functions at essentially the cost of a single evaluation. While such a computation might be manageable for Dirichlet $L$-functions, it is hopeless for cusp form $L$-functions where the time and also the number of Dirichlet coefficients required is $O(|N^{1/2}t|)$, i.e. linear in $t$. Here $N$ is the conductor of the $L$-function. Using present algorithms and hardware, it might be possible to extend these cusp form computations to $t = 10^8$ or $10^9$.

Slight care is needed to normalize these zeros correctly as the formula for the number of zeros of $L(s)$ depends on the degree of the $L$-function and on its conductor. For Dirichlet $L$-functions $L(s, \chi), \chi \mod q$, we
Figure 2. Pair correlation for zeros of all primitive $L(s, \chi), 3 \leq q \leq 17$, the Ramanujan $\tau$ $L$-function, and five elliptic curve $L$-functions
should normalize its zeros $1/2 + i\gamma$ as follows:

$$
\tilde{\gamma} = \gamma \log(|\gamma|q/(2\pi e))
$$

For a cusp form $L$-function of conductor $N$, we should take the following normalization:

$$
\tilde{\gamma} = \gamma \log(|\gamma|N^{1/2}/(2\pi e))
$$

From a graphical point of view, it is hard to display information concerning higher order correlations. Instead one can look at a statistic that involves knowing all the $n$-level correlations for characteristic functions, namely the nearest neighbour spacings distribution.

In Figure 3 we display Odlyzko’s picture for the distribution of the normalized spacings $\delta_j$ for $2 \times 10^8$ zeros of $\zeta(s)$ near the $10^{23}$rd zero. This is computed by breaking up the $x$-axis into small bins and counting how many $\delta_j$’s fall into each bin, and then comparing this against the nearest neighbour spacings distribution of the normalized eigenangles of matrices in $U(N)$, as $N \to \infty$, again averaged according to Haar measure on $U(N)$. The density function for this distribution is given as

$$
\frac{d^2}{dt^2} \prod_n (1 - \lambda_n(t))
$$

where $\lambda_n(t)$ are the eigenfunctions of the integral operator

$$
\lambda(t)f(x) = \int_{-1}^{1} \frac{\sin(\pi t(x-y))}{\pi(x-y)} f(y)dy,
$$

sorted according to $1 \geq \lambda_0(t) \geq \lambda_1(t) \geq \ldots \geq 0$. See [O3] for a description of how the density function can be computed.

In Figure 4 we display the nearest neighbour spacings distribution for the sets of zeros described above, namely millions of zeros of primitive $L(s, \chi)$, with conductors $3 \leq q \leq 17$, and hundreds of thousands of zeros of six cusp form $L$-functions. We also depict the nearest neighbour spacings for the first 500,000 zeros of each of the 16 primitive $L(s, \chi)$ with $\chi \mod 19$ complex, and 1,000,000 zeros for the one primitive real $\chi \mod 19$.

Eight graphs are displayed. The first is for the 4,772,120 zeros of $L(s, \chi)$, $\chi \mod 3$. The second one depicts the average spacings distribution for all 76 primitive $L(s, \chi)$, $\chi \mod q$ with $3 \leq q \leq 19$, i.e. the spacings distribution was computed individually for each of these $L$-functions and then averaged. The next six graphs show the spacings distribution for the Ramanujan $\tau L$-function, and the $L$-functions associated to the elliptic curves of conductors $11, 14, 15, 17, 19$. Again,
Figure 3. The first graph shows Odlyzko’s nearest neighbour spacings distribution for $2 \times 10^8$ zeros of $\zeta(s)$ near the $10^{23}$rd zero. The second graph shows the difference he computed between the histogram and the predicted density function. Recently, Bogomolny, Bohigas and Leboeuf have explained the role of secondary terms in shaping the difference displayed.

the fit is comparable to the fit one gets with the same number of zeros of $\zeta(s)$.

4.3. Density of zeros. Rather than look at statistics of a single $L$-function, we can form statistics involving a collection of $L$-functions. This has the advantage of allowing us to study the behaviour of our collection near the critical point where specific information about the collection may be revealed. This idea was formulated by Katz and Sarnak [KS] [KS2] who studied function field zeta functions and conjectured that the various classical compact groups should be relevant to questions about $L$-functions.

While the eigenvalues of matrices in all the classical compact groups share, on average, the same limiting correlations and spacings distributions, their characteristic polynomials do exhibit distinct behaviour near the point $z = 1$. Using the idea that the unit circle for characteristic polynomials in the classical compact groups corresponds to the critical line, with the point $z = 1$ on the unit circle corresponding...
Figure 4. Nearest neighbour spacings distribution for several Dirichlet and cusp form $L$-functions. The first is for $L(s, \chi), q = 3$. The second is the average nearest neighbour spacing for all primitive $L(s, \chi), 3 \leq q \leq 19$. The last six are for the Ramanujan $\tau$ $L$-function, and five $L$-functions associated to elliptic curves.
to the critical point, Katz and Sarnak were led to formulate conjectures regarding the density of zeros near the critical point for various collections of $L$-functions. This is detailed in section 4.3.1 below.

