CARRIER CONES OF ANALYTIC FUNCTIONALS

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Abstract. We prove that every continuous linear functional on the space $S^0(\mathbb{R}^d)$ consisting of the entire analytic functions whose Fourier transforms belong to the Schwartz space $\mathcal{D}$ has a unique minimal carrier cone in $\mathbb{R}^d$, which substitutes for the support. The proof is based on a relevant decomposition theorem for elements of the spaces $S^0(\mathcal{K})$ associated naturally with closed cones $\mathcal{K} \subset \mathbb{R}^d$. These results, essential for applications to nonlocal quantum field theory, are similar to those obtained previously for functionals on the Gelfand-Shilov spaces $S^0_\alpha$, but their derivation is more sophisticated because $S^0(\mathcal{K})$ are not DFS spaces and have more complicated topological structure.

1. Introduction

The Gelfand-Shilov test-function spaces $S^\beta$ and $S^\alpha_\beta$ with the index $\beta < 1$, consisting of entire analytic functions [1], find an interesting application in quantum field theory (QFT), where they are used as a functional domain of definition of nonlocal fields. This permits the extension of the basic results of the Wightman axiomatic approach [2] to nonlocal interactions, see [3] and references therein. The spaces $S^0$ and $S^0_\alpha$ play a dominant role in these applications because the Fourier transforms of their elements have compact supports and these spaces are hence suitable for fields with arbitrarily singular high-energy behavior. The central problem in constructing nonlocal QFT is an adequate generalization of the microcausality condition. The most natural formulation is attained through the use of the notion of a carrier cone (quasi-support) of non-localizable generalized functions. The existence of a unique minimal carrier cone for every generalized function defined on $S^0_\alpha$ has been proved in [4] with the aid of techniques of the theory of hyperfunctions. The proof relied essentially on the nice topological properties of the spaces $S^0_\alpha$ which belong to the well-studied class [5] of DFS spaces.

The space $S^0$ is of particular interest because it coincides with the Fourier transform of Schwartz’s space $\mathcal{D}$ of infinitely differentiable functions of compact support. Initially, this space was introduced just in the theory of Fourier transformation of distributions and, in this context, it was denoted by $Z$ in [1]. The notation $S^0$ used in physical literature emphasizes that this space is smallest among the spaces $S^\beta$. Furthermore, the original definition of $S^0$ was formulated in terms of real variables and our concern here is precisely with carriers in the real space $\mathbb{R}^d$, although the

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representation in terms of complex variables will be used as an efficient method of the analysis. As noted in [11], the continuous linear functionals defined on $Z$ and composing the dual space $Z'$ are analytic in the sense that their expansions in Taylor series are convergent in the strong topology of $Z'$. The proof of the existence of a quasi-support for every functional of the class $Z' = S^0$ has some subtleties as compared to $S^0_0$, which are elucidated in this paper. It should be noted that the arguments given below admit the extension of the results to multilinear forms, which is important for the applications to nonlocal QFT and to noncommutative field theory, where such test functions also came recently into use [6].

Our construction is based on exploiting spaces related to $S^0$ and associated with open and closed cones in $\mathbb{R}^d$. In Sec. 2, we recall their definition introduced in [7] and describe the topological properties of these spaces. The existence of a smallest carrier cone for every generalized function on $S^0$ is deduced from a decomposition theorem which asserts that each element of the space $S^0(K_1 \cap K_2)$ associated with the intersection of two closed cones can be decomposed into a sum of functions belonging to $S^0(K_1)$ and $S^0(K_2)$. There are three steps in proving this theorem. In Sec. 3, we first perform a decomposition into smooth functions satisfying the required restrictions on the behavior at infinity. The analyticity is restored in Sec. 5 with the use of Hörmander’s classical results [8] concerning solutions of the nonhomogeneous Cauchy-Riemann equations. However, Hörmander’s $L^2$-estimates involve plurisubharmonic functions, whereas the indicator functions defining $S^0(K)$ are not of such a form. Because of this, we run into the problem of approximating the indicator functions by plurisubharmonic ones, which is solved in Sec. 4. The main theorem on carrier cones is proved in Sec. 6. In concluding Sec. 7, we point some possible applications of the obtained results.

