Abstract

We discuss a novel manifestation of the $SU(2)$ global anomaly in an $SU(2)$ gauge theory with an odd number of chiral quark doublets and arbitrary Yukawa couplings. We argue that the massive 4-dim. ($D = 4$) Euclidean Dirac operator is nonhermitean with its spectrum of eigenvalues $(\lambda, -\lambda)$ lying in pairs in the complex plane. Consequently the existence of an odd number of normalizable zero modes of the 5-dim. ($D = 5$) massive Dirac operator is equivalent to a fermionic level exchange phenomenon, level “circling”, under continuous topologically nontrivial deformations of the external gauge field. More generally global anomalies are a manifestation of fermionic level “circling” in any $SP(2n)$ gauge theory with an odd number of massive fermions in the spinor representation and arbitrary Yukawa couplings.
1 Introduction

The phenomenon of fermion level crossing and its relation to the existence of normalizable fermionic zero modes of the Dirac operator in the presence of topologically non-trivial gauge fields is well known [1]. It has been extensively studied and firmly established for the case of massless fermions.

The $SU(2)$ global anomaly [2] is also related to the phenomenon of level crossing of the $D = 4$ Dirac operator. This eigenvalue flow corresponds to the existence of a zero mode for an appropriately defined $D = 5$ Dirac operator in the presence of an external topologically non-trivial gauge field. As a result the fermionic path integral is not gauge invariant and the theory is not self-consistent.

In the realistic electro-weak theory fermions become massive due to the Higgs mechanism and the presence of Yukawa interactions with the Higgs field being a non-vanishing constant at spatial infinity. At first appears surprising the occurrence of level crossing in the spectrum of massive fermions.

We, however, offer an intuitive argument for its existence in this case. Let us first consider the $D = 5$ Dirac operator in an external topologically nontrivial gauge field. Such a configuration is topologically guaranteed by the fact that $\pi^4(SU(2)) = Z_2$ [3]. In our case where Yukawa interactions are present it is tacitly taken that Higgs fields are suitably included as external fields. As a consequence of our construction we take at $\mathbb{R}^5$ infinity our gauge fields to be a pure gauge $A_\mu \propto U^{-1}\partial_\mu U$ with $M \propto U$ being the fermion mass matrix. Here $U$ is a unitary matrix, a noncontractible map from $S^4$ into $SU(2)$ on a sphere at $\mathbb{R}^5$ infinity. It means that one can continuously deform it to the identity everywhere on the sphere except for the region near a single point where $U \neq 1$ (a singularity). As the radius $R$ of the sphere takes value from infinity to zero we form a line in the neighbourhood of which the topological non-triviality is concentrated and the external field changes very rapidly. Therefore at $R = 0$ we should have either zero or singularity. By continuity of the mass matrix the latter is impossible, and hence we should expect the existence of zero in $M$ with level crossing to occur.

We have previously studied [4] the effect of Yukawa interactions for the level crossing phenomenon in the case of Witten’s global anomaly. However our analysis was restricted to the symmetric case of equal up and down quark masses in an $SU(2)$ doublet. Herein we present an analysis for the more general and realistic case of arbitrary Yukawa couplings and nondegenerate up and down quark masses. The main difference with the symmetric case we find to be in the nonexistence of an appropriate hermitean generalization for the $D = 4$ and $D = 5$ Dirac operators. They in fact turn out to be nonhermitean with their spectra of eigenvalues complex. In the present work we formulate generalization of the zero mode index theorems and demonstrate the $SU(2)$ global anomaly to be a manifestation of a generalization of the fermionic level crossing phenomenon. Under continuous topologically nontrivial deformations of the $SU(2)$ external gauge field the fermionic eigenlevels of the $D = 4$ Dirac operators lying in the complex plane exchange their sign with one another $(\lambda, -\lambda)$ pairwise by ‘circling’ around the origin $\lambda = 0$. The level circling is shown.
to be a result of the existence of an odd number of normalizable zero modes of the $D = 5$ massive Dirac operator. Moreover our arguments can be shown to trivially generalize to the case of $SP(2n)$ groups.

2 Global Anomaly for Massive Fermions

It is well known that $SP(2n)$ gauge theories have no local anomalies. Moreover when coupled with an odd number of Weyl fermions they possess a global anomaly and become self-inconsistent. The simplest such example is an $SU(2)$ gauge theory with one fermionic Weyl doublet. This model was previously considered in some detail for the case of massless fermions [4]. In this section we extent Witten’s arguments in order to take into account the presence of Yukawa interactions. We will show that the Atiyah-Singer index theorem mod 2 is sufficient for the presence of the anomaly as a consequence of the existence of a massive normalizable zero mode of the $D = 5$ Dirac operator.

