PARKING ON TRANSITIVE UNIMODULAR GRAPHS

MICHAEL DAMRON, JANKO GRAYNER, MATTHEW JUNGE, HANBAEK LYU,
AND DAVID SIVAKOFF

Abstract. Place a car independently with probability $p$ at each site of a graph. Each initially vacant site is a parking spot that can fit one car. Cars simultaneously perform independent random walks, and when a car encounters an available parking spot, it parks there and the spot becomes unavailable. For a large class of transitive and unimodular graphs, we show that the root is almost surely visited infinitely many times when $p \geq 1/2$, and only finitely many times otherwise.

1. Introduction

We consider a process in which vertices of a graph $G = (V, E)$ are initially independently labeled cars or parking spots with probability $p$ or $1 - p$. The cars perform independent random walks according to a transition kernel $K$ and stop at a parking spot if it is unoccupied. If multiple cars arrive at the same unoccupied parking spot at the same time, one of them is uniformly chosen to park. For each site $v \in V$, let $V_t(v)$ be the number of visits to $v$ up to time $t$ by cars, and let $V(v) = \lim_{t \to \infty} V_t(v)$ be the total number of visits to $v$. We drop the $(v)$ superscript from our notation when the $V_t(v)$ are identically distributed.

At $p = 1/2$ the densities of cars and parking spots are equal. It is natural to guess that a phase transition in $V(v)$ occurs at this balance point. We show that this is true for sufficiently homogeneous graphs, and describe the behavior at criticality under rather general conditions. Before describing these conditions we state a special case of our result on the lattice (see Figure 2).

Theorem 1.1. Consider the parking process on $\mathbb{Z}^d$ with simple symmetric random walks.

(i) If $p \geq 1/2$, then $V$ is infinite almost surely. Moreover, $\mathbb{E}V_t = (2p - 1)t + o(t)$.

(ii) If $p < 1/2$, then $V$ is finite almost surely. Moreover, $\mathbb{E}V < \infty$ if $p < (256d^6e^2)^{-1}$.

Note that a phase transition at $p = 1/2$ does not occur on asymmetric graphs. [GP16, Proposition 3.5] shows that on the directed binary tree the critical probability is in the interval $[1/64, 1/4] \approx [0.02, 0.25]$. Moreover, on graphs with rapid enough degree expansion, there is no phase transition. Consider the tree where each vertex at distance $n$ from the root has degree $2^n$. Even when $p = 1$ the induced drift is so strong that only finitely many cars will visit the root.

Christina Goldschmidt and Michael Przykucki study a similar process on trees in [GP16]. Let Pois$(\alpha)$ denote a Poisson distribution with mean $\alpha$. Their main result concerns rooted Galton-Watson trees with a Pois$(1)$ offspring distribution. They make the graph oriented so all edges point towards the root, and therefore all random walks move a.s. at each step closer to the root. Every vertex is initially a parking spot. Place Pois$(\alpha)$ cars at each site. A car parks if it arrives at an available spot (breaking ties uniformly at random). If a car arrives at an occupied spot, it continues towards the root until it finds a spot or drives off of the tree through the root, which we typically denote by $0$. 

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Abstract. Place a car independently with probability $p$ at each site of a rooted graph. Each initially vacant site is a parking spot that can fit one car. Cars simultaneously perform independent random walks, and when a car encounters an available parking spot, it parks there and the spot becomes unavailable. For a family of graphs that includes oriented and unoriented lattices and regular trees, we show that the total number of times that the root is visited by a car is almost surely infinite when $p \geq \frac{1}{2}$, and otherwise it is finite. This mirrors the recent findings of Goldschmidt and Przykucki on oriented Galton-Watson trees.

Let $X$ be the total number of cars that never find a spot. Since the tree they work on is almost surely finite, it is more natural to consider the expected value of $X$. In [GP16, Theorem 1.2], the authors prove $\mathbf{E}X$ undergoes a sharp phase transition: $\mathbf{E}X = \infty$ for $\alpha > \frac{1}{2}$, and $\mathbf{E}X \leq 1$ for $\alpha \leq \frac{1}{2}$. They also study $X$ when the critical Galton Watson tree is conditioned to be infinite. In [GP16, Theorem 1.3] they prove that, for $\alpha < \frac{1}{2}$, with probability $q_{\alpha} = \sqrt{1 - 2\alpha(1 - \alpha)}$ every car finds a parking spot. Note that $\alpha = \frac{1}{2}$ is not the exact analogue of $p = \frac{1}{2}$ in the parking process we consider here. Because [GP16] has every site initially a parking spot, the probability a parking spot is empty after placing cars is $e^{-\alpha}$, and expected number of cars at a site is $\mathbf{E} \left( \right. \left. \text{Poi}(\alpha) - 1 \right) = e^{-\alpha} + \alpha - 1$. These two quantities are not equal at $\alpha = \frac{1}{2}$, and in fact, there are much more vacant spots than cars.

We will see that this imbalance comes from the fact that the oriented trees they consider are not unimodular. Goldschmidt’s and Przykucki’s work gives a probabilistic intuition for an analogous phase transition observed by Marie-Louise Lackner and Alois Panholzer.

In [LP16a], Lackner and Panholzer consider a parking process on directed uniformly random labeled trees on $n$ vertices. Every vertex is initially a parking spot. Place $m = \lceil \alpha n \rceil$ cars uniformly at random, and have them attempt to park as before. They show that when $\alpha \geq \frac{1}{2}$ the probability every car finds a parking spot goes to 0 (as $n \to \infty$). For $\alpha < \frac{1}{2}$ this probability converges to $q_{\alpha}$. Uniformly random trees are well-approximated by critical Galton-Watson trees. In fact, the result for uniformly random trees is a corollary of [GP16, Theorem 1.3].

For [GP16] and [LP16a], the randomness comes from the environment and the initial placement of cars. Since the underlying graph is a directed tree, the paths cars follow are deterministic. As we have described in the first paragraph, we consider the parking process on graphs where cars have multiple vertices they can move to at each step. It is natural to
have them decide between these options randomly according to a common transition kernel \( K \) of random walks at each site. Depending on this kernel, the trajectory that each car follows may either be deterministic or random. The content of our main result, Theorem 2.1, is that Theorem 1 holds in general if the pair \((G, K)\) has certain transitivity and unimodularity properties.

**Organization of paper.** We give a more precise definition of the model and our results in Section 2. We give some examples and state open questions in Section 3. The proofs of our main results—Theorem 2.1 and Theorem 2.2—are given in Section 4.

2. Definition and Statement of results

Let \( G = (\mathcal{V}, \mathcal{E}) \), \( \mathcal{E} \subseteq \mathcal{V}^2 \) be a locally finite connected simple graph. For \( v \in \mathcal{V} \), let \( N(v) \) be the set of all neighbors of \( v \) in \( G \). For \( x, y \in \mathcal{V} \), we let \( \text{dist}(x, y) \) denote shortest path distance between \( x \) and \( y \). Let \( \mathbb{B}(x, t) = \{ v \in G : \text{dist}(x, v) \leq t \} \) denote the ball of radius \( t \) centered at \( x \), and \( 1\{ \cdot \} \) be an indicator random variable.

