Rainbow cycles for matchings, single edges, and triangles

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Abstract

Given a family \( F = (F_1, \ldots, F_m) \) of sets of edges, a set \( F \) of edges is said to be rainbow for \( F \) if each of the edges in \( F \) is taken from a distinct \( F_i \). The rainbow girth \( \text{rgirth}(F) \) of \( F \) is the minimal length of a cycle that is rainbow with respect to \( F \).

As a generalization of the famous Caccetta-Häggkvist conjecture, Aharoni conjectured that if in the above \( m = n \), \( F_i \subseteq E(K_n) \) and \( |F_i| = k \) for all \( 1 \leq i \leq n \), then \( \text{rgirth}(F) \leq \lceil \frac{n}{k} \rceil \).

In [3] it was shown that if all sets \( F_i \) are matchings of size 2 then \( \text{rgirth}(F) = O(\log n) \), and in [1] the same was shown when all sets \( F_i \) are triangles. In this note we study the mixed case: assuming that \( \alpha n \) sets \( F_i \) are matchings of size 2, \( \beta n \) are triangles, and \( \gamma n \) are single edges—in what range of values of \((\alpha, \beta, \gamma)\), \( \text{rgirth}(F) \) is logarithmic?

1 Introduction

The directed girth \( \text{dgirth}(G) \) of a digraph \( G \) is the minimal length of a directed cycle in it (\( \infty \) if there is no directed cycle). A famous conjecture of Caccetta and Häggkvist [6] (below - CHC) states that any digraph \( G \) on \( n \) vertices satisfies \( \text{dgirth}(G) \leq \left\lceil \frac{n}{\delta^+(G)} \right\rceil \), where \( \delta^+(G) \) is the minimum out-degree of \( G \). See [7, 9, 10, 12] for progress on this problem. In particular it has been shown that

(a) The CHC is true if \( n \geq 2\delta^+(G)^2 - 3\delta^+(G) + 1 \) [14], and

(b) \( \text{dgirth}(G) \leq n/\delta^+(G) + 73 \) for all \( G \) [13].

In [2] a possible generalization of CHC was suggested. Given a family \( F = (F_1, \ldots, F_m) \) of sets of edges, a set \( F \) of edges is said to be rainbow for \( F \) if each of the edges in \( F \) is taken from a distinct \( F_i \) (if the sets \( F_i \) are disjoint, this means that \( |F \cap F_i| \leq 1 \) for each \( i \)). The rainbow girth \( \text{rgirth}(F) \) of \( F \) is the minimal length of a rainbow cycle with respect to \( F \).

**Conjecture 1.** For any family \( F = (F_1, \ldots, F_n) \) of subsets of \( E(K_n) \) such that \( |F_i| = k \) for each \( 1 \leq i \leq n \), we have \( \text{rgirth}(F) \leq \lceil n/k \rceil \).

We may clearly assume that the sets \( F_i \) are disjoint, since otherwise there is a rainbow digon, meaning that the rainbow girth is 2. Given \( F = (F_1, \ldots, F_m) \) with \( \cup_{i=1}^{m} F_i = E(G) \) for some graph \( G \), we shall refer to \( F \) as an edge coloring of \( G \), the indices \( i \in [m] \) as colors, the sets \( F_i \) as color classes, and \( \text{rgirth}(F) \) as the rainbow girth of \( G \) with respect to the edge coloring \( F \). We always assume that every color class \( F_i \) is non-empty.

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Devos et. al. [8] proved Conjecture 1 for $k = 2$. In [1] a stronger version of the conjecture was proved, in which the sets $F_i$ are of size 1 or 2. In [11] it was shown that the order of magnitude is right: there exists a constant $C > 0$ such that for any $k, n$ and $F$ satisfying the assumption of Conjecture 1, we have $\text{rgirth}(F) \leq C n/k$. An explanation why Conjecture 1 implies the CHC can be found in [3] and [1].

All known extreme examples for Conjecture 1 are obtained from those of the CHC, taking the color classes as stars. This suggests looking at the case when the sets of edges are not stars, and trying to improve the upper bound on the girth. Indeed, in [3], it is proved that if an $n$-vertex graph is edge-colored by $n$ colors such that each color class is a matching of size 2, then the rainbow girth is $O(\log n)$, asymptotically improving the conclusion of the conjecture.

