Quantization of branching coefficients for classical Lie groups

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Abstract

We study natural quantizations of branching coefficients corresponding to the restrictions of the classical Lie groups to their Levi subgroups. We show that they admit a stable limit which can be regarded as a $q$-analogue of a tensor product multiplicity. According to a conjecture by Shimozono, the stable one-dimensional sum for nonexceptional affine crystals are expected to occur as special cases of these $q$-analogues.

1 Introduction

The Kostka coefficients and the Littlewood-Richardson coefficients which have many occurrences in the representation theory of $GL_n$ admit interesting $q$-analogues. Giving $\lambda$ a partition of length at most $n$ and $\mu \in \mathbb{Z}^n$, the $q$-analogue of the Kostka coefficient $K_{\lambda,\mu}$ giving the dimension of the weight space $\mu$ in the irreducible finite dimensional $GL_n$-module $V^{GL_n}(\lambda)$ of highest weight $\lambda$ is the Kostka-Foulkes polynomial $K_{\lambda,\mu}(q)$ (also called Lusztig $q$-analogue of weight multiplicity). Consider $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ a $r$-tuple of partitions of lengths summing $n$ and denote by $\mu$ the $n$-tuple obtained by reading the parts of the $\mu^{(p)}$'s from left to right. There exist in the literature different quantizations of the Littlewood-Richardson coefficient $c_{\lambda^{(1)}, \ldots, \lambda^{(r)}}^{\mu^{(1)}, \ldots, \mu^{(r)}}(\mu^{(1)}) \otimes \cdots \otimes V^{GL_n}(\mu^{(r)})$.

In [9] Lascoux Leclerc and Thibon have introduced such a $q$-analogue by mean of certain generalizations of semi-standard Young tableaux called ribbon tableaux. They have proved in [10] that the polynomials obtained belong to a family of parabolic Kazhdan-Lusztig polynomials introduced by Deodhar which have nonnegative integer coefficients [5].

When $\mu^{(1)}, \ldots, \mu^{(r)}$ are rectangular partitions, it is also possible to define $q$-analogues of the coefficients $c_{\lambda^{(1)}, \ldots, \lambda^{(r)}}^{\mu^{(1)}, \ldots, \mu^{(r)}}$ by considering the one-dimensional sums $X^\emptyset_{\lambda,\mu}(q)$ obtained from affine $A_n^{(1)}$-crystals associated to Kirillov-Reshetikhin $U_q'(\hat{sl}_n)$-modules [5].

Consider $\eta = (\eta_1, \ldots, \eta_r)$ a sequence of positive integers summing $n$ and suppose that $\mu^{(p)}$ has length $\eta_p$ for any $p = 1, \ldots, r$. The Littlewood-Richardson coefficient $c_{\lambda^{(1)}, \ldots, \lambda^{(r)}}^{\mu^{(1)}, \ldots, \mu^{(r)}}$ also coincide with the multiplicity of the tensor product $V^{GL_n}(\mu^{(1)}) \otimes \cdots \otimes V^{GL_n}(\mu^{(r)})$ in the restriction of $V^{GL_n}(\lambda)$ to its Levi subgroup $GL_\eta = GL_{\eta_1} \times \cdots \times GL_{\eta_r}$. This duality permits to express $c_{\lambda^{(1)}, \ldots, \lambda^{(r)}}^{\mu^{(1)}, \ldots, \mu^{(r)}}$ in terms of a Kostant-type partition function. By quantifying this partition function, Shimozono and Weyman [18] have introduced another natural $q$-analogue of $c_{\lambda^{(1)}, \ldots, \lambda^{(r)}}^{\mu^{(1)}, \ldots, \mu^{(r)}}$ that we will denote $K_{\lambda,\mu}^{\emptyset}(q)$ ($I$ being the set of the simple roots of $GL_n$). The polynomials $K_{\lambda,\mu}^{\emptyset}(q)$ are Poincaré polynomials and appear in the Hilbert series of the Euler characteristic of certain graded virtual $G$-modules. By a result of Broer [1], they admit nonnegative coefficients providing that the $\mu^{(p)}$'s are rectangular partitions of decreasing heights. In this case Shimozono has proved in [18] that $K_{\lambda,\mu}^{\emptyset}(q)$ coincide with $X^\emptyset_{\lambda,\mu}(q)$. This result which is based on a combinatorial description of the polynomials $K_{\lambda,\mu}^{\emptyset}(q)$, permits in particular to recover that they have nonnegative coefficients independently of the results of Broer. Under the same hypothesis, it is conjectured that $K_{\lambda,\mu}^{\emptyset}(q)$ also coincide with the
LLT quantization of $\ell_{\mu^{(1)},...\mu^{(r)}}$. When the $\mu^{(p)}$'s are simply row partitions, we have $\mu = (\mu^{(1)},...\mu^{(r)}) \in \mathbb{Z}^n$ and $K^{GLn,I}_{\lambda,\mu}(q)$ is the Kostka-Foulkes polynomial associated to the weights $\lambda$ and $\mu$.

Let $G$ be one of the classical groups $GL_n$, $SO_{2n+1}$, $Sp_{2n}$ or $SO_{2n}$ and $R^+_G$ its set of positive roots. Kostka and Littlewood-Richardson coefficients can be regarded as branching coefficients corresponding to the restriction of $GL_n$ to its principal Levi subgroups. This naturally yields to study the branching coefficients corresponding to the restriction to a subgroup $G_0$ (not necessarily of Levi type) of $G$ and their corresponding $q$-analogues. The branching coefficients which are considered in this paper can be expressed in terms of certain partition functions counting the number of way to decompose a weight of $G$ into a linear positive combination of simple roots belonging to a fixed subset of $R^+_G$. To obtain the corresponding $q$-analogues of these coefficients, it suffices to quantify these partition functions. In particular when $G_0 = H_G$ is the maximal torus of $G$, the $q$-analogues obtained in this way are precisely the Lusztig $q$-analogues of weight multiplicities associated to $G$.

Our aim in this paper is two fold. First, we study the natural quantizations of branching coefficients corresponding to the restrictions of $G$ to the Levi subgroups of its standard parabolic subgroups. These polynomials will be denoted $K^{G,I}_{\lambda,\mu}(q)$ where $I$ is the set of simple roots of the Levi subgroup $L_{G,I}$ considered. The polynomials $K^{G,I}_{\lambda,\mu}(q)$ are generalizations of Lusztig $q$-analogues of weight multiplicities which coincide with the $q$-analogues $K^{GLn,I}_{\lambda,\mu}(q)$ for $G = GL_n$. From the results of [11], one can derive that they have nonnegative coefficients when $\mu$ is stable under the action of the Weyl group of $L_{G,I}$. The polynomials $K^{GLn,I}_{\lambda,\mu}(q)$ indexed by pairs of dominant weights $\lambda$ and $\mu$ which contain sufficiently large multiples of the fundamental weight $\kappa = (1,...,1) \in \mathbb{N}^n$ are called stable. This terminology reflects the fact that they are invariant if $\lambda$ and $\mu$ are translated by $\kappa$. When the Levi subgroup $L_{G,I}$ is isomorphic to a direct product of linear groups, we prove that this stable limit decomposes as a negative integer combination of polynomials $K^{GLn,I}_{\lambda,\mu}(q)$ (Theorem 3.2.4). This result can be regarded as a generalization of the decomposition of the stable limit of Lusztig $q$-analogues associated to $G$ as a sum of Kostka-Foulkes polynomials given in [11]. For a general Levi subgroup, we conjecture that the polynomials $K^{G,I}_{\lambda,\mu}(q)$ have nonnegative coefficients providing that $\mu$ is a partition (Conjecture 3.4.1). Note that this condition is in particular fulfilled when $\mu$ is stable under the action of the Weyl group of $L_{G,I}$. While writing this paper, the author was informed that an equivalent statement of this conjecture first appeared in some unpublished notes by Broer.

Next we study Littlewood-Richardson-type coefficients associated to $G = SO_{2n+1}$, $Sp_{2n}$ or $SO_{2n}$ and discuss the problem of their possible quantizations. Note first that the LLT quantization of the Littlewood-Richardson coefficients for $GL_n$ is based on a very special property of the plethysm of the Schur functions with the power sums. Indeed the coefficients appearing in the decomposition of this plethysm on the basis of the Schur functions are, up to a sign, Littlewood-Richardson coefficients. An analogous property for the other classical groups does not exist. Thus it seems impossible to relate $q$-analogues of tensor product multiplicities to Deodhar’s polynomials by proceeding as in [10]. With the above notation for $\mu$ and $\eta$, we define the coefficient $d^{\lambda}_{\mu^{(1)},...\mu^{(r)}}$ as the multiplicity of the finite dimensional irreducible $G$-module $V^G(\lambda)$ of highest weight $\lambda$ in the tensor product $V^G(\mu^{(1)}) \otimes \cdots \otimes V^G(\mu^{(r)})$. We show that this coefficient can be expressed in terms of a partition function (Proposition 4.1.4). This implies in particular that it does not depend on $G$. We also establish a duality result (Proposition 4.1.4) between the coefficients $d^{\lambda}_{\mu^{(1)},...\mu^{(r)}}$ and certain branching coefficients corresponding to the restriction of $SO_{2n}$ to the subgroup $SO_{2n_1} \times \cdots \times SO_{2n_r}$ (which is not isomorphic to a Levi subgroup of $SO_{2n}$). This permits to define $q$-analogues for the coefficients $d^{\lambda}_{\mu^{(1)},...\mu^{(r)}}$ but the polynomials obtained in this way may have negative coefficients even if the $n$-tuple $\mu$ is a partition. Denote by $\mathfrak{D}^G(\lambda)$ the restriction of the irreducible finite dimensional $GL_N$-module of highest weight $\lambda$ to $G$ where $N = 2n + 1$ if $G = SO_{2n+1}$ and $N = 2n$ if $G = Sp_{2n}$ or $SO_{2n}$. By replacing each module $V^G(\mu^{(p)})$ by $\mathfrak{D}^G(\mu^{(p)})$ in the definition of $d^{\lambda}_{\mu^{(1)},...\mu^{(r)}}$, one obtains tensor product coefficients $\mathfrak{D}^{\lambda,G}_{\mu^{(1)},...\mu^{(r)}}$ which can also be expressed in terms of a partition function. Thus, they admit natural quantizations $\mathfrak{D}^{\lambda,G}_{\mu^{(1)},...\mu^{(r)}}(q)$. Note that this time the coefficients $\mathfrak{D}^{\lambda,G}_{\mu^{(1)},...\mu^{(r)}}$ and the polynomials $\mathfrak{D}^{\lambda,G}_{\mu^{(1)},...\mu^{(r)}}(q)$ depend on the Lie group $G$ considered. We obtain a duality between the $q$-analogues $\mathfrak{D}^{\lambda,G}_{\mu^{(1)},...\mu^{(r)}}(q)$ and the stable limit of the polynomials.
$K_{\lambda,\mu}^{G,I}(q)$ associated to the Levi subgroup $GL_{\eta_1} \times \cdots \times GL_{\eta_r}$. In particular the polynomials $D_{\mu(1),\ldots,\mu(r)}(q)$ decomposes as nonnegative integer combination of polynomials $K_{\lambda,\mu}^{G,I}(q)$ (Theorem 1.3.2) and have nonnegative integer coefficients when the $\mu^{(p)}$'s are rectangular partitions of decreasing heights. Within each nonexceptional family of affine algebras, the one-dimensional sums have large rank limits which are called stable one-dimensional sums \cite{16}. There exist four distinct kinds of stable one-dimensional sums $X^\diamond$ labelled by the symbols $\diamond = \emptyset, (1), (2), (1,1)$. The stable one-dimensional sums of kind $\emptyset$ are related to $A_{n-1}$-affine crystals whereas the stable one-dimensional sums of kind $(1), (2), (1, 1)$ are defined from the other nonexceptional families of affine crystals. Then, according to Conjecture 5 of \cite{17} giving the decomposition of $X^\diamond$ in terms of one-dimensional sums $X^\emptyset$, the three families of $q$-analogues $D_{\mu(1),\ldots,\mu(r)}(q)$, $G = SO_{2n+1}, Sp_{2n}$ and $SO_{2n}$ should coincide (up to a simple renormalization) respectively with the stable one-dimensional sums of kind $(1), (1,1)$ and $(2)$ associated to $\mu$ when the $\mu^{(p)}$'s are rectangular partitions of decreasing heights. This means that it should be possible to extend the results of \cite{12} which holds when the $\mu^{(p)}$'s are row partitions of decreasing heights to all stable one-dimensional sums by establishing Conjecture 5 of \cite{17}.

The paper is organized as follows. In Section 2 we review the necessary background on branching multiplicity formulas and Levi-subgroups for classical Lie groups. In particular we introduce the Kostant-type partition functions which permit to compute the branching coefficients we use in the sequel. Section 3 is concerned with the $q$-analogues of branching coefficients corresponding to the restrictions to Levi subgroups. We prove that they admit a stable limit which decompose as nonnegative integer combination of Poincaré polynomials when the Levi subgroup considered is isomorphic to a direct product of linear groups. In section 4, we use the Jacobi-Trudi type determinantal expressions for the Schur functions of classical type to derive a duality between the Littlewood-Richardson coefficients $d_{\mu(1),\ldots,\mu(r)}$ and the branching coefficients corresponding to the restriction of $SO_{2n}$ to $SO_{2\eta_1} \times \cdots \times SO_{2\eta_r}$. This duality and the arguments used to prove it generalize the results of \cite{11} (corresponding to the case when all the $\mu^{(p)}$'s are row partitions). We observe that the natural quantization of the multiplicities $d_{\mu(1),\ldots,\mu(r)}$ may have negative coefficients.

We then introduce the polynomials $D_{\mu(1),\ldots,\mu(r)}(q)$ and show how they are related to the $q$-analogues of the branching coefficients corresponding to the restriction of $G$ to $GL_{\eta_1} \times \cdots \times GL_{\eta_r}$.

