Green's Functions for Neutrino Mixing

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The Green's function formalism for neutrino mixing is presented and the exact oscillation formula is obtained. The usual Pontecorvo formula is recovered in the relativistic limit.

1 Introduction

We report on the Green's function formalism for neutrino field mixing recently presented in [1] (see also [2,3]). The result is an oscillation formula which differs from the usual one in the non-relativistic region. We get, together with the "squeezing" factor of the amplitude found in ref. [2], also an additional term with a different oscillatory frequency. This last feature is particularly important since it shows that resonance is possible also in vacuum for particular values of the masses or of the momentum, thus leading to a suppression or to an enhancement of the conversion probability.

We consider two Dirac neutrino fields $\nu_e$ and $\nu_\mu$ (space-time dependence suppressed). The "flavor mixing" transformations are

$$
\begin{align*}
\nu_e(x) &= \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \\
\nu_\mu(x) &= -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta,
\end{align*}
$$

where $\theta$ is the mixing angle. $\nu_1$ and $\nu_2$ are explicitly given by

$$
\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{k,r} \left[ u_{k,i}^r e^{-i\omega_{k,i} t} \alpha_{k,i}^r + v_{-k,i}^r e^{i\omega_{k,i} t} \beta_{-k,i}^r \right] e^{i k \cdot x}, \quad i = 1, 2.
$$

We use $t \equiv x_0$, when no misunderstanding arises. The vacuum for the $\alpha_i$ and $\beta_i$ operators is denoted by $|0\rangle_{1,2}$: $\alpha_{k,i}^r |0\rangle_{12} = \beta_{k,i}^r |0\rangle_{12} = 0$. The anticommutation relations, the completeness and orthonormality relations are the usual
ones. In order to circumvent the difficulty of the construction of a Fock space for the mixed fields, it is useful to expand the flavor fields $\nu_e$ and $\nu_\mu$ in the same basis as $\nu_1$ and $\nu_2$; e.g. $\nu_e$ is given by

$$\nu_e^0(x) = G^{-1}(\theta, t) \nu_1^0(x) G(\theta, t)$$

$$= \frac{1}{\sqrt{P}} \sum_{k,r} \left[ u_{k,1}^{r} e^{-i\omega_{k,1} t} \alpha_{k,e}^r(t) + v_{k,1}^{r} e^{i\omega_{k,1} t} \beta_{k,1}^r(t) \right] e^{ikx},$$

where $G(\theta, t) = \exp \left[ \theta \int d^3x \left( \nu_1^2(x) - \nu_2^2(x) \right) \right]$, is the generator of the mixing transformations $G$. The flavor annihilation operators can be now explicitly given. For example, the electron neutrino annihilator is

$$\alpha_{k,e}^r(t) = \cos \theta \alpha_{k,1}^r$$

$$\sin \theta \sum_s \left( u_{k,1}^s U_{k,2}^t e^{-i(\omega_{k,2} - \omega_{k,1}) t} \alpha_{k,2}^s + u_{k,1}^t U_{k,2}^s e^{i(\omega_{k,1} + \omega_{k,2}) t} \beta_{k,2}^t \right).$$

Notice that it has contributions from $\alpha_1$, $\alpha_2$ but also from the anti-particle operator $\beta_2^2$ since spinor wave functions for different masses are not orthogonal. In the more traditional treatment of mixing, the $\beta_1^2$ contribution is missed since the non-orthogonality of the spinor wave functions is not considered.

We can show that when the two point Green’s function for the mixed fields $\nu_e$, $\nu_\mu$ are constructed by using the vacuum $|0\rangle_{1,2}$ then the “survival” probability amplitude, say, of an electronic neutrino state in the limit $t \to 0^+$ is computed to be $P_{ee}(k, 0^+) = \cos^2 \theta + \sin^2 \theta |U_{k}|^2 < 1$, which is clearly not acceptable since, of course, it should be $\lim_{t \to 0^+} P_{ee}(t) = 1$.

Here $|U_{k}|^2$ is calculated from the spinor basis and its explicit form is given in. For different masses and $k \neq 0$, $|U_{k}|$ is always $< 1$ and we will also use $|W_k| = \sqrt{1 - |U_{k}|^2}$, $|U_{k}|^2 \to 1$ in the relativistic limit $k \gg m_1 m_2$.

The above contradiction shows that the choice of the state $|0\rangle_{1,2}$ in the computation of the Wightman function is not the correct one. The problem is in the fact that the transformation $G$ does not leave invariant the vacuum $|0\rangle_{1,2}$. The mixing generator induces on it a SU(2) coherent state structure, resulting in a new state, $|0(\theta, t)\rangle_{e,\mu} = G^{-1}(\theta, t) |0\rangle_{1,2}$, which is the flavor vacuum for the flavor operators $\alpha_{e,\mu}$, $\beta_{e,\mu}$. Important features of the flavor vacuum $|0\rangle_{e,\mu}$ (and of the relative Fock space) is its non-perturbative nature, resulting in the unitary inequivalence with the “perturbative” vacuum $|0\rangle_{1,2}$, in the infinite volume limit. Notice that the squared modulus of the survival probability amplitude reproduces the Pontecorvo oscillation formula in the relativistic limit.

