Abstract

We prove an Ornstein-Weiss lemma for amenable unimodular groups containing a uniform lattice and show that averages along Van Hove nets can be obtained by averaging inside the lattice. We use this result to introduce relative topological entropy for actions of amenable unimodular groups that contain a uniform lattice and show that Bowens formula for relative topological entropy is satisfied.

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1 Introduction

In the context of aperiodic order, actions of the euclidean space $\mathbb{R}^d$, or more generally $\sigma$-compact locally compact abelian groups, on compact Hausdorff spaces play a central role. Nevertheless the theory of topological entropy is only developed for actions of countable discrete amenable groups, but important results like the Bowen formula about relative topological entropy are only shown for actions of continuous maps. Wherever such results are needed ad hoc solutions take the place of a general theory.

Several authors, like [BLR07] or [JLO16], define topological entropy of actions of $\mathbb{R}^d$ on compact metric spaces in terms of separated sets and the Bowen metric. The respective averages are taken with respect to the sequence of closed balls $B_n$ of radius $n \in \mathbb{N}$ or closed cubes $C_n$ of side length $2^n$. This approach goes back to [TZ91], which is up to our knowledge the only reference for a systematic treatment of entropy theory of actions of non-discrete groups. Note that [TZ91] only considers the case of $\mathbb{R}^d$ actions and does not consider notions like relative topological entropy with respect to a factor map, i.e. the maximal entropy contained in some fiber. In [FGJO18] relative topological entropy of actions of non-compact, locally compact second countable abelian groups is introduced by averaging along Van Hove sequences, but the question of the dependence of the averages on the Van Hove sequences is left open. We show that there is no such dependence.

In [JLO16] Remark 2.18 a version of Bowens entropy formula for actions of $\mathbb{R}^d$ is used. This formula states that the topological entropy of an action is smaller than the sum of the topological entropy of a factor and the relative topological entropy. This is shown in [Bow71] Theorem 17 for the action induced by a continuous map but open for actions of groups beyond $\mathbb{Z}$ as stated in [FGJO18] Remark 2.9. We will show that this formula holds for all amenable non-unimodular actions on compact Hausdorff spaces that contain a countable lattice and thus in particular for actions of $\mathbb{R}^d$.

Different authors considered topological entropy for actions of countable discrete amenable groups. See for example [OW87] [HYZI0] [Yan13] [ZC16] [Zho16]. These definitions are independent from the choice of the chosen Van Hove sequence. The main tool to establish this is the so called Ornstein-Weiss lemma, which goes back to [OW87]. In [Gro99] an idea of a proof was presented. In [Kri10] it is shown that this proof works for discrete amenable groups. It seems as it does not work in the non-discrete case without a boundedness assumption, which is needed in the last part of the proof and follows trivially in the discrete
case. Note that in [CSCK14] a version of the Ornstein-Weiss lemma for discrete semi-groups is given.

It is thus naturally to ask, whether there is a version of the Ornstein-Weiss lemma for non-discrete groups, like for example \( \mathbb{R}^d \). In Section 5 we will present a proof for amenable unimodular groups that contain a uniform lattice. This proof uses that the Ornstein-Weiss lemma holds true inside the uniform lattice and extends to the whole group. From this proof we also see that the average along a Van Hove net with respect to the whole group can also be obtained by averaging inside the uniform lattice. Thus entropy of actions of \( \mathbb{R}^d \) or the continuous Heisenberg group can be studied by the restriction to \( \mathbb{Z}^d \) and the discrete Heisenberg group respectively. This generalizes the idea that entropy of \( \mathbb{R} \) actions can be studied by restricting to the uniform lattice \( \mathbb{Z} \).

With a version of the Ornstein-Weiss lemma at hand we define relative topological entropy for actions of several non-discrete groups in Section 4. To underline that our treatment directly generalizes to other amenable unimodular groups, we introduce the notion of "Ornstein-Weiss groups", which are the groups in which a suitable version of the Ornstein-Weiss lemma holds true. In order to avoid a unnecessary restriction to metric spaces, we consider compact uniform spaces as phase spaces, following ideas from [YZ16, Hoo74, DSV12]. As an application of the independence of the definition of relative topological entropy from the choice of a Van-Hove net we will present that the approach to relative topological entropy simplifies for positive expanding systems similarly to the case of actions of continuous maps, like considered in [BS02, Proposition 2.5.7]. The link to the notions of topological entropy for metric spaces and in [YZ16] will be given in Section 5.

In this work we do not intend to study (relative) measure theoretic entropy. Nevertheless we will introduce this notion for discrete groups, as studied in [Yan15], in order to show a version of the variational principle in Section 6. In particular we will show that relative topological entropy can be obtained as the supremum over all relative measure theoretic entropies, calculated in some uniform lattice and with respect to some invariant measure, where the invariance only relates to the lattice. We then combine this with a version of the Rokhlin-Abramov theorem for countable discrete groups, as shown in [Yan15], to obtain the Bowen formula for all amenable unimodular groups that contain a countable uniform lattice.

2 Preliminaries

In this section we provide notion and background on topological groups, uniform spaces, topological dynamical systems, amenable groups, Van Hove nets, uniform lattices and Ornstein-Weiss groups.

2.1 Topological groups

Consider a group \( G \). We write \( e_G \) for the neutral element in \( G \). For subsets \( A, B \subseteq G \) the Minkowski product is defined as \( AB := \{ab; (a, b) \in A \times B\} \). For \( A \subseteq G \) and \( g \in G \) one also writes \( Ag := \{a \mid g \} \) and \( gA := \{g \mid A \} \). Furthermore we define the complement \( A^c := G \setminus A \) and the inverse \( A^{-1} := \{a^{-1}; a \in A\} \) of a subset \( A \subseteq G \). We call a set \( A \subseteq G \) symmetric, if \( A = A^{-1} \). In order to omit brackets, we will use the convention, that the operation of taking the Minkowski product of two sets is stronger binding than set theoretic operations, except from taking the complement; and that the inverse and the complement are stronger binding than the Minkowski product. Note that the complement and the inverse commute, i.e. \((A^c)^{-1} = (A^{-1})^c\) for any \( A \subseteq G \).

A topological group is a group \( G \) equipped with a \( T_2 \)-topology, such that the multiplication \( \cdot : G \times G \to G \) and the inverse function \((\cdot)^{-1} : G \to G \) are continuous. With our definition every topological group is regular, hence Hausdorff, as shown in [HR12 Theorem 4.8]. An isomorphism of topological groups is a homeomorphism that is a group homomor-

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\[ A \text{ is a group if for any two distinct points } g, g' \in G \text{ there is an open neighbourhood of } g \text{ that does not contain } g'. \]
phism as well. We write \( \overline{A} \) for the closure and \( \text{int}(A) \) for the interior of a subset \( A \subseteq G \). By \( \mathcal{K}(G) \) we denote the set of all non-empty compact subsets of \( G \).

If \( G \) is a locally compact group, a left (resp. right) Haar measure on \( G \) is a non zero regular Borel measure \( \mu \) on \( G \), which satisfies \( \mu(gA) = \mu(A) \) (resp. \( \mu(Ag) = \mu(A) \)) for all \( g \in G \) and all Borel measurable subsets \( A \subseteq G \). We call a measure on \( G \) a Haar measure, if it is a right and a left Haar measure. A locally compact group that possesses a Haar measure is called a unimodular group. There is \( \mu(U) > 0 \) for all non empty open \( U \subseteq G \) and \( \mu(K) < \infty \) for all compact \( K \subseteq G \). A Haar measure is unique up to scaling, i.e. if \( \mu \) and \( \nu \) are Haar measures on \( G \), then there is \( c > 0 \) such that \( \mu(A) = cv(A) \) for all Borel measurable sets \( A \subseteq G \). If nothing else is mentioned, we denote a Haar measure of a topological group \( G \) by \( \mu \).

If \( G \) is a discrete group, then the counting measure, denoted by \( | \cdot | \), is a Haar measure on \( G \). We will always equip discrete groups with this Haar measure, i.e. scale such that \( |\{g\}| = 1 \) for some (and hence all) \( g \in G \). Another class of examples of unimodular groups are locally compact abelian groups or the Heisenberg group, as presented in subsection 2.2.2 below. For further reference see [Fol13].

2.2 Uniform spaces

2.2.1 Binary relations

Let \( X \) be a set. A binary relation on \( X \) is a subset of \( X \times X \). For binary relations \( \eta \) and \( \kappa \) on \( X \) we denote the inverse \( \eta^{-1} : = \{(y,x); (x,y) \in \eta \} \) and the composition \( \eta \kappa : = \{(x,y); \exists z \in X : (x,z) \in \eta \text{ and } (z,y) \in \kappa \} \). A binary relation is called symmetric, if \( \eta = \eta^{-1} \).

A binary relation is called symmetric, if \( \eta = \eta^{-1} \). For \( \eta \subseteq X \times X \) and \( x \in X \) we write \([x]_{\eta} := \{ y \in X; (x,y) \in \eta \}\) and \( \eta[x] := [x]_{\eta^{-1}} \). For \( M \subseteq X \) we denote \([M]_{\eta} := \bigcup_{x \in M} [x]_{\eta} \) and \( \eta[M] := [M]_{\eta^{-1}} \).

2.2.2 Uniform spaces

A uniformity for a set \( X \) is a non-empty family \( \mathcal{U}_X \) of subsets of \( X \times X \) such that

(a) each member of \( \mathcal{U}_X \) contains the diagonal \( \Delta_X \);

(b) if \( \eta \in \mathcal{U}_X \), then \( \eta^{-1} \in \mathcal{U}_X \);

(c) if \( \eta \in \mathcal{U}_X \), then there is \( \kappa \in \mathcal{U}_X \) such that \( \kappa \kappa \subseteq \eta \);

(d) if \( \eta \) and \( \kappa \) are members of \( \mathcal{U}_X \), then so is \( \eta \cap \kappa \); and

(e) if \( \eta \in \mathcal{U}_X \) and \( \eta \subseteq \kappa \subseteq X \times X \), then \( \kappa \in \mathcal{U}_X \).

The pair \( (X, \mathcal{U}_X) \) is called a uniform space and the members of \( \mathcal{U}_X \) are called entourages. An entourage \( \eta \in \mathcal{U}_X \) is called open (or closed), whenever it is open (or closed) with respect to the product topology on \( X \times X \). A subfamily \( \mathcal{B}_X \subseteq \mathcal{U}_X \) is called a base for \( \mathcal{U}_X \), if every entourage contains a member of \( \mathcal{B}_X \). The family of all open and symmetric entourages form a base of the corresponding uniform space. If \( (X, \mathcal{U}_X) \) is a uniform space the corresponding uniform topology \( T_X \) consists of all subsets \( U \subseteq X \) such that for each \( x \in U \) there exists \( \eta \in \mathcal{U}_X \) with \( \eta[x] \subseteq U \). Topological terminology in the context of uniform spaces refers to this topology.

For a map \( f : X \to Y \) we write \( f \times f : X \times X \to Y \times Y \) for the map with \( (f \times f)(x,y) := (f(x), f(y)) \). A map \( f : X \to Y \) between uniform spaces \( (X, \mathcal{U}_X) \) and \( (Y, \mathcal{U}_Y) \) is called uniformly continuous, if the preimage of every entourage of \( Y \) under \( f \times f \) is an entourage of \( X \). Every uniformly continuous map between uniform spaces is continuous with respect to the corresponding uniform topologies. The reverse holds true, whenever the domain of the map is assumed to be compact, as shown in [Kel17, Theorem 6.31]. For further notions on uniform spaces, like the product of uniform spaces, see [Kel17].

Example 2.1. (i) If \( (X, d) \) is a metric space we define for \( \varepsilon > 0 \)

\[
[d < \varepsilon] := \{(x, y) \in X \times X; d(x, y) < \varepsilon\}.
\]
Then $B_d := \{ [d < \varepsilon]; \varepsilon > 0 \}$ is a base for the uniformity

$$\mathbb{U}_X := \{ \eta \subseteq X \times X; \exists \varepsilon > 0 : [d < \varepsilon] \subseteq \eta \}.$$  

The corresponding topology is the topology of open sets with respect to $d$.

(ii) Every compact Hausdorff space $X$ has a unique uniformity $\mathbb{U}_X$ consisting of all neighbourhoods of the diagonal $\Delta_X$ in $X \times X$. This can be obtained from the combination of [Kel17, Theorem 6.22] with [Mun00, Theorem 32.3].

For $\eta \in \mathbb{U}_X$ and $(x, y) \in \eta$, we say that $x$ is $\eta$-close to $y$. This notion is symmetric, if and only if $\eta$ is symmetric. If $x$ is $\eta$-close to $y$ and $y$ is $\kappa$-close to $z$, then $x$ is $\eta \kappa$-close to $z$. If $(X, d)$ is a metric space, then $x$ is $[d < \varepsilon]$-close to $y$, if and only if $d(x, y) < \varepsilon$.

