Transport properties and first arrival statistics of random searches with stochastic reset times

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Stochastic resets have lately emerged as a mechanism capable of generating finite stationary mean square displacement (MSD) when they are applied to diffusive random searches. Furthermore, walkers which have an infinite search time or mean first arrival time (MFAT) may become finite when they are randomly restarted. In this work we study these emerging phenomena from a unified perspective. Concretely, on one hand we study the existence of a finite MSD in the asymptotic limit when resets are applied to a random walker with movement given by \( \langle x^2(t) \rangle \sim t^p \) with \( 0 < p \leq 2 \).

In the particular case of exponentially distributed reset times, a compact formula is derived for the stationary MSD of the overall process in terms of the mean reset time and the MSD of the search process. On the other hand, we also test the robustness of the MFAT finiteness for random walks subject to resets with different types of processes and reset distributions.

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I. INTRODUCTION

The strategies employed by animals when they seek for food are complex and strongly dependent on the species. A better understanding of their fundamental aspects would be crucial to control some critical situations as the appearance of invading species in a certain region or preventing weak species to extinct, for instance.

In the last decades a lot of effort has been put in the description of the territorial motion of animals [1]. Among the models that have been proposed random walk models as correlated random walks and Lévy walks [2, 3] or Lévy flights [4] are the most commonly used. Nevertheless, in the vast majority of these approaches, only the foraging stage of the animals territorial dynamics is described (i.e. the motion patterns while they are collecting), leaving aside the fact that some species return to their nest after reaching their target.

On this direction, Evans and Majumdar [5] studied the properties of a macroscopic model consisting on a diffusive process subject to resets with constant rate (mesoscopically equivalent to consider exponentially distributed reset times), which introduces this back-to-the-nest stage. For this process, the mean first passage time (MFPT) was found to be finite and the MSD to collapse to a stationary value unlike a simple diffusive process without resets, which has infinite MFPT and the MSD scales linearly with time. The fact that the system achieves a stationary state allows us to define the territoriality of a given species being a quantitative measure of the region that animals occupy around its nest.

From then on, multiple works have been published generalizing this seminal paper keeping a diffusion foraging process [6–18], introducing for instance absorbing states [7] or \( d \)-dimensional diffusion [10]. Some works have been devoted to the study of Lévy flights when they are subject to constant rate resets [19, 20] and others have focused on the analysis of first passage processes subject to general resets [21–28]. Also, stochastic resets have been studied as a new element within the continuous-time random walks (CTRWs) formulation [24–28].

Nevertheless, despite of the amount of works devoted to this topic, the robustness of the properties found in [5] have not been explicitly tested and there is no general study of when they are preserved. In this work we address this issue by analyzing general properties of foraging processes with resets from a general mesoscopic perspective by considering random reset times distributed according to a given probability distribution function (PDF). From all the existing papers, in [29] Eule and Metzger perform a similar study to ours but taking Langevin dynamics as the foraging process description. Our work differs from that one in the fact that we model the underlying foraging process by a propagator \( P(x, t) \) directly, which allows us to derive an elegant and treatable expression for the MSD of the overall process in terms of the foraging process one (see Eq. (2.3)) and it also eases the inclusion of processes which are not trivial to model in the Langevin picture as Lévy flights and Lévy walks.

This paper is organized as follows. In Section II A we find an expression for the propagator of the overall process (i.e. the foraging process with restarts) in the Laplace-position space and a general formula for the MSD of the overall process in terms of the foraging process’ MSD; the first passage properties of the system are studied in Section II B. In Section III we apply the general results to three different free processes and in Section IV we use the formalism to study the transport properties and the first arrival of a random walk which is subject to a potential force. Finally, we conclude the work in Section V. 


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II. GENERAL FORMULATION

In this section we apply a general formalism to study both the transport properties of a random walk and its first arrival behaviour. Concretey, we derive expressions with which the global properties of the system can be found in terms of the same magnitude about the search process. Despite the ideas used herein could be applied to many different measures, we focus in two which are of special interest in random walk processes: the MSD and the FAT.

