Statistical mechanics of Kerr-Newman dilaton black holes
and the bootstrap condition

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The Bekenstein-Hawking “entropy” of a Kerr-Newman dilaton black hole is computed in a
perturbative expansion in the charge-to-mass ratio. The most probable configuration for a gas of
such black holes is analyzed in the microcanonical formalism and it is argued that it does not satisfy
the equipartition principle but a bootstrap condition. It is also suggested that the present results
are further support for an interpretation of black holes as excitations of extended objects.

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In Refs. [1,2] we analyzed gases of black holes and black extended objects with various geometries. We have recently
solved the field equations of the Kerr-Newman dilaton black hole in the small charge-to-mass approximation [3]. In
this letter we show an attempt to use these solutions to calculate the quantum degeneracy of such a black hole and
from it obtain the microcanonical density of states for a gas of such black holes. We also improve previous analyses
of the gas of Reissner-Nordström and Kerr-Newman black holes.

We assume that \( \rho_{BH} \), the quantum degeneracy of a black hole, is given by the inverse probability \( (T^{-1}) \) for a
particle of tunneling out the horizon. In general one has \( T \sim \exp(-S_E) \), where \( S_E \) is the Euclidean action of the black
hole. However, it is known that the Euclidean action of a rotating black hole does not exist and the calculation of
the tunnelling probability must invoke some suitable prescription for getting meaningful results. This has been given
in Ref. [4] for a Kerr-Newman black hole and amounts to \( \rho_{BH} \sim c \exp(A_{BH}/4) \), where \( A_{BH}/4 \) is the Bekenstein-
Hawking “entropy” (\( A_{BH} \) is the surface area of the horizon) and \( c \) represents the quantum field theoretic effects (here
we neglect possible non-local contributions). Note that, in our interpretation [1,2], \( A_{BH}/4 \) is given no statistical
mechanical attribute.

The Einstein-Hilbert action of dilatonic black holes is given by \( (G = 1) \),

\[
S_{BH} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \phi)^2 - e^{-a\phi} F^2 \right] + \Sigma ,
\]

(1)

where the first term on the R.H.S. is the volume contribution obtained by integrating on the whole region outside
the outer horizon, \( R \) is the scalar curvature, \( \phi \) is the dilaton field, \( a \) its coupling constant, \( F \) is the Maxwell field and \( \Sigma \)
contains all surface terms. In Ref. [3] the field equations derived from the action in Eq. (1) were expanded in the
charge-to-mass ratio, \( Q/M \), and the perturbative static solution was found, which is of the form

\[
ds^2 = -\frac{\Delta \sin^2 \theta}{\Psi} dt^2 + \Psi (d\varphi - \omega dt)^2 + \rho^2 \left( \frac{(dr)^2}{\Delta} + (d\theta)^2 \right) .
\]

(2)

The latter can be simplified upon substituting for the (bare) parameters \( M \), \( Q \) and \( J \equiv \alpha M \) the ADM mass, charge
and angular momentum of the hole and also by shifting the radial coordinate, \( r \rightarrow r - a^2 Q^2/6M \) (see Ref. [3] for the
details). One finally obtains that the metric in Eq. (2) coincides (at order \( Q^2/M^2 \)) with the Kerr-Newman solution
\[ [3] \]. This implies that the background dilaton field,

\[
\phi = -a \frac{r}{\rho^2} \frac{Q^2}{M} + O(Q^4) ,
\]

(3)

does not affect the causal structure at order \( Q^2/M^2 \).

Once analytically continued to the quasi-Euclidean section, the surface term in Eq. (1) contributes the surface
action of the Kerr-Newman black hole \[ 3 \],

\[
\Sigma = \frac{\pi M (r_+^2 + a^2)}{\sqrt{M^2 - \alpha^2 - Q^2}} + \Sigma_a = \frac{\beta_H}{2} M ,
\]

(4)

where \( \Sigma_a \) represents (unknown) \( O(Q^4) \) corrections from the dilaton, \( \beta_H = 2\pi/\kappa \) is the period of the complexified
time \( T = it \), \( \kappa \) being the surface gravity of the Kerr-Newman dilaton black hole, and \( r_+ = M + \sqrt{M^2 - \alpha^2 - Q^2} \)
is the horizon of the Kerr-Newman black hole. Neither the dilaton nor the electromagnetic field add new surface
contributions because they both fall off fast enough at infinity and are regular on the horizon, where the measure of integration $\sqrt{\gamma}\sim \Delta = 0$.

