ANALYTICITY OF SOLUTIONS TO PARABOLIC EVOLUTIONS
AND APPLICATIONS

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Abstract. We find new quantitative estimates on the space-time analyticity of solutions to linear parabolic equations with analytic coefficients near the initial time. We apply the estimates to obtain observability inequalities and null-controllability of parabolic evolutions over measurable sets.

1. Introduction

This work is concerned with the study of quantitative estimates up to the boundary of analyticity in the spatial and time variables of solutions to boundary value parabolic problems for small values of the time variable. If \( \Omega \subset \mathbb{R}^n \) is a bounded domain, we obtain new quantitative estimates of analyticity for solutions of

\[
\begin{aligned}
\partial_t u + \mathcal{L} u &= 0, & & \text{in } \Omega \times (0,1], \\
u = D u = \ldots = D^{m-1} u &= 0, & & \text{in } \partial \Omega \times (0,1].
\end{aligned}
\]

Throughout the work \( \mathcal{L} \) is defined by

\[
\mathcal{L} = (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha(x,t) \partial_\alpha^2,
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \) is in \( \mathbb{N}^n \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \); the coefficients of \( \mathcal{L} \) are bounded and satisfy a uniform parabolicity condition, i.e., there is \( \varrho > 0 \) such that

\[
\sum_{|\alpha| = 2m} a_\alpha(x,t) \xi^\alpha \geq \varrho |\xi|^{2m}, \quad \text{for } \xi \in \mathbb{R}^n, \ (x,t) \in \Omega \times [0,1],
\]

\[
\sum_{|\alpha| \leq 2m} \|a_\alpha\|_{L^\infty(\Omega \times [0,1])} \leq \varrho^{-1}.
\]

Our approach to prove quantitative estimates of analyticity is based on an induction process which employs \( W_2^{2,1}(\Omega \times [0,1]) \) Schauder estimates for solutions to parabolic initial-boundary value problems. This estimates were first derived in [34] for parabolic problems with quite general boundary conditions. In order to employ these estimates, we must assume that \( \partial \Omega \) is globally of class \( C^{2m-1,1} \). Thus, the \( W_2^{2m,1}(\Omega \times [0,1]) \) Schauder estimates hold [34, Theorem 6]; i.e., there is \( K > 0 \) such
that
\begin{equation}
\|\partial_t u\|_{L^2(\Omega \times (0,1))} + \sum_{|\alpha| \leq 2m} \|\partial^\alpha u\|_{L^2(\Omega \times (0,1))} \leq K \left( \|u\|_{L^2(\Omega \times (0,1))} + \|u\|_{L^2(\Omega \times (0,1))} \right),
\end{equation}
when \(u\) satisfies
\[
\begin{cases}
\partial_t u + \mathcal{L} u = F, & \text{in } \Omega \times (0,1], \\
u = Du = \ldots = D^{m-1}u = 0, & \text{in } \partial \Omega \times (0,1], \\
u(0) = 0, & \text{in } \Omega.
\end{cases}
\]

In this setting, we improve the quantitative estimates on the space-time analyticity of solutions to \((1.1)\) available in the literature when the coefficients of \(\mathcal{L}\) and the boundary of \(\Omega\) are analytic. As far as we understand, the best quantitative bounds that we can infer or derive for solutions to \((1.1)\) from the reasonings in \([10, 11, 12, 30, 16, 17, 28, 29]\) are the following:

There is \(0 < \rho \leq 1\), \(\rho = \rho(\varrho, m, n, \partial \Omega)\) such that for \((x, t)\) in \(\overline{\Omega} \times (0,1]\), \(\alpha \in \mathbb{N}^n\) and \(p \in \mathbb{N},\)

\begin{equation}
|\partial_x^\alpha \partial_t^p u(x, t)| \leq \rho^{-1-|\alpha|-p} (|\alpha| + p)! t^{-\alpha_1 - 1 + p - |\alpha|-2m \frac{p}{4m}} \|u\|_{L^2(\Omega \times (0,1))},
\end{equation}
where \(2m\) is the order of the evolution and \(|\alpha| = \alpha_1 + \cdots + \alpha_n\).

The later can be seen to hold when the boundary of \(\Omega\) is analytic and the coefficients of the underlying linear parabolic equation satisfy for some \(0 < \varrho \leq 1\) bounds like
\[
|\partial_x^\alpha \partial_t^p A(x, t)| \leq \varrho^{-1-|\alpha|-p} (|\alpha| + p)!, \text{ for all } (x, t) \in \overline{\Omega} \times [0,1], \alpha \in \mathbb{N}^n \text{ and } p \in \mathbb{N}.
\]

A first observation regarding \((1.5)\) is that it blows up as \(t\) tends to zero, something unavoidable since it holds for arbitrary \(L^2(\Omega)\) initial data; however, \((1.5)\) provides a lower bound \(\frac{2\sqrt{\rho}}{t}\) for the radius of convergence of the Taylor series in the spatial variables around any point in \(\overline{\Omega}\) of the solution \(u(\cdot , t)\) at times \(0 < t \leq 1\). This lower bound shrinks to zero as \(t\) tends to zero and does not reflect the infinite speed of propagation of parabolic evolutions. Thus, it would be desirable to prove a quantitative estimate of space-time analyticity which provides a positive lower bound of the spatial radius of convergence for small values of \(t\).

Concerning this and with the purpose to prove the interior and boundary null controllability of parabolic evolutions with time-independent analytic coefficients over bounded analytic domains and with bounded controls acting over measurable sets of positive measure, we derived in \([2, 3, 9]\) the following quantitative estimates on the space-time analyticity of the solutions of such parabolic evolutions: there is \(0 < \rho \leq 1\) such that for \((x, t)\) in \(\overline{\Omega} \times (0,1]\), \(\alpha \in \mathbb{N}^n\) and \(p \in \mathbb{N},\)

\begin{equation}
|\partial_x^\alpha \partial_t^p u(x, t)| \leq \rho^{1/\rho t^{1/(2m-1)}} \rho^{-|\alpha|-p} (|\alpha| + p)! \|u\|_{L^2(\Omega \times (0,1))}.
\end{equation}

This was done by quantifying each step in a reasoning developed in \([19]\), which reduces the study of the strong unique continuation property within characteristic hyperplanes for solutions of time-independent parabolic evolutions to its elliptic counterpart.

The bound \((1.6)\) shows that the space-time Taylor series expansion of solutions converges absolutely over \(B_\rho(x) \times ((1-\rho)t, (1+\rho)t)\), for some \(0 < \rho \leq 1\), when

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\(^1\)We explain in Section 3 our understanding of previous results.
Here, we derive a formal proof of (1.6) valid for all parabolic operators. To carry it out we use the the full time interval of existence of the solutions before time $t$, the $W^{2m,1}_t$ Schauder estimate (1.4), the weighted $L^2$ estimates in Lemmas 9, 12 and 13 and the inequalities (2.6). We mention that the precise behavior of the bounds in (2.6) is key for our reasonings. The novelty of our proof rests on the fact that we use the weighted $L^2$ estimates in Lemmas 9, 12 and 13.

Throughout the work $\nu$ is the exterior unit normal to the boundary of $\Omega$, $d\sigma$ denotes surface measure on $\partial\Omega$, $B_R$ stands for the open ball of radius $R$ centered at 0 and $B_R^+ = B_R \cap \{x_n > 0\}$. To describe the analyticity of a piece of boundary $B_R(q_0) \cap \partial\Omega$ with $q_0$ in $\partial\Omega$, we assume that for each $q$ in $B_R(q_0) \cap \partial\Omega$ we can find, after a translation and rotation, a new coordinate system (in which $q = 0$) and an analytic function $\varphi : B'_{\theta} \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ verifying

$$\varphi(0') = 0, \ |\partial_{x'}^{\alpha} \varphi(x')| \leq |\alpha|! \theta^{-|\alpha|-1}, \ \text{when} \ x' \in B'_{\theta}, \ \alpha \in \mathbb{N}^{n-1},$$

$$B_{\theta} \cap \Omega = B_{\theta} \cap \{(x', x_n) : x' \in B'_{\theta}, \ x_n > \varphi(x')\},$$

$$B_{\theta} \cap \partial\Omega = B_{\theta} \cap \{(x', x_n) : x' \in B'_{\theta}, \ x_n = \varphi(x')\},$$

where $B'_{\theta} = \{x' \in \mathbb{R}^{n-1}, \ |x'| < \theta\}$. Regarding the analytic regularity of the coefficients, we consider the following conditions:
Let \( x_0 \) in \( \Omega \), there is \( \rho > 0 \) such that for any \( \alpha \in \mathbb{N}^n \) and \( p \in \mathbb{N} \),

\[
|\partial_x^\gamma \partial_t^\mu a_\alpha(x,t)| \leq \rho^{-1-|\gamma|} |\gamma|! |\mu|! \quad \text{in } B_R(x_0) \cap \Omega \times [0,1],
\]

(1.9)

\[
|\partial_x^\gamma a_\alpha(x,t)| \leq \rho^{-1-p} |\mu|! \quad \text{in } \Omega \times [0,1].
\]

(1.10)

The main result in this work is the following:

**Theorem 1.** Let \( x_0 \) be in \( \Omega \), \( 0 < R \leq 1 \). Assume that \( L \) satisfies \( (1.6), (1.9), (1.10) \) and \( B_R(x_0) \cap \partial \Omega \) is analytic when it is non-empty. Then, there is \( \rho = \rho(\varrho, m, n), 0 < \rho \leq 1 \), such that the inequality

\[
|\partial_x^\gamma \partial_t^\mu u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-1-|\gamma|} |\gamma|! |\mu|! \left( |\alpha| + p \right) \|u\|_{L^2(\Omega \times (0,1))}
\]

holds for all \( \alpha \in \mathbb{N}^n, p \in \mathbb{N} \) and \( (x,t) \in B_{R/2}(x_0) \cap \Omega \times (0,1) \), when \( u \) solves \( (1.1) \).

**Remark 1.** If we only assume \( (1.9) \) for some \( x_0 \) in \( \Omega \), so that some of the coefficients of \( L \) may not be globally analytic in the time-variable over \( \Omega \), the solutions of \( (1.1) \) are still analytic in the spatial variable over \( B_{R/2}(x_0) \cap \Omega \times [0,1] \) with a lower bound on the radius of analyticity independent of time but only Gevrey of class \( 2m \) in the time-variable; i.e.,

\[
|\partial_x^\gamma \partial_t^\mu u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-1-|\gamma|} |\gamma|! |\mu|! \left( |\alpha| + 2mp \right) \|u\|_{L^2(\Omega \times (0,1))}
\]

when \( (x,t) \in B_{R/2}(x_0) \cap \Omega \times (0,1), \alpha \in \mathbb{N}^n \) and \( p \in \mathbb{N} \).

At the end of Section 2 we give a counterexample showing that solutions can fail to be time-analytic at all points of a hyperplane \( \Omega \times \{t_0\} \), when some of the coefficients are not time-analytic in a proper subdomain \( \Omega' \times \{t_0\} \subset \Omega \times \{t_0\}, t_0 > 0 \). Thus, the lack of time-analyticity of the coefficients in a subset of a characteristic hyperplane \( t = t_0 \) can propagate to the whole hyperplane \( t = t_0 \).

Our motivation to prove Theorem 1 stems from its applications to the null-controllability of parabolic evolutions with bounded controls acting over measurable sets of positive measure. The main tool used to establish null-controllability properties of parabolic evolutions are the observability inequalities from which the null-controllability follows by duality arguments \( (4, 22) \). The reasonings in \( (2, 3, 9, 31, 36, 37) \) make it now clear, that after Theorem 1 is established, most of the results in those works can now be extended to parabolic evolutions with time-dependent coefficients and for general measurable sets with positive measure.

We remark that only \( 31 \) and \( 32 \) have dealt with some operators with time-dependent coefficients and measurable control regions but only for the special case of \( \partial_t - \Delta + c(x,t) \), with \( c \) bounded in \( \mathbb{R}^{n+1} \) and for control regions of the form \( \omega \times E \), with \( \omega \subset \Omega \) an open set and \( E \subset [0,T] \) a measurable set. In particular, \( 31 \), Theorem 1 and \( \S 5 \) and Theorem 1 imply Theorem 2 where \( L^* \) is the adjoint operator of \( L \).

**Theorem 2.** Let \( 0 < T \leq 1, \Omega \) be a bounded open set in \( \mathbb{R}^n \) with analytic boundary, \( D \subset \Omega \times (0,T) \) be a measurable set with positive measure and \( L \) satisfy \( (1.9) \) over \( \Omega \times [0,1] \). Then, there is \( N = N(\Omega, T, D, \varrho) \) such that the inequality

\[
\|\varphi(0)\|_{L^2(\Omega)} \leq N\|\varphi\|_{L^1(D)}
\]
holds for all \( \varphi \) satisfying
\[
\begin{aligned}
-\partial_t \varphi + \mathcal{L}^* \varphi &= 0, & \text{in } \Omega \times [0, T), \\
\varphi &= Du = \ldots = D^{m-1}u = 0, & \text{in } \partial \Omega \times [0, T), \\
\varphi(T) &= \varphi_T, & \text{in } \Omega,
\end{aligned}
\]
with \( \varphi_T \) in \( L^2(\Omega) \). For each \( u_0 \) in \( L^2(\Omega) \), there is \( f \) in \( L^\infty(\mathcal{D}) \) with
\[
||f||_{L^\infty(\mathcal{D})} \leq N||u_0||_{L^2(\Omega)},
\]
such that the solution to
\[
\begin{aligned}
\partial_t u + \mathcal{L} u &= f \text{\chi}_\mathcal{D}, & \text{in } \Omega \times (0, T], \\
u &= Du = \ldots = D^{m-1}u = 0, & \text{in } \partial \Omega \times (0, T], \\
u(0) &= u_0, & \text{in } \Omega,
\end{aligned}
\]
satisfies \( u(T) \equiv 0 \). Also, the control \( f \) with minimal \( L^\infty(\mathcal{D}) \)-norm is unique and has the bang-bang property; i.e., \( |f(x, t)| = \text{const.} \) for a.e. \( (x, t) \) in \( \mathcal{D} \).

