Sequential scheme for locally discriminating bipartite unitary operations without inverses

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Local distinguishability of bipartite unitary operations has recently received much attention. A nontrivial and interesting question concerning this subject is whether there is a sequential scheme for locally discriminating between two bipartite unitary operations, because a sequential scheme usually represents the most economic strategy for discrimination. An affirmative answer to this question was given in the literature, however with two limitations: (i) the unitary operations to be discriminated were limited to act on $d \otimes d$, i.e., a two-qudit system, and (ii) the inverses of the unitary operations were assumed to be accessible, although this assumption may be unrealizable in experiment. In this paper, we improve the result by removing the two limitations. Specifically, we show that any two bipartite unitary operations acting on $d_A \otimes d_B$ can be locally discriminated by a sequential scheme, without using the inverses of the unitary operations. Therefore, this paper enhances the applicability and feasibility of the sequential scheme for locally discriminating unitary operations.

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I. INTRODUCTION

Distinguishability of unitary operations is a fundamental problem in quantum information and has received extensive attention. Discrimination of unitary operations is generally transformed to discrimination of quantum states by preparing an input state and then discriminating the output states generated by different unitary operations. However, distinguishability of unitary operations shows some interesting properties essentially different from that of quantum states, especially in the case of multiple queries.

Two unitary operations $U$ and $V$ are said to be perfectly distinguishable (with a single query), if there exists an input state $|\psi\rangle$ such that $U|\psi\rangle \perp V|\psi\rangle$. It has been shown that $U$ and $V$ are perfectly distinguishable if, and only if $\Theta(U^\dagger V) \geq \pi$, where $\Theta(W)$ denotes the length of the smallest arc containing all the eigenvalues of $W$ on the unit circle [1, 2]. The situation changes dramatically when multiple queries are allowed, since any two different unitary operations are perfectly distinguishable in this case. Specifically, it was shown that for any two unitary operations $U$ and $V$, there exist a finite number $N$ and a suitable state $|\varphi\rangle$ such that $U^\otimes N |\varphi\rangle \perp V^\otimes N |\varphi\rangle$ [1, 2]. Such a discriminating scheme is intuitively called a parallel scheme. Note that in the parallel scheme, an $N$-partite entangled state as an input is required and plays a crucial role. Then, the result was further refined in [3] by showing that the entangled input state is not necessary for perfect discrimination of unitary operations. Specially, [3] showed that for any two different unitary operations $U$ and $V$, there exist an input state $|\varphi\rangle$ and auxiliary unitary operations $w_1, \ldots, w_N$ such that $Uw_N U \ldots w_1 U|\varphi\rangle \perp Vw_N V \ldots w_1 V|\varphi\rangle$. Generally, such a discriminating scheme is called a sequential scheme.

Note that in these researches mentioned above, it was assumed by default that the unitary operations to be discriminated are under the complete control of a single party who can perform any physically allowed operations to achieve an optimal discrimination. Actually, a more complicated case is that the unitary operations to be discriminated are shared by several spatially separated parties. Then, in this case a reasonable constraint on the discrimination is that each party can only make local operations and classical communication (LOCC). Despite this constraint, Refs. [4, 5] independently showed that any two bipartite unitary operations can be perfectly discriminated by LOCC when a finite number of queries are allowed. This implies that LOCC reaches distinguishability that global operations would have, probably with more queries to the unitary operations.

It is worth mentioning that distinguishability of unitary operations has interesting relations with other issues. For instance, it has a closed relation with universality of quantum gates [6] as shown in [5], and it is also related to the analysis of numerical range [7] as presented in [4]. Despite different methods used in [4, 5], the main idea of them, which is depicted in Fig. 1, can be roughly described as follows.

