Variable Selection for Multiply-imputed Data: A Bayesian Framework

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Funding information

Multiple imputation is a widely used technique to handle missing data in large observational studies. For variable selection on multiply-imputed datasets, however, if we conduct selection on each imputed dataset separately, different sets of important variables may be obtained. MI-LASSO, one of the most popular solutions to this problem, regards the same variable across all separate imputed datasets as a group of variables and exploits Group-LASSO to yield a consistent variable selection across all the multiply-imputed datasets. In this paper, we extend the MI-LASSO model into Bayesian framework and utilize five different Bayesian MI-LASSO models to perform variable selection on multiply-imputed data. These five models consist of three shrinkage priors based and two discrete mixture prior based approaches. We conduct a simulation study investigating the practical characteristics of each model across various settings. We further demonstrate these methods via a case study using the multiply-imputed data from the University of Michigan Dioxin Exposure Study. The Python package BMIselect is hosted on Github under an Apache-2.0 license: https://github.com/zjg540066169/Bmiselect

KEYWORDS
Bayesian variable selection; Discrete mixture prior; Missing values; Multiple imputation; Shrinkage prior.
INTRODUCTION

Variable selection is a classic but increasingly popular area in the statistical research. The goal of the variable selection is to identify variables with remarkable statistical significance in data. Followed by the parsimonious principle and Occam’s razor, the model with minimal number of explanatory variables but powerful predictive ability is what scientists prefer. Variable selection models can function as filters to keep important information while remove noise brought from data. This technique is thus widely exploited in various applications, especially in high dimensional data. Generally speaking, most of the variable selection models are primarily based on complete data cases we observed. However, data with missing value is ubiquitous in real life. It is difficult for classic models to balance between variable selection and handling missing data in most of cases. The variable selection problem on missing data setting therefore presents substantial challenges to statistical analysis.

Theoretically speaking, there are mainly three types of missing mechanisms: missing completely at random (MCAR), missing at random (MAR) and not missing at random (NMAR). Various methods have been proposed to cope with different missing situations, such as multiple imputation [1], maximum likelihood estimation by EM algorithm [2], weighted estimating equations [3]. Multiple imputation (MI) is a basic method to deal with missing value, which generates several imputed datasets and pools the results from these “full” datasets into a single analysis. Its simple but powerful effect has been verified in previous studies [4–9].

In some real problems, the dataset with missing values has been already imputed via MI. Therefore, a straightforward strategy is to apply variable selection models on the multiply-imputed data. However, this may have inadequate performance, because different sets of significant variables can be possibly selected for each imputed set, which leads to inconsistency of variable selection across all imputed datasets. In literature, various solutions [10–17] have been proposed to solve this problem. MI-LASSO, proposed by Chen and Wang [17] and acknowledged successful in application [18–21], provides a flexible method to select the same important variables jointly among all imputed datasets. It regards each variable across all datasets as a group and applies Group-LASSO to consistently select these grouped variables, which keeps or discards all variables in the same group together. Empirical study has shown the outstanding performance of MI-LASSO to select significant variables on missing data. Although diverse extensions [14, 22, 23] of MI-LASSO were proposed in recent years, there is still no research that extends MI-LASSO into Bayesian framework.

Compared with classic frequentist methods, the Bayesian framework has several advantages which can facilitate the assembly of group information by specifying a common prior on group parameters and can improve model performance on small-sized data. To combine Bayesian methodology and MI-LASSO, we therefore propose five Bayesian MI-LASSO models and provide a systematical study to demonstrate the performance of these models in selecting important variables on multiply-imputed data in this paper.

The rest of paper is organized as follows. In section 2, we introduce five Bayesian MI-LASSO models. In section 3 we conduct multiple experiments which compare the performance of different Bayesian MI-LASSO models with the original MI-LASSO model using simulations. In section 4, we apply Bayesian MI-LASSO methods to the University of Michigan Dioxin Exposure Study data. We close our paper with a discussion in Section 5.

METHODS

2.1 Notations

Let $X$ and $Y$ stand for the $n \times p$ covariate matrix and $n \times 1$ outcome vector, respectively. We denote set $\{(X_i, Y_i), i = 1, 2, \ldots, n\}$ as $n$ independent and identically distributed units in a sample, with $X_i = (1, X_{i1}, X_{i2}, \ldots, X_{ip})$ for unit $i$. 
Additionally, we let \( \mathbf{X}_j = (X_{1j}, X_{2j}, \ldots, X_{nj}) \) represent the values of the \( j \)-th explanatory variable for all \( n \) units in the sample. We are interested in the linear regression model of \( \mathbf{Y} = \mathbf{X}\beta + \epsilon \), where \( \beta = (\beta_0, \beta_1, \beta_2, \ldots, \beta_p) \) is a vector of coefficients and \( \epsilon \) is a vector of error terms following an independent normal distribution with a constant variance.

With MI, missing values of \( \mathbf{X} \) and \( \mathbf{Y} \) in the sample are imputed multiple times. Suppose that we create \( D \) imputed datasets, denoted as \( \{(\mathbf{X}_d, \mathbf{Y}_d), d = 1, 2, \ldots, D\} \), where \( \mathbf{Y}_d = (Y_{d1}, Y_{d2}, \ldots, Y_{dn}) \) are the values of \( \mathbf{Y} \) in the \( d \)-th imputed dataset, and \( \mathbf{X}_d = (\mathbf{X}_{d1}, \mathbf{X}_{d2}, \ldots, \mathbf{X}_{dn}) \) with \( \mathbf{X}_{di} = (1, X_{d1i}, X_{d2i}, \ldots, X_{dpi})^T \) stands for the values of \( \mathbf{X} \) in the \( d \)-th imputed dataset. For linear regression, we have \( \mathbf{Y}_d = \mathbf{X}_d\beta_d + \epsilon_d, d = 1, 2, \ldots, D \), where \( \beta_d = (\beta_{d0}, \beta_{d1}, \beta_{d2}, \ldots, \beta_{dp}) \) is the coefficients vector for regression on \( d \)-th imputed data and \( \epsilon_d = (\epsilon_{d1}, \epsilon_{d2}, \ldots, \epsilon_{dn}) \) stands for random noise in the \( d \)-th imputed dataset. We also define \( \beta_j = (\beta_{1j}, \beta_{2j}, \ldots, \beta_{Dj}) \) as the group coefficients for the \( j \)-th explanatory variable across all the imputed datasets.

### 2.2 MI-LASSO

Although various models were proposed to solve the inconsistency of variable selection in MI settings, it was the first time that MI-LASSO \[17\] introduced the Group-LASSO \[24\] penalty into this problem. Different from stacking methods \[10, 11, 15\] which "stacks" multiply-imputed data as a single dataset and applies weighted models to select important variables, MI-LASSO treats the same variable across all imputed sets as a group of variables, and adopts Group-LASSO to jointly include or exclude the group of variables together.

To make it clear, the mathematical loss function for Group-LASSO is:

\[
\min_{\beta} \| \mathbf{Y} - \mathbf{X}\beta \|_2^2 + \lambda \sum_{g = 1}^{G} \| \beta_g \|_2, \tag{1}
\]

where \( \| \cdot \|_2 \) is L2-norm, \( \beta_g \) represents the vector of coefficients in \( g \)-th group, and \( \lambda \) is pre-specified to control the shrinkage of group coefficients. By treating the regression coefficients \( \beta_j \) associated with the \( j \)-th explanatory variable in the linear regression models across all the imputed datasets as a group, the loss function of MI-LASSO is written as follows:

\[
\min_{\beta_{Dj}} \sum_{d = 1}^{D} \| \mathbf{Y}_d - \mathbf{X}_d\beta_d \|_2^2 + \lambda \sum_{j = 1}^{p} \| \beta_j \|_2. \tag{2}
\]

The authors also developed algorithm to solve the above optimization problem. The simulation and real case study in \[17\] showed the outstanding performance of MI-LASSO.

### 2.3 Proposed Bayesian MI-LASSO

With the development of Bayesian Group-LASSO in recent years \[25, 30\], it is appealing to extend the MI-LASSO to the Bayesian framework. Compared with the classic MI-LASSO model, the non-flat priors in Bayesian models functions similarly as the regularization term in model \[1\], but Bayesian MI-LASSO has 3 advantages:

1. Bayesian methods can inherently incorporate subjective information into prior distributions. Better results can be obtained if some prior knowledge is known.
2. Bayesian methods perform well in data with small size.
3. Classic MI methods always applied Rubin’s rules(RR) \[1\] to combine the estimators from multiple-imputed data and
analyze them subsequently. However, RR has some restrictions that can only be exploited when the distribution of the repeated estimators from multiple sets is normal. Although further research on different distribution settings is explored, it is too difficult to check the distribution in some problems. Fortunately, Bayesian framework provides another method to pool, that we just need to mix up posterior samples from multiply-imputed datasets [31, 32], and the mixed distribution represents the posterior distribution which can be inferred directly. In our Bayesian MI-LASSO models, we simply mix up the posterior samples of the same variable across all the multiple-imputed data and then select variables. This straightforward and efficient method proved another advantage of Bayesian MI-LASSO.

