Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials

Svetlana Jitomirskaya* and Rajinder Mavi†

December 18, 2014

Abstract

We establish localization type dynamical bounds as a corollary of positive Lyapunov exponents for general operators with quasiperiodic potentials defined by piecewise Hölder functions.

1 Introduction

We will study the quantum dynamical properties of Schrödinger Hamiltonian acting on $\ell^2(\mathbb{Z})$.

$$h_\theta u(n) = u(n-1) + u(n+1) + f(n\omega + \theta)u(n).$$

(1.1)

where $\omega \in \mathbb{R}\setminus\mathbb{Q}$ and $f : T \to \mathbb{R}, T = \mathbb{R}/\mathbb{Z}$, in the regime of positive Lyapunov exponents. The evolution of a wave packet under the Hamiltonian (1.1) is given by the formula

$$u(t) = e^{-ith}u(0)$$

Dynamical localization, i.e. the nonspread as $t \to \infty$ of $u(t)$ with initially localized $u(0)$, is related to various quantities that can be measured in an experiment. It is often assumed by physicists to be a corollary of positivity of Lyapunov exponents, a quantity defined by dynamics of the associated cocycle and easily computable numerically. As mathematicians, we know however, that positive Lyapunov exponents, while implying no absolutely continuous spectrum, can coexist even with almost ballistic transport [26, 12] so one cannot expect dynamical localization in full generality, and for a more general result in the direction that physicists want, one has to tone down the notion of “nonspread” accordingly.

For a nonegative function $A(t)$ of time denote

$$\langle A(t) \rangle_T = \frac{2}{T} \int_0^\infty e^{-2t/T}A(t)dt$$

Let

$$a(n, t) = |\langle e^{-ith}\delta_0, \delta_n \rangle|^2$$

and

$$a_T(n) = \langle a(n, t) \rangle_T$$

Clearly, $\sum_n a_T(n) = \sum_n a(n, t) = 1$ for all $t$. The classical quantities of interest are the moments of the position operator, that can be defined both with time averaging

$$\langle |X|^p \rangle_T = \sum_n (1 + |n|^p)a_T(n)dt.$$ or without

$$\langle |X|^p(T) \rangle = \sum_n (1 + |n|^p)a(n, T)dt.$$ For $p > 0$ define the lower and upper transport exponents

$$\beta^+(p) = \limsup_{t \to \infty} \frac{\ln \langle |X|^p(t) \rangle}{p \ln t}; \quad \beta^-(p) = \liminf_{t \to \infty} \frac{\ln \langle |X|^p \rangle}{p \ln t}.$$
determining the upper/lower power-law rate of growth of the moments along subsequences. Note that for the purposes of this paper we define the upper rate without time averaging, while the lower rate with time averaging.

Dynamical localization is defined as boundedness in $T$ of $(\|X|^p(T))$. This implies pure point spectrum, thus for parameters for which spectrum is singular continuous (known to be generic in many situations with positive Lyapunov exponents) one cannot have dynamical localization in this sense. Then, vanishing of $\beta^+$, or, in absence of that, at least of $\beta^-$ are properties to look for. Moreover, dynamical localization is a property that is often unstable with respect to compact perturbations of the potential or phase shifts. In contrast, vanishing of $\beta$ is always stable with respect to compact perturbations [10] and also with respect to phase shifts in all known examples. It should be noted however that such vanishing cannot be expected in general for operators \((1.1)\) if the Lyapunov exponent is allowed to vanish even on a set of measure zero (as shown by Sturmian potentials), or, in a slightly more general context even if it vanishes at a single point [30].

The fact that $\beta(p)$ may depend nontrivially on $p$ is the signature of intermittency, reflected on an even deeper level in the fact that different parts of the wave packet may spread at different rates. While any kind of upper bound discussed above requires control of the entire wave packet, even control of the spread of a portion of it is an interesting statement.

Set

$$P(N,t) = \sum_{|n| \leq N} a(n,t), \quad P_T(N) = \sum_{|n| \leq N} a_T(n).$$

A bound of the form $P(T^a,T) > c$ shows that at time $T$ a portion of the wave packet is confined in a box of size $T^a$. Thus a bound like that holding for arbitrary $a > 0$ can be considered as a signature of localization. It is natural in this respect to introduce two other scaling exponents:

$$\bar{\xi} = \lim_{\delta \to 0} \lim_{T \to \infty} \sup \frac{\ln(\inf\{L|P_T(L) > \delta\})}{\ln T},$$

and

$$\bar{\xi} = \lim_{\delta \to 0} \lim_{T \to \infty} \inf \frac{\ln(\inf\{L|P_T(L) > \delta\})}{\ln T}.$$

Then vanishing of $\bar{\xi}$ or even $\xi$ is again a localization-type statement.

Various quantities have been used to quantify quantum dynamics, see [3, 10] for a more comprehensive description. In this paper we focus on $\xi$ and $\beta$ only. Our main question is what kind of localization-type statements can be obtained from positivity of the Lyapunov exponents under very mild restrictions on regularity of the potential.

While the last decade has seen an explosion of general results for operators \((1.1)\) with analytic $f$, see e.g. [5, 15] and references therein, and by now even the global theory of such operators is well developed [12], there are very few results beyond the analytic category that do not require energy exclusion [3] (with few recent exceptions [31, 34, 19] only confirming the rule). Indeed, not only the methods of proof usually require analyticity (or at least the Gevrey condition), but certain results fail to hold as long as analyticity is relaxed (see also [18]). It is expected that many recent “analytic” results in fact do require analyticity. In this paper we show that, in contrast to the above, dynamical upper bounds can be obtained as a corollary of positive Lyapunov exponents under surprisingly weak regularity.

Namely, we allow $f$ in \((1.1)\) with only Hölder continuity, and even allow it to have finitely many discontinuities (so only require it to be locally Hölder). Allowing for discontinuities in the class of considered potentials is important for two reasons. First, the main explicit non-analytic operators \((1.1)\) that appear in different contexts in physics literature [14, 31] have $f$ with discontinuities. Several models that are well studied mathematically: Maryland, Fibonacci (or, more generally, Sturmian) operators also belong to this class. Second, while there are few results on positivity of Lyapunov exponents for non-analytic $f$, the Lyapunov exponents of operators \((1.1)\) with discontinuous $f$ are always positive at least a.e. [9], providing us with a large collection of models for which our results are directly applicable. As far as we know, the present paper is the first one holding for a class of potentials that rough. Spectral localization for (continuous) Hölder potentials outside a set of energies of measure zero was established in [8], but there have been no dynamical bounds (see Footnote [1]). In [19] we proved continuity of measure of the spectrum for (continuous) Hölder potentials.

We will say that $f$ is piecewise Hölder if $f$ has a finite set of discontinuities, $J_f$, and there exists $\gamma > 0$ such that $\|f\|_{PL,\gamma} < \infty$ where

$$\|f\|_{PL,\gamma} = \|f\|_{\infty} + \sup_{h > 0} \sup_{t \in T, \text{dist}(t,J_f) > |h|} \frac{|f(t+h) - f(t)|}{|h|^\gamma}.$$

\footnote{It should be noted that exclusion of any energies in localization type results, such as, e.g. [5], make upgrading to dynamical statements very problematic, as even a single energy that does not carry any spectral measure can lead to robust transport [30, 20].}

\footnote{It should be mentioned that our analysis is not relevant to Fibonacci and most Sturmian models as for them the Lyapunov exponent vanishes on the spectrum.}
The functions \( f \) with finite \( \| \cdot \|_{PL} \) norm form the space of piecewise \( \gamma \)-Lipschitz functions, that we denote \( PL_\gamma(\mathbb{T}) \).

