Evolution of Structure Functions with Jacobi Polynomial: Convergence and Reliability

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Abstract

The Jacobi polynomial has been advocated by many authors as a useful tool to evolve non-singlet structure functions to higher $Q^2$. In this work, it is found that the convergence of the polynomial sum is not absolute, as there is always a small fluctuation present. Moreover, the convergence breaks down completely for large $N$.

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The structure functions are the necessary tool in our effort to understand the hadronic structure and strong interaction. The study of $Q^2$ evolution of nucleon structure functions has been an important source of experimental information supporting Quantum Chromodynamics (QCD), which is believed to be the fundamental theory of strong interaction. It has already been shown some time back that QCD is the only theory which can explain the gross features of scaling violations in deep inelastic scattering (DIS). As a result a huge amount of effort is being put, both experimentally as well as theoretically, to understand the nucleon structure functions for different values of $x$ and $Q^2$.

The evolution of quark distribution with $Q^2$ is governed by the Altarelli-Parisi (AP) equation \cite{AP}. To leading order in $\alpha_s$, the AP equation is given by,

$$ \frac{dq(x, Q^2)}{dt} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dy}{y} q(y, Q^2) P_{qq}(\frac{x}{y}) $$ \hspace{1cm} (1)

where $q$ is the quark distribution, $\alpha_s$ is the strong coupling, $t \equiv log Q^2$ and $P_{qq}$ is the quark splitting function which represents the probability of a quark emitting a gluon and so...
becoming a quark with momentum reduced by a fraction \( \frac{x}{y} \). Eq. (1), an integro-differential equation, is not very easy to solve; different methods have been proposed in the literature which can be grouped into three main categories.

One method, already used by Altarelli, Nason and Rudolfi \([3]\) is based on the assumption that, for a small variation of \( t \), one can neglect the \( t \) dependence of the r.h.s in eq. (1) and realize, in steps, the evolution of quark distribution for a given \( \delta t \).

The second method \([4]\) consists of expanding the quark (parton) distribution functions into a truncated series of Chebyshev polynomial. This expanded form is substituted in eq. (1) and then the resulting coupled differential equations are solved self-consistently.

The third method relies on the premise that the moments of the structure function (say \( F_2 \)) depend only on \( Q^2 \). This means that one can expand \( F_2 \) in terms of an orthonormal polynomial (usually Jacobi polynomial) such that the \( x \) dependence is carried by the polynomial whereas the full \( Q^2 \) dependence is confined to the weight factors. The variation of these weight factors with \( t \) can then be extracted from the knowledge of the variation of moments with \( t \). In the present report we confine our attention to the use of Jacobi polynomial in solving the evolution equation of quark distribution functions.

The Jacobi polynomial was first used by Sourlas and Parisi \([5]\) and later on elaborated by Barker et al. \([6]\) and also Chyla et al. \([7]\). We will follow the prescription of ref. \([6]\), though we have found that, both \([5]\) and \([6]\) give similar results. For illustrative purposes, we restrict ourselves to the valence part of \( F_2(x, Q^2) \).

The method of orthogonal polynomials (here Jacobi Polynomial) is based on inverting moments with the help of orthogonal polynomials. The Jacobi polynomial is defined \([6]\) as

\[
\Theta_k^{\alpha,\beta}(x) = \sum_{j=0}^{k} C_{k,j}(\alpha, \beta)x^j
\]  

(2)

satisfying a weighted orthogonality relation,

\[
\int_0^1 dx x^\beta(1-x)^\alpha \Theta_k^{\alpha,\beta}(x) \Theta_l^{\alpha,\beta}(x) = \delta_{kl}
\]  

(3)

Now the structure function \( F_2 \) can be expanded as

\[
F_2(x, Q^2) = x^\beta(1-x)^\alpha \sum_{k=0}^{\infty} a_k^{\alpha,\beta}(Q^2) \Theta_k^{\alpha,\beta}(x)
\]  

(4)
From eq. (4) one can write the expansion coefficients \( a \) in terms of \( F_2 \) as

\[
a_k^{\alpha\beta}(Q^2) = \int_0^1 F_2(x, Q^2) \Theta_k^{\alpha\beta}(x) dx
\]

(5)

where the orthogonality relation of \( \Theta_k^{\alpha\beta}(x) \) (eq. (3)) has been made use of.

