Abstract. We show that any strictly mean convex translator of dimension $n \geq 3$ which admits a cylindrical estimate and a corresponding gradient estimate is rotationally symmetric. As a consequence, we deduce that any translating solution of the mean curvature flow which arises as a blow-up limit of a two-convex mean curvature flow of compact immersed hypersurfaces of dimension $n \geq 3$ is rotationally symmetric. The proof is rather robust, and applies to a more general class of translator equations. As a particular application, we prove an analogous result for a class of flows of embedded hypersurfaces which includes the flow of two-convex hypersurfaces by the two-harmonic mean curvature.

1. Introduction

We are interested in hypersurfaces $X : M^n \to \mathbb{R}^{n+1}$ satisfying the translator equation
\[
\vec{H} = T \perp
\]
for some constant vector $T \in \mathbb{R}^{n+1}$, where, given a local choice of unit normal field $\nu$, $\vec{H} = -H\nu$ is the mean curvature vector of the immersion with respect to the choice of mean curvature $H = \text{div} \nu$, and $\perp$ denotes the projection onto the normal bundle. We call such immersions translators. Up to a time-dependent tangential reparametrization, the family $\{X(\cdot, t)\}_{t \in \mathbb{R}}$ of immersions $X(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ defined by $X(x, t) := X(x) + tT$ satisfies the mean curvature flow
\[
\partial_t X(\cdot, t) = \vec{H}(\cdot, t),
\]
where $\vec{H}(\cdot, t)$ is the mean curvature vector of $X(\cdot, t)$. We therefore also refer to solutions of (T) as translating solutions of the mean curvature flow.

It is well-known that translating solutions arise as blow-up limits of the mean curvature flow about type-II singularities [15, 20]. More precisely, if a solution $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ of (MCF) has type-II curvature blow-up (that is, $\limsup_{t \to T} (T - t) \max_{M^n \times \{t\}} H^2 = \infty$) then there is a sequence of parabolically rescaled solutions of (MCF) which converge locally uniformly in $C^\infty$ to a (non-trivial) translating solution of (MCF).
Probably the most well-known translator is the Grim Reaper\textsuperscript{1} curve $\Gamma$, which is the graph of the function $x \mapsto -\log \cos x$, $x \in (-\pi/2, \pi/2)$. In dimensions $n \geq 2$, there exists a strictly convex, rotationally symmetric translator asymptotic to a paraboloid, which is commonly referred to as the ‘bowl’ \cite{10}. The bowl is the unique rotationally symmetric translating complete graph, and the unique translator with finite genus and a single end asymptotic to a paraboloid \cite{26}. In a remarkable study of convex ancient graphical solutions of the mean curvature flow, X.-J. Wang showed that any strictly convex, entire translator in dimension two is rotationally symmetric, and hence the bowl \cite{28}. Moreover, in every dimension $n \geq 3$, he constructed strictly convex, entire examples without rotational symmetry.

In the setting of two-convex (that is, $\kappa_1 + \kappa_2 > 0$, where $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ denote the principal curvatures) mean curvature flow in dimensions $n \geq 3$, the far-reaching theory of Huisken and Sinestrari \cite{20–22} shows that regions of high curvature are either uniformly convex and cover a whole connected component of the surface, or else they contain regions which are very close, up to rescaling, to cylindrical segments $[-L, L] \times S^{n-1}$. This suggests that the translating blow-up limits which arise at type-II singularities might be rotationally symmetric. We note that this is true (in dimensions $n \geq 3$) for two-convex self-shrinking solutions which arise as blow-up limits of the mean curvature flow with type-I curvature blow-up (that is, $\limsup_{t \to T}(T-t)\max_{M^n \times \{t\}}H^2 < \infty$) since the only possibilities are shrinking spheres $S^n \sqrt{-2nt}$ and cylinders $\mathbb{R} \times S^{n-1} \sqrt{-2(n-1)t}$\cite[Theorem 5.1]{19}.

Recently, Haslhofer \cite{16} proved that this is true in the embedded case (even in dimension 2), his proof relying crucially on the non-collapsing theory of \cite{5} and \cite{17}. In fact, he shows that any strictly convex, uniformly two-convex translator which is non-collapsing is necessarily rotationally symmetric. In the immersed setting, we no longer have a non-collapsing property; however, by the work of Huisken and Sinestrari \cite{22}, we have a cylindrical estimate and a corresponding gradient estimate. Motivated by Haslhofer’s result and the Huisken–Sinestrari theory, we prove the following.

**Theorem 1.1.** Let $X : M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a mean convex translator and $C_1 < \infty$ a constant such that the following hold:

1. **cylindrical estimate:** $|A|^2 - \frac{1}{n-1}H^2 < 0$
2. **gradient estimate:** $|\nabla A|^2 \leq -C_1 \left(|A|^2 - \frac{1}{n-1}H^2\right)H^2$

where $A$ is the second fundamental form of $X$. Then $M^n$ is rotationally symmetric.

In fact (assuming $T = e_{n+1}$), we need only prove that the blow-down of $M^n_t := M^n + te_{n+1}$ is the shrinking cylinder $S^{n-1} \sqrt{2(n-1)(1-t)} \times \mathbb{R}$, since this is enough to deduce rotational symmetry of $M^n$ by §3–5 of Haslhofer’s paper.\footnote{So named because, as it translates, it ‘kills’ any compact solution of curve shortening flow which is unfortunate enough to lie in its path.}
We remark that the cylindrical estimate implies uniform two-convexity, 
\[ \kappa_1 + \kappa_2 \geq \frac{1}{2(n-1)} H \] (see [23, Lemma 5.1]). As a consequence, any type-II blow-up limit of a two-convex mean curvature flow in dimensions \( n \geq 3 \) is rotationally symmetric (even when the mean curvature flow is only immersed).

**Corollary 1.2.** Suppose that \( X : M^n \to \mathbb{R}^{n+1}, n \geq 3, \) is a translator which arises as a proper blow-up limit of a two-convex mean curvature flow of immersed hypersurfaces. Then \( M^n \) is rotationally symmetric.

We note that Corollary 1.2 fails in dimension 2 without some additional assumption, such as non-collapsing, to rule out the Grim plane \( \mathbb{R} \times \Gamma \). This is in accordance with the type-I case, where the non-embedded Abresch–Langer planes \( \mathbb{R} \times \gamma_{k,l} \) can arise [1].

We remark that our proof of Theorem 1.1 also works (in dimensions \( n \geq 2 \)) if assumptions (1) and (2) are replaced by

\[ (1') \text{cylindrical estimate: } \overline{k} - \frac{1}{n-1} H < 0 \] and
\[ (2') \text{gradient estimate: } |\nabla A|^2 \leq C_1 \kappa_1 H^3, \]

where \( \overline{k} \) denotes the inscribed curvature. By work of Brendle [8, Theorem 1] (see also [13]) and Haslhofer and Kleiner [17, Corollary 2.7], these assumptions are met for blow-up limits of type-II singularities of two-convex mean curvature flows of *embedded* hypersurfaces. This provides a slightly different perspective of Haslhofer’s result.

Apart from dealing with blow-up limits of type-II singularities of two-convex mean curvature flows of *immersed* hypersurfaces, a further motivation for removing the (two-sided) non-collapsing assumption in Haslhofer’s result was to study translating solutions of more general curvature flows, where (two-sided) non-collapsing will in general not hold. Let \( F \) be given by \( F(x) = f(\bar{k}(x)) \) for some smooth function \( f : \Gamma^n \subset \mathbb{R}^n \to \mathbb{R} \) of the principal curvatures \( \bar{k} := (\kappa_1, \ldots, \kappa_n) \) defined with respect to some choice of unit normal field \( \nu \). Then we can consider solutions \( X : M^n \to \mathbb{R}^{n+1} \) of the fully non-linear translator equation

\[ F = -\langle \nu, T \rangle \] (FT)

for some \( T \in \mathbb{R}^{n+1} \). We will call the function \( f : \Gamma^n \to \mathbb{R} \) *admissible* if \( \Gamma^n \) is an open, symmetric cone and \( f \) is smooth, symmetric, monotone increasing in each variable and 1-homogeneous. These conditions on \( f \) are very natural: Indeed, smoothness and symmetry are needed to ensure that \( F \) is smooth, monotonicity ensures that (FT) is elliptic, and homogeneity ensures that \( F \) scales like curvature.

