A NOTE ON COMBINATORIAL SPLICING FORMULAS FOR HEegaard Floer Homology

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Abstract. We give a precise description of splicing formulas from a previous paper in terms of knot Floer complex associated with a knot in a homology sphere.

1. Introduction

Heegaard Floer homology has provided topologists with a quite powerful technique in the study of surgery along null-homologous knots inside three-manifolds. After the introduction of surgery formulas for Heegaard Floer homology by Ozsváth and Szabó ([OS3, OS4]), several interesting results have been proved (e.g. [OS4, Hed]). The author has considered another type of surgery formulas which are suggested from the combinatorial approach of Sarkar and Wang to Heegaard Floer homology (see [Ef3], also [SW], also [MOS, MOST]). More generally, there is a formula for Heegaard Floer homology of the three-manifold obtained by splicing two knot complements, provided in [Ef4]. In this paper, we will compare these formulas, and will bring the combinatorial surgery formulas into a form similar to the formulas of Ozsváth and Szabó. More interestingly, this will give a more explicit formula for Heegaard Floer homology of the three-manifold obtained by splicing two knot complements, which is described in terms of knot Floer complexes of the two knots.

Let $K$ be a knot inside the homology sphere $Y$. We may remove a tubular neighborhood of $K$ and glue it back to obtain the three-manifold $Y_{p/q} = Y_{p/q}(K)$, which is the result of $p/q$-surgery on $K$. The core of the solid torus, which is the tubular neighborhood of $K$, will represent a knot in $Y_{p/q}$ which will be denoted by $K_{p/q}$. We may denote $(Y, K)$ by $(Y_{\infty}, K_{\infty})$, as an extension of the above notation. Let $\mathbb{H}_\bullet(K)$ be the Heegaard Floer homology group $\widehat{HF}_\bullet(K_\bullet)$ for $\bullet \in \mathbb{Q} \cup \{\infty\}$. Note that $\widehat{HF}_\bullet$ is defined for knots inside rational homology spheres (see [OS4]), and that $\mathbb{H}_0(K) = \widehat{HF}(Y, K)$ is the longitude Floer homology of $K$ from [Ef1]. In all these cases, we choose the coefficient ring to be $\mathbb{Z}/2\mathbb{Z}$ (for simplicity). If we choose a Heegaard diagram for $Y - K$ and let $\lambda_\bullet$ denote a longitude which has framing $\bullet \in \mathbb{Z} \cup \{\infty\}$ (with $\lambda_\infty = \mu$ the meridian for $K$), one may observe that the pairs $(\lambda_\infty, \lambda_1)$ and $(\lambda_1, \lambda_0)$ have a single intersection point in the Heegaard diagram. Let $(\bullet, \cdot) \in \{(\infty, 1), (1, 0)\}$ correspond to either of these pairs. There are four quadrants around the intersection point of $\lambda_\bullet$ and $\lambda_\cdot$. If we puncture three of these quadrants and consider the corresponding holomorphic triangle map, we obtain an induced map $\mathbb{H}_\bullet \to \mathbb{H}_\cdot$. If the punctures are chosen as in figure 1 the result would be two

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maps $\phi, \overline{\phi} : \mathbb{H}_\infty(K) \to \mathbb{H}_1(K)$ and two other maps $\psi, \overline{\psi} : \mathbb{H}_1(K) \to \mathbb{H}_0(K)$ so that the following two sequences are exact:

$\mathbb{H}_0(K) \xrightarrow{\psi} \mathbb{H}_1(K) \xrightarrow{\overline{\phi}} \mathbb{H}_\infty(K)$, and

$\mathbb{H}_0(K) \xrightarrow{\overline{\psi}} \mathbb{H}_1(K) \xrightarrow{\phi} \mathbb{H}_\infty(K)$.

