Extended Transfer-Function and Pole-Zero Cancellation in Linear Time-Varying Systems

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Abstract: Pole-zero cancellation is a well-known and important concept in linear time-invariant systems. In contrast, transfer functions as well as poles and zeros are not defined for linear time-varying systems. In this paper, we attempt to generalize the concept of pole-zero cancellation to linear time-varying systems. We first introduce the new concept of extended transfer-function for linear time-varying systems in the time domain instead of the frequency domain. We then propose the computational procedure of pole-zero cancellation to linear time-varying systems. We finally discuss the meaning of the proposed computational procedure regardless of the lack of poles and zeros in linear time-varying systems. The proposed concept and computational procedure are illustrated by a numerical example.

Key Words: pole-zero cancellation, linear time-varying system, Kalman canonical decomposition, minimal realization, companion form.

1. Introduction

In classical control theory, a linear time-invariant (LTI) system is transformed to a transfer function by the Laplace transformation. When a transfer function has common poles and zeros, common poles and zeros are canceled out, and then, the degree of the transfer function decreases. This computational procedure is called the pole-zero cancellation [1]. The concept of pole-zero cancellation appears in the context of LTI systems [2]–[11]. For example, a method for selecting the weights in an $H^\infty$-mixed sensitivity design problem which prevents undesirable pole-zero cancellation is discussed [3]. Over-parametrization may arise in a model reference adaptive control when the transfer function of the modeled part of the plant has common zeros and poles [4].

At the beginning age of modern control theory in the 1960s, various system analysis and control problems are formulated using linear time-varying (LTV) systems rather than LTI systems. For example, the concepts of controllability and observability are firstly defined for LTV systems [12]. When we compare the condition of controllability for LTV systems with the well-known condition of controllability for LTI systems, we can understand the essence in system control theory. From this point of view, it is important to re-interpret the several concepts from LTI systems to LTV systems.

In this paper, we attempt to generalize the concept of pole-zero cancellation to LTV systems. As well-known, the Laplace transformation cannot be applied to LTV systems. There is no concepts of poles and zeros as well as transfer functions in LTV systems. So, this generalization is not a trivial problem. We would like to consider how and in what sense the computational procedure of pole-zero cancellation is generalized to LTV systems.

To formulate the pole-zero cancellation in LTV systems, we need to specify the certain standard form in LTV systems. We firstly introduce the new concept of extended transfer-functions for LTV systems. All properties of transfer functions cannot be extended to LTV systems, of course. We focus on the correspondence between transfer functions and certain higher order differential equations, which can be transformed to the companion form from for LTI systems. Our idea is to find a class of higher order differential equations for LTV systems so that the equations can be transformed to the companion form. We then propose the computational procedure of pole-zero cancellation to LTV systems. The proposed computational procedure consists of several steps. In particular, we utilize the minimal realization by using the Kalman canonical decomposition. We should notice that the Kalman controllable-observable canonical decomposition is not suitable for our purpose because the order of minimal realization can be time-dependent [12]–[16]. So, we utilize the Kalman influenceable-visible canonical decomposition so that the order of minimal realization is time-independent [13]. We should also notice that we need to connect the extended transfer-functions and the Kalman canonical decomposition. So, we apply the computational procedure in [17] to convert the minimal realization to companion forms.

Finally, we discuss the meaning of the proposed computational procedure regardless of the lack of poles and zeros in linear time-varying systems. It is shown that input-output relations are preserved in the proposed computational procedure. Because this property is valid in LTI systems, it would be appropriate to say that the proposed computational procedure is a reasonable generalization to LTV systems. The proposed concept and computational procedure are illustrated in detail by a numerical example.

A preliminary version of this paper has been presented in our conference proceeding [18], where the input coefficient of the extended transfer-function is supposed to have no zeros. In this paper, we have removed this assumption by reconsidering the properties of analytic functions.

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2. Extended Transfer-Function for LTV Systems

In this section, we introduce the new concept of extended transfer-function for LTV systems.