(i) For two bipartite unitary operations $U$ and $V$ shared by Alice and Bob that satisfy certain conditions, there exist a finite number $N$ and a product state $|\psi\rangle_A |\varphi\rangle_B$, such that $U^\otimes N |\psi\rangle_A |\varphi\rangle_B \perp V^\otimes N |\psi\rangle_A |\varphi\rangle_B$, where $|\psi\rangle_A$ and $|\varphi\rangle_B$ are two $N$-partite states prepared by Alice and Bob, respectively, and one of which must be an $N$-partite entangled state. Such an entangled state

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held by one party is called local entanglement.

(ii) For any two general bipartite unitary operations \( U \) and \( V \), one can construct a quantum circuit \( f(X) = Xw_1X...w_N \) with \( X \in \{U,V\} \) and a sequence of local unitary operations \( w_1, \ldots, w_n \) (each \( w_i \) has the form \( w_i = u_i \otimes v_i \)), such that \( f(U) \) and \( f(V) \) satisfy the desired condition stated in item (i). Thus \( f(U) \) and \( f(V) \) can be discriminated as in item (i), which means that \( U \) and \( V \) can be perfectly discriminated by LOCC.

In the above procedure, there generally needs to be a mixed scheme which combines the sequential and the parallel schemes to achieve a perfect discrimination. At the same time, one of the two parties who share the bipartite unitary operations must prepare a multipartite entangled state. But, note that a sequential scheme usually represents the most economic strategy for discrimination, since it does not require entanglement as indicated by the sequential scheme \([3]\) compared with the parallel scheme \([1, 2]\). Then a natural question, as proposed in \([4]\), is whether there is a sequential scheme for perfectly discriminating bipartite unitary operations by LOCC.

In Ref. \([3]\) we answered the above question affirmatively by proving that any two bipartite unitary operations acting on \( d \otimes d \) (i.e., a two-qudit system), in principle, can be perfectly discriminated by LOCC with a sequential scheme, when a finite number of queries are allowed. However, there is still room for improvement at least from the following two aspects.

First, the result only applies to the unitary operations acting on \( d \otimes d \), where the two subsystems have the same dimension. Then, how about the general unitary operations acting on \( d_A \otimes d_B \) with \( d_A \neq d_B \)?

Second, in the proof of the result, in order to discriminate \( U \) and \( V \), their inverses \( U^\dagger \) and \( V^\dagger \) were assumed to be accessible as long as \( U \) and \( V \) are accessible. This assumption is also fundamental in \([3]\). One may think that a unitary operation \( U \) can be regarded as a black box with input and output ports, and then the inverse \( U^\dagger \) can be obtained by simply reversing the whole setup. However, by the current experiment technology, such an operation may not be easily realized or even cannot be realized. Then, a natural question is, can we avoid using the inverse \( U^\dagger \)? The answer was shown to be “yes” for the case of \( d = 2 \) in \([2, 3]\), but it was not clear for the case of higher dimensions.

Therefore, in this paper we improve the result of \([3]\) by considering the above two points. Specifically, we show that any two different bipartite unitary operations acting on \( d_A \otimes d_B \), allowed to be queried a finite number of times, can be locally discriminated by a sequential scheme, without using the inverses of the unitary operations. This result rests on universality of quantum gates \([10]\).

The rest of this paper is organized as follows. Section II presents some preliminaries of this paper. The main result is made in Section III. A conclusion is made in Section IV.

II. PRELIMINARIES

We first recall a result regarding the distinguishability of unitary operations in \([2]\).

**Lemma 1.** Let \( U \) and \( V \) be two different unitary operations. Then there exist a finite number \( N \), auxiliary unitary operations \( w_1, \ldots, w_N \), and an input state \( |\psi\rangle \) such that

\[
Uw_N U \cdots w_1 U |\psi\rangle \perp V w_N V \cdots w_1 V |\psi\rangle.
\]

The above scheme is the so-called sequential scheme for discriminating two unitary operations. Also, there was a parallel scheme \([1, 2]\) which claims that for any two different unitary operations \( U \) and \( V \), there exist a finite number \( N \) and a state \( |\varphi\rangle \) such that \( U^\otimes N |\varphi\rangle \perp V^\otimes N |\varphi\rangle \).

