Resonance and rapid decay of exponential sums of Fourier coefficients of a Maass form for $GL_m(\mathbb{Z})$

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Abstract

Let $f$ be a full-level cusp form for $GL_m(\mathbb{Z})$ with Fourier coefficients $A_f(n_1, \ldots, n_{m-1})$. In this paper an asymptotic expansion of Voronoi’s summation formula for $A_f(n_1, \ldots, n_{m-1})$ is established. As applications of this formula, a smoothly weighted average of $A_f(n_1, \ldots, n_{m-1})$ against $e^{|n|^\beta}$ is proved to be rapidly decayed when $0 < \beta < 1/m$. When $\beta = 1/m$ and $\alpha$ equals or approaches $\pm mq^{1/m}$ for a positive integer $q$, this smooth average has a main term of the size of $|A_f(1, \ldots, 1, q) + A_f(1, \ldots, 1, -q)|X^{1/(2m)+1/2}$, which is a manifestation of resonance of oscillation exhibited by the Fourier coefficients $A_f(n_1, \ldots, 1)$. Similar estimate is also proved for a sharp-cut sum.

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1. Introduction

Voronoi’s summation formula is an important technique in analytic number theory. Ivić \[1\] generated the original Voronoi formula to non-cuspidal forms as given by multiple divisor functions. For cuspidal representations of $GL_2(\mathbb{Z})$, a Voronoi-type summation formula was proved by Sarnak \[2\] for holomorphic cusp forms and by Kowalski-Michel-Vanderkam \[3\] for Maass forms. Voronoi’s summation formula for Maass forms for $SL_3(\mathbb{Z})$ was proved by Miller-Schmid \[4\] and Goldfeld-Li \[5\]. For $m \geq 4$, the Voronoi summation formula was proved by Miller-Schmid \[6\]. To state the formula, let $f$ be a full-level cusp form for $GL_m(\mathbb{Z})$ with Langlands’ parameters $\mu_f(j), j = 1, \ldots, m$, and Fourier coefficients $A_f(c_{m-2}, \ldots, c_1, n)$. Let $\psi \in C^\infty_c(\mathbb{R}^+), q$ a positive integer, and $h$ an integer coprime with $q$. Let $h\bar{h} \equiv 1(\text{mod } q)$. Then Voronoi summation formula as proved by Miller-Schmid \[6\] is the following
\[
\sum_{n \neq 0} A_f(c_{m-2}, c_{m-3}, \ldots, c_1, n) e\left( -\frac{nh}{q} \right) \psi(|n|) = q \sum_{d_1|q} \sum_{d_2|q} \cdots \sum_{d_{m-2}|q} \sum_{n \neq 0} A_f(n, d_{m-2}, \ldots, d_1) \frac{d_1^{n-i}}{d_{m-1-1}} \\
\times S(n, \tilde{h}; q, c, d) \Psi\left( \frac{|n|}{q^m} \prod_{i=1}^{m-2} \frac{d_i^{n-i}}{c_i^{m-1}} \right),
\]

(1.1)

where \( c = (c_1, \ldots, c_{m-2}) \), \( d = (d_1, \ldots, d_{m-2}) \), and

\[
S(n, \tilde{h}; q, c, d) = \sum_{x_1 \left( \mod \frac{c_1 \cdot c_{m-2}}{d_1} \right)}^{*} e\left( \frac{d_1 x_1 n}{q} \right) \sum_{x_2 \left( \mod \frac{c_1 \cdot c_2 \cdot d_2}{d_1 \cdot d_{m-2}} \right)}^{*} e\left( \frac{d_2 x_2 \tilde{x}_1}{c_1} \right) \cdots \\
\times \sum_{x_{m-2} \left( \mod \frac{c_1 \cdot c_{m-2}}{d_1 \cdot d_{m-2}} \right)}^{*} e\left( \frac{d_{m-2} x_{m-2} \tilde{x}_{m-3}}{d_1 \cdot d_{m-3}} + \frac{\tilde{h} \tilde{x}_{m-2}}{d_1 \cdot d_{m-2}} \right).
\]

(1.2)

The * in \( \sum_{t \left( \mod r \right)}^{*} \) indicates that \((t, r) = 1\). Here \( \Psi \) is an integral transform of \( \psi \) given by

\[
\Psi(x) = \frac{1}{2\pi i} \int_{\Re s = -\sigma} \tilde{\psi}(s)x^s \hat{F}(1 - s) \frac{F(s)}{F(s)} ds,
\]

(1.3)

where

\[
\tilde{\psi}(s) = \int_{0}^{\infty} \psi(x)x^s \frac{dx}{x}
\]

and

\[
F(s) = \pi^{-ms/2} \prod_{i=1}^{m} \Gamma\left( \frac{s - \mu_f(j)}{2} \right),
\]

(1.4)

and

\[
\hat{F}(s) = \pi^{-ms/2} \prod_{i=1}^{m} \Gamma\left( \frac{s - \tilde{\mu}_f(j)}{2} \right)
\]

(1.5)

with \( \{\mu_f(j)\}_{j=1,\ldots,m} = \{\mu_f(j)\}_{j=1,\ldots,m} \) being the Langlands’ parameters for the dual form \( \tilde{f} \) of \( f \).
A special case of (1.1) for even Maass forms for \( SL_m(\mathbb{Z}) \) was proved by Goldfeld-Li [5, 7, 8]:

\[
\sum_{n \neq 0} A_f(1, 1, \ldots, 1, n) e\left(\frac{nh}{q}\right) \psi(|n|) = q \sum_{d_1|q} \sum_{d_2} \cdots \sum_{d_m} \sum_{n \neq 0} A_f(n, d_{m-2}, \ldots, d_1) \frac{1}{d_1 \cdots d_{m-2}|n|} \\
\times KL(h, n; d, q) \Psi(\frac{n}{q^m} \prod_{i=1}^{m-2} d_i^{m-i}),
\]

(1.6)

where \( KL(h, n; d, q) \) is the Kloosterman sum

\[
KL(h, n; d, q) = \sum_{t_1 (mod \frac{d_1}{q})} e\left(\frac{ht_1}{d_1}\right) \sum_{t_2 (mod \frac{d_2}{d_1})} e\left(\frac{ht_1 t_2}{d_1 d_2}\right) \cdots \sum_{t_{m-2} (mod \frac{d_{m-2}}{d_1 \cdots d_{m-3}})} e\left(\frac{ht_1 t_2 \cdots t_{m-2}}{d_1 \cdots d_{m-2}}\right) \Psi(\frac{n}{q^m} \prod_{i=1}^{m-2} d_i^{m-i}).
\]

(1.7)

Note that (1.7) can be rewritten as (1.2). When \( q = 1 \), the formulas (1.1) and (1.6) become

\[
\sum_{n \neq 0} A_f(1, \ldots, 1, n) \psi(|n|) = \sum_{n \neq 0} A_f(n, 1, \ldots, 1) \frac{1}{|n|} \psi(|n|).
\]

(1.8)

Replacing \( f \) by its dual form \( \tilde{f} \) and noting that \( A_f(n, 1, \ldots, 1) = A_f(1, \ldots, 1, n) \), we have

\[
\sum_{n \neq 0} A_f(n, 1, \ldots, 1) \psi(|n|) = \sum_{n \neq 0} A_{\tilde{f}}(1, \ldots, 1, n) \frac{1}{|n|} \psi(|n|),
\]

(1.9)

where \( \tilde{\Psi}(x) \) is defined as \( \Psi(x) \) in (1.3) by replacing \( \mu_f(j) \) by \( \mu_{\tilde{f}}(j) \) in (1.4) and (1.5).

In applications, asymptotic behavior of \( \Psi(x) \) is often required. An asymptotic expansion for Voronoi’s summation formula were firstly obtained by Ivić [1] for multiple divisor functions. Similar asymptotic formulas were proved and used in subconvexity bounds for \( GL_2 \) \( L \)-functions by Sarnak [2] for holomorphic cusp forms and by Liu-Ye [9] and Lau-Liu-Ye [10] for Maass forms. For Maass forms for \( SL_3(\mathbb{Z}) \), an asymptotic Voronoi formula was proved by Li [11] and Ren-Ye [12] and applied to subconvexity problems for \( L \)-functions attached to a self-dual Maass form for \( SL_3(\mathbb{Z}) \) by Li [13].

In this paper we prove the following asymptotic expansion.

**Theorem 1.1.** Let \( f \) be a full-level cusp form for \( GL_m(\mathbb{Z}) \). Let \( m \geq 3 \) be an integer. Let \( \Psi(x) \) be as defined in (1.3) with \( \psi(y) = \phi(y/X) \), where \( \phi(x) \ll 1 \) is a fixed smooth function of compact support on \([a, b]\) with \( b > a > 0 \). Then for any \( x > 0 \), \( xX \gg 1 \) and \( r > m/2 \), we
have

\[ \Psi(x) = x \sum_{k=0}^r c_k \int_0^\infty (xy)^{1/(2m)-1/2-k/m} \psi(y) \times \left\{ i^{k+(m-1)/2} e\left(m(xy)^{1/m}\right) + (-i)^{k+(m-1)/2} e\left(-m(xy)^{1/m}\right) \right\} dy + O((xX)^{-r/m+1/2+\varepsilon}), \]

(1.10)

where \( c_k, k = 0, \ldots, r, \) are constants depending on \( m \) and \( \{ \mu_f(j) \}_{j=1}^m \) with \( c_0 = -1/\sqrt{m} \), and the implied constant depends at most on \( f, \phi, r, a, b \) and \( \varepsilon \).

We point out that if we replace \( f \) be \( \tilde{f} \) in (1.3), the formula (1.10) for \( \tilde{\Psi}(x) \) has the same leading term, i.e., \( \tilde{c}_0 = c_0 = -1/\sqrt{m} \).

As applications of Theorem 1.1, we consider sums of the Fourier coefficients \( A_f(n, 1, \cdots, 1) \) twisted with the exponential function \( e(\pm \alpha |n|^{\beta}) \). Our results are the following.

