On the denominators of Young’s seminormal basis

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Abstract

We study the seminormal basis \( \{ f_t \} \) for the Specht modules of the Iwahori-Hecke algebra \( H_n(q) \) of type \( A_{n-1} \). We focus on the base change coefficients between the seminormal basis \( \{ f_t \} \) and Murphys’ standard basis \( \{ x_t \} \) with emphasis on the denominators of these coefficients. In certain important cases we obtain simple formulas for these coefficients involving hook lengths. Even for general standard tableaux we obtain new formulas. On the way we prove a new result about submodules of the restricted Specht module at root of unity.

1 Introduction

In this work we are concerned with the representation theory of the (Iwahori)-Hecke algebra of type \( A \).

As is well known, when defined over the fraction field \( K := \mathbb{Z}[q, q^{-1}] \) the Hecke algebra \( H_n^K(q) \) is semisimple and its irreducible modules are the Specht modules \( S^K_q(\lambda) \) with \( \lambda \) varying over the set of integer partitions \( \text{Par}_n \) of \( n \). Each \( S^K_q(\lambda) \) is endowed with the seminormal basis \( \{ f_t \mid t \in \text{Std}(\lambda) \} \), on which the action of the Hecke algebra generators \( T_i \) can be described in terms of simple formulas, Young’s seminormal form, that have been known for a long time.

Our main interest is however rather the representation theory of the Hecke algebras \( H_n^k(q) \), defined over a field \( k \) such that \( q \) is specialized to a root of unity in \( k \). These specialized Hecke algebras are in general not semisimple, and their representation theory is much more complicated than that of \( H_n^K(q) \), with many fundamental problems still open. The group algebra \( F_p \mathfrak{S}_n \) of the symmetric group over the finite field \( F_p \) is a special case of such an \( H_n^k(q) \), and in spite of much progress in recent years there is still no efficient algorithm for calculating the dimensions of the irreducible modules for \( F_p \mathfrak{S}_n \).

Let \( H_n^A(q) \) be the Hecke algebra defined over \( A \). Then there are also Specht modules \( S^A_q(\lambda) \) for \( H_n^A(q) \) but for these integral Specht modules the seminormal basis does not exist and one needs to work with Murphy’s standard basis, \( \{ x_t \mid t \in \text{Std}(\lambda) \} \), on which the action of \( T_i \) can only be described indirectly via a complicated recursion. On the other hand, the standard basis has the advantage that it exists for all specializations, including \( S^K_q(\lambda) \).

In this work we consider \( S^K_q(\lambda) \) and study the very natural question of determining the coefficients of the \( \{ f_t \} \)-basis when expanded in the \( \{ x_t \} \)-basis, that is the base change matrix between the two bases. For certain important standard \( \lambda \)-tableaux, that we call generalized James-Murphy tableaux, we are able to get exact formulas for these coefficients. The denominators that appear in the formulas involve certain hook lengths, and when these denominators are nonzero in \( k \), the corresponding seminormal basis element will also exist in \( S^K_q(\lambda) \), which will have certain consequences for the restriction of \( S^K_q(\lambda) \) from \( H_n^k(q) \) to \( H_{n-1}^k(q) \), that we investigate in the final section of our paper.

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For completely general standard $\lambda$-tableaux our methods do not give rise to a closed formula for the base change matrix, but even in this case we obtain a fast algorithm for expanding the $f_t$’s in terms of the $x_t$’s, that we explain.

This paper has a long story. The first version of it was published on arXiv in 2009, and was formulated only in the symmetric group setting, but in 2010 we uploaded a version of it to the arXiv in which our results were generalized to the Hecke algebra setup. The present version of the paper is essentially the same as the 2010 version, although we have corrected a large number of errors and inaccuracies, and have added some new examples.

The original motivations for our paper were twofold. In [RH] we showed that the coefficients of the quantum group action on the Fock space, a main ingredient in the LLT-algorithm for calculating decomposition numbers for the Hecke $H_n(q)$ with $q \mapsto e^{2\pi i/l}$, see [LLT] and [Ar], are closely related to the $\{f_t\}$-basis. Indeed, let $t_n$ be the $\lambda$-tableau that has $n$ in a removable node and has the remaining numbers $\{1, 2, \ldots, n-1\}$ filled in along rows; here is an example with $n = 26$

$$t_{26} = \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 & 19 \\
20 & 21 & 22 & 23 & 24 & 25 & 26
\end{array}$$

(1)

This kind of $\lambda$-tableaux plays an important role in James and Murphy’s calculation of the Gram matrix of the Specht module, [JM]. Let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $S_n^\lambda$. We then proved in [RH], that the coefficients of the above mentioned quantum group action on the Fock space are the $l$-adic quantum valuation of the $\langle f_{t_n}, f_{t_n} \rangle$’s. This observation made us speculate on a connection between the representation theory of $H_n(q)$ and the $\{f_t\}$-basis, and this was the first motivation for studying in more detail the expansion of $f_{t_n}$.

A second motivation for our work came from the strong analogy with the theory of symmetric functions. The construction of the $\{f_t\}$-basis is parallel to the construction of the Macdonald polynomials. Both are obtained through a Gram-Schmidt process over a partial order which must first be extended to a total order to perform the Gram-Schmidt process. In the case of Macdonald polynomials the initial basis is the basis of monomial symmetric functions, in the case of the seminormal basis the initial basis is the $\{x_t\}$-basis. By Cherednik’s work, see [C], the Macdonald polynomials are independent of this extension because they are eigenvectors of operators coming from the double affine Hecke algebra; in the case of the seminormal basis this role is played by the Jucys-Murphy elements, see [M3]. Finally, the formula for $\langle f_t, f_t \rangle$ has a striking similarity with its Macdonald polynomial analogue, see [JM] and [C]. But in the above picture an analogue for $\{f_t\}$ of the positivity theory for the expansion coefficients of the Macdonald polynomials, due to Haiman and others, see [H], is still missing. Our second motivation for the paper was to explain an attempt on how to fill this gap.

Since the previous version of this paper, a number of papers treating related topics have appeared, although the present paper is still the only one which works at the Hecke algebra level of generality. We here mention the paper by Raicu, [Rai], whose Theorem 1.2 is related to our Theorem 3. Especially relevant are the two papers by Fang, Lim and Tan [FLT1] and [FLT2], in which the question of the splitting over $\mathbb{Z}_p$ of the canonical map $\Delta(\lambda + \mu) \rightarrow \Delta(\lambda) \otimes \Delta(\mu)$ of Weyl modules for the Schur algebra is treated. The authors connect this question to another question about the denominators of Young’s seminormal basis, and are this way able to answer it for certain pairs of partitions $\lambda, \mu$. It should be pointed out that the methods used in all the mentioned papers are different from ours, and different between them.
We finally mention that computational evidence indicates that exact formulas for expressing the inverse coefficients of the \( \{ x_t \}_t \)-basis in terms of the \( \{ f_t \}_t \)-basis are more difficult to get by. On the other hand, these inverse coefficients are the topic of a recent paper by Armon and Halverson, [AH], who use results of Ram to give a fast algorithm for calculating them.

Let us now explain in more detail the contents of the paper. In section 2 we give a precise formulation of the setting in which we shall be working, and recall the relevant results from the literature, the most important being Murphy’s standard basis \( \{ x_t \}_{t \in \text{Std} (\lambda)} \) for the Specht module with associated Hecke algebra action (6) and the Garnir relations (9). We also recall Young’s seminormal basis \( \{ f_t \}_{t \in \text{Std} (\lambda)} \) and its Hecke algebra action, that is Young’s seminormal form, or YSF, see (19). In section 3 we explain a simple algorithm for calculating the expansion of \( f_t \) in terms of \( x_s \)’s, but where \( s \) runs over all row standard tableaux, not just standard tableaux. In section 4 we introduce the generalized James-Murphy tableaux that enter in our main Theorems. In section 5 we describe a formula that results from applying YSF along one row of the generalized James-Murphy tableau. In section 6 we treat the expansion of \( f_t \) for \( t \) a fat hook tableaux. The formula of the previous section 5 has a certain similarity with the Garnir relations and this similarity is a main ingredient of Theorem 2 of that section. In section 7 we treat the expansion of the seminormal basis elements corresponding to all generalized James-Murphy tableaux, in essence, by reducing to the case of fat hook partitions. In the final section 8 we consider the expansion in the case of general standard tableaux. We also give an application of the theory developed in the previous sections to the modular Hecke algebra representation theory, that is the representation theory of \( H_{k_n}^q \), giving a criterion for a certain Specht module to split off from the restricted Specht module \( \text{res} S^\lambda_k \). This criterion was also present in the previous versions of our article, but the statement and proof of it were very inadequate. The present formulation of the criterion is inspired by the formulation of the splitting criterion in [FLT1], but our proof technique is quite different from the one used in [FLT1].

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2 Basic notations and results

In this section we introduce the notation and recall some basic results that shall be used throughout the paper.

Let \( \mathcal{A} := \mathbb{Z}[q, q^{-1}] \) be the ring of Laurent polynomials over \( \mathbb{Z} \) and let \( \mathcal{K} \) be its quotient field.

We denote by \( \mathcal{H}^A_n (q) \) the Iwahori-Hecke algebra of type \( A_{n-1} \) over \( \mathcal{A} \). That is, \( \mathcal{H}^A_n (q) \) is the \( \mathcal{A} \)-algebra on generators \( T_1, T_2, \ldots, T_{n-1} \) subject to the relations

\[
T_i T_j = T_j T_i \quad \text{for } |i - j| > 1
\]

\[
T_i T_j T_i = T_j T_i T_j \quad \text{for } |i - j| = 1
\]

\[
(T_i - q)(T_i + 1) = 0 \quad \text{for } i = 1, 2, \ldots, n - 1.
\]

For an \( \mathcal{A} \)-algebra \( \mathcal{B} \) we introduce the specialized Iwahori-Hecke algebra \( \mathcal{H}^B_n (q) := \mathcal{B} \otimes_\mathcal{A} \mathcal{H}^A_n (q) \). If \( \mathcal{B} = \mathcal{K} \) we also sometimes write \( \mathcal{H}_n = \mathcal{H}^K_n (q) \). In general, we shall refer to \( \mathcal{H}_n^A (q) \) and to the various variations of it simply as the Hecke algebra, omitting the name Iwahori.

Let \( \mathfrak{S}_n \) be the symmetric group on \( n \) letters. There is a natural action of \( \mathfrak{S}_n \) on \( \{ 1, 2, \ldots, n \} \) which we view as a right action. It is well known that \( \mathfrak{S}_n \) is a Coxeter group on the set \( S := \ldots \)
\{s_1, s_2, \ldots, s_{n-1}\} where \( s_i \) is the simple transposition \( s_i := (i, i+1) \). For \( w = s_{i_1}s_{i_2} \cdots s_{i_N} \in S_n \) a reduced expression for \( w \) we set \( T_w := T_{i_1}T_{i_2} \cdots T_{i_N} \in H^A_n(q) \). Then it follows from Matsumoto’s Theorem that \( T_w \) is independent of the choice of reduced expression for \( w \). Moreover, as is also well known, \( \{T_w \mid w \in S_n\} \) is an \( A \)-basis for \( H^A_n(q) \) and \( \{T_w \mid w \in S_n\} \) is a \( B \)-basis for \( H^B_n(q) \).

Any field \( k \) with identity 1 can be made into an \( A \)-algebra via \( q \mapsto 1 \) and in this case we have that \( H^k_n(q) \equiv kS_n \), where \( kS_n \) is the group algebra of the symmetric group. Thus all results that hold for specialized Hecke algebras also hold for \( kS_n \).

We next recall the combinatorial notions of partitions and tableaux, associated with the Hecke algebra. For \( n \) a positive integer we denote by \( Par_n \) the set of integer partitions of \( n \). To be precise, an element \( \lambda \) of \( Par_n \) is a weakly decreasing sequence of positive integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K) \) such that \( \lambda_1 + \lambda_2 + \ldots + \lambda_K = n \). Similarly, we denote by \( Comp_n \) the set of integer compositions of \( n \), defined the same way as \( Par_n \) but allowing each \( \lambda_i \) to be zero and omitting the condition that \( \lambda \) be weakly decreasing. We set \( Par := \bigcup_n Par_n \) and \( Comp := \bigcup_n Comp_n \).

The sets \( Par_n \) and \( Comp_n \) are endowed with order relations that play an important role in the representation theory of the Hecke algebra, and also elsewhere. For \( \lambda, \mu \in Comp_n \) we may assume that there is a \( K \) such that \( \lambda = (\lambda_1, \ldots, \lambda_K) \) and \( \mu = (\mu_1, \ldots, \mu_K) \), by extending with zeros if necessary. Then the dominance order on \( Comp_n \) is defined via \( \lambda \preceq \mu \) if

\[
\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i, \quad \text{for all } 1 \leq j \leq K.
\]

By restriction, it induces an order relation on \( Par_n \), denoted the same way.

