Handbook of normal frames and coordinates

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References. The book is divided into chapters which have a sequential Roman enumeration. The chapters are divided into sections with a sequential Arabic enumeration, which is independent in each chapter. Some sections are divided into subsections.

In each chapter the subsections, equations, propositions, theorems, lemmas, and so on have a double independent enumeration of the form \( m.n \) or \((m.n)\) for the equations, where \( m \) is the number of the section in which the designated item appears and \( n \) is its sequential number in it. So, proposition 4.7 and (3.12) (or equation (3.12)) mean respectively proposition 7 in section 4 and equation 12 in section 3 of the current chapter. A suitable item from a chapter different from the current one is referred as \( R.m \), \( R.m.n \) or \((R.m.n)\) for equations, where \( R=I,II, \ldots \) is the Roman number of the chapter in which the item appears; e.g. remark II.5.3 and IV.4 (or section IV.4) mean respectively remark 3 in section 5 of chapter II and section 4 in chapter IV.

The footnotes are indicated as superscripts in the main text and have independent Arabic enumeration in each section. When we refer to a footnote, it is on the current page if the page on which it appears is not explicitly indicated.

Citations. An Arabic number in square brackets, e.g. [27], directs the reader to the list of references, i.e. in this example [27] means the 27th item from the Bibliography list beginning on page 397.

The ends of the proofs are marked by empty square sign, viz. with \( \square \).

Indices. The Latin indices refer to an arbitrary linear (vector) space, in particular to the tangent and cotangent spaces. If in a given problem are presented the tangent and cotangent spaces to a manifold and other vector space(s), then the indices referring to the first two spaces are denoted with small Greek letters; for the rest one(s) the Latin letters will be used.

Einstein’s summation convention: in a product of quantities or in a single expression, a summation over indices repeated on different levels is assumed over
the whole range in which they change. Any exception of this rule is explicitly stated.

**Symmetrization and antisymmetrization.** On indices included in (or surrounded by) round (resp. square) brackets a symmetrization (resp. antisymmetrization) with coefficient one over the factorial of their number is assumed. If some indices in such a group have to be excluded from this operation, they are included in (surrounded by) vertical bars.

**Matrix of linear mapping** with respect to a given basis, or bases, or field of, possibly local, bases: the same symbol but the kernel letter is in **boldface**. Exception: the matrix of a derivation (derivative operator) is denoted by boldface capital Greek letter gamma, i.e. by $\Gamma$, possibly with some indices.

**Matrix elements.** When the elements of a (two-dimensional) matrix are labeled by superscript and subscript, the superscript is considered as a first index, numbering the matrix’s rows, and the subscript as a second one, numbering the matrix’s columns. In this way the matrix of composition of linear mappings is equal to the product of the matrices of the mappings in the same order in which they appear in the composition and this does not depend on the way the matrix’s indices are situated.

**Free arguments.** If we want to show explicitly the argument(s) of some mapping or to single out it (them) as arbitrary while the other arguments, if any, are considered as fixed ones, we denote it (them) by (centered) dot, i.e. by $\cdot$. E.g., if $f: A \to C$ and $g: A \times B \to C$, then $f(\cdot) \equiv f$, $g(\cdot, \cdot) \equiv g$, and $g(\cdot, b), b \in B$, means $g(\cdot, b): A \to C$ with $g(\cdot, b): a \mapsto g(a, b)$ for all $a \in A$. 
Preface

The main subject of this book is an up-to-date and in-depth survey of the theory of normal frames and coordinates in differential geometry. The existing results, as well as new ones obtained lately by the author, on the theme are presented.

The text is so organized that it can serve equally well as a reference manual, introduction to and review of the current research on the topic. Correspondingly, the possible audience ranges from graduate and post-graduate students to scientists working in differential geometry and theoretical/mathematical physics. This is reflected in the bibliography which consists mainly of standard (text)books and journal articles.

The present monograph is the first attempt for collecting the known facts concerning normal frames and coordinates into a single publication. For that reason, the considerations and most of the proofs are given in details.

Conventionally local coordinates or frames, which can be holonomic or not, are called normal if in them the coefficients of a linear connection vanish on some subset, usually a submanifold, of a differentiable manifold. Until recently the existence of normal frames was known (proved) only for symmetric linear connections on submanifolds of a manifold. Now the problems concerning normal frames for derivations of the tensor algebra over a differentiable manifold are well investigate; in particular they completely cover the exploration of normal frames for arbitrary linear connections on a manifold. These rigorous results are important in connection with some physical applications. They may be applied for rigorous analysis of the equivalence principle. This results in two general conclusions: the (strong) equivalence principle (in its 'conventional' formulations) is a provable theorem and the normal frames are the mathematical realization of the physical concept of 'inertial' frames. The normal frames find other important physical application in the bundle formulation of quantum mechanics. It turns out that in a normal frame the bundle Heisenberg and Shrödinger pictures of motion coincide.