In this section, we adapted different successful Bayesian Group-LASSO methods into the framework of MI-LASSO, and proposed several Bayesian MI-LASSO models. Similar to MI-LASSO, we collected the same variable across all imputed data as group variables and selected or removed them together through a common specified prior distribution. Followed by the category of Bayesian MI-LASSO suggested in [26], we primarily divided our Bayesian MI-LASSO models into two classes: shrinkage Bayesian MI-LASSO and discrete mixture Bayesian MI-LASSO. Moreover, we considered to specify non-informative prior for regression variance for all the Bayesian MI-LASSO models, because little knowledge we can know about the variance in practice.

2.3.1 Shrinkage Bayesian MI-LASSO

Technically speaking, shrinkage Bayesian MI-LASSO imposes the same prior distribution for all coefficients in the same group to keep consistency. Same as the maximum a posteriori estimation interpretation of LASSO, the specified prior distributions can shrink coefficients as small as possible, but not exact zero. This type of models always output smooth posterior estimations of coefficients, which are stable in practice.

The first Bayesian MI-LASSO model specifies Multi-Laplace priors for grouped regression coefficients, which is inspired by the original Bayesian Group-LASSO proposed in the literature [25]. Followed by the two-level hierarchical model derived in original model, a Normal prior with Gamma mixture is adopted to replace Multi-Laplace prior in order to reduce the complexity of MCMC sampling. The mathematical form of the Multi-Laplace Bayesian MI-LASSO can be shown as follows:

\[
\begin{align*}
Y_{d.i} & \sim \text{MVN}(X_{d.i}, \beta_{d,i}, \sigma^2_1), \ d = 1, 2, ..., D \\
\beta_{d,i} & \sim \text{MVN}(0, \lambda^2_{1}), \ d = 1, 2, ..., D \\
\lambda^2_j & \sim \text{Gamma}(D+1, \frac{2}{D}), j = 1, 2, ..., p \\
\rho & \sim \text{Gamma}(r, s) \\
\pi(\sigma) & \propto \frac{1}{\sigma}
\end{align*}
\]

where \( \lambda^2 = (\lambda^2_1, \lambda^2_2, ..., \lambda^2_p) \) and \( \lambda^2_j \) denote the parameters for j-th group coefficients \( \beta_j \). \( r, s \) are hyperparameters of shape and rate for gamma distribution which should be specified in advance. 1 means identity matrix. Although the idea of this model is simple, however, the experiments showed the excellent performance of this prior, which can achieve results with both high sensitivity and high specificity. The disadvantage of this model is that it is sensitive to credible interval selection criteria [33] and we will discuss it in section 4.

The second shrinkage Bayesian MI-LASSO model defined the Horseshoe prior for group coefficients, adjusted from the Bayesian Horseshoe method in [26]. Horseshoe model shows better stability than Multi-Laplace prior in
FIGURE 1  Distribution of $k_j$. It looks like a horseshoe. If $k_j = 0$, there is no shrinkage. If $k_j = 1$, the variable will be removed.

simulations. The mathematical form is stated as follows:

$$
\begin{align*}
Y_{d, i, d} & \sim \text{MVN}(X_{d, i, d}, \beta_{d, i, d}, \sigma^2), \; d = 1, 2, ..., D \\
\beta_{d, i, d} & \sim \text{MVN}(0, \tau^2 \lambda^2), \; d = 1, 2, ..., D \\
\lambda_j & \sim \text{C}(0, 1), \; j = 1, 2, ..., p \\
\tau & \sim \text{C}(0, 1) \\
\pi(\sigma) & \propto \frac{1}{\sigma}
\end{align*}
$$

where $\lambda^2 = (\lambda_1^2, \lambda_2^2, ..., \lambda_p^2)$, $\text{C}^+$ is the half-Cauchy distribution. In the paper [26], the authors stated that Horseshoe prior can take advantage of global shrinkage parameter $\tau$ and local shrinkage parameter $\lambda_j$ to better control shrinkage. In addition, by setting $\tau$ and $\sigma$ as 1, if $X$ is identity matrix with $n = p$, the posterior expectation $E(\beta_j | Y, X)$ is linear function of $k_j = \frac{1}{1+\lambda_j^2}$, where $k_j$ is distributed as $\text{Beta}(0.5, 0.5)$ if $\lambda_j \sim \text{C}^+(0, 1)$. The distribution of $k_j$ looks like a horseshoe (see FIGURE 1), which gives the model ability to clearly discern noises and signals. With the modification into Bayesian MI-LASSO, the horseshoe effect remained the same, which can identify the important variables with stability.

The final shrinkage Bayesian MI-LASSO we proposed is a simple but well-performed model, inspired by [28, 30], which extended the grouped Automatic Relevance Determination(ARD) model into MI-LASSO framework:

$$
\begin{align*}
Y_{d, i, d} & \sim \text{MVN}(X_{d, i, d}, \beta_{d, i, d}, \sigma^2), \; d = 1, 2, ..., D \\
\beta_{d, i, d} & \sim \text{MVN}(0, \frac{1}{\lambda_j^2}), \; d = 1, 2, ..., D \\
\pi(\lambda_j^2) & \propto \frac{1}{\lambda_j^2}, \; j = 1, 2, ..., p \\
\pi(\sigma) & \propto \frac{1}{\sigma}
\end{align*}
$$

where $\lambda = (\lambda_1^2, \lambda_2^2, ..., \lambda_p^2)$. Although this model is rather simple, the experimental results show notable performance with highest robustness in section 3.
2.3.2 | Discrete Mixture Bayesian MI-LASSO

Although shrinkage Bayesian MI-LASSO can function for variable selection, however, the coefficients cannot be shrunk to exact zero, and therefore a threshold or specific criteria need to be specified in advance to determine if the variable is selected or not. On the other hand, discrete mixture prior, Spike-Slab prior for example [34], specifies a binary hidden variable for each group coefficients, which sets coefficients as exact 0 if the related binary variable is 0. We here adapted two Bayesian MI-LASSO methods into MI-LASSO framework, which are derivatives of Spike-Slab prior.

We first introduced Spike-Normal prior proposed in the paper [30] into MI-LASSO, which can be regarded as the combination of Spike-Slab prior and normal distribution. The mathematical form takes the following formula:

\[
\begin{align*}
Y_{d_c} & \sim i.i.d \ MVN(X_{d_c}, \beta_{d_c, \sigma^2 1}), \quad d = 1, 2, \ldots, D \\
\beta_{d_c} & \sim i.i.d \ MVN(0, \nu_0 1) + \delta \beta_{d_c}(1 - \pi), \quad d = 1, 2, \ldots, D \\
\pi_j & \sim Bernoulli(\nu_0), \quad j = 1, 2, \ldots, p \\
\pi(\sigma) & \propto \frac{1}{\sigma^2}
\end{align*}
\]

(6)

where \( \pi = (\pi_1, \pi_2, \ldots, \pi_p) \), \( \delta \) means vector of point probability mass at 0. \( \nu_0 \) and \( \nu_0 \) are hyperparameters. Since Normal prior is specified for the coefficients as well, non-zero-coefficients can be also shrunk.

Finally, we combine MI-LASSO method and Spike-Laplace prior [35] to obtain Spike-Laplace Bayesian MI-LASSO model. With the advantage of discrete mixture prior, it can gain exact 0 coefficients, while the shrinkage effect on coefficients level happens as well:

\[
\begin{align*}
Y_{d_c} & \sim i.i.d \ MVN(X_{d_c}, \beta_{d_c, \sigma^2 1}), \quad d = 1, 2, \ldots, D \\
\beta_{d_c} & \sim i.i.d \ MVN(0, \tau^2 1) + \delta \beta_{d_c}(1 - \pi), \quad d = 1, 2, \ldots, D \\
\tau_j & \sim Gamma\left(\frac{D+1}{2}, \frac{2}{\lambda}\right), \quad j = 1, 2, \ldots, p \\
\pi_j & \sim Beta(\nu, b), \quad j = 1, 2, \ldots, p \\
\pi(\sigma) & \propto \frac{1}{\sigma^2}
\end{align*}
\]

(7)

where \( \pi = (\pi_1, \pi_2, \ldots, \pi_p)^T \), \( \tau^2 = (\tau_1^2, \tau_2^2, \ldots, \tau_p^2) \), \( a, b, \lambda \) are hyperparameters.

3 | SIMULATION

3.1 | Experiments Design

In this section, a series of simulation studies were conducted to evaluate the variable selection effects on multiply-imputed datasets for different methods. The performances of various Bayesian MI-LASSO models discussed in section 2.3 are evaluated and compared. We ran multiple MCMC chains until they were converged with the Gelman-Rubin statistics [36] less than 1.1. For hyperparameters, we followed non-informative principle and recommendation in literature [25, 26, 28, 30, 35] to set. For choice of \( r \) and \( s \) for Multi-Laplace model, we first sampled \( \rho \) and then sampled \( \lambda^2 \) based on sampled \( \rho \), so that the prior mean for \( \lambda^2 \) was about 1, and approximately 1% of \( \lambda^2 \) values exceeded \( 5 \times \text{mean}(\lambda^2) \). For Spike-Normal model, \( \nu_0 \) was set as 0.5 as suggested in [30] and \( \nu_0 = 4 \) to keep unit variance of \( \beta_{d_c} \). As followed by [35], we set \( a = 1, b = 1 \) and \( \lambda = 6/11 \) to stay unit variance as well. So these hyperparameters can be displayed in TABLE 1:

On the other hand, the control groups consisted of three methods from frequentists’ view and two Bayesian
TABLE 1 | Hyperparameters specified for Bayesian MI-LASSO models in simulation.

| Model       | Hyperparameters | Value |
|-------------|-----------------|-------|
| Multi-Laplace | $r$             | 2     |
|             | $s$             | 15    |
| Horseshoe   | No hyperparameter |       |
| ARD         | No hyperparameter |       |
| Spike-Normal | $p_0$           | 0.5   |
|             | $v_0$           | 4     |
| Spike-Laplace | $a$           | 1     |
|             | $b$             | 1     |
|             | $\lambda$     | $6/11$ |

LASSO methods: (i) MI-LASSO method [17], where we tuned the penalty parameter $\lambda$ by minimizing BIC; (ii) original LASSO algorithm applied to full data before missing value adopted; (iii) CC-LASSO which exploits LASSO model on only complete cases with no missing value; (iv) BLASSO, the Bayesian LASSO [37] applied to full data with posterior median estimates; (v) BLASSO(CI), the Bayesian LASSO on full data with symmetric credible interval criterion discussed in section 3.2.

