The volume of the space of holomorphic maps
from $S^2$ to $\mathbb{C}P^k$

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Abstract

Let $\Sigma$ be a compact Riemann surface and $\mathcal{H}_{d,k}(\Sigma)$ denote the space of degree $d \geq 1$ holomorphic maps $\Sigma \rightarrow \mathbb{C}P^k$. In theoretical physics this arises as the moduli space of charge $d$ lumps (or instantons) in the $\mathbb{C}P^k$ model on $\Sigma$. There is a natural Riemannian metric on this moduli space, called the $L^2$ metric, whose geometry is conjectured to control the low energy dynamics of $\mathbb{C}P^k$ lumps. In this paper an explicit formula for the $L^2$ metric on of $\mathcal{H}_{d,k}(\Sigma)$ in the special case $d = 1$ and $\Sigma = S^2$ is computed. Essential use is made of the kähler property of the $L^2$ metric, and its invariance under a natural action of $G = U(k+1) \times U(2)$. It is shown that all $G$-invariant kähler metrics on $\mathcal{H}_{1,k}(S^2)$ have finite volume for $k \geq 2$. The volume of $\mathcal{H}_{1,k}(S^2)$ with respect to the $L^2$ metric is computed explicitly and is shown to agree with a general formula for $\mathcal{H}_{d,k}(\Sigma)$ recently conjectured by Baptista. The area of a family of twice punctured spheres in $\mathcal{H}_{d,k}(\Sigma)$ is computed exactly, and a formal argument is presented in support of Baptista’s formula for $\mathcal{H}_{d,k}(S^2)$ for all $d, k$, and $\mathcal{H}_{2,1}(T^2)$.

1 Introduction

Let $\Sigma$ be a compact Riemann surface of genus $g$ and $Y = \mathbb{C}P^k$ equipped with the Fubini-Study metric of constant holomorphic sectional curvature $c_2 > 0$. Maps $\phi : \Sigma \rightarrow Y$ are classified topologically by an integer degree $d = \int_\Sigma \phi^* \omega_Y$, where $\omega_Y$ is a suitably normalized kähler form on $Y$. It is well known that among all degree $d \geq 1$ maps, the Dirichlet energy

$$E(\phi) = \frac{1}{2} \int_\Sigma ||d\phi||^2$$

is minimized when $\phi$ is holomorphic. Let $\mathcal{H}_{d,k}$ denote the set of degree $d$ holomorphic maps $\Sigma \rightarrow Y$. In physics language, this is the moduli space of charge $d \mathbb{C}P^k$ lumps (or instantons) on $\Sigma$. If we equip $\Sigma$ with a Riemannian metric (note that $E$ requires only a conformal structure

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on $\Sigma$), then $H_{d,k}$ inherits a natural Riemannian metric $\gamma_{L^2}$ defined so that for any curve $\phi(t)$ in $H_{d,k}$,

$$\gamma_{L^2} (\dot{\phi}, \dot{\phi}) = \int_\Sigma \| \dot{\phi} \|^2 .$$

This is usually called the $L^2$ metric, and may be interpreted as (twice) the kinetic energy of the time varying field $\phi(t,x)$. Following Manton’s approach to soliton dynamics [11], it is thought that geodesic motion in $H_{d,k}$ approximates classical low-energy $d$-lump dynamics on $\Sigma$. The quantum energy spectrum of a $d$-lump system can be related to the spectrum of an appropriate Laplace-Beltrami operator on $H_{d,k}$, and the equation of state of a classical gas of lumps can be deduced if one knows the volume growth of $H_{d,k}$ as a function of $d$. So given exact information about the metric $\gamma_{L^2}$ there is a well-developed programme for extracting information about the classical, quantum and statistical mechanics of lumps moving on $\Sigma$.

In this regard, Baptista [2] has recently made a very interesting conjecture for the volume of $H_{d,k}(\Sigma)$, motivated by a certain singular limit of a related abelian Yang-Mills-Higgs theory, namely, provided $d > 2g - 1$,

$$\text{Vol}(H_{d,k}(\Sigma)) = \frac{(k + 1)^g}{N_{d,k,g}} \left( \frac{4\pi}{c_2} \text{Vol}(\Sigma) \right)^{N_{d,k,g}} , \quad N_{d,k,g} = (k + 1)(d + 1 - g) + g - 1 .$$

In this paper we will make a detailed study of $\gamma_{L^2}$ in the case $d = 1$ and $\Sigma = S^2$ (with the round metric), confirming Baptista’s conjecture for all $k \geq 2$ (the case $k = 1$ was already known by earlier work of Baptista himself [1]). What makes this case $(d = 1, \Sigma = S^2)$ tractable is that there is a cohomogeneity 1 isometric group action, built from the isometry groups of $S^2$ and $\mathbb{C}P^k$, so we may decompose $H_{1,k}$ into a one parameter family of homogeneous orbits. The kähler property then almost completely determines $\gamma_{L^2}$: we will see that an arbitrary invariant kähler metric on $H_{1,k}$ is determined by a single function of one variable and a single constant. It is not hard to show that all metrics with this structure have finite volume for $k \geq 2$, and to find a formula for this volume. This is a nontrivial and rather surprising result, given that $H_{1,k}$ is noncompact, and that $H_{1,1}$ is known to admit invariant kähler metrics of infinite volume (one example is the Stenzel metric on $T S^3$, also known as the “deformed conifold” [5, 16, 14]).

