Jordan-Schwinger map, 3D harmonic oscillator constants of motion, and classical and quantum parameters characterizing electromagnetic wave polarization

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Abstract

In this work we introduce a generalization of the Jauch and Rohrlich quantum Stokes operators when the arrival direction from the source is unknown \textit{a priori}. We define the generalized Stokes operators as the Jordan-Schwinger map of a triplet of harmonic oscillators with the Gell-Mann and Ne’eman SU(3) symmetry group matrices. We show that the elements of the Jordan-Schwinger map are the constants of motion of the three-dimensional isotropic harmonic oscillator. Also, we show that generalized Stokes Operators together with the Gell-Mann and Ne’eman matrices may be used to expand the polarization density matrix. By taking the expectation value of the Stokes operators in a three-mode coherent state of the electromagnetic field, we obtain the corresponding generalized classical Stokes parameters. Finally, by means of the constants of motion of the classical three-dimensional isotropic harmonic oscillator we describe the geometric properties of the polarization ellipse.

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1 Introduction

In both classical and quantum optics, Stokes parameters have proven to be intuitive and practical tools for characterizing the polarization state of light [1-6].

A classical or quantum electromagnetic wave propagates, in general, in an arbitrary but fixed direction in space. However, for the study of the polarization properties of the wave, the knowledge of the propagation direction of the wave allows us to choose a coordinate system in such a way that the propagation is along the $z$ axis and the polarization vector rests on the $x-y$ plane (i.e., it has only two polarization components) [10]. Also, the knowledge of the propagation direction allows to use a 2D apparatus (polarizers, wave plate rotators, etc.) placed perpendicular to the wave propagation direction to measure the polarization characteristics of the wave. The works [1-9] were done under the assumption that the arrival direction from the source of the electromagnetic wave was known. In [7] G. Stokes studied the polarization properties of a quasi-monochromatic plane wave of light in an arbitrary polarization state by introducing four quantities, known since then as the Stokes parameters. Wiener used the $2 \times 2$ unit matrix and the Pauli matrices as a basis to expand the coherence tensor [8]. Fano [9] showed that the Stokes parameters are the expansion coefficients of the coherence tensor. Stokes parameters obtained under an a priori knowledge of the propagation direction will be referred in this work as the usual classical or quantum Stokes parameters, and they are well described in references [10, 15] and [11], respectively.

When we do not know a priori the propagation direction of the wave, we already do not get the above adequate choice of the coordinate system, so that, in general, the three components of the polarization vector are nonzero. In this case, the three-dimensional coherence tensor must be used to obtain a complete polarization characterization [12, 13]. Roman [12] used the basis of nine $3 \times 3$ matrices which form the Kemmer algebra to define the generalized Stokes parameters as the expansion coefficients of the correlation matrix. In [13] Carozzi et al. defined the generalized Stokes parameters as the expansion coefficients of the spectral density tensor in terms of the $SU(3)$ Gell-Mann and Ne’eman matrices.

In this work, we introduce a generalization of the Jauch and Rohrlich quantum Stokes operators when the arrival direction from the source is unknown a priori. For simplicity, we study the case of a monochromatic quantized plane electromagnetic wave that propagates in a fixed but arbitrary direction in space. Also, we will use $\hbar = \omega = \mu = 1$, where $\mu$ is the mass of each 1D harmonic oscillator and $\omega$ is the angular frequency of either the electromagnetic wave or each harmonic oscillator. In section 2, we define the generalized quantum Stokes operators as the Jordan-Schwinger map of a triplet of harmonic oscillators with the Gell-Mann and Ne’eman $\lambda_i$ matrices of the $SU(3)$ symmetry group. We show that the elements of the Jordan-Schwinger map are the constants of mo-
tion of the quantum 3D isotropic harmonic oscillator. Also, we show that the generalized Stokes operators together with the $\lambda_i$ matrices may be used to expand the polarization matrix. In section 3, by taking the expectation value of the generalized quantum Stokes operators in a three-mode coherent state of the electromagnetic field, we obtain the corresponding generalized classical Stokes parameters. In section 4, by means of the classical constants of motion of the 3D isotropic harmonic oscillator we describe the geometrical properties of the polarization ellipse. Finally, in section 5, we give some concluding remarks.

