Coloring count cones of planar graphs

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Abstract
For a plane near-triangulation $G$ with the outer face bounded by a cycle $C$, let $n_G^*$ denote the function that to each 4-coloring $\psi$ of $C$ assigns the number of ways $\psi$ extends to a 4-coloring of $G$. The Block-count reducibility argument (which has been developed in connection with attempted proofs of the Four Color Theorem) is equivalent to the statement that the function $n_G^*$ belongs to a certain cone in the space of all functions from 4-colorings of $C$ to real numbers. We investigate the properties of this cone for $|C| = 5$, formulate a conjecture strengthening the Four Color Theorem, and present evidence supporting this conjecture.

KEYWORDS
coloring count cone, four color theorem, graph coloring, planar graphs

1 INTRODUCTION

By the Four Color Theorem [1,2,5], every planar graph is 4-colorable. Nevertheless, many natural follow-up questions regarding 4-colorability of planar graphs are wide open. Even very basic precoloring extension questions, such as the one given in the following problem, are unresolved (a near-triangulation is a connected plane graph in which all faces except for the outer one have length three).

Problem 1. Does there exist a polynomial-time algorithm which, given a near-triangulation $G$ with the outer face bounded by a 4-cycle $C$ and a 4-coloring $\psi$ of $C$, correctly decides whether $\psi$ extends to a 4-coloring of $G$?
Note that there exist infinitely many near-triangulations $G$ with the outer face bounded by a 4-cycle such that not every precoloring of $C$ extends to a 4-coloring of $G$; and we do not have any good guess at how the near-triangulations with this property could be described.

Nevertheless, we do have some information about the precoloring extension properties of plane near-triangulations. For a plane near-triangulation $G$ with the outer face bounded by a cycle $C$, let $n^*_G$ denote the function that to each 4-coloring $\psi$ of $C$ assigns the number of ways $\psi$ extends to a 4-coloring of $G$ if and only if $n^*_G(\psi) \neq 0$. Suppose $C = v_1v_2v_3v_4$ is a 4-cycle and $\psi_1, \psi_2,$ and $\psi_3$ are its 4-colorings such that $\psi_i(v_j) = j$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}, \psi_1(v_3) = \psi_3(v_3) = 1, \psi_2(v_3) = 3, \psi_1(v_4) = \psi_2(v_4) = 2$, and $\psi_3(v_4) = 4$; see Figure 1. A standard Kempe chain argument shows that if $n^*_G(\psi_1) \neq 0$, then $n^*_G(\psi_2) \neq 0$ or $n^*_G(\psi_3) \neq 0$.

Actually, much more information can be obtained along these lines, using the idea of Block-count reducibility [3,4] developed in connection with the attempts to prove the Four Color Theorem: Certain inequalities between linear combinations of $n^*_G(\psi_1), n^*_G(\psi_2),$ and $n^*_G(\psi_3)$ are satisfied for all near-triangulations $G$, or equivalently, the vector $(n^*_G(\psi_1), n^*_G(\psi_2), n^*_G(\psi_3))$ is contained in a certain cone in $\mathbb{R}^3$. The main goal of this note is to present and motivate a conjecture regarding this cone in the case of near-triangulations with the outer face bounded by a 5-cycle; this conjecture strengthens the Four Color Theorem. We also provide evidence supporting this conjecture.

2 DEFINITIONS

To describe the cone we alluded to in Section 1, we need a number of definitions, which we introduce in this section. It is easier to state the idea in the dual setting of 3-edge-colorings of cubic plane graphs, which is well known to be equivalent to 4-coloring of plane triangulations [6].

Some graphs in this paper may have parallel edges or loops. We call two parallel edges a double edge and three parallel edges a triple edge.

2.1 Near-cubic graphs and their edge-colorings

Let $G$ be a connected graph and let $v$ be a vertex of $G$. A half-edge is $(e, u)$, where $e$ is an edge and $u$ is one of its endpoints. If $e = uv$, when we say $u$ is incident with $(e, u)$ but it is not incident with $(e, v)$. We consider each edge $e = uv$ of $G$ as consisting of two half-edges $(e, u)$

**Figure 1** Precolorings $\psi_1, \psi_2,$ and $\psi_3$ of a 4-cycle
and \((e, v)\) even if \(e\) is a loop. Let \(v\) be a bijection between the half-edges incident with \(v\) and \([0, ..., \deg(v) - 1]\) (so, if \(v\) is incident with a loop, each half of the loop is assigned a different number by \(\nu\)). If all vertices of \(G\) other than \(v\) have degree three, we say that \(\tilde{G} = (G, v, \nu)\) is a near-cubic graph. We say that \(\tilde{G}\) is a plane near-cubic graph if \(G\) is a plane graph and the half-edges incident with \(v\) are drawn around it in the clockwise cyclic order \(\nu^{-1}(0), ..., \nu^{-1}(\deg(v) - 1)\). We define \(d(\tilde{G}) = \deg(v)\).

A 3-edge-coloring of \(\tilde{G}\) is an assignment of colors 1, 2, and 3 to edges of \(G\) such that any two edges incident with a common vertex other than \(v\) have different colors. For an integer \(d \geq 3\), a function \(\psi : [0, ..., d - 1] \rightarrow \{1, 2, 3\}\) is a \(d\)-precoloring if \(|\psi^{-1}(1)| \equiv |\psi^{-1}(2)| \equiv |\psi^{-1}(3)| \equiv d \pmod{2}\). This parity condition is necessary, see Observation 2. We say that a 3-edge-coloring \(\varphi\) of \(\tilde{G}\) extends a \(d(\tilde{G})\)-precoloring \(\psi\) if for any edge \(e\) incident with \(v\) and a half-edge \(h\) of \(e\) incident with \(v\), we have \(\varphi(e) = \psi(h)\).

**Observation 2.** For an integer \(d \geq 2\), if a function \(\psi : [0, ..., d - 1] \rightarrow \{1, 2, 3\}\) does not satisfy \(|\psi^{-1}(1)| \equiv |\psi^{-1}(2)| \equiv |\psi^{-1}(3)| \equiv d \pmod{2}\), then there is no \(\tilde{G}\) and 3-edge-coloring \(\varphi\) of \(\tilde{G}\) such that \(\varphi\) extends \(\psi\).