Applying some freedom of language, we can state the general physical idea: the normal frames are the most suitable ones for describing free objects and events, i.e. such that on them do not act any forces. Regardless of the different realizations of that idea in general relativity and its generalizations, quantum mechanics, gauge theories etc., there is an underlying mathematical background for the general description of such situations: the existence (or non-existence) of normal frames.
in vector bundles. This observation fixes to a great extent the mathematical tools required for the description of some fundamental physical theories.

In the book, formally, may be distinguished three parts: The first one includes chapters I–III and deals with a variety of mathematical problems concerning normal frames and coordinates on differentiable manifolds. The second part consists of chapters IV and V and investigates normal frames (and possibly coordinates) in vector bundles and differentiable bundles, respectively. The last part, involving the text after chapter V, contains inquiry material.

The requisite mathematical language required for the description of normal frames is spread over the initial sections of the chapters. In particular, sections I.2–I.4, III.2, IV.9, IV.2, IV.14.1 and V.2–V.5 can be collected into an introductory chapter under the title “Mathematical preliminaries” \(^1\) but this is not done by pedagogical reasons. \(^2\) The normal coordinates and frames, in the case of linear connections on a manifold, are initially introduced in chapter I. It contains our basic preliminary material and a review of the Riemannian coordinates. Chapter II is devoted to the existence, uniqueness, construction and other related problems concerning normal frames and coordinates in manifolds endowed with linear connection. It presents, in historical order, a detailed review of the existing literature as well as generalization of a number of results, e.g. for connections with torsion. Further, in chapter III, problems connected with the existence, uniqueness, holonomicity etc. of normal frames for arbitrary derivations of the tensor algebra over a manifold are investigated. Next (chapter IV), the same range of problems is explored for normal frames for linear transports in vector bundles. This material covers completely the special case of normal frames for linear connections in vector bundles or on a differentiable manifold. The main aim of chapter V is the exploration of normal frames (and coordinates, if any) for general connections on differentiable fibre bundles which, in particular, can be vector ones.

The general approach of the book is essentially coordinate-dependent or basis-dependent. This is due to its basic subject: frames, bases or coordinates with some special properties. However, if possible and suitable, the coordinate-free notation and methods are not neglected.

The basic mathematical prerequisites vary from chapter to chapter but generically they include the grounds of vector (linear) spaces, differentiable manifolds, vector bundles, connection theory, and a firm belief in the existence and uniqueness theorems of ordinary differential equations. Some of the corresponding concepts and results are reproduced in our text but the acquaintance with adequate literature is required. Appropriate references are given in the Introductions to the chapters and directly in the main text.

The material is so organized that a successive chapter generalizes the pre-

\(^1\) As (practically) any ‘preliminary’ knowledge requires for its understanding some other ‘preliminary’ to it knowledge, in the corresponding sections are cited a number of works containing this second kind of mathematical ‘luggage’.

\(^2\) The material is so organized, that the required concepts and results appear in the logical order in which they are necessary for some particular purpose(s).
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7 July, 2006
PREFACE
Acknowledgments

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Chapter I

Manifolds, normal frames and Riemannian coordinates

The basic differential-geometric concepts, such as differentiable manifolds and mappings, tensors and tensor fields, and linear connections, on which the book rests, are introduced. Partially the notation and terminology employed are fixed. The normal frames and coordinates are defined as ones in which the coefficients of a linear connection in them vanish on some set. Certain their general properties are mentioned. The Riemannian coordinates, which are normal at their origin, are described.
1. Introduction

The goal of this chapter is twofold: it introduces most of the basic preliminary definitions and results on which our investigation rests (sections 2, 3, and 4) and it begins the study of the normal frames and coordinates (sections 5 and 6).

The main concepts of differential geometry required for the understanding of the book are: differentiable manifolds and mappings, submanifolds, Riemannian manifolds, tangent vectors and vector fields, tensors and tensor fields, linear connections. The readers acquainted with them may only look over the corresponding sections for our notation, omitting the major text to which they may wish to return later, following the references to it.