2 Jordan-Schwinger map and the harmonic oscillator constants of motion

Usual classical Stokes parameters are defined as the expansion coefficients of the polarization matrix [14, 15] as

$$J_{2D} = \frac{1}{2} \sum_{i=0}^{3} \sigma_i s_i. \quad (1)$$

where $s_i$ are the four Stokes parameters, $\sigma_0 = 1_{2 \times 2}$ and $\sigma_i$, $i = 1, 2, 3$, are the Pauli matrices. Since the $\sigma_i$ matrices are such that $Tr(\sigma_i \sigma_j) = 2\delta_{ij}$ and $Tr(\sigma_0 \sigma_j) = 0$, then

$$Tr(J_{2D} \sigma_j) = s_j. \quad (2)$$

2.1 Usual quantum Stokes Operators

The usual Stokes operators for a quantized plane electromagnetic wave that propagates along the $z$ axis are defined as [11]

$$S_0 = a_1^\dagger \sigma_0 a = a_2^\dagger a_1 + a_2^\dagger a_2, \quad S_1 = a_1^\dagger \sigma_1 a = a_1^\dagger a_2 + a_2^\dagger a_1,$$

$$S_2 = a_1^\dagger \sigma_2 a = i(-a_1^\dagger a_2 + a_2^\dagger a_1), \quad S_3 = a_1^\dagger \sigma_3 a = a_1^\dagger a_1 - a_2^\dagger a_2, \quad (3)$$

where $a_j^\dagger$ and $a_j$, $j = 1, 2$, are the creation and annihilation operators of the $j$-th harmonic oscillator defined as

$$a_j^\dagger = \frac{1}{\sqrt{2}} (x_j - ip_j), \quad a_j = \frac{1}{\sqrt{2}} (x_j + ip_j), \quad (4)$$

with $[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1$ and

$$a^\dagger = (a_1^\dagger, a_2^\dagger), \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (5)$$

We note that equations (3) are a particular case of the Jordan-Schwinger map with two kinematically independent bosons [16].
In the rest of the paper, the following observation is of fundamental importance. The quantities (3) are nothing more than the constants of motion of the 2D isotropic harmonic oscillator, with Hamiltonian $H_{2D} = a_1^\dagger a_1 + a_2^\dagger a_2 + 1$. In fact, we can show that

$$[S_i, H_{2D}] = 0, \quad i = 0, 1, 2, 3. \quad (6)$$

The commutation relations of the Stokes operators are immediately obtained from the properties of the Jordan-Schwinger map [16]. This leads us to the $SU(2)$ Lie algebra

$$\left[ \frac{S_\ell}{2}, \frac{S_m}{2} \right] = i\epsilon_{\ell mn} S_n, \quad \ell, m, n = 1, 2, 3, \quad (7)$$

where $\epsilon_{\ell mn}$ is the well known totally antisymmetric tensor.

We note that the angular momentum and the energy minus the zero point energy of the 2D isotropic harmonic oscillator are equal to

$$L_z = S_2, \quad H_{2D} - 1 = S_0, \quad (8)$$

respectively. According to Jauch and Rorhlich [11], the spin of the photon is given by $S_2$ and it is along the direction of propagation. Therefore, the first equality in (8) means that the angular momentum of the 2D isotropic harmonic oscillator is equal to the spin operator of the photon.

