Influence of topological constraints and topological excitations: Decomposition formulas for calculating homotopy groups of symmetry-broken phases

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A symmetry broken phase of a system with internal degrees of freedom often features a complex order parameter, which generates a rich variety of topological excitations and imposes topological constraints on their interaction (topological influence); yet the very complexity of the order parameter makes it difficult to treat topological excitations and topological influence systematically. To overcome this problem, we develop a general method to calculate homotopy groups and derive decomposition formulas which express homotopy groups of the order parameter manifold $G/H$ in terms of those of the symmetry $G$ of a system and those of the remaining symmetry $H$ of the state. By applying these formulas to general monopoles and three-dimensional skyrmions, we show that their textures are obtained through substitution of the corresponding su(2)-subalgebra for the su(2)-spin. We also show that a discrete symmetry of $H$ is necessary for the presence of topological influence and find topological influence on a skyrmion characterized by a non-Abelian permutation group of three elements in the ground state of an SU(3)-Heisenberg model.

I. INTRODUCTION

Topological excitations create nontrivial spatial structures of the order parameter that cannot be removed by continuous deformation and are characterized by a topological charge. When a system has internal degrees of freedom as in spinor Bose-Einstein condensates (BECs) [1, 2], $p$-wave superfluids and superconductors [3, 4] and multiorbital electron systems [5–9], a symmetry broken phase has a complex order parameter, accommodating a rich variety of topological excitations. Examples include fractional and non-Abelian vortices [10–12], skyrmions [13, 14], Shankar skyrmions [15, 16] and knot solitons [16, 17]. Several different types of topological excitations have been experimentally observed in condensed matter and ultracold atomic systems: skyrmions in chiral magnets [18–20], and quantum Hall ferromagnets [20–22], half-quantum vortices in $p$-wave superconductors [23] and liquid $^3$He [24], and knot solitons in liquid crystals [25]. In particular, ultracold atomic gases offer an ideal playground for the study of topological excitations due to high controllability of experimental parameters; here the controlled generations of vortices [26–28], skyrmions [29–31], monopoles [31, 32], and a knot soliton [33] have been demonstrated. Yet another remarkable feature arising from internal degrees of freedom is the coexistence of different types of topological excitations, which leads to non-conservation of individual topological charges due to topological influence [13–16]. For example, the A phase of superfluid $^3$He can simultaneously accommodate a half-quantum vortex and a monopole. When the latter makes a complete circuit of the former, the topological charge of the latter changes its sign [35].

In mathematical parlance, the set of topological charges classify textures of an order parameter of topological excitations; two textures can continuously transform into each other if and only if their topological charges are the same. While the complexity of $G/H$ leads to the richness of topological excitations, it makes the calculation of homotopy groups involved, the understanding of textures highly nontrivial, and the analysis of topological influence difficult. While topological influence of a vortex on a topological excitation is known to be described by the action of the fundamental group $\pi_1(G/H)$ on the $m$th homotopy group $\pi_m(G/H)$ [34–36], where $m$ is the spatial dimension in which the texture of the topological excitation varies, general conditions for its presence are yet to be clarified. For topological influence on a monopole or a skyrmion, only one type is known, in which the topological influence changes the sign of the topological charge of a monopole and that of a skyrmion [34, 37, 38, 39, 40].

In the present paper, we develop a general method to calculate the homotopy group $\pi_m(G/H)$ of the order parameter manifold $G/H$ by deriving a formula which expresses $\pi_m(G/H)$ in terms of $\pi_m(G)$ and $\pi_m(H)$. Since the homotopy groups can be determined systematically for Lie groups $\pi_m(G/H)$ and the corresponding textures can be determined through the formula. By applying the derived formulas for $m = 2$ and $3$, we show that the texture of a general monopole and that of a general three-dimensional skyrmion are obtained from that of a monopole in a ferromagnet and those of a knot soliton or a Shankar skyrmion, respectively, through substitution of an appropriate su(2)-subalgebra in $G$ for the su(2)-spin. Consequently, their topological charges are described by a set of integers distinguished by co-roots [11, 12] which label different su(2)-subalgebras in $G$.

We also obtain the necessary and sufficient condition for the appearance of non-Abelian vortices and prove the absence of topological influence on a three-dimensional skyrmion. We find that possible types of topological in-
fluence on a monopole or a skyrmion can be identified with the Weyl group \[ \Pi(d) \] of \( G \), where only one type is shown to be allowed if \( G \) is \( U(1) \), \( SU(2) \), \( SO(3) \), or their direct product. Moreover, we find topological influence on skyrmions characterized by a non-Abelian permutation group of three elements in the ground state of an SU(3)-Heisenberg model \[ 7, 8, 43, 44 \], in which three types of skyrmions exchange their types through topological influence.

This paper is organized as follows. In Sec. II we derive a decomposition formula for \( \pi_m(G/H) \) for an arbitrary dimension \( m \). In Sec. III we derive simplified formulas for \( \pi_m(G/H) \) with \( m = 1, 2, \) and \( 3 \), and determine the texture of a general monopole and that of a general three-dimensional skyrmion. In Sec. IV we analyze the conditions for the presence of topological influence. In Sec. V we discuss the non-Abelian topological influence on a skyrmion. In Sec. VI we conclude this paper. Some mathematical proofs are relegated to the appendices to avoid digressing from the main subject. Appendix [A] proves a lemma on the third homotopy group of a compact Lie group used in Sec. II. Appendices [B] and [C] prove formulas for \( \pi_m(G/H) \) and \( \pi_2(G/H) \), respectively, discussed in Sec. III. Appendix [D] proves a theorem concerning topological influence on a general topological excitation discussed in Sec. IV. Appendix [E] proves a corollary concerning topological influence on a monopole or a skyrmion discussed in Sec. V.

II. DECOMPOSITION FORMULA FOR HOMOTOPY GROUPS OF ORDER PARAMETER MANIFOLDS

A. Homotopy groups of a Lie group

We first introduce the Cartan canonical form and the lattices of a compact Lie group, by means of which the first, second, and third homotopy groups are determined. When the parameter space of \( G \) is (not) finite, \( G \) is said to be (non-)compact. If \( G \) includes translational symmetry, \( G \) is non-compact. However, for the calculation of homotopy groups, \( G \) must be (not)finite. If \( G \) is non-compact, \( \pi_2 \) and \( \pi_3 \), the second and third homotopy groups, are determined.

We denote as \( \{ E^{(2)}_r \} \) the \( (2\text{-}) \)-subalgebras in \( \mathfrak{g} \). The Cartan canonical form \( [41, 42] \) of \( \mathfrak{g} \) is a generalization of the basis of the \( su(2) \)-Lie algebra \( \{ S_3, \{ S_1, S_2 \} \} \), and decomposes the generators of the Lie algebra into the off-diagonal matrices \( \{ E^{(2)}_r, E^{(1)}_\alpha \} \), and the diagonal ones \( \{ H_j \} \), where \( \alpha \) is an \( r \)-dimensional real vector known as a positive root and \( R_+ \) denotes the entire set of positive roots. The positive roots are introduced to distinguish different \( su(2) \)-subalgebras in \( \mathfrak{g} \). It is known that any positive root can be expressed as a linear combination of the \( r \) positive roots known as simple roots, which we denote as \( \{ \alpha_j \} \). Two matrices \( E_\alpha \) and \( E_-\alpha = E^{(1)}_\alpha \) are generalizations of the raising and lowering operators \( S_+ := S_1 + i S_2 \) and \( S_- := S_1 - i S_2 \) of the \( su(2) \)-spin vector \( S = (S_1, S_2, S_3) \). Physically \( \alpha \) describes the difference between two quantum numbers. When \( E_+ \alpha \) is applied to a state, its quantum number changes by \( \alpha \) (\(-\alpha \)), as \( S_+ (S_-) \) changes the magnetic quantum number of a spin state by \(+1 \) (\(-1 \)).

Together with the Cartan generator \( H_\alpha \) defined by \( H_\alpha := \sum_{j=1}^r (\alpha)_j H_j \), where \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_r)^T \in \mathbb{R}^r \) (\( T \) denotes the transpose of a vector), the two generators \( E^{(2)}_r \) and \( E^{(1)}_\alpha \) satisfy the following commutation relations:

\[
\begin{align*}
[E^{(2)}_r, E^{(1)}_\alpha] &= i (\alpha, \alpha) H_\alpha, \\
[H_\alpha, E^{(2)}_r] &= i (\alpha, \alpha) E^{(1)}_\alpha, \\
[E^{(1)}_\alpha, H_\alpha] &= i (\alpha, \alpha) E^{(2)}_r.
\end{align*}
\]

We define the co-root \( \alpha^c \) as a dual vector to each positive root \( \alpha \) and the corresponding generator \( H_{\alpha^c} \) as follows:

\[
\alpha^c := \frac{2\alpha}{(\alpha, \alpha)} ,
\]

\[
H_{\alpha^c} := \sum_{j=1}^r (\alpha^c)_j H_j.
\]
One can see from Eq. (1) that a triad $S_\alpha$ defined by
\[ S_\alpha := (S_{\alpha,1}, S_{\alpha,2}, S_{\alpha,3}) := \left( \frac{E^R_\alpha}{\alpha, \alpha}, \frac{E^I_\alpha}{\alpha, \alpha}, \frac{H_\alpha}{2} \right) \] (7)
forms an $su(2)$-subalgebra satisfying the following commutation relations:
\[ [S_{\alpha,a}, S_{\alpha,b}] = i\epsilon_{abc}S_{\alpha,c} \text{ for } a, b, c = 1, 2, 3, \] (8)
where $\epsilon_{abc}$ is the three-dimensional Levi-Civita symbol which is a totally antisymmetric unit tensor of rank three. We refer to $S_\alpha$ as the three co-roots as follows:
\[ H_t \in \mathfrak{g} \exp(2\pi i H_t) = e \}
\[ L_G := \left\{ \sum_{\alpha} n_\alpha H_\alpha^c \in \mathfrak{g} \mid n_\alpha \in \mathbb{Z}, \alpha \in R_+ \right\}, \] (10)
where $H_t \in \mathfrak{g}$ for $t \in \mathbb{R}$ is defined by $H_t := \sum_{j=1}^r t_j H_j$ with $t = (t_1, t_2, \cdots, t_r)^T$ ($T$ denotes the transpose of a vector). Both $L_G$ and $L_G^c$ form Abelian groups under the addition of matrices.

Consider an example of $\mathfrak{g} = su(3)$, which is generated by the following nine generators:
\begin{align*}
S_{RG,1} &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
S_{RG,2} &= \frac{1}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
S_{RG,3} &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
S_{GB,1} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
S_{GB,2} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
S_{GB,3} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
S_{BR,1} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
S_{BR,2} &= \frac{1}{2} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
S_{BR,3} &= \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11}
\end{align*}

The corresponding Cartan canonical form is constituted from the following three generalized $su(2)$-spin vectors
\begin{align*}
S_{RG} &= (S_{RG,1}, S_{RG,2}, S_{RG,3}),
S_{GB} &= (S_{GB,1}, S_{GB,2}, S_{GB,3}),
S_{BR} &= (S_{BR,1}, S_{BR,2}, S_{BR,3}). \tag{12}
\end{align*}

Note that the three diagonal generators $S_{RG,3}, S_{GB,3}, \text{ and } S_{BR,3}$ are not linearly independent because $S_{RG,3} + S_{GB,3} + S_{BR,3} = 0$. Three root vectors $\alpha_{RG}, \alpha_{GB}, \text{ and } \alpha_{BR}$ corresponding to generators $S_{RG}, S_{GB}, \text{ and } S_{BR}$ are given by $E^R_\alpha, E^I_\alpha, \text{ and } H_\alpha$.

From direct calculations using Eqs. (11), (13), and (14), one can show that the integral lattice $L_{SU(3)}$ and the co-root lattice $L_{SU(3)}^c$ coincide and that they are isomorphic to the triangular lattice [see Fig. 1]:
\begin{align*}
L_{SU(3)} &= L_{SU(3)}^c = \left\{ \sum_{a=RG,GB,BR} m_a \alpha_a \mid m_a \in \mathbb{Z}, \sum_{a=RG,GB,BR} \alpha_a = 0 \right\}. \tag{15}
\end{align*}

2. First, second, and third homotopy groups of a compact Lie group

It is known that $L_{SU(3)}^c$ is an Abelian subgroup of $L_G$ and that the quotient group $L_G/L_{SU(3)}^c$ is isomorphic to $\pi_1(G)$
\[ \pi_1(G) \simeq L_G/L_{SU(3)}^c. \tag{16} \]
While $L_G$ describes all loops on $G$, $L_{SU(3)}^c$ describes only those loops on $G$ that can continuously transform into a
trivial one, so the quotient space naturally gives $\pi_1(G)$. To be concrete, let us consider an element $H_\xi$ of $L_G$ corresponding to a loop defined by
\[
g_{1,n}(\phi) := \exp(i\phi H_\xi) \text{ for } \phi \in [0, 2\pi]. \tag{17}
\]
The map $g_{1,n}$ indeed describes a loop on $G$, since $g_{1,n}(0) = g_{1,n}(2\pi) = e$ from Eq. (19). The triviality of loops in $L_G^c$ can be checked by considering one of its generator $H_\alpha^c$ and the corresponding loop $g_{1,c}(\phi) := \exp((i\phi H_\alpha^c)$.

