Learning to Control in Metric Space with Optimal Regret

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Abstract—We study online reinforcement learning for finite-horizon deterministic control systems with arbitrary state and action spaces. Suppose the transition dynamics and reward function is unknown, but the state and action space is endowed with a metric that characterizes the proximity between different states and actions. We provide a surprisingly simple upper-confidence reinforcement learning algorithm that uses a function approximation oracle to estimate optimistic Q functions from experiences. We show that the regret of the algorithm after K episodes is $O(DLKH^{d+1})$, where $D$ is the diameter of the state-action space, $L$ is a smoothness parameter, and $d$ is the doubling dimension of the state-action space with respect to the given metric. We also establish a near-matching regret lower bound. The proposed method can be adapted to work for more structured transition systems, including the finite-state case and the case where value functions are linear combinations of features, where the method also achieve the optimal regret.

I. INTRODUCTION

Reinforcement learning has proved to be a powerful approach for online control of complicated systems [Bertsekas, 1995], [Sutton and Barto, 2018]. Given an unknown transition system with unknown rewards, we aim to learn to control the system on-the-fly by exploring available actions and receiving real-time feedback. Learning to control efficiently requires the algorithm to actively explore the problem space and dynamically update the control policy.

A major challenge with effective exploration is how to generalize past experiences to unseen states. Extensive research has focused on reinforcement learning with parametric models, for examples linear quadratic control [Dean et al., 2018], linear model for value function approximation [Parr et al., 2008], and state aggregation model [Singh et al., 1995]. While these parametric models reduce the complexity or regret of reinforcement learning, their practical performances are at risk of model misspecification.

In this paper, we focus on finite-horizon deterministic control systems without any parametric model. We suppose that the control system is endowed with a metric $\text{dist}$ that characterizes the proximity between state-action pairs. We assume that the transition and reward functions are continuous w.r.t. $\text{dist}$, i.e., states close to each other have similar values. Proximity measures of the state-actions have been extensively studied in literature (e.g. [Ferns et al., 2004], [Ortner., 2007], [Castro and Precup, 2010], [Ferns et al., 2012a], [Ferns et al., 2012b], [Tang and van Breugel, 2016]).

Under this very general assumption, we develop a surprisingly simple upper-confidence reinforcement learning algorithm. It adaptively updates the control policy through episodes of online learning. The algorithm keeps track of an experience buffer, as is common in practical deep reinforcement learning methods [Mnih et al., 2013]. After each episode, the algorithm recomputes the Q-functions by using the updated experience buffer through a function approximation oracle. The function approximation oracle is required to find an upper-confidence Q-function that fits the known data and optimistically estimate the value of unseen states and actions. We show that the oracle can be achieved using a nearest neighbor construction. The optimism nature of the algorithm would encourage exploration of unseen states and actions. We show that for arbitrary metric state-action space, the algorithm achieves the sublinear regret

$$O(DLKH^{d+1}),$$

where $D$ is the diameter of the state-action space, $L$ is some smoothness parameter, and $d$ is the doubling dimension of the state-action space with respect to the metric. This regret is “sublinear” in the number of episodes played. Therefore the average number of mistakes decreases as more experiences are collected. We use an information-theoretical approach to show that this regret is optimal.

The algorithm we propose is surprisingly general and easy to implement. It uses a function approximation oracle to find “optimistic” Q-functions, which are later used to control the system in the next episode. With suitable function approximators, we can adapt our method and analysis to more structured classes of control systems. For example, we show that the method can be adapted to the setting where the value functions are linear combinations of features. In this setting, our method achieves a state-of-art regret upper bound $O(dH)$, where $d$ is the dimension of feature space. This regret is known to be optimal. We believe our method can be adapted to a broader family of function approximators, including both the classical spline methods and deep neural networks. Understanding the regret for learning to control using these function classes is for future research.

II. RELATED LITERATURES

Complexity and regret analysis for reinforcement learning received significant attention. A basic setting is the Markov decision process (MDP), where the transition law at a given state and action is according to some probability distribution. In the case of finite-state-action MDP without structural knowledge, efficient methods typically achieve regrets scaling as $O(\sqrt{HSAK})$, where $S$ and $A$ are the...
numbers of states and actions, \( T = KH \) is number of time steps (e.g. [Jaksch et al., 2010], [Agrawal and Jia, 2017], [Azar et al., 2017], [Osband and Van Roy, 2016]). The work of [Jaksch et al., 2010] provided a lower bound of \( \Omega(\sqrt{HSA^2T}) \) for \( H \)-horizon MDP and bounds for weakly communicating infinite-horizon average reward MDP. The number of sample transitions needed to learn an approximate policy has been considered by for example [Lattimore and Hutter, 2014a], [Lattimore and Hutter, 2014b], [Dann and Brunskill, 2015]. The optimal sample complexity for finding an \( \epsilon \)-optimal policy is \( O\left(\frac{SA}{(1-\epsilon)^2}\right) \) [Sidford et al., 2018].