Using equations (3), we can write the polarization matrix in terms of the constants of motion of the 2D isotropic harmonic oscillator (usual quantum Stokes operators) as

$$J_{2D} = \frac{1}{2} \begin{pmatrix} \langle S_0 \rangle_\alpha + \langle S_3 \rangle_\alpha & \langle S_1 \rangle_\alpha + i \langle S_2 \rangle_\alpha \\ \langle S_1 \rangle_\alpha - i \langle S_2 \rangle_\alpha & \langle S_0 \rangle_\alpha - \langle S_3 \rangle_\alpha \end{pmatrix} \quad (9)$$

where $\langle S_i \rangle_\alpha$ means the classical limit of the Stokes operators, which will be found in section 3 by taking their expectation values when the states of the electromagnetic field are expressed as coherent or semiclassical states.

The physical and geometrical implications of the equality between the Stokes operators and the constants of motion of the 2D isotropic harmonic oscillator are extensively discussed in Ref. [17].

### 2.2 Generalized Stokes Operators

When the direction of arrival from the source is unknown \textit{a priori}, we generalize the quantum Stokes operators as follows. By using the Gell-Mann and Ne’eman $\lambda_i$ matrices of the $SU(3)$ symmetry group [18] and the triplet of independent harmonic oscillators (three independent bosons) $a^\dagger = (a_1^\dagger, a_2^\dagger, a_3^\dagger)$, we define the generalized quantum Stokes operators as the Jordan-Schwinger map $\Sigma_i = a^\dagger \lambda_i a$, which explicitly are given by
\[\Sigma_0 = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3, \quad \Sigma_1 = a_1^\dagger a_2 + a_2^\dagger a_1, \quad \Sigma_2 = i(-a_1^\dagger a_2 + a_2^\dagger a_1),\]
\[\Sigma_3 = a_1^\dagger a_1 - a_2^\dagger a_2, \quad \Sigma_4 = a_1^\dagger a_3 + a_3^\dagger a_1, \quad \Sigma_5 = i(a_3^\dagger a_1 - a_1^\dagger a_3),\] (10)
\[\Sigma_6 = a_2^\dagger a_3 + a_3^\dagger a_2, \quad \Sigma_7 = i(a_3^\dagger a_2 - a_2^\dagger a_3), \quad \Sigma_8 = \frac{1}{\sqrt{3}}(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3),\]
where we have used \(\lambda_0 = 1_{3\times3}\).

From the commutation relations \([a_1, a_1^\dagger] = [a_2, a_2^\dagger] = [a_3, a_3^\dagger] = 1\), we show that the generalized quantum Stokes operators are the constants of motion of the 3D isotropic harmonic oscillator with Hamiltonian \(H_{3D} = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 + \frac{3}{2}\), i.e.
\[\left[\Sigma_i, H_{3D}\right] = 0, \quad i = 0, \ldots, 8.\] (11)

Also, by the properties of the Jordan-Schwinger map [16], we show that generalized quantum Stokes operators commutation rules satisfy the \(SU(3)\) Lie algebra
\[\left[\frac{\Sigma_\ell}{2}, \frac{\Sigma_m}{2}\right] = if_{\ell mn}\frac{\Sigma_n}{2}, \quad \ell, m, n = 1, \ldots, 8\] (12)
where the structure constants \(f_{\ell mn}\) are totally antisymmetric under exchange of any two indices and are given by
\[f_{123} = 1, \quad f_{147} = \frac{1}{2}, \quad f_{156} = -\frac{1}{2},\]
\[f_{246} = \frac{1}{2}, \quad f_{257} = \frac{1}{2}, \quad f_{345} = \frac{1}{2},\] (13)
\[f_{367} = -\frac{1}{2}, \quad f_{458} = \frac{\sqrt{3}}{2}, \quad f_{678} = \frac{\sqrt{3}}{2}.\]

A careful analysis leads us to show that the angular momentum operator \(\hat{L} = \hat{r} \times \hat{p}\) as well as the energy operator of the 3D isotropic harmonic oscillator are contained in the generalized quantum Stokes operators. Explicitly, we can show that
\[L_1 = \Sigma_7, \quad L_2 = -\Sigma_5, \quad L_3 = \Sigma_2, \quad H_{3D} - \frac{3}{2} = \Sigma_0.\] (14)

Because of the first three equalities of this equation, the generalization of what we have written after equation (8) means that the angular momentum of the 3D isotropic harmonic oscillator essentially equal is to the spin operator of the photon.