This loop can continuously transform into a trivial one through $g_{2}^{(2)}(\theta, \phi)$ defined by
\[
g_{2}^{(2)}(\theta, \phi) := e^{-i\theta S_{\alpha_3}e^{i\phi S_{\alpha_3}e^{i\phi S_{\alpha_3}}}} , \tag{18}
\]
where $\theta \in [0, \pi]$ is the parameter of the deformation. In fact, we have
\[
g_{2}^{(2)}(\theta = 0, \phi) = e^{i\phi S_{\alpha_3}} = g_{1,\alpha}(\phi), \tag{19}
g_{2}^{(2)}(\theta = \pi, \phi) = e^{-i\pi S_{\alpha_2}e^{i\phi S_{\alpha_3}e^{i\phi S_{\alpha_3}}}} = e^{i\phi S_{\alpha_3}} = e ^{i\phi S_{\alpha_3}} = e , \tag{20}
\]
where the last equality in Eq. (19) follows from the definition of $S_{\alpha_3}$ and the second line in Eq. (20) is derived from $e^{-i\pi S_{\alpha_2}e^{i\phi S_{\alpha_3}e^{i\phi S_{\alpha_3}}}} = -S_{\alpha_3}$. It is worthwhile to mention that $\pi_1(G)$ is Abelian, which follows from the fact that $L_G$ is Abelian and the fact that a quotient group of an Abelian group is Abelian [47].

The second homotopy group of a compact Lie group is known to vanish identically \[41, 42, 48\]:
\[
\pi_2(G) \simeq 0. \tag{21}
\]

We now discuss the third homotopy group. It is known that the Lie algebra $g$ of a compact Lie group $G$ can be decomposed into the direct sum of one-dimensional Lie algebras $u(1)$ and a set of compact simple Lie algebras $\{g_i\}_{i=1}^a$ [42],
\[
g = u(1)^{a'} \bigoplus \bigoplus_{i=1}^a g_i \tag{22},
\]
where $a$ and $a'$ are the integers which are uniquely determined from $g$, and $u(1)$ is the Lie algebra of $U(1)$, the unitary group of degree one. Let $\alpha_i$, $\alpha'_i$, and $S_{\alpha_i}$ be one of the root vectors in $g_i$, with the largest length, the corresponding co-root, and the corresponding generalized $su(2)$-spin vector defined in Eq. (17), respectively. We define $g_{3}^{(3)} : S^3 \to G$ for $S_{\alpha_i}$ by
\[
g_{3}^{(3)}(\psi, \theta, \phi) := \exp[2i\psi S_{\alpha_i} \cdot \hat{r}(\theta, \phi)], \tag{23}
\]
where $\hat{r}(\theta, \phi)$ is a unit vector on $S^2$ defined by $\hat{r}(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, with $\psi, \theta, \phi$ being the polar coordinates of the three-dimensional sphere $S^3$:
\[
S^3 = \{(\sin \psi \sin \cos \phi, \sin \psi \sin \sin \phi, \sin \psi \cos \theta, \cos \psi) | \psi \in [0, \pi], \theta \in [0, \pi], \phi \in [0, 2\pi] \}. \tag{24}
\]

FIG. 2: (Color online) Schematic illustration of the symmetry transformation $g_{3}^{(3)}(\psi, \theta, \phi)$ defined in Eq. (23). The red arrow indicates the generalized $su(2)$-spin vector parallel to the unit vector $\hat{r}(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ and $g_{3}^{(3)}(\psi, \theta, \phi)$ describes the spin rotation about $\hat{r}(\theta, \phi)$ through angle $2\psi$.

Since $S_{\alpha_i} \cdot \hat{r}(\theta, \phi)$ is the projection of the generalized $su(2)$-spin vector in the direction of $\hat{r}(\theta, \phi)$, $g_{3}^{(3)}(\psi, \theta, \phi)$ describes the rotation of $S_{\alpha_i}$ about $\hat{r}(\theta, \phi)$ through angle $2\psi$ [see Fig. 2]. For example, $g_{3}^{(3)}(\psi, \theta, \phi)$ for $G = SO(3)$ describes rotation in three dimensions about a vector $\hat{r}(\theta, \phi)$ through angle $2\psi$. Let $[f]_M$ be the homotopy class of $f$ on a manifold $M$, which is the set of those maps to $M$ that can continuously transform into $f$, where $f$ is referred to as a representative element. Then, the following lemma holds.

\textbf{Lemma 1} The third homotopy group of $G$ is generated by the set $\{ [g_{3}^{(3)}]_G \}_{i=1}^a$
\[
\pi_3(G) \simeq \left\{ \sum_{i=1}^a m_i [g_{3}^{(3)}]_G \right\} \bigg| m_i \in \mathbb{Z} \right\} \simeq \mathbb{Z}^a, \tag{25}
\]
where we denote the product on $\pi_3(G)$ as the sum since $\pi_3(G)$ is Abelian.

The proof of Lemma 1 is given in Appendix [A]. We note that the homotopy class $[g_{3}^{(3)}]_G$ does not depend on the choice of the root vector $\alpha_i$, since $g_{3}^{(3)}$ and $g_{3}^{(3)}$ can continuously transform into each other if the corresponding root vectors $\alpha_i$ and $\alpha'_i$ in $g_i$ both have the largest length [49].

\section{Homotopy groups of the order parameter manifold $G/H$}

\subsection{Two types of textures on $G/H$}

Let $D^m$ be the surface and the inner region of an $m$-dimensional sphere $S^m$ with radius $\pi$
\[
D^m := \{ x \in \mathbb{R}^m ||x|| \leq \pi \}. \tag{26}
\]
and consider a topological excitation without a defect characterized by $\pi_m(G/H)$, such as two-dimensional ($m = 2$) and three-dimensional ($m = 3$) skyrmions. Assuming that it is localized in $D^m$, we may regard its texture $O(x)$ as a map from $D^m$ to the $G/H$ subject to the boundary condition

$$O(x) = O_0 \text{ for } \|x\| = \pi,$$

(27)

where $O_0$ is a fixed value of the order parameter called the reference order parameter and $\|x\|$ denotes the modulus of $x$. We note that a map $O : D^m \to G/H$ subject to the boundary condition [27] can represent a texture of a topological excitation with a defect through the replacement of $D^m$ by an $m$-dimensional sphere $S^m$ enclosing the defect. A crucial point for obtaining $\pi_m(G/H)$ is to express the texture $O(x)$ in terms of a symmetry transformation $g(x)$ depending on the space coordinate $x$ as follows:

$$O(x) = g(x)O_0,$$

(28)

where $gO_0$ for $g \in G$ denotes the action of $g$ on $O_0$. The expression [25] relates a texture $O(x)$ on $G/H$ to a texture $g(x)$ on $G$ and $\pi_m(G/H)$ to $\pi_m(G)$ and $\pi_m(H)$. Although the texture $O(x)$ is continuous on $D^m$, $g(x)$ may not be continuous because $g(x)$ and $g(x)h(x)$ for any discontinuous function $h$ with $h(x) \in H$ give the same texture $O(x)$ in Eq. (28). As we will see below, two cases arise depending on whether or not $g(x)$ is continuous on the entire region of $D^m$.

Given a subgroup $H$ of $G$, we can define the inclusion map $i : H \to G$ by $i(h) := h$ [see Fig. 3 (a)]. Then, a texture on $H$, i.e., a map $g$ from $D^m$ to $H$, can also be regarded as a texture on $G$, and we define a map $i_m : \pi_m(H) \to \pi_m(G)$ between the homotopy groups as

$$i_m([f_H]) = [i \circ f_G],$$

where $\circ$ denotes the composition of two maps. We construct textures $G/H$ in two ways from two groups Coker $i_m$ and Ker $i_{m-1}$ which are defined as follows:

$$\text{Coker } i_m := \text{Coker } \{i_m : \pi_m(H) \to \pi_m(G)\}$$

$$= \pi_m(G) \big/ \text{Im } i_m,$$

(29)

$$\text{Ker } i_{m-1} := \text{Ker } \{i_{m-1} : \pi_{m-1}(H) \to \pi_{m-1}(G)\}$$

$$= \{x \in \pi_{m-1}(H) | i_{m-1}(x) = e\},$$

(30)

where $\text{Im } F := \{F(g)|g \in G\}$ and Coker $F$ for $F : G \to G'$ is defined by Coker $F := G'/\text{Im } F$ [see Fig. 3 (b)]. An element of Ker $i_{m-1}$ represents a nontrivial texture on $H$ that is trivial as a texture on $G$. While an element of $\text{Im } i_m$ represents a nontrivial texture on $G$ that can be represented as a texture on $H$, that of Coker $i_{m-1}$ represents a nontrivial texture on $G$ that cannot be represented as a texture on $H$. We denote the element of Coker $i_m$ corresponding to $a \in \pi_m(G)$ by $[a]$ and call $a$ the representative element of $[a]$.

Let us construct the texture $O[a]$ on $G/H$ from $[a] \in \text{Coker } i_m$. Since $a$ is a texture of $G$, we can define the texture $O[a]$ through the action of $a$ on $O_0$:

$$O[a](x) := a(x)O_0 \text{ for } x \in D^m.$$

(31)

Equation (31) implies that a nontrivial texture on $G/H$ can be obtained from a nontrivial texture on $G$ [see Fig. 1 (a)]. From the boundary condition for $a$, i.e.,

$$a(x) = e \text{ for } \|x\| = \pi,$$

(32)

we see that $O[a](x)$ satisfies the boundary condition (27). It is worth mentioning two things. First, Coker $i_m$ is Abelian for $m \geq 1$ because the numerator on the right-hand side of Eq. (29) is Abelian. This follows from the fact that $\pi_1(G)$ is Abelian and from the commutativity of higher-dimensional homotopy groups $[50]$. Second, we must consider the quotient space Coker $i_{m-1}$ instead of $\pi_m(G)$, which is the numerator on the right-hand side of Eq. (29), because the denominator $\text{Im } i_m$ gives a uniform texture through Eq. (31). Indeed, for $a_h := i_{m-1}(aH) \in \text{Im } i_m$, we have

$$O[a_h](x) := a_h(x)O_0 = [i(aH)](x)O_0$$

$$= aH(x)O_0 = O_0 \text{ for } x \in D^m,$$

(33)

where we use the invariance of $O_0$ under the transformation in $H$ in obtaining the last equality. The simplest example of the construction is an integer-quantum vortex in a scalar BEC. Let $\Psi$ be the mean-field wave function of the condensate. Then, the texture $\Psi(\phi)$ around a vortex with a unit winding number is given by

$$\Psi(\phi) = \exp(i\phi)\Psi_0,$$

(34)
where $\theta$ and $\phi$ denote the polar angle and the azimuth angle around the vortex, and $\Psi_0$ is the mean-field wave function at $\phi = 0$. Thus, the nontrivial texture $\Psi(\phi)$ of the vortex is expressed in terms of the nontrivial winding $\exp(i\phi)$ on the symmetry group $G = U(1)$. Another example of the construction is a three-dimensional skyrmion in a ferromagnet called a hedgehog [51], whose texture $M(r, \theta, \phi)$ of the spin is written as

$$M(r, \theta, \phi) := \exp \left[ 2i\psi(r)S \cdot \tilde{r}(\theta, m\phi) \right] M_0,$$  

(35)

where $(r, \theta, \phi)$ is the three-dimensional polar coordinates and $m \in \mathbb{Z}$ denotes the topological charge of the vortex. Here, $M_0 := (0, 0, 1)$ and $\psi(r)$ is a function that satisfies $\psi(0) = 0$ and $\psi(\infty) = \pi$. One can see from Eq. (28) that the nontrivial texture is expressed in terms of a nontrivial winding $\exp \left[ 2i\psi(r)S \cdot \tilde{r}(\theta, \phi) \right]$ on a symmetry group $SO(3)$ of spin rotation.