A set of edges not containing a matching of size 2 is either a star or a triangle, hence the next interesting case is that of families of triangles. In [11], it is proved that a family of $n$ triangles in $K_n$ has rainbow girth $O(\log n)$. Furthermore, it was shown there that $\log n$ is the right order of magnitude: an $n$-vertex graph is constructed, consisting of $n$ edge-disjoint triangles whose rainbow girth is $\Omega(\log n)$.

In this note we fine-tune the above results, by finding the threshold proportions of matchings-triangles-singletons (the latter means sets consisting of one edge) that guarantee logarithmic rainbow girth.

2 Graph theoretical and probabilistic tools

As in [11, 3], a key ingredient in the proofs is a result by Bollobás and Szemerédi [4] on the girth of sparse graphs.

**Theorem 2.** For $N \geq 4$ and $k \geq 2$, every $N$-vertex graph with $N + k$ edges has girth at most

$$\frac{2(N + k)}{3k} (\log_2 k + \log_2 \log_2 k + 4).$$

We shall use two well-known concentration inequalities.

**Theorem 3** (Chernoff). Let $X$ be a binomial random variable $\text{Bin}(n, p)$. For any $t \geq 0$, we have

$$\mathbb{P}(X \geq \mathbb{E}X + t) \leq \exp\left(-\frac{t^2}{2(\mathbb{E}X + t/3)}\right).$$

**Theorem 4** (Chebyshev). Let $X$ be a random variable. For any $t > 0$, we have

$$\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq \frac{\text{Var}X}{t^2},$$

where $\text{Var}X$ is the variance of $X$.

3 Main results

3.1 Matchings and single edges

**Theorem 5.** For any $\alpha > 1/2$, there exists $C$ such that for any edge coloring $F = (F_1, \ldots, F_n)$ of an $n$-vertex graph $G$, if at least $\alpha n$ color classes in $F$ are matchings of size 2, then $\text{rgirth}(F) \leq C \log n$. 

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We shall need a slightly stronger result, which allows the number of size-two matchings to be \( \alpha n - o(n) \) and the total number of color classes to be less than \( n \).

**Theorem 6.** For any \( \alpha > 1/2 \), there exist \( \xi = \xi(\alpha) > 0 \) and \( C = C(\alpha) \) such that the following holds. Let \( G \) be an \( n \)-vertex graph and \( \mathcal{F} = (F_1, \ldots, F_m) \) be an edge coloring of \( G \). If \( \mathcal{F} = \mathcal{F}_M \sqcup \mathcal{F}_E \), where

1. every \( F_i \in \mathcal{F}_M \) is a matching of size 2,
2. every \( F_i \in \mathcal{F}_E \) is a single edge,
3. \( |\mathcal{F}_M| \geq (\alpha - \xi)n \) and \( |\mathcal{F}_E| \geq (1 - \alpha - \xi)n \),

then \( \text{rgirth}(\mathcal{F}) \leq C \log n \).

Theorem 6 will follow from Theorem 2, and the following:

**Theorem 7.** For any \( \alpha > 1/2 \), there exist \( \beta, c > 0 \) such that for any large enough \( n \), given an \( n \)-vertex graph \( G \) and an edge coloring of \( G \) satisfying the assumption in Theorem 6, there exists a subset \( S \) of \( V(G) \) of size at most \( \beta n \) containing a rainbow edge set of size at least \( (\beta + c)n \).

Once this is proved, Theorem 6 follows by applying Theorem 2 with \( N = \beta n \) and \( k = cn \).

The idea used to prove Theorem 7 is choosing a random subset \( S \) of \( V(G) \) and considering the induced subgraph \( G[S] \). The crux of the argument is that the expected number of vertices \( E|S| \) is less than the expected number of rainbow edges in \( G[S] \), and their difference is linear in \( n \). Furthermore, these two random numbers are concentrated around their expectations, which follows from the concentration inequalities in Section 2.