**Proof:** Let \(\varphi\) be a 3-edge-coloring of \(\tilde{G}\) extending \(\psi\). Let \(n = |V(\tilde{G})|\). Since \(\tilde{G}\) is cubic except for one vertex of degree \(d\), the handshaking lemma gives \(|E(\tilde{G})| = (3(n - 1) + d)/2\). For \(i \in \{1, 2, 3\}\), then number of edges colored by \(|\varphi^{-1}(i)| = ((n - 1) + |\psi^{-1}(i)|)/2\). Therefore the parities of \(d, |\psi^{-1}(1)|, |\psi^{-1}(2)|,\) and \(|\psi^{-1}(3)|\) are the same.

Let \(n_\varphi(\psi)\) denote the number of 3-edge-colorings of \(\tilde{G}\) which extend \(\psi\). Via the theory of nowhere-zero flows [7], it is easy to establish the following correspondence between 4-colorings of near-triangulations and 3-edge-colorings in their duals. Recall \(n^*_\varphi(\psi)\) denotes the number of 4-colorings of \(G\) which extend \(\psi\).

**Observation 3.** Let \(\tilde{G} = (G, v, \nu)\) be a plane near-cubic graph, and let \(G^*\) be the dual of \(G\) drawn so that the outer face of \(G^*\) corresponds to \(v\). Suppose the outer face of \(G^*\) is bounded by a cycle \(C\). Then there exists a mapping \(f\) from 4-colorings of \(C\) to \(d(\tilde{G})\)-precolorings such that

- \(f\) maps exactly four distinct 4-colorings of \(C\) to each \(d(\tilde{G})\)-precoloring, and
- every 4-coloring \(\psi\) of \(C\) satisfies \(n^*_\varphi(\psi) = n_\varphi(f(\psi))\).

Given two near-cubic graphs \(\tilde{G}_1 = (G_1, v_1, \nu_1)\) and \(\tilde{G}_2 = (G_2, v_2, \nu_2)\) with \(\deg(v_1) = \deg(v_2)\), let \(\tilde{G}_1 \oplus \tilde{G}_2\) denote the graph obtained from \(G_1\) and \(G_2\) by, for \(0 \leq i \leq \deg(v_1) - 1\), removing the half-edges \(\nu_1^{-1}(i)\) and \(\nu_2^{-1}(i)\) and connecting the other halves of the edges. Note that \(\tilde{G}_1 \oplus \tilde{G}_2\) is a cubic graph, and if \(\tilde{G}_1\) and \(\tilde{G}_2\) are plane near-cubic graphs, then \(\tilde{G}_1 \oplus \tilde{G}_2\) is a cubic planar graph. Observe that the number of 3-edge-colorings of \(\tilde{G}_1 \oplus \tilde{G}_2\) is

\[
\sum_{\psi} n_{\tilde{G}_1}(\psi)n_{\tilde{G}_2}(\psi),
\]

(1)
where the sum goes over all $\deg(v_i)$-precolorings $\psi$. For any integer $n \geq 3$, let $\tilde{C}_n$ denote the plane near-cubic graph $(W_n, v, v)$, where $W_n$ is the wheel with the central vertex $v$ adjacent to all vertices of an $n$-cycle; see Figure 2.

2.2 Signatures and Kempe chains

The following definition of $d$-signature is will be used to capture the possible parities of 2-edge-colored cycles containing $v$ in a 3-edge-coloring of $\tilde{G} = (G, v, v)$ distinguished by the half-edges contained in the cycles. In particular, parity will be $s \in \{-1, 1\}$ and the pair of half-edges will be $m$. For an integer $d \geq 2$, a $d$-signature is a set $S$ of pairs $(m, s)$, where $m$ is an unordered pair of integers in $\{0, \ldots, d-1\}$ and $s \in \{-1, 1\}$, satisfying the following conditions:

(i) for any distinct $(m_1, s_1), (m_2, s_2) \in S$ we have $m_1 \cap m_2 = \emptyset$, and

(ii) $S$ does not contain elements $(\{a, b\}, s_1)$ and $(\{c, d\}, s_2)$ such that $a < c < b < d$.

A $d$-precoloring $\psi$ is compatible in (distinct) colors $i, j \in \{1, 2, 3\}$ with a $d$-signature $S$ if

- $\psi^{-1}(\{i, j\}) = \bigcup_{(m, s) \in S} m$, and
- for each $(\{a_1, a_2\}, s) \in S, \psi(a_1) = \psi(a_2)$ holds if and only if $s = -1$.

Now, consider a 3-edge-coloring $\varphi$ of a near-cubic graph $\tilde{G} = (G, v, v)$. Each vertex other than $v$ is incident with edges of all three colors. Hence, for any distinct $i, j \in \{1, 2, 3\}$, the subgraph $G_{ij}$ of $G$ consisting of edges of colors $i$ or $j$ is a union of pairwise edge-disjoint cycles, vertex-disjoint except for possible intersections in $v$. An $ij$-Kempe chain of $\varphi$ is a cycle $C$ in $G_{ij}$ containing $v$; the sign $\sigma(C)$ of the $ij$-Kempe chain $C$ is 1 if the length of $C$ is even and $-1$ if the length of $C$ is odd. If $h_1$ and $h_2$ are the half-edges in $C$ incident with $v$, we let $\mu(C) = \{v(h_1), v(h_2)\}$. The $ij$-Kempe chain signature $\sigma_{ij}(\varphi)$ of $\varphi$ is defined as

$$\{\langle \mu(C) , \sigma(C) \rangle : C \text{ is an } ij\text{-Kempe chain of } \varphi\}.$$ 

Note that if $\tilde{G}$ is plane, then the $ij$-Kempe chains do not cross and the $ij$-Kempe chain signature of $\varphi$ satisfies the condition (ii); and thus $\sigma_{ij}(\varphi)$ is a $d(\tilde{G})$-signature.
3  |  COLORING COUNT CONES

Let $\hat{G} = (G, v, \nu)$ be a plane near-cubic graph and let $\psi$ be a $d(\hat{G})$-precoloring. Suppose that $\psi$ is compatible (in colors $i, j \in \{1, 2, 3\}$) with a $d(\hat{G})$-signature $S$. We define $n_{\hat{G},S}(\psi)$ as the number of 3-edge-colorings $\varphi$ of $\hat{G}$ extending $\psi$ such that $\sigma_j(\varphi) = S$. Note that swapping the colors $i$ and $j$ on any set of $ij$-Kempe chains of $\varphi$ results in another 3-edge-coloring with the same $ij$-Kempe chain signature. Furthermore, clearly for any permutation $\pi$ of colors, we have $n_{\hat{G},S}(\psi \circ \pi) = n_{\hat{G},S}(\psi)$. This establishes bijections implying the following.

