LIMITS LAWS FOR GEOMETRIC MEANS OF FREE POSITIVE RANDOM VARIABLES

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ABSTRACT. Let \( \{a_k\}_{k=1}^{\infty} \) be free identically distributed positive non-commuting random variables with probability measure distribution \( \mu \). In this paper we proved a multiplicative version of the Free Central Limit Theorem. More precisely, let \( b_n = a_1/2 \cdot a_2/2 \cdots a_n/2 \cdot a_1/2 \) then \( b_n \) is a positive operator with the same moments as \( x_n = a_1 a_2 \cdots a_n \) and \( b_n^{1/2n} \) converges in distribution to positive operator \( \Lambda \). We completely determined the probability measure distribution \( \nu \) of \( \Lambda \) from the distribution \( \mu \). This gives us a natural map \( \mathcal{G} : \mathcal{M}_+ \to \mathcal{M}_+ \) with \( \mu \mapsto \mathcal{G}(\mu) = \nu \). We study how this map behaves with respect to additive and multiplicative free convolution. As an interesting consequence of our results, we illustrate the relation between the probability distribution \( \nu \) and the distribution of the Lyapunov exponents for the sequence \( \{a_k\}_{k=1}^{\infty} \) introduced in [13].

1. Introduction

Denote by \( \mathcal{M} \) the family of all compactly supported probability measures defined in the real line \( \mathbb{R} \). We denote by \( \mathcal{M}_+ \) the set of all measures in \( \mathcal{M} \) which are supported on \([0, \infty)\). On the set \( \mathcal{M} \) there are defined two associative composition laws denoted by \( \ast \) and \( \boxplus \). The measure \( \mu \ast \nu \) is the classical convolution of \( \mu \) and \( \nu \). In probabilistic terms, \( \mu \ast \nu \) is the probability distribution of \( X + Y \), where \( X \) and \( Y \) are commuting independent random variables with distributions \( \mu \) and \( \nu \), respectively. The measure \( \mu \boxplus \nu \) is the free additive convolution of \( \mu \) and \( \nu \) introduced by Voiculescu [19]. Thus, \( \mu \boxplus \nu \) is the probability distribution of \( X + Y \), where \( X \) and \( Y \) are free random variables with distribution \( \mu \) and \( \nu \), respectively.

There is a free analogue of multiplicative convolution also. More precisely, if \( \mu \) and \( \nu \) are measures in \( \mathcal{M}_+ \) we can define \( \mu \boxtimes \nu \) the multiplicative free convolution by the probability distribution of \( X^{1/2}YX^{1/2} \), where \( X \) and \( Y \) are free random variables with distribution \( \mu \) and \( \nu \), respectively.

In this paper we proved a multiplicative version of the Free Central Limit Theorem. To be more precise, let \( \{a_k\}_{k=1}^{\infty} \) be free positive identically distributed non-commutative random variables with distribution \( \mu \). Let us define \( x_n := a_1 a_2 \cdots a_n \) and \( b_n := a_1/2 \cdot a_2/2 \cdots a_n/2 \cdot a_1/2 \). Then \( x_n \) and \( b_n \) have the same moments and there exists a positive operator \( \Lambda \) such that

\[
b_n^{1/n} \longrightarrow \Lambda^2 \quad \text{in distribution.}
\]

Moreover, if \( \nu \) is the distribution of \( \Lambda \), then

\[
\nu = \beta \delta_0 + \sigma \quad \text{with} \quad d\sigma = f(t) \mathbf{1}(\|a_1^{-1/2}a_2^{-1/2}a_3^{-1/2}\|) \, dt
\]

(1.1)
where $\beta = \mu(\{0\})$, $f(t) = (F^<t>^-1)'(t)$ and $F_\mu(t) = S_\mu(t-1)^{-1/2}$ ($F^<t>$ is the inverse with respect to composition of $F_\mu$).

