Waiting to Borrow From a 457(b) Plan

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Abstract

This paper formulates and solves the optimal stopping problem for a loan made to one’s self from a tax-advantaged retirement account such as a 401(k), 403(b), or 457(b) plan. If the plan participant has access to an external asset with a higher expected rate of return than the investment funds and indices that are available within the retirement account, then he must decide how long to wait before exercising the loan option. On the one hand, taking the loan quickly will result in many years of exponential capital growth at the higher (external) rate; on the other hand, if we wait to accumulate more funds in the 457(b), then we can make a larger deposit into the external asset (albeit for a shorter period of time). I derive a variety of cutoff rules for optimal loan control; in general, the investor must wait until he accumulates a certain amount of money (measured in contribution-years) that depends on the disparate yields, the loan parameters, and the date certain at which he will liquidate the retirement account. Letting the horizon tend to infinity, the optimal (horizon-free) policy gains in elegance, simplicity, and practical robustness to different life outcomes. When asset prices and returns are stochastic, the (continuous time) cutoff rule turns into a “wait region,” whereby the mean of terminal wealth is rising and the variance of terminal wealth is falling. After his sojourn through the wait region is over, the participant finds himself on the mean-variance frontier, at which point his subsequent behavior is a matter of personal risk preference.

Keywords: Retirement Accounts; Defined Contribution Plans; Tax Planning; Mean-Variance Frontier; 401(k) Plans; Optimal Stopping; Financial Decision Making; Asset Management; Household Finance; Consumer Finance.

JEL Classification Codes: D14; D15; G11; G19; G23; G28; G29; H26.

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“The more the capitalist has accumulated, the more is he able to accumulate.”

—Karl Marx, *Capital* (1867)

Gregor: Can’t we do something?
Sam: We are doing something, we’re sitting here waiting.

—*Ronin* (1998)

So we keep on waiting (waiting)
Waiting on the world to change
No we keep on waiting (waiting)
Waiting on the world to change
We keep on waiting (waiting)
Waiting on the world to change
Waiting on the world to change

—John Mayer,

*Waiting on the World to Change* (2006)
1 Introduction.

Salaried workers in the United States have access to a variety of flexible, tax-advantaged retirement accounts such as 401(k), 403(b), and 457(b) plans, to name just a few. There are powerful forces & incentives to fund these accounts. For one thing, employers who match contributions are providing their participants with a very high, instant, riskless return on their money, just for making ordinary payroll contributions. For individuals who use such plans as a de-facto Traditional IRA, there can be significant tax-deferred savings and benefits. Such tIRA money can trade, transact, and accumulate interest frictionlessly for many years, and then ultimately declare itself as income when the retiree is suddenly in a low tax bracket. The option to convert to the Roth format also has significant hidden value, as it provides a kind of insurance against unexpected under- and un-employment: the participant can declare these monies as income during any opportune year during which his tax bracket is lower than normal. All of these types of accounts also carry significant asset protection features: they are not subject to seizure in tort, divorce, alimony, or child support proceedings. We also have the convenient mechanic of automatic, continuous, mechanical payroll contributions that allow the worker to average into various index funds in a very convenient way.

This paper focuses on a very significant benefit that is offered by such plans: the participant has the option to borrow money from the plan (viz., borrow from his own self), making level repayments over a five year term, at an interest rate that is specified by the plan custodian. Such a feature provides a valuable source of liquidity for salaried employees, who may need to tap their own funds in order to finance unexpected consumption, health care expenses, or purchases of
durable goods such as cars and houses. The present work studies the possibility of taking a loan from the retirement plan, and then immediately investing it in higher-growth (and higher risk) opportunities that are believed to exist externally. Say, the investor could plow the loan principal into a portfolio of leveraged ETFs, i.e., the loan provides a partial workaround for the insufficient access to leverage that is often typical for such accounts.

Thus, we study below the fascinating (and useful) optimal stopping problem: how long should the participant wait in order to exercise his right to borrow his own funds from himself? The economic tension and meaning of that choice is quite clear. Certainly, if we wait too long to pull the trigger on the loan, then we will miss out on the supposed high exponential capital growth rate that is available outside of the account. On the other hand, if we borrow too quickly, then the amount of the resulting external deposit may be too small to move the needle: say, a $100 loan from your 457(b) plan is not going to accomplish much of anything.

We take up the problem in continuous time, in order to avail ourselves of the convenience and simplicity of optimizing over a continuous variable, \( \tau \), rather than a discrete set of waiting times (say, with a monthly frequency). Thus, we will deal with payroll contributions that flow at a continuous rate into the 457(b) plan. At present, the annual maximum contribution limit for savers under 50 is $19,500; workers 50 and older are eligible for annual “catch-up” contributions in the amount of $6,500, so that individuals who are closer to retirement age can contribute $26,000 per annum in total. Figure 1 plots the historical (nominal) contribution limits for the years 1987-2021; figure 2 plots these same limits on an inflation-adjusted basis, in constant, 2021 dollars. The real contribution limits to these accounts have stagnated since the mid 2000s.
However, there is an important restriction on the maximum size of self-loans that are taken from 401(k), 403(b), and 457(b) plans, and their ilk: participants are allowed to borrow 50% of their plan balances, up to a maximum loan of $50,000. Our models below will demonstrate that this is a serious, binding restriction for participants who regularly max out their payroll contributions, who ideally would prefer to wait much longer before they exercise their borrowing rights. As such, once we hit $100,000 in account value, there is no further benefit to waiting, at least in a deterministic model. On the other hand, savers who are unable to contribute the legal maximum will be able to choose very long waiting times, and still stay well within the borrowing limits.

The plan of the paper is as follows. Section 2.1 starts by building a simplified, baseline model that is easily tractable, and yet contains many of the features of the general case. We give exact formulas for the (unique) optimal stopping

Figure 1: Annual 457(b) contribution limits in the United States (1987-2021). Starting in 2002, individuals 50 and older were permitted to make special “catch-up” contributions, thereby raising the effective annual contribution limits for agents who are closer to retirement age.
**Figure 2**: Inflation-adjusted historical 457(b) contribution limits: 1987-2021. All time series are expressed in 2021 dollars. Real contribution limits have been stagnant since the mid 2000s.

rule in terms of Lambert’s W function (cf. with Lambert 1758), also called the product logarithm. The optimal control is then expressed in the form of a cutoff rule, that is, once a specific amount of money is accumulated in the account (as measured in contribution-years), then the participant should take a full loan. In general, the optimal timing must be tailored to a specific investment horizon, e.g., it requires a date certain at which the retirement plan will be liquidated. However, we provide the reader with very elegant and convenient horizon-free cutoff rules, which should be used by investors who have long, but uncertain, time horizons.

Section 2.2 presents an expanded, fully-detailed model of the investor’s aggregate wealth dynamics in continuous time. We give a fine-grained analysis of the properties of the (self) debt covenant, with careful attention given to exact liability matching, the “liability” being the stream of repayments that is owed back to the retirement plan. The key parameter, which we call $\eta_R$, is the per-
percentage of the initial principal that is effectively “externalized” at the instant that the loan is made, where $R$ is the external rate of interest. The remaining $1 - \eta R$ will be set aside to match liabilities. Once the loan term is over, and the “sinking fund” that is set aside for loan repayments is fully exhausted, the externalized funds are free to compound themselves at the high, external rate for many years. Accordingly, we generalize the cutoff rules given in section 2.1; perfect loan timing will depend on the horizon $T$ and the particular parameters of the loan obligation. However, if we let $T \to \infty$, we get an excellent numerical approximation that is universally valid for all investors; the horizon-free decision rule does not care about loan details such as the term or the interest rate.

Finally, section 2.3 builds a two-moment decision model vis-à-vis terminal aggregate wealth in environment, with random, diffusive asset prices that follow the classical log-normal random walks of efficient market theory. In this generality, the agent behaves as a mean-variance optimizer over terminal capital accumulation. Our cutoff rule now consists in a special interval of waiting times during which the mean of terminal wealth is increasing whilst the variance is simultaneously decreasing; such times are dominated in the mean variance plan. Accordingly, we derive exact formulas that are useful to mean-variance investors, and we study the beautiful form of the corresponding efficient frontier, which here obtains as a parametric curve in the waiting time $\tau$. Upon his exit from the dominated region (at which time we have peak expected value), the remaining waiting time is a matter of personal preference: the reward is decreasing, but the risk is also decreasing. At this juncture, the least risky option (other than refusing to take a loan), is to wait until we are exactly 5 (in general, $\phi$) years before the end of the planning period. Based on our plots
and calculations, this is the only rational time for anybody to take a partial loan
(the better to decrease risk even further). Taking a partial loan (e.g., one that
is lower than the allowed maximum) at time before \( r = T - \phi \) will be dominated
by taking a full loan at some subsequent date.