**Remark 2.** When \( \mathcal{D} = \omega \times (0, T) \), the constant in Theorem 2 is of the form \( e^{C/T^{1/(2m-1)}} \), with \( C = C(\Omega, |\omega|, \varrho) \).

**Remark 3.** The proof of Theorem 2 is the same as in [9, Theorem 1 and §5] and requires energy estimates; i.e., we need to be able to solve the initial value problem
\[
\begin{aligned}
\partial_t u + \mathcal{L} u &= 0, & \text{in } \Omega \times (0, 1], \\
u &= Du = \ldots = D^{m-1}u = 0, & \text{in } \partial \Omega \times (0, 1], \\
u(0) &= u_0, & \text{in } \Omega,
\end{aligned}
\]
with data \( u_0 \) in \( L^2(\Omega) \) and with a unique solution \( u \) in the energy class
\[
C([0, 1], L^2(\Omega)) \cap L^2([0, 1], H^m_0(\Omega)).
\]
To make sure that such energy estimates and uniqueness of solutions hold in the later class, we recall that operators \( \mathcal{L} \) as in (1.2), which satisfy the conditions in Theorem 2 can always be written in variational form as
\[
(1.12) \quad \sum_{|\alpha| \leq 2m} a_\alpha(x, t) \partial^\alpha_x = \sum_{|\gamma|, |\beta| \leq m} \partial^\gamma_x \left( A_{\gamma\beta}(x, t) \partial^\beta_x \right),
\]
with
\[
(1.13) \quad \sum_{|\gamma|=|\beta|=m} A_{\gamma\beta}(x, t) |\xi|^\gamma |\xi|^\beta \geq \varrho |\xi|^{2m}, \quad \text{for } \xi \in \mathbb{R}^n, \quad (x, t) \in \Omega \times [0, 1],
\]
\[
\sum_{|\gamma|, |\beta| \leq m} \| A_{\gamma\beta} \|_{L^\infty(\Omega \times [0,1])} \leq \varrho^{-1},
\]
for some possibly smaller \( \varrho > 0 \). The later is claimed without a proof in [12, p. 32]. For the convenience of the reader we add its proof at the end of the Appendix.

Similar results on boundary null-controllability over measurable sets with positive measure can be stated for higher order time-dependent parabolic evolutions, under the same global analyticity conditions as in Theorem 2. This follows from Theorem 1 and the reasonings in [9, Theorem 2 and §5].

For second order parabolic equations the last results hold with less global regularity assumptions on the coefficients and of the boundary of \( \Omega \). In particular, from
Theorem 3 and the telescoping series method one can get the following results for time-dependent second order parabolic equations

\[ \partial_t - \nabla \cdot (A(x,t) \nabla ) + b_1(x,t) \cdot \nabla + \nabla \cdot (b_2(x,t) ) + c(x,t), \]

verifying

\[ \phi I \leq A \leq \phi^{-1} I, \text{ in } \Omega \times [0,1] \]

\[ \| \nabla_x A \|_{L^\infty(\Omega \times [0,1])} + \max_{i=1,2} \| b_i \|_{L^\infty(\Omega \times [0,1])} + \| c \|_{L^\infty(\Omega \times [0,1])} \leq \phi^{-1}. \]

**Theorem 3.** Let \( 0 < T \leq 1, \mathcal{D} \subset B_R(x_0) \times (0,T) \) be a measurable set with positive measure, \( \Omega \) be a bounded \( C^{1,1} \) domain, \( B_{2R}(x_0) \subset \Omega, A, b_i, i = 1, 2 \) and \( c \) also satisfy (1.9) over \( B_{2R}(x_0) \times [0,1] \) and (1.10). Then, there is \( N = N(\Omega, T, \mathcal{D}, g) \) such that the inequality

\[ \| \varphi(0) \|_{L^2(\Omega)} \leq N \| \varphi \|_{L^1(\mathcal{D})}, \]

holds for all \( \varphi \) satisfying

\[
\begin{cases}
- \partial_t \varphi - \nabla \cdot (A \nabla \varphi) - \nabla \cdot (b_1 \varphi) - b_2 \nabla \cdot \varphi + c \varphi = 0, & \text{in } \Omega \times [0,T), \\
\varphi = 0, & \text{in } \partial \Omega \times [0,T), \\
\varphi(T) = \varphi_T, & \text{in } \Omega,
\end{cases}
\]

for some \( \varphi_T \) in \( L^2(\Omega) \). For each \( u_0 \) in \( L^2(\Omega) \), there is \( f \) in \( L^\infty(\mathcal{D}) \) with

\[ \| f \|_{L^\infty(\mathcal{D})} \leq N \| u_0 \|_{L^2(\Omega)}, \]

such that the solution to

\[
\begin{cases}
\partial_t u - \nabla \cdot (A \nabla u) + b_1 \cdot \nabla u + \nabla \cdot (b_2 u) + c u = f \chi_\mathcal{D}, & \text{in } \Omega \times (0,T], \\
u = 0, & \text{in } \partial \Omega \times [0,T], \\
u(0) = u_0, & \text{in } \Omega,
\end{cases}
\]

satisfies \( u(T) \equiv 0 \). Also, the control \( f \) with minimal \( L^\infty(\mathcal{D}) \)-norm is unique and has the bang-bang property; i.e., \( |f(x,t)| = \text{const.} \) for a.e. \((x,t) \) in \( \mathcal{D} \).

**Remark 4.** Theorem 3 also holds when \( b_1 \equiv 0 \), (1.14) holds and \( A, b_2 \) and \( c \) are only analytic with respect to the space-variables over \( B_{2R}(x_0) \times [0,1] \). This follows from Remark 1 and the reasonings in [31, 32, 3]. We outline the proof of this result in Section 3.

**Theorem 4.** Let \( \Omega \) and \( T \) be as above, \( \mathcal{J} \subset \Delta_R(q_0) \times (0,T) \) be a measurable set with positive measure, \( q_0 \in \partial \Omega, \Delta_{2R}(q_0) \) be analytic, \( A, b_i, i = 1, 2 \) and \( c \) also satisfy (1.9) over \( B_{2R}(q_0) \cap \Omega \times [0,1] \) and (1.10). Then, there is \( N = N(\Omega, T, \mathcal{J}, g) \) such that the inequality

\[ \| \varphi(0) \|_{L^2(\Omega)} \leq N \| A \nabla \varphi \|_{L^1(\mathcal{J})}, \]

holds for all \( \varphi \) satisfying

\[
\begin{cases}
- \partial_t \varphi - \nabla \cdot (A \nabla \varphi) - \nabla \cdot (b_1 \varphi) - b_2 \nabla \cdot \varphi + c \varphi = 0, & \text{in } \Omega \times [0,T), \\
\varphi = 0, & \text{in } \partial \Omega \times [0,T), \\
\varphi(T) = \varphi_T, & \text{in } \Omega,
\end{cases}
\]

for some \( \varphi_T \) in \( L^2(\Omega) \). For each \( u_0 \) in \( L^2(\Omega) \), there is \( g \) in \( L^\infty(\mathcal{J}) \) with

\[ \| g \|_{L^\infty(\mathcal{J})} \leq N \| u_0 \|_{L^2(\Omega)}, \]
such that the solution to
\[
\begin{aligned}
&\partial_t u - \nabla \cdot (A \nabla u) + b_1 \cdot \nabla u + \nabla \cdot (b_2 u) + cu = 0, \quad \text{in } \Omega \times (0, T], \\
u = g x_3, \quad &\text{in } \partial \Omega \times [0, T], \\
u(0) = u_0, \quad &\text{in } \Omega,
\end{aligned}
\]
satisfies \(u(T) \equiv 0\). Also, the control \(g\) with minimal \(L^\infty(\Omega)\)-norm is unique and has the bang-bang property; i.e., \(|g(q, t)| = \text{const. for a.e. } (q, t)\) in \(\Omega\).

As in [2, 3, 9], the main tools to derive these results are Theorem 1, the telescoping series method [21] and Lemma 1 below. Lemma 1 was first derived in [32]. See also [26] and [27] for close results. The reader can find a simpler proof of Lemma 1 in [2, §3]. The proof there is built with ideas from [23], [26] and [35].

Lemma 1. Let \(\omega \subset B_R\) be a measurable set, \(|\omega| \geq g|B_R|\), \(f\) be an analytic function in \(B_R\) and assume there are \(M > 0\) and \(0 < p \leq 1\) such that
\[
|\partial^\alpha f(x)| \leq M(Rg)^{-|\alpha|} |\alpha|!, \quad \text{when } x \in B_R \text{ and } \alpha \in \mathbb{N}^n.
\]
Then, there are \(N = N(g)\) and \(\theta = \theta(g), 0 < \theta < 1\), such that
\[
\|f\|_{L^\infty(B_R)} \leq NM^{1-\theta} \left(\frac{1}{|\omega|} \int_{\omega} |f| dx\right)^{\theta}.
\]

The paper is organized as follows: in Section 2 we prove Theorem 1 give an outline for the proof of the second part of Remark 1 after Remark 2 and then, finish Section 2 with the counterexample. Section 3 contains the proofs of Theorems 3 and 4 and Remark 4. Section 4 provides a historical background on previous works. Section 5 is an appendix which contains some Lemmas we use in Section 2.

2. Proof of Theorem 1

We prove Theorem 1 in several steps following the scheme devised in [12, Ch. 3, §3]. Throughout the work \(N\) denotes a constant depending on \(g, n, m\) and \(R\). We also define
\[
\sigma = 1/(2m - 1), \quad b = (2m - 1)/2m,
\]
\[
\|\cdot\| = \|\cdot\|_{L^2((0,1) \times (0,1))}, \quad \|\cdot\|_r = \|\cdot\|_{L^2(B_r(0,1))} \text{ and } \|\cdot\|'_r = \|\cdot\|_{L^2(B^+_r(0,1))}
\]
with \(B^+_r = \{x \in B_r : x_n > 0\}\). We first prove an estimate related to the time-analyticity of global solutions.

Lemma 2. Assume that \(L\) satisfies (1.3) and (1.10). Then, there are \(M = M(g, n, m)\) and \(\rho = \rho(g, n, m), 0 < \rho \leq 1\), such that
\[
|||\mathcal{P}^{p+1} \partial_t^{p+1} u||| + \sum_{l=0}^{2m} |||\mathcal{P}^{p+1} \mathcal{P}_l D^{\rho} u||| \leq M \rho^{-p}(p + 1)! ||u||,
\]
holds for \(p \in \mathbb{N}\) and all solutions \(u\) to (1.1).

Proof. We prove (2.1) by induction on \(p\). For the case \(p = 0\) of (2.1), apply the weighted \(L^2\) estimate in Lemma 9 with \(\theta = 0, k = 2\) and \(F = 0\). It suffices to choose \(M \geq 3N\). By differentiating (1.1), we find that \(\partial_t^p u, p \geq 1\), satisfies
\[
\begin{aligned}
\partial_t^{p+1} u + \mathcal{L} \partial_t^p u = F_p, \quad &\text{in } \Omega \times (0, 1], \\
\partial_t^p u = D \partial_t^p u = \cdots = D^{m-1} \partial_t^p u = 0, \quad &\text{on } \partial \Omega \times (0, 1],
\end{aligned}
\]
with
\[ F_p = (-1)^{m+1} \sum_{|\alpha| \leq 2m} \sum_{q=0}^{p-1} \binom{p}{q} \partial_x^\alpha \partial_t^p u. \]

Assume that (2.1) holds up to \( p - 1 \) for some \( p \geq 1 \) and apply the weighted \( L^2 \) estimate in Lemma 3 with \( \theta = 0 \) and \( k = 2 \) to \( \partial_t^p u \) to obtain
\[
\| t^{p+1} \partial_t^{p+1} u \| + \sum_{l=0}^{2m} \| t^{l+1} D^l \partial_t^p u \| \leq N \left[ 2(p+1) \| t^p \partial_t^p u \| + \| t^{p+1} F_p \| \right] \triangleq I_1 + I_2.
\]

By the induction,
\[
\| t^p \partial_t^p u \| \leq M \rho^{-p+1} \| u \|.
\]

From (1.10) and induction
\[
\| t^{p+1} F_p \| \leq \sum_{|\alpha| \leq 2m} \sum_{q<p} \binom{p}{q} (p-q)! \| t^{p+1+1} \partial_x^\alpha \partial_t^p u \|
\leq \sum_{|\alpha| \leq 2m} \sum_{q<p} \binom{p}{q} (p-q)! M \rho^{-q}(q+1)! \| u \|
\leq N M \rho \sum_{q<p} \binom{p}{q} (p-q)! q! \rho^{-q} \rho^{-q} \| u \|
\leq M \rho^{-p}(p+1)! \| u \| \frac{N \rho}{\rho - \rho},
\]
where the last inequality follows from Lemma 14. Adding \( I_1 \) and \( I_2 \), we get
\[
I_1 + I_2 \leq M \rho^{-p}(p+1)! \| u \| N \left( \rho + \frac{\rho}{\rho - \rho} \right)
\]
and the induction for \( p \) follows after choosing \( \rho = \rho(\varrho, n, m) \) small. \( \square \)

Lemma 3 yields an interior quantitative estimate of spatial analyticity.

