Abstract. We determine for which Coxeter types the associated small quotient of the 2-category of Soergel bimodules is finitary and, for such a small quotient, classify the simple transitive 2-representations (sometimes under the additional assumption of gradability). We also describe the underlying categories of the simple transitive 2-representations. For the small quotients of general Coxeter types, we give a description for the cell 2-representations.

1. Introduction and description of the results

In this paper we fix the ground field $\mathbb{C}$ of complex numbers. Let $(W, S)$ be a finitely generated Coxeter system and $h$ a reflections faithful $W$-module in the sense of [So2, Definition 1.5]. With this datum one associates a 2-category $\mathcal{S} = \mathcal{S}_{W,S,h}$ of Soergel bimodules, see [So2]. By [So1] (for finite Weyl groups) and [EW] (in the general case), the Grothendieck decategorification of $\mathcal{S}$ is isomorphic to the Hecke algebra $H = H_{W,S}$ of $W$.

The paper [KMMZ] studies a certain quotient $\mathcal{S}'$ of $\mathcal{S}$, called the small quotient, in the case when the Coxeter system $(W, S)$ is finite. The main result of [KMMZ] is a classification of all simple transitive 2-representations of $\mathcal{S}'$ in all finite Coxeter types but $I_2(12)$, $I_2(18)$ and $I_2(30)$. Under the additional assumption of gradability, classification of simple transitive 2-representations in these three exceptional cases was completed in [MT].

The setup of finite Coxeter systems, considered in [KMMZ], is motivated by the fact that, in this setup, the 2-category $\mathcal{S}$ (and hence also the 2-category $\mathcal{S}'$) can be defined over the coinvariant algebra of $W$ and is finitary in the sense of [MM1]. At the same time, for infinite Coxeter types, it might still happen that the 2-category $\mathcal{S}'$ is finitary despite the fact that the 2-category $\mathcal{S}$ no longer is.

In this note, we determine for which Coxeter systems the corresponding small quotient of the 2-category of Soergel bimodules is finitary, and, in that case, classify all simple transitive 2-representations of this small quotient. Our argument is an application of the main result of [MMMZ] which reduces our classification problem to the classification results of [KMMZ, MT]. Consequently, for some special cases, we need to work under the additional assumption of gradability. We also determine the quiver and relations for the underlying categories of our 2-representations. In the Coxeter type where the small quotient is not finitary, we do not have a classification for the simple transitive 2-representations but still give the quiver description for a distinguished subclass of it, namely the cell 2-representations. It turns out that, in all cases, the underlying category corresponds to the zig-zag algebra (cf. [HK, Du]) of a certain combinatorially described tree.

The paper is organized as follows: Section 2 studies Kazhdan-Lusztig cell combinatorics of the small Kazhdan-Lusztig cell of an arbitrary Coxeter system. Section 3 is devoted to...
classification of simple transitive 2-representations of finitary small quotients of Soergel bimodules. In Section 4 one finds a description of the quiver and relations for the underlying category of these 2-representations.

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2. Combinatorics of the small Kazhdan-Lusztig cell

2.1. Kazhdan-Lusztig cells. We consider the Hecke algebra $H = H_{W,S}$ of $W$ in the normalization of [So2]. It has the standard basis $\{H_w : w \in W\}$ and the Kazhdan-Lusztig (KL) basis $\{H_w : w \in W\}$, cf [KL].

We define the left preorder $\leq_L$ on $W$ as follows: for $x, y \in W$, we have $x \leq_L y$ provided that there is $w \in W$ such that $H_y$ appears with a non-zero coefficient when expanding $H_w H_x$ in the KL basis. Equivalence classes for this preorder are called left cells. The fact that $x, y \in W$ belong to the same left cell is written $x \sim_L y$. Similarly one defines the right preorder $\leq_R$ and the right cells $\sim_R$ using $H_x H_w$, and also the two-sided preorder $\leq_J$ and the right cells $\sim_J$ using $H_w H_w'$.

2.2. The small cell. The following statement is fundamental for our results. It can be easily obtained from [Lu1, Proposition 3.8], see also [Do, Lu2, Lu3, Lu4]. A detailed argument from [KMMZ, Lemma 3 and Proposition 4], which is formally written in the setup of finite Coxeter groups, works in the general case.

Proposition 1. Let $(W, S)$ be a finitely generated indecomposable Coxeter system.

(i) All simple reflections $s \in S$ belong to the same two-sided cell, called the small cell and denoted $J$.

(ii) The map $L \mapsto L \cap S$ is a bijection between the set of all left cells in $J$ and $S$.

(iii) The map $R \mapsto R \cap S$ is a bijection between the set of all right cells in $J$ and $S$.

(iv) An element $e \neq w \in W$ belongs to $J$ if and only if $w$ has a unique reduced expression.

(v) If $L$ is a left cell in $J$ with $s = L \cap S$ and $R$ is a right cell in $J$ with $t = R \cap S$, then $L \cap R$ consists of all those element $w \in W$ with unique reduced expression for which this unique expression is of the form $t \ldots s$.

For $s \in S$, we denote by $L_s$ and $R_s$ the left and the right cells containing $s$, respectively.

