ŁOJASIEWICZ EXPONENTS OF A CERTAIN ANALYTIC FUNCTIONS

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Abstract. We consider the exponent of Łojasiewicz inequality $\| \partial f(z) \| \geq c|f(z)|^\theta$ for two classes of analytic functions and we will give an explicit estimation for $\theta$. First we consider certain non-degenerate functions which is not convenient. In §3.4, we give an example of a polynomial for which $\theta_0(f)$ is not constant on the moduli space and in §3.5, we show that the behaviors of the Łojasiewicz exponents is not similar as the Milnor numbers by an example.

In the last section (§4), we give also an estimation for product functions $f(z) = f_1(z) \cdots f_k(z)$ associated to a family of a certain convenient non-degenerate complete intersection varieties. In either class, the singularity is not isolated. We will give explicit estimations of the Łojasiewicz exponent $\theta_0(f)$ using combinatorial data of the Newton boundary of $f$.

We generalize this estimation for non-reduced function $g = f_1^{m_1} \cdots f_k^{m_k}$.

1. Holomorphic functions and Łojasiewicz exponents

Consider a germ of an analytic function $f(z)$ at the origin. There are two type of inequalities which are shown by S. Łojasiewicz [16, 17].

(1) $\| \partial f(z) \| \geq c|f(z)|^\theta$, $c \neq 0$, $0 \leq \exists \theta < 1$, $\forall z \in U$,

(2) $\| \partial f(z) \| \geq c\|z\|^\eta$, $c \neq 0$, $\exists \eta > 0$, $\forall z \in U$

where $U$ is a sufficiently small neighborhood of the origin. Here $\partial f(z)$ is the gradient vector $(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})$. These equalities hold in a sufficiently small neighborhood of the origin. For the second inequality, $f(z)$ must have an isolated singularity at the origin. In our previous paper [20], we considered the exponent of the second Łojasiewicz inequality for a non-degenerate holomorphic function $f(z)$ (or a mixed function $f(z, \bar{z})$) with an isolated singularity at the origin. For further information about Łojasiewicz inequality (2), we refer [16 17 1 2 3 5 13 20 21 22].

In this paper, we are interested in the exponent of the type (1) for a certain type of holomorphic functions which may have non-isolated singularities at the origin. This inequality has been originally studied by Łojasiewicz [16, 17] and then by many other authors. Most of the researches have been concentrated for the existence of the inequality in more general setting. For example, in the papers [15 11 12], the authors show the existence of

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Lojasiewicz inequality for the o-minimal situation. We study the exponent of the inequality \((1)\) for a certain type of holomorphic functions which may have non-isolated singularities at the origin. They are either non-degenerate functions or the product of convenient non-degenerate polynomials associated to a non-degenerate complete intersection variety. The existence of Lojasiewicz inequality is well-known for holomorphic functions ([16], [17]).

We are interested in the vanishing speed of the gradient vectors \(\partial f(z)\) near the origin in comparison with that of the absolute value of \(f\) when \(z\) goes to the origin. Thus we are mostly interested in the best possible \(\theta\) which satisfies \((1)\). This number is the infinimum of \(\theta's\) which satisfy \((1)\) and we denote it by \(\theta_0(f)\) hereafter.

Let \(z(t)\) be an analytic curve with \(z(0) = 0\) and \(z(t) \in \mathbb{C}^n \setminus f^{-1}(0)\) for \(t \neq 0\). Then we compare the order of the both side of \((1)\), after substituting \(z = z(t)\) to get the inequality:

\[
\text{ord}_t \| \partial f(z(t)) \| \leq \theta \times \text{ord}_t f(z(t))
\]

or equivalently

\[
\frac{\text{ord}_t \| \partial f(z(t)) \|}{\text{ord}_t f(z(t))} \leq \theta.
\]

Using the Curve Selection Lemma ([14], [8]), \(\theta_0(f)\) can be understood as the supremum of the left side ratios of the above inequality for all possible such curves \(z(t)\) and we call it the *Lojasiewicz exponent* of the function \(f\) for the Lojasiewicz inequality \((1)\).

2. Non-degenerate hypersurfaces

2.1. Dual Newton diagram. We consider an analytic function (or a polynomial)

\[
f(z) = \sum_{\nu} c_{\nu} z^{\nu}
\]

defined in the neighborhood of the origin. Recall that the Newton polyhedron \(\Gamma_+(f)\) is the convex hull of the union \(\bigcup_{\nu \cdot c_{\nu} \neq 0} (\nu + \mathbb{R}_+^n)\). The Newton boundary \(\Gamma(f)\) is defined by the union of compact faces of \(\Gamma_+(f)\). Let \(N^+ \subset \mathbb{R}^n\) be the space of non-negative weight vectors. That is, a weight vector \(P = (p_1, \ldots, p_n)\) is in \(N^+\) if and only if \(p_i \geq 0\). It defines linear function \(\ell_P\) of the Newton polyhedron \(\Gamma_+(f)\) by \(\ell_P(\nu) = \sum_{i=1}^n p_i \nu_i\) for \(\nu \in \Gamma_+(f)\). Its minimal value is denoted by \(d(P, f)\) and the face of \(\Gamma_+(f)\), where this minimal value is taken, is denoted as \(\Delta(P, f)\). If no ambiguity is likely, we simply denote it as \(d(P)\) and \(\Delta(P)\). We recall an equivalence relation in \(N^+\) which gives a polyhedral conical structure in \(N^+\). Two weight vectors \(P, Q\) are equivalent if and only if \(\Delta(P) = \Delta(Q)\) and this equivalence gives a conical polyhedral subdivision of \(N^+\) which we call the *dual Newton diagram* and denote it as \(\Gamma^*(f)\). The equivalence class of \(P\) is denoted as \([P]\).
2.2. Vanishing and non-vanishing weight vectors. For $I \subset \{1, \ldots, n\}$, we put $C^I = \{z \mid z_j = 0, \forall j \notin I\}$. Thus $C^I \subset C^n$. The subspace $C^I$ is called a vanishing coordinate subspace if $f^I \equiv 0$. Note that $f^I$ is the restriction of $f$ to $C^I$. Consider a weight vector $P = (p_1, \ldots, p_n)$. Put $I(P) := \{i \mid p_i = 0\}$. Assume that $I(P) \neq \emptyset$. A weight vector $P$ is called a vanishing weight vector (respectively non-vanishing weight vector if $d(P) > 0$ (resp. if $d(P) = 0$). Thus $C^{I(P)}$ is a vanishing coordinate subspace if $P$ is a vanishing weight vector. We denote the sets of strictly positive weight vectors (i.e. $I(P) = \emptyset$), vanishing weight vectors and non-vanishing weight vectors by $W_+(f), W_v(f)$ and $W_{nv}(f)$ respectively. Hereafter we simply denote them as $W_+, W_v, W_{nv}$, if no ambiguity is likely.

2.3. Convexity of the equivalence class. Let $P$ be a weight vector in $N^+$ and let $[P]$ the set of equivalent weight vectors. The equivalence class $[P]$ is the interior of a polyhedral convex cone in $N^+$ and $\dim [P] = n - \dim \Delta(P)$. This follows from the obvious equality:

$$\Delta(P) \cap \Delta(Q) \neq \emptyset \implies \Delta((1 - t)P + tQ) = \Delta(P) \cap \Delta(Q), \quad 0 < t < 1.$$  

Put $LS(P, Q) := \{(1 - t)P + tQ \mid 0 \leq t \leq 1\}$ and we call $LS(P, Q)$ the line segment with ends $P, Q$. Consider the closure $[P]$ of $[P]$ in the Euclidean topology. Then $Q \in [P]$ if and only if $\Delta(Q) \supset \Delta(P)$ and $[P]$ is also a closed polyhedral convex cone. We say $Q$ is on the boundary of $[P]$ if $[Q] \subset [P] \setminus [P]$ and denote as $Q \succ P$. Note that $Q \succ P$ if and only if $\Delta(Q) \supset \Delta(P)$.

We visualize $\Gamma_+(f)$ by cutting $N^+$ by some transversal hyperplane $\Pi$ to the cone, say $\Pi : \nu_1 + \cdots + \nu_n = 1$ and we see the silhouette. See Figure 2. In the figure, the dimension of the equivalence class is one less. Let $P$ a weight vector. We say that $P$ is a vertex of the dual Newton diagram $\Gamma_+(f)$ if and only if $\dim \Delta(P) = n - 1$ or equivalently $\dim [P] = 1$.

Definition 1. $f$ is called $k$-convenient if $f^I \neq 0$ for any $I \subset \{1, \ldots, n\}$ with $|I| \geq n - k$. We say for simplicity $f$ is convenient if $f$ is $(n - 1)$-convenient ([18]).

2.4. Face function, non-degeneracy and tameness. Let $\Xi$ be a face of $\Gamma_+(f)$. The face function of $\Xi$ is defined by $f_\Xi(z) := \sum_{c_\nu} c_\nu z^\nu$. For a weight vector $P$, we define $f_P(z) := f_{\Delta(P)}(z)$. If $P \in W_+ \cup W_v$, then $d(P) > 0$ and $f_P$ is a weighted homogeneous polynomial of $z_I$, $J = I(P)^c$, of degree $d(P)$ with respect to the weight $P$. Here $I(P)^c = \{1, \ldots, n\} \setminus I(P)$ and $z_I = (z_i \mid i \in I)$. We recall that $f$ is non-degenerate if the mapping $f_P : \mathbb{C}^n \to \mathbb{C}$ has no critical point for any $P \in W_+$. Recall that $\mathbb{C}^n := \{z \in \mathbb{C}^n \mid z_i \neq 0, 1 \leq i \leq n\}$.

