A characterization of 2-threshold functions via pairs of prime segments

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Abstract

In this paper we study 2-threshold functions over a two-dimensional rectangular grid, i.e. the intersections of two threshold functions. We provide a characterization for 2-threshold functions by pairs of oriented prime segments with certain properties, which we call proper. To this end, we first show that any proper 2-threshold function \( f \) can be defined by a proper pair of segments. Then we prove that such a representation is unique, if \( f \) has a true point on the boundary of the grid. Finally, we establish a bijection between almost all proper pairs of segments and almost all 2-threshold functions. Due to this bijection almost all 2-threshold functions admit encoding by ordered sets of 4 integer points.

Keywords: threshold function, \( k \)-threshold function, intersection of halfplanes, integer lattice, rectangular grid

Contents

1 Introduction 1
2 Preliminaries 3
  2.1 Segments, triangles, quadrilaterals and their orientation 3
  2.2 Convex sets and their tangents 4
3 Oriented prime segments and threshold functions 5
4 Proper pairs of oriented prime segments 6
5 Proper pairs of segments and proper 2-threshold functions 10
6 Conclusion 17

1 Introduction

Denote a two-dimensional rectangular grid by \( G_{m,n} = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \). A function \( f \) mapping \( G_{m,n} \) to \( \{0, 1\} \) is called threshold if there exist natural numbers \( a_0, a_1, a_2 \) such that for each \( (x_1, x_2) \in G_{m,n} \)

\[
f(x_1, x_2) = 1 \iff a_1 x_1 + a_2 x_2 \geq a_0.
\]

The inequality \( a_1 x_1 + a_2 x_2 \geq a_0 \) is called a threshold inequality for the function \( f \). We also say that the set of true points \( M_1(f) \) and the set of false points \( M_0(f) \) are separable by the line \( a_1 x_1 + a_2 x_2 = a_0 \).

It is easy to see that \( f \) is threshold if and only if

\[
\text{Conv}(M_0(f)) \cap \text{Conv}(M_1(f)) = \emptyset,
\]

where \( \text{Conv}(S) \) denotes the convex hull of a given set of points \( S \).
For a natural number \( k \geq 2 \), a function \( f : \mathcal{G}_{m,n} \rightarrow \{0, 1\} \) is called \( k \)-threshold if there exist at most \( k \) threshold functions \( f_1, \ldots, f_k \) such that \( f \) coincides with the conjunction of the functions \( f_1, \ldots, f_k \), i.e. \( f = f_1 \land \cdots \land f_k \). We also say that the functions \( f_1, \ldots, f_k \) \textit{define} the \( k \)-threshold function \( f \). A \( k \)-threshold function is called \textit{proper} \( k \)-threshold if it is not \((k-1)\)-threshold.

Threshold functions refer to the linear partitions of a given set of points. One also studies non-linear partitions by circles \([20, 21]\), convex curves \([23]\), arbitrary curves \([38]\) in 2-dimension and spheres \([36]\) and surfaces \([37]\) in higher dimensions. In particular, polynomial threshold functions are considered in \([4,9,17,25]\). It is worth to note, that \( k \)-threshold functions represent the partition of the domain by at most \( k \) straight lines (halfspaces) in \textit{general position}, and hence have richer structure than many other studied partitions by multiple lines or surfaces such as parallel hyperplanes \([16]\) or \( d \)-dimensional spheres centered at the same point.

In machine learning theory learning of Boolean \( k \)-threshold functions was studied, for instance, in \([7, 19, 22, 26]\). Lower bounds on the complexity of learning for threshold, \( k \)-threshold functions, and some related geometric objects were derived in \([29]\). In \([10]\) the authors provided an efficient algorithm of learning with membership queries for \( k \)-threshold functions over the two-dimensional grid. Structural properties of threshold and \( k \)-threshold functions affecting their learning complexity were also studied in \([3,27,28,32,33,35]\).

In digital geometry, the problem of polyhedral separability can be formulated in terms of \( k \)-threshold functions as follows: given a domain \( S \), a finite set of points \( T \subseteq S \), and a positive integer \( k \), does there exist a \( k \)-threshold function \( f \) over \( S \) such that \( T \) is the set of true points of \( f \)? The problem of polyhedral separability is widely investigated (see \([5,6,8,12–15,30]\)). In particular, in \([8]\) the authors studied bilinear separation which is closely related to 2-threshold functions, and the papers \([11,14,15]\) are devoted to the polyhedral separability problem in two- and three-dimensional spaces.

Threshold functions admit various representations and usually the choice of specific description depends on the restrictions of a particular application. The most natural way of defining threshold functions is via threshold inequalities. However, for a given threshold function there are continuously many threshold inequalities, and given two linear inequalities it is not obvious whether they define the same threshold function or not. A useful characterization of two-dimensional threshold functions via oriented prime segments was provided in \([24]\). In that and the subsequent works \([1,2,18]\) the relation between threshold functions and prime segments was applied to estimate the number of threshold functions asymptotically. Since any 2-threshold function can be represented as the conjunction of two threshold functions, it is also possible to define them via pairs of threshold functions or pairs of the corresponding prime segments. A drawback of such approach for representing 2-threshold functions is that the same 2-threshold function, in general, can be defined by many different pairs of threshold functions and therefore by many different pairs of the prime segments. In this paper we deal with this ambiguity and consider the pairs of oriented prime segments with certain properties which we call \textit{proper} pairs of segments. We provide a characterization of 2-threshold functions over \( \mathcal{G}_{m,n} \) by establishing a bijection between almost all proper pairs of segments and almost all 2-threshold functions. Not only this bijection is proved to be useful for the asymptotic estimation of the number of 2-threshold functions but also it provides the space-optimal coding of almost all 2-threshold functions by ordered sets of 4 integer points (endpoints of the corresponding segments in a proper pair). Representing 2-threshold functions in this way, enables, for example, a constant-time comparison of two functions.

The organization of the paper is as follows. All preliminary information can be found in Section 2. In Section 3 we describe and adapt to our purposes the bijection between oriented prime segments and non-constant threshold functions from \([24]\). In Section 4 we introduce proper pairs of segments and show that any proper 2-threshold function can be defined by a proper pair of segments. In Section 5 we show that for almost every proper 2-threshold function there exists a unique proper pair of segments that defines the function. In this way we establish a bijection between almost all proper pairs of segments and almost all 2-threshold functions.
2 Preliminaries

In this paper we denote points on the plane by capital letters $A, B, C$, etc. For two sets of points $S_1, S_2$ we denote by $d(S_1, S_2)$ the distance between the sets, that is, the minimum distance between two points $A \in S_1$ and $B \in S_2$. When a set consists of a single point we omit $\{\}$ and write simply $d(A, S_2)$ or $d(A, B)$ to denote the distance between the point $A$ and set $S_2$ or the distance between the points $A$ and $B$, respectively. For two distinct points $A, B$ we denote by $\ell(AB)$ the line which passes through these points.

A point $A = (x, y)$ is integer, if both of its coordinates $x$ and $y$ are integer. Two points $A, B$ are called adjacent if they are integer and there is no other integer points on $AB$. A segment with adjacent endpoints is called prime.

We say that the points $A_1, A_2, \ldots, A_n$ are in convex position if $\{A_1, \ldots, A_n\} = \text{Vert(Conv(}\{A_1, \ldots, A_n\}\text{))).}$ We also denote by $P(f)$ the convex hull of $M_1(f)$, that is $P(f) = \text{Conv}(M_1(f))$.

2.1 Segments, triangles, quadrilaterals and their orientation

We often denote a convex polygon by a sequence of its vertices in either clockwise or counterclockwise order. For example, by $AB, ABC$, and $ABCD$ we denote, respectively, the segment with endpoints $A, B$, the triangle with vertices $A, B, C$, and the convex quadrilateral with vertices $A, B, C, D$ and edges $AB, BC, CD, DA$. When the order of vertices is important, we call the polygon or segment oriented and add an arrow in the notation, that is, $\overrightarrow{AB}, \overrightarrow{ABC}, \overrightarrow{ABCD}$ denote the oriented segment, the oriented triangle, and the oriented convex quadrilateral, respectively.

Let $A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2)$ be pairwise distinct points on the plane. It is a basic fact that $A, B, C$ are collinear if and only if $\Delta = 0$, where

$$\Delta = \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$ 

The oriented triangle $\overrightarrow{ABC}$ is called clockwise if $\Delta < 0$ and counterclockwise if $\Delta > 0$. Geometrically, an oriented triangle $\overrightarrow{ABC}$ is clockwise (resp. counterclockwise) if its vertices $A, B, C$, in order, rotate clockwise (resp. counterclockwise) around the triangle’s center. Some properties of oriented triangles easily follow from the definition:

**Claim 1.** Let $\ell$ be a line and let $A, B$ be two distinct points on $\ell$. Then for any two points $C, D \notin \ell$ the orientations of the triangles $\overrightarrow{ABC}$ and $\overrightarrow{ABD}$ are the same if and only if $\ell \cap CD = \emptyset$ (see Fig. 1a and 1b).