We show below that the correct definition of the Green’s functions is the one which involves the non-perturbative vacuum $|0\rangle_{e,\mu}$.
2 Green’s functions for flavor neutrinos

In the case of $\nu_e \rightarrow \nu_e$ propagation, the relevant Wightman function is (we use $x_0 = t, y_0 = 0$)

$$iG^{>\alpha\beta}_{ee}(t, x; 0, y) = e_{\mu}(t) \langle 0|\nu_e^\alpha(0)\bar{\nu}_e^\beta(0)|x, \mu \rangle.$$  

It can be conveniently expressed in terms of anticommutators at different times as

$$iG^{>\alpha\beta}_{ee}(k, t) =$$

$$\sum_r \left[ u^r_{k, 1} u^r_{k, 1} \{ \alpha^r_{k, e}(t), \alpha^r_{k, e}(t) \} e^{-i\omega_{k, 1}t} + u^r_{k, 1} u^r_{k, 1} \{ \beta^r_{-k, e}(t), \alpha^r_{k, e}(t) \} e^{i\omega_{k, 1}t} \right].$$

(5)

Here $\alpha^r_{k, e}(t)$ stands for $\alpha^r_{k, e}(0)$. The corresponding transition amplitude is

$$P^r_{ee}(k, t) = \cos^2 \theta + \sin^2 \theta \left[ |U_k|^2 e^{-i(\omega_{k, 2} - \omega_{k, 1})t} + |V_k|^2 e^{i(\omega_{k, 2} + \omega_{k, 1})t} \right].$$

(6)

We thus find that the probability amplitude is now correctly normalized:

$$\lim_{t \to 0^+} P_{ee}(k, t) = 1,$$

and one can show that $P_{ee}$, $P_{\mu e}$, $P_{\mu\mu}$ go to zero in the same limit $t \to 0^+$. Moreover,

$$|P_{ee}(k, t)|^2 + |P^r_{ee}(k, t)|^2 + |P^r_{\mu e}(k, t)|^2 + |P^r_{\mu\mu}(k, t)|^2 = 1,$$

as the conservation of the total probability requires. Notice that in the perturbative case, there were only two non-zero amplitudes, i.e. $P_{ee}$ and $P_{\mu e}$.

At time $t = 0$ the one electronic neutrino state is (momentum and spin indices dropped) $|\nu_e(t)\rangle \equiv \alpha_{\mu}^0|0\rangle_{e, \mu}$. In this state a multiparticle component is present, disappearing in the relativistic limit $k \gg \sqrt{m_1 m_2}$, where the Pontecorvo state is recovered. Its time evolution is given by $|\nu_e(t)\rangle \equiv e^{-iHt}|\nu_e\rangle$ and in the flavor basis this state is found to be

$$|\nu_e(t)\rangle = \eta_1(t) \alpha_e^1 + \eta_2(t) \alpha_e^2 + \eta_3(t) \alpha_e^3 + \eta_4(t) \alpha_e^4 |0\rangle_{e, \mu}. \quad (8)$$

Here the $\eta(t)$ are coefficient satisfying the normalization condition $|\eta_1(t)|^2 + |\eta_2(t)|^2 + |\eta_3(t)|^2 + |\eta_4(t)|^2 = 1$.

Notice that $|0\rangle_{e, \mu}$ is not eigenstate of the free Hamiltonian $H$; it “rotates” under the action of the time evolution generator: $|0(t)\rangle_{e, \mu} \equiv e^{-iH_{1, 2}t}|0\rangle_{e, \mu}$. In fact one finds $\lim_{t \to \infty} e_{\mu}(0) |0(t)\rangle_{e, \mu} = 0$. Thus at different times we have unitarily inequivalent flavor vacua (in the limit $V \to \infty$): this is not surprising since it is direct consequence of the fact that the flavor states are not mass eigenstates and therefore the Poincaré structure of the flavor vacuum is lacking.

Finally, the charge operators are $Q_{e/\mu} \equiv \alpha_{e/\mu}^1 \alpha_{e/\mu}^2 - \beta_{e/\mu}^1 \beta_{e/\mu}^2$. We have $e_{\mu} \langle 0(t)|Q_{e/\mu}|0(t)\rangle_{e, \mu} = 0$ and charge conservation is ensured at any time:
\langle \nu_e(t) | (Q_e + Q_\mu) | \nu_e(t) \rangle = 1$. The oscillation formula for the flavor charges then readily follows

\[ P_{\nu_e \rightarrow \nu_e}(k, t) = \\
1 - \sin^2(2\theta) \left[ |U_k|^2 \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_k|^2 \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right], \quad (9) \]

\[ P_{\nu_e \rightarrow \nu_\mu}(k, t) = \\
\sin^2(2\theta) \left[ |U_k|^2 \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_k|^2 \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right]. \quad (10) \]

This result is exact and includes the previous result of momentum dependent oscillation amplitude of refs.\textsuperscript{2}. Notice that the additional contribution to the usual oscillation formula, does oscillate with a frequency which is the sum of the frequencies of the mass components. In the relativistic limit \( k \gg \sqrt{m_1 m_2} \) the traditional oscillation formula is recovered.

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