The parking process is defined on a pair \((G, K)\) of graph \( G = (\mathcal{V}, \mathcal{E}) \) and a (Markov) kernel on \( G \), which is a function \( K : \mathcal{V} \times \mathcal{V} \to [0, 1] \) such that \( K(u, v) = 0 \) if \( (u, v) \notin \mathcal{E} \) and \( \sum_{v \in N(u)} K(u, v) = 1 \). Initially each \( v \in \mathcal{V} \) is assigned an unparked car, written as \( (v, \text{unparked}) \), independently with probability \( p \). Unassigned sites are parking spots that can fit one car. Unparked cars simultaneously perform independent discrete time random walks according to the kernel \( K \) until they find an available spot. When multiple unparked cars move to the same parking spot, each car generates a uniform \([0, 1]\) variable. The car with smallest value parks and the others remain unparked.

The probability space we work in is \( \Omega := \{\{0, 1\} \times ([0, 1]^N)\}^\mathcal{V} \) with probability measure \( \mathbf{P}_p \) under which all coordinates are independent, and the coordinates themselves for each \( v \in \mathcal{V} \) are distributed as follows: The first coordinate is an independent Bernoulli(\( p \)) variable. These indicate whether there is a car initially at a vertex. The second is an independent random walk started at \( v \) with transition kernel \( K \). This is the path an unparked car placed at \( v \) will follow. The third is a sequence of i.i.d. uniform(0, 1) random variables to break ties if multiple cars arrive at the same parking spot.

We say car \( j \in \mathcal{V} \) visits site \( v \) at time \( t \geq 1 \) if \((j, \text{unparked})\) is at some neighbor \( u \in N(v) \) at time \( t - 1 \) and moves to \( v \) at time \( t \). Define

\[
V_t^{(v)} = \sum_{s=1}^{t} \#\{ j \in \mathcal{V} : \text{car } j \text{ visits site } v \text{ at time } s \},
\]

and let \( V^{(v)} := \lim_{t\to\infty} V_t^{(v)} \) be the total number of visits to \( v \). We write these as \( V_t \) and \( V \) whenever visits are identically distributed over all sites.

The canonical example of our theorem is the parking process with simple random walk on \( \mathbb{Z}^d \) from Theorem 1.1. In this setting we have translation invariance, a mass-transport principle by which we are able to change perspectives between cars and spots, and nice ergodicity properties. These essential features are shared by a broader class of graphs that includes regular trees and Cayley graphs. We describe them in more detail now.

The analogue on \((G, K)\) of translation invariance is a type of transitivity. Denote by \( \Gamma \) the set of all automorphisms of \( G \). \( \Gamma \) acts on our probability space naturally by shifting all vertex labeled variables; that is, for each \( \varphi \in \Gamma \) and \( \omega \in \Omega \), \( \varphi(\omega)(v) := \omega(\varphi^{-1}(v)) \). Let \( \Gamma_K \) be the subgroup of \( \Gamma \) consisting of \( K \)-preserving automorphisms; that is

\[
\Gamma_K = \{ \varphi \in \Gamma : K(u, v) = K(\varphi(u), \varphi(v)) \quad \forall u, v \in \mathcal{V} \}.
\]
For each $u, v \in V$, denote $\Gamma_K(u, v) = \{ \varphi \in \Gamma_K: \varphi(u) = v \}$. We say $(G, K)$ is transitive if for each $u, v \in V$, $\Gamma_K(u, v)$ is nonempty. Note that if $(G, K)$ is transitive, then the parking process on it is translation invariant in law. In particular, for each fixed $t \geq 0$, the law of $V_t(v)$ does not depend on $v \in V$. This is crucial to obtain a recursive relation for $EV_t$ (see Proposition 4.5).

Duality between parking spots and cars on $\mathbb{Z}^d$ is a form of the well-known mass-transport principle, which holds for unimodular graphs. In our case, we say the pair $(G, K)$ is unimodular if $|\Gamma_K(u, u)| = |\Gamma_K(v, v)|$ for each $u, v \in V$. For instance, this property is trivially satisfied when $G$ is a Cayley graph and $K$ is uniform.

A key ingredient for both parts of Theorem 2.1, is a 0-1 law for $V$. This relies on the existence of an ergodic transformation $\varphi \in \Gamma_K$. Furthermore, in order to show $EV < \infty$ for small $p$, we need to estimate the expected exit time of a random walk from a ball of fixed radius. To make these arguments work, we need the following technical assumptions on $(G, K)$. For each $u, v \in V$, we say $u$ is accessible from $v$ if there exists a sequence $v = x_0, x_1, \ldots, x_n = u$ of adjacent nodes such that $\prod_{i=0}^{n-1} K(x_i, x_{i+1}) > 0$. We say $(G, K)$ is infinitely accessible if there exist $\varphi \in \Gamma_K$ and $u \in V$ such that $\{\varphi^n(u) : n \geq 0\}$ is infinite and $u$ is accessible from $\varphi(u)$. Since $\varphi^{-1} \in \Gamma_K$, it follows that $u$ is accessible from all $\varphi^n(u)$. Also, we say $(G, K)$ is infinitely escapable if there exists a sequence $(u_0)_{n \geq 0}$ of nodes in $G$ such that each $u_n$ is accessible from $u_0$ and $\text{dist}(u_0, u_n) = n$ for all $n \geq 0$.

Note that any such $\varphi \in \Gamma_K$ in the definition of infinite accessibility must be ergodic. Indeed, as in the usual proof, one takes an event $A$ which is $\varphi$-invariant, approximates it by a cylinder event $B$, and chooses any $k$ so that $\varphi^k B$ and $B$ depend on disjoint sets of variables so that they are independent (this is possible by the assumed property of $\varphi$ in the definition of infinitely accessible). Since $B$ approximates $A$ and $\varphi^k B$ approximates $\varphi^k A = A$, $A$ is independent of itself.

Now we state the general version of Theorem 1.1 and also give a theorem that describes the probabilities that cars and parking spots remain unparked for all time.

**Theorem 2.1.** Let $G$ be a locally finite graph with a kernel $K$. Suppose $(G, K)$ is transitive, unimodular, and infinitely accessible. Then the following hold:

(i) If $p \geq 1/2$, then $V$ is infinite almost surely. Moreover, $EV_t = (2p - 1)t + o(t)$.

(ii) If $p < 1/2$, then $V$ is finite almost surely. Moreover, $EV < \infty$ if $(G, K)$ is infinitely escapable (instead of being infinitely accessible) and $p$ is sufficiently small.

The proof of Theorem 2.1 hinges on a recursive distributional equation (Proposition 4.5) that expresses $V_{t+1}$ in term of the number of visits to its neighboring sites, and a duality between parking spots and cars coming from the mass-transport principle (Lemma 4.1).

We are also able to describe the probability a given car eventually parks, and that a parking spot remains vacant forever.

