STRONG CONVERGENCE OF A FULLY DISCRETE FINITE ELEMENT APPROXIMATION OF THE STOCHASTIC CAHN–HILLIARD EQUATION

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Abstract. We consider the stochastic Cahn–Hilliard equation driven by additive Gaussian noise in a convex domain with polygonal boundary in dimension $d \leq 3$. We discretize the equation using a standard finite element method in space and a fully implicit backward Euler method in time. By proving optimal error estimates on subsets of the probability space with arbitrarily large probability and uniform-in-time moment bounds we show that the numerical solution converges strongly to the solution as the discretization parameters tend to zero.

1. Introduction

Let $D \subset \mathbb{R}^d$, $d \leq 3$, be a convex spatial domain with polygonal boundary $\partial D$ and consider the stochastic Cahn–Hilliard equation, also known as the Cahn–Hilliard–Cook equation [4, 8, 9], written in the abstract Itô form

$$
\frac{dX}{dt} + A(AX + f(X)) = dW, \quad t \in (0, T]; \quad X(0) = X_0,
$$

where $-A$ is the Laplacian with homogeneous Neumann boundary conditions and where $\{W(t)\}_{t \geq 0}$ is an $H := L_2(D)$-valued $Q$-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$. In order to avoid additional technical difficulties we assume that the initial-value $X_0$ is deterministic. For the weak (and mild) solution $X$ (see, Theorem 3.1) to preserve mass, we assume that the average $|D|^{-1} \int_D W(t) \, dx = 0$ for all $t \geq 0$.

The nonlinear function $f$ is assumed to be of the form $f = F'$, where $F$ has the following structural property:

$$
F \text{ is a polynomial of degree } 4 \text{ with leading term } c_0 s^4 \text{ where } c_0 > 0.
$$

A typical example is $F(s) = \frac{1}{4}(s^2 - 1)^2$, which is a double well potential. Note that $f$ is only locally Lipschitz and does not satisfy a linear growth condition. We also note that the restriction on the polynomial degree of $F$ comes from the fact that we allow $d = 3$. For $d = 1, 2$ the exponent 4 in (1.2) may be replaced by any even integer larger than or equal 4 and the arguments of the paper are still valid.

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with trivial changes. It is easy to see that \((1.2)\) implies the following dissipativity property, with \(\langle \cdot, \cdot \rangle, \|\cdot\|\) denoting the scalar product and norm in \(H = L_2(D)\),
\[(1.3) \quad \langle f(v), v \rangle \geq -C_0, \quad v \in L_2(D),\]
for some \(C_0\). It also follows that \(F''(s) \geq -c_1^2\) for some \(c_1\), which yields
\[(1.4) \quad F(x) - F(y) \leq f(x)(x - y) + \frac{1}{2}c_1^2(x - y)^2, \quad x, y \in \mathbb{R}.\]
Finally, as \(f\) is a polynomial of degree 3 we have, for some \(C > 0\), that
\[(1.5) \quad |f(x) - f(y)| \leq C(1 + x^2 + y^2)|x - y|, \quad x, y \in \mathbb{R}.\]

It is not hard to see that, due to the homogeneous Neumann boundary conditions and since the average of \(W\) equals 0, it follows that \(X\) preserves the average (the total mass), that is, \(\|D\|^{-1} \int_D X(t) dx = \|D\|^{-1} \int_D X_0 dx\), cf. Remark \((1.2)\). Note that for \(s_0 \in \mathbb{R}\) the function \(F(s) = F(s + s_0)\) also has the structural property \((1.2)\). Therefore, one can employ a change of variables \(X \to X - \|D\|^{-1} \int_D X_0 dx\), and hence we will assume that the average \(\|D\|^{-1} \int_D X_0 dx = 0\).

We fix a finite time horizon \(T > 0\) and for \(N \in \mathbb{N}\) consider the fully implicit finite element method
\[(1.6) \quad X_h^j - X_h^{j-1} + kA_h X_h^j + kA_h P_h f(X_h^j) = P_h \Delta W^j, \quad j = 1, 2, \ldots, N, \quad X_h^0 = P_h X_0.\]
Here \(k = T/N\) is the time-step, \(t_j = jk, \Delta W^j = W(t_j) - W(t_{j-1})\), \(-A_h\) is the discrete Laplacian, and \(P_h\) is the orthogonal projector onto the finite element space \(S_h\) with mesh size \(h > 0\); for more details on the finite element method, see Section \((2.2)\). It is easy to see that also \(X_h^j\) preserves the mass, cf. Remark \((5.1)\). An implementation based on the open source finite element software FEniCS can be found in http://www.math.chalmers.se/~estig/code/chc.py.

The main result of the paper, Theorem \((7.4)\) asserts that if the operator composition \(A \hat{Q} \hat{A}\) is Hilbert–Schmidt and the initial data is regular enough; that is, for some \(L > 0\),
\[|X_{h1}^0 + F(X_h^0) + [A_h X_h^0 + P_h f(X_h^0)]]1 + |X_0|^1| \leq L \text{ for all } h > 0,\]
where \(F(v) = \int_D F(v) dx\) and \(|v|^1 = \|A^2 v\| = \|\nabla v\|\), then
\[\lim_{h,k \to 0} \mathbb{E} \sup_{0 \leq n \leq N} \|X(t_n) - X_h^n\|^2 = 0.\]

The key result used in the proof is a maximal type moment bound on \(|X_h^j|^1\), which is established in Theorem \((13)\) after bootstrapping arguments. There are various difficulties in the proofs that are partly due to the finite element method. First, the finite element method is based on approximating the operator \(A\) and not \(A^2\). This is because the standard finite element functions belong only to the domain of \(A\) but are not more regular. Loosely speaking this means that \(A_h^2 \neq (A^2)_h\), which makes already the deterministic finite element analysis more challenging. Second, the presence of the finite element projection \(P_h\) in front of the semilinear term destroys some of the dissipativity properties of \(f\). While \(f\) enjoys the dissipativity property \((1.3)\), and even
\[\langle A \hat{Q} f(v), A \hat{Q} v \rangle = \langle \nabla f(v), \nabla v \rangle \geq -c|v|^2,\]
we only have
\[\langle P_h f(v_h), v_h \rangle = \langle f(v_h), v_h \rangle \geq -C_0, \quad v_h \in S_h,\]
and unfortunately
\[ \langle A_h^{1/2} P_h f(v_h), A_h^{1/2} v_h \rangle = \langle A_h^{1/2} P_h f(v_h), A_h^{1/2} v_h \rangle \geq -c\|v_h\|^2, \quad v_h \in S_h. \]

