THE VIRTUAL FLYPING THEOREM

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Abstract. We extend the flyping theorem to alternating links in thickened surfaces and alternating virtual links. The proof of the former result uses work of Boden–Karimi to adapt the author’s geometric proof of Tait’s 1898 flyping conjecture (first proved in 1993 by Menasco–Thistlethwaite), while the proof of the latter involves a diagrammatic correspondence recently introduced by the author in a related paper. In the process, we also extend a classical result of Gordon–Litherland, establishing an isomorphism between their pairing on a spanning surface and the intersection form on a 4-manifold constructed as a double-branched cover using that surface.

1. Introduction

P.G. Tait asserted in 1898 that all reduced alternating diagrams of a given prime nonsplit link in $S^3$ minimize crossings, have equal writhe, and are related by flype moves (see Figure 1) [Ta1898]. The first proofs came almost a century later, and all involved the Jones polynomial [Ka87, Mu87, Mu87ii, Th87, MT91, MT93]. In 2017, Greene gave the first purely geometric proof of part of the classical Tait conjectures [Gr17], and in 2020, the author gave the first purely geometric proof of Tait’s flyping conjecture [Ki21].

Recently, Boden, Chrisman, Karimi, and Sikora extended much of this to alternating links in thickened surfaces. First, using generalizations of the Kauffman bracket, Boden–Karimi–Sikora proved that Tait’s first two conjectures hold for alternating links in thickened surfaces [BK18, BKS19]. Second, Boden–Chrisman–Karimi extended the Gordon–Litherland pairing to spanning surfaces in thickened surfaces [BCK21]. Third, Boden–Karimi applied this pairing to extend Greene’s characterization of classical alternating links to links $L$ in thickened surfaces $\Sigma \times I$, proving that $L$ bounds connected definite surfaces of opposite signs if and only if $L$ is alternating and $(\Sigma \times I, L)$ is nonstabilized [BK22].

1 Boden–Karimi proved Tait’s first two conjectures for alternating links in thickened surfaces, with a few extra conditions [BK18], and with Sikora they extended those results to adequate links and removed the extra conditions [BKS19].

2 See §2.1 for definitions of stabilized, prime, locally prime, cellular, end-essential, definite, and removably nugatory.
The first main result of this paper combines and adapts several of these recent developments to prove that the flyping theorem extends to alternating links in (nonstabilized) thickened surfaces.

**Theorem 3.5.** Let $D \subset \Sigma$ be a locally prime, cellular alternating diagram of a link $L$ in a thickened surface $\Sigma \times I$. Then any other such diagram of $L$ is related to $D$ by flypes on $\Sigma$.

The approach is parallel to that in [Ki21], and indeed most of the arguments translate directly. For some, which we mark with the symbol $\gamma$, the statements and proofs hold without further comment. Appendix A lists pertinent cross-referencing information for these and other results marked with the symbol $\gamma$. The upshot is a geometric proof of Theorem 3.5 and other generalized Tait conjectures:

**Theorem 3.3** (Part of Tait’s extended first conjecture [BK18, BKS19]). If $D, D' \subset \Sigma$ are alternating diagrams of a link $L \subset \Sigma \times I$, neither containing removable nugatory crossings, then $D$ and $D'$ have the same number of crossings.

**Theorem 3.6** (Tait’s extended second conjecture [BK18, BKS19]). All locally prime, cellular alternating diagrams of a given link $L \subset \Sigma \times I$ have the same writhe.

Section 4 extends Theorems 3.3, 3.5, and 3.6 to virtual links:

**Theorem 4.11.** Any two locally prime, alternating virtual diagrams of a given virtual link $K$ are related by non-classical $R$-moves and (classical) flypes.

To prove this, we establish a new diagrammatic analog to the correspondence established by Kauffman, Kamada–Kamada, and Carter–Kamada–Saito between virtual links, equivalence classes of abstract links, and stable equivalence classes of links in thickened surfaces [Ka98, KK00, CKS02]; also see [Ku03]. Given a virtual link diagram

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3 A virtual link diagram is alternating if its classical crossings alternate between over and under.

4 A (classical) flype on a virtual link diagram appears as in Figure 1 where $T_1$ contains no virtual crossings.
V, let [V] denote its equivalence class under virtual (non-classical) R-moves. We show that such classes [V] correspond bijectively to abstract link diagrams and thus to cellularly embedded link diagrams on closed surfaces.

As corollaries, we extend Theorems 3.3 and 3.6 to virtual diagrams, and we observe that connect sum is not a well-defined operation on virtual knots or links:

**Theorem 4.12.** All locally prime, alternating diagrams of a given virtual link have the same crossing number and writhe.

**Corollary 4.13.** Given any two non-classical, locally prime, alternating virtual links V_1 and V_2, there are infinitely many distinct virtual links that decompose as a connect sum of V_1 and V_2.

Before all this, in §2, we introduce the required background regarding links in thickened surfaces. Some of this reviews the existing literature, and some of it is new.

2. Links and spanning surfaces in thickened surfaces

**Convention 2.1.** Throughout, Σ is a connected, closed, orientable surface with genus \( g(Σ) > 0 \). We denote the intervals \([-1, 1]\) and \([0, 1]\) by \( I \) and \( I_+ \), respectively. In \( Σ × I \), we identify \( Σ \) with \( Σ × \{0\} \) and denote \( Σ × \{±1\} = Σ_± \). For a pair \((Σ, L)\) or \((Σ × I, L)\), \( L \) is a link in \( Σ × I \), and for a pair \((Σ, D)\), \( D \) is a link diagram on \( Σ \).

2.1. Alternating links in thickened surfaces. A pair \((Σ, L)\) is stabilized if, for some circle \( γ \subset Σ \), \( L \) can be isotoped so that it intersects each component of \((Σ × I) \setminus (γ × I)\) but not the annulus \( γ × I \); one can then destabilize the pair \((Σ, L)\) by cutting \( Σ × I \) along \( γ × I \) and attaching two 3-dimensional 2-handles in the natural way (this may disconnect \( Σ \)); the reverse operation is called stabilization. Equivalently, \((Σ, L)\) is nonstabilized if every diagram \( D \) of \( L \) on \( Σ \) is cellularly embedded, meaning that \( D \) cuts \( Σ \) into disks.

A pair \((Σ, L)\) is split if \( L \) has a disconnected diagram on \( Σ \). Note that if \((Σ, L)\) is split then it is also stabilized (as we assume that \( Σ \) is connected). The converse is false. In fact, the number of split components is an invariant of stable equivalence classes.

Kuperberg’s Theorem states that the stable equivalence class of \((Σ, L)\) contains a unique nonstabilized representative; this implies that when \((Σ, L)\) is nonsplit, \((Σ, L)\) is nonstabilized if and only if \( Σ \) has minimal genus in this stable equivalence class.

**Theorem 2.2** (Theorem 1 of [Ku03]). If \((Σ, L)\) and \((Σ' × I, L')\) are stably equivalent and nonstabilized, then there is a pairwise homeomorphism \((Σ × I, L) \to (Σ' × I, L')\).
If $L$ is nonsplit and $g(\Sigma) > 0$, then $(\Sigma \times I) \setminus L$ is irreducible, as
$\Sigma \times I$ is always irreducible, since its universal cover is $\mathbb{R}^2 \times \mathbb{R}$. The
converse of this, too, is false due to the next observation, which follows from a standard
innermost circle argument:

**Observation 2.3.** If $(\Sigma_i \times I) \setminus L_i$ is irreducible for $i = 1, 2$ and $\Sigma = \Sigma_1 \# \Sigma_2$ with
$L = L_1 \cup L_2 \subset \Sigma \times I$, where the annulus $A = \gamma \times I$ separates $L_1$ from $L_2$ in $\Sigma \times I$, then $(\Sigma \times I) \setminus L$ is irreducible.

We call $(\Sigma, D)$ **cellular** if $D$ cuts $\Sigma$ into disks. Note:

**Fact 2.4** ([Oz06] [BK22], Proposition 5.1 of [Ki22]). Suppose $D \subset \Sigma$ is a cellular alternating diagram of a link $L \subset \Sigma \times I$. Then $(\Sigma, D)$ is checkerboard colorable, and $L$ is nullhomologous over $\mathbb{Z}/2$.

We will use this result of Boden–Karimi and the generalization that follows:

**Fact 2.5** (Corollary 3.6 of [BK22]). If $(\Sigma, L)$ has a cellular alternating diagram, then $(\Sigma, L)$ is nonsplit and nonstabilized.

**Corollary 2.6** (Corollary 2.4 of [Ki22]). Suppose $(\Sigma, L)$ has an alternating diagram $D \subset \Sigma$. Then $(\Sigma, L)$ is nonsplit if and only if $D$ is connected, and $(\Sigma, L)$ is nonstabilized if and only if $D$ is cellular.

