Modelling the QCD Phase Transition with an Effective Lagrangian of Light and Massive Hadrons

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Abstract

The temperature dependence of quark and gluon condensates in QCD as precursor of the chiral and deconfining phase transition is modelled with a conformally extended non–linear $\sigma$–model including broken chiral and scale invariance. The model is further enlarged by including (free) heavier hadrons. Within this frame we then study the interplay of QCD scale breaking effects and heavier hadrons in chiral symmetry restoration.

1 Introduction

Quantum Chromodynamics (QCD) with light quarks possesses, at the classical level, the remarkable feature of chiral and conformal symmetry. At low energy, these symmetries are broken by the non–perturbative vacuum structure which is signalled by the appearance of nonvanishing quark and gluon condensates, $\langle 0|\bar{q}q|0 \rangle$ and $\langle 0|G^a_{\mu\nu}G^a_{\mu\nu}|0 \rangle$, respectively \cite{1}. As a result of spontaneous breakdown of chiral symmetry there arise light Goldstone bosons (pions) which are the relevant hadronic degrees of freedom of low–energy QCD. On the other hand, at higher temperature the non–perturbative vacuum structure partially

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disappears which manifests itself in a phase transition from hadronic matter with confined quarks and gluons to an interacting plasma of deconfined quarks and gluons. In particular, it is important to investigate the temperature behaviour of the quark condensate in order to determine the critical temperature $T_c$, at which the chiral symmetry becomes restored. This question has been extensively studied in lattice QCD [2] from first principles. A natural question in the context of particular models is what influence the existence of heavier (non–Goldstone) mesons $\sigma, \rho, \omega, \ldots$ and baryons $N, \Delta, \ldots$ has on the actual chiral symmetry restoration temperature $T_c$. Finally, taking into account also the temperature–dependent gluon condensate, it is interesting to consider the interplay of temperature effects on the condensates and other thermodynamic quantities.

For the study of non–perturbative condensates we use in this paper the effective Lagrangian approach. Effective meson Lagrangians provide a compact and extremely useful method to summarize low–energy theorems of QCD [3]. They incorporate the broken global chiral and scale symmetries of QCD. In particular, to mimic the QCD scale anomaly [4]

$$\left\langle \Theta^{QCD}_{\mu\mu} \right\rangle = \left\langle \frac{\beta(g)}{2g} G^a_{\mu\nu} G^{a\mu\nu} \right\rangle,$$

one introduces a scalar chiral singlet dilaton–glueball field $\chi$ with an interaction potential $V(\chi)$ so that, via Noether’s theorem $\Theta_{\mu\nu} \propto \chi^4$ and (1) is satisfied for $\chi_0 = \langle 0 | \chi | 0 \rangle \neq 0$. As proposed in [6, 7] the kinetic and symmetry breaking mass terms of standard chiral meson Lagrangians have then to be multiplied by suitable powers of $\chi/\chi_0$ in order to reproduce the scaling behaviour of the analogous terms of the QCD Langrangian. These ideas have incited a number of investigations of the chiral and scaling behaviour of low–energy Lagrangians at finite temperature [8, 9, 10, 11, 12, 13].

Here, we will start with a non–linear (scaled) $O(4)$ $\sigma$–model which is isomorphic to a $SU(2) \times SU(2)$ chiral Lagrangian. The model contains one scalar ($\sigma$) and three pseudoscalar $\pi$ fields, constrained by the scaled condition $\sigma^2 + \pi^2 = f_\pi^2 \left( \frac{\chi}{\chi_0} \right)^2$ where $f_\pi$ is the pion decay constant. In the path integral evaluation of the thermodynamic potential this constraint can be rewritten by introducing a Lagrange multiplier field $\lambda(x)$. This allows to perform the remaining Gaussian integration over the pion fields exactly and to apply a saddle point approximation to the $\lambda$ integral. Finally, to model the temperature effects of heavier hadrons, we add the free Lagrangian of these particles. As discussed in [14] (without studying the interplay with the gluon condensate) their influence would lead to a lowering of $T_c$ by about 10 percent. Therefore, a generalization of the model which includes the effect of the gluon condensate on the low–lying hadronic states (with mass, say, $\leq 2$ GeV) is worthwhile. To couple these non–Goldstone particles to the remaining degrees of freedom in our model, we have to adopt a simple recipe defining the scaling properties of their masses with
In the spirit of Ref. [14], we use a simple-minded additive quark model to determine the chiral-symmetric and non-symmetric pieces of hadron masses. This leads to a definite scaling prescription for hadron masses $M_h \rightarrow M_h[\chi/\chi_0]$. Notice that the finite temperature glueball dynamics might influence the chiral symmetry restoration, too [3, 11]. In order to clarify this point and to compare the thermal effect of glueballs with that of the other heavy hadrons, we include quantum fluctuations of the glueball field. The effective masses $M_\chi$ and $M_{eff}$ of glueballs and pions are then obtained in a self-consistent way from saddle point equations derived for the thermodynamical potential (free energy) of the model.

The structure of the paper is as follows. In sect. 2 we introduce the effective low-energy meson Lagrangian with broken chiral and scale invariance discussing in some detail the links between chiral- and scale-symmetry breaking on the basis of the low-energy theorem for the gluon condensate in the presence of non-vanishing quark masses [15]. Moreover, the scaling behaviour of the masses, which is different for Goldstone and non-Goldstone particles, is set up. In sect. 3 we calculate the free energy density of the extended non-linear $\sigma$-model including pions, heavier hadrons and glueballs and determine the temperature dependent quark and gluon condensates $\langle\langle q\bar{q}\rangle\rangle_T \propto \sigma(T)$ (including the effect of heavier hadrons) and $\langle\langle G^2\rangle\rangle_T \propto \chi^4$ from the saddle point equations for the $\sigma$ and $\chi$ fields, respectively. Finally, sects. 4 and 5 contain the discussion of numerical results and the conclusions.

