Classical Vs Quantum Probability in Sequential Measurements

Charis Anastopoulos ,
Department of Physics, University of Patras, 26500 Paras, Greece*

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Abstract

We demonstrate in this paper that the probabilities for sequential measurements have features very different from those of single-time measurements. First, they cannot be modeled by a classical stochastic process. Second, they are contextual, namely they depend strongly on the specific measurement scheme through which they are determined. We construct Positive-Operator-Valued measures (POVM) that provide such probabilities. For observables with continuous spectrum, the constructed POVMs depend strongly on the resolution of the measurement device, a conclusion that persists even if we consider a quantum mechanical measurement device or the presence of an environment. We then examine the same issues in alternative interpretations of quantum theory. We first show that multi-time probabilities cannot be naturally defined in terms of a frequency operator. We next prove that local hidden variable theories cannot reproduce the predictions of quantum theory for sequential measurements, even when the degrees of freedom of the measuring apparatus are taken into account. Bohmian mechanics, however, does not fall in this category. We finally examine an alternative proposal that sequential measurements can be modeled by a process that does not satisfy the Kolmogorov axioms of probability. This removes contextuality without introducing non-locality, but implies that the empirical probabilities cannot be always defined (the event frequencies do not converge). We argue that the predictions of this hypothesis are not ruled out by existing experimental results (examining in particular the "which way" experiments); they are, however, distinguishable in principle.

*Email: anastop@physics.upatras.gr
1 Introduction

1.1 The main theme

The topic of this paper is sequential quantum measurements and their probabilistic description. We show that the construction of probabilities for sequential measurements is rather intricate and in some aspects differs strongly from its analogues in classical probability theory. For example, quantum multi-time probabilities do not define a stochastic process. We can isolate specific points of divergence between quantum and classical probability theory (including hidden variable theories in the latter) and to argue that these differences can be empirically determined, at least in principle.

The motivation for this line of inquiry is two-fold. First, the determination of probabilities in sequential measurements is of interest on its own right. It seems experimentally feasible, as it is nowadays possible to construct sources that emit individual systems. However, the construction of such probabilities from the rules of standard quantum theory is not as straightforward as it may seem, for the relevant probabilities can not be obtained in a natural way from the Hilbert space geometry. Assumptions about the physical implementation of the measurement process are needed, and these touch inevitably upon fundamental interpretational issues.

An immediate result of our analysis is that multi-time probabilities are strongly dependent upon the specific experimental set-up used in their determination. For observables corresponding to operators with discrete spectrum, one may construct a probability distribution rather simply. The same procedure applied to observables with continuous spectrum leads to probabilities that depend very strongly on an additional parameter $\delta$. This parameter can be interpreted as the resolution of the measurement device, but the dependence of the resulting probabilities is so strong as to be highly counter-intuitive. This dependence persists even for samplings coarse-grained at a scale much larger than $\delta$. An interesting corollary of this analysis is that it is impossible to simulate by a stochastic process the probabilities obtained from sequential measurements of a quantum system.

The other motivation for this research is related to basic interpretational issues of quantum theory. Probabilities are introduced in the quantum mechanical formalism through Born’s interpretation of the wave function. Born’s rule is valid for single-time measurement of one observable (or for a family of compatible observables). In that case, quantum theory is reduced to a description in terms of classical probabilistic concepts, which describe successfully the statistical outcomes of experiments.

But once one moves away from this context, the coexistence between quantum theory and classical probability theory becomes less harmonious. This is highlighted by three representative theorems: Bell’s, Wigner’s and Kochen-Specker’s [1, 3].
The violation of Bell’s inequalities (and their generalisations) implies that local hidden variables theories are ruled out by experiment. This may imply either quantum non-locality, or that it is impossible to define a sample space for a physical system in itself, without referring to the specific experiment that is carried out. The latter property is referred to as contextuality of quantum properties (or measurements). Wigner’s theorem is a representative of a more general result: it is not possible to define a joint probability distribution for variables that correspond to non-commuting operators. This can be argued to be a form of contextuality, in the sense that there does not exist a universal sample space to describe the outcomes of all possible measurements that can be performed in an ensemble of quantum systems. The Kochen-Specker theorem demonstrates a stronger form of contextuality: it is impossible to assign definite values to a physical observable without referring to the commuting set that is measured along with it.

While all three theorems above suggest that quantum mechanical properties (and consequently probabilities) are contextual, they do not easily relate to empirical evidence. The observed violation of Bell’s inequalities may be attributed to non-locality rather than contextuality, the measurement of incompatible observables involves distinct experimental situations, whose outcomes cannot be immediately compared, while the Kochen-Specker theorem refers to idealized values of observables that a physical system possesses prior to measurement (hence empirically inaccessible).

Sequential measurements on the other hand provide a ground, on which the idea of contextuality can be explicitly tested. The application of the rules of standard quantum theory suggests that two different measurement schemes will give rise to different value for the probability of the same property of a physical system, even if the initial state is assumed to be the same. Hence the precise statistical study of the outcomes in sequential measurements may in principle reveal unambiguously the contextual character of quantum probability.

The problem is that we obtain much more contextuality than we bargained for. Not only are multi-time probabilities dependent on the measurement scheme through which they are determined, but they seem to depend strongly on rather trivial details of the measurement device. This is unavoidable, at least if we do not abandon the usual rules of quantum theory. It is then questionable whether it is possible to properly define a statistical ensemble for sequential measurements, or even if any physical information can be extracted from them.

This rather disturbing feature of multi-time probabilities provides the motivation to seek an alternative account. We first consider hidden variable theories. We prove that any local hidden variable theory (deterministic or stochastic) that reproduces the single-time probabilities of quantum theory cannot reproduce those for multi-time probabilities. The only way to do so is by assuming a non-local interaction between system and measuring device, similar to the one appearing in Bohmian mechanics.

The other alternative we examine here is related to proposals [4, 5]—see also
that it might be possible to avoid contextuality (the constraints of Bell’s and Kochen-Specker’s theorem) by assuming that quantum theory is described by a "probability" measure that does not satisfy the Kolmogorov axioms—in particular the additivity property. While a non-additive measure is mathematically natural in multi-time probabilities, its physical interpretation is somewhat problematic. A non-additive measure cannot be interpreted in terms of any empirical probabilities, which are obtained by the limit of event frequencies. It only make sense if one assumes that the event frequencies for sequential measurements do not converge to probabilities. We explore further this idea, showing that it is consistent with usual treatment of probabilities in quantum theory, that it is natural from an operational point of view and that it is in principle distinguishable from any alternative that assumes that empirical probabilities for sequential measurements always exist.

1.2 The structure of this paper

The paper is structured as follows.

In Section 2 we briefly review classical and quantum probability theory, in order to set-up our conventions. We also provide some preliminary mathematical arguments about the inequivalence between the classical and quantum descriptions of sequential measurements.

Section 3 contains the central results of this paper. First, we motivate the discussion on probabilities of sequential measurements, focusing in particular on the fact that the quantum mechanical correlation functions are complex-valued and have no immediate correspondence in terms of objects that can be immediately determined. Then we demonstrate that quantum logic cannot be expected to hold in sequential measurements. This is unlike single-time measurements for which the spectral theorem together with Born’s rule guarantee that different measurement schemes lead to the same probability assignment (assuming identical preparation). Multi-time probabilities are therefore highly contextual. We then discuss the description of multi-time probabilities via Positive-Operator-Valued-Measures (POVMs). We prove two theorems that demonstrate that it is not possible to construct POVMs for sequential measurements compatible with the single-time predictions of quantum theory. These results provide a general proof of an often quoted statement that quantum mechanical probabilities cannot be simulated by stochastic processes. We then demonstrate different ways of constructing POVMs for a specific class of multi-time measurements of position. These POVMs exhibit a very strong dependence on properties of the measurement device (its resolution) that persist even in highly coarse-grained samplings. Finally we show that neither the consideration of a fully quantum measuring device or of decoherence due to the environment affect significantly these conclusions.

In section 4 we discuss other interpretational schemes, most notably hidden variable theories and we demonstrate that the predictions of quantum theory
for sequential measurements are not compatible with the assumption of local interactions between measured system and measuring device. Finally in Section 5 we consider the alternative proposal that probabilities for sequential measurements cannot be defined because the relative frequencies do not converge. The motivation for this proposal is analysed in detail. We then demonstrate that it is compatible with the predictions of single time quantum theory, that it is not contradicted by some well-established results and that it is possible to distinguish it unambiguously even in very simple experimental set-ups.

2 Classical Vs quantum probability

2.1 Basic facts

We briefly describe here the mathematical structure of classical and quantum probability, in order to set-up our notations, conventions and terminology for later use.

2.1.1 Classical probability theory

In classical probability one assumes that all possible elementary alternatives lie in a space $\Omega$, the sample space. Observables are functions on $\Omega$, and are usually called random variables. The outcome of any measurement can be phrased as a statement that the system is found in a given subset $C$ of $\Omega$. Hence the set of certain well-behaved (measurable) subsets of $\Omega$ is identified with the set of all coarse-grained alternatives of the system. To each subset $C$, there corresponds an observable $\chi_C(x)$, the characteristic function of the set $C$. It is defined as $\chi_C(x) = 1$ if $x \in C$ and $\chi_C(x) = 0$ otherwise. It is customary to denote the characteristic function of $\Omega$ as 1 and of the empty set as 0.

If an observable $f$ takes values $f_i$ in subsets $C_i$ of $\Omega$

$$f(x) = \sum_i f_i \chi_{C_i}(x)$$

(2. 1)

A state is intuitively thought of as a preparation of a system. Mathematically it is represented by a measure on $\Omega$, i.e a map that assigns to each alternative $C$ a probability $p(C)$. A probability measure satisfies the Kolmogorov conditions

- for all subsets $C$ of $\Omega$, $0 \leq p(C) \leq 1$
- $p(0) = 0; p(1) = 1$.
- for all disjoint subsets $C$ and $D$ of $\omega$, $p(C \cup D) = p(C) + p(D)$

Due to (2. 1) one can define $p(f) = \sum_i f_i p(C_i)$; $p(f)$ is the mean value of $f$. In the case that $\Omega$ is a subset of $\mathbb{R}^n$, the probability measures are defined in terms of a probability distribution, i.e. a positive function on $\Omega$, which we shall
denote as \( p(x) \).

\[
p(f) = \int dx p(x) f(x)
\]  
(2.2)

### 2.1.2 Quantum probability theory

The formalism of quantum mechanics incorporates probability through Born’s rule, which in its initial form asserts that the square modulus \(|\psi(x)|^2\) of Schrödinger’s wave function can be interpreted as a probability density for the particle’s position. In the abstract Hilbert space formulation Born’s interpretation can be implemented through the *spectral theorem*: under rather general conditions we may assign a Projection-Valued-Measure (PVM) \( dE(\lambda) \) to each self-adjoint operator \( \hat{A} \). The PVM is a map assigning to each measurable set \( U \) of \( \hat{A} \)’s spectrum \( \sigma(\hat{A}) \) a projection operator \( \hat{E}(U) = \int_U dE(\lambda) \), such that \( \hat{E}(U) = \chi_U(\hat{A}) \), where \( \chi_U \) is the characteristic function of \( U \). The projectors in the range of the PVM reflect the Boolean algebra of the subsets of \( \sigma(\hat{A}) \) in the sense that

\[
\begin{align*}
- \hat{E}(\emptyset) &= 0, \quad \hat{E}(\sigma(\hat{A})) = \hat{1}, \\
- \hat{E}(U \cup V) &= \hat{E}(U) + \hat{E}(V), \quad U \cap V = \emptyset, \\
- \hat{E}(U \cap V) &= \hat{E}(U) \hat{E}(V).
\end{align*}
\]

The spectral theorem implies that the Hilbert space \( H \) is isomorphic to that of square-integrable functions over \( \sigma(\hat{A}) \), and as such the Born rule may be directly applied: the probability for an event corresponding to \( U \subset \sigma(\hat{A}) \) is

\[
p(U) = \text{Tr} \rho \hat{E}(U).
\]  
(2.3)

Given that \( \hat{A} = \int \lambda d\hat{E}(\lambda) \) the standard relation between probabilities and expectation values can be established.

It follows that for single-time measurements of a single observable (or of many observables represented by mutually commuting operators) quantum theory via the Born rule is completely equivalent to classical probability theory.

The Copenhagen interpretation employs the formalism of quantum theory to account for the outcomes of specific experiments. It presupposes a split between the measured system, which is fully quantum, and the measuring apparatus, which is part of the classical world. While this creates the key problem of explaining the classical description of an object that consists of fundamentally quantum entities, it is fully self-consistent at an operational level, namely if we only employ quantum theory to account for the statistics of measurement outcomes.

We shall adopt an operational stance in most discussions in this paper. The reason for this choice is that the operational description is a core of quantum theory that refers immediately to the concrete experimental situations, and the remarkable success of quantum theory implies that all contending interpretation must accept it, either as a fundamental or as an emergent theory. Still, we shall
find it necessary in the course of the argument to move beyond the operational description and consider quantum measurement theory, namely the assumption that the measuring apparatus is fully or partly quantum mechanical.

### 2.1.3 Probabilities and event frequencies

To apply a specific version of probability theory in a concrete physical system, one needs to be able to relate the numbers obtained by the mathematical formalism to the concrete experimental data. This relation is achieved by the correspondence of probability to relative frequencies of events in statistical ensembles. While it can be argued that relative frequencies do not exhaust the physical content of probability theory, that the latter can be interpreted in a way that refers to individual systems and not only statistical ensembles, and even that a definition of probabilities from frequencies is highly problematic, any sharp quantitative test of a probabilistic theory involves a comparison of theoretical probabilities to empirical probabilities, which are obtained from event frequencies.

Suppose for simplicity that the sample space of our system \( \Omega = \mathbb{R} \). We assume an experiment that determines a value for \( x \in \mathbb{R} \). Repeating the experiment \( n \) times we obtain a sequence \( x_i, i = 1, \ldots, n \) of measured values. We may then consider the relative frequency for the proposition that the variable \( x \) took value in the subset \( U \subset \mathbb{R} \). If \( \chi_U \) is the characteristic function of the set \( U \), we define the relative frequency for the occurrence of an event in \( U \) for the first \( n \) experimental runs

\[
\nu_n(U) = \frac{1}{n} \sum_{i=1}^{n} \chi_U(x_i).
\]  

(2.4)

The probability \( p(U) \) associated to the event \( U \) is the limit

\[
p(U) = \lim_{n \to \infty} \nu_n(U),
\]  

(2.5)

assuming of course that it exists.

Since any actual determination of probabilities involves a finite number of runs, we can never establish the convergence of frequencies. If, however, the description of the physical systems in terms of probabilities is valid, one expects that the relative rate of convergence \( \epsilon_n = \frac{\nu_n(U) - p(U)}{p(U)} \sim \frac{1}{\sqrt{n}} \) by virtue of the central limit theorem. Hence the fall-off of \( \epsilon_n \) for large \( n \) as \( n^{-1/2} \) is a good indication of convergence for the relative frequencies.

The mean value of a random variable \( f(x) \) is similarly identified as

\[
\langle f \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i).
\]  

(2.6)

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1We assume that \( U \) is a sufficiently well-behaved set (like an open set) so that there is no operational problem in ascertaining that \( x \in U \).
2.2 The analogue of stochastic processes

We saw that for the measurements of a single observable at a single moment of time the predictions of quantum theory are fully compatible with those of classical probability. It is then natural inquire whether this correspondence passes through when one considers the values of a single observable at more than one moments of time.

In multi-time measurements the law of time evolution enters explicitly. Let us assume a classical probabilistic system with sample space \( \Omega = \mathbb{R} \), whose probability density evolves in time according to

\[
\frac{\partial}{\partial t} \rho = \mathcal{L} \rho, \tag{2.7}
\]

where \( \mathcal{L} \) is a positive, norm-preserving operator. The formal solution of this equation is

\[
\rho_t = e^{\mathcal{L}t} \rho_0, \tag{2.8}
\]

which can be written in terms of the integral kernel \( g_t(x, x') \) of \( e^{\mathcal{L}t} \)

\[
\rho_t(x) = \int dx' g_t(x, x') \rho_0(x'). \tag{2.9}
\]

One may then define a probability measure \( d\mu[x(\cdot)] \) on the space of paths on \( \Omega \) as a suitable limit of the expression

\[
d\mu(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = \rho_0(x_0) g_{t_1}(x_0, x_1) g_{t_2-t_1}(x_1, x_2) \ldots g_{t_n-t_{n-1}}(x_{n-1}, x_n) \, dx_0 \, dx_1 \, dx_2 \ldots dx_n, \tag{2.10}
\]

which is defined on discrete-time paths.

