A trichotomy for the autoequivalence groups on smooth projective surfaces

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Abstract
We study autoequivalence groups of the derived categories on smooth projective surfaces, and show a trichotomy of types according to the maximal dimension of Fourier–Mukai kernels for autoequivalences. This number is 2, 3 or 4, and we also pose a conjecture on the description of autoequivalence groups if it is 2, and prove it in some special cases.

1 Introduction
The study of derived categories $D(X) = D^b(Coh(X))$ of coherent sheaves on a smooth projective varieties $X$ has become an important topic in algebraic geometry over the last decades. It is an interesting and basic problem to describe the group $Auteq D(X)$ of autoequivalences of $D(X)$. In this article, we consider the autoequivalence group of smooth projective surfaces.

Let us introduce an integer $N_X$, which plays the key role of a trichotomy of types of the autoequivalence group on smooth projective surfaces. First recall that an Orlov’s deep result states that every autoequivalence on a smooth projective variety $X$ is given by a Fourier–Mukai transform $\Phi^P$ with unique kernel $P \in D(X \times X)$ (see §2.1). Let us define

$$\text{Comp}(\Phi^P)$$

to be the set of irreducible components $W_0$ of $\text{Supp}(P)$ which dominates $X$ by the first projection $p_1 : X \times X \to X$, which turns out to be non-empty by §3.1 (i)). Define

$$N_X : = \max\{\dim(W_0) \mid W_0 \in \text{Comp}(\Phi^P) \text{ for some } \Phi^P \in Auteq D(X)\}$$

$$\in \{\dim(X), \dim(X) + 1, \ldots, 2 \dim(X)\}$$

and call it the Fourier–Mukai support dimension of $X$.

For smooth projective surfaces $S$, (conjectural) descriptions of the group $Auteq D(S)$ and the geometry of surfaces are quite different, depending on the value $N_S$. The following is the first main result in this article.

Theorem 1.1 (=Theorem 5.3). We have the following.
(i) \( N_S = 4 \) if and only if \( K_S \equiv 0 \).

(ii) \( N_S = 3 \) if and only if \( S \) has a minimal elliptic fibration and \( K_S \neq 0 \).

(iii) \( N_S = 2 \) if and only if \( S \) has no minimal elliptic fibration and \( K_S \neq 0 \).

In the case \( N_S = 4 \), Theorem 1.1 implies that \( S \) is one of K3, abelian, bielliptic or Enriques surfaces. Bayer and Bridgland describe the autoequivalence group of K3 surfaces with the Picard number 1 \([BB17]\). Orlov finds a description of the autoequivalence group of abelian varieties (not necessarily surfaces) \([Or02]\). Recently, Potter finds a description of the autoequivalence group of bielliptic surfaces \([Po17]\).

Let us consider the case \( N_S = 3 \). In this case, Theorem 1.1 implies that \( S \) has a minimal elliptic fibration \( \pi : S \to C \) and \( K_S \neq 0 \). Suppose furthermore that each reducible fiber of \( \pi \) is non-multiple, and forms a cycle of \((-2)\)-curves, i.e. it is of type \( I_n \) for some \( n > 1 \). Then, the autoequivalence group \( \text{Auteq} D(S) \) is described in \([Ue16]\). See also Conjecture 2.4.

Finally, let us consider the case \( N_S = 2 \). Let us set \( Z \) the union of all \((-2)\)-curves on \( S \), and define

\[
\text{Br}_Z(S) = \langle T_\alpha \mid \alpha \in D_Z(S) \text{ spherical object} \rangle (\subset \text{Auteq} D(S)).
\]

Here, a functor \( T_\alpha \) is a special kind of an autoequivalence, called a \textit{twist functor} (see \S 2.2). Then, we pose the following conjecture:

**Conjecture 1.2** (cf. Conjecture 6.7). If \( N_S = 2 \), then we have

\[
\text{Auteq} D(S) = \langle \text{Br}_Z(S), \text{Pic}(S) \rangle \rtimes \text{Aut}(S) \times \mathbb{Z}[1].
\]

The classical Bondal–Orlov Theorem states that if \( \pm K_X \) is ample for a smooth projective variety \( X \), we have

\[
\text{Auteq} D(X) = \text{Pic}(S) \rtimes \text{Aut}(S) \times \mathbb{Z}[1].
\]

Because there are no \((-2)\)-curves on a smooth projective surface \( S \) with ample \( \pm K_S \) we can regard Conjecture 1.2 as a variant of their result. The following is the second main result of this article.

**Theorem 1.3** (cf. Theorem 6.8). Let \( S \) be a smooth projective surface with \( N_S = 2 \). Then Conjecture 1.2 holds true, if \( Z \) is a disjoint union of configurations of \((-2)\)-curves of type A.

Theorem 1.3 is a generalization of \([IU05, \text{Theorem 1.5}]\) and \([BP14, \text{Theorem 1}]\). We show Theorem 6.8 which is slightly stronger than Theorem 1.3.
Notation and conventions. We follow the notation and terminology of [Ha77] unless otherwise stated. All varieties will be defined over the complex number field $\mathbb{C}$ in this article. A point on a variety will always mean a closed point.

By a minimal elliptic surface, we will always mean a smooth projective surface $S$ together with a smooth projective curve $C$ and a relatively minimal morphism $\pi: S \to C$ whose general fiber is an elliptic curve. Here a relatively minimal morphism means a morphism whose fibers contains no $(-1)$-curves. Such a morphism $\pi$ is called an minimal elliptic fibration.

We denote by $D^b(X)$ the bounded derived category of coherent sheaves on an algebraic variety $X$. For any subset $Z(\subset X)$, we denote the full triangulated subcategory of $D^b(X)$ consisting of objects supported on $Z$ by $D_Z(X)$. Here, the support of an object $\alpha \in D^b(X)$ is, by definition, the union of the set-theoretic supports of its cohomology sheaves $H^i(\alpha)$. Note that the support is always closed subset because $\alpha$ is a bounded complex of coherent sheaves. We denote the dimension of the support of $\alpha$ by $\dim(\alpha)$.

An object $\alpha$ in $D^b(X)$ is said to be rigid if $\Hom^1_{D^b(X)}(\alpha, \alpha) = 0$.

Given a closed embedding of schemes $i: Z \hookrightarrow X$, we denote the derived pullback $L_i^* \alpha$ simply by $\alpha|_Z$.

For algebraic varieties $X, Y$, we denote the diagonal in $X \times X$ by $\Delta_X$, and denote the projections by $p_X: X \times Y \to X$ and $p_Y: X \times Y \to Y$, or $p_1: X \times Y \to X$ and $p_2: X \times Y \to Y$.

For an abelian variety $X$, we denote the dual variety $\text{Pic}^0 X$ by $\hat{X}$.

$\text{Auteq} T$ denotes the group of isomorphism classes of $\mathbb{C}$-linear exact autoequivalences of a $\mathbb{C}$-linear triangulated category $T$.

For a Cartier divisor $D$ on a normal projective variety $X$, we define a graded $\mathbb{C}$-algebra by

$$R(X, D) := \bigoplus_{m \geq 0} H^0(X, O_X(mD)).$$

Recall that the Iitaka dimension $\kappa(X, D) = \kappa(D)(\in \{-\infty, 0, 1, \ldots, \dim(X)\})$ of $D$ is

$$\kappa(D) := \begin{cases} 
\text{the transcendence degree of } R(X, D) - 1 & \text{if } R(X, D) \neq \mathbb{C} \\
-\infty & \text{otherwise.}
\end{cases}$$

We call $\kappa(K_X)$ the Kodaira dimension of $X$, and simply denote it by $\kappa(X)$. Assume furthermore that $D$ is a nef divisor. Then, recall the numerical Iitaka dimension $\nu(X, D) = \nu(D)(\in \{0, 1, \ldots, \dim(X)\})$ of $D$ by

$$\nu(D) := \max\{k \in \mathbb{Z} \mid D^k \cdot H^{\dim(X) - k} \neq 0\},$$

where $H$ is an ample divisor on $X$. In general, it is known that the inequality $\nu(D) \geq \kappa(D)$.
Let $X$ be a minimal model, that is, $X$ is a normal projective variety with $\mathbb{Q}$-factorial terminal singularities and $K_X$ is nef. We call $\nu(K_X)$ the *numerical Kodaira dimension* of $X$, and simply denote it by $\nu(X)$. The *abundance conjecture* states that if $X$ is a minimal model, then the equality $\kappa(X) = \nu(X)$ holds. It is known to be true for surfaces and 3-folds. See [KMM] for these terminology and results.