**Observation 4.** Let $\hat{G}$ be a plane near-cubic graph and let $S$ be a $d(\hat{G})$-signature. Any $d(\hat{G})$-precolorings $\psi_1$ and $\psi_2$ compatible with $S$ satisfy

$$n_{\hat{G},S}(\psi_1) = n_{\hat{G},S}(\psi_2).$$

Hence, we can define an integer $n_{\hat{G},S}$ to be equal to $n_{\hat{G},S}(\psi)$ for an arbitrarily chosen $d(\hat{G})$-precoloring $\psi$ compatible with $S$.

Let $d \geq 2$ be an integer and let $i, j \in \{1, 2, 3\}$ be distinct colors. For a $d$-precoloring $\psi$, let us define $S_{\psi,ij}$ as the set of $d$-signatures compatible with $\psi$ in colors $ij$. Since every 3-edge-coloring of $\hat{G}$ has an $ij$-Kempe chain signature, we have

$$n_{\hat{G}}(\psi) = \sum_{S \in S_{\psi,ij}} n_{\hat{G},S}(\psi) = \sum_{S \in S_{\psi,ij}} n_{\hat{G},S}. \quad (2)$$

Let $P_d$ denote the set of all $d$-precolorings and $S_d$ the set of all $d$-signatures. We will work in the vector spaces $\mathbb{R}^{P_d}$ and $\mathbb{R}^{S_d}$ with coordinates corresponding to the $d$-precolorings and to the $d$-signatures, respectively. For each integer $d \geq 2$, the **coloring count cone** $B_d$ is the set of all $x \in \mathbb{R}^{P_d}$ such that

- $x(\psi) \geq 0$ for every $d$-precoloring $\psi$, and
- there exists $y \in \mathbb{R}^{S_d}$ such that
  - $y(S) \geq 0$ for every $d$-signature $S$, and
  - $x(\psi) = \sum_{S \in S_{\psi,ij}} y(S)$ for every $d$-precoloring $\psi$ and distinct colors $i, j \in \{1, 2, 3\}$.

Note that $B_d$ is indeed a cone, that is, an unbounded polytope closed under linear combinations with nonnegative coefficients. By (2), the vector of precoloring extension counts for any plane near-cubic graph belongs to the corresponding coloring count cone.

**Theorem 5.** For each plane near-cubic graph $\hat{G}$, we have

$$n_{\hat{G}} \in B_d(\hat{G}).$$

Each cone is uniquely determined as the set of nonnegative linear combinations of its rays. For $d \in \{2, 3, 4, 5\}$, the rays of $B_d$ are easy to enumerate by hand or using polytope-manipulation software such as Sage Math or the Parma Polyhedra Library (a program doing so for $d = 5$ can be found at http://lidicky.name/pub/4cone/). For a near-cubic graph $\hat{G}$ such that $n_{\hat{G}}$ is not the zero function, let $\text{ray}(\hat{G})$ denote the set of all nonnegative multiples of $n_{\hat{G}}$. Graphs $\tilde{R}_{2,1}, \ldots, \tilde{R}_{5,12}$ used in the following lemma are depicted in Figure 3.
FIGURE 3  Graphs $\tilde{R}_{2,1}, \ldots, \tilde{R}_{5,12}$. The dashed circle intersects the half-edges incident with the vertex $v$, which is not depicted for the sake of clarity; the values of $v$ are written at the respective half-edges.
Lemma 6. Referring to graphs in Figure 3:

- the cone $B_2$ has exactly one ray equal to $\text{ray}(\bar{R}_{2,1})$;
- the cone $B_3$ has exactly one ray equal to $\text{ray}(\bar{R}_{3,1})$;
- the cone $B_4$ has exactly four rays equal to $\text{ray}(\bar{R}_{4,1}), \ldots, \text{ray}(\bar{R}_{4,4})$; and
- the cone $B_5$ has exactly 12 rays equal to $\text{ray}(\bar{R}_{5,1}), \ldots, \text{ray}(\bar{R}_{5,12})$.

Let us remark that $B_6$ has 208 rays; the direct method we employ is too slow to enumerate all rays for $d \geq 7$ on current workstations.

4 | THE CONE $B_5$ AND THE CONJECTURE

Note that while $\bar{R}_{5,1}, \ldots, \bar{R}_{5,11}$ are planes, $\bar{R}_{5,12}$ is not. Indeed, the following holds.

Lemma 7. The following claims are equivalent.

(a) Every planar cubic 2-edge-connected graph is 3-edge-colorable.
(b) For every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$, if $n_{\tilde{G}} \in \text{ray}(\bar{R}_{5,12})$, then $n_{\tilde{G}}$ is the zero function.

Proof. Let us first prove that (a) implies (b). Consider a plane near-cubic graph $\tilde{G} = (G, v, \nu)$ such that $n_{\tilde{G}} \in \text{ray}(\bar{R}_{5,12})$, and thus for some constant $c \geq 0$, we have $n_{\tilde{G}}(\psi) = c \cdot n_{\bar{R}_{5,12}}(\psi)$ for every 5-precoloring $\psi$. Observe that $n_{\bar{R}_{5,12}}(\psi) n_{\bar{C}_5}(\psi) = 0$ for every 5-precoloring $\psi$ (since $\bar{R}_{5,12} \oplus \bar{C}_5$ is the Petersen graph, which is not 3-edge-colorable; see Figure 4), and thus the number of 3-edge-colorings of $\tilde{G} \oplus \bar{C}_5$ using (1) is

$$\sum_{\psi} n_{\tilde{G}} n_{\bar{C}_5}(\psi) = c \sum_{\psi} n_{\bar{R}_{5,12}} n_{\bar{C}_5}(\psi) = 0.$$ 

Hence, the planar cubic graph $\tilde{G} \oplus \bar{C}_5$ is not 3-edge-colorable. By (a), $\tilde{G} \oplus \bar{C}_5$ has a bridge, and thus $G$ has a bridge. But then a standard parity argument implies that $\tilde{G}$ has no 3-edge-coloring, and thus $n_{\tilde{G}}$ is the zero function.

