Multiplicity of solutions for a class of quasilinear equations involving critical Orlicz-Sobolev nonlinear term

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Abstract

In this work, we study the existence and multiplicity of solutions for a class of problems involving the \( \phi \)-Laplacian operator in a bounded domain, where the nonlinearity has a critical growth. The main tool used is the variational method combined with the genus theory for even functionals.

1 Introduction

In this paper, we consider the existence and multiplicity of solutions for the following class of quasilinear problem

\[
\begin{cases}
- \text{div} (\phi(|\nabla u|)\nabla u) = \lambda \phi_b(|u|)u + f(x,u), & \Omega \\
u = 0, & \partial \Omega
\end{cases}
\]  

\( (P_\lambda) \)

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary, \( \lambda \) is a positive parameter and \( \phi : (0, +\infty) \to \mathbb{R} \) is a continuous function verifying

\[ (\phi(t)t)' > 0 \ \forall t > 0. \]  

\( (\phi_1) \)

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There exist \( l, m \in (1, N) \) such that
\[
l \leq \frac{\phi(|t|) t^2}{\Phi(t)} \leq m \quad \forall t \neq 0,
\]
where \( \Phi(t) = \int_0^{|t|} \phi(s) \, ds \), \( l \leq m < l^* \), \( l^* = \frac{1N}{N - l} \) and \( m^* = \frac{mN}{N - m} \).

Moreover, \( \phi_* (t) \) is such that Sobolev conjugate function \( \Phi_* \) of \( \Phi \) is its primitive, that is, \( \Phi_*(t) = \int_0^{|t|} \phi_*(s) \, ds \).

Related to function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \), we assume that:

\((f_1)\) \( f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \), odd with respect \( t \) and
\[
\begin{align*}
f(x, t) &= o(\phi(|t|) |t|), \quad |t| \to 0 \text{ uniformly in } x; \\
f(x, t) &= o(\phi_*(|t|) |t|), \quad |t| \to +\infty \text{ uniformly in } x;
\end{align*}
\]

\((f_2)\) There is \( \theta \in (m, l^*) \) such that \( F(x, t) \leq \frac{1}{\theta} f(x, t)t \), for all \( t > 0 \) and a.e. in \( \Omega \), where \( F(x, t) = \int_0^t f(x, s) \, ds \).

There often arise the problem \((P_{\lambda})\) associated by a nonhomogenous nonlinearities \( \Phi \) in the fields of physics (see [16]), e.g.

\( i \) nonlinear elasticity: \( \Phi(t) = (1 + |t|^2)^\gamma - 1 \) for \( \gamma \in (1, \frac{N}{N-2}) \).

\( ii \) plasticity: \( \Phi(t) = |t|^p ln(1 + |t|) \) for \( 1 < p_0 < p < N - 1 \) with \( p_0 = \frac{-1 + \sqrt{1 + 4N}}{2} \).

\( iii \) generalized Newtonian fluids: \( \Phi(t) = \int_0^t s^{1-\alpha} ( \sinh^{-1} s )^\beta \, ds \), \( 0 \leq \alpha \leq 1, \beta > 0 \).

Our main results is the following

**Theorem 1.1.** Assume that \((\phi_1) - (\phi_2)\) and \((f_1) - (f_2)\) are satisfied. Then, there exist a sequence \( \{\lambda_k\} \subset (0, +\infty) \) with \( \lambda_k < \lambda_{k+1} \), such that, for \( \lambda \in (\lambda_k, \lambda_{k+1}) \), problem \((P_{\lambda})\) has at least \( k \) pairs of nontrivial solutions.

The main difficulty to prove Theorem 1.1 is related to the fact that the nonlinearity \( f \) has a critical growth, because in this case, it is not clear that functional energy associated with problem \((P_{\lambda})\) satisfies the well known \((PS)\)
condition, once that the embedding $W^{1,p}(\Omega) \hookrightarrow L_{p^*}(\Omega)$ is not compact. To overcome this difficulty, we use a version of the concentration compactness lemma due to Lions for Orlicz-Sobolev space found in Fukagai, Ito and Narukawa [14]. We would like to mention that Theorem 1.1 improves the main result found in [25].

We cite the papers of Alves and Barreiro [3], Alves, Gonçalves and Santos [4], Bonano, Bisci and Radulescu [5], Clément, Garcia-Huidobro and Manásevich [9], Donaldson [10], Fuchs and Li [12], Fuchs and Osmolovski [13], Fukagai, Ito and Narukawa [14, 15], Gossez [17], Mihailescu and Radulescu [19, 20], Mihailescu and Repovs [21], Pohozaev [22] and references therein, where quasilinear problems like ($P_\lambda$) have been considered in bounded and unbounded domains of $\mathbb{R}^N$. In some those papers, the authors have mentioned that this class of problem arises in a lot of applications, such as, nonlinear elasticity, plasticity and non-Newtonian fluids.

This paper is organized in the following way: In Section 2, we collect some preliminaries on Orlicz-Sobolev spaces that will be used throughout the paper, which can be found in [1], [2], [11] and [23]. In Section 3, we recall an abstract theorem involving genus theory that will use in the proof of Theorem 1.1 and prove some technical lemmas, and Section 4 we prove Theorem 1.1.

