NONSTANDARD INTUITIONISTIC INTERPRETATIONS

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ABSTRACT. We present a notion of realizability and a functional interpretation in the context of intuitionistic logic, both incorporating nonstandard principles. The functional interpretation that we present corresponds to the intuitionistic counterpart of an interpretation given recently by Ferreira and Gaspar. It has also some similarities with an interpretation given by B. van den Berg et al. but replacing finiteness conditions by majorizability conditions. Nonstandard methods are often regarded as nonconstructive. Our interpretations intend to seek for constructive aspects in nonstandard methods in the spirit of recent papers on extensions of Peano and Heyting arithmetic.

1. INTRODUCTION

Nonstandard methods are often regarded as nonconstructive. Nevertheless, in the past few years there has been a growing interest in trying to make explicit constructive aspects of nonstandard methods with the use of functional interpretations. Particularly interesting for us are [2] and [7] (building on previous work by Palmgren [14, 15], Moerdijk [13], Avigad and Helzner [1] and recent papers by Van den Berg and Sanders [3, 18]). In that spirit we present a notion of realizability and a functional interpretation in the context of intuitionistic logic, both incorporating nonstandard principles. We also prove soundness and characterization theorems for both the bounded nonstandard realizability and the intuitionistic nonstandard functional interpretation.

Realizability is a method created by Kleene [11], which makes explicit the constructive content of arithmetical sentences by providing witnesses to existential quantifiers and disjunctions. The notion of realizability presented in this paper is based on the bounded modified realizability [8] and includes nonstandard objects and principles. As the name suggests, the bounded modified realizability is a modification of realizability which relies on intensional majorizability and provides (upper) bounds instead of precise witnesses. Our notion of realizability has also similarities with the Herbrand realizability presented in [2, Section 4]. This is due to the fact that the Herbrand realizability provides a finite set in which there is an element that may serve as a witness. As such one may see both majorizability and the finite set of Herbrand realizability as methods that provide nonprecise witnesses.

In a recent paper, Ferreira and Gaspar [7] showed how the bounded functional interpretation [9] can be recast without intensional notions by going to a wider

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nonstandard setting. This was carried out in the classical setting. Both the
bounded modified realizability and the bounded functional interpretation rely on
the Howard/Bezem notion of strong majorizability. The functional interpretation
that we present corresponds to the intuitionistic counterpart of the interpretation
given by Ferreira and Gaspar. It has also some similarities with an interpretation
given in [2] by Van den Berg, Briseid, and Safarik but once again it replaces finite-
ness conditions by majorizability conditions. All the interpretations mentioned
which include nonstandard principles make use of the syntactic approach of E.
Nelson’s to nonstandard analysis called \textit{internal set theory} [16] [17] by extending
the language with a new unary predicate $st(x)$ to the language which is intended
to be read as “$x$ is standard” and giving three axiom schemes called Idealization,
Standardization and Transfer. As we will see in Section 5 the latter presents some
difficulties when dealing with functional interpretations.

In Section 2 we present the basic framework for our interpretations and recall
basic notions regarding majorizability and nonstandard analysis. In Section 3 we
give a our new notion of realizability, which extends the notion given in [8] to
incorporate nonstandard methods. In Section 4 we give our intuitionistic functional
interpretation.

2. Basic framework

Let $E$-$HA^\omega$ be the theory of extensional Heyting arithmetic in all finite types.
The main purpose of this section is to introduce an extension $E$-$HA^\omega_{st}$ of $E$-$HA^\omega$.
The theory $E$-$HA^\omega_{st}$ is the intuitionistic counterpart of the theory $E$-$PA^\omega_{st}$ presented
in [7]. The language of this extension extends the language of $E$-$HA^\omega$ by having
unary predicates $st_\sigma$ for each finite type $\sigma$ (the predicates for standardness). Note
that the terms of both languages remain the same.

The axioms of $E$-$HA^\omega_{st}$ are those of $E$-$HA^\omega$ together with the standardness ax-
ions and the external induction rule. We start by establishing some notations and
making some observations. The Howard/Bezem notion of strong majorizability
(introduced in [10] and [4]) is defined by induction on the finite type:

- $x \leq^*_0 y$ is $x \leq y$
- $x \leq^*_{\rho \rightarrow \sigma} y$ is $\forall v \forall u \leq^*_{\rho} (x u \leq^*_\sigma y v \land y u \leq^*_\sigma y v)$

For further details on this notion see [6] [12]. In [12] the notation $y$-$maj_\sigma x$ is
used instead of our $x \leq^*_y y$. Strong majorizability is transitive but not reflexive
in general (except for the base type 0). An element $x^\sigma$ is said to be monotone if
$x \leq^*_\sigma x$. It can be proved that if $x$ majorizes some element, then $x$ is monotone. With
the exception of types 0 and 1, it is not set-theoretically true that every element
is majorizable. However, an important theorem, called \textit{Howard’s majorizability theorem}, says that for every closed term $t^\sigma$ of the language there is a closed term
$q^\sigma$ such that $t \leq^*_\sigma q$ [10]. This result will play a central role in the interpretations
presented in this paper.

A formula is called internal if it is part of the original language of $E$-$HA^\omega$ (i.e., the
standard predicates $st$ do not occur in the formula). Otherwise it is called external.
We follow the convention of Nelson in [14] and will use lower case Greek letters to
denote internal formulas and upper case Greek letters to denote a formula which
can be internal or external. Therefore, the axioms of $E$-$HA^\omega$ are only constituted
by internal formulas. Note, also, that the equality and majorizability relations
are given by internal formulas. The universal quantifiers $\forall^st x^\sigma$, $\forall^st x^\sigma$ and $\forall^st x^\sigma$ are
abbreviations of the universal quantifier relativized to the standard elements, to the
monotone elements and, simultaneously, to the standard and monotone elements
(respectively). We use similar abbreviations for the existential quantifier. Bounded
quantifications of the form $\forall x \leq^* t (\ldots)$ are defined in the usual way and come in
three varieties as well.

We are now ready to state the axioms of $\mathbf{E-HA}^\omega_{\text{st}}$ that involve external formulas.
The standardness axioms are:

1. $x =^\sigma y \rightarrow (\text{st}^\sigma(x) \rightarrow \text{st}^\sigma(y))$;
2. $\text{st}^\sigma(y) \rightarrow (x \leq^\sigma y \rightarrow \text{st}^\sigma(x))$;
3. $\text{st}^\sigma(t)$, for each closed term $t$ of type $\sigma$;
4. $\text{st}^\sigma \rightarrow^\tau(z) \rightarrow (\text{st}^\sigma(x) \rightarrow \text{st}^\tau(zx))$;

where the types $\sigma$ and $\tau$ are arbitrary. The external induction rule is

• From $\Phi(0)$ and $\forall^\text{st} n^0(\Phi(n) \rightarrow \Phi(n + 1))$, infer $\forall^\text{st} n^0\Phi(n)$.

