Ryūō Nim: A Variant of the classical game of Wythoff Nim

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Abstract

The authors introduce the impartial game of the generalized Ryūō Nim, a variant of the classical game of Wythoff Nim. In the latter game, two players take turns in moving a single queen on a large chess-board, attempting to be the first to put her in the upper left corner, position (0, 0). Instead of the queen used in Wythoff Nim, we use the generalized Ryūō for a given natural number $p$. The generalized Ryūō for $p$ can be moved horizontally and vertically, as far as one wants. It also can be moved diagonally from $(x, y)$ to $(x - s, y - t)$, where $s$, $t$ are non-negative integers such that $1 \leq s \leq x$, $1 \leq t \leq y$ and $s + t \leq p - 1$. When $p$ is 3, the generalized Ryūō for $p$ is a Ryūō, i.e., a promoted hisha piece of Japanese chess. A Ryūō combines the power of the rook and the king in Western chess. The generalized Ryūō Nim for $p$ is mathematically the same as the Nim with two piles of counters in which a player may take any number from either heap, and a player may also simultaneously remove $s$ counters from either of the piles and $t$ counters from the other, where $s + t \leq p - 1$ and $p$ is a given natural number. The Grundy number of the generalized Ryūō Nim for $p$ is given by $\text{mod}(x + y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor)$. The authors also study the generalized Ryūō Nim for $p$ without a pass move. The generalized Ryūō Nim for $p$ without a pass move has simple formulas for Grundy numbers. This is not the case after the introduction of a pass move, but it still has simple formulas for the previous player’s positions. We also study the Ryūō Nim that restricted the diagonal and side movement. Moreover, we extended the Ryūō Nim dimension to the $n$-dimension.

Keyword: Nim, Wythoff Nim, Grundy Number, a Pass Move, n-Dimention

1 Generalized Ryūō (dragon king) Nim

Let $\mathbb{Z}_{\geq 0}$ be the set of non-negative integers and $N$ be the set of natural numbers. Let $p$ be a fixed natural number. For any $x \in \mathbb{Z}_{\geq 0}$, let $\text{mod}(x, p)$ denote the remainder obtained when $x$ is divided by $p$. Here, we introduce the impartial game of the generalized Ryūō Nim for $p$, a variant of the
classical game of Wythoff Nim that was studied in [1]. Instead of the queen used in Wythoff Nim, we use the generalized Ryūō for $p$. When $p$ is 3, the generalized Ryūō for $p$ is a Ryūō (dragon king) of Japanese chess. A Ryūō combines the power of the rook and the king in Western chess. Let us break with chess traditions here and denote fields on the chessboard by pairs of numbers. The field in the upper left corner will be denoted by $(0,0)$, and the other ones will be denoted according to a Cartesian scheme: field $(x, y)$ denotes $x$ fields to the right followed by $y$ fields down (see Figure 1).

**Definition 1.1.** We define the generalized Ryūō Nim for $p$. The generalized Ryūō for $p$ is placed on a chessboard of unbounded size, and two players move it in turns. The generalized Ryūō for $p$ is to be moved to the left or upwards, vertically, as far as one wants. It can also be moved diagonally from $(x, y)$ to $(x-s, y-t)$, where $s, t$ are non-negative integers such that $1 \leq s \leq x, 1 \leq t \leq y$ and $s+t \leq p-1$. The generalized Ryūō for $p$ has to be moved by at least one field in each move. The object of the game is to move the generalized Ryūō for $p$ to the ”winning field” $(0,0)$; whoever moves the generalized Ryūō for $p$ to this field wins the game.

**Definition 1.2.** Suppose that there are two piles of counters. Players take turns removing counters from one or both piles. A player may take any number from either heap, and a player may also simultaneously remove $n$ counters from either of the piles and $m$ from the other, where $n + m \leq p - 1$ and $p$ is a given natural number.

**Remark 1.1.** Moving to the left or upwards, or to the upper left on the chessboard, is mathematically the same as taking counters from either heap or some counters from both, and hence, the game defined in Definition 1.1 is mathematically the same as that defined in Definition 1.2.

In this article, we only treat impartial games (see [2] or [3] for a background on impartial games). For an impartial games without draws, there will only be two outcome classes.
Definition 1.3. (a) \( N \)-positions, from which the next player can force a win, as long as he plays correctly at every stage.

(b) \( P \)-positions, from which the previous player (the player who will play after the next player) can force a win, as long as he plays correctly at every stage.

We use the theory of Grundy numbers to study impartial games without draws. To define the Grundy numbers, we need some definitions.

Definition 1.4. For any position \( p \) of a game \( G \), there is a set of positions that can be reached by making precisely one move in \( G \), which we will denote by \( \text{move}(p) \).

Definition 1.5. We define \( \text{move}((x, y)) \) of the generalized Ryūō for \( p \). For \( x, y \in \mathbb{Z}_{\geq 0} \) such that \( x + y \geq 1 \), let

\[
\begin{align*}
M_{g1} &= \{(u, y) : u < x\} \\
M_{g2} &= \{(x, v) : v < y\},
\end{align*}
\]

where \( u, v \in \mathbb{Z}_{\geq 0} \). For \( x, y \in \mathbb{Z}_{\geq 0} \) such that \( 1 \leq x, y \), let

\[
M_{g3} = \{(x - s, y - t) : 1 \leq s \leq x, 1 \leq t \leq y \text{ and } s + t \leq p - 1\},
\]

where \( s, t \in \mathbb{Z}_{\geq 0} \).

We define \( \text{move}((x, y)) = M_{g1} \cup M_{g2} \cup M_{g3} \).

Remark 1.2. The sets (1), (2), and (3) denote horizontal, vertical, and upper left moves. If \( x = 0 \) or \( y = 0 \), then \( M_{g1} \) or \( M_{g2} \) is empty, respectively. When \( x = 0 \) or \( y = 0 \), then \( M_{g3} \) is empty. \( M_{g1}, M_{g2}, \) and \( M_{g3} \) depend on \( (x, y) \); hence, mathematically, it is more precise to write \( M_{g1}(x, y), M_{g2}(x, y), \) and \( M_{g3}(x, y) \). If we write \( M_{g1}(x, y) \) instead of \( M_{g1} \), the relations and equations appear more complicated. Therefore, we omit \( (x, y) \) for convenience.

Example 1.1. Figure 2, Figure 3, and Figure 4 show the moves of the generalized Ryūō for \( p = 3 \), \( p = 4 \) and \( p = 8 \), respectively. Figure 5 presents the moves of the generalized Ryūō for a natural number \( p \). In these figures, the horizontal move (the set in (1)) and the vertical move (the set in (2)) are denoted by dotted lines, and the upper left move (the set in (3)) is denoted by a set of small circles.

Definition 1.6. (i) The minimum excluded value (mex) of a set, \( S \), of non-negative integers is the least non-negative integer that is not in \( S \).

(ii) Each position \( p \) of an impartial game \( G \) has an associated Grundy number, and we denote it by \( G(p) \).

The Grundy number is calculated recursively: \( G(p) = \text{mex}\{G(h) : h \in \text{move}(p)\} \).
Example 1.2. Examples of calculation of mex.

\[
\text{mex}\{0, 1, 2, 3\} = 4, \ \text{mex}\{1, 1, 2, 3\} = 0, \\
\text{mex}\{0, 2, 3, 5\} = 1, \ \text{and} \ \text{mex}\{0, 0, 0, 1\} = 2.
\]

Theorem 1.1. Let $\mathcal{G}$ be the Grundy number. Then, $h$ is a $\mathcal{P}$-position if and only if $\mathcal{G}(h) = 0$.

For the proof of this theorem, see \cite{2}.

We define nim-sum that is important for combinatorial game theory.

Definition 1.7. Let $x, y$ be non-negative integers, and write them in base 2, so $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0, 1\}$.

We define the nim-sum $x \oplus y$ by

\[
x \oplus y = \sum_{i=0}^{n} w_i 2^i, \tag{4}
\]

where $w_i = x_i + y_i \pmod{2}$. 

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In this section we prove that the Grundy number of the generalized Røyø Nim for $p$ is $G((x, y)) = \text{mod}(x + y, p) + p(\lfloor x/p \rfloor + \lfloor y/p \rfloor)$ in Theorem 1.2. However, we need some lemmas to prove Theorem 1.2. The calculations used in these lemmas may not be easy to understand for some people. Hence, we study Example 1.3. You can skip Example 1.3 and move to Lemma 1.1.

