A NONLINEAR ELLIPTIC PROBLEM INVOLVING THE GRADIENT ON A HALF SPACE

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Abstract. We consider perturbations of the diffusive Hamilton-Jacobi equation
\[
\begin{align*}
- \Delta u &= (1 + g(x))|\nabla u|^p \quad \text{in } \mathbb{R}^N_+, \\
u &= 0 \quad \text{on } \partial \mathbb{R}^N_+,
\end{align*}
\]
for \(p > 1\). We prove the existence of a classical solution provided \(p \in \left(\frac{4}{3}, 2\right)\) and \(g\) is bounded with uniform radial decay to zero.

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1. Introduction

In this work we will investigate perturbations of
\[
\begin{align*}
- \Delta u &= |\nabla u|^p \quad \text{in } \mathbb{R}^N_+, \\
u &= 0 \quad \text{on } \partial \mathbb{R}^N_+,
\end{align*}
\]
where \(\mathbb{R}^N_+ = \{(x_1, \ldots, x_N) \in \mathbb{R}^N, x_N > 0\}\) and \(\frac{4}{3} < p < 2\). In particular we are interested in classical nonzero solutions.

Example 1. For \(t > 0\) set
\[
u_t(x) := \int_0^{x_N} \frac{1}{((p - 1)y + t)^{\frac{1}{p - 1}}} dy.
\]
A computation shows that for \(p > 1\), \(\nu_t\) is a classical solution of \(\mathbf{1}\). For \(p > 2\) the solution is unbounded when \(x_N \to \infty\) and when \(1 < p < 2\) the solution is bounded. Note that this solution has a closed form. Also note that \(\nu_t\) converges to zero as \(t \to \infty\).

A particular perturbation of the above problem will be
\[
\begin{align*}
- \Delta u &= (1 + g(x))|\nabla u|^p \quad \text{in } \mathbb{R}^N_+, \\
u &= 0 \quad \text{on } \partial \mathbb{R}^N_+.
\end{align*}
\]
(2)
In particular we are interested in nonzero solutions of \(\mathbf{2}\) for sufficient smooth functions \(g\) which satisfy needed assumptions. Our approach will be to linearize around \(\nu_t\) to obtain solutions of \(\mathbf{2}\).

We now state our main theorem.

Theorem 1. Suppose \(\frac{4}{3} < p < 2\) and \(g\) is bounded, Hölder continuous and satisfies
\[
\sup_{|x| > R, x_N \geq 0} |g(x)| \to 0 \quad \text{as } R \to \infty.
\]
Then there is a nonzero classical solution of \(\mathbf{2}\).
Remark 1.  (1) The conditions on \( g \) can surely be weakened but our interest was mainly in not making any smallness assumptions on \( g \).

(2) The condition on \( p \) may seem somewhat arbitrary but we mention that the restriction \( \frac{4}{3} < p < 2 \) ensures that \( \mu + 1 - \alpha \in (0, 1) \) (see Section 2.0.1) which is needed for the proof of Liouville-type theorems, Propositions 10 and 11 that arose in the blow up analysis.

1.1. Background. A well studied problem is the existence versus non-existence of positive solutions of the Lane-Emden equation given by

\[
\begin{array}{l}
-\Delta u = u^p \quad \text{in } \Omega, \\
\end{array}
\]

where \( 1 < p \) and \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (where \( N \geq 3 \)) with smooth boundary. In the subcritical case \( 1 < p < \frac{N+2}{N-2} \) the problem is very well understood and \( H_0^1(\Omega) \) solutions are classical solutions; see [31]. In the case of \( p \geq \frac{N+2}{N-2} \) there are no classical positive solutions in the case of the domain being star-shaped; see [44]. In the case of non star-shaped domains much less is known; see for instance [14, 21, 22, 23, 43]. In the case of \( 1 < p < \frac{N+2}{N-2} \) ultra weak solutions (non \( H_0^1 \) solutions) can be shown to be classical solutions. For \( \frac{N}{N-2} < p < \frac{N+2}{N-2} \) one cannot use elliptic regularity to show ultra weak solutions are classical. In particular in [39] for a general bounded domain in \( \mathbb{R}^N \) they construct singular ultra weak solutions with a prescribed singular set, see the book [42] for more details on this.

We now consider

\[
\begin{array}{l}
-\Delta u = |\nabla u|^p \quad \text{in } \Omega, \\
\end{array}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \). The first point is that it is a non variational equation and hence there are various standard tools which are not available anymore. The case \( 0 < p < 1 \) has been studied in [2]. Some relevant monographs for this work include [32, 28, 47]. Many people have studied boundary blow up versions of [1] where one removes the minus sign in front of the Laplacian; see for instance [35, 48]. See [2, 5, 6, 7, 8, 9, 10, 11, 12, 25, 26, 29, 30, 33, 34, 45, 37, 38, 40, 41] for more results on equations similar to (4). In particular, the interested reader is referred to [40] for recent developments and a bibliography of significant earlier work, where the author studies isolated singularities at 0 of nonnegative solutions of the more general quasilinear equation

\[
\Delta u = |x|^\alpha u^p + |x|^\beta |\nabla u|^q \quad \text{in } \Omega \setminus \{0\},
\]

where \( \Omega \subset \mathbb{R}^N \) (\( N > 2 \)) is a \( C^2 \) bounded domain containing the origin \( 0, \alpha > -2, \beta > -1 \) and \( p, q > 1 \), and provides a full classification of positive solutions vanishing on \( \partial \Omega \) and the removability of isolated singularities.

Let us finally mention that for the whole space case, it was proved in [37] that any classical solution of (1) when \( \Omega = \mathbb{R}^N \) with \( p > 1 \) has to be constant. Also, for the half-space problem (1) in the superquadratic case \( p > 2 \), it was proved in [27] a Liouville-type classification, or symmetry result, which asserts that any solution \( u \in C^2(\mathbb{R}^N_+) \cap C(\mathbb{R}^N_+) \) has to be one-dimensional, where the result was obtained by using moving planes technique, combined with Bernstein type estimates and a compactness argument. A similar result in the subquadratic case \( p \in (1, 2] \) was
Before outlining our approach we mention that our work is heavily inspired by the works \[20\] \[30\] \[42\] \[17\] \[18\] \[19\] \[24\]. Many of these works consider variations of \(-\Delta u = u^p\) on the full space or an exterior domain. Their approach is to find an approximate solution and then to linearize around the approximate solution to find a true solution. This generally involves a very detailed linear analysis of the linearized operator associated with approximate solution and then one applies a fixed point argument to find a true solution.

This current work continues the theme of examining \(-\Delta u = |\nabla u|^p\) (or variations) for singular or classical solutions, see \[15\] \[16\] \[4\] \[1\] \[3\].

We also mention the recent work \[27\] where they examine various results, some of which are Liouville theorems related to (1).