\[ ^2 \text{The improved gradient estimate (2') follows from [17 Corollary 2.7] as in the proof of Claim 4.3 in Section 4.} \]
Just as for the mean curvature flow, the family \( \{X(\cdot, t)\}_{t \in \mathbb{R}} \) of immersions
\[
X(\cdot, t) : M^n \to \mathbb{R}^{n+1}
\]
defined by \( X(x, t) := X(x) + tT \) satisfies, up to a time-
dependent tangential reparametrization, the corresponding flow\(^3\)
\[
\partial_t X(\cdot, t) = -F(\cdot, t) \nu(\cdot, t). 
\]
Moreover, if \( F \) admits an appropriate Harnack inequality (which is true
under very mild concavity assumptions for \( f \)) then solutions of \( (FT) \) arise
as blow-up limits of positive speed solutions of \( (F) \) about type-II singularities
in a completely analogous way to the case of mean convex mean curvature
flow. If \( F \) also admits a strong maximum principle for the Weingarten
tensor (which also holds under natural concavity conditions for \( f \), see Section
5) then our proof goes through with minor modification, and we obtain
a result of the following form (where we denote by \( \Gamma^m_+ \) the positive cone
\( \Gamma^m_+ := \{(z_1, \ldots, z_m) \in \mathbb{R}^m : \min_{1 \leq i \leq m} \{z_i\} > 0\} \) in \( \mathbb{R}^m \)).

**Theorem 1.3.** Let \( X : M^n \to \mathbb{R}^{n+1}, n \geq 3, \) be a solution of \( (FT) \), where
\( F \) is given by \( F(x) = f(\vec{\kappa}(x)) \) for some admissible \( f : \Gamma^n \to \mathbb{R} \) such that
\[
\{(0, \vec{z}) : \vec{z} \in \Gamma^n_{n-1}\} \subset \Gamma^n \subset \Gamma^n_2 := \{z \in \mathbb{R}^n : \min_{1 \leq i < j \leq n} \{z_i + z_j\} > 0\}
\]
and either
(i) \( f \) is convex, or
(ii) \( f \) is concave and the function \( f_* : \Gamma^n_{n-1} \to \mathbb{R} \) defined by
\[
f_*(z_2^{-1}, \ldots, z_n^{-1}) := f(0, z_2, \ldots, z_n)^{-1}
\]
is concave.

Suppose that the solution satisfies
(1) a cylindrical estimate, and
(2) a corresponding gradient estimate.

Then \( M^n \) is rotationally symmetric.

The precise form of the assumptions (1) and (2) will be different depending on whether the speed function \( f \) is convex or concave. This is made precise in Section 4.

As a particular application, we find that translating blow-up limits about type-II singularities of the flows of embedded hypersurfaces studied in \([9]\) are rotationally symmetric.

**Corollary 1.4.** Suppose that \( X : M^n \to \mathbb{R}^{n+1}, n \geq 3, \) is a translator which
arises as a blow-up limit of an embedded solution of the flow \( (F) \), where \( F \) is given by \( F(x) = f(\vec{\kappa}(x)) \) for some concave admissible \( f : \Gamma^n \to \mathbb{R} \) such that
\[
\{(0, \vec{z}) : \vec{z} \in \Gamma^n_{n-1}\} \subset \Gamma^n \subset \Gamma^n_2 := \{z \in \mathbb{R}^n : \min_{1 \leq i < j \leq n} \{z_i + z_j\} > 0\},
\]
(i) \( f|_{\partial M^n} = 0 \) and

\(^3\)We have implicitly assumed orientability of solutions of \( (FT) \) and \( (F) \); however, if \( f \) is an odd function, \( (FT) \) and \( (F) \) also admit non-orientable solutions.
(iii) the function \( f_*: \Gamma_n^{-1} \to \mathbb{R} \) defined by
\[
 f_*(z_2^{-1}, \ldots, z_n^{-1}) := f(0, z_2, \ldots, z_n)^{-1}
\]
is concave.

Then \( X \) is rotationally symmetric.

We mention that the class of flows to which the corollary applies includes
the flow of two-convex hypersurfaces by the two-harmonic mean curvature,
\[
 F := \left( \sum_{i<j} \frac{1}{\kappa_i + \kappa_j} \right)^{-1},
\]
and, for \( n = 3 \), the flows of positive scalar curvature hypersurfaces by either
the square root of the scalar curvature or the ratio of scalar to mean curvature.
Corollary 1.4 does not include any convex speeds, because, as yet, it
is not known if they admit an appropriate gradient estimate (although an
appropriate cylindrical estimate was proved in [6]).

2. Preliminaries

Let \( X: M^n \to \mathbb{R}^{n+1} \) be a solution of (1). After performing a rotation
and a dilation, we can arrange that \( T = e_{n+1} \), which we assume from now
on. Introducing the height function \( h: M^n \to \mathbb{R} \),
\[
 h(x) := \langle X(x), e_{n+1} \rangle,
\]
we denote
\[
 V := \nabla h = \text{proj}_{TM^n} e_{n+1} = e_{n+1} + H\nu.
\]

Then the Weingarten curvature \( A \) and the mean curvature \( H \) satisfy (see,
for instance, [15])
\[
 -\Delta A = |A|^2 A + \nabla_V A \tag{2.1}
\]
and
\[
 -\Delta H = |A|^2 H + \nabla_V H. \tag{2.2}
\]

A well-known consequence of (2.1) and the strong maximum principle is the
following splitting theorem (see [20, Theorem 4.1] or the appendix).

**Theorem 2.1** (Splitting Theorem). Let \( X: M^n \to \mathbb{R}^{n+1} \) be a locally weakly
convex solution of (1). Then, either \( \kappa_1 > 0 \) or \( \kappa_1 \equiv 0 \) and \( M^n \) splits as an
isometric product \( M^n \cong \mathbb{R} \times \Sigma^{n-1} \).

We next note that a mean convex translator \( X: M^n \to \mathbb{R}^{n+1} \) which
satisfies the cylindrical estimate must be locally strictly convex. Indeed,
\[
 |A|^2 - \frac{1}{n-1} H^2 = \frac{1}{n-1} \sum_{1<i<j} (\kappa_j - \kappa_i)^2 + \frac{n}{n-1} \kappa_1^2 - \frac{2}{n-1} \kappa_1 H \\
 \geq -\kappa_1 H, \tag{2.3}
\]
so that, wherever the cylindrical estimate holds,
\[ \kappa_1 \geq \frac{1}{H} \left( |A|^2 - \frac{1}{n-1} H^2 \right) > 0. \tag{2.4} \]

Note also that, for a hypersurface satisfying the weak cylindrical estimate
\[ |A|^2 - \frac{1}{n-1} H^2 \leq 0, \]
the only points at which \( \kappa_1 \) can vanish are the cylindrical points, \( \kappa_1 = 0, \kappa_2 = \kappa_n \).

Since \( M^n \) is smooth and \( n \geq 2 \), local convexity implies that \( M^n \) is the boundary of a convex body \([27]\). In particular, \( M^n \) is embedded, so we may drop the parametrization \( X \) and identify \( M^n \) with its image. A further consequence of convexity and the inequality \( \langle \nu, e_{n+1} \rangle = H > 0 \) is the fact that \( M^n \) can be written globally as the graph of a function \( u : \Omega^n := \text{proj}_{\mathbb{R}^n \times \{0\}}(M^n) \to \mathbb{R} \).

Note also that, applied to the gradient estimate, \([23]\) yields
\[ \frac{|\nabla A|^2}{H^4} \leq C \frac{\kappa_1}{H}. \tag{2.5} \]

Thus, the gradient estimate actually improves wherever \( \kappa_1 \) is small compared to \( H \).

We conclude this section by recalling the following well-known consequence of gradient estimates for the curvature (cf. \([22, \text{Lemma 6.6}]\)).

**Lemma 2.2.** Let \( X : M^n \to \mathbb{R}^{n+1} \) be a mean convex hypersurface and \( C < \infty \) a constant such that
\[ \sup_{M^n} \frac{|\nabla H|}{H^2} \leq C. \]

Then
\[ \max_{y \in B_1(\frac{1}{2C H(x)})} H(y) \leq 2H(x), \]
where \( B_1(\frac{1}{2C H(x)}) \) is the intrinsic ball of radius \( \frac{1}{2C H(x)} \) about the point \( x \).

