A simple discharging method for forbidden subposet problems

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Abstract

The poset $Y_{k,2}$ consists of $k + 2$ distinct elements $x_1, x_2, \ldots, x_k, y_1, y_2$, such that $x_1 \leq x_2 \leq \cdots \leq x_k \leq y_1, y_2$. The poset $Y'_{k,2}$ is the dual poset of $Y_{k,2}$. The sum of the $k$ largest binomial coefficients of order $n$ is denoted by $\Sigma(n, k)$. Let $L(\sharp)(n, \{Y_{k,2}, Y'_{k,2}\})$ be the size of the largest family $\mathcal{F} \subset 2^{[n]}$ that contains neither $Y_{k,2}$ nor $Y'_{k,2}$ as an induced subposet. Methuku and Tompkins proved that $L(\sharp)(n, \{Y_{2,2}, Y'_{2,2}\}) = \Sigma(n, 2)$ for $n \geq 3$ and they conjectured the generalization that if $k \geq 2$ is an integer and $n \geq k + 1$, then $L(\sharp)(n, \{Y_{k,2}, Y'_{k,2}\}) = \Sigma(n, k)$. On the other hand, it is known that $L(\sharp)(n, Y_{k,2})$ and $L(\sharp)(n, Y'_{k,2})$ are both strictly more than $\Sigma(n, k)$. In this paper, we introduce a simple discharging approach and prove this conjecture.

Keywords: forbidden subposets, discharging method, poset Turán theory

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1 Introduction

The $n$-dimensional Boolean lattice, denoted $B_n$, is the partially ordered set (poset) $(2^{[n]}, \subseteq)$, where $[n] = \{1, \ldots, n\}$. For any $0 \leq i \leq n$, let $\binom{[n]}{i} := \{A \subseteq [n]: |A| = i\}$ denote the $i$th level of the Boolean lattice. Let $P$ be a finite poset and $\mathcal{F}$ be a family of subsets of $[n]$. We say that $P$ is contained

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in $\mathcal{F}$ as a weak subposet if there is an injection $\alpha : P \rightarrow \mathcal{F}$ satisfying $x_1 <_P x_2 \implies \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. $\mathcal{F}$ is called $P$-free if $P$ is not contained in $\mathcal{F}$ as a weak subposet. We define the corresponding extremal function to be $La(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } P\text{-free}\}$. Analogously, if $P, Q$ are two posets then, let $La(n, \{P, Q\}) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } P\text{-free and } Q\text{-free}\}$.

The linearly ordered poset on $k$ elements, $a_1 < a_2 < \ldots < a_k$, is called a chain of length $k$, and is denoted by $P_k$. Using this notation the well-known theorem of Sperner [16] can be stated as $La(n, P_2) = \left(\binom{n}{\lfloor n/2 \rfloor}\right)$. Let us denote the sum of the $k$ largest binomial coefficients of order $n$ by $\Sigma(n, k)$. Erdős [6] extended Sperner’s theorem by showing that $La(n, P_k) = \Sigma(n, k - 1)$ with equality if and only if the family is union of $k - 1$ largest levels of the Boolean lattice. Notice that, since any poset $P$ is a weak subposet of a chain of length $|P|$, Erdős’s theorem implies that $La(n, P) \leq (|P| - 1)\left(\binom{n}{\lfloor n/2 \rfloor}\right) = O\left(\binom{n}{\lfloor n/2 \rfloor}\right)$. Later many authors, including Katona and Tarján [12], Griggs and Lu [9], and Griggs, Li, and Lu [8] studied various other posets (see the recent survey by Griggs and Li [7] for an excellent survey of all the posets that have been studied). Let $h(P)$ denote the height (maximum length of a chain) of $P$. One of the first general results is due to Bukh who showed that if $T$ is a finite poset whose Hasse diagram is a tree of height $h(T) \geq 2$, then $La(n, T) = (h(T) - 1 + O(1/n))\left(\binom{n}{\lfloor n/2 \rfloor}\right)$. The most notorious poset for which the asymptotic value of the extremal function is still unknown is the diamond $D_2$, the poset on 4 elements with the relations $a < b, c < d$ where $b$ and $c$ are incomparable. The best known bound is $(2.20711 + o(1))\left(\binom{n}{\lfloor n/2 \rfloor}\right)$, due to Grósz, Methuku, and Tompkins [10].