Substituting eq. (2) in the eq. (5) one gets,

\[
a_k^{\alpha\beta}(Q^2) = \sum_{j=0}^k C_{kj}(\alpha, \beta) \mu(j + 2, Q^2)
\]

(6)

where the moments \( \mu \) are given by,

\[
\mu(j, Q^2) = \int_0^1 dx x^{j-2} F_2(x, Q^2)
\]

(7)

We have used the general form of Jacobi polynomial \([6, 7]\),

\[
\Theta_k^{\alpha\beta}(x) = N_k^{\alpha\beta} H_k^{\alpha\beta}(x)
\]

(8)

where the normalization factor is

\[
N_k^{\alpha\beta} = \Theta_k^{\alpha\beta}(0) = (\beta + 1)_k \frac{(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta)}{\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)k!} \frac{1}{2}
\]

(9)

with \((a)_n \equiv a(a+1) \cdots (a+n-1)\) and \(a_0 = 1\). The expansion coefficients \( C \) are then,

\[
C_{kj}(\alpha, \beta) = (-1)^j \binom{k}{j} N_k^{\alpha\beta} \frac{(k + \alpha + \beta + 1)_j}{(\beta + 1)_j}
\]

(10)

The Jacobi Polynomial can be evaluated using eq. (2) or the recurrence relations for the polynomial \( H_k^{\alpha\beta} \) \([3, 4]\).

In the following discussion we will study the reliability and convergence of Jacobi polynomial method for the evolution of structure functions. This is done firstly by starting from an analytical fitted formula for valence part of \( F_2 \) at a particular \( Q^2 \) and then reevaluating the \( F_2 \) at the same \( Q^2 \) using the Jacobi polynomials. We have used three different formulae at \( Q^2 = 3.5, 5 \) and 15 GeV\(^2\) as given below \([8, 9, 10]\),

\[
Q^2_0 = 3.5 \text{ GeV}^2
\]

\[
u(x, Q^2_0) = \sqrt{x}(1 - x^2)^3(0.594 + 0.461(1 - x^2) + 0.621(1 - x^2)^2)
\]

\[
d(x, Q^2_0) = \sqrt{x}(1 - x^2)^3(0.072 + 0.206(1 - x^2) + 0.621(1 - x^2)^2)
\]

(11)
\( Q_0^2 = 5 \text{ GeV}^2 \)

\[
\begin{align*}
    xu_v(x, Q_0^2) &= 1.78 \sqrt{x(1 - x^{1.51})^{3.5}} \\
    xd_v(x, Q_0^2) &= 0.67x^{0.4}(1 - x^{1.51})^{4.5}
\end{align*}
\] (12)

\( Q_0^2 = 15 \text{ GeV}^2 \)

\[
\begin{align*}
    xu_v(x, Q_0^2) &= \frac{2}{B(\alpha_u, \beta_u + 1)} x^{\alpha_u} (1 - x)^{\beta_u} \\
    xd_v(x, Q_0^2) &= \frac{1}{B(\alpha_d, \beta_d + 1)} x^{\alpha_d} (1 - x)^{\beta_d}
\end{align*}
\] (13)

where \( B(\alpha, \beta) \) are the Euler beta functions, \( \alpha_u = 0.588 \pm 0.020 \pm 0.05 \), \( \beta_u = 2.69 \pm 0.13 \pm 0.21 \), \( \alpha_d = 1.03 \pm 0.10 \pm 0.19 \) and \( \beta_d = 6.87 \pm 0.64 \pm 0.80 \) [9].

The variation of \( R \equiv \frac{F_{p}^{p}(\text{calculated})}{F_{2}^{p}(\text{formula})} \) with the \( N \) (number of terms of the Jacobi polynomial) for \( Q^2 = 15, 5 \) and 3.5 GeV\(^2\) is shown in figure 1. For each \( Q^2 \) curves for \( x = 0.05, 0.4 \) and \( 0.75 \) are plotted. We find that for \( Q^2 = 3.5 \text{ GeV}^2 \), there are large fluctuations in the \( R \) for smaller values of \( N \). The value of \( R \) becomes 1 around \( N = 5 \) and stays the same up to a value of \( N \) around 20. But immediately after that the value of \( R \) diverges. For larger values of \( x \), \( R \) diverges for larger \( N \). Similar features are present for higher values of \( Q^2 \) along with the additional feature that there are small oscillations in \( R \) around 1 till it diverges for \( N > 20 \). This behaviour of \( R \) shows that though there is an apparent convergence of \( F_2 \) with \( N \), this convergence may not be as conclusive as claimed by earlier authors [5, 6, 7].