The homology of the mapping cones of $\phi$ (or $\overline{\phi}$) and $\psi$ (or $\overline{\psi}$) are $\mathbb{H}_0(K)$ and $\mathbb{H}_\infty(K)$ respectively (see [Ef4]). With the above notation fixed, we have proved the following surgery formula in [Ef4] (The orientation convention of [Ef4] is different from that of [Ef2] and [OS3], so we have changed the direction of all maps to compensate this difference):

**Theorem 1.1.** Suppose that $K_1$ and $K_2$ are two knots inside homology spheres $Y_1$ and $Y_2$ respectively. Define the maps $\phi^i, \overline{\phi}^i, \psi^i$ and $\overline{\psi}^i$ for $K_i$ as above. Let $\mathbb{M} = \mathbb{M}(K_1, K_2)$ denote the following cube of maps

\[
\begin{array}{ccc}
H^1_\infty \otimes H^2_\infty & \xrightarrow{I \otimes \phi^2} & H^1_\infty \otimes H^1_1 \\
\eta^1 \otimes \eta^2 & \xrightarrow{\phi^1 \otimes \psi^2} & I \otimes H^1_1 \\
H^1_0 \otimes H^2_0 & \xrightarrow{\psi^1 \otimes I} & H^1_0 \otimes H^0_0, \\
\end{array}
\]

where $H^1_\star \otimes H^2_\star = H_\star(K_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} H_\star(K_2)$ for $\star, \bullet \in \{\infty, 1, 0\}$. The differential $d_M$ of the complex $\mathbb{M}$ is defined to be the sum of all the maps that appear in this cube.

**Figure 1.** For defining chain maps between $C_\bullet(K)$ and $C_\star(K)$, the punctures around the intersection point of $\lambda_\bullet$ and $\lambda_\star$ should be chosen as illustrated in the above diagrams.
Then the Heegaard Floer homology of the three-manifold $Y$, obtained by splicing knot complements $Y_1 - K_1$ and $Y_2 - K_2$, is given by

$$\tilde{HF}(Y; \mathbb{Z}/2\mathbb{Z}) = H_*(\mathcal{M}, d_M),$$

where $H_*(\mathcal{M}, d_M)$ denotes the homology of the cube $\mathcal{M}$.

On the other hand, the groups $H_*(K_i)$ for $\bullet \in \mathbb{Z}$ are described in terms of knot Floer complex for $(Y_i, K_i)$ in $\mathbb{Z}$. We have shown that the following is true:

**Theorem 1.2.** Suppose that $(Y, K)$ is a homology sphere and $B$ is the complex $\tilde{CF}(Y, \mathbb{Z}/2\mathbb{Z})$, equipped by the filtration induced by $K$. Let $B(\geq s)$ denote the quotient-complex generated by those generators whose relative Spin$^c$ class in $\mathbb{Z} = Spin^c(Y, K)$ is greater than or equal to $s \in \mathbb{Z}$. The homology group $\tilde{HF}(Y_n(K), K_n; s; \mathbb{Z}/2\mathbb{Z})$ are isomorphic to the homology of the following complex

$$B(\geq -s) \overset{\psi}{\longrightarrow} B(\geq -s),$$

where $\psi$ is the inclusion map.

The question we would like to address in this paper is the description of the maps $\phi, \varphi, \psi$ and $\psi$ in terms of a presentation of $\mathbb{H}_* (K)$ given by the above theorem. This would give surgery and splicing formulas which will look similar to the surgery formulas of Ozsváth and Szabó in $\mathbb{OS3}$, $\mathbb{OS4}$.

Namely, if we denote the part of $\mathbb{H}_* (K)$ in relative Spin$^c$ class $s \in \mathbb{Z} = Spin^c(Y, K)$ by $\mathbb{H}_* (K, s)$ we will prove

**Theorem 1.3.** Under the identification of $\mathbb{H}_\infty (K, s)$ with $B\{s\}$, $\mathbb{H}_1 (K, s)$ with the homology of the complex

$$C_1 (s) = \left( B\{s\} \overset{\psi}{\longrightarrow} B\{\geq -s\} \right),$$

and $\mathbb{H}_0 (K, s)$ with the homology of the complex

$$C_0 (s) = \left( B\{s + 1\} \overset{\psi}{\longrightarrow} B\{\geq -s\} \right),$$

the map $\phi : \mathbb{H}_1 (K) \rightarrow \mathbb{H}_\infty (K)$ is induced by the sum of maps $\phi_s$ which take the quotient complex $B\{s\}$ of the complex $C_1 (s)$ to it quotient $B\{\geq -s\}$.

