PRODUCTS OF LINDELÖF SPACES WITH POINTS $G_δ$

TOSHIMICHI USUBA

Abstract. We show that if CH holds and either (i) there exists an $ω_1$-Kurepa tree, or (ii) $□(ω_2)$ holds, then there are regular $T_1$ Lindelöf spaces $X_0$ and $X_1$ with points $G_δ$ such that the extent of $X_0 × X_1$ is strictly greater than $2^ω$.

1. Introduction

While every product of compact spaces is compact, the product of two Lindelöf spaces need not to be Lindelöf; The Sorgenfrey line is a typical example. The square of two Sorgenfrey lines has the Lindelöf degree $2^ω$, where the Lindelöf degree of the space $X$, $L(X)$, is the minimal cardinal $κ$ such that every open cover of $X$ has a subcover of size $≤ κ$. This fact lead us to the following natural question.

Question 1.1. Are there two Lindelöf spaces whose product has the Lindelöf degree $> 2^ω$?

Some consistent examples are known. Shelah [5] constructed a model of ZFC in which there are two regular $T_1$ Lindelöf spaces with points $G_δ$ whose product has the extent $(2^ω)^+ = ω_2$, where the extent of $X$, $e(X)$, is sup {$|C| | C ⊆ X$ is closed discrete}. It is clear that $L(X) ≥ e(X)$. Gorelic [1] refined and simplified Shelah’s method and got a model in which there are two regular $T_1$ Lindelöf spaces with points $G_δ$ whose product has the extent $2^{ω_1}$ and $2^{ω_1}$ is arbitrary large. The extent of the product of their spaces is bounded by $2^{ω_1}$, and Usuba [8] proved that it is consistent that the extent of the product of two regular $T_1$ Lindelöf spaces can be arbitrary large up to the least measurable cardinal. However it is still open if the existence of such Lindelöf spaces is provable from ZFC.

In this paper, we give new construction of such Lindelöf spaces under some combinatorial principles.

Theorem 1.2. Suppose CH. If there exists an $ω_1$-Kurepa tree, or Todorčević’s square principle $□(ω_2)$ holds, then there are regular $T_1$ Lindelöf spaces $X_0, X_1$ with points $G_δ$ such that $e(X_0 × X_1) > 2^ω$.

2010 Mathematics Subject Classification. 03E35, 54A25, 54D20.

Key words and phrases. Aronszajn tree, Kurepa tree, Lindelöf space, points $G_δ$, square principle.
An $\omega_1$-Kurepa tree is an $\omega_1$-tree having strictly more than $\omega_1$ cofinal branches. We say that $\square(\omega_2)$ holds if there exists a sequence $\langle c_\alpha \mid \alpha < \omega_2 \rangle$ such that for each $\alpha < \omega_2$, $c_\alpha$ is a club in $\alpha$, $c_\beta = c_\alpha \cap \beta$ for every $\beta$ from the limit points of $c_\alpha$, and there is no club $D$ in $\omega_2$ such that $D \cap \alpha = c_\alpha$ for every $\alpha$ from the limit points of $D$.

This theorem has some interesting consequences. It is known that the following hold under $V = L$:

(1) CH holds (Gödel).

(2) There exists an $\omega_1$-Kurepa tree (Solovay, e.g., see Theorem 27.8 in Jech [2]).

Hence we have alternative proof of the following result by Shelah [5]:

**Corollary 1.3** (Shelah [5]). Suppose $V = L$. Then there are regular $T_1$ Lindelöf spaces $X_0, X_1$ with points $G_\delta$ such that $e(X_0 \times X_1) > 2^\omega$.

It is also known that if $\square(\omega_2)$ fails then $\omega_2$ is weakly compact in $L$ (Todorčević, (1.10) in Todorčević [6]).

**Corollary 1.4.** Suppose CH. If $e(X_0 \times X_1) \leq 2^\omega$ for every regular $T_1$ Lindelöf spaces $X_0, X_1$ with points $G_\delta$, then $\omega_2$ is weakly compact in the constructible universe $L$.

This shows that the non-existence of such Lindelöf spaces would have a large cardinal strength (if it is consistent).

A very rough sketch of our construction is as follows. For a certain Hausdorff Lindelöf space, we modify open neighborhoods of each points of the space and construct finer Lindelöf spaces $X_0$ and $X_1$ such that for each $x \in X$, there are open sets $O_0 \subseteq X_0$ and $O_1 \subseteq X_1$ with $O_0 \cap O_1 = \{x\}$. Clearly the diagonal of $X_0 \times X_1$ is a large closed discrete subset of $X_0 \times X_1$. Basic idea of our construction come from Usuba [7].

**Acknowledgements.** The author would like to thank the referee for many useful comments and suggestions. This research was supported by JSPS KAKENHI Grant Nos. 18K03403 and 18K03404.

2. **Modifying points with character $\omega_1$**

**Proposition 2.1.** Let $X$ be a Hausdorff Lindelöf space of size $> 2^\omega$, and $X_0, X_1$ be regular $T_1$ Lindelöf spaces of character $\leq \omega_1$ such that:

(1) $X_0$ and $X_1$ have the same underlying sets to $X$ and topologies of $X_0$ and $X_1$ are finer than $X$.

