Topological and Curvature Effects in a Multi-fermion Interaction Model

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A multi-fermion interaction model is investigated in a compact spacetime with non-trivial topology and in a weakly curved spacetime. Evaluating the effective potential in the leading order of the $1/N$ expansion, we show the phase boundary for a discrete chiral symmetry in an arbitrary dimensions, $2 \leq D < 4$.

Keywords: Eight-fermion interaction; Dynamical symmetry breaking

1. Introduction

It is believed that a fundamental theory with higher symmetry was realized at the early stage of our universe and the symmetry was broken down to the observed symmetry in the present universe. The mechanism of the symmetry breaking may be found in a dynamics of the fundamental theory. Thus the origin of symmetry breaking is quite important to find a consequence of the fundamental theory in cosmological and astrophysical phenomena.

In QCD the chiral symmetry is dynamically broken according to a non-vanishing expectation value for a composite operator constructed by a quark and an anti-quark field, $\bar{q}q$. The chiral symmetry restoration is theoretically predicted in extreme conditions at the QCD scale. The symmetry restoration at high density may be found in phenomena of dense stars. The heavy ion collision at RIHC and LHC provide experimental data for the symmetry restoration at high temperature.

We can apply the dynamical mechanism to symmetry breaking at GUT era. It is one of candidates to describe symmetry breaking of the fundamental symmetry in GUT. It is natural to expect that the broken symmetry is restored in extreme conditions at GUT scale. Thus we have launched a plan to study the symmetry restoration at high temperature, high density and strong curvature. A topological effect is also interesting before inflationally expansion of our spacetime. In this paper we focus on the topological and the curvature effects.

A variety of works has been done in a simple four-fermion interaction model. The topological effect has been investigated in the spacetime with one compactified dimension, $S^1$, and the torus universe. It has been found that the finite size effect restore the broken symmetry if we adopt the anti-periodic boundary condition to the fermion fields. On the other hand, the fermion fields which possess the periodic
boundary condition contribute to break the symmetry. The curvature effects has been studied in two,7,8 three,9 four10 and arbitrary dimensions.11 The broken symmetry is restored if the spacetime curvature is positive and strong enough. However, the symmetry is always broken in a negative curvature spacetime.12 A combined effect has been also discussed in a weakly curved spacetime,13 the maximally symmetric spacetime14,15 and Einstein space.16,17 For a review, see for example Ref. 18.

In these works the four-fermion interaction model is considered to have something essential as a low energy effective model of a fundamental theory. To discuss the model dependence or independence of above results we have to extend the four-fermion interaction model. Here we consider a multi-fermion interaction model19–23 as a simple extension of the four-fermion interaction model and study the contribution from a higher dimensional operator. In Sec. 2 we introduce a multi-fermion interaction model which is considered in this paper. We show an explicit expression of the effective potential in an arbitrary dimensions, $2 \leq D < 4$. We consider a spacetime $R^{D-1} \otimes S^1$ in Sec. 3. Evaluating the effective potential, we study the topological effect. In Sec. 4 we assume that the spacetime curves slowly and investigate the curvature effect. In Sec. 5 we give some concluding remarks.

2. Multi-fermion Interaction Model

As in well-known, the chiral symmetry is dynamically broken by a simple scalar type four-fermion interaction model.24,25 It is a useful low energy effective theory of QCD to describe meson properties. The four-fermion interaction model is also useful as a simple toy model in the study of low energy phenomena of the strong coupling gauge theory at high energy scale in various environments. But there is no reason to neglect higher dimensional operators in extreme conditions at the early universe.

In the present paper we extend the model to include scalar type multi-fermion interactions,

$$ S = \int d^Dx \sqrt{-g} \left[ \sum_{i=1}^{N} \bar{\psi}_i i\gamma^\mu(x) \nabla_\mu \psi_i + \sum_{k=1}^{n} \frac{G_k}{N^{2k-1}} \left( \sum_{l=1}^{N} \bar{\psi}_l \psi_l \right)^{2k} \right], $$

where index $i$ represents flavors of the fermion field $\psi$, $N$ is the number of fermion flavors. We neglect the flavor index below. The multi-fermion interaction is unrenormalizable in four spacetime dimensions. The model depends on regularization methods. In this paper we adopt the dimensional regularization and regards the spacetime dimension, $D$, for the integration of internal fermion lines as one of parameters in the effective theory.26–30 In QCD it can be fixed to reproduce meson properties. Here we leave it as an arbitrary parameter to be fixed phenomenologically.

