The Complex Angle in Normed Spaces

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We consider a generalized angle in complex normed vector spaces. Its definition corresponds to the definition of the well known Euclidean angle in real inner product spaces. Not surprisingly it yields complex values as ‘angles’. This ‘angle’ has some simple properties, which are known from the usual angle in real inner product spaces. But to do ordinary Euclidean geometry real angles are necessary. We show that even in a complex normed space there are many pure real valued ‘angles’. The situation improves yet in inner product spaces. There we can use the known theory of orthogonal systems to find many pairs of vectors with real angles, and to do geometry which is based on the Greeks 2000 years ago.

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1 Introduction

We deal with complex vector spaces $X$ provided with a norm $\| \cdot \|$. To initiate the following constructions we begin with the special case of an inner product space $(X, < . | . >)$ over the complex field $\mathbb{C}$. It is well-known that the inner product can be expressed by the norm, namely for $\vec{x}, \vec{y} \in X$ we can write

$$< \vec{x} | \vec{y} > = \frac{1}{4} \cdot \left[ \| \vec{x} + \vec{y} \|^2 - \| \vec{x} - \vec{y} \|^2 + i \cdot \left( \| \vec{x} + i \cdot \vec{y} \|^2 - \| \vec{x} - i \cdot \vec{y} \|^2 \right) \right], \quad (1.1)$$

where the symbol ‘$i$’ means the imaginary unit.

For two vectors $\vec{x}, \vec{y} \neq \vec{0}$ it holds $< \vec{x} | \vec{y} > = \| \vec{x} \| \cdot \| \vec{y} \| \cdot < \frac{\vec{x}}{\| \vec{x} \|} | \frac{\vec{y}}{\| \vec{y} \|} >$. We use both facts and
an idea in [18] to generate a continuous product in all complex normed vector spaces \((X, \| \cdot \|)\), which is just the inner product in the special case of a complex inner product space.

**Definition 1.1.** Let \(\vec{x}, \vec{y}\) be two arbitrary elements of \(X\). In the case of \(\vec{x} = \vec{0}\) or \(\vec{y} = \vec{0}\) we set \(<\vec{x}|\vec{y}> : = 0\), and if \(\vec{x}, \vec{y} \neq \vec{0}\) (i.e. \(\|\vec{x}\| \cdot \|\vec{y}\| > 0\)) we define the complex number

\[
<\vec{x}|\vec{y}> := \frac{1}{4}\left[ \left( \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right)^2 - \left( \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right)^2 \right] + i \cdot \left( \left( \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right) \cdot \left( \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right) - \left( \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right) \right).
\]

\(\square\)

It is easy to show that the product fulfils the conjugate symmetry \(<\vec{x}|\vec{y}> = <\vec{y}|\vec{x}>\), where \(<\vec{y}|\vec{x}>\) means the complex conjugate, the positive definiteness \(<\vec{x}|\vec{x}> \geq 0\) and also \(<\vec{x}|\vec{x}> = 0\) only for \(\vec{x} = \vec{0}\), and the homogeneity for real numbers, \(<r \cdot \vec{x}|\vec{y}> = r \cdot <\vec{x}|\vec{y}>\), and the homogeneity for pure imaginary numbers, \(<r \cdot i\cdot \vec{x}|\vec{y}> = r \cdot <\vec{x}|\vec{y}>(r \cdot i\cdot \vec{x}|\vec{y})\), for \(\vec{x}, \vec{y} \in X\), \(r \in \mathbb{R}\). Further, for \(\vec{x} \in X\) it holds \(\|\vec{x}\| = \sqrt{<\vec{x}|\vec{x}>}\).

If we switch for a moment to real inner product spaces \((X, <\cdot|\cdot>_{real})\) we have for all \(\vec{x}, \vec{y} \neq \vec{0}\) the usual Euclidean angle

\[
\angle_{Euclid}(\vec{x}, \vec{y}) = \arccos\left( \frac{<\vec{x}|\vec{y}>_{real}}{\|\vec{x}\| \cdot \|\vec{y}\|} \right) = \arccos\left( \frac{1}{4} \cdot \left[ \left( \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right)^2 - \left( \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right)^2 \right] \right),
\]

which can be defined in terms of the norm, too.

For two vectors \(\vec{x}, \vec{y} \neq \vec{0}\) from a complex normed vector space \((X, \| \cdot \|)\) we are able to define an ‘angle’ which coincides with the definition of the Euclidean angle in real inner product spaces.

**Definition 1.2.** Let \(\vec{x}, \vec{y}\) be two elements of \(X\setminus\{\vec{0}\}\). We define the complex number

\[
\angle(\vec{x}, \vec{y}) := \arccos\left( \frac{<\vec{x}|\vec{y}>}{\|\vec{x}\| \cdot \|\vec{y}\|} \right).
\]

This number \(\angle(\vec{x}, \vec{y}) \in \mathbb{C}\) is called the angle of the pair \((\vec{x}, \vec{y})\).

\(\square\)

We state that the angle \(\angle(\vec{x}, \vec{y})\) is defined for all \(\vec{x}, \vec{y} \neq \vec{0}\). Since we deal with complex vector spaces it is not surprising that we get complex numbers as ‘angles’. For the definition we need the extension of the cosine and arccosine functions on complex numbers. We use two subsets \(\mathcal{A}\) and \(\mathcal{B}\) of the complex plane \(\mathbb{C}\), where

\[
\mathcal{A} := \{a + b \cdot i \in \mathbb{C} \mid 0 < a < \pi, \ b \in \mathbb{R}\} \cup \{0, \pi\}, \ \text{and} \ \ \ (1.2)
\]

\[
\mathcal{B} := \mathbb{C}\setminus\{r \in \mathbb{R} \mid r < -1 \ \text{or} \ r > +1\}. \ \ \ (1.3)
\]

We have two known homeomorphisms

\[
\cos: \mathcal{A} \xrightarrow{\cong} \mathcal{B} \ \text{and} \ \ \ \arccos: \mathcal{B} \xrightarrow{\cong} \mathcal{A}.
\]

The cosines of the ‘angles’ are are in the ‘complex square’ \(\mathcal{S} := \{r + s \cdot i \in \mathbb{C} \mid -1 \leq r, s \leq +1\} \subset \mathcal{B}\). The values of the ‘angles’ are from its image \(\arccos(\mathcal{S})\), which forms a symmetric hexagon in \(\mathcal{A}\). Its center is \(\pi/2\), two corners are \(0\) and \(\pi\). A third is, for instance, \(\arccos(1 + i) = (\pi/2) - (1/2) \cdot \left[ \arccos(\sqrt{2}) + i \cdot \log\left(\sqrt{5} + 2 + 2 \cdot \sqrt{5} + 2\right) \right] \approx 0.90 - i \cdot 1.1.\)

This ‘angle’ has eight comfortable properties (An 1) - (An 8) which are known from the Euclidean angle in real inner product spaces.
Theorem 1.3. In a complex normed space \((X, \| \cdot \|)\) the angle \(\angle\) has the following properties:

- (An 1) \(\angle\) is a continuous map from \((X \setminus \{\emptyset\})^2\) onto a subset of \(\arccos(S\Omega) \subset A\).

For elements \(\vec{x}, \vec{y} \neq \vec{0}\) it holds that

- (An 2) \(\angle(\vec{x}, \vec{x}) = 0\),
- (An 3) \(\angle(-\vec{x}, \vec{x}) = \pi\),
- (An 4) \(\angle(\vec{x}, \vec{y}) = \angle(\vec{y}, \vec{x})\),
- (An 5) for real numbers \(r, s > 0\) we have \(\angle(r \cdot \vec{x}, s \cdot \vec{y}) = \angle(\vec{x}, \vec{y})\),
- (An 6) \(\angle(-\vec{x}, -\vec{y}) = \angle(\vec{x}, \vec{y})\),
- (An 7) \(\angle(\vec{x}, \vec{y}) + \angle(-\vec{x}, \vec{y}) = \pi\).
- (An 8) For any two linear independent vectors \(\vec{x}, \vec{y}\) of \((X, \| \cdot \|)\) there is a continuous injective map \(\Theta : \mathbb{R} \rightarrow A\), \(t \mapsto \angle(\vec{x}, \vec{y} + t \cdot \vec{x})\). The limits are
  \[\lim_{t \rightarrow -\infty} \Theta(t) = \pi\] and \(\lim_{t \rightarrow \infty} \Theta(t) = 0\).

In the following table we note some angles and their cosines. We choose two elements \(\vec{x}, \vec{y} \neq \vec{0}\) of a complex normed space \((X, \| \cdot \|)\), and six suitable real numbers \(a, b, r, s, v, w\) with \(-\frac{\pi}{2} \leq a, v \leq \frac{\pi}{2}\) and \(-1 \leq r, s, \leq 1\), such that

\[
\angle(\vec{x}, \vec{y}) =: \frac{\pi}{2} + a + i \cdot b \in \mathbb{A},
\cos(\angle(\vec{x}, \vec{y})) = \cos \left(\frac{\pi}{2} + a + i \cdot b\right) =: r + i \cdot s \in \mathbb{B},
\angle(i \cdot \vec{x}, \vec{y}) =: \frac{\pi}{2} + v + i \cdot w \in \mathbb{A}.
\]

Note that the cosines of all angles in the table (third column) have the same modulus \(\sqrt{r^2 + s^2}\).

| pair of vectors | their angle \(\angle\) | the cosine of \(\angle\) | the angle for \(\vec{x} = \vec{y}\) | its cosine for \(\vec{x} = \vec{y}\) |
|----------------|-------------------------|-------------------------|------------------------|---------------------|
| \((\vec{x}, \vec{y})\) | \(\frac{\pi}{2} + a + i \cdot b\) | \(r + i \cdot s\) | 0 | 1 |
| \((-\vec{x}, \vec{x})\) | \(\frac{\pi}{2} - a - i \cdot b\) | \(-r - i \cdot s\) | \(\pi\) | \(-1\) |
| \((\vec{y}, \vec{x})\) | \(\frac{\pi}{2} + a - i \cdot b\) | \(r - i \cdot s\) | 0 | 1 |
| \((-\vec{y}, \vec{x})\) | \(\frac{\pi}{2} - a + i \cdot b\) | \(-r + i \cdot s\) | \(\pi\) | \(-1\) |
| \((i \cdot \vec{x}, \vec{y})\) | \(\frac{\pi}{2} + v + i \cdot w\) | \(-s + i \cdot r\) | \(\frac{\pi}{2} - i \cdot \log \sqrt{2} + 1\) | \(i\) |
| \((\vec{y}, i \cdot \vec{x})\) | \(\frac{\pi}{2} + v - i \cdot w\) | \(-s - i \cdot r\) | \(\frac{\pi}{2} + i \cdot \log \sqrt{2} + 1\) | \(-i\) |
| \((\vec{x}, i \cdot \vec{y})\) | \(\frac{\pi}{2} - v - i \cdot w\) | \(s - i \cdot r\) | \(\frac{\pi}{2} + i \cdot \log \sqrt{2} + 1\) | \(-i\) |
| \((i \cdot \vec{y}, \vec{x})\) | \(\frac{\pi}{2} - v + i \cdot w\) | \(s + i \cdot r\) | \(\frac{\pi}{2} - i \cdot \log \sqrt{2} + 1\) | \(i\) |

With given two vectors \(\vec{x}, \vec{y} \in X\) we consider as before the angle \(\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + a + i \cdot b\) with suitable real numbers \(a, b\). Now we express the complex number \(\angle(1 \cdot \vec{x}, \vec{y})\) in dependence of \(a\) and \(b\). For the presentation the real valued cosine and hyperbolic cosine are used, and their inverses arccosine and area hyperbolic cosine. We use abbreviations like \(\cos, \cosh, \arccos, \text{ and } \arcosh\).