The map $\phi$ is the sum of maps $\phi_s$ which are induced by using the map $\phi_{-s}$ followed by the isomorphism $B\{s\} \simeq B\{\geq -s\}$. Moreover, the map $\psi$ is a sum of inclusion maps $\psi_s$ from $C_0(s - 1)$ to $C_1(s)$ and the map $\psi$ is induced by the sum of inclusion maps $\psi_s$ from $C_0(s)$ to $C_1(s)$.

2. Surgery on null-homologous knots; A short review

In this section we will review the construction of $\mathbb{EF2}$, at least to the extent relevant for the purposes of this paper. The reader is referred to $\mathbb{EF2}$ for the proofs.

Consider a Heegaard diagram for the pair $(Y, K)$. Suppose that the curve $\mu = \beta_g$ in the Heegaard diagram

$$H = (\Sigma, \alpha = \{\alpha_1, ..., \alpha_g\}, \beta = \{\beta_1, ..., \beta_g\}, p)$$

corresponds to the meridian of $K$ and that the marked point $p$ is placed on $\beta_g$. One may assume that the curve $\beta_g$ cuts $\alpha_g$ once and that this is the only element of $\alpha$.
that has an intersection point with $\beta_g$. Suppose that $\lambda$ represents a longitude for the knot $K$ (i.e. it cuts $\beta_g$ once and stays disjoint from other elements of $\beta$) such that the Heegaard diagram

$$\left(\Sigma, \alpha, \{\beta_1, \ldots, \beta_{g-1}, \lambda\}\right)$$

represents the three-manifold $Y_0(K)$ obtained by zero surgery on $K$. Winding $\lambda$ around $\beta_g$ - if it is done $n$ times - would produce a Heegaard diagram for the three-manifold $Y_n(K)$. More precisely, if the resulting curve is denoted by $\lambda_n$, the Heegaard diagram

$$H_n = (\Sigma, \alpha, \beta_n = \{\beta_1, \ldots, \beta_{g-1}, \lambda_n\}, p_n)$$

would give a diagram associated with the knot $(Y_n(K), K_n)$, where $y = p_n$ is a marked point at the intersection of $\lambda_n$ and $\beta_g$.

The curve $\lambda_n$ intersects the $\alpha$-curve $\alpha_g$ in $n$-points which appear in the winding region (there may be other intersections outside the winding region). Denote these points of intersection by

$$..., x_{-2}, x_{-1}, x_0, x_1, x_2, ..., $$

where $x_1$ is the intersection point with the property that three of its four neighboring quadrants belong to the regions that contain either $p_n$ as a corner. Any generator which is supported in the winding region is of the form

$$\{x_i\} \cup y_0 = \{y_1, \ldots, y_{g-1}, x_i\},$$

and it is in correspondence with the generator

$$y = \{x\} \cup y_0 = \{y_1, \ldots, y_{g-1}, x\}$$

for the complex associated with the knot $(Y, K)$, where $x$ denotes the unique intersection point of $\alpha_g$ and $\beta_g$. Denote the former generator by $(y)_i$, keeping track of the intersection point $x_i$ among those in the winding region.

By taking an intersection point $x$ in relative Spin$^c$ class $s - k \in \text{Spin}^c(Y, K) = \mathbb{Z}$ to $(x)_k$ when $k \leq 0$ and $(x)_{k+n}$ if $k > 0$ we obtain an identification of the complex $\tilde{\text{CFK}}(Y_n(K), K_n; s)$ and the complex

$$\tilde{\text{U}}(s) = \mathbb{B}\{\geq s\} \oplus \mathbb{B}\{\geq n - s\} = \mathbb{B}\{\geq s\}$$

for large values of $n$, having fixed $s \in \mathbb{Z}$.