Consider a high-order differential equation with constant coefficients
\[
y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = b_{n-1}u^{(n-1)}(t) + \cdots + b_0 u(t),
\]
(1)
where \(y^{(n)}(t)\) denotes the \(n\)-th derivative of \(y(t)\). By taking the Laplace transform of both sides of Eq. (1), we have a transfer function from \(u(s)\) to \(y(s)\) as follows:
\[
G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.
\]
(2)
The transfer function in Eq. (2) can be transformed to the companion form
\[
\dot{\zeta} = A\zeta + Bu,
\]
y = Cζ,
(3)
(4)
where the coefficient matrices \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times 1}\), and \(C \in \mathbb{R}^{1 \times n}\) take on the form
\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
-b_{n-1} & -b_{n-2} & \cdots & \cdots & -a_0
\end{bmatrix},
\]
(5)
\[
B = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix},
\]
(6)
\[
C = \begin{bmatrix}
b_0 & b_1 & \cdots & b_{n-1}
\end{bmatrix}^T.
\]
(7)
where \(X^T\) denotes the transpose of \(X\). By differentiating from the first to \(n\)-th order derivatives of \(y\) in Eq. (4), we can recover the form of Eq. (1).

Consider a high-order differential equation with time-varying coefficients
\[
y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_0(t) y(t) = b_{n-1}(t)u^{(n-1)}(t) + \cdots + b_0(t) u(t),
\]
(8)
This equation cannot be directly transformed to a companion form. The transformation from Eq. (8) to the state space representation is not trivial indeed. Hence, this equation cannot be recognized as an extension of the transfer function.

Instead of Eq. (8), we consider a pair of differential equation and a measurement equation of the form
\[
\begin{align*}
x^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)x^{(k)}(t) &= b(t)u(t), \\
y(t) &= \sum_{k=0}^{n-1} c_k(t)x^{(k)}(t),
\end{align*}
\]
(9)
which is called the extended transfer-function for LTV system. Equation (9) is a differential equation but not a function of course, and we intend to express the meaning of an extension of the concept of transfer function by linking to one word of “transfer-function.” It would be reasonable to regard Eq. (9) as the extension of the transfer function because Eq. (9) can be transformed to the companion form as shown below. The order of the extended transfer-function in Eq. (9) is defined by the order, \(n\), of the differential equation. By setting a state \(\zeta\) as
\[
\zeta = \begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\vdots \\
\zeta_n
\end{bmatrix} = \begin{bmatrix}
x^{(0)}(t) \\
x^{(1)}(t) \\
\vdots \\
x^{(n-1)}(t)
\end{bmatrix},
\]
(10)
Equation (9) is transformed to a companion form
\[
\dot{\zeta} = A(t)\zeta + B(t)u,
\]
y = C(t)\zeta,
(11)
(12)
where the coefficient matrices \(A(t) \in \mathbb{R}^{n \times n}\), \(B(t) \in \mathbb{R}^{n \times 1}\), and \(C(t) \in \mathbb{R}^{1 \times n}\) take on the form
\[
A(t) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
-a_{n-1}(t) & -a_{n-2}(t) & \cdots & \cdots & -a_0(t)
\end{bmatrix},
\]
(13)
\[
B(t) = \begin{bmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\vdots & 1 & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & \cdots & \cdots \\
-b_{n-1}(t) & -b_{n-2}(t) & \cdots & \cdots & -a_0(t)
\end{bmatrix},
\]
(14)
\[
C(t) = \begin{bmatrix}
c_0(t) & c_1(t) & \cdots & \cdots & c_{n-1}(t)
\end{bmatrix}^T.
\]
(15)
As shown above, Eq. (9) has been transformed to a companion form in a similar way to the transfer function. We would like to interpret the higher-order differential equation in Eq. (9) as the equivalent representation of the transfer function in Eq. (2).

3. Computational Procedure

In this section, we propose a computational procedure by extending the pole-zero cancellation for LTI systems to LTV systems. A lower-order extended transfer-function can be obtained from a given higher-order extended transfer-function by applying the following procedures:

1. Higher-order extended transfer-function
2. Higher-order companion form
3. Kalman canonical decomposition
4. Minimal realization
5. Lower-order companion form
6. Lower-order extended transfer-function

We will explain each procedure in detail below.

3.1 Higher-Order Extended Transfer-Function

Consider the \(n\)-th order extended transfer-function, which is defined by a pair of \(n\)-th order differential equation and measurement equation of the form of Eq. (9).