We will now introduce the Lyapunov exponent. For a given \( z \in \mathbb{C} \), a formal solution \( u \) of

\[ h u = z u \]  

with operator \( h \) given by \([1.4]\) can be reconstructed from its values at two consecutive points with the transfer matrix

\[ A^{f,z}(\theta) = \begin{pmatrix} z - f(\theta) & -1 \\ 1 & 0 \end{pmatrix}; \quad A^E : \mathbb{T} \to \text{SL}_2(\mathbb{R}) \]  

via the equation

\[ \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A^{f,z}(\theta + n\omega) \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}. \]  

Setting \( R : \mathbb{T} \to \mathbb{T}, \ Rx := x + \omega \), the pair \( (\omega, A^{f,z}) \) viewed as a linear skew-product \((x,v) \to (Rx, A^{f,z}(x)v)\), \( x \in \mathbb{T}, \ v \in \mathbb{R}^2 \), is called the corresponding Schrödinger cocycle. The iterations of the cocycle \((\omega, A^{f,z})\) for \( k \geq 0 \) are given by

\[ A^{f,z}_k(\theta) = A^{f,z}(R^{(k-1)}\theta) \cdots A^{f,z}(R^1\theta)A^{f,z}(\theta), \quad A^{f,z}_0 = I \]  

and

\[ A^{f,z}_k(\theta) = \left(A^{f,z}_{k-1}(R^{k+1}\theta)\right)^{-1}; \quad k < 0. \]  

Therefore, it can be seen from \([1.4]\) that a solution to \([1.2]\) for chosen initial conditions \((u(0),u(-1))\) for all \( k \in \mathbb{Z} \) is given by,

\[ \begin{pmatrix} u(k) \\ u(k-1) \end{pmatrix} = A^{f,z}_k(\theta) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}. \]  

By the general properties of subadditive ergodic cocycles, we can define the Lyapunov exponent

\[ \mathcal{L}(z) = \lim_{k \to \infty} \frac{1}{K} \int \ln \| A^{f,z}_k(\theta) \| d\theta = \inf_{\kappa > 0} \lim_{k \to \infty} \frac{1}{K} \int \ln \| A^{f,z}_k(\theta) \| d\theta, \]  

furthermore, \( \mathcal{L}(z) = \lim_k \frac{1}{K} \ln \| A^{f,z}_k(\theta) \| \) for almost all \( \theta \in \mathbb{T} \).

Finally, we introduce the Diophantine condition. Writing \( \omega \) in the continued fraction form

\[ \omega = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \equiv [a_1, a_2, \ldots], \]

the truncated continued fractions define the approximants \( \frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n] \). We say that \( \omega \) is Diophantine if for some \( \kappa > 0 \)

\[ q_{n+1} < q_n^{1+\kappa} \]  

for all large \( n \).

Our first result is that just positivity of the Lyapunov exponent on a positive measure subset of the spectrum already implies localization bounds for the transport of the bulk of a wave packet.

Let \( \mu_0 \) be the spectral measure of \( h_0 \) and vector \( \delta_1 \), and \( N := \int \mu_0 d\theta \) be the integrated density of states.

**Theorem 1.1** For piecewise Hölder \( f \) and \( \omega \in \mathbb{R} \setminus \mathbb{Q} \), suppose \( \mathcal{L}(E) \) of \([1.1]\) is positive on a Borel subset \( U \) with \( N(U) > 0 \). Then

1. For any irrational \( \omega, \xi = 0 \) for a.e. \( \theta \)
2. If \( \omega \) is Diophantine, then \( \xi = 0 \) for a.e. \( \theta \)
3. For all \( \theta \) for \( \xi > 0 \), \( P(T_k^\xi, T_k) > C\mu_0(U) \) for a sequence \( T_k \to \infty \); moreover if \( \omega \) is Diophantine, \( P(T_k^\xi, T) > C\mu_0(U) \) for all large \( T \).

**Remark 1.2** Positivity of the Lyapunov exponent on a positive IDS measure subset is clearly essential, as the result does not hold for Fibonacci-type models where Lyapunov exponent is positive a.e. but zero on the spectrum.

The Diophantine condition is essential for vanishing of \( \xi \) as the result does not hold for Liouville \( \omega \) (\([21]\)).

**Remark 1.3** The full measure set of \( \theta \) in 1.2 is precisely the set \( \{ \theta : \mu_0(U) > 0 \} \). It is not entirely clear whether there are quasiperiodic examples with \( N(E : \mathcal{L}(E) > 0) > 0 \) and \( \mu_0(E : \mathcal{L}(E) > 0) = 0 \) for some \( \theta \).
Remark 1.4 It is an interesting question whether or not a.e. vanishing of $\xi$ is a general corollary of positive Lyapunov exponents, so holds for all ergodic potentials. This may be reminiscent of the property of zero Hausdorff dimension of spectral measures of operators with positive Lyapunov exponents, which was originally proved for quasiperiodic operators with trigonometric polynomial potentials [17], but then turned out to be a general fact, easily extractable from some deep results of potential theory [28].

Corollary 1.5 Assume $f$ is locally Hölder, has at least one point of discontinuity and that $N$ has an absolutely continuous component. Then the conclusions of Theorem 1.1 hold.

Proof Follows immediately from a.e. positivity of Lyapunov exponents of potentials with discontinuities (was proved in [9] with a conjecture made in [27]).

Remark 1.6 Various examples of operators (1.1) with discontinuous $f$ and absolutely continuous $N$ are given in [16]. Other potentials have been shown to satisfy the conditions of Theorem 1.1 in various regimes in [4, 7, 25, 36]. In all those cases Theorem 1.1 improves on some of the known results since, even if the results potentially allowed for dynamical extensions, unlike Theorem 1.1 spectral localization cannot hold for all $\theta$ at least for continuous $f$ that are even on the hull [20].

Certainly, not every $f$ in (1.1) corresponds to a model relevant to physics, and since our main question is physically motivated, it is natural to impose assumptions that are necessary for physics relevance. In particular, Lyapunov exponent should be continuous in various parameters for operators coming from physics (although such continuity does not hold universally for operators (1.1) even for $f$ in $C^\infty$ [33]). Our next result has this as an assumption.

Theorem 1.7 For piecewise Hölder $f$ and $\omega \in \mathbb{R}\setminus \mathbb{Q}$ suppose $L$ is continuous in $E$ and $L(E) > 0$ for every $E \in \mathbb{R}$. Then

1. $\beta_{\omega, \theta}^-(p) = 0$ for all $\theta \in \mathbb{T}$, $p > 0$;
2. if $\omega$ is Diophantine, then $\beta_{\omega, \theta}^+(p) = 0$ for all $\theta \in \mathbb{T}$, $p > 0$.

Remark 1.8 It is an interesting question whether or not vanishing of $\beta^-$ is a general corollary of uniformly positive Lyapunov exponents in the regime of their continuity, so for all ergodic potentials. The analogy of Remark 1.4 may also apply.

Remark 1.9 The Diophantine condition is essential for vanishing of $\beta^+$ [27].

Corollary 1.10 If $f$ is $C^2$ with exactly two nondegenerate extrema, and $\omega$ is Diophantine, then for $\lambda > \lambda(f, \omega)$, $\beta_{\omega, \theta}^+(p) = 0$ for all $\theta \in \mathbb{T}$, $p > 0$.

Note that this is the first dynamical bound for $C^2$ potentials.

Proof Follows directly from Theorem 1.7 and the results of [35].

Corollary 1.11 If $f$ is analytic, then for $\lambda > \lambda(f)$ both conclusions of Theorem 1.7 hold.

Proof follows from non-perturbative positivity [29] and continuity [6] of the Lyapunov exponent for analytic $f$.

Remark 1.12 This was established in a combination of [10] and [11] for trigonometric polynomial $f$. The result of [10] allows for a weaker Diophantine condition than ours. Namely it holds for $\omega$ such that

$$\lim_{n \to \infty} \frac{\ln q_{n+1}}{q_n} = 0. \quad (1.10)$$

Our current proof does not automatically extend to this condition because of the need to tackle low regularity. A simple modification of the proof allows to obtain this result for $\omega$ satisfying (1.10) and analytic $f$ but not $f \in C^\gamma$.

Corollary 1.13 If $f$ is Gevrey with a transversality condition and $\omega$ is Diophantine, then for $\lambda > \lambda(f, \omega)$, $\beta_{\omega, \theta}^+(p) = 0$ for all $\theta \in \mathbb{T}$, $p > 0$.

Proof Follows from Theorem 1.7 and the results of [25].

Remark 1.14 $\lambda(f, \omega)$ depends on $\omega$ through its Diophantine class. In [27] Anderson localization is established for all $\theta$ and a.e. $\omega$ in this class (depending on $\theta$).
Another immediate corollary can be obtained for a class of discontinuous $f$ monotone on the period as considered in [10]. Then, for large $\lambda$, Anderson localization is established in [16] for Diophantine $\omega$ satisfying an additional full measure arithmetic condition, while continuity and positivity of the Lyapunov exponent is established requiring the latter condition only. Theorem 1.7 immediately implies in this case vanishing of $\beta^-$ for all $\theta$ and all $\omega$ satisfying the above mentioned arithmetic condition.