2 The effective hadron Lagrangian

2.1 Symmetry breaking and scale anomaly

In order to model the approximate chiral and scale invariance of QCD we consider an extended $O(4)$ non-linear $\sigma$-model containing one scalar $\sigma$- and three pseudoscalar $\pi$-fields as well as a scalar isoscalar dilaton (glueball) field $\chi$. The corresponding Lagrangian is given in Euclidean notation by

$$L(\sigma, \vec{\pi}, \chi) = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{1}{2}(\partial_\mu \chi)^2 + V(\chi) + V_{SB}(\sigma, \chi),$$

where the fields satisfy the rescaled chiral constraint

$$\sigma^2 + \vec{\pi}^2 = f_\pi^2 \left( \frac{\chi}{\chi_0} \right)^2,$$

with $f_\pi = 93$ MeV being the pion decay constant, and where $\chi_0 = \langle 0 | \chi | 0 \rangle$ has been introduced for dimensional reasons. This is the actual vacuum expectation value of the field $\chi$ which will slightly depend on the amount of explicit breaking of chiral symmetry.
In order to take into account the gluon contribution to the QCD scale anomaly we have included the dilaton interaction potential

\[ V(\chi) = K \chi^4 \left( \log \left( \frac{\chi}{\chi_q} \right) - \frac{1}{4} \right) \] (4)

which takes its minimum value at \( \chi = \chi_q \). Finally, \( V_{SB}(\sigma, \chi) \) is the scaled chiral symmetry breaking term of scaling dimension 3

\[ V_{SB}(\chi, \sigma) = -c \sigma \left( \frac{\chi}{\chi_0} \right)^2 \] (5)

to be added to \( V(\chi) \). It should be noted that due to the constraint (3) the field \( \sigma = \sigma(\chi, \bar{\theta}) \) is not an independent degree of freedom. For instance, omitting at tree level the pion field (giving rise to the pion loop contributions), we have \( \sigma(\chi, \bar{\theta}) = f_\pi \frac{\chi}{\chi_0} \). The total potential

\[ V_{tot}(\chi) = V(\chi) + V_{SB}(\chi, \sigma(\chi, \bar{\theta})) \] (6)

has then its minimum shifted to

\[ \chi_0 = \chi_q + \frac{3}{4} K \chi_q^3 c f_\pi + O(c^2). \] (7)

The parameters \( K, \chi_0 \) and \( c \) will be specified later by considering the vacuum energy density (bag constant), the glueball mass and the pion mass.

From eqs. (4,5,6) we get the trace anomaly

\[ \Theta_{\mu\mu}^{ef} = 4 V_{tot}(\chi) - \chi \frac{\partial V_{tot}(\chi)}{\partial \chi} \]
\[ = -K \chi^4 - c f_\pi \left( \frac{\chi}{\chi_0} \right)^3, \] (8)

and the vacuum energy density

\[ \epsilon_{vac} = \frac{1}{4} (\Theta_{\mu\mu}^{ef}) = V_{tot}(\chi_0) \]
\[ = -\frac{1}{4} K \chi_0^4 - \frac{1}{4} c f_\pi \]
\[ = -\frac{1}{4} K \chi_q^4 - c f_\pi + O(c^2), \] (9)

where we have used eq. (7) in the last line.

It is instructive to compare this with the QCD expression for \( N_f = 2 \)

\[ \Theta^{QCD}_{\mu\nu} = \frac{\beta}{2g} G^a_{\mu\nu} G^{a}_{\mu\nu} + m \left( \bar{u} u + \bar{d} d \right), \] (10)
where $\beta$ denotes the Gell–Mann–Low function,
\[
\frac{\beta}{g} = -b \frac{\alpha_s}{8 \pi} + \ldots , \quad b = \frac{11}{3} N_c - \frac{2}{3} N_f ,
\]
and where $m = m_u = m_d$ are the light current quark masses.

Taking into account the low-energy theorem for the gluon condensate in the
presence of non–vanishing quark masses [13]
\[
\langle \frac{\beta}{g} G^a_{\mu\nu} G^a_{\mu\nu} \rangle = \langle \frac{\beta}{g} G^a_{\mu\nu} G^a_{\mu\nu} \rangle |_{m=0} + 3 m \langle \bar{u}u + \bar{d}d \rangle \]
which relates the shift of the gluon condensate compared to the massless case
to the quark condensate, one obtains for the vacuum energy density the form
\[
\epsilon_{QCD}^{\text{vac}} = \frac{1}{4} \langle \Theta_{\mu\nu}^{QCD} \rangle = \langle \frac{\beta}{g} G^a_{\mu\nu} G^a_{\mu\nu} \rangle |_{m=0} + m \langle \bar{u}u + \bar{d}d \rangle .
\]
The comparison with eqs. (9) and (13) suggests the following identifications:
\[
\langle \frac{\beta}{g} G^a_{\mu\nu} G^a_{\mu\nu} \rangle |_{m=0} = -K \langle \chi^4 \rangle |_{c=0} = -K \chi^4_q
\]
and
\[
m \langle \bar{u}u + \bar{d}d \rangle = -c f_\pi = -M^2_\pi f^2_\pi .
\]
Eq. (15) is just the well–known Gell–Mann–Oakes–Renner relation. We have
already used that, discarding pion loops, $\langle \sigma \rangle = \sigma_0 = f_\pi$ due to the constraint (8).
Moreover, by expanding in eq. (6) the $\vec{\pi}$ dependent field $\sigma$
\[
\sigma = f_\pi \frac{\chi}{\chi_0} \sqrt{1 - \frac{\vec{\pi}^2}{f^2_\pi (\chi_0)^2}}
\]
to leading order in $\vec{\pi}^2$ one finds that the pion mass, given by $M_\pi = \sqrt{c/f_\pi}$,
scales as
\[
M_\pi \rightarrow M_\pi \sqrt{\frac{\chi}{\chi_0}}.
\]

2.2 The different scaling behaviour of masses

One of the main issues of the present work is to study the interplay of the quark
and gluon condensates at finite temperature (near the chiral phase transition)
and their relation to the hadron spectrum. In particular, we are interested in the
question how the light $u$ and $d$ quark condensate and the gluon condensate will
be affected by the inclusion both of additional massive hadrons $h = \rho, \omega, N, ...$
(represented by bosonic or fermionic fields $\varphi_h$) and a (lowest) glueball state. Note that for the inclusion of non–Goldstone particles we need to know also the scaling properties of the corresponding mass term. Using the relations

$$\langle H(k)|\Theta_{\mu\nu}|H(k)\rangle = 2 M_H^2,$$

$$\langle N(p)|\Theta_{\mu\nu}|N(p)\rangle = M_N \pi(p) u(p)$$

(for bosons $h = H$ and fermions $h = N$) and taking eq. (14) into account, we see that the masses $M_h$ contain chiral–symmetric and non–symmetric pieces. A more detailed investigation of this issue requires the use of low-energy theorems of QCD [15, 17] which is outside the scope of this paper. Here we shall follow a procedure which determines the hadron masses from a simple–minded additive quark model as follows

$$M_h = N_h^u \hat{m}_u + N_h^d \hat{m}_d$$

(20)

where $N_h^\alpha$ is the number of valence quarks (or antiquarks) of type $\alpha = u, \bar{u}, d$ and $\bar{d}$ with masses $\hat{m}_\alpha$ inside hadron $h$. $\hat{m}_\alpha$ denotes the total quark mass of quark type $\alpha$ which decomposes as

$$\hat{m}_\alpha = m_\alpha + m_{\alpha,dyn}$$

(21)

into current and dynamically generated mass. Let, for illustration, the dynamical part of the quark masses $m_{\alpha,dyn}$ be represented by a scale–invariant interaction term of the form

$$L_{int} = - g \sigma \vec{q}q \simeq - g f_\pi \frac{X}{\chi_0} \vec{q}q = - m_{dyn} \frac{X}{\chi_0} \vec{q}q$$