Choose an intersection point between the curves $\lambda_n$ and $\lambda_{m+n}$ in the middle of the winding region, denoted by $q$. We will assume that $m = \ell n$ for an integer $\ell$ which is chosen to be appropriately large. We continue to assume that $\alpha_g$ is the unique $\alpha$-curve in the winding region. From the 4 quadrants around the intersection point $q$, two of them are parts of small triangles $\Delta_0$ and $\Delta_1$ between $\alpha, \beta_n$ and $\beta_{m+n}$. We may assume that the intersection points between $\alpha_g$ and $\lambda_{m+n}$ in the winding region are

$$..., y_{-2}, y_{-1}, y_0, y_1, y_2, ..., $$

and that the intersection points between $\lambda_n$ and $\alpha_g$ are

$$..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...$$
as before. We may also assume that the domain $\Delta_i$ for $i = 0, 1$ is the triangle with vertices $q, x_i$ and $y_i$, and $\Delta_1$ is one of the connected domains in the complement of curves $\Sigma \setminus C$ where

$$C = \alpha \cup \beta_n \cup \beta_{m+n}.$$ 

Other than $\Delta_0$ and $\Delta_1$ there are two other domains which have $q$ as a corner. One of them is on the right-hand-side of both $\lambda_n$ and $\lambda_{m+n}$, denoted by $D_1$, and the other one is on the left-hand-side of both of them, denoted by $D_2$. The domains $D_1$ and $D_2$ are assumed to be connected regions in the complement of the curves $\Sigma \setminus C$. We may assume that the meridian $\mu$ passes through the regions $D_1, D_2$ and $\Delta_0$, cutting each of them into two parts: $\Delta_0 = \Delta_0^R \cup \Delta_0^L$, $D_1 = D_1^R \cup D_1^L$ and $D_2 = D_2^R \cup D_2^L$. Here $\Delta_0^R \subset \Delta_0$ is the part on the right-hand-side of $\mu$ and $\Delta_0^L$ is the part on the left-hand-side. Similarly for the other partitions. Choose the marked points so that $u$ is in $D_1^R$, $v$ is in $D_2^R$, $w$ is in $D_2^L$ and $z$ is in $D_1^L$ (see figure 2). We obtain the Heegaard diagram

$$R_{m,n} = (\Sigma, \alpha, \beta, \beta_n, \beta_{m+n}; u, v, w, z),$$

which determines three chain complexes associated with the pairs $(\alpha, \beta), (\alpha, \beta_n)$, and $(\alpha, \beta_{m+n})$ once we use the marked points $u, v, w, z$. The complexes associated with the last two pairs are simply $\hat{\text{CFK}}(Y_n(K), K_n)$ and $\hat{\text{CFK}}(Y_{m+n}(K), K_{m+n})$, obtained by puncturing the surface at these marked points. The complex associated with the first pair is constructed by making punctures at $u, v$, counting holomorphic disks, and twisting according to the intersection number of the disks with either of the marked points $w, z$. As we have already seen, $\hat{\text{CFK}}(K_{m+n}; s)$ may be identified with the complex $\hat{U}(s)$ if $m$ is large enough. In fact, the holomorphic triangle map gives a long sequence for each relative Spin$^c$ class $s \in \mathbb{Z}$ as

$$\cdots \overset{f^s}{\longrightarrow} \hat{\text{CF}}(Y) \overset{h^s}{\longrightarrow} \hat{\text{CFK}}(K_n, s) \overset{g^s}{\longrightarrow} \hat{E}(s) \overset{f^s}{\longrightarrow} \hat{\text{CF}}(Y) \overset{h}{\longrightarrow} \cdots,$$

such that the induced maps in homology form a long exact sequence, if $m = \ell n$ and $\ell$ is large enough. Here $\hat{E}(s)$ is the complex $\mathbb{B}\{s\} \oplus \mathbb{B}\{n-s\}$. This implies

