Research Article

Bifurcation Characteristics of Fundamental and Subharmonic Impact Motions of a Mechanical Vibration System with Motion Limiting Constraints on a Two-Parameter Plane

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A two-degree-of-freedom periodically forced system with multiple gaps and rigid constraints is studied. Multiple types of impact vibrations occur at each rigid constraint and interact with each other, which results in the emergence of some complex transitions in the system. Through the cosimulation of the key parameters gap value \( \delta \) between the two masses and the excitation force frequency \( \omega \), the types, existence areas, and bifurcation regularities of the periodic and subharmonic motions can be obtained on the \( (\omega, \delta) \)-parameter plane. In the corresponding three-dimensional surface diagram of the maximum impact velocity, the distribution law of the maximum impact velocity at each constraint can be obtained. The transition laws of fundamental impact motions in the low-frequency parameter domain are studied, and two types of transition regions in the transitions of adjacent fundamental impact motions are found: tongue-like regions and hysteresis regions. Moreover, these two types of transition regions show some atypical partitioning and deformation due to the combined effects of impact vibrations at each constraint. By combining the two-parameter plane diagram and the three-dimensional surface diagram, the effect of changing the gap values between each mass and the fixed constraint and the damping coefficient \( \zeta \) on the dynamic characteristics of the system is studied. Combining the existence areas of periodic motions and the distribution of maximum impact velocity can provide guidance for the reasonable selection of system parameters.

1. Introduction

In the mechanical system, there are inevitably gaps and constraints between components and between the components and the fixed boundaries due to the needs of installation and the limitations of machining accuracy. The existence of gaps and constraints makes the mechanical system witness impact vibration when working, which will increase the noise level of the mechanical system and accelerate the wear of components. Due to the existence of constraints and gaps, the system becomes a typical non-smooth system showing rich dynamic characteristics.

People want to reveal the inherent laws of impact vibration systems and use this to guide the effective use of impact vibration characteristics and regularize their potential hazards. A large number of researches in this field can be found in available monographs. Shaw and Holmes [1] considered a one-degree-of-freedom nonlinear oscillator and found period-doubling bifurcation, through which the system can lead to chaos. Aidanpää and Gupta [2] found the characteristics of period doubling, jumping, chaos, and so forth in the two-degree-of-freedom forced vibration system. Peterka and Vacík [3] discussed the law of the transition from periodic motion to chaos and characterized the periodic-impact motion by \( p/n \). As a strong nonlinear factor, dry friction often exists in shock vibration systems, and the coupling of such strong nonlinear factors as gap, constraint, and dry friction was considered in [4–7]. Multi-degree-of-freedom piecewise smooth mechanical system is a multiparameter high-dimensional nonlinear system; the
codimension-two bifurcations have been studied in [8–13]. Some experiments were also used to verify the results of numerical simulation of impact vibration systems in [14–16].

Due to the existence of constraints and gaps, the impact vibration system has bifurcation types that are not available in smooth systems, such as grazing bifurcation. Nordmark [17] studied the singularities caused by grazing impact and found that the motion after the grazing bifurcation may be non-periodic. Whiston [18] used the singularity theory to study the shedding mechanism of the globally stable manifold caused by the discontinuity of the impact map. Foale and Bishop [19, 20] introduced the local discontinuous map under low-frequency excitation force conditions. Nordmark and Piirainen [31] introduced the local discontinuous map based on the impact chattering accumulation and used the map to analyze the stability of the limit cycle with chattering.

In the previous studies of nonlinear dynamics, the periodic motion mode and bifurcation characteristics of the system were mainly studied through single-parameter bifurcation diagrams. Then, most dynamic systems belong to multiparameter systems, and many parameters affect the dynamic characteristics of the system together. Therefore, how to get the existence areas of the periodic and subharmonic motions and the bifurcation characteristics through the double-parameter cosimulation has become a hot spot for scholars. In [12, 22, 32, 33], the double-parameter bifurcations of one-degree-of-freedom impact vibration system were studied. Luo [34, 35] studied the mode types and distribution areas of the periodic motions of the two-degree-of-freedom impact vibration system on the double-parameter plane.

The aim of this paper is to get the types, existence areas, and bifurcation regularities of the periodic and subharmonic motions of the two-degree-of-freedom impact vibration system with multiple constraints and gaps through double-parameter cosimulation. Discover the reasonable matching relationship between the vibroimpact dynamics of the system and the parameters. The paper is organized as follows. In Section 2, the physical model of a two-degree-of-freedom periodically forced system with multiple gaps and multiple rigid constraints is introduced. In Section 3, through the double-parameter numerical simulation, the types, existence areas, and bifurcation regularities of periodic motions are obtained on the $(\omega, \delta)$-parameter plane, and the distribution of the maximum impact velocity at each constraint is studied. In Section 4, the effects of changing the gaps $\delta_i = \delta_i$ between the two masses and the foundation supports and the damping coefficient $\zeta$ on the system dynamics are studied. Conclusions are obtained in Section 5.

### 2. Mechanical Model

A two-degree-of-freedom periodically forced system with multiple gaps and rigid constraints is depicted in Figure 1, where the masses $M_i$ ($i = 1, 2$) displacements are represented by $X_i$ ($i = 1, 2$). The masses $M_j$ ($i = 1, 2$) are connected to the supporting foundation by linear springs $K_i$ ($i = 1, 2$) and linear dampers $C_i$ ($i = 1, 2$). The masses $M_j$ ($i = 1, 2$) are excited by harmonic force of amplitude $P_i$ ($i = 1, 2$), frequency $\Omega$, and phase angle $\tau$. The clearance between the mass $M_1$ and the impact surface $A_1$ is $B_1$. The clearance between the mass $M_2$ and the impact surface $A_2$ is $B_2$. The clearance between $M_1$ and $M_2$ is $B_3$. The stop is assumed to be rigid and the coefficient of recovery after a rigid collision is $R$. The speed before and after the impact of mass $M_i$ ($i = 1, 2$) is represented by $\dot{X}_{i-}$ and $\dot{X}_{i+}$ ($i = 1, 2$), respectively.

