ON CERTAIN $q$-TRIGONOMETRIC IDENTITIES OF GOSPER

BING HE AND RUIMING ZHANG

Abstract. In this paper we establish certain Jacobi theta function identities, from which two $q$-trigonometric identities of Gosper are deduced. These identities include as special cases some interesting results. As an application, we also give a new proof of Jacobi’s four squares theorem.

1. Introduction

Throughout this paper we take $q = \exp(i\pi \tau)$, where $\text{Im} \tau > 0$. We also use $\Omega$ to denote the period lattice $\{m\pi + n\pi \tau | (m, n) \in \mathbb{Z}^2\}$. To carry out our study, we need the definition of the Jacobi theta functions.

Definition 1.1. The Jacobi theta functions $\theta_j(z|\tau)$ for $j = 1, 2, 3, 4$ are defined as

\[
\theta_1(z|\tau) = -iq^z \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k+1)} e^{(2k+1)zi}, \quad \theta_2(z|\tau) = q^z \sum_{k=-\infty}^{\infty} q^{k(k+1)} e^{(2k+1)zi},
\]

\[
\theta_3(z|\tau) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2kzi}, \quad \theta_4(z|\tau) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2kzi}.
\]

The familiar notation are used frequently in the sequel:

\[(a; q) = \prod_{n=0}^{\infty} (1 - aq^n).
\]

With this notation, the well-known Jacobi triple product identity can be written as (see [21] (2.1) and (1.6.1) and [10])

\[
\sum_{n=-\infty}^{\infty} (-1)^n q^n (n+1/2)^n = (q; q) (z; q) (q/z; q), \quad z \neq 0.
\]

Using the Jacobi triple product identity we can deduce the Jacobi infinite product expressions for theta functions:

\[
\theta_1(z|\tau) = 2q^z \sin (q^2; q^2) (q^2; q^2) (q^2 e^{2zi}; q^2) (q^2 e^{-2zi}; q^2),
\]

\[
\theta_2(z|\tau) = 2q^z \cos (q^2; q^2) (-q^2 e^{2zi}; q^2) (-q^2 e^{-2zi}; q^2),
\]

\[
\theta_3(z|\tau) = (q^2; q^2) (-qe^{2zi}; q^2) (-qe^{-2zi}; q^2),
\]

\[
\theta_4(z|\tau) = (q^2; q^2) (qe^{2zi}; q^2) (qe^{-2zi}; q^2).
\]

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The first author is the corresponding author.
With respect to the (quasi) periods $\pi$ and $\pi \tau$, we have
\begin{equation}
\theta_1(z + \pi | \tau) = -\theta_1(z | \tau), \quad \theta_1(z + \pi \tau | \tau) = -q^{-1} e^{-2iz} \theta_1(z | \tau).
\end{equation}

We also have the following relation:
\begin{equation}
\theta_1 \left( z + \frac{\pi \tau}{2} | \tau \right) = i q^{-1/4} e^{-iz} \theta_4(z | \tau).
\end{equation}

Gosper \cite{gosper} introduced $q$-analogues of $\sin_q z$ and $\cos_q z$ which are defined as
\begin{align*}
\sin_q(\pi z) &:= \frac{(q^{2-2z}; q^2)_\infty (q^{2z}; q^2)_\infty}{(q; q^2)^2_\infty} q^{(z-1/2)^2}, \\
\cos_q(\pi z) &:= \frac{(q^{1-2z}; q^2)_\infty (q^{1+2z}; q^2)_\infty}{(q; q^2)^2_\infty} q^z.
\end{align*}

From above we can see that $\cos_q z = \sin_q(\pi/2 - z)$, $\lim_{q \rightarrow 1} \sin_q z = \sin z$ and $\lim_{q \rightarrow 1} \cos_q z = \cos z$. Two relations between $\sin_q$, $\cos_q$ and the functions $\theta_1$ and $\theta_2$ are equivalent to \cite{mezo}:
\begin{align*}
\sin_q z &= \frac{\theta_1(z | \tau')}{\vartheta_2(\tau')}, \\
\cos_q z &= \frac{\theta_2(z | \tau')}{\vartheta_2(\tau')},
\end{align*}

where $\tau' = -\frac{1}{z}$.