2 Background

2.1 Branching multiplicity formulas

In the sequel $G$ is one of the complex Lie groups $GL_n, Sp_{2n}, SO_{2n+1}$ or $SO_{2n}$ and $\mathfrak{g}$ its Lie algebra. We follow the convention of \cite{8} to realize $G$ as a subgroup of $GL_N$ and $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$ where

$$N = \begin{cases} n & \text{when } G = GL_n \\ 2n & \text{when } G = Sp_{2n} \\ 2n+1 & \text{when } G = SO_{2n+1} \\ 2n & \text{when } G = SO_{2n} \end{cases}.$$ 

With this convention the maximal torus $T_G$ of $G$ and the Cartan subalgebra $\mathfrak{h}_G$ of $\mathfrak{g}$ coincide respectively with the subgroup and the subalgebra of diagonal matrices of $G$ and $\mathfrak{g}$. Similarly the Borel subgroup $B_G$ of $G$ and the Borel subalgebra $\mathfrak{b}_G$ of $\mathfrak{g}$ coincide respectively with the subgroup and subalgebra of upper triangular matrices of $G$ and $\mathfrak{g}$.

Let $d_N$ be the linear subspace of $\mathfrak{gl}_N$ consisting of the diagonal matrices. For any $i \in \{1, \ldots, n\}$, write $\varepsilon_i$ for the linear map $\varepsilon_i : d_N \to C$ such that $\varepsilon_i(D) = \delta_i$ for any diagonal matrix $D$ whose $(i,i)$-coefficient is $\delta_i$.

Then $(\varepsilon_1, \ldots, \varepsilon_n)$ is an orthonormal basis of the Euclidean space $\mathfrak{h}^*_G \otimes \mathbb{R}$ (the real part of $\mathfrak{h}_G^*$). Let $R_G$ be the
root system associated to $G$. We can take for the simple roots of $\mathfrak{g}$
\[
\begin{align*}
\Sigma^+_{GL(n)} &= \{ \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ i = 1, \ldots, n-1 \ \text{for the root system } A_{n-1} \} \\
\Sigma^+_{SO_{2n+1}} &= \{ \alpha_n = \varepsilon_n \ \text{and } \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ i = 1, \ldots, n-1 \ \text{for the root system } B_n \} \\
\Sigma^+_{Sp_{2n}} &= \{ \alpha_n = 2\varepsilon_n \ \text{and } \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ i = 1, \ldots, n-1 \ \text{for the root system } C_n \} \\
\Sigma^+_{SO_{2n}} &= \{ \alpha_n = \varepsilon_n + \varepsilon_{n-1} \ \text{and } \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ i = 1, \ldots, n-1 \ \text{for the root system } D_n \} 
\end{align*}
\] .

Then the set of positive roots are
\[
\begin{align*}
R^+_{GL(n)} &= \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \ \text{with } 1 \leq i < j \leq n \} \ \text{for the root system } A_{n-1} \\
R^+_{SO_{2n+1}} &= \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \ \text{with } 1 \leq i < j \leq n \} \cup \{ \varepsilon_i \ \text{with } 1 \leq i \leq n \} \ \text{for the root system } B_n \\
R^+_{Sp_{2n}} &= \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \ \text{with } 1 \leq i < j \leq n \} \cup \{ 2\varepsilon_i \ \text{with } 1 \leq i \leq n \} \ \text{for the root system } C_n \\
R^+_{SO_{2n}} &= \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j \ \text{with } 1 \leq i < j \leq n \} \ \text{for the root system } D_n
\end{align*}
\]

We denote by $R_G$ the set of roots of $G$. The Weyl group of $GL_n$ is the symmetric group $S_n$ and for $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$, the Weyl group $W_G$ of the Lie group $G$ is the subgroup of the permutation group of the set $\{ \pi, \pi, \pi, 1, 2, \ldots, n \}$ generated by the permutations
\[
\begin{align*}
s_i = (i, i+1)(i, i+1), \ i = 1, \ldots, n - 1 \ \text{and } s_n = (n, n) \ \text{for the root systems } B_n \text{ and } C_n \\
s_i = (i, i+1)(n, n) \ \text{for the root system } D_n
\end{align*}
\]
where for $a \neq b \ (a, b)$ is the simple transposition which switches $a$ and $b$. For $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$, we identify the subgroup of $W_G$ generated by $s_i = (i, i+1)(n, n)$, $i = 1, \ldots, n - 1$ with the symmetric group $S_n$. We denote by $\ell$ the length function corresponding to the above set of generators. The action of $w \in W_G$ on $\beta = (\beta_1, \ldots, \beta_n) \in h_G^*$ is defined by
\[
w \cdot (\beta_1, \ldots, \beta_n) = (\beta_1^w, \ldots, \beta_n^w)
\]
where $\beta_i^w = \beta_i w(i)$ if $\sigma(i) \in \{1, \ldots, n\}$ and $\beta_i^w = -\beta_i w(\ell + 1)$ otherwise. We denote by $\rho_G$ the half sum of the positive roots of $R_G^+$. The dot action of $W_G$ on $\beta = (\beta_1, \ldots, \beta_n) \in h_G^*$ is defined by
\[
w \ast \beta = w \cdot (\beta + \rho_G) - \rho_G.
\]

Write $P_G^+$ for the cone of dominant weights of $G$. Denote by $P_n$ the set of partitions with at most $n$ parts. Each partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n$ will be identified with the dominant weight $\sum_{i=1}^n \lambda_i \varepsilon_i$. Then the irreducible finite dimensional representations of $G$ are parametrized by the partitions of $P_n$. For any $\lambda \in P_n$, denote by $V^G(\lambda)$ the irreducible finite dimensional representation of $G$ of highest weight $\lambda$. In the sequel we will also need the irreducible rational representations of $GL_n$. They are indexed by the $n$-tuples
\[
(\gamma^+, \gamma^-) = (\gamma_1^+, \gamma_2^+, \ldots, \gamma_p^+, 0, \ldots, 0, -\gamma_q, \ldots, -\gamma_r^-)
\]
where $\gamma^+$ and $\gamma^-$ are partitions of length $p$ and $q$ such that $p+q \leq n$. Write $\bar{P}_n$ for the set of such $n$-tuples and denote also by $V^{GL_n}(\gamma)$ the irreducible rational representation of $GL_n$ of highest weight $\gamma = (\gamma^+, \gamma^-) \in \bar{P}_n$. Consider a $r$-tuple $\eta = (\eta_1, \ldots, \eta_p)$ of positive integers summing $n$. Given $m = (\mu(1), \ldots, \mu(r))$ a $r$-tuple of partitions such that $\mu^{(p)} \in P_{\eta_p}$ for $p = 1, \ldots, r$, we denote by $\mu$ the $n$-tuple obtained by reading successively the parts of the partitions $\mu^{(1)}, \ldots, \mu^{(r)}$ from left to right.

As customary, we identify $P_G$ the lattice of weights of $G$ with a sublattice of $(\frac{1}{2}Z)^n$. For any $\beta = (\beta_1, \ldots, \beta_n) \in P_G$, we set $|\beta| = \beta_1 + \cdots + \beta_n$. We use for a basis of the group algebra $Z[Z^n]$, the formal exponentials $(e^\beta)_{\beta \in Z^n}$ satisfying the relations $e^{\beta_1}e^{\beta_2} = e^{\beta_1 + \beta_2}$. We furthermore introduce $n$ independent indeterminates $x_1, \ldots, x_n$ in order to identify $Z[Z^n]$ with the ring of polynomials $Z[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ by writing $e^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} = x^\beta$ for any $\beta = (\beta_1, \ldots, \beta_n) \in Z^n$.

For any $\lambda \in P_n$, we denote by $s^\lambda_N$ the universal character of type $G$ associated to $\lambda$ and by $F^G$ the ring of the universal characters of type $G$ defined by Koike and Terada [4].

We now give a technical lemma that we will be led to use in the sequel. Consider $\lambda \in P_n$, $\mu \in Z^n$ and $G$ one of the Lie groups $SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$. Set $k = (1, \ldots, 1) \in N^n$. 

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Lemma 2.1.1 Let $\mathcal{M} : \mathbb{Z}^n \to \mathbb{Z}[q]$ be a map such that for any $\beta \in \mathbb{Z}^n$, $\mathcal{M}(\beta) = 0$ if $|\beta| < 0$. Then for any integer $k \geq \frac{|\lambda| - |\mu|}{2}$ we have:

$$
\sum_{w \in W_G} (-1)^{\ell(w)} \mathcal{M}(w(\lambda + k\kappa + \rho_G) - \mu - k\kappa - \rho_G) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \mathcal{M}(\sigma(\lambda + \rho) - \mu - \rho)
$$

where $\rho = (n, n-1, \ldots, 1)$.

**Proof.** Consider $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n$ and $w \in W_G$. Write $w(\delta) = (\delta_1^w, \ldots, \delta_n^w)$ and set $E_w = \{i \mid w(i) \notin \{1, \ldots, n\}\}$. Define the sum $S_{w,\delta} = \sum_{i \in E_w} \delta_i$. Then $|w(\delta)| = |\delta| - 2S_{w,\delta}$. Now consider $k$ a nonnegative integer and set $\delta = (\lambda + \rho_G + k\kappa)$. We have $|w(\lambda + \rho_G + k\kappa)| = |(\lambda + \rho_G + k\kappa)| - 2S_{w,\delta}$. Let $S_{w,\delta} = S_{w,\lambda + \rho_G + kp}$ where $p = \text{card}(E_w)$. Thus we obtain:

$$
|w(\lambda + \rho_G + k\kappa) - (\mu + \rho_G + k\kappa)| = |(\lambda + \rho_G + k\kappa)| - 2S_{w,\lambda + \rho_G} - |(\mu + \rho_G + k\kappa)| - 2kp = |\lambda| - |\mu| - 2S_{w,\lambda + \rho_G} - 2kp.
$$

When $w \notin S_n$, we have $p \geq 1$ and $S_{w,\lambda + \rho_G} \geq 1$ since the coordinates of $\lambda + \rho_G$ are all positive. Hence $|w(\lambda + \rho_G + k\kappa) - (\mu + \rho_G + k\kappa)| < |\lambda| - |\mu| - 2k$ and is negative as soon as $k \geq \frac{|\lambda| - |\mu|}{2}$. For such an integer $k$ the sum defining the left hand side of the equality can normally running over $W_G$ can be restricted to $S_n$. Moreover we can write $\rho_G = \rho + \varepsilon\kappa$ with $\varepsilon = -\frac{1}{2}, 0$ or $1$ respectively for $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$. Since $\sigma(p\kappa) = p\kappa$ for any $p \in (\frac{1}{2}\mathbb{Z})^n$, this yields to the desired equality.

Let $G_0 \subset G$ be a complex Lie subgroup of $G$ and $\mathfrak{g}_0$ its Lie algebra. We suppose in the sequel that $G_0$ is isomorphic to a product of classical Lie groups whose maximal torus $T_0$ is equal to $T_G$. Let $\mathfrak{h}_0$ be the Cartan subalgebra of $\mathfrak{g}_0$. We have $\mathfrak{h}_0 = \mathfrak{h}_0^c$, thus $\mathfrak{h}_0^c = \mathfrak{h}_0^G$. In particular we can consider the set $R_0$ of roots of $\mathfrak{g}_0$ and its subset of positive roots $R_0^+$ respectively as subsets of $R_G$ and $R_G^+$.

The partition function $P_{G_0}$ associated to $G_0$ is defined by the formal identity

$$
\prod_{\alpha \in R^+ - R_0^+} \frac{1}{1 - e^{\alpha}} = \sum_{\beta \in P_G} P_{G_0}(\beta) e^{\beta}.
$$

Note that $P_{G_0}$ coincide with the Kostant partition function when $G_0 = T_G$ (that is $R_0^+ = \emptyset$). Write $P_{G_0}^\beta$ for the cone of dominant weights of $G_0$. For any $\lambda$ in $P_{G_0}^+$ and $\mu$ in $P_{G_0}^+$ we denote by $[V(\lambda)^G : V(\mu)^{G_0}]$ the multiplicity of the irreducible $G_0$-module $V(\mu)^{G_0}$ of highest weight $\mu$ in the restriction of the $G$-module $V(\lambda)^G$ to $G_0$.

**Theorem 2.1.2** With the above notation we have

$$
[V(\lambda)^G : V(\mu)^{G_0}] = \sum_{w \in W_G} (-1)^{\ell(w)} P_{G_0}(w \circ \lambda - \mu).
$$

We refer the reader to Theorem 8.2.1 of [4] for the proof.

### 2.2 Branching coefficients associated to Levi subgroups

Consider $I$ a subset of $\sum_{i=1}^n$, the set of simple roots associated to the classical Lie algebra $G$. Denote by $\pi_{G,I}$ the standard parabolic subgroup of $G$ (that is containing the Borel subgroup $B_G$) defined by $I$. Recall that the roots of $\pi_{G,I}$ are those of $R_G^I$ together with the negative roots of $R_G$ which are $\mathbb{Z}$-linear combinations of the simple roots contained in $I$. Write $L_{G,I}$ for the Levi subgroup of the parabolic $\pi_{G,I}$ and $I_{G,I}$ its corresponding Lie algebra. Let $R_{G,I}$ be the subsystem of roots spanned by $I$ and $R_{G,I}^+$ the subset of positive roots in $R_{G,I}$. Then $R_{G,I}$ and $R_{G,I}^+$ are respectively the set of roots and the set of positive roots of $I_{G,I}$.
The Levi subgroup $L_{G,I}$ corresponds to the removal, in the Dynkin diagram of $G$, of the nodes which are not associated to a simple root belonging to $I$. Write

$$J = \Sigma_G^+ - I = \{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$$

where for any $k = 1, \ldots, r$, $\alpha_{j_k}$ is a simple root of $\Sigma_G^+$ and $j_1 < \cdots < j_r$. Set $l_1 = j_1, l_k = j_k - j_{k-1}$, $k = 2, \ldots, r$ and $l_{r+1} = n - j_r$. According to [4], the Levi group $L_{G,I}$ is isomorphic to a direct product of classical Lie groups determined by the $(r+1)$-tuple $l_f = (l_1, \ldots, l_{r+1})$ of nonnegative integers summing $n$. We give in the table below the direct product associated to each Levi group $L_{G,I}$.

| $G$ | $L_{G,I}$ | $L_{G,I}$ |
|-----|-----------|-----------|
| $GL_n$ | $GL_{l_1} \times \cdots \times GL_{l_{r+1}}$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SO_{2l_{r+1}+1}$ |
| $SO_{2n+1}$ and $l_{r+1} \geq 2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SO_{2l_{r+1}+1}$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SP_{2l_{r+1}}$ |
| $Sp_{2n}$ and $l_{r+1} \geq 2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SO_{2l_{r+1}+1}$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SO_{2l_{r+1}-1}$ |
| $SO_{2n}$ and $l_{r+1} \geq 4$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_2 \times SL_2$ |
| $SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$ and $l_{r+1} = 0$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_4$ |
| $SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$ and $l_{r+1} = 1$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_4$ |
| $SO_{2n}$ and $l_{r+1} = 2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_4$ |
| $SO_{2n}$ and $l_{r+1} = 3$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_2$ | $GL_{l_1} \times \cdots \times GL_{l_r} \times SL_4$ |

Note that $l_{r+1} = p$ means that $\alpha_{j_r} = \alpha_{n-p}$. The factors of the decomposition of $L_{G,I}$ in a direct product of classical groups are giving by the connected components of the diagram obtained by removing the nodes corresponding to the simple roots $\alpha_{j_1}, \ldots, \alpha_{j_r}$ in the Dynkin diagram of the root system of $G$.