For technical simplicity, we suppose that the coefficients \(a_k(t), k = 0, \ldots, n-1\), and \(b(t)\) are analytic functions for all \(t\). We note that we have imposed an additional assumption that \(b(t)\) has no zeros in our conference proceeding [18]. In this paper, we have removed this additional assumption by re-considering the properties of analytic functions.

3.2 Higher-Order Companion Form

As shown in the previous section, Eq. (9) can be transformed to the companion form in Eqs. (11) and (12). We note that \(A(t)\) and \(B(t)\) are analytic functions for all \(t\).
3.3 Kalman Canonical Decomposition

In this subsection, we transform the higher-order companion form in Eqs. (11) and (12) to the form of the Kalman canonical decomposition.

We first prove that the higher-order companion form in Eqs. (11) and (12) is controllable. Controllability of the system can be evaluated by examining the dimension of the controllable subspace \( C(t) \) [13]. We note that \( C(t) \) is mathematically well-defined but is hard to be computed because of the difficulties in integrations such as the computations of the state transition matrix and the controllability Gramian. Instead, we study the controllability matrix \( Q(t) \), so that controllability can be evaluated by applying the inclusion relation \( C_d(t) \subseteq C(t) \), where \( C_d(t) \) is the differentially controllable subspace [19],[20]. Let \( q_i(t) \) be the \( i \)-th column of \( Q(t) \) in \( \mathbb{R}^{n \times n} \) of the system in Eqs. (11) and (12). As shown in [17], \( q_i(t) \) is generally written as \( q_i(t) = B(t) \) and \( q_{i+1}(t) = A(t)q_i(t) - \dot{q}_i(t), i = 1, \ldots, n - 1 \). By direct computation, a more specific structure of \( q_i(t) \) can be obtained as follows:

**Proposition 1** Consider the controllability matrix \( Q(t) \) in \( \mathbb{R}^{n \times n} \) of the system in Eqs. (11) and (12). The \( i \)-th column vector \( q_i(t) \) of \( Q(t) \) takes the form

\[
q_i(t) = [0 \cdots 0 b(t) f_{i,n-2}(t) \cdots f_{i,n}(t)]^T, 
\]

where \( f_{i,k}, i = 1, \ldots, n, k = 1, \ldots, n \), denotes the \( k \)-th row of \( q_i(t) \). In other words, zeros are repeated from the first to \((n-i)\)-th rows of \( q_i(t) \), the \((n-i+1)\)-th row of \( q_i(t) \) is \( b(t) \), and \((n-i+2)\)-th to \( n \)-th rows of \( q_i(t) \) are not identical zero.

**Proof** Let \( i = 1 \). Then \( q_1(t) = B(t) \). The \( n \)-th row is \( b(t) \) and other rows are 0. The proposition is obviously true for \( i = 1 \).

Assume that the proposition holds for \( i = k \); that is, \( q_i(t) \) takes on the following form:

\[
q_i(t) = [0 \cdots 0 b(t) f_{i,n-k+2}(t) \cdots f_{i,n}(t)]^T. 
\]

Let \( i = k + 1 \). Let \( a_j(t) \) denotes the \( k \)-th column of \( A(t) \)

\[
A(t) = [a_1(t) a_2(t) \cdots a_n(t)], 
\]

then \( q_{k+1}(t) \) is

\[
q_{k+1}(t) = \alpha_{n-k+1}(t) + f_{k,n-k+2}(t)a_{n-k+2}(t) + \cdots + f_{k,n}(t)a_{n}(t) - \dot{q}_i(t). 
\]

It follows from Eq. (13) that the \( (n-k) \)-th row of \( a_{n-k+1}(t) \) is 1 and that the first to \((n-k)\)-th rows of \( a_j(t) \), \( j = n-k+1, \ldots, n \), are 0. Then, we have

\[
q_{k+1}(t) = [0 \cdots 0 b(t) f_{k+1,n-k+1}(t) \cdots f_{k+1,n}(t)]^T. 
\]

Hence, the proposition holds for \( i = k + 1 \).

