Introduction: Charm meson decay dynamics have been studied extensively over the last decade. Recent studies of multi-body decays of charm mesons probe a variety of physics including doubly-Cabibbo suppressed decays [1], searches for $CP$ violation [1,2,3], the properties of established light mesons [4,5,6] and the properties of $\pi\pi$ [1,6,7] and $K\pi$ [8] S-wave states. Future studies could improve sensitivity to $D^0 - \bar{D}^0$ mixing [9].

Weak nonleptonic decays of charm mesons are expected to proceed dominantly through resonant two-body decays in several theoretical models [10]; see Ref. [11] for a review of resonance phenomenology. These amplitudes are calculated with the Dalitz plot analysis technique [12], which uses the minimum number of independent observable quantities. For three-body final states when the parent particle is a scalar, the decay rate [13] is

$$\Gamma = \frac{1}{(2\pi)^3 32\sqrt{s^3}} |M|^2 dm_{12}^2 dm_{23}^2,$$

where $m_{ij}$ is the invariant mass of $i - j$ and the coefficient of the amplitude includes all kinematic factors. The scatter plot in $m_{12}^2$ versus $m_{23}^2$ is called a Dalitz plot. If $|M|^2$ is constant the allowed region of the plot will be populated uniformly with events. Any variation in $|M|^2$ over the Dalitz plot is due to dynamical rather than kinematical effects.

Formalism: The amplitude of the process, $D \to rc, r \to ab$, is given by

$$M_r (L, m_{ab}, m_{bc}) = \sum_{\lambda} \langle ab | r_{\lambda} \rangle T_r (m_{ab}) \langle cr_{\lambda} | D \rangle$$

$$= Z (L, \vec{p}, \vec{q}) B_L^D (|\vec{p}|) B_L^r (|\vec{q}|) T_r (m_{ab}) ,$$

where the sum is over the helicity states $\lambda$ of the intermediate resonance particle $r$, $a$ and $b$ are the daughter particles of the resonance $r$, $c$ is the spectator particle, $L$ is the orbital angular momentum between $r$ and $c$, $\vec{p}$ is the momentum of
c in the r rest frame and \( \tilde{q} \) is momentum of a in the r rest frame, Z describes the angular distribution of final state particles, \( B^D_L \) and \( B'_L \) are the barrier factors for the production of \( r - c \) and \( a - b \), respectively, with angular momentum L and \( T_r \) is the dynamical function describing the resonance r. Usually the resonances are modeled with a Breit-Wigner and the nonresonant contribution \( D \rightarrow abc \) is parameterized as an S-wave with no variation in magnitude or phase across the Dalitz plot. Some more recent analyses have used the K-matrix formalism [14] with the P-vector approximation [15] to describe the \( \pi \pi \) S-wave.

**Barrier Factor \( B_L \):** The maximum angular momentum \( L \) in a strong decay is limited by the linear momentum \( \tilde{q} \). Decay particles moving slowly with an impact parameter (meson radius) \( r \) of order 1 fm have difficulty generating sufficient angular momentum to conserve the spin of the resonance. The Blatt-Weisskopf [16,17] functions \( B_L \), given in Table 1, weight the reaction amplitudes to account for this spin-dependent effect. These functions are normalized to give \( B_L = 1 \) for \( z = (|\tilde{q}| r)^2 = 1 \). Another common formulation \( B'_L \), also in Table 1, is normalized to give \( B'_L = 1 \) for \( z = z_0 = (|\tilde{q}_0| r)^2 \) where \( q_0 \) is the value of \( q \) when \( m_{ab} = m_r \).

**Table 1:** Blatt-Weisskopf barrier factors.

| \( L \) | \( B_L(q) \) | \( B'_L(q, q_0) \) |
|---|---|---|
| 0 | 1 | 1 |
| 1 | \( \sqrt{\frac{2z}{1+z}} \) | \( \sqrt{\frac{1+z_0}{1+z}} \) |
| 2 | \( \sqrt{\frac{13z^2}{(z-3)^2+9z}} \) | \( \sqrt{\frac{(z_0-3)^2+9z_0}{(z-3)^2+9z}} \) |

where \( z = |\tilde{q}| r \) and \( z_0 = |\tilde{q}_0| r \).
Angular Distribution: $Z$ depends on the spin $L$ of resonance $r$. The Zemach formalism [18] describes the angular distributions in terms of Legendre polynomials but is only valid for reactions where $a$, $b$ and $c$ are spin-0. For final state particles with non-zero spin, the helicity formalism is required [19]. The sum over helicity states yields angular distributions that are proportional to Legendre polynomials when transversality is enforced ($m^2 = m^2_{ab}$ rather than $m^2 = m^2_r$ in the helicity sum denominator, see Eq. (4)).

Vector Intermediate States: The angular and barrier factors of Eq. (2) for vector resonances are

$$\langle cr | D \rangle = \left[ f_+ (P_D + P_c) + f_- (P_D - P_c) \right] g_D \epsilon_\lambda^\mu,$$
$$\langle ab | r \rangle = g_r \epsilon_\lambda^\mu (P_a - P_b) \nu,$$

(3)

where $P_i$ is the four-momentum of particle $i$, $\epsilon$ is the polarization vector associated with each decay vertex, $f_+ = f_+(m^2_{ab})$ and $f_- = f_-(m^2_{ab})$ are strong interaction form factors and $g_D$ and $g_r$ are strong interaction coupling constants.

