Construction of solvable potential partner of Generalized Hylleraas potential in one-dimensional Schrodinger system

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Abstract. The higher order potential partner of generalized Hylleraas potential was constructed by using Supersymmetry potential partner equation. The superpotential of shape invariant system was determined from the ground state wave function by applying Supersymmetry operators’ properties. The ground state wave function and the ground state energy were determined from the solution of Schrodinger equation with generalized Hyllerras potential using Nikiforov-Uvarov method. The higher order potential partner was combination of the generalized Hylleraas potential, ground state energy, and super potential. The energy eigenvalue and energy eigen function of potential partner which was solvable potential were also determined using NU method.

1. Introduction
Supersymmetric Quantum Mechanics (SUSY QM) which was developed as a simplest model of SUSY field theory was proposed by Witten.[1] SUSY QM since then has developed as a powerful technique used to solve one dimensional Schrodinger equation without solving the second order differential equation directly.[2,3,4] By applying the properties of SUSY QM, SUSY QM operators, and the ground state wave function of the original potential, the isospectral potential partner which had higher order was constructed.[5,6]

One dimensional Schrodinger equation for some potentials are exactly solved by some methods such as AIM [7], SUSY QM [2,3], Factorization method, Nikiforov-Uvarov (NU) method [8-10], Romanovsky polynomials [11], and others. NU method is one of those methods that has more general application and has been used widely since NU differential equation has been formulated with generalized parametric of generalized hypergeometric like equation [12].

Hylleraas potential is an interesting short range potential that could be used to describe atomic physics, nuclear physics such as an interaction of nucleon – nucleon, particle physics such as in meson-meson and various branches of nuclear physics [13-14]. For wider application, Hylleraas potential has been generalized into the form given as

$$V(x) = A \frac{a + e^{ax}}{(1/p) + e^{ax}} - B \frac{b + e^{ax}}{(1/p) + e^{ax}}$$

with $a$, $b$, and $p$ are the Hylleraas parameters, $A$ and $B$ are proportional to potential depth, and $\alpha$ is proportional to the range of potential.

By using NU method the energy eigen value and the energy eigen function was obtained from the solution of Schrodinger equation with energy potential expressed in equation (1). By knowing the ground state wave function of the system then the construction of potential partner of generalized
Hylleraas potential equation (1) will be performed by using properties of SUSY quantum mechanics algebra.

The proposed of the research is to construct new potential which is potential partner of generalized Hylleraas potential by manipulating the potential partner equations with the ground state wave function of generalized Hylleraas potential. This paper is organized as follows. Brief introduction is presented in section 1, brief review of NU method and the properties of SUSY QM, and reduction of Schrodinger equation for Hylleraas potential into generalized hypergeometric like equation with generalized parametric are presented in section 2. Result and discussion are presented in section 3 and brief conclusion is in section 4.

2. Method

2.1. Review of NU method

The one-dimensional solvable Schrodinger equation is reduced into hypergeometric or confluent hypergeometric type differential equation by suitable variable transformation. The hypergeometric type differential equation, which is solved using Nikiforov-Uvarov method [8], is presented as

\[
\frac{d^2 \psi(s)}{ds^2} + \frac{\tilde{r}(s)}{\sigma(s)} \frac{d\psi(s)}{ds} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)} \psi(s) = 0
\]  

(2)

where \(\sigma(s)\) and \(\tilde{\sigma}(s)\) are mostly second order polynomials, and \(\tilde{r}(s)\) is first order polynomial. Equation (2) is solved using variable separation method by setting

\[
\psi(s) = \phi(s) y(s)
\]

then equation (2) becomes hypergeometric type equation

\[
\frac{d^2 y(s)}{ds^2} + \tau(s) \frac{dy(s)}{ds} + \lambda y(s) = 0
\]

(4)

and \(\phi(s)\) a logarithmic derivative that fulfill the condition that \(\frac{\phi'}{\phi} = \frac{\pi}{\sigma}\) with \(\pi(s), \lambda\), and \(\tau\) are

\[
\pi = \left(\frac{\sigma' - \tilde{r}}{2}\right) \pm \sqrt{\left(\frac{\sigma' - \tilde{r}}{2}\right)^2 - \tilde{\sigma} + k \sigma}
\]

(5)

\[
\tau = \tilde{r} + 2\pi, \quad \lambda = k + \pi' \quad and \quad \lambda = \lambda_n = -n \tau' - \frac{n(n-1)}{2} \sigma^n, \quad n = 0, 1, 2, ...
\]

The value of \(k\) in equation (5) is found from the condition that the expression under the square root of equation (5) must be square of mostly first-degree polynomial and thus the discriminate of the quadratic expression is zero. The new energy eigenvalue from equation (4) is obtained with the condition that \(\tau' < 0\). The solution of the second part of the wave function, \(y_n(s)\), which is connected to Rodrigues relation [10], is given as

\[
y_n(s) = C_n \frac{d^n}{w(s) ds^n} \left(\sigma^n(s) w(s)\right)
\]