## 2 Preliminaries on Orlicz-Sobolev space

First of all, we recall that a continuous function $A : \mathbb{R} \to [0, +\infty)$ is a $N$-function if:

(A1) $A$ is convex.

(A2) $A(t) = 0 \iff t = 0$.

(A3) $\frac{A(t)}{t} \xrightarrow{t \to 0} 0$ and $\frac{A(t)}{t} \xrightarrow{t \to +\infty} +\infty$.

(A4) $A$ is even.

In what follows, we say that a $N$-function $A$ verifies the $\Delta_2$-condition if,
there exists $t_0 \geq 0$ and $k > 0$ such that

$$A(2t) \leq kA(t) \quad t \geq t_0.$$  

This condition can be rewritten in the following way: For each $s > 0$, there exists $M_s > 0$ and $t_0 \geq 0$ such that

$$A(st) \leq M_sA(t) \quad t \geq t_0. \quad (\Delta_2)$$

Fixed an open set $\Omega \subset \mathbb{R}^N$ and a N-function $A$ satisfying $\Delta_2$-condition, the space $L_A(\Omega)$ is the vectorial space of the measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} A(u) < \infty.$$  

The space $L_A(\Omega)$ endowed with Luxemburg norm, that is, with the norm given by

$$|u|_A = \inf \left\{ \alpha > 0 : \int_{\Omega} A\left(\frac{u}{\alpha}\right) \leq 1 \right\},$$

is a Banach space. The complement function of $A$, denoted by $\tilde{A}(s)$, is given by the Legendre transformation, that is

$$\tilde{A}(s) = \max_{t \geq 0} \{ st - A(t) \} \quad \text{for} \ s \geq 0.$$  

The functions $A$ and $\tilde{A}$ are complementary each other. Moreover, we have the Young’s inequality given by

$$st \leq A(t) + \tilde{A}(s) \quad \forall t, s \geq 0. \quad (1)$$

Using the above inequality, it is possible to prove a Hölder type inequality, that is,

$$\left| \int_{\Omega} uv \right| \leq 2|u|_A |v|_{\tilde{A}}, \ \forall u \in L_A(\Omega) \text{ and } v \in L_{\tilde{A}}(\Omega). \quad (2)$$

Another important function related to function $A$, it is the Sobolev’s conjugate function $A^*$ of $A$ defined by

$$A^{-1}_*(t) = \int_0^t \frac{A^{-1}(s)}{s^{(N+1)/N}} ds, \ \text{for} \ t > 0.$$  

When $A(t) = |t|^p$ for $1 < p < N$, we have $A^*(t) = p^* t^{p^*}$, where $p^* = \frac{pN}{N-p}$.  

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Hereafter, we denote by $W^{1,A}_0(\Omega)$ the Orlicz-Sobolev space obtained by the completion of $C_0^\infty(\Omega)$ with respect to norm
\[
\|u\| = |\nabla u|_A + |u|_A.
\]

An important property that we must detach is: If $A$ and $\tilde{A}$ verifying $\Delta_2$-condition, the spaces $L_A(\Omega)$ and $W^{1,A}(\Omega)$ are reflexive and separable. Moreover, the $\Delta_2$-condition also implies that
\[
\begin{align*}
    u_n \to u & \text{ in } L_A(\Omega) \iff \int_\Omega A(|u_n - u|) \to 0 \quad (3) \\
    u_n \rightharpoonup u & \text{ in } W^{1,A}(\Omega) \iff \int_\Omega A(|u_n - u|) \to 0 \text{ and } \int_\Omega A(|\nabla u_n - \nabla u|) \to 0. \quad (4)
\end{align*}
\]

Another important inequality was proved by Donaldson and Trudinger [10], which establishes that for all open $\Omega \subset \mathbb{R}^N$ and there is a constant $S_N = S(N) > 0$ such that
\[
|u|_A \leq S_N |\nabla u|_A, \quad u \in W^{1,A}_0(\Omega). \quad (5)
\]
Moreover, exist $C_0 > 0$ such that
\[
\int_\Omega A(u) \leq C_0 \int_\Omega A(|\nabla u|), \quad u \in W^{1,A}_0(\Omega). \quad (6)
\]
This inequality shows the below embedding is continuous
\[
W^{1,A}_0(\Omega) \hookrightarrow L_{A_*}(\Omega).
\]
If bounded domain $\Omega$ and the limits below hold
\[
\limsup_{t \to 0} \frac{B(t)}{A(t)} < +\infty \quad \text{and} \quad \limsup_{|t| \to +\infty} \frac{B(t)}{A_*(t)} = 0, \quad (7)
\]
the embedding
\[
W^{1,A}_0(\Omega) \hookrightarrow L_{B}(\Omega) \quad (8)
\]
is compact.

The next four lemmas involving the functions $\Phi, \tilde{\Phi}$ and $\Phi_*$ and theirs proofs can be found in [14]. Hereafter, $\Phi$ is the $N$-function given in introduction and $\tilde{\Phi}, \Phi_*$ are the complement and conjugate functions of $\Phi$ respectively.
Lemma 2.1. Assume $(\phi_1) - (\phi_2)$. Then,

$$\Phi(t) = \int_{0}^{t} s\phi(s)ds,$$

is a $N$-function with $\Phi, \tilde{\Phi} \in \Delta_2$. Hence, $L_\Phi(\Omega), W^{1,\Phi}(\Omega)$ and $W_0^{1,\Phi}(\Omega)$ are reflexive and separable spaces.