We say that $P$ is contained in $\mathcal{F}$ as an induced subposet if and only if there is an injection $\alpha : P \rightarrow \mathcal{F}$ satisfying $x_1 <_P x_2 \iff \alpha(x_1) \subset \alpha(x_2)$ for all $x_1, x_2 \in P$. We say that $\mathcal{F}$ is induced-$P$-free if $P$ is not contained in $\mathcal{F}$ as an induced subposet. We define the corresponding extremal function as $La^I(n, P) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } P\text{-free}\}$. Analogously, if $P, Q$ are two posets then let $La^I(n, \{P, Q\}) := \max\{|\mathcal{F}| : \mathcal{F} \text{ is } P\text{-free and } Q\text{-free}\}$.

Despite the considerable progress that has been made on forbidden weak subposets, little is known about forbidden induced subposets (except for $P_k$, where the weak and induced containment are equivalent). The first results of this type are due to Carroll and Katona [3], and due to Katona [11], showing $La^I(n, V_r) = (1 + o(1))\left(\binom{n}{\lfloor n/2 \rfloor}\right)$ where $V_r$ is the $r$-fork poset ($x \leq y_i$ for all $1 \leq i \leq r$). Boehnlein and Jiang [1] generalized this by extending Bukh’s result to induced containment of tree-shaped posets, $T$, proving $La^I(n, T) = (h(T) - 1 + o(1))\left(\binom{n}{\lfloor n/2 \rfloor}\right)$. Only recently, Methuku and Pálvölgyi [14] showed that for every poset $P$, there is a constant $c_P$ depending only on $P$ such that $La^I(n, P) \leq c_P\left(\binom{n}{\lfloor n/2 \rfloor}\right)$.

Even fewer exact results are known for forbidden induced subposets, which is the topic of this paper. Katona and Tarján [12] proved that $La(n, \{V, \Lambda\}) = La^I(n, \{V, \Lambda\}) = 2\left(\binom{n - 1}{\lfloor n/2 \rfloor - 1}\right)$, where $V$ and $\Lambda$ are the 2-fork and its dual, the 2-brush, respectively.

Now we formally define the posets considered in this paper.

**Definition 1.** Let $k, r \geq 2$ be integers. The $r$-fork with a $k$-shaft poset consists of $k + r$ elements $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_{r-1}, y_r$ with $x_1 \leq x_2 \leq \cdots \leq x_k$ and $x_k \leq y_i$ for all $1 \leq i \leq r$, and is denoted by $Y_{k,r}$. Let $Y_{k,r}'$ denote the reversed poset of $Y_{k,r}$, also called the dual poset of $Y_{k,r}$.

For simplicity, we will write $Y_k$ and $Y_k'$ instead of $Y_{k,2}$ and $Y_{k,2}'$ respectively.
The first result about $Y_{k,r}$ was due to Thanh [19] who showed that $\text{La}(n, Y_{k,r}) = (k + o(1)) \binom{n}{\lfloor n/2 \rfloor}$. The lower order term in his upper bound was improved by De Bonis and Katona [4]. Thanh also gave a construction showing that $\text{La}(n, Y_{k,r}) > \Sigma(n, k)$. Methuku and Tompkins [13] showed that if one forbids both $Y_k$ and $Y'_k$, then an exact result can be obtained: $\text{La}(n, \{Y_k, Y'_k\}) = \Sigma(n, k)$.

Using a cycle decomposition method, they also showed the following exact result for induced posets.

**Theorem 2** (Methuku–Tompkins [13]). If $n \geq 3$, then $\text{La}^*(n, \{Y_2, Y'_2\}) = \Sigma(n, 2)$.

Theorem 2 strengthens the result of De Bonis, Katona, and Swanepoel [5] stating that $\text{La}(n, B) = \Sigma(n, 2)$ where $B$ is the butterfly poset which consists of 4 elements $a, b, c, d$ with $a, b \leq c, d$. Indeed if a family does not contain the butterfly as a subposet, then it contains neither $Y_2$ nor $Y'_2$ as an induced subposet. However, a family might contain neither an induced $Y_2$ nor an induced $Y'_2$ while still containing a butterfly.

In Section 3, we establish the following generalization of Theorem 2 by proving a conjecture from [13].