A. Transport properties

Let us consider a search process which we describe by its propagator $P(x,t)$ starting at $x = 0$ and $t = 0$. This process is randomly interrupted and starts anew at random times given by the resetting process. That is, when the particle resets it appears instantaneously at $x = 0$ and performs again a random walk according to $P(x,t)$. After a given time distributed according to the PDF $\varphi_R(t)$, the particles resets again to $x = 0$ and so on. The propagator of the overall process, which we call $\rho(x,t)$, is an iteration of many foraging propagators, with the running time of each determined by $\varphi_R(t)$. This formulation differs from the method most commonly used in the bibliography to study random walk processes with resets, consisting on introducing ad-hoc a reset term to the Master equation of the foraging process (see 5 for instance). Contrarily, it resembles the techniques employed in 20 to study Lévy flights with exponentially distributed resets or in 23 to study from a general perspective the first passage processes with resets.

We start by building a mesoscopic balance equation for $\rho(x,t)$. We must point out that the process $\rho(x,t)$ starts at the origin and until the first reset it can be described by the foraging propagator itself. Under this perspective, the following integral equation is fulfilled:

$$
\rho(x,t) = \varphi_R^*(t)P(x,t) + \int_0^t \varphi_R(t')\rho(x,t-t')dt', \quad (2.1)
$$

where $\varphi_R^*(t) = \int_0^\infty \varphi_R(t')dt'$ is the probability of the first reset happening after $t$. The first term in the r.h.s. accounts for the cases where no reset has occurred until $t$ and therefore the overall process is described by the foraging propagator. The second term accounts for the cases where at least one reset has occurred before $t$ and the first one has been at time $t' < t$, in which case the system is described by the overall propagator starting at $t'$. Notably, we have introduced $\rho(x,t-t')$ as the propagator of the process starting at $x = 0$ at time $t'$ (formally, $\rho(x,t;0,t')$), which can be done independently of the Markovianity of $\rho(x,t)$. This is true as long as the first realization of the process does not affect the following ones because $t'$ refers to the time the system goes back to the origin and therefore the scenario is equivalent to having a system starting at $t_0 = 0$ and having $t-t'$ to reach $x$. Taking Eq. (2.1) to the Laplace space for the time variable, we can isolate the overall propagator to be

$$
\hat{\rho}(x,s) = \frac{\mathcal{L}[\varphi_R^*(t)P(x,t)]}{1 - \varphi_R(s)} \quad (2.2)
$$

where $\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty e^{-st}f(t)dt$ denotes the Laplace transform. We can now derive a general equation for the MSD of the overall process. To do so, we multiply by $x^2$ at both sides of Eq.(2.2) and integrate over all the real axis to get the MSD in the Laplace space

$$
\langle x^2(s) \rangle = \frac{\mathcal{L}[\varphi_R^*(t)(x^2(t))P]}{1 - \varphi_R(s)}, \quad (2.3)
$$

where $\langle x^2(t) \rangle P$ is the MSD of the foraging process. Crucially, this equation is valid for any $\varphi_R(t)$. The importance of Eq. (2.3) lies in the fact that it contains all the information about the behaviour of a general foraging process with any reset time distribution: its long-time diffusivity, the transient regime, whether the system reaches a stationary state or not, etc.

1. Exponentially distributed reset times

Let us study the particular case where reset times are distributed exponentially $\varphi_R(t) = \frac{1}{\tau_m} e^{-t/\tau_m}$ but keeping the search process general. In this scenario, the propagator of the overall process in Eq.(2.2), inverted by Laplace, can be written as

$$
\rho_e(x,t) = e^{-\frac{t}{\tau_m}}P(x,t) + \frac{1}{\tau_m} \int_0^t e^{-\frac{t'-t}{\tau_m}}P(x,t')dt'. \quad (2.4)
$$

