ON PROPERTIES OF THE SOLUTIONS TO THE $\alpha$-HARMONIC EQUATION

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Abstract. The aim of this paper is to establish properties of the solutions to the $\alpha$-harmonic equations: $\Delta_\alpha(f(z)) = \partial z[(1 - |z|^2)^{-\alpha}\partial z f](z) = g(z)$, where $g : \overline{\mathbb{D}} \to \mathbb{C}$ is a continuous function and $\overline{\mathbb{D}}$ denotes the closure of the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. We obtain Schwarz type and Schwarz-Pick type inequalities for the solutions to the $\alpha$-harmonic equation. In particular, for $g \equiv 0$, the solutions to the above equation are called $\alpha$-harmonic functions. We determine the necessary and sufficient conditions for an analytic function $\psi$ to have the property that $f \circ \psi$ is $\alpha$-harmonic function for any $\alpha$-harmonic function $f$. Furthermore, we discuss the Bergman-type spaces on $\alpha$-harmonic functions.

1. Introduction

Let $\mathbb{C}$ denote the complex plane. For $a \in \mathbb{C}$, let $\mathbb{D}(a, r) = \{ z : |z - a| < r \}$ ($r > 0$) and $\mathbb{D}(0, r) = \mathbb{D}_r$, $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{T} = \partial \mathbb{D}$, the boundary of $\mathbb{D}$, and $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$, the closure of $\mathbb{D}$. Furthermore, we denote by $\mathcal{C}^m(\Omega)$ the set of all complex-valued $m$-times continuously differentiable functions from $\Omega$ into $\mathbb{C}$, where $\Omega$ stands for a subset of $\mathbb{C}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, $\mathcal{C}(\Omega) := \mathcal{C}^0(\Omega)$ denotes the set of all continuous functions in $\Omega$. We use $d(z)$ to denote the Euclidean distance from $z$ to $\mathbb{T}$.

1.1. Distributions. Let $\Omega$ be an open set in $\mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}$ be an infinitely many times differentiable function. We may write its partial derivatives in the form

$$\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index and $\partial_j = \partial/\partial x_j$, $j = 1, 2, \ldots, n$. We denote by $\mathcal{C}_0^\infty(\Omega)$ the space of functions which are infinitely many times differentiable and have a compact support in $\Omega$.

A distribution $f$ in $\Omega$ is a linear form on $\mathcal{C}_0^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there exist constants $C$ and $k$ such that

$$|f(\phi)| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \phi|,$$

where $\phi \in \mathcal{C}_0^\infty(K)$. The set of all distributions in $\Omega$ is denoted by $\mathcal{D}'(\Omega)$ (cf. [16]).

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For locally integrable $u$ and $v$ are on $\Omega$ and
\[-\int_\Omega u \partial_j \varphi \, dx = \int_\Omega v \varphi \, dx,
\]
for all $\varphi \in C_0^\infty(\Omega)$, we shall say that $\partial_j u = v$ in the distribution sense (cf. [14]).

1.2. The $\alpha$-harmonic equation. We denote by $\Delta_\alpha$ the weighted Laplace operator corresponding to the so-called standard weight $w_\alpha = (1 - |z|^2)^\alpha$, that is,
\[
\Delta_\alpha(f(z)) = \partial z [(1 - |z|^2)^{-\alpha} \overline{\partial} f](z)
\]
in $\mathbb{D}$, where $\alpha > -1$ (see [20, Proposition 1.5] for the reason for this constraint),
\[
\partial z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \text{ and } \overline{\partial} z = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

The weighted Laplacians of the form (1.1) were first systematically studied by Garabedian in [11]. In [20], Olofsson and Wittsten introduced the operator $\Delta_\alpha$ and gave a counterpart of the classical Poisson integral formula for it.

Let $g \in C(\mathbb{D})$ and $f \in C^2(\mathbb{D})$. Of particular interest to us is the following inhomogeneous $\alpha$-harmonic equation in $\mathbb{D}$:
\[
\Delta_\alpha(f) = g.
\]
We also consider the associated Dirichlet boundary value problem of functions $f$, satisfying the equation (1.2),
\[
\begin{cases}
\Delta_\alpha(f) = g & \text{in } \mathbb{D}, \\
f = f^* & \text{on } \mathbb{T}.
\end{cases}
\]

Here the boundary data $f^*$ is a distribution on $\mathbb{T}$, i.e. $f^* \in \mathcal{D}'(\mathbb{T})$, and the boundary condition in (1.3) is understood as $f_r \to f^* \in \mathcal{D}'(\mathbb{T})$ as $r \to 1^-$, where
\[
f_r(e^{i\theta}) = f(re^{i\theta})
\]
for $e^{i\theta} \in \mathbb{T}$ and $r \in [0, 1)$.

If $g \equiv 0$ in (1.2), the solutions to (1.2) are said to be $\alpha$-harmonic functions. Obviously, $\alpha$-harmonicity coincides with harmonicity when $\alpha = 0$. See [8] and the references therein for the properties of harmonic mappings.

