On a class of Feynman integrals evaluating to iterated integrals of modular forms

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Abstract In this talk we discuss a class of Feynman integrals, which can be expressed to all orders in the dimensional regularisation parameter as iterated integrals of modular forms. We review the mathematical prerequisites related to elliptic curves and modular forms. Feynman integrals, which evaluate to iterated integrals of modular forms go beyond the class of multiple polylogarithms. Nevertheless, we may bring for all examples considered the associated system of differential equations by a non-algebraic transformation to an $\varepsilon$-form, which makes a solution in terms of iterated integrals immediate.

1 Introduction

It is an open and interesting question to which class of transcendental functions Feynman integrals evaluate. At present, we do not have a general answer. However, there are sub-classes of Feynman integrals for which the class of functions is known. First of all, there is the class of Feynman integrals evaluating to multiple polylogarithms. This covers in particular all one-loop integrals. Starting from two-loops, there are Feynman integrals which cannot be expressed in terms of multiple polylogarithms. The simplest example is given by the two-loop equal-mass sunrise integral [1,20]. Integrals, which do not evaluate to multiple polylogarithms are now an active field of research in particle physics [21,42] and string theory [43,48]. In this talk we focus on a class of Feynman integrals which evaluate to iterated integrals of modular forms. Feynman integrals of this class are associated to one elliptic
curve and depend on one scale $x = p^2/m^2$. They can be seen as generalisations of single-scale Feynman integrals evaluating to harmonic polylogarithms \cite{49,50}. We expect that all our examples are equally well expressible in terms of elliptic polylogarithms \cite{14,17,21,23,37,40,51,56}. The representation in terms of iterated integrals of modular forms has certain advantages:

1. It combines nicely with the technique of differential equations, which by now is the main tool for solving Feynman integrals \cite{57,57}. In fact, for all examples considered we are able to bring the system of differential equations into an $\varepsilon$-form.
2. It only involves a finite number of integration kernels. The integration kernels are modular forms.
3. It allows for an efficient numerical evaluation through the $q$-expansion around the cusps \cite{25}.

Let us also mention, that albeit an important sub-class, this class is not the end of the story. Multi-scale integrals beyond the class of multiple polylogarithms may involve more than one elliptic curve, as seen for example in the double box integral relevant to top-pair production with a closed top loop \cite{27,28}.

### 2 Periodic functions and periods

Let us consider a non-constant meromorphic function $f$ of a complex variable $z$. A period $\omega$ of the function $f$ is a constant such that

$$f(z + \omega) = f(z)$$

for all $z$. The set of all periods of $f$ forms a lattice $\Lambda$, which is either

1. trivial: $\Lambda = \{0\}$,
2. a simple lattice, generated by one period $\omega: \Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
3. a double lattice, generated by two periods $\omega_1, \omega_2$ with $\text{Im}(\omega_2/\omega_1) \neq 0$:

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}.$$  

It is common practice to order these two periods such that $\text{Im}(\omega_2/\omega_1) > 0$.

An example for a singly periodic function is given by

$$\exp(z).$$

In this case the simple lattice is generated by $\omega = 2\pi i$. An example for a doubly periodic function is given by Weierstrass’s $\wp$-function. Let $\Lambda$ be the lattice generated by $\omega_1$ and $\omega_2$ Then
\( \wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left( \frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right). \) (4)

\( \wp(z) \) is periodic with periods \( \omega_1 \) and \( \omega_2 \). Of particular interest are also the corresponding inverse functions. These are in general multivalued functions. In the case of the exponential function \( x = \exp(z) \), the inverse function is given by

\[ z = \ln(x). \] (5)

The inverse function to Weierstrass’s elliptic function \( x = \wp(z) \) is an elliptic integral given by

\[ z = \int_{x}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}} \] (6)

with

\[ g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6}. \] (7)

In both examples the periods can be expressed as integrals involving only algebraic functions. For the first example we may express the period of the exponential function as

\[ 2\pi i = 4i \int_{0}^{1} \frac{dt}{\sqrt{1 - t^2}}. \] (8)

For the second example of Weierstrass’s \( \wp \)-function let us assume that \( g_2 \) and \( g_3 \) are two given algebraic numbers. The periods are expressed as

\[ \omega_1 = 2 \int_{t_1}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}, \quad \omega_2 = 2 \int_{t_3}^{t_2} \frac{dt}{\sqrt{4t^3 - g_2t - g_3}}, \] (9)

where \( t_1, t_2 \) and \( t_3 \) are the roots of the cubic equation \( 4t^3 - g_2t - g_3 = 0 \).

The representation of the periods of \( \exp(z) \) and \( \wp(z) \) in the form of eq. (8) and eq. (9) is the motivation for the following generalisation, due to Kontsevich and Zagier [68]:

A numerical period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in \( \mathbb{R}^n \) given by polynomial inequalities with rational coefficients. Domains defined by polynomial inequalities with rational coefficients are called semi-algebraic sets.

