A Necessary Condition for Byzantine $k$-Set Agreement

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Abstract

This short paper presents a necessary condition for Byzantine $k$-set agreement in (synchronous or asynchronous) message-passing systems and asynchronous shared memory systems where the processes communicate through atomic single-writer multi-reader registers. It gives a proof, which is particularly simple, that $k$-set agreement cannot be solved $t$-resiliently in an $n$-process system when $n \leq 2t + \frac{k}{t}$. This bound is tight for the case $k = 1$ (Byzantine consensus) in synchronous message-passing systems.

Keywords: Byzantine process, $k$-Set agreement, Message-passing system, read/write register.

1 Computation Models

Process failure model The system is made up of $n$ sequential processes $p_1, 2, ..., p_n$. A process that executes without deviating from its intended behavior (as defined by its algorithm) is said to be correct or non-faulty. A process that deviates from its intended behavior is faulty. If it can deviate arbitrarily, it is Byzantine. This is the fault model considered in this paper. The parameter $t$ denotes the maximum number of processes which may be Byzantine in the corresponding computation model.

The bad behavior of a Byzantine process can be intentional or not. As examples of Byzantine behaviors, a process may crash, fail to communicate, communicate fake values, communicate correctly with some processes and incorrectly with others, etc. Several Byzantine processes can also collude to “pollute” the computation.

Message-passing communication models We consider here two types of communication media and three computation models. The first communication medium is the classical message-passing model. Each pair of processes is connected by a bi-directional channel. To simplify the presentation, it is assumed that each process has a channel from itself to itself. Moreover, the channels are reliable (there is neither loss, corruption, duplication, nor creation of messages). The message-passing medium gives rise to two computation models.

- The asynchronous model, denoted $BAMP_{n,t}$, considers $n$ asynchronous processes, among which up to $t$ may be Byzantine, and asynchronous channels. “Asynchronous” means that (a) each process proceeds according to its own speed which may vary with time and always remains unknown to the other processes, and (b) the transit time of each message is finite but unbounded.

- The synchronous model, denoted $BSMP_{n,t}$ considers $n$ processes, among which up to $t$ may be Byzantine, which execute a sequence of rounds in a lock-step manner. In every round a process first sends messages, then receives messages, and finally executes a local computation. The important property is that a message sent in a round is received in the very same round.
Read/write register communication model The second communication medium is the shared memory model where the processes communicate through single-writer multi-reader (SWMR) atomic registers. The processes are asynchronous (as in the corresponding message-passing model). An SWMR register is a read/write register that can be written by a single predefined process, and read by any process. It follows that no Byzantine process can write into a register whose writer is a correct process. Let us observe that the use of SWMR atomic registers is natural in the presence of Byzantine processes: using multi-writer multi-reader registers would allow Byzantine processes to corrupt the whole memory, so that no “useful” computation could be done. The corresponding model is denoted $BARMW_{n,t}$.

2 The $k$-Set Agreement Problem

Definition The $k$-set agreement problem is a generalization of the consensus problem, which corresponds to the case $k = 1$. It was introduced by S. Chaudhuri [1] in the context of the process crash failure model. A crash is an unexpected stop without recovery. The aim was to investigate the relation between the maximal number of faulty processes ($t$) and the minimal number of allowed decision values ($k$).

The problem consists in providing the processes with an operation $\text{propose}_k()$, which returns a value to the invoking process. According to the usual terminology, when $p_i$ invokes $\text{propose}_k(v_i)$, we say “$p_i$ proposes value $v_i$”. If the invocation returns $v$, we say “$p_i$ decides $v$”.

The $k$-set agreement problem is defined by the following properties (which means that any algorithm solving the problem must satisfy them).

- Termination. The invocation of $\text{propose}_k()$ by a correct process terminates.
- Agreement. At most $k$ different values are decided by correct processes.
- Validity. If all correct processes propose the same value, no other value can be decided by a correct process.

On the validity property The validity property relates the outputs (values decided by the correct processes) to the inputs (values proposed by the correct processes). Let us notice that the previous validity property is particularly weak. As soon as two correct processes propose different values, any set of at most $k$ (possibly arbitrary) values can be collectively decided by the correct processes [8].

