Fermion Propagator in QED -
Landau-Khalatnikov-Fradkin Transformations

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Abstract.
Gauge theories such as quantum electrodynamics (QED) and quantum chromodynamics (QCD) describe the physical world accurately at the level of fundamental particles. They possess gauge symmetry reflected in terms of several identities and transformation laws which impose tight constraints on all conceivable Green functions which define the theory. In this article, we describe and summarize the role played by the Landau-Khalatnikov-Fradkin (LKF) transformations in this context. Within the set of covariant gauges, these transformations tell us how to construct a Green function in an arbitrary gauge, starting from its explicit expression in a particular gauge. In perturbation theory, these transformations are satisfied at every order of approximation. A non-perturbative description of QED and QCD in the continuum is provided by the Schwinger-Dyson Equations (SDEs). These are the fundamental equations of motion encoding the dynamics of Green functions. These equations provide a unified description of weak and strong coupling regimes and are thus increasingly employed to study strongly interacting theories and their transition to the perturbative limit. As these equations are an infinite set of coupled non-linear equations, a truncation is essential to reduce them to a solvable number. LKF transformations provide a stringent constraint on the acceptable truncations which preserve the original symmetries of the gauge theory involved. Most of these truncations consist in cleverly constructing an Anstaz for the electron-photon vertex in QED and the quark-gluon vertex in QCD. In this article, we review the LKF transformations for the fermion propagator. Very importantly, they imply the gauge invariance of the chiral fermion condensate and the pole mass of a fermion. We provide the first demonstration of the latter in this article. Moreover, we also describe how the LKF transformations of the fermion propagator provide gauge-symmetry constraints on a non-perturbative construction of the three-point fermion-boson vertex.

1. Introduction
In a gauge field theory, Green functions transform in a specific manner when we vary the covariant gauge parameter. In quantum electrodynamics (QED), these transformations are dubbed as the Landau-Khalatnikov-Fradkin (LKF) transformations, [1, 2], derived originally by Landau and Fradkin. These were subsequently re-derived by Johnson and Zumino through functional methods, [3, 4]. In general, LKF transformations and their applications are far from simple. As a result, these transformations have played less significant and practical role in the study of gauge theories. What are simpler to implement or check are the Ward-Green-Takahashi identities (WGTI) in QED, [5, 6, 7], and Slavnov-Taylor identities (STI) in QCD, [8, 9]. These identities also stem from gauge invariance, relating different $n$-point functions to each other. Both the LKF transformations and the WGTI-STI are non-perturbative
nature. Their derivation is independent of the strength of the coupling involved and hence also of all the consequences it entails. WGTI and STI have extensively been invoked to construct non-perturbative Ansatz for the full fermion-boson vertex, both in QED and QCD, see for example [10, 11].

WGTI follow from the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. One can enlarge this symmetry by transforming also the gauge parameter, [12, 13], to arrive at what are known as the Nielsen identities (NI). An advantage of the NI over the conventional Ward identities is that explicit \( \partial/\partial \xi \) is present in the new relations. This can be helpful in proving the gauge independence of physically observable quantities which can be extracted from the Green functions under study, [14, 15]. Such a variation can be straightforwardly implemented in the LKF transformations as shown in the next section. As a result, we explicitly demonstrate the gauge invariance of the fermion pole mass. To the best of our knowledge, this is the first direct proof of this fact through the LKF transformations. Augmented with an earlier proof and verification that the chiral fermion condensate is also gauge invariant, [16], we expect the fermion mass function to be considerably constrained.

The LKF transformation for the three-point vertex is quite involved and hampers direct extraction of analytical restrictions on its structure. However, indirect analytical insight can be obtained on its non-perturbative constructions. This vertex is tied to the fermion propagator through the gap equation of the latter. Thus we can constrain it by demanding correct gauge covariance properties of the fermion propagator which are widely known.

2. The LKF Transformation for the Fermion Propagator
The LKF transformations have the simplest structure in the Euclidean coordinate space. Therefore, we start by defining the Fourier transformations between the scalar propagators in coordinate and momentum spaces as follows:

\[
S(x; \xi) = \int \frac{d^d k}{(2\pi)^d} e^{-ik \cdot x} S(k; \xi), \tag{1}
\]

\[
S(k; \xi) = \int d^d x e^{ik \cdot x} S(x; \xi). \tag{2}
\]

We use the notation \( S \) for the propagator in the coordinate space in order to specify that its functional dependence is different from that of \( S \), the same propagator in the momentum space.