The action possesses the discrete chiral symmetry, $\bar{\psi}\psi \rightarrow -\bar{\psi}\psi$, and the global $SU(N)$ flavor symmetry, $\psi \rightarrow e^{i \sum_{a} \theta_a T^a} \psi$. The discrete chiral symmetry prohibits the fermion mass term. We can adopt the $1/N$ expansion as a non-
perturbative approach to investigate the dynamical symmetry breaking under the 
global $SU(N)$ symmetry.
For practical calculations it is more convenient to introduce the auxiliary fields and start from the action,
\[ S_y = \int d^Dx \sqrt{-g} \left[ \bar{\psi} i \gamma^\mu(x) \nabla_\mu \psi + \sum_{k=1}^n \frac{NG_k \sigma^{2k}}{(2G_1)^{2k}} - \frac{N}{2G_1} s \left( \sigma + \frac{2G_1}{N} \bar{\psi} \psi \right) \right]. \] (2)

The multi-fermion interactions in the original action are replaced by the auxiliary fields $\bar{\psi}$ and $\bar{\psi}$. If the auxiliary field $s$ develops a non-vanishing expectation value, the fermion field acquires a mass term and the chiral symmetry is eventually broken.

In this paper we only consider the case $n = 2$ for simplicity and concentrate on the contribution from the eight-fermion interaction. To study the phase structure of the model we calculate the expectation value for the auxiliary field, $s$. It is obtained by observing the minimum of the effective potential. It should be noted that the expectation value for the composite operator $\bar{\psi} \psi$ is given by the value of the auxiliary field $\sigma$ at the minimum of the effective potential. In the leading order of the $1/N$ expansion we can analytically integrate out the fermion field and get the effective potential,
\[ V(s, \sigma) = -\frac{N}{4G_1} s - \frac{1}{4G_1} s^2 + \frac{NG_2}{16G_1^2} \bar{\psi} \psi + i \text{Tr} \ln \langle x | i \gamma^\mu(x) \nabla_\mu s | x \rangle. \] (3)

The trace for the Dirac operator in Eq.(3) depends on the spacetime structure. The minimum of the effective potential satisfies the gap equation,

\[ \frac{\partial V}{\partial \sigma} \bigg|_{s} = \frac{\partial V}{\partial s} \bigg|_{\sigma} = 0. \] (4)

If we consider the four-fermion interaction model, $G_2 = 0$, in the Minkowski spacetime, $R^D$, the gap equation allows a nontrivial solution only for a negative $G_1$. The solution is give by
\[ \sigma = s = m_0 \equiv \left( \frac{(4\pi)^{D/2}}{\text{tr} \Gamma (1 - D/2)} \frac{1}{2G_1} \right)^{1/(D-2)}. \] (5)

Before investigating topological and curvature effects, we numerically calculate the effective potential in the Minkowski spacetime. We are interested in the model where the primordial symmetry is broken down at low energy scale. Thus we confine ourselves to a case of a negative $G_1$. In this case the effective potential only depends on $m_0$ and the rate of the coupling constants, $G_2/G_1^2$. We normalize all the mass scales by $m_0$ and set $g = G_2 m_0^2/G_1^3$. As is seen in Fig. a positive $g$ suppresses the symmetry breaking, while a negative $g$ enhances it. A new local minimum appears for a negative $g$.

3. Phase Structure in a Spacetime with Non-trivial Topology

One of extreme conditions we have to consider at the early universe is the topological effect. It is expected that the boundary condition for matter fields restricts how to...
compactify the spacetime. In this section we assume that one of space directions is compactified and investigate the multi-fermion interaction model on the cylindrical spacetime, $R^{D-1} \otimes S^1$. It is a flat spacetime with a non-trivial topology.

On $R^{D-1} \otimes S^1$ the effective potential, \( V(s, \sigma) \), is given by

\[
\frac{V(s, \sigma)}{Nm_0^D} = -\frac{\text{tr}1}{2(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \left( \frac{(\sigma - s)^2}{m_0^2} - \frac{s^2}{m_0^2} + \frac{g \sigma^4}{4 m_0^4} \right) + \frac{\text{tr}1}{2(4\pi)^{(D-1)/2}} \Gamma \left( 1 - \frac{D}{2} - \frac{1}{2} \right) \frac{1}{Lm_0} \sum_{n = -\infty}^{\infty} \left[ \left( \frac{(2n + \delta_{p,1})\pi}{Lm_0} \right)^2 + \frac{s^2}{m_0^2} \right]^{(D-1)/2},
\]

where we set $\delta_{p,1} = 0$ for the periodic and $\delta_{p,1} = 1$ for the anti-periodic boundary conditions. In Fig. 2 the effective potential is shown for $\delta_{p,1} = 1$, as $L$ varies. It is clearly seen that the broken symmetry is restored through the second order phase transition as $L$ is decreased. On the other hand only the broken phase is realized for the periodic boundary condition.

To see the situation more precisely we perform a rigorous analysis on the critical length $L_{cr}$ for $\delta_{p,1} = 1$ and find the boundary which divides the symmetric and the broken phases. For a non-negative $g$ only the second order phase transition is realized. In this case we can find the explicit expression for the critical length,\(^{18}\)

\[
L_{cr} m_0 = 2\pi \left[ \frac{\Gamma((3 - D)/2)}{\sqrt{\pi} \Gamma(1 - D/2)} \right]^{1/(D-2)} \frac{2\Gamma((3 - D)/2)}{\sqrt{\pi} \Gamma((3 - D)/2)} (2^{3-D} - 1) \zeta(3 - D).
\]

It is illustrated by the solid line in Fig. 3. The second local minimum appears for a negative $g$. The dashed line in Fig. 3 shows the length $L$ where the effective potential satisfies $V = 0$ at the second local minimum for $g = -0.1$. We plot the effective potential on this dashed line in Fig. 4. The broken symmetry is restored below both the solid and the dashed lines on $D - g$ plane for $g = -0.1$. 

