Abstract. We investigate the transport properties of simple iterated maps with quenched disorder. The dynamics of these systems is mapped to random walks in random environments with next-nearest-neighbour transitions, constituting generalizations of the well-known Sinai model. The non-equilibrium properties are studied numerically by a direct observation of the transport behaviour, by investigating the density of states of the propagator and by considering the system-size dependence of the escape rate. Characteristic exponents associated with each of these quantities are determined and their dependence on the system parameters is evaluated. We find anomalously slow behaviour which in general deviates from the Sinai case and therefore generalizes the latter. These deviations are attributed to the generic absence of detailed balance, which implies that a potential can no longer be assigned.
1. Introduction

Diffusive transport was long considered as a phenomenon arising from stochastic environmental forces, which act, e.g. on a heavy particle. This picture, which requires the action of many degrees of freedom, was supplemented in the 1980s with the insight that, already, low-dimensional dynamical systems can exhibit diffusive transport in connection with chaotic motion [1]–[4]. Subsequently, the phenomenon of normal and anomalous chaotic diffusion was investigated in great detail [5]–[20] (see also [21, 22]), but mostly for systems exhibiting some periodicity in state space. A notable exception is the random Lorentz gas, in which point particles are scattered chaotically between randomly placed reflecting discs leading to normal chaotic diffusion [23]. The fact that static randomness in the equations of motion of dynamical systems can lead to very drastic modifications of the chaotic transport properties was recognized more recently [24]–[26]. These numerical findings generalize the phenomena proven rigorously for certain random walks in random environments, which are known under the heading of ‘Sinai diffusion’ [27, 28] (see [29, 30] for reviews of related work). In this paper, we investigate in more detail how the anomalous slowing down of the dynamics is reflected in certain generalizations of maps with Sinai disorder and the corresponding random walks. For Sinai disorder, the known results concern quantities such as the disorder-averaged mean-square displacement, the density of states of the propagator and the size dependence of the escape rate, or the mean first-passage time, respectively. For each of them, one can define a characteristic exponent. For example, the disorder-averaged second moment of the displacement grows as \( \log^{2 \alpha}(t) \), where \( \alpha = 2 \) for Sinai disorder. In the generalizations considered here, the corresponding characteristic exponents differ from those of the Sinai model. The reason for this difference seems to lie in the generic lack of detailed balance. Therefore, in contrast to the Sinai model, no potential can be assigned to these more general systems, and correspondingly analytical potential methods cannot be applied. The purpose of this paper is to elucidate numerically the connection between these characteristic exponents and their dependence on the system parameters. The paper is organized as follows. In section 2, we introduce the model systems and provide their connection to random walks in random environments. The considered transport properties are defined and discussed in section 3. The spectral properties of the propagator and the escape rates are treated in sections 4 and 5, respectively. Finally, in section 6, the connection between all these quantities and their parameter dependence is provided and discussed, and the results are summarized.
Figure 1. An iterated map with generalized binary disorder leading to next-nearest neighbour transitions is displayed (left panel). \( p \) and \( \bar{p} \) are randomly chosen with equal probability. In cell \( i \), we have plotted for comparison both the cases corresponding to \( p \) (——) and \( \bar{p} \) (······). A magnification of this cell is shown on the right with the corresponding transition probabilities \( p_{ij} \).

2. One-dimensional iterated maps with disorder and associated Markov processes

In this paper, we study piecewise linear, one-dimensional non-invertible maps. Their general form is given by

\[
x_{t+1} = f(x_t) = x_t + F(x_t), \quad x_t \in \mathbb{R}, \quad t \in \mathbb{N}.
\]  

We define cells, respectively semi-open intervals, on the real axis by \( A_i = [i, i+1), i \in \mathbb{Z} \). Within each cell \( A_i \), the mapping \( f(x) \) is assumed to be continuous, monotonically increasing and piecewise linear. Changes in the slope of \( f(x) \) are allowed only at points \( x_i \), which are mapped onto cell boundaries, i.e. \( f(x_i) = i, i \in \mathbb{Z} \). Furthermore, cell boundaries are mapped onto cell boundaries. Disorder is introduced into the system by randomly varying \( F(x) \) in each cell \( A_i \). The resulting cell dependence of \( F(x_t) \) is indicated by the notation \( F^{(i)}(x) \). An example of such a map is given in figure 1.
Iterative maps as defined in equation (1) are equivalently described by the time evolution of ensembles of points, i.e. state space densities \( \rho(x, t) \). The dynamics of \( \rho \) obeys

\[
\rho(x, t + 1) = \mathcal{P}[\rho(x, t)] = \int \rho(y, t) \delta(f(y) - x) \, dy,
\]

where \( \mathcal{P} \) denotes the Frobenius–Perron operator. Note that with the above assumptions for the map \( f(x) \) a piecewise constant density \( \rho(x, t) \), with \( \rho(x, t) = \pi_i(t) \) for \( x \in A_i \), evolves according to equation (2) again into a piecewise constant density \( \rho(x, t + 1) \) with values \( \pi_j(t + 1) \) given by

\[
\pi_j(t + 1) = \sum_i \pi_i(t) p_{ij},
\]

where the \( p_{ij} \) can be regarded as elements of a matrix \( P \). They are given by the derivative \( f'(y)_{ij} \) of that part of \( f(y) \) in the cell \( A_i \) which maps points into cell \( A_j \)

\[
p_{ij} = (P)_{ij} = \frac{1}{|f'_{ij}|}.
\]