**Theorem 2.2.** Let $(G, K)$ and $(\xi_t^p)_{t \geq 0}$ be as in Theorem 2.1. Then for any $0 \in V$,

(3) \[ \mathbb{P}[\text{the car initially at } 0 \text{ parks eventually}|0 \text{ has a car initially}] = \begin{cases} \frac{1-p}{p}, & p > 1/2 \\ 1, & p \leq 1/2 \end{cases} \]

(4) \[ \mathbb{P}[\text{spot at } 0 \text{ is parked in eventually}|0 \text{ is a parking spot initially}] = \begin{cases} 1, & p \geq 1/2 \\ \frac{1}{1-p}, & p < 1/2 \end{cases} \]

The case $p = 1/2$ is especially interesting. Note that Theorem 2.1 says $EV_t$ grows linearly in $t$ for $p > 1/2$, but sublinearly for $p = 1/2$. Moreover, Theorem 2.2 says that when $p = 1/2$, 
3. Examples and Further questions

In this section, we discuss some examples of pairs \((G, K)\) for which our main theorems apply. We also state a few further questions.

Example 3.1 (Unoriented lattices and regular trees). Let \(G\) a locally finite connected graph with a uniform kernel \(K\), that is, \(K(u, v) = K(u, w)\) if \(v, w \in N(u)\). Then \(\Gamma_K = \Gamma\) so \((G, K)\) is transitive (resp., unimodular) if and only if \(G\) is vertex transitive (resp., unimodular).

The class of vertex transitive unimodular graphs contains the \(d\)-dimensional integer lattice \(\mathbb{Z}^d\) and the \((d + 1)\)-regular tree \(T_d^\text{hom}\) (in general, Cayley graphs). Since lattices and regular trees possess natural translation automorphisms, so they also satisfy infinite accessibility and escapability. Hence Theorem 2.1 applies to \(\mathbb{Z}^d\) and \(T_d^\text{hom}\) with uniform kernel. See the right side of Figure 2 for simulation results for the symmetric parking process on \(\mathbb{Z}^2\). For this case, the bound for \(E_p, V < \infty\) in Theorem 1.1 (ii) is \(p < 2^{-14}e^{-2} \approx 8.2602 \times 10^{-6}\). ▲

Example 3.2 (Oriented lattices). Let \(\mathbb{Z}^d\) be the oriented \(d\)-dimensional integer lattice where the orientation is from Northeast to Southwest. In this example we consider the parking process on \(\mathbb{Z}^d\) with symmetric random walks. More precisely, let \((e_i)_{i=1}^d\) be the standard basis for \(\mathbb{Z}^d\), and define a kernel \(K\) on \(\mathbb{Z}^d\) by \(K(x, x - e_i) = 1/d\) for each \(1 \leq i \leq d\). The parking process is on \((\mathbb{Z}^d, K)\). For each \(y \in \mathbb{Z}^d\) let \(\tau_y\) be the translation automorphism \(\tau_y(x) = x + y\). Then \(\Gamma_K = \{\text{id}\} \cup \{\tau_{-e_i}: 1 \leq i \leq d\}\). Since \(x\) is accessible from \(y\) if and only if \(x \leq y\) coordinatewise but \(x \neq y\), \((\mathbb{Z}^d, K)\) is transitive, infinitely accessible, and infinitely escapable. Moreover, \(\Gamma_K(x, x) = \{\text{id}\}\) for each \(x \in \mathbb{Z}^d\) so \((\mathbb{Z}, K)\) is unimodular. Hence Theorem 2.1 applies. See the left side of Figure 2 for simulation results for the symmetric parking process on \(\mathbb{Z}^2\). For this case, the bound for \(E_p, V < \infty\) in Theorem 2.1 (ii) is \(p < 2^{-10}e^{-2} \approx 1.3216 \times 10^{-4}\). ▲

Example 3.3 (Directed \(d\)-ary trees). Let \(G = T_d^\text{hom}\) (see Example 3.1) and define a kernel \(K\) on \(G\) such that for each \(u \in V, K(u, v) = 1\) for a unique \(v \in N(u)\). In such case, we call \(v\) the parent of \(u\) and \(u\) a child of \(v\). Each car follows a deterministic path toward the ‘root at infinity’. Note that \((G, K)\) is transitive but not unimodular. Indeed, let \(y \in V\) and let \(x\) be the parent of \(y\). Then any automorphism \(\varphi\) fixing \(y\) must also fix \(x\), but if it fixes \(x\), then it may permute the children of \(x\). Hence \(|\Gamma_K(y, y)x| = 1\) but \(|\Gamma_K(x, x)y| = d!\). Thus Theorem 2.1 does not apply here. As we have mentioned in the introduction, [GP16, Proposition 3.5] shows that \(P(V = \infty) > 0\) for \(p \geq 1/4\) when \(d = 2\). Indeed, observe that if there is an infinite ray containing more than a \(1/2\) density of initial cars, then \(P(V = \infty) > 0\). A quasi-Bernoulli percolation argument then shows that this occurs when \(p > 0.038\). Thus in this case \(p_c \in [0.02, 0.038]\), contrary to the transitive unimodular case in Theorem 2.1.

For \(d = 1\), the pair \((G, K)\) is the directed one-dimensional lattice where all edges are directed from right to left. This is covered by Example 3.2, and so Theorem 2.1 applies. Further information about the process can be obtained at criticality \(p = 1/2\) due to the restriction on topology and kernel. Namely, we have

\[
E V_t \sim \sqrt{2t/\pi}.
\]

To see this, note that by counting the number of cars against parking spots from right to left, the lifespan of a car until parking (see (9)) equals the first hitting time of 0 of an associated
simple symmetric random walk \((S_n)_{n \geq 0}\). By induction this gives

\[ V_t^d := M_t := \max_{0 \leq k \leq t} S_k \quad \forall t \geq 1, \]

so (5) follows from the well-known asymptotic of the expected running maximum for simple symmetric random walk. ▲

When \(p = 1/2\), \(\mathbf{E}V_t\) grows sublinearly. As we have seen in (5), we suspect a polynomial growth \(\mathbf{E}V_t \sim C t^{\alpha}\) for some constants \(C > 0\) and \(\alpha \in (0,1)\), which may depend on the underlying graph. On the oriented and unoriented two dimensional lattices, our simulations given in Figure 2 suggest \(\mathbf{E}V_t \sim C_1 t^{1/4}\) and \(\mathbf{E}V_t \sim C_2 t^{1/2}\), respectively. However, we do not have definitive statistical support for this behavior, so we leave this as a further question.

![Figure 2](image-url)

**Figure 2.** (left) Simulation of the dynamics on the two-dimensional oriented lattice, which is approximated by an \(L \times L\) square \(\Lambda_L\) with periodic boundary. The expectation \(\mathbf{E}V_t\) is approximated by \(\bar{V}_t = L^{-2} \sum_{x \in \Lambda_L} V_t(x)\), where \(V_t(x)\) is the number of car-visits to \(x\) in the time interval \([0,t]\). The (red) plot of \(\log \bar{V}_t\) vs. \(\log t\) up to time \(t = L\) is consistent with a power law; the (blue) regression line is obtained by the data in the time interval \([L/2,L]\) and has slope 0.2528. This suggests that \(\mathbf{E}V_t\) might increase as \(t^{1/4}\). The oriented case is difficult to simulate due to slow convergence and early onset of finite-size effects (which make simulations past time \(t = L\) questionable), so we do not have a definite statistical support for the \(t^{1/4}\) conjecture.