It turns out that $(H_{1,k}(S^2), \gamma_{L^2})$ is geodesically incomplete. This can be seen immediately from our explicit formula for the metric, but actually follows in considerably more generality from previous work on the $\mathbb{C}P^1$ model. We note that there is a totally geodesic inclusion $\iota : H_{d,1}(\Sigma) \hookrightarrow H_{d,k}(\Sigma)$ induced by the inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^k$, $[z_0, z_1] \mapsto [z_0, z_1, 0, \ldots, 0]$. So it follows from the results of [12] that whenever $H_{d,1}(\Sigma)$ is nonempty, $H_{d,k}(\Sigma)$ is geodesically incomplete with respect to $\gamma_{L^2}$. In particular, $H_{d,k}(S^2)$ is incomplete for all $d, k$. So geodesic motion in $H_{d,k}(S^2)$ can hit the boundary at infinity in finite time, corresponding to one or more lumps collapsing to zero width. It is likely, however, that the boundary at infinity has high codimension, so that generic geodesics never hit it. (We shall see it has codimension $2k$ in the case $H_{1,k}(S^2)$.) If so, this means that Manton’s discussion [10] of the statistical mechanics of geodesic flow on moduli space at large degree still makes sense (the set of bad initial data has measure 0), and we can hope to derive an equation of state for a gas of $d$ lumps moving on a sphere of total area $A$. If Baptista’s conjecture is correct, this equation of state turns out to be the ideal gas equation

$$PA = (k + 1) dT ,$$

2
where $P$ is pressure and $d$ is interpreted as the number of lumps in the gas.

The rest of this paper is structured as follows. In section 2 we analyze the structure of invariant kähler metrics on $\mathcal{H}_{1,k}(S^2)$ and find an explicit formula for $\gamma_{L^2}$. In section 3 we show that the volume of $\mathcal{H}_{1,k}(S^2)$ with respect to an arbitrary invariant kähler metric is finite, and compute the volume with respect to $\gamma_{L^2}$, confirming Baptista’s conjecture in these cases. Finally, section 4 presents more indirect evidence in favour of Baptista’s conjecture for $\mathcal{H}_{d,k}(S^2)$, $d \geq 2$, $k \geq 1$ and $\mathcal{H}_{2,1}(T^2)$.

## 2 The metric

Throughout the next two sections, $\Sigma = S^2$ and $d = 1$. It is convenient to identify the domain $\Sigma$ with $\mathbb{C}P^1$ given the Fubini-Study metric of holomorphic sectional curvature $c_1$ (equivalently, $S^2$ given the round metric of radius $1/\sqrt{c_1}$). A degree 1 holomorphic map $\mathbb{C}P^1 \to \mathbb{C}P^k$ is one which lifts to a rank 2 linear map $\mathbb{C}^2 \to \mathbb{C}^{k+1}$, so we have an identification of $\mathcal{H}_{1,k}$ with the set of projective equivalence classes of rank 2 $(k+1) \times 2$ complex matrices, explicitly,

$$
\phi([z_0, z_1]) = [a_0 z_0 + b_0 z_1, \ldots, a_k z_0 + b_k z_1] \leftrightarrow [M_\phi] = \begin{pmatrix} a_0 & b_0 \\ \vdots & \vdots \\ a_k & b_k \end{pmatrix}.
$$

We may further identify $[M_\phi]$ with $[a_0, \ldots, a_k, b_0, \ldots, b_k] \in \mathbb{C}P^{2k+1}$ to obtain an open inclusion $\mathcal{H}_{1,k} \hookrightarrow \mathbb{C}P^{2k+1}$ whose image is the complement of a complex codimension $k$ variety biholomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^k$ (corresponding to the rank 1 matrices). The biholomorphism is

$$
([x_0, x_1], [y_0, \ldots, y_k]) \mapsto [x_0 y_0, \ldots, x_0 y_k, x_1 y_0, \ldots, x_1 y_k].
$$

The inclusion into $\mathbb{C}P^{2k+1}$ equips $\mathcal{H}_{1,k}$ with a complex structure. By a straightforward extension of the argument in [15], the $L^2$ metric on $\mathcal{H}_{1,k}$ is kähler with respect to this complex structure.

