Renormalization flow for extreme value statistics of random variables raised to a varying power

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Abstract
Using a renormalization approach, we study the asymptotic limit distribution of the maximum value in a set of independent and identically distributed random variables raised to a power $q_n$ that varies monotonically with the sample size $n$. Under these conditions, a non-standard class of max-stable limit distributions, which mirror the classical ones, emerges. Furthermore, a transition mechanism between the classical and the non-standard limit distributions is brought to light. If $q_n$ grows slower than a characteristic function $q^\star_n$, the standard limit distributions are recovered, while if $q_n$ behaves asymptotically as $\lambda q^\star_n$, non-standard limit distributions emerge.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Extreme value statistics, which is the statistics of the largest value in a set of random variables, has attracted a lot of attention in many different fields, from probability theory [1] to physics—where fecund interactions with disordered systems [2–5], as well as random walks and interface fluctuations [6–8] have recently flourished—hydrology [9], finance [10, 11] or engineering [12]. For independent and identically distributed (i.i.d.) random variables, asymptotic distributions have been known for long [1, 12–14]. Depending on the tail of the parent distribution (from which the variables in the set are drawn at random), three different distributions emerge. For parent distributions with a tail decaying faster than any power law, the limit distribution is the well-known Gumbel one (which also found interesting applications outside the field of extreme values [15]). If the parent distribution decays as a power law close to infinity, the so-called Fréchet distribution is obtained, while if it decays algebraically close to an upper bound, the Weibull distribution is reached.

In spirit, these results bear some similarities with the central limit theorem, which addresses a similar issue for the problem of random sums instead of extreme values. Interestingly, it has been recently shown that the limit distribution of sums can be modified...
by raising the summed variables to a power that diverges with the number of terms in the sum [16]. Such a problem is actually motivated by the physics of disordered systems, as it can be interpreted as the partition function of the random energy model [17], one of the simplest disordered models—which led to recent developments in relation to extreme value statistics [3–5]. This problem also exhibits interesting connections to empirical moment estimation in signal processing and multifractal analysis [18]. It is then natural to wonder whether such a procedure, namely raising the random variable to a power increasing with the sample size, could generate some non-standard distributions as far as extreme values are concerned. In terms of random energy model, this would mean considering the statistics of the maximum value of the Boltzmann weights (which add up to the partition function). A related, but perhaps more concrete, physical example is the statistics of the largest trapping time in a trap model [19], in which particles are trapped in deep energy wells and can escape only through thermal activation. These extreme times are known to play an important role in this context. In the limits of low temperature and large number of traps, the statistics of the largest trapping time could depend on the way the two limits are taken.

This issue has been recently addressed in the mathematical literature [20], following the work by Ben Arous and coworkers on the problem of sums, obtaining precise results about a transition between the Gumbel attraction domain and the Fréchet attraction domain for a specific class of distributions. In addition, this problem has some connections with the question of the existence of different limit distributions using power rescaling procedures [21–25].

In this contribution, we address the general issue of the limit distribution of the maximum value in a set of random variables raised to a power exponent diverging with sample size, by generalizing the renormalization group approach recently introduced to deal with finite-size effects in standard extreme value statistics [26–28]. For exponents increasing as a power law of the sample size, we find non-standard limit distributions, which turn out to be related by an exponential change of variables to the standard limit distributions. We clarify this surprising relationship using a simple argument based on the behavior of the rescaling factors.

2. Problem statement

Starting from a set of i.i.d. variables \((W_i)_{i=1,...,n}\), we consider the maximum \(M_n^W\) in the set, namely

\[
M_n^W = \max\{W_1, \ldots, W_n\}. \tag{1}
\]

Classical extreme value theorems yield asymptotic convergence results for \(M_n^W\) as a function of the behavior of the tail of the probability distribution of \(W\). More precisely, there exist two sequences \(\alpha_n\) and \(\beta_n\) such that the cumulative distribution of the rescaled random variable \(Y_n = (M_n^W - \beta_n)/\alpha_n\) converges to the limit cumulative distribution \(F_\zeta(y)\) defined by

\[
F_\zeta(y) = \begin{cases} 
  e^{-(1+\zeta y)^{-1/\zeta}}, & \text{if } \zeta > 0 \quad (y > -1/\zeta), \\
  e^{-\gamma}, & \text{if } \zeta = 0, \\
  e^{-(1+\zeta y)^{-1/\zeta}}, & \text{if } \zeta < 0 \quad (y < -1/\zeta).
\end{cases} \tag{2}
\]

The case \(\zeta > 0\) corresponds to variables \(W_i\) with a distribution decaying as a power law at infinity, while the case \(\zeta < 0\) corresponds to a power-law decay close to an upper bound. Finally, the value \(\zeta = 0\) is obtained for distributions decaying faster than any power law (either at infinity or close to an upper bound) [1, 12–14].

We wish to investigate whether an \(n\)-dependent transformation of the variables \(W_i\) may lead to asymptotic distributions different from the present ones. We are especially interested
in power transformations of the form \( U_{i,n} = W_i^{q_n} \), where \( q_n \) depends on \( n \), but begin by considering the general class of transformations

\[
U_{i,n} = \omega_n^{-1}(W_i), \quad i = 1, \ldots, n, \tag{3}
\]

where \( \omega_n \) consists in an increasing bijective function\(^1\). One can express the cumulative distribution of \( U \) as

\[
F_{U,n}(u) = F_W(\omega_n(u)). \tag{4}
\]

In the power transformation case, this definition of \( \omega_n(u) \) leads to

\[
\omega_n(u) = u^{1/q_n}, \quad u > 0. \tag{5}
\]

This transformation is reminiscent of the study in [16] concerning the behavior of sums of random exponentials and notably the failure of the classical central limit theorem for rapidly growing powers.

On the one hand, if \( \omega_n \) varies sufficiently slowly as a function of \( n \), it is expected that the transformation \( \omega_n \) does not affect the limit distribution of the maximum. On the other hand, for well-chosen transformations, it should be possible to attain new types of limit distributions. Let us consider the transformed maximum \( MU_n = \max\{U_{1,n}, \ldots, U_{n,n}\} \).

The cumulative distribution \( F_n(m) \) of \( MU_n \) can be expressed in terms of the cumulative function \( F_W \) of the variable \( W \) as

\[
F_n(m) = F_W(\omega_n(m))^n. \tag{6}
\]

In the following section, we devise a renormalization group formulation of equation (6), which allows us to derive in a straightforward way the possible fixed point distributions.

3. Renormalization approach

3.1. Renormalization transformation and standardization conditions

Following [27, 28], we introduce the functions \( g_n(m) = -\ln[-\ln F_n(m)] \), as well as \( g_W(w) = -\ln[-\ln F_W(w)] \), and recast equation (6) into the form

\[
g_n(m) = g_W(\omega_n(m)) - \ln n. \tag{7}
\]

As in the case of standard convergence theorems, it is useful in order to converge to a non-degenerate limit distribution to rescale the maximum value \( M_n^k \) through \( X_n = (M_n^k - b_n)/a_n \), where \( a_n \) and \( b_n \) are chosen so as to meet some specific conditions (for instance, fixing the values of the first two moments). In addition, it is also convenient to consider \( n \) as a real variable rather than an integer one and to define the variable \( s = \ln n \). We thus assume that \( \omega_n(m) \) can be extended to real values of \( n \), and we define the function \( \omega(m,s) = \omega_n(m) \).

