Casino \textit{baccara chemin de fer} as a bimatrix game

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Abstract

The casino game of \textit{baccara chemin de fer} is a bimatrix game, not a matrix game, because the house collects a five percent commission on Banker wins. We generalize the game, allowing Banker’s strategy to be unconstrained and assuming a \(100\alpha\)% percent commission on Banker wins, where \(0 \leq \alpha < 2/5\). Assuming for simplicity that cards are dealt with replacement, we show that, with one exception at \(\alpha = \alpha_0 \approx 0.140705\), there is a unique Nash equilibrium, and we evaluate it. Player’s equilibrium mixed strategy depends explicitly on \(\alpha\), whereas Banker’s equilibrium mixed strategy depends only on whether \(\alpha < \alpha_0\) or \(\alpha > \alpha_0\).

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1 Introduction

The classical parlor game of \textit{baccara chemin de fer} (briefly, \textit{baccara}) played a key role in the development of game theory. Bertrand’s (1889, pp. 38–42) analysis of whether Player should draw or stand on a two-card total of 5 was the starting point of Borel’s investigation of strategic games in the 1920s (Dimand and Dimand 1996, p. 132). Nevertheless, a solution of the game would have to wait until the dawn of the computer age. Kemeny and Snell (1957), assuming that cards are dealt with replacement from a single deck and that each of Player and Banker sees the total of his own two-card hand but not its composition, found the unique solution of the resulting \(2 \times 2^{88}\) matrix game.

Here we consider the modern casino game of \textit{baccara}, as it is played at \textit{Le Casino de Monte-Carlo} and elsewhere. It differs in two respects from the classical parlor game. First, Banker’s strategy is highly constrained, and second,
the casino collects a five percent commission on Banker wins. To better understand this game, let us consider a generalization of it. First, we impose no constraints on Banker’s strategy, so as to avoid any constraints that may be suboptimal. Second, we generalize the amount of the commission, assuming that the casino collects a commission of \(100\alpha\) percent on Banker wins, where \(0 \leq \alpha < \frac{2}{5}\). Results are more easily stated if we assume that \(0 \leq \alpha < \frac{1}{15}\), and it can be argued that a commission larger than \(\frac{1}{15}\) (i.e., \(6 \frac{2}{3}\) percent) would be counter-productive. Nevertheless, the more general assumption leads to a more interesting conclusion.

Assuming a one-unit bet, the three-dimensional vector whose components are Player’s payoff, Banker’s payoff, and the casino’s payoff is \((1, -1, 0)\) if Player wins, \((-1, 1 - \alpha, \alpha)\) if Banker wins, and \((0, 0, 0)\) if there is a tie. Thus, we have a three-person zero-sum game. But since the casino has no strategy options, the game is best regarded as a two-person non-zero-sum game (if \(\alpha > 0\)) between Player and Banker, or a bimatrix game.

The rules of baccara are as follows. The game is dealt from a sabot, or shoe, comprising six 52-card decks mixed together. Denominations A, 2–9, 10, J, Q, K have values 1, 2–9, 0, 0, 0, 0, respectively, and suits are irrelevant. The total of a hand, comprising two or three cards, is the sum of the values of the cards, modulo 10. In other words, only the final digit of the sum is used to evaluate a hand. Two cards are dealt face down to Player and two face down to Banker, and each looks only at his own hand. The object of the game is to have the higher total (closer to 9) at the end of play. A two-card total of 8 or 9 is a natural. If either hand is a natural, the game is over. If neither hand is a natural, Player then has the option of drawing a third card. If he exercises this option, his third card is dealt face up. Next, Banker, observing Player’s third card, if any, has the option of drawing a third card. This completes the game, and the higher total wins. Winning bets on Player’s hand are paid at even odds, with Banker, as the name suggests, playing the role of the bank. Losing bets on Player’s hand, less the casino’s commission, are collected by Banker. Hands of equal total result in a tie or push (no money is exchanged). Since multiple players can bet on Player’s hand, Player’s strategy is restricted. He must draw on a two-card total of 4 or less and stand on a two-card total of 6 or 7. When his two-card total is 5, he is free to draw or stand as he chooses. (The decision is usually made by the player with the largest bet.) Banker, on whose hand no one can bet, has no constraints on his strategy in the generalized version of the game considered here.

To identify the relevant bimatrix, we assume, like Kemeny and Snell (1957), that cards are dealt with replacement from a single 52-card deck, and that each of Player and Banker sees only the total of his own two-card hand, not its composition. This is of course a simplifying assumption. In the context of the parlor game, Downton and Lockwood (1975) and Ethier and Gámez (2013) assumed that cards are dealt without replacement from a \(d\)-deck shoe, and obtained results that are very similar to those found by Kemeny and Snell. All previous authors, including, in addition to those just listed, Foster (1964), Kendall and Murchland (1964), Downton and Holder (1972), and Deloche and
Oguer (2007), have regarded baccara as a zero-sum game.

