The small-scale dynamo: breaking universality at high Mach numbers

Dominik R G Schleicher\textsuperscript{1,4}, Jennifer Schober\textsuperscript{2}, Christoph Federrath\textsuperscript{2,3}, Stefano Bovino\textsuperscript{1} and Wolfram Schmidt\textsuperscript{1}

\textsuperscript{1} Institut für Astrophysik, Georg-August-Universität Göttingen, Friedrich-Hund-Platz 1, D-37077 Göttingen, Germany
\textsuperscript{2} Universität Heidelberg, Zentrum für Astronomie, Institut für Theoretische Astrophysik, Albert-Überle-Strasse 2, D-69120 Heidelberg, Germany
\textsuperscript{3} Monash Centre for Astrophysics, School of Mathematical Sciences, Monash University, VIC 3800, Australia
E-mail: dschleic@astro.physik.uni-goettingen.de, schober@stud.uni-heidelberg.de, christoph.federrath@monash.edu, sbovino@astro.physik.uni-goettingen.de and schmidt@astro.physik.uni-goettingen.de

New Journal of Physics 15 (2013) 023017 (18pp)
Received 6 September 2012
Published 8 February 2013
Online at http://www.njp.org/
doi:10.1088/1367-2630/15/2/023017

Abstract. The small-scale dynamo plays a substantial role in magnetizing the Universe under a large range of conditions, including subsonic turbulence at low Mach numbers, highly supersonic turbulence at high Mach numbers and a large range of magnetic Prandtl numbers $Pm$, i.e. the ratio of kinetic viscosity to magnetic resistivity. Low Mach numbers may, in particular, lead to the well-known, incompressible Kolmogorov turbulence, while for high Mach numbers, we are in the highly compressible regime, thus close to Burgers turbulence. In this paper, we explore whether in this large range of conditions, universal behavior can be expected. Our starting point is previous investigations in the kinematic regime. Here, analytic studies based on the Kazantsev model have shown that the behavior of the dynamo depends significantly on $Pm$ and the type of turbulence, and numerical simulations indicate a strong dependence

\textsuperscript{4} Author to whom any correspondence should be addressed.

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of the growth rate on the Mach number of the flow. Once the magnetic field saturates on the current amplification scale, backreactions occur and the growth is shifted to the next-larger scale. We employ a Fokker–Planck model to calculate the magnetic field amplification during the nonlinear regime, and find a resulting power-law growth that depends on the type of turbulence invoked. For Kolmogorov turbulence, we confirm previous results suggesting a linear growth of magnetic energy. For more general turbulent spectra, where the turbulent velocity scales with the characteristic length scale as $u_\ell \propto \ell^{\vartheta}$, we find that the magnetic energy grows as $(t/T_{ed})^{2\vartheta/(1-\vartheta)}$, with $t$ being the time coordinate and $T_{ed}$ the eddy-turnover time on the forcing scale of turbulence. For Burgers turbulence, $\vartheta = 1/2$, quadratic rather than linear growth may thus be expected, as the spectral energy increases from smaller to larger scales more rapidly. The quadratic growth is due to the initially smaller growth rates obtained for Burgers turbulence. Similarly, we show that the characteristic length scale of the magnetic field grows as $t^{1/(1-\vartheta)}$ in the general case, implying $t^{3/2}$ for Kolmogorov and $t^2$ for Burgers turbulence. Overall, we find that high Mach numbers, as typically associated with steep spectra of turbulence, may break the previously postulated universality, and introduce a dependence on the environment also in the nonlinear regime.

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1. Introduction

The small-scale dynamo has been suggested to operate under a large range of different conditions, including the solar surface [1, 2], galaxies and galaxy clusters [3–8], the intergalactic medium [9] and during the formation of the first stars and galaxies [10–16]. It thus operates on a large range of conditions, concerning for instance the magnetic Prandtl number $Pm$, i.e. the ratio of kinematic viscosity $\nu$ to magnetic resistivity $\eta$, the Mach number of the turbulence $M$, i.e. the ratio of turbulent velocities to the sound speed, and, most likely related to the Mach number, the expected type of turbulence in the system.

Most studies of the small-scale dynamo performed so far have focused on incompressible Kolmogorov turbulence [17], assuming a scaling relation $u_\ell \propto \ell^{1/3}$ between turbulent velocity
For Kolmogorov turbulence, it was previously concluded that the magnetic energy grows exponentially in the kinematic regime (e.g. [18–21]) and linearly once the backreactions from the magnetic field become important (e.g. [22–24]). The latter was interpreted by Beresnyak [24] as evidence for universality of the small-scale dynamo, suggesting that a fixed fraction of the global turbulence dissipation rate is converted into magnetic energy.

However, observations of turbulence in molecular clouds (e.g. [25, 26]) and numerical simulations of supersonic turbulence (e.g. [27–30]) often reveal steeper turbulent spectra, typically in between the incompressible Kolmogorov turbulence and the highly compressible Burgers turbulence [31]. So far, only a small number of studies have investigated the turbulent dynamo in this regime. For instance, Haugen et al [32] provided the first study exploring the dependence of the dynamo on the Mach number in simulations of driven turbulence, and Balsara et al [33, 34] and Balsara and Kim [35] explored the amplification of magnetic fields in turbulence produced by strong supernova shocks. The first systematic study covering turbulent Mach numbers from 0.02 to 20 and different types of turbulence driving was undertaken by Federrath et al [36], while the effect of a large range of different turbulence spectra was explored by Schober et al [21] based on the Kazantsev model [18].