**Example 1.3.** Let $p = 3$. We prove \(5\) for \((x, y) = (17, 19)\) using mathematical induction.

\[G((x, y)) = \text{mod}(x + y, p) + p(\lfloor x/p \rfloor \oplus \lfloor y/p \rfloor).\] \(5\)

We assume that \(5\) is valid for \((x, y)\) such that $x \leq 17$, $y \leq 19$ and, $x + y < 36$ and we prove

\[G((x, y)) = \text{mod}(36, 3) + 3(5 \oplus 6).\] \(6\)

By the definition of Grundy number

\[G((17, 19)) = \text{mex}\{G((u, v)) : (u, v) \in \text{move}((17, 19))\}.\] \(7\)

By Definition 1.5, \text{move}((17, 19)) is the union the sets $M_{g1}$, $M_{g2}$ and $M_{g3}$.

\[M_{g1} = \{(16, 19), (15, 19), (14, 19), \cdots, (0, 19)\}.\] \(8\)

\[M_{g2} = \{(17, 18), (17, 17), (17, 16), \cdots, (17, 0)\}.\] \(9\)

\[M_{g3} = \{(16, 18)\}.\] \(10\)

By the hypothesis of mathematical induction, the set \{G((u, v)) \in M_{g1}\} is the union of the sets in \(\{11\}\) and \(\{12\}\).

\[\{\text{mod}(35, 3) + 3(\lfloor 16/3 \rfloor \oplus \lfloor 19/3 \rfloor), \text{mod}(34, 3) + 3(\lfloor 15/3 \rfloor \oplus \lfloor 19/3 \rfloor)\} = \{2 + 3(5 \oplus 6), 1 + 3(5 \oplus 6)\}.\] \(11\)

\[\{\text{mod}(33, 3) + 3(\lfloor 14/3 \rfloor \oplus \lfloor 19/3 \rfloor), \text{mod}(32, 3) + 3(\lfloor 13/3 \rfloor \oplus \lfloor 19/3 \rfloor), \ldots, \text{mod}(19, 3) + 3(\lfloor 0/3 \rfloor \oplus \lfloor 19/3 \rfloor)\} = \{3(4 \oplus 6), 2 + 3(4 \oplus 6), 1 + 3(4 \oplus 6), 3(3 \oplus 6), 2 + 3(3 \oplus 6), 1 + 3(3 \oplus 6), \ldots, 3(0 \oplus 6), 2 + 3(0 \oplus 6), 1 + 3(0 \oplus 6)\}.\] \(12\)

By the hypothesis of mathematical induction, the set \{G((u, v)) \in M_{g2}\} is the union of the sets in \(\{13\}\) and \(\{14\}\).

\[\{\text{mod}(35, 3) + 3(\lfloor 17/3 \rfloor \oplus \lfloor 18/3 \rfloor) = \{2 + 3(5 \oplus 6)\}.\] \(13\)

\[\{\text{mod}(34, 3) + 3(\lfloor 17/3 \rfloor \oplus \lfloor 17/3 \rfloor), \ldots, \text{mod}(17, 3) + 3(\lfloor 17/3 \rfloor \oplus \lfloor 0/3 \rfloor)\} = \{1 + 3(5 \oplus 5), 3(5 \oplus 5), 2 + 3(5 \oplus 5), 1 + 3(5 \oplus 4), 3(5 \oplus 4), 2 + 3(5 \oplus 4), \ldots, 1 + 3(5 \oplus 0), 3(5 \oplus 0), 2 + 3(5 \oplus 0)\}.\] \(14\)
By the hypothesis of mathematical induction, the set \( \{ G((u,v)) \in M_{g3} \} \) is the set in (15).

\[
\{ \text{mod}(34, 3) + 3(\lfloor \frac{16}{3} \rfloor \oplus \lfloor \frac{18}{3} \rfloor) \} = \{1 + 3(5 \oplus 6)\}. \tag{15}
\]

We denote the union of the sets in (12) and (14) by \( A_{3,6} \). Note that this union of sets is \( A_{k,h} \) for \( k=5 \) and \( h=6 \) used in Lemma 1.1. Then,

\[
A_{3,6} = \bigcup_{u=0}^{2} \{3((5-t) \oplus 6) + u : t = 1, 2, \cdots, 5\} \cup \bigcup_{u=0}^{2} \{3(5 \oplus (6-t)) + u : t = 1, 2, \cdots, 6\}. \tag{16}
\]

We denote by \( C_{5,6,2,1} \) the union of the sets in (11), (13) and (15). Note that this union of sets is \( C_{k,h,v,w} \) for \( k=5, h=6, v=2, w=1 \) used in Lemma 1.5. Then

\[
C_{5,6,2,1} = \{3(5 \oplus 6) + w : w = 1, 2\}. \tag{17}
\]

We need to prove (6).

\[
\text{mod}(36, 3) + 3(5 \oplus 6) = \text{mex}\left( \bigcup_{u=0}^{2} \{3((5-t) \oplus 6) + u : t = 1, 2, \cdots, 5\} \cup \bigcup_{u=0}^{2} \{3(5 \oplus (6-t)) + u : t = 1, 2, \cdots, 6\} \cup \{3(5 \oplus 6) + w : w = 1, 2\} \right). \tag{18}
\]

We need the following lemmas to prove (18).

**Lemma 1.1.** Let \( k, h \in \mathbb{Z}_{\geq 0} \). Then,

\[
k \oplus h = \text{mex}\left( \{(k-t) \oplus h : t = 1, 2, \cdots, k\} \cup \{k \oplus (h-t) : t = 1, 2, \cdots, h\} \right). \tag{19}
\]

**Proof.** We omit the proof, because this is a well-known fact about Nim sum \( \oplus \) (see Proposition 1.4. (p.181) of [3]). \( \square \)

**Lemma 1.2.** Let \( A_{k,h} = \bigcup_{u=0}^{p-1} \{p((k-t) \oplus h) + u : t = 1, 2, \cdots, k\} \cup \bigcup_{u=0}^{p-1} \{p(k \oplus (h-t)) + u : t = 1, 2, \cdots, h\} \) for \( k, h \in \mathbb{Z}_{\geq 0} \). Then, the following statements are true.

(a) For any \( v = 1, \cdots, p-1 \),

\[
p(k \oplus h) + v = \text{mex}(A_{k,h} \cup \{p(k \oplus h) + w : w = 0, \cdots, v-1\}). \tag{20}
\]

(b) The argument in (a) is valid for \( v = 0 \), and we get \( p(k \oplus h) = \text{mex}(A_{k,h}) \).
Proof. (a) By Lemma \[1.1\] and Definition \[1.6\] (the definition of mex)

\[ k \oplus h \notin \{(k-t) \oplus h : t = 1, 2, \cdots, k\} \cup \{k \oplus (h-t) : t = 1, 2, \cdots, h\}, \]

and hence, for any \( v = 0, 1, \cdots, p - 1, \)

\[ p(k \oplus h) + v \notin \{p((k-t) \oplus h) + v : t = 1, 2, \cdots, k\} \]
\[ \cup \{p(k \oplus (h-t)) + v : t = 1, 2, \cdots, h\}. \]

(22)

Since \( p(k \oplus h) + v \neq p((k-t) \oplus h) + u, p(k \oplus (h-t)) + u \) for \( u \neq v, \)

\[ p(k \oplus h) + v \notin A_{k,h}. \]

(23)

Therefore, \( p(k \oplus h) + v \notin A_{k,h} \cup \{p(k \oplus h) + w : w = 0, \cdots, v - 1\}. \) Let \( s \) be an arbitrary non-negative integer such that

\[ p(k \oplus h) + v > s \geq 0. \]

(24)

We prove that \( s \in A_{k,h} \cup \{p(k \oplus h) + w : w = 0, \cdots, v - 1\}. \) If \( s \in \{p(k \oplus h) + w : w = 0, \cdots, v - 1\}, \) then

\[ s \in A_{k,h} \cup \{p(k \oplus h) + w : w = 0, \cdots, v - 1\}. \]

Suppose that \( s \notin \{p(k \oplus h) + w : w = 0, \cdots, v - 1\}. \) We choose non-negative integers \( t, u \) such that \( s = pt + u \) and \( 0 \leq u \leq p - 1. \) Clearly, \( k \oplus h > t \geq 0, \) and by Lemma \[1.1\] and Definition \[1.6\] (the definition of mex),

\[ t \in \{(k-t) \oplus h : t = 1, 2, \cdots, k\} \cup \{k \oplus (h-t) : t = 1, 2, \cdots, h\}, \]

(25)

and

\[ s = pt + u \in \{p((k-t) \oplus h) + u : t = 1, 2, \cdots, k\} \]
\[ \cup \{p(k \oplus (h-t)) + u : t = 1, 2, \cdots, h\} \subset A_{k,h}. \]

(26)

Therefore, by \[23, 24, 25, 26\] and Definition \[1.6\] (the definition of mex), \( p(k \oplus h) + v = \text{mex}(A_{k,h} \cup \{p(k \oplus h) + w : w = 0, \cdots, v - 1\}). \)

(b) The argument in (a) is valid for \( v = 0, \) and hence, we get \( p(k \oplus h) = \text{mex}(A_{k,h}). \)

□

Lemma 1.3. Let \( x, k \in \mathbb{Z}_{\geq 0}. \) If \( 0 \leq k < \text{mod}(x, p), \) then

\[ k \in \{\text{mod}(x-r, p) : 1 \leq r \leq x \text{ and } r \leq p - 1\}. \]

(27)

Proof. Let \( k \in \mathbb{Z}_{\geq 0} \) such that \( 0 \leq k < \text{mod}(x, p). \) We consider two cases.

