AN EXAMPLE OF A BOUNDED $\mathbb{C}$-CONVEX DOMAIN WHICH IS NOT BIHOLOMORPHIC TO A CONVEX DOMAIN

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Abstract. We show that the symmetrized bidisc is a $\mathbb{C}$-convex domain. This provides an example of a bounded $\mathbb{C}$-convex domain which cannot be exhausted by domains biholomorphic to convex domains.

1. Introduction

Recall that a domain $D$ in $\mathbb{C}^n$ is called $\mathbb{C}$-convex if any non-empty intersection with a complex line is contractible (cf. [2] [9]). A consequence of the fundamental Lempert theorem (see [12]) is the fact that any bounded $\mathbb{C}$-convex domain $D$ with $C^2$ boundary has the following property (see [8]):

(*) The Carathéodory distance and Lempert function of $D$ coincide.

Any convex domain can be exhausted by smooth bounded convex ones (which are obviously $\mathbb{C}$-convex); therefore, any convex domain satisfies (*), too. To extend this phenomenon to bounded $\mathbb{C}$-convex domains (see Problem 4’ in [14]), it is sufficient to give a positive answer to one of the following questions:

(a) Can any bounded $\mathbb{C}$-convex domain be exhausted by $C^2$-smooth $\mathbb{C}$-convex domains? (See Problem 2 in [14] and Remark 2.5.20 in [2].)

(b) Is any bounded $\mathbb{C}$-convex domain biholomorphic to a convex domain? (See Problem 4 in [14].)

The main aim of this note is to give a negative answer to Question (b).

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Denote by $G_2$ the so-called symmetrized bidisc, that is, the image of the bidisc under the mapping whose components are the two elementary symmetric functions of two complex variables. $G_2$ serves as the first example of a bounded pseudoconvex domain in $\mathbb{C}^2$ with the property (*) which cannot be exhausted by domains biholomorphic to convex domains (see [3, 6]). We shall show that $G_2$ is a $C$-convex domain. This fact gives a counterexample to the question (b) and simultaneously, it supports the conjecture that (cf. Problem 4’ in [14]) any bounded $C$-convex domain has property (*). Note that the answer to the problem (a) for $G_2$ is not known. The positive answer to this question would imply an alternative (to that of [4] and [1]) proof of the equality of the Carathéodory distance and Lempert function on $G_2$ whereas the negative answer would solve Problem 2 in [14].

Some additional properties of $C$-convex domains and symmetrized polydiscs are also given in the paper.

2. Background and results

Recall that a domain $D$ in $\mathbb{C}^n$ is called (cf. [9, 2]):

- **$C$-convex** if any non-empty intersection with a complex line is contractible (i.e. $D \cap L$ is connected and simply connected for any complex affine line $L$ such that $L \cap D$ is not empty);
- **linearly convex** if its complement in $\mathbb{C}^n$ is a union of affine complex hyperplanes;
- **weakly linearly convex** if for any $a \in \partial D$ there exists an affine complex hyperplane through $a$ which does not intersect $D$.

Note that the following implications hold

$C$-convexity $\Rightarrow$ linear convexity $\Rightarrow$ weak linear convexity.

Moreover, these three notions coincide in the case of bounded domains with $C^1$ boundary.

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$. Let $\pi_n = (\pi_{n,1}, \ldots, \pi_{n,n}) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be defined as follows:

$$\pi_{n,k}(\mu) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \mu_{j_1} \ldots \mu_{j_k}, \quad 1 \leq k \leq n, \quad \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n.$$

The set $G_n := \pi_n(\mathbb{D}^n)$ is called the *symmetrized $n$-disc* (cf. [1], [11]).

Recall that $G_2$ is the first example of a bounded pseudoconvex domain with the property (*) which cannot be exhausted by domains biholomorphic to convex ones (see [3, 6]). On the other hand, $G_n$, $n \geq 3$, does not satisfy the property (*) (see [14]). In particular, it cannot be exhausted by domains biholomorphic to convex domains, either.
In this note we shall show the following additional properties of domains $G_n$, $n \geq 2$.