In the study, we introduced the following dimensionless parameters:

\[
\begin{align*}
\tau &= T \frac{K_1}{M_1}, \\
X_i &= \frac{X_i K_i}{P_1 + P_2}, \\
\mu_m &= \frac{M_2}{M_1 + M_2}, \\
\mu_k &= \frac{K_2}{K_1 + K_2}, \\
\mu_c &= \frac{C_2}{C_1 + C_2}, \\
\zeta &= \frac{C_1}{2 \sqrt{K_1 M_1}}, \\
\delta_j &= \frac{B_j K_1}{P_1 + P_2}, \\
f &= \frac{P_2}{P_1 + P_2}, \\
i &= 1, 2, j = 1, 2, 3.
\end{align*}
\]

It can be found that some parameters in the system have a certain value range through dimensionless parameters, such as $\mu_m \in (0, 1)$, $\mu_k \in (0, 1)$, $\mu_c \in (0, 1)$, $f \in [0, 1]$. When the system has no impact on each rigid constraints, the motion of the impact system could be described by the nondimensional equations:

\[
\begin{align*}
\dot{x}_1 + 2\zeta \dot{x}_1 + x_1 &= (1 - f) \sin(\omega t + \tau) - \frac{\mu_m}{1 - \mu_m} \dot{x}_2 + 2(\frac{\mu_k}{1 - \mu_k} \dot{x}_2 + \frac{\mu_c}{1 - \mu_c} x_2) \\
&= f \sin(\omega t + \tau), \\
x_1 &> -\delta_1, \\
x_2 &< \delta_1, \\
x_1 - x_2 &< \delta_2.
\end{align*}
\]

When the mass $M_j$ ($i = 1, 2$) has an impact with the fixed constraints, the velocity of mass $M_i$ ($i = 1, 2$) is changed immediately.
After collision can be determined by momentum conservation.

\[
\begin{align*}
\dot{x}_{1+} &= -R\dot{x}_{1-}, \\
\dot{x}_{1} &= -\delta_1, \\
\dot{x}_{2+} &= -R\dot{x}_{2-}, \\
\dot{x}_{2} &= \delta_2.
\end{align*}
\]  

(3)

where \(\dot{x}_{1-}\) and \(\dot{x}_{1+}\) (\(i = 1, 2\)) represent the velocities before and after the impact, respectively.

For the periodically forced impact system, the mass \(M_i\) (\(i = 1, 2\)) may stay on the constraint surfaces with a speed of zero for a period of time under the resultant force and this phenomenon is called sticking. The impact vibration system in Figure 1 contains multiple gaps and rigid constraints, so the system may exhibit sticking phenomenon on multiple rigid constraint surfaces in a certain motion time section. From the conditions of the occurrence of the sticking phenomenon, the sticking conditions of the mass \(M_i\) (\(i = 1, 2\)) on each rigid constraint surfaces can be analyzed:

(1) The conditions for mass \(M_1\) sticking to the surface of constraint \(A_1\) are \(x_1 = -\delta_1\) and \(\dot{x}_1 = 0\), and the resultant force of the harmonic excitation force, spring, and damping force on the mass \(M_1\) is satisfied:

\[
F_1(t) = (1 - f)\sin(\omega t + \tau) - x_1 - 2\zeta\dot{x}_1 \leq 0.
\]  

(5)

Until \(F_1(t)\) changes direction, the sticking motion is ended.

(2) The conditions for mass \(M_2\) sticking to the surface of constraint \(A_2\) are \(x_2 = -\delta_2\) and \(\dot{x}_2 = 0\), and the resultant force of the harmonic excitation force, spring, and damping force on the mass \(M_2\) is satisfied:

\[
F_2(t) = f \sin(\omega t + \tau) - 2\zeta\frac{\mu_c}{1 - \mu_c}\dot{x}_2 - \frac{\mu_k}{1 - \mu_k}x_2 \geq 0.
\]  

(6)

Until \(F_2(t)\) changes direction, the sticking motion is ended.

(3) The conditions for mass \(M_1\) and mass \(M_2\) sticking to each other at constraint \(A_{12}\) are \(x_1 = -\delta_1\) and \(\dot{x}_1 = -\delta_3\) and \(x_1 = -\delta_3\) and \(\dot{x}_1 = 0\). Whether there is sticking between the two masses depends on the interaction force between the two masses; here, the interaction force is set to \(N(t)\). At this time, the resultant forces of mass \(M_1\) and mass \(M_2\) can be expressed, respectively, as

\[
\begin{align*}
\dot{F}_1(t) &= (1 - f)\sin(\omega t + \tau) - 2\zeta\dot{x}_1 - x_1 - N(t), \\
\dot{F}_2(t) &= f \sin(\omega t + \tau) - 2\zeta\frac{\mu_c}{1 - \mu_c}\dot{x}_2 - \frac{\mu_k}{1 - \mu_k}x_2 + N(t).
\end{align*}
\]  

(7)

According to \(\dot{f}_1(t) = (1 - \mu_m/\mu_m)\dot{f}_2(t)\), \(N(t)\) can be obtained:

\[
\begin{align*}
\dot{f}_1(t) &= (1 - f)\sin(\omega t + \tau) - 2\zeta\frac{\mu_c}{1 - \mu_c}\dot{x}_2 - \frac{\mu_k}{1 - \mu_k}x_2 \\
&= \frac{f}{f_1} \sin(\omega t + \tau) - 2\zeta\frac{\mu_c}{1 - \mu_c}\dot{x}_2 - \frac{\mu_k}{1 - \mu_k}x_2
\end{align*}
\]  

(8)

The Poincaré section \(\sigma_n\) is selected by selecting the minimum displacement of the mass \(M_i\) in the period of the excitation force and it could determine the number. The Poincaré sections \(\sigma_{A1}\), \(\sigma_{A12}\), and \(\sigma_{A2}\) could determine the number of \(p\) on each of the rigid constraints, respectively.
Generally, the impact mapping of the system corresponding to the sections \( \sigma_{A1}, \sigma_{A12}, \) and \( \sigma_{A2} \) can be expressed as
\[
X^{(i+1)} = f \left( X^{(i)}, \mu \right),
\]
(10)

Where, corresponding to sections \( \sigma_{A1} \) and \( \sigma_{A12}, X^{(i)} = (x_{1i}, x_{2i}, x_{2i}^{(2)}, r_{(1)i})^T \) and \( X^{(i+1)} = (x_{1i+1}, x_{2i+1}, x_{2i+1}^{(2)}, r_{(1)i+1})^T \) corresponding to section \( \sigma_{A2}, X^{(i)} = (x_{1i}, x_{1i}^{(1)}, x_{2i}, r_{(1)i})^T \) and \( X^{(i+1)} = (x_{1i+1}, x_{1i+1}^{(1)}, x_{2i+1}, r_{(1)i+1})^T \). 