Let us first sketch Witten’s arguments [4]. The partition function of the euclidean version of the model of a doublet of massless fermions coupled to an $SU(2)$ gauge field reads as follows

$$Z = \int D\psi_L D\psi^+_L \int DA_\mu \exp(-\int d^4x [(1/2g^2) Tr F^2_{\mu\nu} + \psi^+_L i\nabla_L \psi_L ]).$$  \hspace{1cm} (2.1)$$

There $A_\mu$ is an $SU(2)$ gauge field, $\psi_L$ is a left-handed Weyl fermion doublet, $g$ is the gauge coupling constant, $\nabla = \nabla_\mu \gamma_\mu$ is a Dirac operator restricted to act on a Weyl doublet. The fermionic part of the integral eq.(2.1) is ill defined. However it can be formally integrated as the square root of a functional integral over Dirac fermions. As such it implies the doubling of the fermionic degrees of freedom from one to two Weyl left-handed doublets. Because the 1/2 representation of $SU(2)$ is pseudoreal a left-handed doublet can be mapped to the right-handed one. A theory with two left-handed Weyl doublets is thus equivalent to a vector-like one with a single Dirac doublet. The fermionic functional integral is given by $\det (i\nabla)$ and it is well defined. Then we formally have that

$$\int D\psi_L D\psi^+_L \exp \int \psi^+_L i\nabla_L \psi_L = (\det i\nabla)^{1/2}. \hspace{1cm} (2.2)$$

The sign of the square root is ill defined. As a way out Witten defines the root in eq.(2.2) as the product of all positive eigenvalues of a Dirac operator as a start and then continuous analytically.

If for a given $A_\mu$ the sign in eq.(4.2) is arbitrarily fixed as Witten showed there always exists a configuration $A^U_\mu$ that can be reached continuously from $A_\mu$ for which the fermionic determinant has an opposite sign, i.e.

$$(det i\nabla(A_\mu))^{1/2} = -(det i\nabla(A^U_\mu))^{1/2}. \hspace{1cm} (2.3)$$

Here $A_\mu$ is taken to be the gauge transformed configuration of $A_\mu$. This means that the partition function

$$Z = \int DA_\mu (det i\nabla)^{1/2} \exp(-1/2g^2 \int d^4x Tr F^2_{\mu\nu})$$ \hspace{1cm} (2.4)$$
vanishes due to the contribution with an opposite sign from $A_\mu$ and $A^U_\mu$. This comes about because as we continuously vary the external field value from $A_\mu$ to $A^U_\mu$ an odd number of these eigenvalues flow through zero switching their sign. Such a level crossing effect is a reflection of the existence of an odd number of normalizable zero modes for a properly defined $D = 5$ Dirac operator in a topologically nontrivial gauge field. It can happen because of the nontrivial fourth homotopy group of $SU(2)$

$$\pi^4(SU(2)) = \mathbb{Z}_2. \tag{2.5}$$

The $D=5$ topologically non-trivial gauge field must accordingly belong to the nontrivial homotopy class in $\mathbb{Z}_2$ and it is the interpolating configuration between $A_\mu$ and $A^U_\mu$. An appropriate $D = 5$ Dirac operator is actually an extension of the $D = 4$ one that includes evolution in the fifth coordinate $t$. In what follows we generalize Witten’s argument for the case of Yukawa interactions and the presence of fermion mass through the Higgs mechanism. In this context we will prove Witten’s conjecture of the validity of his arguments for this case too. It is worthwhile to also stress that the $SU(2)$ global anomaly can also be understood as a manifestation of the existence of a local anomaly for an $SU(3)$ gauge theory \cite{5}. From this point of view the generalization to the case of massive fermions is of course straightforward. This is due to the fact that the local anomaly is independent of Yukawa couplings which are $SU(3)$ invariant \cite{5}. It seems nevertheless interesting to try to understand the issue in terms of a level crossing phenomenon for the $D = 4$ Dirac operator. This is the aim of what is to follow. We do it by proving a generalization of the index theorem mod 2 for the massive $D = 5$ Dirac operator.

