CREPANT RESOLUTION CONJECTURE IN ALL GENERA FOR
TYPE A SINGULARITIES

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Abstract. We prove an all genera version of the Crepant Resolution Conjecture of Ruan and Bryan-Graber for type A surface singularities. We are based on a method that explicitly computes Hurwitz-Hodge integrals described in an earlier paper and some recent results by Liu-Xu for some intersection numbers on the Deligne-Mumford moduli spaces. We also generalize our results to some three-dimensional orbifolds.

1. Introduction

Let $\mathcal{X}$ be an orbifold with coarse moduli space $X$ and let $\pi : Y \to X$ be a crepant resolution. Under the general principle often referred to as the McKay correspondence in the mathematical literature (suggested also by work in string theory literature on orbifolds, e.g. [17]), it is expected that invariants of $Y$ coincides with suitably defined orbifold invariants of $\mathcal{X}$. See e.g. Reid [34] for an exposition of some examples of classical invariants, e.g. Euler numbers, cohomology groups, K-theory and derived categories, etc. People are also interested in quantum invariants such as Gromov-Witten invariants and expect a quantum McKay correspondence. Gromov-Witten invariants of smooth varieties have been developed for quite some time. Orbifold Gromov-Witten invariants have been developed more recently for symplectic orbifolds by Chen-Ruan [14] and for Deligne-Mumford stacks by Abramovich-Graber-Vistoli [1]. Genus zero (orbifold) Gromov-Witten invariants can be used to define (orbifold) quantum cohomology. A long standing conjecture of Ruan [35] states that the small orbifold quantum cohomology of $\mathcal{X}$ is related to the small quantum cohomology of $Y$ after analytic continuation and suitable change of variables. This version of quantum McKay correspondence is referred to as the Crepant Resolution Conjecture (CRC). Recently, Bryan and Graber [8] conjectured the explicit formula for the change of variables in CRC for $\mathcal{X} = [V/G]$, where $V = \mathbb{C}^2$ or $\mathbb{C}^3$, $G \subset SU(2)$ or $SO(3)$ is a finite subgroup (the binary polyhedral group or the polyhedral group), and $\pi : Y \to V/G$ is the canonical crepant resolution by $G$-Hilbert schemes $G - \text{Hilb}(V)$ (see e.g. [5]). In these cases, both the orbifold and its crepant resolution are noncompact, but both admit natural $\mathbb{C}^*$-actions, and one can define and work with equivariant Gromov-Witten and orbifold Gromov-Witten invariants respectively. By [8], there is a canonical basis for $H^*_\mathbb{C}(Y)$ indexed by $R \in \text{Irr}(G)$, irreducible representations of $G$; on the other hand, there is a canonical basis of $H^*_{\mathbb{C},\text{orb}}([V/G])$ indexed by $[g] \in \text{Conj}(G)$, conjugacy classes of $G$. Denote the corresponding cohomological variables by $\{y_R\}_{R \in \text{Irr}(G)}$ and $\{x_{[g]}\}_{[g] \in \text{Conj}(G)}$ respectively. Let $y_0$ and $x_0$ be the variables corresponding to the trivial representation and the trivial conjugacy classes respectively. The conjecture stated in [8] (attributed to Bryan and Gholampour) is
Conjecture 1. In the case of \( \pi : G - \text{Hilb}(V) \to V/G \), where \( G \) is a polyhedral or binary polyhedral group, the Crepant Resolution Conjecture holds with the change of variables given by

\[
\begin{align*}
y_0 &= x_0, \\
y_R &= \frac{1}{|G|} \sum_{g \in G} \sqrt{\chi_V(g)} - \dim V \chi_R(g)x_{[g]}, \\
q_R &= \exp \left( \frac{2\pi i \dim R}{|G|} \right),
\end{align*}
\]

where \( R \) runs over the nontrivial irreducible representations of \( G \).

By the classical McKay correspondence \[31, 20, 5\], the geometry of \( G - \text{Hilb}(V) \) gives rise to a Dynkin diagram of ADE type, and hence a simply-laced root system. Denote by \( \alpha_1, \ldots, \alpha_n \) the simple roots. For a positive root \( \beta \in R^+ \), let \( \beta = \sum b_k \alpha_k \). Let \( \sum b_k \alpha_k \) be the largest root. Bryan-Gholampour \[6\] reformulate the above conjecture as follows:

Conjecture 2. Let \( F_X(x_1, \ldots, x_n) \) denote the \( \mathbb{C}^* \)-equivariant genus 0 orbifold Gromov-Witten potential of the orbifold \( X = [\mathbb{C}^2/G] \), where we have set the unit parameter \( x_0 \) equal to zero. Let \( R \) be the root system associated to \( G \). Then

\[
\sum_{\beta \in R^+} h(Q_{\beta}),
\]

where \( h(u) \) is a series with

\[
h'''(u) = -\frac{1}{2} \tan \left( \frac{u}{2} \right)
\]

and

\[
Q_{\beta} = \pi + \sum_{k=1}^{n} \frac{b_k}{|G|} \left( 2\pi n_k + \sum_{g \in G} \sqrt{2 - \chi_V(g)} \chi_k(g)x_{[g]} \right),
\]

where \( n_k \) are the coefficients of \( \beta \) and \( n_k \) are the coefficients of the largest root.

It has been shown in \[6\] that Conjecture 2 is equivalent to Conjecture 1 in binary polyhedral case plus an explicit formula for the Gromov-Witten potential function of \( G - \text{Hilb}(\mathbb{C}^2) \). There are a number of earlier results. Ruan’s Crepant Resolution Conjecture was established for \( G = \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) by Perroni \[33\]. Conjecture 2 was proved for \( G = \mathbb{Z}_2 \) by Bryan-Graber \[8\], for \( \mathbb{Z}_3 \) by Bryan-Graber-Pandharipande \[9\], for \( \mathbb{Z}_4 \) by Bryan-Jiang \[10\]. The polyhedral version of Conjecture 2 was proved by Bryan-Gholampour \[7\] for \( G = A_4 \) and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Conjecture 1 was proved for \( G \) of type \( A \) in a version due to Perroni \[33\] by Coates-Corti-Iritani-Tseng \[15\] by mirror symmetry from Givental formalism (see also the related work by Skarke \[36\] and Hosono \[22\]).

Bryan and Graber \[8\] also conjectured the higher genera version of the Crepant Resolution Conjecture. See also the recent paper by Coates and Ruan \[16\] from the point of view of Givental’s formalism. Maulik \[30\] has computed the equivariant Gromov-Witten invariants in all genera of the minimal resolution of \( \mathbb{C}^2/G \), \( G \subset SU(2) \). Therefore, to establish the CRC in all genera in the binary polyhedral case, a key step is to compute the equivariant orbifold Gromov-Witten invariants in all genera of \( [\mathbb{C}^2/G] \). In this paper we will carry out the calculations of the stationary
part of the potential function for $G = \mathbb{Z}_n$. It is straightforward to see that our results match with that of Maulik\footnote{The author thanks Hsian-Hua Tseng for an email correspondence on January 25, 2008 which informed him this before Maulik’s paper was posted on the arxiv.}

For $k \geq 0$ and $1 \leq a \leq n - 1$, define
\[
y_{a,k} = \frac{2i}{n} \sum_{b=1}^{n-1} \sin \frac{b\pi}{n} \cdot \xi_n^{ab} x_{b,k} u_b.
\]

For $k \geq 0$, $1 \leq s \leq t \leq n - 1$, define
\[
y_{s\to t,k} = \sum_{a=s}^{t} y_{a,k}.
\]

Our main result is:

**Theorem 1.** Up to polynomial terms of degree $\leq 3$ in $y_{a,k}$, the stationary potential function $F_{\mathbb{C}^2/\mathbb{Z}_n}(x_1,k,\ldots,x_{n-1},k)_k \geq 0$ (defined in (2.6)) of the equivariant orbifold Gromov-Witten invariants of $[\mathbb{C}^2/\mathbb{Z}_n]$ is equal to the coefficient of $z^g$ of
\[
(-1)^{2g} 2t \sum_{d=1}^{\infty} d^{2g-3} \sum_{1 \leq s \leq t \leq n-1} \left( \xi_n^{t-s+1} \exp \left( \sum_{k \geq 0} y_{s\to t,k} z^k \frac{4k}{(2k+1)!!} \right) \right)^d
\]
after analytic continuations, where $\xi_n = e^{2\pi i/n}$.

We prove this result by explicit computations of the relevant Hurwitz-Hodge integrals by the method described in an earlier paper \cite{12}. The idea is to combine Tseng’s GRR relations for Hurwitz-Hodge integrals with the results of Jarvis-Kimura \cite{24} on $\tilde{\psi}$-integrals, as suggested by Tseng \cite{37}. This follows the strategy of Faber \cite{18} in the case of ordinary Hodge integrals where he combined Mumford’s GRR relations \cite{32} with the results on $\psi$-integrals computed by Witten-Kontsevich theorem \cite{25}. Another key ingredient is the Hurwitz-Hodge version of Mumford’s relations \cite{32} as established by Bryan-Graber-Pandharipande \cite{9}. This is used to convert the relevant Hurwitz-Hodge integrals to a simple one which involves only one Chern character of the Hurwitz-Hodge bundles. For other work on Hurwitz-Hodge integrals, see e.g. \cite{19, 5, 9, 4, 11, 12, 13, 24}.

In our computation of the Hurwitz-Hodge integrals we encounter some intersection numbers on the Deligne-Mumford moduli spaces (see \cite{24} and \cite{25}). They can be computed using some recent results on the $n$-point functions of intersection numbers on $[\mathbb{M}_{g,n}]$ by Liu-Xu \cite{28}. The formula in (2.5) was first discovered and checked using Faber’s Maple program \cite{18}. In an earlier version of this paper, it was stated as a conjecture and was only proved in some low genera cases. The author thanks Professor Jim Bryan for encouraging him to find a proof. The rest of the proof of Theorem 1 is of combinatorial nature: We take a seven-fold summation to arrive at our final answer.

An interesting byproduct is a relationship between Hurwitz-Hodge integrals with polylogarithm function. In the GRR relations of Mumford and Tseng, Bernoulli numbers and Bernoulli polynomials evaluated at rational numbers appear respectively. It is well-known that they are the values at negative integers of Riemann and Hurwitz zeta functions respectively. In the Hurwitz-Hodge integrals case it turns out that we can rewrite the results in terms of polylogarithm functions.
For type $D$ and $E$ binary polyhedral groups, the simplifying trick used in this paper does not apply: We have to compute integrals against Chern classes, not just one Chern character. Nevertheless, the method described in [42] for computing Hurwitz-Hodge integrals can still be applied, but the combinatorics is much more complicated. We hope to address this in a future work.

In this paper, we also consider the CRC for some 3D orbifolds. It is natural to consider the orbifolds of the form $[\mathbb{C}^2/G] \times \mathbb{C}$, where $G \subset SU(2)$ is a finite subgroup. One can use some natural circle actions on these orbifolds to define and study their equivariant orbifold Gromov-Witten invariants by virtual localization [21]. We study the case of $G = \mathbb{Z}_n$ in this paper. In §4.1 we specify some circle actions on the orbifold $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$ used to define the potential function $F [\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$ of equivariant orbifold Gromov-Witten invariant of this orbifold.