2 The Model.

2.1 Baseline Model.

The time \( t \) (measured in years) is continuous, and finite; we have \( t \in [0, T] \),
where \( t = 0 \) is right now and \( t = T \) is the liquidation date of the 457(b) account
(or the 403(b) account, etc.). You best hope your \( T \) is large. We let the function
\( (x(t))_{t \in [0, T]} \) denote the dollar value of the 457(b) plan at time \( t \), where the cur-
rent account value \( x_0 := x(0) \) is an exogenous parameter of the model. Payroll
contributions are assumed to flow into the account at a continuous rate of \( c \) dol-
lars per year, so that the cumulative deposits into the account will be \( ct \) dollars
after \( t \) years. As of this writing, the maximum annual contribution limit for
457(b) and 403(b) plans is $19,500 a year, which is a good value of \( c \). Note well
the ratio \( x_0 / c \), which is the number of years of contributions that have already
accumulated in the account. For the time being, and to fix ideas in the baseline
model, we assume that there is no interest in the 457(b) plan, e.g., the cash just
sits there collecting dust. Thus, we have the 457(b) law of motion \( dx = c \cdot dt \), so
that \( x(t) = x_0 + ct \).

We assume that the plan rules allow for the participant to take a single loan
(at any time), of no more than the percentage \( \theta \in (0, 1] \) of his account value, up
to a maximum of \( L > 0 \). In practice, we presently have the parameters \( \theta := 50\% \)
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Figure 3: The interval \([0, \min(x/2, 50k)]\) of allowable loan sizes in a 457(b) plan, for different account values \(x\).

and \(\bar{L} := 50,000\) for both 457(b) and 403(b) plans in the United States. Thus, we have the relation

\[
\text{Allowed Loan Amount at Time } t = \min \left( \theta x(t), \bar{L} \right).
\]  

We will assume that such a loan is irrevocable and irreversible once taken (if taken); for the time being we will assume that the investor will make a lump sum “repayment” back into the plan (viz., repay what he borrowed from himself back to himself) at time \(T\). More complicated repayment dynamics will be studied below once we have established the main ideas.

Let us suppose that an attractive (deterministic) investment opportunity is available to the investor outside of his 457(b) plan, an opportunity that multiplies his capital at a continuously-compounded rate of \(R > 0\) per year. Say, \(R := 10\%\), or \(R := 20\%\) for a very aggressive or optimistic individual. Thus, let \(y(t)\) denote the value of the investor’s holdings in this asset at time \(t\). We
let \( \tau \in [0, T] \) denote the instant of time at which the 457(b) loan is taken; these monies are assumed to be instantly shifted into the other account. In this environment, it is never correct for the investor to take a partial distribution; obviously, when he does decide to pull the trigger and borrow, he should borrow the full available amount, \( \theta x(\tau) \).

The tension in the agent’s decision problem is just this: one the one hand, an earlier 457(b) distribution will allow for a greater amount of time (viz., \( T - \tau \) years) of exponential capital growth at the rate \( R \); on the other hand, it behooves him to wait a while so that he can make a large (and profitable) deposit into the outside opportunity. Clearly, we cannot have \( \tau^* = T \), but we can certainly have \( \tau^* = 0 \) if the current balance \( x_0 \) is large enough. If \( x_0 \) is huge, then there is scant benefit to waiting for additional payroll contributions to roll in, which will be paltry by comparison. It is also clear that once the maximum dollar limit \( \bar{L} \) is breached (viz., when \( x(t) \geq \bar{L}/\theta \)), a distribution should be taken immediately, since 100% of all marginal contributions will be marooned in the account. Say, when we hit $100,000 in account value (after 5 years and 2 months of maxing out our payroll contributions), we had better “push the button” and make something happen.

Thus, at the instant \( \tau \) when the loan is taken, we will have \( (1-\theta)x(\tau) \) dollars left in the 457(b) plan, and \( \theta x(\tau) \) dollars invested in the outside opportunity. This generates the (outside) account dynamics \( dy = R y \cdot dt \) for \( t > \tau \), so that the terminal value of account \( y \) is

\[
y(T) = \theta x(\tau)e^{R(T-\tau)}.
\]
The terminal value of the 457(b) is then

\[ x(T) = (1 - \theta)x(\tau) + c(T - \tau). \]  (3)

I hope that the reader’s natural instinct is to choose \( \tau \) so as to optimize the terminal net worth\(^1\)

\[ M(\tau) := x(T; \tau) + y(T; \tau). \]  (4)

If we clean out some constants that are irrelevant to the optimization, we get the objective function (maximand)

\[ F(\tau) := \left( \frac{x_0}{c} + \tau \right) \left( e^{R(T-\tau)} - 1 \right). \]  (5)

Note that the product \( x(\tau) \left( e^{R(T-\tau)} - 1 \right) \) is directly proportional to the aggregate dollar profits that will be earned on the outside investment, e.g., maximizing such profit will achieve the aim of optimizing the terminal wealth \( M(\tau) = x(T; \tau) + y(T; \tau) \).

\[ \max_{0 \leq \tau \leq T} \left( \frac{x_0}{c} + \tau \right) \left( e^{R(T-\tau)} - 1 \right), \]  (6)

we have a continuous function on a compact interval, which therefore attains its maximum (cf. with Rosenlicht 1986) somewhere over \([0, T]\). Now, we have the endpoint values \( F(T) = 0 \) and \( F(0) = x_0 (e^{RT} - 1) > 0 \), so that \( \tau^* < T \).

**Proposition 1.** The objective function \( F(\bullet) \) in the stopping problem (6) is strictly log-concave over \([0, T]\), and thereby generates a unique optimum \( \tau^* \) that is completely characterized by first-order conditions. The optimal policy depends only

\(^1\)Here, \( M(\bullet) \) stands for “money.” In the sequel, asset prices will follow a log-normal diffusion process with drift parameter \( R \) and the diffusion parameter, \( \sigma \); \( M(\bullet) \) will then denote the mean of the investor’s aggregate terminal wealth, and the same formulas will obtain for \( M(\tau) \).
on $R$, $T$, and the value-to-contribution ratio $x_0/c$; the particular plan parameter $	heta$ (e.g., the percentage of 457(b) funds that can be borrowed) is irrelevant to the maximizer $\tau^*$ in (6). However, less restricted values of $\theta$ will of course increase the agent’s ultimate welfare.

**Proof.** $\log (F(\tau))$ consists in two terms: $\log (x_0/c + \tau)$, which is obviously strictly concave, and $\log \left( e^{R(T-\tau)} - 1 \right)$, which is also strictly concave, for it amounts to the substitution of the affine function $\tau \mapsto R \times (T - \tau)$ into the function $z \mapsto \log (e^z - 1)$, which is strictly concave over $(0, \infty)$:

$$\frac{d^2}{dz^2} \log (e^z - 1) = -\frac{e^z}{(e^z - 1)^2} < 0. \quad (7)$$

Finally, in the saturated terminal net worth

$$M^*(T) := \theta x_0 \tau^* e^{R(T-\tau^*)} + (1 - \theta)x_0 \tau^* + c(T - \tau), \quad (8)$$

since $\partial \tau^*/\partial \theta \equiv 0$, we have the sensitivity

$$\frac{\partial M^*}{\partial \theta} = x_0 \tau^* \left( e^{R(T-\tau^*)} - 1 \right) > 0 \quad (9)$$

to a favorable change in the plan rules. \hfill \blacksquare

**Theorem 1** (Baseline Stopping Theorem for 457(b) Plans). *In the baseline model, the investor should take a 457(b) loan whenever his account balance $x(t)$ exceeds the cutoff value of

$$x(t) \geq \min \left( \frac{c}{R} \left( 1 - e^{-R(T-t)} \right), \frac{\bar{L}}{\theta} \right) \quad (10)$$

dollars. If he is unsure of the exact value of $T$ (but, perhaps, he does know that
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Figure 4: Strict concavity of the function $z \mapsto \log(e^z - 1)$ over $(0, \infty)$.

For, once (11) is breached, everybody (regardless of their age or their predicted retirement date) should immediately take a loan from their 457(b) plan.

Remark 1. Note well the recurrent nature of the stopping problem that is at hand before us. Right now, it’s time-zero, and the horizon is $T$. Should we borrow? Yes, if the initial account value $x_0$ is large enough. Now, at time $t$, the horizon will be $T - t$. At that time, we will ask ourselves—should we borrow? Yes, if the initial value $x(t)$ is sufficiently large. Thus, we need only solve the problem once at time-zero (for a generic horizon $T$ and a generic initial account balance $x_0$), and then substitute $T \leftarrow T - t$ and $x_0 \leftarrow x(t)$ into the resulting cutoff formula.