Following [Ki22], we call a checkerboard colorable pair $(\Sigma, D)$ **pairwise prime** if any pairwise connect sum decomposition $(\Sigma, D) = (\Sigma_1, D_1) \# (\Sigma_2, D_2)$ has $(\Sigma_i, D_i) = (S^2, \emptyset)$ for each $i = 1, 2$. Likewise, given $(\Sigma, L)$ where $L$ is nullhomologous over $\mathbb{Z}/2$, we call $(\Sigma, L)$ **pairwise prime** if every annular connect sum decomposition $(\Sigma, L) = (\Sigma_1, L_1) \# (\Sigma_2, L_2)$ is trivial: $(\Sigma_i, L_i) = (S^2, \emptyset)$ for either $i = 1, 2$. Thus, such $(\Sigma, L)$ is pairwise prime if and only if, whenever $\gamma \subset \Sigma$ is a separating curve and $L$ is isotoped to intersect the annulus $\gamma \times I$ in two points, $\gamma$ bounds a disk $X \subset \Sigma$ such that $L$ intersects $X \times I$ in a single unknotted arc.

Howie-Purcell call $(\Sigma, D)$ **weakly prime** if, for every pairwise connect sum decomposition $(\Sigma, D) = (\Sigma, D_1) \# (S^2, D_2)$, either $D_2 = \emptyset$ is the trivial diagram of the unknot or $(\Sigma, D_1) = (S^2, \emptyset)$ [HP20]; following [Ki22], we call such $D$ **locally prime**. We call $(\Sigma, L)$ **locally prime** if, for every pairwise connect sum decomposition $(\Sigma, L) = (\Sigma_1, L_1, L_2) \# (\Sigma_2, L_2)$, either $L_2 = \emptyset$ is the trivial diagram of the unknot or $(\Sigma_1, L_1) = (S^2, \emptyset)$ [HP20]; following [Ki22], we call such $\Sigma$ **locally prime**.

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7 For more detail, see Proposition 12 of [BK22]; the proof cites [CSW14].
8 If $(\Sigma_i \times I, L_i)$ is nonsplit (implying that $\Sigma_i \times I \setminus L_i$ is irreducible) for $i = 1, 2$, then choose disks $X_i \subset \Sigma_i$ with $(X_i \times I) \cap L_i = \emptyset$ and construct the connect sum $\Sigma = (\Sigma_1 \setminus \text{int}(X_1)) \cup (\Sigma_2 \setminus \text{int}(X_2)) = \Sigma_1 \# \Sigma_2$. Let $L = L_1 \cup L_2 \subset \Sigma \times I$. Then $(\Sigma, L)$ is split. Yet, $(\Sigma \times I) \setminus L$ is irreducible by Observation 2.3.
9 Annular connect sum $(\Sigma_1, L_1) \# (\Sigma_2, L_2) = (\Sigma, L)$ is a connect sum of surfaces, $\Sigma_1 \# \Sigma_2 = \Sigma$ thickened up, which restricts to a connect sum of 1-manifolds $L_1 \# L_2 = L$. See [Ka08] [Ma12] [Ki22].
10 The definitions of pairwise primeness are more complicated without the assumptions related to checkerboard colorability; see [Ki22].
(Σ, L₁)#(S², L₂), either L₂ = ∅ is the unknot or (Σ, L₁) = (S², ∅).

As in the classical case [Me84], certain diagrammatic conditions constrain an alternating link L as one might wish:

**Theorem 2.7** ([Oz06, BK22, Aetal19, Ki22]). If D ⊂ Σ is a cellular alternating diagram of a link L ⊂ Σ × I, then (i) L is nonsplit, so in particular, (Σ × I) \ L is irreducible if g(Σ) > 0; (ii) if (Σ, D) is locally prime, then (Σ, L) is locally prime; and (iii) if (Σ, D) is pairwise prime, then (Σ, L) is pairwise prime.

Part (i) was proven by Ozawa in [Oz06] and by Boden-Karimi in [BK22]. Part (ii) was proven by Adams et al in [Aetal19] and generalized by Howie-Purcell in [HP20]. Part (iii) is one of the main results of [Ki22], which also gives new proofs of (i)-(ii).

2.1.1. **End-essential spanning surfaces.** Part (i) of Theorem 2.7 implies that L has spanning surfaces: embedded, unoriented, compact surfaces F ⊂ Σ × I with ∂F = L; while we do not require F to be connected, we do require that each component of F has nonempty boundary. By deleting the interior of a regular neighborhood of L from F and Σ × I, one may instead view F as a properly embedded surface in the link exterior (Σ × I) \ νL. We take this view throughout, except in Definition 2.8, Note 20, and § 2.3.1.

If (Σ, D) is a cellular alternating diagram of (Σ, L), then it is possible to orient each disk of Σ \ D so that, under the resulting boundary orientation, over- and under-strands are oriented respectively toward and away from crossings. Since Σ is orientable, these orientations determine a checkerboard coloring of Σ \ D, i.e. a way of shading the disks of Σ \ D black and white so that regions of the same shade abut only at crossings. One can use this checkerboard coloring to construct checkerboard surfaces B and W for L, where B projects into the black regions, W projects into the white, and B and W intersect in vertical arcs which project to the crossings of D. The main result of [Ki23] is that these checkerboard surfaces satisfy several convenient properties:

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11A third notion of primeness for D on Σ also appears in the literature: Ozawa calls (Σ, D) strongly prime if every circle on Σ (not necessarily separating) that intersects D in two generic points also bounds a disk in Σ which contains no crossings of D [Oz06].

12Throughout, given a manifold X and a submanifold Y ⊂ X, νY denotes a closed regular neighborhood of Y in X.

13We also assume that ∂F is transverse on ∂νL to each meridian, where a meridian is the preimage of a point in L under the bundle map νL → L.

14For compact X, Y ⊂ Σ × I, X \ Y denotes the metric closure of X \ Y; see Note 7 of [Ki21] for a precise definition.

15Interestingly, cellular alternating link diagrams on nonorientable surfaces are never checkerboard colorable.
**Definition 2.8.** Let \( F \subset \Sigma \times I \) be a spanning surface for \((\Sigma, L)\). Denote \( M_F = (\Sigma \times I) \setminus F \), and use the natural map \( h_F : M_F \to \Sigma \times I \) to denote \( h_F^{-1}(L) = \tilde{L}, h_F^{-1}(\Sigma_\pm) = \tilde{\Sigma}_\pm \), and \( h_F^{-1}(F) = \tilde{F} \), so that \( h_F : \tilde{L} \to L \) and \( h_F : \tilde{\Sigma}_\pm \to \Sigma_\pm \) are homeomorphisms and \( h_F : \tilde{F} \setminus \tilde{L} \to \text{int}(F) \) is a 2:1 covering map. Then we say that \( F \) is:

(a) **incompressible** if any circle \( \gamma \subset \tilde{F} \setminus \tilde{L} \) that bounds a disk in \( M_F \) also bounds a disk in \( \tilde{F} \setminus \tilde{L} \).

(b) **end-incompressible** if any circle \( \gamma \subset \tilde{F} \setminus \tilde{L} \) that is parallel in \( M_F \) to \( \tilde{\Sigma}_\pm \) bounds a disk in \( \tilde{F} \setminus \tilde{L} \).

(c) \( \partial \)-**incompressible** if, for any circle \( \gamma \subset \tilde{F} \) with \( |\gamma \cap \tilde{L}| = 1 \) that bounds a disk in \( M_F \), \( \gamma \setminus \tilde{L} \) is parallel in \( \tilde{F} \setminus \tilde{L} \) into \( \tilde{L} \).

(d) **essential** if \( F \) satisfies (a) and (c).

(e) **end-essential** if \( F \) satisfies (b) and (c).

A crossing \( c \) of a diagram \( D \subset \Sigma \) is **removably nugatory** if there is a disk \( X \subset \Sigma \) such that \( \partial X \cap D = \{c\} \); in that case, one can remove \( c \) from \( D \) via a flype and a Reidemeister-1 move. No locally prime cellular diagram has removable nugatory crossings. Also, any diagram \((\Sigma, D)\) with a removable nugatory crossing, has at least one \( \partial \)-compressible checkerboard surface. Conversely:

**Theorem 2.9** (Theorem 1.1 of [Ki23]). If \( D \subset \Sigma \) is a cellular alternating diagram without removable nugatory crossings, then both checkerboard surfaces from \( D \) are end-essential.

**Proposition 2.10.** Suppose \( F_\pm \) are definite surfaces of opposite signs spanning a link \( L \subset \Sigma \times I \) and \( F_+ \cap F_- \) consists only of arcs, none of which are \( \partial \)-parallel in both \( F_+ \) and \( F_- \). If \( F_- \) (resp. \( F_+ \)) is \( \partial \)-incompressible, then no arc of \( F_+ \cap F_- \) is \( \partial \)-parallel in \( F_+ \) (resp. \( F_- \)).