Altogether, one obtains from equation (7) the following evolution equation, in terms of the variable \( X \equiv X_{n,s} \):

\[
g(x,s) = g_W(\omega(a(s)x + b(s), s)) - s, \tag{8}
\]

where \( \exp\left[-\exp(-g(x,s))\right] \) is the cumulative distribution of \( X \). In order to determine \( a(s) \) and \( b(s) \), one needs to impose ‘standardization’ conditions on \( g(x, s) \). Such constraints are arbitrary to some extent and may differ depending whether one is interested in practical problems or in theoretical approaches. In practical applications, fixing some moments of the distribution (e.g., the first two moments) may be convenient. In contrast, it turns out that for theoretical

\(^1\) Considering decreasing bijective functions is another possibility, which would require only minor changes.
purposes, fixing the value of $g(x, s)$ and of its derivative $\partial_x g(x, s)$ at a given value of $x$ is an easier condition to implement. We thus choose the same conditions as in [28], namely
\begin{align}
g(0, s) &= 0, \\
\partial_x g(0, s) &= 1.
\end{align}
These conditions imply
\begin{align}
g_w(\omega(b(s), s)) &= s, \\
a(s) &= b'(s) + \frac{\partial_x \omega(b(s), s)}{\partial_s \omega(b(s), s)},
\end{align}
with $b'(s)$ being the derivative of $b(s)$ and $\partial_s \omega$ indicates the derivative with respect to the first argument of $\omega(m, s)$. In order to simplify the expression of $g_w$, we contract the transformation $\omega(x, s)$ and the rescaling operation into a single transformation $\mathcal{T}(x, s)$:
\begin{align}
\mathcal{T}(x, s) &= \omega(a(s)x + b(s), s),
\end{align}
which leads to
\begin{align}
g(x, s) &= g_w(\mathcal{T}(x, s)) - s.
\end{align}

3.2. Partial differential equation for the flow
The functional equation (13) can be converted into a partial differential equation. We first differentiate $g(x, s)$ with respect to $x$ and $s$:
\begin{align}
\partial_x g(x, s) &= g'_w(\mathcal{T}(x, s)) \partial_x \mathcal{T}(x, s), \\
\partial_s g(x, s) &= g'_w(\mathcal{T}(x, s)) \partial_s \mathcal{T}(x, s) - 1.
\end{align}
Reinjecting equation (14) into equation (15) to eliminate $g'_w$, we obtain
\begin{align}
\partial_x g(x, s) &= \mathcal{U}(x, s) \partial_s g(x, s) - 1,
\end{align}
where we have defined
\begin{align}
\mathcal{U}(x, s) = \frac{\partial_x \mathcal{T}(x, s)}{\partial_s \mathcal{T}(x, s)}.
\end{align}
The function $\mathcal{U}(x, s)$ can be expressed explicitly using equations (12) and (17). We start by computing $\partial_x \mathcal{T}$ and $\partial_s \mathcal{T}$ (in order to lighten the notations, we drop in the following the explicit $s$ dependence of the parameters $a(s)$ and $b(s)$):
\begin{align}
\partial_x \mathcal{T}(x, s) &= a \partial_m \omega(ax + b, s), \\
\partial_s \mathcal{T}(x, s) &= (a'x + b') \partial_m \omega(ax + b, s) + \partial_m \omega(ax + b, s).
\end{align}
We finally obtain
\begin{align}
\mathcal{U}(x, s) &= \frac{a' \partial_x \omega(ax + b, s)}{\partial_m \omega(ax + b, s)} + \frac{1}{a} \partial_s \omega(ax + b, s).
\end{align}
From now on, we focus on the case of a power-law transformation $\omega(m, s) = m^{1/q(s)}$, with $m > 0$. One has
\begin{align}
\partial_m \omega(m, s) &= \frac{\omega(m, s)}{q(s)m}, \\
\partial_s \omega(m, s) &= -\frac{q'(s) \ln m}{q(s)^2} \omega(m, s),
\end{align}
Throughout this paper, we use the notations $\partial_x = \partial/\partial x$ and $\partial_s = \partial/\partial s$. 

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and \( U(x, s) \) reads
\[
U(x, s) = \frac{a'}{a} x + \frac{b'}{a} - \frac{q'}{q} \left( x + \frac{b}{a} \right) \ln(ax + b).
\] (23)

Taking into account the standardization condition (11), which reads, in the case of a power-law transformation \( \omega(m, s) \),
\[
a = b' - \frac{q'}{q} b \ln b,
\] (24)
one can rewrite equation (23) as
\[
U(x, s) = \left( \frac{a}{b} + \partial_s \ln \frac{a}{b} \right) x + 1 - \frac{q'}{q} \left( x + \frac{b}{a} \right) \ln \left( \frac{a}{b} x + 1 \right).
\] (25)

If one defines
\[
\lambda(s) = \frac{a(s)}{b(s)}, \quad \delta(s) = \partial_s \ln \lambda(s), \quad \gamma(s) = \lambda(s) + \delta(s), \quad Q(s) = \frac{q'(s)}{q(s)},
\] (26)

it is possible to rewrite equation (25) in a more compact way as
\[
U(x, s) = 1 + \gamma(s)x - Q(s) \left( \lambda(s)x + 1 \right) \ln \left( \lambda(s)x + 1 \right) \left( \frac{\ln(\lambda(s)x + 1)}{\lambda(s)} \right).
\] (27)

### 3.3. Fixed point distributions

A stationary solution, which is a solution of equation (16) satisfying \( \partial_s g(x, s) = 0 \), can be obtained on condition that \( U(x, s) \) is independent of \( s \), namely \( U(x, s) = U(x) \). In this case, the stationary solution \( g(x) \) is determined by integrating the differential equation
\[
g'(x) = \frac{1}{U(x)},
\] (28)

with the condition \( g(0) = 0 \).

We now investigate under which condition \( U(x, s) \) becomes independent of \( s \).

