Reciprocity-gap misfit functional for Distributed Acoustic Sensing, combining teleseismic and exploration data

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Abstract

We use data from Distributed Acoustic Sensing (DAS) to perform quantitative imaging of sub-surface Earth’s properties in elastic media. We design a new misfit functional based upon the \textit{reciprocity-gap}, that takes products of displacement and strain. These products further associate an observation with a simulation. In comparison with other misfits, this has the advantage to only require little a-priori information on the exciting sources. In particular, it enables the use of data from teleseismic events, followed by the exploration data to perform a multi-resolution reconstruction. The teleseismic data contain the low-frequency content which is missing in the exploration ones, allowing for the recovery of the long spatial wavelength. These data are used to build prior models for the subsequent reconstruction from the higher-frequency exploration data. This gives the elastic Full Reciprocity-gap Waveform Inversion (FRgWI) method, and we demonstrate its performance with an elastic isotropic pilot experiment.

1 Introduction

In the Full Waveform Inversion (FWI) method, the seismic imaging for the quantitative recovery of sub-surface Earth’s parameters is recast as a minimization problem, where one optimizes a misfit criterion between the observed data and numerical simulations. It has first been developed for the time-domain wave equation by Bamberger et al. (1977, 1979); Lailly (1983) and Tarantola (1984), while the frequency-domain formulation, that first applies a Fourier transform to the data, has been proposed by Pratt et al. (1996; 1998; 1999). Resolving FWI with an iterative minimization algorithm performs successive updates of the model parameters and needs successive numerical simulations to compare with the measurements. The main difficulty comes from the local minima in the misfit functional, caused by phase shifts between the observations and simulations. These are due to the lack of background velocity information, as observed by Gauthier et al. (1986); Luo and Schuster (1991); Bunks et al. (1995) and Fichtner et al. (2008). This difficulty has motivated the use of increasing frequency selection in the iterative reconstruction, as advocated by Bunks et al. (1995); Sirgue and Pratt (2004) and Faucher et al. (2020). However, the recovery of the background model, that is, the long wavelength profile, is only possible via very low-frequency contents in the data, which are missing (i.e., unusable due to the noise) in applications. As an alternative, different parametrizations of the inverse problem have been proposed by, e.g., Symes

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and Carazzone (1991) and Clément et al. (2001), at the cost of an increased computational complexity (Faucher et al., 2020).

While the misfit functional traditionally relies on the difference between the full observations and the simulations, several alternatives have been investigated. Note also that the misfit functionals usually assume knowledge of the exact locations of the sources, contrary to the one we introduce. We mention, for instance, the criteria based upon the phase and the envelope of the signal (Fichtner et al., 2008), the use of cross-correlation (Luo and Schuster, 1991; Van Leeuwen and Mulder, 2010) and of the $L1$ norm by Brossier et al. (2010). In particular, the misfit based upon the optimal transport distance is shown to enhance the convexity of the functional by Métivier et al. (2016) and Yang et al. (2018). In our work, we propose a new functional for elastic reconstruction, based upon the product of observations and simulations, using the displacement and the normal stress, equivalently obtained from the strain. Our approach uses the reciprocity-gap formula, which originate from Green’s formula and have first been derived in the acoustic setting for inverse scattering, see Kohn and Vogelius (1985) and Colton and Haddar (2005); de Hoop and de Hoop (2000) in seismic. Alessandrini et al. (2019) and Faucher et al. (2019) further apply the method in acoustic marine exploration, upon taking measurements of the pressure field and of the vertical velocity. This functional relates to the family where the data are correlated or convolved, as explored by, e.g., Van Leeuwen and Mulder (2010); Choi and Alkhalifah (2011) and Montagner et al. (2012).

In this work, we first derive the misfit functional based upon the reciprocity-gap in the context of elasticity. The misfit criterion evaluates combinations of observation data with simulations which, indeed, are not build on the assumption of knowing the sources of the elastic displacement, as standard misfit functionals are. Using the flexibility of the resulting minimization problem, we work with both teleseismic and exploration data, performing a multi-resolution reconstruction. The main feature is that, by nature, teleseismic events contain the very low frequencies that are missing in the context of exploration, therefore, they are used to build the initial, long-wavelength models. We illustrate the configurations in Figure 1. Namely, thanks to the reciprocity-gap misfit functional, we do not need to know the passive source location or characteristics (other than the frequency bandwidth), and it is even allowed to be outside of the computational domain: that is why we can use remote earthquake data for low-frequency exploration. These low-frequency data are critical for the high-resolution inversion of complex geological structures as indicated by Gauthier et al. (1986); Mora (1987); Luo and Schuster (1991); Bunks et al. (1995); Fichtner et al. (2008); Virieux and Operto (2009); Ten Kroode et al. (2013) and Faucher et al. (2020).