In the regime of continuous-state MDP, the complexity and regret of online reinforcement learning has been explored under structured assumptions. [Lattimore et al., 2013] studies the complexity when the true transition system belongs to a finite or compact hypothesis class, and shows sample policy that depends polynomially on the cardinality or covering number of the model class. Ortner and Ryabko [Ortner and Ryabko, 2012] develops a model-based algorithm with a regret bound for Lipschitz continuous MDP problems with continuous state spaces and Holder-continuous transition kernels in terms of the total variation divergence. Pazis and Parr [Pazis and Parr, 2013] considers MDP with continuous state spaces, under the assumption that Q-functions are Lipschitz-continuous and establishes the sample complexity bound that involves an approximate covering number. Ok et al. [Ok et al., 2018] studied structured MDP with finite state and action spaces, including the special case of Liptschiz MDP, and provides various regret upper bounds. Yang and Wang [Yang and Wang, 2019] studied structured MDPs that admit a set of state-action features which can linearly express the process probabilistic transition model, and proved an optimal sample complexity for finding an \( \epsilon \)-optimal policy.

In contrast to the vast literatures on reinforcement learning for MDP, online learning for deterministic control has been studied by few. Note that deterministic transition is far more common in applications like robotics and self-driving cars. A closely related and significant result is [Wen and Van Roy, 2017], which studies the complexity and regret for online learning in episodic deterministic systems. Under the assumption that the optimal \( Q \) function belongs to a hypothesis class, it provides an optimistic constraint propagation method that achieves optimal regret. This result of [Wen and Van Roy, 2017] points out a significant observation that the complexity of learning to control depends on complexity of the functional class where the \( Q \) functions reside in. However, the algorithm provided by [Wen and Van Roy, 2017] is rather abstract (which is due to the generality of the method). In comparison to [Wen and Van Roy, 2017], our paper focuses on the setting where the only structural knowledge is the continuity with respect to a metric. In such a setting, [Wen and Van Roy, 2017] would imply an infinite regret as the Euler-dimension can be infinite in this case. We achieve a sublinear regret \( O(K\epsilon^{2/\alpha}) \) w.r.t. the number of episodes played. We show that it is optimal in this setting, and our algorithm is based on a upper-confidence function approximator which is easier to implement and generalize.

III. Problem Formulation

A. Deterministic MDP

Consider a deterministic finite-horizon Markov Decision Process (MDP) \( \mathcal{M} = \{S, A, f, r, H\} \), where \( S \) is an arbitrary set of states, \( A \) is an arbitrary set of actions, \( f : S \times A \rightarrow S \) is a deterministic transition function, \( H \geq 1 \) is the horizon, and \( r : S \times A \rightarrow [0, 1] \) is a reward function. A policy is a function \( \pi : S \times [H] \rightarrow A \) (\([H] \) denotes the set \( \{1, \ldots, H\} \)). The optimal policy \( \pi^* \) maximizes the cumulative reward in \( H \) steps from any fixed initial state \( s_0 \):

\[
\max_{\pi} \sum_{h=1}^{H} r(s_h, a_h)
\]

subject to \( s_{h+1} = f(s_h, a_h), a_h = \pi(s_h, h), s_1 = s_0 \).

Given a policy \( \pi \), the value function \( V^\pi : S \times [H] \rightarrow \mathbb{R} \) is defined recursively as follows.

\[
\forall s \in S, h \in [H-1] : V^\pi_H(s) = r(s, \pi(s, H)) + V^\pi_{h+1}(f(s, \pi(s, h))
\]

An optimal policy \( \pi^* \) satisfies

\[
\forall h : V^\pi_H^* := V^\pi_H = \max_\pi V^\pi_H \text{ entrywise.}
\]

In particular, the optimal value function \( V^* \) satisfies the following Bellman equation

\[
\forall s \in S, h \in [H-1] : V^*_{H}(s) = \max_{a \in A} r(s, a) + V^*_{h+1}(f(s, a))
\]

We also define the Q-functions, \( Q^*_{H} : S \times A \rightarrow \mathbb{R} \), as,

\[
\forall h \in [H-1], Q^*_{H}(s, a) = r(s, a) + V^*_{h+1}(f(s, a))
\]

where \( Q^*_{H}(s, a) = r(s, a) \). We further denote \( Q^*_{h} = Q^*_{h}^* \) for \( h \in [H] \).

B. Episodic Reinforcement Learning and Regret

We focus on the online episodic reinforcement learning problem, in which the learning agent does not know \( f \) or \( r \) to begin with. The agent repetitively controls the system for episodes of \( H \) time, where each episode starts from some initial state \( s_0 \) that does not depend on the history. We denote the total number of episodes played by the agent as \( K \geq 1 \).