**Theorem 3.** If $k \geq 2$ is an integer and $n \geq k + 1$, then $\text{La}^*(n, \{Y_k, Y'_k\}) = \Sigma(n, k)$.

Note that forbidding only one of $Y_k$ and $Y'_k$ is not enough to obtain an exact result. Indeed, again by Thanh’s construction [19] we have $\text{La}^*(n, Y_k) > \Sigma(n, k)$ and $\text{La}^*(n, Y'_k) > \Sigma(n, k)$.

We further obtain the following LYM-type inequality if we assume $\emptyset$ and $[n]$ are not in our family.

**Theorem 4.** Let $k \geq 2$ be an integer and $n \geq k + 1$. If $\mathcal{F} \subset 2^{[n]}$ contains neither $Y_k$ nor $Y'_k$ as an induced subposet and $\emptyset, [n] \notin \mathcal{F}$, then

$$\sum_{F \in \mathcal{F}} \left( \frac{n}{|F|} \right)^{-1} \leq k.$$

In particular, $|\mathcal{F}| \leq \Sigma(n, k)$.

## 2 Preliminaries

The following terminology will be used to prove Theorems 2 and 4. Let $\mathcal{F}$ be a family of subsets of $[n]$ which is induced $Y_k$-free and induced $Y'_k$-free. For sets $U, V \subseteq [n]$, let the interval $[U, V]$ denote the Boolean lattice induced by the collection of all sets that contain $U$ and are contained in $V$. A chain $C$ where $C = \{A_0, \ldots, A_n\}$ and $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = [n]$ is called a full chain or a maximal chain.

A spine $S$ is a chain $A_1 \subset A_2 \subset \cdots \subset A_\ell$ such that $|A_{i+1} \setminus A_i| = 1$ for $1 \leq i \leq \ell - 1$ where there are exactly $k - 1$ members of $\mathcal{F}$ in $\{A_1, \ldots, A_\ell\}$ and where $A_1, A_\ell \in \mathcal{F}$. Note that a spine may contain elements not from $\mathcal{F}$.

Let $\mathcal{C}$ be the set of all full chains and let $\mathcal{S}$ be the set of all spines. We say that a full chain $C \in \mathcal{C}$ is associated with a spine $S \in \mathcal{S}$ or that $C$ contains $S$ as a spine if either
1. $C$ has exactly $k - 1$ members of $\mathcal{F}$, which we name $F_1, \ldots, F_{k-1}$. In this case, $C$ is associated with the spine that is a subchain of $C$ from $F_1$ to $F_{k-1}$; or

2. $C$ has exactly $k + x$ elements of $\mathcal{F}$ (where $x \geq 1$), which we name $F_1, \ldots, F_{k+x}$. In this case, $C$ is associated with $x$ spines, namely $S_{F_i}$ for $2 \leq i \leq x + 1$, where $S_{F_i}$ is the spine that is a subchain of $C$ from $F_i$ to $F_{i+k-2}$. (Notice that a chain $C$ with at least $k + 1$ elements of $\mathcal{F}$ is not associated with the spines that correspond to the first $k - 1$ elements of $\mathcal{F} \cap C$ and to the last $k - 1$ elements of $\mathcal{F} \cap C$.)

Let spine$(C)$ denote the set of all spines that $C$ contains. More precisely,

$$\text{spine}(C) := \{ S : C \text{ contains } S \text{ as a spine} \}.$$

### 2.1 Overview of the discharging method

In order to prove Theorem 4, we use discharging arguments and Lemma 6 below. We then prove Theorem 3 by using Theorem 4 and induction on $k$.

Before proving Lemma 6, we need the following straightforward counting lemma, the proof of which we provide for completeness.

**Lemma 5.** Let $n \geq 2$. If $\mathcal{G} \subset \\{ \{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, 2, 3, \ldots, n-1\} \}$, then the number of full chains in $2^{|n|}$ containing no member of $\mathcal{G}$ is at least the number of full chains that contain at least one member of $\mathcal{G}$.

**Proof.** Let the set of chains that contain at least one member of $\mathcal{G}$ be $X$ and the set of chains that contain no member of $\mathcal{G}$ be $Y$. To show that $|X| \leq |Y|$ we will construct an injection from $X$ to $Y$. Consider any chain $C \in X$. Let $C$ be $\varnothing \subset \{ x_1 \} \subset \{ x_1, x_2 \} \subset \{ x_1, x_2, x_3 \} \subset \ldots \subset \{ x_1, x_2, x_3, \ldots, x_n \}$. For simplicity, we will say the permutation corresponding to $C$ is $x_1x_2x_3\cdots x_n$.

