ELECTROMAGNETIC SELF-DUALITY IN A LATTICE MODEL

Simon Hands
Department of Physics,
University of Wales, Swansea,
Singleton Park,
Swansea SA2 8PP, U.K.

and

John B. Kogut
Department of Physics,
University of Illinois at Urbana-Champaign,
1110 West Green Street,
Urbana, IL 61801-3080, U.S.A.

Abstract
We formulate a Euclidean lattice theory of interacting elementary spin-half electric and magnetic charges, which we refer to as electrons and magnetic monopoles respectively. The model uses the polymer representation of the fermion determinant, and exhibits a self-dual symmetry provided electric charge $e$ and magnetic charge $g$ obey the minimal Dirac quantisation condition $eg = 2\pi$. In a hopping parameter expansion at lowest order, we show that virtual electron and monopole loops contribute radiative corrections of opposite sign to the photon propagator. We argue that in the limit $e \to 0$, fermion mass $\mu \to 0$, the model describes QED together with strongly interacting monopoles whose chiral symmetry is spontaneously broken. Prospects for the existence of an interacting continuum limit at the self-dual point $e = g$ are discussed.

PACS: 11.15.Ha, 1.10.Lm, 14.80.Hv
Keywords: lattice, QED, magnetic monopole
1. Introduction

Ever since Maxwell’s equations were written down, it has been natural to speculate why there exist fundamental electric charges but apparently no magnetically-charged counterparts. At the classical level, it is straightforward to modify Maxwell’s equations for electric and magnetic field strengths (in appropriate units) by the introduction of magnetic charge and current densities \((\rho_m, j_m)\) analogous to the electric quantities \((\rho_e, j_e)\):

\[
\begin{align*}
\partial_0 E &= \nabla \wedge B - j_e, \\
\partial_0 B &= -\nabla \wedge E - j_m, \\
\nabla \cdot E &= \rho_e, \\
\nabla \cdot B &= \rho_m.
\end{align*}
\] (1.1)

The system of equations (1.1) is invariant under a duality transformation:

\[
\begin{align*}
E &\mapsto B; \quad B &\mapsto -E \\
(\rho_e, j_e) &\mapsto (\rho_m, j_m); \quad (\rho_m, j_m) &\mapsto -(\rho_e, j_e).
\end{align*}
\] (1.2)

The cost of this modification is that it is no longer possible to describe the electromagnetic field in terms of an abelian vector potential \(A_\mu\) which is both globally defined and non-singular. This raises difficulties, because the standard quantum-mechanical description of electromagnetism is formulated in terms of the potential rather than the field strengths, and indeed the success of the prototype quantum field theory, QED, depends critically on the local gauge symmetry transformations which relate physically equivalent configurations of the \(A_\mu\) field. Dirac [1] showed how to introduce magnetic monopoles as pointlike sources of magnetic flux by connecting them to line singularities in \(A_\mu\); so-called Dirac strings. Despite the singularity the string has no observable effect on a particle carrying electric charge \(e\), in either classical or quantum mechanics, if the strength of the magnetic charge \(g\) is quantized according to the famous condition

\[
e g = 2n\pi\hbar,
\] (1.3)

where \(n\) is an integer. Planck’s constant will henceforth be set to one. The spatial position of the string can be changed under gauge transformations, but the magnetic flux emanating from the monopole is uneffected. The Dirac quantization condition (1.3), or variations of it, has been central to all subsequent attempts to create a quantum theory which includes monopoles (for a comprehensive review see [2]).
Attempts to construct a quantum field theory containing both particles with electric charge and particles with magnetic charge – so-called Quantum ElectroMagnetoDynamics (QEMD) – began with Cabibbo and Ferrari [3] and Schwinger [4]. The latter formulation is in the Hamiltonian framework, and contains fermionic electron and monopole fields each interacting with its own vector potential field. The two potentials are not independent. The model also has an explicit dependence on a vector $\mathbf{s}$ defining the direction of a Dirac-type singularity, needed to relate the potentials to the current distributions defined by the matter fields. Later an equivalent Lagrangian formulation was given by Zwanziger [5]. In both cases the quantization condition was modified to be

$$eg = 4n\pi.$$  \hspace{1cm} (1.4)

The string-dependence obscures the Lorentz invariance of the results. In addition, since the string’s position is gauge-dependent, insistence on a fixed string vector $s_\mu$ leaves no possibility of a gauge invariant formalism. An alternative approach, beginning with [3], allows the string, rather than being fixed and infinite, instead to run between monopole-antimonopole pairs, resulting in a description in terms of closed particle world-lines spanned by surfaces known as Dirac sheets. This formulation is gauge invariant and stresses the path-dependence of the particle dynamics – and hence is inherently non-local. The requirement of surface-independence once again results in the Dirac quantization condition. The world-line approach was used by Brandt, Neri and Zwanziger [6] to demonstrate that gauge-invariant Green functions in the Lagrangian formulation are independent of the vector $s_\mu$ defining the string.

As we have sketched, a common feature of the various formulations, some of which have been shown to be equivalent [2], has been the impossibility of constructing a manifestly local, covariant and gauge invariant formulation. Another problematic issue has been how to proceed once the model is formulated – various authors have succeeded in finding “Feynman rules”, but these are of limited use due to the condition (1.3): QEMD is strongly interacting. Indeed, the origin of the quantization condition, the requirement that $\exp(ieg) = 1$, is inherently non-perturbative.

Calucci and Jengo [7], realising that the strongly-interacting nature of the problem necessitates new techniques, regularised a version of the worldline formalism [8] on a Euclidean spacetime lattice. A non-perturbative regularisation must be used if the ultraviolet behaviour of the model is to be discussed adequately. Their model has two field strength
tensors, defined on the direct and dual lattices respectively, and requires the more restrictive quantisation condition (1.4). The lattice is used as little more than a formal framework, but enables the use of a proper time technique to calculate the effects of virtual particle loops on the interaction among electric and magnetic charge strengths. Their claim is that if \(e_0\) and \(g_0\) are “bare” charge strengths satisfying (1.4), then the “renormalised” charges \(e_R, g_R\) satisfy

\[e_R g_R = e_0 g_0,\]  

(1.5)
ie, \(eg\) is a renormalisation group invariant. The result follows essentially, as we shall see, because of an extra factor of \(i\) in the electron–monopole interaction which reverses the sign of the vacuum polarisation diagram: monopoles “anti-screen” electric charge, and vice-versa. Although persuasive, this result is in contradiction to that obtained by other authors [9]. Moreover, the Feynman rules developed by Calucci and Jengo contain an ordering ambiguity which requires an artificial distinction to be drawn between valence and virtual current loops, once again reflecting the non-perturbative nature of the quantisation condition and suggesting that any approach based on series expansions in \(e\) and \(g\) will not be fully satisfactory.

As pointed out in [7], the self-dual point \(e = g\) appears to define a fixed point of the renormalisation group, since if the duality symmetry is respected by the regularisation then \(e_R = g_R\) and \(e_R g_R = \text{constant}\) together imply that neither \(e\) nor \(g\) is renormalised. The existence of an ultraviolet fixed point, or continuum limit, for abelian theories such as QEMD or indeed QED has been a matter of speculation for many years [10]. In perturbative QED we know the relation between bare and renormalised electric charge:

\[e_R^2(\mu) = \frac{e_0^2(\Lambda)}{1 + \frac{e_0^2}{6\pi^2} \ln \left(\frac{\Lambda}{\mu}\right)},\]  

(1.6)
where \(\Lambda\) is the cutoff and \(\mu\) a physical scale, implying that \(e_R \to 0\) as \(\Lambda/\mu \to \infty\). This phenomenon is known as triviality – QED can only exist as an interacting theory if the cutoff is finite. An alternative scenario requires the existence of a zero in the \(\beta\)-function describing renormalisation group flow of the interaction strength beyond the perturbative regime. One possibility was advanced by Miranskii and co-workers, who investigated the self-energy of the electron using a truncated system of Schwinger-Dyson equations [11]. For a critical value \(e_c = 2\pi/\sqrt{3}\) the chiral symmetry protecting the electron from acquiring mass is spontaneously broken. A physically meaningful ground state can only be found if
the bare charge $e_0$ obeys
\[ \frac{e_0^2(\Lambda)}{e_c^2} = 1 + \left( \frac{\pi}{\ln(\Lambda/\mu)} \right)^2, \] (1.7)
where $\mu$ is now related to the electron mass in the broken phase. Hence the continuum limit is taken as $e_0 \to e_c$. Note that $e_c$ is greater than either of the “minimal” self-dual values in QEMD $e = \sqrt{2\pi}, \sqrt{4\pi}$.

The Miranskii fixed-point hypothesis received support from numerical simulations of non-compact lattice QED (NCQED) [12], where it was found that for sufficiently strong electric charge the model exhibits a continuous phase transition from the perturbative phase to one in which chiral symmetry is broken, signalled by the non-vanishing of the condensate $\langle \bar{\psi}\psi \rangle$. However, intensive computer simulation has not found the scaling behaviour consistent with Miranskii’s prediction of an essential singularity as $e \to e_{c-}$: although there has been some dispute in the literature, currently the existing numerical data for the equation of state relating $\langle \bar{\psi}\psi \rangle$, $e$ and the bare electron mass $\mu$ are best fitted by a critical coupling $e_c$ in the region 2.2-2.3 (for the case of four dynamical fermion species), that is, less than the value predicted by the Schwinger-Dyson approach, with a power-law singularity with either non-mean field exponents [13], or mean field exponents with logarithmic corrections [14]. In either case the the continuum limit is thought to consist of both electrons and bosonic $e^+e^-$ bound states. The mean field fit is motivated by a picture in which the bound state charge radii and physical interaction strengths vanish so that the model is trivial.

