Qualitative analysis of oscillating magnetomechanical system

T S Todorov¹, R P Mitrev², B N Tudjarov³ and R F Nikolov⁴

¹Department of Theory of Mechanisms and Machines, Technical University of Sofia, Sofia 1000, Bulgaria
²Department of Logistics Engineering, Technical University of Sofia, Sofia 1000, Bulgaria
³Department of Computer Systems, Technical University of Sofia, Sofia 1000, Bulgaria
⁴Department of Precision Engineering and Measurement Instruments, Technical University of Sofia, Sofia 1000, Bulgaria

*tst@tu-sofia.bg

Abstract. The paper deals with mathematical modelling and qualitative analysis of a dynamic system of the type mass-spring-permanent magnet. The mass of the system is suspended on the free end of a cantilever beam and oscillates in a magnetic field created by a permanent magnet. The magnetic force is obtained using the finite element method and approximated by a function similar to the Coulomb’s Law. The dynamical model of the system is built on the basis of the Newton 2nd law. By means of qualitative analysis of a nonlinear differential equation, the bifurcation zones are determined. The phase portraits of the system are investigated depending on its bifurcation behaviour. Based on the performed analysis a method for dynamical synthesis of system parameters is proposed. Besides, some typical properties of the dynamical system are determined.

1. Introduction

In the last three decades, the microelectromechanical systems (MEMS) have attracted increasing interest in the engineering community. Niarchos [1] describes the magnetic MEMS as a new class of conventional MEMS devices with considerable potential for modern science and its practical applications. The extensive investigations in this field have naturally led to creating a novel type of models associated with the scaling laws, with changing the force dominations, time responses and system performance.

Electromechanical and magnetomechanical systems have sophisticated characteristics and the description of their behaviour is based on solving nonlinear ordinary differential equations [2]. In the general case, it relates to an analytical or numerical investigation of complex dynamical processes which are described by partial differential equations [3]. The selection of the parameters of such systems additionally can be included in the tasks of the dynamical synthesis, as far as such a selection can guarantee some requested properties of the real-life design. Dynamic models of the magnetic MEMS containing oscillating cantilever beams with permanent magnets are typically used to investigate the functioning of such devices as sensors, actuators and micro energy harvesters [4]. The studies of the practical engineering applications containing vibrating cantilever beam are extremely diverse. Primarily, they differ by the design of the structure based on the governing physical laws,
defined goals of the study and complexity of the used mathematical models. During the years a number of researchers [5,6] explored the various aspects of the behavior of the cantilever beam with a tip mass. The paper [7] proves the possibility for use of an electromagnetic actuator to control the vibrations of a cantilever beam with a tip mass by different control laws. The author [8] presents a method for enhancement of the power output of energy harvester by applying a simple repulsive magnetic force to a piezoelectric cantilever beam. The complex nature of the MEMS structures uncovers a wide field for application of optimization methods to improve the characteristics of the structures. An example is a recent research [9] proposing a multi-objective optimization model of an electromagnetic actuator applied as a fast tool servo. Despite the large volume of the conducted research studies, the precise description of the complex dynamical behavior of the magnetomechanical systems [10] is still a challenging problem that needs to be addressed through additional research. Some initial investigations of the authors are presented in the paper [11].

The aim of the present paper is the investigation of a nonlinear oscillating system of the type mass-spring-permanent magnet by the use of qualitative analysis methods and determination of system parameters that provide a predefined system behaviour.

2. Magnetic force determination

The essential elements of the considered system are shown in figure 1(a). A ferromagnetic mass 2 is fixed to the free end of a cantilever beam 1 and oscillates in the force field created by a permanent magnet 3. When the cantilever is in undeformed position between the faces of the ferromagnetic mass and magnet, an initial air gap $h$ is established. A linear generalized coordinate $y$ is selected for describing the mass position. Zero position of this coordinate corresponds to the horizontal position of the cantilever beam.

To qualitatively investigate the dynamic behavior of the considered oscillating system, it is necessary to determine the force dependence on the position. This function is worked out for particular cases and it depends on dimensions, shape and material properties of the system elements.

