Theoretical Construction of 1D anyon models

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Abstract

One-dimensional anyon models are renewedly constructed by using path integral formalism. A statistical interaction term is introduced to realize the anyonic exchange statistics. The quantum mechanics formulation of statistical transmutation is presented.

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The theory of anyons is used to describe the particles with fractional exchange statistics\[1, 2, 3\], which has provided a successful explanation for the fractional quantum Hall effect (FQHE)\[4\]. Nowadays, it has been widely applied in condensed matter physics and quantum computation. For two dimensions (2D), the exchange statistics is well defined. But for one dimension (1D), because two particles cannot be interchanged without collision, the intrinsic statistics is inextricably mixed up with the local interactions, this make the exchange statistics for one dimension cannot be uniquely defined. However, experiments\[5\] have demonstrated the possibility of confining atoms to one dimension, so both conceptually and actually, the 2D anyons can be confined to move in one dimension. Recently, a number of researches on 1D anyons have been reported\[6, 7, 8, 9\].

In a seminal work, Kundu\[10\] defined a 1D anyon field operator \(\hat{\psi}_A(x)\) in terms of the Bose field operator \(\hat{\psi}_B(x)\) through a gauge transformation: 
\[\hat{\psi}_A(x) = e^{-i\kappa \int_x^\infty dx' \rho(x')} \hat{\psi}_B(x),\]
where \(\rho(x) = \hat{\psi}_A^\dagger(x) \hat{\psi}_A(x) = \hat{\psi}_B^\dagger(x) \hat{\psi}_B(x)\) is the number density operator. The commutation relations are 
\[\hat{\psi}_A(x) \hat{\psi}_A^\dagger(x') = e^{-i\kappa(x-x')} \hat{\psi}_A(x') \hat{\psi}_A^\dagger(x) + \delta(x-x'),\]
and 
\[\hat{\psi}_A(x) \hat{\psi}_A(x') = e^{i\kappa(x-x')} \hat{\psi}_A^\dagger(x') \hat{\psi}_A(x),\]
where \(\epsilon(x) = +1(-1)\) for \(x > 0(x < 0)\), and \(\epsilon(0) = 0\). An alternative definition proposed by Girardeau\[7\] is in terms of the Fermi field operator: 
\[\hat{\psi}_F(x) = e^{-i\kappa \int_x^\infty dx' \rho(x')} \hat{\psi}_F(x),\]
where \(\rho(x) = \hat{\psi}_A^\dagger(x) \hat{\psi}_A(x) = \hat{\psi}_F^\dagger(x) \hat{\psi}_F(x)\). The commutation relations become 
\[\hat{\psi}_A(x) \hat{\psi}_A^\dagger(x') = e^{-i\kappa(x-x')} \hat{\psi}_A^\dagger(x') \hat{\psi}_A(x) + \delta(x-x'),\]
and 
\[\hat{\psi}_A^\dagger(x') \hat{\psi}_A(x) + e^{i\kappa(x-x')} \hat{\psi}_A(x') \hat{\psi}_A^\dagger(x) = 0.\]
Note that in this case, the exclusion principle \(\hat{\psi}_A^2(x) = [\hat{\psi}_A^\dagger(x)]^2 = 0\) is satisfied automatically.

From the above commutation relations, the basic exchange symmetry of the \(N\)-body anyonic wave function \(\Psi_A(x_1, \ldots, x_N) = \langle 0 | \hat{\psi}_A(x_1) \cdots \hat{\psi}_A(x_N) | N \rangle\) by transposing only the adjacent \(x\)'s can be obtained\[7, 10\]:
\[\Psi_A(\ldots, x_i, x_{i+1}, \ldots) = \pm e^{-i\kappa(x_i-x_{i+1})} \Psi_A(\ldots, x_{i+1}, x_i, \ldots),\]
(1)
where we have written the exchange symmetry of the two kinds of anyon models in one equation, utilizing the sign \(\pm\). The plus sign is for the Kundu’s anyons and the minus sign is for the Girardeau’s. For exchanging two arbitrary particles, the exchange symmetry expression\[6, 7, 10\] is 
\[\Psi_A(\ldots, x_i, \ldots, x_j, \ldots) = \pm e^{-i\theta} \Psi_A(\ldots, x_j, \ldots, x_i, \ldots)\] with \(\theta = \kappa \left[ \sum_{k=i+1}^j \epsilon(x_i-x_k) - \sum_{k=i+1}^{j-1} \epsilon(x_j-x_k) \right]\), which can be derived from the iteration of the basic exchange symmetry Eq. (1).

One sees that the phase factor appearing under the exchange of two particles also depends
on the coordinates of the particles between them. This is a distinct feature of 1D anyons, because two particles in a line can not pass each other without exchange. This fact hindered the early attempts at direct introduction of the 1D anyons as charge-flux composites\cite{11, 12}, or as a flux-carrying boson (or fermion), which is known as statistical transmutation. In two dimensions, this problem has been well solved, the flux is called the Chern-Simons flux\cite{13}.