2. Spaces over cones and their topology

Let $U$ be an open cone in $\mathbb{R}^d$. The space $S^{0,b}(U)$ is defined to be the intersection (projective limit) of the family of Hilbert spaces $H^{0,B}_N(U)$, $B > b$, $N = 0, 1, 2, \ldots$, consisting of entire analytic functions on $\mathbb{C}^d$ and endowed with the scalar products

\[ \langle f, g \rangle_{U,B,N} = \int f(z)g(z) \left(1 + |x|\right)^{2N} e^{-2B|d(x,U)| - 2B|y|} \, d\lambda, \quad z = x + iy, \]

where $d(x,U) = \inf_{\xi \in U} |x - \xi|$ is the distance of $x$ from $U$ and $d\lambda = dx dy$ is the Lebesgue measure on $\mathbb{C}^d$. Hereafter we assume the norm $| \cdot |$ on $\mathbb{R}^d$ to be Euclidean. It is easily seen from Cauchy’s integral formula that the system of norms $\|f\|_{U,B,N}$ determined by scalar products [11] is equivalent to the system

\[ \|f\|_{U,B,N} = \sup_{\mathbb{C}^d} |f(z)| e^{-\rho_{U,B,N}(z)}, \]

where \( \rho_{U,B,N}(z) = -N \ln(1 + |x|) + B d(x, U) + B|y| \).

In particular, if $U = \mathbb{R}^d$, then $d(x, \mathbb{R}^d) = 0$ and $S^{0,b}(\mathbb{R}^d)$ coincides with the space $Z(b)$ which is the Fourier transform of the space of all smooth functions with support in the closed ball of radius $b$, see [11]. However, the employment of Hilbert norms is preferable in some cases, including the proofs given below. The space $S^0(U)$ is defined to be the union (inductive limit) of the family $S^{0,b}(U)$, $b > 0$. Clearly, there are natural inclusion maps $S^0(U_1) \to S^0(U_2)$ for $U_1 \supset U_2$. As stated above, the Fourier operator transforms $S^0(\mathbb{R}^d)$ into the Schwartz space $\mathcal{D}(\mathbb{R}^d)$.
Let \( v \) be a continuous linear functional on \( S^0(\mathbb{R}^d) \). We say that \( v \) is carried by a closed cone \( K \subset \mathbb{R}^d \) if this functional has a continuous extension to every space \( S^0(U) \), where \( U \ni K \). This extension, if it exists, is unique because \( S^0(\mathbb{R}^d) \) is dense in each \( S^0(U) \), see [9, 10]. Therefore the stated property implies that \( v \) extends to the space
\[
S^0(K) = \lim_{U \ni K} S^0(U),
\]
and the set of functionals carried by \( K \) is algebraically identified with the dual space of \( S^0(K) \). It is worth noting that in the case of degenerate closed cone consisting of the origin, the set of open cones \( U \) such that \( U \ni \{0\} \) is not directed. For the sake of unification, the cone \( \{0\} \) can be added to the totality of open cones with its associated space \( S^0(\{0\}) \) defined as the space of all entire analytic functions of exponential type. This stipulation is quite natural, because really we are dealing here with a sheaf of spaces over \((d-1)\)-dimensional sphere \( S^{d-1} \) compactifying \( \mathbb{R}^d \) and the cone \( \{0\} \) corresponds to the empty open set in \( S^{d-1} \). For clearness, we prefer to say about cones instead of subsets of \( S^{d-1} \), bearing physical applications in mind. All the spaces \( S^0(U) \) are embedded in \( S^0(\{0\}) \) and the decomposition theorem derived below shows that the inductive topology on \( S^0(\{0\}) \) determined by the inclusions \( S^0(U) \rightarrow S^0(\{0\}) \), where \( U \neq \{0\} \), coincides with the topology defined on the basis of \( \{1\} \) with \( d(x, 0) = |x| \). Thus definition \((\text{3})\) is applicable to \( \{0\} \) even without this stipulation but we find it to be convenient.