The dominance order is only a partial order on \( Par_n \) and \( Comp_n \) but can be extended to a total order on both sets, for example via \( \lambda < \mu \) if there is a \( j \) such that \( \sum_{i=1}^{j} \lambda_i < \sum_{i=1}^{j} \mu_i \) but \( \sum_{i=1}^{j'} \lambda_i = \sum_{i=1}^{j'} \mu_i \) for all \( j' < j \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_K) \in Par_n \). Then the Young diagram \( Y(\lambda) \) for \( \lambda \) is the graphical representation of \( \lambda \) through boxes, called nodes, arranged in \( K \) left adjusted lines, with \( \lambda_i \) nodes in the first line, \( \lambda_2 \) nodes in the second line just below the first line, and so on. For example, if \( \lambda = (5, 5, 4, 1, 1) \in Par_{10} \) and \( \mu = (2, 0, 3, 4, 1) \in Comp_{10} \) we have that

\[
Y(\lambda) = \\
Y(\mu) = \begin{array}{cccccc}
\mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \\
\mathbf{7} & \mathbf{8} & \mathbf{9} & \mathbf{10} & \mathbf{11} & \mathbf{12} \\
\end{array}
\]

For \( \lambda \in Comp_n \), and in particular for \( \lambda \in Par_n \), we use matrix convention when referring to the nodes of \( Y(\lambda) \). In other words \([r, c]\) refers to the node that occurs in \( Y(\lambda) \) in the \( r \)’th row from the top and the \( c \)’th column from the left. A \( \lambda \)-tableau \( t \) is a filling of the nodes of \( Y(\lambda) \) with the numbers \( 1, 2, \ldots, n \), each number occurring exactly once. We write \( t[r, c] \) for the number that occurs in the \([r, c]\)’th node of \( t \). Formally, \( t \) is a bijection from the nodes of \( Y(\lambda) \) to the numbers \( \{1, 2, \ldots, n\} \) and so \( t^{-1}(i) \) is the node of \( Y(\lambda) \) that contains \( i \). If \( t \) is a \( \lambda \)-tableau we say that \( t \) is row standard if the numbers of each row of \( t \) appear increasingly from left to right, and that \( t \) is column standard if the numbers of each column of \( t \) appear increasingly from top to bottom. If \( t \) is both row and column standard, we say that \( t \) is standard. We let \( t \mapsto \text{shape}(t) = \lambda \) if \( t \) is a \( \lambda \)-tableau. We denote by \( \text{Tab}(\lambda) \) and \( \text{Std}(\lambda) \) the sets of \( \lambda \)-tableaux, and standard \( \lambda \)-tableaux, respectively, and similarly we set \( \text{Std}(n) := \bigcup_{\lambda \in Par_n} \text{Std}(\lambda) \) and \( \text{Tab}(n) := \bigcup_{\lambda \in Par_n} \text{Tab}(\lambda) \).
For \( t \in \text{Std}(\lambda) \) and \( 1 \leq k \leq n \) we let \( t|_{1,2,\ldots,k} \) denote the tableau obtained by deleting the nodes of \( \mathcal{Y}(\lambda) \) containing the numbers \( k+1, k+2, \ldots, n \). With this notation the dominance order on \( \text{Par}_n \) is extended to standard \( \lambda \)-tableaux as follows: \( s \leq t \) if \( \text{shape}(s|_{1,2,\ldots,k}) \leq \text{shape}(t|_{1,2,\ldots,k}) \) for all \( k \). In a similar way, the total order \( < \) is extended to a total order on \( \text{Std}(\lambda) \), that is \( s < t \) if \( \text{shape}(s|_{1,2,\ldots,k}) < \text{shape}(t|_{1,2,\ldots,k}) \) for some \( k \) and \( \text{shape}(s|_{1,2,\ldots,k}) = \text{shape}(t|_{1,2,\ldots,k}) \) for all \( k' < k \).

There is a unique maximal \( \lambda \)-tableau \( t^\lambda \) with respect to both orders; it has the numbers \( 1, 2, \ldots, n \) filled in increasingly along the rows. There is also a unique minimal \( \lambda \)-tableau \( t_\lambda \) with respect to both orders; it has the numbers \( 1, 2, \ldots, n \) filled in increasingly along the columns. Here are examples of \( t^\lambda \) and \( t_\lambda \), where \( \lambda \) is as in (3).

\[
\begin{align*}
t^\lambda &= \begin{array}{cccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 \\
15 & 16 \\
17
\end{array}, &
t_\lambda &= \begin{array}{cccc}
6 & 10 & 13 & 16 \\
2 & 7 & 11 & 14 \\
3 & 8 & 12 & 15 \\
4 & 9 \\
5
\end{array}.
\end{align*}
\]

Since \( \mathfrak{S}_n \) is a Coxeter group it is endowed with the Chevalley-Bruhat order \( < \). We choose the convention that \( 1 \in \mathfrak{S}_n \) is the maximal element with respect to \( < \). The following compatibility between the Chevalley-Bruhat order and the dominance order is known as Ehresmann’s Theorem, see [Ma] for a proof.

**Theorem 1** Let \( \lambda \in \text{Par}_n \) and suppose that \( s, t \) are standard \( \lambda \)-tableaux. Then \( s < t \) if and only if \( d(s) < d(t) \).

We now recall the construction of Murphy’s standard basis for the Hecke algebra. \( \mathfrak{S}_n \) acts on the right on the set of \( \lambda \)-tableaux \( t \) by permuting the numbers inside each \( t \), that is, formally, \( (tw)([r, c]) := t([r, c])w \) for \( t \in \text{Tab}(\lambda) \) and \( w \in \mathfrak{S}_n \). For \( t \) a \( \lambda \)-tableau, we introduce \( d(t) \in \mathfrak{S}_n \) by requiring that \( t = t^\lambda d(t) \) and let \( \mathfrak{S}_\lambda \subseteq \mathfrak{S}_n \) denote the row stabilizer of \( t^\lambda \) under the \( \mathfrak{S}_n \)-action. For a pair \( (s, t) \) of \( \lambda \)-tableaux, Murphy introduced in this context the following elements of \( \mathcal{H}_n^A(q) \)

\[
x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w, \quad x_{st} := T_{d(s)^{-1}} x_\lambda T_{d(t)}.
\]

If \( s \) and \( t \) are standard tableaux we say that \( x_{st} \) is a standard element, otherwise we say that it is a nonstandard element. Murphy proved in [M1] that the standard elements \( x_{st} \) form an \( A \)-basis for \( \mathcal{H}_n^A(q) \), the standard basis. They also induce a basis for the specialized Hecke algebra.

Let \( \lambda \in \text{Par}_n \) and let \( \overline{N}_\lambda \) be the \( A \)-span of \( \{x_{st} \mid s, t \text{ are } \mu \text{-tableaux with } \mu > \lambda \} \). Then Murphy showed that \( \overline{N}_\lambda \) is a two-sided ideal of \( \mathcal{H}_n^A(q) \) and defined the Specht module \( S^A_q(\lambda) \) for \( \mathcal{H}_n^A(q) \) as the right submodule of \( \mathcal{H}_n^A(q)/\overline{N}_\lambda \), generated by \( x_\lambda + \overline{N}_\lambda \). It is a generalization of the Specht module known from the representation theory of \( \mathfrak{S}_n \), or more precisely of the dual Specht module.

In the representation theory of \( \mathfrak{S}_n \), as exposed for example in [J], the Specht modules play a key role. For a field \( k \) that contains \( \mathbb{Q} \), the Specht modules are the simple modules for \( k\mathfrak{S}_n \), and even for arbitrary fields \( k \) they can be used to describe the simple modules for \( k\mathfrak{S}_n \), at least in principle. Indeed, the Specht modules are endowed with certain, combinatorially defined, bilinear forms, and each simple module for \( k\mathfrak{S}_n \) can be realized as a quotient of a unique Specht module by the radical of its bilinear form. Similar statements also hold for the dual Specht modules for \( \mathfrak{S}_n \), but it should be noted that the parametrizations of the simple modules for \( k\mathfrak{S}_n \), using the Specht modules or the dual Specht modules, are different.

Returning to the Specht module \( S^A_q(\lambda) \) for \( \mathcal{H}_n^A(q) \), we have that it is free over \( A \) with basis given by \( x_t := x_{t^\lambda} + \overline{N}_\lambda \) where \( t \) runs over \( \text{Std}(\lambda) \). We refer to this basis as the standard basis for
$S_q^A(\lambda)$. For $\mathcal{B}$ an $A$-algebra, the standard basis induces a basis for the specialized Specht module $S_q^B(\lambda) := \mathcal{B} \otimes_A S_q^A(\lambda)$. If $\mathcal{B} = \mathcal{K}$ we shall also sometimes write $S(\lambda) = S_q^K(\lambda)$.

The standard basis $\{x_{s}\}$ is a cellular basis for $\mathcal{H}_q^{A}(q)$ in the sense of Graham and Lehrer, see [GL] and [Ma]. Thus $S_q^A(\lambda)$ is endowed with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. It is concretely given by $\langle x_s, x_t \rangle = a$ where $a$ is the coefficient of $x_{\lambda}$ in $x_{\lambda} T_{d(s)} T_{d(t)^{-1}} x_{\lambda}$ when expanded in the standard basis. For $\mathcal{B} = \mathcal{K}$ the form $\langle \cdot, \cdot \rangle$ is nondegenerate, but for example for $\mathcal{B} = \mathbb{F}_p$, the finite field of characteristic $p$, the form may be singular. In any case, the radical $\text{rad}(\cdot, \cdot)$ is a submodule of $S_q^B(\lambda)$. For $\mathcal{B} = \mathfrak{k}$ a field, we have that $S_q^K(\lambda)/\text{rad}(\cdot, \cdot)$ is either simple or zero and the nonzero modules that arise this way provide a classification of the simple modules for $\mathcal{H}_n^A(q)$.

We now describe the action of the $T_i$’s on the standard basis for $S_q^A(\lambda)$. Assume that $t \in \text{Std}(\lambda)$ and that $s = ts_i$. The action of $\mathcal{H}_n^A(q)$ on $x_t$ is then given by the following formulas

$$x_t T_i := \begin{cases} qx_t & \text{if } i \text{ and } i+1 \text{ are in the same row of } t \\ x_s & \text{if } i+1 \text{ is in a row below } i \text{ in } t \\ qx_s + (q-1)x_t & \text{if } i+1 \text{ is in a row above } i \text{ in } t. \end{cases}$$

(6)

Unfortunately, $s$ may be nonstandard even if $t$ is a standard and so we need straightening rules to express nonstandard elements in terms of standard elements.

The relevant straightening rules are the $q$-analogues of the Garnir relations, generalizing the usual Garnir relations known from the representation theory of the symmetric group $S_n$. Let $\lambda \in \text{Par}_n$, and choose $(i, j)$ such that $i \geq 1$ and $j \leq \lambda_{i+1}$. Define $\mu := (\lambda_1, \ldots, \lambda_i, j-1, j, j)$ if $i > 1$ and $\mu := (j-1, j)$ if $i = 1$ and suppose that $\mu \in \text{Comp}_m$. Then the $(i, j)$-Garnir tableau $\mathfrak{g}_{ij}$ is the $\lambda$-tableau satisfying that $\mathfrak{g}_{ij}[1,2,\ldots,m] = t^\mu$ and that the numbers $m+1, m+2, \ldots, n$ are filled in increasingly along the rows in the difference $\mathcal{Y}(\lambda) \setminus \mathcal{Y}(\mu)$. In particular, $\mathfrak{g}_{ij}$ is not column standard, since there is a descent between the nodes $[i, j]$ and $[i+1, j]$. Below we give the example $\mathfrak{g}_{2,3}$ using the same partition $\lambda = (5, 5, 4, 2, 1)$ as before.

$$\mathfrak{g}_{2,3} = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 11 & 12 & 13 \\
8 & 9 & 10 & 14 \\
15 & 16 \\
17
\end{array}$$

(7)

Set $k := \mathfrak{g}_{ij}[i+1, 1]$. Then the numbers $\{k, k+1, \ldots, m\}$ are the numbers of the $(i, j)$-Garnir belt, in (7) they are coloured blue. Let $S_{k,m}$ be the subgroup of $S_n$ consisting of the elements that fix $\{1, 2, \ldots, n\} \setminus \{k, k+1, \ldots, m\}$ pointwise. Set

$$S_i := \{w \in \text{Tab}(\lambda) \mid w \in S_{k,m} \text{ and } \mathfrak{g}_{ij}w \text{ is row-standard}\}.$$

(8)

The $q$-analogue of the Garnir relation is now the following relation in $S_q^A(\lambda)$

$$\sum_{s \in S_i} x_s = 0.$$  

(9)

The only nonstandard tableau appearing in $S_i$ is $\mathfrak{g}_{ij}$ itself, and so (9) can be used to express $x_{\mathfrak{g}_{ij}}$ in terms of standard elements. Using arguments explained in [M2], this can be extended to all nonstandard $x_t$.

