On the exterior Dirichlet problem for special Lagrangian equations*

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Abstract

In this paper, we establish the existence and uniqueness theorem of the exterior Dirichlet problem for special Lagrangian equations with prescribed asymptotic behavior at infinity.

Keywords: Dirichlet problem, existence and uniqueness, exterior domain, special Lagrangian equations, Perron’s method, prescribed asymptotic behavior

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1 Introduction

Let $D$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 3$). We consider in this paper the Dirichlet problem for the special Lagrangian equation

$$ \sum_{i=1}^{n} \arctan \lambda_i(D^2u) = \Theta $$

(1.1)

in the exterior domain $\mathbb{R}^n \setminus \overline{D}$, where $\lambda_i(D^2u)$’s denote the eigenvalues of the Hessian matrix $D^2u$, and $\Theta$ is a constant such that $(n - 2)\pi / 2 \leq |\Theta| < n\pi / 2$.

The special Lagrangian equation (1.1) originates in the calibrated geometry [HL82], and plays an important role in the study of string theory (see...
for example [CMMS04]). The left hand side of the equation (1.1) indeed stands for the argument of the complex number $(1 + \sqrt{-1}\lambda_1(D^2u))...(1 + \sqrt{-1}\lambda_n(D^2u))$, which is usually called phase or Lagrangian phase. When the phase is constant, the gradient graph $\{(x, Du(x))\} \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$ then is called special Lagrangian. One can prove that $\{(x, Du(x))\}$ is special Lagrangian, i.e., (1.1) holds for some constant $\Theta \in (-n\pi/2, n\pi/2)$, if and only if it is a volume minimizing minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$. In the literature, the Lagrangian phase $(n-2)\pi/2$ is usually called critical, since the level set $L_\Theta := \left\{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^{n} \arctan \lambda_i = \Theta \right\}$ is convex only if $|\Theta| \geq (n-2)\pi/2$ [Yuan06, Lemma 2.1]. So the main results in this paper are concerning the special Lagrangian equation (1.1) with critical and supercritical phases.

Recently, we proved in [LLY17] that any smooth solution $u$ of (1.1) with supercritical phase in the exterior domain must tend to a quadratic polynomial $Q$ at infinity, and satisfy

$$u(x) = Q(x) + O \left( \frac{1}{|x|^{n-2}} \right) \quad (|x| \to +\infty). \quad (1.2)$$

That is, there exist $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that

$$\limsup_{|x| \to +\infty} |x|^{n-2} \left| u(x) - \left( \frac{1}{2} x^T Ax + b^T x + c \right) \right| < \infty. \quad (1.3)$$

Thus for any given solution of (1.1) in the exterior domain, we know definitely that it obeys the above asymptotic behavior (1.3) at infinity. So the converse problem arises, that is, with any such prescribed asymptotic behavior at infinity, whether there exists a unique solution of the Dirichlet problem of the special Lagrangian equation (1.1) in the exterior domain? The answer is yes (at least partially), and we will mainly focus on this exterior Dirichlet problem in this paper.

To deal with the exterior Dirichlet problem, as a lot of the previous researches suggested (see [CL03, DB11, BLL14, LB14]), Perron’s method is often adopted, and the key point of which is to construct some appropriate subsolutions of the equation. For this purpose, we turn to study the following
algebraic form of the special Lagrangian equation (1.1):

$$\cos \Theta \sum_{0 \leq 2k+1 \leq n} (-1)^k \sigma_{2k+1}(\lambda(D^2u)) - \sin \Theta \sum_{0 \leq 2k \leq n} (-1)^k \sigma_{2k}(\lambda(D^2u)) = 0,$$

where \(\sigma_k(\lambda)\)'s are the elementary symmetric polynomials with respect to \(\lambda\), defined by

\[
\sigma_0(\lambda) \equiv 1 \quad \text{and} \quad \sigma_k(\lambda) := \sum_{1 \leq s_1 < s_2 < \ldots < s_k \leq n} \lambda_{s_1} \lambda_{s_2} \ldots \lambda_{s_k} \quad (\forall 1 \leq k \leq n).
\]

Note that the solution of (1.1) is always a solution of (1.4), but the converse is not always true. Our strategy is to construct some proper subsolutions of (1.4), and then come back to show that they are exactly the desired subsolutions of (1.1). The techniques used here to construct subsolutions of (1.4) are partially inherited from our previous paper LL16 concerning the exterior Dirichlet problem for the Hessian quotient equations, but are much harder than those and have been largely extended.

We would like to remark that, the rigidity theorems for special Lagrangian equations on the whole space have been fully studied by Prof. Y. Yuan, see for instance Yuan02, Yuan06, WY08; for more on the special Lagrangian equations, we refer the readers to HL82, Fu98, CWY09, WY14 and the references therein.

For the results concerning the exterior Dirichlet problems of the Monge-Ampère equations, of the Hessian equations, and of the Hessian quotient equations, see for example CL03, DB11, LD12, BLL14, LB14, LL16 and the references therein.

Define

\[
A_0^\Theta := \left\{ A \in S(n) \left| \lambda(A) \in \Gamma^+ \cup (-\Gamma^+) , \sum_{i=1}^n \arctan \lambda_i(A) = \Theta \right. \right\}, \quad (1.5)
\]

and

\[
A_\Theta := \left\{ A \in A_0^\Theta \left| m(\Theta, \lambda(A)) > 2 \right. \right\}, \quad (1.6)
\]

where

(1) \(S(n)\) denotes the linear space of symmetric \(n \times n\) real matrices;

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(2) $\Gamma^+$ denotes the positive cone

$$\Gamma^+ := \{ \lambda \in \mathbb{R}^n | \lambda_i > 0, \forall i = 1, 2, ..., n \};$$

(3) $m(\Theta, \lambda)$ is a quantity introduced by us, which plays an important role in the study of the special Lagrangian equations. For some special matrices $A \in \mathcal{A}_\Theta$, for example, $A = \tan(\Theta/n) I_n$, we have $m(\Theta, \lambda(A)) = n$. The specific definition of $m(\Theta, \lambda)$ will be given in (3.32) in Subsection 3.2 when all the necessary preparation is done.

Then the main results of this paper can be stated as the following theorems.

**Theorem 1.1.** Let $D$ be a bounded strictly convex domain in $\mathbb{R}^n$, $n \geq 3$, $\partial D \in C^2$ and let $\varphi \in C^2(\partial D)$. Then for any given $A \in \mathcal{A}_\Theta$ with $(n-2)\pi/2 \leq \Theta < n\pi/2$, and any given $b \in \mathbb{R}^n$, there exists a constant $c_*$ depending only on $n, D, \Theta, A, b$ and $\| \varphi \|_{C^2(\partial D)}$, such that for every $c \geq c_*$, there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ of

$$\begin{cases}
\sum_{i=1}^n \arctan \lambda_i (D^2 u) = \Theta & \text{in } \mathbb{R}^n \setminus D, \\
u = \varphi & \text{on } \partial D, \\
\limsup_{|x| \to +\infty} |x|^{m-2} \left| u(x) - \left( \frac{1}{2} x^T A x + b^T x + c \right) \right| < \infty,
\end{cases}
$$

where $m \in (2, n]$ is a constant depending only on $n, \Theta$ and $\lambda(A)$, which actually can be taken as $m(\Theta, \lambda(A))$.

**Theorem 1.2.** Let $D$ be a bounded strictly convex domain in $\mathbb{R}^n$, $n \geq 3$, $\partial D \in C^2$ and let $\varphi \in C^2(\partial D)$. Then for any given $A \in \mathcal{A}_\Theta$ with $-(n-2)\pi/2 < \Theta \leq -(n-2)\pi/2$, and any given $b \in \mathbb{R}^n$, there exists a constant $c^*$ depending only on $n, D, \Theta, A, b$ and $\| \varphi \|_{C^2(\partial D)}$, such that for every $c \leq c^*$, there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ of (1.7), where $m \in (2, n]$ is a constant depending only on $n, \Theta$ and $\lambda(A)$, which actually can be taken as $m(-\Theta, -\lambda(A))$.

**Remark 1.1.** (1) By symmetry, letting $\tilde{u} = -u$, we see easily that Theorem 1.1 and Theorem 1.2 are equivalent. Thus we need only to consider $a = \lambda(A) \in \Gamma^+$ and prove Theorem 1.1 in the rest of this paper.
(2) $\mathcal{A}_\Theta$ is a large class of matrices containing $\tan(\Theta/n) I_n$, which will be clear in Subsection 3.2. For some particular $n$ and $\Theta$, we will see that $\mathcal{A}_\Theta = \mathcal{A}_\Theta^0$, which is the best possible situation we can hope. For example, if $n = 3$, $\Theta = \pm \pi$, we have the special Lagrangian equation in the algebraic form $\sigma_3(\lambda(D^2u)) = \sigma_1(\lambda(D^2u))$ (in three dimension, this indeed is $\det(D^2u) = \Delta u$). Then the $A \in \mathcal{A}_\Theta$ in Theorem 1.1 and Theorem 1.2 can be chosen as any matrix $A \in S(n)$ such that $\lambda(A) \in L_\Theta \cap (\Gamma^+ \cup (-\Gamma^+))$, i.e., any $A \in \mathcal{A}_\Theta^0$. This result is obtained for the first time in our previous paper [LL16], since $\sigma_3(\lambda(D^2u)) = \sigma_1(\lambda(D^2u))$ is also a Hessian quotient equation.

(3) The restrictions on the Lagrangian phase $\Theta$ to be critical or supercritical are subtle and technical. We believe that they can be discarded more or less, since, from [Yuan02], we know that any convex solution of (1.1) (no restriction on the Lagrangian phase) in $\mathbb{R}^n$ must be a quadratic polynomial.

(4) As far as we know, there are only two papers concerning the exterior Dirichlet problem for special Lagrangian equations, but both are highly restricted. For example, [BL13] considered the problem in the cases that $n \leq 4$, $\Theta = \pm \pi$ and $A = \tan(\Theta/n) I_n$, which are just the Hessian quotient equations; [LB14] considered the problem in the cases that $n \geq 3$, $|\Theta| > (n-1)\pi/2$ and $A = \tan(\Theta/n) I_n$. We remark that the results appeared in [BL13] and [LB14] can all be recovered by our Theorem 1.1 and Theorem 1.2 as special cases.

The paper is organized as follows. In Section 2, we first give some specific definitions of terminologies which are standard in the literature, and a list of notations introduced only by us in this paper. Then we collect in Subsection 2.2 some well known lemmas which are mostly used in Section 4. In Section 3, we first introduce quantities $\Xi_k, \xi_k, \xi$ and investigate their properties in Subsection 3.1. Then, in Subsection 3.2 we study polynomials $X, Y, Z, \hat{X}, \hat{Y}$ and $\hat{Z}$ related to the algebraic special Lagrangian equation (1.4), which play fundamental role in this paper. Section 4 is devoted to the proof of the main theorem (Theorem 1.1). To do so, we start in Subsection 4.1 to construct some appropriate subsolutions of the special Lagrangian equation (1.1), by solving some proper ordinary differential equation. Then, after reducing Theorem 1.1 to Lemma 4.3 by normalization in Subsection 4.2
we prove Lemma 4.3 in Subsection 4.3 by applying the Perron’s method to the subsolutions we constructed in Subsection 4.1. The last section, Section 5, is an appendix, which is devoted to prove Lemma 3.3 asserted by us in Subsection 3.2.

2 Preliminary

2.1 Notation

In this paper, $S(n)$ denotes the linear space of symmetric $n \times n$ real matrices, and $I_n$ denotes the identity matrix.

For any $M \in S(n)$, if $m_1, m_2, \ldots, m_n$ are the eigenvalues of $M$ (usually, the assumption $m_1 \leq m_2 \leq \ldots \leq m_n$ is added for convenience), we will denote this fact briefly by $\lambda(M) = (m_1, m_2, \ldots, m_n)$ and call $\lambda(M)$ the eigenvalue vector of $M$. For convenience, we write often $1 := \lambda(I_n) = (1, 1, \ldots, 1)$.

For $A \in S(n)$ and $\rho > 0$, we denote by

$$E_\rho := \{x \in \mathbb{R}^n | x^T Ax < \rho^2\} = \{x \in \mathbb{R}^n | r_A(x) < \rho\}$$

the ellipsoid of size $\rho$ with respect to $A$, where we set $r_A(x) := \sqrt{x^T Ax}$.

For any $p \in \mathbb{R}^n$, we write

$$\sigma_k(p) := \sum_{1 \leq s_1 < s_2 < \ldots < s_k \leq n} p_{s_1} p_{s_2} \ldots p_{s_k} \quad (\forall 1 \leq k \leq n)$$

as the $k$-th elementary symmetric function of $p$. Meanwhile, we will adopt the conventions that $\sigma_{-1}(p) \equiv 0$, $\sigma_0(p) \equiv 1$ and $\sigma_k(p) \equiv 0$, $\forall k \geq n + 1$; and we will also define

$$\sigma_{k;i}(p) := \left(\sigma_k(\lambda)\bigg|_{\lambda_i = 0}\right)_{\lambda_p} = \sigma_k(p_1, p_2, \ldots, \hat{p}_i, \ldots, p_n)$$

for any $-1 \leq k \leq n$ and any $1 \leq i \leq n$, and similarly

$$\sigma_{k;i,j}(p) := \left(\sigma_k(\lambda)\bigg|_{\lambda_i = \lambda_j = 0}\right)_{\lambda_p} = \sigma_k(p_1, p_2, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_n)$$

for any $-1 \leq k \leq n$ and any $1 \leq i, j \leq n$, $i \neq j$, for convenience.