In the experiments, we considered three simulation scenarios: the first two scenarios set all covariates as continuous variables while binary ones were set for all predictors in the third scenario. In each simulation, the outcome variable $Y$ in full data can be modeled as linear regression function:

$$Y = X\beta + \epsilon$$  \hspace{1cm} (8)

where $\beta = (\beta_1, \beta_2, ..., \beta_p)$, $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_p)$, $\epsilon_j \overset{i.i.d.}{\sim} N(0, \sigma^2)$. Here $\sigma$ can be chosen so that $\beta^T \Sigma \beta / \sigma^2 = 1$, $\Sigma$ is the population covariance matrix for the distribution of $X$. The generating process of $X$ with $\Sigma$ will be discussed in the following sections.

In our simulations, we first generated covariates $X$ and outcome $Y$, then applied MAR or MCAR mechanisms to generate missing value in $X$. Once missing value was imposed, we exploited statsmodels.imputation.mice class in Python to multiply impute the missing data. Imputation function in statsmodels.imputation.mice uses all observable covariates $D^{obs}$ and $Y$ as predictors to model incomplete candidate covariates $D^{mis}$. For continuous variables, the default predictive mean matching method was selected to impute while the logistic regression imputation method was applied for dichotomous covariates. In each simulation, the number of multiply imputations was specified as 5. Although only the missing settings on covariates were considered, the situation of missing value in outcome variable $Y$ can be also solved by multiple imputation as well.

3.2 | Evaluation

As discussed in [3], we simply mixed up the posterior samples for the grouped variables to determine whether to include or exclude these variables. For example, a mixture of samples from $\beta_{1j}, \beta_{2j}, ..., \beta_{D,j}$ was exploited to infer $\beta_j$. For shrinkage Bayesian MI-LASSO models, since exact zero estimates cannot be obtained, we chose symmetric credi-
ible interval criterion to determine whether to keep or remove variables, suggested by [33, 38]. A grouped predictor \( \beta_{dj} \) were kept if the \( x\% \) symmetric credible interval with equal tails of mixture posterior distribution excluded 0. We increased \( x\% \) from 5% to 95% with step size 5% and selected the optimal symmetric credible interval based on the following distance criterion with the largest value:

\[
\text{Distance} = \sqrt{\text{SEN}^2 + \text{SPE}^2}
\]  

(9)

The calculation of sensitivity (\( \text{SEN} \)) and specificity (\( \text{SPE} \)) will be discussed later in this section. For discrete mixture Bayesian MI-LASSO, we keep variables if the posterior median of their corresponding binary variable exceed 0.5, and remove these variables if posterior median less than 0.5, similar as selection criteria suggested in [39].

For MI-LASSO, we tuned the regularization parameter \( \lambda \), which was supposed to minimize BIC. In literature [17], the authors proposed a modified version of BIC:

\[
\text{BIC} = \log \left( \sum_{d=1}^{D} \sum_{i=1}^{n} (Y_{d,i} - \hat{\beta}_{d,0} - \sum_{j=1}^{p} \beta_{d,j} x_{d,ij})^2 / (Dn) \right) + df \ast \log (Dn) / (Dn)
\]  

(10)

where \( df \) can be estimated by

\[
df = \sum_{j=1}^{p} \text{I} \left( \sum_{d=1}^{D} \beta_{d,j}^2 > 0 \right) + \sum_{j=1}^{p} \sqrt{\sum_{d=1}^{D} \beta_{d,j}^2} (D - 1)
\]  

(11)

Where \( I \) is indicator function, \( \hat{\beta} \) and \( \hat{\beta} \) represent for coefficients obtained from MI-LASSO and ordinary least square estimation, respectively. We followed this definition of modified BIC to select optimal parameter \( \lambda \) in our experiments.

To evaluate the performance of all models, we regarded selection of each covariate as dichotomous classification problems and considered the following four criteria:

Sensitivity of selection:

\[
\text{SEN} = \frac{\# \text{of} \ (\text{selected variables} \cap \text{true important variables})}{\# \text{of true important variables}}
\]  

(12)

Specificity of selection:

\[
\text{SPE} = \frac{\# \text{of} \ (\text{removed variables} \cap \text{true unimportant variables})}{\# \text{of true unimportant variables}}
\]  

(13)

F1 score:

\[
F1 = 2 \ast \frac{\text{precision} \ast \text{SEN}}{\text{precision} + \text{SEN}}
\]  

(14)

Mean square error:

\[
\text{MSE} = (\hat{\beta} - \beta)^T \Sigma (\hat{\beta} - \beta)
\]  

(15)

where \( \text{precision} = \frac{\# \text{of} \ (\text{selected variables} \cap \text{true important variables})}{\# \text{of selected variables}} \), and \( \hat{\beta} \) is coefficients of refitted linear regression.
using the selected variables. For LASSO, CC-LASSO, we directly refitted two linear regression models using selected variables on corresponding full data or complete cases. For MI-LASSO and the five Bayesian MI-LASSO models, we separately refitted linear regression models on each imputed data using the selected variable, then averaged the obtained coefficients over multiply-imputed data as \( \hat{\beta} \) by Rubin’s rule. In simulation study, since \( \beta \) and \( \Sigma \) are known, MSE can be calculated directly.

3.3 | Simulation A

In this simulation scenario, we tested models’ performance on continuous covariates with compound symmetry covariance matrix. we specified number of samples as \( n = 100 \), and number of covariates as 20. The covariates \( X \) were sampled from multivariate normal distribution \( \text{MVN}(0, \Sigma) \), where \( \Sigma \) has compound symmetry structure, that is \( \Sigma_{ij} = \rho, \forall i \neq j \) and \( \Sigma_{ii} = 1 \). We set \( \rho = 0.1 \) for low correlation and 0.5 for high correlation among all covariates, separately. For the regression model in equation (6), we set the true regression coefficients as \( \beta_j = 1 \) for \( j = 1, 2, 5, 11, 12, 15 \) and \( \beta_j = 0 \) elsewhere. For MACR setting, we separately dropped 5% samples in \( \{X_{.j}, j = 11, 12, \ldots, 20\} \) to guarantee 60% complete cases. For MAR, we considered the following logistic regression model to generate \( R_{ij} \):

\[
\logit(P(R_{ij} = 0|X_{(j-10)}, Y_{i})) = \alpha_0 + 0.5X_{(j-10)} + 0.5Y_{i}, \ j = 11, 12, \ldots, 20
\]

where \( \alpha_0 \) was set to keep 60% complete cases approximately. The selection of \( \alpha_0 \) is attached in Appendix A.

As shown in TABLE 2, Multi-Laplace model had better sensitivity, specificity, F1 and MSE than MI-LASSO and LASSO applied to full data in both MCAR and MAR settings. Horseshoe achieved greater or similar performance scores than MI-LASSO and LASSO, BLASSO, BLASSO(CI) in MCAR and MAR settings as well. On the other hand, ARD and Spike-Laplace had higher specificity at the cost of slightly decreased sensitivity. Spike-Normal shown over-shrinkage problem that it gained the highest specificity in both MCAR and MAR settings but its sensitivity was remarkably low. By deleting incomplete cases without imputation, CC-LASSO displayed the worst performance compared to all other models, as expected. Furthermore, when \( \rho \) increased to 0.5, MI-LASSO achieved the best performances on some criteria, especially in MCAR setting. Its MSE was significantly lower than all models performed on MI data, and was similar to LASSO. Multi-Laplace obtained similar sensitivity and specificity to LASSO in MCAR while its F1 and MSE were lower. Horseshoe got scores slightly better than ARD. Discrete mixture models failed to identify important variables so that they gained high specificity with notably low sensitivity.