## Fourier–Mukai transforms

### 2.1 Fourier–Mukai transforms

Let $X$ and $Y$ be smooth projective varieties. For an object $P \in D(X \times Y)$, we define an exact functor $\Phi_P$, called the *integral functor* with kernel $P$, by

$$\Phi_P := R\mathbb{R}p_{Y*}(P \otimes p_X^*(-)) : D(X) \to D(Y).$$

We also sometimes write $\Phi_P$ as $\Phi_P_{X\to Y}$ to emphasize that it is a functor from $D(X)$ to $D(Y)$.

By the result of Orlov (see [Hu06, Theorem 5.14]), for a fully faithful functor $\Phi: D(X) \to D(Y)$, there is an object $P \in D(X \times Y)$, unique up to isomorphism, such that $\Phi \cong \Phi_P$. If an integral functor $\Phi_P$ is an equivalence, it is called a *Fourier–Mukai transform*.

Note that every autoequivalence is given as an integral functor by the Orlov’s result, and hence let us consider *standard autoequivalences* as examples of Fourier–Mukai transforms. The autoequivalence group $\text{Auteq} D(X)$ always contains the group

$$A(X) := \text{Pic}(X) \rtimes \text{Aut}(X) \times \mathbb{Z}[1],$$

generated by standard autoequivalences, namely the functors of tensoring with line bundles, push forward along automorphisms, and the shift functor $[1]$. Any standard equivalence $\Phi$ are given by the following form:

$$\Phi = \varphi_* \circ ((-) \otimes \mathcal{L}) \circ [i]$$

for an automorphism $\varphi$, an integer $i$ and a line bundle $\mathcal{L}$. Then, $\Phi$ is the Fourier–Mukai transform with the kernel

$$\mathcal{O}_{\Gamma_\varphi} \otimes p_1^* \mathcal{L})[i],$$

whose support is $\Gamma_\varphi$, where $\Gamma_\varphi$ is the graph of $\varphi$.

Every Fourier–Mukai transform $\Phi_P$ induces a *cohomological Fourier–Mukai transform*

$$\Phi^P, H: H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}),$$
which is an isomorphism of the total cohomologies, and the commutativity
\[ \Phi^P \circ v(-) = v(-) \circ \Phi^P \]
holds (see [Hu06, §5.2]). Here we put \( v(-) := \text{ch}(-) \sqrt{\text{td}(X)} \).

If there exists a Fourier–Mukai transform between \( D(X) \) and \( D(Y) \), then we call \( X \) a Fourier–Mukai partner of \( Y \).

2.2 Calabi–Yau objects and spherical objects

Let \( X \) be a smooth projective variety. An object \( \alpha \in D(X) \) is called a Calabi–Yau object if it satisfies
\[ \alpha \otimes \omega_X \cong \alpha. \quad (2) \]

For example, a 0-dimensional sheaf on a smooth projective variety and a line bundle \( L \) on a \((-2)\)-curve \( C \) on a smooth projective surface are Calabi–Yau objects.

Take a Calabi–Yau object \( \alpha \), an autoequivalence \( \Phi \in \text{Auteq} D(X) \) and a closed subscheme \( D \) of \( \text{Supp}(\mathcal{P}) \). Then all cohomology sheaves \( H^i(\alpha) \) and \( \alpha|_D \) are Calabi–Yau objects. It is known that the Serre functor \((\cdot) \otimes \omega_X[\dim(X)]\) commutes with the equivalence \( \Phi \) (cf. [Hu06, Lemma 1.30]), and thus, \( \Phi(\alpha) \) is also a Calabi–Yau object.

Next, let us consider a sheaf \( F \in \text{Coh}(X) \) which is a Calabi–Yau object (we call it a Calabi–Yau sheaf). Then, we have
\[ \text{ch}(F) = \text{ch}(F) \cdot \text{ch}(\omega_X) = \text{ch}(F) \cdot (1 + c_1(\omega_X) + \frac{1}{2}c_1(\omega)^2 + \cdots) \]
and hence,
\[ 0 = \text{ch}(F) \cdot (c_1(\omega_X) + \frac{1}{2}c_1(\omega)^2 + \cdots). \quad (3) \]

For a Calabi–Yau object \( \alpha \in D(X) \) and an irreducible curve \( C \) contained in \( \text{Supp}(\alpha) \), every cohomology sheaf \( \mathcal{H}^i(\alpha|_C) \) is a Calabi–Yau sheaf. Hence, equality (3) yields
\[ K_X \cdot C = 0. \]

If there exists a Calabi–Yau object \( \alpha \) in \( D(X) \) with \( \text{Supp}(\alpha) = X \), we can find \( i \in \mathbb{Z} \) such that \( \text{rank} \mathcal{H}^i(\alpha) > 0 \). Since \( \mathcal{H}^i(\alpha) \) is also a Calabi–Yau sheaf, equality (3) implies that \( c_1(\omega_X) \) is torsion.

Next we introduce an important class of examples of autoequivalences. We say that an object \( \alpha \in D(X) \) is spherical if \( \alpha \) is a Calabi–Yau object and it satisfies
\[
\text{Hom}^k_{D(X)}(\alpha, \alpha) \cong \begin{cases} 
0 & (k \neq 0, \dim(X)) \\
\mathbb{C} & (k = 0, \dim(X))
\end{cases}
\]
For example, a line bundle on a K3 surface $X$ and a line bundle $\mathcal{L}$ on a $(-2)$-curve on a smooth projective surface $X$ are spherical objects in $D(X)$ (see \[Ue16\] §2.2).

Put $X = X_1 = X_2$. For a spherical object $\alpha \in D(X)$, we consider the mapping cone

$$C := \text{Cone}(p_1^*\alpha \otimes \mathcal{L} \otimes p_2^*\alpha \to O_{\Delta X}) \in D(X_1 \times X_2)$$

of the natural evaluation $p_1^*\alpha \otimes \mathcal{L} \otimes p_2^*\alpha \to O_{\Delta X}$. Then the integral functor $T_\alpha := \Phi_C \mid_{X_1 \to X_2}$ defines an autoequivalence of $D(X)$, called the *twist functor* along the spherical object $\alpha$ (cf. \[Hu06\] Proposition 8.6). By (4), there is an exact triangle

$$\mathbb{R}\text{Hom}_{D(X)}(\alpha, \beta) \otimes \mathcal{L} \alpha \to \beta \to T_\alpha(\beta)$$

for $\beta \in D(X)$.

### 2.3 Fourier–Mukai transforms on elliptic surfaces

Refer \[Br98\] to the results in this subsection. Let $\pi: S \to C$ be a minimal elliptic surface. For an object $E$ of $D(S)$, we define the *fiber degree* of $E$ as $d(E) = c_1(E) \cdot F$, where $F$ is a general fiber of $\pi$. Let us denote by $\lambda_S$ the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, $\lambda_S$ is the smallest number $d$ such that there exists a holomorphic $d$-section of $\pi$. Consider an integer $b$ coprime to $\lambda_S$. There exists a smooth, 2-dimensional component $J_S(b) = J_S(b)$ of the moduli space of pure one-dimensional stable sheaves on $S$, the general point of which represents a rank 1, degree $b$ stable vector bundle supported on a smooth fiber of $\pi$. There is a natural morphism $J_S(b) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of $S$ to the point $x$. This morphism is a minimal elliptic fibration. Obviously, $J_S(0) \cong J(S)$, the Jacobian surface associated to $S$, and $J_S(1) \cong S$.

There exists a universal sheaf $\mathcal{U}$ on $J_S(b) \times_C S$ such that the resulting functor $\Phi_{\mathcal{U}} \mid_{J_S(b) \to S}$ is an equivalence.