\( X \in \mathbb{R}^4 \) and \( \mu \) are the system parameters, where \( \mu \in \mathbb{R}^m, m = 8 \).

3. Dynamical Characteristics of the Two-Degree-of-Freedom Periodically Forced System with Multiple Rigid Constraints

In this section, we mainly study the mode types, existence areas, and bifurcation regularities of the periodic motions on the two-parameter plane composed of the exciting force frequency \( \omega \) and the gap value \( \delta \), as well as the corresponding distribution law of the maximum impact velocity at each constraint. In model 1, the gap can be divided into two types, one is the gap between the mass \( M_i \) \((i = 1, 2) \) and the fixed constraints and the other is the gap between \( M_1 \) and \( M_2 \).

Consider here that the gaps \( \delta_1 \) and \( \delta_2 \) are taken to have fixed values, and \( \delta_1 \) is taken as a variable parameter and recorded as \( \delta \). Here \( \mu_{n_1} = 0.5, \mu_{n_2} = 0.5, \mu_c = 0.5, \xi = 0.1, R = 0.8, \) and \( \delta_1 = \delta_2 = 1.0 \) are selected as the basis parameters. The exciting force frequency \( \omega \) and the gap \( \delta \) between the two masses are set as bifurcation parameters. Through double-parameter numerical simulation, the mode types and occurrence areas of periodic motions can be obtained on the \((\omega, \delta)-parameter plane as depicted in Figure 2. The maximum impact velocity of the system on each constraint surface can be obtained in the \((\omega, \delta, x_{\text{max}})-three-dimensional surface program as depicted in Figure 3. On the \((\omega, \delta)-parameter plane, different types of periodic motions are represented by different colors and marked with \( p\text{/}n \). CIS/1 represents the sticking motions. In addition, the unmarked dark gray areas are mainly quasi-periodic motions, chaotic motions, and unidentified periodic-impact motions (mainly including \( n \) or \( p \) is too large and existence areas are extremely small). Moreover, GB, GC, GS, SN, S, and PB represent the grazing, real grazing, bare grazing, saddle-node, sliding, and period doubling bifurcation, respectively.

Figure 2 shows the existence types and distribution areas of the fundamental impact motions and subharmonic motions on the \((\omega, \delta)-parameter plane associated with each constraint. The corresponding three-dimensional surface diagrams of the maximum impact velocity can be observed in Figure 3. As shown in Figure 2(a), on the \((\omega, \delta)-parameter plane associated with constraint \( A_1 \) in the frequency domain of \( \omega \leq 1.4 \) the system mainly exhibits 1/1 motion, and there is a small amount of 2/1 motion in the lower frequency domain. In the frequency domain of \( \omega > 1.4 \) the system mainly exhibits impactless motion 0/1. Subharmonic motions mainly exist in the middle frequency domain, mainly 2/2, 1/3, 3/3, 4/4, and 4/4 motions. The corresponding three-dimensional surface diagrams of the maximum impact velocity are shown in Figure 3(a); the peaks of the maximum impact velocity mainly exist in the middle frequency domain where the motion form is mainly 1/1. In the high-frequency domain, the peaks of the maximum impact velocity also exist in the existence domain which mainly contains subharmonic motions 1/3 and 2/4. In the low-frequency domain, the maximum impact velocity is relatively small. As shown in Figure 2(c), on the \((\omega, \delta)-parameter plane associated with the constraint \( A_2 \), periodic-impact motions mainly exist in the middle frequency domain. The fundamental motion is mainly 1/1 motion, and the rest are mainly subharmonic motions, including 1/2, 1/2, 1/3, 2/3, and 2/4 motions. The corresponding three-dimensional surface diagrams of the maximum impact velocity are shown in Figure 3(c); the peak values of the maximum impact velocity exist in the regions of 1/1, 1/2, and 1/3 motions. Figure 2(b) shows the \((\omega, \delta)-parameter plane associated with constraint \( A_{12} \), which presents richer dynamic characteristics on the entire parameter plane. In the domain of large \( \delta \) and high \( \omega \), it is mainly impactless motion 0/1. The dark gray areas and subharmonic motions (mainly 3/3, 4/3, and 4/4 motions) mainly exist in the domain of middle \( \omega \). In the parameter domain located at the lower left corner of Figure 2(b), the types of periodic motions are more complicated. It mainly presented as the fundamental impact motions \( p/1 (p \geq 1) \), the incomplete chatting motions \( p/1 \), and the sticking motions \( \overline{p/1} \), as observed in Figure 4. The fundamental impact motions \( p/1 (p \geq 1) \) present band-like distribution in the domain of low \( \omega \) and as the number of impacts \( p \) increases, the width of the band-like region gradually narrows. As the exciting force frequency \( \omega \) decreases, \( p/1 \) motion will have a grazing bifurcation, and the number of impacts \( p \) increases by one, and it transitions to \( (p+1)/1 \) motion. When \( p \) keeps increasing, the system will enter incomplete chatting motions \( \overline{p/1} \). At this time, the number of impacts \( p \) is very large, but the minimum impact speed is still not zero, and the masses \( M_1 \) and \( M_2 \) do not stick together and move synchronously. When the number of impacts \( p \) increases sufficiently and the minimum impact velocity decreases to zero, the system will enter the sticking motions \( \overline{p/1} \). At this time, the interaction force between the two masses \( N(t) > 0 \), and the two masses are sticking together for synchronous movement. The three-dimensional surface diagrams of the maximum impact velocity corresponding to Figure 2(b) are shown in Figure 3(b); the fundamental impact motions regions near \( \omega = 0.7 \) and \( \omega = 1.2 \) have larger maximum impact velocity; in the middle frequency domain, the peaks of maximum impact velocity appear in the subharmonic motions 3/3 and 4/4 regions.