**Theorem 2.** Up to polynomial terms in $u_1, \ldots, u_{n-1}$, we have

\[
 F[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}(\lambda; u_1, \ldots, u_{n-1}) = \sum_{d \geq 1} \frac{1}{4 \sin^2(d\lambda)} \sum_{1 \leq s \leq t \leq n-1} \left( e^{i \pi (t-s+1)} e^{v_{t-s}} \right)^d 
\]

after analytic continuation, where

\[
 v_{s-t} = \sum_{a=s}^t v_a, \quad v_j = \left( \frac{i}{n} \sum_{k=1}^{n-1} \sqrt{2 - 2 \cos \frac{2k\pi}{n}} \xi^{jk} u_k \right). 
\]

On the other hand, one can compute the Gromov-Witten invariants of $\hat{\mathbb{C}}^2/\mathbb{Z}_n \times \mathbb{C}$ by localization using the method of [40], or use the theory of topological vertex [2, 26]. One can simplify the expressions by the combinatorial techniques in [41]. Our result is (Theorem 4.2):

\[
 F[\hat{\mathbb{C}}^2/\mathbb{Z}_n] \times \mathbb{C}(\lambda; Q_1, \ldots, Q_{n-1}) = \sum_{1 \leq a \leq b \leq n-1} \prod_{k=1}^b Q_k^d \frac{1}{d \sin^2(d\lambda/2)}. 
\]

Hence CRC in this case take the following form (Theorem 4.3)

\[
 F[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}(\lambda; u_1, \ldots, u_{n-1}) = F[\hat{\mathbb{C}}^2/\mathbb{Z}_n] \times \mathbb{C}(\lambda; Q_1, \ldots, Q_{n-1}) 
\]

after analytic continuation and change of variables

\[
 Q_j = \xi_n e^{v_j}. 
\]

We make the following

**Conjecture 3.** For a finite subgroup $G \subset SU(2)$, the Crepant Resolution Conjecture takes the following form:

\[
 F[\mathbb{C}^2/G] \times \mathbb{C}(\lambda; \{u_\beta\}_{[\beta] \neq [1]}) = F[\mathbb{C}^2/\text{Hilb}(\mathbb{C}^2) \times \mathbb{C}(\lambda; \{Q_R\}_{[R] \text{ nontrivial}} 
\]

after analytic continuation, where

\[
 Q_R = \exp \left( \frac{2\pi i \dim R}{|G|} + v_R \right), 
\]

\[
 v_R = \frac{1}{|G|} \sum_{g \in G} \sqrt{\chi_V(g) - \dim V \chi_R(g)} u_{[g]}. 
\]

In a forthcoming work, we will study CRC for more general three-dimensional Calabi-Yau orbifolds.
2. Equivariant Gromov-Witten Invariant of $[\mathbb{C}^2/\mathbb{Z}_n]$

In this section we define the equivariant Gromov-Witten invariants of $[\mathbb{C}^2/\mathbb{Z}_n]$, and reduce their computations to intersection numbers on the Deligne-Mumford moduli spaces.

2.1. Definition of the equivariant Gromov-Witten invariants of $[\mathbb{C}^2/\mathbb{Z}_n]$. Because $[\mathbb{C}^2/\mathbb{Z}_n]$ is noncompact, we define the Gromov-Witten invariants of $[\mathbb{C}^2/\mathbb{Z}_n]$ by taking suitable torus action on $[\mathbb{C}^2/\mathbb{Z}_n]$ and using the virtual localization [21].

First of all, $\mathbb{Z}_n$ acts on $\mathbb{C}^2$ by:

$$\omega \cdot (z_1, z_2) = (\xi_n \cdot z_1, \xi_n^{-1} \cdot z_2),$$

where $\omega$ is a generator of $\mathbb{Z}_n$, and $\xi_n = e^{2\pi i/n}$. This action has the origin as the only fixed point, with normal bundle $V_1 \oplus V_{-1}$, where $V_{\pm 1}$ are the one-dimensional representations on which the generator $\omega$ of $\mathbb{Z}_n$ acts by multiplication by $\xi_n^\pm 1$.

Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ by multiplications. The fixed locus of the induced action on the orbifold $[\mathbb{C}^2/\mathbb{Z}_n]$ is a copy of $B\mathbb{Z}_n$, the classifying stack of $\mathbb{Z}_n$. For $m \geq 1$ and $a_1, \ldots, a_m \in \{0, 1, \ldots, n - 1\}$ such that

$$\sum_{i=1}^m a_i \equiv 0 \pmod{n},$$

(7) denote by $\overline{\mathcal{M}}_{g,m}([\mathbb{C}^2/\mathbb{Z}_n]; \prod_{i=1}^m [\omega^{a_i}])$ the moduli space of twisted stable maps to the orbifold $[\mathbb{C}^2/\mathbb{Z}_n]$, with monodromy $[\omega^{a_1}], \ldots, [\omega^{a_m}]$ at the $m$ marked points. Here $[\omega^k]$ denotes the conjugacy class of $\omega^k$. See [1] for definitions and notations. The $\mathbb{C}^*$-action on $[\mathbb{C}^2/\mathbb{Z}_n]$ induces a natural $\mathbb{C}^*$-action on the moduli space $\overline{\mathcal{M}}_{g,m}([\mathbb{C}^2/\mathbb{Z}_n]; \prod_{i=1}^m [\omega^{a_i}])$. Its fixed point set can be identified with the moduli space $\overline{\mathcal{M}}_{g,m}(B\mathbb{Z}_n; \prod_{i=1}^m [\omega^{a_i}])$ of twisted stable maps to $B\mathbb{Z}_n$, with monodromy $[\omega^{a_1}], \ldots, [\omega^{a_m}]$ at the $m$ marked points. Denote by $\mathbb{P}^0_{\pm 1}$ the vector bundle on the moduli space associated with the one-dimensional representations $V_{\pm 1}$. The fibers of $\mathbb{P}^0_{\pm 1}$ at a twisted stable map $f : C \rightarrow B\mathbb{Z}_n = (H^i(\tilde{C}, \mathcal{O}_{\tilde{C}}) \otimes V_{\pm 1})\mathbb{Z}_n$, where $\tilde{C} \rightarrow C$ is the admissible cover parameterized by $f$.

Denote by $p$ the number of $a_i$'s which are equal to zero. It is not hard to see that when $a_i > 0$ for some $i$, i.e. $p < m$, one has $\mathbb{P}^0_{\pm 1} = 0$. Therefore, by the dimension formula in [12, Proposition 4.3], when $p < m$,

$$r_1 := \text{rank}(\mathbb{P}^1_1) = g - 1 + \frac{\sum_{i=1}^m a_i}{n},$$

(8) $$\bar{r}_1 := \text{rank}(\mathbb{P}^{-1}_{-1}) = g - 1 + \frac{\sum_{i=1}^m (n - a_i)(1 - \delta_{a_i, 0})}{n},$$

(9) and so

$$r_1 + \bar{r}_1 = 2g - 2 + m - p.$$ 

(10) In this case, we actually have $m - p \geq 2$, and so $r_1 + \bar{r}_1 \geq 0$. If $r_1 + \bar{r}_1 = 0$, then one has $g = 0$ and $m = p = 2$. However, if all $a_i = 0$, i.e., $p = m$, then the situation is complicated. Note $\overline{\mathcal{M}}_{g,m}(B\mathbb{Z}_n; [1]^m)$ has two components: $\overline{\mathcal{M}}^{\text{disc}}_{g,m}(B\mathbb{Z}_n; [1]^m)$ and $\overline{\mathcal{M}}^{\text{conn}}_{g,m}(B\mathbb{Z}_n; [1]^m)$. A point in $\overline{\mathcal{M}}^{\text{disc}}_{g,m}(B\mathbb{Z}_n; [1]^m)$ is represented by a disconnected étale $\mathbb{Z}_n$-cover $\tilde{C}$ of a stable curve $C$ in $\overline{\mathcal{M}}_{g,m}$, consisting of $n$ disjoint copies of
C. It follows that $H^i(\tilde{\mathcal{C}}, \mathcal{O}_{\tilde{\mathcal{C}}}) \otimes V_{\pm 1}$ is isomorphic to $H^i(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ (as a trivial $\mathbb{Z}_n$ representation) tensored with $V_{\pm 1}$ and the regular representation of $\mathbb{Z}_n$, hence

$$(H^i(\tilde{\mathcal{C}}, \mathcal{O}_{\tilde{\mathcal{C}}}) \otimes V_{\pm 1})^{\mathbb{Z}_n} \cong H^i(\mathcal{C}, \mathcal{O}_{\mathcal{C}}).$$

This means $F^1_{\pm 1}$ are just the pullback of the Hodge bundle by the structure forgetting morphism:

$$\overline{\mathcal{M}}_{g,m}(\mathbb{Z}; \mathbb{Z}) \rightarrow \overline{\mathcal{M}}_{g,m},$$

and both $F^0_1$ and $F^0_{-1}$ are isomorphic to the trivial line bundle. On the other hand, a point in $\overline{\mathcal{M}}_{g,m}(\mathbb{Z}; \mathbb{Z})$ is represented by a connected étale $\mathbb{Z}_n$-cover $\tilde{\mathcal{C}}$ of a stable curve $\mathcal{C}$ in $\overline{\mathcal{M}}_{g,m}$. It follows that $H^0(\tilde{\mathcal{C}}, \mathcal{O}_{\tilde{\mathcal{C}}} \otimes V_{\pm 1})$ is isomorphic to $H^0(C, \mathcal{O}_C)$ (as a trivial $\mathbb{Z}_n$ representation) tensored with $V_{\pm 1}$, hence

$$(H^0(\tilde{\mathcal{C}}, \mathcal{O}_{\tilde{\mathcal{C}}}) \otimes V_{\pm 1})^{\mathbb{Z}_n} \cong V_{\pm 1} \otimes V_{\pm 1} = 0.$$

This means both $F^1_1$ and $F^1_{-1}$ have rank $g-1$ by the orbifold Riemann-Roch formula.

Now $F^0_1 + F^0_{-1} - F^1_1$ is the virtual normal bundle of $\overline{\mathcal{M}}_{g,m}(\mathbb{Z}_n; \mathbb{Z})$ in $\overline{\mathcal{M}}_{g,m}(\mathbb{Z}_n; \mathbb{Z})$. The torus $\mathbb{C}^*$ acts on vector bundles $F_{\pm 1}$, and their equivariant top Chern classes are given by their Chern polynomials. Therefore, using virtual localizations [21], the equivariant correlators are given by:

$$\left\langle \prod_{j=1}^m \tau_{k_j}(e_{\xi_j}) \right\rangle_{\mathbb{C}^*}^{\mathbb{Z}_n} = \int_{\overline{\mathcal{M}}_{g,m}(\mathbb{Z}_n; \mathbb{Z})} \frac{c_t(F^1_1)c_t(F^1_{-1})}{c_t(F^0_1)c_t(F^0_{-1})} \prod_{j=1}^m \psi_{k_j},$$

where $k_1, \ldots, k_m \geq 0$ such that $\sum_{i=1}^m k_i = g + p$. We will only consider the case of $m - p > 0$. In this case we have $m - p \geq 2$ and $F^0_1 = F^0_{-1} = 0$.

