Measuring the quartic Higgs self-coupling at a multi-TeV muon collider

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ABSTRACT

Measuring the shape of the Higgs boson potential is of paramount importance and will be a challenging task at current as well as future colliders. While the expectations for the measurement of the trilinear Higgs self-coupling are rather promising, an accurate measurement of the quartic self-coupling interaction is presently considered extremely challenging even at a future 100 TeV proton-proton collider. In this work we explore the sensitivity that a muon collider with a center of mass energy in the multi-TeV range, and luminosities of the order of $10^{35}\text{cm}^{-2}\text{s}^{-1}$, as presently under discussion, might provide thanks to a rather large three Higgs-boson production and to a limited background. By performing a first and simple analysis, we find a clear indication that a muon collider could provide a determination of the quartic Higgs self-coupling that is significantly better than what is currently considered attainable at other future colliders.
1 Introduction

The Higgs boson discovery at the LHC in 2012 and the subsequent campaign of measurements of its properties, have provided a wonderful confirmation of our understanding of elementary particles and their interactions at the weak scale. So far, the predictions of the Standard Model (SM) for the Higgs boson couplings to the vector bosons and to third generation fermions are in spectacular agreement with observations. On the other hand, its interactions with lighter sectors, such as the first and second generation quarks and leptons, are still to be confirmed. In addition, the very existence of a scalar doublet has opened many possibilities for new physics to couple to the SM as well as many new avenues for searching for it. Leading, yet simple examples are Higgs portals to singlet fermions and/or scalars, which could provide a solution to open questions such as that of the nature of dark matter or the origin of matter-anti-matter asymmetry in the universe. Other possibilities could involve extended charged scalar sectors, which in turn could point to the existence of new symmetries, such as supersymmetry. All such possibilities are continuously pushed at higher scales by the accuracy of the measurements of the Higgs-boson couplings to the heavier SM particles, which is presently O(10%). The high luminosity phase (HL-LHC) will improve the corresponding accuracy to a few percents \cite{1} for the vector bosons and third generation, and access for the first time the couplings to the second generation fermions.

One key sector, which is currently very weakly constrained and could very easily hide or be connected to new physics, is the scalar potential \cite{2}. In the SM, the Higgs scalar potential is fixed by just two low energy parameters, the Higgs mass ($m_H \simeq 125$ GeV) and the Fermi constant $G_F$ (or equivalently the vacuum expectation value $v \simeq 246$ GeV). At the weak scale, the potential can be written in terms of the Higgs trilinear ($\lambda_3$) and quartic ($\lambda_4$) self-couplings

$$V(H) = \frac{1}{2} m_H^2 H^2 + \lambda_3 v H^3 + \frac{1}{4} \lambda_4 H^4,$$

where in the SM, $\lambda_3 = \lambda_4 = m_H^2 / 2v^2 \equiv \lambda_{SM}$. In particular, higher-point Higgs boson self interactions are forbidden by the request of renormalisability of the SM up to arbitrarily high scales.

The measurement of the parameters that describe the shape of the Higgs potential are therefore a milestone in the quest of understanding the mechanism of the electroweak symmetry breaking and of exploration of new physics. The relevance of this information on the one hand, and the inherent experimental challenges on the other hand, make this measurement one of the most relevant benchmarks that can be employed to set the physics potential of future high-energy collider projects.

The determination of the trilinear Higgs self-couplings $\lambda_3$ at the LHC and at future colliders has been vastly considered in the literature, from measurements involving Higgs boson pair production and through radiative effects in single Higgs production (see for instance \cite{1} and \cite{3}). At the end of the complete FCC programme one expects to reach a $\mathcal{O}(5\%)$ accuracy on $\lambda_3$ \cite{4}.

Although in SM extensions where new physics is at higher scales, $\lambda_4$ is related to $\lambda_3$, an unbiased determination of the Higgs quartic self-coupling will require a measurement of processes
genuinely depending on $\lambda_4$, like the triple Higgs production$^1$.

The measurement of the triple Higgs production cross section, currently being the most studied handle on the quartic Higgs self-coupling, looks very challenging even at the 100 TeV proton collisions foreseen at the FCC-hh. Quite a number of studies concerning different final states deriving from various combinations of the Higgs decay channels have been considered, see $^3$. The expected constraint on a $\lambda_4$ deviation (for a SM value of $\lambda_3$) is quite poor $^6$, $^9$, $^{10}$, the most optimistic estimate obtained from $HHH$ production with 6$b$ in the final state is $\lambda_4/\lambda_4^{SM} \in [-2,+13]$ (at 2$\sigma$, with $\lambda_3 = \lambda_3^{SM}$) with a significance for SM $HHH$ production of about 2$\sigma$ with 20 ab$^{-1}$ of integrated luminosity (and perfect $b$-tagging) $^{11}$. Indirect bounds on the quartic Higgs self-coupling can be obtained from one-loop contributions in $HH$ final states at future lepton and hadron colliders (see Refs. $^3$, $^5$, $^6$, $^7$ and references within) and these contributions allow to constrain $\lambda_4/\lambda_4^{SM}$ at FCC-hh in the range $[-2.3,+4.3]$ at 1$\sigma$ for $\lambda_3 = \lambda_3^{SM}$ $^3$.