**Proof.** For any unit length geodesic \( \gamma : [0, s] \to M \) joining the points \( y = \gamma(0) \) and \( x = \gamma(s) \), we have
\[ \nabla_{\gamma'} H^{-1} \leq C. \]

Integrating yields
\[ H^{-1}(x) - H^{-1}(y) \leq Cs \]
or, if \( s \leq \frac{1}{2C H(x)} \),
\[ H(y) \leq \frac{H(x)}{1 - CH(x)s} \leq 2H(x). \]

The claim follows. \( \square \)
Corollary 2.3. Let \((X_j : M^n_j \to \mathbb{R}^{n+1}, x_j)_{j \in \mathbb{N}}\) be a sequence of strictly mean convex, weakly locally convex pointed smooth hypersurfaces and \(C < \infty\) a constant satisfying
\[
X_j(x_j) = 0, \quad H_j(x_j) = 1 \quad \text{and} \quad \sup_{M^n_j} \frac{\nabla A_j}{H_j^2} \leq C,
\]
where, for each \(j \in \mathbb{N}\), \(H_j\) and \(A_j\) are the mean curvature and second fundamental form, respectively, of \(M^n_j\). Then there exists a weakly locally convex pointed \(C^2\) hypersurface \((X_\infty : M^n_\infty \to \mathbb{R}^{n+1}, x_\infty)\) such that, after passing to a subsequence, \(X_j|_{B_j} : B_j \to \mathbb{R}^{n+1}\) converge locally uniformly in \(C^2\) to \(X_\infty|_{B_\infty} : B_\infty \to \mathbb{R}^{n+1}\), where \(B_j\) denotes the intrinsic ball in \(M^n_j\) of radius \((2C)^{-1}\) about the point \(x_j\).

3. Proof of Theorem 1.1 and Corollary 1.2

We begin by noting that the mean curvature goes to zero at infinity.

Lemma 3.1. For any sequence of points \(X_j \in M^n\) with \(\|X_j\| \to \infty\),
\[H(X_j) \to 0.\]

Proof. The proof is similar to [16, Lemma 2.1]. Suppose that the lemma does not hold. Then there is a sequence of points \(\{X_j\}_{j=1}^\infty \subset M^n\) satisfying \(\|X_j\| \to \infty\) and \(\limsup_{j \to \infty} H(X_j) > 0\). Passing to a subsequence, we can assume that \(\liminf_{j \to \infty} H(X_j) > 0\). By translational invariance of (1), we can assume, without loss of generality, that \(0 \in M^n\). Furthermore, after passing to a subsequence, \(w_j := X_j/\|X_j\| \to w \in S^n\). Consider the sequence \(M^n_j := M^n - X_j\). Since each \(M^n_j\) satisfies the translator equation (1) and has mean curvature uniformly bounded by 1, it follows from standard regularity theory for solutions of either (1) [13] or (MCF) [11,12] that, after passing to a subsequence, \(M^n_j\) converges locally uniformly in \(C^\infty\) to a weakly convex translator \(M^n_\infty\). We claim that \(M^n_\infty\) contains the line \(\{sw : s \in \mathbb{R}\}\). First note that the closed convex region \(\overline{\Omega}\) bounded by \(M^n\) contains the ray \(\{sw : s \geq 0\}\), since it contains each of the segments \(\{sw_j : 0 \leq s \leq s_j\}\), where \(s_j := \|X_j\|\) and \(w_j := X_j/\|X_j\|\). By convexity, it also contains the set \(\{rsw + (1-r)X_j : s \geq 0, 0 \leq r \leq 1\}\) for each \(j\). It follows that the closed convex region \(\overline{\Omega}_j\) bounded by \(M^n_j\) contains the set \(\{rsw - rsw_j : s > 0, 0 \leq r \leq 1\}\).

In particular, choosing \(s = 2s_j\), \(\{\vartheta(w - w_j) + \vartheta w : 0 \leq \vartheta \leq s_j\} \subset \overline{\Omega}_j\), and choosing \(s = s_j/2\), \(\{\vartheta w_j - \vartheta (w - w_j) : -s_j/2 \leq \vartheta \leq 0\} \subset \overline{\Omega}_j\). Taking \(j \to \infty\), we find \(\{sw : s \in \mathbb{R}\} \subset \overline{\Omega}_\infty\). The claim now follows from convexity of \(\overline{\Omega}_\infty\) since \(0 \in M^n_\infty\). We conclude that \(\kappa_1\) reaches zero somewhere on \(M^n_\infty\). By the splitting theorem, the limit splits as an isometric product \(M^n_\infty \cong \mathbb{R} \times \mathbb{S}^{n-1}\); in particular, \(\kappa_1 \equiv 0\). On the other hand, by the strong maximum principle, we must have \(H > 0\) everywhere (since, by hypothesis, \(H(0) > 0\)). The cylindrical estimate now implies that \(\Sigma^{n-1}\) is umbilical (recall (2.3)) and hence a round sphere. But this contradicts the fact that \(M^n_\infty + te_{n+1}\) satisfies mean curvature flow. \(\square\)
It follows that $H$ attains a maximum at some point $O$, which we call the ‘tip’ of $M^n$. By translational invariance of (T), we can assume, without loss of generality, that $O$ is the origin.

Recall that the gradient field of the height function is given by

$$V = \text{proj}_{TM^n}(e_{n+1}) = e_{n+1} + H\nu.$$  

By the translator equation (T),

$$\|V\|^2 = 1 - H^2$$  

and, differentiating (T),

$$V = -A^{-1}(\nabla H).$$

Moreover, since $A$ is non-degenerate, at any critical point $X$ of $H$ we must have $V(X) = 0$ and hence $\nu(X) = -e_{n+1}$. By strict convexity of $M^n$ (recall (2.4)), we conclude that $H$ has precisely one critical point, the origin, and $H \leq H(0) = 1$.

Next, observe that

$$\nabla V = HA.$$  

Since $A$ is positive definite, it follows from standard ODE theory that we can find, for each $X \in M^n \setminus \{0\}$, a unique integral curve $\phi_X : (0, \infty) \to M^n$ of $V$ through $X$ such that

$$\lim_{s \searrow 0} \phi_X(s) = 0 \quad \text{and} \quad \lim_{s \to \infty} \|\phi_X(s)\| = \infty.$$  

If we parametrize the integral curves by height, so that

$$h(\phi_X(s)) = \langle \phi_X(s) , e_{n+1} \rangle = s,$$

then we obtain

$$\phi_X' = \frac{V \circ \phi_X}{1 - H^2 \circ \phi_X} = \frac{V \circ \phi_X}{\|V \circ \phi_X\|^2}.$$  

Note that, by Lemma 3.1, the reparametrized curves are still defined on $(0, \infty)$.

We will use the improved gradient estimate (2.5) to extract a lower bound for $H$ along the flow of $V$.

**Lemma 3.2 (Lower bound for $H$).** There exists $h_0 > 0$ such that

$$H(X) \geq \frac{1}{\sqrt{4C_1 h(X)}}$$

for all $X \in M^n$ with height $h(X)$ at least $h_0$.

**Proof.** Let $\phi : (0, \infty) \to M$ be an integral curve of $V$ emanating from the tip and parametrized by height. Set $f(s) := H^{-1}(\phi(s))$. Applying the gradient estimate (2.5), we obtain

$$(f')^2 \leq \frac{|\nabla H|^2 |\phi'|^2}{H^4} \leq \frac{C_1 \kappa_1 f}{1 - H^2}.$$  

(3.2)
On the other hand,
\[-\nabla \phi H = A(\phi', V) = \frac{A(V, V)}{\|V\|^2} \geq \kappa_1.\]

That is,
\[
\kappa_1 f \leq \frac{f'}{f}. \tag{3.3}
\]

Putting (3.2) and (3.3) together yields
\[\left( f^2 \right)' \leq \frac{2C_1}{1 - H^2}.\]

By Lemma 3.1, there exists \( h_1 > 0 \) such that \( H(X) \leq \frac{1}{\sqrt{3}} \) for all \( X \in M^n \) with height \( h(X) \) at least \( h_1 \). Thus, for any \( s \geq h_1 \), we obtain \((f^2)' \leq 3C_1\).