We first consider the index theorem mod 2 for a massless $D = 5$ Dirac operator. Instead of making a unitary transformation to a real representation for the $D = 5$ fermions which transform under the $O(4) \times SU(2)$ group \cite{4} we introduce

$$\mathcal{D} = (\gamma_5 \nabla_t + \nabla). \tag{2.6}$$

Here $\nabla_\mu$ is the usual covariant derivative, $\nabla = \nabla_\mu \gamma_\mu$, whereas $\nabla_t$ is the covariant derivative for the fifth coordinate. Eigenvalues and eigenfunctions are defined by the following equation

$$\mathcal{D} \psi = i\lambda \psi, \tag{2.7}$$

where $\lambda$’s are real since $\mathcal{D}$ is antihermitean. The operator $\mathcal{D}$ is real and antisymmetric in the following sense

$$C\epsilon\gamma_5\mathcal{D}C\epsilon\gamma_5 = \mathcal{D}, \tag{2.8}$$

$$C\epsilon\gamma_5\mathcal{D}^T C\epsilon\gamma_5 = -\mathcal{D}. \tag{2.9}$$

Here $C = i\gamma_2\gamma_0$ and $C\gamma_5$ are the usual $D = 4$ and $D = 5$ charge conjugation matrices, $\epsilon$ is a $2 \times 2$ antisymmetric matrix that acts on $SU(2)$ indices. The reality condition implies the pairing up of all the non-zero eigenvalues. More precisely if $\mathcal{D} \psi = i\lambda \psi$ then

$$\mathcal{D}(C\epsilon\gamma_5 \psi^*) = -i\lambda C\epsilon\gamma_5 \psi^*. \tag{2.10}$$

The zero mode wave functions $\psi_0$ (if they exist) can be chosen real satisfying a Majorana-like condition:

$$C\epsilon\gamma_5 \psi_0^* = \psi_0. \tag{2.11}$$
This reality condition means that the number of zero modes is a topological invariant modulo 2 since non-zero modes can cross the zero level only in pairs when external fields vary smoothly. The crucial point is that such a zero mode does exist in the external field due to the index theorem modulo 2 if the external field is topologically nontrivial. The reality of the Dirac operator $D$ guarantees a pairing of its eigenvalues. They correspond to the existence of the nontrivial homotopy class that arises in the nontrivial topology $\pi^4(SU(2)) = Z_2$.

Let us emphasize here that the generalization of the above considerations to the case of $SP(2n)$ gauge groups is straightforward. This is because the spinor representation of $SP(2n)$ group is pseudoreal while the corresponding representation of $O(5) \times SP(2n)$ group is real. Actually all the formulas in this paper are valid for these groups too with a change of the isotopic antisymmetric tensor into a matrix $s$ of charge conjugation of the $SP(2n)$ group which is given by a $2n \times 2n$ antisymmetric matrix with $\epsilon$ blocks on the diagonal while the rest of the matrix elements being zero. A generator $G$ of an $SP(2n)$ algebra in the spinor representation obeys the condition

$$sG^T s = G \quad s^2 = -1.$$  \hspace{1cm} (2.12)

where the index $T$ means a transposition.

We now proceed to examine massive fermions in an external field. In this model we have a left handed chiral fermion $SU(2)$ doublet $q_L$ and a pair of right handed singlet fermions which can be combined into a doublet $q_R$. This is necessary for the introduction of fermionic masses. The problem at hand now is how to generalize the definition of the chiral fermionic determinant for the massive case. We find it convenient and natural to define it as a square root of the fermionic functional integral for two fermionic $(q^i_L, q^i_R), \ i = 1, 2,$ multiplets with the same mass matrix $M$. We follow, to this end, the recipe of ref. [4]: we introduce a copy of the fermion system coupled to the same external gauge field and make a vectorization of the model to get a generalized Dirac operator.

Let us consider a left-handed fermion doublet coupled to the $SU(2)$ gauge field and its copy

$$q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L, \quad q'_L = \begin{pmatrix} u' \\ d' \end{pmatrix}_L,$$  \hspace{1cm} (2.13)

with associated weak singlet right-handed states as $u_R, d_R, u'_R, d'_R$.

When the weak doublets are considered in pairs it is possible to pass to the pure vector interaction of fermions with $W$ bosons. To achieve this we introduce instead of fermion fields $q'$ the charge-conjugated fields:

$$q'_R = \epsilon C q'_L = \begin{pmatrix} C \bar{d}'_L \\ -C \bar{u}'_L \end{pmatrix}, \quad q'_L = \epsilon C q'_R = \begin{pmatrix} C \bar{d}'_R \\ -C \bar{u}'_R \end{pmatrix}.$$  \hspace{1cm} (2.14)