**Lemma 3.** Let \( 0 < \theta \leq 1 \), \( 0 < R_T < R \leq 1 \), \( B_R \subset \Omega \) and \( \mathcal{L} \) satisfy (1.9) for \( p = 0 \) over \( B_R \times [0,1] \). Then, there are \( M = M(\varrho, n, m) \) and \( \rho = \rho(\varrho, n, m), 0 < \rho \leq 1 \), such that for all \( \gamma \in \mathbb{N}^n \), the inequality
\[
(2.2) \quad (R-r)^{2m} \| te^{-\theta t - \varrho} \partial_t \partial_x^2 u \| + \sum_{k=0}^{2m} (R-r)^k \| t^{\frac{k}{2}} e^{-\theta t - \varrho} D^k \partial_x^2 u \| \leq M \left[ \rho^\theta (R-r) \right]^{-|\gamma|} \| u \|_R
\]
holds when \( u \) in \( C^\infty(B_R \times [0,1]) \) satisfies \( \partial_t u + \mathcal{L} u = 0 \) in \( B_R \times [0,1] \).

**Proof.** We prove (2.2) by induction on \( |\gamma| \). When \( |\gamma| = 0 \), by the weighted \( L^2 \) estimate in Lemma 13 with \( k = 2, p = 0, \delta = \frac{R-r}{2} \) and \( F = 0 \), we have
\[
\| te^{-\theta t - \varrho} \partial_t u \| + \| te^{-\theta t - \varrho} D^{2m} u \| \leq N \left[ (R-r)^{-2m} \| te^{-\theta t - \varrho} u \|_{r+\delta} + \| e^{-\frac{\varrho}{2} t - \varrho} u \|_{r+\delta} \right] \leq N (R-r)^{-2m} \| u \|_R.
\]
and Lemma 8 with $M = 2N$ implies

\[(R - r)^{2m} \left\| te^{-\theta t - \sigma} \partial_t u \right\|_r + \sum_{l=0}^{2m} (R - r)^l \left\| t^{\frac{l}{2m}} e^{-\theta t - \sigma} D^l u \right\|_r \leq M \left\| u \right\|_r,\]

Next, assume that (2.2) holds for multi-indices $\gamma$, with $|\gamma| \leq l$, $l \geq 0$, and we show that (2.2) holds for any multi-index of the same form with $|\gamma| = l + 1$. Differentiating (2.13) we find that $\partial_t \partial_t^2 u$ satisfies

$$\partial_t \partial_t^2 u + \mathcal{L} \partial_t^2 u = F_\gamma, \text{ in } B_R \times (0,1],$$

with

\[F_\gamma = (-1)^{m+1} \sum_{|\alpha| \leq 2m, \beta < \gamma} \left( \frac{\gamma}{\beta} \right) \partial_x^{\gamma - \beta} a_\alpha \partial_x^\beta \partial_x^2 u.\]

Applying the weighted $L^2$ estimate in Lemma 10 to $\partial_t^2 u$ with $p = 0$, we get

\[\left\| te^{-\theta t - \sigma} \partial_t^2 u \right\|_r + \left\| te^{-\theta t - \sigma} D^{2m} \partial_t^2 u \right\|_r \leq N \left( k \left\| e^{-\frac{1}{2} \sigma t} \partial_t^2 u \right\|_{r+\delta} + \delta^{-2m} \left\| te^{-\theta t - \sigma} \partial_t^2 u \right\|_{r+\delta} + \left\| te^{-\theta t - \sigma} F_\gamma \right\|_{r+\delta} \right) = I_1 + I_2 + I_3.\]

Estimate for $I_1$: when $1 \leq |\gamma| \leq 2m$, choose $k = 2$ and $\delta = (R - r)/2$ in (2.5). Also observe the bound

\[t^{-\alpha} e^{-\theta t - \sigma} \leq e^{-\frac{\theta}{m} t} \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha}}, \text{ when } \alpha, \beta, \theta > t > 0,\]

which yields

\[t^{-\frac{1}{2m}} e^{-\frac{\theta}{m} t} \leq N \theta^{-b|\gamma|}, \text{ when } |\gamma| \leq 2m \text{ and } t > 0.\]

Thus, we get

\[\left\| e^{-\frac{1}{2} \sigma t} \partial_t^2 u \right\|_{r+\delta} = \left\| t^{-\frac{1}{2m}} e^{-\frac{\theta}{m} t} t^{\frac{1}{2m}} e^{-\frac{1}{2} \sigma t} \partial_t^2 u \right\|_{r+\delta} \leq N \theta^{-b|\gamma|} \left\| t^{\frac{1}{2m}} e^{-\frac{1}{2} \sigma t} D^{\gamma} u \right\|_{r+\delta},\]

when $|\gamma| \leq 2m$. From (2.3)

\[\left\| t^{\frac{1}{2m}} e^{-\frac{1}{2} \sigma t} D^{\gamma} u \right\|_{r+\delta} \leq M (R - r)^{-|\gamma|} \left\| u \right\|_R,\]

this, together with (2.7) shows that

\[\left\| e^{-\frac{1}{2} \sigma t} \partial_t^2 u \right\|_{r+\delta} \leq N M \left[ \rho \theta^h (R - r)^{-|\gamma|} \left\| u \right\|_R \leq M \left[ \rho \theta^h (R - r)^{-|\gamma|} \left\| u \right\|_R N.\right.\]

If $|\gamma| > 2m$, choose $k = |\gamma|$, $\delta = (R - r)/|\gamma|$ in (2.5) and observe that there is a multi-index $\xi$, with $2m + |\xi| = |\gamma|$ and $|\partial_t^2 u| \leq |D^{2m} \partial_t^2 u|$. Hence, from (2.6)

\[\left\| e^{-\frac{1}{2} \sigma t} \partial_t^2 u \right\|_{r+\delta} = \left\| t^{-1} e^{-\frac{1}{2} \sigma t} t^{-\frac{1}{2m} \sigma t} e^{-\frac{1}{4} \theta t} \partial_t^2 u \right\|_{r+\delta} \leq N \theta^{-(2m-1)} |\gamma|^{2m-1} \left\| t e^{-\frac{1}{4} \theta t} \partial_t^2 u \right\|_{r+\delta}.\]
By induction and because \( R - r - \delta = \frac{|\gamma|}{|\gamma|} (R - r) \),

\[(R - r)^{2m} \| \text{te}^{-(\frac{1}{m}) \gamma t - \sigma} D^{2m} \partial^\gamma_2 u \|_{r + \delta} \]

\[\leq M \left[ \rho \left( \frac{1}{|\gamma|} \right)^{2b} \theta^b (R - r - \delta) \right]^{-|\gamma| + 2m} \]

\times (|\gamma| - 2m)! \| u \|

\[= M \left( 1 - \frac{1}{|\gamma|} \right)^{-(2b+1)(|\gamma| - 2m)} \left[ \rho \theta^b (R - r) \right]^{-|\gamma| + 2m} \]

\times (|\gamma| - 2m)! \| u \| R

\leq MN \left[ \rho \theta^b (R - r) \right]^{-|\gamma| + 2m} (|\gamma| - 2m)! \| u \| R,

where the last inequality is a consequence of the estimate

\[(2.10) \quad \left( 1 - \frac{1}{|\gamma|} \right)^{-(2b+1)(|\gamma| - 2m)} \leq N, \text{ for all } \gamma \in \mathbb{N}^n.

Plugging (2.9) into (2.8) and using that \(|\gamma|^{2m} (|\gamma| - 2m)! \leq N|\gamma|!\), we get

\[I_1 \leq N|\gamma|! \| \text{e}^{-\frac{|\gamma| - 1}{2m} \theta t} \theta^b \partial^\gamma_2 u \|_{r + \delta} \]

\[\leq M \left[ \rho \theta^b (R - r) \right]^{-|\gamma|} |\gamma|^{2m} (|\gamma| - 2m)! \| u \| R N \rho^{2m} \]

\[\leq M \left[ \rho \theta^b (R - r) \right]^{-|\gamma|} |\gamma|! \| u \| R (R - r)^{-2m} N \rho. \]

**Estimate for I_2:** when \(|\gamma| \leq 2m\), the term can be handled like the term \(I_1\) in the case \(|\gamma| \leq 2m\), but now one does not need to push inside \(I_1\) the factor \(t^{(|\gamma|)/2m}\) as we did in (2.7). Here, from (2.9) we get

\[I_2 \leq M \left[ \rho \theta^b (R - r) \right]^{-|\gamma|} \| u \| R (R - r)^{-2m} N \rho. \]

When \(|\gamma| > 2m\), again \(|\partial^\gamma_2 u| \leq |D^{2m} \partial^\gamma_2 u|\), for some \(\xi\) such that \(2m + |\xi| = |\gamma|\). By induction (recall that \(\delta = (R - r)/|\gamma|\) was already chosen in the estimate for \(I_1\), when \(|\gamma| > 2m\)) we get

\[I_2 \leq N(R - r)^{-2m} |\gamma|^{2m} \| \text{te}^{-\theta t} \partial^\gamma_2 u \|_{r + \delta} \]

\[\leq N(R - r)^{-2m} |\gamma|^{2m} \| \text{te}^{-\theta t} D^{2m} \partial^\gamma_2 u \|_{r + \delta} \]

\[\leq NM \left[ \rho \theta^b (R - r) \right]^{-|\gamma| + 2m} |\gamma|^{2m} (|\gamma| - 2m)! \| u \| R (R - r)^{-4m} \]

\[\leq M \left[ \rho \theta^b (R - r) \right]^{-|\gamma|} |\gamma|! \| u \| R (R - r)^{-2m} N \rho. \]
Estimate for $I_3$: by the induction hypothesis and Lemma 14

\[
\|te^{-\theta t - \sum} F\|_{r+\delta} \leq N \sum_{|\alpha| \leq 2m, \beta < \gamma} \left( \gamma_{\beta} \right) \theta^{-|\gamma - \beta|} |\gamma - \beta|! |t|^{\sum} e^{-\theta t - \sum} D^{\alpha}| \partial^\beta u|_{r+\delta}
\]

\[
\leq N M \sum_{\beta < \gamma} \left( \gamma_{\beta} \right) \theta^{-|\gamma - \beta|} |\gamma - \beta|! \left[ \rho^b (R - r) \right]^{-|\beta|} |\beta|! |u|_{R(R - r)^{-2m}}
\]

\[
\leq N M \left[ \theta^b (R - r) \right]^{-|\gamma|} |u|_{R(R - r)^{-2m}} \sum_{\beta < \gamma} \left( \gamma_{\beta} \right) |\gamma - \beta|! |\beta|! \theta^{-|\gamma - \beta|} \rho^{-|\beta|}
\]

\[
\leq M \left[ \rho^b (R - r) \right]^{-|\gamma|} |u|_{R(R - r)^{-2m}} \frac{N \rho}{\theta - \rho}.
\]

The bounds for $I_1$, $I_2$ and $I_3$ imply that

(2.11) \[ I_1 + I_2 + I_3 \leq M \left[ \rho^b (R - r) \right]^{-|\gamma|} |u|_{R(R - r)^{-2m}} N \rho \left( 1 + \frac{1}{\theta - \rho} \right). \]

We can write, $\gamma = \xi + \epsilon_i$, for some $\xi \in \mathbb{N}^n$ and $i = 1, \ldots, n$, and from the induction and (2.6)

(2.12) \[ ||e^{-\theta t - \sum} \partial^\gamma u||_r \leq N \theta^{-b} \left[ t^{\sum} e^{-\theta t - \sum} D\partial^\gamma u \right]_r \leq M \left[ \rho^b (R - r) \right]^{-|\gamma|} |u|_{R(R - r)^{-2m}}. \]

Finally, Lemma 8, (2.5), (2.11) and (2.12) imply the desired result when $\rho = \rho(\theta, h, m)$ is small. \hfill \Box

Remark 5. Lemma 3 also holds when the coefficients of $\mathcal{L}$ are measurable in the time variable and satisfy (1.9) for $p = 0$ over $B_R \times [0, 1]$. This follows from the interior $W^{2m, 1} \mathcal{L}$ Schauder estimate in [3, Theorem 2] and the weighted $L^2$ estimate in Lemma 13.

Next we state the quantitative estimates of spatial analyticity in directions locally tangent to the boundary of $\Omega$ that the methods in Lemma 3 yield. For this purpose, we flatten locally $B_R(q_0) \cap \partial \Omega$, with $q_0 \in \partial \Omega$, by means of the analytic change of variables

\[
y_n = x_n - \varphi(x'), \quad y_j = x_j, \quad j = 1, \ldots, n - 1,
\]

where $\varphi$ is the analytic function introduced in (1.5). The local change of variables does not modify the local conditions satisfied by $\mathcal{L}$ and without loss of generality we may assume that a solution to (1.1) verifies

(2.13) \[
\begin{aligned}
\partial_t u + \mathcal{L} u &= 0, & & \text{in } B_R^+ \times (0, 1], \\
u &= Du = \ldots = D^{m-1}u = 0, & & \text{in } \{x_n = 0\} \cap \partial B_R^+ \times (0, 1],
\end{aligned}
\]

with $u$ in $\mathcal{C}^\infty(B_R^+ \times [0, 1])$ and $0 < R \leq 1$.