Because of the latter we can only establish a non-uniform moment bound on \( \|X^j_h\| \) in Lemma 4.2. As
\[ \langle A f(x), A x \rangle \geq -c\|A x\|^2 \quad \text{and} \quad \langle A_h f(v_h), A_h v_h \rangle \geq -c\|A_h v_h\|^2, \]
the proof of the main moment bound in Theorem 4.3 is rather tedious and not entirely straightforward. In the proof we bound the discrete version of the Lyapunov functional for the original deterministic problem, see (4.15). Having maximal-type moment bounds at hand we use the mild formulation of both (1.1) and (1.6) to establish pathwise error bounds on subsets of the probability space with large probability in Theorem 5.5. This turns out to be sufficient, together with some moment bounds, to show strong convergence of the numerical scheme in Theorem 5.6. Our method of proof does not give rates for the strong convergence.

While strong convergence results for numerical schemes for SPDEs with globally Lipschitz coefficients, or at least some sort of linear growth condition, are abundant there are only few results on strong convergence of discretization schemes for SPDEs with superlinearly growing coefficients [1, 14, 15, 18, 20, 21, 24]. Furthermore, these papers dominantly establish strong convergence of various numerical schemes with no rate given (with a few exceptions), under some sort of global monotonicity assumption on the drift term which is not valid for the Cahn–Hilliard–Cook equation (1.1).

The analysis of numerical methods for SPDEs without a global monotonicity assumption is even less explored. For the Cahn–Hilliard–Cook equation (1.1) studied in the present paper, convergence (with rates) in probability is established for a finite difference scheme in [6]. In [16] strong convergence with rates is established for the spatial spectral Galerkin approximation (no time discretization) for (1.1) and the stochastic Burgers equation driven by trace class noise in spatial dimension \( d = 1 \). The analysis is based on a general perturbation result and exponential integrability properties of the approximation process. Strong convergence of the finite element method without rate and without time discretization is proved in [22, 23] under stronger assumptions on the noise than in the present paper. There it is required that the operator composition \( A^{\gamma} Q^\perp \) for \( \gamma > 1 \) is Hilbert–Schmidt, while here we only require this with \( \gamma = 1 \). Therefore, the present work can be viewed as the (non-trivial) extension of [22, 23] to a strongly convergent fully discrete scheme, still without a strong rate, but with improvements on the regularity requirement on the noise. Both here and in [22, 23] the strategy is based on proving a priori moment bounds with large exponents and in higher order norms using energy arguments and bootstrapping followed by a pathwise Gronwall argument in the mild solution setting. Finally, in connection to the Cahn–Hilliard–Cook equation we note that in [19, 25] the linearized Cahn–Hilliard–Cook equation is treated numerically which requires a significantly simpler analysis.

As further related work on numerical approximation of SPDEs without a global monotonicity assumption we mention the pathwise convergence of a spectral Galerkin method for the stochastic Burgers equation studied in [2, 3], while for the same equation convergence in probability is established in [26] for the Backward Euler method. The stochastic Navier–Stokes equation is considered in [5, 7], in particular,
in [7], the authors obtain a result similar to our Theorem 5.3 (stated in a slightly different form). Finally, we mention the recent work [17], where strong convergence is proved, without rate, for a spectral nonlinearity-truncated accelerated exponential Euler-type approximation for the stochastic Kuramoto–Sivashinsky equation driven by space-time white noise in spatial dimension $d = 1$, an equation rather similar in structure to the Cahn–Hilliard–Cook equation.

The paper is organized as follows. In Section 2 we collect some background material from stochastic and functional analysis and introduce the finite element method in Subsection 2.2. In Section 3 some known results on the existence, uniqueness and regularity on the solution of (1.1) are recalled. Section 4 contains moment bounds for the numerical solution, in particular, it contains the main technical result of the paper, Theorem 4.3. In Section 5 we prove a new error estimate for the derivative of the error in the spatial semidiscretization of the linear deterministic Cahn–Hilliard equation, (5.6) in Lemma 5.4. Then we proceed to prove a pathwise error bound in Theorem 5.5 and the main strong convergence result Theorem 5.6.

2. Preliminaries

2.1. Norms and operators. Throughout the paper we will use various norms for linear operators on a Hilbert space $H$ where the latter is endowed with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote by $L(H)$, the space of bounded linear operators on $H$ with the usual operator norm also denoted by $\| \cdot \|$. If for a selfadjoint positive semidefinite operator $T \in L(H)$, the sum $\text{Tr} \ T := \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle < \infty$ for an orthonormal basis (ONB) $\{e_k\}_{k \in \mathbb{N}}$ of $H$, then we say that $T$ is trace class. In this case $\text{Tr} \ T$, the trace of $T$, is independent of the choice of the ONB. If for an operator $T \in L(H)$, the sum $\|T\|_{\text{HS}}^2 := \sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$

for an ONB $\{e_k\}_{k \in \mathbb{N}}$ of $H$, then we say that $T$ is Hilbert–Schmidt and call $\|T\|_{\text{HS}}$ the Hilbert–Schmidt norm of $T$. The Hilbert–Schmidt norm of $T$ is independent of the choice of the ONB. We have the following well-known properties of the trace and Hilbert–Schmidt norms, see, for example, [10, Appendix C],

\begin{align}
&|T| \leq \|T\|_{\text{HS}}, \quad \|TS\|_{\text{HS}} \leq \|T\|_{\text{HS}}\|S\|, \quad \|ST\|_{\text{HS}} \leq \|S\| \|T\|_{\text{HS}}, \\
&\text{Tr} Q = \|Q^\dagger\|_{\text{HS}}^2 = \|T\|_{\text{HS}}^2 = \|T^*\|_{\text{HS}}^2, \quad \text{if } Q = TT^*.
\end{align}