Proof. Right. We should borrow at time-zero if and only if the first-order condi-
tion $F'(0) \leq 0$ is satisfied. Thus, we have the calculation

$$F'(\tau) = e^{R(T-\tau)} - 1 - \frac{X_0}{c} e^{R(T-\tau)} e^{R(T-\tau)}.$$  

(12)

Simplifying, we see that condition $F'(0) \leq 0$ is equivalent to

$$\frac{x_0}{c} \geq \frac{1 - e^{-RT}}{R}.$$  

(13)

Accordingly, when the horizon is $T-t$ and the time is $t$, our cutoff rule is

$$x(t) \geq \frac{c}{R} \left(1 - e^{-R(T-t)}\right).$$  

(14)

If this be the case, then we should borrow; however, as mentioned above, we should also borrow if

$$x(t) \geq \frac{L}{\theta},$$  

(15)

e.g., if it is already possible to borrow the plan maximum of $L$. Thus, we should borrow if at least one of the conditions (14,15) is satisfied, hence the $\min(\bullet, \bullet)$ in the statement of the theorem. Finally, note that the function $T \mapsto 1 - e^{-R(T-t)}$ is increasing in $T$, so that the cutoff (10) is also increasing in $T$. Thus, let us imagine a wealth value $x(t)$ at which everybody would choose to borrow, e.g., assume that (10) holds uniformly for all $T$. Taking the limit as $T \to \infty$, we get the necessary condition

$$x(t) \geq \min \left(\frac{c}{R}, \frac{L}{\theta}\right),$$  

(16)

which is also sufficient (for everyone to borrow) on account of the fact that (10)
is an increasing function of $T$. That is,

$$
\min \left( \frac{c}{R}, \frac{L}{\theta} \right) = \sup_{T>0} \min \left( \frac{c}{R} \left( 1 - e^{-R(T-t)} \right), \frac{L}{\theta} \right),
$$

viz., an increasing bounded function of $T$ converges to its least upper bound as $T \to \infty$. \hfill \blacksquare

From the proof of Theorem 1, we have, immediately, the following corollary.

**Corollary 1.** If our funds $x_0$ do not exceed the loan cutoff, then the future loan date $t^*$ consists in the unique solution (fixed-point) of the equation

$$
t = \frac{1}{R} \left( 1 - e^{-R(T-t)} \right) - \frac{x_0}{c}.
$$

This fixed point consists in the expression

$$
t^* = \frac{1}{R} \left[ 1 - W \left( \exp \left( 1 - R \left( T + \frac{x_0}{c} \right) \right) \right) \right] - \frac{x_0}{c},
$$

where $W(\bullet)$ denotes the Lambert 1758 $W$ function (cf. with Euler 1783; Corless et al. 1993). At the date the account is opened (viz., $x_0 := 0$), the participant can plan on taking a loan in

$$
t^*(R, T) = \frac{1 - W \left( e^{1 - RT} \right)}{R}
$$

or $\frac{L}{(\theta c)}$ years, whichever is earlier.

**Proof.** Re-arranging the first order condition $F'(t) = 0$, as in (12) with $t$ playing the role of $\tau$, we get the fixed-point relation (18). After a slight manipulation,
Figure 5: Baseline, horizon-free 457(b) strategy card. \( R \) is the investor’s (believed) continuously-compounded annual capital growth rate in assets outside of the 457(b) plan. On the vertical axis, we have the horizon-free cutoff value \( \min (19.5k/R, 100k) \). Thus, in the simplified (baseline) model, supreme investment skill (or confidence) is required in order to rationally take a 457(b) loan that is lower than the allowed ($50k) maximum. The kink is at (19.5\%, 100k).

(18) becomes

\[
\left(1 - R \left( t + \frac{x_0}{c} \right) \right) \exp \left(1 - R \left( t + \frac{x_0}{c} \right) \right) = \exp \left(1 - R \left( T + \frac{x_0}{c} \right) \right). \tag{21}
\]

The left-hand side of (21) has the form \( ze^z \), which, by definition, is the inverse of Lambert’s \( W \) function \( z \mapsto W(z) \). Thus, applying \( W(\bullet) \) to both sides of (21), and bearing in mind that \( W(ze^z) = z \), we have

\[
1 - R \left( t + \frac{x_0}{c} \right) = W \left( 1 - R \left( T + \frac{x_0}{c} \right) \right), \tag{22}
\]

which yields (19). Putting \( x_0 := 0 \) gives (20).
2.2 The Fully Detailed 457(b) Model.

Given the endearing success of the baseline model that we formulated in section 2.1, we proceed to develop a fully detailed model of 457(b) loan control in continuous time. First, we assume that interest and capital growth are available inside of the 403(b) plan, albeit at a lower interest rate $r < R$ than is available in the more aggressive plays that exist outside of the retirement account. As $r \downarrow 0$, we recover some of the simplified properties of the baseline model given above. We now include the (important) feature that the loan must be repaid (from the agent’s external account back into his retirement plan) continuously over a period of $\phi$ years. For U.S. retirement accounts such as 401(k), 403(b), and 457(b) plans, the existing parameter value is $\phi := 5$.

Level repayments will be made continuously, at a rate of $p = p(\tau)$ dollars
per year, where we have written \( p(\tau) \) to indicate the fact that the size of the repayments will ultimately depend on the time \( \tau \) that the loan is taken, i.e., since the rate of (continuous) repayment will be directly proportional to the size of the loan, \( \theta x(\tau) \). The investor will be required to repay himself principal and interest at a continuous annual rate, \( \rho \leq R \), which can vary with the custodian and with the type of the retirement plan. As of this writing, a good sample value is \( \rho := \log(1.0425) = 4.16\% \), compounded continuously. In order to make these repayments, the investor will liquidate \( p(\tau)dt \) dollars worth of the high-growth, external asset in order to pay such monies back into his retirement plan. Let us calculate the magnitude of the level payments \( p(\tau) \). The rate of loan repayment must be such that the present value of the repayment stream (valued at the interest rate \( \rho \)) at time \( \tau \) is equal to the initial principal balance \( \theta x(\tau) \):
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Figure 8: Baseline, fixed-horizon strategy card for the timing of loans from 401(k), 403(b), and 457(b) plans in the United States, for different liquidation dates $T$ (in years) and external capital growth rates, $R$. On the vertical axis, we have $\min\left(t^*(R, T), \frac{L}{\theta c}\right)$, which is the number of years after account opening that the loan will be taken. $(\overline{L}, \theta, c) := (50k, 50\%, 19.5k)$.

\[ p(\tau) \int_{\tau}^{\tau+\phi} e^{-\rho(t-\tau)} \, dt = \theta x(\tau). \]  \hfill (23)

That is, the vintage-$t$ repayment in the amount of $p(\tau)\, dt$ has a present value of $e^{-\rho(t-\tau)}\, p(\tau)\, dt$ at time $\tau$. Evaluating the integral (23), we have the loan payment formula

\[ p(\tau) = \frac{\rho}{1 - e^{-\rho\phi}} \theta x(\tau) = \lambda_\rho \theta x(\tau), \]  \hfill (24)

where

\[ \lambda_\rho := \frac{\rho}{1 - e^{-\rho\phi}} = \text{Annual Payment Flow on a $1 Loan at $\rho for $\phi years} \]  \hfill (25)
is an important loan parameter that will be used extensively in the sequel. *Nota bene*: when the investor deposits a dollar of his loan money into the external asset, he is effectively re-loaning it out for $\phi$ years at a higher interest rate $R \geq \rho$, thereby generating a payment stream in the amount of $\lambda_R \geq \lambda_\rho$ dollars a year. This cash flow supports the repayment $\lambda_\rho$, plus annual profits in the amount of $\lambda_R - \lambda_\rho$ for every dollar that was borrowed from the 457(b) plan.

**Lemma 1.** The annual loan repayment $\lambda(\rho; \phi)$ per dollar borrowed at an interest rate of $\rho$ for $\phi$ years is strictly increasing in $\rho$ and it is strictly decreasing in $\phi$.

**Proof.** It will suffice to show that the function $h(z) := z/(1 - e^{-z})$ is strictly increasing, whence we will have $\lambda(R) = h(R\phi) / \phi > h(\rho\phi) / \phi = \lambda(\rho)$ if $R > \rho$. Thus, we have

$$h'(z) = \frac{1 - e^{-z} - ze^{-z}}{(1 - e^{-z})^2} > 0$$

(26)

on account of the fact that $e^z > 1 + z$ for all $z \neq 0$ (since a strictly convex function lies above all of its tangents). It is also clear that $\lambda$ is strictly decreasing in $\phi$, and in fact we have $\partial \lambda / \partial \phi = -\lambda^2 e^{-\rho \phi} < 0$.

Thus, a total of $\phi \times p(\tau)$ dollars will be repaid into the 457(b) plan, for a total payment percentage (TPP) of $\phi \times \lambda$ and a total interest percentage (TIP) of $\phi \lambda - 1$. Say, for $\rho := 4.16\%$ and $\phi := 5$ years, the annual rate of loan repayment will consist in $\lambda = 22.15\%$ of the initial principal balance, for a total payment percentage of $110.75\%$ and a total interest percentage of $10.75\%$ over the life of the loan.

Now, let $\beta(t)$ denote the outstanding principal balance on the plan loan at
time $t$, where we have $\beta(\tau) = \theta x(\tau)$ and $\beta(\tau + \phi) = 0$. We have the formula

$$\beta(t) = e^{\rho(t-\tau)}\theta x(\tau) - p(\tau) \int_{\tau}^{t} e^{\rho(t-s)} ds.$$  \hfill (27)

expresses the outstanding balance $\beta(t)$ via the “sinking fund” method. That is, imagine that we establish a separate escrow account, or sinking fund, that will accept the continuous flow of payments $p(\tau)dt$. This fund is assumed be an ideal bank (cf. with Luenberger 1997) whose interest rate is $\rho$. After the elapse of $t$ years, the difference between the accumulated value of the debt obligation (principal and interest) and the accumulated value of the sinking fund (viz., the right-hand side of (27)) equals the investor’s net debt to his own retirement account. After simplifying (27), we have the relation

$$\% \text{ of Original Principal Outstanding at } t = \frac{\beta(t)}{\theta x(\tau)} = \frac{\lambda \rho}{\rho} - \left(\frac{\lambda \rho}{\rho} - 1\right) e^{\rho(t-\tau)},$$  \hfill (28)

which expresses the percentage of the original principal that is currently outstanding as an affine combination of $e^{\rho(t-\tau)}$ and 100%.