For the proof of Theorem 1.1 we use a criterion from [24], and to prove Theorem 1.7 apply the results of [11]. This is done in Section 2. To apply those results we need to establish certain lower bounds on transfermatrices. To obtain this we build on the technique we introduced in [19]. Our main technical achievement is in extending both the method of [19] to allow discontinuities and in extending the underlying uniform upper bound for uniquely ergodic dynamics to the case of cocycles with zero measure set of discontinuities. The latter is a general result that is of independent interest and of the type that has been important in various proofs of localization/regularity in many recent articles. Our extension has already been used in [10] for their spectral localization theorem.

2 Key lemmas

For any $\delta \geq 1$ and $1 \geq \zeta > 0$ we define, for $E \in \mathbb{C}$ and $T > 0$

$$\Phi_{\zeta, \delta}(E, T) = \inf \left\{ \min \left\{ \frac{\max_{1 \leq j \leq T^\zeta} \|A_j(\theta, z)\|^2}{T^\delta}, \frac{\max_{1 \leq j \leq T^\zeta} \|A_{-j}(\theta, z)\|^2}{T^\delta} \right\} \right\}$$

where the infimum is over all $|z - E| \leq T^{-\zeta}$ and $\theta \in \mathbb{T}$. We will establish

**Proposition 2.1** Suppose $f \in PL_\chi(\mathbb{T})$, $\chi > 0$, and suppose $\mathcal{L}(E) > \chi$ for all $E$ in a Borel set $U \subset \mathbb{R}$. Then, for any $\delta \geq 1$ and $1 \geq \zeta > 0$ we have, for every $E \in U$,

$$\limsup_T \Phi_{\zeta, \delta}(E, T) > 0.$$  \hfill (2.1)

Moreover, if $U$ is compact and $\mathcal{L}$ is continuous, the bound is uniform,

$$\limsup_T \inf_{E \in U} \Phi_{\zeta, \delta}(E, T) > 0.$$  \hfill (2.2)

Finally, if $\omega$ is Diophantine, the last $\limsup_T$ may be replaced by $\liminf_T$ independent of the continuity of $\mathcal{L}$.

This proposition is essentially a corollary of the following Lemma.

**Lemma 2.2** Suppose $f \in PL_\omega$, $\mathcal{L}(E) > 0$. For any $\tau > 0$ there exists $k_\tau < \infty$ so that if $q_n > e^{\frac{\kappa \mathcal{L}(E)}{\tau}}$, then for any $k \in \mathbb{Z}^+$ such that $k_\tau < k < \frac{\ln q_n}{\tau}$ then for any $\theta \in \mathbb{T}$ there is some $0 < x \leq q_n + q_n - 1$ so that for any $z \in \mathbb{C}$ with $|z - E| < \exp\{-\tau k \mathcal{L}(E)\}$

$$\|A_k^z(R^x \theta)\| \geq e^{\kappa (1 - \tau) \mathcal{L}(E)}.$$

We will in fact prove a more general statement, for cocycles defined in a neighborhood of $f$, see Lemma 4.3. The proofs of Proposition 2.1 and Lemma 2.2 are in section 4. They are based on section 3 where we prove convergence results for discontinuous cocycles in a general setting. The remainder of this section is dedicated to proving Theorems 1.1 and 1.7.

For $f : \mathbb{Z} \to H$, where $H$ is some Banach space and $L \geq 1$, the truncated $\ell^2$ norm in the positive direction is defined as

$$\|f\|_L^2 = \sum_{n=1}^{\lfloor L \rfloor} |f(n)|^2 + (L - \lfloor L \rfloor) |f(\lfloor L \rfloor + 1)|^2.$$

The truncated $\ell^2$ norm in both directions, for $L_1, L_2 \geq 1$, will be denoted

$$\|f\|_{L_1, L_2}^2 = \sum_{n=-\lfloor L_1 \rfloor}^{\lfloor L_2 \rfloor} |f(n)|^2 + (L_1 - \lfloor L_1 \rfloor) |f(-\lfloor L_1 \rfloor - 1)|^2 + (L_2 - \lfloor L_2 \rfloor) |f(\lfloor L_2 \rfloor + 1)|^2.$$

With $A_\cdot(\theta, E)$ a function on $\mathbb{Z}$, define $\tilde{L}_i^+(\theta, E) \in \mathbb{R}^+$ by requiring that the truncated $\ell^2$ norm obeys

$$\|A_\cdot(\theta, E)\|_{\tilde{L}_i^+(\theta, E)}^2 = 2 \|A_1(\theta, E)^{-1}\| \epsilon^{-1}.$$

We now recall the following result of Killip, Kiselev and Last,
Lemma 2.3 (Theorem 1.5 of [24]) Let \( h \) be a Schrödinger operator and \( \mu \) the spectral measure of \( h \) and \( \delta_1 \). Let \( T > 0 \) and \( L_1, L_2 > 2 \), then
\[
\langle |e^{-it\delta_1}|^2 \rangle_{L_1,L_2} > C \mu \left( \left\{ E : \tilde{L}_{T-1}^- \leq L_1; \tilde{L}_{T-1}^+ \leq L_2 \right\} \right)
\]
where \( C \) is a universal constant.

Proof of Theorem 1.1. Clearly, for every \( \theta \) with \( \mu_\theta(U) > 0 \) parts 1, 2 follow from part 3. Since the set \( \{ \theta : \mu_\theta(U) > 0 \} \) is shift invariant, \( N(U) > 0 \) implies \( \mu_\theta(U) > 0 \) for a.e. \( \theta \). Thus we only prove part 3. Assume \( \mu_\theta(U) > 0 \). For \( \epsilon > 0 \), let \( \chi > 0 \) be such that
\[
\mu_\theta(\{ E \in U : \mathcal{L}(E) > \chi \}) > \mu_\theta(U) - \frac{\epsilon}{2}.
\]
Let \( \zeta > 0 \). First consider the Diophantine case. Then by Proposition 2.4 with \( \delta = 3 \), for \( E \in U \) we have \( \Phi_{\zeta,3}(E,t) > c_E > 0 \) for \( t > T_E \). Therefore we can find \( M_\epsilon > 0 \), so that outside a set of \( E \) of measure \( \tilde{\chi} \),
\[
\| A(\theta,E) \|_{T^\tilde{\chi}} > T
\]
for \( t > M_\epsilon \). Thus \( \tilde{L}_{T-1}^\pm(\theta,E) < T^\tilde{\chi} \) for all \( T > M_\epsilon \). We have from Lemma 2.3
\[
\langle |e^{-it\delta_1}|^2 \rangle_{T^\tilde{\chi}} > C(\mu_\theta(U) - \epsilon).
\]
If \( \omega \) is not Diophantine, 2.4 is satisfied for a sequence \( T_k \to \infty \), thus 2.5 holds for a sequence \( T_k \). As 2.5 holds for all \( \epsilon \) we can let \( \epsilon \to 0 \).

The following result of Damanik and Tcheremchantsev allows us to control the evolution of the entire wavepacket.

Lemma 2.4 (Corollary 1 of [11] plus Theorem 1 of [10]) Let \( h \) be operator (1.1), with \( v \) real valued and bounded, and \( K \geq 4 \) is such that \( \sigma(h) \subset [-K + 1, K - 1] \). Suppose for all \( \zeta \in (0,1) \), we have
\[
\int_{-K}^{K} \left( \min_{1 \in (-1,1)} \max_{1 \leq n \leq T^\zeta} \| A_n \left( E + \frac{i}{T} \right) \| \right)^2 \, dE = O(T^{-\delta})
\]
for every \( \delta > 1 \). Then \( \beta^+(p) = 0 \) for all \( p > 0 \). If (2.6) is satisfied for a sequence \( T_k \to \infty \), then \( \beta^-(p) = 0 \) for all \( p > 0 \).

Proof of Theorem 1.7 Assume \( \sigma(h) \subset [-K + 1, K - 1] \). Let \( R > K \) and let \( \chi = \inf_{|z| < R} \{ \mathcal{L}(z) \} \). We assume continuity so \( \chi > 0 \) and there exists a large \( M < \infty \) so that (2.2) holds uniformly for all \( T > M \) and \( E \subset \{ E \in \mathbb{C} : |\mathcal{R}(E)| \leq K; |\mathcal{I}(E)| \leq 1 \} \). Thus for large enough \( T \) and \( \omega \) Diophantine we have
\[
\int_{-K}^{K} \left( \max_{1 \leq n \leq T^\zeta} \| A_n \left( E + \frac{i}{T} \right) \| \right)^2 \, dE \leq C K T^{-\delta} = O(T^{-\delta}).
\]
If \( \omega \) is not Diophantine, (2.7) is satisfied for a sequence of \( T_k \to \infty \).