Clearly, Player has two pure strategies, stand on a two-card total of 5 or draw on a two-card total of 5. On the other hand, Banker has a stand-or-draw decision in each of 88 strategic situations: His two-card total is \( j \in \{0, 1, \ldots, 7\} \) and Player’s third card is \( k \in \{0, 1, \ldots, 9, \emptyset\} \) (\( k = \emptyset \) if Player stands). Thus, we have a \( 2 \times 2^{88} \) bimatrix game. Let \( W \) and \( L \) be the \( 2 \times 2^{88} \) matrices with \((i, j)\)th entry being the win and loss probabilities, respectively, for Player when Player adopts pure strategy \( i \) and Banker adopts pure strategy \( j \). Then

\[
A := W - L \quad \text{and} \quad B_\alpha := (1 - \alpha)L - W
\]

are the payoff matrices for Player and Banker, respectively, making the payoff bimatrix \((A, B_\alpha)\). The special case \( \alpha = 0 \) corresponds to the parlor game.

In Section 2 we use strict dominance to reduce the game to \( 2 \times 2^4 \), at least if \( 0 \leq \alpha < 1/15 \), and we evaluate the resulting payoff bimatrix. In Section 3 we show that there is a unique Nash equilibrium, at least if \( 0 \leq \alpha < 1/15 \), and we evaluate it. We also extend this result to \( 0 \leq \alpha < 2/5 \) but find that there is an exception at \( \alpha = \alpha_0 \approx 0.140705 \), in which case multiple Nash equilibria exist. Section 4 considers modern constrained Banker strategies, and Section 5 discusses the origin of the rules of the nonstrategic form of baccara.

2 Evaluating the payoff bimatrix

The matrices \( W, L, A \), and \( B_\alpha \) can all be derived from the same formula. The distribution of the total of a two-card hand is

\[
p_2(i) := \frac{16 + 9\delta_{i,0}}{13^2}, \quad i = 0, 1, \ldots, 9,
\]

where \( \delta_{i,j} \) is the Kronecker delta, and the distribution of the value of a card is

\[
p_1(k) := \frac{1 + 3\delta_{k,0}}{13}, \quad k = 0, 1, \ldots, 9.
\]

Let \( M : \{0, 1, \ldots\} \mapsto \{0, 1, \ldots, 9\} \) be the function \( M(i) := \text{Mod}(i, 10) \), the remainder when \( i \) is divided by 10. For \( S = \{0, 1, 2, 3, 4\} \) or \( S = \{0, 1, 2, 3, 4, 5\} \), let \( S^c := \{0, 1, \ldots, 7\} - S \), and for \( T \subset \{0, 1, \ldots, 7\} \times \{0, 1, \ldots, 9, \emptyset\} \), let \( T^c \) be the complement of \( T \) with respect to this product set. We think of \( S \) as the set of Player two-card totals on which Player draws and \( T \) as the set of pairs of Banker two-card totals and Player third cards on which Banker draws.

Then, given a real-valued function \( f \) on \( \mathbb{Z} \), define the \( 2 \times 2^{88} \) matrix \( C \) to have \((S, T)\) entry

\[
c_{S,T} := \left( \sum_{i=0}^9 \sum_{j=0}^9 - \sum_{i=0}^7 \sum_{j=0}^7 \right) p_2(i)p_2(j)f(i - j) + \sum_{i \in S} \sum_{j=0}^9 \sum_{k=0}^9 \sum_{l=0}^9 1_T((j, k))p_2(i)p_2(j)p_1(k)p_1(l)f(M(i + k) - M(j + l))
\]
Notice that term 1 corresponds to the case in which Player and/or Banker has a natural. Term 2: both Player and Banker draw. Term 3: Player draws and Banker stands. Term 4: Player stands and Banker draws. Term 5: both Player and Banker stand (but without a natural).

If $f$ is the indicator of the set of positive integers, then $C = W$. If $f$ is the indicator of the set of negative integers, then $C = L$. If $f(x) := \text{sgn}(x)$, then $C = A$, and if $f(x) = \begin{cases} 1 - \alpha & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x > 0, \end{cases}$ then $C = B_{\alpha}$.

Before we try to evaluate $A$ and $B_{\alpha}$, let us reduce the size of the game considerably by eliminating strictly dominated columns of the bimatrix $(A, B_{\alpha})$. A column will be strictly dominated if the corresponding column of $-B_{\alpha}$ (regarded as the payoff matrix for a zero-sum game) is strictly dominated. Assuming $0 \leq \alpha < 1/15$, we apply Lemma 5.1.3 of Ethier (2010, p. 166) to $-B_{\alpha}$ and find that Banker’s optimal move does not depend on Player’s strategy in 84 of the 88 strategic situations, the exceptions being $(3, 9), (4, 1), (5, 4), \text{ and } (6, \emptyset)$. See Table 1.