We note that the small-scale dynamo has also been studied in the context of the so-called shell models [37–40]. The latter originate from shell models of hydrodynamical turbulence, which originally considered turbulence in two dimensions (2D) [41–43], but were extended to three dimensions (3D) once a description of kinetic helicity was obtained [44]. The first 2D magneto-hydrodynamical (MHD) shell model was derived by Frik [45], while 3D models have been developed by Brandenburg et al [46], Basu et al [47] and Frick and Sokoloff [48]. More sophisticated processes such as non-local interactions [49, 50], anisotropies [51] and the Hall effect [52] have been incorporated in more recent studies. These approaches allow us to study both the evolution of the power spectrum and the saturated regime, and are highly complementary to the methods presented here.

In the following, we will consider the small-scale dynamo in the kinematic and nonlinear regime, and present evidence from existing and new calculations suggesting a strong dependence on the magnetic Prandtl number, as well as the Mach number of the flow. In section 2, we summarize the evidence and indications of non-universal behavior in the kinematic regime, which has been derived in previous studies. In section 3, we present the first exploration concerning different types of turbulence during the nonlinear phase of the dynamo, where the backreaction of the magnetic field becomes important. We show that linear growth is only obtained in the case of Kolmogorov turbulence, while steeper power laws result from turbulent spectra with $\theta > 1/3$. We discuss the physical implications in section 4, and summarize our main results in section 5.

2. Non-universality in the kinematic regime

The small-scale dynamo is well studied in the kinematic regime, where exponential growth of the magnetic field is expected on the viscous scale. The growth rate of the magnetic field can be calculated in the framework of the Kazantsev model, assuming homogeneous turbulence that is $\delta$-correlated in time, or with 3D magneto-hydrodynamical simulations. In this section, we discuss hints and evidence for non-universal behavior in the kinematic regime.
2.1. Indications of non-universality in the Kazantsev model

The amplification of magnetic fields is governed by the induction equation, which is given as

$$\partial_t \vec{B} = \nabla \times \vec{v} \times \vec{B} + \eta \Delta \vec{B}. \quad (1)$$

We assume, in the following, $\langle \vec{B} \rangle = 0$, although we note that scenarios considering $\langle \vec{B} \rangle \neq 0$ have been recently explored by Boldyrev et al [53] and Malyshkin and Boldyrev [54–56]. In the Kazantsev model, the velocity field and the magnetic field are decomposed into a mean field, denoted with brackets $\langle \rangle$, and a fluctuating component denoted with $\delta$:

$$\vec{v} = \langle \vec{v} \rangle + \delta \vec{v}, \quad \vec{B} = \langle \vec{B} \rangle + \delta \vec{B}. \quad (2)$$

A central input is the correlation function of the turbulent velocity, which is $\delta$-correlated in time and (in the absence of helicity) can be decomposed as

$$\langle \delta v_i(\vec{r}_1, t) \delta v_j(\vec{r}_2, s) \rangle = T_{ij}(r) \delta(t - s),$$

$$T_{ij}(r) = \left( \delta_{ij} - \frac{r_i r_j}{r^2} \right) T_N(r) + \frac{r_i r_j}{r^2} T_L(r), \quad (3)$$

with $r = |\vec{r}_1 - \vec{r}_2|$ and $T_N, T_L$ are the transverse and longitudinal parts of the correlation function, respectively [57]. The same definitions can be applied to the magnetic field, yielding a two-point correlation function $M_{ij}(r, t)$ with transverse and longitudinal components $M_N(r, t)$ and $M_L(r, t)$. Unlike the velocity field, the magnetic field is always divergence-free, leading to the additional constraint

$$M_N = \frac{1}{2r} \frac{d}{dr} (r^2 M_L). \quad (4)$$

As the Kazantsev model assumes that the flow is $\delta$-correlated in time, concepts such as viscosity or the magnetic Prandtl number cannot be directly incorporated into the flow, as the turbulent velocity field is destroyed and regenerated at each instant, leaving no time for viscosity to act. However, it can be indirectly included by adopting turbulent velocity spectra that become steeper below a given viscous scale $\ell_v$. This is the approach employed here. For a given relation of the type

$$u_\ell \propto \ell^\theta \quad (5)$$

in the inertial range, the longitudinal correlation function of turbulence can be parameterized as [21]

$$T_L(r) = \begin{cases} \frac{VL}{3} \left( 1 - Re^{(1-\theta)/(1+\theta)} \left( \frac{r}{L} \right)^2 \right), & 0 < r < \ell_v, \\ \frac{VL}{3} \left( 1 - \left( \frac{r}{L} \right)^{\theta+1} \right), & \ell_v < r < L, \\ 0, & L < r, \end{cases} \quad (6)$$

with $\ell_v$ the viscous scale, $L$ the driving scale of turbulence, $V$ the turbulent velocity on scale $L$, $Re = VL/\nu$ the Reynolds number of the gas and $\nu$ the kinetic viscosity. Similarly, we have

$$T_N(r) = \begin{cases} \frac{VL}{3} \left( 1 - \theta (\theta) Re^{(1-\theta)/(1+\theta)} \left( \frac{r}{L} \right)^2 \right), & 0 < r < \ell_v, \\ \frac{VL}{3} \left( 1 - \theta (\theta) \left( \frac{r}{L} \right)^{\theta+1} \right), & \ell_v < r < L, \\ 0, & L < r, \end{cases} \quad (7)$$
with $\theta(\vartheta) = (21 - 38\vartheta)/5$. As we expect exponential growth of the magnetic energy as a function of time, we make the following ansatz for the kinematic regime:

$$M_L(r, t) \equiv \frac{1}{r^2 \sqrt{\kappa_{\text{diff}}}} \psi(r) e^{2\Gamma t}. \quad (8)$$