**Theorem 1.**  
(i) $G_2$ is a $\mathbb{C}$-convex domain.  
(ii) $G_n$, $n \geq 3$, is a linearly convex domain which is not $\mathbb{C}$-convex.

Theorem 1 (i) together with a result of [3] and [6] gives a negative answer to the following question posed by S. V. Znamenskiǐ (cf. Problem 4 in [14]):

*Is any bounded $\mathbb{C}$-convex domain biholomorphic to a convex domain?*

Moreover, it seems to us that Theorem 1 (ii) gives the first example of a linearly convex domain homeomorphic to $\mathbb{C}^n$, $n \geq 3$, which is not $\mathbb{C}$-convex, is not a Cartesian product and does not satisfy property (*). To see that $G_n$ is homeomorphic to $\mathbb{C}^n$, observe that $\rho_{\lambda}(z) := (\lambda z_1, \lambda^2 z_2, \ldots, \lambda^n z_n) \in G_n$ if $z \in G_n$ and $\lambda \in \mathbb{C}$. Then setting $h(z) = \max \{|\mu_j| : \pi_n(\mu) = z\}$ and $g(z) = \frac{1}{1-h(z)}$, it is easy to see that the function $G_n \ni z \mapsto \rho_{g(z)}(z) \in \mathbb{C}^n$ is the desired homeomorphism.

These remarks also show that $G_n$ is close, in some sense, to a balanced domain, that is, a domain $D$ in $\mathbb{C}^n$ such that $\lambda z \in D$ for any $z \in D$ and $\lambda \in \mathbb{D}$. On the other hand, in spite of the properties of $G_n$, one has the following.

**Proposition 2.** Any weakly linearly convex balanced domain is convex.

This proposition is a simple extension of Example 2.2.4 in [2], where it is shown that any $\mathbb{C}$-convex complete Reinhardt domain is convex.

We may also prove some general property of $\mathbb{C}$-convex domains showing that all non-degenerate $\mathbb{C}$-convex domains, that is, containing no complex lines, are $\mathbb{C}$-finitely compact. For definitions of the Carathéodory distance $c_D$ of the domain $D$, $\mathbb{C}$-finite compactness, $\mathbb{C}$-completeness and basic properties of these notions we refer the Reader to consult [10].

Observe that a degenerate linearly convex domain $D$ is linearly equivalent to $\mathbb{C} \times D'$ (cf. Proposition 4.6.11 in [2]). Indeed, we may assume that $D$ contains the $z_1$-line. Since the complement $^cD$ of $D$ is a union of complex hyperplanes disjoint from this line, then $^cD = \mathbb{C} \times G$ and hence $D = \mathbb{C} \times ^cG$. On the other hand, we have

**Proposition 3.** Any non-degenerate $\mathbb{C}$-convex domain is biholomorphic to a bounded domain and $\mathbb{C}$-finitely compact. In particular, it is $\mathbb{C}$-complete and hyperconvex.

**Remarks.** (i) In virtue of Proposition 3, we claim that one may conjecture more than Question (a) (see [15]), namely, any $\mathbb{C}$-convex domain containing no complex hyperplanes can be exhausted by bounded
\( C^2 \)-smooth \( \mathbb{C} \)-convex domains (this is not true in general without the above assumption); then the Carathéodory pseudodistance and Lempert function will coincide on any \( \mathbb{C} \)-convex domain.

(ii) The hyperconvexity of \( G_n \) is simple and well-known (see [7]). The above proposition implies more in dimension two. Namely, it implies that the symmetrized bidisc is \( c \)-finitely compact. Although the symmetrized polydiscs in higher dimensions are not \( \mathbb{C} \)-convex the conclusion of the above proposition, that is, the \( c \)-finite compactness of the symmetrized \( n \)-disc \( G_n \), holds for any \( n \geq 2 \). In fact, it is a straightforward consequence of Corollary 3.2 in [5].