To a large extent, the conclusions drawn from numerical simulations are subject to theoretical prejudice; the true fixed point theory, if it exists, may contain many other interactions apart from the electron-photon interaction of the minimal gauge coupling (alternatives for the lattice electron-photon interaction have been explored in [15,16]). Constraints on interaction Lagrangians due to the requirement of perturbative renormalisability may not apply at a strongly-coupled fixed point. Operators such as the chirally-symmetric $(\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2$ may acquire large anomalous scaling dimensions through non-perturbative effects and hence become relevant at the fixed point [17]. It is difficult to know a priori how to restrict the space of bare theories which need to be explored before an interacting continuum limit can be excluded. A defensible viewpoint is that the mere existence of an observed regime where perturbative QED applies (as an effective theory) implies the existence of an underlying theory containing massive degrees of freedom and/or short-ranged interactions which are so far imperfectly determined. Of course, it is more usual to consider
electromagnetism as part of a unified description of fundamental interactions, though it is worth recalling that the Standard Model gauge group contains U(1) as a subgroup, so the issue of a continuum limit for abelian gauge theories cannot as yet be ignored.

In this paper we shall explore the notion that the bare theory, and hence the fixed-point theory, are constrained to have the symmetry (1.2) of electromagnetic duality, by modifying lattice QED so that it has both electron and fermionic magnetic monopole degrees of freedom; in other words, we construct a lattice QEMD. We have been motivated in part by the observation that at the large values of $e$ in the vicinity of the observed chiral transition, the lattice photon field may be interpreted as containing line dislocations defined on the dual lattice, which are akin to magnetic currents [18]. The monopole dynamics in NCQED are unconventional, however, since the Dirac sheet in this formulation has non-vanishing action density, which means that in effect the monopoles probably exist in tightly bound dipole pairs with no associated long-ranged electromagnetic fields. Lattice monopoles bear a closer resemblance to Dirac ones when they occur as excitations in compact or periodic abelian gauge theory [19,20], in which the Dirac sheet costs no action and is physically unobservable; in this case the monopoles act as sources for long-ranged Coulomb fields. Banks et al [19] showed how the partition function for the Villain form of U(1) lattice gauge theory may be represented as a Coulomb gas of monopole point charges in three Euclidean dimensions or monopole world line loops in four dimensions by using an exact lattice duality transformation. By combining this formulation of lattice monopoles with the ideas of Calucci and Jengo, we are able to construct a theory with electron and monopole currents interacting via a Coulombic potential, with the charge strengths given by the less restrictive quantisation condition (1.3) with $n = 1$. Field strengths and gauge potentials, with all their associated complications [2], play a secondary role. The cost is that, as in all previous attempts, we are unable to give a local covariant action. Instead, there is a partition function written in terms of “polymer configurations” on both direct and dual lattices, based on the polymer representation of the lattice fermion determinant introduced by Karowski et al [21]. Because the fundamental excitations are fermions, an additional penalty is that the terms in the partition function fluctuate in sign. In the limit $e \to 0, g \to \infty$, we argue that the electron and magnetic sectors decouple, and that the model reduces to a local action describing electrons interacting via photon exchange as in QED. The monopoles’ chiral symmetry is spontaneously broken, so that the only light degrees of freedom in the monopole sector are monopole – antimonopole “magnetopions”
corresponding to the Goldstone mode associated with the broken symmetry. The only conceivable point where both electrons and monopoles can exist as light degrees of freedom is the self-dual point, which is therefore a candidate continuum limit for the model.

The rest of the paper is organised as follows. In Section 2 we introduce lattice QEMD in a heuristic fashion, by first discussing non-compact lattice QED in the polymer representation, and then reviewing the transformation from Villain lattice model to the monopole gas, before finally writing the QEMD partition function. We also briefly review connections with other approaches to QEMD. In section 3 we discuss the possible phase structure of the model, using exact results, analytic approximations and known numerical results. A hopping parameter expansion to lowest order is used to find that electron and monopole loops induce radiative corrections of opposite sign in the photon propagator, in agreement with [7]. Unfortunately it seems that in the minimal model there is no value of the couplings, including the self-dual point, for which both electrons and monopoles are light degrees of freedom; therefore probably no interacting continuum limit exists. The reasons are discussed in the final section, along with possible modifications of the model which may improve the prospects for a continuum limit, and phenomenological consequences.

2. A Lattice QEMD

In this section we build the QEMD partition function in simple stages. We begin with the partition function of non-compact lattice QED (NCQED):

$$Z_{NCQED} = \int \mathcal{D}\theta \det(D / \theta + \mu) \exp \left( -\frac{1}{4} \sum_{x,\mu,\nu} \Theta_{\mu\nu}^2(x) \right), \quad (2.1)$$

where the determinant results from integrating over fermion fields of bare mass $\mu$, leaving the photon fields $\theta_\mu \in (-\infty, +\infty)$ as real dynamical variables defined on the links of a hypercubic lattice. The field strength tensor $\Theta_{\mu\nu}$ is then defined by

$$\Theta_{\mu\nu}(x) \equiv \Delta_\mu^+ \theta_\nu(x) - \Delta_\nu^+ \theta_\mu(x), \quad (2.2)$$

with $\Delta_\mu^+$ the forward lattice difference operator. We choose to use the staggered fermion formulation (this is not crucial in what follows), so the fermion kinetic operator is given by

$$\bar{\psi}(x, y) = \frac{1}{2} \sum_\mu \eta_\mu(x) [\delta_{y, x+\dot{\mu}} \exp(i e \theta_\mu(x)) - \delta_{y, x-\dot{\mu}} \exp(-i e \theta_\mu(x - \dot{\mu}))], \quad (2.3)$$
where the Kawamoto-Smit phases $\eta_\mu(x) \equiv (-1)^{x_1 + \cdots + x_\mu - 1}$. The electric charge $e$ enters only via the gauge covariant connection in the derivative. To avoid a proliferation of summation signs in the formulae which follow we adopt the convention that repeated spacetime indices and repeated spatial arguments are summed over, even in squared quantities. We shall also employ the subscript $\ell$ in connection with oriented variables associated with a particular link.

The partition function in (2.1) is based on an action with a local gauge invariance, of course. For $Z$ to be strictly well-defined the gauge action must be gauge fixed to avoid the partition function from diverging on integration over a flat direction. We now immediately introduce one of the major conceptual tools we will need – the polymer representation of the fermion determinant introduced by Karowski et al [21]. We write

$$Z = \int \mathcal{D}\theta^N \sum_{\{\mathcal{C}\}} \kappa^{N_\mathcal{C}} \text{phase}(\mathcal{C}, \theta) \exp \left( -\frac{1}{4} \Theta_{\mu\nu}(x) \right), \tag{2.4}$$

where $\kappa$ is the hopping parameter $1/2\mu$ and $\mathcal{C}$ is a partition of the lattice in which every site is either isolated (a monomer), joined to a nearest neighbour by a double “bond” (a dimer), or visited once and once only by a closed oriented self-avoiding loop of single bonds (a polymer). The set of polymers in a particular configuration $\mathcal{C}$ will be denoted $\{\Gamma\}$. Each configuration $\mathcal{C}$ corresponds to a unique term in the expansion of the determinant: the $N_m$ monomers correspond to diagonal elements, and hence contribute a factor $\mu^{N_m}$, and the $N_\ell$ bonds correspond to off-diagonal hopping terms. Clearly we have

$$N_m + N_\ell = N \tag{2.5}$$

where $N$ is the lattice volume. The phase factor has a number of different components: a signed permutation factor from the expansion of the determinant; a sign from the path over forward and backward links and the Kawamoto-Smit phases (we shall refer to this as the “signed KS phase”); and finally the gauge connection $\prod_{\ell \in \mathcal{C}} \exp(i \theta_{\ell})$. A moment’s thought reveals that monomers and dimers give trivial contributions to the phase – only the polymers $\{\Gamma\}$ contribute. The smallest polymer which gives a negative sign is a non-planar loop of length six. We can rewrite $Z$ as follows:

$$Z \propto \int \mathcal{D}\theta \sum_{\{\mathcal{C}\}} \left[ \kappa^{-N_m} (-1)^{N_{\Gamma}} \left( \prod_{\ell \in \{\Gamma\}} \eta_\ell \right) \left( \prod_{\Gamma \in \{\Gamma\}} U(\Gamma) \right) \exp \left( -\frac{1}{4} \Theta_{\mu\nu}(x) \right) \right] \tag{2.6}$$

$$\equiv \sum_{\{\mathcal{C}\}} \kappa^{-N_m} (-1)^{N_{\Gamma}} \left( \prod_{\ell \in \{\Gamma\}} \eta_\ell \right) Z_{\text{photon}}[\Gamma],$$
where \( N_\Gamma \) is the number of polymers, \( \eta_\ell \) is the signed KS factor associated with the link \( \ell \), and \( U(\Gamma) = \prod_{\ell \in \Gamma} \exp(ie\theta_\ell) \) is the Wilson loop associated with the polymer \( \Gamma \) (note that since the polymers are oriented, then both \( U(\Gamma) \) and \( U^*(\Gamma) \) appear as separate terms in \( Z \) with equal weight). \( Z_{\text{photon}}[\Gamma] \) is simply the partition function for free photons coupled to a Wilson loop distribution \( \{\Gamma\} \). Note the sign factor associated with the number of loops: the relation with Fermi-Dirac statistics is clear. Note also that despite the superficial similarity this is not the conventional hopping parameter expansion – the sum over \( \{C\} \) is finite on a finite system, and this expression for \( Z \) is valid for all \( \mu \).