In more details, the contents of the chapter is as follows.

The purpose of Sect. 2 is to fix our terminology and notation concerning differentiable manifolds and some typical to them natural structures. This is not a summary of the differential geometry, only certain basic concepts and particular relations between them required for our future aims are presented. At first the concepts of topological and differentiable manifolds are introduced, then tangent vectors, cotangent vectors, and tensors and the corresponding fields of them on a manifold are defined. Also some expressions in local bases (or frames) and coordinates are given. If the reader is acquainted with all this, he/she can simply look over this section for our notation skipping the main text. A reader interested in deeper understanding of these concepts, as well as in differential geometry as a whole, should consult with the specialized literature. Here is a (random) selection of such titles. An elementary introduction to differential geometry, with ‘physical’ orientation, can be found in [1–6]. The same purpose can serve the books [7–10] which are more ‘mathematically’ oriented. Our text follows the excellent (text)books [11,12]. At last, the advanced works [13–16] can be recommended. A brief synopsis of the mathematics preceding the introduction of manifolds is given it [9,14,16] while [12,17] contain an expanded presentation of the ‘preliminary’ to manifolds material. Of course, the reading of all of the above-mentioned serious books is not necessary for the understanding of what follows. For this end, the reading of Sect. 2 is sufficient and the references cited may be consulted for more detains and proofs of some assertions. The knowledge of the tensor analysis in coordinate-dependent language is desirable [18,19]. It is almost sufficient for the most of this and subsequent chapters.

In Sect. 3, we introduce the concept of linear connection on a manifold. The approach chosen is, in a sense, middle between elementary books on general relativity, such as [20,21], and pure mathematical ones on differential geometry, like [11,22]. We have tried to follow closely [9,11,12] but the abstracting material is adapted to the goals of the present book. After a motivation for what the connections are needed for, we introduce the linear connections via a system of axioms for the covariant derivative of the algebra of tensor fields over a given manifold. We employ this method since the theory of vector bundles, which is not required for chapters I–III, will be involve into action only at the beginning of
chapter IV. In this connection, let us mention that the linear connections can be defined only on the algebra of vector fields on a manifold (i.e. to the tangent to it bundle), and then they admit a unique extension on the whole algebra of tensor fields [11, chapter 3, proposition 7.5]. A more advanced and deep treatment of the theory of linear connections on manifolds and vector bundles can be found in [10,13,15,22,23]. We also present the notion of a parallel transport (induced by a linear connection) which will practically step on scene in chapter IV but here is a natural place for it to appear. It will be used in chapters I–III for proving and formulating some results. Sect. 3 ends with a brief consideration of the geodesics and exponential mapping.

The concept of Riemannian metric and Riemannian connection are given in Sect. 4. If the reader is interested in essence of Riemannian geometry, he/she is referred, for example, to [8–12,19,24–27].

In Sect 5, we introduce the main objects of our investigation, the normal frames and coordinates. We define them as ones in which the coefficients of a linear connection vanish on a given set. Some considerations on the uniqueness and (an)holonomicity of the normal frames are presented too.

Sect. 6 contains a complete description of normal frames at a given point on \((C^\infty)\) Riemannian manifolds. This is done on the base of Riemannian coordinates which turn to be normal at their origin. The geodesic coordinates are pointed as other example of coordinates normal at a point. Some general results, proved further in chapter II, concerning the existence of normal frames on submanifolds are quoted. An expanded presentation of the problem of existence of normal coordinates at a point of a \(C^\infty\) Riemannian manifold is given in [19,24], where also a list of original early works on this topic can be found.

Sect. 7 are presented a number of examples and exercises of concrete Riemannian connection and coordinates/frames normal for them on different sets. At first, the (locally) Euclidean and one-dimensional manifolds are considered. The (pseudo)spherical coordinates on (pseudo)spheres are (partially) investigated for sets on which they are normal for the Riemannian connection induced on them by the metric on them generated by the Euclidean one of the Euclidean space in which the (pseudo)spheres are embedded. Similar instance on the two dimensional torus is presented. The cosmological models of Einstein, de Sitter and Schwarzschild are considered (in concrete coordinates) from the view-point of normal frames/coordinates on them. Some peculiarities of the light cone in Minkowski spacetime are pointed too.

Sect. 8 deals with certain terminological problems concerning bases and frames. Some links between these concepts are explicitly formulated and/or derived.