When the Laplace transform of $P(x,t)$ exists at $s = \frac{1}{\tau_m}$, a stationary distribution distribution is reached and can be generally written as

$$
\rho_e(x) = \frac{\hat{P}(x,\frac{1}{\tau_m})}{\tau_m}, \quad (2.5)
$$

where the hat symbol means Laplace transform. The required condition for the stationary distribution to exist (i.e. the existence of the Laplace transform of $P(x,t)$ evaluated at $s = \frac{1}{\tau_m}$) includes a wide range of processes from the most studied in the bibliography: Brownian motion, Lévy flights, etc. Analogously, we can compute from Eq.(2.5) an expression for the stationary MSD of the overall process in terms of the foraging one can be directly derived:

$$
\langle x^2 \rangle_e(\infty) = \frac{\langle x^2(\frac{1}{\tau_m}) \rangle P}{\tau_m}. \quad (2.6)
$$
While Eq. (2.5) tells us about the existence of a stationary distribution when exponentially distributed resets are applied to the foraging process, Eq. (2.6) includes an extra condition on the foraging processes for them to define a finite area around the origin: the Laplace transform of its MSD must exist and be finite at \( s = \frac{1}{\tau_m} \). For instance, even though the Lévy flight reaches a stationary state when it is subject to constant rate resets, since its MSD is infinite, the MSD of the overall process is infinite too and therefore we are not able to define a territoriality. Contrarily, multiple processes can be found in the bibliography that fulfill the conditions herein derived as Lévy walks, ballistic or even turbulent motion \( [30] \). Notably, Eq. (2.5) is also applicable to processes of more than one dimension which are rotational invariant, in which case the system can be described by a one-dimensional propagator \( P(x, t) \) where \( x \) is the distance from the origin. In this case, correlated random-walks and Lévy walks in the plane \( [2] \) or self-avoiding random walks for arbitrary spatial dimension \( [31] \) are significant processes which form a finite size area when they are subject to exponential resets.

**B. First arrival**

The second remarkable result from \( [5] \) is the existence of a finite MFPT when a diffusive process is subject to constant rate resets. Since then, several works have been published focused on the first completion time of general processes subject to resets \( [21-23] \) but none of them have put the focus on how robust the finiteness of the MFPT of diffusion processes is. During the writing of this paper we have realised that a deep analysis of the first passage for search processes has been recently done in \( [32] \). Nevertheless, in this work, besides the general qualitative analysis similar to the one did in \( [32] \), we deeply study particular cases of interest as subdiffusive motion, Lévy flights or random walks in potential landscapes. Moreover, we perform numerical simulations of the process to check our analytical results.

Herein we us the terminology mean first arrival time (MFAT) as a measure of the time taken by the process to arrive to a given position, instead of crossing it as in the MFPT. This new nomenclature is motivated by the fact that, for instance, for Lévy flights the MFPT has an ambiguous interpretation because of the possibility of extremely long jumps in infinitely small time steps. Contrarily, the MFAT can be clearly interpreted and its properties have been deeply studied in \( [33] \).

Let us start with a general formulation to further study particular cases of interest. To determine whether the MFAT is finite or infinite, we build a renewal equation for the survival probability of the overall process \( \sigma_x(t) \) in terms of the survival probability of the foraging process \( Q_x(t) \) and the reset time PDF \( \varphi_R(t) \), similar to what we have done for the propagator in the previous section:

\[
\sigma_x(t) = \varphi_R^*(t)Q_x(t) + \int_0^t \varphi_R(t')Q_x(t')\sigma_x(t-t')dt', \quad (2.7)
\]

where the first term on the r.h.s. is the probability of not having reached \( x \) without resets and the second term is the probability of not having reached \( x \) when at least one reset has happened at time \( t \) and the first one has been at time \( t' < t \). Applying the Laplace transform and isolating the overall survival probability we obtain:

\[
\hat{\sigma}_x(s) = \frac{L[\varphi_R^*(t)Q_x(t)]}{1-L[\varphi_R(t)Q_x(t)]}. \quad (2.8)
\]