The fact that different families of $L$-functions exhibit distinct behavior near the critical point is illustrated in figure 5. This plot depicts the imaginary parts of the zeros of many $L(s, \chi)$ with $\chi$ a generic non-real primitive Dirichlet character for the modulus $q$, with $5 \leq q \leq 10000$. Other than the fact that, at a fixed height, the zeros become more dense proportionally to $\log q$, the zeros appear to be uniformly dense.

This contrasts sharply with the plot in figure 6 which depicts the zeros of $L(s, \chi_d)$ where $\chi_d$ is a real primitive character (the Kronecker symbol), and $d$ ranges over fundamental discriminants with $-20000 < d < 20000$. Here we see the density of zeros fluctuating as one moves away from the real axis.

Other features can be seen in the plot. First, from the white band near the $x$-axis we notice that the lowest zero for each $L(s, \chi_d)$ tends to stay away from the critical point. We can also see the effect of secondary terms on this repulsion. The lowest zero for $d > 0$ tends to be higher than the lowest zero for $d < 0$. This turns out to be related to the fact that the $\Gamma$-factor in the functional equation for $L(s, \chi_d)$ is $\Gamma(s/2)$ if $d > 0$, but is $\Gamma((s + 1)/2)$ when $d < 0$.

We can also see slightly darker regions appearing in horizontal strips. The first one occurs roughly at height 7, half the height of the first zero of $\zeta(s)$. These horizontal strips are due to secondary terms in the density of zeros for this collection of $L$-functions which include $\text{Ru3, CS}$ a term that is proportional to

$$\Re \frac{\zeta'(1 + 2it)}{\zeta(1 + 2it)}.$$ 

This is large when $\zeta(1 + 2it)$ is small. Surprisingly, $\zeta(1 + iy)$ and $\zeta(1/2 + iy)$ track each other very closely, see figure 7 and the minima of $|\zeta(1 + iy)|$ appear close to the zeros of $\zeta(1/2 + iy)$. This is similar to a phenomenon that occurs when we look at secondary terms in the pair correlation of the zero of $\zeta(s)$ which also involves $\zeta(s)$ on the one line $\text{BoK, BK2}$.

4.3.1. $n$-level density. The $n$-level density is used to measure the average density of the zeros of a family of $L$-functions or matrices. It is arranged to be sensitive to the low lying zeros in the family, i.e. those near the critical point if we are dealing with $L$-functions, and those near
Figure 5. Zeros of $L(s, \chi)$ with $\chi$ a generic non-real primitive Dirichlet character for the modulus $q$, with $5 \leq q \leq 10000$. The horizontal axis is $q$ and, for each $L(s, \chi)$, the imaginary parts of its zeros up to height 15 are listed.

The point $z = 1$ on the unit circle if we are dealing with characteristic polynomials from the classical compact groups.

Let $A$ be an $N \times N$ matrix in one of the classical compact groups. Write the eigenvalues of $A$ as $\lambda_j = e^{i\theta_j}$ with

$$0 \leq \theta_1 \leq \ldots \leq \theta_N < 2\pi.$$ 

Let

$$H^{(n)}(A, f) = \sum_{1 \leq j_1, \ldots, j_n \leq N \text{ distinct}} f(\theta_{j_1}N/(2\pi), \ldots, \theta_{j_n}N/(2\pi))$$

with $f : \mathbb{R}^n \to \mathbb{R}$, bounded, Borel measurable, and compactly supported. Because of the normalization by $N/(2\pi)$, and the assumption that $f$ has compact support, $H^{(n)}(A, f)$ only depends on the small $\theta_j$’s.

Katz and Sarnak [KS] proved the following family dependent result:

$$\lim_{N \to \infty} \int_{G(N)} H^{(n)}(A, f)dA = \int_0^\infty \ldots \int_0^\infty W^{(n)}_G(x)f(x)dx$$

for the following families:
Figure 6. Zeros of $L(s, \chi_d)$ with $\chi_d(n) = (\frac{d}{n})$, the Kronecker symbol. We restrict $d$ to fundamental discriminants $-20000 < d < 20000$. The horizontal axis is $d$ and, for each $L(s, \chi_d)$, the imaginary parts of its zeros up to height 30 are listed. A higher resolution image can be obtained from the author's webpage under 'Publications'.
Figure 7. A graph illustrating that, at least initially, the minima of $|\zeta(1 + iy)|$ occur very close the zeros of $|\zeta(1/2 + iy)|$. The dashed line is the graph of the former while the solid line is the graph of the latter.

$$G \quad W_G^{(n)}$$

| $U(N), U_\kappa(N)$ | $\det (K_0(x_j, x_k))_{1 \leq j \leq n, 1 \leq k \leq n}$ |
| $USp(N)$ | $\det (K_{-1}(x_j, x_k))_{1 \leq j \leq n, 1 \leq k \leq n}$ |
| $SO(2N)$ | $\det (K_1(x_j, x_k))_{1 \leq j \leq n, 1 \leq k \leq n}$ |
| $SO(2N + 1)$ | $\det (K_{-1}(x_j, x_k))_{1 \leq j \leq n, 1 \leq k \leq n} + \sum_{\nu=1}^{n} \delta(x_{\nu}) \det (K_{-1}(x_j, x_k))_{1 \leq j \neq \nu \leq n, 1 \leq k \leq n}$ |

with

$$K_\kappa(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)} + \varepsilon \frac{\sin(\pi(x + y))}{\pi(x + y)}.$$

Here

$$U_\kappa(N) = \{ A \in U(N) : \det(A)^\kappa = 1 \}.$$

The delta functions in the $SO(2N + 1)$ case are accounted for by the eigenvalue at 1. Removing this zero from $\text{SO}(2N + 1)$ yields the same $W_G^{(n)}$ as for $USp$.