Here $\epsilon = i\sigma_2$ acts on the isotopic indices, $C$ is the charge-conjugation matrix. By introducing

$$\psi = \psi_L + \psi_R, \quad \psi_L = q_L, \quad \psi_R = C \bar{q}'_L,$$

$$\eta = \eta_R + \eta_L, \quad \eta_R = q_R, \quad \eta_L = C \bar{q}'_R,$$  \hspace{1cm} (2.15)
it becomes obvious that both components of \( \psi, \psi_L \) and \( \psi_R \), are weak doublets while \( \eta_R \) and \( \eta_L \) are singlets. Therefore only the \( \psi \) field has a vector gauge interaction:

\[
L_W = i\bar{\psi} \nabla \psi + i\bar{\eta} \nabla \eta, \quad \nabla = \gamma_\mu \partial_\mu = \gamma_\mu (\partial_\mu - iW_\mu). \tag{2.16}
\]

Clearly the mixing of the quark and antiquark fields in eq.(2.15) is very unnatural with respect to colour and electric charge (weak hypercharge). As those interactions are irrelevant to our problem at hand from now on we only keep the Yukawa couplings:

\[
- L_Y = h_u \bar{q}_L \varepsilon_{ij} u_R \varphi^i_j + h_d \bar{q}_L d_R \varphi^i + H.c. + (u, d, h_u, h_d \rightarrow u', d', h'_u, h'_d). \tag{2.17}
\]

We give here the Yukawa terms for different mass matrices for two fermionic doublets. This is convenient for explanation of properties of \( D = 5 \) Dirac operator which will be introduced below. Actually for a discussion of the global anomaly we should put \( M = M' \). Here the Higgs field is \( \varphi^i = (\varphi^+, \varphi^0) \) and the fermion masses are given by \( m_u = h_u v / \sqrt{2}, m_d = h_d v / \sqrt{2}, m'_{u} = h'_u v / \sqrt{2}, m'_{d} = h'_d v / \sqrt{2} \).

By using the fields \( \psi \) and \( \eta \) of eq.(2.5) one can rewrite the Lagrangian (2.7) as follows:

\[
- L_Y = \bar{\psi}_L M \eta_R + \bar{\eta}_R M^+ \psi_L - \bar{\psi}_R \varepsilon M^T \varepsilon \eta - \bar{\eta}_L \varepsilon M^T \varepsilon \psi_R, \tag{2.18}
\]

where the mass matrix \( M(x) \) is given by

\[
M(x) = \begin{pmatrix}
h_u \varphi^0(x)^*, & h_d \varphi^+(x) \\
-h_u \varphi^+(x)^*, & h_d \varphi^0(x)
\end{pmatrix}. \tag{2.19}
\]

\( M' \) is taken from \( M \) by substituting \( h_u, h_d \rightarrow h'_u, h'_d \).

We now make a Euclidean rotation upon which the fermion fields rotate into

\[
\psi, \eta \rightarrow \psi', \eta', \quad \bar{\psi}, \bar{\eta} \rightarrow -i\psi^+, -i\eta^+. \tag{2.20}
\]

The mass terms in \( L_Y \) contain now the bilinear combinations of the fields with the same chirality. The Lagrangian reads now as follows

\[
L = L_W + L_Y, \tag{2.21}
\]

\[
L_W = -i\psi^+ \nabla \psi - i\eta^+ \bar{\partial} \eta, \tag{2.22}
\]

\[
L_Y = -i\psi^+_R M \eta_R - i\eta^+_L M^+ \psi_L + i\psi^+_L \varepsilon M^T \varepsilon \eta + i\eta^+_R \varepsilon M^T \varepsilon \psi_R. \tag{2.23}
\]

We combine them into the Dirac spinors \( \psi \) and \( \eta \) whereas \( \psi \) is an \( SU(2) \) doublet and \( \eta \) is a couple of \( SU(2) \) singlet spinors. As a result we get the operator in the fermionic kinetic term as before for the particular case where \( M = M' \). This operator acts in the space of pairs \((\psi, \eta)\) and reads as

\[
T(M) = \begin{pmatrix}
\nabla & \bar{\psi}^+ M^+ L - \epsilon M^T \epsilon L \\
M^+ L - \epsilon M^T \epsilon R & \bar{\eta}^+ \nabla
\end{pmatrix}, \tag{2.23}
\]

where \( L(R) = (1 + (-)\gamma_5)/2 \).
In refs. [4] we restricted ourselves to the case of equal fermion masses ($h_u = h_d$). In this case $M = -\epsilon M^* \epsilon$ with $M$ being proportional to an element of the $SU(2)$ group. This assumption was necessary with regard to a definition of an appropriate chiral fermionic determinant and a $D = 5$ antihermitian euclidean Dirac operator. In this present paper we consider the case of an arbitrary mass matrix $M$. It is easy to see that the Dirac operator $T(M)$ has in general complex eigenvalues. Hence when we continuously change the external gauge and Higgs fields the levels move along the complex plane. The non-zero eigenvalues of the $D = 4$ Dirac operator are paired because of anti-commuting of $\Gamma_5$ with $T(M)$, where

