The Contribution of Electron-Positron Pair Production to the Vacuum Energy

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Abstract

The vacuum, defined as the state where no particles can be observed, is interpreted here to imply that the lifetime of the e-p pair should be equal to the Planck time, $t_P$. Concerning the title’s subject, a perfect theory would require that the true vacuum expectation value of the operator associated with pair production, $S$, be compatible with the normalization of the true vacuum, $|0'>$. At present, a calculation of the vacuum energy based on Feynman diagrams reveals a serious difficulty: if only second order terms of the $S$-matrix are retained, and because there are no external lines, it follows that the space-time integrations over the coordinates $x_2, x_1$, required to calculate $<0|S|0>$, give rise to two identical delta functions. Therefore the amplitude is proportional to the integration volume, $L^4$, and, as a result, the square of the amplitude defies any physically meaningful interpretation. One is faced here with two evils: modify the interaction Lagrangian so that the amplitude becomes proportional to $L^2$; abstain from any calculation. It is felt that the first one is the lesser evil. If the square of the amplitude is proportional to $L^4$, this one can be interpreted as being the number, $N$, of events (pairs created), in the volume of integration. In the calculations for $N$ it was assumed that the integral over momentums (rescaled to be dimensionless) was of the order unity, and that processes with small virtual photon-energy are predominant. The pairs’ contribution to the vacuum density is then given by $\rho V = 2mN t_P/V T$, i.e., by the mass of the particles multiplied both, by the number of events per unit volume and time, and by the pairs’ life time. For the value of $\rho V$ it is found that $\rho V \approx 10^{-31}$ g cm$^{-3}$, in surprisingly good agreement with observations (in this type of calculations, powers of ten are epsilons).

1. Introduction

As a consequence of the commutation relations of creation and annihilation operators, $CR$, the vacuum expectation value, $VEV$, of the Hamiltonian of non interacting bosons- or fermions- fields doesn’t vanish, being positive and negative, respectively (c.f. for example Dolgov et al. 1990, p. 112). It is usually argued that this outcome is related with Heisenberg’s uncertainty principle. But this principle wouldn’t demand a negative $VEV$ for fermions. In fact, this result should rather be an incentive to remain open minded about the interpretation of this type of zero point energy. If the central issue is the fulfilment of the uncertainty principle, it is conceivable that the non zero value of the $VEV$ is simply an indication that physical processes (to be distinguished from merely $CR$), leading to a non zero vacuum energy (as for example, particle pair production), are present.
in the concerned theory. Incomparably more serious are the predictions of the vacuum energy generated by symmetry breaking, c.f. Zee, 2010. As Zee writes, "particle physics is built on a series of spontaneous symmetry breaking" each one contributing to the vacuum energy and resulting in a gigantic discrepancy between theoretical expectations and observational reality. Conceivably a solution to this problem will have to wait until the development of an "ultimate" quantum theory playing a role for the microscopic world comparable to the one played by General Relativity for the macroscopic world.

2. Calculation of the S matrix.

In this paper, Mandl and Shaw’s notation will be used, and citations, as number of equations, figures..., will refer to their text, unless explicitly stated. The Feynman diagram for pair creation, at $x_2$, and annihilation, at $x_1$ is displayed in Fig 7.10, p.110 of their text book. In the calculations, c.g.s units will be used, and in the Appendix, the values of some physical constants in these units are displayed. To second order the S matrix relevant to pair creation and annihilation is given by,

$$ S = 2\pi i \alpha \int d^4x_1 d^4x_2 S_F(x_2 - x_1) \gamma^\alpha S_F(x_1 - x_2) \gamma^\beta D_{\alpha\beta}^F(x_1 - x_2) \quad (1) $$

where $\alpha$ is the fine structure constant and $S_F(x), D_{\alpha\beta}^F(x)$ are the Feynman’s, fermion and photon propagators respectively. Their integral representation is (c.f. Eqs (4.63), (5.27)),

$$ S_F(x) = \frac{\hbar}{(2\pi \hbar)^4} \int d^4p \ e^{-ipx/\hbar} \frac{\gamma^\alpha p_i + mc}{p^2 - (mc)^2 + \i \epsilon} \quad (2) $$

$$ D_{\alpha\beta}^F(x) = -\frac{g_{\alpha\beta}}{(2\pi)^4} \int d^4k \ \frac{e^{-ikx}}{k^2 + \i \epsilon} \quad (3) $$

With the help of these equations one obtains,

$$ <0| S |0> = -2\pi i \alpha \frac{\hbar^2}{(2\pi \hbar)^8} \frac{1}{(2\pi)^4} \int d^4x_1 d^4x_2 d^4p \ d^4p' \ d^4k \ \frac{\gamma^\alpha p_i + mc}{p^2 + (mc)^2 + \i \epsilon} \frac{\gamma^\beta p'_i + mc}{p'^2 + (mc)^2 + \i \epsilon} \times $$

$$ \times \ e^ {ix_1(p/h-h'/h+k)} e^{-ix_2(p'/h-h+k)} \quad (4) $$

The integration over the coordinates $x_1, x_2$ gives rise to two delta functions that are identical, resulting in the amplitude being proportional to $L^4$ which precludes any physically meaningful interpretation of the square of the amplitude (proportional to $L^8$). Here, $L^4 = cVT$, where $V$ and $T$ are respectively, the volume and time of integration. As stated in the Abstract a perfect theory would require that the energy input by the pairs be consistent with the true vacuum normalization. In the framework of Feynman diagrams, and confined to work with the bare vacuum normalized to, $<0|0> = 1$, we would like to interpret $|<0| S |0>|^2$ as the number of events (pair created) in the volume of integration, $L^4$. The amplitude must then be proportional to $L^2$. To achieve this goal, we have no other option left than to modify the interaction Lagrangian. Essentially we are assuming something like, $<0| S_0 |0> \sim <0| S |0>$, where $|0>$, and $S_0$ are respectively the true vacuum, and the conventional S matrix. $S$ stands here for the modified S-matrix, defined below.
3. The Vacuum Energy.