Inserting (8) into the induction equation (1), one obtains the Kazantsev equation, which is of the same form as the quantum-mechanical Schrödinger equation

$$-\kappa_{\text{diff}}(r) \frac{d^2 \psi(r)}{dr^2} + U(r) \psi(r) = -\Gamma \psi(r). \quad (9)$$

In this framework, the amplification depends on the effective potential $U(r)$ in equation (9), which is given as

$$U(r) \equiv \frac{k_{\text{diff}}''}{2} - \left(\frac{k_{\text{diff}}'}{4k_{\text{diff}}^2}\right)^2 + \frac{2k_{\text{diff}}}{r} + \frac{2(T_N' - T_N)}{r^2} + \frac{2(T_L - T_N)}{r^2}. \quad (10)$$

As recently shown by Schober et al [21], this form of the potential also accounts for the effect of compressibility by keeping terms related to $\nabla \cdot \vec{v}$ during the derivation. The equation can be solved using the WKB approximation in the limit of $Pm \to \infty$ [18–21]. For Kolmogorov turbulence, one obtains

$$\Gamma_{K, Pm \gg 1} = 1.028 V \frac{1}{L} Re^{1/2}. \quad (11)$$

In a recent study, analytical solutions based on the Wentzel–Kramers–Brillouin (WKB) approximation have been derived in the limit $Pm \ll 1$ by Schober et al [58]. For Kolmogorov, they yield

$$\Gamma_{K, Pm \ll 1} = 0.0268 V \frac{1}{L} Rm^{1/2}, \quad (12)$$

with $Rm = V L / \eta$ the magnetic Reynolds number, and $\eta$ the magnetic diffusivity. We thus observe a fundamental difference between the limiting cases $Pm \ll 1$ and $Pm \gg 1$ in the kinematic regime: for $Pm \gg 1$, magnetic field amplification occurs predominantly on the viscous scale, corresponding to the most negative range of the potential. For $Pm \ll 1$, on the other hand, the resistive scale becomes larger than the viscous scale. Amplification on the viscous scale is thus not possible, and the strongest contribution is close to the resistive scale due to the short eddy-times. Correspondingly, the growth rate of the magnetic field depends on the Reynolds number $Re$ for $Pm \gg 1$ and on the magnetic Reynolds number $Rm$ for $Pm \ll 1$ (see also [59]).

The results can be generalized further for different types of turbulence. In the limit $Pm \gg 1$, one obtains [21]

$$\Gamma_{\vartheta, Pm \gg 1} = \frac{(163 - 304\vartheta)}{60} V \frac{1}{L} Re^{(1-\vartheta)/(1+\vartheta)}. \quad (13)$$

In the regime $Pm \ll 1$, one finds a similar relation [58]

$$\Gamma = \alpha V \frac{1}{L} Rm^{(1-\vartheta)/(1+\vartheta)} \quad (14)$$

with the prefactor $\alpha$ defined through the quantities

$$\alpha(\vartheta) = \vartheta (56 - 103\vartheta), \quad (15)$$
\[ b(\vartheta) = \vartheta (79 - 157 \vartheta), \quad (16) \]
\[ c(\vartheta) = \frac{25 + \sqrt{135 a(\vartheta) + (b(\vartheta) - 25)^2} - b(\vartheta)}{a(\vartheta)} \quad (17) \]
as
\[ \alpha = \frac{a(\vartheta)}{5} c(\vartheta) \frac{a - 1}{\pi} \exp \left( \sqrt{\frac{5}{3 a(\vartheta)}} \pi (\vartheta - 1) - 2 \right). \quad (18) \]

A numerical evaluation shows that these coefficients are smaller by about two orders of magnitude in the limit \( Pm \ll 1 \) compared to the case of high \( Pm \), assuming the same type of turbulence. This can be expected, as the amplification then occurs on larger scales, with larger eddy-turnover times.

Similarly, also the type of turbulence reflected in the parameter \( \vartheta \) may change the amplification rate by about an order of magnitude, in the case of the same value of \( Rm \). The most efficient amplification rate occurs for Kolmogorov turbulence, \( \vartheta = 1/3 \), while it is less efficient for highly compressible Burgers turbulence, \( \vartheta = 1/2 \), for which the turbulent velocities decrease more rapidly with length scale.

The Kazantsev model thus indicates that the behavior of the dynamo depends both on the type of turbulence and the magnetic Prandtl number. A potential restriction of the underlying model is the assumption of \( \delta \)-correlated turbulence, although the characteristic timescales are certainly small compared to the dynamical time. The resulting uncertainties are explored in [60]. The main results from these considerations are thus the following:

- The behavior of the small-scale dynamo depends sensitively on the value of \( Pm \), and in particular whether \( Pm \ll 1 \) or \( Pm \gg 1 \). We note that there is a continuous transition at \( Pm \sim 1 \), as detailed by Bovino \textit{et al} [61].
- The adopted type of turbulence has a significant influence on the efficiency of magnetic field amplification, as turbulent spectra with \( \vartheta > 1/3 \) correspond to larger eddy-turnover times and smaller amplification rates.

2.2. The results of numerical simulations

Due to the numerical viscosity and resistivity, it is difficult to perform magneto-hydrodynamical simulations with \( Pm \) significantly different from 1. However, a limited range of \( Pm \) has nevertheless been explored. For instance, Haugen \textit{et al} [62] investigated magnetic Prandtl numbers between 0.1 and 30. For \( Pm < 1 \), they report that the mimimum magnetic Reynolds number required for dynamo action, \( Rm_c \), scales as
\[ Rm_c \sim 35 \pi Pm^{-1/2}. \quad (19) \]
We note that the factor \( \pi \) in the above is due to their definition of the magnetic Reynolds number. They further report differences in the obtained power spectra, indicating a steeper decrease on small scales for small values of \( Pm \).