**Proposition 2.11.** If an essential surface \( F \) spanning \((\Sigma, L)\) contains an arc \( \beta \) which is parallel in \((\Sigma \times I) \setminus (F \cup \nu L)\) to an arc \( \alpha \subset \partial \nu L \setminus \partial F \), then \( \alpha \) is parallel in \( \partial \nu L \) to \( \partial F \).

**Observation 2.12.** Suppose \( B, W \) are the checkerboard surfaces of a cellular alternating diagram \( D \subset \Sigma \) of a link \( L \subset \Sigma \times I \). Any properly embedded arc in \( W \) that is disjoint from \( B \) and separating in \( W \) is either \( \partial \)-parallel in \( W \) or isotopic in \( W \) to a vertical arc of \( B \cap W \). Likewise with \( B \) and \( W \) reversed.

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\(^{16}\)F is incompressible if and only if \( F \) is \( \pi_1 \)-injective, meaning that inclusion \( \text{int}(F) \hookrightarrow (\Sigma \times I) \setminus L \) induces an injection of fundamental groups (for all possible choices of basepoint).

\(^{17}\)Note that any end-essential surface is essential. Observe moreover that the converse is true when \( \Sigma \) is a 2-sphere.
Figure 2. Left: an *entire flype* of a diagram of the knot $8_{17}$. Right: Corollary 3.7 will imply that these links are non-isotopic; see Example 3.8.

**Remark 2.13.** Observation 2.12 implies in particular that no vertical arc from a locally prime, cellular alternating diagram is $\partial$-parallel in either checkerboard surface.

2.1.2. Flype-related diagrams.

**Definition 2.14.** If $D \subset \Sigma$ is a link diagram and $\gamma \subset \Sigma$ is an inessential circle that intersects $D$ transversally in three points, exactly one of them a crossing point, $c$, then we call the circle $\gamma$ a *flyping circle* for $D$. Up to mirror symmetry, $D$ and $\gamma$ appear as shown far left in Figure 1 ($D$ intersects the disk component of $\Sigma \setminus \nu \gamma$ in a tangle $T_2$ and intersects the other component in a “higher-genus tangle” $T_1$), so one can flype $D$ along $\gamma$ as shown: this move fixes $T_1$, switches which pair of strands cross within $\nu \gamma$, and changes $T_2$ by reflecting the underlying projection and reversing all crossing information.

**Observation 2.15.** If $D \rightarrow D'$ is a flype, then $D$ and $D'$ represent the same link $L$ and have the same number of crossings. If $D$ is oriented then $D$ and $D'$ have the same writhe. If $D$ is cellular alternating (resp. locally prime), then so is $D'$.

**Remark 2.16.** In the classical setting, the tangle $T_1$ in Figure 1 might contain no crossings, in which case the flype has the effect of changing $D$ to its mirror image and then reversing all crossings; one may think of this move as leaving $D$ unchanged and viewing it from the opposite side of $\Sigma$ (in [Ki21], we call such a flype an *entire flype*). By contrast (by an euler characteristic argument), no cellular checkerboard colorable diagram on a surface of positive genus does. Thus, while, as in [Ki21], we regard two diagrams $D, D' \subset \Sigma$ as equivalent iff they are related by planar isotopy and possibly an entire flype, the latter possibility will be vacuous.

2.2. Definite surfaces.

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18The *writhe* of $D$ is $w_D = |X| - |\overline{X}|$. 
2.2.1. Linking numbers and slopes. We adopt the notion of generalized linking numbers which was first defined for arbitrary 3-manifolds with nonempty boundary in [CT07] and applied in the context of thickened surfaces in [BCK21, BK22]. The generalized linking number of disjoint multicurves \(\alpha, \beta \subset \Sigma \times I\) is
\[
\text{lk}_\Sigma(\alpha, \beta) = |\alpha \cdot \Sigma| - |\beta \cdot \Sigma|.
\]
This linking pairing, taken relative to \(\Sigma_+\), is asymmetric: denoting intersection number on \(\Sigma\) by \(\cdot\) and projection \(p_\Sigma : \Sigma \times I \to \Sigma\),
\[
\text{lk}_\Sigma(\alpha, \beta) - \text{lk}_\Sigma(\beta, \alpha) = p_\Sigma(\alpha) \cdot_\Sigma p_\Sigma(\beta).
\]
If \(F\) spans a link \(L = \bigsqcup L_i \subset \Sigma \times I\) and each \(\tilde{L}_i\) is a co-oriented pushoff of \(L_i\) in \(F\), then we call \(s(F) = \sum_i \text{lk}(L_i, \tilde{L}_i)\) the slope of \(F\).

2.2.2. The Gordon–Litherland pairing. Given a surface \(F\) spanning a link \(L \subset \Sigma \times I\), take \(\nu F\) in the link exterior \((\Sigma \times I) \setminus \nu L\) with projection \(p : \nu F \to F\), such that \(p^{-1}(\partial F) = \nu F \cap \partial \nu L\), and denote the frontier \(\tilde{F} = \partial \nu F \setminus \partial \nu L\) and transfer map \(\tau : H_1(F) \to H_1(\tilde{F})\) (see Figure 3). Following Boden–Chrisman–Karimi, the (generalized) Gordon–Litherland pairing (relative to \(\Sigma_+\)) is the symmetric bilinear mapping \(\langle \cdot, \cdot \rangle_F : H_1(F) \times H_1(F) \to \mathbb{Z}\) given by [GL78, BCK21]:
\[
\langle a, b \rangle_F = \frac{1}{2} (\text{lk}_\Sigma(\tau a, b) + \text{lk}_\Sigma(\tau b, a)).
\]
Given a multicurve \(\gamma \subset F\) representing \(g \in H_1(F)\), we denote \(\langle g, g \rangle_F = |g|_F\) and call \(\frac{1}{2} |g|_F\) the framing of \(\gamma\) in \(F\). Given a basis \(B = (a_1, \ldots, a_n)\) for \(H_1(F)\), the Goeritz matrix \(G = (x_{ij}) \in \mathbb{Z}^{n \times n}\), \(x_{ij} = \langle a_i, a_j \rangle_F\), represents \(\langle \cdot, \cdot \rangle_F\) with respect to \(B\). Denoting the signature of \(G\) by \(\sigma(F)\), the quantity
\[
(2.2) \quad \sigma_F(L) = \sigma(F) - \frac{1}{2} s(F),
\]
\(^{19}\)We call a disjoint union of embedded, oriented circles a multicurve.
depends only on the $S^*$ equivalence class of $F$; whenever $(\Sigma, L)$ is non-split with diagram $(\Sigma, D)$ there are exactly two such classes, each represented by a checkerboard surface of $D$ \cite{BCK21}.

2.2.3. **Definiteness characterizes alternating links.** A spanning surface $F$ is positive- (resp. negative-) definite if $\langle \alpha, \alpha \rangle_F > 0$ (resp. $\langle \alpha, \alpha \rangle_F < 0$) for all nonzero $\alpha \in H_1(F)$ \cite{Gr17}.

Adapting work of Greene from the classical setting \cite{Gr17}, Boden–Karimi characterized nonstabilized alternating links in (and diagrams on) thickened surfaces in terms of definite surfaces:

**Fact 2.17** (Proposition 3.8 of \cite{BK22}). A cellular checkerboard colorable link diagram $D \subset \Sigma$ is alternating if and only if its checkerboard surfaces are definite and of opposite signs.

**Theorem 2.18** (Theorem 4.8 of \cite{BK22}). Suppose $(\Sigma, L)$ is non-stabilized. Then $L$ is alternating if and only if it has connected spanning surfaces of opposite signs.

The proof in \cite{BK22} of Theorem 2.18 shows moreover that if $L$ has connected spanning surfaces of opposite signs, then there is a closed surface $S$ in $\Sigma \times I$ on which $L$ has a cellular alternating diagram whose checkerboard surfaces are isotopic to the given surfaces; further, if $(L, \Sigma)$ is non-stabilized, then $S$ is isotopic to $\Sigma$. Formally:

**Corollary 2.19.** If $(\Sigma, L)$ is nonstabilized and $B$ and $W$ are connected spanning surfaces of opposite signs spanning $L$, then $L$ has a cellular alternating diagram on $\Sigma$ whose checkerboard surfaces are isotopic to $B$ and $W$.

**Convention 2.20.** The checkerboard surfaces $B$ and $W$ of any cellular alternating diagram are labeled such that $B$ is positive-definite and $W$ is negative-definite. Likewise for checkerboard surfaces $B'$ and $W'$ (resp. $B_i$ and $W_i$) from such a diagram $D'$ (resp. $D_i$).

