Generalized CRF-structures

by

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ABSTRACT. A generalized F-structure is a complex, isotropic subbundle $E$ of $T_cM \oplus T^*_cM$ ($T_cM = TM \otimes_R C$ and the metric is defined by pairing) such that $E \cap \bar{E}^\perp = 0$. If $E$ is also closed by the Courant bracket, $E$ is a generalized CRF-structure. We show that a generalized F-structure is equivalent with a skew-symmetric endomorphism $\Phi$ of $TM \oplus T^*M$ that satisfies the condition $\Phi^3 + \Phi = 0$ and we express the CRF-condition by means of the Courant-Nijenhuis torsion of $\Phi$. The structures that we consider are generalizations of the F-structures defined by Yano and of the CR (Cauchy-Riemann) structures. We construct generalized CRF-structures from: a classical F-structure, a pair $(\mathcal{V}, \sigma)$ where $\mathcal{V}$ is an integrable subbundle of $TM$ and $\sigma$ is a 2-form on $M$, a generalized, normal, almost contact structure of codimension $h$. We show that a generalized complex structure on a manifold $\tilde{M}$ induces generalized CRF-structures into some submanifolds $M \subseteq \tilde{M}$. Finally, we consider compatible, generalized, Riemannian metrics and we define generalized CRFK-structures that extend the generalized Kähler structures and are equivalent with quadruples $(\gamma, F_+, F_-, \psi)$, where $(\gamma, F_\pm)$ are classical, metric CRF-structures, $\psi$ is a 2-form and some conditions expressible in terms of the exterior differential $d\psi$ and the $\gamma$-Levi-Civita covariant derivative $\nabla_{F_\pm}$ hold. If $d\psi = 0$, the conditions reduce to the existence of two partially Kähler reductions of the metric $\gamma$. The paper ends by an Appendix where we define and characterize generalized Sasakian structures.

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1 Introduction

This paper belongs to the framework of generalized structures on a differentiable manifold $M$ (e.g., Hitchin [10]). These structures are similar to classical structures but they are defined on the big tangent bundle $T^{big} M = TM \oplus T^* M$ with the neutral metric

\begin{equation}
(1.1) \quad g((X, \alpha), (Y, \beta)) = \frac{1}{2} (\alpha(Y) + \beta(X))
\end{equation}

and the Courant bracket [5]

\begin{equation}
(1.2) \quad [(X, \alpha), (Y, \beta)] = ([X, Y], L_X \beta - L_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X))).
\end{equation}

In (1.1) and (1.2) the notation uses the following conventions that will be followed throughout the whole text: $M$ is an $m$-dimensional manifold and $X, Y, \ldots$ are either contravariant vectors or vector fields, $\alpha, \beta, \ldots$ are either covariant vectors or 1-forms. Furthermore, we will denote by $\chi^k(M)$ the space of $k$-vector fields, by $\Omega^k(M)$ the space of differential $k$-forms, by $\Gamma$ spaces of global cross sections of vector bundles, by $d$ the exterior differential and by $L$ the Lie derivative. All the manifolds and mappings are assumed of the $C^\infty$ class.

Most of the work done until now in this framework was on generalized, complex and Kähler structures [8] and this work was motivated by applications to supersymmetry in string theory [13]. Other generalized structures that were considered are the almost product structures [17, 20], the almost tangent structures [17] and the almost contact structures [11, 18].

A generalized, complex structure can be defined as a complex Dirac structure, i.e., a maximal $g$-isotropic subbundle of the complexified big tangent bundle $T^{big}_c M = T^{big} M \otimes \mathbb{C}$ with the space of cross sections closed by the Courant bracket [8]. By dropping the maximality condition, i.e., by replacing the Dirac structure by a big-isotropic structure [19], we will obtain the notion of a generalized F-structure, such that any classical F-structure [21] produces a generalized F-structure. Furthermore, if the big-isotropic structure is integrable, i.e., its space of cross sections is closed by the Courant bracket, we get an integrable, generalized F-structure.

If the generalized F-structure defined by a classical one is integrable, the classical F-structure is such that its $\sqrt{-1}$-eigenbundle (the holomorphic dis-
tribution) is a CR (Cauchy-Riemann) structure \[6\]. Accordingly, we propose the name CRF-structure, which explains the title of this paper\[1\].

The addition of a compatible, generalized, Riemannian metric leads to a notion of generalized CRF-structure that extends the generalized Kähler structures \[8\]. It turns out that a generalized CRF-structure is equivalent with a quadruple \((\gamma,F_+,F_-,\psi)\) where \((\gamma,F_\pm)\) are classical, metric CRF-structures, \(\psi\) is a 2-form and some conditions expressible in terms of the exterior differential \(d\psi\) and the \(\gamma\)-Levi-Civita covariant derivative \(\nabla F_\pm\) hold. If \(d\psi = 0\), the conditions reduce to the existence of two partially Kähler reductions of the metric \(\gamma\).

The connection with the theory of CR-structures is a motivation of the present paper. The motivation is enhanced by the fact that some submanifolds of a generalized, complex manifold may have an induced, generalized F-structure, like in the case of the well known CR-submanifolds of a Hermitian manifold \[2\].

We recall that the real big-isotropic structures \(E \subseteq T^{big}M\) were studied in \[19\]. The integrability condition of the structure (closure by the Courant brackets) is

\[(1.3) \quad [(X,\alpha),(Y,\beta)] \in \Gamma(E), \quad \forall (X,\alpha),(Y,\beta) \in \Gamma(E)\]

and the properties of the Courant bracket imply that condition \((1.3)\) is equivalent with the condition \([E,E'] \subseteq E'\) where \(E' = E_\perp^g\) is the \(g\)-orthogonal bundle of \(E\) in \(T^{big}M\). The simplest examples are:

**Example 1.1.** Let \(F\) be a subbundle of \(TM\) and \(\theta \in \Omega^2(M)\). Then,

\[(1.4) \quad E_\theta = graph(\flat|_F) = \{(X,\flat X = i(X)\theta) / X \in F) \subseteq T^{big}M\]

is a big-isotropic structure on \(M\) with the \(g\)-orthogonal bundle

\[(1.5) \quad E'_\theta = \{(Y,\flat Y + \gamma) / Y \in TM, \gamma \in ann S\}.

The structure \(E_\theta\) is integrable iff \(F\) is a foliation and \(\theta\) satisfies the condition \[8, 19\]

\[(1.6) \quad d\theta(X_1, X_2, Y) = 0, \quad \forall X_1, X_2 \in \Gamma F, Y \in \chi^1(M)\].

\[1\]The names CR-structure and F-structure are well established in the mathematical literature. It is interesting to mention that an equivalent definition of classical F-structures was given in \[1\] where these structures are called hor-complex structures; “hor” comes from “horizontal”. Correspondingly, we should use the name of a generalized hor-complex structure. The “hor-complex” terminology is nice but it was not adopted in the literature.
Example 1.2. Let $\Sigma$ be a subbundle of $T^*M$ and $P \in \chi^2(M)$. Then

$$E_P = \text{graph}(\sharp_P|_\Sigma) = \{(\sharp_P\sigma = i(\sigma)P, \sigma) / \sigma \in \Sigma\}$$

is a big-isotropic structure on $M$ with the $g$-orthogonal bundle

$$E'_P = \{(\sharp_P\beta + Y, \beta) / \beta \in T^*M, Y \in \text{ann } \Sigma\}.$$ 

The structure $E_P$ is integrable iff $\Sigma \subseteq T^*M$ is closed with respect to the bracket of 1-forms

$${\{\alpha, \beta\}}_P = L_{\sharp_P\alpha}\beta - L_{\sharp_P\beta}\alpha - d(P(\alpha, \beta))$$

and the Schouten-Nijenhuis bracket $[P, P]$ (e.g., [16]) satisfies the condition [19]

$$[P, P](\alpha_1, \alpha_2, \beta) = 0, \quad \forall \alpha_1, \alpha_2 \in \Sigma, \beta \in \Omega^1(M).$$

We end the Introduction by a few words about the content of the paper. In Section 2 we define the generalized CRF-structures as complex, big-isotropic structures, prove the equivalence with a skew-symmetric endomorphism $\Phi \in \text{End}(T^{big})M$ such that $\Phi^3 + \Phi = 0$, which may also be presented as a triple of classical tensor fields $(A \in \Gamma(TM \otimes T^*M), \sigma \in \Omega^2(M), \pi \in \chi^2(M))$, and express the integrability condition in terms of $\Phi$. Finally, we consider generalized F-structures induced on some submanifolds of a generalized complex manifold and show their integrability. In Section 3 we discuss particular classes of CRF-structures. We show that a classical F-structure produces a generalized F-structure and get the corresponding integrability conditions, which define a seemingly new class of Yano’s F-structures. Then, we associate a generalized F-structure with a pair $\{V \subseteq TM, \theta \in \Omega^2(M)\}$, where $\theta$ is non degenerate on $V$, and with a pair $\{\Sigma \subseteq T^*M, \pi \in \Omega^2(M)\}$, where $\pi$ is non degenerate on $\Sigma$. In Section 4, we present the generalized Riemannian metrics following [8] and extend the notion of a generalized Kähler manifold to a notion of CRFK-manifold by replacing the generalized complex structure by a generalized CRF-structure. Then we obtain expressions of the CRFK condition by means of the corresponding classical objects and show that Riemannian manifolds $(M, \gamma)$ where the metric $\gamma$ has two partially Kähler reductions are CRFK-manifolds.
2 The basics of generalized CRF-structures

The definition of big-isotropic structures, integrability and other basic properties may be complexified, i.e., transferred to the complexified tangent bundle $T^{big}_cM$ and, in analogy to the case of the generalized complex structures \[8\], we give following definition.

**Definition 2.1.** A complex, big-isotropic structure $E \subseteq T^{big}_cM$ of rank $k$, with the $g$-orthogonal subbundle $E'$, is called a *generalized F-structure* if

\begin{equation}
E \cap \bar{E}' = 0,
\end{equation}

where the bar denotes complex conjugation. Furthermore, if $E$ is integrable $E$ will be called a *generalized CRF-structure*.

If $k = m$, $E$ is a generalized (almost) complex structure. The condition $E \cap \bar{E}' = 0$ is equivalent with $E' \cap \bar{E} = 0$ and, since the orthogonal space of $\bar{E}$ is $E'$ (because the metric $g$ is real), $\bar{E}$ is a generalized F-structure too and it is integrable iff $E$ is integrable. Furthermore, condition (2.1) is equivalent to each of the equalities

\begin{equation}
T^{big}_cM = E \oplus \bar{E}', \quad T^{big}_cM = E' \oplus \bar{E}.
\end{equation}

We obtain an equivalent definition of the generalized F-structures as follows. Let $E$ be a generalized F-structure of rank $k$. Then, $E' \cap \bar{E}' = S_c$ is the complexification of a real subbundle $S \subseteq T^{big}_cM$ of rank

\[\text{rank } S = (2m - k) + (2m - k) - 2m = 2(m - k)\]

(use (2.2) to see that rank$(E' + \bar{E}') = 2m$) and (2.1) implies $E \cap S_c = \bar{E} \cap S_c = 0$. Thus, since dim$(E \oplus \bar{E}) = 2k$, decomposition (2.2) becomes

\begin{equation}
T^{big}_cM = E \oplus \bar{E} \oplus S_c = L_c \oplus S_c,
\end{equation}

where $L$ is the real subbundle of $T^{big}_cM$ such that $L_c = E \oplus \bar{E}$. Notice also that the definition of $S_c$ implies $S_c \bot_g E, S_c \bot_g \bar{E}$.

We can use (2.3) in order to define a real endomorphism

$$\Phi : T^{big}_cM \rightarrow T^{big}_cM$$
with eigenspaces $E, \bar{E}, S$ of corresponding eigenvalues $\sqrt{-1}, -\sqrt{-1}, 0$. This endomorphism has the property

\[(2.4) \quad \Phi^3 + \Phi = 0.\]

Moreover, by checking on arguments in the various terms of \((2.3)\) while using the isotropy of $E, \bar{E}$ and the relations $S_c \perp E, \bar{E}$, we see that $\Phi$ is skew-symmetric in the sense that

\[(2.5) \quad g(\Phi(X, \alpha), (Y, \beta)) + g((X, \alpha), \Phi(Y, \beta)) = 0.\]

If $\mathcal{X} = (X, \alpha) \in T^{big}M$ and

\[(2.6) \quad \mathcal{X} = \mathcal{X}' + \mathcal{X}'' + \mathcal{X}''' , \quad \mathcal{X}' \in E, \mathcal{X}'' \in \bar{E}, \mathcal{X}''' \in S_c,\]

then

\[(2.7) \quad \Phi\mathcal{X} = \sqrt{-1}(\mathcal{X}' - \mathcal{X}''), \quad \Phi^2\mathcal{X} = -(\mathcal{X}' + \mathcal{X}''),\]

whence $ker \Phi = S$, $im \Phi = L$, and $rank \Phi = 2k$. Furthermore, we get the following expressions of the projections on the terms of \((2.3)\) and on $L$:

\[(2.8) \quad pr_E = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi), \quad pr_{\bar{E}} = -\frac{1}{2}(\Phi^2 - \sqrt{-1}\Phi),\]

\[pr_S = Id + \Phi^2, \quad pr_L = -\Phi^2.\]

**Proposition 2.1.** A generalized $F$-structure $E$ is equivalent with an endomorphism $\Phi : T^{big}M \to T^{big}M$ that satisfies the conditions \((2.4), (2.5)\).