The reason it is possible to extend the single-time probability to a stochastic probability measure is that the evolution law is linear with respect to the probability density. In quantum theory this is not the case; the evolution law is linear with respect to the wave function and not the probability density. Hamiltonian evolution mixes the diagonal elements of the density matrix (which correspond to probabilities) with the off-diagonal ones (which have no such interpretation).

It seems therefore not straightforward (if at all possible) to extend the single-time probabilistic description of quantum theory to a stochastic process. This conclusion will be verified by a more rigorous analysis in section 3.3. Nonetheless, we can write stochastic processes that reproduce some of quantum theory’s predictions. One such example is Nelson’s stochastic mechanics [7, 8], which introduces a stochastic differential equation on configuration space that can reproduce the expectation values of the position observable at every moment of time.
2.3 The history formalism

The underlying reason that quantum evolution cannot be described by a stochastic process is that the mathematically natural measure on histories (or paths) does not satisfy the Kolmogorov axioms of probability theory. This is particularly highlighted in the consistent histories approach to quantum theory [9, 10, 11, 12].

The basic object of this formalism is a history, namely a time-ordered sequence of projection operators \( \hat{P}_{t_1}, \ldots, \hat{P}_{t_n} \), and it corresponds to a time-ordered sequence of propositions about the physical system. The indices \( t_1, \ldots, t_n \) refer to the time a proposition is asserted and have no dynamical meaning. Dynamics are related to the Hamiltonian \( \hat{H} \), which defines the one-parameter group of unitary operators \( \hat{U}(s) = e^{-i\hat{H}s} \). In the consistent histories approach a history is thought to correspond to propositions about the physical system, not necessarily associated to acts of measurement. Consistent histories is a generalisation of Copenhagen quantum theory aiming to provide a quantum mechanical description of individual systems.

The quantum rule for conditional probability is that if the property corresponding to the projector \( \hat{P}_1 \) is realized then we may encode the information obtained in a change of the density matrix:

\[
\hat{\rho} \rightarrow \frac{\hat{P}_1 \hat{\rho} \hat{P}_1}{Tr(\hat{\rho} \hat{P}_1)}, \tag{2.11}
\]

hence the conditional probability the \( \hat{P}_2 \) will be realized at time \( t_2 \) given that \( \hat{P}_1 \) was realized at \( t_1 \) equals

\[
Tr \left( \hat{P}_2 e^{-i\hat{H}(t_2-t_1)} \hat{P}_1 e^{-i\hat{H}t_1} \hat{\rho} e^{i\hat{H}t_1} \hat{P}_1 e^{i\hat{H}(t_2-t_1)} \right) \]

\[
Tr(e^{-i\hat{H}t_1} \hat{\rho} e^{i\hat{H}t_1} \hat{P}_1), \tag{2.12}
\]

leading to a probability for the joint realisation of \( \hat{P}_1 \) at \( t_1 \) and \( \hat{P}_2 \) at \( t_2 \)

\[
Tr \left( \hat{P}_2 e^{-i\hat{H}(t_2-t_1)} \hat{P}_1 e^{-i\hat{H}t_1} \hat{\rho} e^{i\hat{H}t_1} \hat{P}_1 e^{i\hat{H}(t_2-t_1)} \right) \tag{2.13}
\]

For a general \( n \)-time history \( \alpha = \{ \hat{P}_{t_1}, \hat{P}_{t_2}, \ldots, \hat{P}_{t_n} \} \) this results generalizes as follows. We define the class operator \( \hat{C}_\alpha \) defined by

\[
\hat{C}_\alpha = \hat{U}^\dagger(t_n) \hat{P}_{t_n} \hat{U}(t_n) \ldots \hat{U}^\dagger(t_1) \hat{P}_{t_1} \hat{U}(t_1), \tag{2.14}
\]

which leads to a probability measure

\[
p(\alpha) = Tr \left( \hat{C}_\alpha \hat{\rho} \hat{C}_\alpha^\dagger \right). \tag{2.15}
\]

\( \text{We shall argue later that this rule cannot be applied freely, at least as far as measurements are concerned.} \)
These probabilities do not define a genuine measure on the space of histories. To see this, we consider two histories $\alpha = \{\hat{P}_{t_1}, \hat{P}_{t_2}, \ldots, \hat{P}_{t_n}\}$ and $\beta = \{\hat{P'}_{t_1}, \hat{P'}_{t_2}, \ldots, \hat{P'}_{t_n}\}$, such that $\hat{P}_{t_i} \hat{P'}_{t_i} = 0$, the history $\{\hat{P}_{t_1} + \hat{P'}_{t_1}, \hat{P}_{t_2}, \ldots, \hat{P}_{t_n}\}$ is the logical join $\alpha \lor \beta$ of the histories $\alpha$ and $\beta$. The probabilities, however, do not satisfy the additivity condition

$$p(\alpha \lor \beta) = p(\alpha) + p(\beta). \quad (2.16)$$

If the histories are interpreted as referring to measurements the failure of the additivity condition is not (at first sight) a problem, because each history corresponds to a different sequence of YES-NO experiment and there is no a priori reason, why all different experiments should be modeled by a common probability measure. However, if histories are thought to correspond to properties of individual system then the lack of a probability measure becomes a problem.

In the consistent histories approach this is taken into account as follows. We define the decoherence functional as a complex-valued function of pairs of histories: i.e. a map $d : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$. For two histories $\alpha$ and $\alpha'$ it is given by

$$d(\alpha, \alpha') = Tr \left( \hat{C}_{\alpha} \hat{\rho}_0 \hat{C}_{\alpha'}^\dagger \right). \quad (2.17)$$

The consistent histories interpretation of this object is that when $d(\alpha, \alpha') = 0$ for $\alpha \neq \alpha'$ in an exhaustive and exclusive set of histories \footnote{By exhaustive we mean that at each moment of time $t_i$, $\sum_{\alpha} \hat{\alpha}_{t_i} = 1$ and by exclusive that $\hat{\alpha}_{t_i} \hat{\beta}_{t_i} = \delta_{\alpha\beta}$. Note that by $\alpha$ we denote the proposition with the corresponding projector written as $\hat{\alpha}$ with a hat.}, then one may assign a probability distribution to this set as $p(\alpha) = d(\alpha, \alpha)$. The value of $d(\alpha, \beta)$ is, therefore, a measure of the degree of interference between the histories $\alpha$ and $\beta$.

We end this section with a remark. We shall employ many mathematical objects appearing in the consistent histories approach throughout this paper (without a change in name or notation). The reader should keep in mind that the focus of this paper is the description of measurement outcomes through the rules of standard quantum theory, hence the context and interpretation of these objects are different from those in consistent histories.

3 Sequential measurements in standard quantum theory

3.1 Multi-time correlation functions

In classical probability theory there is no conceptual distinction between single-time and multi-time measurements of a physical system. If the sample space for
the single-time measurement is $\Omega$, the sample space for $n$-time measurements as a Cartesian product $\times_{n} \Omega_{n}$: the outcome of $n$ measurements of an observable $x$ is an ordered $n$-tuple of values of $x$. In general, one may define an sample space $\Omega^{T}$ of all paths from a time interval $T = [0, t]$ to $\Omega$ and a corresponding stochastic measure $d\mu[x(\cdot)]$. One then immediately transfers the interpretation of probabilities in terms of relative frequencies and reconstructs the statistical behavior of any observable on the multi-time sample space.

The probabilities for the measurements of an observable $x$ is most conveniently incorporated in the (unequal-time) correlation functions of an observable $f(x)$,

$$\langle f_{t_{1}} f_{t_{2}} \ldots f_{t_{n}} \rangle = \int d\mu[x(\cdot)] F_{t_{1}}[x(\cdot)] F_{t_{2}}[x(\cdot)] \ldots F_{t_{n}}[x(\cdot)],$$

(3.1)

in terms of the functions $F$ on $\Omega^{T}$ defined by

$$F_{t}[x(\cdot)] = f(x(t)).$$

(3.2)

From an operational point of view, there is no problem in measuring multi-time probabilities or correlation functions, as long as the corresponding single-time measurements do not destroy the physical system. The same is true for quantum mechanical systems: we may consider for example a succession of Stern-Gerlach devices, or microscopic particles leaving their trace in sharply localized layers of recording material (we shall elaborate on such experiments later). We therefore expect that quantum mechanics should allow us to determine the values of the correlation functions, which can be unambiguously determined from experiment.

The objects we usually call correlation functions in quantum theory are expectation values of products of operators, such that

$$\langle x_{t_{1}} x_{t_{2}} \rangle = \langle \psi | e^{i\hat{H}t_{1}} \hat{x} e^{i\hat{H}(t_{2} - t_{1})} \hat{x} e^{-i\hat{H}t_{2}} | \psi \rangle.$$  

(3.3)

These “correlation functions” are in general complex-valued, and for this reason they have no interpretation in terms of the statistics of measurement outcomes. Clearly, the construction and interpretation of multi-time quantum probabilities involves many more subtleties than their analogue in the single-time case.

In light of the discussion above, there are two questions that must be raised.

–First, what is the physical meaning of the mathematically natural complex-valued correlation functions?
–Second, how can we employ the standard quantum mechanical formalism (or slight generalizations thereof) to construct real-valued correlation functions that would describe the statistics of multi-time measurements?

Before proceeding to address these questions let us comment on a rather naive
answer that can be given to the second one: the physically relevant correlation functions can be obtained by elementary algebraic manipulations on the complex-valued ones, taking for example their real part, or their totally symmetrizes version etc. The immediate objection is that any such choice is completely ad hoc with no justification in terms of the usual principles of quantum theory. Why should we choose the real part of the correlation function, rather than the imaginary part, or their modulus? But even if we decide by fiat that a specific answer is the correct one, the problem persists at the level of the probabilistic interpretations: correlation functions must be related to probabilities for sequential measurements. Hence any determination of correlation functions must deal with the problem that the mathematically natural probability measure for histories is non-additive.

### 3.2 Sequential measurements and quantum logic

We now examine the definition of probabilities for multi-time measurements. There is a substantial literature on this topic–see for example [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]; indeed discussion of this issue can be traced back to the early days of quantum mechanics. Our presentation here aims to highlight the specific quantum mechanical features through comparison with analogous 'experiments' in classical probability.

For ideal measurements, one may employ equation (2.15) for the probabilities. This expression defines a non-additive measure on the space of histories (we use the word histories heuristically here to denote a temporal succession of measurement outcomes). On the other hand, any empirical probability that is constructed by event frequencies should satisfy the additivity condition. This is an apparent contradiction.

As a first step towards an answer we shall elaborate on specific features of single-time measurements. In any well-designed experiment, we need to guarantee that the results do not depend too strongly on specific details the measurement device. The reasons for that are epistemological (experiments must be reproducible) but also practical: minor details of the measurement device should not affect the experimental outcomes significantly. They should ideally be hidden within the sampling or systematic errors of the experiment. Moreover, it would be highly desirable if different measurement schemes for the same observable and with the same preparation procedure should give compatible (if not identical) results.

We may consider for example two different measurement schemes for the position of a particle. In the first, we assume a source emitting electrons with well defined momentum in the z-direction, but with significant spread in the x and y directions. At a specific distance from the source we place a photographic plate that records the electron’s position. This set-up is equivalent to a single-

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For example, the z-degrees of freedom may be represented by the wave function \( \psi(z) = \)
time measurement of the electron’s $x$ and $y$ coordinates. The distribution of electrons on the screen corresponds to a probability distribution, which modulo sampling errors is given by Born’s rule. The number of electrons found in a subset $U$ of the plate is proportional to $p(U) = \int_U dxdy |\psi(x, y)|^2$.

We may also consider a filter measurement of the electron’s position, by placing instead of a photographic plate a curtain with a hole corresponding to the subset $U$. Any detector placed behind the whole will register a number of particles proportional to $Tr \rho \hat{P}(U)$. This type of measurement is known as a YES-NO experiment, because it can only admit two answers: the particle passing through $U$ or not.

The important point is that the value for the probability $p(U)$ in the experiment with the photographic plate coincides with that obtained from the YES-NO experiment. Moreover, if we carry a sufficiently large number of YES-NO experiments differing only in the position of the hole, we will obtain sufficient information to fully reconstruct the probability distribution of the first experiment. In other words, in single-time quantum theory, YES-NO experiments contain the full probabilistic information about a quantum system. The empirical probabilities for a sample set $U$ are the same in all measurement schemes that correspond to the same preparation of the physical system, modulo sampling and systematic errors. This universality is often referred to as defining a logic for quantum measurements, quantum logic. It is in effect a consequence of the spectral theorem.

We now return to the analysis of multi-time measurements. If the only possible multi-time experiment that could be carried out were of the YES-NO type, there would be no downright problem from the non-additivity of (2.15), at least not a worse problem than appearing in any other quantum “paradox”.

\[
\frac{1}{(2\pi \sigma_z)^{1/4}} e^{-\frac{z^2}{4\sigma_z^2} + ip_z z},
\]

such that the spread $\Delta p_z = 1/\sigma_z << p_z$. This set-up corresponds to a measurement at a reasonably well-specified moment of time.
To see this one may consider the following two-time YES-NO experiment measuring the position of a particle. We assume a source of electrons prepared in the same state as in the previous examples. At fixed distances from the source and parallel to the x-y plane we place two curtains with holes corresponding to the subsets $U_1$ and $U_2$ of the $x-y$ plane. Behind the second slit we place a particle detector.

Repeating the experiments above $n$ times, we record the number of times the detector click, thus constructing the sequence of relative frequencies $\nu_n(U_1, t_1; U_2, t_2)$, and from it the corresponding probability $p(U_1, t_1; U_2, t_2)$. To construct the probability $p(U'_1, t_1; U_2, t_2)$, for a different slit corresponding to $U'_1$ we have to change the experimental configuration, and similarly for $p(U_1 \cup U'_1, t_1; U_2, t_2)$. Hence the probabilities $p(U_1, t_1; U_2, t_2)$, $p(U'_1, t_1; U_2, t_2)$ and $p(U_1 \cup U'_1, t_1; U_2, t_2)$ do not refer to the same experimental set-up, and there is no contradiction between Eq. (2.15) and the additive character of relative frequencies.

However, YES-NO measurements are not the only one possible in practice. For measurements at a single moment of time they contain all the probabilistic information of quantum theory, but this does not hold for sequential measurements. To see this, let us consider the following scheme for a two-time measurement of position. We assume a particle source as before, which can be controlled so finely as to emit a single particle at a time. Two thin sheets of penetrable material are placed one after the other in front of the particle source, both parallel to the $x$-$y$ plane. Particles leave tracks as they cross through the sheets, and one may then determine their $x$ and $y$ coordinates.

Each time the source emits a particle we record the readings $(x_1, t_1; x_2, t_2)_n$; $n$ labels the experimental runs and the $y$ coordinate is suppressed for brevity. We thus construct a sequence of measurement outcomes. From this one defines the sequence $\nu_n(U_1, t_2; U_2, t_2)$ for each pair of subsets $U_1$ of the sheet at $t_1$ and $U_2$ of the sheet at $t_2$. One obtains the probability $p(U_1, t_1; U_n, t_n)$ as the limit $\nu_n(U_1, t_1; U_n, t_n)$ as $n \to \infty$—assuming it exists.

Unlike YES-NO experiment the sequences $\nu_n(U_1, t_1; U_2, t_2)$ constructed for
different choices of the sample sets all refer to the same experimental set-up. They should therefore satisfy the additivity condition (modulo sampling and systematic errors)

$$\nu_n(U_1, t_1; U_2, t_2) + \nu_n(U'_1, t_1; U_2, t_2) = \nu_n(U_1 \cup U'_1, t_1; U_2, t_2),$$

(3.4)
since they refer to indivisible and specific measurement events. It follows that the probabilities (2.15) do not describe the outcomes of this experiment. This conclusion holds for any multiple-time measurement, in which any possible alternative of the observable can be recorded at each moment of time, provided that the corresponding operator does not commute with the Hamiltonian. One could consider, for example, a succession of two Stern-Gerlach apparatuses, with different directions of their magnetic fields placed in such a position as to measure the spin of the particle in the direction $n$ at time $t_1$ and in the direction $n'$ at time $t_2$.