(22)

where use has been made of the approximate expression of eq. (16) (omitting the pion field) and of the Goldberger-Treiman relation $m_{dyn} = g f_\pi$. From eq. (22) we deduce the scaling behaviour of the dynamical quark mass

$$m_{dyn} \rightarrow \frac{X}{\chi_0} m_{dyn}.$$  

(23)

In the case of Goldstone bosons (pions) one has the mass formula

$$M_\pi^2 \propto m_{dyn}$$

(24)

which formally reproduces the scaling behaviour of eq. (17). For the non–Goldstone particles, on the contrary, we obtain from eqs. (20, 21, 23) the scaling prescription

$$M_h \rightarrow M_h \left[\frac{\chi}{\chi_0}\right]$$

(25)

with some well–defined functions $M_h$ of the ratio $\chi/\chi_0$ ($M_h$ should take for $\chi = \chi_0$ the known value $M_{h0}$). In order to include non–Goldstone hadrons

\footnote{For similarity and differences compare with Ref. [14].}
we shall simply use a free Lagrangian \( L_h(\phi_h, M_h[\chi/\chi_0]) \) for each particle type. This corresponds to the assumption that interaction terms with pions lead to higher order derivative terms in the effective Lagrangian of the pion sector. Such terms will be neglected in thermodynamical considerations at low temperatures studied here. The same holds for interactions between non–Goldstone particles because of their low thermal density below the critical temperature.

3 Chiral and conformal thermodynamics

3.1 The free energy density of the extended non–linear \( \sigma \)–model

In the following we shall study the thermodynamical properties of the system of pions, glueballs and free non–Goldstone hadrons described by the Lagrangian

\[
L = L(\sigma, \vec{\pi}, \chi) + \sum_h L_h(\phi_h, M_h[\chi/\chi_0]) .
\]

The partition function of the Lagrangian \( (26) \) reads

\[
Z = N \int_{\tau-i\infty}^{\tau+i\infty} D\lambda \int D\mu(\sigma, \vec{\pi}, \chi) \exp \left( -\int_0^\beta d\tau \int d^3x \left( \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} (\partial_\mu \vec{\pi})^2 
+ \lambda \left( \sigma^2 + \vec{\pi}^2 - f_\pi^2 \left( \frac{\chi}{\chi_0} \right)^2 \right) 
+ \frac{1}{2} (\partial_\mu \chi)^2 + V_{tot}(\sigma, \chi) 
+ \sum_h L_h(\phi_h, M_h[\chi/\chi_0]) \right) \right)
\]

where \( D\mu(\ldots) \) denotes the integration measure of the physical fields. \( \beta = 1/T \) is the inverse temperature, and we have rewritten the \( \delta \)-function constraint \( \delta(\sigma^2 + \vec{\pi}^2 - f_\pi^2 (\chi/\chi_0)^2) \) following from eq.(3) in terms of an integral over the auxiliary field \( \lambda(x) \). Periodic (antiperiodic) boundary conditions for boson (fermion) fields as well as the appropriate form of the particles’ Lagrangians \( L_h(\phi_h, M_h[\chi/\chi_0]) \) are understood. In the following we shall evaluate the path integral in eq. \( (27) \) in the saddle point approximation for \( \lambda \) and \( \sigma \) (putting \( \lambda(x) = \lambda = const, \sigma(x) = \sigma = const \) and dropping the corresponding integrations) but shall keep the quantum fluctuations of the glueball field \( \chi(x) = \chi + \tilde{\chi}(x) \) \(^3\). We expand

\(^3\)Note that the saddle point approximation in \( \lambda \) corresponds to the leading order in a \( 1/N \) expansion of the \( O(N) \) \( \sigma \)-model \(^1\).
the integrand up to terms $O(\chi^4)$. Performing first the Gaussian integration over the fields $\pi^\mu$ and $\varphi_h$ we obtain $(x = (\tau, x))$

\[
\begin{align*}
Z &= \mathcal{N}^\prime \int D\tilde{\chi} \exp \left( - \int^\beta_0 d\tau \int d^3x \left( \Phi[\lambda, \sigma, \chi] + \frac{1}{2} (\partial_\mu \chi)^2 \right) 
+ \tilde{\chi} \frac{\partial \Phi}{\partial \chi} + \frac{1}{2} \chi^2 \frac{\partial^2 \Phi}{\partial \chi^2} + \frac{1}{3!} \chi^3 \frac{\partial^3 \Phi}{\partial \chi^3} + \frac{1}{4!} \chi^4 \frac{\partial^4 \Phi}{\partial \chi^4} \right) \right) \quad (28)
\end{align*}
\]

with

\[
\Phi[\lambda, \sigma, \chi] = \lambda \left( \frac{f_\pi^2}{\chi_0^2} \right) + V_{tot}(\sigma, \chi)
\]

\[
+ \frac{3}{2} \left( \log \left( -\partial^2 + 2\lambda \right) \right)_{(x,x)}
\]

\[
+ \sum_{\text{mesons } h} g_h \left( \log \left( -\partial^2 + M_h[\chi/\chi_0]^2 \right) \right)_{(x,x)}
\]

\[
- \sum_{\text{baryons } h} g_h \left( \text{tr Dirac} \log \left( \hat{\partial} + M_h[\chi/\chi_0] \right) \right)_{(x,x)}.
\]

Here, $g_h$ are statistical degeneracy factors ($g_p = g_n = 4, g_\omega = 3, g_\rho = 9$ etc.). The first logarithmic term in eq.(29) leads just to the partition function of the free relativistic Bose gas of pions with an effective (temperature dependent) pion mass

\[
M^2_{eff} = 2\lambda.
\]

In order to get rid of the term linear in $\tilde{\chi}$ in eq.(28) let us choose $\chi$ to satisfy the following saddle point equation

\[
\frac{\partial \Phi}{\partial \chi} = -2\lambda f_\pi^2 \chi_0 \frac{\chi}{\chi_0} + \frac{\partial V_{tot}}{\partial \chi} \pm \frac{\partial}{\partial \chi} \sum_h g_h \text{tr} \log(\ldots M^2_h[\chi/\chi_0]) = 0. \quad (31)
\]