Theorem 2.8 of [3] states that for any \( \gamma > 3 \sqrt{6}/8 \approx 0.9186 \) and \( n \)-vertex graph with an edge coloring with at least \( \gamma n \) colors, if each color class is a matching of size 2, then the rainbow girth is \( O(\log n) \). Note that in Theorem 6 we can take \( \xi(\alpha) > 0 \) arbitrarily small. In particular, if \( \alpha > 3 \sqrt{6}/8 \), we can guarantee that \( \alpha - \xi > 3 \sqrt{6}/8 \). Therefore here we may assume that \( \alpha \leq 3 \sqrt{6}/8 \). Again since \( \xi \) can be chosen arbitrarily small, we may assume that \( \min\{\alpha - \xi, 1 - \alpha - \xi\} \geq 1/40 \) so that \( \min\{|\mathcal{F}_M|, |\mathcal{F}_E|\} \geq n/40 \). And without loss of generality, we may assume that \( \max\{|\mathcal{F}_M|, |\mathcal{F}_E|\} \leq n \).

First we claim that for \( p \in (0,1) \) close enough to 1, we have

\[
\alpha(2p^2 - p^4) + (1 - \alpha)p^2 > p. \tag{1}
\]

In fact, when \( \alpha = 1/2 + \delta \) for some \( \delta > 0 \), the above is equivalent to

\[
(1/2 + \delta)(2p - p^3) + (1/2 - \delta)p > 1.
\]

Writing \( p = 1 - \tau \), we have

\[
(1/2 + \delta)(2p - p^3) + (1/2 - \delta)p = \frac{3}{2} - \tau^2(3/2 - 3\delta + \tau/2 + \delta\tau),
\]

which is greater than 1 for \( \tau = \tau(\delta) > 0 \) small enough.
For any \( p \in (0, 1) \) close enough to 1 satisfying \(^1\), there exist constants \( \xi(\alpha, p), \epsilon(\alpha, p) > 0 \) small enough so that
\[
(\alpha - \xi)(2p^2 - p^4) + (1 - \epsilon)(1 - \alpha - \xi)p^2 > p.
\]
(2)

Fix \( 1/2 < p < 1 \) and \( \xi, \epsilon > 0 \) that satisfy \(^2\).

A vertex \( v \) is called a rich vertex if there are at least \( (\epsilon^2/10^6)n \) edges in \( \cup F \) incident to it. Let \( D \) be the set of all rich vertices. Then
\[
|D| \leq 2|\cup F|/((\epsilon^2/10^6)n) \leq 10^7/\epsilon^2.
\]
(3)

We construct a random vertex set
\[
S := D \cup (V \setminus D)_p
\]
i.e., \( S \) contains the set \( D \) of rich vertices and includes each vertex of \( V \setminus D \) independently with probability \( p \).

Lemma 8. With high probability \(^3\), \( |S| \leq np + n^{2/3} \).

Proof. Note that by construction, \( |S| \) has the same probability distribution as \( |D| + \text{Bin}(n - |D|, p) \).

Set \( Z = \text{Bin}(n - |D|, p) \). Applying Chernoff’s bound (Theorem 3) and (3), we have
\[
P(Z \geq np + n^{2/3} - |D|) \leq P(Z \geq \mathbb{E}Z + n^{2/3} - |D|) \leq \exp(-n^{\Omega(1)}).
\]

Therefore with high probability, \( |S| \leq |D| + Z \leq np + n^{2/3} \).

Lemma 9. With high probability, the number of color classes in \( \mathcal{F}_M \) that have at least one edge contained in \( S \) is at least \( |\mathcal{F}_M| \cdot (2p^2 - p^4) - n^{2/3} \).

Proof. For \( F_i \in \mathcal{F}_M \), let
\[
X_i := \mathbb{1}\{\text{at least one edge in } F_i \text{ is contained in } S\}
\]
be the indicator random variable that some edge \( e \) in \( F_i \) is contained in \( S \). Since each vertex is included in \( S \) independently with probability at least \( p \) and \( X_i \) is an increasing random variable with respect to the probability that a vertex is included in \( S \), by inclusion-exclusion we have
\[
\mathbb{E}X_i \geq 2p^2 - p^4.
\]

Let
\[
X := \sum_{F_i \in \mathcal{F}_M} X_i.
\]
(4)

We have
\[
\mathbb{E}X \geq |\mathcal{F}_M| \cdot (2p^2 - p^4).
\]
(5)

