On the Existence of Frames of Some Extremal Odd Unimodular Lattices and Self-Dual $\mathbb{Z}_k$-Codes

Masaaki Harada* and Tsuyoshi Miezaki†

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Abstract

For some extremal (optimal) odd unimodular lattices $L$ in dimensions $n = 12, 16, 20, 32, 36, 40$ and $44$, we determine all positive integers $k$ such that $L$ contains a $k$-frame. This result yields the existence of an extremal Type I $\mathbb{Z}_k$-code of lengths $12, 16, 20, 32, 36, 40$ and $44$ and a near-extremal Type I $\mathbb{Z}_k$-code of length $28$ for positive integers $k$ with only a few exceptions.

1 Introduction

Self-dual codes and unimodular lattices are studied from several viewpoints (see [12] for an extensive bibliography). Many relationships between self-dual codes and unimodular lattices are known and there are similar situations between two subjects. As a typical example, it is known that a unimodular lattice $L$ contains a $k$-frame if and only if there exists a self-dual $\mathbb{Z}_k$-code $C$ such that $L$ is isomorphic to the lattice obtained from $C$ by Construction A, where $\mathbb{Z}_k$ is the ring of integers modulo $k$.

*Department of Mathematical Sciences, Yamagata University, Yamagata 990–8560, Japan. email: mharada@sci.kj.yamagata-u.ac.jp
†Faculty of Education, Art and Science, Yamagata University, Yamagata 990–8560, Japan. email: miezaki@e.yamagata-u.ac.jp
As described in [37], self-dual codes are an important class of linear codes for both theoretical and practical reasons. It is a fundamental problem to classify self-dual codes of modest lengths and determine the largest minimum weight among self-dual codes of that length. Type II \( \mathbb{Z}_{2k} \)-codes were defined in [2] as a class of self-dual codes, which are related to even unimodular lattices. For binary Type II codes, much work has been done concerning the above fundamental problem (see e.g. [3, 9, 12, 37]). For general \( k \), if \( C \) is a Type II \( \mathbb{Z}_{2k} \)-code of length \( n \leq 136 \) then we have the bound on the minimum Euclidean weight \( d_E(C) \) of \( C \) as follows: \( d_E(C) \leq 4k \lfloor \frac{n}{24} \rfloor + 4k \) for every positive integer \( k \) (see [20]). We say that a Type II \( \mathbb{Z}_{2k} \)-code meeting the bound with equality is extremal for length \( n \leq 136 \). It was shown in [7, 14] that the Leech lattice, which is one of the most remarkable lattices, contains a \( 2k \)-frame for every positive integer \( k \geq 2 \). This result yields the existence of an extremal Type II \( \mathbb{Z}_{2k} \)-code of length 24 for every positive integer \( k \). Recently, the existence of an extremal Type II \( \mathbb{Z}_{2k} \)-code of length \( n = 32, 40, 48, 56, 64 \) has been established by the authors [20] for every positive integer \( k \). This was done by finding a \( 2k \)-frame in some extremal even unimodular lattices in these dimensions \( n \).

Recently, it was shown in [29] that the odd Leech lattice contains a \( k \)-frame for every positive integer \( k \) with \( k \geq 3 \). This motivates our investigation of the existence of a \( k \)-frame in extremal odd unimodular lattices. In this paper, for some extremal (optimal) odd unimodular lattices \( L \) in dimensions \( n = 12, 16, 20, 32, 36, 40 \) and 44, we determine all integers \( k \) such that \( L \) contains a \( k \)-frame. This result yields the existence of an extremal Type I \( \mathbb{Z}_k \)-code of lengths 12, 16, 20, 32, 36, 40 and 44 and a near-extremal Type I \( \mathbb{Z}_k \)-code of length 28 for positive integers \( k \) with only a few small exceptions.

This paper is organized as follows. In Section 2, we give definitions and some basic properties of self-dual codes and unimodular lattices used in this paper. The notion of extremal Type I \( \mathbb{Z}_k \)-codes of length \( n \) is given for \( n \leq 48 \) and \( k \geq 2 \). Lemma 2.1 gives a reason why we consider unimodular lattices in only dimension \( n \equiv 0 \) (mod 4). In Section 3, using the theory of modular forms (see [30] for details), we derive some number theoretical results (Theorem 3.2), which are used in Section 4. In Section 4, we provide a method for constructing \( m \)-frames in unimodular lattices, which are constructed from some self-dual \( \mathbb{Z}_k \)-codes by Construction A (Proposition 4.1). This method is a generalization of Propositions 3.3 and 3.6 in [20]. Using Theorem 3.2 and Proposition 4.1, we give \( k \)-frames in the unique extremal odd unimodular lattice in dimensions \( n = 12, 16 \) and some extremal (op-
imal) odd unimodular lattices $L$ in dimensions $n = 20, 32, 36, 40$ and $44$, which are listed in Table 8 for all positive integers $k$ satisfying the condition $(\star)$ in Table 8 (Lemma 4.3). In Section 5, some extremal (near-extremal) Type I $Z_k$-codes are constructed for some integers $k$. Then we establish the existence of a $k$-frame in the extremal (optimal) unimodular lattices $L$ in dimension $n = 12, 16, 20, 28, 32, 36$, which are listed in Table 8 (except only lattices $A_3(C_{20,3}(D'_{10}))$ and $A_5(C_{20,5}(D''_{10}))$), for every positive integer $k$ with $k \geq 2$. We also discuss the positivity of coefficients of the theta series of some extremal (optimal) unimodular lattices in dimension $n \leq 36$. When $n = 40, 44$, it is shown that there is an extremal odd unimodular lattice in dimension $n$ containing a $k$-frame for every positive integer $k$ with $k \geq 4$. As a consequence, the existence of an extremal Type I $Z_k$-code of lengths $n = 12, 16, 20, 32, 36, 40, 44$ and a near-extremal Type I $Z_k$-code of length $n = 28$ is established for a positive integer $k$, where $k \neq 1, 3$ if $n = 32$ and $k \neq 1$ otherwise. Finally, in Section 6, we investigate the existence of a $k$-frame in optimal odd unimodular lattices in dimension 48.

All computer calculations in this paper were done by Magma [4].

2 Preliminaries

In this section, we give definitions and some basic properties of self-dual codes and unimodular lattices used in this paper.

2.1 Self-dual codes

Let $Z_k$ be the ring of integers modulo $k$, where $k$ is a positive integer. In this paper, we always assume that $k \geq 2$ and we take the set $Z_k$ to be $\{0, 1, \ldots, k - 1\}$. A $Z_k$-code $C$ of length $n$ (or a code $C$ of length $n$ over $Z_k$) is a $Z_k$-submodule of $Z_k^n$. A $Z_2$-code and a $Z_3$-code are called binary and ternary, respectively. The Euclidean weight of a codeword $x = (x_1, \ldots, x_n)$ of $C$ is $\sum_{\alpha=1}^{[k/2]} n_{\alpha}(x)\alpha^2$, where $n_{\alpha}(x)$ denotes the number of components $i$ with $x_i \equiv \pm \alpha \pmod{k}$ ($\alpha = 1, 2, \ldots, [k/2]$). It is trivial that the Euclidean weight is the same as the (usual) Hamming weight for the case $k = 2, 3$. The minimum Euclidean weight $d_E(C)$ of $C$ is the smallest Euclidean weight.
among all nonzero codewords of $C$.

A $\mathbb{Z}_k$-code $C$ is self-dual if $C = C^\perp$, where the dual code $C^\perp$ of $C$ is defined as $C^\perp = \{ x \in \mathbb{Z}_k^n \mid x \cdot y = 0 \text{ for all } y \in C \}$ under the standard inner product $x \cdot y$. A Type II $\mathbb{Z}_2k$-code was defined in [2] as a self-dual code with the property that all Euclidean weights are divisible by $4k$. It is known that a Type II $\mathbb{Z}_k$-code of length $n$ exists if and only if $n$ is divisible by eight and $k$ is even [2]. A self-dual code which is not Type II is called Type I.

Two self-dual $\mathbb{Z}_k$-codes $C$ and $C'$ are equivalent if there exists a monomial $(\pm 1, 0)$-matrix $P$ with $C' = C \cdot P = \{ xP \mid x \in C \}$. The automorphism group $\text{Aut}(C)$ of $C$ is the group of all monomial $(\pm 1, 0)$-matrices $P$ with $C = C \cdot P$.

### 2.2 Unimodular lattices

A (Euclidean) lattice $L \subset \mathbb{R}^n$ in dimension $n$ is unimodular if $L = L^*$, where the dual lattice $L^*$ of $L$ is defined as $\{ x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L \}$ under the standard inner product $(x, y)$. Two lattices $L$ and $L'$ are isomorphic, denoted $L \cong L'$, if there exists an orthogonal matrix $A$ with $L' = L \cdot A$. The norm of a vector $x$ is defined as $(x, x)$. The minimum norm $\text{min}(L)$ of a unimodular lattice $L$ is the smallest norm among all nonzero vectors of $L$.

The theta series $\theta_L(q)$ of $L$ is the formal power series $\theta_L(q) = \sum_{x \in L} q^{(x,x)}$. The kissing number of $L$ is the second nonzero coefficient of the theta series.

A unimodular lattice with even norms is said to be even, and that containing a vector of odd norm is said to be odd. An even unimodular lattice in dimension $n$ exists if and only if $n \equiv 0 \pmod{8}$, while an odd unimodular lattice exists for every dimension. Two lattices $L$ and $L'$ are neighbors if both lattices contain a sublattice of index 2 in common.

Let $L$ be a unimodular lattice. Define $L_0 = \{ x \in L \mid (x, x) \equiv 0 \pmod{2} \}$. Then $L_0$ is a sublattice of $L$ of index 2 if $L$ is odd and $L_0 = L$ if $L$ is even. The shadow $S$ of $L$ is defined as $S = L_0^* \setminus L$ if $L$ is odd and as $S = L$ if $L$ is even [11]. Now suppose that $L$ is an odd unimodular lattice. Then there are cosets $L_1, L_2, L_3$ of $L_0$ such that $L_0^* = L_0 \cup L_1 \cup L_2 \cup L_3$, where $L = L_0 \cup L_2$ and $S = L_1 \cup L_3$. If $L$ is an odd unimodular lattice in dimension divisible by eight, then $L$ has two even unimodular neighbors of $L$, namely, $L_0 \cup L_1$ and $L_0 \cup L_3$.

Rains and Sloane [36] showed that a unimodular lattice $L$ in dimension $n$ has minimum norm $\text{min}(L) \leq 2\lfloor \frac{n}{24} \rfloor + 2$ unless $n = 23$ when $\text{min}(L) \leq 3$ (see [38] for the case that $L$ is even). A unimodular lattice meeting the bound with equality is called extremal. Gaultier [13] showed that any unimodular
lattice in dimension $24k$ meeting the upper bound has to be even, which was conjectured in [36]. Hence, an odd unimodular lattice $L$ in dimension $24k$ satisfies $\min(L) \leq 2k + 1$. We say that an odd unimodular lattice with the largest minimum norm among all odd unimodular lattices in that dimension is optimal.

2.3 Construction $A$ and $k$-frames

We give a method to construct unimodular lattices from self-dual $\mathbb{Z}_k$-codes, which is referred to as Construction $A$ (see [2, 21]). Let $\rho$ be a map from $\mathbb{Z}_k$ to $\mathbb{Z}$ sending $0, 1, \ldots, k-1$ to $0, 1, \ldots, k-1$, respectively. If $C$ is a self-dual $\mathbb{Z}_k$-code of length $n$, then the lattice

$$A_k(C) = \frac{1}{\sqrt{k}}\{\rho(C) + k\mathbb{Z}^n\}$$

is a unimodular lattice in dimension $n$, where $\rho(C) = \{(\rho(c_1), \ldots, \rho(c_n)) \mid (c_1, \ldots, c_n) \in C\}$. The minimum norm of $A_k(C)$ is $\min\{k, d_E(C)/k\}$. Moreover, $C$ is a Type II $\mathbb{Z}_{2k}$-code if and only if $A_{2k}(C)$ is an even unimodular lattice [2].