Thus, for piecewise constant densities, the system dynamics is fully equivalent to a discrete Markov process as given by equation (3). Actually, the cells \( A_i \) introduced above provide a so-called Markov partition [21, 31]. Considering only piecewise constant densities is not a strong restriction since sufficiently smooth initial distributions relax to this form exponentially fast. In the following, we restrict ourselves to cases where the infinite matrix \( P \) is at most penta-diagonal, i.e. the associated random walk contains at most transitions to the next-nearest neighbours and the corresponding maps have the form shown in figure 1. Of course, more general random walks can be considered and, by inversion of the given reasoning, the corresponding dynamical systems can be constructed. We are interested in the cases where \( P \) is random and therefore defines what is called a random walk in a random environment (‘random random walk’) [30].

The first fundamental problem with random random walks containing more than nearest neighbour transitions is to decide whether a system is recurrent or not. There is a criterion by Key [32], but already for next-nearest-neighbour transitions it cannot in general be evaluated analytically. Nevertheless, a vanishing global bias is guaranteed in some special cases, e.g. for the following generalization of binary disorder. We take two sets of transition probabilities: \( p = \{p_{--}, p_{-}, p_0, p_+\} \) and \( \bar{p} = \{p_{++}, p_+, p_0, p_-, p_{--}\} \), and assign them to each cell \( i \) with equal probability, i.e. \( \{p_{ii-2}, p_{ii-1}, p_{ii}, p_{ii+1}, p_{ii+2}\} = \{p_{--}, p_{-}, p_0, p_+, p_{++}\} \) or \( \{p_{ii-2}, p_{ii-1}, p_{ii}, p_{ii+1}, p_{ii+2}\} = \{p_{++}, p_+, p_0, p_-, p_{--}\} \) with probability \( \frac{1}{2} \). This situation is illustrated in figure 1. Due to this statistical symmetry, a global bias is avoided.

The generalized binary disorder we have introduced includes two well-known special cases. Anomalous transport phenomena are well investigated and exact results exist for the so-called Sinai disorder ([27]–[29]) with a typical realization as a dynamical system shown in figure 2(a). Only nearest-neighbour transitions are allowed in this case. Again, the cells \( A_i \) provide a Markov partition, i.e. the segments of length \( p_{ii}, p_{ii+1} \) in each unit cell correspond to the non-zero transition probabilities of the connected Markov chain. For this special case, a zero global bias is guaranteed by the simple requirement \( E[(\ln(p_{ii}/p_{ii+1}))] = 0 \) [27, 28]. The (irreducible) Markov chain is consequently recurrent.

The other special case was introduced in [24]. An example is shown in figure 2(b). Here, the disorder is induced by randomly varying \( F(x) \) in each cell \( A_i \) by random
Figure 2. (a) Iterated map of the Sinai type. Also shown by broken lines are the unit squares of the integer grid along the bisectrix. The indicated intervals $p_{ii}$, $p_{i\pm1}$ mediate the transition from cell $A_i$ to itself, respectively, its neighbours. They vary randomly from cell to cell. (b) A simple piecewise linear map corresponding to a periodic ‘force’ (- - - - , i.e. $\epsilon(i) = 0$) and a random driving ‘force’ (——, i.e. $\epsilon(i) = \pm \frac{1}{2}$).

shifts: $F^{(i)}(x) = P(x) + \epsilon(i)$ with $P(x) = P(x + 1)$. The $\epsilon(i)$ are identically distributed with expectation value $\overline{\epsilon(i)} = 0$ and independent $[\epsilon(i)\epsilon(j)] \propto \delta_{ij}$. This guarantees that there is no global bias in this system. We set $P(x) = a{x} - \frac{a}{2}$, $a = 3$. Here, $\{x\} = x - \lfloor x \rfloor$ stands for the fractional part of $x$. $\epsilon(i)$ is chosen as $\pm \frac{1}{2}$ with equal probability. One can see that the disorder $\epsilon(i)$ does not affect the fraction of points which remain in a cell $A_i$ or which are mapped into the neighbouring cells $A_{i\pm1}$. $\epsilon(i)$ just affects the fraction of points which are mapped into the next-nearest cells $A_{i\pm2}$ such that one quarter of the points is mapped either into $A_{i\pm2}$ or into $A_{i-2}$. For the corresponding Markov model, we obtain as a consequence the non-zero transition probabilities $p_{ii} = p_{ii\pm1} = \frac{1}{4}$ and $p_{ii\pm2} = [\frac{1}{2} \pm \epsilon(i)]\frac{1}{4}$. Because the disorder affects only the connectivity of the corresponding Markov model, this case was called topological disorder in [26]. The properties of the parts of each cell that map into $A_{i\pm2}$ are also the only ones which differ from the ordered situation with $\epsilon(i) = 0$. Note that the induced disorder does not affect the Lyapunov exponent of the system because the slopes are not affected by the disorder.