To reveal how the choice of credible interval affected shrinkage Bayesian MI-LASSO, we displayed the changes of average sensitivity, specificity and distance as \( x \% \) changed for credible interval criteria (see FIGURE 2). Although Multi-Laplace can achieve the best results compared with other Bayesian MI-LASSO models, it is rather sensitive to the choice of credible interval. The selected variables by Multi-Laplace model changed dramatically with the selection of credible interval. So the similar variable sets can be obtained by tuning the credible interval criterion, regardless of different hyperparameters of Multi-Laplace model. On the contrary, ARD is robust if \( x \% \) is not small while Horseshoe showed slightly worse results than ARD. Combining with TABLE 2, ARD was robust enough to the choice of credible interval while it can achieve favourable performance; Horseshoe can produce better results with the well-tuned credible interval selection; the best performance can be gained from Multi-Laplace while the credible interval selection should be carefully tuned.
**TABLE 2**  Performances of different models in simulation A.

| $\rho = 0.1$ | **Full Data** | **MCAR** | **MAR** |
|--------------|---------------|----------|---------|
| **Models**   | **SEN**       | **SPE**  | **F1**  | **MSE** | **SEN**       | **SPE**  | **F1**  | **MSE** |
| LASSO        | 93.8 (1.6)    | 79.8 (1.6) | 78.4 (1.4) | 1.7 (0.1) |
| BLASSO       | 93.5 (1.0)    | 88.7 (0.9) | 85.5 (0.9) | 1.6 (0.1) |
| BLASSO (CI)  | 95.7 (0.8)    | 91.2 (1.0) | 89.2 (0.9) | 1.3 (0.1) |
| LASSO        | 93.2 (2.6)    | 82.1 (1.5) | 68.5 (2.0) | 3.4 (0.2) |
| BLASSO       | 60.8 (1.7)    | 86.9 (0.9) | 63.4 (1.5) | 3.4 (0.2) |
| BLASSO (CI)  | 80.7 (2.0)    | 75.5 (2.7) | 69.4 (1.4) | 3.6 (0.3) |
| LASSO        | 77.2 (2.6)    | 82.1 (1.5) | 68.5 (2.0) | 3.4 (0.2) |
| BLASSO       | 60.8 (1.7)    | 86.9 (0.9) | 63.4 (1.5) | 3.4 (0.2) |
| BLASSO (CI)  | 80.7 (2.0)    | 75.5 (2.7) | 69.4 (1.4) | 3.6 (0.3) |

| $\rho = 0.5$ | **Full Data** | **MCAR** | **MAR** |
|--------------|---------------|----------|---------|
| **Models**   | **SEN**       | **SPE**  | **F1**  | **MSE** | **SEN**       | **SPE**  | **F1**  | **MSE** |
| LASSO        | 77.2 (1.6)    | 79.1 (1.2) | 68.6 (1.3) | 2.4 (0.1) |
| BLASSO       | 60.8 (1.7)    | 86.9 (0.9) | 63.4 (1.5) | 3.4 (0.2) |
| BLASSO (CI)  | 80.7 (2.0)    | 75.5 (2.7) | 69.4 (1.4) | 3.6 (0.3) |
| LASSO        | 79.7 (1.4)    | 74.8 (1.2) | 67.3 (1.2) | 2.4 (0.1) |
| BLASSO       | 79.7 (1.4)    | 74.8 (1.2) | 67.3 (1.2) | 2.4 (0.1) |
| BLASSO (CI)  | 80.7 (2.0)    | 75.5 (2.7) | 69.4 (1.4) | 3.6 (0.3) |

SEN, SPE, F1 scores are scaled by 100
 FIGURE 2  Average sensitivity, specificity and distance for different choices of x% symmetric credible interval, under MAR missing and $\rho = 0.5$ in simulation A. The results from MCAR setting are similar.

### 3.4 Simulation B

Different from simulation A, we here considered the continuous covariates with first order auto-regressive AR(1), i.e. $\Sigma_{ij} = \rho^{|i-j|}$. Six combinations of n and p were set: (n=100, p=20), (n=100, p=40), (n=200, p=20), (n=100, p=20, high missing proportion), (n=50, p=20), (n=30, p=20). The same process to generate $X$ in simulation A was followed, with $\rho = 0.5$ fixed in this simulation scenario. For the regression model in equation (6) and missing mechanism, all setting were kept same as simulation A for p=20. For p=40, we set $\beta_j = 1$, $j = 1, 2, 5, 11, 12, 15, 21, 22, 25, 31, 32, 35$ and $\beta_j = 0$ elsewhere, while the missing fraction was 2.5% for $\{X_{j}, j = 11, 12, ..., 20, 31, 32, ..., 40\}$ in MACR setting. For MAR, the logistic regression model to generate $R_{ij}$ was:

$$\text{logit}(P(R_{ij} = 0|X_{i(j-10)}, Y_i)) = \alpha_0 + 0.5X_{i(j-10)} + 0.5Y_i, j = 11, 12, ..., 20, 31, 32, ..., 40$$  (17)

The fourth simulation setting was to create more missing value in $X$ to yield 35% complete cases. In this setting, we considered n=100 and p=20 for comparability, and regression components were the same as the situation when p=20 above. To keep missing proportion high, we set missing fraction as 5% for MCAR and corresponding $\alpha_0$ for MAR. The choice of $\alpha_0$ is attached in Appendix A.

All numerical results were summarized into TABLE 3-5. Similarly, Multi-Laplace achieved the best results. Horseshoe performed alike to Multi-Laplace in MCAR and MAR settings. These two models had close sensitivity to LASSO and MI-LASSO with higher specificity. ARD and Spike-Laplace were slightly worse than Horseshoe. As p increased to 40, all models except Multi-Laplace failed to identify significant variables. For complete cases, the size of data was about (n = 65, p = 40), which formed a high dimensional problem, so the CC-LASSO obtained the worst results. For MAR, CC-LASSO only got 0.135 sensitivity, where hardly any variables can be selected. When n increased to 200, all models performed extraordinarily due to the large sample size. When missing proportion is high, Multi-Laplace and Horseshoe shown similar results as BLASSO and BLASSO(CI) and over-performed MI-LASSO and LASSO. ARD and Spike-Laplace tended to select fewer variables so that they had better specificity with lower sensitivity.

After the four comparable tests, we decreased sample size to 50 and 30 separately, to show the outstanding performances of Bayesian MI-LASSO models on small sized data. On small data, BLASSO and BLASSO(CI) achieved the best results while LASSO and MI-LASSO tended to lose power to select variables. Multi-Laplace model can still function to distinguish important variables with remarkable sensitivity and specificity. Horseshoe and ARD acted alike to MI-LASSO. Since Spike-Normal and Spike-Laplace shrunk at both group level and coefficient level, resulting in the over-shrinkage effect, so they preferred to choose fewer variables. Their specificity was notably high but sensitivity
was not high enough. On the contrary, CC-LASSO achieved the terribly large MSE due to too few complete cases when \( n = 30 \).

Additionally, FIGURE 3–5 indicated the change of average sensitivity, specificity and distance as the \( x \% \) value changed for credible interval criteria for shrinkage Bayesian MI-LASSO. Surprisingly, even in the situations with more features or fewer samples, ARD and Horseshoe were still in stability. However, Horseshoe and Multi-Laplace became more sensitive to the data with high missing proportion. Thus ARD can serve as the baseline model in application with the robust property for the different credible interval choice.

3.5 | Simulation C

This simulation scenario is to test models’ performance on binary covariates. The combination (\( n = 100, p = 20 \)) was chosen. The generation process of \( X \) is the same as that in Simulation B. To dichotomize the covariates, we simply set \( X_{ij} = 1 \) if \( X_{ij} > 0 \) and \( X_{ij} = 0 \) if \( X_{ij} < 0 \). After the sparse \( X \) was obtained, the linear regression model and missing mechanism functioned the same as Simulation A.

Simulation results were summarized into TABLE 6. With the double sparsity at covariates and missing mechanism, the methods from frequentists tended to select a small set of variables, resulting in remarkably low sensitivity. However, Bayesian MI-LASSO models and BLASSO, BLASSO(CI) can still obtain reasonable results. Multi-Laplace gained sensitivity and specificity with high quality. Horseshoe and ARD outperformed MI-LASSO as well. Spike-Laplace functioned alike to MI-LASSO. Even Spike-Normal can achieve better performance than LASSO applied to full data.

**FIGURE 3** Average sensitivity, specificity and distance for different choices of \( x \% \) symmetric credible interval, under MAR missing and \( n = 100, p = 40 \), in simulation B.

**FIGURE 4** Average sensitivity, specificity and distance for different choices of \( x \% \) symmetric credible interval, under MAR missing and high missing proportion, in simulation B.
TABLE 3  Performances of different models in simulation B (part 1).

| Models       | SEN (SD) | SPE (SD) | F1 (SD) | MSE (SD) |
|--------------|----------|----------|---------|----------|
| Full Data    |          |          |         |          |
| LASSO        | 94.8 (0.8) | 81 (1.4) | 80.4 (1.0) | 1.4 (0.1) |
| BLASSO       | 91.0 (1.0) | 88.2 (0.9) | 83.7 (0.9) | 1.5 (0.1) |
| BLASSO (CI)  | 92.8 (1.0) | 90.9 (0.9) | 87.1 (0.8) | 1.4 (0.1) |
| MCAR         |          |          |         |          |
| CC-LASSO     | 72.8 (3)  | 84.4 (1.6) | 66.7 (2.3) | 3.3 (0.2) |
| MI-LASSO     | 94.5 (1.1) | 80.9 (1.1) | 79.5 (1.0) | 1.4 (0.1) |
| Multi-Laplace| 94.3 (0.9) | 89.6 (1.0) | 87.0 (0.9) | 1.4 (0.1) |
| Horseshoe    | 92.8 (1.0) | 91.4 (0.9) | 87.7 (0.8) | 1.4 (0.1) |
| ARD          | 86.2 (1.2) | 92.2 (0.8) | 84.6 (0.9) | 1.6 (0.1) |
| Spike-Normal | 71.2 (1.6) | 95.6 (0.6) | 78.0 (1.4) | 2.1 (0.1) |
| Spike-Laplace| 82.8 (1.2) | 91.3 (0.8) | 81.8 (1.0) | 1.8 (0.1) |
| MAR          |          |          |         |          |
| CC-LASSO     | 51.8 (3.4) | 90.8 (1.2) | 52.8 (2.9) | 4.3 (0.3) |
| MI-LASSO     | 92.7 (1.1) | 79.7 (1.1) | 77.7 (0.9) | 1.6 (0.1) |
| Multi-Laplace| 92.5 (1.1) | 87.7 (1.1) | 84.3 (0.9) | 1.6 (0.1) |
| Horseshoe    | 89.5 (1.4) | 90.1 (1.0) | 84.5 (1.0) | 1.5 (0.1) |
| ARD          | 84.2 (1.4) | 91.7 (1.0) | 82.9 (1.1) | 1.7 (0.1) |
| Spike-Normal | 66.8 (1.5) | 95.3 (0.6) | 74.7 (1.3) | 2.4 (0.1) |
| Spike-Laplace| 80.2 (1.5) | 91.1 (0.8) | 79.7 (1.2) | 1.9 (0.1) |