We denote the set of numerical periods by \( \mathbb{P} \). The numerical periods \( \mathbb{P} \) are a countable set of numbers. We may replace in the above definition every occurrence
of “rational function” with “algebraic function” and every occurrence of “rational number” with “algebraic number” without changing the set of numbers $\mathbb{P}$. Then it is clear, that the integrals in eq. (8) and eq. (9) are numerical periods in the sense of the above definition, and so is for example $\ln 2$, since

$$\ln 2 = \int_1^2 \frac{dt}{t}.$$  \hspace{1cm} (10)

3 Elliptic curves

A double lattice $\Lambda$ arises naturally from elliptic curves. Let us consider the elliptic curve

$$E : w^2 - (z - z_1)(z - z_2)(z - z_3)(z - z_4) = 0,$$ \hspace{1cm} (11)

where the roots $z_j$ may depend on variables $x = (x_1, ..., x_t)$:

$$z_j = z_j(x), \quad j \in \{1, 2, 3, 4\}. \hspace{1cm} (12)$$

We set

$$Z_1 = (z_3 - z_2)(z_4 - z_1), \quad Z_2 = (z_2 - z_1)(z_4 - z_3), \quad Z_3 = (z_3 - z_1)(z_4 - z_2). \hspace{1cm} (13)$$

Note that we have $Z_1 + Z_2 = Z_3$. We define the modulus and the complementary modulus of the elliptic curve $E$ by

$$k^2 = \frac{Z_1}{Z_3}, \quad \bar{k}^2 = 1 - k^2 = \frac{Z_2}{Z_3}. \hspace{1cm} (14)$$

Note that there are six possibilities of defining $k^2$. Our standard choice for the periods $\psi_1, \psi_2$ is

$$\psi_1 = \frac{4K(k)}{Z_3^2}, \quad \psi_2 = \frac{4iK(\bar{k})}{Z_3^2}, \hspace{1cm} (15)$$

where $K(x)$ denotes the complete elliptic integral of the first kind. These two periods generate a lattice $\Lambda = \{n_1 \psi_1 + n_2 \psi_2 \mid n_1, n_2 \in \mathbb{Z}\}$. We denote the ratio of the two periods and the nome squared by

$$\tau = \frac{\psi_2}{\psi_1}, \quad q = e^{2i\pi \tau}. \hspace{1cm} (16)$$

Let us note that our choice of periods is not unique. Any other choice related to the original one by
Feynman integrals evaluating to iterated integrals of modular forms

On a class of Feynman integrals evaluating to iterated integrals of modular forms

Fig. 1 The periods \((1, \tau)\) and \((1, \tau')\) generate the same lattice.

\[
\begin{pmatrix}
\psi_2' \\
\psi_1'
\end{pmatrix}
= \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\begin{pmatrix}
\psi_2 \\
\psi_1
\end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
\]

(17)
generates the same lattice \(\Lambda\). This is shown in fig. 1. In terms of \(\tau\) and \(\tau' = \psi_2'/\psi_1'\)
the transformation in eq. (17) reads

\[
\tau' = \frac{a\tau + b}{c\tau + d}
\]

(18)
and equals a Möbius transformation. In this talk we are in particular interested in
the situation, where the roots \(z_j\) in eq. (12) depend only on a single variable \(x\). In
this case we may exchange the variable \(x\) for the variable \(\tau\) and study our problem
as a function of \(\tau\).

4 Modular forms

Let us now consider functions of \(\tau\). We are interested in functions with “nice” prop-
erties under transformations of the form as in eq. (18). We denote by \(\mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im}(\tau) > 0 \}\) the complex upper half plane and by \(\overline{\mathbb{H}}\) the extended upper half plane

\[
\overline{\mathbb{H}} = \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}.
\]

(19)
A meromorphic function \(f : \mathbb{H} \to \mathbb{C}\) is a modular form of modular weight \(k\) for
\(\text{SL}(2, \mathbb{Z})\) if

(i) \(f\) transforms under Möbius transformations as

\[
\begin{pmatrix}
\psi_2' \\
\psi_1'
\end{pmatrix}
= \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\begin{pmatrix}
\psi_2 \\
\psi_1
\end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})
\]

(17)
\[ f \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^k \cdot f(\tau) \quad \text{for} \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}) \quad (20) \]

(ii) \( f \) is holomorphic on \( \mathbb{H} \),
(iii) \( f \) is holomorphic at \( \infty \).

We may also look at subgroups of \( \text{SL}(2, \mathbb{Z}) \). The standard congruence subgroups are defined by

\[
\Gamma_0(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}) : c \equiv 0 \mod N \right\}, \\
\Gamma_1(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}) : a, d \equiv 1 \mod N, c \equiv 0 \mod N \right\}, \\
\Gamma(N) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}) : a, d \equiv 1 \mod N, b, c \equiv 0 \mod N \right\}. \quad (21) 
\]

Let us also introduce the following notation: For an integer \( k \) and a matrix \( \gamma \in \text{SL}(2, \mathbb{Z}) \) we define \( f|_k \gamma \) by

\[
(f|_k \gamma)(\tau) = (c \tau + d)^{-k} \cdot f(\gamma(\tau)). \quad (22) 
\]

With this definition we may re-write the condition (i) in eq. (20) as

\[ f|_k \gamma = f \quad \text{for all} \quad \gamma \in \text{SL}(2, \mathbb{Z}). \quad (23) \]

We may now define modular forms for a congruence subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{Z}) \). A meromorphic function \( f : \mathbb{H} \to \mathbb{C} \) is a modular form of modular weight \( k \) for \( \Gamma \) if

(i) \( f \) transforms as

\[ f|_k \gamma = f \quad \text{for all} \quad \gamma \in \Gamma. \quad (24) \]

(ii) \( f \) is holomorphic on \( \mathbb{H} \),
(iii) \( f|_k \alpha \) is holomorphic at \( \infty \) for all \( \alpha \in \text{SL}(2, \mathbb{Z}) \).