2.3. Indecomposable Coxeter systems with finite small cells. Let $(W, S)$ be a finitely generated indecomposable Coxeter system and $\Gamma = (\Gamma_0, \Gamma_1, m)$ is the associated Coxeter-Dynkin diagram. Here $\Gamma_0 = S$ and $m : \Gamma_1 \to \{3, 4, \ldots, \infty\}$ is the labeling function of the Coxeter presentation of $W$, that is, for any $\{s, t\} \in \Gamma_1$, we have $(st)^{m_{s,t}}$ in the Coxeter presentation of $W$. As usual, for $s, t \in S$, we have $\{s, t\} \not\subseteq \Gamma_1$ if and only if $st = ts$. We also omit the label 3 and refer to an edge with label 3 as unlabeled.
Indecomposability of \((W, S)\) is equivalent to the condition that \(\Gamma\) is connected. The following statement can be found in [Lu1 Proposition 3.8(h)] without proof, we include a proof for completeness.

**Proposition 2.** Let \((W, S)\) be a finitely generated indecomposable Coxeter system. Then the small cell \(\mathcal{J}\) is finite if and only if the following conditions are satisfied:

(a) \(\Gamma\) is a tree.

(b) \(\Gamma\) has at most one labeled vertex.

(c) All labels of \(\Gamma\) are finite.

**Proof.** If \(\Gamma\) contains a cycle, we can just take consecutive products of generators walking as long as we wish along this cycle and we never hit any of the relations in \(W\). Therefore all elements of \(W\) obtained in this way would belong to \(\mathcal{J}\) by Proposition 1(iv), making \(\mathcal{J}\) infinite. Therefore condition (a) is necessary for finiteness of \(\mathcal{J}\).

If \(\Gamma\) contains a subgraph \(s \sim^\infty t\), then all elements of the form \(stst\ldots tsts\) contain no relations in \(W\) and hence belong to \(\mathcal{J}\) by Proposition 1(iv). This again makes \(\mathcal{J}\) infinite and means that condition (c) is necessary for finiteness of \(\mathcal{J}\).

If \(\Gamma\) contains a connected subgraph of the form \(s_1 m s_2 s_3 s_4 \ldots s_{k-1} n s_k\) (with distinct vertices), where \(k \geq 3\) and \(m, n > 3\), then all elements of the form \((s_1 s_2 s_3 \ldots s_{k-1} s_k s_{k-1} \ldots s_3 s_2)^l\), where \(l \in \{1, 2, 3, \ldots\}\), contain no relations in \(W\) and hence belong to \(\mathcal{J}\) by Proposition 1(iv). This again makes \(\mathcal{J}\) infinite and means that condition (b) is necessary for finiteness of \(\mathcal{J}\).

Let us now show that conditions (a), (b) and (c) together are sufficient for finiteness of \(\mathcal{J}\). Let \(w \in \mathcal{J}\) with the unique reduced expression \(t_{k+1} t_{k-1} \ldots t_1\) (here a repetition of simple reflections is allowed). To prove our claim we just need to establish some bound on \(k\). As the reduced expression of \(w\) is unique, no pair of consecutive elements in this expression can commute. This means that all pairs \(\{t_2, t_1\}, \{t_3, t_2\}\), and so on, are edges in \(\Gamma\) and hence we can view \(w\) as a walk in \(\Gamma\) starting at \(t_1\) in the obvious way. As \(\Gamma\) contains no cycles by (a), for big \(k\) some edge \(\{a, b\}\) has to be walked along in the way \(aba\). To keep the reduced expression unique, the edge \(\{a, b\}\) then must be labeled.

If \(t_{k+1} t_{k-1} \ldots t_1\) contains a subexpression of the form \(abuaba\), where neither \(a\) nor \(b\) appears in \(u\), then all edges appearing in \(aua\) are unlabeled by (b). Again, as \(\Gamma\) contains no cycles by (b), if \(u \neq e\), one of the edges in \(u\) will have to be walked along in the way \(a' b' a'\), contradicting uniqueness of the reduced expression. Hence \(u = e\) and this shows that all appearances of \(a\) and \(b\) in \(w\) are next to each other.

The length of a subexpression of \(w\) of the form \(abab\ldots\) is bounded by the finite label of \(\{a, b\}\), given by (c). By the argument above, the lengths of the subwords on both sides of this subexpression are bounded by the total number of edges. This implies that \(k\) is bounded, and the claim follows. \(\Box\)

### 2.4. Combinatorics of finite small cells.

In this subsection we assume that \((W, S)\) is a finitely generated indecomposable Coxeter system such that the small cell \(\mathcal{J}\) is finite. Our aim here is to describe the intersections \(\mathcal{L}_s \cap \mathcal{R}_t\), for \(s, t \in S\).
Corollary 3. Let \((W, S)\) be a finitely generated indecomposable Coxeter system. If \(\Gamma\) is an unlabeled tree, then \(|L_s \cap R_t| = 1\), for all \(s, t \in S\).

Proof. We view \(w \in L_s \cap R_t\) as a walk in \(\Gamma\) similarly to the proof of Proposition 2. As all edges in \(\Gamma\) are unlabeled, the argument in the proof of Proposition 2 implies that no edge can be repeated during this walk. Therefore, by Proposition 1(v), the only possibility for \(w\) is to be the unique shortest path from \(s\) to \(t\) of the form \(t \ldots s\). The claim follows.

Let us now assume that \(\Gamma\) is a tree with a unique labeled edge \(\{s, t\}\). Let \(\Gamma^{(s)}\) denote the connected component of \(\Gamma \setminus \{t\}\) containing \(s\) (or, equivalently, the full subgraph of \(\Gamma\) consisting of all vertices \(r\) for which there is a walk from \(r\) to \(s\) which does not pass through \(t\)). Let, similarly, \(\Gamma^{(t)}\) denote the connected component of \(\Gamma \setminus \{s\}\) containing \(t\). Define the function \(\pi : S \to \{s, t\}\) by sending vertices in \(\Gamma^{(s)}\) to \(s\) and vertices in \(\Gamma^{(t)}\) to \(t\).