$$\Gamma_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}.$$ 

Indeed for each non-zero eigenvalue $\lambda$ with the eigenfunction $\psi_\lambda$ the function $\Gamma_5 \psi_\lambda$ corresponds to the eigenvalue $-\lambda$. As a result of such a pairing there exists an index theorem for the operator $T(M)$

$$n_{\text{Re} \lambda = 0} = \text{topological invariant}, \quad (2.24)$$

where $\alpha$ is an arbitrary non-zero complex number and $n_{\text{Re} \lambda = 0}$ is a number of corresponding eigenvalues, and also

$$n_{\lambda = 0} = \text{topological invariant}. \quad (2.25)$$

It is now straightforward to define the square root of the determinant of $T(M)$ as a product of the eigenvalues with positive real part ($\text{Re} \lambda > 0$) for generic gauge and Higgs field configurations. We thus reach to the following definition for the chiral fermionic determinant

$$\Delta_{ch} = (\text{det} T(M))^{1/2}. \quad (2.26)$$

The existence of Witten’s anomaly implies that an odd number of eigenvalues change their sign under a continuous topologically non-trivial deformation of the external fields. The conventional level crossing phenomenon would necessitate such an exchange of fermionic levels to occur through the simultaneous vanishing of the real and imaginary eigenvalues at some value of the external fields. Alas in the space of all possible paths the eigenvalues may choose to follow this one is truly of measure zero. Indeed there is an infintude of pathways where fermionic levels may continuously circle around the origin ($\lambda = 0$) of the complex plane never in fact crossing each other (see fig.1). In this way the adiabatic approximation which is assumed to be valid is fully justified. We believe this to be the mechanism by which the pairs of fermionic levels exchange their sign. Level circling must accordingly be equivalent to the presence of an odd number of zero modes of the $D = 5$ Dirac operator whose existence we proceed to demonstrate.

In order to generalize our discussion of level crossing we now define an appropriate $D = 5$ operator as

$$\hat{D} = \gamma_5 \left( \begin{array}{cc} \nabla_t & 0 \\ 0 & -\partial_t \end{array} \right) + P \hat{T} = \left( \begin{array}{cc} \gamma_5 \nabla_t + \nabla^\mu & M R - \epsilon M^* \epsilon L \\ -M^* L + \epsilon M^T \epsilon R & -\gamma_5 \partial_t - \phi \end{array} \right). \quad (2.27)$$
The matrix $P$ is given by

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

For the case of equal fermionic masses the operator $\hat{D}$ satisfies the following reality condition

$$C\epsilon\gamma_5 \hat{D} C\epsilon\gamma_5 = \hat{D}.$$ \hspace{1cm} (2.28)

This is an important property that was used in refs. \cite{4} in the analysis of level crossing for the case of equal up and down quark masses. In effect there is a pairing up of all of the non-zero eigenvalues in the spectrum of the Dirac operator. This furthermore implies that the number of its zero modes, if they exist, is a topological invariant mod 2.

For a case of arbitrary Yukawa couplings this property is no more valid. But we are still able to prove a pairing of non-zero levels due to the antisymmetry property of the Dirac operator. We have indeed

$$C\epsilon\gamma_5 P\hat{D}^T C\epsilon\gamma_5 P = -\hat{D}.$$ \hspace{1cm} (2.29)

This operation actually corresponds to a permutation of two identical fermionic systems, $q \leftrightarrow q'$ and $M \leftrightarrow M'$. Thus the antisymmetry of the Dirac operator is not accidental but rather the consequence of the doubling used to make a well defined fermionic path integral which is a priori not the case for chiral fermions.

Notice that the antisymmetry of the Dirac operator is well defined. This is due to the reality of the representations of the group $O(5) \times SP(2n)$. As a result the transposed Dirac operator along with an appropriate conjugation with a constant matrix falls again in the same representation. The pairing of levels can thus be formally proved as follows:

Let us consider the characteristic polynomial $\det(\lambda - \hat{D})$. Because of antisymmetry of the operator $\hat{D}$ we have an identity

$$\det(\lambda - \hat{D}) = \det(\lambda + \hat{D}).$$ \hspace{1cm} (2.30)

Hence the non-zero eigenvalues of $\hat{D}$ are paired $(\lambda, -\lambda)$. As a consequence we see that the number of zero levels of $\hat{D}$ is invariant modulo 2. Moreover we have a generalization of the index theorems as given above for the $D = 4$ Dirac operator (eqs. (2.25), (2.26)).