**Proof.** We just have deduced $\Phi$ from $E$. If we start with $\Phi$, \((2.4)\) shows that the eigenvalues of $\Phi$ are $\pm \sqrt{-1}, 0$, hence, $\forall x \in M$, we get a decomposition \((2.3)\) at $x$, where the projections are given by \((2.8)\). In particular, the ranks of all the terms of the derived decomposition \((2.3)\) are lower semicontinuous (i.e., non decreasing in a neighborhood of a point), which cannot happen unless the ranks are constant; we will denote $k = rank E$. Finally, \((2.5)\) implies that $E$ is big-isotropic and $E \perp S_c, \bar{E} \perp S_c$, therefore, $E' = E \oplus S_c$ and $E \cap E' = 0$. \hfill $\square$

Condition \((2.4)\) is equivalent to the condition

\[(2.9) \quad \Phi^2|_{im \Phi} = -Id.\]
Furthermore, let us notice that a generalized F-structure $\Phi$ has a numerical invariant given by the negative inertia index $q$ of the metric induced by $g$ in $S$, to be called the negative index of either $\Phi$ or $E$. Indeed, from the definition of $S$, it follows that $g|_S$ is non degenerate (and so is $g|_L$) and it is known that, then, $q = \text{const.}$ on the connected components of $M$.

Accordingly, we get

**Proposition 2.2.** If $M$ is a connected manifold, a skew-symmetric endomorphism $\Phi \in \text{End}(T^{\text{big}}M)$ is a generalized F-structure of negative index $q$ iff around each point $x \in M$ there exist local, pairwise $g$-orthogonal cross sections $Z_a, Z_{\alpha}(a = 1, \ldots, q, \alpha = 1, \ldots, p)$ of $T^{\text{big}}M$ that satisfy the conditions $g(Z_a, Z_a) = -1, g(Z_{\alpha}, Z_{\alpha}) = +1$ and one has

\[
(2.10) \quad \Phi Z_a = 0, \Phi Z_{\alpha} = 0, \Phi^2 = -\text{Id} + \sum_a (\flat_g Z_{\alpha}) \otimes Z_a - \sum_a (\flat_g Z_a) \otimes Z_a.
\]

**Proof.** The equalities (2.10) imply (2.4) and $\ker \Phi = \text{span}\{Z_a, Z_{\alpha}\}$, which justifies the value of the negative index. Conversely, we know that (2.4) implies the existence of $S$ and the fact that $g|_S$ is non degenerate of a negative index $q$. Then, if we take a local basis $(Z_a, Z_{\alpha})$ of $S$ as required by the proposition, the conditions (2.10) hold. 

Proposition 2.2 and a known classical situation [3, 9] suggest giving the following definition.

**Definition 2.2.** A generalized F-structure defined by an endomorphism $\Phi \in \text{End}(T^{\text{big}}M)$ and a global, $g$-orthogonal frame $Z_a, Z_{\alpha}$ that satisfy all the hypotheses of Proposition 2.2 is a generalized F-structure with complementary frames.

We can also derive an expression of the integrability condition of $E$ in terms of $\Phi$. For this purpose, we consider the Courant-Nijenhuis torsion of $\Phi$ defined by

\[
(2.11) \quad \mathcal{N}_{\Phi}(\mathcal{X}, \mathcal{Y}) = [\Phi \mathcal{X}, \Phi \mathcal{Y}] - \Phi[\mathcal{X}, \mathcal{Y}] - \Phi[\mathcal{X}, \Phi \mathcal{Y}] + \Phi^2[\mathcal{X}, \mathcal{Y}],
\]

where the brackets are Courant brackets. Then we get

**Proposition 2.3.** The generalized F-structure $E$ is integrable iff the corresponding endomorphism $\Phi$ satisfies one of the following equivalent conditions

\[
(2.12) \quad \mathcal{N}_{\Phi}(\mathcal{X}, \mathcal{Y}) = \text{pr}_S[\mathcal{X}, \mathcal{Y}], \quad \forall \mathcal{X}, \mathcal{Y} \in \Gamma L,
\]
\( S_\Phi(\mathcal{X},\mathcal{Y}) = [\Phi\mathcal{X},\Phi\mathcal{Y}] - [\Phi^2\mathcal{X},\Phi^2\mathcal{Y}] + \Phi[\Phi\mathcal{X},\Phi^2\mathcal{Y}] + \Phi[\Phi^2\mathcal{X},\Phi\mathcal{Y}] = 0, \)
\( \forall \mathcal{X},\mathcal{Y} \in \Gamma T^{big}M. \)

**Proof.** The equivalence between the two conditions follows by using arguments of the form \( \Phi\mathcal{X},\Phi\mathcal{Y} \) in (2.12) (remember that \( L = im \Phi \)). Then, the conclusion of the proposition follows from the fact that \( \forall \mathcal{X}',\mathcal{Y}'' \in \Gamma E \forall \mathcal{Y}'' \in \Gamma \bar{E} \) one has
\[
S_\Phi(\Phi\mathcal{X}',\Phi\mathcal{Y}'') = -2([\mathcal{X}',\mathcal{Y}'] + \sqrt{-1}\Phi[\mathcal{X}',\mathcal{Y}'']), \quad S_\Phi(\Phi\mathcal{X}',\Phi\mathcal{Y}'') = 0.
\]

Notice that, under the conditions (2.4) and (2.5), the concomitant \( S_\Phi \) is \( C^\infty(M) \)-linear while \( N_\Phi \) is not. Another interesting fact is given by

**Proposition 2.4.** If \( E \) is an integrable big-isotropic structure with the endomorphism \( \Phi \) then
\( (2.14) \quad N_\Phi(\mathcal{X},\mathcal{Y}) = 0, \quad \forall \mathcal{X} \in L, \mathcal{Y} \in S. \)

**Proof.** The conclusion is equivalent with
\( (2.15) \quad \Phi^2[\mathcal{X},\mathcal{Y}] - \Phi[\Phi\mathcal{X},\mathcal{Y}] = 0, \quad \forall \mathcal{X} \in L, \mathcal{Y} \in S. \)

It suffices to check the result for \( \mathcal{X} \in E \) since the case \( \mathcal{X} \in \bar{E} \) will be obtained by complex conjugation. If \( \mathcal{X} \in E, \) (2.15) becomes
\[
\sqrt{-1}\Phi[\mathcal{X},\mathcal{Y}] - \Phi^2[\mathcal{X},\mathcal{Y}] = 0,
\]
which, by (2.8), is equivalent with \( [\mathcal{X},\mathcal{Y}] \in E \oplus S = E'. \) For an integrable structure \( E, \) the previous equality holds since the integrability of \( E \) is equivalent with \( [E,E'] \subseteq E'. \)

Like in the case of the generalized almost complex structures [8], one can give a representation of a generalized CRF-structure in terms of classical tensor fields. An endomorphism \( \Phi : T^{big}M \to T^{big}M \) that satisfies the skew-symmetry condition (2.5) has the matrix representation
\[
(2.16) \quad \Phi \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} A & \#_\pi \\ \#_\sigma & -A \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}
\]
where \((X, \alpha) \in T^{\text{big}}M, A \in \text{End}(TM), \sigma \in \Omega^2(M), \pi \in \chi^2(M),\) and \(t\) denotes transposition. From (2.16) we get

\[
\Phi(X, \alpha) = (AX + \sharp \pi \alpha, \flat \sigma X - tA\alpha),
\]

(2.17)

\[
\Phi^2(X, \alpha) = (\tilde{A}X + \sharp \tilde{\pi} \alpha, \flat \tilde{\sigma} X + t\tilde{A} \alpha),
\]

where

\[
\tilde{A} = A^2 + \sharp \tilde{\pi} \beta, \tilde{\pi} = A \sharp \pi - \sharp \pi tA, \tilde{\sigma} = \flat \sigma A - tA \flat \tilde{\sigma}.
\]

(2.18)

The first formula (2.17) yields the following interpretation of the entries of the matrix (2.16)

\[
\begin{align*}
\pi(\alpha, \beta) &= 2g(\Phi(0, \alpha), (0, \beta)), \\
\sigma(X, Y) &= 2g(\Phi(X, 0), (Y, 0)), \\
<AX, \alpha> &= 2g(\Phi(X, 0), (0, \alpha)).
\end{align*}
\]

(2.19)

The endomorphism (2.16) is a generalized F-structure (i.e., satisfies (2.4)) iff:

\[
CA = -\sharp \tilde{\pi} \beta, \ C\sharp \pi = A \sharp \pi - \sharp \pi tA, \ C\flat \tilde{\sigma} = \flat \sigma A - tA \flat \tilde{\sigma}.
\]

(2.20)

The expressions of the integrability conditions of the structure \(\Phi\) by means of the tensor fields \(A, \pi, \sigma\) of (2.16) can be obtained by calculating (2.13) in each of the following cases: a) \(X = (0, \alpha), Y = (0, \beta), b) X = (X, 0), Y = (0, \beta), c) X = (X, 0), Y = (Y, 0).\) In the general case, these expressions are complicated and give no hope for applications. For instance, the condition

\[
pr_{TM} S((0, \alpha), (0, \beta)) = 0
\]

produces the integrability condition

\[
[\sharp \pi \alpha, \sharp \pi \beta] - [\sharp \pi \alpha, \sharp \tilde{\pi} \beta] + A[\sharp \pi \alpha, \sharp \pi \beta] + A[\sharp \tilde{\pi} \alpha, \sharp \pi \beta]
\]

\[
= \sharp \pi \{L_{\sharp \pi} \beta (\alpha \circ \tilde{A}) - L_{\sharp \tilde{\pi}} \alpha (\beta \circ \tilde{A}) + L_{\sharp \tilde{\pi}} \beta (\alpha \circ \tilde{A}) - L_{\sharp \tilde{\pi}} \alpha (\beta \circ \tilde{A})
\]

\[
+ \frac{1}{2} d(\pi(\alpha, \beta \circ \tilde{A}) - \pi(\beta, \alpha \circ \tilde{A}) + \tilde{\pi}(\alpha, \beta \circ A) - \pi(\beta, \alpha \circ A))}.
\]

(2.21)

Under the restrictive conditions \(\tilde{A} = -Id, \tilde{\pi} = 0,\) e.g., in the case of a generalized complex structure [8], (2.21) is equivalent with the fact that \(\pi\) is a Poisson bivector field.
Example 2.1. A generalized, almost contact structure of codimension $h$ is a system of tensor fields $(P, \theta, F, Z_a, \xi^a) (a = 1, \ldots, h)$ where $F \in \text{End}(TM)$, $P \in \chi^2(M)$, $\theta \in \Omega^2(M)$, $Z = (Z_a) : T^*M \to \mathbb{R}^h$ is a sequence of $h$ vector fields and $\xi = (\xi^a) : TM \to \mathbb{R}^h$ is a sequence of $h$ 1-forms and the following conditions hold

\begin{align}
& P(\alpha \circ F, \beta) = P(\alpha, \beta \circ F), \quad \theta(FX, Y) = \theta(X, FY), \\
& F(Z_a) = 0, \quad \xi^a \circ F = 0, \quad i(Z_a)\theta = 0, \quad i(\xi^a)P = 0, \quad \xi^a(Z_b) = \delta^a_b, \\
& F^2 = -\text{Id} - \sharp_P \circ \flat_\theta + \sum_{a=1}^h \xi^a \otimes Z_a.
\end{align}

Then, the tensor fields $A = F, \pi = P, \sigma = \theta$ define a generalized F-structure $\Phi$ of matrix (2.10) because the conditions (2.22) imply the conditions (2.20).

It is also easy to check (2.10) with $Z_\alpha = (Z_\alpha, -\xi^\alpha), Z_a = (Z_a, \xi^a)$. Hence, $\Phi$ is a generalized F-structure with complementary frames.

The structure $(P, \theta, F, Z_a, \xi^a)$ is equivalent with the generalized, almost complex structure defined on $M \times \mathbb{R}^h$ by the matrix

$$
\Psi = \begin{pmatrix}
A' & \sharp_{\pi'} \\
\flat_{\sigma'} & -t'A'
\end{pmatrix}
$$

where

$$
A' = F, \quad \pi' = P + \sum_{a=1}^h Z_a \wedge \frac{\partial}{\partial t^a}, \quad \sigma' = \theta + \sum_{a=1}^h \xi^a \wedge dt^a,
$$

and $t^a$ are coordinates on $\mathbb{R}^h$.

Furthermore, the structure $(P, \theta, F, Z_a, \xi^a)$ is said to be normal if $\Psi$ is integrable [13]. We shall prove that, if $(P, \theta, F, Z_a, \xi^a)$ is normal, the corresponding generalized F-structure $\Phi$ is a CRF-structure. For this purpose, we identify

$$
T^{\text{big}}(M \times \mathbb{R}^h) \approx T^{\text{big}}M \oplus \mathbb{R}^{2h}
$$

and write $\Psi$ under the form

$$
\Psi = \begin{pmatrix}
\Phi & Z' \\
Z & 0
\end{pmatrix}, \quad Z = \begin{pmatrix} 0 & Z \\ \xi & 0 \end{pmatrix}, \quad Z' = \begin{pmatrix} 0 & -tZ \\ -t\xi & 0 \end{pmatrix},
$$

where $Z, \xi$ are 1-column matrices. The integrability of $\Psi$ means $N_\Psi(\tilde{X}, \tilde{Y}) = 0, \forall \tilde{X}, \tilde{Y} \in \Gamma T^{\text{big}}(M \times \mathbb{R}^h)$, where we may write

$$
\tilde{X} = (X, u) \oplus (\alpha, v) = (X, u, v) (X = (X, \alpha) \in \Gamma T^{\text{big}}M, \, u, v \in \mathbb{R}^h)
$$
and a similar expression for \( \tilde{Y} \). In particular, we must have
\[
N_{\Phi}(\Psi(\mathcal{X}, 0, 0), \Psi(\mathcal{Y}, 0, 0)) = 0
\]
and this equality coincides with (2.13). The converse may not hold, i.e., the integrability of \( \Phi \) does not imply the normality of \((P, \theta, F, Z_a, \xi^a)\). Notice that if \( \Phi \) is defined by a normal structure \((P, \theta, F, Z_a, \xi^a)\) then \( \pi \) is a Poisson bivector field [18].