### 3.4. Case \( Q = 0 \): recovering standard limit distributions

In the case \( Q = 0 \), one has
\[
U(x, s) = 1 + \gamma(s)x.
\] (29)

The condition \( \partial_s U = 0 \) yields that \( \gamma(s) \) must be equal to a constant \( \gamma \). Consequently,
\[
g'(x) = \frac{1}{1 + \gamma x}.
\] (30)

Taking into account the standardization condition (9), we obtain the fixed point function
\[
g(x) = \frac{1}{\gamma} \ln(1 + \gamma x).
\] (31)

Reformulating this result in terms of the cumulative distribution \( F(x) = \exp[-\exp(-g(x))] \),
one recovers the standard limit distributions given in equation (2):
\[
F(x) = F_{\gamma}(x) = \exp \left[ -\left( 1 + \gamma x \right)^{-\frac{1}{\gamma}} \right], \quad 1 + \gamma x > 0,
\] (32)

where \( \gamma \) plays the role of the parameter \( \xi \). Hence, classical limit distributions are retrieved in the case where the power \( q(s) \) is a constant, namely \( q(s) = q_0 \). This result was expected: if \( X \) belongs to the attraction domain of \( F_{\xi} \), \( X_{q_0} \) either belongs to the attraction domain of \( F_{\xi/q_0} \) for \( \xi > 0 \), or to the attraction domain of \( F_{\xi} \) otherwise.
3.5. Case $Q \neq 0$: emergence of non-standard stable distributions

We now turn to the case $Q \neq 0$. From equation (27), it is clear that $U(x, s)$ is independent of $s$ if $Q(s)$ and $\lambda(s)$ are constants: $Q(s) = Q$ and $\lambda(s) = \lambda > 0$. Hence, stationary solutions only exist if $g(x)$ is of the form $g(x) = Ke^{Qx}$, with $K > 0$ being a real constant. Inserting $Q$ and $\lambda$ in equation (25), $U(x)$ takes the form

$$U(x) = 1 + \lambda x - \frac{Q}{\lambda} (1 + \lambda x) \ln(1 + \lambda x).$$

Combining equation (33) with equation (28) leads to

$$g'(x) = \frac{1}{(1 + \lambda x) \left(1 - \frac{Q}{\lambda} \ln(1 + \lambda x)\right)}.$$  

By definition, $g(x)$ has to be an increasing function of $x$ so that $g'(x) \geq 0$, which implies that $x$ belongs to a restricted range of values, $x_{\min} < x < x_{\max}$. The bounds $x_{\min}$ and $x_{\max}$ are determined by the conditions $1 + \gamma x > 0$ and $1 - \frac{Q}{\lambda} \ln(1 + \lambda x) > 0$. Assuming $Q > 0$, one finds

$$x_{\min} = \frac{1}{\lambda}, \quad x_{\max} = \frac{1}{\lambda} (e^{\lambda/Q} - 1), \quad \lambda > 0.$$  

Similarly, for $Q < 0$,

$$x_{\min} = \frac{1}{\lambda} (e^{\lambda/Q} - 1), \quad x_{\max} = +\infty, \quad \lambda > 0.$$  

Coming back to equation (34), this equation can be integrated into

$$g(x) = -\frac{1}{Q} \ln \left(1 - \frac{Q}{\gamma} \ln(1 + \gamma x)\right), \quad x_{\min} < x < x_{\max},$$

also taking into account the condition $g(0) = 0$. The corresponding cumulative distribution $F_{\lambda,Q}(x)$ reads

$$F_{\lambda,Q}(x) = \exp \left[-\left(1 - \frac{Q}{\gamma} \ln(1 + \lambda x)\right)^{1/Q}\right], \quad x_{\min} < x < x_{\max},$$

which generalizes the standard extreme value distributions; for $Q = 1$, this expression reduces to a power law on the interval $[x_{\min}, x_{\max}]$. Note that expression (38) of $F_{\lambda,Q}(x)$ converges, in the limit $Q \to 0$, to the standard distribution $F_{\lambda,1}(x)$ given in equation (32). In addition, it is interesting to note that in the limit $\lambda \to 0$, the cumulative $F_{\lambda,Q}$ converges to $F_{-Q}$.

Remarkably, these generalized extreme value distributions are closely related to the standard ones. In term of cumulative, expression (38) implies that

$$F_{\lambda,Q}(x) = F_{-Q}(\ln(1 + \lambda x)/\lambda).$$

Incidentally, setting $\zeta = -Q$, equation (39) implies that if $Z$ is a random variable with cumulative $F_{\zeta}$, then the cumulative of $[\exp(\lambda Z) - 1]/\lambda$ is $F_{\lambda,-\zeta}$. Note that equation (39) is also valid for $Q = 0$ and corresponds in this case to a known relation between the Gumbel and the Fréchet distributions.

It should also be noted that for $Q > 0$, these limits distributions are equivalent, up to an affine transformation, to the non-standard distribution obtained in [25] using power transformations as rescaling factors. This suggests that $q(s)$ plays a role akin to a rescaling factor. However, let us emphasize that our approach is different in spirit from that of [25]. In the latter, the power $q_0$ is used as an adjustable rescaling parameter allowing the transformed distribution to converge to a non-degenerate limit. In this paper, we consider the power $q_0$ as a given function, and we study the non-degenerate limit distributions that are obtained through...
a standard affine rescaling of the data. It is then a priori not obvious that the same stable distributions should emerge from the two procedures.

Deriving the probability density function \( p_{\lambda, Q}(x) \) of these generalized extreme value distributions yields the expression

\[
p_{\lambda, Q}(x) = \frac{1}{1 + \lambda x} \left[ 1 - \frac{Q}{\lambda} \ln(1 + \lambda x) \right]^\frac{1}{\lambda - 1} \exp \left[ -\left( 1 - \frac{Q}{\lambda} \ln(1 + \lambda x) \right)^{1/Q} \right]. \tag{40}
\]

As depicted in figure 1, this family of probability density functions has a non-trivial behavior near its bounds. For \( Q > 0 \), the support of \( p_{\lambda, Q}(x) \) is \((-1/\lambda, [\exp(\lambda/Q) - 1]/\lambda)\). If we consider the asymptotic behavior in the limit \( x \equiv -1/\lambda + \epsilon \), with \( \epsilon \to 0 \), we obtain

\[
p_{\lambda, Q} \left( \frac{1}{\lambda} + \epsilon \right) \approx \frac{1}{\epsilon} \left[ \frac{Q}{\lambda} \ln(\epsilon) \right]^\frac{1}{\lambda - 1} \epsilon^{\left( \frac{Q}{\lambda} \ln(\epsilon) \right)^{1/Q}}. \tag{41}
\]

Therefore, we have a crossover at \( Q = 1 \), where the probability density corresponds to a power law of exponent \( 1/\lambda - 1 \). For \( Q > 1 \), \( p_{\lambda, Q}(x) \) diverges when \( x \to -1/\lambda \) faster than any power functions, whereas for \( Q < 1 \) and all \( n \), \( p_{\lambda, Q}(x) \) and all its derivatives converge to 0.

Similarly, in the neighborhood of the upper bound \( x \equiv [\exp(\lambda/Q) - 1]/\lambda - \epsilon \), we have

\[
p_{\lambda, Q} (x_{\text{max}} - \epsilon) \approx e^{-\lambda/Q} \left( Q e^{-\lambda/Q} \epsilon \right)^{\frac{1}{\lambda - 1}}, \tag{42}
\]

and a singularity appears at the upper bound of the distribution for \( Q > 1 \).