3. Anomalous transport and suppression of chaotic diffusion

A drastic effect of static or quenched disorder in the equations of motion of dynamical systems is the possible suppression of normal or anomalous diffusion. This is a nonlinear effect: the mean square deviation $\sigma^2(t)$ typically stays finite despite the fact that chaotic transport is not restricted locally. For the iterated maps of section 2, let us consider random walkers which are homogenously distributed in an initial cell $i$. Figure 3(a) shows the typical evolution of the
Figure 3. (a) Mean-square deviation $\sigma^2(t)$ for one ordered and several disordered cases (Sinai disorder $\mathbb{1}$, topological disorder $\mathbb{2}$ and—with a higher $\sigma^2(t)$—a more general example $\mathbb{3}$ of a system of figure 1). The ordered case $\mathbb{4}$ is hardly to be seen as a straight line near the $y$-axis. (b) For the disordered situation of figure 2(b) ($\mathbf{p} = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0\}$), the ensemble mean of the squared positions $x_t$ averaged over 500 disorder realizations is shown by plotting $\log \left\langle x_t^2 \right\rangle$ versus $\log \log t$. The assumed law is expected to be valid in the limit $t \to \infty$. Therefore the broken line is a fit over the range $5 \times 10^5 < t < 10^6$.

The variance $\sigma^2(t) = \langle x_t^2 \rangle - \langle x_t \rangle^2$ for a single disorder realization for each of the systems introduced in section 2 (see figures 1, 2(a) and (b)) and one ordered case (figure 2(b) with $\epsilon(i) = 0$). While $\sigma^2(t)$ increases linearly with $t$ in the ordered case, it stays restricted in all three disordered cases although each cell can also be reached with positive probability. This kind of dynamical localization was called ‘Golosov phenomenon’ in [29] because of its rigorous proof in [28] for the Sinai random walk. The same behaviour was observed in [24] for various one-dimensional iterated maps and, in [25], also for certain two-dimensional area-preserving maps. So this is a very robust phenomenon largely independent of the details of the system as long as a local asymmetry in the disordered environment or the disordered maps is present. Therefore it is not astonishing that we find this kind of behaviour also in the systems introduced in section 2. The boundedness of $\sigma^2(t)$, however, does not imply that the mean position of an ensemble of states or random walkers stays constant. Actually, an anomalous drift behaviour is observed, which turns out to be more sensitive to details of the considered system. This will be shown explicitly below for the cases of generalized disorder, Sinai disorder and topological disorder. The drift behaviour is captured by $\langle x_t \rangle^2$, the square of the ensemble mean $\langle x_t \rangle$ averaged over several disorder realizations, or alternatively by $\overline{x_t^2}$. In view of the Golosov phenomenon, both quantities are expected to exhibit the same long-term behaviour. An example of the anomalous increase of $\overline{x_t^2}$ is provided in figure 3(b).

To a large extent, the observed anomalous transport properties can be understood in the case of Sinai disorder, especially in its continuum limit. The continuum limit of nearest-neighbour random walks is Brownian motion in a spatially random force field $F(x) = -\frac{\partial V}{\partial x}$ [29]. The dynamics is then described by the Langevin equation:

$$\dot{x}(t) = -\frac{\partial V}{\partial x}[x(t)] + \xi(t).$$

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The associated potential \( V(x) \) can itself be regarded as the realization of a Brownian path. The corresponding statistical self-similarity \( V(Lx) \sim \sqrt{L} V(x) \) implies the appearance of increasing potential barriers as the particle proceeds. One ensemble of initially close particles moves in such a potential in a coherent manner from one potential well to another. In this stepwise process, typically one minimum dominates at a given time, thereby determining the finite extension of the ensemble. Fluctuations in \( \sigma^2(t) \) still to be observed become quite rare for large times \( t \) because it takes longer and longer times for surmounting the increasing potential barriers provided by the environment. The typical time for surmounting a potential barrier increases exponentially with the height of the barrier, so that a state needs a time \( t \sim \exp(b\sqrt{x}) \) for covering a distance \( x \) \[29\]. Solving this relation for \( x \) implies that the typical distance covered by a state during the time \( t \) increases only with \( \log^2 t \). Averaging over the disordered environment yields, for \( t \to \infty \), the so-called Sinai diffusion law \[27\]–\[30\]
\[
\langle x(t)^2 \rangle \sim \log^2 \alpha t,
\] (6)

with \( \alpha = 2 \).

For systems with next-nearest-neighbour transitions, a potential for the random walk does no longer exist and the above picture breaks down. Our numerical results, however, indicate that this essentially changes only the exponent \( \alpha \) in equation (6). As shown in figure 3(b), it can be determined as the slope of a straight line fitted in a double-logarithmic plot of \( \langle x^2_t \rangle \) versus \( \log t \), i.e. by plotting \( \log \langle x^2 \rangle \) as function of \( \log \log t \). We systematically determined \( \alpha \) for the systems introduced in section 2 as a function of the system parameters. For completeness, we also investigated the behaviour of \( \langle x_t \rangle \). These results are given and discussed in section 6 together with the results for the exponents to be introduced in sections 4 and 5.

4. Spectral properties

As indicated before, the dynamics induced by the iterative maps is also described completely by the Frobenius–Perron operator \( \mathcal{P} \), which propagates the density in state space \( \rho(x, t) \) (equation (2)). In this description, information about the static disorder is fully contained in \( \mathcal{P} \). Therefore we investigate in which way the spectral properties of \( \mathcal{P} \), i.e. its eigenvalues and eigenvectors, are influenced.