We end this section by indicating a connection with the theory of submanifolds of a generalized complex manifold. Let \( M \) be a submanifold of a generalized almost complex manifold \((\tilde{M}, J)\) \(((J^2 = -Id, g(J \mathcal{X}, \mathcal{Y}) + g(\mathcal{X}, J \mathcal{Y}) = 0, \forall \mathcal{X}, \mathcal{Y} \in \Gamma T^{big} \tilde{M}))\).

**Definition 2.3.** \( M \) is called an \( F \)-submanifold of \( \tilde{M} \) if there exists a normal bundle \( \nu M \) \((T_M \tilde{M} = TM \oplus \nu M)\) such that
\[
T^{big} M = (T^{big} M \cap J(T^{big} M)) \oplus (T^{big} M \cap J(\nu^{big} M)).
\]

In condition (2.23) we take \( T^{*} M = \text{ann} \nu M, \nu^{*} M = \text{ann} TM, \nu^{big} M = \nu M \oplus \nu^{*} M \). The \( g \)-skew-symmetry of \( J \) implies the \( g \)-orthogonality of the two terms of the sum (2.23). Since the dimension of both terms of (2.23) is upper semicontinuous (it cannot increase in a neighborhood of \( x \in M \)), this dimension is constant and the formula
\[
(2.24) \quad \Phi(\mathcal{X}) = \begin{cases} 
J(\mathcal{X}) & \text{if } \mathcal{X} \in T^{big} M \cap J(T^{big} M), \\
0 & \text{if } \mathcal{X} \in T^{big} M \cap J(\nu^{big} M)
\end{cases}
\]
defines a generalized F-structure \( \Phi \) on \( M \). The structure \( \Phi \) has the associated subbundles \( L = T^{big} M \cap J(T^{big} M), S = T^{big} M \cap J(\nu^{big} M) \).

**Definition 2.4.** If the structure \( \Phi \) defined by formula (2.24) is CRF the submanifold \( M \) will be called a CRF-submanifold of \( \tilde{M} \).

**Remark 2.1.** Definitions[23, 24] are inspired by the notion of a CR-submanifold \( M \) of a Hermitian manifold \( \tilde{M} \) with the complex structure \( J \) and the metric \( \gamma \) [2], where \( TM = (TM \cap J(TM)) \oplus (TM \cap J(TM)^{\perp}) \).

**Proposition 2.5.** Any \( F \)-submanifold \( M \) of a generalized complex manifold \((\tilde{M}, J)\) is a CRF-submanifold.
Proof. We shall check that the integrability condition (2.12) holds. Take cross sections
\[ X = (X, \alpha), Y = (Y, \beta), Z = (Z, \gamma) \in \Gamma T^{big} M \]
and extend \( X, Y, Z, \alpha, \beta, \gamma \) to fields \( \tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \) on \( \tilde{M} \). It is easy to check that the Courant brackets on \( M \) and \( \tilde{M} \) are related by the following equality
\[ g_M([X, Y], Z) = g_{\tilde{M}}([\tilde{X}, \tilde{Y}], \tilde{Z})|_M. \]
(If we take \( Z \in \Gamma \nu^{big} M \) the right hand side of (2.25) depends on the choice of the extensions and (2.25) may not hold.)

For \( X, Y \in \Gamma(T^{big} M \cap J(T^{big} M)) \) and \( Z \in \Gamma T^{big} M \), using (2.24) and (2.25), we get
\[ g_M(\mathcal{N}_\Phi(X, Y), Z) = g_M(pr_S[X, Y], Z) + K, \]
where
\[ K = g_M([\Phi X, \Phi Y] - \Phi([\Phi X, Y] + [X, \Phi Y]) - [X, Y], Z) \]
\[ = g_{\tilde{M}}([J\tilde{X}, J\tilde{Y}] - \Phi([J\tilde{X}, \tilde{Y}] + [\tilde{X}, J\tilde{Y}]) - [\tilde{X}, \tilde{Y}], Z)|_M. \]
Furthermore, \( J \) satisfies the integrability condition
\[ \mathcal{N}_J(X, Y) = [J\tilde{X}, J\tilde{Y}] - J([J\tilde{X}, \tilde{Y}] + [\tilde{X}, J\tilde{Y}]) - [X, Y] = 0 \]
and since \( X, Y \in \Gamma L(\Phi) \subseteq \chi^1(M) \), (2.28) implies
\[ [J\tilde{X}, \tilde{Y}] + [X, J\tilde{Y}] \in \Gamma L(\Phi), \]
therefore, the last appearance of \( \Phi \) in (2.27) may be replaced by \( J \) and we get \( K = 0 \). Finally, (2.26) with \( K = 0 \) is equivalent with the required integrability condition. \( \Box \)

3 CRF-structures of classical type

In this section we discuss the simplest classes of generalized F-structures.

Definition 3.1. A generalized F-structure \( \Phi \) such that \( \Phi(TM) \subseteq TM \) and \( \Phi(T^*M) \subseteq T^*M \) is called a classical F-structure.
The first formula (2.17) shows that the generalized F-structure $\Phi$ is classical iff $\pi = 0$, $\sigma = 0$ in the matrix representation (2.16). Then, (2.20) reduces to (2.4) for the tensor field $F = A$ and the generalized F-structure reduces to a Yano F-structure.

Accordingly, we may write

$$(3.1) \quad T_cM = H \oplus \bar{H} \oplus Q_c = P_c \oplus Q_c,$$

where $H, \bar{H}, Q_c$ are the $(\pm \sqrt{-1}, 0)$-eigenbundles of $F$, respectively, $P_c = P \otimes \mathbb{C} = H \oplus \bar{H}$, $P \subseteq TM$ and $Q_c = Q \otimes \mathbb{C}$, $Q \subseteq TM$. The terms of the decomposition (3.1) have a constant dimension, because all three dimensions are lower semicontinuous functions of $x \in M$, and rank $F = \text{const}$.

It is well known that a decomposition (3.1) is equivalent with the classical F-structure $F$ and the projections on the terms of (3.1) are defined by the formulas (2.8) with $\Phi$ replaced by $F$. Using this fact and formulas (2.17), it follows that the big-isotropic complex bundle equivalent with the generalized F-structure under discussion is given by

$$(3.2) \quad E = H \oplus \text{ann}(H \oplus Q_c)$$

and its orthogonal bundle is

$$(3.3) \quad E' = (H \oplus Q_c) \oplus \text{ann} H.$$ 

Of course, we have $E \cap \bar{E}' = 0$ and, again, we see that a classical F-structure defines a generalized F-structure. The subbundles $S, L$ of the structure (3.2) are given by

$$(3.4) \quad S = Q \oplus \text{ann} P, \quad L = P \oplus \text{ann} Q.$$ 

**Proposition 3.1.** The big-isotropic structure $E$ of a classical F-structure is integrable iff

$$(3.5) \quad [H, H] \subseteq H, \quad [H, Q_c] \subseteq H \oplus Q_c$$

where the bracket is Lie bracket on $M$.

**Proof.** The Courant bracket (1.2) with $(X, \alpha), (Y, \beta) \in \Gamma E$ where $E$ is given by (3.2) belongs to $\Gamma E$ iff (3.5) holds. \qed

**Corollary 3.1.** For a classical F-structure, the holomorphic distribution $H$ is a classical CR-structure.
\textbf{Proof.} A CR-structure is characterized by $H \cap \bar{H} = 0$ plus the first condition (3.5) \[6\].

Another form of the integrability condition is obtained by using the classical Nijenhuis tensor $N_F$ given by formula (2.11) with arguments in $\chi^1(M)$ and Lie brackets instead of the Courant brackets.

\textbf{Proposition 3.2.} The big-isotropic structure $E$ of a classical $F$-structure $F$ is integrable iff

\begin{equation}
N_F(X,Y) = \text{pr}_Q[X,Y], \quad \forall X,Y \in P,
\end{equation}

\begin{equation}
N_F(X,Y) = 0, \quad \forall X \in P, Y \in Q.
\end{equation}

\textbf{Proof.} By using eigenvectors as arguments, we see that the two conditions (3.6) are equivalent to the two conditions (3.5), respectively.

\textbf{Remark 3.1.} In the case of Proposition 3.2 the second condition (3.6) is not a consequence of the first. This is justified by the following example. Let $H \subseteq T_v^cM$ be a CR-structure that is \emph{Nirenberg integrable}, i.e., $H \oplus \bar{H}$ is also involutive, and let $Q$ be a complementary subbundle of $H \oplus \bar{H}$ in $T_v^cM$. Then, it follows from the Nirenberg-Frobenius theorem \[15\] that $M$ has local complex coordinates $z^\alpha$ and real coordinates $y^u$ such that

\begin{equation}
H = \text{span}\{\frac{\partial}{\partial z^\alpha}\}, \quad Q = \text{span}\{Y_u = \frac{\partial}{\partial y^u} + \lambda_u^\alpha \frac{\partial}{\partial z^\alpha} + \bar{\lambda}_u^\alpha \frac{\partial}{\partial \bar{z}^\alpha}\}.
\end{equation}

The first condition in either (3.5) or (3.6) holds but the second is satisfied iff $\partial \lambda_u^\alpha / \partial \bar{z}^\alpha = 0$.

We shall also notice that it is possible to express the integrability of $E$ by a single non-skew-symmetric condition:

\textbf{Proposition 3.3.} The big-isotropic structure $E$ of a classical $F$-structure $F$ is integrable iff

\begin{equation}
N_F(X,Y) = [F^2X,F^2Y] - F([X,FY] + [F^2X,FY])
+ F^2([F^2X,Y] + [F^2X,F^2Y] + [X,Y]) \quad (\forall X,Y \in \chi^1(M)).
\end{equation}

\textbf{Proof.} Check the result by using eigenvectors as arguments.
The following notion seems to be new and might be of an independent interest.

**Definition 3.2.** A classical F-structure that satisfies the conditions (3.5) (equivalently, (3.6), (3.8)) is called a classical CRF-structure.

A classical CRF-structure may be seen either as a normalized CR-structure \((H,Q)\) (i.e., a CR-structure \(H\) with a normal bundle \(Q\) that satisfies (3.5)) or as a CR-flag \(H \subseteq H' \subseteq T_c M\), where the following conditions are satisfied

\[(3.9)\quad H \cap \bar{H}' = 0, \quad H \oplus \bar{H}' = T_c M, \quad [H, H] \subseteq H, \quad [H, H'] \subseteq H'.\]

A pair \((H,Q)\) yields the flag \((H,H' = H \oplus Q)\); conversely, a flag \((H,H')\) yields the pair \((H,Q_c = H' \cap \bar{H}')\).

**Proposition 3.4.** A tensor field \(F \in \text{End}(TM)\) is an F-structure iff around each point \(x \in M\) there exist local vector fields \(Z_a\) and local 1-forms \(\xi^a\) \((a = 1,\ldots,h)\) such that

\[(3.10)\quad \xi^a(Z_b) = \delta^a_b, \quad F(Z_a) = 0, \quad \xi^a \circ F = 0, \quad F^2 = -Id + \sum_{a=1}^h Z_a \otimes \xi^a.\]

Furthermore, \(F\) is a CRF-structure iff the following conditions hold

\[(3.11)\quad N_F(X,Y) = -\sum_{a=1}^h d\xi^a(X,Y)Z_a + \sum_{a,b,c=1}^h \xi^a(X)\xi^b(Y)\xi^c([Z_a, Z_b])Z_c\]

\[+ \sum_{a,b=1}^h (\xi_a(X)(L_{Z_a}\xi^b)(Y)Z_b + \xi_a(Y)(L_{Z_a}\xi^b)(X)Z_b)\]

\[-\frac{1}{2} \sum_{a=1}^h (\xi^a(X)F(L_{Z_a}F)(Y) - \xi^a(X)F(L_{Z_a}F)(Y)),\]

\[(3.12)\quad \sum_{a=1}^h (\xi^a(X)F(L_{Z_a}F)(Y) + \xi^a(Y)F(L_{Z_a}F)(X)) = 0.\]
Proof. The first assertion is known and can be justified as follows. Conditions (3.10) imply (2.4) for $F$. Conversely, we know that (2.4) implies $\text{rank } F = \text{const}$. Then, if we take an arbitrary local basis $(Z_a)_{a=1}^h$ of $Q$, the decomposition (3.1) shows the existence of a unique system of local 1-forms $\xi^a \in \text{ann } L$ such that the first three conditions (3.10) hold. Now, we can separately check the last condition (3.10) on $X \in L$ and on $Y = \sum_{a=1}^h \xi^a(X)Z_a \in Q$. For the new CRF-conditions, we use (3.8), write separately the annulation of the symmetric and skew-symmetric part, insert the local representation (3.10) and do the required technical computations.

Corollary 3.2. A normal $F$-structure with complementary frames is a CRF-structure.

Proof. A normal $F$-structure with complementary frames is a set of global tensor fields $(F, Z_a, \xi^a) (a = 1, ..., h)$ that satisfies (3.10) and the normality condition

$$(3.13) \quad \mathcal{N}_F(X, Y) = -\sum_{a=1}^h d\xi^a(X, Y)Z_a.$$ 

It is known (e.g., [18]) that the normality condition implies the following properties

$$(3.14) \quad [Z_a, Z_b] = 0, \quad L_{Z_a}\xi^b = 0, \quad L_{Z_a}F = 0.$$ 

Technical computations show that conditions (3.13), (3.14) imply (3.11) and (3.12).