Suppose that the learning agent is an algorithm \( K \) (possibly randomized). It can observe all the state transitions and rewards generated by the system and adaptively pick the next action. We define its regret of this algorithm \( K \)

\[
\text{Regret}_K(K) = \mathbb{E}_K \left[ K \cdot V^*(s_0, 1) - \sum_{k=1}^{K} \sum_{h=1}^{H} r(s_h^{(k)}, a_h^{(k)}) \right],
\]

where the action \( a_h^{(k)} \) is generated by algorithm \( K \) at time \( (k, h) \) based on the entire past history, and \( \mathbb{E}_K \) is taken over the randomness of the algorithm \( K \). The regret of \( K \) measures the difference between the total rewards collected by the algorithm and that by the optimal policy after \( K \) episodes.
IV. THE BASIC CASE OF FINITE STATES AND ACTIONS

We provide Algorithm 1 for the case where the state space $S$ and action space $A$ are finite sets with sizes $S$ and $A$, without assuming any structural knowledge. Note although [Wen and Van Roy, 2017] has provided a regret-optimal algorithm for this setting, we provide a simpler algorithm based on upper-confidence bounds. Despite of the simplicity of this setting, we include the result to illustrate our idea, which might be of independent interest.

Algorithm 1 maintains an upper bound of the optimal value function using the past experiences. The algorithm uses the value upper bound to plan the future actions. After each episode, the value upper bound is improved based on the newly obtained data. Since the exploration is based on the upper bound of the value function, it always encourages the exploration of un-explored actions. The value is improved in such a way that once a regret is paid, the algorithm is always able to gain some new information such that the same regret will not be paid again. The guarantee of the algorithm is presented in the following theorem.

**Theorem 1:** After $K$ episodes, the above algorithm obtains a regret bound

$$\text{Regret}(K) \leq SAH.$$  

The proof is presented in the appendix. Theorem 1 matches the regret bounds proved in [Wen and Van Roy, 2017]. For completeness, we include a regret lower bound proof for our setting. See Theorem 8 in the appendix.

V. POLICY EXPLORATION IN METRIC SPACE

Now we consider the more general case where the state-action space $\mathcal{X} = S \times A$ is arbitrarily large. For example, the state in a video game can be a raw-pixel image, and the state of a robotic system can be a vector of positions, velocities and acceleration. In these problems the state space can be considered a smooth manifold in a high-dimensional ambient space.

A. Metric and Continuity

The major challenge with reinforcement learning is to generalize past experiences to unseen states. For the sake of generality, we only assume that a proper distance between states is given, which suggests that states close to each other have similar values. Suppose we have a metric $\text{dist}$ over the state-action space $\mathcal{X} = S \times A$, i.e., $\text{dist}(x, y) = \text{dist}(y, x)$, and $\text{dist}$ satisfies the triangle inequality.

**Assumption 1 (Lipschitz continuity):** Let the optimal action-value function be $Q^* : \mathcal{X} \to \mathbb{R}$. Then there exist constants $L_1, L_2 > 0$ such that $\forall (s, a), (s', a') \in S \times A$ and $\forall h \in [H], a'' \in A$,

$$|Q^*_h(s, a) - Q^*_h(s', a')| \leq L_1 \cdot \text{dist}([s, a], (s', a')) \tag{1}$$

$$\text{dist}([f(s, a), a''], [f(s', a'), a'']) \leq L_2 \cdot \text{dist}([s, a], (s', a')) \tag{2}$$

We further denote $L = (L_2 + 1) \cdot L_1$ for convenience.

B. Optimistic Function Approximation

To handle the curse of dimensionality of general state space, we will use a function approximator for computing optimistic Q-function from experiences. The function approximator needs to satisfy the following conditions.

**Assumption 2 (Function Approximation Oracle):** Let $q : X \to \mathbb{R}$ be a function. Let $B := \{(x_i, y_i)\}_{i=1}^n \subset X \times \mathbb{R}$ be a set of pairs such that $y_i \geq q(x_i), \forall i$. Let $L > 0$ be a parameter. Then there exists a function approximator, FuncApprox, which, on given $B$, outputs a function $\hat{q} : X \to \mathbb{R}$ that satisfies

1) $\hat{q}$ is $L$-Lipschitz continuous;
2) $\forall x \in \mathcal{X}$: $\hat{q}(x) \geq q(x)$;
3) $\forall i \in [N]$: $\hat{q}(x_i) = y_i$.

One way to achieve the conditions required by the function approximator is to use the nearest neighbor approach:

$$\forall x \in \mathcal{X} : \hat{q}(x) := \min_{i \in [N]} \{y_i + L \cdot \text{dist}(x, x_i)\}, \tag{3}$$

where the distance regularization $L \cdot \text{dist}(x, x_i)$ will overestimate the value at an unseen point using its near neighbors.

**Lemma 2:** Suppose $q$ is $L$-Lipschitz. Then the approximator given by (3) is a function-approximator satisfying Assumption 2 with Lipschitz constant $L$.

**Proof:** Firstly, by triangle inequality, for any $x, x' \in \mathcal{X}$,

$$|\hat{q}(x) - \hat{q}(x')| \leq \max_i |y_i - y_i + L \cdot (\text{dist}(x, x_i) - \text{dist}(x', x_i))|$$

$$\leq L \cdot \max_i \text{dist}(x, x_i) - \text{dist}(x', x_i) \leq L \cdot \text{dist}(x, x').$$

Therefore (1) of Assumption 2 holds.

Secondly, since $q$ is Lipschitz continuous, we have $\forall i \in [N]$,

$$q(x) \leq \hat{q}(x_i) + L \cdot \text{dist}(x, x_i) \leq y_i + L \cdot \text{dist}(x, x_i).$$

Thus

$$q(x) \leq \min_i (y_i + L \cdot \text{dist}(x, x_i)) = \hat{q}(x)$$

and (2) of Assumption 2 holds.