That means that there is an odd number of zero modes for the $D = 5$ Dirac operator since it is the case when the masses of fermions are equal.

Below we generalize an explicit construction of the zero modes in terms of zero modes for massless Dirac operator.

The solution $\Psi_0$ to the zero mode equation

$$\hat{D}\Psi_0 = 0$$ \hspace{1cm} (2.31)

can be chosen real such that

$$C\epsilon\gamma_5 \Psi_0^* = \Psi_0.$$ \hspace{1cm} (2.32)
Notice that such a solution is not of any definite chirality. We may now represent $\Psi_0$ as a superposition of an $SU(2)$ doublet $\psi$ and two $SU(2)$ singlets $\chi_1$ and $\chi_2$, i.e. $\Psi_0 = (\psi, \chi)$ with $\chi = (\chi_1, \chi_2)$. The zero mode equations then read as

$$D /\psi + (MR - \epsilon M^* \epsilon L)\chi = 0, \quad (M^+ L - \epsilon M^T \epsilon R)\psi + \mathcal{P}_0 \chi = 0,$$

where

$$\left(\gamma_5 \nabla_t + \nabla_\tau\right) = D /,$$

$$\left(\gamma_5 \partial_t + \partial_\tau\right) = \mathcal{P}_0.$$

By eliminating $\chi$ from eqs.(2.34) we get

$$D /\psi = (MR - \epsilon M^* \epsilon L)(1/\mathcal{P}_0)(M^+ L - \epsilon M^T \epsilon R)\psi.$$

Let us assume that there is exactly one zero mode of the operator $D /$. We want to find the massive zero mode wave function, and moreover express the solution of the above equation in terms of the zero mode wave function $\psi_0$ for the massless case.

We search for a solution iteratively by starting with $\psi_0$ as a zero level approximation. The first level correction satisfies the following equation

$$D ^{(1)} /\psi = (MR - \epsilon M^* \epsilon L)(1/\mathcal{P}_0)(M^+ L - \epsilon M^T \epsilon R)\psi_0.$$

The integrability condition is deduced from the orthogonality of the left hand side of eq.(2.37) to $\psi_0$

$$a_0 = \int d^5 x \psi_0^+(MR - \epsilon M^* \epsilon L)(1/\mathcal{P}_0)(M^+ L - \epsilon M^T \epsilon R)\psi_0 = 0.$$

Indeed if we make a transposition in eq.(2.37) and subsequently take into account the antisymmetry condition for the operator $D /$

$$C \gamma_5 \mathcal{P}_0^T C \gamma_5 = -\mathcal{P}_0,$$

and the reality condition (2.32) we get that $a_0 = -a_0 = 0$.

We may now solve eq.(2.36)

$$\psi^{(1)} = \frac{1}{\mathcal{P}}(MR - \epsilon M^* \epsilon L)(1/\mathcal{P}_0)(M^+ L - \epsilon M^T \epsilon R)\psi_0.$$

This is a well defined expression on the basis of eq.(2.37). The exact solution to eq.(2.35) is given by

$$\psi = \psi_0 + \mathcal{P} \frac{1}{\mathcal{P}^2 + \alpha P - N \mathcal{P}_0^{-1} \mathcal{P}} N \frac{1}{\mathcal{P}_0} \tilde{N} \psi_0,$$

where $P$ is the zero mode subspace projector and $\alpha$ a regularizing parameter. We have denoted

$$N = MR - \epsilon M^* \epsilon L, \quad \tilde{N} = M^+ L - \epsilon M^T \epsilon R.$$

This expression is well defined and does not depend on the regularizing parameter $\alpha$. By substitution of (2.41) for $\psi$ into eq.(2.35) we get the following equation

$$P \frac{1}{\mathcal{P}^2 + \alpha P - N \mathcal{P}_0^{-1} \mathcal{P}} N \frac{1}{\mathcal{P}_0} \tilde{N} \psi_0 = 0.$$
By using the block matrix representation we find an equivalent form of eq.(2.43)

$$
\int d^5x \psi_0^+ \left( N \frac{1}{\not{D}_0} \not{N} + \frac{N}{\not{D}_0} \not{N}(1 - P) \frac{1}{\not{D} - (1 - P)N \not{D}_0^{-1} \not{N}(1 - P)} \times \right.
\times \left. (1 - P)N \frac{1}{\not{D}_0} \not{N} \right) \psi_0 = 0. \quad (2.43)
$$

The above expression is well defined. It is certainly satisfied as a consequence of the antisymmetry of the operators $\not{D}, \not{D}_0$, and $\not{D}$. This can be checked by a transposition similar to the case of eq.(2.38).