Conversely, for \( Q < 0 \), \( (\exp(\lambda/Q) - 1)/\lambda \) becomes the lower bound of this distribution, and using \( x \equiv x_{\text{min}} + \epsilon \) and \( \epsilon \to 0 \) leads to

\[
p_{\lambda, Q} (x_{\text{min}} + \epsilon) \approx e^{-\lambda/Q} \left( |Q| e^{-\lambda/Q} \epsilon \right)^{\frac{1}{\lambda - 1}} \exp \left[ \left( |Q| e^{-\lambda/Q} \epsilon \right)^{\frac{1}{\lambda - 1}} \right]. \tag{43}
\]

![Figure 1. The probability density function \( p_{\lambda, Q}(x) \) associated with the generalized extreme distribution \( F_{\lambda, Q}(x) \). Top left: \( Q = 0.7 \), the solid line \( \lambda = 0.5 \) and the dashed line \( \lambda = 1.2 \). Top right: \( Q = 1 \), the solid line \( \lambda = 1 \) and the dashed line \( \lambda = 0.4 \). Bottom left: \( Q = 1.5 \), the solid line \( \lambda = 2 \) and the dashed line \( \lambda = 1 \). Bottom right: \( Q = -1 \), the solid line \( \lambda = 1 \) and the dashed line \( \lambda = 0.01 \).](image-url)
Since $Q < 0$, the exponential term is dominant and the probability density function and all its
derivatives converge to 0 in the neighborhood of $x_{\text{min}}$. On the other hand, for $x \to +\infty$, $F(x)$
behaves as the exponential of the power of a logarithm:

$$F(x) \approx x \to +\infty \exp \left[ -\frac{Q}{\lambda} \ln x \right], \quad (44)$$

There is therefore four distinct asymptotic behaviors for the function $p_{\lambda, Q}(x)$ depending on
the value of $Q$ ($Q < 0$, $0 < Q < 1$, $Q = 1$, $Q > 1$). The function $p_{\lambda, Q}(x)$ is shown in figure 1
for several values corresponding to these distinct domains.

4. Attraction domain of the non-standard limit laws

Having found non-standard asymptotic forms of the extreme value distribution in section 3,
we now consider their domain of attraction and we use the partial differential equation of the
flow to develop a heuristic description of these attraction domains. This description provides
a good approximation of the standard attraction domains and gives us precise insights into
the non-standard attraction domains. Finally, these heuristic results are confirmed in section 5
using an independent approach, which sheds some light on the relationship between standard
and non-standard laws.

Although the stationary equation resulting from (28) and (33) cannot be used directly to
describe the dynamic of the general equation (16), it underlines at least the existence of two
important control parameters in the stationary case, namely $Q$ and $\lambda$.

If we return to the general flow equation (16) and the general expression of $U$ described by
equation (27), these two variables can be interpreted as an external forcing. Heuristically, the
convergence of equation (16) should be driven by the asymptotic behavior of these parameters.
Notably, if $\gamma(s)$ and $Q(s)$ converge simultaneously toward the respective finite limits $\Gamma$ and $Q$,
it is plausible to expect the transformed maximum to converge toward the limit law $F_{\Gamma, Q}$.
As a corollary, for laws belonging to a classical domain with a parameter $\zeta$ in the case $Q(s) = 0$,
one should obtain $\Gamma = \zeta$, which sets an interesting test case for our heuristic argument.

4.1. Asymptotic behaviors of the forcing parameter

If the term $Q(s)$ clearly denotes the external forcing due to the power transformation, the
interpretation of $\lambda(s)$ and $\delta(s)$ is hazier. From its definition,

$$\lambda(s) = \partial_s \ln b(s) - q'(s) \ln b(s)$$

$$= \frac{q(s)}{g^{-1}(s)g'(g^{-1}(s))} = q(s)\lambda_0(s). \quad (45)$$

It is thus possible to factorize $\lambda(s)$ into the factor $q(s)$ and a term $\lambda_0(s)$ depending only on
the cumulative function of the parent distribution. Using the change of variable $s = g(x)$ and the
relation $g(x) = -\ln(-\ln F(x))$ leads to

$$\lambda_0(g(x)) = \frac{|F(x) \ln F(x)|}{xF'(x)}. \quad (46)$$

One should note that $\lambda_0(s) > 0$ due to the properties of the cumulative function. Moreover,
defining the complementary cumulative function $\bar{F}(x) = 1 - F(x)$ leads to the asymptotic
expression

$$\lambda_0(g(x)) \approx \frac{\bar{F}(x)}{x\bar{F}(x)}, \quad (47)$$
The variable $\lambda_0(s)$ corresponds to the inverse of the local power exponent of the complementary cumulative function $F$ at point $x = g(s)$. If $\lambda_0(s)$ admits a nonzero finite limit $\Lambda_0$ at $+\infty$, there exists a normalized slowly varying function $L$, such that

$$F(x) = L(x)x^{-1/\Lambda_0}.$$  \hfill (48)

In other words, the parameter $\Lambda_0$ describes the power behavior of the tail of the distribution. Since for $Q(s) = 0$, $\lambda(s)$ and $\lambda_0(s)$ are identical, laws belonging to the classical domains of attraction provide enlightening examples of the asymptotic behavior of $\lambda_0(s)$. Notably, if a law belongs to the Fréchet attraction domain of parameter $\zeta$, there is a slowly varying function $L$ such that

$$F(x) = L(x)x^{-1/\zeta}. \hfill (49)$$

With the regularity assumption that $L$ is normalized, we obtain that $\Lambda_0 = \zeta$. Conversely, for a law belonging to the Weibull domain, we have a slowly varying function $L$ such that

$$F(x) = (x_F - x)^{-1/\zeta}L((x_F - x)^{-1}), \hfill (50)$$

where $x_F$ is the end point of the distribution. Under the assumption that $L$ is normalized, we have

$$\lambda_0(g(x)) \sim \frac{\zeta}{x_F - x} \hfill (51)$$

The Gumbel domain of attraction is harder to characterize. However, the ‘exponential power’ laws

$$F(x) = \exp(-x^\alpha) \hfill (52)$$

often constitute an interesting subset of this domain. For such laws, a short calculation leads to

$$\lambda_0(g(x)) \sim x^{1-\alpha} \hfill (53)$$

In the general case, a law belongs to the Gumbel attraction domain, if and only if $1/F(x)$ is a rapidly varying function. Using an integral characterization of rapidly varying functions [29, p 178], it is possible to show that under a regularity assumption, for any law belonging to the Gumbel attraction domain, $\lambda_0(s) \to 0$. Consequently, for laws in the Weibull and Gumbel attraction domains, we have $\Lambda_0 = 0$. In summary, for laws belonging to a standard attraction domain of parameter $\zeta$, one has $\Lambda_0 = \max(0, \zeta)$.