### 2.3 Actions of a group on a topological space

Let $G$ be a topological group and $X$ be a topological space. A continuous map $\pi: G \times X \to X$ is called an action of $G$ on $X$ (also dynamical system or flow), whenever $\pi(e_G, \cdot)$ is the identity on $X$ and for all $g, g' \in G$ there holds $\pi(g, \pi(g', \cdot)) = \pi(gg', \cdot)$. We write $\pi^g := \pi(g, \cdot): \pi \to X$ for all $g \in G$. If $\pi$ and $\varphi$ are actions of a topological group $G$ on topological spaces $X$ and $Y$ respectively, we call a surjective continuous map $p: X \to Y$ a factor map, if $p \circ \varphi^g = \varphi^g \circ p$ for all $g \in G$. We then refer to $\varphi$ as a factor of $\pi$ and write $\pi \xrightarrow{\sim} \varphi$. If $p$ is in addition a homeomorphism, then $p$ is called a topological conjugacy and we call $\pi$ and $\varphi$ topological conjugate.

### 2.4 Amenable groups and Van Hove nets

#### 2.4.1 Nets and convergence of nets

A partially ordered set $(I, \geq)$ is said to be directed, if $I$ is not empty and if every finite subset of $I$ has an upper bound. A map $f$ from a directed set $I$ to a set $X$ is called a net in $X$. We also write $x_i$ for $f(i)$ and $(x_i)_{i \in I}$ for $f$. A net $(x_i)_{i \in I}$ in a topological space $X$ is said to converge to $x \in X$, if for every open neighbourhood $U$ of $x$, there exists $j \in I$ such that $x_i \in U$ for all $i \geq j$. In this case we also write $\lim_{i \in I} x_i = x$.

For a net $(x_i)_{i \in I}$ in $\mathbb{R} \cup \{-\infty, \infty\}$, we define $\limsup_{i \in I} x_i := \sup_{i \geq j} x_i$ and similarly $\liminf_{i \in I} x_i$. Note that $(x_i)_{i \in I}$ converges to $x \in \mathbb{R} \cup \{-\infty, \infty\}$, if and only if there holds $\limsup_{i \in I} x_i = x = \liminf_{i \in I} x_i$. For more details, see [DS58] and [Kel17].

#### 2.4.2 Van Hove nets

Let $G$ be a unimodular group. For $K, A \subseteq G$ we define the $K$-boundary of $A$ as

$$\partial_K A := K \overline{A} \cap \overline{K A}.$$  

We use the convention, that the Minkowski product is stronger binding as the operation of taking the $K$-boundary and that the set theoretic operations, except from complementation, are weaker binding. From the definition we obtain that $K \mapsto \partial_K A$ is monotone. Note that $\partial_K A$ is the set of all elements $g \in G$ such that $K^{-1} g$ intersects both $\overline{A}$ and $\overline{K A}$.

**Lemma 2.2.** For $K, L, A \subseteq G$ compact there holds

(i) $L \partial_K A \subseteq \partial_{L K} A$ and $\partial_K L A \subseteq \partial_{K L} A$.

(ii) $L A \subseteq A \cup \partial_L A$, whenever $e_G \in L$.

**Proof.** Straight forward arguments show the first statement in (i) and (ii). To see $\partial_K L A \subseteq \partial_{K L} A$ we compute $(L A)^c \subseteq (LA)^c = \overline{L A} \subseteq \overline{K L A}$ for any $l \in L$ and obtain $\partial_K L A \subseteq K L A \cap K(\overline{L A})^c \subseteq K \overline{L A} \cap K \overline{L A} = \partial_{K L} A$. \hfill $\square$
A net \((A_i)_{i \in I}\) of measurable subsets of \(G\) is called finally somewhere dense, if there is \(j \in I\) such that for all \(i \geq j\) the set \(A_i\) is somewhere dense\(^2\). A finally somewhere dense net \((A_i)_{i \in I}\) of compact subsets of \(G\) is called a Van Hove net, if for all \(K \subseteq G\) compact, there holds

\[
\lim_{i \in I} \frac{\mu(\partial K A_i)}{\mu(A_i)} = 0.
\]

**Proposition 2.3.** Let \(K, C \subseteq G\) be compact sets and \((A_i)_{i \in I}\) be a Van Hove net in \(G\). Then \((K A_i)_{i \in I}\) and \((C A_i)_{i \in I}\) are Van Hove nets and satisfy \(\lim_{i \in I} \frac{\mu(K A_i)}{\mu(A_i)} = 1\).

**Proof.** Let \(L \subseteq G\) be compact. As \(L K\) is compact, we obtain

\[
0 \leq \frac{\mu(\partial L K A_i)}{\mu(L K A_i)} \leq \frac{\mu(\partial L K A_i)}{\mu(A_i)} \rightarrow 0.
\]

This proofs \((K A_i)_{i \in I}\) to be a Van Hove net. To show \(\lim_{i \in I} \frac{\mu(K A_i)}{\mu(A_i)} = 1\) let \(k \in K^{-1}\) and note that \(e_G \in kK\). Hence \(k K A_i \subseteq A_i \cup \partial_k K A_i\) and

\[
1 \leq \frac{\mu(K A_i)}{\mu(A_i)} = \frac{\mu(k K A_i)}{\mu(A_i)} \leq 1 + \frac{\mu(\partial_k K A_i)}{\mu(A_i)} \rightarrow 1.
\]

Similarly one obtains \((C A_i)_{i \in I}\) to be a Van Hove net and \(\lim_{i \in I} \frac{\mu(A_i)}{\mu(A_i)} = 1\), hence

\[
\lim_{i \in I} \frac{\mu(K A_i)}{\mu(C A_i)} = \left( \lim_{i \in I} \frac{\mu(K A_i)}{\mu(A_i)} \right) \left( \lim_{i \in I} \frac{\mu(A_i)}{\mu(C A_i)} \right) = 1.
\]

\[\square\]

### 2.4.3 Van Hove nets in the literature

We will now link the definitions of Van Hove nets given in [Tem], in [Sch99] and in [FGJO18]. In [PS16] the same definition as our definition of \(K\)-boundary is considered and Van Hove sequences are called "strong Folner sequences".

**Lemma 2.4.** A finally somewhere dense net \((A_i)_{i \in I}\) of compact sets is a Van Hove net if and only if for all symmetric compact sets \(K\) that contain \(e_G\) there holds

\[
\lim_{i \in I} \frac{\mu(\partial K A_i)}{\mu(A_i)} = 0.
\]

**Proof.** To show the non trivial direction, let \(K \subseteq G\) be an arbitrary non empty compact subset and choose \(k \in K\). Then \(K \subseteq k K^{-1} K\), and hence \(\partial K A_i \subseteq \partial_k K^{-1} A_i = k(\partial K^{-1} A_i)\). As \(K^{-1}\) is symmetric and compact we get

\[
0 \leq \frac{\mu(\partial K A_i)}{\mu(A_i)} \leq \frac{\mu(k(\partial K^{-1} A_i))}{\mu(A_i)} = \frac{\mu(\partial K^{-1} A_i)}{\mu(A_i)} \rightarrow 0.
\]

\[\square\]

The next proposition shows that the definitions of the \(K\)-boundary given above; in [Tem] in [Sch99] and in in [FGJO18], coincide, whenever \(e_G \in K = K^{-1}\). As all definitions are monotone in \(K\), we can adapt the proof of Lemma 2.4 to see that all definitions of \(K\)-boundary yield equivalent definitions of Van Hove nets.

**Proposition 2.5.** For \(K \subseteq G\) compact and \(A \subseteq G\) there holds

\[
\partial K A = \overline{K A \setminus \text{int} \left( \bigcap_{k \in K} k A \right)}.
\]

If we assume in addition \(e_G \in K = K^{-1}\), then there holds

\[
\partial K A = \left((\overline{K A}) \setminus \text{int}(A)\right) \cup \left((K^{-1} A) \setminus \text{int}(A)^c\right).
\]

\(^2\)A subset of a topological space is called somewhere dense, if it has nonempty interior. This ensures \(\mu(A_i) > 0\).

\(^3\)Note that in [Tem] the order of multiplication is inverse to our notation.
Proof. We have
\[
\left( \text{int} \left( \bigcap_{k \in K} kA \right) \right)^c = \bigcup_{k \in K} (kA)^c = KA^c.
\]
Thus, as \( K \) is compact, there holds
\[
\partial_K A = K \bar{A} \cap \bar{K} A = K \bar{A} \cap K A^c = \bar{K} A \backslash \left( \text{int} \left( \bigcap_{k \in K} kA \right) \right).
\]
To see the second equality note \( \bar{A} \subseteq K \bar{A} \) and \( A^c \subseteq K A^c \) and calculate
\[
\partial_K A = K \bar{A} \cap K A^c
= G \cap (K \bar{A} \cup \bar{A}) \cap (\bar{A} \cup K A^c) \cap G
= (K \bar{A} \cup K A^c) \cap (K \bar{A} \cup \bar{A}) \cap (\bar{A} \cup K A^c) \cap (\bar{A} \cup \bar{A})
= (\partial W A \backslash \text{int}(A)) \cup ((K \bar{A}) \backslash \text{int}(A^c))
= ((K \bar{A}) \backslash \text{int}(A)) \cup ((K^{-1} \bar{A}) \backslash \text{int}(A^c)).
\]

2.4.4 Følner nets and Van Hove nets

We call a finally somewhere dense net \((A_i)_{i \in I}\) a Følner net, if for every \( g \in G \) there holds
\[
\lim_{i \in I} \frac{\mu(gA_i \Delta A_i)}{\mu(A_i)} = 0,
\]
where \( A \Delta B := (A \setminus B) \cup (B \setminus A) \) is the symmetric difference for \( A,B \subseteq G \). We obtain the link between Følner nets and Van Hove nets from [Tem, Appendix; (3.K)] as follows. Note that Følner nets are called "left ergodic nets" in [Tem].

**Proposition 2.6.** A net \((A_i)_{i \in I}\) is a Van Hove net, if and only if it is a Følner net and satisfies
\[
\lim_{i \in I} \frac{\mu(\partial W A_i)}{\mu(A_i)} = 0
\]
for some neighbourhood \( W \) of \( e_G \).

From this we obtain that every Van Hove net is a Følner net. Furthermore in discrete locally compact groups \( W := \{e_g\} \) is open and there holds \( \partial W A = A \cap A^c = \emptyset \) for all compact \( A \subseteq G \). Thus the notion of Van Hove nets and Følner nets agree for discrete locally compact groups. Note that our definition of \( K \)-boundary and of Van Hove nets is inspired from [Kri10], where it is used to define Følner nets in discrete amenable groups. In [Tem, Appendix; Example 3.4] an example of a Følner net in \( \mathbb{R}^d \) is presented, that is not a Van Hove net.

2.4.5 Amenable groups

It is shown for \( \sigma \)-compact locally compact groups in [Tem, Appendix 3.L] and for second countable unimodular groups in [PS16 Lemma 2.7.] that the existence of a Van Hove net is equivalent to the existence of a Følner net. As it seems open, whether this holds without a countability assumption, we give a proof below. A unimodular group is called amenable, if one of the equivalent conditions in Proposition 2.8 is satisfied.

**Remark 2.7.** Note that the following proposition also shows that our definition is equivalent to the definition of amenability in the monograph [Pie84]. In order to see this compare (iii) with [Pie84] Theorem 7.3(2) in combination with [Pie84] Proposition 7.4. Furthermore it implies that the notion of "left-amenability" in [Tem] is equivalent to our notion of amenability. Examples of amenable groups can be found in [Pie84 Section 12]. Thus all groups in Example 2.10 are amenable.
Proposition 2.8. For a unimodular group $G$ the following statements are equivalent.

(i) $G$ contains a Van Hove net.

(ii) $G$ contains a Følner net.

(iii) For all $\varepsilon > 0$ and all finite $F \subseteq G$ there exists a compact set $A \subseteq G$ such that $\frac{\mu(F \backslash A)}{\mu(A)} < \varepsilon$ for all $f \in F$.

(iv) For all $\varepsilon > 0$ and all compact $K \subseteq G$ with $e_G \in K$ there exists a compact set $A \subseteq G$ such that $\frac{\mu(K \backslash A)}{\mu(A)} < \varepsilon$.

Proof. From Proposition 2.6 we obtain that (i) implies (ii). Assume (ii) and let $(A_i)_{i \in I}$ be a Følner net in $G$. For $F \subseteq G$ finite and $\varepsilon > 0$ there is $i \in I$ such that

$$\frac{\mu(fA_i \setminus A_i)}{\mu(A_i)} \leq \frac{\mu(fA_i \Delta A_i)}{\mu(A_i)} < \varepsilon$$

for all $f \in F$ and (iii) holds. Combining [Pie84, Theorem 7.3(F*)], [Pie84] Proposition 7.4, [Pie84] Theorem 7.9(SF$_*$) and [Pie84] Proposition 7.11 we obtain that (iii) is equivalent to (iv).

It remains to show that (iv) implies (i). If $G$ is compact, then $(G)_{n \in \mathbb{N}}$ is a Van Hove net in $G$. We can thus assume $G$ to be not compact. As $G$ is locally compact there is a compact neighbourhood $W$ of $e_G$ with $W = W^{-1}$. Let $I$ be the set of all finite subsets of $G$ containing $e_G$, ordered by set inclusion. For $i \in I$ define $K_i := \bigcup_{g \in W} gA_i$. We obtain $K_j \subseteq K_i$, whenever $j \leq i$. As $\{W; g \in G\}$ is an open cover of $G$ for all $K \subseteq G$ compact there is $i \in I$ with $K \subseteq \bigcup_{g \in W} gA_i = K_i$. Thus there holds $\lim_{i \to \infty} \mu(K_i) = \infty$, as $\mu$ is regular and $G$ is not compact.