We next explain the Jucys-Murphy elements for $\mathcal{H}_n^A(q)$. For $m = 1, 2, 3, \ldots, n$ we define $L_m \in \mathcal{H}_n^A(q)$ via

$$L_m := q^{-1} T_{(m-1,m)} + q^{-2} T_{(m-2,m)} + \ldots + q^{1-m} T_{(1,m)}$$

(10)
where \((i, j)\) is the element of \(S_n\) given in usual cycle notation, and where by convention we set 
\(L_1 := 0\). These are the Jucys-Murphy elements for \(\mathcal{H}_n^A(q)\). For \(k \in \mathbb{Z}\) we introduce the following Gaussian integer 
\[
[k]_q := \begin{cases} 
1 + q + q^2 + \cdots + q^{k-1} & \text{if } k > 0 \\
0 & \text{if } k = 0 \\
-(q^{-1} + q^{-2} + \cdots + q^{-k}) & \text{if } k < 0.
\end{cases}
\] (11)

Thus for all \(k \in \mathbb{Z}\) we have \([k]_q = \frac{q^k - 1}{q - 1}\) and \(\lim_{q \to 1}[k]_q = k\). With this at hand we have for \(t \in \text{Tab}(\lambda)\) the content function
\[
c_i : \{1, 2, \ldots, n\} \to \mathcal{A}, i \mapsto [c - r]_q \text{ where } t[r, c] = i.
\] (12)

If \(t = t^\lambda\) we often write \(\lambda\) for \(t^\lambda\) as a subscript, for example \(c_t(i) = c_{\lambda}(i)\). In the context of the specialized Hecke algebra \(\mathcal{H}_n^B(q)\), the content function \(c_t\) is given by the same formula (12) but now takes values in \(\mathcal{B}\). For \(\lambda \in \text{Std}(\lambda)\), the \(L_i\)’s satisfy the following JM-triangularity property with respect to the \(c_t\)’s. This key property is due to Murphy, see [M3], to our knowledge it does not appear anywhere before [M3].

\[
L_i x_t = c_t(i) x_t + \sum_{s \in \text{Std}(\lambda), s \triangleleft t} a_s x_s, \quad \text{where } a_s \in \mathcal{A}.
\] (13)

Let \(\mathcal{B} = \mathcal{K}\). Then an important application of the \(L_i\)’s is the construction of idempotents in the Hecke algebra \(\mathcal{H}_n\). Let \(\mathcal{C}_n := \{c_t(i) \mid \lambda \in \text{Par}_n, t \in \text{Std}(\lambda)\}\) and define for a \(\lambda\)-tableau \(t\) the following element of \(\mathcal{H}_n\)
\[
E_t := \prod_{m=1}^{n} \prod_{c \in \mathcal{C}_n \setminus \text{Std}(m)} \frac{L_m - c}{c_t(m) - c}.
\] (14)

Then the \(E_t\)’s are a complete set of primitive idempotents in \(\mathcal{H}_n\), for \(t\) running over all standard tableaux. In case \(t\) is a nonstandard tableau, the formula (14) also defines an idempotent \(E_t\) in \(\mathcal{H}_n\), but one gets nothing new since \(E_t = E_s\) for some standard tableau \(s\). For \(t \in \text{Std}(\lambda)\), the idempotent \(E_t\) gives rise to an element \(f_t\) of \(S^K_q(\lambda)\) as follows
\[
f_t = x_t E_t \in S^K_q(\lambda).
\] (15)

Then \(\{f_t \mid t \in \text{Std}(\lambda)\}\) is the \(q\)-analogue of the seminormal basis for \(S^K_q(\lambda)\). It consists of common eigenvectors for the \(L_i\)’s, with eigenvalues given by the contents, that is
\[
L_i f_t = c_t(i) f_t, \quad i = 1, \ldots, n.
\] (16)

We keep \(\mathcal{B} = \mathcal{K}\) and so we have that \(\langle \cdot, \cdot \rangle_\lambda\) is nondegenerate. Moreover the \(L_i\)’s are selfadjoint with respect to \(\langle \cdot, \cdot \rangle_\lambda\) and for any \(s, t \in \text{Std}(\lambda), s \neq t\), there is an \(i\) such that \(c_s(i) \neq c_t(i)\). From this we deduce that the \(f_t\)’s are orthogonal with respect to \(\langle \cdot, \cdot \rangle_\lambda\). Furthermore we have the following triangular expansion, which is a consequence of (13)
\[
f_t = x_t + \sum_{s \in \text{Std}(\lambda), s \triangleleft t} a_s x_s, \quad \text{where } a_s \in \mathcal{K}.
\] (17)

The seminormal basis \(\{f_t\}\) for \(S^K_q(\lambda)\) can also be constructed using a Gram-Schmidt orthogonalization process on the standard basis \(\{x_t\}\), with respect to \(\langle \cdot, \cdot \rangle_\lambda\). Recall the extension of \(\ll\) to
a total order < on \( \text{Std}(\lambda) \). For the Gram-Schmidt process we first take \( f_\lambda = f_t := x_\lambda = x_t \) and then continue recursively downwards along < as follows

\[
f_t := x_t - \sum_{s \in \text{Std}(\lambda), s > t} \frac{\langle f_s, x_t \rangle_\lambda}{\langle f_s, f_s \rangle_\lambda} f_s,
\]

(18)

Apriori, the orthogonal basis that results from this Gram-Schmidt process may depend on how \(<\) is extended to a total order. On the other hand, using (13) one checks that replacing \(<\) by \(\triangleleft\) in (18) one obtains the same \( f_t \)’s from the Gram-Schmidt process. In other words, in (18) we can use any extension of \(<\) to a total order, without changing the outcome.

This formalism is reminiscent of basic principles in the theory of symmetric functions. A natural basis for the space of symmetric functions \( \text{Sym} \) is given by the monomial symmetric functions \( \{ m_\lambda | \lambda \in \text{Par} \} \). The Macdonald polynomials \( \{ P_\lambda | \lambda \in \text{Par} \} \) are another basis for \( \text{Sym} \) which is constructed via a Gram-Schmidt algorithm on \( \{ m_\lambda | \lambda \in \text{Par} \} \), using a certain inner product on \( \text{Sym} \). The order relation here is an extension of the dominance to a total order on all of \( \text{Par} \). On the other hand the \( P_\lambda \)’s can also be realized as common eigenvectors for a family of selfadjoint operators on \( \text{Sym} \) that have their origin in the Cherednik algebra. As was the case for the seminormal basis, one then concludes that the Gram-Schmidt construction of \( P_\lambda \) does not depend on the choice of extension of the dominance order to a total order on \( \text{Sym} \).

It should be pointed out, however, that in both cases it is difficult to gain information about the orthogonal basis directly from the Gram-Schmidt process, and so the Gram-Schmidt process is more a tool for calculating examples than a tool for deducing theoretical properties of the basis.

Returning to the seminormal basis \( \{ f_t | t \in \text{Std}(\lambda) \} \), the action of the \( T_i \)’s on \( S^K_q(\lambda) \) is much easier to describe than using the standard basis, since the Garnir relations are not needed. In fact this is one of the big advantages of the seminormal basis over the standard basis.

Suppose that \( t \in \text{Std}(\lambda) \) and let \( s := ts_i \). Define the radial distance \( \rho := c_t(i) - c_s(i) = c_t(i) - c_t(i + 1) \). Then we have the following formulas, known as Young’s seminormal form, or simply YSF. They play an important role throughout the paper.

\[
f_t T_i = \begin{cases} 
q f_t & \text{if } i \text{ and } i + 1 \text{ are in the same row of } t \\
-f_t & \text{if } i \text{ and } i + 1 \text{ are in the same column of } t \\
-\frac{1}{[\rho]_q} f_t + f_s & \text{if } s \text{ is standard and } i + 1 \text{ is in a row below } i \text{ in } t \\
-\frac{1}{[\rho]_q} f_t + \frac{q[\rho + 1]_q[\rho - 1]_q}{[\rho]_q^2} f_s & \text{if } s \text{ is standard and } i + 1 \text{ is in a row above } i \text{ in } t.
\end{cases}
\]

(19)

These formulas were proved in Theorem 6.4 of [M1], although the formulation there contained several minor errors, see the discussion in the proof of Theorem 2.3 of [FLT1].

3 An algorithm using Young’s seminormal form

As mentioned in the introduction, our goal is to find a formula for the base change matrix between the \( \{ f_t \} \) basis and the \( \{ x_s \} \) basis for \( S^K_q(\lambda) \), or equivalently to determine the \( a_s \)’s that appear in (17).

A first observation is that Young’s seminormal form, that is formula (19), in fact does give rise to an algorithm for writing \( f_t \) as a linear combination of \( x_s \)’s, but with \( s \) running over all of \( \text{Tab}(\lambda) \), not just \( \text{Std}(\lambda) \). In this section we explain this algorithm.
Suppose that $t \in \text{Std}(\lambda)$ and write $d(t) = s_{i_1}s_{i_2} \cdots s_{i_k}$ in reduced form. From this we get a chain of standard tableaux $\{t_0, t_1, \ldots, t_k\} \subseteq \text{Std}(\lambda)$ as follows

\[ t_0 := t^k, \ t_1 := t_0 s_{i_1}, \ t_2 := t_1 s_{i_2}, \ldots, \ t_k := t_{k-1} s_{i_k} \quad (20) \]

where $t = t_k$. Using Theorem 1 we get that $t_j < t_{j-1}$ for all $j$, and so $i_j + 1$ appears in a row below $i_j$ in $t_{j-1}$ for all $j$. This means that when calculating $f_{t_{j-1}}T_j$ using YSF, we are in the third case of (19).

For our algorithm we start out with the equality $f_\lambda = x_\lambda$. Applying $T_{i_1}$ to each side of this equality, the left hand becomes $f_\lambda T_{i_1}$ which can be calculated using YSF, whereas the right hand side becomes $x_\lambda T_{i_1} = x_{i_1}$, via (6). To be precise, when applying YSF we get

\[ f_0T_{i_1} = x_{i_1} \iff f_0 + a_1f_{i_1} = x_{i_1} \iff f_{i_1} = a_{i_1}^{-1}(-f_0 + x_{i_1}) \iff \]

\[ f_{i_1} = a_2x_{i_0} + a_3x_{i_1} \quad (21) \]

for certain explicit coefficients $a_1, a_2, a_3 \in \mathcal{K}^*$. This is the first step of the algorithm. For the next step we apply $T_{i_2}$ to both sides of the equation $f_{i_1} = a_2x_{i_0} + a_3x_{i_1}$, that we just found. The left hand side is $f_{i_1}T_{i_2}$ that can be calculated using YSF and the right hand can be calculated via (6). In more detail, for the left hand side we get via YSF that $f_{i_1}T_{i_2} = f_{i_2} + b_1f_{i_1}$. For the right hand side $(a_2x_{i_0} + a_3x_{i_1})T_{i_2}$ we have from (6) an expansion of the form $(a_2x_{i_0} + a_3x_{i_1})T_{i_2} = \sum_{s \in \text{Tab}(\lambda)} c_s x_s$

Combining, and using the first step of the algorithm, we get

\[ f_{i_2} + b_1f_{i_1} = \sum_{s \in \text{Tab}(\lambda)} c_s x_s \iff f_{i_2} = -b_1(a_2x_{i_0} + a_3x_{i_1}) + \sum_{s \in \text{Tab}(\lambda)} c_s x_s = \sum_{s \in \text{Tab}(\lambda)} d_s x_s \quad (22) \]

for some $d_s \in \mathcal{K}$. This is the second step of the algorithm. The following steps of the algorithm are essentially identical to the second step: as mentioned above $f_{t_{j-1}}T_{i_j}$ is always calculated using the third case of (19). The $k$’th step of the algorithm gives the promised expansion of $f_t$.

**We call this algorithm for calculating $f_t$ ’repeated use of YSF’**.

**Example.** Let $\lambda := (2, 1, 1)$ and let $t := t_\lambda$, that is

\[ t = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 3 \\ 4 \end{array} \quad (23) \]

We have $d(t) = s_2s_3$ and the associated series of $\lambda$-tableaux $t_0, t_1, t_2$ is then as follows

\[ t_0 = t^4 = \begin{array}{ccc} 1 & 2 \\ 3 & 4 \\ 4 \end{array}, \quad t_1 = \begin{array}{ccc} 1 & 3 \\ 2 & 4 \end{array}, \quad t = t_2 = \begin{array}{ccc} 1 & 4 \\ 2 \\ 3 \end{array}. \quad (24) \]

Step 1 of the algorithm gives

\[ f_{i_1} = x_{i_1} + \frac{1}{[2]_q}x_{i_0}. \quad (25) \]

Step 2 of the algorithm, using (6), then gives

\[ f_{i_2} = \frac{1}{[3]_q} \left( x_{i_1} + \frac{1}{[2]_q}x_{i_0} \right) + \left( x_{i_1} + \frac{1}{[2]_q}x_{i_0} \right) T_3 \iff \]

\[ f_{i_3} = f_{i_2} = \frac{1}{[3]_q} \left( x_{i_1} + \frac{1}{[2]_q}x_{i_0} \right) + x_{i_2} + \frac{q}{[2]_q}x_{i_3} + \frac{q - 1}{[2]_q}x_{i_0} \quad (26) \]
where $t_3$ is the tableau
\[ t_3 := \begin{array}{ccc}
1 & 2 \\
4 \\
3 
\end{array} \]  \hfill (27)

Thus even for this small example the algorithm produces nonstandard tableaux that must be straightened with the Garnir relations, (9). This straightening procedure is in general a complicated combinatorial procedure that often has to be repeated many times until arriving at the desired linear combination of standard tableaux. In the following sections we shall however see that in the context of the above algorithm, the straightening procedure can be controlled, at least in certain nontrivial cases.

4 A generalization of the James-Murphy tableaux

In this section we explain the kind of tableaux in which we shall be particularly interested.