The last sum refers to bosonic (fermionic) hadrons and the arguments ... of the functional logarithms stand for their respective wave operators. Moreover, to simplify further calculations, we shall take into account the interactions of massive hadrons with the glueball field $\tilde{\chi}$ only up to first order, discarding (on the level of first order perturbation theory) also the term $O(\chi^3)$. In this approximation nucleons do not contribute to the coefficient of the $O(\chi^2)$ term due to their Yukawa–type coupling to the glueball field $\tilde{\chi}$, and the massive hadron contribution in the coefficient of the $O(\chi^4)$ term can also be discarded. On the other hand, the interaction term $\frac{1}{4} \chi^4 (\partial^2 V_{tot}(\sigma, \chi)/\partial \chi^4)$ leads, in the mean field approach, to a dynamical contribution to the glueball mass. To see this, it is convenient to rewrite the $O(\chi^4)$ term by introducing an auxiliary field
\( \rho \) giving rise to a Yukawa–type coupling \( \rho \tilde{\chi}^2 \). Indeed, applying a usual Gaussian integration we have

\[
\exp \left( - \int_0^\beta d\tau \int d^3x \frac{3}{4!} V_{tot}^{(4)} \tilde{\chi}^4 \right)
\]

\[
\propto \int D\rho \exp \left( - \int_0^\beta d\tau \int d^3x \left( - \frac{1}{2} \frac{1}{V_{tot}^{(4)}} \rho^2 + \frac{1}{2} \rho \tilde{\chi}^2 \right) \right)
\]

(32)

where the fourth derivative of the potential is given by

\[
V_{tot}^{(4)}(\chi) = 24 K \log \left( \frac{\chi}{\chi_0} \right) + 44 K,
\]

(33)

and the factor 3 in the l.h.s. of eq. (32) accounts for the three possible pairings to express \( \tilde{\chi}^4 \) as a square of \( \tilde{\chi}^2 \). Inserting the identity (32) into eq. (28), we read off an expression for the effective glueball mass

\[
M_\tilde{\chi}^2(\rho) = \frac{\partial^2}{\partial \tilde{\chi}^2} \left( - \lambda f_\pi^2 \left( \frac{\chi}{\chi_0} \right)^2 + V_{tot}(\sigma, \tilde{\chi}) \right)
\]

\[
+ \sum_h g_h \frac{2M_h^2}{\lambda_0} \Pi(M_h^2) \left( \chi/\chi_0 \right) \right) + \rho
\]

\[
= 12 K \tilde{\chi}^2 \log \left( \frac{\chi}{\chi_0} \right) + 4 K \tilde{\chi}^2 - \frac{2 \lambda f_\pi^2}{\chi_0^2} - \frac{2 c}{\chi_0^2}
\]

\[
+ \sum_h g_h \frac{2M_h^2}{\lambda_0} \Pi(M_h^2) \left( \chi/\chi_0 \right) \right) + \rho.
\]

(34)

Here, \( \Pi(M^2) \) is the expression for a closed meson loop given by the sum of \( T \neq 0 \) and \( T = 0 \) contributions

\[
\Pi(M^2) = \Pi_T(M^2) + \Pi_0(M^2),
\]

\[
\Pi_T(M^2) = \int \frac{d^3p}{(2\pi)^2} \frac{1}{E(p)} \exp(\beta E(p)) - 1,
\]

\[
\Pi_0(M^2) = \int \frac{d^3p}{(2\pi)^2} \frac{1}{2E(p)},
\]

(35)

with \( E(p) = (p^2 + M^2)^{1/2} \). Inserting eq.(32) into eq. (28) we can perform the integration over the glueball field \( \tilde{\chi} \). Thus, there remains a functional integral over the \( \rho \) field,

\[
\int D\rho \exp \left( - \int_0^\beta d\tau \int d^3x \left( - \frac{\rho^2}{2 V_{tot}^{(4)}} + \frac{1}{2} \left( \log \left( \frac{-\partial^2}{4 M_\tilde{\chi}^2(\rho)} \right) \right)_{(x,x)} \right) \right)
\]

(36)

\[\text{We use here } M_h^2 \approx 4m_{2n}^2 (\chi/\chi_0)^2 \approx M_{2n}^2 (\chi/\chi_0)^2 \text{ neglecting small terms } O(m), O(m^2) \text{ depending on the current mass.}\]
which will be evaluated at the saddle point solving the gap equation

$$
\rho_0 = \frac{V_{tot}^{(4)}}{2} \Pi(M_\chi^2(\rho_0)). \tag{37}
$$

Finally, we have

$$
Z = \exp(-\beta V_3 F_{eff}(\sigma, \lambda, \chi)) \tag{38}
$$

where $V_3$ is the 3D volume and the free energy density $F_{eff}$ is given by

$$
F_{eff}(\sigma, \lambda, \chi) = V_{tot}(\sigma, \chi) + \lambda \left( \sigma^2 - f_\pi^2 \left( \frac{\chi}{\chi_0} \right)^2 \right) + (continued)
+ F^\pi(M_{eff}^2) + F^\chi(M_\chi^2(\rho_0)) \tag{39}
- \frac{1}{8} V_{tot}^{(4)} (\Pi(M_\chi^2(\rho_0)))^2 + \sum_h F_h(M_h(\chi/\chi_0)).
$$

Here $F^\pi$, $F^\chi$ and $F^h$ are the expressions for the thermal determinants including the zero point energy

$$
F^\pi, F^\chi = F^{\pi,\chi}_T + F^{\pi,\chi}_0 \tag{40}
$$

with

$$
F^\pi_T(M_{eff}^2) = 3T \int \frac{d^3p}{(2\pi)^3} \log \left( 1 - e^{-\sqrt{p^2 + M_{eff}^2}/T} \right), \tag{41}
$$

$$
F^\chi_T(M_\chi^2) = -g_\chi T \int \frac{d^3p}{(2\pi)^3} \log \left( 1 - \sqrt{p^2 + M_\chi^2}/T \right), \tag{42}
$$

$$
F^h_T(M_h) = \pm g_h T \int \frac{d^3p}{(2\pi)^3} \log \left( 1 - \sqrt{p^2 + M_h^2}/T \right)
\approx -g_h \frac{T^2}{2\pi} M_h^2 e^{-M_h/T}. \tag{43}
$$

The upper (lower) signs in eq. (43) correspond to bosons (fermions) with the masses $M_h$ substituted by $M_h(\chi/\chi_0)$ depending on $\chi$. Finally, $F^\pi_0$, $F^\chi_0$ and $F^h_0$ denote the zero temperature contributions to the free energy density of the pion, glueball and hadron fields,

$$
F^\pi_0(M_{eff}^2) = 3 \int^{\Lambda_\pi} d^3p \frac{1}{2(2\pi)^2} \sqrt{p^2 + M_{eff}^2}, \tag{44}
$$

$$
F^\chi_0(M_\chi^2) = \int^{\Lambda_\chi} d^3p \frac{1}{2(2\pi)^2} \sqrt{p^2 + M_\chi^2}, \tag{45}
$$

$$
F^h_0(M_h) = \pm g_h \int^{\Lambda_h} d^3p \frac{1}{2(2\pi)^2} \sqrt{p^2 + M_h^2}. \tag{46}
$$

\textsuperscript{5}Note that $\Pi(M^2) = 2\frac{\beta F}{\alpha(M^2)}$.  