The result is as follows:

Theorem 1.4. In a complex normed space \((X, \| \cdot \|)\) we take two elements \(\vec{x}, \vec{y} \neq \vec{0}\). We assume the angle \(\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + a + i \cdot b \in \mathbb{A}\), i.e. \(-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}\). If \(a = -\frac{\pi}{2}\) or \(a = \frac{\pi}{2}\) since \(\angle(\vec{x}, \vec{y}) \in \mathbb{A}\) it follows \(b = 0\). We get

\[
\angle(i \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + \frac{1}{2} \cdot \left[-\text{sgn}(b) \cdot \arccos(H_-) + i \cdot \text{sgn}(a) \cdot \arcosh(H_+\right),
\]
with the abbreviations $H_-$ and $H_+$, where

$$H_\pm := \sqrt{\left[ \cos^2\left(\frac{\pi}{2} + a\right) + \cosh^2(b) - 2 \right]^2 + 4 \cos^2\left(\frac{\pi}{2} + a\right) \cdot \cosh^2(b)} \pm \left[ \cos^2\left(\frac{\pi}{2} + a\right) + \cosh^2(b) - 1 \right].$$

It is worthwhile to look at special cases. We consider a real angle $\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + a$ (i.e. $b = 0$), and an angle on the vertical axis $x = \frac{\pi}{2}$, this means $\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + \cdot \cdot \cdot$ (i.e. $a = 0$).

**Corollary 1.5.** For a pure real angle $\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + a$ with $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$, i.e. $b = 0$, we get a complex angle $\angle(i \cdot \vec{x}, \vec{y})$ with a real part $\pi/2$,

$$\angle(i \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + i \cdot \frac{1}{2} \cdot \text{sgn}(a) \cdot \arccos\left(2 \cdot \cos^2\left(\frac{\pi}{2} + a\right) + 1\right)$$

$$= \frac{\pi}{2} + i \cdot \text{sgn}(a) \cdot \log \left[ \sqrt{\cos^2\left(\frac{\pi}{2} + a\right) + 1} + \left| \cos\left(\frac{\pi}{2} + a\right) \right| \right].$$

**Corollary 1.6.** For an angle $\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + i \cdot \cdot \cdot$ (i.e. $a = 0$) we get a pure real angle $\angle(i \cdot \vec{x}, \vec{y})$,

$$\angle(i \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} - \frac{1}{2} \cdot \text{sgn}(b) \cdot \arccos[3 - 2 \cdot \cosh^2(b)]$$

$$= \frac{\pi}{2} - \frac{1}{2} \cdot \text{sgn}(b) \cdot \arccos[2 - \cosh(2 \cdot b)] = \frac{\pi}{2} - \arcsin[\sinh(b)] = \arccos[\sinh(b)].$$

Now we look for pairs $(\vec{x}, \vec{y})$ with a real valued angle $\angle(\vec{x}, \vec{y})$. For a complex normed vector space $(X, \| \cdot \|)$ we define the set of pairs

$$\mathcal{R}_X := \{(\vec{x}, \vec{y}) \mid \vec{x}, \vec{y} \in X, \vec{x}, \vec{y} \neq \vec{0}, \text{ and } \angle(\vec{x}, \vec{y}) \in \mathbb{R}\}.$$

Let us take two vectors $\vec{x}, \vec{y} \neq \vec{0}$. We can prove that there is a real number $\varphi \in [0, 2\pi]$ such that $(e^{i\varphi} \cdot \vec{x}, \vec{y}) \in \mathcal{R}_X$, i.e. the pair has a pure real angle $\angle(e^{i\varphi} \cdot \vec{x}, \vec{y})$.

This fact ensures the existence of many real valued angles even in complex normed spaces. The situation will improve further in the special case of complex vector spaces provided with an inner product $\langle . | . \rangle$. We discuss this case in the next section.

The properties of complex inner product spaces $(X, \langle . | . \rangle)$ have been studied extensively, and such spaces have many applications in technology and physics.

To do ordinary Euclidean geometry we need real valued angles. The idea is simple. We take an orthogonal basis $T$ of $(X, \langle . | . \rangle)$, and we generate $\mathcal{L}(\mathbb{R})(T)$, the set of all finite real linear combinations of elements of $T$. Let

$$\mathcal{L}(\mathbb{R})(T) := \left\{ \sum_{i=1}^{n} r_i \cdot \vec{x}_i \mid n \in \mathbb{N}, r_1, r_2, \ldots, r_n \in \mathbb{R}, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in T \right\}.$$

If $X$ is a Hilbert space, i.e. it is complete, we even can use Cauchy sequences (with real coefficients) from elements of $T$. It means that we can take the closure $\overline{\mathcal{L}(\mathbb{R})(T)}$ of $\mathcal{L}(\mathbb{R})(T)$. This creates a real linear subspace $\overline{\mathcal{L}(\mathbb{R})(T)}$ of $X$, in which all angles are real.

After that we consider complex vector spaces $X$ of finite complex dimension $n \in \mathbb{N}$. Their real dimension is $2 \cdot n$, and we state in Proposition 4.4 that the maximal dimension of a real subspace of $X$ with all-real angles is $n$. The real span $\mathcal{L}(\mathbb{R})(T)$ of an orthogonal basis $T \subset X$ yields an example.

Finally we demonstrate two examples of ordinary Euclidean geometry in complex inner product spaces, and to do this we show that real angles are useful.

Note that the idea of this ‘angle’ are treated first for real normed spaces in [19] and [20].
2 General Definitions

Let $X = (X, \tau)$ be an arbitrary complex vector space, that means that the vector space $X$ is provided with a topology $\tau$ such that the addition of two vectors and the multiplication with complex numbers are continuous. Further let $\| \cdot \|$ denote a norm on $X$, i.e. there is a continuous map $\| \cdot \| : X \to \mathbb{R}^+ \cup \{0\}$ which fulfills the following axioms $\| z \cdot \vec{x} \| = |z| \cdot \| \vec{x} \|$ (‘absolute homogeneity’), $\| \vec{x} + \vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \|$ (‘triangle inequality’), and $\| \vec{x} \| = 0$ only for $\vec{x} = 0$ (‘positive definiteness’), for $\vec{x}, \vec{y} \in X$ and $z \in \mathbb{C}$.

Let $<\cdot, \cdot> : X^2 \to \mathbb{C}$ be a continuous map from the product space $X \times X$ into the field $\mathbb{C}$. Such a map is called a product. We notice the following four conditions.

(1): For all $z \in \mathbb{C}$ and $\vec{x}, \vec{y} \in X$ it holds $< z \cdot \vec{x} | \vec{y} > = z \cdot < \vec{x} | \vec{y} >$ (‘homogeneity’),

(2): for all $\vec{x}, \vec{y} \in X$ it holds $< \vec{x} | \vec{y} > = < \vec{y} | \vec{x} >$ (‘conjugate symmetry’),

(3): for $\vec{x} \in X, \vec{x} \not= 0$ we have a real number $< \vec{x} | \vec{x} > > 0$, (‘positive definiteness’),

(4): for all $\vec{x}, \vec{y}, \vec{z} \in X$ it holds $< \vec{x} | \vec{y} + \vec{z} >= < \vec{x} | \vec{y} > + < \vec{x} | \vec{z} >$ (‘linearity in the second component’).

If $<\cdot, \cdot>$ fulfills all properties (1), (2), (3), (4), the map $<\cdot, \cdot>$ is an inner product on $X$. We call the pair $(X, <\cdot, \cdot>)$ a complex inner product space.

It is well known that in a complex normed space $(X, \| \cdot \|)$ its norm $\| \cdot \|$ generates an inner product $<\cdot, \cdot>$ by equation (1.1) if and only if it holds the following parallelogram identity:

For $\vec{x}, \vec{y} \in X$ we have the equation $\| \vec{x} + \vec{y} \|^2 + \| \vec{x} - \vec{y} \|^2 = 2(\| \vec{x} \|^2 + \| \vec{y} \|^2)$.

Assume that the complex vector space $X$ is provided with a norm $\| \cdot \|$, and further there is a product $<\cdot, \cdot> : X \times X \to \mathbb{C}$. We say that the triple $(X, \| \cdot \|, <\cdot, \cdot>)$ satisfies the Cauchy-Schwarz-Bunjakovsky Inequality or CSB inequality if and only if for $\vec{x}, \vec{y} \in X$ there is the inequality

$$| < \vec{x} | \vec{y} > | \leq \| \vec{x} \| \cdot \| \vec{y} \| .$$

In the following section we need some complex valued functions like the cosine, sine, arcus sine, arcsin, et cetera. We abbreviate them by cos, sin, arccos, arcsin.

In the introduction we already defined two subset $B$ and $A$ of the complex plane $\mathbb{C}$, and we asserted that there are two homeomorphisms $\cos : A \xrightarrow{\sim} B$ and $\arccos : B \xrightarrow{\sim} A$. Note that $A$ contains only inner points except two boundary points 0 and $\pi$, while $B$ has 1 and $-1$.

In the next definition we express the complex functions in detail by known real valued cos, sin, arccos, arccosh, log and exp functions.

Recall the three values of the signum function, $\text{sgn}(0) = 0$, $\text{sgn}(x) = 1$ for positive real numbers $x$, and $\text{sgn}(x) = -1$ for negative $x$. We abbreviate the exponential function by $e^x := \exp(x)$. Note that in the next section we prove explicitly that the defined arccosine is truly the inverse function of the cosine.

Definition 2.1. For a number $z = a + i \cdot b \in \mathbb{C}$ its complex cosine and sine can be defined explicitly by

$$\cos(a + i \cdot b) := \frac{1}{2} \left[ \cos(a) \cdot \left( e^b + \frac{1}{e^b} \right) - i \cdot \sin(a) \cdot \left( e^b - \frac{1}{e^b} \right) \right],$$

$$\sin(a + i \cdot b) := \frac{1}{2} \left[ \sin(a) \cdot \left( e^b + \frac{1}{e^b} \right) + i \cdot \cos(a) \cdot \left( e^b - \frac{1}{e^b} \right) \right].$$

For a shorter presentation we can use the real hyperbolic cosine and hyperbolic sine. Their abbreviations are the symbols cosh and sinh, they are defined by

$$\cosh(x) := \frac{1}{2} \left( e^x + \frac{1}{e^x} \right) \quad \text{and} \quad \sinh(x) := \frac{1}{2} \left( e^x - \frac{1}{e^x} \right), \quad \text{for } x \in \mathbb{R}.$$
For \( r \in \mathbb{R} \) and \( s \in \mathbb{R} \) the functions arcsine and arccosine are

\[
\arcsin(r + i \cdot s) := \frac{1}{2} \cdot \left[ \text{sgn}(r) \cdot \arccos(G_{-}) + i \cdot \text{sgn}(s) \cdot \arccosh(G_{+}) \right], \quad \text{and} \\
\arccos(r + i \cdot s) := \frac{1}{2} - \arcsin(r + i \cdot s).
\]

Here we use the abbreviations

\[ G_{-} := \sqrt{(r^2 + s^2 - 1)^2 + 4 \cdot s^2} - (r^2 + s^2), \quad \text{and} \quad G_{+} := \sqrt{(r^2 + s^2 - 1)^2 + 4 \cdot s^2} + (r^2 + s^2), \]

and recall that the symbol \( \arccos \) means the real area hyperbolic cosine, which is the inverse of the real hyperbolic cosine,

\[ \arccosh(x) := \log \left( x + \sqrt{x^2 - 1} \right) \quad \text{for real numbers} \ x \geq 1. \]

\[ \square \]

We mention a few well-known consequences.