Proposition 4. Assume that \(\Gamma\) is a tree with a unique labeled edge \(\{s, t\}\). Then, for \(p, q \in S\), there is a bijection between \(L_{\pi(p)} \cap R_{\pi(q)}\) and \(L_p \cap R_q\) given by sending \(w \in L_{\pi(p)} \cap R_{\pi(q)}\) to \(u_1 w u_2\), where \(u_1\) is the unique shortest path from \(\pi(q)\) to \(q\) and \(u_2\) is the unique shortest path from \(p\) to \(\pi(p)\).

Proof. Using Proposition 1(v), it is easy to see that the map from the set \(L_{\pi(p)} \cap R_{\pi(q)}\) to the set \(L_p \cap R_q\) described in the formulation is well-defined. It is obviously injective. Furthermore, it is surjective due to the argument in the proof of Proposition 2. The claim follows.

In the setup of Proposition 4, let \((W^{\{s,t\}}, \{s, t\})\) be the parabolic Coxeter subsystem of \((W, S)\) corresponding to \(\{s, t\} \subset S\). We will add the superscript \(\{s, t\}\) to objects associated with this Coxeter group, for example, \(L_{\{s,t\}}\) means the left cell of \(W^{\{s,t\}}\) containing \(s\).

Corollary 5. Assume that \(\Gamma\) is a tree with a unique labeled edge \(\{s, t\}\). Then, for \(p, q \in \{s, t\}\), we have 
\[
L_p \cap R_q = L_{\{s,t\}}^{\{s,t\}} \cap R_{\{s,t\}}^{\{s,t\}}.
\]

Proof. That every element in \(L_{\{s,t\}}^{\{s,t\}} \cap R_{\{s,t\}}^{\{s,t\}}\) belongs to \(L_p \cap R_q\) is clear from the definitions. That every element in \(L_p \cap R_q\) belongs to \(L_{\{s,t\}}^{\{s,t\}} \cap R_{\{s,t\}}^{\{s,t\}}\) follows from the argument in the proof of Proposition 2.

2.5. Examples. If \(\Gamma\) is given by

```
|   | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 |  1| 142| 143| 14 | 145|
| 2 |  2| 241| 243| 24 | 245|
| 3 | 341| 342| 3  | 34 | 345|
| 4 | 41 | 42 | 43 | 4  | 45 |
| 5 | 541| 542| 543| 54 | 5  |
```

then the cell \(J\) has the following structure (here columns are left cells and rows are right cells indexed by the corresponding simple reflections):
If \( \Gamma \) is given by
\[
\begin{array}{cccc}
1 & 2 & 4 & 3 \\
\hline
4 & 3 & 4 & 5
\end{array}
\]
then the cell \( \mathcal{J} \) has the following structure (here columns are left cells and rows are right cells indexed by the corresponding simple reflections):
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1,12321 & 12,1232 & 123 & 1234 & 1235 \\
2 & 21,21232 & 2,232 & 23 & 234 & 235 \\
3 & 321 & 32 & 3,323 & 34,3234 & 35,3235 \\
4 & 4321 & 432 & 43,4323 & 43,43234 & 435,43235 \\
5 & 5321 & 532 & 53,5323 & 534,53234 & 535,53235 \\
\hline
\end{array}
\]

3. SMALL QUOTIENTS OF SOERGEL BIMODULES AND THEIR 2-REPRESENTATIONS

3.1. Small quotients of Soergel bimodules. Below, by graded we always mean \( \mathbb{Z} \)-graded. We also work over \( \mathbb{C} \).

Let \( \mathcal{R} \) denote the monoidal category of (graded) Soergel bimodules over the polynomial algebra \( \mathbb{C}[h] \) associated to \( (W, S, h) \), see [So2, EW]. By [So2, EW], the split Grothendieck ring of \( \mathcal{R} \) is isomorphic to \( \mathbb{H} \) and, for \( w \in W \), we denote by \( B_w \) the unique (up to isomorphism) indecomposable Soergel bimodule which corresponds to the element \( \mathbb{H}_w \) under this isomorphism. Our normalization of grading shifts is chosen such that the full subcategory \( \mathcal{P} \) of \( \mathcal{R} \) with objects \( \{ B_w : w \in W \} \) is positively graded. The graded category \( \mathcal{R} \) has finite dimensional graded components, that is, the morphism space between any two indecomposable 1-morphisms is finite dimensional.

Lemma 6. The monoidal category \( \mathcal{R} \) has a unique tensor ideal \( \mathcal{I} \) which is maximal in the set of all graded tensor ideals of \( \mathcal{R} \) having the property that they

\[ \text{do not contain any} \quad \text{id}_{B_w}, \quad \text{where} \quad w \in \mathcal{J}. \]

Proof. Due to positivity of the grading on \( \mathcal{P} \), the sum of any family of ideals having property (I) also has property (I). Therefore \( \mathcal{I} \) is just the sum of all tensor ideals having property (I). \( \square \)

The quotient monoidal category \( \mathcal{R}^\mathcal{I} := \mathcal{R} / \mathcal{I} \) is called the small quotient of \( \mathcal{R} \), cf. [KMMZ]. We denote by \( \mathcal{J} \) and \( \mathcal{J}^\mathcal{I} \) the ungraded versions of \( \mathcal{R}^\mathcal{I} \) and \( \mathcal{R}^\mathcal{I} \), respectively.