The aim of the present study is to explore for the first time the reach of a multi-TeV muon collider for a complete reconstruction of the shape of the Higgs potential. In connection with the discussion on next generation high-energy colliders carried out for the 2020 European Strategy Update on Particle Physics $^{12}$, a very attractive option was given by the possibility of a high-luminosity multi-TeV muon collider $^{13}$. In particular, a collider with c.m. energies in the range 1.5 to 14 TeV, and luminosities up to order $10^{35}$cm$^{-2}$s$^{-1}$ is presently under consideration. Although a long and challenging period of further accelerator research and development is still needed to prove the actual feasibility of such a machine, its physics opportunities seem extremely wide and rich and therefore need to be carefully assessed.

There are a number of immediate and crucial advantages in replacing electrons with muons in lepton collisions, that would allow to amazingly extend the effective collision energy in realistic colliders. For instance, in the LHC tunnel, the same type of dipole magnets could deliver 14 TeV, whose discovery potential in direct searches of heavy (SM charged) states would be roughly similar to the one of a 100 TeV proton collider of similar luminosity $^{13}$. In addition, accelerating muons could offer a very cost-effective way to increase the lepton collision energy reach, while keeping the beam energy spread one order of magnitude smaller than for an electron collider of similar c.m. energy $^{14}$. Finally, progress on long-standing hurdles has been recently achieved. For example, preliminary studies show that potentially serious beam-induced background effects arising from the beam muon decays could be manageable as they become less severe at higher c.m. energies $^{15}$, $^{16}$.

In the following, we assume four hypothetical setups for the c.m. energy and luminosity as references: $\sqrt{s} \simeq [1.5, 3, 6, 14]$ TeV and $\mathcal{L} \simeq [1.2, 4.4, 12, 33] \cdot 10^{34}$cm$^{-2}$s$^{-1}$, respectively. These configurations are based on the parameters characterizing present muon collider designs according to the MAP scheme $^{14}$, $^{17}$, $^{18}$. In addition, we will consider two further collision energies/luminosities, i.e. $\sqrt{s} \simeq [10, 30]$ TeV and $\mathcal{L} \simeq [20, 100] \cdot 10^{34}$cm$^{-2}$s$^{-1}$, respectively, motivated by the required scaling of the luminosity needed to compensate the $1/s$ decrease in the $s$-channel cross sections that are relevant for pair production of new heavy objects $^{13}$. The setups are summarized in Table 1, where for each $\sqrt{s}$ value we also report the integrated luminosity ($L$) collected over a ten-year run (with a conventional year of $10^7$ seconds).

$^1$Double Higgs production is sensitive to $\lambda_4$ through loop effects, see $^5$, $^6$, $^7$. 

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Table 1: Reference muon collision energies $\sqrt{s}$, and instantaneous luminosities $\mathcal{L}$, with corresponding integrated luminosities $L$ for a 10 years run (one year of $\sim 10^7$s). The luminosity values assumed for $\sqrt{s} \approx (1.5, 3, 6, 14)$ TeV are as from [14, 18].

| $\sqrt{s}$ (TeV) | 1.5 | 3 | 6 | 10 | 14 | 30 |
|------------------|-----|---|---|----|----|----|
| $\mathcal{L}$ (10^{34} \text{ cm}^{-2}\text{s}^{-1}) | 1.2 | 4.4 | 12 | 20 | 33 | 100 |
| $L_{10y}$ (ab^{-1}) | 1.2 | 4.4 | 12 | 20 | 33 | 100 |

A high-luminosity multi-TeV muon collider has a physics potential both for direct searches of heavy objects as well as for precision measurements [13]. As a prime example of the latter, in this work, we focus on its capability to constrain the SM Higgs scalar potential. The reach of such a measurement builds up on the clean environment of lepton collisions, where QCD backgrounds are moderate, which also allows events to be recorded in absence of a trigger. A few percent determination of the trilinear Higgs self-coupling $\lambda_3$ via double Higgs production, eventually even better than that achievable at the FCC, seems possible at the moment [13], although only simplified studies are available [19].

In this paper, we provide a first quantitative analysis of the muon collider potential to access information on the quartic Higgs self-coupling $\lambda_4$ as obtained from direct measurements of the cross section for triple Higgs-boson production. We will consider in particular the multi-TeV energy and order 10^{35} cm^{-2}s^{-1} luminosity options considered in Table 1. Since, for $\sqrt{s} \gtrsim 1.5$ TeV, vector-boson-fusion channels (whose cross sections grow as $\log s$) get the upper-hand on the corresponding $s$-channel production mediated by the $\mu^+\mu^- \rightarrow HHHZ^*$ process, our analysis will be mainly focused on the $w$-boson-fusion (WBF) process

$$\mu^+\mu^- \rightarrow W^*W^*\nu_\mu\bar{\nu}_\mu \rightarrow HHH\nu_\mu\bar{\nu}_\mu.$$  

Depending on the particular Higgs decay channel involved, the final signature of triple Higgs events can be quite diverse [8], although a few kinematical common features (like the presence of three systems resonating at the Higgs masses) will be universal. Thanks to these features, even dijet final states, such as the $b$-jets from high-rate $H \rightarrow b\bar{b}$ decays, are expected to be efficiently reconstructed.