A second class of generalized $F$-structures that we shall consider is defined by

Definition 3.3. A generalized F-structure $\Phi$ is said to be skew classical if $\Phi(TM) \subseteq T^*M$ and $\Phi(T^*M) \subseteq TM$.

Proposition 3.5. A generalized $F$-structures $\Phi$ is skew classical iff $A = 0$ in the corresponding matrix (2.16), and the structure is fully determined either by the pair $(\mathcal{V} = \text{im} \, \sharp \pi, \sigma)$ where $\sigma|_{\wedge^2 \mathcal{V}}$ is non degenerate and

$$(3.15) \quad TM = \mathcal{V} \oplus \text{ker } b_\sigma$$

or by the pair $(\Sigma = \text{im} \, b_\sigma, \pi)$ where $\pi|_{\wedge^2 \Sigma}$ is non-degenerate and

$$(3.16) \quad T^*M = \Sigma \oplus \text{ker } u_\pi.$$
Proof. The first assertion is a straightforward consequence of the first formula (2.17). Furthermore, the first formula (2.17) shows that

\[(3.17) \quad L = \text{im} \Phi = \text{im} \sharp_{\pi} \oplus \text{im} \flat_{\sigma}, \quad S = \text{ker} \Phi = \text{ker} \flat_{\sigma} \oplus \text{ker} \sharp_{\pi} \]

and decomposition (2.3) implies

\[(3.18) \quad TM = \text{im} \sharp_{\pi} \oplus \text{ker} \flat_{\sigma}, \quad T^*M = \text{im} \flat_{\sigma} \oplus \text{ker} \sharp_{\pi}. \]

On the other hand, since $\Phi^2|_{L} = -\text{Id}$, the second formula (2.17) shows that

\[(3.19) \quad (\#_{\pi} \circ \flat_{\sigma})|_{\text{im} \sharp_{\pi}} = -\text{Id}, \quad (\flat_{\sigma} \circ \#_{\pi})|_{\text{im} \flat_{\sigma}} = -\text{Id}. \]

Therefore, $\sigma|_{\wedge^2 V}, \pi|_{\wedge^2 \Sigma}$ are non-degenerate and

\[(3.20) \quad \#_{\pi}|_{\Sigma} = -(\flat_{\sigma}|_{V})^{-1}, \quad \flat_{\sigma}|_{V} = (\#_{\pi}|_{\Sigma})^{-1}. \]

The conditions (2.20), which reduce to

\[(3.21) \quad (\text{Id} + \flat_{\sigma} \#_{\pi}) \#_{\pi} = 0, \quad \flat_{\sigma}(\text{Id} + \#_{\pi} \flat_{\sigma}) = 0, \]

are satisfied in view of (3.18) and (3.19).

Now, if we have the pair $(V, \sigma)$ satisfying the required conditions, the decomposition (3.15) yields $T^*M = \Sigma \oplus \text{ann} V$ and $\pi$ is defined by (3.20) on $\Sigma$ and by 0 on $\text{ann} V$. A similar procedure may be used if we start with $(\Sigma, \pi)$. □

**Proposition 3.6.** The skew classical, generalized $F$-structure defined by a pair $(V, \sigma)$ that satisfies the hypotheses of Proposition 3.5 is a generalized CRF-structure iff $V$ is a foliation and $\sigma$ satisfies the condition

\[(3.22) \quad i(X \wedge Y) d\sigma = 0, \forall X, Y \in V. \]

Proof. From (2.8) and (2.17) it follows that the complex distribution of the structure $\Phi$ defined by $(V, \sigma)$ is

\[(3.23) \quad E = \{\#_{\pi} \flat_{\sigma} X + \sqrt{-1} \#_{\pi} \alpha, \flat_{\sigma} \#_{\pi} \alpha + \sqrt{-1} \flat_{\sigma} X\} \]

\[(3.22) \quad \{(X' + \sqrt{-1}Y', \flat_{\sigma}(Y' - \sqrt{-1}X')) / X', Y' \in V\} (X' = \#_{\pi} \flat_{\sigma} X, Y' = \#_{\pi} \alpha). \]
Formula (3.23) shows that \( E = \text{graph}(\nu_{-\sqrt{-1}\sigma}|_{V_c}) \), which is a situation that was discussed in Example 1.1. The corresponding \( g \)-orthogonal subbundle is

\[
E' = \{(Z + \sqrt{-1}U, \nu_{\sigma}(U - \sqrt{-1}Z) + (\lambda + \sqrt{-1}\mu)) / Z, U \in TM, \lambda, \mu \in \text{ann} \ V \}
\]

and, of course, \( E \cap \bar{E}' = 0 \).

Like in Example 1.1, \( E \) is integrable iff \( V_c \) is involutive and

\[
i(X \wedge Y)d(-\sqrt{-1}\sigma) = 0, \forall X, Y \in V_c.
\]

These conditions are equivalent with the integrability conditions of the structure \( E_{\sigma} \) given by (1.4) with \( \sigma \) instead of \( \theta \), which exactly are the conditions required by the proposition.

**Remark 3.2.** Condition (3.22) is equivalent with the pair of conditions: i) \( \sigma \) induces symplectic forms on the leaves of \( V \), ii) \( (L_Z\sigma)|_{V} = 0 \) for any \( V \)-projectable vector field \( Z \in \ker \nu_{\sigma} \). Indeed, (3.22) evaluated on \( U \in V \) is condition i) and (3.22) evaluated on \( Z \in \ker \nu_{\sigma} \) is ii). Since we have the decomposition (3.15), we are done. It suffices to use a projectable vector field \( Z \) since (3.22) is a pointwise condition and any tangent vector at a point \( x \in M \) can be extended to a projectable vector field.

**Example 3.1.** Let \( \pi : M \to N \) be a symplectic fibration and let \( \mathcal{H} \) be the horizontal distribution of a symplectic Ehresmann connection on \( M \) [7]. Then, we get a skew classical, generalized CRF-structure \( E \) associated with the pair \( (V, \sigma) \) where \( V \) is the vertical distribution (tangent to the symplectic fibers) and \( \sigma \) is the fiber-wise symplectic form extended by 0 on \( \mathcal{H} \). Indeed, it is known that conditions i), ii) hold in the indicated situation [7]. The same holds for the symplectic foliation of a regular Poisson structure if it has a complementary distribution whose projectable vector fields are infinitesimal automorphisms of the Poisson structure; this may be called a regular Poisson structure with a Poisson-Ehresmann connection.

It is worth noticing that we may start with a pair \( (V \subseteq TM, \theta \in \Omega^2(M)) \), where \( \theta|_{\wedge^2V} \) is non degenerate but (3.15) may not hold, and still the big-isotropic structure \( E = \text{graph}(\nu_{-\sqrt{-1}\theta}|_{V_c}) \) is a generalized F-structure on \( M \) (it is easy to check that \( E \cap E' = 0 \)).
The corresponding tensor fields of $E$ can be deduced as follows. The non-degeneracy of $\theta$ on $V$ implies the existence of the decompositions

\[(3.25)\]
\[TM = V \oplus V^\perp, \ T^*M = \text{ann} V^\perp \oplus \text{ann} V\]

and also shows that one has an isomorphism $(b_\theta)|_V : V \rightarrow \text{ann} V^\perp$. Then, if we use (3.25) in the expression (1.5) of the orthogonal bundle $E'$ we see that

\[E' \cap \bar{E'} = (V^\perp \oplus \text{ann} V) \otimes \mathbb{C}\]

and the eigenbundle $S$ of eigenvalue 0 is

\[S = V^\perp \oplus \text{ann} V.\]

Accordingly, we get the following projection formulas

\[pr_E(X, \alpha) = \frac{1}{2}(pr_V X + \sqrt{-1}b_\theta^{-1}(pr_{\text{ann} V^\perp} \alpha), \ pr_{\text{ann} V^\perp} \alpha - \sqrt{-1}b_\theta(pr_V X)),\]
\[pr_{\bar{E}}(X, \alpha) = \frac{1}{2}(pr_V X - \sqrt{-1}b_\theta^{-1}(pr_{\text{ann} V^\perp} \alpha), \ pr_{\text{ann} V^\perp} \alpha + \sqrt{-1}b_\theta(pr_V X)),\]
\[pr_{S_c}(X, \alpha) = (pr_{V^\perp} X, pr_{\text{ann} V} \alpha).\]

Now, it is easy to compute $\Phi(X, 0), \Phi(0, \alpha)$, where $\Phi$ is the equivalent endomorphism of $E$, and using (2.19) we deduce

\[A = 0, \ b_\sigma = b_\theta \circ pr_V, \ \sharp_\pi = -b_\theta^{-1} \circ pr_{\text{ann} V^\perp}.\]

Therefore, the structure $E$ defined by the 2-form $\theta$ that does not satisfy (3.15) coincides with the structure $E$ defined by the form

\[\sigma(X, Y) = \theta(pr_V X, pr_V Y)\]

for which (3.15) holds. The integrability conditions for $E$ defined by $\theta$ are again those indicated in Example 1.1.

The structures of the form $\text{graph}(b_{-\sqrt{-1}b_\theta}|_{\nu_c})$ considered above extend the generalized complex structures associated with a symplectic form [8].

A similar discussion applies if we start with a pair $(\Sigma, \pi)$. Then the generalized F-structure $\Phi$ is equivalent with the big-isotropic structure $E_{\sqrt{-1}\pi}$ defined in Example 1.2 and the integrability conditions are those provided in Example 1.2.

Now, we consider another special case:
Definition 3.4. A generalized F-structure Φ such that \( \Phi^2(TM) \subseteq TM \) and \( \Phi^2(T^*M) \subseteq T^*M \) is called a generalized F-structure with classical square.

The second formula \((2.17)\) gives the following characteristic properties of a structure \(\Phi\) with classical square:

\[
\sharp \tilde{\pi} = A \sharp \pi - \sharp \pi \Gamma = 0, \quad \flat \tilde{\sigma} = b_\sigma A - \Gamma A b_\sigma = 0
\]

and the conditions \((2.20)\) become

\[
CA = 0, \quad C \sharp \pi = 0, \quad b_\sigma C = 0, \quad C = A^2 + \sharp \pi b_\sigma + \text{Id}.
\]

Furthermore, the second formula \((2.17)\) implies

\[
\Phi^2|_{TM} = \tilde{A}, \quad \Phi^2|_{T^*M} = \Gamma \tilde{A}
\]

and we deduce that

\[
L = \text{im}(-\Phi^2) = U \oplus U^*,
\]

where

\[
U = \text{im}(A^2 + \sharp \pi b_\sigma), \quad U^* = \text{im}(\Gamma A^2 + \flat \sigma \sharp \pi).
\]

Moreover, since the projection of \(T^{big}M\) on \(L\) is \(-\Phi^2\), we see that \(\Pi = -A^2 - \sharp \pi b_\sigma\) is a projector of TM onto U \((\Pi^2 = \Pi)\), therefore, \(C\) of \((3.27)\) is the complementary projector of \(\Pi\) and \(C^2 = C\). Using \(\Pi^2 = \Pi\), it is easy to check that the natural pairing between \(U\) and \(U^*\) is non degenerate, hence, \(U^*\) may be identified with the dual space of \(U\) and a comparison with the definition of a generalized, almost complex structure shows that \((L, \Phi|_L)\) should be seen as a generalized, complex vector bundle on \(M\).

Conversely, if we start with an almost product structure \(TM = U \oplus V\) and a generalized complex structure \(\Psi\) on the bundle \(L = U \oplus U^*\) we can define a corresponding generalized F-structure with classical square \(\Phi\) on \(M\). Indeed, \(\Psi\) has a matrix \((2.16)\) where the entries are defined on \(U, U^*\). If these entries are extended by 0 for any case where one of the arguments is in \(V, V^*\), the result is a matrix \((2.16)\) that defines a generalized F-structure with classical square \(\Phi\). It is easy to see that the bundle \(U\) of \(\Phi\) is the given one and that \(\Phi|_L\) is the given structure \(\Psi\) (in particular, the projectors of the almost product structure are \(\Pi = -A^2 - \sharp \pi b_\sigma, C = A^2 + \sharp \pi b_\sigma + \text{Id}\) because the two sides of the latter equalities have the same values on \(U\) and \(V\)).

We finish by recalling the notion of a B-field transformation:

\[(X, \alpha) \mapsto (X, \alpha + i(X)B),\]
where $B \in \Omega^2(M)$. Obviously, if a $B$-field transformation is applied to a generalized F-structure, we get a generalized F-structure again. Moreover, if $B$ is closed, the $B$-field transformation preserves the Courant bracket, hence, if the original structure is CRF the transformed structure is CRF as well.

4 Generalized metric CRF-structures

The study of classical F-structures also includes the metric case, i.e., the case where the structure group of $TM$ is reduced to $U(k) \times O(h)$\,\cite{3,21}. Equivalently, a classical metric F-structure $(F,\gamma)$ consists of an F-structure $F$ and a Riemannian metric $\gamma$ on the manifold $M$ that satisfy the following compatibility condition

$$\gamma(X, FY) + \gamma(FX, Y) = 0 \quad (X, Y \in \chi^1(M)).$$

The pair $(F,\gamma)$ defines the fundamental 2-form

$$\Xi(X,Y) = \gamma(FX,Y).$$

In this section we discuss a corresponding generalized case. Generalized Riemannian metrics were defined in\,\cite{8} as reductions of the structure group $O(m,m)$ of $(T^b M, g)$ to $O(m) \times O(m)$. For the reader’s convenience, we recall the basic facts given in\,\cite{8}.