To prove the lemma, we shall apply Chebyshev’s inequality (Theorem 4). For this purpose we have to estimate \( \text{Var} X \). With a look at (4), we have
\[
\text{Var} X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \sum_{F_i, F_j \in \mathcal{F}_M} (\mathbb{E}X_i\mathbb{E}X_j - \mathbb{E}X_i\mathbb{E}X_j).
\]
(6)

\(^1\)An event holds with high probability if the probability of that event tends to 1 as \( n \) tends to infinity.
Note that if the matchings of the colors $i, j$ are vertex-disjoint, then $X_i$ and $X_j$ are independent and $\mathbb{E}X_iX_j - \mathbb{E}X_i\mathbb{E}X_j = 0$.

Applying Claim 2.9 and the observation below (8) in [3], which implies that there are at most $O(n)$ pairs of matchings are not vertex-disjoint, we have

$$\text{Var} X = O(n).$$

Applying Chebyshev’s inequality, we have

$$\mathbb{P}(X \leq |F_M| \cdot (2p^2 - p^4) - n^{2/3}) \leq \mathbb{P}(X \leq \mathbb{E}X - n^{2/3}) \leq \text{Var} X/n^{4/3} = o(1),$$

which completes the proof.

**Lemma 10.** With probability at least 0.9, the number of color classes in $\mathcal{F}_E$ that are contained in $S$ is at least $(1 - \epsilon)|\mathcal{F}_E| \cdot p^2$.

**Proof.** For $F_i \in \mathcal{F}_E$, let

$$Y_i := \mathbb{1}_{\{F_i \subseteq S\}}$$

be the indicator random variable that the edge of color $i$ is contained in $S$. Then $\mathbb{E}Y_i \geq p^2$. Let

$$Y := \sum_{F_i \in \mathcal{F}_E} Y_i.$$

We have

$$\mathbb{E}Y \geq |\mathcal{F}_E| \cdot p^2.$$ 

To prove the lemma, we shall apply Chebyshev’s inequality (Theorem 4). For this purpose we have to estimate $\text{Var} X$. We have

$$\text{Var} Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \sum_{F_i, F_j \in \mathcal{F}_E} (\mathbb{E}Y_i \mathbb{E}Y_j - \mathbb{E}Y_i \mathbb{E}Y_j).$$

Note that if the edges of colors $i, j$ are vertex-disjoint, then $Y_i$ and $Y_j$ are independent and $\mathbb{E}Y_i \mathbb{E}Y_j - \mathbb{E}Y_i \mathbb{E}Y_j = 0$.

Furthermore, if the edges of two distinct color classes $F_i, F_j \in \mathcal{F}_E$, say $uv$ and $uw$, intersect at a rich vertex $u$ that is in $D$, then $\mathbb{E}Y_i Y_j - \mathbb{E}Y_i \mathbb{E}Y_j = \mathbb{P}(v, w \in S) - \mathbb{P}(v \in S)\mathbb{P}(w \in S) = 0$.

Therefore besides the case $F_i = F_j$, the non-zero contribution to $\mathbb{E}Y$ can only come from $F_i, F_j$ that the edges intersect at a non-rich vertex. Note that by definition, such a non-rich vertex is incident to at most $(\epsilon^2/10^6)n$ edges in $\cup \mathcal{F}_M$. Therefore there are at most $2 \cdot |\mathcal{F}_E| \cdot 2 \cdot (\epsilon^2/10^6)n$ such pairs $(F_i, F_j)$.

Therefore using the assumption that $|\mathcal{F}_E| \geq n/40$, for $n$ large enough we have

$$\text{Var} Y \leq |\mathcal{F}_E| + 4|\mathcal{F}_E|((\epsilon^2/10^6)n \leq \frac{8 \cdot 40 \epsilon^2}{10^6} |\mathcal{F}_E| \cdot n/40 \leq \epsilon^2 |\mathcal{F}_E|^2.$$