On the other hand, the benefits of the elastic reciprocity-gap waveform inversion comes at the price of having measurements of both the displacement (or velocity) and of the normal stress (or, equivalently, of the strain tensor). While measurements of the displacement are common with (string) geophones (which can be buried, cf. Drijkoningen et al. (2006)), the possibility to measure the strain is more recent, with Distributed Acoustic Sensing (DAS). It is an acquisition method where fiber optic cables are deployed in the ground: the cables bend when waves propagate and one can extract measurements of the strain from this curvature (Innanen, 2017). Then, depending on the configuration of the cable, different components can be retrieved, as highlighted by Lim Chen Ning and Sava (2018), see in particular their Figure 2. For instance, helix and straight cables provide the necessary features to extract the strain components, see Innanen (2017) and Lim Chen Ning and Sava (2018). Let us also note that, while comparisons between geophones and DAS data are carried out by Mateeva et al. (2013) and Spikes et al. (2019), in our method, we use both instead.

The novelties of our work are (1) the definition of the reciprocity-gap misfit functional
Figure 1: Illustration of the configuration where (a) the source of the teleseismic event is allowed to lie outside of the (b) computational domain in exploration. In order to avoid the free-surface condition that imposes the normal stress to zero, the devices are considered slightly buried, and we assume the knowledge of the physical properties in the near surface area (in grey) for FRgWI.

for elasticity which allows, from the DAS acquisition system, the (2) multi-resolution reconstruction using (2a) observed data from teleseismic events (low-frequency), and (2b) from exploration acquisition (high-frequency). We detail the method in Section 2, which fundamental feature is that it requires only little a-priori knowledge regarding the sources of the observations (neither the positions or functions), only the frequency bandwidth. We demonstrate the performance of an approach based on our functional with a computational experiment in Section 3. Here, the teleseismic data, containing the low-frequency contents are first used to recover the long-wavelength, background profile, of the models. Then, the exploration data, of higher-frequency contents, are used to retrieve the finer scales.

2 Methodology

We consider the non-linear seismic imaging problem for the recovery of sub-surface elastic properties from measurements of waves and rely on an iterative minimization algorithm. We introduce a misfit functional based upon the reciprocity-gap, which uses combinations of measurements and simulations. It uses measurements of the strain, and enables for arbitrary probing sources. Note that while we setup the method in the frequency-domain, it can be similarly applied in the time-domain. One advantage of the time-harmonic formulation is to easily incorporate attenuation in the propagation, by using a complex-valued elasticity tensor (Carcione, 2007).

2.1 Modeling the Data

We consider the propagation of time-harmonic waves in an elastic medium, given in terms of the vectorial displacement field \( u \) and of the stress tensor of order two \( \sigma \) such that, at frequency \( \omega \) and for an (internal) source \( f \),

\[
\nabla \cdot \sigma(x, \omega) + \omega^2 \rho(x) u(x, \omega) = f(x), \quad \text{in the domain } \Omega.
\]

Here, \( \rho \) is the material density and \( x \) represents the space coordinates. In linear elasticity, the stress tensor is related to the strain tensor \( \varepsilon \) by the Hooke’s law:

\[
\sigma(x, \omega) = C(x, \omega) \varepsilon(x, \omega), \quad \text{with} \quad \varepsilon = \frac{1}{2} \left\{ \nabla u + (\nabla u)^T \right\},
\]
where $T$ indicates the transposed of the matrix. The physical properties of the medium are encoded in the elasticity tensor of order four $\mathcal{C}$ which, in the case of isotropy, reduces to depend on the Lamé parameters $\lambda$ and $\mu$, such that equation 2 writes as (with the Voigt notation),

$$\sigma = \lambda \text{Tr}(\varepsilon) I_d + 2 \mu \varepsilon,$$

where $\text{Tr}$ denotes the trace operator, and $I_d$ the identity matrix. We further recall the P- and S-wave speeds, respectively $c_p$ and $c_s$,

$$c_p = \sqrt{\frac{(\lambda + 2\mu)}{\rho}}, \quad c_s = \sqrt{\mu/\rho}.$$  

On the boundary of the domain $\Omega$, we consider a free surface at the interface between the air and the ground where, in the absence of a source, $\mathbf{g} \cdot \mathbf{n} = 0$. Because of the computational restriction, we impose absorbing boundary conditions elsewhere (see Figure 1b), we refer to, e.g., Givoli and Keller (1990) and Higdon (1991).