In [20], Olofsson and Wittsten showed that if an $\alpha$-harmonic function $f$ satisfies
\[
\lim_{r \to 1^-} f_r = f^* \in \mathcal{D}'(\mathbb{T}) \quad (\alpha > -1),
\]
then it has the form of a Poisson type integral
\[
f(z) = P_\alpha[f^*](z) = \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(ze^{-i\theta}) f^*(e^{i\theta}) \, d\theta
\]
in $\mathbb{D}$, where
\[
P_\alpha(z) = \frac{(1 - |z|^2)^{\alpha+1}}{(1 - z)(1 - \overline{z})^{\alpha+1}}
\]
is the $\alpha$-harmonic Poisson kernel in $\mathbb{D}$. See [17] and [19] for related discussions.
In [3], Behm found the Green function for $-\Delta^\alpha$. The $\alpha$-harmonic Green function $G^\alpha$ is given in $D$ by
\begin{equation}
G^\alpha(z,w) = -(1 - z\overline{w})^\alpha h \circ \varphi(z,w), \quad z \neq w,
\end{equation}
where
\begin{equation}
h(s) = \int_0^s \frac{t^\alpha}{1-t} dt = \sum_{n=0}^\infty \frac{s^{\alpha+1+n}}{\alpha + 1 + n}, \quad 0 \leq s < 1,
\end{equation}
and
\begin{equation}
\varphi(z,w) = 1 - \frac{|z - w|^2}{1 - zw} = \frac{(1 - |z|^2)(1 - |w|^2)}{1 - z\overline{w}^2}.
\end{equation}
For convenience, we let
\begin{equation}
G[g](z) = \int_D G^\alpha(z,w)g(w) \, dA(w)
\end{equation}
and
\begin{equation}
P[f^*](z) = \frac{1}{2\pi} \int_0^{2\pi} P(ze^{-i\theta})f^*(e^{i\theta}) d\theta,
\end{equation}
where $dA(w) = (1/\pi) \, dx \, dy$ denotes the normalized area measure in $D$ and
\begin{equation}
P(z) = \frac{1 - |z|^2}{|1 - z|^2}
\end{equation}
is the Poisson kernel in $D$.

By [20, Theorem 5.3] and [3, Theorem 2], we see that all solutions to the $\alpha$-harmonic equation (1.3) are given by
\begin{equation}
f(z) = P^\alpha[f^*](z) + G[g](z).
\end{equation}

2. MAIN RESULTS

2.1. A Schwarz type lemma. The classical Schwarz lemma states that an analytic function $f$ from $D$ into itself, with $f(0) = 0$, satisfies $|f(z)| \leq |z|$ for all $z \in D$. This result is a crucial theme in many branches of research for more than a hundred years.

Heinz [15] proved the following Schwarz lemma of complex-valued harmonic functions: If $f$ is a complex-valued harmonic function from $D$ into itself with $f(0) = 0$, then, for $z \in D$,\begin{equation}
|f(z)| \leq \frac{4}{\pi} \arctan |z|.
\end{equation}
The first purpose of this paper is to consider the results of the above type for the solutions to the $\alpha$-harmonic equation. Our result is the following:

**Theorem 2.1.** Suppose that $g \in C(D)$ and $f^* \in C^1(T)$. If $f \in C^2(D)$ satisfying (1.3) with $\alpha \geq 0$ and $P^\alpha[f^*](0) = 0$, then for $z \in \overline{D}$,
\begin{equation}
|f(z)| \leq 2^\alpha \left[ \frac{4}{\pi} \|f^*\|_{\infty} \arctan |z| + \|g\|_{\infty} (1 - |z|^2)^{\alpha+1} \right],
\end{equation}
where $\|f^*\|_{\infty} = \sup_{z \in T} \{|f^*(z)|\}$, and $\|g\|_{\infty} = \sup_{z \in \overline{D}} \{|g(z)|\}$.
Moreover, if we take $\alpha = 0$, $g(z) \equiv -C$, where $C$ is a positive constant, and
\[ f(z) = C(1 - |z|^2), \]
then we see that the inequality (2.2) is sharp.

Clearly, if $\alpha = 0$, $g \equiv 0$ and $f$ maps $\mathbb{D}$ into itself, then (2.2) coincides with (2.1).

2.2. A Schwarz-Pick type lemma. Suppose $f = u + iv$ is in $C^1(\Omega)$, where $\Omega$ is a domain in $\mathbb{C}$ and $u, v$ are real functions. The Jacobian matrix of $f$ at $z$ is denoted by
\[ D_f(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \]

Then
\begin{align*}
\|D_f(z)\| &= \sup\{|D_f(z)s| : |s| = 1\} = |f_z(z)| + |f_\bar{z}(z)| \\
\text{and} \\
l(D_f(z)) &= \inf\{|D_f(z)s| : |s| = 1\} = |f_z(z)| - |f_\bar{z}(z)|.
\end{align*}

Colonna [7] obtained a sharp Schwarz-Pick type lemma for complex-valued harmonic functions, which is as follows: If $f$ is a complex-valued harmonic function from $\mathbb{D}$ into itself, then, for $z \in \mathbb{D}$,
\[ \|D_f(z)\| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}. \]

The following result establishes a Schwarz-Pick type lemma for the solutions to the $\alpha$-harmonic equation.

**Theorem 2.2.** Suppose that $g \in C(\mathbb{D})$, $f \in C^2(\mathbb{D})$ satisfies (1.3) with $\alpha \geq 0$ and that $f^* \in C(\mathbb{T})$. Then for $z \in \mathbb{D}$,
\[ \|D_f(z)\| \leq (\alpha + 1)^{2\alpha + 1}\|f^*\|_\infty \frac{1}{1 - |z|^2} + (\alpha + \frac{4}{3})^{2\alpha + 1}\|g\|_\infty, \]
where $\|f^*\|_\infty$ and $\|g\|_\infty$ are as in Theorem 2.1.

