Zero Energy States of Reduced Super Yang-Mills Theories in \(d + 1 = 4, 6\) and 10 dimensions are necessarily Spin(\(d\)) invariant

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March 27, 2022

**Abstract**

We consider reduced Super Yang-Mills Theory in \(d + 1\) dimensions, where \(d = 2, 3, 5, 9\). We present commutators to prove that for \(d = 3, 5\) and 9 a possible ground state must be a Spin(\(d\)) singlet. We also discuss the case \(d = 2\), where we give an upper bound on the total angular momentum and show that for odd dimensional gauge group no Spin(\(d\)) invariant state exists in the Hilbert space.

1 Introduction

We consider models, which are obtained by dimensional reduction of Super Yang-Mills theory with gauge group \(SU(N)\) in \(d + 1\) dimensions, where \(d = 2, 3, 5, 9\). These models were used to formulate a quantum theory of supermembranes living in \(d + 2\) dimensions, and for \(d = 9\) they describe \(N\) interacting D0 branes. It is interesting to know, whether these models admit a possible zero energy state and what the properties of such a state are. The general belief, partially proven, is that for \(d = 2, 3, 5\) no zero energy state exists and that for \(d = 9\) there exists a unique ground state.

Let us start with a very simple argument that zero energy states in \(d = 9\) are Spin(\(d\)) invariant: it is well known \([1, 2, 3]\) that the supercharges, \(Q_\beta\), of reduced Yang-Mills theory (for definitions and conventions, see the next section) and

\[
\tilde{Q}_\beta = m \sum_{i=1}^{3} q_{iA} \gamma_{123}^{(i)} \gamma^\beta \Theta_{\alpha A} - \frac{m}{2} \sum_{\mu=4}^{9} q_{\mu A} \gamma^{(\mu)} \gamma^\beta \Theta_{\alpha A},
\]

\(\gamma^{123} = \gamma^1 \gamma^2 \gamma^3\), satisfy anti commutation relations of the form

\[
\{Q_\beta + \tilde{Q}_\beta, Q_{\beta'} + \tilde{Q}_{\beta'}\} = \delta_{\beta\beta'} \tilde{H} + mJ_{ij} \gamma_{123}^{(ij)} \gamma^\beta \gamma^\beta' - \frac{m}{2} J_{\mu\nu} \gamma^{(\mu\nu)} \gamma^\beta \gamma^\beta' + 2 q_{tA} \gamma_{t}^\beta \gamma_{t}^\beta' J_{A},
\]
where $J_A, J_{ij}, J_{\mu\nu}$ are the generators of $SU(N), \text{Spin}(3), \text{Spin}(6)$ respectively and $\tilde{H}$ is an operator, whose form is not important here. As

$$\{\tilde{Q}_\beta, \tilde{Q}_{\beta'}\} = \delta_{\beta\beta'}(m^2 \sum_{iA} q_{iA}^2 + \frac{m^2}{4} \sum_\mu q_{\mu A}^2) =: \left(\delta_{\beta\beta'}\tilde{H}\right),$$

it immediately follows that

$$\{Q_\beta, \tilde{Q}_{\beta'}\} + \{\tilde{Q}_\beta, Q_{\beta'}\} = \delta_{\beta\beta'}(\tilde{H} - H - \tilde{H}) + mJ_{ij}(\gamma^{123}\gamma^{ij})_{\beta\beta'} - \frac{m}{2} J_{\mu\nu}(\gamma^{123}\gamma^{\mu\nu})_{\beta\beta'},$$

so that for $SU(N)$ invariant zero energy states $\phi, \psi$, i.e. states annihilated by the $Q_\beta$ and $J_A$,

$$\langle \phi, J_{ij}\psi \rangle = 0, \quad \langle \phi, J_{\mu\nu}\psi \rangle = 0.$$

(just multiply $[\Box]$ by $(\gamma^{123}\gamma^{ij})_{\beta\beta'}$, respectively $(\gamma^{123}\gamma^{\mu\nu})_{\beta\beta'}$ and sum over $\beta$ and $\beta'$; hence

$$J_{ij}\psi = 0, \quad J_{\mu\nu}\psi = 0,$$

by choosing $\phi = J_{ij}\psi$, respectively $J_{\mu\nu}\psi$. As (123) may be replaced by any other triple $(stu)$, $J_{st}\psi = 0$, for all $s, t = 1, \ldots, 9$, provided $Q_\beta\psi = 0, J_A\psi = 0$.