**Theorem 1.2.** Let \( f \) be a full-level cusp form for \( GL_m(\mathbb{Z}) \) and \( \phi(x) \) a \( C^\infty \) function on \((0, \infty)\) of compact support \([1, 2]\) with \( \phi^{(j)}(x) \ll 1 \) for \( j \geq 1 \). Let \( X > 1 \) and \( \alpha, \beta \geq 0 \).

(i) Suppose

\[ 2 \max\{1, 2^{\beta-1/m}\}(\alpha \beta)^m \leq X^{1-\beta m}. \]

Then the estimate

\[ \sum_{n \neq 0} A_f(n, 1, \cdots, 1) e(\pm \alpha |n|^{\beta}) \phi\left( \frac{|n|}{X} \right) \ll X^{-M} \]

(1.12)

holds for any \( M > 0 \), where the implied constant in (1.12) depends on \( m, f, \beta \) and \( M \).

(ii) Suppose

\[ 2 \max\{1, 2^{\beta-1/m}\}(\alpha \beta)^m > X^{1-\beta m}. \]

Then for \( \beta \neq 1/m \), we have

\[ \sum_{n \neq 0} A_f(n, 1, \cdots, 1) e(\pm \alpha |n|^{\beta}) \phi\left( \frac{|n|}{X} \right) \ll_{m,f,\beta} (\alpha X^\beta)^{m/2}. \]

For \( \beta = 1/m \), we have

\[ \sum_{n \neq 0} A_f(n, 1, \cdots, 1) e(\pm \alpha |n|^{\beta}) \phi\left( \frac{|n|}{X} \right) \ll_{m,f} (\alpha X^\beta)^{(m+1)/2}. \]
Moreover, for $X > \alpha^{m(m-1)/(1-m\varepsilon)}$ with $0 < \varepsilon < 1/m$,

$$
\sum_{n \neq 0} A_f(n, 1, \ldots, 1)e(\pm \alpha|n|^{1/m})\phi\left(\frac{|n|}{X}\right)
= m \{A_f(1, 1, \ldots, 1, n_\alpha) + A_f(1, 1, \ldots, 1, -n_\alpha)\} \sum_{k=0}^{r} \rho_\pm(k, m, \alpha, X)X^{1/(2m)+1/2-k/m}
+ O_{m,r,\varepsilon}(X^{-r/m+1/2+\varepsilon}),
$$

(1.15)

where

$$
\rho_\pm(k, m, \alpha, X) = (\mp i)^k \frac{(m-1)/2}{\tilde{c}_kn_\alpha^{1/(2m)-1/2-k/m}} \int_0^\infty t^{m/2-k-1/2} \phi(t)e\left((\alpha - mn_\alpha^{1/m})X^{1/m}\right) dt
$$

with $n_\alpha$ being the integer satisfying $(\alpha/m)^m - n_\alpha \in (-1/2, 1/2]$, $\tilde{c}_k$ being constants depending on $m$ and $f$.

We remark that the condition (1.11) holds for any fixed $\alpha$ and $\beta \in (0, 1/m)$ when $X$ is large enough in terms of $\alpha$ and $\beta$, and hence the rapid decay in (1.12) holds. For $\beta = 1/m$ this is the case when $0 \leq \alpha \leq m^{2^{-1/m}}$. For $\beta > 1/m$, (1.12) holds for $0 \leq \alpha \leq 2mX^{1/m-\beta}$. When $\alpha = 0$ this result was firstly obtained by Booker [14] via a different approach.

Note that the main term in (1.15) is negligible when $|(\alpha/m)^m - n_\alpha| > X^{\varepsilon-1/m}$ since the integral in $\rho_\pm(k, m, \alpha, X)$ is arbitrarily small by repeated partial integrating by parts. Consequently, (1.15) manifests resonance of Fourier coefficients of $f$ against $e(\pm \alpha n_\alpha^{1/m})$ when $\alpha$ approaches $mq^{1/m}$. In particular, when $\alpha = mq^{1/m}$ we obtain the following.

**Corollary 1.1.** Let $q$ be a positive integer and $0 < \varepsilon < 1/m$. Then for $X > (mq)^{(m-1)/(1-m\varepsilon)}$ we have

$$
\sum_{n \neq 0} A_f(n, 1, \ldots, 1)e(\pm m(q|n|)^{1/m})\phi\left(\frac{|n|}{X}\right)
= \{A_f(1, 1, \ldots, 1, q) + A_f(1, 1, \ldots, 1, -q)\} \sum_{k=0}^{r} \omega_\pm(k, m, q)X^{1/(2m)+1/2-k/m}
+ O_{m,r,\varepsilon}(X^{-r/m+1/2+\varepsilon}),
$$

where

$$
\omega_\pm(k, m, q) = (\mp i)^{k+(m-1)/2} \tilde{c}_kq^{1/(2m)-1/2-k/m} \int_0^\infty x^{1/(2m)-1/2-k/m} \phi(x) dx.
$$

The asymptotic behavior as described in Corollary 1.1 was proved for cusp forms for $SL_2(\mathbb{Z})$ in Iwaniec-Luo-Sarnak [15] and Ren-Ye [16], while for Maass forms for $SL_3(\mathbb{Z})$ it was proved in Ren-Ye [17]. For similar results concerning coefficients of $L$-functions for general Selberg class, see Kaczorowski and Perelli [18] [19].
We will also prove a sharp-cut version of Theorem 1.2.

**Theorem 1.3.** Let \( f \) be a full-level cusp form for \( \text{GL}_m(\mathbb{Z}) \). Let \( X > 1 \) and \( \alpha, \beta \geq 0 \).

(i) Assume that (1.11) holds. Then we have
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\pm \alpha |n|^\beta) \ll_{f, \beta, m} X^{1-1/m}. \tag{1.16}
\]

(ii) Assume that (1.13) holds. Then for \( \beta \neq 1/m \) we have
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\pm \alpha |n|^\beta) \ll_{f, \beta, m} (\alpha X^\beta)^m/2 + X^{1-1/m}. \tag{1.17}
\]

For \( \beta = 1/m \) we have
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\pm \alpha |n|^{1/m})
\ll \alpha^{1/2} X^{1/(2m)+1/2} + \alpha^{m-1/2} X^{1/2-1/(2m)} + X^{1-1/m}. \tag{1.18}
\]

**Theorem 1.4.** Assume the following bound toward the Ramanujan conjecture
\[ A_f(1, \cdots, 1, n) \ll n^\theta, \quad \text{for some } 0 < \theta < \frac{2}{m-1}. \]

(i) Suppose that the parameters \( \alpha, \beta, X \) satisfy (1.11). Then we have
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\alpha |n|^\beta) \ll_m X^{(m-1)(1+\theta)/(m+1)}. \tag{1.19}
\]

(ii) Suppose that the parameters \( \alpha, \beta, X \) satisfy (1.13). Then we have, for \( \beta \neq 1/m \),
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\alpha |n|^\beta) \ll (\alpha X^\beta)^m/2 + X^{(m-1)(1+\theta)/(m+1)}, \tag{1.20}
\]
and for \( \beta = 1/m \),
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\pm \alpha |n|^{1/m})
\begin{align*}
&= -\sqrt{m} X^{1/(2m)+1/2}(-i)^{(m-1)/2} I(m, \alpha, X) \frac{A_f(1, \cdots, 1, n_\alpha) + A_f(1, \cdots, 1, -n_\alpha)}{n_\alpha^{1/2-1/(2m)}} \\
&+ O(\alpha^{m-1/2} X^{1/2-1/(2m)}) + O(X^{(m-1)(1+\theta)/(m+1)}), \tag{1.21}
\end{align*}
\]
where \( n_\alpha \) is the integer satisfying \((\alpha/m)^m - n_\alpha \in (-1/2, 1/2] \) and
\[
I(m, \alpha, X) = \int_1^{2^{1/m}} t^{m/2-1/2} e((\alpha - mn_\alpha^{1/m}) X^1/m t) dt.
\]

Note that for fixed \( \alpha \) and large \( X \), (1.21) is an asymptotic formula for \( \theta < 1/3 \) when \( m = 3 \) and for \( \theta < 1/24 \) when \( m = 4 \). Recall that the best known results are \( \theta = 5/14 \) for \( m = 3 \)
and \( \theta = 9/22 \) for \( m = 4 \), by Kim and Sarnak [20][21] and Sarnak [22] (21). When \( m \geq 5 \), (1.21) implies the following bound

\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\alpha|n|^{1/m}) \lesssim \alpha^{1/2 + (\theta - 1/2)m} X^{1/(2m) + 1/2 + \alpha^{m-1/2} X^{1/2 - 1/(2m)} + X^{(m-1)(1+\theta)/(m+1)}}.
\]

We point out that when \( \theta < 1/m \), the bounds in (1.19)-(1.21) improves the bounds in (1.16)-(1.18), respectively.

We will prove Theorem 1.1 in Sections 2-4, and prove Theorems 1.2-1.4 in Sections 5-6.