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20 $S^*$ equivalence is generated by attaching and deleting tubes and crosscaps \cite{GL78} and thus respects relative homology classes. The checkerboard surfaces $F$ and $F'$ of $D$ satisfy $[F] + [F'] = [\Sigma]$ in $H_2(\Sigma \times I, L; \mathbb{Z}/2)$, so $[F] \neq [F']$; hence, $F$ and $F'$ are not $S^*$ equivalent. For the converse, following the classical approach of Yasuhara \cite{Ya14}, put an arbitrary spanning surface in disk-band form, attach tubes to make it a checkerboard surface for some diagram, and then perform Reidemester moves (requiring more tubing and crosscapping moves).

$21 F$ is positive-definite iff $\sigma(F) = \beta_1(F)$ or equivalently iff each multicurve in $F$ either has positive framing in $F$ or bounds an orientable subsurface of $F$.

22 When $|\partial F| \leq 2$, every primitive $g \in H_1(F)$ is represented by an oriented circle, but this is not true in general: e.g. take $F$ to be an oriented pair of pants and $g$ the sum of two boundary components, one with the boundary orientation.

23 Recall that this implies that $L \subset \Sigma \times I$ is a non-split link.

24 Spanning surfaces are assumed to be connected throughout \cite{BK22}. 
Lemma 2.21 (c.f. [BK22] Lemma 3.7). The checkerboard surfaces $B$ and $W$ of any cellular alternating diagram of a link $(\Sigma, L)$ satisfy

$$\sigma_B(L) - \sigma_W(L) = 2g(\Sigma).$$

Moreover, much of Boden–Karimi’s proof of Theorem 2.18 goes through even if the spanning surfaces of opposite signs for $L$ are disconnected or if $(\Sigma, L)$ is stabilized, or both. In particular, if $L$ has spanning surfaces (not necessarily connected) of opposite signs, then there is a closed surface $S$ (not necessarily connected) in $\Sigma \times I$ on which $L$ has a cellular alternating diagram $D$ whose checkerboard surfaces are isotopic to the given surfaces; further, each component of $S$ either is parallel to $\Sigma$ or is a 2-sphere. In particular:

Fact 2.22. If $F_{\pm}$ are definite surfaces of opposite signs spanning a link $L \subset \Sigma \times I$, then for some (possibly empty) disjoint union of 2-spheres $\Sigma' \subset (\Sigma \times I) \setminus \Sigma$, $L$ has a cellular alternating diagram $D \subset \Sigma \cup \Sigma'$ whose checkerboard surfaces are isotopic to $F_{\pm}$. Thus:

(A) $F_+$ and $F_-$ have the same number of connected components, and this equals the number of split components of $L$.

(B) $L$ has at most one non-local component.

2.2.4. Intersections between definite surfaces. Let $F$ and $F'$ be spanning surfaces for $(\Sigma, L)$ with $F \cap F'$. Orient $L$ arbitrarily, and orient $\partial F$ and $\partial F'$ so that each is homologous in $\nu L$ to $L$. Given an arc $\alpha$ of $F \cap F'$, take $\nu \alpha$ in $\partial \nu L$. Following Howie [Ho18], we call $\alpha$ standard if $i(\partial F, \partial F')_{\nu \partial \alpha} = \pm 2$ and non-standard if $i(\partial F, \partial F')_{\nu \partial \alpha} = 0$.

$$s(F) - s(F') = i(\partial F, \partial F')_{\partial \nu L} = \sum_{\text{arcs } \alpha \text{ of } F \cap F'} i(\partial F, \partial F')_{\nu \partial \alpha}$$

Procedure 2.23. Let $(\Sigma, L)$ be non-stabilized with connected spanning surfaces $S, T$ such that $S \cap T$ consists entirely of standard arcs and $|S \cap T| = \beta_1(S) + \beta_1(T) + 2g(\Sigma)$. Then extending $S, T$ through $\nu L$ so that $\partial S = L = \partial T$ and collapsing $S \cup T$ along each arc of int$(S) \cap \text{int}(T)$ gives a closed surface $Q$ isotopic to $\Sigma'$ on which $L$

---

\footnote{For an arbitrary diagram on $\Sigma$, $|\sigma_W(L) - \sigma_B(L)| \leq 2g(\Sigma)$.}

\footnote{Connectedness and $|S \cap T| = \beta_1(S) + \beta_1(T)$ imply that $g(Q) = g(\Sigma)$. This and the assumption that $(\Sigma, L)$ is non-stabilized imply that $Q$ is isotopic to $\Sigma$.}
collapses to a connected 4-valent graph; recovering crossing information gives a connected link diagram \( D \subset Q \) for which \( S \) and \( T \) are checkerboard surfaces. The initial configuration of \( S \) and \( T \), up to isotopy of \( S \cup T \) in \( (\Sigma \times I) \setminus \delta L \), uniquely determines \( D \) up to isotopy. See Figure \[\text{4}^{\dagger}\]

**Proposition 2.24.** If \((\Sigma, L)\) is local and has positive- and negative-definite connected spanning surfaces \( F_+ \) and \( F_- \), then

\[
s(F_+) - s(F_-) = 2 (\beta_1(F_+) + \beta_1(F_-)).
\]

**Proof.** Because \( L \) is local, the surfaces \( F_+ \) and \( F_- \) are \( S^* \)-equivalent, so \( \sigma_{F_+}(L) = \sigma_{F_-}(L) \), and the result follows from (2.2). \qed

**Proposition 2.25** (c.f. Propositions 2.12 and 2.22 of [Ki21]). If \((\Sigma, L)\) is non-stabilized and has positive- and negative-definite connected spanning surfaces \( F_+ \) and \( F_- \), then

\[
s(F_+) - s(F_-) = 2 \beta_1(F_+) + 2 \beta_1(F_-) + 4g(\Sigma).
\]

Further, if \( F_+ \cap F_- \) is comprised of arcs \( \alpha \) with \( i(\partial F_+, \partial F_-)_{\partial \alpha} = +2 \):

(A) \( |F_+ \cap F_-| = \beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma) \),

(B) \( F_\pm \) yield an alternating diagram \( D \) via Procedure \[2.23\], and

(C) if \( F_+ \) and \( F_- \) are \( \partial \)-incompressible, then \( D \) has no removable nugatory crossings.

**Proof.** Isotope \( F_\pm \) so that each component \( \alpha \) of \( F_+ \cap F_- \) is an arc with \( i(\partial F_+, \partial F_-)_{\partial \alpha} = +2 \). Now

\[
|F_+ \cap F_-| = \frac{1}{2} |\partial F_+ \cap \partial F_-| = \frac{1}{2} (s(F_+) - s(F_-)),
\]

which equals \( \beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma) \) by (2.2) and Lemma 2.21. Therefore, the pair \( F_\pm \) determines a connected diagram \( D \) of \( L \) via Procedure \[2.23\]. The checkerboard surfaces of \( D \) are \( F_\pm \), so \( D \) is alternating by Fact \[2.17\]. Part (C) follows easily. \qed

**Fact 2.26** (c.f. Fact 2.23 of [Ki21], Lemma 3.4 of [Gr17]). If \( F_+ \cap F_- \) are definite surfaces of opposite signs spanning a link \( L \subset \Sigma \times I \), then any circle \( \gamma \subset F_+ \cap F_- \) bounds disks in both \( F_+ \) and \( F_- \).

**Procedure 2.27.** Suppose \( F_+ \cap F_- \) are definite surfaces of opposite signs spanning a link \( L \subset \Sigma \times I \). Fixing \( F_- \), isotope \( F_+ \) via the following hierarchy of moves\[27\]

1. If \( F_+ \cap F_- \) contains circles, then (using Fact \[2.26\]) choose an innermost one \( \gamma \) in \( F_+ \); \( \gamma \) bounds disks \( X_\pm \subset F_\pm \). Using the irreducibility of \( (\Sigma \times I) \setminus L \), isotope \( X_+ \) past \( X_- \) as shown in Figure \[5\]. Meanwhile, fix \( F_+ \) away from \( X_+ \).

2. If any arc \( \alpha \) of \( F_+ \cap F_- \) is parallel in \( F_- \setminus F_+ \) to \( \partial F_- \) and in \( F_+ \setminus F_- \) to \( \partial F_+ \), then remove \( \alpha \) as shown in Figure \[6\].

\[27\]That is, perform (1) whenever possible, perform (2) whenever possible unless (1) is possible, and perform (3) whenever possible unless (1) or (2) is possible.
Figure 5. Removing a circle $\gamma$ of intersection between positive- and negative-definite surfaces $F_+$ and $F_-$. The dashed purple circle bounds a disk in $F_+$.