Let us set $Z$ the union of all $(-2)$-curves on $S$. Define

$$\text{Br}_Z(S) = \{T_\alpha \mid \alpha \in D_Z(S) \text{ spherical object}\} \subset \text{Auteq } D(S),$$

and denote the congruence subgroup of $\text{SL}(2, \mathbb{Z})$ by

$$\Gamma_0(m) := \left\{ \begin{pmatrix} c & a \\ d & b \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid d \in m\mathbb{Z} \right\}$$

for $m \in \mathbb{Z}$. Then, we pose the following conjecture.
Conjecture 2.1. Suppose that a smooth projective surface $S$ has a minimal elliptic fibration $\pi: S \to C$ and $K_S \neq 0$. Then, we have a short exact sequence

$$1 \to \langle \text{Br}_2(S), \otimes \mathcal{O}_S(D) \mid D \cdot F = 0, F \text{ is a fiber} \rangle \times \text{Aut}(S) \times \mathbb{Z}[2] \to \text{Auteq}(D(S)) \to 1.$$ 

Here $\Theta$ is induced by the action of $\text{Auteq}(D(S))$ on the even degree part $H^0(F, \mathbb{Z}) \oplus H^2(F, \mathbb{Z}) \cong \mathbb{Z}_2$ of the integral cohomology group of a smooth fiber $F$.

Suppose that each reducible fiber of $\pi$ is non-multiple, and forms a cycle of $(-2)$-curves, i.e. it is of type $I_n$ for some $n > 1$. Then Conjecture 2.1 is shown to be true in [Ue16]. See also [Ue17].

3 Support of the kernel of Fourier–Mukai transforms

In this section, we consider the support of the kernel of Fourier–Mukai transforms. Many results and ideas are due to Kawamata [Ka02], but for easy reference, we often refer Huybrecht’s book [Hu06].

Let $X$ and $Y$ be smooth projective varieties, and suppose that $\Phi = \Phi^P_{X \to Y}: D(X) \to D(Y)$ is a Fourier–Mukai transform. In this case, we have $\dim(X) = \dim(Y)$ (cf. [Hu06, Corollary 5.21]), and the quasi-inverse of $\Phi$ is given by $\Phi^Q$, where $Q = P^\vee \otimes p_X^*\omega_X$. It is known (cf. [Hu06, Lemma 3.32]) that

$$\text{Supp}(P) = \text{Supp}(Q).$$  \hfill (6)

Let us denote by $\Gamma$ the support of $P$. For $x \in X$, $\Gamma_x$ denotes the fiber over $x$ by $p_X|_x$. Notice that

$$P|_{x \times X} \cong \Phi(O_x) \quad \text{and} \quad \text{Supp}(P|_{x \times X}) = \Gamma_x$$  \hfill (7)

(see [Hu06, Lemma 3.29]), which implies that $\Gamma_x = \text{Supp}(\Phi(O_x))$ as sets. Furthermore, we note the following lemma.

Lemma 3.1. (i) There exists an irreducible component of $\Gamma$ which dominates $X$ by $p_X$, and a similar statement holds for $p_Y$.

(ii) $\text{Supp}(\Phi(O_x))$ is connected for any $x \in X$.

(iii) If $\dim(\Phi(O_x)) = \dim(X)$ holds, then $K_X \equiv 0$ and $K_Y \equiv 0$. 

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(iv) Let $W$ be an irreducible component of $\Gamma$, and $\nu: \tilde{W} \to W$ be the normalization. Then $\nu^* p_X^* \omega_X^\otimes m \cong \nu^* p_Y^* \omega_Y^\otimes m$ for some $m > 0$.

Proof. (i) See [Hu06, Lemma 6.4] for the first statement. Apply the first statement for the quasi-inverse $\Phi^{\mathcal{O}_Y \to \mathcal{X}}$ and use (6). Then, we can show the second.

(ii) See [Hu06, Lemma 6.11].

(iii) If the equality holds, $\text{Supp}(\Phi(\mathcal{O}_x)) = Y$. Since $\Phi(\mathcal{O}_x)$ is a Calabi–Yau object, $c_1(Y)$ is torsion as in (2.2). Then the result follows by Remark 3.4 (ii).

(iv) See [Hu06, Lemma 6.9].

Let us define

$$\text{Comp}(\Phi^{\mathcal{P}_X \to \mathcal{Y}})$$

the set of irreducible components $W_0$ of $\Gamma = \text{Supp}(\mathcal{P})$ which dominates $X$ by $p_X$. Note that $\text{Comp}(\Phi^{\mathcal{P}_X \to \mathcal{Y}}) \neq \emptyset$ by Lemma 3.1 (i).

**Lemma 3.2.** Take an irreducible component $W$ of $\Gamma$.

(i) We see $\dim(W) \leq \dim(p_X(W)) + \dim(p_Y(W))$. If furthermore $\dim(W) = \dim(p_X(W)) + \dim(Y)$ holds, then $K_X \equiv 0$.

(ii) If $\dim(W) = \dim(X)$ and $W \in \text{Comp}(\Phi^{\mathcal{P}_X \to \mathcal{Y}})$, then $W$ is the unique irreducible component dominating $X$ by $p_X$. Furthermore, it also dominates $Y$ by $p_Y$.

(iii) If $\dim(W) = 2 \dim(X)$, then $W = X \times Y$.

Proof. (i) The first result follows from the fact $W \subset p_X(W) \times p_Y(W)$. For the second, denote by $W_x$ the fiber of $p_X|_W: W \to p_X(W)$ over a point $x \in p_X(W)$. Then, $\dim(Y) \leq \dim(W_x) \leq \dim(\Phi(\mathcal{O}_x))$, and hence $\text{Supp}(\Phi(\mathcal{O}_x)) = Y$. Then Lemma 3.1 (iii) completes the proof.

(ii) Note that if $\dim(\Phi(\mathcal{O}_x)) = 0$ for $x \in X$, there is a point $y \in Y$ and an integer $n$ such that $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y[n]$ by [Hu06, Lemma 4.5]. We also notice that there are no other components dominating $X$, since $\text{Supp}(\Phi(\mathcal{O}_x))$ is connected by Lemma 3.1 (ii). Hence, for general points $x \neq x' \in X$, we have $\dim(\Phi(\mathcal{O}_x)) = \dim(\Phi(\mathcal{O}_{x'})) = 0$, and $\text{Hom}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_{x'})) = 0$ for all $i$. Then, $\text{Supp}(\Phi(\mathcal{O}_x)) \cap \text{Supp}(\Phi(\mathcal{O}_{x'})) = \emptyset$. In particular, $W$ also dominates $Y$ by $p_Y$.

(iii) This is obvious, since $\dim(X) = \dim(Y)$ and $W \subset X \times Y$ by definition.

The equation (6) implies that Lemma 3.2 (i) and (ii) still hold after replacing $p_X$ with $p_Y$.

For an irreducible closed subvariety $V$ of $X$, we set $\mathcal{C}_V := \{C \mid C$ is an irreducible curve contained in $V$, satisfying $K_X \cdot C = 0\}$. 

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Lemma 3.3. Let $V$ be an irreducible closed subvariety of $X$, and take $W_0 \in \text{Comp} (\Phi_{X \to Y}^\circ).$

(i) Suppose that
\[
\bigcup_{C \in \mathcal{C}_V} C \subseteq V
\]
holds. Then, we have
\[
\dim(W_0) \leq 2 \dim(X) - \dim(V).
\]

(ii) Suppose that $K_X|_V$ is big. Then, the inequality (9) holds. Assume furthermore that $K_X|_V$ is nef and the equality in (9) holds. Then, $K_X$ is nef.

(iii) Suppose that $-K_X|_V$ is big. Then, the inequality (9) holds. Assume furthermore that $-K_X|_V$ is nef and the equality in (9) holds. Then, $-K_X$ is nef.

Proof. (i) Set $W_{0V} := p_{X}^{-1}(V) \cap W_0(\subset X \times Y)$. Then, we have
\[
\dim(W_0) - \dim(X) \leq \dim(W_{0V}) - \dim(V).
\]
If the projection $p_Y$ contracts a curve $C'$ on $W_{0V}$, then $p_X(C')$ is a curve on $V$. Denote the normalization $\tilde{W}_0 \to W_0$ by $\nu_0$, and take an irreducible curve $\tilde{C}'$ on $\tilde{W}_0$ with $\nu_0(\tilde{C}') = C'$. Then
\[
K_X \cdot p_X(C') = (\nu_0^* p_X^* K_X) \cdot \tilde{C}' = (\nu_0^* p_Y^* K_Y) \cdot \tilde{C}' = 0
\]
holds by Lemma 3.1(iv). Hence, condition (3) implies that $p_Y|_{W_{0V}}$ is generically finite on the image, and hence $\dim(W_{0V}) \leq \dim(Y)$. The result follows from the equality $\dim(X) = \dim(Y)$ and inequality (10).