The chapter ends with some general remarks and conclusions in Sect. 9.
Chapter II

Existence, uniqueness and construction of normal frames and coordinates for linear connections

An in-depth investigation of existence, uniqueness and construction of frames and coordinates normal for linear connections on manifolds is given. Detailed review of the literature dealing with normal coordinates is presented. Some proofs are improved/generalized which entails a number of new results. Similar problems in the case with non-zero torsion are studied. Main results: For arbitrary (resp. torsionless) connections frames (resp. coordinates) normal at a single point and along path exist; they exist on submanifolds of higher dimensions iff the parallel transport along paths lying in them is path-independent. Complete constructive description of all, if any, frames and coordinates normal for arbitrary linear connections.
CHAPTER II. NORMAL FRAMES FOR CONNECTIONS

1. Introduction

This chapter presents a complete exploration of the problems linked to the existence, uniqueness, and construction of normal coordinates and frames for manifolds endowed with a linear connection, with or without torsion. The review of the literature dealing with normal coordinates is mixed with new results. Such are, first of all, the ones concerning normal frames, connections with non-vanishing torsion, and the complete constructive description of the normal coordinates, if any.

The methods for description of normal coordinates/frames on Riemannian manifolds can mutatis mutandis be transferred on arbitrary manifolds, real or complex \((K = \mathbb{R}, \mathbb{C})\), \(^1\) endowed with linear connection. The possibility for this is hidden in the fact that the existence and properties of the normal coordinates/frames on a Riemannian manifold is intrinsically connected with the properties of the Christoffel symbols, i.e. with the Riemannian connection, not with the particular metric generating them. After this situation was clearly understood, somewhere in 1922–1927 [50, 72–74] (see [19, p. 155] for other references), the attention of the mathematicians, working in the field, was completely switched to the exploration of normal coordinates on manifolds with linear connections. Practically only the symmetric (torsionless) case has been investigated (see the comments after remark I. 5.4 on page 41). Some random works, like [44, 75], dealing with the asymmetric case (non-zero torsion) do not add nothing new as they simply note that the symmetric parts (I.3.9) of the connection coefficients (in coordinate frame) are coefficients of a symmetric linear connection to which the known results for torsionless connections are applicable.

Below in this chapter, in more or less modern terms and notation, are reviewed all results concerning the existence of normal coordinates/frames on manifolds endowed with symmetric linear connection. It contains a number of original new results too.

At first (Sect. 2), we concentrate on coordinates or frames normal at a single point. We present the known classical methods in this field [18, 19, 70] and then, modifying the methods that will be given in chapter III in full generality, we present a full description of these coordinates/frames.

In Sect. 3 the attention is turned on the coordinates or frames normal along paths without self-intersections. For symmetric linear connections, we give a detailed description of the Fermi coordinates as the first known coordinates of this kind with [19] being our basic reference. Then, modifying the methods developed for similar but more general problems (see chapter III and [76]), we derive a complete description of all coordinates or frames normal along paths without self-intersections or along locally injective paths in manifolds with symmetric or, respectively, arbitrary linear connections.

Several pages deal with problems concerning normal frames and coordinates on submanifolds with maximum dimensionality (Sect. 4), in particular on neigh-

\(^1\) In the literature is often supposed \(K = \mathbb{R}\) but this does not influence the results.
1. INTRODUCTION

borhoods and on the whole manifold. We prove that such frames or coordinates exist iff the connection is (locally) respectively flat or flat and torsionless. A complete description of the normal frames and coordinates in these cases is presented. We also point to some links between normal frames and parallel transports for flat linear connections.

Section 4 explores the problems of existence, uniqueness, and construction of frames or coordinates normal on arbitrary submanifolds. The classical results of [55] are reproduced in details using modern notation. Meanwhile, the corresponding proofs are improved, some results are generalized for arbitrary connections, with or without torsion, and new ones are presented. Next, we provide a complete constructive description of all frames (resp. coordinates) normal on submanifolds of a manifold with arbitrary (resp. torsionless) linear connection. Amongst a number of general results, we prove that normal on a submanifold frames (resp. coordinates) exist iff the parallel transport is path-independent along paths lying entirely in it (resp. and the connection is torsionless).

Section 5 contains instances and exercises illustrating the general theory of this chapter. Explicit expressions for frames and coordinates normal at a single point in and along a great circle on a two dimensional sphere are presented in a case of the Riemannian connection induced from the Euclidean space in which the sphere is embedded. Some problems connected with frames/coordinates normal for Weyl connections are investigated. All frames/coordinates normal in the one-dimensional case are explicitly described. Similar problem is solved along a geodesic path in 2-dimensional manifold. All coordinates normal at a point in Einstein-de Sitter spacetime are found.