Figure 6. Removing adjacent points of $\partial F_+ \cap \partial F_-$ of opposite sign

Figure 7. Adding positive twists to a spanning surface

(3) If arcs $\alpha_+ \subset \partial F_+ \setminus \partial F_-$ and $\alpha_- \subset \partial F_- \setminus \partial F_+$ are parallel in $\partial \nu L$, then push $\alpha_+$ past $\alpha_-$ as in Figure 6 bottom.

We also recall:

Fact 2.28. If $\alpha$ is a system of disjoint properly embedded arcs in a definite surface $F$, then $F \setminus \nu \alpha$ is definite.
Fact 2.29. If \( F' \) is obtained by adding positive twists to a positive-definite surface \( F \) as in Figure 7, then \( F' \) is positive-definite.

Fact 2.30. If \( F_\pm \) are definite surfaces of opposite signs spanning \( (\Sigma, L) \) and \( \alpha \) is a non-standard arc of \( F_+ \cap F_- \), then denoting \( F'_+ = F_+ \setminus \nu \alpha, L' = \partial F'_+ \), and \( F'_- = F_- \setminus \nu \alpha \), the following are equivalent:

(I) \( \alpha \) is separating on \( F_+ \);

(II) \( \alpha \) is separating on \( F_- \);

(III) \( L' \) has one more split component than \( L \).

The next two facts differ notably from their classical analogs:

Fact 2.31 (c.f. Proposition 6.6 of [Ki21]). Let \( F \) be a positive-definite surface spanning a locally prime alternating link \( L \), and let \( K \) be the kernel of the map \( H_1(F) \to H_1(\Sigma \times I) \) induced by inclusion \( F \hookrightarrow \Sigma \times I \). Then \( F \) is end-essential if and only if every nonzero \( a \in K \) satisfies \( \langle a, a \rangle_F \geq 2 \).

Proof. Take an end-essential negative-definite spanning surface \( W \) for \( L \) with \( W \cap F \), and let \( D \) be an alternating diagram of \( L \) associated to \( F, W \) (via Procedure 2.27 and then 2.23). If \( D \) is locally prime, then both conditions are satisfied, the first by Theorem 2.9 and the second by an argument analogous to the proof of Lemma 4 of [Ki24]. Conversely, if \( D \) admits a removable nugatory crossing \( c \), then neither condition holds, because \( W \) is end-essential.

Proposition 2.32 (c.f. Proposition 6.7 of [Ki21]). Let \( F \) be a positive-definite surface spanning a locally prime alternating link \( L \), and let \( \alpha \subset F \) be a properly embedded arc such that \( F' = F \setminus \nu \alpha \) spans a locally prime alternating link \( L' \). If \( F \) is end-essential, then \( F' \) is also end-essential.

Proof. Letting \( K \) and \( K' \) denote the kernels of the maps \( H_1(F) \to H_1(\Sigma \times I) \) and \( H_1(F') \to H_1(\Sigma \times I) \) induced by inclusion, Fact 2.31 tells us that every nonzero \( c \in K \) satisfies \( \langle c, c \rangle_F \geq 2 \), and Fact 2.28 implies that \( F' \) is positive-definite. Therefore every nonzero \( c \in K' \) satisfies \( \langle c, c \rangle_F \geq 2 \), and so Fact 2.31 implies that \( F' \) is end-essential.

Proposition 2.33. As a result of Procedure 2.27, \( F_+ \cap F_- \) consists only of standard positive arcs.

Proposition 2.34. If \( F_\pm \) are definite surfaces of opposite signs spanning a link \( L \subset \Sigma \times I \) and \( \alpha \) is an arc of \( F_+ \cap F_- \) that is \( \partial \)-parallel in both \( F_+ \) and \( F_- \), then \( \alpha \) is non-standard.

\(^{28}\)Likewise for adding negative twists to a negative-definite surface.

\(^{29}\)Definiteness implies that \( F \) is end-incompressible.

\(^{30}\)Note that Procedure 2.27 always terminates because each move decreases \( |F_+ \cap F_-| + |\partial F_+ \cap \partial F_-| \).
Lemma 2.35 (c.f. Lemma 2.30 of [Ki21]). Suppose \( F_\pm \) are positive- and negative-definite surfaces spanning a non-stabilized link \( L \subset \Sigma \times I \), and \( \alpha \) is an arc of \( F_+ \cap F_- \). Then:

(A) \( i(\partial F_+, \partial F_-)_{\nu \partial_\alpha} \neq -2 \).

(B) If \( \alpha \) is nonseparating on \( F_- \), then \( i(\partial F_+, \partial F_-)_{\nu \partial_\alpha} = 2 \).

(C) In particular, if \( L \) is locally prime, both \( F_\pm \) are essential, and \( \alpha \) is not \( \partial \)-parallel in both \( F_\pm \), then \( i(\partial F_+, \partial F_-)_{\nu \partial_\alpha} = 2 \).

**Proof.** The argument is largely the same as in [Ki21]. For (A) and (B), we just describe the differences: if \( (\Sigma, L') \) is nonstabilized, then replacing \( \beta_1(F_+) + \beta_1(F_-) \) with \( \beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma) \) in (6.1) and (6.2) of [Ki21] contradicts Proposition 2.25 (A); if \( (\Sigma, L') \) is stabilized, then Fact 2.22 (A) (and, for (B), the assumption that \( \alpha \) is nonseparating on \( F_- \)) implies that \( L' \) is local, so Proposition 2.24 gives:

\[
\begin{align*}
-2 &= (s(F_+) - s(F_-)) - (s(F'_+) - s(F'_-)) \\
-2 &= 2(\beta_1(F_+) + \beta_1(F_-) + 2g(\Sigma)) - 2(\beta_1(F'_+) + \beta_1(F'_-)) \\
-1 &= g(\Sigma).
\end{align*}
\]

We prove (C) by contradiction. Apply Procedure 2.27 \( F_+ = F_0 \to F_1 \to \cdots \to F_t \) until it terminates, and consider the last move (3) \( F_s \to F_{s+1} \) in the sequence, which involves two arcs \( \alpha_1, \alpha_2 \) of \( F_s \cap F_- \) and one arc \( \alpha \) of \( F_{s+1} \cap F_- \); perturb \( \alpha_1 \) in \( F_- \) so that it is disjoint from \( F_s \). Parts (A) and (B) imply without loss of generality that \( \alpha_1 \) is non-standard, so \( F_- \setminus \nu \alpha_1 \) and \( F_s \setminus \nu \alpha_1 \) are definite surfaces of opposite sign spanning the same link \( L' \). Observe that, for all \( i = s + 1, \ldots, t \) (c.f. (6.3) of [Ki21]), and each arc \( \alpha' \) of \( F_- \setminus \{ F_i \} \) that separates \( F_- \), either \( \alpha' \) is \( \partial \)-parallel in \( F_- \) or \( \partial(F_- \setminus \nu \alpha') \) is split with no local components. The latter “possibility” uses the assumption that \( L \) is locally prime; it also contradicts Fact 2.22 (B). Therefore, \( \alpha_1 \) is \( \partial \)-parallel in \( F_- \), which contradicts the hierarchy of the moves in Procedure 2.27.

Using Lemma 2.35, the same reasoning as in [Ki21] leads to:

**Theorem 2.36.** Suppose \( (\Sigma, D) \) and \( (\Sigma, D') \) are locally prime, cellular alternating diagrams of \( (\Sigma, L) \) with checkerboard surfaces \( B, W \) and \( B', W' \). Then \( D \) and \( D' \) are equivalent if and only if \( B \) and \( B' \) are isotopic in \( (\Sigma \times I) \setminus \nu L \), as are \( W \) and \( W' \).

**Corollary 2.37.** There is a bijective correspondence between equivalence classes of locally prime, cellular alternating link diagrams on \( \Sigma \) and pairs of isotopy classes of essential definite surfaces of opposite signs spanning the same locally prime, nonstabilized link in \( \Sigma \times I \).\(^{31}\)

\(^{31}\) Example 2.37 of [Ki21] shows that Theorem 2.36 and Corollary 2.37 become false if one removes “locally prime” or “cellular alternating.”
2.3. Plumbing. A plumbing cap for a surface $F$ spanning $(\Sigma, L)$ is an embedded disk $V \subset (\Sigma \times I) \setminus \nu L$ with $V \cap (F \cup \partial \nu L) = \partial V$ where:
- $\partial V$ bounds a disk $\hat{U} \subset F \cup \nu L$,
- $\hat{U} \cap F$ is a disk $U$ called the shadow of $V$, and
- denoting the components of $(\Sigma \times I) \setminus (\hat{U} \cup V)$ by $Y_1, Y_2$, neither subsurface $F_i = F \cap Y_i$ is a disk.

If the first two properties hold but the third fails, we call $V$ a fake plumbing cap for $F$.