(ii) Take the normalization $\mu: \tilde{V} \to V$. Since $\mu^*(K_X|_V)$ is also big, Kodaira’s lemma yields that $\mu^*(K_X|_V)$ is $\mathbb{Q}$-linearly equivalent to $A + B$, where $A$ is an ample $\mathbb{Q}$-divisor and $B$ is an effective $\mathbb{Q}$-divisor on $\tilde{V}$. Then, we have
\[
\bigcup_{C \in \mathcal{C}_V} C \subset \mu(\text{Supp}(B)) \subseteq V,
\]
and hence, $p_Y|_{W_{0V}}$ is generically finite on the image as in the proof of (i) and inequality (9) holds by (i). If equality in (9) holds, then inequality (10) implies $\dim(Y) \leq \dim(W_{0V})$, and thus $\dim(Y) = \dim(W_{0V})$. Hence, it turns out that $p_Y|_{W_{0V}}$ is surjective. Since the linear equivalence
\[
((p_X \circ \nu_0)|_{\nu_0^{-1}(W_{0V})})^*(mK_X|_V) \sim ((p_Y \circ \nu_0)|_{\nu_0^{-1}(W_{0V})})^*(mK_Y)
\]
holds for some $m > 0$ by Lemma 3.1(iv), $K_Y$ is nef by the assumption that $K_X|_V$ is nef. Hence, $K_X$ is nef (see Remark 3.4(ii)). The statement (iii) can be proved in a similar way. \qed
Remark 3.4. (i) If $K X$ is big, i.e. $X$ is of general type, then Lemma 3.3 (ii) for $V = X$ yields $\dim(W_0) = \dim(X)$. In a similar way, if $-K X$ is big, then $\dim(W_0) = \dim(X)$ holds by Lemma 3.3 (iii). These are actually shown by Kawamata in the proof of [Ka02, Theorem 2.3 (2)].

(ii) If $K X$ is nef and $Y$ is a Fourier–Mukai partner of $X$, then $K Y$ is nef and $\nu(X) = \nu(Y)$ holds. A similar statement is true for anticanonical divisors $-K X$ and $-K Y$. See [Ka02, Theorem 2.3] and [Hu06, Propositions 6.15, 6.18] for the proof.

(iii) Let $\{ \varphi_i \}$ be the set of all extremal contractions on $X$. Define $V$ to be a fiber of maximal dimension among all fibers of all $\varphi_i$. Then $-K X|_V$ is ample, and hence, Lemma 3.3 (iii) implies that inequality (9) holds.

Lemma 3.5. Let $D$ be a nef Cartier divisor and $H$ be a very ample divisor on a normal projective variety $X$. Set $V = \bigcap_{i=1}^{\dim(X)-\nu(D)} H_i$ for general members $H_i \in |H|$. Then

$$
\bigcup_{D \cdot C = 0, C \subset V} C \subset V
$$

Proof. It follows from the definition on the numerical Iitaka-Kodaira dimension that $D|_V$ is a nef and big divisor on $V$. Then by Kodaira’s lemma, $D|_V$ is $\mathbb{Q}$-linearly equivalent to $A + B$, where $A$ is an ample $\mathbb{Q}$-divisor and $B$ is an effective $\mathbb{Q}$-divisor on $V$. Hence, $\bigcup_{D \cdot C = 0, C \subset V} C \subset B$, and then the result follows.

Proposition 3.6. Fix $W_0 \in \text{Comp}(\Phi_{X \to Y}^P)$. 

(i) Assume that $K X$ is nef. Then, we have

$$
\dim(W_0) \leq 2 \dim(X) - \nu(X).
$$

(ii) Assume that $-K X$ is nef. Then, we have

$$
\dim(W_0) \leq 2 \dim(X) - \nu(-K X).
$$

(iii) Assume that $\kappa(X) \geq 0$. Suppose that the minimal model conjecture and the abundance conjecture hold. Then, we have

$$
\dim(W_0) \leq 2 \dim(X) - \kappa(X). \quad (11)
$$

Proof. (i) and (ii) are direct consequences of Lemmas 3.3 (i) and 3.5. (iii) We may assume $\kappa(X) > 0$, since otherwise the statement is obvious. Run the minimal model program for $X$. Then, we obtain a birational map $\phi: X \dashrightarrow X_m$, where $X_m$ is a minimal model. Take a common resolution
Then [Ka02, Lemma 4.4] states that there is an integer $n > 0$ such that $D := n(f^*K_X - g^*K_{X_m})$ is effective. Let $H$ be a very ample divisor on $X_m$. Set $V_m := \bigcap_{i=1}^{\dim(X) - \kappa(X)} H_i$ for general members $H_i \in |H|$. Then, take its strict transform on $X$ and denote it by $V$. We see that $V$ satisfies the condition (3). Indeed, assume

\[ \bigcup_{C \in \mathcal{C}_V} C = V \quad \text{(12)} \]

for a contradiction. Let us set

\[ \tilde{\mathcal{C}}_V := \{ C \in \mathcal{C}_V \mid C \cap U \neq \emptyset \}, \]

where $U$ is the open subset of $X$ on which $\phi$ is an isomorphism. Then we know

\[ \emptyset \neq V_m \cap \phi(U) \subset \bigcup_{C \in \tilde{\mathcal{C}}_V} \phi_4 C \quad \text{(13)} \]

by (12) and the choice of $V_m$. Take irreducible curves $C \in \tilde{\mathcal{C}}_V$ and $C'$ on $X'$ satisfying $f(C') = C$. Then, we have

\[ 0 = nK_X \cdot C = nf^*K_X \cdot C' = ng^*K_{X_m} \cdot C' + D \cdot C'. \]

On the other hand, we have $g^*K_{X_m} \cdot C' \geq 0$ and $D \cdot C' \geq 0$, since $K_{X_m}$ is nef, and $f(\text{Supp}(D)) \cap U = \emptyset$. Therefore, we have $g^*K_{X_m} \cdot C' = D \cdot C' = 0$, and thus

\[ K_{X_m} \cdot \phi_4 C = K_{X_m} \cdot g(C') = 0. \]

In particular, we know that

\[ \{ \phi_4 C \mid C \in \tilde{\mathcal{C}}_V \} \subset \mathcal{C}_{V_m}, \]

and hence

\[ V_m \subset \bigcup_{C \in \tilde{\mathcal{C}}_V} \phi_4 C \subset \bigcup_{C_m \in \mathcal{C}_{V_m}} \phi_4 C. \]

Here, the first inclusion follows from (13). This produces a contradiction to

\[ \bigcup_{C_m \in \mathcal{C}_{V_m}} C_m \not\subset V_m \]

obtained by Lemma 3.5 and $\nu(X_m) = \kappa(X_m) = \kappa(X)$. Therefore, the result follows from Lemma 3.3 (i). 

**Corollary 3.7.** Take $W_0 \in \text{Comp}(\Phi^p_{X \rightarrow Y})$.

(i) If $\dim(W_0) = 2 \dim(X)$, then we have $K_X \equiv 0$.  

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(ii) If \( \dim(W_0) = 2 \dim(X) - 1 \) and \( K_X \neq 0 \), then either \( K_X \) is nef and \( \nu(X) = 1 \) or else \(-K_X\) is nef and \( \nu(-K_X) = 1 \).

Proof. (i) We can see \( \text{Supp}(\Phi^P_{X \to Y}(\mathcal{O}_x)) = Y \) by Lemma 3.2 (iii). Then Lemma 3.1 (iii) implies the statement.

(ii) Lemma 3.3 (ii) and (iii) yield that \( K_X \) or \(-K_X\) is nef. Thus, Proposition 3.6 (i) and (ii) imply the result.

\hspace{1cm} \blacksquare

Remark 3.8. Suppose that \( \kappa(X) = 0 \) or 1. Then Corollary 3.7 implies that equality in (11) cannot be attained unless \( X \) is a minimal model. On the other hand, in the case \( \kappa(X) \geq 2 \), equality in (11) may possibly hold for a non-minimal model \( X \) as follows.