3 Rough cocycles

The goal of this section is to establish the uniformity of uppersemicontinuity of the Lyapunov exponent. It is known (see e.g. [19]) the pointwise Lyapunov exponent has uniform upper bounds in small neighborhoods for continuous cocycles. Here we show the requirement of continuity of cocycles can be relaxed. Let \( (X,T,\mu) \) be a uniquely ergodic compact Borel probability space. We will say a function \( f \) is almost continuous if its set of discontinuities has a closure of measure zero. Let \( \mathbb{B}_\infty(X) \) be the space of bounded functions on \( X \) with
\[
\| f \|_\infty = \sup_{x \in X} |f(x)|,
\]
Notice that sets of measure zero are not dismissed by this norm. For a Borel set \( D \subset X \) define a seminorm
\[
\| f \|_{D,\infty} = \sup_{x \in D} |f(x)|.
\]

A subadditive cocycle on \( (X,T) \) is a sequence of functions \( f_1, f_2, \ldots \) on \( X \) so that \( f_{n+m}(x) \leq f_n(x) + f_m(T^n x) \). We use the notation \( \{ f \} \) for a subadditive cocycle \( f_1, f_2, \ldots \). Let \( \Delta(X) \) be the set of all \( \{ f \} \) with \( f_n \in \mathbb{B}_\infty \) for all \( n \). By Kingman’s subadditive ergodic theorem [32], a subadditive cocycle \( f_n(\cdot) \) on \( (X,T) \) obeys, for \( \mu \)-almost all \( x \in X \),
\[
\lim_{n \to \infty} \frac{1}{n} f_n(x) = \lim_{n \to \infty} \int_X f_n(x) \mu(dx) = \Lambda(f)
\]
Let $E_n = E_n(\{f\})$ be the closure of the set of discontinuities of $f_n$. For a set $E \subset X$ define a ball, $B_\delta(E) = \{x \in E : \exists \varepsilon \in E, |x - \varepsilon| < \delta\}$. Then we introduce, for $\delta \geq 0$ the sequence of sets $D_n = X \setminus B_\delta(E_n(\{f\})), and a pseudometric

$$d_\delta(\{g\}, \{f\}) = \sum_{n \geq 1} \frac{1}{2^n} \frac{\|g_n - f_n\|_{D_n, \infty}}{1 + \|g_n - f_n\|_{D_n, \infty}}.$$  

From this pseudometric we define the $\delta$-$\sigma$ neighborhood of $\{f\}$ as,

$$\mathcal{N}_{\delta, \sigma}(\{f\}) = \{\{g\} : d_\delta(\{f\}, \{g\}) < \sigma\}.$$  

**Theorem 3.1** Suppose $\{f\} \in \Delta(X)$ so that $f_n$ is almost continuous for all $n$. Let $\varepsilon > 0$. There exists $\delta > 0$ and $\sigma > 0$ and $K < \infty$ all depending on $f$ and $\varepsilon$ so that for $g \in \mathcal{N}_{\delta, \sigma}(\{f\}) \cap B_\infty$ and $n > K$, implies

$$\frac{1}{n} g_n < \Lambda(f) + \varepsilon \max\{\|g\|_\infty, 1\}.$$  

The result extends the theorem of Furman [13] and our recent extension of it [19] to the case of almost continuous subadditive cocycles. Here is a simple application of the theorem to a single subadditive cocycle.

**Corollary 3.2** Suppose $f_n$ are almost continuous and subadditive and $\|f_1\|_\infty < \infty$. For any $\varepsilon > 0$ there is $K < \infty$ so that for $n > K$ and all $x \in X$ we have

$$\frac{1}{n} f_n(x) < \Lambda(f) + \varepsilon.$$  

A further corollary arises in the application to matrix cocycles for an almost continuous matrix $M : X \to \mathcal{M}_{2,2}(\mathbb{C})$, the two by two matrices over $\mathbb{C}$. Let $E$ be the set of discontinuities of $M$, let $B = B_\delta(E)$. Let $E_n = E_n(\{\ln \|M_n\|\})$ and let $\mathcal{L}(M)$ be the Lyapunov exponent $\Lambda(\{\ln \|M_n\|\})$.

**Corollary 3.3** Let $M : X \to \mathcal{M}_{2,2}(\mathbb{C})$ be almost continuous and bounded. Suppose for all $n > 0$ there exists an $\eta > 0$ so that for $x \in D_n$, $\|M_n(x)\| > \eta$. For any $\varepsilon > 0$, there is $\delta > 0$ and $\rho > 0$, and $K < \infty$ such that $\|M - M\|_{X \setminus B_\delta(E_n, \rho)} < \rho$, and $k > K$ implies

$$\|M_k(\theta) - \overline{M_k(\theta)}\| < \max_{0 \leq i \leq k-1} \{\|M(R_i^i\theta) - \overline{M(R_i^i\theta)}\|\} e^{k(\mathcal{L}(M)) + \max(1, \ln \|M\|_\infty, \ln \|\overline{M}\|_\infty)}$$

For our application we only need the $\delta = 0$ version:

**Corollary 3.4** Let $M : X \to \mathcal{M}_{2,2}(\mathbb{C})$ be almost continuous and bounded. Suppose for all $n > 0$ there exists an $\eta > 0$ so that for $x \in D_n$, $\|M_n(x)\| > \eta$. For any $\varepsilon > 0$, there is $\rho > 0$, and $K < \infty$ such that $\|M - M\|_\infty < \rho$, and $k > K$ implies

$$\|M_k(\theta) - \overline{M_k(\theta)}\| < \max_{0 \leq i \leq k-1} \{\|M(R_i^i\theta) - \overline{M(R_i^i\theta)}\|\} e^{k(\mathcal{L}(M)) + \max(1, \ln \|M\|_\infty, \ln \|\overline{M}\|_\infty)}$$

**Proof of Theorem 3.1** Let $\varepsilon < (1 + 2\|f_1\|_\infty)^{-1}$. $X \setminus E_n$ is an open set of full measure, and for every $x \in X \setminus E_n$, $f_n$ is continuous in a neighborhood of $x$. The set

$$J_n = \left\{ x \in X \setminus E_n : \frac{1}{n} f_n(x) - \Lambda(f) < \varepsilon \right\}$$

is open and by Kingman’s theorem $\mu(J_n^\varepsilon) \to 0$ as $n \to \infty$.

Let $n > 1$ be large enough so that $\mu(J_n^\varepsilon) < \varepsilon$. Let $\delta > 0$ be such that $\mu(B_\delta(E_n)) < \varepsilon$. Define $D_n = X \setminus B_\delta(E_n)$. For any $\{g\} \in \mathcal{N}_{\delta, 2\varepsilon}(\{f\})$, and for $x \in J_n \cap D_n$ we have $f_n(x) < n(\Lambda(f) + \varepsilon)$ which implies

$$g_n(x) \leq |f_n(x)| + |g_n(x) - f_n(x)| < n(\Lambda(f) + \varepsilon) + 2\varepsilon \leq n(\Lambda(f) + 2\varepsilon). \quad (3.1)$$

Note that $J_n^\varepsilon \cup D_n^\varepsilon$ is a closed set of $\mu$ measure less than $2\varepsilon$. We will now follow the idea in the Weiss-Katznelson proof of Kingman’s theorem [23], adapting it to the setting with discontinuities. By regularity of the Borel measure, there is an open set $D$ containing $J_n^\varepsilon \cup D_n^\varepsilon$ of measure less than $3\varepsilon$, and by Urysohn’s lemma there is a continuous function $0 \leq h_1 \leq 1$ so that $h_1|_{J_n^\varepsilon \cup D_n^\varepsilon} = 1$ and $h_1|_{\mathcal{D}_n} = 0$. Since $(X, T, \mu)$ is compact uniquely ergodic there exists some $M_1 < \infty$ so that for $M > M_1$ and all $x, |\frac{1}{M} \sum_{i=1}^{M} h(T^i x) - \int h d\mu| < \varepsilon$. For any $x \in X$ construct a sequence $(x_i)$ in $X$ in the following way. For $i = 1$ let $x_1 = x$ and for subsequent terms let $x_{i+1} = T^{n_i} x_i$; where $n_i$ is defined as

$$n_i = n_i(x) = \begin{cases} n & \text{if } x_i \in J_n \cap D_n, \\ 1 & \text{otherwise.} \end{cases}$$