Similarly, $\delta(s)$ can be decomposed into

$$\delta(s) = Q(s) + \delta_0(s), \quad \delta_0(s) = \partial_s \ln \lambda_0(s). \hfill (54)$$

One interesting consequence of the previous equation is that the asymptotic behaviors of $\delta_0(s)$ and $\lambda_0(s)$ are entwined. If we assume that both $\delta_0(s)$ and $\lambda_0(s)$ admit a finite limit when $s$ tends to $+\infty$, with $\lim_{s \to +\infty} \lambda_0(s) > 0$, then one necessarily has $\lim_{s \to +\infty} \delta_0 = 0$. This situation corresponds to the Fréchet class. Conversely, if $\lim_{s \to +\infty} \lambda_0(s) = 0$ (a typical situation in the Gumbel and Weibull classes), and if $\delta_0(s)$ admits a finite limit $\Delta_0$ when $s \to +\infty$, then $\delta_0 < 0$. After some tedious calculations, one can expand expression (54) of $\delta_0(s)$ as

$$\delta_0(g(x)) = \ln F(x) = \frac{F''(x) \ln F(x)}{F'(x)^2} - 1 - \lambda_0(g(x)) \hfill (55)$$

\[ \sim \frac{F(x) F''(x)}{F'(x)^2} - 1 - \lambda_0(g(x)). \]

3 The slowly varying functions class gathers constants, logarithms and all functions satisfying the asymptotic relation: $\forall x > 0$, $L(ax) \sim L(x)$. A slowly varying function is said to be normalized if $\lim_{x \to +\infty} L'(x)/L(x) = 0$ [29, p 15]. Since we are only interested in the generic properties of slowly varying functions, we most often use the same notation $L$ for any function belonging to this class.
Going on with our study of the classical domains, we consider a law belonging to the Weibull attraction domain with a parameter $\xi$ and slowly varying function $L$. Under the regularity assumption that both $L$ and $L'$ are normalized slowly varying functions, it is possible to verify that $\Delta_0 = \xi$. The same computation for the ‘exponential power’ laws (a typical example of laws belonging to the Gumbel domain) leads to

$$
\delta_0(q(s)) \sim \frac{1}{1+\beta} - x^{-\beta},
$$

and therefore, $\Delta_0 = 0$. In other words, the limits $\Delta_0$ and $\Delta_0$ are, respectively, the positive and negative parts of $\Gamma_0 = \lim_{s \to +\infty}(\delta_0(s) + \gamma_0(s))$, namely $\Delta_0 = \max(0, \Gamma_0)$ and $\Delta_0 = \min(0, \Gamma_0)$. For a law belonging to the standard attraction domain of parameter $\xi$, we have as expected $\Gamma_0 = \xi$. So, for $Q = 0$, we have recovered the classical results concerning the attraction domain (ignoring the difference in regularity assumptions). From this point, it is easy to extend this convergence result to the non-standard case ($Q \neq 0$).

Piecing together equation (45) and (54), $\gamma(s)$ can also be rewritten as

$$
\gamma(s) = \delta_0(s) + Q(s) + q(s)\lambda_0(s).
$$

The previous expression outlines an interesting interplay between the tail behavior of the parent distribution and the power transformation. The tail behavior of the distribution is responsible for the parameter $\delta_0$, whereas $Q(s)$ is directly derived from the choice of $q(s)$. The term $q(s)\lambda_0(s)$ represents the interaction between the two effects.

### 4.2. Changing the attraction domain by varying $q(s)$

If we consider a fixed parent law, the functions $\delta_0(s)$ and $\lambda_0(s)$ are then fixed. Therefore, the only free parameter is $q(s)$. In this situation, the limit $\Gamma$ is determined by the asymptotic behavior of $\lambda(s) = q(s)\lambda_0(s)$. Let us introduce the limit $\Lambda = \lim_{s \to +\infty} \lambda(s)$, when it exists. If we define a characteristic power scale by

$$
q^*(s) = \frac{1}{\lambda_0(s)},
$$

we obtain

$$
\Lambda = \lim_{s \to +\infty} \frac{q(s)}{q^*(s)}.
$$

In other words, taking the limit $s \to +\infty$ in equation (57), one finds that $\Gamma$ is only dependent on $\Delta_0$, $Q$ and on the limit ratio between $q(s)$ and the characteristic power scale $q^*(s)$. From this point, two different situations arise. On the one hand, a special case appears if $\lim_{s \to +\infty} q(s) = 0$. Indeed, if $\lim_{s \to +\infty} \lambda(s) = 0$, the expression of $U(x, s)$ can be linearized as

$$
U(x, s) \approx 1 + \gamma(s)x - Q(s)x.
$$

So if $\gamma(s)$ and $Q(s)$ admit the finite limits $\Gamma$ and $Q$, we obtain a partial differential equation corresponding to a standard limit distribution with the parameter $\xi = \Delta_0$. This implies that the Weibull and Gumbel domains are unaffected by such a transformation. However, for a law belonging to the Fréchet domain of parameter $\xi$, we have $\Delta_0 = \xi$ and $\Delta = 0$. This means that $\Lambda = 0$ is only possible with $\lim_{s \to +\infty} q(s) = 0$ and $Q < 0$. In other words, a decreasing power destabilizes all the Fréchet domains and any law belonging to a Fréchet domain will converge toward a Gumbel distribution once exposed to a vanishing power transformation.

On the other hand, non-standard limit laws appear when $q(s) \sim \Lambda q^*(s)$. In this case, we have

$$
\Gamma = \Delta_0 + Q + \Lambda.
$$
Moreover, the relation $Q(s) = q'(s)/q(s)$ leads to

$$Q(s) = -\delta_0(s) + \frac{\lambda'(s)}{\lambda(s)}. \quad (62)$$

So, if we assume that $\lim_{s\to+\infty} \lambda'(s)/\lambda(s) = 0$, one obtains that

$$\left\{ \begin{array}{l}
Q = -\Delta_0, \\
\Gamma = \lambda.
\end{array} \right. \quad (63)$$

So using a power transformation with $q(s) \sim \lambda q^*(s)$ leads to the non-standard limit law $F_{\lambda,-\Delta_0}$. Moreover, it is possible to compute an analytic representation of $q_n^*$ by defining the error term $\epsilon(s) = Q(s) + \Delta_0$. Using equation (26) leads to the exact differential equation

$$\partial_s \ln q^*(s) = -\Delta_0 + \epsilon(s). \quad (64)$$

Solving this equation results in

$$q_n^* = L^*(n)n^{-\Delta_0}, \quad (65)$$

$$L^*(n) = \exp \left( \int_0^n \frac{\epsilon(t)}{t} \, dt \right). \quad (66)$$

The factor $L^*(n)$ corresponds to a corrective term depending on the fine convergence structure of $\delta_0(s)$ and $\lambda(s)$. Moreover, since $\epsilon(\ln n) \to 0$, equation (66) corresponds exactly to the integral representation of a normalized slowly varying function. In other words, $L^*(n)$ is a slowly varying correction to the power behavior of $q_n^*$, if $\Delta_0 \neq 0$. However, if $\Delta_0 = 0$, this slowly varying term becomes preponderant. Notably for Fréchet distributions, $\Delta_0 = \zeta$ implies that $q_n^* \approx 1/\zeta$. On the other hand, for the Gumbel attraction domain, $\gamma_0(s) \to 0$ implies that $L^*(n) \to +\infty$. For instance, in the case of the exponential power laws, we have $\delta_0(s) = \epsilon(s) \sim 1/n$; therefore, $q_n \approx K \ln n (K > 0)$. Computing $\gamma_0(s)$ yields the more precise expression

$$q_n^* \approx \alpha \ln n. \quad (67)$$

Consequently, using the right power transformation ($q_n = L^*(n)n^{\epsilon(n)}$), a law belonging to the Weibull attraction domain can be forced to converge into a non-standard limit distribution $F_{\lambda,-\Delta_0}$. On the other hand, after applying a slowly varying power transformation ($q_n = \Delta \alpha \ln n$ for the exponential power laws) to a law belonging to the Gumbel attraction domain, one can force the convergence toward a Fréchet limit law. This is perfectly consistent with the (more precise) result obtained in [20] concerning the transition between the Gumbel domain and the Fréchet domain for a specific power transformation.