To construct the texture $O^b$ on $G/H$ from $b \in \text{Ker} \ i_{s^m-1}$, we regard $b$ as a map from $S^{m-1}$. Since $b$ is a trivial texture on $G$ from its definition, there exists a continuous deformation $b_s$ from $b_{s=0} = b$ to the uniform texture $b_{s=\pi} = e$ subject to the boundary condition

$$b_s(\hat{x}_0) = O_0 \text{ for } s \in [0, \pi],$$  

(36)

where $\hat{x}_0$ is a point on $S^{m-1}$ and $s$ is the parameter of the deformation. Hence, we define $O^b$ as

$$O^b(x) := b_{s=\|x\|} \ (\hat{x}) O_0 \text{ for } x \in D^m,$$  

(37)

where $\hat{x}$ is the unit vector parallel to $x$ [see Fig. 4(b)]. We note that $b_{s=\|x\|} (\hat{x})$ in the construction is not continuous at the origin $x = 0$. From the comparison of Eq. (28) with Eq. (31) (Eq. (37)), a texture on $G/H$ is expressed by a texture on $G$ that is not continuous on $D^m$, and is described by an element of $\text{Coker} \ i_{s^m}$ ($\text{Ker} \ i_{s^m-1}$). Examples of the construction include a half vortex in a uniaxial nematic liquid crystal and a monopole in a ferromagnet. The order parameter of a uniaxial nematic liquid crystal is the orientation $d$ of molecules. The texture $d(\phi)$ around a half vortex is given by

$$d(\phi) := \exp(\phi L_2/2) d_0,$$  

(38)

where $d_0 := (0, 0, 1)$, and $\phi$ and $L_2$ are the azimuth angle around the vortex and a generator of rotation about the $y$-axis, respectively. The nontrivial texture $d(\phi)$ is expressed not by a loop on $G = SO(3)$ but by a path from $e$ to $\exp(\pi L_2)$ on $SO(3)$. Due to the discrete $\pi$-rotational symmetry $d \to -d$, the start point $d_0$ and the end point $\exp(\pi L_2) d_0 = -d_0$ should be identified, where the texture is continuous at $\phi = 0$ ($\phi = 2\pi$). The texture $M(\theta, \phi)$ of a monopole in a ferromagnet is described by a hedgehog configuration of the spin, i.e., $M(\theta, \phi) = \hat{r}(\theta, \phi)$, which can be rewritten in terms of the $su(2)$-spin vector $S$ as follows:

$$M(\theta, \phi) := \exp(i\phi S_3) \exp(i\theta S_2) M_0,$$  

(39)

where $\theta$ and $\phi$ denote the polar angle and the azimuth angle around the monopole, respectively. $M_0 := (0, 0, 1)$, and $S_2$ ($S_3$) is a generator of the rotation about the $x$-axis ($z$-axis). As shown in Fig. 5, the hedgehog texture is obtained by the successive applications of spin rotation $\exp(i\theta S_2)$ about the $y$-axis followed by spin rotation $\exp(i\phi S_3)$ about the $z$-axis. Under continuous deformation $M_u(\theta, \phi) = \exp(\theta S_2) \exp(i\phi S_3) \exp(\phi S_2) M_0$, with $u \in [0, 1]$ being the parameter of the deformation, $M_{u=0}(\theta, \phi) = M(\theta, \phi)$ transforms into

$$M_{u=1}(\theta, \phi) = \exp(-\theta S_2) \exp(i\phi S_3) \exp(\phi S_2) M_0,$$  

(40)

where in the second equality we use the invariance of $M_0$ under the rotation about the $z$-axis, and $g^{(2)}(\theta, \phi) := \exp(-\theta S_2) \exp(i\phi S_3) \exp(i\theta S_2) \exp(i\phi S_3)$ is the map [52] with $S_0$ replaced by $S$. The expression $g^{(2)}(\theta, \phi) M_0$ gives the texture of a monopole in the form of Eq. (28). Indeed, since we have

$$g^{(2)}(\theta = 0, \phi) = \exp(2i\phi S_3),$$  

(41)

$$g^{(2)}(\theta = \pi, \phi) = e^{-i\phi S_3} \exp(i\phi S_3) = e^{-i\phi S_3} e^{i\phi S_3} = e,$$  

(42)

$$g^{(2)}(\phi = 0) = e,$$  

(43)

FIG. 4: (Color online) (a) Construction of the texture $O^{|s|}$ from the map $a$ defined in Eq. (31) for $m = 2$, where $a$ is a map from a two-dimensional disk $D^2$ to $G$ that maps the boundary (red dashed line) of $D^2$ to the identity element $e$. The texture $O^{|s|}$ is a map from $D^2$ to $G/H$ that maps the boundary (blue dotted line) of $D^2$ to the reference order parameter $O_0$. (b) Construction of the texture $O^b$ from the map $b$ defined in Eq. (37) for $m = 2$. Here, $b$ is a map from a circle $S^1$ to $H$ and $b_s$ is a continuous deformation from $b$ to the uniform texture $e$ subject to the condition (36), where the points on the red dashed line are mapped to $e$. The texture $O^b$ is a map from $D^2$ to $G/H$ which maps the points on the blue dotted line to $O_0$. 


Let $b$ and $b'$ be elements of $\text{Ker } i_{s_{m-1}}$. Then, there exist continuous deformations $b_s$, $b'_s$, and $(b b')_s$ from $b$, $b'$, and $b b'$, respectively, to a trivial map subject to the boundary condition (30). We fix the parameter $s$, consider $b_s$ and $b'_s$ to be elements of $\pi_{m-1}(G)$, and denote their composition as $b_s \circ b'_s$. Then, we define $f(b, b')$ by

$$f(b, b') := \left\{ \begin{array}{ll} \tilde{f}(b, b') & \text{for } 0 \leq ||x|| \leq \frac{\pi}{2}; \\ \tilde{f}(b_s \circ b'_s) & \text{for } \frac{\pi}{2} \leq ||x|| \leq \pi. \end{array} \right. \quad (48)$$

Since $\tilde{f}$ satisfies the boundary condition (32), which gives $\tilde{f} \in \pi_{m}(G)$, $f$ is indeed a map to $\text{Ker } i_{s_{m}}$. The map $f$ does not depend on the choices of the representative elements of $b$ and $b'$. Let $B$ and $\bar{B}$ be two representative elements of $b$, and $B_s$ ($\bar{B}_s$) be the continuous deformations from $B$ ($\bar{B}$) to the trivial homotopy class. Since $B$ and $\bar{B}$ transform into each other through continuous deformation on $H$, so do $B_s$ and $\bar{B}_s$. Therefore, the map $f$ in Eq. (48) defined from $B$ and $B_s$ and that defined from $B_s$ and $\bar{B}$ transform into each other continuously.

2. A decomposition formula for $\pi_{m}(G/H)$

Let us define the product $\times_f$ on the product set $\text{Coker } i_{s_{m}} \times \text{Ker } i_{s_{m-1}}$ by

$$(a, b) \times_f ([a']', b') := ([a] + [a'] + f(b, b'), b b'), \quad (49)$$

where $f : \text{Ker } i_{s_{m-1}} \times \text{Ker } i_{s_{m-1}} \rightarrow \text{Coker } i_{s_{m}}$ is the map defined in Eq. (48) and we denote the product in $\text{Coker } i_{s_{m}}$ as the sum since $\text{Coker } i_{s_{m}}$ is Abelian. Then the following theorem holds.
Theorem 1 Under the product defined in Eq. (39), Coker \( i_{sm} \times \text{Ker} \ i_{sm-1} \) becomes a group. This group denoted by \( \text{Coker} \ i_{sm} \times \text{Ker} \ i_{sm-1} \) is isomorphic to the nth homotopy group of \( G/H \):

\[
\pi_m(G/H) \simeq \text{Coker} \ i_{sm} \times \text{Ker} \ i_{sm-1}. 
\] (50)

Any topological charge \( ([a], b) \) in \( \pi_m(G/H) \) can be uniquely decomposed into the product of an element of Coker \( i_{sm} \) and that of Ker \( i_{sm-1} \):

\[
([a], b) = ([a], e) \times_f (e, b). \tag{51}
\]

Furthermore, the texture \( O([a]) \) of a topological excitation with topological charge \( ([a], e) \) is given by Eq. (32) (Eq. (34))

The proof of Theorem 1 is given in Appendix B. Equation (50) implies that there are two distinct types of these two topological excitations described by Ker \( i_{sm-1} \) can produce a topological excitation described by Coker \( i_{sm} \), because we have

\[
(e, b) \times_f (e, b') = (f(b, b'), bb') \]

and \( f(b, b') \neq e \) in general. The group Coker \( i_{sm} \times_f \text{Ker} \ i_{sm-1} \) is referred to as the group extension of Coker \( i_{sm} \) by Ker \( i_{sm-1} \) with the factor set \( f \) (see Appendix B or Ref. 17 for detail).

III. FORMULAS FOR HOMOTOPY GROUPS FOR LOW-DIMENSIONAL TOPOLOGICAL EXCITATIONS

A. First homotopy group: vortices

Since Coker \( i_1^* \) is a quotient group of \( \pi_1(G) \) from Eq. (29) and \( \pi_1(G) \) is a quotient group of \( L_G \) from Eq. (20), we write an element of Coker \( i_1^* \) as \( [H_t] \) where \( H_t \in L_G \). Let us define the product \( \times_f \) on Coker \( i_1^* \times \pi_0(H \cap G_0) \) by

\[
([H_t], [\sigma]) \times_f ([H_s], [\tau]) := ([H_t] + [H_s] + f([\sigma], [\tau]), [\sigma][\tau]), \tag{53}
\]

where \( f \) is the map defined in Eq. (18). Then the following corollary holds.

Corollary 1 Under the product defined in Eq. (53), Coker \( i_1^* \times \pi_0(H \cap G_0) \) is isomorphic to the first homotopy group of \( G/H \):

\[
\pi_1(G/H) \simeq \text{Coker} \ i_1^* \times \pi_0(H \cap G_0). \tag{54}
\]

Any topological charge \( ([H_t], [\sigma]) \) in \( \pi_1(G/H) \) can be uniquely decomposed into the product of an element of Coker \( i_1^* \) and that of \( \pi_0(H \cap G_0) \):

\[
([H_t], [\sigma]) = ([H_t], e) \times_f (e, [\sigma]). \tag{55}
\]

Let \( \phi \) and \( O_0 \) be the azimuth angle around a vortex and the reference order parameter, respectively. The texture\( O([H_t], e) \) of a vortex with topological charge \( ([H_t], e) \) is given by

\[
O([H_t], e)(\phi) = \exp(i\phi H_t)O_0, \tag{56}
\]

while the texture \( O(e, [\sigma]) \) of a vortex with topological charge \( (e, [\sigma]) \) is given by

\[
O(e, [\sigma])(\phi) = \gamma_\sigma(\phi)O_0, \tag{57}
\]

where \( \gamma_\sigma(\phi) \) is a path from \( \sigma \in H \cap G_0 \) to the identity element \( e \).

Proof

Due to the isomorphism Ker \( i_{0^*} \simeq \pi_0(H \cap G_0) \) in Eq. (35), we have Eqs. (54) and (55) from Eqs. (53) and (54), respectively. Then, it is sufficient to show Eqs. (56) and (57) to prove Corollary 1. From Eq. (54), the texture of a vortex with topological charge \( ([H_t], e) \) is given by \( O([H_t], e)(\phi) = a(\phi)O_0 \), where \( a(\phi) \) is a loop on \( G \). From Eqs. (40) and (17), we have \( a(\phi) = \exp(i\phi H_t) \) and hence Eq. (56). For a vortex with topological charge \( (e, [\sigma]) \), Eq. (57) can be expressed as \( O(e, [\sigma])(\phi) = b_\sigma O_0 \), where \( b_\sigma \) is a path from \( b_{\sigma=0} = 0 \) to \( b_{\sigma=\pi} = e \). Defining a path \( \gamma_\sigma(\phi) := b_{\sigma=\phi/2} \), we obtain Eq. (57), which completes the proof of Corollary 1.

Equation (54) implies that there are two types of vortices expressed by either Coker \( i_1^* \) or \( \pi_0(H \cap G_0) \), and it follows from Eq. (50) that any vortex can be written as their composition. Examples of the former include an integer-quantum vortex \( [H_t] \), which is obtained from Eq. (57) through substitution of \( \exp(i\phi H_t) \) and \( O([H_t], e)(\phi) \) with \( \exp(i\phi) \) and \( \Psi(\phi) \), respectively. The latter term \( \pi_0(H \cap G_0) \) describes a vortex associated with a discrete symmetry of the state such as a half vortex \( [H_t] \) in a uniaxial nematic liquid crystal, where the discrete symmetry is the \( \pi \)-rotational symmetry of the orientation \( d \). One can reproduce from Eq. (54) the formula \( \pi_1(G/H) \simeq \pi_0(H) \) based on the lift method \( [38] \), where \( G \) and \( H \) are lifted to a simply connected group \( \tilde{G} \) and the corresponding subgroup \( \tilde{H} \), respectively. Since \( \pi_0(G) \simeq 0 \) and \( \pi_1(G) \simeq 0 \), we have Coker \( i_1^* \simeq 0 \) and \( \pi_0(H \cap G_0) \simeq \pi_0(H) \). We thus obtain Eq. (54). In contrast to the lift method, we find the distinction between vortices represented by Coker \( i_1^* \) and \( \pi_0(H \cap G_0) \). In Sec. IV A, this distinction is shown to be crucial since only the latter can be a cause of nontrivial topological influence.
B. Second homotopy group: monopoles and skyrmions

We generalize the texture of a monopole in a ferromagnet through the replacement of \( S \) by a generalized \( su(2) \)-spin vector. Provided that \( S_{\alpha,3} \) is an unbroken generator, we define the mapping \( O^{\alpha^c} : S^2 \to G/H \) for a co-root \( \alpha^c \) by

\[
O^{\alpha^c}(\theta, \phi) := g^{(2)}_{\alpha}(\theta, \phi)O_0, \quad (58)
\]

\[
g^{(2)}_{\alpha}(\theta, \phi) := \exp(i\theta S_{\alpha,3}) \exp(i\phi S_{\alpha,2}). \quad (59)
\]

Comparing Eqs. (58) and (59) with Eq. (39), we find that \( O^{\alpha^c}(\theta, \phi) \) is invariant under spin rotations generated by the generalized \( su(2) \)-spin vector \( S_{\alpha} \). More precisely, \( O^{\alpha^c}(\theta, \phi) \) is expressed in terms of the co-roots corresponding to the \( su(2) \)-spin vectors parallel to \( \hat{r}(\theta, \phi) \):

\[
\exp[i\psi S_{\alpha} \cdot \hat{r}(\theta, \phi)] O^{\alpha^c}(\theta, \phi) = O^{\alpha^c}(\theta, \phi) \quad \forall \psi \in \mathbb{R}. \quad (60)
\]

This follows from the assumption that \( O_0 \) is invariant under unitary transformations generated by \( S_{\alpha,3} \) and from the decomposition \( \exp[i\psi S_{\alpha} \cdot \hat{r}(\theta, \phi)] = g^{(2)}_{\alpha}(\theta, \phi)e^{i\psi S_{\alpha,3}} [g^{(2)}_{\alpha}(\theta, \phi)]^\dagger \), which is derived directly from the commutation relations.