7
We now consider the cocycles for a sufficiently large index. Let \( M > \max\{\frac{n}{\rho}, M_1\} \), and choose \( p \) so that 
\[
n_1 + \cdots + n_{p-1} \leq M < n_1 + \cdots + n_p.
\]
Let \( K = M - (n_1 + \cdots + n_{p-1}) \leq n \). By subadditivity,
\[
g_M(x) \leq \sum_{i=1}^{p-1} g_n(x_i) + g_K(x_p) \leq \sum_{i=1}^{p-1} g_n(x_i) + n\|g_1\|_\infty.
\]
Partition the above sum into \( x_i \in D_n \cap J_n \) and \( x_i \in D_n^c \cap J_n^c \). On the former set use (3.1) and on the latter use the trivial bound \( \|g_1\|_\infty \).
\[
g_M(x) \leq \sum_{i=1}^{p-1} [n_i (\Lambda(f) + 2\epsilon) 1_{J_n \cap D_n}(x_i) + \|g_1\|_\infty \cdot 1_{J_n^c \cup D_n^c}(x_i)] + n\|g_1\|_\infty. \tag{3.2}
\]
Therefore, we have uniformly in \( x \),
\[
\sum_{i=1}^{p-1} \|g_1\|_\infty \cdot 1_{J_n^c \cup D_n^c}(x_i) \leq \sum_{i=1}^{M} \|g_1\|_\infty \cdot 1_{J_n^c \cup D_n^c}(T^i x) \leq \sum_{i=1}^{M} \|g_1\|_\infty h(T^i(x)) < 3\epsilon\|g_1\|_\infty M
\]
Substituting this into the sum on the right hand side of (3.2), we find
\[
\frac{1}{M} g_M(x) \leq \frac{1}{M} \sum_{i=1}^{p-1} n_i (\Lambda(f) + 2\epsilon) 1_{J_n \cap D_n}(x_i) + \frac{1}{M} \sum_{i=1}^{M} \|g_1\|_\infty \cdot 1_{J_n^c \cup D_n^c}(T^i x) < \Lambda(f) + 2\epsilon + 3\epsilon\|g_1\|_\infty + \frac{n}{M} \|g_1\|_\infty
\]
\[
\leq \Lambda(f) + 2\epsilon + 4\epsilon\|g_1\|_\infty.
\]
We now prove Corollary 3.3 for almost continuous matrices.

**Proof** We pursue the usual construction of matrix cocycles albeit with the intent to work within the topology described above, therefore we are more explicit than usual. The cocycle we approximate is the sequence \( f_n(x) = \ln\|M_n(x)\| \). It is approximated by the sequence \( g_n = \ln\|\overline{M}_n\| \). We have for \( x \in D_n \)
\[
\left|\ln\|M_n(x)\| - \ln\|\overline{M}_n(x)\|\right| \leq C_\eta \|M_n(x) - \overline{M}_n(x)\| \leq C_\eta \|M_n(x) - M(x)\|
\]
where \( C_\eta \) only depends on \( \eta \) which is a lower bound of \( \|M_n(x)\| \) for all \( x, n \). Thus for \( \delta, \sigma > 0 \) there exists \( \rho > 0 \) so that \( \|M - M\|_\infty < \rho \) implies \( d_\delta(\{f\}, \{g\}) < \sigma \) and applying Theorem 3.3 we have there exists \( n_\epsilon \) so that for \( n > n_\epsilon \), for any \( x \in X \),
\[
\|\overline{M}_n(x)\| < \exp\{n(\mathcal{L} + \epsilon Q)\}
\]
where \( Q = \max\{1, \ln\|M\|_\infty, \ln\|\overline{M}\|_\infty\} \) and \( \mathcal{L} = \Lambda((f)) \).

We have
\[
\|M_k(\theta) - \overline{M}_k(\theta)\| \leq \sum_{0 \leq \ell \leq k-1} \|\overline{M}_\ell(R^{k-\ell}\theta)(M - M)(R^{k-1-\ell}\theta)M_{k-1-\ell}(\theta)\|
\]

Thus
\[
\|M_k(\theta) - \overline{M}_k(\theta)\| \leq \sup_{0 \leq \ell \leq k-1} \{\|\overline{M}(M - M)(R^{k-1-\ell}\theta)\|\} \sum_{0 \leq \ell \leq k-1} \|\overline{M}_\ell(R^{k-\ell}\theta)\| \|M_{k-1-\ell}(\theta)\|. \tag{3.4}
\]
Let \( k > 2n_\epsilon \). Then we can separate the above sum into \([0, n_\epsilon - 1], [n_\epsilon, k - 1 - n_\epsilon], [k - n_\epsilon, k - 1] \). On the second two intervals \( \ell \geq n_\epsilon \), and on the first two intervals \( k - 1 - \ell \geq n_\epsilon \) so we can apply (3.3) to \( \overline{M}_\ell \) and \( M_{k-1-\ell} \) respectively. That is, using (3.3) for \([k - n_\epsilon, k - 1] \)
\[
\sum_{k-n_\epsilon \leq \ell \leq k-1} \|\overline{M}_\ell(R^{k-\ell}\theta)\| \|M_{k-1-\ell}(\theta)\| \leq \sum_{k-n_\epsilon \leq \ell \leq k-1} \|M\|^{k-1-\ell} \exp\{(k - 1 - \ell)(\mathcal{L} + \epsilon Q)\}
\]
\[
\leq Ce^{n_\epsilon Q} \exp\{(k - 1)(\mathcal{L} + \epsilon Q)\}
\]
Similarly, for \([0, n_\epsilon - 1]\)
\[
\sum_{0 \leq \ell \leq n_\epsilon - 1} \|\overline{M}_\ell(R^{k-\ell}\theta)\| \|M_{k-1-\ell}(\theta)\| \leq \sum_{0 \leq \ell \leq n_\epsilon - 1} \|\overline{M}\|^{\ell} \exp\{(k - 1 - \ell)(\mathcal{L} + \epsilon Q)\}
\]
\[
\leq Ce^{n_\epsilon Q} \exp\{(k - 1)(\mathcal{L} + \epsilon Q)\}
\]
On the center segment \([n, k - 1 - n]\) both cocycles approach the upper Lyapunov limit, so we have
\[
\sum_{n_s \leq \ell \leq k-1-n_s} \| \overline{M}_\ell(R^{k-\ell}\theta) \| \| M_{k-1-\ell}(\theta) \| \leq \sum_{n_s \leq \ell \leq k-1-n_s} \exp\{(k-1)(\mathcal{L} + \epsilon Q)\} \\
\leq (k - 2n_s) \exp\{(k-1)(\mathcal{L} + \epsilon Q)\}
\]
Thus, there is some \(K < \infty\) so that for \(k > K\),
\[
\sum_{0 \leq \ell \leq k-1} \| \overline{M}_\ell(R^{k-\ell}\theta) \| \| M_{k-1-\ell}(\theta) \| < \exp\{k(\mathcal{L} + 2\epsilon Q)\}
\]
which together with (3.4) implies the result.

Finally, an immediate corollary is

**Lemma 3.5** For \(\mathcal{L}\) continuous on a compact set \(K \subset \mathbb{C}\) given \(\epsilon > 0\) there is a \(k_\epsilon < \infty\) so that \(k > k_\epsilon\) implies, for \(z \in K\) and \(\theta \in \mathbb{T}\),
\[
\| A_k^z(\theta) \| \leq e^{k(\mathcal{L}(z) + \epsilon)}.
\]

**Proof** Follows immediately by compactness and Corollary 3.4.

### 4 Proof of the main Lemmas

We will first use Lemma 2.2 to obtain Proposition 2.1.