We evaluate the sum over helicities $\lambda = \pm 1, 0$ as

$$\sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu = -g^\mu\nu + \frac{q^\mu q^\nu}{m^2}.$$

(4)

Transversality is enforced if we take the denominator of the second term to be $m^2_{ab}$ rather than $m^2_r$. This enforces a vector current and $\epsilon_\lambda^\mu q_\mu = 0$ by construction and so the $f_-(m^2_{ab})$ term does not contribute. The product of the remaining three factors $f_+ g_r g_D$, are approximated as a constant $f_+(m^2_{ab} = m^2_r)$ and $g_{r,D}$ are the $L = 1$ Blatt-Weisskopf factors given in Table 1, respectively.

The angular distribution for vector intermediate states is given by

$$Z = (P_D + P_c) \mu \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{m^2_{ab}} \right) (P_a - P_b) \nu$$
$$= (m^2_{bc} - m^2_{ac}) + (m^2_D - m^2_c) \left( m^2_a - m^2_b \right) / m^2_{ab}$$
$$= -2 \vec{p} \cdot \vec{q},$$

(5)
where the Zemach form [18], following the last equality, is only obtained when transversality is enforced.

**Scalar Intermediate States:** There are no polarization vectors associated with the decay vertices so the process is either an S-wave decay or a D-wave decay. The S-wave decay has a uniform angular distribution and the \( L = 0 \) Blatt-Weisskopf factor is also constant. The D-wave decay amplitude is described by

\[
\mathcal{M}_r = \left[ f_+ (m_{ab}^2) (P_D + P_c)_\mu + f_- (m_{ab}^2) (P_D - P_c)_\mu \right] g_D \\
T_r (m_{ab}) [g_+ (m_{ab}^2) (P_a + P_b)_\mu + g_- (m_{ab}^2) (P_a - P_b)_\mu] g_r. \tag{6}
\]

The \( D \)-wave contribution is suppressed, due to small momenta, relative to the \( S \)-wave and has been neglected is charm Dalitz plot analyses.

**Tensor Intermediate States:** The angular and barrier factors of Eq. (2) for tensor resonances are

\[
\langle cr_\lambda|D \rangle = T_1 (m_{ab}^2) (P_D + P_c)_\mu (P_D + P_c)_\nu \ g_D \epsilon^\mu_\lambda \epsilon^\nu_\lambda, \\
\langle ab|r_\lambda \rangle = T_2 (m_{ab}^2) (P_a - P_b)_\alpha (P_a - P_b)_\beta \ g_r \epsilon^*_\alpha \epsilon^\beta_\lambda, \tag{7}
\]

where \( \epsilon \) are the polarization tensors associated with each decay vertex, \( T_1(m_{ab}^2) \) and \( T_2(m_{ab}^2) \) are strong interaction form-factors and \( g_D \) and \( g_r \) are strong interaction coupling constants.

We evaluate the sum over helicities \( \lambda = \pm 2, \pm 1, 0 \) [20] as

\[
\sum_\lambda \epsilon^*_\lambda \epsilon^\lambda_\mu \epsilon^\lambda_\nu = \left( \frac{\kappa^{\mu \alpha \nu \beta} + \kappa^{\mu \beta \nu \alpha}}{2} - \frac{\kappa^{\mu \nu \alpha \beta}}{3} \right) \tag{8}
\]

where \( \kappa^{\mu \nu} = -g^{\mu \nu} + \frac{q^{\mu} q^{\nu}}{m_r^2} \). Transversality is enforced if we take the denominator of the second term to be \( m_{ab}^2 \) rather than \( m_r^2 \). This enforces a tensor current and \( q^{\mu}_a \epsilon^\lambda_\mu \epsilon^\lambda_\nu = q^{\nu}_b \epsilon^\mu_\mu \epsilon^\mu_\nu = 0 \) by construction. The product of the remaining four factors \( T_1 T_2 g_r g_D \), are approximated as a constant \( T_{1,2}(m_{ab}^2 = m_r^2) \) and \( g_r, D \) are the \( L = 2 \) Blatt-Weisskopf factors given in Table 1, respectively.
The angular distribution for tensor intermediate states is

\[ Z = (P_D + P_c) \mu (P_D + P_c) \nu \left[ \left( \frac{\mathcal{N}^{\mu \alpha} \mathcal{N}^{\nu \beta} + \mathcal{N}^{\mu \beta} \mathcal{N}^{\nu \alpha}}{2} \right) - \frac{\mathcal{N}^{\mu \nu} \mathcal{N}^{\alpha \beta}}{3} \right] \]

\[ (P_a - P_b)_{\alpha} (P_a - P_b)_{\beta} \]

\[ = \left( m^2_{bc} - m^2_{ac} + \frac{(m^2_D - m^2_c) (m^2_a - m^2_b)}{m^2_{ab}} \right)^2 \]

\[ - \frac{1}{3} \left( m^2_{ab} - 2m^2_D - 2m^2_c + \frac{(m^2_D - m^2_c)^2}{m^2_{ac}} \right) \]

\[ \times \left( m^2_{ab} - 2m^2_a - 2m^2_b + \frac{(m^2_a - m^2_b)^2}{m^2_{ac}} \right) \]

\[ = \frac{4}{3} \left[ 3 (\vec{p}.\vec{q})^2 - (|\vec{p}| |\vec{q}|)^2 \right] \]

(9)

where the Zemach form [18], following the last equality, is only obtained when transversality is enforced.