The forward problem $\mathcal{F}$ is defined as the map from the model to the data, that is, it gives the solution to the wave equation at the location of the receivers for a given physical model $\mathbf{m} = (\lambda, \mu, \rho)$. We write the forward problem in terms of the displacement and of the strain components that are needed for the reciprocity formula of Subsection 2.2. We denote by $X_{\text{rcv}}$ the line of string receivers such that, for a source $f_k$ at frequency $\omega$,

$$\mathcal{F}(\mathbf{m}, f_k, \omega) = \left\{ \mathcal{F}_u(f_k, \mathbf{m}, \omega); \mathcal{F}_\varepsilon(f_k, \mathbf{m}, \omega) \right\}$$

$$= \left\{ \mathbf{u}(\mathbf{x}, f_k, \mathbf{m}, \omega); \varepsilon(\mathbf{x}, f_k, \mathbf{m}, \omega) \cdot \mathbf{n}; \text{Tr}(\varepsilon(\mathbf{x}, f_k, \mathbf{m}, \omega)) ; \mathbf{x} \in X_{\text{rcv}} \right\}.$$  

Therefore, the indexed notations $\mathcal{F}_u$ and $\mathcal{F}_\varepsilon$ indicate respectively the displacement or the strain data. For the strain, we motivate in the next sections why we do not need all components but only the trace and the normal strain, namely, to relate with the normal stress.

### 2.2 Full Reciprocity-gap Waveform Inversion (FRgWI): Misfit Functional

The quantitative reconstruction of the subsurface elastic properties (i.e., the model parameters $\lambda$, $\mu$ and $\rho$) is recast as an iterative minimization problem following the FWI method. We consider a line of devices that acquire the data, in terms of displacement and strain, and we design a specific misfit functional based upon the reciprocity-gap. Along the line of receivers, we further assume that the medium parameters are known, that is, the Lamé parameters in the elastic isotropic case, and they are referred to as $\lambda_0$ and $\mu_0$. The measurements are referred to by $\mathbf{d}(\mathbf{g})$, and consist of the observations of the displacement fields, $\mathbf{d}_u(\mathbf{g})$, and of the strain, $\mathbf{d}_\varepsilon(\mathbf{g})$, associated with an observational source $\mathbf{g}$. By denoting with $X_{\text{rcv}}$ the position of the receivers where the data are acquired, we define the misfit functional such that,

$$\mathcal{J}(\mathbf{m}, \omega) = \frac{1}{2} \sum_{j=1}^{n_{\text{sim}}} \sum_{k=1}^{n_{\text{obs}}} \left\| \int_{X_{\text{rcv}}} \left[ \mathbf{u}(\mathbf{x}, f_j, \mathbf{m}, \omega) \cdot \left( 2\mu_0(\mathbf{x}) \mathbf{d}_\varepsilon(\mathbf{x}, g_k, \omega) \cdot \mathbf{n} + \lambda_0(\mathbf{x}) \text{Tr}(\mathbf{d}_\varepsilon(\mathbf{x}, g_k, \omega)) \mathbf{n} \right) - \mathbf{d}_u(\mathbf{x}, g_k, \omega) \cdot \left( 2\mu_0(\mathbf{x}) \varepsilon(\mathbf{x}, f_j, \mathbf{m}, \omega) \cdot \mathbf{n} + \lambda_0(\mathbf{x}) \text{Tr}(\varepsilon(\mathbf{x}, f_j, \mathbf{m}, \omega)) \mathbf{n} \right) \right] \, d\mathbf{x} \right\|^2,$$

derived from Green’s formula, considering the sources of the observation above the line of receivers, cf. Appendix A of Faucher et al. (2019). Here, $\mathbf{g}$ and $\mathbf{f}$ stand for the sources
(right-hand sides of equation 1) that generate the observations and the simulations, respectively. That is, \( g \) are the “real” sources, that generate the measurements data, while \( f \) are “virtual” sources, arbitrarily chosen, used for the simulations. Therefore, the real sources \( g \) are not explicitly needed to construct the functional, allowing for a different number of observational sources, \( n_{\text{obs}}^{\text{src}} \), compared to the number of computational ones, \( n_{\text{sim}}^{\text{src}} \). For the discretization of the integral along the line of receivers, we can use a sum over all point-wise receivers, however, one can use a weighted sum (e.g., following a quadrature rule) to account for different configurations (Montagner et al., 2012), in the case where the receivers are not equally distributed.