A precise determination of the magnetic force is carried out by investigation of the magnetic field by using the Finite Elements Method. The system under investigation is shown in Figure 2. It consists of a cylindrical permanent magnet 1 and a ferromagnetic cylinder 2, disposed over the magnet. All dimensions are denoted in figure 1(b). The magnetic field is analyzed as an axi-symmetrical one. The task of analysis is formulated by Poisson’s equation with respect to the magnetic vector potential in the cylindrical coordinate system. The investigated area is determined as a sufficiently large buffer zone (of the order of 20-multiple dimension of the system) with imposed Dirichlet’s boundary conditions. The problem is numerically solved by the software product FEMM (Finite Element Method Magnetics) and calculations are automated by programs in Lua Script® language. For determining the magnetic force, the built-in Maxwell stress tensor is used. The number of the mesh nodes varies for the different tasks between 18000 and 20000. Investigations are carried out for two types of permanent magnets (BaFerrite и NdFeB) with two diameters - 8 и 13 mm. Results for the magnetic field distribution and for the attractive force between the magnet and ferromagnetic cylinder are obtained by a change of the air gap from 1 to 5 mm. Magnetic field lines for BaFerrite and NdFeB
magnets with diameter 8 mm for air gaps 1 and 5 mm are visualized in figures 2(a) and 2(b). As one can see, there is no fundamental difference in the distribution field patterns. However, a significant difference in terms of force is established - for the NdFeB magnet the mean magnetic force is more than 60 times bigger than the same force for BaFerrite.

![Image of field lines](image)

Figure 2. Magnetic field lines distribution for δ=1 mm and δ=5 mm: a) BaFerrite d = 8mm; h_m = 15mm; h_c = 10 mm; b) NdFeB d = 8 mm; h_m = 15 mm; h_c = 10 mm.

The obtained by the numerical experiment force data are approximated by the relation:

$$ F_m = \frac{\kappa}{(\delta_0 + \delta)^2} $$

which is close to the Coulomb’s law \([12]\) when \(\delta_0\) is small. Analogically to this law, here by \(\kappa\) is denoted the imaginary magnetic mass and \(\delta_0\) is an imaginary initial gap. The determined regression coefficients and maximal deviations of the experimental data from the computed values are shown in table 1.

| Material   | \(d/ h_m/ h_c\) | \(\kappa \times 10^{-7}\) [Nm²] | \(\delta_0\) [m] | Max. deviation \(\times 10^{-6}\) [N] |
|------------|-----------------|-------------------------------|-----------------|-------------------------------------|
| BaFerrite  | 8/15/10         | 6.2681702                     | 0.001           | 2348                                |
| NdFeB     | 8/15/10         | 35.921246                     | 0.001           | 0328                                |
| BaFerrite  | 13/8/5          | 1.4354750                     | 0.001           | 5858                                |

3. Mathematical model of the oscillating system

A mathematical model of the system is derived, taking into account the following more important simplifying assumptions: the influence of the distributions of all kind of loads is neglected; the system is considered as a single degree of freedom system with lumped mass and lumped stiffness; it is assumed that the mass moves rectilinearly and its rotation is ignored; all kinds of resistive forces are neglected.

The latter assumption is highly plausible in case of oscillations in a vacuum combined with low energy dissipation materials, used in some microdevices as accelerometers, energy harvesters etc.

Taking into account the adopted assumptions, notation \(\delta=h-y\), the elastic force \(F_r=-cy\) and the weight force \(G=mg\), according to the Newton 2\textsuperscript{nd} law the motion of the mass at the free end of the cantilever beam is described by the following nonlinear differential equation:

$$ y + k^2 y = g + \frac{\kappa}{m(\delta_0 + h - y)^2} $$

where \(k^2=c/m\) is the natural frequency of the system, \(c\) is the spring stiffness coefficient, \(m\) is the mass, and \(g\) is the Earth gravity acceleration. For the stroke constraint of the cantilever beam it follows the constraint \(y \leq h\).
After substituting \( h_0 = \delta_0 + h \), introducing the dimensionless time \( t_{\text{new}} = k t \) and change of the variable \( y = h_0 - \xi \), the differential equation (2) takes the following simplified form:

\[
\ddot{\xi} + \dot{\xi} - \alpha + \frac{\beta}{\xi} = 0
\]