We now verify (3) of Assumption 2. For all $j \in [N]$,

$$y_j \leq \hat{q}(x_j) = \min_i (y_i + L \cdot \text{dist}(x_j, x_i))$$

$$\leq y_j + L \cdot \text{dist}(x_j, x_j) = y_j,$$

as desired.

C. Regret-Optimal Algorithm in Metric Space

Next we provide a regret-optimal algorithm for the metric MDP. The algorithm does not need additional structural assumption other than the Lipschitz continuity of $Q^*$ and $f$. It is a combination of the UCB-type algorithm with a nearest-neighbor search. It measures the confidence by coupling the Lipschitz constant with the distance of a newly observed state to its nearest observed state. The algorithm is formally presented in Algorithm 2.

The algorithm keeps an experience buffer $B^{(k)} = \{(s_{1}^{(1)}, a_{1}^{(1)}), (s_{2}^{(2)}, a_{2}^{(2)}), \ldots, (s_{H}^{(k)}, a_{H}^{(k)})\}$ that grows as new
Algorithm 1 Upper Confidence Reinforcement Learning for Deterministic Finite MDP

1: **Input:** A deterministic MDP.

2: **Initialize:** $Q_{h}^{(1)} \leftarrow H \cdot 1 \in \mathbb{R}^{|S| \times |A|}$ for every $h$; For every $(s, a) \in S \times A$, $\hat{r}(s, a) \leftarrow 1$, $\hat{f}(s, a) \leftarrow \text{NULL}$, $b(s, a) \leftarrow H$;

3: for episode $k = 1, 2, \ldots, K$ do

4: for stage $h = 1, 2, \ldots, H$ do

5: Current state: $s_{h}$;

6: Play action $a_{h} \leftarrow \arg \max_{a \in A} Q_{h}^{(k)}(s, a)$;

7: Observe the state transition $s_{h+1} \leftarrow f(s_{h}, a_{h})$ and obtain reward $r(s_{h}, a_{h})$;

8: Update: $\hat{f}(s_{h}, a_{h}) \leftarrow s_{h+1}$, $\hat{r}(s_{h}, a_{h}) \leftarrow r(s_{h}, a_{h})$, $b(s_{h}, a_{h}) \leftarrow 0$;

9: end for

10: Obtain new value functions $Q_{h}^{(k+1)}$ using $\hat{f}$, $\hat{r}$ by dynamic programming: $\forall s \in S, h \in [H - 1],$

11: $Q_{h}^{(k+1)}(s, a) \leftarrow \max_{a \in A} Q_{h+1}^{(k+1)}[f(s, a), a'] + b(s, a)],$

where we denote $Q_{h}^{(k+1)}[\text{NULL}, a] = 0$.  

sample transitions are observed. It optimistically explores the policy space in online training using upper-estimate of Q-values. These Q-values are computed recursively by using the function approximator according to the dynamic programming principle.

In particular, if the function approximation oracle is given by the nearest neighbor construction (3), Step 10 of the algorithm takes the form of

$$Q_{h}^{(k+1)}(s, a) \leftarrow \min_{(s', a') \in B^{(k+1)}} \left( r(s', a') + L_{1} \cdot \text{dist}[(s, a), (s', a')] \right) \tag{4}$$

$$Q_{h}^{(k+1)}(s, a) \leftarrow \min_{(s', a') \in B^{(k+1)}} \left[ r(s', a') + \sup_{a'' \in A} Q_{h+1}^{(k+1)}(f(s', a'), a'') + L_{1} \cdot \text{dist}[(s', a'), (s, a)] \right] \tag{5}$$

for all $(s, a) \in S \times A$, $h \leq H - 1$. In this case, the nearest-neighbor function approximator will prioritize exploring $(s, a)$’s that are farther away from the seen ones.

Algorithm 2 provides a general framework for reinforcement learning with a function approximator in deterministic control systems. It can be adapted to work with a broad class of function approximators.

VI. REGRET ANALYSIS

A. Main Results

For a metric space $X$, we denote the $\epsilon$-net, $N(\epsilon) \subset X$, as a set such that

$$\forall x \in X : \exists x' \in N(\epsilon), s.t. \text{dist}(x, x') \leq \epsilon.$$ 

If $X$ is compact, we denote $N(\epsilon)$ as the minimum size of an $\epsilon$-net for $X$. We also denote a similar concept, the $\epsilon$-packing, $C(\epsilon) \subset X$, as a set such that

$$\forall x, x' \in C(\epsilon): \text{dist}(x, x') > \epsilon.$$ 

If $X$ is compact, we denote $C(\epsilon)$ as the maximum size of an $\epsilon$-packing for $X$. In general, $N(\epsilon) \leq C(\epsilon)$ and are of the same order. For a normed space (the metric is induced by a norm), we have $C(2\epsilon) \leq N(\epsilon) \leq C(\epsilon)$.

Next we show that the regret till reaching $\epsilon$-optimality is upper bounded by a constant that is proportional to the size of the $\epsilon$-net. We will show later that the regret is lower bounded by a constant proportional to the size of the $\epsilon$-packing.