**Dynamical Function \( T_r \):** The dynamical function \( T_r \) is derived from the \( S \)-matrix formalism. In general, the amplitude that a final state \( f \) couples to an initial state \( i \) through the unitary scattering operator \( S \), \( S_{fi} = \langle f | S | i \rangle \), where the scattering operator \( S \) is unitary and satisfies \( SS^\dagger = S^\dagger S = I \). The transition operator \( \hat{T} \) is defined by separating the probability that \( f = i \) yielding,

\[ S = I + 2i T = I + 2i \{ \rho \}^{1/2} \hat{T} \{ \rho \}^{1/2}, \]

(10)

where \( I \) is the identity operator, \( \hat{T} \) is Lorentz invariant transition operator, \( \rho \) is the diagonal phase space matrix where \( \rho_{ii} = 2q_i/m \) and \( q_i \) is the breakup momentum for decay channel \( i \). In the single channel S-wave scenario \( S = e^{2i \delta} \) satisfies unitarity and implies

\[ \hat{T} = \frac{1}{\rho} e^{i \delta} \sin \delta. \]

(11)

There are three common formulations of the dynamical function. The Breit-Wigner formalism is the simplest formulation. - the first term in a Taylor expansion about a \( T \) matrix pole, The \( K \)-matrix formalism [14] is more general, allowing
more than one $T$ matrix pole and coupled channels while preserving unitarity. The Flatte distribution [21] is used to parameterize resonances near threshold and is equivalent to a one-pole, two-channel $K$-matrix.

**Breit-Wigner Formulation:** The common formulation of the Breit-Wigner decaying to spin-0 particles $a$ and $b$ is

$$ T_r (m_{ab}) = \frac{1}{m_r^2 - m_{ab}^2 - i m_r \Gamma_{ab} (q)} \quad (12) $$

where the “mass dependent” width $\Gamma_{ab}$ is

$$ \Gamma_{ab} = \Gamma_r \left( \frac{q}{q_0} \right)^{2L+1} \left( \frac{m_0}{m_{ab}} \right) B'_L (q, q_0)^2. \quad (13) $$

A Breit-Wigner parametrization best describes isolated, non-overlapping resonances far from the threshold of additional decay channels. Unitarity can be violated when the dynamical function is parameterized as the sum of two or more overlapping Breit-Wigners. The proximity of a threshold to the resonance shape distorts the line shape from a simple Breit-Wigner. This scenario is described by the Flatte formula and is discussed below.

**$K$-matrix Formalism:** The $T$ matrix can be described as

$$ \hat{T} = \left( I - i \hat{K} \rho \right)^{-1} \hat{K}, \quad (14) $$

where $\hat{K}$ is the Lorentz invariant $K$-matrix describing the scattering process and $\rho$ is the phase space factor.

Resonances appear as a sum of poles in the $K$-matrix

$$ \hat{K}_{ij} = \sum_{\alpha} \frac{m_\alpha \Gamma_{\alpha i} (m) m_\alpha \Gamma_{\alpha j} (m)}{(m_\alpha^2 - m^2) \sqrt{\rho_i \rho_j}}. $$

For the special case of a single channel, single pole we obtain

$$ K = \frac{m_0 \Gamma (m)}{m_0^2 - m^2} $$

and

$$ T = K (1 - iK)^{-1} = \frac{m_0 \Gamma (m)}{m_0^2 - m^2 - i m_0 \Gamma (m)} $$
which is the relativistic Breit-Wigner formula. For the special case of a single channel, two poles we have

$$K = \frac{m_\alpha \Gamma_\alpha (m)}{m_\alpha^2 - m^2} + \frac{m_\beta \Gamma_\beta (m)}{m_\beta^2 - m^2}$$

and in the limit that \( m_\alpha \) and \( m_\beta \) are far apart relative to the widths we can approximate the \( T \) matrix as the sum of two Breit-Wigners,

$$T \approx \frac{m_\alpha \Gamma_\alpha (m)}{m_\alpha^2 - m^2 - im_\alpha \Gamma_\alpha (m)} + \frac{m_\beta \Gamma_\beta (m)}{m_\beta^2 - m^2 - im_\beta \Gamma_\beta (m)}.$$

(15)

In the case of two nearby resonance Eq. (15) is not valid and exceeds unity (violates unitarity).