For each congruence subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{Z}) \) there is a smallest positive integer \( N \), such that \( \Gamma(N) \subseteq \Gamma \). The integer \( N \) is called the level of \( \Gamma \). A modular form \( f \) for the congruence subgroup \( \Gamma \) of level \( N \) has the Fourier expansion

\[ f(\tau) = \sum_{n=0}^{\infty} a_n q_N^n \quad \text{with} \quad q_N = e^{2\pi i \tau/N}. \quad (25) \]

\( f \) is called a cusp form, if \( a_0 = 0 \) in the Fourier expansion of \( f|_k \alpha \) for all \( \alpha \in \text{SL}(2, \mathbb{Z}) \).
5 Iterated integrals

We review Chen’s definition of iterated integrals \[69\]: Let \( M \) be a \( t \)-dimensional manifold and \[ \gamma : [0,1] \rightarrow M \] (26) a path with start point \( x_i = \gamma(0) \) and end point \( x_f = \gamma(1) \). Suppose further that \( \omega_1, ..., \omega_k \) are differential 1-forms on \( M \). Let us write \[ f_j(\lambda) d\lambda = \gamma^* \omega_j \] (27) for the pull-backs to the interval \([0,1]\). For \( \lambda \in [0,1] \) the \( k \)-fold iterated integral of \( \omega_1, ..., \omega_k \) along the path \( \gamma \) is defined by \[ I_{\gamma}(\omega_1, ..., \omega_k; \lambda) = \int_0^\lambda f_1(\lambda_1) d\lambda_1 \int_0^\lambda f_2(\lambda_2) d\lambda_2 \ldots \int_0^\lambda f_k(\lambda_k) d\lambda_k. \] (28)

We define the 0-fold iterated integral to be \[ I_{\gamma}(; \lambda) = 1. \] (29)

We have \[ \frac{d}{d\lambda} I_{\gamma}(\omega_1, \omega_2, ..., \omega_k; \lambda) = f_1(\lambda) I_{\gamma}(\omega_2, ..., \omega_k; \lambda). \] (30)

Let us now discuss two special cases: Multiple polylogarithms and iterated integrals of modular forms. Multiple polylogarithms are iterated integrals, where all differential one-forms are of the form \[ \gamma^* \omega_j = \frac{d\lambda}{\lambda - z_j}. \] (31)

For \( z_w \neq 0 \) they are defined by \[70, 74\] \[ G(z_1, ..., z_w; y) = \int_0^{y_1} \frac{dy_1}{y_1 - z_1} \int_0^{y_2} \frac{dy_2}{y_2 - z_2} \ldots \int_0^{y_w} \frac{dy_w}{y_w - z_w}. \] (32)

The number \( w \) is referred to as the weight of the multiple polylogarithm or the depth of the integral representation. Let us introduce the short-hand notation \[ G_{m_1, ..., m_k}(z_1, ..., z_k; y) = G(0, ..., 0, z_1, ..., z_{k-1}, 0, ..., 0, z_k; y), \] (33)
where all $z_j$ for $j = 1, \ldots, k$ are assumed to be non-zero. This allows us to relate the integral representation of the multiple polylogarithms to the sum representation of the multiple polylogarithms. The sum representation is defined by

$$L_{m_1, \ldots, m_k}(x_1, \ldots, x_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 0} \frac{x_1^{n_1}}{n_1^{m_1}} \cdots \frac{x_k^{n_k}}{n_k^{m_k}}. \quad (34)$$

The number $k$ is referred to as the depth of the sum representation of the multiple polylogarithm, the weight is now given by $m_1 + m_2 + \cdots + m_k$. The relations between the two representations are given by

$$L_{m_1, \ldots, m_k}(x_1, \ldots, x_k) = (-1)^k G_{m_1, \ldots, m_k}(1, x_1, \ldots, 1),$$

$$G_{m_1, \ldots, m_k}(z_1, \ldots, z_k; y) = (-1)^k L_{m_1, \ldots, m_k}(\frac{y - z_1}{z_1}, \frac{z_1}{z_2}, \ldots, \frac{z_k - 1}{z_k}). \quad (35)$$

If one further sets $g(z; y) = 1/(y - z)$, then one has

$$\frac{d}{dy} G(z_1, \ldots, z_w; y) = g(z_1; y) G(z_2, \ldots, z_w; y)$$

and

$$G(z_1, z_2, \ldots, z_w; y) = \int_0^y dy_1 g(z_1; y_1) G(z_2, \ldots, z_w; y_1). \quad (37)$$

One can slightly enlarge the set of multiple polylogarithms and define $G(0, \ldots, 0; y)$ with $w$ zeros for $z_1$ to $z_w$ to be

$$G(0, \ldots, 0; y) = \frac{1}{w!} (\ln y)^w. \quad (38)$$

This permits us to allow trailing zeros in the sequence $(z_1, \ldots, z_w)$ by defining the function $G$ with trailing zeros via eq. (37) and eq. (38).