Next, let us prove that (b) implies (a). Suppose for a contradiction that (b) holds, but there exists a plane cubic 2-edge-connected graph that is not 3-edge-colorable, and let $H$.
be one with the smallest number of vertices. By Euler’s formula and possible parallel edges, $H$ has a face $f$ of length $2 \leq d \leq 5$; hence, we can write $H = \tilde{G} \oplus \tilde{C}_d$ for a plane near-cubic graph $\tilde{G}$. By Theorem 5, we have $n_{\tilde{G}} \in B_d$, and by Lemma 6, there exist nonnegative real numbers $c_i$ such that

$$n_{\tilde{G}} = \sum c_i n_{\tilde{R}_{d,i}}.$$

Observe there exists a plane near-cubic graph $\tilde{P}$ with $d - 1$ vertices such that $\tilde{G} \oplus \tilde{P}$ is 2-edge-connected. By the minimality of $H$, $\tilde{G} \oplus \tilde{P}$ is 3-edge-colorable, and in particular $n_{\tilde{G}}$ is not the zero function. By (b), $n_{\tilde{G}}$ is not a positive multiple of $n_{\tilde{R}_{5,12}}$, and thus there exists an index $k \leq 11$ such that $c_k > 0$. Observe that $\tilde{R}_{d,k} \oplus \tilde{C}_d$ is 3-edge-colorable, and thus there exists a $d$-precoloring $\psi_0$ such that $n_{\tilde{R}_{d,k}}(\psi_0)n_{\tilde{C}_d}(\psi_0) > 0$. However, then the number of 3-edge-colorings of $H$ is

$$\sum_{\psi} n_{\tilde{G}}(\psi)n_{\tilde{C}_d}(\psi) \geq c_k \sum_{\psi} n_{\tilde{R}_{d,k}}(\psi)n_{\tilde{C}_d}(\psi) = c_k n_{\tilde{R}_{d,k}}(\psi_0)n_{\tilde{C}_d}(\psi_0) > 0.$$

This contradicts the assumption that $H$ is not 3-edge-colorable. □

Note that (a) from Lemma 7 is well known to be equivalent to the Four Color Theorem [6], and thus indeed there is no plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ such that $n_{\tilde{G}}$ is not the zero function and $n_{\tilde{G}} \in \text{ray}(\tilde{R}_{5,12})$; and furthermore, a direct proof of this fact would imply the Four Color Theorem. Motivated by this observation (and experimental evidence), we propose the following conjecture, a strengthening of the Four Color Theorem. Let $B'_5$ denote the cone in $\mathbb{R}_{\geq 0}^{\tilde{R}_5}$ with rays $\text{ray}(\tilde{R}_{5,1}), \ldots, \text{ray}(\tilde{R}_{5,11})$.

**Conjecture 8.** Every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ satisfies $n_{\tilde{G}} \in B'_5$.

For $i \in \{0, \ldots, 4\}$, let $\psi_i^{5,a}$ and $\psi_i^{5,b}$ denote the 5-precolorings whose values at $j \in \{0, \ldots, 4\}$ are defined by the following table; see also Figure 5. Notice that $i$ is a rotating the coloring.

![Figure 5](image-url)  
**FIGURE 5** Precolorings $\psi_0^{5,a}$ and $\psi_0^{5,b}$
Note that each 5-precoloring is obtained from one of these 10 by a permutation of colors. The cone $B'_5$ has exactly one facet which is not also a facet of $B_5$, giving an equivalent formulation of Conjecture 8.

**Conjecture 9.** Every plane near-cubic graph $\tilde{G}$ with $d(\tilde{G}) = 5$ satisfies

$$3 \sum_{i=0}^{4} n_5^G(\psi_i^{5,a}) \geq \sum_{i=0}^{4} n_5^G(\psi_i^{5,b}).$$

In the rest of the note, we provide some evidence supporting Conjecture 8; in particular, we show there are no counterexamples to the conjecture for plane near-cubic graphs with less than 30 vertices.

## 5 | EVIDENCE

In this section we present experimental evidence for the validity of Conjecture 8. Our goal is to show Corollary 20 stating that Conjecture 8 holds for graphs on at most 30 vertices. The main idea of our approach is to generate larger graphs $\tilde{G}$ from smaller graphs by planarity preserving operations. One such is depicted in Figure 7. We will generate “all” possibilities for $d(\tilde{G}) \leq 7$ and particular ones with $d(\tilde{G}) = 8$. We then argue that all graphs in at most 30 vertices can be generated this way.

We begin by stating a few more definitions. A vector $\mathbf{x} \in \mathcal{P}_d$ is **invariant with respect to permutation of colors** if all $d$-precolorings $\psi$ and $\psi'$ that only differ by a permutation of colors satisfy $\mathbf{x}(\psi) = \mathbf{x}(\psi')$.

See Figure 6 for an illustration of the following definitions. The **rotation by $t$** of a $d$-precoloring $\psi$ is the $d$-precoloring $r_t(\psi)$ such that $r_t(\psi)((i + t) \mod d) = \psi(i)$ for $i \in \{0, ..., d - 1\}$. The **flip** of a $d$-precoloring $\psi$ is the $d$-precoloring $f(\psi)$ such that $f(\psi)(i) = \psi(d - 1 - i)$ for $i \in \{0, ..., d - 1\}$. For $x \in \mathbb{R}_{\mathcal{P}_d}$, let $r_t(x)$ be defined as $y \in \mathbb{R}_{\mathcal{P}_d}$ such that $y(r_t(\psi)) = x(\psi)$ for every $d$-precoloring $\psi$, and let $f(x)$ be defined as $z \in \mathbb{R}_{\mathcal{P}_d}$ such that $z(f(\psi)) = x(\psi)$ for every $d$-precoloring $\psi$. A set $K \subseteq \mathbb{R}_{\mathcal{P}_d}$ is closed under rotations and flips if we have $x \in K$ if and only if $f(x) \in K$ and $r_t(x) \in K$ for all $t \in \{0, 1, ..., d - 1\}$. For a near-cubic graph $G = (G, v, v_1)$ with $\deg(v) = d$, let $r_t(G)$ denote the near-cubic graph $(G, v, v_1)$, where $v_1^{-1}(i + t) \mod d) = v^{-1}(i)$ for $i \in \{0, ..., d - 1\}$, and let $f(G)$ denote the near-cubic graph $(G, v, v_2)$, where $v_2^{-1}(i) = v_2^{-1}(d - 1 - i)$ for $i \in \{0, ..., d - 1\}$.

**Observation 10.** Let $\tilde{G}$ be a near-cubic graph, $d = d(\tilde{G})$ and $t \in \{0, ..., d - 1\}$. Then $n_r(\tilde{G}) = r_t(n_5(\tilde{G}))$ and $n_f(\tilde{G}) = f(n_5(\tilde{G}))$. 