(External induction is formulated as a rule just as a matter of convenience. Since
there is no restriction in the formulas $\Phi$, the rule is equivalent to the corresponding
axiom scheme.)

The standardness axioms are the same as in [7]. They are also the same as
in [2] with exception of the second one which does not exist there. The second
standardness axiom has a clear meaning in type 0, namely that the nonstandard
natural numbers are an end-extension of the standard natural numbers (cf. [7, p.
12]). For variables of higher types the meaning is not so clear and we discuss one
of its consequences in the next subsection.

3. Bounded nonstandard realizability

In order to define the bounded nonstandard realizability we need the notion of
$\exists^\text{st}$-free formula. These formulas have a role similar the one of $\exists$-free formulas in
[8] which by its turn reminds the well-known notion of $\exists$-free formula, with the
difference that both $\exists^\text{st}$-free and $\exists$-free formulas also allow disjunctions.

Definition 1. A formula $\Phi$ is called $\exists^\text{st}$-free if it is built from atomic internal
formulas by means of conjunction, disjunction, implications, quantifications and
monotone universal quantifications.

Definition 2. To each formula $\Phi$ we assign formulas $\Phi^{\text{bn}}$ and $\Phi_{\text{bn}}$ so that $\Phi^{\text{bn}}$ is
of the form $\exists^\text{st} \Phi_{\text{bn}}(c)$, with $\Phi_{\text{bn}}(c)$ an $\exists^\text{st}$-free formula, according to the following
clauses:

1. $\Phi^{\text{bn}}$ and $\Phi_{\text{bn}}$ are simply $\Phi$, for internal atomic formulas $\Phi$,
2. $\text{st}(t)^{\text{bn}}$ is $\exists^\text{st} c[t \leq^* c]$.

For the remaining cases, if we have already interpretations for $\Phi$ and $\Psi$ given
(respectively) by $\exists^\text{st} \Phi_{\text{bn}}(c)$ and $\exists^\text{st} \Psi_{\text{bn}}(d)$ then we define:

3. $(\Phi \land \Psi)^{\text{bn}}$ is $\exists^\text{st} c, d [\Phi_{\text{bn}}(c) \land \Psi_{\text{bn}}(d)]$,
4. $(\Phi \lor \Psi)^{\text{bn}}$ is $\exists^\text{st} c, d [\Phi_{\text{bn}}(c) \lor \Psi_{\text{bn}}(d)]$,
5. $(\Phi \rightarrow \Psi)^{\text{bn}}$ is $\exists^\text{st} f \exists^\text{st} c [\Phi_{\text{bn}}(c) \rightarrow \Psi_{\text{bn}}(fc)]$,
6. $(\forall x \Phi(x))^{\text{bn}}$ is $\exists^\text{st} c [\forall x \Phi_{\text{bn}}(x, c)]$,
7. $(\exists x \Phi(x))^{\text{bn}}$ is $\exists^\text{st} c [\exists x \Phi_{\text{bn}}(x, c)]$,

From the interpretation given above one derives:
8. $(\forall^* x \Phi(x))^{\text{bn}}$ is $\exists^\text{st} f \forall^* c \forall x \leq^* c \Phi_{\text{bn}}(x, fc)$,
9. \((\exists x^{\ast} \Phi(x))^{bn}\) is \(\exists x^{\ast} c \exists x \leq^* c \Phi_{bn}(x, c)\)
10. \((- \Phi)^{bn}\) is \(\forall c [- \Phi_{bn}(c)]\).

**Proposition 1.** Let \(\Phi\) be an \(\exists^{\ast}\)-free formula. Then \((\Phi)^{bn}\) is \(\Phi_{bn}\) and they are both equivalent to \(\Phi\).

**Proof.** By induction on the complexity of the \(\exists^{\ast}\)-free formula \(\Phi\). \qed

The following four principles play an important role in the sequel. Indeed we show that these are the characteristic principles of our realizability notion. The proper formulation of the first three principles should be with tuples of variables. To ease readability, we formulate them with single variables.

1. **Monotone Choice** \(mAC^{\ast}\): \(\forall x^{\ast} \exists^{\ast} y \Phi(x, y) \rightarrow \exists^{\ast} f y^{\ast} x \exists^{\ast} y \leq^* f x \Phi(x, y)\).
2. **Realization** \(R^{\ast}\): \(\forall x^{\ast} \exists^{\ast} y \Phi(x, y) \rightarrow \exists^* z \forall x \exists^* y \leq^* z \Phi(x, y)\).
3. **Independence of Premises** \(IP^{\ast, \text{free}}\): \(\forall x^{\ast} y^{\ast} \Psi(y) \rightarrow \exists^{\ast} g z^{\ast} y \leq^* y \Psi(z)\), where \(A\) is an \(\exists^{\ast}\)-free formula.
4. **Majorizability Axioms** \(MAJ^{\ast}\): \(\forall^{\ast} x \exists^{\ast} y (x \leq^* y)\).

**Theorem 1 (Soundness).** Suppose that

- \(E-HA^{\ast}_w + mAC^{\ast} + R^{\ast} + IP^{\ast, \text{free}} + MAJ^{\ast} \vdash \Phi(z)\),

where \(\Phi\) is an arbitrary formula (it may have free variables). Then there are closed monotone terms \(t\) of appropriate types such that

- \(E-HA^{\ast}_w \vdash \Phi_{bn}(z, t)\).

**Proof.** The proof is by induction on length of the derivation of \(\Phi\).

Logical axioms are dealt with in a way similar to \(S\).

We turn now to the standardness axioms. The interpretations of the first two axioms are \(\exists^{\ast} f y^{\ast} c(x = y \rightarrow (x \leq c \rightarrow y \leq f c))\) and \(\exists^{\ast} f y^{\ast} c(y \leq c \rightarrow (x \leq y \rightarrow x \leq f c))\), respectively. It is clear that the term \(\lambda c.c\) does the job. Let \(t\) be a closed monotone term. The interpretation of the third standardness axiom asks for a closed monotone term \(q\) such that \(E-HA^{\ast}_w\) proves \(t \leq^* q\), which follows from Howard's majorizability theorem. The interpretation of the fourth standardness axiom is

\[\exists^{\ast} f y^{\ast} c \left( z \leq c \rightarrow y^{\ast} b (x \leq b \rightarrow z x \leq f c b) \right)\].

Clearly the term \(\lambda c, b, c b\) does the job.

We turn now to the characteristic principles.