![Figure 2](image-url)  

**Figure 2.** The Heegaard diagram $R_{m,n}$. The shaded triangles are $\Delta_0$ and $\Delta_1$. The marked points $u, v, w$ and $z$ are placed in $D_1^L, D_2^R, D_2^L$ and $D_1^R$ respectively.
that $\hat{\text{HFK}}(Y_n(K), K_n; s)$ may be computed as the homology of the mapping cone of $f^* : \hat{\mathcal{E}}(s) \to \mathcal{B}$, which is the sum of two inclusion maps. One particular case, is the case where $n = 1$ and this later mapping cone is the same as the complex $C_1(s)$ mentioned before.

3. Understanding the Maps

In this section we prove theorem 1.3.

Proof. In the construction of the previous section, note that $A(s) = \hat{\text{CFK}}(Y_n(K), K_n; s)$ may be written as the mapping cone of some chain map $\rho : A_1(s) \to A_2(s)$, where $A_1(s)$ is the part of $A(s)$ generated by the generators of the form $(x)_1$. The differential of $A(s)$ induces a differential on the quotient complex $A_1(s)$ and the sub-complex $A_2(s)$ of it. There is a similar decomposition of $B = \hat{\text{CFK}}(Y_{m+n}(K), K_{m+n})$ as the mapping cone of $\sigma : B_1 \to B_2$, and the image of $A_1$ under $g$ is in $B_1$. Under the identification of $B(s)$ with $\hat{\mathcal{U}}(s)$, the quotient complex $B_1(s)$ corresponds to the quotient complex $\mathcal{B}\{s\}$ of $\mathcal{B}\{\geq s\} = \hat{\mathcal{U}}(s)$.

We will use the diagram $S_1$, together with the marked points $u, w$ and $z$ to define the map $\phi : C_1(s) = \hat{\text{CFK}}(Y_1(K), K_1, s) \to \hat{\text{CFK}}(Y, K, s) = C_\infty(s)$. Note that puncturing the surface $\Sigma$ at these three marked points implies that this chain map will respect the relative $\text{Spin}^c$ classes on the two sides. One can easily see from the Heegaard diagram that the map $\phi$ is defined on the generators by $\phi(y) = x$, if $y = (x)_1 + b_2$ for some generator $x$ of $C_\infty$ and some $b_2 \in B_2$. This means that $\phi$ is trivial on the sub-complex $A_2$ of $A = C_1$. The image of the generators $(x)_1$ of $A_1$ under the map $g$ is the generator $(x)_1$, as a generator for $\hat{\text{CFK}}(Y_{m+1}(K), K_{m+1}, s)$, i.e. $g$ identifies $A_1(s)$ with $B_1(s)$, via the natural isomorphism of both of them with $\mathcal{B}\{s\}$.

On the other hand, since the map $h$ uses punctures at $u$ and $v$, its image does not contain any generator of the form $(x)_1$ in $\hat{\text{CFK}}(Y_1(K), K_1, s)$, i.e. the image of $h$ is in the sub-complex $A_2$ of $A$. This implies that if an element $a$ in $\text{Ker}(g_*) = \text{Im}(h_*)$ is of the form $a_1 \oplus a_2$ in $A = A_1 \oplus A_2$, and $a_1 = (x)_1$, we should have $(x)_1 = d(y)_1 = (dy)_1$ in $A_2$ for some generator $y$ of $C_\infty$, i.e. $x$ is trivial in $H_\infty(K)$.

