Vacuum decay by $p$-branes production

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December 6, 2018

Abstract
We present a generalization to the $N$-dimensional case for the nucleation coefficient of a spherical $p$-brane, separating two (anti-)de Sitter spacetimes. We use a semiclassical approximation based on the analytical continuation to the Euclidean sector of a suitable effective action describing a $p$-brane in General Relativity.

1 Introduction
Vacuum decay can be seen as a phase transition in spacetime and a long time ago the relevance of gravity for the process was studied [1]. The standard treatment of this process makes use of a scalar field, known as the inflaton, that drives the transition between the false and true vacuum states. This situation can be described by instanton calculations as, for instance, the Coleman-de Luccia and the Hawking-Moss instantons.

Here we present a different approach, generalizing past works of one of the authors [2, 3]. In particular we are going to use (anti-)de Sitter solutions in $N$ spacetime dimensions. In this background we put a spherically symmetric $(N-1)$-brane that splits spacetime into two domains. The system can be described

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*To appear in the Proceedings of the 6th International Symposium on Frontiers in Fundamental and Computational Physics (FFP6), September 26-29, Udine, ITALY
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by Israel junction conditions \cite{4}, which provide the equations of motion for the
timelike brane. The associated solutions are of two kinds: the first one consists
of a degenerate brane of zero radius, while the second one consists of a bounce
brane collapsing from infinity towards a finite nonzero turning point, and then
re-expanding. To model vacuum decay we consider the tunnelling from the zero
radius solution to the bounce solution. The corresponding physical picture is
the following: a very small brane\textsuperscript{1} inside a de Sitter geometry with cosmological
constant $\Lambda_+$, due to quantum effects, has a non-vanishing probability to tunnel
into a brane, containing a de Sitter spacetime with a different cosmological
constant $\Lambda_-$. This represents the formation of a bubble of a different vacuum
phase that then expands to infinity, realizing a transition of the whole spacetime
geometry. We can obtain an expression for the probability of such a process
using an effective action for this system.

2 Classical Dynamics

The stress-energy tensor for a distribution of matter localized on an hypersurface
$\Sigma$ (the $p$-brane we mentioned above) can be written in the form $S_{\mu\nu}\delta(\eta)$, where
$\delta$ is a Dirac delta, and $\eta$ can be thought as a transverse coordinate to $\Sigma$. In
$N$-dimensional General Relativity it is possible to write down the equations of
motion for this infinitesimally thin distribution of matter by splitting Einstein
equations in the tangential and transverse part (see \cite{4} for the 4-dimensional
case; it can be extended to higher dimensions). Israel junction conditions, then,
are
$$[K_{ij} - h_{ij}K] \propto S_{ij},$$
where $K_{ij}$ and $K$ are, respectively, the extrinsic curvature tensor and its trace
and $h_{ij}$ is the induced metric on $\Sigma$. Here we introduced the standard notation
$[A] = \lim_{\eta \to 0^+} \{A(\eta) - A(-\eta)\}$. Israel junction conditions describe how the
$(N-1)$-brane is embedded in the (in principle different) geometries of the two
spacetime domains that it separates. For our purposes, we are going to write
down these equations for a spherical brane with surface stress energy tensor
$S_{ij} = kh_{ij}$ separating two de Sitter spacetimes. This can be done explicitly in
terms of the radius $R$ of the brane\textsuperscript{2}. Then Israel junction conditions reduce to the
single differential equation
$$H(R, \dot{R}) = \left[\epsilon \sqrt{\dot{R}^2 + f(R)}\right] R^{(N-3)} - kR^{N-2} = 0; \quad (1)$$

$k$ is the constant tension of the brane, $\epsilon$ are signs to be determined by the
equation itself \cite{5}, and $f(R) = 1 - \Lambda R^2$ is the metric function appearing in the
static line element adapted to the spherical symmetry for the (anti-)de Sitter

\textsuperscript{1}In the mathematical treatment of the classical situation the brane has, in fact, zero radius;
from the physical point of view, with quantum gravity in mind, we can imagine this brane as
a result of zero point quantum fluctuations.

\textsuperscript{2}We are going to consider $R$ as a function $R(\tau)$ of the proper time $\tau$ of an observer sitting
on the brane and denote by an overdot the derivative with respect to $\tau$. 