Let

$$D(X) = \{ d \text{ a fundamental discriminant : } |d| \leq X \}$$

and let $\chi_d(n) = \left( \frac{d}{n} \right)$ be Kronecker’s symbol. Write the non-trivial zeros of $L(s, \chi_d)$ as

$$1/2 + i\gamma_j^{(d)}, \quad j = \pm 1, \pm 2, \ldots.$$
sorted by increasing imaginary part, and
\[ \gamma^{(d)}_{i-j} = -\gamma^{(d)}_{ij}. \]

The author proved [Ru2] that
\[
\lim_{x \to \infty} \frac{1}{|D(X)|} \sum_{d \in D(X)} \sum_{\substack{j \geq 1 \text{ distinct}}} f \left( l_d \gamma^{(d)}_{j_1}, l_d \gamma^{(d)}_{j_2}, \ldots, l_d \gamma^{(d)}_{j_n} \right)
= \int_{0}^{\infty} \cdots \int_{0}^{\infty} f(x) W_{USp}^{(n)}(x) dx,
\]
where
\[ l_d = \frac{\log(|d|/\pi)}{2\pi}. \]

Here, \( f \) is assumed to be smooth, and rapidly decreasing with \( \hat{f}(u_1, \ldots, u_n) \) supported in \( \sum_{i=1}^{n} |u_i| < 1 \). This generalized the \( n = 1 \) case that had been achieved earlier [OzS] [KS2]. Assuming the Riemann Hypothesis for all \( L(s, \chi_d) \), the \( n = 1 \) case has been extended to \( \hat{f} \) supported in \((-2, 2)\) [OzS2] [KS3]. Chris Hughes has an alternate derivation of (59) appearing in the notes of these proceedings.

This result confirms the connection between zeros of \( L(s, \chi_d) \) and eigenvalues of unitary symplectic matrices and explains the repulsion away from the critical point and the fluctuations seen in figure 6 at least near the real axis, because, when \( n = 1 \), the density of zeros is described by the function \( W_{USp}^{(1)}(x) \) which equals
\[ 1 - \frac{\sin(2\pi x)}{2\pi x}. \]

At height \( x \), we therefore also expect, as we average over larger and larger \( |d| \), for the fluctuations to diminish proportional to \( 1/x \). However, if we allow \( x \) to grow with \( d \) then the fluctuations actually persist due to secondary fluctuating terms that can be large if \( x \) is allowed to grow with \( d \) [Ru3] [CS].

The above suggests that the distribution of the lowest zero, i.e. the one with smallest imaginary part, in this family of \( L \)-functions ought to be modeled by the distribution of the smallest eigenangle of characteristic polynomials in \( \text{USp}(N) \), with \( N \to \infty \). Similarly we expect that the distribution, say, of the second lowest zero ought to fit the distribution of the second smallest eigenangle.

The probability densities describing the distribution of the smallest and second smallest eigenangles, normalized by \( N/(2\pi) \), for characteristic polynomials in \( \text{USp}(N) \), with \( N \) even and tending to \( \infty \) are
given \([KS]\) respectively by
\[
\nu_1(USp)(t) = -\frac{d}{dt}E_{-0}(t)
\]
and
\[
\nu_2(USp)(t) = -\frac{d}{dt}(E_{-0}(t) + E_{-1}(t)),
\]
where
\[
E_{-0}(t) = \prod_{j=0}^{\infty}(1 - \lambda_{2j+1}(2t))
\]
\[
E_{-1}(t) = \sum_{k=1}^{\infty}\lambda_{2k+1}(2t) \prod_{j=0}^{\infty}(1 - \lambda_{2j+1}(2t)).
\]
Here, the \(\lambda_j(t)\)'s are the eigenvalues of the integral equation in \((57)\).

This also suggests that the means of the the first and second lowest zeros are given by
\[
\lim_{X \to \infty} \frac{1}{|D(X)|} \sum_{d \in D(X)} \gamma_1(d) l_d = \int_0^{\infty} \nu_1(USp)(t) \, dt = .78 \ldots
\]
\[
\lim_{X \to \infty} \frac{1}{|D(X)|} \sum_{d \in D(X)} \gamma_2(d) l_d = \int_0^{\infty} \nu_2(USp)(t) \, dt = 1.76 \ldots
\]
However, the convergence to the predicted means is logarithmically slow due to secondary terms of size \(O(1/\log(X))\). Consequently, when comparing against the random matrix theory predictions, one gets a better fit by making sure the lowest zero has the correct mean. This can be achieved by rescaling the data, further multiplying, for a set \(D\) of fundamental discriminants, \(\gamma_1(d) l_d\) by
\[
.78 \left( \frac{1}{|D|} \sum_{d \in D} \gamma_1(d) l_d \right)^{-1}
\]
and \(\gamma_2(d) l_d\) by
\[
1.76 \left( \frac{1}{|D|} \sum_{d \in D} \gamma_2(d) l_d \right)^{-1}
\]
In figures 8 and 9, we use the normalization described above. For our data set, the denominator in \((60)\) equals .83, and, in \((61)\) equals 1.84.