Set \( X := S \times \hat{E} \) and \( Y := S \times E \) for an elliptic curve \( E \) and a smooth projective surface \( S \) of general type. Then it satisfies \( \kappa(X) = 2 \). Consider a Poincaré bundle \( \mathcal{P}_E \) on \( \hat{E} \times E \). If \( h: \Delta_S \times \hat{E} \times E \to \hat{E} \times E \) denotes the projection, then \( h^* \mathcal{P}_E \in D(X \times Y) \) gives rise to a Fourier–Mukai transform between \( D(X) \) and \( D(Y) \) (see [Hu06, Exercise 5.20]). Assume furthermore that \( S \) is not minimal. Then \( X \) is not minimal, but equality in (11) holds.

4 Fourier–Mukai support dimensions

Let \( X \) and \( Y \) be smooth projective varieties, and consider a Fourier–Mukai transform

\[ \Phi = \Phi^P_{X \to Y}: D(X) \to D(Y). \]

We give names to some special types of Fourier–Mukai transforms:

Definition 4.1. A Fourier–Mukai transform \( \Phi^P_{X \to Y} \) is said to be K-equivalent type, if there is an element \( W_0 \in \text{Comp}(\Phi^P_{X \to Y}) \) such that \( \dim(W_0) = \dim(X) \). Similarly, it is said to be Calabi–Yau type, if there is an element \( W_0 \in \text{Comp}(\Phi^P_{X \to Y}) \) such that \( \dim(W_0) = 2 \dim(X) \).

Note that in both cases, it turns out that the set \( \text{Comp}(\Phi^P_{X \to Y}) \) consists of the unique element by Lemma 3.2 (ii) and (iii).

Example 4.2. (i) Any standard autoequivalences are K-equivalent type by the description of the kernel object in (1).

(ii) Let \( \alpha \) be a spherical object in \( D(X) \). Note that \( \mathbb{R} \text{Hom}_{D(X)}(\alpha, \mathcal{O}_x) = 0 \) if and only if \( x \not\in \text{Supp}(\alpha) \) by [BM02, Lemma 4.2]. Then, by the triangle (5) for \( \beta = \mathcal{O}_x \), we see that

\[ \text{Supp}(T_\alpha(\mathcal{O}_x)) = \text{Supp}(\alpha) \]

for \( x \in \text{Supp}(\alpha) \), and

\[ T_\alpha(\mathcal{O}_x) = \mathcal{O}_x \]
for $x \not\in \text{Supp}(\alpha)$. Consequently, equations (7) imply
\[
\text{Supp}(\mathcal{C}) = \Delta_X \cup (\text{Supp}(\alpha) \times \text{Supp}(\alpha))(\subset X \times X),
\]
where $\mathcal{C} \in D(X \times X)$ is the kernel object of $T_\alpha$, given in (4).

Let $C$ be a $(-2)$-curve on a smooth projective surface $X$. Then, the twist functor $T_\mathcal{O}_C$ is $K$-equivalent type. On the other hand, the twist functor $T_\mathcal{O}_X$ along the structure sheaf $\mathcal{O}_X$ on a K3 surface $X$ is Calabi–Yau type.

Remark 4.3. (i) Let $\Phi_{P_X \to Y}$ be a Fourier–Mukai transform of $K$-equivalent type, and take the unique element $W_0 \in \text{Comp}(\Phi_{P_X \to Y})$. Then, Kawamata shows in the proof of [Ka02, Theorem 2.3 (2)] that $p_X|_{W_0}$ and $p_Y|_{W_0}$ are birational morphisms, and that $W_0$ is the graph of the birational map $(p_Y|_{W_0}) \circ (p_X|_{W_0})^{-1}$ between $X$ and $Y$. Moreover, if we take a resolution of singularities $f: Z \to W_0$, then the linear equivalence $f^*(p_X|_{W_0})^*(K_X) \sim f^*(p_Y|_{W_0})^*(K_Y)$ holds (use Lemma 3.1 (iv), and we can take $m = 1$ on the resolution $Z$. See [Hu06, Proposition 6.19]). In other words, varieties $X$ and $Y$ are $K$-equivalent.

(ii) For a given Fourier–Mukai transform $\Phi_{P_X \to Y}$, if $\dim(\Phi_{P_X \to Y}(\mathcal{O}_x)) = 0$ for a point $x \in X$, then $\Phi_{P_X \to Y}(\mathcal{O}_x) = \mathcal{O}_y[i]$ for some point $y \in Y$ and $i \in \mathbb{Z}$ by [Hu06, Lemma 4.5]. Moreover, a Fourier–Mukai transform $\Phi_{P_X \to Y}$ is a $K$-equivalent type if and only if $\dim(\Phi_{P_X \to Y}(\mathcal{O}_x)) = 0$ holds for a general point $x \in X$. Consequently, we can see that a composition of Fourier–Mukai transforms of $K$-equivalent type is again $K$-equivalent type.

Next let us consider the case $X = Y$, i.e. $\Phi = \Phi_P \in \text{Auteq}_D(X)$. Then we define
\[
N_X := \max\{\dim(W_0) \mid W_0 \in \text{Comp}(\Phi_P) \text{ for some } \Phi_P \in \text{Auteq}_D(X)\}
\]
and call it by the Fourier–Mukai support dimension of $X$. Obviously, we have
\[
\dim(X) \leq N_X \leq 2\dim(X).
\]
Let us consider two extreme cases below; the case $N_X = \dim(X)$ and the case $N_X = 2\dim(X)$.

4.1 $K$-equivalent type

Define
\[
\text{Auteq}_{K\text{-equiv}} D(X) := \{\Phi \in \text{Auteq}_D(X) \mid \Phi \text{ is } K\text{-equivalent type}\}.
\]

Then, Remark 4.3(ii) tells us that $\text{Auteq}_{K\text{-equiv}} D(X)$ is a subgroup of $\text{Auteq}_D(X)$. It is easy to see by definition and Remark 4.3(i) that the following conditions are equivalent;
• \( \text{Auteq}_{K\text{-equiv}} D(X) = \text{Auteq} D(X) \).

• \( N_X = \dim(X) \).

• For any \( \Phi^P \in \text{Auteq} D(X) \), the set \( \text{Comp}(\Phi^P) \) consists of the unique element \( W_0(\subset X \times X) \), which is the graph of a birational automorphism of \( X \).

If one, and hence, all of these conditions are satisfied, the autoequivalence group \( \text{Auteq} D(X) \) (or, simply \( X \)) is said to be \( K\text{-equivalent type} \).

**Proposition 4.4** (Kawamata). Let \( X \) a smooth projective variety with \( \pm K_X \) big. Then \( \text{Auteq} D(X) \) is \( K\text{-equivalent type} \).

**Proof.** It follows from Remark 3.4 (i).

### 4.2 Calabi–Yau type

By definition, the following conditions are equivalent;

• There is an autoequivalence \( \Phi \) of \( D(X) \) such that \( \Phi \) is Calabi–Yau type.

• \( N_X = 2 \dim(X) \).

If one, and hence, two of these conditions are satisfied, the autoequivalence group \( \text{Auteq} D(X) \) (or, simply \( X \)) is said to be \( \text{Calabi–Yau type} \). Note that Corollary 3.7 yields \( K_X \equiv 0 \) in this case. It is natural to ask whether the converse is true or not.

**Problem 4.5.** Suppose that \( K_X \equiv 0 \). Then, is \( \text{Auteq} D(X) \) Calabi–Yau type?

We give an affirmative answer to Problem 4.5 for abelian varieties in Proposition 4.6, for curves \( X \) in Theorem 4.7, and for surfaces \( X \) in Theorem 5.3.

**Proposition 4.6.** Let \( X \) be an abelian variety. Then, \( \text{Auteq} D(X) \) is Calabi–Yau type.

**Proof.** Let us put \( d := \dim(X) \), and consider the normalizes Poincaré bundle \( P \) on \( X \times \hat{X} \). Then, the integral functor \( \Phi^P_{X \to \hat{X}} \) is an equivalence (cf. [Hau06, Proposition 9.19]), and the cohomological Fourier–Mukai transform \( \Phi^P_{X \to \hat{X}} \) induces an isomorphism between the total cohomologies \( H^*(X, \mathbb{Q}) \) and \( H^*(\hat{X}, \mathbb{Q}) \), which restricts an isomorphism \( H^n(X, \mathbb{Q}) \) and \( H^{2d-n}(\hat{X}, \mathbb{Q}) \) for any \( n \). The last isomorphism coincides with

\[
(-1)^{\frac{n(n+1)}{2}} \cdot \text{PD}_n: H^n(X, \mathbb{Q}) \to H^{2d-n}(\hat{X}, \mathbb{Q}) \cong H^{2d-n}(X, \mathbb{Q})^*, \quad (14)
\]
where PD\(_n\) is Poincaré duality (see [Hu06, Lemma 9.23]). Here note that there is a natural isomorphism between \(H^{2d-n}(\hat{X}, \mathbb{Q})\) and \(H^{2d-n}(X, \mathbb{Q})^*\) by the construction of the dual abelian variety \(\hat{X}\).