This formulation, which applies to \( s \)-channel production in two-body scattering \( ab \rightarrow cd \), can be generalized to describe the production of resonances in processes, such as the decay of charm mesons. The key assumption here is that the two-body system described by the \( K \)-matrix does not interact with the rest of the final state [15]. The quality of this assumption varies with the production process and is appropriate for scattering experiments like \( \pi^- p \rightarrow \pi^0\pi^0 n \), radiative decays such as \( \phi, J/\psi \rightarrow \gamma \pi \pi \) and semileptonic decays such as \( D \rightarrow K \pi \ell \nu \). This assumption may be of limited validity for production processes such as \( p\bar{p} \rightarrow \pi \pi \pi \) or \( D \rightarrow \pi \pi \pi \). In these scenarios the two-body Lorentz invariant amplitude, \( \hat{F} \), is given as

$$\hat{F}_i = \left( I - i\hat{K} \rho \right)_{ij}^{-1} \hat{P}_j = \hat{T} \hat{K}^{-1} \hat{P}$$

(16)

where \( P \) is the production vector which parameterizes the resonance production in the open channels.

For the \( \pi \pi \) S-wave the five channels relevant for \( D \) decays, are \( \pi \pi, KK, \eta \eta, \eta' \eta' \) and \( 4\pi \). In this scenario a common formulation of the \( K \)-matrix is

$$K_{ij}(s) = \sum_\alpha \frac{g_i^{(\alpha)} g_j^{(\alpha)}}{m_\alpha^2 - s + f_{ij}^{sc} \frac{1 - s_{0}^{sc}}{s - s_{0}^{sc}}} \times \frac{s - s_A/2m_\pi^2}{(s - s_A)(1 - s_{0})}. \quad (17)$$

The factor \( g_i^{(\alpha)} \) is the coupling constant of the \( K \)-matrix pole \( m_\alpha \) to meson channel \( i \); the parameters \( f_{ij}^{sc} \) and \( s_{0}^{sc} \) describe a
smooth part of the $K$-matrix elements; the multiplicative factor 
\[
\frac{s - s_A/2m_{\pi}^2}{(s - s_{A0})(1 - s_{A0})}
\]
suppresses a false kinematical singularity near the $\pi\pi$ threshold - the Adler zero; and the number 1 has units GeV$^2$.

The production vector, with $i = 1$ denoting $\pi\pi$, is
\[
P_j(s) = \left\{ \sum_{\alpha} \beta_{\alpha} g_j^{(\alpha)} + f_{1j}^{pr} \right\} \times \frac{s - s_A/2m_{\pi}^2}{(s - s_{A0})(1 - s_{A0})},
\]
where the free parameters of the Dalitz plot fit are the production coupling $\beta_{\alpha}$, and the production vector background parameters $f_{1j}^{pr}$ and $s_{0}^{pr}$. All other parameters are fixed by scattering experiments.

**Flatte Formalism:** The scenario where another channel opens close to the resonance position is described by the Flatte formulation
\[
\hat{T}(m_{ab}) = \frac{\hat{g}}{m_{r}^2 - m_{ab}^2 - i \left( \rho_1 g_{1}^2 + \rho_2 g_2^2 \right)}, \quad g_{1}^2 + g_2^2 = m_{r} \Gamma_{r}.
\]
This situation occurs in the $\pi\pi$ S-wave where the $f_{0}(980)$ is near the $KK$ threshold and in the $\pi\eta$ channel where the $a_{0}(980)$ also lies near $KK$ threshold. For the $a_{0}(980)^{+}$ resonance the relevant coupling constants are $g_{1} = g_{\pi\eta}$ and $g_2 = g_{KK}$. For the $f_{0}(980)$ the relevant coupling constants are $g_{1} = g_{\pi\pi}$ and $g_2 = g_{KK}$ where the charged and neutral $K$ channels are usually assumed to have the same coupling constant but separate phase space factors due to $m_{K^+} \neq m_{K^0}$.

**Branching Ratios from Dalitz Plot Fits:** The fit to the Dalitz plot distribution using either the Breit-Wigner or the $K$-matrix formalism factorizes into a resonant contribution to the amplitude $M_j$ and a complex coefficient, $a_j e^{i\delta_j}$, where $a_j$ and $\delta_j$ are real. The definition of a rate of a single process, given a set of amplitudes $a_j$ and phases $\delta_j$ is the square of the relevant matrix element. In this spirit, the fit fraction is usually defined as the integral over the Dalitz plot ($m_{ab}$ vs $m_{bc}$) of a single amplitude squared divided by the integral over the Dalitz plot of the square of the coherent sum of all amplitudes,
\[
\text{Fit Fraction}_j = \frac{\int |a_j e^{i\delta_j} M_j|^2 \, dm_{ab}^2 \, dm_{bc}^2}{\int \left| \sum_k a_k e^{i\delta_k} M_k \right|^2 \, dm_{ab}^2 \, dm_{bc}^2},
\]
where $\mathcal{M}_j$ is defined by Eq. (2) and described in Ref. [22]. The sum of the fit fractions for all components will in general not be unity due to interference.

**Reconstruction Efficiency and Resolution:** The efficiency for reconstructing an event as a function of position on the Dalitz plot is in general non-uniform. Typically, a signal Monte Carlo sample, utilizing full GEANT [23] detector simulation generated with a uniform distribution in phase space is used to determine the efficiency. The variation in efficiency across the Dalitz plot varies with experiment and decay mode.

Finite detector resolution can usually be safely neglected as most resonances are comparatively broad. Notable exceptions where detector resolution effects must be modeled are $\phi \to K^+K^-$ and $\omega \to \pi^+\pi^-$. Additionally, the momenta of $a, b$, and $c$ can recalculated with a $D$ mass constraint. This forces the kinematical boundaries of the Dalitz plot to be strictly respected.