(right) Simulation of the dynamics on the two-dimensional unoriented lattice. The notation and the value of \(L\) is as in Fig. 2, but now the simulation is run up to time \(t = 4L\) (the larger time is justified by much slower symmetric random walks), and the regression line is obtained by the data in the time interval \([2L,4L]\) and has slope 0.4854. This suggests that \(\mathbf{E}V_t\) increases as \(t^{1/2}\), and that the density of blockades decreases as \(t^{-1/2}\), which would agree with annihilating walk systems in [BL91].

**Open Question 1.** For the parking process on \((G,K)\) as before, at what rate does \(\mathbf{E}V_t\) increase to infinity when \(p = 1/2\)? On the oriented and unoriented \(\mathbb{Z}^2\), is it true that \(\mathbf{E}V_t \sim C_1 t^{1/4}\) and \(\mathbf{E}V_t \sim C_2 t^{1/2}\) for some constants \(C_1, C_2 > 0\), respectively? How do the critical exponents relate to the dimension of the lattice?

In [GP16, Theorem 1.2] the authors observe a discontinuous phase transition for \(\mathbf{E}X\); it is bounded by 1 for \(\alpha \leq 1/2\). For large \(\alpha\) it is infinite. This abrupt transition likely comes...
from the fact that a Poi(1)-Galton-Watson tree is a.s. finite. On infinite graphs this does not occur. Indeed, the function $p \mapsto E_p V$ is left-continuous on any $(G, K)$ because it is monotone nondecreasing (Proposition 4.6) and lower semi-continuous (being an increasing limit of the continuous functions $p \mapsto E_p V$). In particular, we have $\lim_{p \to 1/2} E_p V = E_{1/2} V = \infty$, where the second equality follows from Theorem 2.1 (i). We conjecture that there is a critical exponent $\gamma > 0$ such that $E_p V \sim (1/2 - p)^{-\gamma}$ as $p \to 1/2$ (see Figure 3). Of course, one must first prove that $E_p V$ is finite for all $p < 1/2$. The second part of Theorem 2.1 (ii) shows $E_p V < \infty$ for small $p$, but our current argument only works for small values of $p$.

![Figure 3](image)

**Figure 3.** Estimation of $EV$ for subcritical density $p \in [0, 0.49]$ on the two-dimensional oriented (left) and unoriented (right) two-dimensional lattice. The dynamics are run on a square $\Lambda_L$ with $L = 2,000$, until time $t$ when all blockades are eliminated. Then $EV$ is approximated by $V = V_t$, as defined in the caption of Fig. 2. We conjecture that $EV$ diverges as $(1/2 - p)^{-\gamma}$ as $p \to 1/2$, for some critical exponent $\gamma$. In the unoriented case, $\gamma$ appears to be near 1, while in the oriented case, it appears to be somewhat less than $1/2$.

**Open Question 2.** For $(G, K)$ as in Theorem 2.1, is $E_p V < \infty$ for $p < 1/2$? Also, is $E_p V \sim (1/2 - p)^{-\gamma}$ as $p \to 1/2$, for some critical exponent $\gamma > 0$?

Abandoning the car analogy, let us refer to cars as particles and parking spots as blockades. We are interested in the number of visits to $0$ when particles coalesce. Define the *coalescing parking process* to have the same initial configuration as the parking process, but have particles coalesce into one whenever they occupy the same site. The first (collection of) particles to visit a blockade are all destroyed together with the blockade. Letting $V_{\text{coal}}$ denote the total number of particles to visit the origin, it is easy to see that $V_{\text{coal}} \leq V$ and thus $V_{\text{coal}}$ is finite almost surely for $p < 1/2$. It would be very interesting to find $p$ such that $V_{\text{coal}}$ is infinite.

**Open Question 3.** Is there a value $p < 1$ such that $P[V_{\text{coal}} = \infty] > 0$?

This may be hard. Coalescing particles are not so different from mutually annihilating particles. For example, the density of particles in coalescing random walk on $\mathbb{Z}^d$ only differs from annihilating random walk by a factor of 2 (see [Gri78, Arr83]). Thus, this relates to the notoriously difficult question of fluctuation in ballistic annihilation. The question is as follows. Independently place particles with probability $p$ and blockades with probability $1 - p$ on $\mathbb{Z}$. Without loss of generality suppose $0$ contains a blockade. Assign a direction to each
particle, either left or right, with equal probability. From here, the model is deterministic. Particles move at speed 1 in their assigned direction and annihilate if they meet another particle or blockade. If the blockade at 0 is destroyed almost surely we say the process fluctuates. If not, it fixates. It is conjectured in [EF85] that for $p > 3/4$ the process fluctuates. If not, it fixates. It is conjectured in [EF85] that for $p > 3/4$ the process fluctuates. This is worked out in a nearly rigorous way in [KS01], but does not yield much intuition. Recent papers [ST16, DJK+16] prove results that imply fixation for $p < 2/3 + \epsilon$ for a small $\epsilon > 0$, but currently nothing is known about the fluctuation phase. For instance, whether it exists for some $p < 1$.

The parking process can also be thought of as a degenerate branching random walk with cookies. Think of the parking spots as cookies. A particle “branches” into one particle at each step, which is placed at a randomly chosen neighbor, unless that neighbor is a cookie. In this case the particle does not branch; it and the cookie both disappear. Cookie branching random walks on $\mathbb{Z}$ are studied in [BKK+13]. Their initial configuration is deterministic; one cookie is placed at each negative integer. They consider a variety of different transition kernels and branching rules and precisely characterize the transience/recurrence threshold for these branching random walks. While related, their results do not imply ours, nor do their techniques appear to obviously extend to other graphs.

In the same spirit, the parking process relates to a branching random walk in random environment introduced by János Engländer and Nándor Sieben in [ES11]. Further study was carried out recently by Engländer and Yuval Peres in [EP+17]. They consider critical and subcritical 0-2 branching random walk started at the origin on $\mathbb{Z}^d$. The environment consists of obstacles placed independently at each site with probability $p$. When a particle is at an obstacle it passes through without branching, or perishing. The main result of [GMPV10, Theorem 1] is that the probability the subcritical branching random walk survives up to time $n$ has a sub-exponential tail. This is different then the exponential tail for a subcritical 0-2 branching random walk when there are no obstacles. Thus, the presence of obstacles helps the branching process survive longer.

4. PROOF OF THEOREM 2.1 AND THEOREM 2.2

For the entire section we assume that $G = (V, E)$ is a locally finite infinite graph with a kernel $K$ such that $(G, K)$ is transitive and unimodular. We start by recalling the mass-transport principle, which we will use heavily in many proofs. We state our version in the following lemma, which is a minor modification of Theorem 8.7 in Lyons and Peres [LP16].