By (iv) for every $i \in I$ there exists a compact set $B_i \subseteq G$ such that

$$\frac{\mu(K_i B_i \setminus B_i)}{\mu(B_i)} < \frac{1}{\mu(K_i)}.$$  \hfill (1)

Let $A_i := WWB_i$ for all $i \in I$. To show that $(A_i)_{i \in I}$ is a Van Hove net it is by Proposition 2.6 sufficient to show that $(A_i)_{i \in I}$ is a Følner net and that $\lim_{i \to \infty} \frac{\mu(\partial_W A_i)}{\mu(A_i)} = 0$. To obtain that it is a Følner net take $j \in I$, such that $gWW \cup g^{-1}WW \subseteq K_i$ for $i \geq j$. For $i \geq j$ we get $gA_i \cup g^{-1}A_i \subseteq K_iB_i$ and thus by Equation (1)

$$\mu(gA_i \Delta A_i) \leq \mu((gA_i \setminus A_i) \cup (A_i \setminus gA_i)) \leq \mu(gA_i \setminus A_i) + (A_i \setminus gA_i) = \mu(gA_i \setminus A_i) + \mu(g^{-1}A_i \setminus A_i) \leq 2\mu(K_i B_i \setminus B_i) \leq \frac{2\mu(B_i)}{\mu(K_i)} \leq \frac{2\mu(A_i)}{\mu(K_i)}.$$

To show $\lim_{i \to \infty} \frac{\mu(\partial_W A_i)}{\mu(A_i)} = 0$, note first that $WW(WWB_i)^c \subseteq B_i^c$, as $WW = (WW)^c$. Let now $j \in I$ such that $WWW \subseteq K_j$ for every $i \geq j$. Using Equation (1) we compute

$$\mu(\partial_W A_i) = \mu(WWWB_i \cap W(WWB_i)^c) \leq \mu(K_i B_i \cap WW(WWB_i)^c) \leq \mu(K_i B_i \setminus B_i) \leq \frac{\mu(B_i)}{\mu(K_i)} \leq \frac{\mu(A_i)}{\mu(K_i)}.$$

This shows that (iv) implies (i). \hfill $\square$

2.5 Uniform lattices in locally compact topological groups

Let $G$ be a locally compact topological group. A discrete subgroup $\Lambda \subseteq G$ is called a uniform lattice, if there is a pre-compact and Borel measurable $C$ that contains $e_G$ and satisfies $0 < \mu(C)$ such that each $g \in G$ can be written uniquely as $g = cz$ with $c \in C$ and $z \in \Lambda$. The set $C$ is called a fundamental domain for $\Lambda$ and satisfies $0 < \mu(C) \leq \mu(C) < \infty$. Our definition of uniform lattices implies the quotient $G/\Lambda$ to be compact, i.e. $\Lambda$ to be a uniform lattice in the notion of [DE14]. Thus by [DE14] Theorem 9.1.6) every locally compact group that contains a uniform lattice is unimodular.

A subset $A$ of a topological space $X$ is called pre-compact, whenever the closure $\overline{A}$ is compact in $X$.  

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Remark 2.9. If $G$ is a unimodular group and $F \subseteq \Lambda$ is finite, then the finite union $\bigcup_{z \in F} Cz$ is disjoint and measurable. Thus by the right invariance of the Haar measure there holds

$$\mu(CF) = \sum_{z \in F} \mu(Cz) = \mu(C)|F|.$$ 

Example 2.10. (i) Every compact group $G$ contains the countable uniform lattice $\{e_G\}$ with fundamental domain $G$.

(ii) The Euclidean space $\mathbb{R}^d$ contains the countable uniform lattice $\mathbb{Z}^d$ with fundamental domain $[0,1)^d$.

(iii) Every compactly generated locally compact abelian group $G$ contains a countable uniform lattice. This follows as such a group is isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times H$ for some compact abelian group $H$ and some nonnegative integers $a$ and $b$. For a reference see [HR12, Theorem 9.8].

(iv) The Heisenberg group

$$H_3(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbb{R} \right\}$$

under matrix multiplication is a non-abelian amenable group and contains the uniform lattice $H_3(\mathbb{Z})$ with fundamental domain $H_3([0,1))$. For reference see [EFHN15, Example 2.13] and [Run04, Exercise 1.2.4].

3 Ornstein-Weiss groups

The Ornstein-Weiss lemma is the key tool in order to define entropy for amenable groups. We will thus introduce the following notion. A function $f : \mathcal{K}(G) \to \mathbb{R}$ is called subadditive, if for all $A, B \in \mathcal{K}(G)$ there holds

$$f(A \cup B) \leq f(A) + f(B).$$

Furthermore a mapping $f : \mathcal{K}(G) \to \mathbb{R}$ is said to be right invariant, if for all $A \in \mathcal{K}(G)$ and for all $g \in G$ there holds

$$f(Ag) = f(A).$$

A function $f : \mathcal{K}(G) \to \mathbb{R}$ is called monotone, if for all $A, B \in \mathcal{K}(G)$ with $A \subseteq B$ there holds

$$f(A) \leq f(B).$$

An amenable group $G$ is called an Ornstein-Weiss group, if for any subadditive, right invariant and monotone function $f : \mathcal{K}(G) \to \mathbb{R}$ and for every Van Hove net $(A_i)_{i \in I}$ in $G$ the limit

$$\lim_{i \in I} \frac{f(A_i)}{\mu(A_i)}$$

exists, is finite and does not depend on the choice of the Van Hove net. From [Kri10, Theorem 1.1.] or [CSCK14, Theorem 1.1] we obtain that every discrete amenable group is an Ornstein-Weiss group. In this section we establish the following theorem, which shows in particular all groups in Example 2.10 to be Ornstein-Weiss groups.

Theorem 3.1. Every amenable group containing a uniform lattice is an Ornstein-Weiss group. More precisely, if $f : \mathcal{K}(G) \to \mathbb{R}$ is a subadditive, right invariant and monotone function and $(A_i)_{i \in I}$ is a Van Hove net in $G$, then

$$\lim_{i \in I} \frac{f(A_i)}{\mu(A_i)} = \frac{1}{\mu(C)} \lim_{j \in J} \frac{f(CF_j)}{|F_j|}$$

holds for any Van Hove net $(F_j)_{j \in J}$ in a uniform lattice $\Lambda \subseteq G$ with fundamental domain $C$. 

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Remark 3.2. Apart from this class of groups we do not know, whether there are further Ornstein-Weiss groups. As the results and arguments in the sections\[3\] and \[5\] do only depend on the fact, whether an Ornstein-Weiss Lemma holds, we prefer to introduce this notion.

Note that in [Gro99], the sketch of a proof is presented, that every amenable groups is an Ornstein-Weiss group. The idea of this proof is carried out in [Kri10] and in [CSCK14] for discrete amenable groups. It seems that this proof does not easily generalize to non-discrete amenable groups by the lack of a boundedness condition on $A \mapsto \hat{f}(A)$, which only holds in discrete groups.

3.1 Uniform lattices and Van Hove nets

In order to prove Theorem 3.1 we will first construct Van Hove nets in a uniform lattice $\Lambda \subseteq G$ with certain properties from Van Hove nets in $G$. In particular we will show the following.

Proposition 3.3. Let $G$ be an amenable group and $\Lambda$ be a lattice in $G$ with fundamental domain $C$. Then for every Van Hove net $(A_i)_{i \in I}$ there exist Van Hove nets $(\hat{F}_i)_{i \in I}$ and $(\hat{F}_i^c)_{i \in I}$ in $\Lambda$ such that

(i) $C\hat{F}_i \subseteq A_i \subseteq C\hat{F}_i^c$ for all $i \in I$

(ii) $\lim_{i \in I} |\hat{F}_i|/|\Lambda| = 1$.

Proof. For $i \in I$ let $(A_i)_{i \in I}$ be a Van Hove net in $G$ and set

$$\hat{F}_i := \Lambda \setminus C^{-1}A_i^c$$

and

$$\hat{F}_i^c := \Lambda \cap C^{-1}A_i.$$

The proof of Proposition 3.3 will be done in several lemmas.

Lemma 3.4. There holds $C\hat{F}_i \subseteq A_i \subseteq C\hat{F}_i^c$ for all $i \in I$.

Proof. From $C^{-1}A_i^c = \{z \in G; Cz \cap A_i^c \neq \emptyset\}$, we obtain

$$\hat{F}_i = \Lambda \setminus C^{-1}A_i^c = \{z \in \Lambda; Cz \cap A_i = \emptyset\} = \{z \in \Lambda; Cz \subseteq A_i\},$$

hence $C\hat{F}_i \subseteq A_i$. Furthermore $C^{-1}A_i = \{z \in G; Cz \cap A_i \neq \emptyset\}$ implies

$$\hat{F}_i = \Lambda \cap C^{-1}A_i = \{z \in \Lambda; Cz \cap A_i \neq \emptyset\}$$

and we obtain $A_i \subseteq C\hat{F}_i^c$. \(\square\)

In order to simplify the notation let $\hat{A}_i := C\hat{F}_i$ and $\hat{A}_i := C\hat{F}_i^c$ for $i \in I$. Set furthermore $K := C^{-1}C$ and note that $e_G \in K = K^{-1}$. The complements below are taken with respect to $G$.

Lemma 3.5. For all $i \in I$ we have

(i) $\overline{\hat{A}_i \cap \hat{A}_i^c} \subseteq \partial_K A_i$ and

(ii) $\overline{\hat{A}_i} \subseteq \partial_K A_i \cup \hat{A}_i$ and $\overline{\hat{A}_i^c} \subseteq \partial_K A_i \cup \hat{A}_i^c$.

Proof. We compute $\hat{A}_i = C(\Lambda \cap C^{-1}A_i) \subseteq CA_i \cap KA_i = KA_i$ and similarly

$$(\hat{A}_i)^c = C(\Lambda \setminus \hat{F}_i) = C(\Lambda \cap C^{-1}A_i^c) \subseteq KA_i^c.$$

Combining these statements we obtain

$$\overline{\hat{A}_i \cap \hat{A}_i^c} \subseteq \overline{KA_i \cap KA_i^c} = \partial_K A_i \cup \partial_K A_i^c,$$

hence (i). The statement of (ii) follows from (i). \(\square\)
Lemma 3.6. For $L \subseteq G$ there holds
\[ \partial_L \tilde{A}_i \cup \partial_L \hat{A}_i \subseteq \partial_{LK} A_i \]
for all $i \in I$.

Proof. From Lemma 3.5(ii) and $\tilde{A}_i \subseteq \hat{A}_i$ we obtain
\[ \overline{L \tilde{A}_i} \subseteq L(\partial_K A_i \cup \overline{A_i}) \subseteq L \partial_K A_i \cup L \overline{A_i} \subseteq \partial_{LK} A_i \cup L \overline{A_i} \]
and analogously $\overline{L \hat{A}_i} \subseteq \partial_{LK} A_i \cup L \overline{A_i}$. Furthermore there holds
\[ \overline{L \tilde{A}_i} \subseteq L \overline{A_i} \subseteq \partial_{LK} A_i \cup L \overline{A_i} \]
and similarly $L \overline{A_i} \subseteq \partial_{LK} A_i \cup L \overline{A_i}$. We compute
\[
\partial_L \tilde{A}_i \cup \partial_L \hat{A}_i = \left( \overline{L \tilde{A}_i} \cap L \overline{A_i} \right) \cup \left( \overline{L \hat{A}_i} \cap L \overline{A_i} \right)
\subseteq \left( \partial_{LK} A_i \cup L \overline{A_i} \right) \cap \left( \partial_{LK} A_i \cup L \overline{A_i} \right)
= \partial_{LK} A_i \cup \left( L \overline{A_i} \cap L \overline{A_i} \right) = \partial_{LK} A_i,
\]
where the last equality follows from $L = L\{e_G\} \subseteq LK$.

As the finite union of measurable sets we obtain $\tilde{A}_i$ and $\hat{A}_i$ to be measurable. For the asymptotic quotient of the values of the measures of these sets we have the following.

Lemma 3.7. There holds $\lim_{i \in I} \frac{\mu(\tilde{A}_i)}{\mu(A_i)} = 1$.

Proof. As $(A_i)_{i \in I}$ is a Van Hove net, there holds $\lim_{i \in I} \frac{\mu(\tilde{A}_i)}{\mu(\partial_K A_i)} = \infty$. By Lemma 3.5(ii) for every $i \in I$ we have $A_i \subseteq \tilde{A}_i \subseteq \tilde{A}_i \cup \partial_K A_i$, hence $\mu(\tilde{A}_i) \leq \mu(\tilde{A}_i) + \mu(\partial_K A_i)$ and $\mu(A_i) - \mu(\partial_K A_i) \leq \mu(\tilde{A}_i)$. We therefore obtain the result from the computation
\[
1 \leq \frac{\mu(\tilde{A}_i)}{\mu(A_i)} \leq \frac{\mu(\tilde{A}_i) + \mu(\partial_K A_i)}{\mu(A_i)} = 1 + \frac{\mu(\partial_K A_i)}{\mu(A_i)}
\leq 1 + \frac{\mu(\partial_K A_i)}{\mu(A_i) - \mu(\partial_K A_i)} = 1 + \frac{1}{\frac{\mu(\tilde{A}_i)}{\mu(A_i) - \mu(\partial_K A_i)}.}
\]

The nets $(\tilde{A}_i)_{i \in I}$ and $(\hat{A}_i)_{i \in I}$ are not necessarily nets of compact sets. Nevertheless, they satisfy the limit property of Van Hove nets.