To elaborate on this point, given $(j, k) \in \{0, 1, \ldots, 7\} \times \{0, 1, \ldots, 9, \emptyset\}$, let $b_{u,v}(j, k)$ be the conditional expected profit of Banker, given that Banker’s two-card total is $j$ and Player’s third card is $k$. Here we assume Player draws (resp., stands) on a two-card total of 5 if $u = 1$ (resp., $u = 0$), and Banker draws (resp., stands) if $v = 1$ (resp., $v = 0$). If $k \neq \emptyset$, we can compute

\[ b_{u,0}(j, k) = \frac{\sum_{i=0}^{4+u} p_2(i) f(M(i + k) - j)}{\sum_{i=0}^{4+u} p_2(i)}, \]
\[ b_{u,1}(j, k) = \frac{\sum_{i=0}^{4+u} \sum_{l=0}^{9} p_2(i)p_1(l) f(M(i + k) - M(j + l))}{\sum_{i=0}^{4+u} p_2(i)}, \]

with $f$ as in (1). Then, for example,

\[ b_{0,0}(3, 9) = \frac{23 - 48\alpha}{89}, \quad b_{0,1}(3, 9) = \frac{12(19 - 52\alpha)}{1157}, \]
\[ b_{1,0}(3, 9) = \frac{7 - 48\alpha}{105}, \quad b_{1,1}(3, 9) = \frac{4(49 - 176\alpha)}{1365}. \]
Table 1: Banker’s optimal move, indicated by D (draw) or S (stand), except in the four cases indicated by * in which it depends on Player’s strategy. The commission $\alpha$ on Banker wins is assumed here to satisfy $0 \leq \alpha < \frac{1}{15}$. The shading of D entries is intended to make the table easier to read.

| Banker’s two-card total | Player’s third card (∅ if Player stands) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ∅ |
|-------------------------|----------------------------------------|---|---|---|---|---|---|---|---|---|---|---|
| 0                       | D                                      | D | D | D | D | D | D | D | D | D | D | D |
| 1                       | D                                      | D | D | D | D | D | D | D | D | D | D | D |
| 2                       | D                                      | D | D | D | D | D | D | D | D | D | D | D |
| 3                       | D                                      | D | D | D | D | D | D | D | D | S | * | D |
| 4                       | S                                      | * | D | D | D | D | D | D | S | S | D | D |
| 5                       | S                                      | S | S | S | S | * | D | D | S | S | D | D |
| 6                       | S                                      | S | S | S | S | S | D | S | S | S | S | S |
| 7                       | S                                      | S | S | S | S | S | S | S | S | S | S | S |

\begin{align}
  b_{0,0}(4, 1) &= \frac{41 - 57\alpha}{89}, \\
  b_{0,1}(4, 1) &= \frac{3(142 - 241\alpha)}{1157}, \\
  b_{1,0}(4, 1) &= \frac{25 - 57\alpha}{105}, \\
  b_{1,1}(4, 1) &= \frac{110 - 257\alpha}{455}, \\
  b_{0,0}(5, 4) &= -\frac{23 + 25\alpha}{89}, \\
  b_{0,1}(5, 4) &= \frac{60(5 + 6\alpha)}{1157}, \\
  b_{1,0}(5, 4) &= -\frac{39 + 25\alpha}{105}, \\
  b_{1,1}(5, 4) &= -\frac{4(41 + 30\alpha)}{455}.
\end{align}

If $k = ∅$ (i.e., Player stands), we find that

\begin{align}
  b_{u,0}(j, ∅) &= \sum_{i=5+u}^{7} p_2(i)f(i - j)/\sum_{i=5+u}^{7} p_2(i), \\
  b_{u,1}(j, ∅) &= \sum_{i=5+u}^{7} \sum_{l=0}^{9} p_2(i)p_1(l)f(i - M(j + l))/\sum_{i=5+u}^{7} p_2(i),
\end{align}

with $f$ as in (1). Then, for example,

\begin{align}
  b_{0,0}(6, ∅) &= -\frac{\alpha}{3}, \\
  b_{0,1}(6, ∅) &= -\frac{3 + 4\alpha}{13}, \\
  b_{1,0}(6, ∅) &= -\frac{\alpha}{2}, \\
  b_{1,1}(6, ∅) &= -\frac{3 + 4\alpha}{26}.
\end{align}
In cases 2–5, \( b_{0,0} > b_{0,1} \) and \( b_{1,0} < b_{1,1} \), for at least \( 0 \leq \alpha < 1/6 \). More precisely, these inequalities hold for (3,9) if \( 0 \leq \alpha \leq 1 \), for (4,1) if \( 0 \leq \alpha < 1/6 \), for (5,4) if \( 0 \leq \alpha < 3/7 \), and for (6,∅) if \( 0 \leq \alpha < 2/5 \). These are the four cases in which asterisks appear in Table 1.

The case of (3,9) (Banker total 3, Player third card 9) occurred in the climactic hand in Ian Fleming’s (1953) novel *Casino Royale*, with James Bond in the role of Player, Le Chiffre in the role of Banker, and 32 million francs at stake. As Fleming noted (Chap. 13), “Holding a three and giving nine is one of the moot situations at the game. The odds are so nearly divided between to draw or not to draw.” Le Chiffre chose to draw, drawing 5 for a total of 8. Then Bond’s hand was turned over to reveal two queens for a total of 9 and the win. Technically, the game they played was not quite *baccara chemin de fer*, in which the role of Banker rotates among the players. Instead, Le Chiffre purchased the role of Banker for one million francs (Chap. 9), presumably in lieu of a commission.