Next we introduce spaces and norms associated with the operator $A$, the negative of the Neumann Laplacian. Let $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded convex domain with polygonal boundary $\partial D$. We denote by $\| \cdot \|_{L_p}$ the standard norm in $L_p(D)$. In particular, we define $H = L_2(D)$ with its standard inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and

$$H = \{ v \in H : \int_D v \, dx = 0 \}.$$

Let $P : H \to \breve{H}$ define the orthogonal projector. Then $(I - P)v = |D|^{-1} \int_D v \, dx$ is the average of $v$. We also denote by $H^k(D)$ the standard Sobolev space. We define
A = −Δ, the negative of the Neumann Laplacian with domain of definition
\[ D(A) = \left\{ v \in H^2(D) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial D \right\}. \]

Then A is a positive definite, selfadjoint, unbounded, linear operator on \( \dot{H} \) with compact inverse. When extended to \( H \) as \( Av = A^1v \) it has an orthonormal eigenbasis \( \{ \varphi_j \}_{j=0}^{\infty} \) with corresponding eigenvalues \( \{ \lambda_j \}_{j=0}^{\infty} \) such that
\[ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots, \quad \lambda_j \to \infty, \]
see, for example, [11, Section 7.2]. The first eigenfunction is constant, \( \varphi_0 = |D|^{-\frac{1}{2}} \).

We define
\[
\|v\|_\alpha = \left( \sum_{j=1}^{\infty} \lambda_j^\alpha |\langle v, \varphi_j \rangle|^2 \right)^{\frac{1}{2}}, \quad \langle v, w \rangle_\alpha = \sum_{j=1}^{\infty} \lambda_j^\alpha \langle v, \varphi_j \rangle \langle w, \varphi_j \rangle, \quad \alpha \in \mathbb{R},
\]
and corresponding spaces, for \( \alpha \geq 0 \),
\[ \dot{H}^\alpha = D(A^{\frac{\alpha}{2}}) = \left\{ v \in \dot{H} : |v|_\alpha < \infty \right\}, \quad H^\alpha = \left\{ v \in H : \|v\|_\alpha < \infty \right\}. \]

For negative order \( -\alpha < 0 \) we define \( \dot{H}^{-\alpha} \) by taking the closure of \( \dot{H} \) with respect to \( |\cdot|_{-\alpha} \). For integer order \( \alpha = k = 1, 2 \), the norm \( \|\cdot\|_k \) is equivalent on \( H^k \) to the standard Sobolev norm \( \|\cdot\|_{H^k(D)} \). More precisely,
\[
\|v\|_1 = (|v|^2 + |\langle v, \varphi_0 \rangle|^2)^{\frac{1}{2}} = (\|\nabla v\|^2 + |\langle v, \varphi_0 \rangle|^2)^{\frac{1}{2}},
\]
\[
\|v\|_2 = (|v|^2 + |\langle v, \varphi_0 \rangle|^2)^{\frac{1}{2}} = (\|\Delta v\|^2 + |\langle v, \varphi_0 \rangle|^2)^{\frac{1}{2}},
\]
are equivalent to the standard norms \( \|v\|_{H^k(D)}, k = 1, 2 \), by the Poincaré inequality and the regularity estimate for the elliptic Neumann problem.

We recall the fact the operator \( -A^2 \) is the infinitesimal generator of an analytic semigroup \( E(t) = e^{-tA^2} \) on \( H \),
\[
E(t)v = e^{-tA^2}v = \sum_{j=0}^{\infty} e^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j = \sum_{j=1}^{\infty} e^{-t\lambda_j^2} \langle v, \varphi_j \rangle \varphi_j + \langle v, \varphi_0 \rangle \varphi_0 = P e^{-tA^2}v + (I - P)v.
\]

By expansion in the eigenbasis of \( A \) and using Parseval’s identity we easily obtain
\[
\|A^\alpha E(t)v\| \leq Ct^{-\frac{\alpha}{2}}\|v\|, \quad v \in H, \ \alpha \geq 0.
\]

and
\[
\left( \int_0^t s^{2j} \|A^{2j+1}E(s)v\|^2 ds \right)^{\frac{1}{2}} \leq C\|v\|, \quad v \in H, \ j = 0, 1, 2, \ldots.
\]

Here \( C \) depends on \( \alpha \) and \( j \), respectively.
2.2. The finite element method. Let \( \{ T_h \}_{h>0} \) denote a family of regular triangulations of \( D \) with maximal mesh size \( h \). Let \( S_h \) be the space of continuous functions on \( D \), which are piecewise polynomials of degree \( \leq 1 \) with respect to \( T_h \). Hence, \( S_h \subset H^1 \). We also define \( \dot{S}_h = PS_h \); that is,

\[
\dot{S}_h = \{ v_h \in S_h : \int_D v_h \, dx = 0 \}.
\]

The space \( \dot{S}_h \) is introduced only for the purpose of theory but not for computation. Now we define the discrete Laplacian \(-A_h : \dot{S}_h \to \dot{S}_h\) by

\[
\langle A_h v_h, w_h \rangle = \langle \nabla v_h, \nabla w_h \rangle, \quad \forall v_h, w_h \in \dot{S}_h.
\]

The operator \( A_h \) is selfadjoint, positive definite on \( \dot{S}_h \), positive semidefinite on \( S_h \), and \( A_h \) has an orthonormal eigenbasis \( \{ \varphi_{h,j} \}_{j=0}^{N_h} \) with corresponding eigenvalues \( \{ \lambda_{h,j} \}_{j=0}^{N_h} \). We have

\[
0 = \lambda_{h,0} < \lambda_{h,1} \leq \cdots \leq \lambda_{h,j} \leq \cdots \leq \lambda_{h,N_h}
\]

and \( \varphi_{h,0} = \varphi_0 = |D|^{-\frac{1}{2}} \). Moreover, we define \( E_h(t) = e^{-tA^2_h} : S_h \to S_h \) by

\[
E_h(t)v_h = e^{-tA^2_h}v_h = \sum_{j=0}^{N_h} e^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j}
\]

and the orthogonal projector \( P_h : H \to \dot{S}_h \) by

\[
\langle P_h v, w_h \rangle = \langle v, w_h \rangle \quad \forall v \in H, \, w_h \in \dot{S}_h.
\]