As described in section 2, we can assume a constant density \( \rho(x, t) = \pi_i(t) \) \( x \in A_i \) in each interval \( A_i \). The action of \( \mathcal{P} \) is then equivalent to the action of a Markov transition matrix \( \mathbf{P} \) with elements \( p_{ij} \). \{\( u^{(\alpha)} \)\} and \{\( v^{(\alpha)} \)\} stand for the complete (bi-orthonormal) sets of right and left eigenvectors and \( \lambda^{(\alpha)} \) for the corresponding eigenvalues of \( \mathbf{P} \). In the following, we consider finite systems consisting of \( L \) cells, and the maps in cells \( i = 1 \) and \( i = L \) are modified to yield reflecting boundary conditions. Then the spectral decomposition is that of an \( L \times L \)-matrix given by
\[
p_{ij} = \sum_{\alpha=1}^{L} u^{(\alpha)}_i \lambda^{(\alpha)} v^{(\alpha)}_j.
\] (7)

Because \( \mathbf{P} \) is a stochastic matrix, all its eigenvalues according to the Frobenius–Perron theorem \[37\] lie inside the complex unit circle. The equilibrium distribution \( \rho^*(x) \), respectively \( \pi^* \),
Figure 4. (a) The spectrum of the propagator for generalized binary disorder \((p = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\})\) shows a distinct clustering of the eigenvalues \(\lambda^{(\alpha)}\) near 1. Eigenvalues with Re \(\lambda^{(\alpha)} > 0\) are real. (b) The integrated density of states \(N(\epsilon)\) versus \(|\log \epsilon|\) for a disordered situation \((p = \{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}\})\) averaged over 50 disorder realizations with a system size of \(L = 2000\) is shown. The asymptotic behaviour of \(N(\epsilon)\) is given by a fit over the 500 smallest \(\epsilon^{(\alpha)} > 0\).

then is given by the left eigenvector belonging to the eigenvalue \(\lambda^{(\alpha)} = 1\). Furthermore, we introduce the relaxation rates \(\epsilon^{(\alpha)} = -\log \lambda^{(\alpha)}\). For \(\lambda^{(\alpha)} \to 1\), i.e. \(\epsilon^{(\alpha)} \to 0\), it can be written as \(\epsilon^{(\alpha)} \approx 1 - \lambda^{(\alpha)}\).

Figure 4(a) shows the real part of the eigenvalues for a realization of the generalized binary disorder introduced in section 2. One observes a clustering of the eigenvalues near \(\lambda^{(\alpha)} = 1\). To understand the consequences, we consider the time evolution of a homogeneous density in a cell \(A_k\) corresponding to the initial condition \(\pi_i(0) = \delta_{i,k}\):

\[
\pi_k(t) = \sum_{i=1}^{L} \pi_i(0)(P^\alpha)_{ik} = \sum_{i=1}^{L} \sum_{\alpha=1}^{L} \delta_{ik} u_i^{(\alpha)} e^{-\epsilon^{(\alpha)} t} v_k^{(\alpha)}. \tag{8}
\]

Averaging this over all possible initial cells yields \(\langle \pi_k(t) \rangle = \frac{1}{L} \sum_{k=1}^{L} \pi_k(t) = \frac{1}{L} \sum_{\alpha} \exp \{-\epsilon^{(\alpha)} t\}\). By introducing the density of relaxation rates

\[
\rho(\epsilon) = \frac{1}{L} \sum_{\alpha=1}^{L} \delta(\epsilon - \epsilon^{(\alpha)}), \tag{9}
\]

\(\langle \pi_k(t) \rangle\) can be expressed as the Laplace transform of \(\rho(\epsilon)\):

\[
\langle \pi_k(t) \rangle = \int_0^\infty \rho(\epsilon)e^{-\epsilon t} d\epsilon. \tag{10}
\]

This relationship shows that the process of decay in an initial cell in the limit of large times is determined by the behaviour of \(\epsilon\) near zero, or by the eigenvalues \(\lambda^{(\alpha)}\) near 1.
For the investigation of the quantitative behaviour of the relaxation rates $\epsilon$ near zero, the integrated density of states is appropriate

$$N(\epsilon) = \int_0^\epsilon \rho(\epsilon') \, d\epsilon'.$$

The singular behaviour of the density of the relaxation rates $\epsilon^{(\alpha)}$ near zero, equivalent to the clustering of the eigenvalues $\lambda^{(\alpha)}$ near 1 (figure 4(a)), becomes visible in the log–log plot of $N(\log |\epsilon|)$ (figure 4(b)). The singular behaviour is specified by