A set $\{f_1, \ldots, f_n\}$ of $n$ vectors $f_1, \ldots, f_n$ of a unimodular lattice $L$ in dimension $n$ with $(f_i, f_j) = k\delta_{i,j}$ is called a $k$-frame of $L$, where $\delta_{i,j}$ is the Kronecker delta. It is known that a unimodular lattice $L$ contains a $k$-frame if and only if there exists a self-dual $\mathbb{Z}_k$-code $C$ with $A_k(C) \cong L$ (see [21]).

By the following lemma, it is enough to consider a $p$-frame in an odd unimodular lattice for each prime $p$. The lemma also gives a reason why we consider unimodular lattices in only dimension $n \equiv 0 \pmod{4}$.

**Lemma 2.1** (Chapman [7, Lemma 5.1]). *If a lattice $L$ in dimension $n \equiv 0 \pmod{4}$ contains a $k$-frame, then $L$ contains a $km$-frame for every positive integer $m$.***
2.4 Upper bounds on the minimum Euclidean weights

A self-dual $Z_k$-code $C$ of length $n$ satisfies the bound:

$$d_E(C) \leq \begin{cases} 
4 \left\lfloor \frac{n}{24} \right\rfloor + 4 & \text{if } k = 2, n \not\equiv 22 \pmod{24} \\
4 \left\lfloor \frac{n}{24} \right\rfloor + 6 & \text{if } k = 2, n \equiv 22 \pmod{24} \\
3 \left\lfloor \frac{n}{12} \right\rfloor + 3 & \text{if } k = 3 \\
8 \left\lfloor \frac{n}{24} \right\rfloor + 8 & \text{if } k = 4, n \not\equiv 23 \pmod{24} \\
8 \left\lfloor \frac{n}{24} \right\rfloor + 12 & \text{if } k = 4, n \equiv 23 \pmod{24} 
\end{cases}$$

(1)

[26, 35, 36]. Note that a binary self-dual code of length divisible by 24 meeting the bound must be Type II [35].

Although the following lemmas are somewhat trivial, we give proofs for the sake of completeness.

Lemma 2.2. Let $C$ be a self-dual $Z_k$-code of length $n$. If $n \not\equiv 23$ and $k \geq 2 \left\lfloor \frac{n}{24} \right\rfloor + 3$, then $d_E(C) \leq 2k \left\lfloor \frac{n}{24} \right\rfloor + 2k$. If $n = 23$ and $k \geq 4$ then $d_E(C) \leq 3k$.

Proof. Since both cases are similar, we only consider the case that $n \not\equiv 23$ and $k \geq 2 \left\lfloor \frac{n}{24} \right\rfloor + 3$. Note that the Euclidean weight of a codeword of $C$ is divisible by $k$. Suppose that $d_E(C) \geq 2k \left\lfloor \frac{n}{24} \right\rfloor + 3k$. Since $\min(A_k(C)) = \min\{k, d_E(C)/k\}$, $\min(A_k(C)) \geq 2 \left\lfloor \frac{n}{24} \right\rfloor + 3$, which is a contradiction to the upper bound on the minimum norms of unimodular lattices.

Lemma 2.3. Let $C$ be a self-dual $Z_k$-code of length 48. Then $d_E(C) \leq 6k$ for every positive integer $k$ with $k \geq 2$.

Proof. By the bound (1) and Lemma 2.2, it is sufficient to consider the cases for only $k = 5, 6$. Assume that $k = 5, 6$ and $d_E(C) \geq 7k$. Since $k < d_E(C)/k$, $\min(A_k(C)) = k$ and the kissing number of $A_k(C)$ is 96. Note that unimodular lattices $L$ with $\min(L) = 6$ and 5 are extremal even unimodular lattices and optimal odd unimodular lattices, respectively. However, the kissing numbers of such lattices are 52416000 (see [12, Chap. 7]) and 385024 or 393216 [13], respectively. This is a contradiction.

Hence, if $C$ is a self-dual $Z_k$-code $C$ of length $n \leq 48$ then we have the following bound:

$$d_E(C) \leq \begin{cases} 
3k & \text{if } n = 23 \text{ and } k \geq 4 \\
4 \left\lfloor \frac{n}{24} \right\rfloor + 6 & \text{if } n = 22, 46 \text{ and } k = 2 \\
20 & \text{if } n = 47 \text{ and } k = 4 \\
2k \left\lfloor \frac{n}{24} \right\rfloor + 2k & \text{otherwise.}
\end{cases}$$
We say that a self-dual $\mathbb{Z}_k$-code meeting the bound with equality is extremal\footnote{For $k = 3$, a self-dual code meeting the bound \footnote{1} is usually called extremal. However, we here adopt this definition since we consider the existence of extremal self-dual $\mathbb{Z}_k$-codes for all positive integers $k$ with $k \geq 2$, at once.} for length $n \leq 48$. We say that a self-dual code $C$ is near-extremal if $d_E(C) + k$ meets the bound.

The following lemma shows that an extremal self-dual $\mathbb{Z}_k$-code of lengths 24 and 48 must be Type II for every even positive integer $k$.

**Lemma 2.4.** (a) Let $C$ be a Type I $\mathbb{Z}_k$-code of length 24. Then $d_E(C) \leq 3k$ for every positive integer $k$ with $k \geq 2$.

(b) Let $C$ be a Type I $\mathbb{Z}_k$-code of length 48. Then $d_E(C) \leq 5k$ for every positive integer $k$ with $k \geq 2$.

**Proof.** We give the proof of (b). By the bound \footnote{1}, it is sufficient to consider only $k \geq 4$. Assume that $k \geq 4$ and $d_E(C) = 6k$. If $k \geq 6$ then $A_k(C)$ has minimum norm 6. Hence, $A_k(C)$ must be even, that is, $C$ is Type II, which is a contradiction. Suppose that $k = 5$. Then $A_5(C)$ is an optimal odd unimodular lattice with kissing number 96, which contradicts that the kissing number is 385024 or 393216 \footnote{1}. Finally, suppose that $k = 4$. Since $d_E(C) = 24$, $A_4(C)$ satisfies the condition that $\min(A_4(C)) = 4$, the kissing number is 96 and there is no vector of norm 5. By \footnote{1} (2) and (3)], one can determine the possible theta series of $A_4(C)$ and its shadow $S$ as follows:

\[
\begin{align*}
\theta_{A_4(C)}(q) &= 1 + 96q^4 + (35634176 + 16777216\alpha)q^6 + \cdots, \\
\theta_S(q) &= \alpha + (96 - 96\alpha)q^2 + (-4416 + 4512\alpha)q^4 + \cdots,
\end{align*}
\]

respectively, where $\alpha$ is an integer. From the coefficients of $\theta_S(q)$, it follows that $\alpha = 1$. Hence, since $S$ contains the zero-vector, $A_4(C)$ must be even, that is, $C$ is Type II.

The proof of (a) is similar to that of (b), and it can be completed more easily. So the proof is omitted. \hfill $\square$

The odd Leech lattice contains a $k$-frame for every positive integer $k$ with $k \geq 3$ \footnote{29}. The binary odd Golay code is a near-extremal Type I code of length 24. Hence, there is a near-extremal Type I $\mathbb{Z}_k$-code of length 24 for every positive integer $k$ with $k \geq 2$. 

\footnote{1}For $k = 3$, a self-dual code meeting the bound \footnote{1} is usually called extremal. However, we here adopt this definition since we consider the existence of extremal self-dual $\mathbb{Z}_k$-codes for all positive integers $k$ with $k \geq 2$, at once.
2.5 Negacirculant matrices

An $n \times n$ matrix $M$ is negacirculant if $M$ has the following form:

$$
\begin{pmatrix}
  r_0 & r_1 & \cdots & r_{n-1} \\
  -r_{n-1} & r_0 & \cdots & r_{n-2} \\
  -r_{n-2} & -r_{n-1} & \cdots & r_{n-3} \\
  \vdots & \vdots & \ddots & \vdots \\
  -r_1 & -r_2 & \cdots & r_0
\end{pmatrix}.
$$

Most of matrices constructed in this paper are based on negacirculant matrices. In Section 5, in order to construct self-dual $\mathbb{Z}_k$-codes of length $4n$, we consider a generator matrix of the following form:

$$
\begin{pmatrix}
  I_{2n} & A & B \\
  -B^T & A^T
\end{pmatrix},
$$

where $A$ and $B$ are $n \times n$ negacirculant matrices, $A^T$ denotes the transpose of the matrix $A$ and $I_k$ denotes the identity matrix of order $k$. It is easy to see that the code is self-dual if $AA^T + BB^T = -I_n$.

In Section 4, in order to find $k$-frames in some lattices, we need to construct matrices $M$ satisfying the condition (11) in Proposition 4.1. Suppose that $p$ is a prime and $p \equiv 3 \pmod{4}$. Let $Q_p = (q_{ij})$ be a $p$ by $p$ matrix where $q_{ij} = 0$ if $i = j$, $-1$ if $j - i$ is a nonzero square $\pmod{p}$, and $1$ otherwise. We consider the following matrix:

$$
P_{p+1} = \begin{pmatrix}
  0 & 1 & \cdots & 1 \\
  -1 & & & \\
  \vdots & & Q_p & \\
  -1 & & &
\end{pmatrix}.
$$

Then it is well known that $P_{p+1}P_{p+1}^T = pI_{p+1}$ and $P_{p+1}^T = -P_{p+1}$, and $P_{p+1} + I_{p+1}$ is a Hadamard matrix, which is equivalent to the Paley Hadamard matrix of order $p+1$. Hence, these matrices satisfy (11). In Section 4, we construct more matrices $M$ satisfying (11) using the following form:

$$
\begin{pmatrix}
  A & B \\
  -B^T & A^T
\end{pmatrix},
$$

where $A$ and $B$ are $n \times n$ negacirculant matrices.
2.6 Positivity of coefficients of the theta series

It is important to study the positivity and non-negativity of coefficients of the theta series of extremal unimodular lattices. For example, let \( \sum_{m=0}^{\infty} A_m q^m \) be the theta series of an even unimodular lattice in dimension \( n \). Then it was shown in [38] that the coefficient \( A_{2 \lfloor \frac{n}{24} \rfloor + 2} \) is always positive when \( A_2 = A_4 = \cdots = A_{2 \lfloor \frac{n}{24} \rfloor} = 0 \) (see also [25]). This gives the upper bound of the minimum norm of even unimodular lattices as in Section 2.2.

To discuss the positivity of coefficients of the theta series of extremal (optimal) unimodular lattices listed in Table 8, the following lemma is used.

**Lemma 2.5.** Let \( L \) be a unimodular lattice in dimension \( n \) with theta series \( \sum_{m=0}^{\infty} A_m q^m \). If \( L \) contains a \( k \)-frame then \( A_k \geq 2n \).

**Remark 2.6.** As described in Section 1, the odd Leech lattice contains a \( k \)-frame for every positive integer \( k \) with \( k \geq 3 \) [29]. By the above lemma, \( A_m \geq 48 \) for every positive integer \( m \) with \( m \geq 3 \).

**Remark 2.7.** At dimensions \( n = 20, 28, 32 \), there are other unimodular lattices with the same theta series as one of the unimodular lattices listed in Table 8. Of course, it also holds that \( A_m \geq 2n \) for every positive integer \( m \geq 2, 3, 4 \) if \( n = 20, 28, 32 \), respectively, for the other lattices.