Schekochihin \textit{et al} [63] and Iskakov \textit{et al} [64] report numerical simulations exploring the small-scale dynamo from \( Pm \sim 0.017 \) up to \( Pm = 1 \). In this regime, they find that even
for constant values of $Rm$, the growth rate decreases with decreasing $Pm$. In particular, for $Pm \sim 1$ and $Rm \sim 830$, they report a normalized growth rate of 1.8, which decreases to 0.9 for $Pm \sim 0.2$ and the same magnetic Reynolds number. The simulations further indicate that the value of $Rm_c$ settles to a constant limit for $Pm \ll 1$, $Re \gg 1$ and $Rm \gg 1$, even though this case is hard to numerically explore. The scaling of the growth rate on $Rm$, on the other hand, has not been conclusively explored.

All in all, simulations thus show that the growth rate depends on the magnetic Prandtl number even in the range $Pm \lesssim 1$. Another quantity that was shown to influence the dynamo is the Mach number of the gas. Haugen et al [32] explored Mach numbers in the range of 0.1–2.1 and reported a clear dependence of the critical magnetic Reynolds number for dynamo action on the Mach number $M$. For $Pm \sim 5$, they report $Rm_c \sim 25\pi$ for $M < 1$ and a rapid increase to $Rm_c \sim 45\pi$ for $M > 1$. Similar behavior was found for $Pm \sim 1$, with critical values of $\sim 40\pi$ and $\sim 80\pi$, respectively.

A larger series of simulations has been reported by Federrath et al [36], exploring Mach numbers from 0.02 up to 20, with compressive and solenoidal forcing, respectively. They performed a fit to the growth rate and saturation levels as a function of Mach number, using the function

$$f(M) = \left( p_0 \frac{M^{p_1}}{M^{p_3} + p_4} + p_5 \right) M^{p_6}.$$  \hspace{1cm} (20)

The fit coefficients for the different cases are given in table 1, and the normalized growth rates are reported in figure 1. In the subsonic regime, their results indicate that the growth rate (normalized by the eddy-turnover time $T_{ed}$ on the forcing scale) strongly decreases with decreasing Mach number for compressive driving, while it is almost constant at solenoidal driving. At $M > 1$, there is an initial drop due to the appearance of shocks, but increases as $M^{1/3}$ at larger values. A similar dependence is reported on the saturation level, which is particularly high for solenoidal driving, and decreases in the regime of large, supersonic Mach numbers.

We thus summarize the results of numerical simulations as follows:

- The critical magnetic Reynolds number for dynamo action as well as the resulting spectra for the magnetic field depend on the magnetic Prandtl number.
- Both the growth rates and the saturation levels of the dynamo depend significantly on the turbulent Mach number and the type of forcing that is employed.

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### Table 1. Fit coefficients reported by Federrath et al [36].

| $p_i$ | $\Gamma_{sol}[T_{ed}^{-1}]$ | $\Gamma_{comp}[T_{ed}^{-1}]$ | $(E_m/E_k)_{sol}$ | $(E_m/E_k)_{comp}$ |
|---|---|---|---|---|
| $p_0$ | -18.71 | 2.251 | 0.020 | 0.037 |
| $p_1$ | 0.051 | 0.119 | 2.340 | 1.982 |
| $p_2$ | -1.059 | -0.802 | 23.33 | -0.027 |
| $p_3$ | 2.921 | 25.53 | 2.340 | 3.601 |
| $p_4$ | 1.350 | 1.686 | 1 | 0.395 |
| $p_5$ | 0.313 | 0.139 | 0 | 0.003 |
| $p_6$ | 1/3 | 1/3 | 0 | 0 |
3. Non-universality in the nonlinear regime

The exponential growth phase will come to an end when the tension force of the magnetic field, $\vec{B} \cdot \nabla \vec{B}$, becomes comparable to the inertial term of the flow, $\vec{u} \cdot \nabla \vec{u}$. At this point, magnetic field amplification will stop on the scales that fulfill this condition, and continue to proceed on larger scales. As discussed by Schekochihin et al [22], this condition translates to

$$\frac{B_{\ell_a}^2}{\ell_a} \sim \frac{u_{\ell_a}^2}{\ell_a},$$

where $\ell_a$ denotes the smallest scale where amplification still occurs. In this regime, a linear growth of the magnetic energy has been reported in previous studies, based on the assumption of Kolmogorov turbulence (e.g. [22–24]). In the following, we will generalize these investigations by employing a simplified toy model as well as a more sophisticated Fokker–Planck model previously suggested by Schekochihin et al [22]. As a result, we will show that different types of power-law growth can be expected depending on the adopted type of turbulence.

We further point out that, in the nonlinear regime, we expect the magnetic Prandtl number to play a less critical role, as the amplification scale of the magnetic field is now expected to be larger than both the viscous and the resistive scale, such that no strong dependence on $Re$ or $Rm$ can be expected.