We also notice that
\[
\begin{align*}
b_{0,0}(6,6) &= -\frac{16(2 + \alpha)}{89}, \\
b_{0,1}(6,6) &= -\frac{3(121 + 105\alpha)}{1157}, \\
b_{1,0}(6,6) &= -\frac{16(1 + 2\alpha)}{105}, \\
b_{1,1}(6,6) &= -\frac{203 + 491\alpha}{1365}.
\end{align*}
\]
Here \( b_{0,0} < b_{0,1} \) and \( b_{1,0} < b_{1,1} \) if \( 0 \leq \alpha < 1/15 \). The same kind of calculation can be done in all 88 strategic situations, and we find that Table 1 applies if \( 0 \leq \alpha < 1/15 \), which is the reason we adopt this restriction on \( \alpha \), at least initially.

This reduces the bimatrix game to \( 2 \times 2^4 \), so we now regard \( W, L, A, \) and \( B_\alpha \) as being \( 2 \times 2^4 \) matrices. We can evaluate \( A \) and \( B_\alpha \), but for typographical reasons we prefer to display their transposes. We find that

\[
A^T = (W - L)^T = \begin{pmatrix}
S \text{ on } 5 & D \text{ on } 5 \\
SSSS & -4636 -3585 \\
SSSD & -2764 -4001 \\
SSDS & -4635 -3600 \\
SSDD & -2763 -4016 \\
SDSS & -4529 -3590 \\
SDSD & -2657 -4006 \\
SDDS & -4528 -3605 \\
SDDD & -2656 -4021 \\
DSSS & -4565 -3690 \\
DSSD & -2693 -4106 \\
DSPS & -4564 -3705 \\
DSPD & -2692 -4121 \\
DDSS & -4458 -3695 \\
DDSD & -2586 -4111 \\
DDDS & -4457 -3710 \\
DDDD & -2585 -4126
\end{pmatrix}
\]
where, for example, DSDS means draw on (3, 9), stand on (4, 1), draw on (5, 4), and stand on (6, ∅). This is the payoff matrix for the zero-sum game corresponding to the special case in which \( \alpha = 0 \). It was first found by Kemeny and Snell (1957). Also,

\[
B_\alpha^T = ((1-\alpha)L - W)^T = \\
\begin{pmatrix}
\text{S on 5} & \text{D on 5} \\
SSSS & (9272 - 278353\alpha) 7170 - 276363\alpha \\
SSSD & 5528 - 277937\alpha 8002 - 278443\alpha \\
SSDS & 9270 - 278423\alpha 7200 - 276433\alpha \\
SSDD & 5526 - 278007\alpha 8012 - 278503\alpha \\
SDSS & 9058 - 278317\alpha 7180 - 276423\alpha \\
SDSD & 5314 - 277901\alpha 8032 - 278573\alpha \\
SDDD & 9056 - 278387\alpha 7210 - 276493\alpha \\
DSSS & 5312 - 277971\alpha 8042 - 278603\alpha \\
DSSD & 9130 - 278353\alpha 7380 - 276523\alpha \\
DSDS & 5386 - 277937\alpha 8212 - 278673\alpha \\
DSDD & 9128 - 278423\alpha 7410 - 276593\alpha \\
DDSS & 5384 - 278007\alpha 8242 - 278663\alpha \\
DDSD & 9116 - 278317\alpha 7390 - 276583\alpha \\
DDDD & 5172 - 277901\alpha 8222 - 278663\alpha \\
\end{pmatrix} \\
\begin{pmatrix}
8 \\
(13)^6.
\end{pmatrix}
\]

Continuing to assume that \( 0 \leq \alpha < 1/15 \), we can further reduce these \( 2 \times 16 \) matrices to \( 2 \times 5 \) matrices using strict dominance. Specifically, \( A \) is replaced by the matrix (also denoted by \( A \)) whose transpose is

\[
A^T = \\
\begin{pmatrix}
\text{S on 5} & \text{D on 5} \\
SSSS & -4636 -3585 \\
SSSD & -4635 -3600 \\
SSDS & -4564 -3705 \\
SSDD & -2692 -4121 \\
SDSS & -2585 -4126 \\
\end{pmatrix} \\
\begin{pmatrix}
16 \\
(13)^6.
\end{pmatrix}
\]

and \( B_\alpha \) is replaced by the matrix (also denoted by \( B_\alpha \)) whose transpose is

\[
B_\alpha^T = \\
\begin{pmatrix}
\text{S on 5} & \text{D on 5} \\
SSSS & (9272 - 278353\alpha) 7170 - 276363\alpha \\
SSSD & 9270 - 278423\alpha 7200 - 276433\alpha \\
SSDS & 9128 - 278423\alpha 7410 - 276593\alpha \\
SSDD & 5384 - 278007\alpha 8242 - 278663\alpha \\
SDSS & 5170 - 277971\alpha 8252 - 278733\alpha \\
\end{pmatrix} \\
\begin{pmatrix}
8 \\
(13)^6.
\end{pmatrix}
\]