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Also here the masses \( M_h \) are understood depending on \( \chi/\chi_0 \). Note that, due to the necessary momentum cut–off, a naive regularization (with fixed cut–off) would violate the scaling properties of our effective theory which is constructed to substitute QCD. In order to keep the wanted scaling behaviour, we have introduced a rescaled cut–off \( \Lambda = \chi_0 \Lambda \) following the analogous argumentation in the case of the Nambu–Jona–Lasinio model [10, 12]. Moreover, to avoid additional parameters, we specify the cut–off in the spirit of effective chiral theories by \( \Lambda = 4 \pi f_\pi = 1.2 \text{ GeV for all fields} \).

Following Ref.[8], we now adopt a suitable subtraction procedure which guarantees the following two renormalization conditions at \( T = 0 \),

\[
\begin{align*}
M_{\text{eff}}^2(\lambda_0) & = 2 \lambda_0 = M_\pi^2 = \frac{c}{f_\pi}, \\
\langle \sigma \rangle & = \sigma_0 = f_\pi.
\end{align*}
\]

Here \( \lambda_0 \) and \( \sigma_0 \) are the known values of \( \lambda \) and \( \sigma \) which should extremize \( F_{\text{eff}} \) at \( T = 0 \). In addition, here we have to fulfill a third renormalization condition

\[
\langle \chi \rangle = \chi_0.
\]

For this purpose let us define a subtracted expression of the free energy density \( F_{\text{sub}}(\sigma, \lambda, \chi) \) which is obtained from \( F_{\text{eff}}(\sigma, \lambda, \chi) \) by replacing in the determinantal terms \( F_\pi \rightarrow F_\pi^{\text{sub}} \) with

\[
F_\pi^{\text{sub}}(M_{\text{eff}}^2) = F_\pi^T(M_{\text{eff}}^2) + F_0^\pi(M_{\text{eff}}^2) - F_0^\pi(M_\pi^2) - (M_{\text{eff}}^2 - M_\pi^2) \frac{\partial F_0^\pi}{\partial (M_{\text{eff}}^2)}|_{T=0},
\]

and using analogous subtraction prescriptions for \( F_\chi, F_\theta \) and for the expressions \( \Pi = 2 \frac{\partial F}{\partial (M_\pi^2)} \). The subscript \( T = 0 \) means that the derivatives have to be evaluated at the physical zero temperature masses, i.e. at \( M_{\text{eff}} = M_\pi = 139 \text{ MeV} \) for the pions, for the non–Goldstone hadrons at \( M_h = M_{h0} \) as known from the Rosenfeld table, and at \( M_\chi = M_{\chi_0} \) being the mass of the lowest glueball state. Note that in eqs.(34) and (37) \( \Pi(M^2) \) has to be replaced by the subtracted expressions \( \Pi_{\text{sub}}(M^2) \).

Then, the thermal averages, abbreviated as \( \sigma(T) = \langle \langle \sigma \rangle \rangle_T, \lambda(T) = \langle \langle \lambda \rangle \rangle_T \) and \( \chi(T) = \langle \langle \chi \rangle \rangle_T \), are obtained from solving the following saddle point equations

\[
\frac{\partial F_{\text{sub}}}{\partial \sigma} = 0, \quad \frac{\partial F_{\text{sub}}}{\partial \lambda} = 0, \quad \frac{\partial F_{\text{sub}}}{\partial \chi} = 0,
\]

Strictly speaking, one has also to introduce subtraction terms \(- (\Lambda_\chi - \Lambda) \frac{\partial F_0^\pi}{\partial \chi}|_{M_0} \) (with zero temperature masses \( M_0 \)). Due to the weak dependence of \( \Lambda_\chi \) on \( T \) these terms turn out to be numerically very small. We neglect them putting \( \Lambda_\chi \rightarrow \Lambda \).
taking the derivatives at $\sigma = \sigma(T), \lambda = \lambda(T)$ and $\chi = \chi(T)$. Note that in eq.(51) we have replaced the "classical" value $\chi$ by a function $\chi(T)$ which satisfies a saddle point equation including now the contribution of the glueball loop. [1]

The first two equations take a simple form

$$2 \lambda(T) \sigma(T) = c \left( \frac{\chi(T)}{\chi_0} \right)^2 + \frac{2}{\chi_0} \frac{\partial F_{\text{sub}}}{\partial (M^2 \chi)} \right) \right) \right) \tag{52}$$

and

$$\sigma(T) = f_{\pi} \frac{\chi(T)}{\chi_0} \left( 1 - \frac{2}{f_{\pi}^2} \frac{\chi_0^2}{\chi(T)^2} \frac{\partial F_{\text{sub}}}{\partial (M^2 \chi)} + \frac{2}{\chi(T)^2} \frac{\partial F_{\text{sub}}}{\partial (M^2 \chi)} \right) \frac{1}{2}. \tag{53}$$

No massive (non–Goldstone) hadrons except the glueballs contribute here because their mass does not depend on $\lambda$. Clearly, eq.(53) is just the thermal average of the constraint (3) evaluated in the saddle point approximation. Notice also that the glueball loop contributes here with a different sign than the pion loop.

It is instructive to consider for a moment the chiral limit $c = 0$. Then eq.(52) admits the two possibilities,

$$\text{case (i)} \quad \lambda(T) = 0 , \quad \sigma(T) \neq 0 \tag{54}$$

and

$$\text{case (ii)} \quad \lambda(T) \neq 0 , \quad \sigma(T) = 0 . \tag{55}$$

Clearly, case (i) describes spontaneous breakdown of chiral symmetry with a nonvanishing order parameter $\sigma(T)$. One expects that this holds for temperatures $T < T_c$, where $T_c$ is the critical temperature, with vanishing $\lambda(T)$ reflecting the existence of a massless Goldstone pion. On the contrary, case (ii) should hold for $T > T_c$ where chiral symmetry is restored and pions become massive. The situation should be qualitatively similar also for $c \neq 0$.