We generalize the definition of the polarization matrix as follows
\[J_{3D} = \frac{1}{3} \lambda_0 \langle \Sigma_0 \rangle_\alpha + \frac{1}{2} \sum_{i=1}^{8} \lambda_i \langle \Sigma_i \rangle_\alpha,\] (15)
again, \( \langle \Sigma_i \rangle_\alpha \) is the classical limit of the generalized quantum Stokes operators, which as it will be shown in the next section, are the expectation values of the operators \( \Sigma_i \) in a coherent state of the electromagnetic field.

Since the \( \lambda_i \) matrices are such that \( Tr(\lambda_i \lambda_j) = 2\delta_{ij} \) and \( Tr(\lambda_0 \lambda_i) = 0 \), \( i, j = 1, \ldots, 8 \), then

\[
Tr(\lambda_0 \lambda_j) = 0.
\]

By using equations (10) and (15) the polarization matrix in terms of the 3D isotropic harmonic oscillator constants of motion (or the generalized quantum Stokes operators) takes the form

\[
J_3D = \begin{pmatrix}
\frac{1}{3} \langle \Sigma_0 \rangle_\alpha + \frac{1}{2} \langle \Sigma_3 \rangle_\alpha + \frac{1}{2\sqrt{3}} \langle \Sigma_8 \rangle_\alpha \\
\frac{1}{2} \langle \Sigma_0 \rangle_\alpha + i\frac{1}{2} \langle \Sigma_2 \rangle_\alpha \\
\frac{1}{2} \langle \Sigma_4 \rangle_\alpha + i\frac{1}{2} \langle \Sigma_5 \rangle_\alpha
\end{pmatrix}.
\]

We observe that the \( \lambda_0 \) coefficient in equation (15) is such that equation (17) reduces to \( J_{2D} \) when the propagation direction of the plane electromagnetic wave is selected to be along the \( z \) axis. Also, we note that our definitions of \( J_{3D} \) is such that the trace of \( J_{2D} \) and \( J_{3D} \) remains invariant.

It is important to note that the polarization matrix (15) formally can be defined for purely quantum states. This means that it can be defined without taking the expectation values in a semiclassical state of the electromagnetic field of the Stokes operators \( \lambda_i \). In this way, equation (16) becomes to \( \langle \Sigma_j \rangle_\alpha = \langle \Sigma_j \rangle_0 \). However, the implications of this definition are out of the scope of this work.

### 3 Generalized classical Stokes parameters

We will obtain the classical limit for the generalized quantum Stokes operators. To do this, we proceed as in Ref. [20] to obtain the classical limit of the usual Stokes operators by taking the expectation value of the operators (3) in a two-mode coherent state of the electromagnetic field. In our case, we compute the mean value of the generalized quantum Stokes operators (10) in a three-mode coherent state of the electromagnetic field

\[
|\alpha_1, \alpha_2, \alpha_3\rangle = \sum_{n_1, n_2, n_3=0}^{\infty} \frac{\alpha_1^{n_1} \alpha_2^{n_2} \alpha_3^{n_3}}{\sqrt{n_1!n_2!n_3!}} |n_1, n_2, n_3\rangle.
\]