Before we move on to the business of optimizing the terminal aggregate wealth $M(\tau)$, let us give a brief analysis of the (bond) duration of the promissory note. Taking the external yield $R$ as the appropriate interest rate for valuing the cash flows $p(\tau)dt$, we have the time-$\tau$ present value (or bond price)

$$PV = p(\tau) \int_{\tau}^{\tau+\phi} e^{-R(t-\tau)} dt = \frac{p(\tau)}{R} \left(1 - e^{-R\phi}\right) = \frac{p(\tau)}{\lambda(R)}.$$  \hfill (29)

The Macaulay duration of this bond (cf. with Wilmott 2007) consists in the
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Figure 9: The amortization schedule (under continuous repayment) for a $50k 457(b) loan at 4.25% APR (4.16% compounded continuously) for 5 years. $5,375 of interest will ultimately be paid into the retirement plan from external sources.

expression

$$- \frac{1}{PV} \times \frac{\partial (PV)}{\partial R} = \int_{\tau}^{\tau+\phi} (t - \tau) \frac{p(t)e^{-R(t-\tau)}}{PV} \, dt = \frac{\lambda'(R)}{\lambda(R)} = \frac{1}{R} - \frac{\phi}{e^{R\phi} - 1},$$

wherein the duration \((t - \tau)\) of the vintage-\(t\) cash flow is being weighted by the percentage of total present value that that cash flow represents. Figure 11 plots the Macaulay duration (in months) of a 60-month 401(k) loan for different external interest rates \(R \in [5\%, 30\%]\). By contrast, the weighted average life\(^a\) of the bond (cf. with Fabozzi and Pollack 2005) is simply \((1/\phi) \int_{\tau}^{\tau+\phi} (t - \tau) \, dt = \phi/2\), which is independent of \(R\).

We proceed to calculate the terminal account values \(x(T; \tau)\) and \(y(T; \tau)\).

\(^a\)The weighted average life of the bond is the result of weighting the life of each cash flow by the percentage of total payment flow that it represents, rather than the percentage of total present value that it represents. Since the repayments are level and continuous, such a weighting amounts to the uniform density \(1/\phi\) over the time interval \([\tau, \tau + \phi]\).
Adding up the terminal external wealth, we get

\[
y(T; \tau) = \theta x(\tau) e^{R(T-\tau)} - p(\tau) \int_{\tau}^{\tau+\phi} e^{R(T-t)} dt
\]

\[
= \theta x(\tau) e^{R(T-\tau)} - \frac{p(\tau)}{R} \left( e^{R(T-\tau)} - e^{R(T-\tau-\phi)} \right), \quad (31)
\]

which simplifies to

\[
y(T) = \left(1 - \frac{\lambda_p}{\lambda_R} \right) \theta x(\tau) e^{R(T-\tau)} = \eta_R \theta x(\tau) e^{R(T-\tau)}, \quad (32)
\]

where

\[
\eta_R := 1 - \frac{\lambda_p}{\lambda_R} = \text{% of the Loan That is Permanently Externalized} \quad (33)
\]
That is, from an initial dollar of 457(b) loan money, we effectively lend out $1 - \eta_R = \frac{\lambda_\rho}{\lambda_R}$ dollars into the world outside of the retirement account at an interest rate of $R$, and this loan generates $\lambda_R \times (1 - \eta_R) = \lambda_\rho$ in annual cash flow, which is exactly what we need in order to service the 457(b) loan. Having achieved this exact matching of assets and liabilities, the residual percentage $\eta$ of the initial loan amount will forever escape the clutches of the 457(b), and thereby generating $\eta_R \theta x(\tau) e^{R(T-\tau)}$ dollars of external wealth at $T$.

The integral term in (31) represents the loss of future value that has occurred due to continuous liquidation of the high-growth asset (in order to make payment to the 457(b)). The vintage-$t$ liquidation (of $p(\tau)d\tau$ dollars) will ulti-
Figure 12: The percentage $\eta_R$ of the loan that gets permanently externalized, for different external yields $R \in [0\%, 30\%]$, under the parameters $(\phi, \rho) := (5 \text{ years}, 5\% \text{ a year})$. The remaining $1 - \eta_R$ is set aside to match liabilities, e.g., a sinking fund that invests $1 - \eta_R$ of the principal into the external asset will be exactly sufficient to fund the loan repayments.

Ultimately have destroyed $p(\tau)e^{R(T-\tau)}dt$ dollars of final external wealth, at $T$. This loss of future value must be netted out from the first term, which is the future value of the initial deposit $\theta x(\tau)$ at an interest rate of $R$ per annum.

Remark 2. Lemma 1 means that the expression (32) for the terminal external wealth $y(T)$ is non-negative if and only if $\rho \leq R$. As one might have guessed, if $\rho > R$, then the agent will ultimately default on his obligation to his own self.

Proposition 2 (Self-Default). If the interest rate $\rho$ on the plan loan is greater than the rate of return $R$ on the outside investment, then the agent will default on his loan to himself (or else he will have to repay using separate funds) precisely at time $T_0 \in (\tau, \tau + \phi)$, where

$$T_0 = \tau - \frac{1}{R} \log \left( 1 - \frac{R}{\lambda_\rho} \right).$$

(35)
Figure 13: Survival time before defaulting on a (5-year) 457(b) loan, for different interest rates $\rho$ of the debt covenant. Here, we have assumed that $\phi := 5$, $\tau := 0$, and that the outside asset yields $R := 0\%$. Note that we can expect to survive for at least 3\( \frac{1}{2} \) years, even under unfavorable conditions.

**Proof.** Taking our cue from (31), the default date $T_0$ (at which time all of the original loan monies $\theta x(\tau)$ will have been depleted by the repayments) is characterized by the equation

$$\theta x(\tau) e^{R(T_0-\tau)} = p(\tau) \int_{\tau}^{T_0} e^{R(T_0-t)} dt = \frac{p(\tau)}{R} \left(e^{R(T_0-\tau)} - 1\right).$$  \hspace{1cm} (36)

Using the fact that $p(\tau)/(\theta x(\tau)) = \lambda \rho$, and simplifying, the stated result (35) obtains from (36). Note that the hypothesis $\rho > R$ guarantees that the argument of the logarithm in (35) is strictly positive. \hfill \blacksquare

Having satisfactorily resolved the question of the terminal external wealth $y(T)$, we proceed to account for the final accumulated balance $x(T)$ of the 403(b) plan. First, we have the unloaned account value $(1 - \theta)x(\tau)$, which multiplies itself into $(1 - \theta)x(\tau)e^{r(T-\tau)}$ dollars. Next, we have the ongoing payroll contribu-
tions$^3$ over $[\tau, T]$, which have a future value of $c \int_{\tau}^{T} e^{r(T-t)} dt = \left(\frac{c}{r}\right) \left(e^{r(T-\tau)} - 1\right)$.

Finally, we have the incoming loan repayments over $[\tau, \tau + \phi]$, which have a future value of

$$p(\tau) \int_{\tau}^{\tau + \phi} e^{r(T-t)} dt = \frac{\lambda \rho}{\lambda r} x(\tau) e^{r(T-\tau)}.$$  \hspace{1cm} (37)

Thus, the final value of the 457(b) plan consists in the expression

$$x(T) = (1 - \eta_r \theta) x(\tau) e^{r(T-\tau)} + \frac{c}{r} \left(e^{r(T-\tau)} - 1\right).$$  \hspace{1cm} (38)

Thus, be advised that the 403(b) loan has turned out to be a splendid retirement investment that has perhaps (depending on the parameter values) outperformed other short-term debt securities that were available from the plan administrator. To complete the circle, we have the time-$\tau$ 457(b) balance (e.g., the full substrate for the plan loan)

$$x(\tau) = x_0 e^{rt} + \int_{0}^{\tau} e^{r(T-t)} dt = \left(x_0 + \frac{c}{r}\right) e^{rt} - \frac{c}{r},$$  \hspace{1cm} (39)

where $x(0) = x_0$ is the initial balance at the start of the model (viz., the balance right now), and $x'(0) = rx_0 + c$.

**Proposition 3** (Final Plan Balance). The terminal 457(b) wealth $\tau \mapsto x(T; \tau)$ has the form

$$x(T) = \frac{c}{r} \theta \eta_r e^{r(T-\tau)} + \text{constant},$$  \hspace{1cm} (40)

$^3$In our model, as it is in reality, payroll contributions are never commingled with any ongoing loan repayments, which occur separately, and are drawn from the agent’s checking account. The 403(b) treats the loan as a proper investment in its own right, say, with the same legitimate status that is enjoyed by any other financial product or instrument that is offered by the plan custodian. The fact that the lender and the borrower happen to be the same person is not relevant to the internal investment logic of the plan, and, indeed, all of the principal is going to be paid back, with interest. What a great investment. Note well that the existence of a 403(b) loan fund does not in any way reduce the participant’s annual contribution limit, $c$. 