**Background Parametrization:** The contribution of background to the charm samples varies by experiment and final state. The background naturally falls into four categories: (i) purely combinatoric background containing no resonances, (ii) combinatoric background containing intermediate resonances, such as a real $K^*$ or $\rho$, plus additional random particles, (iii) mistagged decays such as a real $\overline{D}^0$ incorrectly identified as $D^0$ and (iv) misidentified $D$ daughters such as $D^+ \to \pi^-\pi^+\pi^+$ or $D_s^+ \to K^-K^+\pi^+$ reconstructed as $D^+ \to K^-\pi^+\pi^+$.

The contribution from combinatoric background resonances is distinct from the resonances in the signal because the former do not interfere with the latter since they are not from true $D$’s. The usual identification tag of the initial particle as a $D^0$ or a $\overline{D}^0$ is the charge of the distinctive slow pion in the decay sequence $D^{*+} \to D^0\pi^+$ or $D^{*-} \to \overline{D}^0\pi^-$. Another possibility is the identification of one of the $D$’s from $\psi(3770) \to D^0\overline{D}^0$. The mistagged background is subtle and may be mistakenly enumerated in the signal fraction determined by a $D^0$ mass fit. Mistagged decays contain true $\overline{D}^0$’s and so the resonances in the mistagged sample exhibit interference on the Dalitz plot.
Background from mis-identified $D$ daughters - if present - is the most problematic.

**Experimental Results:** The Dalitz plot analysis technique has been applied to the decays $D \rightarrow rc$, $r \rightarrow ab$ where the decay products $a$, $b$ and $c$ are $K$ or $\pi$ and the intermediate state $r$ is a scalar, vector or tensor meson. More generally the decay products could also be the pseudo-scalar $\eta$ or $\eta'$ mesons or narrow vector $\omega$ or $\phi$ mesons. The set of charm Dalitz plot analyses reported by experiments are listed in Table 2.

**Table 2: Reported Dalitz Plot Analyses.**

| Decay | Experiment(s) |
|-------|---------------|
| $D^0 \rightarrow K_S^0 \pi^+ \pi^-$ | Mark II [24], Mark III [25], E691 [27], E687 [26,29] |
|       | ARGUS [28], CLEO [1] |
| $D^0 \rightarrow K^- \pi^+ \pi^0$ | Mark III [25], E687 [29], E691 [27], CLEO [22] |
| $D^0 \rightarrow K^0 K^+ K^-$ | BABAR [30] |
| $D^0 \rightarrow K^0 K^0 K^+$ | BABAR [30] |
| $D^0 \rightarrow \pi^+ \pi^- \pi^0$ | CLEO [3] |
| $D^0 \rightarrow K_S^0 K^+ K^-$ | BABAR [30] |
| $D^+ \rightarrow K^- \pi^+ \pi^+$ | Mark III [25], E687 [29], E691 [27], E791 [8] |
| $D^+ \rightarrow K^+ K^- \pi^+$ | E687 [4], E791 [7], FOCUS [6] |
| $D^+ \rightarrow K^0 K^+ \pi^0$ | Mark III [25] |
| $D^+ \rightarrow \pi^+ \pi^+ \pi^-$ | E687 [4], E791 [5], FOCUS [6] |
| $D^+ \rightarrow K^+ K^- \pi^+$ | E687 [32], FOCUS [33] |
| $D_s^+ \rightarrow K^+ K^- \pi^+$ | E687 [32], FOCUS [33] |
| $D_s^+ \rightarrow \pi^+ \pi^+ \pi^-$ | E687 [4], E791 [5], FOCUS [6] |

$D^0 \rightarrow K_S^0 \pi^+ \pi^-$ — Several experiments have analyzed the decay $D^0 \rightarrow K_S^0 \pi^+ \pi^-$. The earliest analyses, by Mark II [24], Mark III [25], and E687 [26], assumed only two intermediate resonances, $K_S^0 \rho^0$, $K^*(892)^- \pi^+$, and a significant non-resonant component. Additional resonances were considered by E691 [27] but were not found to be statistically significant. ARGUS [28] and E687 [29], with more events, fit the Dalitz plot with six intermediate resonances: $K^*(892)^- \pi^+$, $K_0^*(1430)^- \pi^+$,
The nonresonant contribution was negligible. The early and later E687 results [26, 29] were consistent under similar assumptions. The most precise results are from CLEO [1], which includes three additional resonances: $K^0_S \rho(975)$, $K^*_0 f_2(1270)$, and $K^0_S f_0(1400)$. The early and later E687 results [26, 29] were consistent under similar assumptions.

It is not straightforward to compare or combine results using different descriptions of the angular distributions, barrier factors, resonant parametrizations, and different sets of resonances. Some of the earlier results [25–27], did not include barrier factors [16, 17]. Most of the earlier results [25–27, 29] used the Zemach formalism [18] to describe the angular shape of the decay pattern, while the more recent results [28, 1] use the helicity formalism [19].