(2) For every $x \in X$, $\chi(x, X_0) = \chi(x, X_1)$.

(3) For $x \in X$, if $\chi(x, X_0) = \chi(x, X_1) = \omega_1$ then there exists a sequence $\langle O_\alpha^x : \alpha < \omega_1 \rangle$ with the following properties:

(a) $O_\alpha^x$ is clopen in $X$. 
(b) \( O_\alpha^x \supseteq O_{\alpha+1}^x \).
(c) \( O_\alpha^x = \bigcap_{\beta < \alpha} O_\beta^x \) if \( \alpha \) is limit.
(d) \( \bigcap_{\alpha < \omega} O_\alpha^x = \{ x \} \).

(4) For \( x \in X \), if \( \chi(x, X_0) = \chi(x, X_1) = \omega \) then there are open sets \( O_0 \subseteq X_0 \) and \( O_1 \subseteq X_1 \) respectively with \( O_0 \cap O_1 = \{ x \} \).

Then there are regular \( T_1 \) Lindelöf spaces \( Y_0 \) and \( Y_1 \) with points \( G_\delta \) such that \( e(Y_0 \times Y_1) = |X| > 2^\omega \).

**Proof.** First, fix an injection \( \sigma : \omega_1 \to \mathbb{R} \) where \( \mathbb{R} \) is the real line. Let \( X' = \{ x \in X_0 \mid \chi(x, X_0) = \omega_1 \} = \{ x \in X_1 \mid \chi(x, X_1) = \omega_1 \} \). For a set \( A \subseteq X \), let \( \| A \| = \bigcup \{ \{ x \} \times \mathbb{R} \mid x \in A \cap X' \} \cup (A \setminus X') \).

For \( x \in X' \), \( \alpha < \omega_1 \), and a set \( W \subseteq \mathbb{R} \), let \( O(x, \alpha, W) = \bigcup \{ [O_\beta^x \setminus O_{\beta+1}^x] \mid \beta \geq \alpha, \sigma(\beta) \in W \} \) \cup (\{ x \} \times W) \).

For constructing \( Y_0 \), let \( S \) be the Sorgenfrey line, that is, the underlying set of \( S \) is the real line \( \mathbb{R} \), and the topology is generated by the family \( \{ \{ r, s \} \mid r, s \in \mathbb{R} \} \) as an open base. It is known that \( S \) is a first countable regular \( T_1 \) Lindelöf space.

We define \( Y_0 \) in the following manner. The underlying set of \( Y_0 \) is \( \| X \| \). The topology of \( Y_0 \) is generated by the family \( \{ \{ O \} \mid O \subseteq X_0 \) is open \} \cup \{ O(x, \alpha, W) \mid x \in X', \alpha < \omega_1, W \subseteq S \) is open \} as an open base. We know that \( Y_0 \) is a regular \( T_1 \) Lindelöf space with points \( G_\delta \) (see Proposition 1.2 in [8]).

For \( Y_1 \), let \( S^* \) be the space \( \mathbb{R} \) equipped with the reverse Sorgenfrey topology, that is, the topology generated by the family \( \{ \{ r, s \} \mid r, s \in \mathbb{R} \} \) as an open base. As with \( S \), \( S^* \) is a first countable regular \( T_1 \) Lindelöf space. Then we define \( Y_1 \) by the same way to \( Y_0 \) but replacing \( X_0 \) by \( X_1 \) and \( S \) by \( S^* \). Again, \( Y_1 \) is a regular \( T_1 \) Lindelöf space with points \( G_\delta \).

To show that \( e(Y_0 \times Y_1) = |X| > 2^\omega \), let \( \Delta = \{ (x, x) \mid x \in X \setminus X' \} \cup \{ \langle (x, r), (x, r) \rangle \mid x \in X', r \in \mathbb{R} \} \). We see that \( \Delta \) is closed and discrete.

For the closeness of \( \Delta \), take \( p \in (Y_0 \times Y_1) \setminus \Delta \).

Case 1: \( p = \langle x, y \rangle \) for some \( x, y \in X \setminus X' \). Since \( X \) is Hausdorff, there are disjoint open sets \( O_0, O_1 \subseteq X \) with \( x \in O_0 \) and \( y \in O_1 \). Since \( X_0 \) and \( X_1 \) are finer than \( X \), \( O_0 \) and \( O_1 \) are open in \( X_0 \) and \( X_1 \) respectively. Then \( \{ [O_0] \subseteq Y_0 \) is open with \( x \in [O_0], [O_1] \subseteq Y_1 \) is open with \( y \in [O_1], [O_0] \cap [O_1] = \emptyset \). Hence \( \langle x, y \rangle \in [O_0] \times [O_1] \) and \( \Delta \cap ([O_0] \times [O_1]) = \emptyset \).