Our second example are iterated integrals of modular forms. Let $f_1(\tau), f_2(\tau), \ldots, f_k(\tau)$ be modular forms of a congruence subgroup. Let us further assume that $f_k(\tau)$ vanishes at the cusp $\tau = i\infty$. For iterated integrals of modular forms we set

$$\omega_j = 2\pi i \int f_j(\tau) \, d\tau.$$ 

Thus the $k$-fold iterated integral of modular forms is given by

$$(2\pi i)^k \int_{i\infty}^x d\tau_1 f_1(\tau_1) \int_{i\infty}^{\tau_1} d\tau_2 f_2(\tau_2) \cdots \int_{i\infty}^{\tau_{k-1}} d\tau_k f_k(\tau_k). \quad (40)$$
The case where $f_k(τ)$ does not vanishes at the cusp $τ = i\infty$ is discussed in \[24, 75\] and is similar to trailing zeros in the case of multiple polylogarithms.

6 Precision calculations

Due to the smallness of all coupling constants $g$, we may compute at high energies an infrared-safe observable (for example the cross section $σ$ for a particular process) reliable in perturbation theory:

$$σ = \left(\frac{g}{4\pi}\right)^4 σ_{LO} + \left(\frac{g}{4\pi}\right)^6 σ_{NLO} + \left(\frac{g}{4\pi}\right)^8 σ_{NNLO} + ...$$  \hspace{1cm} (41)

The cross section is related to the square of the scattering amplitude

$$σ \sim |A|^2,$$  \hspace{1cm} (42)

and the perturbative expansion of the cross section follows from the perturbative expansion of the amplitude

$$A = g^2 A^{(0)} + g^4 A^{(1)} + g^6 A^{(2)} + ...,$$  \hspace{1cm} (43)

where $A^{(l)}$ contains $l$ loops. The computation of the tree amplitude $A^{(0)}$ poses no conceptional problem. For loop amplitudes we have to calculate Feynman integrals. Let us write

$$A^{(l)} = \sum_j c_j I_j,$$  \hspace{1cm} (44)

where the $I_j$’s are Feynman integrals and the $c_j$’s are coefficients, whose computation is tree-like. Without loss of generality we may take the set of Feynman integrals $\{I_1, I_2, ...\}$ to consist of scalar integrals \[76, 77\]. Let us now look closer on the Feynman integrals. A Feynman graph $G$ with $n$ external lines, $r$ internal lines and $l$ loops corresponds (up to prefactors) in $D$ space-time dimensions to the family of Feynman integrals, indexed by the powers of the propagators $ν_j$

$$K_{ν_1, ν_2, ..., ν_r} = \frac{\prod_{j=1}^r Γ(ν_j)}{Γ(ν - ID/2)} (μ^2)^{ν-ILD/2} \int \prod_{s=1}^l \frac{d^Dk_s}{iπ^{D/2}} \prod_{j=1}^r \frac{1}{(−q_j^2 + m_j^2)ν_j},$$  \hspace{1cm} (45)

with $ν = ν_1 + ... + ν_r$. The momenta flowing through the internal lines can be expressed through the independent loop momenta $k_1, ..., k_l$ and the external momenta $p_1, ..., p_n$ as

$$q_i = \sum_{j=1}^l λ_{ij} k_j + \sum_{j=1}^n σ_{ij} p_j, \quad λ_{ij}, σ_{ij} \in \{-1, 0, 1\}.$$  \hspace{1cm} (46)
After Feynman parametrisation we obtain
\[ I_{\nu_1 \nu_2 \ldots \nu_r}^G = \int_\Delta \Omega \left( \prod_{j=1}^{r} x_j^{\nu_j-1} \right) \frac{\mathcal{U}^{-l(l+1)/2} \mathcal{F}^{-l(D-2)}}{\mathcal{F}^{-l(D-2)}}. \] (47)

The prefactors in the definition of the Feynman integral in eq. (45) are chosen such that after Feynman parametrisation we obtain an expression without prefactors, as can be seen from eq. (47). In eq. (47) the integration is over
\[ \Delta = \{ [x_1 : x_2 : \ldots : x_r] \in \mathbb{P}^{r-1} | x_i \geq 0 \}. \] (48)