This gives us, naturally, a map

$$G : \mathcal{M}_+ \rightarrow \mathcal{M}_+ \quad \text{with} \quad \mu \mapsto G(\mu) = \nu.$$

The measure $G(\mu)$ is a compactly supported positive measure with at most one atom at zero and $G(\mu)(\{0\}) = \mu(\{0\})$.

We would like to mention that Vladislav Kargin in Theorem 1 of [12] proved an estimate in the norm of the positive operators $b_n$. More precisely, he proved that if $\tau(a_1) = 1$ there exists a positive constant $K > 0$ such that

$$\sqrt{n} \sigma(a_1) \leq \|b_n\| \leq Kn \|a_1\|$$

where $\sigma^2(a_1) = \tau(a^2_1) - \tau(a_1)^2$.

It is interesting to compare this result with the analogous result in the classical case. Let $\{a_k\}_{k=1}^\infty$ be independent positive identically distributed commutative random variables with distribution $\mu$. Applying the Law of the Large Numbers to the random variables $\log(a_k)$, in case $\log(a_k)$ is integrable, or applying Theorem 5.4 in [6] in the general case, we obtain that

$$(a_1a_2\ldots a_n)^{1/n} \rightarrow e^{\tau(\log(a_1))} \in [0, \infty)$$

where the convergence is pointwise.

The Lyapunov exponents of a sequence of random matrices was investigated in the pioneering paper of Furstenberg and Kesten [3] and by Oseledc in [17]. Ruelle [18] developed the theory of Lyapunov exponents for random compact linear operators acting on a Hilbert space. Newman in [14] and [15] and later Isopi and Newman in [11] studied Lyapunov exponents for random $N \times N$ matrices as $N \rightarrow \infty$. Later on, Vladislav Kargin [13] investigated how the concept of Lyapunov exponents can be extended to free linear operators (see [13] for a more detailed exposition).

In our case, given $\{a_k\}_{k=1}^\infty$ be free positive identically distributed non-commutative random variables. Let $\mu$ be the spectral probability distribution of $a^2_k$ and assume that $\mu(\{0\}) = 0$. Then

$$\left(a_1a_2\ldots a_n^2\ldots a_2a_1\right)^{1/2n} \rightarrow \Lambda$$

where $\Lambda$ is a positive operator. The probability distribution of the Lyapunov exponents associated to the sequence $\{a_k\}_{k=1}^\infty$, is the spectral probability distribution $\gamma$ of the selfadjoint operator $L := \ln(\Lambda)$. Moreover, $\gamma$ is absolutely continuous with respect to Lebesgue measure and has Radon–Nikodym derivative given by

$$d\gamma(t) = e^t f(e^t) 1_{\{\ln\|a^{-1}_1\|^{-1}, \ln\|a_1\|\}}(t) \, dt$$
where the function \( f(t) \) is as in equation (1.1).

Now we will describe the content of this paper. In section §2, we recall some preliminaries as well as some known results and fix the notation. In section §3, we prove our main Theorem and study how the map \( \mathcal{G} \) behaves with respect to additive and multiplicative free convolution. In section §4, we present some examples. Finally, in section §5, we derive the probability distribution of the Lyapunov exponents of the sequence \( \{a_k\}_{k=1}^\infty \).

2. Preliminaries and Notation

We begin with an analytic method for the calculation of multiplicative free convolution discovered by Voiculescu. Denote \( \mathbb{C} \) the complex plane and set \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}, \mathbb{C}^- = -\mathbb{C}^+ \). For a measure \( \nu \in \mathcal{M}_+ \setminus \{ \delta_0 \} \) one defines the analytic function \( \psi_\nu \) by

\[
\psi_\nu(z) = \int_0^\infty \frac{zt}{1-zt} d\nu(t)
\]

for \( z \in \mathbb{C} \setminus [0, \infty) \). The measure \( \nu \) is completely determined by \( \psi_\nu \). The function \( \psi_\nu \) is univalent in the half-plane \( i\mathbb{C}^+ \), and \( \psi_\nu(i\mathbb{C}^+) \) is a region contained in the circle with center at \(-1/2\) and radius \( 1/2 \). Moreover, \( \psi_\nu(i\mathbb{C}^+) \cap (-\infty, 0] = (\beta - 1, 0) \), where \( \beta = \nu(\{0\}) \). If we set \( \Omega_\nu = \psi_\nu(i\mathbb{C}^+) \), the function \( \psi_\nu \) has an inverse with respect to composition

\[
\chi_\nu : \Omega_\nu \to i\mathbb{C}^+.
\]