If $\{x_1, x_2, \ldots, x_j\}$ is the last set from $\mathcal{G}$ in $C$ and $x_i = 1$, then $x_1x_2\cdots x_j$ is a permutation of $\{1, 2, \ldots, j\}$. Hence, $x_{j+1} \geq j + 1$ and $1 \leq i \leq j$. Let us consider the chain $C'$ corresponding to the permutation

$$x_1x_2\cdots x_{i-1}x_{j+1}x_i+1x_{i+2}\cdots x_{j}x_ix_{j+2}\cdots x_n,$$

obtained by swapping $x_{j+1}$ with $x_i$ in the permutation corresponding to $C$. If $C'$ contains the set $\{1, 2, \ldots, j+1\}$, then it must be the case that $x_{j+1} = j + 1$. Thus, $C$ contains the set $\{1, 2, \ldots, j + 1\}$, which contradicts the maximality of $j$. Therefore, under this map, the full chain $C'$ does not contain any member of $\mathcal{G}$. If we map $C \in X$ to $C' \in Y$ in this way, the map is an injection, as desired. \qed

For the discharging step, we start by placing a weight on a spine depending on the chains that contain it. More precisely, if $S \in \mathcal{S}$ is a spine and $C \in C$ is a full chain, then we define a weight
function \( w(S, C) \) as follows.

\[
  w(S, C) = \begin{cases} 
    1, & \text{if } S \in \text{spine}(C) \text{ and } C \text{ contains at least } k + 1 \text{ members of } \mathcal{F}, \\
    -1, & \text{if } S \in \text{spine}(C) \text{ and } C \text{ contains exactly } k - 1 \text{ members of } \mathcal{F}, \\
    0, & \text{otherwise.}
  \end{cases}
\]

Note that if \( \mathcal{F} = \Sigma(n, k) \), then \( \sum_{S \in \text{spine}(C)} w(S, C) = 0. \)

**Lemma 6.** Let \( k \geq 2 \) be an integer and \( n \geq k + 1 \). Let \( \mathcal{F} \) be a family in \( \mathcal{B}_n \) with no induced \( Y_k \) and no induced \( Y'_k \) such that \( \emptyset, [n] \notin \mathcal{F} \). Let \( \mathcal{S} \) denote the set of spines of \( \mathcal{F} \) and let \( \mathcal{C} \) denote the set of full chains in \( \mathcal{B}_n \). For any \( S \in \mathcal{S} \),

\[
\sum_{C \in \mathcal{C}} w(S, C) \leq 0.
\]

**Proof.** Let a spine \( S \) be the chain \( A_1 \subset A_2 \subset \cdots \subset A_\ell \) where \( |A_{i+1} \setminus A_i| = 1 \) for \( 1 \leq i \leq \ell - 1 \). (Recall that, by definition of a spine, there are exactly \( k - 1 \) members of \( \mathcal{F} \) in \( \{A_1, \ldots, A_\ell\} \) and that \( A_1, A_\ell \in \mathcal{F} \).) If all the chains \( C \in \mathcal{C} \) that contain \( S \) as a spine have at most \( k \) members of \( \mathcal{F} \) then since \( w(S, C) \in \{0, -1\} \) for each of these chains, our lemma follows trivially. Therefore, we may assume that there is a chain \( C \in \mathcal{C} \) that contains \( S \) as a spine and has at least \( k + 1 \) members of \( \mathcal{F} \); such a chain \( C \) must have sets \( P, Q \in \mathcal{F} \) with \( P \subset A_1 \) and \( A_\ell \subset Q \).

If two sets \( A, B \in \mathcal{F} \) are unrelated to each other and \( A, B \subset A_1 \) then we have an induced copy of \( Y'_k \) consisting of \( A, B \), the \( k - 1 \) members of \( \mathcal{F} \) in \( S \), and \( Q \). Therefore, \( \mathcal{F} \cap [\emptyset, A_1] \) induces a chain \( \mathcal{G}_1 \). By symmetry, \( \mathcal{F} \cap [A_\ell, [n]] \) induces a chain \( \mathcal{G}_2 \) as well. Since by assumption \( \emptyset, [n] \notin \mathcal{F} \), the chains \( \mathcal{G}_1 \setminus \{A_1\} \) and \( \mathcal{G}_2 \setminus \{A_\ell\} \) may be extended to chains that satisfy the hypotheses of Lemma 5 for \( \emptyset, A_1 \) and \( [A_\ell, [n]] \).