Now let’s focus on \( Z_{\text{photon}}[\Gamma] \). This can be written

\[
Z_{\text{photon}}[\Gamma] = \int_{-\infty}^{+\infty} \prod_{\ell} d\theta_\ell \exp \left( -\frac{1}{4} (\Delta^+ \theta_\nu(x) - \Delta^+ \theta_\mu(x))^2 + ie \sum_{\ell \in \{\Gamma\}} j_\ell \theta_\ell \right),
\]

where \( j_\ell \) is the characteristic function of the polymer configuration, ie

\[
j_\ell = \begin{cases} 
+1, & \text{if } \ell \text{ a forward link } \in \{\Gamma\}; \\
-1, & \text{if } \ell \text{ a backward link } \in \{\Gamma\}; \\
0, & \text{otherwise.}
\end{cases}
\]  

(2.8)

Of course, \( j_\ell \) is nothing other than the electric current. By performing a shift of the integration variables \( \theta_\ell \), we can rewrite

\[
Z_{\text{photon}}[j] = Z_{\text{photon}}[0] \exp \left( -\frac{e^2}{2} j_\mu(x) v_{\mu\nu}(x) j_\nu(y) \right).
\]  

(2.9)

Here \( v_{\mu\nu}(x) \) is the lattice Coulomb propagator, whose existence requires a gauge-fixing term in the photon action. In Feynman gauge it satisfies

\[
\Delta^+_\rho \Delta^-_\rho v_{\mu\nu}(x) = -\delta_{\mu\nu}\delta(x),
\]

(2.10)

where \( \Delta^-_\rho \) is the backward difference operator. We have written the arguments \( x, y \) to stress that the interaction between the current loops is non-local in this representation. The lattice Coulomb propagator is finite for zero spatial separation, and for large \( |x| \) behaves as \( |x|^{-2} \). Hence, in the polymer representation of the fermion determinant, lattice QED resembles a Coulomb gas of unit charged non-intersecting loops of variable sign:

\[
Z_{\text{NCQED}} \propto Z_{\text{photon}}[0] \sum_{\{C\}} k^{-N_m} (-1)^{N_\Gamma} \left( \prod_{\ell \in \{\Gamma\}} \eta_\ell \right) \exp \left( -\frac{e^2}{2} j_\ell v_{ll'} j_{l'} \right).
\]  

(2.11)
It is important to note, however, that the role of monomers and dimers is also important, as we shall see.

Next, we review another well-known model – the Villain approximation to $U(1)$ lattice gauge theory \cite{19}.

\[ Z_{\text{Villain}} = \int_{-\pi}^{\pi} \prod_{\ell} \frac{d\theta_{\ell}}{2\pi} \sum_{\{n\}} \exp \left( \frac{i}{2} n_{\mu\nu}(x) \Theta_{\mu\nu}(x) - \frac{e^2}{4} n_{\mu\nu}^2(x) \right), \]  

where $n_{\mu\nu}(x)$ are integers defined on each plaquette of the lattice. The Villain action looks a little strange, but is simply the convolution of the Gaussian non-compact gauge action $\Theta^2$ with an infinite array of delta functions having period $2\pi$. Around any one of these locations the Villain action has approximately the Gaussian form of NCQED, but by construction it is also periodic in $\Theta$, and in this respect is similar to the Wilson form ($1 - \cos \Theta$). Its great virtue is the existence of a sequence of exact transformations which mean its phase structure may be understood in terms of monopole excitations. The integral over $\{\theta\}$ can be performed immediately to leave a constrained action of the $n_{\mu\nu}$:

\[ Z_{\text{Villain}} = \sum_{\{n\}} \delta(\Delta_{\nu} n_{\nu\mu}) \exp \left( -\frac{e^2}{4} n_{\mu\nu}^2(x) \right). \]  

The constraint can be solved by rewriting in terms of another integer variable $l$:

\[ n_{\mu\nu}(x) = \epsilon_{\mu\nu\lambda\kappa} \Delta_{\nu}^+ l_{\mu}(\tilde{x}). \]  

Here, $l$ is defined on the links of the dual lattice, as indicated by the tilde on $x$, and $\epsilon$ is the totally antisymmetric tensor. As a rule, any expression containing an odd number of $\epsilon$ symbols relates variables living on direct and dual lattices. Note that (2.14) does not uniquely specify $l$, or even require it to be integer-valued, since the definition of $n$ remains invariant under

\[ l_{\mu}(\tilde{x}) \mapsto l_{\mu}(\tilde{x}) + \Delta_{\mu}^+ \Lambda(\tilde{x}), \]  

where $\Lambda$ is an arbitrary scalar function. Now we can write

\[ Z_{\text{Villain}} = \sum_{\{l\}} \exp \left( -\frac{e^2}{4} (\Delta_{\mu}^+ l_{\nu}(x) - \Delta_{\nu}^+ l_{\mu}(x))^2 \right). \]  

This expression is very reminiscent of NCQED, except that the dynamical variables are integers, not real numbers. The partition function may now be reexpressed, first by using the Poisson formula

\[ \sum_{\{l\}} f(l) = \sum_{\{m\}} \int_{-\infty}^{\infty} d\phi f(\phi) e^{2\pi im\phi}, \]  

10
where the $m$ are integer, and then by shifting the $\phi$ variable in the resulting Gaussian integral as before. The result is

$$Z_{\text{Villain}} = Z_{\text{photon}} \sum_{\{m\}} \exp \left( \sum_{\ell \ell'} -\frac{g^2}{2} m_{\ell} v_{\ell} m_{\ell'} \right). \quad (2.18)$$

Here, $m_{\ell}$ are integer variables defined on the links of the dual lattice, which obey the constraint $\Delta_m m_\mu (\tilde{x}) = 0$ (since otherwise (2.18) would not be invariant under gauge transformations (2.15) on $\phi$). These two facts constrain the $m$ to form closed loops on the dual lattice, which interact via the Coulomb potential as in NCQED. Once again the partition function may be thought of as a sum over polymer configurations, though this time the polymers need be neither singly charged nor self-avoiding. The coupling strength is $g \equiv 2\pi/e$; the excitations thus resemble magnetic current loops with charge specified by the quantisation condition (1.3).

We can confirm this picture of interacting magnetic current loops, by coupling gauge-covariant fermions to the Villain model, to yield a model we will call “periodic QED” (PQED) to distinguish it from NCQED. From what we already know, we can write

$$Z_{\text{PQED}} = \int \mathcal{D}\theta \det (\mathcal{D}/\theta + \mu) \sum_{\{n\}} \exp \left( \frac{i}{2} n_{\mu\nu}(x) \Theta_{\mu\nu}(x) - \frac{e^2}{4} n_{\mu\nu}^2 (x) \right)$$

$$= \sum_{\{\eta\}} \kappa^{-N_m} (-1)^{N_t} \left( \prod_{\ell \in \{\Gamma\}} \eta_\ell \right) Z_{\text{Villain}}[\Gamma]. \quad (2.19)$$

Here, $Z_{\text{Villain}}[\Gamma]$ is the same partition function as that of (2.12), but this time in the presence of an electric current distribution defined by $\{\Gamma\}$. The steps (2.12-18) can be repeated to yield

$$Z_{\text{PQED}} = Z_{\text{photon}} [0] \sum_{\{\xi\}} \kappa^{-N_m} (-1)^{N_t} \left( \prod_{\ell \in \{\Gamma\}} \eta_\ell \right) \sum_{\{j\}} \delta_{j, \ell \in \{\Gamma\}} \sum_{\{S\}} \delta_{j, \Delta, \xi} S_{\xi \mu}$$

$$\times \sum_{\{m\}} \exp \left( -\frac{e^2}{2} j_\mu (x) v_{\mu\nu} (x - y) j_\nu (y) \right) \exp \left( -\frac{g^2}{2} m_\mu (\tilde{x}) v_{\mu\nu} (\tilde{x} - \tilde{y}) m_\nu (\tilde{y}) \right)$$

$$\times \exp \left( -2\pi i m_\mu (\tilde{x}) v_{\mu\nu} (\tilde{x} - \tilde{y}) \frac{1}{2} \xi_{\nu\lambda\rho\sigma} \Delta^{+}_{\lambda} S_{\rho\sigma} (y + \hat{\nu}) \right). \quad (2.20)$$