Since the Levi group $L_{G,I}$ is isomorphic to a direct product of classical groups and contains the maximal torus $T_G$, we can define the partition function $P_I$ associated to $G_0 = L_{G,I}$ as in [27] by the formal identity

$$\prod_{\alpha \in S_{G,I}} \frac{1}{1 - e^\alpha} = \sum_{\beta \in \mathbb{Z}^n} P_{G,I}(\beta) e^\beta$$

where $S_{G,I} = R_G^+ - R_G^{I+}$. Note that $S_{G,I}$ does not coincide in general with the subset of positive roots of $R_G^+$ obtained as $N$-linear combinations of the simple roots $\alpha_{j_1}, \ldots, \alpha_{j_r}$. We describe in the table below, the sets $S_{G,I}$ corresponding to the decompositions of $L_{G,I}$ given in (6). Set

$$\Theta_G = \begin{cases} \{\xi_i + \xi_j \mid 1 \leq i < j \leq n\} \cup \{\xi_i \mid 1 \leq i \leq n\} & \text{if } G = SO_{2n+1} \\ \{\xi_i + \xi_j \mid 1 \leq i \leq j \leq n\} & \text{if } G = Sp_{2n+1} \\ \{\xi_i + \xi_j \mid 1 \leq i < j \leq n\} & \text{if } G = SO_{2n} \end{cases}$$

and

$$\Theta^*_G = \begin{cases} \Theta_G - \{\xi_n\} & \text{if } G = SO_{2n+1} \\ \Theta_G - \{2\xi_n\} & \text{if } G = Sp_{2n+1} \\ \Theta_G - \{\xi_{n-1} + \xi_{n}\} & \text{if } G = SO_{2n} \end{cases}$$

| $S_{G,I}$ | $\Theta_G$ | $\Theta^*_G$ |
|----------|------------|-------------|
| $1: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\}$ | $\Theta_G$ | $\Theta^*_G$ |
| $2: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\} \cup \{\xi_i + \xi_j \mid 1 \leq i < j \leq n \text{ and } i \leq j_r\} \cup \{\xi_i \mid 1 \leq i \leq j_r\}$ | $\Theta_G$ | $\Theta^*_G$ |
| $3: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\} \cup \{\xi_i + \xi_j \mid 1 \leq i < j \leq n \text{ and } i \leq j_r\}$ | $\Theta_G$ | $\Theta^*_G$ |
| $4: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\} \cup \{\xi_i + \xi_j \mid 1 \leq i < j \leq n \text{ and } i \leq j_r\}$ | $\Theta_G$ | $\Theta^*_G$ |
| $5: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\} \cup \Theta_G$ | $\Theta_G$ | $\Theta^*_G$ |
| $6: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\} \cup \Theta^*_G$ | $\Theta_G$ | $\Theta^*_G$ |
| $7: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\} \cup \Theta^*_G$ | $\Theta_G$ | $\Theta^*_G$ |
| $8: S_{G,I} = \bigcup_{s=1} \{\xi_i - \xi_j \mid 1 \leq i \leq j \leq n\} \cup \{\xi_i + \xi_j \mid 1 \leq i < j \leq n \text{ and } i \leq n - 3\}$ | $\Theta_G$ | $\Theta^*_G$ |
Example 2.2.1 Consider $G = Sp_6$. We give below the 16 possible sets $I$ and $S_{G,I}$ for each Levi subgroup $L_{G,I}$:

| $I$ | $L_{G,I}$ | $S_{G,I}$ |
|-----|------------|----------|
| $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ | $Sp_6$ | $\emptyset$ |
| $\{\alpha_2, \alpha_3, \alpha_4\}$ | $GL_1 \times Sp_6$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, 2\varepsilon_1$ |
| $\{\alpha_1, \alpha_3, \alpha_4\}$ | $GL_2 \times Sp_4$ | $\varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_3, \alpha_4\}$ | $GL_1 \times GL_1 \times Sp_4$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_1, \alpha_2, \alpha_4\}$ | $GL_3 \times SL_2$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_1, \alpha_4\}$ | $GL_2 \times GL_1 \times SL_2$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_2, \alpha_4\}$ | $GL_1 \times GL_2 \times SL_2$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_4\}$ | $GL_1 \times GL_1 \times GL_1 \times SL_2$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_1, \alpha_2, \alpha_3\}$ | $GL_4$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_2, \alpha_3\}$ | $GL_1 \times GL_3$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_1, \alpha_2\}$ | $GL_3 \times GL_1$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_1, \alpha_3\}$ | $GL_2 \times GL_2$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_3\}$ | $GL_1 \times GL_1 \times GL_2$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_2\}$ | $GL_1 \times GL_2 \times GL_1$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\{\alpha_1\}$ | $GL_2 \times GL_1 \times GL_1$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |
| $\emptyset$ | $GL_1 \times GL_1 \times GL_1 \times GL_1$ | $\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4$ |

Note that $GL_1 \times GL_2 \times SL_2 \simeq GL_1 \times GL_3$ and $GL_1 \times GL_1 \times GL_1 \times SL_2 \simeq GL_1 \times GL_1 \times GL_2$.

Suppose that the Levi subgroup $L_{G,I}$ decomposes following (4) in

$$L_{G,I} \simeq G_1 \times \cdots \times G_p$$

(7)

where the $G_k$’s are classical Lie groups. Write $P_{G,I}^+$ for the cone of dominant weights of $L_{G,I}$. The dominant weights of $P_{G,I}^+$ can be regarded as sequences $\mu = (\mu^{(1)}, \ldots, \mu^{(p)})$ such that $\mu^{(s)}$ belongs to $\mathcal{P}_l$ if $G_s = GL_l$ and $\mu^{(p)}$ belongs to $\mathcal{P}_l$ if $G_p = SL_l, SO_{2l+1}, Sp_{2l}$ or $SO_{2l}$. Recall that, for such a dominant weight $\mu = (\mu^{(1)}, \ldots, \mu^{(p)})$, we denote by $\mu \in \mathbb{Z}^n$ the sequence obtained by reading successively the parts of $\mu^{(1)}, \ldots, \mu^{(p)}$ from left to right.

We deduce immediately from Theorem 2.1.2 the branching coefficients for the restriction of $V^G(\lambda)$ to the Levi subgroup $L_{G,I}$.

**Theorem 2.2.2** Consider $\lambda \in \mathcal{P}_n$ and $\mu \in P_{G,I}^+$ then

$$[V(\lambda)^G : V(\mu)^{L_{G,I}}] = \sum_{w \in W_G} (-1)^{f(w)} P_{G,I}^{G,I}(w \circ \lambda - \mu).$$
Note that \( L_{G,I} \simeq GL_n \) when \( I = \{\alpha_1, ..., \alpha_{n-1}\} \). In this case, the branching coefficients \([V(\lambda)^G : V(\gamma)^{GL_n}]\) where \( \gamma \in \mathcal{P}_n \) can be expressed in terms of the Littlewood-Richardson coefficients \( c_{\gamma,\lambda}^\nu \). For each classical group \( SO_{2n+1}, Sp_{2n} \) and \( SO_{2n} \), set

\[
\prod_{\alpha \in \Delta_G} (1 - e^{\alpha})^{-1} = \sum_{\beta \in \mathbb{N}^n} r_G(\beta)x^\beta. \tag{8}
\]

Denote by \( \mathcal{P}^{(2)}_n \) and \( \mathcal{P}^{(1,1)}_n \) the sub-sets of \( \mathcal{P}_n \) containing respectively the partitions with even rows and the partitions with even columns.

**Proposition 2.2.3** Consider \( \nu \in \mathcal{P}_n \) and \( \lambda = (\lambda^+, \lambda^-) \in \mathcal{P}_2 \). Then

1. \([V(\nu)^{SO_{2n+1}} : V(\lambda)^{GL_n}] = \sum_{w \in W_{B_n}} (-1)^{\ell(w)} r_{SO_{2n+1}}(w \circ \nu - (\lambda^+, \lambda^-)) \)
2. \([V(\nu)^{Sp_{2n}} : V(\lambda)^{GL_n}] = \sum_{w \in W_{C_n}} (-1)^{\ell(w)} r_{Sp_{2n}}(w \circ \nu - (\lambda^+, \lambda^-)) \)
3. \([V(\nu)^{SO_{2n}} : V(\lambda)^{GL_n}] = \sum_{w \in W_{D_{2n}}} (-1)^{\ell(w)} r_{SO_{2n}}(w \circ \nu - (\lambda^+, \lambda^-)) \)

**Proof.** The right equalities of the Proposition are obtained by Theorem [2.2.2]. The left follow from a classical result by Littlewood (see [13] appendix p 295). \( \blacksquare \)

**Remark:** When \( \lambda \) is a partition (that is \( \lambda^+ = \lambda \) and \( \lambda^- = \emptyset \)) we have by the above proposition

\[
\begin{align*}
[V(\nu)^{SO_{2n+1}} : V(\lambda)^{GL_n}] &= \sum_{\gamma, \delta \in \mathcal{P}_n} c_{\gamma,\lambda}^{\nu,\lambda} \\
[V(\nu)^{Sp_{2n}} : V(\lambda)^{GL_n}] &= \sum_{\gamma, \delta \in \mathcal{P}^{(2)}_n} c_{\gamma,\lambda}^{\nu,\lambda} \\
[V(\nu)^{SO_{2n}} : V(\lambda)^{GL_n}] &= \sum_{\gamma, \delta \in \mathcal{P}^{(1,1)}_n} c_{\gamma,\lambda}^{\nu,\lambda}
\end{align*}
\]

In particular for \( \kappa = (1, ..., 1) \in \mathbb{N}^n \) and any nonnegative integer \( k \) we obtain

\[
[V(\nu + k\kappa)^G : V(\lambda + k\kappa)^{GL_n}] = [V(\nu)^G : V(\lambda)^{GL_n}].
\]

### 2.3 Branching coefficients associated to an orthogonal decomposition of the root system \( D_n \)

Consider a \( r \)-tuple \( \eta = (\eta_1, ..., \eta_r) \) of positive integers summing \( n \). We associate to \( \eta \) the orthogonal decomposition \( D_\eta = D_{\eta_1} \cup ... \cup D_{\eta_r} \) of the root system \( D_n \) such that for any \( k = 1, ..., r \)

\[
D_{\eta_k} = \{ \pm \varepsilon_i \pm \varepsilon_j \mid \eta_{k-1} + 1 \leq i < j \leq \eta_k \}
\]

with \( \eta_0 = 1 \). Then \( SO_{2n} \) contains a subgroup \( SO_\eta \) such that

\[
SO_\eta \simeq SO_{2\eta_1} \times \cdots \times SO_{2\eta_r}.
\]

Note that \( SO_\eta \) is not a Levi subgroup of \( SO_{2n} \). The dominant weights of \( SO_\eta \) are the \( r \)-tuple of partitions \( \mu = (\mu^{(1)}, ..., \mu^{(r)}) \) such that \( \mu^{(k)} \) belongs to \( \mathcal{P}_{\eta_k} \) for any \( k = 1, ..., r \). Since \( SO_\eta \) contains the maximal torus of \( SO_{2n} \), we can apply Theorem [2.1.2] with \( G_0 = SO_\eta \). The corresponding partition function is defined by the formal identity

\[
\prod_{\langle i,j \rangle \in E_\eta} (1 - \frac{e^{x_i}}{x_j})^{-1} \prod_{\langle r,s \rangle \in E_\eta} (1 - e^{-x_r x_s})^{-1} = \sum_{\beta \in \mathbb{Z}^n} P^\eta(\beta)e^\beta. \tag{10}
\]

where \( E_\eta = \cup_{2 \leq \rho \leq r} \{(i, j) \mid 1 \leq i \leq \eta_1 + \cdots + \eta_{p-1} < j \leq \eta_p\} \).

**Proposition 2.3.1** Consider \( \lambda \) a partition and \( \mu \) a dominant weight of \( SO_\eta \). Then

\[
[V(\lambda)^{SO_{2n}} : V(\mu)^{SO_{2n}}] = \sum_{w \in W_{D_{2n}}} (-1)^{\ell(w)} P^\eta(w \circ \lambda - \mu).
\]

**Remark:** Although it is possible to obtain similar branching coefficients starting from orthogonal decompositions of the root systems \( B_n \) and \( C_n \), we do not use them in the sequel.
3 Generalization of Lusztig $q$-analogues

3.1 Quantization of the partition functions associated to a Levi subgroup

Consider a classical group $G$ and $I$ a subset of $\Sigma_G^+$. We associated to the Levi subgroup $L_{G,I}$ the $q$-partition function $P_{q}^{G,I}$ defined from the formal identity

$$\prod_{\alpha \in S_{G,I}} \frac{1}{1 - q^{e_{\alpha}}} = \sum_{\beta \in \mathbb{Z}^n} P_{q}^{G,I}(\beta)e^{\beta}. \quad (11)$$

**Definition 3.1.1** Let $\lambda$ be partition of $\mathcal{P}_n$ and $\mu$ a weight of $L_{G,I}$. We denote by $K_{\lambda,\mu}^{G,I}(q)$ the polynomial

$$K_{\lambda,\mu}^{G,I}(q) = \sum_{w \in W_G} (-1)^{\ell(w)}P_{q}^{G,I}(w \circ \lambda - \mu). \quad (12)$$

Remark: Since $P_{q}^{G,I}(\beta) = 0$ for any $\beta \in \mathbb{Z}^n$ with $|\beta| < 0$, we have $K_{\lambda,\mu}^{G,I}(q) \neq 0$ only if $|\lambda| \geq |\mu|$.