We say that $f$ is locally tame (or strongly locally tame) if for any weight vector $P \in W_v$, the face function $f_P : \mathbb{C}^{I(P)^c} \to \mathbb{C}$ has no critical point as a function of variables $z_{I(P)^c}$ for any sufficiently small (resp. for any)
$z_I \in \mathbb{C}^{*I(P)}$ fixed ($\mathbb{I}$). Here $\mathbb{C}^{*I} = \{z \in \mathbb{C}^n | z_j = 0, \text{iff } j \notin I\}$. For $I = \{1, \ldots, n\}$, we write simply $\mathbb{C}^{*n}$ instead of $\mathbb{C}^{*I}$.

**Definition 2.** Let $\text{Var}(P) = \{j \mid \frac{\partial f_P}{\partial z_j} \neq 0\}$. That is, $j \in \text{Var}(P)$ if and only if $z_j$ appears in a monomial of $f_P(z)$. We call $\text{Var}(P)$ the variables of $P$ or of $f_P$. Let $I(P) := \bigcup\{I(Q)\mid Q \succ P, Q \in \mathbb{W}_v\}$ and we put $P \in \overline{\text{Var}(P)} := \text{Var}(P) \setminus I(P)$. We call $\{z_j \mid j \in \text{Var}(P)\}$ invulnerable variables for $P$. Note that $I(P) \supset I(Q)$ if $Q \supset P$. We introduce a stronger tameness: $f$ is strongly inv-tame for $P$ if $\text{Var}(P)$ is not empty and $f_P : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical point as a polynomial of the invulnerable variables $\{z_j \mid j \in \text{Var}(P)\}$ for any $z_{I(P)} \in \mathbb{C}^{*I(P)}$ fixed. We say $f$ is strongly inv-tame if any weight vector $P \in \mathbb{W}_v \cup \mathbb{W}_r$ with $\dim \Delta(P) \geq 1$, $f$ is strongly inv-tame for $P$.

For example, consider a weight vector $D$ on the open interval $\text{Int}(\text{RE}_3)$ in $f_1(z)$ (Figure 2). Then $R, E_3 \succ D$. But $E_3 \in \mathbb{W}_v$. Thus $I(D) = \{1\}$ and $f_{1D} = z_1^2$ and we see $f_1$ is strongly inv-tame for $D$.

**Remark 3.** Assume that $f$ is $(n - 2)$-convenient. Take $P \in \mathbb{W}_v \cup \mathbb{W}_r$ with $\dim \Delta(P) \geq 1$. Assume that $Q \in \overline{\{P\}}$ and $Q \in \mathbb{W}_v$. Then $\sharp I(Q) = 1$. (If $\sharp(I(Q)) = 2, Q \in \mathbb{W}_v$.) If $P \in \mathbb{W}_v$, $I(Q) = I(P)$. Thus it is easy to check if $P$ is strongly inv-tame or not.

2.5. **Dimension of $[P]$**. We recall the following relation of the dimension of the equivalence class $[P]$ and the dimension of $\Delta(P)$:

$$\dim [P] = n - \dim \Delta(P).$$

Suppose that $I(P) \neq \emptyset$ and put $I := I(P)$. Consider $f$ as a polynomial $f(z) \in K[z_I]$ with the coefficient ring $K := \mathbb{C}[z_I]$. We use the notation $K[f]$ when we consider $f$ as a polynomial in $K[z_I]$. Note that $\dim \Delta_c(P) = \dim \Delta(P_{I'}, K[f])$ and

$$\dim \Delta(P) = \dim \Delta_c(P) + \sharp I,$$

where $\sharp I$ is the cardinality of $I$. Here $\Delta_c(P)$ is defined by $\Delta(P) \cap \Gamma(f)$.

2.6. **Normalized weight vector**. Take a weight vector $P = (p_1, \ldots, p_n) \in \mathbb{W}_v \cup \mathbb{W}_r$. Then $d(P) > 0$. We consider the rational weight vector $\hat{P} = (\hat{p}_1, \ldots, \hat{p}_n)$ which is defined by $\hat{P} = P/d(P)$. That is, $\hat{p}_i = p_i/d(P)$, $i = 1, \ldots, n$. It is clear that $P$ and $\hat{P}$ are equivalent and $d(\hat{P}) = 1$. We use this notation throughout the paper and we call $\hat{P}$ the normalized weight vector of $P$. If $d(P) = 0$, $P$ does not have any normalized form. Using the normalized weight vector, each monomial in $f_{\hat{P}}(z)$ has weight 1. For given two weight vectors $P$ and $Q$ with $d(P), d(Q) > 0$ and $\Delta(P, Q) := \Delta(P) \cap \Delta(Q) \neq \emptyset$, consider the line segment $LS(P, Q) = \{\hat{P}_t \mid 0 \leq t \leq 1\}$ where $\hat{P}_t := (1 - t)\hat{P} + tQ$. The $i$-component $\hat{p}_i$ of $\hat{P}_t$ is given as $\hat{p}_i = (1 - t)\hat{p}_i + t\hat{q}_i$ and it is monotone (either increasing or constant or decreasing) function in $t$ for any $1 \leq i \leq n$. In particular,
Proposition 4. There is a canonical inequality: 
\[ \hat{p}_i \geq \min \{ \hat{p}_i, \hat{q}_i \} \].

\begin{align*}
 f_1(z) &= z_1^5z_2^2 + z_1^6z_3 + z_2^6z_3^2 + z_3^6z_1^3 \\
 A &= (5, 2, 0) \iff z_1^5z_2^2 \\
 B &= (6, 0, 1) \iff z_1^6z_3 \\
 C &= (0, 6, 2) \iff z_2^6z_3^2 \\
 D &= (3, 0, 6) \iff z_3^6z_1^3
\end{align*}

\textbf{Figure 1.} Newton boundary of \( f_1 \)

\begin{align*}
 E_1 &= (1, 0, 0) \\
 E_2 &= (0, 1, 0) \\
 E_3 &= (0, 0, 1) \\
 P &= \left( \frac{5}{27}, \frac{3}{22}, \frac{1}{11} \right) \\
 Q &= \left( \frac{4}{27}, \frac{7}{54}, \frac{1}{9} \right) \\
 R &= (0, \frac{1}{2}, 1) \\
 S &= \left( \frac{1}{7}, 0, \frac{1}{2} \right) \\
 T &= \left( \frac{1}{3}, \frac{1}{6}, 0 \right) \\
 f_1(z) &= z_1^5z_2^2 + z_1^6z_3 + z_2^6z_3^2 + z_3^6z_1^3
\end{align*}

\textbf{Figure 2.} \( \Gamma^*(f_1) \)
\[ \hat{P} = \left( \frac{4+bc - 2c}{abc+8}, \frac{4+ac - 2a}{abc+8}, \frac{4+ab - 2b}{abc+8} \right) \]
\[ \hat{R} = (0, \frac{1}{2}, \frac{1}{7}), \quad \hat{S} = (\frac{1}{6}, 0, \frac{1}{2}), \quad \hat{T} = (\frac{1}{2}, \frac{1}{7}, 0) \]
\[ f_2 = z_1^a z_2^a + z_2^b z_3^b + z_3^c z_1^c, \quad a, b, c > 2 \]

\textbf{Figure 3. } \Gamma^*(f_2)

\[ \hat{S}_1 = (0, \frac{1}{2}, \frac{1}{2}) \]
\[ \hat{S}_2 = (0, \frac{3}{4}, \frac{1}{2}) \]
\[ \hat{T}_1 = (\frac{1}{4}, 0, \frac{1}{4}) \]
\[ \hat{T}_2 = (\frac{1}{4}, 0, \frac{3}{4}) \]
\[ \hat{R}_1 = (\frac{1}{2}, \frac{1}{2}, 0) \]
\[ \hat{R}_2 = (\frac{3}{4}, \frac{1}{4}, 0) \]
\[ f_3 = z_1^4 z_2^4 + z_2^4 z_3^4 + z_3^4 z_1^4 + z_1 z_2 z_3 \]

\textbf{Figure 4. } \Gamma^*(f_3)

2.7. **Examples.** In the following, polynomials in Example 1, Example 2 and Example 3 are all non-degenerate and strongly inv-tame.
Example 5. Consider \( f_1(z) = z_1^4z_2^2 + z_2^4z_3^2 + z_2^6z_3^2 + z_1^3z_3^6 \). Figure 1 and Figure 2 show the Newton boundary and the dual Newton diagram of \( f_1 \) respectively. In Figure 2, we see 8 equivalence classes which correspond to the vertices \( P, Q \) respectively. In Figure 2, we see 8 equivalence classes which correspond to the interiors of four polyhedra in Figure 2. Here \( E_1, E_2, E_3 \) are the standard basis of \( \mathbb{R}^3 \).

Example 6. (Weighted homogeneous case) Consider the weighted homogeneous polynomial \( f_2(z) = z_1^2z_2^2 + z_2^4z_3^2 + z_3^6z_1^2 \) with \( a, b, c > 2 \). The dual Newton diagram is given in Figure 3.

Example 7. Consider the polynomial \( f_3(z) = z_1^4z_2^2 + z_2^4z_3^2 + z_3^4z_1^2 + z_1z_2z_3 \). This polynomial is special in the sense that its Newton boundary \( \Gamma(f) \) does not have any compact 2-dimensional face. See Figure 4.