$$N(\epsilon) \sim \frac{c}{|\log \epsilon|^\delta}.$$

Let us look again at the well-investigated case of Sinai disorder (figure 2(a)). In [26] it was shown that, in this case, equation (12) with constants $c = 2, \delta = 2$ very well describes the behaviour of $N(\epsilon)$ for small $\epsilon$. In view of equation (11), this implies a singular behaviour of $\rho(\epsilon)$ at the band edge given by $\rho(\epsilon) \sim 4/(\epsilon |\log \epsilon|^3)$. These results were also found analytically by Bouchaud et al [33] for an exactly solvable continuum model of a random walk in a disordered environment. This implies that the discrete Sinai model or the maps with Sinai disorder and this continuum model belong to the same universality class. Using this result for $N(\epsilon)$, one gets the long-time behaviour of $\langle \pi_k(t) \rangle$, by the application of a Tauberian theorem, as $\langle \pi_k(t) \rangle \sim 2/\log^2 t$. Generally, the scaling behaviour $\bar{P}_{kl} = \langle \pi_l(t) \rangle$, $x = |k - l|$, [34, 36] is assumed to be valid. By the additional assumption that, in the limit of large systems averaging over all initial conditions, $\langle \pi_k(t) \rangle$ is equivalent to the disorder average $\overline{\pi_k(t)}$ (self-averaging), and by exploiting the behaviour at $x = 0$, one finds that $y(t) \sim \log^2 t$. Inserting this into the definition of $\langle \bar{x}(t)^2 \rangle$, one easily deduces $\langle \bar{x}(t)^2 \rangle \sim \log^{2\alpha} t$ with $\alpha = 2$ (Sinai diffusion). This shows that, for the Sinai system, the exponents $\alpha$ and $\delta$ are related by $\alpha = \delta = 2$. This is remarkable in so far as it connects equilibrium properties of large but finite systems ($\delta$) with non-equilibrium properties of infinite systems ($\alpha$). In contrast, maps exhibiting normal diffusion are characterized by $\langle \pi_k(t) \rangle \sim 1/\sqrt{t}$ and $N(\epsilon) \sim \sqrt{\epsilon}$.

In the case of generalized binary disorder, the transition matrix is no longer tridiagonal as in the Sinai case and the eigenvalues become complex. But the part of the spectrum near $\lambda = 1$ stays purely real and again appears to produce a singularity in the integrated density of relaxation rates as shown in figure 4(b). So we find a behaviour at the band edge of the form equation (12), but now with values $\delta$ different from $\delta = 2$. This is not too surprising because the next-nearest-neighbour transitions yield probability currents also in the stationary state and therefore the system can no longer be regarded as a potential system. The anomalous drift in systems with generalized binary disorder is consequently determined by other mechanisms than those in the Sinai case.

It is worth discussing the self-averaging property of the density of relaxation rates $\rho(\epsilon)$ (or its integral $N(\epsilon)$) and its Laplace transform $\langle \pi_k(t) \rangle$ in more detail at this stage. Self-averaging means that $\rho(\epsilon)$, and equivalently $\langle \pi_k(t) \rangle$, are non-random quantities in the limit of infinite system size. In other words, in the limit of infinite system size, almost all disorder realisations yield the same result for a self-averaging quantity, its typical value. Practically, this means that the fluctuations of the considered quantity decrease with the system size. So, in principle, there is no need to perform explicit disorder averages as was done in figure 4(b). Of course, averaging helps in reducing the fluctuations, which are still present for finite system sizes. For the Sinai system,
the deterministic nature of $\rho(\epsilon)$ follows rigorously from the well-known self-averaging property of the density of states for random Schrödinger operators [35], to which the Sinai problem can be mapped exactly [26]. For the more general systems considered in this work, such a mapping no longer exists (due to the lack of detailed balance), but the one-dimensional nature of our system, together with the short-range interactions (next-nearest neighbours) and the decay of correlations in the environment, allows for general arguments for self-averaging based on the resulting form of the transition matrix [35]. The general basis for self-averaging is spatial homogeneity in the mean of the random system and, in consequence, the existence of spatial ergodicity. A second requirement is the decay of correlations of the parameters defining the environment. Since our systems are constructed from i.i.d. random variables, these conditions are trivially fulfilled. Self-averaging quantities are then all extensive quantities (after normalization by the system volume or length). An example for a non-self-averaging quantity is the value of an individual eigenvalue of a random Schrödinger operator, or in our case, of the transition matrix. Such quantities vary randomly with the system parameters such as the system length, and therefore may exhibit large (relative) fluctuations also for large system sizes. The mean first-passage time, or its inverse the escape rate, which is treated below, is such a non-self-averaging quantity because the latter corresponds to an individual eigenvalue of the transition matrix.

5. Escape rates

The diffusive behaviour is linked with escape rates, too [21, 22]. Consider escape from a region of size $L$, which is characterized by a length-dependent escape rate $\gamma(L)$. For diffusive systems, one finds the connection between $\gamma(L)$ and the diffusion constant $D$ as $\gamma(L) \sim D(\chi/L)^2$, or $D \sim \gamma(L)(L^2/\chi^2)$ for $L \to \infty$ ($\chi$ is a geometric factor). For the diffusion constant to vanish, as seen for our disordered systems in section 3, $\gamma(L)$ has to decrease faster than $L^{-2}$.

For further investigations, we use again the equivalent description of our dynamical systems in terms of Markov chains. We consider the submatrix $Q = \{p_{ik}\}_{i,k=1}^{L} \in \mathbb{R}^{L \times L}$ of an, in-principle infinitely large system with transition probabilities $(P)_{ik} = p_{ik}$ and a reflecting boundary at $i = 1$. $Q$ is substochastic, i.e. $\sum_k Q_{ik} < 1$ holds for some $i$, namely for $i = L$. Consequently, the largest eigenvalue of $Q$, respectively its real part, is also less than 1. A distribution $\pi(t)$ in the segment $i = 1, \ldots, L$ then decays as $\pi(t) = \pi(0)Q^t$, which results in $\pi(t) \sim e^{-\gamma(L)t}$ for asymptotically long times. With the spectral decomposition of $Q^t = B^{-1}D^tB$, it becomes evident that the escape rates are determined by the maximum eigenvalue $\lambda^{(1)}$ of $Q$, i.e. $\gamma(L) = -\log(\lambda^{(1)}(L))$. In the disordered case, for a given disorder realization the escape rate $\gamma(L)$ varies irregularly with $L$. For a typical disorder realization, one expects for the Sinai case a large $L$ behaviour of the form