Therefore, the number \( a_0 \) of full chains in \( [\emptyset, A_1] \) containing no member of \( \mathcal{G}_1 \setminus \{A_1\} \) is at least the number \( a_1 \) of full chains in \( [\emptyset, A_1] \) that contain a member of \( \mathcal{G}_1 \setminus \{A_1\} \). Similarly, the number \( b_0 \) of full chains in \( [A_\ell, [n]] \) containing no member of \( \mathcal{G}_2 \setminus \{A_\ell\} \) is at least the number \( b_1 \) of full chains in \( [A_\ell, [n]] \) that contain a member of \( \mathcal{G}_2 \setminus \{A_\ell\} \). Now notice that the number of chains \( C \in \mathcal{C} \) associated with spine \( S \) that have exactly \( k - 1 \) members of \( \mathcal{F} \) is \( a_1 \cdot b_0 \) and the number of chains \( C \in \mathcal{C} \) associated with spine \( S \) that have at least \( k + 1 \) members of \( \mathcal{F} \) is \( a_1 \cdot b_1 \). Therefore, since \( a_1 \leq a_0 \) and \( b_1 \leq b_0 \),

\[
\sum_{C \in \mathcal{C}} w(S, C) = a_1 \cdot b_1 - a_0 \cdot b_0 \leq 0. \tag*{\Box}
\]

## 3 Proofs of Theorem 3 and Theorem 4

First we use a folklore lemma that establishes an inequality very similar to the LYM inequality. A proof of this lemma occurs in [18] as part of a proof of Erdős’ theorem. Recall that \( \Sigma(n, k) \) denotes the sum of the sizes of the largest \( k \) levels in the Boolean lattice \( 2^{[n]} \).
Lemma 7 (See [18, Lemma 1]). If \( \mathcal{F} \subseteq 2^n \) satisfies
\[
\sum_{F \in \mathcal{F}} \left( \frac{n}{|F|} \right)^{-1} \leq k,
\]
then \( |\mathcal{F}| \leq \Sigma(n, k) \).

Proof of Theorem 4. Observe that by Lemma 6,
\[
\sum_{S \in \mathcal{S}} \sum_{C \in \mathcal{C}} w(S, C) \leq 0.
\]
Now notice that
\[
\sum_{S \in \mathcal{S}} \sum_{C \in \mathcal{C}} w(S, C) = \sum_{C \in \mathcal{C}} \sum_{S \in \mathcal{S}} w(S, C)
\]
and that for any \( C \in \mathcal{C} \), we have
\[
\sum_{S \in \mathcal{S}} w(S, C) = |\mathcal{F} \cap C| - k.
\]
Therefore, the right-hand side of (3) becomes
\[
\sum_{C \in \mathcal{C}} (|\mathcal{F} \cap C| - k) = \sum_{F \in \mathcal{F}} |F|! \cdot (n - |F|)! - k \cdot n!.
\]
So by (2) and (4), we have
\[
\sum_{F \in \mathcal{F}} |F|! \cdot (n - |F|)! - k \cdot n! \leq 0.
\]
After rearranging, we obtain \( \sum_{F \in \mathcal{F}} \left( \frac{n}{|F|} \right)^{-1} \leq k \). Lemma 7 gives that \( |\mathcal{F}| \leq \Sigma(n, k) \), proving Theorem 4.

Proof of Theorem 3. The statement of Theorem 3 is true for \( k = 2 \) (base case) due to Theorem 2.

If neither \( \emptyset \) nor \([n]\) are in \( \mathcal{F} \), then we may apply Theorem 4 directly to obtain \( |\mathcal{F}| \leq \Sigma(n, k) \).

If both \( \emptyset \) and \([n]\) are in \( \mathcal{F} \), then \( \mathcal{F} \setminus \{\emptyset, [n]\} \) is induced \( Y_k \)-free and induced \( Y_k' \)-free. Therefore, it has size at most \( \Sigma(n, k - 1) \) by the induction hypothesis. Since \( 2 + \Sigma(n, k - 1) \leq \Sigma(n, k) \) for \( n \geq k + 1 \) and \( k \geq 2 \), we are done.