Therefore by Chebyshev’s inequality and $p \geq 1/2$, we have

$$\mathbb{P}(Y \leq |\mathcal{F}_E| \cdot p^2 - \epsilon |\mathcal{F}_E| \cdot p^2) \leq \mathbb{P}(Y \leq \mathbb{E}Y - \epsilon |\mathcal{F}_E| \cdot p^2) \leq \frac{\text{Var} Y}{(\epsilon |\mathcal{F}_E| \cdot p^2)^2} \leq \frac{\epsilon^2 |\mathcal{F}_E|^2}{10^3 \cdot \epsilon^2 |\mathcal{F}_E|^2 \cdot p^2} \leq 1/10,$$

which completes the proof.  

\[ \square \]
\textbf{Proof of Theorem 7.} Combining Lemmas \ref{lem:3} and \ref{lem:4} and taking a union bound, we know that with positive probability (at least 1/2), all of the following hold:

\[ |S| \leq np + n^{2/3}, \quad X \geq |\mathcal{F}_M| \cdot (2p^2 - p^4) - n^{2/3}, \quad \text{and} \quad Y \geq (1 - \epsilon)|\mathcal{F}_E| \cdot p^2. \]

Therefore there is some $S$ such that all of the above hold. Then the number of rainbow edges contained in $S$ is at least

\[ X + Y \geq |\mathcal{F}_M| \cdot (2p^2 - p^4) - n^{2/3} + (1 - \epsilon)|\mathcal{F}_E| \cdot p^2 \]

\[ \geq (\alpha - \xi)n(2p^2 - p^4) - n^{2/3} + (1 - \epsilon)(1 - \alpha - \xi)n \cdot p^2. \]

With a look at (2) and setting $3c := (\alpha - \xi)(2p^2 - p^4) + (1 - \epsilon)(1 - \alpha - \xi)p^2 - p > 0$, for $n$ large enough we have that $|S| \leq \beta n$ for $\beta := p + c$ and the number of rainbow edges contained in $S$ is at least $(\beta + c)n$. Therefore we complete the proof of Theorem 7. \hfill \Box

\subsection{Sharpness of the condition $\alpha > 1/2$}

To get the logarithmic in $n$ bound on the rainbow girth, it is necessary to assume $\alpha > 1/2$ in Theorem 6. The following $n$-vertex graph $F$ with $n/2$ matchings of size 2 in $\mathcal{F}_M$ and $n/2$ single edges in $\mathcal{F}_E$ (so that $\alpha = 1/2$) has rainbow girth linear in $n$.

For simplicity, we may assume that $n$ is divisible by 4. The vertices of $F$ are $v_{i,j}$ for $i = 1, \ldots, n/4$ and $j = 1, \ldots, 4$. For each $1 \leq i \leq n/4$, the four vertices $v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}$ form a 4-cycle: $\{v_{i,1}v_{i,2}, v_{i,3}v_{i,4}\}$ is a matching of size 2 from $\mathcal{F}_M$ and $v_{i,2}v_{i,3}, v_{i,4}v_{i,1}$ are two edges in two colors from $\mathcal{F}_E$. And $\{v_{i,3}v_{i+1,2}, v_{i,4}v_{i+1,1}\}$ (with the first subscripts module $n/4$) for each $1 \leq i \leq n/4$ is a matching of size 2 from $\mathcal{F}_M$. It can be verified that the $n$-vertex graph $F$ satisfies the assumption but has rainbow girth $n/2$.

\subsection{Matchings and triangles}

\textbf{Theorem 11.} There exists a constant $C > 0$ such that for any $n$-vertex graph $G$ and edge coloring of $G$ with $n$ colors, if each color class is either a matching of size 2 or a triangle, then the rainbow girth is at most $C \log n$.

\textbf{Proof.} Assume $t$ color classes are triangles. If $t < 0.4n$, say, we take all the color classes of matchings to form $\mathcal{F}_M$, and we take one edge from each color class of triangle to form $\mathcal{F}_E$. Let $\alpha := \sup_{x \in [0, 1]} |\mathcal{F}_M| \geq xn$. It is easy to verify that $\mathcal{F}_M$ and $\mathcal{F}_E$ satisfy the assumption of Theorem 6 with $\alpha \geq 0.6$, which implies that the rainbow girth is $O(\log n)$.