For $1 \leq i \leq j \leq n$ we introduce $\sigma_{ij} \in S_n$ as follows
\[ \sigma_{ij} := \begin{cases} 
\sigma_i \sigma_{i+1} \cdots \sigma_{j-1} & \text{if } i < j \\
1 & \text{if } i = j.
\end{cases} \] \hfill (28)

In other words, $\sigma_{ij}$ is a one-cycle permutation. We extend this notation to the Hecke algebra via $T_{ij} := T_{\sigma_{ij}}$.

Let us now fix $\lambda \in \text{Par}_n$ and $1 \leq a \leq n$. Suppose that the $a$-node of $t^\lambda$, that is $(t^\lambda)^{-1}(a)$, is a removable node of $Y(\lambda)$, meaning that when it is removed from $Y(\lambda)$ we still get the Young diagram of a partition. For any $a \leq b \leq n$, we define the tableau $t_b$, as follows
\[ t_b := t^\lambda \sigma_{ab}. \] \hfill (29)

Then $t_b$ is always a standard tableau. For example, for $\lambda = (6, 6, 5, 3, 3, 3) \in \text{Par}_{26}$ and $a = 12$, we have $t^\lambda$, $t_{19}$ and $t_{25}$ as follows
\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 \\
18 & 19 & 20 \\
21 & 22 & 23 \\
24 & 25 & 26 \\
\end{array}, \quad
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 19 \\
12 & 13 & 14 & 15 & 16 \\
17 & 18 & 20 \\
21 & 22 & 23 \\
24 & 25 & 26 \\
\end{array}, \quad
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 25 \\
12 & 13 & 14 & 15 & 16 \\
17 & 18 & 19 \\
20 & 21 & 22 \\
23 & 24 & 26 \\
\end{array}.
\] \hfill (30)

We shall be especially interested in the cases where the $b$-node of $t^\lambda$ belongs to the right border of $Y(\lambda)$, that is $b$ and $b+1$ lie in different rows of $t^\lambda$ or $b = n$. For example, keeping $\lambda = (6, 6, 5, 3, 3, 3)$, the values of $b$ for which the $b$-node of $t^\lambda$ belongs to the right border of $Y(\lambda)$ are $b = 17, 20, 23, 26$. Below we depict $t^\lambda$ with $a$ in blue, and $b \neq a$ in red
\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 \\
18 & 19 & 20 \\
21 & 22 & 23 \\
24 & 25 & 26 \\
\end{array}. \] \hfill (31)
The corresponding tableaux \( t_b \) are as follows

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 \\
\end{array}, \quad 
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 \\
\end{array}, \quad 
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 \\
\end{array}, \quad 
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 & 29 & 30 \\
\end{array}.
\]

(32)

In the particular case where \( b = n \), the tableau \( t_b = t_n \) has the following property

\[
t_n|_{1,2,\ldots,n-1} = t^\mu
\]

(33)

where \( \mu := \text{shape}(t_n|_{1,2,\ldots,n-1}) \). This kind of tableaux plays an important role in James and Murphy’s paper [JM], where the determinant of \( \langle \cdot, \cdot \rangle_\lambda \) is calculated, and for this reason we call the \( \lambda \)-tableaux of the form \( t_b \) generalized James-Murphy tableaux. In [JM], the authors prove a formula for \( \langle f_b, f_b \rangle_\lambda \) that gives rise to a branching rule for \( \langle \cdot, \cdot \rangle_\lambda \) over \( \mathcal{K} \). From this branching rule they calculate the determinant itself by induction.

5 YSF along one row

In this section we shall see that when we work along a single row of \( t^\lambda \), in a sense that we shall shortly explain, there is a simple relationship between the corresponding seminormal elements, involving only one denominator.

For \( b \geq a \), we consider the tableau \( t_b \) as in the previous section and write for simplicity

\[
x_b := x_b, \quad f_b := f_b, \quad c_b(i) := c_b(i).
\]

(34)

We also define \( r_b \) via

\[
r_b := c_\lambda(a) - c_\lambda(b).
\]

(35)

Let \( \{b_0, b_1, \ldots, b_m\} \) be the values of \( b \) corresponding to the right border of \( t^\lambda \), ordered increasingly with \( b_0 = a \), as in (31) and (32). We are interested in the relation between \( f_{b_i} \) and \( f_{b_{i+1}} \). We have the following Lemma, that shall be used throughout.

**Lemma 1** With the above notation the following formula holds

\[
f_{b_{i+1}} = f_b \left( T_{b_i,b_{i+1}} + \frac{1}{[r_{b_{i+1}}]_q} (1 + T_{b_i,b_{i+1}} + T_{b_i,b_{i+2}} + \ldots + T_{b_i,b_{i+1}-1}) \right).
\]

(36)

(Note that the occurring \( T_{b_i,b} \)'s have the second index \( \beta \) running over the row of \( t_{b_i} \) that contains \( b_{i+1} \).

**Proof:** Let \( c_i := c_b(b_i) - c_b(b_i+1) \) be the radial distance in \( t_{b_i} \) from the \( b_i \)-node to the \( (b_i+1) \)-node. Note that \( c_i = c_\lambda(a) - c_\lambda(b_i+1) \), see for example the tableaux in (32) of the previous section. The \( (b_i+1) \)-node is situated in a row below the \( b_i \)-node in \( t_{b_i} \) and so when applying YSF we are in the third case of (19), that is

\[
f_{b_i} T_{b_i} = f_{b_{i+1}} - \frac{1}{[c_i]_q} f_{b_i} \iff f_{b_{i+1}} = f_{b_i} \left( T_{b_i} + \frac{1}{[c_i]_q} \right).
\]

(37)
We now combine this with the expression for \( f_{b_i+1} \) found in (37) and get
\[
f_{b_i+2} = f_{b_i} \left( T_{b_i} + \frac{1}{[c_i]_q} \right) T_{b_i+1} + f_{b_i} \left( T_{b_i} + \frac{1}{[c_i]_q} \right) \frac{1}{[c_i-1]_q} = f_{b_i} \left( T_{b_i} + \frac{1}{[c_i]_q} \right) T_{b_i+1} + \frac{1}{[c_i-1]_q} + \frac{1}{[c_i]_q} T_{b_i} + \frac{1}{[c_i-1]_q} \frac{1}{[c_i]_q} = f_{b_i} \left( T_{b_i} + \frac{1}{[c_i]_q} \right) T_{b_i+1} + \frac{1}{[c_i-1]_q} (T_{b_i} + 1)
\]
where for the second equality we used \( f_{b_i} T_{b_i+1} = qf_{b_i} \), which is the first case of (19), and for the third equality the following Gaussian integer identity
\[
q[k-1]_q + 1 = [k]_q
\]
with \( k = c_i \). For the denominator \([c_i-1]_q\) appearing in the final expression in (38) we have that
\[
[c_i - 1]_q = [c_\lambda(a) - c_\lambda(b_i + 2)]_q
\]

This argument is repeated for \( t_{b_{i+3}}, t_{b_{i+4}}, \ldots \) and so on, until \( t_{b_{i+1}} \). When we arrive at \( t_{b_{i+1}} \) the expression for \( f_{b_{i+1}} \) will be as in (36) and the denominator involved will be
\[
[c_\lambda(a) - c_\lambda(b_i + 1)]_q = [r_{b_{i+1}}]_q
\]
which proves the Lemma.

In the setting of (31) and (32) the Lemma gives for example
\[
f_{t_{17}} = f_{t_{12}} \left( T_{12,17} + \frac{1}{[2]_q} (1 + T_{12,13} + T_{12,14} + T_{12,15} + T_{12,16}) \right) \quad \text{and}
\]
\[
f_{t_{20}} = f_{t_{17}} \left( T_{17,20} + \frac{1}{[5]_q} (1 + T_{17,16} + T_{17,19}) \right)
\]

One should observe that it is not possible to have \( r_1 = 1 \) in the Lemma since that would imply that the \( a \)-node of \( t^3 \) is not removable, contrary to the hypothesis of the Lemma. Even so we could still define \( t_{b_i} \) using the same formula (29), but it would be a nonstandard tableau. Note that the sum of Hecke algebra elements in (36) in this 'limiting case' is exactly the same as the sum of Hecke algebra elements of a Garnir relation. This coincidence lies at the heart of the results of the following sections.

We should remark that Theorem 5.5 of [Ram] offers an alternative approach to Lemma 1. Ram’s Theorem relates the Garnir relations to Young’s seminormal form, in fact in the general setting of calibrated representations for affine Hecke algebras. We should however also point out that the arguments of the forthcoming sections only rely on one specific Garnir relation, namely the one mentioned above, and hence the results of [Ram] do not offer an alternative approach to the rest of our article.
6 Fat hook partitions

We assume in this section that \( \lambda \) is a fat hook partition, by which we mean that it has the form

\[
\lambda = (\lambda_1^{k_1}, \lambda_2^{k_2}) := (\lambda_1, \ldots, \lambda_{k_1}, \lambda_2, \ldots, \lambda_{k_2}),
\]

where \( k_1, k_2 \geq 1 \). Then \( \lambda \in \Par_n \) with \( n = k_1\lambda_1 + k_2\lambda_2 \) and \( \lambda \) has exactly two removable nodes. We focus on the rightmost of these, the one with coordinates \([k_1, \lambda_1] \), and set \( a := t^\lambda [k_1, \lambda_1] \). As an example we take \( \lambda = (6^2, 4^3) \) where \( a = 12 \), that is

\[
t^\lambda = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 \\
\end{array}
\]  

(44)

For \( i = 1, 2, \ldots, k_2 \), we associate with the \( i \)th row of \( t^\lambda \), counted from below \([k_1, \lambda_1] \), an element \( R_i \) of \( \mathcal{H}_n^A(q) \) as follows. Let \( \{b_0, b_1, b_2, \ldots, b_k\} \) denote the right border elements of \( t^\lambda \) as before. Then the \( i \)th row of \( t^\lambda \), counted from \([k_1, \lambda_1] \), contains the numbers \( \{b_{i-1} + 1, b_{i-1} + 2, \ldots, b_i - 1, b_i\} \). Partially inspired by Lemma 1, we then define \( R_i \in \mathcal{H}_n^A(q) \) via

\[
R_i := 1 + T_{b_{i-1},b_{i-1}+1} + T_{b_{i-1},b_{i-1}+2} + \ldots + T_{b_{i-1},b_i-1}
\]

(45)

(thus \( b_i \) is skipped). For example, with \( \lambda \) as in (44) we have

\[
R_1 = 1 + T_{12,13} + T_{12,14} + T_{12,15} \quad \text{and} \quad R_2 := 1 + T_{16,17} + T_{16,18} + T_{16,19}.
\]

(46)

We next define \( F_1 := x_\lambda R_1 = x_{b_0} R_1 \) and recursively for \( i = 2, \ldots, k_2 \)

\[
F_i := (x_{b_i} - qF_{i-1}) R_i.
\]

(47)

For example, with \( \lambda \) as in (44) we have

\[
F_3 = (x_{b_2} - q(x_{b_1} - qx_{b_0}R_1)) R_3 = (x_{20} - q(x_{16} - qx_{12}R_1)) R_3.
\]

(48)

Let us consider the expansion of \( F_{k_2} \) in terms of \( x_i \)’s. We first observe the following useful reformulation of \( F_{k_2} \)

\[
F_{k_2} = \sum_{j=1}^{k_2} (-q)^{k_2-j} x_{b_{j-1}} R_j R_{j+1} \cdots R_{k_2}
\]

(49)

that follows directly from the definitions. Note that the \( R_j \)’s commute. Now choosing any summand \( T_{\sigma} \) from each \( R_j \), one checks easily from (49) that

\[
x_{b_{j-1}} T_{\sigma_{j-1} T_{\sigma_{j+1}} \cdots T_{\sigma_{k_2}}} = x_{b_{j-1}} \sigma_{j-1} \sigma_{j+1} \cdots \sigma_{k_2}
\]

(50)

and that this is a standard element. From this we deduce that the expansion of \( F_{k_2} \) consists of standard elements.

With the notation of (35) we set \( r := r_n \). For example, in the above case (44) we have that \( r = 5 \). Our first Theorem is the following surprising formula for the expansion of \( f_n \) in terms of standard elements, involving only one denominator.