Finally, define the \( S \)-transform of \( \nu \) to be

\[
S_\nu(z) = \frac{1+z}{z} \chi_\nu(z) , \quad z \in \Omega_\nu.
\]

See [2] for a more detailed exposition. The following is a classical Theorem originally proved by Voiculescu and generalized by Bercovici and Voiculescu in [3] for measures with unbounded support.

**Theorem 2.1.** Let \( \mu, \nu \in \mathcal{M}_+ \). Then

\[
S_{\mu \boxplus \nu}(z) = S_\mu(z) S_\nu(z)
\]

for every \( z \) in the connected component of the common domain of \( S_\mu \) and \( S_\nu \).

It was shown by Hari Bercovici in [3] that the additive free convolution of probability measures on the real line tend to have a lot fewer atoms. To be more precise.

**Theorem 2.2.** Let \( \mu \) and \( \nu \) be two probability measures supported in \( \mathbb{R} \). The number \( a \) is an atom for the free additive convolution of \( \mu \) and \( \nu \) if and only if \( a \) can be written as \( a = b + c \) where \( \mu(\{b\}) + \nu(\{c\}) > 1 \). In this case, \( \mu \boxplus \nu (\{a\}) = \mu(\{b\}) + \nu(\{c\}) - 1 \).
For measures supported on the positive half-line, an analogous result holds, with a difference when zero is an atom. The following Theorem was proved by Serban Belinschi in [1].

**Theorem 2.3.** Let \( \mu \) and \( \nu \) be two probability measures supported in \([0, \infty)\).

1. The following are equivalent
   a. \( \mu \boxtimes \nu \) has an atom at \( a > 0 \)
   b. there exists \( u \) and \( v \) so that \( uv = a \) and \( \mu(\{u\}) + \nu(\{v\}) > 1 \).
      Moreover, \( \mu(\{u\}) + \nu(\{v\}) - 1 = \mu \boxtimes \nu (\{a\}) \).
   c. \( \mu \boxtimes \nu (\{0\}) = \max\{\mu(\{0\}), \nu(\{0\})\} \).

In [16] Nica and Speicher introduced the class of \( R \)-diagonal operators in a non-commutative \( C^* \)-probability space. An operator \( T \) is \( R \)-diagonal if \( T \) has the same \( \ast \)-distribution as a product \( UH \) where \( U \) and \( H \) are \( \ast \)-free, \( U \) is a Haar unitary, and \( H \) is positive.

The next Theorem and Corollary were proved by Uffe Haagerup and Flemming Larsen in [9] where they completely characterized the Brown measure of an \( R \)-diagonal element.

**Theorem 2.4.** Let \((M, \tau)\) be a non-commutative finite von Neumann algebra with a faithful trace \( \tau \). Let \( u \) and \( h \) be \( \ast \)-free random variables in \( M \), \( u \) a Haar unitary, \( h \geq 0 \) and assume that the distribution \( \mu_h \) for \( h \) is not a Dirac measure. Denote \( \mu_T \) the Brown measure for \( T = uh \). Then

1. \( \mu_T \) is rotation invariant and
   \[
   \text{supp}(\mu_T) = \left[ \|h^{-1}\|_2^{-1}, \|h\|_2 \right] \times [0, 2\pi).
   \]

2. The \( S \)-transform \( S_{h^2} \) of \( h^2 \) has an analytic continuation to neighborhood of
   \( \mu_h(\{0\}) - 1, 0 \), \( S_{h^2}(\mu_h(\{0\}) - 1, 0) = \left[ \|h\|_2^{-2}, \|h^{-1}\|_2^2 \right] \) and \( S'_{h^2} < 0 \) on \( \mu_h(\{0\}) - 1, 0 \).