Inserting the solutions $\sigma(T), \lambda(T)$ and $\chi(T)$ into $F_{\text{sub}}$ yields the actual free energy density as a function of temperature

$$F(T) = F_{\text{sub}}(\sigma(T), \lambda(T), \chi(T)) . \tag{56}$$

By definition, the subtracted quantity $F(T = 0)$ is just the vacuum energy density $\mathcal{E}$, associated to the gluon and quark condensates through eq.(13). Indeed, using $\chi(T = 0) = \chi_0, \lambda(T = 0) = \lambda_0 = \sqrt{M^2_{\pi}/2}, \sigma(T = 0) = f_{\pi}$ and $c = f_{\pi}M^2_{\pi},$ we obtain

$$F(T = 0) = V_{\text{tot}}(f_{\pi}, \chi_0) \approx \frac{1}{4} K \chi_0^4 - \frac{1}{4} M^2_{\pi} f_{\pi}^2 + O(M^4_{\pi}) \tag{57}$$

For a justification of this procedure in terms of quantum (loop) corrections to the effective action we refer to Refs. [20],[11].
We will now determine the parameters \( K \) and \( \chi_0 (\chi_q) \) in eq. (57) taking the bag constant \( B \) and some lowest glueball mass \( M_{gb} \) at \( T = 0 \) as two reference scales \[11\]

\[
B = - \epsilon_{vac} = \frac{1}{4} K \chi_0^4 + \frac{1}{4} M_\pi^2 f_\pi^2
\]

and

\[
M_{gb}^2 = \frac{\partial^2 V_{tot}(f_\pi, \chi)}{\partial \chi^2} |_{\chi_0} - \frac{M_\pi^2 f_\pi^2}{\chi_0^2}
\]

\[
= 4 K \chi_0^2 + 6 \frac{M_\pi^2 f_\pi^2}{\chi_0^2} .
\]

For the bag constant we choose \( B^+ = 240 \) MeV, which is compatible with the gluon condensate at \( T = 0 \) extracted from QCD sum rules \[1\]. For the glueball mass we take \( M_\chi = 1.6 \) GeV, a value motivated by lattice results \[21\].

Experimental searches favour candidates in the mass region of 1.5...1.8 GeV \[22\].

### 3.2 Quark and gluon condensates at \( T \neq 0 \)

Let us next investigate the influence of the glueball field and of the heavier (non-Goldstone) nonstrange hadrons on the chiral condensate of \( u \) and \( d \) quarks. Primarily, the chiral condensate is obtained as the logarithmic derivative of the partition function with respect to the current quark mass \( m \),

\[
\langle \langle \bar{q} q \rangle \rangle_T = - \frac{T}{V} \frac{\partial \log Z}{\partial m} = \frac{\partial F(T)}{\partial m} .
\]

With eq. (13) we have

\[
\frac{\partial}{\partial m} = - \frac{\langle \bar{u} + \bar{d} d \rangle_0}{f_\pi} \frac{\partial}{\partial c} = - \frac{\langle \bar{u} + \bar{d} d \rangle_0}{f_\pi^2} \frac{\partial}{\partial M_\pi^2},
\]

such that

\[
\frac{\langle \langle \bar{q} q \rangle \rangle_T}{\langle \bar{q} q \rangle_0} = \frac{1}{f_\pi} \frac{\partial F(T)}{\partial c},
\]

where use has been made of the relation

\[
\frac{\partial F(T = 0)}{\partial c} = - f_\pi .
\]

Calculating \( \langle \langle \bar{q} q \rangle \rangle_T \) we have to take into account also the \( c- \)dependence of the glueball mass \( M_\chi^2 \) \[14\] and of the heavy hadron masses in eq. (25) via the current quark masses \( \{20\} \) and \( \{21\} \). From eqs. (62) and (53) we obtain

\[
\frac{\langle \langle \bar{q} q \rangle \rangle_T}{\langle \bar{q} q \rangle_0} = \frac{\sigma(T)}{f_\pi} \left( \frac{\chi(T)}{\chi_0} \right)^2 + \Delta ,
\]

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where $\Delta$ contains contributions from the glueball and the other massive hadron states

$$\Delta = \frac{2}{f_\pi} \frac{\sigma(T)}{\chi_0^2} \frac{\partial F_{\text{sub}}^\chi}{\partial (M_\chi^2)} - \frac{1}{f_\pi} \sum_h \frac{\partial M_h^2}{\partial c} \frac{\partial F_{\text{sub}}^h(M_h)}{\partial (M_h^2)} . \tag{65}$$

with $M_h = M_h[\chi(T)/\chi_0]$. Here we have used

$$\frac{\partial (M\chi^2 - \rho_0)}{\partial c} = -\frac{2}{\chi_0} \sigma \chi_0^2 , \tag{66}$$

and

$$\frac{\partial M_h^2[\chi(T)/\chi_0]}{\partial c} = 2 M_h[\chi(T)/\chi_0] \frac{\partial M_h[\chi(T)/\chi_0]}{\partial c} = - (N_u^h + N_d^h) \frac{2 M_h f_\pi}{\langle uu + dd \rangle_0} \frac{\chi(T)}{\chi_0} , \tag{67}$$

where $N_u^h$ and $N_d^h$ are the numbers of light valence $u$, $d$ quarks (or antiquarks) in the hadron $h$. We postpone the discussion of the last term $\Delta$ in eq.(64) and consider the first one at first. Using eq.(53) and discarding there the glueball loop term in the bracket as well we find

$$\frac{\langle \bar{q}q \rangle_T}{\langle \bar{q}q \rangle_0} = \left( \frac{\chi(T)}{\chi_0} \right)^3 \sqrt{1 - \frac{R}{f_\pi^2}} = \left( \frac{\chi(T)}{\chi_0} \right)^3 \left( 1 - \frac{R}{2 f_\pi^2} - \frac{R^2}{8 f_\pi^4} - ... \right) \tag{68}$$

with

$$R = \frac{\chi_0^2}{\chi(T)^2} \left( \frac{\partial F_{\text{sub}}^\chi}{\partial \lambda} |_{\lambda = \lambda(T)} \right) . \tag{69}$$

For the special case $c = 0$, when $M_{\text{eff}}(T) = M_\pi = 0$ (putting $\Lambda_\chi = \Lambda$), we get

$$R = \frac{\chi_0^2}{\chi(T)^2} \frac{3 T^2}{12} . \tag{70}$$

Notice that if we set $\chi(T)/\chi_0 \to 1$ we recover the structure of results obtained in chiral perturbation theory [14] with inclusion of one– and two–loop contributions [14]. In the case $c \neq 0$ our loop expressions contain a thermal pion mass $M_{\text{eff}}(T) = 2 \lambda(T)$ which has to be determined numerically in a selfconsistent way from eqs.(51, 52, 53).

\[ \text{The two–loop contribution } \propto R^2/f_\pi^4 \text{ in eq.(68) differs from the corresponding term in the } SU(2) \times SU(2) \text{ matrix formulation numerically. This deviation is due to next–leading order terms in the } 1/N \text{ expansion not considered here [2].} \]
Let us consider next at which critical temperature $T_c$ the order parameter $\langle \langle \overline{q} q \rangle \rangle_T$ vanishes such that chiral symmetry is restored. From eq. (68) we obtain, assuming $\chi(T_c) \neq 0$, that $T_c$ is determined by

$$R(T_c) = f^2_\pi. \quad (71)$$

Moreover, in the simplified case of massless quarks ($c = 0, m = 0$), we can use eq. (70) to derive the estimate ($\chi_0 = \chi_q$)

$$T_c = 2 f_\pi \frac{\chi(T_c)}{\chi_0} = 186 \frac{\chi(T_c)}{\chi_0} \text{ MeV}. \quad (72)$$