(3)

where \( \alpha = h_0 - g / k^2 \) and \( \beta = \kappa / \left( mk^2 \right) \). The equation (3) is a nonlinear autonomous differential equation which could be solved in terms of elliptic integrals. The change of the variable leads to the following constraint:

\[
\xi \geq \delta_0
\]

(4)

By the energy transformation \( \xi = \frac{1}{2} \frac{d\dot{\xi}}{d\xi} \) and after separating of the variables the first integral of equation (3) is obtained

\[
\frac{1}{2} \dot{\xi}^2 = \alpha \xi - \frac{1}{2} \xi^2 + \beta \frac{\xi}{\xi} + c
\]

(5)

By the expression (5) after substitution of different values of the integration constant \( c \) the phase trajectories in the plane \( (\xi, \dot{\xi}) \) of the oscillating system can be plotted [13].

4. Determination of the bifurcation zones

The bifurcation zones are determined by the values of the parameters \( \alpha \) and \( \beta \) leading to the change of the nature of the system phase states. The function:

\[
f = \alpha \xi - \frac{1}{2} \xi^2 + \beta \frac{\xi}{\xi}
\]

(6)

is considered for determination of the bifurcation values. For this purpose (6) is presented as a sum of two functions:

\[
f = f_1 + f_2
\]

(7)

where

\[
f_1 = \alpha \xi - \frac{1}{2} \xi^2, \quad f_2 = \beta \frac{\xi}{\xi}
\]

(8)

From equation (3) one can see that \( \beta \) is a positive nonzero parameter, but the parameter \( \alpha \) theoretically can accept negative and zero values. The negative values of \( \alpha \) are possible when simultaneously the air gap \( h \) and the imaginary gap \( \delta_0 \) are small and the natural frequency \( k \) is low. Such values are relatively rarely used in real-life designs.

It is apparent that the function \( f_2 \) is discontinuous for \( \dot{\xi} = 0 \) and is strictly monotone decreasing. The function \( f_1 \) is a parabola with a zero root and a second root equal to \( 2\alpha \). Two cases of the sum function \( f \) are possible. The first one corresponds to a strictly monotone decreasing function for \( \dot{\xi} > 0 \) and to a function with one extremum for \( \dot{\xi} < 0 \). The second case corresponds to a function possessing two extrema for \( \dot{\xi} > 0 \) and to a function with one extremum for \( \dot{\xi} < 0 \). At the first case, for \( \dot{\xi} > 0 \) there will not be a singular point in the phase trajectories. At the second case, for \( \dot{\xi} > 0 \) two singular points will occur corresponding to the stable and unstable equilibrium points. The limit case between the two cases corresponds to the occurrence of an inflection point in the function \( f \) leading to the appearance of a cusp point in the phase trajectories. From this limit case the condition of bifurcation is expressed as simultaneously occurring zeroes of the first and second derivatives of \( f \):

\[
\alpha - \xi - \frac{\beta}{\xi^2} = 0, \quad \frac{2\beta}{\xi^3} - 1 = 0
\]

(9)

therefore

\[
\beta = \frac{1}{2} \xi^3, \quad \alpha = \frac{3}{2} \xi
\]

(10)

which represents the parametric equation of the bifurcation curve. After isolating of the parameter \( \xi \), the equation \( \beta = f(\alpha) \) of the bifurcation curve is obtained:
\[
\beta = \frac{4}{27} \alpha^3
\]  

(11)

From the above mentioned it naturally follows that the bifurcation curve divides the plane into two zones with different behaviour of the dynamical system. For values of \(\alpha\) and \(\beta\) over the bifurcation curve, i.e. when
\[
\beta > \frac{4}{27} \alpha^3
\]  

(12)

conditions for oscillations do not exist, and the mass will accomplish only a displacement with a variable acceleration. The closer to the magnet is the mass, the higher is its velocity. If \(\alpha\) and \(\beta\) have values under the bifurcation curve, according to the initial conditions both oscillatory and non-periodic one-way motions are possible.

