ON FREE BOUNDARY MINIMAL ANNULI
EMBEDDED IN THE UNIT BALL

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Abstract. We prove that an embedded free boundary minimal annulus
in the unit three-ball which is invariant under the antipodal map is the
critical catenoid.

1. Introduction

Differential geometers have made tremendous progress in developing exis-
tence theory for free boundary minimal surfaces in the last decade. The rich
variety of methods employed include variational techniques [7, 17], singular
perturbation gluing methods [4, 8, 9], and min-max constructions [3, 14, 11].
On the other hand, uniqueness results for free boundary minimal surfaces
in the Euclidean unit ball $B^n$ are rare. Although Nitsche [19] and Fraser-
Schoen [6] used Hopf-differential methods to show a free boundary minimal
disk in $B^n$ is a flat equator, classifying free boundary minimal annuli is an
important and challenging open problem:

Conjecture 1 (Fraser and Li, [5]). Up to congruence, the critical catenoid
is the only properly embedded free boundary minimal annulus in $B^3$.

Fraser-Schoen established [7] the conjecture provided the first Steklov
eigenvalue $\sigma_1$ of the annulus equals 1, and McGrath [18] showed $\sigma_1 = 1$
holds assuming the annulus is symmetric with respect to reflections through
the coordinate planes. Here we reduce the symmetry assumption as far as
possible short of the full conjecture:

Theorem 2. Let $\Sigma \subset B^3$ be an embedded, antipodally-invariant free bound-
ary minimal annulus. Then $\Sigma$ is congruent to the critical catenoid.

A crucial ingredient in our proof is a two-piece property for free bound-
ary minimal surfaces in $B^3$ recently proven by Lima-Menezes [15]. We apply
this two-piece property in conjunction with a Courant-type nodal domain
argument to conclude that $\sigma_1 = 1$ on an antipodally-symmetric free bound-
ary minimal annulus. Using the above-noted result of Fraser-Schoen, who
parametrize the minimal surface by a conformal harmonic map $\Phi$ from the
round annulus to $B^n$ and exploit $\sigma_1 = 1$ to show its angular derivative $\Phi_\theta$
extends to a rotational Killing field on $B^n$, Theorem 2 follows.

The study of free boundary minimal surfaces in $B^n$ has features in com-
mon with that of closed minimal surfaces in the round sphere $S^n$, and a
natural counterpart to Conjecture 1 in the setting of embedded minimal surfaces in $S^3$—Lawson’s conjecture on the uniqueness of the Clifford torus—was open for decades before being settled by Brendle [11] in 2012. Under the assumption of reflection symmetry with respect to the four coordinate great-spheres, Ros [20] had earlier shown any embedded minimal surface of genus one in $S^3$ is the Clifford torus. He had also established [21] another long-standing conjecture about the Clifford torus—the Willmore conjecture, ultimately settled by Marques-Neves [16]—assuming antipodal symmetry. In the spirit of these analogous results [20,21,1] in $S^3$, Theorem 2 offers further evidence for the validity of Conjecture 1.

2. The Steklov Eigenvalue Problem

Let $(\Sigma^m, \partial\Sigma)$ be a smooth, compact, connected Riemannian manifold with boundary. Let $\eta$ be the unit outward pointing conormal vector field on $\partial\Sigma$. The Steklov eigenvalue problem is

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Sigma \\
\frac{\partial u}{\partial \eta} &= \sigma u \quad \text{on } \partial\Sigma
\end{align*}
\]

and we call a nontrivial harmonic function $u \in \mathcal{H}(\Sigma)$ satisfying (3) a Steklov eigenfunction. The eigenvalues of (3) are the spectrum of the Dirichlet-to-Neumann map $L : C^\infty(\partial\Sigma) \cong \mathcal{H}(\Sigma) \rightarrow C^\infty(\partial\Sigma) \cong \mathcal{H}(\Sigma)$ given by

\[Lu = \frac{\partial u}{\partial \eta}\]

where we have identified the harmonic extension of $u$ to $\Sigma$ with its boundary values. It is well known that $L$ is a self-adjoint pseudodifferential operator with discrete spectrum

\[0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots\]

We denote the $L$-eigenspace corresponding to $\sigma_i$ by $\mathcal{E}_{\sigma_i}$.