When $\mu$ is a dominant weight of $L_{G,I}$, we deduce from Theorem 2.2 that the polynomial $K_{\lambda,\mu}^{G,I}(q)$ is a $q$-analogue of the branching coefficient $[V(\lambda)^G : V(\mu)^{L_{G,I}}]$. When $I = \emptyset$, that is when $S_{G,I}$ contains all the simple roots of $G$, $L_{G,I}$ coincide with the maximal torus of $G$, thus $P_{q}^{G,I} = P_{q}^{G}$ and $K_{\lambda,\mu}^{G,I}(q)$ is the Lusztig $q$-analogue associated to the weight $\mu$ in $V(\lambda)^G$. If we suppose that $\mu$ is a partition, it is known [2] that $K_{\lambda,\mu}^{G,I}(q)$ has nonnegative integer coefficients.

The polynomials $K_{\lambda,\mu}^{G,I}(q)$ can also be defined from the Hilbert series of the Euler characteristic associated to certain graded virtual $G$-modules $\chi_{\mu}$. When $\mu$ is a dominant weight stable under the action of the Weyl group of $L_{G,I}$, Broer has proved in Theorem 2.2 of [1] that the higher cohomology vanishes in the Euler characteristic associated to $\chi_{\mu}$. In this case, by the use of the Borel-Weil-Bott Theorem, its graded formal character has a nonnegative expansion on $\{e^{\lambda} | \lambda \in \mathcal{P}_n\}$. This implies in particular that $K_{\lambda,\mu}^{G,I}(q)$ has nonnegative coefficients. Note that the results of [1] does not require that $G$ is a classical Lie group. In the context of this article, the dominant weight $\mu = (\mu(1),...,\mu(r))$ is stable under the action of the Weyl group of $L_{G,I}$ if and only if the $\mu^{(k)}$’s are rectangular partitions of decreasing heights and $\mu^{(r)} = 0$ when $L_{G,I}$ is not a direct product of linear groups. This yields to the following theorem:

**Theorem 3.1.2** (from [1]) Consider $L_{G,I}$ a Levi subgroup of the classical Lie group $G$. Let $\lambda$ be a partition of $\mathcal{P}_n$ and $\mu = (\mu(1),...,\mu(r))$ a dominant weight of $L_{G,I}$ such that the $\mu^{(k)}$’s are rectangular partitions of decreasing heights with $\mu^{(r)} = 0$ when $L_{G,I}$ is not a direct product of linear groups. Then $K_{\lambda,\mu}^{G,I}(q)$ has nonnegative coefficients.

When $G = GL_n$, the polynomial $K_{\lambda,\mu}^{GL_n,I}(q)$ can also be interpreted as a $q$-analogue of the generalized Littlewood-Richardson coefficient $c_{\lambda,\mu}^{(1),...,\mu^{(r)}}$ giving the multiplicity of $V(\lambda)^{GL_n}$ in $V(\mu)^{GL_n}(\mu(1)) \otimes \cdots \otimes V(\mu)^{GL_n}(\mu(r))$. Suppose that the $\mu^{(k)}$’s are rectangular partitions and denote by $X_{\mu}^{\emptyset}(q)$ the one-dimensional sum defined from the affine $A_{n-1}$-crystal $B_{\mu}$ associated to $\mu$ and the partition $\lambda$ (3). In fact $X_{\mu}^{\emptyset}(q)$ is defined up to a power of $q$ depending on the normalization of the energy function $H_{\mu}$ chosen on the vertices of $B_{\mu}$. By using a Morris-type recurrence formula for the Poincaré polynomials and a combinatorial description of the polynomials $K_{\lambda,\mu}^{GL_n,I}(q)$, Shimozono has obtained the following theorem:

**Theorem 3.1.3** [13] Let $\lambda$ be a partition of $\mathcal{P}_n$ and $\mu = (\mu(1),...,\mu(r))$ a dominant weight of $L_{G,I}$ such that the $\mu^{(k)}$’s are rectangular partitions of decreasing heights. Then

$$K_{\lambda,\mu}^{GL_n,I}(q) = q^* X_{\lambda,\mu}^{\emptyset}(q)$$

where $q^*$ is a power of $q$ depending on the normalization chosen for $H_{\mu}$.
In the sequel we will also led to consider another family of GL to the partitions \( \lambda \). Continuing Example 2.2.1 with \( \chi \) notes by Broer. In the terminology of [1], it is indeed equivalent to say that higher cohomology vanishes in Conjecture 3.1.4 Let \( \lambda \) be partition of \( P_n \) and \( \mu \) a dominant weight of \( L_{G, I} \) such that \( \mu \) is a partition. Then \( K_{\lambda,\mu}^{G,I}(q) \) has nonnegative coefficients.

Remarks:

(i) : Theorem 3.1.3 gives in particular a combinatorial proof of the positivity of the polynomials \( K_{\lambda,\mu}^{G,I}(q) \). In the next paragraph we will use this result to derive the positivity of the stable limits \( \tilde{K}_{\lambda,\mu}^{G,I}(q) \) independently of Theorem 3.1.2

(ii) : Under the hypotheses of Theorem 3.1.3, it is conjectured that the polynomials \( K_{\lambda,\mu}^{G,I}(q) \) coincide with the \( q \)-analogues of the Littlewood-Richardson coefficients introduced by Lascoux Leclerc and Thibon [10].

Numerous computations lead to conjecture that the positivity result of Theorem 3.1.2 can be extended to the case when the \( r \)-tuple \( \mu \) associated to the dominant weight \( \mu \in P^+_G \) is a partition.

**Conjecture 3.1.4** Let \( \lambda \) be partition of \( P_n \) and \( \mu \) a dominant weight of \( L_{G, I} \) such that \( \mu \) is a partition. Then \( K_{\lambda,\mu}^{G,I}(q) \) has nonnegative coefficients.

I was informed that an equivalent statement of this conjecture appeared for the first time in unpublished notes by Broer. In the terminology of [1], it is indeed equivalent to say that higher cohomology vanishes in \( \chi_\mu \) when \( \mu \) is a dominant weight.

**Example 3.1.5** Continuing Example 2.2.1 with \( G = Sp_8 \), \( \lambda = (4, 2, 2, 1) \) and \( \mu = (3, 1, 1, 0) \), we obtain the following polynomials \( K_{\lambda,\mu}^{G,I}(q) \)

| \( L_{G, I} \) | \( K_{\lambda,\mu}^{G,I}(q) \) |
|----------------|------------------|
| \( Sp_8 \)     | 0                 |
| \( GL_1 \times Sp_6 \) | 0         |
| \( GL_2 \times Sp_4 \) | 2q^4        |
| \( GL_1 \times GL_1 \times Sp_4 \) | \( q^3 + 2q^2 \) |
| \( GL_3 \times SL_2 \) | \( q^3 + q^2 \) |
| \( GL_2 \times GL_1 \times SL_2 \) | \( 3q^4 + 4q^3 + q^2 \) |
| \( GL_1 \times GL_2 \times SL_2 \) | \( q^4 + 2q^3 + q^2 \) |
| \( GL_1 \times GL_1 \times SL_2 \times SL_2 \) | \( 2q^5 + 4q^4 + 4q^3 + q^2 \) |
| \( GL_4 \) | \( q^2 \) |
| \( GL_1 \times GL_3 \) | \( q^3 + q^2 \) |
| \( GL_3 \times GL_1 \) | \( q^3 + 2q^2 + q^2 \) |
| \( GL_2 \times GL_2 \) | \( 3q^4 + 4q^3 + q^2 \) |
| \( GL_1 \times GL_1 \times GL_2 \) | \( 2q^3 + 4q^4 + 4q^3 + q^2 \) |
| \( GL_1 \times GL_2 \times GL_1 \) | \( q^3 + 2q^3 + 2q^2 + q^2 \) |
| \( GL_2 \times GL_1 \times GL_1 \) | \( 2q^3 + 2q^5 + 3q^3 + 4q^4 + 3q^2 \) |
| \( GL_1 \times GL_1 \times GL_1 \times GL_1 \) | \( q^8 + 2q^7 + 3q^6 + 4q^5 + 5q^4 + 4q^3 + q^2 \) |

When \( L_{G, I} = GL_1 \times GL_1 \times GL_1 \times GL_1 \), \( K_{\lambda,\mu}^{G,I}(q) \) is the Lusztig \( q \)-analogue for the root system \( C_4 \) associated to the partitions \( \lambda \) and \( \mu \). Note also that the polynomials corresponding to the isomorphic Levi subgroups \( GL_1 \times GL_2 \times SL_2 \cong GL_1 \times GL_3 \) and \( GL_1 \times GL_1 \times GL_1 \times SL_2 \cong GL_1 \times GL_1 \times GL_2 \) are equal.

In the sequel we will also led to consider another family of \( q \)-analogues for the branching coefficients \([V(\lambda)^{SO_{2n+1}} : V(\mu)^{L_{SO_{2n+1}}}]\) obtained from the partition function \( P_{q,h}^{SO_{2n+1}, I} \) defined by the expansion

\[
\prod_{\alpha \in S_{SO_{2n+1}, I}} \frac{1}{1 - q^{h(\alpha)} e^{\alpha}} = \sum_{\beta \in Z^n} P_{q,h}^{SO_{2n+1}, I}(\beta) e^{\beta}
\]

where \( h(\alpha) = 2 \) if \( \alpha = \varepsilon_i, \ i = 1, ..., n \) and \( h(\alpha) = 1 \) otherwise.
Lemma 3.2.1 Consider With the notation of the above paragraph, we define the stable limit of type $B$ (see Conjecture 4.4.3). This situation is analogous to that observed in [12] where the one-dimensional sums $K$ are classical Lie group, but the sum runs over the parabolic subgroup of $W_G$ generated by the $s_i, i = 1, \ldots, n$ which is a copy of the Weyl group of the root system $A_{n-1}$. We have $\tilde{K}_{\mu, \nu}^{GL_n, I}(q) = K_{\mu, \nu}^{GL_n, I}(q)$ since the Weyl group of $GL_n$ is the symmetric group $S_n$.

Lemma 3.2.1 Consider $\lambda, \mu$ two partitions of length $n$ such that $|\lambda| \geq |\mu|$. Let $k$ be any integer such that $k \geq \frac{|\lambda|-|\mu|}{2}$. Then we have

$$\tilde{K}_{\lambda, \mu}^{G, I}(q) = \left\{ \begin{array}{ll} K_{\lambda+\kappa, \mu+\kappa}^{G, I}(q) & \text{when } G = GL_n, Sp_{2n} \text{ or } SO_{2n} \\
K_{\lambda+\kappa, \mu+\kappa}^{G, I}(q) & \text{when } G = SO_{2n+1} \end{array} \right.$$  \hspace{1cm} (13)

where $\kappa = (1, \ldots, 1) \in \mathbb{N}^n$.

**Proof.** Suppose $G = GL_n, Sp_{2n}$ or $SO_{2n}$. For any $\beta \in \mathbb{Z}^n$, we have $P_{q}^{G, I}(\beta) = 0$ if $\beta$ is not a linear combination of the positive roots of $S_{G, I}$ with nonnegative coefficients. This implies that $P_{q}^{G, I}(\beta) = 0$ if $|\beta| < 0$. Then the Lemma is a consequence of Lemma 2.1.4 applied with $M = P_{q}^{G, I}$. We proceed similarly for $G = SO_{2n+1}$ by using $P_{q}^{SO_{2n+1}, I}$ instead of $P_{q}^{SO_{2n+1}, I}$.

Remark: Since $\sigma(\kappa) = \kappa$ for any $\sigma \in S_n$, we have the following stability property

$$\tilde{K}_{\lambda+\kappa, \mu+\kappa}^{G, I}(q) = \tilde{K}_{\lambda, \mu}^{G, I}(q)$$

for any integer $k$ which justifies the above terminology. So we can extend the definition of $\tilde{K}_{\lambda, \mu}^{G, I}(q)$ when $\lambda$ and $\mu$ are decreasing sequences of integers (positive or not).

For any $\xi \in \mathbb{Z}^n$, we define the polynomial $K_{\xi, \mu}^{GL_n, I}(q)$ by replacing, in [12] the partition $\lambda$ by $\xi$. There exists a straightening procedure for the polynomials $K_{\xi, \mu}^{GL_n, I}(q)$ which follows immediately from the fact that the set $\{\sigma \circ \xi | \sigma \in S_n\}$ (see [2]) intersects at most one time the cone of dominant weights of $GL_n$.

Lemma 3.2.2 Consider $\mu$ and $\xi$ in $\mathbb{Z}^n$. Then

$$\left\{ \begin{array}{ll} K_{\xi, \mu}^{GL_n, I}(q) = (-1)^{|\tau|} K_{\tau, \mu}^{GL_n, I}(q) & \text{if } \xi = \tau \circ (\nu) \text{ with } \tau \in S_n \text{ and } \nu \in \tilde{P}_n \\
0 & \text{otherwise} \end{array} \right.$$  \hspace{1cm} (14)

where $\tilde{P}_n = \{ \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n, \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \}$. 

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In the sequel of this paragraph we restrict ourselves to the case when \( G = SO_{2n+1}, Sp_{2n} \) or \( SO_{2n} \) and \( l_{r+1} = 0 \) (with the notation of paragraph 2.2). This corresponds to the decomposition 5 given in table (8), that is we suppose that \( I \) does not contain the simple root \( \alpha_n \). In this case
\[
L_{G,I} \simeq GL_{t_1} \times \cdots \times GL_{t_r},
\]
and \( I \) is also a subset of \( \Sigma^+_{GL_n} \), thus determinates a Levi subgroup of \( GL_n \) which is isomorphic to \( L_{G,I} \). Moreover we have \( S_{G,I} = S_{GL_n,I} \cup \Theta_G \). The \( q \)-partition functions \( P_{q,G,I} \), \( G = Sp_{2n} \) or \( SO_{2n} \) and \( P_{q,SO_{2n+1},I} \) can be expressed in terms of the \( q \)-partition \( P_{q,SO_{2n+1},I} \).