**Claim 2.** Let $\overrightarrow{AB}, \overrightarrow{CD}$ be two collinear segments with the same orientation. Then for any point $E \notin \ell(AB)$ the triangles $\overrightarrow{ABE}$ and $\overrightarrow{CDE}$ have the same orientation (see Fig. 1c).
Claim 3. Let \( A, B, C, D \) be four distinct points such that \( \overrightarrow{ABD}, \overrightarrow{BCD}, \overrightarrow{CAD} \) are clockwise (resp. counterclockwise) triangles. Then \( \overrightarrow{ABC} \) is a clockwise (resp. counterclockwise) triangle.

Proof. We will prove the statement for clockwise triangles, the counterclockwise case is symmetric. Denote \( \mathcal{P} = \text{Conv}(\{A, B, C, D\}) \). First, we show that \( D \) is not a vertex of \( \mathcal{P} \). Suppose, to the contrary, that \( D \) is a vertex of \( \mathcal{P} \), then two of the segments \( CD, BD, AD \) are edges of \( \mathcal{P} \). The triangle \( \overrightarrow{CAD} \) is clockwise, hence the triangle \( \overrightarrow{CD} \) is clockwise and the points \( A \) and \( B \) are separated by \( \ell(\overrightarrow{CD}) \), and therefore \( CD \) is not an edge of \( \mathcal{P} \). Similarly, the opposite orientations of the triangles \( \overrightarrow{ABD} \) and \( \overrightarrow{BCD} \) imply that \( BD \) is not an edge of \( \mathcal{P} \). The above contradicts the assumption that two of the segments \( CD, BD, AD \) are edges of \( \mathcal{P} \), and therefore \( D \) is not a vertex of \( \mathcal{P} \). Let \( P \) be the triangle with vertices \( A, B, C \). Finally, since \( D \) is an interior point of \( \mathcal{P} \), the points \( C \) and \( D \) lie on the same side from \( \ell(\overrightarrow{AB}) \), hence the triangles \( \overrightarrow{ABD} \) and \( \overrightarrow{ABC} \) have the same orientation, i.e. \( \overrightarrow{ABC} \) is clockwise, as required (see Fig. 2).

It is clear, that for a given convex oriented quadrilateral \( \overrightarrow{ABCD} \) the orientation of the triangles \( \overrightarrow{ABC}, \overrightarrow{BCD}, \overrightarrow{CAD}, \) and \( \overrightarrow{DAB} \) is the same and determines the orientation of \( \overrightarrow{ABCD} \). Moreover, the opposite is also true.

Claim 4. Let \( \overrightarrow{ABC}, \overrightarrow{BCD}, \overrightarrow{CAD}, \overrightarrow{DAB} \) be clockwise (resp. counterclockwise) triangles. Then \( \text{Conv}(\{A, B, C, D\}) \) is a quadrilateral with edges \( AB, BC, CD, \) and \( DA \) and the orientation of \( \overrightarrow{ABCD} \) is clockwise (resp. counterclockwise).

Proof. Clearly, \( A, B, C, \) and \( D \) are pairwise distinct points. Let \( \mathcal{P} = \text{Conv}(\{A, B, C, D\}) \). Since \( \overrightarrow{ABC} \) and \( \overrightarrow{DAB} \) are triangles with the same orientation, we conclude that \( C \) and \( D \) lie on the same side of \( \ell(\overrightarrow{AB}) \), and therefore \( \ell(\overrightarrow{AB}) \) is a tangent to \( \mathcal{P} \) and \( AB \) is an edge of \( \mathcal{P} \). By similar arguments each of the segments \( BC, CD, \) and \( DA \) is an edge of \( \mathcal{P} \), hence \( \mathcal{P} \) is a quadrilateral. Finally, the orientation of the triangles implies that \( \overrightarrow{ABCD} \) has the same orientation as the orientation of the triangles.

2.2 Convex sets and their tangents

Let \( C \) be a convex set. A convex polygon \( \mathcal{P} \) is called circumscribed about \( C \) if for every edge \( AB \) of \( \mathcal{P} \) the line \( \ell(\overrightarrow{AB}) \) is a tangent to \( C \) and \( AB \cap C \neq \emptyset \).

Let \( C_1 \) and \( C_2 \) be two disjoint convex sets. A line \( \ell \) is called an inner common tangent to \( C_1 \) and \( C_2 \) if it is a tangent to both of them, and \( C_1 \) and \( C_2 \) are separated by \( \ell \).

Let \( \ell \) be a tangent to a convex set \( C \), and let \( X \) be a point in \( \ell \cap C \). Then \( \ell \) is called a right (resp. left) tangent from \( X \) to \( C \) if for any points \( Y \in C \cap \ell \) and \( Z \in C \setminus \ell \) the triangle \( \overrightarrow{XYZ} \) is counterclockwise (resp. clockwise). The following claim is a simple consequence of the above definition.

Claim 5. Let \( \ell \) be the right (resp. left) tangent from a point \( X \) to a convex set \( C \), and let \( Y \in \ell \). Then \( \ell \) is the right (resp. left) tangent from \( Y \) to \( C \) if and only if \( XY \cap C = \emptyset \).

Let \( \ell \) be an inner common tangent to two disjoint convex sets \( C_1 \) and \( C_2 \), and let \( A, B \) be two points such that \( A \in C_1 \cap \ell \) and \( B \in C_2 \cap \ell \). Then \( \ell \) is called the right (resp. left) inner common tangent to \( C_1 \) and \( C_2 \) if \( \ell \) is the right (resp. left) tangent from \( A \) to \( C_2 \), and the right (resp. left) tangent from \( B \) to \( C_1 \) (see Fig. 3). It is easy to see that any pair of disjoint convex sets has exactly one right and exactly one left inner common tangent.
3 Oriented prime segments and threshold functions

Definition 3.1. Let $A$ and $B$ be two adjacent points in $G_{m,n}$. We say that $\overrightarrow{AB}$ defines a function $f : G_{m,n} \rightarrow \{0, 1\}$ if:

1. $f(A) = 1, f(B) = 0$;
2. for any $X \in G_{m,n} \cap \ell(AB)$ we have $f(X) = 1$ if and only if $d(A, X) < d(B, X)$;
3. for any $X \in G_{m,n} \setminus \ell(AB)$ we have $f(X) = 1$ if and only if $\overrightarrow{ABX}$ is a counterclockwise triangle.

The function defined by $\overrightarrow{AB}$ will be denoted as $f_{\overrightarrow{AB}}$.

The following statement is an immediate consequence of Definition 3.1.

Claim 6. Let $\overrightarrow{AB}$ be a prime segment in $G_{m,n}$ and let $f = f_{\overrightarrow{AB}}$ be the function over $G_{m,n}$ defined by $\overrightarrow{AB}$. Then for any $C \in \ell(AB) \cap G_{m,n}$ we have either $f(C) = 1$ and $A \in BC$ or $f(C) = 0$ and $B \in AC$.

In [24] authors, in different terms, showed that a function $f_{\overrightarrow{AB}}$ defined by an oriented prime segment $\overrightarrow{AB}$ is threshold and the line $\ell(AB)$ is an inner common tangent to the convex hulls of the sets of true and false points of $f$. For the convenience, the following theorem partly repeats the result from [24], thus adapting it to our purposes and making our exposition self-contained.

Theorem 7. Let $A$ and $B$ be two adjacent points in $G_{m,n}$ and let $f = f_{\overrightarrow{AB}}$. Then

(1) $f$ is a threshold function;
Proof. First we prove (1). Indeed, if we consider the line \( \ell(AB) \) and turn it counterclockwise slightly around the middle of the segment \( AB \) to not intersect any integer points then we obtain a separating line for \( f \), hence \( f \) is a threshold function (see Fig. 4).

Let us now prove (2). Consider the line \( \ell(AB) \) and turn it counterclockwise slightly around the point \( A \) to not intersect any integer points except \( A \). The obtained line separates \( M_1(f) \setminus \{A\} \) and \( M_0(f) \cup \{A\} \), and witnesses that the function that differs from \( f \) in the unique point \( A \) is threshold. Therefore, the point \( A \) is essential for \( f \). Similarly, one can show that \( B \) is also essential for \( f \).

Now we prove (3). First, it is easy to see that \( \ell(AB) \) is a tangent to both \( \text{Conv}(M_1(f)) \) and \( \text{Conv}(M_0(f)) \). Furthermore, since \( \text{Conv}(M_1(f)) \) and \( \text{Conv}(M_0(f)) \) are separated by \( \ell(AB) \), we conclude that \( \ell(AB) \) is an inner common tangent for \( \text{Conv}(M_1(f)) \) and \( \text{Conv}(M_0(f)) \). Now, by Definition 3.1, for any \( X \in M_1(f) \setminus \ell(AB) \) the triangle \( BAX \) is clockwise, and for any \( X \in M_0(f) \setminus \ell(AB) \) the triangle \( ABX \) is clockwise. Hence, \( \ell(AB) \) is a left tangent from \( B \) to \( \text{Conv}(M_1(f)) \) and from \( A \) to \( \text{Conv}(M_2(f)) \), i.e. \( \ell(AB) \) is the left inner common tangent for \( \text{Conv}(M_1(f)) \) and \( \text{Conv}(M_0(f)) \).