Stronger validity properties could be considered, such as: a value decided by a correct was proposed by a correct process. The interest of the weaker validity property lies in the fact that it enlarges the scope of our necessary condition on $t$. To be implemented, any stronger validity property requires a constraint on $t$ as strong or even stronger than our condition [2, 3, 5, 6, 7].

3 A Necessary Condition for $k$-Set Agreement in $BAM\mathcal{P}_{n,t}$ and $BSM\mathcal{P}_{n,t}$

Theorem 1. There is no algorithm that solves $k$-set agreement in $BAM\mathcal{P}_{n,t}$ or $BSM\mathcal{P}_{n,t}$ when $n \leq 2t + \frac{t}{k}$.

Proof The proof is made up of two parts.

Part 1 on the proof.

Let $\Sigma$ be an $n$-process system such that $n \leq 2t + \frac{t}{k}$, and $C(F)$ be its set of correct (faulty) processes. Assuming $|C| \leq t + \frac{t}{k}$ and $|F| = t$, let us partition the set $C$ composed of all correct processes into $(k + 1)$ subsets $S_1, \ldots, S_{k+1}$, such that any of these subsets contains $\lfloor \frac{n + 1}{k + 1} \rfloor$ or $\lceil \frac{n + 1}{k + 1} \rceil$ processes (hence, $\forall i, j \in \{1, \ldots, (k + 1)\} : |S_i| - |S_j| \leq 1$). This system is represented in the left part of Figure[1] where a segment connecting two sets means that each process of a set is connected to each process of the other set (remember that the message-passing communication graph is complete). Let $S_i = C \setminus S_i$. 


Claim: $|S_i| \leq t$.

Proof of the claim. Let us assume by contradiction that $|S_i| > t$. As $S_i$ and $S_i$ define a partition of $C$, we have $|S_i| + |S_i| = |C| \leq t + \frac{t}{k}$. Thus $|S_i| > t$, it follows that $|S_i| < \frac{t}{k}$. Moreover, as $S_i$ contains $k$ sets (all subsets of $C$ except $S_i$), and their cardinality differ at most by 1, there is necessarily a subset $S_j \in S_i$ such that $|S_j| > \frac{t}{k}$. For the same cardinality reason, it follows from $|S_j| > \frac{t}{k}$ that $|S_i| \geq \frac{t}{k}$. But we showed that $|S_i| < \frac{t}{k}$, contradiction. Consequently the initial assumption $|S_i| > t$ is incorrect. End of proof of the claim.

Let us assume (for a future contradiction) that there exists an algorithm $A_k$ that solve the $k$-set agreement problem in the system $\Sigma$ where (thanks to the claim) we have $|S_i| \leq t$ for any $i \in [1..(k+1)]$.

![Figure 1: \Sigma (left) and the behavior of the t Byzantine processes (right)](image)

Part 2 of the proof.

To specify the behavior of the Byzantine processes, let us consider the right part of Figure 1. The Byzantine processes of $F$ behave as follows. For each $i \in [1..(k+1)]$, the processes of $F$ simulate $(k+1)$ sets of processes, $F_1, ..., F_{k+1}$, such that each set $F_i$ behaves correctly (i.e., execute $A_k$) with respect to $S_i$. We say that the processes of $F$ “play $(k+1)$ duplicity roles”.

Let us now suppose that the processes in $S_1 \cup \cdots \cup S_{k+1}$ execute algorithm $A_k$, while the processes of $F$ play the $(k+1)$ duplicity roles $F_1, ..., F_{k+1}$ described previously. Moreover, for each $i \in [1..(k+1)]$, both the processes of $S_i$, and the processes of $F$ in their $F_i$ role, propose the same value $v_i$, these values being such that $(i \neq j) \Rightarrow (v_i \neq v_j)$.

As, for each $i$, $|S_i| \leq t$ (see the claim), it follows that, the processes of $S_i$ (which are correct) cannot distinguish the case where the processes of $F$ are Byzantine and play $(k+1)$ different roles while the processes of $S_i$ are correct, from the case where the processes of $F$ are correct while the processes of $S_i$ are Byzantine. Hence, as by assumption algorithm $A_k$ is correct, it follows from its Termination and Validity properties that, for each $i \in [1..(k+1)]$, the processes of $S_i$ decide $v_i$. Hence, $(k+1)$ values are decided by the correct processes, which violates the Agreement property. Consequently, there is no algorithm $A_k$.