A generalized, Riemannian metric is a Euclidean (positive definite) metric $G$ on the bundle $T^b M$, which is compatible with the metric $g$ given by (1.1) in the sense that the musical isomorphism

$$\sharp_G : T^b M = TM \oplus T^* M \to T^* M \oplus TM \approx T^b M,$$

where $\approx$ is the isomorphism $(\alpha, X) \leftrightarrow (X, \alpha)$, satisfies the conditions

$$\sharp_G^2 = Id,$$

$$g(\sharp_G(X, \alpha), \sharp_G(Y, \beta)) = g((X, \alpha), (Y, \beta)).$$

The isomorphism $\sharp_G$ is determined by the formula

$$2g(\sharp_G(X, \alpha), (Y, \beta)) = G((X, \alpha), (Y, \beta)).$$
and, if we ask \([4.4]\) to hold, we see that \(\sharp G\) may be represented in the matrix form
\[
(4.7) \quad \sharp G \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} \varphi & \sharp \gamma \\ \beta & \flat \varphi \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix},
\]
where \(\varphi \in \text{End}(TM)\) and \(\beta, \gamma\) are classical Riemannian metrics on \(M\).
Furthermore, condition \([4.5]\) is equivalent with
\[
(4.8) \quad \varphi^2 = \text{Id} - \sharp \gamma \circ \beta, \quad \gamma(\varphi X, Y) + \gamma(X, \varphi Y) = 0,
\]
\[
\beta(\varphi X, Y) + \beta(X, \varphi Y) = 0.
\]
Since \(\beta, \gamma\) are non degenerate, the first condition \([4.8]\) yields
\[
(4.9) \quad \beta = \beta \gamma \circ (\text{Id} - \varphi^2), \quad \sharp \gamma = (\text{Id} - \varphi^2) \circ \sharp \beta
\]
and we see that the generalized Riemannian metrics of \(M\) are in a bijective correspondence with pairs \((\gamma, \varphi)\) or \((\beta, \varphi)\) where \(\beta, \gamma\) are Riemannian metrics and \(\varphi\) is a \(\gamma, \beta\)-skew-symmetric \((1,1)\)-tensor field.
Notice that, modulo \([4.8]\), the fact that \(\gamma\) is positive definite implies that \(\beta\) and \(G\) are positive definite; indeed, \([4.8], [4.9]\) and \([4.6]\) imply
\[
\beta(X, X) = \gamma(X, X) + \gamma(\varphi X, \varphi X),
\]
\[
G((X, \beta \gamma Y), (X, \beta \gamma Y)) = \gamma(X, X) + \gamma(Y + \varphi X, Y + \varphi X).
\]
Conversely, if \(\beta\) is positive definite so is \(\gamma\) too. In particular, if \(\varphi = 0\), \(G\) reduces to the classical Riemannian metric \(\gamma\).
Furthermore, \(\varphi\) may be replaced by the 2-form \(\psi\) defined by
\[
(4.10) \quad \beta \psi = -\beta \gamma \circ \varphi,
\]
which means that one has a bijective correspondence \(G \leftrightarrow (\gamma, \psi)\).
From \([4.4], [4.5]\) and \([4.6]\), it follows that a generalized, Riemannian metric \(G\) produces a decomposition
\[
(4.11) \quad T^{\text{big}} M = V_+ \oplus V_-,
\]
where \(V_\pm\) are the \((\pm 1)\)-eigenbundles, which simultaneously are \(G\) and \(g\) orthogonal. On \(V_\pm\) one has \(G = \pm 2g\), respectively, hence, \(g\) is positive definite on \(V_+\) and negative definite on \(V_-\), whence, \(\text{dim} V_\pm = m\). Conversely, a
decomposition (4.11) with \(m\)-dimensional, \(g\)-orthogonal, terms that are \(g\)-positive and \(g\)-negative, respectively, defines the generalized, Riemannian metric \(G = 2g_{V_+} - 2g_{V_-}\). This exactly means that the structure group of \(T^{\text{big}}M\) was reduced to \(O(m) \times O(m)\).

The projectors associated with the decomposition (4.11) are given by

\[
pr_{\pm} = \frac{1}{2}(Id \pm \sharp G),
\]

and, if we apply them to pairs \((0, \alpha), (X, 0)\) using (4.7), we see that the projections

\[
\tau_{\pm} = pr_{TM}|_{V_{\pm}}, \tau^*_{\pm} = pr_{T^*M}|_{V_{\pm}}
\]

are surjective hence, isomorphisms. From (4.7) and (4.10), we get

\[
\begin{align*}
\tau_{+}^{-1}(X) &= (X, b_\gamma(X - \varphi X)) = (X, b_{\psi+\gamma}X), \\
\tau_{-}^{-1}(X) &= (X, -b_\gamma(X + \varphi X)) = (X, b_{\psi-\gamma}X),
\end{align*}
\]

therefore,

\[
V_{\pm} = \{(X, b_{\psi\pm\gamma}X) / X \in TM\}.
\]

The isomorphisms \(\tau_{\pm}\) may be used in order to transfer the metrics \(G|_{V_{\pm}}\) to metrics \(G_{\pm}\) of the tangent bundle \(TM\) given by

\[
G_{\pm}(X,Y) = G(\tau_{\pm}^{-1}(X), \tau_{\pm}^{-1}(Y)) = \pm 2g(\tau_{\pm}^{-1}(X), \tau_{\pm}^{-1}(Y)) = 2\gamma(X,Y),
\]

where \(\gamma\) is the metric that appears in (4.7).

The following definition extends the one given in [8] for the generalized complex case.

**Definition 4.1.** A *generalized metric F-structure* is a pair \((\Phi, G)\), where \(\Phi\) is a generalized F-structure and \(G\) is a generalized Riemannian metric, such that the following skew-symmetry condition holds

\[
G(\Phi X, Y) + G(X, \Phi Y) = 0 \quad (X, Y \in \Gamma T^{\text{big}}M).
\]
Using the $g$-skew-symmetry (2.5) of $\Phi$ and formula (4.6), we see that (4.17) is equivalent with the commutation condition

\[(4.18)\quad \sharp G \circ \Phi = \Phi \circ \# G.\]

Condition (4.18) implies that the pair

\[(4.19)\quad (\Phi^c = \# G \circ \Phi, G)\]

is a second generalized metric F-structure that commutes with $\Phi$. We will refer to $\Phi^c$ as the complementary structure. In the generalized almost complex case a commuting pair $(\Phi, \Phi^c)$ defines $G$ by $\sharp_G = -\Phi \circ \Phi^c$.

Let us assume that the structure $\Phi$ has the matrix representation (2.16). The following proposition expresses the compatibility between $\Phi$ and $G$ via the matrices (4.7) and (2.16).

**Proposition 4.1.** The pair $(G, \Phi)$, where $G$ is a generalized, Riemannian metric given by (4.7) and $\Phi$ is a generalized F-structure given by (2.16), is a generalized metric F-structure iff the following two conditions hold

\[(4.20)\quad \gamma(AX, Y) + \gamma(X, AY) = \varpi(\varphi X, Y) - \varpi(X, \varphi Y),\]

\[(4.21)\quad \sigma(X, Y) = \varpi(X, Y) - \varpi(\varphi^2 X, Y) + \gamma([A, \varphi](X), Y),\]

where $[A, \varphi] = A \circ \varphi - \varphi \circ A$ and $\varpi = b_\gamma \pi$ is defined by

\[(4.22)\quad \varpi(X, Y) = \pi(b_\gamma X, b_\gamma Y).\]

**Proof.** The commutation condition (4.18) is equivalent with

\[(4.23)\quad \varphi \circ A + \sharp_A \circ b_\sigma = A \circ \varphi + \sharp_A \circ b_\beta,\]

\[(4.24)\quad \sharp_A \circ t_A \circ t_\gamma \circ t_\varphi = A \circ \sharp_\gamma + \sharp_\pi \circ t_\varphi,\]

\[(4.25)\quad \sharp_\beta \circ A + t_\varphi \circ b_\sigma = b_\sigma \circ \varphi - t_\varphi A \circ b_\beta,\]

\[(4.26)\quad b_\beta \circ t_\varphi - t_\varphi \circ t_\sigma A = b_\sigma \circ \sharp_\gamma - t_\varphi A \circ t_\varphi.\]

Furthermore, the last condition (4.23) is the transposition of the first, and the first condition implies the equivalence between the second and third condition. Indeed, the first condition (4.23) is equivalent with

\[(4.24)\quad b_\sigma = b_\gamma \circ (A \circ \varphi - \varphi \circ A + \sharp_\pi \circ b_\beta).\]
If this expression of $\flat_\sigma$ is inserted in the third condition (4.28), while taking into account the $\beta, \gamma$-skew-symmetry of $\varphi$ and (4.9), and the result is composed by $\sharp_\gamma$ at the left and by $\sharp_\beta$ at the right, one gets

$$\varphi \circ \sharp_\pi - \sharp_\gamma \circ t^f A = A \circ (Id - \varphi^2) \circ \sharp_\beta + \sharp_\pi \circ t^f \varphi.$$  

Then, the first condition (4.8) shows that (4.25) coincides with the second condition (4.23).

Thus, the compatibility between $G$ and $\Phi$ is equivalent with (4.24) together with the second condition (4.23). If (4.24) is evaluated on tangent vectors $X,Y$ and the second condition (4.23) is evaluated on $\flat_\gamma X, \flat_\gamma Y$ the required conclusion is obtained.

We proceed by a recall of Gualtieri’s expression of a generalized, metric, almost complex structure by classical structures while replacing the structure by a generalized, metric F-structure [8].

From (4.11), it follows that (4.18) is equivalent with the $\Phi$-invariance of the subbundles $V_\pm$. Thus, if we also take into account the skew-symmetry (2.5) of $\Phi$, a $G$-compatible, generalized metric F-structure is equivalent with a pair of bundle morphisms $\Phi_\pm \in \text{End} V_\pm$ which satisfy the condition $\Phi_\pm^3 + \Phi_\pm = 0$ and are skew-symmetric with respect to $G|_{V_\pm}$. Furthermore, we have decompositions

$$V_\pm = E_\pm \oplus \bar{E}_\pm \oplus S_\pm$$

where the terms are the $(\pm \sqrt{-1}, 0)$-eigenbundles of $\Phi_\pm$. Hence, the eigenbundles of $\Phi$ are

$$E = E_+ \oplus E_-,$$
$$\bar{E} = \bar{E}_+ \oplus \bar{E}_-,$$
$$S = S_+ \oplus S_-$$

and we have

$$E_\pm = V_\pm \cap E, \quad \bar{E}_\pm = V_\pm \cap \bar{E}, \quad S_\pm = V_\pm \cap S.$$ 

Similar decompositions hold for the complementary structure $\Phi^c$. We shall denote the corresponding vector bundles by means of an upper index $c$ and, if we look at the corresponding eigenvalue and use (4.19), we get

$$E_+ \subseteq E^c, \quad E_- \subseteq \bar{E}^c, \quad S^c = S.$$
Formulas (4.26), (4.27), (4.28) hold for $E, \bar{E}$ replaced by $E^c, \bar{E^c}$. Moreover, using (4.29) it is easy to get

\[(4.30) \quad E_+ = E \cap E^c, \quad E_- = E \cap \bar{E^c}.\]

Finally, notice that (4.19) implies $(\Phi^c)^c = \Phi$, therefore if we consider the decomposition (4.26) for $\Phi_c$ instead of $\Phi$ we get

\[(4.31) \quad E^c_+ = E_+, \quad E^c_- = \bar{E}_-, \quad E^c = E_+ \oplus \bar{E}_-.\]

Furthermore, the structures $\Phi_{\pm}$ may be transferred to $TM$ by

\[(4.32) \quad F_{\pm} = \tau_{\pm} \circ \Phi_{\pm} \circ \tau_{\pm}^{-1} \in \text{End} TM\]

and, also recalling formula (4.16), the conclusion is that the $G$-compatible, generalized F-structure $\Phi$ is equivalent with the pair of classical F-structures $F_{\pm}$ of $TM$ that satisfy the skew-symmetry condition

\[(4.33) \quad \gamma(F_{\pm}X, Y) + \gamma(X, F_{\pm}Y) = 0.\]

It is easy to find the connection between $F_{\pm}$ and the matrix (2.16) of $\Phi$. Using (4.14), it follows that

\[(4.34) \quad F_{\pm} = A + \sharp_\pi \circ b_{\psi \pm \gamma}.\]

Conversely, from (4.34) we get

\[(4.35) \quad \sharp_\pi = \frac{1}{2}(F_+ - F_-) \circ \sharp_\gamma,
A = \frac{1}{2}[F_+ \circ (Id - \sharp_\gamma b_\psi) + F_- \circ (Id + \sharp_\gamma b_\psi)].\]

The remaining entry of the matrix (2.16) is $b_\sigma$ is given by (4.24).

Therefore, we have the same result as in [8]:

**Proposition 4.2.** A generalized metric F-structure $(G, \Phi)$ is equivalent with a quadruple $(\gamma, \psi, F_+, F_-)$, where $\gamma$ is a classical, Riemannian metric on $M$, $\psi$ is a 2-form, and $(F_\pm, \gamma)$ are classical metric F-structures of $M$.

**Remark 4.1.** Assume that the generalized metric F-structure $(G, \Phi)$ has the corresponding quadruple $(\gamma, \psi, F_+, F_-)$. Since the complementary structure $\Phi^c$ satisfies the conditions $\Phi^c_\pm = \pm \Phi_{\pm}$, formula (4.32) shows that $(G, \Phi^c)$ has the corresponding quadruple $(\gamma, \psi, F_+, -F_-)$. 