Let us consider the monotone axiom of choice. The interpretation of \(\forall x^{\ast} \exists^{\ast} y \Phi(x, y)\) is

\[\exists^{\ast} y^{\ast} c \exists^{\ast} x \leq^* c \exists^{\ast} y \leq^* g c \Phi_{bn}(gc, x, y)\]

and the interpretation of \(\exists^{\ast} f \exists^{\ast} x \exists^{\ast} y \leq^* f x \Phi(x, y)\) is

\[\exists^{\ast} f^{\ast} y^{\ast} c \leq^* g \exists^{\ast} x \leq^* c^{\ast} y \leq^* f x \Phi_{bn}(gc, x, y)\].

Then we need to find a closed monotone term \(t\) such that for all monotone standard \(g\) we have

\(\forall x^{\ast} c \exists^{\ast} y \leq^* g c \Phi_{bn}(gc, x, y) \rightarrow \exists^{\ast} f \leq^* t g \exists^{\ast} c \exists^{\ast} y \leq^* f x \Phi_{bn}(tg c, x, y)\). \(\exists^{\ast} y \leq^* f x \Phi_{bn}(tg c, x, y)\).
We now consider realization. Modulo equivalence in $\mathsf{E-HA}_m^{\omega}$ we have that $(\forall x \exists^* y \Phi(x, y))^{bn}$ is $\exists^* c \forall x \exists y \leq^* c \Phi_{bn}(c, x, y)$ and that $(\exists^* z \forall x \exists y \leq^* z \Phi(x, y))^{bn}$ is $\exists^* c \exists^* z \leq^* c \forall x \exists y \leq^* z \Phi_{bn}(c, x, y)$. So, we must find a closed monotone term $t$ such that for all monotone standard $c$ we have

$$\forall x \exists y \leq^* c \Phi_{bn}(c, x, y) \rightarrow \exists^* z \leq^* t c \forall x \exists y \leq^* z \Phi_{bn}(tc, x, y).$$

Let $c$ be standard and monotone. Clearly the term $t := \lambda c.c$ does the job by making $z = c$.

We turn now to the independence of premises. Let $\Psi$ be an $\exists^* t$-free formula. Modulo equivalence in $\mathsf{E-HA}_m^{\omega}$ we have that $(A \rightarrow \exists^* y \Psi(y))^{bn}$ is $\exists^* c \left(A \rightarrow \exists^* y \leq^* c \Psi_{bn}(c, y)\right)$ and that $(\exists^* y \left(A \rightarrow \exists^* z \leq^* y \Psi(z)\right))^{bn}$ is $\exists^* c \exists^* y \leq^* c \left[A \rightarrow \exists^* z \leq^* y \Psi_{bn}(c, z)\right]$. Then we must show that there exists a closed monotone term $t$ such that for all monotone standard $c$ we have

$$\left(A \rightarrow \exists^* y \leq^* c \Psi_{bn}(c, y)\right) \rightarrow \exists^* y \leq^* tc \left(A \rightarrow \exists^* z \leq^* y \Psi_{bn}(tc, z)\right).$$

Let $t := \lambda c.c$. Fix $c$ such that $c$ is standard and monotone. Assume $A \rightarrow \exists^* y \leq^* c \Psi_{bn}(c, y)$. Make $z = y$. Then $\exists^* y \leq^* tc \left(A \rightarrow \exists^* z \leq^* y \Psi_{bn}(tc, z)\right)$.

The majorizability axioms are $\exists^* t$-free and therefore have straightforward interpretations.

**Theorem 2** (Characterization). Let $\Phi$ be an arbitrary formula (possibly with free variables). Then

- $\mathsf{E-HA}_m^{\omega} + \mathsf{mAC}^{\omega} + \mathsf{R}^{\omega} + \mathsf{IP}^{\omega}_{\exists^* t, \text{free}} + \mathsf{MAJ}^{\omega} + \Phi^{bn} \leftrightarrow \Phi$.

**Proof.** The proof is by induction on the complexity of $\Phi$. If $\Phi$ is internal and atomic there is nothing to prove.

If $\Phi$ is $\text{st}(x)$ then $\Phi \rightarrow \Phi^{bn}$ by $\mathsf{MAJ}^{\omega}$ and $\Phi^{bn} \rightarrow \Phi$ by the second standardness axiom.

If $\Phi$ is $A \land B$ or $A \lor B$ the result is obvious.

If $\Phi$ is $A \rightarrow B$. Assume $\Phi^{bn}$. Then $\exists^* t f^\exists^* b (A_{bn}(b) \rightarrow B_{bn}(fb))$. By the fourth standardness axiom $fb$ is standard. Hence $\exists^* t b^\exists^* c (A_{bn}(b) \rightarrow B_{bn}(c))$, with $c = fb$. One concludes $\exists^* t b A_{bn}(b) \rightarrow \exists^* c B_{bn}(c)$. Assume now $\Phi$. Then

$$\exists^* t b \left(A_{bn}(b) \rightarrow \exists^* c B_{bn}(c)\right).$$

By $\mathsf{IP}^{\omega}_{\exists^* t, \text{free}}$ one has

$$\exists^* t b^\exists^* c \left(A_{bn}(b) \rightarrow \exists^* y \leq^* c B_{bn}(y)\right).$$

Then, by monotonicity

$$\exists^* t b^\exists^* c (A_{bn}(b) \rightarrow B_{bn}(c)).$$

Then, by $\mathsf{mAC}^{\omega}$ and monotonicity

$$\exists^* t f^\exists^* b (A_{bn}(b) \rightarrow B_{bn}(fb)).$$

If $\Phi$ is $\exists x A(x)$, the result is obvious.
If $\Phi$ is $\forall x A(x)$, by $R^\omega$ and monotonicity
\[ \Phi \leftrightarrow \forall x \exists^* t A_{bn}(x,c) \]
\[ \leftrightarrow \exists^* t \forall x \exists b \leq^* c A_{bn}(x,b) \]
\[ \leftrightarrow \exists^* t \forall x A_{bn}(x,c). \]

4. Intuitionistic Nonstandard Functional Interpretation

**Definition 3.** To each formula $\Phi$ we assign formulas $\Phi^{1st}$ and $\Phi_{1st}$ so that $\Phi^{1st}$ is of the form $\exists^* t \forall^* x \exists^* c \Phi_{1st}(b,c)$, with $\Phi_{1st}(b,c)$ an internal formula, according to the following clauses:

1. $\Phi^{1st}$ and $\Phi_{1st}$ are simply $\Phi$, for internal formulas $\Phi$,
2. $st(t)^{1st}$ is $\exists^* t | t \leq^* c$.