Because in the calculations for the amplitude, the interaction Lagrangian is used twice, we are led to multiply this one, by a factor proportional to $1/L$, which results in the amplitude being proportional to $L^2$. The dimensions of the Lagrangian must be left unchanged, however. This constraint fully determines the multiplicative factor $M$, which can be no other than $M = \left(\frac{\hbar}{mc}\right) / L$, a dimensionless quantity indeed. Here $m$ is the mass of the electron, and $c$, the velocity of light.

In Eq.(4), the presence of the delta function $\frac{1}{(2\pi)^4} \int d^4x e^{i(x_1 \cdot p - x_1 ' \cdot p' - k_\cdot x_2 - k_\cdot x_3 - k_\cdot x_4)}$ allows the integration over $k$ to be performed. Furthermore it is straightforward to show that $(Tr$ means the trace of the quantity in brackets),

$$Tr\{\gamma^\mu p_\mu + mc\gamma^\alpha \gamma^\beta g_{\alpha\beta}\} = -16(p^\mu p'_\mu - (mc)^2),$$

(5)
clearly, a physically very appealing result. Finally the variables $p$ and $p'$ are transformed as follows $p \to pmc$, $p' \to p' mc$, which guaranties that the momentums $p$ and $p'$ are dimensionless.

For the amplitude, calculated with the conventional interaction Lagrangian multiplied by $M$, it is then found,

$$<0 | S | 0> = \frac{16i\alpha}{(2\pi)^4} L^2 \frac{(mc)^2}{(h)^2} \mathcal{I}$$

(6)

$$\mathcal{I} = \int d^4p d^4p' \frac{p^\mu p'_\mu - 1}{(p^2 + 1 + i\epsilon)(p'^2 + 1 + i\epsilon)((p - p')^2 + i\epsilon)}$$

(7)
The amplitude must be dimensionless: in Eq.(6), $\alpha$ is associated with the factor $e^2$, in the interaction Lagrangian, and the presence of $(mc/h)^2$ is a consequence of the interaction Lagrangian being proportional to $L^2$.

We assume that the integral, $\mathcal{I}$, is of order unity. Then,

$$|<0 | S | 0>|^2 = \frac{(16\alpha)^2}{(2\pi)^4} L^4 \frac{(mc)^4}{(h)^4}$$

(8)

specifies the total number of events in the volume of integration, $L^4$. Therefore, the number of events per unit volume and unit time is given by,

$$\mathcal{N} = \frac{(16\alpha)^2}{(2\pi)^4} \epsilon \frac{(mc)^4}{(h)^4} = 1.23 \times 10^{39} \text{cm}^{-3}\text{s}^{-1}$$

(9)

The contribution of particle-pair creation to the vacuum density is equal to the pairs mass multiplied by $\mathcal{N}$ and by the lifetime of the pairs,

$$\rho_V = 2m\mathcal{N}t_P = 1.2 \times 10^{-31} \text{g cm}^{-3}$$

(10)
The observations (Riess et al. 1998, Peerlmutter et al. 1999) show that, $\Omega_\Lambda = \rho_\Lambda / \rho_{crit} \sim 0.7$, with $\rho_{crit} = 3H_0^2/8\pi = 1.88 \times 10^{-29}\text{h}^2\text{g cm}^{-3}$ (c.f. Hartle, 2003, Eq.18.32), here $H_0$ is the Hubble constant, and $h \sim 0.72$. Then,

$$\rho_\Lambda' = 6.81 \times 10^{-30} \text{g cm}^{-3}.$$  

(11)
The calculated value for the vacuum energy is in amazingly good agreement with the observations.
To summarize: The Interaction Lagrangian is modified to enable the amplitude squared, to be interpreted as the number of events occurring in the volume of integration. Dimensional considerations reveal that the number of events per unit volume and unit time must be proportional to $\alpha^2 c (mc/\hbar)^4$. The proportionality-component is estimated from Feynman diagrams calculated with the modified Lagrangian. It is very remarkable indeed that such apparently unrelated factors as: the electron mass; the contributions from Feynman diagrams; the Planck time!, could contribute to assign to $\rho_V$ a value in such a good approximation with the observations.

4. Appendix

$\hbar = 1.05 \times 10^{-27} \text{g cm}^2 \text{s}^{-1}$, $m = 9.11 \times 10^{-28} \text{g}$, $mc/\hbar = 2.59 \times 10^{10} \text{cm}^{-1}$

$\alpha = e^2/4\pi\hbar c = 1/137.04$, $t_P = 5.39 \times 10^{-44} \text{s}$

5. References.

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