### 2.2. Manipulations with $c_t(F^1_1)c_t(F^1_{-1})$.

**Lemma 2.1.** The product of the Chern polynomials of $F^1_1$ and $F^1_{-1}$ has the following expansion:

\begin{equation}
  c_t(F^1_1)c_t(F^1_{-1}) = (t_1 + t_2)(-1)^{r_1 - 1}(r_1 + r_{-1} - 1)! t_1^{r_1} t_{-1}^{r_{-1}} + \cdots.
\end{equation}

**Proof.** Recall the Hurwitz-Hodge bundles satisfy an analogue of Mumford’s relations [32]:

\begin{equation}
  c_t(F^1_1)c_{-t}(F^1_{-1}) = (-1)^{r_1} t^{r_1} + \cdots.
\end{equation}

In particular,

\begin{align}
  c_{r_1}(F^1_1)c_{-r_1}(F^1_{-1}) &= 0, \\
  c_{r_1-1}(F^1_1)c_{r_1}(F^1_{-1}) &= c_{r_1}(F^1_1)c_{r_1-1}(F^1_{-1}).
\end{align}

Now we recall the relationship between Newton polynomials and elementary symmetric polynomials (see e.g. [29]). Denote by $e_i(u_1, \ldots, u_n)$ the $i$-th elementary symmetric function in $u_1, \ldots, u_n$ and by

$$p_k(u_1, \ldots, u_n) = u_1^k + \cdots + u_n^k$$
the $k$-th Newton symmetric polynomial in $u_1, \ldots, u_n$. Then one has
\[ \log \sum_{i=0}^{n} t^i e_i(x_1, \ldots, x_n) = \sum_{i=1}^{n} \log(1 + tx_i) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t^k}{k} p_k(x_1, \ldots, x_n). \]

Take derivative in $t$ on both sides:
\[ \sum_{k=1}^{\infty} (-t)^{k-1} p_k(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} it^{i-1} e_i(x_1, \ldots, x_n)}{\sum_{i=0}^{n} t^i e_i(x_1, \ldots, x_n)}. \]

Applying this to $c_i(\mathbb{F}^1)$ and $k! \text{ch}_k(\mathbb{F}^1)$ we get:
\[ \sum_{k=1}^{\infty} (-t)^{k-1} k! \text{ch}_k(\mathbb{F}^1) = \frac{\sum_{i=1}^{r_i} it^{i-1} c_i(\mathbb{F}^1)}{\sum_{j=0}^{\bar{r}_1} j! c_j(\mathbb{F}^1)} = \frac{\sum_{i=1}^{\bar{r}_1} it^{i-1} c_i(\mathbb{F}^1)}{\sum_{j=0}^{\bar{r}_1} j! (-t)^j c_j(\mathbb{F}^1_{-1})}, \]

where we have used (12) in the last equality, therefore we have
\[ (15) \quad k! \text{ch}_k(\mathbb{F}^1) = \sum_{i+j=k} (-1)^{i-1} it^{i-1} c_i(\mathbb{F}^1) c_j(\mathbb{F}^1_{-1}). \]

In particular, $\text{ch}_k(\mathbb{F}^1) = 0$ for $k \geq r_1 + \bar{r}_1$, and combining with (14) we have
\[ (16) \quad (r_1 + \bar{r}_1 - 1)! \text{ch}_{r_1 + \bar{r}_1 - 1}(\mathbb{F}^1) = (-1)^{r_1 - 1} c_{r_1 - 1}(\mathbb{F}^1) c_{\bar{r}_1}(\mathbb{F}^1_{-1}). \]

Therefore,
\[
\begin{align*}
    c_{r_1}(\mathbb{F}^1) c_{\bar{r}_1}(\mathbb{F}^1_{-1}) & = (c_{r_1}(\mathbb{F}^1) + t_1 c_{r_1 - 1}(\mathbb{F}^1) + \cdots + t_{r_1}^{r_1}) \cdot (c_{\bar{r}_1}(\mathbb{F}^1_{-1}) + t_2 c_{\bar{r}_1 - 1}(\mathbb{F}^1_{-1}) + \cdots + t_{\bar{r}_1}^{\bar{r}_1}) \\
    & = (t_1 + t_2) c_{r_1 - 1}(\mathbb{F}^1) c_{\bar{r}_1}(\mathbb{F}^1_{-1}) + \cdots \\
    & = (t_1 + t_2) (-1)^{r_1 - 1} (r_1 + \bar{r}_1 - 1)! \text{ch}_{r_1 + \bar{r}_1 - 1}(\mathbb{F}^1) + \cdots.
\end{align*}
\]

\[ \square \]

2.3. Computations of Hurwitz-Hodge integrals. By (11) we have for $\sum_{i=1}^{m} k_i = g + p$,
\[
\begin{align*}
    & \langle \prod_{j=1}^{m} \tau_{k_j} (e_{[\omega^a_i]}) \rangle_{[\mathbb{P}^2/\mathbb{Z}_n]}^{[c^2/\mathbb{Z}_n]} \\
    = & \quad (-1)^{r_1 - 1} 2^t (r_1 + \bar{r}_1 - 1)! \int_{\mathcal{M}_{g,m}(\mathbb{Z}_n; \Pi_{i=1}^{m}[\omega^a_i])} \text{ch}_{r_1 + \bar{r}_1 - 1}(\mathbb{F}^1) \cdot \prod_{j=1}^{m} \psi_j^{k_j}.
\end{align*}
\]
The Hurwitz-Hodge integral on the right-hand side can be computed by the method described in [42]. Write \([m] = \{1, \ldots, m\}\). By Tseng’s GRR relations for Hurwitz-Hodge integrals 37:

\[
\int_{M_{g,m}(\mathcal{BZ}_n; \mathbb{I} \in [m]; \omega^n)} \text{ch}_{r_1 + \tau_1 - 1}(\mathbb{F}_1) \cdot \prod_{j=1}^m \bar{\psi}_j^{k_j} = -\frac{B_{r_1 + \tau_1}}{(r_1 + \tau_1)!} \int_{M_{g,m+1}(\mathcal{BZ}_n; \mathbb{I} \in [m]; \omega^n, \mathbb{I} || 1)} \prod_{j=1}^m \bar{\psi}_j^{k_j} \cdot \bar{\psi}_{r_1 + \tau_1 + 1}^{k_1} \\
+ \sum_{j=1}^m \frac{B_{r_1 + \tau_1}(a_j/n)}{(r_1 + \tau_1)!} \int_{M_{g,m}(\mathcal{BZ}_n; \mathbb{I} \in [m]; \omega^n)} \prod_{j=1}^m \bar{\psi}_j^{k_j} \cdot \bar{\psi}_{r_1 + \tau_1 - 1}^{k_j} \\
- \frac{1}{2} \sum_{g_1 + g_2 = g} \sum_{J = [m]} \frac{B_{r_1 + \tau_1}(c(a_j)/n)}{(r_1 + \tau_1)!} \cdot n \cdot \sum_{l=0}^{r_1 + \tau_1 - 2} (-1)^l \int_{M_{g_1, [r_1 + \tau_1](\mathcal{BZ}_n; \mathbb{I} \in [m]; \omega^n))} \prod_{i \in I} \bar{\psi}_i^{k_i} \cdot \bar{\psi}_{r_1 + \tau_1 - 2 - l}^{k_1} \\
\cdot \bar{\psi}_{r_1 + \tau_1 + 1}^{k_1} \cdot \bar{\psi}_{r_1 + \tau_1 - 1}^{k_1} \\
- \frac{1}{2} \sum_{c=0}^{n-1} \frac{B_{r_1 + \tau_1}(c/n)}{(r_1 + \tau_1)!} \cdot n \cdot \sum_{l=0}^{r_1 + \tau_1 - 2} (-1)^l \int_{M_{g_1, [r_1 + \tau_1](\mathcal{BZ}_n; \mathbb{I} \in [m]; \omega^n))} \prod_{i \in I} \bar{\psi}_i^{k_i} \cdot \bar{\psi}_{r_1 + \tau_1 - 2 - l}^{k_1} \cdot \bar{\psi}_{r_1 + \tau_1 - 1}^{k_1} \\
\cdot \bar{\psi}_{r_1 + \tau_1 + 1}^{k_1} \cdot \bar{\psi}_{r_1 + \tau_1 - 1}^{k_1} 
\]

where \(c(a_j) = 0, 1, \ldots, n - 1\), such that \(c(a_j) \equiv -\sum_i a_i \pmod{n}\). Now by Jarvis-Kimura 29,

\[
\int_{M_{g,m}(\mathcal{BZ}_n; \mathbb{I} \in [m]; \omega^n)} \text{ch}_{r_1 + \tau_1 - 1}(\mathbb{F}_1) \cdot \prod_{j=1}^m \bar{\psi}_j^{k_j} = n^{2g-1} \left( -\frac{B_{r_1 + \tau_1}}{(r_1 + \tau_1)!} \prod_{j=1}^m \tau_{k_j} \cdot \tau_{r_1 + \tau_1} \right)_g \\
+ \sum_{j=1}^m \frac{B_{r_1 + \tau_1}(a_j/n)}{(r_1 + \tau_1)!} \prod_{i=1}^m \tau_{k_i} \cdot \delta_{j_i} \left( (r_1 + \tau_1 - 1) \right)_g \\
- \frac{1}{2} \sum_{g_1 + g_2 = g} \sum_{J = [m]} \frac{B_{r_1 + \tau_1}(c(a_j)/n)}{(r_1 + \tau_1)!} \cdot \left( \prod_{i \in I} \tau_{k_i} \cdot \tau_{r_1 + \tau_1 - 2 - l} \right)_{g_1} \cdot \left( \prod_{j \in J} \tau_{k_j} \right)_{g_2} \\
- \frac{1}{2} \sum_{c=0}^{n-1} \frac{B_{r_1 + \tau_1}(c/n)}{(r_1 + \tau_1)!} \cdot \left( \prod_{i \in [m]} \tau_{k_i} \cdot \tau_{r_1 + \tau_1 - 2 - l} \cdot \tau_l \right)_{g-1} 
\]
Here as usual,
\[
\langle \tau_{a_1} \cdots \tau_{a_k} \rangle_g = \int_{\mathcal{M}_{g,k}} \psi_1^{a_1} \cdots \psi_k^{a_k},
\]
where \(2g - 2 + k > 0\). Because \(\langle \tau_{a_1} \cdots \tau_{a_k} \rangle_g = 0\) except for \(a_1 + \cdots + a_k = 3g - 3 + k\), the summation over \(l\) in the third term on the right-hand side can be reduced to the case of \(l = 3g_2 - 2 - \sum_{j \in J}(k_j - 1)\). Hence,
\[
\int_{\mathcal{M}_{g,m}(\mathbb{Z}^m: \sum_{i \in [m]} [w_i])} \text{ch}_{r_1 + \bar{r}_1 - 1}(\mathbb{P}^1) \cdot \prod_{j=1}^n \psi_j^{k_j}
\]
\[
= n^{2g-1} \left( \frac{B_{r_1 + \bar{r}_1}}{(r_1 + \bar{r}_1)!} \prod_{j=1}^m \tau_{k_j} \cdot \tau_{r_1 + \bar{r}_1} \right)_g
\]
\[
+ \sum_{j=1}^m \frac{B_{r_1 + \bar{r}_1} (a_j / n)}{(r_1 + \bar{r}_1)!} \prod_{i=1}^m \tau_{k_i + \delta_{ij}(r_1 + \bar{r}_1 - 1)} \cdot \tau_{r_1 + \bar{r}_1 - 1}.
\]