The spaces \( S^{0,b}(U) \) belong to the class FN of nuclear Fréchet spaces. This fact, which is essential for applications to QFT, can be established in a manner analogous to that used in [10] for \( S^{a,b}(U) \). As a consequence, \( S^{0,b}(U) \) also belong to the Fréchet-Schwartz class FS and are Montel spaces and hence reflexive. The spaces \( S^0(U) \) and \( S^0(K) \), being inductive limits of sequences of such spaces, inherit nuclearity (see [11]) and are obviously Hausdorff spaces. The spaces \( S^0(U) \) are complete, as is proved in [12] with the use of acyclicity of the injective sequence \( S^{0,v}(U) \), \( v = 1, 2, \ldots \). Together with nuclearity and barrelledness, this implies that they are also Montel spaces, see Exer. 19 in Chap. IV in [11]. Whether \( S^0(K) \) have such properties is still an open question, but their completions have them because nuclearity and barrelledness are preserved after completion. However the spaces \( S^0(U) \) and \( S^0(K) \) are nonmetrizable. This fact is well known for \( S^0(\mathbb{R}^d) \) and follows, e.g., from Proposition 5 in Sec. VII in [13], because \( D(\mathbb{R}^d) \) is a strict inductive limit. In the case \( U \neq \mathbb{R}^d \), one can use Corollary 7.3 in [14], where injective limits of acyclic sequences are considered.

**Proposition 1.** The spaces \( S^0(U) \) and \( S^0(K) \) are not DFS spaces, with one exception \( S^0(\{0\}) \).

**Proof.** It suffices to show that the dual spaces \( S^{0,a}(U) \), \( S^{0,b}(K) \) endowed with the strong topology are nonmetrizable because the class DFS consists of the strong dual spaces of FS spaces and each DFS space is reflexive. For this purpose we use Theorem A in [15]. In the terminology of [15], a locally convex \( E \) space is called an \( LF \) space if it is a Hausdorff space representable as the inductive limit of a sequence of Fréchet spaces \( E_j \) (without the more usual assumption \([11]\) that this limit is strict). Grothendieck’s theorem [15] asserts that if \( u \) is a continuous linear

\(^1\)For arbitrary cones \( V_1, V_2 \), the notation \( V_1 \oplus V_2 \) means that \( \overline{V_1 \setminus \{0\}} \subset V_2 \), where \( \overline{V} \) is the closure of \( V \).
mapping of a Fréchet space $F$ to such an $\mathcal{LF}$ space $E$, then $u(F)$ is contained in one of the spaces $E_i$. In the case under consideration, $S^{0,b}(U)$ is the inductive limit of the sequence of strong dual spaces of the Hilbert spaces $H^{0,b+1/N}_N(U)$ and so has the structure of $\mathcal{LF}$ space. If $S^{0}(U)$ (or $S^0(K)$) were a Fréchet space, then its image under the natural injection to $S^{0,b}(U)$ would be contained in the dual space of an $H^{0,b+1/N}_N(U)$ but such is not the case except for $S^{0}\{0\}$, when the factor $(1 + |x|)^{2N}$ in (4) can be dropped out. This completes the proof.

Remark 1. By the same argument, none of the spaces $S^\beta(\mathbb{R}^d)$, $\beta \geq 0$, is a DFS space.

3. A smooth decomposition

Our objective is to prove the following theorem.

**Theorem 1.** If $U_1$, $U_2$, and $U$ are open cones in $\mathbb{R}^d$ such that $\bar{U}_1 \cap \bar{U}_2 \subseteq U$ and $f \in S^0(U)$, then $f = f_1 + f_2$ with $f_i \in S^0(U \cup U_i)$, $i = 1, 2$.

Using the dilation invariance of the spaces involved, we assume that $\|f\|_{U_1,N} < \infty$ without loss of generality. As noted above, we first decompose $f$ into smooth functions which behave at infinity as elements of $S^0(U \cup U_i)$ and subsequently restore the analyticity. The smooth decomposition can be performed with the help of the simple geometrical lemma.

**Lemma 1.** For every triple of open cones $U_1$, $U_2$, $U$ satisfying $\bar{U}_1 \cap \bar{U}_2 \subseteq U$, there is a smooth function $\chi(x) \geq 0$ with the properties:

\[ |\chi(x)| e^{d(x,U)} \leq Ce^{bd(x,U \cup U_1)}, \]

\[ |1 - \chi(x)| e^{d(x,U)} \leq Ce^{bd(x,U \cup U_2)}, \]

\[ \left| \frac{\partial \chi}{\partial x_j} \right| e^{d(x,U)} \leq Ce^{bd(x,U \cup U_1 \cup U_2)}, \quad j = 1, \ldots, d, \]

where $C$ and $b$ are positive constants.