Now, without loss of generality, suppose that \( \emptyset \in \mathcal{F} \) and \([n]\) \( \notin \mathcal{F} \). Now consider the family \( \mathcal{F}' := \mathcal{F} \setminus \{\emptyset\} \). By Theorem 4, we have
\[
\sum_{F \in \mathcal{F}'} \left( \frac{n}{|F|} \right)^{-1} \leq k
\]
and $|\mathcal{F}'| \leq \Sigma(n, k)$, by Lemma 7

Now suppose $|\mathcal{F}'| = \Sigma(n, k)$. (Otherwise, we are done.) A consequence of the proof of Lemma 7 is that, in order for equality to hold in (1), the quantities $\binom{n}{n/2}$ (for $F$ in $\mathcal{F}'$) must be as large as possible—that is, the sets $F \in \mathcal{F}'$ must have size as close to $n/2$ as possible. More precisely, in order for equality to hold in (1), the list of the quantities $\binom{n}{F}$ (for $F$ in $\mathcal{F}'$) must be as large as possible—that is, the sets $F \in \mathcal{F}'$ must have size as close to $n/2$ as possible. More precisely, in order for equality to hold in (1), the list of the quantities $\binom{n}{F}$ for $F \in \mathcal{F}'$ in decreasing order (with multiplicities) must be the same as the list of the first $\Sigma(n, k)$ quantities $\binom{n}{S}$ for $S \subseteq 2^{[n]}$ in decreasing order (with multiplicities).

First, if $k$ and $n$ have different parities, then $|\mathcal{F}'| = \Sigma(n, k)$ can only occur if

$$\mathcal{F}' = \left(\binom{n}{\frac{n-k}{2}}\right) \cup \left(\binom{n}{\frac{n-k}{2}+1}\right) \cup \cdots \cup \left(\binom{n}{\frac{n-k}{2}+k-1}\right).$$

However, in that case, $Y_k$ is an induced subposet of $\mathcal{F}'$. Hence, adding $\emptyset$ produces an induced copy of $Y_k$ in $\mathcal{F}$, a contradiction.

Second, if $k$ and $n$ have the same parity, then $|\mathcal{F}'| = \Sigma(n, k)$ can only occur if $\mathcal{F}'$ contains

$$\left(\binom{n}{\frac{n-k}{2}+1}\right) \cup \left(\binom{n}{\frac{n-k}{2}+2}\right) \cup \cdots \cup \left(\binom{n}{\frac{n-k}{2}+k-1}\right)$$

plus $\binom{n}{\frac{n-k}{2}}$ sets from $\binom{n}{\frac{n-k}{2}}$. If $\mathcal{F}'$ contains any set from $\binom{n}{\frac{n-k}{2}}$, then it is easy to see that $Y_k$ is an induced subposet of $\mathcal{F}'$ and adding $\emptyset$ produces an induced copy of $Y_k$ in $\mathcal{F}$. Otherwise, $\mathcal{F}'$ must contain all of the sets from $\binom{n}{\frac{n-k}{2}+k}$ and $n \geq k + 2$. But in this case, $Y_k$ is again an induced subposet of $\mathcal{F}'$, giving an induced copy of $Y_k$ in $\mathcal{F}$, again a contradiction.

Therefore, $|\mathcal{F}'| \leq \Sigma(n, k) - 1$, which implies $|\mathcal{F}| \leq \Sigma(n, k)$, as desired. □

4 Concluding Remarks

During the preparation of this article, we have learned that Tompkins and Wang recently proved Theorem 3 independently [17]. Their approach is closer to the method used in [13] and is different from the approach introduced in this article.

In fact, we believe that a more general result than Theorem 3 holds. Recall that $Y_{k,r}$ denotes the $r$-fork with a $k$-shaft poset and $Y'_{k,r}$ denotes its dual.

**Conjecture 8.** For all $k \geq 2$ and $r \geq 2$, there is an $n_0 = n_0(k, r)$ such that if $n \geq n_0$, then $\text{La}^r(n, \{Y_{k,r}, Y'_{k,r}\}) = \Sigma(n, k)$.

Theorem 3 is the case when $r = 2$; note that for all $k \geq 2$, $n_0(k, 2) = k + 1$.

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