Case (a) First, we suppose that \( x \leq p - 1. \) Since \( 0 \leq k < \text{mod}(x, p) = x, \)

\[ k \in \{0, 1, 2, \cdots, x - 1\} = \{x - r : 1 \leq r \leq x\} = \{\text{mod}(x-r, p) : 1 \leq r \leq x \leq p - 1\} = \{\text{mod}(x-r, p) : 1 \leq r \leq x \text{ and } r \leq p - 1\}. \]

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Case (b) Second, we suppose that \( x > p - 1 \). Then, there exist \( q, w \in \mathbb{Z}_{\geq 0} \) such that \( q \leq p - 1 \) and \( x = pw + q \). Then, \( 0 \leq k < \text{mod}(x, p) = q \). \( k \in \{0, 1, 2, \ldots, q-1\} = \{q-u: 1 \leq u \leq q\} = \{\text{mod}(x-u, p): 1 \leq u \leq q\} \subset \{\text{mod}(x-u, p): 1 \leq u \leq p-1\} = \{\text{mod}(x-r, p): 1 \leq r \leq x \text{ and } r \leq p-1\} \), where the last equation is implied by the inequality \( x > p - 1 \). \( \square \)

**Lemma 1.4.** Let \( V \) be a subset of \( \mathbb{Z}_{\geq 0} \), and let \( v \in \mathbb{Z}_{\geq 0} \) such that 

\[
v = \text{mex}(V). \tag{28}
\]

If \( W \) is a subset of \( \mathbb{Z}_{\geq 0} \) such that \( V \subset W \) and \( v \notin W \), then \( v = \text{mex}(W) \).

**Proof.** This lemma is directly obtained from the definition of \( \text{mex} \) (Definition 1.6). \( \square \)

**Lemma 1.5.** Let \( k, h, v, w \in \mathbb{Z}_{\geq 0} \) such that \( 0 \leq v, w \leq p-1 \), and let 

\[
C_{k,h,v,w} = \{p(k \oplus h) + \text{mod}(v + w - t, p): 1 \leq t \leq v\}, \\
\cup \{p(k \oplus h) + \text{mod}(v + w - t, p): 1 \leq t \leq w\}, \\
\cup \{p\left[\frac{pk + v - s}{p}\right] + \left[\frac{ph + w - t}{p}\right]\} + \text{mod}(v + w - s - t, p): 1 \leq s, t \text{ and } s + t \leq p - 1\}. \tag{29}
\]

Then, the following two statements are true.
1. \( p(k \oplus h) + \text{mod}(v + w, p) \notin C_{k,h,v,w} \).
2. \( p(k \oplus h) + u \in C_{k,h,v,w} \) for any \( u \in \mathbb{Z}_{\geq 0} \) such that 

\[
0 \leq u < \text{mod}(v + w, p). \tag{30}
\]

**Proof.** (a) Let \( t \in \mathbb{Z}_{\geq 0} \) such that \( 1 \leq t \leq v \). Then, \( t \leq v \leq p - 1 \), and hence, \( \text{mod}(v + w, p) \neq \text{mod}(v + w - t, p) \). Similarly, for any \( t \in \mathbb{Z}_{\geq 0} \) such that \( 1 \leq t \leq w \), we have \( \text{mod}(v + w, p) \neq \text{mod}(v + w - t, p) \). Since \( 1 \leq s + t \leq p - 1 \), \( \text{mod}(v + w, p) \neq \text{mod}(v + w - s - t, p) \). Therefore, \( p(k \oplus h) + \text{mod}(v + w, p) \notin C_{k,h,v,w} \).

(b) Let \( 0 \leq u < \text{mod}(v + w, p) \). By Lemma 1.3, \( u = \text{mod}(v + w - r, p) \) for some \( r \in \mathbb{Z}_{\geq 0} \) such that \( 1 \leq r \leq v + w \) and \( r \leq p - 1 \). Then, there exist \( s, t \in \mathbb{Z}_{\geq 0} \) such that \( s \leq v, t \leq w \) and \( r = s + t \), and \( u = \text{mod}(v + w - s - t, p) \) and \( s + t \leq p - 1 \). Here, we consider three cases.

Case (b.1) Suppose that \( s = 0 \). Then, \( p(k \oplus h) + u = p(k \oplus h) + \text{mod}(v + w - t, p) \in C_{k,h,v,w} \).

Case (b.2) Suppose that \( t = 0 \). Then, \( p(k \oplus h) + u = p(k \oplus h) + \text{mod}(v + w - s, p) \in C_{k,h,v,w} \).

Case (b.3) Suppose that \( 1 \leq s, t \). Then, \( p(k \oplus h) + u = p(k \oplus h) + \text{mod}(v - s + w - t, p) \) 

\[
= p\left[\left\lfloor\frac{pk + v - s}{p}\right\rfloor + \left\lfloor\frac{ph + w - t}{p}\right\rfloor\right] + \text{mod}(v - s + w - t, p) \in C_{k,h,v,w}. \tag{29}
\]
Theorem 1.2. The Grundy number of the generalized Ryūo Nim for \( p \) is

\[
\mathcal{G}((x, y)) = \mod(x + y, p) + p(\underbrace{\frac{x}{p}}_{x/p} \oplus \underbrace{\frac{y}{p}}_{y/p}).
\] (31)

Here, \( \mod(x + y, p) \) is the remainder obtained when \( x + y \) is divided by \( p \).

Proof. We prove by mathematical induction. We assume that Equation (31) is valid for \((u, v)\) when \( u < x \) or \( v < y \). Let \( (x, y) = (pk + v, ph + w) \) with \( 0 \leq v \leq p - 1 \) and \( 0 \leq w \leq p - 1 \).

\[
\text{move}((x, y)) = \{(pk + v - t, ph + w) : 1 \leq t \leq v\} \quad \text{(32)}
\]

\[
\cup \{(pk + v, ph + w - t) : 1 \leq t \leq w\} \quad \text{(33)}
\]

\[
\cup \{(pk - t, ph + w) : t = 1, 2, \cdots, pk\} \quad \text{(34)}
\]

\[
\cup \{(pk + v, ph - t) : t = 1, 2, \cdots, ph\} \quad \text{(35)}
\]

\[
\cup \{(pk + v - s, ph + w - t) : 1 \leq s, t \text{ and } s + t \leq p - 1\}. \quad \text{(36)}
\]

If \( v = 0 \) or \( w = 0 \), the set (32) or the set (33) is empty, respectively. By the definition of the Grundy number

\[
\mathcal{G}((x, y)) = \text{mex}(\mathcal{G}((pk + v - t, ph + w)) : 1 \leq t \leq v) \quad \text{(37)}
\]

\[
\cup \mathcal{G}((pk + v, ph + w - t)) : 1 \leq t \leq w \quad \text{(38)}
\]

\[
\cup \mathcal{G}((pk - t, ph + w)) : t = 1, 2, \cdots, pk \quad \text{(39)}
\]

\[
\cup \mathcal{G}((pk + v, ph - t)) : t = 1, 2, \cdots, ph \quad \text{(40)}
\]

\[
1 \leq s, t \text{ and } s + t \leq p - 1\}.
\]