Theorem 3 (Regret till $\epsilon$-optimality): Suppose we have an episodic deterministic MDP $M = (S, A, f, r, H)$ that satisfies Assumption 1. Let $X = S \times A$ be a state-action space with diameter $D > 0$, and $L_{1}, L_{2}$ be parameters specified in Assumption 1. Suppose we use the $L_{1}$-continuous function approximator defined in (4). Suppose the state-action space $X$ admits an $\epsilon$-cover $N(\epsilon)$ for any $\epsilon > 0$. Then after $T = KH$ steps, Algorithm 2 obtains a regret bound

$$\text{Regret}(K) \leq H|N(\epsilon)| + 2\epsilon LKH.$$ 

where $L = (L_{2} + 1) \cdot L_{1}$.

Suppose $d$ is the doubling dimension of the state-action space $X$. The doubling dimension of a metric space is the smallest positive integer, $d$, such that every ball can be covered by $2^{d}$ balls of half the radius. Then we can show the following regret bound.

Theorem 4 (Optimal Regret for Metric Space): Suppose the state-action space is compact with diameter $D$ and doubling dimension $d > 0$. Then after $K$ episodes, Algorithm 2 with approximator (4) obtains a regret bound

$$\text{Regret}(K) = O(DLK) \cdot \frac{\pi^{1/2}}{2^{1/4}} \cdot H.$$ 

About Doubling Dimension: The regret depends on the doubling dimension $d$. It is the intrinsic dimension of $X$ - often very small even though the observed state space has high dimensions. For example, the raw-pixel images in a video games often belong to a smooth manifold with a small intrinsic dimension. Our Algorithm 2 uses the nearest-neighbor function approximation. It can be thought of as learning the manifold space at the same time when solving the dynamic program. It does not need any parametric model or feature map to capture the small intrinsic dimension.
Algorithm 2 Upper Confidence Reinforcement Learning with Function Approximator (UCRL-FA)

1: Input: A deterministic metric MDP
2: Initialize: Initialize \( B^{(0)} \leftarrow \emptyset \), \( Q_h^{(0)}(s, a) \leftarrow H \), for all \((s, a) \in S \times A, h \in [H]\);
3: for episode \( k = 1, 2, \ldots \) do
4: for stage \( h = 1, 2, \ldots, H \) do
5: Current state: \( s_h \);
6: Play action \( a_h^{(k)} = \arg \max_{a \in A} Q_h^{(k)}(s_h, a) \)
7: Record the next state \( s_{h+1} \leftarrow f(s_h, a_h) \) and reward \( r(s_h, a_h) \);
8: end for
9: Update \( B^{(k+1)} \leftarrow B^{(k)} \cup \{(s_h^{(k)}, a_h^{(k)}, f(s_h^{(k)}, a_h^{(k)}), r(s_h^{(k)}, a_h^{(k)})), \ldots, (s_H^{(k)}, a_H^{(k)}, f(s_H^{(k)}, a_H^{(k)}), r(s_H^{(k)}, a_H^{(k)}))\} \);
10: Now we update \( Q_h^{(k+1)} \) recursively as following:
11: end for

B. Optimality

Next we establish a regret lower bound for reinforcement learning in deterministic metric MDP.

**Theorem 5 (Minimax Lower Bound):** Let \( \mathcal{M}(H, \epsilon) \) be a family of MDPs with the form \( M = (S, A, f, r, H) \), where \( X := S \times A \) is a metric space that admits an \( \epsilon \)-packing \( C(\epsilon) \) for some \( \epsilon > H/2 \), and \( M \) satisfies Assumption 1. Let \( K \) be any online algorithm that admits input any MDP from \( M \). Let \( \text{Regret}_K^M(K) \) denote the regret for \( K \) on \( M \) after \( K \) episodes.

\[
\max_{M \in \mathcal{M}(H, \epsilon)} \text{Regret}_K^M(K) \geq \Omega \left[ \min \left( |C(\epsilon)|, K \right) \cdot H \right]
\]

The proof is postponed to the appendix. The core idea is to construct a hard instance distribution such that an MDP sampled from the distribution satisfies: (i) every two state-action pairs have distance \( H \); (ii) has absorbing states so that any algorithm can explore at most one state-action pair per episode; (iii) has only one random non-absorbing state-action pair with reward 1 and others with reward 0.

Since the rewarding state-action pair is random, any algorithm is expected to spend \( \Theta(|C(\epsilon)|) \) episodes until it reaches the rewarding state-action pair. But an “oracle” optimal algorithm can pick the rewarding state-action pair for every episode. Therefore, any exploration-based algorithm requires to pay a regret \( \Omega(|C(\epsilon)|/H) \) in this hard instance distribution.

VIII. EXAMPLES AND EXTENSIONS

A. Finite State-Action MDP

In the case of finitely many states and actions without any structural knowledge, one can simply pick the metric to be \( \text{dist}(s, a), (s', a') = H, \forall (s, a) \neq (s', a') \).