2. STIRLING’S ASYMPTOTIC

Changing \( s \) to \( 1 - 2s \) and noting that \( \{\mu_f(j)\} = \{\bar{\mu}_f(j)\} \) for cusp form \( f \), we get

\[
\Psi(x) = i \pi^{-m/2} \int_{Re s = \sigma} \left( \pi^m x \right)^{-2s+1} \tilde{\psi}(-2s + 1) G(s) ds,
\]

where

\[
G(s) = \prod_{j=1}^{m} \frac{\Gamma(s - \frac{\mu_f(j)}{2})}{\Gamma(-s + \frac{1-\mu_f(j)}{2})}.
\]

Note that the \( \Gamma \)-functions in the numerator on the right side of (2.2) are analytic and nonzero for

\[
\sigma > \frac{1}{2} \max \{\text{Re} \bar{\mu}_f(1), \ldots, \text{Re} \bar{\mu}_f(m)\}.
\]

A bound due to Luo, Rudnick and Sarnak [23] asserts that

\[
|\text{Re} \mu_f(j)| \leq \frac{1}{2} - \frac{1}{m^2 + 1}, \quad j = 1, \ldots, m.
\]

Thus we are allowed to take any \( \sigma > 1/4 - 1/(2(m^2 + 1)) \) in (2.1) where the convergence of the integral is guaranteed by the rapidly decay of \( \tilde{\psi}(-2s + 1) \) with respect to \( t \). Let us consider \( s = \sigma + it \) with

\[
\sigma > \frac{1}{4} - \frac{1}{2(m^2 + 1)}.
\]

Let \( r > m/2 \) and define

\[
\sigma(r) = \frac{1}{4} + \frac{r}{2m} - \varepsilon,
\]

where \( \varepsilon > 0 \) is a small number. We will show that in the vertical strip

\[
L_r : |\text{Re } s - \sigma(r)| \leq \frac{\varepsilon}{2},
\]

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\]

where \( \varepsilon > 0 \) is a small number. We will show that in the vertical strip

\[
L_r : |\text{Re } s - \sigma(r)| \leq \frac{\varepsilon}{2},
\]
there holds
\[ G(s) = m^{-2ms+m/2} \cdot \frac{\Gamma\left(ms - \frac{m-1}{2}\right)}{\Gamma\left(-ms + \frac{1}{2}\right)} \cdot \left(1 + \sum_{j=1}^{r} \frac{h_j}{s^j} + E_r(s)\right), \] (2.6)
where \(h_j\)'s are constants depending on \(f\), \(E_r(s)\) is analytic in \(L_r\) and satisfies \(E_r(s) = O(|s|^{-r-1})\) as \(|s| \to \infty\).

By Stirling’s formula \[12\] \[24\], for \(|\text{Im} s| \geq 2|\beta|\) and \(|\text{Im} s| \gg |s|\) one has
\[
\log \Gamma(s + \beta) = \left(s + \beta - \frac{1}{2}\right) \log s - s + \log \sqrt{2\pi} + \sum_{j=1}^{r} \frac{d_j}{s^j} + O(|s|^{-r-1}),
\] (2.7)
where \(d_j\) are constants depending on \(\beta\), and the implied constant depends on \(\beta\) and \(r\). This in combination with the fact that \(\mu_f(1) + \cdots + \mu_f(m) = 0\) show that, for \(s\) satisfying
\[
|\text{Im} s| \gg |s| \quad \text{and} \quad |\text{Im} s| \geq t_0 = 2 + \max_{1 \leq j \leq m} \{|\mu_f(j)|\},
\] (2.8)
there holds
\[
\log G(s) = (ms - \frac{m}{2}) \log s - 2ms + ms \log(-s)
+ \sum_{j=1}^{r} \frac{f_j}{s^j} + O(|s|^{-r-1}),
\] (2.9)
where \(f_j\) are constants depending on \(m\) and \(\mu_f(j), j = 1, \cdots, m\). Similarly, by (2.7), for \(|\text{Im} s| \geq 1\), we have
\[
\log \frac{\Gamma\left(ms - \frac{m-1}{2}\right)}{\Gamma\left(-ms + \frac{1}{2}\right)} = \left(ms - \frac{m}{2}\right) \log ms - 2ms + ms \log(-ms)
+ \sum_{j=1}^{r} \frac{g_j}{s^j} + O(|s|^{-r-1}).
\] (2.10)
Comparing (2.9) with (2.10) we get, for \(s\) satisfying (2.8), that
\[
G(s) = m^{-2ms+m/2} \cdot \frac{\Gamma\left(ms - \frac{m-1}{2}\right)}{\Gamma\left(-ms + \frac{1}{2}\right)} \cdot \left(1 + \sum_{j=1}^{r} \frac{h_j}{s^j} + O(|s|^{-r-1})\right).
\] (2.11)

On the other hand, one can write
\[
G(s) = m^{-2ms+m/2} \cdot \frac{\Gamma\left(ms - \frac{m-1}{2}\right)}{\Gamma\left(-ms + \frac{1}{2}\right)} \cdot A(s),
\] (2.12)
where \(A(s)\) is analytic and non-zero for \(s = \sigma + it\) satisfying (2.4) and
\[
-ms + \frac{m-1}{2} \not\in \mathbb{N}, \quad ms - \frac{1}{2} \not\in \mathbb{N},
-\left(s - \frac{\mu_f(j)}{2}\right) \not\in \mathbb{N}, \quad s - \frac{1 - \mu_f(j)}{2} \not\in \mathbb{N}, \quad 1 \leq j \leq m.
\] (2.13)
Here we have to avoid poles and zeros of the quotients of \(\Gamma\)-functions on both sides of (2.12). Let \(\sigma(r)\) be defined by (2.5). We show that when \(\varepsilon > 0\) is small enough (2.13) is satisfied for \(s = \sigma + it\) with \(|\sigma - \sigma(r)| \leq \varepsilon/2\). Actually, let \(|a|\) denote the distance from \(a\) to the nearest integer. One can easily check that

\[-ms + (m-1)/2 \notin \mathbb{N}, ms - 1/2 \notin \mathbb{N}, \quad -(s - \overline{\sigma}_f(j))/2 \notin \mathbb{N}\]

when \(\varepsilon < 1/(8m)\) and \(|\sigma - \sigma(r)| \leq \varepsilon/2\). Moreover, one has

\[
\left\| \sigma(r) - \frac{1 - \text{Re} \mu_f(j)}{2} \right\| = \left\| \frac{r}{2m} - \frac{1}{4} + \frac{\text{Re} \mu_f(j)}{2} - \varepsilon \right\|. \tag{2.14}
\]

If \(m|r\), the right expression in (2.14) is \(\geq 1/2(m^2 + 1) - \varepsilon > \varepsilon/2\) when \(\varepsilon < 1/4(m^2 + 1)\). For \(m \nmid r\), write

\[
\delta_j = \left\| \frac{r}{2m} - \frac{1}{4} + \frac{\text{Re} \mu_f(j)}{2} \right\|.
\]

Note that when \(\delta_j = 0\), the right expression in (2.14) is \(\geq \varepsilon\) for any \(0 < \varepsilon < 1/2\). If \(\delta_j \neq 0\), the expression is \(> \varepsilon\) when \(\varepsilon < \delta_j/2\). Thus one can choose

\[
0 < \varepsilon < \min_{1 \leq j \leq m, \delta_j \neq 0} \left\{ \frac{1}{4(m^2 + 1)}, \frac{\delta_j}{2} \right\}
\]

so that (2.13) is satisfied for \(|\sigma - \sigma(r)| \leq \varepsilon/2\).

Now we consider the region

\[
D = \left\{ s = \sigma + it \mid |\sigma - \sigma(r)| \leq \frac{\varepsilon}{2}, \, |t| \leq |\sigma(r)| + 2t_0 \right\}.
\]

In this region one can express \(A(s)\) as

\[
1 + \sum_{j=1}^{r} h_j s^j + E_r(s),
\]

where \(E_r(s)\) is analytic in \(D\). Back to (2.12) and (2.11), we see that (2.6) holds in the vertical strip \(L_r = \{ \sigma + it : |\sigma - \sigma(r)| \leq \varepsilon/2 \}\).

Write \(u = \pi^m x\) and define

\[
\mathcal{H}_j = i\pi^{-m/2 - 1} \int_{Re s = \sigma(r)} \frac{\Gamma(ms - \frac{m-1}{2})}{s^{\frac{3}{2}} \Gamma(-ms + \frac{1}{2})} \times m^{-2ms + m/2} u^{-2s+1} \tilde{\psi}(-2s + 1) ds,
\]

\[
\mathcal{E}_r = i\pi^{-m/2 - 1} \int_{Re s = \sigma(r)} \frac{m^{-2ms + m/2} \Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} \times E_r(s) u^{-2s+1} \tilde{\psi}(-2s + 1) ds. \tag{2.16}
\]

Then we get

\[
\Psi(x) = \mathcal{H}_0 + \sum_{j=1}^{r} h_j \mathcal{H}_j + \mathcal{E}_r.
\]
We remark that the integral for $E_r$ is only valid in the vertical strip $L_r$, while the integral path for $H_j$ can be moved freely in the half plane $\text{Re } s > 1/2 - 1/(2m)$. In the following we will replace $\sigma(r)$ by $\sigma_1(r)$ in (2.15) where

\[ \sigma_1(r) = \frac{1}{2} - \frac{1}{2m} + \frac{r}{m} + \varepsilon. \] (2.17)

The estimate of $E_r$ is immediate. By Stirling’s formula, for $s = \sigma + it$ we have

\[ \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} \ll |s|^{2m\sigma - m/2}, \quad \text{as } |s| \to \infty. \]

When $r > m/2$, $\sigma = \sigma(r)$ as defined in (2.5) one has

\[ \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} E_r(s) \ll |s|^{2m\sigma(r) - m/2 - 1} \ll |s|^{-1-\varepsilon}. \]

Note that

\[ \tilde{\psi}(s) = \int_{aX}^{bX} \psi(y) y^\sigma y^s dy \ll X^\sigma \int_a^b |\phi(x)| x^{\sigma-1} dx \ll x X^\sigma. \] (2.18)

Hence

\[ E_r \ll u^{-2\sigma(r)+1} X^{-2\sigma(r)+1} \int_{\text{Re } s = \sigma(r)} |s|^{-1-\varepsilon} |ds| \]

\[ \ll (uX)^{-2\sigma(r)+1} \ll (xX)^{-r/m+1/2 + 2\varepsilon}. \]

This shows

\[ \Psi(x) = \mathcal{H}_0 + \sum_{j=1}^r h_j H_j + O((xX)^{-r/m+1/2+2\varepsilon}). \] (2.19)

To prove Theorem 1.1 it remains to compute $H_j$ for $j = 0, 1, \ldots, r$, this will be carried out in the following two sections according to $m$ is odd or even.