Here, \( \mathbb{P}^{r-1} \) denotes the real projective space with \( r-1 \) dimensions. \( \Omega \) is a differential \( (r-1) \)-form given by
\[ \Omega = \sum_{j=1}^{r} (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \hat{d}x_j \wedge \ldots \wedge dx_r, \] (49)

where the hat indicates that the corresponding term is omitted. The functions \( \mathcal{U} \) and \( \mathcal{F} \) are obtained from first writing
\[ \sum_{j=1}^{r} x_j (-q_j^2 + m_j^2) = -\sum_{a=1}^{l} \sum_{b=1}^{l} k_a M_{ab} k_b + \sum_{a=1}^{l} 2k_a \cdot Q_a - J, \] (50)

where \( M \) is a \( l \times l \) matrix with scalar entries and \( Q \) is a \( l \)-vector with \( D \)-vectors as entries. We then have
\[ \mathcal{U} = \det(M), \quad \mathcal{F} = \det(M) \left( -J + Q M^{-1} Q \right) / \mu^2. \] (51)

\( \mathcal{U} \) and \( \mathcal{F} \) are the first and second graph polynomial of the Feynman graph \( G \) [73].

The Feynman integral defined in eq. (47) has an expansion as a Laurent series in the parameter \( \epsilon = (4-D)/2 \) of dimensional regularisation:
\[ I_{\nu_1 \nu_2 \ldots \nu_r}^G = \sum_{f_j = f_{\min}}^{\infty} f_j \epsilon^j. \] (52)

The coefficients \( f_j \) are in general functions of the Lorentz invariants
\[ s_J = \left( \sum_{j \in J} p_j \right)^2, \] (53)

where the sum runs over a subset \( J \) of the external momenta, the internal masses \( m_i \) and the scale \( \mu \). We are interested in the question, to which class of functions the coefficients \( f_j \) belong. Let us first consider the situation, where we keep all Lorentz invariants, all masses and the scale fixed. Suppose that (i) all kinematical invariants \( s_J \) are negative or zero, (ii) all masses \( m_i \) and \( \mu \) are positive or zero (\( \mu \neq 0 \) and...
(iii) all ratios of invariants and masses are rational, then it can be shown that all coefficients \( f_j \) in eq. (52) are numerical periods \[79\].

Let us now return to the original problem and view the coefficients \( f_j \) as functions of the Lorentz invariants \( s_J \), the internal masses \( m_i \) and the scale \( \mu \). Let us consider a family of Feynman integrals \( I^{G \nu_1 \nu_2 \ldots \nu_r} \), including all its sub-topologies. A sub-topology \( G' \) is obtained by pinching in the graph \( G \) one or several internal lines. In the Feynman integral the corresponding propagators are then absent and the associated exponents \( \nu_j \) are zero. This is shown in fig. 2. Integration-by-parts identities \[80, 81\] allow us to express the Feynman integrals from the family \( I^{G \nu_1 \nu_2 \ldots \nu_r} \) as a linear combination of a few master integrals, which we denote by \( I = \{ I_1, \ldots, I_N \} \). Let us further denote by \( x = (x_1, \ldots, x_t) \) the vector of kinematic variables the master integrals depend on. The method of differential equations \[57–65, 67\] is a powerful tool to find the functions \( f_j \) in eq. (52). Let \( x_k \) be a kinematic variable. Carrying out the derivative \( \partial I_i / \partial x_k \) under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals:

\[
\frac{\partial}{\partial x_k} I_i + \sum_{j=1}^{N} a_{ij} I_j = 0. \tag{54}
\]

Repeating the above procedure for every master integral and every kinematic variable we obtain a system of differential equations of Fuchsian type

\[
(d + A) I = 0, \tag{55}
\]

where \( A \) is a matrix-valued one-form

\[
A = \sum_{i=1}^{t} A_i dx_i. \tag{56}
\]

The matrix-valued one-form \( A \) satisfies the integrability condition \( dA + A \wedge A = 0 \).

Geometrically we have a vector bundle with a flat connection: The base space is parametrised by the coordinates \( x = (x_1, \ldots, x_t) \), the fibre is a \( N \)-dimensional vector space with basis \( I = (I_1, \ldots, I_N) \), the flat connection is given by \( A \) and called the Gauß-Manin connection.
Suppose $A$ is of the form

$$A = \epsilon \sum_j C_j \ln p_j(x), \quad (57)$$

where all $\epsilon$-dependence is in the prefactor, the $C_j$’s are matrices with constant entries and the $p_j(x)$’s are polynomials in the external variables $x$, then the system of differential equations is easily solved in terms of multiple polylogarithms [63].

In this talk we consider the situation, where the master integrals depend only on a single variable $\tau$ and the connection one-form $A$ is of the form

$$A = \epsilon \sum_j F_j (2\pi i) \ d\tau, \quad (58)$$

where as before all $\epsilon$-dependence is in the prefactor and the $F_j$’s are matrices, whose entries are modular forms. In this case the system of differential equations is easily solved in terms of iterated integrals of modular forms.