The energy-momentum tensor $T_{EM}$ is traceless. Then, on using Einstein’s equations, one can prove that $R = (\nabla \phi)^2/2$, and the volume contribution to the action above on shell reduces to

$$S_{EM} = -\frac{1}{16\pi} \int d^4x \sqrt{\gamma} e^{-a \phi} F^2$$

$$= \frac{\beta_H}{4} \int_{-1}^{+1} d\mu \int_{r_+}^{r_+ \infty} dr \frac{1}{\Psi} \left[ e^{-a \phi} \left( \Delta A_\mu^2 + \delta A_\mu^2 \right) - e^a \phi \left( \Delta B_\mu^2 + \delta B_\mu^2 \right) \right].$$

(5)

The electromagnetic potentials $A$ and $B$ are given by 

$$A = Q \frac{r}{\rho^2} \left[ 1 - \left( \frac{1}{2} \frac{r}{\rho^2} \right) \frac{a^2 Q^2}{3 M} \right] + O(Q^5)$$

$$B = -Q \alpha \frac{\mu}{\rho^2} \left[ 1 - \left( \frac{1}{2} \frac{r}{\rho^2} \right) \frac{a^2 Q^2}{3 M} \right] + O(Q^5),$$

(6)

where $\rho^2 = r^2 + \alpha^2 \mu^2$ and the terms proportional to $Q^2$ inside the brackets are corrections to the Kerr-Newman potentials. The integration can be carried out explicitly up to order $Q^4$ to find

$$S_{EM} = -\frac{\beta_H}{2} Q^2 \frac{r_+}{r_+^2 + \alpha^2} + S_a + O(Q^6),$$

(7)

where the first term on the R.H.S. is the contribution from the Kerr-Newman electromagnetic potentials, and

$$S_a \simeq \frac{\beta_H}{2} \frac{a^2 Q^4}{24} \frac{3 r_+^4 + \alpha^2}{M^3 r_+^2},$$

(8)

is a term which vanishes for zero dilaton field.

The total euclidean action for the Kerr-Newman dilaton black hole is thus given by

$$S_{KND}(M, \alpha, Q; a) = \frac{\beta_H}{2} \left[ M - Q^2 \frac{r_+}{r_+^2 + \alpha^2} + \frac{a^2 Q^4}{24} \frac{3 r_+^4 + \alpha^2}{M^3 r_+^2} \right] + O(Q^6).$$

(9)

For $a = 0$ one recovers the expression $S_{KN}$ for the Kerr-Newman black hole [3], which diverges in the extremal case $M^2 = \alpha^2 + Q^2$. But, as stated above, the quantum degeneracy of states makes use of the surface area of the horizon [3],

$$A_{KN}/4 = S_{KN} - \beta_H \Omega J = \pi \left( r_+^2 + \alpha^2 \right),$$

(10)

which is instead well behaved. When $a \neq 0$ and in the limit $|a| \rightarrow M$, $S_a$ in Eq. (8) still diverges. Although our knowledge of the metric does not allow us to compute $\beta_H$ at order $Q^4$, we can guess that the term $-\beta_H \Omega J$ possibly compensates for such a divergence in the same way it was for $S_{KN}$ at order $Q^2$. This must be so since the surface area of the horizon is finite. Further, there are hints that the extremal case have zero Bekenstein-Hawking entropy. This is true, e.g., for the exact dilatonic Reissner-Nordström solution [3], for which it is also well known that the extremal limit does not commute with the limit of vanishing dilaton parameter ($a \rightarrow 0$). Therefore it is not clear whether a residual action (at $a = 0$) should be considered in the extremal case. In this work we shall not make any assumption about the Bekenstein-Hawking entropy for the extremal black holes.

According to previous results [3], the statistical mechanics of a gas of black holes can be consistently formulated only in the microcanonical ensemble. The microcanonical density of states for a dilute gas of black holes described by the Euclidean action in Eq. (9) of total energy $E$, total angular momentum $J$ and total charge $Q$ is given by

$$\Omega(E, J, Q; V, a) = \sum_{n=1}^{\infty} \Omega_n(E, J, Q; V, a),$$

(11)

where $V$ is the volume of the system and the density of states for the configuration with $n$ black holes is
for all

First we note that the high linear momentum states contribute negligibly [1], so that we neglect the extremum is not a maximum, since the corresponding total Bekenstein-Hawking entropy satisfies

where \( \rho_{KND} \sim c \exp(A_{KND}/4) \). We make the working assumption that black holes obey the particle-like dispersion relation \( E_i = \sqrt{m_i^2 + |p_i|^2} \), where \( E_i \) is the energy of the \( i^{th} \) black hole with linear momentum \( p_i \). Also, the integrations over the masses \( m_i \), the angular momenta \( j_i \) and the charges \( q_i \) are constrained to the domain \( m_i^2 \geq \alpha_i^2 + q_i^2, \forall i = 1, \ldots , n \). The mass \( m_0 < M \) is the mass of the least massive black hole in the gas.