Using such a function, we set

\[ f = f_1 + f_2, \quad f_1(z) = f(z)\chi(z), \quad f_2(z) = f(z)(1 - \chi(z)) \]

and then (4) and (5) (with $b \geq 1$) result in the inequalities

\[ \|f_1\|_{U \cup U_1, b, N} \leq C\|f\|_{U,1,N}, \quad \|f_2\|_{U \cup U_2, b, N} \leq C\|f\|_{U,1,N}, \quad N = 0, 1, 2, \ldots \]

**Proof of Lemma 1.** We recall that the intersection of a cone $V$ with the unit sphere is named its projection and denoted by $\text{pr} V$. The condition $\bar{U}_1 \cap \bar{U}_2 \subseteq U$ implies that the closed cones $V_1 = \bar{U}_1 \setminus U$ and $V_2 = \bar{U}_2 \setminus U$ have disjoint projections. Therefore the distance of $\text{pr} V_1$ from $V_2$, as well as that of $\text{pr} V_2$ from $V_1$, is positive. In the case of Euclidean metric, these two distances are equal. In fact, if the first one is attained at the points $x_1 \in \text{pr} V_1$, $x_2 \in V_2$, then the relation $|x_1 - x_2|^2 = |x_1| |x_2| - x_2/|x_2|^2$ shows that $d(\text{pr} V_1, V_2) \geq d(\text{pr} V_2, V_1)$, and the reverse is also true by symmetry. This quantity will be referred to as the angular separation between $V_1$ and $V_2$ and denoted by $\theta$.

Let us introduce the auxiliary open cone

\[ W = \left\{ \xi \in \mathbb{R}^d : d(\xi, V_2) < \frac{\theta}{2} |\xi| \right\} \]
and let \( \chi_0 \) be a nonnegative function of the class \( C^\infty \) with support in the unit ball and whose integral is unity. We define \( \chi(x) \) by
\[
\chi(x) = \int_W \chi_0(x - \xi) \, d\xi.
\]
and claim that this function satisfies inequalities (11)–(13) with \( b \) arbitrarily close to \( 2/\theta \). To demonstrate this, we use further two cones \( W_1, W_2 \) specified in a similar way to \( W \) but with parameters \( \theta_1, \theta_2 \) subject to the conditions \( \theta < \theta_1 < 2\theta \) and \( 0 < \theta_2 < \theta \). It is easy to see that
\[
d(x, V_1) \geq \left( \frac{\theta - \theta_1}{2} \right) |x| \quad \text{for all} \quad x \in W_1.
\]
Indeed, if this is not the case, then \( d(\text{pr} \, W_1, V_1) < \theta - \theta_1/2 \) and there are points \( x_1 \in \text{pr} \, V_1, x \in W_1 \) such that \( d(x_1, x) < \theta - \theta_1/2 \). Besides, \( |x| < 1 \) because the distance of the unit vector \( x_1 \) to \( W_1 \) is attained inside the unit ball. There is also a point \( x_2 \in V_2 \) such that \( d(x, x_2) < \theta_1|x|/2 < \theta_1/2 \) and the triangle inequality gives \( d(x_1, x_2) < \theta \), which contradicts the definition of \( \theta \). The support of \( \chi \) is contained in the 1-neighborhood of \( W \) and hence in the union of \( W_1 \) with a ball of sufficiently large radius \( R \). The condition (11) is trivially satisfied inside the ball with \( C = e^R \) and any \( b \). The inequality (12) ensures the fulfilment of (11) in \( W_1 \) with \( C = 1 \) and \( b = (\theta - \theta_1/2)^{-1} \) because \( d(x, U) \leq |x| \) and \( d(x, U \cup U_1) = \min\{d(x, U), d(x, U_1 \setminus U)\} \), where \( U_1 \setminus U \subset V_1 \). Further, \( 1 - \chi(x) = 0 \) for all the points \( x \in W \) whose distance from the boundary of \( W \) is greater than 1 and, in particular, for the points of \( W_2 \) outside a sufficiently large ball. Considering that
\[
d(x, V_2) \geq \frac{\theta_2}{2} |x| \quad \text{for all} \quad x \notin W_2,
\]
we infer that (13) is satisfied with \( b = 2/\theta_2 \). The derivatives of \( \chi \) are uniformly bounded and their supports are contained in the 1-neighborhood of the boundary of \( W \). Therefore both of inequalities (12), (13) are satisfied at the points of these supports beyond a sufficiently large ball. Consequently, (13) also holds with any \( b > 2/\theta \) as \( \theta_1 \) and \( \theta_2 \) can be taken arbitrarily close to \( \theta \). Lemma 1 is thus proved.