3 Number theoretical results

Let \( k, \ell \) and \( m \) be positive integers with \( k \geq 2 \) and \( \ell \leq k - 1 \). Consider the following lattice in dimension 4:

(4) \( L_{\ell,m,k} = \{(a, b, c, d) \in \mathbb{Z}^4 \mid b \equiv c-\ell d \pmod{k} \text{ and } d \equiv a+\ell b \pmod{k}\} \),

where we consider the inner product \( \langle x, y \rangle \) induced by \( (a^2 + mb^2 + c^2 + md^2)/k \), instead of the standard inner product. The theta series \( \theta_{L_{\ell,m,k}}(q) \) of \( L_{\ell,m,k} \) is \( \sum_{x \in L_{\ell,m,k}} q^{\langle x, x \rangle} \).

**Lemma 3.1.** If \( m + \ell^2 \equiv -1 \pmod{k} \) then \( \theta_{L_{\ell,m,k}}(z) \) is a modular form (of weight 2) for \( \Gamma_0(4m) \), where \( q = e^{2\pi i z} \), \( z \) is in the upper half plane and

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.
\]
Proof. The lattice $L_{\ell,m,k}$ is spanned by $(k,0,0)$, $(0,0,k)$, $(1,0,\ell+1,1)$ and $(0,1,\ell^2+1,\ell)$ with Gram matrix:

$$M = \begin{pmatrix} k & 0 & 1 & 0 \\ 0 & k & \ell & \ell^2+1 \\ 1 & \ell & (\ell^2+m+1)/k & (\ell^2+m+1)\ell/k \\ 0 & \ell^2+1 & (\ell^2+m+1)\ell/k & (\ell^2+m+1)(\ell^2+1)/k \end{pmatrix}.$$  

Since

$$mM^{-1} = \begin{pmatrix} (1+\ell^2+m)/k & 0 & -1-\ell^2 & \ell \\ 0 & (1+\ell^2+m)/k & 0 & -1 \\ -1-\ell^2 & 0 & k+k\ell^2 & -k\ell \\ \ell & -1 & -k\ell & k \end{pmatrix}$$

and $m + \ell^2 \equiv -1 \pmod{k}$, $mM^{-1}$ has integer entries. Since $\det M = m^2$, $\theta_{L_{\ell,m,k}}(z)$ is a modular form of weight 2 for $\Gamma_0(4m)$ [30, Corollary 4.9.2]. $\square$

In order to give infinite families of $k$-frames by Proposition 4.1, we derive the following theorem. Its proof is similar to that in [7, 20, 29], but this is more complicated. Our notation and terminology for modular forms follow from [30] (see [30] for undefined terms).

**Theorem 3.2.** (a) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - d \pmod{3}$, $d \equiv a + b \pmod{3}$ and $p = \frac{1}{3}(a^2 + 25b^2 + c^2 + 25d^2)$ for each prime $p \neq 2, 5, 7, 13, 23$.

(b) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{4}$, $d \equiv a + 2b \pmod{4}$ and $p = \frac{1}{7}(a^2 + 7b^2 + c^2 + 7d^2)$ for each prime $p \neq 2, 7$.

(c) There are integers $a, b, c$ and $d$ satisfying $b \equiv c \pmod{5}$, $d \equiv a \pmod{5}$ and $p = \frac{1}{5}(a^2 + 49b^2 + c^2 + 49d^2)$ for each prime $p \neq 2, 3, 7, 11, 19, 29$.

(d) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{5}$, $d \equiv a + 2b \pmod{5}$ and $p = \frac{1}{5}(a^2 + 25b^2 + c^2 + 25d^2)$ for each prime $p \neq 2, 3, 17$.

(e) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{4}$, $d \equiv a + 2b \pmod{4}$ and $p = \frac{1}{5}(a^2 + 15b^2 + c^2 + 15d^2)$ for each prime $p \neq 2, 3$.

(f) There are integers $a, b, c$ and $d$ satisfying $b \equiv c - 2d \pmod{6}$, $d \equiv a + 2b \pmod{6}$ and $p = \frac{1}{6}(a^2 + 49b^2 + c^2 + 49d^2)$ for each prime $p \neq 2, 3, 5, 7$. 

10
There are integers $a, b, c$ and $d$ satisfying $b \equiv c \pmod{4}$, $d \equiv a \pmod{4}$ and $p = \frac{1}{2}(a^2 + 19b^2 + c^2 + 19d^2)$ for each prime $p \neq 2, 3, 13, 19$.

There are integers $a, b, c$ and $d$ satisfying $b \equiv c \pmod{5}$, $d \equiv a \pmod{5}$ and $p = \frac{1}{5}(a^2 + 39b^2 + c^2 + 39d^2)$ for each prime $p \neq 2, 3, 7, 17$.

Proof. We only give details for Case (a), to save space. The ideas of the proofs of the other cases are similar to that of Case (a), which is the most complicated case, where main different parts are mentioned in Tables 1–6.

Consider the lattice $L_{1,25,3}$ given in (4). We have verified by Magma that it has the following theta series:

$$
\theta_{L_{1,25,3}}(q) = 1 + 4q^3 + 4q^6 + 4q^9 + 8q^{10} + 4q^{11} + \cdots = \sum_{n=0}^{\infty} a(n)q^n \text{ (say)}.
$$

By Lemma 3.1, $\theta_{L_{1,25,3}}(z)$ is a modular form for $\Gamma_0(100)$, where $q = e^{2\pi iz}$, $z$ is in the upper half plane (see Table 1 for the other cases).

| Case | $\Gamma_0(N)$ | $\dim(S_2(\Gamma_0(N)))$ | Genus of $\Gamma_0(4N)$ |
|------|----------------|--------------------------|-------------------------|
| (b)  | $\Gamma_0(28)$ | 2                        | 11                      |
| (c)  | $\Gamma_0(196)$ | 17                       | 89                      |
| (d)  | $\Gamma_0(100)$ | 7                        | 43                      |
| (e)  | $\Gamma_0(60)$  | 7                        | 37                      |
| (f)  | $\Gamma_0(196)$ | 17                       | 89                      |
| (g)  | $\Gamma_0(76)$  | 8                        | 35                      |
| (h)  | $\Gamma_0(156)$ | 23                       | 101                     |

We denote by $S_2(\Gamma_0(N))$ the space of cusp forms of weight 2 for $\Gamma_0(N)$. It is known that $\dim(S_2(\Gamma_0(100)))$ is seven (see Table 1 for the other cases), and using MAGMA we have found some basis $f_1(z), f_2(z), f_3(z), f_4(z), f_5(z), f_7(z), f_9(z)$ such that

$$
f_i(z) = q^i + c_{i,6}q^6 + c_{i,7}q^7 + \cdots,
$$

for $i = 1, 2, 3, 4, 5$, $f_7(z) = q^7 + \cdots$, and $f_9(z) = q^9 + \cdots$. In particular, we
use $f_i(z)$ ($i = 1, 3, 5, 7, 9$), which are explicitly written as:

$$
\begin{align*}
&\begin{cases}
  f_1(z) = q - q^{11} - q^{19} - 2q^{21} + 4q^{29} + \cdots, \\
  f_3(z) = q^3 - 2q^{13} + q^{17} - 3q^{27} + \cdots, \\
  f_5(z) = q^5 - 2q^{15} - q^{25} + \cdots, \\
  f_7(z) = q^7 - q^{13} - 2q^{17} + 3q^{23} - q^{27} + \cdots, \\
  f_9(z) = q^9 + q^{11} - 3q^{19} - 2q^{21} + 2q^{29} + \cdots
\end{cases}
\end{align*}
$$

(see Table 2 for the other cases). For $i = 1, 3, 5, 7, 9$, we denote by $c_{f_i}(n)$ the coefficient of $f_i(z)$ as follows:

$$
f_i(z) = \sum_{n=1}^{\infty} c_{f_i}(n)q^n.
$$

Let $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ be the Dedekind $\eta$-function. Then

$$
\frac{\eta(4z)^8}{\eta(2z)^4} = \sum_{n=1}^{\infty} \sigma_1(2n-1)q^{2n-1}
$$

is a modular form for $\Gamma_0(4)$, where $\sigma_1(n) = \sum_{m|n} m$ (see [23] p. 145, Problem 10). We define a new modular form $h_{100}(z)$ for $\Gamma_0(100)$ as follows (see Table 3 for the other cases):

$$
h_{100}(z) = \frac{4}{15} \left( \frac{\eta(4z)^8}{\eta(2z)^4} + 4\frac{\eta(20z)^8}{\eta(10z)^4} + 25\frac{\eta(100z)^8}{\eta(50z)^4} \right) - f_1(z) + 11f_3(z)
$$

$$
- 10f_5(z) - 8f_7(z) + 2f_9(z) = \sum_{n=0}^{\infty} b(n)q^n \text{ (say)}.
$$

Note that all the degrees of $\eta(4nz)^8/\eta(2nz)^4$ are divided by $n$, namely, $q^n + 4q^{2n} + \cdots$. Hence, $b(p) = \frac{4}{15}(\sigma_1(p) - c_{f_1}(p) + 11c_{f_3}(p) - 10c_{f_5}(p) - 8c_{f_7}(p) + 2c_{f_9}(p))$ for each odd prime $p$ with $p \neq 5$, noting that $b(5) = 0$.

Let

$$
\chi_2(n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{2} \\
1 & \text{otherwise}
\end{cases}
$$

Then

$$
(\theta_{L,25,3}(z))_{\chi_2} = \sum_{n=0}^{\infty} \chi_2(n)a(n)q^n = 4q^3 + 4q^9 + 4q^{11} + \cdots \text{ and}
$$

$$
(h_{100}(z))_{\chi_2} = \sum_{n=0}^{\infty} \chi_2(n)b(n)q^n = 4q^3 + 4q^9 + 4q^{11} + \cdots
$$
Using Theorem 7 in [31] and the fact that the genus of $\Gamma_0(1)$ for the other cases), the verification by Magma that $\chi_2(n)a(n) = \chi_2(n)b(n)$ for $n \leq 43 \times 2$ shows

\[ (\theta_{L_{1,25,3}}(z))_{\chi_2} = (h_{100}(z))_{\chi_2}. \]

Hence, for each odd prime $p$ with $p \neq 5$, we have

\[ a(p) = \frac{4}{15}(p + 1 - (c_{f_1}(p) - 11c_{f_5}(p) + 10c_{f_{10}}(p) + 8c_{f_{15}}(p) - 2c_{f_{19}}(p))). \]

(see Table 4 for the other cases and the following paragraph is unnecessary for the other cases).

