Geometric Class Field Theory
with Bounded Ramification

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Abstract

Let $X$ be a smooth projective geometrically irreducible variety over a perfect field $k$ and $D$ an effective divisor on $X$. We consider a relative Chow group $\text{CH}_0(X, D)$ of modulus $D$ (defined in a geometric way) and describe the kernel of the Abel-Jacobi map with modulus $\text{CH}_0(X, D)^0 \to \text{Alb}_{X, D}(k)$ to the Albanese variety of $X$ of modulus $D$. If $k$ is a finite field and the Néron-Severi group of $X$ is torsion-free, the Abel-Jacobi map with modulus is an isomorphism. We obtain a Reciprocity Law and Existence Theorem for abelian coverings of $X$ with ramification bounded by $D$.

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0 Introduction

Let $X$ be a smooth projective geometrically irreducible variety over a finite field $k = \mathbb{F}_q$, assume that the Néron-Severi group of $X$ is torsion-free. The aim of this paper is to describe abelian coverings of $X$ whose ramification is bounded by an effective divisor $D$. The main result is a Reciprocity Law and Existence Theorem (Theorem 3.1 and Corollary 3.3) in terms of the relative Chow group $\text{CH}_0(X, D)^0$ with modulus from $\text{[Ru2]}$ Def. 3.28. Main ingredient is “Roitman’s Theorem with modulus over a finite field” (Theorem 2.16), which in particular yields the finiteness of $\text{CH}_0(X, D)^0$ (Corollary 2.20).

This is a purely geometric approach to ramified higher dimensional class field theory, basing on Lang’s class field theory of function fields of varieties over a finite field. A Milnor K-theoretic description of global class field theory of arithmetic schemes is given by the work of Kato-Saito [KS3]. A more geometric approach to unramified and tamely ramified higher dimensional class field theory due to Wiesend can be found in [KeS] and [Ker]. A definition of abelian fundamental groups with modulus that is closely related to $\text{CH}_0(X, D)^0$ together with a finiteness result appears in [Hir]. The finiteness of $\text{CH}_0(X, D)^0$ is deduced in [EK] as a consequence of a more general finiteness theorem due to Deligne.

0.1 Results and Leitfaden

Section 1: Prerequisites. Let $X$ be a smooth projective variety over a perfect field $k$, $D$ an effective divisor on $X$ (with multiplicity). We recall some basic notions and properties that will be used in this paper: the modulus of a rational map into a commutative algebraic group mod$(\varphi)$ (Definition 1.1), the relative Chow group with modulus $\text{CH}_0(X, D)$ (Definition 1.3), the functoriality (Proposition 1.6) and the universal quotient (Proposition 1.17) of $\text{CH}_0(X, D)$, the Albanese variety with modulus $\text{Alb}_{X, D}$ (Definition 1.12 and Theorem 1.14), as well as the relations between these notions (Theorem 1.18 and Corollary 1.19): $\text{Alb}_{X, D}$ is identified with the universal quotient of $\text{CH}_0(X, D)^0$ (= the degree 0 part of $\text{CH}_0(X, D)$).

Section 2: Abel-Jacobi map with modulus, which is the quotient map

$$a_{j_{X, D}} : \text{CH}_0(X, D)^0 \longrightarrow \text{Alb}_{X, D}(k).$$

We construct a pairing between the affine part of the relative Chow group with modulus $\text{ACH}_0(X, D) := \ker\left( \text{CH}_0(X, D) \longrightarrow \text{CH}_0(X) \right)$ and the Cartier dual $\mathcal{F}^{0, \text{red}}_{X, D}$ of the affine part $\text{L}_{X, D}$ of $\text{Alb}_{X, D}$ (Proposition 2.6). This allows us to determine the kernel of the affine part $a_{j_{X, D}}^\text{aff} : \text{ACH}_0(X, D) \longrightarrow \text{L}_{X, D}(k)$.
of the Abel-Jacobi map with modulus (Proposition 2.12). Suppose the base field of $X$ is a finite field $k = \mathbb{F}_q$ and the Néron-Severi group of $X$ is torsion-free. Then the Abel-Jacobi map with modulus $\text{aj}_{X,D}$ is an isomorphism (Theorem 2.16). In particular, this implies that $\text{CH}_0(X,D)^0$ is finite under these assumptions (Corollary 2.20). This approach is independent from [EK].

Section 3: Reciprocity Law and Existence Theorem. Assume $k = \mathbb{F}_q$ is a finite field, $\overline{k}$ an algebraic closure of $k$ and $X$ satisfies the assumptions of Theorem 2.16 from above. Let $\varphi : Y \to X$ be an abelian covering of smooth projective geometrically connected varieties over $k$. If $Y \to X$ is unramified outside a closed proper subset $S$ of $X$, there is an effective divisor $D$ on $X$ with support in $S$ such that we obtain a reciprocity law

$$\psi_* \text{CH}_0(Y,D_Y)^0 \sim \frac{\text{Alb}_{X,D}(k)}{\text{Alb}_\varphi \text{Alb}_{Y,D_Y}(k)} \sim \text{Gal}(K_Y \mid K_X)$$

(Theorem 3.1).

In particular, if $X_D \to X$ denotes the pull-back of the “$q$-power Frobenius morphism minus identity” $\varphi = F_q - \text{id} : \text{Alb}_{X,D} \to \text{Alb}_{X,D}$, we have canonical isomorphisms

$$\text{CH}_0(X,D)^0 \cong \text{Alb}_{X,D}(k) \cong \text{Gal}(K_{X_D} \mid K_X)$$

(Existence Theorem, Corollary 3.3). Taking the limit over all effective divisors $D$ on $X$ we obtain

$$\widehat{Z}_0(X)^0 := \varprojlim_D \text{CH}_0(X,D)^0 \cong \varprojlim_D \text{Alb}_{X,D}(k) \cong \text{Gal} \left( \frac{K_{X_D}^{\text{ab}}}{K_X} \right)$$

where $\text{Gal} \left( \frac{K_{X_D}^{\text{ab}}}{K_X} \right)$ is the geometric Galois group of the maximal abelian extension $K_{X_D}^{\text{ab}}$ of the function field $K_X$ of $X$ (Corollary 3.4).

Taking the limit over all effective divisors $D$ with support in a given closed subset $S$ of $X$ of pure codimension 1 we obtain

$$\widehat{Z}_0(X)^{S,0} := \varprojlim_{\mid D \mid \subset S} \text{CH}_0(X,D)^0 \cong \varprojlim_{\mid D \mid \subset S} \text{Alb}_{X,D}(k) \cong \pi_1^{\text{ab}}(X \setminus S)^0$$

where $\pi_1^{\text{ab}}(X \setminus S)^0$ is the abelian geometric fundamental group of $X \setminus S$, which classifies abelian étale covers of $X \setminus S$ that do not arise from extending the base field (Corollary 3.5).

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1 Prerequisites

In this section I recall some notions from [KR] and [Ru2] and show some basic properties that we will need later. I use the notations and conventions from [Ru2]. In particular, by formal group resp. affine group resp. algebraic group I always mean a commutative formal resp. affine resp. algebraic group.

1.1 Modulus of a Rational Map to an Algebraic Group

Let $X$ be a smooth proper variety over a perfect field $k$.

**Definition 1.1.** Let $\varphi : X \to G$ be a rational map from $X$ to a smooth connected algebraic group $G$. Let $L$ be the affine part of $G$ and $U$ the unipotent part of $L$. The modulus of $\varphi$ from [KR, § 3] is the effective divisor

$$ \text{mod} (\varphi) = \sum_{\text{ht}(q)=1} \text{mod}_q(\varphi) D_q $$

where $q$ ranges over all points of codimension 1 in $X$, and $D_q$ is the prime divisor associated to $q$, and $\text{mod}_q(\varphi)$ is defined as follows.

First assume $k$ is algebraically closed. For each $q \in X$ of codimension 1, the canonical map $L(K_X,q)/L(O_X,q) \to G(K_X,q)/G(O_X,q)$ is bijective, see [KR, No. 3.2]. Take an element $l_q \in L(K_X,q)$ whose image in $G(K_X,q)/G(O_X,q)$ coincides with the class of $\varphi \in G(K_X,q)$. If $\text{char}(k) = 0$, let $(u_{q,i})_{1 \leq i \leq s}$ be the image of $l_q$ in $\mathbb{G}_a(K_X,q)^s$ under $L \to U \cong (\mathbb{G}_a)^s$. If $\text{char}(k) = p > 0$, let $(u_{q,i})_{1 \leq i \leq s}$ be the image of $l_q$ in $W_r(K_X,q)^s$ under $L \to U \subset (W_r)^s$.

$$ \text{mod}_q(\varphi) = \begin{cases} 0 & \text{if } \varphi \in G(O_X,q) \\ 1 + \max \{n_q(u_{q,i}) | 1 \leq i \leq s\} & \text{if } \varphi \notin G(O_X,q) \end{cases} $$

where for $u \in \mathbb{G}_a(K_X,q)$ resp. $W_r(K_X,q)$

$$ n_q(u) = \begin{cases} -v_q(u) & \text{if } \text{char}(k) = 0 \\ \min \{n \in \mathbb{N} | u \in \text{fil}_n^W(K_X,q)\} & \text{if } \text{char}(k) = p > 0. \end{cases} $$

Here $\text{fil}_n^W(K_X,q)$ is the filtration of the Witt group from [KR, No. 2.2]. The multiplicity $\text{mod}_q(\varphi)$ is independent of the choice of the isomorphism $U \cong (\mathbb{G}_a)^s$ resp. of the embedding $U \subset (W_r)^s$, see [KR, Thm. 3.3].

For arbitrary perfect base field $k$ we obtain $\text{mod} (\varphi)$ by means of a Galois descent from $\text{mod} (\varphi \otimes_k \overline{k})$, where $\overline{k}$ is an algebraic closure of $k$, see [KR, No. 3.4].
1.2 Relative Chow Group with Modulus

Let $X$ be a smooth projective variety over a perfect field $k$, let $D$ be an effective divisor on $X$. By a curve we mean a 1-dimensional separated reduced scheme of finite type, not necessarily smooth or irreducible.

**Notation 1.2.** If $C$ is a curve in $X$, then $\nu : \widetilde{C} \to C$ denotes its normalization, $\mathcal{K}_C$ is the total quotient ring of $C$. For $f \in \mathcal{K}_C$, we write $\tilde{f} := \nu^* f$ for the image of $f$ in $\mathcal{K}_{\tilde{C}}$. We write $\iota_C : \widetilde{C} \to C \subset X$ for the composition of the normalization of $C$ with the embedding of $C$ into $X$. If $\varphi : X \dasharrow G$ is a rational map, we write $\varphi|_{\tilde{C}} := \varphi \circ \iota_{\tilde{C}}$ for the composition of $\iota_{\tilde{C}}$ and $\varphi$.

If $Y$ is a smooth variety and $\psi : Y \to X$ is a morphism whose image $\psi(Y)$ intersects a Cartier divisor $D$ properly, then $D \cdot Y$ denotes the pullback of $D$ to $Y$. The reduced part of a divisor $D$ is denoted by $D_{\text{red}}$. Then we denote $D_Y := (D - D_{\text{red}}) \cdot Y + (D \cdot Y)_{\text{red}}$.