By Proposition 1, \( Q(t) \) takes on the form

\[
Q(t) = \begin{bmatrix}
0 & \cdots & 0 & b(t) \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & f_{n-2}(t) \\
0 & b(t) & \cdots & f_{n-1}(t) \\
b(t) & f_{n,1}(t) & \cdots & f_{n,n}(t)
\end{bmatrix}. 
\]

By using Eq. (20), we investigate the controllable subspace \( C(t) \) (see [13]) of the higher-order companion form in Eqs. (11) and (12) to examine its controllability.

**Proposition 2** Consider the controllable subspace \( C(t) \) of the system in Eqs. (11) and (12), Then,

\[
C(t) = \mathbb{R}^n. 
\]

Namely, the system is controllable at each \( t \).

**Proof** We first investigate the relation between \( C(t) \) and \( \text{Im } Q(t) \). Consider the differentially controllable subspace \( C_d(t) \) (see [19],[20]). By the definitions of \( C(t) \) and \( C_d(t) \), the following inclusive relations hold:

\[
C_d(t) \subseteq C(t). 
\]

When the coefficient matrices \( A(t) \) and \( B(t) \) are analytic functions, the following relations are satisfied (see [19],[20]):

\[
\text{Im } Q(t) \subseteq C_d(t). 
\]

By Eqs. (23) and (24), we have

\[
\text{Im } Q(t) \subseteq C(t). 
\]

By combining Eqs. (22) and (25), the following relation holds:

\[
\text{Im } Q(t) \subseteq C(t) \subseteq \mathbb{R}^n. 
\]

Then, we have

\[
\dim \text{Im } Q(t) \leq \dim C(t) \leq \dim \mathbb{R}^n = n. 
\]

We then study the dimension of \( \text{Im } Q(t) \). As shown in Eq. (20), the rank of \( Q(t) \) is less than \( n \) at zeros of \( b(t) \) but \( n \) elsewhere. Then, we have

\[
\dim \text{Im } Q(t) \begin{cases}
< n, & t \not\in \mathcal{X}, \\
= n, & t \in \mathcal{X},
\end{cases}
\]

where \( \mathcal{X} \) is the collection of all zeros of \( b(t) \) and is defined as

\[
\mathcal{X} = \{ t \in \mathbb{R} : b(t) = 0 \}. 
\]

We note that the all \( t \in \mathcal{X} \) are isolated by the properties of analytic functions.

Consider any \( t_0 \in \mathcal{X} \). We finally evaluate \( \dim C(t) \) on a neighborhood of \( t_0 \). Select \( s > t_0 \) in the neighborhood of \( t_0 \), i.e., \( s \) satisfies \( s > t_0 \) and is sufficiently close to \( t_0 \). Then, \( s \) is not the zero of \( b(t) \), i.e., \( s \not\in \mathcal{X} \). By combining Eqs. (27) and (28), we have

\[
\dim C(s) = n. 
\]
Here, we suppose that $\dim C(t_0) < n$. Then, $\dim C(s) < n$ because $\dim C(t)$ is monotonically nonincreasing with increasing $t$ (see Proposition 3.5(i) in [13]). This inequality contradicts Eq. (30). Hence, $\dim C(t_0)$ is also $n$; therefore, we have $C(t) = \mathbb{R}^n$ for all $t$. \hfill $\Box$

We have shown that the dimension of $C(t)$ is $n$ in Proposition 2, and therefore, the higher-order companion form in Eqs. (11) and (12) is shown to be controllable. In contrast, the dimension of the observable subspace $O(t)$ is not necessarily $n$. If the dimension of the observable subspace $O(t)$ is $n$, we cannot reduce the order of extended transfer-function. Therefore, we consider the nontrivial case where the dimension of the observable subspace $O(t)$ is not $n$.

We note that the dimension of the observable subspace $O(t)$ is time-dependent [13]. The sizes of the submatrices in the Kalman controllable-observable canonical decomposition also depends on time. The order of the lower-order extended transfer-function would be time-varying.

In this paper, we consider the Kalman influenceable-visible canonical decomposition instead of the Kalman controllable-observable canonical decomposition. By Eq. (21), the influenceable subspace $I(t)$ and $\mathbb{R}^n$ coincide [13]. However, the visible subspace $V(t)$ and $\mathbb{R}^n$ are not always coincident [13]. If $V(t)$ and $\mathbb{R}^n$ coincide, the order of the system in Eqs. (11) and (12) cannot be reduced. Therefore, we suppose that $V(t)$ and $\mathbb{R}^n$ are not coincident.