Integrating this between \( h_1 \) and \( s \) yields
\[
\frac{1}{H^2(\phi(s))} \leq 3C_1(s - h_1) + \frac{1}{H^2(\phi(h_1))} \leq 4C_1 s
\]
with the last inequality being true provided that \( s \) is large enough; in particular, \( s \geq \max\{h_1, (C_1H^2(h_1))^{-1}\} \). Since \( \phi \) is parametrized by height, the lemma then follows with \( H(h_1) := \min\{H(X) : X \in M^n, h(X) = h_1\} > 0 \).

Next, we derive a lower bound for the 'girth' of \( M^n \). This estimate plays a key role in obtaining an upper bound for \( H \).

**Lemma 3.3 (Girth estimate).** There exists \( h_0 < \infty \) such that
\[
\|\text{proj}_{\mathbb{R}^n \times \{0\}} X\| \geq \sqrt{\frac{h(X)}{16C_1}}
\]
for any \( X \in M^n \) with height \( h(X) \) at least \( h_0 \).

**Proof.** The idea of the proof is the following: If the claim did not hold, then, writing \( M = \text{graph} u \), there would be a point where \( u \) is both large and has large gradient (compared to \( \sqrt{h} \)). But this would contradict the lower bound \( H \geq h^{-\frac{2}{3}} \) since \( H = \frac{1}{\sqrt{1 + |Du|^2}} \).

By Lemma 3.2 there exists \( h_1 > 0 \) such that
\[
H(X) \geq \frac{1}{\sqrt{4C_1 h(X)}} \tag{3.4}
\]
for all \( X \in M^n \) with height \( h(X) \geq h_1 \). Suppose, contrary to the claim, that there is a point \( X = (x, u(x)) \in M^n \) with \( h(X) \geq h_0 := 2h_1 \) but
\[
\frac{h(X)}{\|x\|} > \sqrt{16C_1 h(X)}.
\]
Set $\ell := \|x\|$ and consider the curve $\gamma : [0, \ell] \to M^n$ given by $\gamma(s) := (\hat{\gamma}(s), u(\hat{\gamma}(s)))$, where $\hat{\gamma}(s) := s^2$ is the straight line in $\Omega$ joining 0 and $x$. Slightly abusing notation, set $h(s) := h(\gamma(s))$. Then

$$h' = \nabla_\gamma h \leq \|V\| \sqrt{1 + (D\gamma'u)^2} \leq \sqrt{1 - H^2} \sqrt{1 + |Du|^2} = \frac{\sqrt{1 - H^2}}{H} \leq \frac{1}{H}.$$ 

Let $s_1 \in [0, \ell]$ be the point at which $h(\gamma(s_1)) = h_1$. By the mean value theorem, there is a point $s_2 \in [s_1, \ell]$ such that

$$\frac{1}{H(\gamma(s_2))} \geq h'(s_2) = \frac{h(X) - h_1}{\ell - s_1} \geq \frac{h(X)}{2\ell} > \frac{16}{4} C_1 h(X) \geq \sqrt{4C_1 h(\gamma(s_2))}.$$ 

This contradicts (3.4) and we conclude that

$$\|\text{proj}_{\mathbb{R}^n \setminus \{0\}} X\| \geq \sqrt{\frac{h(X)}{16C_1}}$$

for all $X \in M^n$ with height $h(X) \geq h_0 := 2h_1$. \hfill $\Box$

Next, we find at each height $h$ a point with $H \sim h^{-\frac{1}{2}}$.

**Lemma 3.4.** There exists $h_0 < \infty$ such that, for any $h \geq h_0$, there is a point $X \in M \cap \overline{B}_{\sqrt{2nh}}(he_{n+1})$ satisfying

$$H(X) \leq \sqrt{\frac{h_0}{h}},$$

where $\overline{B}_R(X)$ denotes the closed ball in $\mathbb{R}^{n+1}$ of radius $R$ centred at $X$.

**Proof.** We first show that at each height $h$, the ball of radius $\sqrt{2nh}$ centred at $he_{n+1}$ intersects $M^n$.

**Claim 3.5.** For each height $h > 0$, $\overline{B}_{\sqrt{2nh}}(he_{n+1}) \cap M^n \neq \emptyset$.

**Proof.** Under mean curvature flow, the tip of the translator reaches the point $he_{n+1}$ after time $t = h$. On the other hand, under mean curvature flow, the radius of $\partial B_R(he_{n+1})$ shrinks to the point $he_{n+1}$ in time $t = R^2/2n$. Thus, if the ball $\overline{B}_R(he_{n+1})$ does not intersect $M^n$, the avoidance principle necessitates $R^2 < 2nh$. \hfill $\square$

On the other hand, using Lemma 3.3, the ball in the previous claim can be scaled so that it no longer intersects $M^n$.

**Claim 3.6.** There exists $0 < h_0 < \infty$ such that $B_{\sqrt{h/h_0}}(he_{n+1}) \cap M^n = \emptyset$ for all $h \geq h_0$.

**Proof.** By Lemma 3.3, there exists $h_1 > 0$ such that $\|\text{proj}_{\mathbb{R}^n \setminus \{0\}} X\| \geq \sqrt{\frac{h(X)}{16C_1}}$ for any $X \in M^n$ with height $h(X) \geq h_1$. Set $h_2 := \max\{1, 2h_1\}$, $R := \delta \sqrt{h}$, where $\delta := \frac{1}{2} \min\{\frac{1}{\sqrt{16C_1}}, 1\}$, and consider, for any $h \geq h_2$, the
cylinder $Q_R$ centred at the point $he_{n+1}$ with radius $R$ and height $2R$. Then, for any $X \in Q_R \cap M^n$, we have $h(X) \geq h - R \geq h_1$, so that

$$R \geq \| \text{proj}_{\mathbb{R}^n \times \{0\}}(X) \| \geq \sqrt{\frac{h(X)}{16C_1}} \geq \sqrt{\frac{h - R}{16C_1}}.$$ 

Rearranging, this becomes

$$(1 - 16C_1 \delta^2)\sqrt{h} \leq \delta.$$ 

But this implies $h \leq 2/3$. To avoid a contradiction, we must conclude that $Q_R \cap M^n = \emptyset$. The claim then follows with $h_0 := \max\{h_2, \delta^{-2}\}$. \hfill $\square$

By Claim 3.6 there exists $h_1 < \infty$ such that $B_{\sqrt{h}/h_1}(he_{n+1}) \cap M^n = \emptyset$ for all $h \geq h_0$. Set $h_0 := \max\{h_1, \frac{n}{2}\}$ and define, for any $h \geq h_0$,

$$\rho = \inf\{r : B_{r\sqrt{h}}(he_{n+1}) \cap M \neq \emptyset\}.$$ 

By Claims 3.5 and 3.6 we know that $\frac{1}{\sqrt{h_0}} \leq \rho \leq \sqrt{2n}$. Thus, there exists a point $X \in B_{\sqrt{h_0}}(he_{n+1}) \cap M^n$. Since $M^n$ and $B_{\sqrt{h_0}}(he_{n+1})$ are tangent at $X$ and $M^n$ lies outside of $B_{\sqrt{h_0}}(he_{n+1})$, we have $H(X) \leq \frac{n}{\rho\sqrt{h}} \leq \sqrt{\frac{h_0}{h}}$. \hfill $\square$

Finally, we need to show that $\kappa_1/H$ goes to zero as $h \to \infty$.