Here, we use multi-indices of the form $(\gamma_1, \ldots, \gamma_{n-1}, 0) \in \mathbb{N}^n$ and write $\partial_{\gamma}$ instead of $\partial^\gamma$ to emphasize that $\partial_{\gamma}$ does not involve derivatives with respect to the variable $x_n$. Lemma 3 is proved as Lemma 4 but with Lemma 13 replaced by Lemma 12. We omit the proof.
Lemma 4. Let $0 < \theta \leq 1$, $0 < \frac{K}{\theta} < r < R \leq 1$ and assume that $L$ satisfies (1.9) for $p = 0$ over $B_R^+ \times [0, 1]$. Then, there are $M = M(\gamma, n, m)$ and $\rho = \rho(\gamma, n, m)$, $0 < \rho \leq 1$, such that for all $\gamma \in \mathbb{N}^n$ with $\gamma_n = 0$, the inequality

$$(R - r)^{2m} \left\| te^{-\theta t - \sigma} \partial_t \partial_\gamma u \right\|_r + \frac{2m}{k=0} (R - r)^k \left\| \frac{t^{\frac{k}{2m} - \theta t - \sigma}}{k!} D^k \partial_\gamma u \right\|_r \leq M \left( \rho \theta^k (R - r) \right)^{-\gamma} \| u \|_R,$$

holds when $u \in C^\infty(B_R^+ \times [0, 1])$ satisfies (2.13).

Remark 6. Lemma 4 also holds when the coefficients of $L$ are measurable in the time variable and satisfy (1.9) for $p = 0$ over $B_R^+ \times [0, 1]$. It follows from the weighted $L^2$ estimate in Lemma 12 and [5, Theorem 4].

Next, combining Lemmas 2 and 3 one can prove the following.

Lemma 5. Let $0 < \theta \leq 1$, $0 < \frac{K}{\theta} < r < R \leq 1$ and assume that $L$ satisfies (1.9) and (1.10). Then there are $M = M(\gamma, n, m)$ and $\rho = \rho(\gamma, n, m)$, $0 < \rho \leq 1$, such that for all $\gamma \in \mathbb{N}^n$, $\gamma_n = 0$, and $p \in \mathbb{N}$, the inequality

$$(R - r)^{2m} \left\| t^{p+1} e^{-\theta t - \sigma} \partial_t^{p+1} \partial_\gamma u \right\|_r + \frac{2m}{k=0} (R - r)^k \left\| \frac{t^{\frac{k}{2m} - \theta t - \sigma}}{k!} D^k \partial_\gamma u \right\|_r \leq M \rho^{-p} \left( \rho \theta^k (R - r) \right)^{-\gamma} (p + |\gamma| + 1)! \| u \|$$

holds when $u$ is a solution to (1.1) and (2.13).

Proof. We proceed by induction on $p$ and within each $p$-case we proceed by induction on $|\gamma|$. The case $p = 0$ and $\gamma \in \mathbb{N}^n$ with $\gamma_n = 0$ follows from Lemma 4 whereas the case $|\gamma| = 0$ with arbitrary $p \geq 0$ follows from Lemma 2. Thus, we may in what follows assume that $|\gamma| \geq 1$. By differentiation of (2.13), $\partial_t^p \partial_\gamma u$ satisfies

$$\begin{align*}
\partial_t^{p+1} \partial_\gamma u + L \partial_t^p \partial_\gamma u &= F_{(\gamma, p)}, & \text{in } B_R^+ \times (0, T), \\
\partial_t^p \partial_\gamma u &= D \partial_t^p \partial_\gamma u = \ldots = D^{m-1} \partial_t^p \partial_\gamma u = 0, & \text{on } \{x_n = 0\} \cap \partial B_R^+ \times (0, T),
\end{align*}$$

with

$$(2.14) \quad F_{(\gamma, p)} = (-1)^{m+1} \sum_{|\alpha| \leq 2m} \sum_{(q, \beta)} (p) \binom{\gamma}{\beta} \partial_t^{p-q} \partial_\gamma^\beta a_\alpha \partial_t^\alpha \partial_\gamma^\beta u.$$

By the weighted $L^2$ estimate in Lemma 12 applied to $\partial_t^p \partial_\gamma u$,

$$(2.15) \quad \left\| t^{p+1} e^{-\theta t - \sigma} \partial_t^{p+1} \partial_\gamma u \right\|_r + \left\| t^{p+1} e^{-\theta t - \sigma} D^{2m} \partial_t^p \partial_\gamma u \right\|_r \leq N \left( p + k \right) \left\| t^{p+1} e^{-\theta t - \sigma} \partial_t^p \partial_\gamma u \right\|_{r+\delta} + \delta^{-2m} \left\| t^{p+1} e^{-\theta t - \sigma} \partial_t^p \partial_\gamma u \right\|_{r+\delta} + \left\| t^{p+1} e^{-\theta t - \sigma} F_{(\gamma, p)} \right\|_{r+\delta} \equiv I_1 + I_2 + I_3.$$
Estimate for $I_1$: if $|\gamma| \leq 2m$, take $k = 2$ and $\delta = (R-r)/2$ in (2.15). Taking into account that $(p+1)! \leq N(p+|\gamma|)!$, (2.6) and Lemma 2 we obtain
\[
I_1 \leq N(p+|\gamma|)!\|t^p e^{-\frac{\theta}{2} t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}^r
\leq N(p+|\gamma|)!\|t^p e^{-\frac{\theta}{2} t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}
\leq M \rho^{-p} [\rho^b (R-r)]^{-|\gamma|} \|t^p \partial_t \rho^b - 2m \|_{r+\delta} N\rho(R-r)^{-2m}.
\]
In the previous chain of inequalities we used that
\[
\|t^p \partial_t \rho^b - 2m \|_{r+\delta} \leq M \|t^p \partial_t \rho^b - 2m \|_{r+\delta}.
\]
and applied Lemma 2.

If $|\gamma| > 2m$, choose $k = |\gamma|$ and $\delta = (R-r)/|\gamma|$ in (2.16). There is a multi-index $\xi \in \mathbb{N}^n$ with $\xi_n = 0$ such that $2m + |\xi| = |\gamma|$ and $|\partial_t^\xi \rho^b| \leq |D^2 \partial_t^\xi \rho^b|$ and from (2.6)
\[
I_1 \leq N(p+|\gamma|)!\|t^p e^{-\frac{\theta}{2} t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}
\leq N(p+|\gamma|)!\|t^p e^{-\frac{\theta}{2} t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}
\leq N(p+|\gamma|)!\|t^p+1 e^{-\theta t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}
\leq N(p+|\gamma|)!\|t^p+1 e^{-\theta t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}
\leq M \rho^{-p} [\rho^b (R-r)]^{-|\gamma|+2m} \|t^p \partial_t \rho^b - 2m \|_{r+\delta} N\rho(R-r)^{-2m}.
\]
From
\[
|\gamma|^{2m-1}(p+|\gamma|-2m+1)!\|t^p \partial_t \rho^b - 2m \|_{r+\delta} N\rho(R-r)^{-2m}.
\]
(2.16) and (2.17)
\[
I_1 \leq M \rho^{-p} [\rho^b (R-r)]^{-|\gamma|} \|t^p \partial_t \rho^b - 2m \|_{r+\delta} N\rho(R-r)^{-2m}.
\]

Estimate for $I_2$: For $|\gamma| \leq 2m$, we set $\delta = (R-r)/2$ and because $\theta$ and $R \leq 1$, Lemma 2 shows that
\[
I_2 \leq N(R-r)^{-2m} \|t^p+1 e^{-\theta t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}
\leq N(R-r)^{-2m} M \rho^{-p}(p+1)!\|u\|
\leq M \rho^{-p}(R-r)^{-|\gamma|} \|t^p \partial_t \rho^b - 2m \|_{r+\delta} N\rho(R-r)^{-2m}.
\]
If $|\gamma| > 2m$, we have already chosen $\delta = (R-r)/|\gamma|$ and there is $\xi \in \mathbb{N}^n$, with $\xi_n = 0, 2m + |\xi| = |\gamma|$ and $|\partial_t^\xi \rho^b| \leq |D^2 \partial_t^\xi \rho^b|$. By the induction hypothesis and taking into account that
\[
|\gamma|^{2m}(p+|\gamma|-2m+1)! \leq N(p+|\gamma|+1)!,
\]
we get
\[
I_2 \leq N(R-r)^{-2m+1} \|t^p+1 e^{-\theta t^{-\sigma}} D^\gamma \partial_t^n u\|_{r+\delta}
\leq N(R-r)^{-2m} M \rho^{-p} [\rho^b (R-r)]^{-|\gamma|-2m} (p+|\gamma|-2m+1)!\|u\|
\leq M \rho^{-p} [\rho^b (R-r)]^{-|\gamma|} \|t^p \partial_t \rho^b - 2m \|_{r+\delta} N\rho(R-r)^{-2m}.
\]
\[ I_3 = \| (1 + e^{-\theta t}) F_{(\gamma,p)} \|_{r+\delta} \]
\[ \leq N \sum_{|\alpha| \leq 2m} \sum_{(q,\beta) < (p,\gamma)} \binom{p}{q} \binom{\gamma}{\beta} \theta^{p+q-|\gamma|+|\beta|} (p - q + |\gamma| - |\beta|)! \]
\[ \times \| (1 + e^{-\theta t}) D^{\alpha} \phi \|^2_{L^2(\Omega)} \]
\[ \leq NM \left[ \theta^b(R - r) \right]^{-|\gamma|} (p + |\gamma|)! \theta^{p+q-|\gamma|-|\beta|} (p - q + |\gamma| - |\beta|)! \]
\[ \leq M \theta^{p} \left[ \theta^b(R - r) \right]^{-|\gamma|} (p + |\gamma| + 1)! \theta^{p+q-|\gamma|} (p - q + |\gamma|)! \theta^{p+q-|\gamma|-|\beta|} (p - q + |\gamma| - |\beta|)! \]

Thus,
\[ (2.18) \quad I_1 + I_2 + I_3 \leq M \theta^{p} \left[ \theta^b(R - r) \right]^{-|\gamma|} (p + |\gamma|)! \theta^{p+q-|\gamma|} (p - q + |\gamma|)! \theta^{p+q-|\gamma|-|\beta|} (p - q + |\gamma| - |\beta|)! \]

and Lemma 6 follows from (2.15), (2.18), Lemma 5 and the induction hypothesis for \((p - 1, \gamma)\), when \(\rho = \rho(\theta, n, m)\) is small.

Finally, Theorem 4 follows from the embedding [12]
\[ \| \varphi \|_{L^\infty(\Omega)} \leq C(n) \sum_{|\alpha| \leq \left[ \frac{n}{2} \right] + 1} \| D^{\alpha} \varphi \|_{L^2(\Omega)}, \quad \text{for } \varphi \in C^\infty(\Omega), \]
the inequality
\[ \| f \|_{L^\infty(I)} \leq |I|^{\frac{1}{2}} \| f \|_{L^2(I)} + |I|^{-\frac{1}{2}} \| f \|_{L^2(I)}, \quad \text{for } f \in C^1(I), \]
with \(I\) an interval in \(\mathbb{R}\) and Lemma 6.

**Lemma 6.** Let \(0 < \theta \leq 1, 0 < \frac{2}{b} < r \leq R \leq 1\) and \(L\) satisfy (1.9) and (1.10). Then, there are \(M = M(\rho, n, m)\) and \(\rho = \rho(\theta, n, m), 0 < \rho \leq 1, \) such that
\[ (2.19) \quad \| t^p e^{-\theta t} [\partial^p d_{x} \partial_{\gamma} u] \|_{\rho} \]
\[ \leq M \rho^{p-l} \left[ \rho \theta^b(R - r) \right]^{-l-|\gamma|} (p + l + |\gamma| + 1)! \theta^{p+q-|\gamma|-|\beta|} (p - q + |\gamma| - |\beta|)! \]
holds when \(u\) is a solution to (1.1) and (2.13). Here, \(\partial_n\) denotes differentiation with respect to the variable \(x_n\).