For an ample line bundle \(L\) on \(\hat{X}\), consider the Fourier–Mukai transform
\[
\Phi^Q := \Phi^{P, H}_{\hat{X} \to X} \circ ((-) \otimes L) \circ \Phi^{P, H}_{X \to \hat{X}} \in \text{Auteq } D(X)
\]
with a kernel object \(Q \in D(X \times X)\). The cohomological Fourier–Mukai transform induced by the autoequivalence \((-) \otimes L\) of \(D(\hat{X})\) is just multiplying by \(\text{ch}(L)\). For a point \(x \in X\), we have \(\text{ch}(\mathcal{O}_x) = (0, \ldots, 0, 1) \in H^{2*}(X, \mathbb{Q})\), and hence (14) yields
\[
\Phi^Q_H(\text{ch}(\mathcal{O}_x)) = \Phi^{P, H}_{\hat{X} \to X}(\text{ch}(\mathcal{L})) \cdot \Phi^{P, H}_{X \to \hat{X}}((0, \ldots, 0, 1))
\]
\[
= \Phi^{P, H}_{\hat{X} \to X}(\text{ch}(\mathcal{L})) \cdot (1, 0, \ldots, 0)
\]
\[
= \Phi^{P, H}_{\hat{X} \to X}(\text{ch}(\mathcal{L}))
\]
\[
= (\text{ch}_d(L), -\text{ch}_{d-1}(L), \text{ch}_{d-2}(L), \ldots, (-1)^d \text{ch}_0(L)).
\]
Therefore, the 0-th cohomology component of \(\Phi^Q_H(\text{ch}(\mathcal{O}_x))\) is \(\text{ch}_d(L) = \frac{1}{d!} c_1(L)^d\), which is not 0. This means \(\text{Supp}(\Phi^Q(\mathcal{O}_x)) = X\), and hence \(\text{Supp } Q = X \times X\) (see the equations (14)). In particular, we obtain \(N_X = 2 \dim(X)\).

Now, we can show a dichotomy of the autoequivalence groups of smooth projective curves.

**Theorem 4.7 (Dichotomy).** Let \(C\) be a smooth projective curve with the genus \(g(C)\), and \(N_C \in \{1, 2\}\) be its Fourier–Mukai support dimension.

(i) \(N_C = 2\) (Calabi–Yau type) if and only if \(g(C) = 1\), namely \(C\) is an elliptic curve.

(ii) \(N_C = 1\) (K-equivalent type) if and only if \(g(C) \neq 1\), namely \(C\) is a projective line or a curve of general type.

**Proof.** If \(C\) is not an elliptic curve, then \(\pm K_C\) is ample. Hence, Proposition \([4.3]\) tells us that \(N_C = 1\). Since elliptic curves are 1-dimensional abelian varieties, Proposition \([4.6]\) completes the proof.

Theorem \([4.7]\) shows that Fourier–Mukai support dimensions of smooth projective curves reflect their geometry. We obtain a similar result for smooth projective surfaces in Theorem \([5.3]\).
5 Trichotomy of autoequivalence groups on smooth projective surfaces

In this section we show a similar result in the 2-dimensional case to Theorem 4.7.

Lemma 5.1. Let \( S \) be smooth projective surface, and take \( \Phi^P \in \text{Auteq}_D(S) \) and \( W_0 \in \text{Comp}(P) \). Then we have \( \dim(P) = \dim(W_0) \). In particular,
\[
N_S = \max\{\dim(P) \mid \Phi^P \in \text{Auteq}_D(S)\}
\]
holds.

Proof. Set \( \Gamma := \text{Supp}(P) \). Note that \( 2 \leq \dim(W_0) \leq \dim(\Gamma) \leq 4 \). Obviously, \( \dim(\Gamma) = 4 \) is equivalent to \( \dim(W_0) = 4 \).

Suppose that \( \dim(W_0) = 2 \). Then Lemma 3.2 (ii) implies that \( W_0 \) is the unique irreducible component of \( \Gamma \) dominating \( S \) by \( p_1 \), and is also the unique irreducible component dominating \( S \) by \( p_2 \). Hence, Lemma 3.2 (i) forces that there are no 3-dimensional irreducible components. In particular, \( \dim(\Gamma) = 2 \). To the contrary, if \( \dim(\Gamma) = 2 \), then \( \dim(W_0) = 2 \) follows. This completes the proof.

Remark 5.2. Let \( X \) be a Calabi–Yau 3-fold, i.e. it satisfies \( \omega_X \cong \mathcal{O}_X \) and \( H^1(X, \mathcal{O}_X) = 0 \), and suppose that \( X \) contains \( E \cong \mathbb{P}^2 \). Note that the normal bundle \( \mathcal{N}_{E/X} \) is isomorphic to \( \mathcal{O}_{\mathbb{P}^2}(-3) \). Then, we can see that \( \mathcal{O}_E \) is a spherical object of \( D(X) \) (We leave the proof of this fact to readers. Use [Hu06, Proposition 11.8] and the Local-to-Global Ext spectral sequence). The kernel of the twist functor \( T_{\mathcal{O}_E} \) has two irreducible components. One is supported on the diagonal \( \Delta_X \) in \( X \times X \) and the other one on \( E \times E \) (see Example 4.2 (ii)). Hence, Lemma 5.1 is false for higher dimensional varieties.

Now, we are in a position to show a trichotomy of autoequivalence groups on smooth projective surfaces.

Theorem 5.3 (Trichotomy). Let \( S \) be a smooth projective surface and \( N_S \in \{2, 3, 4\} \) be the Fourier–Mukai support dimension of \( S \).

(i) \( N_S = 4 \) (Calabi–Yau type) if and only if \( K_S \equiv 0 \).

(ii) \( N_S = 3 \) if and only if \( S \) has a minimal elliptic fibration and \( K_S \not\equiv 0 \).

(iii) \( N_S = 2 \) (\( K \)-equivalent type) if and only if \( S \) has no minimal elliptic fibration and \( K_S \not\equiv 0 \).

Proof. (i) For each surface \( S \) with \( K_S \equiv 0 \), let us give an example of autoequivalence whose kernel object has 4-dimensional support.
First, take a K3 surface $S$ and let $P$ be the ideal sheaf $I_{\Delta_S}$ of the diagonal $\Delta_S$ in $S \times S$. For $x \in S$, the integral functor $\Phi^P$ satisfies $\Phi^P(O_x) = I_x$, the ideal sheaf of the point $x$, and then [BM01, Corollary 2.8] implies that $\Phi^P$ is an autoequivalence.

For an abelian surface $S$, we have already shown $N_S = 4$ in Proposition 1.6.

Take an Enriques surface $T$. Then there is a K3 surface $S$ with an involution $\iota$ on $S$ such that $T$ is the quotient of $S$ by $\langle \iota \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Then it turns out that the autoequivalence $\Phi^P$ given above for a K3 surface $S$ descends to an autoequivalence of $D(T)$, and its kernel has a 4-dimensional support. See [BM98, Example 5.2] for details.

For a bielliptic surface $S$, there are elliptic curves $E_1, E_2$ and a finite group $G$ acting diagonally on $E_1 \times E_2$ such that $S = (E_1 \times E_2)/G$. Therefore, $S$ has two minimal elliptic fibrations $\pi_i: S \to E_i/G$. Take a universal sheaf $\mathcal{I}_i$ on $J_{S/(E_i/G)}(1) \times E_i/G$ $S$ for $i = 1, 2$ given in (2.3). Fix an isomorphism between $J_{S/(E_i/G)}(1)$ and $S$, and regard $\Phi_i := \Phi_{\mathcal{I}_i}^{J_{S/(E_i/G)}(1) \to S}$ as an autoequivalence of $D(S)$. Then the kernel of the composition $\Phi_1 \circ \Phi_2$ is 4-dimensional.