**Lemma 3.7 (Asymptotics for $\frac{\kappa_1}{H}$).** For any sequence of points $X_j$ with $h(X_j) \to \infty$,

$$\frac{\kappa_1}{H}(X_j) \to 0.$$ 

**Proof.** Suppose there exists a sequence of points $X_j \in M^n$ with $h_j := h(X_j) \to \infty$ but $\limsup_{j \to \infty} \frac{\kappa_1}{H}(X_j) > 0$. Passing to a subsequence, we can assume that $\liminf_{j \to \infty} \frac{\kappa_1}{H}(X_j) > 0$. We may choose another sequence of points $Y_j \in M^n$ such that

$$\frac{\kappa_1}{H}(Y_j) = \min_{U_j} \frac{\kappa_1}{U_j},$$ 

where $U_j := \{X \in M^n : h(X) \leq h_j\}$. Since $M^n$ is non-compact, we know that $\frac{\kappa_1}{U_j}(Y_j) \to 0$ \cite{[14]}. Moreover, the strong maximum principle implies $h(Y_j) = h_j \to \infty$ since, combining (2.1) and (2.2), the tensor $Z := A/H$ satisfies

$$-\Delta Z(u, u) = \nabla V Z(u, u) + 2 \left\langle \nabla Z(u, u), \frac{\nabla H}{H} \right\rangle.$$ 

Now set $\lambda_j := H(Y_j)$ and consider the sequence $M^n_j := \lambda_j (M^n - Y_j)$. Then

$$H_j(0) = 1 \quad \text{and} \quad \frac{\kappa_1}{H_j}(0) \to 0,$$

where $H_j$ and $\kappa_1^j$ are the mean curvature and smallest principal curvature, respectively, of $M^n_j$. It now follows from the gradient estimate (2.5) (see
Corollary 2.3 that, after passing to a subsequence, the sequence $M^n_j \cap B_j$ converges locally uniformly in $C^2$ to a non-empty limit $M^n_{\infty} \cap B_{\infty}$, where $B_j$ is the intrinsic ball in $M^n_{\infty}$ of radius $(2C_1)^{-1}$ about the origin. But since the sequence $M^n_j \cap B_j$ satisfies

$$H_j(X) = \lambda_j^{-1} \langle \nu_j(X), e_{n+1} \rangle,$$

where $\nu_j(X)$ is the normal to $M^n_j$, the limit $M^n_{\infty} \cap B_{\infty}$ satisfies

$$\langle \nu_{\infty}, e_{n+1} \rangle \equiv 0,$$  \hspace{1cm} (3.5)

where $\nu_{\infty}$ is the normal to $M^n_{\infty}$. In particular, $\kappa_1^{\infty} \equiv 0$ in $B_{\infty}$, and we conclude from the cylindrical estimate that $M^n_{\infty} \cap B_{\infty}$ lies in a cylinder of radius $(n - 1)$. But this implies that the ratio $|\nabla H_j|/H_j^2$ goes to zero on all of $B_j$, and, iterating Corollary 2.3 and passing to a diagonal subsequence, we deduce that $M^n_j$ converges locally uniformly in $C^2$ to a round orthogonal cylinder of radius $(n - 1)$. Moreover, by (3.5), the axis of the cylinder is parallel to $e_{n+1}$. By compactness of the constant height slices, a subsequence of $X_j$ converges to a point in the limit of height zero. But this contradicts $\liminf_{j \to \infty} \mathcal{H}(X_j) > 0$. \hfill \Box

We are now ready to prove that the blow-down of our translator is the shrinking cylinder.

**Lemma 3.8.** Denote by $M^n_t := M^n + te_{n+1}$. Given any sequence $h_j \to \infty$, the sequence $\{M^n_{t,j}\}_{j=1}^{\infty}$ of mean curvature flows

$$M^n_{t,j} := h_j^{-\frac{1}{2}} \left( M^n_{h_j t} - h_j e_{n+1} \right), \hspace{1cm} t \in (-\infty, 1)$$  \hspace{1cm} (3.6)

converges locally uniformly in $C^\infty$ to the shrinking cylinder $S^{n-1}_{r(t)} \times \mathbb{R}$, where $r(t) := \sqrt{2(n-1)(1-t)}$.

**Proof.** By Lemmas 3.1, 3.2, 3.3 and 3.4, there is a sequence $X_j \in M^n$ with $h_j := h(X_j) \to \infty$, $H(X_j) \sim h_j^{-\frac{1}{2}}$ and $\text{proj}_{\mathbb{R}^n \times \{0\}} X_j \sim h_j^{-\frac{1}{2}}$. Moreover, by Lemma 3.7, $\mathcal{H}(X_j) \to 0$. As in the proof of Lemma 3.7, we can use Corollary 2.3 and the cylindrical estimate to deduce that, after passing to a subsequence, $M^n_j := h_j^{-\frac{1}{2}}(M^n - h_j e_{n+1})$ converges locally uniformly in $C^2$ to a limit $M^n_{\infty}$ which is congruent to a round, orthogonal cylinder. Since the limit encloses the ray $\{se_{n+1} : s > 0\}$, its axis must be parallel to $e_{n+1}$. It follows that $H \sim h_j^{-\frac{1}{2}}$. We can now conclude, by the same argument, that for any sequence $\lambda_j \to \infty$ and any $R > 0$, the sequence

$$M^n_j = R \lambda_j^{-\frac{1}{2}} (M^n - \lambda_j e_{n+1})$$  \hspace{1cm} (3.7)

converges subsequentially to a round orthogonal cylinder with axis parallel to $e_{n+1}$. Setting $\lambda_j := R^2 h_j$ where $R := \sqrt{1-t}$, and applying standard regularity theory (see [12] or [11]), we deduce, after passing to a subsequence, that (3.7) converges locally uniformly in $C^\infty$ to a shrinking cylinder with
axis parallel to \( e_{n+1} \). It is also clear from (3.7) that the radius of the limit goes to zero as \( t \to 1 \). We conclude that the limit is \( S^{n-1}_{\tau(t)} \times \mathbb{R} \). Since the limit is the same for any convergent subsequence, the convergence holds for the entire sequence. \( \square \)

**Corollary 3.9** (Asymptotics for \( H \)).

\[
H = \sqrt{\frac{n-1}{2}} h^{-\frac{1}{2}} + o \left( h^{-\frac{1}{2}} \right).
\]

The remainder of the proof of Theorem 1.3 now follows from the work of Haslhofer [16, Sections 3–5].

Finally, we show, briefly, how Corollary 1.2 follows from Theorem 1.1.

**Proof of Corollary 1.2.** The cylindrical estimate follows immediately from the cylindrical estimate of Huisken and Sinestrari [22, Theorem 1.5],

\[
|A|^2 - \frac{1}{n-1}H^2 \leq \varepsilon H^2 + C \varepsilon,
\]

as the lower order term is annihilated under the rescaling no matter how small we take \( \varepsilon \) (see [21, Section 4]). The strong maximum principle then gives the strict inequality.

The gradient estimate follows from the gradient estimate of Huisken and Sinestrari [22, Theorem 6.1 and Remark 6.2],

\[
|\nabla A|^2 \leq C g_1 g_2,
\]

where \( g_1 := 2C_{\varepsilon} + \varepsilon H^2 - \left( |A|^2 - \frac{1}{n-1}H^2 \right) \) for arbitrary \( \varepsilon > 0 \) and \( g_2 = C_\delta + \delta H^2 - \left( |A|^2 - \frac{1}{n-1}H^2 \right) \) with \( \delta = \delta(n) \) fixed. In particular, \( g_1 \leq c_n H^2 + C_n \), so that

\[
|\nabla A|^2 \leq C \left( c_n + \frac{C_n}{H^2} \right) \left( \frac{2C_{\varepsilon}}{H^2} + \varepsilon - \frac{|A|^2 - \frac{1}{n-1}H^2}{H^2} \right).
\]

Under the rescaling, all lower order terms are annihilated, and the claim follows, as for the cylindrical estimate, by taking \( \varepsilon \to 0 \). \( \square \)

### 4. Flows by non-linear functions of curvature

We now consider solutions of (FT) and prove Theorem 1.3 and Corollary 1.4. Let us begin with a discussion of the conditions (1)–(2) of Theorem 1.3 which will replace the corresponding conditions in Theorem 1.1.

#### 4.1. Flows by convex speeds.

For speeds \( F = f(\vec{r}) \) given by convex admissible \( f : \Gamma^n \to \mathbb{R} \), the cylindrical estimate takes the form

\[
\kappa_1 + \kappa_2 - \beta_1^{-1} F > 0,
\]

where \( \beta_1 = f(0,1,\ldots,1) \) is the value \( F \) takes on the cylinder \( \mathbb{R} \times S^{n-1} \).

We claim that \( \kappa_1 \) is bounded from below by \( \kappa_1 + \kappa_2 - \beta_1^{-1} F \), and that the
only points at which both $\kappa_1$ and $\kappa_1 + \kappa_2 - \beta_1^{-1}F$ vanish are the cylindrical points: $\kappa_1 = 0$, $\kappa_2 = \kappa_n$.