The following Corollary 2 shows that the topological charge and the texture of a general monopole are described by co-roots and the hedgehog configuration of the generalized \( su(2) \)-spin vector, respectively. Reflecting the fact that a general compact Lie algebra includes more than one \( su(2) \)-Lie algebra in contrast to \( su(2) \), the topological charge should be described by a set of co-roots. The connection with the co-roots and the generalized \( su(2) \)-spin vectors are pointed out in Refs. [52, 53], where non-Abelian gauge theories are considered and \( H \) is assumed to include a maximal Abelian subgroup of \( G \) [11, 12]. We here generalize their results to arbitrary systems with arbitrary patterns of symmetry breaking.

Corollary 2 Let \( L_H \) and \( L^c_H \) (\( L^c_G \)) be the integral lattice of \( H \) and the co-root lattice of \( H (G) \), respectively. Then, \( L^c_H \) is an Abelian subgroup of \( L_H \cap L^c_G \), and the quotient space of \( L_H \cap L^c_H \) by \( L^c_H \) is isomorphic to \( \pi_2(G/H) \):

\[
\pi_2(G/H) \simeq (L_H \cap L^c_G)/L^c_H. \quad (61)
\]

Therefore, the topological charge \( n \) of a monopole can be expressed in terms of the co-roots corresponding to the simple roots as

\[
n = \sum_{j=1}^{r} m_j \alpha_j^c, \quad (62)
\]

where \( \{m_j\}_{j=1}^{r} \) is the set of integers, and its texture \( O(\theta, \phi) \) is given by

\[
O(\theta, \phi) = g^{(2)}_{\alpha_1}(\theta, m_1\phi)g^{(2)}_{\alpha_2}(\theta, m_2\phi) \cdots g^{(2)}_{\alpha_r}(\theta, m_r\phi)O_0. \quad (63)
\]

The proof of Corollary 2 is given in Appendix C. One can reproduce from Eq. (61) the formula \( \pi_2(G/H) \simeq \pi_1(H) \) based on the lift method [38]. Indeed, we have \( \pi_1(G) \simeq 0 \) and hence \( L^c_C \simeq L^c_G \). Then, we obtain \( \pi_2(G/H) \simeq \pi_1(H) \). However, the texture of each topological excitation is described by a deformable loop on \( G \); in Ref. [38] the existence of the texture is shown but no explicit form is given. We here explicitly determine the texture as shown in Eq. (63).

C. Third homotopy group: three-dimensional skyrmions

Two prototypical examples of three-dimensional skyrmions are a Shankar skyrmion and a knot soliton, which are characterized by the homotopy groups \( \pi_3(S^3) \simeq \mathbb{Z} \) and \( \pi_3(S^2) \simeq \mathbb{Z} \), respectively. Both of their textures are expressed in terms of the \( su(2) \)-spin vector \( S \) as

\[
O(\psi, \theta, \phi) = \exp[2i\psi S \cdot \hat{r}(\theta, m\phi)]O_0, \quad (64)
\]

where \( (\psi, \theta, \phi) \) is the polar coordinates on \( S^3 \) and \( m \in \mathbb{Z} \) denotes the topological charge of the three-dimensional skyrmion. This unified description is based on the isomorphism \( \pi_3(S^3) \simeq \pi_3(S^2) \) derived from the Hopf fibration [11, 50, 54]. When all of the generators in \( S \) are broken, Eq. (64) describes a Shankar skyrmion [2, 55]; otherwise it describes a knot soliton [33, 56]. We generalize the texture through the replacement of \( S \) by \( S_{\alpha} \) and define the mapping \( O^{\alpha^c} : S^3 \to G/H \) for co-root \( \alpha^c \) by

\[
O^{\alpha^c}(\psi, \theta, \phi) := g^{(3)}_{\alpha}(\psi, \theta, m\phi)O_0, \quad (65)
\]

\[
g^{(3)}_{\alpha}(\psi, \theta, \phi) := \exp[2i\psi S_{\alpha} \cdot \hat{r}(\theta, \phi)], \quad (66)
\]

where \( g^{(3)}_{\alpha} \) is defined in Eq. (23).

The following two corollaries show that a general three-dimensional skyrmion may be regarded as the composition of several different types of three-dimensional skyrmions whose topological charges and textures are described by co-roots and the corresponding textures [65], respectively.

Corollary 3 The third homotopy group \( \pi_3(G/H) \) is given as follows:

\[
\pi_3(G/H) \simeq \text{Coker}\{i^*_3 : \pi_3(H) \to \pi_3(G)\}. \quad (67)
\]

Proof Since any subgroup \( H \) of a compact Lie group \( G \) is compact, \( \pi_3(H) \) vanishes. Therefore, we obtain \( \text{Ker} \; i^*_3 \simeq 0 \) and hence Eq. (67) from Theorem 1 which completes the proof of Corollary 3.

We next analyze a topological charge and a texture. Let \( \alpha^c_i \) be a co-root of the Lie algebra \( g_i \) defined in
Table 1: Two types of topological excitations and their examples. Ker $i_{m}$ and Coker $i_{m}$ are the cokernel of $i_{m}$ and the kernel of $i_{m}$ defined in Eqs. (23) and (30), respectively. The entry "absent" means the absence of examples.

| $m$ | Coker $i_{m}$ | Ker $i_{m}$ |
|-----|---------------|-------------|
| 1   | integer-quantum vortex | half vortex |
| 2   | absent | monopole |
| 3   | knot soliton | absent |
|     | Shanker skyrmion | |

Eq. (22). Since the numerator $\pi_{3}(G)$ of Eq. (67) is generated by $\left\{ b^{(3)}_{\alpha_{k}} \right\}_{i=1}^{a}$ from Lemma 1, the quotient space $\pi_{3}(G/H)$ is generated by $\left\{ [O^{a}_{\alpha_{k}}]_{G/H} \right\}_{i=1}^{a}$ for a suitable choice of the subset $\left\{ \alpha_{k} \right\}_{k=1}^{a}$ of $\left\{ \alpha_{i} \right\}_{i=1}^{l}$. Thus, we obtain the following corollary.

Corollary 4 The topological charge $n$ of a three-dimensional skyrmion can be written in terms of co-roots as

$$n = \sum_{k=1}^{a} m_{k} [\alpha_{k}]^{3},$$

where $\left\{ m_{k} \right\}_{k=1}^{a}$ is a set of integers and $[\alpha_{c}]^{3}$ represents the topological charge of the texture $O^{\alpha_{c}}$ defined in Eq. (66).

The results of this section are summarized in Table I.

IV. GENERAL CONDITIONS FOR THE PRESENCE OF TOPOLOGICAL INFLUENCE

A. Topological influence on a general topological excitation

When a topological excitation with topological charge $n \in \pi_{m}(G/H)$ makes a complete circuit of a vortex with topological charge $l \in \pi_{1}(G/H)$, the resulting topological charge $\lambda_{m}^{l}(n)$ is given by the action of $l$ on $n$, where the corresponding texture $O^{\lambda_{m}^{l}(n)}(x)$ is defined as follows

$$O^{\lambda_{m}^{l}(n)}(x) := \begin{cases} O^{n}(2x) & \text{for } 0 \leq \|x\| \leq \frac{\pi}{2}; \\ O^{l}(4\|x\| - 2\pi) & \text{for } \frac{\pi}{2} \leq \|x\| \leq \pi, \end{cases}$$

where $x \in D^{m}$ and $O^{n} : D^{m} \to G/H$ is the texture of a topological excitation (vortex) with topological charge $n$ ($l$). We can express the topological charges $n$ and $l$ as $n = ([a], \sigma^{-1}b\sigma)$ and $l = ([H_{l}], \sigma)$ from Theorem 2 and Corollary 1 respectively. Then, the following theorem holds.

Theorem 2 The topological charge $\lambda_{m}^{l}(n)$ is given by

$$\lambda_{m}^{l}(n) = ([a], \sigma^{-1}b\sigma),$$

where the homotopy class $\sigma^{-1}b\sigma$ is defined as $[\sigma^{-1}b\sigma](x) := \sigma^{-1}b(x)\sigma_{0}$ for $x \in D^{m}$.

The proof of Theorem 2 is given in Appendix 1. The result of Theorem 2 is summarized in Table II. As shown in Table II, only vortices characterized by discrete symmetries can have nontrivial topological influence. To understand this, let us consider a situation in which a topological excitation with texture $O(x)$ makes a complete circuit of a vortex with texture $O^{(c, [\sigma])}(\phi) = \gamma_{\sigma}(\phi)O_{b}$, where $\gamma_{\sigma}(\phi)\sigma^{-1}$ describes a path from $c$ to $\sigma^{-1}$. When the former goes around the latter by angle $\phi$, it undergoes a nontrivial texture produced by the latter, changing its texture from $O(x)$ to $\gamma_{\sigma}(\phi)\sigma^{-1}O(x)$. The final texture is given by $\sigma^{-1}O(x)$. A crucial observation here is that the final texture $\sigma^{-1}O(x)$, in general, does not coincide with the initial one $O(x)$. On the other hand, when the topological excitation goes around a vortex characterized by Coker $i_{m}^{1}$ by angle $\phi$, its texture changes from $O(x)$ to $\exp(i\sigma H_{l})O(x)$. Therefore, the initial and final textures coincide because we have $\exp(2\pi iH_{l}) = e$ from Eq. (10).

B. Topological influence on low-dimensional topological excitations

1. Topological influence on a vortex

The necessary and sufficient condition for the presence of topological influence on a vortex is the non-Abelianness of the first homotopy group $\pi_{1}$. It is known that non-Abelian vortices behave differently from Abelian ones in the collision dynamics, quantum turbulence, and the coarsening dynamics due to the tangling between vortices. However, the conditions for their appearances are yet to be understood from a unified point of view. The following corollary shows that their presence is solely determined by discrete symmetries, where the non-Abelian property is shown to...
emerge only between pairs of vortices characterized by \( \pi_0(H \cap G_0) \).

**Corollary 5** The first homotopy group \( \pi_1(G/H) \) is Abelian if and only if \( \pi_0(H \cap G_0) \) is Abelian and \( f \) defined in Eq. (40) satisfies

\[
f([\sigma], [\tau]) = f([\tau], [\sigma]) \quad \text{for } \forall [\sigma], [\tau] \in \pi_0(H \cap G_0).
\]

**Proof** Comparing the following two equations

\[
([a], [\tau]) \times_f ([b], [\tau]) = ([a] + [b] + f([\sigma], [\tau]) [\sigma][\tau]),
\]

\[
([a], [\tau]) \times_f ([a], [\sigma]) = ([a] + [b] + f([\tau], [\sigma]), [\tau][\sigma]),
\]

we find that \( \pi_1(G/H) \) is Abelian if and only if

\[
\begin{align*}
[\sigma][\tau] &= [\tau][\sigma]; \\
f([\sigma], [\tau]) &= f([\tau], [\sigma])
\end{align*}
\]

for \( \forall [\sigma], [\tau] \in \pi_0(H \cap G_0) \).

The first equation in Eq. (73) implies that \( \pi_0(H \cap G_0) \) is Abelian and we have Eq. (74) from the second equation of Eq. (73), which completes the proof of Corollary 4.

2. **Topological influence on a monopole, a skyrmion, and a three-dimensional skyrmion**

Since one topological charge changes into another due to topological influence, \( \lambda^c_2 \) is an automorphism on \( \pi_2(G/H) \), i.e., a one-to-one map from \( \pi_2(G/H) \) to itself satisfying the homomorphic relation \( \lambda^c_2(nn') = \lambda^c_2(n)\lambda^c_2(n') \). Therefore, topological influence is characterized by the action of the automorphism group \( G_2 \) on \( \pi_2(G/H) \) defined by

\[
G_2 := \{ \lambda^c_2(l) | l \in \pi_1(G/H) \}.
\]

From Corollary 2 \( \pi_2(G/H) \) is described by a co-root lattice \( L^c_G \). Let us define the Weyl reflection \( w_\alpha : L^c_G \to L^c_G \) for \( \alpha \in R_+ \) by

\[
w_\alpha(H_t) := H_{t'} := t - \frac{2(\alpha, t)}{(\alpha, \alpha)} \alpha,
\]

where \( w_\alpha \) describes the reflection across the plane perpendicular to \( \alpha \). It is known that \( w_\alpha \) is an automorphism of \( L^c_G \) \([41, 42]\). The Weyl group \( W_G \) of \( G \) is defined as the automorphism group of \( L^c_G \) generated by the Weyl reflections:

\[
W_G := \text{Gen} \{ w_\alpha | \alpha \in R_+ \},
\]

where Gen \( S \) for a set \( S \) is defined as the group generated by the elements of \( S \). It is instructive to consider an example of \( g = su(2) \). Since \( g \) is constituted from only one \( su(2) \)-subalgebra, its co-root lattice is a one-dimensional lattice \( L^c_G = \{ mH_{\alpha} | m \in \mathbb{Z} \} \). The Weyl reflection acts on \( L^c_G \) as its inversion: \( w_\alpha(mH_{\alpha}) = -mH_{\alpha} \); thus, \( W_G \) is Abelian.