For the remaining cases, if we have already interpretations for $\Phi$ and $\Psi$ given (respectively) by $\exists^* t \forall^* x \exists^* c \Phi_{1st}(b,c)$ and $\exists^* t \forall^* x \exists^* c \Psi_{1st}(d,e)$ then we define:

3. $(\Phi \land \Psi)^{1st}$ is $\exists^* t \forall^* x \exists^* c [\Phi_{1st}(b,c) \land \Psi_{1st}(d,e)]$,
4. $(\Phi \lor \Psi)^{1st}$ is $\exists^* t \forall^* x \exists^* c [\forall^* x \leq^* c \Phi_{1st}(b,c') \lor \forall^* x \leq^* c \Psi_{1st}(d,e')]$,
5. $(\Phi \rightarrow \Psi)^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \exists^* x \Phi_{1st}(a,b,c) \rightarrow \Psi_{1st}(f,b,c) \right]$,
6. $(\exists x \Phi(x))^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \exists^* x \Phi_{1st}(x,b,c) \right]$,
7. $(\forall x \Phi(x))^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \forall^* x \Phi_{1st}(x,b,c) \right]$.

From the interpretation given above one derives:

8. $(\forall x \Phi(x))^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \forall^* x \Phi_{1st}(x,b,c) \right]$,
9. $(\forall^* x \Phi(x))^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \forall^* x \leq^* a \Phi_{1st}(x,f,a,c) \right]$,
10. $(\exists^* x \Phi(x))^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \exists^* x \leq^* a \Phi_{1st}(x,f,a,c) \right]$,
11. $(\exists x \Phi(x))^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \exists^* x \leq^* a \Phi_{1st}(x,b,c') \right]$,
12. $(\exists^* x \Phi(x))^{1st}$ is $\exists^* t \forall^* x \exists^* c \left[ \exists^* x \leq^* a \Phi_{1st}(x,b,c') \right]$,
13. $(\forall^* x \Phi(x))^{1st}$ is $\forall^* c [\forall^* c \Phi_{1st}(x,c)]$.
14. $(-\Phi)^{1st}$ is $\forall^* c [-\Phi_{1st}(x,c)]$.

The following six principles play an important role in the sequel. The proper formulation of the first five principles should be with tuples of variables. To ease readability, we formulate them with single variables:

I. **Monotone Choice** $\text{mAC}^\omega$: $\forall^* x \exists^* y \Phi(x,y) \rightarrow \exists^* t \forall^* x \exists^* y \leq^* x \Phi(x,y)$.

II. **Realization** $\text{R}^\omega$: $\forall x \exists^* y \Phi(x,y) \rightarrow \exists^* z \forall^* x \exists^* y \leq^* z \Phi(x,y)$.

III. **Idealization** $\text{I}^\omega$: $\forall^* z \forall^* x \exists^* y \leq^* z \phi(x,y) \rightarrow \exists^* x \forall^* x \phi(x,y)$.

IV. **Independence of Premises** $\text{IP}^\omega$: $(\Phi \rightarrow \exists^* y \Theta(y)) \rightarrow \exists^* y \left( \Phi \rightarrow \exists^* y \leq^* \Theta(z) \right)$, where $\Phi$ is of the form $\forall^* x \phi(x,y)$.

V. **Markov’s Principle** $\text{MP}^\omega$: $(\forall^* x \phi(x) \rightarrow \psi) \rightarrow \exists^* y \left( \forall^* x \leq^* \phi(x) \rightarrow \psi \right)$.

VI. **Majorizability Axioms** $\text{MA}^\omega$: $(\forall^* x \exists^* y(x \leq^* y))$. 
We show that the standard version $\text{LLPO}^\text{st}$ of the well known lesser limited principle of omniscience $\text{LLPO}$ is a consequence of Idealization.

\[ \text{LLPO}^\text{st} : \forall x, y (\forall x' \leq^* x \phi(x') \lor \forall y' \leq^* y \psi(y')) \rightarrow \forall x \phi(x) \lor \exists^* y \psi(y). \]

Clearly $\text{LLPO}^\text{st}$ implies the bounded version $\text{LLPO}^\text{b}$.

(4.1) $\text{LLPO}^\text{st} : \forall x, y (\forall x' \leq^* x \phi(x') \lor \forall y' \leq^* y \psi(y')) \rightarrow \forall x \phi(x) \lor \exists^* y \psi(y)$.

**Proposition 2.** In $\text{E-HA}_m^\omega$, the Idealization principle $(I^\omega)$ implies $\text{LLPO}^\text{st}$ for any type.

**Proof.** Assume $\forall^* x, y (\forall x' \leq^* x \phi(x') \lor \forall y' \leq^* y \psi(y'))$. Then

\[ \forall^* x, y \exists n^0 [n = 0 \rightarrow (\forall x' \leq^* x \phi(x')) \land (n \neq 0 \rightarrow \forall y' \leq^* y \psi(y'))]. \]

This implies

\[ \forall^* x, y \exists n^0 \forall x' \leq^* x \forall y' \leq^* y [n = 0 \rightarrow \phi(x') \land n \neq 0 \rightarrow \psi(y')]. \]

Then by $(I^\omega)$

\[ \exists n^0 \forall x', y' [n = 0 \rightarrow \phi(x') \land n \neq 0 \rightarrow \psi(y')]. \]

In $\text{E-HA}_m^\omega$, one has $n = 0 \lor n \neq 0$, because $n$ has type $0$. If $n = 0$, then one concludes $\forall^* x', y' \phi(x')$, hence $\forall^* x' \phi(x')$. If $n \neq 0$ then $\forall^* x', y' \psi(y')$, hence $\forall^* y' \psi(y')$. \(\square\)

**Proposition 3.** In $\text{E-HA}_m^\omega$, the principle $\text{LLPO}^\text{st}$ in type $0$ is equivalent to

1. $\forall^* m, n (\phi(m) \lor \psi(n)) \rightarrow \forall^* m \phi(m) \lor \exists^* n \psi(n)$.

**Proof.** In type $0$ the principle $\text{LLPO}^\text{st}$ states that

\[ \forall^* m, n (\forall m' \leq^* m \phi(m') \lor \forall n' \leq^* n \psi(n')) \rightarrow \forall^* m \phi(m) \lor \exists^* n \psi(n). \]

It is enough to show that

\[ \forall^* m, n (\forall m' \leq^* m \phi(m') \lor \forall n' \leq^* n \psi(n')) \leftrightarrow \forall^* m, n (\phi(m) \lor \psi(n)). \]

For the direct implication just put $m' = m$ and $n' = n$. Assume $\forall^* m, n (\phi(m) \lor \psi(n))$. Let $m, n$ be standard. We show first, by external induction on $n$, that

\[ (4.2) \phi(m) \lor \forall n' \leq n \psi(n'). \]