By construction, the images of \( B_w \), where \( w \in \mathcal{J} \cup \{ e \} \), form a complete and redundant list of representatives of the isomorphism classes of indecomposable objects in \( \mathcal{J}^\mathcal{I} \).

3.2. 2-cATEGORIES AND 2-REPRESENTATIONS. Although it is natural to define \( \mathcal{R}^\mathcal{I} \) as a tensor category, we would like to adapt to the setup of 2-representations of 2-categories considered in [MM1, MM3]. For this we use the coherence theorem for monoidal categories and consider a strictification of \( \mathcal{R}^\mathcal{I} \) which we will denote by the same symbol, abusing notation. We also identify a strict monoidal category with the corresponding 2-category with one object.

Recall, from [MM1], that a finitary 2-category (over \( \mathbb{C} \)) is a 2-category \( \mathcal{C} \) such that
• \( \mathcal{C} \) has finitely many objects;
• each \( \mathcal{C}(i, j) \) is equivalent to the category of projective modules over some finite dimensional \( \mathbb{C} \)-algebra;
• all compositions are biadditive and \( \mathbb{C} \)-bilinear whenever this makes sense and all identity 1-morphisms are indecomposable.

The 2-category \( \mathcal{S} \) is never finitary as homomorphism spaces between Soergel bimodules are infinite dimensional vector spaces. However, the small quotient \( \mathcal{S} \) is finitary in some cases, as we will see in Subsection 3.4 below.

A 2-representation of a 2-category \( \mathcal{C} \) is a 2-functor from \( \mathcal{C} \) to an appropriate 2-category, see [MM3]. This can also be viewed as a functorial action of \( \mathcal{C} \) on suitable categories. All 2-representation of \( \mathcal{C} \) form a 2-category in a natural way, see [MM3] for details. For every object \( i \in \mathcal{C} \), we have the corresponding principal 2-representation \( \mathcal{C}(i, -) \).

3.3. A version of graded “abelianization”. There is a natural diagrammatic abelianization 2-functor for finitary 2-categories, see [MM2, Subsection 4.2]. Here we will need a slight modification of this functor due to the fact that \( \mathcal{S}^{gr} \) is not finitary. We only need to abelianize the principal 2-representation of \( \mathcal{S}^{gr} \), so we will try to present the object we need with a minimum amount of technicalities. For \( i \in \mathbb{Z} \), we denote by \( \langle i \rangle \) the corresponding functor of grading shift (which maps elements of degree \( n \) to elements of degree \( n - i \), for all \( n \in \mathbb{N} \)). Here we view \( \mathcal{S}^{gr} \) as a monoidal category rather than a 2-category and define its abelianization as a 1-category.

We denote by \( \mathcal{S}^{gr}_{ab} \) the category whose endomorphisms are diagrams of the form

\[
\left( \prod_{i \leq i_0} F_i(i) \right) \rightarrow G, \text{ where } i_0 \in \mathbb{Z},
\]

in \( \mathcal{S}^{gr} \), and whose morphisms are given by the commutative solid squares in \( \mathcal{S}^{gr} \)

\[
\begin{array}{ccc}
( \prod_{i \leq i_0} F_i(i) ) & \rightarrow & G \\
\downarrow \alpha & & \downarrow \beta \\
( \prod_{j \leq j_0} F'_j(j) ) & \rightarrow & G'
\end{array}
\]

modulo those in which \( \alpha \) factors through some \( \beta \). It is important to note that, due to our positive grading and the bound on the grading shift, for each \( i \), the 1-morphism \( F_i(i) \) (as well as the 1-morphism \( G \)) has a non-zero homomorphism only to finitely many 1-morphisms \( F'_j(j) \).

By construction, the category \( \mathcal{S}^{gr}_{ab} \) is equivalent to the category of finitely generated graded \( \mathcal{P}^{op} \)-modules. In particular, \( \mathcal{S}^{gr}_{ab} \) is abelian whenever \( \mathcal{P}^{op} \) is noetherian. We do not know when \( \mathcal{P}^{op} \) is noetherian, but we do not need \( \mathcal{S}^{gr}_{ab} \) to be abelian in what follows.

The left regular action of \( \mathcal{S}^{gr} \) on itself extends, in the obvious way, to an action of \( \mathcal{S}^{gr}_{ab} \) on \( \mathcal{S}^{gr}_{ab} \).
3.4. Finitary small quotients of Soergel bimodules.

**Proposition 7.** Assume that \((W,S)\) is indecomposable. Then the 2-category \(\mathcal{Z}\) is finitary if and only if \(\mathcal{J}\) is finite.

**Proof.** If \(\mathcal{Z}\) is finitary, it must contain only finitely many isomorphism classes of indecomposable objects. By construction, the latter are indexed by \(\mathcal{J} \cup \{e\}\). Therefore the condition \(|\mathcal{J}| < \infty\) is necessary.

To prove that the condition \(|\mathcal{J}| < \infty\) is sufficient, we assume that this condition is satisfied and we need to show that all homomorphism spaces in \(\mathcal{Z}\) are finite dimensional.