**Lemma 4.1** (The mass-transport principle). Let $Z: V \times V \rightarrow [0, \infty)$ be a collection of random variables such that $E Z(x, y) = E Z(\varphi(x), \varphi(y))$ for all $\varphi \in \Gamma_K$ whenever $y$ is accessible from $x$ or $x$ is accessible from $y$, and $E Z(x, y) = 0$ otherwise. Then we have

$$E \left[ \sum_{y \in V} Z(0, y) \right] = E \left[ \sum_{y \in V} Z(y, 0) \right].$$

Next, we establish a 0-1 law for $V$.

**Lemma 4.2.** For all $p \in [0, 1]$, we have $P[V = \infty] \in \{0, 1\}$.

**Proof.** Assume that $P(V = \infty) > 0$; we will show it must be 1. Since $(G, K)$ is infinitely accessible, there exist $\varphi \in \Gamma_K$ and (a deterministic site) $x \in V$ such that $\gamma^+: = \{\varphi^n(x) : n \geq 0\}$ is infinite and $x$ is accessible from $\varphi(x)$. As noted in the introduction, $\varphi$ is ergodic. Therefore since we have assumed that $P(V = \infty) > 0$, we can a.s. find a (random) integer
n_0 such that z := ϕ^{n_0}(x) satisfies V^{(z)} = ∞. Because x is accessible from z, we see that it suffices to show that if x, z ∈ V satisfy K(z, x) > 0, then a.s., if V^{(z)} = ∞, then V^{(x)} = ∞. This follows as usual from the Markov property.

The next proposition provides a sufficient condition for V to be infinite. Namely, if EV_t grows linearly and EV_t^2 grows at most quadratically, then V is almost surely infinite. Under our assumption on (G, K), we will then show that this is the case for all p > 1/2, and use a more nuanced argument in the critical case p = 1/2.

**Proposition 4.3.** If there exist c, C > 0 such that EV_t ≥ ct and EV_t^2 ≤ Ct^2 for all t ≥ 1, then V is infinite almost surely.

**Proof.** The Paley-Zygmund inequality yields

\[ P[V_t > EV_t/2] ≥ \frac{1}{4} \left( \frac{EV_t^2}{EV_t^2} \right) = \frac{c^2}{4C} \]

for some δ > 0. Since EV_t → ∞ it follows that P[V = ∞] ≥ c^2/4C. Lemma 4.2 implies that V is almost surely infinite.

In fact, the quadratic upper bound on EV_t^2 in Proposition 4.3 holds for all p ∈ [0, 1]. We write X ≤ Y to denote X is stochastically dominated by Y in the usual sense: P[X ≥ z] ≤ P[Y ≥ z] for all z ≥ 0.

**Proposition 4.4.** EV_t^2 ≤ p(p + 1)t^2.

**Proof.** We dominate the parking process by a system of independent random walks with no parking. Namely, put a particle on each site if and only if there is a car initially; particles perform independent random walks indefinitely with the same transition kernel K used for the parking process. We couple this new process with the original parking process by letting each car follow the path of its matched particle. Let V_t' be the number of visits to the origin up to time t in this system (counting multiple visits by the same particle multiple times). Then by the coupling we have

(6) V_t ≤ V_t'.

Now for each x, y ∈ V and t ≥ 0, let W_t(x, y) be the number of visits of a particle at x to y up to time t. That is,

\[ W_t(x, y) = \sum_{s=1}^{t} 1\{ \text{a particle starts at } x \text{ and is at } y \text{ at time } s \}. \]

Then we can write

\[ V_t' = \sum_{y \in V} W_t(y, 0). \]

A key observation is

\[ \sum_{y \in V} W_t(0, y) = t 1\{0 \text{ has a particle initially}\}. \]

Hence Lemma 4.1 yields

\[ EV_t' = E \left[ \sum_{y \in V} W_t(y, 0) \right] = E \left[ \sum_{y \in V} W_t(0, y) \right] = pt, \]
and
\[
\sum_{y \in V} \mathbb{E}[W_t^2(y, 0)] = \sum_{y \in V} \mathbb{E}[W_t^2(0, y)] \leq \mathbb{E} \left[ \left( \sum_{y \in V} W_t(0, y) \right)^2 \right] = pt^2.
\]

Now using the independence between random walk trajectories of particles starting at different sites, we have
\[
\mathbb{E}[(V_t^1)^2] = \sum_{y \in V} \mathbb{E}[W_t^2(y, 0)] + \sum_{x \neq y} \mathbb{E}[W_t(x, 0)] \mathbb{E}[W_t(y, 0)]
\leq p(t + 1)^2 + \left( \sum_{x \in V} \mathbb{E}[W_t(x, 0)] \right) \left( \sum_{y \in V} \mathbb{E}[W_t(y, 0)] \right)
= pt^2 + (\mathbb{E}V_t^1)^2 = p(t + 1)t^2.
\]

Hence the assertion follows from (6).

The foundation of our analysis is a recursive formula satisfied by $\mathbb{E}V_t$. In the next result, we write “$y$ has a car” for “a car initially starts at $y$.”

**Proposition 4.5.** For all $t \geq 0$,
\[
(7) \quad \mathbb{E}V_{t+1} - \mathbb{E}V_t = 2p - 1 + \mathbb{P}[0 \text{ is a spot and } V_t^{(0)} = 0].
\]

**Proof.** First we write
\[
(8) \quad \mathbb{E}V_{t+1}^{(0)} = \sum_{y \in N(0)} \sum_{s=1}^{t+1} \mathbb{E}\#\{\text{cars that visit } 0 \text{ at time } s \text{ through } y\}.
\]

By conditioning on the “information up until time $s-1$” and partitioning the space according to whether $y$ is an available spot at time $s-1$, we see that each unparked car at $y$ at time $s-1$ visits $0$ independently with probability $K(y, 0)$. This gives
\[
\sum_{y \in N(0)} \sum_{s=1}^{t+1} \mathbb{E}\left[ K(y, 0) \#\{\text{cars visiting } y \text{ at time } s-1\} 1_{\{y \text{ has a car or occupied spot at time } s-1\}} \right]
+ \sum_{y \in N(0)} \sum_{s=1}^{t+1} \mathbb{E}\left[ K(y, 0) (\#\{\text{cars visiting } y \text{ at time } s-1\} - 1) 1_{\{y \text{ is parked at time } s-1\}} \right]
= \sum_{y \in N(0)} K(y, 0) \mathbb{E}\#\{\text{cars visiting } y \text{ at times } \leq t\} - \mathbb{P}(y \text{ is parked at at a time } \leq t)
= \left( \sum_{y \in N(0)} K(y, 0) \right) \left( \mathbb{E}V_t^{(0)} + 1 \mathbf{1}_{\{0 \text{ has a car}\}} + \mathbb{E}V_t^{(0)} \mathbf{1}_{\{0 \text{ is a spot}\}} - \mathbb{P}(V_t^{(0)} > 0, 0 \text{ is a spot}) \right)
= \left( \sum_{y \in N(0)} K(y, 0) \right) \left( \mathbb{E}V_t + 2p - 1 + \mathbb{P}(V_t^{(0)} = 0, 0 \text{ is a spot}) \right).
\]

Note that by the transitivity of $(G, K)$ and Lemma 4.1, the sum of in-probabilities at $0$ equals $1$:
\[
\sum_{y \in N(0)} K(y, 0) = \sum_{y \in V} K(0, y) = \sum_{y \in V} K(0, y) = 1.
\]

Combining this with the above identity yields the desired recursion. □
For the following discussions, it is convenient to introduce a quantity which describes the lifespan of an initial car until parking. Namely, for each \( v \in V \), define \( \tau(v) \) by
\[
(9) \quad \tau(v) := \sum_{s=1}^{\infty} 1\{ \text{a car starts at } v \text{ and is unparked at time } s \}.
\]

By translation invariance of the process, the law of \( \tau(v) \) does not depend on \( v \), so we may drop the dependence on \( v \).