In figure 8 we depict the 1-level density of the zeros of \(L(s, \chi_d)\) for 7243 prime \(|d|\) lying in the interval \((10^{12}, 10^{12} + 200000)\). These zeros were computed in 1996 as part of the authors PhD thesis \([Ru]\). Here we
divide the $x$-axis into small bins, count how many normalized zeros of $L(s, \chi_d)$ lie in each bin, divide that count by the number of $d$, namely 7243, and compare that to the graph of $1 - \sin(2\pi x)/(2\pi x)$.

**Figure 8.** Density of zeros of $L(s, \chi_d)$ for 7243 prime values of $|d|$ lying in the interval $(10^{12}, 10^{12} + 200000)$. Compared against the random matrix theory prediction, $1 - \sin(2\pi x)/(2\pi x)$.

In figure 9 we depict the distribution of the lowest and second lowest normalized zero for the set of zeros just described. These are compared against $\nu_1$ and $\nu_2$ which were computed using the same program, obtained from Andrew Odlyzko, that was used in [O3].

In figure 10 we depict the 1-level density and distribution of the lowest zeros for quadratic twists of the Ramanujan $\tau$ $L$-function, $L_{\tau}(s, \chi_d)$, $d > 0$. For this family of $L$-functions, one can prove a result similar to [Ru2] but with $W_{USp}$ replaced with $W_{SO(\text{even})}$, and the support of $\hat{f}$ reduced to $\sum_{i=1}^n |u_i| < 1/2$. The 1-level density is therefore given by $1 + \sin(2\pi x)/(2\pi x)$ and the probability density for the distribution of the smallest eigenangle, normalized by $2N/(2\pi)$, for matrices in $SO(2N)$, with $N \to \infty$, is given [KS] by

$$\nu_1(\text{SO(even)})(t) = -\frac{d}{dt} \prod_{j=0}^\infty (1 - \lambda_{2j}(2t)),$$

whose mean is .32. The figure uses 11464 prime values of $|d|$ lying in $(350000, 650000)$, and the zeros were normalized by $2l_d$, and then
Figure 9. Distribution of the lowest and second lowest zero of $L(s, \chi_d)$ for 7243 prime values of $|d|$ lying in the interval $(10^{12}, 10^{12} + 200000)$. Compared against the random matrix theory predictions.

rescaled so as to have mean .32 rather than .29. The choice of using $2l_d$ for normalizing the zeros is the correct one up to leading term, but is slightly ad hoc and by now a better understanding of a tighter normalization up to lower terms has emerged [CFKRS] [Ru3].

4.4. Value distribution of L-functions. Keating and Snaith initiated the use of random matrix theory to study the value distribution of L-functions with their important paper [KeS] where they consider moments of characteristic polynomials of unitary matrices and conjecture the leading-order asymptotics for the moments of $\zeta(s)$ on the critical line. This was followed by a second paper [KeS2] along with a paper by Conrey and Farmer [CF] which provide conjectures for the leading-order asymptotics of moments of various families of L-functions by examining analogous questions for characteristic polynomials of the various classical compact groups.

Keating and Snaith’s technically impressive work also represents a philosophical breakthrough. Until their paper appeared, one would compare, say, statistics involving zeros of $\zeta(s)$ to similar statistics for eigenvalues of $N \times N$ unitary matrices, with $N \to \infty$. However, their work compares the average value of $\zeta(1/2 + it)$ to the average value of $N \times N$ unitary characteristic polynomials evaluated on the unit circle, with $N \sim \log(t/(2\pi))$. This choice of $N$ is motivated by comparing
Figure 10. One-level density and distribution of the lowest zero of even quadratic twists of the Ramanujan $\tau$ $L$-function, $L_\tau(s, \chi_d)$, for 11464 prime values of $d > 0$ lying in the interval (350000, 650000).

local spacings of zeros, for example [55] v.s. [56]. A slightly different approach to this choice of $N$ proceeds by comparing functional equations of $L$-functions to functional equations of characteristic polynomials [CFKRS].

At first sight, it seems strange to compare the Riemann zeta function which has infinitely many zeros to characteristic polynomials of finite size matrices. However, this suggests that a given height $t$, the Riemann zeta function can be modeled locally by just a small number of zeros, as well as by more global information that incorporates the role played by primes. Recently, Gonek, Hughes, and Keating have developed such a model [GHK].

Below we describe three specific examples where random matrix theory has led to important advances in our understanding of the value distribution of $L$-functions. These concern the families:

1. $\zeta(1/2 + it)$, where we average over $t$.
2. $L(1/2, \chi_d)$, where we average over fundamental discriminants $d$.
3. $L_E(1/2, \chi_d)$, quadratic twists of the $L$-function associated to an elliptic curve $E$ over $\mathbb{Q}$, where we average over fundamental discriminants $d$. 
These three are examples of unitary, unitary symplectic, and even orthogonal families respectively \[\text{[KS2]}\text{ CFKRS}.\] Note that in the last example, we normalize the Dirichlet coefficient of the \(L\)-function as in \([37]\) so that the functional equation of \(L_E(1/2, \chi_d)\) brings \(s\) into \(1 - s\) with the critical point being \(s = 1/2\).