The factor $\chi(T_c)/\chi_0$ can be expressed by the gluon condensate. Indeed, taking the thermal average of eqs. (8, 10) we obtain for $m = 0, c = 0$

$$- K \langle \langle \chi^4 \rangle \rangle_T = \langle \langle \frac{\beta}{2 g} G^a_{\mu\nu} G^a_{\mu\nu} \rangle \rangle_T, \quad (73)$$

or

$$\frac{\langle \langle G^2 \rangle \rangle_T}{\langle G^2 \rangle_0} = \frac{\langle \langle \chi^4 \rangle \rangle_T}{\chi_0^4} \approx \left( \frac{\chi(T)}{\chi_0} \right)^4, \quad (74)$$

where $G^2 = G^a_{\mu\nu} G^a_{\mu\nu}$ and the approximation $\langle \langle \chi^4 \rangle \rangle_T \approx \langle \langle \chi \rangle \rangle_T^4 = \chi(T)^4$ has been made. From these considerations we expect

$$T_c = \left( \frac{\langle \langle G^2 \rangle \rangle_T}{\langle G^2 \rangle_0} \right)^{\frac{1}{4}} 186 \text{ MeV}. \quad (75)$$

As we shall find below, $\langle \langle G^2 \rangle \rangle_T/\langle G^2 \rangle_0 \leq 1$ in the region where $\langle \langle \overline{q} q \rangle \rangle_T \neq 0$. The gluon condensate depends only very weakly on $T$ in this region. Therefore, the estimate $2 f_\pi$ for the chiral transition temperature remains practically unchanged by it. Let us next consider the positive glueball term in eq. (53). Although it is small in comparison with the pion loop, its contribution is expected to be non-negligible in the region $T \sim 2 f_\pi$ where the first two terms in eq. (53) approximately cancel, shifting thus $T_c$ upwards. On the other hand, as was argued in Ref. [14] (without consideration of gluon condensation and glueballs), the inclusion of massive hadron states alone leads to a negative contribution to the quark condensate and gives rise to a lowering of $T_c$ by about 10 percent. In order to estimate the strength of this effect in our model we have to consider the additional terms in eq. (64) omitted until now. Notice first that the arising glueball contribution in eq. (65) is - in addition to the exponential suppression with the high mass - suppressed by the small factor $\sigma(T)$ compared to the contribution of other heavier hadrons and can thus be safely neglected. Clearly, the remaining negative hadron contributions will have a tendency of lowering $T_c$ and thus counteract its increase originating from the positive glueball term in eq. (53). The numerical investigation of the interplay of both effects will be done in Section 4.

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Finally, the temperature dependence of the gluon condensate \( \langle \bar{q}q \rangle \) is numerically determined from the third saddle point equation \( \partial F_{\text{sub}} / \partial \chi(T) = 0 \) in eq. (51) together with eqs. (52) and (53).

4 Discussion of numerical results

![Figure 1: Temperature dependence of the partial pressures \( p_i(T) \) for pions, for heavier mesons, for baryons and the contribution from the dilaton potential \( V(\chi) \).](image)

In this section we present the results of the solution of the saddle point equations. In Fig. 1 we show for sake of illustration, the temperature dependence of the thermal pressure(s) \( p_i(T) \) of pions, all heavy mesons and baryons as well as the tree (potential) term, for the chiral limit \( m \to 0 \). Since the \( \sigma \) model taken here describes only \( N_f = 2 \) light flavours, we have included only non–strange heavier hadrons.

In Fig. 2 the thermal average \( \sigma(T) \) is presented together with the quark condensate as function of temperature in the chiral limit. One finds that the gluon condensate (included to mimic the breaking of scale invariance in QCD) has only a negligible influence on the behaviour of \( \sigma(T) \) and on the quark condensate. The vanishing of \( \sigma(T) \) at some \( T_c \) is dominated by the intrinsic chiral dynamics of pions modified by an increase of \( T_c \) by about 10 MeV (6 percent) due to glueballs, but is not related to the small reduction of \( \chi(T) \) with rising temperature. This conclusion is in accord with Ref. [11] and does not confirm the corresponding conjecture of Ref. [7]. We see that heavier non–Goldstone...
Figure 2: Temperature dependence of the thermal average $\sigma(T)$ of our model (compared with the $\sigma$–model for pions only) and the quark condensate $\langle\langle \overline{q}q \rangle\rangle_T$ for the chiral limit using a bag constant $B^{1/4} = 240$ MeV. All quantities are normalized to their values at $T = 0$.

hadrons indeed influence the quark condensate in the region of $T \leq 186$ MeV due to the current quark mass dependence of the hadron masses. They yield an effective lowering by about 10 MeV of the increased value of $T_c$ obtained for the $\sigma$–model with pions and glueballs leading to a restoration of the value $T_c \approx 2f_\pi = 186$ MeV obtained for pions alone. These results are in qualitative agreement with the conclusions of Ref. [14] where pions and heavier hadrons have been included and contributions to the thermodynamical potential have been calculated up to three loop level for pions (while no interplay with the gluonic degrees of freedom has been considered).

Fig. 3 exhibits the same quantities for the case of finite quark masses $m = m_u = m_d \approx 7$ MeV (explicit chiral symmetry breaking) which are adjusted to give the pion a mass $M_\pi = 139$ MeV. Without QCD scale breaking effects taken into account $\sigma$ is now shifted upward throughout all temperatures and does not play anymore the role of an order parameter. With the coupling to the gluon condensate and, through the glueball mass, to the glueball loop, $\sigma$ even bends upward for temperature $T > 220$ MeV. This can be interpreted as a kind of stabilization of $\sigma$ by the gluonic degrees of freedom. A similar behaviour has been found in Ref. [12].

If we neglect the smaller glueball loop contributions we see that the temperature dependence of $\sigma(T)$ is directly related to the ratio $\chi(T)/M_{eff}(T)$ (see eq. (52). Finally, including the negative contributions of the other heavier

Figure 3: Same as Fig. 2 for a finite pion mass $M_\pi = 139$ MeV ($m = 7$ MeV) using the same bag constant.