If $n = 0$ then $\phi(m) \lor \psi(0)$ is true by hypothesis. By hypothesis one has $\phi(m) \lor \psi(n + 1)$ because if $n$ is standard then $n + 1$ is also standard. Then, by the induction hypothesis, $\phi(m) \lor \forall n' \leq n + 1 \psi(n')$. Hence $\phi(m) \lor \forall n' \leq n \psi(n')$. We now show $\forall m' \leq^* m \phi(m') \lor \forall n' \leq^* n \psi(n')$ by external induction on $n + m$. If $m + n = 0$ then $m = 0$ and $n = 0$ and one has $\phi(0) \lor \psi(0)$ by hypothesis. If $m + n = k + 1$ at least one of the integers $m, n$ is different from $0$. Assume without loss of generality $m \neq 0$. Then $m(n + 1) = n = k$. By induction hypothesis

\[ (4.3) \forall m' \leq^* m - 1 \phi(m') \lor \forall n' \leq^* n \psi(n'). \]

From $(4.2), (4.3)$ there are four cases. If $\forall m' \leq^* m - 1 \phi(m')$ and $\phi(m)$ then $\forall m' \leq^* m \phi(m')$ and one concludes the result. The remaining cases trivially imply the result. \(\square\)
Proposition 4. The theory $\mathsf{E-HA}^\omega_{\text{st}} + \mathsf{mAC}^\omega + \mathsf{IP}_{\text{st}}^\omega + \mathsf{MAJ}^\omega$ proves the Standard Bounded Collection Principle

$$\mathsf{BC}_{\text{st}} : \exists^* c \left( \forall z \leq^* e \exists^* y \Phi(y, z) \rightarrow \exists^* b \exists^* y \leq^* c \exists^* y \leq^* b \Phi(y, z) \right).$$

Proof. The proof follows closely the argument given in [9, Proposition 3]. Let $c$ be standard and monotone. Assume that $\exists^* c \left( \forall z \leq^* e \exists^* y \Phi(y, z) \right)$. Then, by $\mathsf{IP}_{\text{st}}^\omega$,

$$\exists^* c \left( \forall z \leq^* e \exists^* y \leq^* b \Phi(y, z) \right).$$

Then, by $\mathsf{mAC}^\omega$,

$$\exists^* f \exists^* z \exists^* b \leq^* f z \left( z \leq^* c \rightarrow \exists y \leq^* f z \Phi(y, z) \right).$$

Hence

$$\exists^* f \exists^* z \left( z \leq^* c \rightarrow \exists y \leq^* f z \Phi(y, z) \right).$$

Taking $z := c$, we obtain $\exists^* f \exists^* y \leq^* c \exists^* y \leq^* f c \Phi(y, z)$. Since $f$ and $c$ are standard, by the fourth standardness axiom $f \Phi$ is also standard. Then, by $\mathsf{MAJ}^\omega$, there exists a standard $b$ such that $f c \leq^* b$. Hence $\exists^* f \exists^* y \leq^* c \exists^* y \leq^* b \Phi(y, z)$. \qed

Theorem 3 (Soundness). Suppose that

- $\mathsf{E-HA}^\omega_{\text{st}} + \mathsf{mAC}^\omega + \mathsf{R}^\omega + \mathsf{I}^\omega + \mathsf{MP}^\omega + \mathsf{IP}_{\text{st}}^\omega + \mathsf{MAJ}^\omega \vdash \Phi$,

where $\Phi$ is an arbitrary formula (it may have free variables). Then there are closed monotone terms $t$ of appropriate types such that

- $\mathsf{E-HA}^\omega_{\text{st}} + \exists^* t \Phi_{\text{st}}(b, t)$.

Proof. The proof is by induction on length of the derivation of $\Phi$. The standardness axioms have the same interpretations as in the realizability case above. The verifications that deserve special attention are the ones dealing with disjunction and the characteristic principles.

$A \lor A \rightarrow A$. Since all closed terms are standard, to interpret this axiom we need to find closed monotone terms $q$, $r$ and $s$ such that for all monotone standard $b$, $d$ and $v$ we have

$$\exists^* c \leq^* q b d v \exists^* e \leq^* r b d v \left( \exists^* c \leq^* A_{\text{st}}(b, c) \lor \exists^* e \leq^* A_{\text{st}}(d, e) \right) \rightarrow A_{\text{st}}(s b d, v)$$

The terms $q := \lambda b, d, v; r := \lambda b, d, v$ and $s := \lambda b, d, m(b, d)$ do the job.

To interpret $A \rightarrow A \lor B$ we must find closed monotone terms $q, r, s$ such that for all monotone standard $a$, $b$ and $v$ we have

$$\left( \exists^* c \leq^* q b a v A_{\text{st}}(b, c) \rightarrow \left( \exists^* a \leq^* A_{\text{st}}(r b, a') \lor \exists^* v \leq^* v B_{\text{st}}(s b, v') \right) \right).$$

Clearly, the terms $q := \lambda b, a, v, a$, $r := \lambda b, b$ and $s := \lambda b, O$ do the job.

$A \lor B \rightarrow B \lor A$. We need to find closed monotone terms $q, r, s, t$ such that for all monotone standard $b$, $d$, $z$ and $w$ we have

$$\left( \exists^* c \leq^* q b d z w \exists^* e \leq^* r b d z w \exists^* e \leq^* c A_{\text{st}}(b, c') \lor \exists^* e \leq^* e B_{\text{st}}(d, e') \rightarrow \left( \exists^* z \leq^* z B_{\text{st}}(s b d, z') \lor \exists^* w \leq^* w A_{\text{st}}(t b d, w') \right) \right)$$

The terms $q := \lambda b, d, z, w; r := \lambda b, d, z, w, z; s := \lambda b, d, d$ and $t := \lambda b, d, b$ do the job.
A \rightarrow B \Rightarrow C \lor A \rightarrow C \lor B. Suppose we have monotone terms \( q, r \) such that
\[
\tilde{\nu}^{st}b, e \left( \tilde{\nu}c \leq^* qbe A_{1st} (b, c) \rightarrow B_{1st} (rb, e) \right).
\]
We need to find closed monotone terms \( s, t, u, v \) such that for all monotone standard \( b, d, z \) and \( w \) we have
\[
\tilde{\nu}c \leq^* sbdzw\tilde{\nu}e \leq^* tbdzw \left( \tilde{\nu}c' \leq^* c C_{1st} (b, c') \lor \tilde{\nu}e' \leq^* e A_{1st} (d, e') \right) \rightarrow \\
\tilde{\nu}z' \leq^* z C_{1st} (uwdz, z') \lor \tilde{\nu}w' \leq^* w B_{1st} (vbdw, w')
\]
We show that the terms \( s := \lambda b, d, z, w, z; t := \lambda b, d, z, w, qdw; u := \lambda b, d, b \) and \( v := \lambda b, d, d \) do the job. Fix monotone standard \( b, d, z \) and \( w \). Assume that
\[
\tilde{\nu}c' \leq^* x\tilde{\nu}e \leq^* qdw \left( \tilde{\nu}c' \leq^* c C_{1st} (b, c') \lor \tilde{\nu}e' \leq^* e A_{1st} (d, e') \right).
\]
In particular,
\[
\tilde{\nu}c' \leq^* v\tilde{\nu}e \leq^* qdw A_{1st} (b, c')
\]
Now, if \( \tilde{\nu}c' \leq^* v C_{1st} (b, c') \) holds we are done. If \( \tilde{\nu}c' \leq^* qdw A_{1st} (d, e') \) then for any monotone \( w' \leq^*, \) if \( c' \leq^* qdw' \) then \( e' \leq^* qdw' \). Hence \( A_{1st} (d, e') \). By hypothesis we conclude the result.