We assume the real cosine and sine functions and the complex \( e\)xp function, all defined by its power series. Since it holds \( e^{ix} = \cos(x) + i \cdot \sin(x) \) for real numbers \( r \), with the above Definition (2.1) of the complex sine and cosine we can deduce Euler’s formula \( e^{iz} = \cos(z) + i \cdot \sin(z) \) for all \( z \in \mathbb{C} \). After that it is easy to prove the identities

\[ \cos(z) = \frac{1}{2} \cdot \left[ e^{iz} + e^{-iz} \right] \quad \text{and} \quad \sin(z) = \frac{1}{2i} \cdot \left[ e^{iz} - e^{-iz} \right]. \]

We notice the equations \( \arccosh \circ \cosh(x) = x \) for real \( x \geq 0 \), and \( \cosh \circ \arccosh(x) = x \) for \( x \geq 1 \). Further, the complex cosine can be written as \( \cos(a + i \cdot b) = \cos(a) \cdot \cosh(b) - i \cdot \sin(a) \cdot \sinh(b) \), while the complex sine function is \( \sin(a + i \cdot b) = \sin(a) \cdot \cosh(b) + i \cdot \cos(a) \cdot \sinh(b) \).

### 3 Complex Normed Spaces

First we prove that the cosine and arccosine as they are written in Definition (2.1) are actually inverse functions.

**Lemma 3.1.** For all \( z \in \mathbb{B} \) it holds \( \cos \circ \arccos(z) = z \).

**Proof.** We take an arbitrary element \( z = r + i \cdot s \in \mathbb{B} \). We use the abbreviations \( G_{-} \) and \( G_{+} \) of Definition (2.1), and with easy calculations we get

\[
(1 - G_{-}) \cdot (G_{+} + 1) = 4 \cdot r^2, \quad \text{and} \quad (1 + G_{-}) \cdot (G_{+} - 1) = 4 \cdot s^2.
\]

We assume for \( z = r + i \cdot s \in \mathbb{B} \) that both \( r \) and \( s \) are positive. We use Definition (2.1) and elementary calculus, and we compute

\[
\cos(\arccos(z)) = \cos \left[ \arccos(r + i \cdot s) \right] = \cos \left[ \frac{\pi}{2} - \frac{1}{2} \cdot \arccos(G_{-}) - i \cdot \frac{1}{2} \cdot \arccosh(G_{+}) \right]
\]

\[
= \cos \left( \frac{\pi}{2} - \frac{1}{2} \cdot \arccos(G_{-}) \right) \cdot \cosh \left( -\frac{1}{2} \cdot \arccosh(G_{+}) \right) - i \cdot \sin \left( \frac{\pi}{2} - \frac{1}{2} \cdot \arccos(G_{-}) \right) \cdot \sinh \left( -\frac{1}{2} \cdot \arccosh(G_{+}) \right) \quad \text{(note \( \sin(-x) = -\sin(x) \))}
\]

\[
= \sin \left( \frac{1}{2} \cdot \arccos(G_{-}) \right) \cdot \cosh \left( \frac{1}{2} \cdot \arccosh(G_{+}) \right) + i \cdot \cos \left( \frac{1}{2} \cdot \arccos(G_{-}) \right) \cdot \sinh \left( \frac{1}{2} \cdot \arccosh(G_{+}) \right)
\]

\[
= \sqrt{\frac{1}{2} \cdot (1 - G_{-}) \cdot \sqrt{\frac{1}{2} \cdot (G_{+} + 1) + i \cdot \sqrt{\frac{1}{2} \cdot (1 + G_{-}) \cdot \sqrt{\frac{1}{2} \cdot (G_{+} - 1)}})}
\]

\[
= \frac{1}{2} \cdot \sqrt{(1 - G_{-}) \cdot (G_{+} + 1)} + i \cdot \frac{1}{2} \cdot \sqrt{(1 + G_{-}) \cdot (G_{+} - 1)} = r + i \cdot s = z.
\]
The other three cases are \( r \cdot s = 0 \), \( r \cdot s < 0 \), and both \( r \) and \( s \) are negative. By noting the signs they work in the same manner, and the lemma is established.

**Proposition 3.2.** We have two homeomorphisms

\[
\cos : A \xrightarrow{\cong} B \quad \text{and} \quad \arccos : B \xrightarrow{\cong} A,
\]

and we get two identities

\[
\cos \circ \arccos = \text{id}_B \quad \text{and} \quad \arccos \circ \cos = \text{id}_A.
\]

**Proof.** By Definition (2.1) we have two continuous maps, \( \cos : A \to B \) and \( \arccos : B \to A \). We just proved \( \cos \circ \arccos(z) = z \) for all \( z \in B \). This shows that the arccosine function is injective on its domain \( B \), and the cosine function is surjective on its codomain \( B \). If we consider all possible cases for \( a \) and \( b \), for \( a + i \cdot b \in A \), we see that the cosine is injective on the domain \( A \). It follows that the cosine function is a bijective map \( A \to B \). We have \( \cos \circ \arccos = \text{id}_B \).

Therefore it holds \( \arccos = \cos^{-1} \circ \text{id}_B = \cos^{-1} \), and we get that the arccosine is also a bijective map, and it is the inverse of the cosine. Because both functions are continuous, they are both homeomorphisms.

We describe that the cosine and arccosine functions have a ‘central symmetrical behavior’, they respect the ‘center points’ \( \pi/2 \) of \( A \) and \( 0 \) of \( B \), respectively. Note that for each complex number \( z \) we can write \( z = \frac{\pi}{2} + a + i \cdot b \), with a suitable real number \( a \). It means that the real part of \( z \) is \( \frac{\pi}{2} + a \).

**Proposition 3.3.** For each complex number \( z \) in \( A \) we write \( z = \frac{\pi}{2} + a + i \cdot b \), with a suitable real number \( -\frac{\pi}{2} \leq a \leq \frac{\pi}{2} \). If \( \cos \left( \frac{\pi}{2} + a + i \cdot b \right) = r + i \cdot s \) it follows

\[
\cos \left( \frac{\pi}{2} - a - i \cdot b \right) = -r - i \cdot s.
\]

Correspondingly, for a number \( r + i \cdot s \in B \) with \( \arccos(r + i \cdot s) = \frac{\pi}{2} + a + i \cdot b \) it holds

\[
\arccos(-r - i \cdot s) = \frac{\pi}{2} - a - i \cdot b.
\]

**Proof.** For the arccosine we see this immediately from Definition (2.1). Note that both the real arccosine and the real area hyperbolic cosine have a non-negative image. Since the cosine is the inverse function of the arccosine, it must act as it is claimed in the proposition.

Let \( (X, \| \cdot \|) \) be a complex normed vector space. In Definition (1.1) we defined a continuous product \( < . | . > \) on \( X \). In the introduction we already mentioned that this is the inner product in the case that the norm \( \| \cdot \| \) generates an inner product by equation (1.1). Also we noticed some properties of this product. We discuss them now.

**Proposition 3.4.** For all vectors \( \bar{x}, \bar{y} \in (X, \| \cdot \|) \) and for real numbers \( r \) the product \( < . | . > \) has the following properties

(a) \( < \bar{x} | \bar{y} > = < \bar{y} | \bar{x} > \) \hspace{1cm} (conjugate symmetry),
(b) \( < \bar{x} | \bar{x} > \geq 0 \), \( < \bar{x} | \bar{x} > = 0 \) only for \( \bar{x} = \bar{0} \) \hspace{1cm} (positive definiteness),
(c) \( < r \cdot \bar{x} | \bar{y} > = r \cdot < \bar{x} | \bar{y} > \) \hspace{1cm} (homogeneity for real numbers),
(d) \( < r \cdot i \cdot \bar{x} | \bar{y} > = r \cdot i \cdot < \bar{x} | \bar{y} > \) \hspace{1cm} (homogeneity for pure imaginary numbers),
(e) \( \| \bar{x} \| = \sqrt{< \bar{x} | \bar{x} >} \) \hspace{1cm} (the norm can be expressed by the product).

**Proof.** The proofs for (a) and (b) are trivial. The point (c) is trivial for positive \( r \). The proof of \( < \bar{x} | \bar{y} > = - r \cdot i \cdot < \bar{x} | \bar{y} > \) is easy, and (c) follows immediately. The point (d) is similar to (c), and (e) is clear.
In Definition 3.5 we defined the angle $\angle(\vec{x}, \vec{y})$, we wrote $\angle(\vec{x}, \vec{y}) := \arccos \left( \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \|\vec{y}\|} \right)$

$$\angle(\vec{x}, \vec{y}) = \arccos \left( \frac{1}{2} \cdot \left[ \left( \|\vec{x}\| + \|\vec{y}\| \right)^2 - \|\vec{x} - \vec{y}\|^2 + \mathbf{i} \cdot \left( \left( \|\vec{x}\| + \|\vec{y}\| \right)^2 - \|\vec{x} - \vec{y}\|^2 \left( \|\vec{x}\| - \mathbf{i} \cdot \|\vec{y}\| \right)^2 \right) \right] \right).$$

This complex number $\angle(\vec{x}, \vec{y})$ is called the complex angle of the pair $(\vec{x}, \vec{y})$.

**Lemma 3.5.** For a pair $\vec{x}, \vec{y} \neq \vec{0}$ in a complex normed space $(X, \|\cdot\|)$ it holds that both the real and imaginary part of $\cos(\angle(\vec{x}, \vec{y}))$ are in the interval $[-1, 1]$.

**Proof.** The lemma can be proven easily with the triangle inequality. \qed

**Corollary 3.6.** Lemma 3.5 means that $\{ r + s + t \in \mathbb{C} \mid -1 \leq r, s, t \leq +1 \}$, i.e. $|\cos(\angle(\vec{x}, \vec{y}))| \leq \sqrt{2}$. It follows $\cos(\angle(\vec{x}, \vec{y})) \in B$ and $\angle(\vec{x}, \vec{y}) \in A$, i.e. the angle $\angle(\vec{x}, \vec{y})$ is defined for each pair $\vec{x}, \vec{y} \neq \vec{0}$.

**Corollary 3.7.** The values of the ‘angles’ are from the image $\arccos(SQ)$, which forms a symmetric hexagon in $A$. Its center is $\pi/2$, two corners are 0 and $\pi$. A third is, for instance, $\arccos(1+i) = (\pi/2) - (1/2) \cdot \arccos(\sqrt{5} - 2) + 1 \cdot \log \left( \sqrt{5} + 2 + 2 \cdot \sqrt{5} + 2 \right) \approx 0.90 - i \cdot 1.1$.

We get the other three corners easily with the aid of Proposition 3.3.

Now we prove that a stronger property fails.

**Theorem 3.8.** Generally the CSB inequality is not fulfilled. To show this we construct examples of complex normed spaces $X$ with elements $\vec{x}, \vec{y}$ such that

$$|\langle \vec{x}, \vec{y} \rangle| > \|\vec{x}\| \cdot \|\vec{y}\| \text{ or, in other words, } 1 < |\cos(\angle(\vec{x}, \vec{y}))|.$$  

**Proof.** The construction needs some preparations. First, let $S \subset X$ be any subset of the complex vector space $X \neq \{\vec{0}\}$. Note that in the beginning X carries no norm.