In this paper, we try to seek for a theoretical construction of 1D anyon models in order to obtain a unitive description for 1D anyon models, at least, for Kundu’s and Girardeau’s, and we have found an analogical quantum mechanics formulation of the statistical transmutation for 1D particles as in two dimensions, including an auxiliary field which plays the role of the Chern-Simons field in 2D case. The key step in our theoretical construction is to correctly give a 1D statistical interaction term that can realize the anyonic exchange statistics presented above.

To make our construction of the statistical interaction term look to be natural, it is necessary to write the exchange symmetry of the anyonic wave function in a compact form. The more general form of exchange symmetry in principle can be obtained from the iteration of the basic relation, but it is still not very convenient. Here we present a most general exchange symmetry expression written in a compact form:

\[ \Psi_A(\{x\}) = F_S(\{x\})\Psi_A(\{x\}'), \]

where \(\{x\}\) is a permutation of \(x_1, \ldots, x_N\), and \(S\) is a permutation operator that takes \(\{x\}\) into \(\{x\}'\), i.e., \(\{x\}' = S\{x\}\). The phase factor \(F_S(\{x\})\) can be obtained from a rule: write down \(\{x\}\) in a horizontal line, and under it write down \(\{x\}'\), use a straight line to connect both the \(x_j\)'s (\(j = 1, \ldots, N\)) in the two rows, then every crossing point of the two lines corresponding to \(x_i\) and \(x_j\) contributes a phase factor \(\pm e^{-i\kappa(\epsilon_{ij})}\), where \(x_i\) stands on the left side of \(x_j\) in the \(\{x\}\) row. Then \(F_S(\{x\})\) is the product of all these factors:

\[ F_S(\{x\}) = e^{-i\kappa \sum_{(i,j)} S (\epsilon_{ij})} \]

for Kundu’s anyons, and

\[ F_S(\{x\}) = (-1)^n e^{-i\kappa \sum_{(i,j)} S (\epsilon_{ij})} \]

for Girardeau’s anyons. Here, the summation \(\sum_{(i,j)} S\) means only counting the crossing points produced by permutation \(S\), and \(n\) is the number of the crossing points. We mention
here that the rule above is very similar to the rule put forward by Lieb and Liniger\cite{14} for writing the scattering amplitude in their Bethe ansatz solution of 1D $\delta$-Bose gas. In Fig.1, an example of writing the phase factor is given. One can check that using this rule, the above phase factor $\pm e^{-i\theta}$ can be conveniently obtained. A reasonable explanation for this rule will be given in the following.

\[
\pm e^{-i\kappa\epsilon(x_1-x_2)}, \pm e^{-i\kappa\epsilon(x_1-x_4)}, \pm e^{-i\kappa\epsilon(x_3-x_4)}
\]

and $\pm e^{-i\kappa\epsilon(x_1-x_3)}$ respectively, so their product is $F_S(\{x\}) = e^{-i\kappa\left[\sum_{i=2}^{4}\epsilon(x_1-x_i)+\epsilon(x_3-x_4)\right]}$.

\[\text{FIG. 1: The crossing points 1, 2, 3 and 4 give the factor } \pm e^{-i\kappa\epsilon(x_1-x_2)}, \pm e^{-i\kappa\epsilon(x_1-x_4)}, \pm e^{-i\kappa\epsilon(x_3-x_4)} \text{ and } \pm e^{-i\kappa\epsilon(x_1-x_3)} \text{ respectively, so their product is } F_S(\{x\}) = e^{-i\kappa\left[\sum_{i=2}^{4}\epsilon(x_1-x_i)+\epsilon(x_3-x_4)\right]}.
\]

From the compact expression of the exchange symmetry, one can see that the exchange phase factor is only determined by the permutation of the particles' coordinates. In fact, there are many different paths in the configuration space leading to the transposition from $\{x\}$ to $\{x\}' = S\{x\}$, but all the different paths leading to $\{x\}'$ will give the same exchange phase factor. There are two reasons for this fact: (a) According to the basic exchange symmetry, if two particles are exchanged for even times, they will contribute nothing to the exchange phase factor. If they are exchanged for odd times, the result is the same as their being exchanged only once. (b) During the exchange process, changing the order of the adjacent exchanges will not affect the final phase factor. Based on (a) and (b), we can only consider the direct path comprised by a series of adjacent exchanges, in which two particles exchange at most once. All the other paths leading to $\{x\}'$ are equivalent to this direct path in view of producing the same exchange phase factor. So to obtain the phase factor, what all we need to know is how many and what adjacent exchanges happened during the permutation, which give a reasonable explanation of the above rule for writing the phase factor.