This equation is the cornerstone from which the existence of the MFAT is studied. If in the asymptotic limit the survival probability behaves as

\[
\sigma_x(t) \sim t^{-\beta}, \quad (2.9)
\]

for \( \beta > 1 \) the MFAT is finite, while for \( \beta \leq 1 \) it diverges. Since we have the expression of the survival probability in the Laplace space, it is convenient to rewrite these conditions for \( \hat{\sigma}_x(s) \) instead. Let us consider the following situations:

i) When \( \beta > 1 \), the Laplace transform of the survival probability tends to a constant value for small \( s \). The MFAT is finite and can be found as

\[
T_F = \langle t \rangle_q(t) = \lim_{s \to 0} \hat{\sigma}_x(s) \quad (2.10)
\]

where \( q(t) = -\frac{\partial \sigma(t)}{\partial t} \) is the first arrival time distribution of the overall process. Concretely, the MFAT can be found in terms of the distributions defined above as

\[
T_F = \int_0^\infty \frac{\varphi_R^*(t)Q_x(t)dt}{1-\int_0^\infty \varphi_R(t)Q_x(t)dt}. \quad (2.11)
\]

ii) When \( \beta = 1 \), the Laplace transform of the survival probability tends to a constant value for small \( s \). However, in this case the MFAT is infinite since

\[
\lim_{s \to 0} \hat{\sigma}_x(s) = \infty
\]

so

\[
T_F = \int_0^\infty tq(t)dt = \lim_{s \to 0} \hat{\sigma}_x(s) - \lim_{t \to \infty} t\sigma(t) = \infty
\]

iii) When \( \beta < 1 \), the Laplace transform of the survival probability diverges as \( \hat{\sigma}_x(s) \sim s^{\beta-1} \) for small \( s \). The MFAT is infinite and the survival probability decays as \( \sigma_x(t) \sim t^{-\beta} \) with time.

Notably, when the reset times are exponentially distributed \( (\varphi_R(t) = \frac{1}{\tau_m}e^{-\frac{t}{\tau_m}}) \), the MFAT of the overall process is always finite for foraging survival probabilities.
which decay as a power law for long times. Concretely, in this particular case, Eq. (3.9) reduces to
\[ T_P = \frac{\tau_m \hat{Q}_x(\frac{1}{\tau_m})}{\tau_m - \hat{Q}_x(\frac{1}{\tau_m})}, \] (2.12)
where \( \hat{Q}_x(s) \) depends on the particular election of the model for the foraging process.

### III. FREE SEARCH

In order to get a deeper intuition about the results in the previous section, let us take specific (rather generic) forms for both the reset time distribution and the foraging process. In first place we study well-known processes which diffuse without environmental constraints (potential landscapes, barriers, etc.). These processes arise in a natural way from the continuous-time random walk formalism, from which the propagator can be straightforwardly found.

#### A. Transport properties

Let us start by studying the transport properties of the global process with a search process MSD scaling as
\[ \langle x^2(t) \rangle_P \sim t^p, \] (3.1)
with \( 0 < p \leq 2 \), and a the reset time distribution
\[ \varphi_R(t) = \frac{t^{\gamma_R - 1}}{\tau_m^{\gamma_R}} E_{\gamma_R, \gamma_R} \left[ - \left( \frac{t}{\tau_m} \right)^{\gamma_R} \right], \] (3.2)
with \( 0 < \gamma_R \leq 1 \) and
\[ E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha n + \beta)} \]
the generalised Mittag-Leffler function. This choice allows us to recover the exponential distribution for \( \gamma_R = 1 \) and also study power law behaviours for \( \gamma_R < 1 \). With these assumptions, the MSD of the overall process has two possible behaviours for large \( t \):
\[ \langle x^2(t) \rangle \sim \begin{cases} t^p, & \text{for } \gamma_R < 1 \\ t^0, & \text{for } \gamma_R = 1 \end{cases}. \] (3.3)

Therefore, for power law reset time PDF with any exponent \( \gamma_R < 1 \), the MSD of the overall process scales as that of the foraging process (FIG. [1]), so that a fat-tailed reset PDF does not change the transport regime. Nevertheless, for exponential reset time distributions the MSD of the overall process reaches a stationary value and there is transport failure, regardless of the foraging transport regime. In an ecological context, this tells us that any search process with MSD of this type and with a constant probability of going back to the nest during the foraging process allows us to define a territoriality for the animal. Another way to see it is as follows: for animals which return to the nest with a constant rate, all the processes which at long times diffuse as \( \langle x^2(t) \rangle_P \sim t^p \) for any \( 0 < p \leq 2 \) are plausible candidates for the description of its foraging process.