In this paper, we work under two main hypotheses. First, we assume that a number of potential machine and detector issues will be solved after detailed studies, possibly involving innovative technologies, and discuss the potential consequences of just having at disposal signal event statistics for triple Higgs bosons corresponding to such high c.m. energies and luminosities as envisaged in the MAP project. Second, we assume that the bulk of the different $HHH$ final states corresponding to the dominant Higgs decay channels can be reconstructed with high efficiency. Correspondingly, we estimate the muon collider sensitivity to detect a deviation in the Higgs $\lambda_3$ and $\lambda_4$ self-couplings through the full statistics of the triple Higgs production. On the other hand, the final detection efficiency could be strongly affected by the machine-induced background and the machine detector interface that could seriously impact the final detector acceptance [15, 16]. In any case, it is clear that further research and development of accelerator, detector, and analysis technologies for a multi-TeV muon collider will be needed to reach robust conclusions on the physics potential of such a machine.
The other hand, as far as the Higgs trilinear self-coupling $\lambda_3$ is concerned, we do not consider here the stronger direct constraints that presumably can be obtained through the scrutiny of the higher-rate double Higgs production.

The plan of the paper is as follows. In Section 2 we present the results of our Monte Carlo simulations for the signal cross sections and distributions in the standard model. In Section 3, we parametrise the cross sections dependence in new physics scenarios as a quartic polynomial of the deviations $\delta_3$ and $\delta_4$ of the self-couplings with respect to the SM predictions and study the sensitivity of representative distributions to them. Finally, we determine the constraining potential (considering different energy and luminosity setups) of a future muon collider. In the last section, we present our conclusions and the outlook.

2 Triple Higgs production in the standard model

In this section, we present the cross sections and a few kinematical distributions for the process

$$\mu^+\mu^- \rightarrow HHH \nu\bar{\nu},$$

in the SM and in scenarios where the Higgs self-couplings are modified, at muon collider energies in the range [1.5, 30] TeV.

In Figure 1, we show a few representative Feynman diagrams of the process. By inspection, one can quickly conclude that at the tree level, each diagram can be at most linearly dependent on the quartic self-coupling $\lambda_4$, and linearly or quadratically dependent on $\lambda_3$. In fact, the majority of diagrams are independent from Higgs self-couplings. This observation leads to the expectation that on the one hand, the cross section sensitivity to self-couplings in general and to the quartic coupling in particular, will be quite mild and on the other hand, a very precise knowledge of the $WH$ and $WWHH$ couplings will be needed in order to pin down the Higgs potential.

Triple Higgs production proceeds through two main classes of diagrams: the WBF channel

$$\mu^+\mu^- \rightarrow W^*W^*\nu\bar{\nu} \rightarrow HHH \nu\bar{\nu},$$

and the s-channel

$$\mu^+\mu^- \rightarrow HHH Z^* \rightarrow HHH \nu_e\bar{\nu}_e\nu_\mu\bar{\nu}_\mu.$$ Both sets contribute at the amplitude level to $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ yet, as we will discuss in the following, mostly in different phase space regions.

In order to compute the $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ cross sections and distributions, including the complete self-coupling dependence, we have used two Monte Carlo event generators: Whizard \cite{20, 21} (version 2.6.4) and MadGraph5_aMC@NLO \cite{22}. Even though the SM implementation in both codes does not allow the user to change $\lambda_3$ and $\lambda_4$ from the from the input cards, it is sufficiently easy to do that directly accessing the source codes. \cite{23} The Higgs and gauge

\footnote{The corresponding cross sections for $Z$ boson fusion, $\mu^+\mu^- \rightarrow Z^*Z^*\mu^+\mu^-$ amount to 15–20% of the ones for $W$ boson fusion, and therefore relevant. We leave their inclusion to future work.}

\footnote{In MadGraph5_aMC@NLO is also possible to use the SMEFT@NLO model.}
Figure 1: Representative Feynman diagrams contributing to the process $\mu^+\mu^- \rightarrow HHH\nu\overline{\nu}$ that do not involve self-couplings (top-left and bottom-right), involve the trilinear twice (top-right) and once (central), and the quartic (bottom-left) couplings. $s$-channel diagrams (bottom-right) contribute but become negligible at high energy (note that in this case $\nu = \nu_e, \nu_\mu, \nu_\tau$).
boson widths as well as the muon mass (and Yukawa) are set to zero, in order to avoid issues with gauge cancellations at very high energy.

For all the results discussed in the following, we impose a technical generation cut $M_{\nu\nu} > 150$ GeV on the neutrino pair invariant mass $M_{\nu\nu}$, to prevent the singularity arising from a vanishing $Z$-boson width in the s-channel. The latter cut effectively takes away most of the s-channel contribution. The s-channel effects tend anyhow to be strongly suppressed at multi-TeV collision energies. After removing the $Z$-resonance contributions by the $M_{\nu\nu} > 150$ GeV cut, we find that the relative off-shell contribution of the $\mu^+\mu^- \rightarrow HHHZ^* \rightarrow HHH \nu\nu$ to the total cross section is about 2.5% at $\sqrt{s} \approx 1.5$ TeV, $1.4 \cdot 10^{-3}$ at 3 TeV, and $10^{-4}$ at 6 TeV. With the present LO accuracy, our complete results for $\mu^+\mu^- \rightarrow HHH \nu\nu$ will then match the ones for the WBF process $\mu^+\mu^- \rightarrow W^*W^*\nu\nu \rightarrow HHH \nu\nu$ with excellent accuracy in the energy range considered.