Finally, note that by the exactness of the long sequence in homology we have

$$\hat{\text{HFK}}(K_1, s) = \text{Ker}(g_*) \oplus \text{Im}(g_*)$$
$$= \text{Im}(h_*) \oplus \text{Ker}(f_*)$$
$$= \frac{H_*(\mathcal{B})}{\text{Im}(f_*)} \oplus \text{Ker}(f_*) \oplus H_*(\frac{\hat{\mathcal{E}}(s)}{\text{Ker}(f_*)} \cong \text{Im}(f_*))$$
$$= H_*(\frac{\mathcal{B}}{\text{Im}(f_*)} \xrightarrow{d_*} \text{Im}(f_*)) \cong H_*(\frac{\hat{\mathcal{E}}(s)}{\text{Ker}(f_*)} \xrightarrow{d_*} \text{Ker}(f_*))$$
$$= \hat{\mathcal{E}}(s) f_* \xrightarrow{} \mathcal{B}.$$
In these equalities, $H_\star(\bullet)$ denotes the homology of the complex $\bullet$. The above discussion shows that $\text{Ker}(g_\star)$ is in the kernel of the induced map $\phi$. We thus need to understand the image of $\text{Ker}(f_\star)$ under the map $\phi$ in the above description, which are represented in the last line of the above equality by the closed elements $e$ in $\widehat{\mathcal{E}}(s)$ such that $f^\star(e)$ is exact in $\mathcal{B}$. Any such element is the sum of an exact element in $\widehat{\mathcal{E}}(s)$ and an element of the form $g(a_1 \oplus a_2)$ in $B_1$, where $a_1 \in A_1$ is closed and $\rho(a_1) = da_2$ for some $a_2 \in A_2$. Under the identification of $B_1$ with $A_1$, this image is of the form $a_1 \oplus e_2 \in B_1 \oplus B_2$. The discussion implies that for a closed element $e \oplus b \in \widehat{\mathcal{E}} \oplus \mathcal{B}$ (i.e. such that $e$ is closed and $f(e) = db$) such that $e = a_1 \oplus e_2 \in B_1 \oplus B_2$ we have $\phi_\star(e \oplus b) = a_1 \in \widehat{\text{HF}}(Y,K) = \mathbb{H}_\infty(K)$. This proves the first assertion in theorem 1.3. The other claims follow similarly.

References

[Ef1] Eftekhary, E., Longitude Floer homology for knots and the Whitehead double, Alg. and Geom. Topology 5 (2005), also available at math.GT/0407211

[Ef2] Eftekhary, E., Heegaard Floer homology and knot surgery, preprint, available at math.GT/0603171

[Ef3] Eftekhary, E., A combinatorial approach to surgery formulas in Heegaard Floer homology, preprint, available at math.GT/0802.3623

[Ef4] Eftekhary, E., Floer homology and splicing knot complements, preprint, available at math.GT/0802.2874

[Ef5] Eftekhary, E., Floer homology and existence of incompressible tori in homology spheres, preprint

[Hed] Hedden, M., On Floer homology and Berge conjecture on knots admitting lens space surgeries, preprint, available at math.GT/0710.0357v2

[MOS] Manolescu, C., Ozsváth, P., Sarkar, S., A combinatorial description of knot Floer homology, preprint, available at math.GT/0607691

[MOST] Manolescu, C., Ozsváth, P., Szabó, Z., Thurston, D., On combinatorial link Floer homology, preprint, available at math.GT/0610559

[Ni] Ni, Y., Link Floer homology detects the Thurston norm, preprint, available at math.GT/0604360

[OS1] Ozsváth, P., Szabó, Z., Holomorphic disks and knot invariants, Advances in Math. 189 (2004) no.1, also available at math.GT/0209056

[OS2] Ozsváth, P., Szabó, Z., Holomorphic disks and genus bounds, Geom. Topol. 8 (2004), 311-334

[OS3] Ozsváth, P., Szabó, Z., Knot Floer homology and integer surgeries, preprint, available at math.GT/0410300

[OS4] Ozsváth, P., Szabó, Z., Knot Floer homology and rational surgeries, , preprint, available at math.GT/0504404

[SW] Sarkar, S., Wang, J., A combinatorial description of some Heegaard Floer homologies, preprint, available at math.GT/0607777

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