To confirm this, we note that \( B_\alpha = (1 - 15\alpha)B_0 + 15\alpha B_{1/15} \) for \( 0 \leq \alpha < 1/15 \), so it is enough to confirm it with \( \alpha = 0 \) and with \( \alpha = 1/15 \). Labeling the columns from 0 to 15, columns 4, 6, and 12 are strictly dominated by column 10, and columns 5, 7, and 13 are strictly dominated by column 11. In addition,
columns 1, 3, 8, 9, and 14 are strictly dominated by mixtures of columns 0 and 11, 0 and 10, 10 and 11, and 10 and 11, respectively. Thus, only columns 0, 2, 10, 11, and 15 remain, which results in the $2 \times 5$ matrices whose transposes are displayed above.

3 Nash equilibrium

We begin by evaluating the safety levels for this $2 \times 5$ bimatrix game. The safety level is the amount a player can assure himself, regardless of what his opponent does. For Player it is

$$v_p := \text{val}(A) = \frac{-679568}{11(13)^6} \approx -0.0127991,$$

(6)

where $\text{val}(A)$ denotes the value of the zero-sum game with payoff matrix $A$, and this is a result of Kemeny and Snell (1957). (Columns DSDS and DSDD specify the kernel.) Player’s maxmin strategy, which assures the safety level, is a $(1 - p, p)$ mixture of standing on 5 and drawing on 5, where $p := 9/11$.

For Banker, the safety level is

$$v_B := \text{val}(B^T_\alpha) = \frac{8(84946 - 3099233\alpha + 1668708\alpha^2)}{(11 - 6\alpha)(13)^6}.$$ (11 - 6\alpha)(13)^6)

(7)

(Again, rows DSDS and DSDD of $B^T_\alpha$ specify the kernel.) Banker’s maxmin strategy is $q := (0, 0, 1 - q, q, 0)$, where

$$q := \frac{859 - 915\alpha}{2288 - 1248\alpha}.$$ (8)

These assertions can be confirmed by checking that $qB^T_\alpha \geq v_B(1, 1)$ and $B^T_\alpha p^T \leq v_B(1, 1, 1, 1, 1)^T$ for $0 \leq \alpha < 1/15$, where $p := (1 - p, p)$ and

$$p := \frac{9 - \alpha}{11 - 6\alpha}.$$ (9)

If $\alpha = 1/20$, then $p = 179/214 \approx 0.836449$,

$$q = \frac{16265}{44512} \approx 0.365407, \quad \text{and} \quad v_B = \frac{-2633752}{535(13)^6} \approx -0.0101991.$$

We know that there exists a Nash equilibrium, that is, a pair of mixed strategies, one for Player and one for Banker, such that each is a best response to the other. We use a method of Dickhaut and Kaplan (1991), explained in more detail by Avis et al. (2010, Algorithm 1), namely the support enumeration algorithm, to find a Nash equilibrium and show it is unique.

First, it is easy to check that there is no pure Nash equilibrium, that is, for each $\alpha \in [0, 1/15)$, no entry of the bimatrix $(A, B_\alpha)$ is such that its first component is a column maximum of $A$ and its second component is a row.
maximum of $B_\alpha$. Next we consider Nash equilibria $(p,q)$ with $p$ and $q$ both having two nonzero components (supports of size two). There are $\binom{3}{2} = 10$ such cases, and only one of them results in a Nash equilibrium, namely the case in which the support of $q$ comprises DSDS and DSDD. With $q := (0, 0, 1 - q, q, 0)$, we choose $q$ to make Player indifferent as to which of his two pure strategies he adopts, that is, $-4564(1 - q) - 2692q = 3705(1 - q) - 4121q$ or $q = 859/2288$.

With $p := (1 - p, p)$, we choose $p$ to make Banker indifferent as to which of DSDS and DSDD he adopts while preferring none of the other three pure strategies to these two. Specifically,

$$(1 - p)(9128 - 278423\alpha) + p(7410 - 276593\alpha)$$

$$= (1 - p)(5384 - 278007\alpha) + p(8242 - 278673\alpha),$$

which has solution (9), and the DSDS and DSDD components of $pB_\alpha$ are maximal, at least for $0 \leq \alpha < 1/15$. For the other nine supports of size two, the maximality condition fails in five cases with $q \in [0, 1]$ holding, $q \in [0, 1]$ fails in three cases with the maximality condition holding, and both conditions fail in one case.

This suffices for the uniqueness of Nash equilibria, provided our game is nondegenerate, which means that no mixed strategy of support size $s \geq 1$ has more than $s$ pure best responses. This hypothesis can be verified. For $s = 1$, it amounts to checking that $A$ has no equal entries in the same column and $B_\alpha$ has no equal and maximal entries in the same row, assuming $0 \leq \alpha < 1/15$. For $s = 2$, it amounts to checking that, for no $p \in (0, 1)$ and $\alpha \in [0, 1/15)$ does $(1 - p, p)B_\alpha$ have three or more of its five entries equal and maximal. We summarize our result as follows.