The LKF transformation relating the coordinate space propagator in a given gauge \( \xi_0 \) to the one in an arbitrary covariant gauge \( \xi \) reads:

\[
S(x; \xi) = S(x; \xi_0)e^{-i(\Delta(0)-\Delta(x))}, \tag{3}
\]

where

\[
\Delta(x) = -i(\xi - \xi_0)e^2(\mu x)^{4-d} \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik \cdot x}}{k^4} = -\frac{i(\xi - \xi_0)e^2}{16(\pi)^{d/2}}(\mu x)^{4-d}\Gamma\left(\frac{d}{2} - 2\right). \tag{4}
\]

Here, \( \mu \) is a mass scale introduced for dimensional purposes; it ensures that in every space-time dimension \( d \), the electromagnetic coupling \( e \) is dimensionless.

3. The Chiral Fermion Condensate
The LKF transformation leaves SDEs and WGTI functionally invariant in different gauges, and thus imposes strong constraints on the gauge covariance of the fermion propagator. It is trivial
to see that corresponding LKF transformation ensures the gauge invariance of the chiral fermion condensate:

\[
\langle \bar{\psi} \psi \rangle_\xi = -\text{Tr} S(x = 0; \xi),
\]

\[
= -\text{Tr} S(x = 0; \xi_0) e^{-i[\Delta(0) - \Delta(x = 0)]},
\]

\[
= -\text{Tr} S(x = 0; \xi_0) = \langle \bar{\psi} \psi \rangle_{\xi_0}.
\]

This was first demonstrated explicitly in [16] in the context of three-dimensional QED (QED3). SDE for the fermion propagator was explicitly solved in the Landau gauge for quenched QED. The result for the mass function \( M(p^2) \) and the wave-function renormalization \( F(p^2) \) was then numerically transformed to various other covariant gauges. An explicit calculation of the chiral fermion condensate revealed its gauge independence. Interestingly, the Euclidean pole mass, defined as \( p^2 = M^2(p^2) \), was also found to be approximately gauge invariant. A subsequent article, [17], studied unquenched QED3 for different truncations and varying number of massless fermion flavors \( N_f \). As is well known for several truncation schemes, chiral symmetry is restored and deconfined phase is reached above a critical value of \( N_f \). The implementation of LKF transformation revealed that these phenomena of chiral symmetry restoration and deconfinement are coincidental and gauge invariant.

4. The Pole Mass

The gauge independence of the fermion mass is usually established by invoking the Nielsen identity (NI), [12], for the fermion propagator. Following the derivation in [14], as mentioned before, the NI is derived by exploiting an extension of the usual BRST symmetry. A variation of the gauge parameter \( \xi \) is included, \( (\delta \xi = \chi) \), which leaves the QED Lagrangian invariant. Note that this is achieved when the Lagrangian is augmented with an extra term with no physical effects, namely \( \frac{1}{2} \chi \bar{c}(x) B(x) \). \( \chi \) is a new Grassmannian parameter that forms a BRST doublet with \( \delta \chi = 0 \). Moreover \( \bar{c} \) and \( B \) are the usual anti-ghost and Nakanishi-Lautrup fields, respectively. The NI reads:

\[
\frac{\partial S^{-1}(p; \xi)}{\partial \xi} = S^{-1}(p; \xi) \left[ F(p) + \bar{F}(-p) \right],
\]

where \( F(p) \) and \( \bar{F}(p) \) are the Fourier transforms of the following one particle irreducible (1PI) functions:

\[
\frac{\partial}{\partial \chi} \delta^2 \Gamma \bigg|_{\chi = 0} = i \int \frac{d^4q}{(2\pi)^4} e^{iq(y-x)} F(q),
\]

\[
\frac{\partial}{\partial \chi} \delta K \bigg|_{\chi = 0} = i \int \frac{d^4q}{(2\pi)^4} e^{iq(y-x)} \bar{F}(q),
\]

where \( \delta^2 \Gamma \) and \( \delta K \) are the external sources introduced to couple the non linear BRST variations of the \( \psi \) and \( \bar{\psi} \) fields respectively. Assuming that the \( F \) and \( \bar{F} \) functions are regular at the pole mass \( M \), Eq. (5) implies the gauge independence of the inverse fermion propagator at the pole position \( p^2 = -M^2 \):

\[
\frac{\partial S^{-1}(p; \xi)}{\partial \xi} \bigg|_{p^2 = -M^2} = 0,
\]

from which the gauge independence of the pole mass itself is easily inferred, [14].
One is tempted to imagine LKF transformations as the integrated version of the NI. Whether an equivalence between LKF transformations and NI exists is an interesting question which we are unable to address and answer as yet. A first step towards this endeavor might be to show that Eq. (8) is also implied by the LKF transformation for the fermion propagator.