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where the constant term is independent of \( \tau \). Thus, either:

- \( x(T; \tau) \) is strictly increasing and strictly concave in \( \tau \) (when \( r < \rho \))
- \( x(T; \tau) \) is strictly decreasing and strictly convex in \( \tau \) (when \( r > \rho \))
- \( x(T; \tau) \) is a constant, independent of \( \tau \) (when \( r = \rho \)).

Proposition 3 is proved by substituting the formula (39) for \( x(\tau) \) into (38), and simplifying. When \( r < \rho \) (i.e., when the 403(b) loan is a more attractive investment than the custodian’s own bond fund), waiting longer will make for a larger deposit into the (\( \phi \) year) loan down the line, resulting in a higher terminal value of the retirement plan. On the other hand, if \( r > \rho \), since the plan loan will underperform the internal asset, waiting longer to disburse the loan will be deleterious to the final 457(b) value, since (given the way that \( x(T; \tau) \) is structured) a greater deposit will have been made into the poorly performing investment. If the loan performs identically with the internal offerings (\( r = \rho \)), then the disbursement date is irrelevant to the value of \( x(T) \).

**Proposition 4 (Final External Balance).** The accumulated outside balance \( \tau \mapsto y(T; \tau) \) is a linear combination of the (decaying) exponentials \( e^{-(R-r)\tau} \) and \( e^{-R\tau} \):

\[
y(T) = \eta_R \theta e^{RT} \left\{ \left( x_0 + \frac{c}{r} \right) e^{-(R-r)\tau} - \frac{c}{r} e^{-R\tau} \right\}.
\]

The function \( \tau \mapsto y(T; \tau) \) is strictly log-concave (unimodal) over \([0, \infty)\), the mode occurring at the abscissa

\[
\tau_y^* = \arg \max_{\tau \geq 0} y(T; \tau) = \frac{1}{r} \log \left( \frac{Rc}{(R-r)(rx_0 + c)} \right),
\]

which is independent of \( T \).
Figure 14: Terminal 457(b) wealth $x(T; \tau)$ for different waiting times $\tau \in [0, 25]$ and different interest rates $\rho \in \{4\%, 6\\%\}$ on the 457(b) loan. Here, we have used the parameters $(x_0, T, r, \theta, c, \phi) := ($20k, 30 years, 5% a year, 50%, $19.5k, 5 years). The latest date that the loan can be taken is $T - \phi = 25$ years from now. When the interest rate on the plan loan is higher than the rate of return on internal investments, $\tau \mapsto x(T; \tau)$ will be concavely increasing in $\tau$; otherwise it will be decreasing and convex.

Proof. The linear combination (41) is found by substituting (39) into (32), and simplifying. Solution of the first order condition $\partial y(T; \tau)/\partial \tau = 0$ yields the (horizon-free) expression (42). As to the log-concavity (viz., the second order condition), note that $\tau \mapsto y(T; \tau)$ has the form $Ae^{-R\tau} (e^{r\tau} - B)$ where $A$ and $B$ are positive constants. Thus,

$$\frac{\partial \log (y (T; \tau))}{\partial \tau} = \frac{r}{1 - Be^{-r\tau}} - R \tag{43}$$

is strictly decreasing, as promised. 

Thus, after forming the sum $M(\tau) = x(T; \tau) + y(T; \tau)$, and cleaning and
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Figure 15: The final externalized wealth $y(T; \tau)$ for different waiting times $\tau \in [0, 25]$, under the parameters $(x_0, T_1, r, \theta, c, \phi, R, \rho) := (\$0, 30 \text{ years}, 4\% \text{ a year}, 50\%, \$19.5k, 5 \text{ years}, 15\% \text{ a year}, 4.16\% \text{ a year})$.

Factoring out some irrelevant constants, we get the simplified objective function

$$F(\tau) := \eta_r e^{-\tau r} + \eta R e^{(R-r)T} \left\{ \left(1 + \frac{r x_0}{c} \right) e^{-(R-r)\tau} - e^{-\tau r} \right\},$$

which is a linear combination $k_1 e^{-\tau r} + k_2 e^{-(R-r)\tau} - k_3 e^{-\tau r}$ of three decaying exponentials.

**Theorem 2** (Optimal Control of 401(k), 403(b) and 457(b) Loans). The terminal final wealth $x(T; \tau) + y(T; \tau)$ is optimized by adherence to the following policy: we should take a 457(b) loan right now if our current plan balance $x_0$ exceeds the cutoff:

$$x_0 \geq \frac{c}{R - r} \left(1 - \frac{\eta_r e^{(R-r)T}}{\eta R} \right),$$

where $\lambda(z) := z/(1 - e^{-\phi z})$. 

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Proof. The first order condition \( F'(0) \leq 0 \) consists in the inequality

\[
-r \eta_r + \eta_r e^{(R-r)T} \left[ -(R-r) \left(1 + \frac{rx_0}{c}\right) + R \right] \leq 0.
\]

(46)

After simplifying (46) and solving it for \( x_0 \), we get (45).

Theorem 3 (Externalization Theorem). The horizon-free cutoff rule \( x_0 \geq c/(R-r) \), which is gotten by letting \( T \to \infty \) in the horizon-\( T \) cutoff rule (45), is in fact the unique policy that maximizes the external wealth \( y(T) \).

Proof. If we wait until such time \( \tau \) as we have accumulated exactly \( c/(R-r) \) dollars in the retirement plan, then \( \tau \) is characterized by the equation

\[
\left(x_0 + \frac{c}{r}\right) e^{\tau r} - \frac{c}{r} = \frac{c}{R-r}.
\]

(47)

Solving (47) for \( \tau \), we get the same horizon-free waiting time

\[
\tau^* = \frac{1}{r} \log \left( \frac{Rc}{(R-r)(rx_0 + c)} \right) = \arg \max_{\tau \geq 0} \left(x(\tau) - \frac{c}{R-r} \right) = \arg \max_{\tau \geq 0} y(T; \tau)
\]

(48)

that maximized the terminal external wealth in Proposition 4.

Finally, let us work to free up the model a little by removing the restriction \( \rho \leq R \). Initially, this condition was imposed in order to guarantee that the agent does not default on his retirement loan. In practice, however, if \( \rho > R \) it can still be very possible to make payment, if the agent uses his current income and assets to fund the difference. Clearly, since the interest \( \rho \) is paid by the investor to his own self, the 457(b) loan interest does not constitute a permanent loss of
capital. On the other hand, said interest monies will be permanently marooned in the low-growth asset, which can be very costly over a long horizon. Thus, when $\rho > R$, we will treat the situation mathematically as an ongoing debt to the external world, at an interest rate of $R$. This is reasonable, because if the investor uses his salary or other assets in order to make payment in full, the difference will crowd out investment in an asset that compounds at an exponential rate of $R$ per year.

**Theorem 4** (Self-Usury Theorem). Assuming that the deficit can be made up using the participant’s salary (or other funding sources), the maximum acceptable interest rate $\bar{\rho}$ on a 457(b) loan is given by the formula\footnote{A very good minimax approximation to $\lambda^{-1}(\bullet)$, with negligible error $\epsilon(z)$, is given by $\lambda^{-1}(z) = (z - 0.19971) / (0.57167z + 0.40154) + \epsilon(z)$. This simple rational function can be used when computing the self-usury rate (49) for a $\phi = 5$ year loan term.}

$$
\bar{\rho} = \lambda^{-1} \left( \frac{\theta e^{RT} + (1 - \theta)e^{rT} - 1}{\theta (e^{RT} \lambda_r - e^{rT} \lambda_r)} \right) \geq R. \quad (49)
$$

As the event horizon $T \to \infty$, the maximum acceptable interest rate on a 457(b) loan converges to $R$.

**Proof.** If the cutoff rule (45) is satisfied as of right now ($\tau^* = 0$), then taking a plan loan at an interest rate of $\rho$ will generate aggregate terminal wealth (not counting external salary) in the amount of

$$
M(0) = (1 - \eta_r \theta) x_0 e^{rT} + \eta_R \theta x_0 e^{RT} + \frac{c}{r} \left( e^{rT} - 1 \right). \quad (50)
$$

Note that in (50), the outside balance $y(T;0)$ will be negative if $\rho > R$; this number represents the future value of the repayment monies that had to be funded from the agent’s salary. On the other hand, if we refuse to take a loan
over $[0, T]$, our final accumulated 457(b) balance would be $(x_0 + c/r)e^{rT} - c/r$. Thus, the total net profit $\pi$ on the loan (measured in terminal dollars) is given by

$$\pi = M(0) - \left\{ \left( x_0 + \frac{c}{r} \right) e^{rT} - \frac{c}{r} \right\}.$$  

(51)

After simplifying the zero-profit condition $\pi = 0$, and solving it for $\rho$, the result is (49). Finally, $R$ itself is obviously an acceptable rate on a 457(b) loan (so that the greatest such is therefore $\geq R$), since $\rho := R$ creates a situation with zero terminal external wealth, but a higher 457(b) balance at maturity (since the retirement loan outperformed the plan’s internal bond fund). The fact that $\rho \geq R$ can also be proved by direct manipulation of (49). ■

In order to calculate the investor’s overall XIRR (that is, the internal\(^5\) rate

\(^5\)Throughout the paper, we have used the terms “internal” and “external” in order to discuss the goings-on both within and without of the 457(b) plan. Here, we are instead referring to
Figure 17: The usurer’s overall XIRR for different external yields $R \in [5\%, 30\%]$, under the parameter values $(c, x_0, \rho, \theta, \phi, T, r) := ($19.5k, $0, 6\% a year, 50\%, 5 years, 30 years, 5\% a year). Say, if the external yield is 20\% per annum, then the de facto CAGR of the whole shebang is 10\% a year.