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Example 4.1. Let \((F, \gamma)\) be a classical metric F-structure, \(\Phi\) the corresponding generalized F-structure given by (3.2) and \(G\) the generalized Riemannian metric defined by a classical Riemannian metric \(\gamma\) (with \(\varphi = 0\)). Then, Proposition 4.1 shows that \((\Phi, G)\) is a generalized metric F-structure. By (4.10), this structure has the 2-form \(\psi = 0\), by (4.15), \(V_\pm = \{(X, b_{\pm} \gamma X)\}\), by (4.16) the metric to be considered on \(TM\) is \(2\gamma\) and by (4.32) we have \(F_+ = F_- = F\). Furthermore, with the notation of the first part of Section 3, we have

\[
S_\pm = \{(X, b_{\pm} \gamma) / X \in Q\}, \quad E_\pm = \{(X, b_{\pm} \gamma) / X \in H\}.
\]

Now, if \(\Phi\) is a generalized CRF-structure that is skew-symmetric with respect to the generalized Riemannian metric \(G\) then \((\Phi, G)\) is a generalized metric CRF-structure. We will extend the notion of a generalized Kähler manifold \([8]\) by means of the following definition.

Definition 4.2. A generalized, metric F-structure \((\Phi, G)\) with the associated eigenbundles \((E_\pm, S_\pm)\) is a generalized CRFK-structure (and \((M, \Phi, G)\) is a generalized CRFK manifold) iff the following Courant bracket closure conditions hold:

\[
[E_+, E_+] \subseteq E_+, \quad [E_+, S_+] \subseteq E_+ \oplus S_+,
\]

\[
[E_-, E_-] \subseteq E_-, \quad [E_-, S_-] \subseteq E_- \oplus S_-.
\]

The label K used above comes from the name of Kähler.

The relation between this definition and the definition of a generalized Kähler manifold given in \([8]\) is shown by the following proposition.

Proposition 4.3. The generalized, metric F-structure \((\Phi, G)\) is a CRFK-structure iff \(\Phi\) and its complementary structure \(\Phi^c\) are CRF-structures and

\[
[S_+, S_-] \subseteq S.
\]

Proof. We shall use the following property of the Courant bracket ([14], axiom (v) of Courant algebroids):

\[
X(g(\mathcal{X}, \mathcal{Z})) = g([X, \mathcal{Y}], Z) + g(\mathcal{Y}, [X, Z]) + \frac{1}{2}(Z(g(\mathcal{X}, \mathcal{Y}) + Y(g(\mathcal{X}, \mathcal{Z}))), \quad \forall \mathcal{X} = (X, \alpha), \mathcal{Y} = (Y, \beta), \mathcal{Z} = (Z, \gamma) \in \Gamma T^{big}M.
\]
The integrability (CRF condition) of $\Phi, \Phi^c$ means $[E, E] \subseteq E$, $[E^c, E^c] \subseteq E^c$. From (4.27), (4.30) and (4.31), it follows straightforwardly that these integrability conditions are equivalent with the conditions:

\begin{equation}
[E_+, E_+] \subseteq E_+, [E_-, E_-] \subseteq E_-, [E_+, E_-] \subseteq E, [E_+, E_-] \subseteq E^c.
\end{equation}

We will show that conditions (4.40) are equivalent with the conditions (4.41)

\begin{equation}
[E_+, E_+] \subseteq E_+, [E_-, E_-] \subseteq E_-, [E_+, E_-] \perp g S, [E_+, E_-] \perp g S.
\end{equation}

Indeed, the third and fourth condition (4.40) imply the third and fourth condition (4.41), respectively, since $E \perp g S, E^c \perp g S^c = S$. Conversely, if the first two conditions of (4.40) hold and if we take $X, Z \in E_\pm, Y \in E_\mp$ in (4.39), we get

\begin{equation}
[E_+, E_-] \subseteq E^+ = E' = E \oplus S.
\end{equation}

Since, by the third condition (4.41), $[E_+, E_-] \subseteq E \oplus \bar{E}$, it follows that

\begin{equation}
[E_+, E_-] \subseteq (E \oplus S) \cap (E \oplus \bar{E}) = E.
\end{equation}

With the same argument for $E^c$ instead of $E$ we see that the last condition (4.41) implies the last condition (4.40).

Furthermore, we show that the conditions (4.41) are equivalent with either

\begin{equation}
[E_+, E_+] \subseteq E_+, [E_-, E_-] \subseteq E_-, [E_+, S] \subseteq E_+ \oplus S
\end{equation}

or

\begin{equation}
[E_+, E_+] \subseteq E_+, [E_-, E_-] \subseteq E_-, [E_-, S] \subseteq E_- \oplus S.
\end{equation}

Indeed, let us use (4.39) with $X \in E_\pm, Y \in E_\mp, Z \in S$. The result is that

\begin{equation}
[E_+, E_-] \perp g S \iff [E_+, S] \perp g E_- \iff [E_-, S] \perp g E_+.
\end{equation}

Similarly, but changing $E_-$ to $\bar{E}_-$, we get

\begin{equation}
[E_+, \bar{E}_-] \perp g S \iff [\bar{E}_-, S] \perp g E_+ \iff [E_+, S] \perp g \bar{E}_-.
\end{equation}

The relations (4.45) and (4.46) show that (4.41) implies

\begin{equation}
[E_+, S] \perp g (E_- \oplus \bar{E}_-), [E_-, S] \perp g (E_+ \oplus \bar{E}_+).
\end{equation}
Accordingly, and since (4.41) is equivalent to the integrability of \( E \), we get

\[
\begin{align*}
[E_+, S] &\subseteq (E \oplus S) \cap (E_+ \oplus \bar{E}_+ \oplus S) = E_+ \oplus S, \\
[E_-, S] &\subseteq (E \oplus S) \cap (E_- \oplus \bar{E}_- \oplus S) = E_- \oplus S,
\end{align*}
\]

which shows that (4.41) implies both (4.43) and (4.44). Conversely, the condition \([E_+, S] \subseteq E_+ \oplus S\) implies both \([E_+, S] \perp_g E_-\) and \([E_+, S] \perp_g \bar{E}_-\) and (4.45), (4.46) show that the conditions (4.43) imply (4.41). Similarly, \([E_-, S] \subseteq E_- \oplus S\) implies both \([E_-, S] \perp_g E_+\) and \([E_-, S] \perp_g \bar{E}_+\), equivalently, \([\bar{E}_-, S] \perp_g E_+\), and (4.45), (4.46) show that the conditions (4.44) imply (4.41).

Finally, if we use (4.39) with \( X \in S_\pm, Y \in E_\mp, Z \in S_\mp \) we get

(4.47) \[
[S_+, S_-] \perp_g E_+ \iff [E_+, S_+] \perp_g S_-, [S_+, S_-] \perp_g E_- \iff [E_-, S_-] \perp_g S_+.
\]

Therefore, the addition of hypothesis (4.38) to (4.43), (4.44) leads to

\[
\begin{align*}
[E_+, S_+] &\subseteq (E_+ \oplus S) \cap (V_+ \oplus E_- \oplus \bar{E}_-) = E_+ \oplus S_+,
[E_-, S_-] &\subseteq (E_- \oplus S) \cap (V_- \oplus E_+ \oplus E_+) = E_- \oplus S_-.
\end{align*}
\]

Conversely, from (4.47) we get

\[
\begin{align*}
[E_+, S_+] &\subseteq E_+ \oplus S_+ \Rightarrow [E_+, S_+] \perp_g S_- \Rightarrow [S_+, S_-] \perp_g E_+,
[E_-, S_-] &\subseteq E_- \oplus S_- \Rightarrow [E_-, S_-] \perp_g S_+ \Rightarrow [S_+, S_-] \perp_g E_-.
\end{align*}
\]

Since \( S_\pm \) are real subbundles, by complex conjugation, we also get \([S_+, S_-] \perp_g \bar{E}_+, [S_+, S_-] \perp_g \bar{E}_-\) and (4.38) follows. \( \square \)

**Remark 4.2.** During the proof of Proposition 4.3 we have obtained several characterizations of the integrability of the pair of structures \((\Phi, \Phi^c)\); (4.40), (4.41), (4.43), (4.44). In the generalized Kähler case \( S = 0 \), condition (4.38) does not appear and (4.41) shows that the last two conditions (4.40) are superfluous. This is a simple proof of results that appeared in Proposition 6.10 and Theorem 6.28 of [8].

Like in the case of the generalized Kähler structures [8], it is possible to obtain conditions that are equivalent with (4.37) and are expressed in terms of the projected structures \( F_\pm \) given by (4.32).
Proposition 4.4. The generalized metric CRF-structure \((\Phi, G)\) with the associated structures \((F_{\pm}, \gamma, \psi)\) is a CRFK-structure iff \(F_{\pm}\) are classical metric CRF-structures and the equalities

\[
(i(X \wedge Y)d\psi = \pm(i(X)L_Y\gamma - L_Xi(Y)\gamma)
\]
hold for either \(X, Y \in H_{\pm}\) or \(X \in H_{\pm}, Y \in Q_{\pm}\).

Proof. The restriction of the Courant bracket to the subbundles \(V_{\pm}\) defined by \((4.14)\) is given by the formula (see also [8]):

\[
((X, b_{\psi\pm}\gamma X), (Y, b_{\psi\pm}\gamma Y)) = ([X, Y], b_{\psi\pm}\gamma [X, Y])
\]
\[+ i(X \wedge Y)d\psi \pm (L_Xi(Y)\gamma - i(X)L_Y\gamma)) \quad (X, Y \in \chi^1(M)).
\]
This formula follows from the general expression \((1.2)\) of the Courant bracket by evaluating the 1-form component on a vector field \(Z\).

Let us denote by \(H_{\pm}, \bar{H}_{\pm}, Q_{\pm}\) the \((\pm\sqrt{-1}, 0)\)-eigenbundles of \(F_{\pm}\). From \((4.49)\), it follows that the CRFK-conditions \((4.37)\) are equivalent with the conditions

\[
[H_{\pm}, H_{\pm}] \subseteq H_{\pm}, [H_{\pm}, Q_{\pm}] \subseteq H_{\pm} \oplus Q_{\pm}
\]
together with the equalities \((4.48)\). \qed

In what follows we produce some more equivalent CRFK-conditions. Any connection on the principal bundle of frames of \(TM\) given by the reduction of the structure group to \(U(rank H_{\pm}) \times O(rank Q_{\pm})\) defined by the metric \(F\)-structures \((F_{\pm}, \gamma)\), will be called an adapted connection. (Of course, we have different plus-adapted and minus-adapted connections and all the formulas where we write a double sign \(\pm\) include two different formulas.) The parallel translations of adapted connections preserve the structure \((F_{\pm}, \gamma)\) and the associated covariant derivative \(\nabla^\pm\) is characterized by the conditions

\[
\nabla^\pm \gamma = 0, \quad \nabla^\pm F_{\pm} = 0.
\]
Furthermore, the difference tensor

\[
\Theta^\pm(X, Y) = \nabla^\pm_X Y - \nabla_X Y,
\]
where $\nabla$ is the Levi-Civita connection of the metric $\gamma$ will be called the *Levi-Civita difference* of the adapted connection. Since $\nabla \gamma = 0$, we must have

$$\gamma(\Theta^\pm(X,Y),Z) + \gamma(Y,\Theta^\pm(X,Z)) = 0. \quad (4.53)$$

On the other hand, the condition $\nabla^\pm F_\pm = 0$ is equivalent with

$$\Theta^\pm(X,F^\pm_\pm Y) - F^\pm_\pm \Theta^\pm(X,Y) = -(\nabla_X F^\pm_\pm)(Y), \quad (4.54)$$

which also implies

$$\Theta^\pm(X,F^2_\pm Y) - F^2_\pm \Theta^\pm(X,Y) = -(\nabla_X F^2_\pm)(Y). \quad (4.55)$$

Thus, the adapted connections are obtained from the Levi-Civita connection by the addition of a difference tensor that satisfies conditions (4.53) and (4.54).

**Proposition 4.5.** Let $(\Phi, G)$ be a generalized metric CRF-structure with the associated structures $(F_\pm, \gamma, \psi)$ and let $\nabla^\pm$ be adapted connections with the Levi-Civita differences $\Theta^\pm$. Then, $(\Phi, G)$ is a CRF$\kappa$-structure iff $F_\pm$ are classical metric CRF-structures and the equalities

$$\gamma(\Theta^\pm(Z,Y),X) = \pm\frac{1}{2}d\psi(X,Y,Z) \quad (4.56)$$

hold for any $Z \in \chi^1(M)$ and either $X,Y \in H_\pm$ or $X \in H_\pm, Y \in Q_\pm$. 