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spacetime\(^3\). For suitable values of the cosmological constants equation (1) has two types of solutions: the first is \(R \equiv 0\), while the second represents a brane collapsing and re-expanding from and to infinity. For our purpose it is also important to note that equation (1) can also be obtained by an effective action, which in the \(N\)-dimensional case can be written as

\[
S_{\text{eff}} = \int \left\{ R^{N-3} \dot{R} \left[ \frac{\dot{R}}{\sqrt{\dot{R}^2 + f(R)}} \right] - \mathcal{H}(R, \dot{R}) \right\} \, d\tau, \tag{2}
\]

with the additional constraint \(\mathcal{H} = 0\) that has to be imposed on the solutions of the corresponding Euler-Lagrange equation \([5]\).

### 3 Tunnelling

The action (2) is crucial in our semiclassical quantization program, since it can be used to quantize the system via a path integral approach. Here we are going to consider the tunnelling process from the \(R \equiv 0\) solution to the bounce solution, within the saddle-point approximation. This gives the possibility to estimate the following approximated amplitude

\[
A_{\text{sp.}} \propto \exp \left( -S_{\text{eff}}^{(e)} \right), \tag{3}
\]

where \(S_{\text{eff}}^{(e)}\) is the Euclidean effective action obtained by analytically continuing the action (2) to the Euclidean sector. In order to simplify some expressions, we introduce the following adimensional quantities:

\[
x = kR \quad , \quad t = k\tau \quad , \quad \alpha = \frac{\Lambda_- + \Lambda_+}{k^2} \quad , \quad \beta = \frac{\Lambda_- - \Lambda_+}{k^2}.
\]

Moreover it is a well known result that the adimensional version of the equation of motion (1) can be cast in the following form

\[
(x')^2 + V(x) = 0,
\]

where the prime now denotes the derivative with respect the adimensional time \(t\). The potential \(V(x)\) is given by

\[
V(x) = 1 - \frac{x^2}{x_0^2}, \tag{4}
\]

where \(x_0 = 2/\sqrt{(1 + 2\alpha + \beta^2)}\) is the adimensional turning radius, provided that the argument of the square root is positive. If this condition holds, there is a bounce trajectory, otherwise we have only the \(x = 0\) solution. Starting from

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\(^3\)Since the compact notation could be misleading, we remember that we have two spacetimes with different cosmological constants \(\Lambda_{\pm}\) and, thus, two metric functions \(f_{\pm}(R)\); please, also remember the meaning of the square brackets defined above.
it is possible to evaluate the Euclidean action on the tunnelling trajectory, i.e. on the segment $[0, x_0]$. The final result can be expressed as

$$S_{\text{eff}}^{(e)} = \frac{x_0 ^{(N-3)/2}}{2(N-2)k^{(N-2)}} [(\beta - \epsilon)J(N, C_\sigma)],$$

(5)

where $J(N, p)$ is a function of the dimension of spacetime, $N$, and of

$$C_\sigma = 1 - \left(\frac{\beta - \sigma}{2}\right)^2 x_0^2,$$

with

$$\sigma_\pm = \pm 1.$$

A detailed description of the functional form of $J(N, p)$ is beyond the scope of this contribution and can be found elsewhere [7]. We just remark one important feature of it, namely that $J(N, p)$ is defined only when $p < 1$, a condition that, according to the form of $C_\sigma$, is always satisfied in our case.

4 Conclusions

In this contribution we have summarized how it is possible to compute the tunnelling amplitude for a spherical $p$-brane: the process, in view of our short discussion in the introduction, can be used to model the transition between two different vacuum phases, one being the de Sitter spacetime with cosmological constant $\Lambda_+$ and the other being the inflating inside of the $p$-brane, which is characterized by the cosmological constant $\Lambda_-$. This computation was already performed in four dimensions in the seminal work by Coleman and de Luccia [1] and that result was reproduced in [3]. Here we have carried out its generalization to arbitrary dimensions (referring to [7] for a more detailed analysis).

ACKNOWLEDGMENTS One of us (L.S.) wants to thank the Physics Department of University of Udine for financial support.

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