Let \( \psi_1 \) be a \( d_1 \)-precoloring and \( \psi_2 \) a \( d_2 \)-precoloring. For an integer \( k \leq \min(d_1, d_2) \), we say that \( \psi_k \) matches \( \psi_2 \) if \( \psi_k(d_1 - k + i) = \psi_2(d_2 - 1 - i) \) for \( i \in \{0, 1, ..., k - 1\} \). By \( \gamma_{k}(\psi_1, \psi_2) \), we denote the \( (d_1 + d_2 - 2k) \)-precoloring \( \gamma \) such that \( \gamma(i) = \psi_k(i) \) for \( i \in \{0, 1, ..., d_1 - k - 1\} \) and \( \gamma(i) = \psi_2(i - (d_1 - k)) \) for \( i \in \{d_1 - k, ..., d_1 + d_2 - 2k - 1\} \). For \( x_1 \in \mathbb{R}^{d_1} \) and \( x_2 \in \mathbb{R}^{d_2} \), we define \( \gamma_k(x_1, x_2) \) as the vector \( y \in \mathbb{R}^{d_1 + d_2 - 2k} \) such that

\[
y(\psi) = \sum_{\psi, \psi_2 : \gamma_{k}(\psi_1, \psi_2) = \psi} x_1(\psi_1) x_2(\psi_2),
\]

where the sum is over all \( k \)-matching \( d_1 \)-precolorings \( \psi_1 \) and \( d_2 \)-precolorings \( \psi_2 \). For near-cubic graphs \( \bar{G}_1 = (G_1, v_1, v_1) \) with \( \deg(v_1) = d_1 \) and \( \bar{G}_2 = (G_2, v_2, v_2) \) with \( \deg(v_2) = d_2 \), let \( \gamma_k(\bar{G}_1, \bar{G}_2) \) denote the near-cubic graph \( (G, v, \nu) \), where \( G \) is obtained from \( G_1 \) and \( G_2 \) by identifying \( v_1 \) with \( v_2 \) to a single vertex \( v \) and for \( i \in \{0, 1, ..., k - 1\} \) removing the half-edges \( \nu_1^{-1}(d_1 - k + i) \) and \( \nu_2^{-1}(d_2 - 1 - i) \) and connecting the other halves of the edges; and \( \nu^{-1}(i) = \nu_i^{-1}(i) \) for \( i \in \{0, ..., d_1 - k - 1\} \) and \( \nu^{-1}(i) = \nu_2^{-1}(i - (d_1 - k)) \) for \( i \in \{d_1 - k, ..., d_1 + d_2 - 2k - 1\} \). See Figure 7 for an illustration.

**Observation 11.** Let \( \bar{G}_1 \) and \( \bar{G}_2 \) be near-cubic graphs. For every integer \( k \in \{0, ..., \min(d(\bar{G}_1), d(\bar{G}_2))\} \), we have \( n_{k}(\bar{G}_1, \bar{G}_2) = \gamma_k(n_{\bar{G}_1}, n_{\bar{G}_2}) \).

By a computer-assisted enumeration, we verified the following claim.
Lemma 12. There exist cones $K_d \subseteq \mathbb{R}^d$ for $d = 2, \ldots, 8$ such that the following claims hold.

(a) $K_d = B_d$ when $d \leq 4$ and $K_5 = B'_5$.
(b) For all $d \in \{2, \ldots, 8\}$, the elements of $K_d$ are invariant with respect to permutation of colors.
(c) For $d \in \{2, \ldots, 7\}$, the cone $K_d$ is closed under rotations and flips.
(d) If $2 \leq d_1 \leq d_2$ and $d_1 + d_2 \leq 7$, then for all $x_1 \in K_{d_1}$ and $x_2 \in K_{d_2}$ we have $\gamma_0(x_1, x_2) \in K_{d_1 + d_2}$.
(e) If $2 \leq d \leq 5$, then for all $x \in K_d$ we have $\gamma_1(n_{\tilde{r}_d}, x) \in K_{d+1}$.
(f) If $3 \leq d \leq 7$, then for all $x \in K_d$ we have $\gamma_2(n_{\tilde{r}_d}, x) \in K_{d-1}$.
(g) If $2 \leq d_1 \leq 6$ and $1 \leq c \leq d_1/2$, then for all $x_1 \in K_{d_1}$ and $x_2 \in K_{7+2c-d_1}$, we have $\gamma_c(x_1, x_2) \in K_7$.
(h) For every $x_1 \in K_8$ and $x_2 \in K_7$, we have $\gamma_4(x_1, x_2) \in K_7$.
(i) For every $x_1, x_2 \in K_6$, we have $r_2(\gamma_2(x_1, x_2)) \in K_8$.

Proof: The proof and the program to verify the proof can be found at http://lidicky.name/pub/4cone/. The cones are described by their rays, enumerated in the file. Cone $K_6$ has 102 rays, $K_7$ has 22605 rays, and $K_8$ has 4330 rays. It suffices to verify all the claims for $x, x_1, x_2$ being the rays of the cones specified in the claims; the inclusion of the resulting vectors in the appropriate cone is certified by expressing them as a linear nonnegative combination of the rays of the cone.

Parts (e) and (f) of Lemma 12 have the following corollary.

Lemma 13. Let $\tilde{G} = (G, v, v)$ be a plane near-cubic graph and let $d = d(\tilde{G})$. If $d \in \{2, \ldots, 7\}$ and $n_{\tilde{G}} \notin K_d$, then there exists a plane near-cubic graph $\tilde{G}_0 = (G_0, v_0, v_0)$ such that $d(\tilde{G}_0) = 7$, $n_{\tilde{G}_0} \notin K_7$, $G_0 - v_0$ is an induced subgraph of $G - v$, and $|V(G_0)| \leq |V(G)| - (7 - d)$. 

![Figure 8](https://example.com/figure8.png)

**Figure 8** Graph $\tilde{G}$ from Lemma 19. Edges incident to $v$ are crossing the dashed circle and $v$ is not depicted. (A) Cycles $C$ and $C'$ are depicted by thick red and dotted blue, respectively. The gray faces belong to $Y(G)$. The white face in the center belongs to $X(G)$. (B) A construction of $\tilde{G}$ from $\tilde{G}_1, \tilde{G}_2,$ and $\tilde{G}_3$ is indicated by the dotted lines.
Proof. We prove the claim by induction on the number of vertices of \( G \). When \( d \leq 4 \), the claim is vacuously true by Theorem 5, since \( K_d = B_d \). When \( d = 7 \), we can set \( \tilde{G}_0 = \tilde{G} \). Hence, suppose that \( d \in \{5, 6\} \). Since \( n_{\tilde{G}} \notin K_d \), the function \( n_{\tilde{G}} \) is not identically zero.