(7)

where \(C_n\) is normalization constant, and the weight function \(w(s)\) must satisfy the condition

\[
\frac{\partial (\sigma w)}{\partial s} = \tau(s) w(s)
\]

(8)

If in the parametric generalization, the hypergeometric type equation in equation (2) is written as [12]

\[
\frac{d^2 \psi(s)}{ds^2} + \left(c_1 - c_2 s\right) \frac{d\psi(s)}{ds} + \frac{\left(-c_1 s^2 + e_1 s - e_1\right)}{s(1-c_3 s)} \psi(s) = 0
\]

(9)

By comparing equation (2) and equation (9) then the energy value equation and eigen function obtained from equation (9) are respectively given as
\[ c_{2n} - (2n + 1)c_5 + (2n + 1)\left(\sqrt{c_9} + c_1\sqrt{c_8}\right) + n(n - 1)c_3 + c_7 + 2c_2c_8 + 2\sqrt{c_8c_9} = 0 \]  \hfill (10)

and

\[ \psi(s) = N_n s^{c_3} \left(1 - c_3s\right)^{-c_{12} - (c_{13}/c_8)} P_n^{(c_{10} - 1, (c_{11}/c_8) - c_{10} - 1)} \left(1 - 2c_3s\right) \]  \hfill (11)

with

\[ c_4 = \frac{1}{2} (1 - c_5), \quad c_5 = \frac{1}{2} (c_2 - 2c_3), \quad c_6 = c_2^2 + c_4, \quad c_7 = 2c_4 c_5 - c_2^2, \quad c_8 = c_4^2 + c_3 \]
\[ c_9 = c_4 c_7 + c_3 c_8 + c_6, \quad c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, \quad c_{12} = c_4 + \sqrt{c_8} \]
\[ c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right), \quad c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right) \]  \hfill (12)

The ground state energy and wave function are obtained from equations (10) and (11) for \(n = 0\).

2.2. Brief review of SUSY QM

As proposed by Witten [1] the pair of SUSY Hamiltonians \(H_0\) for two charge operators that are commute with \(H_0\) is given by

\[ H_0 = \begin{pmatrix} -\frac{d^2}{dx^2} + \frac{d\phi(x)}{dx} + \phi^2(x) & 0 \\ 0 & -\frac{d^2}{dx^2} - \frac{d\phi(x)}{dx} + \phi^2(x) \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \]  \hfill (13)

with partner Hamiltonian \(H_- = H_1\) and \(H_+ = H_2\), and potential partners \(V_- = V_1\) and \(V_+ = V_2\) are

\[ V_-(x) = V_1 = \phi^2(x) - \phi'(x) \quad \text{and} \quad V_+(x) = V_2 = \phi^2(x) + \phi'(x) \]  \hfill (14)

The relation between the effective potential \(V_{ef}\) and the first potential partner is

\[ V_{ef}(x) = V_-(x; a_0) + E_0 = V_1(x; a_0) + E_0 \]  \hfill (15)

where \(E_0\) is the ground state energy for effective potential \(V_{ef}\). By setting the new SUSY operators, lowering operator \(A\) and raising operator \(A^+\) as

\[ A^+ = -\frac{d}{dx} + \phi(x) \quad \text{and} \quad A = \frac{d}{dx} + \phi(x) \]  \hfill (16)

then the SUSY Hamiltonians in equation (13) becomes

\[ H_-(x) = H_1 = A^+ A, \quad \text{and} \quad H_+(x) = H_2 = AA^+ \]  \hfill (17)

and so

\[ A\psi_0(-) = A\psi_0 = 0 \]  \hfill (18)

The potential partner \(V_2\) of Hylleraas potential is obtained from equations (14) and (18) given as

\[ V_2(x) = V_1 - 2\frac{d^2}{dx^2}\ln\psi_0 \]  \hfill (19)

Here \(\psi_0\) is the ground state wave function of Schrodinger with potential of \(V_1\) whose its potential partner is \(V_2\).