In the case where the system attains a stationary MSD \((\gamma_R = 1)\) we can compute the stationary distributions for different search processes by employing Eq. (2.5). For example, if we consider the foraging propagator in the Fourier-Laplace space
\[ P(k, s) = \frac{1}{s + D_2 \delta^{1-\gamma R^2}} \] (3.4)
which describes a subdiffusive movement for \( \gamma < 1 \) and diffusive movement for \( \gamma = 1 \), the stationary distribution...
FIG. 2: Stationary distribution of the overall process with subdiffusive (SD) with $\gamma = 0.5$, diffusive (D) and Lévy flight (LF) with $\alpha = 1.5$, foraging propagator, all with $D = 0.1$, and exponential reset times with $\tau_m = 10$. The stochastic simulation of the process (solid dots) is compared to the corresponding analytical expressions (3.5) and (3.7) (solid lines). Given by (2.5), becomes a symmetric exponential distribution

$$\rho_e(x) = \frac{1}{\sqrt{4D\tau_m}} e^{-\frac{|x|}{\sqrt{4D\tau_m}}}, \quad (3.5)$$

where for $\gamma = 1$ we recover the stationary distribution found in [5]. If instead of the subdiffusive propagator we consider the corresponding propagator for a Lévy flight $P(k,s) = \frac{1}{s + D|k|^\alpha}$ (3.6) with $\alpha < 2$, the stationary distribution of the overall process is from (2.5)

$$\rho_e(x) = 2\int_0^\infty \frac{\cos(kx)}{1 + \tau_m Dk^\alpha} dk. \quad (3.7)$$

In figure 2, we compare both analytical results (3.5) and (3.7) with numerical Monte-Carlo simulations of the foraging process with resets. The agreement is seen to be excellent.

B. First Arrival

Let us now study the MFAT for a general foraging survival probability of the form

$$Q_x(t) \sim t^{-q}, \quad q > 0$$

for large $t$ and the reset time distribution defined in (1.2). Under these assumptions, the asymptotic behaviour of the overall survival probability is

$$\sigma_x(t) \sim t^{\gamma_R + q - 1}, \quad \text{if } \gamma_R + q < 1, \quad (3.8)$$

as shown in [32], which implies that $T_F = \infty$ as in the frontier case $\gamma_R + q = 1$. However, when $\gamma_R + q > 1$ the MFAT is finite and can be exactly expressed as

$$T_F(x) = \frac{\int_0^\infty E_{\gamma_R,1}^{-\gamma_R,1} \left[ -\left( \frac{t}{\tau_m} \right)^{\gamma_R} \right] Q_x(t) dt}{1 - \int_0^\infty \left( \frac{t}{\tau_m} \right)^{\gamma_R + 1} E_{\gamma_R,\gamma_R}^{-\gamma_R,\gamma_R} \left[ -\left( \frac{t}{\tau_m} \right)^{\gamma_R} \right] Q_x(t) dt} \quad (3.9)$$

The two regions where the MFAT is finite and infinite for a subdiffusive and a Lévy flight foraging processes are shown in figure 3. As shown in [32], for a subdiffusive foraging process the survival probability in the long time limit decays as

$$Q_x(t) \sim t^{-\gamma_R} \quad (3.10)$$
with $0 < \gamma < 1$. For $\gamma = 1$ we have the survival probability of a diffusion process. We can identify $q = \frac{1}{2}$ and consequently the overall process' survival probability decays as

$$\sigma_x(t) \sim t^{-\gamma + \frac{1}{2} - 1},$$  \hspace{1cm} (3.11)