However, considering parent distributions from standard domains only leads to non-standard distributions with positive $Q$. The domains of attraction of non-standard laws with negative $Q$ correspond to parent laws, which do not belong to the standard attraction domains but can be ‘renormalized’ using the decreasing power law. One example of such laws would be the logarithm-power law with

$$F(x) = 1 - L(\ln x)(\ln x)^{-\alpha}, \quad (68)$$

where both $L$ and its derivative $L'$ are normalized slowly varying functions and $\alpha > 0$. A short calculation shows that $\lambda_0(g(x)) \sim \alpha \ln x$ and $\Delta_0 = \alpha$. Thus, these laws belong to the attraction domain of $F_{\lambda,-\Delta_0}$ with $\alpha > 0$ and $\lambda > 0$. The associated power scale correspond to a decreasing power functions $q_n^* = L^*(n)n^{-\alpha}$. Going further, with a power transformation decreasing faster than $n^{-\alpha}$, these logarithm-power law converge toward a Fréchet distribution of parameter $\alpha$. 

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Table 1. Classification of the limit distributions according to the functional dependence of the exponent $q_n$. The function $L_d(n)$ is a slowly varying function that tends to infinity with $n$. The slowly varying function $L^*(n)$ characterizes the parent law—see equation (65). ‘None’ means that no (non-degenerate) limit distribution emerges.

| Parent law           | Functional dependence of $q_n$ | $L(n)n^Q$, $Q < 0$ | $L(n)n^Q$, $Q > 0$ | $L(n)n^Q$, $Q = |ζ|$, $L = λL^*$ | $L(n)n^Q$, $Q = |ζ|$, $L = λL^*$ |
|----------------------|--------------------------------|---------------------|---------------------|--------------------------------|--------------------------------|
| Weibull domain       |                                | $F_ζ$               | $F_ζ$               | None                            | None                            |
| $ζ < 0$, $L^*(n)$    |                                | $L = λL^*$          | $F_λL$              | None                            | None                            |
| Gumbel domain $L^*(n)$ | $F_0$                        | $L = λL^*$          | $F_λ$               | None                            | None                            |
| Fréchet domain $ζ > 0$ | $F_0$                        | $L = λL^*$          | $F_λ$               | None                            | None                            |
| Logarithmic power $α > 0$, $L^*(n)$ | $F_α$                        | $Q = -α$, $L = λL^*$ | $F_λ(-|Q|)$         | None                            | None                            |

Table 2. Classification of the limit distributions according to the ratio $q_n/q^*_n$ when $n \to \infty$.

| Limit of the ratio $q_n/q^*_n$ | $q_n/q^*_n \to 0$ | $q_n/q^*_n \to A > 0$ | $q_n/q^*_n \to +\infty$ |
|-------------------------------|-------------------|------------------------|---------------------------|
| Fréchet domain, $ζ > 0$      | Gumbel $F_0$      | Fréchet $F_λ$          | None                      |
| Gumbel domain, $F_0$         | Gumbel $F_0$      | Fréchet $F_λ$          | None                      |
| Weibull domain, $ζ < 0$      | Weibull $F_α$     | $F_λ(-|Q|)$            | None                      |
| Logarithmic power, $α > 0$   | Fréchet $F_α$     | Non-standard $F_λ(-|Q|)$ | None                      |

In conclusion, our heuristic analysis brought to light the existence of transitions between the standard and the non-standard attraction domains when modifying the $n$-dependence of the power transformation $q_n$ (or $q(s)$). As a byproduct, we also obtained that power transformations can be used to ‘renormalize’ laws beyond the standard attraction domains (a typical example being logarithmic-power laws) in such a way that they converge to a non-degenerate distribution. This set of transitions is summarized in table 1. A somehow more compact presentation can be obtained by using the limit of the ratio $q_n/q^*_n$ in order to classify the limit distributions, as seen in table 2.

5. Alternative approach to the characterization of non-standard attraction domains

5.1. Rescaling factors for transformed maximum

In the previous section, heuristic arguments have shown that only very specific transitions are possible between the standard and the non-standard attraction domains. In order to verify this result, this section will present an alternative approach to the study of the transformed maximum in which we shall tie back the convergence behavior of the transformed and the non-transformed variables.

Let us consider a random variable $W$, which belongs to the domain of attraction of the limit law $F_ζ$. This means that there exists a renormalization sequence $(α_n, β_n)$ such that

$$\forall x, \quad F_W(α_n x + β_n)^α \to F_ζ(x). \quad (69)$$

These renormalization sequences are well known and expressions are available in the literature [10, 1].
Similarly, the \(\omega\)-transformed variable \(M_n^U\) converges in distribution if and only if there exist \(F\) and a sequence \((a_n, b_n)\), such that
\[
\forall x, \quad F_n(\omega_n(a_nx + b_n))^a \xrightarrow{n \to +\infty} \tilde{F}(x). \tag{70}
\]
However, in this extension of the previous problem, no general conditions of convergence are known and the choice of the renormalization sequence becomes more difficult. We propose in the next section to exploit the striking similarity between the two previous equations to obtain convergence conditions for the power transformation in specific cases.

First, one can remark that the only difference between the two previous equations lies in the term \((\alpha_n x + \beta_n)\) in equation (69), which becomes \(\omega_n(a_nx + b_n)\) in equation (70). This similarity suggests a simple way to obtain the convergence in distribution of the transformed maximum by exploiting our knowledge of the renormalization factor of the original distribution. If there exists a renormalization sequence \((a_n, b_n)\), such that
\[
\forall x, \quad \omega_n(a_nx + b_n) \sim_{n \to +\infty} \alpha_n v(x) + \beta_n, \tag{71}
\]
and further assuming that
\[
\lim_{n \to +\infty} F_n(\omega(a_nx + b_n))^a = \lim_{n \to +\infty} F_n(\alpha_n v(x) + \beta_n)^a, \tag{72}
\]
one obtains
\[
\tilde{F}(x) = F(\nu(x)). \tag{73}
\]
Therefore, if conditions (72) and (71) are satisfied it is possible to link the transformed limit distribution and the standard limit distribution.