In particular, if $f$ maps $\mathbb{D}$ into $\mathbb{D}$ and $g \equiv 0$, then
\[ \|D_f(z)\| \leq (\alpha + 1)^{2\alpha + 1} \frac{1}{1 - |z|^2}. \]

We remark that the estimate of (2.6) is sharper than the estimate given by [17, Theorem 1.1].

2.3. Compositions of $\alpha$-harmonic functions. Although the composition $f \circ \phi$, where $f$ is harmonic and $\phi$ is analytic, is known to be harmonic, an $\alpha$-harmonic function ($\alpha \neq 0$) precomposed with an analytic function may not be $\alpha$-harmonic. This can be seen by the following example.
Example 2.1. Let $\alpha \in (-1, 0) \cup (0, +\infty)$ and let $k \geq 1$ be an integer. Denote by $f$ the function

$$f(z) = \left( \sum_{n=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)} (1-|z|^2)^n \right)
\left(- (1-|z|^2)^{\alpha+1} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+2)\Gamma(k)} (1-|z|^2)^n \right)z^k$$

for $z \in \mathbb{D} \setminus \{0\}$, where $\Gamma(s) = \int_{0}^{\infty} t^{s-1}e^{-t}dt \ (s > 0)$ is the Gamma function. Let $\psi(z) = z^2$. Then

1. $f$ is an $\alpha$-harmonic function in $\mathbb{D} \setminus \{0\}$;
2. $f \circ \psi$ is not an $\alpha$-harmonic function in $\mathbb{D} \setminus \{0\}$.

To see this note that, by [20, Lemma 1.6], $f$ is an $\alpha$-harmonic function in $\mathbb{D} \setminus \{0\}$. By letting $\eta = \psi(z) = z^2$, we have

$$\frac{\partial}{\partial \overline{z}} f \circ \psi(z) = \frac{\partial}{\partial \overline{\eta}} f(\eta) \cdot \frac{\partial \overline{\eta}}{\partial \overline{z}} = 2 \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)(k-1)!} (1-|z|^2)\alpha(1+|z|^2)^{\alpha-2k+1},$$

and then

$$\frac{\partial}{\partial z} \left( (1-|z|^2)^{-\alpha} \frac{\partial}{\partial \overline{z}} f \circ \psi(z) \right) \neq 0.$$ 

Hence, it follows that $f \circ \psi$ is not an $\alpha$-harmonic function in $\mathbb{D} \setminus \{0\}$.

Now we are ready to state our result.

Theorem 2.3. Let $\psi$ be an analytic function in $\mathbb{D}$. Then for any $\alpha$-harmonic function $f$ with $\alpha \in (-1, 0) \cup (0, +\infty)$, $f \circ \psi$ is $\alpha$-harmonic if and only if $\psi(z) = e^{\mu z}$, $t \in [0, 2\pi]$.

2.4. Bergman-type spaces. For $\nu, \mu, t \in \mathbb{R}$,

$$\mathcal{D}_f(\nu, \mu, t) = \int_{\mathbb{D}} d^\nu |f(z)|^\mu \|Df(z)||^t dA(z) < \infty$$

is called Dirichlet-type energy integral of the complex-valued function $f$ (cf. [1, 2, 4, 5, 6, 10, 12, 13, 21]). In particular, for $\nu > -1$, $0 < \mu < \infty$ and $t = 0$, we denote by $b_{\nu, \mu}(\mathbb{D})$ the Bergman-type space, consisting of all $f \in \mathcal{C}^0(\mathbb{D})$ with the norm

$$\|f\|_{b_{\nu, \mu}} = |f(0)| + (\mathcal{D}_f(\nu, \mu, 0))^\frac{1}{\nu} < \infty.$$

We refer to [6, 9, 12, 13, 22] for basic characterizations of analytic or harmonic Bergman-type spaces and Dirichlet-type spaces. However, very few related studies can be found from the literature for the general complex-valued functions. The following is a characterization of $\alpha$-harmonic functions in Bergman-type spaces.

Theorem 2.4. Let $f \in \mathcal{C}^2(\mathbb{D})$ be an $\alpha$-harmonic function in $\mathbb{D}$ with $\alpha > -1$, $\Re(\Delta f) \geq 0$ and $\sup_{z \in \mathbb{D}} |f(z)| \leq M$, where $M$ is a constant. Then for $p \geq 2$, $f \in b_{\nu, \mu}(\mathbb{D})$.

We will give proofs of Theorem 2.1 and Theorem 2.2 in Section 3. Theorem 2.3 and Theorem 2.4 will be proved in Section 4.
3. SCHWARZ TYPE AND SCHWARZ-PICK TYPE LEMMAS FOR THE SOLUTIONS OF THE $\alpha$-HARMONIC EQUATION

The aim of this section is to prove Theorems 2.1 and 2.2. First, we show Theorem 2.1. For this, we need some lemmas.