**Definition 3.9.** If $S$ has the property that for any vector $\vec{x} \in S$, $\vec{x} \neq \vec{0}$ there is a real number $\omega > 0$ such that $\omega \cdot \vec{x} \notin S$ for all $\omega > 0$, we call the set $S$ ‘locally bounded’.

If $S$ has the property that for any vector $\vec{x} \in X$, $\vec{x} \neq \vec{0}$ there is a real number $\varepsilon > 0$ such that $\varepsilon \cdot \vec{x} \in S$, the set $S$ is called ‘absorbing’.

**Definition 3.10.** We say that $S \subset X$ has the ‘MINKOWSKI-property’ if and only if $S$ is both locally bounded and absorbing. In this case we can define a positive definite functional $\|\cdot\|_S$ on $X$, i.e. $\|\cdot\|_S : X \to \mathbb{R}^+ \cup \{0\}$, the so-called ‘MINKOWSKI-functional’. We define

$$\|\vec{x}\|_S := \inf \{ \eta > 0 \mid \eta^{-1} \cdot \vec{x} \in S \} \text{ for } \vec{x} \neq \vec{0}, \text{ and } \|\vec{0}\|_S := 0. \quad (3.1)$$

If in addition the set $S$ is balanced (i.e. for $\vec{x} \in S$ and $\phi \in [0, 2\pi]$ it follows $e^{i\phi} \cdot \vec{x} \in S$) and convex (i.e. for two elements $\vec{x}, \vec{y} \in S$ the line segment $\text{SEG}$ between $\vec{x}$ and $\vec{y}$, i.e. the set $\text{SEG} := \{ s \cdot \vec{x} + t \cdot \vec{y} \mid 0 \leq s, t \leq 1, \text{ and } s + t = 1 \}$, is a subset of $S$), the just constructed MINKOWSKI-functional $\|\cdot\|_S$ is a norm. This means that the pair $(X, \|\cdot\|_S)$ is a complex normed vector space. (Compare 15 or 21.)

We need some maps which extend subsets of $X$.

**Definition 3.11.** Let $S \neq \emptyset$ be any subset of the complex vector space $X$. We define

$$\text{twist}(S) := \left\{ e^{i\phi} \cdot \vec{x} \mid \phi \in \mathbb{R} \text{ and } \vec{x} \in S \right\},$$

$$\text{conv}(S) := \left\{ \sum_{j=1}^{k} t_j \cdot \vec{x}_j \mid k \in \mathbb{N}, \ 0 \leq t_j \leq 1, \ \vec{x}_j \in S \text{ for all } 1 \leq j \leq k, \text{ and } \sum_{j=1}^{k} t_j = 1 \right\}.$$  

We complete the definition with $\text{twist}(\emptyset) := \text{conv}(\emptyset) := \{\vec{0}\}$. 

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Lemma 3.12. The set $S$ contains four elements.

- $S \subset \text{twist}(S) \cap \text{conv}(S)$,
- $\text{twist} \circ \text{twist} = \text{twist}, \text{conv} \circ \text{conv} = \text{conv}$,
- $\text{twist} \circ \text{conv}(S) \subset \text{conv} \circ \text{twist}(S)$,
- $\text{twist} \circ \text{conv} \circ \text{twist} = \text{conv} \circ \text{twist}$,
- $\text{conv}(S)$ is convex, $\text{twist}(S)$ is balanced, $\text{conv} \circ \text{twist}(S)$ is both convex and balanced.
- If $S$ is locally bounded, $\text{twist}(S)$ is locally bounded, too.

The reader may look for a locally bounded set $S$ such that $\text{conv}(S)$ is not locally bounded.

We are constructing examples of complex normed vector spaces which do not fulfill the CSB inequality to prove Theorem (3.8). As the vector space we take the easiest non-trivial space, the two-dimensional space $\mathbb{C}^2 = \{(v \cdot (1,0) + w \cdot (0,1) \mid v, w \in \mathbb{C})\}$. For each positive real number $r$, i.e. $r > 0$, we construct a norm $\| \cdot \|_r$ on $\mathbb{C}^2$. We define the finite subset $S_r$ of $\mathbb{C}^2$, the set $S_r$ contains four elements

$$S_r := \left\{ \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} r \\ -r \end{array} \right), \left( \begin{array}{c} r \\ -i \cdot r \end{array} \right) \right\}.$$  

We build the convex and balanced set

$$M_r := \text{conv} \circ \text{twist} (S_r).$$  

Lemma 3.12. The set $M_r$ fulfils the MINKOWSKI-property of Definition (3.10).

Proof. Let $\vec{x} \neq \vec{0}$. The subset $\{(1,0),(0,1)\} \subset S_r$ is already a basis of $\mathbb{C}^2$. Hence there is a representation $\vec{x} = v \cdot (1,0) + w \cdot (0,1)$ with suitable complex numbers $v, w$. We can write this as $\vec{x} = a \cdot e^{i \phi} \cdot (1,0) + b \cdot e^{i \psi} \cdot (0,1)$ with non-negative real numbers $a, b$, and $\phi, \psi \in [0,2\pi]$. We take the sum $\chi := a + b > 0$, and we write

$$\vec{x} = \chi \cdot \left[ \frac{a}{\chi} \cdot e^{i \phi} \cdot \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \frac{b}{\chi} \cdot e^{i \psi} \cdot \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right].$$  

Since $(a/\chi) \cdot e^{i \phi} \cdot (1,0) + (b/\chi) \cdot e^{i \psi} \cdot (0,1) \in M_r$ it follows $(1/\chi) \cdot \vec{x} \in M_r$, i.e. $M_r$ is absorbing.

Since $S_r$ is finite, it is locally bounded. This property will be transported by $\text{twist}$ and in this case also by $\text{conv}$, since $S_r$ is finite. Hence $M_r$ is locally bounded, too. The lemma is proven.  

Now we define for each number $r \in \mathbb{R}^+$ a norm on $\mathbb{C}^2$. We use the MINKOWSKI-functional of Definition (3.10). Let for $\vec{x} \in \mathbb{C}^2$

$$\| \vec{x} \|_r := \inf \{ \eta > 0 \mid \eta^{-1} \cdot \vec{x} \in M_r \} \text{ for } \vec{x} \neq \vec{0}, \text{ and } \| \vec{0} \|_r := 0.$$  

The set $M_r$ is convex and balanced, and by Lemma (3.12) it satisfies the MINKOWSKI-property. We get that the pair $(\mathbb{C}^2,\| \cdot \|_r)$ is a normed space, please see the remark after Definition (3.10).
Now we take two vectors \( \vec{a} := (1, 0) \) and \( \vec{b} := (0, 1) \), both are unit vectors in the spaces \( \mathbb{C}^2, \| \cdot \|_r \), for all \( r \in \mathbb{R}^+ \). We consider the product \( \langle \vec{a} | \vec{b} \rangle_r := \langle \vec{a} | \vec{b} \rangle \). By Definition [1.1] we have
\[
\langle \vec{a} | \vec{b} \rangle_r = \frac{1}{4} \left[ \| \vec{a} + \vec{b} \|_r^2 - \| \vec{a} - \vec{b} \|_r^2 + i \cdot \left( \| \vec{a} + i \cdot \vec{b} \|_r^2 - \| \vec{a} - i \cdot \vec{b} \|_r^2 \right) \right] \quad (3.2)
\]
\[
= \frac{1}{2} \left( \| (1, 1) \|_r^2 - \| (1, -1) \|_r^2 + i \cdot \left( \| (1, i) \|_r^2 - \| (1, -i) \|_r^2 \right) \right) . \quad (3.3)
\]
We consider the norms of four vectors \( \| (1, 1) \|_r, \| (1, -1) \|_r, \| (1, i) \|_r, \| (1, -i) \|_r \). Since we have convex combinations
\[
\frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in M_r, \quad \frac{1}{2} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 0 \\ i \end{pmatrix} \in M_r,
\]
it follows \( \| (1, 1) \|_r = 2 = \| (1, i) \|_r \).

By definition of the set \( M_r \) it contains two vectors \( (r, -r) \), \((r, -i \cdot r)\). For \( r \geq 1/2 \) both are unit vectors in \( \mathbb{C}^2, \| \cdot \|_r \). We have \( r \cdot (1, -1) = (r, -r) \) and \( r \cdot (1, -i) = (r, -i \cdot r) \). If \( r \) is sufficient positive, i.e. \( r \geq 1/2 \), we get the two norms \( \| (1, 1) \|_r = 1/r = \| (1, -i) \|_r \). Finally we get for the product \( \langle \vec{a} | \vec{b} \rangle_r \) the result
\[
\langle \vec{a} | \vec{b} \rangle_r = \frac{1}{4} \cdot \left[ 2^2 - \left( \frac{1}{r} \right)^2 + i \cdot \left( 2^2 - \left( \frac{1}{r} \right)^2 \right) \right], \quad (3.4)
\]
this means for the modulus
\[
| \langle \vec{a} | \vec{b} \rangle_r | = \sqrt{\left[ 1 - \left( \frac{1}{2 \cdot r} \right)^2 \right]^2 + \left[ 1 - \left( \frac{1}{2 \cdot r} \right)^2 \right]^2} = \sqrt{2} \cdot \left[ 1 - \left( \frac{1}{2 \cdot r} \right)^2 \right]. \quad (3.5)
\]
Obviously, we get the limit \( \lim_{r \to \infty} (| \langle \vec{a} | \vec{b} \rangle_r |) = \sqrt{2} \). Since \( \| \vec{a} \|_r = 1 = \| \vec{b} \|_r \) for all \( r \), this is sufficient to prove Theorem [3.8].

Because the expression in equation [3.2] yields a characteristic property of complex normed spaces, we add a definition and a proposition.

**Definition 3.13.** Let \( (X, \| \cdot \|) \) be a complex normed vector space. We define a positive number \( D(X, \| \cdot \|) \), the ‘Deformation’ of \( (X, \| \cdot \|) \). Let \( D(\{ \vec{0} \}, \| \cdot \|) := 1 \), and for \( X \neq \{ \vec{0} \} \) we define
\[
D(X, \| \cdot \|) := \sup \left\{ \text{the modulus of } \left( \langle \vec{a} | \vec{b} \rangle \right) \mid \vec{a}, \vec{b} \text{ are unit vectors in } (X, \| \cdot \|) \right\} \nonumber
\]
\[
= \sup \left\{ \frac{1}{4} \cdot \sqrt{\left[ \| \vec{a} + \vec{b} \|_r^2 - \| \vec{a} - \vec{b} \|_r^2 \right]^2 + \left[ \| \vec{a} + i \cdot \vec{b} \|_r^2 - \| \vec{a} - i \cdot \vec{b} \|_r^2 \right]^2} \mid \vec{a}, \vec{b} \in X, \| \vec{a} \| = 1 = \| \vec{b} \| \right\} .
\]

**Proposition 3.14.** Let the pair \( (X, \| \cdot \|) \) be a complex normed vector space. It holds
\[
1 \leq D(X, \| \cdot \|) \leq \sqrt{2} .
\]

**Proof.** With \( \vec{b} := \vec{a} \) or \( \vec{b} := i \cdot \vec{a} \) we get \( 1 \leq D(X, \| \cdot \|) \). From Lemma [3.5] it follows \( D(X, \| \cdot \|) \leq \sqrt{2} \).

The above considerations suggest the following equivalence. One direction is trivial.

**Conjecture 3.15.** In a complex normed vector space \( (X, \| \cdot \|) \) the norm \( \| \cdot \| \) is generated by an inner product by Definition [1.1] if and only if \( D(X, \| \cdot \|) = 1 \).
In the introduction we stated Theorem (1.3). Here we catch up the proof.