Case 2: \( p = \langle x, \langle y, r \rangle \rangle \) for some \( x \in X \setminus X', y \in X', \) and \( r \in \mathbb{R} \). Again, take open sets \( O_0, O_1 \subseteq X \) such that \( x \in O_0, y \in O_1, \) and \( O_0 \cap O_1 = \emptyset \). Then \( x \in [O_0], \langle y, r \rangle \in [O_1], [O_0] \cap [O_1] = \emptyset \). So \( p \in [O_0] \times [O_1] \) and \( \Delta \cap ([O_0] \times [O_1]) = \emptyset \).

Case 3: \( p = \langle \langle x, r \rangle, y \rangle \) for some \( x \in X', y \in X \setminus X', \) and \( r \in \mathbb{R} \). Similar to Case 2.

Case 4: \( p = \langle \langle x, r \rangle, \langle y, s \rangle \rangle \) for some \( x, y \in X' \) and \( r, s \in \mathbb{R} \). If \( x \neq y \), we can take open sets \( O_0, O_1 \subseteq X \) with \( x \in O_0, y \in O_1, \) and \( O_0 \cap O_1 = \emptyset \). Then \( [O_0] \times [O_1] \) is a required set. If \( x = y \) and \( r \neq s \), take open sets \( W_0, W_1 \subseteq \mathbb{R} \).
with \( r \in W_0, s \in W_1, \) and \( W_0 \cap W_1 = \emptyset. \) Now \( \langle x, r \rangle \in O(x, 0, W_0), \langle y, s \rangle \in O(y, 0, W_1), \) and \( O(x, 0, W_0) \cap O(y, 0, W_1) = \emptyset. \) Hence \( p \in O(x, 0, W_0) \times O(y, 0, W_1) \) and \( \Delta \cap (O(x, 0, W_0) \times O(y, 0, W_1)) = \emptyset. \)

Next we see that \( \Delta \) is discrete. For \( x \in X \setminus X', \) by the assumption, there are open sets \( O_0 \subseteq X_0 \) and \( O_1 \subseteq X_1 \) respectively with \( O_0 \cap O_1 = \{x\}. \) Then it is clear that \([O_0] \cap [O_1] = \{x\}, \) hence \( \Delta \cap ([O_0] \times [O_1]) = \{x\}. \) For \( x \in X' \) and \( r \in \mathbb{R}, \) consider open sets \( W_0 = [r, r + 1) \) in \( S \) and \( W_1 = (r - 1, r) \) in \( S^* \). Trivially \( W_0 \cap W_1 = \{r\}. \) Then, by the definitions of \( O(x, 0, W_0) \subseteq Y_0 \) and \( O(x, 0, W_1) \subseteq Y_1, \) we have \( O(x, 0, W_0) \cap O(x, 0, W_1) = \{\langle x, r \rangle\}. \) Thus \( \Delta \cap (O(x, 0, W_0) \times O(x, 0, W_1)) = \{\langle x, r \rangle\}, \) as required. \( \square \)

A space \( X \) is said to be a \( P\)-space if every \( G_{\delta} \) subset of \( X \) is open. If \( X \) is a regular \( T_1 \) Lindelöf \( P\)-space of character \( \leq \omega_1, \) then every point \( x \in X \) with \( \chi(x, X) = \omega \) is isolated in \( X. \) Hence \( X = X_0 \cap X_1 \) satisfy the assumptions of the previous proposition.

**Corollary 2.2.** If there exists a regular \( T_1 \) Lindelöf \( P\)-space of character \( \leq \omega_1 \) and size \( > 2^\omega, \) then there are regular \( T_1 \) Lindelöf spaces \( Y_0, Y_1 \) with points \( G_\delta \) such that \( e(Y_0, Y_1) > 2^\omega. \)

It is known that such a \( P\)-space exists under \( V = L \) (Juhász-Weiss [3]), so this fact yields one more another proof of Corollary [13].

### 3. Modifying points with character \( \omega \)

For our convenience, we fix some notations and definitions. For an ordinal \( \alpha, \) let \( 2^\alpha \) be the set of all functions from \( \alpha \) to \( 2, \) and \( 2^{<\alpha} (2^{\leq \alpha}, \) respectively) be \( \bigcup_{\beta<\alpha} 2^\beta \) \( (\bigcup_{\beta<\alpha} 2^\beta, \) respectively). We say that \( T \) is a tree if \( T \) is a subset of \( 2^{<\alpha} \) for some ordinal \( \alpha \) such that \( T \) is downward closed, that is, for every \( s \in T \) and \( t \in 2^{<\alpha}, \) if \( t \subset s \) then \( t \in T. \) For \( s, t \in T, \) define \( s \leq t \iff s \subseteq t, \) and \( s < t \iff s \subset t. \) A branch of a tree \( T \) is a maximal chain of \( T. \) If \( B \) is a branch, then \( \bigcup B \) is a function with \( \bigcup B \in 2^{\beta}, \) and \( B = \{\bigcup B \upharpoonright \beta \mid \beta < \operatorname{dom}(\bigcup B)\}. \) Because of this reason, we identify a branch \( B \) as the function \( \bigcup B. \) Cantor tree is the tree \( 2^{<\omega}. \) We say that \( \sigma : 2^{<\omega} \to 2^{<\alpha} \) is an embedding if \( s < t \iff \sigma(s) < \sigma(t) \) for every \( s, t \in 2^{<\omega}. \) Every embedding \( \sigma : 2^{<\omega} \to 2^{<\alpha} \) canonically induces the map \( \sigma^*: 2^\omega \to 2^{\leq \alpha} \) as \( \sigma^*(f) = \bigcup_{n<\omega} \sigma(f | n). \) Note that a tree \( T \) does not contain an isomorphic copy of Cantor tree if and only if for every embedding \( \sigma : 2^{<\omega} \to T \) there is \( f \in 2^\omega \) with \( \sigma^*(f) \notin T. \)