Even though the first three terms on the right-hand side look differently, they can be written in a uniform way if we use the following conventions:
\[
\langle \tau_{l} \rangle_0 = \begin{cases} 1, & l = -2, \\ 0, & \text{otherwise} \end{cases}
\]
and
\[
\langle \tau_{k \cdot \tau_{l}} \rangle_0 = \begin{cases} (-1)^{k}, & l = -k - 1, k \geq 0, \\ (-1)^{l}, & k = -l - 1, l \geq 0, \\ 0, & \text{otherwise} \end{cases}
\]
To simplify the notations further, we introduce for \(I \prod J = [m]\):
\[
\left\{ \prod_{i \in I} \tau_{k_i} \prod_{j \in J} \tau_{k_j} \right\}_g
\]
\[
= (-1)^{\sum_{j \in J} k_j} \sum_{g_1 + g_2 = g} (-1)^{g_2} \left\{ \prod_{i \in I} \tau_{k_i} \cdot \tau_{3g_2 + 2 + |I| - \sum_{j \in J} k_j} \right\}_{g_1}
\]
\[
\cdot \left\{ \prod_{j \in J} \tau_{k_j} \cdot \tau_{3g_2 + 2 + |J| - \sum_{j \in J} k_j} \right\}_{g_2},
\]
and
\[
\left[ \prod_{i \in [m]} \tau_{k_i} \right]_{g-1}^K = \sum_{l=0}^K (-1)^l \langle \prod_{i \in [m]} \tau_{k_i} \cdot \tau_{K-l} \cdot \tau_l \rangle_{g-1}.
\]
Recall \(r_1 + \bar{r}_1 = 2g - 2 + m - p\). Then we have the following:
Proposition 2.1. For \( g \geq 0 \):

\[
\prod_{i,j \in [m]} \mathcal{L}_{r_i+1}([X_{ij}]) \cdot \prod_{j=1}^{n} \psi_j^k = \frac{1}{2^g} \sum_{l=0}^{g-2} \left( \sum_{l=0}^{g-2} B_{2g-2+m-p}(c/n) \cdot \left( \prod_{i \in I} \prod_{j \in J} \tau_{k_i} \mid \tau_{l_j} \right)_g \right)
+ \sum_{c=0}^{n-1} B_{2g-2+m-p}(c/n) \left( \prod_{i \in [m]} \tau_{k_i} \right)^{2g-4+m-p}.
\]

2.4. Results on \( \prod_{i \in [m]} \tau_{k_i} \mid g-1 \mid \). Recent work of Liu-Xu [28] contains explicit formula for \( \prod_{i \in [m]} \tau_{k_i} \mid g-1 \mid \). We are concerned with the case of \( m - p \geq 2 \). For \( m - p = 2 \), first assume \( k_i > 0 \) for \( i = 1, \ldots, m \), then we are in the situation of [28, Theorem 2.1]:

\[
\prod_{i \in [m]} \tau_{k_i} \mid g-1 \mid^{2g-2} = \sum_{l=0}^{2g-2-1} \left( \prod_{i \in [m]} \tau_{k_i} \cdot \tau_{2g-2-l\tau_{l_i}} \right)_g - 1
= \frac{(2g - 3 + m)!}{4^{g-1}(2g - 1)!} \cdot \prod_{i=1}^{m} (2k_i - 1)!!\]

otherwise, by string equation and induction we get:

\[
\prod_{i=1}^{m-a} \tau_{k_i} \mid g-1 \mid^{2g-2} = \sum_{l=0}^{2g-2} \left( \prod_{i=1}^{m-a} \tau_{k_i} \cdot \tau_{2g-2-l\tau_{l_i}} \right)_g - 1
= \frac{(2g - 3 + m - a)!}{4^{g-1}(2g - 1)!} \cdot \prod_{i=1}^{m-a} (2g - 4 + m - a + 2j) \prod_{j=1}^{m} (2k_i - 1)!!.
\]

where \( k_i > 0 \) for \( i = 1, \ldots, m - a \). For \( m - p > 2 \), we are in the situation of [28, Theorem 2.3]:

\[
\prod_{i \in [m]} \tau_{k_i} \mid g-1 \mid^{2g-4+m-p} = 0,
\]

where \( k_i \geq 0 \) for \( i = 1, \ldots, m \).

2.5. Results on \( \prod_{i \in I} \tau_{k_i} \mid \prod_{j \in J} \tau_{l_j} \mid g \). For \( p = 0 \), we have

Proposition 2.2. For \( g \geq 0 \) and \( k_1, \ldots, k_m \geq 0 \) such that \( k_1 + \cdots + k_m = g \), and \( I \prod J = [m] \), the following identities holds:

\[
\prod_{i \in I} \tau_{k_i} \mid \prod_{j \in J} \tau_{l_j} \mid g = \frac{1}{4^g \prod_{j=1}^{m} (2k_j + 1)!!}
\]

Proof. This can be again proved by the recent results of Liu-Xu [28]. Consider the normalized n-point function:

\[
G(x_1, \ldots, x_n; \lambda) = \exp \left( -\frac{\sum_{j=1}^{n} x_j^2 \lambda^2}{24} \right) \cdot F(x_1, \ldots, x_n; \lambda),
\]
where

\[
F(x_1, \ldots, x_n; \lambda) = \sum_{g=0}^{\infty} \lambda^{2g} \sum_{d_1, \ldots, d_n \geq 0} \frac{\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{j=1}^{n} x_j^{d_j}}{\prod_{j=d_j=3g-3+n}^{\infty}} + \delta_{n,1} \langle \tau_{-2} \rangle_0 x_1^{-2} + \delta_{n,2} \sum_{l=0}^{\infty} \langle \tau_{-l-1} \tau_l \rangle_0 x_1^{-l-1} x_2^l
\]

is the \(n\)-point function. Denote by \(C(\prod_{i=1}^{n} x_i^{d_i}, P(x_1, \ldots, x_n))\) the coefficient of \(\prod_{i=1}^{n} x_i^{d_i}\) in a polynomial or a formal power series \(P(x_1, \ldots, x_n)\). It is easy to see from the definitions that for \(d_1, \ldots, d_m \geq 0, d_1 + \cdots + d_m = g\), we have

\[
C(\lambda^{2g}z^{g-4+n} \prod_{i=1}^{m} x_i^{d_i}, G(z, x_1; \lambda)G(-z, x_2; \lambda)) = \sum_{g_1 + g_2 = g} (-1)^{3g_2 - 2 + \sum_{i \in J}(d_i - 1)} \prod_{i \in I} \tau_{d_i} \cdot \tau_{3g_2 - 2 - \sum_{i \in J}(d_i - 1)} g_1 \cdot \prod_{i \in J} \tau_{g_2} \cdot \tau_{3g_2 - 2 - \sum_{i \in J}(d_i - 1)} g_2.
\]

On the other hand, by [28 Theorem 3.2], for \(d_1, \ldots, d_n \geq 0, \sum_{i=1}^{n} d_i = 3g - 2 + n - k\) one has

\[
C(\lambda^{2g}z^{k} \prod_{j=1}^{n} x_j^{d_j}, G(z, x_1, \ldots, x_n; \lambda)) = \begin{cases} 0, & k > 2g - 2 + n, \\ \frac{1}{(2g+1)^{n}}, & k = 2g - 2 + n. \end{cases}
\]

The proof is completed by applying this to \(G(z, x_1; \lambda) \cdot G(-z, x_2; \lambda)\). \(\square\)

Unfortunately we do not have a closed formula for the \(p > 0\) case.

2.6. Potential function. Define the genus \(g\) equivariant orbifold Gromov-Witten potential function by:

\[
F^g_{[[\mathbb{C}/\mathbb{Z}_n]]}(\{e_{i,k}\}_{0 \leq i \leq n-1, k \geq 0}; u)
= \sum_{m \geq 1} \frac{1}{m!} \sum_{0 \leq a_1, \ldots, a_n \leq m-1} \sum_{\sum_{j \equiv 0 \pmod{n}} a_j \equiv g+p} \langle \prod_{j=1}^{m} \tau_{a_j} (e^{[\omega_{a_j}]}) \rangle_{g}^{[[\mathbb{C}/\mathbb{Z}_n]]} \cdot \prod_{j=1}^{m} x_{a_j, k_j} \cdot \prod_{a_i > 0} u_{a_i},
\]

where \(x_{i,k} (0 \leq i \leq n-1, k \geq 0)\) and \(u_j (1 \leq j \leq n-1)\) are formal variables. We understand \(\{x_{i,k}\}\) as linear coordinates on \(H^*_{orb}([[\mathbb{C}/\mathbb{Z}_n]])\). The variables \(u_1, \ldots, u_{n-1}\) are referred to as the degree tracking variables. We understand that \(u_0 = 1\). Then
the stationary potential functions
tion in the expression for the stationary potential functio
3.1. Modified stationary potential function.

\[ F_{g}^{[C^2/Z_n]}(\{x_{i,k}\}_{0 \leq i \leq n-1, k \geq 0}; u) \]
\[ = \frac{1}{4g-1} \sum_{c=0}^{n-1} \frac{B_{2g}(c/n)}{2g} \prod_{k=0}^{m} x_{a,k} \prod_{i=1}^{m} u_{a_i} \]
\[ \cdot (-1)^{[J]} \cdot \frac{1}{4g} \prod_{j=1}^{m} \frac{1}{(2k_j + 1)!!} \cdot \prod_{j=1}^{m} x_{a_j,k_j} \cdot \prod_{i=1}^{m} u_{a_i} \]
\[ \cdot (-1)^{[J]} \cdot \frac{1}{4g} \prod_{j=1}^{m} \frac{1}{(2k_j + 1)!!} \cdot \prod_{j=1}^{m} x_{a_j,k_j} \cdot \prod_{i=1}^{m} u_{a_i} \]
\[ + \frac{1}{2} \sum_{n=1}^{n-1} (-1)^n \sum_{k_j=g}^{n-1} \sum_{c=0}^{n-1} \frac{B_{2g}(c/n)}{2g} \prod_{k_i \geq 0} \left(2k_i + 1\right)!! \cdot x_{a,1} \cdot x_{n-a,2} \cdot u_{a,n-a} \cdot \prod_{i=1}^{m} u_{a_i} \]

3. Crepant Resolution Conjecture for Type A Surface Singularities in All Genera

In this section we perform the change of variables and analytic continuations to the stationary potential functions \( F_{g}^{[C^2/Z_n]} \). We will compare with results of Maulik [30] to establish the CRC in all genera for the stationary potential functions.

3.1. Modified stationary potential function. To simplify the four-fold summation in the expression for the stationary potential function \( F_{g}^{[C^2/Z_n]}(\{x_{i,k}\}_{0 \leq i \leq n-1, k \geq 0}; u) \), we will first convert it to a seven-fold summation.

We first group the terms with \( c(I) = c \) to get a constrained summation

\[ \sum_{c=0}^{n-1} \sum_{x(I) = c,} \]

we then replace it by an equivalent arbitrary summation \( \sum_{c=0}^{n-1} \frac{1}{n} \sum_{I=0}^{n-1} \xi_{n} \xi_{I} \sum_{i=1}^{a_i} \).