In order to further study the bifurcation characteristics of the system, a single-parameter bifurcation diagram that crosses the \((\omega, \delta)-parameter plane along a horizontal scan can be taken. The global bifurcation diagrams of the system when the gap value \( \delta = 0.1 \) is taken can be seen in Figure 5, and the relation between single-parameter bifurcation diagrams and periodic motions on the \((\omega, \delta)-parameter plane can be observed. In the middle frequency domain, impact motions occur in all three constraints. For example, at the point \((\omega, \delta) = (0.85, 0.1)\), the number of cycles \( n \) is two, the number of impacts \( p \) at constraint \( A_1 \) is two, the number of
impacts at constraint $A_1$ is six, and the number of impacts $p$ at constraint $A_2$ is one, as shown in Figure 6. In the low-frequency range, impact motions mainly occur at constraints $A_1$ and $A_{12}$. The impact motions at constraint $A_1$ are mainly fundamental 1/1 and 2/1 motions, while the impact motions at constraint $A_{12}$ are more complicated, which include $p/1$ ($p \geq 1$), $(p/1)$, and $(\bar{p}/1)$ motions. Figure 7 shows a partial enlarged bifurcation diagram of the low-frequency part of Figure 5(b), in which the red part is the sticking motions. It can be seen that, with the decrease of $\omega$, the 2/1 motion passes through the bare-grazing bifurcation point $G_{2/1}^b$ ($\omega = 0.668$) and enters the region containing the subharmonic motion 10/4 and then transfers to the 3/1 motion. The phase diagram at $G_{2/1}^b$ is shown in Figure 8(a); it can be seen that the impact velocity at grazing bifurcation point is zero. The phase diagram at $G_{3/1}^b$ ($\omega = 0.543$) is shown in Figure 8(b), and 3/1 motion transfers to 4/1 motion. The phase diagram at $G_{4/1}^b$ ($\omega = 0.505$) is shown in Figure 8(c), and 4/1 motion transfers to 5/1 motion. So far, it can be found that, with the decrease of $\omega$ in the low-frequency region, the $p/1$ motion passes through real grazing bifurcation, and the number of impacts $p$ increases once and directly enters $(p + 1)/1$ motion. When passing through bare-grazing bifurcation, the system will enter an unstable region that may include subharmonic motions, quasi-periodic motions, and chaotic motions, and then it will enter $(p + 1)/1$ motion. When $\omega$ is further reduced, the number of impacts $p$ increases one by one, and the system enters $(\bar{p}/1)$. The time series of the system when $\omega = 0.3619$ is shown in Figure 9. At this time, $\omega$ is very close to the sliding bifurcation point, and the system is in the incomplete chatting motion. It can be seen that the relative displacement $x_1 - x_2$ is
infinitely close to $\delta = 0.1$ with the increase of $\omega t$ in Figures 9(a) and 9(b). The relative velocity $\dot{x}_1 - \dot{x}_2$ is infinitely close to zero with the increase of $\omega t$ in Figures 9(c) and 9(d). When $\omega$ continues to decrease, the system enters the sticking motions through the sliding bifurcation. Time series of sticking motions with $\omega = 0.35$ are shown in Figure 10. It can be seen that, after the system has undergone a chattering-impact sequence in which the relative displacement $x_1 - x_2$.

Figure 5: Bifurcation diagrams of the system when $\delta = 0.1$. (a) Impact bifurcation diagram at constraint $A_1$; (b) impact bifurcation diagram at constraint $A_1'$; (c) impact bifurcation diagram at constraint $A_2$; (d) periodic bifurcation diagrams.

Figure 6: Phase plane portraits of periodic motion when $\delta = 0.1$ and $\omega = 0.85$. (a) Phase plane portraits of $M_1$, 2/2 motion; (b) phase plane portraits of relative displacement and relative velocity of $M_1$ and $M_2$, 6/2 motion; (c) phase plane portraits of $M_2$, 1/2 motion.
and the relative velocity $\mathbf{\dot{x}}_1 - \mathbf{\dot{x}}_2$ are gradually reduced, $\mathbf{\dot{x}}_1 - \mathbf{\dot{x}}_2$ is reduced to zero, and $x_1 - x_2$ is equal to the gap value $\delta = 0.1$. The interaction force between the two masses $N(t) > 0$, and the two masses are sticking together and moving synchronously. In the low-frequency domain, as $\omega$ decreases, the transition law of the system on the $(\omega, \delta)$-parameter plane associated with constraint $A_{12}$ can be summarized as

**Figure 7:** Bifurcation diagrams of the low-frequency region of the system at constraint $A_{12}$. (a) Impact bifurcation diagram; (b) periodic bifurcation diagram.

**Figure 8:** Phase diagrams at grazing bifurcation point. (a) $G_{b}^{(2/1)}$, $\omega = 0.668$; (b) $G_{r}^{(3/1)}$, $\omega = 0.543$; (c) $G_{r}^{(3/1)}$, $\omega = 0.505$.

**Figure 9:** Time series near the sliding bifurcation point when $\omega = 0.3619$. (a) Time series of $x_1 - x_2$; (b) local enlargement of Figure 9(a); (c) time series of $\mathbf{\dot{x}}_1 - \mathbf{\dot{x}}_2$; (d) local enlargement of Figure 9(c).
From the \((\omega, \delta)\)-parameter plane associated with constraint \(A_{12}\), it can be found that the fundamental impact motions \(p/1\ (p \geq 1)\) are in bands distribution. It can also be found that there are transition regions in the adjacent fundamental impact motions in Figures 2(b), 4, 11(a), and 11(b). This transition regions look like the shape of the tongue, so these regions can be named as tongue-like regions, marked as \(\text{TR}_{(p+1)/1}\). When the numerical simulation was performed with decreasing \(\omega\) and increasing \(\omega\), respectively, and depicted on the same \((\omega, \delta)\)-parameter plane, another type of transition regions, the hysteresis regions, is found, marked as \(\text{HR}_p\). Due to the difference of initial values, there is coexistence of \(p/1\) and \((p+1)/1\) in \(\text{HR}_p\), as shown in the dark gray areas in Figure 11(c).