1.2. Outline of approach. First we note that by a scaling argument, instead of finding a nonzero solution of (2), it is sufficient to find a nonzero solution of

\[
\begin{align*}
-\Delta u(x) &= (1 + g(\lambda x))|\nabla u(x)|^p \quad \text{in } \mathbb{R}_+^N, \\
u &= 0 \quad \text{on } \partial \mathbb{R}_+^N,
\end{align*}
\]

for some \(\lambda > 0\). We will look for a solution of (5) of the form \(u(x) = u_t(x) + \phi(x)\) (where \(t = 1\); but we leave \(t > 0\) arbitrary for now) where \(\phi\) is unknown. Then \(\phi\) must satisfy

\[
\begin{align*}
\tilde{L}_t(\phi) &= g(\lambda x)|\nabla u_t + \nabla \phi|^p + |\nabla u_t + \nabla \phi|^p - |\nabla u_t|^p - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla \phi \\
\phi &= 0 \\
\end{align*}
\]

in \(\mathbb{R}_+^N\), on \(\partial \mathbb{R}_+^N\),

where the arguments for all the functions are \(x\) except for \(g\) and where a computation shows that

\[
\tilde{L}_t(\phi) := -\Delta \phi - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla \phi = -\Delta \phi - \frac{p\phi_{x_N}}{(p-1)x_N + t}.
\]

We will develop a linear theory for the mapping \(L_t\), a rescaled version of \(\tilde{L}_t\). We will show for all \(t > 0\) there is some \(C_t > 0\) such that for all \(f \in Y\) there is some \(\phi \in X\) (see Section 2.11 for the definition \(X\) and \(Y\)) which satisfies \(L_t(\phi) = f\) in \(\mathbb{R}_+^N\) with \(\phi = 0\) on \(\partial \mathbb{R}_+^N\). Moreover one has \(\|\phi\|_X \leq C_t\|f\|_Y\). Using this we will find a solution of (6) using a fixed point argument. Toward this we define a nonlinear mapping on \(B_R\) (the closed ball of radius \(R\) centered at the origin in \(X\)) by \(J_\lambda(\phi) = J_{\lambda,t}(\phi) = \psi\), where

\[
\begin{align*}
\tilde{L}_t(\psi) &= g(\lambda x)|\nabla u_t + \nabla \phi|^p + |\nabla u_t + \nabla \phi|^p - |\nabla u_t|^p - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla \phi \\
\psi &= 0 \\
\end{align*}
\]

in \(\mathbb{R}_+^N\), on \(\partial \mathbb{R}_+^N\).

2. The linear theory

We begin by collecting the various parameters and function spaces for the reader’s convenience.
2.0.1. The parameters, spaces and linear operators. Let $p \in \left(\frac{1}{2}, 2\right)$, $\alpha = \frac{1}{p - 1} > 1$, $\gamma = \frac{p}{p - 1} > 1$, $\mu = \frac{2}{p}$ (note this implies that $\mu + 1 - \alpha \in (0, 1)$) and $\sigma > 0$ small (chosen small enough so that our solution in the end is a classical solution after applying elliptic regularity). We introduce the norms

$$
\|\phi\|_X := \sup_{0 < x_N \leq 1} |x_N|^{\sigma} |\nabla \phi(x)| + \sup_{x_N \geq 1} |x_N|^\alpha |\nabla \phi(x)|,
$$

$$
\|f\|_Y := \sup_{0 < x_N \leq 1} |x_N|^{\sigma + 1} |f(x)| + \sup_{x_N \geq 1} |x_N|^\alpha |f(x)|,
$$

where for $\phi \in X$ we require $\phi = 0$ on $\partial \mathbb{R}_+^N$. The linear operator we deal with is,

$$
L_t(\phi) = \Delta \phi + \frac{\gamma \phi_x}{x_N + t},
$$

and note that

$$
\tilde{L}_t(\phi) = -L \frac{\mu}{x_N + t}(\phi).
$$

After considering the operator $L_t$ it is natural to consider a slight modification of the space $X$ (call it $\tilde{X}$) whose norm is given by

$$
\|
\phi\|_{\tilde{X}} := \sup_{0 < x_N \leq 1} \left\{ |x_N|^\sigma |\nabla \phi(x)| + |x_N|^\sigma |\Delta \phi(x)| \right\}
+ \sup_{x_N \geq 1} \left\{ |x_N|^\alpha |\nabla \phi(x)| + |x_N|^\alpha |\Delta \phi(x)| \right\},
$$

so we are defining $\tilde{X} := \{ \phi : \|\phi\|_{\tilde{X}} < \infty$ and $\phi = 0$ on $\partial \mathbb{R}_+^N \}$.

We will use a change of variables $\psi(x) = (x_N + t)^\mu \phi(x)$ and set $L^t$ by

$$
L^t(\psi) := -\Delta \psi + \frac{\mu(\mu - 1)\psi}{(x_N + t)^{2\mu}}.
$$

Then $L_t(\phi) = f$ in $\mathbb{R}_+^N$ if $L^t(\psi) = (x_N + t)^\mu f(x)$ in $\mathbb{R}_+^N$. The natural function spaces for $\psi$ are endowed with the norms

$$
\|\psi\|_{X_\psi} := \sup_{0 < x_N < 1} |x_N|^{\sigma - 1} |\psi(x)| + \sup_{x_N > 1} |x_N|^\alpha - 1 - \mu |\psi(x)|,
$$

where as before we take $\psi = 0$ on $\partial \mathbb{R}_+^N$, the $Y_\psi$ norm is given by

$$
\|h\|_{Y_\psi} := \sup_{0 < x_N < 1} |x_N|^{\sigma + 1} |h(x)| + \sup_{x_N > 1} |x_N|^\alpha + 1 - \mu |h(x)|.
$$

Again it is natural to consider the modified $X_\psi$ norm given by

$$
\|\psi\|_{\tilde{X}_\psi} := \sup_{0 < x_N < 1} \left\{ |x_N|^{\sigma - 1} |\psi(x)| + |x_N|^\sigma |\nabla \psi(x)| + |x_N|^\sigma |\Delta \psi(x)| \right\}
+ \sup_{x_N > 1} \left\{ |x_N|^\alpha - 1 - \mu |\psi(x)| + |x_N|^{\alpha - \mu} |\nabla \psi(x)| + |x_N|^{\alpha + 1 - \mu} |\Delta \psi(x)| \right\},
$$

where we are imposing $\psi = 0$ on $\partial \mathbb{R}_+^N$.

2.1. The linear theory. We need to consider the following equation

$$
\begin{cases}
L_t(\phi) = f(x) & \text{in } \mathbb{R}_+^N, \\
\phi = 0 & \text{on } \partial \mathbb{R}_+^N.
\end{cases}
$$

(8)

Theorem 2. For all $t \geq 1$ there is some $C = C_t$ such that for all $f \in Y$ there is some $\phi \in X$ which satisfies (8) and $\|\phi\|_X \leq C\|f\|_Y$. 

Instead of working directly with $\phi$ we prefer to use a change of variables. If we set $\psi(x) = (x_N + t)^\mu \phi(x)$ and set $L^t$ by

$$L^t(\psi) := -\Delta \psi + \frac{\mu(\mu - 1)\psi}{(x_N + t)^2},$$

then it is sufficient to develop a theory for

$$\begin{cases}
L^t(\psi) = h(x) & \text{in } \mathbb{R}^N_t, \\
\psi = 0 & \text{on } \partial \mathbb{R}^N_t.
\end{cases}$$

(9)

A computation shows that if $\psi$ satisfies (9) with $h(x) = h_f(x) = - (x_N + t)^\mu f(x)$ then $\phi$ satisfies (8). The result relating the two problems is given by

**Proposition 1.** Suppose there is some $C > 0$ such that for all $h \in Y_\psi$ there is some $\psi \in \hat{X}_\psi$ that solves (8) and $\|\psi\|_{\hat{X}_\psi} \leq C\|h\|_{Y_\psi}$. If we set $\phi := (x_N + t)^{-\mu} \psi$ and put $h(x) = h_f(x) = - (x_N + t)^\mu f(x)$, where $f \in Y$ with $\|f\|_Y = 1$, then $\phi$ satisfies (8) and $\|\phi\|_X \leq C_t$.