Note also that this thought-experiment presupposes that we record the trace of each particle individually on the sheets. It is, therefore, essential that in each individual run of the experiment the source emits only a single particle. If we perform this experiment with beams of particles, we will not have sufficient statistical information to construct the two-time probabilities. We would not be able to ascertain that the particle found recorded in $x_1$ at time $t_1$ is the same with the particle recorded in $x_2$ at time $t_2$. The most we could obtain would be the two marginal probability distributions for the probability density at $t_1$ and the probability density at $t_2$.

Experiments like that of Fig. 2 involve only a single act of detection. Hence, even if they are formally a two-time YES-NO experiment they can also be described as a single-time measurement of a system, whose wave function satisfies specific boundary conditions on the
There is one point that needs to be highlighted in our discussion. It is well accepted in quantum theory that the presence of an intermediate measurement affects the state of the system and for this reason the experimental outcomes depend strongly on the whether an intermediate measurement has been carried out. However, the same statement could be made for sequential measurements in a classical probabilistic system. However, as we shall explicitly prove in section 4, classical probability cannot give rise to the degree of contextuality inherent in quantum theory even if the coupling to measurement devices is taken into account. To demonstrate this in detail, we need first to expand on the construction of probabilities corresponding to the thought-experiments of Fig. 3.

3.3 POVMs and their applicability

3.3.1 POVMs and their properties

Unlike single-time measurements, sequential measurements cannot be described by the spectral projectors of a self-adjoint operator. It is therefore necessary to employ a generalisation of the notion of quantum mechanical observables, namely the Positive-Operator-Valued Measures (POVMs).

A POVM is a map that assigns to each measurable subset $U$ of a sample space $\Omega$ a positive operator $\hat{\Pi}(U)$, such that

- $\hat{\Pi}(\Omega) = 1$, $\hat{\Pi}(\emptyset) = 0$
- $\hat{\Pi}(U \cup V) = \hat{\Pi}(U) + \hat{\Pi}(V)$, $U \cap V = \emptyset$.

A POVM can therefore define a probability density on $\Omega$ by

$$p(U) = Tr \left( \hat{\rho} \hat{\Pi}(U) \right). \quad (3.5)$$

POVMs are generalisations of PVMs, usually thought to correspond to unsharp measurements. Indeed, if we denote by $\lambda$ the points of the spectrum of a self-adjoint operator $\hat{A}$, we may define a POVM as

$$\hat{\Pi}(U) = \int d\lambda \chi^0_U(\lambda) |\lambda\rangle\langle\lambda|, \quad (3.6)$$

in terms of a family of smeared characteristic functions $\chi^0_U$. (For smeared characteristic functions and their properties see appendix A).

For sufficiently coarse sets $U$, the positive operators $\hat{\Pi}(U)$ are close to true projectors. One may estimate that

$$|Tr \hat{\rho}(\hat{\Pi}(U) - \hat{\Pi}(U)^2)| \leq Tr |\hat{\rho}(\hat{\Pi}(U) - \hat{\Pi}(U)^2)| \leq \int d\lambda |\chi^0_U(\lambda) - [\chi^0_U(\lambda)]^2| < c\delta, \quad (3.7)$$

walls. This is the reason we shall ignore them and study exclusively measurement schemes similar to that in Fig. 3.
with \( c \) a constant of order unity. Also for states \( \rho \) with spreads in \( \hat{A} \) much larger than \( \delta \) we may compute (see the Appendix A)

\[
|\text{Tr} \hat{\rho} (\hat{\Pi}(U) - \hat{\Pi}(U)^2)| < c' \frac{\delta}{L} \text{Tr} \hat{\rho} \hat{\Pi}(U),
\]

where \( L \) is the size of \( U \).

When a POVM \( \hat{\Pi} \) is defined on a sample space \( \Omega = \Omega_1 \times \Omega_2 \), we denote the POVMs \( \hat{\Pi}(\Omega_1, \cdot) \) and \( \hat{\Pi}(\cdot, \Omega_2) \), defined on \( \Omega_2 \) and \( \Omega_1 \) respectively, as the marginal POVMs of \( \hat{\Pi} \).

### 3.3.2 POVMs for sequential measurements: non-go theorems

One possibility that should be first considered is that the arguments leading to equation (2. 15) are somehow inadequate to account for the multi-time experiment we considered earlier and that a different procedure should allow us to define proper probabilities for multi-time measurements.

The most general way to define a probability distribution that is linear with respect to the density matrix is through POVMs. One could therefore conjecture the existence of a POVM on the sample space \( \bigotimes_n \Omega_n \) for the n-time measurements. There are, however, limitations.

**Proposition 1.** There exists no POVM for n-time measurements of an observable \( \hat{x} \) compatible with the single-time predictions of quantum theory, unless \( \hat{x} \) commutes with the system’s Hamiltonian \( \hat{H} \).

We consider without loss of generality a POVM for a two-time measurement. We denote by \( \Omega \) the spectrum of \( \hat{x} \), and by \( \hat{P}(U) \) the spectral projectors of \( \hat{x} \), \( U \subset \Omega \). The POVM \( \hat{E}(\cdot, t_1; \cdot, t_2) \) assigns to each pair of sample sets \( U_1, U_2 \subset \Omega \) a positive operator \( \hat{E}(U_1, t_1; U_2, t_2) \). It should be compatible with the single-time predictions of quantum theory, namely

\[
\text{Tr} \left( \hat{\rho} \hat{E}(U_1, t_1; \Omega, t_2) \right) = \text{Tr} \left( \hat{\rho} e^{i\hat{H}t_1} \hat{P}(U_1) e^{-i\hat{H}t_1} \right),
\]

\[
\text{Tr} \left( \hat{\rho} \hat{E}(\Omega, t_1; U_2, t_2) \right) = \text{Tr} \left( \hat{\rho} e^{i\hat{H}t_2} \hat{P}(U_2) e^{-i\hat{H}t_2} \right).
\]

Since this should hold for all \( \hat{\rho} \), the marginals of the POVM \( \hat{E} \) are PVM’s, namely

\[
\hat{E}(U_1, t_1; \Omega, t_2) = e^{i\hat{H}t_1} \hat{P}(U_1) e^{-i\hat{H}t_1},
\]

\[
\hat{E}(\Omega, t_1; U_2, t_2) = e^{i\hat{H}t_2} \hat{P}(U_2) e^{-i\hat{H}t_2}.
\]

6The results implied from propositions 1 and 2 seem to be well accepted in the consideration of sequential measurements. Even though they are rather elementary, we are not aware of any explicit proof in the literature, and for this reason we include the proof in the text. They are essential for the development of the arguments in Section 4.
There is a general result (see e.g. Theorem 2.1 of reference [15]) that any POVM, whose marginals are PVMs, is itself a PVM, it commutes with its marginals and can be written as the marginals’ product. Hence

\[ e^{i\hat{H}t_1} \hat{P}(U_1) e^{-i\hat{H}t_1}, e^{i\hat{H}t_2} \hat{P}(U_2) e^{-i\hat{H}t_2} = 0, \]  

(3. 12)

and since this property holds for all \( t_1, t_2 \) and subsets \( U_1, U_2 \), it follows that \([\hat{x}, \hat{H}] = 0\). The probability measure (2. 15) is additive in that case and the correlation functions are real-valued. It follows that in the generic case the probabilities for \( n \)-time measurements cannot be modeled by a stochastic process, because the latter can only be defined if a compatibility condition of the form (3. 10) is satisfied (see the discussion in section 3.5.1).

One, however, may object that the requirement that the single-time marginals of the POVM’s are projectors is too stringent. The physical set-up of a two-time measurement is different from that of a single-time measurement, and there is no a priori reason for the marginals of the POVM to reduce to those of the single-time measurement. One cannot argue so much against equation (3. 10).

If \( t_1 < t_2 \) the measurement outcomes at \( t_1 \) should not depend on whether or not we choose to perform a second measurement later. However, Eq. (3. 11) may very well be problematic, because the physical system has already interacted with a measuring device, while in the single-time measurement the evolution has been purely unitary.

Still, even this less restrictive case (namely only equation (3. 10) being satisfied) leads to the same conclusions. The proof involves only a few small changes from the earlier one, but we reproduce it here for concreteness.

We consider without loss of generality \( t_1 = 0, t_2 = t \). For the sample sets \( U_1, U_2 = \Omega - U_1, V_1, V_2 = \Omega - V_1 \) we define the positive operators

\[
\hat{E}_{ij} = \hat{E}(U_i, 0; V_j, t), \\
\hat{K}_i = \hat{E}(U_i, 0; \Omega, t), \\
\hat{L}_i = \hat{E}(\Omega, 0; V_i, t).
\]

(3. 13)  

(3. 14)  

(3. 15)

By assumption \( \hat{K}_i \) is a projector, while \( \hat{L}_i \) is a general positive operator.

By definition \( 0 \leq \hat{E}_{ij} \leq \hat{K}_j \), for both values of \( i \). Since \( \hat{K}_i \) is a projector, \( \hat{E}_{ij} \) lies in the closed linear subspace corresponding to \( \hat{K}_j \), with every \( j \) taken separately. Hence \( \hat{E}_{ij} \) commutes with \( \hat{K}_1 \) and \( \hat{E}_{ij} \) commutes with \( \hat{K}_2 \). Since \( \hat{K}_2 = 1 - \hat{K}_1 \), also \([\hat{E}_{i1}, \hat{E}_{i2}] = 0\). Since \( \hat{L}_i = \hat{E}_{i1} + \hat{E}_{i2} \), the operators \( \hat{E}_{i1}, \hat{E}_{i2} \) also commute with \( \hat{L}_i \). Again by definition \( 0 \leq \hat{E}_{ij} \leq \hat{L}_i \) and since \( \hat{E}_{ij} \) lies in the closed-linear subspace corresponding to \( \hat{K}_j \) we obtain \( 0 \leq \hat{E}_{i1} \leq \hat{K}_j \hat{L}_i \hat{K}_j = \hat{K}_j \hat{L}_i \). Since \( \hat{1} = \sum_{i,j} \hat{E}_{ij} \leq \sum_j \hat{K}_j \hat{L}_i \leq \hat{1} \), we obtain that \( \hat{E}_{ij} = \hat{L}_i \hat{K}_j \). We therefore conclude

\(^7\hat{A} \leq \hat{B} \) means that \( \hat{B} - \hat{A} \) is a positive operator.
Proposition 2. A POVM for sequential measurements satisfies (3. 10), only if its marginals commute.

This implies in particular that the marginal $E(\Omega, t_1; U_2, t_2)$ cannot be a POVM of type (3. 6) corresponding to an unsharp measurement of $\hat{x}$, unless $[\hat{x}, \hat{H}] = 0$.

The assumption that equation (3. 10) holds is valid for ideal measurements, like for instance the ones corresponding to measurements of observables with discrete spectrum. The generalisation of this result for non-ideal measurements is straightforward.

We conclude that we cannot construct POVMs that provide the probabilities for multi-time measurements in quantum systems, if we require that they reproduce faithfully (or even approximately) the predictions of single-time quantum theory. This, however, does not imply that we cannot construct any such probabilities in a way compatible with the predictions of single-time quantum theory. POVMs provide the most general way to construct probability densities on a sample space as a linear map of the quantum state $\hat{\rho}$. If we break linearity (and hence assume that the resulting construction will not respect the convexity properties of the space of states) such an assignment may be possible. However, probabilities defined through such a procedure cannot be obtained from a measure of the form (2. 10) (corresponding to a Markov process), because such a measure would be linear with respect to the initial density matrix. We shall take up this issue again in section 4.2.

3.4 Constructing POVMs for sequential measurements

Propositions 1 and 2 above demonstrate the degree of contextuality in sequential quantum measurements. They do not imply, however, that no POVMs exist that provide the probabilities of sequential measurements. Indeed, probabilities for sequential measurements have been considered extensively in the literature. We shall construct such POVMs in detail, in order to demonstrate that they are not only mathematically natural, but also physically reasonable.

3.4.1 Ideal measurements

We first consider the case of measuring an observable $\hat{x} = \sum_i \lambda_i \hat{P}_i$ with discrete spectrum. Writing $\hat{Q}_i = e^{i\hat{H}t_1} \hat{P}_i e^{-i\hat{H}t_1}$, we construct the probabilities for the most-fine grained two-time results

$$p(i, 0; j, t) = Tr(\hat{Q}_j \hat{P}_i \hat{\rho}_0 \hat{P}_i) = |\langle i | \hat{\rho}_0 | i \rangle |^2 e^{-i\hat{H}t} |j\rangle^2$$  \hspace{1cm} (3. 16)

Irrespective of the interpretation of the measurement process, the probabilities (3. 16) refer to the most elementary alternatives that can be unambiguously determined in the experimental set-up corresponding to the sequential measurement of $\hat{x}$. Therefore, they can be employed to construct probabilities for general
sample sets $U_1, U_2$ on the spectrum $\Omega$ of $\hat{x}$, namely

$$p(U_1, 0; U_2, t) = \sum_{i \in U_1} \sum_{j \in U_2} p(i, 0; j, t).$$  (3.17)

The total probability is normalized

$$p(\Omega, 0; \Omega, t) = \sum_{ij} Tr(\hat{Q}_j \hat{P}_i \hat{\rho}_0 \hat{P}_i) = 1.$$  (3.18)

Hence Eq. (3.17) defines a POVM for two-time measurements.

Note that as a result of the construction above, the probabilities $p(U_1, 0; U_2, t)$ for general samplings do not depend on $U_1$ and $U_2$ through the corresponding projectors $\hat{P}_{U_1}$ and $\hat{P}_{U_2}$. This strengthens the conclusion of section 3.2 that there is no quantum logic interpretation for multi-time measurements. In classical probability we use the same mathematical object (a characteristic function for a subset of the sample space) to represent both a concrete measurement outcome and a statement about a measurement outcome. In multi-time quantum measurements this is no longer the case: a coarse-grained projector $\hat{P}_U$ cannot represent a proposition that the outcome of the corresponding measurement lies within $U$: it can only represent a genuine physical event [23].

### 3.4.2 Continuous spectrum

The situation is more complex when one considers observables with continuous spectrum, such as position. In that case there are no fine-grained projectors and the choice of the elementary quantum probabilities, from which one may build the general probabilities for measurement outcomes cannot be made uniquely. We shall see that this implies that the probabilities are very strongly dependent on minor properties of the measurement device, so strongly in fact as to put into question whether the definition of a statistical ensemble is practically meaningful.

The immediate generalisation of Eq. (3.16) for the measurement of an operator with a continuous spectrum is

$$p(x_1, 0; x_2, t) = |\langle x_1|\hat{\rho}_0|x_1\rangle|^2 |\langle x_1|e^{-i\hat{H}t}|x_2\rangle|^2.$$  (3.19)

This, however, does not define a proper probability density, because it is not normalized to unity

$$\int dx_1 \int dx_2 p(x_1, 0; x_2, t) = \infty.$$  (3.20)

This is due to the fact that there can be no measurements of infinite accuracy. One has, therefore, to take into account the finite width of any position measurement, say $\delta$. This quantity depends on the properties of the measuring device—for example the type of the material that records the particle’s position.
The simplest procedure (but not the most natural one) is to consider the measurement of a self-adjoint operator \( \hat{x}_\delta = \sum_i x_i \hat{P}_i^\delta \), where \( \hat{P}_i^\delta \) is a projection operator corresponding to the interval \([x_i - \frac{\delta}{2}, x_i + \frac{\delta}{2}]\). In that case we may immediately construct the fine-grained probabilities
\[
p_\delta(i, 0; j, t) = Tr(\hat{Q}_j^\delta \hat{P}_i^\delta \rho_0 \hat{P}_i^\delta), \tag{3.21}
\]
from which we may construct probabilities for general sample sets \( U_1 \) and \( U_2 \):
\[
p_\delta(U_1, 0; U_2, t) = \sum_{i \in U_1} \sum_{j \in U_2} Tr(\hat{Q}_j^\delta \hat{P}_i^\delta \rho_0 \hat{P}_i^\delta). \tag{3.22}
\]
Strictly speaking one may only consider sample sets that are unions of the elementary sets that define our lattice. If, however, the size of the sample sets \( L \) is much larger than \( \delta \), we may approximate the summation with an integral. This amounts to defining the continuous version of probabilities (3.21)
\[
p_\delta(x_1, t_1; x_2, t_2) = \int_{x_1 - \delta/2}^{x_1 + \delta/2} dx_1 \int_{x_2 - \delta/2}^{x_2 + \delta/2} dx_2 Tr\left(e^{i\hat{H}(t_2 - t_1)} \hat{P}_x^\delta e^{-i\hat{H}(t_2 - t_1)} \hat{Q}_x \hat{P}_x^\delta \hat{P}_x \rho_0 \hat{P}_x^\delta\right), \tag{3.23}
\]
where we denoted \( \hat{P}_x^\delta = \int_{x - \delta/2}^{x + \delta/2} dy |y\rangle \langle y| \).