This leads us to the generalized classical Stokes parameters

\[
\langle \Sigma_0 \rangle_\alpha = |\alpha_0|^2 + |\alpha_2|^2 + |\alpha_3|^2, \quad \langle \Sigma_1 \rangle_\alpha = 2|\alpha_0||\alpha_2| \cos \Delta_{21},
\]
\[ \langle \Sigma_2 \rangle_\alpha = 2|\alpha_01||\alpha_02| \sin \Delta_{21}, \quad \langle \Sigma_3 \rangle_\alpha = |\alpha_01|^2 - |\alpha_02|^2, \]
\[ \langle \Sigma_4 \rangle_\alpha = 2|\alpha_01||\alpha_03| \cos \Delta_{31}, \quad \langle \Sigma_5 \rangle_\alpha = 2|\alpha_01||\alpha_03| \sin \Delta_{31}, \]
\[ \langle \Sigma_6 \rangle_\alpha = 2|\alpha_02||\alpha_03| \cos \Delta_{32}, \quad \langle \Sigma_7 \rangle_\alpha = 2|\alpha_02||\alpha_03| \sin \Delta_{32}, \]
\[ \langle \Sigma_8 \rangle_\alpha = |\alpha_01|^2 + |\alpha_02|^2 - 2|\alpha_03|^2, \]

where \( \alpha_i = |\alpha_{0i}| \exp(i\phi_i) \) and \( \Delta_{ij} = \phi_i - \phi_j \) is the classical phase difference.

As we will show in the next section, equation (19) represents the Stokes parameters for three classical oscillations of amplitudes \( |\alpha_{0i}| \) and phases \( \phi_i \). It is immediate to note that these equalities reduce to the usual classical Stokes parameters as a particular case when the amplitude and phase of the third oscillation vanishes.

4 Classical Stokes parameters, classical 3D isotropic harmonic oscillator constants of motion and the geometric properties of the polarization ellipse

The classical three-dimensional isotropic harmonic oscillator is a particle that moves under the force
\[ \mathbf{F} = -\mathbf{r}. \] (20)

By solving the Newton’s second law and imposing the initial conditions \( \mathbf{r}_{t=0} = \mathbf{x}_0 \) and \( \mathbf{v}_{t=0} = \mathbf{v}_0 \) we obtain the solutions
\[ x_i = a_i \cos t + b_i \sin t, \quad i = 1, 2, 3 \] (21)

where \( a_i = x_{ai} \) and \( b_i = v_{ai} \). It is easy to see that these solutions satisfy the ellipsoid equation
\[ x_1^2(a_1^2 + b_1^2 + a_2^2 + b_2^2) + x_2^2(a_1^2 + b_1^2 + a_3^2 + b_3^2) + x_3^2(a_2^2 + b_2^2 + a_3^2 + b_3^2) + 2x_1x_2x_3(a_2a_3 + b_2b_3) - 2x_1x_3(a_1a_3 + b_1b_3) - 2x_1x_2(a_1a_2 + b_1b_2) = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2. \] (22)

This means that the orbit of the classical 3D isotropic harmonic oscillator is contained in the ellipsoid. Moreover, since the classical 3D isotropic harmonic oscillator potential has spherical symmetry, its orbit is restricted to be in the perpendicular plane to the classical angular momentum \( \mathbf{L}_{cl} = \mathbf{r} \times \mathbf{p} \). Thus, the elliptic orbit of the classical 3D isotropic harmonic oscillator is given by the intersection curve of the ellipsoid (22) and the orthogonal plane to \( \mathbf{L}_{cl} \), which contains the origin of coordinates.