More generally, we consider topological influences on a monopole, a skyrmion, and a three-dimensional skyrmion. It is known that a larger group, in general, has a larger Weyl group, it is natural to ask whether other forms appear when we consider a group larger than \( G = SU(2) \) or \( SO(3) \). We answer this affirmatively in Sec. 5 by showing an example of \( G_2 \).

Finally, there is no topological influence for the case of a three-dimensional skyrmion as stated in the following corollary.

**Corollary 6** For each discrete symmetry \( [\sigma] \in \pi_0(H \cap G_0) \), there exists a Weyl reflection \( w_\sigma \in W_G \) that satisfies

\[
\lambda^{c([\sigma])}(H_t + L^c_G) = w_{\sigma}(H_t) + L^c_G,
\]

and the automorphism group \( G_2 \) is a subgroup of \( W_G \).

The proof of Corollary 6 is given in Appendix B. For all the examples studied so far, \( g = u(1), su(2), so(3) \), or their direct sum \([34, 36, 37, 39, 40]\). Therefore, it follows from Corollary 6 that \( G_2 \) is either trivial or a direct sum of \( \mathbb{Z}_2 \), where a possible form of nontrivial topological influence is essentially the sign change of a topological charge. Since a larger group, in general, has a larger Weyl group, it is natural to ask whether other forms appear when we consider a group larger than \( G = SU(2) \) or \( SO(3) \). We answer this affirmatively in Sec. 5 by showing an example of \( G_2 \).

**Corollary 7** The topological influence on a three-dimensional skyrmion is trivial.

**Proof** From Corollary 6 we have \( \pi_3(G/H) \simeq \text{Coker} \ i^*_G \). Then, Corollary 7 follows directly from Theorem 2.

V. **Non-Abelian Topological Influence on a Skyrmion in an SU(3)-Heisenberg Model**

Since topological influence on a monopole and that on a skyrmion are the same in that they are characterized by the action \([70]\) of \( \pi_1(G/H) \) on \( \pi_2(G/H) \), we consider topological influence on a skyrmion in the two-dimensional space. We include in \( G \) and \( H \) the space symmetry and the lattice symmetry, respectively, because dislocations and disclinations, which result from the breaking of the space symmetry, play a vital role in the topological influence analyzed below.
A. Vortices and skyrmions in a 3-CDW state

1. SU(3)-Heisenberg model and its ground state

The Hamiltonian of the SU(3)-Heisenberg model on a triangular lattice \( L \) is given by

\[
H = J \sum_{\langle i,j \rangle} \sum_{a=1}^{8} T_{a,i} T_{a,j},
\]

where \( \langle i,j \rangle \) denotes a pair of nearest-neighbor sites \( i \) and \( j \), and \( \{ T_{a,i} \}_{a=1}^{8} \) is a set of the generators of \( su(3) \) at site \( i \). On each site, there are three degenerate states, which we refer to as red, green, and blue, and write them as

\[
|R\rangle = (1,0,0)^T, \quad |G\rangle = (0,1,0)^T, \quad |B\rangle = (0,0,1)^T.
\]

We call these internal degrees of freedom as color. This model is expected to be realized in an ultracold atomic gas of alkaline-earth atoms in an optical lattice \([63-68]\) and can be regarded as a spin-1 bilinear-biquadratic model with equal bilinear and biquadratic couplings \([8,69]\). For the case of an antiferromagnetic interaction \((J < 0)\), the ground state \( |\Psi\rangle_{\text{GS}} \) is described by the three-sublattice ordering with a periodic alignment of three colors \([7,8,43]\), and known as the 3-color density-wave state (3-CDW state) \([43,44]\) [see Fig. 7].

\[
|\Psi\rangle_{\text{GS}} = \bigotimes_{i \in L_R} |R\rangle_i \otimes \bigotimes_{i \in L_G} |G\rangle_i \otimes \bigotimes_{i \in L_B} |B\rangle_i,
\]

where \( L_R, \ L_G, \) and \( L_B \) denote the three sublattices. For the triangular lattice, they are given by

\[
L_R := \{(m_1 - m_2) \mathbf{a}_1 + (m_1 + 2m_2) \mathbf{a}_2 | m_1, m_2 \in \mathbb{Z}\},
\]

\[
L_G := \{ x + \mathbf{a}_2 | x \in L_R \},
\]

\[
L_B := \{ x + \mathbf{a}_1 | x \in L_R \},
\]

where \( \mathbf{a}_1 = (1,0)^T \) and \( \mathbf{a}_2 = (1/2, \sqrt{3}/2)^T \) are the primitive vectors of the triangular lattice in units of the lattice constant \( a = 1 \). The 3-CDW state appears as the ground state of the SU(3)-Heisenberg model on various lattices including triangular, square, and cubic lattices \([7,8,43,71,72]\).

2. Symmetries of the system and the state

When we include the space symmetry, the symmetry of the system is given by

\[
G = \text{SU}(3) \times \text{E}(2),
\]

where \( \text{E}(2) := \mathbb{R}^2 \times \text{SO}(2) \) is the two-dimensional Euclidean group generated by the two-dimensional translation group \( \mathbb{R}^2 \) and the two-dimensional rotational group SO(2), where the semidirect product on \( H \rtimes N \) is defined by \((h,n) \rtimes (h',n') := (hnh^{-1},nn')\). The ground state \( |\Psi\rangle_{\text{GS}} \) has the continuous symmetry \( H_0 \) generated by diagonal matrices:

\[
H_0 = \left\{ \exp \left( i \sum_{a = \text{RG,GB,Br}} c_a H_a \phi \right) \right\} c_a \in \mathbb{R}.
\]

Also, \( |\Psi\rangle_{\text{GS}} \) has the discrete symmetries that exchange the three colors R, G, and B and three sublattices \( L_R, L_G, \) and \( L_B \) simultaneously. The permutations of the colors are described by the symmetry group \( S_3 \):

\[
S_3 \simeq \text{Gen} \{ \sigma_{\text{RG}}, \sigma_{\text{GB}}, \sigma_{\text{Br}} \} \equiv \{ I_3, \sigma_{\text{RG}}, \sigma_{\text{GB}}, \sigma_{\text{Br}}, \sigma_{\text{RG}} \sigma_{\text{GB}}, \sigma_{\text{GB}} \sigma_{\text{Br}} \},
\]

where \( I_3 \) is the identity matrix with size three and the generators \( \sigma_{\text{RG}} := e^{i\pi S_{\text{RG}}}, \sigma_{\text{GB}} := e^{i\pi S_{\text{GB}}}, \) and \( \sigma_{\text{Br}} := e^{i\pi S_{\text{Br}}} \) are given by

\[
\sigma_{\text{RG}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_{\text{GB}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
\sigma_{\text{Br}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

The permutations of the sublattices are described by the symmetry \( H_{\text{lat}} \) of the lattice:

\[
H_{\text{lat}} = \{ (m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2, R(n \pi /3)) | m_1, m_2 \in \mathbb{Z}, n = 0, 1, \ldots, 5 \},
\]

where \( m_1 \mathbf{a}_1 + m_2 \mathbf{a}_2 \) and \( R(\phi) \) describe translation and rotation, respectively. The discrete symmetry is isomorphic to the lattice symmetry: \( \pi_0 (H \cap G_0) \simeq H_{\text{lat}} \). Since
$h \ (\in H_{\text{lat}})$ induces a color exchange, we define $\sigma_h$ as the corresponding matrix in $S_3$. From the above discussion, $H$ is generated by the continuous symmetry $H_0$ and the discrete symmetry $H_{\text{lat}}$:

$$H \simeq H_0 \ltimes H_{\text{lat}},$$

(89)

where $H_0 \ltimes H_{\text{lat}}$ is the semidirect product defined by $(h_0, h) \ltimes (h'_0, h') := (h_0 \sigma_h h'_0 (\sigma^{-1}_h), hh').$

3. Vortices in a 3-CDW state

Vortices and skyrmions in the 3-CDW state are determined in Ref. [23], where $E(2)$ and its symmetry breaking are not considered. Here we show that the vortices characterized by $S_3$ indeed emerge. From Eq. (54), we have $\pi_1(G) \simeq \mathbb{Z}$, and $\pi_0(H \cap G_0) \simeq H_{\text{lat}}$. Therefore, we obtain $\text{Im} \ i_1^{\phi} \simeq 0$ and $\text{Coker} \ i_1^{\phi} \simeq \mathbb{Z}$ and hence

$$\pi_1(G/H) \simeq \mathbb{Z} \times f H_{\text{lat}}.$$  

(90)

For a topological charge $(m, h)$, we refer to $\sigma_h$ defined above as a spin topological charge. For the analysis of topological influence, only a spin topological charge is necessary for the two reasons. First, vortices described by $Z$ cannot have nontrivial topological influence from Theorem 2. Second, since skyrmions are shown to be characterized by $S_3$, the color exchange is solely determined by the color exchange $\sigma_h$. When we focus on spin topological charges, vortices are characterized by $S_3$:

$$\{\sigma_h \in S_3 | (m, h) \in \pi_1(G/H)\} \simeq S_3.$$  

(91)

We refer to vortices with spin topological charges $\sigma_{RG}$, $\sigma_{GB}$, and $\sigma_{BR}$ as RG-, GB-, and BR-vortices, respectively, according to their exchanges of the colors and sublattices. An example of an RG-vortex is the disclination with the Frank angle $\pi/3$ around a site belonging to the sublattice $L_B$. Here, the Frank angle describes the angle over which the lattice sites are missing [see Fig. 8 (a) and (d)]. Let $\phi$ be the azimuth angle around the vortex. Due to the exchange of the sublattices, there is an ambiguity in the correspondence between a site and the sublattice it belongs to. We therefore fix the range of $\phi$ to $[0, 2\pi]$ by assigning a sublattice to each site. We take the reference order parameter as the expectation value of $S_{RG}$ with respect to $|\Psi\rangle_{GS}$ in Eq. (90):

$$\langle S_{RG} \rangle_R(\phi) = \langle e^{-i\frac{\phi}{2}S_{RG,1}}S_{RG}e^{i\frac{\phi}{2}S_{RG,1}} \rangle_{R,0}$$

$$= \left[0, \sin \left(\frac{\phi}{2}\right), \cos \left(\frac{\phi}{2}\right)\right],$$

(93)

$$\langle S_{RG} \rangle_G(\phi) = \langle e^{-i\frac{\phi}{2}S_{RG,1}}S_{RG}e^{i\frac{\phi}{2}S_{RG,1}} \rangle_{G,0}$$

$$= \left[0, \sin \left(\frac{\phi}{2}\right), \cos \left(\frac{\phi}{2}\right)\right],$$

(94)

$$\langle S_{RG} \rangle_B(\phi) = \langle e^{-i\frac{\phi}{2}S_{RG,1}}S_{RG}e^{i\frac{\phi}{2}S_{RG,1}} \rangle_{B,0}$$

$$= (0, 0, 0).$$

(95)

One can see from Eqs. (93) and (94) that $S_{RG}$ rotates by angle $\pi$ around the vortex. We note that Eqs. (93) and (94) indeed give a continuous map to the OPM because the sublattices $L_R$ and $L_G$ are exchanged at $\phi = 0$ and $2\pi$.