**Lemma 3.1.** For $\alpha \geq 0$, the function $h(s)$ given by (1.6) satisfies the estimate
\[ h(s) \leq s^\alpha \log \frac{1}{1-s}. \]

**Proof.** By (1.6), we obtain that
\[ h(s) = \sum_{n=0}^{\infty} \frac{s^{\alpha+n}}{\alpha + 1 + n} = s^\alpha \sum_{n=1}^{\infty} \frac{s^n}{\alpha + n} \leq s^\alpha \sum_{n=1}^{\infty} \frac{s^n}{n} = s^\alpha \log \frac{1}{1-s}, \]
and the result follows. □

**Lemma 3.2.** For $\alpha \geq 0$, the function $G_\alpha(z,w)$ given by (1.5) satisfies the estimate
\[ |G_\alpha(z,w)| \leq 2^\alpha (1 - |z|^2)^\alpha \log \frac{1 - \overline{z}w}{z - w}. \]

**Proof.** Applying Lemma 3.1, shows that
\[ |G_\alpha(z,w)| \leq \left| 1 - \overline{z}w \right|^\alpha \left( \frac{(1 - |z|^2)^\alpha (1 - |w|^2)^\alpha}{1 - \overline{z}w} \right) \log \left| \frac{1 - \overline{z}w}{z - w} \right| \]
\[ \leq 2^\alpha (1 - |z|^2)^\alpha \log \left| \frac{1 - \overline{z}w}{z - w} \right|^2, \]
and the result follows. □

**Proof of Theorem 2.1.** For a given $g \in C(\mathbb{D})$, by (1.4), we have
\[ |f(z)| \leq |\mathcal{P}_\alpha[f^*](z)| + |G[g](z)|. \]

Since
\[ |\mathcal{P}_\alpha[f^*](z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} \cdot \frac{(1 - |z|^2)^\alpha}{(1 - \overline{z}e^{i\theta})^\alpha} f^*(e^{i\theta}) d\theta \right| \]
\[ \leq (1 + |z|)^\alpha \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-i\theta}|^2} |f^*(e^{i\theta})| d\theta \]
\[ \leq 2^\alpha \mathcal{P}[|f^*|](z) \]
and $\mathcal{P}[|f^*|](z)$ is harmonic in $\mathbb{D}$, by (2.1), we know that, for $z \in \mathbb{D}$
\[ |\mathcal{P}_\alpha[f^*](z)| \leq 2^\alpha \mathcal{P}[|f^*|](z) \leq 2^\alpha \cdot \frac{4}{\pi} \|f^*\|_\infty \arctan |z|. \]

On the other hand, by Lemma 3.2, we get
\[ |G[g](z)| = \left| \int_{\mathbb{D}} G_\alpha(z,w) g(w) \, dA(w) \right| \leq 2^\alpha \|g\|_\infty (1 - |z|^2)^\alpha J_1, \]
where
\[ J_1 = \int_D \log \left| \frac{1 - \overline{z}w}{z - w} \right|^2 dA(w). \]

In order to estimate \( J_1 \), we let \( w \mapsto \zeta = \phi(w) = \frac{z - w}{1 - \overline{w}z} = re^{i\theta} \)
so that \( \phi' = \phi^{-1} \),
\[ w = \frac{z - \zeta}{1 - \zeta z}, \quad \phi'(w) = \frac{1 - |z|^2}{(1 - w\overline{z})^2}, \]
and thus,
\[ dA(w) = |(\phi^{-1})'(\zeta)|^2 dA(\zeta) = \frac{(1 - |z|^2)^2}{|1 - \zeta z|^4} dA(\zeta). \]
Consequently, by switching to the polar coordinates and using \([18, (2.3)]\), we obtain
\[
J_1 = \int_D \frac{(1 - |z|^2)^2}{|1 - \zeta z|^4} \log \frac{1}{|\zeta|^2} dA(\zeta)
\]
\[
= \frac{(1 - |z|^2)^2}{\pi} \int_0^{2\pi} \int_0^1 r \frac{1}{|1 - r e^{i\theta}z|^4} \log \frac{1}{r^2} dr d\theta
\]
\[
= 2(1 - |z|^2)^2 \sum_{n=0}^\infty (n + 1)^2 |z|^{2n} \int_0^1 r^{2n+1} \log \frac{1}{r^2} dr
\]
\[
= 1 - |z|^2,
\]
which implies that
\[
|G[g](z)| \leq 2^\alpha \|g\|_{\infty} (1 - |z|^2)^{\alpha+1}.
\]
Hence, it follows from (3.1) and (3.2) that (2.2) holds, and the proof of the theorem is complete. 

In order to prove Theorem 2.2, we need some auxiliary lemmas. The following result is from \([17]\).

**Lemma A.** ([17, Lemma 2.1]) If \( \alpha > -1 \) and \( f^* \in C(\mathbb{T}) \), then
\[
\frac{\partial}{\partial z} \int_0^{2\pi} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \int_0^{2\pi} \frac{\partial}{\partial z} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta
\]
and
\[
\frac{\partial}{\partial \overline{z}} \int_0^{2\pi} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \int_0^{2\pi} \frac{\partial}{\partial \overline{z}} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta.
\]

**Lemma 3.3.** Assume that \( f^* \in C(\mathbb{T}) \) and \( \alpha \geq 0 \). Then
\[
\|DP_{\alpha}(f^*)(z)\| \leq (\alpha + 1)2^{\alpha+1} \|f^*\|_{\infty} \frac{1}{1 - |z|^2}.
\]
Proof. By elementary calculations, we obtain
\[
\frac{\partial}{\partial z} P(ze^{-i\theta}) = \frac{e^{-i\theta}}{(1 - ze^{-i\theta})^2}
\]
and
\[
\frac{\partial}{\partial \bar{z}} P(ze^{-i\theta}) = \frac{e^{i\theta}}{(1 - \bar{z}e^{i\theta})^2}.
\]