**Proof.** A solution to (2.13) satisfies
\[ (2.20) \quad \partial_{x}^{p+1} \partial_{\gamma}^l \partial_{\gamma}^\gamma u + L \partial_{x}^{p} \partial_{\gamma}^l \partial_{\gamma}^\gamma u = F_{(p,l,\gamma)}, \quad \text{in } B_R^+ \times (0, 1], \]
with
\[ F_{(p,l,\gamma)} = (-1)^{n+1} \sum_{|\alpha| \leq 2m} \sum_{(q,j,\beta) < (p,l,\gamma)} \binom{p}{q} \binom{\gamma}{\beta} \theta^{p+q-|\gamma|-|\beta|} (p - q + |\gamma| - |\beta|)! \theta^{p+q-|\gamma|-|\beta|} a_{\alpha} \theta^{p+q-|\gamma|-|\beta|} \theta^{p+q-|\gamma|-|\beta|} u. \]
Because of (1.3), $a_{2m_{\varepsilon_n}} \geq \varrho > 0$ in $\Omega \times [0,1]$, and one can solve for $\partial_t^k \partial_{x}^{l+1} \partial_{x}^{\gamma} u$ in (2.20). Substituting into that formula $l$ by $l - 2m + 1$, when $l \geq 2m$, we have

$$
(2.21) \quad |\partial_t^k \partial_{n}^{l+1} \partial_{x}^{\gamma} u| \leq \frac{1}{a_{2m_{\varepsilon_n}}} \left[ |\partial_t^k \partial_{n}^{l-2m+1} \partial_{x}^{\gamma} u| + |F(p,l-2m+1,\gamma)| \right]
+ \frac{1}{a_{2m_{\varepsilon_n}}} \sum_{|\alpha| \leq 2m} \sum_{\alpha_{s} \leq 2m-1} ||a_{\alpha}||_{L^{\infty}(\Omega \times (0,1))} |\partial_{n}^{l-2m+1} \partial_{x}^{\gamma} \partial^{\alpha}_{x} u|.
$$

We prove (2.19) by induction on the quantity $2mp + l + |\gamma|$ with $M$ the same constant as in Lemma 5. If $2mp + l + |\gamma| \leq 2m$, then $l \leq 2m$ and (2.40) and Lemma 5 show that

$$
\left\| t^p e^{-\theta t^{-\sigma}} \partial_t^k \partial_{n}^{l+1} \partial_{x}^{\gamma} u \right\|_r \leq \left\| t^p e^{-\theta t^{-\sigma}} \partial_t^k \partial_{n}^{l-2m+1} \partial_{x}^{\gamma} u \right\|_r + \left\| t^p e^{-\theta t^{-\sigma}} \partial_t^k \partial_{n}^{l-2m+1} \partial_{x}^{\gamma} u \right\|_r
\leq N\theta^{-l} (1 + |\gamma|) \left\| t^p e^{-\theta t^{-\sigma}} \partial_t^k \partial_{n}^{l-2m+1} \partial_{x}^{\gamma} u \right\|_r
\leq N\theta^{-l} (1 + |\gamma|)^{l} (R - r)^{-l} M\rho^{-p} \rho^{b} (R - r)^{-1} (p + |\gamma| + 1)! \|u\|\leq M\rho^{-p-l} \left\| \rho^{b} (R - r)^{-1} (p + l + |\gamma| + 1)! \|u\| N\rho^{2i},
$$

where the last inequality holds because

$$(1 + |\gamma|)^{l} (p + |\gamma| + 1)! \leq N (p + l + |\gamma| + 1)! .$$

Also, (2.19) holds when $l = 0$ from Lemma 5. Thus, (2.19) holds, when $2mp + l + |\gamma| \leq 2m$, and $l \leq 2m$, provided that $\rho$ is small.

Assume now that (2.19) holds when $2mp + l + |\gamma| \leq k$, for some fixed $k \geq 2m$, and we shall prove it holds for $2mp + l + |\gamma| = k + 1$.

In the same way as for the case $k = 2m$, Lemma 5 shows that (2.19) holds, when $2mp + l + |\gamma| = k + 1$ and $l \leq 2m$, provided that $\rho$ is small. So, let us now assume that (2.19) holds for $2mp + j + |\gamma| = k + 1$ and $j = 0, \ldots, l$, for some $l \geq 2m$, and prove that it holds for $2mp + j + |\gamma| = k + 1$ with $j = l + 1$. Let then $\gamma$ and $p$ be such that $2mp + l + |\gamma| = k + 1$. From (2.21) and because $a_{2m_{\varepsilon_n}} \geq \varrho$, we obtain

$$
\left\| t^p e^{-\theta t^{-\sigma}} \partial_t^k \partial_{n}^{l+1} \partial_{x}^{\gamma} u \right\|_r' \leq \varrho^{-1} \left[ \left\| t^p e^{-\theta t^{-\sigma}} \partial_t^k \partial_{n}^{l-2m+1} \partial_{x}^{\gamma} u \right\|_r + \left\| t^p e^{-\theta t^{-\sigma}} F(p,l-2m+1,\gamma) \right\|_r \right]
+ \varrho^{-1} \sum_{|\alpha| \leq 2m} \sum_{\alpha_{s} \leq 2m-1} ||a_{\alpha}||_{L^{\infty}(Q)} \left\| t^p e^{-\theta t^{-\sigma}} \partial_t^k \partial_{n}^{l-2m+1} \partial_{x}^{\gamma} \partial^{\alpha}_{x} u \right\|_r
\triangleq H_1 + H_2 + H_3.
$$

**Estimate for $H_1$:** the multi-indices involved in this term satisfy

$$2m(p + 1) + l - 2m + 1 + |\gamma| = k + 1$$
and the total number of $x_n$ derivatives involved is less or equal than $l$. From the induction hypothesis and (2.24), we can estimate $H_1$ as follows

\[\|t^p e^{-\theta t} \partial_x^p \partial_{x_1} \partial_{x_2} \partial_{x_n}^m u\|_r \]

\[= \|t^{-1} e^{-\theta t} \partial_x^p \partial_{x_1} \partial_{x_2} \partial_{x_n}^m u\|_r \]

\[\leq N\theta^{-2(2m-1)} (l + |\gamma|)^{2m-1} \|t^p e^{-\theta t} \partial_x^p \partial_{x_1} \partial_{x_2} \partial_{x_n}^m u\|_r \]

\[\leq N\theta^{-2(2m-1)} (l + |\gamma|)^{2m-1} M\rho^{-p-1-(l-2m+1)} [(\rho \theta^b(R - r)]^{-l-1-|\gamma|} \]

\[\times (p + l - 2m + |\gamma| + 3)! \|u\| \]

\[\leq M\rho^{-p-(l+1)} [\rho \theta^b(R - r)]^{-(l+1)-|\gamma|} \]

\[\times (p + l + |\gamma| + 2)! \|u\| N\rho^{4m-1}, \]

where the last inequality holds because

\[(l + |\gamma|)^{2m-1} (p + l - 2m + |\gamma| + 3)! \leq N (p + l + |\gamma| + 2)!, \]

when $p + l + |\gamma| + 2 \geq 2m$. Thus,

\[H_1 \leq M\rho^{-p-(l+1)} [\rho \theta^b(R - r)]^{-(l+1)-|\gamma|} \]

\[\times (p + l + |\gamma| + 1)! \|u\| N\rho^{4m-1}. \]

**Estimate for $H_2$:** we expand this term and obtain

\[H_2 \leq N \sum_{|\alpha| \leq 2m} \sum_{p,l-2m+1,|\gamma| < (p, l-2m+1, |\gamma|)} \left( \begin{array}{c} p \\ q \\ j \\ \beta \end{array} \right) \left( \begin{array}{c} l - 2m + 1 \\ \gamma \\ |\beta| \end{array} \right) \]

\[\times e^{-1-(p-q)-(l-2m+1)-|\gamma-\beta|} (p - q + l - 2m + 1 - j + |\gamma - \beta|)! \]

\[\times \|t^p e^{-\theta t} \partial_x^p \partial_{x_1} \partial_{x_2} \partial_{x_n}^m u\|_r. \]

The multi-indices involved in the derivatives of $u$ that appear in (2.23) satisfy $2mq + j + |\alpha| + |\beta| < 2mp + l + 1 + |\gamma| = k + 1$ and we already know how to control these derivatives by the first induction hypothesis. In fact, if we write $\alpha = (\alpha', \alpha_n)$ and because $\alpha_n$ is related to normal derivatives, we get

\[\|t^p e^{-\theta t} \partial_x^p \partial_{x_1} \partial_{x_2} \partial_{x_n}^m u\|_r \]

\[\leq M\rho^{-2(p-q)-j-|\alpha|} \rho^{l-1-|\beta|-|\alpha|} (q + j + |\beta| + |\alpha| + 1)! \|u\|. \]

The sum in (2.23) runs over $\{(q, j, \beta) < (p, l-2m+1, |\gamma|)\}$ and $|\alpha| \leq 2m$ and inside the sum (2.23), $j + \alpha_n + |\alpha| \leq l + 2m + 1$, $j + |\beta| + |\alpha| \leq l + 1 + |\gamma|$ and $q + j + |\beta| \leq p + l - 2m + |\gamma|$. Also,

\[\frac{(q + j + |\beta| + |\alpha| + 1)!}{(q + j + |\beta|)!} \leq \frac{(p + l + |\gamma| + 1)!}{(p + l - 2m + |\gamma|)!}. \]

These and (2.24) show that for all such $(q, j, \beta)$ and $\alpha$

\[\|t^p e^{-\theta t} \partial_x^p \partial_{x_1} \partial_{x_2} \partial_{x_n}^m u\|_r \leq M\rho^{-l-2m-1} [\theta^b(R - r)]^{-l-1-|\gamma|} \rho^{-q-j-|\beta|} \]

\[\times \frac{(p + l + |\gamma| + 1)!}{(p + l - 2m + |\gamma|)!} \|u\|. \]
Plugging (2.25) into (2.23) yields
\begin{equation}
H_2 \leq N M \rho^{-l-2m-1} \left[ \theta^b(R - r) \right]^{-(l+1)-|\gamma|} \frac{(p + l + |\gamma| + 1)!}{(p + l - 2m + |\gamma|)!} \|u\| \\
\times \sum_{(q,j,\beta)} \left( \frac{p}{q} \right) \left( l - 2m + 1 \right) \left( \gamma \right) \left( j \right) \left( \beta \right) \\
\times (p - q + l - 2m + 1 - j + |\gamma - \beta|)! \rho^{q + j + |\beta|} \\
\times \theta \rho^{-l-2m-1-|\gamma|}(p + l - 2m + 1 + |\gamma|)! \frac{\rho}{\theta - \rho}.
\end{equation}
(2.26)

The later and (2.26) imply
\begin{equation}
H_2 \leq M \rho^{-p-(l+1)} \left[ \theta^b(R - r) \right]^{-(l+1)-|\gamma|} (p + (l + 1) + |\gamma|)! \|u\| N \rho \frac{N \rho}{\theta - \rho}.
\end{equation}

Estimate for $H_3$: the multi-indices involved in the sum run over
\[\{ \alpha : |\alpha| \leq 2m : \alpha_n \leq 2m - 1\},\]
the multi-indices involved in the derivatives of $u$ which appear in $H_3$ satisfy
\[2mp + (l - 2m + 1 + \alpha_n) + |\gamma| + |\alpha'| \leq k + 1,\]
with a total number of $x_n$ derivatives equal to $\alpha_n + l - 2m + 1 \leq l$, so we are within previous steps of the induction process and $0 < \rho < 1$. Accordingly, applying the second induction hypothesis one gets
\begin{equation}
H_3 \leq M \rho^{-p-(l+1)} \left[ \theta^b(R - r) \right]^{-(l+1)-|\gamma|} (p + (l + 1) + |\gamma|)! \|u\| N \rho.
\end{equation}
(2.27)

Now, (2.19) when $2mp + (l + 1) + |\gamma| = k + 1$, follows from (2.22), (2.27) and (2.28), when $\rho = \rho(x, n, m)$ is chosen small. \hfill \square

Remark 7. Choosing $\theta = t^\sigma$ in Lemma 6 one recovers (1.5).

Next we give a proof of the claim in the second paragraph in Remark 1. In Lemma 4 we give details only for the interior case. Lemma 4 also holds near the boundary when the boundary is flat and for tangential derivatives $\gamma \in N^n$ with $\gamma_n = 0$. Then, as in Lemma 4 one can extend the result to all the derivatives by showing that there are $M = M(x, n, m)$ and $\rho = \rho(x, n, m)$, $0 < \rho \leq 1$, such that
\[
\|te^{-\theta t}\partial_t^\nu \partial_x^k \partial_x^\kappa u\|_{L^1} \leq M \rho^{-l} \left[ \theta^b(R - r) \right]^{-2mp-|\gamma|-l} (2mp + |\gamma| + l)! \|u\|_{L^1}
\]
when $u$ satisfies (2.13), $\frac{R}{2} < r < R$ and (1.10) holds over $B_R(x_0) \cap \Omega$.

Lemma 7. Let $0 < \theta \leq 1$, $0 < \frac{R}{2} < r < R \leq 1$ and $\mathcal{L}$ satisfy (1.19). Then there are $M = M(x, n, m)$ and $\rho = \rho(x, n, m)$, $0 < \rho \leq 1$, such that for any $\gamma \in N^n$ and $p \in N$,
\begin{equation}
(R - r)^2m \|te^{-\theta t}\partial_t^{p+1} \partial_x^k u\|_{L^1} \leq M \left[ \rho^b(R - r) \right]^{-2mp-|\gamma|} (2mp + |\gamma|)! \|u\|_{L^1}
\end{equation}
holds when $u$ in $C^\infty(B_R \times [0, 1])$ satisfies $\partial_t u + \mathcal{L} u = 0$ in $B_R \times [0, 1]$. 