To construct the probabilities \( p_\delta(U_1, 0; U_2, t) \), we split each set \( U_i \) into mutually exclusive cells \( u_{\alpha i} \) of size \( \delta \), such that
\[
\bigcup_{\alpha} u_{\alpha i} = U_i \tag{3.24}
\]
\[
u_{\alpha i} \cap u_{\beta i} = \emptyset, \alpha \neq \beta. \tag{3.25}
\]
If we denote select points \( x_{\alpha i} \in u_{\alpha i} \), for all \( i \) \( (x_{\alpha i} \) may be the midpoint of \( u_{\alpha i} \)), we may construct the probability \( p_\delta(U_1, 0; U_2, t) \)
\[
p_\delta(U_1, 0; U_2, t) = \sum_{\alpha} \sum_{\beta} p_\delta(x_{\alpha i}, 0; x_{\beta j}, t) \tag{3.26}
\]
In the limit that the typical size of the sets \( U_1, U_2 \) is much larger than \( \delta \), we obtain
\[
p_\delta(U_1, t_1|U_j, t_2) = \frac{1}{\delta^2} \int_{U_1} dx_1 \int_{U_2} dx_2 p_\delta(x_1, t_1; x_2, t_2), \tag{3.27}
\]
In other words, the objects \( \frac{1}{\delta^2} p_\delta(U_1, t_1|U_j, t_2) \) play the role of probability densities.

Equation (3.27) suggests that the two-time probabilities are given by the positive operators
\[
\Pi(U_1, 0; U_2, t) = \frac{1}{\delta^2} \int_{U_1} dx_1 \int_{U_2} dx_2 \hat{P}_x^\delta \hat{Q}_x \hat{P}_x^\delta, \tag{3.28}
\]
21
which fail to define a POVM, because they are not normalized to unity: they differ from 1 by a term of order $O(\delta)$. This is an artefact of the way we implemented the continuous limit in going from probabilities (3.27) to those of (3.28). An error of the order of $O(\delta)$ is reasonable, since the sampling error is itself of the order of $\delta$.

It is easy to remedy this problem by working with POVM’s for the single-time probabilities. We consider a POVM $\hat{\Pi}^\delta_x(U) = \int_U dx \hat{\Pi}_x^\delta$ for position that satisfies the following properties

$$\int dx \hat{\Pi}_x^\delta = \hat{1}, \quad \int dx x \hat{\Pi}_x^\delta = \hat{x}. \quad (3.29)$$

For example one may consider the Gaussian POVM

$$\hat{\Pi}_x^\delta = \int \frac{1}{\sqrt{2\pi\delta}} e^{-(x-\bar{x})^2/2\delta^2} |\bar{x}\rangle\langle \bar{x}|. \quad (3.30)$$

Then the operators

$$\hat{R}^\delta(U_1,0;U_2,t) = \int_{U_1} dx_1 \int_{U_2} dx_2 \sqrt{\hat{\Pi}_{x_1}} e^{i\hat{H}t} \hat{\Pi}_{x_2} e^{-i\hat{H}t} \sqrt{\hat{\Pi}_{x_1}} \quad (3.31)$$

satisfy all properties of a POVM including the normalization condition. It is easy to check that within an error of $O(\delta)$ the probabilities defined by the POVM (3.31) coincide with those of (3.28) (with $\sqrt{2\pi\delta}$ in place of $\delta$). The generalisation to $n$-time measurements is straightforward

$$\hat{R}^\delta(U_1,t_1;U_2,t_2;\ldots;U_n,t_n) = \int_{U_1} dx_1 \int_{U_2} dx_2 \ldots \int_{U_n} dx_n e^{i\hat{H}t_1} \sqrt{\hat{\Pi}_{x_1}} e^{i\hat{H}(t_2-t_1)} \sqrt{\hat{\Pi}_{x_2}} e^{i\hat{H}(t_3-t_2)} \sqrt{\hat{\Pi}_{x_3}} \ldots e^{i\hat{H}(t_n-t_{n-1})} \sqrt{\hat{\Pi}_{x_n}} e^{-i\hat{H}(t_{n-1} - t_{n-2})} \sqrt{\hat{\Pi}_{x_2}} e^{-i\hat{H}(t_{n-2} - t_{n-3})} \sqrt{\hat{\Pi}_{x_2}} e^{-i\hat{H}(t_{n-3} - t_{n-4})} \ldots \sqrt{\hat{\Pi}_{x_1}} e^{-i\hat{H}t_1}. \quad (3.32)$$

3.5 Basic features of the constructed POVMs

3.5.1 The inequivalence with stochastic processes

The probability densities defined by the sequence of the POVMs (3.32) for all values of $n$ as

$$p_n^\delta(x_1,t_1;x_2,t_2;\ldots;x_n,t_n) = Tr[\rho_0 \hat{R}^\delta(U_1,t_1;U_2,t_2;\ldots;U_n,t_n)]. \quad (3.33)$$

This result in conjunction with the theorems of section 3.3, demonstrate the inequivalence of quantum probabilities for multi-time measurements with those that can be obtained by a classical stochastic processes.
33) does not define a probability measure on the space of paths, because the compatibility condition necessary for the definition of such a measure

\[ p_{n-1}^\delta(x_1, t_1; \ldots; x_{i-1}, t_{i-1}; \ldots; x_{i+1}, t_{i+1}; \ldots; x_n, t_n) = \int dx_i p_n^\delta(x_1, t_1; \ldots; x_i, t_i; \ldots; x_n, t_n) \]  

(3.34)

for all possible \( i = 1, 2, \ldots, n \), is not satisfied.

A weaker version is satisfied instead,

\[ p_{n-1}^\delta(x_1, t_1; x_2, t_2; \ldots; x_{n-1}, t_{n-1}) = \int dx_n p_n^\delta(x_1, t_1; x_2, t_2; x_{n-1}, t_{n-1}; x_n, t_n), \quad t_n > t_{n-1} > \ldots > t_2 > t_1 \]  

(3.35)

namely only if we integrate over the variables defined at the final moment of time in the \( n \)-time distribution, do we obtain the \( n-1 \)-time probability distribution.

### 3.5.2 Strong dependence on the apparatus’s resolution

Since the functions (3.33) provide a well defined system of joint probability densities, one could consider defining an generalisation of stochastic processes that would reproduce the predictions of quantum measurements. There is however a problem: the POVM (3.32) and the corresponding probability densities depend very strongly on the parameter \( \delta \).

This is a direct consequence of the fact that the probabilities (3.33) arise out of the non-additive measure (2.15). Suppose we consider two different measurement devices, one characterized by a value \( \delta \) and another by a value \( 2\delta \). In any reasonable measurement scheme one would expect that the two-time probabilities for sample sets \( U_1 \) and \( U_2 \) would not be appreciably different if their size is much larger than \( \delta \). However, this turns out not to be the case. It is easier to see this in the discriminated expression (3.21).

A projection operator \( \hat{P}_x^{2\delta} \) centered around \( x \) with width \( 2\delta \) can be written as the sum \( \hat{P}_{x-\delta/2}^\delta + \hat{P}_{x+\delta/2}^\delta \). When we construct the elementary probabilities corresponding to \( \hat{P}_x^{2\delta} \), which correspond to sets of width \( 2\delta \) they will differ from the probabilities for the same sets, when the latter are constructed by sets of width \( \delta \). Their difference will be the interference term

\[ 2 \text{Re} \int dx_1 \delta(x_1 + \delta/2, x_1 - \delta/2, t_1 : x_2, t_2) = \]  

\[ 2 \text{Re} Tr \left( e^{i\hat{H}(t_2-t_1)} \hat{P}_{x_2}^\delta e^{-i\hat{H}(t_2-t_1)} \hat{P}_{x_1}^\delta \hat{P}_{x_1+\delta/2}^\delta \hat{P}_{x_1-\delta/2}^\delta \right). \]  

(3.36)

The modulus of the 'interference' term is, in general, of the same order of magnitude with the probabilities \( p_\delta \) and \( p_{2\delta} \) themselves. Hence, when we sum (or integrate) over the probabilities corresponding to the cells of width \( \delta \) or \( 2\delta \) to construct the probabilities \( p_\delta(U_1, t_1 | U_2, t_2) \) and \( p_{2\delta}(U_1, t_1 | U_2, t_2) \) for generic large sample sets \( U_1 \) and \( U_2 \) the results differ by an amount of
\[ \epsilon_\delta(U_1, t_1; U_2, t_2) = \text{Re} \int_{U_1} dx_1 \int_{U_2} dx_2 \, d_\delta(x_1 + \delta/2, x_1 - \delta/2, t_1 : x_2, t_2) \] (3.37)

This term is of the same order as the probabilities themselves [23], a fact pointing to the strong dependence of the results on the resolution \( \delta \). Different values of \( \delta \) lead to very different probabilities.

We may also see that in the POVM (3.31). For the special case of a free particle \( \mathcal{H} = \frac{p^2}{2m} \) (3.31) equals

\[ \langle x|\hat{R}^\delta(U_1, 0; U_2, t)|x'\rangle = \frac{m}{(2\pi)^{d/2}\delta} \int_{U_1} dx_1 \int_{U_2} dx_2 \, \text{exp} \left( -i \frac{m}{t}(x - x')(x_2 - x + x')/2 \right) \]
\[ \times \exp \left( -\frac{1}{2} \left( \frac{m^2x'^2}{t^2} + \frac{1}{4\delta^2}(x - x')^2 - \frac{(x_1 - \frac{x_1 + x'}{2})^2}{2\delta^2} \right) \right) \] (3.38)

This can be written as

\[ \langle x|\hat{R}^\delta(U_1, 0; U_2, t)|x'\rangle = \frac{m}{t} \tilde{\chi}_{(x, x') \xi_{U_2}}(x - x') \chi_{U_1}(x + x')/2 \]
\[ \times \exp \left( -\frac{1}{2} \left( \frac{m^2x'^2}{t^2} + \frac{1}{4\delta^2}(x - x')^2 \right) \right), \] (3.39)

where \( T_x \) denotes the translation operator on \( \mathbb{R} \), \( \tilde{\chi} \) is the Fourier transform of the characteristic function \( \chi \) and \( \chi_{U_1} \) is a smeared characteristic function of position. If the size of \( U_1 \) is much larger than \( \delta \), then one may approximate \( \chi_{U_1} \), by an exact characteristic function, thus obtaining

\[ \langle x|\hat{R}^\delta(U_1, 0; U_2, t)|x'\rangle \simeq \frac{m}{t} \tilde{\chi}_{(x, x') \xi_{U_2}}(x - x') \chi_{U_1}(x + x')/2 \]
\[ \times \exp \left( -\frac{1}{2} \left( \frac{m^2x'^2}{t^2} + \frac{1}{4\delta^2}(x - x')^2 \right) \right). \] (3.40)

The matrix elements of the POVM involve a product of two terms that depend on the sample sets (and not on \( \delta \)) with a Gaussian term that is very sensitive on \( \delta \) and does not depend on the sample sets. This clearly demonstrates the strong dependence of the POVM (3.38) on \( \delta \), which persists even for very-coarse sample sets. It is easy to verify that the norm of the difference between two positive operators \( \hat{R}^\delta(U_1, 0; U_2, t) \) and \( \hat{R}^{\delta'}(U_1, 0; U_2, t) \), for different values \( \delta \) and \( \delta' \) respectively, is of the order of the norm of the operators themselves. For most states therefore the probabilities will be very sensitive on the resolution \( \delta \).

Note also that as \( \delta \to 0 \) the POVM (3.32) does not converge to a PVM that provides an ideal value for probabilities, as is the case in single-time measurements. Instead, \( \lim_{\delta \to 0} \hat{R}^\delta(U_1, 0; U_2, t) = 0 \). This behavior is a consequence of the use of the square root of the POVM II in (3.32). This is necessary in order to guarantee that (3.32) is a proper POVM, normalized to unity and with the correct dimensions to define a probability density on \( \mathbb{R}^n \).
3.6 Quantum measuring device

Our previous derivation of the probabilities for multi-time measurements was based on an operational description of the quantum measurement process, namely the assumption that the measuring device is classical and that the effect of the measurement is the "reduction of the wave packet rule", either corresponding to ideal measurements, or to non-ideal ones (in which case we employ POVM’s).

One may however object that a full quantum mechanical treatment of the measuring device may presumably lead to a different result. We shall argue here that this is not the case. It is well known that the standard treatment of a quantum measurement device (together with von Neumann’s reduction rule) is equivalent to the description of the probabilities for a quantum system with a PVM [24]. For observables with continuous spectrum measurements are usually unsharp and the sampling of the quantum system turns out to be equivalent to a POVM that depends on specific properties of the interaction between the quantum system and the measurement device. This dependence is, however, relatively weak as the probability for sufficiently coarse-grained sets is largely insensitive to such details [25, 26].

The generalisation of the results above for sequential measurements is straightforward. One only needs to introduce a different measurement device for each measurement. If the devices are initially uncorrelated, it is easy to demonstrate that in ideal measurements (and discrete pointers) probabilities are provided by a POVM of the type (3.17). However, if we employ a discrete pointer for the measurement of position the dependence of probabilities on the resolution arises out of the explicit correlation between the continuous variable \( \hat{x} \) and the discrete basis for the pointer. Effectively we return to a POVM like (3.21) with the same strong dependence on \( \delta \).

A continuous pointer can be shown to lead to an equation of the form (3.31), with the smearing function determined by the initial state of the apparatus. To see this, we consider the following toy model. Let \( \hat{x} \) be the position of the particle we want to determine, and let the particle be prepared in a state \( |\psi_0\rangle \). We assume two identical measurement devices each in state \( |\Psi_0\rangle \), initially uncorrelated with the particle and with each other. The state of the total system will then be initially \( |\psi_0\rangle|\Psi_0\rangle|\Psi_0\rangle \). The pointer variables are \( \hat{q}_1 \) and \( \hat{q}_2 \) for each device. We assume that the self-dynamics of the devices is negligible, that the particle’s Hamiltonian is \( \hat{H} \) and that the interaction Hamiltonian is

\[
H_{int} = f_{t_1}(t)\hat{x} \otimes \hat{k}_1 \otimes \hat{1} + f_{t_2}(t)\hat{x} \otimes \hat{1} \otimes \hat{k}_2, \tag{3.41}
\]

where \( \hat{k}_i \) are the conjugate momenta of \( \hat{q}_i \), and \( f_{t_i}(t) \) is a function of time sharply concentrated around \( t_i \). At the limit of instantaneous measurements the state of the system at time \( t = t_2 \) is

\[
|\psi_{tot}(t_2)\rangle = \int dk_1 \int dk_2 \left( e^{-ik_2\hat{\hat{x}}} e^{-i\hat{H}(t_2-t_1)} e^{-ik_1\hat{\hat{x}}} e^{-i\hat{H}t_1} |\psi_0\rangle \right) \\
\otimes |k_1\rangle \langle k_1| \otimes |k_2\rangle \langle k_2| |\Psi_0\rangle. \tag{3.42}
\]
The probability distribution for the pointer variables is

\[ \int dx \, |\langle x, q_1, q_2 | \psi_{\text{tot}}(t_2) \rangle|^2. \]  

This coincides with that given by the POVM (3. 31), if we identify \( \sqrt{\Pi_y} = \int \frac{dk}{\sqrt{2\pi}} e^{-ik(x-y)} \langle k | \Psi_0 \rangle \). It is easy to verify that the corresponding \( \Pi_y \) defines a POVM for a single-time measurement of position. In this simple model the resolution \( \delta \) of the device is determined by the spread in momentum of the apparatus’s initial state. In a more realistic model, the resolution \( \delta \) receives contributions from the self-dynamics of the detector (and of a possible environment), from the finite duration of the measurement interaction and from the accuracy in the readings of \( \hat{q}_1, \hat{q}_2 \).

The presence of an environment does not change the essence of the arguments presented previously. The consideration of an enlarged Hilbert space that also contains the degrees of freedom of the environment does not make any difference to the mathematical formulation of the issue. The only possible way to cancel the strong dependence of the probabilities on the resolution \( \delta \), is to assume that the effect of the environment causes interference terms like (3. 36) (where now the projectors refer to the values of the pointer rather than those of the measured particle) to become rapidly small\(^8\).