Lemma 3.8. For every compact $L \subseteq G$ there holds
\[ \lim_{i \in I} \frac{\mu(\partial_L \tilde{A}_i)}{\mu(A_i)} = 0 = \lim_{i \in I} \frac{\mu(\partial_L \hat{A}_i)}{\mu(A_i)}. \]

Proof. Let $L \subseteq G$ be a compact subset. By Lemma 3.7 there is $j \in I$ such that for all $i \geq j$ there holds
\[ 1 \leq \frac{\mu(\tilde{A}_j)}{\mu(A_j)} \leq 2. \]
Hence $\mu(\tilde{A}_i) \leq \mu(A_i) \leq 2 \mu(\tilde{A}_i)$. Using Lemma 3.6 we compute for $i \geq j$ that
\[ 0 \leq \frac{\mu(\partial_L \tilde{A}_j)}{\mu(\tilde{A}_j)} \leq \frac{\mu(\partial_L \tilde{A}_j)}{\mu(A_j)} \quad \text{and} \quad 0 \leq \frac{\mu(\partial_L \hat{A}_j)}{\mu(A_j)} \leq 2 \frac{\mu(\partial_L \tilde{A}_j)}{\mu(A_j)}. \]
As $(A_i)_{i \in I}$ is a Van Hove net and $LK$ is compact the claimed statement follows.

We will now relate the boundaries, taken in $G$ and in $\Lambda$ respectively. We thus specify the notion of $K$-boundary for $K, A \subseteq G$ by $\partial_K^G(A) := K \overline{A} \cap K \overline{G \setminus A}$.
Lemma 3.9. Let $F,E \subseteq \Lambda$ be finite sets. Then $\partial F(E) \subseteq \partial F(CE)$.

Proof. As $C$ is a fundamental domain of $\Lambda$ in $G$ there holds $G \setminus CE = C(\Lambda \setminus E)$ and $e_G \in C$. We compute

$$\partial F(E) = FE \cap F(\Lambda \setminus E) \subseteq FC(\Lambda \setminus E) \cap FC \subseteq \partial F(\Lambda \setminus E) \cap FC \subseteq \partial F(CE).$$

We now finish the proof of Proposition 3.3. Let $F \subseteq \Lambda$ be compact, i.e. finite. For $i \in I$ Lemma 3.9 implies

$$C \partial F(\hat{F}_i) \subseteq C \partial F(\hat{F}_i) \subseteq \partial F(\hat{F}_i) \subseteq \partial F(\hat{F}_i).$$

Thus we compute

$$0 \leq \frac{|\partial F(\hat{F}_i)|}{|\hat{F}_i|} = \mu(C \partial F(\hat{F}_i)) \leq \frac{\mu(C \partial F(\hat{F}_i))}{\mu(\hat{F}_i)} = \frac{\mu(C \partial F(\hat{F}_i))}{\mu(\hat{F}_i)} = \frac{\mu(C \partial F(\hat{F}_i))}{\mu(\hat{F}_i)}.$$

As $\overline{CF}$ is compact in $G$ we obtain from Lemma 3.3 that the last term tends to 0. Similarly one shows $(\hat{F}_i)_{i \in I}$ to be a Van Hove net in $\Lambda$. We obtain (i) from Lemma 3.3 and (ii) from Lemma 3.7 and Remark 2.9.

3.2 Proof of Theorem 3.1

Proof. Let $C$ be a fundamental domain of $\Lambda$ in $G$ and note that $\mathcal{K}(\Lambda)$ is the set of finite subsets of $\Lambda$. In order to use that every discrete amenable group is an Ornstein-Weiss group, we define

$$f^A : \mathcal{K}(\Lambda) \to \mathbb{R} : F \mapsto f(\overline{CF}).$$

It is straightforward to see, that $f^A$ is right invariant and monotone. In order to show, that $f^A$ is subadditive let $F,F' \in \mathcal{K}(\Lambda)$. As $\overline{CF \cup CF'} \subseteq \overline{CF} \cup \overline{CF'}$ we obtain from the monotonicity and the subadditivity of $f$ that

$$f^A(F \cup F') \leq f(\overline{CF \cup CF'}) \leq f^A(F) + f^A(F').$$

Let now $(A_i)_{i \in I}$ be a Van Hove net in $\Lambda$ and $(\hat{F}_j)_{j \in J}$ be a Van Hove net in $\Lambda$. By Proposition 3.3 there are Van Hove nets $(\hat{F}_i)_{i \in I}$ and $(\hat{F}_j)_{j \in J}$ such that $\overline{CF_i} \subseteq A_i \subseteq \overline{CF_i}$ for all $i \in I$ and $\lim_{i \in I} |\hat{F}_i| |\hat{F}_i|^{-1} = 1$. As $A_i$ is closed, we get furthermore $\overline{CF_i} \subseteq A_i \subseteq \overline{CF_i}$ and hence

$$f^A(\hat{F}_i) \leq f(\Lambda) \leq f^A(\hat{F}_i).$$

Note that $\Lambda$ is a discrete amenable group and thus an Ornstein-Weiss group. This implies the existence of the following limits and

$$\lim_{i \in I} f^A(\hat{F}_i) = \lim_{i \in I} \frac{f^A(\hat{F}_i)}{|\hat{F}_i|} = \lim_{j \in J} \frac{f^A(\hat{F}_j)}{|\hat{F}_j|} = \lim_{j \in J} \frac{f(\overline{CF_j})}{|\hat{F}_j|}. \quad (2)$$

Let $\varepsilon > 0$. As $\lim_{i \in I} |\hat{F}_i| |\hat{F}_i|^{-1} = 1$ and $|\hat{F}_i| \leq |\hat{F}_j|$ for all $i \in I$ there is $j \in J$, such that for all $i \geq j$ there holds $|\hat{F}_i| \leq (1 + \varepsilon) |\hat{F}_j|$ and hence

$$\frac{1}{1 + \varepsilon} |\hat{F}_i| \mu(C) \leq \mu(C \hat{F}_i) \leq \mu(C) |\hat{F}_i| \leq \mu(C) |\hat{F}_i| \leq (1 + \varepsilon) |\hat{F}_i| \mu(C).$$

Thus for $j \geq i$ there holds

$$\frac{1}{1 + \varepsilon} f^A(\hat{F}_i) \leq \mu(C) f(\Lambda) \leq (1 + \varepsilon) f^A(\hat{F}_i).$$
We obtain for every $\varepsilon > 0$ that
\[
\frac{1}{(1 + \varepsilon)} \lim_{i \to \infty} \frac{f^A(\hat{F}_i)}{|F_i|} \leq \mu(C) \liminf_{i \to \infty} \frac{f(A_i)}{\mu(A_i)} \leq \mu(C) \limsup_{i \to \infty} \frac{f(A_i)}{\mu(A_i)} \leq (1 + \varepsilon) \frac{f^A(\hat{F}_i)}{|F_i|}.
\]
This shows that the limit $\mu(C) \lim_{i \to \infty} \frac{f(A_i)}{\mu(A_i)}$ exists and that it equals the limits in Equation 2. In particular it does not depend on the choice of $(A_i)_{i \in I}$.

\section{Entropy theory for Ornstein-Weiss groups}

In this section we introduce the relative topological entropy for actions of Ornstein-Weiss groups on compact uniform spaces and relate this approach to the better known approach via the Bowen metric for actions on compact metric spaces \cite{BS02, Bow71} and the approach for compact metric spaces by Tagi-Zade \cite{TZ91} using finite open covers for scaling.

\subsection{Bowen action}

For an action $\pi: G \times X \to X$ on a compact uniform space we define the corresponding Bowen action $\hat{\pi}: \mathcal{K}(G) \times \mathcal{U}_X \to \mathcal{U}_X$ by $\hat{\pi}(A, \eta) := \eta_A$, where
\[
\eta_A := \{(x, y) \in A \times X : (\pi(x), \pi(y)) \in \eta\} = \bigcap_{g \in A} (\pi g \times \pi g)^{-1} \eta.
\]

We will show in Lemma 4.2 below that the image of the Bowen action is indeed contained in $\mathcal{U}_X$. In order to omit brackets we will use the convention, that the Bowen action is a stronger operation than the product of entourages.

Remark 4.1. The definition of the Bowen action is inspired by the definition of Bowen metric, as defined in \cite{Bow71} for actions of $\mathbb{Z}$. Let $\pi: G \times X \to X$ be a flow on a metric space $(X, d)$ and for $A \subseteq G$ compact define the Bowen metric by $d_A(x, y) := \max_{g \in A} d(\pi^g(x), \pi^g(y))$ for $x, y \in X$. It is straightforward to show, that $d_A$ is a metric and that $[d_A < \varepsilon] = (d < \varepsilon)_A$ for all compact $A \subseteq G$ and $\varepsilon > 0$.

Lemma 4.2. Let $\pi: G \times X \to X$ be a flow on a compact uniform space. For every $\eta \in \mathcal{U}_X$ and $A \subseteq G$ compact there holds $\eta_A \in \mathcal{U}_X$.

Proof. Note that $\pi: A \times X \to X$ is uniformly continuous as a continuous mapping of a compact uniform space to a uniform space. Thus $(\pi \times \pi)^{-1}(\eta)$ is contained in the uniformity of $A \times X$. For $\kappa \in \mathcal{U}_A$ and $\vartheta \in \mathcal{U}_X$ we set $\kappa \Box \vartheta := \{(x, y, x', y') \in (A \times X) \times (A \times X) : (g, g') \in \kappa$ and $(x, x') \in \vartheta\}$. As $\kappa \Box \vartheta; \kappa \in \mathcal{U}_A$ and $\vartheta \in \mathcal{U}_X$ is a base for the product uniformity on $A \times X$ there are $\kappa \in \mathcal{U}_A$ and $\vartheta \in \mathcal{U}_X$ with
\[
\kappa \Box \vartheta \subseteq (\pi \times \pi)^{-1}(\eta) = \{(g, x, x', y) \in (A \times X) \times (A \times X); (\pi^g(x), \pi^g(x')) \in \eta\}.
\]
For $(x, x') \in \vartheta$ and $g \in A$ there holds $(g, x, x', y) \in \kappa \Box \vartheta$ and we obtain $(\pi^g(x), \pi^g(x')) \in \eta$. This proves $\vartheta \subseteq \eta_A$ and hence $\eta_A \in \mathcal{U}_X$.

Remark 4.3. Let $\pi$ be an action on a compact metric space. Note that the previous lemma can be seen as the natural generalization of the fact that all Bowen metrics with respect to $\pi$ are equivalent, i.e. they induce the same topology. Indeed, they induce the same uniformity. To see this let $A \subseteq G$ be compact. Observe that $[d_A < \varepsilon] = ([d < \varepsilon])_A$ is contained in the uniformity generated by $d$ for $\varepsilon > 0$, as seen in the previous lemma. Furthermore $[d < \varepsilon] \subseteq [d_A < \varepsilon]$ is contained in the uniformity generated by $d_A$. Hence the uniformities of $d_A$ and of $d$ coincide.

We close this introduction of the concept of the Bowen action by stating the following calculation rules, which are straightforward to prove. Note that (i) justifies that we can omit brackets and write $\eta_{AB}$ for $\eta_{(AB)} = (\eta_A)_B$.

Proposition 4.4. For $\eta, \kappa \in \mathcal{U}_X$ and $A, B \subseteq G$ compact there holds
\[(i) \ \eta(AB) = (\eta A)_B, \]
\[(ii) \ \eta_{A \cup B} = \eta_A \cap \eta_B \text{ and} \]
\[(iii) \ \eta_{A \cap B} \subseteq (\eta \cap \eta)_A. \]

4.2 Relative topological entropy for actions of Ornstein-Weiss groups

The following approach to relative topological entropy is inspired by the approach to topological entropy of \(Z\)-actions on compact metric spaces via sets of small diameter, given in [BS02 Section 2.5]. Consider first a compact uniform space \((X, \mathbb{U}_X)\) and \(\eta \in \mathbb{U}_X\).

**Definition 4.5.** For \(\eta \in \mathbb{U}_X\) we say that a subset \(M \subseteq X\) is \(\eta\)-small, if any \(x \in M\) is \(\eta\)-close to any \(y \in M\), i.e. iff \(M^2 \subseteq \eta\). We say, that a set \(U\) of subsets of \(X\) is of scale \(\eta\), if \(U\) is \(\eta\)-small for every \(U \in U\).