**Proof.** Let $f \in Y$ with $\|f\|_Y = 1$ and set $h(x) = h_f(x) = - (x_N + t)^\mu f(x)$. Then there is some $C_t$ such that $\|h\|_{Y_\psi} \leq C_t$ and hence there is some $C_{1,t}$ and $\psi \in \hat{X}_\psi$ which solves (8) and $\|\psi\|_{\hat{X}_\psi} \leq C_{1,t}$. A direct computation shows that $\phi$ satisfies the needed equation. Also note that

$$\nabla \phi(x) = \frac{\nabla \psi(x)}{(x_N + t)\mu} - \frac{\mu e_N \psi(x)}{(x_N + t)^{\mu+1}},$$

where $e_N$ is the $N^{th}$ coordinate vector. Since $\psi \in \hat{X}_\psi$ one easily sees that $\phi \in \hat{X}$ and there is some $C_1$ depending only on $t, p, N$ such that $\|\phi\|_X \leq C_1\|\psi\|_{\hat{X}_\psi}$. This gives the desired result. □

To prove the needed linear theory for $L^t$ we will use a continuation argument and to start the process we will need some results for Laplacian.

**Proposition 2.** Assuming the earlier assumptions on the parameters we have $\Delta : \hat{X}_\psi \to Y_\psi$ is a homomorphism.

**Proof.** Intu. Let $\psi \in \hat{X}_\psi$ with $\Delta \psi = 0$ in $\mathbb{R}^N_t$. Note that for $0 < x_N < 1$ we have $|\psi(x)| \leq C x_N^{1-\sigma}$ and so $\psi = 0$ on $\partial \mathbb{R}^N_t$. Let $1 \leq i \leq N - 1$ and for any fixed $h \in \mathbb{R} \setminus \{0\}$ set

$$\psi^h(x) = \frac{\psi(x + he_i) - \psi(x)}{h},$$

and note that $\psi^h$ is also harmonic in $\mathbb{R}^N_t$. Note also that there is some $C_h$ such that $|\psi^h(x)| \leq C_h x_N^{1-\sigma}$ for $0 < x_N < 1$. Also for $x_N > 1$ we have

$$|\psi^h(x)| \leq \int_0^1 |\nabla \psi(x + the_i)| dt \leq C|x_N|^{\mu-\alpha},$$

where $C$ is independent of $h$ and also note the exponent $\mu - \alpha$ is negative since $p < 2$. We can extend $\psi^h$ oddly across $x_N = 0$ to see that the extension is harmonic and bounded on $\mathbb{R}^N_t$ and hence is constant. Taking into account the boundary condition of $\psi^h$ we see $\psi^h = 0$ and hence $\psi(x) = \psi(x_N)$ and recalling $\psi$ is harmonic and the bound near $x_N = 0$ we see that $\psi(x_N) = Ax_N$. Now recalling for $x_N > 1$ we have $|\psi(x_N)| \leq C x_N^{\mu+1-\alpha}$ and since this exponent is in $(0, 1)$ we get $\psi = 0$. 
Onto. We will find a supersolution on finite domains and then pass to the limit.
To construct our supersolution we will first consider a one dimensional problem. Firstly consider the one dimensional analogs of the $X_{\psi}$ and $Y_{\psi}$ norms (written $X_{\psi}^{1}, Y_{\psi}^{1}$) on $(0, \infty)$. For $\tilde{h} \in Y_{\psi}^{1}$ we want to find an $\tilde{H} \in X_{\psi}^{1}$ which solves
\[- \tilde{H}''(x_N) = \tilde{h}(x_N) \quad \text{for } x_N \in (0, \infty), \quad \text{with } \tilde{H}(0) = 0. \tag{10}\]
A direct computation shows that
\[\tilde{H}(x_N) = \int_{0}^{x_N} \tau \tilde{h}(\tau) d\tau - x_N \int_{\infty}^{x_N} \tilde{h}(\tau) d\tau,\]
satisfies (10). Additionally one sees there is some $C$ such that $||\tilde{H}||_{X_{\psi}^{1}} \leq C ||\tilde{h}||_{Y_{\psi}^{1}}$. Set
\[\tilde{h}_0(x_N) = \frac{\chi_{(0,2]}(x_N)}{x_N^\alpha} + \frac{\chi_{(1,\infty)}(x_N)}{x_N^\beta} \]
and let $\tilde{H}_0$ denote the corresponding solution as defined above and set $\overline{\psi}(x) = \tilde{H}_0(x_N)$; this will be our supersolution on a truncated domain. Now let $h \in Y_{\psi}$ with $||h||_{Y_{\psi}} = 1$ and for $R > 1$ (big) and $\varepsilon > 0$ (small) consider $Q_{R,\varepsilon} := B_R \times (\varepsilon, R) \subset \mathbb{R}^{N-1} \times \mathbb{R}$. Let $C$ be from the 1 dimensional problem. Let $\psi_{R,\varepsilon}$ denote a solution of
\[- \Delta \psi_{R,\varepsilon}(x) = h(x) \quad \text{in } Q_{R,\varepsilon} \quad \psi_{R,\varepsilon} = 0 \quad \text{on } \partial Q_{R,\varepsilon}.\]
Then by comparison principle we have $\overline{\psi}(x) \geq \psi_{R,\varepsilon}(x)$ in $Q_{R,\varepsilon}$ and one can argue similarly to get $|\psi_{R,\varepsilon}(x)| \leq \overline{\psi}(x)$ in $Q_{R,\varepsilon}$. Hence there is some $C_1 > 0$ such that for all $R > 1$ and $0 < \varepsilon$ small (and independent of $h$) we have
\[\sup_{0 < x_N < 1; x \in Q_{R,\varepsilon}} x_N^{\sigma - 1} |\psi_{R,\varepsilon}(x)| + \sup_{x_N > 1; x \in Q_{R,\varepsilon}} x_N^{\alpha - 1 - \mu} |\psi_{R,\varepsilon}(x)| \leq C_1.\]
By taking $\varepsilon = \frac{1}{2R}$ and using a diagonal argument and compactness we see that we can pass to the limit to find some $\psi$ such that $- \Delta \psi(\cdot) = h(\cdot)$ in $\mathbb{R}^N_\psi$. Also by fixing $x$ we can pass to the limit in the quantities in the norm and see that $\psi \in X_{\psi}$ (hence $\psi = 0$ on $\partial \mathbb{R}^N_{\psi}$). Additionally we have $||\psi||_{X_{\psi}} \leq C_1$. A standard argument now gives the desired bound in $\hat{X}_{\psi}$; we will include the argument for the sake of the reader.
For $0 < x_N < 1$ consider $\tilde{\psi}(y) := x_N^{1-\sigma} \psi(x + x_N y)$ for $y \in B_+$. Fix $N < q < \infty$ and then by local regularity there is some $C = C(q, N)$ such that
\[||\tilde{\psi}||_{W^{2,q}(B_+)} \leq C ||\Delta \tilde{\psi}||_{L^q(B_+)} + C ||\tilde{\psi}||_{L^q(B_+)} \tag{11},\]
and note the bounds on $h$ and $\psi$ show that the norms on the right are bounded (independent of $x$ in the allowable range). One can now use the Sobolev imbedding to see that
\[\sup_{B_+} |\nabla \hat{\psi}| \leq C_\psi ||\hat{\psi}||_{W^{2,q}(B_+)} \]
and hence we have the gradient bounded; writing this out in terms of $\psi$ gives the desired bound on the gradient of $\hat{\psi}$. To get the second order bound we directly use the equation for $\psi$.
A similar argument gives the desired estimate for $x_N > 1$. Combining these results gives the desired $\hat{X}_{\psi}$ bounds. \qed
We first assume that the zero order term in the norm of $\psi$ get the desired result it is sufficient to get estimates on this mapping $u$ uniformly in $x,y$ write $(\|\|_{\infty})$