As shown in Figure 11(a), the tongue-like region \(\text{TR}_{(p+1)/1}\) between the impactless motion and the fundamental \(1/1\) motion contains complex subharmonic motions \((2/3, 2/2, 1/3, 1/4, 2/4, 4/4, 4/8, 5/8)\) motions, and \(2/4\) motion occupies the largest area. Figure 12 shows that the single-parameter bifurcation diagrams cross the tongue-like region \(\text{TR}_{(p+1)/1}\) longitudinally when \(\omega = 0.9\); the transition law with decreasing of \(\delta\) is as follows: \(0/1\) motion \(\rightarrow\) \(G_{p/1}^\text{Bif} \rightarrow\) \(1/4\) motion \(\rightarrow\) chaos \(\rightarrow\) \(1/3\) motion \(\rightarrow\) \(2/4\) motion \(\rightarrow\) chaos \(\rightarrow\) \(1/1\) motion. Figure 11(b) depicts the band-like regions of \(p/1\ (p \geq 2)\) motions and the tongue-like regions \(\text{TR}_{(p+1)/1}\) in the adjacent fundamental \(p/1\) and \((p+1)/1\) motions. Figure 11(c) is a partial enlarged view of Figure 11(b), and the hysteresis regions \(\text{HR}_p\) are drawn by numerical simulation through decreasing and increasing of \(\omega\) (represented by \(\omega\)). At this time, there are two types of transition regions: tongue-like regions and hysteresis regions. Due to the mutual influence of complex impact motions caused by multiple gaps and constraints, the hysteresis regions only appear on the right side of the tongue-like regions, and those on the left side are replaced by the subharmonic motions embedded in the tongue-like region. It can be found that the upper boundary of the tongue-like region \(\text{TR}_{(p+1)/1}\) is bare-grazing bifurcation line \(G^\text{p/1}_{(p+1)/1}\) of \(p/1\) motion, and the lower boundary is the saddle-node bifurcation line \(\text{SN}_{(1+p)/1}\) of \((p+1)/1\) motion. Figures 13(a) and 13(b) depict the single-parameter bifurcation diagrams crossing tongue-like regions of Figure 11(b) along a vertical scan for \(\omega = 0.455\); \(p/1\) motion passes through the bare-grazing bifurcation into the tongue-like region including subharmonic motions, quasi-periodic motions, and chaotic motions and then enters \((p+1)/1\) motion. Figures 13(c) and 13(d) are partial enlargement of Figures 13(a) and 13(b); the \(5/1\) motion crossing \(G^\text{p/1}_{(p+1)/1}\) at \(\delta = 0.198811\) (the phase diagram of \(G^\text{p/1}_{(p+1)/1}\) is depicted in Figure 14) enters subharmonic \(13/3\) motion and then enters \(18/4\) motion, which is the main part of the tongue-like region, and then it enters chaotic motion and finally withdraws to \(6/1\) motion. It can be further observed in Figures 11(b) and 11(c) that the main subharmonic motions in the domains of \(\text{TR}_{(3/1)\cap(4/1)}\), \(\text{TR}_{(4/1)\cap(5/1)}\), \(\text{TR}_{(5/1)\cap(6/1)}\), \(\text{TR}_{(6/1)\cap(7/1)}\), \(\text{TR}_{(7/1)\cap(8/1)}\), \(\text{TR}_{(8/1)\cap(9/1)}\), and \(\text{TR}_{(9/1)\cap(10/1)}\) are 14/4, 18/4, 22/4, 26/4, 30/4, 34/4, and 38/4 motions, respectively. The main subharmonic motions contained in the tongue-like regions on this parameter domain can be summarized as \(((n+2)/n)\ (p = 1, 2, \ldots; \ n = 2, 3, \ldots)\). It can be seen in Figure 4 that, with the decrease of \(\omega\), the boundaries of the tongue-like regions are further destroyed. At this time, the main subharmonic motions in the tongue-like regions \(\text{TR}_{(2/1)\cap(3/1)}\), \(\text{TR}_{(3/1)\cap(4/1)}\), \(\text{TR}_{(4/1)\cap(5/1)}\), \(\text{TR}_{(5/1)\cap(6/1)}\), \(\text{TR}_{(6/1)\cap(7/1)}\), \(\text{TR}_{(7/1)\cap(8/1)}\), \(\text{TR}_{(8/1)\cap(9/1)}\), and \(\text{TR}_{(9/1)\cap(10/1)}\) are 5/2, 7/2, 9/2, 11/2, and 13/2, respectively. Therefore, the main form of subharmonic motions contained in the tongue-like regions in this parameter domain is \(((n+1)/n)\ (p = 1, 2, \ldots; \ n = 2, 3, \ldots)\).

In Figure 11(c), it can be seen that hysteresis region \(\text{HR}_p\) exists between the upper bound \(\text{SN}_{(1+p)/1}\) and the lower bound \(G^\text{p/1}_{(p+1)/1}\). Figure 15 is a single-parameter bifurcation diagram synthesized by two numerical simulation modes of increasing \(\omega\) and decreasing \(\omega\) when \(\delta = 0.09\) crosses the hysteresis region \(\text{HR}_3\) laterally. With the decrease of \(\omega\), \(5/1\) motion enters 6/1 motion through \(G^\text{p/1}_{(5+1)/1}\) with the increase of \(\omega\), 6/1 motion enters 5/1 motion through \(\text{SN}_{(6/1)}\). It can be seen that the 5/1 motion and the 6/1 motion coexist in the hysteresis region \(\text{HR}_3\) (5/1 motion is represented by black, and 6/1 motion is represented by red). Take the point \(\omega = 0.4853\) in the hysteresis regions \(\text{HR}_3\) in Figure 15 to draw the phase diagrams. When the initial values

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**Figure 10**: Time series of sticking motion when \(\omega = 0.35\). (a) Time series of \(x_1 - x_2\); (b) local enlargement of Figure 10(a); (c) time series of \(x_1 - x_2\); (d) local enlargement of Figure 10(c).
Figure 11: The partial enlargement of Figure 2(b). (a) Tongue-like region: $TR_{0/1 \cap 1/1}$; (b) tongue-like region: $TR_{p/1 \cap (p+1/1)}$ ($p \geq 2$); (c) tongue-like region: $TR_{p/1 \cap (p+1/1)}$ ($p \geq 4$) and hysteresis region $HR_p$ ($p \geq 4$).

