On walk-regular graphs and graphs with symmetric hitting times

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Abstract

Aldous [1] asked whether every graph in which the distribution of the return time of random is independent of the starting vertex must be transitive. We remark that this question can be reduced into a purely graph-theoretic one that had already been answered Godsil & McKay [6] and ask some questions motivated by this.

1 Walk-regular graphs

Aldous [1] posed the following problem

Problem 1.1 ([1]). If a graph $G$ satisfies

$$Pr_x(Z(n) = x) = Pr_y(Z(n) = y) \text{ for every } x, y \in V(G),$$

is $G$ necessarily vertex-transitive?

Here, $Pr_x(Z(n) = x)$ denotes the probability that simple random walk on $G$ started at $x$ will be at its starting point $x$ after $n$ steps.

A graph satisfying condition (1) is necessarily regular, for that condition implies that the expected return time to a vertex $x$ is independent of $x$, and it is known that this time equals $2m/d(x)$ [2].

As observed by Aldous [1], condition (1) also implies that for any two vertices $x, y$, if $X$ is the (random) time it takes for random walk from $x$ to visit $y$ and, conversely, $Y$ is the (random) time it takes for random walk from $y$ to visit $x$, then $X$ and $Y$ have the same distribution. To see this, note that if $d(x) = d(y)$ and $P$ is an $x$-$y$ walk whose interior does not visit $x$ or $y$, then the probability that random walk from $x$ will traverse $P$ in its first $|P|$ steps equals the probability that random walk from $y$ will traverse $P$ in the converse direction in its first $|P|$ steps, see [5].

This implies a further proof of the fact that condition (1) implies regularity: note that for any two neighbours $x, y$, the probability of visiting the other in just one step is the reciprocal of the degree.

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The above remark also implies in particular that if a graph satisfies condition (1), then hitting times are symmetric — the hitting time $H_{xy}$ from $x$ to $y$ is the expected value of the variable $X$ defined above — that is, $H_{xy} = H_{yx}$ for every $x, y \in V(G)$.

In this note we show that Problem 1.1 can be reduced to a purely graph-theoretic question which has long been known to have a negative answer. This yields a negative answer to Problem 1.1. A graph $G$ is called walk-regular, if for every $n \in \mathbb{N}$, the number of closed walks in $G$ of length $n$ starting at a vertex $x$ is independent of the choice of $x$.

**Observation 1.1.** A graph is walk-regular if and only if it satisfies (1).

*Proof.* Applying the definition of walk-regular for $n = 1$ we see that every walk-regular graph is regular. Recall that any graph that satisfies (1) must be regular too. Now note that in a $k$-regular graph, given a closed walk $W$ of length $n$, the probability that the first $n$ steps of a random walk coincide with $W$ is $k^{-n}$, and the probability to return to the starting vertex $x$ after $n$ steps is that number multiplied by the number of closed walks of length $n$ starting at $x$.

This reduces Problem 1.1 to the question of whether every walk-regular graph is transitive. This is however not the case, as already observed by Godsil & McKay [6]: any distance-regular graph [3] is walk-regular, but not necessarily transitive; in fact, there are many distance-regular graphs that have a trivial automorphism group [4]; see also http://mathoverflow.net/questions/106589/is-every-distance-regular-graph-vertex-transitive. Godsil & McKay [6] have constructed a walk-regular graph that is neither transitive nor distance-regular.

## 2 Symmetry of hitting times

Let us call a graph $G$ reversible if hitting times are symmetric, that is, if $H_{xy} = H_{yx}$ holds for every $x, y \in V(G)$. The aforementioned discussion implies that every walk-regular graph is reversible [5]. This motivates the following

**Problem 2.1.** Is every reversible graph regular? If yes, is it even walk-regular?

I suspect that the answer is no.

It is shown in [5] that a graph is reversible if and only if the sum $R_d(v) := \sum_{w \in V(G)} d(w)r(v, w)$ is independent of the choice of the vertex $v$, where $r(v, w)$ is the effective resistance between $v$ and $w$ when $G$ is considered as an electrical network (with unit resistances). In this case, we can think of $R_d(G) = R_d(v)$ as an invariant of the graph. It is natural to consider the normalised version $R_\pi(G) := R_d(G)/2m$, where $m$ is the number of edges of $G$; note that $R_\pi(G)$ is the expected effective resistance between an arbitrary vertex and a randomly chosen vertex chosen by picking an edge uniformly at random and choosing each of its endvertices with probability a half. These numbers are always rational because $r(v, w)$ is the solution of a linear system with integer coefficients. This motivates the following problem

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1 There is an error in the printed version; see the authors’ website.
Problem 2.2. Which (rational) numbers appear as \( R_\pi(G) \) for some reversible graph \( G \)? Are they dense in the positive reals?

For the complete graphs \( K_n \), an easy calculation yields \( R_\pi(K_n) = \frac{2}{n^2} \). For the cycles \( C_n \), we have \( R_\pi(C_n) = \Theta(n) \). This shows that \( R_\pi(G) \) can be arbitrarily small or large. It would be interesting to have constructions that combine simple reversible graphs into more complicated ones.

The fact that random walk on expander graphs has desirable properties (e.g. rapid mixing) \cite{Lovasz} motivates

Problem 2.3. Construct reversible graphs that are good expanders.

References

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