Figure 4: Temperature dependence of $\sigma(T)$, the effective pion mass $M_{\text{eff}}(T)$, the gluon condensate $\langle\langle GG\rangle\rangle_T$ and the glueball mass $M_{\chi}(T)$ for the chiral limit using the same bag constant as before. The effective pion mass is normalized to $M_\pi = 139$ MeV, all other quantities to their respective values at $T = 0$.

hadrons, the total quark condensate is again strongly decreased and vanishes at
$T_c \approx 220$ MeV. In Fig. 4 we show the temperature dependence of the effective pion mass $M_{\text{eff}}(T)$, the gluon condensate $\langle G^2 \rangle|_{T} \propto \chi^4$ and the effective glueball mass $M_{\chi}(T)$ in the case of the chiral limit. All quantities are normalized to their respective values at $T = 0$, with the only exception of $M_{\text{eff}}$, which is always normalized as $M_{\text{eff}}/M_\pi$, whether $c \rightarrow 0$ or not. Notice that for $T > T_c$ there arises a non–vanishing pion mass from the $\sigma$–model, whose temperate behaviour is now influenced by the gluon condensate. The plot ends when $\lambda$ hits the constraint $\lambda \geq 0$.

Fig. 5 presents the same quantities as Fig. 4 in the case of finite quark and pion mass. Clearly, for temperatures $T \geq 180$ MeV the above discussion and the figures must be taken with caution with respect to real QCD. From the point of view of the effective hadronic Lagrangian one has to recall that in this temperature region higher order derivative terms have to be included into the action and higher loop contributions are expected to contribute. Therefore our results are at best qualitative ones in the temperature region $T > T_c$.

5 Conclusions

In this paper we tried to give a qualitative description of various important aspects of temperature–dependent QCD using an effective Lagrangian approach. Our analysis is based on a conformally extended non–linear $\sigma$–model which describes light pions coupled to the gluon condensate and, eventually, to glueball thermal loop effects. Most importantly, this model embodies explicit and dynamical breaking of chiral and conformal symmetry by construction. Heavier
non–Goldstone mesons and baryons have further been added in order to model the additional dependence of the condensates via heavy hadron masses.

The primary motivation was to investigate in this framework the mutual influence of quark and gluon condensates. However, if the parameters are fixed using information from zero temperature like the vacuum energy density and the glueball mass, we find that the gluon condensate is rather temperature independent in this physical frame up to temperature \( T \approx 200 \text{ MeV} \). At somewhat higher temperature, in the interval up to \( T = 250 \text{ MeV} \), it decreases to \( \approx 0.75 \) of the zero temperature value. This effect is accompanied by a (roughly inverse) increase of the glueball mass. If the bag constant is taken smaller \( (B = (140\text{MeV})^4) \) instead of \( (240\text{MeV})^4 \) this ”melting” and the stronger rise of the glueball mass with temperature set in at slightly lower temperature.

In the chiral limit, the temperature dependence of the \( \sigma \) field, the quark condensate etc. is almost independent of the gluon condensate, but slightly changed by the glueball loop contributions. It is strongly influenced by the explicit chiral symmetry breaking through the quark mass term. Let us first summarize the case of the chiral limit \( (m_u = m_d = M_\pi = 0) \). Discarding loop effects of glueballs and heavy hadrons, the critical temperature of chiral symmetry restoration turns out to be \( T_c = 2 \int_0^{\chi(T_c)} \frac{\chi(T_c)}{f_\pi} \approx 186 \text{ MeV} \), in view of \( \chi(T_c)/\chi_0 \approx 1 \). The inclusion of the glueball loop in \( \sigma(T) \) then increases this value of \( T_c \) by about 10 MeV. Next, heavy hadrons and dynamical glueballs contribute with different signs to the extra term \( \Delta \) \( (\text{cf. eq. (65)}) \) in the quark condensate. Due to the large glueball mass and the small value of the factor \( \sigma(T) \) near \( T_c \), the respective glueball term is here numerically too small to counteract the decrease of the quark condensate (and of \( T_c \)) caused by heavy hadrons. In accord with Ref. [14], we find a 6 percent (10 MeV) downward shift of \( T_c \) which just compensates the corresponding increase arising from \( \sigma(T) \) in our model. In particular, our results confirm the conclusion of Ref. [11] that in the chiral limit the vanishing of the quark condensate is, above all, dictated by the internal pion dynamics. This behaviour is only slightly corrected by glueballs and the heavier non-Goldstone mesons and baryons and has nothing to do with a decrease (not to mention vanishing !) of the gluon condensate. Thus, the conjecture of Ref. [7] is not supported by our analysis. Note also that at \( T > T_c \) the effective pion mass becomes non–zero, grows to a maximum and decreases to zero again. This behaviour is related to the beginning decrease of the gluon condensate in the temperature interval \( 200 \text{ MeV} < T_c < 250 \text{ MeV} \). However, beyond the temperature of chiral symmetry restoration the predictions of the \( \sigma \)–model (present here in its simplest version) have to be taken with caution and are at best qualitative ones.

In the more realistic case of soft explicit breaking of chiral symmetry by finite quark and pion masses, the curve of the pion loop contributions to the quark condensate (represented by the saddle point solution \( \sigma(T) \)) is shifted upwards and does not clearly define a phase transition. (This is the case even for the
pure \( \sigma \) model.) Discarding the small glueball loop contributions, this behaviour can be directly related to the behaviour of the ratio \( \chi(T)/M_{\text{eff}}(T) \) as can be seen from eq. (52).

Nevertheless, including also non-Goldstone hadrons, their contribution is sufficient to make the quark condensate vanish, however at somewhat higher temperature. This happens at \( T_c \approx 220 \) MeV. Again, we have to emphasize that this temperature region is beyond the region of applicability of the simple non-linear \( \sigma \)–model used here as a cornerstone of our extended model. From chiral perturbation theory [14 24] it is known that higher order derivative terms in the Lagrangian and higher loops cannot be neglected at these temperatures.

Summarizing, our model exhibits in the chiral limit, for temperatures \( T \leq 200 \) MeV, competing effects from glueballs (increase of \( T_c \)) and heavier hadrons (decrease of \( T_c \)) which, because of cancellation, have however only a minor influence on the chiral symmetry restoration. The gluon condensate shows also very weak temperature dependence in this temperature range. On the other hand, in the more realistic case of finite quark and pion masses, it is just the quark mass dependence of the heavier hadron masses which leads to a significant lowering of the quark condensate and, finally, to its vanishing.

Moreover, noticeable variations of the gluon condensate down to 75 percent of the zero temperature value are only seen in a temperature region \( 200 \text{MeV} < T_c < 250 \) MeV where the simple non-linear \( \sigma \)–model is not realistic anymore.

Modelling the implicitly temperature dependent heavy hadron masses by another dependence on the order parameters might lead to stronger and yet more interesting to analyse effects for the detailed temperature dependence of chiral symmetry restoration. Such models need dynamical input from hybrid models including quark degrees of freedom simultaneously with the Goldstone fields. These mechanisms will be the subject of future studies.

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