We turn now to the characteristic principles.

**mAC**\(^w\). Modulo equivalence in E-HA\(^{w}_{st}\), the interpretation of \( \forall x \exists^w y \Phi (x, y) \) is
\[
\exists^w f, g\tilde{\nu}^w d, c\tilde{\nu}x \leq^* d \exists^w y \leq^* f \tilde{\nu}c' \leq^* c \Phi_{1st} (x, y, gd, c')
\]
and the interpretation of \( \exists^w f \tilde{\nu}^w x \exists^w y \leq^* f x \Phi (x, y) \) is
\[
\exists^w g, h\tilde{\nu}^w a, c\tilde{\nu}f \leq^* h\tilde{\nu}a' \leq^* a\tilde{\nu}c'' \leq^* c\tilde{\nu}x \leq^* a\exists^w y \leq^* f x \tilde{\nu}c' \leq^* c'' \Phi_{1st} (x, y, ga', c').
\]
Hence we need to find closed monotone terms \( s, t, u \) and \( v \) such that for all monotone standard \( f, g, a \) and \( c \) we have
\[
\tilde{\nu}d \leq^* s \exists^w g \tilde{\nu}c' \leq^* t \exists^w f \tilde{\nu}c' \leq^* e \Phi_{1st} (x, y, gd, c') \rightarrow \\
\exists^w f, g, a, c, a; t := \lambda f, g, a, c, c; u := \lambda f \text{ and } v := \lambda g, a', g'
\]

**R**\(^w\). Modulo equivalence in E-HA\(^{w}_{st}\), the interpretation of \( \forall x \exists^w y \Phi (x, y) \) is
\[
\exists^w a, b\tilde{\nu}^w c\tilde{\nu}x \exists^w y \leq^* a\tilde{\nu}c' \leq^* c \Phi_{1st} (x, y, b, c')
\]
and the interpretation of \( \exists^w z \exists^w x \exists^w y \leq^* z \Phi (x, y) \) is
\[
\exists^w a, b\tilde{\nu}^w x \exists^w z \leq^* a\tilde{\nu}c'' \leq^* c\tilde{\nu}x \exists^w y \leq^* z \tilde{\nu}c' \leq^* c'' \Phi_{1st} (x, y, b, c').
\]
Hence we need to find closed monotone terms \( q, r \) and \( s \) such that for all monotone standard \( a, b \) and \( c \) we have
\[
\tilde{\nu}e \leq^* qabc \left( \forall x \exists^w y \leq^* a\tilde{\nu}c' \leq^* e \Phi_{1st} (x, y, b, e') \right) \\
\rightarrow \exists^w z \leq^* rab\tilde{\nu}c'' \leq^* c\tilde{\nu}x \exists^w y \leq^* z \tilde{\nu}c' \leq^* c'' \Phi_{1st} (x, y, sab, c')
\]
The terms \( q := \lambda a, b, c, r := \lambda a, b, a, s := \lambda a, b, b \) do the job.

**I**\(^w\). Modulo equivalence in E-HA\(^{w}_{st}\), the interpretation of \( \exists^w z \exists^w x \forall y \leq^* z \Phi (x, y) \) is
\[
\exists^w z \exists^w x \forall y \leq^* z \Phi (x, y)
\]
and the interpretation of \( \exists^w x \forall y \Phi (x, y) \) is
\[
\tilde{\nu}^w y \exists^w x \forall y' \leq^* y \Phi (x, y').
\]
Hence the same terms satisfy both interpretations.

\[ \text{IP}_{\text{st}}. \] Modulo equivalence in \( \text{E-HA}^w_{\text{st}} \), the interpretation of \( \Phi \rightarrow \exists^* y \Psi(y) \) is

\[
\exists^* f, g, h \forall^* x \left( \forall^* y \leq^* h \phi(x) \rightarrow \exists^* y \leq^* g \forall^* c \leq^* c \Psi_{\text{st}}(y, f, c) \right)
\]

and the interpretation of \( \exists^* y \left( \Phi \rightarrow \exists z \leq^* y \Psi(z) \right) \) is

\[
\exists^* f, g, a \forall^* x \exists y \leq^* a \forall^* c \leq^* c \left( \forall^* y \leq^* g \forall^* c \leq^* c \Psi_{\text{st}}(y, f, c) \right).
\]

Hence we need to find closed monotone terms \( q, r, s \) and \( t \) such that for all monotone standard \( f, g, h \) and \( c \) we have

\[
\forall^* y \leq^* q f g h c \left( \forall^* y \leq^* h d \phi(x) \rightarrow \exists^* y \leq^* g d e f \leq^* d \Psi_{\text{st}}(y, f, d') \right)
\]

\[
\rightarrow \exists y \leq^* r f g h \forall^* c \leq^* c \left( \forall^* y \leq^* s f g h \forall^* \phi(x) \rightarrow \exists^* y \forall^* c \leq^* c \Psi_{\text{st}}(y, f, c') \right).
\]

The terms \( q := \lambda f, g, h, c, r := \lambda f, g, h, g, s := \lambda f, g, h, c' \), \( t := \lambda f, g, h, c' \), \( f \) do the job by putting \( d = c \)

\[ \text{MP}_{\text{st}}. \] \text{Trivial.}

\[ \text{MAJ}^w. \] The interpretation of \( \text{MAJ}^w \) is, modulo equivalence in \( \text{E-HA}^w_{\text{st}} \),

\[
\exists^* f \forall^* b \forall^* x \leq^* b \exists y \leq^* f b (x \leq^* y).
\]

Then we need to find a closed monotone term \( t \) such that

\[
\forall^* b \forall^* x \leq^* b \exists y \leq^* f b (x \leq^* y).
\]

It is easy to see that \( t := \lambda b. b \) does the job. \( \square \)

**Theorem 4** (Characterization). Let \( \Phi \) be an arbitrary formula (possibly with free variables). Then

- \( \text{E-HA}^w_{\text{st}} + \text{mAC}^w + \text{R}^w + \text{I}^w + \text{MP}_{\text{st}} + \text{IP}_{\text{st}}^w + \text{MAJ}^w \vdash \Phi^\text{lst} \iff \Phi. \)

**Proof.** If \( \Phi \) is internal and atomic the result is obvious.

If \( \Phi \) is \( \text{st}(x) \). Assume \( \Phi \) then one concludes \( \Phi^\text{lst} \) by \( \text{MAJ}^w \). Assume \( \Phi^\text{lst} \). Then \( \Phi \) follows from the second standardness axiom.