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**Fig. 1.** Behavior of the effective potential in the three dimensional Minkowski spacetime $R^3$. We set $g = 0.1, 0$ and $-0.1$ and draw the solid, dotted and dashed lines, respectively.

**Fig. 2.** Behavior of the effective potential on $R^2 \otimes S^1$ for $\delta_{p,1} = 1$, as $L$ varies. We set $g = 0.1, 0$ and $-0.1$ and draw the solid, dotted and dashed lines, respectively.
Fig. 3. Critical length. The dashed line shows the length to satisfies $V = 0$ at the second local minimum for $g = -0.1$.

Fig. 4. Behavior of the effective potential on the dashed line in Fig.3 for $D = 2.25, 2.3, 2.35$ and $2.4$.

4. Phase Structure in a Weakly Curved Spacetime

At the early universe the symmetry breaking may be induced under the influence of the spacetime curvature. In this section we assume that the spacetime curved slowly and keep only terms independent of the curvature $R$ and terms linear in $R$. We discuss the curvature induced phase transition by observing the minimum of the effective potential. Following the procedure developed in Ref. 11, we obtain the effective potential up to linear in $R$,

$$V(s, \sigma) = -\frac{\text{tr} 1}{2(4\pi)^{D/2}} \Gamma \left( 1 - \frac{D}{2} \right) \left( \frac{(\sigma - s)^2}{m_0^2} - \frac{s^2}{4 m_0^2} + g \frac{\sigma^4}{4 m_0^2} \right) - \frac{\text{tr} 1}{(4\pi)^{D/2} D} \Gamma \left( 1 - \frac{D}{2} \right) \frac{|s|^D}{m_0^D} - \frac{\text{tr} 1}{(4\pi)^{D/2} D} \Gamma \left( 1 - \frac{D}{2} \right) \frac{R}{24 m_0^D} \Gamma \left( 1 - \frac{D}{2} \right) \frac{|s|^{D-2}}{m_0^{D-2}}. \tag{8}$$

In the four-fermion interaction model, $g = 0$, the phase transition takes place by varying the curvature $R$. The broken symmetry is restored for a large positive curvature $R > R_{cr} \geq 0$. The phase transition is of the first order for $2 < D < 4$. We evaluate the solution of the gap equation and find the analytic expression for the critical curvature,$^{11}$

$$\frac{R_{cr}}{m_0^2} = 6(D - 2) \left( \frac{(4 - D)D}{4} \right)^{(4-D)/(D-2)}. \tag{9}$$

It is drawn as a function of the spacetime curvature in Fig.6.

The critical curvature $R_{cr}$ is modified by the eight-fermion interaction. For a positive $g$ we find a smaller $R_{cr}$ except for two and four dimensions. The dashed line shows the curvature to satisfy $V = 0$ at the second minimum for $g = -0.1$. Thus the symmetric phase is realized above both the solid and the dashed lines for $g = -0.1$. In Fig.6 we show the behavior of the effective potential near the point A.
on the solid line in Fig. 5. It is observed that the first local minimum disappears at the point A. Finally, we noted that only the broken phase is realized in a spacetime with a negative curvature.\textsuperscript{12}

5. Conclusion

We have investigated the multi-fermion interaction model under the influence of the spacetime topology and curvature. In the practical calculation the four- and the eight-fermion interactions is studied. Evaluating the effective potential in the leading order of the $1/N$ expansion, we show the phase structure of the model in an arbitrary dimension, $2 \leq D < 4$.

On $R^{D-1} \otimes S^1$ the finite size effect restores the broken chiral symmetry for fermion fields with the anti-periodic boundary condition. In this case the theory is equivalent to the finite temperature field theory. The phase transition is of the second order for a non-positive $g$. We found that the phase boundary is not modified by the eight-fermion interaction with a positive $g$. For a negative $g$ an additional local minimum contributes to the phase boundary. A new boundary appears in lower dimensions and the first order phase transition takes place.

The spacetime curvature also contributes to the phase structure of the theory. For $2 \leq D < 4$ the broken symmetry is restored through the first order phase transition as increasing the curvature $R$. The eight-fermion interaction with a positive $g$ suppresses the chiral symmetry breaking and changes the phase boundary. For a negative $g$ we observe a contribution from an additional local minimum to enhance the broken symmetry in lower dimensions.

In the present paper our study is restricted on the analysis of the phase structure of the theory. We are interested in applying our results to critical phenomena at the early universe. We will continue our work and hope to report on these problems.
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