Lemma 2.2. The functions $\Phi, \Phi_*, \tilde{\Phi}$ and $\tilde{\Phi}_*$ satisfy the inequality

$$\tilde{\Phi}(\phi(|t|)t) \leq \Phi(2t) \text{ and } \tilde{\Phi}_*(\phi_*(|t|)t) \leq \Phi_*(2t) \; \forall t \geq 0. \quad (9)$$

Lemma 2.3. Assume that $(\phi_1) - (\phi_2)$ hold and let $\xi_0(t) = \min\{t^l, t^m\}$, $\xi_1(t) = \max\{t^l, t^m\}$, for all $t \geq 0$. Then,

$$\xi_0(\rho)\Phi(t) \leq \Phi(\rho t) \leq \xi_1(\rho)\Phi(t) \quad \text{for } \rho, t \geq 0$$

and

$$\xi_0(|u|_{\Phi}) \leq \int_{\Omega} \Phi(u) \leq \xi_1(|u|_{\Phi}) \quad \text{for } u \in L_\Phi(\Omega).$$

Lemma 2.4. The function $\Phi_*$ satisfies the following inequality

$$l^* \leq \frac{\Phi'_*(t)}{\Phi_*(t)} \leq m^* \quad \text{for } t > 0.$$

As an immediate consequence of the Lemma 2.4, we have the following result

Lemma 2.5. Assume that $(\phi_1) - (\phi_2)$ hold and let $\xi_2(t) = \min\{t^{l^*}, t^{m^*}\}$, $\xi_3(t) = \max\{t^{l^*}, t^{m^*}\}$ for all $t \geq 0$. Then,

$$\xi_2(\rho)\Phi_*(t) \leq \Phi_*(\rho t) \leq \xi_3(\rho)\Phi_*(t) \quad \text{for } \rho, t \geq 0$$

and

$$\xi_2(|u|_{\Phi_*}) \leq \int_{\Omega} \Phi_*(u)dx \leq \xi_3(|u|_{\Phi_*}) \quad \text{for } u \in L_{A_*}(\Omega).$$

Lemma 2.6. Let $\tilde{\Phi}$ be the complement of $\Phi$ and put

$$\xi_4(s) = \min\{s^{\frac{1}{l^*}}, s^{\frac{m^*}{l^*}}\} \text{ and } \xi_5(s) = \max\{s^{\frac{1}{l^*}}, s^{\frac{m^*}{l^*}}\}, \; s \geq 0.$$

Then the following inequalities hold

$$\xi_4(r)\tilde{\Phi}(s) \leq \tilde{\Phi}(rs) \leq \xi_5(r)\tilde{\Phi}(s), \; r, s \geq 0$$

and

$$\xi_4(|u|_{\tilde{\Phi}}) \leq \int_{\Omega} \tilde{\Phi}(u)dx \leq \xi_5(|u|_{\tilde{\Phi}}), \; u \in L_{\tilde{\Phi}}(\Omega).$$
3 An abstract theorem and technical lemmas

In this section we recall an important abstract theorem involving genus theory, which will use in the proof of Theorem 1.1. After, we prove some technical lemmas that will use to show that the energy functional associated with problem \( (P) \) verifies the hypotheses of the abstract theorem.

3.1 An abstract theorem

Let \( E \) be a real Banach space and \( \Sigma \) the family of sets \( Y \subset E\setminus\{0\} \) such that \( Y \) is closed in \( E \) and symmetric with respect to 0, that is,

\[
\Sigma = \{ Y \subset E\setminus\{0\}; Y \text{ is closed in } E \text{ and } Y = -Y \}.
\]

Hereafter, let us denote by \( \gamma(Y) \) the genus of \( Y \in \Sigma \) (see [24, pp. 45]). Moreover, we set

\[
K_c = \{ u \in E; I(u) = c \text{ and } I'(u) = 0 \}
\]

and

\[
A_c = \{ u \in E; I(u) \leq c \}.
\]

Next, we recall a version of the Mountain Pass Theorem for even functional. For details of the proof, see [24].

**Theorem 3.1.** Let \( E \) be an infinite dimensional Banach space with \( E = V \oplus X \), where \( V \) is finite dimensional and let \( I \in C^1(E,\mathbb{R}) \) be a even function with \( I(0) = 0 \) and satisfying:

1. \( (I_1) \) there are constants \( \beta, \rho > 0 \) such that \( I(u) \geq \beta > 0 \), for each \( u \in \partial B_\rho \cap X \);
2. \( (I_2) \) there is \( \Upsilon > 0 \) such that \( I \) satisfies the \( (PS)_c \) condition, for \( 0 < c < \Upsilon \);
3. \( (I_3) \) for each finite dimensional subespace \( \tilde{E} \subset E \), there is \( R = R(\tilde{E}) > 0 \) such that \( I(u) \leq 0 \) for all \( x \in \tilde{E}\setminus B_R(0) \).