If \( \Phi \) is \( A \land B \) the result is obvious.

If \( \Phi \) is \( A \lor B \). Modulo equivalence in \( \text{E-HA}^w_{\text{st}} \), \( (A \lor B)^\text{lst} \) is

\[
\exists^* b, d \forall^* c, e \left[ \forall^* c \leq^* c \forall^* e \leq^* e \Psi_{\text{st}}(b, c) \lor \forall^* e \leq^* e \Psi_{\text{st}}(d, e) \right].
\]

By (11) this is equivalent to

\[
(A \lor B)^\text{lst} \iff \exists^* b, d \left( \forall^* c \forall^* e \Psi_{\text{st}}(b, c) \lor \forall^* e \Psi_{\text{st}}(d, e) \right)
\]

\[
\iff \exists^* b \forall^* c \forall^* e \Psi_{\text{st}}(b, c) \lor \exists^* d \forall^* e \Psi_{\text{st}}(d, e)
\]

\[
\iff A^\text{lst} \lor B^\text{lst}.
\]

If \( \Phi \) is \( A \rightarrow B \). One has, modulo equivalence in \( \text{E-HA}^w_{\text{st}} \),

\[
A^\text{lst} \rightarrow B^\text{lst} \iff \exists^* b \forall^* c \forall^* e \Psi_{\text{st}}(b, c) \rightarrow \exists^* d \forall^* e \Psi_{\text{st}}(d, e)
\]

\[
\iff \forall^* b \left( \forall^* c \forall^* e \Psi_{\text{st}}(b, c) \rightarrow \exists^* d \forall^* e \Psi_{\text{st}}(d, e) \right).
\]
Then by \( \text{IP}^{\omega}_{\text{std}} \),
\[
A^{\text{std}} \to B^{\text{std}} \iff \tilde{\nu}^{\text{std}}b \tilde{\exists}^{\text{std}} d \left( \tilde{\nu}^{\text{std}} c A_{\text{std}}(b, c) \to \tilde{\nu}^{\text{std}} e B_{\text{std}}(d, e) \right)
\]
\[
\iff \tilde{\nu}^{\text{std}}b \tilde{\exists}^{\text{std}} d\tilde{\nu}^{\text{std}} e \left( \tilde{\nu}^{\text{std}} c A_{\text{std}}(b, c) \to B_{\text{std}}(d, e) \right).
\]

By \( \text{MP}^{\text{std}} \) one has
\[
A^{\text{std}} \to B^{\text{std}} \iff \tilde{\nu}^{\text{std}}b \tilde{\exists}^{\text{std}} d\tilde{\nu}^{\text{std}} e c \left( \tilde{\nu}^{\text{std}} c \leq c A_{\text{std}}(b, c') \to B_{\text{std}}(d, e) \right).
\]

By \( \text{mAC}^{\omega} \) one concludes
\[
A^{\text{std}} \to B^{\text{std}} \iff \tilde{\nu}^{\text{std}}b \tilde{\exists}^{\text{std}} d\tilde{\nu}^{\text{std}} e \exists c \leq c f \left( \tilde{\nu}^{\text{std}} c \leq c A_{\text{std}}(b, c') \to B_{\text{std}}(d, e) \right)
\]
\[
\iff \exists g, F\tilde{\nu}^{\text{std}}b \tilde{\exists} \leq gb \exists f \leq c f \left( \tilde{\nu}^{\text{std}} c \leq c A_{\text{std}}(b, c') \to B_{\text{std}}(d, e) \right)
\]

Hence, by monotonicity
\[
A^{\text{std}} \to B^{\text{std}} \iff \exists g, F\tilde{\nu}^{\text{std}}b, e \left[ \tilde{\nu}^{\text{std}} c \leq c Fb A_{\text{std}}(b, c) \to B_{\text{std}}(gb, e) \right]
\]
\[
\iff (A \to B)^{\text{std}}.
\]

If \( \Phi \) is \( (\forall x A(x)) \). By \( \text{R}^{\omega} \) and monotonicity, modulo equivalence in \( \text{E-HA}^{\omega}_{\text{std}} \), one derives
\[
\forall x (A(x))^{\text{std}} \iff \forall x \exists^g x \exists x \exists x' \leq x A_{\text{std}}(x', b, c)
\]
\[
\iff \exists^g x \exists x' \leq x A_{\text{std}}(x', b, c)
\]
\[
\iff (\forall x A(x))^{\text{std}}.
\]

If \( \Phi \) is \( (\exists x \Phi(x)) \). By \( \text{I}^{\omega} \) and monotonicity, modulo equivalence in \( \text{E-HA}^{\omega}_{\text{std}} \), one derives
\[
\exists x (A(x))^{\text{std}} \iff \exists^g x \exists x' \leq x A_{\text{std}}(x, b, c)
\]
\[
\iff \exists^g x \exists x' \leq x A_{\text{std}}(x, b, c)
\]
\[
\iff (\exists x A(x))^{\text{std}}.
\]
\[\square\]

5. Remarks on Transfer

Nonstandard theories usually include a Transfer principle. The main reason to do so is that this principle allows for a connection between the standard and the nonstandard universes. Typically, one reasons using nonstandard objects, which is usually simpler, and then use Transfer to obtain a standard result (see for example [5] for many applications of this kind of reasoning). Transfer also implies that all uniquely defined objects are standard (see [16], p. 1166). The price to pay is that one can only apply Transfer to internal assertions and all parameters must be standard. Nelson called the violation of this rule illegal Transfer. As we will discuss below, in the context of (nonstandard) arithmetic theories further restrictions must be made (cf. [13] [11]).

We consider the following two transfer principles. Let \( \varphi \) be an internal formula whose free variables are \( L \).
The principles \((TP\forall)\) and \((TP\exists)\) are classically (but not intuitionistically) equivalent. As we shall see, in both classic and intuitionistic settings, adding any of the Transfer principles to finiteness theories lead to nonconservativeness and adding any of the Transfer principles to majorizability theories leads to inconsistency.

Alternatively, Moerdijk [13] suggested to use Transfer as a rule instead of a principle. One needs to add the following two rules:

\[
\begin{align*}
\forall x \varphi(x) & \quad \vdash \forall x \varphi(x) \quad (TR\forall) \\
\exists x \varphi(x) & \quad \vdash \exists x \varphi(x) \quad (TR\exists)
\end{align*}
\]

This elegant solution allows to incorporate Transfer in the theory and was followed by other authors, for example Van den Berg et al. Indeed Van den Berg et al. proved that their system is closed under both Transfer rules [2, Proposition 5.12]. However, the two rules are not equivalent, not even classically.