In Figure 2, we plot the $\mu^+\mu^- \rightarrow HHH \nu\nu$ cross section versus $\sqrt{s}$ in the SM. On the right axis we include the expected number of triple Higgs final states produced for an integrated luminosity $L=100$ ab$^{-1}$. The left-hand plot corresponds to the cross-section results in a linear scale for two anomalous scenarios as obtained in Whizard, while on the right-hand side the MadGraph5_aMC@NLO results for the yield are plotted in a log-scale, also for two additional scenarios. We have carefully verified that the results from the two MC’s agree within uncertainties for SM as well as in presence of anomalous interactions. We define $\delta_{3,4}$ and $\kappa_{3,4}$, through the following relations

$$\lambda_3 = \lambda_{SM}(1 + \delta_3) = \kappa_3 \lambda_{SM},$$

$$\lambda_4 = \lambda_{SM}(1 + \delta_4) = \kappa_4 \lambda_{SM},$$

which imply that the SM values for the couplings are recovered for $\delta_{3,4} = 0$, or equivalently for $\kappa_{3,4} = 1$. We point out that, for the sake of both simplicity and generality, we phrase our results in terms of the anomalous couplings above. At the perturbative level of our predictions, i.e., at the tree level, one can easily link the deformations of the $\lambda$’s to the coefficients of higher dimensional operators, see for instance [7]. The simplest instance is that of adding just one operator of dimension six, $c_6(\Phi^\dagger \Phi)^3/\Lambda^2$. In this case, one finds that the shifts in the trilinear and quartic couplings are related, i.e.,

$$\delta_4 = 6 \delta_3, \quad \text{(SMEFT at dim = 6)}. \quad \text{(8)}$$

This constraint can be lifted by further adding operators of higher dimension, i.e., $c_8(\Phi^\dagger \Phi)^4/\Lambda^4$. As special case of the latter situation, one can fix the couplings of the six and eight dimensional operators, to only have the quartic coupling modified, $\delta_3 = 0$ and $\delta_4 \neq 0$. However, it is important to remind that this is not what is generically expected from the SMEFT and it implies a fine tuning, which is valid only at a given scale.

In order to get a first feeling of the cross section sensitivity to variations of the Higgs quartic coupling, in Figs 2 we also show the cross section obtained by keeping the SM value for $\lambda_3$ and

\[ Note that interference effects between the WBF and s-channel diagrams are negligible due to the non-overlapping typical kinematics of the two configurations. For the reasons above, in MadGraph5_aMC@NLO we find it easier to directly exclude the s-channel contributions by actually simulating $e^+\mu^- \rightarrow HHH \nu\nu e$. We have explicitly checked that this approximation is excellent and make the simulations faster.\]
switching off $\lambda_4$ ($\delta_3 = 0, \delta_4 = -1$ or $\kappa_3 = 1, \kappa_4 = 0$). The effect is an increase, as expected from general arguments on unitarity cancellation, of production rates of about 20%–30% in the $\sqrt{s}$ range considered here. On the right-hand plot, we show the corresponding results as obtained from MG5aMC also including two scenarios of interest: the $\delta_3 = \pm 1, \delta_4 = \pm 6$ cases, corresponding to relative shift between $\delta_3$ and $\delta_4$ consistent with an EFT approach, and a scenario $\delta_3 = 0, \delta_4 = +1$ with no change in $\lambda_3$, yet a 100% increase of $\lambda_4$. It is interesting to note that, as far as total rates are concerned, the latter case turns out to be hardly distinguishable from the scenario where $\lambda_3 = \lambda_{SM}$ and $\lambda_4 = 0$.

A second set of relevant information is provided in Table 2 where we report the $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ total cross sections and event numbers for the reference set of collision energies and integrated luminosities of Table 1. In addition to total cross sections, also the number of events close to threshold, i.e., with a requirement on the $HHH$-invariant-mass ($M_{HHH}$) to be less than 1 and 3 TeV is given. As we will discuss in the following, the sensitivity to the quartic coupling depends rather strongly on the phase space region occupied by the Higgs bosons in the final state, being the strongest close to threshold.

In Figs. 3, 4, 5 we plot the inclusive Higgs transverse momentum, the Higgs rapidity and the Higgs-pair $\Delta R$ distributions, with and without an upper cut of 1 TeV on the $HHH$ invariant mass, respectively. We note that peak value of the transverse momentum is around 100 GeV, a value that turns out to be rather independent on the collider energy. The invariant mass cut at

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\[^{6}\text{A cut } M_{\nu\bar{\nu}} \geq 150 \text{ GeV will be implicit from now on.} \]
Table 2: Cross sections and (in squared brackets) event numbers for triple Higgs production via the process $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$, at collision energies and integrated luminosities as from Table 1. A cut $M_{\bar{\nu}\nu} \gtrsim 150$ GeV is applied. The effect of imposing an upper cut on the $HHH$ invariant mass is also detailed. Cross sections and corresponding event numbers refer to the SM case.