| Case | $f_i(z)$ |
|------|----------|
| (b)  | $f_1(z) = q - 2q^4 + q^7 + q^9 + \cdots$ |
| (c)  | $f_1(z) = q + q^{25} - q^{29} + \cdots$ |
|      | $f_3(z) = q^3 + q^{2q} + \cdots$ |
|      | $f_4(z) = q^4 - 2q^{19} + q^{27} + \cdots$ |
|      | $f_7(z) = q^7 - 2q^{21} + \cdots$ |
|      | $f_9(z) = q^9 - q^{23} + \cdots$ |
|      | $f_{11}(z) = q^{11} + q^{23} + 2q^{29} + \cdots$ |
|      | $f_{13}(z) = q^{13} + q^{19} - 2q^{27} + \cdots$ |
|      | $f_{15}(z) = q^{15} + q^{23} - q^{29} + \cdots$ |
| (d)  | $f_1(z) = q - q^{11} + \cdots$ |
|      | $f_3(z) = q^3 - 2q^{13} + \cdots$ |
|      | $f_7(z) = q^7 - q^{13} + \cdots$ |
| (e)  | $f_1(z) = q - q^4 + \cdots$ |
|      | $f_3(z) = q^3 - 2q^4 + \cdots$ |
|      | $f_7(z) = q^7 - q^9 + \cdots$ |
| (f)  | $f_1(z) = q + q^{25} - q^{29} + \cdots$ |
|      | $f_3(z) = q^3 - q^{2q} + \cdots$ |
|      | $f_7(z) = q^7 - 2q^{21} + \cdots$ |
|      | $f_9(z) = q^9 - q^{23} + \cdots$ |
|      | $f_{11}(z) = q^{11} + q^{23} + 2q^{29} + \cdots$ |
|      | $f_{13}(z) = q^{13} + q^{19} - 2q^{27} + \cdots$ |
|      | $f_{15}(z) = q^{15} + q^{23} - q^{29} + \cdots$ |
| (g)  | $f_1(z) = q - 2q^9 + \cdots$ |
|      | $f_3(z) = q^3 - 2q^9 + \cdots$ |
|      | $f_7(z) = q^7 - 2q^9 + \cdots$ |
| (h)  | $f_1(z) = q + q^{25} - q^{29} + \cdots$ |
|      | $f_3(z) = q^3 + q^{25} - q^{2q} - q^{29} + \cdots$ |
|      | $f_7(z) = q^7 - 2q^{2q} + \cdots$ |
|      | $f_9(z) = q^9 + 2q^{23} - 2q^{25} - 3q^{27} - 2q^{29} + \cdots$ |
|      | $f_{11}(z) = q^{11} - q^{27} + q^{29} + \cdots$ |
|      | $f_{13}(z) = q^{13} - 2q^{25} - 6q^{29} + \cdots$ |
|      | $f_{15}(z) = q^{15} + q^{23} - q^{25} - q^{27} - q^{29} + \cdots$ |
|      | $f_{17}(z) = q^{17} + 2q^{23} - 2q^{25} - 2q^{27} - 2q^{29} + \cdots$ |
|      | $f_{19}(z) = q^{19} - 2q^{25} - 2q^{29} + \cdots$ |
|      | $f_{21}(z) = q^{21} + q^{2q} - q^{29} + \cdots$ |
|      | $f_{23}(z) = 3q^{23} - 3q^{25} - 2q^{27} - 2q^{2q} + \cdots$ |
Hence, for each odd prime $p$

Now take the unique normalized cusp form $h_N(z)$ in (10)

| Case | $h_N(z)$ |
|------|----------|
| (b)  | $\frac{1}{4} (\frac{\eta(4z)^4}{\eta(2z)^2}) - \frac{\eta(28z)^4}{\eta(14z)^2} - 7f_1(z)$ |
| (c)  | $\frac{4}{21} (\frac{\eta(4z)^4}{\eta(2z)^2}) - \frac{7}{2} \frac{\eta(28z)^4}{\eta(14z)^2} + 49 \frac{\eta(196z)^4}{\eta(98z)^2} - f_1(z) - 4f_3(z) + 15f_5(z) - 13f_9(z)$ |
|      | $-12f_{11}(z) + 7f_{13}(z) + 18f_{15}(z) + 3f_{17}(z) + 11f_{25}(z)$ |
| (d)  | $\frac{4}{15} (\frac{\eta(4z)^4}{\eta(2z)^2}) + 4 \frac{\eta(20z)^4}{\eta(10z)^2} + 25 \frac{\eta(100z)^4}{\eta(50z)^2} - f_1(z) - 4f_3(z) + 5f_5(z) + 7f_7(z) + 2f_9(z))$ |
| (e)  | $\frac{2}{7} (\frac{\eta(4z)^4}{\eta(2z)^2}) - 3 \frac{\eta(12z)^4}{\eta(6z)^2} + 5 \frac{\eta(20z)^4}{\eta(10z)^2} - 15 \frac{\eta(60z)^4}{\eta(30z)^2} - f_1(z) - f_3(z) + f_5(z) + 4f_7(z)$ |
| (f)  | $\frac{4}{21} (\frac{\eta(4z)^4}{\eta(2z)^2}) - \frac{7}{2} \frac{\eta(28z)^4}{\eta(14z)^2} + 49 \frac{\eta(196z)^4}{\eta(98z)^2} - f_1(z) - 4f_3(z) - 6f_5(z) + 8f_9(z)$ |
|      | $+ 9f_{11}(z) + 7f_{13}(z) - 3f_{15}(z) + 3f_{17}(z) - 10f_{25}(z)$ |
| (g)  | $\frac{4}{7} (\frac{\eta(4z)^4}{\eta(2z)^2}) - 19 \frac{\eta(76z)^4}{\eta(38z)^2} - f_1(z) - 4f_3(z) + 3f_5(z) + f_7(z)$ |
| (h)  | $\frac{2}{7} (\frac{\eta(4z)^4}{\eta(2z)^2}) - 3 \frac{\eta(12z)^4}{\eta(6z)^2} + 13 \frac{\eta(52z)^4}{\eta(26z)^2} - 39 \frac{\eta(156z)^4}{\eta(78z)^2} - f_1(z) - f_3(z) + 8f_5(z) - 8f_7(z)$ |
|      | $- f_9(z) + 2f_{11}(z) + f_{13}(z) + 8f_{15}(z) - 18f_{17}(z) + 8f_{19}(z) - 8f_{21}(z) + 3f_{23}(z)$ |

Now take the unique normalized cusp form $f_5^*(z) = \sum_{n=1}^{\infty} c_{f_5}(n)q^n \in S_2(\Gamma_0(20))$. The verification by MAGMA that $c_{f_5}(5n) = c_{f_5}(n)$ for $n \leq 43 \times 2$ shows that $f_5(z) = f_5^*(5z)$. Thus, for each prime $p$ with $p \neq 5$, $c_{f_5}(p) = 0$. Hence, for each odd prime $p$ with $p \neq 5$, we have

$$a(p) = \frac{4}{15} (p + 1 - (c_{f_1}(p) - 11c_{f_5}(p) + 8c_{f_7}(p) - 2c_{f_9}(p))).$$

Table 4: $a(p)$ in (17)

| Case | $a(p)$ |
|------|--------|
| (b)  | $\frac{4}{15} (p + 1 - (c_{f_1}(p)))$ |
| (c)  | $\frac{4}{15} (p + 1 - (c_{f_1}(p) + 4c_{f_3}(p) - 15c_{f_5}(p) + 13c_{f_9}(p) + 12c_{f_{11}}(p) - 7c_{f_{13}}(p) - 18c_{f_{15}}(p) - 3c_{f_{17}}(p) - 11c_{f_{25}}(p)))$ |
| (d)  | $\frac{4}{15} (p + 1 - (c_{f_1}(p) + 4c_{f_3}(p) - 5c_{f_5}(p) - 7c_{f_7}(p) - 2c_{f_9}(p))$ |
| (e)  | $\frac{4}{15} (p + 1 - (c_{f_1}(p) + c_{f_5}(p) - c_{f_7}(p) - 4c_{f_9}(p))$ |
| (f)  | $\frac{4}{15} (p + 1 - (c_{f_1}(p) + 4c_{f_3}(p) + 3c_{f_5}(p) - 8c_{f_7}(p) - 9c_{f_9}(p) - 3c_{f_{11}}(p) - 3c_{f_{13}}(p) + 10c_{f_{25}}(p)))$ |
| (g)  | $\frac{4}{15} (p + 1 - (c_{f_1}(p) + 4c_{f_3}(p) - 3c_{f_5}(p) - c_{f_7}(p))$ |
| (h)  | $\frac{4}{15} (p + 1 - (c_{f_1}(p) + c_{f_5}(p) - 8c_{f_7}(p) + 8c_{f_9}(p) + c_{f_9}(p) - 2c_{f_{11}}(p) - c_{f_{13}}(p) - 8c_{f_{15}}(p) + 18c_{f_{17}}(p) - 8c_{f_{19}}(p) + 8c_{f_{21}}(p) - 3c_{f_{23}}(p))$ |
Table 5: Matrices in (9)

| Case | Matrices |
|------|----------|
| (b)  | \[
\begin{array}{cccc}
-2 & 0 & \frac{1}{2} & (1 - \sqrt{2}) \\
1 & -2 & 0 & \frac{1}{2} (1 + \sqrt{2}) \\
1 & 0 & 0 & 0 \\
1 & -1 & -3 & 0 \\
1 & 1 & 3 & 0 \\
1 & -\sqrt{2} & 2\sqrt{2} & 0 \\
1 & \sqrt{2} & -2\sqrt{2} & 0 \\
1 & 2 & 0 & 0 \\
1 & 2\sqrt{2} & -\sqrt{2} & 0 \\
1 & -2\sqrt{2} & \sqrt{2} & 0 \\
\end{array}
\] |
| (c)  | \[
\begin{array}{cccc}
1 & -1 & -2 & 2 \\
1 & 1 & 2 & 1 \\
1 & -2 & -1 & 1 \\
1 & -1 + \sqrt{2} & -1 & 2 \\
1 & -2 & 0 & \frac{1}{2} (1 - \sqrt{2}) \\
1 & -2 & 0 & \frac{1}{2} (1 + \sqrt{2}) \\
1 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 \\
1 & -\sqrt{2} & 2\sqrt{2} & 0 \\
1 & \sqrt{2} & -2\sqrt{2} & 0 \\
1 & 2 & 0 & 0 \\
1 & 2\sqrt{2} & -\sqrt{2} & 0 \\
1 & -2\sqrt{2} & \sqrt{2} & 0 \\
\end{array}
\] |
| (d)  | \[
\begin{array}{cccc}
1 & -1 & -2 & 2 \\
1 & 1 & 2 & 1 \\
1 & -2 & -1 & 1 \\
1 & -1 + \sqrt{2} & -1 & 2 \\
1 & -2 & 0 & \frac{1}{2} (1 - \sqrt{2}) \\
1 & -2 & 0 & \frac{1}{2} (1 + \sqrt{2}) \\
1 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 \\
1 & -\sqrt{2} & 2\sqrt{2} & 0 \\
1 & \sqrt{2} & -2\sqrt{2} & 0 \\
1 & 2 & 0 & 0 \\
1 & 2\sqrt{2} & -\sqrt{2} & 0 \\
1 & -2\sqrt{2} & \sqrt{2} & 0 \\
\end{array}
\] |
| (e)  | \[
\begin{array}{cccc}
1 & -1 & -2 & 2 \\
1 & 1 & 2 & 1 \\
1 & -2 & -1 & 1 \\
1 & -1 + \sqrt{2} & -1 & 2 \\
1 & -2 & 0 & \frac{1}{2} (1 - \sqrt{2}) \\
1 & -2 & 0 & \frac{1}{2} (1 + \sqrt{2}) \\
1 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 \\
1 & -\sqrt{2} & 2\sqrt{2} & 0 \\
1 & \sqrt{2} & -2\sqrt{2} & 0 \\
1 & 2 & 0 & 0 \\
1 & 2\sqrt{2} & -\sqrt{2} & 0 \\
1 & -2\sqrt{2} & \sqrt{2} & 0 \\
\end{array}
\] |
| (f)  | \[
\begin{array}{cccc}
1 & -1 & -2 & 2 \\
1 & 1 & 2 & 1 \\
1 & -2 & -1 & 1 \\
1 & -1 + \sqrt{2} & -1 & 2 \\
1 & -2 & 0 & \frac{1}{2} (1 - \sqrt{2}) \\
1 & -2 & 0 & \frac{1}{2} (1 + \sqrt{2}) \\
1 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 \\
1 & -\sqrt{2} & 2\sqrt{2} & 0 \\
1 & \sqrt{2} & -2\sqrt{2} & 0 \\
1 & 2 & 0 & 0 \\
1 & 2\sqrt{2} & -\sqrt{2} & 0 \\
1 & -2\sqrt{2} & \sqrt{2} & 0 \\
\end{array}
\] |
| (g)  | \[
\begin{array}{cccc}
1 & -1 & -2 & 2 \\
1 & 1 & 2 & 1 \\
1 & -2 & -1 & 1 \\
1 & -1 + \sqrt{2} & -1 & 2 \\
1 & -2 & 0 & \frac{1}{2} (1 - \sqrt{2}) \\
1 & -2 & 0 & \frac{1}{2} (1 + \sqrt{2}) \\
1 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 \\
1 & -\sqrt{2} & 2\sqrt{2} & 0 \\
1 & \sqrt{2} & -2\sqrt{2} & 0 \\
1 & 2 & 0 & 0 \\
1 & 2\sqrt{2} & -\sqrt{2} & 0 \\
1 & -2\sqrt{2} & \sqrt{2} & 0 \\
\end{array}
\] |
| (h)  | \[
\begin{array}{cccc}
1 & -1 & -2 & 2 \\
1 & 1 & 2 & 1 \\
1 & -2 & -1 & 1 \\
1 & -1 + \sqrt{2} & -1 & 2 \\
1 & -2 & 0 & \frac{1}{2} (1 - \sqrt{2}) \\
1 & -2 & 0 & \frac{1}{2} (1 + \sqrt{2}) \\
1 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 \\
1 & -\sqrt{2} & 2\sqrt{2} & 0 \\
1 & \sqrt{2} & -2\sqrt{2} & 0 \\
1 & 2 & 0 & 0 \\
1 & 2\sqrt{2} & -\sqrt{2} & 0 \\
1 & -2\sqrt{2} & \sqrt{2} & 0 \\
\end{array}
\] |
Set \( \hat{h}_i(z) \) \((i = 1, 3, 5, 7, 9)\) as follows (see Table 5 for the other cases):