**Definition 1.3.** Let $Z_0(X \setminus D)$ be the group of 0-cycles on $X \setminus D$, set $R_0(X,D) = \{ (C,f) \mid C \text{ a curve in } X \text{ intersecting } D \text{ properly } \}
\begin{align*}
&\quad f \in \mathcal{K}_C^* \text{ s.t. } \tilde{f} \equiv 1 \mod D_{\tilde{C}} \}
\end{align*}

and let $R_0(X,D)$ be the subgroup of $Z_0(X \setminus D)$ generated by the elements $\text{div}(f)_C$ with $(C, f) \in R_0(X,D)$. Then the **relative Chow group of $X$ of modulus $D$ from [Ru2, Def. 3.28]** is defined as

\[
\text{CH}_0(X,D) = \frac{Z_0(X \setminus D)}{R_0(X,D)}
\]

$\text{CH}_0(X,D)^0$ is the subgroup of $\text{CH}_0(X,D)$ of cycles $\zeta$ with $\text{deg} \zeta|_W = 0$ for all irreducible components $W$ of $X \setminus D$.

**Remark 1.4.** Note that for $D = 0$ the relative Chow group with modulus becomes the usual Chow group:

\[
\text{CH}_0(X,0) = \text{CH}_0(X).
\]

**Remark 1.5.** If we denote

\[
R_0(X,0D) = \{ (C,f) \mid C \text{ a curve in } X \text{ intersecting } D \text{ properly } \}
\begin{align*}
&\quad f \in \mathcal{K}_C^* \quad f \in \mathcal{O}_{C,p}^* \quad \forall p \in C \cap D \}
\end{align*}

then by Bloch’s so called “easy” moving lemma [Blo, Prop. 2.3.1] it holds

\[
\text{CH}_0(X) = \text{CH}_0(X,0D) = \frac{Z_0(X \setminus D)}{R_0(X,0D)}.
\]
**Proposition 1.6.** The relative Chow group with modulus $\text{CH}_0(X, D)$ is a covariant functor in $X$ and a contravariant functor in $D$. More precisely: If $\psi : Y \rightarrow X$ is a morphism of smooth projective varieties, then for any effective divisor $E \geq D_Y$ on $Y$ there is an induced homomorphism of groups

$$\psi_* : \text{CH}_0(Y, E) \rightarrow \text{CH}_0(X, D).$$

**Proof.** The contravariant functoriality of $\text{CH}_0(X, D)$ in $D$ is clear since obviously $Z_0(X \setminus E) \subset Z_0(X \setminus D)$ and $R_0(X, E) \subset R_0(X, D)$ for $E \geq D$.

Let $\psi : Y \rightarrow X$ be a smooth projective variety over $X$. We have the following presentations of $\text{CH}_0(X, D)$, where $N(f) \in \mathcal{K}_{\psi(C)}^*$ is the norm of $f$ (see [Ful, Chap. 1, Prop. 1.4]). One checks that $\tilde{f} \equiv 1 \mod \tilde{D}_{\tilde{C}}$ implies $\tilde{N}(\tilde{f}) \equiv 1 \mod \tilde{D}_{\psi(\tilde{C})}$ (cf. [Se2, III, No. 2, proof of Prop. 4]). Thus the push-forward of cycles $\psi_* : Z_0(Y \setminus D_Y) \rightarrow Z_0(X \setminus D)$, as it maps $R_0(Y, D_Y) \rightarrow R_0(X, D)$, induces the homomorphism $\text{CH}_0(Y, D_Y) \rightarrow \text{CH}_0(X, D)$. □

**Definition 1.7.** Define $\text{ACH}_0(X, D)$ to be the kernel of the homomorphism $\text{CH}_0(X, D) \rightarrow \text{CH}_0(X)$.

$$\text{ACH}_0(X, D) = \ker \left( \text{CH}_0(X, D) \rightarrow \text{CH}_0(X) \right)$$

$$= \ker \left( \text{CH}_0(X, D)^0 \rightarrow \text{CH}_0(X)^0 \right)$$

$$= R_0(X, 0D)/R_0(X, D).$$

Due to Remark [12] the map $\text{CH}_0(X, D) \rightarrow \text{CH}_0(X)$ is surjective. We have thus exact sequences

$$0 \rightarrow \text{ACH}_0(X, D) \rightarrow \text{CH}_0(X, D) \rightarrow \text{CH}_0(X) \rightarrow 0$$

and

$$0 \rightarrow \text{ACH}_0(X, D) \rightarrow \text{CH}_0(X, D)^0 \rightarrow \text{CH}_0(X)^0 \rightarrow 0.$$

**Proposition 1.8.** Let $k$ be an algebraically closed field or a finite field. Let $X$ be a smooth projective variety over $k$. Let $D$ and $E$ be effective divisors on $X$. We have the following presentations of $\text{CH}_0(X, D)$ and $\text{ACH}_0(X, D)$ as inductive limits:

$$\text{CH}_0(X) = \lim_{\rightarrow} \text{CH}_0(C)$$

$$\text{CH}_0(X, D) = \lim_{\rightarrow} \iota_{\tilde{C}_*} \text{CH}_0(\tilde{C}, D_{\tilde{C}})$$

$$\text{ACH}_0(X, D) = \lim_{\rightarrow} \iota_{\tilde{C}_*} \text{ACH}_0(\tilde{C}, D_{\tilde{C}})$$

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where $C$ ranges over all curves in $X$ that meet $S := \text{Supp}(D + E)$ properly (and we use Notation \[1.2\]). This forms a directed system with transition maps given by inclusion: for two curves $C_1$ and $C_2$ the curve $Z = C_1 \cup C_2$ in $X$ satisfies $C_1 \subset Z$, $C_2 \subset Z$.

**Proof.** For all 0-cycles $z,t$ on $X$ there is a smooth hypersurface $H$ containing $z$ and avoiding $t$. This is obvious for $k$ algebraically closed. For $k$ finite see [Gab, Cor. 1.6]. If $t$ contains a point of every irreducible component of $S$, then this implies that $H$ meets $S$ properly. By induction we find for every $z \in Z_0(X \setminus D)$ a smooth curve $C$ containing $z$ and intersecting $S$ properly. This implies the statement for $\text{CH}_0(X,D)$, and it shows that the canonical map $\lim \text{CH}_0(C) \rightarrow \text{CH}_0(X) = Z_0(X)/R_0(X)$ is surjective. By Remark \[1.5\] it holds moreover $\text{CH}_0(X) = Z_0(X \setminus D + E)/R_0(X, 0(D + E))$. Thus the relations of $\text{CH}_0(X)$ are generated by curves that intersect $S = \text{Supp}(D + E)$ properly, hence this canonical map is injective as well. This yields the statement for $\text{CH}_0(X)$.

Suppose $\kappa \in \text{ACH}_0(X,D)$, let $z \in Z_0(X \setminus D)$ be a representative of $\kappa$. Since $\text{CH}_0(X) = \lim \text{CH}_0(C)$ as above, there is a curve $C$ in $X$, meeting $D$ properly, with irreducible components $C_i$ such that $z = \sum \text{div}(f_i) \in Z_0(C)$ for certain $f_i \in K_{C_i}$, i.e. $0 = [z]_C \in \text{CH}_0(C)$ and $0 = [\nu^* z]_{\tilde{C}} \in \text{CH}_0(\tilde{C})$ (Not. \[1.2\]). Then $\kappa = \iota_{\tilde{C}}* [\nu^* z]_{\tilde{C},D_{\tilde{C}}} \in \iota_{\tilde{C}*} \ker(\text{CH}_0(\tilde{C}, D_{\tilde{C}}) \rightarrow \text{CH}_0(\tilde{C}))$.

**Lemma 1.9.** (a) The functoriality map $\text{CH}_0(X,E) \rightarrow \text{CH}_0(X,D)$ is surjective for $E \geq D$.

(b) The map $\text{CH}_0(X,D + E) \rightarrow \text{CH}_0(X,D) \times_{\text{CH}_0(X)} \text{CH}_0(X,E)$

$$[z]_{X,D+E} \mapsto ([z]_{X,D}, [z]_{X,E})$$

is surjective if the divisors $D$ and $E$ are coprime, i.e. do not have any common prime divisors.

(c) $\text{CH}_0(X, \sup\{D,E\}) \rightarrow \text{CH}_0(X,D) \times_{\text{CH}_0(X,\inf\{D,E\})} \text{CH}_0(X,E)$ is surjective, where $\inf\{D,E\}$ is the largest divisor $\leq D,E$ and $\sup\{D,E\}$ is the smallest divisor $\geq D,E$.

**Proof.** Due to the decomposition of $\text{CH}_0(X,?)$ as extension of $\text{CH}_0(X)$ by $\text{ACH}_0(X,?)$ and the Five Lemma, the assertions follow from the corresponding statements for $\text{ACH}_0(X,?)$ instead of $\text{CH}_0(X,?)$.

By Proposition \[1.8\] and the exactness of the inductive limit, we can reduce to the case that $X$ is a smooth curve $C$:

The map $\iota_{\tilde{C}*} \text{CH}_0(\tilde{C}, E_{\tilde{C}}) \rightarrow \iota_{\tilde{C}*} \text{CH}_0(\tilde{C}, D_{\tilde{C}})$ is surjective if the map
\[
\text{CH}_0(\tilde{C}, E) \to \text{CH}_0(\tilde{C}, D)
\]
is, since we have a commutative square

\[
\begin{array}{ccc}
\text{CH}_0(\tilde{C}, E) & \to & \text{CH}_0(\tilde{C}, D) \\
\downarrow & & \downarrow \\
\text{CH}_0(X, E) & \to & \text{CH}_0(X, D),
\end{array}
\]

and corresponding statements hold for (b) and (c).

By the Approximation Lemma it holds for each irreducible component \( C \) of a smooth curve

\[
\text{ACh}_0(C, D) = \prod_{q \in |D|} \mathcal{O}^*_C q = \left( \prod_{q \in |D|} \frac{\mathcal{O}^*_C q}{1 + m_{C, q}} \right) / l^*.
\]

where \( l := H^0(C, \mathcal{O}_C) \). Then the statements of (a), (b), (c) become obvious, where (a) is the case \( E = \sum_{q \in |E|} m_{q} q \geq \sum_{q \in |D|} n_{q} q = D, \) (b) is the case \( |E| \cap |D| = \emptyset \), and in (c) we have \( \inf \{ D, E \} = \sum_{q \in |D| \cap |E|} \min \{ n_{q}, m_{q} \} q \) and \( \sup \{ D, E \} = \sum_{q \in |D| \cup |E|} \max \{ n_{q}, m_{q} \} q \).