3. Proof of Theorem 1.1 when $m$ is odd

By definition we have

\[ \mathcal{H}_0 = i\pi^{-m/2 - 1} \int_{\text{Re } s = \sigma_1(r)} \frac{\Gamma(ms - \frac{m-1}{2})}{\Gamma(-ms + \frac{1}{2})} m^{-2ms + m/2} u^{-2s + 1} \tilde{\psi}(-2s + 1) ds. \]

Move the integral path to $\text{Re } s = -\infty$ and note that the only poles of the integrand are from $\Gamma(ms - (m - 1)/2)$ and hence are simple at $s = -n/m + (m - 1)/(2m)$ for $n = 0, 1, \ldots$ with
the residue \((-1)^n/(n!m)\). Thus

\[
\mathcal{H}_0 = -2\pi^{-m/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n + 1 - \frac{m}{2})} m^{2n-m/2} u^{(2n+1)/m} \psi \left( \frac{2n+1}{m} \right)
\]

\[
= -2\pi^{-m/2} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n + 1 - \frac{m}{2})} m^{2n-m/2} u(y)^{(2n+1)/m} \psi(y) dy.
\]

Recall the power series definition of the Bessel function

\[
J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + 1 + \nu)} \left( \frac{z}{2} \right)^{2n+\nu}, \quad (3.1)
\]

we get

\[
\mathcal{H}_0 = -2\pi^{-m/2} u \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n + 1 - \frac{m}{2})} \times (uy)^{1/m-1/2} (m(uy)^{1/m})^{2n-m/2} \psi(y) dy
\]

\[
= -2\pi^{-m/2} u \int_0^{\infty} (uy)^{1/m-1/2} J_{-m/2} (2m(uy)^{1/m}) \psi(y) dy. \quad (3.2)
\]

Note that if \(\nu = -q - 1/2\) and \(q \geq 1\), the half integral-order Bessel function is elementary. Applying (8.462.2) in [25], we have

\[
J_{-q-1/2}(z) = \frac{1}{\sqrt{2\pi}z} \sum_{j=0}^{q} \frac{(q+j)!}{j!(q-j)!} \left( \frac{z}{2\pi} \right)^j \times \left\{ e^{i(q+j)} \cos \left( \frac{z}{2\pi} \right) + e^{-i(q+j)} \sin \left( \frac{z}{2\pi} \right) \right\}. \quad (3.3)
\]

When \(m = 2\ell + 1\), we apply (3.3) with \(z = 2m(uy)^{1/m} = 2\pi m(xy)^{1/m}\), \(q = \ell\) and then substitute it back to (3.2) to get

\[
\mathcal{H}_0 = -2\pi^{m/2} x \int_0^{\infty} \pi^{m} xy \left( \frac{xy}{m} \right)^{1/m-1/2} \frac{1}{\sqrt{4\pi^2 m(xy)^{1/m}}} \times \sum_{j=0}^{\ell} \frac{(\ell+j)!}{\ell!(\ell-j)! \left( \frac{4\pi m(xy)^{1/m}}{2\pi} \right)^j}
\]

\[
\times \left\{ e^{i(q+j)} (m(xy)^{1/m}) + e^{-i(q+j)} (-m(xy)^{1/m}) \right\} \psi(y) dy.
\]

Denoting

\[
a_j^{(0)} = \frac{1}{\sqrt{m}} \cdot \frac{(\ell+j)!}{j!(\ell-j)!(4\pi m)^j}, \quad j = 0, 1, \ldots, \ell,
\]
we get

\[ \mathcal{H}_0 = x \sum_{j=0}^{\ell} a_j^{(0)} \int_0^\infty (xy)^{-(j+\ell)/m} \psi(y) \]
\[ \times \left\{ i^{j+\ell} e(m(xy)^{1/m}) + (-i)^{j+\ell} e(-m(xy)^{1/m}) \right\} dy. \]  
(3.4)

Now we turn to the estimate of \( \mathcal{H}_1 \). We have

\[ \mathcal{H}_1 = i \pi^{-m/2-1} \int_{\text{Re} s = \sigma_1(r)} \frac{\Gamma(ms - \ell)}{s \Gamma(-ms + \frac{1}{2})} \times m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s + 1) ds. \]

Using the formula \( \Gamma(s + 1) = s \Gamma(s) \) repeatedly we obtain

\[ \frac{\Gamma(ms - \ell)}{s} = m \left\{ \sum_{b=0}^{r-1} \frac{(\ell + b)!}{\ell!} (-1)^b \Gamma(ms - (\ell + b + 1)) \right\} \]
\[ + (-1)^r (\ell + 1) \cdots (\ell + r) \frac{\Gamma(ms - (\ell + r))}{s}. \]

This gives us

\[ \mathcal{H}_1 = i \pi^{-m/2-1} \left( \sum_{b=0}^{r-1} \frac{(\ell + b)!}{\ell!} (-1)^b m \mathcal{I}_{b+1} \right) \]
\[ + i (-1)^r \pi^{-m/2-1} (\ell + 1) \cdots (\ell + r) \mathcal{I}_r^*, \]  
(3.5)

where for \( d = 1, \ldots, r \)

\[ \mathcal{I}_d = \int_{\text{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + d))}{\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s + 1) ds, \]  
(3.6)

and

\[ \mathcal{I}_r^* = \int_{\text{Re} s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + r))}{s \Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \tilde{\psi}(-2s + 1) ds. \]  
(3.7)

Moving the integral contour in (3.6) to \( \text{Re}(s) = -\infty \) and picking up residues \((-1)^n/(mn!)\)

at \( s = (-n + \ell + d)/m \) for \( n = 0, 1, 2, \ldots \), we get

\[ m \mathcal{I}_d = 2 \pi i \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n - (\ell + d) + \frac{1}{2})} \]
\[ \times m^{2(n-\ell-d)+m/2} u^{2(n-\ell-d)/m+1} \tilde{\psi} \left( \frac{2(n - \ell - d)}{m} + 1 \right) \]
\[ = 2 \pi i \sum_{n=0}^\infty \frac{(-1)^n}{n! \Gamma(n - (\ell + d) + \frac{1}{2})} \]
\[ \times m^{2(n-\ell-d)+m/2} u^{2(n-\ell-d)/m+1} \int_0^\infty y^{2(n-\ell-d)/m} \psi(y) dy. \]
Collecting factors and using (3.1), we have

\[ mI_d = 2\pi i^{m/2+1-d}m^{1/2-d} \int_0^\infty (uy)^{1/2m-(\ell+d)/m} \times \sum_{n=0}^\infty \frac{(-1)^m}{n!}\Gamma(n-(\ell+d)+\frac{1}{2}) \left(m(uy)^{1/m}\right)^{2n-\ell-d-1/2} \psi(y)dy \]

\[ = 2\pi i^{m/2+1/2-(\ell+d)} \times \int_0^\infty (uy)^{1/2m-\ell/(d+1)} J_{\ell-\ell-1/2}(2m(uy)^{1/m}) \psi(y)dy. \]

Using (3.3) with \( z = 2m(uy)^{1/m} = 2\pi m(xy)^{1/m} \) and \( q = \ell + d \) and then following the computation leading to (3.4), we get

\[ mI_d = i\pi^{m/2+1-d}m^{1/2-d} \sum_{j=0}^{\ell+d} \frac{(\ell+d+j)!}{j!(\ell+d-j)!(4\pi m)^j} \int_0^\infty (xy)^{-(\ell+d+j)/m} \psi(y) \ times \ \left\{ e^{\ell+d+j} \left( m(xy)^{1/m} \right) + (-i)^{\ell+d+j} e^{-m(1/m)} \right\} \psi(y)dy. \]  

(3.8)

To estimate \( I_r^* \), we move the integral contour in (3.7) from \( \text{Re}(s) = \sigma_1(r) \) back to \( \text{Re}(s) = \sigma(r) = 1/4+r/(2m)-\varepsilon. \) Then the real part of \( ms-(\ell+r) \) moves from \( \varepsilon \) to \( 1/4-(\ell+r)/2-m\varepsilon. \) We pick up residues at \( s = (-n+\ell+r)/m \) for \( n = 0, \ldots, [(\ell+r)/2-1/4] \) to get

\[ I_r^* = 2\pi i^{m/2} \sum_{n=0}^{[(\ell+r)/2-1/4]} (-1)^n \frac{(\ell+r-n)n!\Gamma(n-\ell-r+\frac{1}{2})}{(\ell+r-n)\Gamma(n-\ell-r+\frac{1}{2})} \int_0^\infty (uy)^{1+m}\Gamma(ms-\ell+r) \times \frac{(ms-\ell+r)}{s^{\Gamma(-ms+\frac{1}{2})}} \psi(y)dy \]

\[ + \int_{\text{Re}s=\sigma(r)} \frac{1}{s^{\Gamma(-ms+\frac{1}{2})}} \psi(-2s+1)ds. \]  

(3.9)

Since \( \psi(y) = \phi(y/X) \) is supported on \([aX, bX] \), the first term on the right side above is

\[ \ll (uX)^{1+\frac{2(\ell+r)/2-1/4}{m}} \ll (X)^{1-(\ell+r)/m-1/(2m)} = (X)^{r/m+1/2}, \]

where the implied constant depends on \( \phi, a \) and \( b. \) By Stirling’s formula, for \( \text{Re}(s) = \sigma(r) \) one has

\[ \frac{\Gamma(ms-\ell+r)}{s^{\Gamma(-ms+\frac{1}{2})}} \ll |s|^{2m\sigma(r)-\ell-r-\frac{3}{2}} \ll |s|^{-1-\varepsilon}. \]

By (2.18) the last integral in (3.9) is

\[ \ll (uX)^{-2\sigma(r)+1} \ll (X)^{-r/m+1/2+\varepsilon}. \]

Therefore

\[ I_r^* \ll (uX)^{-2\sigma(r)+1} \ll (X)^{-r/m+1/2+\varepsilon}. \]  

(3.10)
This together with (3.8) and (3.5) show that

\[ \mathcal{H}_1 = -x m^{1/2} \sum_{b=0}^{r-1} (\pi m)^{-b-1} \frac{(\ell + b)!}{\ell!} (-1)^b \]

\[ \times \sum_{j=0}^{\ell+r} \frac{(\ell + b + 1 + j)!}{j!(\ell + b + 1 - j)!} \int_0^\infty (xy)^{-(\ell+b+1+j)/m} \]

\[ \times \left\{ \ell^{\ell+b+1+j} e(m(xy)^{1/m}) + (-i)^{\ell+b+1+j} e(-m(xy)^{1/m}) \right\} \psi(y) dy \]

\[ + \ O\left((xX)^{-r/m+1/2+\varepsilon}\right). \]