In [24] authors also proved the bijection between oriented prime segments and non-constant threshold functions:

**Theorem 8.** [24] There is one-to-one correspondence between oriented prime segments in \( G_{m,n} \) and non-constant threshold functions over \( G_{m,n} \).

**Corollary 9.** Let \( f \) be a non-constant threshold function over \( G_{m,n} \). Then there exists a unique prime segment \( AB \) with \( A, B \in G_{m,n} \) such that \( f = f_{AB} \).

## 4 Proper pairs of oriented prime segments

Since a 2-threshold function is the conjunction of two threshold functions, the defining threshold functions via oriented prime segments can be naturally extended to 2-threshold functions.

**Definition 4.1.** We say that a pair of oriented prime segments \( \overrightarrow{AB}, \overrightarrow{CD} \) in \( G_{m,n} \) defines a 2-threshold function \( f \) over \( G_{m,n} \) if

\[
f = f_{\overrightarrow{AB}} \land f_{\overrightarrow{CD}}.
\]

A 2-threshold function can be expressed as the conjunction of different pairs of threshold functions, therefore there is no bijection between pairs of oriented prime segments and non-constant 2-threshold functions. However, we may impose some restrictions on the pairs of oriented prime segments to exclude redundant pairs of segments defining the same function.

**Definition 4.2.** We say that a pair of oriented segments \( \overrightarrow{AB}, \overrightarrow{CD} \) is proper if the segments are prime and

\[
f_{\overrightarrow{CD}}(A) = f_{\overrightarrow{CD}}(B) = f_{\overrightarrow{AB}}(C) = f_{\overrightarrow{AB}}(D) = 1.
\]

**Claim 10.** Let \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) be a proper pair of segments. Then \( A \neq D, C \neq B, \) and \( B \neq D \).

**Proof.** The statement follows from the inequalities \( f_{\overrightarrow{CD}}(A) \neq f_{\overrightarrow{CD}}(D), f_{\overrightarrow{AB}}(C) \neq f_{\overrightarrow{AB}}(B), \) and \( f_{\overrightarrow{AB}}(B) \neq f_{\overrightarrow{AB}}(D) \).

The following theorem provides the criteria for a pair of oriented prime segments to be proper.

**Theorem 11.** The pair of prime segments \( \overrightarrow{AB}, \overrightarrow{CD} \) is proper if and only if one of the following holds:

1. \( AC \subset BD \);

2. \( A \in BD \) and \( CDB \) is a counterclockwise triangle or \( C \in BD \) and \( ABD \) is a counterclockwise triangle;
(a) $AC \subset BD$

(b) $C \in BD$ and $\overrightarrow{ABD}$ is a counterclockwise triangle

(c) $\overrightarrow{ABCD}$ is a convex counterclockwise quadrilateral.

Figure 5: Black points are the true points of $f = f_{\overrightarrow{AB}} \wedge f_{\overrightarrow{CD}}$ where $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ is a proper pair of segments.

(3) $\overrightarrow{ABCD}$ is a counterclockwise quadrilateral.

Proof. Clearly $\text{Conv}(\{A, B, C, D\})$ has at least 2 and at most 4 vertices. The proof of the theorem is split up into Lemmas 12, 13, and 14 according to the number of vertices of $\text{Conv}(\{A, B, C, D\})$.

The following lemmas treat the cases where $\text{Conv}(\{A, B, C, D\})$ is a segment, triangle, and quadrilateral.

Lemma 12. A pair of collinear prime segments $\overrightarrow{AB}, \overrightarrow{CD}$ is proper if and only if $AC \subset BD$.

Proof. Let $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ be a proper pair of collinear prime segments (see Fig. 5a). Then using Claim 6 we derive from $f_{\overrightarrow{AB}}(D) = \ell_{\overrightarrow{CD}}(B) = 1$ the inclusion $A, C \in BD$.

Conversely, let $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ be a pair of collinear prime segments with $AC \subset BD$. The primality of the segments implies that $A \in BC$ and $C \in AD$. Therefore, by Claim 6, we have $f_{\overrightarrow{AB}}(C) = f_{\overrightarrow{CD}}(A) = f_{\overrightarrow{AB}}(D) = f_{\overrightarrow{CD}}(B) = 1$, and hence the pair $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ is proper, as required.

Lemma 13. Let $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ be a pair of prime segments such that $\text{Conv}(\{A, B, C, D\})$ is a triangle. Then the pair is proper if and only if either $\overrightarrow{CDB}$ is a counterclockwise triangle with $A \in BD$ or $\overrightarrow{ABD}$ is a counterclockwise triangle with $C \in BD$.

Proof. First assume $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ is a proper pair of prime segments with Conv($\{A, B, C, D\}$) being a triangle. There are four cases to consider:

1. $D \in \overrightarrow{ABC}$. We claim that this case is impossible. Indeed, if $D$ belongs to the triangle $\overrightarrow{ABC}$, then $D$ belongs neither to $BC$ nor to $AC$, as otherwise, by Claim 6, at least one of $f_{\overrightarrow{CD}}(A)$ and $f_{\overrightarrow{CD}}(B)$ would be zero, contradicting the assumption that $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ is proper. Therefore, $\ell(CD)$ separates $A$ and $B$, which contradicts $f_{\overrightarrow{CD}}(A) = f_{\overrightarrow{CD}}(B)$.

2. $B \in \overrightarrow{CD\overrightarrow{A}}$. This case is impossible by similar arguments as in case 1.
and we omit the details. Since

\[ \text{orientation of triangle.} \]

Proof. By Claim 6, we have

\[ \overrightarrow{BCD} \]

for, by Claim 4, \( \ell(\overrightarrow{BD}) = 1 \). Then the pair is proper if and only if \( \overrightarrow{ABD} \) is not an edge of \( \text{Conv}(\overrightarrow{ABD}) \). We show that \( \overrightarrow{ABD} \) is a counterclockwise triangle and \( C \in BD \) (see Fig. 5b). The former follows from \( f_{\overrightarrow{AB}}(D) = 1 \). To prove the latter, suppose to the contrary that \( C \notin BD \). Then \( \ell(\overrightarrow{BD}) \) does not intersect \( AC \), and hence, by Claim 1, the orientations of the triangles \( BDC \) and \( BD\overrightarrow{A} \) are the same. Since the orientation of \( \overrightarrow{BD\overrightarrow{A}} \) is the same as that of \( \overrightarrow{ABC} \), we conclude that the orientation of \( \overrightarrow{BCD} \) is counterclockwise, and therefore the orientation of \( \overrightarrow{CDB} \) is clockwise, which contradicts \( f_{\overrightarrow{CDB}}(B) = 1 \).

4. \( A \in \overrightarrow{CDB} \). In this case arguments similar to the analysis of case 3 show that \( \overrightarrow{CDB} \) is a counterclockwise triangle and \( A \in BD \).

Assume now that \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) is a pair of prime segments such that \( \overrightarrow{ABD} \) is a counterclockwise triangle and \( C \in BD \). The case where \( \overrightarrow{CDB} \) is a counterclockwise triangle with \( A \in BD \) is symmetric and we omit the details. Since \( C \in BD \), the orientation of \( \overrightarrow{ABC} \) and \( \overrightarrow{CDA} \) is the same as the orientation of \( \overrightarrow{ABC} \), i.e. counterclockwise. Consequently, \( f_{\overrightarrow{AB}}(D) = f_{\overrightarrow{AB}}(C) = f_{\overrightarrow{CD}}(A) = 1 \). Furthermore, by Claim 6, we have \( f_{\overrightarrow{CBD}}(B) = 1 \), and therefore the pair \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) is proper.

Lemma 14. Let \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) be a pair of prime segments, such that \( A, B, C, \) and \( D \) are in convex position. Then the pair is proper if and only if \( AB, BC, CD, DA \) are edges of \( \text{Conv}(\{A, B, C, D\}) \) and the orientation of \( \overrightarrow{ABCD} \) is counterclockwise.

Proof. First let \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) be a proper pair of prime segments. It follows from \( f_{\overrightarrow{AB}}(C) = f_{\overrightarrow{AB}}(D) = f_{\overrightarrow{CD}}(A) = f_{\overrightarrow{CD}}(B) = 1 \) that the triangles \( \overrightarrow{ABC}, \overrightarrow{ABD}, \overrightarrow{CDA}, \) and \( \overrightarrow{CDB} \) are counterclockwise. Therefore, by Claim 4, \( AB, BC, CD, DA \) are edges of \( \text{Conv}(\{A, B, C, D\}) \) and the orientation of \( \overrightarrow{ABCD} \) is counterclockwise, as required (see Fig. 5c).

Conversely, let \( \overrightarrow{ABCD} \) be a counterclockwise quadrilateral. By definition, the triangles \( \overrightarrow{ABC}, \overrightarrow{BCD}, \overrightarrow{CDA}, \overrightarrow{DAB} \) are counterclockwise. Therefore

\[ f_{\overrightarrow{CDB}}(B) = f_{\overrightarrow{CDB}}(A) = f_{\overrightarrow{AB}}(C) = f_{\overrightarrow{AB}}(D) = 1, \]

and hence the pair \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) is proper.