4. Skyrmions in a 3-CDW state

Since $H$ does not include $su(2)$-subalgebras from Eq. (54), $L_H^G$ vanishes. Hence, from Corollary 2 the second homotopy group is isomorphic to the triangular lattice, which, in turn, is isomorphic to the co-root lattice.
of SU(3):

\[ \pi_2(G/H) \simeq (L_R \cap L_G)/L_R \simeq L_H \cap L_G \]

\[ \simeq \left\{ \sum_{a=RG,GB,GR} m_a \alpha_a c \middle| m_a \in \mathbb{Z}, \sum_{a=RG,GB,GR} \alpha_a c = 0 \right\} \]

\[ \simeq L^c_{SU(3)}. \] (96)

Reflecting the triangular geometry of \( L^c_{SU(3)} \) [see Fig. 4], the 3-CDW state has three types of skyrmions [see Fig. 10 (c)]. Let \((r, \phi)\) be the polar coordinates in \( \mathbb{R}^2 \), and \( \theta(r) \) be a real function that satisfies \( \theta(0) = 0 \) and \( \theta(\infty) = \pi \). From Corollary 2, the texture of a skyrmion with topological charge \( \alpha_{RG} \) is obtained by operating \( g_{RG}^2(\theta(r), \phi) \) on the reference order parameter in Eq. (92):

\[
\langle S_{RG} \rangle_R (\theta, \phi) = \left( g_{RG}^2(\theta(r), \phi) \right)^{(*)}_{R,0} = \bar{r}(\theta(r), \phi), \]

\[
\langle S_{RG} \rangle_G (\theta, \phi) = \left( g_{RG}^2(\theta(r), \phi) \right)^{(*)}_{G,0} = -\bar{r}(\theta(r), \phi), \]

\[
\langle S_{RG} \rangle_B (\theta, \phi) = \left( g_{RG}^2(\theta(r), \phi) \right)^{(*)}_{B,0} = 0. \] (99)

Thus, this skyrmion is described by a hedgehog configuration of \( S_{RG} \) with winding number +1 (−1) on \( L_R \) (\( L_G \)) as shown in Fig. 9. We refer to this skyrmion as an RG-skyrmion. Similarly, the texture of a skyrmion with topological charge \( \alpha_{GB} \) is described by a hedgehog configuration of \( S_{GB} \) with winding number +1 (−1) on \( L_G \) (\( L_B \)) and we refer to it as a GB-skyrmion [see Fig. 10 (b)]. Also, there exists a skyrmion with topological charge \( \alpha_{BR} \) described by a hedgehog configuration of \( S_{BR} \) with winding number +1 (−1) on the sublattice \( L_B \) (\( L_R \)), and we refer to it as a BR-skyrmion [see Fig. 10 (c)]. It follows from the relation \( \alpha_{RG}^0 + \alpha_{GB}^2 + \alpha_{BR}^0 = 0 \) [see Fig. 4] that these three skyrmions are not independent; the composition of all of them results in a trivial texture.

### B. Topological influence in a 3-CDW state

From Eq. (96) and Fig. 4, \( \pi_2(G/H) \) is isomorphic to the triangular lattice and \( \pi_1(G/H) \) is isomorphic to \( S_3 \) as far as the spin topological charge is considered. We will see below that \( G_2 \) defined in Eq. (74) is isomorphic to \( S_3 \), where three skyrmions with \( \alpha_{RG}, \alpha_{GB}, \) and \( \alpha_{BR} \), together with their anti-skyrmions with \( -\alpha_{RG}, -\alpha_{GB}, \) and \( -\alpha_{BR} \) are exchanged through topological influence, reflecting the \( S_3 \)-symmetry of the triangular lattice. From Theorem 2, the topological influence of a vortex with spin topological charge \( \sigma \) on a skyrmion with topological charge \( n \) is described by the conjugation by \( \sigma \):

\[ \lambda_\sigma^n(n) := \sigma^{-1}n\sigma. \] (100)
For example, for $\sigma = \sigma_{RG}$ and $n = \alpha_{RG}, \alpha_{GB}, \text{and} \alpha_{BR}$, Direct calculations of the matrices in Eqs. (11) and (17) give
\[
\begin{align*}
\lambda_2^\sigma(\alpha_{RG}) &= -\alpha_{RG}, \quad \lambda_2^\sigma(\alpha_{GB}) = -\alpha_{BR}, \\
\lambda_2^\sigma(\alpha_{BR}) &= -\alpha_{GB}.
\end{align*}
\] (101)

A crucial observation here is that this vortex acts on the triangular lattice, inverting it about the line perpendicular to $\alpha_{RG}$ [see Fig. 11 (a) and (d)]. Similarly, a vortex with topological charge $\sigma_{GB}$ ($\sigma_{BR}$) acts on the triangular lattice inverting it about the line perpendicular to $\alpha_{RG}$ ($\alpha_{GB}$) as shown in Figs. 11 (b) and (c) ((c) and (f)). Since $S_3$ is generated by $\sigma_{RG}$, $\sigma_{GB}$, and $\sigma_{BR}$, we have $G_2 \simeq S_3$.

The non-Abelian property of $G_2$ emerges when we consider topological influence of two vortices. Let $\sigma$ and $\tau$ be the topological charges of the vortices and $n$ be that of a skyrmion. Suppose that the skyrmion goes around the vortex with $\sigma$ clockwise, goes around the vortex with $\tau$ clockwise, goes around the vortex with $\sigma$ anticlockwise, and finally goes around the vortex with $\tau$ anticlockwise [see Fig. 12]. Since the third (fourth) process is the inverse process of the first (second) one, the change in the topological charge is given by
\[
\lambda_2^{-1} \left( \lambda_2^{\sigma^{-1}} \{ \lambda_2^{\tau} \{ \lambda_2^\sigma(n) \} \} \right) = \lambda_2^\sigma(n) = \rho = \tau^{-1} \sigma^{-1} \tau \sigma.
\] (102)

While the final topological charge coincides with the initial one for an Abelian $G_2$ because we have $\rho = e$ for any pair of vortices, it does not for a non-Abelian $G_2$ because $\rho \neq e$ in general. For the case of $\sigma = \sigma_{RG}, \tau = \sigma_{GB}$, and $n = \alpha_{RG}$, we have $\rho = \sigma_{BR} \sigma_{GB}$ and $\lambda_2^\sigma(\alpha_{RG}) = \alpha_{GB} \neq \alpha_{RG}$.

VI. CONCLUSION AND DISCUSSION

In the present paper, we have developed a general method to determine the homotopy group $\pi_m(G/H)$ of the order parameter manifold $G/H$ by deriving the formula (100) which expresses $\pi_m(G/H)$ in terms of $\pi_m(G)$ and $\pi_m(H)$. Since the homotopy group of a Lie group and each texture on it can be calculated systematically by means of the Cartan canonical forms (3) and the lattices defined in Eqs. (11) and (17), the obtained formulas allow us to calculate $\pi_m(G/H)$ and the texture $O$ of each topological excitation systematically. We find that the textures of a monopole and that of a three-dimensional skyrmion are obtained by the replacement of the $\text{su}(2)$-spin vector $S$ by the generalized $\text{su}(2)$-spin vector $S_{\alpha}$ defined in Eq. (17), and that their topological charges are described by a set of co-roots, reflecting the fact that a Lie algebra $g$, in general, includes multiple $\text{su}(2)$-subalgebras. We have also shown the necessity of a discrete symmetry $\pi_0(H \cap G_0)$ for the presence of nontrivial topological influence. Moreover, we derive the necessary and sufficient condition for the presence of non-Abelian vortices and prove the absence of topological influence on a three-dimensional skyrmion. As for topological influence on a monopole or a skyrmion, we prove that the automorphism group $G_2$ of topological influence is a subgroup of the Weyl group $W_G$, clarifying why only one type of topological influence is known so far. Seeking for other types, we find that topological influence characterized by a non-Abelian group $S_3$ emerges in the 3-color density-wave state of the SU(3)-Heisenberg model, where three types of skyrmions and vortices characterized by $S_3$ appear. These skyrmions change their types through the topological influence, giving $G_2 = S_3$.

Finally, we raise three problems for future study. First, the dynamical stability of the textures of topological excitations derived in Sec. III needs to be clarified. These textures and their variations have widely been used as...
candidates for dynamically stable textures of topological excitations[1, 50, 54, 74, 76]. In fact, the dynamically stability has been demonstrated in a number of examples[54, 73, 77, 81]. The texture $O(\theta, \phi) := g^{(2)}_{\alpha}(\theta, \phi)O_0$ and its variation $O(r, \phi) := g^{(2)}_{\alpha}(\theta(r), \phi)O_0$ are widely used as candidates for the textures of a monopole and that of a skyrmion, respectively[1, 50, 54, 74, 76], where $\theta(r)$ is a function subject to the boundary conditions $\theta(0) = 0$ and $\theta(\infty) = \pi$. Moreover, they indeed give stable textures[54, 73, 77, 81] for an appropriate choice of $\theta(r)$. It merits further study to clarify their dynamical stability. Second, we represent in Corollary 5 the necessary and sufficient condition for non-Abelian vortices in terms of the map $f$ defined in Eq. (46). However, its physical implication is yet to be clarified. Considering the growing interest in the dynamics of non-Abelian vortices[54, 74, 84], it is of interest to understand whether we can simplify the condition (73). Third, analogous concepts of topological influence in topological insulators and superconductors have recently been discussed in specific examples[83–85], where the domain $\pi_3(G/H)$ of $\pi_3(G)$ is of interest to understand whether we can simplify the condition (73). Third, analogous concepts of topological influence in topological insulators and superconductors and clarify its difference from topological influence in topological excitations by using the general formulas developed in the present paper.

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Appendix A: Proof of Lemma 1

Lemma 1 follows from the following theorem on the third homotopy group of a simple compact Lie group[49].

Theorem 3 Let $G$ and $S_\alpha$ be a simple compact Lie group and the generalized $\text{su}(2)$-spin vector for the co-root $\alpha^c$ with shortest length in $G$, respectively. Then, we have

$$\pi_3(G) \simeq \left\{ m \left[ g^{(3)}_{\alpha} \right]_G \mid m \in \mathbb{Z} \right\},$$

where $\left[ g^{(3)}_{\alpha} \right]_G$ denotes the homotopy class of $G$ with representative element $g^{(3)}_{\alpha}$ defined in Eq. (25). The isomorphism in Eq. (A1) is given by $i_{3, H} : \pi_3(H) \to \pi_3(G/H)$, where $H' = SU(2)$ or $SO(3)$ is the subgroup of $G$ generated by $S_\alpha$. The right-hand side of Eq. (A1) does not depend on the choice of the co-root since $\left[ g^{(3)}_{\alpha} \right]_G = \left[ g^{(3)}_{\alpha^c} \right]_G$ for two co-roots $\alpha_1$ and $\alpha_2$ with the shortest length.

Proof of Lemma 1

Let Eq. (22) and $\alpha_1^c$ be the decomposition of the Lie algebra $\mathfrak{g}$ of $G$ and a co-root in $\mathfrak{g}_1$, respectively. If we denote the universal covering group of $G'$ by $\tilde{G}$, $G$ is given by $\mathbb{R}^a \times \tilde{G}_1 \times \cdots \times \tilde{G}_a$ and hence we obtain

$$\pi_3(G) \simeq \pi_3\left( \mathbb{R}^a \times \tilde{G}_1 \times \cdots \times \tilde{G}_a \right) \simeq \bigoplus_{i=1}^a \pi_3(\tilde{G}_k)$$

$$\simeq \left\{ \sum_{i=1}^a m_i \left[ g^{(3)}_{\alpha_i^c} \right] \mid m_i \in \mathbb{Z} \right\},$$

where the first isomorphism follows from the relation $\pi_m(G') \simeq \pi_m(G''')$ for $m \geq 2$, the second one from the relation $\pi_m(X \times Y) \simeq \pi_m(X) \oplus \pi_m(Y)$, and the third one from Theorem 1 which completes the proof of Lemma 1

Appendix B: Proof of Theorem 1

Theorem 1 is proved by applying the theory of a group extension with an Abelian kernel[47].

1. Group extension with an Abelian kernel

Definition 1 (Group extension) (a) Let $Q$ and $N$ be two groups. Then $G$ is a group extension of $Q$ by $N$ if $N$ is a normal subgroup of $G$ and $Q$ is a quotient group of $G$ by $N$, i.e., $Q = G/N$. In particular, if $N$ is Abelian, $G$ is referred to as a group extension of $Q$ by an Abelian kernel $N$. (b) Let $G$ and $G'$ be group extensions of $Q$ by $N$. We denote the projection from $G$ to $G'$ as $\pi(Q) = T(T')$. If there exists an isomorphism $F : G \to G'$ such that $T' \circ F = T$, the two group extensions $G$ and $G'$ are regarded as equivalent.

Let $G$, $N$, and $Q$ be a group, a normal subgroup of $G$, and the quotient group of $G$ by $N$, respectively, and consider a situation in which we know $N$ and $Q$ but do not know $G$. The problem of constructing $G$ from $N$ and $Q$ is referred to as an group extension problem. As we will see below, there is a general theory to solve the group extension problem if $N$ is Abelian.
Let $T$ be the projection from $G$ to $Q$. A map $s : Q \to G$ that satisfies $T \circ s = \text{id}_Q$ is referred to as a section of $T$, where $\text{id}_Q$ denotes the identity map on $Q$. We assume that $N$ is Abelian and that a section $s$ of $T$ is given. We define the map $f : Q \times Q \to N$ referred to as the factor set of $G$ associated with $s$ by

$$f(q, q') := s(q)s(q')s(q'q)^{-1}. \quad (B1)$$

Since $T [f(q, q')] = e$ and hence $f(q, q') \in N$, $f$ is indeed a map to $N$. Since $N$ is a normal subgroup of $G$, $N$ is invariant under the inner isomorphism $g' \mapsto gg'g^{-1}$. Moreover, the inner isomorphisms of $N$ that satisfies with $\theta_q(n) := gng^{-1}$, where $g \in G$ satisfies $T(g) = T(q)$. Then, the following theorem holds [17].

**Theorem 4 (a)** Let the product $\times f$ on the product set $N \times Q$ be defined by

$$(n, q) \times f (n', q') := (n + \theta_q(n') + f(q, q'), qq'), \quad (B2)$$

where we write the product on $N$ by the sum because $N$ is Abelian. Then, $N \times Q$ becomes a group, where the identity element is given by $(f(0, e)^{-1}, e)$ and the inverse of $(n, q)$ is given by $(n^{-1} + f(q, q^{-1}), q^{-1})$. This group denoted by $N \times f Q$ is a group extension of $Q$ by $N$ and isomorphic to $G$ under this product. (b) Let $\tilde{s}$ and $\tilde{f}$ be another section of $T$ and the factor set associated with $\tilde{s}$, respectively. If there exists a map $\alpha : Q \to N$ that satisfies $\tilde{s}(q) = \alpha(q)s(q)$, the two group extensions $N \times f Q$ and $N \times \tilde{f} Q$ are equivalent.