So we can see Algorithm 2 contains Algorithm 1 as a special case. For discrete space, if we take \( \epsilon = H - \delta \) for an arbitrary \( \delta > 0 \), then the covering size is \( N(\epsilon) = SA \). Then Theorem 3 implies that the \( K \)-episode regret is \( SAH \) regardless of \( H \), which matches Theorem 1.  

B. Linear Model with Feature Map

An important family of structured MDP is the family where the reward and transition \( r, f \) are linear with respect to some feature map \( \phi(s, a) \in \mathbb{R}^d \). In this case, \( Q_h^\epsilon \) and \( Q_h^\epsilon \) are all linear in the feature space.

Let us adapt Algorithm 2 to the linear model. To do so, we use a different function approximator to capture the class of linear Q functions. Given a data set \( \{(x_i, y_i)\} \) \( i = 1 \), we let \( \Phi_X \) be the \( N \times d \) matrix whose \( i \)-th row is \( \phi(x_i)^T \) and let \( y \in \mathbb{R}^N \) be the vector whose \( i \)-th entry is \( y_i \). We use the following function approximation oracle

\[
\text{FuncApprox}_{\{(x_i, y_i)\}}(x) = \begin{cases} \phi(x)^T (\Phi_X^T \Phi_X)^{-1} \Phi_X^T y & \phi(x) \in \text{Span}\{\phi(x_i)\} \\ \phi(x) & \phi(x) \notin \text{Span}\{\phi(x_i)\} \end{cases}
\]

The above approximator fits a linear function on the observed \( \{(x_i, y_i)\} \). When it is queried at a point \( x \) that does not belong to the subspace spanned by the observations \( x_i \)'s, it will output an upper bound \( H \). Then we can show that the regret depends linearly on the feature dimension:

\[
\text{Regret}(K) \leq HD.
\]

The proof is similar to that of Theorem 2: the instant regret when visiting \( (s, a) \) is bounded by \( b(s, a) \), which is 0 if \((s, a)\) is the linear combination of seen states in the feature space and is \( H \) when \( \phi(s, a) \notin \text{Span}\{\phi(s_i, a_i)\} \). This would happen at most \( d \) times since the feature space has dimension \( d \). The result matches the optimal regret bound established in [Wen and Van Roy, 2017]. It is a \( O(1) \) regret that does not depend on the episode number \( K \). It scales linearly (instead of exponentially) with respect to dimension of feature space.

VIII. CONCLUSION

This paper provides an upper-confidence reinforcement learning algorithm for episodic deterministic system. Given a metric over the state-action space that captures continuity of...
the rewards and transition functions, the algorithm achieves sublinear regret that depends on the doubling dimension of the space. We show that this regret is non-improvable in general. Our method can be adapted to achieve the state-of-art $O(1)$ regret in the setting where the value functions can be represented by a linear combination of features.

**IX. Missing Proofs**

**A. Proofs of the Main Theorems**

To prove Theorem 3, we need several core lemmas. The following lemma shows that the approximated Q-function is always an upper bound of the optimal Q-function.

**Lemma 6 (Optimism):** Suppose Assumption 2 holds for the FuncApprox in Algorithm 2. Then, for any $k \in [K]$, $(s, a) \in S \times A, h \in [H]$, and $k \in [K]$, we have

$$Q_h^*(s, a) \leq Q_h^k(s, a)$$

**Proof:** By the properties of FuncApprox, we have $r \leq \hat{r}^k$ entriwise. We prove the result by induction: when $h = H$, we have

$$Q_H^*(s, a) = \hat{r}(s', a')$$

Suppose the relation holds for $h + 1$, then we have

$$Q_h^*(s, a) \leq Q_h^k(s', a')$$

Thus we have

$$Q_h^*(s, a) \leq Q_h^k(s, a) + \sup_{a' \in A} Q_h^{k+1}(f(s', a'))$$

This completes the proof. 

Given a finite set $B = \{(s, a, f(s, a), r(s, a))\}$, for any $(s, a) \in S \times A$, we define the nearest neighbor operator $NN$ and function $b_B$ as follows,

$$NN(B, (s', a', h)) = \arg \min_{(s', a') \in B} \text{dist}((s, a), (s', a'))$$

$$b_B(s, a) = \text{dist}((s, a), NN(B, (s', a', h)))$$

The next lemma shows that Algorithm 2 with function approximator (4) does not incur too much per-step error.

**Lemma 7 (Induction):** Suppose the FuncApprox in Algorithm 2 is (4). Then for any $k \in [K], (s, a) \in S \times A, h \in [H]$, we have

$$Q_h^k(s, a) \leq r(s, a) + \sup_{a' \in A} Q_h^{k+1}(f(s, a), a')$$

**Proof:** Let $$(s', a') = \arg \min_{(s', a') \in B} \text{dist}((s, a), (s', a'))$$. By definition of $Q_h^k(s', a')$, we have

$$Q_h^k(s', a') = \arg \min_{(s', a') \in B} \text{dist}((s, a), (s', a'))$$

$$Q_h^k(s, a) \leq Q_h^k(s', a') \leq \hat{r}(s', a') + \sup_{a' \in A} Q_h^{k+1}(f(s', a'), a')$$

Next we consider $Q_h^k(s', a') - Q_h^k(s, a') \leq L \cdot b_B^k(s', a')$.