A brief recapitulation of the above items can be found in Sect 6.
Chapter III

Normal frames and coordinates for derivations on differentiable manifolds

The existence, uniqueness, and construction of frames and coordinates normal for derivations (along vector fields, fixed vector field, paths, and fixed path) of the tensor algebra over a manifold are explored in details. For arbitrary vector fields or paths, normal frames (resp. coordinates) exist always (resp. if the torsion vanishes); on other submanifolds or along more general mappings necessary and sufficient conditions for such existence are derived. For derivations along fixed vector field or path normal frames and coordinates exist always. With a few exceptions, a complete constructive description of the normal frames and coordinates, if any, is presented. Frames simultaneously normal for two derivations are studied. With respect to the normal frames, the unique role of the linear connections amongst the other derivations is pointed out.
1. Introduction

The aim of this chapter is the investigation of frames and coordinates normal for different kinds of derivations of the tensor algebra over a differentiable manifold. Since the linear connections are a particular example of such derivations, the presented here material is a direct continuation and generalization of the one in chapter II. But, as we shall see, a number of problems concerning normal frames and charts for general derivations are ‘locally’ reduced to the same problems for linear connections and, consequently, their (local) solutions could be found, in more or less ready form, in chapter II.

Some of the results in the present chapter are partially based on the ones in the series of works [76, 80, 83–87] and are completely revised and generalized their versions. But most of the material is new and original.

Sect. 2 has an introductory character. The concepts of derivations and derivations along vector fields of the tensor algebra over a manifold are introduced. Their components, coefficients (if they are linear), curvature, and torsion are defined. Next, in section 3, the normal frames and coordinates are defined as ones in which the components of a derivation along vector fields vanish (on some set). The equations describing the transition to normal frames or coordinates are derived and the linearity of a derivation along vector fields is pointed as a necessary conditions for their existence.

In Sect. 4 (resp. Sect. 5) is proved that at a single point (resp. along a (locally injective) path) frames normal for a linear at it (resp. along it) derivation along vector fields always exist and their complete descriptions are given. Besides, if the derivation is torsionless, all normal coordinates are found. In sections 6–8, the problems of existence, uniqueness, and complete description of frames and local charts (or coordinates) on neighborhoods, on submanifolds, and along (injective or locally injective) mappings, respectively, for derivations along vector fields are studied in details and solved.

To the problems concerning frames or coordinates normal for derivations along fixed vector field is devoted Sect. 9. The existence of normal frames and coordinates in this case is proved. A complete description of the frames normal at a single point, along a path, and on the whole manifold are presented. The local charts (or coordinates) normal at a point are completely described. Along a path the explicit system of differential equations, which the normal coordinates must satisfy and which always have (local) solutions, is derived. A method for obtaining the coordinates (locally) normal on the whole manifold is pointed in the $C^\infty$ case.

Normal frames for derivations along paths are investigated in section 10. After the introduction of the basic definitions and notation, it is proved that frames normal for a derivation along a given (fixed) path always exist and their general form is found. A (local) holonomic extension of such frames, as well as of any other frame defined only along a path, is constructed. For derivations along arbitrary paths is proved that they admit normal frames iff they are covariant derivatives along paths induced by linear connections for which normal frames
exist. Since the normal frames for the derivations and connections turn to be identical, all problems for these frames are transferred to similar ones considered in chapter II.

Section 11 deals with problems connected with frames simultaneously normal for two derivations along arbitrary/fixed vector field or path. Necessary and sufficient conditions for the existence of such frames are found. In particular, in the case of arbitrary vector field or path, they exist iff the two derivations coincide. Normal frames for mixed linear connections are explored. It is shown that this range of problems is completely and equivalently reduced to similar one for two, possibly identical, linear connections, the contra- and co-variant ‘parts’ of the initial mixed connection.

In section 12 are collected and commented some results concerning linear connections obtained in the preceding sections of this chapter.

Section 13 illustrates the theory of the preceding sections with concrete examples.

Section 14 contains a discussion of some terminological problems linked to the normal frames or coordinates.