For the reader’s convenience, we give the following list of notations which are only introduced by us in this paper.
The reasons for why we define them will be clear in future sections. We just give a few comments about them here now. To establish the existence of the solution of the special Lagrangian equation (1.1) by the Perron’s method, the key point is to construct some appropriate subsolutions of its algebraic form (1.4). Since (1.4) is a fully nonlinear equation consisting of polynomials with respect to the eigenvalues of the Hessian matrix $D^2 u$ (i.e., polynomials $\sigma_k(\lambda(D^2 u))$, $k = 1, 2, \ldots, n$) of different order of homogeneities, to solve it we need to strike a balance among them. It will turn out to be clear that the quantities $\Xi_k, \xi_k, \overline{\xi}_k$ and $\xi_k$ are very natural and perfectly fit for this purpose. Roughly speaking, $\Xi_k$ originates from the computation of $\sigma_k(\lambda(D^2 \Phi(x)))$ where $\Phi(x) = \phi(r_A(x))$ is a generalized radially symmetric function (see Lemma 2.1 and the proof of Lemma 4.2); $\xi_k, \overline{\xi}_k$ and $\xi_k$ result from the comparison among different $\sigma_k(\lambda)$’s in the attempt to derive an ordinary differential equation from the original equation (1.4) (see the proof of Lemma 4.2); $m(\Theta, a)$ arises in the process of solving this ordinary differential equation (see (4.4) and (4.5) in the proof of Lemma 4.1), so do the polynomials $X, Y, Z, \hat{X}, \hat{Y}, \hat{Z}$ and their coefficients $c_k$, which offer us deeper
understanding of the algebraic special Lagrangian equation (1.4) and play crucial roles in the construction of the subsolutions. By $\Xi_k, \xi_k$, we get a good balance among different $\sigma_k(\lambda)$’s, which can be viewed as being measured by $m(\Theta, a)$. Furthermore, we will find that $m(\Theta, a)$ has also some special meaning related to the decay and asymptotic behavior of the solution (see Lemma 4.1(ii), Corollary 4.1 and Theorem 1.1).

2.2 Some preliminary lemmas

In this subsection, we collect some well known preliminary lemmas which will be mainly used in Section 4.

We first give a lemma to compute $\sigma_k(\lambda(M))$ with $M$ of certain type. If $\Phi(x) := \phi(r)$ with $\phi \in C^2$, $r = \sqrt{x^T A x}$, $A \in S(n) \cap \Gamma^+$ and $a = \lambda(A)$ (we may call $\Phi$ a generalized radially symmetric function with respect to $A$, according to [BLL14]), one can conclude that

$$\partial_{ij} \Phi(x) = \frac{\phi'(r)}{r} a_i \delta_{ij} + \frac{\phi''(r)}{r^2} (a_i x_i)(a_j x_j), \forall 1 \leq i, j \leq n,$$

provided $A$ is normalized to a diagonal matrix (see Subsection 4.2 and the proof of Lemma 4.2 for details). As far as we know, there is generally no explicit formula for $\lambda(D^2 \Phi(x))$ of this type, but luckily we have a method to calculate $\sigma_k(\lambda(D^2 \Phi(x)))$ for each $1 \leq k \leq n$, which can be represented as the following lemma.

**Lemma 2.1.** If $M = (p_i \delta_{ij} + sq_{ij})_{n \times n}$ with $p, q \in \mathbb{R}^n$ and $s \in \mathbb{R}$, then

$$\sigma_k(\lambda(M)) = \sigma_k(p) + s \sum_{i=1}^n \sigma_{k-1; i}(p) q_i^2, \forall 1 \leq k \leq n.$$

**Proof.** See [BLL14].

To process information on the boundary we need the following lemma.

**Lemma 2.2.** Let $D$ be a bounded strictly convex domain of $\mathbb{R}^n$, $n \geq 2$, $\partial D \in C^2$, $\varphi \in C^0(D) \cap C^2(\partial D)$ and let $A \in S(n)$, det $A \neq 0$. Then there exists a constant $K > 0$ depending only on $n$, diam $D$, the convexity of $D$. 

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$\|\varphi\|_{C^2(D)}$, the $C^2$ norm of $\partial D$ and the upper bound of $A$, such that for any $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying

$$|\bar{x}(\xi)| \leq K \quad \text{and} \quad Q_{\xi}(x) < \varphi(x), \quad \forall x \in \overline{D} \setminus \{\xi\},$$

where

$$Q_{\xi}(x) := \frac{1}{2} (x - \bar{x}(\xi))^T A (x - \bar{x}(\xi)) - \frac{1}{2} (\xi - \bar{x}(\xi))^T A (\xi - \bar{x}(\xi)) + \varphi(\xi), \quad \forall x \in \mathbb{R}^n.$$ 

Proof. See [CL03] or [BLL14].

Remark 2.1. It is easy to check that $Q_{\xi}$ satisfy the following properties.

1. $Q_{\xi} \leq \varphi$ on $\overline{D}$ and $Q_{\xi}(\xi) = \varphi(\xi)$.

2. If $A \in \mathcal{A}_\Theta^0$, then

$$\sum_{i=1}^n \arctan \lambda_i(D^2 Q_{\xi}) = \Theta \quad \text{in} \quad \mathbb{R}^n.$$

3. There exists $\bar{c} = \bar{c}(D, A, K) > 0$ such that

$$Q_{\xi}(x) \leq \frac{1}{2} x^T A x + \bar{c}, \quad \forall x \in \partial D, \quad \forall \xi \in \partial D.$$

Now we introduce the following well known lemmas about the comparison principle and Perron’s method. These lemmas are adaptations of those appeared in [CNS85] [Jen88] [Ish89] [Urb90] and [CIL92]. For specific proof of them one may also consult [BLL14] and [LB14].

Lemma 2.3 (Comparison principle). Assume $f \in C^1(\mathbb{R}^n)$ and $f_{\lambda}(\lambda) > 0$, $\forall \lambda \in \mathbb{R}^n$, $\forall i = 1, 2, ..., n$. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\underline{u}, \overline{u} \in C^0(\overline{\Omega})$ satisfying

$$f(\lambda(D^2 \underline{u})) \geq 0 \geq f(\lambda(D^2 \overline{u}))$$

in $\Omega$ in the viscosity sense. Suppose $\underline{u} \leq \overline{u}$ on $\partial \Omega$ (and additionally

$$\lim_{|x| \to +\infty} (\underline{u} - \overline{u})(x) = 0$$

provided $\Omega$ is unbounded). Then $\underline{u} \leq \overline{u}$ in $\Omega$. 

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Lemma 2.4 (Perron’s method). Assume $f \in C^1(\mathbb{R}^n)$ and $f_{\lambda_{i}}(\lambda) > 0$, $\forall \lambda \in \mathbb{R}^n$, $\forall i = 1, 2, ..., n$. Let $\Omega \subset \mathbb{R}^n$ be a domain, $\varphi \in C^0(\partial \Omega)$ and let $u, \overline{u} \in C^0(\overline{\Omega})$ satisfying

$$f(\lambda(D^2 u)) \geq 0 \geq f(\lambda(D^2 \overline{u}))$$

in $\Omega$ in the viscosity sense. Suppose $u \leq \overline{u}$ in $\Omega$, $u = \varphi$ on $\partial \Omega$ (and additionally

$$\lim_{|x| \to +\infty} (u - \overline{u})(x) = 0$$

provided $\Omega$ is unbounded). Then

$$u(x) := \sup \left\{ v(x) \mid v \in C^0(\Omega), \; u \leq v \leq \overline{u} \text{ in } \Omega, \; f(\lambda(D^2 v)) \geq 0 \text{ in } \Omega \right\}$$

is the unique viscosity solution of the Dirichlet problem

$$\begin{cases}
    f(\lambda(D^2 u)) = 0 & \text{in } \Omega, \\
    u = \varphi & \text{on } \partial \Omega.
\end{cases}$$

Remark 2.2. Clearly, by letting

$$f(\lambda) := H(\lambda) - \Theta \triangleq \sum_{i=1}^{n} \arctan \lambda_i - \Theta,$$

the special Lagrangian equation (1.1) satisfies the above two lemmas.

3 Quantities and polynomials related to the special Lagrangian equations

3.1 Quantities $\Xi_k$, $\xi_k$, $\overline{\xi}_k$ and their properties

The quantities $\Xi_k$, $\xi_k$, and $\overline{\xi}_k$ originate in the calculation of $\sigma_k(\lambda(D^2 \Phi))$ and in the comparison among different $\sigma_k(\lambda(D^2 \Phi))$'s (see Lemma 2.1 and the proof of Lemma 4.2). They were partially given for the first time in our previous paper [LL16]. For completeness and convenience, we introduce them again in this subsection.
For any $0 \leq k \leq n$ and any $a \in \mathbb{R}^n \setminus \{0\}$, let
\[ \Xi_k := \Xi_k(a, x) := \frac{\sum_{i=1}^{k-1} \sigma_{k-1;i}(a) a_i^2 x_i^2}{\sigma_k(a) \sum_{i=1}^{n} a_i x_i^2}, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad (3.1) \]
and define
\[ \bar{\xi}_k := \bar{\xi}_k(a) := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \Xi_k(a, x), \quad (3.2) \]
and
\[ \underline{\xi}_k := \underline{\xi}_k(a) := \inf_{x \in \mathbb{R}^n \setminus \{0\}} \Xi_k(a, x). \quad (3.3) \]

It is easy to see that
\[ \bar{\xi}_k(qa) = \bar{\xi}_k(a), \quad \underline{\xi}_k(qa) = \underline{\xi}_k(a), \quad \forall q \neq 0, \quad \forall a \in \mathbb{R}^n \setminus \{0\}, \quad \forall 0 \leq k \leq n, \]
and
\[ \underline{\xi}_k(q1) = \frac{k}{n} = \bar{\xi}_k(q1), \quad \forall q \neq 0, \quad \forall 0 \leq k \leq n. \quad (3.4) \]
Furthermore, we have the following lemma.

**Lemma 3.1.** Suppose $a = (a_1, a_2, ..., a_n)$ with $0 < a_1 \leq a_2 \leq ... \leq a_n$. Then
\[ 0 < \frac{a_1 \sigma_{k-1;1}(a)}{\sigma_k(a)} = \underline{\xi}_k(a) \leq \frac{k}{n} \leq \bar{\xi}_k(a) = \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)} \leq 1, \quad \forall 1 \leq k \leq n; \quad (3.5) \]
\[ 0 = \bar{\xi}_0(a) < \frac{1}{n} \leq \frac{a_n}{\sigma_1(a)} = \underline{\xi}_1(a) \leq \underline{\xi}_2(a) \leq ... \leq \underline{\xi}_{n-1}(a) < \bar{\xi}_n(a) = 1; \quad (3.6) \]
and
\[ 0 = \underline{\xi}_0(a) < \frac{a_1}{\sigma_1(a)} = \underline{\xi}_1(a) \leq \underline{\xi}_2(a) \leq ... \leq \underline{\xi}_{n-1}(a) < \underline{\xi}_n(a) = 1. \quad (3.7) \]
Moreover,
\[ \underline{\xi}_k(a) = \frac{k}{n} = \bar{\xi}_k(a) \quad (3.8) \]
for some $1 \leq k \leq n - 1$, if and only if $a = C \mathbf{1}$ for some $C > 0$. 

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Proof. (1°) By the definitions of $\sigma_k(a)$ and $\sigma_{k;i}(a)$, we see that

$$\sigma_k(a) = \sigma_{k;i}(a) + a_i \sigma_{k-1;i}(a), \quad \forall 1 \leq i \leq n;$$

and

$$\sum_{i=1}^{n} \sigma_{k;i}(a) = \frac{nC_{n-1}^{k}}{C_n^{k}} \sigma_k(a) = (n-k) \sigma_k(a).$$

Hence we obtain

$$\sum_{i=1}^{n} a_i \sigma_{k-1;i}(a) = k \sigma_k(a).$$

(3.10)

Now we show that

$$a_1 \sigma_{k-1;1}(a) \leq a_2 \sigma_{k-1;2}(a) \leq \ldots \leq a_n \sigma_{k-1;n}(a).$$

(3.11)

In fact, for any $i \neq j$, similar to (3.9), we have

$$a_i \sigma_{k-1;i}(a) = a_i \left( \sigma_{k-1;i,j}(a) + a_j \sigma_{k-2;i,j}(a) \right),$$

$$a_j \sigma_{k-1;j}(a) = a_j \left( \sigma_{k-1;i,j}(a) + a_i \sigma_{k-2;i,j}(a) \right),$$

and hence

$$a_i \sigma_{k-1;i}(a) - a_j \sigma_{k-1;j}(a) = (a_i - a_j) \sigma_{k-1;i,j}(a).$$

Therefore, if $a_i \leq a_j$, then

$$a_i \sigma_{k-1;i}(a) \leq a_j \sigma_{k-1;j}(a).$$

(3.12)

By the definition of $\xi_k$, we have

$$\xi_k(a) = \sup_{x \neq 0} \frac{\sum_{i=1}^{n} \sigma_{k-1;i}(a)a_i^2x_i^2}{\sigma_k(a) \sum_{i=1}^{n} a_i x_i^2}$$

\begin{align*}
&\geq \sup_{x_1=\ldots=x_{n-1}=0, x_n \neq 0} \frac{\sum_{i=1}^{n} \sigma_{k-1;i}(a)a_i^2x_i^2}{\sigma_k(a) \sum_{i=1}^{n} a_i x_i^2} \\
&= \sup_{x_n \neq 0} \frac{\sigma_{k-1;n}(a)a_n^2x_n^2}{\sigma_k(a) a_n x_n^2} \\
&= \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)}
\end{align*}
and
\[
\bar{\xi}_k(a) = \sup_{x \neq 0} \frac{\sum_{i=1}^{n} \sigma_{k-1;i}(a) a_i x_i}{\sigma_k(a) \sum_{i=1}^{n} a_i x_i^2}
\leq \sup_{x \neq 0} \frac{a_n \sigma_{k-1;n}(a) \sum_{i=1}^{n} a_i x_i^2}{\sigma_k(a) \sum_{i=1}^{n} a_i x_i^2}
\text{ (according to (3.11))}
= \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)}.
\]

Hence we obtain
\[
\bar{\xi}_k(a) = \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)}.
\tag{3.13}
\]

Similarly
\[
\xi_k(a) = \frac{a_1 \sigma_{k-1;1}(a)}{\sigma_k(a)}.
\tag{3.14}
\]

From (3.10), we have
\[
\sum_{i=1}^{n} \frac{a_i \sigma_{k-1;i}(a)}{\sigma_k(a)} = k.
\]

Combining this with (3.11), (3.13) and (3.14), we deduce that
\[
\xi_k(a) \leq \frac{k}{n} \leq \bar{\xi}_k(a).
\]

Thus the proof of (3.5) is complete, and (3.8) is also clear in view of (3.12).