There is a natural left action of $G = U(k+1) \times U(2)$ on $\mathcal{H}_{1,k}$ given by

$$
(U_1, U_2) : [M_\phi] \mapsto [U_1 M_\phi U_2^{-1}].
$$

This maps $\phi$ to $i_2 \circ \phi \circ i_1$ where $i_1, i_2$ are isometries of $\mathbb{C}P^1, \mathbb{C}P^k$ respectively. Hence the $L^2$ metric is invariant under this $G$ action. The action has cohomogeneity 1, meaning that generic orbits are codimension 1 submanifolds of $\mathcal{H}_{1,k}$. Each orbit contains a unique map of the form

$$
\phi_\mu([z_0, z_1]) = [\mu z_0, z_1, 0, \ldots, 0], \quad \mu \geq 1
$$

so the action decomposes $\mathcal{H}_{1,k}$ into a one parameter family of orbits parametrized by $\mu \in [1, \infty)$, each orbit diffeomorphic to $G/K$, where $K$ is the isotropy group of $\phi_\mu$. For $\mu > 1$ (the single exceptional orbit $\mu = 1$, which has codimension 3, will not concern us), one finds that $K$ is isomorphic to $T^3 \times U(k-1)$, one isomorphism being

$$
(e^{i\xi}, e^{i\alpha}, e^{i\beta}, U) \mapsto \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & U \end{pmatrix}, \begin{pmatrix} e^{i(\alpha+\xi)} & 0 \\ 0 & e^{i(\beta+\xi)} \end{pmatrix}.
$$
From now on, let $\gamma$ be any $G$-invariant hermitian metric on $\mathcal{H}_{1,k}$. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of $G, K$ respectively, and denote by $\langle , \rangle$ the natural $Ad(G)$ invariant inner product on $\mathfrak{g}$, namely

$$\langle (A, B), (A', B') \rangle = -\frac{1}{2}(tr AA' + tr BB').$$

(6)

We may identify the tangent space to the orbit through $\phi_\mu$ with any $Ad(K)$ invariant subspace $\mathfrak{p}$ complementary to $\mathfrak{k}$ in $\mathfrak{g}$. We choose $\mathfrak{p} = \mathfrak{k}^\perp$, the orthogonal complement to $\mathfrak{k}$ with respect to $\langle , \rangle$. Any other tangent space $T_{gK}(G/K)$ may be identified with $\mathfrak{p}$ by left translation by $g^{-1}$, but this identification is only well-defined modulo the adjoint action of $K$ on $\mathfrak{p}$, since the element $u \in \mathfrak{p}$ with which $X \in T_{gK}(G/K)$ is identified moves on an $Ad(K)$ orbit as $g$ takes all values in $gK$. It follows that the metric $\gamma$ on $\mathcal{H}_{1,k}$ is uniquely determined by the one-parameter family of symmetric bilinear forms

$$\gamma_\mu : V_\mu \times V_\mu \rightarrow \mathbb{R}$$

(7)

where $V_\mu = T_{\phi_\mu} \mathcal{H}_{1,k} = \langle \partial/\partial \mu \rangle \oplus \mathfrak{p}$, and each of these bilinear forms must be invariant under the adjoint action of $K$ on $\mathfrak{p}$. Since $\gamma$ is assumed to be hermitian, $\gamma_\mu$ must also be invariant under the action of the almost complex structure $J$ on $V_\mu$.

This leads us to decompose $\mathfrak{p}$ into subspaces

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_\mu \oplus \tilde{\mathfrak{p}}_\mu \oplus \hat{\mathfrak{p}} \oplus \check{\mathfrak{p}}$$

(8)

defined by

$$\mathfrak{p}_0 = \{ \lambda(\text{diag}(i, -i, 0, \ldots, 0), \text{diag}(-i, i)) : \lambda \in \mathbb{R} \} \equiv \mathbb{R}$$

(9)

$$\mathfrak{p}_\mu = \left\{ \begin{pmatrix} 0 & x & 0 & \cdots \\ -\bar{x} & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} 0 & \mu x \\ -\mu \bar{x} & 0 \end{pmatrix} : x \in \mathbb{C} \right\} \equiv \mathbb{C}$$

(10)

$$\tilde{\mathfrak{p}}_\mu = \left\{ \begin{pmatrix} 0 & -\mu y & 0 & \cdots \\ \mu y & 0 & 0 & \cdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} : \begin{pmatrix} 0 & -\bar{y} \\ y & 0 \end{pmatrix} : y \in \mathbb{C} \right\} \equiv \mathbb{C}$$

(11)

$$\hat{\mathfrak{p}} = \left\{ \begin{pmatrix} 0 & 0 & -u \dagger \\ 0 & 0 & 0 \\ u & 0 & 0 \end{pmatrix} : u \in \mathbb{C}^{k-1} \right\} \equiv \mathbb{C}^{k-1}$$

(12)

$$\check{\mathfrak{p}} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -v \dagger \\ 0 & v & 0 \end{pmatrix} : v \in \mathbb{C}^{k-1} \right\} \equiv \mathbb{C}^{k-1}$$

(13)