Let \(\mathcal{F}^{gr}\) be the diagrammatic abelianization of \(\mathcal{F}^{gr}\) as in Subsection 3.3. For \(w \in W\), we denote by \(L_w\) the simple top of the projective object \(B_w\) in \(\mathcal{F}^{gr}\). Fix some \(s \in S\) and consider the full subcategory \(A\) of \(\mathcal{F}^{gr}\) given by the additive closure of the objects of the form \(B_wL_s\), where \(w \in L_s\) (up to graded shift). Similarly to [MM1, Lemma 12], one shows that, for \(w \in W\), the inequality \(B_wL_s \neq 0\) implies \(w \in L_s \cup \{e\}\). Consequently, the action of \(\mathcal{F}^{gr}\) on \(\mathcal{F}^{gr}\) restricts to \(A\).

Our next observation is that each \(B_xL_s\), where \(x \in L_s\), is finite dimensional in the sense that the direct sum

\[
\bigoplus_{y \in W, i \in \mathbb{Z}} \text{Hom}_{\mathcal{F}^{gr}}(B_y\langle i \rangle, B_xL_s)
\]

is finite dimensional. Indeed, for \(y \in W\) and \(i \in \mathbb{Z}\), using adjunction we have

\[
\text{Hom}_{\mathcal{F}^{gr}}(B_y\langle i \rangle, B_xL_s) \cong \text{Hom}_{\mathcal{F}^{gr}}(B_{x-i}B_y\langle i \rangle, L_s).
\]

if the right hand side is non-zero, then \(y \in L_s \cup \{e\}\) and \(|i|\) must be bounded by the length of \(x\). This leaves us with finitely many choices for both \(x\) and \(i\). Furthermore, all graded components of homomorphism spaces in \(\mathcal{F}^{gr}\) are finite dimensional.

As each \(B_xL_s\), where \(x \in L_s\), is finite dimensional, it has finitely many indecomposable summands. Therefore, up to grading shift, \(A\) has finitely many indecomposable objects. In other words, the action of \(\mathcal{F}^{gr}\) on \(A\) is an action on some category which is equivalent to the category of graded projective modules over a finite dimensional graded algebra.

Let \(\mathcal{J}\) be the kernel (in \(\mathcal{F}^{gr}\)) of this action. Note also that this action is given by exact functors as all \(1\)-morphisms in \(\mathcal{F}^{gr}\) have biadjoints. This implies that all morphism spaces between \(1\)-morphisms in the ungraded version of \(\mathcal{F}^{gr}/\mathcal{J}\) are finite dimensional.

Note that the ideal \(\mathcal{J}\) is graded by construction and that the identity \(2\)-morphism on \(B_s\) does not belong to \(\mathcal{J}\) as \(B_s(B_sL_s) \neq 0\). Hence \(\mathcal{J} \subset \mathcal{J}\) by the maximality of \(\mathcal{J}\). Consequently, all morphism spaces between \(1\)-morphisms in \(\mathcal{Z}\) are finite dimensional. This completes the proof. \(\square\)

3.5. Simple transitive 2-representations of finitary small quotients of Soergel bimodules. Following [MM5], we are interested in classification of simple transitive 2-representations of \(\mathcal{Z}\) in case the latter 2-category is finitary. Recall, from [MM5], that a simple transitive 2-representation of \(\mathcal{Z}\) is a functorial action of \(\mathcal{Z}\) on a small category \(\mathcal{C}\) equivalent to \(B\)-proj, for some finite dimensional algebra \(B\), such that \(\mathcal{C}\) has no proper \(\mathcal{Z}\)-invariant ideals.

Examples of simple transitive 2-representations of \(\mathcal{Z}\) include the so-called cell 2-representation \(C_L\) associated to a left cell \(L\), cf. [MM1, MM2].

**Proposition 8.** Assume that \(\Gamma\) is an unlabeled tree. Then every simple transitive 2-representations of \(\mathcal{Z}\) is equivalent to a cell 2-representation.
Proof. Thanks to Corollary 3 [MM5, Proposition 1] and [KM, Corollary 19], in case \( \Gamma \) is an unlabeled tree, the 2-category \( \mathcal{S} \) satisfies the assumptions of [MM5, Theorem 18] and hence the assertion of the proposition follows from [MM5, Theorem 18]. \( \square \)

The case when \( \Gamma \) is a tree with one labeled edge requires some preparation. Assume that the labeled edge of gamma is

\[
(2) \quad s \rightarrow t ,
\]

where \( 3 < n < \infty \). Let \( \tilde{\mathcal{S}} = \{s, t\} \) and \( \tilde{W} \) be the parabolic Coxeter subgroup of \( W \) generated by \( \tilde{S} \). Let \( \mathfrak{h} \) be the 2-dimensional subspace of \( \mathfrak{h} \) generated by the unique (up to scalar) eigenvector of \( s \) with eigenvalue \(-1\) and the unique (up to scalar) eigenvector of \( t \) with eigenvalue \(-1\). Then we have the corresponding 2-categories \( \tilde{S}^{\mathfrak{h}} \) and \( \mathcal{S} \).

**Theorem 9.** Assume that \( \Gamma \) is a tree with one labeled edge of the form \((2)\). Then there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories \( \mathcal{S} \) and \( \tilde{S}^{\mathfrak{h}} \).

Note that, for \( n \neq 12, 18, 30 \), simple transitive 2-representations of \( \tilde{S}^{\mathfrak{h}} \) are classified in [KMMZ, Theorem 1]. For \( n = 12, 18, 30 \), simple transitive 2-representations of \( \tilde{S}^{\mathfrak{h}} \) are classified in [MT, Theorem II] (with some classes of 2-representations constructed in [KMMZ, Theorem 1]).