To implement the steps (2.12-18) it has been necessary to introduce the oriented characteristic function $S_{\mu\nu}$ for a surface which spans the self-avoiding loops in the polymer.
expansion, but is otherwise unconstrained. Due to the Kronecker δ’s both \( j \) and \( S \) may be regarded as auxiliary fields: a given configuration is uniquely specified by \( \mathcal{C} \) and \( m^\dagger \). However, in the form (2.20) the similarity between the electric and magnetic currents is suggestive. The model has a new term – the last exponential in (2.20) – which describes the interaction between electric and magnetic currents (the spacetime argument of \( S \) is determined by the requirement that the Coulomb propagator couples to a conserved current at either end). To see that this is indeed an interaction between magnetic and electric current distributions, consider the distribution of a static magnetic charge \( m_0(\tilde{x}) \) with a spacelike surface \( S_{ij}(y + \hat{0}) \), with \( i, j \) ranging from 1 to 3. The interaction may be written in the form \( igm_0(\tilde{x})V_{\text{mag}}(\tilde{x}) \) where \( V_{\text{mag}} \) is the magnetic scalar potential:

\[
V_{\text{mag}}(\tilde{x}) = \sum_y \left[ -\frac{1}{2} \varepsilon_{ijk} \Delta^\dagger_i v_3 d(\tilde{x} - \tilde{y}) S_{jk}(y + \hat{0}) \right]. \tag{2.21}
\]

The factor of \( i \) is a consequence of the formulation in Euclidean space. In the long wavelength limit the contribution to \( V_{\text{mag}} \) from a current loop spanned by a surface \( S \) assumes its textbook form:

\[
V_{\text{mag}}(x, S) = \frac{e}{4\pi} \int_S \frac{\mathbf{r} \cdot d\mathbf{S}}{r^3} = e \frac{\Omega}{4\pi}, \tag{2.22}
\]

where \( \Omega \) is the solid angle subtended by the circuit at \( x \). Hence the interaction only depends on the boundary of the loop \( \partial S \). The magnetic scalar potential is not single valued, since \( \Omega \) is only defined modulo \( 4\pi \): however, this has no physical consequence in the present case due to the Dirac quantisation condition, as we shall now discuss.

To see explicitly that the interaction only depends on the edge of the surface, consider the following sequence of transformations, called the “disentangling theorem” in ref. [8], on the monopole-electric current interaction term in Feynman gauge; we define \( D_{\mu\nu} \) to be the integer-valued characteristic function for a Dirac sheet which spans the \( m \) loop as \( S \)

\[ \dagger \text{This is not true in a finite volume – we are grateful to John Stack for pointing this out} \]
spans the $j$ loop:

$$m_\mu(\tilde{x}) v_{\mu\nu}(\tilde{x} - \tilde{y}) \frac{1}{2} \epsilon_{\nu\lambda \rho \sigma} \Delta^+ S_{\rho \sigma}(y + \hat{\nu})$$

$$= \Delta^- D_{\tau \mu}(\tilde{x}) v(\tilde{x} - \tilde{y}) \frac{1}{2} \epsilon_{\mu \lambda \rho \sigma} \Delta^+ S_{\rho \sigma}(y + \hat{\mu})$$

$$= -\frac{1}{2} D_{\tau \mu}(\tilde{x}) \Delta^- \Delta^+ v(\tilde{x} - \tilde{y} + \hat{\tau} + \hat{\mu}) \epsilon_{\mu \lambda \rho \sigma} S_{\rho \sigma}(y)$$

$$= -\frac{1}{2} \Delta^+ D_{\tau \mu}(\tilde{x}) \Delta^- \Delta^+ v(\tilde{x} - \tilde{y} + \hat{\tau} + \hat{\mu}) \epsilon_{\tau \mu \rho \sigma} S_{\rho \sigma}(y) \delta(\tilde{x} - \tilde{y} + \hat{\tau} + \hat{\mu})$$

$$= -\frac{1}{2} \Delta^- D_{\tau \mu}(\tilde{x} - \hat{\lambda} - \hat{\mu} - \hat{\tau}) v(x - y) \epsilon_{\tau \mu \alpha \lambda} \Delta^- S_{\rho \sigma}(y) - \frac{1}{4} D_{\tau \mu}(\tilde{x}) \epsilon_{\tau \mu \rho \sigma} S_{\rho \sigma}(x + \hat{\tau} + \hat{\mu})$$

$$= - j_\sigma(y) v(y - x) \frac{1}{2} \epsilon_{\sigma \lambda \tau \mu} \Delta^+ D_{\tau \mu}(\tilde{x} - \hat{\lambda} - \hat{\mu} - \hat{\tau}) - \frac{1}{4} D_{\tau \mu}(\tilde{x}) \epsilon_{\tau \mu \rho \sigma} S_{\rho \sigma}(x + \hat{\tau} + \hat{\mu}).$$

(2.23)

We have used translation invariance $v(\tilde{x}) = v(x) = v(-x)$, and the identity

$$[\epsilon_{\lambda \tau \rho \sigma} \Delta^- \Delta^+ + \epsilon_{\mu \lambda \rho \sigma} \Delta^- \Delta^+ + \epsilon_{\mu \tau \rho \sigma} \Delta^+ \Delta^+ + \epsilon_{\mu \tau \rho \sigma} \Delta^- \Delta^+] v(x) = -\epsilon_{\mu \tau \rho \sigma} \delta(x).$$

(2.24)

We have also made the lattice coordinates explicit, resulting in a slightly unwieldy expression. This is due to the offset of the dual lattice origin from the direct one. More compact notations, which make use of the language of differential forms, are available (e.g. [7,22]).

The first thing to notice is that the expression $\frac{1}{4} D dS$ is an integer (since $D$ and $S$ are themselves integer) – the intersection number of the two surfaces multiplied by the monopole and electron charges. Since the coefficient of the term is $-2\pi i \equiv -ieg$, this term contributes unity to $Z_{\text{PQED}}$ and hence has no dynamical influence. Therefore the electron-monopole interaction is manifestly independent of the interior of either of the surfaces $S$ or $D$ – which are unconstrained by the dynamics – but only their boundaries, respectively the electric current $j$ or the magnetic current $m$. It is important to note that these currents are constrained to lie on direct and dual lattices, and that the surface independence, which is a necessary requirement for the recovery of a local limit for the quantum field theory, depends on the Dirac quantisation condition (1.3). Essentially the same argument was put forward originally in [6].

The interaction between electric and magnetic charges is best described on the lattice by a non-local interaction between conserved currents. However, it is useful at this stage to pause and review the connection between the formulation appearing in (2.20) and the more familiar description in terms of local potentials and field strengths. From the discussion preceding (2.21), we recall that magnetic charge has an interaction energy depending on
a magnetic scalar potential $V_{mag}$. By treating $V_{mag}$ as the zeroth component of a four vector $\tilde{A}_\mu$, it is possible to write the interaction term for electric current as $e j_\mu A_\mu$, and the interaction for magnetic current as $g m_\mu \tilde{A}_\mu$, where the two potential fields $A_\mu$ and $\tilde{A}_\mu$ are given, in continuum notation and in Feynman gauge, by

$$A_\mu = e^{\frac{-1}{\partial^2}} j_\mu + ig \frac{1}{2} \epsilon_{\mu\lambda\rho\sigma} \partial_\lambda \frac{-1}{\partial^2} D_{\rho\sigma};$$

$$\tilde{A}_\mu = g \frac{-1}{\partial^2} m_\mu - i e \frac{1}{2} \epsilon_{\mu\lambda\rho\sigma} \partial_\lambda \frac{-1}{\partial^2} S_{\rho\sigma}.$$

(2.25)

Once again, the factors of $i$ associated with the $\epsilon$ symbols are due to the Euclidean space formulation. Field strength tensors $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ can now be defined in the usual way by taking the four-dimensional curl of the potentials; using (2.24) we find the relation between them:

$$-i \frac{1}{2} \epsilon_{\mu\nu\lambda\kappa} (F_{\lambda\kappa} + e S_{\lambda\kappa}) = \tilde{F}_{\mu\nu} - g D_{\mu\nu}.$$  

(2.26)

Hence both $F$ and $\tilde{F}$ contain ambiguities due to Dirac sheets, which in the continuum theory must be resolved by the introduction of compensation terms supported only on the sheets [2]. Of course, both surfaces $S$ and $D$ may be moved around at will by gauge transformations, and hence have no physical meaning. In continuum formulations of QEMD, in order to avoid double-counting, it is customary to express the ambiguity solely in terms of $D_{\mu\nu}$, but relations (2.25-26) expose the duality symmetry to the full. On the lattice problems never arise so long as an action which is a periodic function of the field strength, such as (2.12), is chosen; the ambiguity then simply relates physically equivalent configurations of the $F$ field (for the case of the Wilson action for U(1) lattice gauge theory see [20]).