Equation (21) can be written in oscillation form as
\[ x_i = |\alpha_{0i}| \sin(\omega t + \phi_i), \] (23)
with
\[ a_i = |\alpha_0| \sin \phi_i, \quad b_i = |\alpha_0| \cos \phi_i. \]  \hfill (24)

These equalities imply that
\[ a_0^2 = a_i^2 + b_i^2, \quad \sin \phi_i = \frac{a_i}{\sqrt{a_i^2 + b_i^2}}, \quad \cos \phi_i = \frac{b_i}{\sqrt{a_i^2 + b_i^2}}. \]  \hfill (25)

The amplitudes and phases of the three classical oscillations of equation (23) depend on the initial conditions \( a_i \) and \( b_i \) according to equations (25). Thus, if we substitute equation (25) into equation (19), we incorporate the initial conditions in the generalized classical Stokes parameters (constants of motion of the classical 3D isotropic harmonic oscillator). In particular, at \( t = 0 \), the constant of motion of the angular momentum vector is
\[ \mathbf{L}_{cl} = (\langle \Sigma_7 \rangle_\alpha, -\langle \Sigma_5 \rangle_\alpha, \langle \Sigma_2 \rangle_\alpha) = \mathbf{a} \times \mathbf{b}, \quad (26) \]
where \( \mathbf{a} = (a_1, a_2, a_3) \), \( \mathbf{b} = (b_1, b_2, b_3) \), and the ellipsoid equation (22) results to be
\[ \begin{align*}
x_1^2 & \left( \frac{4\langle \Sigma_0 \rangle_\alpha - \langle \Sigma_8 \rangle_\alpha}{6} - \frac{1}{2} \langle \Sigma_5 \rangle_\alpha \right) + x_2^2 \left( \frac{4\langle \Sigma_0 \rangle_\alpha - \langle \Sigma_8 \rangle_\alpha}{6} + \frac{1}{2} \langle \Sigma_5 \rangle_\alpha \right) \\
& + x_3^2 \left( \frac{2\langle \Sigma_0 \rangle_\alpha + \langle \Sigma_8 \rangle_\alpha}{3} \right) - x_1x_2\langle \Sigma_1 \rangle_\alpha - x_2x_3\langle \Sigma_6 \rangle_\alpha - x_1x_3\langle \Sigma_4 \rangle_\alpha \\
& = \frac{1}{4} \left( \langle \Sigma_7 \rangle_\alpha^2 + \langle \Sigma_5 \rangle_\alpha^2 + \langle \Sigma_2 \rangle_\alpha^2 \right). \hfill (27)
\end{align*} \]

Following the definition of the Euler angles of Ref. [21], we perform a rotation such that the direction of the new \( x_1 \) axis coincide with that of the line of nodes, and the direction of the new \( x_3 \) axis coincide with that of \( \mathbf{L}_{cl} \). The direction of the line of nodes (direction of the intersection line between the orbit and the \( x_1 - x_2 \) plane) is found by a unitary vector in the \( x_1 - x_2 \) plane, perpendicular to \( \mathbf{L}_{cl} = (\langle \Sigma_7 \rangle_\alpha, -\langle \Sigma_5 \rangle_\alpha, \langle \Sigma_2 \rangle_\alpha) \). This leads us to
\[ \begin{align*}
\sin \phi &= n_x = \pm \frac{\langle \Sigma_7 \rangle_\alpha}{\sqrt{\langle \Sigma_7 \rangle_\alpha^2 + \langle \Sigma_5 \rangle_\alpha^2}}. \\
\cos \phi &= n_y = \mp \frac{\langle \Sigma_5 \rangle_\alpha}{\sqrt{\langle \Sigma_7 \rangle_\alpha^2 + \langle \Sigma_5 \rangle_\alpha^2}}. \hfill (28)
\end{align*} \]

The orthogonality between \( \mathbf{L}_{cl} \) and the ellipse plane leads to
\[ \cos \theta = \frac{\langle \Sigma_2 \rangle_\alpha}{\sqrt{\langle \Sigma_7 \rangle_\alpha^2 + \langle \Sigma_5 \rangle_\alpha^2 + \langle \Sigma_2 \rangle_\alpha^2}}. \hfill (29) \]
On the other hand, it is well known that the constants of motion of the classical 3D isotropic harmonic oscillator, in addition to the energy and the angular
momentum are given by the symmetric Runge-type tensor [22]

$$A_{ij} = \frac{1}{2}(p_ip_j + \omega^2 x_ix_j), \quad i, j = 1, 2, 3$$

(30)