So we suppose the result is false and hence there are sequences $m$ and $z$ uniformly in $x,y$ such that $\|\psi_m\|_{X^m} = 1$ and $\|h_m\|_{Y_\psi} \rightarrow 0$ and $L^t_{\tau_m}(\psi_m) = h_m$ in $\mathbb{R}^N_+$. We first assume that the zero order term in the norm of $\psi_m$ is bounded away from zero; so after renormalizing we can assume that $\|\psi_m\|_{X_\psi} = 1$ and we still have $\|h_m\|_{Y_\psi} \rightarrow 0$. For ease of notation now we will slightly switch notation; we will write $(x,y) \in \mathbb{R}^{N-1} \times (0,\infty)$ instead of $x \in \mathbb{R}^N_+$.

We consider three cases:

(i) there is $y^m \rightarrow 0$ such that $(y^m)^{\alpha-1}|\psi_m(x^m, y^m)| \geq \frac{1}{T}$,

(ii) there is some $y^m \rightarrow \infty$ such that $(y^m)^{\alpha-1-\mu}|\psi_m(x^m, y^m)| \geq \frac{1}{T}$,

(iii) there is some $y^m$ bounded and bounded away from zero such that $|\psi_m(x^m, y^m)|$ is bounded away from zero.

In all three cases we write $\overline{x^m} = (x^m, y^m)$.

Case (i). Set $\psi^m(z) = (y^m)^{\alpha-1} \psi_m(\overline{x^m} + y^m z)$ for $z^m > -1$. Then $|\psi^m(0)|$ is bounded away from zero and

$\left|\psi^m(z)\right| \leq (1 + z^N)^{1-\sigma}$ for $0 < y^m(1 + z^N) < 1$,

and a computation shows that

$-\Delta \psi^m(z) + \frac{\tau_m \mu(\mu - 1) \psi^m(z)}{(z^N + 1 + (y^m)^{-1} t)^2} = \hat{h}_m(z)$ in $z^N > -1$,

with $\psi^m = 0$ on $z^N = -1$ where $\hat{h}_m(z) = (y^m)^{\sigma+1} h_m(\overline{x^m} + y^m z)$. Note that

$|\hat{h}_m(z)| \leq \frac{\|h_m\|_{Y_\psi}}{(1 + z^N)^{\sigma+1}}$ for $0 < y^m(1 + z^N) < 1$,

and hence $\hat{h}_m \rightarrow 0$ uniformly away from $z^N = -1$. By a standard compactness and diagonal argument (and after passing to a subsequence) $\psi^m \rightarrow \psi$ locally in $C^{1,\delta}_{loc}(z^N > -1)$ and $\psi$ satisfies $\Delta \psi(z) = 0$ in $z^N > -1$, $|\psi(0)| \neq 0$, $|\psi(z)| \leq (1 + z^N)^{1-\sigma}$. Using a similar argument as in the proof of the previous proposition we see that we must have $\psi = 0$ which is a contradiction.

Case (ii). Set $\psi^m(z) = (y^m)^{\alpha-1-\mu} \psi_m(\overline{x^m} + y^m z)$ for $z^m > -1$. Then $|\psi^m(0)|$ is bounded away from zero and

$|\psi^m(z)| \leq (1 + z^N)^{\mu+1-\alpha}$ for $y^m(1 + z^N) > 1$,

and recall that $\mu + 1 - \alpha \in (0,1)$. One should note there is an estimate valid for $z^N$ near $-1$ but we won’t need this. A computation shows that

Theorem 3. For all $t \geq 1$ there is some $C_t$ such that for all $h \in Y_\psi$ there is some $\psi \in \hat{X_\psi}$ such that (3) holds and $\|\psi\|_{X_\psi} \leq C_t \|h\|_{Y_\psi}$.

Proof. Since $\Delta : \hat{X_\psi} \rightarrow Y$ is a homomorphism we can use a continuation argument to get the desired result. So towards this we consider

$L^t_{\tau}(\psi) := -\Delta \psi + \frac{\tau \mu(\mu - 1) \psi}{(x^N + t)^2}$.

Then note that $(\tau, \psi) \rightarrow L^t_{\tau}(\psi)$ is a continuous mapping from $[0,1] \times \hat{X_\psi}$ to $Y$. So to get the desired result it is sufficient to get estimates on this mapping uniformly in $\tau$. So we suppose the result is false and hence there are sequences $\tau_m \in (0,1]$, $\psi_m \in \hat{X_\psi}$ and $h_m \in Y_\psi$ such that $\|\psi_m\|_{X^m} = 1$ and $\|h_m\|_{Y_\psi} \rightarrow 0$ and $L^t_{\tau_m}(\psi_m) = h_m$ in $\mathbb{R}^N_+$. We first assume that the zero order term in the norm of $\psi_m$ is bounded away from zero; so after renormalizing we can assume that $\|\psi_m\|_{X_\psi} = 1$ and we still have $\|h_m\|_{Y_\psi} \rightarrow 0$. For ease of notation now we will slightly switch notation; we will write $(x,y) \in \mathbb{R}^{N-1} \times (0,\infty)$ instead of $x \in \mathbb{R}^N_+$.
\[-\Delta \psi^m(z) + \frac{\tau m(\mu - 1)\psi^m(z)}{(z_N + 1 + (y^m)^{-1}t)^2} = \hat{h}_m(z) \quad \text{in } z_N > -1,\]

with \(\psi^m = 0\) on \(z_N = -1\), where \(\hat{h}_m(z) = (y^m)^{\alpha - \mu + 1} h_m(x^m + y^m z)\). A computation shows that

\[|\hat{h}_m(z)| \leq \frac{\|h_m\|_{Y^\alpha}}{(1 + z_N)^{\alpha - \mu + 1}}, \quad \text{for } y^m(1 + z_N) > 1,\]

and hence \(\hat{h}_m \rightarrow 0\) uniformly away from \(z_N = -1\). Again by compactness and a diagonal argument we can assume \(\hat{h}_m \rightarrow \hat{\psi}\) and hence \(\hat{\psi}\). Using compactness and a diagonal argument we can now apply Proposition 4 to get the desired contradiction.