Then
\[
\frac{\partial}{\partial z} \alpha P(ze^{-i\theta}) = (1 - |z|^2)^\alpha \left[ e^{-i\theta}(1 - |z|^2) - (\alpha + 1)\bar{z}(1 - ze^{-i\theta}) \right] \frac{\partial}{\partial z} P(ze^{-i\theta})
\]
and
\[
\frac{\partial}{\partial \bar{z}} \alpha P(ze^{-i\theta}) = \frac{(\alpha + 1)(1 - |z|^2)^\alpha e^{i\theta}}{(1 - \bar{z}e^{i\theta})^\alpha + 2} = \frac{(\alpha + 1)(1 - |z|^2)^\alpha}{(1 - \bar{z}e^{i\theta})^\alpha} \frac{\partial}{\partial \bar{z}} P(ze^{-i\theta}).
\]

Since
\[
|1 - |z|^2 - (\alpha + 1)\bar{z}e^{i\theta}(1 - ze^{-i\theta})| = |(1 - ze^{-i\theta})(-\alpha\bar{z}e^{i\theta}) + (1 - \bar{z}e^{i\theta})| \leq (\alpha + 1)|1 - \bar{z}e^{i\theta}|,
\]
we see from (3.3) that
\[
\left| \frac{\partial}{\partial z} P_{\alpha}(ze^{-i\theta}) \right| \leq (\alpha + 1)2^\alpha \left| \frac{\partial}{\partial z} P(ze^{-i\theta}) \right|.
\]

Hence, by combining the above with (3.4), we conclude
\[
\left| \frac{\partial}{\partial z} P_{\alpha}(z) \right| + \left| \frac{\partial}{\partial \bar{z}} P_{\alpha}(z) \right| \leq (\alpha + 1)2^\alpha \left( \left| \frac{\partial}{\partial z} P(ze^{-i\theta}) \right| + \left| \frac{\partial}{\partial \bar{z}} P(ze^{-i\theta}) \right| \right).
\]

Consequently, from Lemma A and the identities
\[
\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial}{\partial z} P(ze^{-i\theta}) \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\partial}{\partial \bar{z}} P(ze^{-i\theta}) \right| d\theta = \frac{1}{1 - |z|^2},
\]
it follows that
\[
\|D_{P_{\alpha}[f^*]}(z)\| \leq (\alpha + 1)2^{\alpha + 1}\|f^*\|_\infty \frac{1}{1 - |z|^2},
\]
which is what is needed.

Identify the complex plane \(\mathbb{C}\) with \(\mathbb{R}^2\), and denote by \(L^1_{\text{loc}}(\mathbb{D})\) the space of locally integrable functions in \(\mathbb{D}\). Functions \(\psi \in L^1_{\text{loc}}(\mathbb{D})\) with distributional partial derivatives in \(\mathbb{D}\), we have the action
\[
\langle \psi_z, \varphi \rangle = -\int_{\mathbb{D}} \psi \varphi_z dA, \ \varphi \in C_0^\infty(\mathbb{D}),
\]
for the distributional partial derivative \(\psi_z\) have, and similarly for the distribution \(\psi_\bar{z}\) (cf. [3, 14, 16]).
Lemma 3.4. For \( \alpha \geq 0 \) and \( g \in C(\overline{D}) \), the function \( G_\alpha(z, w) \), given by (1.5), satisfies the following inequality:

1. \( \int_D |G_\alpha(z, w)g(w)| \, dA(w) \leq 2^\alpha \|g\|_\infty \) and \( \int_D |G_\alpha(z, w)| \, dA(z) < \infty \);
2. For fixed \( w \in D \),
   \[
   \frac{\partial G_\alpha(z, w)}{\partial z} = \alpha \overline{m}(1 - \overline{z} w)^{\alpha - 1} h \circ g(z, w) - \frac{(1 - |z|^2)^\alpha (1 - |w|^2)^{\alpha + 1}}{(1 - \overline{z} w)^\alpha (1 - \overline{z} w)} \cdot \frac{1}{z - w}
   \]
   in the sense of distributions in \( D \);
3. (a) \( \frac{\partial G[g](z)}{\partial z} = \int_D \frac{\partial G_\alpha(z, w)}{\partial z} g(w) \, dA(w) \), and
   (b) \( \left| \frac{\partial G[g](z)}{\partial z} \right| \leq \int_D \left| \frac{\partial G_\alpha(z, w)}{\partial z} g(w) \right| \, dA(w) \leq (\alpha + \frac{2}{3}) 2^{\alpha + 1} \|g\|_\infty \),
   in the sense of distributions in \( D \);
4. (a) \( \frac{\partial G[g](z)}{\partial \overline{z}} = \int_D \frac{\partial G_\alpha(z, w)}{\partial \overline{z}} g(w) \, dA(w) \), and
   (b) \( \left| \frac{\partial G[g](z)}{\partial \overline{z}} \right| \leq \int_D \left| \frac{\partial G_\alpha(z, w)}{\partial \overline{z}} g(w) \right| \, dA(w) \leq \frac{2^{\alpha + 2}}{3} \|g\|_\infty \),
   in the sense of distributions in \( D \).