(iii) Finally, we mention that, for \( n \geq 2 \), \( G_n \) is starlike with respect to the origin if and only if \( n = 2 \). This observation gives the next difference in the geometric shape of the 2-dimensional and higher dimensional symmetrized discs. Recall that the fact that \( G_2 \) is starlike is contained in [1]. For the converse just take the point \((3,3,1,\ldots,0)\).

3. Proofs

**Proof of Theorem 1** (i). We shall make use of the following description of \( \mathbb{C} \)-convex domains. For \( a \in \partial D \), denote by \( \Gamma(a) \) the set of all hyperplanes through \( a \) and disjoint from \( D \). Then a bounded domain \( D \) in \( \mathbb{C}^n \), \( n > 1 \), is \( \mathbb{C} \)-convex if and only if any \( a \in \partial D \) the set \( \Gamma(a) \) is non-empty and connected as a set in \( \mathbb{C} P^n \) (cf. Theorem 2.5.2 in [2]).

So we have to check that \( \Gamma(\pi_2(\mu)) \) is a singleton.

To show the connectedness of \( \Gamma(\pi_2(\mu)) \), we shall check the simple-connectedness of \( A \). Let us recall that the mapping \( \frac{z-\alpha}{\bar{z}-\beta} \), where \( |\beta| > 1 \),
Proof of Proposition 2. Let $D$ weakly linearly convex if and only if

$$\{ \frac{\lambda + \lambda_1 - 2x}{\lambda \lambda_1 - 1} : \lambda \in \mathbb{D} \} = \Delta(\frac{2x - 2 \text{Re}\lambda_1}{1 - |\lambda_1|^2}, \frac{|2x\lambda_1 - \lambda_1|}{1 - |\lambda_1|^2}) =: A_{\lambda_1}. $$

Consequently the set $A = \bigcup_{\lambda_1 \in \mathbb{D}} A_{\lambda_1} \subset \mathbb{C}$ is simply connected. \hfill \Box

Proof of Theorem 1 (ii). For the proof of the linear convexity of $G_n$ consider the point $z = \pi_n(\lambda) \in \mathbb{C}^n \setminus G_n$. We may assume that $|\lambda_1| \geq 1$. Then the set

$$B := \{ \pi_n(\lambda_1, \mu_1, \ldots, \mu_{n-1}) : \mu_1, \ldots, \mu_{n-1} \in \mathbb{C} \}$$

is disjoint from $G_n$. On the other hand, it is easy to see that

$$B = \{ (\lambda_1 + z_1, \lambda_1z_1 + z_2, \ldots, \lambda_1z_{n-2} + z_{n-1}, \lambda_1z_{n-1}) : z_1, \ldots, z_{n-1} \in \mathbb{C} \},$$

so $B$ is a complex affine hyperplane. Hence $G_n$ is linearly convex.

To show that $G_n$ is not C-convex for $n \geq 3$, consider the points

$$a_t := \pi_n(t, t, t, 0, \ldots, 0) = (3t, 3t^2, t^3, 0, \ldots, 0),$$

$$b_t := \pi_n(-t, -t, -t, 0, \ldots, 0) = (-3t, 3t^2, -t^3, 0, \ldots, 0), \quad t \in (0, 1).$$

Obviously $a_t, b_t \in G_n$. Denote by $L_t$ the complex line passing through $a_t$ and $b_t$, that is,

$$L_t = \{ c_{t,\lambda} := (3t(1 - 2\lambda), 3t^2, t^3(1 - 2\lambda), 0, \ldots, 0) : \lambda \in \mathbb{C} \}.$$