Proof. The property (An 1) is a consequence of Lemma (3.3). In Corollary (3.4) it is said that the angle is always defined, and that the image of the map $\angle$ is situated in $A$.

The five points (An 2) - (An 6) are rather trivial. We use properties of the product $\langle \cdot, \cdot \rangle$ from Proposition (3.4), and properties of the arccosine. The next point (An 7) is also easy.

We want to prove $\angle(x, y) + \angle(-x, y) = \pi$, for $x, y \neq 0$. We use Proposition (3.3) and $\angle(-x, y) = -\angle(x, y)$ from Proposition (3.4). If $\angle(x, y) = \arccos(r + i \cdot s) = \frac{\pi}{2} + a + i \cdot b$, we have $\angle(-x, y) = \arccos(-r - i \cdot s) = \frac{\pi}{2} - a - i \cdot b$. It follows (An 7).

To prove the last point (An 8) we use [20]. We take the two linear independent vectors $\vec{x}, \vec{y}$ and we build the set $U$ of all real linear combinations, $U := \{r \cdot \vec{x} + s \cdot \vec{y} \mid r, s \in \mathbb{R}\}$. The set $U$ is a real subspace of $X$ with the real dimension two. The norm in $U$ is the induced norm of $(X, \| \cdot \|)$, this makes the pair $(U, \| \cdot \|)$ to a real subspace of $(X, \| \cdot \|)$. Instead of $\angle(x, y + t \cdot \vec{x})$ we consider the real part of the complex number $\cos(\angle(x, y + t \cdot \vec{x})) \in B$. We define the map

$$
\tilde{\Theta}(t) := \frac{1}{4} \left[ \frac{2}{\| x \|^2} \right]^2 - \frac{2}{\| y + t \cdot \vec{x} \|^2} \right) , \text{ for } t \in \mathbb{R}.
$$

By the triangle inequality we can regard this as a map $\tilde{\Theta} : \mathbb{R} \to [-1, 1]$. The main theorem in [20] states that the map $\tilde{\Theta}$ is an increasing homeomorphism onto the open interval $(-1, 1) \subset B$. Since the complex arccosine function is a homeomorphism with domain $B$ and Codomain $A$, the first claim of (An 8) is true.

The limits are mentioned in the proof of the theorem in [20], or we can find one directly in [4], which was the main source of [20].

**Lemma 3.16.** In a complex normed vector space $(X, \| \cdot \|)$ we take two vectors $\vec{x}, \vec{y} \neq 0$. It holds $\| \cos(\angle(x, y)) \| = \| \cos(\angle(-x, y)) \| = \| \cos(\angle(i \cdot \vec{x}, \vec{y})) \| = \| \cos(\angle(x, i \cdot \vec{y})) \| .

**Proof.** This fact follows easily with Proposition (3.1).

We assume two elements $\vec{x}, \vec{y} \neq 0$ of a complex normed space $(X, \| \cdot \|)$, and we repeat the table from the introduction. After the table some explanations were given.

| pair of vectors | their angle $\angle$ | the cosine of $\angle$ | the angle for $\vec{x} = \vec{y}$ | its cosine for $\vec{x} = \vec{y}$ |
|---------------|-----------------|-----------------|-----------------|-----------------|
| $(\vec{x}, \vec{y})$ | $\frac{\pi}{2} + a + i \cdot b$ | $r + i \cdot s$ | 0 | 1 |
| $(-\vec{x}, \vec{y})$ | $\frac{\pi}{2} - a - i \cdot b$ | $-r - i \cdot s$ | $\pi$ | $-1$ |
| $(\vec{y}, \vec{x})$ | $\frac{\pi}{2} + a - i \cdot b$ | $r - i \cdot s$ | 0 | 1 |
| $(-\vec{y}, \vec{x})$ | $\frac{\pi}{2} - a + i \cdot b$ | $-r + i \cdot s$ | $\pi$ | $-1$ |
| $(i \cdot \vec{x}, \vec{y})$ | $\frac{\pi}{2} + v + i \cdot w$ | $-s + i \cdot r$ | $\frac{\pi}{2} - i \cdot \log \sqrt{2} + 1$ | $i$ |
| $(\vec{y}, i \cdot \vec{x})$ | $\frac{\pi}{2} - v - i \cdot w$ | $s - i \cdot r$ | $\frac{\pi}{2} + i \cdot \log \sqrt{2} + 1$ | $-i$ |
| $(\vec{x}, i \cdot \vec{y})$ | $\frac{\pi}{2} + v - i \cdot w$ | $s + i \cdot r$ | $\frac{\pi}{2} + i \cdot \log \sqrt{2} + 1$ | $-i$ |
| $(i \cdot \vec{y}, \vec{x})$ | $\frac{\pi}{2} - v + i \cdot w$ | $s + i \cdot r$ | $\frac{\pi}{2} - i \cdot \log \sqrt{2} + 1$ | $i$ |

Note that in the table the values $\angle(x, y) = \frac{\pi}{2} + a + i \cdot b$ and $\cos(\angle(x, y)) = r + i \cdot s$ are by definition, as well the expression $\angle(i \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + v + i \cdot w$.

**Proof.** The final column comes directly from Definition (1.2), e.g. $\cos(\angle(x, x)) = \langle x, x \rangle / \|x\|^2$, etc. We get the values of the fourth column by using the fifth column and applying the arccosine function from Definition (2.1). The other columns have to be discussed.
We show, for instance, the final line \( \angle(\vec{i} \cdot \vec{y}, \vec{x}) = \frac{\pi}{2} - v + i \cdot w \), and \( \cos(\angle(\vec{i} \cdot \vec{y}, \vec{x})) = s + i \cdot r \). We compute
\[
\angle(\vec{i} \cdot \vec{y}, \vec{x}) = \arccos \left( \frac{\vec{i} \cdot \vec{y} \cdot \|\vec{x}\|}{\|\vec{y}\| \cdot \|\vec{x}\|} \right) = \arccos \left( \frac{\vec{i} \cdot \vec{y} \cdot \|\vec{x}\|}{\|\vec{y}\| \cdot \|\vec{x}\|} \right),
\]
hence it follows
\[
\cos(\angle(\vec{i} \cdot \vec{y}, \vec{x})) = \frac{\vec{i} \cdot \vec{x} \cdot \|\vec{y}\|}{\|\vec{y}\| \cdot \|\vec{x}\|} = i \cdot (r + i \cdot s) = s + i \cdot r.
\]
With a similar argumentation we get
\[
\cos(\angle(\vec{i} \cdot \vec{x}, \vec{y})) = -s + i \cdot r.
\]
We had defined \( \angle(\vec{i} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + v + i \cdot w \), hence it follows \( \frac{\pi}{2} + v + i \cdot w = \arccos(-s + i \cdot r) \).

In Definition (2.1) we introduced the arccosine function. By the different signs of \( s \) we can deduce \( \arccos(s + i \cdot r) = \frac{\pi}{2} - v + i \cdot w \). With equation (3.6) we get \( \angle(\vec{i} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} - v + i \cdot w \), and the final line of the table is shown. The other lines can be proven with similar considerations.

With the help of the table we can conclude other values, for instance we get \( \angle(-\vec{i} \cdot \vec{y}, \vec{x}) = \frac{\pi}{2} + v - i \cdot w = \angle(\vec{y}, \vec{i} \cdot \vec{x}) \) with its cosine \( \cos(\angle(-\vec{i} \cdot \vec{y}, \vec{x})) = -s - i \cdot r \).

We refer to the above table, where we had assumed \( \angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + a + i \cdot b \). If we want to express \( \cos(\angle(\vec{x}, \vec{y})) = r + i \cdot s \) in coordinates of \( a \) and \( b \), we get at once from Definition (2.1)
\[
r + i \cdot s = \cos \left( \frac{\pi}{2} + a + i \cdot b \right) = \frac{1}{2} \cdot \left[ \cos \left( \frac{\pi}{2} + a \right) \cdot \left( e^{i b} - \frac{1}{e^{i b}} \right) \right].
\]
To express the complex number \( \angle(\vec{i} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + v + i \cdot w \) in dependence of \( a \) and \( b \) we have to make some more effort. In the introduction in Theorem (1.4) we already have presented the result. We need to prove it.

Proof. Here we also use the real sine and hyperbolic sine functions, abbreviated by \( \sin \) and \( \sinh \), please see Definition (2.1). To shorten the presentation of the proof it is useful to introduce more abbreviations.

Let \( \cpi2a := \cos \left( \frac{\pi}{2} + a \right) \), \( \spi2a := \sin \left( \frac{\pi}{2} + a \right) \).

From equation (3.7) we have
\[
r + i \cdot s = \cos(\angle(\vec{x}, \vec{y})) = \cpi2a \cdot \cosh(b) - i \cdot \spi2a \cdot \sinh(b),
\]
and we have two real numbers \( r = \cpi2a \cdot \cosh(b) \) and \( s = -\spi2a \cdot \sinh(b) \). Note \( -\frac{\pi}{2} \leq a \leq \frac{\pi}{2} \).

Since \( \frac{\pi}{2} + a + i \cdot b \in \mathcal{A} \) it follows from the special cases \( a = -\frac{\pi}{2} \) or \( a = \frac{\pi}{2} \) that the imaginary part \( b \) vanished, i.e. \( 0 = b = s = \spi2a = \sinh(b) \).

We get from the above table and Definition (2.1)
\[
\angle(\vec{i} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + v + i \cdot w = \arccos(-s + i \cdot r) \tag{3.8}
\]
\[
= \arccos[(\spi2a \cdot \sinh(b)) + i \cdot (\cpi2a \cdot \cosh(b))] \tag{3.9}
\]
\[
= \frac{\pi}{2} - \arcsin[(\spi2a \cdot \sinh(b)) + i \cdot (\cpi2a \cdot \cosh(b))] \tag{3.10}
\]
\[
= \frac{\pi}{2} - \frac{1}{2} \cdot \left[ \arccos(K_-) + \arccos(K_+) \right] \tag{3.11}
\]
\[
= \frac{\pi}{2} - \frac{1}{2} \cdot \left[ \arccos(K_-) + \arccos(K_+) \right]. \tag{3.12}
\]
with the abbreviations $K_-$ and $K_+$,

$$K_\pm := \sqrt{[\text{sp}^2 a \cdot \text{sh}^2(b) + \text{cp}^2 a \cdot \text{ch}^2(b) - 1]^2 + 4 \cdot \text{cp}^2 a \cdot \text{ch}^2(b) \mp [\text{sp}^2 a \cdot \text{sh}^2(b) + \text{cp}^2 a \cdot \text{ch}^2(b)]}.$$  

With the aid of the well-known equations

$$\sin^2(x) + \cos^2(x) = 1 = \cosh^2(x) - \sinh^2(x)$$

we finally reach the identities $H_- = K_- \text{ and } H_+ = K_+$, which was the last step to prove Theorem (1.4).

Now we prove Corollary (1.5).

**Proof.** Since $b = 0$ we have $\sinh(b) = 0$ and $s = 0$, and $\cosh(b) = 1$. It follows

$$\angle(i \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + v + i \cdot w = \arccos(i \cdot r) = \arccos(i \cdot \text{cp}2a \cdot \cosh(b))$$

$$= \frac{\pi}{2} - \frac{1}{2} \cdot [i \cdot \text{sgn}(-a) \cdot \arccosh(K_+)]$$

$$= \frac{\pi}{2} + \frac{1}{2} \cdot i \cdot \text{sgn}(a) \cdot \arccosh\left[2 \cdot \cos^2\left(\frac{\pi}{2} + a\right) + 1\right].$$

Please see the definition of the arccosh function in Definition (2.1), and note that the equation

$$\log \left[\sqrt{\cos^2(x) + 1} + |\cos(x)|\right] = \frac{1}{2} \cdot \log \left[2 \cdot \cos^2(x) + 1 + 2 \cdot |\cos(x)| \cdot \sqrt{\cos^2(x) + 1}\right]$$

holds for all real numbers $x$, which concludes the proof.