**Lemma 3.2.3** For any \( \beta \in \mathbb{Z}^n \) we have
\[
\begin{cases}
(i) : P_{q,G,I}^{\beta} = q^{[\beta]/2} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\beta|} r_G(\delta) P_{q,SO_{2n+1}}^{\beta - \delta} & \text{for } G = Sp_{2n} \text{ or } SO_{2n} \\
(ii) : P_{q,SO_{2n+1},I}^{\beta} = q^{[\beta]/2} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\beta|} r_{SO_{2n+1}}(\delta) P_{q,SO_{2n+1,I}}^{\beta - \delta}
\end{cases}
\]

**Proof.** (i) : The \( q \)-partition function \( P_{q,SO_{2n+1,I}}^{\beta} \) is defined by
\[
\prod_{\alpha \in S_{GL_n,I}} (1 - qe^{\alpha})^{-1} = \sum_{\gamma \in \mathbb{Z}^n} P_{q,SO_{2n+1,I}}^{\gamma} e^\gamma
\]
and since \( S_{G,I} = S_{GL_n,I} \cup \Theta_G \) the \( q \)-partition function \( P_{q,G,I}^{\beta} \) verifies
\[
\sum_{\beta \in \mathbb{Z}^n} P_{q,G,I}^{\beta} e^\beta = \prod_{\alpha \in \Theta_G} (1 - qe^{\alpha})^{-1} \prod_{\alpha \in S_{GL_n,I}} (1 - qe^{\alpha})^{-1}
\]
and we derive from (8)
\[
\prod_{\alpha \in \Theta_G} (1 - qe^{\alpha})^{-1} = \sum_{\delta \in \mathbb{N}^n} q^{[\delta]/2} r_G(\delta) e^\xi
\]
(14)
since the number of roots appearing in a decomposition of \( \delta \in \mathbb{N}^n \) as a sum of positive roots \( \varepsilon_r + \varepsilon_s \) with \( 1 \leq r < s \leq n \) or \( 2\varepsilon_i \) with \( 1 \leq i \leq n \) is always equal to \( |\delta|/2 \). Thus we obtain
\[
\sum_{\beta \in \mathbb{Z}^n} P_{q,G,I}^{\beta} e^\beta = \sum_{\gamma \in \mathbb{Z}^n} q^{[\gamma]/2} r_{SO_{2n+1}}(\gamma) e^{\delta + \gamma}
\]
We derive the equality \( P_{q,G,I}^{\beta} = \sum_{\gamma + \delta = \beta} r_G(\delta) q^{[\delta]/2} P_{q,SO_{2n+1,I}}^{\gamma} \). Since the set \( S_{GL_n,I} \) contains only positive roots \( \alpha \) with \( |\alpha| = 0 \), we will have \( P_{q,SO_{2n+1,I}}^{\gamma} = 0 \) when \( |\gamma| \neq 0 \). So we can suppose \( |\gamma| = 0 \) and \( |\delta| = |\beta| \) in the previous sum.
(ii) : Since \( h(\alpha) = 2 \) when \( |\alpha| = 1 \) we can also write
\[
\prod_{\alpha \in \Theta_{SO_{2n+1}}} (1 - q^{h(\alpha)} e^{\alpha})^{-1} = \sum_{\delta \in \mathbb{N}^n} q^{[\delta]/2} r_{SO_{2n+1}}(\delta) e^\xi
\]
Then we derive (ii) by proceeding as in (i). ■

**Remark:** A similar result for the \( q \)-partition function \( P_{q,SO_{2n+1},I}^{\beta} \) does not exit. Indeed the number of roots appearing in a decomposition of \( \delta \in \mathbb{N}^n \) as a sum of positive roots \( \varepsilon_r + \varepsilon_s \) with \( 1 \leq r < s \leq n \) and \( \varepsilon_i \) with \( 1 \leq i \leq n \) does not depend only of \( |\delta| \) since \( |\varepsilon_r + \varepsilon_s| \neq |\varepsilon_i| \).

**Theorem 3.2.4** Suppose \( G = SO_{2n+1}, Sp_{2n} \) or \( SO_{2n} \) and \( l_{r+1} = 0 \) and consider \( \lambda, \mu \in \mathcal{P}_n \) such that \( |\lambda| \geq |\mu| \). Then for any integer \( k \geq \frac{|\lambda| - |\mu|}{2} \) we have:
\[
\overline{K}_{\lambda,\mu}^{G,I}(q) = q^{|\lambda| - |\mu|/2} \sum_{\gamma \in \mathcal{P}_n} [V(\lambda + k\gamma)^G : V(\gamma + k\mu)^{GL_n}] K_{\gamma,\mu}^{GL_n,I}(q).
\]
We deduce from Lemma 2.1.1 applied with $K_{\lambda, \mu}^{G, I}(q) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} P_q^{G, I}(\sigma(\lambda + \rho) - (\mu + \rho)).$

Hence from the previous lemma we derive
\[
K_{\lambda, \mu}^{G, I}(q) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\beta|} r_G(\delta) q^{1/2} P_q^{GL_n, I}(\sigma(\lambda + \rho) - (\mu + \delta + \rho))
\]
where $\beta = \sigma(\lambda + \rho) - (\mu + \rho)$ in the second sum. Since $|\beta| = |\lambda| - |\mu|$, we obtain
\[
K_{\lambda, \mu}^{G, I}(q) = q^{\frac{1}{2}} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\lambda| - |\mu|} r_G(\delta) P_q^{GL_n, I}(\sigma(\lambda + \rho - \sigma^{-1}(\delta)) - (\mu + \rho))
\]
For any $\sigma \in S_n$, we have $\sigma^{-1}(\mathbb{N}^n) = \mathbb{N}^n$ and $r_G(\delta) = r_G(\sigma(\delta))$ since $\sigma(\Theta_G) = \Theta_G$. Thus
\[
K_{\lambda, \mu}^{G, I}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\lambda| - |\mu|} r_G(\delta) P_q^{GL_n, I}(\sigma(\lambda + \rho - (\mu + \rho)) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\lambda| - |\mu|} r_G(\delta) K_{\lambda - \delta, \mu}^{GL_n, I}(q). \tag{15}
\]
Now by Lemma 3.2.2 $K_{\lambda - \delta, \mu}^{GL_n, I}(q) = 0$ or there exits $\sigma \in S_n$ and $\gamma \in \tilde{P}_n$ such that $\gamma = \sigma^{-1} \circ (\lambda - \delta)$. Then we have $|\gamma| = |\lambda| - |\delta| = |\mu|$ and $\delta = \lambda + \rho - \sigma(\gamma + \rho)$. It follows that
\[
K_{\lambda, \mu}^{G, I}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\gamma \in \tilde{P}_n} r_G(\lambda + \rho - \sigma(\gamma + \rho)) K_{\gamma, \mu}^{GL_n, I}(q).
\]
Since $c(\delta) = c(\sigma(\delta))$ for any $\sigma \in S_n$ and $\delta \in \mathbb{N}^n$, we obtain the equality
\[
K_{\lambda, \mu}^{G, I}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\gamma \in \tilde{P}_n} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} r_G(\lambda + \rho - (\gamma + \rho)) K_{\gamma, \mu}^{GL_n, I}(q).
\]
We deduce from Lemma 2.1.1 applied with $\mathcal{M} = r_G$ and from Proposition 2.2.2 that the equality
\[
\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} r_G(\sigma(\lambda + \rho) - (\gamma + \rho)) = [V(\lambda + k\kappa)^G : V(\gamma + k\kappa)^{GL_n}]
\]
holds for any integer $k \geq \frac{|\lambda| - |\gamma|}{2} = \frac{|\lambda| - |\mu|}{2}$. This yields to the desired equality. \[\Box\]

By using Theorem 3.1.1 we obtain immediately

**Corollary 3.2.5** Suppose $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$ and $t_{r+1} = 0$. Consider $\lambda \in \tilde{P}_n$, $\mu^k = (\mu^{(1)}, ..., \mu^{(p)}) \in \tilde{P}_{G, I}^k$ such that the $\mu^{(k)}$’s are rectangular partitions of decreasing heights. Then $K_{\lambda, \mu}^{SO_{2n+1, I}}(q)$ have nonnegative coefficients.

**Remark:** When $G = Sp_{2n}$ or $SO_{2n}$, Lemma 3.2.1 implies that the above corollary can be regarded as a particular case of Theorem 3.1.1. Since the polynomials $K_{\lambda, \mu}^{SO_{2n+1, I}}(q)$ are generalized Lusztig $q$-analogues defined by using the parameters $q$ and $q^2$, we can not deduce their positivity from the results of Broer.
4 Some dualities between tensor product and branching coefficients

4.1 Determinantal identities and operators on formal series

Consider \( k, m \in \mathbb{Z} \) such that \( m > 0 \). When \( k \) is a nonnegative integer, write \( (k)_n = (k, 0, \ldots, 0) \) for the partition of length \( n \) with a unique non-zero part equal to \( k \). Then set \( h_k^G = s_k^G \) if \( k \geq 0 \) and \( h_k^G = 0 \) otherwise where \( s_k^G \) is the universal character of Koike and Terada associated to \( (k)_n \) for the Lie group \( G \). Given \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m \) define

\[
u^G_{\alpha} = \det \begin{pmatrix}
    h_{\alpha_1}^G & h_{\alpha_1+1}^G & \cdots & h_{\alpha_1+m-1}^G \\
    h_{\alpha_2}^G & h_{\alpha_2+1}^G & \cdots & h_{\alpha_2+m-1}^G \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{\alpha_m}^G & h_{\alpha_m+1}^G & \cdots & h_{\alpha_m+m-1}^G \\
    h_{\alpha_m-m+1}^G & h_{\alpha_m-m+2}^G & \cdots & h_{\alpha_m+2m-1}^G
\end{pmatrix}.
\]

(16)

The following proposition is a well known analogue of the Jacobi-Trudi determinantal formula for \( G = SO_{2n+1}, Sp_{2n} \) or \( SO_{2n} \).

Proposition 4.1.1 (see [3] §24.2) Consider \( \lambda \) a partition with at most \( m \) nonzero parts. Then for \( G = SO_{2n+1}, Sp_{2n} \) or \( SO_{2n} \) we have \( u^G_{\lambda} = s^G_{\lambda} \).

By using elementary permutations on rows in the determinant (16) we obtain the straightening law for \( u^G_{\alpha} \):

Lemma 4.1.2 Consider \( \alpha \in \mathbb{Z}^m \) then

\[
u^G_{\alpha} = \begin{cases} (-1)^{\ell(\sigma)} s^G_{\lambda} & \text{if there exists } \sigma \in \mathcal{S}_m \text{ and } \lambda \in \mathcal{P}_m \text{ such that } \sigma \circ \alpha = \lambda \\ 0 & \text{otherwise} \end{cases}
\]

Denote by \( \mathcal{L}_n = \mathbb{K}[\{x_1, x_n^{-1}, \ldots, x_n, x_n^{-1}\}] \) the vector space of formal Laurent series in the indeterminates \( x_1, x_n^{-1}, \ldots, x_n, x_n^{-1} \). We identify the ring of Laurent polynomials \( L_n = \mathbb{K}[\{x_1, x_n^{-1}, \ldots, x_n, x_n^{-1}\}] \) with the sub-space of \( \mathcal{L}_n \) containing the finite formal series. The vector space \( L_n \) is not a ring since the formal series are in the two directions. More precisely, the product \( F_1 \cdots F_r \) of the formal series \( F_i = \sum_{i \in E_i} x^{\beta_i} \) \( i = 1, \ldots, r \) is defined if and only if, for any \( \gamma \in \mathbb{Z}^n \), the number \( N_{\gamma} \) of decompositions \( \gamma = \beta_1 + \cdots + \beta_r \) such that \( \beta_i \in E_i \) is finite and in this case we have

\[
F_1 \cdots F_r = \sum_{\gamma \in \mathbb{Z}^n} N_{\gamma} x^\gamma.
\]

In particular the product \( P \cdot F \) with \( P \in L_n \) and \( F \in \mathcal{L}_n \) is well defined.

Consider \( \alpha \in \mathbb{Z}^m \). We set \( h_{\alpha}^G = h_{\alpha_1}^G \cdots h_{\alpha_m}^G \). Let \( K = [k_1, \ldots, k_m] \subset \{1, \ldots, n\} \) be the interval containing the \( m \) consecutive integers \( k_1 < \cdots < k_m \). Denote by \( \mathcal{L}_K \subset \mathcal{L}_n \) the vector space of formal Laurent formal series in the indeterminates \( x_{k_1}, x_{k_1}^{-1}, \ldots, x_{k_m}, x_{k_m}^{-1} \). We define the determinant

\[
\delta_K(\alpha) = \det \begin{pmatrix}
    x_{k_1}^{\alpha_1} & x_{k_1}^{\alpha_1+1} & \cdots & x_{k_1}^{\alpha_1+m-1} \\
    x_{k_1}^{\alpha_2+1} & x_{k_2}^{\alpha_2} & \cdots & x_{k_2}^{\alpha_2+m-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{k_m}^{\alpha_m-m+1} & x_{k_m}^{\alpha_m-m+2} & \cdots & x_{k_m}^{\alpha_m-2m+2}
\end{pmatrix}
\]

Set

\[
\delta_K = \prod_{1 \leq i < j \leq m} \left( 1 - \frac{x_{k_i}}{x_{k_j}} \right) \prod_{1 \leq r < s \leq m} \left( 1 - \frac{1}{x_{k_r} x_{k_s}} \right)
\]
Then $\delta_K(\alpha)$ and $\delta_K$ belong to $\mathcal{L}_K$. From a simple computation we derive the equality:

$$
\delta_K(\alpha) = \delta_K \cdot x^{\alpha_1}_1 \cdots x^{\alpha_m}_m.
$$

(17)

Consider $\eta = (\eta_1, \ldots, \eta_r)$ a $r$-tuple of positive integers summing $n$. We define from $\eta$ the intervals $K_1, \ldots, K_r$ of $\{1, \ldots, n\}$ by setting $K_1 = [1, \ldots, \eta_1]$ and for any $p = 2, \ldots, r$, $K_p = [\eta_1 + \cdots + \eta_{p-1} + 1, \ldots, \eta_1 + \cdots + \eta_p]$. Write $\delta_\eta = \delta_{K_1} \cdots \delta_{K_r}$.