With respect to this decomposition, the action of $J$ on $(\lambda, x, y, u, v) \in \mathfrak{p}$ is

$$J : (\lambda, x, y, u, v) \mapsto 4\mu \lambda \frac{\partial}{\partial \mu} + (0, ix, iy, iu, iv),$$

(14)
and the adjoint action of $K \equiv T^3 \times U(k - 1)$ is
\[
(e^{i\xi}, e^{i\alpha}, e^{i\beta}, U) : (\lambda, x, y, u, v) \mapsto (\lambda, e^{i(\alpha-\beta)}x, e^{-i(\alpha-\beta)}y, e^{-i\alpha}Uu, e^{-i\beta}Uv).
\]
(15)

Hence, any $G$-invariant hermitian metric $\gamma$ on $\mathcal{H}_{1,k}$ has
\[
\gamma_\mu = A_0(\mu)(d\mu^2 + 8\mu^2\langle\cdot, \cdot\rangle) + A_1(\mu)\langle\cdot, \cdot\rangle_\mu + A_2(\mu)\langle\cdot, \cdot\rangle_\tilde{\mu} + A_3(\mu)\langle\cdot, \cdot\rangle_\tilde{\nu} + A_4(\mu)\langle\cdot, \cdot\rangle_\hat{\nu}
\]
where $A_0, \ldots, A_4$ are smooth positive functions of $\mu$.

Now let us assume further that the metric $\gamma$ is kähler. By previous work of Dancer and Wang [7], this implies the following constraints on the kähler form $\omega(\cdot, \cdot) = \gamma(J\cdot, \cdot)$:
\[
\omega([X, Y]_\mu, Z) + \omega([Y, Z]_\mu, X) + \omega([Z, X]_\mu, Y) = 0
\]
(17)
\[
\frac{\partial}{\partial \mu}\omega(X, Y) - \omega(\frac{\partial X}{\partial \mu}, Y) - \omega(X, \frac{\partial Y}{\partial \mu}) + \omega(\frac{\partial}{\partial \mu}, [X, Y]_\mu) = 0
\]
(18)

where $X, Y, Z$ are any (possibly $\mu$ dependent) elements of $\mathfrak{p}$. Constraint (17) in the case $X = (0, 1, 0, 0, 0) \in \mathfrak{p}_\mu, Y = (0, 0, 1, 0, 0) \in \tilde{\mathfrak{p}}_\mu$ and $Z = (1, 0, 0, 0, 0) \in \mathfrak{p}_0$ implies that
\[
A_1(\mu) = A_2(\mu).
\]
(19)

In the case $X = (0, 1, 0, 0, 0) \in \mathfrak{p}_\mu, Y = (0, 0, 0, (1, 0, \ldots, 0), 0) \in \tilde{\mathfrak{p}}$ and $Z = (0, 0, 0, 0, (i, 0, \ldots, 0)) \in \hat{\mathfrak{p}}$, constraint (17) implies
\[
A_4(\mu) - \frac{\mu^2 + 1}{\mu^2 - 1}A_1(\mu) - A_5(\mu) = 0.
\]
(20)

Turning to constraint (18), the choice $X = (0, 1, 0, 0, 0) \in \mathfrak{p}_\mu, Y = JX = (0, i, 0, 0, 0) \in \mathfrak{p}_\mu$, yields
\[
A_0 = \frac{1}{4\mu} \frac{d}{d\mu} \left( \frac{\mu^2 + 1}{\mu^2 - 1}A_1 \right),
\]
(21)

while $X = (0, 0, 0, (1, 0, \ldots, 0), 0) \in \tilde{\mathfrak{p}}, Y = JX = (0, 0, 0, (i, 0, \ldots, 0), 0) \in \tilde{\mathfrak{p}}$, yields
\[
\frac{dA_4}{d\mu} - 2\mu A_0 = 0
\]
(22)

and $X = (0, 0, 0, 0, (1, 0, \ldots, 0)) \in \hat{\mathfrak{p}}, Y = JX = (0, 0, 0, 0, (i, 0, \ldots, 0)) \in \hat{\mathfrak{p}}$, yields
\[
\frac{dA_5}{d\mu} + 2\mu A_0 = 0.
\]
(23)

Assembling these constraints, we see that any $G$-invariant kähler metric on $\mathcal{H}_{1,k}$ takes the form prescribed in (16) with the coefficient functions $A_0, \ldots, A_4$ uniquely determined by a smooth positive function $A(\mu), \mu > 1$ and a positive constant $B$, so that
\[
A_0 = \frac{1}{4\mu} \frac{dA}{d\mu}, \quad A_1 = A_2 = \frac{\mu^2 - 1}{\mu^2 + 1}A, \quad A_3 = B + \frac{1}{2}A, \quad A_4 = B - \frac{1}{2}A
\]
(24)
Given that $\gamma$ is positive definite, $A$ must be strictly increasing and bounded above by $2B$. Hence $A$ has a limit $A(\infty) \leq 2B$. Also, since $\gamma$ extends to the exceptional orbit $\mu = 1$, one finds that $A(1) = \lim_{\mu \to 1} A(\mu) = 0$.