In the parking process adding more cars can only increase the lifespan of cars and the number of visits to a fixed site. While this is intuitively obvious, a possible concern is that introducing a new car may change the manner in which we break ties at a spot. This could potentially cause different cars to park in different places, thus shortening the paths of some cars. Of course this is not the case. We explain why in the following lemma.

**Proposition 4.6.** Write an arbitrary element of our probability space \( \Omega = (\{0,1\} \times (\mathbb{N}) \times ([0,1]^V)) \) as
\[
\omega = (\omega_1, \omega_2, \omega_3)_v \in V = (\eta(v), (X_v(n))_{n \geq 1}, (\epsilon_v(n))_{n \geq 1})_v \in V.
\]

Let \( \omega, \omega' \in \Omega \) be such that \( \omega_1(v) \leq \omega'_1(v), \omega_2(v) = \omega'_2(v) \), and \( \omega_3(v) = \omega'_3(v) \) for all \( v \in V \). Then for all \( v \in V \) and \( t \geq 0 \), we have \( \tau(v)(\omega) \leq \tau(v)(\omega') \) and \( V^t(v)(\omega) \leq V^t(v)(\omega') \).

**Proof.** It suffices to show the assertion for the case when \( \eta(\omega) = \eta(\omega') \) except at one vertex, say, \( x \in V \). Observe that the second assertion is implied by the first. Namely, suppose a site \( y \neq x \) has a car initially which visits \( v \in V \) at some time \( t \geq 1 \) in the \( \omega \)-trajectory. Then \( \tau(y)(\omega) \geq t \), and by the first assertion, this yields \( \tau(y)(\omega') \geq t \). Since the trajectory of the car started at \( y \) is shared in both \( \omega \) and \( \omega' \)-trajectories, this implies that the car started at \( y \) still visits \( v \) at time \( t \) in the \( \omega' \)-trajectory. Hence \( V^t(v)(\omega) \leq V^t(v)(\omega') \), as desired.

Now we show \( \tau(v)(\omega) \leq \tau(v)(\omega') \) for all \( v \in V \). For a contradiction, let \( t \geq 1 \) be the smallest time for which there is a \( y \) such that \( t = \tau(y)(\omega') < \tau(y)(\omega) \). Let \( z \) be the site that the car \( (y,\text{unparked}) \) visits at time \( t \). Since \( z \) is a spot and \( (y,\text{unparked}) \) does not park at time \( t \) in the \( \omega \)-trajectory, some other car \( (u,\text{unparked}) \) parks at \( z \) at some time \( s \leq t \). If \( s \leq t - 1 \), then by the minimality of \( t \), \( \tau(u)(\omega') \geq \tau(u)(\omega) = s \), so \( (u,\text{unparked}) \) visits site \( z \) at time \( s \) in the \( \omega' \)-trajectory. Since \( s \leq t - 1 \), this means that the spot at site \( z \) is already occupied before time \( t \) in the \( \omega' \)-trajectory, which is a contradiction. Hence we may assume \( s = t \), which means that \( z \) is an open spot at time \( t \) in the \( \omega \)-trajectory as well. Since \( (y,\text{unparked}) \) parks at \( z \) at time \( t \) in the \( \omega' \)-trajectory, \( \epsilon_y(t) \) must be minimum among all cars at site \( z \) at time \( t \) in the \( \omega' \)-trajectory. But adding a new car at site \( x \) only adds a new tie breaking variable \( \epsilon_z(t) \) to be compared at site \( z \) at time \( t \), so this would imply that the car \( (y,\text{unparked}) \) also parks at site \( z \) at time \( t \) in the \( \omega \)-trajectory, which is a contradiction. This shows \( \tau(v)(\omega) \leq \tau(v)(\omega') \) for all \( v \in V \), as desired. \( \square \)

A consequence of monotonicity is that if at least one car parks at a fixed site with probability one, then infinitely many do so with probability one.

**Lemma 4.7.** If \( \mathbb{P}[V = 0] = 0 \), then \( V \) is almost surely infinite.

**Proof.** We prove the contrapositive. Suppose that \( \mathbb{P}[V < \infty] = \delta > 0 \). Let \( T \) be the smallest time after which no car visits \( 0 \), so that \( \mathbb{P}[T < \infty] \geq \delta \). Accordingly, let \( t_0 \) be such that \( \mathbb{P}[T < t_0] \geq \delta/2 \). Let \( A \) be the event that the initial configuration has no cars in \( \mathbb{E}(0,t_0) = \{ x : \text{dist}(0,x) \leq t_0 \} \). The event \( \{ T < t_0 \} \) implies that no car initially outside of \( \mathbb{E}(0,t_0) \) ever visits \( 0 \), so \( \{ T < t_0 \} \cap A \subset \{ V = 0 \} \). Write an arbitrary element \( \omega \) in \( \Omega \) as in the statement of Proposition 4.6. By the same proposition, for every fixed realization of the
random walk paths \((X_v(n))\) and tie breakers \((\epsilon_v(n))\) for all \(v \in V\), the variables \(1\{T < t_0\}\) and \(1\{A\}\) are nonincreasing functions of the variables \((\eta(v))\) (which determine the vertices initially occupied by cars), so

\[
P(V = 0) \geq P \{\{T < t_0\} \cap A\} = E[P \{\{T < t_0\} \cap A \mid ((X_v(n))_{n \geq 1}, (\epsilon_v(n))_{n \geq 1})_{v \in V}\}] \geq E[P(T < t_0 \mid (X_v(n)))P(A \mid ((X_v(n))_{n \geq 1}, (\epsilon_v(n))_{n \geq 1})_{v \in V}\}] = P(A)E[P(T < t_0 \mid ((X_v(n))_{n \geq 1}, (\epsilon_v(n))_{n \geq 1})_{v \in V}\)] = P(A)P(T < t_0) \geq (\delta/2)(1 - p)^{[0, t_0]},
\]

where we have used the FKG inequality (see [Hol74]) and the fact that \(A\) is independent of the random walk paths. 

The last ingredient is another monotonicity statement that relates the probability of no visits to 0 conditioned on different starting configurations.