In this paper, we focus on unitary operations acting on a bipartite system \( AB \). Assume that each subsystem \( X (X \in \{A, B\}) \) has a \( d_X \)-dimensional Hilbert space \( H_X \). Then the whole state space of \( AB \) is \( H_A \otimes H_B \), and we will use \( d_A \otimes d_B \) as an abbreviation for it. Let \( U(d_A \otimes d_B) \) denote the set of all unitary operations acting on \( d_A \otimes d_B \), and let \( U_M \) denote the set of all unitary operations acting on a \( d \)-dimensional Hilbert space. \( U \in U(d_A \otimes d_B) \) is said to be primitive if there exist \( |\varphi\rangle \in H_A \) and \( |\phi\rangle \in H_B \) such that the state \( U|\varphi\rangle |\phi\rangle \) is entangled. Otherwise, it is primitive. Equivalently, as mentioned in \([2, 10]\), \( U \) is primitive if it can be written as \( U_A \otimes U_B \) when \( d_A \neq d_B \), or can be written as \( U_A \otimes U_B \) or \( (U_A \otimes U_B) P \) when \( d_A = d_B \), where \( U_A \in U_{d_A} \), \( U_B \in U_{d_B} \), and \( P \) is a swapping operation, i.e., \( P|x\rangle |y\rangle = |y\rangle |x\rangle \).

Let \( P \) denote a subset of \( U(d_A \otimes d_B) \) as

\[
P \equiv \{ U \in U(d_A \otimes d_B) : U_A = U_B \}.
\]

Harrow \([10]\) obtained a result concerning universality of quantum gates as follows.
FIG. 2: A sequential scheme for locally discriminating unitary operations $U$ and $V$ acting on $d_A \otimes d_B$. Here $X$ represents the unknown unitary operation $U$ or $V$, $N+1$ is the finite number of applying $X$, $\{u_i\}_{i=1}^N$ and $\{v_i\}_{i=1}^N$ are unitary operations acting on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $|\varphi_A\rangle|\phi_B\rangle$ is the input state. The output state $|\Phi_U\rangle$ and $|\Phi_V\rangle$ are orthogonal, and thus can be perfectly discriminated by LOCC.

Lemma 2. $\mathcal{P}$ together with an imprimitive $V$ can generate any unitary operation acting on $d_A \otimes d_B$. More specifically, there exists an integer $N$ such that for any $U \in \mathcal{U}(d_A \otimes d_B)$ there is $U = X_N \cdots X_2 X_1$ for $X_i \in \mathcal{P} \cup \{V\}$ with $i = 1, \ldots, N$.

The above result improves the one in [6], since the inverse of $V$ is not used in the above, whereas it was required in [6]. The result will be a base of this paper. The following technical lemma is also required in order to proving the main result of this paper. A detailed proof of the lemma for the case of $d_A = d_B$ was given in [6], and one can easily extend the proof to the general case of $d_A \neq d_B$.

Lemma 3. For unitary operation $U \in \mathcal{U}(d_A \otimes d_B)$, $U^\dagger = W U W^\dagger$ holds for all $W \in \mathcal{S} \equiv \{\sigma_x \otimes I, \sigma_y \otimes I, \sigma_z \otimes I, \sigma_y \otimes \sigma_y, I \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z\}$ if, and only if $U$ has the form $U = e^{ix u_1 \otimes u_2}$ for some real number $x$, where $u_1 = \sigma_x \otimes 0_{(d_A - 2)}$, $u_2 = \sigma_x \otimes 0_{(d_B - 2)}$, with $\sigma_x, \sigma_y, \sigma_z$ being Pauli operators.

III. SEQUENTIAL SCHEME FOR LOCAL DISCRIMINATION WITHOUT INVERSES

Now, we are in a position to give our main result. We show that any two different unitary operations acting on $d_A \otimes d_B$, allowed with a finite number of queries, can be locally discriminated by a sequential scheme without using the inverses. The result is depicted in Fig 2 and formally presented in Theorem 1 below. In the rest of this paper, we will use the notation $f(X)$ to denote a sequential circuit of the form depicted in Fig 2.