The essence of this approach is twofold: first, it does not compare the observations and simulations directly but instead works with their products (correlation). Secondly, this product is made of different fields (i.e., displacement and strain). Consequently, the set of sources for the data \( (g_k) \) and for the simulations \( (f_j) \) is separated, allowing for different setups. For instance, the positions of the sources that generate the measurements are not needed to assemble the misfit, because the simulations can use different ones.

2.3 From Distributed Acoustic Sensing (DAS) to Reciprocity-gap

Our misfit functional is defined in terms of the displacement and of the strain. As mentioned in the introduction, Lim Chen Ning and Sava (2018) show that the components of the strain tensor can be retrieved from DAS acquisition system, providing a specific helix configuration of the deployed cables. Here, we show that the functional equation 6 is equivalent to working with the normal stress.

Using the Hooke’s law equation 2, the coefficients of the stress tensor are given by,

\[
\sigma_{ij} = \sum_k \sum_l C_{ijkl} \epsilon_{kl},
\]

(7)

where the indices refer to the coefficients of the tensors \( \sigma, C \) and \( \epsilon \). Then, the component of the normal stress in the direction \( d \) (i.e., \( d = \{x,y,z\} \)) is written as

\[
[\sigma \cdot n]_d = \sum_j \sigma_{dj} n_j = \sum_j \sum_k \sum_l C_{dijkl} \epsilon_{kl} n_j.
\]

(8)

Therefore, to retrieve the normal stress from the measurements of the strain in an anisotropic medium, one needs all of the coefficients of the strain and the knowledge of the physical parameters \( \{C_{ijkl}\} \) at the position of the receivers.

In the case of elastic isotropy, equation 7 is simplified as the elasticity tensor \( C \) is defined from the Lamé parameters only, from the Hooke’s law given with equation 3, hence we have,

\[
[\sigma \cdot n]_d = 2 \mu \sum_j \epsilon_{dj} n_j + \lambda Tr(\epsilon) n_d = 2 \mu [\epsilon \cdot n]_d + \lambda \left( \sum_j \epsilon_{jj} \right) n_d.
\]

(9)

In order to write the reciprocity-gap formula that works with the normal stress, we require, in terms of the strain (i.e., the observables in DAS acquisition), from equation 9:

1. measurements of the normal strain \( \epsilon \cdot n \),

2. measurements of the trace of the strain tensor (or the diagonal coefficients \( \epsilon_{jj} \)),

3. the physical parameters \( \lambda_0(x) \) and \( \mu_0(x) \), at the position of the receivers, i.e., for \( x \in X_{\text{rcv}} \).
We note that from inserting the normal stress given by equation 9 in the functional derived in equation 6, we can equivalently write,

\[ J_\sigma(m, \omega) = \frac{1}{2} \sum_{j=1}^{n_{\text{src}}} \sum_{k=1}^{n_{\text{obs}}} \left\| \int_{X_{\text{rec}}} \left( u(x, f_j, m, \omega) \cdot d_{\sigma}(x, g_k, \omega) - d_u(x, g_k, \omega) \cdot (\sigma(x, f_j, m, \omega) \cdot n) \right) d x \right\|^2, \]

where \( d_{\sigma} \) denotes the normal stress data. This relates to the Green’s identity, see, e.g., in the acoustic settings, Colton and Haddar (2005) and the Appendix A of Faucher et al. (2019). Let us note that, as an alternative to the measurements of the strain, we can also imagine to work with a very fine lattice of receivers that record only the displacements. If the lattice is sufficiently refined, we can deduce the stress (and strain) using, e.g., a spatial finite-differences approximation for the derivative, together with the knowledge of the medium parameters at the position of the devices, with equation 2. This is further possible as the acquisitions in seismic exploration usually maps densely the surface.