In the next section we will treat $d = 2, 3, 5, 9$ on equal footing and, similar to $[\Box]$, look for anti-commutators to prove that zero-energy states have to be invariant under $\text{Spin}(d)$. We do find such anti-commutators for $d = 3, 5$ and $9$. For $d = 2$, we give an upper bound on the total angular momentum and show that if $SU(N)$ is odd dimensional, i.e. $N$ even, no $\text{Spin}(d)$ invariant state exists in the Hilbert space. The discussion below generalizes to other gauge groups.

### 2 Model and Results

Let $d = 2, 3, 5, 9$, and let $(\gamma^i)_{\alpha\beta}$ denote the real irreducible representation of smallest dimension, called $s_d$, of the $\gamma$-matrices in $d$ dimensions, i.e. the relations $\{\gamma^s, \gamma^t\} = 2\delta^{st}\mathbb{I}$. We have $s_d = 2, 4, 8, 16$. The model, which we are discussing, contains the self adjoint bosonic degrees of freedom $q_{sA}, p_{sA}$ ($s = 1, \ldots, d, A = 1, \ldots, N^2 - 1$) and the self adjoint fermionic degrees of freedom $\Theta_{sA}$ ($\alpha = 1, \ldots, s_d, A = 1, \ldots, N^2 - 1$) satisfying

$$[q_{sA}, p_{tB}] = i\delta_{st}\delta_{AB}, \quad \{\Theta_{sA}, \Theta_{tB}\} = \delta_{s\alpha}\delta_{AB}, \quad [q_{sA}, \Theta_{sB}] = [p_{sA}, \Theta_{sB}] = 0.$$

More precisely, we consider the Schrödinger representation $(p_{sA} = -i\partial_{sA})$ of (2) on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^{d(N^2 - 1)}) \otimes \mathcal{F},$$

where $\mathcal{F} \cong (\mathbb{C}^2)^{(s_d/2)(N^2 - 1)}$ is the irreducible representation space of (3). The infinitesimal generators of the gauge group $SU(N)$ read

$$J_A = -if_{ABC}(q_{tB}\partial_{tC} + \frac{1}{2}\Theta_{tB}\Theta_{tC}),$$

2
where $f_{ABC}$ are real, antisymmetric structure constants of $SU(N)$. The physical Hilbert space $\mathcal{H}_{\text{phys}}$, given by the $SU(N)$ invariant states in $\mathcal{H}$, is the Hilbert space of the model. We have a representation of Spin($d$) on $\mathcal{H}$ ($\mathcal{H}_{\text{phys}}$), with infinitesimal generators

$$J_{st} = i(q_{sA}\partial_{tA} - q_{tA}\partial_{sA}) - \frac{i}{4}\Theta_{\alpha\alpha'}\gamma_{\alpha\beta}^{st}\Theta_{\beta\alpha} \equiv L_{st} + M_{st},$$

where $\gamma^{st} = \frac{1}{2}[\gamma^{s}, \gamma^{t}]$. The supercharges are given by

$$Q_{\beta} = \Theta_{\alpha\alpha'}(-i\gamma_{\alpha\beta}^{t}\partial_{iA} + \frac{1}{2}f_{ABC}q_{sB}q_{tC}\gamma_{\alpha\beta}^{st}) ,$$

and the Hamiltonian by

$$H = -\Delta + \frac{1}{2}f_{ABC}q_{sB}q_{tC}f_{ADE}q_{sD}q_{tE} + iq_{sA}f_{ABC}\Theta_{\alpha\alpha'}\Theta_{\beta\alpha'}\gamma_{\alpha\beta}^{s}.$$ 