| $\sqrt{s}$ (TeV) / L (ab$^{-1}$) | $\sigma_{SM}$ (ab) | $N_{\nu\bar{\nu}}$ |
|---------------------------------|---------------------|---------------------|
| 1.5 / 1.2                       | 3 / 4.4             | 6 / 12              | 10 / 20             | 14 / 33              | 30 / 100             |
| $\sigma^{tot}$                  | 0.03 [0]            | 0.31 [1]            | 1.65 [20]           | 4.18 [84]           | 7.02 [232]          | 18.51 [1851]         |
| $\sigma(M_{HHH} < 3$ TeV)      | 0.03 [0]            | 0.31 [1]            | 1.47 [18]           | 2.89 [58]           | 3.98 [131]          | 6.69 [669]           |
| $\sigma(M_{HHH} < 1$ TeV)      | 0.02 [0]            | 0.12 [1]            | 0.26 [3]            | 0.37 [7]            | 0.45 [15]           | 0.64 [64]            |

Figure 3: Inclusive Higgs transverse momentum distributions (normalized) for the $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ process, in the SM, at different collision energies. A technical cut of $M_{\bar{\nu}\nu} \gtrsim 150$ GeV is included. The plot on the right includes an upper cut of 1 TeV on the $HHH$ invariant mass.
Figure 4: Inclusive Higgs rapidity distributions (normalized) for the $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ process, in the SM, at different collision energies. A technical cut of $M_{\nu\nu} \gtrsim 150$ GeV is included. The plot on the right includes an upper cut of 1 TeV on the $HHH$ invariant mass.

Figure 5: Inclusive $\Delta R$ distributions (normalized) for the $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ process, in the SM, at different collision energies. A technical cut of $M_{\nu\nu} \gtrsim 150$ GeV is included. The plot on the right includes an upper cut of 1 TeV on the $HHH$ invariant mass.
1 TeV has a mild effect and only on the shapes of the distributions at higher energy collisions. On the other hand, the rapidity distributions are found to have a rather strong dependence on the collision energy and also on being at threshold. At high collision energy the rapidity range become quite large reaching more than five units in rapidity. To be detected, such Higgs bosons would need a very wide rapidity coverage of the detector. Finally, Fig.5 shows that the most probable distance between two Higgs bosons is around $\pi$, extending to larger values at high energy, due to forward-backward Higgs production. At threshold, there is a very mild dependence on the collision energy.

In order to have a more complete understanding of the dynamics of a $HHH$ event, in Fig. 6 we present the rapidity and $\Delta R$ distributions of each of the Higgs bosons ordered in $p_T$. The solid curves represent the inclusive sample with no lower or upper cut of 1 TeV on the $M_{HHH}$. By inspecting the two plots one concludes that at threshold the ordering of the Higgs in $p_T$ has mild effect as the Higgs have comparable momenta. On the other hand, in far from threshold configurations, which dominate inclusive cross sections, two Higgs bosons are typically rather central and back-to-back, while the softest one is forward.

3 Triple Higgs production with anomalous self-couplings

We can now pass to consider in detail how modifications of the trilinear and quartic couplings can modify cross sections and distributions. As already mentioned, the Feynman diagrams contributing to the process $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ can involve one quartic Higgs vertex or up to two Higgs trilinear vertices, see Fig. 1.
As a result, the most general expression for the cross section as a function of the deviations from the SM cubic and quartic Higgs couplings can be expressed in terms of a polynomial which is quartic in δ₃ and quadratic in δ₄:

\[ \sigma = c₁ + c₂δ₃ + c₃δ₄ + c₄δ³₃ + c₅δ₃δ₄ + c₆δ₄² + c₇δ₄³ + c₈δ₃²δ₄ + c₉δ₃⁴, \]  

(9)

where the coefficients cᵢ can be obtained once for all from a MC simulation and they are collected in Tab. 3 for the total cross sections with and without an upper cut on the HHH invariant mass of 1 TeV. This parametrization is useful for at least two reasons. The first is that it can be used to extract sensitivities to different scenarios without the need to rerun MC simulations for each benchmark point. The second advantage is that it is possible to directly gauge the sensitivity to new physics effects by comparing the value of the SM coefficient (c₁), with the linear terms c₂, c₃, which are dominant for δ₃,n, ∼ 1, and the quadratic (mixed or diagonal) terms (c₄,5,6), the cubic (c₇,8) and finally the quartic terms (c₉). First, the SM coefficient, as we had already seen in Fig. 2, grows faster than linearly, yet tends to flatten at high energy. As also seen before, the increase of the cross section is clearly provided by configurations which are far from threshold, and where at least one Higgs boson is soft and can be very forward. In fact, once an upper cut on the HHH invariant mass of 1 TeV is set, the increase on the cross sections is less than linear and very mild. Second, at the linear level and for total cross sections, the sensitivity to δ₄ is from 2 to 100 smaller than that of δ₃. On the other hand, if one focuses on events at threshold, there is a rather uniform difference of only a factor of two, the sign being opposite. This generically implies that positively correlated changes of the δ₄ and δ₃ will be more difficult to constrain than variations in opposite directions. For example, in the SMEFT case where δ₄ = 6 δ₃, there will be a cancellation, yet with the δ₄ contribution being dominating. More in general, the difference between the sensitivity at the inclusive level and at threshold, entails the possibility for flat directions in the parameter space to be lifted. Third, in presence of larger deviations, the higher-order terms in the polynomial could become the dominant effects. In this case, one notices that c₆, corresponding to the δ₃² term, is always smaller than c₄, the coefficient of the δ₃δ₄ term. This means that a joint departure of the trilinear and quartic term will be in general easier to detect, than that of the quartic alone.