Next, we study the set of Grundy numbers in (36) and (37). Since

\[
\{\mod(v, p), \mod(v + 1, p), \cdots, \mod(v + p - 1, p)\} = \{\mod(w, p), \mod(w + 1, p), \cdots, \\
\mod(w + p - 1, p)\} = \{0, 1, \cdots, p - 1\},
\]
by the assumption of mathematical induction, we have
\( \{G((pk - t, ph + w)) : t = 1, 2, \ldots, pk\} \cup \{G((pk + v, ph - t)) : t = 1, 2, \ldots, ph\} \)
\(= \{\mod(pk - t + ph + w, p) + p(\frac{pk - t}{p} \oplus \frac{ph + w}{p}) : t = 1, 2, \ldots, pk\} \)
\(\cup \{\mod(pk + v + ph - t, p) + p(\frac{pk + v}{p} \oplus \frac{ph - t}{p}) : t = 1, 2, \ldots, ph\} \)
\(= \{p((k - 1) \oplus h) + \mod(w + p - 1, p), p((k - 1) \oplus h) + \mod(w + p - 2, p), \)
\(\cdots, p((k - 1) \oplus h) + \mod(w, p), \cdots, p(0 \oplus h) + \mod(w + p - 1, p), p(0 \oplus h) + \mod(w + p - 2, p), \)
\(\cdots, p(0 \oplus h) + \mod(w, p)\} \cup \{p(k \oplus (h - 1)) + \mod(v + p - 1, p), p(k \oplus (h - 1)) + \mod(v + p - 2, p), \cdots, p(k \oplus 0) + \mod(v + p - 1, p), p(k \oplus 0) + \mod(v + p - 2, p), \cdots, p(k \oplus 0) + \mod(v, p)\} \)
\(= \bigcup_{u=0}^{p-1} \{p((k - u) \oplus h) + u : t = 1, 2, \ldots, k\} \)
\(\cup \bigcup_{u=0}^{p-1} \{p(k \oplus (h - u)) + u : t = 1, 2, \ldots, h\} = A_{k,h}. \quad (39) \)

Here, \(A_{k,h}\) is the set defined in Lemma 1.2. Next, we study the set of
Grundy numbers in (34), (35), and (38). By the assumption of mathematical
induction,
\(\{G((pk + v - t, ph + w)) : 1 \leq t \leq v\} \cup \{G((pk + v, ph + w - t)) : 1 \leq t \leq w\} \)
\(\cup \{G((pk + v - 1, ph + w - 1))\} \)
\(= \{p(k \oplus h) + \mod(v - t + w, p) : 1 \leq t \leq v\} \cup \{p(k \oplus h) + \mod(v + w - t, p) : 1 \leq t \leq w\} \)
\(\cup \{p(\frac{pk + v - s}{p} \oplus \frac{ph + w - t}{p}) + \mod(v - s + w - t, p) : 1 \leq s, t \)
and \(s + t \leq p - 1\} = C_{k,h,v,w}. \quad (40) \)

Note that \(C_{k,h,v,w}\) is used in Lemma 1.5. By (34), (35), (36), (37), (38),
(39), and (40).
\(G((x, y)) = \text{mex}(A_{k,h} \cup C_{k,h,v,w}). \quad (41) \)

By Lemma 1.5
\(C_{k,h,v,w} \supset \{p(k \oplus h) + u : 0 \leq u < \mod(v + w, p)\}. \quad (42) \)

By Lemma 1.5, \(p(k \oplus h) + \mod(v + w, p) \notin C_{k,h,v,w}\), and it is clear that
\(p(k \oplus h) + \mod(v + w, p) \notin A_{k,h}\). Therefore,
\(p(k \oplus h) + \mod(v + w, p) \notin A_{k,h} \cup C_{k,h,v,w}. \quad (43) \)
By Lemma 1.2
\[ p(k \oplus h) + \text{mod}(v + w, p) = \text{mex}(A_{k,h} \cup \{p(k \oplus h) + u : 0 \leq u < \text{mod}(v + w, p)\}). \quad (44) \]

Since \( A_{k,h} \cup \{p(k \oplus h) + u : 0 \leq u < \text{mod}(v + w, p)\} \subset A_{k,h} \cup C_{k,h,v,w}, \) by (41), (43), (44), and Lemma 1.4

\[ p(k \oplus h) + \text{mod}(v + w, p) \]
\[ = \text{mex}(A_{k,h} \cup \{p(k \oplus h) + u : 0 \leq u < \text{mod}(v + w, p)\}) \]
\[ = \text{mex}(A_{k,h} \cup C_{k,h,v,w}) = \mathcal{G}((x, y)). \]
\[ \square \]

2 Relation Between Grundy Number and Move

By Theorem 1.2 the Grundy number of the generalized Ruyū Nim for \( p \) is
\[ \mathcal{G}((x, y)) = \text{mod}(x + y, p) + p\left(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor\right). \quad (45) \]

In this section, we consider a necessary condition for a Nim with a new chess piece to obtain the Grundy number expressed by (45).

We define the move set in Definition 2.1 and present some examples of move sets in Example 2.1. Although \( \text{move}((x, y)) \) depends on \( (x, y) \), the move set depends only on the chess piece.

**Definition 2.1.** Let \( \text{move}((x, y)) \) be the set of positions that can be reached by a chess piece from the position \( (x, y) \). Then, a subset \( M \) of \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \) is said to be the move set of the chess piece when the following hold:

(i) \((0, 0) \notin M\).

(ii) \( \text{move}((x, y)) = \{(x - s, y - t) : (s, t) \in M, s \leq x \text{ and } t \leq y\} \).

**Remark 2.1.** Since \((0, 0) \notin M\), the move of a chess piece reduces at least one of the coordinates.

**Example 2.1.** (a) The move set of a rook on a chess board is \( M = \{(s, 0) : s \in \mathcal{N}\} \cup \{(0, t) : t \in \mathcal{N}\} \), and \( \text{move}((x, y)) = \{(u, y) : 0 \leq u < x \text{ and } u \in \mathbb{Z}_{\geq 0}\} \cup \{(x, v) : 0 \leq v < y \text{ and } v \in \mathbb{Z}_{\geq 0}\} = \{(x - s, y - t) : (s, t) \in M, s \leq x \text{ and } t \leq y\} \).

(b) The move set of a queen on a chess board is \( M = \{(s, 0) : s \in \mathcal{N}\} \cup \{(0, t) : t \in \mathcal{N}\} \cup \{(r, r) : r \in \mathcal{N}\} \), and \( \text{move}((x, y)) = \{(u, y) : 0 \leq u < x \text{ and } u \in \mathbb{Z}_{\geq 0}\} \cup \{(x, v) : 0 \leq v < y \text{ and } v \in \mathbb{Z}_{\geq 0}\} \cup \{(x - r, y - r) : 1 \leq r \leq x, y \text{ and } r \in \mathbb{Z}_{\geq 0}\} = \{(x - s, y - t) : (s, t) \in M, s \leq x \text{ and } t \leq y\} \).

(c) The move set of the generalized Ruyū Nim for \( p \) is \( \{(s, 0) : s \in \mathcal{N}\} \cup \{(0, t) : t \in \mathcal{N}\} \cup \{(s, t) : s, t \in \mathbb{Z}_{\geq 0}, 1 \leq s + t \leq p - 1\} \), \( \text{move}((x, y)) = \{(u, y) : u < x \text{ and } u \in \mathbb{Z}_{\geq 0}\} \cup \{(x, v) : v < y \text{ and } v \in \mathbb{Z}_{\geq 0}\} \cup \{(x - s, y - t) : 1 \leq s \leq x, 1 \leq t \leq y, s, t \in \mathbb{Z}_{\geq 0} \text{ and } s + t \leq p - 1\} = \{(x - s, y - t) : (s, t) \in M, s \leq x \text{ and } t \leq y\} \).
Theorem 2.1 shows that a new chess game has to have the move set that include the move set of the generalized Ryūo Nim for $p$ to have the Grundy number $G^\prime((x, y)) = \text{mod}(x + y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor)$.

**Theorem 2.1.** Suppose that we make a variant of Wythoff Nim using a new chess piece with the restriction that, by moving, no coordinate increases, and at least one of the coordinates reduces. We also suppose that the Grundy number of this game satisfies Equation (46).

$$G^\prime((x, y)) = \text{mod}(x + y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor).$$  \hspace{1cm} (46)

Then, the move set $M$ of this new chess piece satisfies (47).

$$M \supset \{(s, 0) : s \in \mathbb{N}\} \cup \{(0, t) : t \in \mathbb{N}\} \cup \{(s, t) : s, t \in \mathbb{Z}_{\geq 0}, 1 \leq s + t \leq p - 1\}. \hspace{1cm} (47)$$

**Proof.** By (46), we have

- $G^\prime((0, 0)) = 0$,
- $G^\prime((1, 0)) = 1, G^\prime((0, 1)) = 1$,
- $G^\prime((2, 0)) = 2, G^\prime((1, 1)) = 2, G^\prime((0, 2)) = 2$,
- $\ldots$
- $G^\prime((p - 1, 0)) = p - 1, G^\prime((p - 2, 1)) = p - 1, \ldots$
- and $G^\prime((0, p - 1)) = p - 1$. \hspace{1cm} (48)

For any $s, t$ such that $0 < s + t \leq p - 1, G^\prime((s, t)) = s + t > 0$, and hence, by the definition of the Grundy number, $\text{move}((s, t))$ contains a position whose Grundy number is 0. In this game, no coordinate increases when players move the chess piece; thus, $\text{move}((s, t)) \subset \{(u, v) : u + v < s + t \leq p - 1\}$. By (48), $(0, 0)$ is the only position in $\text{move}((s, t))$ whose Grundy number is 0.