**Definition 1.10.** We denote by \( X^n \) the set of points of codimension \( n \) in \( X \) for \( n \in \mathbb{N} \). Let \( d = \dim X \). We define

\[
\begin{align*}
\text{C}_1(X) &= \bigoplus_{C \subset X} k(C)^* \xrightarrow{\text{div}_{X}^{-1}} \bigoplus_{x \in X^d} \mathbb{Z} = \text{Z}_0(X) \\
\text{C}_1(X, 0D) &= \bigoplus_{C \subset X} \bigcap_{q \in D} \mathcal{O}_C q \xrightarrow{\text{div}_{X}^{-1}} \bigoplus_{x \in (X \setminus D)^d} \mathbb{Z} = \text{Z}_0(X \setminus D) \\
\text{C}_1(X, D) &= \bigoplus_{C \subset X} \bigcap_{q \in D \setminus \tilde{C}} \left( 1 + m_{C, q} \right) \xrightarrow{\text{div}_{X}^{-1}} \bigoplus_{x \in (X \setminus D)^d} \mathbb{Z} = \text{Z}_0(X \setminus D)
\end{align*}
\]

where \( C \) ranges over irreducible curves, for \( \text{C}_1(X, 0D) \) and \( \text{C}_1(X, D) \) those which intersect \( D \) properly, and \( \text{div}_{X}^{-1} \) is the boundary map, i.e.

\[
\text{div}_{X}^{-1}(C, f) = \text{div}(f)_{C} \quad \text{for } C \subset X \text{ a curve and } f \in k(C)^*,
\]

where \( (C, f) \) denotes the element in \( \text{C}_1(X) \) associated with \( C \) and \( f \), and \( \text{div}_{X, 0D}^{-1} \) resp. \( \text{div}_{X, D}^{-1} \) are the induced maps on subgroups.
We set resp. have by Definition 1.3 and Remark 1.5
\[ N_1(X) = \ker \left( \partial_{X}^{d-1} \right), \quad R_0(X) = \text{im} \left( \partial_{X}^{d-1} \right), \]
\[ N_1(X,0D) = \ker \left( \partial_{X,0D}^{d-1} \right), \quad R_0(X,0D) = \text{im} \left( \partial_{X,0D}^{d-1} \right), \]
\[ N_1(X,D) = \ker \left( \partial_{X,D}^{d-1} \right), \quad R_0(X,D) = \text{im} \left( \partial_{X,D}^{d-1} \right). \]

Lemma 1.11. Assume the base field \( k \) is finite. The affine part \( \text{ACH}_0(X,D) \) of the relative Chow group with modulus is compatible with the action of the absolute Galois group \( \text{Gal}(\overline{k}/k) \):
\[ \text{ACH}_0(X,D) = \text{ACH}_0(\overline{X},\overline{D})^{\text{Gal}(\overline{k}/k)} \]
where \( \overline{k} \) is an algebraic closure of \( k \), \( \overline{X} = X \otimes_k \overline{k} \), \( \overline{D} = D \otimes_k \overline{k} \).

Proof. Step 1: We have
\[ \text{ACH}_0(X,D) = \ker \left( \text{CH}_0(X,D) \to \text{CH}_0(X) \right) \]
\[ = \ker \left( \frac{Z_0(X \setminus D)}{R_0(X,D)} \to \frac{Z_0(X \setminus D)}{R_0(X,0D)} \right) \]
\[ = \frac{R_0(X,0D)}{R_0(X,D)} \]
\[ = \frac{C_1(X,0D)/N_1(X,0D)}{C_1(X,D)/N_1(X,D)} \]
\[ = \frac{C_1(X,0D)}{C_1(X,D) + N_1(X,0D)}. \]

Step 2: Let \( l|k \) be a finite field extension, \( G = \text{Gal}(l/k) \). In order to show that \( \text{ACH}_0(X,D) = \text{ACH}_0(X_l,D_l)^G \), due to Step 1 it comes down to showing that the natural map
\[ \frac{C_1(X,0D)}{C_1(X,D) + N_1(X,0D)} \to \left( \frac{C_1(X_l,0D_l)}{C_1(X_l,D_l) + N_1(X_l,0D_l)} \right)^G \]
is an isomorphism. If \( k = \mathbb{F}_q \) is finite of order \( q \), \( G \) is cyclic, generated by the \( q \)-power Frobenius, which acts on \( C_1(X) \) by raising coefficients in \( l \) to the \( q \)-th power. Then the assertion is a straightforward computation.

Step 3: As \( \text{ACH}_0(X_{\overline{k}},D_{\overline{k}}) = \lim_{l|k} \text{ACH}_0(X_l,D_l) \),
where the limit ranges over all finite extensions \( l|k \), it holds
\[ \text{ACH}_0(X_{\overline{k}},D_{\overline{k}})^{\text{Gal}(\overline{k}/k)} = \lim_{l|k} \text{ACH}_0(X_l,D_l)^{\text{Gal}(l/k)} = \text{ACH}_0(X,D). \]
1.3 Albanese Variety with Modulus

Let $X$ be a smooth proper variety over a perfect field $k$, let $D$ be an effective divisor on $X$. In the context of generalized Albanese varieties we can reduce, by means of a Galois descent, to the situation of an algebraically closed base field $k$, which we will assume in the following.

**Definition 1.12.** The category $\text{Mr}(X, D)$ from [Ru2, Def. 3.12] is the category of those rational maps $\varphi$ from $X$ to smooth connected commutative algebraic groups such that $\text{mod} (\varphi) \leq D$. The universal object of $\text{Mr}(X, D)$ (if it exists) is denoted by $\text{Alb}_{X,D}$, called the Albanese of $X$ of modulus $D$.

**Definition 1.13.** $F_{X,D} = (F_{X,D})_{\text{et}} \times (F_{X,D})_{\text{inf}}$ from [Ru2, Def. 3.13] is the formal subgroup of the sheaf of relative Cartier divisors $\text{Div}_X$ defined by the conditions

$$(F_{X,D})_{\text{et}} = \{ B \in \text{Div}_X(k) \mid \text{Supp}(B) \subset \text{Supp}(D) \}$$

and if $\text{char}(k) = 0$

$$(F_{X,D})_{\text{inf}} = \exp \left( \widehat{\mathbb{G}}_a \otimes_k \Gamma(X, \mathcal{O}_X (D - D_{\text{red}}) / \mathcal{O}_X) \right)$$

where $\exp$ is the exponential map and $\widehat{\mathbb{G}}_a$ is the completion of $\mathbb{G}_a$ at 0, if $\text{char}(k) = p > 0$

$$(F_{X,D})_{\text{inf}} = \text{Exp} \left( \sum_{r>0} \widehat{\mathbb{W}} \otimes_{W(k)} \Gamma \left( X, \text{fil}^r W_{D-D_{\text{red}}} W_r(K_X) / W_r(\mathcal{O}_X) \right) \right)$$

where $\text{Exp}$ denotes the Artin-Hasse exponential, $\widehat{\mathbb{W}}$ is the kernel of the $r^{th}$ power of the Frobenius on the completion $\widehat{W}$ of the Witt group $W$ at 0 and $\text{fil}^r W_r(K_X)$ is the filtration of the sheaf of Witt groups from [Ru2, Def. 3.2].

$F_{X,D}^{0,\text{red}} = F_{X,D} \times_{\text{Pic}_X} \text{Pic}_X^{0,\text{red}}$ is the part of $F_{X,D}$ that is mapped to the Picard variety $\text{Pic}_X^{0,\text{red}}$ under the class map $\text{Div}_X \rightarrow \text{Pic}_X$.

**Theorem 1.14.** The Albanese $\text{Alb}_{X,D}$ of $X$ of modulus $D$ exists and is dual (in the sense of 1-motives) to the 1-motive $F_{X,D}^{0,\text{red}} \rightarrow \text{Pic}_X^{0,\text{red}}$.

**Proof.** See [Ru2] Thm. 3.19. \hfill $\blacksquare$

**Definition 1.15.** Let $\text{Mr}^{CH}(X, D)$ be the category of morphisms $\varphi : X \setminus D \rightarrow G$ whose associated map on 0-cycles of degree 0

$$Z_0(X \setminus D)^0 \rightarrow G(k)$$

$$\sum l_i p_i \mapsto \sum l_i \varphi(p_i), \quad l_i \in \mathbb{Z}, \ p_i \in \subset X \setminus D$$
factors through a homomorphism of groups $\text{CH}_0(X, D)^0 \rightarrow G(k)$.

We refer to the objects of $\text{Mr}^{\text{CH}}(X, D)$ as rational maps from $X$ to algebraic groups factoring through $\text{CH}_0(X, D)^0$. (Cf. [Ru2, Def. 3.29].)

The universal object of $\text{Mr}^{\text{CH}}(X, D)$ (if it exists) is denoted by $\text{Alb}_{X, D}$ and called the universal quotient of $\text{CH}_0(X, D)^0$.

**Definition 1.16.** Let $\mathcal{F}^{\text{CH}}_{X,D}$ be the formal subgroup of $\text{Div}_X$ defined as follows:

$$\mathcal{F}^{\text{CH}}_{X,D} = \bigcap C \left( \varphi \cdot \tilde{C} \right)^{-1} \mathcal{F}_{\tilde{C}, D_{\tilde{C}}}$$

where $C$ ranges over all curves in $X$ that intersect $D$ properly, $\text{Div}_{X, \tilde{C}}$ is the subfunctor of $\text{Div}_X$ consisting of those relative Cartier divisors on $X$ whose support (see [Ru2, Def. 2.2]) meets $C$ properly and $\varphi : \text{Div}_{X, \tilde{C}} \rightarrow \text{Div}_{\tilde{C}}$ is the pull-back of relative Cartier divisors from $X$ to $\tilde{C}$.

**Proposition 1.17.** The universal quotient $\text{Alb}^{\text{CH}}_{X,D}$ of $\text{CH}_0(X, D)^0$, if it exists (as an algebraic group), is dual (in the sense of 1-motives) to the 1-motive $[(\mathcal{F}^{\text{CH}}_{X,D})^{\text{red}} \rightarrow \text{Pic}^{\text{red}}_0]$.  

**Proof.** It is sufficient to show that the category $\text{Mr}^{\text{CH}}(X, D)$ is equal to the category $\text{Mr}^{\text{red}}_{\mathcal{F}^{\text{CH}}_{X,D}}$ of those rational maps that induce a transformation to $\mathcal{F}^{\text{CH}}_{X,D}$ (see [Ru2, Thm. 2.12 and Rmk. 2.14]). For this aim we show that for a morphism $\varphi : X \setminus D \rightarrow G$ the following conditions are equivalent:

(i) $\varphi(\text{div}(f)_C) = 0$ \quad $\forall (C, f) \in \mathcal{R}_0(X, D)$,
(ii) $\left( \varphi|_{\tilde{C}}, f \right)_q = 0$ \quad $\forall (C, f) \in \mathcal{R}_0(X, D), \forall q \in \text{Supp}(D_{\tilde{C}})$,
(iii) $\text{mod}(\varphi|_{\tilde{C}}) \leq D_{\tilde{C}}$ \quad $\forall C$ intersecting $D$ properly,
(iv) $\text{im}(\tau_{\varphi|_{\tilde{C}}}) \subset \mathcal{F}_{\tilde{C}, D_{\tilde{C}}}$ \quad $\forall C$ intersecting $D$ properly,
(v) $\text{im}(\tau_{\varphi}) \cdot \tilde{C} \subset \mathcal{F}_{\tilde{C}, D_{\tilde{C}}}$ \quad $\forall C$ intersecting $D$ properly,
(vi) $\text{im}(\tau_{\varphi}) \subset \mathcal{F}^{\text{CH}}_{X,D}$.