Set

$$
\delta = \prod_{1 \leq i < j \leq n} (1 - \frac{x_i}{x_j}) \prod_{1 \leq r < s \leq n} (1 - \frac{1}{x_r x_s}).
$$

Giving $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, set $\beta^{(1)} = (\beta_1, \ldots, \beta_{\eta_1})$ and $\beta^{(p)} = (\beta_{\eta_1 + \cdots + \eta_{p-1} + 1}, \ldots, \beta_{\eta_1 + \cdots + \eta_p})$ for any $p = 2, \ldots, r$. The number of decompositions

$$
\beta = \sum_{1 \leq i < j \leq n} a_{i,j}(\varepsilon_i - \varepsilon_j) - \sum_{1 \leq r < s \leq n} b_{r,s}(\varepsilon_r + \varepsilon_s)
$$

with $a_{i,j}$ and $b_{r,s}$ some positive integers is finite. Thus $\delta^{-1}$ is well defined and belongs to $\mathcal{L}_n$. We introduce the linear maps

$$
\begin{align*}
\Delta_\eta : \mathcal{L}_n \to \mathcal{L}_n, \\
\nabla : \mathcal{L}_n \to \mathcal{L}_n
\end{align*}
$$

and

$$
\begin{align*}
\{ U_{G,\eta} : \mathcal{L}_n \to \mathcal{F}^G, x^\beta \mapsto u_{\beta^{(1)}}^G \cdots u_{\beta^{(r)}}^G \}, \\
\{ U_G : \mathcal{L}_n \to \mathcal{F}^G, x^\beta \mapsto u_{\beta}^G \}, \\
\{ H_G : \mathcal{L}_n \to \mathcal{F}^G, x^\beta \mapsto h_{\beta}^G \}
\end{align*}
$$

(18)

Note that these maps are not ring homomorphisms.

**Lemma 4.1.3** Let $K = [k_1, \ldots, k_m] \subset \{1, \ldots, n\}$ be the interval containing the $m$ consecutive integers $k_1 < \cdots < k_m$. Then for any $\alpha \in \mathbb{Z}^n$ we have $H_G(\delta_K x^{\alpha_1}_1 \cdots x^{\alpha_m}_m) = u_{\alpha}^G$.

**Proof.** To simplify the notation we set $x_{k_i} = y_i$ for any $k = 1, \ldots, m$. The linear map $H_G$ is not a ring homomorphism. Nevertheless, if $P_1, \ldots, P_k$ are polynomials respectively in the indeterminates $y_1, \ldots, y_m$, we have

$$
H_G(P_1(y_1) \cdots P_k(y_m)) = H_G(P_1(y_1)) \cdots H_G(P_k(y_m))
$$

by linearity of $H_G$. We can write

$$
\delta_K(\alpha) = \sum_{\sigma \in S_m} (-1)^{\ell(\sigma)} y_{\alpha_1 - \sigma(1)}(1) \cdot \cdots \cdot y_{\alpha_2 - \sigma(2)}(2) \cdot \cdots \cdot y_{\alpha_n - \sigma(n)}(n) + \sum_{\sigma \in S_m} (-1)^{\ell(\sigma)} x_{\alpha_1 - \sigma(1)}(1) \cdot \cdots \cdot x_{\alpha_2 - \sigma(2)}(2) \cdot \cdots \cdot x_{\alpha_n - \sigma(n)}(n)
$$

and by the previous argument

$$
H_G(\delta_K(\alpha)) = \sum_{\sigma \in S_m} (-1)^{\ell(\sigma)} h_{\alpha_1 - \sigma(1)} + \cdots + h_{\alpha_m - \sigma(n)} = u_{\alpha}^G
$$

where the last equality follows from (16). By (18) we have $\delta_K(\alpha) = \delta_K y^\alpha$. Thus by applying $H_G$ to this equality we obtain $H_G(\delta_K y^\alpha) = u_{\alpha}^G$. ■

**Proposition 4.1.4** We have

(i) : $U_{G,\eta} = H_G \circ \Delta_\eta$, (ii) : $H_G = U_G \circ \nabla$ and (iii) : $U_{G,\eta} = U_G \circ \nabla \circ \Delta_\eta$

**Proof.** (i) : Consider $\beta \in \mathbb{Z}^n$. If $P_1, \ldots, P_k$ are polynomials in the indeterminates belonging respectively to the sets $\{x_i \mid i \in K_1\}, \ldots, \{x_i \mid i \in K_r\}$ we have as in the proof of the previous lemma

$$
H_G(P_1 \cdots P_k) = H_G(P_1) \cdots H_G(P_k).
$$
Since $\Delta_\eta(x^\beta) = \delta_{K_1}x^{\beta(1)} \cdots \delta_{K_r}x^{\beta(r)}$ where the polynomials $\delta_{K_p}x^{\beta(p)}$, $p = 1, \ldots, r$ are respectively in the variables $\{x_i \mid i \in K_1\}, \ldots, \{x_r \mid i \in K_r\}$ we can write

$$H_G \circ \Delta_\eta(x^\beta) = H_G(\delta_{K_1}x^{\beta(1)}) \cdots H_G(\delta_{K_r}x^{\beta(r)}).$$

By applying Lemma 4.1.3 we derive $H_G \circ \Delta_\eta(x^\beta) = u_G^{\beta(1)} \cdots u_G^{\beta(r)} = U_G, \eta(x^\beta)$.

(ii) : Consider the linear map

$$\Delta : L_n \to L_n$$

Then by Lemma 4.1.3 applied with $K = \{1, \ldots, n\}$ we will have $H_G \circ \Delta = U_G$. Now for any $\beta \in \mathbb{Z}^n$ it is clear that $\Delta \circ \nabla(x^\beta) = x^\beta$. This implies that $U_G \circ \nabla(x^\beta) = H_G \circ \Delta \circ \nabla(x^\beta) = H_G(x^\beta)$ for any $\beta \in \mathbb{Z}^n$ and (ii) is proved.

(iii) is a straightforward consequence of (i) and (ii).

Now we have the equality

$$\delta^{-1} \cdot \delta_\eta = \prod_{(i,j) \in E_\eta} (1 - \frac{x_i}{x_j})^{-1} \prod_{(r,s) \in E_\eta} (1 - \frac{1}{x_r \cdot x_s})^{-1}$$

(19)

where $E_\eta = \cup_{2 \leq p \leq r} \{(i,j) \mid 1 \leq i \leq \eta_1 + \cdots + \eta_{p-1} < j \leq n\}$. In particular $\delta^{-1} \cdot \delta_\eta$ belongs to $L_n$. Set

$$\delta^{-1} \cdot \delta_\eta = \sum_{\beta \in \mathbb{Z}^n} Q^\eta(\beta)x^\beta.$$ (20)

Let $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ be a $r$-tuple of partitions such that $\mu^{(k)}$ belongs to $P_{\eta_k}$ for any $k = 1, \ldots, r$. For $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$, the Littlewood-Richardson coefficients $d^{\lambda}_{\mu^{(1)}, \ldots, \mu^{(r)}}$ are defined by the equality

$$s_{\mu^{(1)}}^{G} \cdots s_{\mu^{(r)}}^{G} = \sum_{\lambda \in P_n} d^{\lambda}_{\mu^{(1)}, \ldots, \mu^{(r)}} s_{\lambda}^{G}. $$

This means that the coefficient $d^{\lambda}_{\mu^{(1)}, \ldots, \mu^{(r)}}$ gives the multiplicity of the irreducible $G$-module $V^G(\lambda)$ in the tensor product $V^G(\mu^{(1)}) \otimes \cdots \otimes V^G(\mu^{(r)})$ thus is a nonnegative integer.

**Proposition 4.1.5** With the above notation we have

$$d^{\lambda}_{\mu^{(1)}, \ldots, \mu^{(r)}} = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} Q^\eta(\sigma \circ \lambda - \mu)$$

where $\mu \in \mathbb{N}^n$ is obtained by reading successively the parts of the partitions $\mu^{(1)}, \ldots, \mu^{(r)}$ defining $\mu$ from left to right.

**Proof.** By (iii) of Proposition 4.1.1 we have $U_{G, \eta} = U_G \circ \nabla \circ \Delta_\eta$. Since $\nabla \circ \Delta_\eta(x^\mu) = \delta^{-1} \cdot \delta_\eta \cdot x^\mu$, we obtain by (20)

$$U_{G, \eta}(x^\mu) = u_{\mu^{(1)}}^{G} \cdots u_{\mu^{(r)}}^{G} = \sum_{\beta \in \mathbb{Z}^n} Q^\eta(\beta) u_{\beta + \mu}^{G}. $$

Now by using Lemma 4.1.2 we derive

$$s_{\mu^{(1)}}^{G} \cdots s_{\mu^{(r)}}^{G} = \sum_{\lambda \in P_n} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} Q^\eta(\sigma \circ \lambda - \mu) s_{\lambda}^{G}$$

and the proposition is proved.

**Remark:** When $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ is a $r$-tuple of partitions such that $\mu \in \mathbb{N}^n$, we recover from the above proposition that the coefficients $d^{\lambda}_{\mu^{(1)}, \ldots, \mu^{(r)}}$ do not depend on the Lie group $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$ considered.
4.2 A duality for the coefficients $d_{\mu}^{\lambda}$

We define the involution $\iota$ on $\mathbb{Z}^n$ by setting $\iota(\beta_1, \ldots, \beta_n) = (\beta_n, \ldots, \beta_1)$ for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$. Let $\eta = (\eta_1, \ldots, \eta_r)$ be a $r$-tuple of positive integers summing $n$. Set $\overline{\eta} = (\eta_r, \ldots, \eta_1)$.

**Lemma 4.2.1** For any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ we have

$$Q^n(\beta) = P^\overline{\eta}(\iota(\beta))$$

where $P^\overline{\eta}$ and $Q^n$ are respectively the partition functions defined in (10) and (20).

**Proof.** By abuse of notation we also denote by $\iota$ the ring automorphism of $\mathcal{L}_n$ defined by $\iota(x^i) = x^{\iota(i)}$.

By applying $\iota$ to the identity

$$\prod_{(i,j) \in E_\eta} (1 - \frac{x_i}{x_j})^{-1} \prod_{(r,s) \in E_\eta} (1 - \frac{1}{x_r x_s})^{-1} = \sum_{\beta \in \mathbb{Z}^n} Q^n(\beta) x^\beta$$

we obtain

$$\prod_{(i,j) \in E_{\overline{\eta}}} (1 - \frac{x_i}{x_j})^{-1} \prod_{(r,s) \in E_{\overline{\eta}}} (1 - \frac{1}{x_r x_s})^{-1} = \sum_{\beta \in \mathbb{Z}^n} Q^n(\beta) x^{\iota(\beta)} = \sum_{\beta \in \mathbb{Z}^n} P^\overline{\eta}(\beta) x^\beta$$

where $E_{\overline{\eta}} = \bigcup_{2 \leq p \leq r} \{(i, j) \mid 1 \leq i \leq \overline{n}_1 + \cdots + \overline{n}_{p-1} < j \leq n\}$. This implies $Q^n(\beta) = P^\overline{\eta}(\iota(\beta))$ for any $\beta \in \mathbb{Z}^n$.

---

Given $\sigma \in S_n$, denote by $\overline{\sigma}$ the permutation defined by

$$\overline{\sigma}(k) = \sigma(n - k + 1).$$

For any $i \in \{1, \ldots, n - 1\}$, we have $\overline{\iota}_i = s_{n-i}$. The following Lemma is straightforward:

**Lemma 4.2.2** The map $\sigma \to \overline{\sigma}$ is an involution of the group $S_n$. Moreover we have $\sigma(\iota(\beta)) = \iota(\overline{\sigma}(\beta))$ and $\ell(\sigma) = \ell(\overline{\sigma})$ for any $\beta \in \mathbb{Z}^n$, $\sigma \in S_n$.

**Lemma 4.2.3** Let $\lambda, \mu$ two partitions of length $n$ and $\sigma \in S_n$. Then

$$(-1)^{\ell(\sigma)} Q^n(\sigma(\lambda + \rho) - (\mu + \rho)) = (-1)^{\ell(\overline{\sigma})} P^\overline{\lambda}(\iota(\lambda) + \rho) - (\iota(\mu) + \rho))$$

**Proof.** It suffices to prove the identity

$$Q^n(\sigma(\lambda + \rho) - (\mu + \rho)) = P^\overline{\lambda}(\iota(\lambda) + \rho) - (\iota(\mu) + \rho))$$

Set $P = P^\overline{\lambda}(\iota(\lambda) + \rho) - (\iota(\mu) + \rho))$. From the above Lemma we deduce

$$P = P^\overline{\lambda}(\iota(\sigma(\lambda)) + \overline{\sigma}(\rho) - \iota(\mu) - \rho).$$

Now an immediate computation shows that $\overline{\sigma}(\rho) - \rho = \iota(\sigma(\rho) - \rho)$. Thus we derive

$$P = P^\overline{\lambda}(\iota(\sigma(\lambda + \rho) - \mu - \rho)) = Q^n(\sigma(\lambda + \rho) - \mu - \rho)$$

where the last equality follows from Lemma 4.2.1.