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Proof: We prove (2.29) by induction on \( p \) and then by induction on \(|\gamma|\). When \( p = 0 \), (2.29) is the estimate in Lemma 3. Assume (2.29) holds up to \( p - 1 \) for some \( p \geq 1 \). Then,

\[
\partial_t^{p+1} \partial_x^\gamma u + \mathcal{L} \partial_t^p \partial_x^\gamma u = F_{\gamma,p}, \quad \text{in } B_R \times (0,1],
\]

with

\[
F_{\gamma,p} = (-1)^{m+1} \sum_{|\alpha| \leq 2m} \sum_{(q,\beta) < (p,\gamma)} \binom{p}{q} \binom{\gamma}{\beta} \partial_t^{p-q} \partial_x^{\gamma-\beta} a_\alpha \partial_t^\alpha \partial_x^\beta \partial_x^\gamma u.
\]

Apply the weighted \( L^2 \) estimate in Lemma 13 with \( p = 0 \), \( k = p + |\gamma| + 1 \) and \( \delta = (R - r)/(p + |\gamma| + 1) \) to \( \partial_t^p \partial_x^\gamma u \). It gives,

\[
\|te^{-\theta t - \sigma} \partial_t^{p+1} \partial_x^\gamma u\|_r + \|te^{-\theta t - \sigma} D^{2m} \partial_t^p \partial_x^\gamma u\|_r \leq N \left( |\gamma| + p \right) \|e^{-\theta t - \sigma} \partial_t^p \partial_x^\gamma u\|_{r+\delta} + \frac{(|\gamma| + p + 1)^{2m}}{(R - r)^{2m}} \|te^{-\theta t - \sigma} \partial_t^p \partial_x^\gamma u\|_{r+\delta} + \|te^{-\theta t - \sigma} F_{\gamma,p}\|_{r+\delta} \triangleq I_1 + I_2 + I_3.
\]

**Estimate for \( I_1 \):** by induction hypothesis for \( p - 1 \) and (2.6)

\[
\|e^{-\theta t - \sigma} \partial_t^p \partial_x^\gamma u\|_{r+\delta} = \|t^{-1} e^{-\theta t - \sigma} \partial_t^p \partial_x^\gamma u\|_{r+\delta} \leq N \theta^{-2mb} (|\gamma| + p)^{2m-1} \|te^{-\theta t - \sigma} \partial_t^p \partial_x^\gamma u\|_{r+\delta} \leq N \theta^{-2mb} (|\gamma| + p)^{2m-1} M \left[ \rho_\theta (R - r) \right]^{-2m(p-1) - |\gamma|} (2m(p - 1) + |\gamma|)! \times \left( 1 + \frac{1}{|\gamma| + p} \right)^{6m-2(p+|\gamma|)} \|u\|_{R(N R^p)}.
\]

This and \((|\gamma| + p)^{2m} (2m(p - 1) + |\gamma|)! \leq N(2mp + |\gamma|)!\), give

\[
I_1 \leq M \left[ \rho_\theta (R - r) \right]^{-2m(p-1) - |\gamma|} (2mp + |\gamma|)! \|u\|_{R N \rho} (R - r)^{-2m}.
\]

**Estimate for \( I_2 \):** by induction hypothesis for \( p - 1 \)

\[
\|te^{-\theta t - \sigma} \partial_t^p \partial_x^\gamma u\|_{r+\delta} \leq M \left[ \rho_\theta (R - r) \right]^{-2m(p-1) - |\gamma|} (2m(p - 1) + |\gamma|)! \|u\|_{R N \rho}
\]

and

\[
I_2 \leq M \left[ \rho_\theta (R - r) \right]^{-2m(p-1) - |\gamma|} (2mp + |\gamma|)! \|u\|_{R N \rho} (R - r)^{-2m}.
\]
Estimate for $I_3$: by induction on $(q, \beta) < (p, \gamma)$ and Lemma 14 for $\mathbb{N}^{n+1}$

$$
\|te^{-\delta}F_{\gamma,p}\|_{r+\delta} \leq N \sum_{|\alpha| \leq 2m} \sum_{(q,\beta) < (p,\gamma)} \left( \frac{p}{q} \right)^{\beta} e^{-p+q-|\gamma-\beta|(p-q+|\gamma|-|\beta|)!} \\
\times \|t^{1+2me^{-\delta}}D^{|\alpha|}q^\alpha q^\beta u\|_{r+\delta} \\
\leq NM \left[\rho^h(R-r)\right]^{-2mp-|\gamma|} \frac{(2mp + |\gamma|)!}{(p+|\gamma|)!} \rho^{-2(p-1)m} \|u\|_R(R-r)^{-2m} \\
\times \sum_{(q,\beta) < (p,\gamma)} \left( \frac{p}{q} \right)^{\beta} (p-q+|\gamma|-|\beta|)! \frac{(q+|\gamma|)!}{q^{p+q-|\gamma|+|\beta|} \rho^{-q-|\beta|}} \\
\leq M\rho^h(R-r)^{-2mp-|\gamma|}(2mp + |\gamma|)! \|u\|_R(R-r)^{-2m} \frac{N\rho}{\varrho - \rho}.
$$

Hence

$$
\tag{2.30} I_1 + I_2 + I_3 \leq M\rho^h(R-r)^{-2mp-|\gamma|}(2mp + |\gamma|)! \|u\|_R(R-r)^{-2m} \frac{N\rho}{\varrho - \rho}.
$$

Lemma 8, the induction hypothesis and (2.30) finish the proof. \hfill \Box

Here we describe the counterexample alluded at the end of Remark 1 let $\omega \subset \Omega$ be an open set and $\varphi \in C_0^\infty(\omega)$, $0 \leq \varphi \leq 1$, with $\varphi \equiv 1$ somewhere in $\omega$. Define

$$
V(x,t) = \begin{cases} 
\varphi(x)e^{-\frac{|x|^2}{2t^2}}, & t > \frac{1}{2}, \\
0, & t \leq \frac{1}{2},
\end{cases}
$$

which is identically zero outside $\omega$ for all times and not time-analytic inside $\omega \times \left(\frac{1}{2}\right)$. Let $u$ be the solution to

$$
\begin{cases} 
\partial_t u - \Delta u + V(x,t)u = 0, & \text{in } \Omega \times (0,1], \\
u = 0, & \text{on } \partial \Omega \times (0,1], \\
u(0) = u_0, & \text{in } \Omega,
\end{cases}
$$

with $u_0 \in C_0^\infty(\Omega)$, $u_0 \equiv 0$ in $\Omega$. The strong maximum principle 21 shows that $u > 0$ in $\Omega \times (0,1]$ and $e^{t\Delta}u_0$ coincides with $u$ over $\Omega \times \left(0,\frac{1}{2}\right)$. If $u$ was analytic in the $t$ variable at some point $(x_0,\frac{1}{2})$ with $x_0$ in $\Omega$, because all the time derivatives of $u$ and $e^{t\Delta}u_0$ coincide at $(x_0,\frac{1}{2})$, one gets $e^{t\Delta}u_0(x_0,t) = u(x_0,t)$ in $[0,1]$. But $v = u - e^{t\Delta}u_0$ satisfies

$$
\begin{cases} 
\partial_t v - \Delta v \leq 0, & \text{in } \Omega \times \left(\frac{1}{2},1\right], \\
v = 0, & \text{on } \partial \Omega \times (0,1], \\
v(0) = 0, & \text{in } \Omega,
\end{cases}
$$

and the weak maximum principle implies, $v \leq 0$ in $\Omega \times \left[\frac{1}{2},1\right]$. Because $v$ attains its maximum inside $\Omega \times \left(\frac{1}{2},1\right]$, the strong maximum principle gives, $u = e^{t\Delta}u_0$ in $\Omega \times [0,1)$, which is a contradiction. Thus, $u$ fails to be analytic in the time variable at all points in $\Omega \times \left(\frac{1}{2}\right)$. 
3. Observability inequalities

Here we give a proof of the observability inequalities in Theorems 3 and 4. We choose to do it for the equivalent case of the forward parabolic equation. The second parts of the Theorems follow from standard duality arguments and the reasonings in [3, §5] or [4, §5].

Proof. From [13, 15] and (1.14), the observability inequalities

\begin{align}
\|u(T)\|_{L^2(\Omega)} &\leq Ne^{N/(1-\epsilon)T}\|u\|_{L^2(B_R(x_0)\times(\epsilon T, T))}, \\
\|u(T)\|_{L^2(\Omega)} &\leq Ne^{N/(1-\epsilon)T}\|A\nabla u \cdot \nu\|_{L^2(\Delta_R(q_0)\times(\epsilon T, T))},
\end{align}

for solutions to

\begin{equation}
\begin{cases}
\partial_t u - \nabla \cdot (Au) + b_1 \cdot \nabla u + \nabla \cdot (b_2 u) + cu = 0, & \text{in } \Omega \times (0, T], \\
u = 0, & \text{in } \partial \Omega \times [0, T], \\
 u(0) = u_0, & \text{in } \Omega,
\end{cases}
\end{equation}

with \(u_0\) in \(L^2(\Omega)\), \(0 \leq \epsilon < 1\), \(B_{2R}(x_0) \subset \Omega\), \(q_0 \in \partial \Omega\), \(\Delta_R(q_0) = B_R(q_0) \cap \partial \Omega\) and \(N = N(\Omega, R, \theta)\), hold when \(\partial \Omega\) is \(C^{1,1}\). We may assume that \(D\) satisfies \(|D| \geq \theta |B_R(x_0)|\) and define

\[D_t = \{ x \in \Omega : (x, t) \in D \} \quad \text{and} \quad E = \{ t \in (0, T) : |D_t| \geq |D|/(2T) \}.
\]

By Fubini’s theorem, \(D_t\) is measurable for a.e. \(0 < t < T\), \(E\) is measurable in \((0, T)\) with \(|E| \geq \theta T/2\). Next, let \(z > 1\) to be determined later and \(0 < l < T\) be a Lebesgue point of \(E\). From [3, Lemma 2], there is a monotone decreasing sequence \(\{l_k\}_{k \geq 1}\), \(l < \cdots < l_{k+1} < l_k < \cdots < l_1 \leq T\), such that

\begin{equation}
l_k - l_{k+1} = z(l_{k+1} - l_{k+2}) \quad \text{and} \quad |E \cap (l_{k+1}, l_k)| \geq \frac{1}{z} (l_k - l_{k+1}), \quad \text{for } k \geq 1.
\end{equation}

Define \(\tau_k = l_k - l_{k+1}\). From (3.1),

\begin{equation}
\|u(l_k)\|_{L^2(\Omega)} \leq Ne^{N/(l_k-l_{k+1})}\|u\|_{L^2(B_R(x_0)\times(\tau_k, l_k))},
\end{equation}

Theorem 4 shows that the solution \(u\) to (3.3) verifies

\begin{equation}
|\partial_x^\alpha \partial_t^p u(x, t)| \leq e^{N/(l_k-l_{k+1})} \rho^{-1-|\alpha|-p} R^{-|\alpha|} (l_k - l_{k+1})^{-p} |\alpha|! |\beta|! \|u(l_k)\|_{L^2(\Omega)},
\end{equation}

for \(\alpha \in \mathbb{N}^n\), \(p \in \mathbb{N}\), \(x \in B_R(x_0)\) and \(\tau_k \leq t \leq l_k\). Then, from (3.5), (3.6) and two consecutive applications of Lemma 1, the first with respect to the time-variable and the second with respect to the space-variables, show that

\[\|u(l_k)\|_{L^2(\Omega)} \leq Ne^{N/(l_k-l_{k+1})} \int_{E \cap (l_{k+1}, l_k)} \|u(t)\|_{L^1(D_t)} dt \|u(l_k)\|_{L^2(\Omega)}^{1-\theta},\]

holds for any choice of \(z > 1\) and \(k \geq 1\), with \(N = N(\Omega, R, \theta)\), \(0 < \theta < 1\) and \(\theta = \theta(q)\). Proceeding with the telescoping series method, the later implies

\[\epsilon^{1-\theta} e^{-N/(l_k-l_{k+1})}\|u(l_k)\|_{L^2(\Omega)} - \epsilon e^{-N/(l_k-l_{k+1})}\|u(l_k)\|_{L^2(\Omega)} \leq N \int_{E \cap (l_{k+1}, l_k)} \|u(t)\|_{L^1(D_t)} dt, \quad \text{when } \epsilon > 0.
\]
Choosing \( \epsilon = e^{-1/(l_k - l_{k+1})} \) and (3.4) yield
\[
e^{-\frac{N+1-\theta}{l_k-1+k+1}} \| u(l_k) \|_{L^2(\Omega)} - e^{-\frac{N+1-\theta}{l_{k+1} - l_k+2}} \| u(l_{k+1}) \|_{L^2(\Omega)}\]
\[
\leq N \int \limits_{E \cap (l_{k+1}, l_k)} \| u(t) \|_{L^1(\mathcal{D}, t)} \, dt, \text{ when } z = \frac{N+1}{N+1-\theta}.
\]
The addition of the above telescoping series and the local energy inequality for solutions to (3.3) leads to
\[
\| u(T) \|_{L^2(\Omega)} \leq N \| u \|_{L^1(\mathcal{D})},
\]
with \( N = N(\Omega, T, \mathcal{D}, \theta) \).