**Proposition 4.8.** For all \(t \geq 0\) and \(p \in (0, 1)\), we have

\[
P(V^{(0)}_t = 0 \mid 0 \text{ has a car initially}) \leq P(V_t = 0) \leq P(V^{(0)}_t = 0 \mid 0 \text{ is a spot})
\]

and

\[
P(V^{(0)} = 0 \mid 0 \text{ has a car initially}) \leq P(V = 0) \leq P(V^{(0)} = 0 \mid 0 \text{ is a spot})
\]

**Proof.** It suffices to show the first part. By Proposition 4.6,

\[
P(V^{(0)}_t = 0 \mid 0 \text{ is a spot}) \geq P(V^{(0)}_t = 0 \mid 0 \text{ has a car initially}).
\]

Hence the assertion follows from

\[
P(V^{(0)}_t = 0) = P(V^{(0)}_t = 0 \mid 0 \text{ is a spot})(1 - p) + P(V^{(0)}_t = 0 \mid 0 \text{ has a car initially})p.
\]

We now have what we need to prove Theorem 2.1 (i).

**Proof of Theorem 2.1 (i).** Note that (7) with \(p \in [1/2, 1)\) and Proposition 4.8 gives

\[
EV_{t+1} \geq EV_t + (1 - p)P(V_t = 0).
\]

If \(P(V_t = 0) \to 0\), then \(P(V = 0) = 0\) and \(V\) is almost surely infinite by Lemma 4.7. If \(P(V_t = 0) \to \delta > 0\), then (10) implies that \(EV_t \geq \delta(1 - p)t\) for all \(t \geq 1\). Thus, Proposition 4.3 implies \(V\) is infinite almost surely.

To show the second part, let \(p \geq 1/2\). Then

\[
P(V^{(0)}_t = 0, 0 \text{ is a spot}) \leq P(V_t = 0) \to P(V = 0) \leq 1 - P(V = \infty) = 0.
\]

Thus the recursion in (7) gives

\[
EV_t = (2p - 1)t + EV_0 + \sum_{s=0}^{t-1} P(V^{(0)}_s = 0, 0 \text{ is a spot}),
\]

where the summation is of order \(o(t)\). 

Before we prove Theorem 2.1 (ii) we use unimodularity to relate the probability a car eventually parks to the probability a parking spot is eventually parked at.

**Lemma 4.9.** For any \(p \in (0, 1)\),

\[
P[\text{a car starts at 0 and eventually parks}] = P[0 \text{ is a spot and it is eventually parked in}].
\]
Proof. For any two sites $x, y \in \mathcal{V}$, let

$$Z(x, y) = 1\{\text{a car starts at } x \text{ and it parks at } y\}.$$ 

Then by Lemma 4.1

$$\mathbb{E} \left[ \sum_{y \in G} Z(0, y) \right] = \mathbb{E} \left[ \sum_{y \in G} Z(y, 0) \right].$$

Since at most one car parks in each spot, the left hand side equals the probability that a car starts at $0$ and it eventually parks. On the other hand, the right hand side equals the probability that $0$ is a parking spot and some car parks there. This proves the assertion. □

The formulas for the probability a car parks and the probability $V = 0$ are quick consequences of Lemma 4.9 and Theorem 2.1.

Proof of Theorem 2.2. First let $p \in [1/2, 1)$. Then Theorem 2.1 (i) implies $\mathbb{P}[V = \infty] = 1$, and so $\mathbb{P}(V(0) = 0, 0 \text{ is a spot}) \leq \mathbb{P}(V = 0) = 0$. This implies $\mathbb{P}(V(0) > 0 \mid 0 \text{ is a spot}) = 1$, which is the first part of (4). Moreover, applying this to the relation in Lemma 4.9 gives

$$\mathbb{P}\{\text{car at } 0 \text{ parks } \mid 0 \text{ has a car initially} \} = \frac{1-p}{p}$$

for $1/2 \leq p \leq 1$, which is the first part of (3). Note that this probability is 1 at $p = 1/2$. Monotonicity of the process ensures that the probability remains 1 for $p < 1/2$. This establishes the second part of (3). It remains to show (4) for $p \in (0, 1/2]$. In this case, (3) and Lemma 4.9 yields

$$\mathbb{P}[V > 0 \mid 0 \text{ is a spot}] = \frac{p}{1-p}.$$

□

Now we turn our attention to Theorem 2.1 (ii). An easy application of the mass-transport principle allows us to write $\mathbb{E}V_t$ in terms of survival probabilities $\mathbb{P}(\tau \geq s)$ of a car.

Proposition 4.10. For any $t \geq 1$,

$$\mathbb{E}V_t = \sum_{s=0}^{t} \mathbb{P}[\tau \geq s].$$

Proof. For each $x, y \in \mathcal{V}$, define a random variable $Z_t(x, y)$ by

$$Z_t(x, y) = 1\{\text{a car is at } x \text{ initially and visits } y \text{ at time } t\}.$$ 

Fix $0 \in \mathcal{V}$. Observe that

$$\sum_{y \in \mathcal{V}} Z_t(y, 0) = \#\{\text{cars visiting } 0 \text{ at time } t\} = V_t^{(0)} - V_{t-1}^{(0)}.$$ 

On the other hand, since each car parks at at most one spot,

$$\sum_{y \in \mathcal{V}} Z_t(0, y) = 1\{\text{a car starts at } 0 \text{ and is unparked at time } t\}.$$ 

Hence by Lemma 4.1, for all $t \geq 1$,

$$\mathbb{E}V_t - \mathbb{E}V_{t-1} = \mathbb{P}\{\text{a car starts at } 0 \text{ and is unparked at time } t\} = \mathbb{P}[\tau \geq t].$$

Thus the assertion follows. □
Next, we need an estimate of the expected time that a random walk spends in a ball of fixed radius. Given a transitive pair \((G, K)\) and \(v \in \mathcal{V}\), let \((X_t^{(v)})_{t \geq 0} \subseteq \mathcal{V}\) be an independent random walk trajectory given by the kernel \(K\) that a particle initially at \(v \in \mathcal{V}\) follows. For each \(j \geq 1\), define
\[
    t^{(v)}(j) = \inf\{t \geq 0 : \text{dist}(X_0^{(v)}, X_t^{(v)}) > j\},
\]
the first exit time of \(B(v, j)\). Since \((G, K)\) is transitive, the law of \(t^{(v)}(j)\) does not depend on \(v\). Define the following generating function
\[
    F(s) = \sum_{j=0}^{\infty} \mathbb{E}[t^{(v)}(j)] s^j.
\]
Finally, define \(K_{\min}\) to be the minimum (non-zero) transition probability over all \((G, K)\):
\[
    K_{\min} = \min\{K(x, y) : x, y \in \mathcal{V} \text{ and } K(x, y) > 0\}.
\]
Note that since \((G, K)\) is transitive, and \(G\) is locally finite, we have \(K_{\min} = \min\{K(0, y) : K(0, y) > 0\} > 0\).

**Proposition 4.11.** \(F(s) < \infty\) for all \(|s| < (K_{\min})^2\).

**Proof.** For each \(k, j \geq 0\), define the hitting probability
\[
    a_{k, j} = \mathbb{P}\left[ \max_{0 \leq i \leq k} \text{dist}(X_0^{(v)}, X_i^{(v)}) = j \right]
\]
and its generating function
\[
    Q(u, s) = \sum_{k, j \geq 0} a_{k, j} u^k s^j.
\]
Note that \(Q\) is well-defined on \((-1, 1)^2\).