**Theorem 1.** The bimatrix game with payoff bimatrix $(A, B_\alpha)$ has a unique Nash equilibrium for $0 \leq \alpha < 1/15$. Player’s equilibrium mixed strategy is to draw on 5 with probability (9). Banker’s equilibrium mixed strategy is as in Table 2, except that he draws on $(3, 9)$, stands on $(4, 1)$, draws on $(5, 4)$, and mixes on $(6, \emptyset)$, drawing with probability $q = 859/2288$.

If both Player and Banker use their equilibrium strategies, their expected payoffs are their safety levels. The expected payoffs must be at least as large as the safety levels, but in fact they are no larger. Banker’s safety level $v_B$ is greater than or equal to Player’s safety level $v_P$ if and only if $0 \leq \alpha \leq \alpha'$, where

$$\alpha' := \frac{34601239 - \sqrt{1060031672799697}}{36711576} \approx 0.0556531.$$ 

A commission on Banker wins greater than $\alpha'$ would be counter-productive because there would then be no incentive for players to take the role of Banker.

Despite this, let us extend the results of this section to $0 \leq \alpha < 2/5$. The number

$$\alpha_0 := \frac{1159 - \sqrt{974841}}{1220} \approx 0.140705$$

plays a role in the extension.
Theorem 2. The bimatrix game with payoff bimatrix $(A, B, \alpha)$ has a unique Nash equilibrium for $\alpha \in [0, \alpha_0) \cup (\alpha_0, 2/5)$. Player’s equilibrium mixed strategy is to draw on 5 with probability $p$, Banker’s equilibrium mixed strategy depends on whether $\alpha < \alpha_0$ or $\alpha > \alpha_0$. In the former case, it is as in Table 4, except that Banker draws on (3, 9), stands on (4, 1), draws on (5, 4), and mixes on (6, 2), drawing with probability $q = 859/2288$. In the latter case, it is as in Table 4, except that Banker draws on (3, 9), stands on (4, 1), draws on (5, 4), and mixes on (6, 2), drawing with probability $q = 811/2288$. If $\alpha = \alpha_0$, then there are exactly four extreme Nash equilibria, as described in the proof.

Proof. The first step is to extend Table 4 to $0 \leq \alpha < 2/5$. Only two changes are needed. The (6, 6) entry is D for $0 \leq \alpha < 1/15$ and * for $\alpha \geq 1/15$; and the (4, 1) entry is * for $0 \leq \alpha \leq 1/6$ and S for $\alpha > 1/6$. On the $\alpha$ interval $[1/15, 1/6]$, we use strict dominance to reduce the game to $2 \times 2^5$, labeling Banker’s pure strategies 0–31 corresponding to Banker’s moves on (3, 9), (4, 1), (5, 4), (6, 6), (6, $\emptyset$), and then to $2 \times 8$, with pure strategies 2, 6, 20, 21, 22, 23, 29, 31 remaining. On the $\alpha$ interval $(1/6, 2/5)$, we use strict dominance to reduce the game to $2 \times 2^4$, labeling Banker’s pure strategies 0–15 corresponding to Banker’s moves on (3, 9), (5, 4), (6, 6), (6, $\emptyset$), and then to $2 \times 9$, with pure strategies 0, 2, 6, 8, 10, 12, 13, 14, 15 remaining.

The bimatrix is nondegenerate except when $\alpha = 1/15$, $\alpha = \alpha_0$, or $\alpha = 1/6$, and except in four other cases:

$$\alpha = \frac{64 - \sqrt{3526}}{57}, \quad \alpha = \frac{6}{35}, \quad \alpha = \frac{21 - \sqrt{301}}{14}, \quad \alpha = \frac{130 - 3\sqrt{998}}{107}. \quad (10)$$

For $\alpha \in [1/15, 2/5]$ we apply the algorithm used for $\alpha \in [0, 1/15)$, namely the support enumeration algorithm. For $\alpha \in (1/15, \alpha_0)$, we have a unique Nash equilibrium, namely $p := (1 - p, p)$ with $p$ as in (5) and $q := (0, 0, 0, 0, 1 - q, q, 0, 0)$ with $q = 859/2288$. The support of $q$ consists of Banker’s pure strategies 22 and 23, or DSDDS and DSDDD on (3, 9), (4, 1), (5, 4), (6, 6), (6, $\emptyset$). For $\alpha \in (\alpha_0, 1/6)$, we can use strict dominance to eliminate one of Banker’s eight remaining strategies (namely 31), and we have a unique Nash equilibrium, namely $p := (1 - p, p)$ with $p$ as in (5) and $q := (0, 0, 1 - q, q, 0, 0)$ with $q = 811/2288$; the support of $q$ consists of Banker’s pure strategies 20 and 21, or DSDSS and DSDSD on (3, 9), (4, 1), (5, 4), (6, 6), (6, $\emptyset$). For $\alpha \in (1/6, 2/5)$, we have a unique Nash equilibrium, namely $p := (1 - p, p)$ with $p$ as in (5) and $q := (0, 0, 0, 0, 1 - q, q, 0, 0)$ with $q = 811/2288$; the support of $q$ consists of Banker’s pure strategies 12 and 13, or DDSS and DDSD on (3, 9), (5, 4), (6, 6), (6, $\emptyset$).