The final step in the construction of QEMD is to endow the monopole excitations with specific dynamical properties. In eqn. (2.20) the only difference between monopoles and electrons are the constraints imposed on the electric current loops by the polymer representation of the fermion determinant, ie. that $C$ be a partition of the lattice into monomers, dimers and self-avoiding (and thus singly-charged) polymers. Suppose we now insist that the dual lattice be similarly partitioned by specifying $\tilde{C}$ consisting of dual
monomers, dimers and polymers, which define loops of magnetic current. We then write

\[
Z_{QEMD} = Z_{\text{photon}}[0] \sum_{\{C\}} \sum_{\{\bar{C}\}} \kappa^{-N_m} \bar{\kappa}^{-\bar{N}_{\bar{m}}} (-1)^{N_{\Gamma} + \bar{N}_{\bar{\Gamma}}} \left( \prod_{\ell \in \{\Gamma\}} \eta_{\ell} \right) \left( \prod_{\bar{\ell} \in \{\bar{\Gamma}\}} \eta_{\bar{\ell}} \right) \\
\times \sum_{\{j\}} \delta_{j,\ell \in \{\Gamma\}} \sum_{\{S\}} \delta_{j,\Delta_{\mu}} S_{\nu \mu} \sum_{\{m\}} \delta_{m,\bar{\ell} \in \{\bar{\Gamma}\}} \sum_{\{D\}} \delta_{m,\Delta_{\nu}} D_{\nu \mu} \\
\times \exp \left( -\frac{e^2}{2} j_{\mu}(x) v_{\mu \nu}(x - y) j_{\nu}(y) \right) \exp \left( -\frac{g^2}{2} m_{\mu}(\bar{x}) v_{\mu \nu}(\bar{x} - \bar{y}) m_{\nu}(\bar{y}) \right) \\
\times \exp \left( 2\pi i m_{\mu}(\bar{x}) v_{\mu \nu}(\bar{x} - \bar{y}) \frac{1}{2} \epsilon_{\nu \lambda \rho \sigma} \Delta_{\lambda}^{+} S_{\rho \sigma}(y + \hat{\nu}) \right).
\]  

(2.27)

Each configuration is completely specified by \(C, \bar{C}\); the variables \(j, m, S\) and \(D\) are auxiliary. Note that the monopole fields have been given the same hopping parameter \(\kappa\) as the original fermion fields. Now, the crucial point is that the model defined by (2.27) is invariant under the following duality transformation:

\[
j_{\mu} \leftrightarrow m_{\mu} \quad e \leftrightarrow g \quad D \leftrightarrow S \quad Z \leftrightarrow Z^* \quad (2.28)
\]

The manipulations (2.23) play a crucial role in demonstrating this. Note further that under the interchange \(m_{\mu} \leftrightarrow -m_{\mu}\), every term in \(Z_{QEMD}\) remains unchanged (since the number of links in every polymer is even, the factors \(\prod_{\bar{\ell}} \eta_{\bar{\ell}}\) remain unaltered) except the argument of the last exponential, which changes sign. Since this last term is a phase, and every \(m\)-polymer loop appears with equal weight going in either direction, we find that

\[
Z_{QEMD} \equiv Z_{QEMD}^*.
\]  

(2.29)

Therefore the model is self-dual, respecting the symmetry (1.2): in particular at the self-dual point \(e = g = \sqrt{2\pi}\) the electron and monopole degrees of freedom behave identically.

To complete the formal exposition of the model, three final comments are needed. Firstly, although the role of the gauge potential has been suppressed in our treatment, notice that the interactions in the model (2.27) are all described in terms of conserved integer-valued currents; hence the model is clearly also gauge invariant. Next, although in the forms (2.6), (2.20) and (2.27) the various partition functions \(Z_{NCQED}\), \(Z_{PQED}\) and \(Z_{QEMD}\) consist of terms with fluctuating sign, in the first two cases of NCQED and PQED we know that the partition functions can also be expressed via a positive definite (albeit non-local) effective action, and hence are themselves positive definite on a finite lattice.
No such argument can be used for QEMD, since it is not expressible simultaneously in terms of both local electron and monopole fields. A demonstration of the positivity of $Z_{QEMD}$ remains an open question, and is desirable for the discussion of the possible phase structure of the model (see next section). Thirdly, in the form (2.27) the magnetic current $m_\mu$ transforms as a vector, by analogy with the electric current. Even in the classical theory descibed by (1.1), it is impossible to satisfy invariance simultaneously under discrete parity (P) and charge-conjugation (C) transformations. For a vector $m_\mu$, under P:

$$E \mapsto -E; \quad B \mapsto B;$$

$$\rho_\epsilon \cdot j_\epsilon \mapsto (\rho_\epsilon \cdot -j_\epsilon); \quad (\rho_m \cdot j_m) \mapsto (\rho_m \cdot -j_m),$$

whereas under C:

$$E \mapsto -E; \quad B \mapsto -B;$$

$$\rho_\epsilon \cdot j_\epsilon \mapsto (-\rho_\epsilon \cdot -j_\epsilon); \quad (\rho_m \cdot j_m) \mapsto (-\rho_m \cdot -j_m).$$

The QEMD Maxwell equations (1.1) are invariant under C but not P, or hence CP. If $m_\mu$ is defined to be an axial vector, then under P:

$$\rho_m \cdot j_m \mapsto (-\rho_m \cdot j_m),$$

and under C:

$$\rho_m \cdot j_m \mapsto (\rho_m \cdot j_m),$$

in which case (1.1) are now invariant under P, but not C or CP. A Euclidean lattice analogue of this latter case can also be constructed if the monopole conserved current is derived using a kinetic term of the form

$$\langle \Phi + \mu \rangle (x, y) = \frac{1}{2} \sum_\mu \eta_\mu(x) \varepsilon(x) \left[ \delta_{y,x+\hat{\mu}} \exp(i e \theta_\mu(x)) - \delta_{y,x-\hat{\mu}} \exp(-i e \theta_\mu(x-\hat{\mu})) \right]$$

$$+ \mu \delta_{y,x} \varepsilon(x).$$

Here, $\varepsilon(x)$ is the alternating phase $(-1)^{x_1+x_2+x_3+x_4}$. The result of this modification in the polymer expansion is to reverse the sign of every loop $\Gamma$ containing $(2n+2)$ links.

3. Phase Structure

In this section we will try to discuss the lattice model described in section 2 more quantitatively, with particular emphasis on its phase structure. We shall draw insight
and information from what is already known about NCQED, Villain QED, and PQED. First, however, we shall consider the limit of large fermion bare mass $\mu$, in which case an expansion in powers of $1/\mu$ is viable [23].

For large $\mu$, the partition function in (2.27) is dominated by configurations in which $N_m$ and $N_{\tilde{m}}$, the numbers of direct and dual monomers respectively, are large. Both dimers and polymers must be considered as excitations suppressed by powers of $\kappa = 1/2\mu$. Only polymers, however, interact with the electromagnetic field, and hence with other loops, via the exponential terms of (2.27). Consider the smallest monopole loop excitation, simply four dual links bounding a dual plaquette. By considering the photon exchange diagrams of figure 1, and using the Feynman gauge expression

$$v_{\mu\nu}(\tilde{x} - y) = \delta_{\mu\nu} \int_p \frac{\exp(ip.(x - y))}{S(p)},$$

with

$$\int_p \equiv \int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4}$$

and

$$S(p) = 4 \sum_{\mu} \sin^2 \left( \frac{1}{2} p_{\mu} \right),$$

we obtain the result that a single loop excitation is suppressed by the factor $\kappa^4 \exp(-g^2/4)$ [23]. We have folded in the other factors from (2.27): -1 for a fermion loop, -1 from the signed KS phase, and $\kappa^4$ from the hopping parameters. Now, consider the effect of such a loop on a photon propagating between electric currents on the direct lattice at sites 0 and $y$. From the disentangling transformation (2.23) we know that electric current interacts with the four-dimensional curl of the Dirac sheet, which for the simple dual plaquette excitation is a plaquette on the direct lattice. We can then consider the process shown in figure 2.

Summing over all possible positions and orientations of the loop, we find for the contribution $v_{\mu\nu}^{[1]}$ to the “dressed” propagator:

$$v_{\mu\nu}^{[1]}(0, y) = 2g^2 \sum_x \sum_{\alpha > \beta} v_{\mu\alpha}(0, x) \times [2v_{\alpha\nu}(x, y) - v_{\alpha\nu}(x + \hat{\beta}, y) - v_{\alpha\nu}(x - \hat{\beta}, y) - v_{\beta\nu}(x, y) + v_{\beta\nu}(x + \hat{\alpha}, y) + v_{\beta\nu}(x - \hat{\beta}, y) - v_{\beta\nu}(x + \hat{\alpha} - \hat{\beta}, y)].$$

It is most convenient to evaluate (3.2) in Landau gauge;

$$v_{\mu\nu}(x, y) = \int_p \frac{\exp(ip.(x - y))}{S(p)} P_{\mu\nu}(p),$$

(3.3)
with the lattice transverse projection operator $P_{\mu\nu}(p)$ given by

$$P_{\mu\nu}(p) = \delta_{\mu\nu} - \sum_{\rho} \sin \left( \frac{1}{2}p_\rho \right) \sin \left( \frac{1}{2}p_{\mu} \right) \sin \left( \frac{1}{2}p_{\nu} \right).$$

(3.4)

We find

$$v_{[1]}{_{\mu\nu}}(0,y) = g^2 \int \frac{e^{-ip\cdot y}}{S(p)} \left\{ P_{\mu\nu}(p) - \sum_{\alpha\beta} P_{\mu\alpha}(p)P_{\beta\nu}(p) \frac{(1 - e^{-ip\alpha})(1 - e^{-ip\beta})}{S(p)} \right\}.$$  