Clearly, \( P_h : \dot{H} \to \dot{S}_h \) and

\[
E_h(t)P_h v = PE_h(t)P_h v + (I - P) v.
\]

Likewise, for its time discrete analog \( R^n_{k,h} = (I + k A_h^2)^{-n} \), we have

\[
R^n_{k,h}P_h v = PR^n_{k,h}P_h v + (I - P) v.
\]

We have the discrete analogs of \( (2.9) \),

\[
\| A_h^0 E_h(t)P_h v \| \leq Ct^{-\frac{n}{2}} \| v \|, \quad \| A_h^0 R^n_{k,h}P_h v \| \leq Ct_{\frac{n}{2}} \| v \|, \quad v \in H, \ \alpha \geq 0,
\]

where the constants \( C \) depend on \( \alpha \) but not on \( h \) and \( k \). Similarly to \( (2.0) \), these are proved by expansion in the eigenbasis of \( A_h \) and Parseval’s identity. For example, the first constant is \( C = (\sup_{s \in [0, \infty)} s^\alpha e^{-2s})^{\frac{1}{2}} \).

Finally, we define the Ritz projector \( R_h : \dot{H}^1 \to \dot{S}_h \) by

\[
\langle \nabla R_h v, \nabla w_h \rangle = \langle \nabla v, \nabla w_h \rangle, \quad \forall v \in \dot{H}^1, \, w_h \in \dot{S}_h.
\]

We extend it as \( R_h : H^1 \to S_h \) by

\[
R_h v = R_h P v + (I - P) v, \quad v \in H^1.
\]

We then have the following bound for \( R_h v - v = (R_h - I)P v \) (cf. [27, Chapt. 1])

\[
\| (R_h - I)P v \| \leq C h^2 \| Av \|, \quad v \in \dot{H}^2.
\]

Finally, we define norms on \( \dot{S}_h \), analogous to the norms \( |v|_\alpha \) on \( \dot{H}^\alpha \):

\[
|v_h|_{\alpha,h} = \| A_h^{\alpha/2} v_h \| = \left( \sum_{j=1}^{N_h} \lambda_{h,j}^\alpha |\langle v, \varphi_{j,h} \rangle|^2 \right)^{\frac{1}{2}}, \quad v_h \in \dot{S}_h, \ \alpha \in \mathbb{R}.
\]
The corresponding scalar products are denoted $\langle \cdot , \cdot \rangle_{\alpha,h}$. We note that
\begin{equation}
|v_h|_1 = \|A^{1/2}v_h\| = \|\nabla v_h\| = \|A^{1/2}v_h\| = |v_h|_{1,h}, \quad v_h \in \hat{S}_h.
\end{equation}
We assume that $P_h$ is bounded with respect to the $H^1$ norm
\begin{equation}
|P_h v|_1 \leq C|v|_1, \quad v \in H^1.
\end{equation}
This holds, for example, if the mesh family $\{T_h\}_{h>0}$ is quasi-uniform. By combining this with (2.14), we obtain
\begin{equation}
\|A^{1/2}P_h v\| = |P_h v|_1 \leq C|v|_1 = C\|A^{1/2}v\|.
\end{equation}

2.3. Useful inequalities. We will use the Burkholder–Davis–Gundy inequality for Itô-integrals of the form $\int_0^t (\eta(s), d\tilde{W}(s))$, where $\eta$ is a predictable $H$-valued stochastic process and $\tilde{W}$ is a $\tilde{Q}$-Wiener process in $H$. For this kind of integral, the Burkholder–Davis–Gundy inequality, [10] Lemma 7.2, takes the form
\begin{equation}
E \sup_{t \in [0,T]} \left| \int_0^t (\eta(s), d\tilde{W}(s)) \right|^p \leq CE \left( \int_0^T \|\tilde{Q}^{1/2} \eta(s)\|_H^2 \, ds \right)^{\frac{p}{2}}, \quad p \geq 2,
\end{equation}
where $C$ depends on $p$.

Also, if $Y$ is an $H$-valued centered Gaussian random variable with covariance operator $\tilde{Q}$, then, by [10] Corollary 2.17, we can bound its $p$-th moments via its covariance operator as
\begin{equation}
E\|Y\|^{2p} \leq C(E\|Y\|^2)^p = C(\text{Tr} \tilde{Q})^p = C\|\tilde{Q}^{1/2}\|_{\text{HS}}^2, \quad p \geq 1,
\end{equation}
where $C$ depends on $p$. In particular, for an Itô integral $Y = \int_s^t R \, dW(r) = R(W(t) - W(s))$, where $R$ is a constant, possibly unbounded, operator on $H$ and $W$ is a $Q$-Wiener process, the inequality (2.18) reads
\begin{equation}
E\left\| \int_s^t R \, dW(r) \right\|^{2p} \leq C(t-s)^p \|RQ^{1/2}\|_{\text{HS}}^{2p}.
\end{equation}
If $p = 1$, the inequality in (2.19) becomes an equality with $C = 1$. The inequality
\begin{equation}
\left| \sum_{j=K}^M a_j \right|^p \leq |M - K + 1|^{p-1} \sum_{j=K}^M |a_j|^p, \quad p \geq 1,
\end{equation}
will be frequently utilized; it is a direct consequence of Hölder’s inequality.

3. Existence, uniqueness and regularity

Existence, uniqueness, and regularity of weak solutions to (1.1) has been studied in [11] with some minor improvements in [22]. Note that here we assume that $X_0$ is deterministic and that $X_0 \in \hat{H}$, so that $X(t) \in \hat{H}$. We summarize the results:

**Theorem 3.1.** If $\|A^{1/2}Q^{1/2}\|_{\text{HS}} < \infty$, $|X_0|_1 < \infty$, and $T < \infty$, then there is a unique weak solution $X$ of (1.1); that is, an adapted $H$-valued process $X$, which is continuous almost surely and satisfies the equation
\begin{equation}
(X(t), v) - (X_0, v) + \int_0^t \left( \langle X(s), A^2 v \rangle + \langle f(X(s)), Av \rangle \right) \, ds = \langle W(t), v \rangle.
\end{equation}
almost surely for all $v \in \dot{H}^4 = D(A^2)$, $t \in [0, T]$. Furthermore, there is $C > 0$ such that
\begin{equation}
\mathbb{E} \sup_{t \in [0, T]} |X(t)|^2 + \mathbb{E} \sup_{t \in [0, T]} \|X(t)\|^4_{L_4} \leq C.
\end{equation}
In addition, $X$ is also a mild solution; that is, it satisfies the equation
\begin{equation}
X(t) = E(t)X_0 - \int_0^t E(t - s)Af(X(s)) \, ds + \int_0^t E(t - s) \, dW(s),
\end{equation}
almost surely.