For the cases $\alpha = 1/15$ and $\alpha = 1/6$, three additional support possibilities must be ruled out to confirm the uniqueness of Nash equilibria, owing to the degeneracy of the bimatrix in these two cases. A similar argument is needed to complete the proof at the four values of $\alpha$ in (10). For example, if $\alpha = 6/35 \in (1/6, 2/5)$, $p = 7/16$, and $p := (1 - p, p)$, then $pB$ has four equal and maximal entries (pure strategies 2, 6, 10, 14). For $p$ to be part of a Nash equilibrium, we would need $q_2, q_6, q_{10}, q_{14} \in [0, 1]$ with $q_2 + q_6 + q_{10} + q_{14} = 1$ and $Aq^T$ having
its two components equal, where \( q := (0, q_6, 0, q_{10}, 0, 0, q_{14}, 0) \). But then, for example, \( q_{14} = \frac{875}{16} + 11q_2 + 10q_6 \geq \frac{875}{16} > 1 \), a contradiction.

Finally, at \( \alpha = \alpha_0 \), we find exactly four extreme Nash equilibria. (Any convex combination of these four is itself a Nash equilibrium.) Again \( p := (1 - p, p) \) with \( p \) as in \((3)\) with \( \alpha = \alpha_0 \) and \( q \) as specified by one of the following four mixtures:

\[
\begin{align*}
(1429/2288, 859/2288) \text{ mixture of DSDDS and DSDDD}, \\
(1477/2288, 811/2288) \text{ mixture of DSDSS and DSDSD}, \\
(1429/2240, 811/2240) \text{ mixture of DSDSS and DSDDD}, \\
(1477/2336, 859/2336) \text{ mixture of DSDDS and DSDSD},
\end{align*}
\]

on \((3, 9), (4, 1), (5, 4), (6, 6), (6, \emptyset)\). Notice that the first two equilibria mix on \((6, \emptyset)\) only, whereas the last two mix on both \((6, 6)\) and \((6, \emptyset)\).

Finally, we extend results \((7)\) and \((8)\) to \(0 \leq \alpha < \frac{2}{5}\). There are only two cases, \(0 \leq \alpha < \alpha_0\) and \(\alpha_0 \leq \alpha < \frac{2}{5}\). The results for \(0 \leq \alpha < \alpha_0\) are as in \((7)\) and \((8)\), except that the kernel is specified by rows DSDDS and DSDDD of \(B^T_\alpha\). If \(\alpha_0 \leq \alpha < \frac{2}{5}\), then Banker’s safety level is

\[
v_B := \text{val}(B^T_\alpha) = \frac{8(84644 - 3096915\alpha + 1667488\alpha^2)}{(11 - 6\alpha)(13)^6}. \tag{12}\]

(Here rows DSDSS and DSDDSD of \(B^T_\alpha\) specify the kernel.) Banker’s maxmin strategy is the \((1 - q, q)\) mixture of these two pure strategies, where

\[
q := \frac{811 - 883\alpha}{2288 - 1248\alpha}.
\]

At \(\alpha = \alpha_0\), the two safety levels coincide, but there are two maxmin strategies.

If both Player and Banker use their equilibrium strategies, Banker’s expected payoff is his safety level \((7)\) if \(\alpha \leq \alpha_0\), \((12)\) if \(\alpha \geq \alpha_0\). Player’s expected payoff is his safety level \((8)\) if \(\alpha < \alpha_0\) but is larger, namely \(-677152/[11(13)^6] \approx -0.01277536\), if \(\alpha > \alpha_0\). At \(\alpha = \alpha_0\), Player’s expected payoff is

\[
\begin{align*}
-\frac{679568}{11(13)^6} & \approx -0.0127991, \\
-\frac{677152}{11(13)^6} & \approx -0.0127536, \\
-\frac{8629431}{140(13)^6} & \approx -0.0127701, \\
-\frac{9007929}{146(13)^6} & \approx -0.0127824,
\end{align*}
\]

depending on which of the four Nash equilibria in \((11)\) is used. Each is at least as large as \(v_P\).
4 Constrained Banker strategies

As noted previously, in modern casino baccara, Banker’s strategy is highly constrained. Specifically, the 84 of the 88 strategic situations that require a draw or stand decision in Table 1 are all part of the modern Banker strategy. In addition, Banker stands on (4, 1) (Banker total 4, Player third card 1), perhaps because the improvement in expected profit from drawing instead of standing when Player draws on 5 is very small (about 0.0025641 when $\alpha = 1/20$); see (3). The remaining three cases, (3, 9), (5, 4), and (6, $\varnothing$), were optional at Crockford’s Club in London in the early 1960s, as noted by Kendall and Murchland (1964). Under these rules, we have a $2 \times 8$ bimatrix game whose unique Nash equilibrium is exactly as in Theorem 1 if $0 \leq \alpha < 1/15$