\[
\begin{pmatrix}
\hat{h}_1(z) \\
\hat{h}_3(z) \\
\hat{h}_7(z) \\
\hat{h}_9(z)
\end{pmatrix} =
\begin{pmatrix}
1 & -1 & -2 & -2 \\
1 & 1 & 2 & -2 \\
1 & 2 & -2 & 1 \\
1 & -2 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
f_1(z) \\
f_3(z) \\
f_7(z) \\
f_9(z)
\end{pmatrix}.
\]

For \( i = 1, 3, 7, 9\), we denote by \( c_{\hat{h}_i}(n) \) the coefficient of \( \hat{h}_i(z) \) as follows:

\[
\hat{h}_i(z) = \sum_{n=1}^{\infty} c_{\hat{h}_i}(n) q^n.
\]

Let \( T(n) \) be the Hecke operator considered on the space of modular forms for \( \Gamma_0(100) \) (see [23, p. 161, Proposition 37]). Then, \( \hat{h}_i(z) \) \((i = 1, 3, 5, 7, 9)\) are eigen forms for \( T(3) \). Since the algebra of Hecke operators is commutative [30, Theorem 4.5.3], \( \hat{h}_i(z) \) \((i = 1, 3, 7, 9)\) are normalized Hecke eigen forms. In addition, for each prime \( p \) and \( i = 1, 3, 7, 9\),

\[
|c_{\hat{h}_i}(p)| \leq 2\sqrt{p}
\]

(see [23, p. 164]).

| Case | \( \frac{p}{n} \) | \( a(p) > 0 \) |
|------|----------------|----------------|
| (b)  | \( \frac{p + 1 - 2\sqrt{p}}{m} \) | \( p > 0 \) |
| (c)  | \( \frac{p + 1 - (8 + 4\sqrt{2})\sqrt{p}}{m} \) | \( p > 181 \) |
| (d)  | \( \frac{p + 1 - 7\sqrt{p}}{m} \) | \( p > 43 \) |
| (e)  | \( \frac{p + 1 - 2\sqrt{p}}{m} \) | \( p > 0 \) |
| (f)  | \( \frac{p + 1 - (2 + 6\sqrt{2})\sqrt{p}}{m} \) | \( p > 107 \) |
| (g)  | \( \frac{p + 1 - 6\sqrt{p}}{m} \) | \( p > 31 \) |
| (h)  | \( \frac{p + 1 - 4\sqrt{2}\sqrt{p}}{m} \) | \( p > 29 \) |

By (9), we have

\[
\begin{pmatrix}
f_1(z) \\
f_3(z) \\
f_7(z) \\
f_9(z)
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
-\frac{1}{6} & \frac{1}{6} & -\frac{1}{12} & \frac{1}{12} \\
-\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6}
\end{pmatrix}
\begin{pmatrix}
\hat{h}_1(z) \\
\hat{h}_3(z) \\
\hat{h}_7(z) \\
\hat{h}_9(z)
\end{pmatrix}.
\]
Hence, we have
\[ f_1(z) - 11f_3(z) + 8f_7(z) - 2f_9(z) = \hat{h}_1(z) - \frac{5}{2}\hat{h}_7(z) + \frac{5}{2}\hat{h}_9(z). \]

For each odd prime \( p \) with \( p \neq 5 \), \(|c_{f_1}(p) - 11c_{f_3}(p) + 8c_{f_7}(p) - 2c_{f_9}(p)|\) is bounded above by
\[ \left(1 + \frac{5}{2} + \frac{5}{2}\right)2\sqrt{p} = 12\sqrt{p}. \]

Using (8) (7) for the other cases), \( a(p) \) is bounded below by
\[ \frac{4}{15}(p + 1 - 12\sqrt{p}), \]
(see Table 6 for the other cases). Hence, (10) is positive for \( p > 139 \), namely, \( a(p) > 0 \) for \( p > 139 \) (see Table 6 for the other cases). We have verified by MAGMA that \( a(p) > 0 \) for each prime \( p \) with \( p \leq 139 \) and \( p \neq 2, 5, 7, 13, 23 \). This completes the proof of Case (a).

\[ \square \]

4 Construction of \( m \)-frames in some unimodular lattices

In this section, we provide a method for constructing \( m \)-frames in unimodular lattices, which are constructed from some self-dual \( \mathbb{Z}_k \)-codes by Construction A. Combined Theorem 3.2 with the method, we construct \( m \)-frames in some extremal (optimal) odd unimodular lattices.

The following method is a generalization of Propositions 3.3 and 3.6 in [20]. Also, the cases \((k, m, \ell) = (4, 11, 2)\) and \((4, 11, 0)\) of the following method can be found in [7] and [29], respectively.

Proposition 4.1. Let \( k \) be a positive integer with \( k \geq 2 \), and let \( \ell \) be a nonnegative integer with \( \ell \leq k - 1 \). Let \( M \) be an \( n \times n \) matrix over \( \mathbb{Z} \) satisfying
\[ M^T = -M \text{ and } MM^T = mI_n, \]
where \( m + \ell^2 \equiv -1 \pmod{k} \). Let \( C_{2n,k}(M) \) be the self-dual \( \mathbb{Z}_k \)-code of length \( 2n \) with generator matrix \( \left( I_n \ M + \ell I_n \right) \), where the entries of the matrix are regarded as elements of \( \mathbb{Z}_k \). Let \( a,b,c \) and \( d \) be integers with \( b \equiv c - \ell d \)
and \( d \equiv a + \ell b \pmod{k} \). Then the set of \( 2n \) rows of the following matrix

\[
F(M) = \frac{1}{\sqrt{k}} \begin{pmatrix} aI_n + bM & cI_n + dM \\ -cI_n + dM & aI_n - bM \end{pmatrix}
\]

forms a \( \frac{1}{k}(a^2 + mb^2 + c^2 + md^2) \)-frame in the unimodular lattice \( A_k(C_{2n,k}(M)) \).

Proof. Since \( MM^T = mI_n \) with \( m + \ell^2 \equiv -1 \pmod{k} \), \( C_{2n,k}(M) \) is a self-dual \( \mathbb{Z}_k \)-code of length \( 2n \). Thus, \( A_k(C_{2n,k}(M)) \) is a unimodular lattice. Since \( C_{2n,k}(M) \) is self-dual and \( M^T = -M \), both \( G = \begin{pmatrix} I_n & M + \ell I_n \end{pmatrix} \) and \( H = \begin{pmatrix} M - \ell I_n & I_n \end{pmatrix} \) are generator matrices of \( C_{2n,k}(M) \).

Let \( s, t \) be integers. Here, we regard the entries of the matrices \( G, H \) as integers. Then

\[
\begin{pmatrix} sG + tH \\ -tG + sH \end{pmatrix} = \begin{pmatrix} (s - \ell t)I_n + tM & (\ell s + t)I_n + sM \\ -(\ell s + t)I_n + sM & (s - \ell t)I_n - tM \end{pmatrix}.
\]

By putting

\[
a = s - \ell t, \quad b = t, \quad c = \ell s + t, \quad d = s,
\]

we have the form of \( F(M) \). Thus, if \( b \equiv c - \ell d \pmod{k} \) and \( d \equiv a + \ell b \pmod{k} \) then all rows of the matrix \( F(M) \) are vectors of \( A_k(C_{2n,k}(M)) \). Since \( F(M)F(M)^T = \frac{1}{k}(a^2 + mb^2 + c^2 + md^2)I_{2n} \), the result follows.

Remark 4.2. It follows from the assumption that \( a^2 + mb^2 + c^2 + md^2 \equiv 0 \pmod{k} \).
Table 7: Matrices $M$

| $M$  | $k$ | $m$ | $\ell$ | $r_A$          | $r_B$          |
|------|-----|-----|--------|----------------|----------------|
| $D_6$| 3   | 25  | 1      | (0, 2, 2)      | (0, 1, −4)     |
| $P_8$| 4   | 7   | 2      |                |                |
| $D_{10}$| 3 | 25  | 1      | (0, 0, 2, 2, 0)| (1, 2, 2, −2, 2)|
| $D'_{10}$| 3 | 25  | 1      | (0, 0, 0, 0, 0)| (−3, −2, 2, −2, 2)|
| $D''_{10}$| 5 | 49  | 0      | (0, 0, 3, 3, 0)| (−2, −3, 4, −1, 1)|
| $D_{14}$| 3 | 25  | 1      | (0, 2, 1, 0, 0, 1, 2)| (−1, −2, 1, −2, 2, 1, 0)|
| $D'_{14}$| 5 | 25  | 2      | (0, 0, 2, −1, −1, 2, 0)| (−2, −1, −2, 0, −1, −1, −2)|
| $D_{16}$| 4 | 15  | 2      | (0, 1, 1, 0, 1, 0, 1, 1)| (1, 1, 1, −1, −1, 2, −1, 0)|
| $D_{18}$| 6 | 49  | 2      | (0, 1, −3, 0, 2, 2, 0, −3, 1)| (−2, 2, −1, 2, 1, 2, 1, 1)|
| $P_{20}$| 4 | 19  | 0      |                |                |
| $D_{22}$| 5 | 25  | 2      | (0, 0, −1, 1, 0, 0, 0, 1, −1, 0)| (1, 0, −2, 1, 1, 1, 2, 1, 0, 2, −2)|
| $D_{24}$| 5 | 39  | 0      | (0, 1, 1, 1, 2, −1, 1, −1, 2, 1, 1, 1)| (−2, −1, 2, −1, −1, −2, 0, 1, 0, 2, −1, −1)|
The matrices $P_{p+1}$ ($p = 7, 19$), which are given in Section 2.5, satisfy the assumptions in Proposition 4.1 for the integers $k, m$ and $\ell$ listed in Table 7. Using the form (3), we have found more matrices $D_n$ ($n = 6, 10, 14, 16, 18, 22, 24$), $D'_n$ ($n = 10, 14$) and $D''_{10}$, satisfying the assumptions in Proposition 4.1, where the integers $k, m$ and $\ell$ and the first rows $r_A$ and $r_B$ of negacirculant matrices $A$ and $B$ are also listed in Table 7.