4. GOING TO PLURISUBHARMONIC FUNCTIONS

We go on to prove Theorem 1. To obtain an analytic decomposition, we write
\[
f = f_1 + f_2, \quad f_1 = f_1 - \psi, \quad f_2 = f_2 + \psi
\]
and subject \( \psi \) to the equations
\[
\frac{\partial \psi}{\partial \bar{z}_j} = \eta_j,
\]
where
\[
\eta_j := \frac{\partial \chi}{\partial \bar{z}_j} - \frac{i}{2} \frac{\bar{\partial} \chi}{\partial x_j}, \quad j = 1, \ldots, d.
\]
The functions \( \eta_j(z) \) satisfy the estimate
\[
|\eta_j(z)| \leq C_{f,N} \|f\|_{U,1,N} e^{\rho_{U,V \cup U_1 \cup U_2, b,N}(z)},
\]
where notation (4) is used. In fact, from Cauchy’s integral formula we have
\[
|f(z)| \leq C \|f\|_{L^2(B)}.
\]
where $\mathcal{B}$ is any bounded neighborhood of the point $z$ in $\mathbb{C}^d$. Taking the unit ball for $\mathcal{B}$ and applying the triangle inequality to every item on the right-hand side of (2), we infer that $-\rho_{U,B,N}(z) \leq -\rho_{U,B,N}(\zeta) + N \ln 2 + 2B$ for all $\zeta \in \mathcal{B}$. Therefore

$$|f(z)|^2 e^{-2\rho_{U,1,N}(z)} \leq C_N \int_{\mathcal{B}} |f(\zeta)|^2 e^{-2\rho_{U,1,N}(\zeta)} d\lambda \leq C_N \|f\|_{L^2,1,N}^2$$

and, coupled with (6), this gives (12). It remains to show that there exist a solution of Eqs. (10) with the required behavior at infinity. For this purpose we will use Hörmander’s $L^2$-estimates but first prove another lemma.

Lemma 2. Let $V$ be an open cone in $\mathbb{R}^d$ and $B > 2edb$. For each function $\eta(z)$, $z \in \mathbb{C}^n$, satisfying the estimate

$$|\eta(z)| \leq C_N e^{\rho_{V,B,N}(z)}$$

there is a plurisubharmonic function $\varrho(z)$ with values in $(-\infty, +\infty)$ such that

$$|\eta(z)| \leq e^{\varrho(z)} \leq C'_N e^{\rho_{V,B,N}(z)}.$$

Proof. For brevity, we set $b = 1$. As shown in [10], for each $\alpha > 1$ and $\sigma > 2$, there is a sequence of functions $\varphi_n \in S_{\alpha}^0(\mathbb{R})$, $n = 0, 1, 2, \ldots$, with the properties:

$$|\varphi_n(z)| \leq C \exp\{\sigma|y| - |x|^{1/\alpha}\},$$

$$\ln |\varphi_n(iy)| \geq |y|,$$

$$\ln |\varphi_n(z)| \leq \sigma|y| - n \ln^+(|x|/n) + A,$$

where $\ln^+ r = \max(0, \ln r)$ and the constants $C, A$ are independent of $n$. \(^2\)