We also have pathwise Hölder regularity in time:

**Proposition 3.2.** Let $\|A^{1/2}Q\|_{\text{HS}} < \infty$ and $|X_0|_1 < \infty$. Then, for all $\gamma \in [0, \frac{1}{2})$, there is an almost surely finite nonnegative random variable $K$ such that, almost surely,
\begin{equation}
\sup_{t \neq s \in [0, T]} \frac{\|X(t) - X(s)\|}{|t - s|^\gamma} \leq K.
\end{equation}

We omit the proof as it is analogous to the proof of [20, Proposition 3.2].

### 4. Moment bounds for the discrete solution

We start by proving a preliminary moment bound which will be used later on in a bootstrapping argument. Throughout the proofs, $C$ denotes a generic non-negative constant that is independent of the discretization parameters $h$ and $k$ and may change from line to line.

We recall our assumption that $X_0 \in \dot{H}$, so that $X_0^0 = P_hX_0 \in \dot{S}_h$ and hence $X^j_h \in \dot{S}_h$ for $0 \leq j \leq N$.

**Lemma 4.1.** Let $p \geq 1$. If $\|A_{h}^{-1/2}P_hQ^{1/2}\|_{\text{HS}} \leq K$ and $|X^0_h|_{-1,h} \leq L$ for all $h > 0$, then there exists $C > 0$ depending on $p, T, K, L$, such that, for all $h, k > 0$,
\begin{align}
&\mathbb{E} \left( \sup_{1 \leq j \leq N} |X^j_h|^{2p}_{-1,h} \right) \leq C, \\
&\mathbb{E} \left( \sum_{j=1}^{N} |X^j_h - X^{j-1}_h|^{2}_{-1,h} + k|X^j_h|^{2}_{1} \right)^p \leq C.
\end{align}

**Proof.** Since $X^j_h \in \dot{S}_h$, we may multiply by $A_{h}^{-1}X^j_h$ in (1.6) to get
\begin{align}
\frac{1}{2} \left( |X^j_h|^{2}_{-1,h} - |X^{j-1}_h|^{2}_{-1,h} + |X^j_h - X^{j-1}_h|^{2}_{-1,h} \right) + k|X^j_h|^{2}_{1} + k\langle f(X^j_h), X^j_h \rangle = \langle P_h\Delta W^j, A_{h}^{-1}X^j_h \rangle,
\end{align}
where we have used the selfadjointness of $A_{h}$, (2.14), and the identity
\begin{equation}
\langle X - Y, X - Y \rangle_{-1,h} = \frac{1}{2} \left( |X|^{2}_{-1,h} - |Y|^{2}_{-1,h} + |X - Y|^{2}_{-1,h} \right).
\end{equation}
Next, the dissipativity inequality (1.3) yields
\begin{align}
\frac{1}{2} \left( |X^j_h|^{2}_{-1,h} - |X^{j-1}_h|^{2}_{-1,h} + |X^j_h - X^{j-1}_h|^{2}_{-1,h} \right) + k|X^j_h|^{2}_{1} \\
&\leq C_0k + \langle P_h\Delta W^j, X^j_h - X^{j-1}_h \rangle_{-1,h} + \langle P_{h}\Delta W^j, X^j_h \rangle_{-1,h}.
\end{align}
Furthermore, \( \langle P_h \Delta W^j, X_h^j - X_h^{j-1} \rangle_{-1,h} \) ≤ \( C' \| P_h \Delta W^j \|^2 \| X_h^j - X_h^{j-1} \|^2 \) \( \leq \| \). Hence,

\[
\frac{1}{2} \left( |X_h^j|_{-1,h}^2 - |X_h^{j-1}|_{-1,h}^2 \right) + c|X_h^j - X_h^{j-1}|_1^2 + k|X_h^j|_1^2
\leq C \left( k + \| P_h \Delta W^j \|_{-1,h}^2 + \langle P_h \Delta W^j, X_h^{j-1} \rangle_{-1,h} \right).
\]

Summing with respect to \( j \) in (4.4), thus yields

\[
|X_h^n|_{-1,h}^2 + \sum_{j=1}^n \left( |X_h^j - X_h^{j-1}|_{-1,h}^2 + k|X_h^j|_1^2 \right) \leq C \left( T + \| X_h^0 \|_{-1,h}^2 \right)
\]

\[
+ \sum_{j=1}^n \| P_h \Delta W^j \|_{-1,h}^2 + \sum_{j=1}^n \langle P_h \Delta W^j, X_h^{j-1} \rangle_{-1,h} \right) ^p
\]

\[
E \sup_{1 \leq n \leq N} \| X_h^n \|_{-1,h}^{2p} \leq E \sup_{1 \leq n \leq N} \left\{ T + \| X_h^0 \|_{-1,h}^2 \right\}^p
\]

\[
+ \sum_{j=1}^n \| P_h \Delta W^j \|_{-1,h}^2 \right\}^p + \sup_{1 \leq n \leq N} \left\{ \sum_{j=1}^n \langle P_h \Delta W^j, X_h^{j-1} \rangle_{-1,h} \right\}^p
\]

\[
\leq C \left\{ T^p + \| X_h^0 \|_{-1,h}^{2p} \}
\]

\[
+ \left( \sum_{j=1}^n \| P_h \Delta W^j \|_{-1,h}^2 \right) \right\}^p + \sup_{1 \leq n \leq N} \left\{ \sum_{j=1}^n \langle P_h \Delta W^j, X_h^{j-1} \rangle_{-1,h} \right\}^p
\]