We add the proof of Corollary (1.6).

**Proof.** First a lemma.

**Lemma 3.17.** In the case of $\angle(\vec{x}, \vec{y}) = \frac{\pi}{2} + ib$, i.e. $a = 0$, the range of $b$ is

$$-\log\left(\sqrt{2} + 1\right) \leq b \leq +\log\left(\sqrt{2} + 1\right) \approx 0.88.$$  

**Proof.** By Lemma (3.5), there is a suitable $-1 \leq s \leq +1$ with $\cos\left(\frac{\pi}{2} + ib\right) = i \cdot s$. By using the arccosine of Definition (2.1) it follows for the modulus of $b$

$$|b| = \frac{1}{2} \cdot \arccosh(G_+) = \frac{1}{2} \cdot \arccosh(2 \cdot s^2 + 1) = \log\left(\sqrt{2 \cdot s^2 + 1 + 2 \cdot |s| \cdot \sqrt{s^2 + 1}}\right),$$

and note $\sqrt{3 + 2 \cdot \sqrt{2}} = \sqrt{2} + 1$, and the lemma is proven.

We apply Theorem (1.4), and since $a = 0$ we get

$$\angle(i \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + \frac{1}{2} \cdot [-\text{sgn}(b) \cdot \arccosh(H_-)] \text{, with } H_- = \sqrt{[\cosh^2(b) - 2]^2 - [\cosh^2(b) - 1]}.$$  

A consequence of Lemma (3.17) is the fact $\cosh^2(b) \leq 2$, it follows

$$H_- = [2 - \cosh^2(b)] - [\cosh^2(b) - 1] = 3 - 2 \cdot \cosh^2(b),$$

and the first line of Corollary (1.6) is proven. The next line is some calculus.
We choose $\vec{e}$ in a complex normed space $(X, \| \cdot \|)$ for $\vec{x} \in X$ and real $\varphi$ there is the identity
\[ < e^{i \varphi} \cdot \vec{x} | \vec{x} > = e^{i \varphi} \cdot < \vec{x} | \vec{x} > . \]

**Proof.** Straightforward. Write $e^{i \varphi} = \cos(\varphi) + i \cdot \sin(\varphi)$, and use Definition (1.1).

**Corollary 3.19.** For an unit vector $\vec{e} \in (X, \| \cdot \|)$ we have that the set $\{ < e^{i \varphi} \cdot \vec{x} | \vec{x} > \mid \varphi \in [0, 2\pi) \}$ is the complex unit circle, since $< e^{i \varphi} \cdot \vec{x} | \vec{x} > = e^{i \varphi} \cdot < \vec{x} | \vec{x} > = e^{i \varphi}$. 

**Remark 3.20.** The next example shows that in a complex normed space $(X, \| \cdot \|)$ generally we have the inequality
\[ < e^{i \varphi} \cdot \vec{x} | \vec{y} > \neq e^{i \varphi} \cdot < \vec{x} | \vec{y} > . \]
This means that in this case the set of products $\{ < e^{i \varphi} \cdot \vec{x} | \vec{y} > \mid \varphi \in [0, 2\pi) \}$ does not generate a proper Euclidean circle (with radius $| < \vec{x} | \vec{y} > |$) in $\mathbb{C}$. But with Proposition (3.4) we can be sure that we have three identities
\[ < -\vec{x} | \vec{y} > = -< \vec{x} | \vec{y} > , \quad < i \cdot \vec{x} | \vec{y} > = i \cdot < \vec{x} | \vec{y} > , \quad \text{and} \quad < -i \cdot \vec{x} | \vec{y} > = -i \cdot < \vec{x} | \vec{y} > . \]

**Lemma 3.21.** In a complex normed space $(X, \| \cdot \|)$ generally it holds the inequality
\[ < e^{i \varphi} \cdot \vec{x} | \vec{y} > \neq e^{i \varphi} \cdot < \vec{x} | \vec{y} > , \quad \text{even their moduli are different.} \]

**Proof.** The lemma can be deduced by Theorem (3.8), but we make a direct proof. We use the most simple non-trivial example of a complex normed space, let $(X, \| \cdot \|) := (\mathbb{C} \times \mathbb{C}, \| \cdot \|_{\infty})$, where for two complex numbers $r + i \cdot s, \quad v + i \cdot w \in \mathbb{C}$ we get its norm $\| \cdot \|_{\infty}$ by
\[ \left\| \left( \begin{array}{c} r + i \cdot s \\ v + i \cdot w \end{array} \right) \right\|_{\infty} = \max \{ \sqrt{r^2 + s^2}, \sqrt{v^2 + w^2} \} . \]

We define two unit vectors $\vec{x}, \vec{y}$ of $(\mathbb{C} \times \mathbb{C}, \| \cdot \|_{\infty})$, let
\[ \vec{x} := \frac{1}{4} \cdot \left( \begin{array}{c} 1 + i \cdot \sqrt{15} \\ 2 + i \cdot 2 \end{array} \right) \quad \text{and} \quad \vec{y} := \frac{1}{4} \cdot \left( \begin{array}{c} 2 + i \\ 3 + i \cdot \sqrt{7} \end{array} \right) . \]

Some calculations yield the complex number
\[ < \vec{x} | \vec{y} > = \frac{1}{64} \cdot \left( 19 + 4 \cdot \sqrt{7} + 2 \cdot \sqrt{15} + i \cdot \left[ 7 - 4 \cdot \sqrt{7} + 4 \cdot \sqrt{15} \right] \right) \approx 0.583 + i \cdot 0.186 . \]

We choose $e^{i \varphi} := 1/2 \cdot (1 + i \cdot \sqrt{3})$ from the complex unit circle, and we get approximately $e^{i \varphi} \cdot < \vec{x} | \vec{y} > \approx 0.130 + i \cdot 0.598$. After that we take the unit vector
\[ e^{i \varphi} \cdot \vec{x} = \frac{1}{8} \cdot \left( \begin{array}{c} 1 - \sqrt{45} + i \cdot \left[ \sqrt{3} + \sqrt{15} \right] \\ 2 - 2 \cdot \sqrt{3} + i \cdot \left[ 2 + 2 \cdot \sqrt{3} \right] \end{array} \right) , \]
and we compute the product $< e^{i \varphi} \cdot \vec{x} | \vec{y} >$. We get the result
\[ < e^{i \varphi} \cdot \vec{x} | \vec{y} > = \frac{1}{64} \cdot (p + i \cdot q) \approx 0.113 + i \cdot 0.628, \quad \text{where} \quad p \quad \text{and} \quad q \quad \text{abbreviate the real numbers} \]
\[ p = 11 + 2 \cdot \left( \sqrt{7} + \sqrt{21} - \sqrt{45} \right) - 5 \cdot \sqrt{3} + \sqrt{15} , \]
\[ q = 8 + 2 \cdot \left( 4 \cdot \sqrt{3} - \sqrt{7} + \sqrt{15} + \sqrt{21} \right) + \sqrt{45} . \]

This proves the inequality $< e^{i \varphi} \cdot \vec{x} | \vec{y} > \neq e^{i \varphi} \cdot < \vec{x} | \vec{y} >$, and the lemma is confirmed. 

The above lemma suggests the following conjecture. One direction is trivial.
Conjecture 3.22. In a complex normed space \((X, \| \cdot \|)\) for all \(\vec{x}, \vec{y} \in X\) it holds the equality
\[
< e^{i\varphi} \cdot \vec{x} \mid \vec{y} > = e^{i\varphi} \cdot < \vec{x} \mid \vec{y} >
\]
if and only if its product \(< \cdot \mid \cdot >\) from Definition (1.1) is actually an inner product, i.e. \((X, < \cdot \mid \cdot >)\) is an inner product space.

Up to now we had defined for each complex normed space \(X\) an ‘angle’ which generally has complex values. The geometrical meaning of a complex angle is unclear. But to do the usual known ‘Euclidean’ geometry we need real valued angles. During the following consideration it turns out that although we deal with complex vector spaces actually ‘a lot’ of our angles are pure real. The situation will even improve in inner product spaces, which will be investigated in the next section. First we create a name for these ‘good’ pairs with a real product, which include all pairs with a real angle.

Definition 3.23. For a complex vector space \(X\) provided with a norm \(\| \cdot \|\) we define the set of pairs \(\mathcal{R}_X \subseteq X \times X\) with real products by \(\mathcal{R}_X := \{ (\vec{x}, \vec{y}) \mid \vec{x}, \vec{y} \in X \text{ and } < \vec{x} \mid \vec{y} > \in \mathbb{R} \} \).

Further we denote by \(\mathcal{R}_X^\bullet\) those pairs \((\vec{x}, \vec{y}) \in \mathcal{R}_X\) with a pure real angle, i.e. \(\angle(\vec{x}, \vec{y}) \in \mathbb{R}\).

Note that the ‘diagonal’ \(\{ (\vec{x}, \vec{x}) \mid \vec{x} \in X \}\) is a subset of \(\mathcal{R}_X\).

The following proposition shows that the set \(\mathcal{R}_X^\bullet\) of pairs with real angles is ‘rather large’.

Proposition 3.24. Let us take two vectors \(\vec{x}, \vec{y} \neq \vec{0}\). It holds
\[
\{(e^{i\varphi} \cdot \vec{x}, \vec{y}) \mid \varphi \in [0, 2\pi]\} \cap \mathcal{R}_X^\bullet \neq \emptyset .
\]

Of course, the parts of \(\vec{x}\) and \(\vec{y}\) can be exchanged. The proposition means that for \(\vec{x}, \vec{y} \neq \vec{0}\) we have to ‘twist’ either \(\vec{x}\) or \(\vec{y}\) by a suitable complex factor \(e^{i\varphi}\) to generate a pure real angle.

Proof. Please see both Proposition (3.3) and Proposition (3.4). Let us assume a complex angle
\[
\angle(\vec{x}, \vec{y}) = \arccos \left( \frac{< \vec{x} \mid \vec{y} >}{\| \vec{x} \| \cdot \| \vec{y} \|} \right) = \frac{\pi}{2} + a + i \cdot b \in \mathbb{A}, \text{ with } b \neq 0 .
\]

From Proposition (3.4) we have \(< -\vec{x} \mid \vec{y} > = - < \vec{x} \mid \vec{y} >\), i.e. with Proposition (3.3) it follows
\[
\angle(-\vec{x}, \vec{y}) = \arccos \left( \frac{- < \vec{x} \mid \vec{y} >}{\| \vec{x} \| \cdot \| \vec{y} \|} \right) = \frac{\pi}{2} - a - i \cdot b .
\]

We know \(e^{i\pi} = -1\). The set \(\text{Oval}(\vec{x}, \vec{y}) := \{ \angle(e^{i\varphi} \cdot \vec{x}, \vec{y}) \mid \varphi \in [0, 2\pi]\} \subset \mathbb{A}\) is the continuous image of the connected complex unit circle \(\{ e^{i\varphi} \mid \varphi \in [0, 2\pi]\}\), or the interval \([0, 2\pi]\), respectively, therefore it has to be connected. This means that \(\text{Oval}(\vec{x}, \vec{y})\) is connected, i.e. it must cross the real axis. \(\square\)

Let’s turn to inner product spaces.