when $\gamma_R + \frac{1}{2} < 1$ (i.e. the MFAT is infinite) and the MFAT is finite when $\gamma_R + \frac{1}{2} > 1$. This result has been corroborated by stochastic simulations of the process (figure 3A), where the limiting curve $\gamma_R = 1 - \frac{1}{2}$ and the tail exponent for the overall survival probability are in clear agreement with the analytical results. We have also studied the case when the foraging process is a Lévy flight. In this case, the survival probability decays as

$$Q_x(t) \sim t^{\frac{1}{\alpha} - 1},$$  \hspace{1cm} (3.12)

with $1 < \alpha < 2$, as shown in \[33\]. In this case we can also recover the diffusive behavior for $\alpha = 2$. Identifying $q = 1 - \frac{1}{\alpha}$, the overall survival probability reads

$$\sigma_x(t) \sim t^{\gamma R - \frac{1}{\alpha}}$$  \hspace{1cm} (3.13)

in the asymptotic limit and for $\gamma_R - \frac{1}{\alpha} < 0$. In this case the MFAT is infinite while for $\gamma_R - \frac{1}{\alpha} > 0$ it is finite. In figure 3B we present the results on which we see that these two regions have been also found in a stochastic simulation of the overall process.

Unlike the existence of a stationary MSD, the finiteness of the MFAT is not drastically broken when the reset time distribution goes from short to long-tailed. A remarkable property we can see in figure 3 is that both the reset time distribution and the first arrival time distribution for the foraging process can have an infinite mean value and still the mean value of the overall process can be finite. This property has been tested by computing the simulated MFAT for parameters in the white region in figure 3 for both the subdiffusive and the Lévy flight cases, and also for the diffusive limiting case. This simulated MFAT is compared to the one obtained from numerical integration of Eq.(3.9) and the results match (FIG.4).

IV. BIASED SEARCH: RANDOM WALK IN AN HARMONIC POTENTIAL

The strength of the formalism presented in Section II lies in the fact that it does not depend on the fundamental method from which the survival probability and the MSD (or the propagator) are obtained, contrarily to what happens with the most extended formalism employed to study stochastic resets \[5\]. To illustrate this, we use the general results developed herein to study the MSD and the FAT of random walks in an harmonic potential which is fundamentally studied from the Langevin equation formalism. The MSD of a Brownian particle starting at $x = 0$ with white, gaussian noise with diffusion constant $D$, subjected to a non-centered harmonic potential $V(x) = \frac{1}{2}k(x - x_0)^2$ and with drift $\gamma$ can be easily derived (see \[35\] for instance). When this system is constrained by constant rate resets (i.e. exponentially distributed reset times), an equilibrium distribution is attained as shown in \[12\] by means of introducing the reset mechanism to the Fokker-Plank equation ad-hoc. With our formulation, the starting point is the Langevin equation for the Brownian particle in an harmonic potential \[34\] without taking into account the resets:

$$\frac{dx}{dt} = -\frac{D}{\gamma} \frac{\partial V(x)}{\partial x} + \sqrt{2D}\eta(t)$$  \hspace{1cm} (4.1)

where the overdamped limit has been implicitly taken and $\eta(t)$ is a Gaussian noise so that $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$ (i.e. a white noise). If we introduce the harmonic potential we get

$$\frac{dx}{dt} = -DF(x - x_0) + \sqrt{2DF}\eta(t)$$  \hspace{1cm} (4.2)

with $F = k/\gamma$. From this equation the MSD of the particle can be straightforwardly derived \[35\] to be

$$\langle x^2(t) \rangle_p = \left(1 + \frac{1}{F} + x_0^2 \right) \left(1 - e^{-2DFt} \right),$$  \hspace{1cm} (4.3)

which we take as the starting point to study the system when it is subject to resets. Introducing this expression to the main equation for the global MSD (Eq.(2.3) we
can obtain the dynamics of the MSD in the Laplace space in terms of the survival probability:

\[ \langle \dot{x}^2(s) \rangle = \left( \frac{1}{F} + x_0^2 \right) \left( \frac{1}{s} - \frac{\hat{\varphi}_R^*(s + 2DF)}{1 - \hat{\varphi}_R(s)} \right) \]  

(4.4)