### 2.4 Inversion using the Combination of the Data

Data obtained for seismic exploration are generated by vibroseis trucks, thus controlled by the experimenter. It usually consists in several hundreds or thousands of independent point-sources, which provide a good illumination of the domain from the surface: these are reflection data. On the other hand, in the case of a teleseismic event, the source can be several kilometers below the Earth’s surface but these are yet sparser because the sources are not controlled. The nature of the sources also differs between the two situations: in practice, the source of a teleseismic event is characterized by the moment tensor while the vibroseis truck imposes a Neumann boundary condition (via the traction), see, e.g., Aki and Richards (2002); Carcione (2007); Baeten (1989) and Shi et al. (2019). Let us first note that the FRgWI method does not require the characterization of the observational sources and, in addition, one can use a dense set of computational sources to compensate for a sparse observational set, as highlighted by Faucher et al. (2019).

In exploration acquisition, the typical frequency peak of the source function is of 15–20 Hz, resulting in unusable (noisy) low-frequency content. On the contrary, the teleseismic data contain signal of low frequencies (as low as 0.1 Hz). Therefore, the iterative minimization is conducted following the two steps:

1. we minimize \( J \) using the teleseismic data for \( d_u \) and \( d_\sigma \). This corresponds to a few sources relatively far from the domain of interest, but where the low-frequency content is usable.

2. From the low-frequency model built after step 1, we minimize \( J \) using the exploration data for \( d_u \) and \( d_\sigma \). Here, the acquisition is denser and the frequency content higher, to recover the finer scales.

It is crucial that, numerically speaking, the steps 1 and 2 do not require a different computational domain. FRgWI works with arbitrary observational sources and, for instance, here, they are taken outside of the computational area. That is, the computational domain only consists in the exploration part, even if all reflections are not taken into account, as illustrated in Figure 1. As an alternative, one can perform local updates after a computation is done on the global domain, as carried out by Robertsson and Chapman (2000) and Masson and Romanowicz (2017). In our work, we avoid the preliminary computation and remain on the small domain.
2.5 Gradient Computation

For the computation of the gradient of the misfit functional with respect to the parameter coefficients, we use the adjoint-state method, originally derived by Chavent (1974), and reviewed in the context of geophysics by Plessix (2006). Namely, the gradient is computed from the combination of the solution of the forward problem with the solution of a backward problem, thus avoiding the formation of a dense matrix $DF$. The backward problem is the adjoint of the forward problem and, using its symmetry, the same numerical machinery is re-used (that is, the matrix factorization of the time-harmonic discretized system), however with different right-hand sides, referred to as the ‘adjoint-sources’. The method is also at the heart of the adjoint-tomography technique in seismology, we refer to, for instance, Tromp et al. (2005); Fichtner et al. (2006a,b); Tape et al. (2007) and Bozdag et al. (2016). It can also be employed for second-order derivations, see, e.g., Wang et al. (1992); Fichtner and Trampert (2011) and M´ etivier et al. (2013).

For the sake of conciseness, we shall only detail the adjoint-sources, which are specific to the misfit functional, and refer the readers to, e.g., Pratt et al. (1998); Plessix (2006); Chavent (2010); Alessandrini et al. (2019) and Barucq et al. (2019) for details on the derivation of the adjoint-state method in seismic applications. Namely, the adjoint-state solves the backward/adjoint problem where the adjoint-sources are given by the derivatives of the misfit functional with respect to each component of the wavefield (here, $u$ and $\sigma$). We further refer to Alessandrini et al. (2019) for the adjoint-state associated with acoustic FRgWI, and to Faucher and Scherzer (2020) for its specificity with hybridizable discontinuous Galerkin discretization (see Remark 2).