The significance of the nonresonant component in the smaller data samples has been attributed to the presence of the broad scalar resonances $K^*_0(1430)^-$ and $f_0(1370)$ that were later observed in the larger data samples. The observation of a small but significant nonresonant component in the largest data samples suggests the presence of additional broad scalar resonances, the $\kappa(800)$ and $\sigma(500)$. The CLEO analysis could accommodate the $\sigma(500)$ in lieu of the nonresonant component, but found no evidence for the $\kappa(800)$.

**Table 3:** Dalitz fit results of $D^0 \to \pi^+ \pi^- \pi^0$.

| Resonance | Amplitude | Phase($^\circ$) | Fit fraction(%) |
|-----------|-----------|----------------|----------------|
| $\rho^+$  | 1. (fixed)| 0. (fixed)     | 76.5 ± 1.8 ± 4.8|
| $\rho^0$  | 0.56 ± 0.02 ± 0.07 | 10 ± 3 ± 3     | 23.9 ± 1.8 ± 4.6|
| $\rho^-$  | 0.65 ± 0.03 ± 0.04 | -4 ± 3 ± 4    | 32.3 ± 2.1 ± 2.2|
| nonresonant| 1.03 ± 0.17 ± 0.31 | 77 ± 8 ± 11  | 2.7 ± 0.9 ± 1.7  |
The BABAR [30] results for $D^0 \rightarrow K^0 K^+ K^-$ are given in Table 4. The non-$\phi$ resonant substructure in $K^+ K^-$ is significant. Resonant contributions from $a_0(980)^0$, $a_0(980)^+$, and $f_0(980)$ are observed. The nonresonant and the doubly Cabibbo-suppressed contributions are consistent with zero.

**Table 4:** Dalitz fit results of $D^0 \rightarrow K^0 K^+ K^-$.  

| Resonance          | Phase(°)     | Fit fraction(%) |
|--------------------|--------------|-----------------|
| $K^0 \phi$         | 0. (fixed)   | 45.4 ± 1.6 ± 1.0|
| $K^0 a_0(980)$     | 109 ± 5      | 60.9 ± 7.5 ± 13.3|
| $K^0 f_0(980)$     | -161 ± 14    | 12.2 ± 3.1 ± 8.6|
| $a_0(980)^+ K^-$   | -53 ± 4      | 34.3 ± 3.2 ± 6.8|
| $a_0(980)^- K^+$   | -13 ± 15     | 3.2 ± 1.9 ± 0.5 |
| nonresonant        | 40 ± 44      | 0.4 ± 0.3 ± 0.8 |

Charm Dalitz plot analyses might be useful for calibrating tools used in $B$ decays: specifically, to extract $\alpha$ from $B^0 \rightarrow \pi^+ \pi^- \pi^0$, $\beta$ from $B^0 \rightarrow K^0 K^+ K^-$, and $\gamma$ from $B^\pm \rightarrow DK^\pm$ followed by $D \rightarrow K^0 K^+ K^-$ or $D \rightarrow K^0 \pi^+ \pi^-$ [31].

**$D^+ \rightarrow \pi^+ \pi^+ \pi^-$: a $\sigma(500)$ or $f_0(600)$** — The decay $D^+ \rightarrow \pi^+ \pi^+ \pi^-$ has been studied by the E687 [4], E791 [7] and FOCUS [6] experiments. The E687 experiment considered the $\rho(770)^0 \pi^+$, $f_0(980) \pi^+$, $f_2(1270) \pi^+$, and a nonresonant component. The E791 experiment included in addition $f_0(1370) \pi^+$ and $\rho(1450)^0 \pi^+$. Both analyses found a very large fraction ($\sim 50\%$) for the nonresonant contribution, perhaps indicating a broad scalar contribution. E791 found the nonresonant amplitude to be consistent with zero if a broad scalar resonance was included in the fit. FOCUS analyzed its data sample using both the Breit-Wigner formalism and the $K$-matrix formalism. The Breit-Wigner analysis included $\rho(770), f_0(980), f_2(1270), f_0(1500), \sigma(500)$, and a nonresonant contribution. Applying the $K$-matrix formalism to the S wave and parameterizing the $\rho(770)$ and $f_2(1270)$ with the Breit-Wigner functions also described the FOCUS data well.
None of these analyses has modeled the dynamics of the $\pi^+\pi^+$ interaction. Consideration of the $I = 2$ S-wave and D-wave phase shifts, also measured in $\pi^+p \rightarrow \pi^+\pi^+n$ [34], could affect the $\pi^+\pi^-$ S-wave result.

E791 finds additional evidence that the low mass $\pi\pi$ feature is resonant by examining the phase of the $\pi\pi$ amplitude in the vicinity of the reported $\sigma(500)$ mass. A phase variation with invariant $\pi\pi$ mass is consistent with a resonant contribution [35].

Table 5 gives the parameters of the $\sigma(500)$ determined in charm Dalitz plot analyses. A consistent relative phase between the $\sigma(500)$ and $\rho(770)$ resonances is observed.