The nodal set of an eigenfunction $u$ is $N_u := \{ p \in \Sigma : u(p) = 0 \}$ and a nodal domain of $u$ is a connected component of $\Sigma \setminus N_u$. The following Courant-type nodal domain theorem is standard [13]:

**Lemma 4.** Each nonzero $u \in \mathcal{E}_{\sigma_1}$ has exactly two nodal domains.

3. Free Boundary Minimal Submanifolds in $B^n$ and Symmetries

Consider a free boundary minimal submanifold of the ball, that is a properly immersed minimal submanifold $\Sigma^m$ of $B^n \subset \mathbb{R}^n$ such that $\Sigma$ meets $\partial B^n$ orthogonally along $\partial\Sigma$. We denote by $\mathcal{C}$ the span of the coordinate functions on $\Sigma$, which are well-known to be Steklov eigenfunctions with eigenvalue 1, and we define

\[\mathcal{C}^\perp = \left\{ u \in \mathcal{E}_{\sigma_1} : \int_{\partial\Sigma} uv = 0, \ v \in \mathcal{C} \right\}.\]

Note that if $\mathcal{C}^\perp = \{0\}$, then $\mathcal{E}_{\sigma_1} = \mathcal{C}$ and $\sigma_1 = 1$.

We now study free boundary minimal submanifolds of $B^n$ which are preserved by a reflection of the Euclidean space $\mathbb{R}^n$ whose fixed-point set is a
given vector subspace $V$. Denote by $V^\perp$ the orthogonal complement of $V$ in $\mathbb{R}^n$, and define the reflection $R_V : \mathbb{R}^n \to \mathbb{R}^n$ with respect to $V$, by

$$R_V := \Pi_V - \Pi_{V^\perp},$$

where $\Pi_V$ and $\Pi_{V^\perp}$ are the orthogonal projections of $\mathbb{R}^n$ onto $V$ and $V^\perp$ respectively. Note in particular that $R_{\{0\}}$ is the antipodal map, where $\{0\}$ is the trivial subspace of $\mathbb{R}^n$.

Suppose $\Sigma$ is $R_V$-symmetric, that is, $R_V(\Sigma) = \Sigma$. Then $R_V$ induces an involutive isometry of each $E_{\sigma_i}$ by the map $u \mapsto u \circ R_V$. It follows that $C^\perp$ has an orthogonal direct sum decomposition $C^\perp = A_V(C^\perp) \oplus S_V(C^\perp)$ into anti-symmetric and symmetric parts

$$A_V(C^\perp) := \{u \in C^\perp : u \circ R_V = -u\} \quad \text{and} \quad S_V(C^\perp) := \{u \in C^\perp : u \circ R_V = u\}.$$

The following boundary nodal domain principle will be useful in the proof of Theorem 2.

**Lemma 5.** Suppose $R_V(\Sigma) = \Sigma$. There do not exist $u \in C^\perp$ and $\varphi \in C$ satisfying the following properties:

1. $u \circ R_V = -u$ and $u$ has exactly two nodal domains;
2. $\varphi = 0$ on $V$ and $\partial \Sigma \cap \Omega = \partial \Sigma \cap \Omega_\varphi$, where $\Omega$ is one of the nodal domains of $u$, and where $\Omega_\varphi$ is one of the nodal domains of $\varphi$.

**Proof.** Suppose that $u \in E_{\sigma_1}$ satisfies (a) and $\varphi \in C$ satisfies (b). Since $u \circ R_V = -u$, the nodal domains of $u$ are $\Omega$ and $R_V(\Omega)$. Also, $\varphi \circ R_V = -\varphi$ because $\varphi = 0$ on $V$. Thus,

$$\int_{\partial \Sigma} \varphi u = \int_{\partial \Sigma \cap \Omega} \varphi u + \int_{\partial \Sigma \cap R_V(\Omega)} \varphi u = 2 \int_{\partial \Sigma \cap \Omega} \varphi u.$$

But $\varphi u$ has a sign on $\partial \Sigma \cap \Omega = \partial \Sigma \cap \Omega_\varphi$, so $\int_{\partial \Sigma} \varphi u \neq 0$ and hence $u \notin C^\perp$. \qed

Note that the condition $\partial \Sigma \cap \Omega = \partial \Sigma \cap \Omega_\varphi$ in Lemma 5 can be relaxed to the condition that the equality holds almost everywhere.

4. The Two-Piece Property and Radial Graphs

An important property of the coordinate functions for free boundary minimal surfaces in $B^3$ is the following result of Lima-Menezes [15]:

**Theorem 6** (Two-piece property [15, Theorem A]). Every nontrivial linear function $\varphi \in C$ on an embedded free boundary minimal surface $\Sigma \subset B^3$ has exactly two nodal domains.

We are grateful to W. Kusner for pointing out [12] that (i) implies (ii) in the following extension of [11, Prop. 8.1]:

**Corollary 7.** Let $\Sigma \subset B^3$ be an embedded free boundary minimal surface of genus zero. Then the following hold:

1. $\Sigma$ is a radial graph, in the sense that $0 \notin \Sigma$ and each ray from 0 transversally intersects $\Sigma$ at most once;
2. each component of $\partial \Sigma$ is a convex curve in $S^2$. 