Finally, we investigate the discriminating power of differential distributions, focusing our attention on the HHH invariant mass. In Fig. 7 we plot the ratio between the \( M_{hhh} \) distribution in a scenario where δ₃ = 0, for δ₄ = −0.5, −0.2, −0.05 (left plot) and for δ₄ = 0.5, 0.2, 0.05 (right plot) for different c.m. energies. The first observation is the size as well as the dependence of the corrections on the \( M_{hhh} \) are very different between positive and negative values of δ₄. The main reason can be traced back to the fact that even at the total integrated level the linear coefficient c₃ is negative while the quadratic coefficient c₆ is positive. For negative values of δ₄ the contributions sum and the final result is always larger than the SM, the larger effects being at threshold. For positive values of δ₄, cancellations take place between the differential version of c₃ and c₆, leading to a final non trivial pattern shown on the right plot: corrections start negative very close to threshold, and then become positive above about 600-800 GeV. In Figure 8 we show the results of an analogous study, assuming \( δ₃ = −0.5, −0.2, −0.05 \) (left plot) and \( δ₃ = 0.5, 0.2, 0.05 \) with δ₄ = 6 δ₃, i.e., in the SMEFT scenario. Also in this case the shape changes are larger at threshold and deviations with respect to SM predictions can be quite significant.
\[ \sigma = c_1 + c_2 \delta_3 + c_3 \delta_4 + c_4 \delta_3 \delta_4 + c_5 \delta_3^2 + c_6 \delta_4^2 + c_7 \delta_3^2 \delta_4 + c_8 \delta_3^4 \delta_4 + c_9 \delta_4^4 \]

| \( \sqrt{s} \) (TeV) | 3  | 6  | 10 | 14 | 30 |
|-----------------------|----|----|----|----|----|
| \( c_1 \) (ab)       |    |    |    |    |    |
|                      | 0.3127 | 1.6477 | 4.1820 | 7.0200 | 18.5124 |
| \( c_2 \)            | -0.1533 | -1.7261 | -4.4566 | -7.1000 | -15.9445 |
| \( c_3 \)            | -0.0753 | -0.1159 | -0.1166 | -0.1147 | -0.1117 |
| \( c_4 \)            | -2.0566 | -6.3052 | -11.4981 | -15.9807 | -29.2794 |
| \( c_5 \)            | 4.7950 | 14.9060 | 27.1081 | 37.4658 | 67.7539 |
| \( c_6 \)            | 0.2772 | 0.8637 | 1.5992 | 2.2455 | 4.2038 |
| \( c_7 \)            | -1.8353 | -4.3210 | -6.6091 | -8.3962 | -13.0964 |
| \( c_8 \)            | 0.5032 | 1.1861 | 1.8173 | 2.2967 | 3.5217 |
| \( c_9 \)            | 0.2943 | 0.5954 | 0.8946 | 1.1611 | 1.9349 |

\( \bar{c}_i \equiv c_i(M_{HHH} < 1 \text{ TeV}) \) (ab)

| \( \bar{c}_1 \) | 0.1165 | 0.2567 | 0.3743 | 0.4541 | 0.6404 |
| \( \bar{c}_2 \) | 0.1667 | 0.3003 | 0.4046 | 0.3545 | 0.6972 |
| \( \bar{c}_3 \) | -0.0768 | -0.1510 | -0.2105 | -0.2285 | -0.3519 |
| \( \bar{c}_4 \) | -1.3604 | -2.8996 | -4.1522 | -5.0582 | -6.9538 |
| \( \bar{c}_5 \) | 3.1017 | 6.6033 | 9.4721 | 11.4547 | 15.9505 |
| \( \bar{c}_6 \) | 0.1842 | 0.3954 | 0.5679 | 0.6931 | 0.9543 |
| \( \bar{c}_7 \) | -1.5210 | -3.0591 | -4.3186 | -4.8398 | -7.3196 |
| \( \bar{c}_8 \) | 0.4222 | 0.8550 | 1.2103 | 1.3906 | 2.0398 |
| \( \bar{c}_9 \) | 0.2691 | 0.5482 | 0.7720 | 0.9702 | 1.2482 |

Table 3: Coefficients \( c_i \), ruling the \( \mu^+ \mu^- \rightarrow HHH\nu_\mu\bar{\nu}_\mu \) cross-section dependence on the Higgs anomalous self-couplings \( \delta_3 \) and \( \delta_4 \) (as defined in the first row of the table), at different c.m. energies. The coefficients \( \bar{c}_i \), entering the residual cross sections after applying a 1-TeV upper cut on the \( HHH \) invariant mass, are also detailed.
Figure 7: Dependence of the $M_{HHH}$ distributions on a variation of the quartic Higgs coupling, for three energy setups, assuming $\delta_3 = 0$ (i.e., a SM trilinear self-coupling).