**Proof** Fix \(f \in \text{PL}_\gamma(\mathbb{T})\), \(\delta \geq 1\) and \(1 > \gamma > 0\) and \(\theta \in \mathbb{T}\). Boundedness of the Lyapunov exponent on compact sets in \(\mathbb{C}\) follows from upper semicontinuity, so we may define
\[
\bar{\chi} = \sup\{\mathcal{L}(z) \in \mathbb{C} : |\Re(z)| \leq K; |\Im(z)| \leq 1\}.
\]

We consider arbitrary irrationals, and make a separate argument for the Diophantine case at the end. If \(\omega\) is Diophantine let \(\xi = 1 + 2\kappa\) where \(\kappa > 0\) is as described in (1.9), otherwise, let \(\xi = 1\). Let \(1 > \tau > 0\) be such that
\[
\frac{\tau}{1 - \tau} < \frac{\gamma \bar{\chi}}{\delta \xi}\chi
\]
and choose \(\sigma\) so that
\[
\frac{\chi \xi}{\delta \xi}(1 - \tau) > \sigma > \tau / \gamma.
\]

Then from Lemma 2.2 for \(k_\tau < k < \frac{1}{\sigma \chi} \ln q_n\) there is some \(0 \leq j \leq q_n + q_{n-1} - 1\) so that for \(|E - z| < e^{-\tau\bar{\chi}k}\)
\[
\| A_k^{jz}(\theta + j\omega) \| \geq \exp\{(1 - \tau)k\mathcal{L}\}.
\]

By definition,
\[
A_k^{jz}(\theta) = A_k^z(\theta + j\omega)A_j^z(\theta)
\]
and, as \(A_k^{jz}\) is an \(\text{SL}_2(\mathbb{R})\)-cocycle, we have
\[
\max_j \left\{ \| A_j^{jz}(\theta) \|, \| A_{j+1}^{j+1z}(\theta) \| \right\} \geq \exp\left\{ \frac{1}{2}(1 - \tau)k\mathcal{L} \right\}
\]
for \(|z - E| < e^{-\tau\bar{\chi}k}\). By (4.1) we can choose \(t\) so that
\[
\frac{\sigma \bar{\chi}}{\zeta} < t < \frac{(1 - \tau)\chi}{\delta \xi}.
\]

Finally, let \(M_k = e^{tk}\). The first inequality in (4.3) guarantees that for sufficiently large \(n\) there exists \(k < \frac{1}{\sigma \chi} \ln q_n\) such that \(M_k^\xi \geq q_n + q_{n-1} - 1 + k\). By (4.1), \(M_k^\xi < e^{-\tau k \bar{\chi}}\) so we have, for \(|z - E| < M_k^{-\xi}\)
\[
\max_{1 \leq j \leq M_k^\xi} \left\| A_j^{jz}(\theta) \right\|^2 \geq e^{(1 - \tau)k \chi} = M_k^{(1 - \tau)k \chi} > M_k^{\delta},
\]
This settles the general case.

For the Diophantine case for sufficiently large $T > 0$, let $k = t^{-1} \log T$. For large $n$, $q_{n+1} < q_n^{1+\kappa}$ so there exists $q_n$ such that $T^{-\kappa}k < 2q_n + k < T^{\kappa}$. Let $M_k$ be chosen so that

$$T^{-\kappa}k < M_k^{-\kappa} < 2q_n + k < M_k^{\kappa} \leq T^{\kappa}.$$  

By construction and \[1.1\], $\delta(1+2\kappa) < (1-\tau)\frac{k}{T}$. It follows that, for any large $T$, and $|z - E| \leq T^{-\kappa}$

$$\max_{1 \leq j \leq T^{\kappa}} \|A_j(\theta, z)\|^2 \geq \max_{1 \leq j \leq M_k^{\kappa}} \|A_j(\theta, z)\|^2 \geq e^{(1-\tau)k\kappa} \geq M_k^{\delta(1+2\kappa)} > T^\delta. \tag{4.5}$$

To complete the proof, it remains to show the transfer matrices grow on comparable lengths in the positive and negative directions. Note that for an ergodic invertible cocycle, the Lyapunov exponent of the forward cocycles equals the Lyapunov exponent of the backward cocycles. Moreover, if $A_k^\omega$ is the cocycle over rotations by $\omega$, then the relation $A_k^{-\omega}(\theta) = A_k^{-\omega}(\theta + \omega)$ holds. Since $\omega$ and $-\omega$ have the same sequence of denominators $q_n$ from the continued fraction approximants, we have that for $k$ large, $M_k$ may be chosen exactly the same for $A_k^\omega$ and $A_k^{-\omega}$.

We will obtain approximating polynomials for the rough potentials using Fejer’s summability kernel

$$K_N(\theta) = \frac{1}{N+1} \left( \frac{\sin \left( \frac{N+1}{2} \theta \right)}{\sin \left( \frac{1}{2} \theta \right)} \right)^2 = \sum_{-N \leq j \leq N} \left( 1 - \frac{j}{N+1} \right) e^{ij\theta}. \tag{4.6}$$

We have

$$\sigma_N(f)(\theta) := K_N \ast f(\theta) = \sum_{-N \leq j \leq N} \left( 1 - \frac{|j|}{N+1} \right) \hat{f}(j) e^{ij\theta},$$

is a $2N + 1$st degree trigonometric polynomial. Moreover, from the general theory, for $f \in L^1(\mathbb{T})$, $\sigma_n(f) \to f$ in $L^1(\mathbb{T})$. The following is another standard result on the pointwise rate of convergence at well behaved points.

The $\gamma$-Lipschitz function space $L^{\gamma}(\mathbb{T})$ is defined as the set functions on $\mathbb{T}$ with the norm

$$\|f\|_{L^{\gamma}} = \|f\|_{\infty} + \sup_{t \in \mathbb{T}, |h| > 0} \frac{|f(t + h) - f(t)|}{|h|^\gamma}.$$

**Lemma 4.1** Suppose $f \in L^{\gamma}(\mathbb{T})$ and for $\theta \in \mathbb{T}$ and $n \in \mathbb{N}$ we have

$$|K_n \ast f(\theta) - f(\theta)| < K\|f\|_{L^{\gamma}} n^{-\gamma}$$

where $K$ does not depend on $n$.

**Proof** Observe $K_n$ has the following property,

$$|K_n(\theta)| \leq \min \left\{ n+1, \frac{\pi^2}{(n+1)^2} \right\}. \tag{4.7}$$

Assume $f$ is $\gamma$-Lipschitz on $\mathbb{T}$ with constant $C$, then using \[4.7\] and $\sigma = \frac{1}{(n+1)^{\gamma}}$

$$|K_n \ast f(\theta) - f(\theta)| = \left| \int_{\mathbb{T}} K_n(\tau) (f(\tau - \theta) - f(\theta)) d\tau \right|$$

$$\leq \int_{[0,\pi]} |K_n(\tau)| 2C\tau^\gamma d\tau$$

$$\leq 2C \int_{[0,\pi]} (n+1)\tau^\gamma d\tau + 2C \int_{[\sigma,\pi]} \frac{\pi}{n+1} \tau^{\gamma-2} d\tau$$

$$\leq CK'_n n^{-\gamma},$$

Here $K$ does not depend on $f$ or $n$, and $C$ is the Lipschitz constant at $\theta$. \[4.8\]
Let $I$ be the set of intervals in $T$. We say $f \in \mathcal{I} \ast L_\gamma(T)$, if for $f_i \in L_\gamma(T)$ and $I_i \in \mathcal{I}$ for $i = 1, \ldots, r$.

$$f(\theta) = \sum_{i=1}^{r} 1_{I_i} f_i(\theta).$$

**Lemma 4.2**

$$\mathcal{I} \ast L_\gamma(T) = PL_\gamma(T)$$

**Proof** One inclusion is clear. Suppose $f \in PL_\gamma(T)$ where $f$ is continuous on $T \setminus J_f$ for $\infty > |J_f| \geq 2$ (if $f$ is continuous everywhere there is nothing to show, if $f$ is discontinuous at only one point $x$ add $x + \pi$ to $J_f$). Let $I_i = (a_i, b_i)$ for $1 \leq i \leq |J_f|$ be largest intervals in $T \setminus J_f$ so that $\bigcup I_i = T \setminus J_f$. The Lipschitz conditions ensure that limits $\lim_{x \to a^+} f(a_i + \epsilon) = f(a_i + 0)$ and $\lim_{x \to b_0^-} f(b_i - \epsilon) = f(b_i - 0)$ exist. Now define $f_i$ to be equal to $f$ on $I_i$ and linearly interpolate the points $(b_i, f(b_i - 0))$ and $(a_i, f(a_i + 0))$ on $I_i^c$, which clearly defines a $\gamma$-Lipschitz function. Then $f = \sum_{i=1}^{r} 1_{I_i} f_i(\theta)$ so $f \in \mathcal{I} \ast L_\gamma(T)$.

We will now show uniform upper bounds for cocycles in a neighborhood of $f \in \mathcal{I} \ast L_\gamma(T)$.