If \( G - v \) is disconnected, we can by symmetry assume that \( \tilde{G} = \gamma(\tilde{G}_1, \tilde{G}_2) \) for plane near-cubic graphs \( \tilde{G}_1 \) and \( \tilde{G}_2 \) such that \( d(\tilde{G}_1) = d(\tilde{G}_2) + 1 \). Since \( n_{\tilde{G}} \) is not the zero function, \( n_{\tilde{G}_1} \) is not the zero function either, and thus \( d(\tilde{G}_1) \neq 1 \). Hence \( d(\tilde{G}_1) \geq 2 \), and thus \( 2 \leq d(\tilde{G}_2) \leq 4 \). By Lemma 14, we have \( n_{\tilde{G}_1} \notin K_d \) and \( n_{\tilde{G}_2} \notin K_d \), and \( n_{\tilde{G}} \notin K_d \) by Lemma 12(d), which is a contradiction.

Hence, \( G - v \) is connected (and the same argument as for disconnected \( G - v \) shows that no loop is incident with \( v \)). Consequently, \( v \) is not incident with a triple edge. If \( v \) is incident with a double edge, then we can by symmetry assume that \( \tilde{G} = \gamma(\tilde{G}_1, \tilde{G}_2) \) for a plane near-cubic graph \( \tilde{G}_1 = (G_1, v_1, v_1) \) with \( d(\tilde{G}_1) = d - 1 \leq 5 \). By Lemma 12(e), since \( n_{\tilde{G}} \notin K_d \), we have \( n_{\tilde{G}_1} \notin K_{d-1} \). By the induction hypothesis, there exists a plane near-cubic graph \( \tilde{G}_0 = (G_0, v_0, v_0) \) with \( d(\tilde{G}_0) = 7 \), such that \( n_{\tilde{G}_0} \notin K_7 \), and \( G_0 - v_0 \) is a proper minor of \( G_1 - v_1 \), and thus also of \( G - v \), and \( |V(G_0)| \leq |V(G)| - (7 - (d - 1)) < |V(G)| - (7 - d) \), as required.

Hence, we can assume \( v \) is not incident with a double edge. Consequently, we can by symmetry assume that \( \tilde{G} = \gamma(\tilde{G}_1, \tilde{G}_2) \) for a plane near-cubic graph \( \tilde{G}_1 = (G_1, v_1, v_1) \) with \( d(\tilde{G}_1) = d + 1 \). By Lemma 12(f), since \( n_{\tilde{G}} \notin K_d \), we have \( n_{\tilde{G}_1} \notin K_{d+1} \). By the induction hypothesis, there exists a plane near-cubic graph \( \tilde{G}_0 = (G_0, v_0, v_0) \) with \( d(\tilde{G}_0) = 7 \), such that \( n_{\tilde{G}_0} \notin K_7 \), and \( G_0 - v_0 \) is an induced subgraph of \( G_1 - v_1 \), and thus also of \( G - v \), and \( |V(G_0)| \leq |V(G)| - (7 - (d + 1)) = |V(G)| - (7 - d) \). Hence, the claim of the lemma follows. \( \square \)

We will say that a plane near-cubic graph \( \tilde{G} = (G, v, v) \) is extremal if \( d(\tilde{G}) = 7 \), \( n_{\tilde{G}} \notin K_7 \), and there does not exist any plane near-cubic graph \( \tilde{G}_0 = (G_0, v_0, v_0) \) with \( d(\tilde{G}_0) = 7 \) such that \( n_{\tilde{G}_0} \notin K_7 \) and \( G_0 - v_0 \) is a proper minor of \( G - v \).

**Lemma 14.** If \( \tilde{G} = (G, v, v) \) is an extremal plane near-cubic graph and \( \tilde{G}' = (G', v', v') \) is a plane near-cubic graph with \( d(\tilde{G}') \leq 7 \) such that \( G' - v' \) is a proper minor of \( G - v \), then \( n_{\tilde{G}'} \notin K_d(\tilde{G}') \).

**Proof.** If \( n_{\tilde{G}'} \notin K_d(\tilde{G}') \), then by Lemma 13 there would exist a plane near-cubic graph \( \tilde{G}_0 = (G_0, v_0, v_0) \) such that \( d(\tilde{G}_0) = 7 \), \( n_{\tilde{G}_0} \notin K_7 \), and \( G_0 - v_0 \) is an induced subgraph of \( G' - v' \). However, then \( G_0 - v_0 \) would be a proper minor of \( G - v \), contradicting the assumption that \( \tilde{G} \) is extremal. \( \square \)

Next, let us explore the consequences of part (g) of Lemma 12.

**Lemma 15.** If \( \tilde{G} = (G, v, v) \) is an extremal plane near-cubic graph, then \( v \) is not incident with loops or parallel edges and \( G - v \) is 2-edge-connected.

**Proof.** Analogously to the proof of Lemma 13, if \( v \) were incident with a loop or a parallel edge or if \( G - v \) were not 2-edge-connected, we would have \( \tilde{G} = \gamma(\tilde{G}_1, \tilde{G}_2) \) for plane near-cubic graphs \( \tilde{G}_1 \) and \( \tilde{G}_2 \) such that \( 2 \leq d(\tilde{G}_1) \leq d(\tilde{G}_2), d(\tilde{G}_1) + d(\tilde{G}_2) = 7 + 2c \), and \( c \leq 1 \); in particular, \( d(\tilde{G}_2) \leq 7 \) and \( d(\tilde{G}_1) \leq (7 + 2c)/2 \) \leq 4. By Lemma 14, we have \( n_{\tilde{G}_i} \in K_d(\tilde{G}_i) \) for \( i \in \{1, 2\} \). By Lemma 12(g), we conclude \( n_{\tilde{G}} \in K_7 \), which is a contradiction. \( \square \)
Suppose \( A \) and \( B \) form a partition of the vertex set of a graph \( H \), and let \( S \) be the set of edges of \( H \) with one end in \( A \) and the other end in \( B \). In this situation, we say \( S \) is an edge cut of \( H \) with sides \( A \) and \( B \).

**Lemma 16.** If \( \tilde{G} = (G, v, \nu) \) is an extremal plane near-cubic graph, then \( G - v \) does not contain an edge cut \( S \) such that \( v \) has at least \( |S| \) neighbors in each side of the cut.