We first illustrate that these three examples exhibit distinct behaviour by contrasting their value distributions. The first-order asymptotics for the moments of \(|\zeta(1/2 + it)|\) are conjectured by Keating and Snaith \[\text{[KeS]}\] to be given by

\[
\frac{1}{T} \int_0^T |\zeta(1/2 + it)|^r \, dt \sim a_{r/2} \prod_{j=1}^N \frac{\Gamma(j) \Gamma(j + r)}{\Gamma(j + r/2)^2}, \quad \Re r > -1,
\]

with \(N \sim \log(T)\) and \(a_{r/2}\) defined by \([18]\).

For quadratic Dirichlet \(L\)-functions Keating and Snaith \[\text{[KeS2]}\] conjecture that

\[
\frac{1}{|D(X)|} \sum_{d \in D(X)} L(1/2, \chi_d)^r \sim b_r 2^{2N_r} \prod_{j=1}^N \frac{\Gamma(N + j + 1) \Gamma(j + 1/2 + r)}{\Gamma(N + j + 1 + r) \Gamma(j + 1/2)}, \quad \Re r > -3/2,
\]

with \(N \sim \log(X)/2\), where the sum runs over fundamental discriminants \(|d| \leq X\), and, as suggested by Conrey and Farmer \[\text{[CF]}\],

\[
b_r = \prod_p \frac{(1 - \frac{1}{p})^{(r+1)/2}}{1 + \frac{1}{p}} \left( \frac{(1 - \frac{1}{\sqrt{p}})^{-r} + (1 + \frac{1}{\sqrt{p}})^{-r}}{2} + \frac{1}{p} \right).
\]

Next, let \(q\) be the conductor of the elliptic curve \(E\). Averaging over fundamental discriminants and restricting to discriminants for which \(L_E(s, \chi_d)\) has an even functional equation, the conjecture asserts \[\text{[CF]}\text{ [KeS2]}\] that

\[
\frac{1}{|D(X)|} \sum_{\substack{d \in D(X) \\ (d,q) = 1}} L_E(1/2, \chi_d)^r \sim c_r 2^{2N_r} \prod_{j=1}^N \frac{\Gamma(N + j - 1) \Gamma(j - 1/2 + r)}{\Gamma(N + j - 1 + r) \Gamma(j - 1/2)}, \quad \Re r > -1/2,
\]

with \(N \sim \log(X)\) and

\[
c_r = \prod_p \left(1 - \frac{1}{p}\right)^{k(k-1)/2} R_{r,p},
\]

where the sum runs over fundamental discriminants \(|d| \leq X\), and, as suggested by Conrey and Farmer \[\text{[CF]}\],

\[
b_r = \prod_p \frac{(1 - \frac{1}{p})^{(r+1)/2}}{1 + \frac{1}{p}} \left( \frac{(1 - \frac{1}{\sqrt{p}})^{-r} + (1 + \frac{1}{\sqrt{p}})^{-r}}{2} + \frac{1}{p} \right).
\]
where, for $p \nmid q$,

$$R_{k,p} = \left(1 + \frac{1}{p}\right)^{-1} \left(\frac{1}{p} + \frac{1}{2} \left((1 - \frac{a_p}{p} + \frac{1}{p})^{-k} + (1 + \frac{a_p}{p} + \frac{1}{p})^{-k}\right)\right).$$

In the above equation, $a_p$ stands for the $p$th coefficient of the Dirichlet series of $L_E$.

In the case of the Riemann zeta function we take absolute values, $|\zeta(1/2 + it)|$, otherwise the moments would be zero. In the other two cases, the $L$-values are conjectured to be non-negative real numbers, hence we directly take their moments.

We should also observe that, while statistics such as the pair correlation or density of zeros involving zeros of $L$-functions have arithmetic information appearing in the secondary terms, moments already reveal such behaviour at the level of the main term. This reflects the global nature of the moment statistic as compared to the local nature of statistics of zeros that have been discussed.

Using the above conjectured asymptotics we can naively plot value distributions. Figure 11 compares numerical value distributions for data in these three examples against the counterpart densities from random matrix theory. Notice that these three graphs behave distinctly near the origin. The solid curves are computed by taking inverse Mellin transforms, as in (19), of the right hand sides of equations (62), (63), and (64), but without the arithmetic factors $a_k$, $b_k$, $c_k$. Shifting the inverse Mellin transform line integral to the left, the location of the first pole in each integrand dictates the behaviour of the corresponding density functions near the origin. The locations of these three poles are at $r = -1$, $-3/2$, and $-1/2$ respectively. Taking $t$ to be the horizontal axis, near the origin the first density is proportional to a constant, the second to $t^{1/2}$, and the third to $t^{-1/2}$. In forming these graphs one takes $N$ as described above, so that the proportionality constants do depend on $N$. As $N$ grows, these graphs tend to get flatter.