3. Result and discussion

The Schrodinger equation with generalized Hylleraas potential which expressed in equation (1) is given as

\[ \frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + \left(A - \alpha e^{\alpha x} - (1/l) + e^{\alpha x} - B - \beta e^{\beta x} - (1/l) + e^{\beta x}\right)\psi(x) = E\psi(x) \]  \hfill (20)

By setting \(\hbar = 1, 2m = 1, \text{ and } e^{\alpha x} = s\) in equation (20) then we have...
\[ \frac{\partial^2 \psi}{\partial s^2} + \frac{(1-\alpha)s}{s(1-\alpha)} \frac{\partial \psi}{\partial s} + \left(-p(p_1 - p_2 - pE)s^2 + (a_1 + p + p_1 - p_2 - 2Ep)s - (\alpha + b_1 + E)\right) \psi(s) = 0 \]  

with

\[ a_1 = Ap, b_1 = Bp, p_1 = Ap, p_2 = Bp \]

By comparing equations (9) and (21) we obtain

\[ c_1 = 1, c_2 = c_3 = p, \quad \varepsilon_1 = \frac{p(p_1 - p_2 - pE)}{\alpha^2}, \quad \varepsilon_2 = \frac{(\alpha + b_1)p + p_1 - p_2 - 2Ep}{\alpha^2} \]

By manipulating equations (12) and (23) we get

\[ c_4 = 0, c_5 = -\frac{p}{2}, c_6 = p^2 \left(1 - \frac{E}{\alpha^2}\right) + \frac{p(p_1 - p_2)}{\alpha^2}, \quad c_7 = -\frac{(\alpha + b_1 - 2Ep)p + p_1 - p_2}{\alpha^2}, \]

\[ c_8 = \frac{(-\alpha + b_1 + E)}{\alpha^2}, \quad c_9 = p^2 \left(-2a_1 + 2E\right) + \frac{1}{4}, \quad c_{10} = 1 + 2\sqrt{\frac{(-\alpha + b_1 + E)}{\alpha^2}} \]

\[ c_{11} = p + p + 2 \left(p \sqrt{\frac{(-2a_1 + 2E)}{\alpha^2} + \frac{1}{4}} + p \sqrt{\frac{(-\alpha + b_1 + E)}{\alpha^2}}\right), \]

\[ c_{12} = \sqrt{\frac{(-\alpha + b_1 + E)}{\alpha^2}}, \quad c_{13} = -\frac{p}{2} \left(p \sqrt{\frac{(-2a_1 + 2E)}{\alpha^2} + \frac{1}{4}} + p \sqrt{\frac{(-\alpha + b_1 + E)}{\alpha^2}}\right) \]

The ground state wave function generated from equations (11) and (24) is given by

\[ \psi_0(x) = N_0 s^{c_1} \left(1 - c_3 s\right)^{-c_1} = N_0 e^{c_1 \alpha x} \left(1 - c_3 e^{\alpha x}\right)^{-c_1} \]

with

\[ c_{14} = c_{12} + \frac{c_{13}}{c_3} = -\frac{1}{2} \sqrt{\frac{-2a_1 + 2E}{\alpha^2} + \frac{1}{4}} \]

In equation (26) \( E = E_0 \) is the ground state energy of generalized Hylleraas potential and is obtained numerically from equation (28). By using equations (10) and (21-24) we obtain the energy eigen value equation given as

\[ pn + (2n+1) \frac{p}{2} + (2n+1) \left(-\frac{2a_1}{4} + \frac{1}{4}\right) p^2 + p \sqrt{(-\alpha + b_1 + E)} + n(n-1) p \]

\[ -((\alpha + b_1 - 2E)p + p_1 - p_2) + 2p(-a_1 + b_1 + E) + 2 \sqrt{(-\alpha + b_1 + E)(-2a_1 - 2E + \frac{1}{4})} p^2 = 0 \]

By setting \( n = 0 \) in equation (27) we get the ground state energy eigen value equation which is given as

\[ \frac{p}{2} + p \left(\sqrt{\frac{-2a_1 - 2E_0}{4} + \frac{1}{4}} + \sqrt{(-\alpha + b_1 + E_0)}\right) + \]

\[ (\sqrt{-3a_1 + b_1 + 4E_0}p - p(A - B)) + 2p \sqrt{(-\alpha + b_1 + E_0)(-2a_1 - 2E_0 + \frac{1}{4})} = 0 \]

The ground state energy is calculated numerically from equation (28). The corresponding wave function generated from equations (11) and (24) is given by

\[ \psi(s) = N_n s^{c_1} \left(1 - c_3 s\right)^{-c_1} e^{c_1 \alpha s} P_n^{(c_{0-1}, (c_1/c_3) - c_{0-1})} \left(1 - 2c_3 s\right) \]
3.1. Construction of the potential partner of Hylleraas potential