(2°) Since it follows from (3.9) that
\[
a_i \sigma_{k-1;i}(a) < \sigma_k(a), \ \forall 1 \leq i \leq n, \ \forall 1 \leq k \leq n - 1,
\]
we obtain
\[
\xi_k(a) \leq \bar{\xi}_k(a) < 1, \ \forall 0 \leq k \leq n - 1.
\]

On the other hand, we have \(\bar{\xi}_n(a) = \xi_n(a) = 1\) which follows from
\[
a_i \sigma_{n-1;i}(a) = \sigma_n(a), \ \forall 1 \leq i \leq n.
\]

Combining (3.13) and (3.9), we observe that
\[
\bar{\xi}_k(a) = \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)} = \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a) + a_n \sigma_{k-1;n}(a)}
\]

\[
= \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a) + a_n \sigma_{k-1;n}(a)}
\]

\[13\]
\[
\leq \frac{a_n\sigma_{k:n}(a)}{\sigma_{k+1;n}(a) + a_n\sigma_{k;n}(a)} = \frac{a_n\sigma_{k:n}(a)}{\sigma_{k+1}(a)} = \bar{\xi}_{k+1}(a),
\]
where we used the inequality
\[
\frac{\sigma_{k-1;n}(a)}{\sigma_{k;n}(a)} \leq \frac{\sigma_{k;n}(a)}{\sigma_{k+1;n}(a)}
\]
which is a variation of the famous Newton inequality (see [HLP34]).
\[
\sigma_{k-1}(\lambda)\sigma_{k+1}(\lambda) \leq (\sigma_k(\lambda))^2, \quad \forall \lambda \in \mathbb{R}^n.
\]
This completes the proof of (3.6), and similarly of (3.7).

3.2 Polynomials \(X, Y, Z, \hat{X}, \hat{Y}, \hat{Z}\) and their properties

The polynomials appeared in (1.4) play crucial roles in this paper. For convenience, we give their names by setting
\[
X(\lambda) := \sum_{0 \leq 2j \leq n} (-1)^j \sigma_{2j}(\lambda)
= 1 - \sigma_2(\lambda) + \sigma_4(\lambda) - ..., \quad (3.15)
\]
and
\[
Y(\lambda) := \sum_{0 \leq 2j+1 \leq n} (-1)^j \sigma_{2j+1}(\lambda)
= \sigma_1(\lambda) - \sigma_3(\lambda) + \sigma_5(\lambda) - ..., \quad (3.16)
\]
for any \(\lambda \in \mathbb{R}^n\). Write for short also
\[
H(\lambda) := \sum_{i=1}^{n} \arctan \lambda_i, \quad \forall \lambda \in \mathbb{R}^n. \quad (3.17)
\]

We get the following elementary relations according to the trigonometry:
\[
\frac{Y(\lambda)}{X(\lambda)} = \tan H(\lambda) = \frac{\sin H(\lambda)}{\cos H(\lambda)}, \quad (3.18)
\]
and
\[
\frac{X(\lambda)}{Y(\lambda)} = \cot H(\lambda) = \frac{\cos H(\lambda)}{\sin H(\lambda)}, \quad (3.19)
\]
provided the denominators are not zero. Moreover, we see that
Lemma 3.2. For any $\lambda \in \mathbb{R}^n$, $\cos H(\lambda)$ and $X(\lambda)$ (respectively, $\sin H(\lambda)$ and $Y(\lambda)$) have the same sign. That is

$$\cos H(\lambda) \leq 0 \iff X(\lambda) \leq 0,$$

and

$$\sin H(\lambda) \leq 0 \iff Y(\lambda) \leq 0.$$  

Proof. By (3.18), it is clear that (3.20) and (3.21) are equivalent, thus we need only to prove the former. To do this, the following assertions (which can also be used as sketches of the proof) will be adequate.

(1°) $\{\lambda \in \mathbb{R}^n | \cos H(\lambda) \neq 0\}$ has exactly $\tilde{n} + 1$ connected components, and in any two adjacent such components $\cos H(\lambda)$ have different signs. Here

$$\tilde{n} := \tilde{n}(n) := \begin{cases} n - 1, & \text{when } n \text{ is odd}, \\ n, & \text{when } n \text{ is even}, \end{cases}$$

which will be used throughout this proof and will not be mentioned again.

(2°) Since $X(\lambda) = 0 \iff \cos H(\lambda) = 0$, $\{\lambda \in \mathbb{R}^n | X(\lambda) \neq 0\}$ also has exactly $\tilde{n} + 1$ connected components, as same as those of $\{\lambda \in \mathbb{R}^n | \cos H(\lambda) \neq 0\}$. Because, for any $\lambda \in \Gamma^+$, $X(t\lambda)$ is a polynomial with respect to $t$ of $\tilde{n}$ order, and it has exactly $\tilde{n}$ different real roots, we know that $\{\lambda \in \mathbb{R}^n | X(\lambda) \neq 0\} \cap \Gamma^+ \cap (-\Gamma^+)$ also has exactly $\tilde{n} + 1$ connected components, and in any two adjacent such components $X(\lambda)$ have different signs. Thus, by smoothness of $X(\lambda)$ and $\cos H(\lambda)$, we conclude that $X(\lambda)$ have different signs in any two adjacent components of $\{\lambda \in \mathbb{R}^n | X(\lambda) \neq 0\}$.

(3°) $X(0) = 1 = \cos H(0)$.

To confirm (1°), we need only to note that

$$|DH(\lambda)| = \left| \left( \frac{1}{1 + \lambda_1^2}, \ldots, \frac{1}{1 + \lambda_n^2} \right) \right| > 0,$$

$$\cos H(\lambda) = 0 \iff H(\lambda) = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}.$$  

(3.22)
and there are exactly $\tilde{n}$ different $k \in \mathbb{Z}$ such that

$$\frac{\pi}{2} + k\pi \in \left(-\frac{n\pi}{2}, \frac{n\pi}{2}\right).$$

Now consider $(2^\circ)$. One can easily check that $X(t\lambda)$ is a polynomial with respect to $t$ of $\tilde{n}$ order. Thus it has at most $\tilde{n}$ different complex roots. In particular, when $\lambda \in \Gamma^+$, $X(t\lambda)$ has exactly $\tilde{n}$ different real roots. Indeed, since

$$\lim_{t \to -\infty} H(t\lambda) = -n\pi/2, \quad \lim_{t \to +\infty} H(t\lambda) = n\pi/2$$

and

$$\partial_t(H(t\lambda)) = \sum_{i=1}^{n} \frac{\lambda_i}{1 + (t\lambda_i)^2} > 0, \ \forall t \in \mathbb{R},$$

we see that the line $\{t\lambda\}_{t \in \mathbb{R}}$ intersects with all the level surfaces $L_\Theta$ ($\Theta \in (-n\pi/2, n\pi/2)$) once and only once. This implies that there are exactly $\tilde{n}$ different real $t$ such that $\cos H(t\lambda) = 0$. On the other hand, since it follows from (3.19) that

$$X(\lambda) = 0 \iff \cos H(\lambda) = 0,$$

we thus conclude that the $\tilde{n}$-order polynomial $X(t\lambda)$ has exactly $\tilde{n}$ different real roots, which lie right in the $\tilde{n}$ different level sets $L_\Theta$ ($\Theta = \pi/2 + k\pi \in (-n\pi/2, n\pi/2)$, $k \in \mathbb{Z}$), respectively.

Hence, $\{\lambda \in \mathbb{R}^n | X(\lambda) \neq 0\} \cap \Gamma^+ \cap (-\Gamma^+)$ has exactly $\tilde{n} + 1$ connected components, and in any two adjacent such components $X(\lambda)$ have different signs. Therefore, $(2^\circ)$ is clear.

Assertion $(3^\circ)$ is easy, thus the proof of this lemma is completed. $\square$

Now set

$$\widehat{X}(\lambda) := \sum_{0 \leq 2j \leq n} (-1)^j \cdot (2j) \cdot \sigma_{2j}(\lambda)$$

$$= -2\sigma_2(\lambda) + 4\sigma_4(\lambda) - ..., \quad (3.23)$$

and

$$\widehat{Y}(\lambda) := \sum_{0 \leq 2j+1 \leq n} (-1)^j \cdot (2j + 1) \cdot \sigma_{2j+1}(\lambda)$$

$$= \sigma_1(\lambda) - 3\sigma_3(\lambda) + 5\sigma_5(\lambda) - ..., \quad (3.24)$$

for any $\lambda \in \mathbb{R}^n$. As a corollary of Lemma 3.2, we obtain
Corollary 3.1. For any $a \in \mathbb{R}^n$, we have

$$X(a)\hat{Y}(a) - Y(a)\hat{X}(a) > 0 \Rightarrow \cos H(a) \hat{Y}(a) - \sin H(a) \hat{X}(a) > 0.$$  

Proof. (1°) If $\cos H(a) \neq 0$, then it follows from Lemma 3.2 that

$$\frac{\cos H(a)}{X(a)} > 0.$$  

Thus, by (3.18), we can deduce that

$$\frac{\cos H(a)}{X(a)} (X(a)\hat{Y}(a) - \tan H(a) X(a)\hat{X}(a))$$

$$= \frac{\cos H(a)}{X(a)} (X(a)\hat{Y}(a) - Y(a)\hat{X}(a))$$

$$> 0.$$  

(2°) If $\cos H(a) = 0$, then $\sin H(a) = \pm 1 \neq 0$. Thus we can employ a procedure similar to the above one. That is, by using

$$\frac{\sin H(a)}{Y(a)} > 0$$

according to Lemma 3.2, we can now deduce from (3.19) that

$$\frac{\sin H(a)}{Y(a)} (\cot H(a) Y(a)\hat{Y}(a) - Y(a)\hat{X}(a))$$

$$= \frac{\sin H(a)}{Y(a)} (X(a)\hat{Y}(a) - Y(a)\hat{X}(a))$$

$$> 0.$$  

This finishes the proof of the lemma.

We discussed in Corollary 3.1 the sign of $X\hat{Y} - Y\hat{X}$ and its application. In fact, for $X\hat{Y} - Y\hat{X}$, we have established the following explicit formula, which shows immediately that $X(a)\hat{Y}(a) - Y(a)\hat{X}(a) > 0$ for all $a \in \Gamma^+$. 

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Lemma 3.3. For any $a \in \mathbb{R}^n$, we have

$$X(a)\hat{Y}(a) - Y(a)\hat{X}(a) = \sum_{k=1}^{n} \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n} a_{i_1}^2 a_{i_2}^2 \ldots a_{i_k-1}^2 a_{i_k}.$$  

Proof. See the Appendix in Section 5 (Since it is too lengthy, we attach it in the end of this paper.) \(\qed\)

The notations $X, Y, \hat{X}$ and $\hat{Y}$ we introduced above can all be viewed as “global” notations. To understand better the special Lagrangian equation (1.4), we need to introduce the following “local” notations $c_k, \xi_k$ and their combinations.

For any $\Theta \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$ and any $k = 0, 1, \ldots, n$, let $c_k = c_k(\Theta)$ be the coefficients such that

$$\sum_{k=0}^{n} c_k(\Theta)\sigma_k(\lambda) = \cos \Theta Y(\lambda) - \sin \Theta X(\lambda), \forall \lambda \in \mathbb{R}^n;$$

that is, we set

$$c_k := c_k(\Theta) := \begin{cases} c_{2j}(\Theta) := (-1)^{j+1} \sin \Theta, & \text{if } k = 2j, \\ c_{2j+1}(\Theta) := (-1)^j \cos \Theta, & \text{if } k = 2j + 1. \end{cases} \quad (3.25)$$

For any $\Theta \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$, any $a \in \mathbb{R}^n$ and any $k = 0, 1, \ldots, n$, we define

$$\xi_k := \xi_k(a) := \xi_k(\Theta, a) := \begin{cases} \xi_k(a), & \text{if } c_k(\Theta) > 0, \\ \xi_k(a), & \text{if } c_k(\Theta) \leq 0. \end{cases} \quad (3.26)$$

By definition and Lemma 3.1, it is easy to see that

$$\Xi_k(a, x)c_k(\Theta) \leq \xi_k(\Theta, a)c_k(\Theta), \quad (3.27)$$

and

$$\xi_k(\Theta, a)c_k(\Theta) \geq \frac{k}{n} c_k(\Theta), \quad (3.28)$$

for all $k = 0, 1, \ldots, n$.

For any $\Theta \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$ and any $\lambda \in \mathbb{R}^n$, set

$$Z(\lambda) := Z(\Theta, \lambda) := \sum_{k=0}^{n} c_k(\Theta)\sigma_k(\lambda).$$
\[ \sum_{0 \leq 2j+1 \leq n} (-1)^j \sigma_{2j+1}(\lambda) \]

and

\[
\tilde{Z}(\lambda) := \tilde{Z}(\Theta, \lambda) := \sum_{k=0}^{n} k c_k(\Theta) \sigma_k(\lambda) \\
:= \cos \Theta \tilde{Y}(\lambda) - \sin \Theta \tilde{X}(\lambda) \\
= \cos \Theta \sum_{0 \leq 2j+1 \leq n} (-1)^j (2j+1) \sigma_{2j+1}(\lambda) \\
- \sin \Theta \sum_{0 \leq 2j \leq n} (-1)^j (2j) \sigma_{2j}(\lambda). \tag{3.30}
\]

We see, for any \( a \in \mathbb{R}^n \), that

\[
Z(ta) = \sum c_k \sigma_k(ta) = \sum c_k \sigma_k(a) t^k = \cos \Theta Y(ta) - \sin \Theta X(ta), \tag{3.31}
\]

\[
\tilde{Z}(ta) = \sum k c_k \sigma_k(a) t^k = \cos \Theta \tilde{Y}(ta) - \sin \Theta \tilde{X}(ta),
\]

and

\[
\tilde{Z}(ta) = t \frac{d}{dt} (Z(ta)).
\]

As a corollary of Lemma 3.3, we have

**Corollary 3.2.** Let \( a \in \Gamma^+ \cap L_\Theta \) with \( 0 < \Theta < n\pi/2 \). Then

\[
\sum_{k=0}^{n} k c_k(\Theta) \sigma_k(a) > 0,
\]

and

\[
\sum_{k=0}^{n} \xi_k(\Theta, a) c_k(\Theta) \sigma_k(a) > 0.
\]

**Proof.** By Lemma 3.3 we have

\[
X(a) \tilde{Y}(a) - Y(a) \tilde{X}(a) > 0.
\]
Since \( a \in L_\Theta \) means that \( H(a) = \Theta \), by (3.28) and Corollary 3.1, we thus deduce that
\[
\sum \xi_k c_k \sigma_k \geq \frac{1}{n} \sum k c_k \sigma_k = \frac{1}{n} \left( \cos \Theta \hat{Y}(a) - \sin \Theta \hat{X}(a) \right) > 0.
\]
The proof thereby is completed. \( \Box \)

For any \( \Theta \in (0, n\pi/2) \) and any \( a \in L_\Theta \cap \Gamma^+ \), we now define
\[
m(\Theta, a) := \frac{\sum k c_k \sigma_k}{\sum \xi_k c_k \sigma_k} := \frac{\sum_{k=0}^{n} k c_k(\Theta) \sigma_k(a)}{\sum_{k=0}^{n} \xi_k(a) c_k(\Theta) \sigma_k(a)}.
\]
(3.32)

By the proof of Corollary 3.2, we see immediately that
\[
0 < m(\Theta, a) \leq n, \quad \forall a \in L_\Theta \cap \Gamma^+, \quad \forall \Theta \in (0, n\pi/2).
\]
(3.33)

Thus for any \( A \in \mathcal{A}_\Theta \), we have
\[
2 < m(\Theta, a) \leq n.
\]

To construct subsolutions of (1.1), we need \( m(\Theta, \lambda(A)) > 2 \). Though a large class of matrices satisfying this, there are still some exceptions. To see this, we now give some examples.