We derive the implementation with respect to the normal stress, following equation 10, which can equivalently be replaced by the strain using equation 9. we denote with indices the component of the vector fields, such that $d_u = \{(d_u)_x, (d_u)_y, (d_u)_z\}$ and similarly for the measurements of the normal stress. Then, for each of the computational source $f_j$ in equation 10 corresponds a backward problem for which each of the unknowns (displacement and stress tensor) has a right-hand side. We refer to these right-hand sides by $W_\bullet$, using $d = \{x, y, z\}$ to denote the direction (e.g., $W_{ux}$ for the right-hand side of the backward problem associated with $u_x$), they are given by, for $x \in X_{rcv}$,

$$
W_{ud}(f_j, x) := \sum_{k=1}^{N_{src}} \eta(f_j, g_k) (d_u(g_k, x))_d,
$$

$$
W_{gd}(f_j, x) := -\sum_{k=1}^{N_{src}} \eta(f_j, g_k) (d_u(g_k, x))_d n_d,
$$

$$
W_{gd_{1}d_{2}}(f_j, x) := -\sum_{k=1}^{N_{src}} \eta(f_j, g_k) \sum_{d_1 \neq d_2} (d_u(g_k, x))_{d_1} n_{d_2}, \quad \text{for } d_1 \neq d_2,
$$

with

$$
\eta(f_j, g_k) = \int_{X_{rcv}} (u(f_j, x, m) \cdot d_u(g_k, x) - d_u(g_k, x) \cdot (\vec{\sigma}(x, f_j, m) \cdot n)) \, dx,
$$

using $\bar{\cdot}$ to denote the complex conjugation. We see that each of the adjoint-source takes the contribution from all the measurement sources ($g_k$) that is, from all the observed data.

Upon discretization, the solution of the wave problem made of equations 1 and 2 results in the solution of the linear system $AU = F$, where $U = [u, \sigma]$ represents the discretized
unknowns and $F = [f, 0]$ the right-hand side. The adjoint-problem consists in finding $V$ solution to the system $A^* V = W$, where $*$ denotes the adjoint. The gradient of the misfit functional with respect to the model parameter $m = \{\lambda, \mu, \rho\}$ is then given by (Pratt et al., 1998; Plessix, 2006; Barucq et al., 2019) $\nabla_m J = \langle \partial_m(A)U, V \rangle$, using the angle brackets to denote the complex inner product.

**Remark 1** (Reciprocity-gap at the surface). In the exploration acquisition, the source of the direct problem is given by the traction imposed by the vibroseis: $\sigma \cdot n = f$ at the surface $\Gamma$, while the right-hand side of equation 1 is set to zero (Baeten, 1989). In this case, with displacement data acquired by surface geophones (e.g., Figure 1b), it gives the Neumann-to-Dirichlet map (Shi et al., 2019), which graph forms the necessary Cauchy data for reciprocity-gap (i.e., displacement and normal stress). For surface acquisition, the misfit functional equation 10 is written by replacing the values of the normal stress by the imposed source traction, such that,

\[
J_{\text{surface}}(m) = \frac{1}{2} \sum_{j=1}^{n_{\text{src}}} \left\| \int_{X_{\text{rcv}} \subset \Gamma} (u(f_j, x, m) \cdot g_k(x) - d_u(g_k, x) \cdot f_j(x)) \, dx \right\|^2.
\]

To obtain the gradient, the backward problem correspond to a boundary value problem where the adjoint-sources, following the steps prescribed by Shi et al. (2019), are

\[
\sigma \cdot n = \sum_{k=1}^{n_{\text{obs}}} g_k(x) \int_{X_{\text{rcv}} \subset \Gamma} (u(f_j, x, m) \cdot g_k(x) - d_u(g_k, x) \cdot f_j(x)) \, dx,
\]

boundary value adjoint-sources for surface reciprocity-gap formula.

Nonetheless, as the source traction is imposed only at the position of the vibroseis base-plate (with zero elsewhere), it is unclear if the misfit will perform well in this configuration.

In the following experiment, we only consider, per convenience, the sub-surface functional equation 6 for the exploration acquisition. That is, we remain in the context of buried devices.

**Remark 2** (Numerical implementation). We need to evaluate both the components of the displacement fields and of the stress tensor (or the strain). In our implementation, we use the Hybridizable Discontinuous Galerkin (HDG) method (Arnold et al., 2002; Cockburn et al., 2009) for the discretization of equation 1. The motivation is that HDG solves the first-order problem (i.e., the system made of equations 1 and 2), hence gives access to both the displacement and stress tensor with similar accuracy. Nonetheless, it designs a global matrix for the linear system that only accounts for the degrees of freedoms on the faces of the elements, for one variable (the displacement vector field). Therefore, the size of the linear system remains relatively small compared to other discretization methods such as Finite Element or Internal Penalty Discontinuous Galerkin discretizations in which, if one wants to solve the first-order system, one has to generate a linear system which contains all degrees of freedom of all unknowns (stress and displacement). Here, the linear system is smaller, and only takes the traces of one of the unknowns. For similar reason, HDG is also used in the acoustic settings for FRgWI with the Euler’s equation by Faucher et al. (2019). We further refer to Kirby et al. (2012); Bonnasse-Gahot et al. (2017) and Faucher and Scherzer (2020) for more details on the HDG discretization and its performance.
3 Computational Experiment