The following claim is related to the property of non-proper pairs of oriented prime segments.

Claim 15. Let \( \overrightarrow{AB}, \overrightarrow{CD} \) be distinct prime segments in \( G_{m,n} \) such that \( f_{\overrightarrow{AB}}(C) = 1, f_{\overrightarrow{AB}}(D) = 0, \) and \( f_{\overrightarrow{CD}}(A) = 1 \). Then \( f_{\overrightarrow{CDB}}(B) = 1 \), the points \( B, C, D \) are not collinear, and \( A \in BC\overrightarrow{D} \).
Proof. First we claim that the points $A$, $B$, $C$, $D$ are not collinear. Suppose to the contrary, that they are collinear. Then, by Claim 6, we have $A \in BC$, $B \in AD$, and $C \in AD$, which imply that either $A = C$ or $A = B$. The latter is not possible as $AB$ is a prime segment. Therefore $A = C$ and $B \in CD$. Since $CD$ is prime and $C = A \neq B$, we conclude that $B = D$ and $\overrightarrow{AB} = \overrightarrow{CD}$, which contradicts the assumption of the statement.

Assume now that $A, B, C, D$ do not lie on the same line. From $f_{\overrightarrow{AB}}(C) \neq f_{\overrightarrow{AB}}(D)$ it follows that $\ell(AB)$ intersects $CD$. Suppose three of the points $A, B, C, D$ are collinear. We will consider four cases:

1. $A, C, B$ are collinear, i.e. $CD \cap \ell(AB) = C$ (see Fig. 6b). By Claim 6, we have $A \in BC$ and hence $A \in BCD$. To show $f_{\overrightarrow{CD}}(B) = 1$ we observe that the segments $\overrightarrow{AB}$ and $\overrightarrow{CB}$ are collinear and have the same orientation, and therefore, by Claim 2, the triangles $\overrightarrow{ABD}$ and $\overrightarrow{CBD}$ have the same orientation. Since $f_{\overrightarrow{AB}}(D) = 0$, the triangle $\overrightarrow{ABD}$ is clockwise, and hence $\overrightarrow{CBD}$ is counterclockwise and $f_{\overrightarrow{CD}}(B) = 1$.

2. $A, B, D$ are collinear, i.e. $CD \cap \ell(AB) = D$. We will prove that this case is impossible by showing that $\overrightarrow{CDA}$ is a clockwise triangle, which contradicts $f_{\overrightarrow{CD}}(A) = 1$. By Claim 6, we have $B \in AD$, and therefore the segments $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are collinear and have the same orientation. Hence, by Claim 2, the triangles $\overrightarrow{ABC}$ and $\overrightarrow{ACD}$ have the same orientation. Namely, since $f_{\overrightarrow{AB}}(C) = 1$, we conclude that both triangles are counterclockwise. Consequently, $\overrightarrow{CD}$ is clockwise, as desired.

3. $A, C, D$ are collinear, i.e. $CD \cap \ell(AB) = A$. Since $CD$ is prime and $f_{\overrightarrow{CD}}(A) = 1$, we conclude that $A = C$ and hence the first case takes place.

4. $C, B, D$ are collinear, i.e. $CD \cap \ell(AB) = B$. Since $CD$ is prime and $f_{\overrightarrow{AB}}(D) = 0$, we conclude that $B = D$ and hence the second case takes place.

Assume finally that $A, B, C, D$ are in general position and denote $P = \text{Conv}(\{A, B, C, D\})$ (see Fig. 6a). We consider the oriented triangles $\overrightarrow{CD}, \overrightarrow{AB}$, $\overrightarrow{BAD}$, and $\overrightarrow{CDB}$. It follows from the assumptions of the claim that the first three triangles are counterclockwise. Therefore, by Claim 3, the triangle $\overrightarrow{CDB}$ is also counterclockwise, and hence $f_{\overrightarrow{CD}}(B) = 1$.

It remains to show that $A$ belongs to the triangle $BCD$, i.e. $P = BCD$. Suppose, to the contrary, $P \neq BCD$. Then $A$ is a vertex of $P$ and two of the segments $AC$, $AB$, and $AD$ are edges of $P$. We will arrive to a contradiction by showing that neither $\overrightarrow{AB}$ nor $\overrightarrow{AD}$ can be an edge of $P$. Indeed, if $\overrightarrow{AB}$ is an edge of $P$, then $C$ and $D$ are not separated by $\ell(AB)$, which contradicts $f_{\overrightarrow{AB}}(C) \neq f_{\overrightarrow{AB}}(D)$. Furthermore, if $\overrightarrow{AD}$ is an edge of $P$, then $B$ and $C$ are not separated by $\ell(AD)$, and hence the triangles $\overrightarrow{DAC}$ and $\overrightarrow{DAB}$ have the same orientation. However, the triangle $\overrightarrow{DAC}$ is counterclockwise as $f_{\overrightarrow{CD}}(A) = 1$, and the triangle $\overrightarrow{DAB}$ is clockwise as $f_{\overrightarrow{AB}}(D) = 0$. Contradiction.

Corollary 16. Under the conditions of Claim 15 the intersection $\ell(AB) \cap CD$ is a point $X$ such that $A \in X B$.

Theorem 11 implies a sequence of useful statements about 2-threshold functions. The first of them leads to the conclusion that the 2-threshold function defined by a pair of oriented segments is proper whenever the pair is proper.

Claim 17. Let $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ be a proper pair of segments. Then $AC \cap BD \neq \emptyset$.

Proof. By Theorem 11, for a proper pair of segments $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ one of the following statements is true:

1. $AC \subset BD$; in this case $AC \cap BD = AC$.

2. $A \in BD$ and $\overrightarrow{CDB}$ is a counterclockwise triangle or $C \in BD$ and $\overrightarrow{ABD}$ is counterclockwise triangle; then $AC \cap BD = A$ or $AC \cap BD = C$ respectively.
In all cases we have Corollary 18. Every proper pair of segments intersect, and hence the function is not threshold.

Let Corollary 19. A proper 2-threshold function over

\[ f \]

Let Corollary 20. For any proper 2-threshold function \( f \) over \( G_{m,n} \). Then \( M_1(f) = AC \cap G_{m,n} \) (see Fig. 5a).

Corollary 20. Let \( \{AB, CD\} \) be a proper pair of segments that define a 2-threshold function \( f \) over \( G_{m,n} \). Then \( M_1(f) = \{A\} \) (see Fig. 7).

Corollary 21. Let \( \{AB, CD\} \) be a proper pair of segments that define a 2-threshold function \( f \) over \( G_{m,n} \). Then \( AB \cap CD \neq \emptyset \) if and only if \( M_1(f) = \{A\} \) (see Fig. 7).

5 Proper pairs of segments and proper 2-threshold functions

In the following statements we will show that any proper 2-threshold function \( f \) can be defined by a proper pair of segments, and such a pair is unique if \( f \) has a true point on the boundary of the grid. We start with the existence of a proper pair of segments for \( f \).

Theorem 22. For any proper 2-threshold function \( f \) over \( G_{m,n} \) there exists a proper pair of segments in \( G_{m,n} \) that defines \( f \).

Proof. Since every proper 2-threshold function is a conjunction of two non-constant threshold functions, it follows from Corollary 9 that there exists a pair of oriented prime segments that defines \( f \). Let \( \{AB, CD\} \) be a pair of oriented prime segments defining \( f \) such that \( |M_1(f_{\overrightarrow{AB}})| + |M_1(f_{\overrightarrow{CD}})| \) is minimized. We claim that \( f_{\overrightarrow{CD}}(A) = f_{\overrightarrow{AB}}(C) = 1 \). For the sake of contradiction, assume without loss of generality that \( f_{\overrightarrow{CD}}(A) = 0 \). By Theorem 7, the point \( A \) is essential for \( f_{\overrightarrow{AB}} \), hence the function \( f' \), that differs from \( f_{\overrightarrow{AB}} \) in the unique point \( A \), is threshold. Since \( A \in M_0(f_{\overrightarrow{CD}}) \) and \( M_1(f') = M_1(f_{\overrightarrow{AB}}) \setminus \{A\} \), we have

\[
M_1(f') \cap M_1(f_{\overrightarrow{CD}}) = M_1(f_{\overrightarrow{AB}}) \cap M_1(f_{\overrightarrow{CD}}) = M_1(f),
\]
and therefore \( f = f' \wedge f_{\overrightarrow{CD}} \). By assumption \( f \) is proper, and hence \( f' \) is a non-constant threshold function. Consequently, by Corollary 9, there exists an oriented prime segment \( \overrightarrow{A'B'} \) that defines \( f' \). Therefore, the pair \( \{A'B', CD\} \) defines \( f \). But \( |M_1(f')| < |M_1(f_{\overrightarrow{AB}})| \), which contradicts the choice of \( \{AB, CD\} \).