(i)$\iff$(ii) see [Se2, III, § 1],
(ii)$\iff$(iii) see [KR, No. 6.1–3],
(iii)$\iff$(iv) is [Ru2, Lem. 3.17],
(iv)$\iff$(v) is evident,
(v)$\iff$(vi) by Definition 1.16.

**Theorem 1.18.** The category $\text{Mr}(X, D)$ of rational maps of modulus $\leq D$ is equal to the category $\text{Mr}^{\text{CH}}(X, D)$ of rational maps factoring through $\text{CH}_0(X, D)^0$. Thus it holds $\text{Alb}_{X,D} = \text{Alb}^{\text{CH}}_{X,D}$ and $\mathcal{F}_{X,D} = \mathcal{F}^{\text{CH}}_{X,D}$.

**Proof.** See [Ru2, Thm. 3.30].
Corollary 1.19. The group of $k$-rational points of the Albanese variety with modulus $\text{Alb}_{X,D}(k)$ is a quotient of the degree 0 part of the Chow group with modulus $\text{CH}_0(X,D)^0$, which is compatible with the universal map $\text{alb}_{X,D}$ of $\text{Mr}(X,D)$, for any perfect base field $k$.

Proof. The universal map $\text{alb}_{X,D} : X \to \text{Alb}_{X,D}$ of $\text{Mr}(X,D)$ factors through $\text{CH}_0(X,D)$ by [Ru2, Lem. 3.31, (i) $\Rightarrow$ (ii)] (for this direction we did not use the assumption that $k$ is algebraically closed). More precisely, the induced map on $\mathbb{Z}_0(X \setminus D)^0$ factors as $\mathbb{Z}_0(X \setminus D)^0 \to \text{CH}_0(X,D)^0 \to \text{CH}_0(X,D)^0 \to \text{Alb}_{X,D}(\overline{k})$. The image of any $z \in \mathbb{Z}_0(X \setminus D)^0$, as a point of $\text{Alb}_{X,D}(\overline{k})$, is geometrically irreducible, and defined over $k$, hence an element of $\text{Alb}_{X,D}(k)$. Now $\text{Alb}_{X,D}$, as the universal object of $\text{Mr}(X,D)$, is generated by $X$, hence $\mathbb{Z}_0(X \setminus D)^0 \to \text{Alb}_{X,D}(k)$ is surjective (cf. the first paragraph of [Se1]). Thus there is an epimorphism $\text{CH}_0(X,D)^0 \to \text{Alb}_{X,D}(k)$. ■

2 Abel-Jacobi Map with Modulus

Let $X$ be a smooth proper variety over a perfect field $k$ and $D$ an effective divisor on $X$. In this section we study the quotient map from Cor. 1.19

$$a_{j_{X,D}} : \text{CH}_0(X,D)^0 \to \text{Alb}_{X,D}(k),$$

called Abel-Jacobi map of $X$ of modulus $D$. The goal is to show that $a_{j_{X,D}}$ is an isomorphism when $k$ is a finite field and the Néron-Severi group of $X$ is torsion-free (Theorem 2.16).

2.1 Duality

In this No. we assume that the base field $k$ is algebraically closed.

Point 2.1. Let $C$ be a smooth proper irreducible curve over $k$. Let $D = \sum_{q \in S} n_q q$ be an effective divisor on $C$, where $S$ is a finite set of closed points on $C$ and $n_q$ are integers $\geq 1$ for $q \in S$. The Albanese with modulus $\text{Alb}_{C,D}$ of $C$ of modulus $D$ coincides with the Jacobian with modulus $\text{Jac}_{C,D}$ of Rosenlicht-Serre (see [Ru2, Thm. 3.26]), which is an extension

$$0 \to \text{L}_{C,D} \to \text{Jac}_{C,D} \to \text{Jac}_C \to 0$$

of the classical Jacobian $\text{Jac}_C \cong \text{Pic}_C^0$ of $C$, an abelian variety, by an affine algebraic group $\text{L}_{C,D}$ (see [Se2, V, § 3]). Taking $k$-valued points, this exact
sequence becomes the short exact sequence from Definition 1.7:

\[
\text{Jac}_C(k) = \text{CH}_0(C)^0,
\]
\[
\text{Jac}_{C,D}(k) = \text{CH}_0(C, D)^0,
\]
\[
\text{L}_{C,D}(k) = \text{ACH}_0(C, D) = \prod_{q \in S} \mathcal{O}_{C,q}^* \times \prod_{q \in S} (1 + m_q^*)^{k^*},
\]

where \(m_q\) denotes the maximal ideal at \(q \in C\) (see [Se2, I, No. 1]). Our duality construction of \(\text{Alb}_{C,D}\) (Theorem 1.14) yields that \(\text{Jac}_{C,D}\) is dual to \([\mathcal{F}_{C,D}^{\text{0,red}} \to \text{Pic}_{C,\text{red}}^0]\), in particular \(\text{L}_{C,D}\) is Cartier dual to \(\mathcal{F}_{C,D}^{\text{0,red}}\).

**Definition 2.2.** Let \(F\) be a formal group and \(A\) be an abstract abelian group. A pairing

\[
F \times A \to \mathbb{G}_m
\]

is a collection of bi-homomorphisms \(F(R) \times A \to \mathbb{G}_m(R)\). Such a pairing is called **perfect**, if it induces an isomorphism \(A \cong \text{Hom}_{\text{Ab}/k}(F, \mathbb{G}_m)\).

If \(F\) is dual to an affine group-variety and \(A = L(k)\) is the group of \(k\)-valued points of an affine group-variety \(L\) over an algebraically closed field \(k\), the condition above is equivalent to saying that the pairing induces an isomorphism of sheaves \(F \cong \text{Hom}_{\text{Ab}/k}(L, \mathbb{G}_m)\), according to Cartier duality (see e.g. [Ru2, Thm. 1.13]) and the fact that varieties over an algebraically closed base field \(k\) are determined by its \(k\)-valued points.

**Proposition 2.3.** In the curve case, the Cartier duality \(\mathcal{F}_{C,D}^{\text{0,red}} = L_{C,D}^\vee\) is expressed by the perfect pairing

\[
\mathcal{F}_{C,D}^{\text{0,red}} \times \text{ACH}_0(C, D) \to \mathbb{G}_m
\]

\[
\langle \mathcal{D}, [\text{div}(f)] \rangle_{C,D} = \prod_{q \in S} (\mathcal{D}, f)_q
\]

induced by the local symbols \((?, ?)_q : \hat{\text{Div}}_C \times \hat{\mathcal{O}}_{C,q}^* \to \mathbb{G}_m\) at \(q \in S\), in the sense of [Ru3, Prop. 2.6]. (For \(\mathcal{D} \in \hat{\text{Div}}_C(R)\), \(R\) a finite ring, one chooses a local section of \(\mathcal{D}\) in a neighbourhood of \(q\) in \(C\), this defines a rational map \(\phi^D : C \to \mathbb{G}_m(\hat{\mathcal{O}}_{C,q})\) which can be inserted into the local symbol. Here the local symbol \((\phi^D, f)_q\) is independent of the choice of the local section of \(\mathcal{D}\), if \(f \in \hat{\mathcal{K}}_C^*\) is regular at \(q\).)

**Proof.** Due to [Ru3, 4.3] the local symbol induces a monomorphism of formal groups

\[
\hat{\text{Div}}_C^{S,\text{0,red}} \to \bigoplus_{q \in S} \text{Hom}_{\text{Ab}/k}(\hat{\mathcal{O}}_{C,q}^*, \mathbb{G}_m) = \text{Hom}_{\text{Ab}/k}(\prod_{q \in S} \hat{\mathcal{O}}_{C,q}^*, \mathbb{G}_m),
\]
where $\hat{\text{Div}}_{\overline{S},0,\text{red}}$ denotes the subfunctor of $\hat{\text{Div}}_{\overline{S},\text{red}}$ of relative Cartier divisors with support in $S$. By construction of $\mathcal{F}_{C,D,0,\text{red}}$ (Definition 1.13) one sees, according to [KR, § 6, Prop. 6.4 (3)] or by the equivalence of conditions (ii) and (iv) in the proof of Proposition 1.17, that $\mathcal{F}_{C,D,0,\text{red}}$ is the annihilator in $\hat{\text{Div}}_{\overline{S},0,\text{red}}$ of $k^* \times \prod_{q \in S} (1 + m_q^n)$ with respect to the local symbol:

$$\mathcal{F}_{C,D,0,\text{red}} = \text{Ann} \left( R_0(C, D) \right).$$

Thus the local symbol induces a monomorphism

$$\mathcal{F}_{C,D,0,\text{red}} \to \text{Hom}_{A_b/k} \left( \prod_{q \in S} O_{C,q}, \mathbb{G}_m \right) = \text{Hom}_{A_b/k} \left( L_{C,D}, \mathbb{G}_m \right)$$

the image of which is again a formal group that is isomorphic to the Cartier dual of $L_{C,D}$, which shows that this map is an isomorphism. ■

Now we are looking at curves $C$ in $X$ which are not necessarily smooth or irreducible.