**Theorem 2** We have \( f_n = e_n + \frac{1}{\prod_{i=1}^{k_2} F_{k_2}} \). Moreover, the expansion of \( F_{k_2} \) gives rise to a linear combination of standard \( x_i \)’s as explained above.
Proof: The second statement was proved in (50) so let us concentrate on the first statement, that is the formula for \( f_n \). We prove by induction on \( j \) that

\[
f_{b_j} = x_{b_j} + \frac{1}{|r_{b_j}|_q} F_j
\]

from which the formula follows by setting \( j = k_2 \). The colour blue is only meant to help visualizing the cancellations that take place. The induction basis \( j = 1 \) follows directly from Lemma 1 and the definitions, so let us prove the induction step from \( j - 1 \) to \( j \). Thus we assume that \( f_{b_{j-1}} = x_{b_{j-1}} + \frac{1}{|r_{b_{j-1}}|_q} F_{j-1} \). From this we deduce via Lemma 1 that

\[
f_{b_j} = (x_{b_j} + \frac{1}{|r_{b_j}|_q} F_{j-1})T_{b_{j-1}, b_j} + \frac{1}{|r_{b_j}|_q} (x_{b_{j-1}} + \frac{1}{|r_{b_{j-1}}|_q} F_{j-1}) R_j
\]

\[
= x_{b_j} + \frac{1}{|r_{b_j}|_q} x_{b_{j-1}} R_j + \frac{1}{|r_{b_{j-1}}|_q} (F_{j-1} T_{b_{j-1}, b_j} + \frac{1}{|r_{b_{j-1}}|_q} F_{j-1} R_j).
\]

(52)

We consider the two terms of the last parenthesis. Let \( x_t \) be a standard element occurring in the expansion of \( F_{j-1} \), in the sense explained in (50). Then the action of \( R_j \) on \( x_t \) only involves the 'easy' second case of (6). To be more precise, let \( \sigma^1, \sigma^2, \ldots, \sigma^{\lambda_2} \) be the permutations involved in the definition of \( R_j \), corresponding to the numbers of the \( j \)th row of \( t^\lambda \) counted from \([k_1, \lambda_1]\), see (45). Then we have that

\[
x_t R_j = x_{t\sigma^1} + x_{t\sigma^2} + \ldots + x_{t\sigma^{\lambda_2}}.
\]

(53)

We keep \( x_t \), but now focus on the \( x_t T_{b_{j-1}, b_j} \) term of the last parenthesis of (52). Recall that by definition \( T_{b_{j-1}, b_j} = T_{b_{(j-1)}, b_{j-1}} T_{b_{j-1}, b_j} \). Using this we get once again via (6) that

\[
x_t T_{b_{(j-1)}, b_j} = x_{t\sigma_{b_{(j-1)}, b_j}}
\]

(54)

but this \( x_{t\sigma_{b_{(j-1)}, b_j}} \) is not a standard element. To illustrate, we take \( \lambda \) as in (44), \( j = 2 \), and \( x_t \) occurring in the expansion of \( F_{j-1} = F_1 \) with \( t \) and \( t\sigma_{b_{(j-1)}, b_j} = t\sigma_{16,20} \) as follows

\[
t = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 15 \\ 12 & 13 & 14 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix}, \quad t\sigma_{16,20} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 15 \\ 12 & 13 & 14 & 20 \\ 16 & 17 & 18 & 19 \\ 21 & 22 & 23 & 24 \end{bmatrix}.
\]

(55)

The numbers where \( t \) and \( t\sigma_{16,20} \) differ have been coloured red, they form a Garnir belt.

The last comment holds in general. Let therefore \( g_{k_1, j-1, \lambda_2} \) be the Garnir \( \lambda \)-tableau, as introduced in the section above (7). Then we have that \( d(g_{k_1, j-1, \lambda_2}) = \sigma_{b_{(j-1), b_j}} \) which is a \( S_n \)-element commuting with \( d(t) \) and so the Garnir relation (9) gives that

\[
x_{t\sigma_{b_{(j-1), b_j}}} + x_{t\sigma^1} + x_{t\sigma^2} + \ldots + x_{t\sigma^{\lambda_2}} = 0
\]

(56)

where the \( \sigma^i \)'s are as in (53). Combining (56), (52) and (54), and using once again the Gaussian identity (40), we arrive at

\[
f_{b_j} = x_{b_j} + \frac{1}{|r_{b_j}|_q} x_{b_{j-1}} R_j - \frac{q}{|r_{b_j}|_q} F_{j-1} R_j = x_{b_j} + \frac{1}{|r_{b_j}|_q} (x_{b_{j-1}} - q F_{j-1}) R_j = x_{b_j} + \frac{1}{|r_{b_j}|_q} F_j.
\]

(57)

This completes the inductive step of the proof of the Lemma. \( \square \)
Let us illustrate the formula on the partition $\lambda = (3, 2^2)$ of 7. In that case we have $r = 3$ and the formula for $f_7$ becomes

$$
\begin{pmatrix}
1 & 2 & 7 \\
3 & 4 & \\
5 & 6 & \\
\end{pmatrix} + \frac{1}{3!} \left( \begin{pmatrix}
1 & 2 & 6 \\
3 & 4 & \\
5 & 7 & \\
\end{pmatrix} - q \begin{pmatrix}
1 & 2 & 3 \\
3 & 4 & \\
5 & 7 & \\
\end{pmatrix} - q \begin{pmatrix}
1 & 2 & 4 \\
3 & 5 & \\
6 & 7 & \\
\end{pmatrix} - q \begin{pmatrix}
1 & 2 & 3 \\
3 & 6 & \\
5 & 7 & \\
\end{pmatrix} \right)
$$

(58)

where we identify $t$ and $x_i$.

Let us now illustrate the Theorem on the bigger example $t_{24}$ for $\lambda = (6^2, 4^3)$, that is

$$
t_{24} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 24 \\
12 & 13 & 14 & 15 & \\
16 & 17 & 18 & 19 & \\
20 & 21 & 22 & 23 & \\
\end{pmatrix}
$$

(59)

which will also give an indication on how to work with the Theorem in general, via formula (49) for $F_{k_2}$. In this case $k_2 = 3$ and so there will be three summands $x_{b_{j-1}} R_j R_{j+1} \cdots R_{k_2}$ in (49) with the following 'leading terms' $x_{b_{j-1}}$

$$
x_{b_0} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & \\
17 & 18 & 19 & 20 & \\
21 & 22 & 23 & 24 & \\
\end{pmatrix}, \quad x_{b_1} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 16 \\
12 & 13 & 14 & 15 & \\
17 & 18 & 19 & 20 & \\
21 & 22 & 23 & 24 & \\
\end{pmatrix}, \quad x_{b_2} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & \\
17 & 18 & 19 & 20 & \\
21 & 22 & 23 & 24 & \\
\end{pmatrix}.
$$

(60)

By definition $x_{b_0} R_1 R_2 R_3$ is the sum of all the products of the 'monochromatic' cycles of $x_{b_0}$ where each cycle starts in 12, 16 or 20, in other words products of a 'red' cycle 1, (12, 13), (12, 13, 14), (12, 13, 14, 15), a 'blue' cycle 1, (16, 17), (16, 17, 18), (16, 17, 18, 19) and a 'green' cycle 1, (20, 21), (20, 21, 22) and (20, 21, 22, 23). Note that this can also be described as the sum of all standard elements obtained from $x_{b_0}$ by shuffling monochromatic numbers. Thus $x_{b_0} R_1 R_2 R_3$ is the sum of $4 \times 3 \times 3 = 36$ standard elements, that all enter in $F_{k_2}$ with coefficient $q^2$, and similarly $x_{b_2} R_2 R_3$ is the sum of 12 standard elements that all enter in $F_{k_2}$ with coefficient $-q$ whereas $x_{b_2} R_3$ is the sum of 4 standard elements that all enter in $F_{k_2}$ with coefficient 1.

One also checks that the standard elements in $x_{b_0} R_1 R_2 R_3$, $x_{b_2} R_2 R_3$ and $x_{b_2} R_3$ are all different. For example, if $x_1$ is a standard element appearing in both $x_{b_0} R_1 R_2 R_3$ and $x_{b_2} R_2 R_3$, then looking at the distribution of blue numbers in $x_{b_0}$ and $x_{b_1}$ one sees that 16 would have to be in the fourth row of $t$ and therefore either 17, 18 or 19 would have to be in the second row of $t$, which is impossible.

### 7 Expansion of $f_n$ for general partitions

Our next aim is to show that the results from the previous section can be extended to arbitrary partitions $\lambda$. In this section we determine the expansion of the seminormal element $f_n$ for the generalized James-Murphy tableaux in terms of standard elements. Although this extension does not require substantially new ideas compared to the previous section, the notational technicalities are more involved and so we prefer to treat this extension separately.

Let us set up the relevant notation. Let $\lambda$ be a partition of $n$. We fix a removable node $[\alpha_0, \beta_0]$ for $\lambda$ and let the removable nodes below $[\alpha_0, \beta_0]$ be $[\alpha_j, \beta_j]$, $j = 1, 2, \ldots, N$, from top to bottom. In (61) we give the example $\lambda = (6^2, 4^3, 3^2, 1)$, where we choose $[\alpha_0, \beta_0] := [2, 6]$ and
where we indicate with arrows the \([\alpha_j, \beta_j]'s. We set \(d_j := t^k[\alpha_j, \beta_j]\) and so have in (61) that \(d_0 = 12, d_1 = 24, d_2 = 30, d_3 = 31.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 \\
25 & 26 & 27 \\
28 & 29 & 30 \\
31
\end{array}
\]

(61)

With the notation from the previous sections we have \(t_n = t^k\sigma_{d_0,d_N}\) and our aim is to determine the expansion of \(f_n = f_1\) in terms of standard elements \(x_i\). Here is \(t_n\) for the same \(\lambda\) as in (61).

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 31 \\
12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 \\
20 & 21 & 22 & 23 \\
24 & 25 & 26 \\
27 & 28 & 29 \\
30
\end{array}
\]

(62)

Now \(\lambda\) determines a series of subpartitions \(\text{hook}_j(\lambda), j = 1, 2, \ldots, N\) of \(\lambda\) via

\[
\mathcal{Y}(\text{hook}_j(\lambda)) := \text{shape}(t_{1,2,\ldots,d_j}) \setminus \text{shape}(t_{1,2,\ldots,d_{j-2}})
\]

(63)

where \(d_{-1} := 0\) and \(\text{shape}(t_{1,2,\ldots,0}) := \emptyset\). We then define the \(\text{hook}_j(\lambda)\)-tableau \(t_{\text{hook}_j(\lambda)}\) to be the restriction of \(t^k\) to \(\text{hook}_j(\lambda)\). Thus the numbers appearing in \(t_{\text{hook}_j(\lambda)}\) are \(\{d_{j-2} + 1, d_{j-2} + 2, \ldots, d_j\}\) and so, strictly speaking, \(t_{\text{hook}_j(\lambda)}\) is not a tableau according to the definition in section 2, but still we shall refer to it as a \(\text{hook}_j(\lambda)\)-tableau. Below we give the \(t_{\text{hook}_j(\lambda)}'s\) for \(\lambda\) as in (61).

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 \\
25 & 26 & 27 \\
28 & 29 & 30
\end{array}
\]

(64)

Note that in general \(\text{hook}_j(\lambda)\) is a fat hook partition, except possibly for \(\text{hook}_1(\lambda)\).

Set \(b_0^j := d_{j-1}\) and let \(\{b_0^j, b_1^j, \ldots, b_k^j\}\) be the right border of \(t_{\text{hook}_j(\lambda)}\) below the \(b_0^j\)-node. Generalizing (45) we set

\[
R_i^j := 1 + T_{b_{i-1}^j,b_{i-1}^j+1} + T_{b_{i-1}^j,b_{i-1}^j+2} + \ldots + T_{b_{k-1}^j,b_{k-1}^j+1}.
\]

(65)

We next define \(\mathcal{F}_j \in \mathcal{H}_n(q)\), generalizing the element \(F_{k_2}\) of (47). Set first \(\mathcal{F}_1^j := R_1^j\) and then recursively

\[
\mathcal{F}_i^j := (T_{b_0^j,b_i^j} - q\mathcal{F}_{i-1}^j)R_i^j.
\]

(66)

Finally set

\[
\mathcal{F}^j := \mathcal{F}_k^j.
\]

(67)
Generalizing (49), we have the following reformulation of $F^j$

$$F^j = \sum_{i=1}^{k} (-q)^{k-i} T_{b_i b_{i+1}} R_i^j R_{i+1}^j \cdots R_k^j. \quad (68)$$

Note that there is a difference between $F^j$ and $F_k$: the former belongs to $H^A_q(q)$ whereas the latter belongs to $S^A_q(\lambda)$.

To illustrate these definitions we write down the $R_i^j$'s and $F^j$'s, for $\lambda$ as in (61). Using (68) we get

$$R_1^1 = 1 + T_{12,13} + T_{12,14} + T_{12,15}, \quad R_2^1 = 1 + T_{16,17} + T_{16,18} + T_{16,19}$$
$$R_3^1 = 1 + T_{20,21} + T_{20,22} + T_{16,23}$$
$$F^1 = q^2 R_1^1 R_2^1 R_3^1 - q T_{12,16} R_2^1 R_3^1 + T_{12,20} R_3^1$$
$$R_1^2 = 1 + T_{24,25} + T_{24,26}, \quad R_2^2 = 1 + T_{27,28} + T_{27,29}$$
$$F^2 = -q R_1^2 R_2^2 + T_{24,27} R_2^2$$
$$R_3^1 = 1$$
$$F^3 = 1.$$

We have that $r_{d_i} := c_\lambda(d_0) - c_\lambda(d_i)$, see (35). We set $P^0 := 1$ and recursively for $j = 1, 2, \ldots, N$

$$P^j := P^{i-1} \left( T_{d_i-1, d_i} + \frac{1}{[r_{d_i}]_q} F^i \right). \quad (70)$$

We are interested in $x_\lambda P^N \in S^K_q(\lambda)$. For example, using once again $\lambda$ as in (61) we get that

$$x_\lambda P^3 = x_\lambda \left( T_{12,24} + \frac{1}{[5]_q} F^1 \right) \left( T_{24,30} + \frac{1}{[8]_q} F^2 \right) \left( T_{30,31} + \frac{1}{[11]_q} F^3 \right). \quad (71)$$

where the $F^j$'s are as in (69).