| Models       | SEN (SD) | SPE (SD) | F1 (SD) | MSE (SD) |
|--------------|----------|----------|---------|----------|
| Full Data    |          |          |         |          |
| LASSO        | 53.0 (2.8) | 90.5 (1.0) | 56.1 (2.2) | 8.9 (0.4) |
| BLASSO       | 66.3 (1.5) | 89.1 (0.5) | 68.7 (1.2) | 6.9 (0.3) |
| BLASSO (CI)  | 77.7 (1.8) | 80.7 (2.0) | 70.5 (1.0) | 6.8 (0.3) |
| MCAR         |          |          |         |          |
| CC-LASSO     | 29.6 (2.9) | 92.1 (1.3) | 32.5 (2.5) | 14.2 (0.5) |
| MI-LASSO     | 57.5 (2.4) | 89.0 (0.9) | 59.6 (2.0) | 8.0 (0.4) |
| Multi-Laplace| 77.5 (1.7) | 80.7 (1.7) | 70.1 (0.9) | 7.3 (0.3) |
| Horseshoe    | 52.6 (1.6) | 91.6 (1.2) | 60.6 (1.3) | 8.5 (0.3) |
| ARD          | 61.1 (1.7) | 86.0 (2.0) | 63.5 (1.2) | 8.1 (0.3) |
| Spike-Normal | 45.3 (1.2) | 93.7 (0.5) | 56.3 (1.2) | 9.0 (0.3) |
| Spike-Laplace| 54.9 (1.3) | 89.2 (0.6) | 60.7 (1.1) | 8.7 (0.3) |
| MAR          |          |          |         |          |
| CC-LASSO     | 11.0 (2.0) | 93.6 (1.9) | 12.1 (1.6) | 17.1 (0.5) |
| MI-LASSO     | 53.0 (2.4) | 88.6 (1.0) | 55.7 (2.0) | 8.8 (0.4) |
| Multi-Laplace| 69.2 (2.1) | 80.6 (2.2) | 64.6 (1.1) | 9.2 (0.4) |
| Horseshoe    | 48.9 (2.0) | 89.0 (1.7) | 55.1 (1.3) | 10.0 (0.4) |
| ARD          | 53.7 (1.8) | 88.0 (1.6) | 58.7 (1.3) | 9.6 (0.3) |
| Spike-Normal | 42.3 (1.2) | 92.0 (0.5) | 52.2 (1.3) | 10.7 (0.4) |
| Spike-Laplace| 51.6 (1.3) | 87.1 (0.7) | 56.6 (1.2) | 10.2 (0.4) |

SEN, SPE, F1 scores are scaled by 100.
### TABLE 4 Performances of different models in simulation B (part 2).

#### n = 200, p = 20

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| LASSO        | 99.8 (0.1) | 83.8 (0.8) | 85.1 (0.7) | 0.5 (0.0) |
| BLASSO       | 99.2 (0.3) | 91.8 (0.5) | 91.3 (0.5) | 0.5 (0.0) |
| BLASSO (CI)  | 98.7 (0.3) | 95.1 (0.5) | 94.3 (0.5) | 0.5 (0.0) |

#### MCAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 96.7 (0.6) | 82.4 (0.9) | 82.3 (0.7) | 1.0 (0.0) |
| MI-LASSO     | 99.8 (0.1) | 82.0 (0.8) | 83.4 (0.6) | 0.6 (0.0) |
| Multi-Laplace| 98.7 (0.3) | 96.0 (0.4) | 95.2 (0.5) | 0.5 (0.0) |
| Horseshoe    | 99.0 (0.3) | 96.2 (0.4) | 95.6 (0.4) | 0.4 (0.0) |
| ARD          | 97.2 (0.4) | 94.8 (0.5) | 93.2 (0.6) | 0.5 (0.0) |
| Spike-Normal | 86.8 (1.0) | 97.1 (0.4) | 89.5 (0.9) | 0.9 (0.1) |
| Spike-Laplace| 97.2 (0.5) | 94.8 (0.5) | 93.2 (0.6) | 0.5 (0.0) |

#### MAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 84.2 (1.7) | 84.8 (0.9) | 75.4 (1.2) | 2.0 (0.1) |
| MI-LASSO     | 99.5 (0.2) | 79.9 (0.7) | 81.6 (0.6) | 0.6 (0.0) |
| Multi-Laplace| 98.5 (0.3) | 95.1 (0.5) | 94.2 (0.5) | 0.5 (0.0) |
| Horseshoe    | 97.0 (0.5) | 95.1 (0.5) | 93.5 (0.6) | 0.5 (0.0) |
| ARD          | 85.7 (0.9) | 96.9 (0.4) | 88.6 (0.8) | 0.9 (0.0) |
| Spike-Normal | 96.3 (0.5) | 93.9 (0.4) | 91.7 (0.6) | 0.6 (0.0) |
| Spike-Laplace| 88.0 (1.3) | 92.7 (0.9) | 85.4 (1.2) | 2.0 (0.1) |

#### n = 100, p = 20 (HM)

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| LASSO        | 94.8 (0.8) | 81.0 (1.4) | 80.7 (1.0) | 1.3 (0.1) |
| BLASSO       | 91.0 (1.0) | 88.2 (0.9) | 83.7 (0.9) | 1.5 (0.1) |
| BLASSO (CI)  | 92.8 (1.0) | 90.9 (0.9) | 87.1 (0.8) | 1.4 (0.1) |

#### MCAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 60.7 (2.9) | 79.3 (2.2) | 55.4 (2.3) | 5.8 (0.3) |
| MI-LASSO     | 93.8 (1.1) | 81.2 (1.1) | 79.5 (1.0) | 1.4 (0.1) |
| Multi-Laplace| 94.5 (0.9) | 89.9 (1.2) | 87.6 (1.0) | 1.4 (0.1) |
| Horseshoe    | 92.2 (1.1) | 92.6 (0.8) | 88.3 (0.9) | 1.3 (0.1) |
| ARD          | 83.7 (1.3) | 92.7 (1.0) | 83.7 (1.1) | 1.6 (0.1) |
| Spike-Normal | 68.3 (1.5) | 95.1 (0.5) | 75.6 (1.3) | 2.2 (0.1) |
| Spike-Laplace| 82.0 (1.5) | 92.0 (0.6) | 81.5 (1.2) | 1.7 (0.1) |

#### MAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 22.2 (2.9) | 89.3 (2.1) | 22.4 (2.4) | 8.4 (0.9) |
| MI-LASSO     | 88.2 (1.4) | 76.9 (1.4) | 73.5 (1.2) | 2.0 (0.1) |
| Multi-Laplace| 90.5 (1.3) | 83.3 (1.7) | 80.2 (1.1) | 2.0 (0.1) |
| Horseshoe    | 87.2 (1.5) | 85.4 (1.8) | 80.1 (1.3) | 2.0 (0.1) |
| ARD          | 79.0 (1.8) | 87.1 (1.6) | 76.1 (1.4) | 2.2 (0.1) |
| Spike-Normal | 62.7 (1.7) | 93.8 (0.6) | 70.2 (1.5) | 2.7 (0.1) |
| Spike-Laplace| 76.3 (1.7) | 87.3 (1.0) | 74.2 (1.4) | 2.4 (0.1) |

SEN, SPE, F1 scores are scaled by 100
### TABLE 5  Performances of different models in simulation B (part 3).