Proof. Obviously, it follows from (3.2) and the proof of [3, Proposition 4] that (1) holds. Therefore, \( G_\alpha(z, w) \in L^1(D) \) so that its derivative has the action

\[
\left\langle \frac{\partial G_\alpha(z, w)}{\partial z}, \varphi(z) \right\rangle = -\int_D G_\alpha(z, w) \varphi_z(z) \, dA(z), \quad \varphi \in C_0^\infty(D).
\]

By Lebesgue’s dominated convergence theorem we get

\[
\int_D G_\alpha(z, w) \varphi_z(z) \, dA(z) = \lim_{\varepsilon \to 0} \int_{D \setminus D(w, \varepsilon)} G_\alpha(z, w) \varphi_z(z) \, dA(z).
\]

For \( \varepsilon > 0 \), let \( D_\varepsilon = D(w, \varepsilon) \). Partial integration gives

\[
\int_{D \setminus D_\varepsilon} G_\alpha(z, w) \varphi_z(z) \, dA(z) = -\int_{\partial D_\varepsilon} G_\alpha(z, w) \varphi(z) v(z) \, ds(z)
\]

\[
-\int_{D \setminus D_\varepsilon} \frac{\partial G_\alpha(z, w)}{\partial z} \varphi(z) \, dA(z),
\]

where \( v \) is the unit exterior normal of \( D \setminus D_\varepsilon \), that is, the inward directed unit normal of \( D_\varepsilon \) and \( ds \) denotes the normalized arc length measure.

It follows from Lemma 3.2 that

\[
|G_\alpha(z, w)| \leq 2^\alpha (1 - |z|^2)^\alpha \log \frac{1 - \overline{z} w}{|z - w|}.
\]
Then
\[ \left| \int_{\partial D_\varepsilon} G_\alpha(z, w) \varphi(z) v(z) \, ds(z) \right| \leq C_\varphi \int_{\partial D_\varepsilon} |G_\alpha(z, w)| \, ds(z) \]
\[ \leq 2^\alpha C_\varphi \varepsilon \log \varepsilon \xrightarrow{\varepsilon \to 0} 0, \]
as \( \varepsilon \to 0 \), where \( C_\varphi \) is a constant depending only on \( \sup \varphi \), and the supremum is taken over all functions \( \varphi \). A straightforward computation shows that for \( z \neq w \) we have
\[ \frac{\partial G_\alpha(z, w)}{\partial z} = H(z, w) = \alpha \overline{w}(1 - z \overline{w})^{\alpha - 1} h \circ g(z, w) + \frac{(1 - |z|^2)^\alpha (1 - |w|^2)^\alpha}{(1 - z \overline{w})^\alpha (1 - \overline{z}w)} \cdot \frac{1}{z - w}. \]
It follows from Lemma 3.1 that
\[ \left| \frac{\partial G_\alpha(z, w)}{\partial z} \right| \leq \alpha \cdot 2^{\alpha + 1} (1 - |z|^2)^{\alpha - 1} \log \frac{1 - \overline{w} z}{|z - w|} \]
\[ + 2^\alpha (1 - |z|^2)^\alpha \frac{1 - |w|^2}{|1 - z \overline{w}| \cdot |z - w|}. \]
Therefore
\[ \left\langle \frac{\partial G_\alpha(z, w)}{\partial z}, \varphi(z) \right\rangle = \lim_{\varepsilon \to 0} \int_{\partial D_\varepsilon} \frac{\partial G_\alpha(z, w)}{\partial z} \varphi(z) \, dA(z) \]
\[ = \int_\mathbb{D} H(z, w) \varphi(z) \, dA(z) = \langle H(z, w), \varphi(z) \rangle. \]
By a similar reasoning as above, one obtains that
\[ \frac{\partial G_\alpha(z, w)}{\partial \overline{z}} = \frac{(1 - |z|^2)^\alpha (1 - |w|^2)^{\alpha + 1}}{(1 - \overline{z}w)^{\alpha + 1} (\overline{z} - \overline{w})} \]
in the sense of distributions in \( \mathbb{D} \). Hence, the assertion (2) of the lemma holds.