The main result of this section, Theorem 3, is the identity $f_n = x_\lambda P^N$. We need two auxiliary lemmas. Here is the first one.

Lemma 2 For each $j = 1, \ldots, N$ there is a $p_j \in H^K_n(q)$ such that $f_{d_j} = f_{d_{j-1}} p_j$. It satisfies $x_{d_{j-1}} p^j = x_{d_j} + \frac{1}{[r_{d_j}]_q} x_{d_{j-1}} F^j$.

Proof: The existence of $p_j$ follows from repeated applications of Lemma 1. It is a product of factors, each one of the form

$$T_{b_i b_{i+1}} + \frac{1}{[r_{b_i b_{i+1}+1}]_q} (1 + T_{b_i b_{i+1}} + T_{b_i b_{i+2}} + \cdots + T_{b_i b_{i+1}+1}). \quad (72)$$

The formula $x_{d_{j-1}} p_j = x_{d_j} + \frac{1}{[r_{d_j}]_q} x_{d_{j-1}} F^j$ follows from arguments identical to those in the proof of Theorem 2. Note that these arguments in fact show that $f_n = e_\lambda p_1$. The cancellations of Theorem 2 depend only on the Garnir relations and in particular they do not require that the nodes above the relevant Garnir belt, which is always of the form as in (55), are those of a fat hook partition. Hence the cancellations will also occur in the present setting. This proves the last statement of the Lemma. □
For \( x \in \mathbb{Z} \) we define \( p_j^x \) the same way as \( p_j \), but replacing each factor (72) with
\[
T_{b_i,b_{i+1}} + \frac{1}{[r_{b_i} + x]_q} (1 + T_{b_i,b_{i+1}} + T_{b_i,b_{i+2}} + \ldots + T_{b_i,b_{i+1-1}})
\]
and thus we have \( p_j^0 = p_j \). We next introduce \( f_j^x \in S_q^K(\lambda) \) via
\[
f_j^x := x_\lambda p_j^x.
\]
It is a generalization of the seminormal basis, in fact \( f_n = f_n^0 \) in the case where \( \lambda \) is a fat hook partition.

We need a slightly more general version of this construction. Let \( t \) be a \( \lambda \)-tableau that coincides with \( t^\lambda \) in all nodes below the node \([\alpha_j, \beta_j]\), that is
\[
t[\alpha, \beta] = t^\lambda[\alpha, \beta] \text{ if } \alpha > \alpha_j.
\]
For example, using \( \lambda \) as in (61), and \( j = 2 \), the following \( \lambda \)-tableau \( t \) could be used, since \( t^\lambda \) and \( t \) coincide below the red line.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 & & & & & & & & \\
17 & 18 & 19 & 20 & & & & & & & & \\
21 & 22 & 23 & 24 & & & & & & & & \\
25 & 26 & 27 & & & & & & & & & & \\
28 & 29 & 30 & & & & & & & & & & \\
31 & & & & & & & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 4 & 5 & 7 & 8 & 9 & 2 & 6 & 10 & 12 & 11 & 24 \\
3 & 13 & 14 & 15 & & & & & & & & \\
16 & 17 & 18 & 19 & & & & & & & & \\
20 & 21 & 22 & 23 & & & & & & & & \\
25 & 26 & 27 & & & & & & & & & & \\
28 & 29 & 30 & & & & & & & & & & \\
31 & & & & & & & & & & & \\
\end{array}
\]

For such a \( \lambda \)-tableau \( t \) and \( x \in \mathbb{Z} \) we define
\[
f_j^{x,t} := x_t p_j^x.
\]

Our second auxiliary Lemma is as follows.

**Lemma 3** In the above setup we have
1. \( f_j^x = x_{d_j} + \frac{1}{[r_{d_j} + x]_q} x_{d_j-1} F^j \).
2. \( f_j^{x,t} = x_t T_{\sigma_d-1,d_j} + \frac{1}{[r_{d_j} + x]_q} x_t F^j \).

*Proof:* Just as in the proof of Lemma 2, we recycle the proof of Theorem 2. That proof depends on the formula given in Lemma 1, and on cancellations that arise from the Garnir relations of the form (55). These cancellations also take place in the present setting, when we replace the \( r_{b_j} \)'s of that proof with \( r_{b_j} + x \). This proves the Lemma. \( \square \)

We are now in position to prove the main Theorem of this section, which generalizes Theorem 2 to arbitrary partitions.

**Theorem 3** Let \( P^N \in \mathcal{H}_q^K(\lambda) \) be the element given by the recursion (70). Then \( x_\lambda P^N \in S_q^K(\lambda) \) satisfies \( f_n = x_\lambda P^N \). Moreover, the \( x_t \)'s arising from the expansion of \( x_\lambda P^N \) are all standard elements.
Proof: We proceed by induction on $N$. The basis case $N = 1$ is $f_{d_1} = x_{d_1} + \frac{1}{[r_{d_1}]q} x_\lambda F^1$, and it corresponds to Theorem 2. Let us consider the step from $N = 1$ to $N = 2$. We consider the standard tableaux $t$ such that $x_i$ appear in the expansion of $f_{d_1} = x_{d_1} + \frac{1}{[r_{d_1}]q} x_\lambda F^1$. In $t_{d_1}$, corresponding to the first term $x_{d_1}$ of this expansion, we have that $d_1$ is located in position $[\alpha_0, \beta_0]$, whereas in all the other $t$’s, we have that $d_1$ is located in position $[\alpha_1, \beta_1]$.

Now

$$f_{b_2} = f_{b_1} p_2 = x_{d_1} p_2 + \frac{1}{[r_{d_1}]q} x_\lambda F^1 p_2.$$  \hfill (78)

With regards to $x_{d_1} p_2$ we get by Lemma 2 that

$$x_{d_1} p_2 = x_{d_2} + \frac{1}{[r_{d_2}]q} x_{d_1} F^2 = x_{d_1} \left( T_{d_1,d_2} + \frac{1}{[r_{d_2}]q} F^2 \right)$$  \hfill (79)

and hence we must prove that the same formula holds for all $x_i$ involved in $\frac{1}{[r_{d_1}]q} x_\lambda F^1 p_2$, i.e. that

$$x_i p_1 = x_i \left( T_{d_1,d_2} + \frac{1}{[r_{d_2}]q} F^2 \right)$$  \hfill (80)

holds for these $x_i$. As already mentioned, for each such $x_i$-term we have that $t[\alpha_1, \beta_1] = d_1$. We now apply part b) of Lemma 3 with $x = r_2 - r_1$, and conclude that (80) is correct. The general induction step is treated the same way.

We now give a characterization of the standard tableaux that appear in the expansion of $f_n = x_\lambda P^N$ in Theorem 2, extending equation (60) for fat hook partitions and the comments below it.

The characterization relies on a set of rules that allow us to produce a set of standard tableaux $\mathcal{CS}_\lambda$, in which all the numbers are coloured from a large colour set, that includes black. The standard tableaux that appear in the expansion of $f_n$ are those tableaux that are obtained from $\mathcal{CS}_\lambda$ by shuffling in all possible ways the monochromatic numbers, except the black ones, such that the results are still standard.

Let therefore the situation be as above, with $\lambda \in \text{Par}_n$ arbitrary, having removable nodes $[\alpha_j, \beta_j], j = 0, 1, 2, \ldots, N$, see (61). In the following the words 'before', 'after', 'next' and so on, refer to the natural total order on the nodes of $\lambda$ given by $[r_1, c_1] < [r_2, c_2]$ iff $t^\lambda[r_1, c_1] < t^\lambda[r_2, c_2]$; this is 'the row reading order' on the nodes for $\lambda$. Moreover, the word 'coloured' refers to a colour different from black.

We consider $t$ as a filling of the nodes of $\lambda$ with the numbers $1, 2, \ldots, n$, in this order. Then $\mathcal{CS}_\lambda$ is the set of $\lambda$-tableaux $t$ that can be obtained by applying the following rules.

**Rule 1** Let $i := t^\lambda[r, c]$ where $[r, c]$ is a node strictly before $[\alpha_0, \beta_0]$. Then $i$ should also be placed in $[r, c]$ in $t$ and should be coloured black.

**Rule 2** Let $i := t[r, c]$ where neither $[r, c]$ nor $[r, c + 1]$ belongs to the right border of $\lambda$. Then $i + 1$ should be placed in $[r, c + 1]$ and should be coloured with the same colour as $i$.

These two rules are visible in (60). It follows from them that in $t$, the node and colour of $i + 1$ is uniquely determined by the node and colour of $i$, unless the node of $i$ is situated after $[\alpha_0, \beta_0]$, and belongs either to the right border of $\lambda$ or is a node one step before the right border of $\lambda$. The remaining rules consider these cases.
Rule 3 Suppose that \( i = t[\alpha_0, \beta_0 - 1] \). Then \( i + 1 \) should be placed either in \([\alpha_0, \beta_0]\), with a colour different from black, or in \([\alpha_0, \beta_0]\) with colour black. Here an example with \( i = 12 \).

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 \\
12 \\
\end{array}
\]

Rule 4 Let \([r, c]\) be a node one step before the right border of \( \lambda \) such that \( r > \alpha_0 \) and suppose that \( i := t[r, c] \) is black. Then \( i + 1 \) should be placed in \([r, c + 1]\) and should be black. Here an example with \( i = 11 \).

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 \\
12 \\
\end{array}
\]

Rule 5 Let \([r, c]\) be a node one step before the right border of \( \lambda \) such that \( r \not\in \{\alpha_0, \ldots, \alpha_N\} \) and suppose that \( i := t[r, c] \) is coloured. Then \( i + 1 \) should be placed in \([r, c + 1]\) and should have a previously unoccupied colour. Here is an example.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
10 & 11 & 12 \\
13 \\
\end{array}
\]

Rule 6 Let \([r, c]\) be a node one step before the right border of \( \lambda \) such that \( r \in \{\alpha_1, \ldots, \alpha_N\} \) and suppose that \( i := t[r, c] \) is coloured. Then \( i + 1 \) should either be placed in \([r, c + 1]\) and have a previously unoccupied colour or should be placed in \([r + 1, 1]\) with colour black.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12 \\
13 & 14 & 15 \\
\end{array}
\]

Rule 7 Suppose that \( i := t[r, c] \) is black and that \([r, c]\) is a node belonging to the right border of \( \lambda \) such that \( r > \alpha_0 \). Then \( i + 1 \) should be placed either in the first node of the \( r + 1 \)’st row of \( \lambda \), with colour black, or in the unique previously unoccupied node \([\alpha_i, \beta_i]\) where \( \alpha_i < r \), with a previously unoccupied colour, in particular different from black. Here is an illustration with
Rule 8 Suppose that \( i := t[r, c] \) where \([r, c]\) is a node belonging to the right border of \( \lambda \), such that \( r \geq \alpha_0 \) and such that \( i \) is coloured. Then \( i + 1 \) should be placed in the first node of the first unoccupied row of \( \lambda \), with the same colour as \( i \), unless that node belongs to the right border of \( \lambda \) in which case it should have a new colour. We illustrate with \( i = 13 \).

Observe that applying these rules, filling in a black \( i \) in the \( r \)'th row where \( r > \alpha_0 \) there will always be a unique unoccupied node before \( i \), whereas filling in a coloured \( i \) in row \( r \), there will be no such unoccupied node.

The rules can be read off directly from (70). Indeed, the black numbers of the rules correspond to the factors \( T_{d_i}^{d_i} \) of (70) and the coloured numbers correspond to the factors \( F_i \) of (70). The connection with (70) also allows us to the determine the coefficient of \( x_{s} \) in \( f_n \), since black numbers correspond to \( T_{d_i}^{d_i} \), for some \( i \), that only contributes with 1 to \( f_n \), whereas \( j \) coloured numbers that correspond to \( F_i \), for some \( i \), contribute with \( \frac{(-q)^{j-1}}{[r_d]_q} \) to \( f_n \). Note that for standard tableaux \( s \) and \( t \) related via a shuffling of monochromatic numbers, the coefficients of \( x_{s} \) and \( x_{t} \) are the same.

Setting \( \mu := \lambda \setminus [\alpha_0, \beta_0] \), the set \( \mathcal{CS}_\lambda \) also appears in [FLT2] where it is denoted the set of colour semistandard tableaux \( SStd(\mu; 1) \), although the formulation in [FLT2] is different from ours.