We illustrate the performance of FRgWI with a two-dimensional isotropic elastic experiment, where we consecutively use exploration and teleseismic data. The generation of the data takes a domain of size $28 \text{ by } 5 \text{ km}^2$ and we consider a domain of size $22 \text{ by } 4.5 \text{ km}^2$ for the reconstruction. Therefore, absorbing boundary conditions are positioned on the sides of the computational box (except at the surface) for inversion, while an extended medium is used for the generation of the data, as illustrated in Figure 1. Namely, we consider a synthetic experiment but consider reflections from outside of the area of interest. The elastic properties are pictured in Figures 2a, 4a and 4c, with the density, P-wave speed and S-wave speed respectively, and where we indicate by white dashes the restriction to the inversion domain. This domain for inversion corresponds to the section from 4.5 to 22 km in $x$ and from 0 to 4.5 km in depth of the domain used to generate the data. The model is composed of a body of high contrast in its center, with layer structures on the sides and below. In addition, there also is an important contrast in speeds between the layers.

For the reconstruction, we follow the situation described in Subsection 2.4 and work with both exploration and teleseismic data.

1. First, we minimize $J$ using the data from ten teleseismic sources which are located around the area of interest such that none of these sources are in computational domain, see Figure 1a. We consequently have $n_{\text{src}}^{\text{obs}} = 10$ in equation 6: this is a relatively small set of data but it contains usable low-frequency, and we use contents from $0.2 \text{ Hz}$ to $2 \text{ Hz}$.

2. Next, we use data from an acquisition setup that corresponds with an exploration configuration, with $n_{\text{src}}^{\text{obs}} = 89$ sources located at the surface. This set of data is dense but does not contain the low-frequency, below $2 \text{ Hz}$.

Despite using synthetic data, we incorporate noise in the measurements and use a different numerical setup (e.g., different meshes and order of the polynomials). We assume the measurements are acquired by 359 receivers, located below the surface in a regular distribution, every 50 m. The initial models for the reconstruction are shown in Figures 2b, 4b and 4d. In our experiment, we further assume the knowledge of the near surface area where the parameters are taken constant (see Figure 1). This allows us to reduce the effect of the free-surface condition that generates reflections, increasing the nonlinearity of the inversion procedure (Brossier et al., 2009). As an alternative, the use of a regularization term or a smoothing filter applied onto the gradient can also be used to balance the contributions of the free-surface (Guitton et al., 2012; Trinh et al., 2017).

With FRgWI, the choice of the sources for the computational acquisition is arbitrary (compared to more traditional misfit criterion which must respect the observational sources) and we take $n_{\text{src}}^{\text{sim}} = 89$ computational sources, similarly to the exploration setup, for simplicity only. We refer to Faucher et al. (2019) for more details on the flexibility in the choice of the numerical acquisition, and investigation on the efficiency with respect to shot summation. In Figures 3a and 3b, we compare the gradient of the misfit functional with respect to the Lamé parameter $1/\mu$ for the teleseismic and the exploration data set, using the starting models of Figure 4, where we force to zero the upper (known) area. In both cases, the acquisition for the simulation (which is arbitrary with FRgWI) is using sources near the surface only, as mentioned above. For the teleseismic data, we use a frequency of $0.2 \text{ Hz}$: we see in Figure 3a that the gradient consequently shows the long wavelength variation. We notice higher amplitudes on the sides and bottom regions, that is, near where the teleseismic sources are located, even if the computational ones are positioned near the surface. With the
exploration data set, Figure 3b, we use frequency 2 Hz: we observe that the structures are smaller (higher frequency), with higher amplitudes in the upper part of the domain, where the exploration sources are located.

Figure 2: Target (a) density model of size 28 by 5 km\(^2\), and (b) starting model for reconstruction, which cover the area marked with the white dashes, of size 22 by 4.5 km\(^2\). During the iterative reconstruction, the density remains fixed at its starting representation.