#### n = 50, p = 20

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| Full Data    |      |      |      |      |
| LASSO        | 66.8 (4.4) | 83.9 (2.3) | 61.4 (3.4) | 4.0 (0.4) |
| BLASSO       | 70.2 (2.3) | 87.9 (1.2) | 70.5 (1.8) | 3.3 (0.2) |
| BLASSO (CI)  | 84.0 (2.5) | 82.1 (2.5) | 75.1 (1.8) | 3.0 (0.2) |

#### MCAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 54.5 (4.6) | 70.5 (4.3) | 44.8 (3.2) | 10.6 (3.1) |
| MI-LASSO     | 66.2 (4) | 83.6 (2.2) | 61.8 (3.1) | 4.0 (0.3) |
| Multi-Laplace| 80.7 (2.8) | 82.5 (2.6) | 73.3 (1.4) | 3.4 (0.2) |
| Horseshoe    | 75.8 (3.1) | 83.7 (2.7) | 71.0 (2.0) | 3.5 (0.3) |
| ARD          | 64.2 (2.9) | 86.2 (2.9) | 65.7 (2.1) | 3.9 (0.3) |
| Spike-Normal | 46.7 (2.3) | 93.5 (0.8) | 57.0 (2.4) | 4.4 (0.3) |
| Spike-Laplace| 59.7 (2.3) | 87.9 (1.3) | 63.4 (2.1) | 4.1 (0.3) |

#### MAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 80.2 (3.5) | 26.9 (3.4) | 45.4 (1.7) | 181.6 (113.1) |
| MI-LASSO     | 58.7 (4.3) | 77.5 (3.7) | 54.2 (3.4) | 6.5 (0.8) |
| Multi-Laplace| 74.0 (4.3) | 74.1 (5.5) | 64.6 (2.4) | 7.4 (1.2) |
| Horseshoe    | 55.5 (4.9) | 83.5 (4.6) | 56.0 (3.3) | 6.8 (0.8) |
| ARD          | 57.7 (4.1) | 83.9 (3.9) | 58.6 (2.9) | 6.6 (0.6) |
| Spike-Normal | 39.0 (3.0) | 92.8 (1.4) | 49.3 (3.3) | 6.7 (0.6) |
| Spike-Laplace| 46.7 (3.0) | 87.1 (1.9) | 52.8 (3.0) | 6.9 (0.7) |

#### n = 30, p = 20

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| Full Data    |      |      |      |      |
| LASSO        | 59.8 (4.7) | 71.6 (5.4) | 52.9 (3.2) | 8.1 (1.1) |
| BLASSO       | 51.7 (3.1) | 89.8 (1.6) | 58.5 (3.0) | 5.8 (0.6) |
| BLASSO (CI)  | 75.2 (4.6) | 75.9 (5.2) | 65.6 (2.8) | 6.5 (0.8) |

#### MCAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 82.0 (3.5) | 26.9 (3.4) | 45.4 (1.7) | 181.6 (113.1) |
| MI-LASSO     | 58.7 (4.3) | 77.5 (3.7) | 54.2 (3.4) | 6.5 (0.8) |
| Multi-Laplace| 74.0 (4.3) | 74.1 (5.5) | 64.6 (2.4) | 7.4 (1.2) |
| Horseshoe    | 55.5 (4.9) | 83.5 (4.6) | 56.0 (3.3) | 6.8 (0.8) |
| ARD          | 57.7 (4.1) | 83.9 (3.9) | 58.6 (2.9) | 6.6 (0.6) |
| Spike-Normal | 39.0 (3.0) | 92.8 (1.4) | 49.3 (3.3) | 6.7 (0.6) |
| Spike-Laplace| 46.7 (3.0) | 87.1 (1.9) | 52.8 (3.0) | 6.9 (0.7) |

#### MAR

| Models       | SEN  | SPE  | F1   | MSE  |
|--------------|------|------|------|------|
| CC-LASSO     | 82.7 (4.6) | 22.6 (4.9) | 44.2 (2.0) | 234.8 (120.2) |
| MI-LASSO     | 51.7 (4.9) | 79.7 (3.5) | 49.1 (3.9) | 6.7 (0.6) |
| Multi-Laplace| 70.7 (4.9) | 74.1 (5.4) | 61.3 (2.8) | 7.6 (1.1) |
| Horseshoe    | 52.8 (5.5) | 82.1 (4.7) | 51.5 (3.6) | 7.4 (0.7) |
| ARD          | 51.5 (4.5) | 83.5 (4.0) | 52.6 (3.1) | 7.3 (0.8) |
| Spike-Normal | 36.0 (2.7) | 92.4 (1.3) | 45.9 (2.8) | 7.0 (0.6) |
| Spike-Laplace| 42.2 (2.8) | 86.4 (1.9) | 48.3 (2.8) | 7.3 (0.6) |

SEN, SPE, F1 scores are scaled by 100
FIGURE 5  Average sensitivity, specificity and distance for different choices of $x\%$ symmetric credible interval, under MAR missing and $n = 30$, $p = 20$, in simulation B.

TABLE 6  Performances of different models in simulation C.

| Models     | Full Data | MCAR | MAR |
|------------|-----------|------|-----|
|            | SEN       | SPE  | F1  | MSE   | SEN  | SPE  | F1  | MSE   |
| LASSO      | 29.2 (2.2)| 95.9 (0.8)| 38.4 (2.0)| 6.8 (0.2)| 6.8 (1.1)| 96.2 (0.9)| 9.8 (1.5)| 8.5 (0.1) |
| BLASSO     | 52.8 (1.8)| 89.1 (0.9)| 58.6 (1.6)| 4.4 (0.1)| 36.3 (2.6)| 92.4 (1.1)| 42.1 (2.6)| 5.8 (0.2) |
| BLASSO (CI)| 84.0 (2.1)| 66.2 (3.1)| 66.0 (1.3)| 3.8 (0.1)| 77.3 (2.6)| 70.1 (2.8)| 62.7 (1.4)| 4.3 (0.1) |

SEN, SPE, F1 scores are scaled by 100
4.1 | Data Description

We followed literature [17] to investigate our models’ performance on University of Michigan Dioxin Exposure Study (UMDES) data. Dioxin is a chemical pollutant with high toxicity and long life-cycle in the environment. The UMDES data came from the first population based dioxin exposure study of residents living in the Midland, Michigan area, which is provided to explore the association between important exposure factors in the specific circumstances and serum dioxin concentration in human blood. The whole UMDES data is composed of the data measuring dioxins from participants’ serum in the donated blood and their household dust and soil samples, along with their information on demographics, health, residence, activities, work history, lifetime food consumption, current diet, etc [40–42].

The data of the 448 residents who live in or close to the flood plain of the Tittabawassee between the Dow Chemical Company plant in Midland and the confluence of the Tittabawassee and Shiawassee rivers in Saginaw is what we used in this study. The outcome variable is the log-10 transformed serum concentration of 2,3,7,8-tetrachlorodibenzop-dioxin (TCDD), the dioxin compound with highest toxicity. Entirely, the covariates consist of 63 continuous and 68 dichotomous variables of the related demographics and other information discussed above. Most of them are moderately correlated, with very few variables have high correlation (see FIGURE 6). Like other epidemiological data, there also exist missing values in covariates due to nonresponse in survey questionnaire and rejection to provide related sample data. However, in literature [43], Olson et al applied a sequential regression imputation method [44] and then generated five imputed sets of the UMDES data. Therefore, our analysis was conducted on these five imputed datasets.

4.2 | Performances

In real data, we have no ground truth. Therefore information based criterion was used to select $x\%$ credible interval for shrinkage Bayesian MI-LASSO and evaluated performances. We took advantages of the modified version of Bayesian Information Criterion (BIC, referring to formula (10) and (11)) to assess different selection of credible interval. Also,
For the hyperparameters, a sensitivity analysis was conducted in Appendix B. Here we exploited different strategies for Multi-Laplace model and discrete mixture models. Since Multi-Laplace prior is robust to different hyperparameters (see Section 3.3 and Appendix B), we used the default settings. For discrete models, we introduced a novel tuning method developed in recent years, Bayesian optimization. Bayesian optimization is a heuristic black-box optimization algorithm, which is widely used to find excellent hyperparameters with low computational cost. More discussions of Bayesian optimization on our models can be found in Appendix C. In this analysis, we ran 20 rounds of Bayesian optimization for discrete mixture models to select best hyperparameters.

As exhibited in TABLE 7, all the five Bayesian MI-LASSO models achieved notable results with approximately 15 selected variables and BIC less than -2.4. Compared with shrinkage methods, the discrete mixture models tend to select fewer variables. Among all the selected variables, Age, BMI, Female, BMI change, Years of living in Midland: 1960-79, months of breast-feeding, ever smoke, Fishing in the Tittabawassee River below the Tridge after 1980 more than once per month, Years of living in a property with crops live stock or poultry: 1960-79, Years of working in foundry after 1980, Years of spraying chemicals to kill plants: 1960-79, Years of living with somebody worked at The Dow Chemical Company: 1940-59, Age * female were selected from most of models. Comparably, most of these variables were also selected from the original MI-LASSO [17]. FIGURE 7 shows the posterior distributions of Age, Female and Years of living in Midland: 1960-79 from Bayesian MI-LASSO models. We also plotted the estimates and 95% confidence intervals obtained from the original MI-LASSO in FIGURE 7. Since the 95% confidence intervals were calculated by refitting a linear regression model after variable selection process, the estimates were slightly different from those obtained from Bayesian MI-LASSO models. Although few variables were chosen by only one or two models, we can compare the results from other models to reduce the inaccuracy. On the other hand, there were totally 19 variables selected from MI-LASSO. However, all Bayesian MI-LASSO models selected less than 19 variables with remarkable BIC, which showed these five Bayesian MI-LASSO models can overpower the original MI-LASSO in selecting variables on UMDES data.
TABLE 7 Performance of all models on UMDES data. For shrinkage models, credible interval criteria is applied to
determine the selection of variables. For discrete mixture models, hyperparameters are selected by running 20 turns
of Bayesian optimization.