By computing the subspaces $I(t) \cap V(t)$ and $I(t) \cap V(t)$, the system can be transformed to the Kalman influenceable-visible canonical decomposition. Define $n_1$ and $n_2$ as follows:

$$ \dim[I(t) \cap V(t)] = \text{const.} =: n_1, $$

$$ \dim[I(t) \cap V(t)] = \text{const.} =: n_2. $$

By applying the method of [13], we have the system of the Kalman influenceable-visible canonical decomposition

$$ \dot{\xi} = F(t) \xi + G(t)u, $$

$$ y = H(t) \xi, $$

where the coefficient matrices $F(t)$, $G(t)$, and $H(t)$ take on the form

$$ F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}, $$

$$ G(t) = \begin{bmatrix} G_1(t) \\ G_2(t) \end{bmatrix}, $$

$$ H(t) = \begin{bmatrix} 0 \\ H_2(t) \end{bmatrix}, $$

where $F_{11}(t) \in \mathbb{R}^{n_1 \times n_1}$ and $F_{22}(t) \in \mathbb{R}^{n_2 \times n_2}$.

### 3.4 Minimal Realization

By omitting the invisible mode of the system in Eqs. (33) and (34), we have the minimal realization

$$ \dot{\xi}_2 = F_{22}(t) \dot{\xi}_2 + G_2(t)u, $$

$$ y = H_2(t) \xi_2, $$

where $\xi_2 \in \mathbb{R}^{n_2}$.

### 3.5 Lower-Order Companion Form

Here we suppose that the controllability matrix of the system in Eqs. (38) and (39) is invertible. Then, we can apply the procedure in [17] to obtain the lower-order companion form as follows:

$$ \ddot{\eta} = \hat{A}(t) \eta + \hat{B}(t)u, $$

$$ y = \hat{C}(t) \eta, $$

where the coefficient matrices $\hat{A}(t)$, $\hat{B}(t)$, and $\hat{C}(t)$ take on the form

$$ \hat{A}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\hat{a}_{n-1}(t) & -\hat{a}_{n-2}(t) & \cdots & \cdots & -\hat{a}_0(t) \end{bmatrix}, $$

$$ \hat{B}(t) = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}^T, $$

$$ \hat{C}(t) = \begin{bmatrix} \hat{c}_d(t) \\ \hat{c}_1(t) \\ \cdots \\ \hat{c}_{n-1}(t) \end{bmatrix}. $$

### 3.6 Lower-Order Extended Transfer-Function

By direct computation, the lower-order companion form in Eqs. (40) and (41) can be transformed to the lower-order extended transfer-function of the form:

$$ x^{(n)}(t) + \sum_{k=0}^{n-1} \hat{a}_k(t)x^{(k)}(t) = u(t), $$

$$ y(t) = \sum_{k=0}^{n-1} \hat{c}_k(t)x^{(k)}(t). $$

We should notice that the order $n$ of Eq. (9) is reduced to the order $n_2$ of Eq. (45).

We also notice that, if the dimension of $V(t)$ is $n$, the order $n$ of Eq. (9) cannot be reduced. In this way, we have succeeded to generalize the pole-zero cancellation for LTI systems to LTV systems in terms of the reduction of order. The state which appears in the output is characterized by $I(t) \cap V(t)$. The difference in the states which does not appear in the output is characterized by $I(t) \cap V(t)^\perp$; see Proposition 4.21 (iv) in [13] for more detail about $V(t)^\perp$. Hence, it might be possible to interpret $I(t) \cap V(t)^\perp$ as the characterization of canceled pole-zero property. The input-output relation will be discussed in the next section.

### 4. Discussion

In the previous section, we have proposed the computational procedure for reducing the order of an equivalent transfer function. By comparing the orders of equivalent transfer functions, the proposed computational procedure can be recognized as the generalization of the pole-zero cancellation for LTI systems to LTV systems.

Needless to say, poles and zeros are not defined for LTV systems. We discuss the meaning of the proposed computational procedure regardless of the concepts of poles and zeros in this section.