Conversely, it follows from Corollary 5.1 (i) that the equality $N_S = 4$ implies the equality $K_S \equiv 0$.

(ii) First note that the equality $N_S = 3$ implies that either $K_S$ is nef and $\nu(S) = 1$, or that $-K_S$ is nef and $\nu(-K_S) = 1$ by Corollary 5.1 and Lemma 5.1. Moreover, if $K_S$ is nef, it is known that $S$ has a minimal elliptic fibration (cf. [Be96, Proposition IX.2]). Therefore, we consider the only case $-K_S$ is nef and $\nu(-K_S) = 1$. Note that in this case, there is a smooth rational curve $C$ on $S$ with $K_S \cdot C < 0$.

Take an autoequivalence $\Phi = \Phi^P$ of $D(S)$ with $\dim(P) = 3$. Then Lemma 5.1 implies that there is a 3-dimensional irreducible component $W_0$ of $\text{Supp}(P)$ dominating $S$ by $p_1$. Let us denote by $W_{0x}(\subset \{x\} \times S)$ the fiber of the morphism $p_1_{|W_0}: W_0 \to S$ over a point $x \in S$, and regard it as a divisor on $S$ by the isomorphism $\{x\} \times S \cong S$. If $\dim W_{0x} = 2$ for some $x$, Lemma 5.1 (iii) supplies a contradiction to $K_S \neq 0$. Hence, every fiber of $p_1_{|W_0}$ is 1-dimensional, and therefore $p_1_{|W_{0x}}$ is flat (cf. [Ha77, Exercise III.10.9]). Take points $x, y \in C$. Since $C$ is isomorphic to $\mathbb{P}^1$, the points $x$ and $y$ are rationally equivalent 0-cycles on $S$. Hence, the divisors $W_{0x}$ and $W_{0y}$ on $S$ are linearly equivalent (see [Fu98, Theorems 1.1.4, 1.1.7]).

If $\bigcap_{x \in C} W_{0x} \neq \emptyset$, then we see $C \subset \text{Supp}(\Phi^{-1}(O_x))$ for a point $z \in \bigcap_{x \in C} W_{0x}$. This contradicts $K_S \cdot C \neq 0$, since $\Phi^{-1}(O_x)$ is a Calabi–Yau object. Hence, we conclude $\bigcap_{x \in C} W_{0x} = \emptyset$, and therefore the complete linear system $\delta := |W_{0x}|$ is base point free. Moreover, note that $K_S \cdot W_{0x} = 0$ for each $x \in C$, since $W_{0x}$ is contained in $\text{Supp}(\Phi(O_x))$. Furthermore, the Hodge index theorem implies $W_{0x} \cdot W_{0x} = 0$. Then we can see that $\delta$ defines a minimal elliptic fibration, after taking the Stein factorization if necessary.
Conversely, if $S$ has a minimal elliptic fibration, take a universal sheaf $U$ on $J_S(1) \times S$. Then $\Phi_U^{J_S(1) \to S}$ is a Fourier–Mukai transform. Since $J_S(1) \cong S$ and $\dim(U) = 3$, we obtain $N_S = 3$.

(iii) The result follows from (i) and (ii).

For a smooth projective curve $C$, every Fourier–Mukai partner of $C$ is isomorphic to $C$. Therefore, it is obvious that Fourier–Mukai support dimension is a derived invariant for smooth projective curves. In the surface case, a similar result holds.

**Corollary 5.4.** Fourier–Mukai support dimension is a derived invariant for smooth projective surfaces, i.e. if smooth projective surfaces $S$ and $T$ are Fourier–Mukai partners, then $N_S = N_T$.

**Proof.** Let $T$ be a Fourier–Mukai partner of a smooth projective surface $S$. Suppose that $T$ is not isomorphic to $S$. Then $S$ is either a K3 surface, an abelian surface or a minimal elliptic surface, and moreover, $T$ is also a surface of the same type as $S$ ([BM01, Ka02]). Therefore, Theorem 5.3 implies the conclusion.

**Conjecture 5.5.** (i) Fourier–Mukai support dimension is a derived invariant for smooth projective varieties, i.e. if smooth projective varieties $X$ and $Y$ are Fourier–Mukai partners, then $N_X = N_Y$.

(ii) Let $X$ and $Y$ be a smooth projective varieties. Assume that $X$ is of K-equivalent type and that there is a Fourier–Mukai transform $\Phi_X^{P \to Y}$. Then, $\Phi_X^{P \to Y}$ is K-equivalent type.

**Remark 5.6.** (i) Conjecture 5.5 (ii) implies Conjecture 5.5 (i) for smooth projective varieties of K-equivalent type. In fact, suppose that Conjecture 5.5 (ii) is true. Then $Y$ in Conjecture 5.5 (ii) is also K-equivalent type, since a composition of Fourier–Mukai transforms of K-equivalent type is K-equivalent type. In particular, Conjecture 5.5 (i) is true for $X$ of K-equivalent type.

(ii) Kawamata predicts in [Ka02, Conjecture 1.2] that birationally equivalent, derived equivalent smooth projective varieties are K-equivalent, but a counterexample to his conjecture is discovered by the author in [Ue04]. Conjecture 5.5 (ii) is a special version of Kawamata’s conjecture, since $X$ and $Y$ are K-equivalent by Remark 4.3 (i) when a Fourier–Mukai transform $\Phi_X^{P \to Y}$ is K-equivalent type.

### 6 Autoequivalence groups of K-equivalent type

Let $S$ be smooth projective surface and take $\Phi = \Phi^P \in \text{Auteq}_{\text{K-equiv}} D(S)$. Set

$$\Gamma := \text{Supp}(P).$$
Then $\dim \Gamma = 2$ by Lemma 5.1, and Lemma 3.2 (ii) yields that there is the unique component $W_0$ of $\Gamma$ dominating $S$ by both of $p_1$ and $p_2$. Note that $\Gamma_x$ is at most 1-dimensional for $x \in S$.

Let us denote by $Z$ the union of $(-2)$-curves on $S$. The set $Z$ has finitely many connected components, since the Picard number $\rho(S)$ is finite. But it can possibly have infinitely many irreducible components. If a K3 surface $S$ contains $(-2)$-curves, and $S$ admits the infinite automorphism group, then the set $Z$ on $S$ is an example of such.

We first show Proposition 6.4 below. We need several claims to prove it. Take a $(-2)$-curve $C$ on $S$ and $L \in \text{Pic}(C)$. We regard $L$ as an object of $D(S)$ in a natural way.

**Claim 6.1.** We have $\dim(\Phi(L)) = 1$. Moreover, every cohomology sheaf $H^i(\Phi(L))$ is rigid and pure 1-dimensional.

**Proof.** Note that $\text{Supp}(\Phi(L))$ is contained in $p_2(p_1^{-1}(C) \cap \Gamma)$. Then, we see $\dim(\Phi(L)) \leq 1$, since $W_0$ is the unique component dominating $S$ by $p_2$. Since $L$ is rigid on $S$, but $O_x$ is not rigid for any $x \in S$, we have $\dim(\Phi(L)) > 0$ by [Hu06, Lemma 4.5]. Moreover, [IU05, Proposition 3.5] implies that $H^i(\Phi(L))$ is rigid, and it is pure 1-dimensional by [IU05, Lemma 3.9].

**Claim 6.2.** We have $\text{Supp}(\Phi(L)) \subset Z$.

**Proof.** Take an irreducible component $D$ of $\text{Supp}(\Phi(L))$. Then, we see $D \cdot K_S = 0$, since $\Phi(L)$ is a Calabi–Yau object.

Take an integer $i$ such that $\text{Supp}(H^i(\Phi(L)))$ contains the irreducible curve $D$. Let us set $M := H^i(\Phi(L))$ and $\text{Supp}(M) = E \cup D$, where the closed subset $E$ does not contain $D$. Then consider the short exact sequence

$$0 \to H^0_E(M) \to M \xrightarrow{\phi} K \to 0$$

in $\text{Coh}(S)$, where $H^0_E(M)$ is the subsheaf with supports in $E$ (cf. [Ha77, Exercise II.1.20]).