While the previous reasoning relies on the fact that communication is by message-passing (Byzantine processes send different messages to each set $S_i$), it is independent of the fact that communication is synchronous or asynchronous. Hence, the proof is valid for both $BAMP$ and $BSMP_{n,t}$. $\square$

4 The Condition is Also Necessary in $B\!A\!R\!W_{n,t}$

**Theorem 2.** There is no algorithm that solves $k$-set agreement in $B\!A\!R\!W_{n,t}$ when $n \leq 2t + \frac{t}{k}$. 

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Proof  Considering the proof of Theorem 1, the proof consists in showing that the duplicity behavior of the Byzantine processes can be produced in $\text{BARW}_{n,t}$. The theorem then follows from the previous proof.

Let $p_y \in F$, $\text{REG}[y]$ a register that can be written only by $p_y$, and $p_x$ a correct process of a set $S_i$. The duplicity behavior of $p_y$ with respect to $p_x$ is produced as follows. Just before $p_x$ reads $\text{REG}[y]$, $p_y$ writes in $\text{REG}[y]$ the corresponding value produced by its execution of $A_k$ in its $F_i$ role. It follows that, for each $i \in [1..(k + 1)]$, the processes of $F$ appear as correct processes to the processes of $S_i$, which concludes the proof.

$\blacksquare$

Theorem 2

The Byzantine Consensus Case ($k = 1$) in $\text{BSMP}_{n,t}$

When considering consensus, the necessary condition $n > 2t + \frac{t}{k}$ boils down to $n > 3t$, which has been shown to be both necessary and sufficient in the model $\text{BSMP}_{n,t}$ [4]. It follows that, for $k = 1$ and the model $\text{BSMP}_{n,t}$, the proof of Theorem 1 constitutes a new proof of the necessity of $n > 3t$.

A noteworthy feature of this proof lies in the fact that it is a direct proof. Differently, the proofs of the condition $n > 3t$ encountered in the literature (see the first proof given in [4] or classic proofs given thereafter in textbooks, e.g., [5][9]) are decomposed in two steps: (a) first a proof showing that there is no consensus algorithm in $\text{BSMP}_{3,1}$, (b) followed by a simulation, on top of $\text{BSMP}_{3,1}$, of an algorithm assumed to solve consensus in $\text{BSMP}_{n,t}$ where $n \leq 3t$. In addition of being direct and based on a classic indistinguishability argument, the proof of Theorem 1 is more general as it considers a generic agreement problem, namely $k$-set agreement whose consensus is only its more constrained instance.

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References

[1] Chaudhuri S., More choices allow more faults: Set consensus problems in totally asynchronous systems. Information and Computation, 105(1):132158, 1993.

[2] Correia M., Ferreira Neves N., and Verissimo P., From consensus to atomic broadcast: time-free Byzantine-resistant protocols without signatures. Computer Journal, 49(1):82-96, 2006.

[3] Herlihy M.P., Kozlov D., and Rajsbaum S., Distributed computing through combinatorial topology, Morgan Kaufmann/Elsevier, 336 pages, 2014 (ISBN 9780124045781).

[4] Lamport L., Shostack R., and Pease M., The Byzantine generals problem. ACM Transactions on Programming Languages and Systems, 4(3)-382-401, 1982.

[5] Lynch N.A., Distributed algorithms. Morgan Kaufmann Pub., San Francisco (CA), 872 pages, 1996 (ISBN 1-55860-384-4).

[6] Mostéfaoui A., Moumen H., and Raynal M., Modular randomized Byzantine k-set agreement in asynchronous message-passing systems. Proc. 17th Int’l ACM Conference on Distributed Computing and Networks (ICDCN’16), ACM press, 10 pages, Singapore, January 2016.

[7] Mostéfaoui A. and Raynal M., Signature-free broadcast-based intrusion tolerance: never decide a Byzantine value. Proc. 14th Int’l Conference On Principles Of Distributed Systems (OPODIS'10), Springer LNCS 6490, pp. 144-159, 2010.

[8] de Prisco R., Malkhi D., and Reiter M.K., On k-set consensus problems in asynchronous systems. IEEE Transactions on Parallel Distributed Systems, 12(1):7-21, 2001.

[9] Raynal M., Fault-tolerant agreement in synchronous message-passing systems. Morgan & Claypool, 165 pages, 2010 (ISBN 978-1-60845-525-6).