Suppose \( V \) is \( k \) dimensional and \( V = \text{span}\{e_1,\ldots,e_k\} \). For \( m \geq k \), inductively choose \( e_{m+1} \notin E_m := \text{span}\{e_1,\ldots,e_m\} \). Let \( R_m = R(E_m) \) and \( D_m = B_{R_m} \cap E_m \). Define

\[
G_m := \{ h \in C(D_m, E); h \text{ is odd and } h(u) = u, \forall u \in \partial B_{R_m} \cap E_m \} \quad (10)
\]
and
\[ \Gamma_j := \left\{ h(D_m \setminus Y) ; h \in G_m, m \geq j, Y \in \Sigma, \text{ and } \gamma(Y) \leq m - j \right\}. \] (11)

For each \( j \in \mathbb{N} \), let
\[ c_j = \inf_{K \in \Gamma_j} \max_{u \in K} I(u). \] (12)

Then, \( 0 < \beta \leq c_j \leq c_{j+1} \) for \( j > k \), and if \( j > k, c_j < \Upsilon \) and \( c_j \) is critical value of \( I \). Moreover, if \( c_j = c_{j+1} = \cdots = c_{j+l} = c < \Upsilon \) for \( j > k \), then \( \gamma(K_c) \geq l + 1 \).

### 3.2 Technical lemmas

Associated with the problem \([P_\lambda]\), we have the energy functional \( J_\lambda : W_0^{1,\Phi}(\Omega) \to \mathbb{R} \) defined by
\[
J_\lambda(u) = \int_\Omega \Phi(|\nabla u|) - \lambda \int_\Omega \Phi^*(u) - \int_\Omega F(x,u).
\]

By conditions \((f_1) - (f_2)\), \( J_\lambda \in C^1 \left( W_0^{1,\Phi}(\Omega), \mathbb{R} \right) \) with
\[
J'_\lambda(u) \cdot v = \int_\Omega \phi(|\nabla u|) \nabla u \nabla v - \lambda \int_\Omega \phi^*(|u|) uv - \int_\Omega f(x,u)v,
\]
for any \( u, v \in W_0^{1,\Phi}(\Omega) \). Thus, critical points of \( J_\lambda \) are weak solutions of problem \([P_\lambda]\).

**Lemma 3.2.** Under the conditions \((f_1) - (f_2)\), \( J_\lambda \) satisfies \((I_1)\).

**Proof.**

On the other hand, from \((f_1) - (f_2)\), given \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that
\[
|F(x,t)| \leq \epsilon \Phi(t) + C_\epsilon \Phi^*(t), \quad \forall (x,t) \in \bar{\Omega} \times \mathbb{R}.
\] (13)

Combining \((6)\) with \((13)\),
\[
J_\lambda(u) \geq (1 - \epsilon C_0) \int_\Omega \Phi(|\nabla u|) - (1 + C_\epsilon) \int_\Omega \Phi^*(u).
\]
For $\epsilon$ is small enough and $\|u\| = \rho \simeq 0$, it follows from (5) and Lemma 2.4

$$J_\lambda(u) \geq C_1 \left| \nabla u \right|^m_\Phi - C_2 S_N^r \left| \nabla u \right|^r_\Phi.$$ 

for some positive constants $C_1$ and $C_2$. Once that, $m < l^*$, if $\rho$ is small enough, there is $\beta > 0$ such that

$$J_\lambda(u) \geq \beta > 0 \forall u \in \partial B_\rho(0),$$

finishing the proof. \hfill \blacksquare

**Lemma 3.3.** Under the conditions $(f_1)-(f_2)$, $J_\lambda$ satisfies $(I_3)$.

**Proof.** Suppose $(I_3)$ does not hold. Then, there is a finite dimensional subspace $\widetilde{E} \subset W^{1,\Phi}(\Omega)$ and a sequence $(u_n) \subset \widetilde{E} \backslash B_n(0)$ verifying:

$$J_\lambda(u_n) > 0, \quad \forall n \in \mathbb{N}.$$ 

(14)

A direct computation shows that given $\epsilon > 0$, there is a constant $M > 0$ such that

$$F(x,t) \geq -M - \epsilon \Phi_*(t) \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}.$$ 

(15)

Consequently,

$$J_\lambda(u_n) \leq \int_\Omega \Phi(|\nabla u_n|)dx - \lambda \int_\Omega \Phi_*(u_n) + \epsilon \int_\Omega \Phi_*(u_n) + M |\Omega|.$$ 

Fixing $\epsilon = \frac{\lambda}{2}$, and using Lemma 2.5 we get

$$J_\lambda(u_n) \leq \int_\Omega \Phi(|\nabla u_n|) dx - \frac{\lambda}{2} \xi_3(|u_n|_\Phi) + M |\Omega|.$$ 

(16)

Once that $\dim \widetilde{E} < \infty$, we know that any two norms in $\widetilde{E}$ are equivalent. Then, using that $\|u_n\| \to \infty$, we can assume that $|u_n|_\Phi > 1$. Thereby, from Lemmas 2.3 and 2.5

$$J_\lambda(u_n) \leq |\nabla u_n|^m_\Phi - \frac{\lambda}{2} |u_n|^r_\Phi + M |\Omega|.$$ 

Using again the equivalence of the norms in $\widetilde{E}$, there is $C > 0$ such that

$$J_\lambda(u_n) \leq \|u_n\|^m - \frac{\lambda}{2} C \|u_n\|^r + M |\Omega|.$$ 

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Recalling that \( m < l^* \), the above inequality implies that there is \( n_0 \in \mathbb{N} \) such that

\[
J_\lambda(u_n) < 0, \quad \forall n \geq n_0,
\]

which contradicts (14).

\[\blacksquare\]

**Lemma 3.4.** Under the conditions \((f_1) - (f_2)\), any \((PS)\) sequence for \( J_\lambda \) is bounded in \( W_0^1, \Phi(\Omega) \).