Proof. Our proof of this theorem is based crucially on an application of [MMMZ, Theorem 15]. Note that \( \mathcal{S} \) is finitary by Propositions 2 and 7. Define \( C \) as the quotient of the 2-subcategory of \( \tilde{\mathcal{S}} \) generated by \( B_w \), where \( w \in L_s \cap R_s \), modulo the unique maximal two-sided 2-ideal which does not contain any non-zero identity 2-morphisms. By [MMMZ, Theorem 15], there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories \( \mathcal{S} \) and \( C \).

Define \( \tilde{C} \) as the quotient of the 2-subcategory of \( \tilde{\mathcal{S}} \) generated by \( B_w \), where \( w \in L_s \cap R_s \), modulo the unique maximal two-sided 2-ideal which does not contain any non-zero identity 2-morphisms. By [MMMZ, Theorem 15], there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories \( \mathcal{S} \) and \( \tilde{C} \).

At the same time, from Corollary 5 and the definition of Soergel bimodules it follows easily that the 2-categories \( C \) and \( \tilde{C} \) are biequivalent. Therefore there is a bijection between the sets of equivalence classes of simple transitive 2-representations of the 2-categories \( C \) and \( \tilde{C} \). The claim follows. \( \square \)

The best way to explicitly define a bijection given by Theorem 9 is to use the approach to simple transitive 2-representations via (co)algebra objects as developed in [MMM1, MMMZ].

### 4. Quiver and relations for the underlying category

#### 4.1. Zig-zag categories associated to graphs

Let \( \Omega \) be an unoriented graph without loops. Let \( \tilde{\Omega} \) be the oriented graph obtained from \( \Omega \) by replacing each unoriented edge of \( \Omega \) by a pair of oppositely oriented edges between the same vertices. Here is an example:

\[
\Omega = 1 \longrightarrow 2 \longrightarrow 3, \quad \tilde{\Omega} = 1 \overrightarrow{\longrightarrow} 2 \overleftarrow{\longrightarrow} 3 .
\]

We denote by \( A^{\tilde{\Omega}} \) the quotient of the path category of \( \tilde{\Omega} \) by the following relations:
any path of length three is zero;
any path of length two between different vertices is zero;
all paths of length two that start and end at the same vertex are equal.

The category $\mathcal{A}^\Omega$ corresponds to the classical zig-zag algebra associated to $\bar{\Omega}$, cf. [HK, Du, ET]. The category $\mathcal{A}^\Omega$ is graded by path lengths. Note that the path algebra of $\mathcal{A}^\Omega$ is not unital if $\Omega$ has infinitely many vertices.

4.2. Cell 2-representations. In this subsection we assume that $(W, S)$ is of any type. For a fixed $s \in S$, consider an unoriented graph $\Lambda(s)$ whose set of vertices is $\Lambda_0(s) := L_s$ and whose set of unoriented arrows is

\[ \Lambda_1(s) := \{ \{u, v\} \in L_s \times L_s : u = tv > v, \text{ for some } t \in S \}. \]

Note that the graph $\Lambda(s)$ is an unlabeled tree which might be infinite. We denote by $\mathcal{A}(s)$ the category $\mathcal{A}^{\Lambda(s)}$. For example, if $\Gamma$ is the graph $b \rarrow a$, then the associated graph $\Lambda(s)$ is as follows:

```
  b \rarrow s \rarrow t \rarrow c,
```

then the associated graph $\Lambda(s)$ is as follows:

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  b \rarrow s \rarrow t \rarrow c
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Proposition 10. The category $\mathcal{A}(s)$ is isomorphic to the underlying category of the cell 2-representation $C_{L_s}$ of $\mathcal{C}$ (and hence also of $\mathcal{A}$).

Proof. Let $B(s)$ denote the underlying category of $C_{L_s}$. For $w \in L_s$, we denote by $P_w$ the indecomposable $B(s)$-projective corresponding to $w$. We also denote by $L_w$ the simple top of $P_w$.

Recall (see e.g. [Ir, Corollary 5.2.4]) that, for $t \in S$ and $w \in W$, we have

\[
H_t H_w = \begin{cases} 
  wH_w + w^{-1}H_w, & tw < w; \\
  H_t w + \sum_{x < w, tx < x} \mu(x, w) H_x, & tw > w;
\end{cases}
\]

where $\mu(x, w)$ is the Kazhdan-Lusztig $\mu$-function, see [KL Subsection 2.2]. By construction of $C_{L_s}$, this implies that, for $t \in S$ and $x, w \in L_s$, the multiplicity $m_{x, w}^{(t)}$ of $P_x$ as a direct summand of $B_t \cdot P_w$ is given by

\[ m_{x, w}^{(t)} = \begin{cases} 
  2, & x = w \text{ and } tw < w; \\
  1, & x = tw > w; \\
  \mu(x, w), & tw > w, x < w, tx < x; \\
  0, & \text{otherwise}.
\end{cases} \]

In the graded picture, we additionally have that, for $x = w$ and $tw < w$, the two summands $P_w$ appearing in $B_t \cdot P_w$ are, in fact, $P_w(-1)$ and $P_w(1)$ and all other appearing summands have no grading shift. By adjunction, see e.g. [AM Lemma 8], $m_{x, w}^{(t)}$ coincides with the the composition multiplicity of $L_w$ in $B_t \cdot L_x$. 
Now, if \( x \in L_x \), then there is a unique \( t \in S \) such that \( tx < x \). The computation above implies that \( B_t \cdot L_x \) has two simple subquotients isomorphic to \( L_x(-1) \) and \( L_x(1) \) while all other summands \( \text{Hom}(B_t \cdot L_x, L_w) = 0 \), thus \( L_x(1) \) is the simple top of \( B_t \cdot L_x \). Similarly, \( L_x(-1) \) is the simple socle of \( B_t \cdot L_x \).