Since \( f \) is non-threshold, there exist \( X, Y \in M_0(f) \) such that \( XY \cap \text{Conv}(M_1(f)) \neq \emptyset \). Indeed, otherwise \( \text{Conv}(M_0(f)) \) and \( \text{Conv}(M_1(f)) \) would be disjoint, and
(a) All integer points of the stripped region are exactly the true points of $f^*$. $Z$ is chosen outside of $\text{Conv}(M_1(f))$ and such that $f^*(Z) = 1$.

(b) The pair $A^*B^2$, $C^*D^2$ is proper. The stripped region is $P$. $S_1$ and $S_2$ have the different pattern orientation. The segment $D^*Z$ intersects $A^*C^*$.

Figure 8: The white polygons are $\text{Conv}(M_X)$ and $\text{Conv}(M_Y)$. The grey polygon is $\text{Conv}(M_1(f))$.

therefore separable by a line. Hence, for any pair of prime segments $\overrightarrow{AB}, \overrightarrow{CD}$ that defines $f$, neither $f_{\overrightarrow{AB}}$ nor $f_{\overrightarrow{CD}}$ can be false in both $X, Y$. Furthermore, since $X, Y \in M_0(f)$, we conclude that one of the points is a false point of $f_{\overrightarrow{AB}}$ and a true point of $f_{\overrightarrow{CD}}$, and the other point is a true point of $f_{\overrightarrow{AB}}$ and a false point of $f_{\overrightarrow{CD}}$.

Let $X$ be the family of ordered pairs of segments $\overrightarrow{AB}, \overrightarrow{CD}$ defining $f$ such that $X \in M_0(f_{\overrightarrow{AB}}) \cap M_1(f_{\overrightarrow{CD}})$ and $Y \in M_1(f_{\overrightarrow{AB}}) \cap M_0(f_{\overrightarrow{CD}})$. Denote

$$M_X = \bigcap_{(\overrightarrow{AB}, \overrightarrow{CD}) \in X} M_0(f_{\overrightarrow{AB}}) \cap M_1(f_{\overrightarrow{CD}}).$$

$$M_Y = \bigcap_{(\overrightarrow{AB}, \overrightarrow{CD}) \in X} M_1(f_{\overrightarrow{AB}}) \cap M_0(f_{\overrightarrow{CD}}).$$

Notice that each of $M_X$ and $M_Y$ is the intersection of convex sets that have a common element, and therefore both $M_X$ and $M_Y$ are non-empty and convex. Moreover, since $M_X, M_Y \subset M_0(f)$, both $\text{Conv}(M_X)$ and $\text{Conv}(M_Y)$ are disjoint from $\text{Conv}(M_1(f))$.

Let $\ell_X$ be the left inner common tangent to $\text{Conv}(M_1(f))$ and $\text{Conv}(M_X)$. Let $A^* \in \text{Conv}(M_1(f)) \cap \ell_X$, $B^* \in \text{Conv}(M_X) \cap \ell_X$ be such that $A^*B^*$ is of minimum length. We claim that $A^*B^*$ is a prime segment. To prove this, we show first that $\text{Conv}(M_1(f) \cup M_X)$ contains no integer points other than points in $M_1(f) \cup M_X$. Indeed, let $(\overrightarrow{AB}, \overrightarrow{CD})$ be a pair of segments from $X$ and suppose there exists an integer point $Z$ in $\text{Conv}(M_1(f) \cup M_X)$ that belongs neither to $M_1(f)$ nor to $M_X$. Notice, by definition, $M_X \subset M_1(f_{\overrightarrow{CD}})$ and $M_1(f) \subset M_1(f_{\overrightarrow{AB}})$, which implies that $\text{Conv}(M_1(f) \cup M_X) \subset \text{Conv}(M_1(f_{\overrightarrow{CD}}))$. Consequently, if $f_{\overrightarrow{AB}}(Z) = 1$ we have $Z \in M_1(f)$, and if $f_{\overrightarrow{AB}}(Z) = 0$ we have $Z \in M_X$, a contradiction. Now, any segment with endpoints in $M_1(f) \cup M_X$ belongs to $\text{Conv}(M_1(f) \cup M_X)$, hence if there is an integer point $Z$ in the interior of $A^*B^*$ then $Z \in M_1(f) \cup M_X$, which contradicts the minimality of $A^*B^*$. Similarly, considering the left inner common tangent $\ell_Y$ to $\text{Conv}(M_1(f))$ and $\text{Conv}(M_Y)$, the two points $C^* \in M_1(f) \cap \ell_Y$, $D^* \in M_Y \cap \ell_Y$ at minimum distance define a prime segment $C^*D^*$. Fig. 8 illustrates $M_X, M_Y, A^*, B^*, C^*$, and $D^*$.

Let now $f^* = f_{A^*B^2} \land f_{C^*D^2}$ be the 2-threshold function defined by $\{A^*B^2, C^*D^2\}$. In the rest of the proof we will show that $f = f^*$ and the pair $\{A^*B^2, C^*D^2\}$ is proper. To establish the former we will prove that $M_1(f) = M_1(f^*)$.

First we show that $M_1(f) \subseteq M_1(f^*)$. Indeed, by definition, $\ell(A^*B^2) = \ell_X$ is a left tangent from $B^*$ to $\text{Conv}(M_1(f))$, and therefore $M_1(f) \subseteq M_1(f_{A^*B^2})$. Similarly, we have $M_1(f) \subseteq M_1(f_{C^*D^2})$, and therefore $M_1(f) \subseteq M_1(f_{A^*B^2}) \cap M_1(f_{C^*D^2}) = M_1(f^*)$.

Now, let us show that $M_1(f^*) \subseteq M_1(f)$. Assume, to the contrary, $M_1(f^*) \setminus M_1(f) \neq \emptyset$ and let $Z$ be a point in $M_1(f^*) \setminus M_1(f)$. In particular, we have $Z \notin M_X \cup M_Y$. We observe that $f(Z) = 0$.
Figure 9: The pair \(\overrightarrow{A B^*}, \overrightarrow{C^* D^*}\) is not proper and \(f_{\overrightarrow{A B^*}}(D^*) = 0\). The grey region is \(\text{Conv}(M_1(f))\). The stripped region is \(P\). \(S_1\) and \(S_2\) have different patterns.

and \(Z \notin M_Y\) imply that there exists a pair \((\overrightarrow{A B}, \overrightarrow{C D})\) such that \(Z \in M_0(f_{\overrightarrow{A B}})\), and therefore \(M_X \cap \{Z\} \subseteq M_0(f_{\overrightarrow{A B}})\) and

\[
\text{Conv}(M_X \cup \{Z\}) \cap \text{Conv}(M_1(f)) = \emptyset. \tag{1}
\]

Similarly, it can be shown that

\[
\text{Conv}(M_Y \cup \{Z\}) \cap \text{Conv}(M_1(f)) = \emptyset. \tag{2}
\]

We will consider two cases depending on whether \(\overrightarrow{A B^*}, \overrightarrow{C^* D^*}\) is a proper pair or not. We start with the case of proper pair, in which case we have \(f_{\overrightarrow{A B^*}}(D^*) = f_{\overrightarrow{C D^*}}(B^*) = 1\) (see Fig. 8a). First we claim that \(A^* \neq C^*\). Indeed, otherwise, by Corollary 20, we would have \(M_1(f^*) = \{A^*, C^*\}\), and therefore since \(M_1(f) \subseteq M_1(f^*)\) and \(M_1(f^*) \setminus M_1(f) \neq \emptyset\), we would conclude that \(f\) is the constant-zero function, contradicting the assumption that \(f\) is a proper 2-threshold function. Let us now denote \(P = \text{Conv}(M_1(f^*) \cup \{B^*, D^*\})\). From \(M_1(f) \cup \{D^*\} \subseteq M_1(f_{\overrightarrow{A B^*}})\) and \(A^*, B^* \in \ell_X\) it follows that \(\ell_X\) is a tangent to \(P\) where \(A^*\) is a tangent point. Analysis similar to the above implies that \(\ell_Y\) is a tangent to \(P\) and \(C^*\) is a tangent point. Consequently, all points of \(P \setminus A^*C^*\) are separated by the segment \(A^*C^*\) into two parts, which we denote as \(S_1 \) and \(S_2\) (see Fig. 8b). By Claim 17, the segments \(A^*C^*\) and \(B^*D^*\) intersect, and hence \(B^*\) and \(D^*\) are in different parts, say \(B^* \in S_1\) and \(D^* \in S_2\). We now claim that \(Z\) belongs to one of the parts \(S_1\) and \(S_2\). To see this, we first observe that \(Z \in M_1(f^*) \subseteq P\). Furthermore, since \(Z\) belongs to \(M_0(f)\), it does not belong to \(A^*C^*\), and hence the claim.

Now, assume without loss of generality \(Z \in S_1\), and therefore \(D^*Z\) intersects \(A^*C^*\). Since \(D^* \in M_Y\) and \(A^*C^* \subseteq \text{Conv}(M_1(f))\), we conclude that \(\text{Conv}(M_Y \cup \{Z\}) \cap \text{Conv}(M_1(f)) = \emptyset\), which contradicts (2).