First we know from (3.4) that \( \xi_k(\tan(\Theta/n) 1) = k/n \) and therefore
\[
m(\Theta, \lambda(\tan(\Theta/n) I_n)) = n > 2.
\]

Next we consider (1.1) in \( \mathbb{R}^3 \) or \( \mathbb{R}^4 \) with the Lagrangian phase \( \Theta = \pi \). Since \( A \in \mathcal{A}_\pi \) means that \( a = \lambda(A) \) satisfies \( H(a) = \pi \), we have \( \sin H(a) = 0 \). By Lemma 3.2 we see that \( c_1(a)\sigma_1(a) - c_3(a)\sigma_3(a) = Y(a) = 0 \), Thus we obtain
\[
m(\pi, a) = \frac{c_1 \sigma_1 - 3 c_3 \sigma_3}{\xi_1 c_1 \sigma_1 - \xi_3 c_3 \sigma_3} = \frac{1 - 3}{\xi_1 - \xi_3} > 2,
\]
which implies that \( \mathcal{A}_\pi = \mathcal{A}_\pi^0 \) in three or four dimension.

Finally, we give an example in \( \mathbb{R}^5 \) to show how the distribution of the values of \( m(\Theta, a) \) looks like in general. Let
\[
a_\varepsilon := \left( \tan \left( \frac{\pi}{3} - 2\varepsilon \right), \tan \left( \frac{\pi}{3} - \varepsilon \right), \tan \left( \frac{\pi}{3} \right), \tan \left( \frac{\pi}{3} + \varepsilon \right), \tan \left( \frac{\pi}{3} + 2\varepsilon \right) \right),
\]
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for any $0 < \varepsilon < \pi/12$. Then $a_\varepsilon \in \Gamma^+$ and $H(a_\varepsilon) = 5\pi/3 \in (3\pi/2, 2\pi)$. Furthermore, we can calculate (with the help of the computer, for example) that

$$m(\varepsilon) := m(5\pi/3, a_\varepsilon) = \frac{4\sqrt{3}\cos(4\varepsilon) + 4\sqrt{3}\cos(2\varepsilon) + 2\sqrt{3}}{2\sqrt{3}\cos(4\varepsilon) + 2\sin(6\varepsilon) + 2\sin(2\varepsilon) + 3\sin(4\varepsilon)}.$$  

Thus $m(\varepsilon)$ is deceasing in $[0, \pi/12]$ with $m(0) = 5$, $m(0.2068) \approx 2$ and $m(0.2618) \approx m(\pi/12) = 16 + 4\sqrt{3}/13 \approx 1.7637$.

This indicates that in a wide strip around the ray $\{k1\}$, $m(\Theta, a)$ is larger than 2 in general; But when $a$ is close to the boundary of $\Gamma^+$, $m(\Theta, a)$ appears to be less than 2. We would like to mention that we have done a lot of experiments like the one above, all the results of them give us similar conclusions even when the Lagrangian phase is subcritical. For brevity, we will not add them here any more.

Now we study the properties of the polynomial $Z(ta)$, which will play a crucial role in the construction of the subsolutions of (1.1). First, we have the following basic lemma.

**Lemma 3.4.** For any fixed $a \in L_\Theta \cap \Gamma^+$ with $(n - 2)\pi/2 \leq \Theta < n\pi/2$, $Z(ta)$ is a polynomial with respect to $t$ of $N$ order, and with the leading coefficient $c_N > 0$, where

$$N := N(n, \Theta) := \begin{cases} n - 1, & \text{when } \Theta = (n - 2)\pi/2, \\ n, & \text{when } (n - 2)\pi/2 < \Theta < n\pi/2, \end{cases} (3.34)$$

which will be used only in this sense throughout this paper.

**Proof.** The proof is elementary and hidden in the definition of $c_k$:

$$
\begin{align*}
  c_{2j}(\Theta) &:= (-1)^{j+1}\sin \Theta, & \text{if } k = 2j, \\
  c_{2j+1}(\Theta) &:= (-1)^j\cos \Theta, & \text{if } k = 2j + 1.
\end{align*}
$$

For the reader’s convenience, we check them case by case.

(1°) For $\Theta = (n - 2)\pi/2$, if $n = 2j$, then $\Theta = (n - 2)\pi/2 = (j - 1)\pi$ and

$$
\begin{align*}
  c_n(\Theta) &= c_2(\Theta) = (-1)^{j+1}\sin \Theta = 0, \\
  c_{n-1}(\Theta) &= c_2(j-1+1)(\Theta) = (-1)^{j-1}\cos \Theta = 1 > 0;
\end{align*}
$$
if $n = 2j + 1$, then $\Theta = (n - 2)\pi/2 = \pi/2 + (j - 1)\pi$ and

\[
\begin{align*}
    c_n(\Theta) &= c_{2j+1}(\Theta) = (-1)^j \cos \Theta = 0, \\
    c_{n-1}(\Theta) &= c_{2j}(\Theta) = (-1)^{j+1} \sin \Theta = 1 > 0.
\end{align*}
\]

(2º) For $(n - 2)\pi/2 < \Theta < n\pi/2$, if $n = 2j$, then $(j - 1)\pi < \Theta < j\pi$ and

\[c_n(\Theta) = c_{2j}(\Theta) = (-1)^{j+1} \sin \Theta > 0;\]

if $n = 2j + 1$, then $\pi/2 + (j - 1)\pi < \Theta < \pi/2 + j\pi$ and

\[c_n(\Theta) = c_{2j+1}(\Theta) = (-1)^j \cos \Theta > 0.\]

This completes the proof of the lemma. \qed

By definition, it is easy to see that $Z(a) = 0$. This is to say that $t = 1$ is a root of $Z(ta)$. The following lemma asserts that it is indeed a simple root.

**Lemma 3.5.** For any fixed $a \in \Gamma^+$, $t = 1$ is a simple root of $Z(ta) = 0$.

**Proof.** Invoking the following lemma from algebra:

*(Derivative criterion for simple root)* Let $p(x)$ be a polynomial with real coefficients. Then $x_0$ is a simple root of $p(x)$ if and only if

\[p(x_0) = 0 \quad \text{and} \quad p'(x_0) \neq 0;\]

since $Z(a) = 0$, to show that $t = 1$ is a simple root of $Z(ta) = 0$, we now need only to check that

\[\frac{d}{dt}(Z(ta))\bigg|_{t=1} \neq 0.\]

This is evidently true by Corollary 3.2, since

\[\frac{d}{dt}(Z(ta))\bigg|_{t=1} = \sum kc_k\sigma_k(a).\]

\qed

In fact, we can prove easily the following lemma.

**Lemma 3.6.** For any fixed $a \in L_\Theta \cap \Gamma^+$ with $(n - 2)\pi/2 \leq \Theta < n\pi/2$, all the roots of $Z(ta) = 0$ are real and simple.
Proof. Because \( Z(ta) \triangleq \cos \Theta Y(ta) - \sin \Theta X(ta) \), we see that
\[
Z(ta) = 0 \iff \tan H(ta) = \tan \Theta \iff H(ta) = \Theta + k\pi.
\]
Thus it is clear that there are exactly \( N \) different \( t \in \mathbb{R} \) such that
\[
-n\pi/2 < H(a) = \Theta + k\pi < n\pi/2
\]
for some \( k \in \mathbb{Z} \), and that each such \( t \) is the root of \( Z(ta) = 0 \). Since \( Z(ta) \) is a polynomial with respect to \( t \) of \( N \) order, according to Lemma 3.4, this shows that all the roots of \( Z(ta) = 0 \) are real and simple.

We remark that, using the same method, Lemma 3.6 can be generalized as: for any fixed \( a \in \Gamma^+ \), all the roots of \( Z(ta) = 0 \) are real and simple.

The positiveness of \( Z(ta) \) and its derivatives, for each \( t \in (1, +\infty) \), is crucial in constructing the subsolutions of (1.1). This is the main reason why we restrict ourselves in the cases that the Lagrangian phases \( \Theta \) are critical and supercritical. We will now discuss these problems in the following lemma.

Lemma 3.7. Suppose \( a \in L_\Theta \cap \Gamma^+ \) with \((n-2)\pi/2 \leq \Theta < n\pi/2\). Then
\[
\frac{d}{dt}(Z(ta)) = \sum_{k=0}^{n} kc_k(\Theta)\sigma_k(a)t^{k-1} > 0, \ \forall t \geq 1.
\]
Furthermore, we have
\[
Z(a) = \sum_{k=0}^{n} c_k(\Theta)\sigma_k(a) = 0,
\]
\[
Z(ta) = \sum_{k=0}^{n} c_k(\Theta)\sigma_k(a)t^k > 0, \ \forall t > 1,
\]
and
\[
\frac{d^k}{dt^k}(Z(ta)) > 0, \ \forall t \geq 1, \ \forall 1 \leq k \leq N.
\]

To prove Lemma 3.7, we first introduce the following simple principle concerning polynomials.
Lemma 3.8. Suppose $P(x)$ is a polynomial in $\mathbb{R}$, $P(1) > 0$ and all the roots of $P(x)$ are real and less than 1. Then $P'(x) > 0$, $\forall x \geq 1$. Furthermore, we have $\frac{d^k}{dx^k}P(x) > 0$, $\forall k = 1, 2, \ldots, n$, $\forall x \geq 1$.

Proof. (This simple proof was established with the help of Shanshan Ma.) Let $a_1, a_2, \ldots, a_n < 1$ be the roots of $P(x)$. Then there exists a constant $K$ such that

$$P(x) = K(x - a_1)(x - a_2)\cdots(x - a_n) = K \prod_{j=1}^{n} (x - a_j).$$

Since $P(1) > 0$, we know that $K > 0$. Thus we have

$$P'(x) = K \sum_{i=1}^{n} \prod_{j \neq i} (x - a_j) > 0$$

for all $x \geq 1$, since $x - a_j \geq 1 - a_j > 0$.

Similarly, we can also obtain $\frac{d^k}{dx^k}P(x) > 0$, $\forall k = 1, 2, \ldots, n$, $\forall x \geq 1$. 

We now prove Lemma 3.7

Proof of Lemma 3.7. $Z(a) = 0$ is already known. We now show that

$$Z(ta) \triangleq \cos \Theta Y(ta) - \sin \Theta X(ta) > 0, \forall t > 1.$$ 

For any $t > 1$, we have $(n - 2)\pi/2 \leq \Theta = H(a) < H(ta) < n\pi/2$ and

(A) if $\Theta = (n - 2)\pi/2$ and

(a) if $n$ is even, then $\sin \Theta = 0$, $\cos \Theta \neq 0$, and $\sin H(ta)$ and $\cos \Theta$ have the same sign. Since by Lemma 3.2 we know that $\sin H(ta)$ and $Y(ta)$ have the same sign, we thus obtain $Z(ta) = \cos \Theta Y(ta) > 0$; 

(b) if $n$ is odd, then $\cos \Theta = 0$, $\sin \Theta \neq 0$, and $\cos H(ta)$ and $-\sin \Theta$ have the same sign. Since by Lemma 3.2 we know that $\cos H(ta)$ and $X(ta)$ have the same sign, we thus get $Z(ta) = -\sin \Theta X(ta) > 0$.

(B) if $(n - 2)\pi/2 < \Theta < n\pi/2$ and
(a) if \( n \) is even, then \( \sin \Theta \neq 0 \), \( \sin H(ta) \neq 0 \), \( \sin \Theta \) and \( \sin H(ta) \) have the same sign, and \( -\cot \theta \) is strictly increasing in \( ((n-2)\pi/2, n\pi/2) \). Since, by Lemma 3.2, \( \sin H(ta) \) and \( Y(ta) \) have the same sign, we thus know that \( Y(ta) \neq 0 \) and has the same sign with \( \sin \Theta \). Hence we have

\[
-\frac{X(ta)}{Y(ta)} = -\cot H(ta) > -\cot H(a) = -\cot \Theta = \frac{-\cos \Theta}{\sin \Theta},
\]

and therefore \( Z(ta) = \cos \Theta Y(ta) - \sin \Theta X(ta) > 0 \);

(b) if \( n \) is odd, then \( \cos \Theta \neq 0 \), \( \cos H(ta) \neq 0 \), \( \cos \Theta \) and \( \cos H(ta) \) have the same sign, and \( \tan \theta \) is strictly increasing in \( ((n-2)\pi/2, n\pi/2) \). Since, by Lemma 3.2, \( \cos H(ta) \) and \( X(ta) \) have the same sign, we thus know that \( X(ta) \neq 0 \) and has the same sign with \( \cos \Theta \). Hence we have

\[
\frac{Y(ta)}{X(ta)} = \tan H(ta) > \tan H(a) = \tan \Theta = \frac{\sin \Theta}{\cos \Theta},
\]

and therefore \( Z(ta) = \cos \Theta Y(ta) - \sin \Theta X(ta) > 0 \).

Thus we have proved, in any case, that \( Z(ta) > 0 \), \( \forall t > 1 \). This is to say that all the roots of \( Z(ta) \) are less than or equal to 1. Invoking Lemma 3.6 and by a slightly modified version of Lemma 3.8 (i.e., we need to consider \( 1 + \epsilon \) for any small \( \epsilon > 0 \) rather than 1 in the deducing of \( \frac{d}{dt}(Z(ta)) > 0 \), \( \forall t > 1 \); as for \( \frac{d}{dt}(Z(ta))|_{t=1} > 0 \), it has already been proved in Corollary 3.2, or actually can be viewed as a corollary of Lemma 3.5 or Lemma 3.6. Once \( \frac{d}{dt}(Z(ta)) > 0 \) (\( \forall t \geq 1 \)) is established, the higher order derivative inequalities can be obtained directly by Lemma 3.8) the rest assertions of this lemma are clear.