Therefore, $(0, 0) \in \text{move}((s, t))$ for any $s, t \in \mathbb{Z}_{\geq 0}$ such that $1 \leq s + t \leq p - 1$. \hspace{1cm} (49)

By (49), $M \supset \{(s, t) : s, t \in \mathbb{Z}_{\geq 0}, 1 \leq s + t \leq p - 1\}$. Let $s$ be an arbitrary non-negative integer. Then, there exist $s', u \in \mathbb{Z}_{\geq 0}$ such that $s = ps' + u$ and $0 \leq u < p$. Then, $G^\prime((s, 0)) = \text{mod}(ps' + u, p) + p(\lfloor \frac{ps' + u}{p} \rfloor \oplus \lfloor \frac{0}{p} \rfloor) = u + p\lfloor \frac{u}{p} \rfloor = u + ps' = s$, and hence, $G^\prime((s, 0)) = s > 0$ for any natural number $s$. By the definition of the Grundy number, $\text{move}((s, 0))$ contains a position whose Grundy number is 0. Therefore, $(0, 0) \in \text{move}((s, 0))$ for any $s \in \mathbb{Z}_{\geq 0}$ such that $0 < s$. \hspace{1cm} (50)
By (50), \( M \supset \{(s, 0) : s \in \mathcal{N}\} \). Similarly,

\[
(0, 0) \in \text{move}((0, t)) \text{ for any } t \in \mathbb{Z}_{\geq 0} \text{ such that } 0 < t,
\]

and \( M \supset \{(0, t) : t \in \mathcal{N}\} \).

Therefore, we have (47). \( \Box \)

3 Generalized Ryūō Nim with a Pass Move

An interesting, albeit extremely difficult, problem in combinatorial game theory is to determine what happens when standard game rules are modified so as to allow for a one-time pass, i.e., a pass move which may be used at most once in a game, and not from a terminal position. Once the pass move has been used by either player, it is no longer available. In this section, we study a generalized Ryūō Nim with a pass move. In a generalized Ryūō Nim with a pass move, a position is represented by three coordinates \((x, y, \text{pass})\), where \text{pass} denotes the pass move. When \text{pass} = 1 or \text{pass} = 0, then a pass move is available or not available, respectively. Since a pass move may not be used from a terminal position, a player cannot move from \((0, 0, 1)\) to \((0, 0, 0)\) by using a pass move. Although there does not seem to be a simple formula for Grundy numbers of Ryūō Nim with a pass move, there are simple formulas for the \(\mathcal{P}\)-position of Ryūō Nim and the generalized Ryūō Nim for \(p\) with a pass move.

**Definition 3.1.** We define the move of the generalized Ryūō Nim for \(p\) with a pass move. Let \(x, y \in \mathbb{Z}_{\geq 0}\) such that \(1 \leq x + y\). We have two cases.

(a) We define the move of the generalized Ryūō Nim for \(p\) when a pass move is not available.

Let

\[
M_{\text{pass}1} = \{(u, y, 0) : u \in \mathbb{Z}_{\geq 0} \text{ and } u < x\},
\]

\[
M_{\text{pass}2} = \{(x, v, 0) : v \in \mathbb{Z}_{\geq 0} \text{ and } v < y\}
\]

and

\[
M_{\text{pass}3} = \{(x - s, y - t, 0) : s, t \in \mathbb{Z}_{\geq 0}, 1 \leq s \leq x, 1 \leq t \leq y
\]

\(\text{and } s + t \leq p - 1\}.
\]

We define

\[
\text{move}(x, y, 0) = M_{\text{pass}1} \cup M_{\text{pass}2} \cup M_{\text{pass}3}.
\]

(b) We define the move of the generalized Ryūō Nim for \(p\) when a pass move is available.

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Let
\[
M_{\text{pass}4} = \{(u, y, 1) : u \in \mathbb{Z}_{\geq 0} \text{ and } u < x\}, \tag{56}
\]
\[
M_{\text{pass}5} = \{(x, v, 1) : v \in \mathbb{Z}_{\geq 0} \text{ and } v < y\}, \tag{57}
\]
\[
M_{\text{pass}6} = \{(x - s, y - t, 1) : s, t \in \mathbb{Z}_{\geq 0}, 1 \leq s \leq x, 1 \leq t \leq y \text{ and } s + t \leq p - 1\} \tag{58}
\]
and
\[
M_{\text{pass}7} = \{(x, y, 0)\}. \tag{59}
\]

We define
\[
\text{move}(x, y, 1) = M_{\text{pass}4} \cup M_{\text{pass}5} \cup M_{\text{pass}6} \cup M_{\text{pass}7}. \tag{60}
\]

Remark 3.1. In Figure 5, the sets (52) and (56), (53) and (57), and (54) and (58) denote the horizontal, vertical, and upper left moves, and the set (59) denotes a pass move. \(M_{\text{pass}1}, M_{\text{pass}2}, \text{ and } M_{\text{pass}3}\) depend on \((x, y, 0)\), and \(M_{\text{pass}4}, M_{\text{pass}5}, M_{\text{pass}6}, \text{ and } M_{\text{pass}7}\) depend on \((x, y, 1)\). Hence, it is more precise to write \(M_{\text{pass}1}(x, y, 0), \ldots, M_{\text{pass}7}(x, y, 1)\). If we write \(M_{\text{pass}1}(x, y, 0)\) instead of \(M_{\text{pass}1}\), the relations and equations appear more complicated. Therefore, we omit \((x, y, 0)\) and \((x, y, 1)\) for convenience.

Example 3.1. When \(\text{pass} = 0\), i.e., a pass move is not available, the game becomes the same as the game we studied in Section 1. Table 1 lists the Grundy numbers \(G(x, y, 1)\) for \(x, y = 0, 1, 2, \ldots, 12\) of the general Ryūo Nim for \(p = 3\) with a pass move. Since the third coordinate \(\text{pass} = 1\), these are the Grundy numbers of positions with an available pass move. It seems that there is no simple formula for the Grundy numbers of the general Ryūo Nim for \(p = 3\) with a pass move in Table 1.

The Grundy number of positions in red squares is zero, and hence, they are \(P\)-positions (these positions are in dark gray when this article is printed in black and white.) There seems to be a certain pattern in the set of \(P\)-positions, and clearly, these \(P\)-positions can be divided into three groups of positions. The first group consists of three adjoining \(P\)-positions, and Table 2 shows this group. We denote this by \(P_{1,1}\).

The second group consists of positions that appear diagonally with a certain interval, and Table 3 summarizes this group. We denote this group by \(P_{1,2}\).

The third group consists of smaller groups that appear with a certain interval, and each group has two adjoining positions. Table 4 summarizes this group. We denote this group by \(P_{1,3}\).

In Definition 3.2 we define sets \(P\) and \(N\), and in Theorem 3.1 we prove that \(P\) and \(N\) are the sets of \(P\)-positions and \(N\)-positions of the generalized Ryūo Nim for \(p\) with a pass move, respectively. The set \(P\) consists of four sets: \(P_0, P_{1,1}, P_{1,2}, \text{ and } P_{1,3}\).
Table 1: Grundy numbers $G(x, y, 1)$ for $x, y = 0, 1, 2, \ldots, 12$ of the general Ryūō Nim for $p = 3$ when a pass move is available.

Table 2: The set $P_{1,1}$ for $p = 3$

Definition 3.2. Let $P_0 = \{(x, y, 0) : x + y = 0 \pmod{p} \text{ and } \lfloor \frac{x}{p} \rfloor = \lfloor \frac{y}{p} \rfloor \}$, $P_{1,1} = \{(m + 1, p - m, 1) : m \in \mathbb{Z}_{\geq 0} \text{ and } 0 \leq m \leq p - 1 \}$, $P_{1,2} = \{(pn + 1, pm + 1, 1) : n \in \mathcal{N} \}$, and $P_{1,3} = \{(k + pn, p + 2 - k + pm, 1) : n \in \mathcal{N}, 2 \leq k \leq p \text{ and } k \in \mathcal{N} \}$. Let $P_1 = P_{1,1} \cup P_{1,2} \cup P_{1,3}$ and $P = P_0 \cup P_1$.