2.8. Ratio maps for curves in the coordinate subspaces. Assume that \( \mathbb{C}^f \) is a non-vanishing coordinate subspace. Let \( \mathcal{P}_I \) be the set of analytic curves \( C: z = z(t), 0 \leq t \leq 1 \) such that its image is in \( \mathbb{C}^f \subset \mathbb{C}^n \), \( z(0) = 0 \) and \( z(t) \in \mathbb{C}^f \setminus V(f^I) \) for \( t \neq 0 \). For a curve \( z(t) \in \mathcal{P}_I \), we consider the ratio map

\[
\theta: \mathcal{P}_I \to [0, 1), \quad \theta(C) = \theta(z(t)) = \frac{\text{ord}_t \| \partial f(z(t)) \|}{\text{ord}_t f(z(t))}.
\]

Note that \( \| \partial f(z(t)) \| \) is measured in \( \mathbb{C}^n \). Consider a modified curve \( \tilde{z}(t) \) defined as \( \tilde{z}_i(t) = z_i(t) \) for \( i \in I \) and \( = t^N \) for \( i \notin I \). Note that \( \tilde{z}(t) \in \mathbb{C}^n \) for \( t \neq 0 \). Let \( \tilde{P} = (\tilde{p}_1, \ldots, \tilde{p}_n) \) be the weight vector of \( \tilde{z}(t) \). Thus \( \tilde{p}_i = p_i \) for \( i \in I \) and \( \tilde{p}_i = N \) for \( i \notin I \).

Proposition 8. Taking \( N \) sufficiently large we have that

\[
\text{ord}_t \partial f(\tilde{z}(t)) = \text{ord}_t \partial f(z(t)) \leq \text{ord}_t f(\tilde{z}(t)),
\]

\[
\text{ord}_t f(z(t)) = \text{ord}_t f(\tilde{z}(t)).
\]

Thus we have the equality:

\[
\frac{\text{ord}_t \partial f(z(t))}{\text{ord}_t f(z(t))} = \frac{\text{ord}_t \partial f(\tilde{z}(t))}{\text{ord}_t f(\tilde{z}(t))}.
\]

Proof. First observe that the difference

\[
f(z) - f(z_1), \quad \frac{\partial f}{\partial z_i}(z) - \frac{\partial f}{\partial z_i}(z_1) \equiv 0 \mod (z_j)_{j \notin I}.
\]

Here \( (z_j)_{j \notin I} \) is the ideal generated by \( z_j, j \notin I \). Therefore taking \( N \) sufficiently large we may assume that

(i) \( \Delta(P, f) = \Delta(P, f^I) \) and

(ii) \( \text{ord}_t \left( \frac{\partial f}{\partial z_j}(z(t)) - \frac{\partial f}{\partial z_j}(\tilde{z}(t)) \right) \geq N \)

and (iii) \( \text{ord}_t f(z(t)) = \text{ord}_t f(\tilde{z}(t)) \). Then the assertion follows immediately. \( \square \)
Thus we have

**Corollary 9.** \( \theta(z(t)) = \theta(\tilde{z}(t)) \).

Therefore for the estimation of \( \theta_0(f) \), it is enough to consider the case \( I = \{1, \ldots, n\} \), i.e. \( z(t) \in \mathbb{C}^n \) for \( t \neq 0 \).

**Lemma 10.** Consider a weight vector \( P \in \mathcal{W}_+ \) and assume that \( f_P(z) = cz^\nu, c \neq 0 \). Consider a normalized weight \( \hat{P} \). Put \( \hat{p}_{\text{max}} := \max\{\hat{p}_i | i \in \text{Var}(P)\} \) and \( |\nu| := \sum_{i \in \text{Var}(P)} \nu_i \). Then \( \hat{p}_{\text{max}} \geq \frac{1}{|\nu|} \).

**Proof.** The assertion is immediate from the equality
\[
1 = \sum_{i \in \text{Var}(P)} \hat{p}_i \nu_i \leq \hat{p}_{\text{max}} |\nu|.
\]

\( \square \)

**Corollary 11.** Consider a curve \( C \) parametrized as in Lemma 10. Then
\[
\theta(C) = 1 - \hat{p}_{\text{max}} \leq 1 - \frac{1}{|\nu|}.
\]

The assertion follows from the observation: \( \frac{\partial f_P}{\partial z_j}(a) \neq 0 \) for any \( j \in \text{Var}(P) \) and \( a \in \mathbb{C}^n \).

### 2.9. Ratio maps for weight vectors

We assume that \( f \) is strongly inv-tame. Let \( \mathcal{W}_+ \) be the set of strictly positive weight vectors and let \( \mathcal{W}_v \) be the subset of positive weight vectors such that \( I(P) \neq \emptyset \) and \( d(P) > 0 \). Let \( C = \{z = z(t)\} \in \mathcal{P} \) and consider the Taylor expansion
\[
z_i(t) = a_i t^{p_i} + (\text{higher terms}), \quad a_i \neq 0, \quad 1 \leq i \leq n.
\]

We consider the weight map \( wt : \mathcal{P} \to \mathcal{W} \) by \( wt(z(t)) = P \) where \( P = (p_1, \ldots, p_n) \). We want to estimate the ratio \( \theta(z(t)) \) using the weight vector \( wt(z(t)) \).

**Definition 12.** We define the ratio maps for weight vector \( P \in \mathcal{W}_+ \cup \mathcal{W}_v \) as follows. First we put
\[
\hat{p}_{\text{min}} := \min\{\hat{p}_j | j \in \text{Var}(P)\},
\]
\[
\hat{p}'_{\text{min}} := \min\{\hat{p}_j | j \in \tilde{\text{Var}}(P)\}.
\]

We define
\[
\theta_i(P) = 1 - \hat{p}_i, \quad i \in \tilde{\text{Var}}(P), \quad \dim \Delta(P) \geq 1,
\]
\[
\theta(P) = 1 - \hat{p}_{\text{min}},
\]
\[
\theta(P)' = \begin{cases} 1 - \hat{p}'_{\text{min}}, & \text{dim } \Delta(P) \geq 1, \\ 1 - \frac{1}{|\nu|}, & \text{dim } \Delta(P) = 0, \quad f_P(z) = c_e z^\nu. \end{cases}
\]

Here \( P = (p_1, \ldots, p_n) \) and \( \hat{P} = (\hat{p}_1, \ldots, \hat{p}_n) \) is the normalized weight vector of \( P \).
As a special case, we have

**Proposition 13.** Assume that \( \dim \Delta(P) = n - 1 \). Then \( I(P) = \overline{I}(P) \) and \( \hat{p}_{\min} = \min \{ \hat{p}_j \mid j \in \text{Var}(P) \setminus I(P) \} \). In particular, \( \hat{p}_{\min} = \hat{p}_0 \) if \( P \in \mathcal{W}_+ \).

### 2.10. Admissible line segments.

We assume that \( f \) is strongly inv-tame and non-degenerate. Consider a weight vector \( R \in \mathcal{W}_+ \) with \( \dim \Delta(R) \geq 1 \). Consider a line segment \( LS(P, Q) \) passing through \( R \) with two weight vectors \( P, Q \) on the boundary of \( [R] \). Recall that \( LS(P, Q) = \{ P_s = (1 - s)P + sQ \mid 0 \leq s \leq 1 \} \). By the assumption, \( R = P_{s_0} \), \( 0 < \exists s_0 < 1 \). We divide the situation into three cases depending the end points \( P, Q \).

#### 2.10.1. Strictly positive line segment.

Let \( R \) be as above. We say that the boundary of \( [R] \) is strictly positive if the closure of the equivalence class \( [R] \) contains only strictly positive weight vectors. Thus \( P, Q \in \mathcal{W}_+ \). We use the line segment expression using the normalized vectors \( \hat{P}_s = (1 - s)\hat{P} + s\hat{Q} \).

Then \( R = P_{s_0} \) for some \( s_0 \). As the normalized weight vector \( \hat{P}_s \) is given as \( \hat{p}_{s_j} = (1 - s)\hat{p}_j + s\hat{q}_j \) is a monotone linear function in \( s \), it is easy to see that

\[
\begin{align*}
\theta_j(\hat{R}) &\leq \max\{\theta_j(\hat{P}), \theta_j(\hat{Q})\}, \quad j \in \text{Var}(\hat{R}), \\
\theta(\hat{R})' &\leq \max\{\theta(\hat{P}), \theta(\hat{Q})\}.
\end{align*}
\]

Note that in this case, \( \overline{I}(R) = \emptyset \) and \( \theta(\hat{R})' = \theta(\hat{R}) \), \( \theta(\hat{P})' = \theta(\hat{P}) \) and \( \theta(\hat{Q})' = \theta(\hat{Q}) \), as \( \overline{I}(R) = \emptyset \). For example, take \( R \) on the line segment \( LS(P, Q) \) in Example 1 (Figure 2).

**Remark 14.** Note that \( \text{Cone}(LS(P, Q)) = \text{Cone}(LS(\hat{P}, \hat{Q})) \), though \((1 - t)\hat{P} + t\hat{Q} \) is not necessarily the normalized vector of \((1 - t)P + tQ \). Here for a subset \( K \subset N^+ \), we put \( \text{Cone}(K) := \{ rP \mid P \in K, r > 0 \} \).