By using equations (15), (19), and (25) we get the potential partner $V_2$ of generalized Hylleraas effective potential given as

$$V_2(x) = V_1 - 2 \frac{d^2}{dx^2} \ln \psi_0 = \left(- \frac{a_1 + p s}{1 - p s} + B \frac{b_1 + p s}{1 - p s}\right) - E_0 - 2 \left( \frac{c_{14} p \alpha^2 s}{(1 - p s)^2} \right)$$  \hspace{1cm} (30)

with $c_{14} = - \frac{1}{2} \sqrt{\frac{-2a_1 + 2E_0}{\alpha^2} + \frac{1}{4}}$. It is seen in equation (30) that the potential partner shows higher order function potential compared to the generalized Hylleraas potential in equation (1). It will also be shown that the potential partner in equation (30) is solvable. By writing the Schrodinger equation for potential partner in equation (30) as

$$\frac{\partial^2 \psi}{\partial s^2} + (1 - p s) \frac{\partial \psi}{\partial s} \left(\frac{p(p_1 - p_2 - pE_{2n} + pE_0)s^2}{\alpha^2 s^2 (1 - p s)^2}\right) \psi = 0$$  \hspace{1cm} (31)

then we can see that equation (31) is similar with equation (21), therefore equation (31) can also be solved using NU method. So the one dimensional Schrodinger equation with potential partner $V_2$ which is expressed in equation (31) are exactly solvable by NU method with parameter given as

$$c_1' = 1, \quad c_2' = c_3' = p, \quad c_4' = \frac{p(p_1 - p_2 - pE_{2n} + pE_0)}{\alpha^2}, \quad c_5' = \frac{-a_1 + b_1 + E + E_0}{\alpha^2}, \quad c_6' = \frac{-a_1 + b_1 + E + E_0}{\alpha^2}, \quad c_7' = \frac{-a_1 + b_1 + E + E_0}{\alpha^2}, \quad c_8' = \frac{-a_1 + b_1 + E + E_0}{\alpha^2}$$

By manipulating equations in equation (32) and together with equation (12) we generate

$$c_9' = \frac{2p^2(E_{2n} + c_{14} \alpha^2 - 2a_1)}{\alpha^2} + \frac{p^2}{4}, \quad c_{10}' = 1 + 2 \sqrt{\frac{-a_1 + b_1 + E + E_0}{\alpha^2}}, \quad c_{11}' = 2p + 2p \left(\sqrt{\frac{2(E_{2n} + c_{14} \alpha^2 - 2a_1)}{\alpha^2} + \frac{1}{4}} + p \sqrt{\frac{-a_1 + b_1 + E + E_0}{\alpha^2}}\right),$$

$$c_{12}' = \sqrt{\frac{-a_1 + b_1 + E + E_0}{\alpha^2}}, \quad c_{13}' = -\frac{p}{2} - p \left(\sqrt{\frac{2(E + c_{14} \alpha^2 - 2a_1)}{\alpha^2} + \frac{1}{4}} + \sqrt{\frac{-a_1 + b_1 + E + E_0}{\alpha^2}}\right)$$

By using equations (10) and (11) and together with equations in equation (33) we get the energy eigen value equation and energy eigen function for potential partner given as
\[
(2n^2 + 2n + 1) \frac{p^2}{2} + (2n + 1) \left( \sqrt{\frac{2p^2(E + c_1\alpha^2 - 2a_1)}{\alpha^2}} \right)
+ p \frac{(-3a_1 + b_1 + 4E - A + B + 2c_1\alpha^2)}{\alpha^2}
+ 2 \left( \sqrt{\frac{2p^2(E + c_1\alpha^2 - 2a_1)}{\alpha^2}} \right) \right) \right) = 0
\]

(34)

and

\[
\psi(s) = N_n s^{c_{12}'-1} \left( 1 - c_{3}'s \right)^{(c_{12}'-1)\gamma_{3}'} \frac{P_n^{(c_{10}'-1, c_{11}'\gamma_{3}'-c_{10}'-1)}}{1 - 2c_{3}'s}
\]

(35)

with all \(c\)'s parameters are listed in equation (33).

It was shown that from exactly solvable potential in quantum system could be generated another potential that was also solvable potential in quantum system. This new solvable potential is simply generated by using the properties of SUSY QM. Due to the differences in parameters between equation (21) and (31) then it is expected that the potential partner has different energy with the original ones. This potential partner has different energy spectra compared to the energy spectra of the original potential. The result of this work is in agreement with the result of the research about the simple construction of potential partner for Hulthen potential in Ref. (6)

4. Conclusion

The new exactly solvable potential in quantum system was generated from generalized Hylleraas potential using SUSY QM. By applying the properties of a pair of potential partner \(V_1\) and \(V_2\) and SUSY operators, potential partner \(V_2\) was generated from another potential partner \(V_1\) and with known ground state wave function of Hylleraas potential. Both ground state energy and wave functions of generalized Hylleraas potential were determined by using parametric generalization NU method. The energy eigenvalue equation and wave function of potential partner \(V_2\) was also analytically obtained using NU method. The potential partner has different energy spectra compared to the energy spectra of the original potential and so the wave function. This potential partner was considered to be new potential that could be applied to describe the interaction model in atomic system. It would be worthy to explore more potential partner of solvable potential system which could be used to describe the behavior of particles in atomic system.

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