Figure 8: Dependence of the $M_{HHH}$ distributions on a variation of the trilinear Higgs coupling, for three energy setups, assuming $\delta_4 = 6 \delta_3$. 
Constraints on $\delta_4$ (with $\delta_3 = 0$)

| $\sqrt{s}$ (TeV) | Lumi (ab$^{-1}$) | $\sigma$ | $\sigma$ + threshold + $M_{HHH} > 1$ TeV |
|------------------|-----------------|-------|----------------------------------|
| 6                | 12              | $[-0.60, 0.75]$ | $[-0.90, 1.00]$ |
| 10               | 20              | $[-0.50, 0.55]$ | $[-0.70, 0.80]$ |
| 14               | 33              | $[-0.45, 0.50]$ | $[-0.60, 0.65]$ |
| 30               | 100             | $[-0.30, 0.35]$ | $[-0.45, 0.45]$ |
| 3                | 100             | $[-0.35, 0.60]$ | $[-0.50, 0.80]$ |

Table 4: Summary of the constraints on the quartic deviations $\delta_4$, assuming $\delta_3 = 0$, for various muon collider energy/luminosity options, as obtained from the total expected cross sections (1$\sigma$ and 2$\sigma$ CL). The third column shows the bounds obtained from the combination of the constraints corresponding to the setups $M_{HHH} < 1$ TeV and $M_{HHH} > 1$ TeV.

4 Sensitivity to the Higgs self-coupling deviations

We are now ready to perform the first exploration of the sensitivity of a future muon collider to deviations of the Higgs self-couplings.

For the sake of simplicity, we restrict the presentation to two possibly relevant scenarios:

A) $\delta_3 = 0$, $\delta_4 \neq 0$, i.e., deviations only in the quartic Higgs coupling;

B) $\delta_4 = 6\delta_3$, i.e., the pattern of deviations as expected from the SMEFT at dim=6.

Scenario A assumes that no deviations on the trilinear coupling have been detected (and/or exist) and explores the possibility that new physics effects appear for the first time in the quartic self-coupling. Scenario B, on the other hand, assumes the SMEFT scaling between the two couplings. This scenario would fit the case where a deviation in the trilinear coupling is observed in other observables, such as in $HH$ production. In this situation, an interesting question would be whether the deviation in $\delta_4$ would follow the linear SMEFT pattern or not.

To provide a first estimation of the sensitivity, we focus on the signal process $\mu^+\mu^- \rightarrow HHH\nu\bar{\nu}$ and disregard possible backgrounds. In so doing, we are clearly setting an optimal target for more detailed future phenomenological and experimental investigations. We define the sensitivity to the non-SM Higgs couplings as:

$$\frac{|N - N_{SM}|}{\sqrt{N_{SM}}}$$

where $N_{SM}$ is the number of events assuming $\delta_3 = \delta_4 = 0$, while $N$ is the number of events obtained for the values of $\delta_3$ and $\delta_4$ under consideration.

In Figure 9, we show the dependence on $\delta_4$ and $\delta_3$ of the total cross section in two different bins, inclusive and for $M_{HHH} < 1$ TeV, and for the A (left) and B (right) scenarios (under the SM hypothesis), respectively. In both scenarios, one finds that the highest sensitivity comes from the threshold region and that the energy dependence is actually rather weak on the sensitivity. That means that increasing the energy brings only an advantage in the statistics.
The results corresponding to independent variations of $\delta_3$ and $\delta_4$ are shown in Figs. 10, 11 and 12, where the red shaded areas correspond to the constraints obtained from threshold region, while the blue shaded areas correspond to the full sample. The plots on the right are blowups of the region close to the SM point (0, 0). First, we note that as the energy increases, the blue areas tend to the shape of a ring in the plot range, showing the relevance of the quadratic terms and the fact that bounds are obtained from upper as well as lower limit in the number of events with respect to the SM expectations. As expected from the arguments given above, the constraints improve as the energy/luminosity increase mostly for the blue areas. In addition, the linear flat direction in the case of same sign variations of $\delta_3$ and $\delta_4$ are resolved by using two different regions and the higher terms in the $c_i$ expansion. Figure 12 indicates that low energy runs, around 3 TeV, yet with a luminosity of 100 ab$^{-1}$ could provide a determination in the range $-0.3 < \delta_4 < 0.6$ (with $\delta_3 = 0$). Finally, Fig. 13 presents the sensitivity in terms of number of standard deviations. The constraints that can be obtained from the various energy/luminosity scenarios by using only information on the total cross section at 1$\sigma$ and 2$\sigma$ and by combining events in the regions $M_{HHH} < 1$ TeV and $M_{HHH} > 1$ TeV (1$\sigma$) are summarized in Tab. 4.

The underlying assumption for the setup $\delta_3 = 0$ is that no deviations are measured from the SM triple Higgs self-coupling. However, if the study of $HH$ production at the muon collider or at other machines would discover deviations from $\delta_3 = 0$, it would be interesting to search for possible deviations of $\delta_4$ from its expectation value in the SMEFT ($\tilde{\delta}_4 = \delta_4 - 6\delta_3$). As an example, plots in Fig. 14 show the sensitivity to $\tilde{\delta}_4$ under the assumptions $\delta_3 = \mp 0.2$. 