Collecting like terms, we get

\[ \mathcal{H}_1 = x \sum_{t=1}^{\ell+2r} a^{(1)}_t \int_0^\infty (xy)^{-(\ell+t)/m} \]

\[ \times \left\{ \ell^{\ell+t} e(m(xy)^{1/m}) + (-i)^{\ell+t} e(-m(xy)^{1/m}) \right\} \psi(y) dy \]

\[ + \ O\left((xX)^{-r/m+1/2+\varepsilon}\right), \]

where for \( t = 1, 2, \ldots, \ell + 2r \),

\[ a^{(1)}_t = -\frac{4\sqrt{m}}{(4\pi m)^t} \frac{(\ell + t)!}{\ell!} \sum_{\max\{0, \ell+t-1\} \leq b \leq \min(r-1,t-1)} \frac{(-4)^b (\ell + b)!}{(t - b - 1)! (\ell + 2b - t + 2)!}. \]

Note that when \( j \geq r + 1 \),

\[ x \int_0^\infty (xy)^{-(\ell+j)/m} \left\{ \ell^{\ell+j} e(m(xy)^{1/m}) + (-i)^{\ell+j} e(-m(xy)^{1/m}) \right\} \psi(y) dy \]

\[ \ll (xX)^{1-(\ell+r+1)/m} = (xX)^{-r/m+1/2-1/(2m)}. \]

Thus we finally obtain

\[ \mathcal{H}_1 = x \sum_{t=1}^r a^{(1)}_t \int_0^\infty (xy)^{-(\ell+t)/m} \]

\[ \times \left\{ \ell^{\ell+t} e(m(xy)^{1/m}) + (-i)^{\ell+t} e(-m(xy)^{1/m}) \right\} \psi(y) dy \]

\[ + \ O\left((xX)^{-r/m+1/2+\varepsilon}\right). \]  

(3.11)

To finish the proof of Theorem 1.1, we need to estimate \( \mathcal{H}_j \) for \( 2 \leq j \leq r \). This is similar to the case when \( j = 1 \) and so we will briefly describe the idea. By (2.15) we have

\[ \mathcal{H}_j = i \pi^{-m/2-1} \int_{\text{Res}=\sigma_1(r)} \frac{\Gamma(ms-\ell)}{s^{2} \Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \psi(-2s+1) ds. \]
By repeated use of the formula $\Gamma(s + 1) = s\Gamma(s)$, we have

$$\frac{\Gamma(ms - \ell)}{s^j} = m\left\{ \frac{\Gamma(ms - (\ell + 1))}{s^{j-1}} \right\} + \sum_{k=1}^{h-1} (\ell + 1) \cdots (\ell + k)(-1)^k \frac{\Gamma(ms - (\ell + k + 1))}{s^{j-1}} \right\} + \cdots \frac{\Gamma(ms - (\ell + h))}{s^j},$$

where $h = r - j + 1$. Applying this process repeatedly, we can finally decompose $\frac{\Gamma(ms - \ell)}{(s^j \Gamma(-ms + 1/2))}$ into finite sums of

$$\frac{\lambda \Gamma(ms - (\ell + p))}{\Gamma(-ms + \frac{1}{2})}, \quad 2 \leq p \leq r,$$

and

$$\frac{\mu \Gamma(ms - (\ell + p))}{s^q \Gamma(-ms + \frac{1}{2})}, \quad p + q = r + 1, \quad 2 \leq p \leq r, \quad 1 \leq q \leq r$$

with some constants $\lambda = \lambda_p, \mu = \mu_{p,q}$. So it remains to estimate integrals of the form

$$I_p = \int_{\text{Res}=\sigma_1(r)} \frac{\Gamma(ms - (\ell + p))}{\Gamma(-ms + \frac{1}{2})} m^{-2ms + m/2} u^{-2s + 1/2} \tilde{\psi}(-2s + 1) ds, \quad 2 \leq p \leq r,$$

and

$$I^* = \int_{\text{Res}=\sigma_1(r)} \frac{\Gamma(ms - (\ell + p))}{s^q \Gamma(-ms + \frac{1}{2})} m^{-2ms} u^{-2s} \tilde{\psi}(-2s + 1) ds$$

with $p + q = r + 1, \quad 2 \leq p \leq r, \quad 1 \leq q \leq r$.

The integral $I_p$ has been treated in (3.8). The integral $I^*$ can be estimated in a similar way as $I^*_r$, and it satisfies $I^* \ll (uX)^{-r/m + 1/2 + \varepsilon}$. So one finally obtains for $j \geq 2$,

$$H_j = x \sum_{t=j}^r a_t^{(j)} \int_0^\infty (xy)^{-(\ell+\ell)/m}$$

$$\times \left\{ i^{\ell+\ell} e(m(xy)^{1/m}) + (-i)^{\ell+\ell} e(-m(xy)^{1/m}) \right\} \tilde{\psi}(y) dy$$

$$+ O((uX)^{-r/m + 1/2 + \varepsilon}).$$

Collecting estimates in (3.4), (3.11) and (3.12), and substituting them into (2.19), we finishes the proof of Theorem 1.1 when $m$ is odd by writing $c_k = \sum_{j=0}^k a_k^{(j)}$. In particular we have $c_0 = a_0^{(0)} = -1/\sqrt{m}$.
By (3.2), when $m = 2l$ one has
\[ H_0 = -2\pi x \int_0^\infty (xy)^{1/m-1/2} \psi(y)J_{-\ell}(2m\pi(xy)^{1/m}) dy. \] (4.1)

Note that for positive integer $k$, $J_{-k}(z) = (-1)^k J_k(z)$. By (4.8) in [9],
\[ \begin{aligned}
J_k(z) &= \frac{1}{\sqrt{2\pi z}} e^{i(z-(2k+1)\pi/4)} \sum_{0 \leq j < 2L} \frac{i^j \Gamma(k + j + \frac{1}{2})}{j! \Gamma(k - j + \frac{1}{2}) (2z)^j} (2z)^{-j} \\
&\quad + \frac{1}{\sqrt{2\pi z}} e^{-i(z-(2k+1)\pi/4)} \sum_{0 \leq j < 2L} \frac{i^j \Gamma(k + j + \frac{1}{2})}{j! \Gamma(k - j + \frac{1}{2}) (2z)^j} (-2z)^{-j} \\
&\quad + O(|z|^{-2L-1/2}).
\end{aligned} \]

Since $e^{-i(2k+1)\pi/4} = (-i)^k i^{-1/2}$ and $e^{i(2k+1)\pi/4} = i^k (-i)^{-1/2}$ the above formula can be rewritten as
\[ \begin{aligned}
J_{-k}(z) &= \frac{1}{\sqrt{2\pi z}} \sum_{0 \leq j < 2L} \frac{\Gamma(k + j + \frac{1}{2})}{j! \Gamma(k - j + \frac{1}{2}) (2z)^j} \\
&\quad \times \left\{ i^{j+k-1/2} e\left(\frac{z}{2\pi}\right) + (-i)^{j+k-1/2} e\left(-\frac{z}{2\pi}\right) \right\} \\
&\quad + O(|z|^{-2L-1/2}). \quad (4.2)
\end{aligned} \]

Let $z = 2m\pi(xy)^{1/m}$ and $k = \ell$ in (4.2), we get
\[ \begin{aligned}
H_0 &= -\frac{x}{\sqrt{m}} \sum_{0 \leq j < 2L} \frac{\Gamma(\ell + j + \frac{1}{2})}{j! \Gamma(\ell - j + \frac{1}{2}) (4\pi m)^j} \int_0^\infty (xy)^{1/(2m)-(j+\ell)/m} \psi(y) dy \\
&\quad \times \left\{ i^{j+\ell-1/2} e(m(xy)^{1/m}) + (-i)^{j+\ell-1/2} e(-m(xy)^{1/m}) \right\} dy \\
&\quad + O((X)^{1/2-(2L)/m+1}/m).)
\end{aligned} \]

For any $r \geq 2$ and $L = \lceil r/2 \rceil + 1$, the error term above is $O((X)^{1/2-r/m})$, and this gives
\[ \begin{aligned}
H_0 &= x \sum_{j=0}^{2[r/2]+1} b_j^{(0)} \int_0^\infty (xy)^{1/(2m)-(j+\ell)/m} \psi(y) dy \\
&\quad \times \left\{ i^{j+\ell-1/2} e(m(xy)^{1/m}) + (-i)^{j+\ell-1/2} e(-m(xy)^{1/m}) \right\} dy \\
&\quad + O((X)^{1/2-r/m}),
\end{aligned} \]
where
\[ b_j^{(0)} = -\frac{1}{\sqrt{m}} \frac{\Gamma(\ell + j + \frac{1}{2})}{j! \Gamma(\ell - j + \frac{1}{2}) (4\pi m)^j}. \]
Now we turn to the estimate of $H_j$ for $j \geq 1$. By (2.15) we have

$$H_j = i\pi^{-\ell-1} \int \frac{\Gamma(ms - (\ell - \frac{1}{2}))}{s\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \psi(-2s+1) ds.$$  

We first consider $H_1$. Using the formula $\Gamma(s+1) = s\Gamma(s)$, one obtains

$$\frac{\Gamma(ms - (\ell - \frac{1}{2}))}{s} = m\Gamma(ms - (\ell + \frac{1}{2})) - (\ell + \frac{1}{2}) \frac{\Gamma(ms - (\ell + \frac{1}{2}))}{s}.$$  

Applying this formula to the quotient on its right side repeatedly, we get

$$\frac{\Gamma(ms - (\ell - \frac{1}{2}))}{s} = m\left\{\Gamma(ms - (\ell + \frac{1}{2}))\right\}$$

$$+ \sum_{b=1}^{r-1} (-1)^b (\ell + \frac{1}{2}) \cdots (\ell + \frac{2b-1}{2}) \Gamma(ms - (\ell + \frac{2b+1}{2}))\right\}$$

$$+ (-1)^r (\ell + \frac{1}{2}) \cdots (\ell + \frac{2r-1}{2}) \frac{\Gamma(ms - (\ell + \frac{2r-1}{2}))}{s}.$$  