**Remark 3.1.** To see \( Z(ta) > 0 \) (\( \forall t > 1 \)) quickly, there is an easy point of view which will be stated as below, although we still believe that the above proof are of more value. By Lemma 3.4 and the proof of Lemma 3.6, we see that \( t \) is a root of the \( N \)-order polynomial \( Z(ta) \) if and only if \( H(ta) = \Theta + k\pi \). Since we now take \( \Theta \in [(n-2)\pi/2, n\pi/2) \), there are exactly \( N \) different such roots \( t \) and \( t = 1 \) is the largest one. By Lemma 3.4, we know that the leading coefficient \( c_N \) of \( Z(ta) \) is positive, thus \( Z(ta) > 0 \) (\( \forall t > 1 \)) is evident.

The following corollary of Lemma 3.7 can be viewed as an extension of Corollary 3.2.

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Corollary 3.3. Suppose $a \in L_\Theta \cap \Gamma^+$ with $(n-2)\pi/2 \leq \Theta < n\pi/2$. Then

$$\sum_{k=0}^{n} \xi_k(\Theta, a)c_k(\Theta)\sigma_k(a)t^{k-1} > 0, \forall t \geq 1.$$  

Proof. By (3.28) and Lemma 3.7 it is clear that

$$\sum \xi_k c_k \sigma_k t^{k-1} \geq \frac{1}{n} \sum c_k \sigma_k t^{k-1} > 0.$$  

\[\Box\]

4 Proof of the main theorem

4.1 Construction of the subsolutions

In this subsection, we prove the following key lemma and then use it to construct subsolutions of the special Lagrangian equation (1.1). Note that for the generalized radially symmetric subsolution $\Phi(x) = \phi(r)$ that we want to construct, the solution $\psi(r)$ discussed in the following lemma actually is corresponding to $\phi'(r)/r$, informally those major eigenvalues of the Hessian $D^2\Phi$ (see the proof of the Lemma 4.2).

Lemma 4.1. Assume $A \in A_\Theta$ with $(n-2)\pi/2 \leq \Theta < n\pi/2$ and $n \geq 3$. Suppose $a := (a_1, a_2, ..., a_n) := \lambda(A), 0 < a_1 \leq a_2 \leq ... \leq a_n$ and $\beta \geq 1$. Then the problem

$$\begin{align*}
\left\{ \begin{array}{l}
 r\psi^\prime(r) \sum_{k=0}^{n} \xi_k(\Theta, a)c_k(\Theta)\sigma_k(a)(\psi(r))^{k-1} \\
 + \sum_{k=0}^{n} c_k(\Theta)\sigma_k(a)(\psi(r))^k = 0, \quad r > 1, \\
 \psi(1) = \beta,
\end{array} \right. \tag{4.1}
\end{align*}$$

has a unique smooth solution $\psi(r) = \psi(r, \beta)$ on $[1, +\infty)$, which satisfies

(i) $1 \leq \psi(r, \beta) \leq \beta$, $\partial_r \psi(r, \beta) \leq 0$, $\forall r \geq 1$, $\forall \beta \geq 1$. More specifically, $\psi(r, 1) \equiv 1$, $\psi(1, \beta) \equiv \beta$; and $1 < \psi(r, \beta) < \beta$, $\forall r > 1$, $\forall \beta > 1$.  

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(ii) $\psi(r) \to 1 \ (r \to +\infty)$ and $r\psi'(r) \to 0 \ (r \to +\infty)$. Furthermore, $\psi(r, \beta) = 1 + O(r^{-m}) \ (r \to +\infty)$, where $m = m(\Theta, a) \in (2, n]$ and the $O(\cdot)$ depends only on $n, \Theta, \lambda(A)$ and $\beta$.

(iii) $\psi(r, \beta)$ is continuous and strictly increasing with respect to $\beta$ and 

$$\lim_{\beta \to +\infty} \psi(r, \beta) = +\infty, \ \forall r \geq 1.$$

Proof. For simplicity of notation, we will often write $\psi(r)$ or $\psi(r, \beta)$ (respectively, $c_k(\Theta)$, $\xi_k(\Theta, a)$, $\sigma_k(a)$) simply as $\psi$ (respectively, $c_k$, $\xi_k$, $\sigma_k$), when there is no confusion. The proof of this lemma now will be divided into three steps.

Step 1. In this step, we largely follow the strategy of the one in our previous paper [LL16], although the foundations of the statements are totally different.

We deduce from (4.1) that

$$\frac{d\psi}{dr} = -\frac{1}{r} \sum_{k=0}^{n} c_k(\Theta) \psi^k \frac{\xi_k(\Theta, a) \sigma_k(a) \psi^k}{r} = \frac{g(\psi)}{r},$$

where we set

$$g(\nu) := \frac{\sum_{k=0}^{n} c_k(\Theta) \psi^k \nu^k}{\sum_{k=0}^{n} \xi_k(\Theta, a) c_k(\Theta) \sigma_k(a) \psi^k \nu^{k-1}}.$$

Hence the problem (4.1) is equivalent to the following problem

$$\begin{cases} 
\psi'(r) = \frac{g(\psi(r))}{r}, \ r > 1, \\
\psi(1) = \beta.
\end{cases}$$

If $\beta = 1$, then $\psi(r) \equiv 1$ is a solution of the problem (4.3), since $g(1) = 0$ according to Lemma 3.7. Thus, by the uniqueness theorem for the solution of the ordinary differential equation, we know that $\psi(r, 1) \equiv 1$ is the unique solution satisfies the problem (4.3).

Now if $\beta > 1$, since

$$h(r, \nu) := \frac{g(\nu)}{r} \in C^\infty((1, +\infty) \times (\nu_0, +\infty)),$$
where $0 < \nu_0 < 1$ (note that $\nu_0$ exists, since we have
\[\sum_{k=0}^{n} \xi_k(\Theta, a)c_k(\Theta)\sigma_k(a)\nu^{k-1} > 0, \quad \forall \nu \geq 1,\]
according to Corollary 3.3, by the existence theorem (i.e., the Picard-Lindelöf theorem) and the theorem of the maximal interval of existence for the solution of the initial value problem of the ordinary differential equation, we know that the problem (4.3) has a unique smooth solution $\psi(r) = \psi(r, \beta)$ locally around the initial point and can be extended to a maximal interval $[1, \zeta)$, in which $\zeta$ can only be one of the following cases:

1. $\zeta = +\infty$;
2. $\zeta < +\infty$, $\psi(r)$ is unbounded on $[1, \zeta)$;
3. $\zeta < +\infty$, $(r, \psi(r))$ converges to some point on $\{\nu = \nu_0\}$ as $r \to \zeta$.

By Lemma 3.7 and Corollary 3.3, we see that
\[\frac{g(\psi(r))}{r} < 0, \quad \forall \psi(r) > 1.\]
Thus $\psi(r) = \psi(r, \beta)$ is strictly decreasing with respect to $r$, this excludes the case (2°) above.

We assert that the case (3°) can also be excluded. Otherwise, the solution curve must intersect with $\{\nu = 1\}$ at some point $(r_0, \psi(r_0))$ on it and then tends to $\{\nu = \nu_0\}$ after crossing it. But $\psi(r) \equiv 1$ is also a solution through $(r_0, \psi(r_0))$ which contradicts the uniqueness theorem for the solution of the initial value problem of the ordinary differential equation.

Thus we complete the proof of the existence and uniqueness of the solution $\psi(r) = \psi(r, \beta)$ of the problem (4.1) on $[1, +\infty)$.

According to the same reason, that is, $\psi(r, \beta)$ is strictly decreasing with respect to $r$ and the solution curve can not cross $\{\nu = 1\}$ provided $\beta > 1$, we see easily that $1 < \psi(r, \beta) < \beta, \forall r > 1, \forall \beta > 1$. Thus, assertion (i) of the lemma now is clear.

Step 2. In view of Lemma 3.6 and Lemma 3.7, all the roots of the polynomial $\sum_{k=0}^{n} c_k \sigma_k \psi^{k-1}$ are real (less than or equal to one) and simple. Suppose they are $1 > \psi_2 > ... > \psi_N$, where
\[N = N(n, \Theta) \triangleq \begin{cases} n - 1, & \text{when } \Theta = (n - 2)\pi/2, \\ n, & \text{when } (n - 2)\pi/2 < \Theta < n\pi/2, \end{cases}\]
which has been defined in (3.34) in Lemma 3.4. We then have

\[ \sum_{k=0}^{n} c_k \sigma_k \psi^{k-1} = K(\psi - 1)(\psi - \psi_2)...(\psi - \psi_N) \]

for some \( K > 0 \) (according to Lemma 3.7), and

\[ \frac{\sum_{k=0}^{n} \xi_k c_k \sigma_k \psi^{k-1}}{\sum_{k=0}^{n} c_k \sigma_k \psi^k} = \frac{K_1}{\psi - 1} + \frac{K_2}{\psi - \psi_2} + ... + \frac{K_N}{\psi - \psi_N} \tag{4.4} \]

for some \( K_1, K_2, ..., K_N \in \mathbb{R} \). It is easy to show that \( K_1 = 1/m \), where

\[ m = m(\Theta, a) \triangleq \frac{\sum_{k=0}^{n} k c_k \sigma_k}{\sum_{k=0}^{n} \xi_k c_k \sigma_k} = -g'(1), \]

which has been defined in (3.32) in Subsection 3.2 (for \( g'(1) \) see also (4.7)). Indeed, if there are polynomials \( P, Q, R \) on \( \mathbb{R} \) and \( K \in \mathbb{R} \) such that \( Q(1) \neq 0 \) and

\[ \frac{P(x)}{(x - 1)Q(x)} = \frac{K}{x - 1} + \frac{R(x)}{Q(x)}, \]

we then have

\[ P(x) = KQ(x) + (x - 1)R(x). \]

Letting \( x = 1 \), we conclude that

\[ K = \frac{P(1)}{Q(1)} = \frac{P(1)}{\frac{d}{dx}((x - 1)Q(x))|_{x=1}}, \]

which gives formula to calculate \( K_1 \).

Combining (4.2) and (4.4), we have

\[ -d \ln r = -\frac{dr}{r} = \frac{\xi_k c_k \sigma_k \psi^{k-1}}{\sum_{k=0}^{n} c_k \sigma_k \psi^k} d\psi \]

\[ = \left( \frac{K_1}{\psi - 1} + \frac{K_2}{\psi - \psi_2} + ... + \frac{K_N}{\psi - \psi_N} \right) d\psi. \]

Since \( K_1 = 1/m \), we deduce that

\[ d \ln (r^{-m}) = -md \ln r = \left( \frac{1}{\psi - 1} + \frac{mK_2}{\psi - \psi_2} + ... + \frac{mK_N}{\psi - \psi_N} \right) d\psi \]
\[ d \ln \left( (\psi - 1)(\psi - \psi_2)^{mK_2} \cdots (\psi - \psi_N)^{mK_N} \right). \]  

Integrating it from 1 to \( r \) and recalling \( \psi(1) = \beta \geq 1 \), we obtain

\[
\ln \left( (\psi(r) - 1)(\psi(r) - \psi_2)^{mK_2} \cdots (\psi(r) - \psi_N)^{mK_N} \right) = \ln \left( (\beta - 1)(\beta - \psi_2)^{mK_2} \cdots (\beta - \psi_N)^{mK_N} \right) + \ln(r^{-m}),
\]

and hence

\[
(\psi(r) - 1)(\psi(r) - \psi_2)^{mK_2} \cdots (\psi(r) - \psi_N)^{mK_N} = (\beta - 1)(\beta - \psi_2)^{mK_2} \cdots (\beta - \psi_N)^{mK_N} r^{-m} =: (\beta - 1)B(\beta)r^{-m},
\]

where we set

\[
B(\nu) := (\nu - \psi_2)^{mK_2} \cdots (\nu - \psi_N)^{mK_N}.
\]

Thus

\[
\frac{\psi(r, \beta) - 1}{r^{-m}} = \frac{(\beta - 1)B(\beta)}{B(\psi(r, \beta))}.
\]

Since \( \psi_N < \psi_{N-1} < \cdots < \psi_2 < 1 \) and \( 1 \leq \psi(r, \beta) \leq \beta, \forall r \geq 1 \), we have

\[
0 \leq \frac{\psi(r, \beta) - 1}{r^{-m}} \leq C(n, \Theta, \lambda(A), \beta), \forall r \geq 1.
\]

Thus we get

\[
\psi(r, \beta) \to 1 (r \to +\infty), \forall \beta \geq 1.
\]

Substituting it into (4.6), we deduce that

\[
\frac{\psi(r, \beta) - 1}{r^{-m}} \to \frac{(\beta - 1)B(\beta)}{B(1)} (r \to +\infty), \forall \beta \geq 1.
\]

Therefore

\[
\psi(r, \beta) = 1 + \frac{(\beta - 1)B(\beta)}{B(1)} r^{-m} + o(r^{-m}) = 1 + O(r^{-m}) (r \to +\infty),
\]

where \( o(\cdot) \) and \( O(\cdot) \) depend only on \( n, \Theta, \lambda(A) \) and \( \beta \). Note also that \( r\psi'(r) = g(\psi(r)) \to g(1) = 0 (r \to +\infty) \). Thus the assertion (ii) of the lemma is proved.
Step 3. By the theorem of the differentiability of the solution with respect to the initial value, we can differentiate $\psi(r, \beta)$ with respect to $\beta$ as below:

$$
\begin{align*}
\frac{\partial \psi(r, \beta)}{\partial r} &= \frac{g(\psi(r, \beta))}{r}, \\
\psi(1, \beta) &= \beta;
\end{align*}
$$

$$
\Rightarrow \begin{cases}
\frac{\partial^2 \psi(r, \beta)}{\partial \beta \partial r} &= \frac{g'(\psi(r, \beta))}{r} \cdot \frac{\partial \psi(r, \beta)}{\partial \beta}, \\
\frac{\partial \psi(1, \beta)}{\partial \beta} &= 1.
\end{cases}
$$

Let

$$v(r) := \frac{\partial \psi(r, \beta)}{\partial \beta}.$$

We have

$$
\begin{align*}
\frac{dv}{dr} &= \frac{g'(\psi(r, \beta))}{r} \cdot v, \\
v(1) &= 1.
\end{align*}
$$

Therefore we can deduce that

$$
\frac{dv}{v} = \frac{g'(\psi(r, \beta))}{r} dr,
$$

and hence

$$
\frac{\partial \psi(r, \beta)}{\partial \beta} = v(r) = \exp \int_{1}^{r} \frac{g'(\psi(\tau, \beta))}{\tau} d\tau.
$$

By calculation, we have

$$
g'(\nu) = -\frac{1}{\left(\sum_{k=0}^{n} \xi_k c_k \sigma_k \nu^{k-1}\right)^2} \left( \sum_{k=0}^{n} k c_k \sigma_k \nu^{k-1} \sum_{k=0}^{n} \xi_k c_k \sigma_k \nu^{k-1} - \sum_{k=0}^{n} c_k \sigma_k \nu^{k} \sum_{k=0}^{n} (k-1) \xi_k c_k \sigma_k \nu^{k-2} \right).
$$

\(4.7\)

Since $\sum_{k=0}^{n} \xi_k c_k \sigma_k \nu^{k-1} > 0 \ (\forall \nu \geq 1)$, $1 \leq \psi(r, \beta) \leq \beta \ (\forall r \geq 1, \ \forall \beta \geq 1)$, $g'(1) = -m \in [-n, -2)$ and

$$
\lim_{\nu \to +\infty} g'(\nu) = -\frac{1}{\xi_N} \in \left[-\frac{n}{n-1}, -1\right],
$$

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we obtain
\[ |g'(r, \beta)| \leq C(n, \Theta, \lambda(A)) < +\infty, \forall r \geq 1, \]
and hence
\[ 0 < \frac{\partial \psi(r, \beta)}{\partial \beta} \leq r^C, \forall r \geq 1. \]
In particular, we see that \( \psi(r, \beta) \) is strictly increasing with respect to \( \beta \).