#### 2.10.2. Vanishing line segment.

We say that \( LS(P, Q) \) is a vanishing line segment if \( P, Q \) are in \( \mathcal{W}_+ \cup \mathcal{W}_v \) and at least one of \( P \) or \( Q \) is in \( \mathcal{W}_v \).

**Lemma 15.** Assume that \( LS(P, Q) \) is a vanishing line segment. We assume that \( P \in \mathcal{W}_v, Q \in \mathcal{W}_+ \cup \mathcal{W}_v \). Consider the family of the normalized weight vectors \( \hat{P}_s \) for the line segment \( LS(P, Q) \) which is defined as \( \hat{P}_s = (1 - s)\hat{P} + s\hat{Q} \) for \( 0 \leq s \leq 1 \) and \( \hat{R} = \hat{P}_{s_0} \) \((0 < \exists s_0 < 1) \). Then

\[
\begin{align*}
\theta_j(\hat{R}) &\leq \max\{\theta_j(\hat{P}), \theta_j(\hat{Q})\}, \quad j \in \overline{\text{Var}}(R), \\
\theta(\hat{R})' &\leq \max\{\theta(\hat{P})', \theta(\hat{Q})'\}.
\end{align*}
\]

**Proof.** By the strong inv-tameness, \( \overline{\text{Var}}(R) \neq \emptyset \) and there exists a \( j \in \overline{\text{Var}}(R) \) so that \( \frac{\partial f}{\partial z_j}(a) \neq 0 \). This implies \( \theta(\hat{R})' \leq 1 - \hat{p}_{s_0,j} \). The assertion follows from the monotonicity of \( \hat{p}_{tj} \). \( \square \)
2.10.3. Non-vanishing line segment. A line segment LS($P, Q$) is called non-vanishing line segment if one of $P, Q$ is a non-vanishing weight vector. Assume that $P \in W_{nw}$ and $Q \in W_+ \cup W_v$ so that $\Delta(Q) \cap \Delta(P) \supset \Delta(R)$. Recall that LS($P, Q$) is defined by $\{P_s | 0 \leq s \leq 1\}$ where $P_s := (1-s)P + sQ$ and $R = (1-s_0)P + s_0Q$ as before. The normalized weight vectors of this family is written as $\tilde{P}_s := \tau P + Q$ with $\tau = \frac{1-s}{s}$ for $s \neq 0$. In this parameter $\tau$, $0 \leq \tau < \infty$ and $\tilde{R} = \tilde{P}_{\tau_0}$, $\exists \tau_0 > 0$. Note that $\hat{p}_{r,j}$ is monotone increasing (or constant) in $\tau$ for any $j$. That is $\hat{p}_{r,j} \geq \hat{p}_{0,j} = q_j$.

Lemma 16. We have the inequality:

\[ \theta_j(\tilde{R}) \leq \theta_j(\tilde{Q}), \quad j \in \text{Var}(R), \]
\[ \theta(\tilde{R})' \leq \theta(\tilde{Q})'. \]

Remark 17. We do not need to consider the case where $P, Q$ are both non-vanishing, as $R$ is assumed to be in $W_+ \cup W_v$. In the inductive argument on $\dim[R]$, if the line segment is as in Lemma 16, we continue to work only for $Q$.

3. Main result on non-degenerate functions

3.1. Convenient case. Assume that $f(z) = \sum c_v z^\nu$ is a convenient non-degenerate analytic function in the sense of Kouchnirenko [10] and let $b_j$ be the point $\Gamma(f) \cap \{j-th \text{ coordinate axis}\}$. Then $V(f)$ has an isolated singularity at the origin (Theorem (3.4), [15], Corollary 20, [19]). Consider an analytic curve $z(t)$, $0 \leq t \leq 1$ as in the previous section. Namely $z(0) = 0$ and $z(t) \in \mathbb{C}^n \setminus f^{-1}(0)$ for $t \neq 0$. We first assume that $z(t) \in \mathbb{C}^n$ for $t \neq 0$. Assume that $z(t)$ has the following Taylor expansion:

\[ z_j(t) = a_j t^\nu_j + \text{(higher terms)}, \quad a_j \neq 0, \quad j = 1, \ldots, n, \]
\[ \frac{\partial f}{\partial z_j}(z(t)) = \frac{\partial f_P}{\partial z_j}(a) t^{d(P)-p_j} + \text{(higher terms)}, \]
\[ f(z(t)) = b d' + \text{(higher terms)}, \quad b \neq 0, \quad d' \geq d(P). \]

where $P = (p_1, \ldots, p_n)$, $a = (a_1, \ldots, a_n)$. We use the same notation as in [15]. Put $p_{\text{min}} = \min\{p_j \mid j \in \text{Var}(P)\}$. Choose index $1 \leq \alpha \leq n$ so that $p_{\alpha} = p_{\text{min}}$. Note that $\alpha$ may not unique but we fix it. If $f(z)$ is non-degenerate, then there exists $j_0$ so that $\frac{\partial f}{\partial z_{j_0}}(a) \neq 0$. Thus

\[ \text{ord}_t \frac{\partial f(z(t))}{f(z(t))} \leq \frac{d(P) - p_{\text{min}}}{d'} \leq \frac{d(P) - p_{\text{min}}}{d(P)}. \]

Use the normalized weight $\tilde{P} = (\hat{p}_1, \ldots, \hat{p}_n)$ where $\hat{p}_j := p_j / d(P)$. The right side of (7) is equal to $1 - 1/b'_{\alpha}$ where $b'_{\alpha}$ is the $\alpha$-coordinate so that $\hat{p}_{\alpha} b'_{\alpha} = 1$. This may not be an integer but we know that $b'_{\alpha} \leq b_{\alpha}$. Thus we obtain

\[ \text{ord}_t \frac{\partial f(z(t))}{f(z(t))} \leq 1 - \frac{1}{b_{\alpha}}, \quad \forall \alpha, \quad p_{\alpha} = p_{\text{min}}. \]
Proposition 19. If there exists an analytic curve \( z \) and thus \( z \) is tame function. Then the Lojasiewicz exponent of type (1) has the estimation:
\[
\theta(z(t)) = \frac{\text{ord}_t f(z(t))}{\text{ord}_t f(z(t))} \leq 1 - 1/B.
\]

Definition 18. The monomial \( z_j^{b_j} \) is called Lojasiewicz monomial if \( b_j = B \), i.e. \( j \in I_B \). The monomial \( z_j^{b_j} \) is called Lojasiewicz exceptional if \( j \in I_B \) and there exists \( k \neq j \) and a monomial \( z_j^{B'} z_k \) in \( f(z) \) with \( B' < B - 1 \). Otherwise \( z_j^B \) is called a non-exceptional Lojasiewicz monomial (20).

Proposition 19. If \( f \) has a non-exceptional Lojasiewicz monomial, there exists an analytic curve \( z(t) \) so that the equality, \( \theta(z(t)) = 1 - 1/B \), holds and thus
\[
\theta_0(f) = 1 - \frac{1}{B}.
\]

Proof. To see this, assume that \( z_j^{b_j} \) is a non-exceptional Lojasiewicz monomial. Consider the analytic curve \( z(t) \) which is defined by \( z_j(t) = t \) and \( z_j(t) = t^N \) for any \( j \neq j_0 \) where \( N \) is a sufficiently large positive integer. Then it is easy to see that \( \frac{\partial f}{\partial z_j}(z(t)) = ct^{B-1} + (\text{higher terms}), c \neq 0 \). If the derivative \( \frac{\partial f}{\partial z_j}(z) \), \( j \neq j_0 \) contains a monomial \( z_j^{a_j} \), it comes from the monomial \( z_j z_j^{a_j} \) in \( f(z) \). By the assumption, \( a \geq B - 1 \). Thus \( \text{ord}_t \frac{\partial f}{\partial z_j}(z(t)) \geq B - 1 \) for \( j \neq j_0 \) and \( f(z(t)) = ct^B, c \neq 0 \). Therefore it is easy to see that the equality is satisfied.

Theorem 20. Assume that \( f(z) \) is a convenient non-degenerate function. Then \( \theta_0(f) \leq 1 - 1/B \). Furthermore if there exists a non-exceptional Lojasiewicz monomial, the equality holds.

Example 21. Consider \( f(z) = z_1^5 + z_1^3 z_2 + z_2^4 + z_3^4 \). Then \( B = 5 \) but \( z_1^5 \) is an exceptional Lojasiewicz monomial. In fact, Lojasiewicz exponent is given by \( \theta_0(f) = 1 - \frac{1}{4} = 3/4 \).

3.2. Non-convenient case. We assume that \( f(z) \) is non-degenerate and strongly inv-tame but we do not assume the convenience of \( f \). The singularity is not necessarily isolated. Let \( \mathcal{D} \) be the set of equivalent classes \([P]\) with \( \dim[P] = n \).

For a \([P]\) \( \in \mathcal{D} \), the face function is given as a monomial function \( f_P(z) = cz^{\nu(P)} \) and we associate the total degree \( |\nu(P)| \) to \([P]\).