From the above analysis, one see that all the paths from $\{x\}$ to $\{x\}'$ can be thought of as belonging to an equivalent class, which can be denoted by $S$ and give the same exchange phase factor. This implies that the anyonic statistics here may be possible to be elucidated.
by Feynman’s path integral formalism\cite{15} just as done in Ref.\cite{16,17} for two-dimensional fractional statistics. Let’s consider a one-dimensional $N$-body system with Lagrangian $L = L_0 + L_s$. Here $L_s$ is constructed as

$$L_s = \frac{\hbar \kappa}{2} d \frac{d}{dt} \left[ \sum_{i<j} \epsilon(x_i - x_j) + l(x_1, ..., x_N) \right],$$

(5)

where the $l(x_1, ..., x_N)$ is a function whose contribution to the path integral is only determined by the permutation of the coordinates, as well as the function $\sum_{i<j} \epsilon(x_i - x_j)$. We will show below that this form of $L_s$ is the statistical interaction term for 1D anyons. For Kundu’s anyons, $l(x_1, ..., x_N)$ can be any constant or zero. And for Girardeau’s anyons, we make $l(x_1, ..., x_N) = \frac{\pi}{\kappa} A(x_1, ..., x_N)$, where $A(x_1, ..., x_N) = \prod_{i<j} \text{sgn}(x_i - x_j)$ is just the mapping function introduced by Girardeau for the theory of Fermi-Bose mapping\cite{18}, and takes the value $\pm 1$ according to the parity of the permutation of the coordinates. Theoretically, other forms of $l(x_1, ..., x_N)$ is also allowed, which means that there maybe exist some other kinds of anyons besides Kundu’s and Girardeau’s.

According to Feynman’s path integral formalism\cite{15}, the transition amplitude from $(x, t)$ to $(x', t')$ in the configuration space is given by\cite{13}

$$\langle x', t' \mid x, t \rangle \propto \sum_{\text{all paths}} \exp \left( \frac{i}{\hbar} \int_{t}^{t'} L dt \right),$$

(6)

where $x$ is an initial configuration point with fixed permutation of the particles, and $x'$ is the same configuration point, however, with an uncertain permutation of the particles, because of the indistinguishability of the particles during the evolvement, which means that $x'$ can be $S x$ for any element $S$ in the permutation group $S_N$. Without any loss of generality, we can make $x = (x_1, x_2, ..., x_N)$.

As we have stated above, the contribution of $L_s$ to the the path integral, i.e., the right side of Eq.(6) is only determined by the final permutation of the coordinates of the particles, so Eq.(6) can be rewritten as a weighted sum over all the permutations, following the formulation given in\cite{17}:

$$\langle x', t' \mid x, t \rangle = \sum_{S \in S_N} \chi(S, x) \sum_{\text{path} \in S} \exp \left( \frac{i}{\hbar} \int_{t}^{t'} L dt \right),$$

(7)

where $\sum_{\text{path} \in S}$ means the summation takes over all the paths leading to the same permuta-
tion $Sx$, and the weight factor is

$$\chi(S,x) = \exp \left( \frac{i}{\hbar} \int t' L_s dt' \right)$$

$$= \exp \left[ i\frac{\kappa}{2} \sum_{i<j} \epsilon(x_i - x_j)|Sx|^{Sx} + i\kappa l(x_1, ..., x_N)|Sx|^{Sx} \right].$$

(8)

From Eq.(7), it looks as if that the permutation group $S_N$ is isomorphic to the first homotopy group of the configuration space of this 1D $N$-body system. But it is not the fact, because in the concept of homotopy group, the paths are classified into homotopy classes by whether they can be deformed into each other, but here all the paths are classified by whether they can lead to the same permutation. However, it is clear that the permutation group $S_N$ takes the equivalent role in this one-dimensional case as the homotopy group in two-dimensional case[17], of course under the condition of a given permutation $x$ of the particles’ coordinates as the initial configuration. So we hope that $\chi(S,x)$ be a phase factor and equivalent to $F_S(|\{x\}|)$, provided $x = \{x\}$ as an initial permutation, i.e.:

$$\chi(S,x) = F_S(|\{x\}|).$$

(9)

It is not difficult to check the correctness of this equation. In fact, this is the original idea of our construction of the statistical term $L_s$. Here, we give a brief explanation of Eq.(9) in the following. For Kundu’s anyons, we let $l(x_1, ..., x_N)$ be any constant or zero. Inspecting the Eq.(8) tell us that if two particles with initial coordinates, e.g. $x_i, x_j$ with $i < j$, are not exchanged during the process to the new permutation $Sx$, they will contribute nothing to $\chi(S,x)$. But if they are exchanged, their contribution will be $\exp{i\kappa l(x_i - x_j)}$. This coincides with our rule for writing the phase factor $F_S(|\{x\}|)$. While for Girardeau’s anyons, we take $l(x_1, ..., x_N) = \frac{\kappa}{A}(x_1, ..., x_N)$, which is responsible for the factor $(-1)^n$ in the right side of Eq.(4), because the number of crossing points $n$ can be replaced by the parity of the permutation.