In the same paper, Gosper listed without proofs many identities involving $\sin_q z$ and $\cos_q z$. These identities were conjectured by him using a computer program called MACSYMA. Using the method of logarithmic differentiation, Mező in \cite{mezo} confirmed the conjecture ($q$-Double$_2$):
\begin{equation}
\sin_q(2z) = \frac{\Pi_q}{\Pi_{q^2}} \sin_{q^2} z \cos_{q^2} z,
\end{equation}

where
\begin{equation}
\Pi_q := q^{1/4} \frac{(q^2; q^2)_\infty}{(q; q^2)^2_\infty}.
\end{equation}

Applying the theory of elliptic functions, M. El Bachraoui \cite{bachraoui} gave proofs of the conjecture ($q$-Double$_2$) and the conjecture ($q$-Double$_3$):
\begin{equation}
\cos_q(2z) = (\cos_{q^2} z)^2 - (\sin_{q^2} z)^2.
\end{equation}

In \cite{abotouk} S. Abo Touk, Z. Al Houchan and M. El Bachraoui proved the $q$-trigonometric identities:
\begin{align*}
\frac{\sin_{q^4} x}{\sin_q x} (\cos_q y)^2 + \frac{\cos_{q^4} x}{\cos_q x} (\sin_q y)^2 &= \frac{\sin_{q^4} y}{\sin_q y} (\cos_q x)^2 + \frac{\cos_{q^4} y}{\cos_q y} (\sin_q x)^2, \\
\frac{\sin_{q^2} x}{\sin_q x} (\cos_q y)^2 + \frac{\cos_{q^2} x}{\cos_q x} (\sin_q y)^2 &= \frac{\sin_{q^2} y}{\sin_q y} (\cos_q x)^2 + \frac{\cos_{q^2} y}{\cos_q y} (\sin_q x)^2.
\end{align*}

See \cite{abotouk} and \cite{liu} for proofs of some other $q$-trigonometric identities of Gosper and Liu \cite{liu1} \cite{liu2} \cite{liu3} \cite{liu4}, Shen \cite{shen1} \cite{shen2} and Whittaker, Watson \cite{whittaker} for more information dealing with formulas of theta functions using the theory of elliptic functions.
In [8] p. 100] Gosper stated without proofs the following \( q \)-trigonometric identities:

\[
\sin_q \left( \sum_{1 \leq j \leq n} c_j \right) = \prod_{1 \leq i \leq n} \frac{\sin_q a_i}{\sin_q (a_i - c_i)} \sum_{1 \leq i \leq n} \frac{\sin_q c_i}{\sin_q a_i} \left( a_i - \sum_{1 \leq j \leq n} c_j \right) \prod_{1 \leq j \leq n, j \neq i} \frac{\sin_q (c_j + a_i - a_j)}{\sin_q (a_i - a_j)}
\]

where

\[
\text{ssn}_q \frac{z}{\overline{z}} = \frac{\sin_q z}{\sin_q \overline{z}}.
\]

In this paper we shall prove the above \( q \)-trigonometric identities. Actually, we will establish the following theta function identities:

**Theorem 1.1.** Let \( a_1, a_2, \cdots, a_n, c_1, c_2, \cdots, c_n \) be \( 2n \) complex numbers such that \( a_i \not\equiv 0 (\mod \Omega) \) for \( 1 \leq i \leq n \) and \( a_j - a_k \not\equiv 0 (\mod \Omega) \) for \( j \neq k \). Then

\[
\theta_1 \left( \sum_{1 \leq j \leq n} c_j \bigg| \tau \right) \prod_{1 \leq i \leq n} \frac{\theta_1(a_i - c_i | \tau)}{\theta_1(a_i | \tau)} = \sum_{1 \leq i \leq n} \frac{\theta_1(c_i | \tau)}{\theta_1(a_i | \tau)} \theta_1 \left( a_i - \sum_{1 \leq j \leq n} c_j \bigg| \tau \right) \prod_{1 \leq j \leq n, j \neq i} \frac{\theta_1(c_j + a_i - a_j | \tau)}{\theta_1(a_i - a_j | \tau)}
\]

and

\[
\theta_1 \left( \sum_{1 \leq j \leq n} c_j \bigg| \tau \right) \prod_{1 \leq i \leq n} \frac{\theta_4(a_i - c_i | \tau)}{\theta_4(a_i | \tau)} = \sum_{1 \leq i \leq n} \frac{\theta_1(c_i | \tau)}{\theta_4(a_i | \tau)} \theta_4 \left( a_i - \sum_{1 \leq j \leq n} c_j \bigg| \tau \right) \prod_{1 \leq j \leq n, j \neq i} \frac{\theta_4(c_j + a_i - a_j | \tau)}{\theta_4(a_i - a_j | \tau)}.
\]

**Corollary 1.1.** The \( q \)-trigonometric identity (1.4) is true.

The \( q \)-trigonometric identities (1.4) has the following consequences.