We also replace the constrained summation

\[ \sum_{1 \leq a_1, \ldots, a_m \leq n-1} \sum_{\sum_{j=0} a_j = 0} (\text{mod } n) \]
by an equivalent arbitrary summation

\[ \sum_{1 \leq a_1, \ldots, a_m \leq n-1} \frac{1}{n} \sum_{b=0}^{n-1} \xi_n^b \sum_{i} a_i, \]

now we get

\[ F_g^{[\mathbb{C}^2/\mathbb{Z}_n]}(\{x_{i,k}\}_{1 \leq i \leq n-1, k \geq 0}; u) \]

\[ = tn^{2g-1} \left( \sum_{m \geq 1} \frac{1}{m!} \sum_{1 \leq a_1, \ldots, a_m \leq n-1} \frac{(-1)^{g+1} + \sum a_i}{n} \sum_{b=0}^{n-1} \xi_n^b \sum_{i} a_i \sum_{k_1, \ldots, k_m \geq 0} \right) \]

\[ \times \sum_{l \prod J = \{m\}} n \sum_{1 \leq a_1, \ldots, a_m \leq n-1} \frac{(-1)^{g+1} + \sum a_i}{n} \sum_{b=0}^{n-1} \xi_n^b \sum_{i} a_i \sum_{k_1, \ldots, k_m \geq 0} \]

\[ \prod_{j=1}^{m} (x_{a_j,k} u_{a_j}) \]

whose coefficient of \( z^g \) is equal to \( F_g^{[\mathbb{C}^2/\mathbb{Z}_n]}(\{x_{i,k}\}_{0 \leq i \leq n-1, k \geq 0}; u) \), up to some quadratic terms.

Finally the summation over \( k_i \)'s is constrained by the condition that \( \sum_{i} k_i = g \). To convert to an arbitrary summation we introduce an extra genus tracking variable \( z \), and consider the modified stationary potential function:

\[ \tilde{F}_g^{[\mathbb{C}^2/\mathbb{Z}_n]}(\{x_{i,k}\}_{1 \leq i \leq n-1, k \geq 0}; u; z) \]

\[ = tn^{2g-1} \sum_{m \geq 1} \frac{1}{m!} \sum_{1 \leq a_1, \ldots, a_m \leq n-1} \frac{(-1)^{g+1} + \sum a_i}{n} \sum_{b=0}^{n-1} \xi_n^b \sum_{i} a_i \sum_{k_1, \ldots, k_m \geq 0} \]

\[ \cdot \sum_{l \prod J = \{m\}} n \sum_{1 \leq a_1, \ldots, a_m \leq n-1} \frac{(-1)^{g+1} + \sum a_i}{n} \sum_{b=0}^{n-1} \xi_n^b \sum_{i} a_i \sum_{k_1, \ldots, k_m \geq 0} \]

\[ \prod_{j=1}^{m} (x_{a_j,k} u_{a_j}) \]

\[ \prod_{j=1}^{m} \xi_n^{a_j} - 1 \cdot \prod_{j=1}^{m} \frac{x_{a_j,k} u_{a_j} z^{k_j}}{4^{k_j}(2k_j + 1)!!} \]

\[ \sum_{c=0}^{n-1} \xi_n^c \cdot \frac{B_{2g+2+m}(c/n)}{2g - 2 + m}. \]
Then we take the summations\[\sum_{1 \leq a_1, \ldots, a_m \leq n-1} \sum_{k_1, \ldots, k_m \geq 0} t^{2g-3} \sum_{b=0}^{n-1} \sum_{l=1}^{m} \frac{1}{m!} \left( \sum_{1 \leq a \leq n-1} \sum_{k \geq 0} \xi_2^{a} s_n^{(\xi_n - 1)} \cdot \frac{x_{a,k} u_{a} z^{k}}{4^k(2k+1)!!} \right)^{m} \]
\[\cdot \sum_{c=0}^{n-1} \xi_n^{lc} \frac{B_{2g-2+m}(c/n)}{2g - 2 + m}.\]

3.2. Reformulation in terms of the polylogarithm function. In this subsection we take care of summations\[\sum_{c=0}^{n-1}.\] The result will not be used below but it may have some independent interest. Recall Hurwitz zeta function is defined by:

\[\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}.\]

It is well-known that

\[\zeta(-n, a) = -\frac{B_{n+1}(a)}{n + 1}.\]

The polylogarithm function is defined by:

\[\text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}.\]

Lemma 3.1. For \(0 \leq l \leq n - 1\),

\[\sum_{a=0}^{n-1} \xi_n^{al} \zeta(s, a/n) = n^s \cdot \text{Li}_s(\xi_n^l).\]

Proof.

\[\sum_{a=0}^{n-1} \xi_n^{al} \zeta(s, a/n) = \sum_{m=1}^{\infty} \frac{1}{m^s} + \sum_{a=1}^{n-1} \sum_{m=0}^{\infty} \frac{\xi_n^{al}}{(m + a/n)^s} = n^s \sum_{m=1}^{\infty} \frac{1}{(nm)^s} + n^s \sum_{a=1}^{n-1} \sum_{m=0}^{\infty} \frac{\xi_n^{al}}{(nm + a)^s} = n^s \sum_{m=1}^{\infty} \frac{\xi_n^{ml}}{m^s} = n^s \cdot \text{Li}_s(\xi_n^l).\]

Combining (20), (22) and (24) we get:

Proposition 3.1. For \(n \geq 2\) the following identity holds:

\[\tilde{F}_{g}^{[C_2/Z_n]}(\{x_{i,k}\}_{0 \leq i \leq n-1, k \geq 0}: u; z)\]
\[= (-1)^{g-1} t^{2g-3} \sum_{b=0}^{n-1} \sum_{l=1}^{m} \frac{1}{m!} \left( \sum_{0 \leq a \leq n-1} \sum_{k \geq 0} \xi_2^{a} s_n^{(\xi_n - 1)} \cdot \frac{x_{a,k} u_{a} z^{k}}{4^k(2k+1)!!} \right)^{m} \]
\[\cdot n^{-m} \cdot \text{Li}_{-2g+2+m}(\xi_n^l).\]
3.3. Change of variables. When $G = \mathbb{Z}_n$, the change of variables given by Bryan-Graber [8] is:

$$q_1 = \cdots = q_{n-1} = \xi_n,$$

$$y_j = \frac{i}{n} \sum_{k=1}^{n-1} \sqrt{2 - 2 \cos \frac{2k\pi}{n}} \xi_n^{jk} x_k = \frac{2i}{n} \sum_{k=1}^{n-1} \frac{k\pi}{n} \cdot \xi_n^{jk} x_k.$$

Here $q_1, \ldots, q_n$ are degree tracking variables for the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_n$, and they have set all the degree tracking variables on $[\mathbb{C}^2/\mathbb{Z}_n]$ side to be 1. In the above we have considered the degree tracking variables in our partition function, so we will take instead:

$$y_j = \frac{2i}{n} \sum_{k=1}^{n-1} \sin \frac{k\pi}{n} \cdot \xi_n^{jk} x_k u_k.$$

**Lemma 3.2.** For $a = 1, \ldots, n-1$ we have

$$(26) \quad x_a u_a = -\frac{i}{2 \sin \frac{a\pi}{n}} \sum_{j=1}^{n-1} (\xi_n^{-aj} - 1)y_j.$$

**Proof.** First notice that

$$\sum_{j=1}^{n-1} \xi_n^{-ja} y_j = \sum_{j=1}^{n-1} \frac{2i}{n} \sum_{k=1}^{n-1} \frac{k\pi}{n} \cdot \xi_n^{jk} x_k u_k = \frac{2i}{n} \sum_{k=1}^{n-1} \frac{k\pi}{n} \cdot (-1 + n\delta_{kl}) x_k u_k.$$

It is more transparent when written in matrix form:

$$\left( \begin{array}{cccc} \xi_n^{-1} & \xi_n^{-2} & \cdots & \xi_n^{-(n-1)} \\ \xi_n^{-2} & \xi_n^{-2} & \cdots & \xi_n^{-(2n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_n^{-(n-1)} & \xi_n^{-(n-1)^2} & \cdots & \xi_n^{-(n-1)^2} \end{array} \right) \cdot \left( \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{array} \right) = \frac{2i}{n} \left( \begin{array}{cccc} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{array} \right) \cdot \left( \begin{array}{c} \sin \frac{\pi}{n} \cdot x_1 u_1 \\ \sin \frac{\pi}{n} \cdot x_2 u_2 \\ \vdots \\ \sin \frac{(n-1)\pi}{n} \cdot x_{n-1} u_{n-1} \end{array} \right).$$

Notice that

$$\left( \begin{array}{cccc} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{array} \right)^{-1} = \frac{1}{n} \left( \begin{array}{cccc} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{array} \right).$$
and

\[
\begin{pmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 2
\end{pmatrix}
\begin{pmatrix}
\xi_n^{-1} & \xi_n^{-2} & \cdots & \xi_n^{-(n-1)} \\
\xi_n^{-2} & \xi_n^{-2} & \cdots & \xi_n^{-(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_n^{-(n-1)} & \xi_n^{-(n-1)} & \cdots & \xi_n^{-(n-1)}
\end{pmatrix}
\begin{pmatrix}
\xi_n^{-1} - 1 \\
\xi_n^{-2} - 1 \\
\vdots \\
\xi_n^{-(n-1)} - 1
\end{pmatrix}
\begin{pmatrix}
\xi_n^{-2} - 1 \\
\xi_n^{-2} - 1 \\
\vdots \\
\xi_n^{-(n-1)} - 1
\end{pmatrix}
\begin{pmatrix}
\xi_n^{-(n-1)} - 1 \\
\xi_n^{-(n-1)} - 1 \\
\vdots \\
\xi_n^{-(n-1)} - 1
\end{pmatrix}
\begin{pmatrix}
\xi_n^{-(n-1)} - 1 \\
\xi_n^{-(n-1)} - 1 \\
\vdots \\
\xi_n^{-(n-1)} - 1
\end{pmatrix}
\]

Now it is straightforward to get \((26)\).

\[\square\]

### 3.4. A summation formula.

We now take care of the summation \(\sum_{n=0}^{n-1}\) in \((20)\).

**Lemma 3.3.** For \(0 \leq b \leq n - 1, 1 \leq l \leq n - 1\), we have

\[
(27) \quad \sum_{k=1}^{n-1} \xi_n^{b k} \xi_n^{k l} (\xi_n^{kl} - 1) x_k u_k = \begin{cases} 
ny_{b+l-b+l}, & b + l < n, \\
-ny_{b+l-b+l}, & b + l \geq n,
\end{cases}
\]

where for \(1 \leq s \leq t \leq n - 1\), we define

\[
y_s \rightarrow t = y_s + \cdots + y_t.
\]

**Proof.** By \((20)\),

\[
\sum_{k=1}^{n-1} \xi_n^{b k} \xi_n^{k l} (\xi_n^{kl} - 1) x_k u_k = -\sum_{k=1}^{n-1} \xi_n^{b k} \xi_n^{k l} (\xi_n^{kl} - 1) \frac{i}{2 \sin \frac{k \pi}{n}} \sum_{j=1}^{n-1} (\xi_n^{-j k} - 1) y_j
\]

\[
= -\sum_{j=1}^{n-1} K(b, l, j) y_j,
\]

where

\[
K(b, l, j) = \sum_{k=1}^{n-1} \xi_n^{b k} \xi_n^{k l} (\xi_n^{kl} - 1) \frac{i}{2 \sin \frac{k \pi}{n}} (\xi_n^{-j k} - 1).
\]
For $a \in \mathbb{Z}$, let $r_n(a)$ be the integer such that $0 \leq r_n(a) < n$ and $r_n(a) \equiv a \pmod{n}$.

$$K(b, l, j) = \sum_{k=0}^{l-1} \frac{\xi_{bn} \xi_{kn}}{\xi_{2n}} (\xi_{nk} - 1) - \sum_{c=0}^{l-1} \frac{\xi_{kn}}{\xi_{kn}} (\xi_{nk} - 1)$$