(3.5)

In the long wavelength limit $p \to 0$ the second term in braces is $O(p^2)$ and hence irrelevant. Combining this result with the suppression factor of figure 1, we obtain for the full photon propagator $v_{\mu\nu} = v_{[0]}{_{\mu\nu}} + v_{[1]}{_{\mu\nu}}$:

$$\lim_{y \to \infty} v_{\mu\nu}(0,y) = \int_p \frac{e^{-ip\cdot y}}{p^2} P_{\mu\nu}(p) \left( 1 + g^2 \kappa^4 \exp \left( -\frac{g^2}{4} \right) \right).$$

(3.6)

Expression (3.6) is of the same form as the bare continuum propagator, except that the overall strength of the interaction has been rescaled, that is, electric charge has been renormalised: $e_R^2 = Z_g e^2$, with the renormalisation constant $Z_g$ given by

$$Z_g = 1 + \frac{g^2}{8\mu^4} \exp \left( -\frac{g^2}{4} \right).$$

(3.7)

This expression is the first term of an expansion in $1/\mu$, but is non-perturbative (and exact) in $g$ (it is also worth noting that since the fermion determinant is polynomial in $\mu$, the expansion in $1/\mu$ is finite, and there are no non-perturbative contributions). The most interesting aspect is the $+$ sign of the correction, which is in contrast to the charge renormalisation due to an electric current loop found originally for NCQED in [23]:

$$Z_e = 1 - \frac{e^2}{8\mu^4} \exp \left( -\frac{e^2}{4} \right).$$

(3.8)

Of course, the difference has its origin in the factor $ieg$ in the interaction between electric and magnetic currents, and supports the claim made in the introduction that monopoles anti-screen electric charge. Indeed, at the self-dual point $e = g$, $Z_e = Z_g^{-1}$ and bare charges remain unrenormalised to this order. This agrees with the results of Calucci and Jengo [7], who considered an expansion not in inverse mass but in number of fluctuating loops. Because they considered scalar matter fields, they were able to use a proper time formalism and consider the current loops as continuum worldlines. They also found that
a charge renormalisation was caused by the effect of small intermediate loops (ie. from the limit $T \to 0$, where $T$ parametrises the length of the worldline loop), although the precise details of their regularisation are left hazy. It would be interesting to pursue the inverse mass expansion further to see if the self-duality persists (indeed, the expansion was originally suggested as a means of treating fermion fields in [7]). For larger magnetic loop excitations, the dual loop interacting with the electric current is no longer the same shape; indeed, a large planar monopole loop corresponds to an array of elementary electric current plaquettes in the orthogonal directions. It seems highly plausible, however, that Calucci and Jengo’s geometrical arguments that electric and magnetic loops of the same size and shape will induce opposite charge renormalisations will continue to hold in the limit that loop size is much less than photon wavelength, although the fluctuating signs due to the fermionic nature of the currents may present a complication.

Now, in order to define a continuum field theory, we need to approach the continuum limit, in which all particles are light; the $1/\mu$ expansion cannot help in this regard. Next we consider the limit $e \to 0$, $g \to \infty$. By self-duality, our findings will also hold in the dual limit $e \to \infty$, $g \to 0$. For $g$ sufficiently large (see below), the exponential term $\exp(-\frac{g^2}{2}m_\ell v_{\ell\ell}m_{\ell\ell}')$ in (2.27) alone will suppress monopole polymers, and the monopole sector will be described entirely in terms of monomers and dimers, which do not interact with the electromagnetic field. In this limit the argument of the third exponential term in (2.27) describing the monopole-electron interaction also vanishes; only loops which subtend a non-zero solid angle feel this term. Therefore $Z_{QEMD}$ factorises:

$$\lim_{e \to 0, g \to \infty} Z_{QEMD} = Z_{NCQED}(e)Z_{NCQMD}(g). \quad (3.9)$$

By reversing the sequence of transformations leading to eqn. (2.11) we see that the electron sector is now governed solely by NCQED (2.1), which may be adequately treated in perturbation theory. The staggered lattice fermion formulation used here is known to describe four physical fermion species; in the long wavelength limit we recover continuum QED with $N_f = 4$. The monopole sector, on the other hand, is saturated by monomers and dimers. It is known rigorously [24] that in the chiral limit $\mu \to 0$ ($\kappa \to \infty$) the chiral symmetry of the monopole degrees of freedom is spontaneously broken. In continuum field theory, this is signalled by a non-vanishing condensate $\langle \bar{\psi} \psi \rangle$, where $\psi, \bar{\psi}$ are monopole field operators. In the polymer language this symmetry-breaking order parameter is given by
\[ \lim_{\mu \to 0} \langle \bar{\psi} \psi(\mu) \rangle = \frac{1}{V} \left. \frac{\partial \ln Z_{NCQMD}}{\partial \mu} \right|_{\mu=0} = \lim_{\mu \to 0} \frac{1}{V} \langle N_{\bar{m}} \rangle. \] (3.10)

Symmetry breaking is signalled by the equilibrium concentration of monomers vanishing with some power of \( \mu \) less than or equal to one.

In the staggered fermion formulation, global chiral transformations are defined by a U(1) rotation:

\[ \psi(x) \mapsto \exp(i\alpha \varepsilon(x))\psi(x) \; ; \; \bar{\psi}(x) \mapsto \exp(i\alpha \varepsilon(x))\bar{\psi}(x). \] (3.11)

Hence in the polymer representation these rotations leave both polymers and dimers unaffected, since each contain an equal number of odd and even sites. Electron and monopole fields can be independently rotated – so in the \( \mu \to 0 \) limit \( Z_{QEMD} \) has a \( U(1)_V \otimes U(1)_A \otimes U(1)_{\bar{V}} \otimes U(1)_{\bar{A}} \) symmetry, with \( V \) and \( A \) standing for vector and axial vector symmetries respectively on both direct and dual lattices. Thus for \( e \to 0 \; g \to \infty \), the condensate \( \langle N_{\bar{m}} \rangle \) breaks the symmetry to \( U(1)_V \otimes U(1)_A \otimes U(1)_{\bar{V}} \). Of course, by Goldstone’s theorem there must in this case be a massless boson in the spectrum – this will be a tightly bound monopole – anti-monopole state which we shall refer to as a “magnetopion”. In the \( g \to \infty, \mu \to 0 \) limit in which the monopole physics is described by dimer configurations, the magnetopions have vanishing spatial extent, and do not couple either to photons or electrons.

For \( g \) large but finite, one may once again estimate the effect of small monopole loop excitations on charge renormalisation, using a mean field argument. If the vacuum consists of a fraction \( \rho \) of monomers (\( \rho = 1 \) in the \( 1/\mu \) expansion), then the excitation cost of a monopole loop changes to \((2 \mu)^4 \rho\). Using (3.10), the expression for \( Z_g \) becomes

\[ Z_g = 1 + 2g^2 \exp \left( -\frac{g^2}{4} + 4\langle \bar{\psi}\psi \rangle \mu \ln \frac{1}{\mu} \right). \] (3.12)

Note that the correction involving \( \langle \bar{\psi}\psi \rangle \) vanishes in the chiral limit. For electric current loops we can use the perturbative QED result

\[ Z_e = 1 - \frac{e^2}{6\pi^2} \ln \frac{1}{\mu}. \] (3.13)

Since the loop suppression is now due to the factor \( \exp(-g^2/4) \), eqn. (3.12) is no longer the first term in a systematic expansion; the next smallest loop is suppressed by a factor \( \exp(-0.4311g^2) \) [23].

[21,24]
Before turning to the behaviour of QEMD for intermediate values of $e$, $g$, it will be useful to discuss and contrast the phase structure of the two other models we have introduced, NCQED (2.1), and Villain QED (2.12). As mentioned above, in the long wavelength limit NCQED resembles continuum QED with $N_f = 4$. Numerical simulations of this model reveal that in the limit $\mu \to 0$ it exhibits a continuous chiral symmetry breaking phase transition for $e$ in the range 2.2 - 2.3 [13,14]. The Villain model, which has no fermion fields, has a phase transition, possibly weakly first order, at $e \simeq 1.25$ [25], or $g \simeq 5.0$, separating confinement and Coulomb phases. There is no local order parameter due to Elitzur’s theorem.

At first sight, it is not clear in this language why either model has a phase transition. For large $x$, $v(x) \propto 1/x^2$, so that any change in the coupling strength in (2.9) or (2.18) can simply be absorbed by a change of length scale. Thus for weak coupling, Villain QED resembles continuum QED, and electric charges interact via Coulomb’s law. However, for small $x$, the lattice cutoff breaks this scaling symmetry; eg. $v(0)$ is actually a finite quantity. This means that in the Villain model, once $g$ is sufficiently small the monopole excitations of like sign no longer have a strong repulsive interaction, and those of unlike sign are no longer strongly attracted. At this point hitherto small monopole loops become free to grow and spread over large distances, forming a plasma which screens the long-range Coulomb forces. If a Wilson loop of electric current is introduced into the plasma, the interaction term $\exp(-i2\pi m_\ell v_{\ell\ell'}(e\Delta S_{\ell\ell'}))$ has the effect of disordering its phase, and causing its expectation value to decay as its area, signalling confinement. This is a non-perturbative generalisation of the anti-screening effect discussed above. For a loop of length $L$ the activation energy $\sim g^2L v(0)$, and the entropy $\sim L \ln 7$ – so the critical $g$ can be estimated by an energy/entropy argument [19]. This “monopole condensation” is now widely accepted as being the mechanism behind the phase transition in not only the Villain model but also U(1) lattice gauge theory with the Wilson action – there are similarities with the Kosterlitz-Thouless-Berezinsky mechanism in the two dimensional X-Y model.

