Stratified Monte Carlo Quadrature for Continuous Random Fields

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Abstract We consider the problem of numerical approximation of integrals of random fields over a unit hypercube. We use a stratified Monte Carlo quadrature and measure the approximation performance by the mean squared error. The quadrature is defined by a finite number of stratified randomly chosen observations with the partition generated by a rectangular grid (or design). We study the class of locally stationary random fields whose local behaviour is like a fractional Brownian field in the mean square sense and find the asymptotic approximation accuracy for a sequence of designs for large number of the observations. For the Hölder class of random functions, we provide an upper bound for the approximation error. Additionally, for a certain class of isotropic random functions with an isolated singularity at the origin, we construct a sequence of designs eliminating the effect of the singularity point.

Keywords Numerical integration · Random field · Sampling design · Stratified sampling · Monte Carlo methods

AMS 2000 Subject Classifications 60G60 · 65D30

1 Introduction

Let \( X(t) \), \( t \in [0, 1]^d, d \geq 1 \), be a continuous random field with finite second moment. We consider the problem of numerical approximation of the integral of \( X \) over the
unit hypercube using finite number of observations. The approximation accuracy is measured by the mean squared error. We use a stratified Monte Carlo quadrature (sMCQ) for the integral approximation introduced for deterministic functions by Haber (1966). The quadrature is defined by stratified random observations with the partition generated by a rectangular grid (or design). We use cross regular sequences of designs, generalizing the well known regular sequences pioneered by Sacks and Ylvisaker (1966). We focus on random fields satisfying a local stationarity condition proposed for stochastic processes by Berman (1974) and extended for random fields in Abramowicz and Seleznjev (2011a). Approximation of random functions from this class is studied in, e.g., Seleznjev (2000); Hüsler et al. (2003); Abramowicz and Seleznjev (2011a, b). For quadratic mean (q.m.) continuous locally stationary random functions, we derive an exact asymptotic behavior of the approximation accuracy. We propose a method for the asymptotically optimal sampling point distribution between the mesh dimensions. We also study optimality of grid allocation along coordinates and provide asymptotic optimality results in the one-dimensional case. For q.m. continuous fields satisfying a Hölder type condition, we determine an upper bound for the approximation accuracy. Furthermore, we investigate a certain class of random fields with different q.m. smoothness at the origin (isolated singularity), and construct sequences of designs eliminating the effect of the singularity point.

Approximation of integrals of random functions is an important problem arising in many research and applied areas, like environmental and geosciences (Ripley 2004), communication theory and signal processing (Masry and Vadrevu 2009). Regular sampling designs for estimating integrals of stochastic processes are studied in Benhenni and Cambanis (1992). Random designs of sampling points, including stratified sampling for stochastic processes, are investigated in Cambanis and Masry (1992); Schoenfelder and Cambanis (1982). Multivariate numerical integration of random fields satisfying Sacks-Ylvisaker conditions is studied in Ritter et al. (1995). Ritter (2000) contains a survey of various random function approximation and integration problems.

The paper is organized as follows. First we introduce a basic notation. In Section 2, we consider a stratified Monte Carlo quadrature for continuous random fields which local behavior is like a fractional Brownian field in the mean square sense. We derive an exact asymptotics and a formula for the optimal interdimensional sampling point distribution. Further, we provide an upper bound for the approximation accuracy for q.m. continuous fields satisfying Hölder type conditions. In the second part of this section, we study random fields with an isolated singularity at the origin and construct sequences of designs eliminating the effect of the singularity. In Section 3, we present the results of numerical experiments, while Section 4 contains the proofs of the statements from Section 2.

1.1 Basic Notation

Let $X = X(t), t \in D := [0, 1]^d, d \geq 1$, be a random field defined on a probability space $(\Omega, \mathcal{F}, P)$. Assume that for every $t$, the random variable $X(t)$ lies in the normed linear space $L^2(\Omega) = L^2(\Omega, \mathcal{F}, P)$ of random variables with finite second moment and identified equivalent elements with respect to $P$. We set $||\xi|| := (E\xi^2)^{1/2}$.