Figure 9: Dependence of the $\mu^+\mu^- \to HHH\nu\bar{\nu}$ cross section on the anomalous Higgs self-couplings in two different scenarios: A ($\delta_3 = 0$) on the left and B ($\delta_4 = 6\delta_3$) on the right. In the latter case the ratio of the cross sections is expressed in terms of $\delta_3$. 
Figure 10: Left: 1-σ exclusion plots for the anomalous Higgs self-couplings in terms of the standard deviations \(|N - N_{SM}|/\sqrt{N_{SM}}\) from the SM (green dot), where the event numbers \(N\) refer either to \(\sigma(\mu^+\mu^- \rightarrow HHH\nu\nu)\), for \(M_{\nu\nu} \gtrsim 150\text{GeV}\) (blue area), or to the same cross section with an upper cut of 1 TeV on the \(HHH\) invariant mass (red area). Right: same plots zoomed around the SM configuration.
Figure 11: Left: 1-σ exclusion plots for the anomalous Higgs self-couplings in terms of the standard deviations $|N - N_{SM}|/\sqrt{N_{SM}}$ from the SM (green dot), where the event numbers $N$ refer either to $\sigma(\mu^+ \mu^- \rightarrow HHH_1 \nu \bar{\nu})$, for $M_{\nu \bar{\nu}} \gtrsim 150$GeV (blue area), or to the same cross section with an upper cut of 1 TeV on the $HHH$ invariant mass (red area). Right: same plots zoomed around the SM configuration.
Figure 12: Left: 1-σ exclusion plot for the anomalous Higgs self-couplings in terms of the standard deviations $|N - N_{SM}|/\sqrt{N_{SM}}$ from the SM (green dot), where the event numbers $N$ refer either to $\sigma(\mu^+ \mu^- \rightarrow HHH\nu\bar{\nu})$, for $M_{\nu\bar{\nu}} \gtrsim 150\text{GeV}$ (blue area), or to the same cross section with an upper cut of 1 TeV on the $HHH$ invariant mass (red area). Right: same plot zoomed around the SM configuration. The integrated luminosity assumed is about 20 times larger than the reference luminosity in Table 1.
Figure 13: Sensitivity to the quartic Higgs self-coupling in terms of standard deviations $|N - N_{SM}|/\sqrt{N_{SM}}$ with respect to the SM configuration, where the event numbers $N$ refer to $\sigma(\mu^+\mu^- \rightarrow HHH\nu\overline{\nu})$, for $M_{\nu\nu} \gtrsim 150\text{GeV}$, for $\delta_3 = 0$ (left), and $\delta_4 = 6\delta_3$ (right). Results are obtained considering deviations from the inclusive cross sections only.
Figure 14: Sensitivity to $\tilde{\delta}_4 = \delta_4 - 6\delta_3$ in terms of standard deviations $|N(\delta_3, \tilde{\delta}_4 + 6\delta_3) - N(\delta_3, 6\delta_3)|/\sqrt{N(\delta_3, 6\delta_3)}$ with respect to the SMEFT configuration, where the event numbers $N$ refer to $\sigma(\mu^+\mu^- \rightarrow HHH\nu\bar{\nu})$, for $M_{\bar{\nu}\nu} \gtrsim 150\text{GeV}$, for $\delta_3 = -0.2$ (left), and $\delta_3 = 0.2$ (right). Results are obtained considering deviations from the inclusive cross sections only.
5 Conclusions

Dreaming about a muon collider as a future option to study fundamental interactions of elementary particles at the energy frontier is becoming a widespread reality in the high-energy community. Technical obstacles that were previously thought as insurmountable are turned into formidable challenges worth to be investigated, wild expectations into ambitious goals at an increasing pace. In keep with the progress in understanding what could be really achieved at the accelerator and detector level in a not-too-far future, theoretical and phenomenological investigations are mandatory to fully establish the physics reach of a very high energy lepton collider.

In this work, we have considered one of the most important and challenging task ahead of us in the on going exploration and verification of the standard model, i.e., the characterization of the Higgs potential at low energy. Many studies exist on the perspectives to measure the trilinear Higgs self-coupling at future hadron and (up to 3 TeV) lepton colliders and there is a general expectation that a precision at a few percent level could be reached at some point. For this first exploration, we have therefore focused on the fourth derivative, the quartic self-coupling, whose determination is expected to be extremely difficult at all foreseen colliders.

We have considered in detail weak boson fusion production of three Higgs bosons, studying the sensitivity of total rates as well as of distributions on the Higgs boson self couplings. We have found the most sensitive region to be at threshold, yet the high-energy tail to provide the most of the statistics at highest c.m. energy currently foreseen. We have then considered various possible scenarios attainable in different energy/luminosity configurations, and by adopting very simplifying assumptions, determined the limits on the trilinear and quartic couplings in two motivated reference scenarios. Even though, we have made many simplifying assumptions, theoretical as well as experimental, we think that the most important features have been correctly identified, and imagine that worsening of the sensitivity from a more realistic analysis including, for instance, backgrounds and systematic uncertainties, could be offset by the many possible improvements in future analyses and detectors.

Our results provide a first indication that a leptonic collider at several TeV’s of c.m. energy and with integrated luminosities of the order of several tens of attobarns, could provide enough events to allow a determination (a SM) quartic Higgs self-coupling with an accuracy in the tens of percent. For example, assuming $\lambda_3 = \lambda_{SM}$, and a (14 TeV/33 ab$^{-1}$) scenario, one could constraint $\lambda_4$ with a 50% uncertainty at 1σ, i.e., significantly better than what is currently expected to be attainable at the FCC-ee with a similar luminosity.

To finally assess the reach of a multi-TeV muon collider many more (and more detailed) studies will be necessary. This first work on the determination of the quartic self-coupling of the Higgs suggests that such studies are certainly worth to be undertaken.

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