Now we come to prove
\[ \lim_{\beta \to +\infty} \psi(r, \beta) = +\infty, \forall r \geq 1, \]
by contradiction. Suppose not. There would exist \( r_0 \geq 1, M > 1 \) and \( \{\beta_k\}_{k=1}^\infty \) such that \( \psi(r_0, \beta_k) \leq M, \forall k \in \mathbb{Z}^+ \). Note that there are infinitely many \( \beta_k > M \) satisfying \( 1 \leq \psi(r_0, \beta_k) \leq M < \beta_k \).

Since
\[ \frac{d\psi}{dr} = \frac{g(\psi)}{r}, \]
where
\[ g(\nu) = -\sum c_k \sigma_k \nu^k \sum \xi_k c_k \sigma_k \nu^{k-1}, \]
satisfies \( g(1) = 0 \) and \( 0 < -g(\nu) < C\nu (\forall \nu > 1) \) with \( C = C(n, \Theta, \lambda(A)) > 0 \), we have
\[ \frac{d\psi}{C\psi} \leq \frac{d\psi}{-g(\psi)} = -\frac{dr}{r}. \]
Integrating it from \( M \) to \( \beta_k \) and recalling \( \psi(1, \beta_k) = \beta_k \), we get
\[ \int_M^{\beta_k} \frac{d\psi}{C\psi} \leq \int_M^{\beta_k} \frac{d\psi}{-g(\psi)} \leq \int_{\psi(r_0, \beta_k)}^{\beta_k} \frac{d\psi}{-g(\psi)} = -\int_{r_0}^{1} \frac{dr}{r} = \ln r_0 < +\infty. \]

Let \( \beta_k \to +\infty \), we have
\[ \int_M^{\beta_k} \frac{d\psi}{C\psi} \to +\infty, \]
which is a contradiction. Hence the assertion (iii) of the lemma is proved, and we thus complete the proof of the whole lemma. \[\square\]

Set
\[ \mu_R(\beta) := \int_R^{+\infty} \tau(\psi(\tau, \beta) - 1) d\tau, \quad \forall R \geq 1, \forall \beta \geq 1. \]
Note that the integral on the right hand side is convergent, in view of Lemma 4.1 (ii). Moreover, as an application of Lemma 4.1, we have the following corollary.

**Corollary 4.1.** \( \mu_R(\beta) \) is nonnegative, continuous and strictly increasing with respect to \( \beta \). Furthermore,

\[
\mu_R(\beta) = O(R^{-m+2}) \ (R \to +\infty), \ \forall \beta \geq 1;
\]

and

\[
\mu_R(\beta) \to +\infty \ (\beta \to +\infty), \ \forall R \geq 1. \quad (4.8)
\]

**Proof.** The proof is straightforward in light of Lemma 4.1 (ii), (iii). \( \square \)

For any \( \alpha, \beta, \gamma \in \mathbb{R}, \beta, \gamma \geq 1 \) and for any diagonal matrix \( A \in \mathcal{A}_\Theta \), let

\[
\phi(r) := \phi_{\alpha,\beta,\gamma}(r) := \alpha + \int_\gamma^r \tau \psi(\tau, \beta) d\tau, \ \forall r \geq \gamma, \quad (4.9)
\]

and let

\[
\Phi(x) := \Phi_{\alpha,\beta,\gamma,A}(x) := \phi(r) := \phi_{\alpha,\beta,\gamma}(r_A(x)), \ \forall x \in \mathbb{R}^n \setminus E_\gamma, \quad (4.10)
\]

where \( r = r_A(x) = \sqrt{x^T A x} \). Then

\[
\phi_{\alpha,\beta,\gamma}(r) = \int_\gamma^r \tau (\psi(\tau, \beta) - 1) d\tau + \frac{1}{2} r^2 - \frac{1}{2} \gamma^2 + \alpha
\]

\[
= \frac{1}{2} r^2 + \left( \mu_\gamma(\beta) + \alpha - \frac{1}{2} \gamma^2 \right) - \mu_r(\beta) \quad (4.11)
\]

\[
= \frac{1}{2} r^2 + \left( \mu_\gamma(\beta) + \alpha - \frac{1}{2} \gamma^2 \right) + O(r^{-m+2}) \ (r \to +\infty), \quad (4.12)
\]

according to Corollary 4.1. Furthermore, we have

**Lemma 4.2.** \( \Phi \) is a smooth subsolution of (1.1) in \( \mathbb{R}^n \setminus \overline{E_\gamma} \), that is,

\[
\sum_{i=1}^n \arctan \lambda_i \left( D^2 \Phi(x) \right) \geq \Theta, \ \forall x \in \mathbb{R}^n \setminus \overline{E_\gamma}. \quad (4.13)
\]
Proof. It is clear that $\phi'(r) = r\psi(r)$ and $\phi''(r) = \psi(r) + r\psi'(r)$. Since

$$r^2 = x^T Ax = \sum_{i=1}^{n} a_i x_i^2,$$

we have

$$2r\partial_x r = \partial_x (r^2) = 2a_i x_i \quad \text{and} \quad \partial_x r = \frac{a_i x_i}{r}.$$

Thus

$$\partial_x \Phi(x) = \phi'(r) \partial_x r = \frac{\phi'(r)}{r} a_i x_i,$$

and

$$\partial_{x_i x_j} \Phi(x) = \frac{\phi'(r)}{r} a_i \delta_{ij} + \frac{\phi''(r) - \psi'(r)}{r^2} (a_i x_i)(a_j x_j)$$

$$= \psi(r) a_i \delta_{ij} + \frac{\psi'(r)}{r} (a_i x_i)(a_j x_j).$$

Therefore

$$D^2 \Phi(x) = \left( \psi(r) a_i \delta_{ij} + \frac{\psi'(r)}{r} (a_i x_i)(a_j x_j) \right)_{n \times n}.$$  \hfill (4.14)

Applying the formula Lemma 2.1, we compute

$$\sigma_k(\lambda(D^2\Phi))(x) = \sigma_k(a) \psi(r)^k + \frac{\psi'(r)}{r} \psi(r)^{k-1} \sigma_{k-1}(a) a_i^2 x_i^2$$

$$= \sigma_k(a) \psi^k + \Xi_k(a, x) \sigma_k(a) r\psi^{k-1}\psi'.$$

Thus we deduce, for all $x \in \mathbb{R}^n \setminus E$, that

$$Z(\lambda(D^2\Phi))(x) = \cos \Theta \; Y(\lambda(D^2\Phi)) - \sin \Theta \; X(\lambda(D^2\Phi))$$

$$= \cos \Theta \sum_{0 \leq 2j+1 \leq n} (-1)^j \sigma_{2j+1}(\lambda(D^2\Phi)) - \sin \Theta \sum_{0 \leq 2j \leq n} (-1)^j \sigma_{2j}(\lambda(D^2\Phi))$$

$$= \sum_{k=0}^{n} c_k(\Theta) \sigma_k(\lambda(D^2\Phi))$$

$$= \sum_{k=0}^{n} c_k(\Theta) \left( \sigma_k(a) \psi^k + \Xi_k(a, x) \sigma_k(a) r\psi^{k-1}\psi' \right).$$

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\[
\sum_{k=0}^{n} c_k(\Theta) \sigma_k(a) \psi^k + \sum_{k=0}^{n} \Xi_k(a, x) c_k(\Theta) \sigma_k(a) r \psi^{k-1} \psi' \\
\geq \sum_{k=0}^{n} c_k(\Theta) \sigma_k(a) \psi^k + \sum_{k=0}^{n} \xi_k(\Theta, a) c_k(\Theta) \sigma_k(a) r \psi^{k-1} \psi' \\
= \sum_{k=0}^{n} c_k(\Theta) \sigma_k(a) \psi^k + \sum_{k=0}^{n} \xi_k(\Theta, a) c_k(\Theta) \sigma_k(a) \psi^{k-1} \\
= 0,
\]

in which we have used the facts that \(\Xi_k(a, x) c_k(\Theta) \leq \xi_k(\Theta, a) c_k(\Theta)\) for all \(k \geq 0\), according to (3.27), and that \(\psi(r) \geq 1 > 0\) and \(\psi'(r) \leq 0\) for all \(r \geq 1\), according to Lemma 4.1 (i).

Since \(\psi(r) \to 1 (r \to +\infty)\) and \(r \psi'(r) \to 0 (r \to +\infty)\) by Lemma 4.1 in light of (4.14), we have

\[
\begin{align*}
D^2\Phi(x) &\to \text{diag}\{a_1, a_2, \ldots, a_n\} \\
\lambda(D^2\Phi(x)) &\to a \\
H(\lambda(D^2\Phi(x))) &\to H(a) = \Theta
\end{align*}
\]

as \(r_A(x) \to +\infty\). (4.15)

Note that, for any \(\Theta \in [(n-2)\pi/2, n\pi/2)\), \(L_\Theta := \{\lambda \in \mathbb{R}^n \mid H(\lambda) = \Theta\}\) is one of the \(N = N(n, \Theta)\) mutually disjoint components of

\(L^*_\Theta := \{\lambda \in \mathbb{R}^n \mid Z(\lambda) = \Theta\}\),

and it is also the last one (or say, the outermost one) in the \(1\)-direction. Since \(Z(t1)\) is a polynomial of \(N\) order, which has exactly \(N\) real and simple roots, and approaches to +\(\infty\) as \(t \to +\infty\), we conclude that

\[
Z(\lambda) \begin{cases} 
> 0 & \text{if } H(\lambda) > \Theta, \\
< 0 & \text{if } \Theta + \pi < H(\lambda) < \Theta.
\end{cases}
\] (4.16)

Combing (4.15), (4.16) with \(Z(\lambda(D^2\Phi)) \geq 0\) we derived above, by the continuity of \(\Phi\), we obtain

\[
\sum_{i=1}^{n} \arctan \lambda_i(D^2\Phi(x)) = H(\lambda(D^2\Phi(x))) \geq \Theta, \quad \forall x \in \mathbb{R}^n \setminus \overline{E_\gamma}.
\]

This completes the proof of Lemma 4.2.
4.2 Lemma 4.3 implies Theorem 1.1

In this subsection, we introduce the following lemma which is a special and simple case of Theorem 1.1 with the additional condition that the matrix $A$ is diagonal and the vector $b$ vanishes.

**Lemma 4.3.** Let $D$ be a bounded strictly convex domain in $\mathbb{R}^n$, $n \geq 3$, $\partial D \in C^2$ and let $\varphi \in C^2(\partial D)$. Then for any given diagonal matrix $A \in \mathcal{A}_\Theta$ with $(n-2)\pi/2 \leq \Theta < n\pi/2$, there exists a constant $c_*$ depending only on $n, D, \Theta, A$ and $\|\varphi\|_{C^2(\partial D)}$, such that for every $c \geq c_*$, there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ of

$$
\begin{cases}
\sum_{i=1}^n \arctan \lambda_i \left(D^2 u \right) = \Theta & \text{in } \mathbb{R}^n \setminus \overline{D}, \\
\frac{1}{2} x^T A x + c & \text{on } \partial D,
\end{cases}
$$

where $m = m(\Theta, A) \in (2, n]$.

To prove Theorem 1.1, it suffices to prove Lemma 4.3. Indeed, suppose that $D, \varphi, A$ and $b$ satisfy the hypothesis of Theorem 1.1. Consider the decomposition $A = Q^T \Lambda Q$, where $Q$ is an orthogonal matrix and $\Lambda$ is a diagonal matrix which satisfies $\lambda(\Lambda) = \lambda(A)$. Let

$$
\tilde{x} := Qx, \quad \tilde{D} := \{Qx | x \in D\}
$$

and

$$
\tilde{\varphi}(\tilde{x}) := \varphi(x) - b^T x = \varphi(Q^T \tilde{x}) - b^T Q^T \tilde{x}.
$$

By Lemma 4.3, we conclude that there exists a constant $c_*$ depending only on $n, D, \Theta, \Lambda$ and $\|\tilde{\varphi}\|_{C^2(\partial \tilde{D})}$, such that for every $c \geq c_*$, there exists a unique viscosity solution $\tilde{u} \in C^0(\mathbb{R}^n \setminus \tilde{D})$ of

$$
\begin{cases}
\sum_{i=1}^n \arctan \lambda_i \left(D^2 \tilde{u} \right) = \Theta & \text{in } \mathbb{R}^n \setminus \overline{\tilde{D}}, \\
\tilde{u} = \tilde{\varphi} & \text{on } \partial \tilde{D},
\end{cases}
$$

where $m = m(\Theta, \lambda(A)) \in (2, n]$. 