Note that $\text{Supp}(K) = D$ and hence $\dim(H^0_E(M)) \cap D \leq 0$. Assume for a contradiction that $K$ is not pure 1-dimensional. Then, there is a local section $s$ of $K$ such that $s(x) \neq 0$ for some point $x \in S$, but $s(y) = 0$ for all point $y \in S$ except $x$. Let $t$ be a local section of $M$ which is a lift of $s$. If $x \notin E$, then $\phi$ is an isomorphism around the point $x$, and hence $t$ generates a 0-dimensional subsheaf of $M$, which contradicts Claim 6.1. Suppose that $x \in E$. Then, $t$ gives a local section of $H^0_E(M)$, and hence $s = \phi(t)$ should
be 0, which also gives a contradiction. Therefore we can conclude that \( K \) is pure 1-dimensional. Thus, Serre duality yields

\[
\text{Ext}^2_S(K, \mathcal{H}^0_E(M)) = \text{Hom}_S(\mathcal{H}^0_E(M), K) = 0,
\]

and then, [KO95, Lemma 2.2 (2)] implies that \( K \) is rigid. Therefore, we have

\[
2 \leq \dim \text{Hom}_S(K, K) + \dim \text{Ext}^2_S(K, K) = \chi(K, K) = -c_1(K) \cdot c_1(K).
\]

Consequently, we have \( D^2 < 0 \) and hence, \( D \) is a \((-2)\)-curve.

Claim 6.3. Supp(φ(Ox)) is contained in \( Z \) for any point \( x \in Z \).

Proof. Take a \((-2)\)-curve \( C \) containing \( x \). Then, there is an exact triangle

\[
\Phi(O_C(-1)) \rightarrow \Phi(O_C) \rightarrow \Phi(O_x),
\]

which implies

\[
\text{Supp}(\Phi(O_x)) \subset \text{Supp}(\Phi(O_C(-1))) \cup \text{Supp}(\Phi(O_C)).
\]

This completes the proof by Claim 6.2.

Proposition 6.4. Let \( S \) be a smooth projective surface. Then, there is a group homomorphism

\[
\iota_Z : \text{Auteq}_{K\text{-equiv}} D(S) \rightarrow \text{Auteq} D_Z(S) \quad \Phi \mapsto \Phi|_{D_Z(S)}.
\]

Proof. Claim 6.3 and [Ue16, Lemma 2.4] complete the proof.

We define the group \( \text{Br}_Z(S) \) generated by twist functors along spherical objects supported in \( Z \):

\[
\text{Br}_Z(S) = \langle T_\alpha \mid \alpha \in D_Z(S) \text{ spherical object} \rangle (\subset \text{Auteq}_{K\text{-equiv}} D(S)).
\]

The following is crucial to show Theorems 6.6 and 6.8.

Lemma 6.5. Let \( S \) be a smooth projective surface.

(i) For a point \( x \in S \) and \( \Phi \in \text{Auteq} D(S) \), suppose that \( \text{Supp}(\Phi(O_x)) \) is at most 1-dimensional, and contains no \((-2)\)-curves. Then, \( \Phi(O_x) \) is a shift of a sheaf.

(ii) Suppose that \( \Phi \in \text{Auteq} D_Z(S) \) preserves the cohomology class \( \text{ch}(O_x) \in H^1(S, \mathbb{Q}) \) for all points \( x \in Z \). Then, there is an autoequivalence \( \Psi \in \text{Br}_Z(S) \), an integer \( i \) and a point \( y \in Z \) satisfying \( \Psi \circ \Phi(O_x) \cong O_y[i] \).

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Proof. (i) Recall that every 1-dimensional component \( C \) of \( \text{Supp}(H^i(\Phi(O_x))) \) satisfies \( K_z \cdot C = 0 \) and is not a \((-2)\)-curve, and then we obtain \( C^2 \geq 0 \). Thus the Riemann–Roch theorem yields
\[
\chi(H^i(\Phi(O_x)), H^i(\Phi(O_x))) = -c_1(H^i(\Phi(O_x))) \cdot c_1(H^i(\Phi(O_x))) \leq 0
\]
(see \([Ue16] \S 2.2\)). In particular,
\[
\dim \text{Ext}^1_S(H^i(\Phi(O_x)), H^i(\Phi(O_x))) \geq 2
\]
holds. Now the statement follows from \([BM01] \text{Lemma 2.9}\].

(ii) Notice that the proof of \([IU05] \text{Key proposition}\) works in our situation, and that the assumption \( \Phi^H \) preserves the class \( \text{ch}(O_x) \) is needed in the proof of \([IU05] \text{Condition 7.5}\) (see also \([Ue16] \text{footnote in pp. 572}\)). Then the result follows.

Let us define
\[
\text{Auteq}_{K\text{-equiv}} D_Z(S) := \text{Im} \iota_Z,
\]
\[
\text{Aut}_Z(S) := \{ \varphi \in \text{Aut}(S) \mid \varphi|_Z = \text{id}_Z \}
\]
\[
\text{Pic}_Z(S) := \{ L \in \text{Pic}(S) \mid L|_Z \cong O_Z \}.
\]

**Theorem 6.6.** Let \( S \) be a smooth projective surface. Then, there is a short exact sequence
\[
1 \rightarrow \text{Pic}_Z(S) \times \text{Aut}_Z(S) \rightarrow \text{Auteq}_{K\text{-equiv}} D_Z(S) \xrightarrow{\iota_Z} \text{Auteq}_{K\text{-equiv}} D_Z(S) \rightarrow 1.
\]

**Proof.** Take \( \Phi = \Phi^P \in \text{Auteq}_{K\text{-equiv}} D(S) \) and a point \( x \in S \setminus Z \). Then, we know by Claim 6.3 that \( \text{Supp}(\Phi(O_x)) \cap Z = \emptyset \) as in \([Ue16] \text{Corollary 3.5}\]. Moreover Lemma 6.5 (i) implies that \( \Phi(O_x) \) is a shift of a sheaf. Hence, \([Br99] \text{Lemma 4.3}\) implies that \( P|_{p_1^{-1}(S \setminus Z)} \) is a shift of a sheaf, flat over \( S \setminus Z \) by \( p_1 \). Consequently, we see \( \dim(\Phi(O_x)) = 0 \). Furthermore, assume \( \Phi \in \ker \iota_Z \). Then, we can say \( \dim(\Phi(O_x)) = 0 \) for all point \( x \in S \), and thus it follows from \([Ue16] \text{Lemma 2.2}\) that
\[
\Phi \cong \phi_o \circ ((-\otimes L)
\]
for \( L \in \text{Pic}_Z(S) \), \( \phi \in \text{Aut}_Z(S) \). Therefore, we obtain the result. \(\square\)

**Conjecture 6.7.** Let \( S \) be a smooth projective surface. Then
\[
\text{Auteq}_{K\text{-equiv}} D_Z(S) = \langle \text{Br}_Z(S), (\text{Aut}(S)/\text{Aut}_Z(S)) \times (\text{Pic}(S)/\text{Pic}_Z(S)) \times \mathbb{Z}[1] \rangle.
\]
Consequently,
\[
\text{Auteq}_{K\text{-equiv}} D(S) = \langle \text{Br}_Z(S), \text{Pic}(S) \rangle \times \text{Aut}(S) \times \mathbb{Z}[1].
\]
Theorem 6.8. Let $S$ be a smooth projective surface. Then Conjecture 6.7 holds true, if $Z$ is a disjoint union of configurations of $(-2)$-curves of type $A$.

Proof. Note that $\Phi \in \text{Auteq}_{K\text{-equiv}} DZ(S)$ preserves the class $\text{ch}(\mathcal{O}_x)$ for $x \in Z$. Then for any point $x \in Z$, Lemma 6.5 (ii) assures that there is an autoequivalence $\Psi \in \text{Br}_Z(S)$, an integer $i$ and a point $y \in Z$ satisfying $\Psi \circ \Phi(\mathcal{O}_x) \cong \mathcal{O}_y[i]$. Now the result follows from [Hu06, Corollary 5.23].

Recall that if $\text{Auteq}_KD(S)$ is $K$-equivalent type, then $\text{Auteq}_KD(S) = \text{Auteq}_{K\text{-equiv}} D(S)$. Therefore, Theorem 6.8 implies Theorem 1.3.

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