The analysis above applies to all $G$-invariant kähler metrics on $H_{1,k}$, of which the $L^2$ metric is an example. To construct $\gamma_{L^2}$ completely, it remains to compute $A(\mu)$ and $B$. An economical way to do this is to compute the squared lengths of $X = (0,0,0,(1,0,\ldots,0),0 \in \hat{\mathfrak{p}}$ and $Y = (0,0,0,0(1,0,\ldots,0)) \in \hat{\mathfrak{p}}$. For the $L^2$ metric, one finds that

$$A_{L^2}(\mu) = \frac{16\pi}{c_1 c_2} \frac{\mu^4 - 4\mu^2 \log \mu - 1}{(\mu^2 - 1)^2}, \quad B_{L^2} = \frac{8\pi}{c_1 c_2}, \quad (25)$$

where $c_1, c_2$ are the holomorphic sectional curvatures of the domain, $\mathbb{C}P^1$, and target, $\mathbb{C}P^k$, respectively [8]. An elementary estimate shows that the length of the curve $\phi_\mu$, $\mu \geq 1$ is finite, whence it follows that this metric has finite diameter and is incomplete.

Recall that $H_{1,k}$ sits naturally as an open subset of $\mathbb{C}P^{2k+1}$, whence it inherits an alternative kähler metric $\gamma_{FS}$, induced by the Fubini-Study metric of holomorphic sectional curvature $c$, say, on $\mathbb{C}P^{2k+1}$. This metric is invariant under the natural action of $G' = U(2k+2)$, which contains $G$ as a subgroup, $G \hookrightarrow G'$, $(U_1, U_2) \mapsto U_1 \otimes U_2^{-1}$. Hence this metric is also $G$-invariant, so has the structure prescribed by equations (16), (24). In this case

$$A_{FS}(\mu) = \frac{4}{c} \left( \frac{\mu^2 - 1}{\mu^2 + 1} \right), \quad B_{FS} = \frac{2}{c}. \quad (26)$$

Note that for both $\gamma_{L^2}$ and $\gamma_{FS}$, $A(\mu)$ is an increasing function from $A(1) = 0$ to $A(\infty) \leq 2B$, as required for positivity and regularity. In fact, $A(\infty) = 2B$ in both cases, so as $\mu \to \infty$ the subspace $\langle \partial/\partial \mu \rangle \oplus \mathfrak{p}_0 \oplus \mathfrak{p}$ of $T_{\phi_\mu}H_{1,k}$ collapses, and the boundary of $H_{1,k}$ at infinity has (real) codimension $2k$. This is consistent with our earlier observation that the rank 1 matrices form a complex codimension $k$ submanifold of $\mathbb{C}P^{2k+1}$.

3 The volume of $H_{1,k}$

In this section we will show that every $G$-invariant kähler metric on $H_{1,k}$ ($k \geq 2$) has finite total volume, a result in stark contrast to the previously considered case $k = 1$, where invariant kähler metrics of infinite volume certainly exist [14, 11]. We will also find a formula for this volume in terms of $B$ and $k$ under the extra assumption that $A(\infty) = 2B$, as holds for the $L^2$ metric. This formula confirms Baptista’s conjecture in the cases under consideration.

We start by computing the volume form of any $G$-invariant hermitian metric on $H_{1,k}$. Denote by $vol_{G/K}$ the volume form on $G/K$ induced by the $Ad(G)$ invariant metric $\langle \cdot, \cdot \rangle$, and $vol_0$ the volume form on $(1, \infty) \times G/K$ induced by the product metric $\gamma_0 = d\mu^2 + \langle \cdot, \cdot \rangle_p$, so $vol_0 = d\mu \wedge vol_{G/K}$. Then, the volume form induced by $\gamma$ is, for some smooth positive function $F(\mu)$,

$$vol = F(\mu)vol_0, \quad (27)$$

and we seek to deduce $F(\mu)$. To do this, we construct orthonormal bases for $\gamma$ and $\gamma_0$. Let $e_{ij}$ denote a square matrix (the size will be either $2 \times 2$ or $(k+1) \times (k+1)$, which being clear
from context) with \((i,j)\) entry 1 and all others 0. Then an orthonormal basis for \(\gamma_0\) is given by

\[
Y_1 = \frac{\partial}{\partial \mu}, \quad Y_2 = \frac{1}{\sqrt{2}}(ie_{11} - ie_{22}, -ie_{11} + ie_{22}),
\]

\[
Y_3 = (e_{12} - e_{21}, 0), \quad Y_4 = (ie_{12} + ie_{21}, 0), \quad Y_5 = (0, -e_{12} + e_{21}), \quad Y_6 = (0, ie_{12} + ie_{21}),
\]

\[
\hat{Y}_{2j - 1} = (-e_{1,j+2} + e_{j+1,1}, 0), \quad \hat{Y}_{2j} = (ie_{1,j+2} + ie_{j+1,1}, 0), \quad j = 1, \ldots, k - 1
\]