Proof. Item (i) follows as in [7, Prop. 8.1] by replacing the condition that \( \sigma_1 = 1 \) with the two-piece property. Next, observe that (i) and the two-piece property imply each great circle divides the radial projection of \( \Sigma \) to \( S^2 \) into precisely two components. Hence each great circle has at most two transverse intersections with each component of \( \partial \Sigma \), which yields (ii). \( \square \)

5. Nodal Lines and the Proof of Theorem 2

It is a standard fact [2, 7, 10] that when \( m = 2 \), the nodal set \( \mathcal{N}_u \) of a Steklov eigenfunction is a piecewise-\( C^1 \) graph consisting of finitely many edges where an even number of edges meet each vertex at equal angles. We call a proper \( C^1 \) curve in \( \mathcal{N}_u \) a nodal line.

Proof of Theorem 2. Note that embeddedness of the annulus \( \Sigma \) implies the antipodal map \( R_{\{0\}} \) exchanges the components of \( \partial \Sigma \). We shall prove \( E_{\sigma_1} = C \) by arguing (indirectly in both cases) that \( A_{\{0\}}(C^\perp) \) and \( S_{\{0\}}(C^\perp) \) are each trivial; our theorem then follows immediately, by the result of Fraser-Schoen [7, Theorem 6.6].

First, consider a nonzero \( u \in S_{\{0\}}(C^\perp) \) with nodal domains \( \Omega_1 \) and \( \Omega_2 \). By the symmetry, \( \Omega_1 = R_{\{0\}}(\Omega_1) \) and \( \Omega_2 = R_{\{0\}}(\Omega_2) \). Choose embedded \( R_{\{0\}} \)-invariant loops \( \gamma_1 \subset \Omega_1 \) and \( \gamma_2 \subset \Omega_2 \). But now \( \gamma_1 \) and \( \gamma_2 \) must meet, which contradicts that \( \Omega_1 \cap \Omega_2 = \emptyset \).

Next, consider the two alternatives for a nonzero \( u \in A_{\{0\}}(C^\perp) \).

Case 1: \( u \) does not change sign on either component of \( \partial \Sigma \). Then since \( \int_{\partial \Sigma} u = 0 \), \( u \) has opposite signs on the two boundary components. Choose a point \( p \in \partial \Sigma \), and let \( P \) be the plane through 0 which contains \( p \) and is tangent to \( \partial \Sigma \) at \( p \). By symmetry, \( -p \in \partial \Sigma \cap P \). Choose a nontrivial \( \varphi \in C \) vanishing on \( P \). By the free boundary condition, \( P \) is tangent to \( \Sigma \) at \( p \), so two nodal lines emanate from \( p \). By the two-piece property, \( \mathcal{N}_\varphi \) must consist of these lines, which join again at \( -p \) and divide \( \Sigma \) into two connected sets. It follows that \( \varphi \) cannot change sign on either component of \( \partial \Sigma \). But then \( u \) and \( \varphi \) satisfy the conditions in Lemma 5, which is impossible.

Case 2: \( u \) changes sign on a component of \( \partial \Sigma \). Then there exist \( p \) and \( q \) in this component of \( \partial \Sigma \) such that \( u |_{\partial \Sigma} \) changes sign at \( p \) and at \( q \). Since \( u \) has only two nodal domains, \( \Sigma \) is an annulus, and \( \mathcal{N}_u = R_{\{0\}}(N_u) \), it follows that \( \mathcal{N}_u = \ell \cup R_{\{0\}}(\ell) \), where \( \ell \) is a nodal line joining \( p \) to \( -q \). Let \( P \) be the plane containing 0, \( p \), and \( q \) and choose a nonzero \( \varphi \in C \) which vanishes on \( P \). Now \( \mathcal{N}_\varphi \) contains nodal lines connecting \( p \) to \( -q \) and \( -p \) to \( q \) which divide \( \Sigma \) into two topological disks. By the two-piece property, \( \mathcal{N}_\varphi \) contains no more nodal lines, and again \( u \) and \( \varphi \) satisfy the conditions in Lemma 5, which is impossible. \( \square \)

In a forthcoming paper, we use these techniques in relation to a conjecture of Fraser-Li that \( \sigma_1 = 1 \) for any embedded free boundary minimal surface in \( B^3 \); in particular we show that \( \sigma_1 = 1 \) for families of surfaces constructed by Kapouleas-Li [8] and Kapouleas-Wiygul [9].
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