$$\gamma(L) \sim \exp(-bL^\beta)$$

(13)

with $\beta = \frac{1}{2}$. This matches well with the picture of potential barriers growing with $\sqrt{L}$ in disordered environments and with analytical results of [38] for the length dependence of the typical mean first-passage time $\tau(L)$, which corresponds to the inverse of $\gamma(L)$. This behaviour should be contrasted with the ordered case with $\gamma(L) \sim L^{-2}$, figure 5(b). The behaviour $\gamma(L) \sim \exp(-bL^{1/2})$ implies, for the Sinai case, that the relation

$$\alpha = 1/\beta = \delta$$

(14)

is valid. Of course, this poses the question whether equation (14) is true also for our more general next-nearest-neighbour maps of section 2, where the potential picture breaks down. In order to
Figure 5. (a) The log–log plot of $-\log \gamma(L)$ versus $L$ for 50 disorder realizations each with $p = \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}$ and a maximum system size $L = 1100$ confirms the law (16). Only points with $L > 30$ are taken into account for the fit $-\log \gamma(L) \sim L^{\beta}$. (b) In the ordered situation $p = \{1/4, 1/4, 0, 1/4, 1/4\}$, the escape rate behaves as $\gamma(L) \sim L^{-2}$.

check this, we have to extract the typical escape rate $\gamma(L)$. It is tempting to take the disorder average $\overline{\gamma(L)}$ and to check whether a law of the form (13) holds also for $\overline{\gamma(L)}$. We found already in [26] that such a law appears to be valid also in systems with generalized binary disorder. It turned out, however, that the exponent $\tilde{\beta}$ defined by

$$\log \overline{\gamma(L)} \sim L^{\tilde{\beta}},$$

which can be determined from log–log plots of $\log \overline{\gamma(L)}$, differs from $1/\delta$ and thus violates relation (14). This violation is confirmed also in the more extensive investigations of this paper (see section 6). A plausible explanation is that the disorder average $\overline{\gamma(L)}$ is dominated by untypical rare configurations of the environment and not by the typical ones. An analogous discrepancy was found in the Sinai model for the length dependence of disorder-averaged mean first-passage times versus typical mean first-passage time [38]. The techniques for calculating, e.g., the typical free energy for disordered systems suggest that, instead of $\log \gamma(L)$, one should better consider the disorder average $\overline{\log \gamma(L)}$ in order to capture the typical escape rates. In order to illustrate this procedure, consider a random, length-dependent variable $X(L)$. Its disorder average $\overline{X(L)}$ is assumed to originate from two kinds of events $X_1(L)$ and $X_2(L)$, each occurring with probability $p_1(L)$ and $p_2(L)$, respectively. Assume now that $X_1(L) = O(1)$, $X_2(L) = O(L^2)$, $p_1(L) = 1 - O(1/\sqrt{L})$ and $p_2(L) = O(1/\sqrt{L})$. In this example, $\overline{X(L)}$ increases with $L^{3/2}$ and therefore is dominated by events, whose probability vanishes in the limit $L \to \infty$, i.e. $\overline{X(L)}$ is dominated by atypical events. In contrast, the averaged logarithm is given by $\log \overline{X(L)} = (1 - O(1/\sqrt{L})) \log O(1) + O(1/\sqrt{L}) \log O(L^2) = O(1)$ for large $L$, and therefore is dominated by the typical events. This is, by the way, exactly the scenario which is supposed to hold for $X(L)$ being the variance $\sigma(L)$ of the equilibrium distribution for the Sinai case [39]. A similar example was given in [40] for the Sinai model to illustrate the difference between typical and disorder-averaged mean first-passage times. Actually, with the results of [38] for the Sinai model, one can verify easily that averaging the logarithm leads to the correct typical result. The
Figure 6. The exponent $\delta$ of the integrated density of states $N(\epsilon)$, the inverse of the exponent $\beta$ of the escape rate $\gamma(L)$ and the exponent $\alpha$ of the drift $\langle x^2(t) \rangle$ are shown to depend on $\pi_{ii} + 2$ for two different sets of transition probabilities’ (a) $p_2 = \{1/4, p_0(\lambda_2), 1/4, \lambda_2\}$ and (b) $p_3 = \{\lambda_3, 5/12, p_0(\lambda_3), 1/12, \lambda_3\}$.

same reasoning can be applied to the escape rate, which means that we determine the exponent $\beta$ via

$$\log \gamma(L) \sim L^\beta.$$  \hspace{1cm} (16)

Therefore we averaged $-\log \gamma(L)$ over several disorder realizations. The procedure for determining the escape rates is shown in figure 5(a). In a double-logarithmic plot, $-\log \gamma(L)$, with $\gamma(L) = (-\log(\lambda^{(1)}(L)))$, is depicted as a function of $L$. Thus $\beta$ is obtained from the slope of the straight regression line, which can be placed through the points for large $L$ values.