Figure 3: Comparison of the FRgWI gradient for the parameter \(1/\mu\) (a) using the teleseismic data \(n_{\text{obs}}^{\text{src}} = 10\) sources at 0.2 Hz and (b) using the exploration data \(n_{\text{obs}}^{\text{src}} = 89\) sources at 2 Hz: both use the same acquisition for the simulations \(n_{\text{sim}}^{\text{src}} = 89\) sources and we force the values to zero in the layer where the parameters are known, according to Figure 1.

We follow a sequential progression of frequency, as advocated by Faucher et al. (2020), and perform 30 minimization iterations per frequency. Our reconstruction focuses on the P- and S-wave speeds, see equation 4, as the density is known to be more difficult to retrieve. In our experiment, the density remains as its initial representation of Figure 2b. Nonetheless, this lack of information should not prevent us from recovering the other parameters and we further select the parameter \(1/\lambda\) and \(1/\mu\) for the inversion, cf. Faucher (2017). The results after the iterations with the low-frequency teleseismic data are pictured in Figures 4g and 4h. The low-frequency reconstruction provides the background profile of the parameters, and is able to discover the contrasting body in the center. As expected, the parameters are smooth at this stage, with only the long wavelength investigated. Despite the use of data from few sources, located outside of the numerical area, the inversion procedure produces the expected outcome, giving access to the low-frequency profiles of the physical parameters.

We continue the reconstruction procedure consisting in iterative updates of the model parameters, now working with the exploration data, using nine frequencies from 2 Hz to 10 Hz (every 1 Hz), performing 30 iterations per individual frequency. It starts from the initial models built from the teleseismic data. The P-wave speed and S-wave speed that are
Figure 4: Target (a) P-wave speed and (c) S-wave speed models of size 28 by 5 km\(^2\) from which the observed measurements are generated. The inverse problem is conducted on a restricted area marked with the white dashes, of size 22 by 4.5 km\(^2\). The initial (b) P-wave speed and (d) S-wave speed cover this area. The reconstructed (e) P-wave speed and (f) S-wave speed use the exploration data from 2 Hz to 10 Hz starting from the (g) P-wave speed and (h) S-wave speed models built with the low-frequency teleseismic data. (i) Vertical sections of the target, starting and reconstructed models in \(x = 16.5\) km.

reconstructed with FRgWI are respectively shown in Figures 4e and 4f. We see that the first stage, using the low-frequency teleseismic data, allows for the reconstruction of the finer scales when we switch to the exploration data. The upper part of the contrast is correctly
anticipated, and some of the underneath layers appear, even if the deep and side parts of the models are still harder to detect due to the limited illumination of the deeper area. We picture a vertical section in \( x = 16.5 \text{ km} \) in Figure 4i where we compare the target, starting and reconstructed speeds. It highlights the accuracy of the upper part of the model, while the deeper area are more difficult to obtain.

4 Discussion

The main feature of FRgWI is to allow the separation of the computational and observational sources as in equation 6, every simulation is tested against each observation, and the source employed to generate the data does \textit{not} have to match the one used for the simulation (respectively \( g_k \) and \( f_j \)), which can thus be taken arbitrarily. In this work, we have been able to combine exploration data with teleseismic ones to perform seismic imaging. In particular, we have experimented the use of data from events (observational sources) located outside of the computational domain, without requiring any pre-processing. Indeed, with FRgWI, we do \textit{not} need to know the precise origin of the source (here, the teleseismic event) and we do \textit{not} have to generate simulations on the large domain where the event has occurred because we only illuminate from simulations within the area of interest. Here, the use of teleseismic data compensates for the lack of low-frequency contents in the exploration ones, and are used as a first step to build initial models, that is, to recover the long spatial wavelength.

Applications of the method is directly related to the improvement of the acquisition techniques with fiber optic cable in DAS, in order to obtain the strain. A future task concerns addressing the accuracy of required DAS technology, and the available bandwidth of such measurements. Then, the implementation from the strain also requires the values of the physical parameters (except the density) along the fiber optic cable. For future applications, the following steps should be employed with (1) reconstruction of the very near-surface elastic parameters using the (surface) exploration data, (2) minimization of the sub-surface reciprocity functional using the teleseismic data to recover the coarse scale of the models and (3) reconstruction of the finer scales with the exploration data from vibroseis. In this work, we carried out a pilot computational experiment to study the capability of our approach and to illustrate how to exploit DAS in an essential way.

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