To prove the assertion (3), it follows from (3.5) that
\[ \int_{\mathbb{D}} \left| \frac{\partial G_\alpha(z, w)}{\partial z} \right| \, dA(w) \leq \alpha 2^{\alpha + 1} (1 - |z|^2)^{\alpha - 1} \int_{\mathbb{D}} \log \left| \frac{1 - \overline{w} z}{z - w} \right|^2 \, dA(w) \]
\[ + 2^\alpha (1 - |z|^2)^\alpha \int_{\mathbb{D}} \frac{1 - |w|^2}{|1 - z \overline{w}| \cdot |z - w|} \, dA(w). \]
Moreover, as before, by using the transformation
\[ w \mapsto \zeta = \phi(w) = \frac{z - w}{1 - w \overline{z}} = r e^{i\theta} \]
we obtain after computation that
\[
\int_D \frac{1 - |w|^2}{|z - w| \cdot |1 - wz|} dA(w) = \int_D \frac{(1 - |z|^2)(1 - |\zeta|^2)}{|\zeta| \cdot |1 - \zeta \bar{z}|^4} dA(\zeta)
\]
\[
= \frac{(1 - |z|^2)}{\pi} \int_0^1 \int_{0}^{2\pi} \frac{1 - r^2}{|1 - \bar{z}re^{i\theta}|^4} \, d\theta \, dr
\]
\[
= 2(1 - |z|^2) \sum_{n=0}^{\infty} (n + 1)^2 |z|^{2n} \int_0^1 r^{2n}(1 - r^2) \, dr
\]
\[
= 4(1 - |z|^2) \sum_{n=0}^{\infty} \frac{(n + 1)^2}{(2n + 1)(2n + 3)} |z|^{2n}
\]
\[
\leq 4(1 - |z|^2) \sum_{n=0}^{\infty} |z|^{2n} = \frac{4}{3}.
\]
Thus, we conclude that
\[
\int_D \left| \frac{\partial G_\alpha(z, w)}{\partial z} \right| \, dA(w) \leq \alpha 2^{\alpha+1}(1 - |z|^2)^{\alpha-1} J_1 + 2^\alpha \frac{4}{3}(1 - |z|^2)^\alpha
\]
\[
= \alpha 2^{\alpha+1}(1 - |z|^2)^\alpha + 2^\alpha \frac{4}{3}(1 - |z|^2)^\alpha
\]
\[
\leq (\alpha + \frac{2}{3}) 2^{\alpha+1}.
\]
Hence
\[
(3.6) \quad \int_D \left| \frac{\partial G_\alpha(z, w)}{\partial z} g(w) \right| \, dA(w) \leq (\alpha + \frac{2}{3}) 2^{\alpha+1} \|g\|_\infty.
\]
By the assertion (1) and (3.6), we see that
\[
\left\langle \frac{\partial}{\partial z} \int_D G_\alpha(z, w) g(w) dA(w), \varphi(z) \right\rangle = - \int_D \left( \int_D G_\alpha(z, w) g(w) dA(w) \right) \varphi(z) \, dA(z)
\]
\[
= - \int_D \left( \int_D G_\alpha(z, w) \varphi(z) \, dA(z) \right) \, dA(w)
\]
\[
= \int_D \left( \int_D \frac{\partial}{\partial z} G_\alpha(z, w) g(w), \varphi(z) \right) \, dA(w)
\]
\[
= \int_D \left( \int_D \frac{\partial}{\partial z} G_\alpha(z, w) g(w) \varphi(z) \, dA(z) \right) \, dA(w)
\]
\[
= \int_D \left( \int_D \frac{\partial}{\partial z} G_\alpha(z, w) g(w) \, dA(w) \right) \varphi(z) \, dA(z)
\]
\[
= \left\langle \int_D \frac{\partial}{\partial z} G_\alpha(z, w) g(w) \, dA(w), \varphi(z) \right\rangle
\]
Hence, we conclude that
\[
\frac{\partial G}{\partial z} = \int_D \frac{\partial G_\alpha(z, w)}{\partial z} g(w) \, dA(w),
\]
and then the assertion (3) holds.

For the assertion (4), it follows from the assertion (2) that
\[
\left| \frac{\partial G_\alpha(z, w)}{\partial z} \right| \leq 2^\alpha (1 - |z|^2)^\alpha \frac{1 - |w|^2}{|1 - \overline{w}z| |z - w|},
\]
and then
\[
\int_D \left| \frac{\partial G_\alpha(z, w)}{\partial z} g(w) \right| dA(w) \leq 2^\alpha (1 - |z|^2)^\alpha \|g\|_\infty \int_D \frac{1 - |w|^2}{|z - w| \cdot |1 - \overline{w}z|} dA(w) \leq \frac{2^{\alpha+2}}{3} \|g\|_\infty.
\]
The proof of the remaining assertion (4) is similar to the proof of the assertion (3). Hence the proof of the lemma is complete. \qed

**Proof of Theorem 2.2.** The result follows from Lemma 3.3 and Lemma 3.4 together with (2.3). \qed

4. SOME PROPERTIES OF $\alpha$-HARMONIC FUNCTIONS

In this section, we will prove Theorem 2.3 and Theorem 2.4.

In [20], the authors obtained the following homogeneous expansion of $\alpha$-harmonic functions (see [20, Theorem 1.2]):

A function $f$ in $\mathbb{D}$ is $\alpha$-harmonic if and only if it has the following convergent power series expansion:

\[
f(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} P_{\alpha,k}(|z|^2) \overline{z}^k,
\]
where

\[
P_{\alpha,k}(x) = \int_0^1 t^{k-1} (1 - tx)^\alpha dt,
\]
$-1 < x < 1$, and $\{c_k\}_{k=-\infty}^{\infty}$ denotes a sequence of complex numbers with

\[
\lim_{|k| \to \infty} \sup |c_k|^{|k|} \leq 1.
\]

Now, we are ready to prove Theorem 2.3.

**Proof of Theorem 2.3.** The sufficiency is obvious, and now we prove the necessity. Suppose that $f \circ \psi$ is $\alpha$-harmonic. By (4.1), we get

\[
f \circ \psi(z) = \sum_{k=0}^{\infty} c_k \psi(z)^k + \sum_{k=1}^{\infty} c_{-k} P_{\alpha,k}(|\psi(z)|^2) \overline{\psi(z)}^k.
\]

Take the derivative of (4.2), we obtain

\[
P_{\alpha,k}'(x) = -\int_0^1 t^{k-1} x (1 - tx)^{\alpha-1} dt,
\]
and an integration by parts gives

(4.3) \[ xP'_{\alpha,k}(x) + kP_{\alpha,k}(x) = (1 - x)^\alpha. \]

Let \( \tau(z) = P_{\alpha,k}(|\psi(z)|^2 \bar{\psi}(z)) \). Then, by (4.3), we obtain

\[ \tau(z) = \frac{\psi'(z) \cdot \psi(z)}{2} - \frac{1 - |\psi(z)|^2}{1 - |z|^2} \alpha. \]