The first graph is reproduced from [KeS]. In the second and third graphs displayed, a slight cheat was used to get a better fit. The histograms were rescaled linearly along both axis until the histogram matched up nicely with the solid curves. We must ignore the arithmetic factors when taking inverse Mellin transforms since these factors are known [CGo] to be functions of order two and cause the inverse Mellin transforms to diverge. To properly plot the correct value distributions we would need to use more than just the leading-order asymptotics. Presently, our knowledge of the moments of various families of $L$-functions extends beyond the first-order asymptotics, but only for...
positive integer values of \( r \) (even integer in the case of \( |\zeta(1/2 + it)|^r \)), however, one would need to apply full asymptotics for complex values of \( r \). The paper by Conrey, Farmer, Keating, the author, and Snaith [CFKRS] conjectures the full asymptotics, for example, of the three moment problems above, but for integer \( r \), with corresponding theorems in random matrix theory given in [CFKRS2]. The paper of Conrey, Farmer and Zirnbauer goes even beyond this stating conjectures for the full asymptotics of moments of ratios of \( L \)-functions, and, using methods from supersymmetry, proving corresponding theorems in random matrix theory [CFZ]. Another paper, by Conrey, Forrester, and Snaith, uses orthogonal polynomials to obtain alternative proofs of the random matrix theory theorems for ratios [CFS].

4.4.1. Moments of \( |\zeta(1/2 + it)| \). Next we describe the full moment conjecture from [CFKRS] for \( |\zeta(1/2 + it)| \). In that paper, the conjecture is derived heuristically by looking at products of zetas shifted slightly away from the critical line and then setting the shifts equal to zero.

The formula is written in terms of contour integrals and involve the Vandermonde:

\[
\Delta(z_1, \ldots, z_m) = \prod_{1 \leq i < j \leq m} (z_j - z_i).
\]

Suppose \( g(t) = f(t/T) \) with \( f : \mathbb{R}^+ \to \mathbb{R} \) non-negative, bounded, and integrable. The conjecture of [CFKRS] states that, as \( T \to \infty \),

\[
\int_0^\infty |\zeta(1/2 + it)|^{2k} g(t) \, dt \sim \int_0^\infty P_k(\log(t/(2\pi))) \, g(t) \, dt,
\]

where \( P_k \) is the polynomial of degree \( k^2 \) given by the 2\( k \)-fold residue

\[
P_k(x) = \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint G(z_1, \ldots, z_{2k}) \Delta^2(z_1, \ldots, z_{2k}) e^{\frac{x}{2} \sum_{j=1}^{2k} z_j - z_{k+j}} dz_1 \cdots dz_{2k},
\]

where one integrates over small circles about \( z_i = 0 \), with

\[
G(z_1, \ldots, z_{2k}) = A_k(z_1, \ldots, z_{2k}) \prod_{i=1}^k \prod_{j=1}^k \zeta(1 + z_i - z_{k+j}),
\]
Figure 11. Value distribution of L-functions compared to the random matrix theory counterparts. The first picture, depicts the value distribution of $|\zeta(1/2 + it)|$, with $t$ near $10^6$, the second of $L(1/2, \chi_d)$ with $800000 < |d| < 10^6$, and the third of $L_{E_{11}}(1/2, \chi_d)$, with $-85000000 < d < 0$, $d = 2, 6, 7, 8, 10 \mod 11$.

and $A_k$ is the Euler product

$$A_k(z) = \prod_{p} k \prod_{i=1}^{k} \prod_{j=1}^{k} \left(1 - \frac{1}{p^{1 + z_i - z_k + j}}\right) \int_{0}^{1} \prod_{j=1}^{k} \left(1 - \frac{e^{2\pi i \theta}}{p^{\frac{1}{z_j} + z_j}}\right)^{-1} \left(1 - \frac{e^{-2\pi i \theta}}{p^{\frac{1}{z_k} - z_k + j}}\right)^{-1} d\theta$$

$$= \prod_{p} k \sum_{m=1}^{k} \prod_{j=1}^{k} \frac{1 - \frac{1}{p^{1 + z_j - z_k + i}}}{1 - p^{\frac{1}{z_k + i} - z_k + m}}.$$
When $k = 1$ or 2, this conjecture agrees with theorems for the full asymptotics as worked out by Ingham \[11\] and Heath-Brown respectively \[H\]. In the first case $A_1(z) = 1$ and in the second case $A_2(z) = \frac{2}{\zeta(2 + z_1 + z_2 - z_3 - z_4)}$, and one can write down the coefficients of the polynomials $P_k(x)$ in terms of known constants. When $k = 3$ the product over primes becomes rather complicated. However, one can numerically evaluate \[CFKRS\] the coefficients of $P_3(x)$ and the polynomial is given by:

$$P_3(x) = 0.000005708527034652788398376841445252313 x^9 + 0.0004050213308411440331215332025984 x^8 + 0.1107245521524699835041040826667 x^7 + 0.14840073080150272680851401518774 x^6 + 1.0459251779054883439385323798059 x^5 + 3.984385094823534724747964073429 x^4 + 8.60731914578120675614834763629 x^3 + 10.274330830703446134183009522 x^2 + 6.59391302064975810465713392 x + 0.9165155076378930590178543.$$

In the $k = 3$ case the moments of $|\zeta(1/2 + it)|$ have not been proven, and it makes sense to test the moment conjecture numerically. Table \[3\] reproduced from \[CFKRS\], depicts

\[
\int_C^{D} |\zeta(1/2 + it)|^6 dt \tag{65}
\]
as compared to

\[
\int_C^{D} P_3(log(t/2\pi)) dt, \tag{66}
\]
along with their ratio, for various blocks $[C, D]$ of length 50000, as well as a larger block of length 2,350,000.