**Claim 4.1.** Set

$$\Lambda := \left\{ z \in \Gamma^n : \min_{1 \leq i < j \leq n} (z_i + z_j) - \beta_1^{-1}f(z) > 0 \right\}.$$ 

Then

1. $\Lambda \subset \Gamma^+$, and
2. $\partial \Lambda \cap \partial \Gamma^+ = \cup_{\sigma \in P_n} \{ k(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)}) : k \geq 0 \}$, where $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_n = 1$ and $P_n$ denotes the set of permutations of the set $\{1, \ldots, n\}$.

In particular, there is a constant $\beta_2 > 0$ such that

$$\min_{1 \leq i < j \leq n} (z_i + z_j) - \beta_1^{-1}f(z) \leq \beta_2 \min_{1 \leq i \leq n} z_i .$$

**Proof.** Note that, as a super-level set of a concave function, $\Lambda$ is convex. Note also that $(0, 1, \ldots, 1) \in \partial \Lambda$. Thus, by symmetry and convexity, we have $(1, \ldots, 1) \in \Lambda$. Finally, by homogeneity and strict monotonicity of $f$, the only points in $\overline{\Lambda}$ of the form $(0, z_2, \ldots, z_n)$ for $0 < z_i$ are those with $z_2 = \cdots = z_n$. Claims (1) and (2) follow. The existence of $\beta_2$ then follows from compactness of the set $\Lambda \cap \{ \| z \| = 1 \}$ and homogeneity of $f$. \qed

The gradient estimate then takes the form

$$\frac{|\nabla A|^2}{F^4} \leq C_1 \frac{\kappa_1}{F} . \tag{4.2}$$

We remark that (4.1) holds on blow-up limits of two-convex flows by convex admissible speeds $[6]$; however, it is unknown (to the authors) whether a gradient estimate of the form (4.2) holds, except when $F$ is the mean curvature. We note that, by a similar argument as in Claim 4.3 below, the estimate

$$\frac{|\nabla A|^2}{F^4} \leq C_1$$

would suffice.

**4.2 Flows by concave speeds.** For speeds $F = f(\vec{R})$ given by concave admissible $f : \Gamma^n \to \mathbb{R}$, the cylindrical estimate takes the form

$$\kappa_n - \beta_1^{-1}F < 0 , \tag{4.3}$$

where $\beta_1 = f(0, 1, \ldots, 1)$ is the value $F$ takes on the cylinder $\mathbb{R} \times S^{n-1}_1$. We claim that $\kappa_1$ is bounded from below by $\beta_1^{-1}F - \kappa_n$, and that the only points at which both $\kappa_1$ and $\kappa_n - \beta_1^{-1}F$ vanish are the cylindrical points (cf. [6] Proposition 3.6 and Theorem 3.8)].
Claim 4.2. Set
\[
\Lambda := \{ z \in \Gamma^n : \max_{1 \leq i \leq n} z_i - \beta_1^{-1} f(z) < 0 \}.
\]
Then

1. \( \Lambda \subset \Gamma^+ \), and

2. \( \partial \Lambda \cap \partial \Gamma^+ = \bigcup_{\sigma \in P_n} \{ k(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)}) : k \geq 0 \} \), where \( \lambda_1 = 0 \) and \( \lambda_2 = \cdots = \lambda_n = 1 \) and \( P_n \) denotes the set of permutations of the set \( \{1, \ldots, n\} \).

In particular, there is a constant \( \beta_2 > 0 \) such that

\[
\beta_1^{-1} f(z) - \max_{1 \leq i \leq n} z_i \leq \beta_2 \min_{1 \leq i \leq n} z_i.
\]

Proof. The proof is the same as the proof of Claim 4.1. \( \square \)

The gradient estimate then takes the form

\[
\frac{|\nabla A|^2}{F^4} \leq C_1 \kappa_1 \frac{F}{F}
\]

(4.4)

We remark that (4.3) holds on blow-up limits of two-convex flows by concave admissible speeds [25] (cf. [9, Theorem 3.1]). Moreover, making use of [9, Theorem 6.1], we find that the gradient estimate also holds if the underlying flow is embedded.

Claim 4.3. Let \( X : M^n \times [0, T) \to \mathbb{R}^{n+1} \) be an embedded solution of (F), where \( F \) is given by \( F = f(\bar{\kappa}) \) for some admissible \( f : \Gamma^n \to \mathbb{R} \) satisfying the conditions of Corollary 1.4. Then there is a constant \( C = C(n, M_0) \) and, for any \( \varepsilon > 0 \), a constant \( F_\varepsilon = F(\varepsilon, n, M_0) \) such that

\[
\frac{|\nabla A|^2}{F^4} \leq \varepsilon + C \kappa_1 \frac{F}{F}
\]

wherever \( F > F_\varepsilon \).

Proof. We will make use of the gradient estimate of [9, Theorem 6.1], which provides a constant \( \Lambda = \Lambda(n, M_0) \) such that

\[
|\nabla A|^2 \leq \Lambda F^4.
\]

(4.5)

We note that the interior non-collapsing estimate [7] and Sections 5 and 6 of [9] apply to embedded flows satisfying the conditions of Corollary 1.4.

So suppose that the claim does not hold. Then there is a constant \( \varepsilon_0 > 0 \) and a sequence of points \( (x_j, t_j) \in M^n \times [0, T) \) with \( F(x_j, t_j) \to \infty \) such that

\[
\frac{|\nabla A|^2}{F^4}(x_j, t_j) > \varepsilon_0 + j \kappa_1 \frac{F}{F}(x_j, t_j).
\]

If

\[
\limsup_{j \to \infty} \frac{\kappa_1}{F}(x_j, t_j) > 0
\]
then we would obtain a contradiction to (4.3). Otherwise, passing to a subsequence, translating in space and time, and rescaling by 
\[ \lambda_j = F(x_j, t_j), \]
we obtain a sequence of flows \( X_j : \mathbb{R}^n \times (-\lambda_j^2 t_j, 0) \rightarrow \mathbb{R}^{n+1} \) with
\[ X_j(x_j, 0) = 0, \quad F_j(x_j, 0) = 1, \quad \kappa_1(x_j, 0) \rightarrow 0 \quad \text{and} \quad |\nabla A|^2(x_j, 0) > \frac{\varepsilon_0}{2}. \]

By [9, Theorem 6.1], this sequence converges in a uniform parabolic neighbourhood of \((x_j, 0)\) locally uniformly in \(C^2\) to some non-empty smooth limit flow. By the cylindrical estimate [25] (cf. [9, Theorem 3.1]), this limit must satisfy \( \kappa_n - \beta^{-1} F \leq 0 \). By Claim 4.2, this implies \( \kappa_1 \geq 0 \). Since \( \kappa_1 \) reaches zero at the origin, we can now conclude from the splitting theorem and Claim 4.2 that the limit is contained in a shrinking cylinder. But this contradicts the fact that \( |\nabla A|^2_F \geq \frac{\varepsilon_0}{2} \) at some point on the limit.

It follows that blow-up limits of (F) with speeds satisfying the conditions of Corollary 1.4 satisfy
\[ \frac{|\nabla A|^2}{F^4} \leq C\kappa_1 F. \]

Note that flows by concave speeds are interior non-collapsing [7]. Moreover, the non-collapsing estimate improves at a singularity [25]. Thus, we can replace the cylindrical estimate by
\[ \kappa_n - \beta^{-1} F < 0. \]

This formulation of the cylindrical estimate is non-trivial in dimension \( n = 2 \), but stronger than (4.3) when \( n \geq 3 \).

Armed with these facts, and the splitting theorem of the Appendix, we can proceed almost exactly as in Section 3 to show (assuming, without loss of generality, that \( f(0, 1, \ldots, 1) = n - 1 \), that the blow-down of \( M^n_t := M^n + t e_{n+1} \) is the shrinking cylinder \( S^{n-1}_{\sqrt{2(n-1)(1-t)}} \times \mathbb{R} \).

By the conditions on \( F \), the remainder of the proof differs only slightly from [16, Sections 3-5]. Indeed, the linearization of (F) is the equation
\[ (\partial_t - \Delta_F)u = |A|^2 F u, \quad (4.6) \]
where, in an orthonormal frame of eigenvectors for \( A \), \( \Delta_F := \partial^l_i \nabla_i \nabla^l \) and \( |A|^2_F := \partial^l_i \kappa^2 \). Solutions of the linearized flow on a translating solution of (F) correspond to solutions of the linearized translator equation
\[ -\Delta_F u = \nabla V u + |A|^2 F u \quad (4.7) \]
on the corresponding solution of (F T). Since the speed \( F \) satisfies this equation, the strong maximum principle implies that
\[ \sup_{h \leq h_0} \frac{|u|}{F} \leq \sup_{h = h_0} \frac{|u|}{F} \]
for any \( u \) satisfying (4.7) on a strictly convex solution of (F T).
By the invariance of \( F \) under ambient isometries, the functions
\[
u_{J,O}(x,t) := \langle J(X(x,t) - O), \nu(x,t) \rangle
\]
satisfy (4.6) for any rotation generator \( J \in \mathfrak{so}(n+1) \) and translation generator \( O \in \mathbb{R}^{n+1} \).