**Proof.** Suppose for a contradiction \( G - v \) contains such an edge cut \( S \) of size \( c \), and thus \( \tilde{G} = \nu_c(\tilde{G}_1, \tilde{G}_2) \) for plane near-cubic graphs \( \tilde{G}_1 \) and \( \tilde{G}_2 \) such that \( 2c \leq d(\tilde{G}_1) \leq d(\tilde{G}_2) \) and \( d(\tilde{G}_1) + d(\tilde{G}_2) = 7 + 2c \). Since \( v \) has seven neighbors and at least \( c \) of them are contained in each of the sides of the cut, we have \( c \leq 3 \). Note that \( d(\tilde{G}_2) \leq 7 \) and \( d(\tilde{G}_1) \leq \lfloor (7 + 2c)/2 \rfloor \leq 6 \). By Lemma 14, we have \( n_{\tilde{G}_i} \in K_{d(\tilde{G}_i)} \) for \( i \in \{1, 2\} \). By Lemma 12(g), we conclude \( n_{\tilde{G}} \in K_7 \), which is a contradiction. \( \square \)

An edge cut \( S \) of size at most five in a near-cubic graph \( \tilde{G} = (G, v, \nu) \) is essential if the side of \( S \) containing \( v \) contains at least one other vertex and the other side \( B \) of \( S \) induces neither a tree nor a 5-cycle.

**Lemma 17.** If \( \tilde{G} = (G, v, \nu) \) is an extremal plane near-cubic graph, then \( \tilde{G} \) does not contain an essential edge cut \( S \) of size at most five.

**Proof.** Suppose for a contradiction \( \tilde{G} \) contains an essential edge-cut \( S \) of size \( k \leq 5 \), and choose one with minimum \( k \), and subject to that one for which the side \( B \) not containing \( v \) is minimal. We claim \( G[B] \) is 2-edge-connected. Otherwise, \( B \) is a disjoint union of nonempty sets \( B_1 \) and \( B_2 \), where \( G \) contains \( r \leq 1 \) edges with one end in \( B_1 \) and the other end in \( B_2 \). For \( i \in \{1, 2\} \), let \( S_i \) denote the set of edges of \( G \) with exactly one end in \( B_i \). Since \( \tilde{G} \) is extremal, \( n_{\tilde{G}} \notin K_7 \) is not identically zero, and thus \( G \) is 2-edge-connected, implying \( |S_i| \geq 2 \). Hence, \( |S| = k + 2r - |S_{3-r}| \leq k \). By the minimality of \( B \), we conclude that \( B_i \) induces a tree or a 5-cycle, and thus \( |S_i| \geq 3 \). Hence \( 5 \geq k = |S_1| + |S_2| - 2r \geq 6 - 2r \), and thus \( r = 1 \) and \( |S_1|, |S_2| \leq 4 \). This implies that neither \( B_1 \) nor \( B_2 \) induces a 5-cycle, and thus both of them induce trees; and \( G \) contains an edge between them, implying that \( B \) induces a tree, contrary to the assumption that \( S \) is an essential edge cut.

Since \( G[B] \) is 2-edge-connected and subcubic, each face of \( G[B] \) is bounded by a cycle. Let \( C_j \) denote the boundary cycle bounding the face \( f \) of \( G[B] \) whose interior contains \( v \). Observe that all edges of \( S \) are drawn inside \( f \). Otherwise, the set \( S' \) of edges of \( S \) drawn inside \( C \) forms an edge cut of order smaller than \( k \) and by the minimality of \( k \), its side \( B' \supseteq B \) induces a tree or a 5-cycle; this is not possible, since \( G[B] \) is 2-edge connected and not a tree.

Let \( \tilde{G}_i \) be the plane near-cubic graph obtained from \( G \) by contracting the side of the cut containing \( v \) to a single vertex. By Lemma 14, we have \( n_{\tilde{G}_i} \in K_k \). Since \( K_d = B_d \) for \( d \leq 4 \) and \( K_5 = B'_5 \),

\[
n_{\tilde{G}_i} = \sum_i c_i n_{\tilde{G}_i},
\]

where \( i \leq 11 \) if \( k = 5 \) and the coefficients \( c_i \) are nonnegative. Let \( \tilde{G}_i = (G_i, v_i, \nu_i) \) denote the plane near-cubic graph obtained from \( \tilde{G} \) by replacing the side of the cut \( S \) not
containing \( v \) by \( \tilde{R}_k \). Note that \( n_{\tilde{G}} = \sum_i c_in_{\tilde{G}_i} \), and since \( K_7 \) is a cone and \( n_{\tilde{G}} \notin K_7 \), there exists \( i \) such that \( n_{\tilde{G}_i} \notin K_7 \). Because \( B \) contains the cycle \( C_S \) and all edges of \( S \) are incident with vertices of \( C_S \), we see \( G_i - v_i \) is a proper minor of \( G - v \), contradicting the extremality of \( \tilde{G} \). \[ \square \]

In Lemma 15, we argued that if \( \tilde{G} = (G, v, v) \) is an extremal plane near-cubic graph, then \( G - v \) is 2-edge-connected, and thus its face containing \( v \) is bounded by a cycle \( C \). Let us now argue that the graph stays 2-edge-connected after removing \( V(C) \) as well.

**Lemma 18.** Let \( \tilde{G} = (G, v, v) \) be an extremal plane near-cubic graph and let \( C \) be the cycle bounding the face of \( G - v \) containing \( v \). The cycle \( C \) is induced, no two neighbors of \( v \) in \( C \) are adjacent, and the graph \( G - (V(C) \cup \{v\}) \) is 2-edge-connected and has more than one vertex.

**Proof.** Consider a simple closed curve \( c \) in the plane intersecting \( G \) in two edges of \( C \), \( b \leq 4 \) edges incident with \( v \), and \( r \leq 1 \) edges of \( E(G - v) \setminus E(C) \), where each edge is intersected at most once. The curve \( c \) separates the plane into two parts; let \( A \) and \( B \) be the corresponding partition of vertices of \( G \), where \( v \in A \), and let \( S \) be the edge cut in \( G \) consisting of the edges with one end in \( A \) and the other end in \( B \). By Lemma 16 applied to the edge cut in \( G - v \) obtained from \( S \) by removing the edges incident with \( v \), it follows that \( b \leq r + 1 \), and thus \( |S| \leq 3 + 2r \leq 5 \). By Lemma 17 we conclude that the edge cut satisfies one of the following conditions.

- \( r = 0 \), \( b = 1 \), \( |S| = 3 \), and \( B \) consists of a single vertex of \( C \), or
- \( r = 1 \) and \( G[B] \) is a subpath of \( C \), or
- \( r = 1 \), \( b = 2 \), and \( G[B] \) is a 5-cycle containing exactly one vertex not in \( V(C) \).