By Proposition 4.1, for matrices $M$ given in Table 7, the odd unimodular lattice $A_k(C_{2n,k}(M))$, which is constructed from the Type I $\mathbb{Z}_k$-code $C_{2n,k}(M)$, contains a $1$-frame for integers $a, b, c$ and $d$ with $b \equiv c - \ell d$ (mod $k$) and $d \equiv a + \ell b$ (mod $k$). The minimum norms $\min(L)$ of the lattices $L = A_k(C_{2n,k}(M))$ listed in Table 8, which have been determined by Magma, are also listed in the table.

**Lemma 4.3.** Suppose that $L$ is any of the lattices listed in Table 8. Then $L$ contains a $k$-frame for a positive integer $k$ satisfying the conditions $(\ast)$ listed in Table 8, where $m_i$ in $(\ast)$ is a non-negative integer.

**Proof.** All cases are similar, and we only give details for the lattice $A_3(C_{12,3}(D_6))$. Let $a, b, c$ and $d$ be integers with $b \equiv c - d$ (mod 3) and $d \equiv a + b$ (mod 3). By Proposition 4.1, $A_3(C_{12,3}(D_6))$ contains a $1_3(a^2 + 25b^2 + c^2 + 25d^2)$-frame. By Theorem 3.2 (a), there are integers $a, b, c$ and $d$ satisfying $b \equiv c - d$ (mod 3), $d \equiv a + b$ (mod 3) and $p = \frac{1}{3}(a^2 + 25b^2 + c^2 + 25d^2)$ for each prime $p \neq 2, 5, 7, 13, 23$. The result follows from Lemma 2.1. For the other lattices, Table 8 lists the cases of Theorem 3.2 which are used in the proof. \hfill $\square$

## 5 Frames of some extremal odd unimodular lattices and extremal Type I $\mathbb{Z}_k$-codes

In this section, we establish the existence of a $k$-frame in some extremal (optimal) unimodular lattices for every positive integer $k$ with $k \geq \min(L)$. These results yield the existence of an extremal Type I $\mathbb{Z}_k$-code of lengths $n = 12, 16, 20, 32, 36, 40, 44$ and a near-extremal Type I $\mathbb{Z}_k$-code of length $n = 28$ for a positive integer $k$, where $k \neq 1, 3$ if $n = 32$ and $k \neq 1$ otherwise.

### 5.1 Frames of $D_{12}^+$ and Length 12

There is a unique extremal odd unimodular lattice in dimension 12, up to isomorphism (see [12, Table 16.7]), where the lattice is denoted by $D_{12}^+$. There
is a unique binary extremal Type I code of length 12, up to equivalence \[33\], where the code is denoted by \(B_{12}\) in \[33\] Table 2. It is known that \(D_{12}^+\) is constructed as \(A_2(B_{12})\). Hence, by Lemma \[4.3\] we investigate the existence of a \(k\)-frame in \(D_{12}^+\) for \(k = 5, 7, 13, 23\).

There are 16 inequivalent Type I \(\mathbb{Z}_5\)-codes of length 12 \[24\]. We have verified by Magma that the \(i\)th code in \[24\] Table III gives \(D_{12}^+\) by Construction A \((i = 8, 11, 13, 16)\). There are 64 inequivalent Type I \(\mathbb{Z}_7\)-codes of length 12 \[22\]. We have verified by Magma that the code \(C_{12,i}\) in \[22\] Table 1] gives \(D_{12}^+\) by Construction A \((i = 11, 12, 15, 17, 20, 38, 42, 43, 47, 49, 51, 54, 55, 57, \ldots, 62)\). For \(k = 13\) and 23, let \(C_{k,12}\) be the \(\mathbb{Z}_k\)-code with generator matrix of the form \[2\], where the first rows \(r_A\) and \(r_B\) of \(A\) and \(B\) are as follows:

\[(r_A, r_B) = ((0, 1, 6), (2, 3, 1))\) and \(((0, 1, 18), (7, 4, 0))\), respectively. Since \(AA^T + BB^T = -I_3\), these codes are Type I. Moreover, we have verified by Magma that \(A_k(C_{k,12}) (k = 13, 23)\) is isomorphic to \(D_{12}^+\). Hence, combined with Lemma \[4.3\] we have the following:

**Theorem 5.1.** \(D_{12}^+\) contains a \(k\)-frame if and only if \(k\) is a positive integer with \(k \geq 2\).
Hence, there is a Type I $\mathbb{Z}_k$-code $C$ with $A_k(C) \cong D_{12}^+$ for every positive integer $k$ with $k \geq 2$. Since $D_{12}^+$ has minimum norm 2, $C$ must be extremal.

**Corollary 5.2.** There is an extremal Type I $\mathbb{Z}_k$-code of length 12 for every positive integer $k$ with $k \geq 2$.

By Lemma 2.5, we have the following:

**Corollary 5.3.** Let $\sum_{m=0}^{\infty} A_m q^m$ denote the theta series of $D_{12}^+$. Then $A_k \geq 24$ for every positive integer $k$ with $k \geq 2$.

### 5.2 Frames of $D_8^2$ and Length 16

There is a unique extremal odd unimodular lattice in dimension 16, up to isomorphism (see [12, Table 16.7]), where the lattice is denoted by $D_8^2$. There is a unique binary extremal Type I code of length 16, up to equivalence [33], where the code is denoted by $F_{16}$ in [33, Table 2]. It is known that $D_8^2$ is constructed as $A_2(F_{16})$. Hence, by Lemma 4.3 we investigate the existence of a 7-frame in $D_8^2$.

Let $C_{7,16}$ be the $\mathbb{Z}_7$-code with generator matrix of the form (2), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are $r_A = (0, 0, 1, 1)$ and $r_B = (1, 3, 1, 0)$, respectively. Since $AA^T + BB^T = -I_4$, $C_{7,16}$ is Type I. We have verified by MAGMA that $A_7(C_{7,16})$ is isomorphic to $D_8^2$. Hence, combined with Lemma 4.3 we have the following:

**Theorem 5.4.** $D_8^2$ contains a $k$-frame if and only if $k$ is a positive integer with $k \geq 2$.

**Corollary 5.5.** There is an extremal Type I $\mathbb{Z}_k$-code of length 16 for every positive integer $k$ with $k \geq 2$.

By Lemma 2.5, we have the following:

**Corollary 5.6.** Let $\sum_{m=0}^{\infty} A_m q^m$ denote the theta series of $D_8^2$. Then $A_k \geq 32$ for every positive integer $k$ with $k \geq 2$.

### 5.3 Frames of $D_4^5$, $A_5^4$, $D_{20}$ and Length 20

There are 12 non-isomorphic extremal odd unimodular lattices in dimension 20 (see [12, Table 2.2]). We have verified by MAGMA that the odd unimodular lattices $A_3(C_{20,3}(D_{10}))$, $A_3(C_{20,3}(D_{10}'))$ and $A_5(C_{20,5}(D_{10}''))$ in Table...
are isomorphic to the \( i \)th lattices \( (i = 11, 12, 1) \) in dimension 20 in [12, Table 16.7], where we denote the lattices by \( D_4^5 \), \( A_5^4 \) and \( D_{20} \), respectively. By Lemma [4,3] we investigate the existence of a \( k \)-frame in \( D_4^5 \) and \( A_5^4 \) for \( k = 2, 5, 7, 13, 23 \), and a \( k \)-frame in \( D_{20} \) for \( k = 2, 3, 7, 11, 19, 29 \).

| Codes | \( r_A \) | \( r_B \) | Codes | \( r_A \) | \( r_B \) |
|-------|----------|----------|-------|----------|----------|
| \( C_{5,20} \) | (0, 0, 0, 1, 1) | (1, 4, 2, 1, 0) | \( C_{7,20} \) | (0, 0, 0, 1, 6) | (3, 0, 1, 1, 0) |
| \( C_{13,20} \) | (0, 0, 0, 1, 1) | (10, 3, 2, 1, 0) | \( C_{23,20} \) | (0, 0, 0, 1, 18) | (7, 4, 0, 0, 0) |
| \( C'_{5,20} \) | (0, 0, 0, 1, 4) | (3, 1, 4, 1, 0) | \( C'_{7,20} \) | (0, 0, 0, 1, 5) | (1, 5, 3, 1, 0) |
| \( C'_{13,20} \) | (0, 0, 0, 1, 4) | (4, 0, 3, 3, 0) | \( C'_{23,20} \) | (0, 0, 0, 1, 12) | (3, 5, 7, 1, 0) |
| \( C'_{9,20} \) | (0, 0, 0, 1, 4) | (1, 3, 2, 3, 1) | \( C''_{9,20} \) | (0, 0, 0, 1, 3) | (1, 2, 4, 2, 6) |
| \( C'_{11,20} \) | (0, 0, 0, 1, 8) | (5, 6, 6, 3, 2) | \( C''_{19,20} \) | (0, 0, 0, 1, 12) | (14, 12, 11, 1, 0) |
| \( C_{29,20} \) | (0, 0, 0, 1, 21) | (7, 11, 16, 1, 0) |  |

\[
\begin{pmatrix}
A & B_1 + 2B_2
\end{pmatrix} =
\begin{pmatrix}
11 & 220113303 \\
00 & 021012300 \\
10 & 222120030 \\
01 & 031321330 \\
01 & 232220201 \\
11 & 231021312 \\
00 & 023031002 \\
10 & 230133321 \\
11 & 333130022
\end{pmatrix},
\quad
2D =
\begin{pmatrix}
202202200 \\
220022200
\end{pmatrix}
\]

Figure 1: A generator matrix of \( C'_{4,20} \)

There are 7 binary extremal Type I code of length 20, up to equivalence [33]. The unique code with 5 (resp. 45) codewords of weight 4 is denoted by \( M_{20} \) (resp. \( J_{20} \)) in [33] Table 2]. We have verified by MAGMA that \( A_2(M_{20}) \) (resp. \( A_2(J_{20}) \)) is isomorphic to \( D_4^5 \) (resp. \( D_{20} \)). It is known that there is no binary Type I code \( C \) such that \( A_2(C) \) is isomorphic to \( A_5^4 \). There are 6 inequivalent ternary self-dual codes of length 20 and minimum weight 6 [34]. We have verified by MAGMA that \( L \) is obtained from some ternary self-dual code of length 20 and minimum weight 6 by Construction A if and only if \( L \) is \( D_4^5 \) or \( A_5^4 \).
Let $C_{k,20}, C'_{k,20} (k = 5, 7, 13, 23)$ and $C''_{k,20} (k = 7, 9, 11, 19, 29)$ be the $\mathbb{Z}_k$-codes with generator matrices of the form (2), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 9. Since $AA^T + BB^T = -I_5$, these codes are Type I. Let $C'_{4,20}$ be the $\mathbb{Z}_4$-code with generator matrix of the following form:

$$\begin{pmatrix}
I_9 & A & B_1 + 2B_2 \\
O & 2I_2 & 2D
\end{pmatrix},$$

where we only list in Figure 1 the matrices $(A \ B_1 + 2B_2)$ and $2D$ in order to save space. Here, $A$, $B_1$, $B_2$ and $D$ are $(1, 0)$-matrices and $O$ denotes the zero matrix. The self-dual $\mathbb{Z}_4$-code $C'_{4,20}$ has been found by directly finding a 4-frame in $A_4^4$ using Magma. Also, some other (new) self-dual $\mathbb{Z}_4$-codes are constructed in a similar way.

We have verified by Magma that $A_k(C_{k,20})$ is isomorphic to $D_5^5$, $A_k(C'_{k,20})$ is isomorphic to $A_4^4 (k = 4, 5, 7, 13, 23)$, and $A_k(C''_{k,20})$ is isomorphic to $D_{20} (k = 7, 9, 11, 19, 29)$. Hence, combined with Lemma 4.3 we have the following:

**Theorem 5.7.** $D_5^5$ contains a $k$-frame if and only if $k$ is a positive integer with $k \geq 2$. $A_4^4$ contains a $k$-frame if and only if $k$ is a positive integer with $k \geq 3$. $D_{20}$ contains a $k$-frame if and only if $k$ is a positive integer $k \geq 2$, $k \neq 3$.