It is then easy to see that when $n - b + 1 > l - 1$, i.e., $b + l < n$,

$$K(b, l, j) = \begin{cases} 
-n, & b + 1 \leq j \leq l + b, \\
0, & \text{otherwise};
\end{cases}$$

when $n - b + 1 \leq l - 1$, i.e., $b + l \geq n$,

$$K(b, l, j) = \begin{cases} 
n, & b + l - n + 1 \leq j \leq b, \\
0, & \text{otherwise}.
\end{cases}$$

3.5. Crepant Resolution Conjecture for type A surface singularities in all genera. For $k \geq 0$ and $1 \leq a \leq n - 1$, define

$$y_{a,k} = \frac{2}{n} \sum_{b=1}^{n-1} \sin \frac{b\pi}{n} \cdot \xi_{bn} x_{b,k} u_b.$$
Proof. Combining (20) with Lemma 3.3 one gets:

\[
\frac{F^c}{y^{c/2}}(\{x_{i,k}\}_{1 \leq i \leq n-1, k \geq 0}; u; z) = (-1)^{g-1}2^{2g-3} \sum_{m \geq 0} \left( \sum_{0 \leq b \leq n-1, 1 \leq l \leq n-1} \left( \sum_{k \geq 0} y_{b+1-b+l,k} \frac{z^k}{4^k \cdot (2k+1)!!} \right) \right)^m 
\]

\[+ \sum_{0 \leq b \leq n-1, 1 \leq l \leq n-1} \left( -n \sum_{k \geq 0} y_{b+l-n+1-b,k} \frac{z^k}{4^k \cdot (2k+1)!!} \right)^m \]

\[+ \sum_{c=0}^{n-1} c^{(t+1)} \cdot \frac{B_{2g-2+m}(c/n)}{(2g-2+m) \cdot m!} \sum_{c=0}^{n-1} c^{(t+1)} \cdot \frac{B_{2g-2+m}(c/n)}{(2g-2+m) \cdot m!} \]

where in the second equality we have used:

\[(28) \quad B_n(1-x) = (-1)^n B_n(x). \]

On the other hand, for \(0 < l < n\) and \(u < 0\),

\[1 + \sum_{d=1}^{\infty} \left( \xi_n^l e^u \right)^d = \frac{1}{1 - \xi_n^l e^u} = -\sum_{c=0}^{n-1} \xi_n^c \sum_{k=1}^{\infty} \frac{B_k(c/n) m^m}{k!} (nu)^k. \]

Integrating three times:

\[(29) \quad u + \sum_{d=1}^{\infty} \left( \frac{(\xi_n^l e^u)^d}{d} \right) - \text{Li}_1(\xi_n^l) = -\sum_{c=0}^{n-1} \xi_n^c \sum_{m=1}^{\infty} \frac{B_m(c/n)}{m \cdot m!} n^{m-1} u^m, \]

\[\frac{u^2}{2} + \sum_{d=1}^{\infty} \left( \frac{(\xi_n^l e^u)^d}{d^2} \right) - \text{Li}_2(\xi_n^l) - \text{Li}_1(\xi_n^l) u = -\sum_{c=0}^{n-1} \xi_n^c \sum_{m=2}^{\infty} \frac{B_{m-1}(c/n)}{(m-1) \cdot m!} n^{m-2} u^m, \]

\[\frac{u^3}{6} + \sum_{d=1}^{\infty} \left( \frac{(\xi_n^l e^u)^d}{d^3} \right) - \text{Li}_3(\xi_n^l) - \text{Li}_2(\xi_n^l) u - \frac{1}{2} \text{Li}_1(\xi_n^l) u^2 \]

\[= -\sum_{c=0}^{n-1} \xi_n^c \sum_{m=3}^{\infty} \frac{B_{m-2}(c/n)}{(m-2) \cdot m!} n^{m-3} u^m. \]

Also, by differentiating \(2g-3\) (for \(g > 1\) times:

\[(31) \quad \sum_{d=1}^{\infty} d^{2g-3} \left( \xi_n^l e^u \right)^d = -\sum_{c=0}^{n-1} \xi_n^c \sum_{m=0}^{\infty} \frac{B_{2g-2+m}(c/n)}{(2g-2+m) \cdot m!} n^{2g-3+m} u^m. \]
Therefore up to polynomial terms of degree \( \leq 3 \), \( \hat{F}_g^{[\mathcal{C}^2/\mathbb{Z}_n]}(\{x_{i,k}\}_{1 \leq i \leq n-1, k \geq 0}; u; z) \) is equal to

\[
(-1)^{g_2 t} \sum_{d=1}^{\infty} d^{2g-3} \sum_{1 \leq s \leq t \leq n-1} \left( c_{t-s+1}^{s} \exp \left( \sum_{k \geq 0} y_{s-t,k} \frac{z^k}{4^k \cdot (2k+1)!!} \right) \right)^d
\]

\[
= (-1)^{g_2 t} \sum_{1 \leq s \leq t \leq n-1} \text{Li}_{3-2g} \left( c_{t-s+1}^{s} \exp \left( \sum_{k \geq 0} y_{s-t,k} \frac{z^k}{4^k \cdot (2k+1)!!} \right) \right)
\]

after analytic continuations. This completes the proof. \( \square \)

Hence by taking the coefficient of \( z^g \) we get:

**Theorem 3.2.** Up to polynomial terms of degree \( \leq 3 \) in \( y_{a,k} \), the stationary potential function \( \hat{F}_g^{[\mathcal{C}^2/\mathbb{Z}_n]}(\{x_{i,k}\}_{1 \leq i \leq n-1, k \geq 0}; u) \) of the equivariant Gromov-Witten invariants of \( [\mathcal{C}^2/\mathbb{Z}_n] \) is equal to

\[
(-1)^{g_2 t} \sum_{d=1}^{\infty} d^{2g-3} \sum_{1 \leq s \leq t \leq n-1} \xi_n^{(t-s+1)d} \sum_{\sum k \geq 0 \ k_m=g k \geq 0} \prod_{m} \frac{d_m \cdot y_{a-t,k}}{m_k \cdot [(2k+1)!!]^{m_k}}
\]

after analytic continuations, where \( y_{s-t,k} = \sum_{a=s}^{t} y_{a,k} \).

Denote by \( \pi : \mathbb{C}^2/\mathbb{Z}_n \rightarrow \mathbb{C}^2/\mathbb{Z}_n \) the minimal resolution of \( \mathbb{C}^2/\mathbb{Z}_n \). Let \( E_1, \ldots, E_{n-1} \) be the exceptional divisors. Let \( (g_{ij}) = (E_i \cdot E_j) \) be the intersection matrix, and let \( (g^{ij}) \) be the inverse matrix of \( (g_{ij}) \). Then \( \{C^i = g^{ij} E_j : i = 1, \ldots, n-1\} \) is the dual basis to the basis \( E_1, \ldots, E_{n-1} \). By [20], for each nontrivial representation \( V_i \) of \( \mathbb{Z}_n \) on which \( \xi_n \) acts as multiplication by \( e^{2\pi i \sqrt{-1}/n} \), there is a holomorphic line bundle \( L_i \) on \( \mathbb{C}^2/\mathbb{Z}_n \) whose first Chern class is \( C^i \). One can use virtual localization to define the Gromov-Witten invariants of \( \mathbb{C}^2/\mathbb{Z}_n \). By [20] Theorem 1.1, for curve classes of the form \( \beta = d(E_a + E_{i+1} + \cdots + E_b) \), \( 1 \leq a \leq b \leq n-1 \), and consider integers \( a \leq l_1, \ldots, l_m \leq b \) and \( k_1, \ldots, k_m \geq 0 \) such that \( k_1 + \cdots + k_m = g \),

\[
\left( \prod_{i=1}^{m} \tau_{C^i}(\mathbb{C}^2/\mathbb{Z}_n) \right)_{[\mathbb{G},\beta]} = (-1)^{g_2 t} d^{2g-3+m} \prod_{i=1}^{m} \frac{1}{4^{k_i} \cdot (2k_i+1)!!}.
\]

Other correlators vanish. Therefore, the instanton part of stationary partition function of \( \mathbb{C}^2/\mathbb{Z}_n \) is

\[
\sum_{m} \frac{1}{m!} \sum_{\beta \in H_2(\mathbb{C}^2/\mathbb{Z}_n; \mathbb{Z}) - \{0\}} \sum_{k_1 = g l_1, \ldots, l_m = 1}^{n-1} \prod_{i=1}^{m} \tau_{C^i}(\mathbb{C}^2/\mathbb{Z}_n) q^k \prod_{i=1}^{m} y_{i,k_i}
\]

\[
= \sum_{m} \frac{1}{m!} \sum_{d=1}^{\infty} d^{2g-3} \sum_{1 \leq a \leq b \leq n-1} \sum_{k_1 = g l_1, \ldots, l_m = b} \sum_{a \leq l_1, \ldots, l_m \leq b} (-1)^{g_2 t} d^{2g-3+m} \prod_{i=1}^{m} \frac{1}{4^{k_i} \cdot (2k_i+1)!!} \cdot \prod_{i=a}^{b} q_i \prod_{i=1}^{m} y_{i,k_i}
\]

\[
= (-1)^{g_2 t} \sum_{d=1}^{\infty} d^{2g-3} \prod_{1 \leq a \leq b \leq n-1} \prod_{i=a}^{b} q_i \sum_{k_1 = g l_1, \ldots, l_m = b} \prod_{k \geq 0 \ k_m=g k \geq 0} \frac{d_m \cdot y_{a-t,k}}{m_k \cdot [(2k+1)!!]^{m_k}}.
\]
Hence we have established the Crepant Resolution Conjecture in all genera for type A resolutions for the stationary part of the partition functions, because one now only has to take $q_i = e^{2\pi \sqrt{-1}/n}$.

4. CREPANT RESOLUTION CONJECTURE FOR $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$

In this section we establish a version of the Crepant Resolution Conjecture for $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$ in all genera.

4.1. The equivariant Gromov-Witten invariants of $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$.

4.1.1. The circle actions on $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$. Fix an integer $a \in \mathbb{Z}$, consider the following circle action $\mathbb{C}^3$:

$$e^{i \theta} \cdot (z_1, z_2, z_3) = (e^{ia \theta} z_1, e^{-i(a+1) \theta} z_2, e^{i \theta} z_3).$$

This action commutes with the following $\mathbb{Z}_n$-action:

$$\xi_n \cdot (z_1, z_2, z_3) = (\xi_n \cdot z_1, \xi_n^{-1} \cdot z_2, z_3).$$

Hence we get an induced circle action on the orbifold $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$. Note both the circle action and the $\mathbb{Z}_n$-action preserve the holomorphic volume form $dz_1 \wedge dz_2 \wedge dz_3$ on $\mathbb{C}^3$.

4.1.2. Definition of the equivariant Gromov-Witten invariants. As in the $[\mathbb{C}^2/\mathbb{Z}_n]$ case, one can define the Gromov-Witten invariants of $[\mathbb{C}^2/\mathbb{Z}_n] \times \mathbb{C}$ equivariantly using the above circle actions and virtual localizations. It is straightforward to see that the equivariant correlators are given by

$$\langle \prod_{j=1}^{m} \tau_0(e^{[\omega^a_j]}) \rangle_{\mathbb{C}^2/\mathbb{Z}_n} = \int_{\mathcal{M}_{g,m}(\mathbb{BZ}_n;[\omega^a_1],\ldots,[\omega^a_m])} \frac{1}{t} c_t(F_0)c_{at(t)}(F_1),$$

where $F_0$ is the vector bundle whose fiber at a twisted stable map $f : \Sigma \to \mathbb{BZ}_n$ is $H^1(\Sigma, f^*V_0)$, where $V_0$ is the trivial representation of $\mathbb{Z}_n$.