Let $\mathcal{N}_0 = \{(x, y, 0) : x, y \in \mathbb{Z}_{\geq 0} \} - P_0$, $\mathcal{N}_1 = \{(x, y, 1) : x, y \in \mathbb{Z}_{\geq 0} \} - P_1$ and $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$.

Remark 3.2. Tables 3, 4 and 5 summarize the following groups of positions: $P_{1,1}$, $P_{1,2}$, and $P_{1,3}$ for $p = 3$.

Example 3.2. Table 6 lists the Grundy numbers of the generalized Ryūō Nim for $p = 4$. Tables 7, 8 and 9 summarize the sets $P_{1,1}$, $P_{1,2}$, and $P_{1,3}$ for $p = 4$. 

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Lemma 3.1. \( \mathcal{P}_0 \) and \( \mathcal{N}_0 \) are the set of \( \mathcal{P} \)-positions and the set of \( \mathcal{N} \)-positions when a pass move is not available, respectively.

Proof. Since \( \text{pass} = 0 \), by Theorem 1.1, a position \((x, y, 0)\) of a generalized Ryūō Nim for \( p \) is a \( \mathcal{P} \)-position if and only if
\[
\mathcal{G}((x, y)) = \text{mod}(x + y, p) + p(\lfloor \frac{x}{p} \rfloor \oplus \lfloor \frac{y}{p} \rfloor) = 0.
\]

By the definition of \( \mathcal{P}_0 \), (61) is true if and only if \((x, y, 0) \in \mathcal{P}_0\).
Since \( \mathcal{P}_0 \) is the set of \( \mathcal{P} \)-positions, \( \mathcal{N}_0 \) is the set of \( \mathcal{N} \)-positions. ⌣

We need three lemmas to prove Theorem 3.1.
Table 5: Grundy numbers \( G(x, y, 1) \) for \( x, y = 0, 1, 2, \cdots, 12 \) of the general Ryūō Nim for \( p = 4 \) when a pass move is available.

Table 6: The set \( \mathcal{P}_{1,1} \) for \( p = 4 \)

Lemma 3.2. (a) If \((x, y, 1) \in \mathcal{P}_1\), then \( x + y \geq p + 1 \).
(b) If \((x, y, 1) \in \mathcal{P}_{1,1}\), then \( x + y = p + 1 \), \( x + y = 1 \ (\text{mod} \ p) \), and \( 1 \leq x, y \leq p \).
(c) If \((x, y, 1) \in \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}\), then \( x + y = 2 \ (\text{mod} \ p) \) and \( x - y, y - x \leq p - 2 \).
(d) If \((x, y, 1) \in \mathcal{P}_{1,2}\), then \( x = y = 1 \ (\text{mod} \ p) \).
(e) If \((x, y, 1) \in \mathcal{P}_{1,3}\), then \( x + y = p + 2 \ (\text{mod} \ 2p) \).

Proof. (a), (b), (c), (d), and (e) follow directly from Definition 3.2.

Lemma 3.3. \( \text{move}((x, y, 1)) \cap \mathcal{P} = \emptyset \) for any \((x, y, 1) \in \mathcal{P}_1\).

Proof. By Definition 3.1

\[
\text{move}(x, y, 1) = M_{\text{pass}4} \cup M_{\text{pass}5} \cup M_{\text{pass}6} \cup M_{\text{pass}7},
\]
Table 7: The set $\mathcal{P}_{1,2}$ for $p = 4$

Table 8: The set $\mathcal{P}_{1,3}$ for $p = 4$

where

$$M_{\text{pass4}} = \{(u, y, 1) : u \in \mathcal{N} \text{ and } u < x\},$$  \hspace{1cm} (62)

$$M_{\text{pass5}} = \{(x, v, 1) : v \in \mathcal{N} \text{ and } v < y\},$$  \hspace{1cm} (63)

$$M_{\text{pass6}} = \{(x - s, y - t, 1) : s, t \in \mathcal{N} \text{ and } s + t \leq p - 1\},$$  \hspace{1cm} (64)

and

$$M_{\text{pass7}} = \{(x, y, 0)\}.$$  \hspace{1cm} (65)

We prove that $M_{\text{pass4}} \cup M_{\text{pass5}} \cup M_{\text{pass6}} \cup M_{\text{pass7}} \cap \mathcal{P} = \emptyset$. For any $(x, y, 1) \in \mathcal{P}_1$, by (b) and (c) of Lemma \ref{lem}, we have $x + y = 1$ or $2 \pmod{p}$, and hence, $(x, y, 0) \notin \mathcal{P}_0$. Therefore, $M_{\text{pass7}} \cap \mathcal{P} = \emptyset$, i.e., one cannot reach an element of $\mathcal{P}$ by using a pass move. Since $\mathcal{P}_1$ is unchanged when we switch $x$ and $y$, $M_{\text{pass4}} \cap \mathcal{P}_1 = \emptyset$ implies that $M_{\text{pass5}} \cap \mathcal{P}_1 = \emptyset$. Therefore, we only
have to prove that \( (M_{pass4} \cup M_{pass6}) \cap P_1 = \emptyset \). We consider three cases.

Case (a) Suppose that \((x, y, 1) = (1 + m, p - m, 1) \in P_{1,1}\) for a non-negative integer \(m\) such that \(0 \leq m \leq p - 1\). We consider subcases (a.1) and (a.2). Subcase (a.1) Let \((m + 1 - u, p - m, 1)\) be an arbitrary element of \(M_{pass4}\) such that \(u \in \mathcal{N}\) and \(u \leq m + 1\). Since \((m + 1 - u) + (p - m) = p + 1 - u < p + 1\), by (a) of Lemma \[3.2\] \((m + 1 - u, p - m, 1 - v) \notin P_1\). Therefore, \(M_{pass4} \cap P_1 = \emptyset\), i.e., from \((m + 1, p - m, 1)\), one cannot reach an element in \(P_1\) by reducing the first coordinate.

Subcase (a.2) Let \((m + 1 - u, p - m - v, 1)\) be an arbitrary element of \(M_{pass6}\) such that \(s, t \in \mathcal{N}\) and \(s + t \leq p - 1\). Since \((m + 1 - u) + (p - m - v) \leq p\), by (a) of Lemma \[3.2\] \(M_{pass6} \cap P_1 = \emptyset\).

By subcases (a.1) and (a.2), for any \((x, y, 1) \in P_{1,1}\),

\[
(M_{pass4} \cup M_{pass6}) \cap P_1 = \emptyset. 
\] (66)

Case (b) Let \((1 + m, 1 + m, 1) \in P_{1,2}\). We consider subcases (b.1) and (b.2). Subcase (b.1) Let \((1 + m - u, 1 + m, 1)\) be an arbitrary element of \(M_{pass4}\).

Since \(1 + m > p\), the second coordinate is greater than \(p\). Therefore, by (b) of Lemma \[3.2\] \((1 + m - u, 1 + m, 1) \notin P_{1,1}\). Since \((1 + m - u < 1 + m, (1 + m - u, 1 + m, 1) \notin P_{1,2}\).

There exists no \(u \in \mathcal{N}\) such that \((1 + m - u) + (1 + m) = 2 \mod p\) and \((1 + m) - (1 + m - u) \leq p - 2\). Therefore, by (c) of Lemma \[3.2\] \((1 + m - u, 1 + m, 1) \notin P_{1,1}\). Since \(2np + 2 > (1 + m - u) + (1 + m - v) \geq (2n - 1)p + 3, (pn + 1 - u) + (pn + 1 - v) \not\equiv 2 \mod p\). Therefore, by (c) of Lemma \[3.2\] \((pn + 1 - u, pn + 1 - v) \notin P_{1,2} \cup P_{1,3}\), and \(M_{pass6} \cap P_1 = \emptyset\).

By subcases (b.1) and (b.2), for any \((x, y, 1) \in P_{1,2}\)

\[
(M_{pass4} \cup M_{pass6}) \cap P_1 = \emptyset. 
\] (67)

Case (c) Let \((k + m, p + 2 - k + m, 1) \in P_{1,3}\) such that \(n \in \mathcal{N}, 2 \leq k \leq p\) and \(k \in \mathcal{N}\). We consider subcases (c.1) and (c.2)

Subcase (c.1) Let \((k + m - u, p + 2 - k + m, 1) \notin P_{1,1}\). There exists no \(k \in \mathbb{Z}_{\geq 0}\) and \(u \in \mathcal{N}\) such that \(2 \leq k \leq p\) and \(k + m - u = p + 2 - k + m = 1 \mod p\). Therefore, by (d) of Lemma \[3.2\] \((k + m - u, p + 2 - k + m, 1) \notin P_{1,2}\). There exists no \(u\) such that \((k + m - u) + (p + 2 - k + m) = p + 2 \mod 2p\) and \((p + 2 - k + m) - (k + m - u) \leq p - 2\). Therefore, by (c) and (e) of Lemma \[3.2\] \((k + m - u, p + 2 - k + m, 1) \notin P_{1,3}\). Therefore, \(M_{pass4} \cap P_1 = \emptyset\).