Moreover, we immediately have

$$Q_h^k(s', a') - Q_h^k(s, a') \leq L \cdot b_B^k(s', a')$$

Therefore,

$$\text{Regret}(K) \leq \sum_{k=1}^{K} \min_{h \in [H]} \left\{ L \cdot \sum_{h=1}^{H} b_B^{k+1}(s', a') \right\}.$$
At episode $k \geq 1$, if for some $k' < k$ and some $h' \in [H]$ there is $NN_{\epsilon}(s^{(k')}_{h'}, a^{(k')}_{h'}) = (s, a)$, we call $(s, a)$ has been visited. Thus if $NN_{\epsilon}(s^{(k)}, a^{(k)}) = (s, a)$, we can upper bound
\[
b^{(k)}(s^{(k)}, a^{(k)}) \leq \text{dist}(s^{(k)}, a^{(k)}, s^{(k)}, a^{(k)}) \\
\leq \text{dist}(s^{(k)}, a^{(k)}), |s, a| + \text{dist}(s^{(k)}, a^{(k)}), |s, a| \leq 2c.
\]
On the other hand, if $NN_{\epsilon}(s^{(k)}, a^{(k)})$ has not been visited, we upper bound the regret of the entire episode by $H$. However, such case can only happen at most $|\mathcal{N}(\epsilon)|$ times as for the next episode, $NN_{\epsilon}(s^{(k)}, a^{(k)})$ will become visited. Therefore,
\[
\text{Regret}(K) \leq H|\mathcal{N}(\epsilon)| + 2\epsilon LKH
\]
as desired.

We are now ready to prove Theorem 4.

Proof: [Proof of Theorem 4] Since the metric space has a doubling dimension $d$, we can have an $c$-net $\mathcal{N}(\epsilon)$ with size $|\mathcal{N}(\epsilon)| = \Theta\left(\frac{D}{\epsilon} \right)^d$. By Theorem 3, the regret is upper bounded by
\[
\text{Regret}(K) \leq H \cdot \Theta\left(\frac{D}{\epsilon} \right)^d + 2\epsilon LKH
\]
When $\epsilon = D^{\frac{1}{2d}} \cdot (LK)^{-\frac{1}{2d}}$, we can bound the regret as
\[
\text{Regret}(K) = O(DLK^{\frac{1}{d}}) \cdot H.
\]
as desired.

Proof: [Proof of Theorem 1] Note that $Q_h^k \leq H \cdot 1$ for any $h \in [H]$. Denote $\bar{r}(k)$ as the $\bar{r}$ at the beginning of the $k$th episode. Similarly we denote $\tilde{f}(k)$ and $b^{(k)}$. By definition of the algorithm, we have
\[
\forall k \in [K], Q_h^k \geq r \leq \bar{r}(k) := Q_h^k.
\]
We then can show inductively that $\forall k \in [K], (s, a) \in S \times A, Q_h^k[s, a] \leq \min \{H, \bar{r}(k)(s, a) + \max_{a' \in A} Q_h^{k+1}[\tilde{f}(k)(s', a') + b^{(k)}(s', a')], Q_h^k[s, a]\}$.

Denote $s_h^k$ as the state at time $(k, h), a_h^k = \pi(s_h^k, h)$, $h = \arg \max_{a \in A} Q_h^k(s, a)$, and the policy at episode $k$ as $\pi^*(k)$. We can write the the regret as
\[
\text{Regret}(K) := \sum_{k=1}^K [V_1^*[s_0] - \sum_{h=1}^H r(s_h^k, a_h^k)] \\
= \sum_{k=1}^K [V_1^*[s_0] - V_1^*(s_0^k)].
\]
Denote $V_h^{(k)}(s) = \max_{a \in A} Q_h^k(s, a), \forall s \in S$, then $\forall s \in S : V^*(s) = \max_{a \in A} Q^*(s, a) \leq V_h^k(s)$.

We can thus upper bound $V_h^k(s) - V_h^{(k)}(s)$ as follows.
\[
\forall s \in S, h \in [H] : V_h^*(s) - V_h^{(k)}(s) \leq V_h^k(s) - V_h^{(k)}(s) \\
\text{Note that for all } h \in [H - 1], s \in S, \text{ we have} \\
V_h^k[s_h^k] \leq \bar{r}(k)(s_h^k, a_h^k) + V_h^{k+1}(s_h^{k+1}) + b(k)(s_h^k, a_h^k) \\
\leq \bar{r}(k)(s_h^k, a_h^k) + V_h^{k+1}(s_h^k) + b(k)(s_h^k, a_h^k).$$