However, current (and older, e.g., Scarne 1949, p. 206) rules do not permit Banker to draw on (6, $\varnothing$) either. This is, we believe, a misguided rule, because the improvement in expected profit from drawing instead of standing when Player draws on 5 is significant (about 0.0673077 when $\alpha = 1/20$); see (5). But let us consider its consequences anyway. Under this rule, the only optional cases for Banker are (3, 9) and (5, 4). Thus, we have a $2 \times 4$ bimatrix game with payoff bimatrix $(A, B)$, where

\[
A^T = \begin{pmatrix}
S on 5 & D on 5 \\
SSSS & -4636 -3585 \\
SSDS & -4635 -3600 \\
DSSS & -4565 -3690 \\
DSDS & -4564 -3705
\end{pmatrix} \frac{16}{(13)^6}
\]

and

\[
B_\alpha^T = \begin{pmatrix}
S on 5 & D on 5 \\
SSSS & 9272 -278353\alpha & 7170 - 276363\alpha \\
SSDS & 9270 -278423\alpha & 7200 - 276433\alpha \\
DSSS & 9130 -278353\alpha & 7380 - 276523\alpha \\
DSDS & 9128 -278423\alpha & 7410 - 276593\alpha
\end{pmatrix} \frac{8}{(13)^6}.
\]

The safety level for Player is

\[
v_P := \text{val}(A) = (-3705)^\frac{16}{(13)^6} \approx -0.0122814
\]

because $A$ has a saddle point at (D on 5, DSDS), and the safety level for Banker is

\[
v_B := \text{val}(B^T) = (7410 - 276593\alpha)^\frac{8}{(13)^6}
\]

because $B^T$ has a saddle point at (DSDS, D on 5) when $0 \leq \alpha < 2/5$. If $\alpha = 1/20$, then $v_B = -256786/[5(13)^6] \approx -0.0106400$.

For $0 \leq \alpha < 1/15$, there is a pure Nash equilibrium, in which Player draws on 5 and Banker uses Table 1 and DSDS on (3, 9), (4, 1), (5, 4), (6, $\varnothing$), and it is the unique Nash equilibrium. Again, both Player and Banker achieve their safety levels with these strategies.
5 Nonstrategic baccarat

The most widely played form of baccara today (usually spelled “baccarat” and often called punto banco) is not a strategic game because both Player and Banker have mandated strategies: Player draws on 5 or less and stands on 6 and 7, and Banker uses Table 1 and DSDS on (3, 9), (4, 1), (5, 4), (6, ∅). Baccarat is said to have originated in Argentina (most likely in Mar del Plata, whose Casino Central opened in 1939) or in pre-Castro Havana before coming to Las Vegas in November 1959. It currently enjoys its greatest popularity in Macau, where annual baccarat revenue is over US$35 billion.

There is one other important distinction between baccara chemin de fer and baccarat. In baccarat, one can bet either on Player (paid at even odds) or on Banker (paid at 19 to 20, equivalent to even odds with a five percent commission on winning bets). Both bets are banked by the casino.

The only issue we want to address here is a historical one. How did the rules originate? There are three potential explanations, which we list in increasing order of plausibility.

1. The Player and Banker strategies mandated by the rules of baccarat are the closest pair of pure strategies to the optimal mixed strategies of Kemeny and Snell (1957) for the parlor game of baccara.

2. The Player and Banker strategies mandated by the rules of baccarat coincide with the pure Nash equilibrium at modern casino baccara, in which Banker has options only on (3, 9) and (5, 4); see Section 2 (When α = 0, the pure Nash equilibrium is a saddle point.)

3. The Banker strategy mandated by the rules of baccarat coincides with Banker’s best response to Player’s (1/2, 1/2) mixed strategy (that is, Player draws on 5 with probability 1/2).

Since baccarat almost certainly predates the Kemeny–Snell paper, the first explanation is not credible. The second explanation is at least a possibility but still implausible because game-theoretic concepts were not well known in the casino industry in the 1950s and earlier; furthermore, there is no mention of this property in the baccara literature. The third explanation is the most plausible because this property was implicitly noted in a well-known book by Le Myre (1935)1, which almost certainly predates baccarat. Of course, it would have been more logical if Banker’s mandated strategy at baccarat coincided with his best response (namely, Table 1 and DDDD on (3, 9), (4, 1), (5, 4), (6, ∅)) to Player’s mandated strategy of drawing on 5, but that is easy to say with hindsight.

1Although Le Myre’s mixture was intended to be (1/2, 1/2) (p. 37), it was actually (89/194, 105/194) when Player draws (p. 85) and (3/5, 2/5) when Player stands (p. 51). 89/169 (resp., 105/169) is the probability that Player’s two-card total is 4 or less (resp., 5 or less).
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