Therefore we may assume $t \geq 0.4n$. We take one edge from each matching and two edges from each triangle and we get an $n$-vertex graph with at least $(n - t) + 2t \geq 1.4n$ edges. Theorem 2 implies there is a cycle of length at most $C \log n$. If such a cycle is not rainbow, then any two edges of the same color must come from a triangle. We can replace two edges in the same color by the other edge in the triangle to get a shorter cycle. Do it repeatedly until we obtain a rainbow cycle, which is of length at most $C \log n$. We complete the proof. \hfill \Box

\subsection{Triangles and single edges}

\textbf{Theorem 12.} For any constants $0 \leq \alpha < 1$ and $0 \leq \xi \leq \alpha$ with $\xi < (1 - \alpha)/3$, if an $n$-vertex graph $G$ and edge coloring of $G$ satisfying that at least $(\alpha - \xi)n$ color classes consisting of a single
edge and at least \((1 - \alpha - \xi)n\) color classes consisting of a triangle, then the rainbow girth is at most \(C \log n\) for some constant \(C(\alpha, \xi) > 0\).

Proof. The proof is similar as Theorem 11: we take each of the single edge and two edges from each triangle. Then we have at least \((\alpha - \xi)n + 2 \cdot (1 - \alpha - \xi)n = (2 - \alpha - 3\xi)n\) edges. Since \(2 - \alpha - 3\xi > 1\), Theorem 2 implies that there is a cycle of length at most \(C \log n\) for some constant \(C(\alpha, \xi) > 0\). If this cycle is not rainbow, we can replace two edges of the same color, which comes from a same triangle, by the other edge in the triangle to get a shorter cycle. Do it repeatedly until we obtain a rainbow cycle, which is of length at most \(C \log n\). We complete the proof.

3.3.1 Sharpness of the condition \(\alpha < 1\)

To get a logarithmic in \(n\) bound on the rainbow grith, we need \(\alpha < 1\) in Theorem 12. Otherwise for \(\alpha = 1\), an \(n\)-cycle with \(n\) edges in distinct colors has rainbow girth \(n\).

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References

[1] R. Aharoni, E. Berger, M. Chudnovsky, H. Guo, and S. Zerbib. Non-uniform degrees and rainbow versions of the Caccetta-Häggkvist conjecture. Preprint (2021). [arXiv:2110.11183]
[2] R. Aharoni, M. DeVos, and R. Holzman. Rainbow triangles and the Caccetta-Häggkvist conjecture. J. Graph Theory 92 (2019), 347–360.
[3] R. Aharoni and H. Guo. Rainbow cycles for families of matchings. Israel Journal of Mathematics, to appear. [arXiv:2110.14332]
[4] B. Bollobás and E. Szemerédi. Girth of sparse graphs. J. Graph Theory 39 (2002), 194–200.
[5] J.A. Bondy. Counting subgraphs: A new approach to the Caccetta-Häggkvist conjecture. Discrete Math. 165/166 (1997), 71–80.
[6] L. Caccetta and R. Häggkvist. On minimal digraphs with given girth. Congress. Numer. 21 (1978), 181–187.
[7] V. Chvátal and E. Szemerédi. Short cycles in directed graphs. J. Combin. Theory Ser. B 35 (1983), 323–327.
[8] M. DeVos, M. Drescher, D. Funk, S. González Hermosillo de la Maza, K. Guo, T. Huynh, B. Mohar, and A. Montejano. Short rainbow cycles in graphs and matroids. J. Graph Theory 96 (2021), 192–202.
[9] Y.O. Hamideh. A note on minimal directed graphs with given girth. J. Combin. Theory Ser. B 43 (1987), 343–348.
[10] C.T. Hoang and B. Reed. A note on short cycles in diagraphs. Discrete Math. 66 (1987), 103–107.
[11] P. Hompe and S. Spirkl. Further approximations for Aharoni’s rainbow generalization of the Caccetta-Häggkvist conjecture. Electron. J. Combin. 29 (2022), #P1.55.
[12] T. Nishimura. Short cycles in digraphs. Discrete Math. 38 (1988), 295–298.
[13] A.A. Razborov. On the Caccetta-Häggkvist conjecture with forbidden subgraphs. J. Graph Theory 74 (2013), 236–248.
[14] J. Shen. On the girth of digraphs. Discrete Math. 211 (2000), 167–181.
[15] J. Shen. On the Caccetta-Häggkvist conjecture. Graphs Combin. 18 (2002), 645–654.