If $V$ is a plumbing cap for $F$ with shadow $U$, then the operation $F \to (F \setminus U) \cup V$ is called re-plumbing. See Figure 8. The same operation along a fake plumbing cap, a “fake re-plumbing,” is an isotopy move. Two spanning surfaces are plumb-related if they are related by re-plumbing and isotopy moves.

2.3.1. The 4-dimensional perspective.

Proposition 2.38 (c.f. Proposition 2.36 of [Ki21]). Given surfaces $F_1, F_2$ spanning $(\Sigma, L)$, let $F'_i$ be properly embedded surfaces in $\Sigma \times I \times I_+$ obtained by perturbing $\text{int}(F_i)$, while fixing $\partial F_1 = L = \partial F_2$.

If $F_1 \setminus \nu L$ and $F_2 \setminus \nu L$ are plumb-related, then:
- (A) $F'_1$ and $F'_2$ are related by an ambient isotopy of $\Sigma \times I \times I_+$ which fixes $\Sigma \times I \supset L$;
- (B) there is an isomorphism $\phi : H_1(F_1) \to H_1(F_2)$ satisfying $\langle \alpha, \beta \rangle_{F_1} = \langle \phi(\alpha), \phi(\beta) \rangle_{F_2}$ for all $\alpha, \beta \in H_1(F_1)$;
- (C) if $F_1$ is definite, then $F_2$ is definite of the same sign;
- (D) in particular, if $F_1$ is a checkerboard surface from an alternating diagram of $L$ on $\Sigma$, then so is $F_2$;
- (E) $F_1$ and $F_2$ are $S^*$ equivalent, and thus $\sigma_{F_1}(L) = \sigma_{F_2}(L)$.

Proof. Part (A) is the same as in [Ki21]. For (B), construct the desired isomorphism $\phi : H_1(F_1) \to H_1(F_2)$ as follows. Given $a \in H_1(F_1)$, take a multicurve $\alpha \subset F_i$ representing $a$, replace each arc of $\alpha \cap U$ with an arc in $V$ (with the same initial and terminal points), and denote the resulting multicurve by $\alpha'$; set $\phi(a) = [\alpha']$. This

---

32 The decomposition $F = F_1 \cup F_2$ is a de-plumbing of $F$ along $U$ and $V$, denoted $F = F_1 \ast F_2$. The reverse operation, in which one obtains $F$ by gluing $F_1$ and $F_2$ along $U$, is called generalized plumbing or Murasugi sum.
immediately gives (C) and (D), and (E) now follows from the observation that \([F_1] + [F_2] = 0 \in H_2(\Sigma \times I; \mathbb{Z}/2)\), since the union of any plumbing cap and its shadow is nullhomologous.

Next, we extend Theorem 3 of [GL78] to the context of thickened surfaces. Let \(F\) be a spanning surface of a link \(L \subset \Sigma \times I\). Isotope \(F\) so that \(F \subset (\Sigma \setminus x) \times I\) for some point \(x \in \Sigma\). Let \(F'\) be a properly embedded surface in \((\Sigma \setminus \hat{x}) \times I \times I_+\) obtained by perturbing the interior of \(F\) while fixing \(\partial F\). One can construct the double-branched cover \(M_{\hat{F}}\) of \((\Sigma \setminus \hat{x}) \times I \times I_+\) by cutting \(\Sigma \times I \times I_+\) along the trace of this isotopy, taking two copies, and gluing. Yet, these two copies are homeomorphic to \(\Sigma \times I \times I_+\), and the gluing region corresponds to a regular neighborhood \(N\) of \(F\) in \(\Sigma \times I\). Therefore, one may instead construct \(M_{\hat{F}}\) as follows. Let \(\iota: N \to N\) be involution given by reflection in the fiber, take two copies \(\Sigma_1\) and \(\Sigma_2\) of \((\Sigma \setminus \hat{x}) \times I \times I_+\), and define

\[
M_{\hat{F}} = (\Sigma_1 \cup \Sigma_2) / (y \in N \subset \partial \Sigma_1 \sim \iota(y) \in N \subset \partial \Sigma_2).
\]

Consider the Mayer-Vietoris sequence for \(M_{\hat{F}}\):

\[
0 = H_2(\Sigma_1^4) \oplus H_2(\Sigma_2^4) \to H_2(M_{\hat{F}}) \xrightarrow{\varphi} H_1(N) \xrightarrow{\psi} H_1(\Sigma_1^4) \oplus H_1(\Sigma_2^4) \to \cdots
\]

If \(g(\Sigma) = 0\), as in [GL78], then both \(\Sigma_i^4\) are 4-balls, so \(\varphi\) is an isomorphism; Gordon–Litherland then use the inverse map to compare the intersection form \(\cdot\) on \(M_{\hat{F}}\) with their pairing \(\mathcal{G}_F\) on \(F\). After restricting appropriately, the same ideas work here:

**Theorem 2.39** (c.f. Theorem 3 of [GL78]). With the setup above, let \(i_*: H_1(F) \to H_1(N)\) be the isomorphism induced by inclusion, and denote \(K = i_\ast^{-1}(\text{ker}(\psi))\). Then there is an isomorphism \(S: (K, \mathcal{G}_F) \to (H_2(M_{\hat{F}}), \cdot)\).

**Proof.** Consider the following map \(S: K \to H_2(M_{\hat{F}})\). Given \(A \in K\), choose a multicurve \(\alpha \subset F\) with \([\alpha] = A\). Then \(\alpha\) bounds properly embedded oriented surfaces \(s_i \subset \Sigma_i^4\) for \(i = 1, 2\). Define \(S(A) = [s_1] - [s_2] \in H_2(M_{\hat{F}})\).

To see that this is the required isomorphism \((K, \mathcal{G}_F) \to (H_2(M_{\hat{F}}), \cdot)\), let \(A, B \in K\), represented respectively by multicurves \(\alpha, \beta \subset F\). Then \(\alpha\) and \(\beta\) are disjoint multicurves in \(N\) with \([\alpha] = 2A\), \([\beta] = 2B\).

\[\text{33To see that this is always possible, consider isotoping } F \text{ into disk-band form.}\]
THE VIRTUAL FLYPING THEOREM

Figure 9. A flype move corresponds to an isotopy of one checkerboard surface (here, $W$) and a re-plumbing of the other.

\[ \iota(\alpha) = \alpha, \text{ and } \iota(\tilde{\beta}) = \tilde{\beta}. \] Hence:

\[
S(A) \cdot S(B) = \frac{1}{4} \left( S([\alpha]) \cdot S([\beta]) + S([\tilde{\beta}]) \cdot S([\alpha]) \right)
\]

\[
= \frac{1}{4} \left( \text{lk}_\Sigma(\tilde{\alpha}, \beta) + \text{lk}_\Sigma(\iota\tilde{\alpha}, \iota\beta) + \text{lk}_\Sigma(\tilde{\beta}, \alpha) + \text{lk}_\Sigma(\iota\tilde{\beta}, \iota\alpha) \right)
\]

\[
= \frac{1}{2} \left( \text{lk}_\Sigma(\tilde{\alpha}, \beta) + \text{lk}_\Sigma(\tilde{\beta}, \alpha) \right)
\]

\[ = \mathcal{G}_F(A, B). \quad \square \]

2.3.2. Flyping caps. Let $D \subset \Sigma$ be a locally prime, cellular alternating link diagram with checkerboard surfaces $B, W$. Say that a plumbing cap $V$ for $B$ is a **flyping cap** if $V$ appears as in Figure 10, left-center. There is then a corresponding flype move as shown in Figures 10 and 9. Namely, denoting the shadow of $V$ by $U$, the flype move proceeds along an annular neighborhood of a circle $\gamma \subset \Sigma$ comprised of the arc $V \cap W$ together with an arc in $U \cup \nu L$. (The resulting link diagram might be equivalent to $D$.) More formally:

**Proposition 2.40** (c.f. Proposition 2.37 of [Ki21]). Let $V$ be a flyping cap for $B, D \rightarrow D'$ the flype move corresponding to $V, B'$ and $W'$ the checkerboard surfaces from $D'$, and $B''$ the surface obtained by re-plumbing $B$ along $V$. Then $B'$ and $B''$ are isotopic, as are $W'$ and $W$. Hence, $D'$ is equivalent to the diagram determined by $B'', W$ via Theorem 2.36. \[34\]

**Proof.** As in [Ki21], Figure 9 demonstrates the isotopies. \[ \square \]

Conversely, if $\gamma$ is a flyping circle for $(\Sigma, D)$, then there is an flyping cap $V$ for $B$ (or $W$) with $V \cap W \subset \nu \gamma$ (resp. $V \cap B \subset \nu \gamma$).

3. THE FLYPING THEOREM IN THICKENED SURFACES

The arguments in §§3-5 and 7-8 of [Ki21] have been revised so that they apply directly in the context of this paper (with the obvious replacements of $S^3$ with $\Sigma \times I, S^2$ with $\Sigma$, essential with end-essential, and prime with locally prime): $B, W$ are the checkerboard surfaces.

\[34\] An analogous statement holds for flyping caps for $W$. 