**Remark 4.6.** Let \((X, d)\) be a metric space. The diameter of a subset \(M\) is defined as \(\text{diam}(M) := \sup_{(x, y) \in M^2} d(x, y)\). The diameter of a set \(U\) of subsets of \(X\) is \(\text{diam}(U) = \sup_{U \in U} \text{diam}(U)\). For \(\varepsilon > 0\) a subset \(M \subseteq X\) satisfies \(\text{diam}(M) \leq \varepsilon\) if and only if it is \([d \leq \varepsilon]\)-small, and an open cover satisfies \(\text{diam}(U) \leq \varepsilon\) if and only if it is of scale \([d \leq \varepsilon]\).

As \(X\) is compact there is a finite open cover of \(X\) of scale \(\eta\). Thus for every \(M \subseteq X\) there exists a finite open cover of scale \(\eta\) as well.

**Definition 4.7.** For \(M \subseteq X\) and \(\eta \in \mathbb{U}_X\) we denote by \(\text{cov}_M(\eta)\) the minimal cardinality of an open cover of \(M\) of scale \(\eta\). If \(p: X \to Y\) is a map to some set \(Y\), we define

\[
\text{cov}_p(\eta) := \sup_{y \in Y} \text{cov}_{p^{-1}(y)}(\eta).
\]

It is immediate that \(\eta \mapsto \text{cov}_p(\eta)\) is decreasing. As the Bowen action is decreasing in the second argument, we obtain

\[
\mathcal{K}(G) \ni A \mapsto \log(\text{cov}_p(\eta_A))
\]

to be monotone for every \(\eta \in \mathbb{U}_X\). In Lemma \ref{lem:4.11} below we present that this mapping is also subadditive and right invariant, whenever \(p\) is a factor map. Thus the limit in the following definition of relative topological entropy exists and is independent of the Van Hove net.

**Definition 4.8.** Let \(\pi: G \times X \to X\) be an action of an Ornstein-Weiss group \(G\) on a compact uniform space \(X\) and \(\varphi\) be a factor of \(\pi\) via factor map \(p\). For any Van Hove net \((A_i)_{i \in I}\) and \(\eta \in \mathbb{U}_X\), we define

\[
E(\eta|\pi \xrightarrow{p} \varphi) := \lim_{i \in I} \frac{\log(\text{cov}_p(\eta_{A_i}))}{\mu(A_i)}.
\]

We furthermore define the relative topological entropy of \(\pi\) under the condition \(\varphi\) as

\[
E(\pi \xrightarrow{p} \varphi) := \sup_{\eta \in \mathbb{U}_X} E(\eta|\pi \xrightarrow{p} \varphi).
\]

The topological entropy of \(\pi\) is defined as the relative topological entropy under the condition of the one point flow. Note that in this case \(\text{cov}_p(\eta)\) is the minimal cardinality of an open covering of \(X\) of scale \(\eta\).

**Remark 4.9.** (i) As \(\eta \mapsto E(\eta|\pi \xrightarrow{p} \varphi)\) is decreasing, we obtain

\[
E(\pi \xrightarrow{p} \varphi) := \sup_{\eta \in \mathbb{U}_X} E(\eta|\pi \xrightarrow{p} \varphi)
\]

for any base \(\mathcal{B}_X\) of \(\mathbb{U}_X\).
Taking the supremum over all $y$ of $\eta$ after we equip $G$ with the discrete topology. If $\varphi: G \times X \to Y$ is a further set theoretic action we call a map $\mu: X \to Y$ a set theoretic map.

Proof. For a group $G$ and a set $Y$, we call a mapping $\varphi: G \times Y \to Y$ a set theoretic action of $G$ on $Y$, whenever $\varphi$ is (continuous) action after we equip $G$ and $Y$ with the discrete topology. If $\varphi: G \times X \to Y$ is a further set theoretic action we call a map $p: X \to Y$ a set theoretic factor map, if it is a (continuous) factor map after we equip $G$, $X$ and $Y$ with the discrete topology.

Example 4.10. An action $\pi$ of a topological group $G$ on a compact uniform space $X$ is called equicontinuous, whenever for all $\eta \in \mathbb{U}_X$ there is $\vartheta \in \mathbb{U}_X$ such that for all $g \in G$ there holds $|g| \leq |\vartheta| \cdot |\pi(x)|$, which reformulates as $\vartheta \subseteq \eta_x$. If we assume $G$ to be a non-compact Ornstein Weiss group there is

$$E(\pi \varphi_\eta) = 0$$

for every factor $\varphi$ of $\pi$. Indeed, the equicontinuity implies that for all $A \in K(G)$ there holds $\vartheta \subseteq \eta_A$. Hence for every Van Hove net $(A_i)_{i \in I}$ there holds

$$\lim_{i \in I} \frac{\log(\text{cov}_p(\eta_A))}{\mu(A)} = \frac{\log(\text{cov}_p(\vartheta))}{\mu(A)} = 0.$$

Taking the supremum over all $\eta \in \mathbb{U}_X$ proves the statement.

The non-compactness of the acting group is necessary to obtain that every equicontinuous action has zero entropy. In Remark 4.22 below it is shown that an action $\pi$ of a compact group on a compact Hausdorff space $X$ has non-zero topological entropy as soon as $X$ is not a single point.

Lemma 4.11. For $A, B \subseteq G$ compact, $g \in G$, and $\eta \in \mathbb{U}_X$ there holds

(i) $\text{cov}_p(\eta_{A \cup B}) \leq \text{cov}_p(\eta_A) \cdot \text{cov}_p(\eta_B)$ and

(ii) $\text{cov}_p(\eta_{Ag}) = \text{cov}_p(\eta_A)$.

Proof. To show (i) let $y \in Y$ and $\mathcal{U}$ and $\mathcal{V}$ be open covers of $p^{-1}(y)$ by $\eta_A$-small (respectively $\eta_B$-small) sets. Then $\mathcal{W} := \{U \cap V; U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$ is an open cover of $p^{-1}(y)$ by $\eta_{A \cup B}$-small sets and satisfies $|\mathcal{W}| \leq |\mathcal{U}| \cdot |\mathcal{V}|$. Thus

$$\text{cov}_{p^{-1}(y)}(\eta_{A \cup B}) \leq \text{cov}_{p^{-1}(y)}(\eta_A) \cdot \text{cov}_{p^{-1}(y)}(\eta_B) \leq \text{cov}_p(\eta_A) \cdot \text{cov}_p(\eta_B).$$

Taking the supremum over all $y \in Y$ we obtain (i). To show (ii) it suffices to show $\text{cov}_p(\eta_{Ag}) \leq \text{cov}_p(\eta_A)$ for all $g \in G$. Let $y \in Y$ and $\mathcal{U}$ be an open cover of $p^{-1}(\pi(y))$ of scale $\eta_A$. A straightforward computation shows $\{(\pi^{-1}(U); U \in \mathcal{U})\}$ to be an open cover of $p^{-1}(y)$ of scale $\eta_{Ag}$ and we obtain

$$\text{cov}_{p^{-1}(y)}(\eta_{Ag}) \leq \text{cov}_{p^{-1}(\pi(y))}(\eta_A) \leq \text{cov}_p(\eta_A).$$

Taking the supremum over all $y \in Y$ yields the claim.

4.3 Relative topological entropy of lattices in amenable groups

It is standard to define the topological entropy of an action of $\mathbb{R}$ as the restriction to the action of $\mathbb{Z}$. In fact one can always obtain the relative topological entropy of an action as the scaled entropy of the restricted action to a uniform lattice. For a map $f: A \to B$ and $M \subseteq A$ we denote by $f|_M$ the restriction $f|_{h_M}: M \to B: a \mapsto f(a)$.

---

5 For a group $G$ and a set $Y$, we call a mapping $\varphi: G \times Y \to Y$ a set theoretic action of $G$ on $Y$, whenever $\varphi$ is a (continuous) action after we equip $G$ and $Y$ with the discrete topology. If $\varphi: G \times X \to Y$ is a further set theoretic action we call a map $p: X \to Y$ a set theoretic factor map, if it is a (continuous) factor map after we equip $G$, $X$ and $Y$ with the discrete topology.
Theorem 4.12. Let $\pi$ be an action of an amenable group on a compact uniform space $X$. Let furthermore $\varphi$ be a factor of $\pi$ via factor map $p: X \to Y$. Let $\Lambda$ be a uniform lattice in $G$ with fundamental domain $\Lambda$. Then there holds

$$\mu(C) E(\pi \xrightarrow{p} \varphi) = E\left(\pi^{\Lambda \times X} \xrightarrow{p} \varphi^{\Lambda \times Y}\right).$$

Proof. By Theorem 3.1 there holds

$$\mu(C) E(\pi \xrightarrow{p} \varphi) = \sup_{\eta \in U_X} \lim_{i \to \infty} \frac{\log(cov_p(\eta_{\Lambda i }))}{|F_i|}.$$

for any Van Hove net $(F_i)_{i \in I}$ in $\Lambda$. It thus remains to show that

$$E\left(\pi^{\Lambda \times X} \xrightarrow{p} \varphi^{\Lambda \times Y}\right) = \sup_{\eta \in U_X} \lim_{i \to \infty} \frac{\log(cov_p(\eta_{\Lambda i }))}{|F_i|}.$$

Let $\eta \in U_X$. For $i \in I$ we obtain from $F_i \subseteq \overline{G F_i}$, that $\log(cov_p(\eta_{F_i})) \leq \log(cov_p(\eta_{\Lambda F_i}))$. For every $\eta \in U_X$ there holds $\eta_{\Lambda F_i} \in U_X$ and we compute

$$E\left(\pi^{\Lambda \times X} \xrightarrow{p} \varphi^{\Lambda \times Y}\right) = \sup_{\eta \in U_X} \lim_{i \to \infty} \frac{\log(cov_p(\eta_{F_i}))}{|F_i|} \leq \sup_{\eta \in U_X} \lim_{i \to \infty} \frac{\log(cov_p(\eta_{\Lambda F_i}))}{|F_i|}$$

$$= \sup_{\eta \in U_X} \lim_{i \to \infty} \frac{\log(cov_p(\eta_{\Lambda F_i}))}{|F_i|} \leq E\left(\pi^{\Lambda \times X} \xrightarrow{p} \varphi^{\Lambda \times Y}\right).$$

Remark 4.13. Note that for $n \in \mathbb{N}$ the set $\{1, \ldots, n\}$ is a fundamental domain for the uniform lattice $n\mathbb{Z}$ in $\mathbb{Z}$. We thus obtain from Theorem 4.12 for every homeomorphism $f: X \to X$ the well known formula

$$n E(f) = E(f^n).$$

Here $E(f)$ abbreviates the topological entropy of the flow $\pi: \mathbb{Z} \times X \to X$ with $\pi(n, x) = g^n(x)$ for a homeomorphism $g: X \to X$. Thus $E(f^n)$ is the entropy of the flow $\left(m, x \mapsto f^m(x)\right)$ restricted to $n\mathbb{Z} \times X \to X$.

4.4 Relative topological entropy via spanning and separating sets

As shown in [BS02] Section 2.5] one can also define topological entropy of $\mathbb{Z}$-actions on compact metric spaces in terms of separated and of spanning sets. In [Hoo74] this approach is generalized to $\mathbb{Z}$-actions of compact uniform spaces. We will now present a similar approach to relative topological entropy of Ornstein-Weiss groups acting on compact uniform spaces using the Bowen action. Consider a compact uniform space $(X, U_X)$.

Definition 4.14. For $\eta \in U_X$ a subset $S \subseteq X$ is called $\eta$-separated, if for every $s \in S$ there is no further element in $S$ that is $\eta$-close to $s$. Furthermore we say that $S \subseteq X$ is $\eta$-spanning for $M \subseteq X$, if for all $m \in M$ there is $s \in S$ such that $s$ is $\eta$-close to $m$ or $m$ is $\eta$-close to $s$.

Remark 4.15. A subset $S$ of a metric space $(X, d)$ is $[d < \varepsilon]$-separated, if any two distinct points in $S$ are at least $\varepsilon$ apart, i.e. $d(x, y) > \varepsilon$ for all $x, y \in S$ with $x \neq y$. Furthermore $S$ is $[d < \varepsilon]$-spanning for $M \subseteq X$, iff for every $m \in M$ there is $s \in S$ such that $d(s, m) < \varepsilon$.

Lemma 4.16. For $\eta \in U_X$ and $M \subseteq X$ the cardinality of every $\eta$-separated subset $S \subseteq M$ is bounded from above by $\text{cov}_M(\eta) < \infty$. In particular there are finite $\eta$-separated subsets of $M$ of maximal cardinality.
Proof. Let \( \mathcal{U} \) be an open cover of \( M \) by \( \eta \)-small sets and assume \( \mathcal{U} \) to have minimal cardinality. To obtain a contradiction let \( S \subseteq M \) be an \( \eta \)-separated set with \( |S| > |\mathcal{U}| \). Thus by the pigeon hole principle there is \( U \in \mathcal{U} \) such that \( S \cap U \) contains at least two distinct elements \( x \) and \( y \). As \( U \) is \( \eta \)-small we know \( x \) to be \( \eta \)-close to \( y \). This contradicts the \( \eta \)-separation of \( S \).

Lemma 4.17. Let \( \eta \in \mathbb{U}_X \) and \( M \subseteq X \). Then every \( \eta \)-separated subset \( S \subseteq M \) of maximal cardinality is \( \eta \)-spanning for \( M \). In particular there are finite subsets of \( M \) that are \( \eta \)-spanning for \( M \).