Let us now turn to the issue of normalizability of $\psi$ in eq.(2.41). Let us demonstrate that the wave function (2.41) is normalizable. For that it is sufficient to check that expression (2.41) decreases rapidly enough at large distances $x \to \infty$. This we can more conveniently do in the regular gauge for the instanton configuration. At infinity we have that

$$
D_\mu \to U \partial_\mu U^{-1}, M \to U M_0, \quad (2.44)
$$

where $U$ is an element of the $SU(2)$ group that corresponds to the instanton configuration and $M_0$ is a constant matrix. By a direct substitution of expressions (2.45) back into eq.(2.41) we easily find that

$$
\psi_L \propto U \frac{\partial^2}{\partial^2 - M_0^2} U^{-1} \psi_{0L}. \quad (2.45)
$$

It becomes obvious from the above that $\psi_L$ behaves better at infinity than $\psi_{0L}$ itself a normalizable wave function. Indeed we trivially deduce from eq.(2.46) that

$$
\psi_L \leq U \frac{1}{x^2 M_0^2} U^{-1} \psi_{0L}. \quad (2.46)
$$

Hence $\psi_L$ is normalizable. It is easy to see that this wave function decreases more rapidly at infinity provided that the mass matrix $M$ is an asymptotically covariant constant similar to the case of the $D = 4$ instanton + Higgs configurations. This necessitates that the gauge field is asymptotically a pure gauge $A_\mu \propto U \partial_\mu U^{-1}$ with the mass matrix being $M \propto U M_0$ at infinity where $U$ is an element of the $SU(2)$ group and $M_0$ is a constant. In this case the wave function (2.41) is normalizable. It is to be emphasized at this point that by changing $M$ from zero to a nonzero value we can obtain additional zero modes but always in pairs. It is of some significance to note that this condition implies that in the presence of Yukawa interactions the interpolation between two $D = 4$ configurations of gauge fields $A_\mu$ and $A_\mu^U$ which is given by a $D = 5$ gauge field configuration should also include Higgs fields.

We now turn to the case where the operator $\not{D}$ has a number of normalizable zero modes $\psi_{0i}, i = 1, \ldots, N$. These zero modes can be chosen to obey the reality condition

$$
C \gamma_5 \epsilon \psi_{0i}^* = \psi_{0i}. \quad (2.47)
$$

The possibility of such a choice seems to be crucially necessary for the following since allows to prove the integrability of the equation for zero mode.
In order to solve eq.(2.36) we start the iteration procedure with a linear combination
\[
\psi_0 = c_i \psi_{0i}
\]  
(2.48)
as a zero level approximation. At the first step we get eq.(2.37). The integrability condition reads now as follows for all \(i\)
\[
\int d^5x \bar{\psi}_{0i}^+ N \mathcal{D}_0^{-1} \tilde{N} \psi_0 = 0.
\]  
(2.49)
This means that the vector \(c_i\) should be annihilated by the matrix
\[
a_{ij}^{(0)} = \int d^5x \bar{\psi}_{0i}^+ N \mathcal{D}_0^{-1} \tilde{N} \psi_j.
\]  
(2.50)
A solution of eq.(2.37) and hence of eq.(2.36) exists only if the matrix \(a_{ij}^{(0)}\) has zero eigenvalues. By considering the Hermitian conjugate and the complex one of \(a_{ij}\) we can easily check that this matrix is antisymmetric but complex in general in contrast to the case of equal fermionic masses [4]
\[
a_{ij}^{(0)} = -a_{ji}^{(0)}.
\]  
(2.51)
In general this matrix is non-zero and belongs to the complexification of \(O(N)\) algebra. Let us first consider the case of an odd number of \(\psi_0\)'s, i.e. \(N = 2n + 1\). It is easy to see that the matrix \(a_{ij}^{(0)}\) has at least one zero eigenvalue.

In order to show that it is sufficient to consider the characteristic polynomial \(\det(\lambda - a^{(0)})\). Because of the antisymmetry of the matrix \(a^{(0)}\) we have
\[
\det(\lambda - a^{(0)}) = \det(\lambda + a^{(0)}).
\]  
(2.52)
Hence the non-zero eigenvalues of \(a^{(0)}\) are paired \((\lambda, -\lambda)\). Moreover we see that the determinant of \(a^{(0)}\) is zero for any odd dimension of \(a^{(0)}\). As a consequence we see that the number of zero levels of \(a^{(0)}\) is invariant modulo 2. Therefore since for the real \(a^{(0)}\) of odd number there is an odd number of zero eigenvalues the same is correct for the complex \(a^{(0)}\).