In general, the interference terms (3. 36) can only be suppressed (for a sufficiently generic initial state), if the environment causes the reduced density matrix of particle+apparatus to be diagonalisable in the factorized basis \(|i⟩|a_i⟩\), where \(|i⟩\) are the eigenstates of the measured observable and \(|a_i⟩\) the pointer basis in the apparatus’s Hilbert space. This is in general not possible as can be seen from a very general theorem [27] (and in a different but related context by [25]). Indeed if this diagonalization took place we would have a full resolution of the so-called macroobjectification problem by environment-induced decoherence, which is known not to be the case\(^9\) — see the discussion in [28, 29, 30].

Hence we conclude that the strong dependence of the multi-time probabilities on the properties of the measurement device is unavoidable, whether one considers the formalism of quantum theory as an operational description, or whether one considers any minimal generalisations that involve a quantum mechanical treatment of the measuring device.

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8The environment is coupled to the measuring apparatus and not directly to the particle. If that were not the case the particle would exhibit fully classical behavior and its measurement would not be different from that of a classical probabilistic system.

9Whether environment-induced decoherence solves the full measurement problem in interpretations other than the Copenhagen one (e.g. many-worlds or consistent histories) is a different issue, unrelated to the aims of this paper. For the purposes of the present argument, we are only interested in the mathematical statement that the diagonalization in the basis \(|i⟩|a_i⟩\) cannot implemented in any closed system (however large) that evolves unitarily.
3.7 Consequences

3.7.1 Contextuality of measurements

The first result of our analysis of sequential measurements is the breakdown of quantum logic. Unlike the single-time case, a proposition about a measurement outcome is not represented by a projection operator. The probabilities $p(U_1, t_1; U_2, t_2)$ do not depend on the sample sets through projectors and are very different in different experimental setups. A two-time YES-NO experiment will lead to different probabilities from those obtained by the experiment of Fig. 3.

This result is complementary to the Kochen-Specker theorem: it is in general not possible to attribute specific values to sets of observables, even commuting ones, without specifying the context, namely the concrete experimental set-up. In other words, one cannot define a sample space for the possible outcomes of an observable, without referring to the specific measurement being implemented.

The relevance of contextuality in sequential measurements (both factual and counterfactual) has been studied extensively in the literature. Albert, Aharonov and D’Amato employed the result of Ref. [13] concerning an ensemble that is both pre- and post-selected through measurements at times $t_i$ and $t_f$ [31]. They showed that it is possible to retrodict the results of specific measurements that could have been carried out at any moment of time in the time interval $[t_i, t_f]$.

It is important to remark that this retrodiction can be applied to incompatible measurements, i.e. ones corresponding to non-commuting observables. A similar result is also obtained by Kent [32], who argues that retrodiction in a quantum theory that purports to describe individual systems (consistent histories) leads generically to contrary inferences.

The discussion in this paper is within a slightly different context than the ones of the references above. We are only interested in providing a probability measure for the results of sequential measurements that have actually taken place. The intermediate measurement device is part of a specific experimental set-up and we do not consider any counterfactual statements (about retrodiction). In any case, the situations studied here typically involve probabilities that are spread over different alternatives: typically only trivial inferences can be made.

The standard proof of the Kochen-Specker theorem assumes that it is possible to assign definite values to commuting physical observables in individual systems prior to measurement, which, while reasonable, is not an statement amenable to empirical verification. Moreover, it involves an interpretation-dependent assumption that it is possible to extrapolate the rules of quantum theory from the description of statistical ensembles to that of individual systems. In sequential measurements however the dependence of the measurement outcomes on the specific experiment is direct in terms of concrete empirical data, and makes no assumptions other than that quantum theory provides the
correct probabilities for the measurement outcomes in statistical ensembles.

Contextuality can also be inferred from Bell’s theorem and its generalisations or from Wigner’s theorem about the lack of a joint probability distribution for non-commuting observables. However, in the cases above one may provide alternative explanations: in the former case one may attribute the failure of Bell’s inequalities to non-locality, while in the second one may invoke the inability to perform simultaneously measurements of incompatible observables. There are no such limitations, when the argument for quantum contextuality is phrased in terms of the probabilities for sequential measurements. It is in principle possible to measure multi-time probabilities in different experimental set-ups. If the results of our analysis are correct, these probabilities will differ strongly, thus providing irrefutable empirical evidence about the contextuality of quantum events.

3.7.2 Inferences and conditional probability

Conditioning it is a very important part of classical probability; it is the mathematical implementation of the idea that when we obtain information from an experiment, we need to modify our description of the system (i.e. the probability distribution) in order to account for the new information. The prototype of conditioning is the notion of conditional probability, i.e. the probability $p(A|B)$ that $A$ will take place when we have verified that $B$ occurred

$$p(A|B) = \frac{p(A \cap B)}{p(B)} \quad (3.44)$$

It is sometimes suggested that the "wave packet reduction rule" can be interpreted as a quantum version of conditional probability. Our results suggest that this is not the case. Such an interpretation is only possible, when 'conditioning' refers to the most fine-grained recordings of a physical system’s properties. If we attempt to employ this rule to account for coarser alternatives, we inevitably lose information in the process and cannot obtain correct physical predictions.

In the classical theory conditional probability can be employed to define logical implication. If the conditional probability $p(A|B)$ for an event $A$ given that $B$ was realized equals 1. If in an experiment we verify the property $B$, we may expect that the property $A$ will be almost surely satisfied. In quantum theory the situation concerning implication is more subtle. There is a strong distinction between prediction and retrodiction.

The fact that Eq. (3.34) does not hold implies that retrodiction is problematic. A two-time measurement involves a different experimental set-up from that of a single-time measurement. The single-time probability for a measurement at time $t$ is different from any marginal obtained by tracing out the results of any measurement at any time $t' < t$. It is, therefore, impossible to make any inference (or any probabilistic statement) about what would have happened if a measurement had taken place earlier, solely from the data obtained at time $t$. 

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It is necessary to provide the specifics of the intermediate measurement scheme. On the other hand, the validity of Eq. (3.35) implies that prediction works the same as in the classical case. Tracing out the results of a later measurement yields the same probability distribution as if the measurement had not taken place, hence it is in principle possible to make inferences from the data at time $t$ about the results of measurements that takes place at time $t' > t$.

### 3.7.3 Strong dependence on the measuring device’s resolution

In single-time quantum theory we know that different experiments of the same type are expected to yield identical results, up to sampling and systematic errors. This is guaranteed by the spectral theorem, and it is epistemologically very desirable because the results of similar experiments can be immediately compared. But in multi-time measurements even two experiments that are identical in all details (preparation of the measuring device, source of particles, design of the experiment) but the resolution of the measuring device, will lead to different probabilities and correlation functions. This is a very stronger effect and in principle observable. It is a source of doubt about whether any meaningful information can be obtained from such experiments.

At a practical level we know that experiments yield more reliable results, when we have expended time and effort to minimise the errors, which may arise from either sampling inaccuracies or from the finite resolution of the measurement device. Copenhagen quantum theory assumes that any results we obtain will make reference to the specific set-up, but even when we restrict our expectations to that case, common sense suggests that the smaller the error, the more reliable our experimental results will be. The event frequencies should converge to some ideal values that would characterize, if not the measured system in itself, at least the general design of the experiment. This expectation is fulfilled in single-time quantum theory. The dependence of the typical POVMs for unsharp measurements on the error (or resolution) $\delta$ is rather weak, and for sufficiently coarse-grained samplings the corresponding positive operators are close to true projectors. The probabilities corresponding to such samplings probabilities will therefore be the same in all measurement devices, and they will coincide with the probabilities obtained from YES-NO experiments. But in multi-time measurements this is no longer the case. The POVM’s dependence on $\delta$ persists even for very coarse samplings. When we increase the resolution, we do not obtain ”better” results, we simply obtain different results.
4 Sequential measurement in hidden-variable theories

4.1 Bohmian mechanics

The results of section 3.5 suggest that the probabilities for sequential measurements are very sensitive to even minute changes of the measuring procedure. Hence even if we accept that such probabilities can be defined from the statistical data, a question is immediately raised. Why are multi-time quantum measurements so different from single-time ones? Clearly, an answer to this question cannot be provided within standard quantum theory, because the issue itself arises as a consequence of the theory’s basic postulates. One would have to enlarge the domain of standard quantum theory and essentially work with hidden variables.

The most important hidden variables theory, both because of its long history and its intrinsic strength, is Bohmian mechanics [33]. In this theory the additional variables are the particles’ position, which evolve according to the modified Newton’s equations

\[ m \ddot{x} = \text{Im} \frac{\partial_x \Psi(x,t)}{\Psi(x,t)}, \quad (4.1) \]

where the wave function \( \Psi(x,t) \) is a solution of Schrödinger’s equation.

The dynamical equations (4.1) are usually supplemented by the condition of quantum equilibrium, namely that in a statistical ensemble of particles the probability density for the particle’s position is given by Born’s rule:

\[ \rho(x,t) = |\Psi(x,t)|^2. \]

Assuming quantum equilibrium, it is easy to construct a probability density for the outcomes of sequential measurements. The particle’s position and the wave function satisfy a set of differential equations and are fully deterministic. Hence any trajectory can be fully specified by the knowledge of the initial conditions: the position \( x_0 \) and the wave function \( \Psi_0(x) \). Let us denote as \( x(t; x_0, \Psi_0) \) the solutions to (4.1); they can be viewed as functions of the random variable \( x_0 \).

The probability that the particle lies in the set \( U_1 \) at time \( t_1 \), in \( U_2 \) at time \( t_2 \),... and in \( U_n \) at time \( t_n \) then equals

\[ p^n(U_1, t_1; U_2, t_2; \ldots; U_n, t_n)) = \int dx_0 |\Psi_0(x_0)|^2 \times \chi_{U_1} [x(t_1; x_0, \Psi_0)] \chi_{U_2} [x(t_2; x_0, \Psi_0)] \ldots \chi_{U_n} [x(t_n; x_0, \Psi_0)]. \quad (4.2) \]

The tower of all \( n \)-time probabilities satisfy by construction the compatibility condition and thus defines a measure on the space of paths \( d\mu[x(\cdot)] \), which depends only on the initial wave-function and the Hamiltonian operator of the

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wave function’s evolution. This measure fully reproduces the predictions of standard quantum theory at a single moment of time.

There is no contradiction with our results at section 3.3. The non-go theorems proved there refer to probabilities that can be constructed via POVMs: the n-time probability densities are linear functionals of the initial density matrix. Clearly, this is not the case here as the initial state enters in a non-trivial way in the definition of the random variables \( x(t; x_0, \Psi_0) \). For the same reason the stochastic process corresponding to (4.2) is non-Markovian\(^{10}\). It is however local-in-time, because the densities corresponding to (4.2) factories

\[
p^n(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = |\Psi(x, t_1)|^2 \times \delta(x_1, g_{t_1, t_2}[\Psi_0](x_2)) \delta(x_2, g_{t_2, t_3}[\Psi_0](x_3)) \ldots \delta(x_{n-1}, g_{t_{n-1}, t_n}[\Psi_0](x_n)), \quad (4.3)
\]

where \( g_{t,t'}[\Psi_0], t < t' \) is the backwards-in-time evolution operator corresponding to the equation of motion (4.1).

It would seem from the above expression that multi-time probabilities in Bohmian mechanics are different from those of standard quantum theory. The multi-time probability distributions in standard quantum theory cannot be obtained from a probability distribution. A difference in the probabilistic outcomes of multi-time measurements between Bohmian mechanics and quantum theory has been suggested before in [34, 35]. In these references the predictions of Bohmian mechanics were compared with correlation functions of the form (3.3), which have no immediate operational interpretation, while here we compare them with the probabilities for sequential measurements, which can in principle be determined empirically. Our analysis also shares some features with that of Hartle [36].

An immediate objection can be raised to the analysis above. Bohmian mechanics refers to the properties of things in themselves and not to measurement outcomes. To obtain the measured probabilities one would have to model the interaction of the quantum system with a measuring device. The Bohmian description of quantum measurements has been developed in [37]—see also a related discussion about Stochastic Mechanics in [38]. In these references it is argued that the reduction of the wave packet rule can be obtained from Bohmian mechanics after the interaction of a system with a measuring device has been taken into account. One would therefore expect that in sequential measurements the predictions of quantum theory should be reproduced.

We next examine this issue in more detail. We consider a two time measurement. Let \( x \) be the particle’s position, \( Q^1 \) and \( Q^2 \) the variables for the first and second measurement device respectively, and \( X \) a pointer function of \( Q_1 \) or \( Q_2 \), the range of which is the space \( \Omega \) of possible alternatives in each measurement. The pointer function may be either continuous or discrete. The total system

\(^{10}\)If we distinguish the two roles of the wave function as probability distribution and as agent of dynamical evolution, then the stochastic process is Markovian. It is however not time-homogeneous, unless the wave function is an eigenstate of the Hamiltonian operator.
will be effectively described by a stochastic process analogous to the one defined by (4.2). Assuming quantum equilibrium for the total system, the probabilities that the pointer \( X \) is found in a set \( U_1 \) at \( t_1 \), and in a set \( U_2 \) at \( t_2 \) equals

\[
p^2(U_1, t_1; U_2, t_2) = \int dx_0 dQ_1^2 dQ_2^2 |\Psi_0(x_0, Q_0^1, Q_0^2)|^2 \\
\times \chi_{U_1} \left[ X(Q_1^1(t_1; x_0, Q_1^1, Q_0^2, \Psi_0)) \right] \chi_{U_2} \left[ X(Q_2(t_2; x_0, Q_1^1, Q_0^2, \Psi_0)) \right] \tag{4.4}
\]

where \( Q_i(t, x_0, Q_0, \Psi) \) are the solutions to the deterministic equations of motion for the variables \( Q \) written in terms of the initial condition. The marginal probability, in which the results of the first measurement have been traced out equals

\[
p^2(\Omega, t_1; U_2, t_2) = \int dx_0 dQ_2^2 |\Psi_0(x_0, Q_0^1, Q_0^2)|^2 \\
\times \chi_{U_2} \left[ X(Q_2(t_2; x_0, Q_1^1, Q_0^2, \Psi_0)) \right] . \tag{4.5}
\]

On the other hand the probability of a single-time measurement at time \( t_2 \) is

\[
p^1(U_2, t_2) = \int dx_0 dQ_0^2 |\Psi_0(x_0, Q_0^2)|^2 \chi_{U_2} \left[ X(Q_2(t_2; x_0, Q_0^2, \Psi_0)) \right] . \tag{4.6}
\]

The crucial difference lies in the equations of motion for the pointer variable—there are no \( Q_1 \) variables in the expression for \( p^1 \), because there is no measuring device at time \( t_1 \). The probabilities (4.5) and (4.6) refer to different physical systems and as such they correspond to a different stochastic process.

### 4.2 Contextuality from non-locality

The analysis of multi-time probabilities in Bohmian mechanics above does not guarantee that the predictions of Bohmian mechanics coincide with those of standard quantum theory. We shall demonstrate now that the key property that permits that is the inherent non-locality of Bohm’s theory. For a different derivation of the constraints from local realism to the conditional probabilities of sequential measurements the reader is referred to [39, 40]. Also related are the constraints that can be expressed in terms of "temporal Bell inequalities" [41, 42, 43]. The derivation we provide here refers to the most general case.

We consider a general deterministic hidden variable theory. The probabilistic description arises from an initial probability distribution for a statistical ensemble, which is related to the wave functions by Born’s rule. To model a sequential measurement we assume the same variables \( x, Q_1 \) and \( Q_2 \) as in section 4.2. The wave function \( \Psi \) at \( t = 0 \) is assumed factorized, namely \( \Psi_0(x_0, Q_1^0, Q_0^2) = \psi_0(x_0)\phi_1(Q_1^0)\phi_2(Q_0^2) \). We also assume that the degrees of freedom \( Q_1 \) and \( Q_2 \) do not interact directly, and that at time \( t_1 \) the particle has
not interacted with the degrees of freedom of the second measurement device. This implies that the value of \( x \) and \( Q^1 \) at time \( t_1 \) does not depend on the initial value \( Q^2_0 \). The conditions above are natural in any measurement process.