4 Complex Inner Product Spaces

In the introduction we construct in Definition (1.1) a continous product for all complex normed spaces \((X, \| \cdot \|)\). There we already mentioned that in a case of an inner product space \((X, < \cdot \mid \cdot >)\) the product from Definition (1.1) coincides with the given inner product \(< \cdot \mid \cdot >\), which can be described as in equation (1.1). For all complex normed spaces \((X, \| \cdot \|)\) we introduced an ‘angle’ \(\angle\) in Definition (1.2). Now we investigate its properties in the special case of an inner product space \((X, < \cdot \mid \cdot >)\).

We deal with complex vector spaces \(X\) provided with an inner product \(< \cdot \mid \cdot >\), i.e. it has the properties (1), (2), (3), (4). Further, the parallelogram identity \(\| \vec{x} + \vec{y} \|^2 + \| \vec{x} - \vec{y} \|^2 =\)
Let us take two vectors \( \vec{x}, \vec{y} \in X \) with a pure real angle. Proposition \([3.21]\) ensures the existence of many of such pairs. But now we want more. Roughly spoken we seek for subsets \( U \subset X \), such that \( U \) is a real subspace of the complex vector space \( X \) and in addition \( U \times U \) is a subset of \( \mathcal{R}_{X} \), i.e. all products in \( U \) are real.

First we take a second look on Proposition \([3.24]\) and its proof. For two elements \( \vec{x}, \vec{y} \) of a complex normed space \( (X, \| \cdot \|) \) we know from Proposition \([3.24]\) that there is at least one \( \varphi \in [0, 2\pi] \) such that the angle of the pair \( (e^{i\varphi} \cdot \vec{x}, \vec{y}) \) is real.

**Lemma 4.1.** Let us take two vectors \( \vec{x}, \vec{y} \) from an inner product space \( (X, \langle \cdot | \cdot \rangle) \) with \( \langle \vec{x} | \vec{y} \rangle \neq 0 \). We have that there exists one number \( 0 \leq \varphi < 2 \cdot \pi \) such that the set \( \text{Oval}(\vec{x}, \vec{y}) \) has exactly two real angles

\[
\angle (e^{i\varphi} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} + a, \quad \angle (e^{i(\varphi+\pi)} \cdot \vec{x}, \vec{y}) = \frac{\pi}{2} - a, \quad \text{with a suitable number } 0 < a \leq \frac{\pi}{2}.
\]

**Proof.** For an inner product \( \langle \cdot | \cdot \rangle \) it holds \( \langle e^{i\varphi} \cdot \vec{x} | \vec{y} \rangle = e^{i\varphi} \cdot \langle \vec{x} | \vec{y} \rangle \), hence the cosines \( \cos(\text{Oval}(\vec{x}, \vec{y})) \subset \mathcal{B} \) build an Euclidean circle with radius \( \| \vec{x} \| = \| \vec{y} \| /\| \langle \vec{x} | \vec{y} \rangle \| \). We map this set with the arccosine function, and by Proposition \([3.33]\) the image \( \text{Oval}(\vec{x}, \vec{y}) \subset \mathcal{A} \) is symmetrical to \( \frac{\pi}{2} \), it crosses the real axis exactly two times. (Note that \( \text{Oval}(\vec{x}, \vec{y}) \) is no Euclidean circle.)

Albeit we deal with complex inner product spaces we are interested in real subspaces. In a complex normed space \( (X, \| \cdot \|) \) let \( U \neq \emptyset \) be any non-empty subset of \( X \). We define \( \mathcal{L}(\mathbb{R})(U) \) as the set of all finite real linear combinations of elements from \( U \), while \( \mathcal{L}(\mathbb{C})(U) \) is the set of complex linear combinations. In formulas we define

\[
\mathcal{L}(\mathbb{R})(U) := \left\{ \sum_{i=1}^{n} r_i \cdot \vec{x}_i \mid n \in \mathbb{N}, \ r_1, r_2, \ldots, r_n \in \mathbb{R}, \ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in U \right\}, \quad \text{while}
\]

\[
\mathcal{L}(\mathbb{C})(U) := \left\{ \sum_{i=1}^{n} z_i \cdot \vec{x}_i \mid n \in \mathbb{N}, \ z_1, z_2, \ldots, z_n \in \mathbb{C}, \ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in U \right\}.
\]

This definitions mean that \( \mathcal{L}(\mathbb{C})(U) \) is a \( \mathbb{C} \)-linear subspace of the complex vector space \( X \), while \( \mathcal{L}(\mathbb{R})(U) \) only is a real linear subspace of \( X \), which is a real vector space, too.

For both spaces we regard the closure in \( X \). Let \( \overline{\mathcal{L}(\mathbb{R})(U)} \) and \( \overline{\mathcal{L}(\mathbb{C})(U)} \) are the closures of \( \mathcal{L}(\mathbb{R})(U) \) and \( \mathcal{L}(\mathbb{C})(U) \), respectively. Of course, for a finite set \( U \subset X \) it holds \( \overline{\mathcal{L}(\mathbb{R})(U)} = \overline{\mathcal{L}(\mathbb{R})(U)} \) and \( \overline{\mathcal{L}(\mathbb{C})(U)} = \overline{\mathcal{L}(\mathbb{C})(U)} \). If we assume an infinite set \( U \), an element \( \vec{y} \in X \) belongs to \( \overline{\mathcal{L}(\mathbb{R})(U)} \) if and only if there is a countable set \( \{ \vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots \} \subset U \) and there are real numbers \( r_1, r_2, r_3, r_4, \ldots \) such that

\[
\lim_{k \to \infty} \left\| \vec{y} - \sum_{i=1}^{k} r_i \cdot \vec{x}_i \right\| = 0. \quad \text{We can write } \vec{y} = \sum_{i=1}^{\infty} r_i \cdot \vec{x}_i. \tag{4.1}
\]

The set \( \overline{\mathcal{L}(\mathbb{C})(U)} \) is constructed similarly, but we can use complex numbers \( z_1, z_2, z_3, \ldots \). Again we get two subspaces of \( X \), \( \overline{\mathcal{L}(\mathbb{R})(U)} \) is a real subspace, while \( \overline{\mathcal{L}(\mathbb{C})(U)} \) is a complex subspace. We have inclusions \( \mathcal{L}(\mathbb{R})(U) \subset \overline{\mathcal{L}(\mathbb{R})(U)} \), and \( \mathcal{L}(\mathbb{C})(U) \subset \overline{\mathcal{L}(\mathbb{C})(U)} \), respectively, and generally both inclusions are proper. Note \( \overline{\mathcal{L}(\mathbb{R})(U)} \subset \overline{\mathcal{L}(\mathbb{C})(U)} \).
In a complex inner product space \((X, \langle \cdot | \cdot \rangle)\) we can use the well-known theory of orthogonal systems. Informations about this topic can be found in [15] or [24], or any other book about functional analysis.

**Definition 4.2.** Let \((X, \langle \cdot | \cdot \rangle)\) be a complex Hilbert space, i.e. a complex inner product space which is complete. A subset \(\emptyset \neq T \subset X\) is called an orthonormal system if and only if for each pair of distinct elements \(\vec{x}, \vec{y} \in T\) (i.e. \(\vec{x} \neq \vec{y}\)) it holds \(\langle \vec{x} | \vec{y} \rangle = 0\), and all \(\vec{x} \in T\) are unit vectors, i.e. \(\| \vec{x} \| = 1\).

An orthonormal system \(T\) is called an orthonormal basis if and only if \(T\) is maximal. This means that if we have a second orthonormal system \(V\) with \(T \subset V\) it has to be \(T = V\).

Note that an orthonormal basis generally is not a vector space basis.

Each unit vector \(\vec{x}\) provides an orthonormal system \(\{\vec{x}\}\). It is well known that there is an orthonormal basis \(T\) with \(\{\vec{x}\} \subset T \subset X\). This shows that there are orthonormal bases in all Hilbert spaces \(X \neq \{0\}\). Further note that an orthonormal system \(T \subset X\) is an orthonormal basis in \(\mathcal{L}(\mathbb{C})(T)\).

The next proposition describes real subspaces which have only real inner products.

**Proposition 4.3.** Let \((X, \langle \cdot | \cdot \rangle)\) be a complex Hilbert space. Let \(T \subset X\) be an orthonormal system. The set \(\mathcal{L}(\mathbb{R})(T)\) is a real subspace of \(X\), and we get that its square \(\mathcal{L}(\mathbb{R})(T) \times \mathcal{L}(\mathbb{R})(T)\) is a subset of \(RX\), i.e. each pair \(\vec{y}, \vec{z} \in \mathcal{L}(\mathbb{R})(T)\), \(\vec{y}, \vec{z} \neq 0\), has a real angle, i.e. \(\angle(\vec{y}, \vec{z}) \in \mathbb{R}\).

**Proof.** The set \(\mathcal{L}(\mathbb{R})(T)\) is a real subspace of \(X\) by construction, with a vector space basis \(T\). The real vector space \(\mathcal{L}(\mathbb{R})(T)\) has the closure \(\overline{\mathcal{L}(\mathbb{R})(T)}\) in \(X\). For \(\vec{y}, \vec{z} \in \overline{\mathcal{L}(\mathbb{R})(T)}\) and \(r \in \mathbb{R}\) it is easy to verify \(\vec{y} + \vec{z} \in \mathcal{L}(\mathbb{R})(T)\) and \(r \cdot \vec{y} \in \mathcal{L}(\mathbb{R})(T)\). This shows that \(\mathcal{L}(\mathbb{R})(T)\) is a real subspace of \(X\).

Now we take two vectors \(\vec{y}, \vec{z} \in \overline{\mathcal{L}(\mathbb{R})(T)}\) with \(\vec{y}, \vec{z} \neq 0\). We want to show \(\angle(\vec{y}, \vec{z}) \in \mathbb{R}\). With line (4.1) we have a countable set \(\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \ldots\} \subset T\) and two sequences \(r_1, r_2, r_3, \ldots\) and \(s_1, s_2, s_3, \ldots\) of real numbers such that \(\vec{y} = \sum_{i=1}^{\infty} r_i \cdot \vec{x}_i\) and \(\vec{z} = \sum_{i=1}^{\infty} s_i \cdot \vec{x}_i\).

We compute the cosine of the angle of the pair \((\vec{y}, \vec{z})\).

\[
\cos(\angle(\vec{y}, \vec{z})) = \cos \left( \arccos \left( \frac{\langle \vec{y} | \vec{z} \rangle}{\| \vec{y} \| \cdot \| \vec{z} \|} \right) \right) = \frac{1}{\| \vec{y} \| \cdot \| \vec{z} \|} \cdot \langle \vec{y} | \vec{z} \rangle \quad (4.2)
\]

\[
= \frac{1}{\| \vec{y} \| \cdot \| \vec{z} \|} \cdot \left( \sum_{i=1}^{\infty} r_i \cdot \vec{x}_i \right) \cdot \left( \sum_{j=1}^{\infty} s_j \cdot \vec{x}_j \right) \quad (4.3)
\]

\[
= \frac{1}{\| \vec{y} \| \cdot \| \vec{z} \|} \cdot \lim_{k \to \infty} \left( \sum_{i=1}^{k} r_i \cdot s_j \cdot < \vec{x}_i | \vec{x}_j > \right) \quad (4.4)
\]

\[
= \frac{1}{\| \vec{y} \| \cdot \| \vec{z} \|} \cdot \lim_{k \to \infty} \left( \sum_{i=1}^{k} r_i \cdot s_i \cdot < \vec{x}_i | \vec{x}_i > \right) \quad (4.5) \text{ (T is orthonormal)}
\]

\[
= \frac{1}{\| \vec{y} \| \cdot \| \vec{z} \|} \cdot \lim_{k \to \infty} \left( \sum_{i=1}^{k} r_i \cdot s_i \right) = \frac{1}{\| \vec{y} \| \cdot \| \vec{z} \|} \cdot \sum_{i=1}^{\infty} r_i \cdot s_i \quad (4.6)
\]

We get \(\cos(\angle(\vec{y}, \vec{z})) = \frac{1}{\| \vec{y} \| \cdot \| \vec{z} \|} \cdot \sum_{i=1}^{\infty} r_i \cdot s_i\), and we confirm that indeed the cosine of \(\angle(\vec{y}, \vec{z})\) is real, this means that the angle \(\angle(\vec{y}, \vec{z})\) is real, too. Therefore we have proven that \(\mathcal{L}(\mathbb{R})(T) \times \mathcal{L}(\mathbb{R})(T)\) is a subset of \(RX\), and the proof of Proposition (4.3) is finished.