The chapter ends with certain general remarks in Sect. 15.
CHAPTER III. NORMAL FRAMES ON MANIFOLDS
Chapter IV

Normal frames in vector bundles

The theory of linear transports along paths in vector bundles, generalizing the parallel transports generated by linear connections, is developed. The normal frames for them are defined as ones in which their matrices are the identity one. A number of results, including theorems of existence and uniqueness, concerning normal frames are derived. Special attention is paid to the case when the bundle’s base is a manifold. The normal frames are defined and investigated also for derivations along paths and along tangent vector fields in the last case. Frames normal at a single point or along a given path always exist. On other subsets normal frames exist only in the curvature free case. The privileged role of the parallel transports is pointed out in this context.
1. Introduction

The analysis of corollary II. 4.4 on page 123 reveals that the properties of the parallel transport assigned to a linear connection, not directly the ones of the connection itself, are responsible for the existence of frames normal on a submanifold for the connection. \(^1\) This observation forms the groundwork of the idea the ‘normal’ frames to be defined directly for (parallel) transports without referring to the concept of a (linear) connection (or some other derivation along vector fields). The main obstacle for the realization of such an approach to ‘normal frames’ is that, ordinary, the concept of a parallel transport is a secondary one, it is introduced on the base of the concept of a (linear) connection. To the solution of the last problem and the development of the mentioned approach to normal frames (in finite dimensional vector bundles) is devoted the present chapter of the book. As we shall demonstrate below, the consistent realization of the above idea leads to a completely new look on the ‘normal frames’, which is self-contained and incorporates as special cases all of the results of the preceding chapters.

The material in sections 3–6 and 8 is based on the work \([102]\) and the one after them is practically new and written especially for the present book. \(^2\)

In the present chapter is studied a wide range of problems concerning frames normal for linear transports and derivations along paths in vector bundles and for derivations along tangent vector fields in the case when the bundle’s base is a differentiable manifold. In the last case, when tangent bundles are concerned, the only general result, known to the author and regarding normal frames, is \([23, p. 102, \text{theorem 2.106}]\).

The structure of this chapter is as follows.

Sect 2 introduces some basic concepts from the theory of (fibre) bundles, in particular of the one of vector bundles, required for the investigations in this chapter. After the concepts of bundle, section, and vector bundle are fixed, a special attention to the ones of liftings of paths and derivations along paths, which will play an important role further, is paid. The tensor bundles over a manifold are pointed as particular examples of vector bundles. Details on these and many other concepts regarding (fibre) bundles, the reader can find in the monographs \([7, 11, 106–110]\).

Sect. 3 is devoted to the general theory of linear transports along paths in vector fibre bundles which is a far reaching generalization of the theory of parallel transports generated by linear connections. \(^3\) The general form and other properties of these transports are studied. A bijective correspondence between them and derivations along paths is established. In Sect. 4, the normal frames are defined

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\(^1\) Here the situation is similar to the one described in the second paragraph of Sect. II. 1 on page 74: the properties of the Christoffel symbols, not directly the ones of the Riemannian metric generating them, are fully responsible for the existence of coordinates normal at a single point in a Riemannian manifold.

\(^2\) Although, some initial ideas and results are borrowed from the papers \([103–105]\).

\(^3\) This section is based on the early works \([101, 105, 111–115]\) of the author. For some more general results, see chapter V.
1. INTRODUCTION

as ones in which the matrix of a linear transport along paths is the identity (unit) one or, equivalently, in which its coefficients, as defined in Sect. 3, vanish 'locally'. A number of properties of the normal frames are found. In Sect. 5 is explored the problem of existence of normal frames. Several necessary and sufficient conditions for such existence are proved and the explicit construction of normal frames, if any, is presented.

Sect. 6 concentrates on, possibly, the most important special case of frames normal for linear transports or derivations along smooth paths in vector bundles with a differentiable manifold as a base. A specific necessary and sufficient condition for existence of normal frames in that case is proved. In particular, normal frames may exist only for those linear transports or derivations along paths whose (2–index) coefficients linearly depend on the vector tangent to the path along which they act. Obviously, this is a generalization of the derivation along curves assigned to a linear connection. Sect. 8 is devoted to problems concerning frames normal for derivations along tangent vector fields in a bundle with a manifold as a base. Necessary and sufficient conditions for the existence of these frames are derived. The conclusion is made that there is a one-to-one onto correspondence between the sets of linear transports along paths, derivations along paths, and derivations along tangent vector fields all of which admit normal frames.

In the first part of Sect. 9, based on [103], the concept of a curvature of a linear transport along paths is introduced and some its properties are explored. In its second part, relations between the curvature of a linear transports along paths and the frames normal for them are studied. The main result is that only the curvature free transports admit normal frames. The concept of a torsion of a linear transport along paths in the tangent bundle over a manifold is introduced in Sect. 10 (cf. the early paper [103]). Links between the torsion and holonomic normal frames are investigated. The vanishment of the torsion is pointed as a necessary and sufficient condition for existence of normal coordinates on submanifolds. If such coordinates exist, their complete description is given.