If \( C \) had a chord \( e \), this would give a contradiction by considering a curve \( c \) (with \( r = 0 \)) drawn next to the chord so that \( e \in E(G[B]) \) and \( b \leq 3 \); hence, \( C \) is an induced cycle. If two neighbors of \( v \) in \( C \) were adjacent, we would obtain a contradiction by considering a curve \( c \) (with \( r = 0 \) and \( b = 2 \)) drawn around them. If the graph \( G - (V(C) \cup \{v\}) \) were not connected, we would obtain a contradiction by considering a curve \( c \) (with \( r = 0 \) and \( b \leq 3 \)) chosen so that both \( A \) and \( B \) contain a vertex of \( G - (V(C) \cup \{v\}) \). Finally, if the graph \( G - (V(C) \cup \{v\}) \) were not 2-edge-connected, then we could choose \( c \) so that \( r = 1 \), \( b \leq 3 \), and \( B \) contains a vertex of \( G - (V(C) \cup \{v\}) \). But then \( G[B] \) would be a 5-cycle containing exactly one vertex not in \( V(C) \) and consequently two adjacent vertices of \( C \) would be neighbors of \( v \), which is a contradiction.

Therefore, the graph \( G - (V(C) \cup \{v\}) \) is 2-edge-connected. Since no two neighbors of \( v \) in \( C \) are adjacent, \( G \) contains at least 7 edges between \( V(C) \) and \( V(G) \setminus V(C) \cup \{v\} \), and thus \( G - (V(C) \cup \{v\}) \) has more than one vertex. \[ \square \]

Finally, let us apply the parts (h) and (i) of Lemma 12.

**Lemma 19.** If \( \tilde{G} = (G, v, v) \) is an extremal plane near-cubic graph, then \( G \) has at least 28 vertices.
Proof. Recall that by the definition of extremal, \(d(\tilde{G}) = 7\). By Lemma 15, the face of \(G - v\) containing \(v\) is bounded by a cycle \(C\). Let \(v_1, ..., v_7\) be the neighbors of \(v\) in \(C\) in order. For \(i \in \{1, ..., 7\}\), let \(P_i\) denote the subpath of \(C\) from \(v_i\) to \(v_{i+1}\) (where \(v_8 = v_1\)).

By Lemma 18, the cycle \(C\) is induced, no two neighbors of \(v\) in \(C\) are adjacent, and the graph \(\bigcup G V C v - ((\ ))\) is 2-edge-connected and has more than one vertex. Hence, the face of \(\bigcup G V C v - ((\ ))\) containing \(v\) is bounded by a cycle \(C'\). For a subgraph \(\bigcap G G'\) containing \(C\), let \(X_G(P)\) denote the set of faces of \(G'\) separated from \(v\) by \(C\) and \(Y_G(P)\) denote the set of faces of \(G'\) separated from \(v\) by \(C\) but not by \(C'\). See Figure 8A for an example. For \(i \in \{1, ..., 7\}\), we say that a face \(f \in X_G(P)\) sees \(P_i\) if there exists a face \(f' \in Y_G(P)\) such that \(f\) is incident with an edge of \(P_i\) and the boundaries of \(f\) and \(f'\) share at least one edge.

If for some \(\in i \in \{1, ..., 7\}\), some face of \(X_G(P)\) sees \(P_i\), \(P_{i+2}\), and \(P_{i+4}\) (with indices taken cyclically) then \(\tilde{G} = \gamma(\gamma(\tilde{G}_1, \tilde{G}_2)), \tilde{G}_3)\) for plane near-cubic graphs \(\tilde{G}_1, \tilde{G}_2,\) and \(\tilde{G}_3\) with \(d(\tilde{G}_1) = d(\tilde{G}_2) = 6\) and \(d(\tilde{G}_3) = 7\) (see Figure 8B). Lemma 14 would imply \(n_{G_j} \in K_{d(\tilde{G})}\) for \(j \in \{1, 2, 3\}\), and by Lemma 12(h) and (i), we would have \(n_{G_j} \in K_j\), which is a contradiction. Hence,

\[
\text{no face of } X(G) \text{ sees } P_i, P_{i+2}, \text{ and } P_{i+4}. \tag{3}
\]

Let \(b_1\) be the number of edges of \(G\) with one end in \(C\) and the other end in \(C'\), let \(b_2\) be the number of chords of \(C'\), let \(b_3\) be the number of edges with one end in \(C'\) and the other end in \(\bigcap G V C - ((\ ))\), and let \(b_4\) be the number of edges of \(\bigcup G V C - ((\ ))\). Note that \(b_1 \geq 7, b_3\) is at least three times the number of components of \(G - v - V(C \cup C')\), \(|E(C)| = 7 + b_1, |E(C')| = b_1 + 2b_2 + b_3,\) and

\[
|E(G)| = 7 + (7 + b_1) + b_1 + (b_1 + 2b_2 + b_3) + b_2 + b_3 + b_4 = 14 + 3b_1 + 3b_2 + 2b_3 + b_4.
\]

A case analysis shows that since (3) holds, one of the following conditions holds:

- \(b_1 \geq 8\) and \(b_2 \geq 2\), or
- \(b_1 \geq 8\) and \(b_3 \geq 3\), or
- \(b_3 \geq 6\), or
- \(b_3 \geq 4\) and \(b_4 \geq 1\).

Hence \(3b_1 + 3b_2 + 2b_3 + b_4 \geq 30\), and thus \(G\) has at least 44 edges. Consequently, \(|V(G)| \geq (2|E(G)| - 4)/3 \geq 28\).

As a consequence, this verifies Conjecture 8 for small graphs.

Corollary 20. Conjecture 8 holds for all plane near-cubic graphs with less than 30 vertices.

Proof. Let \(\tilde{G} = (G, v, v)\) be a counterexample to Conjecture 8, and in particular \(n_{\tilde{G}} \notin B' = K_5\). By Lemma 13, there exists a plane near-cubic graph \(\tilde{G}_0 = (G_0, v_0, v_0)\) such that \(d(\tilde{G}_0) = 7, n_{\tilde{G}_0} \notin K_7,\) and \(|V(G_0)| \leq |V(G)| - 2\). Hence, there exists an extremal
plane near-cubic graph $\tilde{G}_1 = (G_1, v_1, \nu_1)$ such that $|V(G_1)| \leq |V(G_0)|$. By Lemma 19, we have $|V(G_1)| \geq 28$, and thus $|V(G)| \geq 30$.

Note that the analysis at the end of the proof of Lemma 19 can be improved. By a computer-assisted enumeration, one can show that to ensure that (3) holds, $G - \nu$ must contain one of 38 specific graphs as a minor; the smallest is depicted in Figure 9. Hence, every counterexample to Conjecture 8 must contain one of these 38 as a minor. The list of these 38 graphs is available at http://lidicky.name/pub/4cone/.

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