The anti-commutation relations for the supercharges are

$$\{Q_{\alpha}, Q_{\beta}\} = \delta_{\alpha\beta}H + 2\gamma_{\alpha\beta}^{s}q_{sA}J_{A} ,$$

On $\mathcal{H}_{\text{phys}}$ this reads

$$\{Q_{\alpha}, Q_{\beta}\} = \delta_{\alpha\beta}H.$$ 

We note that the Operators $Q_{\alpha}$ and $H$ are self adjoint on their maximal domain, i.e.

$$\mathcal{D}(Q_{\alpha}) = \{\psi \in \mathcal{H}|(Q_{\alpha}\psi)_{\text{dist}} \in \mathcal{H}\},$$

$$\mathcal{D}(H) = \{\psi \in \mathcal{H}|(H\psi)_{\text{dist}} \in \mathcal{H}\},$$

where $(\cdot)_{\text{dist}}$ is understood in the sense of distributions. The restrictions of $H$ and $Q_{\alpha}$ to $\mathcal{H}_{\text{phys}}$ are also self adjoint. We are only interested in $SU(N)$ invariant states, i.e. states in $\mathcal{H}_{\text{phys}}$. By definition $\psi$ is a zero energy state iff $\psi \in \mathcal{H}_{\text{phys}} \cap \text{Ker}H$. We want to prove the following

**Theorem 1.**

(a) For $d = 3, 5, 9$, a possible zero energy state is a Spin($d$) singlet.

(b) For $d = 2$, a possible zero energy state $\psi$ satisfies

$$\|J_{st}\psi\| \leq \frac{3}{2} \cdot \text{dim} \cdot SU(N) \|\psi\|.$$ 

We start with

**Lemma 2.** We have the following (formal) anti-commutator relations.

(a) For $d = 2, 3, 5, 9$, we have with $b_{\alpha}^{uv} := \frac{1}{s_{d}}(q_{uA}\gamma_{\alpha\epsilon}^{v}\Theta_{\epsilon A} - q_{vA}\gamma_{\alpha\epsilon}^{u}\Theta_{\epsilon A})$,

$$\{Q_{\alpha}, b_{\alpha}^{uv}\} = J_{uv} + (8/s_{d} - 1)M_{uv} .$$
(b) For \( d = 2, 3, 9 \), we have with \( c^w_\alpha := \frac{1}{s_d} (\gamma^w \gamma^u)_{\alpha \epsilon} q_w D \Theta_{\epsilon D} \),

\[
\{ Q_\alpha, c^w_\alpha \} = J_{uw} + (4d/s_d - 1)M_{uv}
\]

(c) For \( d = 3, 5, 9 \), we have with \( a^w_\alpha = \frac{1}{4d-8} \cdot ((4d-s_d)b^w_\alpha - (8-s_d)c^w_\alpha) \),

\[
\{ Q_\alpha, a^w_\alpha \} = J_{uv}.
\]

Proof.

By a straight forward calculation we find for \( d = 2, 3, 5, 9 \)

\[
\{ Q_\alpha, q_u D \Theta_{\epsilon F} \} = -i \gamma^u_\beta \epsilon D \Theta_{\epsilon D} + s_d q_u D (-i \partial_{\epsilon D}) + \frac{1}{2} f_{DBC} q_u D q_s B q_t C \gamma^u_{\alpha \epsilon} .
\]

(a) We have, using (4),

\[
\{ Q_\alpha, q_u D \gamma^u_{\alpha \epsilon} \Theta_{\epsilon D} \} = -i (\gamma^u \gamma^v)_{\beta \epsilon} \Theta_{\beta \epsilon D} + s_d q_u D (-i \partial_{\epsilon D}) + \frac{1}{2} f_{DBC} q_u D q_s B q_t C \gamma^u_{\alpha \epsilon} \gamma^v_{\alpha \epsilon} .
\]