Subcase (c.2) Let \((k + m - u, p + 2 - k + m - v, 1)\) be an arbitrary element
of $M_{\text{pass}6}$ such that $u, v \in \mathcal{N}$ and $u + v \leq p - 1$. Then,

$$2pn + 3 \leq (k + pn - u) + (p + 2 - k + pn - v) \leq (2n + 1)p + 1. \tag{68}$$

By (68), $(k + pn - u) + (p + 2 - k + pn - v) > p + 1$, and hence, by (b) of Lemma 3.2, $(k + pn - u, p + 2 - k + pn - v) \notin \mathcal{P}_{1,1}$.

By (68), $(k + pn - u) + (p + 2 - k + pn - v) \not\equiv 2 \pmod{p}$, and hence, by (c) of Lemma 3.2 $(k + pn - u, p + 2 - k + pn - v) \notin \mathcal{P}_{1,2} \cup \mathcal{P}_{1,3}$.

Therefore, $M_{\text{pass}6} \cap \mathcal{P}_{1} = \emptyset$.

By (66), (67), and (69), $(M_{\text{pass}4} \cup M_{\text{pass}6}) \cap \mathcal{P}_{1} = \emptyset$ for any $(x, y, 1) \in \mathcal{N}_{1}$. \hfill \Box

**Lemma 3.4.** move((x, y, 1)) $\cap \mathcal{P} \neq \emptyset$ for any $(x, y, 1) \in \mathcal{N}_{1}$.

**Proof.** We consider six cases.

Case (a) Let $(x, y, 1) = (k +pn, h + pn', 1) \notin \mathcal{P}_{1}$ such that $n, n' \in \mathcal{N}$, $n' > n$, $k, h \in \mathbb{Z}_{\geq 0}$, and $k, h \leq p - 1$. We consider subcases (a.1), (a.2), (a.3), and (a.4).

Subcase (a.1) If $k > 2$, then $(k + pn, p + 2 - k + pn) \in M_{\text{pass}5} \cap \mathcal{P}_{1,3} \subset \text{move}((x, y, 1)) \cap \mathcal{P}_{1}$.

Subcase (a.2) Suppose that $k = 2$. We consider subsubcases (a.2.1) and (a.2.2).

Subsubcase (a.2.1) If $h > 0$, $(k + pn, p + 2 - k + pn) \in M_{\text{pass}5} \cap \mathcal{P}_{1,3} \subset \text{move}((x, y, 1)) \cap \mathcal{P}_{1}$.

Subsubcase (a.2.2) Suppose that $h = 0$. If $n' = n + 1$, then $(x, y, 1) = (k + pn, h + pn', 1) = (pn + 2, pm + p, 1) \in \mathcal{P}_{1,3}$. This contradicts the assumption that $(x, y, 1) \notin \mathcal{P}_{1}$. Therefore, we assume that $n' > n + 1$. Then, $(k + pn, p + 2 - k + pn) \in M_{\text{pass}5} \cap \mathcal{P}_{1,3} \subset \text{move}((x, y, 1)) \cap \mathcal{P}_{1}$.

Subcase (a.3) If $k = 1$, then $(k + pn, 2 - k + pn) = (1 + pn, 1 + pn) \in M_{\text{pass}5} \cap \mathcal{P}_{1,2} \subset \text{move}((x, y, 1)) \cap \mathcal{P}_{1}$.

Subcase (a.4) Suppose that $k = 0$. We consider subsubcases (a.4.1) and (a.4.2).

Subsubcase (a.4.1) If $n \geq 2$, then $(pn, 2 + p/(n - 1)) = (p + p/(n - 1), p + 2 - p + p/(n - 1), 1) \in M_{\text{pass}5} \cap \mathcal{P}_{1,3} \subset \text{move}((x, y, 1)) \cap \mathcal{P}_{1}$.

Subsubcase (a.4.2) If $n = 1$, then $(p, 1, 1) \in M_{\text{pass}5} \cap \mathcal{P}_{1,1} \subset \text{move}((x, y, 1)) \cap \mathcal{P}_{1}$.

Case (b) Let $(x, y, 1) = (k + pn', k + pn, 1) \notin \mathcal{P}_{1}$ such that $n' > n$, $k, h \in \mathbb{Z}_{\geq 0}$, and $k, h \leq p - 1$. Then, we use a method that is similar to the one used in (a).

Case (c) Let

$$(x, y, 1) = (k + pn, h + pn, 1) \in \mathcal{N}_{1} \tag{70}$$

such that $k, h \in \mathbb{Z}_{\geq 0}$ and $k, h \leq p - 1$. We consider Subcase (c.1), (c.2) and (c.3).
Subcase (c.1) Suppose that \( k = 0 \). We consider subsubcases (c.1.1) and (c.1.2).

Subsubcase (c.1.1) If \( n \geq 2 \), then \((pn, 2 + p(n - 1)) = (p + p(n - 1), p + 2 - p + p(n - 1), 1) \in M_{pass5} \cap \mathcal{P}_{1,3} \subset move((x, y, 1)) \cap \mathcal{P}\).

Subsubcase (c.1.2) If \( n = 1 \), then \((p, 1, 1) \in M_{pass5} \cap \mathcal{P}_{1,1} \subset move((x, y, 1)) \cap \mathcal{P}\).

Subcase (c.2) Suppose that \( h = 0 \). Then, we use the method that is similar to the one used in (c.1).

Subcase (c.3) Suppose that \( k, h \geq 1 \). We consider subsubcases (c.3.1), (c.3.2), and (c.3.3).

Subsubcase (c.3.1) Let \( k = 1 \). If \( h = 1 \), then \((k + mn, h + pn, 1) = (pm + 1, pn + 1) \in \mathcal{P}_{1,2}\). This contradicts (70). We assume that \( h \geq 2 \). Then, \((pn + 1, pm + 1) \in M_{pass5} \cap \mathcal{P}_{1,2} \subset move((x, y, 1)) \cap \mathcal{P}\).

Subsubcase (c.3.2) If \( h = 1 \), then we use a method that is similar to the one used in (c.3.1).

Subsubcase (c.3.3) Suppose that \( k, h \geq 2 \). We consider subsubcases (c.3.3.1) and (c.3.3.2).

Subsubsubcase (c.3.3.1) If \( k + h \leq p + 1, (k - 1) + (h - 1) \leq p - 1 \). Then, \((1 + pn, 1 + pm) \in M_{pass6} \cap \mathcal{P}_{1,2} \subset move((x, y, 1)) \cap \mathcal{P}\).

Subsubsubcase (c.3.3.2) If \( k + h = p + 2, (x, y, 1) = (k + mn, h + pm, 1) \in \mathcal{P}_{1,3}\). This contradicts (70). Therefore, we assume that \( p + 3 \leq k + h \leq 2p - 2 \). Since \( 1 \leq k + h - (p + 2) \leq p - 4 \), there exist \( u, v \) such that \( u, v \in \mathbb{Z}_{\geq 0}, 1 \leq u + v \leq p - 1, u \leq k, v \leq h \) and \((k - u) + (h - v) = p + 2 \). Then, \((k - u + mn, h - v + pm) \in M_{pass6} \cap \mathcal{P}_{1,3} \subset move((x, y, 1)) \cap \mathcal{P}\).

Case (d) Let

\[
(x, y, 1) = (k, h, 1) \in \mathcal{N}_1
\]  

(71)

such that \( k, h \in \mathbb{Z}_{\geq 0} \) and \( k, h \leq p - 1 \). By (71), \((k, h, 1) \notin \mathcal{P}_{1,1} \), and hence, \( k + h \neq p + 1 \). We consider subcases (d.1) and (d.2).

Subcase (d.1) If \( k = 0 \) or \( h = 0 \), then \((0, 0, 1) \in M_{pass4} \) or \( M_{pass5} \). In other words, we reach \((0, 0, 1)\) by reducing the positive coordinate to zero.

Subcase (d.2) We suppose that \( k, h \geq 1 \). We consider subsubcases (d.2.1), (d.2.2), and (d.2.3).

Subsubcase (d.2.1) If \( k + h \leq p - 1 \), then \((0, 0, 1) \in M_{pass6} \).

Subsubcase (d.2.2) If \( k + h = p \), then \((k, h, 0) \in M_{pass7} \cap \mathcal{P}_0\).