In order to make connection with the NI, one takes the Fourier transform of Eq. (3) and differentiates it with respect to the gauge parameter \( \xi \), obtaining

\[
\frac{\partial S(p; \xi)}{\partial \xi} = -e^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4} \left[ S(p; \xi) - S(p - q; \xi) \right].
\]

We have assumed that the starting value of the gauge parameter is zero. This corresponds to the well known Landau gauge. Since the variation of the inverse propagator is required, one uses the identity

\[
\frac{\partial S^{-1}(p; \xi)}{\partial \xi} = -S^{-1}(p; \xi) \frac{\partial S(p; \xi)}{\partial \xi} S^{-1}(p; \xi),
\]

thus obtaining

\[
\frac{\partial S^{-1}(p; \xi)}{\partial \xi} = e^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4} \left[ S^{-1}(p; \xi) - S^{-1}(p; \xi) S(p - q; \xi) S^{-1}(p; \xi) \right].
\]

It is clear that the right side of Eq. (11) vanishes at the mass pole, even if the first term multiplies the logarithmically ultraviolet divergent integral, rendered finite by, for instance, the dimensional regularization procedure.

### 5. LKF Transformation and the Fermion-Boson Vertex

Two completely general Lorentz decompositions of the massless fermion propagator in momentum and coordinate space are, respectively:

\[
S(p; \xi) = \frac{F(p; \xi)}{p^\nu},
\]

\[
S(x; \xi) = \not{x} \chi(x; \xi).
\]

Let us start by supposing the Landau gauge fermion propagator to be merely the tree level one: \( F(p; 0) = 1 \). We Fourier transform this free fermion propagator to the coordinate space and apply the gauge transformation law of Eq. (3) to find the fermion Green function \( S(x; \xi) \) in an arbitrary covariant gauge. Its inverse Fourier transform yields the fermion propagator back in momentum space:

\[
F(p; \xi) = \frac{1}{2^{2\nu}} \frac{\Gamma(1 - \nu)}{\Gamma(2 + \nu)} (p^2 x_{\text{min}}^2)^\nu.
\]

where \( \nu = \alpha \xi/(4\pi) \), \( \alpha = e^2/(4\pi) \) and \( x_{\text{min}} \propto 1/\Lambda \) is a cut-off regularization. The renormalized wave-function renormalization is thus

\[
F_R(p^2/\mu^2; \xi) = 2^{-1} (\mu^2/\Lambda^2) F(p^2/\Lambda^2; \xi) = \left( \frac{p^2}{\mu^2} \right)^\nu,
\]

where we have imposed the renormalization condition of restoring the tree level fermion propagator at sufficiently large renormalization scale \( \mu^2 \). Subscript \( R \) refers to the renormalized quantity and a slightly modified notation is to emphasize the difference between the regularized
We compare different vertex Ansätze as regards the LKF transformation for the massless fermion propagator. The first two columns define the vertex we consider. The letters correspond to the names of the authors in the order they appear in the publication. The third column shows whether the fermion propagator is MR or not. The last column states the exact exponent of the fermion propagator to determine if the vertex complies with the exact prediction of the LKF transformation for the leading log series, namely $\nu = \alpha\xi/(4\pi)$. Some Ansätze lead to MR fermion propagator but the exponent can only be calculated numerically, as indicated in the last column.

$Z_2(\mu^2; \Lambda^2) = F(\mu^2/\Lambda^2; \xi)$. The relation (14) gets generalized to the non-abelian case of QCD if $\nu \rightarrow c_F\nu$, $\alpha \rightarrow \alpha_s$ and $e \rightarrow g_s$, the strong coupling constant. Taking into account the gluon self interactions could imply a more involved expression or exponent $\nu$ beyond the leading order in $\alpha_s$, as compared to the abelian case of QED. Note that Eq. (14) encompasses an all order resummation of the leading logarithms in the perturbative expansion of the massless fermion propagator in QED.

This gauge covariant solution is achievable through the fermion SDE of the fermion propagator only if the fermion-boson vertex employed is carefully chosen. Neither the bare vertex nor the Ball-Chiu vertex, [10], comply with this fundamental requirement of gauge covariance. In short, ensuring the WGTI for the three-point vertex provides no guarantee that the LKF transformation for the fermion propagator will be preserved. Note that the vertex itself must satisfy its LKF transformation. This is a relatively harder problem to undertake. However, string inspired proper time formalism may provide a realistic possibility in this connection, [25]. With a varying degree of realistic connection with one-loop perturbation theory in the weak coupling regime, several vertex Ansätze have been put forward. These efforts are important for QCD phenomenology, especially because cleaner probes of hadrons are electromagnetic in nature. For example, without an adequate choice of the transverse quark-photon vertex, abelian anomaly is not satisfied in the study of the pion transition form factor to two photons through SDEs, [22]. Moreover, the experimentally observed mass difference between hadrons and their parity partners owes itself to the anomalous chromomagnetic term in the quark-gluon vertex, [23]. Similarly, the momentum-squared dependence of the proton elastic form factor crucially depends on the appropriate construction of the transverse quark-gluon vertex, [24].

With the additional constraint of the gauge-invariance of the chiral fermion condensate and the fermion pole mass, as dictated by the LKF transformations and explicitly demonstrated in this article, we believe that the choice of the truncations would get substantially narrowed down. This is an important step forward.

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