5. Topological analysis

The study of the number and type of the equilibrium points will facilitate the study of the dynamic behaviour of the system and defining the zones allowing the realization of stable oscillations around a certain position. The equilibrium or singular points [14] of the oscillating system follow from the system of algebraic equations
\[
\ddot{\xi} = 0, \quad \ddot{\xi} = 0
\]  

(13)

Taking into account equation (3) it follows
\[
\xi^3 - \alpha \xi^2 + \beta = 0
\]  

(14)

The number of the equilibrium points of the system is equal to the number of the real roots of equation (14) which discriminant is:
\[
D = 4\alpha^3 \beta - 27\beta^2
\]  

(15)

and according to its values, three cases are considered.

**Case I:** For \(D > 0\) the equation has three distinct real roots and the following condition is satisfied:
\[
\beta < \frac{4}{27} \alpha^3
\]  

(16)

and the value of the largest positive root is:
\[
\xi_1 = \left( \alpha^3 + \frac{\alpha^2}{\sqrt{3}} + \sqrt[3]{A} \right)
\]  

(17)

where
\[
A = \left( \alpha^3 + \frac{3}{2} \sqrt[3]{\beta(27\beta - 4\alpha^3)} - 9\beta \right)
\]

**Case II:** For \(D = 0\) the equation has three real roots and a repeated root:
\[
\xi_1 = -\alpha / 3, \quad \xi_2 = \xi_3 = 2\alpha / 3
\]  

(18)

This is achieved when
\[
\beta = \frac{4}{27} \alpha^3
\]  

(19)

**Case III:** For \(D < 0\) the equation has one real root and two complex conjugate roots. This gives the following condition:
\[
\beta > \frac{4}{27} \alpha^3
\]  

(20)

By the substitutions \(\xi = z_1\) and \(\dot{\xi} = z_2\), equation (3) is converted to a system of two first order equations:
\[
\begin{cases}
\dot{z}_1 = z_2 \\
\dot{z}_2 = \alpha - z_1 - \beta / z_1^2
\end{cases}
\]  

(21)

To find the equilibrium points for the system, we solve
\[
\begin{align*}
\dot{z}_1 &= z_2 = 0 \\
\dot{z}_2 &= \alpha - z_1 - \beta / z_1^2 = 0
\end{align*}
\]  

(22)

simultaneously and a set of \(i\) solutions \(\left( z_1^i, z_2^i \right) \) is obtained. According to Hartman’s theorem [15], the classification of the equilibrium points is performed by the use of the Jacobian matrix \(J\) with elements \(j_{11} = j_{22} = 0, \ j_{12} = 1, \ j_{21} = 2\beta / z_1^3 - 1\). Evaluation of the Jacobian matrix \(J_{\left( z_1, z_2 \right)}\) at the equilibrium points and computation of its eigenvalues allows determining the equilibrium point type. The obtained numerical results for the three discussed cases are shown in table 2 together with the corresponding phase portraits with denoted equilibrium points.

**Table 2.** Number and classification of the equilibrium points.

| Equilibrium point | Coordinates \(\left( z_1^i, z_2^i \right)\) | Jacobian \(J_{\left( z_1^i, z_2^i \right)}\) | Eigenvalues | Phase portrait |
|-------------------|--------------------------------|---------------------------------|-------------|----------------|
| p.A Stable center | \(0.00944,0\) \(_1\) \([0 \ -0.88108 \ 1]\) \(\lambda_{1,2} = \pm 0.93866i\) | | | |
| p.B Unstable saddle | \(0.00259,0\) \(_2\) \([0 \ 1 \ 4.6945 \ 0]\) \(\lambda_{1,2} = \pm 2.1667\) | | | |
| p.C Stable center | \((-0.00204,0)\) \(_3\) \([0 \ -12.813 \ 1]\) \(\lambda_{1,2} = \pm 3.5796i\) | | | |