It can be shown that the contraction of this equation with the components of $L_{cl}$ yields to zero. This means that all the geometric characteristics of the orbit must be determined by $A_{ij}$. In fact, we can show that

$$2A_{11} = \frac{2\langle \Sigma_0 \rangle_\alpha + \langle \Sigma_8 \rangle_\alpha}{6} + \frac{\langle \Sigma_3 \rangle_\alpha}{2}, \quad 2A_{22} = \frac{2\langle \Sigma_0 \rangle_\alpha + \langle \Sigma_8 \rangle_\alpha - \langle \Sigma_3 \rangle_\alpha}{6},$$

$$2A_{33} = \frac{\langle \Sigma_0 \rangle_\alpha - \langle \Sigma_8 \rangle_\alpha}{3}, \quad 2A_{12} = \frac{1}{2}\langle \Sigma_1 \rangle_\alpha, \quad 2A_{13} = \frac{1}{2}\langle \Sigma_4 \rangle_\alpha,$$

$$2A_{23} = \frac{1}{2}\langle \Sigma_6 \rangle_\alpha.$$  (31)

This shows that the geometric properties of the polarization ellipse exclusively depend on the generalized classical Stokes parameters. It can be shown that the eigenvalues of $A_{ij}$ depend only on the energy and the magnitude of the angular momentum of the classical 3D isotropic harmonic oscillator [22]. $A_{ij}$ has an eigenvector in the direction of the angular momentum, and the other two eigenvectors are in the directions of the minor and major axis of the elliptical orbit [22]. Also, the eigenvectors of a symmetric rank two tensor are determined by its eigenvalues as well as its components [23]. All this leads us to conclude that the principal axis directions on the elliptical orbit are completely determined by the generalized classical Stokes parameters. Also, since $L_{cl}$ is orthogonal to the polarization ellipse, then $L_{cl}$ points along the propagation direction of the electromagnetic wave.

5 Concluding Remarks

This work links the quantum optics to classical optics by means of quantum mechanics and it is a useful extension of the generalized classical Stokes parameters into the quantum domain.

Although already there are treatments of the classical Stokes parameters in the case of an $a \ priori$ unknown of the electromagnetic propagation [12, 13], our treatment results to be novel in the following aspects. We have introduced a generalization of the quantum Stokes parameters of Jauch et. al. [11] using the Jordan-Schwinger map, three independent bosons and the Gell-Mann and Ne’eman $SU(3)$ symmetry group matrices. It was shown that the generalized quantum Stokes operators result to be the expansion coefficients of the polarization matrix in terms of the Gell-Mann and Ne’eman $SU(3)$ matrices. The semiclassical limit of the generalized Stokes operators were achieved by taking their expectation values in a three-mode coherent state of the electromagnetic field. We have shown that the resulting generalized classical Stokes parameters
are consistent with the generalized classical Stokes parameters which have recently been published [13]. Thus, our treatment is more general than those of references [12, 13], which are restricted to the classical aspects of electromagnetic polarization.

We described by means of the classical 3D isotropic harmonic oscillator constants of motion the geometrical properties of the polarization ellipse. Particularly, we showed that the ellipsoid coefficients and the symmetric Runge-type tensor of the classical 3D isotropic harmonic oscillator are completely determined by the generalized classical Stokes parameters. Also, we showed that the first two Euler angles are intimately related to the components of the orbital angular momentum of the classical 3D isotropic harmonic oscillator.

Finally, we emphasize that the number of independent generalized classical Stokes parameters are six. This is because in going from (22) to (27), all of them were written in terms of the six parameters, $a_i$ and $b_i$, $i = 1, 2, 3$ which contain the initial conditions of the classical 3D isotropic harmonic oscillator.

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