**Proposition 2.4.** Let $C$ be a curve in $X$ which intersects $D$ properly. In the following diagram, the upper row is a well-defined pairing (induced by the pairing of the lower row from Proposition 2.3):

$$\begin{array}{ccc}
\iota_{\overline{C}}^* \mathcal{F}_{X,D,0,\text{red}} \times \iota_{\overline{C}}^* \text{ACH}_0(\overline{C}, D_{\overline{C}}) & \longrightarrow & \mathbb{G}_m \\
\cap \downarrow & & \\
\mathcal{F}_{\overline{C},D_{\overline{C}},0,\text{red}} \times \text{ACH}_0(\overline{C}, D_{\overline{C}}) & \longrightarrow & \mathbb{G}_m.
\end{array}$$

**Proof.** The lower row is a perfect pairing by Proposition 2.3. This reduces the proof to showing that $\iota_{\overline{C}}^* \mathcal{F}_{X,D,0,\text{red}}$ lies in the annihilator of $\ker(\iota_{\overline{C}}^*)$:

$$\iota_{\overline{C}}^* \mathcal{F}_{X,D,0,\text{red}} \subset \text{Ann} \left( \ker \left( \text{ACH}_0(\overline{C}, D_{\overline{C}}) \xrightarrow{\iota_{\overline{C}}^*} \text{ACH}_0(X, D) \right) \right).$$

By Theorem 1.18 Definition 1.16 and Proposition 2.3 we have

$$\mathcal{F}_{X,D,0,\text{red}} = \bigcap_Z \left( t_Z^* \left[ \text{Div}_{\overline{X},0,\text{red}}^0 \right]^{-1} \mathcal{F}_{Z,D_{\overline{Z}},0,\text{red}} \right)$$

$$= \bigcap_Z \left( t_Z^* \left[ \text{Div}_{\overline{X},0,\text{red}}^0 \right]^{-1} \text{Ann} \left( R_0(\overline{Z}, D_{\overline{Z}}) \right) \right).$$
and by definition of $\text{ACH}_0(X,D)$

$$
\ker (\iota_{\overline{C}}) = (\iota_{\overline{C}})^{-1} \sum \mathbb{Z} R_0(\overline{Z}, D\overline{Z}).
$$

Then for $D \in \mathcal{F}_{X,D}^{0,\text{red}}$ and $\operatorname{div}(f)_Z \in \iota_{\overline{Z}} R_0(\overline{Z}, D\overline{Z})$ we obtain by Lemma 2.5

$$
\left< D \cdot \overline{C}, \operatorname{div}(f)_Z \cdot \overline{C} \right>_{\overline{C}, D\overline{C}} = \left< D \cdot \overline{Z}, \operatorname{div}(f)_Z \cdot \overline{Z} \right>_{\overline{Z}, D\overline{Z}} = 1
$$
since $D \cdot \overline{Z} \in \text{Ann}(\operatorname{R}_0(\overline{Z}, D\overline{Z}))$ and $\operatorname{div}(f)_Z \cdot \overline{Z} \in \operatorname{R}_0(\overline{Z}, D\overline{Z})$. This shows $\iota_{\overline{C}}^* \mathcal{F}_{X,D}^{0,\text{red}} \subset \text{Ann}(\ker(\iota_{\overline{C}}))$. ■

**Lemma 2.5.** Let $D \in \mathcal{F}_{X,D}(R)$, where $R$ is a finite ring. Suppose the two pairs $(C, f), (Z, g) \in \mathcal{R}_0(X, 0D)$ satisfy $\operatorname{div}(f)_C = \operatorname{div}(g)_Z \in \text{ACH}_0(X, D)$. Then

$$
\left< D \cdot \overline{C}, \operatorname{div}(\tilde{f}) \right>_{\overline{C}, D\overline{C}} = \left< D \cdot \overline{Z}, \operatorname{div}(\tilde{g}) \right>_{\overline{Z}, D\overline{Z}}.
$$

**Proof.** If $[\operatorname{div}(f)_C] = [\operatorname{div}(g)_Z] \in \text{ACH}_0(X, D)$, then there are $f' \in \mathcal{K}_C^*$, $g' \in \mathcal{K}_Z^*$ with $[\operatorname{div}(f')_C] = [\operatorname{div}(f)_C]$ resp. $[\operatorname{div}(g')_Z] = [\operatorname{div}(g)_Z]$ such that $\operatorname{div}(f')_C = \operatorname{div}(g')_Z \in \mathbb{Z}_0(X)$. Therefore we may assume that we have representatives as cycles in $X$ which coincide.

Let $G(D) \in \operatorname{Ext}_{\text{Ab}/k} (\text{Alb}_X, \mathbb{L}_R) \cong \text{Pic}_{\text{Ab}}^0(XR) \cong \text{Pic}^0(XR)$ be the algebraic group corresponding to $\mathcal{O}_{X\otimes R}(D)$, where $\mathbb{L}_R := \mathbb{G}_{m,R}$ is the Weil restriction of $\mathbb{G}_{m,R}$ to $k$. The canonical 1-section of $\mathcal{O}_{X\otimes R}(D)$ induces a rational map $\varphi^D : X \dasharrow G(D)$, which is regular away from $S$. If $h \in \mathcal{O}_{\overline{C}, q}$ for some $q \in \overline{C}(k)$, then by [Ru1, Lemma 3.16] the local symbol $(\varphi^D|_{\overline{C}}, h)_q$ lies in the fibre of $G(D)$ over $0 \in \text{Alb}_X$, which is $\mathbb{L}_R$, and $(\varphi^D|_{\overline{C}}, h)_q = (D \cdot \overline{C}, h)_q$. Then

$$
\left< D \cdot \overline{C}, [\operatorname{div}(\tilde{f})] \right>_{\overline{C}, D\overline{C}} = \prod_{q \in S} \left( \varphi^D|_{\overline{C}}, \tilde{f} \right)_q
$$

$$
= \prod_{q \in \operatorname{div}(\tilde{f})} \left( \varphi^D|_{\overline{C}}, \tilde{f} \right)_q^{-1}
$$

$$
= \prod_{q \in \operatorname{div}(\tilde{f})} \varphi^D|_{\overline{C}}(q)^{-\varphi_q(\tilde{f})}
$$

$$
= \prod_{p \in \operatorname{div}(\tilde{f})} \varphi^D|_{\overline{C}}(p)^{-\varphi_{\overline{C}}(p)}
$$

$$
= \varphi^D(\operatorname{div}(f)_C)^{-1}
$$

i.e. the expression depends only on the cycle $\operatorname{div}(f)_C \in \mathbb{Z}_0(X)$ in $X$. ■
Proposition 2.6. There is a canonical pairing
\[ \mathcal{F}_{X,D} \times \text{ACH}_0(X, D) \rightarrow \mathbb{G}_m \]
\[ \left\langle D, [\text{div}(f)_C] \right\rangle_{X,D} = \left\langle D \cdot \tilde{C}, [\text{div}(\tilde{f})] \right\rangle_{\tilde{C}, \tilde{D}} \]
\[ \left\langle D, [z] \right\rangle_{X,D} = \varphi^D(z)^{-1} \]

for \( z \in \mathcal{R}_0(X, 0D) \subset \text{Z}_0(X \setminus D) \) and where \( \varphi^D \) is the rational map associated with \( D \), i.e. induced by the 1-section of \( \mathcal{O}_{X \otimes \mathbb{R}}(D) \) (see Proof of Lemma 2.5).

The induced map \( \mathcal{F}_{X,D}^{0,\text{red}}(R) \rightarrow \text{Hom}_{\text{Grp}}(\text{ACH}_0(X, D), \mathbb{G}_m(R)) \) is injective for all \( R \in \text{Alg}/k \).

**Proof.** Since \( \text{ACH}_0(X, D) = \lim_{\leftarrow C} \iota_{\tilde{C}*} \text{ACH}_0(C, D_{\tilde{C}}) \) by Proposition 1.8, the pairing \( \left\langle ?, ? \right\rangle_{X,D} \) can be defined by restriction to curves, using the pairing from Proposition 2.4. Due to Lemma 2.5, the definition of \( \left\langle D, \kappa \right\rangle_{X,D} \) for \( \kappa \in \text{ACH}_0(X, D) \) is independent of the choice of the representative \( (C, f) \in \mathcal{R}_0(X, 0D) \) of the cycle class \( \kappa \). The formula \( \left\langle D, [z] \right\rangle_{X,D} = \varphi^D(z)^{-1} \) was shown in the proof of Lemma 2.5.

Suppose \( \left\langle D, ? \right\rangle_{X,D} = 0 \) for \( D \in \mathcal{F}_{X,D}^{0,\text{red}}(R) \). Then \( \left\langle D \cdot \tilde{C}, ? \right\rangle_{\tilde{C}, D_{\tilde{C}}} = 0 \) for all curves \( C \) in \( X \) intersecting \( D \) properly. As \( \left\langle ?, ? \right\rangle_{\tilde{C}, D_{\tilde{C}}} \) is perfect, this implies \( D \cdot \tilde{C} = 0 \) for all \( C \), hence \( D = 0 \).

From now on we assume that the base field \( k \) is countable, e.g. \( k = \overline{\mathbb{F}_q} \) an algebraic closure of a finite field.

Proposition 2.7. There is a (non-canonical) embedding of abelian groups
\[ \zeta : \text{Z}_0(X) \longrightarrow \widehat{\text{Z}_0(X)} := \lim_{\leftarrow D} \text{CH}_0(X, D) \]

with \( p^D(\zeta(z)) = [z] \in \text{CH}_0(X, D) \) for \( z \in \text{Z}_0(X) \) and all \( D \) with \( |z| \cap |D| = \emptyset \), where \( p^D : \widehat{\text{Z}_0(X)} = \lim_{\leftarrow E} \text{CH}_0(X, E) \longrightarrow \text{CH}_0(X, D) \) is the canonical map.

The image of \( \text{Z}_0(X) \) is dense in \( \widehat{\text{Z}_0(X)} \).

**Proof.** To \( z \in \text{Z}_0(X) \) there is a naturally associated family \( ([z]_{X,D})_D \in \lim_{\leftarrow D} \text{CH}_0(X, D) \). We have to show that this can be extended to an element of \( \widehat{\text{Z}_0(X)} = \lim_{\leftarrow D} \text{CH}_0(X, D) \), compatible with the group structure. Since \( \text{Z}_0(X) \) is the free abelian group generated by \( \{x \mid x \in X\} \), we do this for these 1-cycles \( x \in \text{Z}_0(X) \) and extend by linearity.
By assumption $k$ is countable and $X$ is of finite type over $k$, thus the set of effective divisors on $X$ is countable. Let $\mathbb{N} \rightarrow \text{PDiv}(X)$, $n \mapsto P_n$ be an enumeration of the set of prime divisors on $X$. As for $E \geq D$ there is a map $\text{CH}_0(X,E) \rightarrow \text{CH}_0(X,D)$, $[z]_{X,E} \mapsto [z]_{X,D}$, the element $[z]_{X,D}$ is determined by $[z]_{X,E}$. Thus it is sufficient to determine the values $[z]_n := [z]_{X,E_n}$, $n \in \mathbb{N}$ for

$$E_n := \sum_{i=0}^{n} (n-i) P_i.$$  

Let $E_n^{2z} := \sum_{i \neq 0 P \twoheadrightarrow z} (n-i) P_i$ be the divisor obtained from $E_n$ by omitting all prime divisors $P$ with $|P| \cap |z| \neq \emptyset$. Then obviously $E_n^{2z} \leq E_n$ and $[z]_n^{2z} := [z]_{X,E_n^{2z}}$ is determined by $z \in \text{Z}_0(X)$.

Now suppose $[z]_i$ is determined for all $i < n$. Then choose $[z]_n$ such that

(a) $[z]_n \mapsto [z]_n^{2z}$,

(b) $[z]_n \mapsto [z]_{n-1}$.

Such an element $[z]_n \in \text{CH}_0(X,E_n)$ exists according to Lemma 1.9 (a) and (c), since by construction it holds $[z]_n^{2z} \mapsto [z]_{n-1}^{2z} \mapsto [z]_{n-1}$.

The compositions $p_0 \circ \zeta : \text{Z}_0(X) \rightarrow \text{CH}_0(X,D)$ yield a compatible system of surjections, and $\emptyset \neq \lim \text{CH}_0(X,D) \ni 0$. This implies that the image of $\zeta$ is dense in $\text{Z}_0(X)$, according to [RZ] Lem. 1.1.7.