Recalling that we have normalized \( f \) so that 
\[
f(0,1,\ldots,1) = n-1,
\]
observe that (modulo a time-dependent tangential reparametrization) the shrinking cylinders
\[
C : S^{n-1} \times \mathbb{R} \times (-\infty,1) \to S^{n-1}_r(t) \times \mathbb{R} \subset \mathbb{R}^{n+1}
\]
with \( r(t) := \sqrt{2(n-1)(1-t)} \) satisfy (1.6). By symmetry and homogeneity of \( f \), we find, for each \( j = 2, \ldots, n \), that
\[
\frac{\partial f}{\partial \kappa_j} = \frac{r}{n-1} \sum_{i=2}^{n} \frac{\partial f}{\partial \kappa_i} \kappa_i = \frac{r}{n-1} \sum_{i=1}^{n} \frac{\partial f}{\partial \kappa_i} \kappa_i = \frac{r}{n-1} F = 1
\]
on the shrinking cylinder, so that
\[
\Delta_F = \frac{\partial f}{\partial \kappa_1} \nabla_h \nabla_h + \frac{1}{r^2} \Delta S^n
\]
and
\[
|A|^2 = \frac{1}{2(1-t)}.
\]
It is now clear that the decay estimate [16, Proposition 4.1] and the contradiction argument in [16, Section 5] apply in the non-linear setting. This proves Theorem 1.3. Corollary 1.4 then follows, since, by [25] (cf. [9, Theorem 3.1]) and Claim 4.3, the assumptions of Theorem 1.3 hold on blow-up limits of solutions of (F).

5. Appendix: The splitting theorem

We include here a proof of the splitting theorem for solutions of (F).

**Theorem 5.1** (Splitting Theorem). Let \( X : M^n \times (0,t_0] \to \mathbb{R}^{n+1}, n \geq 2, \) be a weakly convex solution of (F), where \( F \) is given by \( F(x) = f(\tilde{\kappa}(x)) \) for some admissible \( f : \Gamma^n \to \mathbb{R} \) such that

(i) \( \{ (0, \bar{z}) : \bar{z} \in \Gamma^{n-1}_+ \} \subset \Gamma^n \) and the function \( f_s : \Gamma^{n-1}_+ \to \mathbb{R} \) defined by
\[
f_s(z_2^{-1},\ldots,z_n^{-1}) := f(0, z_2, \ldots, z_n)^{-1}
\]
is concave.

Suppose also that

(ii) \( \tilde{\kappa}(M^n \times (0,t_0]) \subset \Gamma_0^n \) for some cone \( \Gamma_0^n \) satisfying \( \Gamma_0^n \setminus \{0\} \subset \Gamma_2^n \), where \( \Gamma_2^n := \{ z \in \mathbb{R}^n : \min_{1 \leq i < j \leq n} \{ z_i + z_j \} > 0 \} \).

Then \( \kappa_1(x_0,t_0) = 0 \) for some \( x_0 \in M^n \) only if \( \kappa_1 \equiv 0 \) and \( M^n \) splits isometrically as a product \( M^n \cong \mathbb{R} \times \Sigma^{n-1} \).
Proof. This was established for convex speeds in [24, Theorem 4.21]. The proof for speeds satisfying the weaker inverse-concavity condition is similar: Suppose that \( \kappa_1 \) reaches zero at an interior space-time point \((x_0, t_0)\). By hypothesis, \( \kappa_1 < \kappa_2 \) at this point. Let \( U \) be the largest space-time neighbourhood of \((x_0, t_0)\) in \( M^n \times (0, t_0] \) such that \( \kappa_1 < \kappa_2 \). Then \( U \) is open, \( \kappa_1 \) has a unique principal direction field \( e_1 \) in \( U \), and both are smooth in \( U \).

Differentiating \( \kappa_1 = A(e_1, e_1) \) yields
\[
\nabla_k \kappa_1 = \nabla_k A_{11} + 2A(\nabla_k e_1, e_1),
\]
so that
\[
\nabla_k A_{11} = \nabla_k \kappa_1 = 0 \quad (5.1)
\]
at \((x_0, t_0)\) for each \( k \). Note that \( \nabla_k e_1 \perp e_1 \) since \( e_1 \) has constant length. Differentiating the eigenvalue identity \( A(e_1) = \kappa_1 e_1 \) yields the remaining components:
\[
(A - \kappa_1 I)(\nabla_k e_1) = (\nabla_k \kappa_1 I - \nabla_k A)(e_1),
\]
so that
\[
\nabla_k e_1 = -R(\nabla_k A(e_1)), \quad (5.2)
\]
where \( R := (A - \kappa_1 I)|_{e_1}^{-1} \circ \text{proj}_{e_1} \). Next, consider the time derivative
\[
\partial_t \kappa_1 = \nabla_t A_{11} + 2A(\nabla_t e_1, e_1),
\]
where the covariant time derivative \( \nabla_t \) is defined on vector fields \( v \) via \( \nabla_t v = [\partial_t, v] - HA(v) \), and extended to tensor fields by the Leibniz rule. This yields
\[
\partial_t \kappa_1 = \nabla_t A_{11}
\]
at \((x_0, t_0)\). Finally, we compute the Hessian,
\[
\nabla_k \nabla_l \kappa_1 = \nabla_k \nabla_l A_{11} + 4\nabla_k A(\nabla_l e_1, e_1) + 2A(\nabla_k \nabla_l e_1, e_1) + 2A(\nabla_k e_1, \nabla_l e_1).
\]
Applying (5.2) and the Codazzi identity, we obtain
\[
\nabla_k \nabla_l \kappa_1 = \nabla_k \nabla_l A_{11} - 2R(\nabla_l A(e_k), \nabla_1 A(e_l))
\]
at \((x_0, t_0)\).

In an orthonormal frame of eigenvectors of \( A \), we have the evolution equation
\[
(\nabla_t - \Delta_F)A_{ij} = |A|^2_{F}A_{ij} + \frac{\partial^2 F}{\partial A_{pq} \partial A_{rs}} \nabla_i A_{pq} \nabla_j A_{rs}, \quad (5.3)
\]
where \( \Delta_F := \frac{\partial F}{\partial A_{kl}} \nabla_k \nabla_l \) and \( |A|^2_{F} := \frac{\partial F}{\partial A_{kl}} A_{kl}^2 \), and we conclude
\[
(\partial_t - \Delta_F)\kappa_1 = |A|^2_{F} \kappa_1 + N(A, \nabla A),
\]
where
\[
N(A, \nabla A) := 2A((\nabla t - \Delta)e_1, e_1) + \frac{\partial^2 F}{\partial A_{pq} \partial A_{rs}} \nabla_1 A_{pq} \nabla_1 A_{rs} \\
+ 2 \frac{\partial F}{\partial A_{kl}} \left[ 2R(\nabla_1 A_k, \nabla_1 A_l) - A(R(\nabla_1 A_k), R(\nabla_1 A_l)) \right].
\]

Observe that, at any boundary point \(Z \in \text{Sym} \Gamma_n \cap \partial \Gamma^+_n\), the space of symmetric \(n \times n\) matrices with eigenvalues \(z\) in \(\Gamma_n \cap \partial \Gamma^+_n\), we have, for any totally symmetric \(T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n\),
\[
N(Z, T) = B^p(Z, T)T_{p11} + \sum_{p,q,r,s>1} Q^{pq,rs}(Z)T_{pq}T_{rs},
\]
where
\[
B^1(Z, T) := \left( \frac{\partial^2 F}{\partial A_{11} \partial A_{11}} T_{111} + 2 \sum_{p,q>1} \frac{\partial^2 F}{\partial A_{pq} \partial A_{11}} T_{1pq} \right) |_{Z},
\]
\[
B^p(Z, T) := R^{pq} \left( \frac{\partial F}{\partial A_{11}} T_{11q} + 2 \sum_{k>1} \frac{\partial F}{\partial A_{1k}} T_{k1q} \right) |_{Z}
\]
for \(p > 1\)
and
\[
Q^{pq,rs}(Z) := \left( \frac{\partial^2 F}{\partial A_{pq} \partial A_{rs}} + \frac{\partial F}{\partial A_{pr}} R^{qs} \right) |_{Z}.
\]