6. Summary of the results and discussion

In this section, we present results for the parameter dependence of the exponents introduced in sections 3–5. The most general model considered here is the system with generalized binary disorder of figure 1 in section 2. This model contains five parameters $\{p_{--}, p_{-}, p_0, p_+, p_{++}\}$, which by normalization reduces to four independent ones. The parameter space is a four-simplex. We varied parameters in small steps $\Delta \lambda_i$ along three lines in parameter space $p_1(\lambda_1) = \{1/6, 1/6, p_0(\lambda_1), 1/6, \lambda_1\}$ with $0 \leq \lambda_1 \leq 1/2$, $p_2(\lambda_2) = \{1/4, 1/4, p_0(\lambda_2), 1/4, \lambda_2\}$ and $p_3(\lambda_3) = \{\lambda_3, 5/12, p_0(\lambda_3), 1/12, \lambda_3\}$ with $0 \leq \lambda_2, \lambda_3 \leq 1/4$. The entry $p_0(\lambda_i)$ is always determined by normalization. The exponents obtained by the procedures described in the previous sections are plotted as functions of $\lambda_i$ in figures 6 and 7. The set $p_1(\lambda_1)$ contains, for $\lambda_1 = 1/3$, a case considered previously in [26]; the case of topological disorder of figure 2(b) corresponds to the point $p_2(\lambda_2 = 0)$; and the case of Sinai disorder corresponds to $p_3(\lambda_3 = 0)$. The exponents $\beta$, respectively $\tilde{\beta}$, and $\delta$ (equations (16), (15) and (12), respectively) were determined for each value of $\lambda$ by averaging over 50 disorder realizations. In the determination of $\beta$ and $\tilde{\beta}$, the system size is varied up to $L = 1100$ cells and, for determining $\delta$, a size of $L = 2000$ was chosen. The assignment of $\alpha$ followed from evaluating $\langle x^2(t) \rangle$ according to equation (6) from $10^4$ random walkers over a time period of $10^6$ time steps and by averaging over 500 disorder realizations.
Figure 7. Both figures are made of the same results for the parameter set \( p_1 = \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \lambda_1\} \). The asymptotic exponent \( \delta \) of the integrated density of states \( N(\epsilon) \) is depicted as a function of \( \lambda_1 \) in (a), with the asymptotic exponent \( \alpha \) of \( \langle x(t)^2 \rangle \) and the inverse of asymptotic exponent \( \beta \) of \( -\log \gamma(L) \). In (b), it is depicted with \( 1/\tilde{\beta} \) (determined by \( -\log \gamma(L) \)) and \( \tilde{\alpha} \) (determined by drift \( \langle x_i^2 \rangle \)).

Let us first consider the path \( p_2(\lambda_2) \). The results are shown in figure 6(a). The ordered case with symmetric transition probabilities lies at \( p_2(\lambda_2 = \frac{1}{4}) \), i.e. at the right edge of the figure. Here, the assumed dependences \( \langle x(t)^2 \rangle \sim \log^2 t \), \( \log \gamma(L) \sim L^\beta \) and \( N(\epsilon) \sim \log |\epsilon|^{-\delta} \) are no longer valid because the behaviour is diffusive in this limit. Instead, one has \( \langle x(t)^2 \rangle = \sigma^2(t) \sim t \), \( \log \gamma(L) \sim \log L \) and \( N(\epsilon) \sim \epsilon^{1/2} \). Thus \( \alpha \), \( 1/\beta \) and \( \delta \) are no longer well defined; at best one can assign to them the value infinity. It is clear that, also in the neighbourhood of this point, the determination of \( \alpha \), \( 1/\beta \) and \( \delta \) is numerically difficult and unreliable because one needs unaccessible large systems and/or simulation times. This explains the considerable discrepancy between the values for \( \alpha \), \( 1/\beta \) and \( \delta \) in this parameter regime. This implies that it is impossible to obtain by these methods the behaviour for weak disorder. In the regime below \( \lambda_2 \simeq 0.14 \), the results for \( 1/\beta \) and \( \delta \) coincide within numerical accuracy and can therefore be regarded as reliable. The values of \( \alpha \) follow \( 1/\beta \) and \( \delta \) in tendency but still lie systematically above those of \( 1/\beta \) or \( \delta \). We found that this discrepancy decreases with increasing size of the disorder ensemble, but obviously the ensemble is still too small. We remark that, at the same time, determination of \( \alpha \) is the most time-consuming among these exponents.

In figure 6(b), the behaviour of the exponents along the path \( p_3(\lambda_3) \) is plotted. This dataset includes for \( \lambda_3 = 0 \) the Sinai case, where we know that \( \alpha = 1/\beta = \delta = 2 \) is true. In the neighbourhood of this point, we find a clear increase of all exponents and the coincidence of the values for \( \alpha \), \( 1/\beta \) and \( \delta \) is given. With further increase of \( \lambda_3 \), the determination of \( \alpha \) apparently becomes unreliable again, whereas the other two exponents remain identical within numerical accuracy over the whole \( \lambda_3 \)-range. This makes sense since along this path there exists no ordered case, where the exponents are expected to diverge. So we conclude that the true values of \( 1/\beta \) and \( \delta \) are well approximated in the whole parameter regime and that, with improved calculations for \( \alpha \) also, this exponent would coincide with \( 1/\beta \) and \( \delta \).