The map $\zeta$ is injective, since $\ker(\zeta) = \bigcap_D \text{R}_0(X,D) = 0$: For every $0 \neq z \in \text{Z}_0(X)$ one finds a rational map $\varphi : X \dashrightarrow G$ (e.g. $G = \mathbb{G}_m$), defined on $\text{Supp}(z)$, such that $\varphi(z) \neq 0$. But $\varphi$ factors through $\text{CH}_0(X,D)$ for $D := \text{mod}(\varphi)$, according to Theorem 1.18. Hence $z \not\in \text{R}_0(X,D)$.

**Remark 2.8.** By left-exactness of the projective limit we have

$$\text{R}_0(X) := \lim_{\leftarrow D} \text{ACH}_0(X,D) = \ker \left( \lim_{\leftarrow D} \text{CH}_0(X,D) \rightarrow \text{CH}_0(X) \right).$$

**Proposition 2.9.** Let $S$ be a closed subset of $X$ and let $\text{Div}^S_{X,0,\text{red}}$ be the subfunctor of $\text{Div}^0_{X,\text{red}}$ of relative Cartier divisors with support in $S$. Then $\text{Div}^S_{X,0,\text{red}} = \lim_{\leftarrow D \subseteq S} \text{Div}^0_{X,D}$, and let $\text{R}_0(X)^S = \lim_{\leftarrow D \subseteq S} \text{ACH}_0(X,D)$, where in both limits $D$ ranges over all effective divisors on $X$ with support in $S$. Then the pairing of Proposition 2.7 induces a canonical pairing

$$\text{Div}^S_{X,0,\text{red}} \times \text{R}_0(X)^S \rightarrow \mathbb{G}_m.$$
If $X = C$ is a curve, this pairing is perfect, yielding an isomorphism

$$\text{Div}_{X,C}^{\text{red}}, D \ni \text{Hom}_{\hat{A}/k} \left( \prod_{q \in S} \hat{O}_{C,q} \right) / k^* \cong \mathbb{G}_m.$$ 

**Proof.** For $D \in \text{Div}_{X,C}^{\text{red}}, D$, the rational map associated with $D$ as in Lemma 2.5 is well-defined since every curve $X$ holds $\langle \hat{D} \rangle$ with respect to $\langle \hat{D} \rangle$. Set $D = \text{mod}(\varphi^D)$. Then $D \in \mathcal{F}_{X,D}^{\text{red}}$ by [Ru2, Lemma 3.17]. This shows $\text{Div}_{X,C}^{\text{red}} = \lim_{D \in S} \mathcal{F}_{X,D}^{\text{red}}$.

The expression $\langle D, [z] \rangle_{X,D} = \varphi^D(z)^{-1}$ is independent of the choice of $D \geq \text{mod}(\varphi^D)$, as the pairing is induced by the local symbol. Thus for $D \in \text{Div}_{X,C}^{\text{red}}, D$ and $\kappa \in R_0(X)^S$ we can define

$$\langle D, \kappa \rangle_{X,S} := \langle D, p_D(\kappa) \rangle_{X,D},$$

where $D \geq \text{mod}(\varphi^D)$ and $p_D : R_0(X)^S = \lim_{D \in S} \text{ACH}_0(X,E) \longrightarrow \text{ACH}_0(X,D)$ is the canonical map. This is well-defined since for every curve $C$ in $X$ it holds $\mathcal{F}_{X,D}^{\text{red}} \cdot C \subseteq \mathcal{F}_{C,D}^{\text{red}} = \text{Ann} \left( R_0(C,D) \right)$, hence

$$\mathcal{F}_{X,D}^{\text{red}} = \text{Ann} \left( R_0(X,D) \right)$$

with respect to $\langle ?, ? \rangle_{X,S}$.

If $X = C$ is a curve, this pairing is perfect by Proposition 2.3 and it holds $R_0(X)^S = \lim_{\rightarrow} \text{ACH}_0(X,D) \cong \lim_{\rightarrow} \frac{\prod_{q \in S} \hat{O}_{C,q}}{k^* \times \prod_{q \in S} (1 + m_q)} = \left( \prod_{q \in S} \hat{O}_{C,q}^* \right) / k^*$.

**Lemma 2.10.** For the pairing $\text{Div}_{X,C}^{\text{red}}, D \ni \text{R}_0(X) \longrightarrow \mathbb{G}_m$ from Prop. 2.7, it holds $\bigcap_{D \in \mathcal{F}_{X,D}^{\text{red}}} \text{Ann} \left( \mathcal{F}_{X,D}^{\text{red}} \right) = 0$, where $D$ ranges over all effective divisors on $X$.

**Proof.** We have

$$\bigcap_{D} \text{Ann} \left( \mathcal{F}_{X,D}^{\text{red}} \right) = \text{Ann} \left( \lim_{D} \mathcal{F}_{X,D}^{\text{red}} \right) = \text{Ann} \left( \text{Div}_{X,C}^{\text{red}} \right).$$

Let $z = \sum l_i p_i \in \text{R}_0(X)$ with $l_i \in \mathbb{Z}$, $p_i \in X$. Then for $D \in \text{Div}_{X,C}^{\text{red}}, D$ with $|D| \cap |z| = \emptyset$ we have

$$\langle D, \zeta(z) \rangle_X = \varphi^D(z) = \sum l_i \varphi^D(p_i).$$

There are divisors $D_i \in \text{Div}_{X,C}^{\text{red}}, D$ such that $\varphi^{D_i}(p_i) \neq 0$ and $\varphi^{D_i}(p_j) = 0$ for all $j \neq i$. Therefore $z \in \text{Ann} \left( \text{Div}_{X,C}^{\text{red}} \right)$ if and only if $z = 0$. As $\text{R}_0(X)$ is dense in $\text{R}_0(X)$ by Prop. 2.7 this shows that $\text{Ann} \left( \text{Div}_{X,C}^{\text{red}} \right) = 0$. \qed
2.2 Albanese Kernel

Point 2.11. Let $L_{X,D}$ be the affine part of the Albanese $\text{Alb}_{X,D}$ of $X$ of modulus $D$. The Abel-Jacobi map $\text{aj}_{X,D}$ of $X$ of modulus $D$ induces a commutative diagram with exact rows

\[
\begin{array}{cccccccccc}
0 & \longrightarrow & ACH_0(X, D) & \longrightarrow & CH_0(X, D)^0 & \longrightarrow & CH_0(X)^0 & \longrightarrow & 0 \\
& & \downarrow \text{aj}_{X,D} & & \downarrow \text{aj}_{X,D} & & \downarrow \text{aj}_X & & \\
0 & \longrightarrow & L_{X,D}(k) & \longrightarrow & \text{Alb}_{X,D}(k) & \longrightarrow & \text{Alb}_X(k) & \longrightarrow & 0.
\end{array}
\]

Here $\text{aj}_{X,D}$ is surjective by Corollary [1.19]. According to [KS2, Prop. 9, (1)] the map $\text{aj}_X$ is surjective, and an isomorphism if the Néron-Severi group of $X$ is torsion-free. Then by the Snake Lemma the map $\text{aj}_{X,D}^{\text{aff}}$ is surjective.

In what follows we assume the Néron-Severi group of $X$ is torsion-free.

Proposition 2.12. Assume that the base field $k$ is algebraically closed. The kernel of the quotient map

\[
\text{aj}_{X,D}^{\text{aff}} : ACH_0(X, D) \longrightarrow L_{X,D}(k)
\]

is given by the annihilator of $F_{X,D}^{0,\text{red}}$ with respect to the pairing $\langle ?, ? \rangle_{X,D} : F_{X,D}^{0,\text{red}} \times ACH_0(X, D) \longrightarrow \mathbb{G}_m$ from Proposition [2.6]:

\[
\ker (\text{aj}_{X,D}^{\text{aff}}) = \text{Ann} (F_{X,D}^{0,\text{red}}).
\]

Proof. By construction of $\text{Alb}_{X,D} = \text{Alb}_{X,D}^{\text{CH}}$, the Abel-Jacobi map is compatible with curves, i.e. the diagram

\[
\begin{array}{cccccccc}
\text{CH}_0(\tilde{C}, D_{\tilde{C}})^0 & \longrightarrow & \text{Alb}_{\tilde{C}, D_{\tilde{C}}}(k) \\
\downarrow & & \downarrow \\
\text{CH}_0(X, D)^0 & \longrightarrow & \text{Alb}_{X,D}(k)
\end{array}
\]

commutes, and hence so does the diagram

\[
\begin{array}{cccccccc}
ACH_0(\tilde{C}, D_{\tilde{C}}) & \longrightarrow & L_{\tilde{C}, D_{\tilde{C}}}(k) \\
\downarrow & & \downarrow \\
ACH_0(X, D) & \longrightarrow & L_{X,D}(k).
\end{array}
\]
Now $L_{C,D}^\ast$ resp. $L_{X,D}$ is the Cartier dual of $F_{C,D}^{0,\text{red}}$ resp. $F_{X,D}^{0,\text{red}}$. Therefore the map $a{j}^{\text{aff}}_{C,D} : ACH_0(\tilde{C}, D_{\tilde{C}}) \xrightarrow{\sim} L_{C,D}^\ast(k)$ is given by $\kappa \mapsto \langle ?, \kappa \rangle_{C,D}$, where $\langle ?, ? \rangle_{C,D}$ is the pairing from Prop. 2.3. As $ACH_0(X, D)$ is the inductive limit of the $\iota_{C,*} ACH_0(\tilde{C}, D_{\tilde{C}})$ (Prop. 1.8) and the pairing $\langle ?, ?, ? \rangle_{X,D}$ from Prop. 2.6 was constructed via restriction to curves, this shows that the map $a{j}^{\text{aff}}_{X,D} : ACH_0(X, D) \rightarrow L_{X,D}(k)$ is given by $\kappa \mapsto \langle ?, \kappa \rangle_{X,D}$. Here $\langle ?, ?, ? \rangle_{X,D}$ induces a perfect pairing $F_{X,D}^{0,\text{red}} \times ACH_0(X, D)/\text{Ann}(F_{X,D}^{0,\text{red}}) \rightarrow \mathbb{G}_m$, since the map $F_{X,D}^{0,\text{red}} \rightarrow \text{Hom}_{\text{Grp}}(ACH_0(X, D), \mathbb{G}_m(?)$ is injective by Prop. 2.6. Thus $L_{X,D}(k)$ is isomorphic to the quotient $ACH_0(X, D)/\text{Ann}(F_{X,D}^{0,\text{red}})$. This implies the statement. ■

**Lemma 2.13.** Assume that the base field $k$ is countable and algebraically closed or finite. There is a smooth affine algebraic $k$-group $\Lambda_{X,D}$ such that

$$ACH_0(X, D) = \Lambda_{X,D}(k).$$

**Proof. Step 1:** Assume that the base field $k$ is algebraically closed. We have $ACH_0(X, D) = R_0(X, 0D)/R_0(X, D) = \widehat{R_0(X,D)}$, where $R_0(X, D)$ is the completion of $R_0(X, D)$ in $\widehat{R_0(X)}$. By Proposition 2.9 there is a pairing $\text{Div}_X^{0,\text{red}} \times R_0(X) \rightarrow \mathbb{G}_m$, and $\bigcap_E \text{Ann}(F_{X,E}^{0,\text{red}}) = 0$ by Lemma 2.10 where $E$ ranges over all effective divisors on $X$. Thus there is a divisor $E \geq D$ such that $R_0(X, D) \supset \text{Ann}(F_{X,E}^{0,\text{red}})$. By Proposition 2.12 $ACH_0(X, E)/\text{Ann}(F_{X,E}^{0,\text{red}}) \cong L_{X,E}(k)$ carries the structure of an algebraic group. Let $\pi_{X,E} : R_0(X) \rightarrow ACH_0(X, E) \rightarrow L_{X,E}(k)$ be the composition map. Then

$$ACH_0(X, D) = \frac{\widehat{R_0(X)}}{R_0(X, D)} = \frac{L_{X,E}(k)}{\pi_{X,E} R_0(X, D)}.$$