For each \( n \), the corresponding density of states can be approximated by taking the most probable configuration \( \mathcal{P}_n = \{ (m_i', q_i', j_i') \}, i = 1, \ldots , n \) which satisfies the constraints expressed by the delta functions in the integrand above. First we note that the high linear momentum states contribute negligibly [3], so that we neglect \( |p_i| \) with respect to \( m_1 \) everywhere and set \( M = E \). Then we argue that \( \mathcal{P}_n \) does not satisfy the equipartition principle, that is \( \mathcal{P}_n \neq \mathcal{E}_n \equiv \{ (m_i = M/n, q_i = Q/n, j_i = J/n), i = 1, \ldots , n \} \).

It is possible to prove the statement above for a gas of \( n \) Reissner-Nordström black holes \( (a = j_i = 0, i = 1, \ldots , n) \), whose total Bekenstein-Hawking entropy is given by the sum

\[
S_{tot}(M, Q) = \sum_{i=1}^{n} S_{RN}(m_i, q_i),
\]

together with the constraints \( \sum_{i=1}^{n} m_i = M, \sum_{i=1}^{n} q_i = Q, m_i \geq |q_i|, i = 1, \ldots , n \), and \( S_{RN} = \pi (M + \sqrt{M^2 - Q^2})^2 \).

By using the Lagrange multiplier technique, one finds that the extrema of \( S_{tot} \) are given by solutions of the following equations

\[
\begin{align*}
 r_i^2 & = \lambda_m \sqrt{m_i^2 - q_i^2} \\
 r_i q_i & = -\lambda_Q \sqrt{m_i^2 - q_i^2} & i = 1, \ldots , n,
\end{align*}
\]

where \( r_i = m_i + \sqrt{m_i^2 - q_i^2} \) and \( \lambda_m, \lambda_Q \) are Lagrange multipliers corresponding respectively to the mass and charge constraint. These are algebraic equations of order four, thus one expects a certain number of different solutions to be available for each black hole in the configuration. However, since the black holes in the gas are supposed to interact negligibly with each others, the equations in Eq. (14) decouple in the index \( i \). Further, they are exactly the same for all \( i = 1, \ldots , n \) and one solution certainly exists which corresponds to equipartition of mass and charge. But this extremum is not a maximum, since the corresponding total Bekenstein-Hawking entropy satisfies

\[
S_{tot}(\mathcal{E}_n) = \sum_{i=1}^{n} S_{RN}(M/n, Q/n) = \frac{1}{n} S_{RN}(M, Q),
\]

where \( S_{RN}(M, Q) \) is also the total Bekenstein-Hawking entropy for a configuration in which one black hole carries all the mass and charge (assuming \( m_0 = 0 \) and \( M > \langle |Q| \rangle \)). Indeed, for \( n = 2 \) we are able to show graphically that \( S_{RN}(M, Q) \) is actually the absolute maximum \( S_{tot}(\mathcal{P}_n) \), while \( \mathcal{E}_n \) is only a saddle point (see Fig. 3). Moreover, if \( m_0 \neq 0 \), the most favored configuration is the one in which the light black hole is extremal, \( q_2 = m_2 = m_0 \) (see Fig. 3).

We are presently presenting on a numerical study which seems to support the conjecture that this is the case \( \forall n > 1 \) and that one has \( \mathcal{P}_n = \{ (M - (n - 1) m_0, Q - (n - 1) m_0), (m_i' = m_0, q_i' = m_0), i = 2, \ldots , n \} \), where the choice of \( i = 1 \) being the most massive black hole is arbitrary.