A system of differential equations, where the only $\epsilon$-dependence is in a prefactor like in eq. (57) or eq. (58) is said to be in $\epsilon$-form. Clearly, it is advantageous to have the system in $\epsilon$-form. There are two operations at our disposal to transform a system of differential equations, which follow from the geometric picture described above: We may change the variables in the base manifold and/or we may change the basis of the vectorspace in the fibre. A change of variables in the base manifold introduces a Jacobian: If $\tau' = \gamma(\tau)$ (for simplicity we consider the case where the base manifold is one-dimensional) we have

$$A' = A \frac{\partial \tau'}{\partial \tau}. \quad (59)$$

A change of the basis of the vectorspace in the fibre

$$I' = UI \quad (60)$$

transforms the connection into

$$A' = UAU^{-1} + UdU^{-1}. \quad (61)$$

7 Picard-Fuchs operators

An extremely helpful tool for Feynman integral computations within the approach based on differential equations are the factorisation properties of Picard-Fuchs operators [65]. Let us consider an (unknown) function $f(\lambda)$ of a single variable $\lambda$, which obeys a (known) homogeneous differential equation of order $r$
On a class of Feynman integrals evaluating to iterated integrals of modular forms

\[ \sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j} f(\lambda) = 0, \quad (62) \]

where the \( p_j \)'s are polynomials in \( \lambda \), such that the differential equation is of Fuchsian type. We call the differential operator

\[ L = \sum_{j=0}^{r} p_j(\lambda) \frac{d^j}{d\lambda^j} \quad (63) \]

a Picard-Fuchs operator. Suppose that this operator factorises into linear factors:

\[ L = (a_1(\lambda) \frac{d}{d\lambda} + b_1(\lambda)) \cdots (a_{r}(\lambda) \frac{d}{d\lambda} + b_{r}(\lambda)) \left( a_1(\lambda) \frac{d}{d\lambda} + b_1(\lambda) \right). \quad (64) \]

Such a differential equation is easily solved. Let us denote the homogeneous solution of the \( j \)-th factor by

\[ \psi_j(\lambda) = \exp \left( -\int_0^{\lambda} \frac{b_j(\kappa)}{a_j(\kappa)} d\kappa \right). \quad (65) \]

Then the full solution is given by iterated integrals as

\[ f(\lambda) = C_1 \psi_1(\lambda) + C_2 \psi_2(\lambda) \int_0^{\lambda} d\lambda_1 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1) \psi_1(\lambda_1)} + C_3 \psi_1(\lambda) \int_0^{\lambda_1} d\lambda_2 \frac{\psi_2(\lambda_1)}{a_1(\lambda_1) \psi_1(\lambda_1)} \int_0^{\lambda_2} d\lambda_3 \frac{\psi_3(\lambda_2)}{a_2(\lambda_2) \psi_2(\lambda_2)} + \ldots \quad (66) \]

From eq. (66) we see that multiple polylogarithms are of this form, i.e. have Picard-Fuchs operators, which factorise into linear factors.

The next more complicated situation is the case, where the Picard-Fuchs operator contains one irreducible second-order differential operator

\[ a_j(\lambda) \frac{d^2}{d\lambda^2} + b_j(\lambda) \frac{d}{d\lambda} + c_j(\lambda). \quad (67) \]

As an example consider the differential equation

\[ \left[ \lambda \left( 1 - \lambda^2 \right) \frac{d^2}{d\lambda^2} + (1 - 3\lambda^2) \frac{d}{d\lambda} - \lambda \right] f(\lambda) = 0 \quad (68) \]

This second-order differential operator is irreducible. The solutions of the differential equation are \( K(\lambda) \) and \( K(\sqrt{1 - \lambda^2}) \), where \( K(\lambda) \) is the complete elliptic integral of the first kind:
\[ K(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - \lambda^2 x^2)}} \]  

(69)

Let us now return to a system of differential equations as in eq. (55). In general, such a system may depend on several kinematic variables \( x = (x_1, \ldots, x_t) \). We may reduce a multi-scale system to a single-scale system by setting \( x_i(\lambda) = \alpha_i \lambda \) with \( \alpha = [\alpha_1 : \ldots : \alpha_t] \in \mathbb{CP}^{t-1} \) and by viewing the master integrals as functions of \( \lambda \). For the derivative with respect to \( \lambda \) we have

\[ \frac{d}{d\lambda} I = BI, \quad B = \sum_{i=1}^{t} \alpha_i A_i. \]  

(70)

In addition we may assume that the \( \epsilon \)-dependence of the matrices \( A \) and \( B \) is polynomial, if this is not the case, a rescaling of the master integrals with \( \epsilon \)-dependent prefactors will achieve this situation. Let us write

\[ B = B^{(0)} + \sum_{j>0} \epsilon^j B^{(j)}. \]  

(71)

A system of ordinary first-order differential equations is easily converted to a higher-order differential equation for a single master integral. We may work modulo subtopologies, therefore the order of the differential equation is given by the number \( N_s \) of master integrals in this sector. In order to find the required transformation we work in addition modulo \( \epsilon \)-corrections, i.e. we focus on \( B^{(0)} \). Let \( I \) be one of the master integrals \( \{I_1, \ldots, I_{N_s}\} \). We determine the largest number \( r \), such that the matrix which expresses \( I, (d/d\lambda)I, \ldots, (d/d\lambda)^{r-1}I \) in terms of the original set \( \{I_1, \ldots, I_{N_s}\} \) has full rank. It follows that \( (d/d\lambda)^r I \) can be written as a linear combination of \( I, \ldots, (d/d\lambda)^{r-1}I \). This defines the Picard-Fuchs operator \( L \) for the master integral \( I \) with respect to \( \lambda \):

\[ LI = 0, \quad L = \sum_{k=0}^{r} p_k(\lambda) \frac{d^k}{d\lambda^k}. \]  