Substituting this into the integral defining $H_1$, we have

$$H_1 = i\pi^{-\ell-1} \frac{\Gamma(ms - (\ell + \frac{1}{2}))}{\Gamma(\ell + \frac{1}{2})} \sum_{b=0}^{r-1} (-1)^b \Gamma(\ell + b + \frac{1}{2}) m\mathcal{J}_{b+1}$$

$$+ i\pi^{-\ell-1} (-1)^r (\ell + \frac{1}{2}) \cdots (\ell + r - \frac{1}{2}) \mathcal{J}_r,$$  

(4.3)

where for $d = 1, \ldots, r$

$$\mathcal{J}_d = \int_{\Re s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + d - \frac{1}{2}))}{\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \psi(-2s+1) ds,$$

and

$$\mathcal{J}_r^* = \int_{\Re s = \sigma_1(r)} \frac{\Gamma(ms - (\ell + r - \frac{1}{2}))}{s\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \psi(-2s+1) ds.$$
For $\mathcal{J}_d$, we move the integral line to $\Re s = -\infty$ and pick up residues $(-1)^n/(n!m)$ at $s = (-n + \ell + d - 1/2)/m$ with $n = 0, 1, 2, \ldots$. Consequently,

$$m\mathcal{J}_d = 2\pi i \sum_{n=0}^{\infty} \frac{(-1)^n m^{2n-\ell-d+m/2+1}}{n!\Gamma(n+1-\ell-d)}$$

\[ \times \quad u^{2n-\ell-d/m+1} \psi \left( \frac{2(n-\ell-d)}{m} + \frac{1}{m} + 1 \right) \]

$$= 2\pi i u m^{1-d} \int_0^\infty (uy)^{1-m-(\ell+d)/m}$$

\[ \times \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1-(\ell+d)/m)} (m(uy)^{1/m})^{2n-(\ell+d)} \psi(y)dy. \]

By the series definition of the Bessel-function in (3.1), we have

$$m\mathcal{J}_d = 2\pi i u m^{1-d} \int_0^\infty (uy)^{1-m-(\ell+d)/m} J_{-(\ell+d)}(2m(uy)^{1/m}) \psi(y)dy.$$

Applying the asymptotic expansion in (4.2) with $z = 2m(uy)^{1/m} = 2\pi m(xy)^{1/m}$, $k = \ell + d$, we get

$$m\mathcal{J}_d = i^{1+\ell-d} m^{1/2-d} \sum_{0 \leq j < 2L} \frac{\Gamma(\ell + d + j + \frac{1}{2})}{j!\Gamma(\ell + d - j + \frac{1}{2})(4\pi m)^j}$$

\[ \times \quad \int_0^\infty (xy)^{(1/(2m))-(\ell+d+j)/m} \]

\[ \times \quad \left\{ i^{j+\ell+d-1/2} e \left( m(xy)^{1/m} \right) + (-i)^{\ell+d-j-1/2} e \left( - m(xy)^{1/m} \right) \right\} \psi(y)dy + O \left( (xX)^{-2L/m+1/(2m)-d/m+1/2} \right). \]

Let $L = [(r-d)/2] + 1$. Since $L \geq (r-d)/2 + 1/2$, the error term above is $O((xX)^{-r/m+1/2})$, and this finishes the estimate for $m\mathcal{J}_d$.

To estimate $\mathcal{J}_r$, we move the integral contour from $\Re(s) = \sigma_1(r)$ in (2.17) to $\Re(s) = \sigma(r) = 1/4 + r/(2m) - \varepsilon$ in (2.5). Then $\Re(ms-(\ell+r-1/2))$ moves from $m\varepsilon$ to $-(\ell+r-1)/2 - m\varepsilon$. Picking up residues in this region at $ms - (\ell + d - 1/2) = -n$ for $0 \leq n \leq (\ell + r - 1)/2$, we get

$$\mathcal{J}_r = 2\pi i u \sum_{n=0}^{[(\ell+r-1)/2]} \frac{(-1)^n m^{2(n-\ell-r)+1}}{(\ell + r - n - \frac{1}{2})n!\Gamma(n-\ell-r+1)}$$

\[ \times \quad \int_0^\infty (uy)^{1/m+2(n-\ell-r)/m} \psi(y)dy \]

\[ + \quad \int_{\Re s = \sigma(r)} \frac{\Gamma(ms - (\ell + r - \frac{1}{2}))}{s\Gamma(-ms + \frac{1}{2})} m^{-2ms+m/2} u^{-2s+1} \psi(-2s + 1)ds. \quad (4.4) \]
By Stirling’s formula, for $\Re s = \sigma(r)$ one has
\[
\frac{\Gamma(ms - (\ell + r - \frac{1}{2}))}{s\Gamma(-ms + \frac{1}{2})} \ll |s|^{2m\sigma(r) - \ell - r - 1} \ll |s|^{-1-\varepsilon}.
\]

By (2.18) the last integral in (4.4) is $O((xX)^{-r/m+1/2+2\varepsilon})$. The first quantity on the right of (4.4) is
\[
\ll (uX)^{1+\frac{1}{m}} + \frac{2[(\ell+1)/2]-2\ell}{m} \ll (xX)^{-r/m+1/2}.
\]

Back to (4.3) and collecting our results on $mJ_d$, $J_\varepsilon$ and their error terms, we get
\[
\mathcal{H}_1 = -\frac{\sqrt{m}}{(4\pi m)^{t/2}} \sum_{b=0}^{r-1} (-1)^b \frac{\Gamma(\ell + b + \frac{1}{2})}{(\pi m)^b} \left( \Gamma(\ell + b + 1 + j + \frac{1}{2}) \int_0^\infty (xy)^{1/(2m)-(\ell+b+1+j)/m} \right)
\times \frac{\psi(y)dy}{j!\Gamma(\ell + b + 1 - j + \frac{1}{2})(4\pi m)^{j}}
\times \left\{ i^{\ell+b+j+1/2}e(m(xy)^{1/m}) + (-i)^{\ell+b+j+1/2}e(-m(xy)^{1/m}) \right\} \psi(y)dy
+ O((xX)^{-r/m+1/2+\varepsilon}).
\]

We write $t = b + j + 1$ and collect like terms with coefficients
\[
b_t^{(1)} = -\frac{4\sqrt{m}}{(4\pi m)^{t/2}} \cdot \frac{\Gamma(\ell + t + \frac{1}{2})}{\Gamma(\ell + \frac{1}{2})} \sum_{0 \leq b < \min(r,t)} \frac{(-4)^b \Gamma(\ell + b + \frac{1}{2})}{\Gamma(t - b)\Gamma(\ell + 2b - t + \frac{3}{2})}.
\]

Then we obtain
\[
\mathcal{H}_1 = x \sum_{t=1}^{r+1} b_t^{(1)} \int_0^\infty (xy)^{1/(2m)-(t+\ell)/m}
\times \left\{ i^{\ell+t-1/2}e(m(xy)^{1/m}) + (-i)^{\ell+t-1/2}e(-m(xy)^{1/m}) \right\} \psi(y)dy
+ O((xX)^{-r/m+1/2+\varepsilon}).
\]

One can estimates $\mathcal{H}_j \ (j \geq 2)$ in a similar way, and finally obtain
\[
\mathcal{H}_j = x \sum_{t=j}^{r+1} b_t^{(j)} \int_0^\infty (xy)^{1/(2m)-(t+\ell)/m}
\times \left\{ i^{\ell+t-1/2}e(m(xy)^{1/m}) + (-i)^{\ell+t-1/2}e(-m(xy)^{1/m}) \right\} \psi(y)dy
+ O((xX)^{-r/m+1/2+\varepsilon}).
\]
Substituting these estimates into (2.19) and putting the term with \( t = r + 1 \) into the error term we finish the proof of Theorem 1.1 for even \( m \) by letting \( c_k = \sum_{j=0}^{k} b_j^{(j)} \) where \( c_0 = b_0^{(0)} = -1 / \sqrt{m} \).

5. Proof of Theorem 1.2

By (1.9) with \( \psi(x) = \phi(x/X)e(\alpha x^\beta) \), we get

\[
\sum_{n \neq 0} A_f(n, 1, \ldots, 1)e(\alpha |n|^\beta)\phi\left(\frac{|n|}{X}\right) = \sum_{n \neq 0} A_f(1, \ldots, 1, n)\frac{\phi(|n|)}{|n|},
\]

where by Theorem 1.1, for \( r > m/2 \) one has

\[
\Phi(x) = x \sum_{k=0}^{r} \tilde{c}_k \sum_{\pm} (\pm i)^{k+(m-1)/2} \int_0^\infty (xy)^{1/(2m)-1/2-k/m} \phi\left(\frac{y}{X}\right)e(\alpha y^\beta \pm m(xy)^{1/m})dy
\]

\[+ O((xX)^{-r/m+1/2+\epsilon}).\]

Making change of variable \( y = t^m X \) and putting in (5.1), we get

\[
\sum_{n \neq 0} A_f(n, 1, \ldots, 1)e(\alpha |n|^\beta)\phi\left(\frac{|n|}{X}\right) = m \sum_{k=0}^{r} \tilde{c}_k X^{1/(2m)+1/2-k/m} \sum_{n=1}^\infty \frac{A_f(1, \ldots, 1, n) + A_f(1, \ldots, 1, -n)}{n^{1/2+k/m-1/(2m)}}
\]

\[\times \sum_{\pm} (\pm i)^{k+(m-1)/2} I_k(n; \pm)
\]

\[+ O\left(X^{-r/m+1/2+\epsilon}\sum_{n=1}^\infty \frac{|A_f(1, \ldots, 1, n)| + |A_f(1, \ldots, 1, -n)|}{n^{r/m+1/2-\epsilon}}\right),
\]

where

\[
I_k(n; \pm) = \int_0^\infty t^{m/2-k-1/2} \phi(t^m) e(\alpha X^\beta t^m \pm (nX)^{1/m} mt) dt.
\]

By Rankin-Selberg method for \( GL(n) \times GL(n) \) convolution [26] (Remark 12.1.8), one has

\[
\sum_{1 \leq |n| \leq X} |A_f(1, \ldots, 1, n)|^2 \ll X.
\]

Therefore the \( O \)-term in (5.2) is \( O(X^{-r/m+1/2+\epsilon}) \) for \( r > m/2 \), where the implied constant depends on \( r, m \) and \( \epsilon \). To estimate the integral in (5.3) we consider integral of the form

\[
\int_0^\infty h(t)e(f(t))dt.
\]
where \( h, f \in C^\infty(\mathbb{R}) \) and \( h \) is supported on \([a, b] \subseteq (0, \infty)\). Suppose \( f'(x) \neq 0 \) for \( x \in [a, b] \).