Subsubcase (d.2.3) We suppose that \( p + 1 \leq k + h \leq 2p - 2 \). By (71), \( k + h \neq p + 1 \). Then, there exist \( u, v \in \mathcal{N} \) such that \( u \leq k, v \leq h, u + v \leq p - 1 \), and \((k - u) + (h - v) = p + 1 \), and \((k - u, h - v) \in M_{pass6} \cap \mathcal{P}_{1,1}\).

Case (e) Let

\[
(x, y, 1) = (k, h + np, 1) \in \mathcal{N}_1
\]

(72)

such that \( n \in \mathcal{N}, k, h \in \mathbb{Z}_{\geq 0} \) and \( k, h \leq p - 1 \). We consider subcases (e.1) and (e.2).

Subcase (e.1) If \( k = 0 \), then \((0, 0, 1) \in M_{pass4} \).
Subcase (e.2) We suppose that $k \geq 1$. We consider subsubcase (e.2.1) and (e.2.2).

Subcase (e.2.1) If $n \geq 2$, then $2 \leq p - k + 1 < h + np$. By reducing $h + np$ to $p - k + 1$, we have $(k, p - k + 1) \in M_{pass4} \cap P_{1,1}$.

Subcase (e.2.2) We suppose that $n = 1$. We consider subsubcases (e.2.2.1) and (e.2.2.2).

Subsubcase (e.2.2.1) If $h \geq 1$, then $p - k + 1 < h + np = h + p$. By reducing $h + np$ to $p - k + 1$, $(k, p - k + 1) \in M_{pass4} \cap P_{1,1}$.

Subsubcase (e.2.2.2) We suppose that $h = 0$. If $k = 1$, then $(k, h + np, 1) = (1, p, 1) \in P_{1,1}$. This contradicts (72). Therefore, we assume that $k \geq 2$. Then, by reducing $k$ to 1, $(1, p) \in M_{pass4} \cap P_{1,1}$.

Case (f) Let

$$(x, y, 1) = (k + np, h, 1) \in N_1$$

such that $n \in N$, $h, k, h \in \mathbb{Z}_0$ and $k, h \leq p - 1$. Then, we prove by the method that is similar to the one used in Case (e).

Theorem 3.1. The sets $P$ and $N$ defined in Definition 3.2 are the sets of $P$-positions and $N$-positions of the Generalized Ryūō Nim for $p$ respectively.

Proof. Suppose that we start the game from a position $\{x, y, z, pass\} \in P = P_0 \cup P_1$. If $(x, y, pass) = (x, y, 0) \in P_0$, then by Lemma 3.1 $move((x, y, pass)) \subset N_0$. If $(x, y, pass) = (x, y, 1) \in P_1$, then by Lemma 3.3 $move((x, y, pass)) \subset N$. Therefore, any option we take leads to a position $(x', y', z', pass') \in N$. If $(x', y', z', pass') \in N_0$, then by Lemma 3.1 our opponent can choose a proper option that leads to a position in $P_0$. If $(x', y', z', pass') \in N_1$, then by Lemma 3.4 our opponent can choose a proper option that leads to a position in $P$. Note that any option reduces some of the numbers in the coordinates. In this way, our opponent can always reach a position in $P$, and will finally win by reaching $\{0, 0, 0\}$ or $\{0, 0, 1\} \in P$. Therefore, $P$ is the set of $P$-position. If we start the game from a position in $N$, then Lemma 3.1 and Lemma 3.4 mean that we can choose a proper option that leads to a position in $P$. Any option from this position taken by our opponent leads to a position in $N$. In this way, we win the game by reaching $\{0, 0, 0\}$ or $\{0, 0, 1\}$. Therefore, $N$ is the set of $N$-positions.

4 Generalized Ryūō (dragon king) Nim that Restricted the Diagonal and Side Movement

Restrict the diagonal movement by $p \in \mathbb{Z}_{\geq 1}$ and side movement by $q \in \mathbb{Z}_{\geq 1}$. It is possible to take up to a total of $p$ tokens when taking them at once and up to $q$ tokens when taking them from one heaps.

In this case, Grundy Number is known only in the following cases:
Theorem 4.1. If $\mod(q,p) = 0$, then we have

$$G(x, y) = \mod(\mod(x, q) + \mod(y, q), p) + p\left(\frac{\mod(x, q)}{p} \oplus \frac{\mod(y, q)}{p}\right).$$

Theorem 4.2. If $\mod(q, p) = 1$, then we have

$$G(x, y) = \begin{cases} q & (\text{where } \mod(x, q) = 0, \mod(y, q) = 0, x \neq 0, y \neq 0) \\ \mod(\mod(x, q) + \mod(y, q), p) + p\left(\frac{\mod(x, q)}{p} \oplus \frac{\mod(y, q)}{p}\right) & (\text{otherwise}) \end{cases}.$$ 

In other case, it becomes complicated and generally difficult.

Next, instead of the restriction of side movement by $q \in \mathbb{Z}_{>1}$, we consider the restriction of horizontal movement by $q \in \mathbb{Z}_{>1}$ and the restriction of vertical movement by $r \in \mathbb{Z}_{>1}$.

Theorem 4.3. If $\mod(q, p) = 0$ and $\mod(r, p) = 0$, then we have

$$G(x, y) = \mod(\mod(x, q) + \mod(y, r), p) + p\left(\frac{\mod(x, q)}{p} \oplus \frac{\mod(y, r)}{p}\right).$$

5 n-dimensional Ryūō Nim

Definition 5.1 (3-dimensiontoal Ryūō Nim). 3-dimensional Ryūō Nim is an impartial game with three heaps of tokens. The rules are as follows:

- The legal move is to remove any number of tokens from a single heap (as in Nim) or
Figure 7: Restricted the Diagonal and Side Movement (2)

- remove one token from any two heaps or remove one token from all the three heaps.

- The end position is the state of no tokens in the three heaps.

We could not get the indication of Grundy Number for 3-dimensional Ryūō Nim but we get the $P$-positions as shown in this theorem.

**Theorem 5.1.** Let $(x, y, z)$ be a 3-dimensional Ryūō Nim position. The $P$-positions of 3-dimensional Ryūō Nim are given as follows:

\[ \text{mod}(x + y + z, 3) = 0, \text{ and moreover} \]

(A) If \( \text{mod}(x, 3) = \text{mod}(y, 3) = \text{mod}(z, 3) = 1 \) then

\[ \left\lfloor \frac{x}{3} \right\rfloor \oplus \left\lfloor \frac{y}{3} \right\rfloor \oplus \left\lfloor \frac{z}{3} \right\rfloor \oplus 1 = 0 \]

(B) Otherwise

\[ \left\lfloor \frac{x}{3} \right\rfloor \oplus \left\lfloor \frac{y}{3} \right\rfloor \oplus \left\lfloor \frac{z}{3} \right\rfloor = 0. \]

**Definition 5.2** (Modified 3-dimensional Ryūō Nim). We also consider the modified rule of 3-dimensional Ryūō Nim as follows:

- The legal move is to remove any number of tokens from a single heap (as in Nim) or
- remove one token from any two heaps or
- remove one token from all the three heaps.

- The end position is the state of no tokens in the three heaps.
Then, we can obtain the Grundy Number of this game as follows:

**Theorem 5.2.** Let \((x, y, z)\) be a Modified 3-dimensional Ryūo Nim position, then we have

\[
G(x, y, z) = \text{mod}(x + y + z, 3) + 3(\lfloor \frac{x}{3} \rfloor \oplus \lfloor \frac{y}{3} \rfloor \oplus \lfloor \frac{z}{3} \rfloor).
\]

Next, we also consider \(n\)-dimensional Ryūo Nim.

**Definition 5.3** (\(n\)-dimensional Ryūo Nim). \(n\)-dimensional Ryūo Nim is an impartial game with \(n\) heaps of tokens. The rules are as follows:

- The legal move is to remove any number of tokens from a single heap (as in Nim) or
- The total number of tokens removed from the \(k \in \mathbb{Z}(1 < k \leq n)\) heaps at once must be less than \(p \in \mathbb{Z}_{>1}\).
- The end position is the state of no tokens in the \(n\) heaps.

**Theorem 5.3.** Let \((x_1, \ldots, x_n)\) be a \(n\)-dimensional Ryūo Nim position, then we have

\[
G(x_1, \ldots, x_n) = \text{mod}(x_1 + \cdots + x_n, p) + p(\lfloor \frac{x_1}{p} \rfloor \oplus \cdots \oplus \lfloor \frac{x_n}{p} \rfloor). \quad (p \in \mathbb{Z}_{>1}).
\]

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