**Lemma 4.3** Suppose $f \in PL_\gamma(T)$, and $E \subseteq \mathbb{C}$ so that $\mathcal{L}(E) > 0$. For any $0 < \tau < \|f\|^{-1}_\infty$ there exists a $\kappa_\tau < \infty$ so that if $q_n > e^{k_\tau \mathcal{L}(E)/\gamma}$ then for any $k \in \mathbb{Z}^+$ such that $\kappa_\tau < k < \tau \mathcal{L}(E)/\log q_n$ and any $\theta \in T$ there is some $0 < x \leq q_n + q_n - 1$ so that for $z \in \mathbb{C}$ with $|z - E| < \exp\{1-\tau k \mathcal{L}(E)\}$ and $g \in B_{\infty}(T)$ with $\|g - f\|_{\infty} < e^{\tau k \mathcal{L}(E)}$, $\|g\|_{\infty} < \tau^{-1}$, we have

$$\|A^{f, z}_{k}(R^g \theta)\| \geq e^{k(1-\tau)\mathcal{L}(E)}.$$ 

**Proof** It is clearly enough to prove the Lemma for $\tau < 1$. To begin we first fix some parameters for the proof. Let

$$\tau/2 > \nu = \tau/4, \quad 1 - \tau/16 > a > b > c > 1 - \tau/8.$$ 

Finally, let $\eta > 0$ be so small that $\eta < \mathcal{L}(E)\tau/16$.

Write $f = f_11_{I_1} + \cdots + f_r1_{I_r}$ for Lipschitz functions $f_i \in L_\gamma(T)$ and intervals $I_i$. There is no loss of generality if we assume $r \geq 2$. Let $J(I_i)$ be the set of discontinuities of $1_{I_i}$; then the set of discontinuities of $f$ is $J_f = J(f) = \bigcup_{i=1}^{r} J(I_i)$. In practice, we will use a simple bound for the supremum norm of $f$

$$\|f\|_{\infty} \leq \|f_1\|_{\infty} + \cdots + \|f_r\|_{\infty} =: M.$$ 

Observe, for $h \in L_{\infty}(T)$, we have $\|K_h \ast h\|_{\infty} \leq \|h\|_{\infty}$. For $f \in \mathcal{I} \ast L_\gamma(T)$, we write $f_N = \sigma_N(f_1)1_{I_1} + \cdots + \sigma_N(f_r)1_{I_r}$, so we have $\|f_N\|_{\infty} \leq M$. It is clear that

$$\|A^{f, z}_{k}(\theta)\| \leq 1 + \|h\|_{\infty} + |E|,$$

so we easily have uniform bounds for the cocycle matrices over bounded energies and uniformly bounded potentials.

Let $c > 0$. There is some $\rho_c > 0$ and $K_c < \infty$ so that for $k > K_c$, and $|z - E| + \|g - f\|_{\infty} < \rho_c$ we have, from Corollary 3.4 with $\eta = 1$,

$$\|A^{f, E}_{k}(\theta) - A^{f, z}_{k}(\theta)\| < \rho e^{k(\mathcal{L}(E)/\gamma)}.$$ \hspace{1cm} (4.8)

where $M = \max\{1, \ln[1 + M + \rho_c + |E|]\}$.

In particular if $g = f_N$, we have by Lemma 4.1

$$\|f_N(R^g \theta) - f(R^g \theta)\|_{\infty} < C_f N^{-\gamma}.$$ \hspace{1cm} (4.9)

Set

$$N = \exp \left\{ \mathcal{L}(E)k \frac{\nu}{\gamma} \right\}.$$ 

Let $\epsilon < \eta/\ln$. Then there is $M_\epsilon < \infty$ so that for $k > M_\epsilon$, we have $C_f N^{-\gamma} < 1/2\rho_c$ and for $|z - E| < C_f N^{-\gamma} \epsilon$ we have

$$\|A^{f, E}_{k}(\theta) - A^{f, z}_{k}(\theta)\| < C_f N^{-\gamma} e^{k(\mathcal{L}(E)/\gamma)}.$$ \hspace{1cm} (4.10)

Let $A^{f, E}_{k}$ be the cocycle matrix defined by the potential determined by the sampling function $f_N$, which is a piecewise polynomial on $|J_f|$ intervals, each supporting a continuous polynomial of order $(2N + 1)$.

For a map $B : T \to SL_2(\mathbb{R})$ and associated cocycle set

$$V_k(t, B) = \left\{ \theta \in T : \frac{1}{k} \ln \|B_k(\theta)\| > t \right\} \subset T.$$ \hspace{1cm} (4.11)
The measure of this set for $B = A^{f,E}$ for large enough $k$ can be bounded below using the fact that $\mathcal{L}(E) = \inf_k \int \frac{1}{k} \ln \|A_k^{f,E}(\theta)\| d\theta$. Indeed, by Corollary 1.2 there is $k_f < \infty$ so that for $k > k_f$ we have for all $\theta$, $\frac{1}{k} \ln \|A_k^{f,E}(\theta)\| < \mathcal{L}(E) + \eta$, thus,

$$
\mathcal{L}(E) \leq \int \frac{1}{k} \ln \|A_k^{f,E}(\theta)\| d\theta \leq |V_k (a\mathcal{L}(E), A^{f,E})| (\mathcal{L}(E) + \eta) + |V_k^c (a\mathcal{L}(E), A^{f,E})| a\mathcal{L}(E) \leq |V_k (a\mathcal{L}(E), A^{f,E})| (|1-a|\mathcal{L}(E) + \eta) + a\mathcal{L}(E).
$$

By the choice of $\eta$ we have $\eta < (1-a)\mathcal{L}(E)$ so for $k > k_f$,

$$
\frac{1}{2} \leq \frac{(1-a)\mathcal{L}(E)}{(1-a)\mathcal{L}(E) + \eta} \leq |V_k (a\mathcal{L}(E), A^{f,E})|.
$$

(4.12)

Furthermore, we make the following claim regarding the sets $V_k(\cdot, \cdot)$ for $k > k_f = \max \{k_f, k_{a,b,c}\}$, and $|E - z| < \exp\{-\mathcal{L}(E)\tau k\}$,

$$
V_k(a\mathcal{L}(E), A^{f,E}) \subset V_k(b\mathcal{L}(E), A^{f,N,E}) \subset V_k(c\mathcal{L}(E), A^{g,\cdot}) \subset V_k(c\mathcal{L}(E), A^{g,\cdot}).
$$

(4.13)

First note from the assumption,

$$
\mathcal{L}(E)(1-\nu) + \eta < \mathcal{L}(E)(1-\tau/4) + \tau\mathcal{L}(E)/16 = \mathcal{L}(E)(1-3\tau/16) < \mathcal{L}(E)c
$$

to show the left inclusion, for $\theta \in V_k (a\mathcal{L}(E), A^{f,E})$ write

$$
\left\|A_k^{f,N,E}(\theta)\right\| \geq \left\|A_k^{f,E}(\theta)\right\| - \left\|A_k^{f,N,E}(\theta) - A_k^{f,E}(\theta)\right\|
$$

from (4.8) and (4.9) we have

$$
\left\|A_k^{f,N,E}(\theta) - A_k^{f,E}(\theta)\right\| \leq C \cdot N^{-\gamma}e^{k(\mathcal{L}(E)+\eta)} \leq Ce^{k(\mathcal{L}(E)+\eta-\mathcal{L}(E)\nu)}
$$

having in the last step also used the definition of $N$. Putting this together, we have, using $\mathcal{L}(E)(1-\nu) + \eta < a\mathcal{L}(E)$,

$$
\left\|A_k^{f,E}(\theta)\right\| > e^{ak\mathcal{L}(E)} - Ce^{k(\mathcal{L}(E)+\eta-\mathcal{L}(E)\nu)} > e^{bk\mathcal{L}(E)}
$$

The right inclusion of (4.13) is similar: for $\theta \in V_k (b\mathcal{L}(E), A^{f,N,E})$

$$
\left\|A_k^{g,\cdot}(\theta)\right\| > \left\|A_k^{f,N,E}(\theta)\right\| - \left\|A_k^{f,N,E}(\theta) - A_k^{f,E}(\theta)\right\| - \left\|A_k^{f,E}(\theta) - A_k^{g,\cdot}(\theta)\right\|
$$

The second term on the right can be bounded as above, the last term on the right is bounded similarly by the assumptions on $g$ and $z$ so

$$
\left\|A_k^{f,E}(\theta) - A_k^{g,\cdot}(\theta)\right\| \leq (\|g - f\|_\infty + |E - z|)e^{k(\mathcal{L}(E)+\eta)} \leq Ce^{k(\mathcal{L}(E)+\eta-\mathcal{L}(E)\nu)}
$$

Altogether, using $b\mathcal{L}(E) > \mathcal{L}(E)(1-\nu) + \eta$ we have,

$$
\left\|A_k^{g,\cdot}(\theta)\right\| > e^{bk\mathcal{L}(E)} - (CN^{-\gamma} + \|g - f\|_\infty + |E - z|)e^{k(\mathcal{L}(E)+\eta)} > e^{bk\mathcal{L}(E)} - Ce^{k(\mathcal{L}(E)+\eta-\mathcal{L}(E)\nu)} > e^{bk\mathcal{L}(E)}
$$