This sequence is the main tool of our proof and we also use the auxiliary function

$$H(\xi) = \sup_y (\ln |\eta(\xi + iy)| - |y|).$$

By condition (13) it satisfies the inequality

$$H(\xi) \leq \ln C_N - N \ln(1 + |\xi|) + d(\xi, V).$$

First we consider the simplest one-dimensional case $V = \mathbb{R}_-$, $d(\xi, V) = \theta(\xi)|\xi|$, where $\theta(x)$ is Heaviside’s step-function. Let $\Phi_n(\zeta) = \ln |\varphi_n(\xi)|$. The function $\Phi_n$ is subharmonic according to [10], §II.9.12. As a candidate for the desired function $\varrho$, we take the upper envelope of the family $\Phi_n(z - \xi) + H(\xi)$, allowing the index $n$ to depend on the point $\xi$ running $\mathbb{R}$. The functions of this family are locally uniformly bounded from above and hence their upper envelope is also subharmonic (ibid, §II.9.6). Furthermore, it obviously dominates $\ln |\eta(z)|$ because (16) and (18) imply

$$\sup_{\xi} \{\Phi_n(z - \xi) + H(\xi)\} \geq \Phi_n(iy) + H(x) \geq \ln |\eta(z)|.$$ \(^2\)

Let us show that $n(\xi)$ can be chosen so that to ensure the second of inequalities (14). For $\xi < 0$, when $d(\xi, \mathbb{R}_-) = 0$, there is no problem and we may set $n(\xi) = 0$ because (13), together with the elementary inequalities

$$-|x - \xi|^{1/\alpha} \leq \tilde{C}_N - N \ln(1 + |x - \xi|), \quad \ln(1 + |x - \xi|) + \ln(1 + |\xi|) \geq \ln(1 + |x|),$$

\(^2\)Here we mean that $n \ln^+(|x|/n) = 0$ for $n = 0$. 
implies that
\[
\sup_{\xi<0} \{ \Phi_0(z - \xi) + H(\xi) \} \leq C'_N + \epsilon_0 |y| - N \ln(1 + |x|).
\]

By using (21), we also obtain the estimate
\[
(22) \quad \beta \Phi_0(z - \xi) - N \ln(1 + |\xi|) \leq C''_N + \beta \epsilon_0 |y| - N \ln(1 + |x|),
\]
which holds for any \( \beta > 0 \) and for all \( \xi \) and shows that difficulties emerge only from the linear growth of the term \( d(\xi, V) \) in (19).

Let \( \xi \geq 0 \) and so \( d(\xi, \mathbb{R}^_) = \xi \). Let, in addition, \( e|x - \xi| > n \). Then
\[
(23) \quad n \ln(e|x - \xi|/n) + cd(x, \mathbb{R}^_) \geq n \ln(e\xi/n).
\]
This is obvious for \( |x - \xi| > \xi \). For the other points, it suffices to use the inequality \( \theta(x) x \geq \xi - |x - \xi| \) and take into account that the function \( n \ln(\lambda/n) - \lambda \) is monotone decreasing in the interval \( \{n, e\xi\} \). Applying (17) and (23), we get
\[
\Phi_n(z - \xi) + \xi \leq A + \sigma e|y| + cd(x, \mathbb{R}^_) - n \ln(e\xi/n) + \xi.
\]
We take \( n(\xi) \) to be the integral part of the number \( \xi \). Then \( n(\xi) = n > \xi - 1 \) and
\[
(24) \quad \Phi_n(\xi)(z - \xi) + \xi \leq A + \sigma e|y| + cd(x, \mathbb{R}^_).
\]
An analogous inequality holds for \( e|x - \xi| \leq n \), when \( \ln^+(e|x - \xi|/n) = 0 \). Indeed, then \( \xi \leq \theta(x) x \geq \xi - |x - \xi| \leq \theta(x) x + \xi/\epsilon \) and hence \( \xi \leq cd(x, \mathbb{R}^-) \). Thus, inequality (24) (with a proper constant) is valid for all \( \xi \geq 0 \). In combination with estimate (22), where we set \( \beta = B/(e\epsilon) - 1 \), it shows that the upper envelope
\[
(25) \quad \tilde{\Phi}(z) = \lim_{z \to z} \sup_{\xi} \{ \beta \Phi_0(z' - \xi) + \Phi_n(\xi)(z' - \xi) + H(\xi) \}
\]
satisfies all the requirements. 3