Now consider NCQED (2.9). This model has the same driving term describing a Coulomb gas of current loops, but with three important modifications. First, the loops are constrained to be singly-charged and self-avoiding. Secondly, there is the sign factor. Thirdly, even for a given loop configuration $\{\Gamma\}$ there will be many different background monomer/dimer configurations, each carrying its own weight $\kappa^{-N_m}$. In the limit $e \to \infty$ all loops are suppressed, and we are left with a pure monomer/dimer system which is known
to break chiral symmetry [24]. In the limit $e \to 0$ we are left with free massless fermions – hence the correlation length must diverge. In this limit large loops are unsuppressed, although it is far from clear whether they must dominate $Z_{NCQED}$ – indeed since the contributions to $Z$ contain fluctuating signs an entropy argument cannot be constructed. Thus, it is unclear whether the phase transition in this model coincides with loops growing without bound – so much is hidden in the sign factor. Even continuum QED can potentially exhibit singular behaviour, via the unbounded $\sigma_{\mu\nu} F_{\mu\nu}$ term which appears in the expression for the inverse propagator [26].

Now we return to QEMD; for intermediate values of $e, g$, the interaction between electrons and monopoles governed by the final exponential in (2.27) becomes crucial. As discussed above, the standard lore is that monopole loops disrupt the phases of electron current loops, causing contributions from larger electric loops to cancel in the partition function, and hence suppressing these loops, leading to electric charge confinement. Similarly, electron loops will disrupt magnetic current loop phases. We may then question the existence of a phase in which large polymers are present simultaneously on both direct and dual lattices (which would correspond to the coexistence of light electrons and monopoles). Even though the loops each carry a sign from the polymer expansion of the determinant, making each term in the partition function of indefinite sign, the additional random phase due to the interaction must surely effectively cancel all contributions to $Z_{QEMD}$ of this form. This physical interpretation is based on the idea that the polymers are the nearest we can get to a “fermion worldline”. We should also not forget the technical obstacle, discussed in section 2, namely a demonstration that $Z_{QEMD}$ is positive for real values of $e$.

To summarise our conclusions so far: there is probably no extended region of the phase diagram where both electrons and magnetic monopoles are light propagating particles. For $e \to 0$ the only light degrees of freedom are electrons and decoupled magnetopions, and vice versa for $g \to 0$. For $g$ large but finite, the monopoles’ chiral condensate $\langle \bar{\psi}\psi \rangle$ is non-vanishing in lattice units, and hence divergent in physical units. By analogy with hadronic physics, therefore, we expect the decay constant $f_\pi$ to diverge and the pion screening length $m_\pi/f_\pi^2$ to vanish in the chiral limit, leaving a theory of non-interacting bosons in the monopole sector. The question of the magnetopions’ coupling to photons and hence electrons is more delicate, depending on the spatial extent of the polymer configurations dominating $Z_{QEMD}$, but from the previous discussion there is no particular reason to
suppose that the coupling will remain non-vanishing in the continuum limit if \( \langle \bar{\psi}\psi \rangle > 0 \). The only scenario which supports an interacting continuum limit for QEMD, therefore, is one in which the chiral symmetry of the electrons is spontaneously broken precisely at the self-dual point \( e = g = \sqrt{2\pi} \); by duality the monopoles’ chiral symmetry will be restored at precisely the same point. It may thus be possible to approach the self-dual point from either phase to take a continuum limit, in which the electrons are in their chirally symmetric phase (say), with physical mass vanishing in the limit \( \mu \to 0 \), and the monopoles are in a chirally broken state, with the condensate \( \langle \bar{\psi}\psi \rangle \) vanishing non-analytically as \( e \to (\sqrt{2\pi})_- \), and some related mass scale defining an inverse correlation length, giving dimensional transmutation à la Miranskii [11]. Of course, the self-dual point is also a candidate for a zero of the \( \beta \)-function using the arguments following (3.8).

Unfortunately, the attractive scenario of a fixed point at the self-dual point appears to be excluded by current numerical data. We have already noted that in the limit \( g \to \infty \) the monopole loops decouple to leave standard lattice NCQED. As discussed above, NCQED is known to exhibit a chiral symmetry breaking phase transition at \( e \simeq 2.2 - 2.3 \), which is some way short of the self-dual value \( e = 2.5066 \ldots \). It is difficult to believe (and indeed runs contrary to the arguments presented above) that the effect of monopole - electron interactions will revise this value upwards: if anything monopoles make the \( e^+e^- \) interaction stronger, and would thus reduce the value of the critical coupling needed for chiral symmetry breakdown. Hence we must conclude that at the self-dual point electrons and monopoles probably both have their chiral symmetry spontaneously broken. The resulting model is necessarily massive, the correlation length thus finite, and the full \( \beta \)-function non-vanishing at this point. Instead there must be a region of non-zero width centred on the self-dual point, separating two phases where light particles can exist at two critical couplings \( e_c, 2\pi/e_c \).

One final point should be made. As mentioned in the introduction, the considerations of this paper were originally motivated by the observation that monopole-like excitations, defined on the dual lattice, become dense in a geometric sense and percolate very close to or actually at the point where chiral symmetry is broken [13,18,27]. This observation encouraged us to speculate that monopole condensation was the agent behind chiral symmetry breaking in NCQED [26]. The truth cannot be so simple – as pointed out in [28], the percolation transition does not correspond to a condensation in the Bose-Einstein sense (assuming bosonic monopoles), and indeed the monopole excitations of NCQED do not
have the long range potentials that could have the effects discussed here. Only formulations in which the action is periodic in the field strength, and hence in which the Dirac string is invisible, can give rise to the monopole condensation phenomenon usually discussed. Moreover the monopole-like excitations of [18] will still be present as dislocations in the photon field of the NCQED sector of the lattice QEMD model, independent of the dynamical monopoles we have introduced explicitly. Therefore it seems unlikely that the dynamical monopoles described here could be the primary agent of chiral symmetry breaking in QEMD, at least for $N_f = 4$. It is interesting to note that the numerical coincidence of chiral and monopole percolation transitions at higher values of $N_f$ remains impressive [29]. Moreover, the critical exponent $\nu$ derived from the size of the largest cluster is consistent with the critical indices found in powerlaw fits of the chiral equation of state of NCQED [29]. Therefore it may well be that the monopoles seen in NCQED are connected in some way with chiral symmetry breaking. Quenched simulations using alternative actions for the lattice photon field manage to separate the transitions, but still suggest that chiral symmetry breakdown is due to fluctuations on the scale of the lattice spacing [16].

4. Discussion

We have succeeded in formulating QEMD using an explicit regularisation which preserves an electromagnetic duality symmetry. This has been done in effect by introducing an interaction term into the expansion of the partition function, rather than directly into the action. Our formulation (2.27) shares the non-local nature of previous approaches, but is more economical in the sense that the Dirac quantisation condition necessary to ensure covariance of Green functions is minimal, viz:

$$\epsilon g = 2\pi.$$  \hspace{1cm} (4.1)

This follows from the natural appearance of monopole-like excitations in abelian lattice gauge theory [19]. The lattice formulation admits non-perturbative approaches such as the $1/\mu$ expansion discussed in section 3. Although we have not pursued this in detail, it seems plausible that higher order calculations will prove to be consistent with the claim of Calucci and Jengo that monopole and electron virtual loops give radiative corrections to the photon propagator of opposite sign, and hence that there will be no charge renormalisation at the self-dual point $e = g$. 

24
On the negative side, the fermionic nature of the monopoles, which we impose precisely in order to have a self-dual symmetry, means that it remains to be proved whether \( Z_{QEMD}(e) > 0 \) for \( e \) real. Moreover, the fluctuating signs in the terms of \( Z_{QEMD} \) mean that Monte Carlo simulations are probably impracticable; numerical studies of low-dimensional fermionic theories using the polymer representation, and even free fermions, have proved difficult [21,30]. Despite this, one can argue that results from simulations of NCQED suggest that QEMD in the form (2.27) does not have a continuum limit at the self-dual point, and that there is no value of \( e \) for which both electrons and monopoles are light particles.

The reason for the difficulty of finding a continuum limit is probably the excitation energies for polymer loops discussed in section 3. Even the smallest loop, a single plaquette, is suppressed by a factor \( \exp(-e^2/4) \simeq 0.21 \) at the self-dual point – large loops necessary for the description of light particles are even further suppressed. The suppression factors are intimately bound up in the geometry of the lattice regularisation; it is difficult to see how the lattice could be modified without also destroying the duality symmetry.