Figure 12: Bifurcation diagrams that cross the tongue-like region $TR_{0/1 \cap 1/1}$ longitudinally at $\omega = 0.9$. (a) Impact bifurcation diagram; (b) periodic bifurcation diagram.

Figure 13: Continued.
Figure 14: Phase diagrams at grazing bifurcation point $\theta_{(3/1)}$ when $\delta = 0.19811$ and $\omega = 0.668$. (a) Overall; (b) local enlargement of Figure 14(a).

Figure 15: Bifurcation diagrams that cross the hysteresis region $HR_3$ with the forward and backward of $\omega$ when $\delta = 0.09$. 
X^{(0)} = (r^{(0)}, x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0.0, -0.097587, 0.690129, -0.087587, -0.089877) are used for numerical simulation, the system is in 5/1 motion, as shown in Figures 16(a) and 16(b); when the initial values X^{(0)} = (r^{(0)}, x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0.0, -0.080625, 0.705641, -0.170624, -0.068216) are used for numerical simulation, the system is in 5/1 motion, as shown in Figure 16(c).

4. The Correlation between Dynamics and System Parameters

This section mainly studies the relationship between system dynamics and system parameters. The effect of changing the fixed gap value δ_1 = δ_2 and damping coefficient ζ on dynamics is studied, respectively. In the following analysis, only the parameters that need to be studied are changed; other parameters are the same as the basis parameters.

4.1. The Correlation between Dynamic Performance and Fixed Gap Value δ_1 = δ_2. In the analysis of basis parameters in the previous chapter, the gaps δ_1 = δ_2 = 1.0 are considered. Here, the larger gap value δ_1 = δ_2 = 1.5 and the smaller gap value δ_1 = δ_2 = 0.5 are used for analysis.

When taking a larger gap value δ_1 = δ_2 = 1.5, Figure 17 shows the existence types and distribution areas of periodic motions on the (ω, δ) parameter plane associated with each constraint. The corresponding three-dimensional surface diagrams of the maximum impact velocity can be observed in Figure 18. As for low ω, there is no impact vibration at constraint A_1 and constraint A_2; the impact vibration of the system is mainly represented as the impact between mass M_1 and mass M_2 at constraint A_12, and its mode types are fundamental impact motions, the incomplete chatting motions, and the sticking motions. Figure 19 is a partial enlarged view of Figure 17(b) in the low fundamental impact motions, the incomplete chatting motions, and the sticking motions which are mainly distributed like several vertical strips. At constraint A_2, the system mainly presented as 1/1 motion, as well as a small amount of 2/1 motion. On the (ω, δ) parameter plane associated with constraint A_12, due to the complex impact vibrations occurring at constraint A_1, the band-like region of fundamental p/1 motion is deformed and segmented in low-frequency domain. Figure 21 is a partial enlarged view of Figure 20(b) in the low ω and small δ domain, the band-like region of fundamental p/1 motion is disrupted, the existence areas and boundaries of the tongue-like regions are no longer clear, and more unrecognized gray areas are embedded in the domain of small δ. For example, the subharmonic motion 7/2 that appears during the transition of 3/1 motion and 4/1 motion appears as an interval, and its shape no longer appears as a tongue; the subharmonic motion 9/2 that appears during the transition of 4/1 motion and 5/1 motion also showed a similar distribution law. It can be seen that although the band-like region of fundamental p/1 motion is divided, the subharmonic motions between the adjacent p/1 and (p + 1)/1 motions still conform to the law of (np + 1)/n (p = 1, 2, . . . ; n = 2, 3, . . .). More unrecognized gray areas are embedded in the high-frequency domain of the system.

The three-dimensional surface diagrams of the maximum impact velocity can be observed in Figure 22. The maximum impact velocity at constraint A_1 increases compared to the basis parameters, and its distribution area is roughly the same as that of the basis parameters. The peaks of the maximum impact velocity appear in the existence areas of 1/1 and 2/1 motions in the large δ domain. At constraint A_12, the maximum impact velocity reduces compared to the basis parameters, and the peak appears in the subharmonic 4/2 motion region distributed in the domain of small δ and low ω. At constraint A_2, the maximum impact velocity increases compared with the basis parameters, and the peak appears in the fundamental 1/1 motion region distributed in the domain of middle ω.

By comparison with the parameter changes, for large δ_1 = δ_2, the system has no impact vibration at constraint A_1 and constraint A_2 at low frequency, and impact vibration only occurs at constraint A_12. On the (ω, δ) parameter plane associated with constraint A_12, the band-like regions of fundamental p/1 motions and the tongue-like regions of fundamental p/1 motions have complete boundaries. The maximum impact velocity increases at constraint A_12 and decreases at
constraint $A_1$ and constraint $A_2$. With the decrease of $\delta_1 = \delta_2$, the impact vibration at constraint $A_1$ and constraint $A_2$ gradually intensified. At constraint $A_1$, there is a transition from $p/1$ motion to $\overline{p}/1$ motion. At constraint $A_{12}$, the mode types and distribution areas of the periodic-impact vibrations have not changed much. Due to the interaction of the impact vibrations of each constraint, the band-like regions of fundamental $p/1$ motions are cut by the gray unidentified motions; the boundaries of the tongue-like regions $TR_{p/1\cap(p+1)/1}$ are also destroyed, but the laws of the main subharmonic motions mode types remain the same. The system embeds a large amount of gray unrecognized motions in the middle frequency domain, which reduces the existence areas of periodic motions. The distribution domains of the peaks of the maximum impact velocity have little change. The maximum impact velocity increases at constraints $A_1$ and $A_2$ and decreases at constraint $A_{12}$.
Figure 19: Local enlargement of Figure 17(b).