Lemma 4.1. One has

$$\frac{1}{t} \int_{\mathcal{M}_{g,m}(\mathbb{BZ}_n;[\omega^a_1],\ldots,[\omega^a_m])} \cdot (F_0)c_{at(t)}(F_1),$$

where $\lambda_{g,0} = (-1)^j c_j(F_0)$. 
Proof. By (11) we have
\[
\frac{1}{t} \int_{\mathcal{M}_{g,m}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{nm}])} c_t(F_0)c_{at}(F_1)c_{-(a+1)t}(F_{-1})
\]
\[= \frac{1}{t} \int_{\mathcal{M}_{g,m}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{nm}])} (t^g + t^{g-1}c_1(F_0) + \cdots + c_g(F_0))
\]
\[\cdot ((at)^{r_1} + (at)^{r_1-1}c_1(F_1) + \cdots + c_{r_1}(F_1))
\]
\[\cdot (\sum_t (-1)^{-t(1+\sum_i a_i/n)}((2g - 3 + m)! \text{ch}_{2g-3+m}(F_1) + \cdots))
\]
\[= (-1)^{\sum_i a_i/n-1}(2g - 3 + m)! \int_{\mathcal{M}_{g,m}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{nm}])} \lambda_{g,0} \cdot \text{ch}_{2g-3+m}(F_1).
\]

Proposition 4.1. For \(m \geq 2\) and \(g \geq 0\), we have
\[
\int_{\mathcal{M}_{g,m}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{nm}])} \lambda_{g,0} \cdot \text{ch}_{2g-3+m}(F_1)
\]
\[= - \frac{1}{2} 2^{g-2} k_g \sum_{i=0}^{m-1} \prod_{i=1}^{n-1} (\xi_i - 1) \cdot \sum_{a=0}^{n-1} \xi_a B_{2g-2+m}(a/n) (2g - 2 + m)!
\]

where
\[
\lambda_{g,0} \cdot \text{ch}_{2g-3+m}(F_1)
\]

Proof. By Tseng’s GRR relations for Hurwitz-Hodge integrals [37]:
\[
\int_{\mathcal{M}_{g,m}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{nm}])} \lambda_{g,0} \cdot \text{ch}_{2g-3+m}(F_1)
\]
\[= - \frac{B_{2g-2+m}}{(2g - 2 + m)!} \int_{\mathcal{M}_{g,m+1}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{nm}],[1])} \lambda_{g,0} \cdot \text{ch}_{2g-2+m}
\]
\[+ \sum_{i=1}^{m} \frac{B_{2g-2+m}(a_i/n)}{(2g - 2 + m)!} \int_{\mathcal{M}_{g,m}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{nm}])} \lambda_{g,0} \cdot \text{ch}_{2g-3+m}
\]
\[= - \frac{1}{2} \sum_{\sum_{i=1}^{m} \sum_{g_1+g_2=g} (c(a_1)/n)} (2g - 2 + m)!
\]
\[\cdot \sum_{l=0}^{r_1+r_2-2} (-1)^l \int_{\mathcal{M}_{g_1+r_1+1}(\mathcal{B}Z_{n};[\omega_{n1}],...,[\omega_{n1}], [\omega^{-\sum_{i \in [l,n]} a_i}])} \lambda_{g_1,0} \cdot \text{ch}_{2g-3+m}(F_1)
\]
\[\cdot \sum_{l=0}^{r_1+r_2-2} (-1)^l \int_{\mathcal{M}_{g_2+1}[l+1]} \lambda_{g_2,0} \cdot \text{ch}_{2g-3+m}(F_1).
\]

Here the prime sign in the summation \(\sum_{\sum_{i=1}^{m} \sum_{g_1+g_2=g}}\) in the third term on the right-hand side means the following stability conditions are satisfied:
\[2g_1 - 1 + |l| > 0, \quad 2g_2 - 1 + |l| > 0.
\]
Our convention is that $\lambda_{0,0} = 1$. Now we use the morphism

$$\varphi : \overline{M}_{g_1, |J|+1}(BZ_n; \prod_{i \in J} [\omega^{a_i}], [\omega^{\sum_{i \in J} a_i}]) \rightarrow \overline{M}_{g, \sum_i m_i+1}$$

that forgets the orbifold structure. Because $\lambda_{g_1,0} = \varphi^* \lambda_{g_1}, \psi_{\sum_i, j_i+1} = \varphi^* \psi_{\sum_i, j_i+1}$ and $\varphi$ is of degree $n^{2g_1-1}$, we have for $2g_2 - 2 + |J| > 0$:

$$\int_{\overline{M}_{g_2, |J|+1}(BZ_n; \prod_{i \in J} [\omega^{a_i}], [\omega^{\sum_{i \in J} a_i}])} \lambda_{g_2, 0} \psi_{|J|+1} = n^{2g_2-1} \delta_{t, 2g_2-2+|J|} \int_{\overline{M}_{g_2, |J|+1}} \lambda_{g_2} \psi_{|J|+1} = n^{2g_2-1} \delta_{t, 2g_2-2+|J|} b_{g_2}.$$  

For $2g_2 - 2 + |J| \leq 0$, we have used the conventions (17) and (18) to understand the following two cases:

$$\int_{\overline{M}_{g_2, |J|+1}(BZ_n; \prod_{i \in J} [\omega^{a_i}], [\omega^{\sum_{i \in J} a_i}])} \lambda_{g_2, 0} \psi_{|J|+1} = \begin{cases} \delta_{t, -2}, & g_2 = 0, J = \emptyset, \\ -\delta_{t, -1}, & g_2 = 0, |J| = 1. \end{cases}$$

Therefore, one finds

$$\int_{\overline{M}_{g, m}(BZ_n; [\omega^{a_1}], \ldots, [\omega^{a_m}])} \lambda_{g, 0} \, \text{ch}_1 + \bar{r}_1 - 1(\mathbb{F}_1) = -\frac{1}{2} n^{2g-1} \sum_{m} \sum_{|J|+g_1+g_2 = g} (-1)^{|J|} \frac{B_{2g-2+m}(c(a))}{(2g-2+m)!} b_g b_{g_2},$$

$$= -\frac{k_g}{2} n^{2g-1} \sum_{|J|} \sum_{m} (-1)^{|J|} \frac{B_{2g-2+m}(c(a_j))}{(2g-2+m)!} b_g,$$

$$= -\frac{1}{2} n^{2g-2} k_g \sum_{i=0}^{n-1} \prod_{i=1}^{m} (\xi_{a_i} - 1) \cdot \sum_{c=0}^{n-1} \varepsilon_{c} B_{2g-2+m}(c/n),$$

$$\square$$

4.2. Crepant Resolution Conjecture for $[\mathbb{C}^2/Z_n] \times \mathbb{C}$. Define the instanton part of genus $g$ equivariant orbifold Gromov-Witten potential function by:

$$F_{g}^{\mathbb{C}^2/Z_n \times \mathbb{C}}(u_1, \ldots, u_{n-1}) = \sum_{m \geq 1} \frac{1}{m!} \sum_{1 \leq a_1, \ldots, a_m \leq n-1 \atop \sum_j a_j \equiv 0 \mod n} \left( \prod_{j=1}^{m} \tau_0(e^{[\omega^{a_j}]}) \right) \prod_{j=1}^{m} u_{a_j}.$$  

Define

$$F_{g}^{[\mathbb{C}^2/Z_n] \times \mathbb{C}}(\lambda; u_1, \ldots, u_{n-1}) = \sum_{g \geq 0} \lambda^{2g-2} F_{g}^{[\mathbb{C}^2/Z_n] \times \mathbb{C}}(u_1, \ldots, u_{n-1}).$$
Theorem 4.1. Up to polynomial terms of degree \( \leq 3 \) in \( u_1, \ldots, u_{n-1} \), we have

\[
F_{C^2/Z_n}^{[C^2/Z_n] \times \mathbb{C}}(\chi; u_1, \ldots, u_{n-1}) = \sum_{d \geq 1} \frac{1}{d!} \frac{4d \sin^2(d\lambda/2)}{1 - \cos^2(d\lambda)} \sum_{1 \leq s \leq t \leq n-1} \left( \xi_n^{t-s+1}e^{v_{s-t}} \right)^d
\]

where

\[
v_{s-t} = \sum_{a=s}^{t} v_a, \quad v_j = \frac{1}{n} \sum_{k=1}^{n-1} \sqrt{2 - 2 \cos \frac{2k\pi}{n}} \xi_n^{jk} u_k.
\]

Proof. By Lemma 3.3 and Proposition 4.1 we have

\[
F_{C^2/Z_n}^{[C^2/Z_n] \times \mathbb{C}}(u_1, \ldots, u_{n-1})
\]

\[
= \frac{1}{2} n^{2g-2} k_g \sum_{m \geq 1} \frac{1}{m!} \sum_{1 \leq a_1, \ldots, a_m \leq n-1} (-1)^{m-1} \prod_{i=1}^{m} \left( \xi_n^{a_i} - 1 \right) u_{a_i}
\]

\[
\cdot \sum_{c=0}^{n-1} \xi_n^{lc} \frac{B_{2g-2+m}(c/n)}{2g - 2 + m}
\]

\[
= \frac{1}{2} n^{2g-3} k_g \sum_{m \geq 1} \frac{1}{m!} \sum_{b=0}^{n-1} \sum_{a_1, \ldots, a_m=1}^{n-1} \xi_n^{b} \sum_{i=1}^{a_1} \xi_n^{a_i} \sum_{i=1}^{a_2} \cdots \sum_{i=1}^{a_m} \left( \sum_{1 \leq a \leq n-1} \xi_n^{a} \xi_n^{ba} \left( \xi_n^{a} - 1 \right) u_{a} \right)^m
\]

\[
\cdot \sum_{c=0}^{n-1} \xi_n^{lc} \frac{B_{2g-2+m}(c/n)}{2g - 2 + m}.
\]

By Lemma 3.3

\[
F_{C^2/Z_n}^{[C^2/Z_n] \times \mathbb{C}}(u_1, \ldots, u_{n-1})
\]

\[
= \frac{1}{2} n^{2g-3} k_g \sum_{m \geq 1} \left( \sum_{0 \leq b \leq n-1, 1 \leq t \leq n-1} \left( n v_{b+1-t} \right)^m \right)
\]

\[
+ \sum_{0 \leq b \leq n-1, 1 \leq t \leq n-1} \left( - n v_{b+t-n-1} \right)^m \cdot \sum_{c=0}^{n-1} \xi_n^{lc} \frac{B_{2g-2+m}(c/n)}{2g - 2 + m} \cdot m!
\]

\[
= k_g \sum_{m \geq 1} n^{2g-3+m} \sum_{1 \leq s \leq t \leq n-1} \xi_n^{m} v_{s-t} \cdot \sum_{c=0}^{n-1} \xi_n^{c(t-s+1)} \frac{B_{2g-2+m}(c/n)}{2g - 2 + m} \cdot m!
\]

Therefore, by (39), (40), (41), up to polynomials terms of degree \( \leq 3 \), the potential function \( F_{C^2/Z_n}^{[C^2/Z_n] \times \mathbb{C}}(u_1, \ldots, u_{n-1}) \) is equal to

\[
k_g \sum_{d=1}^{\infty} \sum_{1 \leq s \leq t \leq n-1} \left( \xi_n^{t-s+1} e^{v_{s-t}} \right)^d
\]

after analytic continuations. The proof is completed by (33). \( \square \)
The potential function of $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$ can be defined and computed also by virtual localization. For $\beta = d_1 E_1 + \cdots + d_{n-1} E_{n-1} \neq 0$,

$$
(1)^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}_{g,\beta} = \int_{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}^\times_{\mu,\nu} 1.
$$

Introduce degree tracking variable $Q_1, \ldots, Q_{n-1}$. We define

$$
F^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}_g(Q_1, \ldots, Q_{n-1}) = \sum_{\beta \neq 0} (1)^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}_{g,\beta} Q^\beta,
$$

where for $\beta = d_1 E_1 + \cdots + d_{n-1} E_{n-1}$, $Q^\beta = Q_1^{d_1} \cdots Q_{n-1}^{d_{n-1}}$. Define

$$
F^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}(\lambda; Q_1, \ldots, Q_{n-1}) = \sum_{g \geq 0} \lambda^{2g-2} F^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}_g(Q_1, \ldots, Q_{n-1}),
$$

where $\lambda$ is the genus tracking variable.