Now, when looking at the inverse Laplace transform of \( \langle \dot{x}^2(t) \rangle \) for long times (i.e. small \( s \)), the second term in the second parenthesis can be neglected w.r.t. the first when the distribution \( \varphi(t) \) is long tailed since for small \( s \) the numerator remains finite while the denominator \( 1 - \hat{\varphi}_R(s) \sim s^{\gamma_R} \), \( \gamma_R < 1 \). Therefore, for long tailed reset time distributions, the equilibrium MSD remains as in the case without resets. On the other hand, for exponentially distributed resets (i.e. as in Eq.(3.2) with \( \gamma_R = 1 \)) we have

\[ \hat{\varphi}_R^*(s + 2DF) = \frac{1}{s(1 + 2DF\tau_m)} + O(s^0). \]

Therefore, the global MSD at the equilibrium state can be expressed as

\[ \langle x^2(\infty) \rangle_x = \left( \frac{1}{F} + x_0^2 \right) \left( 1 - \kappa_{\gamma_R,1} \frac{1}{1 + 2DF\tau_m} \right) \]  

(4.5)

with \( \kappa_{\gamma_R,1} \) for \( \gamma_R = 1 \) and \( \kappa_{\gamma_R,1} = 0 \) for \( \gamma_R < 1 \). This difference between long-tailed and exponentially decaying distributions has been tested by means of a stochastic simulation of the Langevin equation (see Fig.5). We can also study the FAT for the random walker on the harmonic potential. For this system, the survival probability at the minimum of the potential has been found to be [36]

\[ Q_{x_0}(t) = \frac{2De^{-DFt}|x_0|}{\sqrt{2\pi\sigma_t^2}} \exp \left\{ -\frac{(x_0e^{-DFt})^2}{2\sigma_t} \right\} \]  

(4.6)

with \( \sigma_t = (1 - e^{-DFt})/F \). In the asymptotic limit the survival probability decays as \( Q_x(t) \sim e^{-DFt} \), which makes the global survival probability have a short tail. This can be easily seen by looking at Eq.(2.5); when the asymptotic limit of \( Q_x(t) \) is exponential, the expression of the global survival probability in the Laplace space tends to a finite value for small \( s \); which is in fact the first arrival time of the global process. In Fig.5 we compare the analytical result predicted by Eq.(3.9), taking the survival probability in Eq.(4.6) instead of the ones studied in Section III with Monte-Carlo simulations of the Langevin equation in Eq.(4.2), showing a perfect agreement.

V. CONCLUSIONS

In this work we have derived the MSD and the MFAT of territorial searches with resets from a unified, renewal equation based formulation. Concretely, with Eq.(2.3) and Eq.(3.9), we can find the overall MSD and MFAT respectively in terms of a general resetting mechanism and magnitudes of the foraging process. This opens the analysis of resets acting on the vast range of search processes for which the survival distribution or the MSD is well-known without the need of building a particular motion model for each case.

In order to show the power of the formalism herein employed, the existence of an equilibrium MSD and a finite MFAT have been tested for general stochastic processes subject to random resets. The first property turns to be quite sensitive with respect to the reset time distribution. When it is long-tailed, the MSD of the overall process scales in time as the foraging process (Eq.(3.3) in the text). Nevertheless, for exponential distributions for the resets, a wide range of foraging processes lead to a stationary MSD. Regarding the second feature, we have
seen that the overall process with exponential reset time distributions applied to foraging processes with survival probability of the form $Q_x(t) \sim t^{-q}$, $q > 0$ for large $t$ has a finite MFAT. In this case, long-tailed distributions for the reset time can generate a process with finite MFAT too. Consequently, the sufficient conditions for which both properties are preserved for free foraging processes are the following: the reset time distribution is exponential and the foraging processes has a finite, Laplace transformable MSD and a power-law survival probability in the long $t$ limit.

Finally, in Section [IV] we show another usage of the formulation. Concretely, we study the effect of resets to a diffusive random walk in an harmonic potential. With this, we show that the renewal formalism is very useful to obtain general equations which can be applied to a wide range of movement processes that have been already studied in detail without resets.

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