Figure 20: Periodic motions on the \((\omega, \delta)\)-parameter plane associated with each constraint when \(\delta_1 = \delta_2 = 0.5\). (a) Constraint \(A_1\); (b) constraint \(A_{12}\); (c) constraint \(A_2\).

Figure 21: Local enlargement of Figure 20(b).
4.2. The Correlation between Dynamic Performance and Damping Coefficient $\zeta$. The damping coefficient $\zeta$ is the precoefficient of the velocity term in the differential equation of the system, which affects the impact absorption characteristics of the dynamic system. The damping coefficient is $\zeta = 0.1$ in the basis parameters; the larger value $\zeta = 0.2$ and the smaller value $\zeta = 0.05$ are selected to analyze its influence on the dynamics.

When a larger damping coefficient $\zeta = 0.2$ is taken, the mode types and existence regions of the periodic motions on the $(\omega, \delta)$-parameter plane associated with each constraint are depicted in Figure 23. The impact vibrations of the system only occur at constraint $A_1$ and constraint $A_{12}$, and there is no impact vibration at constraint $A_2$. Figure 23(a) is the $(\omega, \delta)$-parameter plane associated with constraint $A_1$; similar to the basis parameter, it is mainly $1/1$ motion in the domain of $\omega \leq 1.4$, and the domains of $2/1, 2/2, 3/3$, and $4/4$ motions are significantly reduced. There is subharmonic motion $4/2$ in the middle frequency domain. The parameter domain of $\omega > 1.4$ is impactless motion $0/1$. The $(\omega, \delta)$-parameter plane associated with constraint $A_{12}$ is shown in Figure 23(b), the existence domain of the fundamental impact motions $p/1$ ($p \geq 1$) increases, and the embedded subharmonic motions and unrecognized gray areas decrease significantly. The existence domain of the fundamental impact motions $p/1$ ($p \geq 1$) extends to the domain of large $\omega$, its bulge forms a semicircular domain, and Figure 24 is a partial enlarged view of it. In the semicircular parameter domain, the transition process with decreasing of $\omega$ can be summarized as

$$\omega^1_1: \frac{4}{1} SN Bif \frac{5}{1} SN Bif \cdots \frac{SN Bif p}{1} \frac{(p + 1)}{1} \frac{SN Bif p}{1} \frac{Rising Bif p}{1} \frac{\bar{G} Bif}{1}$$

Tongue-like regions appear between adjacent $p/1$ and $(p + 1)/1$ motions in the semicircular parameter domain. The tongue-like region $TR_{3/17/4/1}$ contains the main forms of subharmonic motions as $2/5$, $7/3$, and $8/3$ motions. The tongue-like region $TR_{4/17/3/1}$ contains the main forms of subharmonic motions as $10/3$ and $11/3$ motions. It can be found that the main forms of subharmonic motions in the tongue-like regions $TR_{p/17(p+1)/1}$ are $(np + 1)/n$ and $(np + 2)/n$ $(p = 1, 2, \ldots; n = 2, 3, \ldots)$. The three-dimensional surface diagrams of the maximum impact velocity can be observed in Figure 25. The maximum impact velocities at constraint $A_1$ and constraint $A_{12}$ are both reduced compared to the basis parameters. At constraint $A_1$, the peaks of the maximum impact velocity are similar to those in basis parameters, appearing in the domain of $1/1$ motion. At constraint $A_{12}$, the peaks of the maximum impact velocity appear in the region of $1/1$ motion in the small $\delta$ domain and the region of subharmonic motion $4/4$.

When a smaller damping coefficient $\zeta = 0.05$ is taken, periodic motions on the $(\omega, \delta)$-parameter plane associated with each constraint are depicted in Figure 26. It can be seen that, on the $(\omega, \delta)$-parameter plane associated with each constraint, the types and distribution areas of periodic motions are roughly the same as those in the basis parameters, but the existence regions of subharmonic motions and the unidentified gray areas increase and are embedded to the domains of fundamental impact motions. The distribution and transition laws of the fundamental impact motions $p/1$ ($p \geq 1$) on $(\omega, \delta)$-parameter plane associated with constraint $A_{12}$ are mainly analyzed. Figure 27 is a partial enlarged view of Figure 26(b) in the low $\omega$ and small $\delta$ domain; compared with the basis parameters, the wave-like undulations of the $p/1$ ($p \geq 1$) motions band-like regions increase. The area of tongue-like regions increases, but, due to the embedding of the gray unrecognized area, the boundaries of the tongue-like regions are destroyed. The tongue-like regions in the parameter domain of $\omega \in (0.4, 0.5)$ are similar to those in Figure 11(b) with the basis parameter, but the area of the tongue-like regions increases. The subharmonic motions in tongue-like regions $TR_{3/17/4/1}$ and $TR_{3/17/4/1}$ are $10/4$ and $14/4$, respectively. It can be seen that the main subharmonic motions in the tongue-like regions in this parameter domain are still $(np + 2)/n$ $(p = 1, 2, \ldots; n = 2, 3, \ldots)$. In the parameter domain of $\omega \in (0.3, 0.4)$, the shape of the tongue-like regions is seriously deformed, and its boundaries are no longer clear due to the embedding of more gray areas. The subharmonic motions in tongue-like regions $TR_{21/13/1}$ and $TR_{31/14/1}$ are $5/2$ and $7/2$, respectively. It can be seen that the main subharmonic motions in the tongue-like regions in this parameter domain are $(np + 1)/n$ $(p = 1, 2, \ldots; n = 2, 3, \ldots)$.

In three-dimensional surface diagrams of the maximum impact velocity shown in Figure 28, the maximum impact velocity associated with each constraint has increased compared to the basis parameters. At constraint $A_1$, peaks of
the maximum impact velocity appear in the areas where unrecognized gray area and the subharmonic motion 2/2 exist in the domain of middle $\omega$ and large $\delta$. At constraint $A_{12}$, peaks of the maximum impact velocity appear in the regions of periodic-impact motions 3/1, 2/1, and 1/1, and subharmonic motion 4/2 exists in the small $\delta$ domain. At constraint $A_2$, peaks of the maximum impact velocity appear in the areas where 1/1 motion exists in the domain of small $\delta$, as well as the subharmonic motion 2/2 and the gray unrecognized area in the domain of large $\delta$. 