The above proposition proves the existence of many real angles in each complex inner product space. The following statement is in the opposite direction. It shows that the subsets of \((X, \langle \cdot | \cdot \rangle)\) with real angles can not be ‘arbitrary large’. We formulate the precise statement.
Proposition 4.4. Let \((X, < \cdot , \cdot >)\) be a complex inner product space. Let \(n \in \mathbb{N}\) be a natural number. The following two properties are equivalent.

1. There is an orthonormal system \(T := \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\} \subset X\) of \(n\) vectors.

2. There exists a set \(B := \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subset X\) of \(n\) elements such that both \(B\) is \(\mathbb{R}\)-linear independent and all angles in \(B\) are real.

Before we make the easy proof we formulate a corollary.

Corollary 4.5. Let \(n \in \mathbb{N}\). We assume any \(n\)-dimensional complex inner product space \((X, < \cdot , \cdot >)\). (Hence the real dimension of the real vector space \(X\) is \(2 \cdot n\)).

It follows that it does not be possible to find a set \(B := \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \vec{v}_{n+1}\} \subset X\) of \(n + 1\) elements such that both \(B\) is \(\mathbb{R}\)-linear independent, and all angles in \(B\) are real.

Proof. There are at most \(n\) \(\mathbb{C}\)-linear independent vectors in \(X\).

Now we prove Proposition (4.4), which is easy.

Proof. From (1) trivially it follows (2), because for the orthonormal system \(T = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}\) it holds that it is \(\mathbb{C}\)-linear independent, hence \(\mathbb{R}\)-linear independent, and the inner product of two elements out of \(T\) is either 0 or 1.

Also, for complex inner product spaces of infinite dimension the conclusion from (2) to (1) is trivial, since in such a space the method of Gram-Schmidt yields an orthonormal system with the cardinality of \(\mathbb{N}\).

We assume a complex inner product space \((X, < \cdot , \cdot >)\) with finite complex dimension and a set \(B \subset X\) with the properties of (2), i.e. all inner products in \(L(\mathbb{R})(B)\) are real. We use the very well-known method of Gram-Schmidt to generate a set \(\hat{B} := \{\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n\}\) of \(n\) vectors. This method yields a set \(\hat{B}\) which spans the same \(n\)-dimensional real subspace as \(B\), i.e. \(L(\mathbb{R})(B) = L(\mathbb{R})(\hat{B})\). By construction, the set \(\hat{B}\) consists of \(n\) \(\mathbb{R}\)-linear independent vectors. Because their inner product \(< \hat{v}_i | \hat{v}_j >\) is either 0 or 1, the set \(\hat{B}\) even is an orthonormal system as defined in Definition (4.2). This was required in (1), and Proposition (4.4) is proven.

At last we demonstrate that real angles are very useful to do classical Euclidean geometry. We are still investigating complex inner product spaces \((X, < \cdot , \cdot >)\), and we consider the desirable equation

\[\angle(\vec{x}, \vec{y}) = \angle(\vec{x}, \vec{x} + \vec{y}) + \angle(\vec{x} + \vec{y}, \vec{y}).\]  

(4.7)

Note that inner product spaces have the property (4), the linearity. By using the known formula

\[\arccos(r) + \arccos(s) = \arccos\left(\sqrt{1-r^2} \cdot \sqrt{1-s^2} + r \cdot s \cdot \sqrt{1-r^2} \cdot \sqrt{1-s^2}\right)\]

for real numbers \(-1 \leq r, s \leq +1\) we can show that equation (4.7) is fulfilled for a real angle \(\angle(\vec{x}, \vec{y})\), i.e. \(< \vec{x} | \vec{y} > \in \mathbb{R}\). To demonstrate that a real angle is also necessary, it is sufficient to consider the special case of unit vectors \(\vec{x}, \vec{y}\). The use of unit vectors simplifies the proof.

Proposition 4.6. In a complex inner product space \((X, < \cdot , \cdot >)\) let \(\vec{x}, \vec{y}\) be two unit vectors. Then it holds equation (4.7) if and only if \((\vec{x}, \vec{y}) \in \mathbb{R}_X\), i.e. their angle \(\angle(\vec{x}, \vec{y})\) is real.

For the proof first we show a lemma

Lemma 4.7. For all numbers \(z \in \mathbb{C}\) we have the identity \(2 \cdot \arccos(z) = \arccos(2 \cdot z^2 - 1)\).

Proof. After Definition (2.1) we wrote \(\cos(z) = \frac{1}{2} \left[e^{iz} + e^{-iz}\right]\), and then we can prove easily \(\cos(2 \cdot w) = 2 \cdot \cos^2(w) - 1\), for \(w \in \mathbb{C}\). We set \(w := \arccos(z)\), and another application of the arccosine function gives the desired equation of Lemma (4.7). 

\[\square\]
Proof. To prove Proposition (4.6) we assume a complex number \( r + i \cdot s := \cos(\angle(x, y)) \) with two unit vectors \( x, y \), i.e. \( \|x\| = 1 = \|y\| \). By Definition (4.2) of the angle \( \angle(x, y) \) this means \( r + i \cdot s = \langle x \mid y \rangle \). We consider the right hand side of equation (4.7), and we calculate

\[
\angle(x, x + y) = \arccos \left( \frac{\langle x \mid x + y \rangle}{\|x + y\|} \right) + \arccos \left( \frac{\langle x + y \mid y \rangle}{\|x + y\|} \right)
\]

\[
= \arccos \left( \frac{2 \cdot \left[ 1 + \langle x \mid y \rangle \right]^2}{\|x + y\|^2} - 1 \right) \quad \text{(by Lemma (4.7))}
\]

\[
= \arccos \left( \frac{2 \cdot \left[ 1 + 2 \cdot r + i \cdot s \right]}{\|x + y\|^2} \right) - 1 \quad \text{(note } \langle y \mid x \rangle = \langle x \mid y \rangle \text{)}
\]

\[
= \arccos \left( r - \frac{s^2 + 2 \cdot r \cdot s}{1 + r} \right) = \arccos \left( r - \frac{s^2 + 2 \cdot r \cdot s}{1 + r} \right). \]

Obviously, the last term is equal \( \arccos(r + i \cdot s) = \angle(x, y) \) only in the case of \( s = 0 \), i.e. if and only if the angle \( \angle(x, y) \) is real. The proof of Proposition (4.6) is finished.

With the properties of Theorem (1.3) and equation (4.7) we can do ordinary Euclidean geometry even in complex Hilbert spaces in those parts where real angles occur. We make two examples. As a first example we prove that the sum of inner angles in a triangle is \( \pi \).

**Proposition 4.8.** Let us assume a real angle \( \angle(x, y) \) in a complex Hilbert space. We get

\[
\angle(x, y) + \angle(-x, y - x) + \angle(-y, x - y) = \pi.
\]

**Proof.** We use equation (4.7), and Theorem (1.3) (An 4),(An 6),(An 7). If we regard (4), the linearity, we see that all angles in the following equation are real. We compute

\[
\angle(x, y) + \angle(-x, y - x) = \angle(-y, x - y) + \angle(-y, x) - \angle(-y, x + y) + \angle(-y, x) - \angle(-y + x, x) = \pi + 0 = \pi.
\]

As a second example we consider the ‘Law of Cosines’.

**Proposition 4.9.** Let \( x, y \neq \emptyset \) be two vectors in a complex Hilbert space. It holds the ‘Law of Cosines’

\[
\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \cdot \|x\| \cdot \|y\| \cdot \cos(\angle(x, y)) \tag{4.8}
\]

if and only if the angle \( \angle(x, y) \) is real, i.e. \( (x, y) \in R_X \), or in other words \( \langle x \mid y \rangle \in \mathbb{R} \).

**Proof.** If the angle \( \angle(x, y) \) is real, the proof of the law of cosines is straightforward. In the case of a proper complex angle \( \angle(x, y) \) the right hand side of equation (4.8) is complex, while the left hand side is real.
Note that the above theorems can be adapted to the complex situation, e.g. the ‘law of cosines’ can be described by
\[ \| \mathbf{x} - \mathbf{y} \|^2 = \| \mathbf{x} \|^2 + \| \mathbf{y} \|^2 - 2 \| \mathbf{x} \| \cdot \| \mathbf{y} \| \cdot [\cos(\angle(\mathbf{x}, \mathbf{y})) + \cos(\angle(\mathbf{y}, \mathbf{x}))], \]
or alternatively \[ < \mathbf{x} | \mathbf{y} > = \cos(\angle(\mathbf{x}, \mathbf{y})) \cdot \| \mathbf{x} \| \cdot \| \mathbf{y} \|. \]

At the end we should mention that in the last decades other concepts of generalized angles in complex inner product spaces have been considered. Note that for the following concepts of angles (except \( \angle_3 \), see Theorem 4.8) one can use all complex normed spaces, provided with the product of Definition 1.1.

There were attempts to create pure real angles by the definitions \( \angle_1, \angle_2, \) and \( \angle_3. \)

\[ \angle_1(\mathbf{x}, \mathbf{y}) := \text{the real part of } (\angle(\mathbf{x}, \mathbf{y})) = \text{the real part of } \left( \arccos \left( \frac{< \mathbf{x} | \mathbf{y} >}{\| \mathbf{x} \| \cdot \| \mathbf{y} \|} \right) \right), \]
or alternatively

\[ \angle_2(\mathbf{x}, \mathbf{y}) := \text{the arccosine of the real part of } \left( \frac{< \mathbf{x} | \mathbf{y} >}{\| \mathbf{x} \| \cdot \| \mathbf{y} \|} \right). \]

The angle \( \angle_3 \) is defined by

\[ \angle_3(\mathbf{x}, \mathbf{y}) := \arccos(\varrho), \text{ for } \frac{< \mathbf{x} | \mathbf{y} >}{\| \mathbf{x} \| \cdot \| \mathbf{y} \|} = \varrho \cdot e^{i \varphi} \in \mathbb{C}. \]

For more information and references see a paper [16] by Scharnhorst.

An interesting position is held by Froda in [6]. For the complex number \( \frac{< \mathbf{x} | \mathbf{y} >}{\| \mathbf{x} \| \cdot \| \mathbf{y} \|} = r + i \cdot s \) he defined the complex angle

\[ \angle_4(\mathbf{x}, \mathbf{y}) := \arccos(r) + i \cdot \arcsin(s). \]

Note that in the case of a pure real non-negative product \( < \mathbf{x} | \mathbf{y} > \) all four angles coincide with our angle \( \angle(\mathbf{x}, \mathbf{y}). \) More investigations about these constructions are necessary.

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