We note that the models considered in this section have previously been motivated in the context of the incompressible induction equation, given as

$$\partial_t \vec{B} + (\vec{v} \cdot \nabla) \vec{B} = (\vec{B} \cdot \nabla)\vec{v} + \eta \Delta \vec{B}.$$  

However, they can be naturally extended into the compressible regime with the replacement

$$\vec{B} \rightarrow \frac{\vec{B}}{\rho}.$$
Inserting this replacement as well as the continuity equation
\[ \dot{\rho} = -\nabla \cdot (\rho \vec{v}), \]  
(24)
it is straightforward to show that one obtains the compressible form of the induction equation
\[ \partial_t \vec{B} + (\vec{v} \cdot \nabla) \vec{B} = (\vec{B} \cdot \nabla) \vec{v} - \vec{B}(\nabla \cdot \vec{v}) + \eta \Delta \vec{B}, \]
(25)
equivalent to equation (1). As long as the mean density \( \langle \rho \rangle \) in the box is constant, a significant growth of the quantity \( \langle B/\rho \rangle \) nevertheless implies a corresponding growth of the magnetic energy, assuming that the density distribution function will not change significantly over time. In the case of well-developed driven turbulence, one indeed expects a characteristic log-normal density probability distribution function, which naturally complies with these requirements [65–67]. Strictly speaking, the following considerations apply to the quantity \( \vec{B} = B/\rho \) and \( \vec{W} = W/\rho^2 \), with \( W \) the magnetic energy. In the following, the \( \sim \) is, however, dropped for simplicity.

3.1. First considerations based on a toy model

In the toy model previously proposed by Schekochihin et al [22], the dominant fraction of the magnetic energy resides on the scale \( \ell_a \), the smallest scale where magnetic field amplification still occurs (thus yielding the shortest amplification timescale). On that scale, the magnetic energy is expected to be already close to saturation. The magnetic energy \( W(t) \) can thus be related to the amplification scale \( \ell_a \) by the approximate relation
\[ W(t) \sim \frac{1}{2} \langle \rho \rangle u_{\ell_a(t)}^2. \]
(26)
The magnetic energy is evaluated here at the mean density \( \langle \rho \rangle \) of the turbulent box, as we are interested only in the magnetic field amplification by shear. Adopting the eddy-turnover rate on the scale \( \ell_a \) as the growth rate for the magnetic field, i.e.
\[ \Gamma(t) \sim \frac{u_{\ell_a(t)}}{\ell_a(t)}, \]
(27)
the magnetic energy evolves as
\[ \frac{d}{dt} W = \Gamma(t) W(t) - 2\eta k_{\text{rms}}^2 W(t) \]
(28)
with
\[ k_{\text{rms}}^2(t) = \frac{1}{W} \int_0^\infty dk k^2 M(t, k) \]
(29)
and
\[ M(t, k) = \frac{1}{2} \int d\Omega_k |\vec{B}(t, \vec{k})|^2. \]
(30)
Now, we have \( \Gamma(t) W(t) \sim \langle \rho \rangle u_{\ell_a(t)}^3/\ell_a(t) =: \epsilon(t) \). Inserting into equation (28) yields
\[ \frac{d}{dt} W = \chi \epsilon(t) - 2\eta k_{\text{rms}}^2(t) W(t), \]
(31)
where \( \chi \) is a constant of the order of unity. For Kolmogorov turbulence, the quantity \( \epsilon(t) = \langle \rho \rangle u_{\ell_a(t)}^3/\ell_a(t) \) is a constant [17]. In this case, and as long as magnetic energy dissipation is negligible, \( dW/dt \) = const, implying a phase of linear growth. In this limit, we obtain the result
of Beresnyak [24], where a constant fraction of the turbulence dissipation rate is converted into magnetic energy.

In the general case with \( u_\ell a \propto \ell \theta a \), \( \epsilon(t) \) is however not constant, but varies as \( \ell^{3\theta-1} \). In the case of Burgers turbulence, we thus obtain \( \epsilon \propto \ell^{0.5} \). In this case, the growth of the magnetic energy is no longer linear, as the turbulent energy dissipation rate is not independent of scale!

For comparison, we note that the quantity \( \tilde{\epsilon} = \rho e_{\text{SGS}}/\ell \), with \( e_{\text{SGS}} \) the specific energy density of subgrid-scale turbulence, is practically independent of \( \ell \). It, however, has a weak dependence on the Mach number, and a strong dependence on the type of forcing [68]. As the density fluctuations will, however, not contribute to the shearing, we will adopt \( \epsilon \) as the quantity of interest here.

To quantify the expected behavior, we need to solve equation (26) for \( \ell_\alpha \). For this purpose, we recall that \( u_\ell a \) is related to the turbulence driving scale \( L \) and the velocity \( V \) on that scale via

\[
\frac{u_\ell a}{L} = \frac{V}{(L/\rho)^{1/2}} \quad (32)
\]

From (26), we thus obtain

\[
\ell_\alpha = L \left( \frac{2W}{\langle \rho \rangle V^2} \right)^{1/(2\theta)} \quad (33)
\]

We can now evaluate (27) and (28), yielding

\[
\frac{dW}{dt} \sim W V L^{-\theta} \left( \frac{2W}{\langle \rho \rangle V^2} \right)^{1/(2\theta)} \propto W^{1+(\theta-1)/(2\theta)} \quad (34)
\]

For Kolmogorov turbulence (\( \theta = 1/3 \)), we confirm that \( dW/dt = \text{const} \), while in the more general case, this quantity will increase with increasing \( W \). This can be intuitively understood, as the steep spectra for \( \theta > 1/3 \) imply a more modest increase of the eddy timescale with length scale, suggesting that the amplification rate remains larger with increasing the scale. We re-assess these results with the Fokker–Planck model below and explore the physical implications in more detail.