By repeated partial integrating by parts, one obtains for \( j \geq 0 \) that

\[
\int_0^\infty h(t)e(f(t))dt = \left(\frac{-1}{2\pi i}\right)^j \int_0^\infty h_j(t)e(f(t))dt,
\]

where \( h_0(t) = h(t) \) and

\[
h_j(t) = \left(\frac{h_{j-1}(t)}{f'(t)}\right)' := \frac{g_j(t)}{(f'(t))^{2j}}, \quad j \geq 1.
\]

Let \( h(t) = t^{m/2-k-1/2}\phi(t^m) \) and \( f(t) = \alpha X^\beta m^n + m(nX)^{1/m}t \) in (5.5) one easily obtains

\[
I_k(n;+) \ll_{m,j} (nX)^{-j/m}, \quad \text{for} \quad n \geq 1.
\]

Set \( j = r + 1 \). Then the contribution of the terms in \( \sum_+ \) to (5.2) is

\[
\ll X^{-1/(2m)+1/2-r/m} \sum_{n=1}^\infty \frac{|A_f(1, \cdots, 1, n)| + |A_f(1, \cdots, 1, -n)|}{n^{r/m+1/2+1/(2m)}} \ll_{r,m} X^{-1/(2m)+1/2-r/m}.
\]

Here we have used (5.4). To estimate the contribution of the terms concerning \( \sum_- \), we write

\[
n_0 = \frac{1}{2} \min\{1, 2^{3-1/m}\} (\alpha \beta X^\beta)^m X^{-1}, \quad (5.6)
n_1 = 2 \max\{1, 2^{3-1/m}\} (\alpha \beta X^\beta)^m X^{-1}. \quad (5.7)
\]

Then for \( n \not\in (n_0, n_1) \) one has \( I_k(n; -) \ll (nX)^{-j/m} \). Therefore the contribution of the terms with \( n \not\in (n_0, n_1) \) in \( \sum_- \) is \( O(X^{-1/(2m)+1/2-r/m}) \). This shows that

\[
\sum_{n \neq 0} A_f(n, 1, \cdots, 1)e(\alpha|n|^\beta)\phi\left(\frac{|n|}{X}\right)
= m \sum_{k=0}^r \tilde{c}_k(-i)^{k+(m-1)/2}X^{1/(2m)+1/2-k/m}
\times \sum_{n_0 < n < n_1} \frac{A_f(1, \cdots, 1, n) + A_f(1, \cdots, 1, -n)}{n^{1/2+k/m-1/(2m)}} I_k(n; -)
+ O_{r,m,\varepsilon}(X^{-r/m+1/2+\varepsilon}). \quad (5.8)
\]

If \( 2 \max\{1, 2^{3-1/m}\} (\alpha \beta)^m \leq X^{-1-\beta m} \), then \( n_1 \leq 1 \). Hence the main term in (5.8) disappears and the estimate

\[
\sum_{n \neq 0} A_f(n, 1, \cdots, 1)e(\alpha|n|^\beta)\phi\left(\frac{|n|}{X}\right) \ll_{m,\beta,r} X^{-r/m+1/2+\varepsilon} \ll_{m,\beta,M} X^{-M}
\]

holds for any \( M > 0 \) by taking \( r \) sufficiently large in terms of \( M \). This proves Theorem 1.2 (i).
If $2 \max \{1, 2^{\beta-1/m}\} (\alpha \beta)^m > X^{1-\beta m}$, then $n_1 > 1$. We distinguish two cases according to $\beta \neq 1/m$ or not. For $\beta \neq 1/m$ we have
\[
(aX^\beta t^{m\beta} - (nX)^{1/m} mt)^n = \alpha(m\beta)(m\beta - 1)X^\beta t^{m\beta - 2} \gg_{m, \beta} \alpha X^\beta.
\]
By the second derivative test one has $I_k(n; -) \ll_{\beta, m} (\alpha X^\beta)^{-1/2}$. Thus the main term in (5.8) is
\[
\ll_{m, \beta} X^{1/(2m) + 1/2}(\alpha X^\beta)^{-1/2} \sum_{n_0 < n < n_1} \left| A_f(1, \ldots, 1, n) + A_f(1, \ldots, 1, -n) \right| \frac{n^{1/2 - 1/(2m)}}{n^{1/2 - 1/(2m)}}
\]
\[
\ll_{m, \beta} (n_1 X)^{1/(2m) + 1/2}(\alpha X^\beta)^{-1/2} \ll_{m, \beta} (\alpha X^\beta)^{m/2}.
\]
Choosing $r = [(m + 1)/2]$ we get
\[
\sum_{n \neq 0} A_f(n, 1, \ldots, 1) e(\alpha n^\beta) \phi \left( \frac{|n|}{X} \right) \ll_{m, \beta} (\alpha X^\beta)^{m/2}.
\]
For $\beta = 1/m$, we use the obvious estimate $I_k(n; -) \ll 1$ in (5.8) to get
\[
\sum_{n \neq 0} A_f(n, 1, \ldots, 1) e(\alpha n^\beta) \phi \left( \frac{|n|}{X} \right) \ll (n_1 X)^{1/2 + 1/(2m)} \ll_{m} (\alpha X^\beta)^{(m+1)/2}.
\]
This proves (1.14).
Moreover, when $\beta = 1/m$, one has $I = (n_0, n_1) = (\alpha/m)^m/2, 2(\alpha/m)^m)$ with $2(\alpha/m)^m \geq 1$, and
\[
I_k(n; -) = \int_0^\infty t^{m/2 - k - 1/2} \phi(t^m) e((\alpha - mn^{1/m}) X^{1/m}) dt.
\]
Since $(\alpha/m)^m > 1/2$, there is an unique integer $n_\alpha \geq 1$ such that
\[
(\alpha/m)^m = n_\alpha + \lambda, \quad -1/2 < \lambda \leq 1/2.
\]
For $n \in I$, $n \neq n_\alpha$, one has $|n^{1/m} - \alpha m| \gg_{m} |n - n_\alpha| \alpha^{1-m}$. By repeated partial integrating by parts we get
\[
I_k(n; -) \ll_{m, j} \frac{1}{(|n - n_\alpha| \alpha^{1-m} X^{1/m})^j}, \quad j \geq 0.
\]
Putting in (5.8) and applying (5.4), the main terms except the term with $n = n_\alpha$ produce the contribution which is, for $j \geq 1,$
\[
\ll_{m} X^{1/(2m) + 1/2}(\alpha^{m-1} X^{-1/m})^{j} \sum_{n_0 < n < n_1 \neq n_\alpha} \left| A_f(1, \ldots, 1, n) + A_f(1, \ldots, 1, -n) \right| \frac{n^{1/2 - 1/(2m)}}{n^{1/2 - 1/(2m)}} \frac{1}{|n - n_\alpha|^j}
\]
\[
\ll_{m} X^{1/(2m) + 1/2}(\alpha^{m-1} X^{-1/m})^{j} n_1^{1/(2m)} \ll_{m} X^{1/(2m) + 1/2}(\alpha^{m-1} X^{-1/m})^{j} \alpha^{1/2}.
\]
(5.9)
Here we have used (5.4). Let $0 < \varepsilon < 1/m$. If $X > \alpha^{m(m-1)/(1-m\varepsilon)}$, one has $\alpha^{m-1} X^{-1/m} < X^{-\varepsilon}$. Thus the last expression in (5.9) is $\ll X^{-r/m + 1/2 + \varepsilon}$ by taking $j$ sufficiently large in
terms of \( r \). This proves
\[
\sum_{n \neq 0} A_f(n, 1, \cdots, 1)e(\alpha|n|^\beta)\phi\left(\frac{|n|}{X}\right)
\]
\[
= m \{ A_f(1, \cdots, 1, n_0) + A_f(1, \cdots, 1, -n_0)\} \sum_{k=0}^{r} \rho_+(k, m, \alpha, X)X^{1/(2m)+1/2-k/m}
\]
\[
+ O_{r,m,\varepsilon}(X^{-r/m+1/2+\varepsilon})
\]  
(5.10)

with
\[
\rho_+(k, m, \alpha, X) = \tilde{c}_k(-i)^{k+(m-1)/2} \frac{I_k(n; -)}{n_\alpha^{1/2+k/m-1/(2m)}}.
\]

In particular, suppose \((\alpha/m)^m = q\) is an integer, that is \(\alpha = mq^{1/m}\). Then \(n_\alpha = q\) and
\[
I_k(n; -) = \int_0^\infty t^{m/2-k-1/2} \phi(t^m) = \frac{1}{m} \int_0^\infty x^{1/(2m)-1/2-k/m} \phi(x) dx.
\]

This finishes the proof of Theorem 1.2 and Corollary 1.1 with the exponential function being \(e(\alpha|n|^\beta)\). Proof for the case of \(e(-\alpha|n|^\beta)\) is analogous.