**Case (iii).** Here we set \(\psi^m(z) = \psi_m(x^m + y^m z)\) for \(z_N > -1\). Then \(|\psi^m(0)|\) is bounded away from zero and there is some \(C\) (independent of \(m\)) such that

\[|\psi^m(z)| \leq C\chi_{(-1,1)}(z_N)(1 + z_n)^{1 - \sigma} + C\chi_{(0,\infty)}(z_N)(1 + z_n)^{1 + \mu - \alpha},\]

for \(z_N > -1\). A computation shows that

\[-\Delta \psi^m(z) + \frac{\tau m(\mu - 1)\psi^m(z)}{(1 + z_N + (y^m)^{-1}t)^2} = \hat{h}_m(z) \quad \text{in } z_N > -1,\]

where \(\hat{h}_m(z) = (y^m)^2 h_m(x^m + y^m z)\) and \(\hat{h}_m \rightarrow 0\) uniformly away from \(z_N = -1\). Using compactness and a diagonal argument we have \(\psi^m \rightarrow \psi\) in \(C^{1,\delta}_{loc}(z_N > -1)\), hence \(\psi\) satisfies

\[-\Delta \psi(z) + \frac{\tau \mu(\mu - 1)\psi(z)}{(1 + z_N + T)^2} = 0 \quad \text{in } z_N > -1\]

with \(T = \frac{t}{(y^m)^2}\), where \(y^m \rightarrow y^\infty \in (0,\infty)\). Note also that \(|\psi(0)| \neq 0\) and \(\psi\) also satisfies the pointwise bound for \(\psi^m\) given in \([13]\). We can now apply Proposition 3 to get the desired contradiction.

We have proven the desired estimates on \(\|\psi_m\|_{X^\alpha}\), i.e., \(\|\psi_m\|_{X^\alpha} \rightarrow 0\). To see that in fact \(\|\psi_m\|_{X^\alpha} \rightarrow 0\) one can now use a standard scaling argument, see the end of the proof of Proposition 2 for an idea of the needed scaling argument. \(\square\)

### 2.2. Liouville theorems

In this section we prove the needed Liouville theorems that arose in the blow up analysis.

**Proposition 3.** Let \(t > 0\), \(\tau \in [0,1]\) and \(\psi \in \hat{X}_\psi\) be such

\[-\Delta \psi(x) + \frac{\tau \mu(\mu - 1)\psi(x)}{(x_N + t)^2} = 0 \quad \text{in } \mathbb{R}_+^N.\]

Then \(\psi = 0\).
Proof. The case of $\tau = 0$ has already been handled since this is just the Laplacian. Again we will switch notation to $x = (x, y)$. For $1 \leq i \leq N - 1$ and $0 < |h| \leq 1$ we consider

$$
\psi^h(x, y) = \frac{\psi((x, y) + he_i) - \psi(x, y)}{h},
$$

and note that $\psi^h$ satisfies the same equation as $\psi$. Also note that since $t > 0$ the equation has no singularities in it at $y = 0$ and hence $\psi$ is in fact smooth up to the boundary. Also there is some $C > 0$ (independent of $h$) such that $|\psi^h(x, y)| \leq Cy^{\mu - \alpha}$ for $x_N > 1$ and note $\mu - \alpha < 0$. Also there is some $C_h$ such that $|\psi^h(x, y)| \leq C_h y^{1 - \sigma}$ for $0 < y < 1$ and again we have $\psi^h$ is in fact smooth. Using the above bounds we see that $\psi^h$ is bounded and so if we assume its not identically zero we can then assume (after multiplying by $-1$ if needed) that $\sup_{\mathbb{R}^N} \psi^h = T \in (0, \infty)$. If this is attained at some $(x^0, y^0)$ (with $y^0 \in (0, \infty)$) we get a contradiction via the maximum principle. Hence there must be some $(x^m, y^m)$ such that $\psi^h(x^m, y^m) \to T$ and that we must have $y^m$ bounded and bounded away from zero after considering the pointwise bound. For $z_N < 1$, we set $\zeta_m(z) = \psi^h(x^m, y^m + m z)$ and note $\zeta_m(0) \to T$ and $\zeta_m \leq T$. Also note that

$$
|\zeta_m(z)| \leq C(y^m)^{\mu - \alpha}(1 + z_N)^{\mu - \alpha} \quad \text{for } y^m(1 + z_N) > 1,
$$

and

$$
|\zeta_m(z)| \leq C_h(y^m)^{1 - \sigma}(1 + z_N)^{1 - \sigma} \quad \text{for } 0 < y^m(1 + z_N) < 1.
$$

By a compactness and diagonal argument we see there is some $\zeta_{\mu} \to \zeta$ in $C^{1,\beta}_{loc}(z_N > -1)$ and $\zeta$ satisfies

$$
-\Delta \zeta(z) + \frac{\tau \mu (\mu - 1) \zeta(z)}{(1 + z_N + \frac{1}{y^m})^2} = 0 \quad \text{in } z_N > -1,
$$

where $y^m \to y^\infty \in (0, \infty)$ and $\zeta$ satisfies the same pointwise bounds as $\zeta_m$ and hence $\zeta$ is nonconstant on $z_N > -1$ and attains its maximum at the origin which contradicts the maximum principle. From this we see that $\psi^h$ is zero and hence $\psi(x) = \psi(x_N)$. Returning to the equation for $\psi$ we see it is now an ode of Euler type and hence has solutions of the form

$$
\psi(x_N) = C_1(x_N + t)^{\beta_+(\tau)} + C_2(x_N + t)^{\beta_-(\tau)},
$$

where

$$
\beta_\pm(\tau) = \frac{1}{2} \pm \sqrt{1 + 4\tau \mu^2 - 4\tau \mu}.
$$

A computation shows that $\beta_+^*(\tau) > 0$ for $\tau \in (0, 1)$ and hence for $\tau \in (0, 1]$ one has $\beta_+^*(\tau) > \beta_+(0) = 1$. Note that

$$
\alpha - 1 - \mu + \beta_+(\tau) > \alpha - 1 - \mu + \beta_+(0) = \alpha - \mu > 0,
$$

and hence writing out limit sup$_{x_N \to \infty} x_N^{\alpha - 1 - \mu} |\psi(x_N)| \leq C$ gives that $C_1 = 0$. To satisfy the boundary condition one sees they must have $C_2 = 0$ and hence $\psi = 0$. □

Proposition 4. Suppose $\tau \in [0, 1]$ and $\psi$ satisfies

$$
-\Delta \psi(x) + \frac{\tau \mu (\mu - 1) \psi(x)}{x_N^{2\tau}} = 0 \quad \text{in } \mathbb{R}^N_+,
$$

with $|\psi(x)| \leq Cx_N^{\mu + 1 - \alpha}$ for $x \in \mathbb{R}^N_+$. Then $\psi = 0$. 

Proof. The case of \( \tau = 0 \) is handled in the proof of a previous result. As in the previous proof, for \( 1 \leq i \leq N - 1 \) and \( 0 < |h| \leq 1 \), we consider
\[
\psi^h(x, y) = \frac{\psi((x, y) + he_i) - \psi(x, y)}{h},
\]
and note that \( \psi^h \) satisfies the same equation as \( \psi \). Note this time the equation is singular on the boundary.