The last term in (4) vanishes since the trace over the \( \gamma \)-matrices equals zero. We find

\[
\{ Q_\alpha, q_u A \gamma^u_{\alpha \epsilon} \Theta_{\epsilon A} - q_v A \gamma^u_{\alpha \epsilon} \Theta_{\epsilon A} \} = -is_d (q_u A \partial_{\epsilon A} - q_v A \partial_{\epsilon A}) - i \Theta_{\alpha \alpha} 2 \gamma^u_{\alpha \epsilon} \Theta_{\epsilon A}
\]

\[
= s_d J_{uv} + (8 - s_d) M_{uv} .
\]

(b) We have, using (4),

\[
\{ Q_\alpha, (\gamma^w \gamma^u)_{\alpha \epsilon} q_w D \Theta_{\epsilon D} \} = d(-i) \gamma^w_{\beta \epsilon} \Theta_{\beta \epsilon D} + s_d q_u D (-i q_u D \partial_{\epsilon D} - q_v D \partial_{\epsilon D})
\]

\[
- \frac{1}{2} Tr(\gamma^w \gamma^u \gamma^v) f_{DBC} q_u D q_s B q_t C
\]

\[
= s_d J_{uv} + (4d - s_d) M_{uv} ,
\]

where the term in the second line is zero, as the trace over the five \( \gamma \)-matrices vanishes.

(c) follows by a linear combination of (a) and (b).

We note that the action of Spin(\( d \)) leaves the kernel of \( H \) invariant. Let \( \varphi, \psi \in \text{Ker}H \cap \mathcal{H}_{\text{phys}} \). Then \( \varphi, \psi \in \text{Ker}Q_\beta \) for all \( \beta \) and by elliptic regularity \( \varphi, \psi \in C^\infty \). We assume that \( \psi \) lies in an irreducible representation space of Spin(\( d \)). Hence \( J_{uv} \psi \in \text{Ker}H \cap \mathcal{H}_{\text{phys}} \). Let \( d = 3, 5, 9 \). By Lemma 2 (c), we have

\[
\{ Q_\alpha, a^w_\alpha \psi \} = J_{uv} \psi \in \mathcal{H} .
\]

Taking the scalar product with \( \varphi \), we want to bring \( Q_\alpha \) to the other side, i.e. integrate by parts. Therefore we regularize as in (4). There exists a function \( \chi : [0, \infty) \to \mathbb{R} \) in \( C^\infty \), such that

\[
\chi(r) = \begin{cases} 
1 & r \leq 1 \\
\in [0, 1] & 1 < r < 3 \\
0 & 3 \leq r
\end{cases}
\]
and $|\chi'(r)| \leq 1$. Define $g_n(q) \equiv \chi(|q|/n)$. By dominated convergence,
\[
(\varphi, J_{uv} \psi) = \lim_{n \to \infty} (\varphi, g_n Q_{\alpha} a^w_{\alpha} \psi) = \lim_{n \to \infty} (\varphi, [g_n, Q_{\alpha}] a^w_{\alpha} \psi) + \lim_{n \to \infty} (\varphi, Q_{\alpha} g_n a^w_{\alpha} \psi) .
\] (6)

The second term in (6) vanishes since $g_n a^w_{\alpha} \psi \in C^\infty$ is in the domain of $Q_{\alpha}$ and $Q_{\alpha}$ is self adjoint. By
\[
||[Q_{\beta}, g_n]||_{F} \leq \text{const} \cdot 1 \cdot \frac{1}{n} \chi(|q|/n),
\]
where $| \cdot |_{F}$ stands for the the norm in $F$ or the operator norm in $L(F)$, the first term in (6) vanishes using the following estimate.
\[
|(\varphi, [g_n, Q_{\alpha}] a^w_{\alpha} \psi)| \leq \text{const} \cdot \int_{n \leq |q| \leq 3n} n \cdot \frac{1}{n} |\varphi|_{F} |\psi|_{F} dq \to 0 \quad \text{for} \quad n \to \infty.
\]