Of return of his aggregate capital in the model, we must solve the equation

$$
\left( x_0 + \frac{\beta}{c} \right) e^{\beta T} - \frac{\beta}{c} = \max_{\tau \in [0, T-\phi]} M(\tau)
$$

(52)

for the XIRR that will be achieved over $[0, T]$, here denoted by $\beta \in (r, R)$. Figure ?? plots the XIRR against the external yield $R$ in a practical example. The defining equation (52) has a unique solution because the function $\beta \mapsto (x_0 + \beta/c) \exp(\beta T) - \beta/c$ is strictly increasing over $[0, \infty)$. Putting $\beta = r$ corresponds to refusing to take a 457(b) loan; accordingly, it gives a value that is (by definition) less than the optimized terminal aggregate wealth. Similarly, the financial concept (cf. with Sengupta 2009; Jackson and Staunton 2006) called “internal rate of return,” or the investor’s “money-weighted rate of return,” that corresponds to the Excel worksheet function XIRR(•).
putting $\beta = R$ gives a value of the left-hand side that would have obtained under continuous growth of all contributions at the external rate, which is clearly greater than $\max_{0 \leq \tau \leq T - \phi} M(\tau)$. Since the investor’s (optimized) final aggregate balance is strictly increasing in $R$, any increase in the value of $R$ must be accompanied by a commensurate increase in $\beta$.

### 2.3 Stochastic Asset Prices.

In the prequel, we gave a complete solution of the problem of optimal control of a 457(b) loan in the event of a deterministic yield $R > r$ that is available outside of the retirement account. This being done, we proceed to generalize the existing framework in order to model the extra risk that must obviously be present in order to earn the higher (expected) yield. Accordingly, in this subsection the external asset price $S(t) = S_t$ will be assumed to follow a log-normal diffusion process (cf. with Wilmott 1998)

$$\frac{dS_t}{S_t} := Rdt + \sigma dW_t,$$

where $R > 0$ is the expected instantaneous yield $\mathbb{E}[dS_t/S_t]/dt$, but that yield is now subject to the continuous influence of a white noise term, $\sigma \times dW_t$, where $\sigma^2 = \operatorname{Var}(dS_t/S_t)/dt$ is the variance per unit time. As $\sigma \downarrow 0$ we will get the deterministic model that was developed in section 2.2. Here, $(W_t)_{t \geq 0}$ is a standard Brownian motion, and we make note of the covariance function

$$(s, t) \mapsto \mathbb{E}[W_sW_t] = \operatorname{Cov}[W_s, W_t] = \min(s, t).$$

The introduction of random, diffusive fluctuations of the external asset price
already introduces some complications into the model; accordingly, we will keep the goings-on inside of the retirement plan deterministic, at a risk-free annual yield of \( r \) per annum, as before. Thus, we will continue to use the same notation \( x(T; \tau) \) and formulas for the terminal 457(b) balance that were used in the prequel.

In the event that the investor ultimately fails to make payment on his loan at some future date (viz., because the random asset price \( S_t \) declined significantly over the loan term), then such payments will be made out of the agent’s income stream. As this diversion of income will effectively crowd out some of the agent’s investments in the risky asset, the vintage-\( t \) loan repayment of \( p(\tau)dt \) dollars will amount to a loss of \( p(\tau) \times (S_T/S_t) \) \( dt \) dollars of future value, where \( S_T/S_t \) is the capital growth factor that the risky asset will achieve over the interval \( [t, T] \). At any time during the interval \( [\tau, \tau + \phi] \), in the event that loan payments cannot be made in full, we will treat the repayment as a short sale (of the risky asset). Effectively, such shares (all \( p(\tau)dt/S_t \) of them) will have been borrowed from a third pool of stockholdings that are funded with the agent’s uncontributed salary.

Let us assume that the asset price \( S_t \) starts to drop precipitously in the immediate aftermath of the loan. Then, we are continuously liquidating shares (for a song) in order to make payment; eventually we reach the point where we must start to borrow shares. Thus, a negative terminal external value \( y(T; \tau) \) means that the investor wound up with a net short position in the risky asset at \( \tau + \phi \); if the aggregate wealth \( M(\tau) \) is itself negative, it means that the future value of that share debt was so great that it dwarfed the final 457(b) accumulation. Thus, in our model we will tolerate the (very remote) mathematical possibility that the aggregate terminal wealth is negative; the investor will
of course seek to avoid this fate through a judicious choice of \( \tau \).

Solving the stochastic differential equation (53), we have the standard price formula (cf. with Mikosch 1998; Black and Scholes 1973)

\[
S_t = S_0 \exp (\nu t + \sigma W_t),
\]

where \( \nu := R - \sigma^2/2 \) is the (geometric mean) growth rate of the risky asset, in contrast to its arithmetic mean, which is \( R = \nu + \sigma^2/2 \) (cf. with Björk 2009). In this connection, the final externalized wealth \( y(T) \) now consists in the formula

\[
y(T) = \theta(x(\tau) \frac{S_T}{S_T}) - p(\tau) \int_{\tau}^{\tau+\phi} \frac{S_T}{S_t} dt
\]

future value of time \( \tau \) investment future value of loan repayments

\[
= \theta(x(\tau) \left\{ \frac{S_T}{S_T} - \lambda \rho \int_{0}^{\phi} \frac{S_T}{S(\tau + u)} du \right\},
\]

where we have made the change of variable \( u := t - \tau \) in the definite integral \( S_T \int_{\tau}^{\tau+\phi} dt/S(t) \). Now, the capital growth factor \( S_T/S_t \) (that we are integrating with respect to \( t \)) is given by the expression

\[
\frac{S_T}{S_t} = \exp (\nu (T-t) + \sigma (W_T - W_t)),
\]

which depends on the behavior of the process \((W_T - W_t)_{t \geq 0}\). Accordingly, we have the expected value

\[
\mathbb{E} \left[ \frac{S_T}{S_t} \right] = e^{R(T-t)},
\]

so that the mean aggregate terminal wealth is given by the same formula \( M(\tau) \) that we used in the prequel, e.g., section 2.2 solved the problem of maximizing the aggregate terminal value of the investor’s accounts.
Now, the covariance function of the stochastic process \((W_T - W_t)_{t \geq 0}\) amounts to (cf. with Papoulis 1989)

\[
(s, t) \mapsto \text{Cov} [W_T - W_s, W_T - W_t] = T - \max(s, t).
\]

(59)

Next, we have the joint moment (cf. with Soong 1973)

\[
(s, t) \mapsto \mathbb{E} \left[ \frac{S_T^2}{S_s S_t} \right] = e^{v(2T-s-t)} \mathbb{E} \left[ \exp \left( \sigma \left(W_T - W_s + W_T - W_t\right)\right) \right]
= e^{v(2T-s-t)} \exp \left\{ \frac{\sigma^2}{2} \text{Var} \left[W_T - W_s + W_T - W_t\right] \right\}
= e^{v(2T-s-t)} \exp \left\{ \frac{\sigma^2}{2} (4T - s - t - 2 \max(s, t)) \right\}
= \exp \left( R(2T - s - t) + \sigma^2(T - s \vee t) \right),
\]

(60)

where \(s \vee t\) denotes \(\max(s, t)\). Using the joint moment (60) in conjunction with the expected growth factor (58), we finally arrive at our key formula for the covariance function \(\Gamma(s, t)\) of the process \((S_T/S_t)_{t \geq 0}\):

\[
\Gamma(s, t) := \text{Cov} \left[ \frac{S_T}{S_s}, \frac{S_T}{S_t} \right] = e^{R(2T-s-t)} \left( e^{\sigma^2(T-s \vee t)} - 1 \right).
\]

(61)

Thus, for instance, we have the variance \(\Gamma(t, t) = e^{2R(T-t)} \left( e^{\sigma^2(T-t)} - 1 \right)\).