Now we have $R_0(X, D) = \sum_C t_{C,*} R_0(\tilde{C}, D_{\tilde{C}})$, therefore it holds $R_0(\tilde{X}, D) = \lim_{E \geq D} \pi_{X,E} \zeta \left( \sum_C t_{C,*} R_0(\tilde{C}, D_{\tilde{C}}) \right)$, thus

$$\pi_{X,E} R_0(\tilde{X}, D) = \pi_{X,E} \zeta \left( \sum_C t_{C,*} R_0(\tilde{C}, D_{\tilde{C}}) \right) = \sum_C \pi_{X,E} t_{C,*} R_0(\tilde{C}, D_{\tilde{C}})$$

where the sum ranges over all curves $C$ that intersect $D$ properly. For every such $C$ the group $R_0(\tilde{C}, D_{\tilde{C}}) = \ker \left( \widehat{R_0(C)} \rightarrow L_{C,D_{\tilde{C}}} \right)$ is an affine group: $\widehat{R_0(C)} = \lim_{D} L_{\tilde{C},D_{\tilde{C}}}$ is the projective limit of affine groups, hence an affine
group. The explicit description of $L_{\tilde{C},D_{\tilde{C}}}$ from Point 2.1 shows that $R_0(\tilde{C}, D_{\tilde{C}})$ is reduced and connected. As the diagram

$$
\begin{array}{ccc}
R_0(C) & \xrightarrow{\pi_{\tilde{C},E_{\tilde{C}}}} & L_{\tilde{C},E_{\tilde{C}}}(k) \\
\downarrow{\iota_{\tilde{C},*}} & & \downarrow{\text{Alb}_{\tilde{C}}} \\
R_0(X) & \xrightarrow{\pi_{X,E}} & L_{X,E}(k) \\
\end{array}
$$

commutes, the composition $\pi_{X,E} \iota_{\tilde{C},*} = \text{Alb}_{\tilde{C}} \pi_{\tilde{C},E_{\tilde{C}}}$ is a homomorphism of affine groups. Hence $\pi_{X,E} \iota_{\tilde{C},*} R_0(\tilde{C}, D_{\tilde{C}}) = \text{Alb}_{\tilde{C}} \pi_{\tilde{C},E_{\tilde{C}}} R_0(\tilde{C}, D_{\tilde{C}})$ is an affine group, and since $\pi_{\tilde{C},E_{\tilde{C}}} R_0(\tilde{C}, D_{\tilde{C}}) = \ker (L_{\tilde{C},E_{\tilde{C}}} \to L_{\tilde{C},D_{\tilde{C}}})$ is algebraic, it is algebraic. For any finite set $E$ of curves $C$ that meet $D$ properly, the finite sum $\sum_{C \in E} \pi_{X,E} \iota_{\tilde{C},*} R_0(\tilde{C}, D_{\tilde{C}}) =: \Sigma_E$ is an affine algebraic subgroup of $L_{X,E}$. As $\Sigma_E$ is constructible, it is closed (see [Bor, I, 1.3 (c)]); since it is reduced, it is smooth; since it is in addition connected, it is irreducible. Then for any increasing sequence of finite sets of curves $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ the sequence $\{\Sigma_{E_n}\}_{n \in \mathbb{N}}$ terminates (for dimension reasons). Hence there exist finitely many curves $C_1, \ldots, C_n$ such that $\pi_{X,E} R_0(X, D) = \sum_{i=1}^n \pi_{X,E} \iota_{\tilde{C}_i,*} R_0(\tilde{C}_i, D_{\tilde{C}_i})$, which is hence an affine algebraic group-variety. This shows that $\text{ACH}_0(X, D) = L_{X,E}(k)/\pi_{X,E} R_0(X, D)$ has the structure of an affine algebraic group-variety, in particular it is smooth.

**Step 2:** For finite base field $k$, we make a base change to an algebraic closure $\overline{k}$. Then the statement follows from Step 1 and Lemma 1.11 via Galois descent.

**Corollary 2.14.** If $k$ is a finite field, then $\text{ACH}_0(X, D)$ is finite.

**Proof.** $\text{ACH}_0(\tilde{C}, D_{\tilde{C}}) = \Lambda_{\tilde{C},D_{\tilde{C}}}(k)$ by Lemma 2.13 which is finite if $k$ is a finite field.

**Corollary 2.15.** There is a curve $C$ in $X$ such that $\text{ACH}_0(X, D)$ is a quotient of $\text{ACH}_0(\tilde{C}, D_{\tilde{C}})$.

**Proof.** Since $\text{ACH}_0(X, D) = \varinjlim \iota_{\tilde{Z},*}$, $\text{ACH}_0(\tilde{Z}, D_{\tilde{Z}})$ by Lemma 1.8, then if $\text{ACH}_0(X, D)$ is finite, there are finitely many curves $Z_1, \ldots, Z_r$ such that $\text{ACH}_0(X, D) = \sum_{i=1}^r \iota_{\tilde{Z}_i,*} \text{ACH}_0(\tilde{Z}_i, D_{\tilde{Z}_i})$. Thus the curve $C := \bigcup_{i=1}^r Z_i$ has the required property.
Theorem 2.16. Let \( X \) be a smooth projective geometrically irreducible variety over a finite field \( k \) and \( D \) an effective divisor on \( X \). Assume that the Néron-Severi group of \( X \) is torsion-free. Then the Abel-Jacobi map of \( X \) of modulus \( D \) is an isomorphism:

\[
aj_{X,D} : CH_0(X, D)^0 \xrightarrow{\sim} Alb_{X,D}(k).
\]

Leitfaden of Proof. Decomposing the Abel-Jacobi map \( aj_{X,D} \) as in Point 2.11, the theorem follows from the corresponding fact for the Abel-Jacobi map \( aj_X : CH_0(X)^0 \xrightarrow{\sim} Alb_X \) without modulus due to [KS2] on the one hand, as well as from Lemma 2.13 in combination with the universal property of \( Alb_{X,D} \).

I break the proof down into several steps.

Lemma 2.17. The relative Chow group with modulus is compatible with the action of the absolute Galois group of the base field \( k \):

\[
CH_0(X, D) = CH_0(X, \mathcal{D})^{\text{Gal}(\overline{k}|k)}
\]

where \( \overline{k} \) is an algebraic closure of \( k \), \( X = X \otimes_k \overline{k}, \mathcal{D} = D \otimes_k \overline{k} \).

Lemma 2.18. There is a smooth algebraic group \( G_{X,D} \) such that

\[
CH_0(X, D)^0 = G_{X,D}(k).
\]

Lemma 2.19. \( G_{X,D} \cong Alb_{X,D} \), the isomorphism given by \( aj_{X,D} \).

which concludes the proof of Theorem 2.16.

It remains to prove the listed lemmata.

Proof of Lemma 2.17. Let \( G = \text{Gal}(\overline{k}|k) \), the absolute Galois group of \( k \). Consider the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & ACH_0(X, D) & \longrightarrow & CH_0(X, D) & \longrightarrow & CH_0(X) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ACH_0(X, \overline{k})^G & \longrightarrow & CH_0(X, \overline{k})^G & \longrightarrow & CH_0(X)^G &
\end{array}
\]

obtained from the functoriality of \( ACH_0(?, ?) \), \( CH_0(?, ?) \), \( CH_0(?) \) for \( X \rightarrow X \) via pull-back, and the left-exactness of the functor \((?)^G\). The assumptions on \( X \) imply that \( CH_0(X)^0 \cong Alb_X \), according to [KS2] No. 10, Prop. 9 (1)], and likewise \( CH_0(X)^0 \cong Alb_X(\overline{k}) = (Alb_X \otimes_k \overline{k})(\overline{k}) = Alb_X(\overline{k}) \). As \( Alb_X(\overline{k}) = Alb_X(\overline{k})^G \), we have \( CH_0(X) = CH_0(X)^G \). Thus in order to show
that \( \text{CH}_0(X, D) = \text{CH}_0(X, D)^G \), due to the Five Lemma it is sufficient to show \( \text{ACH}_0(X, D) = \text{ACH}_0(X, D)^G \), which is Lemma 2.11

**Proof of Lemma 2.18.** \( \text{CH}_0(X, D)^0 \) is an extension of \( \text{CH}_0(X)^0 \) by \( \text{ACH}_0(X, D) \). Now \( \text{CH}_0(X)^0 = \text{Alb}_X(k) \) and \( \text{ACH}_0(X, D) = \Lambda_{X,D}(k) \) for some smooth affine algebraic group \( \Lambda_{X,D} \), see Lemma 2.13. Due to Lemma 2.10 there is a divisor \( E \geq D \) such that \( \Lambda_{X,D} \) is a quotient of \( L_{X,E} \), cf. Proof of Lemma 2.13. Let \( \pi : L_{X,E} \to \Lambda_{X,D} \) be the quotient map. But \( \text{CH}_0(X, D)^0 \) is the push-out of \( \text{CH}_0(X, E)^0 \) via \( \text{ACH}_0(X, E) \to \text{ACH}_0(X, D) \), and the last map factors through \( \text{aj}\text{aff}_{X,E} : \text{ACH}_0(X, E) \to L_{X,E}(k) \), thus the functoriality map \( \text{CH}_0(X, E)^0 \to \text{CH}_0(X, D)^0 \) factors through \( \text{Alb}_{X,E}(k) = \text{aj}\text{aff}_{X,E} \text{CH}_0(X, E)^0 \). We obtain the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \text{ACH}_0(X, E) & \to & \text{CH}_0(X, E)^0 & \to & \text{CH}_0(X)^0 & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_{X,E}(k) & \to & \text{Alb}_{X,E}(k) & \to & \text{Alb}_X(k) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda_{X,D}(k) & \to & \text{CH}_0(X, D)^0 & \to & \text{Alb}_X(k) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{ACH}_0(X, D) & \to & \text{CH}_0(X, D)^0 & \to & \text{CH}_0(X)^0 & \to & 0.
\end{array}
\]

Then the push-out \( G_{X,D} := \pi_* \text{Alb}_{X,E} = \Lambda_{X,D} \Pi_{L_{X,E}} \text{Alb}_{X,E} \) of \( \text{Alb}_{X,E} \) via \( \pi \) is a smooth algebraic group that satisfies \( \text{CH}_0(X, D)^0 = G_{X,D}(k) \). By construction and Lemma 2.17 it holds for every finite field extension \( l/k \):

\[
G_{X,D}(l) = G_{X,D}(k)^{\text{Gal}(\overline{k}/l)} = (\text{CH}_0(X, D)^0)^{\text{Gal}(\overline{k}/l)} = \text{CH}_0(X_l, D_l)^0.
\]

**Proof of Lemma 2.19.** Let \( E \geq D \) be a divisor such that \( G_{X,D} \) is a quotient of \( \text{Alb}_{X,E} \), as in the Proof of Lemma 2.18. Choose any base point \( x_0 \in (X \setminus E)(k) \) and consider the maps \( \gamma_{X,E} : (X \setminus E)(k) \to \text{CH}_0(X, E)^0, \ x \mapsto [x]_{X,E} - [x]_{X,E} \) and \( \gamma_{X,D} \) defined analogously. According to the sandwich from the Proof of Lemma 2.18 \( \gamma_{X,D} \) factors as

\[
\begin{array}{ccc}
(X \setminus E)(k) & \xrightarrow{\gamma_{X,E}} & \text{CH}_0(X, E)^0 \\
\xrightarrow{\text{aj}_{X,E}} & & \xrightarrow{\rho} \text{Alb}_{X,E}(k) \\
& & \xrightarrow{\rho} \text{CH}_0(X, D)^0 \\
\end{array}
\]

\[\xrightarrow{G_{X,D}(k)}\]
where $a_{j,X,E} \circ \gamma_{X,E} = \text{alb}_{X,E}$ up to translation by a constant (cf. Cor. 1.19).