By (2.18), we have

\[
E \left( \sum_{j=1}^n \| P_h \Delta W^j \|_{-1,h}^2 \right) \leq C N^{p-1} \sum_{j=1}^n \| P_h \Delta W^j \|_{-1,h}^{2p}
\]

\[
\leq C N^{p-1} \sum_{j=1}^n k^p \| A_h^{-1/2} P_h Q^{1/2} \|_{HS}^{2p} \leq CT^p \| A_h^{-1/2} P_h Q^{1/2} \|_{HS}^{2p} \leq C.
\]
Moreover, by Cauchy’s inequality and \[2.17\],

\[
\left| \sum_{j=1}^{n} \langle P_h \Delta W^j, X_h^{j-1} \rangle_{-1,h} \right|^p = \mathbb{E} \sup_{1 \leq n \leq N} \left| \sum_{j=1}^{n} \langle \Delta W^j, A_h^{-1} X_h^{j-1} \rangle \right|^p \\
\leq C \left( \sum_{j=1}^{N} k \| Q^{1/2} A_h^{-1} X_h^{j-1} \|^2 \right)^{p/2} \\
\leq C \left\{ 1 + \mathbb{E} \left( \sum_{j=1}^{N} k \| Q^{1/2} A_h^{-1} X_h^{j-1} \|^2 \right) \right\}^{p/2} \\
\leq C \left\{ 1 + (Nk)^{p-1} \sum_{j=1}^{N} k \| Q^{1/2} A_h^{-1} P_h X_h^{j-1} \|^2 \right\} \\
\leq C \left\{ 1 + (Nk)^{p-1} \sum_{j=1}^{N} k \| Q^{1/2} A_h^{-1/2} P_h \|^{2p} \mathbb{E} |X_h^{j-1}|^{2p} \right\} \\
= C \left\{ 1 + (Nk)^{p-1} \sum_{j=1}^{N} k \| A_h^{-1/2} P_h Q^{1/2} \|^{2p} \mathbb{E} |X_h^{j-1}|^{2p} \right\}.
\]

(4.8)

As \( Nk = T \), \( \| A_h^{-1/2} P_h Q^{1/2} \| \leq \| A_h^{-1/2} P_h Q^{1/2} \|_{\text{HS}} \leq K \), and \( |X_{-1,h}^0| \leq L \), by inserting \([4.7]\) and \([4.8]\) into \([4.6]\), we see that

\[
\mathbb{E} \sup_{0 \leq n \leq N} |X_h^n|^{2p} \leq C(p, T, K, L) \left( 1 + \sum_{j=0}^{N-1} k \mathbb{E} |X_h^j|^{2p} \right)
\]

(4.9)

By induction, \([4.9]\) shows that the quantity \( \mathbb{E} \sup_{0 \leq j \leq n} |X_h^j|^{2p} \) is finite for all \( n = 1, \ldots, N \) and hence the inequality \([4.11]\) follows from Gronwall's lemma. Having this result at hand one may return to \([4.5]\) to prove \([4.2]\) by a similar procedure but without the Gronwall argument at the end. \( \Box \)

In the sequel, in many places the quantity

\[
Y_h^j := A_h X_h^j + P_h f(X_h^j)
\]

(4.10)

plays a crucial role. It can be regarded as the discrete version of the “chemical potential” \( Y = AX + f \). With this notation, the scheme \([4.4]\) can be rewritten as

\[
X_h^j - X_h^{j-1} = k A_h Y_h^j = P_h \Delta W^j, \quad j = 1, 2, \ldots, N; \quad X_h^0 = P_h X_0.
\]

(4.11)

We continue to prove a stronger moment bound. Note that it is not “closed”, because \( Y_h^j \) remains on the right hand side.

**Lemma 4.2.** Suppose that \( \| A^{1/2} Q^{1/2} \|_{\text{HS}} \leq K \) and \( |X_h^0|_{-1,h} \leq L \) for all \( h > 0 \). Then, for every \( \epsilon, \delta > 0 \) and \( p \geq 1 \), there are \( C_1 > 0 \) depending on \( T, \epsilon, \delta, p, K \) and
Proof. By taking inner products with $X_j^1 \in \dot{S}_h$ in (4.11) we get
\[
\frac{1}{2} \left( \|X_h^j\|^2 - \|X_h^{j-1}\|^2 \right) + k\langle Y_h^j, X_h^j \rangle_1 = \langle \Delta W^j, X_h^j \rangle,
\]
where we recall that $\langle x, y \rangle_1 = \langle \nabla x, \nabla y \rangle$. Summing with respect to $j$, using analogous arguments as in the previous proof, thus yields for $1 \leq n \leq N$
\[
\|X_h^n\|^2 + \sum_{j=1}^n \|X_h^j - X_h^{j-1}\|^2 \leq C \left( \|X_h^0\|^2 + \sum_{j=1}^n k \langle Y_h^j, X_h^j \rangle_1 \right) + \sum_{j=1}^n \|\Delta W^j\|^2 + \sum_{j=1}^n \langle \Delta W^j, X_h^{j-1} \rangle.
\]
Therefore,
\[
\mathbb{E} \left( \sup_{1 \leq j \leq N} \|X_h^j\|^{2p} + \mathbb{E} \left( \sum_{j=1}^N \|X_h^j - X_h^{j-1}\|^2 \right)^p \right) \\
\leq C \left( \|X_h^0\|^{2p} + \mathbb{E} \left( \sum_{j=1}^N k \langle Y_h^j, X_h^j \rangle_1 \right)^p \right) + \mathbb{E} \left( \sum_{j=1}^N \|\Delta W^j\|^2 \right)^p + \mathbb{E} \sup_{1 \leq n \leq N} \left\| \mathcal{A}_h^{1/2} \mathcal{P}_h \Delta W^j, A_h^{1/2} X_h^{j-1} \right\|^p.
\]
The next to last term can be bounded, similarly to (4.7) in the previous proof, by $C T^p \|Q^{1/2}\|_{HS}^{2p}$. Using the (4.1) and a calculation similar to (4.8) we obtain
\[
\mathbb{E} \left( \sup_{1 \leq n \leq N} \left\| \mathcal{A}_h^{1/2} \mathcal{P}_h \Delta W^j, A_h^{1/2} X_h^{j-1} \right\|^p \right) \leq C \left( 1 + T^p \|A_h^{1/2} \mathcal{P}_h Q^{1/2}\|^{2p} \right) \\
\leq C \left( 1 + T^p \|A_h^{1/2} Q^{1/2}\|^{2p} \right),
\]
where we also used (4.11). Finally,
\[
\mathbb{E} \left( \sum_{j=1}^N k \langle Y_h^j, X_h^j \rangle_1 \right)^p \leq \mathbb{E} \left( \sum_{j=1}^N k |Y_h^j|_1 |X_h^j|_1 \right)^p \\
\leq \mathbb{E} \left\{ \left( \sum_{j=1}^N k |Y_h^j|_1^2 \right)^{p/2} \left( \sum_{j=1}^N k |X_h^j|_1^2 \right)^{p/2} \right\} \\
\leq \mathbb{E} \left\{ \delta \left( \sum_{j=1}^N k |Y_h^j|_1^2 \right)^{p/2} + C \delta \left( \sum_{j=1}^N k |X_h^j|_1^2 \right)^{p/2} \right\},
\]
and the proof is complete in view of (4.2) by noting that
\[
\|A_h^{-1/2} P_h Q^{1/2}\|_{\text{HS}} = \|A_h^{-1/2} P_h A^{-1/2} A^{1/2} Q^{1/2}\|_{\text{HS}} \\
\leq \|A_h^{-1/2} P_h A^{-1/2} P\| \|A^{1/2} Q^{1/2}\|_{\text{HS}} \leq C \|A^{1/2} Q^{1/2}\|_{\text{HS}}.
\]