**Proof.** By a simple computation and using $\nabla^\pm \gamma = 0$ we get

$$(L_X \gamma)(Y,Z) = \gamma(\nabla^\pm_X Z, Y) + \gamma(Y,\nabla^\pm_X Z)$$

$$+ \gamma(T^\pm(X,Y),Z) + \gamma(Y,T^\pm(X,Z)), \quad (4.57)$$

where $T^\pm$ is the torsion of $\nabla^\pm$. Then, if we evaluate (4.48) on $Z \in \chi^1(M)$ and use (4.51) and (4.57) we get the following equivalent form of (4.48):

$$d\psi(X,Y,Z) = \pm \gamma(X,\nabla^\pm_Z Y) - \gamma(Y,\nabla^\pm_Z X)$$

$$+ \gamma(X,T^\pm(Y,Z)) + \gamma(Y,T^\pm(Z,X)) - \gamma(Z,T^\pm(X,Y)), \quad (4.58)$$

where the first two terms of the right hand side vanish if either $X,Y \in H_\pm$ or $X \in H_\pm, Y \in Q_\pm$. If we insert

$$T^\pm(X,Y) = \Theta^\pm(X,Y) - \Theta^\pm(Y,X),$$

in (4.58), we get (4.56). \qed
The last form of the CRFK-conditions that we will prove is

**Proposition 4.6.** The generalized metric CRF-structure \((\Phi, G)\) with the associated structures \((F_\pm, \gamma, \psi)\) is a CRFK-structure iff \(F_\pm\) are classical metric CRF-structures and the following conditions hold for any \(X, Y, Z \in \chi^1(M)\):

\[
(4.59) \quad \gamma(F_\pm X, (\nabla_Z F_\pm)(Y)) = \pm \frac{1}{2} [d\psi(F_\pm^2 X, Y, Z) + d\psi(F_\pm X, F_\pm Y, Z)].
\]

**Proof.** We replace \((4.56)\) by conditions with general arguments \(X, Y, Z \in \chi^1(M)\) by replacing the arguments in \(H_\pm\) by arguments of the form \((F_\pm^2 + \sqrt{-1} F_\pm)X\) and the arguments in \(Q_\pm\) by \((Id + F_\pm^2)Y\). After this replacements, the conditions present a real and an imaginary part, which are equivalent via the change \(X \mapsto F_\pm X\), and we get the following characteristic conditions of the CRFK-structures

\[
(4.60) \quad \gamma(\Theta^\pm(Z, F_\pm Y), F_\pm^2 X) + \gamma(\Theta^\pm(Z, F_\pm^2 Y), F_\pm X) = \mp \frac{1}{2}[d\psi(F_\pm X, F_\pm^2 Y, Z) + d\psi(F_\pm^2 X, F_\pm Y, Z)],
\]

\[
\gamma(\Theta^\pm(Z, Y + F_\pm^2 Y), F_\pm X) = \mp \frac{1}{2}d\psi(F_\pm X, Y + F_\pm^2 Y, Z).
\]

Now, if we use \((4.54), (4.55)\) and the equality

\[
\nabla F_\pm^2 = F_\pm \circ \nabla F_\pm + \nabla F_\pm \circ F_\pm,
\]

and then subtract the first condition from the second, we get the following system that is equivalent to \((4.60)\):

\[
(4.61) \quad \gamma(F_\pm X, (\nabla_Z F_\pm)(F_\pm Y)) = \pm \frac{1}{2} [d\psi(F_\pm X, F_\pm^2 Y, Z) + d\psi(F_\pm^2 X, F_\pm Y, Z)],
\]

\[
\gamma(F_\pm X, F_\pm (\nabla_Z F_\pm)(Y)) = \pm \frac{1}{2} [d\psi(F_\pm X, Y, Z) - d\psi(F_\pm^2 X, F_\pm Y, Z)],
\]

which does not contain the difference tensor \(\Theta\) anymore. Moreover, by replacing \(X \mapsto F_\pm X, Y \mapsto F_\pm Y\) in the second condition \((4.61)\), it follows that the second condition implies the first. Finally, by means of the change \(X \mapsto F_\pm X\) we see that the (only remaining) second condition \((4.61)\) is equivalent with the required condition \((4.59)\).

**Remark 4.3.** In the generalized Kähler case, \(F_\pm = J_\pm\) are complex structures and the following formula, where \(\omega_\pm\) are the Kähler forms of the Hermitian structures \((\gamma, J_\pm)\), holds

\[
(4.62) \quad \gamma((\nabla_X J_\pm)(Y), Z) = \frac{1}{2} [d\omega_\pm(X, Y, Z) - d\omega_\pm(X, J_\pm Y, J_\pm Z)].
\]
(e.g., Proposition IX.4.2 of [12]; in (4.62), the constant factors are different from those in [12] because we use different factor conventions for the exterior differential and product and a different sign of the fundamental form). Accordingly, (4.59) becomes

\begin{equation}
\begin{aligned}
d\psi(X, Y, Z) - d\psi(J_{\pm}X, J_{\pm}Y, Z) \\
= \pm[d\omega_{\pm}(J_{\pm}X, Y, Z) + d\omega_{\pm}(X, J_{\pm}Y, Z)].
\end{aligned}
\end{equation}

In (4.63), if we replace $X \mapsto J_{\pm}X$, then subtract the first cyclic permutation of $(X, Y, Z)$ and add the second cyclic permutation, we get

\begin{equation}
\begin{aligned}
d\psi(X, Y, Z) = \pm \frac{1}{2}[d\omega_{\pm}(JX, JY, JZ) \\
+ d\omega_{\pm}(JX, Y, Z) + d\omega_{\pm}(JY, Z) + d\omega_{\pm}(X, Y, JZ)].
\end{aligned}
\end{equation}

Finally, by using arguments in the eigenbundles of $J_{\pm}$ and since $\omega_{\pm}$ is of the complex type $(1, 1)$ and $d\omega_{\pm}$ has no $(3, 0)$ and $(0, 3)$ type components, it is easy to check that the last three terms of (4.64) add up to the first term. Therefore, we get Gualtieri’s characteristic conditions for generalized Kähler structures [8]

\begin{equation}
\begin{aligned}
d\psi(X, Y, Z) = \pm d\omega_{\pm}(J_{\pm}X, J_{\pm}Y, J_{\pm}Z).
\end{aligned}
\end{equation}

**Proposition 4.7.** A generalized CRFK-manifold with a closed 2-form $\psi$ is a triple $(M, \gamma, \psi)$ where $\psi$ is a closed 2-form and $\gamma$ is a Riemannian metric that has two partially Kähler reductions.

**Proof.** A Riemannian metric $\gamma$ is said to have a partially Kähler reduction if there exists an atlas of local coordinates $(z^{a}, y^{u})$, where $z^{a}$ are complex and $y^{u}$ are real, which has smooth, local transition functions

\begin{equation}
\begin{aligned}
\tilde{z}^{a} = \tilde{z}^{a}(z), \quad \tilde{y}^{u} = \tilde{y}^{u}(y)
\end{aligned}
\end{equation}

with complex analytic functions $\tilde{z}^{a}(z)$, and

\begin{equation}
\begin{aligned}
\gamma = \gamma_{ab}(z)dz^{a}d\bar{z}^{b} + \gamma_{uv}(y)dy^{u}dy^{v} \quad (\gamma_{b\bar{a}} = \gamma_{ab}),
\end{aligned}
\end{equation}

where the first term is a Kähler metric.

If $d\psi = 0$, the CRFK-conditions (4.59) reduce to the condition

\begin{equation}
\begin{aligned}
\gamma(F_{\pm}X, (\nabla_{Z}F_{\pm})(Y)) = 0,
\end{aligned}
\end{equation}

Proposition 4.7.
which is equivalent with \((\nabla_Z F_\pm)(Y)\) \(\in Q_\pm\) for any \(Y, Z \in \chi^1(M)\).

If \(Y \in Q_\pm\) the previous condition gives \(F_\pm(\nabla_Z Y) \in Q_\pm \cap (\text{im} F_\pm)\), therefore, \(F_\pm \nabla_Z(Y) = 0\) and \(\nabla_Z Y \in Q_\pm\). Thus, the distribution \(Q_\pm\) is \(\nabla\)-parallel, the same must hold for its \(\gamma\)-orthogonal distribution \(P_\pm = \text{im} F_\pm\), the distributions \(P_\pm, Q_\pm\) are foliations and \(\gamma\) is a reducible metric in two ways.

Furthermore, we get
\[
(\nabla_Z F_\pm)(Y) = \nabla_Z(F_\pm Y) - F_\pm(\nabla_Z Y) \in P_\pm, \quad \forall Y \in P_\pm, Z \in \chi^1(M),
\]
hence, \(\gamma(F_\pm X, (\nabla F_\pm)Y) = 0\) implies
\[(4.68) \quad (\nabla_Z F_\pm)(Y) = 0\]
for all \(Y \in P_\pm\). By looking at the case \(Z \in Q_\pm\) we can see that the structures \(F_\pm|_{P_\pm}\) are projectable onto the space of leaves of the foliation \(Q_\pm\). Indeed, projectability means that for any projectable vector field (an infinitesimal automorphism of the foliation) \(Y \in P_\pm\) the field \(F_\pm Y\) is projectable too; this holds since, \(\forall Z \in \Gamma Q_\pm\)
\[
[Z, F_\pm Y] = \nabla_Z(F_\pm Y) - \nabla_{F_\pm Y}Z \quad \overset{(4.68)}{=} \quad F_\pm(\nabla_Z Y) - \nabla F_\pm Y Z
\]
(If \(Z \in Q_\pm\) then \(\nabla_Y Z \in Q_\pm\) and \(F\nabla_Y Z = 0\). On the other hand, the projectability of \(Y\) means that \([Z, Y] \in Q_\pm\) and \(F[Z, Y] = 0\).) Then, condition \[(4.68)\] for \(Z \in P_\pm\) exactly means that \((\gamma|_{P_\pm}, F_\pm|_{P_\pm})\) is a Kähler structure on the leaves of foliation \(P_\pm\), hence, the reductions of the metric \(\gamma\) mentioned above are partially Kähler reductions. The chain of arguments may be reversed. This leads from the partial Kähler reductions and the closed form \(\psi\) to a CRFK-structure.

\(\square\)

**Example 4.2.** If \((F, \gamma)\) is a classical metric F-structure and \((\Phi, G)\) the corresponding generalized structure given in Example 4.1, the latter is CRFK iff the classical structure \((F, \gamma)\) comes from a partial Kähler manifold. Accordingly, the generalized CRFK-manifolds should be seen as generalized, partially Kähler manifolds.

**Example 4.3.** Take \(M = \mathbb{R}^{2n+h}\) with the canonical coordinates \((x^i, y^u)\) \((i = 1, \ldots, 2n; u = 1, \ldots, h)\) and with the Euclidean metric
\[
\gamma = \sum_{i=1}^{2n} (dx^i)^2 + \sum_{u=1}^{h} (dy^u)^2.
\]
If we define complex coordinates $z^a = x^a + \sqrt{-1}x^{n+a}$, $a = 1, ..., n$, $\gamma$ gets the partially Kähler reduction

$$\gamma = \sum_{a=1}^{n} dz^a dz^b + \sum_{u=1}^{h} (dy^u)^2.$$ 

On the other hand, if we define complex coordinates $w^u = x^u + \sqrt{-1}y^u$, we get a second partially Kähler reduction

$$\gamma = \sum_{a=1}^{n} dw^a dw^b + \sum_{a=h+1}^{2n} (dx^a)^2.$$ 

Therefore, for any closed 2-form $\psi$, $(M, \gamma, \psi)$ is endowed with a generalized CRFK-structure. By the usual quotientizing, the previous structure a generalized CRFK-structure on the torus $T^{2n+h} = \mathbb{R}^{2n+h}/\mathbb{Z}^{2n+h}$.

## 5 Appendix: Generalized Sasakian Structures

The structures discussed in Section 4 do not include properly generalized Sasakian manifolds [4]. In view of the importance of the latter we define such a generalization in this Appendix. With the notation of Example 2.1 consider a generalized almost contact structure of codimension $h = 1$. This structure may be identified with the generalized almost complex structure of $M \times \mathbb{R}$ that has the classical tensor fields

$$\tag{5.1} A' = F, \pi' = P + Z \wedge \frac{\partial}{\partial t}, \sigma' = \theta + \xi \wedge dt.$$ 

More exactly, it is easy to see that the generalized almost complex structures $\Phi$ of $M \times \mathbb{R}$ that can be obtained in this way are those that satisfy the following two properties:

(i) $\Phi$ is invariant by translations along $\mathbb{R}$,
(ii) $\Phi(TM \oplus 0) \subseteq 0 \oplus T^*M$, $\Phi(0 \oplus T^*R) \subseteq TM \oplus 0$.

The addition of the property

(iii) $\Phi(TM \oplus 0) \subseteq TM \oplus T^*R$, $\Phi(0 \oplus T^*M) \subseteq TR \oplus T^*M$,

takes us to the case $P = 0, \theta = 0$, which is the case of a classical almost contact structure. This is an interpretation of the classical almost contact
structures of \( M \) by non-classical, generalized, almost complex structures of \( M \times \mathbb{R} \).

In classical geometry the almost contact structure \((F, Z, \xi)\) is identified with the classical almost complex structure of \( M \times \mathbb{R} \) defined by

\[
J = F + dt \otimes Z - \xi \otimes \frac{\partial}{\partial t}.
\]

The structures \( J \) obtained in this way are characterized by the properties

(i) \( J \) is translation invariant,

(ii) \( J(T\mathbb{R}) \subseteq TM \).

We also notice the important fact that the integrability of \( J \) is equivalent with the normality of \((F, Z, \xi)\) and we know from Example 2.1 that normality is also equivalent with the integrability of the corresponding generalized almost complex structure \( \Phi_0 \) given by \((5.1)\) with \( P = 0, \theta = 0 \).

The addition of a Riemannian metric \( \gamma \) of \( M \) such that

\[
\gamma(FX, FY) = \gamma(X, Y) - \xi(X)\xi(Y)
\]

(which implies \( \xi = b \gamma Z, g(Z, FX) = 0, g(Z, Z) = 1 \)) yields an almost contact metric structure \((F, Z, \xi, \gamma)\). It is easy to check that \((5.3)\) is equivalent with the fact that the metric

\[
\Gamma = e^t(\gamma + dt^2)
\]

is Hermitian for the almost complex structure \( J \) of \( M \times \mathbb{R} \) defined by \((5.2)\). The factor \( e^t \) is superfluous here but essential for the notion of a Sasakian structure. Thus, the almost contact metric structures of \( M \) may be seen as translation invariant, almost Hermitian structures \((\Gamma, J)\) of \( M \times \mathbb{R} \) (property (ii) of \( J \) necessarily holds since \( J(\partial/\partial t) \perp \partial/\partial t \)).