Putting \( b_j = a_j - c_j \) and \( \sum_{1 \leq j \leq n} c_j = 0 \) in the first identity of (1.4) we can readily get

\[
\sum_{1 \leq i \leq n} \sin_q (a_i - b_i) \prod_{1 \leq j \leq n, j \neq i} \frac{\sin_q (a_i - b_j)}{\sin_q (a_i - a_j)} = 0.
\]

Actually, this identity is equivalent to [13 (1.12)]. If \( n = 2m \) and \( c_j = (-1)^j \pi/2 \), then the above identity reduces to

\[
\sum_{1 \leq i \leq 2m} \prod_{1 \leq j \leq 2m, j \neq i} \frac{\cos_q (a_i - a_j)}{\sin_q (a_i - a_j)} = 0.
\]
When $m = 2$, $a_1 = w$, $a_2 = x$, $a_3 = y$, $a_4 = z$, the above identity becomes

$$
\begin{align*}
\cos_q(z - x)\cos_q(x - w)\cos_q(x - y)\sin_q(y - w)\sin_q(y - z)\sin_q(z - w) \\
+ \sin_q(z - x)\sin_q(x - w)\cos_q(x - y)\cos_q(y - w)\cos_q(y - z)\sin_q(z - w) \\
+ \cos_q(z - x)\sin_q(x - w)\sin_q(y - w)\cos_q(y - z)\cos_q(z - w) \\
+ \sin_q(z - x)\cos_q(x - w)\sin_q(y - w)\sin_q(y - z)\cos_q(z - w) = 0.
\end{align*}
$$

Let $q \to 1$ in (1.7), (1.8) and (1.9). We have

$$
\begin{align*}
\sum_{1 \leq i \leq n} \sin(a_i - b_i) \prod_{1 \leq j \leq n \atop j \neq i} \sin(a_i - a_j) &= 0, \\
\sum_{1 \leq i \leq 2m} \prod_{1 \leq j \leq 2m \atop j \neq i} \cos(a_i - a_j) / \sin(a_i - a_j) &= 0
\end{align*}
$$

and

$$
\begin{align*}
\cos(z - x)\cos(x - w)\cos(x - y)\sin(y - w)\sin(y - z)\sin(z - w) \\
+ \sin(z - x)\sin(x - w)\cos(x - y)\cos(y - w)\cos(y - z)\sin(z - w) \\
+ \cos(z - x)\sin(x - w)\sin(y - w)\cos(y - z)\cos(z - w) \\
+ \sin(z - x)\cos(x - w)\sin(y - w)\sin(y - z)\cos(z - w) = 0.
\end{align*}
$$

We set $q \to 1$ in the second identity of (1.4) to get

$$
\sin\left( \sum_{1 \leq j \leq n} c_j \right) = \sum_{1 \leq i \leq n} \sin c_i \prod_{1 \leq j \leq n \atop j \neq i} \sin(c_j + a_i - a_j) / \sin(a_i - a_j).
$$

In order to prove Theorem 1.1 we need the following result.

**Theorem 1.2.** (See [15] Theorem 1.2) Let $z_1, z_2, \ldots, z_r$ be $r$ complex numbers such that for $j \neq k$, $z_j - z_k \not\equiv 0 \pmod{\Omega}$, and let $f(z)$ be an entire function of $z$ satisfying the functional equations

$$
\begin{align*}
f(z) &= (-1)^r f(z + \pi) = (-1)^r q^r \exp\left( 2irz - 2i \sum_{j=1}^r z_j \right) f(z + \pi r).
\end{align*}
$$

Then $f(z)$ and $\theta_1(z|\tau)$ satisfy the identity

$$
\begin{align*}
\sum_{i=1}^r \prod_{1 \leq j \leq r \atop j \neq i} \theta_1(z_i - z_j|\tau) = 0.
\end{align*}
$$