Also there is some \( C > 0 \) (independent of \( h \)) such that \( |\psi^h(x, y)| \leq Cy^{\mu - \alpha} \) for all \( y > 0 \) and note \( \mu - \alpha < 0 \). Also there is some \( C_h \) such that \( |\psi^h(x, y)| \leq C_h y^{\mu - \alpha + 1} \) for all \( y > 0 \) and this exponent is positive. Combining the pointwise estimates we see there is some \( \varepsilon > 0 \) such that
\[
\sup_{(x, y) \in \mathbb{R}_+^N} |\psi^h(x, y)| = \sup_{(x, y) \in \mathbb{R}^{N-1} \times (\tau, \varepsilon^{-1})} |\psi^h(x, y)|.
\]

We can argue exactly as in the previous case to see that \( \psi(x) = \psi(x_N) \) (we have switched notation back to just \( x \in \mathbb{R}_+^N \)). So we have
\[
\psi(x_N) = C_1 x_N^{\beta_+ (\tau)} + C_2 x_N^{\beta_- (\tau)},
\]
where \( \beta_+ (\tau) \) is from the previous proof. Provided we have both \( \beta_+ (\tau), \beta_- (\tau) \) different from \( \mu + 1 - \alpha \) then by sending \( x_N \to 0, \infty \) we can see \( C_1 = C_2 = 0 \). From the previous proof we know that \( \beta_+ (\tau) > \mu + 1 - \alpha \) for \( \tau > 0 \). By using monotonicity in \( \tau \) one sees that \( \beta_- (\tau) < \mu + 1 - \alpha \) and this gives us the desired result.

\[\square\]

3. The fixed point argument

We now will fix \( t = 1 \). The following lemma includes some fairly standard inequalities that are needed to prove the nonlinear mapping is a contraction. Note there are no smallness assumptions on the \( y \) and \( z \) terms. See, for instance, \[3, 34\] for a proof.

Lemma 1. Suppose \( 1 < p \leq 2 \). Then there is some \( C = C_p \) such that for all vectors \( x, y, z \in \mathbb{R}_+^N \) one has
\[
0 \leq |x + y|^p - |x|^p - p|x|^{p-2} x \cdot y \leq C |y|^p, \quad (16)
\]
\[
|x + y|^p - p|x|^{p-2} x \cdot y - |x + z|^p + p|x|^{p-2} x \cdot z \leq C \left(|y|^{p-1} + |z|^{p-1}\right) |y - z|. \quad (17)
\]
\[
|x + y|^p - |x + z|^p \leq C \left(|y|^{p-1} + |z|^{p-1} + |x|^{p-1}\right) |y - z|. \quad (18)
\]

We will now prove Theorem \[1\] and for the readers convenience we restate the theorem.

Theorem 1. Suppose \( \frac{1}{3} < p < 2 \) and \( g \) is bounded, Hölder continuous and satisfies
\[
\sup_{|x| > R, \ x_N \geq 0} |g(x)| \to 0 \quad \text{as} \quad R \to \infty.
\]
Then there is a nonzero classical solution of \[2\].

Proof of Theorem \[1\] We will show that \( J_\lambda \) is a contraction mapping on \( B_R \) as we outlined in the outline. In what follows \( C \) is a constant that can change from line to line but is independent of \( \lambda \) and \( R \).
Into. Let $0 < R \leq 1$, $\phi \in B_R$ and let $\psi = J_\lambda(\phi)$. Then $\psi$ satisfies (7) and by the linear theory (Theorem 2) and using (16) we see that

$$\|J_\lambda(\phi)\|_X = \|\psi\|_X \leq C\|g(\lambda x)|\nabla u_1|^p|_Y + C\|g(\lambda x)|\nabla \phi|^p|_Y + C\|\nabla u_1 + \nabla \phi\|_Y$$

since $g$ is bounded. Using the bound on $\phi$ we see that $\|\nabla \phi\|_Y \leq CR^p$ and we now examine the other term. So towards this we set

$$I_\lambda^1 := \sup_{0 < x_N < 1} x_N^{\sigma + 1}|g(\lambda x)|\nabla u_1|^p$$

and

$$I_\lambda^2 := \sup_{x_N > 1} x_N^{\sigma + 1}|g(\lambda x)|\nabla u_1|^p$$

and note that $\|g(\lambda x)|\nabla u_1|^p|_Y \leq \|I_\lambda^1 + I_\lambda^2\|$. Let $0 < \delta < 1$ (small). Set $A(T) := \sup_{x_N > 0, |x| > T}|g(x)|$ and recall that $A(T) \to 0$ as $T \to \infty$. Then

$$I_\lambda^1 \leq C \sup_{0 < x_N < 1} x_N^{\sigma + 1} \frac{|g(\lambda x)|}{(x_N + t)^{\alpha p}} \leq C \sup_{0 < x_N < 1} x_N^{\sigma + 1} \frac{|g(\lambda x)|}{(x_N + t)^{\alpha p}}$$

and

$$I_\lambda^2 \leq C \sup_{x_N > 1} x_N^{\sigma + 1} \frac{|g(\lambda x)|}{(x_N + t)^{\alpha p}} \leq C \sup_{x_N > 1} x_N^{\sigma + 1} \frac{|g(\lambda x)|}{(x_N + t)^{\alpha p}}$$

Similarly one sees that $I_\lambda^2 \leq CA(\lambda)$. Combining the above results and using the fact that $A$ is monotonic we see that

$$\|J_\lambda(\phi)\|_X \leq C \{R^p + \delta^{\sigma + 1} + A(\lambda \delta)\}.$$ 

So for $J_\lambda(B_R) \subset B_R$ it is sufficient that

$$C \{R^p + \delta^{\sigma + 1} + A(\lambda \delta)\} \leq R.$$ (19)

Contraction. Let $0 < R \leq 1$, $\phi_i \in B_R$ and $\psi_i = J_\lambda(\phi_i)$, $i = 1, 2$. Writing out the equations for $\psi_2$ and $\psi_1$ and taking a difference and using (17) and (18) we arrive at

$$\|J_\lambda(\phi_2) - J_\lambda(\phi_1)\|_X = \|\psi_2 - \psi_1\|_X \leq CH_\lambda + CK_\lambda,$$

where

$$H_\lambda = \|g(\lambda x) \{\|\nabla u_1|^{p-1} + |\nabla \phi_2|^{p-1} + |\nabla \phi_1|^{p-1}\} |\nabla \phi_2 - \nabla \phi_1\|_Y,$$

and

$$K_\lambda = \|\{\|\nabla \phi_2|^{p-1} + |\nabla \phi_1|^{p-1}\} |\nabla \phi_2 - \nabla \phi_1\|_Y.$$ 

We first estimate $K_\lambda$. So using the bound on $\phi_2$ one can see

$$\sup_{0 < x_N < 1} x_N^{\sigma + 1} |\nabla \phi_2|^{p-1} |\nabla \phi_2 - \nabla \phi_1| \leq R^{p-1} \sup_{0 < x_N < 1} x_N^{\sigma + 1 - \sigma(p-1)} |\nabla \phi_2 - \nabla \phi_1|$$

provided $\sigma + 1 - \sigma(p-1) - \sigma \geq 0$, which is satisfied after recalling we are taking $\sigma > 0$ very small. A similar argument shows that

$$\sup_{x_N > 1} x_N^{\sigma + 1} |\nabla \phi_2|^{p-1} |\nabla \phi_2 - \nabla \phi_1| \leq R^{p-1} |\phi_2 - \phi_1|_X \sup_{x_N > 1} x_N^{\sigma + 1 - \sigma(p-1)},$$

and so here we need the exponent to be less or equal zero and note that this exponent is zero. Combining these two results we see that $K_\lambda \leq CR^{p-1} |\phi_2 - \phi_1|_X$. 