With the covariance function (61) in hand, we proceed to calculate the variance of the investor’s terminal wealth, and study the properties of the resulting mean-variance frontier (cf. with Markowitz 1952; Markowitz 1959).\(^6\) We let

\(^6\)All of the randomness inherent in the participant’s final accumulation resides in the externalized term (56). The internal 457(b) dynamics are completely deterministic, as there is no risk of default. If the initial external investment \(\theta x(t)\) cannot self-finance the loan repayments, then the (share-equivalent) difference will be paid out of the agent’s uncontributed salary.
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Figure 18: The autocorrelation function \((s, t) \mapsto \Gamma(s, t)/\sqrt{\Gamma(s, s)\Gamma(t, t)}\) of the stochastic process \((S_T/S_t)_{t \geq 0}\). For the parameters \((T, \sigma, R) := (30 \text{ years}, 40\% \text{ a year}, 15\% \text{ a year})\)

\(V(\tau)\) denote the variance that is created by a $1 loan from the defined contribution plan. That is,

\[
V(\tau) := \text{Var} \left[ \frac{S_T}{S_\tau} - \lambda_\rho \int_0^\phi \frac{S_T}{S(\tau + u)} du \right]
\]

\[
= \text{Var} \left[ \frac{S_T}{S_\tau} \right] - 2\lambda_\rho \text{Cov} \left( \frac{S_T}{S_\tau}, \int_0^\phi \frac{S_T}{S(\tau + u)} du \right) + \lambda_\rho^2 \text{Var} \left[ \int_0^\phi \frac{S_T}{S(\tau + u)} du \right]
\]

\[
= \Gamma(\tau, \tau) - 2\lambda_\rho \int_0^\phi \Gamma(\tau, \tau + u) du + \lambda_\rho^2 \int_0^\phi \int_0^\phi \Gamma(\tau + u, \tau + v) du dv. \quad (62)
\]

We must dispose of the integrals in (62). First, we have

\[
\int_0^\phi \Gamma(\tau, \tau + u) du = e^{2R(T-\tau)} \left\{ e^{\sigma^2(T-\tau)} \int_0^\phi e^{-(R+\sigma^2)u} du - \int_0^\phi e^{-Ru} du \right\}
\]

\[
= e^{2R(T-\tau)} \left\{ e^{\sigma^2(T-\tau)} \lambda_{R+\sigma^2}^{-1} - \lambda_{R}^{-1} \right\}. \quad (63)
\]
Next, we compute the double integral (cf. with Hoel and Stone 1972)

\[
\int_0^\phi \int_0^\phi \Gamma(\tau + u, \tau + v) \, du \, dv
\]

\[
= e^{2R(T-\tau)} \left\{ e^{\sigma^2(T-\tau)} \int_0^\phi \int_0^\phi e^{-R(u+v)-\sigma^2\max(u,v)} \, du \, dv - \int_0^\phi \int_0^\phi e^{-R(u+v)} \, du \, dv \right\}
\]

\[
= e^{2R(T-\tau)} \left\{ 2e^{\sigma^2(T-\tau)} \int_0^\phi \int_0^\phi e^{-R(u+v)-\sigma^2u} \, du \, dv - \lambda_R^{-2} \right\}
\]

\[
= e^{2R(T-\tau)} \left\{ \frac{2}{R} e^{\sigma^2(T-\tau)} \left( \lambda_R^{-1} - \lambda_R^{-2} \right) - \lambda_R^{-2} \right\}.
\]

Combining (63) and (64), and simplifying, we have the following formula for the terminal variance that is created by a $1 loan from the retirement plan:

\[
V(\tau) = e^{2R(T-\tau)} \left( \psi e^{\sigma^2(T-\tau)} - \eta_R^2 \right),
\]

where

\[
\psi := 2\eta_{R+\sigma^2} - 1 + \frac{2\lambda_\rho}{R} \left( \eta_{2R+\sigma^2} - \eta_{R+\sigma^2} \right),
\]

\[
\eta(z) := 1 - \lambda(\rho)/\lambda(z) \text{ and } \lambda(z) := z/(1 - e^{-\phi z}).
\]

**Theorem 5** (Cutoff Rule for Mean-Variance Investors). If the current plan balance $x_0$ lies in the interval $[x^*_V, x^*_M]$, where

\[
x^*_V := c \left( R - r + \frac{\sigma^2}{2} \left( 1 - \frac{\eta_R^2}{\psi} e^{-\sigma^2T} \right)^{-1} \right)^{-1}
\]

and

\[
x^*_M := \frac{c}{R - r} \left( 1 - \frac{\eta_r}{\eta_R} e^{-(R-r)T} \right),
\]

then every mean-variance investor should wait to take a loan, since the mean of
Figure 19: The mean $(= M(\tau))$ and the standard deviation $(= \theta x(\tau) \sqrt{V(\tau)})$ of terminal wealth for different waiting times $\tau \in [0, 25]$, under the parameters $(\phi, \rho, R, r, T, \theta, \sigma, x_0, c) := (5\% , 5\% , 15\% , 4\% , 30\% , 50\% , 40\% , \$0, \$19.5k)$. The worst possible terminal variance occurs for $\tau^*_V = 4.77$ years, and the maximum terminal mean occurs for $\tau^*_M = 7.82$ years. Thus, any $\tau \in [\tau^*_V, \tau^*_M]$ is dominated by $\tau^*_M$: waiting until $\tau^*_M$ guarantees both a higher terminal mean and a lower terminal variance.

terminal wealth is presently increasing in $\tau$ and the variance of terminal wealth is decreasing in $\tau$. As $T \to \infty$, we get the horizon-free wait region

$$\left[ \frac{c}{R - r + \sigma^2/2}, \frac{c}{R - r} \right],$$

(69)

which is independent of the loan parameters $\phi$, $\rho$, and $\theta$.

Proof. We take up the first order condition

$$\frac{\partial}{\partial \tau} \left( x(\tau)^2 V(\tau) \right) \bigg|_{\tau = 0} \leq 0,$$

(70)

which would imply that the variance of terminal wealth is decreasing in the
Figure 20: The feasible (risk, reward) pairs, given by the parametric curve \( \tau \mapsto c^{-1}\left(\theta x(\tau)\sqrt{V(\tau)}, M(\tau)\right) \) for \( 0 \leq \tau \leq T - \phi \). The scale factor \( 1/c \) puts the terminal mean and standard deviation in units of contribution-years, i.e., “80” means an amount of money equal to 80 years of contributions. This plot uses the parameters \((\theta, \phi, T, \rho, r, R, \sigma, x_0) := (50\%, 5y, 30y, 6\%, 5\%, 15\%, 40\%, \$0)\).

locale of \( \tau = 0 \). This condition amounts to

\[
x_0 [V'(0) + 2rV(0)] \leq -2cV(0).
\]

(71)

Now, when we divide (71) through by the (negative) factor \([V'(0) + 2rV(0)]\), the sense of that inequality will be reversed. The negativity obtains from the inequalities

\[
V'(0) + 2rV(0) \leq V'(0) + 2RV(0) = -\sigma^2 \psi e^{(2R + \sigma^2)(T-\tau)} < 0,
\]

(72)

since \( \psi > 0 \). That is, \( \psi = e^{-\sigma^2(T-\tau)} \left(e^{-2R(T-\tau)}V(\tau) + \eta_R^2\right) > 0 \) since \( V(\tau) \geq 0 \) for all \( \tau \), since \( V(\bullet) \) is a variance. Thus, evaluating the logarithmic derivative
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Figure 21: The interval \([\tau_V^\ast, \tau_M^\ast]\) for different expected external yields \(R \in [5\%, 30\%]\). Here, we used the parameters \((\phi, \rho, r, \sigma, c, x_0, T) := (5y, 5\%, 4\%, 55\%, \$19.5k, \$0, 30y)\). The waiting times \(\tau \in [\tau_V^\ast, \tau_M^\ast]\) in this interval are dominated for all mean-variance investors, since the terminal mean is increasing in \(\tau\) and the terminal variance is decreasing in \(\tau\).

\[
d (\log V(\tau)) / d\tau \text{ at } \tau = 0, \text{ we have}
\]

\[
x_0 \geq -c \left( r + \frac{V'(0)}{2V(0)} \right)^{-1} = c \left( R - r + \frac{\sigma^2}{2} \left( 1 - \frac{\eta^2 R}{\psi \sigma^2} \right) e^{-\sigma^2 T} \right)^{-1},
\]

as promised. Letting \(T \to \infty\) yields the horizon-free result. As to second order conditions, the terminal variance of the investor’s wealth is log-concave, i.e., we have \(2 \log(x(\tau)) + \log(V(\tau))\), where \(x(\bullet)\) and \(V(\bullet)\) are both log-concave. Thus, if the first order condition (70) is satisfied, then the terminal variance will forever be decreasing in the waiting time, for all \(\tau \in [0, T - \phi]\).
3 Conclusions.

This paper formulated and solved an optimal stopping problem for retirement plan loans taken from 401(k), 403(b), and 457(b) accounts in the United States. A 457(b) loan allows the participant to borrow (over a five year term) from his own self at a rate of interest that is specified by the plan custodian, say, an annual percentage rate (APR) of 4.25%. The account owner repays himself with level (monthly) payments, which in this paper are made continuously (e.g., in continuous time, rather than at discrete dates) for the sake of mathematical elegance and simplicity. Tax-advantaged retirement accounts that permit such loans typically allow their participants to borrow up to 50% of the account balance, up to a maximum loan of $50,000. The resulting debt covenant has a weighted average life of 30 months and a Macaulay duration of 25-30 months, depending on the particular discount rate that is chosen.