Theorem 1. For any two different operations $U, V$ acting on $d_A \otimes d_B$, there exist two finite sequences of unitary operations $\{u_i\}_{i=1}^N \subseteq \mathcal{U}_A$ and $\{v_i\}_{i=1}^N \subseteq \mathcal{U}_B$, and a product state $|\varphi\rangle|\phi\rangle \in d_A \otimes d_B$ such that

\[
\begin{align*}
U(u_N \otimes v_N) \cdots U(u_1 \otimes v_1)U(\varphi)|\phi\rangle \\
\perp V(u_N \otimes v_N) \cdots V(u_1 \otimes v_1)V(\varphi)|\phi\rangle.
\end{align*}
\]

Remark. The above result improves the one in [2] in two aspects. First, the inverses $U^\dagger$ and $V^\dagger$ are not used here, whereas they were required in [2] as well as in [3]. Actually, it is not easy to obtain $U^\dagger$ from $U$ in experiment. Second, the result here holds for the general case of $d_A \neq d_B$, but it was required that $d_A = d_B$ in [9].

Proof of Theorem 1. We prove the result by considering three cases: (i) both $U$ and $V$ are primitive, (ii) one of them is primitive, and (iii) neither of them is primitive.

Case (i): Both $U$ and $V$ are primitive. Then it suffices to consider the following three subcases.

Case (i-a): $U = U_A \otimes U_B$ and $V = V_A \otimes V_B$. Without loss of generality, assume that $U_A \neq V_A$. Then by Lemma 1 $U_A$ and $V_A$ can be discriminated sequentially, and thus $U$ and $V$ can be locally discriminated by a sequential scheme as in Fig 2.

Case (i-b): $U = U_A \otimes U_B$ and $V = (V_A \otimes V_B)P$. Note that this case occurs only if $d_A = d_B$. In this case, it is easy to locally discriminate $U, V$ by applying them once, since by letting $|\Phi_X\rangle = X|\varphi\rangle|\phi\rangle$ with $X \in \{U, V\}$, we have

\[
\langle \Phi_U|\Phi_V\rangle = \langle \varphi|U_A^\dagger V_A|\phi\rangle|\phi\rangle|\phi\rangle,
\]

which can be zero by setting $|\phi\rangle = V_A^\dagger U_A|\varphi\rangle$.

Case (i-c): $U = (U_A \otimes U_B)P$ and $V = (V_A \otimes V_B)P$. This case also occurs only if $d_A = d_B$. Without loss of generality, assume that $U_A \neq V_A$. Let $f(X) = X(u \otimes v)X$, where $X \in \{U, V\}$ and $u, v$ are two given single-particle unitary operations. Then it is straightforward to get that

\[
\begin{align*}
f(U) &= U_A v U_B \otimes U_B v U_A, \\
f(V) &= V_A v V_B \otimes V_B v V_A.
\end{align*}
\]

It can be found that there always exists $v$ such that $U_A v U_B \neq V_A v V_B$. By contradiction, suppose $U_A v U_B = V_A v V_B$ holds for all $v$. Then we have $v U_B V_B^\dagger = U_A^\dagger V_A v$ for all $v$, which holds only if $U_A = V_A$ and $U_B = V_B$. This contradicts the premise that $U, V$ are different.

Therefore, $f(U)$ and $f(V)$ can be locally discriminated by a sequential scheme as in subcase (i-a), and so for $U$ and $V$.

Case (ii): One of $U$ and $V$ is primitive. Without loss of generality, assume that $V$ is primitive. We have the following discussion.

Case (ii-a): $U$ is imprimitive and $V = V_A \otimes V_B$. By Lemma 2 we can construct a sequential circuit $f(X)$ consisting of local unitary operations and $X \in \{U, V\}$, such that

\[
f(U) = P_A \otimes U_B + P_A' \otimes U_B',
\]

where $U_B \neq U_B'$, and $P_A$ and $P_A'$ are two projectors satisfying $P_A + P_A' = I_A$. In other words, $f(U)$ is a controlled unitary transformation. At the same time, it is clear that $f(V) \in \mathcal{P}$. Thus we let $f(V) = V_B \otimes V_B'$. Note that it holds that either $V_B \neq U_B$ or $V_B \neq U_B'$.