We first consider a discrete pointer \( X \), that takes values in the finite set \( \Omega \) of elementary alternatives. We assume that after a measurement the pointer \( X \) reveals the value of the function \( f \) of the variable \( x \), so that at the time \( t \) when the measurement interaction has finished

\[
X[Q(t, x_0, Q_0)] = f[x(t, x_0, Q_0)],
\]

where \( x_0, Q_0 \) are the initial values of the configuration variables of the system and apparatus respectively. We simplify the notation by writing

\[
X[Q(t, x_0, Q_0)] = X_t(x_0, Q_0)
\]

and

\[
f[x(t, x_0, Q_0)] = f_t(x_0, Q_0).
\]

The variables \( x \) need not only refer to particle positions, but may in principle refer to other degrees of freedom, e.g. spin.

We next assume that the single-time predictions of this theory coincide with those of standard quantum mechanics for ideal measurements, namely ones corresponding to a PVM \( \hat{F}_U \) on \( \Omega \) defined on the Hilbert space of the system’s wave functions.

\[
p(U, t) = \int dx_0 dQ_0 |\psi_0|^2(x_0) |\phi|^2(Q_0) \chi_U[X_t(x_0, Q_0)] = \int dx_0 |\psi_0|^2(x_0) \chi_U[f^{sys}_t(x_0)] = \langle \psi_0 | e^{i\hat{H}_0 t} \hat{F}_U e^{-i\hat{H}_0 t} |\psi\rangle,
\]

where \( f^{sys}_t(x_0) \) refers to the evolution of the measured system in absence of the measurement device (equivalent to the quantum mechanical evolution with a Hamiltonian \( \hat{H} \)). Equation (4.8) holds in Bohmian mechanics when \( f \) is a function of position, but may be valid for more general configurational variables. The key assumption is that the operational predictions of quantum theory are valid, namely that the probabilities for measurements can be obtained by an application of Born’s rule in the wave function of the system alone. This holds for ideal measurements.

The crucial assumption is that after the first measurement has been completed the system does not interact any more with the first device. The variables \( Q^1 \) do not appear any more in its equation of motion. This is essentially an assumption of \textit{locality} for the interaction of the system with the measurement device. It implies that

\[
\begin{align*}
x_{t_2}(x_0, Q^1_0, Q^2_0) &= x_{t_1}(x_{t_1}(x_0, Q^1_0), Q^2_0) \\
Q^2_{t_2}(x_0, Q^1_0, Q^2_0) &= Q^2_{t_1}(x_{t_1}(x_0, Q^1_0), Q^2_0),
\end{align*}
\]

namely the values of \( x \) and \( Q_2 \) after the second measurement depend on \( Q_1 \) only through the value of \( x \) immediately after the first measurement. Using the
locality condition the probability (4.4) is written as

\[ p^2(U_1, t_1; U_2, t_2) = \int dx_0 \int dQ_0^1 \int dQ_0^2 \left| \psi(x_0) \right|^2 \left| \phi_1(x_0) \right|^2 \left| \phi_2(x_0) \right|^2 \]

\[ \times \chi_{U_1} \left[ X_{t_1}(x_0, Q_0^1) \right] \chi_{U_2} \left[ X_{t_2}(x_1(x_0, Q_0^1), Q_0^2) \right] = \]

\[ \int dx_0 \int dQ_0^1 \left| \psi(x_0) \right|^2 \left| \phi_1(x_0) \right|^2 \chi_{U_1} \left[ X_{t_1}(x_0, Q_0^1) \right] \chi_{U_2} \left[ x_{t_2}^{sys}(f_{t_1}(x_0, Q_0^1)) \right] = \]

\[ \int dx_0 \int dQ_0^1 \left| \psi(x_0) \right|^2 \left| \phi_1(x_0) \right|^2 \chi_{U_1} \left[ X_{t_1}(x_0, Q_0^1) \right] \chi_{U_2} \left[ x_{t_2}^{sys}(f_{t_1}(x_0, Q_0^1)) \right] = \]

\[ \int dx_0 \int dQ_0^1 \left| \psi(x_0) \right|^2 \left| \phi_1(x_0) \right|^2 \chi_{U_1 \cup (x^{sys} - 1U_2)} \left[ X_{t_1}(x_0, Q_0^1) \right] = \]

\[ \int dx_0 \int dQ_0^1 \left| \psi(x_0) \right|^2 \left| \phi_1(x_0) \right|^2 \chi_{U_1 \cup (x^{sys} - 1U_2)} \left[ f_{t_1}(x_0, Q_0^1) \right] = \]

\[ \int dx_0 \int dQ_0^1 \left| \psi(x_0) \right|^2 \chi_{U_1 \cup (x^{sys} - 1U_2)} \left[ f_{t_1}^{sys}(x_0) \right] = \]

\[ \int dx_0 \int dQ_0^1 \left| \psi(x_0) \right|^2 \chi_{U_1 \cup (x^{sys} - 1U_2)} \left[ f_{t_1}^{sys}(x_0) \right] . \]

In the above derivation we employed Eq. (4.8) in going from the second to the third line; in going from the third to the fourth line we used Eq. (4.7) and denoted by \( (x^{sys})^{-1} \) the inverse of the deterministic law of motion that takes \( x_{t_1} \) to \( x_{t_2} \) in absence of apparatus; in going from the fourth to the fifth line we used the fact that \( \chi_{U_1 \cap U_2} = \chi_{U_1 \cup U_2} \) if \( U_1 \cup U_2 = \emptyset \). In going from the fifth to the sixth line we used Eq. (4.7). Note that \( f_{t_1}^{sys}(x_0) \) stands for \( f(x_{t_1}^{sys}(x_0)) \).

We proved therefore that the two-time probabilities coincide with those obtained for a stochastic process constructed solely from the degrees of freedom of the measured system. Hence a hidden variables theory that satisfies (4.9) and reproduces the single-time predictions of standard quantum theory exhibits no contextuality in sequential measurements. Obviously \( p^2 \) satisfies the compatibility condition (3.34) and thus differs from the corresponding predictions of standard quantum theory.

The result above does not apply to Bohmian mechanics, because the latter does not satisfy the locality condition (4.9). As the wave function evolves, it becomes entangled in the variables \( x \) and \( Q_1 \) after the first measurement—see the discussions in [44]. As a result of Eq. (4.1), the equation of motion for \( x \) after the first measurement will explicitly involve \( Q_1 \) and hence the Eq. (4.9) will not be satisfied. Due to entanglement the measured system continues to be affected by the degrees of freedom of the first measurement device even if it is far away from it. We see therefore that what appears as strong contextuality of the empirical probabilities in standard quantum theory, in Bohmian mechanics it arises as a consequence of the role of the wave function as a carrier of non-
locality in entangled systems.

In the Appendix B we provide a generalisation of this result for unsharp measurements, and also for hidden variable theories that can be modeled by a Markov process. It is important to remark that in deriving these results we need make no assumptions about the explicit form of the POVM for sequential measurements: any POVM that satisfies the assumption of Proposition 2 in Section 3.3.2 is adequate for this purpose.

4.3 Other hidden variable models

Is it possible to write hidden variable theories that reproduce the predictions of quantum theory for sequential measurements, without violating some form of the locality assumption? This is the same question that may be asked about hidden variable theories that violate the Bell inequalities. If such theories are to account for the single-time predictions of quantum theory one needs to introduce a probability density for these variables (either fundamental or emergent) and the usual calculus of probabilities almost guarantees that sequential measurements will be described by a stochastic process.

The only conceivable hidden variable theories compatible with standard quantum theory in sequential measurements would be ones that introduce additional variables, other than the ones necessary to obtain agreement with the predictions of quantum theory. They would correspond to degrees of freedom fundamentally different from those of classical mechanics (e.g. ’t Hooft’s deterministic quantum theory [45]). One may then assume that these variables are highly uncontrollable (or exhibit a kind of ”coherence” within the elements of an ensemble) so that their statistical effect cannot be modeled by any probability density. Hence it would not be possible to write a stochastic process for the multi-time probabilities of the theory, and consequently the arguments of section 4.2 would not follow. At the moment this is just a conjecture, for no such model has been explicitly constructed. Indeed, how could one effect the statistical descriptions of systems that are not described by probabilities? However, the existence of this possibility suggests that one might avoid both contextuality and non-locality by relaxing the condition that the full system is described by a probability theory that satisfies the Kolmogorov axioms. We shall consider this issue in the next section.

5 Beyond probabilities

5.1 Motivation

In single-time measurements probabilities are defined naturally in terms of the projective geometry of the Hilbert space, through the spectral theorem. This is not the case for sequential quantum measurements. A choice of basis is necessary
to take into account the effect of the intermediate measurements. This results in a probability assignment that is not natural with respect to the Hilbert space geometry (i.e. it does not preserve quantum logic). Contextuality of probabilities follows. If one attempts to explain it away by hidden variable models, one needs to introduce non-local features in the interaction of the measured system with the apparatus.

In standard quantum theory contextuality is generic, as witnessed by the Kochen-Specker theorem. In sequential measurements however the dependence on probabilities on even minor details of the measurement scheme appears as rather too extreme. It is a natural question then whether one can introduce an interpretative scheme that avoids it, without assuming on the same time non-locality at the fundamental level. The only way to do this, would be if the multi-time probabilities respected quantum logic, or in other words if they could be expressed in a way that respects the projective geometry of the Hilbert space.

The consistent histories approach preserves quantum logic not at the level of measurement outcomes but at the level of individual quantum systems. The non-additivity of the measure (2.15) is sidestepped by assuming that probabilities can only be defined for specific sets of histories (consistent sets), in which (2.15) is reduced to an additive probability measure. Still, consistent histories does not avoid contextuality, whenever one attempts to make logical inferences based on the probabilities obtained by different consistent sets. This is natural from a mathematical point of view, because any probability assignment depends not only on the projectors representing the relevant properties of the system, but also on the consistent set.

The only conceivable way to obtain uncontextual predictions for quantum theory would be if the non-additive measure (2.15) could be employed for any sampling of measurement outcomes described by the corresponding projectors. Indeed, a key assumption the derivation of Bell’s and Kochen-Specker theorems is that a probability distribution satisfying the Kolmogorov axioms can be employed to model probabilities in a statistical ensemble. Hence a quantum theory based on a non-additive measure for multi-time measurements may potentially avoid the consequences of those theorem–see the discussion in references [4, 5]. However, empirical probabilities (that refer to the same measurement set-up) are always additive as they correspond to relative frequencies. The only way to relax the Kolmogorov probability conditions is to assume that frequencies do not always define probabilities, i.e. that they generically do not converge. The non-additivity of the probability measure (2.15) would then provide an estimate of this lack of convergence.

We shall explore this rather unconventional alternative in this section. It may seem rather contrary to the standard use of probabilities in quantum theory, but we believe it is worth studying, because it is the only conceivable alternative to the strong contextuality of standard quantum theory and non-locality. In any case its predictions are in principle distinguishable from those of standard
quantum theory.

Before proceeding in the further examination of this hypothesis of non-converging frequencies, we shall first provide a different motivation for its introduction. We shall argue that it may be a natural description of the statistical data of sequential measurements, solely from an operational point of view.

5.2 Sequential measurements: stability of the sample space

While classical probability has been remarkably successful in modeling various physical systems, its applicability to a specific situation is not \textit{a priori} guaranteed. One needs to provide \textit{physical} arguments why probability theory can model the outcomes of a specific experiment. These arguments involve an explanation of the choice of the variables that define the sample space and a justification why the different runs of the experiment define a proper statistical ensemble.

The relation of probabilities to event frequencies suggests that the experiment can be repeated a large number of times with the same preparation, or at least in such a way that variations in the preparation procedure affect little the results of the experiment. If this condition cannot be satisfied, then we cannot talk about a statistical ensemble and have no reason to expect that the measured frequencies would in any way allow us to determine meaningful empirical probabilities. Small variations in the preparation and execution of the experiment are not so much a problem: there is always a sampling error and intrinsic uncertainty in the determination of any experiment and we shall see in the next section how this can be taken into account by the consideration of unsharp measurements. If, however, the uncertainties in an experiment are too large, there is little hope of extracting meaningful probabilistic information from it. In other words, the use of probability theory in modeling a physical system requires a condition of \textit{stability of the sample space}, i.e. the assumption that the relation of the mathematical space of physical alternatives to the experimental outcomes remains the same in all elements of the statistical ensemble.

In classical physics at least, the above condition is a necessary requirement for any meaningful experiment—if it is not satisfied then one usually asserts that the corresponding experiment is ill-designed.

A key feature of the probabilities for sequential measurements we derived in section 3 is the strong dependence on the parameter $\delta$ that quantifies the fuzziness of single-time measurements. In classical probability the fuzziness includes contributions of very different physical origins: sampling and systematic error, specific features of the measurement device and the effect of uncontrollable parameters, whose effect cannot be reproduced identically in all measurements of the statistical ensemble. It is usually unnecessary to distinguish between the different contributions: $\delta$ may be taken as an \textit{upper limit} of all possible sources of error, as it does not affect the probabilities of sufficiently coarse samplings. In sequential measurements this is no longer the case. If $\delta$ is considered as a measure of the uncontrollable parameters in the system, the sensitivity of probabilities
on its value, implies that the probability density relevant to each different run of the experiment will be substantially different from each other. It is difficult to see, how a statistical ensemble of reproducible experiments is meaningful, if their outcomes depend so strongly on the values of uncontrollable parameters. This suggests strongly that the sample space for sequential measurements is not stable. One therefore may question whether empirical probabilities can be constructed in that case. It is quite likely, on operational grounds alone, that the frequencies obtained do not exhibit the needed convergence properties to define genuine probabilities.

5.3 Non-convergent frequencies

5.3.1 The case of classical probability

While the relation of probabilities to event frequencies is the basic principle in any statistical manipulation of data, its application is not straightforward. All statistical samples are obtained from a finite number of experimental runs, while the probabilities are defined from event frequencies in the limit that the number of runs goes to infinity. This has been traditionally a very strong argument against the definition of probabilities through frequencies. However, for practical purposes it suffices that we consider a sufficiently large ensemble so that the frequencies seem to stabilize. The central limit theorem guarantees that if the description of a system by classical probability theory is valid, then the error in the determination of probabilities after \( n \) runs will fall like \( n^{-1/2} \) for sufficiently large \( n \).

More relevant to the present discussion is the behavior of relative frequencies in unsharp measurements, namely when there is an error of \( \delta \) (sampling error or effect of uncontrollable parameters) in the specification of probabilities. In that case the sequence \( \nu_n(U) \) of event frequencies cannot be expected to stabilize to a probability even after a large number of runs, if the coarse-graining scale \( L \) of \( U \) is of the order of magnitude of \( \delta \): sampling is simply unreliable at this scale. There will be a region of convergence: no matter how many experimental runs we take into consideration the sequence will take values in a region of finite size.

We may define a quantitative measure for the failure of a sequence to converge to a specific value. If a sequence \( \nu_n \) does not converge then for \( n, m > N \), where \( N \) may be a large integer, we cannot find a number \( \epsilon \), such that \( \nu_n - \nu_m < \epsilon \). This suggests defining the degree of non-convergence of \( \nu_n \) as the limit

\[
\epsilon[\nu_n] = \lim_{N \to \infty} \sup_{n, m > N} |\nu_n - \nu_m|.
\]  

If \( \nu_n \) is a sequence of relative frequencies it is easy to verify that \( \epsilon[\nu_n] \leq 1 \) and that for a converging sequence \( \epsilon[\nu_n] = 0 \). Since in practice a sequence never

\[\text{If } \delta \text{ is an upper limit for the effect of uncontrollable parameters, different sub-ensembles may be characterized by different values of } \delta \text{ and hence different probability assignments.}\]
converges, we need to establish a rather more heuristic criterion: we say that \( \nu_n \) converges to a probability \( p \), if the parameter \( \epsilon \) is much smaller than the value of \( p \). In that case it defines the size of error (or ambiguity) in the determination of \( p \).