**Proof.** Let \( \{u_n\} \) be a \((PS)_d\) sequence of \( J_\lambda \). Then, \( J_\lambda(u_n) \to d \) and \( J'_\lambda(u_n) \to 0 \) as \( n \to +\infty \).

We claim that \( \{u_n\} \) is bounded. Indeed, note that

\[
J_\lambda(u_n) - \frac{1}{\theta} J'_\lambda(u_n) u_n = \int_\Omega \Phi(|\nabla u_n|) - \frac{1}{\theta} \int_\Omega \phi(|\nabla u_n|) |\nabla u_n|^2
\]

\[
- \lambda \int_\Omega \Phi_*(u_n) + \frac{\lambda}{\theta} \int_\Omega \phi_*(|u_n|) u_n^2
\]

\[
- \int_\Omega F(x, u_n) + \frac{1}{\theta} \int_\Omega f(x, u_n) u_n.
\]

Consequently,

\[
\lambda \int_\Omega \left( \frac{1}{\theta} \phi_*(|u_n|) u_n^2 - \Phi_*(u_n) \right) = J_\lambda(u_n) - \frac{1}{\theta} J'_\lambda(u_n) u_n - \int_\Omega \Phi(|\nabla u_n|)
\]

\[
+ \frac{1}{\theta} \int_\Omega \phi(|\nabla u_n|) |\nabla u_n|^2
\]

\[
+ \int_\Omega \left( F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right).
\]

Then, by \((\phi_2), (f_2)\) and Lemma 2.4 for \( n \) sufficiently large

\[
\lambda \left( \frac{l}{\theta} - 1 \right) \int_\Omega \Phi_*(u_n) \leq C + 1 + \|u_n\| + \left( \frac{m}{\theta} - 1 \right) \int_\Omega \Phi(|\nabla u_n|),
\]

which implies that

\[
\left[ \lambda \left( \frac{l}{\theta} - 1 \right) \right] \int_\Omega \Phi_*(u_n) \leq C + \|u_n\|,
\]

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where $C$ is a positive constant, and so
\[
\int_{\Omega} \Phi_s(u_n) dx \leq C (1 + \|u_n\|).
\] (17)

By (15) and (17)
\[
\int_{\Omega} \Phi(|\nabla u_n|) \leq J_\lambda(u_n) + \lambda \int_{\Omega} \Phi_s(u_n) + \int_{\Omega} G(x, u_n) dx
\]
\[
\leq C + o_n(1) + (\lambda + \epsilon) \int_{\Omega} \Phi_s(u_n)
\]
\[
\leq C(1 + \|u_n\|) + o_n(1).
\]

Therefore, for $n$ sufficiently large
\[
\int_{\Omega} \Phi(|\nabla u_n|) \leq C (1 + \|u_n\|).
\]

If $\|u_n\| > 1$, it follows from Lemma 2.5
\[
\|u_n\|^l \leq C (1 + \|u_n\|).
\]

Once that $l > 1$, the above inequality gives that $\{u_n\}$ is bounded in $W_0^{1, \Phi}(\Omega)$. \hfill $\blacksquare$

As a consequence of the last result, if $\{u_n\}$ is a ($PS$) sequence for $J_\lambda$, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ and $u \in W_0^{1, \Phi}(\Omega)$, such that

- $u_n \rightharpoonup u$ in $W_0^{1, \Phi}(\Omega)$;
- $u_n \to u$ in $L_{\Phi_s}(\Omega)$;
- $u_n \to u$ in $L_{\Phi}(\Omega)$;
- $u_n(x) \to u(x)$ a.e. in $\Omega$;
- $u_n(x) \to u(x)$ a.e. in $\Omega$.

From the concentration compactness lemma of Lions in Orlicz-Sobolev space found in [14], there exist two nonnegative measures $\mu, \nu \in M(\Omega)$,
a countable set $\mathcal{J}$, points $\{x_j\}_{j \in \mathcal{J}}$ in $\Omega$ and sequences $\{\mu_j\}_{j \in \mathcal{J}}, \{\nu_j\}_{j \in \mathcal{J}} \subset [0, +\infty)$, such that

$$\Phi(\|\nabla u_n\|) \to \mu \geq \Phi(\|\nabla u\|) + \sum_{j \in \mathcal{J}} \mu_j \delta_{x_j} \text{ in } \mathcal{M}(\Omega) \quad (18)$$

$$\Phi_*(u_n) \to \nu = \Phi_*(u) + \sum_{j \in \mathcal{J}} \nu_j \delta_{x_j} \text{ in } \mathcal{M}(\Omega) \quad (19)$$

and

$$\nu_j \leq \max\{S_N^* \mu_j^{\frac{l}{m}}, S_N^* \mu_j^{\frac{m}{l}}, S_N^* \mu_j^{\frac{l}{m}}, S_N^* \mu_j^{\frac{m}{l}}\}, \quad (20)$$

where $S_N$ verifies (5).