Figure 10. A flyping cap and the associated flype move

from a locally prime, cellular alternating diagram $D \subset \Sigma$ of a link $L \subset \Sigma \times I$, $F$ is an end-essential positive-definite surface spanning $L$, $v_F$ is comprised of the vertical arcs at the crossings where $F$ has crossing bands, and $D_{F,W}$ is the diagram determined via Theorem 2.36 by $F, W$. One then implements Menasco’s crossing ball setup, isotopes $F$ into fair position, and performs a sequence of isotopy and re-plumbing moves according to a hierarchy: one only performs each move $k$ if $F$ is in $(k-1)$-good position, meaning that $F$ is in fair position and none of Moves 1 through $k-1$ are possible. See [Ki21] for the notations $C, v, \tilde{W}, S \pm$ etc. associated with the crossing ball setup and for the precise definitions of fair position and Moves 1-10. Moves 1-9, all of which are isotopy moves, appear in Figure 11. Move 10 is a re-plumbing move and is more complicated; see [Ki21].

A few details are worth noting. First, one must be more careful with push-through moves (see Definition 3.10 of [Ki21]) in thickened surfaces than in $S^3$. The definition is the same (because it was written with this paper in mind!), but in addition to the three pictures shown top in Figure 19 of [Ki21], three more pictures are possible. See Figure 12. In any case, if we wish to perform (or observe the possibility of) a push-through move along an arc $\alpha$ whose endpoints lie on a circle $\gamma$, we must now check that $\alpha$ is parallel in $S_+ \cup S_- \approx S^2$ into $\gamma$; in [Ki21], this was free. Importantly, however, this is always the case.

Second, whereas in [Ki21] every circle of $F \cap S_\pm$ was inessential in $S_\pm \approx S^2$, this property holds here only because the assumption that $F$ is end-incompressible allows us to require that $S_+ \cup S_- \approx S^2$ cuts $F$ into disks (c.f. Definition 3.2 (h) and Lemma 3.3 of [Ki21]).

Third, Sublemma 5.2 of [Ki21] implies there that the circles of $F \cap S_+$ are mutually nested, but this is less clear here. The proof of Lemma 5.3 of [Ki21] is thus written with this paper in mind, and is slightly more complicated as a result.

Adapting the arguments from §§3-5, 7-8 of [Ki21] thus gives:

---

35 In [Ki21], see the definition of Move 2 and the proofs of Lemmas 3.22, 4.1, and 5.3 and of Propositions 8.2 and 8.3.
Figure 11. Moves 1-9
**Theorem 3.1.** If $D = D_{B,W}$ is a locally prime, cellular alternating diagram of $(\Sigma, L)$, then any end-essential, positive definite surface $F$ spanning $L$ is plumpl-relat-ed to $B$; likewise for end-essential negative-definite surfaces and $W$. \(\dagger\)

**Corollary 3.2.** With $F$ and $D$ as in Theorem \(3.1\), $\beta_1(B) = \beta_1(F)$ and $s(B) = s(F')$. \(\dagger\)

**Theorem 3.3** (Part of Tait’s extended first conjecture \[Gr17, Ka87, Mu87, Th87, Tu87\]). If $D, D' \subset \Sigma$ are alternating diagrams of a link $L \subset \Sigma \times I$, neither containing removable nugatory crossings, then $D$ and $D'$ have the same number of crossings. \(\dagger\)

**Theorem 3.4.** If $F$ is in 9-good position, then $F$ contains no saddle disks: $F \cap C = v_F$; hence, every circle $\gamma$ of $F \cap S_+$ is a flyping circle, and $D_{F,W}$ is related to $D$ by a sequence of flypes that preserve the isotopy class of $W$. \(\dagger\)

**Theorem 3.5** (Tait’s extended flyping conjecture). All locally prime, cellular alternating diagrams $D = D_{B,W}$ and $D' = D'_{B',W'}$ of the same link $L \subset \Sigma \times I$ are related by a sequence of flypes $D \to \cdots \to D'' \to \cdots \to D'$ in which $D \to \cdots \to D''$ preserves the isotopy class of $W$ and $D'' \to \cdots \to D'$ preserves the isotopy class of $B'$.\(\dagger\)

Since writhe is invariant under flypes (recall Observation \[2.15\] and additive under diagrammatic connect sum and disjoint union, we obtain a new geometric proof of Tait’s second conjecture:

**Theorem 3.6** (Tait’s extended second conjecture \[BK18, BKS19\]). All locally prime, cellular alternating diagrams of a given link $L \subset \Sigma \times I$ have the same writhe.

---

\(36\) This is also true if $L$ is non-stabilized and/or locally prime.

\(37\) When $D$ and $D'$ are locally prime and cellular alternating, Fact \[2.17\] implies this Theorem 3.4 and Corollary 3.2 immediately imply this. The general case then follows, as the number of crossings is additive under (de)stabilization, diagrammatic connect sum, and split union.
Theorem 3.5 implies that, unlike a classical link and a link in $S^2 \times I$, a link in a thickened surface of positive genus is not necessarily isotopic to the link obtained by reflecting horizontally (in the projection surface) and then vertically. More precisely, let $D \subset \Sigma$ be a locally prime, cellular alternating diagram of a link $L \subset \Sigma \times I$; let $\phi : \Sigma \to \Sigma$ be an orientation-reversing involution; let $D' \subset \Sigma$ be the diagram obtained from $\phi(D)$ by reversing all crossing information; and let $L' \subset \Sigma \times I$ be the link represented by $D'$. Note that $L'$ is the image of $L$ under the map $\Sigma \times I \to \Sigma \times I$ given by $(x, t) \mapsto (\phi(x), -t)$.

**Corollary 3.7.** With the setup above, if $D$ is locally prime and cellular alternating, then the links $L$ and $L'$ are isotopic in $\Sigma \times I$ if and only if the diagrams $D$ and $D'$ are flype-related on $\Sigma$. In particular, this is always true if $g(\Sigma) = 0$, but not necessarily if $g(\Sigma) > 0$.

**Example 3.8.** The diagrams on $T^2$ shown right in Figure 2 admit no non-trivial flypes and are non-isotopic; thus, by Corollary 3.7, they represent non-isotopic links in $T^2 \times I$.

### 4. Virtual links

A virtual link diagram is the image of an immersion $\bigcup S^1 \to S^2$ in which all self-intersections are transverse double-points, some of which are labeled with over-under information. These labeled points are called classical crossings, and the other double-points are called virtual crossings. Traditionally, virtual crossings are marked with a circle, as in Figure 13. A virtual link is an equivalence class of such diagrams under generalized Reidemeister moves (R-moves), as shown in Figure 13. There are seven types of such moves, the three classical moves and four non-classical moves.

**Notation 4.1.** Given a virtual link diagram $V \subset S^2$, let $[V]$ denote the set of all virtual diagrams related to $V$ by planar isotopy and non-classical R-moves.

**Definition 4.2.** A virtual link diagram $V$ is nonsplit if every $V' \in [V]$ is connected.
4.1. **Correspondences.** By work of Kauffman [Ka98], Kamada–Kamada [KK00], and Carter–Kamada–Saito [CKS02], there is a bijective correspondence between virtual links and stable equivalence classes of links in thickened surfaces. In [Ki22], the author introduces the following correspondence between equivalence classes of the associated diagrams:

**Correspondence 4.3.** The following gives a bijection from equivalence classes $[V]$ of nonsplit virtual link diagrams to cellular link diagrams on connected closed surfaces:

Choose $V \in [V]$, take a regular neighborhood $\nu V$ of $V$ in $S^2$, modify $\nu V$ near each virtual crossing of $V$ as shown in Figure 14, and cap off each boundary component (abstractly) with a disk.

There is an important subtlety in Correspondences 4.3 and 4.4 pertaining to how one constructs a virtual link diagram from a given cellular pair $(\Sigma, D)$. The idea of this construction is to embed $(\Sigma, D)$ into $S^3$ and project. The subtlety is that, under this embedding, all crossings of $D$ must lie on the front of $\Sigma$. See §4.2 of [Ki22] for details.

Correspondence 4.3 gives a new diagrammatic perspective on a well-known correspondence [Ka98, KK00, CKS02]:

**Correspondence 4.4.** There is a correspondence between virtual links and stable equivalence classes of links in thickened surfaces: choose any representative diagram and apply the diagrammatic Correspondence 4.3.