2. Proof of Theorem 1.1

**Proof of Theorem 1.1** We first prove (1.10). Set $r = n + 1$, $z_1 = a_1$, $z_2 = a_2, \ldots, z_n = a_n$ and $z_{n+1} = 0$ in Lemma 1.2. Let

$$
f(z) = \theta_1\left( z - \sum_{1 \leq j \leq n} c_j |\tau \right) \prod_{1 \leq j \leq n} \theta_1(z - a_j + c_j|\tau).
$$
Then \( f(z) \) is an entire function of \( z \). By the functional equations (1.1), we verify that \( f(z) \) satisfies the functional equations (1.10). It is easily seen that for \( 1 \leq i \leq n \),

\[
f(z_i) = f(a_i) = \theta_1 \left( a_i - \sum_{1 \leq j \leq n} c_j \middle| \tau \right) \prod_{1 \leq j \leq n} \theta_1(a_i - a_j + c_j | \tau)
\]

and

\[
\prod_{1 \leq j \leq n+1} \theta_1(z_i - z_j | \tau) = \theta_1(a_i | \tau) \prod_{1 \leq j \leq n} \theta_1(a_i - a_j | \tau).
\]

This means that

\[
\frac{f(z_i)}{\prod_{1 \leq j \leq n+1} \theta_1(z_i - z_j | \tau)} = \frac{\theta_1(c_i | \tau)}{\theta_1(a_i | \tau)} \theta_1 \left( a_i - \sum_{1 \leq j \leq n} c_j \middle| \tau \right) \prod_{1 \leq j \leq n} \frac{\theta_1(c_j + a_i - a_j | \tau)}{\theta_1(a_i - a_j | \tau)}
\]

for \( 1 \leq i \leq n \). We also know that

\[
\frac{f(z_{n+1}^\ast)}{\prod_{1 \leq j \leq n+1} \theta_1(z_{n+1} - z_j | \tau)} = \frac{f(0)}{\prod_{1 \leq j \leq n} \theta_1(-a_j | \tau)}
\]

\[
= -\theta_1 \left( \sum_{1 \leq j \leq n} c_j \middle| \tau \right) \prod_{1 \leq i \leq n} \frac{\theta_1(a_i - c_i | \tau)}{\theta_1(a_i | \tau)}.
\]

Substituting (2.1) and (2.2) into (1.10) we can easily obtain (1.11).

The identity (1.0) follows readily by replacing \( a_i \) by \( a_i + \frac{\tau}{n} \) in (1.5) and using (1.2). This completes the proof of Theorem 1.1. □

**Proof of Corollary 1.1** Dividing both sides of (1.5) by \( \vartheta_2(\tau) \prod_{1 \leq i \leq n} \frac{\theta_1(a_i - c_i | \tau)}{\theta_1(a_i | \tau)} \), replacing \( \tau \) by \( \tau' \) in the resulting identity and then applying (1.3) we can easily deduce the first identity.

It can be seen from the Jacobi infinite product expressions for theta functions that

\[
2\theta_1(z|2\tau)\vartheta_4(z|2\tau) = \vartheta_2(\tau)\theta_1(z|\tau),
\]

\[
2\theta_2(z|2\tau)\vartheta_3(z|2\tau) = \vartheta_2(\tau)\theta_2(z|\tau).
\]

We put \( z = 0 \) in (2.3) to arrive at

\[
2\vartheta_2(2\tau)\vartheta_3(2\tau) = \vartheta_2^2(\tau).
\]

Combining (1.8), (2.3) and (2.4) we have

\[
\sin_{q}z = \frac{\sin_{q,2}z}{\sin_{q}z} = \frac{\theta_1(z|\tau'/2)}{\theta_1(z|\tau')}, \quad \frac{\vartheta_2(\tau)}{\vartheta_2(\tau'/2)} = \frac{2\vartheta_2(\tau')\vartheta_4(z|\tau')}{\vartheta_2^2(\tau'/2)} = \frac{\theta_4(z|\tau')}{\vartheta_3(\tau')}.
\]

Thus, the second identity follows readily by dividing both sides of (1.6) by \( \vartheta_2(\tau) \) \cdot \prod_{1 \leq i \leq n} \frac{\theta_4(a_i - c_i | \tau)}{\theta_1(a_i | \tau)} \), replacing \( \tau \) by \( \tau' \) in the resulting identity and employing (2.6) and (1.3). This finishes the proof of Corollary 1.1. □
3. A NEW PROOF OF JACOBI’S FOUR SQUARES THEOREM

Given two positive integers \( n \) and \( k \), we denote by \( r_k(n) \) the number of representations of \( n \) as a sum of \( k \) squares, where the orders and signs are also counted. Jacobi’s four squares theorem says that for each positive integer \( n \), we have

\[
r_4(n) = 8 \sum_{d|n} d.
\]

The first proof of this formula was given by Jacobi \([9]\). See \([2]\) and \([6]\) for other proofs of this formula. We now give a new proof of Jacobi’s four squares theorem using a special case of (1.6).