Next, we will show an important estimate from below for $\{\nu_i\}$. We have to prove a technical lemma.

**Lemma 3.5.** Under the conditions of Lemma 3.4. If $\{u_n\}$ is a $(PS)$ sequence for $J_\lambda$ and $\{\nu_j\}$ as above, then for each $j \in \mathcal{J}$,

$$\nu_j \geq \left( \frac{l}{\lambda m^*} \right)^{\frac{m}{l}} S_N^{\frac{m}{l} \frac{l}{m}} \text{ or } \nu_j = 0,$$

for some $\alpha \in \{l^*, m^*\}$ and $\beta \in \{\frac{m}{l}, \frac{l}{m}, \frac{l}{m}, \frac{m}{l}\}$.

**Proof.** Let $\psi \in C^\infty_0(\mathbb{R}^N)$ such that

$$\psi(x) = 1 \text{ in } B_\frac{1}{2}(0), \quad \text{supp } \psi \subset B_1(0) \text{ and } 0 \leq \psi(x) \leq 1 \quad \forall x \in \mathbb{R}^N.$$

For each $j \in \Gamma$ and $\epsilon > 0$, let us define

$$\psi_\epsilon(x) = \psi \left( \frac{x - x_j}{\epsilon} \right), \quad \forall x \in \mathbb{R}^N.$$

Then $\{\psi_\epsilon u_n\}$ is bounded in $W^{1, \Phi}_0(\Omega)$. Since $J'_\lambda(u_n) \to 0$, we have

$$J'_\lambda(u_n)(\psi_\epsilon u_n) = o_n(1),$$

or equivalently,

$$\int_\Omega \phi(\|\nabla u_n\|) \nabla u_n \nabla (u_n \psi_\epsilon) = o_n(1) + \lambda \int_\Omega \phi_*(\|u_n\|) u_n^2 \psi_\epsilon + \int_\Omega f(x, u_n) u_n \psi_\epsilon$$

$$\leq o_n(1) + \lambda m^* \int_\Omega \Phi_*(u_n) \psi_\epsilon + \int_\Omega f(x, u_n) u_n \psi_\epsilon. \quad (21)$$
By the compactness Lemma of Strauss [8]

\[ \lim_{n \to \infty} \int_{\Omega} f(x, u_n) u_n \psi_\epsilon = \int_{\Omega} f(x, u) u \psi_\epsilon. \]  \hspace{1cm} (22)

On the other hand, by \( (\phi_2) \)

\[ \int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \nabla (u_n \psi_\epsilon) = \int_{\Omega} \phi(|\nabla u_n|) |\nabla u_n|^2 \psi_\epsilon + \int_{\Omega} \phi(|\nabla u_n|) (\nabla u_n \nabla \psi_\epsilon) u_n \]
\[ \geq l \int_{\Omega} \Phi(|\nabla u|) \psi_\epsilon + \int_{\Omega} \phi(|\nabla u_n|) (\nabla u_n \nabla \psi_\epsilon) u_n. \]  \hspace{1cm} (23)

By Lemmas 2.2 and 2.6, the sequence \( \{ \phi(|\nabla u_n|) \nabla u_n \nabla \}_{\tilde{\Phi}} \) is bounded. Thus, there is a subsequence \( \{ u_n \} \) such that

\[ \phi(|\nabla u_n|) \nabla u_n \rightharpoonup \tilde{w}_1 \text{ weakly in } L_{\tilde{\Phi}}(\Omega, \mathbb{R}^N), \]

for some \( \tilde{w}_1 \in L_{\tilde{\Phi}}(\Omega, \mathbb{R}^N) \). Since \( u_n \to u \) in \( L_{\Phi}(\Omega) \),

\[ \int_{\Omega} \phi(|\nabla u_n|)(\nabla u_n \nabla \psi_\epsilon) u_n \to \int_{\Omega} (\tilde{w}_1 \nabla \psi_\epsilon) u. \]

Thus, combining (21), (22), (23) and letting \( n \to \infty \), we have

\[ l \int_{\Omega} \psi_\epsilon d\mu + \int_{\Omega} (\tilde{w}_1 \nabla \psi_\epsilon) u \leq \lambda m^* \int_{\Omega} \psi_\epsilon d\nu + \int_{\Omega} f(x, u) u \psi_\epsilon. \]  \hspace{1cm} (24)

Now we show that the second term of the left-hand side converges 0 as \( \epsilon \to 0 \).

First, we show the claim:

**Claim 1:** \( \{ f(x, u_n) \} \) is bounded in \( L_{\tilde{\Phi}_*}(\Omega) \).

In fact, by \( (f_1) \) and Lemma 2.2 we have

\[ \int_{\Omega} \tilde{\Phi}_*(f(x, u_n)) \leq c_1 \int_{\Omega} \tilde{\Phi}_*(\phi(|u_n|) u_n) + c_2 \int_{|u_n| > 1} \tilde{\Phi}_*(\phi(|u_n|) u_n) \]
\[ + c_3 \int_{|u_n| \leq 1} \tilde{\Phi}_*(\phi(|u_n|) u_n) \]
\[ \leq c_1 \int_{\Omega} \Phi_*(u_n) + c_2 \int_{|u_n| > 1} \tilde{\Phi}_*(\phi(|u_n|) u_n) + c_3 |\Omega|. \]
Hence, by (φ2), Lemma 2.3 and $m < l^*$,

$$\int_{\Omega} \tilde{\Phi}_*(f(x, u_n)) \leq C_1 \int_{\Omega} \tilde{\Phi}_*(|u_n|) u_n + C_2 \int_{|u_n| > 1} \tilde{\Phi}_*(|u_n|^{m-1}) + C_3 |\Omega|$$

$$\leq C_1 \int_{\Omega} \Phi_*(u_n) + C_2 \int_{|u_n| > 1} \tilde{\Phi}_*(|u_n|^{l^*-1}) + C_3 |\Omega|.$$ 