Sect. 11 deals with parallel transports in the tangent bundles over manifolds and frames normal for these transports. It is shown that the parallel transport assigned to a linear connection is a special kind of a linear transport in tangent bundles. As a side result, an axiomatic definition of a parallel transport is obtained, on the base of which a new definition of a linear connection, equivalent to the usual one, is given. The flat parallel transports are pointed as the only linear transports along paths in tangent bundles which transports admit normal frames. The coordinates normal for flat and torsionless parallel transports are explicitly presented.

Sect. 12 concerns a special type of normal frames in which the 3-index coefficients, if any, of a linear transport along paths vanish.

Sect. 13 is similar to Sect. 11, but it deals with the interrelations between different types of derivations along vector fields over a manifold and the linear transports along paths in the tangent bundle over it. As examples, particular derivations or transports, such as Fermi-Walker, Jaumann, etc., are considered.
CHAPTER IV. NORMAL FRAMES IN VECTOR BUNDLES

The aim of Sect. 14 is twofold. On one hand (Subsections 14.1–14.3), the rigorous relations between the theory of linear transports along paths in vector bundles and the one of parallel transports and connections in these bundles are investigated. On the base of the axiomatic approach to the theory of parallel transports, as presented in [23], we show how it (and hence the one of connections) is incorporated as a special case in the general theory of linear transports along paths. On another hand (subsections 14.4 and 14.5), we demonstrate how the results concerning normal frames and derived for linear connections on manifolds and linear transports along paths are almost in extenso applicable to the theory of parallel transports and connections on vector bundles.

In Sect. 15 is introduced the notion of autoparallel paths in manifolds whose tangent bundle is endowed with a linear transport along paths. If this transport is a parallel one, it is proved that the autoparallels coincide with the geodesics of the linear connection generating the transport.

The chapter ends with some notes in Sect. 17.

All fibre bundles in this chapter are vectorial ones. The base and total bundle space of such bundles can be general topological spaces. However, if some kind of differentiation in one/both of these spaces is needed to be introduced (considered), it/they should possess a smooth structure; if this is the case, we require it/they to be smooth, of class \( C^1 \), differentiable manifold(s). Starting from Sect. 6, the base and total bundle space are supposed to be \( C^1 \) manifolds. Sections 3–5 do not depend on the existence of a smoothness structure in the bundle’s base. Smoothness of the bundle space is partially required in sections 2–5. \(^4\)

\(^4\) The bundle space is required to be a \( C^1 \) manifold in Sect 2 (starting from definition 2.1), in definition 4.1', in proposition 4.1–4.2, if (4.1c) and (4.1d) are taken into account, in theorem 5.2, and in proposition 5.6.
Chapter V

Normal frames for connections on differentiable fibre bundles

The general connection theory on differentiable fibre bundles, with emphasis on the vector ones, is partially considered. The theory of frames normal for general connections on these bundles is developed. Links with the theory of frames normal for linear connections in vector bundles are revealed. Existence of bundle coordinates normal at a given injective horizontal point and/or along injective horizontal mappings is proved. The concept of a transport along paths in differentiable bundles is introduced. Different links between connections, parallel transports (along paths) and transports along paths are investigated.
1. Introduction

All connections considered until now, on manifolds and on vector bundles, were linear. It is well known that there exist non-linear connections on vector bundles as well as on non-vector ones. Can normal frames (and/or coordinates) be introduced for such more general connections? The positive solution of that problem is the main goal of the present chapter of this book. For the purpose and for a comparison with the definitions and results already obtained is required some preliminary material on general connection theory on differentiable bundles, which is collected in sections 2–5. On its base, the normal frames for connections on such bundles are studied in sections 6 and 7.

Sections 2–5 follow the work [137], sections 6 and 7 are a slightly revised version of [139], and section 8 reproduces in a modified form the paper [118].

The work is organized as follows.

In Sect. 2 is collected some introductory material, like the notion of Lie derivatives and distributions on manifolds, needed for our exposition. Here some of our notation is fixed too.