**Proposition 3.1.** Assume \( CH. \) Suppose there exists a tree \( T \subseteq 2^{<\omega^2} \) such that:

(1) Each level of \( T \) has cardinality at most \( \omega_1. \)
(2) \( T \) has no branch of size \( \omega_2. \)
(3) \( |T| > 2^\omega, \) or \( T \) has strictly more than \( 2^\omega \) many branches.
(4) $T$ does not contain an isomorphic copy of Cantor tree.

Then there exist zero-dimensional $T_1$ Lindelöf spaces $X, X_0, X_1$ which satisfy the assumptions of Proposition 2.1.

Now Theorem 1.2 follows from Propositions 2.1 and 3.1. If there exists an $\omega_1$-Kurepa tree $T \subseteq 2^{<\omega_1}$, by CH, $T$ satisfies the assumptions of Proposition 3.1. If $\square(\omega_2)$ holds, then there is an $\omega_2$-Aronszajn tree $T \subseteq 2^{<\omega_2}$ which does not contain an isomorphic copy of Cantor tree (Todorčević, (1.11) in [6]. See also Corollary 3.10 in König [4]). It is clear that $T$ fulfills the assumptions of Proposition 3.1.

We start the proof of Proposition 3.1. Fix a tree $T$ satisfying the assumptions. We may assume that every $t \in T$ has two immediate successors $t^-(0), t^-(1)$ in $T$.

Let $T^* = \{ t \in T \mid \text{cf(} \operatorname{dom}(t) ) = \omega_1 \}$. For $i = 0, 1$, let $B_i$ be the set of all branches $B$ of $T$ with $\text{cf(} \operatorname{dom}(B) ) = \omega_i$. For $t \in T$, let $[t] = \{ B \in B_0 \cup B_1 \mid t \in B \} \cup \{ s \in T^* \mid t \leq s \}$ and $[t]^+ = [t^-(0)] \cup [t^-(1)]$. Note that if $t \in T \setminus T^*$ then $[t] = [t]^+$.

First we define the space $X$. The underlying set of $X$ is $B_0 \cup B_1 \cup T^*$. The topology is generated by the family

$$\{ [t] \mid t \in T \setminus T^* \} \cup \{ [s] \setminus [t]^+ \mid t \in T^*, s \notin T^*, s < t \}$$

as an open base. It is routine to check that $X$ is a zero-dimensional $T_1$ space of size $> 2^\omega$. For $t \in T^*$, the family $\{ [t] \setminus [t]^+ \mid \text{cf(} \operatorname{dom}(t) ) \neq \omega_1 \}$ is a local base for $t$, and $\chi(t, X) = \omega_1$. For $B \in B_0 \cup B_1$, the family $\{ [B \setminus t] \mid t \in B, \text{cf(} \operatorname{dom}(B) ) \neq \omega_1 \}$ is a local base for $B$. It is clear that $\chi(B, X) = \omega_1 \iff B \in B_i$.

We prove that $X$ is Lindelöf.

**Claim 3.2.** $X$ is Lindelöf.

**Proof.** Let $U$ be an open cover of $X$. Let $T_U$ be the set of all $t \in T$ such that there is no countable subfamily $V \subseteq U$ with $[t] \subseteq \bigcup V$. If $T_U = \emptyset$, then $[\emptyset] \subseteq V$ for some countable $V \subseteq U$, and $V$ is a countable cover of $X$. Thus it is enough to see that $T_U = \emptyset$.

Suppose to the contrary that $T_U \neq \emptyset$. We note that for $t \in T_U$ and $s \in T$, if $s \leq t$ then $s \in T_U$. Hence $T_U$ is a subtree of $T$.

First we check that $T_U$ has no maximal element. Suppose not and take $t \in T_U$ which is a maximal element of $T_U$. Then $t^-(0), t^-(1)$ are elements of $T$ but not of $T_U$. Thus there are countable subfamilies $V_0, V_1 \subseteq U$ with $[t^-(i)] \subseteq \bigcup V_i$ for $i = 0, 1$. If $t \notin T^*$, then $[t] = [t]^+ \subseteq \bigcup (V_0 \cup V_1)$, thus we have $t \in T_U$. This is a contradiction. If $t \in T^*$, pick $O \in U$ with $t \in O$. Then $[t] = \{ t \} \cup [t]^+ \subseteq O \cup \bigcup (V_0 \cup V_1)$, this is a contradiction too.