We claim that, as quadratic forms on the space of \((n-1) \times (n-1)\) symmetric matrices,
\[
Q \geq 2 \frac{DF \otimes DF}{F} \tag{5.4}
\]
at any \(Z \in \text{Sym} \Gamma^+_n \cap \partial \Gamma^+_n\). Indeed, embedding the space \(\text{Sym} \Gamma^+_n\) of positive definite \((n-1) \times (n-1)\) symmetric matrices into the space \(\text{Sym} \Gamma^+_n\) of non-negative definite \(n \times n\) symmetric matrices via the natural inclusion, the inverse-concavity condition is equivalent to concavity of the function \(F_* : \text{Sym} \Gamma^+_n \to \mathbb{R}\) defined by \(F_*(Z^{-1}) := F(Z)^{-1}\), where \(F(Z) := f(z)\) and \(z\) is the \(n\)-tuple of eigenvalues of \(Z\). Differentiating this identity in the direction of \(B \in \text{Sym} \Gamma^+_n\), we find
\[
-D_X F_*|_{Z^{-1}} = -\frac{1}{F^2(Z)} DBF|_Z,
\]
where \(X := Z^{-1}BZ^{-1}\). Differentiating once more yields
\[
D_X D_X F_*|_{Z^{-1}} + 2D_X Z X F_*|_{Z^{-1}} = \frac{2}{F^3(Z)} (DBF|_Z)^2 - \frac{1}{F^2(Z)} DBDBF|_Z
\]
and we conclude
\[
0 \leq - D_X D_X F_s |_{Z^{-1}} = \left. \frac{1}{F^2(Z)} \left( D^2 F - \frac{2DF \otimes DF}{F} + 2DF \ast Z^{-1} \right) \right|_{Z} (B, B),
\]
where $\ast$ denotes the product $(R \ast S)^{pq,rs} := R^{pr} S^{qs}$. This implies [5.4].

We now return to the evolution equation for $\kappa_1$. Note that $N$ is Lipschitz with respect to $A$. Thus, denoting by $\mathbf{A}$ the projection of $A$ onto $\partial \text{Sym}_{Γ^n}^+$, we obtain
\[
(\partial_t - \Delta F) \kappa_1 + B^k \nabla_k \kappa_1 \geq - | N(A, \nabla A) - N(\mathbf{A}, \nabla A) | \geq - C \| A - \mathbf{A} \| = - C \kappa_1,
\]
where $C$ is the worst Lipschitz constant of $N(\cdot, \nabla A)$ on the set $U$. Note that $C$ is bounded on any compact subset of $U$. The strong maximum principle now implies that $\kappa_1 \equiv 0$ on $K$ for any compact subset $K$ of $U$. It follows that $U \subset \{(x, t) \in M^n \times (0, t_0] : \kappa_1(x, t) = 0\} \subset U$ and we deduce that $U$ is closed, and hence equal to $M^n \times (0, t_0]$. But in that case, we must have, by [5.3],
\[
0 \equiv \partial F \frac{\partial}{\partial A_{pq}} \nabla_1 A_{pq}.
\]
By monotonicity of $F$, we conclude that $\nabla_1 A \equiv 0$.

Using standard arguments, we can now deduce the splitting: Observe that, for any $v \in \Gamma(\ker(A))$,
\[
0 \equiv \nabla_k (A(v)) = \nabla_k A(v) + A(\nabla_k v) = A(\nabla_k v).
\]
Thus, $\nabla_k v \in \Gamma(\ker(A))$ whenever $v \in \Gamma(\ker(A))$; that is, $\ker(A) \subset TM^n$ is invariant under parallel translation in space. Since, for any $v \in \Gamma(\ker(A))$ and any $u \in TM^n$, we have
\[
\nabla^X_D_u X_s v = X_s \nabla_u v - A(u, v) v = X_s \nabla_u v \in X_s \ker A,
\]
where $\nabla^X$ is the pull-back of the Euclidean connection along $X$, we deduce that $X_s \ker A \subset T\mathbb{R}^{n+1}$ is parallel (in space) with respect to $\nabla^X_D$.

Moreover, using the evolution equation [5.3] for $A$, we obtain
\[
\nabla_t A(v) = \Delta_F A(v) = \frac{\partial F}{\partial A_{kl}} [\nabla_k (\nabla_t A(v)) - \nabla_t (A(\nabla_k v)) - A(\nabla_t \nabla_k v)] = 0,
\]
so that
\[
A(\nabla_t v) = \nabla_t (A(v)) - \nabla_t A(v) \equiv 0;
\]
that is, \( \ker A \) is also invariant with respect to \( \nabla_t \). Since, for any \( v \in \Gamma(\ker(A)) \), we have \( \nabla_v F = \frac{\partial F}{\partial A_{kl}} \nabla_v A_{kl} \equiv 0 \), this implies that
\[
X D_t X_s v = (\nabla_v F) v + X_s \nabla_t v = X_s \nabla_t v,
\]
and we deduce that \( X_s \ker A \) is also parallel in time. We conclude that the orthogonal compliment of \( X_s \ker(A) \) is a constant (in space and time) subspace of \( \mathbb{R}^{n+1} \).

Now consider any geodesic \( \gamma : \mathbb{R} \to M^n \times \{t\}, \ t \in (0, t_0], \) with \( \gamma'(0) \in \ker(A) \). Then, since \( \ker(A) \) is invariant under parallel translation, \( \gamma'(s) \in \ker(A) \) for all \( s \), so that
\[
X D_s X_s \gamma' = X_s \nabla_s \gamma' - A(\gamma', \gamma') v = 0.
\]
Thus, \( X \circ \gamma \) is geodesic in \( \mathbb{R}^{n+1} \). We can now conclude that \( X \) splits off a line, \( M^n \cong \mathbb{R} \times \Sigma^{n-1} \), such that \( \mathbb{R} \) is flat (\( T \mathbb{R} \) is spanned by \( \ker(A) \)) and \( \Sigma^{n-1} \) is strictly convex (\( T \Sigma^{n-1} \) is spanned by the rank space of \( A \)) and maps into the constant subspace \( (X_s \ker(A))^1 \cong \mathbb{R}^n \).

It follows that \( X \mid_{\{0\} \times \Sigma^{n-1} \times (0, t_0]} \) satisfies
\[
\partial_t \bar{X}(\bar{x}, t) = -\bar{F}(\bar{x}, t)\bar{v}(\bar{x}, t), \quad (5.5)
\]
for all \( (\bar{x}, t) \in \{0\} \times \Sigma^{n-1} \times (0, t_0] \), where \( \bar{v} = v \mid_{\{0\} \times \Sigma \times (0, t]} \) and \( \bar{F} \) is given by the restriction of \( f \) to \( \Gamma^{n-1}_+ \cong \{z \in \bar{\Gamma}_+ : z_1 = 0, z_2 > 0, \ldots, z_n > 0\} \). \( \square \)

**Remarks 5.2.**

1. By condition (ii), the cross-section \( \Sigma^{n-1} \) in the splitting must be compact \( [\Gamma] \), and we conclude, by uniqueness of solutions of \( [5.5] \), that the isometric splitting persists until the maximum time.
2. Flows by convex admissible speeds defined on the faces of \( \Gamma^1_+ \) automatically satisfy condition (i).
3. If \( n = 2 \), flows by admissible speeds defined on the faces of \( \Gamma^2_+ \) automatically satisfy condition (i).
4. Condition (ii) can be arranged if the flow preserves any form of uniform two-convexity. This is the case for flows by convex speeds, which preserve \( \kappa_1 + \kappa_2 \geq \alpha F \), flows of surfaces (trivially) and flows by concave speeds satisfying \( f \mid_{\partial \Gamma^n} \equiv 0 \), \( \Gamma^n \subset \Gamma^2_2 \), which preserve \( \kappa_n \leq CF \) or \( H \leq CF \).

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