Now, by Lemmas 2.4 and 2.2

$$\int_{\Omega} \tilde{\Phi}_*(f(x, u_n)) \leq K_1 \int_{\Omega} \Phi_*(u_n) + K_2 |\Omega| < +\infty.$$ 

From Claim 1, there is a subsequence $\{u_n\}$ such that

$$\phi_*(|u_n|) u_n + f(x, u_n) \rightharpoonup \tilde{w}_2$$ weakly in $L_{\tilde{\Phi}_*}(\Omega),$ for some $\tilde{w}_2 \in L_{\tilde{\Phi}_*}(\Omega).$ Since

$$J'_\lambda(u_n)v = \int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \nabla v - \int_{\Omega} (\phi_*(|u_n|) u_n + f(x, u_n)) v$$

$$\rightarrow 0,$$

as $n \rightarrow \infty$ for any $v \in W^{1,\Phi}_0(\Omega),$ 

$$\int_{\Omega} (\tilde{w}_1 \nabla v - \tilde{w}_2 v) = 0,$$

for any $v \in W^{1,\Phi}_0(\Omega).$ Substituting $v = u\psi_\epsilon$ we have

$$\int_{\Omega} (\tilde{w}_1 \nabla (u\psi_\epsilon) - \tilde{w}_2 u\psi_\epsilon) = 0.$$ 

Namely,

$$\int_{\Omega} (\tilde{w}_1 \nabla \psi_\epsilon) u = - \int_{\Omega} (\tilde{w}_1 \nabla u - \tilde{w}_2 u) \psi_\epsilon.$$ 

Noting $\tilde{w}_1 \nabla u - \tilde{w}_2 u \in L^1(\Omega),$ we see that right-hand side tends to 0 as $\epsilon \rightarrow 0.$ Hence we have

$$\int_{\Omega} (\tilde{w}_1 \nabla \psi_\epsilon) u \rightarrow 0,$$

as $\epsilon \rightarrow 0.$

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Letting $\epsilon \to 0$ in (24), we obtain

$$l\mu_j \leq \lambda m^* \nu_j.$$ 

Hence,

$$S_N^{-\alpha} \nu_j \leq \mu_j^\beta \leq \left(\frac{l \lambda}{m^*}\right)^\beta \nu_j^\beta,$$

for some $\alpha \in \{l^*, m^*\}$, $\beta \in \{l^* \ell, m^* \ell, l^* m, m^* m\}$, and so

$$\nu_j \geq \left(\frac{l}{\lambda m^*}\right)^\frac{\beta}{\beta-1} S_N^{-\frac{\beta}{\beta-1}} \text{ or } \nu_j = 0.$$

\[ \blacksquare \]

**Lemma 3.6.** Assume that $(f_1)-(f_2)$. Then, $J_\lambda$ satisfies $(PS)_d$ for $d \in (0, d_\lambda)$ where

$$d_\lambda = \min \left\{ \frac{l^* - \theta}{\theta S_N^{-\frac{\beta}{\beta-1}}} \left(\frac{l}{m^*}\right)^\frac{\beta}{\beta-1} \right\}; \alpha \in \{l^*, m^*\}, \beta \in \{l^* \ell, m^* \ell, l^* m, m^* m\}.$$

**Proof.** Once that

$$J_\lambda(u_n) = d + o_n(1) \text{ and } J'_\lambda(u_n) = o_n(1),$$

$$d = \lim_{n \to \infty} I(u_n) = \lim_{n \to \infty} \left( J_\lambda(u_n) - \frac{1}{\theta} J'_\lambda(u_n) u_n \right)$$

$$\geq \lim_{n \to \infty} \left[ \left( 1 - \frac{m}{\theta} \right) \int_\Omega \Phi(|\nabla u_n|) + \lambda \left( \frac{l^*}{\theta} - 1 \right) \int_\Omega \Phi_*(u_n) \right]$$

$$- \int_\Omega \left( F(x, u_n) - \frac{1}{\theta} f(x, u_n) u_n \right)$$

$$\geq \lambda \left( \frac{l^*}{\theta} - 1 \right) \int_\Omega \Phi_*(u_n). \quad (25)$$