**Corollary 5.8.** There is an extremal Type I $\mathbb{Z}_k$-code of length 20 for every positive integer $k$ with $k \geq 2$.

By Lemma 2.5 we have the following:

**Corollary 5.9.** Let $\sum_{m=0}^{\infty} A_m q^m$ denote the theta series of $D_5^5$ or $A_4^4$. Then $A_k \geq 40$ for every positive integer $k$ with $k \geq 2$. Let $\sum_{m=0}^{\infty} A_m q^m$ denote the theta series of $D_{20}$. Then $A_k \geq 40$ for every positive integer $k \geq 2$, $k \neq 3$.

**Remark 5.10.** $D_5^5$ and $A_4^4$ have the identical theta series $1 + 120q^2 + 5120q^3 + 67320q^4 + 503808q^5 + \cdots$, and $D_{20}$ has theta series $1 + 760q^2 + 77560q^4 + 524288q^5 + \cdots$.

### 5.4 Length 28

The largest minimum norm among odd unimodular lattices in dimension 28 is 3. There are 38 non-isomorphic optimal odd unimodular lattices in dimension...
The 38 lattices are denoted by $R_{28,1}^{(0)}, R_{28,2}^{(0)}, \ldots, R_{28,36}^{(0)}, R_{28,37e}^{(0)}, R_{28,38e}^{(0)}$. We have verified by Magma that $A_3(C_{28,3}(D_{14}))$ and $A_5(C_{28,5}(D_{14}))$ in Table 8 are isomorphic to $R_{28,32}^{(0)}$ and $R_{28,15}^{(0)}$, respectively. By Lemma 4.3, we investigate the existence of a $k$-frame in $R_{28,32}^{(0)}$ for $k = 4, 5, 7, 13, 23$ and a $k$-frame in $R_{28,15}^{(0)}$ for $k = 3, 4, 17$.

**Table 10: Near-extremal Type I $\mathbb{Z}_k$-codes of length 28**

| Codes  | $r_A$   | $r_B$  |
|--------|---------|--------|
| $C_{5,28}$ | (0, 0, 0, 1, 3, 4, 2) | (3, 1, 2, 0, 3, 4, 0) |
| $C_{7,28}$ | (0, 1, 2, 2, 4, 2, 3) | (2, 2, 4, 0, 4, 1, 2) |
| $C_{13,28}$ | (0, 0, 0, 1, 0, 9, 1) | (5, 1, 3, 7, 7, 1, 4) |
| $C_{21,28}$ | (0, 0, 0, 1, 12, 1, 1) | (3, 19, 7, 5, 14, 21, 17) |
| $C_{17,28}$ | (0, 0, 0, 1, 13, 14, 2) | (10, 1, 1, 9, 16, 11, 15) |

Let $C_{k,28}$ ($k = 5, 7, 13, 23$) and $C'_{17,28}$ be the $\mathbb{Z}_k$-codes with generator matrices of the form [2], where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 10. Since $AA^T + BB^T = -I_7$, these codes are Type I. Let $C_{4,28}$ and $C'_{4,28}$ be the $\mathbb{Z}_4$-codes with generator matrices of the following form:

$$
\begin{pmatrix}
I_{13} & A & B_1 + 2B_2 \\
O & 2I_2 & 2D
\end{pmatrix},
$$

where we list in Figure 2 the matrices $\begin{pmatrix} A & B_1 + 2B_2 \\ 2I_2 & 2D \end{pmatrix}$. Then these codes are Type I. For $k = 4, 5, 7, 13, 23$, we have verified by Magma that $A_k(C_{k,28})$ is isomorphic to $R_{28,32}^{(0)}$. For $k = 4, 17$, we have verified by Magma that $A_k(C'_{k,28})$ is isomorphic to $R_{28,15}^{(0)}$. It is known that $R_{28,15}^{(0)}$ contains a 3-frame (see [21] for the classification of 3-frames in the 38 lattices). Hence, combined with Lemma 4.3, we have the following:

**Theorem 5.11.** $R_{28,i}^{(0)}$ ($i = 15, 32$) contains a $k$-frame if and only if $k$ is a positive integer with $k \geq 3$.

**Lemma 5.12.** Let $C$ be a Type I $\mathbb{Z}_k$-code of length 28. Then $d_E(C) \leq 3k$ for every positive integer $k$ with $k \geq 2$.

**Proof.** As described above, the largest minimum norm among odd unimodular lattices in dimension 28 is 3. Assume that $k \geq 4$ and $d_E(C) = 4k$. Since
\[
\min(A_k(C)) = \min\{k, d_E(C)/k\}, \min(A_k(C)) = 4, \text{ which is a contradiction.}
\]
For \(k = 2, 3\), it follows that \(d_E(C) \leq 3k\) (see \([9, 26]\)).

By the above lemma, there is no extremal Type I \(\mathbb{Z}_k\)-code of length 28 for every positive integer \(k\) with \(k \geq 2\). There is a binary Type I code of length 28 and minimum weight 6 (see \([9]\)). Hence, we have the following:

**Corollary 5.13.** There is no extremal Type I \(\mathbb{Z}_k\)-code of length 28 for every positive integer \(k\) with \(k \geq 2\). There is a near-extremal Type I \(\mathbb{Z}_k\)-code of length 28 for every positive integer \(k\) with \(k \geq 2\).

Since \(R_{28,i}(\emptyset)\) \((i = 1, 2, \ldots, 36)\) have the identical theta series, by Lemma \([25]\) we have the following:

**Corollary 5.14.** Let \(\sum_{m=0}^{\infty} A_m q^m\) denote the theta series of \(R_{28,i}(\emptyset)\) \((i = 1, 2, \ldots, 36)\). Then \(A_k \geq 56\) for every positive integer \(k\) with \(k \geq 3\).

\[
\begin{pmatrix}
00 & 3221032113010 \\
00 & 2312302202000 \\
01 & 101113132031 \\
01 & 2021011201031 \\
10 & 303332032202 \\
00 & 220031132311 \\
00 & 1130232110223 \\
11 & 2213122020013 \\
01 & 3200201111201 \\
01 & 313320202230 \\
10 & 3110000202123 \\
10 & 301332120200 \\
10 & 3331011112112 \\
20 & 2220022000000 \\
02 & 0022222000000
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
01 & 1023301203302 \\
01 & 1022200000021 \\
01 & 113020322312 \\
11 & 1202303012212 \\
00 & 232132113032 \\
01 & 0002112332213 \\
11 & 113323310300 \\
00 & 3321000023111 \\
00 & 1210231221321 \\
11 & 2012013002211 \\
11 & 1010001123020 \\
11 & 2203101320001 \\
00 & 3302011030033 \\
20 & 0002202020200 \\
02 & 2220222202200
\end{pmatrix}
\]

Figure 2: Generator matrices of \(C_{4,28}\) and \(C'_{4,28}\)

### 5.5 Length 32

There are 5 non-isomorphic extremal odd unimodular lattices in dimension 32, and these 5 lattices are related to the 5 inequivalent binary extremal Type II codes of length 32 \([11]\), where the 5 codes are denoted by \(C_81, \ldots, C_85\) in \([9, \text{Table A}]\). We denote the extremal odd unimodular
lattice related to $C_i$ by $L_{32,i}$ ($i = 81, \ldots, 85$). We have verified by MAGMA that the odd unimodular lattice $A_4(C_{32,4}(D_{16}))$ in Table 8 is isomorphic to $L_{32,82}$. Since $A_4(C_{32,4}(D_{16}))$ contains a 4-frame, we investigate the existence of a $k$-frame in $L_{32,82}$ for $k = 6, 9$ by Lemma 4.3.

| Codes  | $r_A$          | $r_B$          |
|--------|----------------|----------------|
| $C_{6,32}$ | (0, 0, 1, 2, 2, 1, 2) | (1, 0, 5, 5, 1, 1, 3, 3) |
| $C_{9,32}$ | (0, 0, 1, 5, 0, 6, 0, 1) | (0, 6, 2, 2, 7, 6, 1, 7) |

Table 11: Extremal Type I $\mathbb{Z}_k$-codes of length 32

For $k = 6, 9$, let $C_{k,32}$ be the $\mathbb{Z}_k$-code with generator matrix of the form (2), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 11. Since $AA^T + BB^T = -I_8$, these codes are Type I. For $k = 6, 9$, we have verified by MAGMA that $A_k(C_{k,32})$ is isomorphic to $L_{32,82}$. Hence, combined with Lemma 4.3 we have the following:

**Theorem 5.15.** $L_{32,82}$ contains a $k$-frame if and only if $k$ is a positive integer with $k \geq 4$.

There are three inequivalent binary extremal Type I codes of length 32 [10]. Any ternary self-dual code of length 32 has minimum weight at most 9 [27]. Hence, we have the following:

**Corollary 5.16.** There is an extremal Type I $\mathbb{Z}_k$-code of length 32 for every positive integer $k$ with $k \neq 1, 3$.

Since the 5 non-isomorphic extremal odd unimodular lattices have the identical theta series [11], by Lemma 2.5 we have the following:

**Corollary 5.17.** Let $\sum_{m=0}^{\infty} A_m q^m$ denote the theta series of an extremal odd unimodular lattice in dimension 32. Then $A_k \geq 64$ for every positive integer $k$ with $k \geq 4$.

For each extremal odd unimodular lattice in dimension 32, one of the even unimodular neighbors is extremal [11]. Moreover, it follows from the construction in [11] that the extremal even unimodular neighbor of $L_{32,82}$ is the 32-dimensional Barnes–Wall lattice $BW_{32}$ (see e.g. [12, Chapter 8, Section 8] for $BW_{32}$). Since the even sublattice of $L_{32,82}$ contains a $2k$-frame for every positive integer $k$ with $k \geq 2$ by Theorem 5.15 we have the following:
Proposition 5.18. BW\(_{32}\) contains a 2\(k\)-frame if and only if \(k\) is a positive integer with \(k \geq 2\).

Then we have an alternative proof of the following:

Corollary 5.19 (Harada and Miezaki [20]). There is an extremal Type II \(\mathbb{Z}_{2k}\)-code of length 32 for every positive integer \(k\).

5.6 Length 36

Since \(A_6(C_{36,6}(D_{18}))\) contains a 6-frame, we investigate the existence of a \(k\)-frame in \(A_6(C_{36,6}(D_{18}))\) for \(k = 4, 5, 7, 9\) by Lemma 4.3. For \(k = 5, 7, 9\), let \(C_{k,36}\) be the \(\mathbb{Z}_k\)-code with generator matrix of the form (2), where the first rows \(r_A\) and \(r_B\) of \(A\) and \(B\) are listed in Table 12. Since \(AA^T + BB^T = -I_9\), these codes are Type I. Let \(C_{4,36}\) be the \(\mathbb{Z}_4\)-code with generator matrix of the following form:

\[
\begin{pmatrix}
I_{16} & A & B_1 + 2B_2 \\
O & 2I_4 & 2D
\end{pmatrix},
\]

where we only list in Figure 3 the matrices \((A B_1 + 2B_2)\) and \(2D\). For \(k = 4, 5, 7, 9\), we have verified by MAGMA that \(A_k(C_{k,36})\) is isomorphic to \(A_6(C_{36,6}(D_{18}))\). Hence, combined with Lemma 4.3 we have the following:

Theorem 5.20. \(A_6(C_{36,6}(D_{18}))\) contains a \(k\)-frame if and only if \(k\) is a positive integer with \(k \geq 4\).

Remark 5.21. We have verified by MAGMA that \(A_6(C_{36,6}(D_{18}))\) has theta series \(1 + 42840q^4 + 1916928q^5 + 42286080q^6 + \cdots\) and automorphism group of order 288.