As we have just established that \( B_t \cdot L_x \) has simple top, we obtain \( B_t \cdot L_x \cong P_1(1) \).

As \( P \) is positively graded, the degree zero part of \( B_t \cdot L_x \) is semi-simple. Hence the number of arrows from \( x \) to \( w \) in \( B^{(s)} \) coincides with the number of arrows from \( w \) to \( x \) in \( B^{(s)} \) and this also coincides with the multiplicity of \( L_w \) in \( B_t \cdot L_x \). If \( w = rx \), for some \( r \in S \), then the multiplicity of \( L_w \) in \( B_t \cdot L_x \) is equal to 1 by formula (4).

If the multiplicity of \( L_w \) in \( B_t \cdot L_x \) is non-zero and \( w \neq rx \), for any \( r \in S \), then \( x < w \) by formula (4). In that case, the multiplicity of \( L_x \) in \( B_t' \cdot L_w \), where \( t' \in S \) is the unique element such that \( B_{t'} \cdot L_w \neq 0 \), is also non-zero. Then \( w < x \) by formula (4) implying \( w = x \), a contradiction.

To sum up, we established that the only \( L_w \) which appear in \( B_t \cdot L_x \) are those for which \( w = rx \), for some \( r \in S \), and these simples appear with multiplicity 1. This implies that the Cartan matrices of \( B^{(s)} \) and \( A^{(s)} \) coincide and now it is easy to construct inductively an explicit isomorphism between \( B^{(s)} \) and \( A^{(s)} \) by rescaling, if necessary, all arrows, starting the induction from the initial vertex \( s \). The claim follows.

\[ \square \]

4.3. Rooted graphs and their pointed union. Here we introduce the notion of the one point union of rooted graphs, which we use in Subsection 4.4. By a rooted graph, we mean a pair \((\Xi, a)\) of a graph \(\Xi\) and a root \(a \in \Xi_0\). Let \((\Xi, a)\) and \((\Xi', a')\) be two rooted graphs. The one point union \((\Xi, a) \vee (\Xi', a')\) is the graph obtained from the disjoint union \((\Xi, a) \bigsqcup (\Xi', a')\) of \((\Xi, a)\) and \((\Xi', a')\) by identifying the roots \(a\) and \(a'\). The graph \((\Xi, a) \vee (\Xi', a')\) is naturally rooted with the root being the identified vertex \(a = a'\). Here is an example, where \(\bullet\) denotes a root and \(\circ\) an ordinary vertex:

\[ \text{Example} \]

4.4. Simple transitive 2-representations for finitary small quotients. We assume that \((W, S)\) is indecomposable and that \(\mathcal{A}\) is finitary. The purpose of the subsection is to describe the underlying category of each gradable simple transitive 2-representation that is not covered in Subsection 4.2. They still corresponds to the zig-zag algebras of certain graphs. We explicitly determine this graph in terms of the Coxeter-Dynkin diagram \(\Gamma\) of \((W, S)\).

If \(\Gamma\) is an unlabeled tree, then any simple transitive 2-representation of \(\mathcal{A}\) is a cell 2-representation by Proposition [3] and its underlying category is described by Proposition [10]. Because of this, in what follows we assume that \(\Gamma\) has a full subgraph of the form \([2]\). Then we have the 2-category \(\mathcal{A}\) as in Subsection 3.5.

Let \(M\) be a gradable simple transitive 2-representation of \(\mathcal{A}\) and \(N\) the corresponding simple transitive 2-representation of \(\mathcal{A}\) given by Theorem [9]. From the proofs of
Theorem 9 and [MMMZ, Theorem 15] it follows that the underlying category of $N$ is isomorphic to a full subcategory of the underlying category of $M$. As $N$ is gradable, the underlying category of $N$ is explicitly described in [MMMZ, MT], and is known to be of the form $A^\Omega$, where $\Omega$ is a simply laced Dynkin diagram. There is also an additional datum on $\Omega$, given by considering $\Omega$ as a bipartite graph $\Omega_0 = \Omega_0^{(s)} \coprod \Omega_0^{(t)}$ where, for $r \in \{s, t\}$, we have $u \in \Omega_0^{(r)}$ if and only if $B_r \cdot L_u \neq 0$.

By deleting the labeled edge $\{s, t\}$, the connected graph $\Gamma$ splits into a disjoint union of two connected graphs: the graph $\Gamma^{(s)}$ which is the connected component containing $s$, and the graph $\Gamma^{(t)}$ which is the connected component containing $t$. For $r \in \{s, t\}$, we denote by $S^{(r)}$ the set of vertices in $\Gamma^{(r)}$, by $W^{(r)}$ the parabolic subgroup of $W$ generated by $S^{(r)}$, and by $\mathcal{Z}^{(r)}$ the small quotient 2-category of Soergel bimodules associated with $(W^{(r)}, S^{(r)})$. Let $\mathcal{L}^{(r)}$ be the left cell in $W^{(r)}$ containing $r$.