**Case I:** \(\beta < \frac{4}{27} \alpha^3, \ \alpha = 0.01, \ \beta = 0.5 \times 10^{-7}\)

| Equilibrium point | Coordinates \(\left( z_1^i, z_2^i \right)\) | Jacobian \(J_{\left( z_1^i, z_2^i \right)}\) | Eigenvalues | Phase portrait |
|-------------------|--------------------------------|---------------------------------|-------------|----------------|
| p.A Degenerate, Unstable | \(0.00666,0\) \(_1\) \([0 \ 1 \ 0 \ 0]\) \(\lambda_{1,2} = 0\) | | | |
| p.B Stable center | \((-0.00333,0)\) \(_2\) \([0 \ 1 \ -9 \ 0]\) \(\lambda_{1,2} = \pm 3i\) | | | |

**Case II:** \(\beta = \frac{4}{27} \alpha^3, \ \alpha = 0.01, \ \beta = 1.4815 \times 10^{-7}\)

| Equilibrium point | Coordinates \(\left( z_1^i, z_2^i \right)\) | Jacobian \(J_{\left( z_1^i, z_2^i \right)}\) | Eigenvalues | Phase portrait |
|-------------------|--------------------------------|---------------------------------|-------------|----------------|
| p.A Stable center | \((-0.0056,0)\) \(_1\) \([0 \ -6.5386 \ 1]\) \(\lambda_{1,2} = \pm 2.5571i\) | | | |

**Case III:** \(\beta > \frac{4}{27} \alpha^3, \ \alpha = 0.01, \ \beta = 5 \times 10^{-7}\)
For all cases when the eigenvalues are pure imaginary complex numbers, the equilibrium point is center or focal. Applying global theorem due to Poincare [16] one determines that the equilibrium points are of type center.

As was already established, a possibility for stable vibrations in the vicinity of a value of $\xi$, satisfying constraint (4) exists only in Case I – p.A. This is feasible if the initial conditions of motion are within the area enclosed by the separatrix curve with the equation:

$$z_4 = \pm \sqrt{\frac{2\beta + 2c_{sep}z_1 + 2\alpha z_1^2 - z_1^4}{z_4}}$$

which follows directly from equation (5). In equation (23) by $c_{sep}$ is denoted the energy constant, corresponding to the separatrix curve - the phase trajectory, passing through the saddle point B. For initial conditions outside of the area enclosed by the separatrix curve, only non-periodical single displacements with variable mass velocities and accelerations are possible.

The phase portrait for Case II corresponds to the bifurcation value (19). In this case, there are two equilibrium points. The unstable point A satisfies the constraint (4) and it is a result of merging of two equilibrium points – unstable saddle and stable center. The type of the phase lines shows that for all possible initial conditions the mass approaches the axis $z=0$ with increasing velocity.

The phase portrait for Case III shows that there is one equilibrium point and it doesn’t satisfy constraint (4). The shape of the trajectories in the right half plane shows that in this case, the periodic motion is impossible.

6. Dynamical synthesis of an oscillating system

The aim of the dynamical synthesis of the considered magnetomechanical system is expressed here in finding such parameters combination, which guarantees periodic oscillations. For convenience the condition for stable oscillations (16) is rewritten in the following form:

$$\alpha > 3\frac{\beta}{\sqrt{4}}$$

The value of the positive parameter $\beta = \kappa / (mk^2)$ is completely determined by predefined values of $k$ and $m$ or their combination $\kappa = mk^2$. Taking into account that $\alpha = h_0 - g / k^2$ and using (24) one is able to determine the size of the air gap $h$:

$$h > 3\sqrt{\frac{\kappa}{4mk^2}} + \frac{g}{k^2} - \delta_0$$

The obtained numerical values are sufficient for choosing the rest of the geometrical parameters and material properties of the designed system but they do not guarantee oscillations with the prescribed natural frequency $k$. The reason for that is the presence of the imaginary magnetic mass which changes the natural frequency of the vibrating system.