4.4.2. Moments of $L(1/2, \chi_d)$. Another conjecture listed in \[CFKRS\] concerns the full asymptotics for the moments of $L(1/2, \chi_d)$. We quote the conjecture here:

Suppose $g(t) = f(t/T)$ with $f: \mathbb{R}^+ \to \mathbb{R}$ non-negative, bounded, and integrable. Let $X_d(s) = |d|^{1/2-s}X(s,a)$ where $a = 0$ if $d > 0$ and $a = 1$ if $d < 0$, and

$$X(s,a) = \pi^{s-\frac{1}{2}} \Gamma \left( \frac{1 + a - s}{2} \right) / \Gamma \left( \frac{s + a}{2} \right).$$
Table 3. Sixth moment of $\zeta$ versus the conjecture. The ‘reality’ column, i.e. integrals involving $\zeta$, were computed using Mathematica.

| $[C, D]$ | conjecture | reality | ratio |
|---------|------------|---------|-------|
| $[0,5000]$ | 7236872972.7 | 7231005642.3 | .999189 |
| $[50000,100000]$ | 15696470555.3 | 15723919113.6 | 1.001749 |
| $[100000,150000]$ | 21568672884.1 | 21536840937.9 | .998524 |
| $[150000,200000]$ | 26381397608.2 | 26246250354.1 | .994877 |
| $[200000,250000]$ | 31568572884.1 | 31536840937.9 | .998524 |
| $[250000,300000]$ | 37695829854.3 | 37683495193.0 | .999673 |
| $[300000,350000]$ | 43783216365.2 | 43907511751.1 | 1.002839 |
| $[350000,400000]$ | 49166313161.9 | 49136264678.2 | .999627 |
| $[400000,450000]$ | 54035153255.1 | 53962410634.2 | .998654 |
| $[450000,500000]$ | 56541799179.3 | 56541799179.3 | 1.004024 |
| $[500000,550000]$ | 58365383245.2 | 58365383245.2 | .997577 |
| $[550000,600000]$ | 60870809317.1 | 60870809317.1 | 1.00138 |
| $[600000,650000]$ | 62765207086.6 | 62765207086.6 | 1.001663 |
| $[650000,700000]$ | 64227164326.1 | 64227164326.1 | .993665 |
| $[700000,750000]$ | 65994874052.2 | 65994874052.2 | .999673 |
| $[750000,800000]$ | 68961250798.2 | 68961250798.2 | 1.000155 |
| $[800000,850000]$ | 70233931770.0 | 70233931770.0 | 1.002839 |
| $[850000,900000]$ | 72919426905.7 | 72919426905.7 | 1.001298 |
| $[900000,950000]$ | 76752487052.2 | 76752487052.2 | .996927 |
| $[950000,100000]$ | 80320710380.9 | 80320710380.9 | 1.001663 |
| $[100000,105000]$ | 83782957374.3 | 83782957374.3 | 1.000017 |
| $[105000,110000]$ | 8626103164.6 | 8626103164.6 | 1.00138 |
| $[110000,115000]$ | 8870809317.1 | 8870809317.1 | 1.00138 |
| $[115000,120000]$ | 9136264678.2 | 9136264678.2 | 1.00138 |
| $[120000,125000]$ | 93782957374.3 | 93782957374.3 | 1.000017 |
| $[125000,130000]$ | 9626103164.6 | 9626103164.6 | 1.00138 |
| $[130000,135000]$ | 9870809317.1 | 9870809317.1 | 1.00138 |
| $[135000,140000]$ | 101138 | 101138 | 1.00138 |
| $[140000,145000]$ | 1036264678.2 | 1036264678.2 | 1.00138 |
| $[145000,150000]$ | 106138 | 106138 | 1.00138 |

That is, $X_d(s)$ is the factor in the functional equation $L(s, \chi_d) = X_d(s)L(1-s, \chi_d)$. Summing over negative fundamental discriminants $d$ we have, as $T \to \infty$,

$$\sum_{d<0} L(1/2, \chi_d)^k g(|d|) \sim \sum_{d<0} Q_k(\log |d|)g(|d|)$$
where $Q_k$ is the polynomial of degree $k(k + 1)/2$ given by the $k$-fold residue

$$Q_k(x) = \frac{(-1)^{k(k-1)/2} 2^k}{k!} \frac{1}{(2\pi i)^k} \oint \ldots \oint G(z_1, \ldots, z_k) \Delta(z_1^2, \ldots, z_k^2)^2 e^{\sum_{j=1}^k x_j z_j} \prod_{j=1}^k z_j^{2k-1} \, dz_1 \ldots dz_k,$$