We now examine the $H_\lambda$ term.

- First we examine the term $\sup_{0 < x_N < 1} |x_N | g(\lambda x) | \nabla u_t |_{p-1} | \nabla \phi_2 - \nabla \phi_1 |$. Using an argument as before one has
  \[
  \sup_{0 < x_N < 1} x_N |g(\lambda x) | \leq C\delta + CA(\lambda\delta). \]

A computation shows that
  \[
  \sup_{0 < x_N < 1} \frac{|g(\lambda x) | \nabla u_t |_{p-1} | \nabla \phi_2 - \nabla \phi_1 |}{x_N} \leq C\|\phi_2 - \phi_1\|_X \sup_{0 < x_N < 1} x_N |g(\lambda x) | \leq C(\delta + A(\lambda\delta)) \|\phi_2 - \phi_1\|_X.
  \]

We now examine the outer portion of the norm,
  \[
  \sup_{x_N > 1} x_N^{\sigma - 1} |g(\lambda x) | \nabla \phi_2 |_{p-1} | \nabla \phi_2 - \nabla \phi_1 | = C \sup_{x_N > 1} \frac{x_N}{x_N + t} |g(\lambda x) | \{x_N^{\sigma - 1} |\nabla \phi_2 - \nabla \phi_1 | \} \leq CA(\lambda) \|\phi_2 - \phi_1\|_X.
  \]

Combining the results gives
  \[
  \|g(\lambda x) | \nabla u_t |_{p-1} | \nabla \phi_2 - \nabla \phi_1 | \leq C \{\delta + A(\lambda\delta)\} \|\phi_2 - \phi_1\|_X \tag{20}
  \]
  after using monotonicity of $A$.

- We now examine the term $\|g(\lambda x) | \nabla \phi_2 |_{p-1} | \nabla \phi_2 - \nabla \phi_1 | \|_Y$. Using the estimate for $\phi_2$ one sees that
  \[
  \sup_{0 < x_N < 1} x_N^{\sigma + 1} |g(\lambda x) | \nabla \phi_2 |_{p-1} | \nabla \phi_2 - \nabla \phi_1 | \leq R^{p-1} \|\phi_2 - \phi_1\|_X \sup_{0 < x_N < 1} x_N^{\sigma - 1} |g(\lambda x) | \leq R^{p-1} \frac{x_N}{x_N + t} |g(\lambda x) | \{x_N^{\sigma - 1} |\phi_2 - \phi_1 | \} \leq CA(\lambda) \|\phi_2 - \phi_1\|_X.
  \]

A computation as before shows that
  \[
  \sup_{0 < x_N < 1} x_N^{\sigma - 1} |g(\lambda x) | \leq C\delta^{1 - \sigma(p-1)} + CA(\lambda\delta),
  \]
  and hence
  \[
  \sup_{0 < x_N < 1} x_N^{\sigma + 1} |g(\lambda x) | \nabla \phi_2 |_{p-1} | \nabla \phi_2 - \nabla \phi_1 | \leq CR^{p-1} \{\delta^{1 - \sigma(p-1)} + A(\lambda\delta)\} \|\phi_2 - \phi_1\|_X.
  \]

Similarly the outer portion of the norm gives
  \[
  \sup_{x_N > 1} x_N^{\sigma - 1} |g(\lambda x) | \nabla \phi_2 |_{p-1} | \nabla \phi_2 - \nabla \phi_1 | \leq R^{p-1} \sup_{x_N > 1} |g(\lambda x) | x_N^{\sigma - 1} |\nabla \phi_2 - \nabla \phi_1 | \leq R^{p-1} A(\lambda) \|\phi_2 - \phi_1\|_X,
  \]
  and hence combining these two results gives
  \[
  \|g(\lambda x) | \nabla \phi_2 |_{p-1} | \nabla \phi_2 - \nabla \phi_1 | \|_Y \leq CR^{p-1} \{\delta^{1 - \sigma(p-1)} + A(\lambda\delta)\} \|\phi_2 - \phi_1\|_X, \tag{21}
  \]
  where again we have used the monotonicity of $A$.

Combining with the previous results gives
  \[
  H_\lambda \leq C \{\delta + A(\lambda\delta) + R^{p-1} \left(\delta^{1 - \sigma(p-1)} + A(\lambda\delta)\right)\} \|\phi_2 - \phi_1\|_X.
  \]

Combining the estimates for $H_\lambda$ and $K_\lambda$ shows that
  \[
  \|J_\lambda(\phi_2) - J_\lambda(\phi_1)\|_X \leq C \left\{R^{p-1} + \delta + A(\lambda\delta) + R^{p-1} \left(\delta^{1 - \sigma(p-1)} + A(\lambda\delta)\right)\right\} \|\phi_2 - \phi_1\|_X.
  \]

Hence, $J_\lambda$ is a contraction on $B_R$ provided
  \[
  C \left\{R^{p-1} + \delta + A(\lambda\delta) + R^{p-1} \left(\delta^{1 - \sigma(p-1)} + A(\lambda\delta)\right)\right\} \leq \frac{3}{4}. \tag{22}
  \]
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So for $J_\lambda$ to be a self-map and contraction mapping on $B_R$ we need both (19) and (22) to hold. To pick the $R, \delta, \lambda$ one first chooses $R > 0$ very small but fixed, then fixes $\delta$ very small and finally picks $\lambda$ very big. Once $J_\lambda$ is a contraction we can use Banach’s Contraction Mapping Principle to see there is a fixed point $\phi \in B_R$ and hence we see that $u(x) = u_t(x) + \phi(x)$ is a solution of (17). Note that $u_t$ is smooth and the gradient of $\phi$ can have slight blow up at $x_N = 0$; depending on $\sigma > 0$. By taking $\sigma > 0$ very small one can apply elliptic regularity to see that $u$ is a classical solution. To see that $u$ is not identically zero one needs to choose $R > 0$ sufficiently small (relative to $\|u_t\|_X$) and then one sees that $|\nabla u(x)| > 0$ for $x_N > 1$ (for instance).

\[\Box\]