Assume that the set $G_n \cap L_t$ is connected. Since $a_t = c_{t,0}$ and $b_t = c_{t,1}$, then $c_{t,\lambda} \in G_n$ for some $\lambda = \frac{1}{2} + i\tau$, $\tau \in \mathbb{R}$. It follows that

$$c_{t,\lambda} = (-6i\tau t, 3t^2, -2i\tau t^3, 0, \ldots, 0).$$

We may choose $\mu \in \mathbb{D}^n$ such that $\mu_j = 0$, $j = 4, \ldots, n$, and $c_{t,\lambda} = \pi_n(\mu)$, $\mu \in \mathbb{D}^n$. Then $-36\tau^2 t^2 = (\mu_1 + \mu_2 + \mu_3)^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 + 6t^2$ and hence

$$t^2 = \frac{\mu_1^2 + \mu_2^2 + \mu_3^2}{36\tau^2 + 6} < \frac{3}{36\tau^2 + 6} \leq \frac{1}{2}.$$

Therefore, $G_n \cap L_t$ is not connected if $t \in \left[ \frac{1}{\sqrt{2}}, 1 \right]$ and so $G_n$ is not a C-convex domain. \hfill \Box

Proof of Proposition 2. Set $D^* := \{ w \in \mathbb{C}^n : < z, w > \neq 1, \forall z \in D \}$. We shall use the fact that a domain $D$ in $\mathbb{C}^n$ containing the origin is weakly linearly convex if and only if $D$ is a connected component of $D^{**}$ (cf. Proposition 2.1.4 in [2]).

Since our domain $D$ is balanced, it is easy to see that $D^*$ is balanced. We shall show $D^*$ is convex. Then, applying this fact to $D^*$, we conclude that $D^{**}$ is a convex balanced domain. On the other hand,
it follows by our assumption that $D$ is a component of $D^{**}$ and hence $D^{**} = D$.

To see that $D^*$ is convex, suppose the contrary. Then we find points $w_1, w_2 \in D^*$, $z \in D$ and a number $t \in (0, 1)$ such that $<z, tw_1 + (1 - t)w_2> = 1$. We may assume that $|<z, w_1>| \geq 1$. Since $D$ is balanced, we get $\tilde{z} : = \frac{z}{<z, w_1>} \in D$ and $<\tilde{z}, w_1> = 1$, a contradiction. \hfill \Box

Proof of Proposition 3. Let $D$ be non-degenerate $\mathbb{C}$-convex domain in $\mathbb{C}^n$. For any point $z \in \partial D$ consider a hyperplane $L_z$ through $z$ and disjoint from $D$. Let $l_z$ be the orthogonal line through 0 and orthogonal to $L_z$. Denote by $\pi_z$ the orthogonal projection of $\mathbb{C}^n$ onto $l_z$ and set $\alpha_z = \pi_z(a)$. Observe that $D_z = \pi_z(D)$ is biholomorphic to $\mathbb{D}$, since it is connected, simply connected (cf. Theorem 2.3.6 in [2]) and $\pi_z(z) \notin \pi_z(D)$. Moreover, since $D$ is a non-degenerate linearly convex domain, it is easy to see that there are $n \mathbb{C}$-independent $l_z$'s. We may assume that these $l_z$ are the set $C$ of coordinate planes. Then $D \subset G : = \prod_{l_z \in C} \pi_z(D)$ and $G$ is biholomorphic to the polydisc $\mathbb{D}^n$. In particular, $D$ is biholomorphic to a bounded domain, hence it is $c$-hyperbolic.

Further, we may assume that $0 \in D$. To see that $D$ is $c$-finitely compact, it is enough to show that $\lim_{a \rightarrow z} c_D(0; a) = \infty$ for any $z \in \partial D$ and, if $D$ is unbounded, $z = \infty$. But the last one follows by the fact that $G$ is $c$-finitely compact. On the other hand, if $a \rightarrow z \in \partial D$, then $a_z \rightarrow \pi_z(z) \in \partial D_z$ and hence $c_D(0; a) \geq c_{D_z}(0; a_z) \rightarrow \infty$. \hfill \Box

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