**Table 5:** Parameters of the $\sigma(500)$ resonance.

| Experiment | E791 [7] | CLEO [1] | FOCUS [6] |
|------------|----------|----------|-----------|
| Decay Mode | $D^+ \rightarrow \pi^+\pi^+\pi^-$ | $D^0 \rightarrow K_S^0\pi^+\pi^-$ | $D^+ \rightarrow \pi^+\pi^+\pi^-$ |
| Amplitude  | 1.17 $\pm$ 0.13 $\pm$ 0.06 | 0.57 $\pm$ 0.13 | 0  |
| Phase(°)   | 205.7 $\pm$ 8.0 $\pm$ 5.2 | 214 $\pm$ 11 | 200 $\pm$ 31 |
| $m$(MeV/c$^2$) | 478$^{+24}_{-23}$ $\pm$ 17 | 513 $\pm$ 32 | 443 $\pm$ 27 |
| $\Gamma$(MeV/c$^2$) | 324$^{+12}_{-10}$ $\pm$ 21 | 335 $\pm$ 67 | 443 $\pm$ 80 |

$D^+ \rightarrow K^-\pi^+\pi^+$: a $\kappa(800)$? — Indication of a broad $K\pi$ scalar intermediate resonance has been reported by E791 in the decay $D^+ \rightarrow K^-\pi^+\pi^+$ [8]. Fitting the Dalitz plot with $\bar{K}^+(892)^0\pi^+$, $\bar{K}_0^+(1430)^0\pi^+$, $\bar{K}_2^+(1430)^0\pi^+$, and $\bar{K}^+(1680)^0\pi^+$, plus a constant nonresonant component, E791 finds results consistent with earlier results from E691 and E687 with a nonresonant fit fraction of over 90%. Having reconstructed more events than the other experiments, E791 was led to include an extra low-mass S-wave $K\pi$ resonance to account for the poor fit already seen by earlier experiments: A $\kappa(800)$ with $m = 797 \pm 19 \pm 43$ MeV/c$^2$ and $\Gamma = 410 \pm 43 \pm 87$ MeV/c$^2$ much improved the fits. The $\kappa(800)$ is now the dominant resonance and the nonresonant fit fraction is reduced from 90.9 $\pm$ 2.6% to 13.0 $\pm$ 5.8 $\pm$ 4.4%. As discussed with the $\sigma(500)$, the $K^-\pi^+$
S-wave result could be affected by modeling the dynamics of the $I = 2 \pi^+\pi^+$ interaction.

E791 also modeled the $K\pi$ S-wave phase variation as a function of $K\pi$ mass with the $K_0^+(1430)$ resonance only and a nonresonant component following the parameterization of LASS [36]. It was necessary to relax the unitarity constraint to describe the E791 data [37]. The $K\pi$ S-wave phase behavior in this model is consistent with the model that includes the $\kappa$ resonance.

CLEO allowed scalar $K\pi$ resonances in the fit to $D^0 \to K^-\pi^+\pi^0$ [22] and $D^0 \to K^0_S\pi^+\pi^-$ [1] and observed a significant contribution for only $K^*_0(1430)$ [38]. BABAR has analyzed the decay $D^0 \to K^0K^-\pi^+$ and $D^0 \to \overline{K}^0K^+\pi^-$ [30]. They fit the former Dalitz plot with both positively charged and neutral $K(892), \overline{K}_0(1430), \overline{K}_2(1430), \overline{K}^*(1680)$ and $a_0(980)^-, a_0(1450)^-, a_2(1310)^-$ resonances, and a nonresonant component. The second Dalitz plot is fit with the identical resonances except for the $a_2(1310)^-$. A good fit is obtained in both cases without including the $\kappa$.

$f_0(980), f_0(1370)$ and $f_0(1500)$ — The proximity of the $K\overline{K}$ threshold requires a coupled-channel or Flatté parametrization [21] of the $f_0(980)$ in charm Dalitz plot analyses. The width of the $f_0(980)$ is poorly known. E791 used a coupled-channel Breit-Wigner function, following the parametrization of Ref. [39], to describe the $f_0(980)$ in $D_s^+ \to \pi^+\pi^+\pi^- [5]$, and measured $m_r = 977\pm3\pm2\text{ MeV}/c^2, g_{\pi\pi} = 0.09\pm0.01\pm0.01$, and $g_{KK} = 0.02\pm0.04\pm0.03$. Results similar to these are desirable for input to the analysis of the $D_s^+ \to K^+K^-\pi^+$ [33], which includes the $f_0(980)$ and $a_0(980)$.

The quark content of the $f_0(1370)$ and $f_0(1500)$ can perhaps be inferred from how they populate various Dalitz plots. The E791 analysis of $D^+ \to \pi^+\pi^+\pi^- [7]$ finds a contribution from the $f_0(1370)$ but not the $f_0(1500)$. The FOCUS analysis [6] of this decay does not find a significant contribution from the $f_0(1370)$. For the $D_s^+ \to \pi^+\pi^+\pi^-$, E687 [4] and FOCUS [6] do not see the $f_0(1370)$ but do see a resonance with parameters similar to the $f_0(1500)$, while E791 [5] observes a $\pi\pi$ resonance ($m = 1434\pm18\pm9\text{ MeV}/c^2$ and $\Gamma = 172\pm32\pm6\text{ MeV}/c^2$).
that is not consistent with either meson. BABAR has found no evidence for either the $f_0(1370)$ or the $f_0(1500)$ in $D^0 \to \overline{K}^0 K^+ K^-$ [30], while CLEO has observed the $f_0(1370)$ in $D^0 \to K_S^0 \pi^+ \pi^-$ [1]. Future analyses will present a clearer picture only if the same resonances and model of decay amplitudes are applied to all Dalitz plot fits.