Proof. Let \( S \) be an \( \eta \)-separated subset of \( M \) of maximal cardinality and assume that \( S \) is not \( \eta \)-spanning for \( M \). Thus there is \( m \in M \) such that for all \( s \in S \) we know that \( s \) is not \( \eta \)-close to \( m \) and \( m \) is not \( \eta \)-close to \( s \). Hence no two distinct elements in \( S \cup \{m\} \) are \( \eta \)-close. We have shown \( S \cup \{m\} \subseteq M \) to be \( \eta \)-separated, which contradicts the maximality of \( S \).

Definition 4.18. For \( \eta \in \mathbb{U}_X \) and \( M \subseteq X \) we define \( \text{sep}_M(\eta) \) as the maximal cardinality of a subset of \( M \) that is \( \eta \)-separated and \( \text{spa}_M(\eta) \) as the minimal cardinality of a subset of \( M \) that is \( \eta \)-spanning for \( M \). For a map \( p: X \to Y \) to some set \( Y \) we define

\[
\text{sep}_p(\eta) := \sup_{y \in Y} \text{sep}_{p^{-1}(y)}(\eta) \quad \text{and} \quad \text{spa}_p(\eta) := \sup_{y \in Y} \text{spa}_{p^{-1}(y)}(\eta).
\]

Lemma 4.19. For symmetric entourages \( \eta, \vartheta \in \mathbb{U}_X \) and a subset \( M \subseteq X \) there holds \( \text{cov}_M(\vartheta \eta \vartheta) \leq \text{spa}_M(\eta) \), whenever \( \vartheta \) is open.

Proof. Let \( S \subseteq M \) be \( \eta \)-spanning for \( M \). As \( \vartheta \) is open and \( \eta \) is symmetric we obtain \( \{\vartheta \eta[s]; s \in S\} \) to be an open cover of \( M \). It suffices to show that \( \vartheta \eta[s] \) is \( (\vartheta \eta \vartheta) \)-small for any \( s \in S \). For \( x, y \in \vartheta \eta[s] = \vartheta^{-1} \eta^{-1}[s] = (\eta \vartheta)^{-1}[s] \) we know \( x \) to be \( \vartheta \eta \)-close to \( s \), and \( s \) to \( \eta \vartheta \)-close to \( y \), hence \( x \) to be \( \vartheta \eta \vartheta \)-close to \( y \).

In order to link these notions to the definition of relative topological entropy, we need the following.

Lemma 4.20. Let \( \eta \in \mathbb{U}_X \) and \( p: X \to Y \) be a map to a set \( Y \). Then there exists an entourage \( \vartheta \in \mathbb{U}_X \) with \( \vartheta \subseteq \eta \) such that for every compact \( A \subseteq G \) there holds

\[
\text{cov}_p(\eta_A) \leq \text{spa}_p(\vartheta_A) \leq \text{sep}_p(\vartheta_A) \leq \text{cov}_p(\vartheta_A).
\]

Proof. The second and the third inequality follow from Lemma 4.17 and Lemma 4.16 respectively. In order to show the first let \( \vartheta \in \mathbb{U}_X \) be symmetric and such that \( \vartheta \eta \vartheta \subseteq \eta \). For \( A \subseteq G \) compact we calculate \( \vartheta_A \vartheta_A \vartheta_A \vartheta_A \subseteq (\vartheta \vartheta \vartheta \vartheta) = \eta_A \). Let now \( \kappa \in \mathbb{U}_X \) be open and symmetric such that \( \kappa \subseteq \vartheta_A \). As \( \vartheta_A \) is symmetric Lemma 4.19 yields for every \( y \in Y \)

\[
\text{cov}_{p^{-1}(y)}(\eta_A) \leq \text{cov}_{p^{-1}(y)}(\vartheta_A \vartheta_A \vartheta_A \vartheta_A) \leq \text{cov}_{p^{-1}(y)}(\kappa \vartheta_A \vartheta_A) \leq \text{spa}_{p^{-1}(y)}(\vartheta_A) \leq \text{spa}_p(\vartheta_A).
\]

Taking the supremum over all \( y \in Y \) we obtain \( \text{cov}_p(\eta_A) \leq \text{spa}_p(\vartheta_A) \).

Theorem 4.21. Let \( \pi: G \times X \to X \) be an action of an Ornstein-Weiss group \( G \), on a compact uniform space \( X \). Let furthermore \( \varphi \) be a factor of \( \pi \) with factor map \( p \). There holds

\[
E(\pi \xrightarrow{p} \varphi) = \sup_{\eta \in \mathbb{B}_X} \lim_{\mu} \inf_{\mu} \frac{\log(\text{spa}_p(\eta A_i))}{\mu(A_i)} = \sup_{\eta \in \mathbb{B}_X} \lim_{\mu} \sup_{\mu} \frac{\log(\text{spa}_p(\eta A_i))}{\mu(A_i)} = \sup_{\eta \in \mathbb{B}_X} \lim_{\mu} \inf_{\mu} \frac{\log(\text{sep}_p(\eta A_i))}{\mu(A_i)} = \sup_{\eta \in \mathbb{B}_X} \lim_{\mu} \sup_{\mu} \frac{\log(\text{sep}_p(\eta A_i))}{\mu(A_i)},
\]

for any Van Hove net \( (A_i)_{i \in I} \) in \( G \) and any base \( \mathbb{B}_X \) of \( \mathbb{U}_X \).
Proof. As $\eta \mapsto \liminf_{i \in I} \frac{\log(spa_{\eta}(\varphi_A))}{\mu(A_i)}$ and the other similar terms are antitone, it suffices to show the statement for $B_X = \bigcup X$. By Lemma 4.20 it is immediate that for any $\eta \in U_X$ there holds

$$E(\eta|\pi \overset{p}{\rightarrow} \varphi) \leq \sup_{\delta \in U_X} \lim_{i \in I} \sup_{i \in I} \frac{\log(spa_{\delta}(\varphi_A))}{\mu(A_i)} \leq \sup_{\delta \in U_X} \lim_{i \in I} \sup_{i \in I} \frac{\log(sep_{\delta}(\varphi_A))}{\mu(A_i)} \leq E(\pi \overset{p}{\rightarrow} \varphi).$$

Taking the supremum over $\eta$ yields the result about the limit superior. A similar argument shows the result for the limit inferior. \qed

Remark 4.22. Assume $\pi$ to be an action of a compact group $G$ on a compact Hausdorff space $X$. There holds

$$E(\pi) = \frac{\log(|X|)}{\mu(G)},$$

where we denote $\log(\infty) := \infty$. To show this we assume $G = \{e_G\}$. This can be done as Theorem 4.12 can be applied to the lattice $\{e_G\}$ with fundamental domain $G$. As $X$ is assumed to be Hausdorff for every finite set $F \subseteq X$ there is $\eta \in U_X$ such that $F$ is $\eta$-separated. Thus there holds

$$\sup_{\eta \in U_X} \text{sep}_{X}(\eta) = \begin{cases} |X|, & \text{if } X \text{ is finite} \\ \infty, & \text{otherwise} \end{cases}$$

and we compute along the trivial Van Hove net $(\{e_G\})_{n \in \mathbb{N}}$

$$E(\pi) = \sup_{\eta \in U_X} \lim_{n \to \infty} \frac{\log(\text{sep}_X(\eta))}{\log(|\{e_G\}|)} = \sup_{\eta \in U_X} \log(\text{sep}_X(\eta)) = \log(|X|).$$

4.5 Relative topological entropy of positive expanding systems

In this subsection we show that a simplification for expansive actions, as presented for actions of continuous maps in [BS02, Proposition 2.5.7.] can be obtained from the independence of the definition of relative topological entropy from the choice of a Van Hove net.

Definition 4.23. An action $\pi$ of a topological group $G$ on a uniform space $X$ we call expansive, if there is an entourage $\delta \in U_X$ such that for all distinct $x, y \in X$ there is $g \in G$ such that $\pi^g(x)$ is not $\delta$-close to $\pi^g(y)$. In this case, we call $\delta$ an expansiveness entourage for $\pi$.

Remark 4.24. An action on a metric space $(X, d)$ is called expansive, if there is an $\delta > 0$ such that for all distinct $x, y \in X$ there is $g \in G$ such that $d(\pi^g(x), \pi^g(y)) \geq \delta$. In this case, we call $\delta$ an expansiveness constant. A straightforward argument shows that this is the case, if and only if $[d < \varepsilon]$ is an expansiveness entourage. As $[|d| < \varepsilon; \varepsilon > 0]$ is a base for the corresponding uniformity, the uniform approach to expansive systems is consistent with the metric approach.

Lemma 4.25. An entourage $\delta \in U_X$ is an expansiveness entourage for $\pi$, if and only if $\bigcap_{g \in G} \delta_{\pi^g} = \Delta_X$.

Proof. An entourage $\delta$ is an expansiveness entourage, iff for all $x, y \in X$ that satisfy $(\pi^g(x), \pi^g(y)) \in \delta$ for every $g \in G$, we have $(x, y) \in \Delta_X$. This is equivalent to $\Delta_X \subseteq \bigcap_{g \in G} \delta_{\pi^g} \subseteq \Delta_X$, as for $x, y \in X$ there is $(\pi^g(x), \pi^g(y)) \in \delta$ if and only if $(x, y) \in \delta_g$. \qed

Lemma 4.26. Let $\delta$ be a closed expansiveness entourage for an action $\pi$ on a compact uniform space $X$ and $\eta$ be an open entourage that satisfies $\eta \subseteq \delta$. Then there is a finite set $F \subseteq G$ such that $\delta_F \subseteq \eta$.
Proof. As \( \delta \) is an expansiveness entourage, there holds
\[
\bigcup_{g \in G} (X^2 \setminus \delta(g)) = X^2 \setminus \left( \bigcap_{g \in G} \delta(g) \right) = X^2 \setminus \Delta_X.
\]
Thus \( \{X^2 \setminus \delta(g); g \in G\} \) is an open cover of \( X^2 \setminus \eta \). Note that \( X^2 \setminus \eta \) is an closed subset of the compact set \( X^2 \), hence compact. Thus there is a finite subcover \( \{X^2 \setminus \delta(g); g \in F\} \) and we compute
\[
\delta_F = \bigcap_{g \in F} \delta(g) = X^2 \setminus \left( \bigcup_{g \in G} (X^2 \setminus \delta(g)) \right) \subseteq X^2 \setminus (X^2 \setminus \eta) = \eta.
\]
\( \square \)

**Theorem 4.27.** Let \( \pi \) be an expansive action of an Ornstein-Weiss group \( G \) on a compact uniform space \( X \) and \( \varphi \) be a factor of \( \pi \) with factor map \( p \). Let furthermore \( \delta \) be a closed expansiveness entourage.

(i) For every open entourage \( \eta \subseteq \delta \) there holds
\[
E(\pi \xrightarrow{\delta} \varphi) = E(\eta| \pi \xrightarrow{\delta} \varphi).
\]
(ii) For all open and symmetric entourages \( \eta \) and \( \vartheta \) with \( \delta \eta \vartheta \subseteq \delta \) the following limits exist and there holds
\[
E(\pi \xrightarrow{\delta} \varphi) = \lim_{i \in I} \log(\text{spa}_p(\eta_{A_i})) \mu(A_i) = \lim_{i \in I} \log(\text{sep}_p(\eta_{A_i})) \mu(A_i),
\]
for any Van Hove net \( (A_i)_{i \in I} \) in \( G \).