\[\Box\]

We next prove the main stability result of the paper. It is well known that, for the deterministic Cahn–Hilliard equation,
\[\dot{u} + A v = 0, \quad t > 0; \quad v = A u + f(u),\]
the functional
\[J(u) := \frac{1}{2} |u|^2 + F(u), \quad \text{where } F(u) = \int_D F(u) \, dx,\]
is a Ljapunov functional, that is, \(J(u(t)) \leq J(u_0), \quad t \geq 0.\) This leads to a uniform bound for \(|u(t)|^2 + \|u(t)\|_{L^2}^2.\) The proof proceeds by multiplication of the equation by the chemical potential \(v\) and noting that \(J'(u) = A u + f(u) = v\) and \(\langle \dot{u}, v \rangle = \frac{1}{2} \frac{d}{dt} J(u)\) and
\[J(u(T)) + \int_0^T |v(t)|^2 dt = J(u_0),\]
which is the desired result.

This was imitated for the spatially semidiscrete Cahn–Hilliard–Cook equation in [22] by applying the Itô formula to \(J(X_h(t)).\) For the fully discrete equation (1.6) we do not have an Itô formula, so we must use a more direct imitation of the above calculation in the proof of the following theorem, which contains our main moment bounds.

In the proof we denote by \(P_\alpha(x), \quad x = (x_1, ..., x_m), \quad x_i \geq 0,\) any nonnegative quantity such that
\[(4.13) \quad P_\alpha(x) \leq C \left(1 + \sum_{i=1}^m x_i^\alpha\right).\]

**Theorem 4.3.** Let \(p \geq 1.\) If \(\|A^{1/2} Q^{1/2}\|_{\text{HS}} \leq K\) and
\[(4.14) \quad |X_h^0|_{-1, h} + J(X_h^0) + |Y_h^0|_1 \leq L \quad \text{for all } h > 0,\]
then there exist \(C, k_0 > 0,\) depending on \(p, K, L, \) and \(T,\) such that for all \(h > 0\) and \(0 < k < k_0,\)
\[E \sup_{1 \leq j \leq N} J(X_h^j) + E \left(\sum_{j=1}^N k |Y_h^j|_1^2\right)^p \leq C.\]