The behaviour along the path \( p_4(\lambda_3) \) is shown in figure 7. Similar to figure 6(a), this path encounters a symmetric situation, which now lies at \( \lambda_3 = \frac{1}{6} \), i.e. in the interior of the \( \lambda_3 \)-interval,
The fact that unlike $1$ for $\alpha$ seem to fulfil the scaling relation $\alpha = \tilde{\alpha}$ should hold. For the Sinai case, the difference $\sigma^2(t) = \langle x(t)^2 \rangle - \langle x(t) \rangle^2$ is known to be only of the order $\log^3 t$ for $t \to \infty$ [36, 41, 42]. Since $\langle x(t)^2 \rangle$ increases asymptotically with $\log^4 t$ in the Sinai case, the leading divergence of $\langle x(t)^2 \rangle$ has to cancel the former, i.e. it has to be also of the order $\log^3 t$ and therefore $\tilde{\alpha} = \tilde{\alpha} = 2$. This is to be expected also in view of the Golosov phenomenon, and since the latter seems to hold also for the generalizations considered here, we expect $\alpha = \tilde{\alpha}$ in the whole parameter regime. This is roughly fulfilled for most $\lambda_3$ values except at the point $\lambda_3 = \frac{1}{6}$, where $\alpha$ and $\tilde{\alpha}$ are not well defined anyway. The observed rough coincidence of $\tilde{\alpha}$ (and therefore also of $\alpha$) with $1/\tilde{\beta}$ indicates that $\alpha$ and $\tilde{\alpha}$ are also sensitive to untypical disorder configurations and that large-disorder samples are needed to get the correct behaviour.

In summary, we have shown that disordered iterated maps can show dynamical localization and an anomalously slow drift behaviour, which generalizes the results known for Sinai disorder. The drift behaviour is captured in the disorder-averaged mean-square displacement, the density of states of the propagator, and the size dependence of the escape rate. The corresponding laws are of the form $\langle x(t)^2 \rangle \sim \log^{2\alpha} t$, $N(\epsilon) \sim |\log \epsilon|^{-\delta}$ and $\log \gamma(L) \sim L^\beta$, respectively, where $\alpha$, $\beta$ and $\delta$ depend on the system parameters. Nevertheless, for a given system parameter, the exponents seem to fulfil the scaling relation $\alpha = 1/\beta = \delta$. This is to be expected in view of the dynamical localization (Golosov phenomenon) which characterizes the evolution of ensembles of states. The numerical values of $\alpha$, $1/\beta$ and $\delta$ were reliably estimated up to values twice as large as those in the Sinai case.

The deviations from the Sinai case indicate that a new aspect enters the mechanisms for transport. We attribute these deviations to the generic absence of detailed balance in our systems, implying that, in contrast to potential systems, there exist probability currents also in equilibrium. To support this view, we also considered cases where one has transitions to next-nearest neighbours with detailed balance being fulfilled. We first remark that this situation cannot be achieved within the class of systems defined in section 2. To see this, observe that the uniqueness of the equilibrium distribution $\{\pi_i\}$ and the conditions for detailed balance $\pi_i \pi_{i+1} = \pi_{i+1} \pi_{i+1}$ and $\pi_i \pi_{i+2} = \pi_{i+2} \pi_{i+3}$ imply the following constraint for the transition probabilities $p_{i,i+2}/p_{i+2,i} = (p_{i,i+1} p_{i+1,i+2})/(p_{i+2,i+1} p_{i+1,i+1})$ (see also [43]). For the parameters of our systems with generalized binary disorder, it is easy to see that the latter relation implies $p_- = p_+$. Thus the binary disorder degenerates to an ordered situation with symmetric transition probabilities, where in consequence one observes normal diffusion. One way to implement detailed balance for systems with next-nearest neighbour transitions lies in defining a potential $V_i$ on the sites $i$ and to assign transition probabilities analogous to the rules for Glauber dynamics [44] $p_{i,i+\pm 1} = (1 + \exp \left[ \pm b(V_{i+1} - V_i) \right])^{-1}$ and

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\( p_{i,i+2} = \{1 + \exp[ \pm b(V_{i+2} - V_i)]\}^{-1} \) with appropriate modifications at the system boundaries. It is easy to check that detailed balance is fulfilled with \( \pi_i \sim \exp(-bV_i) \), the Gibbs distribution. We defined the \( V_i \) as the sum of i.i.d. binary random variables \( r_i \), i.e. \( V_i = \sum_{k=0}^{i} r_i \). Thus \( V_i, i = 0, 1, 2, \ldots \), corresponds to a discrete random walk as in the case of the Sinai model (see section 3). Note, however, that in this system the \( \{p_{i,i+2}\} \) for neighbouring sites \( i \) are no longer uncorrelated. Our numerical results obtained for this system clearly show that the exponents \( \alpha, \beta \) and \( \delta \) are those of the Sinai class. Obviously, the lack of detailed balance in our systems with generalized binary disorder is the reason for observing new parameter-dependent exponents.

As a final remark, we note that from a numerical point of view the determination of the exponents \( \alpha, \beta \) and \( \delta \) is quite different, although they capture the same physical phenomenon. For instance, \( \delta \) is determined from a self-averaging quantity and therefore one is not plagued with increasing fluctuations due to the underlying distribution as in the case \( \alpha \) and \( \beta \). But also the determination of \( \delta \) has its limitations, since diagonalization of large matrices is needed with increasing accuracy as the system size is increased. This work provides also a perspective for further generalization, where dynamical systems will be considered which are not piecewise linear and do not allow for Markov partitions.

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