Next we check that $T_U$ is branching. Suppose not, and take $t_0 \in T_U$ such that every $t \in T_U$ with $t_0 \leq t$ has only one immediate successor in $T_U$. Let $C = \{ t \in T_U \mid t_0 \leq t \}$. $C$ is a chain of $T$. By the assumption, we have that $|C| \leq \omega_1$. Let $\langle t_\alpha \mid \alpha < \gamma \rangle$ be the increasing enumeration of $C$. We know that $\gamma$ is a limit ordinal.
with $\gamma < \omega_2$. By induction on $\alpha < \gamma$, we claim that there is a countable $V \subseteq U$ with $[t_0] \setminus [t_\alpha] \subseteq \bigcup \mathcal{V}$. The case $\alpha = 0$ is trivial. If $\alpha = \beta + 1$ and $\text{cf}(\beta) = \omega_1$, then $t_\beta \in T^*$ and $[t_0] \setminus [t_\alpha] = ([t_0] \setminus [t_\beta]) \cup [t_\beta \setminus t_\alpha (\text{dom}(t_\beta)))] \cup \{t_\beta\}$. Take a countable $V \subseteq U$ with $[t_0] \setminus [t_\beta] \subseteq \bigcup \mathcal{V}$. Because $t_\alpha \setminus (1 - t(\text{dom}(t_\beta))) \notin T_U$, there is a countable $V' \subseteq U$ with $[t_\alpha \setminus (1 - t_\alpha (\text{dom}(t_\beta)))] \subseteq \mathcal{V}'$. Then $[t_0] \setminus [t_\alpha] \subseteq O \cup \bigcup \mathcal{V}$ for some $O \in U$ with $t_\beta \in O$. The case that $\alpha = \beta + 1$ and $\text{cf}(\beta) \neq \omega_1$ is similar. Suppose $\alpha$ is a limit ordinal. If $\text{cf}(\alpha) = \omega$, take an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ with limit $\alpha$. By the induction hypothesis, for $n < \omega$ there is a countable $V_n \subseteq U$ with $[t_0] \setminus [t_\alpha_n] \subseteq \bigcup \mathcal{V}_n$. $[t_0] \setminus [t_\alpha] = \bigcup_{n < \omega} ([t_0] \setminus [t_\alpha_n])$, hence $[t_0] \setminus [t_\alpha] \subseteq \bigcup_{n < \omega} \mathcal{V}_n$. Finally suppose $\text{cf}(\alpha) = \omega_1$. Then $t_\alpha \in T^*$. Pick $O \in U$ with $t_\alpha \in O$. By the definition of the topology of $X$, there is some $s < t_\alpha$ such that $s \notin T^*$ and $[s] \setminus [t_\alpha]^+ \subseteq O$. Fix $\beta < \alpha$ with $s \leqslant t_\beta$, and take a countable $V \subseteq U$ with $[t_0] \setminus [t_\beta] \subseteq \bigcup \mathcal{V}$. Then

$$[t_0] \setminus [t_\alpha] \subseteq ([t_0] \setminus [t_\beta]) \cup ([s] \setminus [t_\alpha]^+) \subseteq O \cup \bigcup \mathcal{V}.$$ 

Let $t_\gamma = \bigcup_{\alpha < \gamma} t_\alpha$. We know $t_\gamma \notin T_U$. If $t_\gamma \in T$, by the same argument as before, we can find a countable $V \subseteq U$ with $[t_0] \setminus [t_\gamma] \subseteq \bigcup \mathcal{V}$. Since $t_\gamma \notin T_U$, there is a countable $V' \subseteq U$ such that $[t_\gamma] \subseteq \mathcal{V}'$. Then $[t_0] \subseteq \bigcup \mathcal{V}$, this is a contradiction. If $t_\gamma \notin T$, then $t_\gamma \in B_0 \cup B_1$. Pick $O \in U$ with $t_\gamma \in O$. Then there is $t \in T \setminus T^*$ with $t < t_\gamma$ and $[t] \subseteq O$. Fix $\beta < \gamma$ with $t \leqslant t_\beta$. We have $[t_0] = ([t_0] \setminus [t_\beta]) \cup [t]$, and we can derive a contradiction as before.

Now we know that $T_U$ has no maximal element and is branching. Hence we can take an embedding $\sigma : 2^{<\omega} \to T_U$. By the assumption on $T$, there is some $f \in 2^{<\omega}$ with $\sigma^*(f) \notin T$. Then $B = \sigma^*(f)$ is a branch of $T$ and $B \in B_0$. Fix an open set $O \in U$ with $B \subseteq O$. There is some $t \in B$ with $[t] \subseteq O$, and we can choose $n < \omega$ with $t < \sigma(f \upharpoonright n)$. However then $[\sigma(f \upharpoonright n)] \subseteq O$, this contradicts to $\sigma(f \upharpoonright n) \in T_U$.

**Remark 3.3.** The place where we use the assumption that “Cantor tree $2^{<\omega}$ cannot be embedded into $T$” is the proof of this claim, and the referee pointed out us that, for proving this claim, the Cantor tree assumption can be weakened to that “the tree $2^{<\omega_1}$ cannot be embedded into $T$”.