**Theorem 4.2.** We have

$$
(36) \quad F^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}(\lambda; Q_1, \ldots, Q_{n-1}) = \sum_{1 \leq a \leq b \leq n-1} \sum_{d=1}^{\infty} \prod_{k=a}^{b} Q_k^d \frac{1}{4 \sin^2(d\lambda/2)}.
$$

**Proof.** The circle action \([32]\) induces a circle action on $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$, with fixed points $p_1, \ldots, p_n$. The tangents weights at $p_i$ are $(na+i-1)t$, $-(na+i)t$, and $t$. By virtual localization one encounters two-partition Hodge integrals which have been studied in \([39, 27]\). By the method of \([40]\), we get the following expression for the potential function:

$$
F^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}(\lambda; Q_1, \ldots, Q_{n-1}) = \log \sum_{\mu^0, \ldots, \mu^{n-1}} \prod_{i=1}^{n} (W_{\mu^{i-1}, \mu^i}(q)q^{-\kappa_{\mu^i}}Q_i^{[\mu^i]}),
$$

where $\mu^0 = \mu^n = 0$, $q = e^{\sqrt{-1}\lambda}$. See notations see \([40]\). One can rewrite this as in \([41]\) by Schur calculus. Indeed, we have \([39]\):

$$
(37) \quad W_{\mu, \nu}(q) = (-1)^{|\mu|+|\nu|}q^{(\kappa_{\mu}+\kappa_{\nu})/2} \sum_{\eta} s_{\mu/\eta}(q^{-\rho}) \cdot s_{\nu/\eta}(q^{-\rho}),
$$

where $q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots)$. Hence

$$
F^{\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}}(\lambda; Q_1, \ldots, Q_{n-1})
= \log \sum_{\mu^0, \ldots, \mu^{n-1}} \sum_{\eta^0, \ldots, \eta^{n-2}} \prod_{i=1}^{n-1} s_{\mu^i/\eta^{i-1}}(q^{-\rho})Q_i^{[\mu^i]} \cdot s_{\mu^i/\eta^i}(q^{-\rho}).
$$
Therefore, by [11, Lemma 3.1],
\begin{align*}
F_{\mathbb{C}^2/\mathbb{Z}_n} & (\lambda; Q_1, \ldots, Q_{n-1}) \\
& = \log \prod_{1 \leq a \leq b \leq n-1} \prod_{i,j=1}^{b} (1 - \prod_{k=a}^{b} Q_k \cdot q^{i+j-1}) \\
& = - \sum_{1 \leq a \leq b \leq n-1} \sum_{d=1}^{\infty} \frac{\prod_{k=a}^{b} Q_k^d}{d} \sum_{m=1}^{\infty} m q^{md} \\
& = - \sum_{1 \leq a \leq b \leq n-1} \sum_{d=1}^{\infty} \frac{\prod_{k=a}^{b} Q_k^d}{d} \frac{q^d}{(1 - q^d)^2} \\
& = \sum_{1 \leq a \leq b \leq n-1} \sum_{d=1}^{\infty} \frac{\prod_{k=a}^{b} Q_k^d}{d} \frac{1}{4 \sin^2(d\lambda/2)}.
\end{align*}
\hfill \Box

By combining the above two Theorems, we get

**Theorem 4.3.** Up to polynomial terms of degree ≤ 3 in \( u_1, \ldots, u_{n-1} \), we have
\begin{equation}
F_{\mathbb{C}^2/\mathbb{Z}_n} (\lambda; u_1, \ldots, u_{n-1}) = F_{\mathbb{C}^2/\mathbb{Z}_n} (\lambda; Q_1, \ldots, Q_{n-1})
\end{equation}
after analytic continuation, where
\[ Q_j = \xi_n e^{v_j}, \quad v_j = \frac{\sqrt{-1}}{n} \sum_{k=1}^{n-1} \sqrt{2 - 2 \cos \frac{2k\pi}{n}} \xi_{jk} u_k. \]

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**References**

[1] D. Abramovich, T. Graber, A. Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, arXiv:math.AG/0603151, 2006.
[2] M. Aganagic, A. Klemm, M. Mariño, C. Vafa, *The topological vertex*, Comm. Math. Phys. 254 (2004), 425-478, arXiv:hep-th/0305132.
[3] A. Bayer, C. Cadman, *Quantum cohomology of \([\mathbb{C}^n/\mu_k]\)*, arXiv:0705.2160, 2007.
[4] V. Bouchard, C. Cavalieri, *On the mathematics and physics of high genus invariants of \([\mathbb{C}^3/\mathbb{Z}_3]\)*, arXiv:0709.3805, 2007.
[5] T. Bridgeland, A. King, M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. 14 (3), 534-555 (electronic), 2001.
[6] J. Bryan, A. Gholampour, *Root systems and the quantum cohomology of ADE resolutions*, arXiv:0707.1337, 2007.
[7] J. Bryan, A. Gholampour, *Hurwitz-Hodge integrals, the E6 and D4 root systems, and the Crepant Resolution Conjecture*, arXiv:0708.3244, 2007.
[8] J. Bryan, T. Graber, *The crepant resolution conjecture*, to appear in *Algebraic Geometry–Seattle 2005 Proceedings*, arXiv:math.AG/0610129.
[9] J. Bryan, T. Graber, R. Pandharipande, *The orbifold quantum cohomology of \(\mathbb{C}^2/\mathbb{Z}_3\) and Hurwitz-Hodge integrals*, J. Alg. Geom. 17 (2008), 1-28, arXiv:math/0510335.
[10] J. Bryan, Y. Jiang, in preparation.
[11] C. Cadman, R. Cavalieri, *Gerby localization, $\mathbb{Z}_3$-Hodge integrals and the GW theory of $[\mathbb{C}^3/\mathbb{Z}_3]$, arXiv:0705.2158* [math.AG], 2007.
[12] R. Cavalieri, *Hodge-type integrals on moduli spaces of admissible covers*, arXiv:math/0411500, 2004.
[13] R. Cavalieri, *Generating functions for Hurwitz-Hodge integrals*, arXiv:math/0506859, 2006.
[14] W. Chen, Y. Ruan, *Orbifold Gromov-Witten theory*, in Orbifolds in mathematics and physics (Madison, WI, 2001), volume 310 of Contemp. Math., pages 25-85. Amer. Math. Soc., Providence, RI, 2002 [arXiv:math.AG/0103150].
[15] T. Coates, A. Corti, H. Iritani, H.-H. Tseng, *The crepant resolution conjecture for the type $A$ surface singularities*, arXiv:0704203, 2007.
[16] T. Coates, Y. Ruan, *Quantum cohomology and crepant resolutions: A conjecture*, arXiv:0710.5901.
[17] L. Dixon, J.A. Harvey, C. Vafa, E. Witten, *Strings on orbifolds*, Nucl. Phys. B 261 (1985), no. 4, 678-686.
[18] C. Faber. *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of jacobians*, in New trends in algebraic geometry (Warwick 1996). London Math. Soc. Lecture Note Ser., 264:93-109, 1999 [arXiv:alg-geom/9706006].
[19] C. Faber, R. Pandharipande, *Logarithmic series and Hodge integrals in the tautological ring*, Michigan Math. J., 48:215-252, 2000 [arXiv:math.AG/0002112]. With an appendix by Don Zagier, Dedicated to William Fulton on the occasion of his 60th birthday.
[20] G. Gonzalez-Sprinberg, J.L. Verdier, *Construction g´eom´etrique de la correspondence de McKay*, Ann. Sci. ´Ecole Norm. Sup(4), 16 (3) (1984), 409-449.
[21] T. Graber, R. Pandharipande, *Localizations of virtual classes*, Invent. Math. 135 (2006), 487-518.
[22] S. Hosono, *Central charges, symplectic forms, and hypergeometric series in local mirror symmetry*, arXiv:hep-th/0404043, 2004.
[23] T. J. Jarvis, T. Kimura, *Orbifold quantum cohomology of the classifying space of a finite group*, in Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 123-134.
[24] T. J. Jarvis, T. Kimura, *A relative Riemann-Hurwitz theorem, the Hurwitz-Hodge bundle, and orbifold Gromov-Witten theory*, arXiv:0810.2488.
[25] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. 147 (1992), no. 1, 1-23.
[26] J. Li, C.-C. Liu, K. Liu, J. Zhou, *A mathematical theory of topological vertex*, arXiv:math/0408426.
[27] C.C. Liu, K. Liu, J. Zhou, *A formula of two-partition Hodge integrals*, JAMS 20 (2007), no. 1, 149-184, [arXiv:math/0510272]
[28] K. Liu, H. Xu, *The n-point functions for the intersection numbers on moduli spaces of curves*, arXiv:math/0701319.
[29] I.G. MacDonald, *Symmetric functions and Hall polynomials*, 2nd edition. Claredon Press, 1995.
[30] D. Maulik, *Gromov-Witten theory of A-resolutions*, arXiv:0802.2681.
[31] J. McKay, *Graphs, singularities, and finite groups*, in The Santa Cruz Conference on Finite Groups (Univ. California, Snata Cruz, Calif., 1979), vol. 37 of Proc. Sympos. Pure Math., 183-186. Amer. Math. Soc, Providence, R.I., 1980.
[32] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhauser Boston, Boston, MA, 1983, pp. 271-328.
[33] F. Perroni, *Chen-Ruan cohomology of ADE singularities*, arXiv:math/0605207.
[34] M. Reid, *La correspondance de McKay*, Séminaire Bourbaki, Vol. 1999/2000. Astérisque No. 276 (2002), 53-72.
[35] Y. Ruan, *The cohomology ring of crepant resolutions of orbifolds*, in Gromov-Witten theory of spin curves and orbifolds, 117-126, Contemp. Math., 403. Amer. math. Soc., Providence, RI, 2006.
[36] H. Skarke, *Non-perturbative gauge groups and local mirror symmetry*, JHEP 0111 (2001) 013, arXiv:hep-th/0109164.
[37] H.-H. Tseng, *Orbifold Quantum Riemann-Roch, Lefschetz and Serre*, arXiv:math.AG/0506111, 2005.
[38] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry, vol.1, (1991) 243-310.

[39] J. Zhou, *A conjecture on Hodge integrals*, arXiv:math/0310282

[40] J. Zhou, *Localizations on moduli spaces and free field realizations of Feynman rules*, arxiv:math/03110283

[41] J. Zhou, *Curve counting and instanton counting*, arXiv:math/0311237

[42] J. Zhou, *On computations of Hurwitz-Hodge integrals*, arXiv:0710.1679

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