Recalling that

$$\lim_{n \to \infty} \int_\Omega \Phi_*(u_n) dx = \left[ \int_\Omega \Phi_*(u) + \sum_{j \in J} \nu_j \right] \geq \nu_j,$$
we derive that
\[
    d \geq \lambda \left( \frac{l^*}{\theta} - 1 \right) \left( \frac{l}{\lambda m^*} \right)^{\frac{\beta}{\beta - 1}} S_N^{-\frac{\alpha}{\beta - 1}} \left( l^* - \theta \right) \left( \frac{l}{m^*} \right)^{\frac{\beta}{\beta - 1}} S_N^{-\frac{\alpha}{\beta - 1}} \lambda^{1 - \frac{\beta}{\beta - 1}},
\]
for some \( \alpha \in \{ l^*, m^* \} \), \( \beta \in \{ \frac{l^*}{l}, \frac{m^*}{m}, \frac{m^*}{l}, \frac{l^*}{m} \} \), which is an absurd. From this, we must have \( \nu_j = 0 \) for any \( j \in J \), leading to
\[
    \int_{\Omega} \Phi^*(u_n) \to \int_{\Omega} \Phi^*(u).
\]
(27)
Combining the last limit with Brézis and Lieb [6], we obtain
\[
    \int_{\Omega} \Phi^*(u_n - u) \to 0 \quad \text{as} \quad n \to \infty,
\]
from where it follows by Lemma 2.5
\[
    u_n \to u \quad \text{in} \quad L_{\Phi^*}(\Omega).
\]
(28)
Now, as \( J^*_1(u_n)u_n = o_n(1) \), the last limit gives
\[
    \int_{\Omega} \phi(|\nabla u_n|) |u_n|^2 = \lambda \int_{\Omega} \phi(|u_n|)u_n^2 + \int_{\Omega} f(x, u_n)u_n + o_n(1).
\]
In what follows, let us denote by \( \{ P_n \} \) the following sequence,
\[
    P_n(x) = \langle \phi(|\nabla u_n(x)||\nabla u_n(x)) - \phi(|\nabla u(x)||\nabla u(x)), \nabla u_n(x) - \nabla u(x) \rangle.
\]
Since \( \Phi \) is convex in \( \mathbb{R} \) and \( \Phi(|.|) \) is \( C^1 \) class in \( \mathbb{R}^N \), has \( P_n(x) \geq 0 \). From definition of \( \{ P_n \} \),
\[
    \int_{\Omega} P_n = \int_{\Omega} \phi(|\nabla u_n|) |\nabla u_n|^2 - \int_{\Omega} \phi(|\nabla u_n|) \nabla u_n \cdot \nabla u - \int_{\Omega} \phi(|\nabla u|) \nabla u \cdot \nabla (u_n - u).
\]
Recalling that \( u_n \rightharpoonup u \) in \( W^{1, \Phi}_0(\Omega) \), we have
\[
    \int_{\Omega} \phi(|\nabla u|) \nabla u \cdot \nabla (u_n - u) \to 0 \quad \text{as} \quad n \to \infty,
\]
(29)
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which implies that
\[ \int_{\Omega} P_n = \int_{\Omega} \phi(|\nabla u_n|)|\nabla u_n|^2 - \int_{\Omega} \phi(|\nabla u_n|)\nabla u_n \nabla u + o_n(1). \]

On the other hand, from \( J'_\lambda(u_n)u_n = o_n(1) \) and \( J'_\lambda(u_n)u = o_n(1) \), we derive
\[ 0 \leq \int_{\Omega} P_n = \lambda \int_{\Omega} \phi_*(|u_n|)|u_n|^2 - \lambda \int_{\Omega} \phi_*(|u_n|)u_n u 
+ \int_{\Omega} f(x, u_n)u_n - \int_{\Omega} f(x, u_n)u + o_n(1). \]

Combining (27) with the compactness Lemma of Strauss [8], we deduce that
\[ \int_{\Omega} P_n \to 0 \quad \text{as} \quad n \to \infty. \]

Applying a result due to Dal Maso and Murat [18], have that
\[ u_n \to u \text{ in } W^{1,\Phi}_0(\Omega). \]  

The next lemma is similar to [25, Lemma 5] and its proof will be omitted.

**Lemma 3.7.** Under the conditions \((f_1)-(f_2)\), there is sequence \( \{M_m\} \subset (0, +\infty) \) independent of \( \lambda \) with \( M_m \leq M_{m+1} \), such that for any \( \lambda > 0 \)
\[ c^\lambda_m = \inf_{K \in \Gamma_m} \max_{u \in K} J_\lambda(u) < M_m. \]  

4 Proof of Theorem 1.1

For each \( k \in \mathbb{N} \), choose \( \lambda_{k+1} \) such that
\[ M_k < d_{\lambda_k}. \]

Thus, for \( \lambda \in (\lambda_k, \lambda_{k+1}) \),
\[ 0 < c^\lambda_1 \leq c^\lambda_2 \leq \cdots \leq c^\lambda_k < M_k \leq d_{\lambda}. \]
By Theorem 3.1 the levels \( c^\lambda_1 \leq c^\lambda_2 \leq \cdots \leq c^\lambda_k \) are critical values of \( J_\lambda \). Thus, if
\[
c^\lambda_1 < c^\lambda_2 < \cdots < c^\lambda_k ,
\]
functional \( J_\lambda \) has at least \( k \) critical points. Now, if \( c^\lambda_j = c^\lambda_{j+1} \) for some \( j = 1, 2, \cdots, k \), it follows from Theorem 3.1 that \( K_{c^\lambda_j} \) is an infinite set \[21\] Cap. 7. Then, in this case, problem \( (P) \) has infinite solutions.

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