3. \( \mu_T(\{0\}) = \mu_h(\{0\}) \) and \( \mu_T(B(0, S_{h^2}(t - 1)^{-1/2}) = t \) for \( t \in (\mu_h(\{0\}), 1] \).

4. \( \mu_T \) is the only rotation symmetric probability measure satisfying (3).

**Corollary 2.5.** With the notation as in the last Theorem we have

1. the function \( F(t) = S_{h^2}(t - 1)^{-1/2} : (\mu_h(\{0\}), 1] \rightarrow (\|h^{-1}\|_2^{-1}, \|h\|_2) \) has an analytic continuation to a neighborhood of its domain and \( F' \) is positive on \( (\mu_h(\{0\}), 1] \).

2. \( \mu_T \) has a radial density function \( f \) on \((0, \infty)\) defined by
   \[
   g(s) = \frac{1}{2\pi s} (F'^{-1}(s)')\left(F(\mu_h(\{0\})), F(1)\right)(s) \]
   Therefore, \( \mu_T = \mu_h(\{0\})\delta_0 + \sigma \) with \( d\sigma = g(|\lambda|)dm_2(\lambda) \).
3. Main Results

Let \( \{a_k\}_{k=1}^{\infty} \) be a sequence of free positive equally distributed non-commuting operators. Let us define the operators \( x_n \) and \( b_n \) by

\[
x_n := a_1a_2\ldots a_n \quad \text{and} \quad b_n := a_1^{1/2}a_2^{1/2}\ldots a_n^{1/2}a_1^{1/2}.
\] (3.1)

It is easy to see that \( b_n \) is positive for all \( n \geq 1 \). In the next Lemma we will show that \( b_n \) and \( x_n \) have the same moments.

**Lemma 3.1.** For all \( n \geq 1 \) and for all \( k \) we have that \( \tau(x_n^k) = \tau(b_n^k) \).

**Proof.** The case \( n = 1 \) it is clear. For \( n = 2 \) we have that \( x_2 = a_1a_2 \) and \( b_2 = a_1^{1/2}a_2a_1^{1/2} \) then

\[
\tau(b_2^k) = \tau(a_1^{1/2}a_2(a_1a_2)^{k-1}a_1^{1/2}) = \tau(a_1a_2(a_1a_2)^{k-1}) = \tau(x_2^k).
\]

Now we proceed by induction. Assume that \( \tau(x_n^k) = \tau(b_n^k) \) for all \( k \geq 1 \). Let

\[
x_{n+1} = a_1y_n \quad \text{and} \quad b_{n+1} = a_1^{1/2}c_n a_1^{1/2}
\]

where \( y_n = a_2\ldots a_{n+1} \) and \( c_n = a_2^{1/2}a_3^{1/2}\ldots a_{n+1}^{1/2}a_2^{1/2} \). We know by the induction hypothesis that \( \tau(y_n^m) = \tau(c_n^m) \) for all \( m \geq 1 \). Now we observe that

\[
\tau(x_{n+1}^k) = \tau((a_1y_n)^k)
\] (3.2)

and

\[
\tau(b_{n+1}^k) = \tau((a_1^{1/2}c_n a_1^{1/2})^k) = \tau(a_1^{1/2}c_n(a_1c_n)^{k-1}a_1^{1/2}) = \tau((a_1c_n)^k).
\] (3.3)

Finally, using the fact that \( a_1 \) is free from \( c_n \) and \( y_n \) and equations (3.2) and (3.3) we conclude the proof.

Now we are ready to prove our main result.