Suppose now that the pair \(\overrightarrow{A B^*}, \overrightarrow{C^* D^*}\) is not proper, which implies that \(f_{\overrightarrow{A B^*}}(D^*) = 0\) or \(f_{\overrightarrow{C D^*}}(B^*) = 0\). There is no loss of generality in assuming \(f_{\overrightarrow{A B^*}}(D^*) = 0\) (see Fig. 9a). Then Claim 15 yields \(f_{\overrightarrow{C D^*}}(B^*) = 1\). Let \(A^* \neq C^*\), the case \(A^* = C^*\) will be considered separately. From \(f_{\overrightarrow{A B^*}}(C^*) \neq f_{\overrightarrow{B^* A}^*}(D^*)\) it follows that \(\ell_X\) intersects \(C^*D^*\). We denote \(O = \ell_X \cap C^*D^*\) and consider \(P = \text{Conv}(M_1(f) \cup \{B^*, O\})\). As in the previous case it can be verified that \(\ell_X\) and \(\ell_Y\) are tangents to \(P\), and therefore \(A^*\) and \(C^*\) are tangent points. Thus the points of \(P \setminus A^*C^*\) are separated by \(A^*C^*\) into two parts, which we denote as \(S_1\) and \(S_2\). We next prove that \(O \) and \(B^*\) are in different parts. For this purpose, we consider the triangle \(B^*C^*D^2\), and Claim 15 implies \(A^* \in B^*C^*D^2\). It is easily seen that \(OB^* = B^*C^*D^2 \cap \ell(A^*B^*)\), hence \(A^* \in OB^*\), and therefore \(O \) and \(B^*\) belong to the different parts, say \(B^* \in S_1\) and \(O \in S_2\). Clearly, \(Z \in P \setminus A^*C^*\), and therefore either \(Z \in S_1\) or \(Z \in S_2\). The latter would contradict (1), so we assume the former holds, which in turn implies \(OZ \cap A^*C^* \neq \emptyset\).

To obtain a contradiction with (2) we will show \(OZ \subseteq \text{Conv}(M_Y \cup \{Z\})\). To this end we first observe that \(\ell_Y\) intersects \(YZ\) because \(f_{\overrightarrow{C D^*}}(Y) \neq f_{\overrightarrow{C D^*}}(Z)\). Let \(V\) be the intersection point of \(YZ\) and \(\ell_Y\). Now from \(f_{\overrightarrow{A B^*}}(Y) = f_{\overrightarrow{A B^*}}(Z) = 1\) it follows that \(V \in \text{Conv}(M_1(f_{\overrightarrow{A B^*}}))\). Since \(D^* \in M_0(f_{\overrightarrow{A B^*}}), \)
Lemma 23. Let that either

Theorem 24. For any proper 2-threshold function $f$ is a 2-threshold function. From Theorem 11 and Corollary 20 it follows that the pair in all the cases imply that $G$ defines it, if $A^* = C^*$ (see Fig. 9b). By replacing $O$ with $A^*$, and using arguments similar to the above one can show that $A^* \in YD^Z$ and $A^*Z \subseteq \text{Conv}(OY \cup \{Z\})$, which contradicts (2). The contradictions in all the cases imply that $M_1(f^*) \setminus M_1(f) = \emptyset$, and hence $f = f^*$.

We have shown that $\{A^*B^2, C^*D^2\}$ defines $f$. It remains to prove that $\{A^*B^2, C^*D^2\}$ is a proper pair of segments. Since $B^* \in M_X$ and $B^* \in M_0(f_{A^*B^2})$, the definition of $M_X$ implies that $f_{A^*B^2}(B^*) = 1$. Similarly, from $D^* \in M_Y$ and $D^* \in M_0(f_{C^*D^2})$ we conclude $f_{A^*B^2}(D^*) = 1$. Finally, the equality $f_{A^*B^2}(C^*) = f_{C^*D^2}(A^*) = 1$ follows from $A^*, C^* \in M_1(f)$. Hence $\{A^*B^2, C^*D^2\}$ is a proper pair of segments that defines $f$, as claimed.

Now we will show that a proper 2-threshold function $f$ has a unique proper pair of segments that defines it, if $f$ has true points on the boundary of the grid. In the following lemma we consider the case where $f$ is a singleton-function, and then proceed with general case.

Lemma 23. Let $f$ be a $\{0,1\}$-valued function over $G_{m,n}$ with a unique true point $X = (x_1, x_2)$ such that either $x_1 \in \{0, m-1\}$ or $x_2 \in \{0, n-1\}$, but not both. Then $f$ is a proper 2-threshold function with a unique proper pair of segments defining $f$.

Proof. Due to symmetry it is enough to consider the case $x_1 = 0$ and $x_2 \in \{1, \ldots, n-2\}$. We will show that $\{XY, XZ\}$, where $Y = (0, x_2 - 1)$, $Z = (0, x_2 + 1)$, is the desired pair (see Fig. 10). In [32] it was proved that any $\{0,1\}$-function containing one true point is $k$-Threshold for any $k \geq 2$, hence $f$ is a 2-threshold function. From Theorem 11 and Corollary 20 it follows that the pair $\{XY, XZ\}$ is proper and defines $f$, and therefore $f$ is non-threshold. Now, let us prove that there is no other proper pair of segments that defines $f$.

Let $\{XY, XZ\}$ be a proper pair segments that defines $f$. We will show that $\{Y', Z'\} = \{Y, Z\}$. First, $f(Z) = 0$ implies that $f_{XY}(Z) = 0$ or $f_{XZ}(Z) = 0$. Without loss of generality we assume $f_{XZ}(Z) = 0$. Since both $XZ$ and $XZ'$ are prime, we conclude that either $Z' = Z$ or $XZ'Z$ is a clockwise triangle. For the sake of contradiction, let us assume the latter holds. By definition of a clockwise triangle,

$$
\begin{pmatrix}
0 & x_2 & 1 \\
z_1 & z_2 & 1 \\
0 & x_2 + 1 & 1
\end{pmatrix} = z_1 < 0,
$$

where $Z' = (z_1, z_2)$. But this contradicts $z_1 \geq 0$, hence $Z' = Z$. Now let us show that $Y' = Y$. Indeed, as $\{XY, XZ\}$ is a proper pair, by definition, $Y' \in M_1(f_{XZ}) = \{(0,0), (0,1), \ldots, (0,x_2)\}$, and therefore, since $XY$ is prime and $X = (0, x_2)$, we conclude that $Y' = (0, x_2 + 1) = Y$.

Theorem 24. For any proper 2-threshold function $f$ over $G_{m,n}$ there exists a unique proper pair of segments in $G_{m,n}$ that defines $f$.

Proof. By Theorem 22, there exists at least one proper pair of segments that defines $f$. Suppose, for the sake of contradiction, that there are two different proper pairs of segments defining $f$, which we denote as $\{AB, CD\}$ and $\{A'B', C'D'\}$ respectively.

First we will prove that

$$\{AB, CD\} \cap \{A'B', C'D'\} = \emptyset.$$ (3)
Suppose, to the contrary, that \( \overrightarrow{AB} = \overrightarrow{A'B'} \), then \( \overrightarrow{CD} \neq \overrightarrow{C'D'} \). Since \( f(\overrightarrow{AB}) = f(\overrightarrow{A'B'}) = 1 \) and \( f(\overrightarrow{D}) = f(\overrightarrow{D'}) = 0 \), we have \( f(\overrightarrow{CD}) = f(\overrightarrow{C'D'}) = 0 \). Furthermore, \( f(C) = f(C') = 1 \) implies \( f(\overrightarrow{C'D'})(C) = f(\overrightarrow{C'D'})(C') = 1 \). On the other hand, by Claim 15, the equations \( f(\overrightarrow{C'D'})(C) = f(\overrightarrow{C'D'})(C') = 1 \), \( f(\overrightarrow{C'D'})(D') = 0 \) imply \( f(\overrightarrow{C'D'})(D) = 1 \), a contradiction.

Now we will look more closely at the functions \( f(\overrightarrow{AB}), f(\overrightarrow{C'D'}), \) and \( f(\overrightarrow{A'B'}) \). Since \( f(B) = 0 \), we have either \( f(\overrightarrow{AB})(B) = 0 \) or \( f(\overrightarrow{A'B'})(B) = 0 \). Without loss of generality we assume \( f(\overrightarrow{A'B'})(B) = 0 \). From \( f(\overrightarrow{AB})(A') = 1, f(\overrightarrow{A'B'})(A) = 1, f(\overrightarrow{A'B'})(B) = 0 \), and Claim 15 it follows that the points \( A, B, B' \) are not collinear and \( f(\overrightarrow{A'B'})(B') = 1 \). The latter together with the fact that \( f(B) = 0 \) imply \( f(\overrightarrow{C'D'})(B') = 0 \). By Corollary 16, the line \( \ell(\overrightarrow{A'B'}) \) intersects \( AB \) in a unique point, which we denote by \( X \), and \( A' \in X B' \).