4.2. **Prime virtual link(diagram)s.** We adopt the following definitions from [Ki22]:

**Definition 4.5.** A diagrammatic connect sum decomposition $V = V_1 \# V_2$ of a virtual link diagram is nontrivial if both $V_1$ and $V_2$ have classical crossings. A virtual link diagram $V$ is prime if it has no nontrivial connect sum decomposition. A virtual link $K \neq \emptyset$ is prime if for every diagram $V$ of $K$ and every decomposition $V = V_1 \# V_2$, one diagram $V_i$ represents the classical unknot and the other represents $K$.

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38 At this intermediate stage, we have an *abstract link diagram*, which we will not need again.
Primeness of virtual links corresponds as follows to primeness of nonstabilized pairs \((\Sigma, L)\). For simplicity, we add an assumption of checkerboard colorability, which does not appear in the full version of the theorem in [Ki22].

Theorem 4.6 (c.f. Theorem 5.7 of [Ki22]). Given a nonsplit checkerboard colorable virtual link \(K\) and the corresponding nonstabilized link \(L\) in a thickened surface \(\Sigma \times I\),

1. \((\Sigma, L)\) is locally prime if and only if \(K\) admits no nontrivial connect sum decomposition \(K = K_1 \# K_2\) in which \(K_1\) is a classical link and \(g(K_2) = g(K)\), and
2. \(K\) is prime if and only if \((\Sigma, L)\) is pairwise prime.

4.3. Determining primeness of alternating virtual link diagrams. Theorem 2.7 states that an alternating pair \((\Sigma, L)\) is composite (in some sense) if and only if it is “obviously so” in a given reduced alternating diagram. To translate this result to alternating virtual link diagrams \(V\), one must describe what it means for such \(V\) to be “obviously” composite (in either the local or pairwise sense). The solution in [Ki22] is to use a new tool called lassos to capture the salient features of all diagrams in \([V]\) and find some suitable \(V' \in [V]\).

A lasso for \(V\) is a disk \(X \subset S^2\) that contains all classical crossings of \(V\) and no virtual ones, and \(X\) is acceptable if the part of the diagram in \(X\) is connected and the part of the diagram outside \(X\) does not admit an “obvious” simplification (see [Ki22] for details). In [Ki22], the author proves (the proof of the first result is constructive):

Proposition 4.7 (Proposition 3.8 of [Ki22]). Given a nonsplit virtual link diagram \(V\) some \(V' \in [V]\) admits an acceptable lasso.

Fact 4.8 (c.f. Proposition 7.3 of [Ki22]). Let \(V \subset S^2\) be a nonsplit virtual link diagram corresponding to a cellular pair \((\Sigma, D)\). Choose \(V' \in [V]\) which admits an acceptable lasso \(X\). Given a crossing \(c' \in V\) corresponding to a crossing \(c \in D\), \(c\) removably nugatory if and only if some disk \(U \subset X\) has \(\partial U \cap V' = \{c'\}\);

In other words, \(V'\), chosen constructively from \([V]\) so that it admits an acceptable lasso, has an “obvious” removable nugatory crossing whenever \((\Sigma, D)\) has a removable nugatory crossing. Moreover, \(V'\) is (resp. locally) prime if and only if every \(V'' \in [V]\) is (resp. locally prime); see Observations 5.14-5.14 of [Ki22]. With this context, the culminating result of [Ki22] is that, in the alternating case, one can determine by inspecting \(V'\) whether or not it represents prime or locally prime virtual link:

Theorem 4.9 (Theorem 7.8 of [Ki22]). Let \(V\) be an alternating diagram of a nonsplit virtual link \(K\), and consider a diagram \(V' \in [V]\) which admits an acceptable lasso \(X\). Assume that \(V'\) has at least one virtual crossing. Then:
(1) When \( V' \) has no nugatory crossings, \( K \) is prime if and only if \( V' \) is prime.

(2) When \( V' \) has no removably nugatory crossings, \( K \) is locally prime if and only if \( V' \) is locally prime.

4.4. Tait’s conjectures for virtual links.

Definition 4.10. A (classical) flype on a virtual link diagram appears as in Figure 1, where \( T_1 \) contains no virtual crossings.

Theorem 4.11. Any two locally prime, alternating diagrams of a given virtual link \( \tilde{L} \) are related by non-classical \( R \)-moves and (classical) flypes.

Proof. Let \( V \) and \( V' \) be two such diagrams, and let \( (\Sigma, D) \) and \( (\Sigma', D') \) be the associated pairs under Correspondence 4.3. By Kuperburg’s theorem, we may identify \( \Sigma \equiv \Sigma' \), and by Theorem 3.5 there is a sequence of flype moves on \( \Sigma \) taking \( D \) to \( D' \):

\[
D = D_0 \to D_1 \to \cdots \to D_n = D'.
\]

We will show for each \( i = 1, \ldots, n \) that there are virtual diagrams \( V_{i-1}^2 \) and \( V_i^1 \) which correspond to \( (\Sigma, D_{i-1}) \) and \( (\Sigma, D_i) \) and which are related by a flype. This will produce a sequence of virtual diagrams

\[
V = V_0^1 \to V_0^2 \to V_1^1 \to V_1^2 \to \cdots \to V_n^1 \to V_n^2 = V'
\]

where each \( V_i^1 \to V_i^2 \) comes from a sequence of virtual \( R \)-moves and each \( V_{i-1}^2 \to V_i^1 \) comes from a flype.

Consider a flype \( D_{i-1} \to D_i \); it is supported within a disk \( X \subset \Sigma \). Denote the quotient map \( q : \Sigma \to \Sigma/X \equiv \Sigma \), and denote the underlying graph of \( D_{i-1} \) by \( G \). Choose a spanning tree \( T \) for the 4-valent graph \( q(G) \subset \Sigma/X \), and take a regular neighborhood \( \nu T \). Denote \( U = q^{-1}(\nu T) \), and observe that \( U \) is a disk in \( \Sigma \) that contains \( X \) and all crossings of \( D_{i-1} \).

Choose an embedding \( \phi : \Sigma \to \tilde{S}^3 \) such that \( \pi|_{\phi(U)} \) has no critical points and \( \pi \circ \phi(\Sigma) \cap \pi \circ \phi(D \setminus U) = \emptyset \). Denote \( f = \pi \circ \phi \) and \( f(D_{i-1}) = V_{i-1}^2 \). Observe that \( f|_X \) is a homeomorphism onto its image, and so the disk \( f(X) \) supports a flype \( V_{i-1}^2 \to V_i^1 \) where \( V_i^1 \) corresponds to \( (\Sigma, D_i) \).

Thus, as needed, each \( V_{i-1}^2 \to V_i^1 \) comes from a flype. To complete the proof, we note that each \( V_i^1 \to V_i^2 \) comes from a sequence of virtual \( R \)-moves, due to Correspondence 4.3 since both \( V_i^1 \) and \( V_i^2 \) correspond to the same cellulary embedded diagram \( D_i \) on \( \Sigma \).

Since crossing number and writhe are invariant under flypes, we can also extend more parts of Tait’s conjectures to virtual links:

39That is, take \( X \) to be the oval-shaped disk shown left in Figure 1.

40Thus, \( U \) is a lasso for \( (\Sigma, D_{i-1}) \).

\[
\]
Figure 15. There are infinitely many different ways to take minimal-genus connect sums of any two non-classical alternating links.

**Theorem 4.12.** All locally prime, alternating diagrams of a given virtual link have the same crossing number and writhe.

4.5. **Non-uniqueness of minimal-genus connect sum.** Finally, as an additional corollary, we observe that, whereas minimal-genus connect sum is a well-defined operation for classical knots and for any classical knot with any virtual knot, minimal-genus connect sum is *not* a well-defined operation for virtual knots:

**Corollary 4.13.** Given any two non-classical, locally prime, alternating virtual links $K_1$ and $K_2$, there are infinitely many distinct virtual links $K$ with $g(K) = g(K_1) + g(K_2)$ that decompose as a connect sum of $K_1$ and $K_2$.

This follows immediately from Theorem 4.12 using the construction suggested in Figure 15. We conjecture that the same construction works more generally:

**Conjecture 4.14.** Given any two non-classical, locally prime, checkerboard colorable virtual links $K_1$ and $K_2$, there are infinitely many distinct virtual links $K$ with $g(K) = g(K_1) + g(K_2)$ that decompose as a connect sum of $K_1$ and $K_2$. 
Appendix A: Cross-referencing with [Ki21]

| here in [Ki21] | here in [Ki21] |
|----------------|----------------|
| Prop. 2.10     | Prop. 2.5      |
| Ob. 2.12       | Fact 2.7       |
| Def. 2.14      | Def. 2.9       |
| Proc. 2.23     | Proc. 2.23     |
| Fact 2.28      | Subl. 6.3      |
| Fact 2.30      | Prop. 6.5      |
| Prop. 2.34     | Prop. 6.9      |
| Cor. 2.37      | Cor. 2.36      |
| Cor. 3.2       | Cor. 4.6       |

Table 1. Cross-listing information with [Ki21]

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