Similarly, we may assume that \( |\beta| \geq \theta(\Omega, T, \mathcal{D}, \theta) \) and setting
\[
\mathcal{D}_t = \{ q \in \partial \Omega : (q, t) \in \mathcal{D} \} \quad \text{and} \quad E = \{ t \in (0, T) : |\beta| \geq |\beta|/ (2T) \},
\]
we get from (3.2), Theorem 1 with \( x_0 = q_0 \) and the obvious generalization of Lemma 1 for the case of analytic functions defined over analytic hypersurfaces in \( \mathbb{R}^n \) that
\[
\| u(l_k) \|_{L^2(\Omega)} \leq \left( N e^{-N/((l_k - l_{k+1})} \int \limits_{E \cap (l_{k+1}, l_k)} \| \mathbf{A} \nabla u(t) \cdot \nu \|_{L^1(\beta_1)} \, dt \right)^{\theta} \| u(l_{k+1}) \|_{L^2(\Omega)}^{-1/\theta}
\]
for all \( k \geq 0, z > 1 \), with \( N = N(\Omega, R, \theta) \), \( 0 < \theta < 1 \) and \( \theta = \theta(\theta) \). Again, after choosing \( z > 1 \), the telescoping series method implies
\[
\| u(T) \|_{L^2(\Omega)} \leq N \| \mathbf{A} \nabla u \cdot \nu \|_{L^1(\beta)},
\]
with \( N = N(\Omega, T, \beta, \theta) \).

Finally, Remark 1 holds because under (1.14) with \( b_2 = 0 \), the Carleman inequalities and reasonings in [7] and [8 §3] can be used to prove the following global interpolation inequality: there are \( N = N(\Omega, R, \theta) \) and \( 0 < \theta < 1 \), \( \theta = \theta(\theta, \theta) \) such that
\[
(3.7) \quad \| u(t) \|_{L^2(\Omega)} \leq \left( N e^{-N/((l_k - l_{k+1})} \| u(t) \|_{L^1(B_R(x_0))} \right)^{\theta} \| u(s) \|_{L^2(\Omega)}^{-1/\theta},
\]
holds, when \( 0 \leq s < t \leq 1 \) and \( u \) satisfies (3.3). Also, from Remark 1 the solution \( u \) to (3.3) verifies
\[
(3.8) \quad |\partial_x^\alpha u(x, t)| \leq e^{-N/((l_k - l_{k+1})} \mu^{-1-|\alpha|} R^{-|\alpha|} \| u(l_{k+1}) \|_{L^2(\Omega)},
\]
for \( \alpha \in \mathbb{N}^n \), \( x \in B_R(x_0) \) and \( l_k \leq t \leq l_{k+1} \). Then, replace respectively (3.1) and (3.6) by (3.7) and (3.8) in the proof of Theorem 3.

4. Historical remarks and comments

It is worth mentioning that the main result Theorem 1 in this paper has not been indicated in the existing literature related to analyticity properties of solutions to parabolic equations. It is motivated by the problem of establishing the null-controllability over measurable sets with positive measure and to obtain the bang-bang property of optimal controls for general parabolic evolutions.

With the purpose to extend the estimates of the form (1.6) to time-dependent parabolic evolutions, we studied the literature concerned with analyticity properties of solutions to parabolic equations and found the following: most of the works [10, 12, 30, 12, 6, 16, 17, 28, 29] make no precise claims about lower bounds for the radius of convergence of the spatial Taylor series of the solutions for small
values of the time-variable; the authors were likely more interested in the qualitative behavior.

If one digs into the proofs, one finds the following: [10] considers local in space interior analytic estimates for linear parabolic equations and finds a lower bound comparable to $t$. [11] is a continuation of [10] for quasi-linear parabolic equations and contains claims but no proofs. The results are based on [10]. Of course, one can after the rescaling of the local results in [10] for the growth of the spatial-derivatives over $B_1 \times \left[\frac{1}{2}, 1\right]$ for solutions living in $B_2 \times (0, 1]$, to derive the bound (1.5) for the spatial directions. [30] finds a lower bound comparable to $t$. [12, ch. 3, Lemma 3.2] gets close to make a claim like (1.6) but the proof and claim in the cited Lemma are not correct, as the inequalities (3.5), (3.6) in the Lemma and the last paragraph in [12, ch. 3, §3] show when comparing them with the following fact: an exponential factor of the form $e^{1/\rho t^{1/(2m-1)}}$ in the right hand side of (1.6) is necessary and should also appear in the right hand side of the inequality (3.6) of the Lemma, for the Gaussian kernel, $G(x, t + \epsilon)$, $t \geq 0$, satisfies $G(iy, 2\epsilon) = (2\epsilon)^{-\frac{m}{2}} e^{y^2/(8\epsilon)}$ and (3.5) in the Lemma independently of $\epsilon > 0$, but the conclusion (3.6) in the Lemma would bound $G(iy, 2\epsilon)$, for $y$ small and independently of $\epsilon > 0$, by a fixed negative power of $\epsilon$, which is impossible. The approach in [12, ch. 3, Lemma 3.2], which only uses the existence of the solution over the time interval $[t/2, t]$ to bound all the derivatives at time $t$, cannot see the exponential factor and find a lower bound for the spatial radius of convergence independent of $t$. On the contrary, the methods in [12 ch. 3] are easily seen to imply (1.5), [16] and [17] deal with non-linear parabolic second order evolutions and find a lower bound comparable to $t$. [28, 29] consider linear problems and find a lower bound comparable to $t^{1/2m-\epsilon}$, for all $\epsilon > 0$. See also [28, §6] and [29, §9] for a historical discussion.

Finally, [3, p. 178 Th. 8.1 (15)] builds a holomorphic extension in the space-variables of the fundamental solution for high-order parabolic equations or systems. This holomorphic extension is built upon the assumptions of local analyticity of the coefficients in the spatial-variables and continuity in the time-variable. The later provides an alternative proof of (1.6) with $p = 0$ at points in the interior of $\Omega$. As far as we know, Eidelman’s School did not work out similar estimates for the complex holomorphic extension of the Green’s function with zero lateral Dirichlet conditions for $\mathcal{L}$ over $\Omega$ up to the boundary. If they had done so, it would provide another proof of (1.7) up to the boundary. We believe that such approach is more complex than the one in this work.

On the other hand, the motivation to prove the estimates of the form (1.6) comes from its applications to the null-controllability of parabolic evolutions with bounded controls acting over measurable sets of positive measure. To describe these results we begin with a report of the progresses made on the null-controllability and observability of parabolic evolutions over measurable sets. In what follows, $\omega$, $\gamma$ and $E$ denote subsets of $\Omega$, $\partial\Omega$ and $(0, T)$ respectively: except for the 1997 work [25] - where the authors proved the one-sided boundary observability of the heat equation in one space dimension over measurable sets - up to 2008 the control regions considered in the literature were always of the type $\omega \times (0, T)$ or $\gamma \times (0, T)$, with $\omega$ and $\gamma$ open. Then, [30] showed that the heat equation is observable over sets $\omega \times E$, with $\omega$ open and $E$ measurable with positive measure. [2] showed that second order parabolic equations with time-independent Lipschitz coefficients associated to self-adjoint elliptic operators with local analytic coefficients in a neighborhood
Lemma 8. Let $\Omega$ be a Lipschitz domain. Then, there is $N = N(m,n,\Omega)$ such that
\begin{equation}
\|t^{p+\frac{1}{2m}}e^{-\theta t^{-\sigma}} D^k u\| \leq N \left[ \|t^{p+1}e^{-\theta t^{-\sigma}} D^{2m} u\|_{L^2(\Omega \times (0,1))} + \|t^{p}e^{-\theta t^{-\sigma}} u\|_{L^2(\Omega \times (0,1))} \right]
\end{equation}
holds for all $k = 1, \ldots, 2m - 1$, $p \geq 0$ and $u$ in $C^{\infty}(\overline{\Omega} \times [0,1])$.

Remark 8. When $\Omega$ is either $B_R$ or $B^+_R$, $R > 0$, then
\begin{equation}
\|t^{p+\frac{1}{2m}}e^{-\theta t^{-\sigma}} D^k u\|_{L^2(\Omega \times (0,1))} \leq N \left[ \|t^{p+1}e^{-\theta t^{-\sigma}} D^{2m} u\|_{L^2(\Omega \times (0,1))} + R^{-k}\|t^{p}e^{-\theta t^{-\sigma}} u\|_{L^2(\Omega \times (0,1))} \right],
\end{equation}
with $N = N(m,n)$.

Proof. By the interpolation inequality [1] Theorems 4.14, 4.15, there is $N = N(m,\Omega)$ such that
\begin{equation}
\|D^k u(t)\|_{L^2(\Omega)} \leq N \left[ \|u(t)\|_{L^2(\Omega)} \|D^{2m} u(t)\|_{L^2(\Omega)} + \|u(t)\|_{L^2(\Omega)} \right],
\end{equation}
when $1 \leq k < 2m$. Now, multiply (5.3) by $t^{p+\frac{1}{2m}}e^{-\theta t^{-\sigma}}$ and Hölder’s inequality over $[0,1]$ yields (5.1). \hfill \Box

Lemma 9. Let $u$ in $C^{\infty}(\overline{\Omega} \times [0,1])$ satisfy
\begin{align*}
&\partial_t u + L u = F, \quad \text{in } \Omega \times (0,1), \\
&u = D u = \ldots = D^{m-1} u = 0, \quad \text{in } \partial \Omega \times (0,1).
\end{align*}
Then, there is \( N = N(\Omega, n, m, l) \) such that

\[
\|t^{p+1}e^{-\theta t^{-\sigma}} \partial_t u\| + \sum_{i=0}^{2m} \left( \|t^{p+1/2m}e^{-\theta t^{-\sigma}} D^i u\| + \|t^{p+1} e^{-\theta t^{-\sigma}} F\| \right) \leq N \left( (p + k + 1) \|t^p e^{-\frac{k-1}{m} \theta t^{-\sigma}} u\| + \|t^{p+1} e^{-\theta t^{-\sigma}} F\| \right),
\]

holds for any \( \theta \geq 0, p \geq 0 \) and \( k \geq 2 \).

**Proof.** Define \( v = t^{p+1}e^{-\theta t^{-\sigma}} u \), then \( v \) satisfies \( \partial_t v + \mathcal{L} v = G \) in \( \Omega \times (0, 1) \), with

\[
G = t^{p+1}e^{-\theta t^{-\sigma}} F + \left[ (p + 1) t^p e^{-\theta t^{-\sigma}} + \sigma \theta^{-\sigma} e^{-\theta t^{-\sigma}} \right] u.
\]

For \( t > 0 \) and \( k \geq 2 \),

\[
\theta t^{-\sigma} e^{-\theta t^{-\sigma}} = \frac{\theta}{k} t^{-\sigma} e^{-\frac{k-1}{m} \theta t^{-\sigma}} \leq k e^{-\frac{k-1}{m} \theta t^{-\sigma}}.
\]

By the \( W^{2m, 1}_2 \) Schauder estimate (1.4),

\[
\|\partial_t v\| + \|D^{2m} v\| \leq N \|v\| + \|G\|,
\]

with \( N = N(\Omega, n, m, l) \) and (5.4) follows from (5.7), (5.6), (5.5) and Lemma \( 8. \)

**Lemma 10.** Let \( u \) in \( C^\infty(B_R^+ \times [0, 1]) \) verify

\[
\begin{align*}
&\partial_t u + \mathcal{L} u = F, \quad \text{in } B_R^+ \times (0, 1), \\
&u = Du = \ldots = D^{m-1} u = 0, \quad \text{in } \{x_n = 0\} \cap \partial B_R^+ \times (0, 1), \\
&u(0) = 0, \quad \text{in } B_R^+.
\end{align*}
\]

and \( 0 < r < r + \delta < R \leq 1 \). Then, there is \( N = N(n, m, l) \) such that

\[
\|\partial_t u\|_r + \|D^{2m} u\|_r \leq N \left[ \delta^{-2m} \|u\|_{r + \delta} + \|F\|_{r + \delta} \right].
\]

**Proof.** Let \( \eta \) in \( C_0^\infty(B_R) \) be such that for \( 0 < \lambda \leq 1 \)

\[
\eta(x) = \begin{cases} 1, & \text{in } B_{r+\lambda \delta}, \\
0, & \text{in } B_{r}^{e+\frac{1}{2} \lambda \delta},
\end{cases}
\]

and \( |D^k \eta| \leq C_m [(1 - \lambda) \delta]^{-k} \), for \( k = 0, \ldots, 2m \). Define \( v = u \eta \), then

\[
\partial_t v + \mathcal{L} v = \eta F + (-1)^m \sum_{|\alpha| \leq 2m} \alpha_{\alpha} \sum_{\gamma} \left( \frac{\alpha}{\gamma} \right) \partial^{\alpha - \gamma} \eta \partial^{\gamma} u.
\]

By the \( W^{2m, 1}_2 \) Schauder estimate over \( B_R^+ \times (0, T) \) applied to \( v \),

\[
\|\partial_t u\|_r + \|D^{2m} u\|_{r + \lambda \delta} \leq N \left[ \|F\|_{r + \frac{1}{2} \lambda \delta} + \sum_{k=0}^{2m-1} [(1 - \lambda) \delta] \|D^k u\|_{r + \frac{1}{2} \lambda \delta} \right].
\]

Define the semi-norms

\[
|u|_{k, \delta} = \sup_{\mu \in (0, 1)} \|(1 - \mu \delta)^k D^k u\|_{r + \mu \delta}, \quad k = 0, \ldots, 2m.
\]
Estimate (5.9) can be rewritten in terms of these semi-norms as follows

\[
(5.10) \quad \delta^{2m} \| \partial_t u \|_r + |u|_{2m, \delta} \leq N \left[ \sum_{k=0}^{2m-1} |u|_{k, \delta} + \delta^{2m} \| F \|_{r+\delta} \right].
\]

To eliminate the terms \(|u|_{k, \delta}\) from the right hand side of (5.10), recall that the semi-norms interpolate ([1, Theorem 4.14] and [14, p. 237]); i.e., there is \(c = c(n, m)\) such that

\[
|u|_{k, \delta} \leq \epsilon |u|_{2m, \delta} + ce^{-\frac{k}{\sqrt{m}n}} \| u \|_{r+\delta},
\]

for any \(\epsilon \in (0, 1)\), so

\[
\sum_{k=0}^{2m-1} |u|_{k, \delta} \leq 2mc |u|_{2m, \delta} + c \sum_{k=0}^{2m-1} e^{-\frac{k}{\sqrt{m}n}} \| u \|_{r+\delta}.
\]

Choose then \(\epsilon = \frac{1}{4mN}\) and from (5.10)

\[
\delta^{2m} \| \partial_t u \|_r + |u|_{2m, \delta} \leq N \left[ \| u \|_{r+\delta} + \delta^{2m} \| F \|_{r+\delta} \right],
\]

which yields (5.8). \(\square\)

Lemma 11 is the interior analogue of Lemma 10 but now using [3, Theorem 2].