**Transfer and finiteness.** We start by discussing the transfer principles in the setting of Benno et al. (cf. [2] for the notation). As discussed in the appendix of [7] (see also [3]), in the classical setting, adding either of the transfer principles to \(E-PA_{\omega+1} + 1 + \text{HA} \text{int} \) leads to a nonconservative extension of \(PA\). However, as shown in [3], a conservative extension of Peano Arithmetic can still be obtained by restricting \((TP\forall)\) to the following parameter-free version

\[(PF-TP\forall) \quad \forall^x \varphi(x) \Rightarrow \forall \varphi(x)\]

where the internal formula \(\varphi(x)\) does not involve parameters. But, even in this restricted version, Transfer has undesirable consequences, namely instances of the law of excluded middle and is therefore unsuitable for intuitionistic purposes. For example, the existence of type 0 nonstandard elements together with \((TP\forall)\) implies the law of excluded middle for internal arithmetical formulas (see the discussion in [2, Section 3.3] for more details).

In both [3] and the appendix of [7] the argument used to prove nonconservativeness uses, in an essential way, a strong form of the law of excluded middle and is therefore nonconstructive. However it is also possible to prove nonconservativeness in the intuitionistic setting, i.e., that adding any of the two Transfer principles to \(E-\text{HA}^\omega_{\omega+1} + \text{R} + \text{HGMP}^\text{0} \) leads to a nonconservative extension of \(\text{HA} \). To do so we recall the (type 0) Underspill principle

\[(US_0) : \forall z^0 (\neg \text{st}(x) \rightarrow \varphi(x)) \rightarrow \exists z^0 \varphi(x)\]

It had already been pointed out by Avigad and Helzner [1] that the combination of any of the transfer principles with \((US_0)\) leads to a nonconservative extension of

\footnote{Both [3] and the appendix of [7] present examples where the parameter which leads to nonconservativeness is a parameter of type 1. It seems that type 0 parameters are benign but so far, to our knowledge, it is still an open question if type 0 parameters also lead to nonconservativeness.}
HA (see also [2, p. 1973]). By [2, Proposition 5.12] the type 0 Underspill principle can be eliminated by their functional interpretation, i.e.

\[ E - HA^\omega_{st} + R + HGMP^st - US_0. \]

Hence \( E - HA^\omega_{st} + R + HGMP^st + TP_\forall \) and \( E - HA^\omega_{st} + R + HGMP^st + TP_\exists \) are nonconservative extensions of HA.

**Transfer and majorizability.** In the context of (nonstandard) majorizability theories, both \((TP_\forall)\) and \((TP_\exists)\) are incompatible with the second standardness axiom. Therefore none of the Transfer principles can be consistently added to any extension of Heyting (or Peano) arithmetic which possesses the second standardness axiom. In particular, neither \((TP_\forall)\) nor \((TP_\exists)\) can be added to the theory in [7]. Also, neither of these principles can be added to the bounded realizability presented in Section 3 or to the intuitionistic nonstandard interpretation presented in Section 4. To see this, let \(a\) be a nonstandard natural number and let \(\alpha\) be the sequence defined by

\[
\alpha(n) \begin{cases} 
0, & \text{if } n < a \\
1, & \text{if } n \geq a
\end{cases}
\]

By the second standardness axiom \(\alpha\) is standard because \(\alpha \leq^* 1\). Then \(\alpha(n) = 0\) is an internal formula. We have that, for all standard \(n\), \(\alpha(n) = 0\) and \(\neg (\forall n (\alpha(n) = 0))\) hence the second standardness axiom is incompatible with \((TP_\forall)\). Moreover, the element \(a\) is unique because it can be defined as the smallest element \(\alpha(n)\) such that \(\alpha(n) = 1\). Observe that this would imply that \(a\) is standard, a contradiction. Hence the second standardness axiom is also incompatible with \((TP_\exists)\).

**References**

[1] J. Avigad and J. Helzner. Transfer principles in nonstandard intuitionistic arithmetic. Archive for Mathematical Logic, 41(6):581–602, 2002.

[2] B. van den Berg, E. Briseid, and P. Safarik. A functional interpretation for nonstandard arithmetic. Annals of Pure and Applied Logic, 163(12):1962 – 1994, 2012.

[3] B. van den Berg, S. Sanders. Transfer equals comprehension. Available on arXiv: http://arxiv.org/abs/1409.6881.

[4] M. Bezem. Strongly majorizable functionals of finite type: a model for bar recursion containing discontinuous functionals. The Journal of Symbolic Logic, 50:652–660, 1985.

[5] F. and M. Diener (eds.), Nonstandard Analysis in Practice, Springer Universitext (1995).

[6] F. Ferreira. Proof interpretations and majorizability. Logic Colloquium’07, Françoise Delon et al. org., Cambridge University Press, 32–81, 2010.

[7] F. Ferreira and J. Gaspar. Nonstandardness and the bounded functional interpretation. Annals of Pure and Applied Logic, 166:665–740, 2015.

[8] F. Ferreira and A. Nunes. Bounded modified realizability. The Journal of Symbolic Logic, 71:329–345, 2006.

[9] F. Ferreira and P. Oliva. Bounded functional interpretation. Annals of Pure and Applied Logic, 135:73–112, 2005.

[10] W. A. Howard. Hereditarily majorizable functionals of finite type. In A. S. Troelstra, editor, Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, volume 344 of Lecture Notes in Mathematics, pages 454–461. Springer-Verlag, Berlin, 1973.

[11] S. Kleene. On the interpretation of intuitionistic number theory. The Journal of Symbolic Logic, 10:109–124, 1945.

[12] U. Kohlenbach. Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2008.

[13] I. Moerdijk. A model for intuitionistic non-standard arithmetic, Annals of Pure and Applied Logic 73(1):37–51, 1995.

[14] E. Palmgren. A constructive approach to nonstandard analysis, Annals of Pure and Applied Logic 73:297–325, 1995.
[15] E. Palmgren. Developments in constructive nonstandard analysis, Bull. Symbolic Logic 4, 3:233–272, 1998.
[16] E. Nelson. Internal set theory: A new approach to nonstandard analysis Bulletin of the American Mathematical Society, 83(6):1165–1198, 11 1977.
[17] E. Nelson. The syntax of nonstandard analysis. Annals of Pure and Applied Logic, 38(2):123 – 134, 1988.
[18] S. Sanders. Non-standard Nonstandard Analysis and the computational content of standard mathematics. Available on arXiv: [http://arxiv.org/abs/1508.07434](http://arxiv.org/abs/1508.07434)

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