(72)

\( L \) is easily found by transforming to a basis which contains \( I, \ldots, (d/d\lambda)^{r-1}I \). Although the Picard-Fuchs operator is a differential operator of order \( r \), it is very often the case that this operator factorises. The factorisation can be obtained with standard algorithms [82]. Let us write for the factorisation into irreducible factors

\[ L = L_1 L_2 \ldots L_s, \]  

(73)

where the differential operators \( L_i \) are irreducible. Since we started from the \( \epsilon \)-independent matrix \( B^{(0)} \), the differential operators \( L_i \) are \( \epsilon \)-independent.
8 Feynman integrals evaluating to iterated integrals of modular forms

Let us now consider a few examples. We consider the Feynman integrals shown in

![Fig. 3](image-url) Examples of Feynman integrals evaluating to iterated integrals of modular forms. Internal solid lines correspond to a propagator with mass $m^2$, internal dashed lines to a massless propagator. External dashed lines indicate a light-like external momentum.

fig. 3 These are two-loop two-point or three-point integrals, depending on a single dimensionless variable

$$x = \frac{p^2}{m^2}. \quad \text{(74)}$$

All examples shown in fig. 3 contain the equal-mass sunrise graph as a subtopology and are – as we will see – expressible in terms of iterated integrals of modular forms. In order to proceed we would like to

1. verify that the integrals depend only on a single elliptic curve,
2. identify the elliptic curve,
3. change the variable of the base manifold from $x$ to the modular parameter $\tau$,
4. change the basis of master integrals such that the transformed system of differential equations is in $\varepsilon$-form.

These steps can be done systematically. Let us start with the first step. In order to verify that the integrals depend only on a single elliptic curve we construct for all integrals (including all sub-topologies) the Picard-Fuchs operators as described in
the previous section. We recall that for a specific integral we work modulo sub-topologies and modulo $\varepsilon$-corrections. We then look at the factorisations of the various Picard-Fuchs operators and verify, that there is only one second-order irreducible factor. All other factors are first order. The irreducible second-order differential operator is associated with the sunrise graph.

In the second step we identify the elliptic curve. For the sunrise graph this can be done either from the maximal cuts \[83–89\] or from the Feynman parameter representation. The former method generalises easily to more complicated Feynman integrals \([27, 28]\) and we discuss it here. One finds for the sunrise integral in two space-time dimensions

$$\text{MaxCut}_\mathcal{C} I = \frac{u}{\pi^2} \int_{\mathcal{C}} \frac{dz}{z^\frac{1}{2} (z+4)^\frac{1}{2} \left[z^2 + 2(1+x)z + (1-x)^2\right]^\frac{1}{2}},$$

(75)

where $u$ is an (irrelevant) phase and $\mathcal{C}$ an integration contour. The denominator of the integrand defines an elliptic curve, which we denote by $E_x$:

$$E_x : w^2 - z(z+4)\left[z^2 + 2(1+x)z + (1-x)^2\right] = 0.$$  

(76)

We denote the roots of the quartic polynomial in eq. (76) by

$$z_1 = -4, \quad z_2 = -(1+\sqrt{x})^2, \quad z_3 = -(1-\sqrt{x})^2, \quad z_4 = 0.$$  

(77)

We consider a neighbourhood of $x = 0$ without the branch cut of $\sqrt{x}$ along the negative real axis. The correct physical value is specified by Feynman’s $i\delta$-prescription: $x \to x + i\delta$. The periods $\psi_1, \psi_2$ and the modular parameter $\tau$ are then defined by eq. (15) and eq. (16), respectively.

In the third step we change the variable of the base manifold from $x$ to the modular parameter $\tau$. We recall that $\tau$ as a function of $x$ is given by eq. (16):

$$\tau = \frac{\psi_2}{\psi_1}.$$  

(78)

In a neighbourhood of $x = 0$ we may invert eq. (78). This gives

$$x = 9\frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(2\tau)^8\eta(3\tau)^4},$$  

(79)

where $\eta$ denotes Dedekind’s eta-function. For the Jacobian we have

$$\frac{d\tau}{dx} = \frac{W}{\psi_1},$$  

(80)

where the Wronskian $W$ is given by
\[ W = \psi'_1 - \psi'_2 - \psi_2 - \psi_1 = -\frac{6 \pi i}{x(x-1)(x-9)}. \] (81)