Consider $\lambda \in \mathcal{P}_n$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ a $r$-tuple of partitions such that $\mu^{(k)}$ belongs to $\mathcal{P}_{n_k}$ for any $k = 1, \ldots, r$. Recall that $\mu \in \mathbb{N}^n$ is the $n$-tuple obtained by reading successively the parts of the partitions $\mu^{(1)}, \ldots, \mu^{(r)}$ defining $\mu$ from left to right. Let $a$ be the minimal integer such that

$$\hat{\lambda} = (a - \lambda_n, \ldots, a - \lambda_1)$$

and $\hat{\mu} = (a - \mu_n, \ldots, a - \mu_1)$

(21)
belong to \( \mathbb{N}^n \). Then \( \hat{\lambda} \) is a partition of length \( n \). Set \( \mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_r) \) and denote by \( \hat{\mu} = (\hat{\mu}^{(1)}, \ldots, \hat{\mu}^{(r)}) \) the \( r \)-tuple of partitions such that \( \hat{\mu}^{(1)} = (\mu_1, \ldots, \mu_{\mathfrak{p}_1}) \in \mathcal{P}_{\mathfrak{p}_1} \) and \( \hat{\mu}^{(p)} = (\mu_{\mathfrak{p}_1 + \cdots + \mathfrak{p}_{p-1} + 1}, \ldots, \mu_{\mathfrak{p}_1 + \cdots + \mathfrak{p}_p}) \in \mathcal{P}_{\mathfrak{p}_p} \) for any \( k = 2, \ldots, r \). The following proposition shows that the Littlewood-Richardson coefficients \( d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} \) defined above are branching coefficients associated to the restriction from the orthogonal group \( SO_{2n} \) to the subgroup \( SO_{\mathfrak{p}_1} \cong SO_{\mathfrak{p}_1} \times \cdots \times SO_{\mathfrak{p}_r} \) defined in \( \text{Lemma } 4.2.2 \).

**Proposition 4.2.4** With the above notation, we have for any integer \( k \geq \frac{|\mu| - |\lambda|}{2} \)

\[
d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} = [V(\hat{\lambda} + k\kappa)^{SO_{2n}} : V(\hat{\mu} + k\kappa)^{SO_{\mathfrak{p}}}] \tag{22}
\]

**Proof.** It follows from the definition of \( d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} \) and the above lemma that

\[
d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\ell(\sigma)} Q^\mathfrak{p}(\sigma(\lambda + \rho) - \mu - \rho) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\ell(\sigma)} P^\mathfrak{p}(\sigma(\lambda(\lambda) + \rho)) - (\lambda(\mu) + \rho)).
\]

Then by Lemma 4.2.2 we obtain

\[
d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\ell(\sigma)} P^\mathfrak{p}(\sigma(\lambda(\lambda) + \rho)) - (\lambda(\mu) + \rho)).
\]

We have \( \sigma(\lambda(\lambda) + \rho + a\kappa) = \sigma(\lambda(\mu) + \rho) + a\kappa \) since \( \sigma \in \mathcal{S}_n \) and \( \kappa = (1, \ldots, 1) \). So we can write

\[
d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\ell(\sigma)} P^\mathfrak{p}(\sigma(\lambda(\lambda) + a\kappa + \rho)) - (\lambda(\mu) + a\kappa + \rho)).
\]

Since \( \hat{\lambda} = \lambda(\lambda) + a\kappa \) and \( \hat{\mu} = \lambda(\mu) + a\kappa \) we derive

\[
d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\ell(\sigma)} P^\mathfrak{p}(\sigma(\hat{\lambda}(\rho) + a\kappa + \rho)) - (\hat{\lambda} + a\kappa + \rho)).
\]

Now by using Lemma 4.1.1 with \( M = P^\mathfrak{p} \) and Proposition 2.3.1 we obtain

\[
\sum_{\sigma \in \mathcal{S}_n} (-1)^{\ell(\sigma)} P^\mathfrak{p}(\sigma(\hat{\lambda}(\rho) + a\kappa + \rho)) = [V(\hat{\lambda} + k\kappa)^{SO_{2n}} : V(\hat{\mu} + k\kappa)^{SO_{\mathfrak{p}}}]\]

for any integer \( k \geq \frac{|\hat{\lambda}| - |\hat{\mu}|}{2} = \frac{|\mu| - |\lambda|}{2} \) and the Proposition is proved. \( \blacksquare \)

**Remarks:**

(i) : The previous proposition can be regarded as an analogue for the Lie groups \( SO_{2n+1}, Sp_{2n} \) and \( SO_{2n} \) of the duality

\[
d_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} = [V(\lambda)^{GL_n} : V(\mu)^{GL_n}] \tag{23}
\]

between the Littlewood-Richardson coefficient \( c_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda} \) giving the multiplicity of \( V(\lambda)^{GL_n} \) in the tensor product \( V^{GL_n}(\mu^{(1)}) \otimes \cdots \otimes V^{GL_n}(\mu^{(r)}) \) and the branching coefficient of the restriction of \( V(\lambda)^{GL_n} \) to the Levi subgroup \( GL_\eta = GL_{\eta_1} \times \cdots \times GL_{\eta_r} \). Note that this duality can be proved by similar methods than those used in this paragraph starting from Jacobi-Trudi’s determinantal expression for the Schur function \( S_\lambda^{GL_n} = \text{char}(V^{GL_n}(\lambda)) \) instead of \( \text{Lemma } 4.2.2 \).

(ii) : When all the \( \mu^{(k)} \)'s are row partitions, \( \text{Lemma } 4.2.2 \) simply express the Schur-Weyl duality between the dimension of the weight space \( \mu \) in \( V^{GL_n}(\lambda) \) and the multiplicity of \( V^{GL_n}(\lambda) \) in the tensor product of the symmetric powers of the vector representation of \( GL_n \) associated to \( \mu \). Similarly, in this particular case, \( \text{Proposition } 4.2.4 \) reduce to the duality already observed in \( \text{Lemma } 4.2.2 \).
4.3 Quantization of the coefficients $d_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda$

Consider $\lambda \in \mathcal{P}_n$, $\eta = (\eta_1, \ldots, \eta_p)$ a $p$-tuple of positive integers summing $n$ and $\mu = (\mu_1^{(1)}, \ldots, \mu_r^{(r)})$ a $r$-tuple of partitions such that $\mu^{(k)}$ belongs to $\mathcal{P}_{\eta_k}$ for any $k = 1, \ldots, r$. We have seen in paragraph 3.13 that it is possible to define natural $q$-analogues of the Littlewood-Richardson coefficients $c_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda$ from the duality (23) by setting

$$c_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda (q) = K_{\lambda, \mu}^{GL_n, I}(q).$$

where the polynomials $K_{\lambda, \mu}^{GL_n, I}(q)$ are the generalized Lusztig $q$-analogues of Definition 3.13. When the $\mu^{(k)}$'s are rectangular partitions of decreasing heights, the polynomials $c_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda (q)$ have nonnegative coefficients by Theorem 3.12. It is tempting to define $q$-analogues of the coefficients $d_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda$ by setting

$$d_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda (q) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \mathcal{Q}_\sigma^\eta (\sigma \circ \lambda - \mu)$$

where the $q$-partition function $\mathcal{Q}_\sigma^\eta$ verifies

$$\prod_{(i, j) \in E_\eta} (1 - q x_i^{-1} x_j^{-1}) \prod_{(r, s) \in E_q} (1 - q x_r x_s)^{-1} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{Q}_\sigma^\eta (\beta) x^{\beta}.$$  

Unfortunately the polynomials $d_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda (q)$ have not nonnegative coefficients in general. This is in particular the case for $\lambda = (1, 1, 1, 0, 0), \mu^{(1)} = (5), \mu^{(2)} = (4, 4)$ and $\mu^{(3)} = (2, 2)$ where we have $d_{\mu_1^{(1)}, \mu_2^{(2)}, \mu_3^{(3)}}^\lambda (q) = q^{11} - q^8$.

4.4 The $q$-analogues $\mathfrak{D}_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda G$

For $G = SO_{2n+1}, Sp_{2n}$ or $SO_{2n}$, there exist coefficients associated to the decomposition of a tensor product of $G$-modules into its irreducible components admitting a natural quantization with nonnegative coefficients. For any partition $\nu \in \mathcal{P}_n$, write $\mathfrak{V}^G(\nu)$ for the restriction from the irreducible $GL_n$-module of highest weight $\nu$ to $G$. Consider $\eta = (\eta_1, \ldots, \eta_p)$ a $p$-tuple of positive integers summing $n$ and $\mu = (\mu_1^{(1)}, \ldots, \mu_r^{(r)})$ a $r$-tuple of partitions such that $\mu^{(k)}$ belongs to $P_{\eta_k}$ for any $k = 1, \ldots, r$. Given $\lambda \in \mathcal{P}_n$, the coefficients $\mathfrak{D}_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda G$ are defined as the multiplicities of $V^G(\lambda)$ in $\mathfrak{V}^G(\mu_1^{(1)}) \otimes \cdots \otimes \mathfrak{V}^G(\mu_r^{(r)})$, that is we have

$$\mathfrak{V}^G(\mu_1^{(1)}) \otimes \cdots \otimes \mathfrak{V}^G(\mu_r^{(r)}) \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V^G(\lambda) \otimes \mathfrak{D}_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda G.$$

Contrary to the coefficients $d_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda$, the coefficients $\mathfrak{D}_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda G$ depend on the Lie group $G$ considered. Set

$$\Omega^G = \left\{ \prod_{1 \leq r < s \leq n} (1 - x_r^{-1} x_s^{-1}) \text{ for } G = Sp_{2n}, \prod_{1 \leq r < s \leq n} (1 - x_r^{-1} x_s^{-1}) \prod_{1 \leq i \leq n} (1 - x_i^{-1} x_i^{-1}) \text{ for } G = SO_{2n}, \prod_{1 \leq r < s \leq n} (1 - x_r^{-1} x_s^{-1}) \prod_{1 \leq i \leq n} (1 - x_i^{-1} x_i^{-1}) \prod_{1 \leq r < s \leq n} (1 - x_r^{-1} x_s^{-1}) \right\} \text{ for } G = SO_{2n+1} \right\}.$$

Denote by $\Omega^\eta G$ the partition function defined for $G = Sp_{2n}, SO_{2n}$ and $SO_{2n+1}$ by the identities

$$\prod_{(i, j) \in E_\eta} (1 - x_i^{-1} x_j^{-1}) \Omega G = \sum_{\beta \in \mathbb{Z}^n} \Omega^\eta G (\beta) x^{\beta} \quad \text{(25)}$$

where $E_\eta = \cup_{2 \leq p \leq r} \{(i, j) \mid 1 \leq i \leq \eta_1 + \cdots + \eta_{p-1} < j \leq n\}$.

**Proposition 4.4.1** With the above notation we have

$$\mathfrak{D}_{\mu_1^{(1)}, \ldots, \mu_r^{(r)}}^\lambda G = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \Omega^\eta G (\sigma \circ \lambda - \mu)$$
\textbf{Proof.} Consider }m \in \mathbb{N}\text{ and for any }\alpha = (\alpha_1, \ldots, \alpha_m)\text{ set } v_{\alpha} = \det \begin{pmatrix} h_{\alpha_1}^{GL_n} & h_{\alpha_1+1}^{GL_n} & \cdots & h_{\alpha_1+m-1}^{GL_n} \\ h_{\alpha_2-1}^{GL_n} & h_{\alpha_2}^{GL_n} & \cdots & h_{\alpha_2+m-2}^{GL_n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\alpha_m-m+1}^{GL_n} & h_{\alpha_m-m+2}^{GL_n} & \cdots & h_{\alpha_m}^{GL_n} \end{pmatrix}.

As in Lemma 11.1.2 we have from Jacobi-Trudi’s determinantal expression of the Schur function }s_{\lambda}^{GL_n}\text{ we have:}

\begin{equation}
 v_{\alpha} = \begin{cases} (-1)^{\ell(\sigma)} s_{\lambda}^{GL_n} & \text{if there exists } \sigma \in S_n \text{ and } \lambda \in \mathcal{P}_n \text{ such that } \sigma \circ \alpha = \lambda \\ 0 & \text{otherwise} \end{cases} \quad (26)
\end{equation}

Let }K = [k_1, \ldots, k_m] \subset \{1, \ldots, n\}\text{ be the interval containing the }m\text{ consecutive integers }k_1 < \cdots < k_m.\text{ Consider the Laurent polynomial }d_K = \prod_{1 \leq i < j \leq m} (1 - \frac{x_i}{x_j})\text{ (see paragraph 4.1). Then we derive as in Lemma 4.1.3 the equality } H_{GL_n}(d_K \cdot x^n) = v_{\alpha}^{GL_n}.\text{ Now observe that for any integer }k,\text{ we have }h_k^{Sp_2n} = h_k^{GL_{2n}}, h_k^{SO_2n} = h_k^{GL_{2n}} - h_{k-2}^{GL_{2n}}\text{ and }h_k^{SO_{2n+1}} = h_k^{GL_{2n+1}} - h_{k-1}^{GL_{2n+1}}. \text{ This permits to express the determinant } (10) \text{ in terms of the } h_k^{GL_{2n}}, k \in \mathbb{Z}. \text{ Set:}

\begin{equation}
 s_G = \prod_{1 \leq i < j \leq n} (1 - \frac{x_i}{x_j})\Omega_G.
\end{equation}

For any }\beta = (\beta_1, \ldots, \beta_n)\in \mathbb{Z}^n\text{ and for each Lie group }Sp_{2n}, SO_{2n+1}\text{ and }SO_{2n}\text{ we will have }

\begin{equation}
 u_{\beta}^{G} = H_{GL_n}(\partial^G \cdot x^\beta).
\end{equation}