Combining (4.13) and (4.12) yields

$$
\left\|V_k(b\mathcal{L}(E), A^{f,N,E})\right\| \geq \frac{1}{2}
$$

The set

$$
V = \left\{ \theta \in \mathbb{T} : \frac{1}{k} \ln \|A_k^{f,N,E}(\theta)\| > b\mathcal{L}(E) \right\} = V_k(b\mathcal{L}(E), A^{f,N,E})
$$

is defined by a piecewise polynomial function. That is $T$ is partitioned into $k|J|$ intervals and on each interval $\|A_k^{f,N,E}(\theta)\|^2$ is a polynomial of degree $2k(2N + 1)$. At least one interval in the partition must have an intersection with $V$ of size $\frac{1}{k}|J|^{-1}$, and therefore $V$ must contain an interval of length $\frac{1}{k\exp\{\tau\mathcal{L}(E)k\}} \frac{1}{4k(2N + 1)|J|}$, which is bounded below by $\exp\{-\mathcal{L}(E)k\}$. It follows from (4.13) that $V_k(c\mathcal{L}(E), A^{g,\cdot})$ also contains this interval. We will now use the following fact:

**Lemma 4.4** (e.g. [17]) For an interval $I \subset \mathbb{T}$, if $n$ is such that $|I| > \frac{1}{q_n}$ then for any $\theta \in \mathbb{T}$ there is $0 \leq j \leq q_n - 1$ so that $\theta + j\omega \in I$.

For any $k \leq \frac{\gamma \ln q_n}{\mathcal{L}}$ we have that $V_k(c\mathcal{L}(E), A^{g,\cdot})$ contains an interval of length $\frac{1}{q_n}$ so for some $0 \leq x \leq q_n - 1$ we obtain the result. 

12
5 Acknowledgement

S.J. is a 2014-15 Simons Fellow. This research was partially supported by NSF DMS-1101578 and DMS-1401204.

References

[1] Artur Avila. Global theory of one-frequency Schrödinger operators I: stratified analyticity of the Lyapunov exponent and the boundary of nonuniform hyperbolicity. preprint, 2009. arXiv:0905.3902[math.DS].

[2] Artur Avila. Global theory of one-frequency Schrödinger operators II: Acriticality and finiteness of phase transitions for typical potentials. preprint, 2011.

[3] Jean-Marie Barbaroux, F. Germinet, and Serguei Tcheremchantsev. Fractal dimensions and the phenomenon of intermittency in quantum dynamics. Duke, 1:161 –193, 2001.

[4] Kristian Bjerklov. Dynamical properties of quasi-periodic Schrödinger equations. PhD thesis, KTH Matematik, 2003.

[5] Jean Bourgain. Green’s function estimates for lattice Schrödinger operators and applications. Princeton Univ. Press, Princeton, 2005.

[6] Jean Bourgain and Svetlana Jitomirskaya. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. Journal of statistical physics, 108:1203 – 1218, 2001.

[7] Jackson Chan. Method of variations of potential of quasi-periodic Schrödinger equations. Geometric and Functional Analysis, 17(5):1416 – 1478, 2008.

[8] Jackson Chan, Micheal Goldstein, and Wilhem Schlag. On non-perturbative anderson localization for $C^\alpha$ potentials generated by shifts and skew-shifts. ArXiv:0607302, 1–39, 2006.

[9] David Damanik and Rowan Killip. Ergodic potentials with a discontinuous sampling function are non-deterministic. Math. Res. Letters, 21:191 – 204, 1999.

[10] David Damanik and Serguei Tcheremchantsev. Upper bound in quantum transport. Journal of the American Mathematical Society, 20(3):700 – 827, 2007.

[11] David Damanik and Serguei Tcheremchantsev. Quantum dynamics via complex analysis methods: General upper bounds without time-averaging and tight lower bounds for the strongly coupled Fibonacci Hamiltonian. Journal of functional analysis, 255(10):2872 – 2887, 2008.

[12] R. del Rio, Svetlana Jitomirskaya, Yoram Last, and Barry Simon. Operators with singular continuous spectrum. J. d’Analyse Math, 69:153–200, 1996.

[13] Alex Furman. On the multiplicative ergodic theorem for uniquely ergodic systems. Probabilities et Statistiques, 33(6):797– 815, 1997.

[14] S. Ganesan, K. Kechedzhi, and S. Das Sarma. Critical integer quantum Hall topology and the integrable Maryland model as a topological quantum critical point. Phys. Rev. B, 90:041405, 2014.

[15] Svetlana Jitomirskaya. Ergodic Schrödinger operators (on one foot). In Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon’s 60th birthday, pages 613 – 647. Amer. Math. Soc., 2007.

[16] Svetlana Jitomirskaya and Ilya Kachkovskiy. Anderson localization for discrete 1d quaziperiodic operators with piecewise monotonic sampling functions. preprint, 2014.

[17] Svetlana Jitomirskaya and Yoram Last. Power law subordinacy and singular spectra. II. Line operators. Communications in Mathematical Physics, 211:643 – 658, 2000.

[18] Svetlana Jitomirskaya and Chris Marx. Analytic quasi-periodic cocycles with singularities and the Lyapunov exponent of extended Harper’s model. Communications in mathematical physics, 2011. to appear.

[19] Svetlana Jitomirskaya and Rajinder Mavi. Continuity of spectral measure for quasiperiodic Schrödinger operators with rough potentials. Comm. Math. Phys., 325:585–601, 2014.

[20] Svetlana Jitomirskaya and Herman Schulz-Baldes. Upper bounds on wavepacket spreading for random Jacobi matrices. Comm. Math. Phys., 273:601–618, 2007.
[21] Svetlana Jitomirskaya and Shiwen Zhang. Lower spectral dimensional and dynamical bounds for quasiperiodic operators with Liouville frequencies. *preprint*, 2014.

[22] Yitzhak Katznelson. *Harmonic Analysis*. publisher, 2002.

[23] Yitzhak Katznelson and Benjamin Weiss. A simple proof of some ergodic theorems. *Israel Journal of Mathematics*, 42:291–296, 1982.

[24] Rowan Killip, Alexander Kiselev, and Yoram Last. Dynamical upper bounds on wavepacket spreading. *American Journal of Mathematics*, 125(5):1165–1198, 2003.

[25] Silvius Klein. Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey class function. *Journal of Functional Analysis*, 218:255–292, 2005.

[26] Yoram Last. Quantum dynamics and decompositions of singular continuous spectra. *Journal of Functional Analysis*, 142(2):406 – 445, 1996.

[27] V.A. Mandel’shtam and S. Ya. Zhitomirskaya. 1d-quasiperiodic operators. Latent symmetries. *Commun. Math. Phys.*, 39:589–604, 1991.

[28] Barry Simon. Equilibrium measures and capacities in spectral theory. *Inverse Problems and Imaging*, 1(4):713 – 772, 2007.

[29] Eugene Sorets and Thomas Spencer. Discrete one-dimensional quasi-periodic Schrödinger operators with pure point spectrum. *Communications in Mathematical Physics*, 142(3):543 – 566, 1991.

[30] Herman Schulz-Baldes Svetlana Jitomirskaya and Günter Stolz. Delocalization in random polymer models. *Comm. Math. Phys.*, 233:27–48, 2003.

[31] Mor Verbin, Oded Zilberberg, Yaacov E. Kraus, Yoav Lahini, and Yaron Silberberg. Observation of topological phase transitions in photonic quasicrystals. *Phys. Rev. Lett.*, 110:076403, 2013.

[32] Peter Walters. *An introduction to Ergodic Theory*. Springer, 1982.

[33] Y. Wang and J. You. Examples of discontinuity of Lyapunov exponent in smooth quasi-periodic cocycles. *Duke, to appear*, 2014.

[34] Y. Wang and Z. Zhang. Cantor spectrum for a class of $C^2$ quasiperiodic Schrödinger operators. *Preprint*, 2014.

[35] Y. Wang and Z. Zhang. Uniform positivity and continuity of Lyapunov exponents for a class of $C^2$ quasiperiodic Schrödinger cocycles. *JFA, to appear*, 2014.

[36] Zhenghe Zang. Positive Lyapunov exponents for quasiperiodic Szegö cocycles. *Nonlinearity*, 25:1771, 2012.