Main Theorem 22. Assume that \( f \) is non-degenerate and strongly inv-tame function. Then the Lojasiewicz exponent of type (1) has the estimation:
\[
\theta_0(f) \leq \max \left\{ L, \tilde{\theta} \right\}
\]
where
\[
\tilde{\theta} := \max \{1 - \nu_{\text{min}} | P \in \mathcal{W}_+ \cup \mathcal{W}_e, \dim \Delta(P) = n - 1 \},
\]
\[
L = \max \{ 1 - 1/|\nu(P)| | [P] \in \mathcal{D} \}.
\]
Proof. Consider an analytic curve $C \subset \mathcal{P}$ defined by $z = z(t)$, $0 \leq t \leq 1$ and $f(z(t)) \neq 0$ for $t \neq 0$. We may assume that $z(t) \in \mathbb{C}^n$ for $t \neq 0$ by Proposition 8. Assume that $z(t)$ has the following expansion:

\begin{equation}
z_j(t) = a_j t^{r_j} + \text{(higher terms)}, \quad a_j \neq 0, \, j = 1, \ldots, n,
\end{equation}

\begin{equation}
\frac{\partial f}{\partial z_j}(z(t)) = \frac{\partial f_R}{\partial z_j}(a) t^{d(R) - p_j} + \text{(higher terms)},
\end{equation}

\begin{equation}
f(z(t)) = bt^d + \text{(higher terms)}, \quad d' \geq d(R).
\end{equation}

where $R = (r_1, \ldots, r_n)$, $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. We use the same notation as in [18]. If $\Delta(R)$ is a vertex, $[R] \in \mathcal{D}$ and it is clear that $\frac{\partial f_R}{\partial z_j}(a) \neq 0$ for any $j \in \text{Var}(R)$ and $\theta(z(t)) \leq 1 - \frac{1}{r}$ by Lemma 10.

Now we assume that $\dim \Delta(R) \geq 1$. Assume that $\frac{\partial f_R}{\partial z_j}(a) \neq 0$ for some $j \in \overline{\text{Var}}(P)$, it is clear from the definition that $\theta(C(t)) \leq 1 - \tilde{p}_j$. Thus $\theta(z(t)) \leq \theta(R)'$ by the non-degeneracy and the strong inv-tameness. Thus the proof is reduced to the following assertion.

**Assertion 23.** Assume that $\dim \Delta(R) \geq 1$. Then

$$
\theta(R)' \leq \max\{\theta(P)' \mid P > R, \, \dim \Delta(P) = n - 1, \, P \in \mathcal{W}_+ \cup \mathcal{W}_0\}.
$$

This is proved easily on the induction on $\dim [R]$, using Lemma 15 and Lemma 16. If $\dim [R] = 1$, $R$ is a vertex and there are nothing to be proved. Take an admissible line segment $LS(P, Q)$, $P_s = (1-s)P + sQ$ and $R = P_{s_0}$ for some $0 < s_0 < 1$. By Lemma 15 and Lemma 16, we have the estimation

$$
\theta(R)' \leq \max\{\theta(P)' \mid P > R, \, \dim \Delta(P) = n - 1, \, P \in \mathcal{W}_+ \cup \mathcal{W}_0\}
$$

for $P, Q \in \mathcal{W}_+ \cup \mathcal{W}_0$ and the assertion holds for $R$ by the strong local inv-tameness. The assertion holds by the induction’s hypothesis for $P$ and $Q$ if $P, Q \in \mathcal{W}_+ \cup \mathcal{W}_0$. If $P$ is non-vanishing, the estimation is simply replaced by $\theta(R)' \leq \theta(Q)'$. \qed

**3.3. Examples of the estimation of $\theta_0(f)$.**

**Example 24.** Consider the polynomial $f_1(z) = z_1^4 z_2^2 + z_1^6 z_3 + z_2^6 z_3^2 + z_3^3 z_1^6$ considered in Example 5. We have

$$
\theta(P)' = \frac{10}{11}, \quad \theta(Q)' = \frac{8}{9}, \quad \theta(R)' = \frac{1}{2}, \quad \theta(S)' = \frac{4}{5}, \quad \theta(T)' = \frac{5}{6}
$$

The region $A, B, C, D$ corresponds to the monomials $z_1^5 z_2^2, z_1^6 z_3, z_2^6 z_3^2, z_3^3 z_1^6$ respectively and these region give the bounds 6/7, 6/7, 7/8, 8/9 respectively. Thus $\theta_0(f_1) \leq \frac{10}{11}$ by Theorem 22.

**Remark 25.** The estimation by Main Theorem 22 is not always sharp. In fact, the equality in the above estimation can not be obtained. For the weight vector $P$, $f_1 P(z) = z_1^6 z_3 + z_1^4 z_2 + z_1^4 z_3^6$ and we see that $\frac{\partial f_1 P}{\partial z_2} \neq 0$ on $\mathbb{C}^3$. As $\hat{P} = (\frac{5}{33}, \frac{3}{22}, \frac{1}{11})$, the real contribution for $P$ is from $\frac{\partial f_1 P}{\partial y}$. Thus
\[ \theta(z(t)) \leq 1 - 3/22 = 19/22 \] for any \( z(t) \) with \( \text{wt}(z(t)) = P \). The contribution from \( Q \) is in fact sharp. Note that \( f_{1Q} \) is given by \( z_1^5 z_2^2 + z_1^6 z_3 + z_2^2 z_3^2 + z_1^2 z_1^2 \) and one can find \( z(t) \) with the coefficient vector \( \mathbf{a} \) satisfies \( \frac{\partial f_{1Q}}{\partial z_1}(\mathbf{a}) = \frac{\partial f_{1Q}}{\partial z_2}(\mathbf{a}) = 0 \). Thus \( \theta(z(t)) = \frac{8}{9} \). As \( \frac{8}{9} > \frac{19}{22} \), we conclude \( \theta_0(f_1) = \frac{8}{9} \).

3.4. Is \( \theta_0(f) \) a moduli invariant? For a given non-degenerate function \( f(z) \), we ask if the Łojasiewicz exponents are constant or not on the moduli space. For this purpose, we consider the branched poly-cyclic covering \( \varphi_2 : \mathbb{C}^n \to \mathbb{C}^n \) and its lift of \( f \), defined by \( \varphi_2(w) = z, z_i = w_i^2 (1 \leq i \leq n) \) and put \( f^{(2)}(w) := \varphi^*f(w) = f(w_1^2, \ldots, w_n^2) \). More precisely we consider \( f_1(z) = z_1^5 z_2^2 + z_1^6 z_3 + z_2^2 z_3^2 + z_1^2 z_1^2 \) in Example 1. Put \( f_1^{(2)}(w) := f_1(\varphi_2(w)) = w_1^{10} w_2^4 + w_1^{12} w_3^4 + w_2^6 w_3^4 + w_3^6 z_3^4 \). The dual Newton diagram \( \Gamma^*(f^{(2)}) \) is given by the same diagram of \( \Gamma^*(f_1) \). Only change is that \( d(K, f^{(2)}) = 2d(K, f_1) \) for any weight vector \( K \). Thus in the normalized vectors, \( P, Q, R, S, T, \) are to be divided by 2. By the same discussion as in the above Remark, we see that \( \theta_0(f_{1_0}^{(2)}) = \frac{12}{18} \). Let \( g(w) := f_1^{(2)}(w) + w_1^2 w_2^8 w_3^8 \). Note that the new monomial \( w_3^4 w_6^2 w_3^8 \) corresponds to the midpoint of the edge \( C^{(2)}D^{(2)} \). Here \( C^{(2)}, D^{(2)} \) are the lift of \( C, D \) in \( \Gamma(f^{(2)}) \). Thus the dual Newton diagram of \( g \) is the same with \( \Gamma^*(f^{(2)}) \). By the result of [3], the family \( g_t(w) := f_1^{(2)}(w) + t w_1^2 w_2^8 w_3^8 \) is non-degenerate and strongly locally inv-tame except a finite exceptional \( t \)'s. The exceptional set \( S \) is the union of \( \{ \pm 2 \} \) from the non-degenerated of \( g_t, T \) and possibly some more \( t \)'s from the non-degeneracy of \( g_{1_0} \). Actually \( f_{1_0} = 0 \) is non-singular in \( \mathbb{C}^* \) for \( t \neq \pm 2 \) as we can see by a direct calculation, \( \frac{\partial g_{1_0}}{\partial z_1} = \frac{\partial g_{1_0}}{\partial z_2} = \frac{\partial g_{1_0}}{\partial z_3} = 0 \) has no solution in \( \mathbb{C}^* \).

Thus \( S = \{ \pm 2 \} \). This family has a canonical Whitney regular stratification and \( V(f^{(2)}) \) and \( V(g) \) are topologically equivalent for any \( t \in \mathbb{C} \setminus S \). We assert \( \theta_0(g) = \frac{21}{22} \) (\( = 1 - \frac{6}{132} \)) which comes from the vertex \( P = (10, 9, 6) \) with \( d(P, g) = 132 \). Thus \( \theta_0(g) > \theta_0(f^{(2)}) \) and \( \theta_0 \) is not constant on the moduli space of \( g \). Here we mean by moduli the space of functions with fixed Newton boundary and local tameness. For example, we can choose the following curve which satisfies \( \frac{\partial g}{\partial w_1} = \frac{\partial g}{\partial w_2} = 0 \):

\[ w_1(s) = \frac{1}{2} \sqrt[3]{3} \sqrt[6]{2} \sqrt[9]{-1} s^{10}, \quad w_2(s) = \sqrt[3]{-1} \sqrt[6]{3} \sqrt[9]{s^9}, \quad w_3(s) = s^6. \]

We can see that

\[ \text{ord}_s\theta(g(w(s))) = 126, \quad \text{ord}_s\theta(g(w(s))) = 132 \]

\[ \theta(w(s)) = \frac{126}{132} = \frac{21}{22}. \]

Unfortunately \( g_t \) has 1-dimensional singularity.