Suppose for example that \( \lambda = (4, 3, 2, 2) \) with \((\alpha_0, \beta_0) = (1, 4)\). Then applying the rules we get the following 6 elements of \( \mathcal{CS}_\lambda \), with corresponding coefficients.

\[
\begin{align*}
\mathbf{s}_0 & = \begin{array}{cccc}
1 & 2 & 3 & 11 \\
4 & 5 & 6 & 10 \\
7 & 8 & 9 & 10 \\
\end{array} & \mathbf{s}_1 & = \begin{array}{cccc}
1 & 2 & 3 & 13 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
\end{array} & \mathbf{s}_2 & = \begin{array}{cccc}
1 & 2 & 3 & 5 \\
4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 \\
\end{array} & \mathbf{s}_3 & = \begin{array}{cccc}
1 & 2 & 3 & 7 \\
4 & 5 & 6 & 8 \\
10 & 11 & 12 & 13 \\
\end{array} & \mathbf{s}_4 & = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
\end{array} & \mathbf{s}_5 & = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
\end{array}
\end{align*}
\]

This is the 'running example' of [FLT2] and one sees that the tableaux \( \mathbf{s}_0, \ldots, \mathbf{s}_5 \) in (87) coincide with the tableaux in Example 3.3 in [FLT2], denoted the same way, up to a shuffling of the coloured monochromatic numbers. Using Theorem 3.5 in [FLT2], the corresponding coefficients \( a_{s} \) are calculated in Example 3.6 in [FLT2] and are

\[
a_{s_0} = 1, \ a_{s_1} = \frac{1}{2}, \ a_{s_2} = \frac{1}{5}, \ a_{s_3} = -\frac{1}{5}, \ a_{s_4} = \frac{1}{10}, \ a_{s_5} = -\frac{1}{10}.
\]
(Note that we have here corrected the values of $a_{52}$ and $a_{53}$ that are indicated wrongly in [FLT2]). One sees that upon specializing $q = 1$, the coefficients in (87) become the coefficients in (88).

We finally remark that in [FLT2] the approach to Theorem 3 is the converse of ours, in the sense that it is based on the characterization of the tableaux in $\mathcal{CS}_\lambda$, whereas in our approach the characterization of the tableaux in $\mathcal{CS}_\lambda$ is a consequence of Theorem 3. The approach in [FLT2] allows the authors to determine the greatest common divisor of the coefficients of the expansion of $f_n$.

8 Restricted Specht modules and $f_t$ for $t$ a general standard tableau

In this section we give an application of the results and methods of the previous sections to the modular representation theory of the Hecke algebra, that is the representation theory of $\mathcal{H}_n^A(q)$, where $k$ is a field made into an $\mathcal{A}$-algebra via $q \mapsto \xi \in k^\times$. We also study the problem of expanding $f_t$ in terms of standard elements, for $t$ a general standard $\lambda$-tableau which is not necessarily a generalized James-Murphy tableau. In this general case we are unfortunately not able to produce an expansion in terms of standard elements, but still we get some interesting results.

There is a natural embedding $\mathcal{H}_n^A(q) \subset \mathcal{H}_n^\ell(q)$ which gives rise to a restriction functor res from the category of $\mathcal{H}_n^A(q)$-modules to the category of $\mathcal{H}_n^A(q)$-modules.

In particular, for $\lambda \in \text{Par}_n$ we obtain an $\mathcal{H}_n^A(q)$-module $\text{res} S_q^A(\lambda)$. By the branching rule, it is known that $\text{res} S_q^A(\lambda)$ has a Specht filtration, that is an $\mathcal{H}_n^A(q)$-module filtration in which the subquotients are $\mathcal{H}_{n-1}^A(q)$ Specht modules. Let us explain a combinatorial construction of this filtration.

Let $[\alpha_i, \beta_i], i = 0, \ldots, N$ be all the removable nodes of $\lambda$, read from top to bottom as for example in (61). For $i = 0, \ldots, N$ we define $E_i \subseteq S_q^A(\lambda)$ via

$$E_i := \text{span}_A \{ x_i \mid t \in \text{Std}(\lambda) \text{ and } n \text{ appears in or below the } \alpha_i \text{'th row of } t \}.$$  \hfill (89)

Then we have that

$$0 \subset E_N \subset E_{N-1} \subset \cdots \subset E_0 = \text{res} S_q^A(\lambda)$$  \hfill (90)

is a filtration of $\mathcal{H}_n^A(q)$-modules and that

$$E_i/E_{i+1} \cong S_q^A(\lambda^i) \text{ where } \lambda^i := \lambda \setminus [\alpha_i, \beta_i].$$  \hfill (91)

This is a well-known fact, that relies on the Garnir relations, that is (9). From this filtration we get in particular that $\text{res} S_q^A(\lambda)$ contains $S_q^A(\lambda^N)$ as a submodule and has $S_q^A(\lambda^0)$ as a quotient module.

Since the $E_i$’s and $S_q^A(\lambda^i)$’s are free over $A$ there is a similar filtration for the specialized Specht module $\text{res} S_q^k(\lambda)$, which in particular contains $S_q^k(\lambda^N)$ as a submodule and has $S_q^k(\lambda^0)$ as a quotient module.

Let us now set up some further notation. Suppose that $k$ is a field which is made into an $\mathcal{A}$-algebra via $q \mapsto \xi \in k^\times$. Let $A_m$ be the localization of $A$ at the maximal ideal $m := \ker(A \to k)$. Then we have that $A_m \subseteq K$. Let $\mathcal{H}_n^{A_m}(q)$ be the Hecke algebra defined over $A_m$ instead of $A$. All constructions for $\mathcal{H}_n^A(q)$ can also be carried out for $\mathcal{H}_n^{A_m}(q)$ and in particular we have Specht modules $S_q^{A_m}(\lambda)$ for $\mathcal{H}_n^{A_m}(q)$. Note that $S_q^A(\lambda) \subseteq S_q^{A_m}(\lambda) \subseteq S_q^k(\lambda)$ and that $S_q^{A_m}(\lambda) \otimes_{A_m} k = S_q^A(\lambda) \otimes_A k = S_q^k(\lambda)$.
As in the previous sections we let \( f_n \in S^K_q(\lambda) \) be the seminormal basis element, corresponding to the James-Murphy tableau \( t_n \), for example as in (62). We have \( H_n^A(q) \subseteq H_n^K(q) \) and so we may introduce an \( H_{n-1}(q) \)-module \( U^A_n(\lambda^0) \) as follows

\[
U^A_n(\lambda^0) := f_n H_{n-1}(q) \subseteq S^K_q(\lambda).
\]  

(92)

Assume now further that \([r_d],\xi \neq 0\) for \( i = 1, 2, \ldots, N \). Then by Theorem 3 we have that \( f_n \in S^A_n(\lambda) \) and so we can define an \( H_{n-1}(q) \)-submodule \( U^A_n(\lambda^0) \) of \( \text{res}S^A_n(\lambda) \) via

\[
U^A_n(\lambda^0) := f_n H_{n-1}(q) \subseteq \text{res}S^A_n(\lambda).
\]  

(93)

Note that \( U^A_n(\lambda^0) \) is not defined as a specialization of \( U^A_n(\lambda^0) \) since that would not be a submodule of \( \text{res}S^A_n(\lambda) \).

We now have the following Theorem.

**Theorem 4**  

a) There is an \( H_{n-1}(q) \)-isomorphism \( S^A_n(\lambda^0) \rightarrow U^A_n(\lambda^0) \) given by \( x_0 \mapsto f_n \) and similarly, when \([r_d],\xi \neq 0\) for \( i = 1, 2, \ldots, N \), there is an \( H_{n-1}(q) \)-isomorphism \( S^A_n(\lambda^0) \rightarrow U^A_n(\lambda^0) \) given by \( x_0 \mapsto f_n \).

b) Suppose that \([r_d],\xi \neq 0\) for \( i = 1, 2, \ldots, N \). Then \( S^k_q(\lambda^0) \) splits off from \( \text{res}S^k_q(\lambda) \) with splitting homomorphism given by specializing the composition \( S^A_n(\lambda^0) \rightarrow U^A_n(\lambda^0) \subseteq \text{res}S^A_n(\lambda) \).

**Proof:** To show a) we first verify that \( f_n \mapsto x_0 \) defines an \( H_{n-1}(q) \)-homomorphism \( U^A_n(\lambda^0) \rightarrow S^A_n(\lambda^0) \). This is not completely obvious, since \( f_n \) and \( x_0 \) may apriori have different annihilators in \( H_{n-1}(q) \). We resolve this problem as follows. By definition \( U^A_n(\lambda^0) \) is an \( H_{n-1}(q) \)-submodule of the \( H_{n-1}(q) \)-module \( U^K_n(\lambda^0) := f_n H_{n-1}(q) \). Similarly, the Specht module \( S^A_n(\lambda^0) = x_0 \cdot H_{n-1}(q) \) is an \( H_{n-1}(q) \)-submodule of \( S^K_n(\lambda^0) = x_0 \cdot H_{n-1}(q) \). Define

\[
\text{Std}^0(\lambda) := \{ t \in \text{Std}(\lambda) \mid t[0_0, 0_0] = n \}.
\]  

(94)

Using YSF, that is Theorem 19, we have that \( U^K_n(\lambda^0) \) is generated by \( \{ f_i \mid t \in \text{Std}^0(\lambda) \} \) and since \( \{ f_i \mid t \in \text{Std}^0(\lambda) \} \subseteq \{ f_i \mid t \in \text{Std}(\lambda) \} \subseteq S^K_n(\lambda) \) we also have that \( \{ f_i \mid t \in \text{Std}^0(\lambda) \} \) is \( \mathcal{K} \)-linearly independent. Hence it is a \( \mathcal{K} \)-basis for \( U^K_n(\lambda^0) \). On the other hand, \( \{ f_s \mid s \in \text{Std}(\lambda) \} \) is a \( \mathcal{K} \)-basis for \( S^k_q(\lambda^0) \), and so we obtain a \( \mathcal{K} \)-linear bijection \( \varphi : U^K_n(\lambda^0) \rightarrow S^k_q(\lambda^0) \), via

\[
\varphi(f_i) = f_s \text{ where } t \in \text{Std}^0(\lambda) \text{ and } s = t[1, 2, \ldots, n-1]
\]  

(95)

Using YSF on \( f_s \) as well as on \( f_s \), we conclude that \( \varphi \) is in fact an \( H_{n-1}(q) \)-isomorphism, since the action of \( T_i \) on both cases is ‘the same’, given only by radial lengths. The restriction of \( \varphi \) to \( U^A_n(\lambda^0) \) is the inverse of the \( H_{n-1}(q) \)-isomorphism that is postulated in a). The second part of a), involving the ground ring \( \mathcal{A}_m \), is proved the same way.

To show b), letting \( \psi : S_n^A(\lambda^0) \rightarrow U^A_n(\lambda^0) \) be the isomorphism from a), we have that \( \psi(x_0) = f_n \). Since \( U^A_n(\lambda^0) \subseteq \text{res}S_n^A(\lambda) \) we obtain by specializing an \( H_{n-1}(q) \)-homomorphism \( (\iota \circ \psi) \otimes 1 : S^k_q(\lambda) \rightarrow \text{res}S_n^A(\lambda) \), where \( \iota : U^A_n(\lambda^0) \rightarrow \text{res}S_n^A(\lambda) \) is the inclusion homomorphism. Note that apriori \( (\iota \circ \psi) \otimes 1 \) may not be injective, since the functor \( \otimes \mathbb{k} \) is not left exact. But letting \( \pi \otimes 1 : \text{res}S^k_q(\lambda) \rightarrow S^k_q(\lambda) \) be the quotient map from the specialization of the filtration (90), we get that \(( (\pi \otimes 1) \circ (\iota \circ \psi) \otimes 1)(x_0) \otimes 1) = x_0 \otimes 1 \) and hence \(( (\pi \otimes 1) \circ (\iota \circ \psi) \otimes 1) \) is the identity map on \( S^k_q(\lambda^0) \). This proves b).

Our next Theorem establishes a converse of b) of the previous Theorem, but over \( \mathcal{A}_m \) instead of \( \mathbb{k} \). Its formulation was influenced by Proposition 3.11 of [FLT1]. Note however that the authors of
Theorem 5 Let $\pi$ be the $\mathcal{H}_n^{\lambda}(q)$-quotient map $\pi : \text{res}S_q^{\lambda}(\lambda) \rightarrow S_q^{\lambda}(\lambda^0)$. Then $\pi$ admits a splitting if and only if $[r_d, \zeta] \neq 0$ for $i = 1, 2, \ldots, N$.

Proof: Suppose that $[r_d, \zeta] \neq 0$ for $i = 1, 2, \ldots, N$. Then we actually showed in b) of Theorem (4) that $\pi$ has a splitting as an $\mathcal{H}_n^{\lambda}(q)$-homomorphism and so we only need to prove the 'only if' part of the Theorem.

Suppose therefore that $\psi : S_q^{\lambda}(\lambda^0) \rightarrow \text{res}S_q^{\lambda}(\lambda)$ is a splitting homomorphism for $\pi$ and set $f'_n := \psi(x_\lambda)$. Then $f'_n := \psi(x_\lambda)$ has an expansion

$$f'_n = x_{t_n} + \sum_{s \in \text{Std}(\lambda)} a_s x_s; \quad a_s \in A_m. \quad (96)$$

By the splitting property $\pi \circ \psi = Id$, we have that for all $s$ occurring in the sum, $n$ appears in $s$ in a node strictly below $[\alpha_0, \beta_0]$.

Extending scalars from $A_m$ to $K$ we now consider $f'_n$ as an element of $\text{res}S_q^K(\lambda)$ and similarly we consider $\pi$ and $\psi$ as $\mathcal{H}_n^K(q)$-homomorphisms. Suppose now that $t \in \text{Std}(n - 1)$ and let $E_t \in \mathcal{H}_n^K(q)$ be the idempotent corresponding to $t$, as introduced in (14). Then we have that

$$f'_n E_t = \psi(x_\lambda) E_t = \psi(x_\lambda) E_t = \left\{ \begin{array}{ll} \psi(x_\lambda) = f'_n & \text{if } t = t^0 \\ 0 & \text{otherwise.} \end{array} \right. \quad (97)$$

We are interested in the idempotent $E_{t_n} \in \mathcal{H}_n^K(q)$ since we must show that $x_{t_n} E_{t_n} = f'_n$.