6. PROOF OF THEOREMS 1.3 AND 1.4

Let \(\Delta > 1\) and \(\phi : \mathbb{R}^+ \to [0, 1]\) be a \(C^\infty\) function supported on \([1 - \Delta^{-1}, 2 + \Delta^{-1}]\) such that \(\phi(x) \equiv 1\) for \(x \in [1, 2]\) and satisfies
\[
\phi^{(j)}(x) \ll \Delta^j, \quad \text{for any integer} \quad j \geq 0.
\]  
(6.1)

Then by (5.4) and Cauchy’s inequality we get
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1)e(\alpha|n|^\beta)
\]
\[
= \sum_{n \neq 0} A_f(n, 1, \cdots, 1)e(\alpha|n|^\beta)\phi\left(\frac{|n|}{X}\right) + O(X\Delta^{-1/2}).
\]  
(6.2)

By (5.2) and applying (5.4) again we have
\[
\sum_{n \neq 0} A_f(n, 1, \cdots, 1)e(\alpha|n|^\beta)\phi\left(\frac{|n|}{X}\right)
\]
\[
= m \sum_{k=0}^{r} \tilde{c}_k X^{1/(2m)+1/2-k/m} \sum_{n > 0} A_f(1, \cdots, 1, n) + A_f(1, \cdots, 1, -n)
\]
\[
\times \sum_{\pm} (\pm i)^{k+(m-1)/2} I_k(n; \pm) + O_m(1),
\]  
(6.3)

where \(r = [(m + 1)/2]\) and \(I_k(n; \pm)\) is defined as in (5.3). By (5.5) and (6.1), for \(j \geq 1\) we have
\[
I_k(n; +) \ll_{m,k,j} (nX)^{-j/m} \Delta^{-1}, \quad \text{for} \quad n \geq 1.
\]
Set \( j = r \) for \( n > H = \Delta^m X^{-1} \) and \( j = 1 \) for \( n \leq H \). The contribution of \( I(n; +) \) to (6.3) is

\[
\ll m X^{1/2 - 1/(2m)} \sum_{n \leq H} \left| A_f(1, \cdots, 1, n) \right| + \left| A_f(1, \cdots, 1, -n) \right|\n\]

\[
+ m X^{1/(2m) + 1/2 - r/m} \Delta^{-1} \sum_{n > H} \left| A_f(1, \cdots, 1, n) \right| + \left| A_f(1, \cdots, 1, -n) \right|\n\]

\[
\ll m (X H)^{1/2 - 1/(2m)} + m (X H)^{1/(2m) + 1/2 - r/m} \Delta^{-1}
\]

\[
\ll m \Delta^{(m-1)/2}.
\]

Next, let \( n_1 \) be given by (5.7). Then for \( j \geq 1 \),

\[
I_k(n; -) \ll_{m,k,j} (n X)^{-j/m} \Delta^{j-1}, \quad \text{for} \quad n \geq n_1.
\]

Applying (6.4) with \( j = 1 \) for \( n_1 \leq n \leq H = \Delta^m X^{-1} \) and \( j = r \) for \( n > H \), then the contribution of \( I_k(n; -) \) with \( n \geq n_1 \) to (6.3) is \( O(\Delta^{(m-1)/2}) \). This together with (6.2) shows that

\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1) e(\alpha |n|^\beta)
\]

\[
= m \sum_{k=0}^r \hat{c}_k X^{1/(2m) + 1/2 - k/m} \sum_{1 \leq n < n_1} \frac{A_f(1, \cdots, 1, n) + A_f(1, \cdots, 1, -n)}{n^{1/2 + k/m - 1/(2m)}}
\]

\[
\times \sum_{\pm} \pm^{k+(m-1)/2} I_k(n; \pm) + O(\Delta^{(m-1)/2}) + O(X \Delta^{-1/2}).
\]

Suppose that the parameters \( \alpha, \beta, X \) satisfy (1.11). Then \( n_1 < 1 \) and the main term above disappears. Setting \( \Delta = X^{2/m} \) we get

\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1) e(\alpha |n|^\beta) \ll_{m} \Delta^{(m-1)/2} + \Delta X^{-1/2} \ll_{m} X^{1-1/m}.
\]

Suppose that the parameters \( \alpha, \beta, X \) satisfy (1.13), then \( n_1 > 1 \). Now we have

\[
I_k(n; -) = \int_1^{2^{1/m}} t^{n/2-k-1/2} e(\alpha X^\beta t^m - (n X)^{1/m} m t) dt + O(\Delta^{-1}).
\]

For \( \beta \neq 1/m \), the second derivative test shows that

\[
I_k(n; -) \ll_{m, \beta, k} (\alpha X^\beta)^{-1/2} + \Delta^{-1}.
\]

Thus the main term of (6.5) is

\[
\ll X^{1/2 + 1/(2m)} \left( \Delta^{-1} + (\alpha X^\beta)^{-1/2} \right) \sum_{1 \leq n \leq n_1} \frac{|A_f(1, \cdots, 1, n)| + |A_f(1, \cdots, 1, -n)|}{n^{1/2 - 1/(2m)}}
\]

\[
\ll (X n_1)^{1/(2m) + 1/2} \left( \Delta^{-1} + (\alpha X^\beta)^{-1/2} \right)
\]

\[
\ll (\alpha X^\beta)^{(m+1)/2} \Delta^{-1} + (\alpha X^\beta)^{m/2}.
\]
Setting $\Delta = \max\{\alpha X^\beta, X^{2/m}\}$, we yield
\[\sum_{X < |n| \leq 2X} A_f(n, 1, \ldots, 1)e(\alpha|n|^\beta) \ll (\alpha X^\beta)^{m/2} + X^{1-1/m}.
\]

For $\beta = 1/m$, we let $n_\alpha \geq 1$ be the integer such that $(\alpha/m)^m - n_\alpha \in (-1/2, 1/2]$. By (6.6) for $n \neq n_\alpha$,
\[I_k(n; -) \ll m \frac{1}{|n - n_\alpha|^{1-\theta}} + O(\Delta^{-1}).
\]

Choosing $\Delta = \max\{\alpha X^\beta, X^{2/m}\}$, the contribution of the terms with $n \neq n_\alpha$ in (6.5) is
\[\ll X^{1/2-1/(2m)} \alpha^{(m-1)} \sum_{1 \leq n \leq n_1} \frac{|A_f(1, \ldots, 1, n)| + |A_f(1, \ldots, 1, -n)|}{n^{1/2-1/(2m)}} \cdot \frac{1}{|n - n_\alpha|} + X^{1/(2m)+1/2} \Delta^{-1} \sum_{1 \leq n \leq n_1} \frac{|A_f(1, \ldots, 1, n)| + |A_f(1, \ldots, 1, -n)|}{n^{1/2-1/(2m)}}
\[\ll \alpha^{m-1/2} X^{1/2-1/(2m)} + (n_1X)^{1/2+1/(2m)} \Delta^{-1}
\[\ll \alpha^{m-1/2} X^{1/2-1/(2m)}.
\]

Thus we get
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \ldots, 1)e(\alpha|n|^\beta) = m\tilde{c}_0 X^{1/(2m)+1/2} A_f(1, \ldots, 1, n_\alpha) + A_f(1, \ldots, 1, -n_\alpha) (-i)^{(m-1)/2} I_0(n_\alpha; -) + O_m(\alpha^{m-1/2} X^{1/2-1/(2m)}) + O_m(X^{1-1/m}).
\]

By (5.4) the above fraction is $O(\alpha^{1/2})$. Therefore the main term in (6.7) is $O(\alpha^{1/2} X^{1/(2m)+1/2})$. This proves (1.18) and hence finishes the proof of Theorem 1.3.

Now we assume the following
\[A_f(1, \ldots, 1, n) \ll n^\theta, \quad \text{for some} \quad 0 < \theta < \frac{2}{m-1}.
\]

Then the error term $O(X\Delta^{-1/2})$ in (6.2) and (6.5) becomes $O(X^{1+\theta}\Delta^{-1})$. Suppose that the parameters $\alpha, \beta, X$ satisfy (1.11), then we can take $\Delta = X^{(1+\theta)/(m+1)}$ and get
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \ldots, 1)e(\alpha|n|^\beta) \ll X^{(m-1)(1+\theta)/(m+1)}.
\]

Suppose the parameters $\alpha, \beta, X$ satisfy (1.13), then we can set $\Delta = \max\{\alpha X^\beta, X^{2(1+\theta)/(m+1)}\}$ to yield, for $\beta \neq 1/m$,
\[\sum_{X < |n| \leq 2X} A_f(n, 1, \ldots, 1)e(\alpha|n|^\beta) \ll (\alpha X^\beta)^{m/2} + X^{(m-1)(1+\theta)/(m+1)},
\]

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and for $\beta = 1/m$,
\[
\sum_{X < |n| \leq 2X} A_f(n, 1, \cdots, 1) e(\alpha |n|^\beta)
= m\tilde{c}_0 X^{1/(2m)+1/2} \frac{A_f(1, \cdots, 1, n_\alpha) + A_f(1, \cdots, 1, -n_\alpha)}{n_\alpha^{1/2-1/(2m)}} (-i)^{(m-1)/2} I_0(n_\alpha; -)
+ O_m(\alpha^{m-1/2} X^{1/2-1/(2m)}) + O_m(X^{(m-1)(1+\theta)/(m+1)}).
\]
(6.8)
Note that $I_0(n_\alpha; -) = I(m, \alpha, X) + O(\Delta^{-1})$, where
\[
I(m, \alpha, X) = \int_1^{2^{1/m}} t^{m/2-1/2} e((\alpha - mn_\alpha^{1/m})X^{1/m} t) dt.
\]
Thus the main term in (6.8) can be rewritten as
\[
m\tilde{c}_0 X^{1/(2m)+1/2} \frac{A_f(1, \cdots, 1, n_\alpha) + A_f(1, \cdots, 1, -n_\alpha)}{n_\alpha^{1/2-1/(2m)}} (-i)^{(m-1)/2} I(m, \alpha, X).
\]
This finishes the proof of Theorem 1.4.

7. Conclusion and discussion

Two important features of smoothly weighted sums of Fourier coefficients of a Maass form for $GL_m(\mathbb{Z})$ against $e(\alpha n^\beta)$ have been discovered. They are rapid decay and resonance for various combinations of $\alpha$ and $\beta$. These features capture the vibration behavior of the Fourier coefficients of a Maass form. It is interesting to see whether these two features can be used to characterize Fourier coefficients of Maass forms. On the other hand, when $\beta$ is large, our methods failed to derive a non-trivial bound for the smoothly weighted sum. These are subjects of our subsequent research.

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