Tax advantaged retirement accounts such as the 457(b) have a large number of attractive features from the standpoint of estate, tax, and retirement planning. These accounts allow for participants to make tax deferred payroll contributions that are not counted toward current (taxable) income, so that the account functions similarly to a traditional IRA, albeit one with a comparatively high contribution limit ($19,500 a year for individuals under 50, with an additional $6,500 in “catch-up” contributions permitted for workers who are 50 years of age or older, for a combined limit of $26,000). Employees and plan participants who expect to be in a lower tax bracket in retirement than they are at present can reap significant tax benefits from such “tIRA” style contributions; in the present paper, contribution flow is assumed to happen at a continuous rate (rather than monthly), so that the cumulative deposits are described by a
linear function of time. Many U.S. states (i.e., Illinois) do not levy state income taxes on distributions from such retirement accounts.

In the United States, salaried workers are subject to powerful deposit incentives for 401(k), 403(b), and 457(b) accounts, including employer match policies. These accounts are also useful from the standpoint of asset protection, as retirement accounts are immune from seizure in divorce proceedings, or, say, by debt collectors. The option to convert existing “traditional” money into “Roth” funds at any time (subject to the proviso that every converted dollar counts toward current income) can also be a valuable benefit: whenever a worker is unemployed or underemployed, he or she can make a Roth conversion in that year and declare the income, temporarily shielded by a lower tax bracket.

The ability to borrow from tax-advantaged retirement plans can be an important, convenient, and reliable source of liquidity during trying times, and it can be a crucial source of (lump sum) funds for individuals who need to make large purchases, say, of homes or automobiles. One drawback of such accounts, which is what we studied in this paper, may be a lack of high-growth (and correspondingly high risk) investments that are suitable for younger, more sophisticated, or longer term investors who have a high risk tolerance. Ostensibly for the participant’s own protection, the investment offerings in 401(k), 403(b), and 457(b) plans can be severely restricted, say, to a list of index or target date funds containing broadly diversified portfolios of stocks and bonds. The present work studies the problem of an investor who believes that he or she can earn a significantly higher rate of return outside of the plan, albeit at the cost of significant volatility risk. What is the correct time for such an individual to take a retirement loan? On the one hand, we should clearly wait a while in order to take a loan that is large enough in order to make a substantial (external) profit.
On the other hand, the longer that we wait for contributions to accumulate in the 457(b), the longer that we have failed to share in the exponential capital growth of the high-return, external asset.

We started with a simplified, baseline model of the situation that assumes that there is zero interest available within the confines of the retirement plan; if a loan is taken, the agent will repay himself via a single, lump sum (balloon) payment at the end of the planning horizon. In this pleasant environment, which already has many of the features of the general case, we found that there is a unique optimal waiting time, \( \tau^* \), that optimizes the investor’s aggregate terminal wealth. We gave a simple and pleasant formula for \( \tau^* \) in terms of Lambert’s W function, also known as the “product logarithm.” We found that the optimal policy is independent of the fact that we are restricted to borrowing only \( \theta = 50\% \) of the available monies in the plan; whatever the value of the parameter \( \theta \), the correct time to pull the trigger is the same, and at such time we should borrow every available dollar.

We found that the optimal control is most conveniently expressed in terms of a cutoff rule for the accumulated balance in the retirement account: once the cutoff is breached, whatever be the time, the participant should immediately take a full loan. In general, the optimal stopping rule must be tailored to the specific investment horizon of the participant, e.g., he must have a date certain at which he will liquidate his retirement account and add up his winnings. However, we found that the correct behavior (for everybody) is very well approximated by letting the horizon tend to infinity. Thus, we found the horizon-free rule of thumb: once you have accumulated an amount of money (in contribution-years) that exceeds \( 1/R \), where \( R \) is the external rate of return, then you should borrow. Say, if I believe that I can make \( R := 25\% \) a year in
some (very risky) biotech or semiconductor stocks that exist outside of the account, then I should take a loan once I accumulate an amount of money that is equivalent to \(\frac{1}{0.25} = 4\) years worth of contributions.

According to this model, whose tractability is well worth the slight lack of realism, most investors (barring those who believe in very spectacular rates of external return) should optimally prefer to wait for an accumulated balance that is north of $100,000. However, the loan size restrictions that are in effect (50% of the balance up to a maximum of $50,000) mean that there is no point in waiting once we have accumulated $100,000 in the 457(b) plan; an individual who is contributing the maximum will breach this cutoff in 5 years and 2 months. Thus, such restrictions prove severe (and binding) for people who regularly max out their retirement contributions. On the other hand, it means that workers whose contributions are more modest should wait for a very long time before taking loans from their retirement plans. Say, if I contribute $7,000 a year, and I think that I can earn 10% a year outside of the plan, then I should (approximately) wait until I accumulate ten contribution-years, or $70,000, in the account.

Buoyed by the elegance and success of the baseline model, we proceeded to build a fully detailed (deterministic) model of optimal loan control in continuous time. The more general model assumes an internal interest rate of \(r < R\) inside of the retirement plan, and it requires level, continuous loan repayments over the \(\phi\)-year loan term.

For the sake of parsimony, we identified a few key parameters that play a pivotal role in the structure of the optimal cutoff rule. At the time that the loan is made, a certain percentage (that we called \(\eta_R\)) of the loan money must be immediately set aside in order to match the liability that is represented by the
repayment stream. This money is effectively invested in a sinking fund (at the higher external rate \( R \)) that is exactly sufficient to fund the level payments back into the 457(b) plan. The remaining percentage \( 1 - \eta_R \) of the initial principal is therefore “permanently externalized,” compounding itself at \( R \) per year until the liquidation date is reached. In this (more fully realistic) environment, we again found that the optimal policy has the “cutoff” form, which we derived in earnest.

Taking \( T \to \infty \) in the horizon-\( T \) policy, we found the lovely rule of thumb: younger participants should only take a loan when they accumulate \( 1/(R - r) \) contribution-years worth of money in the account. Say, if I think that my external investments can outperform by 10% a year, then I will borrow when I hit a retirement balance of 10 contribution years. Note well that, due to the compound interest that is available inside of the plan, this achievement will require less than 10 calendar years. For instance, if I can compound at 5% internally, then I will hit the horizon-free cutoff in 8.1 calendar years; the horizon-\( T \) cutoff (if I know \( T \)) will be breached a little bit earlier. Thus, on a long enough horizon, the particular details of the loan covenant (interest rate, term, percentage of plan balance, etc.) are irrelevant to the optimal loan timing. In order to behave perfectly vis-á-vis a particular horizon, they are relevant, of course, but the arithmetical difference from the case \( T = \infty \) is quite small, as we demonstrated in practical examples. The intuitive reason for this phenomenon is that the interest payments on the 457(b) loan are not irretrievably lost, since the participant is just moving money from one of his pockets into another. Rather, the drawback of the loan interest is that such monies will be marooned in the low-growth asset until the end of the planning period. Accordingly, we derived a practical formula for the greatest possible interest rate that the participant
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is willing to pay himself on a plan loan; we found realistic examples where this “self usury rate” is 30% or more.

Finally, we built a stochastic loan model whereby the external asset price follows a log-normal random walk, or diffusion process (cf. with Cootner 1964; Malkiel 1999) whose expected rate of return is $R$ per year and whose annual volatility, or diffusion parameter, is $\sigma > 0$. In this environment, the expected terminal aggregate wealth corresponds exactly to the formulas that were derived and optimized in the prequel; however, the random nature of the returns introduces significant (and painful) variance into the investor’s final bankroll. We derived (through so many integrations) exact formulas for the terminal variance and studied the quantitative nature of the mean-variance frontier.

In this connection, our cutoff rule takes the following form: there exist a pair of critical wealth values, $x_V^*$ and $x_M^*$, at which peak variance and peak mean are achieved, respectively. Once we hit an account balance of $x_V^*$, all mean-variance investors should wait, since we are in a region where the terminal variance is decreasing and the terminal mean is increasing. Once we exit this region (viz., at peak mean), the exact stopping time becomes a matter of personal preference, depending on the precise quantitative trade-off that the individual is willing to make between risk and return. At such time, we are comfortably on the efficient frontier: the terminal mean is decreasing but the terminal variance is also decreasing. Thus, waiting too long to borrow is no great cause for concern, but borrowing too early is wrong-headed and can cause a world of pain. On a long horizon, the critical value $x_V^*$ converges to $1/(R - r + \sigma^2/2)$ contribution years. Say, if we expect the external asset to outperform by 10% a year, but its volatility is 60% a year, then peak terminal variance is achieved at just 3.6 contribution years. That is a very, very bad time to take a loan, and the participant should
wait until he has accumulated at least 10 contribution years worth of money in the account, maybe more, according to his risk preference. Somebody who is sufficiently intolerant of risk could rationally choose to take the loan at the last possible moment (exactly 5 years before liquidation of the 457(b)), or not at all. According to our model here, then, the only rational time to take a partial loan (less than the plan maximum) is when the participant has already waited until $\tau = T - 5$, at which time a partial loan is the only remaining way to achieve a risk reduction. A partial loan that is taken at any other time will constitute dominated behavior in the mean-variance plane, as we showed above. Good luck with that.

Delphi, Greece, Summer 2021.

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