In the general case of several variables and arbitrary open cone \( V \subset \mathbb{R}^d \), we set \( \Phi_n(z) = \sum_{j=1}^d l \ln |v_n(e\sqrt{d} z_j)| \). Then inequality (20) holds true and (22) is replaced by
\[
(26) \quad \beta \Phi_0(z - \xi) - N \ln(1 + |\xi|) \leq C''_N + \beta \epsilon_0 |y| - N \ln(1 + |x|),
\]
because \( \sum_{j=1}^d |y_j| \leq \sqrt{d} |y| \). For \( x \notin V \) and \( e|x - \xi| > n \), we have
\[
\sum_{j=1}^d \ln \left( \frac{\epsilon \sqrt{d}}{n} |x_j - \xi_j| \right) + cd(x, V) \geq n \ln \left( \frac{\epsilon}{n} d(\xi, V) \right).
\]
To make sure of this, it suffices to consider that
\[
\sum_{j=1}^d \ln^+ |x_j| \geq \ln^+ \frac{|x|}{\sqrt{d}} \quad \text{and} \quad d(\xi, V) = \inf_{\xi' \in V} |\xi' - \xi| \leq d(x, V) + |x - \xi|.
\]
This time we take \( n(\xi) \) to be the integral part of \( d(\xi, V) \) and then (24) is changed for the inequality
\[
(27) \quad \Phi_n(\xi)(z - \xi) + d(\xi, V) \leq A' + \epsilon_0 |y| + cd(x, V),
\]

3Taking the upper limit ensures the upper semicontinuity of the resulting function and enters into the definition (10) of upper envelope.
which holds for all \( x \). Combining (27) with (26), we conclude that the plurisubharmonic function defined by (25) with \( \xi \in \mathbb{R}^d \) and \( \beta = B/(e\sigma d) - 1 \) satisfies the conditions (14). So we have proved Lemma 2.

5. The use of \( L^2 \)-estimates

End of the proof of Theorem 1. We set

\[
\tilde{\varrho}(z) = 2\varrho(z) + (d + 1)\ln(1 + |z|^2),
\]

where \( \varrho \) is the upper envelope of the plurisubharmonic functions assigned to \( \eta_j(z) \), \( j = 1, \ldots, d \) by Lemma 2 with \( V = U \cup U_1 \cup U_2 \). Inequalities \( |\eta_j(z)| \leq e^{\varphi(z)} \) implies that the functions \( \eta_j \) belong to \( L^2(\mathbb{C}^d, e^{-\varrho}d\lambda) \). By definition (11), the compatibility conditions \( \partial \eta_j / \partial z_k = \partial \eta_k / \partial z_j \) are fulfilled. Therefore we can apply Theorem 15.1.2 in [8], which shows that the system of equations (10) has a solution \( \psi \) such that

\[
2 \int |\psi|^2 e^{-\tilde{\varrho}}(1 + |z|^2)^{-2}d\lambda \leq \int |\eta|^2 e^{-\tilde{\varrho}}d\lambda.
\]

From (11) and (28), it follows that

\[
\psi \in L^2(\mathbb{C}^d, e^{-2\rho_{V,B,N} - d - 2}d\lambda)
\]

for each \( N \). When coupled with (7), this gives \( f_1 \in S^0(U \cup U_1) \) and \( f_2 \in S^0(U \cup U_2) \), which completes the proof.

Remark 2. If \( U_1 \cap U_2 = \{0\} \), then every element \( f \in S^0(\{0\}) \) allows the decomposition \( f = f_1 + f_2 \), where \( f_i \in S^0(U_i) \), \( i = 1, 2 \). This follows from the above arguments with \( d(x,0) = |x| \) and is included in the formulation of Theorem 1 under the stipulation of Sec. 2 that the cone \( \{0\} \) is open in the sense that its projection is open.

6. The existence of smallest carrier cones

Theorem 2. For every continuous linear functional on \( S^0(\mathbb{R}^d) \), there exists a unique minimal closed carrier cone.

Proof. By Theorem 1,

\[
S^0(K_1 \cap K_2) = S^0(K_1) + S^0(K_2),
\]

for each pair of closed cones in \( \mathbb{R}^d \). Indeed, if \( f \in S^0(U) \), where \( U \ni K_1 \cap K_2 \neq \{0\} \), then there are \( U_i \) satisfying the condition \( U_i \cap U_i = U \). In the trivial case \( K_1 \cap K_2 = \{0\} \), above Remark is applicable because then there are open cones \( U_i \) such that \( K_i \in U_i \). Let \( K_1 \cup K_2 = \{0\} \).