Since \((G, K)\) is infinitely escapable, there exists a sequence \((u_n)_{n \geq 0}\) such \(u_n\) is accessible from \(u_0\) and \(\text{dist}(u_0, u_n) = n\) for every \(n\). Using the Markov property of the random walk \(X_t = X_t^{(u_0)}\) on \((G, K)\) it holds for any \(\alpha \in \Gamma_G\) that
\[
    \mathbb{P}[X_n^{(\alpha(u_0))} = \alpha(u_n)] = \mathbb{P}[X_n = u_n] \geq (K_{\min})^n.
\]
By the triangle inequality and the Markov property, this yields
\[
    a_{k, j} \leq \prod_{\ell=0}^{\lfloor k/(2j+1) \rfloor - 1} \mathbb{P}\left[ \text{dist}(X_{(2j+1)\ell}^{(v)}, X_{(2j+1)(\ell+1)}^{(v)}) \leq 2j \right] \\
    \leq (1 - (K_{\min})^{2j+1})^k \left( \frac{2j+1}{2j+1} \right) \leq 3 \exp\left( - \frac{(K_{\min})^{2j+1}k}{(2j+1)} \right)
\]
for any \(k, j \geq 0\). Hence by dominated convergence
\[
    Q(1, s) = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{k, j} \right) s^j \leq 3 \sum_{j=0}^{\infty} \frac{s^j}{1 - \exp(- (K_{\min})^{2j+1}k/(2j+1))}.
\]
By the ratio test, the power series on the right converges whenever \(|s| < (K_{\min})^2\). Now observe that
\[
    \sum_{j=0}^{\infty} \mathbb{E}[t^{(v)}(j)] s^j = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} \mathbb{P}(t^{(v)}(j) > k) \right) s^j.
\]
This shows $F(s) < \infty$ whenever $s \in (-K_{\min}^2, (K_{\min})^2)$. \hfill \square

We are now ready to show Theorem 2.1 (ii).

Proof of Theorem 2.1 (ii). Suppose $0 \leq p < 1/2$. Rewrite the relation in Lemma 4.9 as

\[
\mathbb{P}[V > 0 | v \text{ is a spot}] = \frac{p \mathbb{P}[\text{car at } v \text{ parks } | v \text{ has a car}]}{1 - p} \leq \frac{p}{1 - p} < 1.
\]

So the complement event has positive probability: \( \mathbb{P}[V = 0 | v \text{ is a spot}] > 0 \). By Proposition 4.8, this implies \( \mathbb{P}[V = 0] > 0 \). The 0-1 law in Lemma 4.2 then requires that \( \mathbb{P}[V = \infty] = 0 \).

Next, we show \( EV < \infty \) when \( p \) is small. For any connected subgraph \( H \subseteq G \), say \( H \) is busy if initially there are at least as many cars as spots on \( H \). For each \( x, y \in V \), denote by \( \{x \xrightarrow{t} y\} \) the event that \( x \) has a car initially which visits \( y \) at time \( t \). We claim that the event \( \{v_1 \xrightarrow{t} x_1\} \) implies the existence of a busy subgraph \( H \subseteq B(v_1, 2t) \). This will imply that \( \{\tau^{(v)} \geq t\} \) also implies that there exists a busy connected subgraph \( H \subseteq B(v, 2t) \). To show the claim, first note that the event \( \{v_1 \xrightarrow{t} x_1\} \) does not depend on sites outside of \( B(v_1, 2t) \), since two sites interact by contact in space-time and cars move at speed 1. Let \( \gamma_1 \) be the induced subgraph consisting of all sites that the car \((v_1, \text{unparked})\) visits through time \( t \). If \( \gamma_1 \) is busy, then we are done. Otherwise, since each spot takes at most one car, there is some spot at site \( x_2 \in V(\gamma_1) \) such that \( \{v_2 \xrightarrow{t} x_2\} \) occurs for some \( v_2 \notin V(\Gamma_1) \). Let \( \gamma_2 \) be the induced subgraph consisting of all sites that the car \((v_2, \text{unparked})\) visits through time \( t \). By an earlier remark, \( \gamma_2 \subseteq B(v_1, 2t) \). If \( \gamma_1 \cup \gamma_2 \) is not busy, then find another pair of spot and external car and add a new subgraph \( \gamma_3 \subseteq B(v_1, 2t) \), and so on. The resulting union is always contained in \( B(v, 2t) \), so this process terminates with a busy subgraph. This shows the claim.

Note that \( G \) has at most \((e\Delta)^j\) connected subgraphs of size \( j \) containing \( v \), and by a Chernoff bound for Binomial\((j, p)\) variable, the probability of a connected subgraph of size \( j \) being busy is at most \((2\sqrt{p(1-p)})^j\) when \( p < 1/2 \). Moreover, \(|B(v, r)| \leq (\Delta - 1)^{r+1}\).

Let \( (X_t)_{t \geq 0} \) be an independent random walk trajectory on \((G, K)\) with \( X_0 = v \). Applying a union bound gives

\[
\mathbb{P}[\tau^{(v)} \geq t] \leq \sum_{j=1}^{\Delta^{2t+1}} \sum_{|H|=j} (2\sqrt{p(1-p)})^j \mathbb{P}[X_k \text{ is in } H \text{ for all } 0 \leq k \leq t]
\]

\[
\leq \sum_{j=1}^{\Delta^{2t+1}} (2e\Delta \sqrt{p(1-p)})^j \mathbb{P}[X_k \text{ is in } B(v, j) \text{ for all } 0 \leq k \leq t].
\]

Let \( \tau^{(v)}(j) \) and \( F(t) \) be as defined in (11) and (12), respectively. Then Proposition 4.10 and the above observation yield

\[
EV^{(v)} \leq \sum_{t=0}^{\infty} \sum_{j=1}^{\Delta^{2t+1}} (2e\Delta \sqrt{p(1-p)})^j \mathbb{P}[X_k \text{ is in } B(v, j) \text{ for all } 0 \leq k \leq t]
\]
\[
\sum_{j=1}^{\infty} (2e\Delta e \sqrt{p(1-p)})^j \sum_{t: t+1 \geq \log \Delta j} P[X_k \text{ is in } \mathcal{B}(v, j) \text{ for all } 0 \leq k \leq t]
\]
\[
\leq \sum_{j=0}^{\infty} (2e\Delta e \sqrt{p(1-p)})^j E[\tau^{(v)}(j)] = F(s_p),
\]

where \( s_p = 2e\Delta e \sqrt{p(1-p)} \). Hence by Proposition 4.11, \( EV^{(v)} < \infty \) whenever \( |s_p| < (K_{\text{min}})^2 \). Observe that for \( \mathbb{Z}^d \) with simple symmetric random walks, it is sufficient that \( p < (256d^6e^2)^{-1} \). On the oriented lattice \( \mathbb{Z}^d \), \( p < (64d^6e^2)^{-1} \) yields \( E_p V < \infty \). \( \square \)

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