Without loss of generality, assume that $V_B \neq U_B'$. Then as shown below, local discrimination between $f(U)$ and $f(V)$ can be reduced to discrimination between $V_B$ and $U_B$.
Let $|\alpha\rangle_A \in \mathcal{H}_A$ satisfy $P_A^\dagger |\alpha\rangle_A = |\alpha\rangle_A$. Then for any $|\phi\rangle_B \in \mathcal{H}_B$, and $w_1, \ldots, w_N$ acting on $\mathcal{H}_B$, we have

$$f(U)(I \otimes w_N)f(U)\ldots(I \otimes w_1)f(U)|\alpha\rangle_A|\varphi\rangle_B = |\alpha\rangle_A \otimes (U'w_NU'B \ldots w_1U')|\phi\rangle_B$$

and

$$f(V)(I \otimes w_N)f(V)\ldots(I \otimes w_1)f(V)|\alpha\rangle_A|\varphi\rangle_B = V_A^{\dagger}N|\alpha\rangle_A \otimes (V'_Bw_NV_B' \ldots w_1V'_B)|\phi\rangle_B$$

According to Lemma 1 there exist a state $|\phi\rangle_B$ and unitary operations $w_1, \ldots, w_N$ such that the two output states of $B$ in the above are orthogonal. Therefore, $f(U)$ and $f(V)$ can be locally discriminated by a sequential scheme, and so for $U$ and $V$.

Case (ii-b): $U$ is imprimitive and $V = (V_A \otimes V_B)P$. As we did in subcase (ii-a), construct a sequential circuit $f(X)$ such that $f(U)$ is in the form of Eq. 11. In this case, $f(V)$ is still primitive. Thus, if $f(V) = V_A' \otimes V_B'$, then $f(U)$ and $f(V)$ can be discriminated as in subcase (ii-a). If $f(V) = (V_A' \otimes V_B')P$, then it is easy to discriminate $f(U)$ and $f(V)$, since by letting $|\Phi_X\rangle = f(X)|\alpha\rangle_A|\phi\rangle_B$, we find that $\langle \Phi_U|\Phi_V\rangle = \langle \alpha|V]\phi\langle\phi|U'V'_B|\alpha\rangle$ which can be zero by choosing $|\phi\rangle$.

Case (iii): Neither $U$ nor $V$ is primitive, i.e., they are both imprimitive. Firstly, by Lemma 2 we can construct a sequential circuit $f(X)$ consisting of local unitary operations and $X \in \{U, V\}$, such that $f(U) = e^{i\pi u_1 \otimes u_2}$ with $u_1 = \sigma_x \oplus 0_{d_A - 2}$ and $u_2 = \sigma_x \oplus 0_{d_B - 2}$. Thus, $f(U)$ is imprimitive. Now, if $f(V)$ is primitive, then according to case (ii), we know that $f(U)$ and $f(V)$ can be locally discriminated by a sequential scheme. Otherwise, based on Lemma 3 we have the following discussion.

Case (iii-a): $f(V) \neq e^{i\pi u_1 \otimes u_2}$. Let $F(X) = Wf(X)W^{\dagger}f(X)$ for $W \in S$. Then in terms of Lemma 3 we have $F(U) = I$ and $F(V) \neq I$ for some $W$. Therefore, by the previous cases, $F(U)$ and $F(V)$ can be locally discriminated by a sequential scheme, and so for $U$ and $V$.