Next, by modifying open neighborhoods of points in $B_0$, we construct finer spaces $X_0$ and $X_1$. Let us say that an embedding $\sigma$ is *good* if $\text{dom}(\sigma^*(f)) = \text{dom}(\sigma^*(g))$ for every $f, g \in 2^{<\omega}$.

**Claim 3.4.** For every embedding $\sigma$, there is a good embedding $\tau$ such that $\text{Range}(\tau) \subseteq \text{Range}(\sigma)$.

**Proof.** First note that the set $D = \{\text{dom}(\sigma^*(f)) \mid f \in 2^{<\omega}\}$ is at most countable, because $D$ is a subset of all limit points of the countable set $\{\text{dom}(\sigma(t)) \mid t \in 2^{<\omega}\}$.

Now we have $2^{<\omega} = \bigcup_{\alpha \in D} \{f \in 2^{<\omega} \mid \text{dom}(\sigma^*(f)) = \alpha\}$. $D$ is countable, thus there is some $\alpha \in D$ such that $E = \{f \in 2^{<\omega} \mid \text{dom}(\sigma^*(f)) = \alpha\}$ is uncountable. It is clear that $\alpha$ is a limit ordinal with countable cofinality. Take an increasing
sequence \( \langle \alpha_i \mid i < \omega \rangle \) with limit \( \alpha \). Then for every \( t \in 2^{<\omega} \) and \( i < \omega \), if \( \{ f \in E \mid t \subseteq f \} \) is uncountable, then there are two \( s_0, s_1 \in 2^{<\omega} \) such that \( t < s_0, s_1, \) \( \text{dom}(\sigma(s_0)), \text{dom}(\sigma(s_1)) \geq \alpha_i \), and both \( \{ f \in E \mid s_0 \subseteq f \} \), \( \{ f \in E \mid s_1 \subseteq f \} \) are uncountable; Since the Cantor space \( 2^\omega \) is compact, we can find two \( f_0, f_1 \in 2^\omega \) such that \( f_0, f_1 \supseteq t \), and for every open neighborhood \( O \) of \( f_0 \) or \( f_1 \) in \( 2^\omega \), the set \( O \cap E \) is uncountable. Take a large \( n < \omega \) with \( \text{dom}(\sigma(f_0)) \), \( \text{dom}(\sigma(f_1)) \geq \alpha_i \) and \( f_0 \upharpoonright n \neq f_1 \upharpoonright n \). Let \( s_0 = f_0 \upharpoonright n \) and \( s_1 = f_1 \upharpoonright n \). Then we have that the sets \( \{ f \in E \mid s_0 \subseteq f \} \) and \( \{ f \in E \mid s_1 \subseteq f \} \) are uncountable.

Using the above observation, we can take an embedding \( \rho : 2^{<\omega} \to 2^{<\omega} \) such that for every \( t \in 2^{<\omega} \) with \( \text{dom}(t) = n \), we have \( \alpha_n \leq \text{dom}(\sigma(\rho(t))) < \alpha \). Let \( \tau = \sigma \circ \rho \). It is easy to check that \( \tau : 2^{<\omega} \to T \) is a required embedding. \( \square \)

**Claim 3.5.** Let \( \sigma : 2^{<\omega} \to T \) be a good embedding. Then the set \( \{ f \in 2^\omega \mid \sigma^*(f) \notin T \} \) is uncountable.

**Proof.** If it is countable, we can take an enumeration \( \{ f_n \mid n < \omega \} \) of it. Then we can take an embedding \( \tau : 2^{<\omega} \to 2^{<\omega} \) such that \( \sigma(\tau(t)) \neq \sigma(f_{\text{dom}(t)} \upharpoonright \text{dom}(\tau(t))) \). Let \( \rho = \sigma \circ \tau \). \( \rho \) is an embedding, \( \text{Range}(\rho) \subseteq \text{Range}(\sigma) \), and \( \text{Range}(\rho^*) \cap \{ \sigma^*(f) \mid f \in 2^\omega, \sigma^*(f) \notin T \} = \emptyset \). Because \( T \) does not contain an isomorphic copy of Cantor tree, there is some \( f \in 2^\omega \) such that \( \rho^*(f) \notin T \). \( \text{Range}(\rho) \subseteq \text{Range}(\sigma) \), hence \( \text{Range}(\rho^*) \subseteq \text{Range}(\sigma^*) \) and there is \( n \) with \( \rho^*(f) = \sigma^*(f_n) \), this is a contradiction. \( \square \)

Let \( G \) be the set of all good embeddings.

**Claim 3.6.** There is an injection \( \varphi \) from \( G \) into \( B_0 \) such that \( \varphi(\sigma) \in \text{Range}(\sigma^*) \) for every \( \sigma \in G \).

**Proof.** For \( \sigma \in G \), let \( \alpha_\sigma \) be the ordinal such that \( \text{dom}(\sigma^*(f)) = \alpha_\sigma \) for every \( f \in 2^\omega \). \( \alpha \) is a limit ordinal with countable cofinality.