### 3.2. Implications of the Fokker–Planck model for universality

The starting point for our investigations is the Fokker–Planck model of Schekochihin et al [22]. Here, the time evolution of the magnetic-energy spectrum is given as

\[
\partial_t M = \frac{\partial}{\partial k} \left[ D(k) \frac{\partial M}{\partial k} - V(k)M \right] + 2\Gamma(t)M - 2\eta k^2 M, \quad (35)
\]

with the diffusion coefficient \( D(k) = \Gamma(t)k^2/5 \) and the drift velocity in \( k \)-space \( V(k) = 4\Gamma(t)k/5 \). We recall that the magnetic-energy spectrum \( M \) is related to the magnetic energy \( W \) via

\[
W(t) = \int_0^\infty dk M(t, k). \quad (36)
\]

To describe the evolution in the nonlinear regime, Schekochihin et al [22] postulated the following expressions:

\[
\Gamma(t) = c_1 \left[ \int_0^{k_1(t)} dk k^2 E(k) \right]^{1/2}, \quad (37)
\]
\[ W(t) = c_2 \int_{k_s(t)}^{\infty} dk E(k). \]  
(38)

The constants \( c_1 \) and \( c_2 \) are of the order of unity, \( E(k) \) is the hydrodynamic energy spectrum neglecting the influence of the magnetic field and the wave vector \( k_s(t) \) is defined via equation (38). It corresponds to the smallest scale where amplification efficiently occurs. As the input for the Fokker–Planck model, we require an energy spectrum of the turbulence. As before, we assume that the velocity in the inertial range scales as

\[ u_\ell \propto \ell^\theta. \]  
(39)

The hydrodynamic energy spectrum is then approximately given as

\[ E(k) = \begin{cases} C_t \epsilon^{2/3} k^{-2\theta - 1} & \text{for } k \in [k_f, k_v], \\ 0 & \text{elsewhere,} \end{cases} \]  
(40)

with \( C_t \) a constant which depends on the type of turbulence, and \( k_f \) and \( k_v \) the wave vectors describing the injection scale of turbulence and the viscous scale, respectively. The value of \( k_v \) is set to enforce the condition \( \epsilon = 2\nu \int_0^\infty dk k^2 E(k) \). Unlike in (6) and (7), we do not explicitly model the turbulent spectra in the viscous regime, as these no longer contribute during the nonlinear stage. With these input data, equation (38) can be evaluated as

\[ W(t) = c_2 C_t \epsilon^{2/3} 2\theta \left[ k_s^{-2\theta} - k_v^{-2\theta} \right]. \]  
(41)

We further introduce the quantities

\[ W_0 = c_2 \int_0^\infty dk E(k) = \frac{c_2 C_t \epsilon^{2/3}}{2\theta} \left[ k_f^{-2\theta} - k_v^{-2\theta} \right], \]  
(42)

\[ W_v = \frac{c_2 C_t \epsilon^{2/3}}{2\theta} k_v^{-2\theta}. \]  
(43)

We note that in the above expressions, the integral \( \int_0^\infty dk \) corresponds to an integration from \( k_f \) to \( k_v \), as the turbulent energy is non-zero only in this regime (see (40)).

Using these definitions, the wave vectors \( k_v, k_s \) and \( k_f \) can be expressed as

\[ k_s = \left( \frac{2\theta}{c_2 C_t \epsilon^{2/3}} \right)^{-1/(2\theta)} \left[ W(t) + W_v \right]^{-1/(2\theta)}, \]  
(44)

\[ k_f = \left( \frac{2\theta}{c_2 C_t \epsilon^{2/3}} \right)^{-1/(2\theta)} \left[ W_0 + W_v \right]^{-1/(2\theta)}, \]  
(45)

\[ k_v = \left( \frac{2\theta}{c_2 C_t \epsilon^{2/3}} \right)^{-1/(2\theta)} W_v^{-1/(2\theta)}. \]  
(46)

Integrating equation (37) now yields the following:

\[ \Gamma(t) = c_1 \left[ \left( \frac{C_t \epsilon^{2/3}}{2 - 2\theta} \right) (k_s^{-2\theta}(t) - k_f^{-2\theta}) \right]^{1/2}. \]  
(47)
Substituting equations (44)–(46) into (47) yields the expression
\[
\Gamma(t) = c_1 \left( \frac{C_t \epsilon^{2/3}}{2 - 2\vartheta} \right)^{1/2} \left( \frac{c_2 C_t \epsilon^{2/3}}{2\vartheta} \right)^{1/2} \left[ (W(t) + W_\nu)^{1-\frac{1}{2\vartheta}} - (W_0 + W_\nu)^{1-\frac{1}{2\vartheta}} \right]^{1/2}.
\] (48)

Considering turbulence models between Kolmogorov and Burgers, we have \(1/3 \leq \vartheta \leq 1/2\). We further assume that \(W(t) \ll W_0\), implying that the magnetic field is far from saturation on the current amplification scale. In this case, we can neglect the second term in square brackets. As we focus here on the nonlinear regime, we can further neglect \(W_\nu\) compared to \(W(t)\), and obtain the expression
\[
\Gamma(t) = c_1 \left( \frac{C_t \epsilon^{2/3}}{2 - 2\vartheta} \right)^{1/2} \left( \frac{c_2 C_t \epsilon^{2/3}}{2\vartheta} \right)^{1/2} \left( \vartheta \right)^{1/(2\vartheta)} W^{(\vartheta-1)/(2\vartheta)}(t).
\] (49)

As in our toy model, the growth of the magnetic energy thus scales as
\[
\frac{d}{dt} W \sim W(t) \Gamma(t) \propto W^{1+(\vartheta-1)/(2\vartheta)}.
\] (50)

For Kolmogorov turbulence, the growth is thus linear, while it grows faster than linear for \(\vartheta > 1/3\). Integrating equation (50), we obtain
\[
W(t) = (\tilde{C} t)^{2\vartheta/(1-\vartheta)},
\] (51)
with
\[
\tilde{C} = c_1 \left( \frac{C_t \epsilon^{2/3}}{2 - 2\vartheta} \right)^{1/2} \left( \frac{c_2 C_t \epsilon^{2/3}}{2\vartheta} \right)^{(1-\vartheta)/(2\vartheta)} \left( \frac{1 - \vartheta}{2\vartheta} \right).
\] (52)

From this expression, we already see that the energy grows linearly in \(t\) for Kolmogorov, while it grows as \(t^2\) for Burgers turbulence. For a physical interpretation, the normalization in terms of the eddy-turnover time \(T_{ed}\) on the forcing scale is still required, which we perform below.