The almost contact metric structure \((F, Z, \xi, \gamma)\) has the associated fundamental 2-form \( \Xi(X, Y) = g(FX, Y) \) while the corresponding almost Hermitian structure \( J \) has the Kähler form \( \omega \). A simple calculation gives

\[
\omega = e^t(\Xi - \xi \wedge dt), \quad d\omega = e^t[d\Xi + (\Xi - d\xi) \wedge dt].
\]

The most usual definition of a Sasakian structure (e.g., \cite{4}) requires it to be a normal, contact, metric structure \((F, Z, \xi, \gamma)\) where the use of the term contact instead of almost contact means the requirement \( \Xi = d\xi \). Thus,
in view of (5.5), a Sasakian structure is characterized by the fact that the corresponding structure $(\Gamma, J)$ is Kähler [4].

The previous remark suggests the question of determining the almost contact metric structures $(F, Z, \xi, \gamma)$ such that the corresponding structure $\Phi_0$ is generalized Kähler for a convenient metric, which turns out to be $\bar{\Gamma} = \gamma + dt^2$.

**Proposition 5.1.** The almost contact metric structure $(F, Z, \xi, \gamma)$ corresponds to a generalized Kähler structure $(\bar{\Gamma}, \Phi_0)$ iff the structure is cosymplectic in the sense of Blair.

**Proof.** Here, $\bar{\Gamma}$ is to be seen as the generalized Riemannian metric of $M \times \mathbb{R}$ given by (4.7) with $\varphi = 0, \gamma = \beta = \bar{\Gamma}$. Then, using Proposition 4.1, it is easy to check that the metric conditions (5.3) are equivalent with the fact that $(\bar{\Gamma}, \Phi_0)$ is a generalized almost Hermitian structure.

The generalized almost Hermitian structure $(\bar{\Gamma}, \Phi_0)$ has two associated classical almost complex structures defined by formula (4.34) with $\psi = 0$. These structures are

$$J_{\pm} = A' \pm \sharp_{\pi'} \circ \flat_{\bar{\Gamma}} = F \mp dt \otimes Z \pm \xi \otimes \frac{\partial}{\partial t}$$

($J_-$ is exactly the structure (5.2)). The Kähler forms of $(\bar{\Gamma}, J_{\pm})$ are

$$\omega_{\pm} = \Xi \pm \xi \wedge dt.$$

With (4.65), the structure $(\bar{\Gamma}, \Phi_0)$ is generalized Kähler iff $J_{\pm}$ are integrable and $d\omega_{\pm} = 0$. Therefore, $(\bar{\Gamma}, \Phi_0)$ is generalized Kähler iff the structure $(F, Z, \xi, g)$ is normal and satisfies the conditions

$$d\xi = 0, \quad d\Xi = 0.$$  (5.6)

This exactly is the definition of a cosymplectic structure in the sense of Blair [4].

It is known that a cosymplectic structure in the sense of Blair may be identified with a partially Kähler metric of the form (4.66) where the second term is $dt^2$ [4]. Therefore, as noticed in Example 4.2, the generalized CRFK-structures included adequate, generalized, Blair-cosymplectic structures.

Earlier, we saw that a Sasakian manifold is a Riemannian manifold $(M, \gamma)$ endowed with a translation invariant complex structure $J$ of $M \times \mathbb{R}$ such
that $(M \times \mathbb{R}, \Gamma, J)$ is a Kähler manifold. Now, let us assume that $M$ is endowed with a generalized Riemannian metric $G$, equivalently, with a pair $(\gamma, \psi)$ where $\gamma$ is a classical Riemannian metric and $\psi \in \Omega^2(M)$. Then, the generalized Riemannian metrics $\tilde{G}$ of $M \times \mathbb{R}$ that are related to $G$ in the way $\Gamma$ was related to $\gamma$ are those defined by pairs

$$\tilde{G} \Leftrightarrow (\Gamma, \Psi = e^t(\psi + \kappa \wedge dt))$$

where $\kappa$ is an arbitrary 1-form on $M$. Accordingly, we give

**Definition 5.1.** A generalized Sasakian manifold is a generalized Riemannian manifold $(M, (\gamma, \psi))$ endowed with a translation invariant generalized complex structure $\Phi$ of $M \times \mathbb{R}$ such that, for some $\kappa \in \Omega^1(M)$, $(M \times \mathbb{R}, \tilde{G}, \Phi)$ is a generalized Kähler manifold.

In this definition $\tilde{G}$ is defined by (5.7) and the invariance of $\Phi$ by translations means $L_{\partial/\partial t} \Phi = 0$, where the Lie derivative is defined like for an endomorphism of $TM$ and acts on both the $TM$ and $T^*M$ components. In the next proposition we will also use the following notation: $\forall \lambda \in \Omega^k(M), \lambda^c$ will be the form obtained by evaluating $\lambda$ on arguments of the form $F \pm X$.

**Proposition 5.2.** A generalized Sasakian structure of a manifold $M$ is equivalent with a pair of classical, normal, almost contact, metric structures $(F^\pm, Z^\pm, \xi^\pm, \gamma)$ complemented by a pair of forms $\psi \in \Omega^2(M), \kappa \in \Omega^1(M)$ that satisfy the conditions

$$i(Z^\pm)(\psi + d\kappa) = 0, (\psi + d\kappa)^c = -L_{\xi^\pm}[L_{Z^\pm}(\psi + d\kappa)]^c,$$

$$i(Z^\pm)(\psi + d\kappa)^c = -i(Z^\pm)[\xi^\pm \wedge d[L_{Z^\pm}(\psi + d\kappa)]^c],$$

and

$$\Xi^\pm = d\xi^\pm \mp [L_{Z^\pm}(\psi + d\kappa)]^c.$$  

**Proof.** The structure $\Phi$ of Definition 5.1 is equivalent with a pair of $\Gamma$-compatible, translation invariant, classical, complex structures $J^\pm$ of $M \times \mathbb{R}$ and we have seen that, in turn, this pair is equivalent with a pair of normal, almost contact metric structures $(F^\pm, Z^\pm, \xi^\pm, \gamma)$ of $M$. The remaining conditions to be discussed are Gualtieri’s conditions (4.65) for the Kähler forms of $(\Gamma, J^\pm)$ given by (5.5).
These conditions are
\begin{equation}
(5.10)\quad d\omega_{\pm}(J_{\pm}(X + a\frac{\partial}{\partial t}), J_{\pm}(Y + b\frac{\partial}{\partial t}), J_{\pm}(U + u\frac{\partial}{\partial t})) = \pm d\Psi(X + a\frac{\partial}{\partial t}, Y + b\frac{\partial}{\partial t}, U + u\frac{\partial}{\partial t}).
\end{equation}

Using (5.2) and (5.5), (5.10) becomes
\begin{equation}
(5.11)\quad d\Xi_{\pm}(F_{\pm}X, F_{\pm}Y, F_{\pm}U) + \sum_{Cyel} u[i(Z_{\pm})d\Xi_{\pm}](F_{\pm}X, F_{\pm}Y) - \sum_{Cyel} \xi_{\pm}(U)(\Xi_{\pm} - d\xi_{\pm})(F_{\pm}X + aZ_{\pm}, F_{\pm}Y + bZ_{\pm}) = \pm \{d\psi(X, Y, U) + \sum_{Cyel} u(\psi + d\kappa)(X, Y)\},
\end{equation}

where the cyclic permutations in the sums are on the arguments \((X, a), (Y, b), (U, u)\).

Since \(T(M \times \mathbb{R}) = TM \oplus TR\) and (5.10) is the equality of 3-forms, it follows that (5.11) holds iff it holds in the cases 1) \(a = 0, b = 0, u = 1, U = 0\) and 2) \(a = b = u = 0\). In case 1), (5.11) reduces to
\begin{equation}
(5.12)\quad [i(Z_{\pm})d\Xi_{\pm}](F_{\pm}X, F_{\pm}Y) + \{\xi_{\pm} \wedge [i(Z_{\pm})(\Xi_{\pm} - d\xi_{\pm})] \circ F_{\pm}\}(X, Y) = \pm (\psi + d\kappa)(X, Y),
\end{equation}

which, by taking into account \(i(Z_{\pm})\Xi_{\pm} = 0\) and the fact that \(\xi_{\pm}(Z_{\pm}) = 1\) and normality imply \(i(Z_{\pm})d\xi_{\pm} = L_{Z_{\pm}}\xi_{\pm} = 0\) [4], becomes
\begin{equation}
(5.13)\quad [L_{Z_{\pm}}\Xi_{\pm}](F_{\pm}X, F_{\pm}Y) = \pm (\psi + d\kappa)(X, Y).
\end{equation}

In case 2), (5.11) reduces to
\begin{equation}
(5.14)\quad d\Xi_{\pm}(F_{\pm}X, F_{\pm}Y, F_{\pm}U) - \sum_{Cyel} \xi_{\pm}(U)[(\Xi_{\pm} - d\xi_{\pm})(F_{\pm}X, F_{\pm}Y)] = \pm d\psi(X, Y, U).
\end{equation}

Furthermore, since \(TM = imF_{\pm} \oplus \text{span}\{Z_{\pm}\}\) we may decompose each condition (5.13), (5.14) into the cases (i) one of the arguments is \(Z_{\pm}\) and the others belong to \(imF_{\pm}\), (ii) all the arguments are in \(imF_{\pm}\). In case (i) we have to write (5.13) for \((X, Y) \mapsto (F_{\pm}X, Z_{\pm})\) and the result is
\begin{equation}
(5.15)\quad [i(Z_{\pm})(\psi + d\kappa)]^c = 0.
\end{equation}

Since \(<i(Z_{\pm})(\psi + d\kappa), Z_{\pm}> = 0\), (5.15) is the first condition (5.8). In case (ii) we have to write (5.13) for \((X, Y) \mapsto (F_{\pm}X, F_{\pm}Y)\) and the result is
\begin{equation}
(5.16)\quad L_{Z_{\pm}}\Xi_{\pm} = \pm (\psi + d\kappa)^c.
\end{equation}
Similarly, we have to express (5.14) for the cases (i) \((X, Y, U) \mapsto (F_\pm X, F_\pm Y, Z_\pm)\) (ii) \((X, Y, U) \mapsto (F_\pm X, F_\pm Y, F_\pm U)\). For (i) we get

\[
\Xi_\pm - d\xi_\pm = \mp [i(Z_\pm)d\psi]^c
\]

and for (ii) we get

\[
d\Xi_\pm - \xi_\pm \wedge (i(Z_\pm)d\Xi_\pm) = i(Z_\pm)(\xi_\pm \wedge d\Xi_\pm) = \mp (d\psi)^c,
\]

where the first equality follows from the properties of the operator \(i(Z_\pm)\).

Furthermore, we may calculate \(L_{Z_\pm} \Xi_\pm\) and \(d\Xi_\pm\) from (5.17) and insert the corresponding values in (5.16), (5.18). The result will be a system of conditions that look exactly like (5.8), (5.9) except for the fact that instead of the form \(L_{Z_\pm}(\psi + d\kappa)\) one has \(i(Z_\pm)d\psi\). But, modulo (5.15), we may replace the former by the latter.

**Corollary 5.1.** If \(\psi\) is closed the structures \((F_\pm, Z_\pm, \xi_\pm, \gamma)\) of a generalized Sasakian manifold \(M\) are classical Sasakian structures.

**Proof.** Use \(L_{Z_\pm}(\psi + d\kappa) = i(Z_\pm)d\psi\) in (5.8), (5.9).

**Corollary 5.2.** Let \((F_\pm, Z_\pm, \xi_\pm, \gamma)\) be a pair of normal almost contact metric structures and let \(\psi \in \Omega^2(M), \kappa \in \Omega^1(M)\) be forms such that

\[
i(Z_\pm)(\psi + d\kappa) = 0, \quad L_{Z_\pm}(\psi + d\kappa) = 0.
\]

Then, these data define a generalized Sasakian structure on \(M\) iff the structures \((F_\pm, Z_\pm, \xi_\pm, \gamma)\) are classical Sasakian structures.

**Proof.** Insert (5.19) in (5.8), (5.9).

We should notice that a generalized Sasakian structure in the sense of Definition 5.1 may not be a generalized almost contact structure as defined in the first part of this Appendix. Recall that the latter was characterized by the properties (i), (ii) mentioned after formula (5.1) and if we use formula (4.35) for the structures \(J_\pm\) of a generalized Sasakian manifold (instead of \(F_\pm\)) to reconstruct the generalized complex structure \(\Phi\) we see that \(\Phi\) may not satisfy the required property (ii). However, \(\Phi\) will satisfy property (ii) if \(\kappa = 0, Z_- = -Z_+, \xi_- = -\xi_+\). A similar terminological difficulty is encountered if one defines the notion of a generalized, metric, almost contact structure by a translation invariant, generalized, almost complex structure \(\Phi\) of \(M \times \mathbb{R}\) that is Hermitian with respect to the generalized metric \(\tilde{G}\), equivalently, by a pair of classical metric, almost contact structures \((F_+, Z_+, \xi_+), (F_-, Z_-, \xi_-)\) with the same metric \(\gamma\) and a pair of forms \((\psi \in \Omega^2(M), \kappa \in \Omega^1(M))\).
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