**Proof.** When \( n = 2 \), the identity (1.6) becomes

\[
\theta_1(a_1 - a_2|\tau)\theta_1(c_1 + c_2|\tau)\theta_4(a_1 - c_1|\tau)\theta_4(a_2 - c_2|\tau) = \theta_4(a_2|\tau)\theta_1(c_1|\tau)\theta_4(a_1 - c_1|\tau)\theta_1(c_2|\tau)\theta_4(a_2 - c_2|\tau)\theta_1(a_2 - a_1|\tau).
\]

Differentiating this equation with respect to \( a_1 \) using logarithmic differentiation and then setting \( a_1 = a_2 \) we find that

\[
\frac{\theta_1'(a_1 - c_1 - c_2|\tau)}{\theta_1} + \frac{\theta_1'(c_1|\tau)}{\theta_1} + \frac{\theta_1'(c_2|\tau)}{\theta_1} - \frac{\theta_4'(a_1|\tau)}{\theta_4} = \frac{\theta_4'(c_1|\tau)\theta_1(c_1 + c_2|\tau)\theta_4(a_1 - c_1|\tau)\theta_4(a_1 - c_2|\tau)}{\theta_4(a_1|\tau)\theta_1(c_1|\tau)\theta_4(c_2|\tau)\theta_4(a_1 - c_1 - c_2|\tau)}.
\]

Differentiating the above identity with respect to \( c_1 \) and then taking \( c_1 = -c_2 \) we have

\[
\left( \frac{\theta_1'}{\theta_1} \right)'(a_1|\tau) - \left( \frac{\theta_1'}{\theta_1} \right)'(c_1|\tau) = \frac{(\theta_1'(\tau))^2 \theta_4(a_1 - c_1|\tau)\theta_4(a_1 + c_1|\tau)}{\theta_1^2(c_1|\tau)\theta_4^2(a_1|\tau)}.
\]

In particular,

\[
(3.1) \quad \left( \frac{\theta_1'}{\theta_1} \right)'(z|\tau) - \left( \frac{\theta_1'}{\theta_1} \right)'(z|\tau) = \frac{(\theta_1'(\tau))^2 \theta_4(2z|\tau)}{\theta_1^2(z|\tau)\theta_4^2(z|\tau)}.
\]

Recall from \([16]\) the trigonometric expansions for logarithmic derivatives of theta functions:

\[
\frac{\theta_1'}{\theta_1}(z|\tau) = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin(2nz),
\]

\[
\frac{\theta_4'}{\theta_4}(z|\tau) = 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \sin(2nz).
\]