Analysis similar to above shows that \( f(\overrightarrow{C'D'})(B') = 0 \) implies \( f(\overrightarrow{C'D'})(D) = 0 \) and that the line \( \ell(\overrightarrow{CD}) \) intersects \( A'B' \) in a unique point, which we denote by \( Y \), and \( C \in YD \). In turn, the equation \( f(\overrightarrow{C'D'})(D) = 0 \) implies \( f(\overrightarrow{A'B'})(D') = 0 \) and the intersection of \( \ell(\overrightarrow{CD}) \) and \( CD \) in a unique point denoted by \( Z \), and \( C' \in ZD' \). Finally, the equation \( f(\overrightarrow{A'B'})(D') = 0 \) implies that \( \ell(\overrightarrow{AB}) \) intersects \( C'D' \) in a unique point denoted by \( U \), and \( A \in UB \).

In the rest of the proof we will show that \( M_1(f) \subseteq \text{Conv}\{X, Y, Z, U\} \) and that \( X, Y, Z, U \) are interior points of \( \text{Conv}(g_{m,n}) \), which will lead to a contradiction (see Fig. 11). We will consider four different cases.

**Case 1.** The points \( X, Y, Z, U \) are pairwise distinct. First we will show that \( \text{Conv}\{X, Y, Z, U\} \) is a counterclockwise quadrilateral with the edges \( XY, YZ, ZU, \) and \( UX \) (see Fig. 11). Applied to \( f(\overrightarrow{AB}), f(\overrightarrow{C'D'}) \), Claim 15 yields \( A \in BC'D' \), and hence \( A \in UB \). The latter together with \( X \in AB \) imply that \( \overrightarrow{AB} \) and \( \overrightarrow{UX} \) have the same orientation. By similar arguments, \( \overrightarrow{A'B'} \) and \( \overrightarrow{XY} \), \( \overrightarrow{CD} \) and \( \overrightarrow{YZ} \), and \( \overrightarrow{C'D'} \) and \( \overrightarrow{ZU} \) have the same orientation respectively. Now we observe that the assumption \( Y \neq Z \) implies \( Z \notin \ell(\overrightarrow{A'B'}) \). Therefore, since \( f(\overrightarrow{A'B'})(C) = f(\overrightarrow{A'B'})(D) = 1 \) and \( Z \in CD \), the triangle \( A'B'Z \) is counterclockwise. Hence, by Claim 2, the triangle \( XYZ \) is counterclockwise. By similar arguments, the triangles \( YZU \), \( ZUX \), \( UXY \) are counterclockwise. Consequently, by Claim 4, \( \text{Conv}\{X, Y, Z, U\} \) is a quadrilateral \( XYZU \) with edges \( XY, YZ, ZU, \) and \( UX \).

Next, the inclusion \( \text{Conv}(M_1(f)) \subseteq XYZU \) follows from the fact that \( XYZU \) is a polygon circumscribed about \( \text{Conv}(M_1(f)) \). Indeed, each of the lines \( \ell(\overrightarrow{A'B'}) = \ell(\overrightarrow{XY}), \ell(\overrightarrow{CD}) = \ell(\overrightarrow{YZ}), \ell(\overrightarrow{C'D'}) = \ell(\overrightarrow{ZU}), \) and \( \ell(\overrightarrow{AB}) = \ell(\overrightarrow{UX}) \) is a tangent to \( \text{Conv}(M_1(f)) \), and \( A' \in XY \cap \text{Conv}(M_1(f)) \), \( C \in YZ \cap \text{Conv}(M_1(f)) \), \( C' \in ZU \cap \text{Conv}(M_1(f)) \), \( A \in UX \cap \text{Conv}(M_1(f)) \).

It remains to prove that all the points \( X, Y, Z, \) and \( U \) are interior points of \( \text{Conv}(g_{m,n}) \), i.e.
that A, B, C, D vertices. Then, by Theorem 11, either it follows that \( X \) with \( Y \). First we will show that \( \text{Conv}(M) \) and \( f \). Indeed, from \( \ell(AB) \) separates \( D \) and \( D' \). This contradiction proves that \( X \) does not belong to \( B(\mathcal{G}_{m,n}) \).

**Case 2.** \( X = Z \) or \( Y = U \). Suppose \( X = Z \). Then from \( X \in AB \) and \( Z \in CD \) it follows that \( AB \) and \( CD \) intersect. However, \( \{AB, CD\} \) is a proper pair of segments, and, by Corollary 21, we have \( M_1(f) = \{A\} \) (see Fig. 12a). Since \( f \) is a proper 2-threshold function, \( A \) is not a vertex of \( \text{Conv}(\mathcal{G}_{m,n}) \), and therefore Lemma 23 implies \( A \in \{1, \ldots, m - 2\} \times \{1, \ldots, n - 2\} \), as required. The case \( Y = U \) is symmetric and we omit the details.

**Case 3.** \( |\{X, Y, Z, U\}| = 3, X \neq Z, \text{ and } Y \neq U \). Let \( X = Y \), using the same arguments as in Case 1 it can be shown that \( XZU \) is a triangle circumscribed about \( \text{Conv}(M_1(f)) \), and that none of \( X, Z, \text{ and } U \) lies on the boundary of \( \mathcal{G}_{m,n} \). The cases \( X = U, Y = Z, \text{ and } Z = U \) are symmetric and we omit the details.

**Case 4.** \( |\{X, Y, Z, U\}| = 2 \) and \( X \neq Z, Y \neq U \). Then either \( X = Y \) and \( U = Z \) or \( X = U \) and \( Y = Z \). The two cases are symmetric and therefore we consider only one of them, namely, \( X = Y, U = Z \). First we will show that \( \text{Conv}(M_1(f)) = AC \). Indeed, from \( X \in AB, Y \in AB', \text{ and } AB' \) it follows that \( X = Y = A' \), and hence \( A = A' \) as \( AB \) is prime. Moreover, \( Y \in (CD) \) together with \( Y = A \) imply that \( A, C, D \) are collinear points, and hence \( \text{Conv}(\{A, B, C, D\}) \) has at most three vertices. Then, by Theorem 11, either \( A \in BD \) or \( C \in BD \) or both. All cases lead to the conclusion that \( A, B, C, D \) are collinear, and, by Corollary 19, we have \( \text{Conv}(M_1(f)) = AC \) (see Fig. 12b).

Now, it remains to show that \( A, C \notin B(\mathcal{G}_{m,n}) \). Conversely, suppose \( A \in B(\mathcal{G}_{m,n}) \) or \( C \in B(\mathcal{G}_{m,n}) \). Without loss of generality we assume the former, which in turn implies that \( \ell(AB) \) is a tangent to \( \text{Conv}(\mathcal{G}_{m,n}) \) as \( A \) is an interior point of \( BD \) and \( B, D \in \mathcal{G}_{m,n} \). We will arrive to a contradiction by showing that \( \ell(AB) \) separates \( B' \) and \( D' \). For this we observe that neither \( AB' \) nor \( CD' \) belongs to \( \ell(AB) \). Indeed, as by Theorem 11 \( AC \subset BD \), the inclusion \( AB' \subset \ell(AB) \) would imply that \( A'B' \) coincides either with \( AB \) or with \( CD \), and the inclusion \( CD' \subset \ell(AB) \) would imply that \( CD' \)

\[ X, Y, Z, U \notin B(\mathcal{G}_{m,n}), \] where
\[ B(\mathcal{G}_{m,n}) = \{0, m - 1\} \times \{0, n - 1\} \cup \{0, m - 1\} \times \{0, n - 1\}. \]

We will prove that \( X \notin B(\mathcal{G}_{m,n}) \), for the other three points the arguments are similar. Suppose, to the contrary, that \( X \in B(\mathcal{G}_{m,n}) \). Since \( X \in AB \) and \( A \in UB \), we have \( X \in UB \). We claim that \( X \) is an interior point of \( UB \). Indeed, \( X \neq U \) by the assumption. Furthermore, the equality \( X = B \) would imply \( A' \in BB' \), which is not possible as \( f_{AB'} \) is a threshold function and \( f_{AB'}(B) = 0, f_{AB'}(A') = 1, f_{AB'}(AB') = 0 \). Now, since both \( U \) and \( B \) belong to \( \text{Conv}(\mathcal{G}_{m,n}) \), and \( X \) is an interior point of \( UB \) and a boundary point of \( \text{Conv}(\mathcal{G}_{m,n}) \), we conclude that \( \ell(UB) = \ell(AB) \) is a tangent to \( \text{Conv}(\mathcal{G}_{m,n}) \). We will arrive to a contradiction by showing that \( \ell(AB) \) separates \( D \) and \( D' \). First, we observe that \( D' \neq \ell(AB) \), as otherwise we would have \( U \in D' \) and \( A \in D'B \), which is not possible as \( f_{AB} \) is threshold and \( f_{AB}(B) = 0, f_{AB}(A) = 1, f_{AB}(D') = 0 \). Consequently, \( ABD' \) is a clockwise triangle. On the other hand, the triangle \( ABD' \) is counterclockwise as the pair \( \{AB, CD\} \) is proper. Therefore, \( \ell(AB) \) separates \( D \) and \( D' \). This contradiction proves that \( X \) does not belong to \( B(\mathcal{G}_{m,n}) \).
Figure 13: $A = (4, 3)$, all the proper pairs of segments belong to the subgrid with the dashed boundary and $A$ in the center. The possible choices of $B$ are drawn on the left half of the subgrid.

coincides either with $AB$ or with $CD$. In each of the cases we would have a contradiction with (3). This observation together with the fact that $A', C' \in M_1(f) \subseteq AC \subset \ell(AB)$ imply that neither $B'$ nor $D'$ belongs to $\ell(AB)$. Consequently, as $f_{\overrightarrow{AB}}$ takes different values in $B'$ and $D'$ we conclude that $\ell(AB)$ separates $B'$ and $D'$, as required.