The magnet force (1) is linearized in the vicinity of the equilibrium point with coordinate $y_1$, corresponding to the value of the largest positive root (17):

$$F_m \approx \frac{\kappa}{h_0 - y_1} + \frac{2\kappa}{(h_0 - y_1)^2}$$

After inserting (26) in (2) and rearranging some terms one obtains the linearized differential equation describing the motion of the mass in the vicinity of the equilibrium point:

$$\dot{y} + \left( k^2 - \frac{2\kappa}{m(h_0 - y_1)^2} \right) y = g + \frac{\kappa}{m(h_0 - y_1)} - \frac{2\kappa y_1}{m(h_0 - y_1)^3}$$

As one can see the term

$$k_i = \sqrt{\frac{k^2 - \frac{2\kappa}{m(h_0 - y_1)^2}}{m(h_0 - y_1)^3}}$$
represents the natural frequency of the linearized magnetomechanical system and its value is less than the value of the prescribed natural frequency \( k \) of the mechanical system. Equation (28) points out the possibility for tuning the system resonant frequency by adjusting the magnetic system parameters [17]. For larger amplitudes due to the lack of isohronity of the system the frequency depends on the initial conditions.

7. Conclusion

Due to the iterative nature of the design process, in most cases, many iterations are needed to carefully design a device satisfying predefined requirements. On the basis of the presented qualitative analysis of the considered nonlinear oscillating system, a preliminarily dynamical synthesis is reasonably achieved and thus the design process is considerably facilitated. The developed mathematical model of the system under study is investigated by the use of the qualitative analysis methods of dynamical systems. Such type of analysis is the only mean of studying similar systems in the cases when the differential equation is nonlinear and cannot be solved exactly, or it leads to solutions which cannot be analyzed. The bifurcation zones corresponding to different behavior of the system are determined. The study of the system phase portraits shows that only for a certain ratio of the parameters it is possible to have stable oscillation around an equilibrium point. In addition, some typical properties of the considered dynamical system are determined.

Acknowledgements

The authors would like to thank the Research and Development Sector at the Technical University of Sofia for the financial support.

References

[1] Niarchos D 2003 Magnetic MEMS: Key issues and some applications Sensors Actuators A 109 166-73.
[2] Timoshenko S and Woinovsk-Krieger S 1959 Theory of Plates and Shells (New York: McGraw-Hill) pp 63-76
[3] Nguyen V, Zehn M and Marinković D 2016 An efficient co-rotational fem formulation using a projector matrix Facta Universitatis Series: Mechanical Engineering 14 227 – 240
[4] Liu S, Davidson A and Lin Q 2004 Simulation studies on nonlinear dynamics and chaos in a MEMS cantilever control system J. Micromech. Microeng. 14 1064
[5] Schafer B E and Holzach H 1985 Experimental researches on flexible beam modal control J. Guid. Control Dyn. 8 597–604.
[6] Ju M S, Chung Y Y and Tsuei Y G 1993 Development of electromagnetic actuators for structure control Journal of Control Systems and Technology 1 235–246.
[7] Fung R F, Liu Y T, Wang C C 2005 Dynamic model of an electromagnetic actuator for vibration control of a cantilever beam with a tip mass J. Sound Vib. 288 957-980.
[8] Lin J T 2010 The magnetic coupling of a piezoelectric cantilever for enhanced energy harvesting efficiency Smart Mater. Struct. 19 045012
[9] Nie Y, Du Y and Xu Z 2017 Optimization Design of Electromagnetic Actuator Applied as Fast Tool Servo Actuators 6 25.
[10] Kim P, Bae S and Seok J 2012 Resonant behaviors of a nonlinear cantilever beam with tip mass subject to an axial force and electrostatic excitation Int J Mech Sci 64 232-57
[11] Todorov T and Nikolov R 2008 Dynamical modeling of mechanical oscillations in magnetic field \( \text{RECENT} 9 \) 96-9.
[12] Walker J, Halliday D and Resnick R 2014 Fundamentals of Physics (New Jersey; Wiley) p 614.
[13] Manevitch L I, Kovaleva A S and Manevitch E L 2010 Limiting Phase Trajectories and Resonance Energy Transfer in a System of Two Coupled Oscillators Math Probl Eng 2010
[14] Khalil H K 2002 Nonlinear Systems (New Jersey: Prentice Hall) pp 35-88
[15] Lynch S 2010 Dynamical Systems with Applications Using Maple (Boston: Birkhauser)
[16] Ens R 2011 It’s a Nonlinear World (New York: Springer) p 384
[17] Challa V R, Prasad M G and Fisher F T 2011 Towards an autonomous self-tuning vibration energy harvesting device for wireless sensor network applications Smart Mater. Struct. 20 025004