Since $\text{alb}_{X,E}$ and $\rho$ are morphisms of algebraic varieties, the composition $\gamma_{X,D} = \rho \circ \text{alb}_{X,E}$ is a rational map of algebraic varieties. Then by definition, $G_{X,D} \otimes \overline{k}$ satisfies the universal property for the category $\text{Mr}^{\text{CH}}(X, D)$, hence coincides (up to unique isomorphism) with the universal object of $\text{Mr}^{\text{CH}}(X, D)$, which is $\text{Alb}_{X,D} = \text{Alb}_{X,D} \otimes \overline{k}$. By construction, the isomorphism between $G_{X,D} \otimes \overline{k}$ and $\text{Alb}_{X,D} \otimes \overline{k}$ is given by $a_{j,X,D} \otimes \overline{k}$. Galois descent yields that $a_{j,X,D} : G_{X,D} \xrightarrow{\sim} \text{Alb}_{X,D}$ is an isomorphism. □

**Corollary 2.20.** Under the assumptions of Theorem 2.16, $\text{CH}_0(X, D)^0$ is finite.

**Proof.** $\text{CH}_0(X, D)^0 = \text{Alb}_{X,D}(k)$ by Theorem 2.16 and this is finite when $k$ is a finite field. □

**Corollary 2.21.** Under the assumptions of Theorem 2.16, the pairing from Proposition 2.6 is a perfect pairing (Definition 2.2).

**Proof.** According to Proposition 2.6 and compatibility with Galois descent (Lemma 2.17) the pairing

$$ \mathcal{F}_{X,D}^{0,\text{red}} \times \text{ACH}_0(X, D) / \text{Ann}(\mathcal{F}_{X,D}^{0,\text{red}}) \rightarrow \mathbb{G}_m $$

is perfect. $\text{Ann}(\mathcal{F}_{X,D}^{0,\text{red}})$ is the kernel of $a_{j,X,D}^{\text{eff}} : \text{ACH}_0(X, D) \rightarrow L_{X,D}(k)$ by Proposition 2.12 and this kernel is trivial by Theorem 2.16. □

3 Reciprocity Law and Existence Theorem

Let $k = \mathbb{F}_q$ be a finite field, $\overline{k}$ an algebraic closure and $X$ a smooth projective geometrically irreducible variety over $k$. Assume that the Néron-Severi group of $X$ is torsion-free. Let $S$ be a closed proper subset of $X$.

The following theorem relies basically on Theorem 2.16 and on Lang’s class field theory for function fields over finite fields, as described in [Se2].

**Theorem 3.1 (Reciprocity Law).** Let $\psi : Y \rightarrow X$ an abelian covering of smooth projective geometrically connected varieties over the finite field $k$, unramified outside of $S$. Then there are an effective divisor $D$ on $X$ with support in $S$ and canonical isomorphisms

$$ \frac{\text{CH}_0(X, D)^0}{\psi_* \text{CH}_0(Y, D_Y)^0} \xrightarrow{\sim} \frac{\text{Alb}_{X,D}(k)}{\text{Alb}_\psi \text{Alb}_{Y,D_Y}(k)} \xrightarrow{\sim} \text{Gal}(Y|X) $$

where $\text{Gal}(Y|X)$ denotes the Galois group of the extension of function fields $K_Y | K_X$. 24
Proof. The first isomorphism is due to Theorem 2.16 and functoriality of \( \text{Alb}_{X,D} \) and \( \text{CH}_0(X,D) \).

Every abelian covering (which arises from a “geometric” situation, i.e. does not arise from an extension of the base field) is the pull-back of a separable isogeny \( H \to G \) via a rational map \( \varphi : X \to G \) (see Se2 VI, No. 8, Cor. of Prop. 7]). Using the universal mapping property of the Albanese with modulus, there is a homomorphism \( g : \text{Alb}_{X,D} \to G \) such that \( \varphi \) factors as \( X \to \text{Alb}_{X,D} \to G \), where \( D = \text{mod}(\varphi) \). Replacing \( H \to G \) by the pull-back \( g^*H = \text{Alb}_{Y,X} \to \text{Alb}_{X,D} \), we may thus assume that \( \psi : Y \to X \) is the pull-back of a separable isogeny \( i_{Y|X} : H_{Y|X} \to \text{Alb}_{X,D} \). (Note that by definition \( D \) is the smallest modulus for \( \varphi \) in the sense of Se2 III, No. 1, Def. 1.) The same arguments as for Se2 VI, No. 12, Prop. 11 show that the support of \( D \) is equal to the branch locus of \( \psi \), i.e. the locus of \( X \) that ramifies in \( Y \), hence is contained in \( S \). We have \( \text{Gal}(Y|X) = \ker(i_{Y|X}) \).

The isogeny \( i_{Y|X} \) is a quotient of the “\( q \)-power Frobenius minus identity” \( \varphi := \text{F}_q - \text{id} \) (see Se2 VI, No. 6, Prop. 6]), i.e. there is a homomorphism \( h_{Y|X} : \text{Alb}_{X,D} \to H_{Y|X} \) with \( i_{Y|X} = h_{Y|X} \circ h_{Y|X} \). Let \( X_D \to X \) be the pull-back of \( \varphi : \text{Alb}_{X,D} \to \text{Alb}_{X,D} \). Then \( \text{Gal}(X_D|X) = \ker(\varphi) = \text{Alb}_{X,D}(k) \).

The universal property of \( \text{Alb}_{Y,D_Y} \) implies that the map \( Y \to H_{Y|X} \) factors through \( \text{alb}_{Y,D_Y} : Y \to \text{Alb}_{Y,D_Y} \). Let \( H_{X_D|Y} \) be the fibre product of \( \text{Alb}_{X,D} \) and \( \text{Alb}_{Y,D_Y} \) over \( H_{Y|X} \). Due to the universal property of the fibre-product, the commutative square

\[
\begin{array}{ccc}
X_D & \longrightarrow & \text{Alb}_{X,D} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & H_{Y|X} \\
\end{array}
\]

factors as

\[
\begin{array}{ccc}
X_D & \longrightarrow & H_{X_D|Y} & \longrightarrow & \text{Alb}_{X,D} \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & \text{Alb}_{Y,D_Y} & \longrightarrow & H_{Y|X} \\
\end{array}
\]

The functoriality of \( \text{Alb}_{X,D} \) yields a commutative square

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{alb}_{Y,D_Y}} & \text{Alb}_{Y,D_Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\text{alb}_{X,D}} & \text{Alb}_{X,D} \\
\end{array}
\]

1 The only point left to show is that \( \ker(L_{X,D} \to L_{X,D}) \), \( E \geq D \) is connected. This follows from the structure of the formal groups \( F_{X,D}^{0,\text{red}} \) resp. \( F_{X,E}^{0,\text{red}} \) and Cartier duality.
We obtain the following commutative diagram

\[ \begin{array}{ccc}
X_D & \xrightarrow{h_{X_D}} & Y_D \\
\psi_D & \searrow & \swarrow \psi \\
X & \xrightarrow{h_{X}} & Y \\
\end{array} \]

\[ \xrightarrow{Alb_{Y,D}(k)} \]

Claim 3.2. \( j_{Y|X} \circ h_{X_D}|_{Y} = Alb_{\psi} \)

Proof of Claim 3.2. As the diagram commutes, we have

\[ i_{Y|X} h_{Y|X} j_{Y|X} h_{X_D}|_{Y} = Alb_{\psi} \]

\[ i_{X_D}|_{Y} h_{X_D}|_{Y} = Alb_{\psi} \]

Now \( \varphi := F_{q} - id \) commutes with polynomial maps defined over \( k = \mathbb{F}_{q} \), Thus \( \varphi \) in particular commutes with \( j_{Y|X} h_{X_D}|_{Y} \), i.e. we have

\[ j_{Y|X} h_{X_D}|_{Y} \varphi = Alb_{\psi} \]

and \( \varphi \) is surjective (see [Se2, VI, No. 4, Prop. 3]). Hence \( j_{Y|X} h_{X_D}|_{Y} = Alb_{\psi} \), which proves the claim.

The following special case of Thm. 3.1 was already treated in its proof:

**Corollary 3.3 (Existence Theorem).** There are canonical isomorphisms

\[ CH_0(X, D)^0 \cong Alb_{X,D}(k) \cong Gal(X_D|X) \]

where \( X_D \rightarrow X \) is the pull-back of “\( q \)-power Frobenius morphism minus identity” \( \varphi = F_{q} - id : Alb_{X,D} \rightarrow Alb_{X,D} \).
Corollary 3.4. Taking the limit over all effective divisors $D$ on $X$ we obtain

$$
\lim_{D} \text{CH}_0(X, D)^0 \cong \lim_{D} \text{Alb}_{X, D}(k) \cong \text{Gal} \left( K_X^{ab} \mid K_X \right)
$$

where $\text{Gal} \left( K_X^{ab} \mid K_X \right)$ is the geometric Galois group of the maximal abelian extension $K_X^{ab}$ of the function field $K_X$ of $X$.

Corollary 3.5. Taking the limit over all effective divisors $D$ on $X$ with support in $S$ we obtain

$$
\lim_{\text{Supp}(D) \subset S} \text{CH}_0(X, D)^0 \cong \lim_{\text{Supp}(D) \subset S} \text{Alb}_{X, D}(k) \cong \pi_1^{ab}(X \setminus S)^0
$$

where $\pi_1^{ab}(X \setminus S)^0$ is the abelian geometric fundamental group of $X \setminus S$ which classifies abelian étale covers of $X \setminus S$ that do not arise from extending the base field.

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