We next examine how the ratio of convergence for a sequence of events is related to the fuzziness of a measurement scheme. Let us denote by \( \bar{\nu}_n \) sequence of relative frequencies constructed from the experimental data (hence being inaccurate due to sampling errors), and by \( \nu_n \) the ideal frequency that converges to some probability \( p \), we see that

\[
|\bar{\nu}_n - p| \leq |\bar{\nu}_n - \nu_n| + |\nu_n - p|.
\] (5.2)

The second term falls with \( n^{-1/2} \) for large \( n \), since it is assumed to converge ideally. The first term in the right-hand-side converges for large \( n \) to

\[
\int dx \rho(x)|\chi_{\delta U}(x) - \chi_U(x)|,
\] (5.3)

for some smeared characteristic function for \( U \) that takes into account the effect of sampling errors. We essentially assume that \( \bar{\nu}_n = \frac{1}{n} \sum_{i=1}^{n} \chi_{\delta U}(x_n) \), hence the ambiguity in the sampling of \( x_n \) is transferred in the smeared characteristic function. Hence, we have

\[
|\bar{\nu}_n - p| < c \frac{\delta}{L} \quad n \to \infty,
\] (5.4)

or for probability distributions with spread much larger than \( \delta \) we have the more stringent estimation (see Appendix A)

\[
|\bar{\nu}_n - p| < c \frac{\delta}{L^p}
\] (5.5)

5.4 Quantitative estimation

In this section we explore the theoretical possibility that the lack of a non-contextual probability measure for multi-time histories is indicative of a failure of the event frequencies to stabilize after a large number of runs. To elaborate on this proposal we first need to guarantee that this assumption is compatible with the single-time predictions of quantum theory. This follows trivially from the fact that (2.15) is additive for single-time measurements. The same holds for multi-time measurements, for which the projectors \( \hat{P}_{\hat{U}_i} \) commute with the Hamiltonian.

We would expect the lack of convergence appear in all \( n \)-time measurements, for which (2.15) is genuinely non-additive. The lack of additivity is quantified by the object (2.17), namely the decoherence functional in the consistent histories approach. The decoherence functional should be a measure of the degree
of non-convergence for histories. There is another argument that lends plausibility to this expectation. The absolute values of the off-diagonal elements of the decoherence functional often become very small, when the selected histories are sufficiently coarse-grained. It then becomes a good approximation to assign probabilities to such histories. This situation can be compared with the behavior of relative frequencies under coarse-graining. If our sampling is of the order of the measurement error $\delta$, frequencies do not stabilize to probabilities. If we coarse-grain sufficiently however, so that the sample set is of size $L \gg \delta$, the relative error falls like $\frac{\delta}{L}$ and reasonable probabilities can be approximately defined.

The analogy above is only mathematical, as the postulated lack of convergence in quantum theory cannot be explained away as measurement error. It strongly argues however that the decoherence functional should encode the information of the frequency's non-convergence. In effect, one may assign a probability for a specific sampling $U_1, U_2, \ldots, U_n$, if the decoherence functional between the corresponding history $\alpha$ and its negation $\bar{\alpha}$ (corresponding to the subset $\Omega^n - U_1 \times U_2 \times \ldots \times U_n$) is much smaller in magnitude than the probability associated to $U_1, U_2, \ldots, U_n$, or in other words if

$$2 \Re d(\alpha, \bar{\alpha}) \ll d(\alpha, \alpha).$$

(5.6)

This suggests that the proper measure for the relative rate of convergence $\epsilon[\nu_n]/p$ should be identified with the ratio $\frac{\Re d(\alpha, \bar{\alpha})}{d(\alpha, \alpha)}$.

We need to comment at this point on the difference of the present proposal from the consistent histories approach. The first difference lies in the context. The consistent histories approach describes individual systems, without making special reference to measurement outcomes. Here we are only interested in empirical probabilities, as determined by measurements. The second difference is more important: the consistent histories approach places no interpretation on the decoherence functional or the objects (2.15), unless the former vanishes, in which case (2.15) define a genuine probability measure. Here, in our search to preserve the quantum logic at the level of measurements, we need to find an interpretation of the mathematical object (2.15) in terms of observable quantities, namely the statistical behavior of the sequence of relative frequencies $\nu_n$. Hence even if there are many structural similarities between the present hypothesis and consistent histories both the context and the physical interpretation of the mathematical objects is conceptually distinct.

### 5.5 Experimental distinction

We mentioned two motivations for the hypothesis of non-converging frequencies (a third more tentative one arising from the study of frequency operators [46, 47] can be found in Appendix C). The main one was the preservation of the quantum logic structure of sequential measurements. The decoherence functional for a
history depends on the sample sets only through the projectors. Hence the statistical behavior of the relative frequencies that it incorporates would be the same (modulo sampling and systematic errors) in all different experiments that measure multi-time probabilities. It would not depend, in particular, on whether the experimental set-up is that of a sequential YES-NO experiment or of a sequential measurement of position.

If this is true then it is very easy to distinguish the predictions of this hypothesis from that of standard quantum theory. If the quantum logic structure is preserved, probabilities for multi-time samplings are not definable, but they are for single-time measurements. In particular, the single-time marginals in a multi-time measurement should always coincide with those of single-time quantum theory. In a two-time measurement of an observable \( \hat{A} = \sum_i a_i \hat{P}_i \), the frequencies \( \nu_n(U_1, 0; U_2, t) \) do not converge for generic sample sets \( U_1 \) and \( U_2 \), but the coarse-grained frequencies \( \nu_n(\Omega, 0; U_2, t) \) should correspond to the single-time probabilities \( \sum_{i \in U_2} Tr(\hat{\rho} \hat{P}_i) \), where \( \hat{Q}_i = e^{i\hat{H}_t} \hat{P}_i e^{-i\hat{H}_t} \). On the other hand standard quantum theory predicts for the same probabilities that

\[
p(\Omega, 0; U_2, t) = \sum_{j \in U_2} \sum_i Tr(\hat{\rho} \hat{P}_j \hat{P}_i) . \tag{5.7}
\]

The results are clearly different. For successive Stern-Gerlach measurements, the first in the \( x \) and the second in the \( z \) direction of spin, the former hypothesis yields a probability density \( p_i = Tr(\hat{\rho} \hat{P}_z^i) \) with \( \hat{P}_z^i \) the spectral projectors of spin, while standard quantum theory yields a constant probability density \( p_i = \frac{1}{2} \). Since the distinction arises at the level of the marginals, it is not necessary to perform experiments with individual, distinguishable runs, but it suffices to employ particle beams.

Note that the local hidden variable theories of the type considered in section 4.3 also satisfy equation (5.7), since their measurement outcomes are described effectively by a stochastic process by virtue of equations (4.10, B.8).

"Welcher-Weg experiments." One may contend that the behavior above can be excluded on the basis of the so-called "welcher Weg experiments", in which a detector placed immediately behind the holes of a two slit experiments destroy the interference pattern. One may consider for example the treatment of [50], which involves the interference of two neutron beams. One places a micromaser cavity in the course of each beam. The photons in the cavity interact with the neutrons’ spin degrees of freedom. If the field in the cavities is prepared in a number state, the interaction reveals unambiguously that the neutron passed through one cavity or the other, and hence provides information about the neutron’s position. As a result the interference pattern observed in a screen behind the detectors is destroyed. Clearly, the intermediate measurement leads to different results for the probability distribution on the screen and a violation of (5.7).
There is however a flaw in this argument. Within standard quantum theory, any measurement involves the separation between a quantum system and a classical apparatus. This is not an easy distinction to make, but it is necessary in order to specify the level at which the reduction procedure is implemented. In the experiment discussed above the electromagnetic field in the cavity is described by quantum mechanics. In the assumed splitting between quantum and classical, it falls within the quantum domain. Hence, the quantum system in consideration is not the neutron, but the combined system of neutron and electromagnetic field, which interact non-trivially through the spin degrees of freedom. However, in absence of the micromaser the quantum system is only the neutron. It is therefore not possible to verify a violation of (5. 7) by comparing the probabilities in these different experiments. Indeed, if the total system of electromagnetic field and neutron is treated as quantum mechanical, the loss of interference is expected whether or not the photon number has been measured in the microcavity. The distribution of particle positions is obtained by the reduced density matrix of the neutron interacting with the electromagnetic field. This is naturally expected to exhibit a loss of coherence, arising solely from unitary evolution of the total system. To test equation (5. 7) one would have to compare the probability distribution of neutron positions between an experiment that includes a measurement of the photons in the cavity and one that does not.

In any attempt to verify the validity of (5. 7) one has to compare experimental set-ups for which the split between quantum and classical occurs at the same level. This is the case of the two-time position measurement sketched in Fig.3. In that case one also has to take into account all possible sources of error. In the original Bohr-Einstein debate that led to the formulation of the complementarity principle, the demonstration that "which-way" information destroys the interference pattern came essentially from classical arguments about the inherent limitation in the precision of the first measurement [51]. Quantum theory was only introduced, in order to place an upper limit in the measuring accuracy. Bohr’ argument is therefore very different in character than that of [50]. It states essentially that the uncertainty at the level of the classical device affects the quantum phases randomly and thus leads to a destruction of the interference pattern.

The same argument can be invoked in the present context in relation to the assumption that measurements take place at a sharply specified moment of time. This is not the case in a realistic experiment. One may of course incorporate this uncertainty in the error $\delta$ of an unsharp measurement. However, in multi-time measurements there appears an additional source of randomness: the presence of the first measuring device makes it more difficult to specify the moment of time, at which the second measurement takes place.

To see this, we may consider for example the thought-experiment of Fig. 3. The detection of a particle can be assumed to take place at a specific moment of time (the same for all runs of the experiment) if the particle’s momentum is sharply defined: $p_z$ is essentially treated as a classical variable. In a two-time
measurement however the value of $p_z$ changes randomly after the interaction with the first measurement device, say by an amount of $\delta p_z = d^{-1}$, for some parameter $d$ with the units of length. This implies a fuzziness in the time that the particle arrives in the second detector of the order of $\frac{1}{\delta p_z d}$, hence an additional uncertainty in the specification of the particle’s position of an order of $\frac{1}{m \delta}$. The uncertainty $d$ can be expected to be of the order of $\delta$, hence the uncertainty is of the order of magnitude of $\frac{t}{m \delta}$.

To see that the randomness induced by the uncertainty in the specification of the measurement time is by itself responsible for the loss of the interference pattern, we consider the marginal probability density induced by the POVM (3.39). For an initial state corresponding to the set-up of a two-slit experiment

$$\psi_0(x) = \frac{1}{\sqrt{2(2\pi \sigma^2)^{1/4}}} \left[ e^{-\frac{(x-L/2)^2}{2\sigma^2}} + e^{-\frac{(x+L/2)^2}{2\sigma^2}} \right],$$

(5.8)

where $\sigma$ is the width of the slit, $L$ the distance between the slits and the mean momentum in the $x$-direction is for simplicity set equal to zero $^{12}$ we obtain

$$p(\Omega, 0; U, t) = \int_U dx \sqrt{\frac{m}{\pi t}} \left[ e^{-\frac{(x-L/2)^2}{4(\gamma + \beta)}} + e^{-\frac{(x+L/2)^2}{4(\gamma + \beta)}} + 2e^{-\frac{1}{4\sigma^2}(1-\frac{m^2}{\sigma^2(\gamma + \beta)})} e^{-\frac{m^2}{4\sigma^2(\gamma + \beta)} x^2} \cos \left( \frac{mL}{t\sigma^2(\beta + \gamma)} x \right) \right],$$

(5.9)

where $\beta = \frac{1}{2\beta^2} + \frac{2m^2\delta^2}{t^2}$ and $\gamma = \frac{1}{\sigma^2} + \frac{m^2\sigma^2}{t^2}$. Hence, even the probability constructed from the consideration of a two-time measurement in standard quantum mechanics exhibits terms that describe a distinguishable interference pattern. The interference pattern is washed out only when an additional consideration of the error due to the time uncertainty is taken into account. The period of the interference pattern is $c\frac{L}{m \delta}$, where $c$ is a constant of order unity. The fuzziness due to the uncertainty relation is of the order of $\frac{1}{m \delta}$. For the two-slit experiment to make sense the distance between the slits has to be much larger than the resolution of the measurement device—hence $L >> \delta$. It follows that the interference pattern is hidden beneath the effect of the time uncertainty.

It is important to stress the procedure we followed to obtain the result above. Following Bohr’s argumentation, the proof that the interference pattern disappears does not arise from the operational rules of quantum theory about sequential measurements, but by physical considerations that apply to our inherent inability to establish with arbitrary precision the time a measurement takes place. This effect cannot be obtained through formal manipulations in standard quantum theory: at an operational level the formalism allows one to talk only about measurements at sharp moment of time $^{13}$, while if we attempt to treat

$^{12}$Note the $x$ direction is transverse to the particle’s direction of motion.

$^{13}$This problem is related to issue of constructing time-of-arrival probabilities in quantum theory, and is explored further in [52].
the measuring apparatus as fully quantum mechanical we come face to face with
the measurement problem (is the reduction of the wave packet instantaneous, if
a physical process at all?).

The argument that proves that interferences are washed out in a two-time
position measurement works the same way in standard quantum theory and
within the non-converging frequency hypothesis. In the latter case one also
obtains an interference pattern with period of the order \( \frac{\hbar}{mL} \), which is hidden
beneath the effects of the randomness due to time uncertainty. Even though
the right-hand-side is different from (5. 9) the form of the terms is the same
and the difference only lies in the exact value of the coefficients. Since this
difference is drown from the effects of the time uncertainty, we conclude that
it is very difficult (if not impossible) to distinguish the predictions of quantum
theory from those of the non-converging frequency hypothesis (or of local hidden
variable theories for that matter) by means of equation (5. 7).

Hence the only way to distinguish between those theories would be by di-
rectly measuring the frequencies \( \nu_n(U_1, t_1; U_2, t_2) \) and trying to establish whether
they converge or not. In the Appendix B we prove that this is in principle feas-
able, i.e. the suggested failure of the frequencies to converge is much stronger
than any sources of error (such as the time uncertainty considered earlier), and
for this reason it is in principle detectable.

6 Conclusions

We conclude with a brief summary of the paper’s results.

We studied the issue of constructing probabilities for sequential measure-
ments. In Section 3 we demonstrated that these probabilities are highly con-
textual, namely they depend very strong on seemingly trivial details of the
apparatus (the parameter that determine its resolution). We noted that this is
a case of contextuality that does not involve counterfactual reasoning: it may
be determined in a direct measurement set-up.

A key step for our analysis is the proof of a general theorem that there is
no way to reproduce the probability distribution for the results of quantum
theory from a stochastic process for the measured system’s degrees of freedom.
We elaborated on this point in section 4, where we demonstrated that hidden
variable theories can reproduce the predictions of standard quantum theory only
if they include non-local interactions.

Finally, in section 5 we explored a rather unconventional alternative that
could allow the preservation of quantum logic in sequential measurements: that
probabilities are not defined, because the corresponding frequencies do not con-
verge. We demonstrated that the predictions of this proposal can be unambigu-
ously distinguished from those of quantum theory.
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A Unsharp measurements and smeared characteristic functions

In any measurement there are systematic errors, uncertainties in the specification of the initial state or the preparation of the apparatus, fuzziness in the sampling of the results, dependence on uncontrollable properties of the measurement device etc. For this purpose it is necessary to consider the description of unsharp or fuzzy measurements.

In unsharp measurements there will be outcomes for which we will not be able to state unambiguously that “the system was found in the subset $U$ of the sample space $\Omega$” or its negation. Such assertions can be made only with a degree of confidence characterized by the relative size $\delta$ of the measuring uncertainty. This can be implemented by substituting the characteristic functions that represent the propositions about the measurement outcomes with smeared characteristic functions $\chi_\delta(U)$, which differ from true characteristic functions on the scale of $\delta$. Given the fact that a characteristic function for $\Omega = \mathbb{R}$ is written as

$$\chi_U(x) = \int_U dx' \delta(x - x'), \quad (A. 1)$$

a smeared characteristic function may be written as

$$\chi_\delta(U)(x) = \int_U dx' f_\delta(x - x'), \quad (A. 2)$$

where $f_\delta$ is an one-parameter family of smooth functions converging weakly to the delta function as $\delta \to 0$. Any real-valued function that falls to zero rapidly outside $U$, takes values close to unity well inside $U$ and interpolates continuously between 1 and 0 in a region of size $\delta$ around the boundary of $U$ is an adequate smeared characteristic function. We may use for example a Gaussian family

$$f_\delta(x) = \frac{1}{2\pi\delta^2} e^{-x^2/2\delta^2}. \quad (A. 3)$$

If the size of the sample set $U$ is $L$, then the difference of the smeared characteristic function from a true characteristic function is of the order of $O(\delta)$. This difference may be quantified by a norm in the space of functions—usually the $L^1$ norm—of the difference $\chi_U - \chi_\delta(U)$. Indeed, it is easy to estimate that for the Gaussian smearing function (A. 3)
where $c$ is a positive number of order unity. This equation can also be employed as a definition of a smeared characteristic function. This implies that the relative difference between the smeared and the genuine characteristic function is

$$
\epsilon = \frac{\int dx |\chi_U - \chi_U^\delta|}{\int_U dx} < c \frac{\delta}{L}.
$$

(A. 5)

It is not possible to obtain an equation analogous to (A. 4) for the difference $\chi_U - \chi_U^\delta$ weighted by the probability. However, if we define the margin $M$ of the sample set $U$ as the region of $\Omega$ in which $|\chi_U - \chi_U^\delta|$ is appreciably larger than zero (say larger than a fixed small number $r << 1$), we may estimate that

$$
\int \rho(x)|\chi_U - \chi_U^\delta| < \int_M dx \rho(x) + O(r),
$$

(A. 6)

hence for suitable choice of $r$ we can always find a constant $c$ of the order of unity such that

$$
\int \rho(x)|\chi_U - \chi_U^\delta| < c \int_M dx \rho(x).
$$

(A. 7)

For general $\rho$ we cannot improve the above inequality. The physically interesting case is one for which $\rho$ varies at a scale much larger than the size $\delta$ of the margin, otherwise any probabilistic information would be completely lost beneath the sampling error. In that case one may estimate that

$$
\int \rho(x)|\chi_U - \chi_U^\delta| < c' \frac{\delta}{L} \rho(U)
$$

(A. 8)

For the most general case the following estimation is relevant

$$
\int \rho(x)|\chi_U - \chi_U^\delta| < c' \frac{\delta}{R} \rho(U),
$$

(A. 9)

where $R$ is the size of the area of support of $\rho(x)\chi U(x)$.