### 2. Proof of Theorem 4

We start from the relation derived in Ref. [46]:

$$\frac{\pi_m(G/H)}{\text{Coker } i_{s_m}} \simeq \text{Ker } i_{s_m-1}, \quad (B3)$$

which follows from the homotopy exact sequence [46, 80]. From the homotopy lifting theorem [80], any homotopy class $O$ of $\pi_t G/H$ can be written as

$$O(x) = g(x)O_0 \quad \forall x \in D^m, \quad (B4)$$

where $g$ is a map from $D^m$ to $G$ subject to the boundary condition

$$g(x) = e \quad \text{for } ||x|| = \pi. \quad (B5)$$

Then, the projection map $T : \pi_m(G/H) \to \text{Ker } i_{s_m-1}$ in Eq. [B3] is given by

$$\{T(O)\} \tilde{x} := \lim_{r \to 0} g^O(r \tilde{x}) \quad \text{for } \tilde{x} \in S^{m-1}, \quad (B6)$$

where $T(O)$ is a map from $S^{m-1}$.

We first prove that the inner isomorphism of $\pi_m(G/H)$ on $\text{Coker } i_{s_m}$ is trivial:

$$n[a]n^{-1} = [a] \quad \forall n \in \pi_m(G/H), \forall [a] \in \text{Coker } i_{s_m}. \quad (B7)$$

For $m \geq 2$, Eq. (B7) follows from the commutativity of higher-dimensional homotopy groups. For $m = 1$, from Eq. (B4), we can express the loop $I^m([\tilde{a}])$ corresponding to $n(a) |_{[\tilde{a}]} = \gamma_n(a)g_0$ to $n(a)O_0$, where $\gamma_n$ is a path from $\gamma_n(1) = \sigma_0$ to $\gamma_n(2\pi a) = e$ and $a$ is a loop on $G$. Then, defining $a_s(\tilde{a})$ for $s, \tilde{a} \in [0, 2\pi]$ by

$$a_s(\tilde{a}) := \begin{cases} \gamma_n(2\pi - 3\phi)O_0, & 0 \leq \phi \leq \phi_0 - \frac{s}{3}; \\ a\left(\frac{3(\phi - s)}{3\pi - s}\right), & \phi_0 - \frac{s}{3} \leq \phi \leq 2\pi - \frac{s}{3}; \\ \gamma_n(3\phi - 4\pi)O_0, & 2\pi - \frac{s}{3} \leq s \leq 2\pi, \end{cases} \quad (B8)$$

we find that $a_s$ is a continuous deformation from $a_{s=0} = [a]$ to $a_{s=2\pi} = n[a]n^{-1}$, which completes the proof of Eq. (B7).

We next apply Theorem 4 to derive Eqs. (49), (50), and (51). Let $b$ and $[O^b] G/H$ be an element of Ker $i_{s_m-1}$ and that of $\pi_m(G/H)$ defined in Eq. (B7), respectively. When we define $S : \text{Ker } i_{s_m-1} \to \pi_m(G/H)$ by $S(b) := [O^b] G/H$, it is a section of $T$ since

$$\{T \circ S (b)\} (x) = [T([O^b] G/H)] (x) = b_{s=0} (\tilde{x}) = b(x) \quad \text{for } x \in S^{m-1}. \quad (B9)$$

We define $f : \text{Ker } i_{s_m-1} \times \text{Ker } i_{s_m-1} \to \text{Coker } i_{s_m}$ by Eq. (B1) with substitution of $S$ for $s$. Since $f_{s_m}$ is Abelian for any $m$, Theorem 4 gives the isomorphism:

$$\pi_m(G/H) \simeq \text{Coker } i_{s_m} \times f \text{Ker } i_{s_m-1}, \quad (B10)$$

where the product on the right-hand side of Eq. (B10) is defined by

$$[[a], b] \times f ([a'], b) := ([a] + \theta_b([a'])) + f((b, b'), (b', b')). \quad (B11)$$

From Eq. (B10), the inner isomorphism of $\pi_m(G/H)$ on $\text{Coker } i_{s_m}$ is trivial: $\theta_b([a']) := b[a']b^{-1} = [a']$, which gives Eqs. (49) and (50). Since we take the section of the identity element as $S(e) := [O^e] G/H = e$, we obtain $f(e, b) = S(e)S(b) [S(eb)]^{-1} = e$ and hence Eq. (51).

We finally prove that any choice of the section gives an equivalent group extension. Let $[O^b] G/H$ be another element of $\pi_m(G/H)$ corresponding to another deformation $\tilde{b}_s$ of $b \in \text{Ker } i_{s_m-1}$ to the trivial homotopy class. Then, the map $S : b \to [O^b] G/H$ provides another section. From the relation

$$[O^b]^{-1} O^b (x) = \begin{cases} b_{s=\pi/2} [x] (x) O_0 & 0 \leq \|x\| \leq \frac{\pi}{2}; \\ b_{s=\pi} [x] \| \pi (x) O_0 & \frac{\pi}{2} \leq \|x\| \leq \pi, \end{cases} \quad (B12)$$
we have
\[ \tilde{b}_{n} = e^{n\pi} = e, \quad \tilde{b}_{n} = 0, \]
and hence \([O^h]_f \in \text{Coker } i_{n-1}. It follows from Theorem 1 that
\[ \pi_2(G/H) \cong \text{Ker } \{ \pi_i : \pi_1(H) \to \pi_1(G) \}. \]

Appendix C: Proof of Corollary 2

Since \( \pi_2(G) \) vanishes from Eq. (21), \( \text{Coker } \pi_2^* \) vanishes. It follows from Theorem 1 that
\[ \pi_2(G/H) \cong \text{Ker } \{ i^* : \pi_1(H) \to \pi_1(G) \}. \]

Appendix D: Proof of Theorem 2

It follows from Eq. (19) that the action of \([H_k, [\sigma]]\) can be decomposed into the action of \([H_k, e]\) and that of \([e, [\sigma]]\) as follows:
\[ \lambda_m^l(n) = \lambda_{m[e,\sigma]}^{l} \{ \lambda_{m[H_k,e]}^{l}([a,b]) \}. \]
From Eq. (29), we can write the texture \( O_\lambda(\phi) \) of a vortex with topological charge \([H_k, e]\), as \( O_\lambda(\phi) = g_1(\phi) O_0 \), where \( g_1(\phi) = \exp(i\phi H_k) \) and \( \phi \in [0, 2\pi] \) is the azimuth angle around the vortex. Let \( O([a,b])(x) \) be the texture of a topological excitation with topological charge \([a,b]\) and let us define \( \lambda_s \) for \( s \in [0, 2\pi] \) by
\[ \lambda_s(a) = \begin{cases} g_1(s) O([a,b]) \left( \frac{\pi}{2}, \frac{\pi}{2} + \frac{s}{4} \right) x & \text{for } 0 \leq ||x|| \leq \frac{\pi}{2} + \frac{s}{4}, \\ g_1(4||x|| - 2\pi) O_0 & \text{for } \frac{\pi}{2} + \frac{s}{4} \leq ||x|| \leq \pi. \end{cases} \]
\[ \lambda_{s=2\pi}(a) = O([a,b])(x) = O([a,b])(x), \]
subject to the boundary condition (27). We thus have \( \lambda_{m[H_k,\sigma]}^{l}([a,b]) \) for \( [a,b] \). Then, Eq. (D1) reduces to
\[ \lambda_{m[H_k,\sigma]}^{l}([a,b]) = \lambda_{m(H_k,\sigma)}^{l}([a,b]) x \times f \lambda_{m(e,\sigma)}^{l}([e,b]), \]
where we use the homomorphic property of \( \lambda_m^l \) in the third equality. Let \( \gamma_\sigma \) be a path from \( \gamma_\sigma(0) = \sigma \) to \( \gamma_\sigma(2\pi) = e \) and we write the texture of a topological excitation with topological charge \([a,b] \) \((e,b)\) by
$O'(x) = g(x)O_0$ with $g(x) = a(x)$ ($g(x) = b_{||x||}(x)$). Let us define $\lambda'_s$ for $s \in [0, 2\pi]$ by

$$
\lambda'_s(x) := \begin{cases} 
\gamma(s)^{-1}O' \left( \frac{\pi}{2 + \frac{\pi}{2}} x \right) & \text{for } 0 \leq \|x\| \leq \frac{\pi}{2} + \frac{\pi}{4}, \\
\gamma(s) \left( 4 \|x\| - 2\pi \right) O_0 & \text{for } \frac{\pi}{2} + \frac{\pi}{4} \leq \|x\| \leq \pi.
\end{cases}
$$

(D5)

Then, $\lambda'_s$ is a continuous deformation from $\lambda'_{s=0}(x) = \lambda^{(c,|\alpha|)}(x)$ to

$$
\lambda'_{s=2\pi}(x) = \sigma^{-1}O'(x) = \sigma^{-1}g(x)\sigma O_0 = [\sigma^{-1}g\sigma](x),
$$

where we use $\sigma \in H \cap G_0$ and hence $\sigma O_0 = O_0$ in the second equality in Eq. (D6). When the topological charge is $[(a), e]$, $[\gamma(s)]^{-1} a \gamma(s)$ for $s \in [0, 2\pi]$, describes a continuous deformation from $\sigma^{-1}a\sigma$ to a subject to the boundary condition (D7). We therefore have

$$
\lambda^{(c,|\alpha|)}[[[a], e]] = [(a), e] \text{ and hence Eq. (D7)}:
$$

$$
\lambda^{(c,|\alpha|)}[[[a], e]] \times \lambda^{(c,|\alpha|)}[[e, b]] = [[[a], e] \times [\alpha, \sigma]^{-1}b\sigma] = [[[a], \sigma]^{-1}b\sigma],
$$

(D7)

which completes the proof of Theorem [2]

Appendix E: Proof of Corollary [6]

Let $g_C$ be the Cartan subalgebra of $G$ and we define two subalgebras of $g$ by

$$
\mathfrak{h}_C := \{ h_u \in g_C | (u, t) = 0 \text{ for all } t \in L_H \cap L_G \},
$$

$$
\mathfrak{h}^\perp := \mathfrak{h}_C \oplus \text{Span}\{ E_\alpha^R, E_\alpha^I | \alpha \in R_+ \},
$$

$$
(\alpha, t) = 0 \text{ for all } t \in L_H \cap L_G,
$$

(E1)

where $\text{Span}S$ denotes the vector space spanned by the elements of $S$.

We first prove that $\mathfrak{h}^\perp$ is a subalgebra of $g$ that commutes with $L_H \cap L_G$. From the commutation relations of the Cartan canonical form, the commutators among $H_u \in \mathfrak{h}_C$ and $E_\alpha^R, E_\alpha^I \in \mathfrak{h}^\perp$ are spanned by $H_u$ and $E_\alpha^R, E_\alpha^I$ that satisfy $(u, t) = (\alpha, t) = 0$ for all $t \in L_H \cap L_G$. Therefore, $\mathfrak{h}^\perp$ forms a subalgebra of $g$. Since both $\mathfrak{h}_C$ and $L_H \cap L_G$ are generated by the Cartan generators, $\mathfrak{h}^\perp$ commutes with $L_H \cap L_G$. For $t \in L_H \cap L_G$ and $E_\alpha^R, E_\alpha^I \in \mathfrak{h}^\perp$, we have $[H_t, E_\alpha^R, E_\alpha^I] = \pm i(\alpha, t)E_\alpha^R, E_\alpha^I = 0$ from Eq. (E1), indicating that $L_H \cap L_G$ commutes with $\mathfrak{h}^\perp$.

Let $\sigma$ be a representative element of $[\sigma] \in \pi_0(H \cap G_0)$. We next prove $Ad(\sigma)h_C \subset \mathfrak{h}_C$, where $Ad(\sigma) : g \rightarrow g$ for $g \in G$ is defined by $[Ad(\sigma)](X) := gXg^{-1}$ for $X \in g$. For $H_u \in \mathfrak{h}_C$, we expand $Ad(\sigma)H_u$ in terms of the Cartan canonical form as

$$
Ad(\sigma)H_u = H_u + \sum_{\alpha \in R_+} (c_\alpha E_\alpha^R + d_\alpha E_\alpha^I),
$$

(E2)

where $c_\alpha$ and $d_\alpha$ are real numbers. Let $H_t$ be an element of $L_H \cap L_G$. Since $Ad(\sigma)$ is an automorphism on $(L_H \cap L_G)/L_G$, from Theorem [2] we have $Ad(\sigma^{-1})H_t \in L_H \cap L_G$, and hence it can be written as $H_t = Ad(\sigma)H_t'$ for some $H_t' \in L_H \cap L_G$. From Eq. (E1), we have

$$
(t, u') = Tr[H_tH_u'] = Tr[Ad(\sigma)H_tAd(\sigma)H_u] = (t', u) = 0.
$$

(E3)

Hence we obtain $H_u' \in \mathfrak{h}_C$. Also, it follows from Eq. (E1) that

$$
0 = Ad(\sigma)[H_t, H_u] = [H_t', H_u' + \sum_{\alpha \in R_+} (c_\alpha E_\alpha^R + d_\alpha E_\alpha^I)]
$$

$$
= \sum_{\alpha \in R_+} i(\alpha, t')(c_\alpha E_\alpha^I - d_\alpha E_\alpha^R).
$$

(E4)

This gives $c_\alpha = d_\alpha = 0$ if $\alpha$ satisfies $(\alpha, t') \neq 0$ for some $t' \in L_H \cap L_G$, resulting in $Ad(\sigma)H_u \in \mathfrak{h}_C$, which completes the proof of $Ad(\sigma)\mathfrak{h}_C \subset \mathfrak{h}_C$.