**Doubly Cabibbo-Suppressed Decays** — There are two classes of multibody doubly Cabibbo-suppressed (DCS) decays of charm mesons. The first consists of those in which the DCS and corresponding Cabbibo-favored (CF) decays populate distinct Dalitz plots: the pairs $D^0 \to K^+ \pi^- \pi^0$ and $D^0 \to K^- \pi^+ \pi^0$, or $D^+ \to K^+ \pi^+ \pi^-$ and $D^+ \to K^- \pi^+ \pi^+$, are examples. CLEO [2] has reported $B(D^0 \to K^+ \pi^- \pi^0) / B(D^0 \to K^- \pi^+ \pi^0) = (0.43^{+0.11}_{-0.10} \pm 0.07)\%$.

The second class consists of decays where the DCS and CF modes populate the same Dalitz plot: for example, $D^0 \to K^{*-} \pi^+$ and $D^0 \to K^{*+} \pi^-$ both contribute to $D^0 \to K_S^0 \pi^+ \pi^-$. In this case, the potential for interference of DCS and CF amplitudes increases the sensitivity to the DCS amplitude. CLEO [1] has reported the relative amplitudes and phases to be $(7.1 \pm 1.3^{+2.6}_{-0.6} \pm 0.6)\%$ and $(189 \pm 10 \pm 3^{+15}_{-5})^\circ$, respectively, corresponding to $B(D^0 \to K^*(892)^+ \pi^-) / B(D^0 \to K^*(892)^- \pi^+) = (0.5 \pm 0.2^{+0.5}_{-0.1} \pm 0.4)\%$.

**CP Violation** — In the limit of $CP$ conservation, charge conjugate decays will have the same Dalitz plot distribution. The $D^{*\pm}$ tag enables the discrimination between $D^0$ and $\overline{D}^0$. The integrated $CP$ violation across the Dalitz plot is determined from

$$A_{CP} = \frac{\int |M|^2 - |\overline{M}|^2}{\int |M|^2 + |\overline{M}|^2} \frac{d m_{ab}^2 \ dm_{bc}^2}{\int d m_{ab}^2 \ dm_{bc}^2},$$

where $M$ and $\overline{M}$ are the $D^0$ and $\overline{D}^0$ Dalitz plot amplitudes. This expression is less sensitive to $CP$ violation than the individual resonant submodes reported in Ref. [40]. Table 6 reports the results for $CP$ violation. No evidence of $CP$ violation has been observed.
Table 6: Dalitz-plot-integrated $CP$ violation.

| Experiment | Decay mode | $\mathcal{A}_{CP}$(%) |
|------------|------------|------------------------|
| CLEO [22]  | $D^0 \rightarrow K^-\pi^+\pi^0$ | $-3.1 \pm 8.6$         |
| CLEO [2]   | $D^0 \rightarrow K^+\pi^-\pi^0$ | $+9^{+22}_{-25}$       |
| CLEO [40]  | $D^0 \rightarrow K^0_S\pi^+\pi^-$ | $-0.9 \pm 2.1^{+1.0+1.3}_{-4.3-3.7}$ |
| CLEO [3]   | $D^0 \rightarrow \pi^+\pi^-\pi^0$ | $+1^{+9}_{-7} \pm 9$  |

The possibility of interference between $CP$–conserving and $CP$–violating amplitudes provides a more sensitive probe of $CP$ violation. The constraints on the square of the $CP$–violating amplitude obtained in the resonant submodes of $D^0 \rightarrow K^0_S\pi^+\pi^-$ range form $(3.5$ to $28.4) \times 10^{-4}$ at $95\%$ confidence level [40].

Guidance for Future Analyses: It is essential that a consistent formalism be adopted by all experiments that report Charm Dalitz plot analyses. This is necessary for the Particle Data Group to sensibly combine results from different experiments. Differences in the parametrizations of the angular distribution $Z$, the barrier factors $B_L$ and the dynamical function $T_r$, as well as the set of resonances $r$, complicate the comparison of results from different experiments.

As the sizes of the data samples increase, the dynamical model which describes the intermediate resonances must be improved. Replacing the sum of Breit-Wigner amplitudes with a unitary parametrization such as the $K$-matrix is important. Of course, other dynamical models should be studied. In particular, the phase shift associated with $\pi^+\pi^+$ in $D^+_s \rightarrow K^-\pi^+\pi^+, \pi^-\pi^+\pi^+$ should not be neglected and momentum-dependent form-factors in Eq. (3) should be considered.

Finally, for a consistent picture of the $\pi\pi$ and $K\pi$ scalar mesons to emerge from charm Dalitz plot analyses many decays must be evaluated simultaneously. This will allow resonances decaying to different final states to be analyzed together. The same set of intermediate states can be used for all final states and input (masses, widths, couplings, spectra) from scattering experiments like $\pi^-p \rightarrow \pi^0\pi^0n$, radiative decays such as $\phi, J/\psi \rightarrow \gamma\pi\pi$ and semileptonic decays such as $D \rightarrow K\pi\ell\nu$ can be consistently incorporated.
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