**Proof.** Let \( \mathcal{B}_X \) be the base of \( U_X \) consisting of all open entourages that are contained in \( \delta \). For (i) it is sufficient to show that \( E(\eta| \pi \xrightarrow{\delta} \varphi) \leq E(\delta| \pi \xrightarrow{\delta} \varphi) \) for every \( \eta \in \mathcal{B}_X \), as \( \eta \mapsto E(\eta| \pi \xrightarrow{\delta} \varphi) \) is decreasing and \( E(\pi \xrightarrow{\delta} \varphi) = \sup_{\eta \in \mathcal{B}_X} E(\eta| \pi \xrightarrow{\delta} \varphi) \). Let \( \eta \in \mathcal{B}_X \). By Lemma 4.20 there is a finite set \( F \subseteq G \) such that \( \delta_F \subseteq \eta \), hence \( \delta_{F,A_i} \subseteq \eta_{A_i} \) for every \( i \in I \). Thus by Proposition 4.22 and the independence of the Van Hove net, we obtain
\[
E(\eta| \pi \xrightarrow{\delta} \varphi) = \lim_{i \in I} \log \left( \frac{\text{cov}_p(\eta_{A_i})}{\mu(A_i)} \right) \leq \lim_{i \in I} \log \left( \frac{\text{cov}_p(\delta_{F,A_i})}{\mu(A_i)} \right) = E(\delta| \pi \xrightarrow{\delta} \varphi).
\]

To show (ii) define \( \kappa := \eta \vartheta \). Similarly to the proof of Lemma 4.20 one shows, that for every \( i \in I \) there holds
\[
\text{cov}_p(\kappa_{A_i}) \leq \text{spa}_p(\eta_{A_i}) \leq \text{sep}_p(\eta_{A_i}) \leq \text{cov}_p(\eta_{A_i}).
\]
As \( \kappa \) is an open entourage that is contained in \( \delta \) we obtain from (i) that
\[
E(\pi \xrightarrow{\delta} \varphi) = E(\kappa| \pi \xrightarrow{\delta} \varphi) \leq \liminf_{i \in I} \log \left( \frac{\text{spa}_p(\eta_{A_i})}{\mu(A_i)} \right) \leq \limsup_{i \in I} \log \left( \frac{\text{spa}_p(\eta_{A_i})}{\mu(A_i)} \right) \leq E(\eta| \pi \xrightarrow{\delta} \varphi) \leq E(\pi \xrightarrow{\delta} \varphi).
\]
A similar argument shows the statement for \( \text{sep}_p \). \( \square \)
5 Actions on compact metric and compact Hausdorff spaces

5.1 Relative topological entropy of actions on compact metric spaces

In this section let π be an action of an Ornstein-Weiss group G on a compact metric space (X, d). As already defined in Remark 4.4 we define the Bowen metric of a compact subset A ⊆ G by d_A(x, y) := \max_{g \in A} d(\pi^g(x), \pi^g(y)) for x, y ∈ X. It is presented in Remark 4.3 that all Bowen metrics induce the same uniformity and hence topology as d. Recall the diameter of an open cover from Remark 4.6. For ε > 0, A ⊆ G compact and M ⊆ X we denote by cov(ε, A, M) the minimum cardinality of an open cover of M with sets of d_A-diameter less than ε. From the Remarks 4.1 and 4.6 we obtain this notion to be well defined and cov(ε, A, M) = cov_M((d < ε)_A). As \{(d < ε); ε > 0\} is a base of \(\bigcup_X\) we conclude the following formula for relative topological entropy of an action on a compact metric space from Remark 4.9.

Theorem 5.1. If π is an action of an Ornstein-Weiss group G on a compact metric space (X, d) and ϕ is a factor of π via factor map p, there holds

\[
E(\pi \xrightarrow{p} \varphi) = \sup_{\varepsilon > 0} \lim_{\epsilon \to 0} \frac{\log(\sup_{y \in Y} \text{cov}(\varepsilon, A_i, p^{-1}(y)))}{\mu(A_i)}
\]

for every Van Hove net \((A_i)_{i \in I}\).

A subset S ⊆ X is called ε-separated (with respect to d), if any two distinct points in S are at least ε apart. Furthermore S is said to be ε-spanning for a subset M ⊆ X (with respect to d), if for every m ∈ M there is s ∈ S such that d(m, s) < ε. For ε > 0, A ⊆ G compact and M ⊆ X we denote by sep(ε, A, M) the maximum cardinality of an ε-separated subset of M with respect to d_A. Furthermore spa(ε, A, M) denotes the minimum cardinality of an ε-spanning set for M with respect to d_A. From Remark 4.15 we obtain these notions to be well defined and the equalities spa(ε, A, M) = spa_M((d < ε)_A) and sep(ε, A, M) = sep_M((d < ε)_A). Thus similarly as above we obtain from Theorem 4.21 the well known approach of Bow71 to (relative) topological entropy.

Theorem 5.2. If π is an action of an Ornstein-Weiss group G on a compact metric space (X, d) and ϕ is a factor of π via factor map p, there holds

\[
E(\pi \xrightarrow{p} \varphi) = \sup_{\varepsilon > 0} \lim_{\epsilon \to 0} \frac{\log(\sup_{y \in Y} \text{spa}(\varepsilon, A_i, p^{-1}(y)))}{\mu(A_i)}
\]

for every Van Hove net \((A_i)_{i \in I}\) in G. The limit superior can also be taken as a limit inferior.

Remark 5.3. Theorem 5.2 shows that the relative topological entropy considered in Yan15 for actions of countable discrete groups on compact metric spaces is equivalent to our definition. In order to see this combine Yan15 Lemma 2.4 with Yan15 Proposition 4.3.

In Remark 4.24 the notion of an expansiveness constant is introduced for actions on compact metric spaces. We can now easily obtain the following version of Theorem 4.27.

Theorem 5.4. Let π be an expanding action of a Ornstein-Weiss group G on a compact metric space (X, d) and ϕ be a factor of π with factor map p. Let furthermore δ be an expansiveness constant.

(i) For every ε < δ there holds

\[
E(\pi \xrightarrow{p} \varphi) = \lim_{\epsilon \to 0} \frac{\log(\sup_{y \in Y} \text{cov}(\varepsilon, A_i, p^{-1}(y)))}{\mu(A_i)}
\]

for any Van Hove net \((A_i)_{i \in I}\) in G.
(ii) For every \( \varepsilon < \frac{1}{2} \delta \) the following limits exist and there holds

\[
E(\pi, \varphi) = \lim_{i \in I} \frac{\log(\sup_{y \in Y} \text{spa}(\varepsilon, A_i, p^{-1}(y)))}{\mu(A_i)}
\]

\[
= \lim_{i \in I} \frac{\log(\sup_{y \in Y} \text{sep}(\varepsilon, A_i, p^{-1}(y)))}{\mu(A_i)}
\]

for any Van Hove net \((A_i)_{i \in I}\) in \(G\).

Proof. "(i)" As \( \varepsilon < \delta \), we obtain \([d \leq \varepsilon]\) to be a closed expansiveness entourage. The statement follows as the open entourage \([d < \varepsilon]\) is contained \([d \leq \varepsilon]\).

"(ii)" As \(2^{-1}\delta + \varepsilon < \delta\), we obtain \([d < 2^{-1}\delta + \varepsilon] \subseteq [d < \delta]\) to be a closed expansiveness entourage. Furthermore with \(s := 4^{-1}\delta + 2^{-1}\varepsilon\) there holds

\[
[d < s][d < \varepsilon][d < s] \subseteq [d \leq 2(4^{-1}\delta + 2^{-1}\varepsilon) + 2\varepsilon] = [d \leq (2^{-1}\delta + \varepsilon)].
\]

As \([d < \varepsilon]\) and \([d < s]\) are open and symmetric entourages, the second claim follows from Theorem 4.2(ii).

5.2 Relative topological entropy of actions on compact Hausdorff spaces

In [TZ91] a definition of entropy for actions of \(\mathbb{R}\) on compact metric spaces along the Van Hove sequence of hypercubes \((C_n)_{n \in \mathbb{N}}\) is provided. Furthermore a topological definition of \((\mathcal{U}, C_n)\)-separated and \((\mathcal{U}, C_n)\)-spanning sets for an action of \(\mathbb{R}\) on compact Hausdorff spaces, using finite open covers \(\mathcal{U}\) as a scale, is given. We now intend to generalize this approach to actions of Ornstein-Weiss groups on compact Hausdorff spaces and give a third approach by defining \((\mathcal{U}, A)\)-small sets with respect to a finite open cover \(\mathcal{U}\) and a compact subset \(A \subseteq G\). Let \(\pi : G \times X \to X\) be an action of an Ornstein-Weiss group \(G\) on a compact Hausdorff space \(X\).

**Definition 5.5.** We denote \(\mathcal{C}_{\text{fin}}(X)\) for the set of all finite open covers of \(X\). For \(\mathcal{U} \in \mathcal{C}_{\text{fin}}(X)\), \(A \subseteq G\) compact and \(S \subseteq X\) a subset \(S \subseteq X\) is said to be \((\mathcal{U}, A)\)-small, if for all \(x, y \in S\) and all \(g \in A\) there is \(U \in \mathcal{U}\), such that \(\pi^g(x), \pi^g(y) \in U\). For \(M \subseteq X\) a subset \(S \subseteq M\) is called \((\mathcal{U}, A)\)-dense in \(M\), if for all \(m \in M\) there exists \(s \in S\) such that for all \(g \in A\) there is \(U \in \mathcal{U}\) with \(\pi^g(m), \pi^g(s) \in U\). Furthermore \(S \subseteq X\) is said to be \((\mathcal{U}, A)\)-separated, if for any distinct \(x, y \in S\) there exists \(g \in A\) such that for no \(U \in \mathcal{U}\) there holds \(\pi^g(x) \in U\) and \(\pi^g(y) \in U\).

Let \(\mathcal{U} \in \mathcal{C}_{\text{fin}}(X), A \subseteq G\) compact and \(M \subseteq X\). The minimal cardinality of an open cover of \(M\) consisting of \((\mathcal{U}, A)\)-small sets is denoted by \(\text{cov}(\mathcal{U}, A, M)\). Furthermore \(\text{sep}(\mathcal{U}, A, M)\) is defined as the maximal cardinality of a \((\mathcal{U}, A)\)-separated set for \(M\) and \(\text{spa}(\mathcal{U}, A, M)\) as the minimal cardinality of a \((\mathcal{U}, A)\)-dense subset of \(M\).

Similarly to the metric case this approach can also be seen as the restriction to a certain base of \(U_X\). For \(\mathcal{U} \in \mathcal{C}_{\text{fin}}(X)\) we denote \(\langle \mathcal{U} \rangle := \bigcup_{U \in \mathcal{U}} U^2\).

**Lemma 5.6.** The set \(\mathcal{B}_X := \{\langle \mathcal{U} \rangle ; \mathcal{U} \in \mathcal{C}_{\text{fin}}(X)\}\) is a base of \(U_X\).

Proof. First note that \(\langle \mathcal{U} \rangle\) is a neighbourhood of the diagonal in \(X \times X\), hence \(\langle \mathcal{U} \rangle \in U_X\). For \(\eta \in U_X\) there is an open and symmetric entourage \(\kappa\) with \(\kappa \subseteq \eta\). Thus \(\{\kappa[x]; x \in X\}\) is an open cover of \(X\) and contains a finite subcover \(\mathcal{U} = \{\kappa[f]; f \in F\} \in \mathcal{C}_{\text{fin}}(X)\). For \((x, y) \in \langle \mathcal{U} \rangle\) there is \(f \in F\) with \(x, y \in \kappa[f]\). Thus the symmetry of \(\kappa\) implies \((x, f), (f, y) \in \kappa\), hence \((x, y) \in \kappa \subseteq \eta\). This shows \(\langle \mathcal{U} \rangle \subseteq \eta\).

It is thus possible to use the notion of Bowen-action. A straightforward proof shows the following link between the approach in [TZ91] and our uniform approach.

**Lemma 5.7.** For \(\mathcal{U} \in \mathcal{C}_{\text{fin}}(X), A \subseteq G\) compact and \(S, M \subseteq X\) there holds

(i) \(S\) is \((\mathcal{U}, A)\)-small, if and only if \(S\) is \((\mathcal{U})_A\)-small.
(iii) $S$ is $(U, A)$-dense in $M$, if and only if $S$ is $(U, A)$-spanning for $M$.

From the definition of relative topological entropy and Theorem 4.21 we obtain the following.

**Theorem 5.8.** If $\pi$ is an action of an Ornstein-Weiss group $G$ on a compact metric space $(X, d)$ and $\varphi$ is a factor of $\pi$ via factor map $p$, there holds

$$E(\pi \rightarrow p, \varphi) = \sup_{U \in C_{fin}(X)} \lim_{I} \log(\sup_{y \in Y} \text{cov}(U, A, p^{-1}(y))) \frac{\mu(A)}{\mu(A)}$$

for every Van Hove net $(A_i)_{i \in I}$. Furthermore $\text{spa}(\cdot)$ can be replaced by $\text{sep}(\cdot)$ and the limit superior can also be taken as a limit inferior.

**Remark 5.9.** Using finite open covers $U$ of a compact metric space $X$ as a scale, we can define the following. For an action $\pi$ on a compact Hausdorff space a finite open cover $U \in C_{fin}(X)$ is called an expansiveness cover for $\pi$, if for any distinct $x, y \in X$ there is $g \in G$ such that no $U \in U$ contains $\pi^g(x)$ and $\pi^g(y)$. It is straight forward to show, that this is equivalent to $(U)$ being an expansiveness entourage for $\pi$. Furthermore by Lemma 5.6 every expansiveness entourage contains an entourage of the form $(U)$. Thus an action on a compact Hausdorff space is expansive, if and only if there exists an expansiveness cover for it. We say that an open cover $V$ is strongly finer than $U$, if for all $V \in V$ there is $U \in U$ with $\cap V \subseteq U$. In particular this implies $(V) \subseteq (U) \subseteq (U)$. Thus by Theorem 4.21 we have for every finite open cover $V$ that is strongly finer than an expansiveness cover for $\pi$ there holds

$$E(\pi \rightarrow p, \varphi) = \lim_{i \in I} \log(\sup_{y \in Y} \text{cov}(V, A, p^{-1}(y))) \frac{\mu(A)}{\mu(A)}.$$

### 6 The Bowen entropy formula for actions of groups that contain a uniform lattice

In this section we show the following.