| Model               | Multi-Laplace | Shrinkage Models | Discrete Mixture Models |
|---------------------|---------------|------------------|------------------------|
|                     |               | Horseshoe        | ARD                    | Spike-Normal | Spike-Laplace |
| Hyperparameters     |               |                  |                        |             |              |
| r                   | 2             |                  |                        | 0.959        | 9.230        |
| 2                   |               |                  |                        | 0.095        | 0.955        |
| Modified BIC        | -2.413 (95% CI)| -2.461 (90% CI)  | -2.433 (95% CI)        | -2.402       | -2.456       |
| Variables           |               |                  |                        |             |              |
| Age                 | 0.0145        | (0.0111, 0.0179) | 0.0151 (0.0125, 0.0173) | 0.0155       | (0.0135, 0.0183) | 0.0134 (0.0104, 0.0160) | 0.0153 (0.0127, 0.0180) |
| Months of breast-feeding | -0.0057       | (-0.0093, -0.0020) | -0.0056 (-0.0094, -0.0019) | -0.0051 (-0.0086, -0.0019) | -0.0057 (-0.0092, -0.0021) |
| BMI                 | 0.0096        | (0.0046, 0.0145) | 0.0110 (0.0061, 0.0160) | 0.0100 (0.0046, 0.0146) | 0.0123 (0.0075, 0.0171) | 0.0119 (0.0071, 0.0156) |
| Female              | 0.2267        | (0.1560, 0.2998) | 0.1904 (0.1378, 0.2471) | 0.2118 (0.1654, 0.2594) | 0.1668 (0.1107, 0.2288) | 0.1918 (0.1372, 0.2466) |
| BMI change          | -0.0157       | (-0.0286, -0.0031) | -0.0214 (-0.0335, -0.0099) | -0.0220 (-0.0344, -0.0081) | -0.0189 (-0.0314, -0.0061) | -0.0221 (-0.0344, -0.0099) |
| Ever smoker         | -0.0950       | (-0.1556, -0.0328) | -0.0938 (-0.1465, -0.0456) | -0.0982 (-0.1365, -0.0556) | -0.0894 (-0.1407, -0.0387) | -0.0961 (-0.1407, -0.0387) |
| Log10 transformed Soil dioxin concentrations for soil contact 0-6 inch | 0.1763 | (0.0310, 0.3284) | 0.1314 (0.0392, 0.2135) | 0.1315 (0.0172, 0.2426) | 0.0048 (0.0012, 0.0082) | 0.0041 (0.0007, 0.0075) |
| Days of hunting in the surrounding areas of the Saginaw River and bay 1960-1979 | 0.0841 | (-0.1551, -0.0148) | 0.0521 (-0.1110, 0.0035) | 0.0551 (-0.1326, -0.0009) | -0.0713 (-0.1338, -0.0072) | -0.1338 (-0.0072) |
| Eating vegetables and fruits raised in Saginaw river and bay 1960-1979 | 0.0071 | (0.0003, 0.0139) | 0.0062 (0.0016, 0.0122) | 0.0053 (0.0004, 0.0107) | 0.0093 (0.0027, 0.0154) | -0.0009 (-0.0000, 0.0009) |
| Years of living in a property with crops livestock or poultry 1960-1979 | 0.0082 | (0.0038, 0.0127) | 0.0092 (0.0060, 0.0125) | 0.0089 (0.0053, 0.0156) | 0.0087 (0.0047, 0.0127) | 0.0103 (0.0068, 0.0137) |
| Years of working in foundry after 1980 | 0.0229 | (0.0044, 0.0424) | 0.0156 (0.0017, 0.0307) | 0.0199 (0.0063, 0.0336) | -0.0093 (-0.0153, -0.0039) | -0.0095 (-0.0156, -0.0042) | -0.0129 (-0.0196, -0.0063) | -0.0114 (-0.0188, -0.0047) |
| Years of spraying chemicals to kill plants 1960-1979 | 0.0197 | (0.0188, 0.0207) | 0.0196 (0.0101, 0.0204) | 0.0196 (0.0060, 0.0231) | -0.0093 (-0.0153, -0.0039) | -0.0095 (-0.0156, -0.0042) | -0.0129 (-0.0196, -0.0063) | -0.0114 (-0.0188, -0.0047) |
| Years of working at the Dow Chemical company after 1980 | 0.0076 | (0.0027, 0.0127) | 0.0092 (0.0017, 0.0173) | 0.0093 (0.0027, 0.0154) | 0.0093 (0.0027, 0.0154) | 0.0093 (0.0027, 0.0154) |
| Years of working at the Dow Chemical company after 1990 | 0.0082 | (0.0027, 0.0127) | 0.0092 (0.0060, 0.0125) | 0.0089 (0.0053, 0.0156) | 0.0087 (0.0047, 0.0127) | 0.0103 (0.0068, 0.0137) |
| Years of living in a property with a garden in the past 5 years | -0.0097 | (-0.0188, -0.0012) | -0.0111 (-0.0217, -0.0025) | -0.0111 (-0.0217, -0.0025) | -0.0111 (-0.0217, -0.0025) | -0.0111 (-0.0217, -0.0025) |
| Years worked in waste disposal, metal scrap yards, or water treatment facilities after 1980 | 0.0066 | (0.0038, 0.0094) | 0.0092 (0.0017, 0.0173) | 0.0093 (0.0027, 0.0154) | 0.0093 (0.0027, 0.0154) | 0.0093 (0.0027, 0.0154) |
| Number of selected variables | 15 | 15 | 17 | 12 | 11 | 11 |

The other variables not listed are excluded by all the Bayesian MI-LASSO models.
Although missing data is a common problem in statistical analysis, multiple imputation is widely exploited to handle missing values. However, the inconsistent variable selection on multiply-imputed data is an increasingly popular problem in practice. MI-LASSO, proposed in 2013, is the most successful solution to this problem. It regards the same variable across all the imputed sets as a group of variables and applies Group-LASSO to achieve consistent variable selection on multiply-imputed datasets. In this paper, we extended the classic MI-LASSO model into Bayesian framework and proposed totally five novel Bayesian MI-LASSO models. Unlike the frequentists’ method, Bayesian MI-LASSO models yield the grouped penalty by specifying a common regularization prior distribution to the whole group of variables. Among these five models, three are based on shrinkage prior, while the other two are built on discrete mixture prior. Through the numeric simulation study, the Bayesian MI-LASSO models can perform better or equally than the original MI-LASSO, especially on the small sized data. On the case study using the University of Michigan Dioxin Exposure Study data, the five Bayesian MI-LASSO models can achieve notable BIC with fewer number of selected variables than MI-LASSO. Therefore, these proposed approaches can be promptly exploited to variable selection problems on multiply-imputed datasets.

Several considerations should be taken into account in using Bayesian MI-LASSO methods. Firstly, investigators should carefully decide the evaluation of credible interval criterion for shrinkage models. We recommend the modified version of BIC as the evaluation criterion as it performed well in the case study on UMDES data. Secondly, Horseshoe and ARD models have no hyperparameter, and are robust to the credible interval selection. So investigators can quickly take advantages of these two models to explore the variable selection problems when little knowledge about data is known. On the other hand, the Multi-Laplace model is robust to hyperparameters and is able to select fluctuating sets of variables with different choices of $\alpha$% credible interval criterion, as it can possibly select the best variable set with some choices of $\alpha$%. The remarkable upper limit of its performance can be achieved if the credible interval selection is thoughtfully tuned. Thirdly, the discrete mixture models tend to select few variables compared with shrinkage models. However, this class of models are sensitive to the hyperparameter selection although no extra criterion is needed. In order to obtain well-performed models, we recommend to apply Bayesian optimization to discover excellent hyperparameters. For the Spike-Laplace model, a notable result can be returned with few number of turns of Bayesian optimization, while more turns are needed for the Spike-Normal model.

Compared with the original MI-LASSO from frequentists' view, its Bayesian versions provide a set of flexible models to achieve better performances for the variable selection problems on multiply-imputed data. Through the comparison among results from different models, a reasonable set of important variables can be eventually determined. One advantage of Bayesian MI-LASSO is the well-performed results on the data with small size. Another advantage is that they can be straightforwardly extended into the framework of generalized linear models, by adding some transformations on the distribution of the response variable. One main limitation of Bayesian MI-LASSO is the computational cost of running MCMC chains on large data sets. A challenging but efficient avenue for future research is to explore inference-based Bayesian MI-LASSO models with Variational Bayesian methods developed in recent years \cite{45,46}. Additionally, MI-LASSO and its Bayesian versions are all built on multiply-imputed data while the process of MI can be integrated into Bayesian framework. Therefore, further research in simultaneously imputing missing data and selecting important variables can be a worthwhile topic.
Acknowledgements

Acknowledgements should include contributions from anyone who does not meet the criteria for authorship (for example, to recognize contributions from people who provided technical help, collation of data, writing assistance, acquisition of funding, or a department chairperson who provided general support), as well as any funding or other support information.

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Appendix A. Values of $\alpha_0$

In experimental design, we test Bayesian MI-LASSO models on generated data. To make the simulation replicable, the values of $\alpha_0$ were listed in TABLE 8.

Appendix B. Sensitivity Analysis

This appendix provided a detailed sensitivity analysis for the hyperparameters selection for Bayesian MI-LASSO models on UMDES data. Horseshoe and ARD models were excluded in this section. We tried different combination of hyperparameters via grid search, and calculated the modified BIC. Since Spike-Laplace model had three hyperparameters, we searched the grid by stratifying $\lambda$. The contour plots of the model performance of the three models were displayed in FIGURE 8-10. The results from best models via grid search were summarized into TABLE 9.