4. Physical implications

To explore the physical implications of the above-mentioned results, we now perform a normalization in terms of the eddy-turnover time \(T_{ed}\) on the forcing scale \(k_f^{-1}\). For this purpose, we note that the expression within the central brackets of equation (52) is identical to \(W_\nu k_v^{2\vartheta}\), and it is straightforward to show that
\[
W_\nu k_v^{2\vartheta} = W_0 k_v^{2\vartheta}.
\] (53)

If we normalize equation (51) in terms of \(T_{ed} \sim (k_f \sqrt{W_0})^{-1}\), we thus obtain
\[
W(t) = C \left( \frac{t}{T_{ed}} \right)^{2\vartheta/(1-\vartheta)},
\] (54)
with
\[
C = c_1 W_0^{(1-\vartheta)/(2\vartheta)} \left( \frac{c_2 (1 - \vartheta)}{2(1 - \vartheta)} \right)^{1/2} \left( \frac{1 - \vartheta}{2\vartheta} \right).
\]

Adopting a system of units with \(W_0 = 1\) and \(k_f = 1\), it is evident that \(E_f \sim 1\), \(v(k_f) \sim 1\) and thus \(T_{ed} \sim 1\). We also assume \(c_1 \sim 1\). From equation (40), we also expect \(\epsilon \sim 1\). In these units, our evolution equations simplify as
\[
C = \frac{1}{2} \left( \frac{1 - \vartheta}{\vartheta} \right).
\] (55)
Figure 2. The power-law growth of magnetic energy for different types of turbulence in the nonlinear regime, following the evolution equation (55).

Table 2. The power-law behavior of the small-scale dynamo for different types of turbulence in the nonlinear regime.

| The model and reference                          | $\theta$ | $W \propto \ell_a \propto$ |
|-------------------------------------------------|----------|-----------------------------|
| Kolmogorov [17]                                 | $1/3$    | $t^{1/3}$ $t^{1/2}$        |
| Intermittency of Kolmogorov turbulence [69]     | $0.35$   | $t^{1.077}$ $t^{1.54}$    |
| Driven supersonic MHD turbulence [27]          | $0.37$   | $t^{1.17}$ $t^{1.59}$     |
| Observation in molecular clouds [25]           | $0.38$   | $t^{1.23}$ $t^{1.61}$     |
| Solenoidal forcing of turbulence [30]           | $0.43$   | $t^{1.51}$ $t^{1.75}$     |
| Compressive forcing of turbulence [30]          | $0.47$   | $t^{1.77}$ $t^{1.89}$     |
| Observation in molecular clouds [26]           | $0.47$   | $t^{1.77}$ $t^{1.89}$     |
| Burgers turbulence [31]                         | $1/2$    | $t^{2}$ $t^{2}$           |

We illustrate the behavior for the different types of turbulence in figure 2 for $Re = 10^4$, and summarize the power-law behavior in table 2. We thus obtain a steeper power-law growth for steeper turbulent spectra, implying that saturation can be reached in approximately the same time, in spite of the initially lower saturation level on smaller scales. The latter is fully consistent with our expectations for the kinematic regime, where the growth rates are higher for Kolmogorov turbulence, and a larger amount of magnetic energy may build up before the nonlinear regime is reached (due to the increased amount of turbulent energy that is available on the same scale). We note that in the final stage close to saturation, the evolution may start to deviate from the power-law behavior reported here, providing a transition to the regime where $W(t) = \text{const}$.

From the relation derived above, we further calculate the characteristic scaling of the current amplification scale $\ell_s$ as a function of time $t$. Adopting equation (26), we have $W(t) \sim \langle \rho \rangle u_{a(t)}^2 \propto \ell_a^{2\theta}$, thus

$$\ell_a(t) \propto W^{1/(2\theta)}(t) \propto t^{1/(1-\theta)}.$$  \hspace{1cm} (56)
For Kolmogorov turbulence, the characteristic length scale of the magnetic field thus grows as $t^{3/2}$, while it grows as $t^2$ for Burgers turbulence. The results are summarized for all types of turbulence in table 2.

The power laws derived here depend on the type of turbulence due to the different eddy-turnover timescales as a function of scale, as we sketch in figure 3. We summarize the main ingredients based on the toy model developed in section 3.1.

Considering a driving scale $L$ with a turbulent velocity $V$ on that scale, the ratio of the eddy-turnover times on scale $l \ll L$ for Kolmogorov and Burgers turbulence is given as

$$\frac{t_K}{t_B} = \left(\frac{\ell}{L}\right)^{1-1/3} \left(\frac{\ell}{L}\right)^{1-1/2} = \left(\frac{L}{l}\right)^{1/6}.$$  \hspace{1cm} (57)

During the growth of the magnetic energy, the relevant length scale, however, shifts to larger scales. According to equation (57), the ratio of the eddy timescales approaches unity for $\ell \to L$. For Burgers turbulence, the magnetic field amplification is thus initially delayed with respect to Kolmogorov, and catches up later, resulting in the nonlinear behavior and the power-law growth described here.