Figure 23: Periodic motions on the $(\omega, \delta)$-parameter plane associated with each constraint when $\zeta = 0.2$. (a) Constraint $A_1$; (b) constraint $A_{12}$; (c) constraint $A_2$.

Figure 24: Local enlargement of Figure 23(b).

Figure 25: Three-dimensional surface diagrams of the maximum impact velocity of the system associated with each constraint when $\zeta = 0.2$. (a) Constraint $A_1$; (b) constraint $A_{12}$; (c) constraint $A_2$. 
By comparison with the basis parameters, changing the damping coefficient $\zeta$ has a greater influence on the mode types and existence areas of the periodic-impact motions. The large $\zeta$ makes the impact vibrations only occur in constraint $A_1$ and constraint $A_{12}$. At constraint $A_1$, the medium low $\omega$ domain is mainly 1/1 motion, and the medium high $\omega$ domain is mainly impactless motion 0/1. At constraint $A_{12}$, the band-shaped regions of the fundamental
impact motions extend to the domain of large $\delta$, and the unrecognized gray areas are sharply reduced. The area of the tongue-like regions decreased during the transition process, and its pattern types also changed. The maximum impact velocity at each constraint is smaller, and the number of peaks is reduced. The peaks only appear in the existence domain of 1/1 motion. When the damping coefficient $\zeta$ decreases, the impact vibration at constraint $A_1$ and constraint $A_2$ is enhanced, and the domain of the fundamental impact motions is embedded with a large number of sub-harmonic motions and gray unidentified areas. At constraint impact motions is embedded with a large number of sub-harmonic motions and gray unidentified areas. At constraint $A_1$, the wave-like undulations of the $p/1$ ($p \geq 1$) motions band-like regions increase, and the gray area divides the band-like regions, and its integrity is destroyed. The boundaries of the tongue-like regions are no longer complete, but the mode types of the subharmonic motions are consistent with the reference parameters. The maximum impact velocity at each constraint increases; the number of peaks increases, mainly distributed in the occurrence domains of 1/1 motion and subharmonic motions, as well as the unidentified gray areas.

5. Conclusions

In this paper, we consider the dynamic characteristics of a two-degree-of-freedom periodically forced system with multiple gaps and multiple rigid constraints. On the $(\omega, \delta)$-parameter plane associated with each constraint, the mode types, existence areas, and bifurcation regularities of the periodic motions are obtained. The distribution area of the maximum impact velocity and the correlation between the periodic motions distributions are acquired.

1. Under the basis parameters, on the $(\omega, \delta)$-parameter planes associated with constraints $A_1$ and $A_2$, the mode types of impact vibrations are relatively simple. On the $(\omega, \delta)$-parameter plane associated with constraint $A_{12}$, the system exhibits rich dynamic characteristics. For low $\omega$, the system presents the fundamental impact motions $p/1$ ($p \geq 1$), the incomplete chatting motions $\bar{p}/1$, and the sticking motions $\overline{p}/1$. There are two types of transition regions between adjacent fundamental impact motions $p/1$ and $(p + 1)/1$: tongue-like regions $\text{TR}_{p/1 \cap (p+1)/1}$ and hysteresis regions $\text{HR}_p$. $\text{TR}_{p/1 \cap (p+1)/1}$ is bounded by the upper boundary $G^p_{p/1}$ and the lower boundary $\text{SN}_{(1+p)/1}$. Subharmonic motions dominate in $\text{TR}_{p/1 \cap (p+1)/1}$ and their mode types are $(np + 1)/n$ or $(np + 2)/n$ ($p = 1, 2, \ldots; n = 2, 3, \ldots$). $\text{HR}_p$ exists between the upper boundary $\text{SN}_{(1+p)/1}$ and the lower boundary $G^p_{p/1}$. Due to the different initial values, $p/1$ and $(p + 1)/1$ motions coexist in $\text{HR}_p$. The peaks of the maximum impact velocity generally appear in the existence domains of 1/1 motion, partial sub-harmonic motions, and a small part of identified gray areas.

2. The influence of changing the gap value $\delta_1 = \delta_2$ and the damping coefficient $\zeta$ on the system dynamics is separately considered. With the decrease of $\delta_1 = \delta_2$, the impact vibrations at each constraint are strengthened and the mutual reaction is increased. At constraint $A_1$, the low-frequency region appears as vertical strips of $p/1$ ($p \geq 1$), $\overline{p}/1$, and $\overline{p}/1$ motions. On the $(\omega, \delta)$-parameter plane associated with constraint $A_{12}$, the band-like domains of $p/1$ ($p \geq 1$) are divided by the embedded gray areas, and the boundaries of the tongue-like regions $\text{TR}_{p/1 \cap (p+1)/1}$ are partially destroyed. The maximum impact velocities at constraints $A_1$ and $A_2$ increase, and the maximum impact velocity at constraint $A_{12}$ decreases. The increase in the number of subharmonic impact vibrations and unidentified gray areas leads to an increase in the peak number of maximum impact velocities.

The damping coefficient $\zeta$ has a great influence on the types and distribution areas of the periodic motions. With the decrease of the damping coefficient $\zeta$, the domains of fundamental impact motions decrease, and the domains of subharmonic motions and the unrecognized gray areas increase sharply. The wave-like undulations of the $p/1$ ($p \geq 1$) motions band-like regions increase, and the area of the tongue-like regions increases. At each constraint, the maximum impact velocity increases and the number of peaks increases.

Many mechanical vibration systems can be described by a two-degree-of-freedom periodically forced system with multiple gaps and rigid constraints. In incomplete chatting motions and sticking motions, since a lot of impacts occur in one exciting force cycle, it may accelerate the wear of components and increase noise level. The larger impact velocity mostly appears in the domains of 1/1 motion and partial subharmonic motions, which may aggravate the pitting fatigue in the mechanical system. Therefore, multi-parameter cosimulation can better provide theoretical guidance for the parameters design of impact vibration systems.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors’ Contributions

Shijun Wang contributed to conceptualization, formal analysis, writing original draft, writing review, and editing. Guanwei Luo supervised the study.

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