This together with the fact that \( f \circ \psi \) is \( \alpha \)-harmonic, shows that

\[ \frac{\partial}{\partial z} \left[ (1 - |z|^2)^{-\alpha} (1 - |\psi(z)|^2)^\alpha \right] = 0, \]

and then

(4.4) \[ z\psi'(z)\bar{\psi}(z) = \frac{1 - |\psi(z)|^2}{1 - |z|^2} |z|^2 \]

is real valued. This implies that \( \psi(z) = a_k z^k, k = 1, 2, \ldots \), where \( a_k \) are constants. Therefore, it follows from (4.4) that, for \( z \in \mathbb{D} \setminus \{0\} \),

\[ |a_k|^2 = \frac{1}{k|z|^{2k-2} - (k-1)|z|^{2k}}. \]

Hence, for \( k \geq 2 \), \( a_k \) are not constants. This contradicts with the assumption that \( a_k \) are constants. Therefore, \( \psi(z) = a_1 z \). By (4.4), we have that

\[ |a_1|^2 |z|^2 = \frac{1 - |a_1|^2 |z|^2}{1 - |z|^2} |z|^2. \]

Therefore, we conclude that \( |a_1|^2 = 1 \), which completes the proof. \( \Box \)

**Proof of Theorem 2.4.** By [6, Theorem 3], we only need to prove

(4.5) \[ \int_\mathbb{D} (1 - |z|^2)^{p+1} \Delta(|f(z)|^p) dA(z) < \infty. \]

From Theorem 2.2 we know that

(4.6) \[ \|D_f(z)\| \leq \frac{C_1}{1 - |z|^2}, \]

where \( C_1 = (\alpha + 1)2^{\alpha+1} \). Since \( f \) is \( \alpha \)-harmonic, the function \( (1 - |z|^2)^{-\alpha} f_\tau \) is antianalytic. Therefore it has a power series expansion of the form

\[ (1 - |z|^2)^{-\alpha} f_\tau(z) = \sum_{k=0}^\infty a_k z^k, \quad z \in \mathbb{D}. \]

By calculations, we have

\[ \Delta f(z) = 4f_\tau(z) = \frac{-4\alpha \tau}{1 - |z|^2} f_\tau(z). \]

Hence, it follows from (4.6) that

(4.7) \[ |\Delta f(z)| \leq \frac{4\alpha}{1 - |z|^2} \|D_f(z)\| \leq \frac{C_2}{(1 - |z|^2)^2}, \]
where $C_2 = \alpha(\alpha + 1)2^{\alpha+3}$. Furthermore,

$$|f(z)| \leq |f(0)| + \left| p \int_{[0,z]} df(\xi) \right|$$

$$\leq |f(0)| + \int_{[0,z]} \|Df(\xi)\| |d\xi|$$

$$\leq |f(0)| + \frac{C_1}{1 - |z|^2},$$

where $[0, z]$ denotes the line segment from 0 to $z$. Then, for $z \in \mathbb{D}$,

$$|f(z)|^{p-1} \leq 2^{p-2} \left( |f(0)|^{p-1} + \frac{C_1^{p-1}}{(1 - |z|^2)^{p-1}} \right),$$

and

$$|f(z)|^{p-2} \leq 2^{p-2} \left( |f(0)|^{p-2} + \frac{C_1^{p-2}}{(1 - |z|^2)^{p-2}} \right).$$

We divide the remaining part of the proof into two cases, namely, $p \in [4, \infty)$ and $p \in [2, 4)$. For the case $p \in [4, \infty)$, a straightforward computation gives

$$\Delta(|f|^p) = p(p-2) |f|^{p-4} |f\bar{f}_z + f_z\bar{f}|^2 + 2p |f|^{p-2} (|f_z|^2 + |f\bar{f}|^2) + p |f|^{p-2} \text{Re}(\bar{f}\Delta f)$$

$$\leq p^2 |f|^{p-2} \|Df\|^2 + p |f|^{p-1} |\Delta f|.$$
ON PROPERTIES OF THE SOLUTIONS TO THE $\alpha$-HARMONIC EQUATION

For $r \in (0, 1)$, $\Delta(F_p^n)$ and $F$ are integrable in $\mathbb{D}_r$, and $\Delta(F_p^n) \leq F$. By Lebesgue’s dominated convergence theorem, we have

$$
\lim_{n \to \infty} \int_{\mathbb{D}_r} (1 - |z|^2)^{p+1} \Delta(F_p^n(z)) \, dA(z) = \int_{\mathbb{D}_r} (1 - |z|^2)^{p+1} \lim_{n \to \infty} \left( \Delta(F_p^n(z)) \right) \, dA(z)
$$

$$
\leq \int_{\mathbb{D}_r} (1 - |z|^2)^{p+1} \left[ p^2 |f(z)|^{p-2} \|Df(z)\|^2 + p|f(z)|^{p-1} |\Delta f(z)| \right] \, dA(z)
$$

$$
\leq \int_{\mathbb{D}_r} \left[ p^2 2^{p-2} (|f(0)|^{p-2} + C_1^{p-2}) C_1^2 (1 - |z|^2)ight.
+ p 2^{p-2} (|f(0)|^{p-1} + C_1^{p-1}) C_2 \left. \right] \, dA(z)
$$

$$
< \infty.
$$

Therefore, (4.5) follows from the above two estimates and so the proof of the theorem is complete. □

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