Substituting these two identities into (3.1) gives

\[
(3.2) \quad \frac{1}{\sin^2 z} + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1 + q^n} \cos(2nz) = \frac{(\theta_1'(\tau))^2 \theta_4(2z|\tau)}{\theta_1^2(z|\tau)\theta_4^2(z|\tau)}.
\]
Appealing to the Jacobi infinite product expressions for theta functions we can deduce
\[
\vartheta'_1(\tau) = 2q^{\frac{1}{4}}(q^2; q^2)^3_{\infty},
\]
\[
\vartheta_3(\tau) = (q^2; q^2)_\infty(-q; q^2)^2_{\infty},
\]
\[
\vartheta_4(\tau) = (q^2; q^2)_\infty(q; q^2)^2_{\infty},
\]
\[
\theta_1(\pi/4|\tau) = \sqrt{2}q^{\frac{1}{4}}(q^2; q^2)_\infty(-q^4; q^4)^2_{\infty},
\]
\[
\theta_4(\pi/4|\tau) = (q^2; q^2)_\infty(-q^2; q^4)^{2}_{\infty}.
\]
We put \( z = \pi/4 \) in (3.2), employ the above five identities in the resulting identity and then replace \( q^2 \) by \( q \) to get
\[
1 + 8 \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 + q^n} = \vartheta'_3(\tau).
\]
Replacing \( q \) by \(-q\) in the above identity we are led to
\[
\left( \sum_{k=-\infty}^{\infty} q^{k^2} \right)^4 = 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1 + (-q)^n}
\]
\[
= 1 + 8 \sum_{n=1}^{\infty} \frac{2n(2n-1)q^{2n-1}}{1 - q^{2n-1}} + 8 \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1 + q^{2n}}
\]
\[
= 1 + 8 \sum_{n=1}^{\infty} \frac{2n(2n-1)q^{2n-1}}{1 - q^{2n-1}} + 8 \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1 - q^{2n}}
\]
\[
+ 8 \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1 + q^{2n}} - 8 \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1 - q^{2n}}
\]
\[
= 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - 32 \sum_{n=1}^{\infty} \frac{nq^{4n}}{1 - q^{4n}}
\]
\[
= 1 + 8 \sum_{n=1}^{\infty} \left( \sum_{d|n} d - \sum_{d|n} \frac{n}{4d} \right) q^n.
\]
Then the formula for \( r_4(n) \) follows readily by comparing the coefficients of \( q^n \) on both sides of this identity. \( \square \)

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REFERENCES

[1] S. Abo Touk, Z. Al Houchan and M. El Bachraoui, Proofs for two \( q \)-trigonometric identities of Gosper. J. Math. Anal. Appl. 456(1) (2017), 662-670.
[2] G.E. Andrews, S.B. Ekhad and D. Zeilberger, A short proof of Jacobi’s formula for the number of representations of an integer as a sum of four squares, Amer. Math. Monthly 100(3) (1993), 274–276.

[3] M. El Bachraoui, Confirming a q-trigonometric conjecture of Gosper, Proc. Amer. Math. Soc. 146(4) (2018), 1619–1625.

[4] M. El Bachraoui, Proving some identities of Gosper on q-trigonometric functions, arXiv:1801.03654.

[5] M. El Bachraoui, Solving some q-trigonometric conjectures of Gosper. J. Math. Anal. Appl. 460(2) (2018), 610–617.

[6] S. Bhargava and C. Adiga, Simple proofs of Jacobi’s two and four square theorems, Inter. J. Math. Ed. Sci. Tech. 19 (1988), 779–782.

[7] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[8] R.W. Gosper, Experiments and discoveries in q-trigonometry, in: F.G. Garvan, M.E.H. Ismail (Eds.), Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics, Kluwer, Dordrecht, Netherlands, 2001, pp.79–105.

[9] C.G.J. Jacobi, Fundamenta Nova Theoriae Functionum Ellipticarum, Bornträger, Regiomonti, 1829.

[10] S. Kongsiriwong and Z.-G. Liu, Uniform proofs of q-series-product identities, Results Math. 44(3–4) (2003), 312–339.

[11] Z.-G. Liu, A theta function identity and its implications, Trans. Amer. Math. Soc. 357(2) (2005), 825–835.

[12] Z.-G. Liu, A three-term theta function identity and its applications. Adv. Math. 195(1) (2005), 1–23.

[13] Z.-G. Liu, Addition formulas for Jacobi theta functions, Dedekind’s eta functions, and Ramanujan’s congruences, Pacific J. Math. 240(1) (2009), 135–150.

[14] Z.-G. Liu, An addition formula for the Jacobian theta function and its applications, Adv. Math. 212(1) (2007), 389–406.

[15] Z.-G. Liu, Elliptic functions and the Appell theta functions, Int. Math. Res. Not., IMRN 11 (2010), 2064–2093.

[16] Z.-G. Liu, Residue theorem and theta function identities. Ramanujan J. 5(2) (2001), 129–151.

[17] I. Mező, Duplication formulae involving Jacobi theta functions and Gosper’s q-trigonometric functions, Proc. Amer. Math. Soc. 141(7) (2013), 2401–2410.

[18] L.-C. Shen, On some modular equations of degree 5, Proc. Amer. Math. Soc. 123(5) (1995), 1521–1526.

[19] L.-C. Shen, On the additive formulae of the theta functions and a collection of Lambert series pertaining to the modular equations of degree 5, Trans. Amer. Math. Soc. 345(1) (1994), 323–345.

[20] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis. 4th ed., Cambridge University Press, Cambridge, 1990.