Table 12: Extremal Type I \(\mathbb{Z}_k\)-codes of length 36

| Codes | \(r_A\)            | \(r_B\)            |
|-------|--------------------|--------------------|
| \(C_{5,36}\) | \(0,1,1,2,3,2,0,2,3\) | \(1,1,0,2,0,3,4,0,4\) |
| \(C_{7,36}\) | \(0,1,6,2,3,6,4,5\) | \(4,3,3,6,2,4,3,0,3\) |
| \(C_{9,36}\) | \(0,1,0,5,5,0,0,0,3\) | \(0,2,3,3,4,5,5,7,3\) |

There are 41 inequivalent binary extremal Type I codes of length 36 [28]. There is a ternary extremal Type I code of length 36 [32]. Hence, we have the following:
\[
(A \quad B_1 + 2B_2) = \begin{pmatrix}
0100 & 1203132123101121 \\
1011 & 1011202100200000 \\
1010 & 202022122311322 \\
0101 & 131122322101123 \\
1110 & 002223222133220 \\
0110 & 0102101300313130 \\
0100 & 100313123103103 \\
0001 & 0212210231101002 \\
1101 & 331110322131110 \\
0101 & 30331233020103 \\
0101 & 1320133200323130 \\
0100 & 200221022321133 \\
0101 & 3211333002312322 \\
0101 & 103111322033320 \\
0101 & 0103111200301112 \\
1110 & 222020200331300
\end{pmatrix}, \quad 2D = \begin{pmatrix}
0020020000222022 \\
0200202000000220 \\
2220200000200000 \\
2022000000000000
\end{pmatrix}
\]

Figure 3: A generator matrix of $C_{4,36}$

**Corollary 5.22.** There is an extremal Type I $\mathbb{Z}_k$-code of length 36 for every positive integer $k$ with $k \geq 2$.

By Lemma 2.5, we have the following:

**Corollary 5.23.** Let $\sum_{m=0}^{\infty} A_m q^m$ denote the theta series of $A_6(C_{36,6}(D_{18}))$. Then $A_k \geq 72$ for every positive integer $k$ with $k \geq 4$.

### 5.7 Length 40

Since $A_4(C_{40,4}(P_{20}))$ contains a 4-frame, we investigate the existence of a $k$-frame in extremal odd unimodular lattices in dimension 40 for $k = 6, 9, 13, 19$ by Lemma 4.3. For $k = 9, 13, 19$, let $C_{k,40}$ be the $\mathbb{Z}_k$-code with generator matrix of the form (2), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 13. Since $AA^T + BB^T = -I_{10}$, these codes are Type I. Moreover, we have verified by MAGMA that $A_k(C_{k,40})$ is extremal ($k = 9, 13, 19$). An extremal Type I $\mathbb{Z}_6$-code of length 40 can be found in [15]. Hence, combined with Lemma 4.3, we have the following:

**Lemma 5.24.** There is an extremal odd unimodular lattice in dimension 40 containing a $k$-frame if and only if $k$ is a positive integer $k$ with $k \geq 4$.

**Remark 5.25.** The possible theta series of an extremal odd unimodular lattice in dimension 40 is given in [5]:

$$\theta_{40,\alpha}(q) = 1 + (19120 + 256\alpha)q^4 + (1376256 -$$
4096\alpha q^5+\cdots$, where $\alpha$ is even with $0 \leq \alpha \leq 80$. We have verified by MAGMA that $A_4(C_{40,4}(P_{20}))$ has theta series $\theta_{40,80}(q)$ and automorphism group of order 7172259840, and $A_k(C_{k,40})$ ($k = 9, 13, 19$) have theta series $\theta_{40,0}(q)$ and automorphism group of order 40. Also, we have verified by MAGMA that three lattices $A_k(C_{k,40})$ ($k = 9, 13, 19$) are non-isomorphic.

For $k = 2, 3$, there is an extremal Type I $\mathbb{Z}_k$-code of length 40. Hence, we have the following:

**Theorem 5.26.** There is an extremal Type I $\mathbb{Z}_k$-code of length 40 for every positive integer $k$ with $k \geq 2$.

### Table 13: Extremal Type I $\mathbb{Z}_k$-codes of length 40

| Codes  | $r_A$                  | $r_B$                  |
|--------|------------------------|------------------------|
| $C_{9,40}$ | (0, 0, 1, 0, 5, 8, 3, 0, 4, 4) | (0, 5, 0, 0, 5, 6, 7, 2, 5, 8) |
| $C_{13,40}$ | (0, 0, 1, 4, 10, 5, 1, 10, 11, 4) | (11, 4, 4, 6, 7, 12, 11, 7, 2, 8) |
| $C_{19,40}$ | (0, 0, 1, 2, 14, 16, 17, 1, 0, 13) | (10, 2, 15, 2, 18, 16, 9, 15, 12, 0) |

We have verified by MAGMA that at least one of the even unimodular neighbors of $L$ is extremal for $L = A_4(C_{40,4}(P_{20})), A_9(C_{9,40}), A_{13}(C_{13,40})$ and $A_{19}(C_{19,40})$. There are binary extremal Type II codes of length 40 (see [3] for their classification). Then we have an alternative proof of the following:

**Proposition 5.27** (Harada and Miezaki [20]). There is an extremal Type II $\mathbb{Z}_{2k}$-code of length 40 for every positive integer $k$.

### 5.8 Length 44

By Lemma 4.3, we investigate the existence of a $k$-frame in extremal odd unimodular lattices in dimension 44 for $k = 4, 6, 9, 17$. For $k = 9, 17$, let $C_{k,44}$ be the $\mathbb{Z}_k$-code with generator matrix of the form (2), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 14. Since $AA^T + BB^T = -I_{11}$, these codes are Type I. Moreover, we have verified by MAGMA that $A_k(C_{k,44})$ is extremal ($k = 9, 17$). For $k = 4, 6$, an extremal Type I $\mathbb{Z}_k$-code of length 44 can be found in [18, Table 1] and [15], respectively. Hence, combined with Lemma 4.3 we have the following:
Lemma 5.28. There is an extremal odd unimodular lattice in dimension 44 containing a $k$-frame if and only if $k$ is a positive integer $k$ with $k \geq 4$.

Remark 5.29. The possible theta series of an extremal odd unimodular lattice in dimension 44 is given in [17]:

\[
\theta_{44,1,\beta}(q) = 1 + (6600 + 16\beta)q^4 + (811008 - 128\beta)q^5 + \cdots,
\]

\[
\theta_{44,2,\beta}(q) = 1 + (6600 + 16\beta)q^4 + (679936 - 128\beta)q^5 + \cdots,
\]

where $\beta$ is an integer. We have verified by Magma that $A_5(C_{44,5}(D_{22}))$ and $A_k(C_{k,44}) (k = 9, 17)$ have theta series $\theta_{44,1,\beta}(q) (\beta = 0, 88, 176)$, and automorphism groups of orders 44, 88 and 44, respectively.

Table 14: Extremal Type I $\mathbb{Z}_k$-codes of length 44

| Codes   | $r_A$      | $r_B$        |
|---------|------------|--------------|
| $C_{9,44}$ | (0, 0, 0, 0, 1, 0, 1, 4, 0, 8, 0) | (7, 0, 7, 1, 8, 8, 2, 8, 1, 5, 1) |
| $C_{17,44}$ | (0, 0, 0, 0, 1, 13, 7, 13, 11, 16, 13) | (12, 14, 8, 14, 7, 12, 14, 7, 14, 14, 7) |

For $k = 2, 3$, there is an extremal Type I $\mathbb{Z}_k$-code of length 44. Hence, we have the following:

Theorem 5.30. There is an extremal Type I $\mathbb{Z}_k$-code of length 44 for every positive integer $k$ with $k \geq 2$.

6 Remarks

We end this paper with some remarks about the existence of a $k$-frame in optimal odd unimodular lattices in dimension 48.

By Lemma 4.3, we investigate the existence of a $k$-frame in optimal odd unimodular lattices in dimension 48 for $k = 6, 7, 8, 9, 17$. It was shown in [19] that an extremal even unimodular lattice in dimension 48 has an optimal odd unimodular neighbor. Using this result, we have the following:

Lemma 6.1. There is an optimal odd unimodular lattice in dimension 48 containing an $8k$-frame for every positive integer $k$.

Proof. Let $\Lambda$ be an extremal even unimodular lattice in dimension 48. Let $x$ be a vector of $\Lambda$ with $(x, x) = 8$. Put $\Lambda^+_x = \{v \in \Lambda \mid (x, v) \equiv 0 \pmod{2} \}$. Since there is a vector $y$ of $\Lambda$ such that $(x, y)$ is odd, the following lattice

\[
\Lambda_x = \Lambda^+_x \cup \left(\frac{1}{2}x + y\right) + \Lambda^+_x
\]

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is an optimal odd unimodular neighbor of $\Lambda$ \cite{19}.

Some extremal even unimodular lattice in dimension 48 containing an 8-frame can be found in \cite{8} Corollary 1. We take this lattice as $\Lambda$ in the above construction. Let $\{f_1, \ldots, f_{48}\}$ be an 8-frame in $\Lambda$. Then $\Lambda_{f_1}$ is an optimal odd unimodular neighbor containing $\{f_1, \ldots, f_{48}\}$. The result follows from Lemma 2.1.

Some near-extremal Type I $\mathbb{Z}_k$-codes of length 48 can be found in \cite{19}. For $k = 7, 9$, let $C_{k,48}$ be the $\mathbb{Z}_k$-code with generator matrix of the form (2), where the first rows $r_A$ and $r_B$ of $A$ and $B$ are listed in Table 15. Since $AA^T + BB^T = -I_{12}$, these codes are Type I. Moreover, we have verified by MAGMA that $A_k(C_{k,48})$ is optimal ($k = 7, 9$). Hence, we have the following:

**Lemma 6.2.** There is an optimal odd unimodular lattice in dimension 48 containing a $k$-frame for every positive integer $k$ with $k \geq 5$ and $k \neq 2^{m_1}3^{m_2}17^{m_3}$, where $m_i$ are non-negative integers $(i = 1, 2, 3)$ with $(m_1, m_2) \in \{(0, 0), (0, 1), (1, 0), (2, 0)\}$ and $m_3 \geq 1$.

**Remark 6.3.** $A_6(C_{6,48})$ has kissing number 393216 \cite{19} p. 553]. In addition, we have verified by MAGMA that $A_5(C_{48,5}(D_{24}))$, $A_7(C_{7,48})$ and $A_9(C_{9,48})$ have kissing number 393216.

There are at least 264 inequivalent binary near-extremal Type I code of length 48 \cite{6}. There are at least two inequivalent ternary near-extremal Type I code of length 48 \cite{32}. It is not known whether there is a near-extremal Type I $\mathbb{Z}_4$-code of length 48 (see \cite{18}).

**Proposition 6.4.** There is a near-extremal Type I $\mathbb{Z}_k$-code of length 48 for integers $k = 2, 3$ and for integers $k$ with $k \geq 5$, $k \neq 2^{m_1}3^{m_2}17^{m_3}$, where $m_i$ are non-negative integers $(i = 1, 2, 3)$ with $(m_1, m_2) \in \{(0, 0), (0, 1), (1, 0), (2, 0)\}$ and $m_3 \geq 1$.

| Codes | $r_A$ | $r_B$ |
|-------|-------|-------|
| $C_{7,48}$ | (0, 1, 6, 3, 0, 2, 0, 2, 4, 2, 5, 3) | (3, 6, 1, 5, 4, 6, 0, 5, 0, 5, 1, 5) |
| $C_{9,48}$ | (0, 1, 2, 4, 6, 1, 6, 2, 2, 0, 3, 0) | (7, 2, 5, 1, 6, 8, 4, 1, 2, 8, 4) |
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