We now show that (24) leads to the following dual relation for functionals

\[
S'^0(K_1 \cap K_2) = S^0(K_1) \cap S^0(K_2),
\]

where all the spaces are considered as subspaces of \( S^0(\mathbb{R}^d) \). The nontrivial part of (30) is the assertion that if a functional \( \varphi \in S^0(\mathbb{R}^d) \) is carried both by \( K_1 \) and by \( K_2 \), then \( K_1 \cap K_2 \) is also its carrier cone. Let \( v_i \) be continuous extensions of \( v \) to \( S^0(K_i) \) and let \( f \in S^0(K_1 \cap K_2) \). Using the decomposition \( f = f_1 + f_2 \), where \( f_i \in S^0(K_i) \), we put \( \hat{\varphi}(f) = v_1(f_1) + v_2(f_2) \) and claim that this extension of \( v \) to \( S^0(K_1 \cap K_2) \) is well defined. Indeed, if \( f = f'_1 + f'_2 \) is another decomposition, then

\[
f_1 - f'_1 = f'_2 - f_2 \in S^0(K_1) \cap S^0(K_2) = S^0(K_1 \cup K_2)
\]
and \( v_1(f_1 - f'_1) = v_2(f'_2 - f_2) \) because \( S^0(\mathbb{R}^d) \) is dense in \( S^0(K_1 \cup K_2) \) by Theorem 2 in [9]. The functional \( \hat{v} \) is obviously continuous under the inductive topology \( \mathcal{T} \) determined by the injections \( S^0(K_i) \to S^0(K_1 \cap K_2), \) \( i = 1, 2, \) and this topology coincides with the original topology \( \tau \) on \( S^0(K_1 \cap K_2) \) by the open mapping theorem. Indeed, \( \tau \) is not stronger than \( \mathcal{T} \) merely by the definition of \( \mathcal{T} \) and we can apply Grothendieck’s version [15] of the open mapping theorem, because \((S^0(K_1), \tau), (S^0(K_2), \tau)\) is an \( \mathcal{LF} \) space. This is the case because either of \( S^0(K_i) \) is an \( \mathcal{LF} \) space and \( \mathcal{T} \) coincides with the quotient topology of the sum \( S^0(K_1) \oplus S^0(K_2) \) modulo a closed subspace, see [13].

A formula analogous to (30) holds for the intersection of each finite family of closed cones and now the existence of the smallest carrier cone for \( v \) can be established by standard compactness arguments. Indeed, let \( K \) be the intersection of all its carrier cones and let \( U \) be an open cone such that \( K \subset U \). The projections of the cones complementary to these carriers cover the compact set \( \text{pr} \mathcal{C} U \) and we can choose a finite subcovering \( \text{pr} \mathcal{C} K \) of these open (in the topology of the unit sphere) covering. Consequently, the functional \( v \) is continuous in the topology of \( S^0(U) \) and \( K \) is its carrier cone, which completes the proof.

7. Conclusion

Theorem 2 provides a basis for the development of the theory of Fourier-Laplace transformation for the functional class \( S^0 \) and, in particular, for the derivation of a Paley-Wiener-Schwartz-type theorem analogous to Theorem 4 in [10] established for \( S^0_\alpha \). It is worth noting that the proof in Sec. 6 uses only the decomposability of \( f \in S^0(U) \) into a sum of elements of \( S^0(U_i) \) under the condition \( \bar{U}_1 \cap \bar{U}_2 \subset U \). The more precise formulation given in Theorem 1 becomes essential in studying the carrier properties of multilinear forms on \( S^0 \times \cdots \times S^0 \). A similar formulation was introduced in [17] in the framework of DFS spaces \( S^0_\alpha \). The generalization to multilinear forms is important for applications to QFT because the vacuum expectation values of quantum fields are such forms. Specifically, this development is necessary to an understanding of the interrelation between the asymptotic commutativity condition [3, 9], which ensures the normal spin-statistics connection and CPT invariance of nonlocal QFT, and the regularity properties [13] of retarded Green’s functions in momentum space that are required for the construction of scattering theory. The relation between the carriers of multilinear forms with respect to their arguments and the carrier cones of those functionals that are generated by these forms by the kernel theorem is also essential in using the analytic test functions in noncommutative field theory, where a causality condition [19] is formulated with a light wedge instead of the light cone. These questions will be discussed at length in a subsequent paper.

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