For the Multi-Laplace model, since it is sensitive to the tuning of credible interval percentage instead of hyperparameters, similar remarkable results can be achieved regardless of different combination of hyperparameters (see Figure 8). Compared with the default settings, five variables were excluded and three new variables were added in the best model via grid search, thus more accurate with lower BIC. The three variables years of living with somebody

| Parameter | Scenario                      | Values |
|-----------|-------------------------------|--------|
| $\alpha_0$| low missing proportion       | -4     |
| $\alpha_0$| high missing proportion      | -1.8   |
worked at The Dow Chemical Company after 1980, years of living with burned trash in 1940-1959 and 1960-1979 were only selected by the default Multi-Laplace model among all models.

On the other hand, the Spike-Normal model is rather sensitive to hyperparameters. When $v_0$ is small and $\rho_0$ is large, the results are typically terrible. Compared with the model from Bayesian optimization, the best model discovered by grid search has similar BIC but distinct variable sets. There are only five common variables determined by both two models. The variable sets selected are sensitive to hyperparameters. Therefore, the tuning of hyperparameters of this model should be carefully considered. Further discussion about Bayesian optimization with Spike-Normal model can be found in Appendix C.

Finally, for the Spike-Laplace model, the stratified contour plots showed the model can achieve better results if $a$ and $b$ are not different significantly, especially when $\lambda$ has large value (displayed in Figure 10). This phenomenon is consistent with the searched hyperparameter via Bayesian optimization in TABLE 7. Compared with the model from Bayesian optimization, there were totally 3 new variables added and no variable excluded in the best model via grid search. With much less computational cost, Bayesian optimization can achieve similar result as grid search, which is more suitable for discrete mixture models.

Appendix C. Bayesian Optimization

Bayesian optimization is a black-box optimization method which sequentially searches for optimal points via Gaussian process. At each iteration, it models all the searched points as a Gaussian process, and then finds a potential optimal solution and evaluates it. Unlike other tuning methods which discard previous searched results, Bayesian optimization leverages the information obtained from previous turns, to achieve a better point in the next turn. Thus Bayesian optimization can return an excellent solution after a small number of iterations, making it widely used in the hyperparameter selection for expensive-to-evaluate functions and models. More technical details about Bayesian optimization can be found in literature [49–52].

As we can see in FIGURE 11, the Spike-Normal model is sensitive to hyperparameter, since the performance can be dramatically improved with more turns of Bayesian optimization. At the beginning of Bayesian optimization, the randomly searched model was inaccurate. With more turns, the performance of the model was improved significantly.
**FIGURE 9** Sensitivity analysis for Spike-Normal model. Modified BIC is the criteria.

(a) $\lambda = 0.1$

(b) $\lambda = 1$

(c) $\lambda = 10$

**FIGURE 10** Sensitivity analysis for Spike-Laplace model, stratified by $\lambda$. Modified BIC is the criteria.
TABLE 9  Performance of the best models selected by grid search.

| Model          | Multi-Laplace | Spike-Normal | Spike-Laplace |
|----------------|---------------|--------------|---------------|
| Hyperparameters| r 0.01        | y0 0.1       | λ 1           |
|                | s 10          | p0 0.1       | a 10          |
|                |               | b 1          |               |
| Modified BIC   | -2.446 (95% CI) | -2.408 | -2.448 |
| Variables      |               | Posterior Estimates with Symmetric 95% Credible Interval |
| Age            | 0.0171 (0.0152, 0.0186) | 0.0140 (0.0115, 0.0162) | 0.0145 (0.0118, 0.0172) |
| Months of breast-feeding | -0.0061 (-0.0090, -0.0020) | -0.0060 (-0.0095, -0.0023) | -0.0061 (-0.0097, -0.0024) |
| BMI            | 0.0104 (0.0060, 0.0147) | 0.0110 (0.0064, 0.0159) | 0.0110 (0.0065, 0.0161) |
| Female         | 0.2150 (0.1615, 0.2682) | 0.1807 (0.1286, 0.2508) | 0.1926 (0.1349, 0.2492) |
| BMI change     | -0.0194 (-0.0326, -0.0051) | -0.0226 (-0.0333, -0.0107) | -0.0222 (-0.0341, -0.0102) |
| Ever smoker    | -0.1068 (-0.1485, -0.0599) | -0.0900 (-0.1525, -0.0475) | -0.0987 (-0.1472, -0.0502) |
| Age * female   | ~             | 0.0045 (0.0016, 0.0081) | 0.0043 (0.0009, 0.0077) |
| Log_{10} transformed Soil dioxin concentrations for soil contact 0-6 inch | 0.1441 (0.0085, 0.2934) | 0.0794 (-0.1177, -0.0220) | 0.0700 (-0.1303, -0.0119) |
| Fishing in the Tittabawassee River below the Tridge after 1980 (> 1/month) | ~ | 0.1177 (0.0222, 0.2041) | 0.1267 (0.0363, 0.2171) |
| Days of hunting in the surrounding areas of the Saginaw River and bay: 1960-79 | ~ | 0.1177 (0.0222, 0.2041) | 0.1267 (0.0363, 0.2171) |
| Eating vegetables and fruits raised in Saginaw river and bay (> 0, <1/month) | 0.1087 (0.0191, 0.2002) | ~ | 0.1178 (0.0205, 0.2199) |
| Years of living in Midland: 1960-79 | 0.0088 (0.0055, 0.0122) | 0.0094 (0.0052, 0.0131) | 0.0092 (0.0057, 0.0127) |
| Years of working in foundry: 1960-79 | ~ | -0.0162 (-0.0254, -0.0066) | ~ | 0.1178 (0.0205, 0.2199) |
| Years of working in foundry after 1980 | -0.0010 (-0.0159, -0.0043) | ~ | ~ | -0.0101 (-0.0160, -0.0041) |
| Years of spraying chemicals to kill plants: 1960-79 | 0.0153 (0.0016, 0.0287) | 0.0247 (0.0116, 0.0405) | 0.0234 (0.0086, 0.0382) |
| Years of living with somebody worked at The Dow Chemical Company: 1940-59 | 0.0197 (0.0049, 0.0339) | 0.0202 (0.0060, 0.0331) | 0.0185 (0.0047, 0.0331) |
| Years of using wood burning stoves: 1960-79 | 0.0043 (0.0004, 0.0093) | 0.0063 (-0.0073, 0.0108) | ~ | ~ |
| Number of selected variables | 13 | 14 | 14 |

The other variables not listed are excluded by these three Bayesian MI-LASSO models.
| Model                      | Spike-Normal | Spike-Laplace |
|----------------------------|--------------|---------------|
| Hyperparameters            |              |               |
| $v_0$                      | 697.063      | $\lambda$     |
| $\rho_0$                   | 0.065        | $a$           |
|                            |              | $b$           |
|                            |              | 837.712       |
|                            |              | 818.815       |
| Modified BIC               | -2.320       | -2.448        |
| Variables                 |              |               |
| Posterior Estimates with Symmetric 95% Credible Interval |
| Age                       | 0.0165       | 0.0176        |
|                           | (0.0143, 0.0187) | (0.0157, 0.0196) |
| Months of breast-feeding  | -0.0060      | -0.0246       |
|                           | (-0.0096, -0.0024) | (-0.0366, -0.0128) |
| BMI                       | -0.0060      | 0.0125        |
|                           | (-0.0096, -0.0024) | (0.0077, 0.0173) |
| Female                    | -0.0060      | 0.2051        |
|                           | (-0.0096, -0.0024) | (0.1521, 0.2595) |
| BMI change                | -0.0060      | -0.0246       |
|                           | (-0.0096, -0.0024) | (-0.0366, -0.0128) |
| Ever smoker               | -0.0060      | -0.0246       |
|                           | (-0.0096, -0.0024) | (-0.0366, -0.0128) |
| Fishing in the Tittabawassee River below the Tridge after 1980 (> 1/month) | -0.0969 |                          |
|                           | (-0.1622, -0.0313) |                          |
| Eating vegetables and fruits raised in Saginaw river and bay (> 0, <1/month) |                          | 0.1243 |
|                           |                          | (0.0234, 0.2254) |
| Years of living in a property with crops livestock or poultry: 1960-79 |                          | 0.0091 |
|                           |                          | (0.0035, 0.0148) |
| Years of living in Midland: 1960-79 | 0.0113       | 0.0096        |
|                           | (0.0073, 0.0153) | (0.0061, 0.0132) |
| Years of working in foundry after 1980 |                          | -0.0098 |
|                           |                          | (-0.0159, -0.0038) |
| Years of living with somebody worked at The Dow Chemical Company: 1940-59 |                          | 0.0178 |
|                           |                          | (0.0033, 0.0327) |
| Number of selected variables | 3               | 11            |

The other variables not listed are excluded by the discrete mixture Bayesian MI-LASSO models.
On the other hand, the Spike-Laplace model is more robust. The performance was remarkable initially, while little enhancement was made with more turns. This difference of these two discrete mixture models was verified via the comparison between results from 10 turns and 20 turns of Bayesian optimization. From what was displayed in Table 10, for Spike-Normal model, there were only three variables selected after only 10 turns. This model was too simple with bad BIC, different from that from 20 turns. On the contrary, for the Spike-Laplace model, there were nice common variables chosen by both models with 10 turns and 20 turns, steadily. To conclude, more turns of Bayesian optimization should be ran for the Spike-Normal model to achieve remarkable result in practice. For Spike-Laplace model, however, excellent results can be obtained with only few turns.

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