In the case of even \(N\), i.e. \(N = 2n\), all eigenvalues of \(a_{ij}^{(0)}\) are generically non-zero and are arranged in pairs \((i\lambda, -i\lambda)\) as above.

We get the zero level approximate solution to eq.(2.36) by picking the eigenvector of \(a_{ij}^{(0)}\) that corresponds to the zero eigenvalue for the vector \(c_i\) in eq.(2.49). It is obvious from the above construction that the ‘oddness’ of the number of the fermionic zero modes does not change. In particular there is at least one normalizable zero mode of massive fermion in the \(D = 5\) topologically nontrivial configuration since it has exactly one zero mode for massless ones.

The exact solution to eq.(2.36) is given by (2.41) where \(P\) is the projector onto the total zero mode subspace. We must be more careful here than with the case of the one massless zero mode \((N = 1)\) since in general the function \(N \mathcal{D}_0^{-1} \tilde{N} \psi_0\) is not orthogonal to the zero mode subspace. By the use of the block matrix representation
it is easy to see that the expression (2.41) obeys eq.(2.36) provided that the following holds true for all \(i\)

\[
\int d^5x \psi^+_0 \left( N \frac{1}{\bar{\partial}_0} \bar{N} + N \frac{1}{\bar{\partial}_0} \bar{N}(1 - P) \times \frac{1}{\bar{\partial} - (1 - P)N \bar{\partial}_0^{-1} \bar{N}(1 - P)(1 - P)N \frac{1}{\bar{\partial}_0} \bar{N}} \right) \psi_0 = 0.
\]

The above expression is well defined and implies that the vector \(c_j\) should be the eigenvector that corresponds to the zero eigenvalue of the following matrix

\[
a_{ij} = \int d^5x \psi^+_0 \left( N \frac{1}{\bar{\partial}_0} \bar{N} + N \frac{1}{\bar{\partial}_0} \bar{N}(1 - P) \times \frac{1}{\bar{\partial} - (1 - P)N \bar{\partial}_0^{-1} \bar{N}(1 - P)(1 - P)N \frac{1}{\bar{\partial}_0} \bar{N}} \right) \psi_{0j},
\]

i.e.

\[
\sum_j a_{ij} c_j = 0.
\]

In the leading order in the Yukawa couplings this condition reproduces eq. (2.49). It can be easily seen that it is equivalent to the integrability condition analogous to eq.(2.33). One may observe that the matrix \(a_{ij}\) is antisymmetric by using the reality condition (2.32) and the antisymmetry of the operators \(\bar{\partial}\) and \(\bar{D}\). By repeating here the procedure we followed for the leading approximation we find that the matrix \(a_{ij}\) has exactly an odd number of zero eigenvalues for odd \(N = 2n + 1\) and correspondingly an even number of zero eigenvalues for even \(N = 2n\). In the latter case the zero eigenvalues can be absent. Hence there is an invariance for the odd/even number of fermionic zero modes under smooth deformations of the mass matrix \(M\).

At this point our arguments that generalize Witten’s observation of an \(SU(2)\) anomaly for odd number of massless fermion doublets in the presence of Yukawa interactions are complete. The existence of the zero mode for massive \(D = 5\) Dirac operator was demonstrated for the general case of a nondegenerate mass matrix.

**Conclusions**

In the present paper we looked at the Witten \(SU(2)\) global anomaly for the case of an odd number of \(SU(2)\) Weyl doublets in an \(SU(2)\) gauge theory with arbitrary Yukawa couplings and fermion masses. The \(D = 4\) Dirac operator appears to be antisymmetric and nonhermitean. As a consequence its spectrum of nonzero eigenvalues appears in pairs with opposite sign and is complex. We showed explicitly the existence of an odd number of normalizable zero modes for the \(D = 5\) Dirac operator. Their presence is equivalent to a level exchange phenomenon, level ”circling”, between the fermionic level pairs under continuous deformations of the external gauge field. This is a direct generalization of the well established level crossing phenomenon and applies to all \(SP(n)\) gauge theories with arbitrary Yukawa couplings and fermion masses coming in an odd number in its spin representation.
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Figure Caption

Fig 1. The level circling phenomenon is depicted for an odd number of fermionic level pairs $(\lambda, -\lambda)$ on the complex plane. Under continuous gauge field deformations the lowest lying pair switches sign by circling around the origin.
This figure "fig1-1.png" is available in "png" format from:

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