There are other possible modifications. Recall that the reason a continuum limit appears elusive is that the critical coupling \( e_c \) of NCQED is somewhat smaller than \( \sqrt{2\pi} \). Could NCQED be modified to yield a larger \( e_c \)? One could introduce a repulsive four-fermi interaction to delay the onset of spontaneous chiral symmetry breaking. The suggested lattice interaction is [31]

\[
S_{rep} = \lambda \sum_{xy} \bar{\chi}(x)M(x,y)\chi(y),
\]

\[
M(x,y) = \sum_{\mu} \delta_{y,x+\hat{\mu}} \exp(i\phi_{\mu}(x)) + \delta_{y,x-\hat{\mu}} \exp(-i\phi_{\mu}(x - \hat{\mu})),
\]

where \( \chi, \bar{\chi} \) are generic fermion fields and \( \phi_{\mu} \) is a vector auxiliary field, defined on the lattice links, which is freely integrated over. The + sign in the definition of \( M \) is responsible for the repulsion; it renders \( M \) hermitian rather than antihermitian, and hence forces the eigenvalues of the full fermion kinetic operator to be complex. Repulsive fermion – anti-fermion interactions are not usually considered in isolation due to Dyson’s argument about vacuum stability [32] – in the context of a strongly-coupled theory these arguments are hopefully inappropriate.

A second possibility to increase \( e_c \) is to increase the number \( N_f \) of fermion flavors in the model. Simulations with varying \( N_f \) [29,33] show clear evidence that \( e_c \) increases with \( N_f \). For sufficiently large \( N_f \) (estimated between 13 [33] and 24 [29]) the phase transition
becomes first order. However, it is conceivable that there is a window of $8 \leq N_f \leq 15$ where both the transition is continuous and $e_c > \sqrt{2\pi}$ (i.e., $1/e_c^2 \equiv \beta_c^2 = 0.159 \ldots$). In this case introduction of a dual sector of magnetic monopoles could force the existence of a phase transition, and hence a fixed point, at the self-dual point.

Of course, even if no interacting continuum limit can be found, the model presented here may be of interest as an effective theory of strongly interacting and massive fermionic abelian monopoles arising from some more fundamental model. It is possible in this case to relax the condition that $\mu_{\text{monopole}} = \mu_{\text{electron}}$. It is both interesting and amusing to contemplate the phenomenological implications of lattice QEMD. A recent survey of experimental and observational limits on monopole matter, and in particular the effect of virtual monopole loops, has been given in [34]. The novelty of the present approach is that monopoles are strongly interacting fermions, which means that in our world of small $e \simeq 0.3$, monopole chiral symmetry is spontaneously broken, and there will be at least one light Goldstone particle in the monopole sector, the magnetopion $\pi_m$. Can magnetopions interact with ordinary matter? The scattering of photons off tiny magnetic dipole moments will depend on the details of the magnetic charge distribution in the magnetopion, which in turn depends on the (non-perturbative) dynamics. However, we note that magnetopion production via virtual photon decay $\gamma \rightarrow n\pi_m$ is forbidden by charge-conjugation symmetry, just as strong interaction decays such as $\rho^0 \rightarrow n\pi^0$ are forbidden. The main interaction of magnetopions, just as for axions, will be via the coupling $\pi_m \gamma \gamma$; indeed, massive magnetopions will decay to two photons. Since the lattice regularisation presented here does not admit an axial anomaly, magnetopion decay will be suppressed by a factor of $O(m_{\pi_m}^2)$ by the usual arguments [35]. This raises the interesting question of whether there is a range of bare mass $\mu$ such that the magnetopion is both sufficiently long-lived and sufficiently heavy to be of cosmological significance. Such issues cannot be resolved without a further understanding of the detail of the strongly interacting monopole dynamics.

Acknowledgements

The work of SJH was supported partly by a CERN Fellowship and partly by a PPARC Advanced Fellowship. JBK is supported in part by the National Science Foundation, NSF PHY92-00148.
References

[1] P.A.M. Dirac, Proc. R. Soc. A133 (1931) 60.
[2] M. Blagojević and P. Senjanović, Phys. Rep. 157 (1988) 233.
[3] N. Cabibbo and E. Ferrari, Nuovo Cimento 23 (1962) 1647.
[4] J. Schwinger, Phys. Rev. 144 (1966) 1087.
[5] D. Zwanziger, Phys. Rev. D3 (1971) 880.
[6] R.A. Brandt, F. Neri and D. Zwanziger, Phys. Rev. D19 (1979) 1153.
[7] G. Calucci and R. Jengo, Nucl. Phys. B223 (1983) 501.
[8] G. Calucci, R. Jengo and M.T. Vallon, Nucl. Phys. B211 (1983) 77.
[9] J. Schwinger, Phys. Rev. 151 (1966) 1048, 1055;
R. Brandt and F. Neri, Phys. Rev. D18 (1978) 2080;
C. Panagiotakopoulos, J. Phys. A16 (1983) 133.
[10] L.D. Landau and I.Ya. Pomeranchuk, Dokl. Akad. Nauk. SSSR 102 (1955) 489;
L.D. Landau, in Niels Bohr and the Development of Physics (ed. W. Pauli) p.52, Pergamon
Press, London (1955).
[11] P.I. Fomin, V.P. Gusynin, V.A. Miranskii and Yu.A. Sitenko, Riv. Nuovo Cimento 6
(1983) 1;
V.A. Miranskii, Nuovo Cimento 90A (1985) 149.
[12] J.B. Kogut, E. Dagotto and A. Kocić, Phys. Rev. Lett. 60 (1988) 772; Nucl. Phys.
B317 (1989) 271.
[13] A. Kocić, J.B. Kogut and K.C. Wang, Nucl. Phys. B398 (1993) 405.
[14] M. Göckeler, R. Horsley, P.E.L. Rakow, G. Schierholz and R. Sommer, Nucl. Phys.
B371 (1992) 713.
[15] S.J. Hands, J.B. Kogut and J.H. Sloan, Nucl. Phys. B344 (1990) 255; Erratum, Nucl.
Phys. B352 (1991) 528.
[16] S.J. Hands, Nucl. Phys. B(Proc. Suppl.)42 (1995) 663.
[17] C.N. Leung, S.T. Love and W.A. Bardeen, Nucl. Phys. B273 (1986) 649.
[18] S.J. Hands and R.J. Wensley, Phys. Rev. Lett. 63 (1989) 2169.
[19] T. Banks, R. Myerson and J.B. Kogut, Nucl. Phys. B129 (1977) 493.
[20] T.A. DeGrand and D. Toussaint, Phys. Rev. D22 (1980) 2478.
[21] M. Karowski, R. Schrader and H.J. Thun, Commun. Math. Phys. 97 (1985) 5.
[22] A.H. Guth, Phys. Rev. D21 (1980) 2291;
L. Polley and U.-J. Wiese, Nucl. Phys. B356 (1991) 629.
[23] S.W. de Souza and R.D. Kenway, Phys. Lett. **B248** (1990) 423; Nucl. Phys. **B354** (1991) 39.
[24] P. Rossi and U. Wolff, Nucl. Phys. **B248** (1984) 105;
M. Salmhofer and E. Seiler, Commun. Math. Phys. **139** (1991) 395.
[25] R.J. Wensley, Ph.D. thesis, University of Illinois (1989);
Z. Schram and M. Teper, Phys. Rev. **D48** (1993) 2881.
[26] S.J. Hands, J.B. Kogut and A. Kocić, Nucl. Phys. **B357** (1991) 467.
[27] S.J. Hands, A. Kocić, J.B. Kogut, R.L. Renken, D.K. Sinclair and K.C. Wang, Phys. Lett. **B261** (1991) 294; Nucl. Phys. **B413** (1994) 503.
[28] P.E.L. Rakow, Nucl. Phys. **B**(Proc. Suppl.)**30** (1992) 591;
M. Göckeler, R. Horsley, P.E.L. Rakow and G. Schierholz, preprint DESY-93-025, [hep-lat/9303001](http://arxiv.org/abs/hep-lat/9303001) (1993).
[29] J.B. Kogut and K.C. Wang, Illinois preprint ILL-TH-95-#28, [hep-lat/9501021](http://arxiv.org/abs/hep-lat/9501021) (1995).
[30] I. Montvay, Phys. Lett. **B227** (1989) 260; in proceedings of the 1989 Cargèse Workshop *Probabilistic Methods in Quantum Field Theory and Quantum Gravity*, p.87 (1989).
[31] J.B. Kogut and J.-F. Lagaë, Nucl. Phys. **B**(Proc. Suppl.)**42** (1995) 681.
[32] F.J. Dyson, Phys. Rev. **85** (1952) 631.
[33] E. Dagotto, A. Kocić and J.B. Kogut, Phys. Lett. **B232** (1989) 235;
V. Azcoiti, G. Di Carlo and A.F. Grillo, Phys. Lett. **B305** (1993) 275.
[34] A. De Rújula, Nucl. Phys. **B435** (1995) 257.
[35] D.G. Sutherland, Nucl. Phys. **B2** (1967) 433;
M. Veltman, Proc. R. Soc. **A301** (1967) 107;
S.L. Adler, Phys. Rev. **177** (1969) 2426.
**Figure 1**
Photon corrections to the smallest loop excitation

**Figure 2**
Loop correction to the photon propagator