\[
\hat{Y}_{2j - 1} = (-e_{2,j+2} + e_{j+2,1}, 0), \quad \hat{Y}_{2j} = (ie_{2,j+2} + ie_{j+2,1}, 0), \quad j = 1, \ldots, k - 1 \tag{28}
\]

and an orthonormal basis for \(\gamma\) is given by

\[
X_1 = \frac{1}{\sqrt{A_0}} Y_1, \quad X_2 = \frac{1}{\mu\sqrt{8A_0}} Y_2,
\]

\[
X_3 = \frac{Y_3 - \mu Y_5}{\sqrt{(1 + \mu^2)A_1}}, \quad X_4 = \frac{Y_4 + \mu Y_6}{\sqrt{(1 + \mu^2)A_1}}, \quad X_5 = \frac{-\mu Y_3 + Y_5}{\sqrt{(1 + \mu^2)A_2}}, \quad X_6 = \frac{\mu Y_4 + Y_6}{\sqrt{(1 + \mu^2)A_2}}.
\]

\[
\hat{X}_j = \frac{1}{\sqrt{A_3}} \hat{Y}_j, \quad \hat{Y}_j = \frac{1}{\sqrt{A_4}} \hat{Y}_j, \quad j = 1, \ldots, 2k - 2. \tag{29}
\]

Evaluating both sides of (27) on the orthonormal basis for \(\gamma\) gives

\[
1 = F(\mu) \text{vol}_0(X_1, \ldots, X_6, \hat{X}_1, \ldots, \hat{X}_{2k-2}, \hat{Y}_1, \ldots, \hat{Y}_1, \ldots)
\]

\[
= \frac{F(\mu) \text{vol}_0(Y_1, Y_2, Y_3 - \mu Y_5, Y_4 + \mu Y_6, -\mu Y_3 + Y_5, \mu Y_4 + Y_6, \hat{Y}_1, \ldots, \hat{Y}_1, \ldots)}{\sqrt{8\mu(1 + \mu^2)^2 A_0 A_1 A_2 (A_3 A_4)^k-1}}
\]

\[
= \frac{(1 - \mu^2)^2 F(\mu)}{\sqrt{8\mu(1 + \mu^2)^2 A_0 A_1 A_2 (A_3 A_4)^k-1}}. \tag{30}
\]

Hence, the volume form of a general \(G\)-invariant hermitian metric on \(H_{1,k}\) is

\[
\text{vol} = \sqrt{8\mu} \left(\frac{\mu^2 + 1}{\mu^2 - 1}\right)^2 A_0 A_1 A_2 (A_3 A_4)^k-1 d\mu \wedge \text{vol}_{G/K}. \tag{31}
\]

Assume now that the metric is kähler. Then, owing to (24),

\[
\text{vol} = \frac{1}{\sqrt{2}} A^2 (B^2 - \frac{A^2}{4})^{k-1} dA \frac{dA}{d\mu} d\mu \wedge \text{vol}_{G/K}, \tag{32}
\]

whence we find that the total volume of \(H_{1,k}\) is

\[
\text{Vol}(H_{1,k}) = \frac{1}{\sqrt{2}} \int_1^\infty A^2 (B^2 - \frac{A^2}{4})^{k-1} \frac{dA}{d\mu} d\mu \int_{G/K} \text{vol}_{G/K}
\]

\[
= 4\sqrt{2} B^{2k+1} \text{Vol}(G/K) \int_0^{A(\infty)/2B} t^2 (1 - t^2)^{k-1} dt, \tag{33}
\]

which is finite. Hence, every \(G\)-invariant kähler metric on \(H_{1,k}\) has finite volume (for \(k \geq 2\).
Let us assume further that $A(\infty) = 2B$, as holds for both the $L^2$ metric and the Fubini-Study metric. Then

$$\text{Vol}(\mathcal{H}_{1,k}) = B^{2k+1} \alpha_k$$

(34)

where

$$\alpha_k = 4\sqrt{2}\text{Vol}(G/K) \int_0^1 t^2(1-t^2)^{k-1} dt$$

(35)

depends only on $k$. It is not hard to compute the above integral exactly. However, computing $\text{Vol}(G/K)$ (which also depends on $k$, of course) is not so easy, so we deduce $\alpha_k$ indirectly, as follows. Let $\gamma_{FS}$ be the Fubini-Study metric on $\mathcal{H}_{1,k}$ of holomorphic sectional curvature $c = 1$, and hence with $B_{FS} = 2$. Since the complement of $\mathcal{H}_{1,k}$ in $\mathbb{C}P^{2k+1}$ has measure 0 (it is a codimension $k$ submanifold), the volume of $\mathcal{H}_{1,k}$ with respect to $\gamma_{FS}$ coincides with the volume of $\mathbb{C}P^{2k+1}$. But this volume is well known to be $(4\pi/2k+1)!$ (this follows [10] from the fact that the integral of the kähler form over any $\mathbb{C}P^1$ submanifold generating $H_2(\mathbb{C}P^{2k+1}, \mathbb{Z})$ is $4\pi$). Hence, for this particular Fubini-Study metric

$$\text{Vol}(\mathcal{H}_{1,k}) = 2^{2k+1} \alpha_k = (4\pi)^{2k+1}/(2k+1)!$$

(36)

whence we deduce that

$$\alpha_k = (2\pi)^{2k+1}/(2k+1)!.$$  

(37)