For \( a = 0 \) but \( j_i \neq 0 \), the proof that \( \mathcal{E}_n \) does not extremize the entropy follows along the same lines. One just needs to notice that the equations defining the possible extrema of the total Bekenstein-Hawking entropy of a gas of Kerr-Newman black holes,

\[
\frac{1}{4} A_{tot}(M, a, Q; a) = \frac{1}{4} \sum_{i=1}^{n} A_{KN}(m_i, \alpha_i, q_i),
\]

are still of the kind in Eq. (14) and do not contain cross terms in the index \( i \). However, \( A_{KN}(M/n, J/n, Q/n; a) \) is not well defined for \( n > J/M^2 \) since one would have \( m_i^2 = (M/n)^2 < (J/M)^2 = \alpha_i^2 \). Thus, in general, \( \mathcal{E}_n \) is not even an acceptable configuration. Further, a numerical analysis for \( n = 2 \) shows that \( \mathcal{P}_n \approx \{ (M - m_0, J, Q - m_0), (m_0, 0, m_0) \} \) for \( M > J/M \), \( M > |Q| \). By setting \( q_1 = q_2 = 0 \) one obtains plots of \( A_{tot} \) as function of \( m_1 \) and \( j_1 \) which look qualitatively the same as the ones in Figs. 4 and 5.
If $A_{KND}$ is regular in the limit $|\alpha| \to M$, then in the general case one is led to consider a configuration with one massive black hole which carries most of the charge and angular momentum and is surrounded by $n - 1$ lighter, extremal black holes. Then the density of states can be approximated by

$$\Omega_n(M, J, Q; V, a) \simeq \left[\frac{eV}{(2\pi)^3}\right]^n \frac{1}{n!} e^{\frac{n-1}{2} A_{KND}(m_0, \gamma m_0, \sqrt{1-\gamma^2} m_0; a)} \times e^{\frac{1}{4} A_{KND}(M-(n-1)m_0, J-(n-1)\gamma m_0^2, Q-(n-1)\sqrt{1-\gamma^2} m_0; a)},$$

where $0 \leq \gamma \leq 1$. Preliminary numerical calculations seem to suggest that $\gamma \ll 1$. We intend to perform a complete numerical treatment for several (large) values of $n$ in a future publication.

Now we can determine the most probable number $N$ of black holes in the gas, for which $d\Omega_n/dn|_{n=N} = 0$, and further approximate

$$\Omega(M, J, Q; V, a) \simeq \Omega_N(M, J, Q; V, a).$$

We can now check whether the gas of black holes we have been describing satisfies the bootstrap condition [8],

$$\lim_{M \to \infty} \frac{\Omega(M, J, Q; V, a)}{\rho_{KND}(M, J, Q; a)} = 1.$$

Indeed, it does, provided $m_0 = 0$ and

$$e^{N \Psi(N+1)/N!} = c.$$

As in the case of a gas of Schwarzschild black holes [1], this equation relates the constant $c$ to the volume $V$. The number $N$ is then given by $\Psi(N+1) \simeq \ln|eV/(2\pi)^3|$, where $\Psi$ is the psi function. Correspondingly, one obtains the inverse microcanonical temperature $\beta = d\ln\Omega/dE \simeq d\ln\Omega_N/dM$,

$$\beta = \beta_H \left(M - (N - 1)m_0, J - (N - 1)\gamma m_0^2, Q - (N - 1)\sqrt{1-\gamma^2} m_0; a\right),$$

which gives exactly the Hawking temperature $1/\beta_H$ when the bootstrap condition ($m_0 = 0$) is satisfied.

Our results show that the equilibrium state for a gas of Kerr-Newman dilaton black holes is very far from thermal equilibrium. Not only does one black hole acquire nearly all of the mass as in the Schwarzschild case, but it also acquires most of the charge and of the angular momentum of the whole gas, the other black holes in the gas being much lighter and extremal. Further, when the mass of the lighter black holes vanishes, the bootstrap condition is satisfied at high energy. Thus the interpretation of the inverse of the tunneling probability as obtained from the WKB approximation as the quantum mechanical degeneracy of states rather than as the statistical mechanical density of states holds for a gas of black holes each of whose members may be given arbitrary mass, charge and angular momentum in some initial state. Of course, the final equilibrium configuration of the gas is given by the inhomogeneous distribution described above.

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FIG. 1. The total Bekenstein-Hawking entropy $S_{rn}$ for a system of two Reissner-Nordström black holes with total mass $M = 4$ and total charge $Q = 1$ as a function of $m_1$ and $q_1$.

FIG. 2. When the lower limit for the mass of each black hole is $m_0 = 0.5$, the action $S_{rn}$ in Fig. 1 has a maximum for $m_1 = 4 - m_0$ and $q_1 = 1 - m_0$, meaning that the second black hole is extremal ($m_2 = q_2 = m_0$).