Condition (72) corresponds to a quite technical convergence problem. For now, we assume that this condition is satisfied. We show in the appendix that this condition holds for our proposed choice of \((a_n, b_n)\). Condition (71) is more interesting and through \(\nu(x)\) defines the kind of transition. For a power transformation, it can be read
\[
\forall x, \quad (a_nx + b_n)^{1/q_n} \sim_{n \to +\infty} \alpha_n v(x) + \beta_n. \tag{74}
\]
Considering table 2, four distinct transitions should be possible. For laws belonging to the Gumbel or the Weibull domain, one should have either \(v(x) = x\) for \(\lim_{n \to -\infty} q_n/q_n^* = 0\) or \(v(x) = \ln(1 + \Lambda x)/\Lambda\) for \(\lim_{n \to -\infty} q_n/q_n^* = \Lambda > 0\). Similarly, in the Fréchet domain with parameter \(\zeta\), transitions should appear for exponents \(q_n\) converging to a finite value, considering \(v(x) = [(1 + \Lambda x)^{1/\Lambda} - 1]/\Lambda\) (with \(\Lambda = \lim_{n \to -\infty} q_n/\zeta\)), and for vanishing exponents \(q_n\) considering \(v(x) = [\exp(\zeta x) - 1]/\zeta\).

So, using the insight gained from section 4, it is natural to study separately the behavior of \((a_nx + b_n)^{1/q_n}\) for diverging, vanishing and converging power \(q_n\).

5.2. Diverging powers \(q_n\)

In the case where \(\lim_{n \to -\infty} q_n = +\infty\), the expression \((a_nx + b_n)^{1/q_n}\) has two distinct asymptotic behaviors, which lead to two different convergence regimes.

First, if \(a_n/b_n \to 0\), one has the asymptotic behavior
\[
(a_nx + b_n)^{1/q_n} \sim_{n \to +\infty} b_n^{1/q_n} \left(1 + \frac{a_n}{b_n q_n} x\right). \tag{75}
\]
Condition (74) is then satisfied if
\[
\begin{align*}
\{ & b_n = \beta_n^{q_n}, \\
& a_n = b_n q_n \alpha_n/\beta_n\}
\end{align*} \tag{76}
\]
This choice of \((a_n, b_n)\) is compatible with the assumption \(a_n/b_n \to 0\) if
\[
q_n \frac{\alpha_n}{\beta_n} \to 0.
\] (77)

The term \(\alpha_n/\beta_n\) corresponds to the parameter \(\lambda_0(n)\) defined in section 4. So using the definition of the characteristic exponent \(q_n^* = 1/\lambda_0(n)\) introduced in equation (58), the previous result states that if \(q_n\) is negligible with respect to \(q_n^*\) (i.e. \(\lim_{n \to \infty} q_n/q_n^* = 0\)), then the transformed maximum converges toward the same limit distribution as the original maximum. This is the expected result from section 4.

The second asymptotic behavior arises when \(a_n/b_n\) is a constant. By factorizing \(b_n\), one obtains
\[
(a_n x + b_n)^{1/q_n} \sim b_n^{1/q_n} \left(1 + \frac{\ln (1 + q_n x)}{q_n}\right).
\] (78)

Equation (74) is satisfied if
\[
\begin{align*}
b_n &= (\beta_n)^{q_n}, \\
\alpha_n &= \lambda b_n, \\
q_n &= q_n \to +\infty \lambda.
\end{align*}
\] (79)

In this case, \(\nu(x) = (1 + \lambda \ln x)/\lambda\) implies that \(X_n\) converges toward the non-standard limit laws. More precisely, if \(q_n\) is asymptotically equivalent to \(\lambda q_n^*\), then the maximum converges in distribution toward \(F_{\lambda, -\zeta}\). Once again, we recover the results of section 4 for the Weibull and Gumbel domains.

### 5.3. Vanishing powers \(q_n\)

The next interesting transition appears for vanishing moments. From table 2, we know that an exponential term should appear in \((a_n x + b_n)^{1/q_n}\). The easiest way to obtain this term is to assume that \(a_n = b_n q_n \zeta\). Then, we have
\[
(a_n x + b_n)^{1/q_n} = (b_n)^{1/q_n} (1 + q_n \zeta x)^{1/q_n} \sim_{n \to +\infty} (b_n)^{1/q_n} e^{\zeta x}.
\]

Substituting \(\nu(x) = [\exp(\zeta x) - 1]/\zeta\), one may also satisfy equation (74) by assuming
\[
\begin{align*}
b_n &= (\beta_n)^{q_n}, \\
\alpha_n &= q_n^* n \to +\infty \frac{1}{\zeta}.
\end{align*}
\] (80)

It is therefore possible to go from the Fréchet domain to the Gumbel domain using any decreasing power transformation, as expected from our heuristic analysis in the previous section. This confirms that the Fréchet domains are very unstable under power transformation. Any vanishing power is enough to change a distribution belonging to the Fréchet domain to converge toward the Gumbel distribution.

### 5.4. Converging powers \(q_n\)

In the case of converging powers, one expects to observe only transitions between Fréchet domains with distinct parameters \(\zeta\). Considering the possible translation and dilation, one can define without loss of generality \(a_n = \lambda b_n\) and \(b_n = \beta_n^n\). Then, one obtains
\[
(a_n x + b_n)^{1/q_n} = \beta_n (1 + \lambda x)^{1/q_n}.
\] (81)
and choosing \( v(x) = [(1 + \lambda x)^{\frac{1}{\zeta}} - 1]/\zeta \) leads to the condition

\[
\begin{cases}
q_n^* & \rightarrow \frac{1}{\zeta} \\
q_n & \rightarrow \lambda.
\end{cases}
\]

(82)

We observe as expected a transition between the Fréchet domain of parameter \( \zeta \) and the Fréchet domain of parameter \( \lambda \). If the choice of \( v(x) \) can appear to be quite unnatural, it should be noted that it is merely a consequence of the representation chosen for the Fréchet limit laws. A different choice of representation \( (F(x) = \exp(-x^{1/\zeta})) \) leads to the far simpler \( v(x) = x^{\lambda/\zeta} \).

However, it is compelling that our renormalization methods have allowed us to shed light on this transition even in this convoluted settings.

With this transition between Fréchet domains, we have recovered all the possible transitions from standard attraction domains to other domains described in table 2 using only the insight obtained from our analysis of the partial equation of the renormalization flow and elementary arguments on the renormalization coefficients. As shown in the appendix, these arguments lead to a rigorous proof of the convergence of the transformed maximum.

6. Conclusion

In this contribution, the renormalization approach of the problem of maximum developed in [26–28] has been extended to the case where the underlying variables \( W_i \) are subjected to a transformation \( \omega_n \), which depends on the sample size. The reduction of the problem of maximum to a partial differential equation turns out to be a rather straightforward generalization of the standard case and leads in the case of the power transformation \( U_{i,n} = W_{i,n}^{q_n} \), to a quite short categorization of the limit distributions.

Using this categorization, non-standard max-stable laws mirroring the standard limit laws have been brought to light. These new limit laws are closely related to the standard ones. However, the behavior of the partial differential equation describing the evolution of the distribution of the maximum is more complex and involves some intriguing interactions between the rate of growth of the power transformation and the tail of the distribution.