Section 3 is devoted to the general connection theory on bundles whose base and bundles spaces are differentiable manifolds. From different view-points, this theory can be found in many works, like [6, 7, 10–13, 16, 28, 60, 98, 106, 107, 117, 138, 140–146]. In Subsect. 3.1 are reviewed some coordinates and frames/bases on the bundle space which are compatible with the fibre structure of a bundle. Subsect. 3.2 deals with the general connection theory. A connection on a bundle is defined as a distribution on its bundle space which is complimentary to the vertical distribution on it. The notions of parallel transport generated by connection and of specialized frame are introduced. The fibre coefficients and fibre components of the curvature of a connection are defined via part of the components of the anholonomicity object of a specialized frame. Frames adapted to local bundle coordinates are introduced and the local (2-index) coefficients in them of a connection are defined; their transformation law is derived and it is proved that a geometrical object with such transformation law uniquely defines a connection.

In Sect. 4, the general connection theory from Sect. 3 is specified on vector bundles. The most important structures in/on them are the ones that are consistent/compatible with the vector space structure of their fibres. The vertical lifts of sections of a vector bundle and the horizontal lifts of vector fields on its base are investigated in more details in Subsect. 4.1. Subsect. 4.2 is devoted to linear connections on vector bundles, i.e. connections such that the assigned to them parallel transport is a linear mapping. It is proved that the 2-index coefficients of a linear connection are linear in the fibre coordinates, which leads to the introduction of the (3-index) coefficients of the connection; the latter coefficients being defined on the base space. The transformations of different objects under changes of vector

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1 The presentation of the material in sections 2–4 is according to some of the main ideas of [138, chapters 1 and 2], but their realization here is quite different and follows the modern trends in differential geometry.
bundle coordinates are explored. The covariant derivatives are introduced and investigated in Subsect. 4.3. They are defined via the Lie derivatives [138] and a mapping realizing an isomorphism between the vertical vector fields on the bundle space and the sections of the bundle. The equivalence of that definition with the widespread one, defining them as mappings on the module of sections of the bundle with suitable properties, is proved. In Subsect. 4.4, the affine connections on vector bundles are considered briefly.

In Section 5, some of the results of the previous sections are generalized when frames more general than the ones generated by local coordinates on the bundle space are employed. The most general such frames, compatible with the fibre structure, and the frames adapted to them are investigated. The main differential-geometric objects, introduced in the previous sections, are considered in such general frames. Particular attention is paid on the case of a vector bundle. In vector bundles, a bijective correspondence between the mentioned general frames and pairs of bases, in the vector fields over the base and in the sections of the bundle, is proved. The (3-index) coefficients of a connection in such pairs of frames and their transformation laws are considered. The covariant derivatives are also mentioned on that context.

The theory of normal frames for connections on bundles is considered in Section 6. Subsect. 6.1 deals with the general case. Loosely said, an adapted frame is called normal if the 2-index coefficients of a connection vanish in it (on some set). It happens that a frame is normal if and only if it coincides with the frame it is adapted to. The set of these frames is completely described in the most general case. The problems of existence, uniqueness, etc. of normal frames adapted to holonomic frames, i.e., adapted to local coordinates, are discussed in Subsect. 6.2. If such frames exist, their general form is described. The existence of frames normal at a given point and/or along an injective horizontal path is proved. The flatness of a connection on an open set is pointed as a necessary condition of existence of (locally) holonomic frames normal on that set. Some links between the general theory of normal frames and the one of normal frames in vector bundles, presented in chapter IV, are given in Subsect. 6.3. It is proved that a frame is normal on a vector bundle with linear connection if and only if in it vanish the 3-index coefficients of the connection. The equivalence of the both theories on vector bundles is established.

In section 7 is formulated and proved a necessary and sufficient condition for existence of coordinates normal along injective mappings with non-vanishing horizontal component, in particular along injective horizontal mappings.

Section 8 is devoted to some aspects of the axiomatical approach to parallel transport theory [17, 23, 30–33, 91, 147–150] and its relations to connection theory; it is based on the paper [118]. It starts with a definition of a transport along paths in a bundle and a result stating that, under some assumptions, it defines a connection. The most important properties of the parallel transports generated by connections are used to be (axiomatically) defined the concept of a parallel transport (irrespectively to some connection on a bundle). In a series of results
are constructed bijective mappings between the sets of transports along paths satisfying some additional conditions, connections, and parallel transports. In this way, two different, but equivalent, systems of axioms defining the concept “parallel transport” will be established.

The chapter ends with some remarks and conclusions in Sect. 9.
CHAPTER V. NORMAL FRAMES FOR CONNECTIONS ON BUNDLES
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