Note that for any $t \in \text{Std}(n)$ we have the following formula which can be read off from (14)

$$E_t = E_{t|1,2,\ldots,n-1} E_{t|n} \quad \text{where } E_{t|n} := \prod_{c \in C_n \setminus C_t(n)} \frac{L_n - c}{c_t(n) - c}. \quad (98)$$

For all $s$ occurring in (96) we have that

$$x_s E_{t_n} = 0. \quad (99)$$

Indeed, we have the general triangularity property

$$x_u E_v \neq 0 \implies u \trianglelefteq v \quad (100)$$

which follows from inverting the expansion (17) and using the orthogonality of the $E_v$'s, and so if $x_u E_v$ were nonzero, we would have $s \trianglelefteq t_n$. But removing the $n$-node from both sides we have $\text{shape}(s_{1,2,\ldots,n-1}) \triangleright \text{shape}(t_n_{1,2,\ldots,n-1}) = \lambda$, in contradiction with $s \trianglelefteq t_n$.

Similarly to (100) we have that

$$x_u E_v \neq 0 \implies \text{shape}(u) = \text{shape}(v). \quad (101)$$

Combining (99), (98), (100) and (101) and using that $\sum_{t \in \text{Std}(n)} E_t = 1$ we now get

$$f'_n = \sum_{t \in \text{Std}(n)} E_t = \sum_{t \in \text{Std}(\lambda)} E_t = f'_n E_{t_n} = x_{t_n} E_{t_n} = f_n \quad (102)$$

which proves the Theorem. \qed
Remark For $M, N$ general $\mathcal{H}^A_n(q)$-modules it is not true that any $\mathcal{H}^k_n(q)$-homomorphism $\varphi$ between the specialized modules $M \otimes k$ and $N \otimes k$ can be lifted to an $\mathcal{H}^A_n(q)$-homomorphism between $M$ and $N$. On the other hand, if $M$ and $N$ are $\mathcal{H}^{A_\mu}(q)$-modules, then it appears plausible that $\varphi : M \otimes k \rightarrow N \otimes k$ can be lifted. This issue is discussed in [FLT1] for the splitting of the top Specht quotient from the induced Specht module in the symmetric group case, where in particular it is pointed out that there are no known counterexamples to the lifting property in this case. If the lifting property holds for $\mathcal{H}^{A_\mu}(q)$-modules, then we could improve the criterion of Theorem 5 to a criterion for $\mathcal{H}^k_n(q)$-splittings.

We now return to our main object of study, namely that of expanding $f_t$ in terms of standard elements.

We can generalize $a)$ of Theorem 4 as follows. Let $t$ be a standard $\lambda$-tableau. Then $t$ can be viewed as a chain of partitions $\{\lambda^{\leq j}\}_{j=1,\ldots,n}$ where $\lambda^{\leq j} := \text{shape}(t|_{1,2,\ldots,j})$. For $1 \leq j \leq n$ we define the $\lambda$-tableau $t_{\leq j}^{\leq j}$ via

$$
(t_{\leq j}^{\leq j})^{-1}(k) := \begin{cases} 
(t^{\leq j})^{-1}(k) & \text{if } k \leq j \\
 t^{-1}(k) & \text{if } k > j.
\end{cases}
$$

(103)

For simplicity we also write $t^{\leq j} = t_{\leq j}^{\leq j}$. Below is an example of a tableau $t$ with corresponding $t^{\leq 10}$. The numbers $k \leq 10$, corresponding to the first case in (103), have been coloured blue.

$$
t = \begin{array}{cccccc}
1 & 3 & 7 & 11 & 12 \\
2 & 5 & 9 & 13 \\
4 & 8 & 14 & 18 \\
6 & 10 & 17 \\
15 & 16
\end{array}, \quad t^{\leq 10} = \begin{array}{cccc}
1 & 2 & 3 & 11 & 12 \\
4 & 5 & 6 & 13 \\
7 & 8 & 14 & 18 \\
9 & 10 & 17 \\
15 & 16
\end{array}
$$

(104)

We then introduce the $\mathcal{H}_j^A(q)$-submodule $U_q^A(t^{\leq j})$ of $S^K_\ell(q, \lambda)$ via $U_q^A(t^{\leq j}) := f_{t^{\leq j}} \mathcal{H}_j^A(q)$. This is a generalization of $U_q^A(\lambda^n)$ from Theorem 4 which is recovered by setting $j := n - 1$. We now have the following Theorem, generalizing $a)$ of Theorem 4.

**Theorem 6** There is an isomorphism of $\mathcal{H}_j^A(q)$-modules

$$
\varphi : U_q^A(t^{\leq j}) \rightarrow S^K_\ell(q, \lambda^{\leq j}), \quad f_{t^{\leq j}} \mapsto x_{\lambda^{\leq j}}.
$$

(105)

**Proof:** The proof is essentially the same as the proof of $a)$ of the previous Theorem. Let us briefly indicate the necessary modifications. By definition $U_q^A(t^{\leq j})$ is an $\mathcal{H}_j^A(q)$-submodule of the $\mathcal{H}_j^K(q)$-module $U^K_q(t^{\leq j}) := f_{t^{\leq j}} \mathcal{H}_j^K(q) \subseteq S^K_\ell(q, \lambda)$. Using YSF, we see that $U_q^K(t^{\leq j})$ has $K$-basis

$$
\{ f_s | t \in \text{Std}(\lambda) \text{ and } s^{-1}(k) = t^{-1}(k) \text{ for } k = j + 1, \ldots, n \}.
$$

(106)

But then, using the $K$-basis $\{ f_u | u \in \text{Std}(\lambda^{\leq j}) \}$ for $S^K_\ell(\lambda^{\leq j})$, we get via YSF that $f_s \mapsto f_{s|_{1,\ldots,j}}$ induces an $\mathcal{H}_j^K(q)$-isomorphism $U^K_q(t^{\leq j}) \rightarrow S^K_\ell(\lambda^{\leq j})$. As the required $\varphi$ we can then use the restriction of this isomorphism to $U_q^K(t^{\leq j})$. \( \square \)

With this result established, we now finally consider the problem of determining the expansion in standard elements of $f_t$ where this time $t$ is a completely general standard $\lambda$-tableau.

Given $t$, we define an element $P_t \in \mathcal{H}_n$ via the following formula

$$
P_t := P_n P_{n-1} \cdots P_1
$$

(107)
where $P_j \in \mathcal{H}_j \subseteq \mathcal{H}_n$ is the element $P^N$ given by the recursion (70), but with respect to the $\lambda \leq j$-tableau $t_{n,j}^{j-1}$.

We now have the following generalization of Theorem 3.

**Theorem 7** In the above setup we have $f_t = x_\lambda P_t$.

**Proof:** By Theorem 3 we have that $f_{t,n} = f_{t,n-1} = x_\lambda P_n$ and also $f_{t,n-2} = x_\lambda (n-1)P_{n-1}$. In view of Theorem (6) we deduce from this that

$$f_{t,n} = f_{t,n-1}P_{n-1} = x_\lambda P_nP_{n-1}. \quad (108)$$

This argument is now repeated until arriving at the formula claimed in the Theorem. \qed

Let us illustrate the Theorem on the partition $\lambda = (3, 1^2)$ of 5 and the $\lambda$-tableau

$$t := \begin{array}{c}
1 & 4 & 5 \\
2 & 3 \\
\end{array}. \quad (109)$$

We have by the Theorem that $f_t = x_\lambda P_5P_4P_3P_2P_1$. Let us work out the $P_i$'s. Since $t_{1,2,3}$ is the largest, in fact the only, standard $\lambda \leq 3$-tableau we find immediately that $P_3 = P_2 = P_1 = 1$. Let us then consider $P_5$. By Theorem 2 we have for $f_{t_5}^{t_5}$ the following expansion in terms of standard $x_i$'s.

$$f_{t_5}^{t_5} = \begin{array}{c}
1 & 2 & 5 \\
3 & 4 \\
\end{array} + \frac{1}{[4]q} \begin{array}{c}
1 & 2 & 4 \\
3 & 5 \\
\end{array} - \begin{array}{c}
1 & 2 & 3 \\
4 & 5 \\
\end{array}. \quad (110)$$

Thus we have $P_5 = T_{3,5} + \frac{1}{[4]q}(T_3 - q)$. We next work out $P_4$. Using Theorem 2 once more we have for $f_{t_4}^{t_4}$ the following expansion

$$f_{t_4}^{t_4} = \begin{array}{c}
1 & 4 \\
2 & 3 \\
\end{array} + \frac{1}{[3]q} \begin{array}{c}
1 & 3 \\
2 & 4 \\
\end{array} - \begin{array}{c}
2 & 1 \\
3 & 4 \\
\end{array}. \quad (111)$$

and so we get $P_4 = T_{2,4} + \frac{1}{[3]q}(T_2 - q)$. Combining, we get an expression for $f_t = x_\lambda P_5P_4P_3P_2P_1 = x_\lambda P_5P_4$ in terms of 9 $x_i$'s, which however unfortunately are not all standard. After straightening we get

$$f_t = \begin{array}{c}
1 & 4 & 5 \\
2 & 3 \\
\end{array} + \frac{1}{[3]q} \begin{array}{c}
1 & 3 \\
2 & 4 \\
\end{array} - \begin{array}{c}
1 & 2 & 4 \\
3 & 5 \\
\end{array} - \begin{array}{c}
1 & 2 & 5 \\
3 & 4 \\
\end{array} - \begin{array}{c}
1 & 3 & 5 \\
2 & 4 \\
\end{array}. \quad (112)$$

For comparison, repeated use of Young's seminormal form twice, that is the algorithm explained in section 3 on $f_t$ would have given $4 \cdot 3 = 12$ $x_i$'s instead of 9 (that after straightening would have reduced to the above expression, of course). In general, as actually already follows from Theorem 3, this algorithm will in general produce more than just one denominator, i.e. the above example with the only denominator $[3]_q$ is special in this respect.

**Remark** In general, as we just saw on the example (109), the expansion of $f_t$ using Theorem 7 does not always produce standard elements $x_i$. It is an interesting open problem to find an efficient algorithm that does give such an expansion.

In spite of this remark, we can still use Theorem 7 to deduce the following consequence for the coefficient of $x_s$ of $f_t$, which is valid for general standard tableaux $s$ and $t$. We are grateful to the referee for pointing this out to us.
Corollary 1 Let \( \{ f_t \} \) and \( \{ x_t \} \) be the seminormal and standard bases for \( \mathcal{H}_n^{C(q)}(q) \) and let \( f_t = \sum_{s \in \text{Std}(\lambda)} c_{st} x_s \) be the expansion of \( f_t \), with \( c_{st} \in \mathbb{C}(q) \). Then for any \( s, t \in \text{Std}(\lambda) \), the poles of \( c_{st} \) are roots of unity in \( \mathbb{C} \).

**Proof:** The poles of \( \frac{1}{[k]_q} = \frac{q - 1}{q^k - 1} \) are roots of unity and so via Theorem 7, together with the definitions in (70) and (107), we get that the coefficient of \( x_s \) in the expansion of \( f_t \), where \( s \in \text{Tab}(\lambda) \), has poles that are roots of unity. But the expansion of \( x_s \) in terms of standard elements, using the Garnir relations, can be carried out over \( A \) and will therefore not introduce new poles. This proves the Corollary. \( \square \)

Let us do a rudimentary complexity analysis of the two algorithms for calculating \( f_t \), that is 'repeated use of Young’s seminormal form' versus the algorithm given by Theorem 7. Suppose \( \lambda = (\lambda_1, \lambda_2^k) \) is a fat hook partition with the first block of rows of width one, that is \( \lambda_1 > \lambda_2 \). We have \( n = \lambda_1 + \lambda_2 k \) and \( a = \lambda_1 \), that is \( \sigma_{a,n} \) is of Coxeter length \( \lambda_2 k \). Thus, the repeated use of Young’s seminormal form to calculate \( f_n \) produces a linear combination \( 2^{\lambda_2 k} \) (standard and nonstandard) elements \( x_t \)'s, whereas the algorithm contained in Theorem 2 produces

\[
(\lambda_2 - 1)^{k_2} + \ldots + (\lambda_2 - 1)^2 + (\lambda_2 - 1) = \frac{(\lambda_2 - 1)^{k_2 + 1} - \lambda_2 - 1}{\lambda_2 - 2} \quad (113)
\]

such elements \( x_t \). Thus, with respect to \( \lambda_2 \) we see that Theorem 2 has polynomial complexity whereas repeated use of Young’s seminormal form has exponential complexity. Thus the algorithm contained in Theorem 2 is much more efficient. This relationship carries over to the general algorithm of Theorem 7. We have implemented the algorithms using the GAP system.

**Remark** As already mentioned there is no known algorithm for expanding a general \( f_t \) in terms of standard elements. On the other hand, our calculations for small (but nontrivial) \( t \), using the Gram-Schmidt algorithm explained in (18), indicate that the coefficients of the expansion are 'nice' expressions involving radial lengths. Unfortunately, we are at this point unable to state the exact meaning of 'nice'.

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