Fix a limit ordinal \( \alpha \) with countable cofinality. We define \( \varphi \upharpoonright \{ \sigma \in G \mid \alpha_\sigma = \alpha \} \). We have that \( \text{Range}(\sigma) \subseteq T \cap 2^{<\alpha} \) for every \( \sigma \in G \) with \( \alpha_\sigma = \alpha \). By the assumption on \( T \), we have that \( T \cap 2^{<\alpha} \) has cardinality at most \( \omega_1 \), so there are at most \( (\omega_1)^\omega = \omega_1 \) many good embeddings \( \sigma \) with \( \alpha_\sigma = \alpha \). In addition, by Claim 3.5 for every \( \sigma \in G \) with \( \alpha_\sigma = \alpha \), the set \( \{ f \in 2^\omega \mid \sigma^*(f) \notin T \} \) is uncountable, hence has cardinality \( \omega_1 \). Combining these observations, we can easily take an injection \( \varphi \upharpoonright \{ \sigma \in G \mid \alpha_\sigma = \alpha \} \) into \( B_0 \) with \( \varphi(\sigma) \in \text{Range}(\sigma^*) \). \( \square \)

Fix an injection \( \varphi : G \to B_0 \) with \( \varphi(\sigma) \in \text{Range}(\sigma^*) \). For \( B \in B_0 \), let \( \delta_B = \text{dom}(B) \). We define an increasing sequence \( \langle \delta^n_B \mid n < \omega \rangle \) with limit \( \delta_B \) as follows. If \( B \notin \text{Range}(\varphi) \), then \( \langle \delta^n_B \mid n < \omega \rangle \) is an arbitrary increasing sequence with limit \( \delta_B \) and \( \text{cf}(\delta^n_B) \neq \omega_1 \). If \( B \in \text{Range}(\varphi) \), there is a unique \( \sigma \in G \) with \( \varphi(\sigma) = B \). Take \( f \in 2^\omega \) with \( \sigma^*(f) = B \). Then take an increasing sequence \( \langle \delta^n_B \mid n < \omega \rangle \) with limit \( \delta_B \) such that \( \text{cf}(\delta^n_B) \neq \omega_1 \) and for each \( n < \omega \) there is \( m < \omega \) with
\[ B \upharpoonright \delta_n^B < s < B \upharpoonright \delta_{n+1}^B, \text{ where } s \text{ is a maximal element of } T \text{ with } s < \sigma(f \upharpoonright m + 1), \sigma(f \upharpoonright m - 1 - f(m))). \]

Now we are ready to define \( X_0 \) and \( X_1 \). For \( B \in \mathcal{B}_0 \) and \( m < \omega \), let \( W_0(B, m) = \{B\} \cup \bigcup\{ [B \upharpoonright \delta_n^B] \mid [B \upharpoonright \delta_{n+1}^B] \mid n : \text{even}, n > m \} \) and \( W_1(B, m) = \{B\} \cup \bigcup\{ [B \upharpoonright \delta_n^B] \mid [B \upharpoonright \delta_{n+1}^B] \mid n : \text{odd}, n > m \} \). The topology of \( X_0 \) is generated by the family
\[
\{[t] \mid t \in T \setminus T^*\} \cup \{[s] \mid [s]^+ \setminus t \in T^*, s \notin T^*, s < t\} \cup \{W_0(B, m) \mid B \in \mathcal{B}_0, m < \omega\}
\]
as an open base. The topology of \( X_1 \) is generated by the family
\[
\{[t] \mid t \in T \setminus T^*\} \cup \{[s] \mid [s]^+ \setminus t \in T^*, s \notin T^*, s < t\} \cup \{W_1(B, m) \mid B \in \mathcal{B}_0, m < \omega\}
\]
as an open base. It is not hard to check that \( X_0 \) and \( X_1 \) are zero-dimensional \( T_1 \) spaces finer than \( X \). We have to check that \( X_0 \) and \( X_1 \) satisfy the assumptions in Proposition 2.1.

For \( B \in \mathcal{B}_0 \), the family \( \{W_0(B, m) \mid m < \omega\} \) forms a local base for \( B \) in \( X_0 \), and \( \{W_1(B, m) \mid m < \omega\} \) forms a local base for \( B \) in \( X_1 \). Moreover \( W_0(B, 0) \cap W_1(B, 0) = \{B\} \).

For \( B \in \mathcal{B}_1 \), take an increasing continuous sequence \( \langle \delta_\alpha \mid \alpha < \omega_1 \rangle \) with limit \( \text{dom}(B) \) and \( \text{cf}(\delta_\alpha) \neq \omega_1 \). Then \( \{[B \upharpoonright \delta_\alpha] \mid \alpha < \omega_1\} \) is a continuously decreasing sequence of clopen sets in \( X \) with \( \bigcap_{\alpha < \omega_1}[B \upharpoonright \delta_\alpha] = \{B\} \). Similarly, for \( t \in T^* \), take an increasing continuous sequence \( \langle \delta_\alpha \mid \alpha < \omega_1 \rangle \) with limit \( \text{dom}(t) \) and \( \text{cf}(\delta_\alpha) \neq \omega_1 \). Then the sequence \( \{[t \upharpoonright \delta_\alpha] \mid [t]^+ \setminus \alpha < \omega_1\} \) is a required one.