Subsequently, let’s consider the quantum mechanics of 1D anyons following the formalism in Ref.[19] for 2D anyons. For simplicity, we only present the formulation for Kundu’s anyons in the following. The formulation for other anyons can be obtained in the same way by considering the additional term $l(x_1, ..., x_N)$. The Lagrangian for ordinary bosons can be written as:

$$L_0 = \frac{m}{2} \sum_{i=1}^{N} \dot{x}_i^2 - V(x_1, ..., x_N).$$

(10)
The properties of Kundu’s anyons can be obtained if the statistical term $L_s$ is included:

$$L = L_0 + \hbar \kappa \sum_{i<j} \delta(x_i - x_j)(\dot{x}_i - \dot{x}_j), \quad (11)$$

where the use of identity $\epsilon(x_i - x_j) = \theta(x_i - x_j) - \theta(x_j - x_i)$ has been made. $\theta(x)$ is the Heaviside’s step function. The conjugate momentum is

$$p_k = mx_k + eA_k, \quad (12)$$

where

$$eA_k = \hbar \kappa \left[ \sum_{j=k+1}^{N} \delta(x_k - x_j) - \sum_{j=1}^{k-1} \delta(x_j - x_k) \right] \quad (13)$$

is introduced for brevity. But soon, we will see that $eA_k$ has further significance: the parameter $e$ can be regarded as a charge, and $A_k$ can be regarded as an auxiliary field potential like the Chern-Simons field potential in two dimensions (see the Hamiltonian below). Using the Legendre transformation $H = \sum_k p_k \dot{x}_k - L$, and after a straightforward derivation, one can finally obtain the Hamiltonian in a compact form:

$$H = \frac{1}{2m} \sum_k (p_k - eA_k)^2 + V(x_1, \ldots, x_N), \quad (14)$$

The first quantization form of which can be obtained just by taking $p_k \rightarrow -i\hbar \partial_{x_k}$. It is the Hamiltonian of a flux-carrying bosons system with complicated interaction, however, it also can be regarded as a free anyon system apart from the interaction $V(x_1, \ldots, x_n)$. This can be quickly seen if we operate a phase transformation on the Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m} \sum_k (\partial_{x_k} - \frac{ie}{\hbar}A_k)^2 + V(x_1, \ldots, x_N) \right] \Psi(x_1, \ldots, x_N) = E \Psi(x_1, \ldots, x_N). \quad (15)$$

The phase transformation is

$$\tilde{\Psi} = e^{ief(x_1, \ldots, x_N)/\hbar} \Psi, \quad (16)$$

where $f(x_1, \ldots, x_N)$ is requested to satisfy $\partial_{x_k} f = -A_k$, and can be chosen as

$$f(x_1, \ldots, x_N) = -\frac{\hbar \kappa}{2e} \sum_{i<j} \epsilon(x_i - x_j). \quad (17)$$

Consequently the transformed Hamiltonian is

$$\tilde{H} = -\frac{\hbar^2}{2m} \sum_k \partial_{x_k}^2 + V(x_1, \ldots, x_N), \quad (18)$$

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which represents a pure 1D anyons system. The corresponding wave function $\tilde{\Psi}$ has anyonic exchange symmetry. Because $\Psi$ is a Bose wave function, the phase transformation Eq. (16) is just the anyon-boson mapping presented in Kundu’s paper [10]. We note here that in an earlier work [11], the attempt to establish this kind of statistical transmutation failed, because the Hamiltonian there didn’t include the potential like $A_k$ here.

For Girardeau’s anyons, because of the intrinsic property of exclusion principle, the interaction term including $\delta$ function must vanish, which make the formation of Hamiltonian be of no difference from the Bose or Fermi one [1]. So the transformation of the Hamiltonian is a trivial problem, knowing the formulation of anyon-fermion (or boson) mapping is enough for statistical transmutation. If $\Psi$ is a Fermi (or Bose) wave function, the phase transformation Eq. (16) is just the anyon-fermion (or boson) mapping for Girardeau’s anyons [2].

In conclusion, we have obtained a theoretical construction for 1D anyon models based on a form of statistical interaction term. According to the path integral formalism, this interaction term can successfully recover the anyonic exchange symmetry of the known anyons defined by Kundu and Girardeau. Furthermore, it theoretically provide a unitive description for Kundu’s and Girardeau’s, and perhaps for more extensive 1D anyon models beyond them. The statistical transmutation in quantum mechanics formalism are also presented, the known anyon-fermion (or boson) mapping relations are successfully recovered.

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