Due to these results, it is clear that the growth rate of the dynamo is not a fixed fraction of the global turbulence dissipation rate, as previously proposed by Beresnyak [24]. Due to the dependence on the turbulent spectrum, such a consideration may only hold locally, i.e. on a given scale, where the growth rate of the field is indeed related to the local eddy timescale. From a more global perspective, however, the turbulence dissipation rate changes as a function of scale for models different from Kolmogorov, such that the previously postulated universal behavior cannot be expected. From equation (55), it is further evident that the evolution depends on the

\textbf{Figure 3.} A sketch of Kolmogorov versus Burgers turbulence. While the turbulent energy is considerably smaller for Burgers spectra ($\vartheta = 1/2$) on small scales, it approaches the values for Kolmogorov turbulence ($\vartheta = 1/3$) on larger scales. As a result, the magnetic energy grows faster than linear for Burgers turbulence, as the growth rates gradually approach the Kolmogorov values at later times.
Reynolds number of the gas, and that larger Reynolds numbers imply stronger magnetic fields at earlier times.

5. Discussion and conclusions

In this paper, we have explored both the kinematic regime of the small-scale dynamo, where an exponential growth of the magnetic energy is generally observed, and the nonlinear regime, where backreactions start occurring on small scales and shift the amplification scale of the magnetic field to larger scales.

In the kinematic regime, analytical studies based on the Kazantsev model suggest a fundamental dependence on the magnetic Prandtl number. In particular, for \( Pm \ll 1 \), the growth rate of the dynamo is a function of the magnetic Reynolds number \( Rm \), while for \( Pm \gg 1 \), it depends on the kinematic Reynolds number \( Re \). In addition, the amplification rates significantly depend on the adopted type of turbulence. For \( Pm \gg 1 \), it scales as \( Re^{1/2} \) for Kolmogorov turbulence and as \( Re^{1/3} \) for Burgers turbulence. The same scaling relations, with a different normalization, were found for \( Pm \ll 1 \), with the replacement \( Re \rightarrow Rm \).

Numerical simulations confirm the dependence on \( Pm \) also in the range \( Pm \sim 1 \), and find a strong dependence of the growth rate and the saturation level on the turbulent Mach number \( M \) and the type of turbulence forcing. Magnetic field amplification is particularly efficient for solenoidal forcing and low Mach numbers, but also occurs for high Mach numbers and solenoidal/compressive forcing. If the Mach numbers are very small, compressive forcing is hardly able to trigger magnetic field amplification, as the presence of density gradients is required for the production of solenoidal turbulence in this case.

To investigate the nonlinear regime of the dynamo, we employed the Fokker–Planck model of Schekochihin et al. [22] and explored the effect of different turbulent spectra on the magnetic field amplification rate. We find that the previously known linear growth only occurs for Kolmogorov turbulence, while in the general case with \( u_1 \propto \ell^\vartheta \), we expect the magnetic energy to scale as \( t^{2\vartheta/(1-\vartheta)} \). The energy growth is thus faster than linear, and may even become quadratic for Burgers turbulence (\( \vartheta = 1/2 \)). However, we note that the growth rate is initially smaller for Burgers turbulence, as the turbulent energy available for amplification is initially much smaller on small scales. While magnetic field amplification is shifted to larger scales, the difference in the turbulent energy decreases, implying the reported power-law behavior as a function of time.

We have further shown that also the scaling of the characteristic length scale \( \ell_a \) for magnetic field amplification depends on the turbulent slope. Specifically, we find a scaling as \( t^{1/(1-\vartheta)} \), corresponding to \( t^{3/2} \) for Kolmogorov and \( t^2 \) for Burgers turbulence. The change of length scales proceeds thus in a fashion analogous to the inverse cascade in the case of helicity (e.g. [70, 71]). The evolution of this quantity may thus provide another relevant diagnostic for a comparison with numerical simulations.

Due to the above considerations, we point out that the nonlinear stage of the small-scale dynamo does not generally correspond to converting a fixed fraction of the turbulence dissipation rate into magnetic energy, as previously suggested by Beresnyak [24]. While their results agree with our model for the case of Kolmogorov turbulence (low Mach numbers), steeper power laws may occur in the highly compressible regime. Universality in the sense of a uniform behavior under all the conditions can thus not be expected. Nevertheless, we note that there are still universal laws governing the behavior of the dynamo, which relate the growth of
the magnetic energy to the eddy-turnover time on the current amplification scale. This quantity, in general, does depend on the Mach number and the type of turbulence involved, such that the breaking of universality is a result of the properties of different environments. We propose to explore these effects in more detail with numerical simulations in the near future, in order to improve our understanding of such non-universal behavior.

Acknowledgments

We thank Robi Banerjee and Ralf Klessen for stimulating discussions on the topic. DRGS, JS and SB acknowledge funding from the Deutsche Forschungsgemeinschaft (DFG) in the Schwerpunktprogramm SPP 1573 ‘Physics of the Interstellar Medium’ under grant numbers KL 1358/14-1 and SCHL 1964/1-1. DRGS and WS acknowledge funding via the SFB 963/1 on ‘Astrophysical flow instabilities and turbulence’, projects A12 and A15. JS acknowledges support from the IMPRS HD, the HGSFP and the SFB 881 ‘The Milky Way System’. CF acknowledges funding provided by the Australian Research Council under the Discovery Projects scheme (grant number DP110102191). We acknowledge support by the German Research Foundation and the Open Access Publication Funds of Göttingen University. We thank the anonymous referees for valuable suggestions that helped to improve the manuscript.

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