In the fourth step we change the basis of master integrals such that the transformed system of differential equations is in $\varepsilon$-form. The essential new ingredient is the appropriate definition of the master integrals corresponding to the second-order irreducible differential operator. We need two master integrals for this case. The first master integral may be taken as the sunrise integral in $D = 2 - 2\varepsilon$ space-time dimensions divided by the $\varepsilon^0$-term of its maximal cut. This is familiar from the case of Feynman integrals, which evaluate to multiple polylogarithms. The difference lies in the fact, that for Feynman integrals, which evaluate to multiple polylogarithms, the maximal cut is an algebraic function, while in the case of the sunrise integral it is given by a complete elliptic integral. We thus set

\[ I_1 = \varepsilon^2 \frac{\pi}{\psi_1} S_{111}(2 - 2\varepsilon, x), \] (82)

where $S_{111}(2 - 2\varepsilon, x)$ denotes the sunrise integral in $D = 2 - 2\varepsilon$ space-time dimensions with $\nu_1 = \nu_2 = \nu_3 = 1$. Let us turn to the second master integral: It is well-known in mathematics, that the first cohomology group for a family of elliptic curves $E_x$, parametrised by $x$, is generated by the holomorphic one form $dz/w$ and its $x$-derivative. This motivates an ansatz, consisting of $I_1$ and its $\tau$-derivative. One finds for the second master integral in the elliptic sector

\[ I_2 = \frac{1}{\varepsilon} \frac{d}{d\tau} I_1 + \frac{1}{24} (3x^2 - 10x - 9) \frac{\psi^2_1}{\pi^2} I_1. \] (83)

The full set of master integrals is completed by transforming in addition the master integrals in the non-elliptic sectors. The entries on the diagonal of the transformation matrix for the non-elliptic sectors can be read off from the linear factors appearing in the factorisation of the Picard-Fuchs operators [66]. The non-diagonal entries are obtained from an ansatz along the lines of [90, 91].

Let us look at a specific example. We denote the two-loop tadpole integral by

\[ I_0 = 4\varepsilon^2 S_{110}(2 - 2\varepsilon, x). \] (84)

Then we have for $I = (I_0, I_1, I_2)$

\[ \frac{1}{2\pi i} \frac{d}{d\tau} I = \varepsilon A I, \] (85)

where the matrix $A$ is $\varepsilon$-independent and is given by

\[ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -f_2 & 1 \\ \frac{1}{2} f_3 & f_4 - f_2 \\ \end{pmatrix}. \] (86)

The entries of $A$ are given by
One checks that $f_2$, $f_3$ and $f_4$ are modular forms of $\Gamma_1(6)$ of modular weight 2, 3 and 4, respectively. We introduce a basis $\{e_1, e_2\}$ for the modular forms of modular weight 1 for the Eisenstein subspace $E_1(\Gamma_1(6))$:

$$e_1 = E_1(\tau; \chi_0, \chi_1), \quad e_2 = E_1(2\tau; \chi_0, \chi_1),$$

where $E_1(\tau, \chi_0, \chi_1)$ and $E_1(2\tau, \chi_0, \chi_1)$ are generalised Eisenstein series [92] and $\chi_0$ and $\chi_1$ denote primitive Dirichlet characters with conductors 1 and 3, respectively. The integration kernels may be expressed as polynomials in $e_1$ and $e_2$:

$$f_2 = -6 \left( e_1^2 + 6e_1e_2 - 4e_2^2 \right),$$

$$f_3 = 36\sqrt{3} \left( e_1^3 - e_1^2e_2 - 4e_1e_2^2 + 4e_2^3 \right),$$

$$f_4 = 324e_1^4.$$

The solution for these Feynman integrals in terms of iterated integrals of modular forms follows now directly from the differential equation (85). The $q$-expansion of the iterated integrals provides an efficient method for the numerical evaluation [25,93].

Let us close this paragraph with the observation that the integration kernels

$${\omega}_0 = \frac{dx}{x}, \quad {\omega}_0 = \frac{dx}{x-1}$$

may also be expressed as modular forms:

$${\omega}_0 = g_{2,0} 2\pi i d\tau, \quad {\omega}_0 = g_{2,1} 2\pi i d\tau.$$

The modular forms $g_{2,0}$ and $g_{2,1}$, both of modular weight 2, are given by

$$g_{2,0} = \frac{1}{2i\pi} \frac{\psi_1^2}{Wx} = -12 \left( e_1^2 - 4e_2^2 \right),$$

$$g_{2,1} = \frac{1}{2i\pi} \frac{\psi_1^3}{Wx-1} = -18 \left( e_1^3 + e_1e_2 - 2e_2^3 \right).$$

This shows that the harmonic polylogarithms [49,50] in the letters 0 and 1 are a subset of the iterated integrals of modular forms discussed in this talk.
9 Conclusions

In this talk we considered a class of Feynman integrals, which evaluate to iterated integrals of modular forms. These Feynman integrals are beyond the class of Feynman integrals, which evaluate to multiple polylogarithms. However, several important properties, known from the case of multiple polylogarithms, carry over: The system of differential equations can be brought into an $\varepsilon$-form, the iterated integrals satisfy a shuffle algebra and there is an efficient method for the numerical evaluation of the iterated integrals of modular forms based on the $q$-expansion. We considered single-scale integrals. We may view these Feynman integrals, which evaluate to iterated integrals of modular forms as generalisations of Feynman integrals, which may be expressed in terms of harmonic polylogarithms in the letters 0 and 1.

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