**Theorem 3.2.** Let \( \{a_k\}_k \) be a sequence of free, positive, equally distributed non-commuting operators with distribution \( \mu \) in \( \mathcal{M}_+ \). Let \( b_n \) be as in (3.1). The sequence of positive operators \( b_n^{1/2n} \) converges in distribution to a positive operator \( \Lambda \) with distribution \( \nu \) in \( \mathcal{M}_+ \). Moreover,

\[
\nu = \beta \delta_0 + \sigma \quad \text{with} \quad d\sigma = f(t) 1_{\{a_1^{-1/2}\|z\|^{-1},a_1^{1/2}\|z\|\}}(t) dt
\]

where \( \beta = \mu(\{0\}) \), \( f(t) = (F_{\mu}^{-1})'(t) \) and \( F_{\mu}(t) = S_{\mu}(t-1)^{-1/2} \).

**Proof.** Let \( u \) a Haar unitary \(*\)-free with respect to the family \( \{a_k\}_k \) and let \( h = a_1^{1/2} \). Let \( T \) be the \( R \)-diagonal operator defined by \( T = uh \). It is easy to see, by the freeness assumptions, that \( (T^*)^nT^n \) and \( b_n \) have the same distribution. Moreover, by [10] the sequence \( \{(T^*)^nT^n\}^{1/2n} \) converges in the strong operator topology to a positive
operator $\Lambda$. Let $\nu$ be the probability measure distribution of $\Lambda$.

If the distribution of $a_k$ is a Dirac delta, $\mu = \delta_\lambda$, then $h = \sqrt{\lambda}$ and
\[
\left[(T^*)^n T^n\right]^{1/2n} = \left[\lambda^n (u^n)^n u^n\right]^{1/2n} = \sqrt{\lambda}.
\]
Therefore, $b_n^{1/2n}$ has the Dirac delta distribution distribution $\delta_{\sqrt{\lambda}}$ and $\nu = \delta_{\sqrt{\lambda}}$.

If the distribution of $a_k$ is not a Dirac delta, let $\mu_T$ the Brown measure of the operator $T$. By Theorem 2.5 in [10] we know that
\[
\int_{C} |\lambda|^p d\mu_T(\lambda) = \lim_n \|T^n\|_{\frac{p}{n}} = \lim_n \tau\left(\left[(T^*)^n T^n\right]^{\frac{p}{n}}\right) = \tau(\Lambda^p) = \int_0^\infty t^p \nu(t). \tag{3.4}
\]
We know by Theorem 2.4 and Corollary 2.5 that
\[
\mu_T = \beta \delta_0 + \rho \quad \text{with} \quad d\rho(r, \theta) = \frac{1}{2\pi} f(\mu(t)) \mathbf{1}_{(F_{\mu(\beta)}, F_{\mu(1)})}(r) dr d\theta \tag{3.5}
\]
where $f(t) = (F_{\mu(t)}^{<1})'(t)$ and $F_{\mu}(t) = S_{\mu}(t-1)^{-1/2}$.

Hence, using equation (3.4) we see that
\[
\int_0^\infty t^p \nu(t) = \int_0^{2\pi} \int_{F_{\mu(\beta)}}^{F_{\mu(1)}} \frac{1}{2\pi} t^p f(r) dr d\theta = \int_{F_{\mu(\beta)}}^{F_{\mu(1)}} t^p f(r) dr
\]
for all $p \geq 1$. Using the fact that if two compactly supported probability measures in $\mathcal{M}_+$ have the same moments then they are equal, we see that
\[
\nu = \beta \delta_0 + \sigma \quad \text{with} \quad d\sigma = f(t) \mathbf{1}_{(F_{\mu(\beta)}, F_{\mu(1)})}(t) dt.
\]
By Corollary 2.5, we know that
\[
F_{\mu}(1) = \|a_{1/2}\|_2^2 \quad \text{and} \quad \lim_{t \to \beta^+} F_{\mu}(t) = \|a_{1/2}^{-1/2}\|_2^{-1}
\]
concluding the proof. 