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where $m = m(\Theta, \lambda(A)) = m(\Theta, \lambda(A)) \in (2, n]$. Let

$$u(x) := \tilde{u}(\tilde{x}) + b^T x = \tilde{u}(Qx) + b^T x = \tilde{u}(\tilde{x}) + b^T Q^T \tilde{x}. \tag{1.7}$$

We assert that $u$ is the solution of (1.7) in Theorem 1.1. To show this, we need only to note that

$$D^2 u(x) = Q^T D^2 \tilde{u}(\tilde{x}) Q, \quad \lambda(D^2 u(x)) = \lambda(D^2 \tilde{u}(\tilde{x}));$$

and

$$\eta = \inf_{x \in E \setminus D} Q_\xi(x), \quad Q(x) := \sup_{\xi \in \partial D} Q_\xi(x)$$

and

$$\Phi_\beta(x) := \eta + \int_{\tau}^{r_A(x)} \tau \psi(\tau, \beta) d\tau, \quad \forall r_A(x) \geq 1, \forall \beta \geq 1,$$

where $Q_\xi(x)$ and $\psi(\tau, \beta)$ are given by Lemma 2.2 and Lemma 4.1, respectively. Then we have

4.3 Proof of Lemma 4.3

Now we use the Perron’s method to prove Lemma 4.3. The procedures employed here are standard and quite similar to the ones in our previous paper [LL16]. For the reader’s convenience, we give the full details.

Proof of Lemma 4.3. We may assume without loss of generality that $E_1 \subset D \subset E \subset E$ and $a := (a_1, a_2, ..., a_n) := \lambda(A)$ with $0 < a_1 \leq a_2 \leq ... \leq a_n$. The proof now will be divided into three steps.

Step 1. Let

$$\eta := \inf_{x \in E \setminus D} Q_\xi(x), \quad Q(x) := \sup_{\xi \in \partial D} Q_\xi(x)$$

and

$$\Phi_\beta(x) := \eta + \int_{\tau}^{r_A(x)} \tau \psi(\tau, \beta) d\tau, \quad \forall r_A(x) \geq 1, \forall \beta \geq 1,$$

where $Q_\xi(x)$ and $\psi(\tau, \beta)$ are given by Lemma 2.2 and Lemma 4.1, respectively. Then we have
(1) Since $Q$ is the supremum of a collection of smooth solutions $\{Q_{\xi}\}$ of (1.1), it is a continuous subsolution of (1.1), i.e.,

$$\sum_{i=1}^{n} \arctan \lambda_i (D^2 Q) \geq \Theta$$

in $\mathbb{R}^n \setminus \bar{D}$ in the viscosity sense (see [Ish89, Proposition 2.2]).

(2) $Q = \varphi$ on $\partial D$. To prove this we need only to show that for any $\xi \in \partial D$, $Q(\xi) = \varphi(\xi)$. This is obvious since $Q_{\xi} \leq \varphi$ on $\bar{D}$ and $Q_{\xi}(\xi) = \varphi(\xi)$, according to Remark 2.1-(1).

(3) By Lemma 4.2, $\Phi_\beta$ is a smooth subsolution of (1.1) in $\mathbb{R}^n \setminus \bar{D}$.

(4) $\Phi_\beta \leq \varphi$ on $\partial D$ and $\Phi_\beta \leq Q$ on $\bar{E}_r \setminus D$. To show them we note that $\Phi_\beta(x)$ is strictly increasing with respect to $r_A(x)$ since $\psi(r, \beta) \geq 1 > 0$ by Lemma 4.1-(i). Invoking $\Phi_\beta = \eta$ on $\partial E_r$ and $\eta \leq Q$ on $\bar{E}_r \setminus D$ by their definitions, we have $\Phi_\beta \leq \eta \leq Q$ on $\bar{E}_r \setminus D$. On the other hand, according to Remark 2.1-(1), we have $Q_{\xi} \leq \varphi$ on $\bar{D}$ which implies that $\eta \leq \varphi$ on $\bar{D}$. Combining these two aspects we deduce that $\Phi_\beta \leq \eta \leq \varphi$ on $\partial D$.

(5) $\Phi_\beta(x)$ is strictly increasing with respect to $\beta$ and

$$\lim_{\beta \to +\infty} \Phi_\beta(x) = +\infty, \quad \forall r_A(x) \geq 1,$$

(4.19)

by the definition of $\Phi_\beta(x)$ and Lemma 4.1-(iii).

(6) As showed in (4.11) and (4.12), for any $\beta \geq 1$, we have

$$\Phi_\beta(x) = \eta + \int_{\bar{r}}^{r_A(x)} \tau \psi(\tau, \beta) d\tau$$

$$= \eta + \frac{1}{2}(r_A(x)^2 - r^2) + \int_{\bar{r}}^{r_A(x)} \tau (\psi(\tau, \beta) - 1) d\tau$$

$$= \frac{1}{2} r_A(x)^2 + \left( \eta - \frac{1}{2} r^2 + \mu_\psi(\beta) \right) - \mu_{r_A(x)}(\beta)$$

$$= \frac{1}{2} r_A(x)^2 + \mu(\beta) - \mu_{r_A(x)}(\beta)$$

$$= \frac{1}{2} x^T A x + \mu(\beta) + O \left( |x|^{-m+2} \right) \ (|x| \to +\infty),$$
where we set
\[ \mu(\beta) := \eta - \frac{1}{2} r^2 + \mu_r(\beta), \]
and used the fact that \( x^T Ax = O(|x|^2) \ (|x| \to +\infty) \) since \( \lambda(A) \in \Gamma^+ \).

**Step 2.** For fixed \( \hat{r} > \bar{r} \), there exists \( \hat{\beta} > 1 \) such that
\[
\min_{\partial E_{\hat{r}}} \Phi_{\hat{\beta}} > \max_{\partial E_{\hat{r}}} \Phi > Q,
\]
in light of (4.19). Thus we obtain
\[
\Phi_{\hat{\beta}} > Q \quad \text{on } \partial E_{\hat{r}}. \tag{4.20}
\]

Let
\[ c_* := \max \{ \eta, \mu(\hat{\beta}), \bar{c} \}, \]
where the \( \bar{c} \) comes from Remark 2.1 (3), and hereafter fix \( c \geq c_* \).

By Lemma 4.1 and Corollary 4.1 we deduce that
\[
\psi(r, 1) \equiv 1 \Rightarrow \mu_r(1) = 0 \Rightarrow \mu(1) = \eta - \frac{1}{2} r^2 < \eta \leq c_* \leq c,
\]
and
\[
\lim_{\beta \to +\infty} \mu_r(\beta) = +\infty \Rightarrow \lim_{\beta \to +\infty} \mu_r(\beta) = +\infty.
\]
On the other hand, it follows from Corollary 4.1 that \( \mu_r(\beta) \) is continuous and strictly increasing with respect to \( \beta \) (which indicates that the inverse of \( \mu_r(\beta) \) exists and \( \mu_r^{-1} \) is strictly increasing). Thus there exists a unique \( \beta(c) \) such that \( \mu_r(\beta(c)) = c \). Then we have
\[
\Phi_{\beta(c)}(x) = \frac{1}{2} r_x(x)^2 + c - \mu_r(x)(\beta(c)) = \frac{1}{2} x^T A x + c + O(|x|^{-m+2}) \ (|x| \to +\infty),
\]
and
\[
\beta(c) = \mu_r^{-1}(c) \geq \mu_r^{-1}(c_*) \geq \hat{\beta}.
\]
Invoking the monotonicity of \( \Phi_{\beta} \) with respect to \( \beta \) and (4.20), we obtain
\[
\Phi_{\beta(c)} \geq \Phi_{\hat{\beta}} > Q \quad \text{on } \partial E_{\hat{r}}. \tag{4.21}
\]
Note that we already know
\[
\Phi_{\beta(c)} \leq Q \quad \text{on } \overline{E_{\hat{r}}} \setminus D,
\]
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from (4) of Step 1.

Let
\[
\underline{u}(x) := \begin{cases} 
\max \{ \Phi_{\beta(c)}(x), Q(x) \}, & x \in E_\beta \setminus D, \\
\Phi_{\beta(c)}(x), & x \in \mathbb{R}^n \setminus E_\beta.
\end{cases}
\]

Then we have

(1) \( \underline{u} \) is continuous and satisfies
\[
\frac{\sigma_k(\lambda(D^2\underline{u}))}{\sigma_1(\lambda(D^2\underline{u}))} \geq 1
\]
in \( \mathbb{R}^n \setminus \overline{D} \) in the viscosity sense, by (1) and (3) of Step 1.

(2) \( \underline{u} = Q = \varphi \) on \( \partial D \), by (2) of Step 1.

(3) If \( r_A(x) \) is large enough, then
\[
\underline{u}(x) = \Phi_{\beta(c)}(x) = \frac{1}{2} x^T A x + c + O \left( |x|^{-m+2} \right) \ (|x| \to +\infty).
\]

Step 3. Let
\[
\overline{u}(x) := \frac{1}{2} x^T A x + c, \ \forall x \in \mathbb{R}^n.
\]

Then \( \overline{u} \) is obviously a supersolution and
\[
\lim_{|x| \to +\infty} (\underline{u} - \overline{u})(x) = 0.
\]

To use the Perron’s method to establish Lemma 4.3 we now need only to prove that
\[
\underline{u} \leq \overline{u} \ \text{in} \ \mathbb{R}^n \setminus D.
\]

In fact, since
\[
\mu_{r_A(x)}(\beta) \geq 0, \ \forall x \in \mathbb{R}^n \setminus E_1, \ \forall \beta \geq 1,
\]
according to Corollary 4.1 we have
\[
\Phi_{\beta(c)}(x) = \frac{1}{2} x^T A x + c - \mu_{r_A(x)}(\beta(c)) \leq \frac{1}{2} x^T A x + c = \overline{u}(x), \ \forall x \in \mathbb{R}^n \setminus D.
\]

(4.22)
On the other hand, for every $\xi \in \partial D$, since
\[
Q_\xi(x) \leq \frac{1}{2} x^T Ax + \bar{c} \leq \frac{1}{2} x^T Ax + c^\star \leq \frac{1}{2} x^T Ax + c = \overline{u}(x), \ \forall x \in \partial D,
\]
and
\[
Q_\xi \leq Q < \Phi_{\beta(c)} \leq \overline{u} \quad \text{on } \partial E_{\hat{r}}
\]
follows from (4.21) and (4.22), we obtain
\[
Q_\xi \leq \overline{u} \quad \text{on } \partial (E_{\hat{r}} \setminus D).
\]
In view of
\[
\sum_{i=1}^n \arctan \lambda_i \left(D^2 Q_\xi \right) = \Theta = \sum_{i=1}^n \arctan \lambda_i \left(D^2 \underline{u} \right) \quad \text{in } E_{\hat{r}} \setminus D,
\]
we deduce from the comparison principle that
\[
Q_\xi \leq \overline{u} \quad \text{in } E_{\hat{r}} \setminus D.
\]
Hence
\[
Q \leq \overline{u} \quad \text{in } E_{\hat{r}} \setminus D. \tag{4.23}
\]
Combining (4.22) and (4.23), by the definition of $\underline{u}$, we get
\[
\underline{u} \leq \overline{u} \quad \text{in } \mathbb{R}^n \setminus D.
\]
This finishes the proof of Lemma 4.3.

5 Appendix: proof of Lemma 3.3

Lemma 5.1 (Lemma 3.3). For any $a \in \mathbb{R}^n$,
\[
\hat{Z}_\star(a) := X(a)\hat{Y}(a) - Y(a)\hat{X}(a) = \sum_{k=1}^n \sum_{1 \leq i_1, i_2, \ldots, i_k \leq n} a_{i_1}^2 a_{i_2}^2 \cdots a_{i_{k-1}}^2 a_{i_k}^2. \tag{5.1}
\]
Recall that we have defined
\[
X(\lambda) := \sum_{0 \leq 2j \leq n} (-1)^j \sigma_{2j}(\lambda)
\]
\[ Y(\lambda) := \sum_{0 \leq 2j+1 \leq n} (-1)^j \sigma_{2j+1}(\lambda) \]
\[ = \sigma_1(\lambda) - \sigma_3(\lambda) + \sigma_5(\lambda) - \ldots, \]

\[ \hat{X}(\lambda) := \sum_{0 \leq 2j \leq n} (-1)^j \cdot (2j) \cdot \sigma_{2j}(\lambda) \]
\[ = -2\sigma_2(\lambda) + 4\sigma_4(\lambda) - \ldots \]

and

\[ \hat{Y}(\lambda) := \sum_{0 \leq 2j+1 \leq n} (-1)^j \cdot (2j + 1) \cdot \sigma_{2j+1}(\lambda) \]
\[ = \sigma_1(\lambda) - 3\sigma_3(\lambda) + 5\sigma_5(\lambda) - \ldots, \]

**Proof of Lemma [5.1]** The proof will be divided into four steps.

**Step 1.** We have

\[ \hat{Z}_s(a) = \sum_{0 \leq 2i \leq n} (-1)^i \sigma_{2i}(a) \sum_{0 \leq 2j+1 \leq n} (-1)^j (2j + 1) \sigma_{2j+1}(a) \]
\[ - \sum_{0 \leq 2i \leq n} (-1)^i (2i) \sigma_{2i}(a) \sum_{0 \leq 2j+1 \leq n} (-1)^j \sigma_{2j+1}(a) \]
\[ = \sum_{0 \leq 2i \leq n} \sum_{0 \leq 2j+1 \leq n} (-1)^{i+j} (2j - 2i + 1) \sigma_{2i}(a) \sigma_{2j+1}(a) \]
\[ = \sum_{p=0}^{\hat{n}} \sum_{q=0}^{n-1} (-1)^q (2q + 1) \sigma_{p-q}(a) \sigma_{p+q+1}(a) \]
\[ + \sum_{p=\hat{n}+1}^{n-1} \sum_{q=0}^{n-1-p} (-1)^q (2q + 1) \sigma_{p-q}(a) \sigma_{p+q+1}(a), \] (5.2)

where

\[ \hat{n} := \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd}, \\ \frac{n-2}{2} & \text{if } n \text{ is even}, \end{cases} \] (5.3)
and the last equality follows from the rearrangement (see for example Figure 1) realized by

\[
\begin{cases}
    p = i + j, \\
    q = \begin{cases}
        j - i & \text{if } i \leq j, \\
        i - j - 1 & \text{if } i > j,
    \end{cases}
\end{cases}
\]  

(5.4)

or, equivalently, by

\[
\begin{cases}
    i = \frac{p - q}{2} \\
    j = \frac{p + q}{2}
\end{cases}
\]  

if \( p + q \) is even,

(5.5)

in the opposite direction. Note that if \( 0 \leq i \leq j \), then \( p := i + j \geq 0, q := j - i \geq 0, 2j + 1 > 2i, (-1)^{i+j} = (-1)^{j-i} = (-1)^q, 2j - 2i + 1 = 2q + 1, 2i = p - q \) and \( 2j + 1 = p + q + 1 \); if \( 0 \leq j < i \), then \( p := i + j \geq 0, q := i - j - 1 \geq 0, 2j + 1 < 2i, (-1)^{i+j} = (-1)^{i-j-1} = -(-1)^q, 2j - 2i + 1 = -(2i - 2j - 1) = -(2q + 1), 2i = p + q + 1 \) and \( 2j + 1 = p - q \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{rearrangement.png}
\caption{Rearrangement for odd and/or even dimension}
\end{figure}
Thus, in order to calculate \( \tilde{Z}_s(a) \), we need only to consider

\[
\sum_{q=0}^{\hat{p}} (-1)^q (2q + 1) \sigma_{p-q}(a) \sigma_{p+q+1}(a) \tag{5.6}
\]

for all \( 0 \leq p \leq n - 1 \), where

\[
\hat{p} := \begin{cases} p, & p \leq \frac{n-1}{2}, \\ n - 1 - p, & p > \frac{n-1}{2}. \end{cases}
\]

Note that the \( \hat{p} \) here, unlike the \( \hat{n} \) in (5.3), is independent of the parity (oddness or evenness) of the dimension \( n \).