**Theorem 6.1.** Let $\pi, \varphi$ and $\psi$ be actions of an amenable group containing a countable uniform lattice on compact Hausdorff spaces $X, Y$ and $Z$ respectively. Let $\varphi$ be a factor of $\pi$ via factor map $p$ and $\psi$ be a factor of $\pi$ via factor map $q$, i.e. $\pi \rightarrow p, \varphi \rightarrow q, \psi$. Then there holds

$$\max\{E(\pi \rightarrow p, \varphi), E(\varphi \rightarrow q, \psi)\} \leq E(\pi \rightarrow q, p, \psi) \leq E(\pi \rightarrow p, \varphi) + E(\varphi \rightarrow q, \psi).$$

If we take $\psi$ as the action on a one point space, we obtain Bowen’s formula for the entropy of factors from the second inequality.

**Corollary 6.2.** Let $\pi, \varphi$ and $p$ as above. Then there holds

$$E(\pi) \leq E(\varphi) + E(\pi \rightarrow p, \varphi).$$

**Remark 6.3.** We obtain $E(\pi) = E(\varphi)$, whenever $E(\pi \rightarrow p, \varphi) = 0$; and $E(\pi) = E(\pi \rightarrow p, \varphi)$, whenever $E(\varphi) = 0$. If we assume $G$ to be non-compact and $X$ and $Y$ to be compact metric spaces we obtain $E(\pi \rightarrow p, \varphi) = 0$ under one of the following conditions.

(i) $p$ is a distal factor map, i.e. for $y \in Y$ all pairs of distinct points in $p^{-1}(y)$ are distal.

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6 We call two points $x, x' \in X$ distal, whenever there is $\eta \in \mathbb{U}_X$ such that for all $g \in G$ there is $(x, x') \notin \eta_g$. 

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(ii) \( p \) is a countable to one factor map, i.e. for \( y \in Y \) \( p^{-1}(y) \) is countable.

Proof. The statement in (i) implies \( p \) to be a distal factor map from the action \( \pi|_{\Lambda \times X} \) to \( \varphi|_{\Lambda \times Y} \) for every countable uniform lattice \( \Lambda \subseteq G \). We thus obtain the statement from [Yan15 Corollary 6.7] and Theorem 4.12. We get (ii) directly from [Yan15 Theorem 5.7] and Theorem 4.12. Note that we restrict to metric spaces, as the statements in [Yan15] are only given for compact metric spaces. \( \square \)

### 6.1 Measure theoretic Relative Entropy for actions of countable discrete amenable groups

In this subsection we give a brief introduction into a special case of the theory of measure theoretical relative entropy, presented in [Yan15], in order to state the variational principle and the Rohlin-Abramov theorem. Let \( X \) be a compact Hausdorff space. By \( B_X \) we denote the Borel \( \sigma \)-algebra. Furthermore we denote the set of all regular Borel probability measures by \( M(X) \). A family \( \alpha \) of pairwise disjoint Borel-measurable non empty subsets of \( X \) with union \( X \) is called a measurable partition of \( X \). We denote the set of all partitions of \( X \) by \( P_X \).

The refinement of two partitions \( \alpha, \beta \in P_X \) is the partition \( \alpha \cup \beta := \left\{ A \cap B; A \in \alpha \text{ and } B \in \beta \right\} \setminus \{\emptyset\} \). Similarly we define the refinement of a finite number of partitions. Let \( \pi: \Lambda \times X \rightarrow X \) be an action of a countable discrete amenable group \( \Lambda \) on a compact Hausdorff space. For a finite subset \( F \subseteq \Lambda \) we denote by \( \alpha_F \) the refinement of the partitions \( \left\{ (\pi^g)^{-1}(A); A \in \alpha \right\} \), where \( g \) ranges over \( F \). A measure \( \nu \in M(X) \) is called \( \pi \)-invariant, if \( \nu(A) = \nu(\pi^g(A)) \) for every \( g \in G \). We denote by \( M_\pi \) the set of all \( \pi \)-invariant \( \nu \in M(X) \). Every continuous map \( p: X \rightarrow Y \) to some compact Hausdorff space is measurable with respect to the Borel \( \sigma \)-algebras and \( p^{-1}(B_Y) \) is a sub-\( \sigma \)-algebra of \( B_X \). For \( \Lambda \in B_X \) and \( \nu \in M_\pi \) let \( E_{\nu,p}(\chi_A) \) be the conditional expectation of the characteristic function \( \chi_A \) of \( A \) with respect to \( p^{-1}(B_X) \). For \( \alpha \in P_X \) we define

\[
H_{\nu,p}(\alpha) := - \sum_{A \in \alpha} \int_X \log(E_{\nu,p}(\chi_A))d\nu.
\]

As presented in [Yan15] the Ornstein-Weiss lemma can be applied to \( F(\Lambda) \ni F \mapsto H_{\nu,p}(\alpha_F) \) for every \( \alpha \in P_X \) to obtain that

\[
E_\nu(\alpha|\pi \xrightarrow{p} \varphi) := \lim_{i \in I} \frac{H_{\nu,p}(\alpha_{F_i})}{|F_i|}
\]

exists and that is independent of the choice of the Van Hove net \( (F_i)_{i \in I} \) in \( \Lambda \). The relative measure theoretical entropy of \( \pi \) under the condition \( \varphi \) is given by

\[
E_\nu(\pi \xrightarrow{p} \varphi) := \sup_{\alpha \in P_X} E_\nu(\alpha|\pi \xrightarrow{p} \varphi).
\]

The following can be found in [Yan15 Theorem 3.1].

**Proposition 6.4.** (Rohlin-Abramov theorem) Let \( \pi, \psi \) and \( \varphi \) be actions of a countable discrete amenable group \( \Lambda \) on compact Hausdorff spaces \( X, Y \) and \( Z \) respectively and \((\nu_X, \nu_Y, \nu_Z) \in M_\pi(X) \times M_\psi(Y) \times M_\psi(Z)\). Let \( \varphi \) be a factor of \( \pi \) via factor map \( p \) and \( \psi \) be a factor of \( \varphi \) via factor map \( q \), i.e. \( \pi \xrightarrow{p} \varphi \xrightarrow{q} \psi \). Then there holds

\[
E_{\nu_X}(\pi \xrightarrow{q \circ p} \psi) = E_{\nu_X}(\pi \xrightarrow{p} \varphi) + E_{\nu_Y}(\varphi \xrightarrow{q} \psi).
\]

Using the variational principle, shown in [Yan15] for discrete countable groups, we obtain the following.

**Theorem 6.5.** Let \( \pi \) be an action of an amenable group \( G \), containing a countable uniform lattice \( \Lambda \), on a compact Hausdorff space \( X \) and let \( \varphi \) be a factor of \( \pi \) via \( p: X \rightarrow Y \). Then

\[
E(\pi \xrightarrow{p} \varphi) = \mu(C) \sup_{\nu \in M_\pi} E_\nu \left( \pi|_{\Lambda \times X} \xrightarrow{p} \varphi|_{\Lambda \times Y} \right),
\]
Proof. By Theorem 4.12 it remains to show that
\[ E(\pi|_{\Lambda \times X} \overset{p}{\to} \varphi|_{\Lambda \times Y}) = \sup_{\nu \in M_{\Lambda}} E_{\nu}(\pi|_{\Lambda \times X} \overset{p}{\to} \varphi|_{\Lambda \times Y}). \]

In Remark 5.3 it is presented, that our definition of relative topological entropy is equivalent to the definition given in [Yan15]. As [Yan15, Lemma 5.4] is also valid in the context of compact Hausdorff spaces the proof given in [Yan15, Theorem 5.1] easily generalizes to actions on compact Hausdorff spaces.

For the proof of Theorem 6.1 we need a further ingredient.

Definition 6.6. Let \( X \) and \( Y \) be compact Hausdorff spaces. For a continuous surjective mapping \( p: X \to Y \) and a \( \nu \in M(X) \) we define the push forward measure \( p_* \nu \in M(Y) \) by \( p_* \nu(M) := \nu(p^{-1}(M)) \) for every measurable \( M \subseteq Y \).

For a compact Hausdorff space we can identify the set of all (positive) Borel measures with the cone of all positive functionals on the Banach space \( C(X) \) by the Riesz-Markov theorem. For a reference see [EFHN15, Theorem E.11]. The set \( M(X) \) can be identified with the convex cone base of all positive functionals in \( C(X) \) that map the unit \( (X \to \mathbb{R}; x \mapsto 1) \) to 1. As this set is closed and contained in the weak*-compact unit ball, we equip \( M(X) \) with the restricted weak*-topology, to obtain a compact topological space. If we interpret \( \nu \in M(X) \) as a positive linear functional on \( C(X) \), then \( p_* \nu \) is a positive linear functional on \( C(Y) \) and satisfied \( p_* \nu(f) = \nu(f \circ p) \) for all \( f \in C(Y) \). A straight forward calculation shows \( p_*: M(X) \to M(Y) \) to be affine and continuous with respect to the weak*-topologies. As we assume \( p: X \to Y \) to be surjective \( C(Y) \) can be seen as a subspace of \( C(X) \) and the Hahn-Banach theorem implies \( p_*: M(X) \to M(Y) \) to be surjective.

Proposition 6.7. Let \( \pi \) be an action of a discrete amenable group \( \Lambda \) on a compact topological space \( X \) and \( \varphi \) be a factor of \( \pi \) via factor map \( p \). Then the restricted push forward operation \( p_* : M_{\pi} \to M_{\varphi} \) is surjective.

Proof. As \( p \) is a factor map we have \( p_* \nu_X \in M_{\varphi} \) for every \( \nu_X \in M_{\pi} \). Let \( \nu \in M_{\varphi} \). By the surjectivity of \( p_* : M(X) \to M(Y) \) there is \( \omega \in M(X) \) such that \( \nu = p_* \omega \). Let \( (F_i)_{i \in I} \) be a Følner net in \( \Lambda \). A standard argument (see [EW13, Theorem 4.1]) shows, that every weak*-limit point of the net \( \left( \frac{1}{|F_i|} \sum_{g \in F_i} (\pi^n \omega) \right)_{i \in I} \) is \( \pi \)-invariant. For any \( g \in \Lambda \) there holds furthermore \( p_* (\pi^n \omega) = \varphi^n (p_* \omega) = \varphi^n \nu = \nu \). Thus as \( p_* \) is affine and continuous every weak*-limit point \( \pi \nu \) of \( \left( \frac{1}{|F_i|} \sum_{g \in F_i} (\pi^n \omega) \right)_{i \in I} \) satisfies \( p_* \pi \nu = \nu \). As \( M(X) \) is compact with respect to the weak*-topology such a limit point exists.

6.2 Proof of Theorem 6.1

Proof of Theorem 6.1. Let \( \pi, \varphi \) and \( \psi \) be actions of an amenable group containing a countable uniform lattice \( \Lambda \) on compact Hausdorff spaces \( X, Y \) and \( Z \) respectively. Assume \( \pi^X \overset{p}{\to} \varphi^Y \overset{q}{\to} \psi \). Let \( C \) be a fundamental domain of \( \Lambda \). We abbreviate \( \pi^\Lambda := \pi|_{\Lambda \times X} \), \( \varphi^\Lambda := \varphi|_{\Lambda \times Y} \) and \( \psi^\Lambda := \psi|_{\Lambda \times Z} \) for the restrictions of the actions to the lattice. From Proposition 6.3 we obtain
\[
E(\pi \overset{p}{\to} \varphi) = \mu(C) \sup_{\nu \in M_{\Lambda}} E_{\nu}(\pi^\Lambda \overset{p}{\to} \varphi^\Lambda) \\
\leq \mu(C) \sup_{\nu \in M_{\Lambda}} \left( E_{\nu}(\pi^\Lambda \overset{p}{\to} \varphi^\Lambda) + E_{p_* \nu}(\varphi^\Lambda \overset{q}{\to} \psi^\Lambda) \right) \\
= \mu(C) \sup_{\nu \in M_{\Lambda}} E_{\nu}(\pi^\Lambda \overset{qp}{\to} \psi^\Lambda) = E(\pi \overset{qp}{\to} \psi).
\]

By Proposition 6.7 there holds
\[
E(\varphi^\Lambda \overset{q}{\to} \psi^\Lambda) = \mu(C) \sup_{\omega \in M_{\Lambda}} E_{\omega}(\varphi^\Lambda \overset{q}{\to} \psi^\Lambda) = \mu(C) \sup_{\nu \in M_{\Lambda}} \left( E_{p_* \nu}(\varphi^\Lambda \overset{q}{\to} \psi^\Lambda) \right).
\]
Thus an analogue argument yields $E(\varphi \to \psi) \leq E(\pi \to \psi)$. To see the second inequality we calculate
\[
E(\pi \to \psi) = \mu(C) \sup_{\nu \in \mathcal{M}_{\nu,\lambda}} E(\nu(\pi \to \psi)) \\
= \mu(C) \sup_{\nu \in \mathcal{M}_{\nu,\lambda}} \left( E_{\nu}(\pi_{\lambda} \to \varphi_{\lambda}) + E_{\nu}(\psi_{\lambda} \to \psi_{\lambda}) \right) \\
\leq \mu(C) \sup_{\nu \in \mathcal{M}_{\nu,\lambda}} \left( E_{\nu}(\pi_{\lambda} \to \varphi_{\lambda}) \right) + \mu(C) \sup_{\omega \in \mathcal{M}_{\omega,\lambda}} \left( E_{\omega}(\psi_{\lambda} \to \psi_{\lambda}) \right) \\
= E(\pi \to \varphi) + E(\varphi \to \psi).
\]

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