where

$$G(z_1, \ldots, z_k) = B_k(z_1, \ldots, z_k) \prod_{j=1}^k X(1/2 + z_j, 1)^{-1/2} \prod_{1 \leq i \leq j \leq k} \zeta(1 + z_i + z_j),$$

and $B_k$ is the Euler product, absolutely convergent for $|\Re z_j| < 1/2$, defined by

$$B_k(z_1, \ldots, z_k) = \prod_p \prod_{1 \leq i \leq j \leq k} \left(1 - \frac{1}{p^{1+z_i+z_j}}\right) \times \left(\frac{1}{2} \left(\prod_{j=1}^k \left(1 - \frac{1}{p^{1/2+z_j}}\right)^{-1}\right) + \prod_{j=1}^k \left(1 + \frac{1}{p^{1/2+z_j}}\right)^{-1}\right) + \frac{1}{p} \left(1 + \frac{1}{p}\right)^{-1}.$$

We can also sum over $d > 0$ but then need to replace $X(1/2 + z_j, 1)$ with $X(1/2 + z_j, 0)$.

This conjecture agrees with theorems in the case of $k = 1, 2, 3$ (only the leading term has been checked in the case of $k = 3$, but in principle the lower terms could be verified).

Figure 12, reproduced from [CFKRS], depicts, for $k = 1, \ldots, 8$ and $X = 10000, 20000, \ldots, 10^7$,

$$\sum_{0<d \leq X} L(1/2, \chi_d)^k$$

divided by

$$\sum_{0<d \leq X} Q_k(\log d).$$

4.4.3. Vanishing of $L_E(1/2, \chi_d)$. In [CFKRS], Conrey, Keating, the author, and Snaith apply the moment conjecture (64) to the problem of predicting asymptotically the number of vanishings of $L_E(1/2, \chi_d)$. Using the fact that these $L$-values are discretized, for example via the Birch and Swinnerton-Dyer conjecture or the theorem of Kohnen-Zagier [KZ], and by studying, up to leading term and for small values,
Figure 12. Horizontal axis in each graph is $X$. These graphs depict the first eight moments, sharp cutoff, of $L(1/2, \chi_d)$, $0 < d \leq X$ divided by the conjectured value, sampled at $X = 10000, 20000, \ldots, 10^7$. We see the graphs fluctuating above and below one. Notice that the vertical scale varies from graph to graph.
the density function predicted by (64) they conjectured that
\[
\sum_{d \in D(X)} 1 \sim \alpha_E X^{3/4} \log(X)^{\beta_E}.
\]

The power on the logarithm depends on the underlying curve \(E\) because, in the Birch Swinnerton Dyer conjecture, the Tamagawa factors can contribute powers of 2 depending on the prime factors of \(d\) and on \(E\) and this affects the discretization. The constant \(\alpha_E\) depends on \(c_{-1/2}\) and the real period of \(E\), but also on some extra subtle arithmetic information that seems to be related to Delaunay’s heuristics for Tate-Shafarevich groups [De] and is not yet fully understood. Numerical evidence in favour of this conjecture is presented in [CKRS2]. One can skirt these delicate issues, the power on the logarithm and the constant \(\alpha_E\), as follows.

Let \(p \nmid q\) be prime. Sort the \(d\)'s for which \(L_E(1/2, \chi_d) = 0\) by residue classes mod \(p\), according to whether \(\chi_d(p) = 1\) or \(-1\), and consider the ratio
\[
R_p(X) = \frac{1}{\sum_{d \in D(X)} \sum_{\chi_d(p) = 1} 1} \sum_{d \in D(X)} \sum_{\chi_d(p) = -1} 1.
\]

One can formulate [CKRS2] [CFKRS] conjectures for the moments in these two subfamilies and the moments agree except for a factor that depends on \(p\). By considering this ratio, the powers of \(X\), of \(\log X\), and the constant \(\alpha_E\) should all cancel out, except for a single factor that depends on \(p\). This leads to a conjecture [CKRS] for \(R_p(X)\):
\[
R_p = \lim_{X \to \infty} R_p(X) = \sqrt{\frac{p + 1 - a_p}{p + 1 + a_p}},
\]
where \(a_p\) denotes the \(p\)th coefficient of the Dirichlet series for \(L_E\). The square root in this conjecture is a consequence of the moments having a pole at \(r = -1/2\).

We end this paper with a plot that substantiates this conjecture. Figure 13 compares, for one hundred elliptic curves \(E\), the predicted value of \(R_p\) to the actual value \(R_p(X)\), with \(X = 10^8\) and the set of \(d\)'s restricted to certain residue classes depending on \(E\) as described in [CKRS2]. The \(L\)-values were computed in this special case by exploiting their connection to the coefficients of certain weight three
halves modular forms and using a table of Rodriguez-Villegas and Tornaria [RT].

The horizontal axis is $p$. For each $p$, and each of the one hundred elliptic curves $E$ we plot $R_p(X) - R_p$. We see the values fluctuating about zero, most of the time agreeing to within about two percent. The convergence in $X$ is predicted from secondary terms to be logarithmically slow and one gets a better fit by including more terms [CKRS2].

![Graph](image)

**Figure 13.** A plot for one hundred elliptic curves of $R_p(X) - R_p$ for $2 \leq p < 3500$, $X = 10^8$.

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