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References

[1] A. Aghajani and C. Cowan, Some elliptic problems involving the gradient on general bounded and exterior domains, submitted.
[2] A. Aghajani, C. Cowan and S. H. Lui, Existence and regularity of nonlinear advection problems, Nonlinear Analysis, 166 (2018) 19–47.
[3] A. Aghajani and C. Cowan, Singular solutions of elliptic equations on a perturbed cone, J. Differential Equations, 266, 6, (2019), 3328-3366.
[4] A. Aghajani, C. Cowan and S. H. Lui, Singular solutions of elliptic equations involving nonlinear gradient terms on perturbations of the ball, J. Diff. Eqns., 264 (2018) 2865–2896.
[5] D. Arcoya, L. Boccardo, T. Leonori and A. Porretta, Some elliptic problems with singular natural growth lower order terms, J. Diff. Eqns., 249 (2010) 2771-2795.
[6] D. Arcoya, J. Carmona, T. Leonori, P. J. Martinez-Aparicio, L. Orsina and F. Petitta, Existence and non-existence of solutions for singular quadratic quasilinear equations, J. Diff. Eqns., 246 (2009) 4006-4042.
[7] D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka, Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions, J. Math. Anal. Appl., 420 (2014) 772-780.
[8] D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient, J. Funct. Anal., 268 (2015) 2296-2335.
[9] A. Bensoussan, L. Boccardo and F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solution, Ann. Inst. Henri Poincare, 5 (1988) 347-364.
[10] M.-F. Bidaut-Veron, M. Garcia-Huidobro and L. Veron, Remarks on some quasilinear equations with gradient terms and measure data, Contemp. Math, 595 (2013) 31-53.
[11] M.-F. Bidaut-Veron, M. Garcia-Huidobro and L. Veron, Local and global properties of solutions of quasilinear Hamilton-Jacobi equations, J. Funct. Anal., 267 (2014) 3294-3331.
[12] M.-F. Bidaut-Veron, M. Garcia-Huidobro and L. Veron, Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations, Calc. Var., 54 (2015) 3471-3515.
[13] J. Ching and F. C. Cirstea, Existence and classification of singular solutions to nonlinear elliptic equations with a gradient term, Anal. PDE, 8 (2015) 1931-1962.
[14] J.M. Coron, Topologie et cas limite des injections de Sobolev. C.R. Acad. Sc. Paris, 299, Series I, (1984) 209-212
[15] C. Cowan and A. Razani, Singular solutions of a Lane-Emden system, Discrete & Continuous Dynamical Systems-A 41(2) (2021) 621-656.
[16] C. Cowan and A. Razani, *Singular solutions of a p-Laplace equation involving the gradient*, accepted J. Differ. Equ. 269 (4), (2020) 3914-3942.

[17] J. Dávila, M. del Pino and M. Musso, *The Supercritical Lane–Emden–Fowler Equation in Exterior Domains*, Communications in Partial Differential Equations, 32:8, (2007) 1225-1243.

[18] J. Dávila, M. del Pino, M. Musso and J. Wei, *Fast and slow decay solutions for supercritical elliptic problems in exterior domains*, Calculus of Variations and Partial Differential Equations, 32, 4 (2008) 453-480.

[19] J. Dávila, Manuel del Pino, M. Musso and J. Wei, *Standing waves for supercritical nonlinear Schrödinger equations*, Journal of Differential Equations 236, 1(2007) 164-198.

[20] J. Dávila and L. Dupaigne, *Perturbing singular solutions of the Gelfand problem*. Commun. Contemp. Math. 9, 5 (2007) 639-680.

[21] M. del Pino and M. Musso, *Super-critical bubbling in elliptic boundary value problems*, Variational problems and related topics (Kyoto, 2002). 130 7 (2003), 85-108.

[22] M. del Pino, P. Felmer and Monica Musso, *Two bubble solutions in the supercritical Bahri-Coron’s problem*, Calculus of Variations and Partial Differential Equations 16, 2 (2003) 113-145.

[23] M. del Pino, P. Felmer and M. Musso, *Multi-bubble solutions for slightly supercritical elliptic problems in domains with symmetries*, Bull. London Math. Society 35, 4 (2003) 513-521.

[24] M. del Pino and J. Wei, *Supercritical elliptic problems in domains with small holes*, Annales de l’Institut Henri Poincare, Non Linear Analysis 24, 4 (2007) 507-520.

[25] V. Ferone and F. Murat, *Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small*, Nonlinear Anal., 42 (2000) 13309-1326.

[26] V. Ferone, M. R. Posteraro and J. M. Rakotoson, *$L^\infty$-estimates for nonlinear elliptic problems with $p$-growth in the gradient*, J. Inequal. Appl., 3 (1999) 109-125.

[27] R. Filippucci, P. Pucci and P. Souplet, *A Liouville-type theorem in a half-space and its applications to the gradient blow-up behavior for superquadratic diffusive Hamilton–Jacobi equations*, Communications in Partial Differential Equations 45,4 (2019) 321-349.

[28] M. Gherga and V. Radulescu, *Nonlinear PDEs*, Springer-Verlag, Berlin Heidelberg, 2012.

[29] D. Giachetti, F. Petitta and S. Segura de Leon, *Elliptic equations having a singular quadratic gradient term and a changing sign datum*, 11 (2012) 1875-1895.

[30] D. Giachetti, F. Petitta and S. Segura de Leon, *A priori estimates for elliptic problems with a strongly singular gradient term and a general datum*, Diff. Integral Eqns., 226 (2013) 913-948.

[31] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*. Comm. Pure Appl. Math., 34, 4 (1981)525-598.

[32] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[33] N. Grenon, F. Murat and A. Porretta, *Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms*, C. R. Acad. Sci. Paris, Ser. I, 342 (2006) 23-28.

[34] N. Grenon and C. Trombetti, *Existence results for a class of nonlinear elliptic problems with p-growth in the gradient*, Nonlinear Anal., 52, (2003) 931-942.

[35] J. M. Lasry and P. L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints*, Math. Ann., 283 (1989) 583-630.

[36] P. Lindqvist, *Notes on the p-Laplace equation*, University of Jyväskylä, Report 102, (2006).

[37] P. L. Lions, *Quelques remarques sur les problemes elliptiques quasilinéaires du second ordre*, J. Anal. Math., 45 (1985) 234-254.

[38] M. Marcus and P. T. Nguyen, *Elliptic equations with nonlinear absorption depending on the solution and its gradient*, Proc. London Math. Soc., 111 (2015) 205-239.
[39] R. Mazzeo and F. Pacard. A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis, J. Diff. Geom. 44 (1996) 331-370.
[40] P.T. Nguyen, Isolated singularities of positive solutions of elliptic equations with weighted gradient term, Analysis & PDE, 9, 7 (2016) 1671-1692.
[41] P.T. Nguyen and L. Veron, Boundary singularities of solutions to elliptic viscous Hamilton-Jacobi equations, J. Funct. Anal., 263 (2012) 1487-1538.
[42] F. Pacard and T. Rivi`ere, Linear and nonlinear aspects of vortices: the Ginzburg Landau model, Progress in Nonlinear Differential Equations, 39, Birkauser. 342 pp. (2000).
[43] D.Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains, J. Funct. Anal. 114, 1 (1993) 97-105.
[44] S. Pohozaev, . Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Soviet. Math. Dokl. 6 (1965) 1408–1411.
[45] A. Porretta and S. Segura de Leon, Nonlinear elliptic equations having a gradient term with natural growth, J. Math. Pures Appl., 85 (2006) 465-492.
[46] A. Porretta and L. Veron, Asymptotic behavior for the gradient of large solutions to some nonlinear elliptic equations, Adv. Nonlinear Stud. 6 (2006) 351-378.
[47] Struwe, M. (1990). Variational Methods – Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Berlin: Springer-Verlag.
[48] Z. Zhang, Boundary blow-up elliptic problems with nonlinear gradient terms, J. Diff. Equns., 228 (2006) 661-684.

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