Hence, the volume of $\mathcal{H}_{1,k}$ with respect to any $G$-invariant kähler metric with $A(\infty) = 2B$ is

$$\text{Vol}(\mathcal{H}_{1,k}) = (2B\pi)^{2k+1}/(2k+1)!.$$ 

(38)

For the $L^2$ metric, one sees from (25) that for any $k \geq 2$.

$$\text{Vol}_{L^2}(\mathcal{H}_{1,k}) = \frac{1}{(2k+1)!} \left(\frac{4\pi}{c_1} \frac{4\pi}{c_2}\right)^{2k+1}.$$ 

(39)

This confirms Baptista’s conjecture for these moduli spaces. We note that the same formula is known to hold in the case $k = 1$ (see [1], which uses the convention $c_1 = c_2 = 4$).

In fact, we can compute exactly the volume of any $G$-invariant kähler metric on $\mathcal{H}_{1,k}$, even if $A(\infty) < 2B$. Since we know $\alpha_k$, and

$$\int_0^1 t^2(1-t^2)^{k-1} dt = \frac{(k-1)!2^{k-1}}{(2k+1)!!}$$

(40)

we deduce that

$$\text{Vol}(G/K) = \frac{2^k\pi^{2k+1}}{\sqrt{2}(k-1)!k!},$$

(41)

and hence

$$\text{Vol}(\mathcal{H}_{1,k}) = \frac{2^{k+2}(B\pi)^{2k+1}}{(k-1)!k!} \int_0^{A(\infty)/2B} t^2(1-t^2)^{k-1} dt.$$ 

(42)

The last integral can easily be computed explicitly using the binomial theorem, but the final answer is not very instructive. It is interesting to note that the total volume depends only on the asymptotic behaviour of the metric close to the boundary at infinity.
4 Higher degree and genus

For higher degree $d$, or domain $\Sigma$ of higher genus $g$, the $L^2$ geometry of $H_{d,k}(\Sigma)$ is much less accessible. Nonetheless, we can make one exact calculation which, while not directly confirming Baptista’s conjecture, seems to support it.

Let $W$ be a meromorphic function of degree $d$ on a compact Riemann surface $\Sigma$. Associated to $W$ is a cylindrical submanifold $C_W$ of $H_{d,k}(\Sigma)$, consisting of the holomorphic maps $\phi_\mu : \Sigma \rightarrow \mathbb{C}P^k$ defined locally by $\phi_\mu(z) = [\mu W(z), 1, 0, \ldots, 0]$, $\mu \in \mathbb{C} \setminus \{0\}$. Physically, this is the orbit of a fixed $d$-lump under dilation (changing $|\mu|$) and isorotation (changing $\arg(\mu)$). The induced $L^2$ metric on $C_W$ is

$$\gamma|_{C_W} = F(\mu) d\mu d\bar{\mu}, \quad F(\mu) = \frac{4}{c_2} \int_{\Sigma} \frac{|W|^2}{(1 + |\mu|^2|W|^2)^2},$$

where the measure on $\Sigma$ is the one defined by its Riemannian metric. Hence, the total volume of $C_W$ is the integral

$$\text{Vol}(C_W) = \int_{\mathbb{C}^\times} F = \frac{4}{c_2} \int_{\mathbb{C}^\times} \left( \int_{\Sigma} \frac{|W|^2}{(1 + |\mu|^2|W|^2)^2} \right)$$

if this exists (i.e., is finite). By Fubini’s theorem [6], the integral exists if and only if

$$\int_{\Sigma} \left( \int_{\mathbb{C}^\times} \frac{|W|^2}{(1 + |\mu|^2|W|^2)^2} \right)$$

exists, in which case they are equal. But (45) is trivial:

$$\int_{\Sigma} \left( \int_{\mathbb{C}^\times} \frac{|W|^2}{(1 + |\mu|^2|W|^2)^2} \right) = 2\pi \int_{\Sigma} \left( \int_{0}^{\infty} d|\mu| \frac{|\mu||W|^2}{(1 + |\mu|^2|W|^2)^2} \right) = 2\pi \int_{\Sigma} \frac{1}{2}.$$

Hence, the cylinder $C_W^\times$ has total volume

$$\text{Vol}(C_W) = \frac{4\pi}{c_2} \text{Vol}(\Sigma).$$

Note that this is independent of the meromorphic function $W$. The above calculation of $\text{Vol}(C_W)$ generalizes the result in [9], which considered the case $k = 1$, $\Sigma = S^2$ and $W = z^d$ (where $z$ is a stereographic coordinate on $S^2$). In that case, $C_W$ is a totally geodesic submanifold of $H_{d,k}$, but in general there is no reason why $C_W^\times$ should be totally geodesic.