Note that the last Theorem gives us a map $\mathcal{G} : \mathcal{M}_+ \to \mathcal{M}_+$ with $\mu \mapsto \mathcal{G}(\mu) = \nu$. The measure $\mathcal{G}(\mu)$ is a compactly supported positive measure with at most one atom at zero and $\mathcal{G}(\mu)(\{0\}) = \mu(\{0\})$.

Since
\[
\mathcal{G}(\mu) = \beta \delta_0 + \sigma \quad \text{with} \quad d\sigma = f(t) \mathbf{1}_{(F_{\mu(\beta)}, F_{\mu(1)})}(t) dt
\]
and $f(t) = (F_{\mu(t)}^{<1})'(t)$ where $F_{\mu}(t) = S_{\mu}(t-1)^{-1/2}$ for $t \in (\beta, 1]$. The function $S_{\mu}(t-1)$ for $t \in (\beta, 1]$ is analytic and completely determined by $\mu$. If $\mu_1, \mu_2 \in \mathcal{M}_+$ and $S_{\mu_1}(t-1) = S_{\mu_2}(t-1)$ in some open interval $(a, b) \subseteq (0, 1]$ implies that $\mu_1 = \mu_2$. Therefore, the map $\mathcal{G}$ is an injection.
Remark 3.3. A measure $\mu$ in $\mathcal{M}_+$ is said $\boxtimes$-infinitely divisible if for each $n \geq 1$ there exists a measure $\mu_n$ in $\mathcal{M}_+$ such that

$$\mu = \mu_n \boxtimes \mu_n \ldots \boxtimes \mu_n \quad (n \text{ times}).$$

We would like to observe that the image of the map $G$ is not contained in the set of $\boxtimes$-infinitely divisible laws since an $\boxtimes$-infinitely divisible law cannot have an atom at zero (see Lemma 6.10 in [4]).

The next Theorem investigates how the map $G$ behaves with respect to additive and multiplicative free convolution.

**Theorem 3.4.** Let $\mu$ be a measure in $\mathcal{M}_+$ and $n \geq 1$. If $G(\mu) = \beta \delta_0 + \sigma$ with $d\sigma = f(t) 1_{(F_{\mu}(\beta),F_{\mu}(1))}(t) \, dt$ then

$$G(\mu^{\boxtimes n}) = \beta_n \delta_0 + \sigma_n \quad \text{with} \quad d\sigma_n = \sqrt{n} f(t/\sqrt{n}) 1_{(\sqrt{n} F_{\mu}(\beta_n),\sqrt{n} F_{\mu}(1))}(t) \, dt$$

where $\beta_n = \max\{0,n\beta - (n-1)\}$ and

$$G(\mu^{\boxtimes n}) = \beta \delta_0 + \rho_n \quad \text{with} \quad d\rho_n = \frac{1}{n} t^{1-n} f(t^{1/n}) 1_{(F_{\mu}(\beta)^n),F_{\mu}(1)^n)}(t) \, dt.$$

**Proof.** Recall the relation between the $R$–transform and $S$–transform (see [9]),

$$\left(z R_\mu(z)\right)^{-1} = z S_\mu(z).$$

By the fundamental property of the $R$–transform we have $R_{\mu^{\boxtimes n}}(z) = n R_\mu(z)$. Therefore,

$$\left(zn R_\mu(z)\right)^{-1} = z S_{\mu^{\boxtimes n}}(z).$$

Hence

$$\frac{z}{n} S_\mu(z/n) = z S_{\mu^{\boxtimes n}}(z)$$

thus

$$S_{\mu^{\boxtimes n}}(z) = \frac{1}{n} S_\mu(z/n). \quad (3.6)$$

Then

$$F_{\mu^{\boxtimes n}}(t) = S_{\mu^{\boxtimes n}}(t-1)^{-1/2} = \left(\frac{1}{n} S_\mu\left(\frac{t-1}{n}\right)\right)^{-1/2} = \sqrt{n} F_\mu\left(\frac{t+n-1}{n}\right)$$

it is a direct computation to see that

$$F_{\mu^{\boxtimes n}}^{-1}(t) = n F_\mu^{-1}(t/\sqrt{n}) - n + 1. \quad (3.7)$$

By iterating Theorem 2.2 we see that $\mu^{\boxtimes n}(\{0\}) = \max\{0,n\beta - (n-1)\} = \beta_n$. 