These interactions received further investigations by studying the asymptotic behavior of the forcing parameters appearing in the partial differential equation of the flow. Using a heuristic argument, it was possible to recover a slight approximation of the standard attraction domain. Moreover, the same argument leads to an interesting description of the attraction domain of a non-standard law, illustrating the existence of specific transitions between the classical limit laws and their mirrors laws, when varying the functional dependence of the power \( q_n \). These transitions are associated with a characteristic power scale \( q^*_n \). If \( q_n/q^*_n \rightarrow 0 \), the power transformation \( q_n \) is too slow to influence the convergence of the maximum toward the standard limit distributions. In contrast, if \( q_n \sim \lambda q^*_n \), the distribution converges toward a non-standard limit distribution. Using insights gained from the partial differential equation of flow, it was then possible to confirm the existence of these transitions and to investigate their mechanisms using a more direct approach based on a study of the rescaling factors.

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Appendix. Convergence problems

The convergence results obtained in section 5 are dependent on the condition given in equation (72). This condition is equivalent to \( F((a_n x + b_n)^{1/q_n})^n - F(\alpha_n v(x) + \beta_n)^n \to 0 \). This is immediately true if \( g_W \) is uniformly continuous. However, this condition is superfluous. Let us define

\[
\begin{align*}
x_n &= \alpha_n v(x) + \beta_n, \\
\epsilon_n &= \frac{(a_n x + b_n)^{1/q_n} - x_n}{\alpha_n},
\end{align*}
\]

(A.1)

(A.2)

It is then possible to show that for the four cases presented in section 5, \( \lim_{n \to \infty} \epsilon_n = 0 \).

For diverging \( q_n \) and \( q_n/q_n^* \to 0 \), equation (76) leads to

\[
\epsilon_n \sim \frac{1}{q_n} \frac{q_n}{q_n^*}, \quad n \to \infty.
\]

(A.3)

In the case \( q_n \sim \lambda q_n^* \), we have

\[
\epsilon_n = \ln(1 + \lambda x) \left( \frac{1}{\lambda} - \frac{q_n}{q_n^*} \right) + O(q_n).
\]

(A.4)

Similarly for vanishing \( q_n \), equation (80) yields

\[
\epsilon_n = \left( \frac{q_n}{q_n^*} - \frac{1}{\xi} \right) (e^{\xi} - 1) + O(q_n).
\]

(A.5)

Since \( q_n \to 0 \) in this case and \( q_n^* \to 1/\xi \), \( \epsilon_n \) is therefore a vanishing quantity. And finally, for converging moment and \( q_n \sim \lambda q_n^* \)

\[
\epsilon_n \approx \frac{1}{\xi} \left( (1 + \lambda x)^{1/q_n} - (1 + \lambda x)^{1/\lambda} \right) + \frac{1}{\xi} = q_n^*.
\]

(A.6)

Using \( q_n^* \to 1/\xi \), this confirms that \( \lim_{n \to \infty} \epsilon_n = 0 \).

Moreover, by construction \( \lim_{n \to \infty} n\mathbb{F}(x_n) \in \mathbb{R} \). It is possible to show that \( F((a_n x + b_n)^{1/q_n})^n - F(\alpha_n v(x) + \beta_n)^n \to 0 \) is equivalent to

\[
\lim_{n \to \infty} \frac{\mathbb{F}(x_n + \alpha_n \epsilon_n)}{\mathbb{F}(x_n)} = 1.
\]

(A.7)

If equation (A.7) is satisfied, then the convergence in distribution of \( M_n^u \) is ensured. In appendices A.1, A.2 and A.3, we verify that this condition holds for all the standard domains. Consequently, the transition described earlier is always valid.

A.1. The Gumbel domain

The Gumbel attraction domain is the harder to characterize. In order to prove the convergence in the general settings, we will temporarily use the standard renormalization factors

\[
\begin{align*}
\beta_n &= \mathbb{F}^{-1}(1/n), \\
\alpha_n &= E(\beta_n),
\end{align*}
\]

(A.8)

(A.9)

where the function \( E(x) \), defined as \( E(x) = \frac{1}{\mathbb{F}(x)} \int_x^{+\infty} 1 - F(t) dt \), satisfies

\[
\begin{align*}
\lim_{x \to +\infty} E'(x) &= 0, \\
\forall r > 0, \quad \lim_{x \to +\infty} \frac{1 - F(x + rE(x))}{1 - F(x)} &= e^{-r}.
\end{align*}
\]

(A.10)
One useful property of $E$ is that for any positive real $r$,
\[
\frac{E(x + rE(x))}{E(x)} \xrightarrow{x \to +\infty} 1.
\]  
(A.11)

Combining equations (A.9) and (A.11) leads to
\[
E(x_n) = E(\alpha_n x + \beta_n) \\
= E(\beta_n + xE(\beta_n)) \sim E(\beta_n),
\]  
(A.12)

so that $E(x_n) \sim \alpha_n$. Consequently, $\alpha_n \epsilon_n / E(x_n) \to 0$, and using this result with equation (A.10)

\[
F(x_n + \alpha_n \epsilon_n) = F(x_n + \alpha_n \epsilon_n) \\
\sim e^{-\alpha_n x \epsilon_n / E(x_n)} \\
\sim F(x_n)
\]  
(A.13)

So, equation (A.7) holds for the Gumbel attraction domain.

A.2. The Weibull domain

As stated in equation (50), we have for the Weibull domain
\[
F(x) = L((x_F - x)^{-1}) (x_F - x)^{-1/\xi}.
\]  
(A.14)

Moreover, using the definition of the renormalization factor leads to
\[
\frac{x_F - \beta_n}{\alpha_n} \sim \frac{(x_F - \beta_n)F(\beta_n)}{F(\beta_n)}.
\]  
(A.15)

Hence, with the assumption that $L'/L \to 0$, we have
\[
\lim_{n \to \infty} \frac{x_F - \beta_n}{\alpha_n} = \xi.
\]  
(A.16)

Therefore, it is possible to show that
\[
x_F - (x_n + \alpha_n \epsilon_n) \sim \alpha_n x \left(1 + \frac{\xi}{x}\right).
\]  
(A.17)

The properties of slowly varying functions yield, for $n \to \infty$,
\[
F(x_n + \alpha_n \epsilon_n) = L \left(\frac{1}{x_F - x_n - \alpha_n \epsilon_n}\right) (x_F - x_n - \alpha_n \epsilon_n)^{-\xi} \\
= L \left(\frac{1}{\alpha_n x}\right) (\alpha_n x(1 + \xi / x))^{-\xi} \\
\sim F(x_n).
\]  
(A.18)

Hence, equation (A.7) holds for the Weibull domain.

A.3. The Fréchet domain

Within the Fréchet domain, the proof is immediate using $\alpha_n \epsilon_n / x_n \to 0$, which is directly implied by $\lim_{n \to \infty} \epsilon = 0$ and $\lim_{n \to \infty} q_n > 0$. We have from equation (49)
\[
F(x_n + \alpha_n \epsilon_n) = (x_n + \alpha_n \epsilon_n)^{-1/\xi} L(x_n + \alpha_n \epsilon_n) \\
\sim x_n^{-1/\xi} L(x_n) = F(x_n).
\]  
(A.19)

As a result, equation (A.7) is also satisfied for the Fréchet class.
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