Since $V_h^{(k)}[s_h^k] = r(k)(s_h^k, a_h^k) + V_{h+1}^{(k)}(s_{h+1}^k).$ Thus
\[
V_h^k(s_h^k, h) - V_h^{(k)}(s_h^k, h) \\
\leq V_{h+1}^{(k)}(s_{h+1}^k) - V_h^{(k)}(s_1^k) + b(k)(s_h^k, a_h^k),
\]
where $\hat{b}(k)(s_h^k, a_h^k) := \bar{r}(k)(s_h^k, a_h^k) - r(k)(s_h^k, a_h^k) + b(k)(s_h^k, a_h^k)$. Now we can recursively bound
\[
V_1^k(s_0^k) - V_1^*(s_0^k) \\
\leq V_2^k(s_1^k) - V_2^*(s_1^k) + \hat{b}(k)(s_1^k, a_1^k) \\
\leq \ldots \leq \sum_{h'=1}^H \hat{b}(k)(s_{h'}^k, a_{h'}^k)
\]
Moreover, we can immediately bound $V_1^k(s_0^k) - V_1^*(s_0^k) \leq H$. Therefore,
\[
\text{Regret}(K) \leq \sum_{k=1}^K \min_{h' = 1}^H \sum_{h'=1}^H \hat{b}(k)(s_{h'}^k, a_{h'}^k)
\]
It remains to bound $\sum_{h'=1}^H \hat{b}(k)(s_{h'}^k, a_{h'}^k)$. For each $(s_h^k, a_h^k)$, if it is visited in the past $k - 1$ episodes, then $b(k)(s_h^k, a_h^k) = 0$.

If it is visited for the first time, then the regret of the entire episode can be bounded by $H$. Since there are $SA$ number of such $(s, a)$ pairs, we have
\[
\sum_{k=1}^K [V_1^*[s_0] - V_1^*(s_0^k)] \\
\leq \sum_{k=1}^H \sum_{k=1}^K \sum_{h'=1}^H \hat{b}(k)(s_{h'}^k, a_{h'}^k)
\]
\[
\leq \sum_{k=1}^K \max_{M \in \mathcal{M}(S, A, H)} \text{Regret}^M_K = \Omega(\min(|S|, |A|, K)H)
\]
as desired.
Next, we show that for any deterministic algorithm $K_{\text{det}}$, the expected regret on distribution $\mu$ is $\Omega(|S||A|H)$. Note that every MDP $M \in \text{supp}(\mu)$ has the same initial structures. For a deterministic algorithm $K_{\text{det}}$, we consider a particular instance $\bar{M} \in M(S,A,H)$: the initial structure of $\bar{M}$ is $T$, but very state-action pair $(s,a) \in S \times A$ transitions to $s_n$. Consider $K_{\text{det}}$ runs on $\bar{M}$. Suppose we have run $K_{\text{det}}$ for $K = p|S||A|$ episodes with $p < 1$. Denote the state-action pair reached at the end of episode $k \in [K]$ as $(\tilde{s}^*, \tilde{a})(K_{\text{det}}, k) = (\tilde{s}'(K_{\text{det}}, k), \tilde{a}(K_{\text{det}}, k))$.

Suppose we now run $K_{\text{det}}$ on an instance sampled from $\mu$, then with probability at least $1 - p$, $K_{\text{det}}$ has paid regret at least $K(H - O(\log(|S|)))$. Indeed, for an instance $M \sim \mu$, if for all $k$, $(\tilde{s}^*, K_{\text{det}}, k, \tilde{a}(K_{\text{det}}, k))$ on $M$ does not equal to $(s^*, a^*)$, which happens with probability $1 - K/(|S||A|) = 1 - p$, then $K_{\text{det}}$ would have the exact same history on $M$ as it runs on $M$. Therefore, it pays regret $K(H - O(\log(|S|)))$ on $M$. Hence,

$$\min_{K\in M \sim \mu} \mathbb{E}[\text{Regret}_{K_{\text{det}}}^M(K)] = (1 - p) \cdot K(H - O(\log(|S|))) = \Omega(|S||A|H),$$

as long as $K = \Omega(|S||A|)$. Denote $\nu$ as an distribution on $M(S,A,H)$, then we have,

$$\sup_{\nu \in M \sim \nu} \mathbb{E}[\text{Regret}_{K_{\text{det}}}^M(K)] \geq \min_{K \in M \sim \nu} \mathbb{E}[\text{Regret}_{K_{\text{det}}}^M(K)] = \Omega(|S||A|H).$$

By Yao’s minimax [Yao, 1977] theorem, we have,

$$\min_{K \in M(S,A,H)} \max_{\nu \in M \sim \nu} \mathbb{E}[\text{Regret}_{K_{\text{det}}}^M(K)] \geq \sup_{\nu \in M \sim \nu} \min_{K \in M \sim \nu} \mathbb{E}[\text{Regret}_{K_{\text{det}}}^M(K)] = \Omega(|S||A|H).$$

This completes the proof. 

**Proof:** [Proof of Theorem 5] We will use the same distribution as in the proof of Theorem 8 to prove the theorem. Note that the space $S \times A$ in the proof of Theorem 8 is not a metric space yet. To convert it to a metric space, we assign a naive metric by setting

$$\text{d}(s,a, (s',a')) = H \cdot I((s,a) = (s',a')).$$

Since the optimal action-value function $Q^*$ is upper bounded by $H$ uniformly, the Lipschitz continuity conditions in Assumption 1 can be verified for any $L_1 \geq 1$ and $L_2 \geq 1$.

Then Theorem 5 is proved the same way as Theorem 8.

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