We emphasize that all the results of sections 2 and 3 and the present section up to this point, are mathematically rigorous. The remaining paragraphs of this section are suggestive, rather than rigorous.

The formula (47) supports Baptista’s conjecture as follows. Let $\Sigma = \mathbb{C}P^1$ (with any metric). Then points in $H_{d,k}$ may be identified with projective equivalence classes of $(k+1) \times (d+1)$ complex matrices via

$$\phi([z_0, z_1]) = [a_{00} z_0^d + a_{01} z_0^{d-1} z_1 + \cdots + a_{kd} z_1^d, \ldots, a_{k0} z_0^d + a_{k1} z_0^{d-1} z_1 + \cdots + a_{kd} z_1^d] \leftrightarrow [(a_{ij})].$$
This gives an open inclusion $\mathcal{H}_{d,k} \hookrightarrow \mathbb{C}P^{dk+d+k}$ whose complement is again an algebraic variety of high codimension. Suppose that the $L^2$ metric on $\mathcal{H}_{d,k}$ extends smoothly to $\mathbb{C}P^{dk+d+k}$. Then the $L^2$ volume of $\mathcal{H}_{d,k}$ coincides with the $L^2$ volume of $\mathbb{C}P^{dk+d+1}$,

$$\text{Vol}(\mathcal{H}_{d,k}) = \int_{\mathbb{C}P^{dk+d+k}} \frac{\omega_{L^2}^{dk+d+k}}{(dk + d + k)!} = \frac{1}{(dk + d + k)!} \left( \int_X \omega_{L^2} \right)^{dk+d+k}$$

(49)

where $X$ is any 2-cycle generating $H_2(\mathbb{C}P^{dk+d+k}, \mathbb{Z})$ [10]. One such 2-cycle is the cylinder $C_W$ completed by adding the points $\mu = 0, \mu = \infty$, where $W$ is the meromorphic function $(z_0/z_1)^d$. This is a complex submanifold of $\mathbb{C}P^{dk+d+k}$, so

$$\int_X \omega_{L^2} = \text{Vol}(X) = \frac{4\pi}{c_2} \text{Vol}(\Sigma)$$

(50)

by (47). Hence, if $\gamma_{L^2}$ extends smoothly to $\mathbb{C}P^{dk+d+k}$ then, when $g = 0$,

$$\text{Vol}(\mathcal{H}_{d,k}) = \frac{1}{(dk + d + k)!} \left( \frac{4\pi}{c_2} \text{Vol}(\Sigma) \right)^{dk+d+k},$$

(51)

which agrees with Baptista’s conjecture. Unfortunately, it is probable that $\gamma_{L^2}$ never extends smoothly to $\mathbb{C}P^{dk+d+1}$. Certainly it cannot when $d = 1$, as the scalar curvature of $\gamma_{L^2}$ is unbounded in this case. So the above argument is purely formal. It may be, however, that the metric extends sufficiently regularly for the crucial step (49) above to make sense.

Finally, let us consider the case of genus $g = 1$, that is, $\Sigma = T^2$, with $k = 1$ (target $\mathbb{C}P^1$) and degree $d = 2$. It is known [13] that there is a four-fold covering map

$$\text{Rat}_1 \times T^2 \rightarrow \mathcal{H}_{2,1}(T^2), \quad (R, z_0) \mapsto \phi(z) = [R(\phi(z) - z_0)], 1$$

(52)

where $\phi$ is the Weierstrass P function, and $\text{Rat}_1$ denotes the space of degree 1 rational maps (fractional linear transformations). The $L^2$ metric on $\mathcal{H}_{2,1}(T^2)$ lifts to a kähler product metric on $\text{Rat}_1 \times T^2$, the $T^2$ factor being $8\pi/c_2$ (the rest mass of a charge 2 lump) times the metric on $\Sigma$. So,

$$\text{Vol}(\mathcal{H}_{2,1}(T^2)) = \frac{1}{4} \frac{8\pi}{c_2} \text{Vol}(\Sigma) \text{Vol}(\text{Rat}_1).$$

(53)

It is not known whether $\text{Rat}_1$ has finite volume in this geometry. However, repeating the formal argument above, we can compactify $\text{Rat}_1$ to obtain $\mathbb{C}P^3$, and compute its volume as $(\text{Vol}(C_W))^3/3!$, where $W = \phi$. But, by (47), $\text{Vol}(C_W) = 4\pi c_2^{-1} \text{Vol}(\Sigma)$. Hence, we are led to expect that

$$\text{Vol}(\mathcal{H}_{2,1}(T^2)) = \frac{2}{4!} (4c_2^{-1} \pi \text{Vol}(T^2))^4,$$

(54)

which again coincides with Baptista’s conjecture.

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