For isolated singularity case, this does not happen. In fact, Brzostowski proved the Łojasiewicz exponent \( \eta_0(f) \) of the Łojasiewicz inequality of type (2) is constant on the moduli space of functions with fixed Newton boundary.
and an isolated singularity at the origin (Theorem 1, [4]). On the other hand, \(\theta_0(f)\) and \(\eta_0(f)\) are related by \(\theta_0(f) = \eta_0(f)/(1 + \eta_0(f))\) by Teissier [23]. I thank Professor Tadeusz Krasiński for this information.

**Example 26.** Consider the simplicial weighted homogeneous polynomial
\[
f_2(z) = z_1^2 z_2^3 + z_2 z_3^2 + z_3 z_1^2 + z_1 z_2 z_3
\]
in Example 2. Then normalized weight is given as \(\hat{P} = (\frac{4-2c+8}{abc+8}, \frac{ac+4-2a}{abc+8}, \frac{ab+1-2b}{abc+8})\). Suppose \(c \geq a, b\). Then the contribution from \(\hat{P}\) is \(1 - \frac{ab-2bc+4}{abc+8}\). The contribution from \(\hat{R}, \hat{S}, \hat{T}\), which are \(\theta(\hat{R})', \theta(\hat{S})', \theta(\hat{T})'\), are given by \(1 - 1/c, 1 - 1/b, 1 - 1/a\) respectively by Theorem 22 but these estimation is not sharp. For example, \(f_{\hat{R}} = z_1^7 z_2^2 + z_2^2 z_1^3\) and \(\frac{\partial f_{\hat{R}}}{\partial z_2}\) can not be zero on \(\mathbb{C}^*\). Thus the real contribution is \(1 - 1/2 = 1/2\). The same is true for \(\hat{S}, \hat{T}\). Thus \(\theta_0(f) = 1 - \frac{ab-2bc+4}{abc+8}\).

**Example 27.** Consider \(f_3(z) = z_1^4 z_2^2 + z_2^4 z_3^2 + z_3^4 z_1^2 + z_1 z_2 z_3\) of Example 3 (See Figure 4). This polynomial has no compact face of dimension 2 in \(\Gamma(f)\). We observe that \(\theta(S_1)' = \theta(T_1)' = \frac{1}{7}\) and \(\theta(S_2)' = \theta(T_2)' = \theta(R_2)' = \frac{3}{7}\). \(\mathcal{D}\) contains 4 regions. The pentagon with vertices \(S_2, S_1, T_2, T_1, R_2, R_1\) contribute by \(\frac{3}{7}\). The other triangles contribute by \(\frac{5}{6}\). Thus we have \(\theta_0(f_3) \leq \frac{5}{6}\). In fact, \(\theta_0(f_3) = \frac{5}{6}\). To see this, consider the triangle region \(S_1 T_2 E_3\) in Figure 4 and take an analytic curve, for example, \(z(t) = (t, t, t^N)\) for a sufficiently large. Then the weight vector is given by \(P = (1, 1, N)\) or \(\hat{P} = (\frac{1}{6}, \frac{1}{6}, \frac{N}{6})\) and \(f_{3P} = z_1^4 z_2^2\) and \(\theta(z(t)) = \frac{5}{6}\).

3.5. \(\theta_0(f)\) does not behave like Milnor numbers. We give another example of a delicate behavior of \(\theta_0(f)\). Assume that \(f(z), g(z)\) have isolated singularities at the origin and they are non-degenerate. Let \(\Gamma_-(f), \Gamma_-(g)\) be the cones of \(\Gamma(f), \Gamma(g)\) with the origin. Assume that \(\Gamma_-(g) \supseteq \Gamma_-(f)\) and \(\Gamma(f) \cap \mathbb{R}^I = \Gamma(g) \cap \mathbb{R}^I\) for any \(I \subseteq \{1, \ldots, n\}\). Then by Kouchnirenko’s theorem ([10]), the Milnor numbers satisfies the inequality: \(\mu(g) > \mu(f)\). This is not always true for \(\theta_0(g)\) and \(\theta_0(f)\).

As an example, consider \(g_4(z) = z_1^7 z_2 + z_2^3 z_3 + z_1 z_2 z_3\) and \(f_4(z) = g_4(z) + z_1^2 z_2^2 z_3\). Note that \(\Gamma_-(g_4) \supsetneq \Gamma_-(f_4)\). First, their Milnor numbers are given as \(\mu(g_4) = 990\) and \(\mu(f_4) = 543\). Their dual Newton boundaries are given as Figure 5 and Figure 6. Lojasiewicz exponent is given as \(\theta_0(g_4) = \frac{910}{991} = 0.91\ldots\). On the other hand, \(\theta_0(f_4) \leq \frac{95}{101} = 0.94\ldots\) which comes from \(T_3 = (\frac{35}{101}, \frac{19}{202}, \frac{6}{101})\). In fact, we can show that the equality \(\theta_0(f) = \frac{95}{101}\) is taken by the following curve:
\[
z_1(t) = b_1 t^{70}, \quad z_2(t) = b_2 t^{19}, \quad z_3(t) = b_3 t^{12}
\]
where \(b \in \mathbb{C}^3\) satisfies the equality
\[
\frac{\partial f_{T_3}}{\partial z_1}(b) = \frac{\partial f_{T_3}}{\partial z_2}(b) = 0
\]
\[
f_{T_3}(z) = z_2^{10} z_3 + z_3^{11} z_1 + z_1^2 z_2^2 z_3.
\]
Then we see that $\text{ord}_t \partial f_4(z(t)) = 190$, $\text{ord}_t f_4(z(t)) = 202$. For example, we can take

$$b_1 = \sqrt[6]{\frac{5}{16}}, b_2 := \sqrt[12]{\frac{1}{20}}, b_3 = -1.$$ 

Thus we have the inequality: $\theta_0(f_4) > \theta_0(g_4)$, while $\mu(f_4) < \mu(g_4)$.

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure5}
\caption{$\Gamma^*(g_4)$}
\end{figure}
4. Lojasiewicz exponents of non-irreducible functions

In this section, we study Lojasiewicz exponents of reducible functions which are associated with non-degenerate complete intersection varieties.

4.1. A function associated with a convenient non-degenerate complete intersection variety. We consider a family of functions $F := \{f_1, \ldots, f_k\}$. We say $F$ is a defining family of convenient non-degenerate complete intersection varieties if each $f_\alpha$ is a convenient non-degenerate function such that for each $I \subset \{1, \ldots, k\}$, $V(I) := \{z \in \mathbb{C}^n | f_i(z) = 0, i \in I\}$ is a non-degenerate complete intersection variety in the sense of Khovanskii (\cite{9, 18}). Namely for any strictly positive weight vector $P$, the variety $V_I(P)^* := \{z \in \mathbb{C}^n | f_jP(z) = 0 | j \in I\}$ is a smooth complete intersection variety in $\mathbb{C}^n$. We consider the product function $f = f_1 \ldots f_k$ and we call $f$ as the function associated with the family $F$. Note that $f$ is also a convenient function. In \cite{7}, we considered a similar family, which we called a Newton-admissible family $F = \{f_{1s}, \ldots, f_{ks}\}$ with one parameter $s \in \mathbb{C}$ and the associated product function $f_s = f_{1s} \ldots f_{ks}$ in $s \in \mathbb{C}$ to study the topological stability of the one-parameter family of hypersurfaces $V_s = f_s^{-1}(0)$.

In this paper, we assume that each function $f_j$ is convenient and has no parameter $s$. However the argument below can be done in the exact same way for one parameter family if their Newton boundary $\Gamma(f_{js})$ is independent of $s$. As in §3.1, we put $b_{\alpha,j}, 1 \leq \alpha \leq k, 1 \leq j \leq n$ be the unique point of
\( \Gamma(f_0) \) on \( z_j \)-axis. So \( z_j^{b_0} \) is a monomial of \( f_0(z) \) with non-zero coefficient.

We will give an explicit estimation for \( \theta_0(f) \). The hypersurface \( V(f) \) has \( k \) irreducible components \( V(f_\alpha), \alpha = 1, \ldots, k \) and \( V(f) \) has non-isolated singularities if \( k \geq 2 \). The singular locus of \( V(f) \) is the union \( \cup_{i \neq j} V(f_i, f_j) \).

Here \( V(f_i, f_j) := \{ f_i = f_j = 0 \} \).