Now using Theorem 3.2 we obtain
\[ G(\mu^{\boxplus n}) = \beta_n \delta_0 + \sigma_n \] with
\[ d\sigma_n = \sqrt{n} f(t/\sqrt{n}) 1_{(\sqrt{\pi F_{\mu}(\beta_n)^n}, \sqrt{\pi F_{\mu}(1)^n})}(t) \, dt. \]

Now let us prove the multiplicative free convolution part, let \( \mu^{\boxplus n} \) then
\[ S_{\mu^{\boxplus n}}(z) = S_{\mu}^n(z). \]

Then \( F_{\mu^{\boxplus n}}(t) = F_{\mu}^n(t) \) and therefore,
\[ F_{\mu^{\boxplus n}}^{< -1} = F_{\mu}^{< -1}(t^{1/n}). \] (3.8)

By Theorem 2.3 we now that \( \mu^{\boxplus n}(\{0\}) = \mu(\{0\}) = \beta. \) Therefore, using Theorem 3.2 again we obtain
\[ G((\mu^{\boxplus n} = \beta \delta_0 + \rho_n \] with
\[ d\rho_n = \frac{1}{n} t^{1-n} f(t^{1/n}) 1_{(\sqrt{\pi F_{\mu}(\beta)^n}, \sqrt{\pi F_{\mu}(1)^n})}(t) \, dt. \]

4. Examples

In this section we present some examples of the image of the map \( G. \)

**Example 4.1.** (Projection) Let \( p \) be a projection with \( \tau(p) = \alpha. \) Then the spectral probability measure of \( p \) is \( \mu_p = (1 - \alpha) \delta_0 + \alpha \delta_1. \) We would like to compute \( G(\mu_p). \)

Recall that
\[ S_p(z) = \frac{z + 1}{z + \alpha}. \]

Therefore,
\[ F_{\mu}(t) = \left( \frac{t - 1 + \alpha}{t} \right)^{1/2} \] and \( F_{\mu}^{< -1} = \frac{1 - \alpha}{1 - t^2}. \)

Hence,
\[ G(\mu_p) = (1 - \alpha) \delta_0 + \sigma \] with
\[ d\sigma = \frac{2t(1 - \alpha)}{(t^2 - 1)^2} 1_{[0, \sqrt{\alpha}]}(t) \, dt. \]

**Example 4.2.** Let \( h \) be a quarter-circular distributed positive operator,
\[ d\mu_h = \frac{1}{\pi} \sqrt{4 - t^2} 1_{[0, 2]}(t) \, dt. \]

A simple computation shows that
\[ S_{h^2}(z) = \frac{1}{z + 1} \]

hence by Theorem 3.2 we see that
\[ dG(\mu_{h^2}) = 2t 1_{[0, 1]}(t) \, dt. \]
Example 4.3. (Marchenko – Pastur distribution)

Let $c > 0$ and let $\mu_c$ be the Marchenko Pastur or Free Poisson distribution given by

$$d\mu_c = \max\{1 - c, 0\}\delta_0 + \frac{\sqrt{(t - a)(b - t)}}{2\pi t}\mathbf{1}_{(a,b)}(t)\,dt$$

where $a = (\sqrt{c} - 1)^2$ and $b = (\sqrt{c} + 1)^2$.