The calculation of (5.6) will be given in Step 4. To do this, we need some other preparations which will be presented in the following two steps.

Step 2. We now prove that

\[
\sum_{q=0}^{Q} (-1)^q (2q + 1) C_{2Q+1}^{Q-q} = \begin{cases} 0 & \text{if } Q \geq 1, \\ 1 & \text{if } Q = 0. \end{cases} \tag{5.7}
\]

For \( Q = 0 \), (5.7) is obviously true. So we may assume that \( Q \geq 1 \). Then

\[
\sum_{q=0}^{Q} (-1)^q (2q + 1) C_{2Q+1}^{Q-q}
\]

\[
= (2Q + 1) \left( \sum_{q=0}^{Q} (-1)^q C_{2Q}^{Q+q} + \sum_{q=0}^{Q} (-1)^{q+1} C_{2Q}^{Q+q+1} \right)
\]

\[
= (2Q + 1) \left( C_{2Q}^{Q} + \sum_{q=1}^{Q} (-1)^q C_{2Q}^{Q+q} + \sum_{q=0}^{Q-1} (-1)^{q+1} C_{2Q}^{Q+q+1} \right)
\]

\[
= (2Q + 1) \left( C_{2Q}^{Q} + 2 \sum_{q=1}^{Q} (-1)^q C_{2Q}^{Q+q} \right)
\]

\[
= 0,
\]

where we have used the formulas

\[
(2q + 1) C_{2Q+1}^{Q-q} = (2Q + 1) \left( C_{2Q}^{Q+q} - C_{2Q}^{Q+q+1} \right), \forall 0 \leq q \leq Q, \tag{5.8}
\]

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and

\[ \sum_{q=1}^{Q} (-1)^q C_{2Q}^{Q+q} = -\frac{1}{2} C_Q^{2Q}, \quad \forall Q \geq 1. \quad (5.9) \]

In order to complete the proof of (5.7), we must prove (5.8) and (5.9). To verify assertion (5.8), we check directly that

\[
(2Q + 1) \left( C_{2Q}^{Q+q} - C_{2Q}^{Q+q+1} \right) \\
= (2Q + 1) \left( \frac{(2Q)!}{(Q + q)! (Q - q)!} - \frac{(2Q)!}{(Q + q + 1)! (Q - q - 1)!} \right) \\
= \frac{(2Q + 1)! (Q + q + 1 - (Q - q))}{(Q + q + 1)! (Q - q)!} \\
= \frac{(2Q + 1)! (2q + 1)}{(Q + q + 1)! (Q - q)!} \\
= (2q + 1) C_{2Q+1}^{Q-q}.
\]

To prove assertion (5.9), we first note that this is equivalent to

\[ \sum_{q=0}^{Q-1} (-1)^q C_{2Q}^q = \frac{1}{2} (-1)^{Q+1} C_{2Q}^Q, \quad \forall Q \geq 1, \quad (5.10) \]

since

\[
\sum_{q=1}^{Q} (-1)^q C_{2Q}^{Q+q} = \sum_{q=1}^{Q} (-1)^q C_{2Q}^{Q-q} = \sum_{j=0}^{Q-1} (-1)^{Q-j} C_{2Q}^{j} \\
= (-1)^Q \sum_{j=0}^{Q-1} (-1)^j C_{2Q}^{j} = (-1)^Q \sum_{q=0}^{Q-1} (-1)^q C_{2Q}^q.
\]

Next, to show (5.10), we need only to note that

\[ 0 = (1 + x)^{2Q} \big|_{x=-1} = \sum_{q=0}^{2Q} (-1)^q C_{2Q}^q \\
= \sum_{q=0}^{Q-1} (-1)^q C_{2Q}^q + (-1)^Q C_{2Q}^Q + \sum_{q=Q+1}^{2Q} (-1)^q C_{2Q}^q \]
\[ Q - 1 \sum_{q=0}^{Q} (-1)^q C_{2Q}^q + (-1)^Q C_{2Q}^Q, \]

where the last equality holds since
\[ \sum_{q=Q+1}^{2Q} (-1)^q C_{2Q}^q = \sum_{j=0}^{Q-1} (-1)^{2Q-j} C_{2Q}^{2Q-j} = \sum_{q=0}^{Q-1} (-1)^q C_{2Q}^q. \]

Thus the proof of (5.7) is complete.

**Step 3.** For any \(0 \leq j \leq k \leq n\) and any \(a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n\), set
\[ S_j^k := S_j^k(a) := \sum_{1 \leq i_1, i_2, ..., i_k \leq n} a_{i_1} a_{i_2}^2 ... a_{i_j} a_{i_{j+1}} ... a_{i_k}. \]

Note that \(S^0_j = \sigma_k\) and \(S^j_k(1) = C_n^k C_j^k\), where \(1 := (1, 1, ..., 1) \in \mathbb{R}^n\), which means that \(S_j^k\) is a polynomial of \(C_n^k C_j^k\) terms and can be viewed as a generalization of the elementary symmetric polynomial \(\sigma_k\).

The main purpose in this step is to prove the following elementary decompositions

\[ \sigma_j(a) \sigma_k(a) = \sum_{h=0}^{j} C_{j+k-2h}^j S_{j+k-h}^h(a), \quad \forall 0 \leq j \leq k \leq n, \quad j + k \leq n, \quad (5.11) \]

and

\[ \sigma_j(a) \sigma_k(a) = \sum_{h=0}^{n-k} C_{2n-j-k-2h}^{n-j-h} S_{n-h}^{j+k-n+h}(a), \quad \forall 0 \leq j \leq k \leq n, \quad j + k \geq n, \quad (5.12) \]

for any \(a \in \mathbb{R}^n\).

To do this, we first observe that \(\sigma_j(a) \sigma_k(a)\) is a polynomial with terms of the \(S_L^M(a)\) types, where \(0 \leq L \leq M \leq n\) and \(L + M = j + k\). Thus for \(j + k \leq n\), any term of \(\sigma_j \sigma_k\) can only be one of the elements of
\[ \{ S^0_{j+k}, S^1_{j+k-1}, S^2_{j+k-2}, ..., S^h_{j+k-h}, ..., S^{j-1}_{j+k-1}, S^j_k \}, \]

and, for \(j + k \geq n\), any term of \(\sigma_j \sigma_k\) can only be one of the elements of
\[ \{ S_j^{j+k-n}, S_{j+k-n+1}, S_{j+k-n+2}, ..., S_{j+k-n+h}, ..., S_{j+k-1}^{j+k-n}, S^j_k \}. \]
Hence it remains to determine the coefficient of each \( S^M_L(a) \). To do so, we first recognize that, for any fixed \( a = (a_1, a_2, ..., a_n) \in \mathbb{R}^n \), there are \( N \) terms of \( S^M_L(a) \) in the expansion of \( \sigma_j(a) \sigma_k(a) \), if and only if there are \( N \) terms of \( a_1^2 a_2^2 ... a_M^2 a_{M+1} ... a_L \) in the expansion of \( \sigma_j(a) \sigma_k(a) \). But we know that, for each \( a_1^2 a_2^2 ... a_M^2 a_{M+1} ... a_L \), its divisor \( a_1 a_2 ... a_M a_i_1 a_i_2 ... a_{i_j-M} \) must come from the terms in the expansion of \( \sigma_j(a) \), where

\[ i_1, i_2, ..., i_{j-M} \in \{ M + 1, M + 2, ..., L \} \]

and are different from each other. This indicates that the number of \( S^M_L(a) \) in the expansion of \( \sigma_j(a) \sigma_k(a) \) is exactly \( C_{j-M}^{j-M} \). (Note that we can also consider the divisor

\[ a_1 a_2 ... a_M a_{s_1} a_{s_2} ... a_{s_{k-M}} = \frac{a_1^2 a_2^2 ... a_M^2 a_{M+1} ... a_L}{a_1 a_2 ... a_M a_i_1 a_i_2 ... a_{i_j-M}} \]

of \( a_1^2 a_2^2 ... a_M^2 a_{M+1} ... a_L \), comes from the terms in the expansion of \( \sigma_k(a) \), to yield a result \( C_{L-M}^{j-M} \). But since \( j + k = L + M \) and hence \( (j - M) + (k - M) = j + k - 2M = L - M \), we actually have \( C_{j-M}^{j-M} = C_{L-M}^{j-M} \). Thus, both of these two observations are equivalent.) Therefore, the coefficient of \( S^h_{j+k-h} \) in the decomposition of \( \sigma_j(a) \sigma_k(a) \) is

\[ C_{j-k-h}^{j-k-h} = C_{j+k-2h}^{j-k-h} \]

and the coefficient of \( S^{j+k-n+h}_{n-h} \) in the decomposition of \( \sigma_j(a) \sigma_k(a) \) is

\[ C_{(n-h)-(j+k-n+h)}^{j-(j+k-n-h)} = C_{2n-j-k-2h}^{n-k} = C_{2n-j-k-2h}^{n-j-h} \]

Thus the proof of (5.11) and (5.12) is complete.

**Step 4.** Since

\[ (p - q) + (p + q + 1) = 2p + 1 \leq n \iff p \leq (n - 1)/2, \]

substituting (5.11) and (5.12) into (5.6), we get

(a) For each \( p \leq (n - 1)/2 \),

\[ \sum_{q=0}^{p} (-1)^q (2q + 1) \sigma_{p-q} \sigma_{p+q+1} \]
\[
(5.7) \quad C^p_{2p+1-2h} = S_h^{p+1} > 0,
\]

where we used \( C^{p-q-h}_{2p+1-2h} \equiv 0 \) (\( \forall h > p-q \)), \( C^{(p-h)-q}_{2(p-h)+1} \equiv 0 \) (\( \forall q > p-h \)) and (5.7) in the second, fourth and fifth equality, respectively.

(b) For each \( p > (n-1)/2 \),

\[
\sum_{q=0}^{n-1-p} (-1)^q(2q + 1)C^{n-p-q-h}_{2n-2p-1-2h} S_{n-h}^{2h+1-n+h} = 0,
\]

where we used \( C^{n-1-p-q-h}_{2n-2p-1-2h} \equiv 0 \) (\( \forall h > n-1-p-q \)), \( C^{(n-1-p-h)-q}_{2(n-1-p-h)+1} \equiv 0 \) (\( \forall q > n-1-p-h \)) and (5.7) in the second, fourth and fifth equality, respectively.
∀ q > n − 1 − p − h) and (5.7) in the third, fifth and sixth equality, respectively.

Combining (a), (b) and (5.2), we obtain

\[ \hat{Z}_s(a) = \sum_{p=0}^{n-1} S_{p+1}(a), \]

that is, the desired formula (5.1). Thus the proof is completed.

Remark 5.1. (1) The formula (5.1) in Lemma 5.1 gives us an elementary way to prove the following two important inequalities in this paper:

\[ \hat{Z}_s(a) \triangleq (1 - \sigma_2(a) + \sigma_4(a) - ...)(\sigma_1(a) - 3\sigma_3(a) + 5\sigma_5(a) - ...) - (2\sigma_2(a) + 4\sigma_4(a) - ...)(\sigma_1(a) - \sigma_3(a) + \sigma_5(a) - ...) > 0, \]

and

\[ \hat{Z}(a) \triangleq \cos \Theta (\sigma_1(a) - 3\sigma_3(a) + 5\sigma_5(a) - ...) - \sin \Theta (-2\sigma_2(a) + 4\sigma_4(a) - ...) > 0, \text{ if } H(a) = \Theta. \]

(See Corollary 3.1 and Corollary 3.2.)

(2) For the special case that \( a = 1 \), (5.1) in Lemma 5.1 is exactly

\[ \hat{Z}_s(1) = (1 - C_n^2 + C_n^4 - ...)(C_n^1 - 3C_n^3 + 5C_n^5 - ...) - (-2C_n^2 + 4C_n^4 - ...)(C_n^1 - C_n^3 + C_n^5 - ...) = \sum_{0 \leq 2i \leq n} (-1)^i C_n^{2i} \sum_{0 \leq 2j + 1 \leq n} (-1)^j (2j + 1) C_n^{2j+1} - \sum_{0 \leq 2i \leq n} (-1)^i (2i) C_n^{2i} \sum_{0 \leq 2j + 1 \leq n} (-1)^j C_n^{2j+1} = \sum_{p=0}^{n-1} S_{p+1}(1) = \sum_{p=0}^{n-1} C_n^{p+1} C_{p+1}^p = \sum_{p=0}^{n-1} C_n^{p+1} C_{p+1}^1 = \sum_{p=0}^{n-1} (p + 1) C_n^{p+1} = \sum_{k=0}^{n} k C_n^k = n2^{n-1}. \]
where for the last equality we just used the following standard tricks

\[
J(1 + x)^{J-1} = \frac{d}{dx} \left((1 + x)^J\right) = \frac{d}{dx} \left(\sum_{j=0}^{J} C_j x^j\right) = \sum_{j=0}^{J} jC_j x^{j-1}.
\]

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