We are interested in the estimation of the Lojasiewicz component of \( f \) on the \( - \)-axis. So
\[
\sum_{i=1}^k \theta_i \leq \theta_0(f) = \max \{ \theta_i \mid \theta_i \neq 0 \}.
\]

(Actually we also consider the case \( z(t) \in \mathbb{C}^n \setminus V(f) \) for \( t \neq 0 \) and we consider its Taylor expansion:
\[
(11) \quad z_j(t) = a_j t^{p_j} + \text{(higher terms)}, \quad p_j > 0, a_j \neq 0, j = 1, \ldots, n.
\]

We use this expansion assumption throughout this paper. Note that \( d = \sum_{\alpha=1}^k d_\alpha \) and \( c = c_1 \cdots c_k \). Put \( e_\alpha := d(P, f_\alpha) \) and \( e = \sum_{\alpha=1}^k e_\alpha \). We observe that
\[
\frac{\partial f_\alpha}{\partial z_j}(z(t)) = \frac{\partial f_\alpha P}{\partial z_j}(a) t^{e_\alpha - p_j} + \text{(higher terms)}
\]
\[
\partial f(z(t)) = \sum_{\alpha=1}^k \left( \prod_{\beta \neq \alpha} f_\beta(z(t)) \right) \partial f_\alpha(z(t)),
\]

Therefore by (12) and (13),
\[
(15) \quad \frac{\partial f}{\partial z_j}(z(t)) = \sum_{\alpha=1}^k \left( \prod_{\beta \neq \alpha} f_\beta(z(t)) \right) \frac{\partial f_\alpha}{\partial z_j}(z(t))
\]
\[
= \sum_{\alpha=1}^k \left\{ \frac{\partial f_\alpha P}{\partial z_j}(a) c_\alpha \right\} t^{d_\alpha + e_\alpha - p_j} + \text{(higher terms)}
\]

Put \( B_j := \sum_{\alpha=1}^k b_{\alpha, j} \) and \( B = \max \{ B_j \mid j = 1, \ldots, n \} \). Note that \( z_j^{B_j} \) has a non-zero coefficient in the expansion of \( f(z) \) and it corresponds to the unique point of \( \Gamma(f) \) on the \( z_j \)-axis. Let \( p_{\min} := \min \{ p_j \mid j = 1, \ldots, n \} \), \( I_{\min} = \{ j \mid p_j = p_{\min} \} \).
4.1.1. **Case 1.** \( \mathbf{a} \) is generic. We consider the case \( \mathbf{a} \) is generic so that \( f_P(\mathbf{a}) = \prod_{\alpha=1}^{k} f_{\alpha P}(\mathbf{a}) \neq 0 \). This implies that \( d_{\alpha} = e_{\alpha} \) for any \( \alpha = 1, \ldots, k \). As \( f_P(z) \) is a weighted homogeneous polynomial of degree \( e = \sum_{\alpha} e_{\alpha} \) with respect to the weight vector \( P \), 0 is the only possible critical value of \( f_P \). Thus \( d = e = \sum_{\alpha=1}^{k} e_{\alpha} \) and \( \partial f_P(\mathbf{a}) \neq 0 \) and

\[
\frac{\text{ord}_t \partial f(z(t))}{\text{ord}_t f(z(t))} \leq \frac{e - p_{\text{min}}}{e} = 1 - \frac{p_{\text{min}}}{e}.
\]

We define a rational number \( b'_{\alpha,j} \) by \( b'_{\alpha,j} p_j = e_{\alpha} \) for \( j \in I_{\text{min}} \). Then \( p_j B_j' = e \) where \( B_j' = \sum_{\alpha=1}^{k} b'_{\alpha,j} \) and \( j_0 \) is fixed in \( I_{\text{min}} \). Thus \( b'_{\alpha,j_0} \leq b_{\alpha,j_0} \). Note the equality \( p_j \sum_{\alpha=1}^{k} b'_{\alpha,j_0} = p_j B_j' = e \). Thus the above estimation implies

\[
\text{ord}_t \partial f(z(t)) \leq 1 - \frac{1}{B_{j_0}} \leq 1 - \frac{1}{B}.
\]

4.1.2. **Case 2.** \( \mathbf{a} \) is not generic. This case is more complicated. We consider non-generic coefficient vector \( \mathbf{a} \). So we assume that there exists \( \alpha, 1 \leq \alpha \leq k \) such that \( f_{\alpha P}(\mathbf{a}) = 0 \). Consider the defect numbers \( d'_j := d_j - e_j, 1 \leq j \leq k \) and changing the numbering of \( f_j, 1 \leq j \leq k \) if necessary, we assume for simplicity

\[
d'_1 \leq d'_2 \leq \cdots \leq d'_{\ell-1} < d'_\ell = \cdots = d'_k
\]

for some \( \ell, 1 \leq \ell \leq k \). Therefore \( \ell := \min \{ \alpha \mid d_{\alpha} = d'_{k} \} \). Note that \( f_{\alpha P}(\mathbf{a}) \neq 0 \) if and only if \( d'_{\alpha} = 0 \). In particular \( f_{\alpha P}(\mathbf{a}) = 0 \) as \( d'_{\alpha} > 0 \) for \( \ell \leq \alpha \leq k \).

We have the estimation:

\[
\text{ord}_t \left\{ \left( \prod_{\beta \neq \alpha} f_{\beta}(z(t)) \right) \frac{\partial f_{\alpha}}{\partial z_j}(z(t)) \right\} \geq d - d_{\alpha} + (e_{\alpha} - p_j) = d - d'_{\alpha} - p_j, 1 \leq j \leq n.
\]

Note that

\[
d - d'_1 - p_j \geq \cdots \geq d - d'_{\ell-1} - p_j \\
> d - d'_\ell - p_j = \cdots = d - d'_k - p_j.
\]

and finally we have the expression:

\[
\frac{\partial f}{\partial z_j}(z(t)) = \sum_{\alpha=1}^{k} \frac{c}{c_{\alpha}} \frac{\partial f_{\alpha P}}{\partial z_j}(\mathbf{a}) t^{d - d'_j - p_j} + \text{(higher terms)}.
\]

**Assertion 28.** There exists some \( j_0 \) such that \( \text{ord}_t \frac{\partial f}{\partial z_{j_0}}(z(t)) = d - d'_\ell - p_{j_0} \).

**Proof.** By the assumption, we have \( d'_\ell = \cdots = d'_k \). For the proof of the assertion, we use the non-degeneracy assumption of the complete intersection variety \( V_I(P)^* := \{ z \in \mathbb{C}^n \mid f_{\alpha P}(z) = 0, \alpha \in I \} \) with \( I = \{ \ell, \ldots, k \} \). By the assumption, \( \mathbf{a} \in V_I(P)^* \). Assume that \( \sum_{\alpha=\ell}^{k} \frac{c}{c_{\alpha}} \frac{\partial f_{\alpha P}}{\partial z_j}(\mathbf{a}) = 0 \) for any \( j \). This implies \( \sum_{\alpha=\ell}^{k} \frac{c}{c_{\alpha}} \partial f_{\alpha P}(\mathbf{a}) = 0 \) and it gives a non-trivial linear relation.
among gradient vectors $\partial f_{\alpha P}(a), \ldots, \partial f_{kP}(a)$ which is a contradiction to the non-degeneracy of the complete intersection assumption $V_I(P)^*$. \hfill \Box$

Thus there exists $j_0, 1 \leq j_0 \leq n$ so that
\[ \sum_{\alpha = \ell}^k c_\alpha \frac{\partial f_{\alpha P}}{\partial z_j}(a) \neq 0, \text{ that is, } \ord_t \frac{\partial f}{\partial z_{j_0}}(z(t)) = d - d'_\ell - p_{j_0}. \]

Consider the integers:
\[ D'_{\alpha - 1} := d'_1 + \cdots + d'_{\alpha - 1}, 1 \leq \alpha \leq k. \]

We have
\[
\ord_t f(z(t)) = d_1 + \cdots + d_k = D'_{\ell - 1} + (k - \ell)d'_\ell + e,
\]
\[
\ord_t f(z(t)) = \inf \{ \ord_t \frac{\partial f}{\partial z_j}(z(t)) | 1 \leq j \leq n \}
\leq \sup \{ d - d'_\ell - p_j | 1 \leq j \leq n \}
\leq D'_{\ell - 1} + (k - \ell - 1)d'_\ell + e - \frac{e}{B}.
\]

Here we have used the equality $p_j \geq e/B, j \geq e/B$ and $d_\alpha = d'_\alpha + e_\alpha$, in particular $d_\alpha = d''_\alpha + e_\alpha$ for $\alpha \geq \ell$. Thus we have
\[
\theta(z(t)) = \frac{\ord_t f(z(t))}{\ord_t f(z(t))} \leq F_{\ell}
\]
where $F_{\ell}$ is defined by the following:
\[
F_{\ell} := \frac{D'_{\ell - 1} + (k - \ell - 1)d'_\ell + e - e/B}{D'_{\ell - 1} + (k - \ell)d'_\ell + e}.
\]

Note that under the assumption that $z(t)$ has the weight vector $P$ and (17), $d''_1, \ldots, d''_{\ell}$ are not constant but the other numbers are constant. Originally $d'_\ell$ is an integer but we extend to real numbers so that $F_{\ell}$ is a function of $d'_\ell$ on the interval $[d'_{\ell - 1}, \infty)$, fixing $d'_1, \ldots, d'_{\ell - 1}$. Note that $F_0 = 1 - 1/B$.