Note that $\Gamma^{(r)}$ is an unlabeled tree and hence any simple transitive 2-representation of $\mathcal{Z}^{(r)}$ is a cell 2-representation of $\mathcal{Z}^{(r)}$ isomorphic to a full subcategory of the underlying category of $\mathcal{Z}^{(r)}$. Let $\mathcal{L}^{(r)}$ be the left cell in $W^{(r)}$ containing $r$.

Consider $\Lambda^{(r)}$ as a rooted graph with root $r$. Letting $\Omega_0^{(s)} = \{u_1, u_2, \ldots, u_p\}$ and $\Omega_0^{(t)} = \{w_1, w_2, \ldots, w_q\}$, we have $u \in \Omega_0^{(r)}$ if and only if $B_r \cdot L_u \neq 0$.

Informally, the graph $\Theta$ is obtained from $\Omega$ by attaching at each vertex in $\Omega_0^{(s)}$ a copy of $\Lambda^{(s)}$ and attaching at each vertex in $\Omega_0^{(t)}$ a copy of $\Lambda^{(t)}$ at $t$.

As an example, below we rearrange the graph (3) such that the solid part depicts $\Omega$, the dashed parts show the attached copies of $\Lambda^{(s)}$ and the dotted part shows the attached copy of $\Lambda^{(t)}$.

---

**Theorem 11.** The category $A^\Omega$ is isomorphic to the underlying category of the 2-representation $M$ of $\mathcal{Z}$.
Proof. Let $\mathcal{B}$ be the the underlying category of the 2-representation $\mathcal{M}$ of $\mathcal{L}$. Since $\mathcal{B}$ is graded, the same argument as the one used in the proof of Proposition 10 implies that $\mathcal{B}$ is isomorphic to $\mathcal{A}^{\Theta'}$, for some graph $\Theta'$. So, we just need to check that this $\Theta'$ is isomorphic to $\Theta$ constructed above.

The subgraph of $\Theta'$ which is isomorphic to $\Omega$ is uniquely determined by the fact that $\mathcal{N}$ is isomorphic to a full subcategory of the underlying category of $\mathcal{M}$. See above.

Take now some vertex $u \in \Omega_0^{(s)}$, viewed as a vertex of $\Theta'$ and consider the additive closure $\mathcal{C}$ of all $B_w \cdot L_u$, where $w \in \mathcal{L}_w^{(s)}$. The action of $\mathcal{L}_w^{(s)}$ preserves $\mathcal{C}$ by construction and it is easy to see that the corresponding 2-representation of $\mathcal{L}_w^{(s)}$ is the cell 2-representation corresponding to $\mathcal{L}_x^{(s)}$. Note that the underlying category of this 2-representation is isomorphic to $\mathcal{A}^{\Lambda^{(s)}}$. This argument works for any $u \in \Omega_0^{(s)}$ and a similar argument (with $\mathcal{A}^{\Lambda^{(s)}}$ replaced by $\mathcal{A}^{\Lambda^{(s)}}$) works for any $u \in \Omega_0^{(t)}$. Consequently, there is a natural embedding of $\Theta$ into $\Theta'$, and we are left to show that $\Theta'$ has no extra edges.

Let $u_1, u_2 \in \Omega_0^{(s)}$ be two different vertices and $C_1$ and $C_2$ the corresponding 2-representations of $\mathcal{L}_w^{(s)}$ constructed in the previous paragraph. Existence of an edge in $\Theta'$ connecting a vertex in $C_1$ and a vertex in $C_2$ implies existence of a non-trivial discrete self-extension for the cell 2-representation of $\mathcal{L}_x^{(s)}$ in the sense of [CM] Subsection 5.2. This is, however, prohibited by [CM] Theorem 25.

Let $u \in \Omega_0^{(s)}$ and $v \in \Omega_0^{(t)}$, and let $p$ be a vertex in the copy of $\Gamma^{(s)}$ in $\Theta$ attached to $u$ and $q$ be a vertex in the copy of $\Gamma^{(t)}$ attached to $v$. Suppose there is an edge in $\Theta'$ between $p$ and $q$. Letting $p'$ be the element in $\mathcal{L}_x$ corresponding to $p$ and $q'$ be the elements in $\mathcal{C}_x$ corresponding to $q$, we have

$$\text{Hom}(B_{\langle q' \rangle}^{-1} B_{\langle p' \rangle} L_u, L_v) \cong \text{Hom}(B_{\langle p' \rangle} L_u, B_{\langle q' \rangle} L_v) \supset \text{Hom}(P_p, P_q) \neq 0,$$

since $B_{\langle q' \rangle} L_u$ contains $P_p$, the projective at $p$ in $\mathcal{B}$ and $B_{\langle p' \rangle} L_v$ contains $P_q$. In particular, $B_{\langle q' \rangle}^{-1} B_{\langle p' \rangle} L_u \neq 0$ and $\langle q' \rangle^{-1} p' \in \mathcal{J}$. Writing $\langle q' \rangle^{-1} p' = txys$, where $x$ is in the parabolic subgroup generated by $\mathcal{S}^{(t)} \setminus t$ and $y$ is in the parabolic subgroup generated by $\mathcal{S}^{(s)} \setminus s$, we see that $\langle q' \rangle^{-1} p' \in \mathcal{J}$ implies $x = y = e$. Therefore, the edge between $p$ and $q$ comes from an edge in $\Omega$. This completes the proof. \qed

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