The unreliability of the present method of evolution of structure functions is illustrated in figure 2. Here we have plotted the evolved value of only the valence part of \( F_{p}^{p} \) from \( Q_0^2 = 3.5 \text{ GeV}^2 \) to \( Q^2 = 5 \) and 15 GeV\(^2\), for \( x = 0.05, 0.4 \) and 0.75. Here we find that \( F_{p}^{p} \) diverges for \( N \geq 12 \) which is much lower than the one observed \( (N \geq 20) \) in figure 1. Here again we find that the \( F_{p}^{p} \) diverges earlier for lower values of \( x \). The implication of these observations is that even the apparent convergence of \( R \) does not say anything conclusive regarding the number of terms needed in the Jacobi Polynomial.

In figures 3 and 4, we have plotted the evolved value of \( \Delta F = F_{p}^{p} - F_{n}^{p} \) starting from \( Q_0^2 = 3.5 \) and 5 GeV\(^2\) respectively. The figures show that the \( \Delta F \) gives same values for \( N \leq 14 \). For \( N = 17 \) there is a large oscillation in the value of \( \Delta F \). The oscillations are
larger for lower values of $x$ and they decrease with increase in $x$. Furthermore, the evolved $\Delta F$ at $Q^2 = 7, 12$ and $15 \text{ GeV}^2$ have similar values. The above observations are valid for both $Q_0^2 = 3.5$ as well as $Q_0^2 = 5 \text{ GeV}^2$; fluctuations being less for $Q_0^2 = 5 \text{ GeV}^2$.

In conclusion we have studied the reliability and convergence of the method of Jacobi polynomials for the evolution of structure functions. We find that the convergence of this method is not wholly reliable in the sense that for large $N$, the valence part of $F_2$ diverges and there are large fluctuations in $F_2^p - F_2^n$. Furthermore, the final results at higher $Q^2$ are strongly dependent on the initial fitted formula used, even for the value of $N$ where fluctuations are not problematic. This is exemplified by the fact that even for $N =10$, the final value of $F_2^p - F_2^n$ at $Q^2 = 15 \text{ GeV}^2$ is strongly dependent on whether one starts from $Q^2 = 3.5 \text{ GeV}^2$ or $5 \text{ GeV}^2$ (figs. 3(c) and 4(c)). Thus, the method of Jacobi Polynomials, its simplicity and apparent success notwithstanding, is seen to be of limited validity in evaluating the $Q^2$ evolution of structure functions. It may be interesting and worthwhile to explore if there are any other polynomials which could perform better in this respect.

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FIGURE CAPTIONS

Fig. 1. The ratio of the calculated value of the valence part of $F_2^p$ and the corresponding value from the fitted formula (eq.(11-13)) , starting from a given value of $Q^2$ and reconstructing it back at the same $Q^2$ using the Jacobi polynomial. Three curves in each plot correspond to $x=0.05$ (straight line), $x=0.4$ (long dash) and $x=0.75$ (short dash), respectively.

Fig. 2. Evolved value of only the valence part of $F_2^p$ (a) from 3.5 GeV$^2$ to 5 GeV$^2$ and (b) from 3.5 GeV$^2$ to 15 GeV$^2$. 1, 2 and 3 in (a) and (b) correspond to $x=0.05$, 0.4 and 0.75 respectively.

Fig. 3. The evolved value of $F_2^p - F_2^n$ from $Q_0^2 = 3.5$ GeV$^2$ to $Q^2 = (a)$ 7 GeV$^2$, (b) 12 GeV$^2$ and (c) 15 GeV$^2$. The evolution for $N=6$, 10 and 14 give the same value (solid line) whereas $N=17$ (dotted line) shows fluctuations for all the cases.

Fig. 4. The evolved value of $F_2^p - F_2^n$ from $Q_0^2 = 5$ GeV$^2$ to $Q^2 = (a)$ 7 GeV$^2$, (b) 12 GeV$^2$ and (c) 15 GeV$^2$. The evolution for $N=6$, 10 and 14 give the same value (solid line) whereas $N=17$ (dotted line) shows fluctuations for all the cases.