Case (iii-b): $f(V) = e^{i\pi u_1 \otimes u_2}$. When $x = 1$, $f(U)$ and $f(V)$ are the same and imprimitive. Thus by Lemma 2 we can construct a quantum circuit $h(\cdot)$ such that $h(f(U)) = U^{\dagger}$, and then we have $Uh(f(U)) = I$ and $Vh(f(V)) = VU^{\dagger}$. Therefore, they can be locally discriminated from the previous cases. When $x \neq 1$, discriminating $f(U)$ and $f(V)$ can be reduced to discriminating $e^{ix_1}$ and $e^{ix_2}$ as follows. By inputting $|\varphi\rangle_A|\alpha\rangle_B$ where $|\alpha\rangle_B$ is an eigenvector of $u_2$ associated with the eigenvalue 1, it is easy to check that $e^{ix_2 \otimes iu_2}|\alpha\rangle_B = (e^{ix_2} \otimes I)|\varphi\rangle_A|\alpha\rangle_B$. Furthermore, we have

$$|\Phi_U\rangle \equiv f(U)(w_N \otimes I)f(U)\ldots(w_1 \otimes I)f(U)|\alpha\rangle_A|\varphi\rangle_B = (e^{ix_2}w_Ne^{ix_1} \ldots w_1e^{ix_1})|\varphi\rangle_A \otimes |\alpha\rangle_B,$$

$$|\Phi_V\rangle \equiv f(V)(w_N \otimes I)f(V)\ldots(w_1 \otimes I)f(V)|\alpha\rangle_A|\varphi\rangle_B = (e^{ix_1}w_Ne^{ix_1} \ldots w_1e^{ix_1})|\varphi\rangle_A \otimes |\alpha\rangle_B.$$

Therefore, in terms of Lemma 1 by choosing a suitable input state $|\varphi\rangle_A$ and auxiliary operations $w_i$, we can get $|\Phi_U\rangle \perp |\Phi_V\rangle$. Thus, $f(U)$ and $f(V)$ can be locally discriminated, and so for $U$ and $V$.

Therefore, we have completed the proof of Theorem 1.

**IV. CONCLUSION**

A sequential scheme usually represents the most economic strategy for (locally) discriminating two unitary operations. In this paper we have proved that any two bipartite unitary operations $U$ and $V$ acting on $d_A \otimes d_B$ can be locally discriminated by a sequential scheme without using the inverses of the unitary operations. Compared with the existing related work, the improvement of this paper is twofold. First, the result here applies to the general case of $d_A \otimes d_B$, whereas Ref. [9] only considered the special case of $d \otimes d$. Second, the sequential scheme here does not use the inverses of $U$ and $V$, while the inverses were required to construct a sequential scheme in [9]. Note that when $U$ and $V$ are not identified, how to obtain their inverses $U^{\dagger}$ and $V^{\dagger}$ is not easy and even not realizable in experiment. Therefore, this paper enhances the applicability and feasibility of the sequential scheme for locally discriminating unitary operations.

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[1] A. Acín, Phys. Rev. Lett. 87, 177901 (2001).
[2] G. M. D’Ariano, P. Lo Presti, and M. G. A. Paris, Phys. Rev. Lett. 87, 270404 (2001).
[3] R. Y. Duan, Y. Feng, and M. S. Ying, Phys. Rev. Lett. 98, 100503 (2007).
[4] R. Y. Duan, Y. Feng, and M. S. Ying, Phys. Rev. Lett. 100, 020503 (2008).
[5] X.F. Zhou, Y.S. Zhang, and G.C. Guo, Phys. Rev. Lett. 99, 170401 (2007).
[6] J.-L. Brylinski and R. Brylinski, Mathematics of Quan-
quantum Computation, edited by R. Brylinski and G. Chen (CRC Press, Boca Raton, 2002). Also see arXiv: quant-ph/0108062.

[7] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, 1991).

[8] J. Walgate, A. J. Short, L. Hardy and V. Vedral, Phys. Rev. Lett. 85, 4972 (2000).

[9] L. Z. Li, D. W. Qiu, Phys. Rev. A 77, 032337 (2008).

[10] A. W. Harrow, Quantum Information and Computation, 9(9), 773-777 (2009). Also see arXiv: 0806.0631.