To this end, we compare the input-output relations for each system description. The output of Eq. (9) and that of the system in Eqs. (11) and (12) is given by
The output of the system in Eqs. (33) and (34) is given by
\[
y(t) = H(t)\Theta(t, 0)\xi(0) + \int_0^t H(t)\Theta(t, \tau)G(\tau)u(\tau)d\tau. \quad (46)
\]
The output of the system in Eqs. (33) and (34) is given by
\[
y(t) = H(t)\Theta(t, 0)\xi(0) + \int_0^t H(t)\Theta(t, \tau)G(\tau)u(\tau)d\tau. \quad (47)
\]
where \( \Theta \) is the state transition matrix of the system in Eqs. (33) and (34). The matrices \( H(t), \Theta(t, \tau), \) and \( G(t) \) take on the form
\[
H(t) = \begin{bmatrix} 0 & H_2(t) \end{bmatrix}, \quad (48)
\]
\[
\Theta(t, \tau) = \begin{bmatrix} \Theta_{11}(t, \tau) & \Theta_{12}(t, \tau) \\ 0 & \Theta_{22}(t, \tau) \end{bmatrix}, \quad (49)
\]
\[
G(t) = \begin{bmatrix} G_1(t) \\ G_2(t) \end{bmatrix}, \quad (50)
\]
where \( \Theta_{11}(t, \tau) \in \mathbb{R}^{n_1 \times n_1} \) and \( \Theta_{22}(t, \tau) \in \mathbb{R}^{n_2 \times n_2} \). Then, Eq. (47) can be rewritten as follows:
\[
y(t) = H(t)\Theta(t, 0)\xi(0) + \int_0^t H(t)\Theta_{22}(t, \tau)G_2(\tau)u(\tau)d\tau. \quad (51)
\]
The output of the system in Eqs. (38) and (39) is given by
\[
y(t) = H_2(t)\Theta_{22}(t, 0)\xi_2(0) + \int_0^t H_2(t)\Theta_{22}(t, \tau)G_2(\tau)u(\tau)d\tau.\quad (52)
\]
We note the state transition matrix of the system in Eqs. (38) and (39) is equal to the submatrix \( \Theta_{22}(t, \tau) \) in Eq. (49). The output of Eq. (45) and that of the system in Eqs. (40) and (41) is given by
\[
y(t) = \hat{C}(t)\hat{\phi}(t, 0)\eta(0) + \int_0^t \hat{C}(t)\hat{\phi}(t, \tau)\hat{B}(\tau)u(\tau)d\tau. \quad (53)
\]
where \( \hat{\phi}(t, \tau) \) is the state transition matrix of the system in Eqs. (40) and (41).

The system in Eqs. (11) and (12) and the system in Eqs. (33) and (34) are related by the coordinate transformation. The system in Eqs. (38) and (39) and the system in Eqs. (40) and (41) are also related by the coordinate transformation. Therefore, the outputs in Eqs. (46) and (51) are equivalent. The outputs in Eqs. (52) and (53) are also equivalent.

In contrast, the system in Eqs. (33) and (34) and the system in Eqs. (38) and (39) are not related by the coordinate transformation but by the projection. Therefore, the outputs in Eqs. (51) and (52) are not equivalent. They are equivalent if the initial states \( \xi(0) \) and \( \eta(0) \) are 0.

In summary, the input-output relation between the higher-order extended transfer-function in Eqs. (11) and (12) and the lower-order extended transfer-function in Eqs. (38) and (39) are equivalent when the initial states are zero. This statement is valid for pole-zero cancellation in LTI systems because transfer functions are obtained by computing Laplace transform with zero initial values. Therefore, we have succeeded to generalize the concept of pole-zero cancellation in terms of the equivalence of the input-output relation.

The pole-zero cancellation in this paper can also be recognized as a special case of model reduction. In a typical setting, a model reduction is formulated by an approximation problem that causes an approximation error. In contrast, procedures from 3.2 to 3.5 do not cause an approximation error. When the extended transfer-function and the companion form are identified from 3.1 to 3.2 and from 3.5 to 3.6, the whole procedure in Section 3 can be recognized as a model reduction which does not cause an approximation error in terms of input-output relations with zero initial conditions.