Let $T_G$ be a maximum Abelian group of $G$. Let $H^\perp$ be the connected Lie group generated by $\mathfrak{h}^\perp$ including the identity element. Since $Ad(\sigma)\mathfrak{h}_C \subset \mathfrak{h}_C$, $\mathfrak{h}_C$ and $Ad(\sigma)\mathfrak{h}_C$ are maximum Abelian subgroups of $H^\perp$. Since any two maximum Abelian subgroups are conjugate to each other [11, 12], there exists an element $h^\perp in H^\perp$ such that $Ad(h^\perp)\mathfrak{h}_C = \mathfrak{h}_C$. On the other hand, $Ad(h^\perp)$ acts on $L_H \cap L_G$ trivially from the commutativity between $h^\perp$ and $L_H \cap L_G$. We therefore have $Ad(h^\perp)\mathfrak{h}_C \cap L_H \cap L_G = L_H \cap L_G$. Since $g_C$ is generated by $L_H \cap L_G$ and $\mathfrak{h}_C$, we have

$$
h^\perp \sigma = H_t \in N_W := \{ g \in G | Ad(g)X \in g_C \text{ for all } X \in g_C \}.
$$

(E5)

It is known that $T_G$ is a normal subgroup of $N_W$ and that the quotient group $N_W/T_G$ is isomorphic to $W_G/T_G$. We define $w_{[\sigma]} \in W_G$ as the projection of $h^\perp \sigma \in N_W$ to $W_G \cong N_W/T_G$. From Theorem [2] the action of $[H_t], [\sigma] \in \pi_1(G/H)$ on $H_t + L_H \in \pi_2(G/H)$ can be written as

$$
\lambda^\perp_2([H_t], [\sigma]) = Ad(\sigma)H_t + L_H
$$

$$
= \lambda^\perp_2((H_t), [\sigma]) = w_{[\sigma]}(H_t) + L_H.
$$

(E6)

We note that the right-hand side does not depend on the choice of a representative element since $Ad(\sigma)$ acts on $L_H$ trivially. Thus we have

$$
G_2 \simeq \{ \lambda^\perp_2([H_t], [\sigma]) | ([H_t], [\sigma]) \in \pi_1(G/H) \}
$$

$$
\simeq \{ w_{[\sigma]} \in W_G | [\sigma] \in \pi_0(H \cap G_0) \}.
$$

(E7)

Since the right-hand side is a subgroup of $W_G$, $G_2$ is also a subgroup of $W_G$, which completes the proof of Corollary [6].
[1] Y. Kawaguchi and M. Ueda, Phys. Rep. 520, 253 (2012).
[2] M. Ueda, Fundamentals and New Frontiers of Bose-Einstein Condensation (World Scientific, Singapore, 2010).
[3] A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).
[4] A. P. Mackenzie and Y. Maeno, Rev. Mod. Phys. 75, 657 (2003).
[5] B. Coqblin and J. R. Schrieffer, Phys. Rev. 185, 847 (1969).
[6] K. I. Kugel and D. I. Khomskii, Zh. Eksp. Teor. Fiz. 64, 1429 (1973).
[7] H. Tsunetsugu and M. Arikawa, J. Phys. Soc. Japan 75, 083701 (2006).
[8] A. Läuchli, F. Mila, and K. Penc, Phys. Rev. Lett. 97, 087205 (2006).
[9] M. a. Cazalilla and A. M. Rey, Reports Prog. Phys. 77, 124301 (2014).
[10] M. M. Salomaa and G. E. Volovik, Phys. Rev. Lett. 98, 1184 (2007).
[11] G. W. Semenoff and F. Zhou, Phys. Rev. Lett. 98, 100401 (2007).
[12] H. Kleinert, Gauge Fields in Condensed Matter: Vol. 1: Superflow and Vortex Lines (Disorder Fields, Phase Transitions) Vol. 2: Stresses and Defects (Differential Geometry, Crystal Melting) (World Scientific, Singapore, 1989).
[13] T. H. R. Skyrme, Proc. R. Soc. London A Math. Phys. Eng. Sci. 260, 127 (1961).
[14] T. Skyrme, Nucl. Phys. 31, 55 (1962).
[15] R. Shankar, J. Phys. 38, 1184 (1985).
[16] G. E. Volovik and V. P. Mineev, Sov. J. Exp. Theor. Phys. 46, 401 (1977).
[17] L. Faddeev and A. J. Niemi, Nature 387, 58 (1997).
[18] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni, Science 323, 915 (2009).
[19] X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui, N. Nagaosa, and Y. Tokura, Nature 465, 901 (2010).
[20] S. E. Barrett, G. Dabbagh, L. N. Pfeiffer, K. W. West, and R. Tycko, Phys. Rev. Lett. 74, 5112 (1995).
[21] A. Schmeller, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. 75, 4290 (1995).
[22] E. H. Aifer, B. B. Goldberg, and D. A. Broido, Phys. Rev. Lett. 76, 680 (1996).
[23] J. Jang, D. G. Ferguson, V. Vakaryuk, R. Budakian, S. B. Chung, P. M. Goldbart, and Y. Maeno, Science (80-. ). 331, 186 (2011).
[24] S. Autti, V. V. Dmitriev, J. T. Mäkinen, A. A. Soldatov, G. E. Volovik, A. N. Yudin, V. V. Zavjalov, and V. B. Eltsov, Phys. Rev. Lett. 117, 255301 (2016).
[25] B.-g. Chen, P. J. Ackerman, G. P. Alexander, R. D. Noja, and I. I. Smalyukh, Phys. Rev. Lett. 110, 237301 (2013).
[26] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science (80-. ). 292, 476 (2001).
[27] S. W. Seo, S. Kang, W. J. Kwon, and Y.-i. Shin, Phys. Rev. Lett. 115, 015301 (2015).
[28] S. W. Seo, W. J. Kwon, S. Kang, and Y. Shin, Phys. Rev. Lett. 116, 185301 (2016).
[29] L. S. Leslie, A. Hansen, K. C. Wright, B. M. Deutsch, and N. P. Bigelow, Phys. Rev. Lett. 103, 250401 (2009).
[30] J.-Y. Choi, W.-J. Kwon, and Y.-i. Shin, Phys. Rev. Lett. 108, 035301 (2012).
[31] M. W. Ray, E. Ruokokoski, S. Kandel, M. Mottonen, and D. S. Hall, Nature 505, 657 (2014).
[32] M. W. Ray, E. Ruokokoski, K. Tiurev, M. Mottonen, and D. S. Hall, Science (80-. ). 348, 544 (2015).
[33] D. S. Hall, M. W. Ray, K. Tiurev, E. Ruokokoski, A. H. Gheorghe, and M. Mottonen, Nat. Phys. 12, 478 (2016).
[34] G. Volovik and V. Mineev, Zh. Eksp. Teor. Fiz. 72, 2257 (1977).
[35] M. Kléman, L. Michel, and G. Toulouse, J. Phys. Lettres 38, 195 (1977).
[36] S. Kobayashi, M. Kobayashi, Y. Kawaguchi, and M. Ueda, Nucl. Phys. B 856, 577 (2012).
[37] U. Leonhardt and G. E. Volovik, J. Exp. Theor. Phys. Lett. 74, 46 (2000).
[38] N. D. Mermin, Rev. Mod. Phys. 51, 591 (1979).
[39] N. D. Mermin, J. Math. Phys. 19, 1457 (1978).
[40] A. S. Schwarz, Nucl. Phys. B 208, 141 (1982).
[41] T. Bröcker and T. tom Dieck, Representations of compact Lie groups, vol. 98 (Springer-Verlag Berlin Heidelberg, 1985), 1st ed.
[42] B. C. Hall, Lie groups, Lie algebras, and representations: an elementary introduction, vol. 222 (Springer International Publishing, Basel, 2015), 2nd ed.
[43] T. A. Toth, A. M. Läuchli, F. Mila, and K. Penc, Phys. Rev. Lett. 105, 265301 (2010).
[44] B. Bauer, P. Corboz, A. M. Läuchli, F. Mila, K. Penc, M. Troyer, and F. Mila, Phys. Rev. B 85, 125116 (2012).
[45] H. M. Georgi, Lie algebras in particle physics; 2nd ed., Frontiers in Physics (Perseus, Cambridge, 1999).
[46] S. Higashikawa and M. Ueda, Phys. Rev. A 94, 013613 (2016).
[47] D. Robinson, A Course in the Theory of Groups, vol. 80 (Springer-Verlag New York, 1996).
[48] E. Cartan, Oeuvres Complet. 1, 2 (1936).
[49] A. L. Onishchik, Topology of transitive transformation groups (Huthig Pub Ltd, Basel, 1994).
[50] M. Nakahara, Geometry, topology and physics (CRC Press, 2003).
[51] I. E. Dzyaloshinskii and B. A. Ivanov, JETP Lett. 29, 540 (1979).
[52] E. J. Weinberg, Nucl. Phys. B 167, 500 (1980).
[53] E. J. Weinberg and P. Yi, Phys. Rep. 348, 65 (2007).
[54] N. Manton and P. Sutcliffe, Topological solitons (Cambridge University Press, Cambridge, 2004).
[55] U. A. Khawaja and H. T. C. Stoof, Phys. Rev. A 64, 043612 (2001).
[56] Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. 100, 180403 (2008).
[57] S. Kobayashi, N. Tarantino, and M. Ueda, Phys. Rev. A 89, 033603 (2014).
[58] V. Poenaru and G. Toulouse, J. Phys. 38, 887 (1977).
[59] M. Kobayashi, Y. Kawaguchi, M. Nitta, and M. Ueda, Phys. Rev. Lett. 103, 115301 (2009).
[60] M. O. Borgh and J. Ruostekoski, Phys. Rev. Lett. 117, 275302 (2016).
[61] M. Kobayashi and M. Ueda, arXiv Prepr. arXiv1606.07190 (2016).
[62] N. V. Priezjev and R. A. Pelcovits, Phys. Rev. E 66,
P. McGraw, Phys. Rev. D **57**, 3317 (1998).

D. Spergel and U.-L. Pen, Astrophys. J. Lett. **491**, L67 (1997).

T. Fukuhara, Y. Takasu, M. Kumakura, and Y. Takahashi, Phys. Rev. Lett. **98**, 030401 (2007).

B. J. DeSalvo, M. Yan, P. G. Mickelson, Y. N. MartinezdeEscoba, and T. C. Killian, Phys. Rev. Lett. **105**, 030402 (2010).

M. A. Cazalilla, A. F. Ho, and M. Ueda, New J. Phys. **11**, 103033 (2009).

A. V. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. S. Julienne, J. Ye, P. Zoller, E. Demler, M. D. Lukin, and A. M. Rey, Nat. Phys. **6**, 289 (2010).

N. Papanicolaou, Nucl. Phys. B **305**, 367 (1988).

C. Honerkamp and W. Hofstetter, Phys. Rev. Lett. **92**, 170403 (2004).

A. Rapp and A. Rosch, Phys. Rev. A **83**, 053605 (2011).

A. Sotnikov and W. Hofstetter, Phys. Rev. A **89**, 063601 (2014).

H. T. Ueda, Y. Akagi, and N. Shannon, Phys. Rev. A **93**, 021606 (2016).

N. Nagaosa and Y. Tokura, Nat. Nano. **8**, 899 (2013).

D. Vollhardt and P. Wolfe, *The superfluid phases of helium 3* (Courier Corporation, 2013).

G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon Press; Oxford University Press, Oxford, United Kingdom, 2003).

J. Tong, arXiv Prepr. hep-th/0509216 (2005).

S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, Phys. Rev. B **47**, 16419 (1993).

K. Yang, S. Das Sarma, and A. H. MacDonald, Phys. Rev. B **74**, 075423 (2006).

U. K. Roszler, A. N. Bogdanov, and C. Pfleiderer, Nature **442**, 797 (2006).

J. Sampayo, V. Cros, S. Rohart, A. Thiaville, and A. Fert, Nat. Nano. **8**, 839 (2013).

V. Pietila and M. Mottonen, Phys. Rev. Lett. **102**, 080403 (2009).

J. E. Moore, Y. Ran, and X.-G. Wen, Phys. Rev. Lett. **101**, 186805 (2008).

R. Kennedy and C. Guggenheim, Phys. Rev. B **91**, 245148 (2015).

R. Kennedy, Phys. Rev. B **94**, 035137 (2016).

A. Hatcher, *Algebraic topology* (Cambridge University Press, Cambridge, 2002).