**B Generalization of the results of section 4.2**

**Unsharp measurements.** The result (4. 10) can be reproduced even for unsharp measurements of a continuous pointer function $X$. In that case one
substitutes equations (4.7) and (4.8) with

\[ X[Q_t(x_0, Q_0)] = f(x_t(x_0, Q_0)) + O(\delta) \]  

\[ p(U, t) = \int dx_0, dQ_0 |\psi_0|^2(x_0)|\phi|^2(Q_0) \chi_U[X_t(x_0, Q_0)] = \int dx_0 |\psi_0|^2(x_0) \chi^\delta_U[x^\text{sys}_t(x_0)] = \langle \psi_0|e^{i\hat{H}_0 t}\hat{F}_\delta e^{-i\hat{H}_0 t}|\psi \rangle, \]  

where now \( \hat{F}_\delta \) is a POVM for the variable \( f \) characterized by a parameter \( \delta \), which incorporates the effects of the interaction with the measurement device. The proof follows the same steps as the discrete variable case, the only difference being that we substitute the characteristic functions with smeared ones. Since for a family of characteristic functions \( \chi^\delta \) labeled by \( \delta \)

\[ \chi^\delta_{U_1}\chi^\delta_{U_2} = \chi^\delta(U_1 \cap U_2) + O(\delta), \]

it is easy to conclude that if the locality condition (4.9) the probabilities \( p^\gamma \) coincide up to an error of order \( O(\delta) \) with those obtained by a stochastic process constructed from the degrees of freedom of the system by itself. Hence for samplings of size much larger than \( \delta \) the probabilities do not depend on properties of the measurement device, something that contrasts the results of standard quantum theory like Eq. (3.32).

**Markov process.** The same arguments may be applied for non-deterministic hidden variable theories that reproduce the predictions of quantum theory through a Markov process. We denote by \( g(x, Q, t|x', Q', t') \) the propagator corresponding to the interacting dynamics between system and apparatus, by \( h(Q, t|Q', t') \) the one corresponding to the self-dynamics of the apparatus and by \( g^\text{sys}(x, t|x', t') \) the one corresponding to the self-dynamics of the system *in absence of apparatus.*

Assuming an initially factorized state the conditions that the stochastic process reproduces the operational predictions for single-time quantum theory for a discrete pointer \( X \) are

\[ X(Q_t) = f(x_t), \]  

\[ p(U, t) = \int dx_0 dQ_0 dx_t dQ_t |\psi_0|^2(x_0)|\phi|^2(Q_0) \times g(x_0, Q_0, 0|x_t, Q_t, t) \chi_U[X(Q_t)] = \int dx_0 dx_t |\psi_0|^2(x_0)g^\text{sys}(x_0, 0|x_t, t)\chi_U(f(x_t)) \]

where \( t \) denotes the time that the measurement has been completed.

In a two-time measurement the total propagator for the degrees of freedom of the measured system and the two measuring devices factorized as \( g(x_0, Q_0, 0; x_t^1, Q_t^1, t) \)
\[ x g(Q^2, 0|Q_1^2, t) \] for all times prior to the first measurement, as the two devices are assumed non-interacting. The key assumption, analogue to the locality postulate (4.9) in deterministic systems is that the propagator for the total system factorized after the first measurement as

\[ g(x_t, Q_1^2, t|x_{t'}, Q_{1'}^2, t') h(Q_{i}, t|Q_{i'}, t'), \quad t, t' > t_1, \quad (B.6) \]

which states that the particle’s stochastic evolution is not affected by the degrees of freedom of the first apparatus after the interaction has been completed.

With the assumptions above, it is easy to show following steps analogous to those of (4.10) that the two-time probability

\[ p(U_1, t_1; U_2, t_2) = \int dx_0 dQ_0^2 dx_{t_1} dQ_{t_1}^1 dQ_{t_2} dQ_{t_2}^1 \times |\psi_0|^2(x_0) |\phi_1|^2(Q_0^1) |\phi_2|^2(Q_2^1) g(x_0, Q_0^1, 0|x_{t_1}, Q_{t_1}^1, t_1) h(Q_0^2, 0|Q_{t_1}^1, t_1) \times \chi_{U_1}[X_{t_1}] g(x_{t_1}, Q_{t_1}^2, t_1|x_{t_2}, Q_{t_2}^2, t_2) h(Q_{t_1}, t_1|Q_{t_2}^1, t_2) \chi_{U_2}[X_{t_2}] \quad (B.7) \]

equals

\[ p(U_1, t_1; U_2, t_2) = \int dx_0 dx_1 dx_{t_2} |\psi_0|^2(x_0) g^{\psi_0}(x_0, 0|x_{t_1}, t_1) \times \chi_{U_1}(f_{t_1}) g^{\psi_0}(x_{t_1}, t_1|x_{t_2}, t_2) \chi_{U_2}(f_{t_2}). \quad (B.8) \]

Hence probabilities are again described by a stochastic processes for the measured system’s degrees of freedom, in conflict with the predictions of quantum theory. To reproduce the predictions of quantum theory with a Markov process, one would need to assume a violation of Eq. (4.9).

## C Frequency operators

In some interpretations of quantum theory, it is often asserted that the Born’s rule from probabilities can be obtained from a weaker postulate, namely that if an observable \( A \) is measured on a system in one of its eigenstates, the outcome is the corresponding eigenstate. The idea is to construct a Hilbert space for the statistical ensemble \( \mathcal{H}_{\text{ens}} \) as (ideally an infinite) tensor product \( \otimes_n \mathcal{H}_n \) of the Hilbert space \( \mathcal{H} \) of the single system [46, 47] (see also [48] and references therein). Assuming a projection operator \( \hat{P} \) corresponding to a state \( |i\rangle \) of \( \mathcal{H} \), we may construct a PVM corresponding to the different values of the frequencies \( f \) for the event corresponding to \( \hat{P} \) in the statistical ensemble. For a finite ensemble consisting of \( N \) copies this PVM reads

\[ \hat{\Pi}_f(f = n/N) = \sum_{k_1+k_2+\ldots+k_n=n} \hat{P}_{k_1} \otimes \ldots \otimes \hat{P}_{k_n}, \quad (C.1) \]

where \( k_i = 1 \) corresponds to \( \hat{P}_{k_i} = \hat{P} \) and \( k_i = 1 \) corresponds to \( \hat{P}_{k_i} = \hat{1} - \hat{P} \).
One then may (attempt to) prove that the vectors ⊗\_n |\_ψ\_n⟩ are eigenstates of the frequency operators $\hat{F}_p = \sum_{n=0}^{N} \frac{n}{N} \hat{P}_\psi(f = n/N)$ at the limit $N \to \infty$ with eigenvalues coinciding to the standard probabilities $|\langle i | \psi \rangle|^2$.

The underlying concept in this approach is that the Born rule may be derived solely from the projective geometry of the Hilbert space by making reference to the ensemble as an individual quantum system. This approach faces some problems in its mathematical implementation [49], but it is interesting to see whether it can be applied to sequential measurements.

The key obstacle is that one cannot assign projection operators corresponding to the outcomes of sequential measurements. Even for ideal measurements the best we can do is to construct a POVM like (3. 17), in which two successive readings $i$ and $j$ of the variable $\hat{x}$ will correspond to the positive operators $\hat{K}_{ij} = \hat{P}_i e^{i\hat{H}t} \hat{P}_j e^{-i\hat{H}t} \hat{P}_i$. The analogue of the PVM (C. 1) would therefore be a POVM $\hat{\Pi}_K$, in which $\hat{K}_{ij}$ would be inserted in place of the projector $\hat{P}$. It is easy to verify that the failure of the idempotency condition $\hat{K}_{ij}^2 \neq \hat{K}_{ij}$ implies that

$$\hat{\Pi}_K(f_{ij} = n/N)\hat{\Pi}_K(f_{ij} = n'/N) \neq 0 \text{ if } n \neq n', \quad (C. 2)$$

and that this property persists even at the limit $N \to \infty$. These POVMs cannot distinguish between different values of the frequency in the ensemble. It is, therefore, not possible to obtain probabilities solely from the geometry of Hilbert space, because the positive operators $\hat{P}_\psi(f = n/N)$ cannot be associated with specific values of frequency in a way that respects the projective character of Hilbert space geometry.

Alternatively one could define a POVM corresponding to frequencies $f_1 = n_1/N$ for the outcome $i$ of the first measurement and $f_2 = n_2/N$ for the second

$$\Pi(f_1 = n_1/N, 0; f_2 = n_2/N, t) = \hat{\Pi}(f_1 = n_1/N) \otimes_n e^{i\hat{H}t} \hat{\Pi}(f_2 = n_2/N) \otimes_n e^{-i\hat{H}t} \hat{\Pi}(f_1 = n_1/N). \quad (C. 3)$$

This POVM is different from the one obtained from the modification of (C. 1), because different fine-grained alternatives are used in its construction. It is still subject to Eq. (C. 2) and cannot distinguish between different values of frequency.

The results above imply that the programme of defining probabilities through frequencies (without a priori assuming Born’s rule) cannot account for sequential measurements. The only way to salvage it, is to take Eq. (C. 2) at face value and assume that different values of the frequency cannot be distinguished in sequential measurements, implying in effect that multi-time probabilities are ill-defined–or they do not converge.

53
Distinguishability of non-converging frequencies

We consider a two-time measurement of position as described in section 3.2. We assume that a beam of free particles with mass \( m \) is prepared in a state \( \psi_0 \), centered around \( x = 0 \), with zero mean momentum \( p_x \) and a spread \( L \) in the \( x \) direction. We consider for simplicity only two samplings at each moment of time, corresponding to the sets \( U_+ = [0, \infty) \) and \( U_- = (-\infty, 0] \). The corresponding projectors are \( \hat{P}_+ \) and \( \hat{P}_- \). We then consider the candidate probability that the particle is detected in \( U_+ \) at time \( t_1 \) and in \( U_- \) at time \( t_2 \),

\[
p_{++} = \langle \psi_0 | \hat{P}_+ e^{i \hat{H} t} \hat{P}_+ e^{-i \hat{H} t} \hat{P}_+ | \psi_0 \rangle, \tag{D. 1}
\]

while the obstruction to additivity equals

\[
b = 2 \text{Re} \langle \psi_0 | \hat{P}_+ e^{i \hat{H} t} \hat{P}_+ e^{-i \hat{H} t} \hat{P}_- | \psi \rangle. \tag{D. 2}
\]

\[\sqrt{2} L \chi_{[-L,L]}(x), \tag{D. 3}\]

where \( \chi_{[-L,L]} \) is the characteristic function of the set \([ -L, L ]\) – corresponding for instance to a slit of width \( 2L \) placed immediately before the first detector. We then obtain

\[
p_{++} = \frac{1}{\pi} \int_0^1 dz S \left( \frac{z^2}{r} \right)
\]

Figure 4: The particles pass through a slit of width \( 2L \) and are registered in two successive screens.

The details of the wave-function’s shape do not significantly affect the result. For calculational convenience we consider

\[
\psi_0(x) = \frac{1}{\sqrt{2L}} \chi_{[-L,L]}(x), \tag{D. 3}
\]

where \( \chi_{[-L,L]} \) is the characteristic function of the set \([ -L, L ]\) – corresponding for instance to a slit of width \( 2L \) placed immediately before the first detector. We then obtain

\[
p_{++} = \frac{1}{\pi} \int_0^1 dz S \left( \frac{z^2}{r} \right)
\]
Figure 5: A plot of the ratio $b/p_{++}$ versus dimensionless time $r = \frac{t}{mL^2}$ for the system of Fig. 2. This ratio measures the failure of additivity in the natural probability assignment and estimates the relative size of the area of non-convergence for event frequencies.

$$b = \frac{r}{\pi} \int_{0}^{1} \frac{1 - \cos(\frac{z^2}{r})}{z^2},$$

(D. 4)

where $r = \frac{t}{mL^2}$, the dimensionless time-scale; $\text{Si}$ stands for the sin-integral function. In Fig. 6 we plot the ratio $b/p_{++}$ as a function of $r$: it starts from 0 at $r = 0$ (in which case we have a single-time measurement), it increases rapidly, and for $r \sim 1$ reaches the asymptotic value $1/2$. In other words, the assumed non-convergence of probabilities is manifested very strongly even for the highly coarse-grained sample sets $U_+$ and $U_-$. In any statistical sample there exists an extended region of convergence for relative frequencies due to sampling errors or systematic uncertainties, or due to the finite number of experimental runs. These errors can be accounted for by positive operators that approximate projectors within an order of the error. If $d$ is the size of the error, we need to employ operators of the form $\int x U(x)|x\rangle \langle x|$, defined by the smeared characteristic functions $\chi_U$. The related error equals $|\text{Tr}(\hat{P}_U - \hat{P}_U^2)| \sim \frac{\pi}{R} \text{Tr}(\hat{\rho}\hat{P}_U)$, where $R$ is the size of the support of $\rho(x, x)\chi_U(x)$.

For the configuration of Fig. 4, $R \sim L$ and $d$ is the width of the particle’s trace, so the error is at most of the order of $d/L$.

The operators describing the sampling of measurement outcomes have also to incorporate the indeterminacy in measurement time—see the discussion in section V.5. One should therefore employ a time-averaged projector, on an interval of width $\tau$ around the moment of time $t$. This corresponds to the proposition that the measurement took place at any time within the interval $[t - \tau/2, t + \tau/2]$,

$$\hat{\Pi}_U = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} ds e^{iHs} \hat{P}_U e^{-iHs}.$$  

(D. 5)

To leading order in $\tau$, the spread of $\hat{\Pi}_U$ is $\epsilon = \frac{\tau}{mL^2}$. The indeterminacy in time $\tau$ is related to the interaction of the system with the first sheet and should be of the order $\frac{\delta p_z}{p_z} \tau$, where $\delta p_z$ is the momentum transfer in the $z$ direction as the
particle crossed the sheet. Hence we estimate that $\epsilon \sim \delta p_{z} \frac{t}{m v_{z} d}$. The uncertainty relation suggests that $\delta p_{z}$ is of the order of $1/d$, so $\epsilon \sim \frac{t}{m^2 v_{z} d^2}$, where $v_{z}$ is the mean velocity of the particles in the $z$ direction. Since the non-additivity of probabilities is manifested for $r \sim 1$, we may substitute the corresponding value of $t$ to obtain $\epsilon \sim \frac{L}{m d^2 v_{z}}$. Hence the total uncertainty in the measurement is of the order of $c_1 \frac{L}{d} + c_2 \frac{L}{m d v_{z}}$, with $c_1, c_2$ numbers of order unity. For realistic values of $L = 1 \text{ cm}$, $d = 10^{-2} \text{ cm}$, $v_{z} = 10^4 \text{ m/s}$, the error due to time indeterminacy is of the order of $10^{-4}$, much smaller than the ratio $d/L \sim 10^{-2}$ of relative error in position sampling. It follows that the non-convergence of probabilities—if present— is more pronounced than the measurement uncertainties and can be in principle detected.