Hence $(\varphi, J_{uv} \psi) = 0$. Choosing $\varphi = J_{uv} \psi$, we find
\[
(J_{uv} \psi, J_{uv} \psi) = 0.
\]

By linear combination, it follows that for $d = 3, 5, 9$ all states in $\text{Ker} H \cap \mathcal{H}_{\text{phys}}$ are Spin($d$) singlets. For $d = 2$, we use Lemma 2 (a) or (b) and find by an analogous argument $(\varphi, (J_{st} + 6 M_{st}) \psi) = 0$. Choosing $\varphi = J_{st} \psi$, we obtain
\[
(J_{12} \psi, J_{12} \psi) \leq 6 |(J_{12} \psi, M_{12} \psi)| \leq 6 \|J_{12}\| \cdot \|M_{12}\| .
\]

A real irreducible representation of the $\gamma$-matrices in 2 dimensions is given by $\gamma^1 = \sigma^1$, $\gamma^2 = -\sigma^3$. In this representation we have $\gamma^{12} = \frac{1}{2}[\gamma^1, \gamma^2] = i \sigma^2$. It follows that
\[
\|J_{12}\| \leq 6 \|M_{12}\|
\]
\[
= 6 \|\frac{i}{4} \Theta_{\alpha A} \gamma^{12}_{\alpha \beta} \Theta_{\beta A} \psi\|
\]
\[
= 6 \|\frac{i}{2} \Theta_{1A} \Theta_{2A} \psi\|
\]
\[
\leq \frac{3}{2} \text{dim } SU(N) \|\psi\|.\]

By linear combination the above equation holds for all states in $\text{Ker} H \cap \mathcal{H}_{\text{phys}}$. Hence Theorem 3 follows.

The case $d = 2$ is special as the following theorem shows.

**Theorem 3.** For $d = 2$ and odd dimensional gauge group $SU(N)$ no Spin($d$) invariant state exists in $\mathcal{H}$.

**Proof.** By definition
\[
J_{12} = -i(q_{1A} \partial_{2A} - q_{2A} \partial_{1A}) - \frac{i}{4} \Theta_{\alpha A} \gamma^{12}_{\alpha \beta} \Theta_{\beta A} .
\]

As above, we choose $\gamma^1 = \sigma^1$, $\gamma^2 = -\sigma^3$. We define the following annihilation and creation operators
\[
\frac{\partial}{\partial \lambda_A} = \frac{1}{\sqrt{2}}(\Theta_{1A} + i \Theta_{2A}), \quad \lambda_A = \frac{1}{\sqrt{2}}(\Theta_{1A} - i \Theta_{2A}).
\]
We find
\[ J_{12} = L_{12} - \frac{i}{2} \Theta_{1A} \Theta_{2A} = L_{12} - \frac{1}{2} \lambda_A \frac{\partial}{\partial \lambda_A} + \frac{1}{4} \cdot \dim SU(N). \]

Assume \( \psi \) is Spin\((d)\)-invariant, i.e. \( J_{12} \psi = 0 \). Then
\[
\left( L_{12} - \frac{1}{2} \lambda_A \frac{\partial}{\partial \lambda_A} \right) \psi = -\frac{1}{4} \cdot \dim SU(N) \psi.
\]

If \( \dim SU(N) \) is odd this contradicts that the spectrum of \( L_{12} - \frac{1}{2} \lambda_A \frac{\partial}{\partial \lambda_A} \) only takes values in \( \frac{1}{2} \mathbb{Z} \). Hence the claim follows. \( \square \)

**Acknowledgments.** We thank J. Fröhlich and G.M. Graf for useful discussions, and S. Sethi for pointing out to us reference [4].

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