5. Example

In this section, we demonstrate that the proposed computational procedure can be applicable to LTV systems indeed. Consider the second-order extended transfer-function
\[
\begin{align*}
x(2)(t) - x(0)(t) &= \sin(t)u(t), \\
y(t) &= -\sin(t)x(1)(t) + \sin(t)x(t).
\end{align*} \quad (54)
\]

This can be transformed to the companion form
\[
\begin{align*}
x(t) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\
y(t) &= -\sin(t)x(t).
\end{align*} \quad (55)
\]

Although the input coefficient, \( \sin(t) \), of the extended transfer-function has zeros, the system is controllable for all \( t \) as shown in Proposition 2. According to the definition of [13], \( \mathcal{V}(t) \) and \( \mathcal{V}(t)^+ \) are given by
\[
\mathcal{V}(t) = \text{Im} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad (57)
\]
\[
\mathcal{V}(t)^+ = \text{Im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (58)
\]

By computing intersection subspaces, we have
\[
\dim(I(t) \cap \mathcal{V}(t)^+) = 1, \quad (59)
\]
\[
\dim(I(t) \cap \mathcal{V}(t)) = 1. \quad (60)
\]

With \( Z(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \) and \( \xi = Z(t)\zeta \), the system in Eqs. (55) and (56) can be transformed to the Kalman canonical decomposition
\[
\begin{align*}
\dot{\xi} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \xi + \begin{bmatrix} \sin(t) \\ -\sin(t) \end{bmatrix} u, \\
y &= \begin{bmatrix} 0 & -2 \sin(t) \end{bmatrix} \xi.
\end{align*} \quad (61)
\]

According to Section 3.4, the minimal realization is given by
\[
\begin{align*}
\dot{\xi}_2 &= -\dot{\xi}_2 - \frac{\sin(t)}{2}u, \\
y &= -2\sin(t)\dot{\xi}_2. \quad (64)
\end{align*}
\]

Since the order of the minimal realization is 1, we need not transform it to a companion form. By replacing \( \xi \) with \( x(t) \), we have
\[
\begin{align*}
x(1)(t) + x(0)(t) &= \frac{\sin(t)}{2}u(t), \\
y(t) &= -\sin(t)x(0)(t).
\end{align*} \quad (65)
\]

Hence, we have succeeded to reduce the second order extended transfer-function in Eq. (54) to first-order extended transfer-function in Eq. (65).
We also check the input-output relations of Eq. (54) and Eq. (65). The output of Eq. (54) is given by

\[ y(t) = e^{-t} \sin(t)x^{(1)}(0) - e^{-t} \sin(t)x^{(0)}(0) + \int_{0}^{t} \sin^{2}(\tau)e^{(r-\gamma)t}u(\tau)d\tau, \tag{66} \]

and that of Eq. (65) is given by

\[ y(t) = -2e^{-t} \sin(t)x^{(0)}(0) + \int_{0}^{t} \sin^{2}(\tau)e^{(r-\gamma)t}u(\tau)d\tau. \tag{67} \]

If the initial values are zero, \( x^{(1)}(0) = x^{(0)}(0) = 0 \), the outputs of Eq. (66) and Eq. (67) are equivalent.

6. Conclusion

In this paper, we have attempted to generalize the concept of pole-zero cancellation to LTV systems. We have introduced the concept of extended transfer-function, which is a counterpart of the transfer function in LTI systems. Based on the representation of extended transfer-functions, we have proposed the computational procedure of pole-zero cancellation to LTV systems. The proposed computational procedure consists of several steps. A higher-order extended transfer-function is converted to a companion form. We have shown that the obtained companion form is controllable.

The companion form is, therefore, transformed to the Kalman canonical decomposition when the companion form is not visible. The minimal realization is obtained from the Kalman canonical decomposition, and then, is converted to the lower-order extended transfer-function. The computational procedure can be recognized as the generalization of pole-zero cancellation because the order of extended transfer-function is decreased. In addition, we have shown that the input-output relation of extended transfer-function is preserved via the computational procedure. Therefore, even though the concepts of poles and zeros do not appear in LTV systems, it would be appropriate to say that the proposed computational procedure is a reasonable generalization of pole-zero cancellation. Further research on system analysis and control would be challenging future work of this paper.

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