4.2. Comparison with $F_{\ell - 1}$. We want to show $F_{\ell} \leq 1 - 1/B$. We assert that $F_{\ell}$ is monotone decreasing function of $d''_\ell$, fixing $d'_1, \ldots, d'_{\ell - 1}$ where $d''_j := d_j - e_j$. Here $d''_\ell$ moves on the interval $[d'_{\ell - 1}, \infty)$. In fact, the differential of the right hand side in $d''_\ell$ is given as
\[
\frac{\partial F_{\ell}}{\partial d''_\ell} = \frac{-D'_{\ell - 1} - e + (k - \ell)e/B}{(D'_{\ell - 1} + (k - \ell)d'_\ell + e)^2}.
\]

We assert that

Lemma 29. $F_{\ell}$ is monotone decreasing function of $d''_\ell$ as
\[
\frac{\partial F_{\ell}}{\partial d''_\ell} \leq 0.
\]
Proof. The numerator of the differential \(\partial F_\ell / \partial d_\ell'\) can be estimated as

\[-D'_{\ell-1} - e + (k - \ell)e/B = -D'_{\ell-1} - e(1 - \frac{k - \ell}{B}) \leq 0.\]

Here we have used the obvious inequality \(B \geq k\). \(\square\)

Thus putting \(d_\ell' = d_{\ell-1}' - 1\), we get the estimation \(\theta_0(f) \leq F_\ell \leq F_{\ell-1}\) where \(F_\ell\) is obtained by substituting \(d_\ell' = d_{\ell-1}'\):

\[F_{\ell-1} := F_\ell|_{d_\ell' = d_{\ell-1}'} = \frac{D'_{\ell-2} + (k - \ell)d_{\ell-1}' + e - e/B}{D'_{\ell-2} + (k - \ell + 1)d_{\ell-1}' + e}\]

where \(D'_{\ell-2} = d_1' + \cdots + d_{\ell-2}'\), \(e = e_1 + \cdots + e_k\).

4.3. Lojasiewicz exponents for the product functions. Now we are ready to state our main result for the product function.

Main Theorem 30. Let \(f = f_1 \cdots f_k\) be the product function associated to a generating family of convenient non-degenerate complete intersection varieties \(F = \{f_1, \ldots, f_k\}\). Then the Lojasiewicz exponent of type (1) satisfies the inequality: \(\theta_0(f) \leq 1 - 1/B\) and the equality holds if \(f\) has a Lojasiewicz non-exceptional monomial.

Proof. Continuing the above argument repeatedly, \(\theta_0(f)\) can be estimated by assuming \(d_\ell' = \cdots = d_1'\) as follows:

\[F_\ell \leq F_{\ell-1} \leq \cdots \leq F_1 = \frac{(k - 1)d_1' + e - e/B}{(k - 1)d_1' + e}.\]

As \(F_1\) is also a monotone increasing function of \(d_1'\), putting \(d_1' = 0\), we conclude \(F_\ell \leq F_0 = 1 - 1/B\). This implies

\[\theta_0(f) \leq \frac{e - e/B}{e} = 1 - \frac{1}{B}.\]

For the existence of the curve attaining the equality under the assumption of the existence of Lojasiewicz non-exceptional monomial, we do the same argument as in §3.1. \(\square\)

4.4. Generalization to non-reduced functions. We will show that Lojasiewicz exponent of a non-reduced expression is determined by the reduced one. First suppose that \(f\) is a reduced function and let \(g(z) = f^m(z)\). Let \(z(t), 0 \leq t \leq 1\) be an analytic curve starting from the origin and \(z(t) \in \mathbb{C}^n \setminus V(f)\) for \(t \neq 0\) as before. Then we observe that

\[(21) \quad \text{ord}_z \frac{\partial g}{\partial z_j}(z(t)) = (m - 1)\text{ord}_z f(z(t)) + \text{ord}_z \frac{\partial f}{\partial z_j}(z(t)).\]

To distinguish two Lojasiewicz exponents of \(f\) and \(g\), we put

\[\theta_f(z(t)) := \frac{\text{ord}_z \partial f(z(t))}{\text{ord}_z f(z(t))}, \quad \theta_g(z(t)) := \frac{\text{ord}_z \partial g(z(t))}{\text{ord}_z g(z(t))}.\]
In particular, we have the equality
\[
\theta_g(z(t)) = \frac{\operatorname{ord}_t \frac{\partial g}{\partial z_j}(z(t))}{\operatorname{ord}_t g(z(t))} = \left( m - 1 \right) \operatorname{ord}_t f(z(t)) + \operatorname{ord}_t \frac{\partial f}{\partial z_j}(z(t)) \]
\[
= \frac{m - 1}{m} + \frac{1}{m} \theta_f(z(t)).
\]
Thus we have

Proposition 31. The Lojasiewicz exponents of \( f \) and \( g = f^m \) are related by the equality:
\[
\theta_0(g) = \frac{m - 1}{m} + \frac{1}{m} \theta_0(f).
\]

This observation can be generalized to our product function \( f \) discussed in §4.1. We consider a defining family \( \mathcal{F} = \{f_1, \ldots, f_k\} \) of convenient non-degenerate complete intersection varieties as §4.1. Let \( f(z) := f_1(z) \cdots f_k(z) \). We consider also non-reduced product function
\[
g(z) = f_1^{m_1}(z) \cdots f_k^{m_k}(z)
\]
where \( m_1, \ldots, m_k \) are positive integers. Let \( z(t) \) be an analytic curve expanded as (11), (12) and (13). We use the same notations of numbers \( e_j, d_j, d'_j, e \). We define new integers \( \tilde{d}_j, \tilde{e}_j, \tilde{d}, \tilde{e} \) and complex numbers \( \tilde{c}_\alpha, \tilde{c}, \tilde{d} \) as
\[
\tilde{d}_\alpha = m_\alpha d_\alpha, \quad \tilde{e}_\alpha = m_\alpha e_\alpha, \quad \tilde{d} = \sum_{\alpha=1}^k m_\alpha d_\alpha, \quad \tilde{e} = \sum_{\alpha=1}^k m_\alpha e_\alpha
\]
\[
\tilde{c}_\alpha = c_\alpha^{m_\alpha}, \quad \tilde{c} = \prod_{\alpha=1}^k c_\alpha^{m_\alpha}
\]
so that
\[
g_\alpha(z(t)) = f_\alpha(z(t))^{m_\alpha} = \tilde{c}_\alpha t^{\tilde{d}_\alpha} + \text{(higher terms)}, \quad 1 \leq \alpha \leq k
\]
\[
g(z(t)) = \tilde{c} t^{\tilde{d}} + \text{(higher terms)}.
\]

We work under the same assumption (17):
\[
d'_1 \leq d'_2 \cdots \leq d'_{l-1} < d'_l = \cdots = d'_k.
\]

We proceed by the exact same argument as the one in the reduced case. The equality (19) is replaced as
\[
(22) \quad \frac{\partial g}{\partial z_j}(z(t)) = \sum_{\alpha=1}^k \left( \left( \frac{m_\alpha}{f_\alpha} \right) g(z(t))^{\frac{\partial f_\alpha}{\partial z_j}(z(t))} \right)
\]
\[
= \left( \sum_{\alpha=\ell}^k \frac{m_\alpha \tilde{c} \partial f_\alpha P}{\partial z_j}(a) \right) t^{\tilde{d}'_\ell - p_j} + \text{(higher terms)}.
\]
Define $\tilde{D}_\alpha$ in the same manner as in §4.1:

$$\tilde{D}_\alpha = \tilde{d}_1 + \cdots + \tilde{d}_\alpha = \sum_{i=1}^{\alpha} m_i d_i.$$  

$\tilde{B}_j$ corresponds the point $\Gamma(g) \cap \{z_j\text{-axis}\}$ which is equal to $\sum_{\alpha=1}^{k} m_\alpha b_{\alpha,j}$ and $\tilde{B}$ is the maximum of $\{\tilde{B}_1, \ldots, \tilde{B}_n\}$. So $p_{\min} \tilde{B} \geq \tilde{e}$. As the gradient vectors

$$\left\{ \frac{\partial f_\alpha}{\partial z_j}(a) \mid \alpha = \ell, \ldots, k \right\}$$

are linearly independent by the non-degeneracy of the intersection variety $V_\ell^* (P)$ (see the proof of Assertion 28), we have

$$\text{ord } g(z(t)) = \tilde{d}_1 + \cdots + \tilde{d}_k = \tilde{D}'_{\ell-1} + (m_\ell + \cdots + m_k) d'_\ell + \tilde{e},$$

$$\text{ord } \partial g(z(t)) \leq \tilde{d} - d'_\ell - p_{\min} \leq \tilde{D}'_{\ell-1} + (m_\ell + \cdots + m_k - 1) d'_\ell + \tilde{e} - \frac{\tilde{e}}{\tilde{B}}.$$  

Thus we can modify equality (20) as:

$$\theta_g(z(t)) = \frac{\text{ord}_t g(z(t))}{\text{ord}_t g(z(t))} \leq \tilde{F}_\ell$$

where

$$\tilde{F}_\ell = \frac{\tilde{D}'_{\ell-1} + (m_\ell + \cdots + m_k - 1) d'_\ell + \tilde{e} - \tilde{e}/\tilde{B}}{\tilde{D}'_{\ell-1} + (m_\ell + \cdots + m_k) d'_\ell + \tilde{e}}.$$  

and we have

$$\frac{\partial \tilde{F}_\ell}{\partial d'_\ell} = \frac{-\tilde{D}_\ell - \tilde{e} + (m_\ell + \cdots + m_k) \tilde{e}/\tilde{B}}{(\tilde{D}'_{\ell-1} + (m_\ell + \cdots + m_k) d'_\ell + \tilde{e})^2} < 0$$

where the negativity is derived from the fact $\tilde{B} \geq m_1 + \cdots + m_k$. By the exact same argument, we get the generalization of Theorem 30:

**Theorem 32.** The Lojasiewicz exponent of $g = f_1^{m_1} \cdots f_k^{m_k}$ can be estimated as

$$\theta_0(g) \leq 1 - \frac{1}{\tilde{B}}.$$  

Furthermore, if $g$ has a non-exceptional Lojasiewicz monomial, the equality holds.

We comment that $\tilde{B} = \max \{m_1 b_{1j} + \cdots + m_k b_{kj} \mid j = 1, \ldots, n\}$.

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