**Lemma 11.** Let \(u\) in \(C^\infty(B_R \times [0, 1])\) verify

\[
\begin{align*}
\partial_t u + \mathcal{L} u &= F, & \text{in } B_R \times (0, 1], \\
u(0) &= 0, & \text{in } B_R,
\end{align*}
\]

and \(0 < r < r + \delta < R \leq 1\). Then, there is \(N = N(n, q, m)\) such that

\[
\| \partial_t u \|_r + \| D^{2m} u \|_r \leq N \left[ \delta^{-2m} \| u \|_{r+\delta} + \| F \|_{r+\delta} \right].
\]

Lemmas 10 and 9 imply Lemma 12.

**Lemma 12.** Let \(u\) in \(C^\infty(B_R^+ \times [0, 1])\) satisfy

\[
\begin{align*}
\partial_t u + \mathcal{L} u &= F, & \text{in } B_R^+ \times (0, 1], \\
u &= 0, & \text{in } \{x_n = 0\} \cap \partial B_R^+ \times (0, 1]
\end{align*}
\]

and \(0 < r < r + \delta < R \leq 1\). Then, there is \(N = N(n, q, m)\) such that

\[
\| t^{p+1} e^{-\theta t - \sigma} \partial_t u \|_r + \| t^{p+1} e^{-\theta t - \sigma} D^{2m} u \|_r \\
\leq N \left[ (p + k) \| t^p e^{-\frac{k-1}{2} \theta t - \sigma} u \|_{r+\delta} \right. \\
+ \delta^{-2m} \| t^{p+1} e^{-\theta t - \sigma} u \|_{r+\delta} + \| t^{p+1} e^{-\theta t - \sigma} F \|_{r+\delta} \],
\]

for \(0 < \theta \leq 1\), \(p \geq 0\) and \(k \geq 2\).

Similarly, Lemmas 9 and 11 imply the weighted \(L^2\) estimate in Lemma 13.

**Lemma 13.** Let \(u\) in \(C^\infty(B_R \times [0, 1])\) satisfy

\[
\partial_t u + \mathcal{L} u = F \quad \text{in } B_R \times (0, 1]
\]
and $0 < r < r + \delta < R \leq 1$. Then there is $N = N(n, q, m)$ such that
\[
\left\| t^{p+1} e^{-\theta t^{-\sigma}} \partial_t u \right\|_r + \left\| t^{p+1} e^{-\theta t^{-\sigma}} D^{2m} u \right\|_r \\
\leq N \left[ (p+k) \left\| t^{p} e^{-\frac{k}{k-p} \theta t^{-\sigma}} u \right\|_{r+\delta} + \delta^{-2m} \left\| t^{p+1} e^{-\theta t^{-\sigma}} u \right\|_{r+\delta} + \left\| t^{p+1} e^{-\theta t^{-\sigma}} F \right\|_{r+\delta} \right],
\]
holds for $0 < \theta \leq 1$, $p \geq 0$ and $k \geq 2$.

**Lemma 14.** If $\gamma \in \mathbb{N}^n$, $0 < t < s$,
\[
\sum_{\beta < \gamma} \binom{\gamma}{\beta} |\gamma - \beta|! |\beta|! s^{-|\gamma|+|\beta|} t^{-|\beta|} \leq |\gamma| t^{1-|\gamma|} \frac{s}{s-t},
\]
Proof. Let $f(x) = \varphi(u)$, with $u = (x_1 + \cdots + x_n)$ and $\varphi(u) = (1 - u)^{-1}$. Then, \(\frac{\partial^\gamma f}{\partial x^\gamma}(x) = \varphi^{(\gamma)}(u) = |\gamma|! u^{-|\gamma|-1}\). Now let, $f_t(x) = f(x)$, $\frac{\partial^\gamma f}{\partial x^\gamma}(x) = t^{-|\gamma|} \frac{\partial^\gamma f}{\partial x^\gamma}(\frac{x}{t})$, and taking $x = 0$, we have $\frac{\partial^\gamma f}{\partial x^\gamma}(0) = |\gamma|! t^{-|\gamma|}$. Now, set $g(x) = f_s(x)f_t(x) = \psi(u)$, with
\[
\psi(u) = \frac{1}{(1 - \frac{u}{s})(1 - \frac{u}{t})}.
\]
Let $|u| < t$. Then
\[
\psi(u) = \sum_{i=0}^{+\infty} (u/s)^i \sum_{j=0}^{+\infty} (u/t)^j = \sum_{i,j=0}^{+\infty} \frac{u^{i+j}}{s^i t^j} = \sum_{k=0}^{+\infty} u^k \sum_{i+j=k} \frac{1}{s^i t^j}
\]
and
\[
\psi^{(k)}(0) = k! t^{-k} \sum_{i=0}^{k} (t/s)^i, \text{ for } k \geq 0.
\]
Thus,
\[
\frac{\partial^\gamma g}{\partial x^\gamma}(0) = \psi^{(\gamma)}(0) = |\gamma|! t^{-|\gamma|} \sum_{i=0}^{+\infty} (t/s)^i, \text{ for } \gamma \in \mathbb{N}^n.
\]
From Leibniz’s rule
\[
\frac{\partial^\gamma g}{\partial x^\gamma}(0) = \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} \partial^{\gamma - \beta} f_s(0) \partial^\beta f_t(0)
\]
\[
= \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} |\gamma - \beta|! |\beta|! s^{-|\gamma|+|\beta|} t^{-|\beta|}.
\]
It implies that
\[
\sum_{\beta \leq \gamma} \binom{\gamma}{\beta} |\gamma - \beta|! |\beta|! s^{-|\gamma|+|\beta|} t^{-|\beta|} = |\gamma|! t^{-|\gamma|} \sum_{i=0}^{+\infty} \left( \frac{t}{s} \right)^i,
\]
where dropping the term corresponding to $\beta = \gamma$,
\[
\sum_{\beta < \gamma} \binom{\gamma}{\beta} |\gamma - \beta|! |\beta|! s^{-|\gamma|+|\beta|} t^{-|\beta|} = |\gamma|! t^{-|\gamma|} \sum_{i=1}^{+\infty} \left( \frac{t}{s} \right)^i \leq |\gamma|! t^{-|\gamma|} \sum_{i=1}^{+\infty} \left( \frac{t}{s} \right)^i = |\gamma|! t^{1-|\gamma|} \frac{s}{s-t},
\]
Lemma 15. Let $\mathcal{L} = (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha(x) \partial^\alpha$ be an elliptic operator in non-variational form whose coefficients satisfy (1.3) and $a_\alpha$ belong to $C^{(|\alpha| - m)}$, for $m < |\alpha| \leq 2m$. Then, $\mathcal{L}$ can be written in variational form as

$$\mathcal{L} = (-1)^m \sum_{|\gamma|, |\beta| \leq m} \partial^\gamma (A_{\gamma,\beta}(x) \partial^\beta),$$

with

$$\sum_{|\gamma|, |\beta| \leq m} \|A_{\gamma,\beta}\|_{L^\infty(\Omega)} \leq \varrho^{-1}, \quad \sum_{|\gamma| = |\beta| = m} A_{\gamma,\beta}(x) \xi^\gamma \xi^\beta \geq \varrho |\xi|^{2m},$$

for $x$ in $\overline{\Omega}$ and $\xi$ in $\mathbb{R}^n$.

Proof. We write

$$\sum_{|\alpha| \leq 2m} a_\alpha \partial^\alpha = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha + \sum_{m+1 \leq |\alpha| \leq 2m} a_\alpha \partial^\alpha \equiv I + J.$$ 

Then

$$I = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha = \sum_{|\gamma|, |\beta| \leq m} \partial^\gamma (A_{\gamma,\beta} \partial^\beta),$$

with $A_{\gamma,\beta} = 0$ if $|\gamma| \neq 0$ and $A_{\gamma,\beta} = a_\beta$, if $\gamma = 0$; and $I$ can be written in non-variational form. For each $\alpha \in \mathbb{N}^n$ with $m + 1 \leq |\alpha| \leq 2m$, define

$$c_\alpha = \# \{(\gamma, \beta) : \gamma + \beta = \alpha \text{ and } |\beta| = m\}.$$

Then,

$$J = \sum_{j = m+1}^{2m} \sum_{|\gamma| = j - m, |\beta| = m} \frac{1}{c_{\gamma+\beta}} A_{\gamma+\beta} \partial^{\gamma+\beta} = \sum_{j = m+1}^{2m} \sum_{|\gamma| = j - m, |\beta| = m} A_{\gamma,\beta} \partial^{\gamma+\beta}.$$ 

Since $A_{\gamma,\beta} \in C^{(|\gamma|+|\beta|-m)}$, we can apply Lemma 16 below to each of the terms $A_{\gamma,\beta} \partial^{\gamma+\beta}$ and we are done. It remains to check that the new expression satisfies the ellipticity condition: we notice that

$$\sum_{|\alpha| = 2m} a_\alpha(x_0) \partial^\alpha = \sum_{|\gamma| = |\beta| = m} A_{\gamma,\beta}(x_0) \partial^{\gamma+\beta},$$

for each fixed $x_0$ in $\overline{\Omega}$. Therefore, if we apply these constant coefficients operators to any rapidly decreasing function and take Fourier transform, we get

$$\sum_{|\alpha| = 2m} a_\alpha(x_0) \xi^\alpha = \sum_{|\gamma| = |\beta| = m} A_{\gamma,\beta}(x_0) \xi^{\beta+\gamma}, \text{ for any } \xi \in \mathbb{R}^n,$$

which implies the uniform ellipticity of the operator in variational form, when the operator in non-variational form is uniformly elliptic. □

Lemma 16. For $m \in \mathbb{N}$ and $j = m + 1, \ldots, 2m$ the following holds: if $a \in C^{j-m}$ and $\alpha \in \mathbb{N}^n$ is a multi-index with $|\alpha| \leq j$, we can write

$$(5.11) \quad a \partial^\alpha = \sum_{|\gamma|, |\beta| \leq m} \partial^\gamma (b_{\gamma,\beta} \partial^\beta)$$

for some $b_{\gamma,\beta} \in C^{|\gamma|}$. 

Proof. For each fixed $m \in \mathbb{N}$ we prove Lemma 16 by induction on $j$.

Case $j = m + 1$: if $|\alpha| \leq m$, $a\partial^\alpha$ already has the required form. Then, we only need to check the statement for multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| = m + 1$. If $j = m + 1$ and $a \in C^l$, $a = e_i + \beta$ for some multi-index $\beta$ with $|\beta| = m$. Then,

$$a\partial^\alpha = \partial^{e_i}(a\partial^\beta) - \partial^{e_i}a\partial^\beta,$$

which already has the required form because $|e_i|$ and $|\beta| \leq m$.

Now, assuming that the statement holds for $j = m + 1, \ldots, k$, we prove it for $j = k + 1$. Let $a \in C^{k+1-m}$. If $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, the induction hypothesis shows that $a\partial^\alpha$ can be written as (5.11); hence, we only need to check that the statement holds for multi-indices $\alpha$ with $|\alpha| = k + 1$. In this case, we can write $\alpha = \gamma + \beta$ with $|\gamma| = k + 1 - m$ and $|\beta| = m$; then, applying Leibniz’s rule we get

$$a\partial^\alpha = a\partial^{|\gamma|+\beta} = \partial^\gamma(a\partial^\beta) - \sum_{0 < \sigma \leq \gamma} \binom{\gamma}{\sigma} \partial^\sigma a\partial^{\beta+\gamma-\sigma}$$

and we want to apply the induction hypothesis to each of the terms of the sum in (5.12). Thus we only need to check that each operator $\partial^\sigma a\partial^{\beta+\gamma-\sigma}$ satisfies hypothesis which fall into one of the previous steps of the induction hypothesis:

- Because $a \in C^{k+1-m}$, we have $\partial^\sigma a \in C^{k+1-|\sigma|-m}$.
- Because the sum in (5.12) runs for multi-indices $\sigma$ with $1 \leq |\sigma| = |\gamma| \leq k + 1 - m$, we have that $|\beta + \gamma - \sigma| = k + 1 - |\sigma| \leq k$.

This finishes the proof of Lemma 16. \qed

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