It can be shown (see for example [9]) that

$$S_{\mu_c}(z) = \frac{1}{z + c}.$$ 

Therefore,

$$F_{\mu_c}(t) = \sqrt{t - 1 + c} \quad \text{and} \quad F_{\mu_c}^{< -1>} = t^2 + 1 - c.$$ 

Hence,

$$G(\mu_c) = \max\{1 - c, 0\}\delta_0 + \sigma \quad \text{with} \quad d\sigma = 2t\mathbf{1}_{(\sqrt{\max\{c - 1, 0\}}, \sqrt{c})}(t)\,dt.$$ 

5. Lyapunov exponents of free operators

Let $\{a_k\}_{k=1}^\infty$ be free positive identically distributed operators. Let $\mu$ be the spectral probability measure of $a_k^2$ and assume that $\mu(\{0\}) = 0$. Using Theorem 3.2 we know that the sequence of positive operators

$$\left(a_1a_2\ldots a_n^2\ldots a_2a_1\right)^{1/2n}$$

converges in distribution to a positive operator $\Lambda$ with distribution $\nu$ in $\mathcal{M}_+$. Since $\mu(\{0\}) = 0$, this distribution is absolutely continuous with respect to the Lebesgue measure and has Radon–Nikodym derivative

$$d\nu(t) = f(t)\mathbf{1}_{(\ln\|\alpha_1^{-1}\|_2^{-1}, \|\alpha_1\|_2)}(t)\,dt$$

where $f(t) = (F_{\mu}^{< -1>}t)'$ and $F_{\mu}(t) = S_{\mu}(t - 1)^{-1/2}$.

Let $L$ be the selfadjoint, possibly unbounded operator, defined by $L := \ln(\Lambda)$, and let $\gamma$ be the spectral probability distribution of $L$. It is a direct calculation to see that $\gamma$ is absolutely continuous with respect to Lebesgue measure and has Radon–Nikodym derivative

$$d\gamma(t) = e^t f(e^t)\mathbf{1}_{(\ln\|\alpha_1^{-1}\|_2^{-1}, \ln\|\alpha_1\|_2)}(t)\,dt.$$ 

The probability distribution $\gamma$ of $L$ is what is called the distribution of the Lyapunov exponents (see [14], [15] and [18] and [13] for a more detailed exposition on Lyapunov exponents in the classical and non–classical case).
Theorem 5.1. Let \( \{a_k\}_{k=1}^{\infty} \) be free positive identically distributed invertible operators. Let \( \mu \) be the spectral probability measure of \( a_k^2 \). Let \( \gamma \) be probability distribution of the Lyapunov exponents associated to the sequence. Then \( \gamma \) is absolutely continuous with respect to Lebesgue measure and has Radon–Nikodym derivative

\[
d\gamma(t) = e^t f(e^t) 1_{(\ln \|a_1^{-1}\|_2^{-1}, \ln \|a_1\|_2]}(t) \, dt.
\]

where \( f(t) = (F_{\mu}^{<1>})'(t) \) and \( F_{\mu}(t) = S_{\mu}(t - 1)^{-1/2} \).

Remark 5.2. Note that if the operators \( a_k \) are not invertibles in the \( \| \cdot \|_2 \) then the selfadjoint operator \( L \) is unbounded. See in the next example the case \( \lambda = 1 \).

The following is an example done previously in [13] using different techniques.

Example 5.3. (Marchenko – Pastur distribution) Let \( \{a_k\}_{k=1}^{\infty} \) be free positive identically distributed operators such that \( a_k^2 \) has the Marchenko–Pastur distribution \( \mu \) of parameter \( \lambda \geq 1 \). Then as we saw in the Example 4.3, in the last section

\[
d\nu(t) = 2t 1_{(\sqrt{\lambda - 1}, \sqrt{\lambda}]}(t) \, dt.
\]

Therefore, we see that the probability measure of the Lyapunov exponents is \( \gamma \) with

\[
d\gamma(t) = 2e^{2t} 1_{(\frac{1}{2}\ln(\lambda - 1), \frac{1}{2}\ln(\lambda)]}(t) \, dt.
\]

If \( \lambda = 1 \), this law is the exponential law discovered by C.M. Newman as a scaling limit of Lyapunov exponents of large random matrices. (See [14], [15] and [11]). This law is often called the “triangle” law since it implies that the exponentials of Lyapunov exponents converge to the law whose density is in the form of a triangle.

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