Finally we have to check that \( X_0 \) and \( X_1 \) are Lindelöf.

**Claim 3.7.** \( X_0 \) and \( X_1 \) are Lindelöf.

**Proof.** We only show that \( X_0 \) is Lindelöf. One can check that \( X_1 \) is also Lindelöf by the same way.

Let \( \mathcal{U} \) be an open cover of \( X_0 \). As before, let \( T_{\mathcal{U}} \) be the set of all \( t \in T \) such that there is no countable \( V \subseteq \mathcal{U} \) with \( [t] \subseteq V \). It is enough to see that \( T_{\mathcal{U}} = \emptyset \). Suppose to the contrary that \( T_{\mathcal{U}} \neq \emptyset \). We can see that \( T_{\mathcal{U}} \) has no maximal element. Next we check that \( T_{\mathcal{U}} \) is branching. If not, then we can take a chain \( \langle t_\alpha \mid \alpha < \gamma \rangle \) in \( T_{\mathcal{U}} \). By the same argument as before, we know that for every \( \alpha < \gamma \) there is a countable \( V \subseteq \mathcal{U} \) with \( [t_\alpha] \subseteq V \). Let \( t_\gamma = \bigcup_{\alpha < \gamma} t_\alpha \). If \( t_\alpha \in \mathcal{B}_1 \) or \( t_\alpha \in \mathcal{B}_2 \), then one can derive a contradiction as before. If \( t_\gamma \in \mathcal{B}_0 \), take an increasing sequence \( \langle \alpha_n \mid n < \omega \rangle \) with limit \( \gamma \). For \( n < \omega \), take a countable \( V_n \subseteq \mathcal{U} \) with \( [t_{\alpha_n}] \subseteq V_n \). Pick an open set \( O \in \mathcal{U} \) with \( t_\gamma \in O \). Then \( [t_0] = \bigcup_{n<\omega}[t_0] \setminus [t_{\alpha_n}] \cup \{t_\gamma\} \subseteq O \cup \bigcup_{n<\omega} V_n \), this is a contradiction.

Now we have that \( T_{\mathcal{U}} \) has no maximal element and is branching. Hence there is an embedding \( \sigma : 2^{<\omega} \rightarrow T_{\mathcal{U}} \). By Claim 3.4 there is a good embedding \( \tau \) with \( \text{Range}(\tau) \subseteq \text{Range}(\sigma) \). Consider \( B = \varphi(\tau) \in \mathcal{B}_0 \). Take \( f \in 2^\omega \) with \( \tau^*(f) = B \). Fix an open set \( O \in \mathcal{U} \) with \( B \in O \). Then there is \( m < \omega \) such that \( W_0(B, m) \subseteq O \), so there is an odd number \( n^* \) with \( [B \upharpoonright \delta_{n^*}^B] \setminus [B \upharpoonright \delta_{n^*+1}^B] \subseteq O \). By the choice of
δ_{n^r}, there is some l < ω with B \upharpoonright δ_{n^r} < s < B \upharpoonright δ_{n^r+1}, where s is a maximal element of T with s < τ(f \upharpoonright l + 1), τ(f \upharpoonright l \langle 1 - f(l) \rangle). This means that [τ(f \upharpoonright l \langle 1 - f(l) \rangle)] \subseteq [B \upharpoonright δ_{n^r}] \setminus [B \upharpoonright δ_{n^r+1}], hence [τ(f \upharpoonright l \langle 1 - f(l) \rangle)] \subseteq O. This contradicts to τ(f \upharpoonright l \langle 1 - f(l) \rangle) \in T_U. \quad \square [Claim]

**Remark 3.8.** As in the proof of Claim 3.2, we used the assumption that “Cantor tree 2^{ω_1} cannot be embedded into T” in the proof of this claim. However, unlike Claim 3.2, the author does not know whether it can be weakened to that “the tree 2^{<ω_1} cannot be embedded into T”.

**References**

[1] I. Gorelic, *On powers of Lindelöf spaces*. Comment. Math. Univ. Carol. Vol. 35, No. 2 (1994), 383–401.

[2] T. Jech, *Set theory*. The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[3] I. Juhász, W. Weiss, *On a problem of Sikorski*. Fund. Math. 100 (1978), 223–227.

[4] B. König, *Local coherence*. Ann. Pure Appl. Logic 124, No. 1-3 (2003), 107–139.

[5] S. Shelah, *On some problems in general topology*. Contemp. Math. 192 (1996), 91–101.

[6] S. Todorcević, *Partitioning pairs of countable ordinals*. Acta Math. 159 (1987), 261–294.

[7] T. Usuba, *Large regular Lindelöf spaces with points G_{δ},* Fund. Math. 237 (2017), 249–260.

[8] T. Usuba, *G_{δ}-topology and compact cardinals*. To appear in Fundamenta Mathematicae.

(T. Usuba) Faculty of Science and Engineering, Waseda University, Okubo 3-4-1, Shinjyuku, Tokyo, 169-8555 Japan

E-mail address: usuba@waseda.jp