Now we define the intervals }K_1, \ldots, K_r\text{ of }\{1, \ldots, n\}\text{ from }\eta\text{ by setting }K_1 = [1, \ldots, \eta_1]\text{ and for any }p = 2, \ldots, r, K_p = [\eta_1 + \cdots + \eta_{p-1} + 1, \ldots, \eta_1 + \cdots + \eta_p].\text{ Consider the linear maps }

\begin{equation}
 V_\eta : L_n \to F_n^{GL_n} \quad \Phi_\eta : L_n \to L_n \quad \text{and} \quad \Psi_\eta : L_n \to L_n
\end{equation}

\begin{equation}
 V_\eta : L_n \to F_n^{GL_n} \quad \Phi_\eta : L_n \to L_n \quad \text{and} \quad \Psi_\eta : L_n \to L_n
\end{equation}

where }d_\eta = d_{K_1} \cdots d_{K_r}\text{ and } \beta^{(1)}, \ldots, \beta^{(r)}\text{ are defined as in (18). Then we obtain as in Proposition 4.1.3 the identities } V_\eta = H_{GL_n} \circ \Phi_\eta, H_{GL_n} = U_G \circ \Psi_G \text{ which lead to } V_\eta = U_G \circ \Psi_G \circ \Phi_\eta. \text{ By (25) we have}

\begin{equation}
 (\partial^G)^{-1} \cdot V_\eta = \prod_{(i,j) \in E_n} (1 - \frac{x_i}{x_j})^{-1}\Omega^G = \sum_{\beta \in \mathbb{Z}^n} \Omega^{\eta,G}(\beta)x^\beta
\end{equation}

\begin{equation}
 (\partial^G)^{-1} \cdot V_\eta = \prod_{(i,j) \in E_n} (1 - \frac{x_i}{x_j})^{-1}\Omega^G = \sum_{\beta \in \mathbb{Z}^n} \Omega^{\eta,G}(\beta)x^\beta
\end{equation}

thus we can write } V_\eta(x^\mu) = U_G(\sum_{\beta \in \mathbb{Z}^n} \Omega^{\eta,G}(\beta)x^{\beta+\mu}) = \sum_{\beta \in \mathbb{Z}^n} \Omega^{\eta,G}(\beta)u_{\beta+\mu}. \text{ Then by using the straightening law (20) we derive}

\begin{equation}
 s_{\mu^{(1)}}^{GL_n} \cdots s_{\mu^{(r)}}^{GL_n} = \sum_{\lambda \in \mathcal{P}_n} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)}\Omega^{\eta,G}(\sigma \circ \lambda - \mu)s_{\lambda}^{G}
\end{equation}

which establishes the proposition. \hfill \blacksquare

The coefficients }D_{\mu^{(1)}, \ldots, \mu^{(r)}}^{(\lambda)}\text{ admit the natural quantization

\begin{equation}
 D_{\mu^{(1)}, \ldots, \mu^{(r)}}^{(\lambda)}(q) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)}\Omega^{q,G}(\sigma \circ \lambda - \mu)
\end{equation}

where }\Omega^{q,G}\text{ is defined for }G = Sp_{2n}, SO_{2n}\text{ and }SO_{2n+1}\text{ respectively by the identities}

\begin{equation}
 \begin{cases} 
 \prod_{(i,j) \in E_n} (1 - q\frac{x_i}{x_j})^{-1} \prod_{1 \leq r < s \leq n} (1 - q\frac{1}{x_r x_s})^{-1} = \sum_{\beta \in \mathbb{Z}^n} \Omega^{q,Sp_{2n}}(\beta)x^\beta \\
 \prod_{(i,j) \in E_n} (1 - q\frac{x_i}{x_j})^{-1} \prod_{1 \leq r < s \leq n} (1 - q\frac{1}{x_r x_s})^{-1} = \sum_{\beta \in \mathbb{Z}^n} \Omega^{q,SO_{2n}}(\beta)x^\beta \\
 \prod_{(i,j) \in E_n} (1 - q\frac{x_i}{x_j})^{-1} \prod_{1 \leq r < s \leq n} (1 - q^2\frac{1}{x_i x_s})^{-1} = \sum_{\beta \in \mathbb{Z}^n} \Omega^{q,SO_{2n+1}}(\beta)x^\beta
\end{cases} \quad . \quad (27)
\end{equation}
There exists a duality between the polynomials $D^{\lambda,G}_{\mu(1),\ldots,\mu(r)}(q)$ and the $q$-analogues of Theorem 3.2.4. Surprisingly this duality does not relate the polynomials $D^{\lambda,G}_{\mu(1),\ldots,\mu(r)}(q)$ to the polynomials $K^{\lambda,G}_{\alpha,\beta}(q)$ but to the polynomials $K^{\lambda,G}_{\alpha,\beta}(q)$ where

\[
\hat{G} = \begin{cases}
    Sp_{2n} & \text{if } G = SO_{2n} \\
    SO_{2n} & \text{if } G = Sp_{2n} \\
    SO_{2n+1} & \text{if } G = SO_{2n+1}
\end{cases}.
\]

The polynomials $D^{\lambda,G}_{\mu(1),\ldots,\mu(r)}(q)$ also decompose as nonnegative combinations of Poincaré polynomials. Set $\pi = (\pi_1, \ldots, \pi_r)$. Then $G$ and $GL_n$ contain Levi subgroups $L_{G,I}$ and $L_{GL_n,I}$ isomorphic to $GL_{\pi_1} \times \cdots \times GL_{\pi_r}$. We have $I = \{0 < \alpha_i < \pi_1 \cup \ldots \cup \pi_r \} \in \{\alpha_i | \pi_1 + \cdots + \pi_r < \pi_1 + \cdots + \pi_{r+1}\}$.

**Theorem 4.4.2** With the previous notation we have

\[
D^{\lambda,G}_{\mu(1),\ldots,\mu(r)}(q) = K^{\lambda,G}_{\alpha,\beta}(q) = q^{\frac{\lambda^2 - |\lambda|}{2}} \sum_{\nu \in \mathbb{F}_n} [V(\nu)_{\hat{G}} : V(\lambda)^{GL_n}] K^{GL_n, I}_{\nu, \delta}(q)
\]

where $\lambda, \beta$ are defined as in Theorem 4.2.4.

**Proof.** We use the notation of paragraph 4.2. We only give the proof for $G = Sp_{2n}$ or $G = SO_{2n}$. The proof is essentially the same for $G = SO_{2n+1}$. By proceeding as in the proof of Lemma 4.2.1, we establish, for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, the identity

\[
D^\eta_{G}(\beta) = P^\hat{G}_{q,I}(\beta).
\]

Then the equality $D^{\lambda,G}_{\mu(1),\ldots,\mu(r)}(q) = K^{\lambda,G}_{\alpha,\beta}(q)$ is obtained by using essentially the same arguments as in the proof of theorem 4.2.4. We deduce from (8) the expansions

\[
\prod_{1 \leq r < s \leq n} \left(1 - q \frac{1}{x_r x_s}\right)^{-1} = q^{\frac{1}{2}} \sum_{\delta \in \mathbb{N}^n} r_{SO_{2n}}(\delta)x^{-\delta} \quad \text{and} \quad \prod_{1 \leq r < s \leq n} \left(1 - q \frac{1}{x_r x_s}\right)^{-1} = q^{\frac{1}{2}} \sum_{\delta \in \mathbb{N}^n} r_{Sp_{2n}}(\delta)x^{-\delta}.
\]

Consider $\beta \in \mathbb{Z}^n$. By proceeding as for Lemma 3.2.3, we derive from (27) the equality

\[
D^\eta_{G}(\beta) = q^{\frac{1}{2}} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\beta|} r_{G}(\delta)P^GL_n,I(\beta + \delta).
\]

This implies the decomposition

\[
D^{\lambda,G}_{\mu(1),\ldots,\mu(r)}(q) = \sum_{\sigma \in S_n} (-1)^{f(\sigma)} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\beta|} r_{G}(\delta)q^{-|\beta|/2}P^GL_n,I(\sigma(\lambda + \rho + \delta) - (\mu + \rho))
\]

where $\beta = \sigma(\lambda + \rho) - (\mu + \rho)$ in the second sum. Since $|\beta| = |\lambda| - |\mu|$, we have

\[
D^{\lambda,G}_{\mu(1),\ldots,\mu(r)}(q) = q^{\frac{|\lambda| - |\mu|}{2}} \sum_{\sigma \in S_n} (-1)^{f(\sigma)} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\mu| - |\lambda|} r_{G}(\delta)P^GL_n,I(\sigma(\lambda + \rho + \sigma^{-1}(\delta)) - (\mu + \rho)).
\]
For any $\sigma \in S_n$, we have $\sigma^{-1}(\mathbb{N}^n) = \mathbb{N}^n$ and $r_{\hat{G}}(\delta) = r_{\hat{G}}(\sigma(\delta))$. Thus we obtain

$$\mathcal{D}_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda, G} (q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\mu| - |\lambda|} r_{\hat{G}}(\delta) p_{\mu^{G}, I} q^{r_{\hat{G}}(\sigma(\lambda + \rho + \delta) - (\mu + \rho))} =$$

$$q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\delta \in \mathbb{N}^n, |\delta| = |\mu| - |\lambda|} r_{\hat{G}}(\delta) K_{\lambda + \delta, \mu}^{G, I}(q).$$

Now by Lemma 3.2.2, $K_{\lambda + \delta, \mu}^{G, I}(q) = 0$ or there exists $\sigma \in S_n$ and $\nu \in \mathbb{P}^n$ such that $\nu = \sigma^{-1} \circ (\lambda + \delta)$. Since $\lambda + \delta$ has positive coordinates, $\nu$ cannot have negative coordinates and thus $\nu \in \mathbb{P}_n$. Then we have $|\nu| = |\lambda| + |\delta| = |\mu|$ and $\delta = \sigma(\nu + \rho) - \rho - \lambda$ and it follows that

$$\mathcal{D}_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda, G} (q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathbb{P}_n} \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} r_{\hat{G}}(\sigma(\nu + \rho) - (\rho - \lambda)) K_{\nu, \mu}^{G, I}(q).$$

Now we deduce from Lemma 3.2.2 applied with $M = r_{\hat{G}}$ and Proposition 3.2.2 that the equality

$$\sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} r_{\hat{G}}(\sigma(\nu + \rho) - (\rho - \lambda)) = [V(\nu + k\rho \hat{G}) : V(\lambda + k\rho)]^{G, I}$$

holds for any integer $k \geq \frac{|\mu| - |\lambda|}{2} = \frac{|\mu| - |\lambda|}{2}$. Since $\lambda$ and $\nu$ are partitions, we can write $[V(\nu + k\rho \hat{G}) : V(\lambda + k\rho)]^{G, I} = [V(\nu \hat{G}) : V(\lambda)^{G, I}]$ (see Remark after Proposition 3.2.2), this yields to the desired equality.

**Remark:** By setting $q = 1$ in the above identity, we obtain the identity

$$\mathcal{D}_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda, G} = \sum_{\nu \in \mathbb{P}_n} [V(\nu)^{G, I}] c_{\nu}^{G, I}$$

which can also be deduced from the decompositions

$$\mathcal{Y}_{\mu^{(1)}} \otimes \cdots \otimes \mathcal{Y}_{\mu^{(r)}} \simeq \bigoplus_{\nu \in \mathbb{P}_n} V (\nu^{G, I} \otimes c_{\mu^{(1)}, \ldots, \mu^{(r)}}^{G, I})$$

and

$$V (\nu^{G, I}) \simeq \bigoplus_{\lambda \in \mathbb{P}_n} [V(\nu)^{G, I}]^{G, I}$$

since we have the duality $[V(\nu)^{G, I}]^{G, I} = [V(\nu)^{G, I}]^{G, I}$ (see Theorem 3.1.1 of [7]).

When the $\mu^{(k)}$’s are rectangular partitions of decreasing heights, $\mathcal{D}_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda, G} (q)$ has nonnegative coefficients by (23) and Theorem 3.1.3. From the above theorem and (9) we obtain

$$\mathcal{D}_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda, G} (q) = q^{\frac{|\mu| - |\lambda|}{2}} \sum_{\nu \in \mathbb{P}_n} c_{\nu, \mu}^{\lambda, G, I}(q)$$

(29)

where

$$\hat{\gamma} = \begin{cases} (1, 1) & \text{if } G = SO_{2n} \\ (2) & \text{if } G = Sp_{2n} \\ (1) & \text{if } G = SO_{2n+1} \end{cases}$$

For completion, set $\mathcal{D}_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\nu, G, I} (q)$ and $\hat{\delta}_{G, I} = \emptyset$. Then the four families of polynomials $\mathcal{D}_{\mu^{(1)}, \ldots, \mu^{(r)}}^{\lambda, G, I}(q)$ can be classified by using the same symbols $\hat{\gamma} = \emptyset, (1, 1), (1, 1), (2)$ labelling the four families.
of stable one-dimensional sums. Write $X^{\lambda, \hat{G}_{\mu}(1), \ldots, \mu(r)}(q)$ for the one-dimensional sum defined from the affine crystal of kind $\hat{G}$ associated to $\mu(1), \ldots, \mu(r)$ and $\lambda$. By Theorem 3.1.3 we know that $X^{\nu, \emptyset}(q) = K_{\nu, \emptyset}^L(q)$. Then we derive from (29) and Theorem 4.4.2 that Conjecture 5 of [17] giving the decomposition of the one-dimensional sums of kind $\hat{G} = (1), (1, 1), (2)$ in terms of those of kind $\emptyset$ can be reformulated as follows:

**Conjecture 4.4.3** The $q$-analogue $D^{\lambda, G_{\mu(1)}, \ldots, \mu(r)}(q)$ coincide with the stable one-dimensional sum $X^{\lambda, \hat{G}_{\mu(1)}, \ldots, \mu(r)}(q)$ up to the multiplication by a power of $q$:

$$D^{\lambda, G_{\mu(1)}, \ldots, \mu(r)}(q) = K_{\lambda, \beta}^G(q) = q^* X^{\lambda, \hat{G}_{\mu(1)}, \ldots, \mu(r)}(q)$$

where $q^*$ is a power of $q$ depending on the normalization of the energy (or co-energy) function chosen on this crystal.

**Remark:** This conjecture has been proved in [12] when $\mu(1), \ldots, \mu(r)$ are row partitions of decreasing lengths. In this case the stable one-dimensional sums are stable limits of Lusztig $q$-analogues. Accordingly to the previous conjecture, the stable one-dimensional sums associated to nonexceptional affine Lie algebras are stable limits of generalized Lusztig $q$-analogues.

5 Question

In Theorem 4.4.2 the partitions $\mu(1), \ldots, \mu(r)$ are not supposed rectangular. Conjecture 3.1.4 suggests that it should exists, for all classical Lie groups, nonnegative $q$-analogues $D^{\mu(1), \ldots, \mu(r)}(q)$ of tensor product coefficients defined from the $r$-tuples $\mu = (\mu(1), \ldots, \mu(r))$ such that $\mu$ is a partition. It would be interesting to find a combinatorial interpretation of the $D^{\lambda, G_{\mu(1)}, \ldots, \mu(r)}(q)$ when $\mu(1), \ldots, \mu(r)$ are not rectangular.

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