Fig. 11 provides the examples of proper 2-threshold functions with at least two distinct proper pairs of segments related to them. Moreover, the following statement shows that the number of distinct proper pairs of segments defining the same function can be as high as $O(mn)$:

**Claim 25.** Let $f$ be a $\{0, 1\}$-valued function over $\mathcal{G}_{m,n}$ with a unique true point $A = (a_1, a_2)$ such that $a_1 \in \{1, \ldots, m-2\}$ and $a_2 \in \{1, \ldots, n-2\}$. Then $f$ is a 2-threshold function, and the number of proper pairs of segments defining $f$ is at most

$$\frac{3}{\pi^2}mn + O(m \log n).$$

**Proof.** Without loss of generality we assume

$$a_1 \leq \frac{m-1}{2}, \quad a_2 \leq \frac{n-1}{2}. \quad (4)$$

Let $\overrightarrow{AB}$ and $\overrightarrow{AD}$ be distinct prime segments. By Theorem 11, the pair $\{\overrightarrow{AB}, \overrightarrow{AD}\}$ is proper if and only if both segments belong to the same line. Hence, if $\{\overrightarrow{AB}, \overrightarrow{AD}\}$ is proper, then $d(\overrightarrow{AB}) = d(\overrightarrow{AD})$, and therefore all the considered pairs of segments belong to a subgrid of size $(2a_1+1) \times (2a_2+1)$. Next, we notice that for any given proper pair $\{\overrightarrow{AB}, \overrightarrow{AD}\}$ the points $B$ and $D$ are symmetric to each other with respect to $A$. Therefore it is enough to estimate the number of choices for $B$. Let $B = (b_1, b_2), D = (d_1, d_2)$. The only proper pair with $b_1 = d_1$ is the pair where $\{B, D\} = \{(a_1, a_2 + 1), (a_1, a_2 - 1)\}$, so we can exclude this case and assume $b_1 \neq d_1$. By symmetry, we may also assume $b_1 < d_1$.

Putting all together and using a standard number-theoretical formula

$$\sum_{p=1}^{m} \sum_{q=1}^{n} \sum_{q \perp p} 1 = \frac{6}{\pi^2}mn + O(m \log n)$$

we derive the number of possible choices for $B$ (see Fig. 13):

$$\sum_{b_1=0}^{a_1-1} \sum_{b_2=0}^{2a_2+1} 1 = \sum_{p=1}^{a_1} \sum_{q=-a_2}^{a_2} \sum_{p \perp q} 1 = \frac{12}{\pi^2}a_1a_2 + O(a_1 \log a_2).$$

The target estimation follows from the latter by replacing $a_1, a_2$ with their upper bound (4).
Despite the above statement, it turns out that the number of 2-threshold functions is asymptotically equal to the number of proper pairs of segments. To prove this, we start with the following two claims.

Claim 26. Let \( \overrightarrow{AB}, \overrightarrow{CD} \) be a proper pair of segments in \( \mathcal{G}_{m,n} \), and let \( f \) be the 2-threshold function defined by \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \). If \( f \) does not have true points on the boundary of the grid, i.e. \( M_1(f) \subseteq \{1, \ldots , m-2\} \times \{1, \ldots , n-2\} \), then the distances \( d(A, \ell(\overrightarrow{CD})) \) and \( d(B, \ell(\overrightarrow{CD})) \) do not exceed one.

Proof. The statement is obvious for \( \ell(AB) = \ell(CD) \), so we assume that \( AB \) and \( CD \) are not collinear. Let us first assume that \( \ell(AB) \) and \( \ell(CD) \) are not parallel and denote by \( O \) the intersection point of the two lines. We start by showing that there exists a point \( X \in \ell(AB) \cap B(\mathcal{G}_{m,n}) \) such that \( AB \subseteq OX \). Indeed, since \( f(A) = 1 \), the point \( A \) is an interior point of \( \text{Conv}(\mathcal{G}_{m,n}) \), and hence the line \( \ell(AB) \) intersects \( B(\mathcal{G}_{m,n}) \) in exactly two points, which we denote by \( X \) and \( Y \). Furthermore, as \( \ell(CD) \) does not separate \( A \) and \( B \), we have either \( AB \subseteq OX \) or \( AB \subseteq OY \). Without loss of generality assume \( AB \subseteq OX \). Let \( Z \in B(\mathcal{G}_{m,n}) \) be the closest point to \( X \) such that \( f_{\overrightarrow{AB}}(Z) = 1 \). Clearly, \( d(X, Z) \leq 1 \). The assumption \( M_1(f) \subseteq \{1, \ldots , m-2\} \times \{1, \ldots , n-2\} \) implies that \( f(Z) = 0 \), and therefore \( f_{\overrightarrow{CD}}(Z) = 0 \). Hence, either \( Z \in \ell(CD) \) or the triangle \( CDZ \) is clockwise. The former implies that \( d(X, \ell(CD)) \leq 1 \). The latter leads to the same conclusion, if we notice that the triangle \( CDX \) is counterclockwise as \( X \) and \( A \) lie on the same side of \( \ell(CD) \), and hence \( \ell(CD) \) intersects \( XZ \). Finally, since \( A, B \in OX \), we conclude that \( \max\{d(A, \ell(CD)), d(B, \ell(CD))\} \leq d(X, \ell(CD)) \leq 1 \), as required.

The proof for parallel \( \ell(AB) \) and \( \ell(CD) \) is similar and uses the fact that the distance from any point of \( \ell(AB) \) to \( \ell(CD) \) is the same. \( \square \)

Claim 27. There are \( O(m^2n^2(m+n)^2) \) proper pairs of segments \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) in \( \mathcal{G}_{m,n} \) such that the 2-threshold function defined by \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) does not have true points on the boundary of \( \mathcal{G}_{m,n} \).

Proof. There are at most \( mn \) ways to choose each of \( C \) and \( D \). Given the segment \( CD \), by Claim 26, each of \( A \) and \( B \) lies at distance at most one from \( \ell(CD) \). Since there are \( O(m+n) \) such points, we conclude that there are \( O(m^2n^2(m+n)^2) \) desired pairs of segments. \( \square \)

We are now in a position to prove the main result of the section. Denote by \( q(m, n) \) the number of proper pairs of segments in \( \mathcal{G}_{m,n} \).

Theorem 28.

\[
t_2(m, n) = q(m, n) + O(m^2n^2(m+n)^2). \tag{5}
\]

Proof. Let \( t'_2(m, n) \) denote the number of proper 2-threshold functions over \( \mathcal{G}_{m,n} \). Since \( t_2(m, n) = t'_2(m, n) + t(m, n) \) and \( t(m, n) = O(m^2n^2) \), to prove (5), it is enough to show that

\[
t'_2(m, n) = q(m, n) + O(m^2n^2(m+n)^2). \tag{6}
\]

For this, we first notice that, by Corollary 18, every proper pair of oriented segments in \( \mathcal{G}_{m,n} \) defines a proper 2-threshold function. Furthermore, by Claim 27, only \( O(m^2n^2(m+n)^2) \) of these pairs define 2-threshold functions with no true points on the boundary of \( \mathcal{G}_{m,n} \). Finally, by Theorem 24, for any proper 2-threshold function that contains true points on the boundary of \( \mathcal{G}_{m,n} \) there exists a unique proper pair of segments in \( \mathcal{G}_{m,n} \) that defines the function, and equation (6) follows. \( \square \)

6 Conclusion

In this paper we introduced the notion of proper pairs of segments and revealed the relation between these objects and proper 2-threshold functions. We proved that a 2-threshold function with a true point on the boundary of the grid has a unique proper pair of segments that defines the function. Moreover, we showed that the number of 2-threshold functions is asymptotically equal to the number of the proper pairs of segments. This latter number is estimated asymptotically in the subsequent paper of the authors [34], which together with the results of the current paper implies the first asymptotic formula for the number of 2-threshold functions.
It is natural to wonder whether the approach we used to characterize 2-threshold functions can be generalized to higher order threshold functions, say to 3-threshold functions. One difference between 2-threshold and 3-threshold functions that might be an obstacle towards such a generalization is an observation that while almost all 2-threshold functions have true points on the boundary of the grid, this might not hold for 3-threshold. This is an issue for future research to explore.

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