**Proof.** Following the procedure from the deterministic case, we multiply (4.11) by the discrete chemical potential \(Y_h^j = A_h X_h^j + P_h f(X_h^j):\)
\[\langle Y_h^j, \Delta X_h^j \rangle + k |Y_h^j|_1 = \langle Y_h^j, P_h \Delta W^j \rangle,\]
where \(\Delta X_h^j = X_h^j - X_h^{j-1}.\) From (1.4) it follows that, for \(X, Z \in \hat{H},\)
\[\mathcal{F}(X) - \mathcal{F}(Z) \leq \langle f(X), X - Z \rangle + \frac{1}{2} \mathcal{E}_2^2 \|X - Z\|^2.\]
Hence,
\[
\langle p_h f(X_h^j), \Delta X_h^j \rangle = \langle f(X_h^j), \Delta X_h^j \rangle \geq \mathcal{F}(X_h^j) - \mathcal{F}(X_h^{j-1}) - \frac{1}{2}c_1^2 \| \Delta X_h^j \|^2.
\]
As in (4.3) we have
\[
\langle A_h X_h^j, \Delta X_h^j \rangle = \langle X_h^j, \Delta X_h^j \rangle_1 = \frac{1}{2} \left( \| X_h^j \|^2_1 - \| X_h^{j-1} \|^2_1 + \| \Delta X_h^j \|^2_1 \right).
\]
By adding the latter two relations, we obtain
\[
\langle Y_h^j, \Delta X_h^j \rangle = \langle A_h X_h^j + p_h f(X_h^j), \Delta X_h^j \rangle \geq J(X_h^j) - J(X_h^{j-1}) + \frac{1}{2} \| \Delta X_h^j \|^2_1 + \frac{1}{2}c_1^2 \| \Delta X_h^j \|^2.
\]
This is the discrete analog of \( \langle v, u \rangle = \frac{1}{4} \mathcal{J}(u) \). By (4.11), we now have
\[
(4.15) \quad J(X_h^j) - J(X_h^{j-1}) + \frac{1}{2} \| \Delta X_h^j \|^2_1 + k \| Y_h^j \|^2_1 \leq \langle Y_h^j, P_h \Delta W^j \rangle - \frac{1}{2}c_1^2 \| \Delta X_h^j \|^2.
\]
The remaining challenge is to deal with the term
\[
\langle Y_h^j, P_h \Delta W^j \rangle = \langle Y_h^{j-1}, P_h \Delta W^j \rangle + \langle Y_h^j - Y_h^{j-1}, P_h \Delta W^j \rangle.
\]
We begin by
\[
\langle Y_h^j - Y_h^{j-1}, P_h \Delta W^j \rangle = \langle A_h \Delta X_h^j, P_h \Delta W^j \rangle + \langle f(X_h^j) - f(X_h^{j-1}), P_h \Delta W^j \rangle.
\]
Here, by (2.10),
\[
(4.16) \quad \langle A_h \Delta X_h^j, P_h \Delta W^j \rangle \leq c \| \Delta X_h^j \|^2_1 + C_1 \| \Delta W^j \|^2_1.
\]
and, as \( P_h \Delta W^j \in \tilde{S}_h \subseteq \tilde{H} \),
\[
|\langle f(X_h^j) - f(X_h^{j-1}), P_h \Delta W^j \rangle| = |\langle f(X_h^j), P_h \Delta W^j \rangle - \langle f(X_h^{j-1}), P_h \Delta W^j \rangle| \leq \| A_h^{-1/2} P_h f(X_h^{j-1}) \| \| \Delta W^j \|_1.
\]
By using Hölder’s and Sobolev’s inequalities \((d \leq 3)\) we show
\[
\| A_h^{-1/2} P_h f \| = \sup_{v_h \in \tilde{S}_h} \frac{\langle f, v_h \rangle}{\| v_h \|_1} \leq \sup_{v_h \in \tilde{S}_h} \frac{\| f \|_{L_{a/5}(D)} \| v_h \|_{L_a(D)}}{\| v_h \|_1} \leq C \| f \|_{L_{a/5}(D)}.
\]
Therefore, (1.5) implies
\[
(4.18) \quad \| A_h^{-1/2} P_h f(X_h^j) - f(X_h^{j-1}) \| \leq C \| f(X_h^j) - f(X_h^{j-1}) \|_{L_{a/5}(D)} \leq C \left( \int_D |X_h^j - X_h^{j-1}|^{6/5} (1 + (X_h^j)^2 + (X_h^{j-1})^2)^{6/5} \, dx \right)^{5/6} \leq C \left( \int_D |X_h^j - X_h^{j-1}| \, dx \right)^{1/6} \left( \int_D (1 + (X_h^j)^2 + (X_h^{j-1})^2)^{3/2} \, dx \right)^{2/3} \leq C \| X_h^{j-1} \|_{L_a(D)} \left( 1 + \| X_h^j \|^2_{L_a(D)} + \| X_h^{j-1} \|^2_{L_a(D)} \right).
\]
Further, with \( p < q < r \) and \( \lambda = \frac{2 - \frac{r}{q}}{\frac{r}{q} - \frac{r}{p}} \), we have
\[
\| X \|_{L_q(D)} \leq \| X \|_{L_{a/5}(D)} \| X \|_{L_{a/5}(D)}^{1-\lambda},
\]
see [13, Proposition 6.10]. We take \( p = 2, \, q = 3, \, r = 4 \), and hence \( \lambda = \frac{1}{3} \), to conclude that

\[
\|X_h^j\|_{L^3(\Omega)}^2 \leq \|X_h^j\|_{L^2(\Omega)}^{2/3}\|X_h^j\|_{L^4(\Omega)}^{4/3},
\]

(4.19)

Thus, from (4.17), (4.18), (4.19), and since by Sobolev’s inequality we have \( \|X_h^j - X_h^{j-1}\|_{L^4(\Omega)} \leq C|X_h^j - X_h^{j-1}|_1 \), it follows that

\[
|\langle f(X_h^j) - f(X_h^{j-1}), P_h\Delta W^j \rangle| \leq C|\Delta W^j|_1|X_h^j - X_h^{j-1}|_1
\]

(4.20)

\[
\times P_{2/3}(\|X_h^j\|_{L^2(\Omega)}), \|X_h^{j-1}\|_{L^2(\Omega)}), \|X_h^{j-1}\|_{L^4(\Omega)})
\]

\[
\leq \epsilon |X_h^j - X_h^{j-1}|_1^2 + C|\Delta W^j|^2 P_{4/3}(\|X_h^j\|_{L^2(\Omega)}), \|X_h^{j-1}\|_{L^2(\Omega)})
\]

\[
\times P_{8/3}(\|X_h^j\|_{L^4(\Omega)}), \|X_h^{j-1}\|_{L^4(\Omega)})
\]

where we used the notation \( P_\alpha \) from (4.13). This means that we have bounded the Lipschitz constant in (4.18) by powers of \( \|X_h^j\|_{L^4(\Omega)} \), \( q = 2, 4 \), which we shall be able to control. It will be important that the exponent on \( \|X_h^j\|_{L^4(\Omega)} \), is strictly less than 4 so that it can be controlled in terms of \( F(\epsilon/4) \). Therefore we cannot simplify by multiplying the polynomials together. The exponent on \( \|X_h^j\|_{L^2(\Omega)} \) can be arbitrarily large because of Lemma 4.2.

Thus, with \( 0 < \epsilon < \frac{1}{4} \) we get, after inserting (4.16) and (4.20) into (4.15) and rearranging, that

\[
J(X_h^j) - J(X_h^{j-1}) + c|X_h^j - X_h^{j-1}|_1^2 + k|Y_h^j|_1^2
\]

\[
\leq C|\Delta W^j|_1^2 P_{4/3}(\|X_h^j\|_{L^2(\Omega)}), \|X_h^{j-1}\|_{L^2(\Omega)}), \|X_h^{j-1}\|_{L^4(\Omega)})
\]

\[
+ \langle Y_h^{j-1}, \Delta W^j \rangle + C|X_h^j - X_h^{j-1}|_1^2.
\]

Summing with respect to \( j \) then yields

\[
J(X_h^n) + \sum_{j=1}^n \left( |X_h^j - X_h^{j-1}|_1^2 + k|Y_h^j|_1^2 \right)
\]

\[
\leq C \left( J(X_h^0) + \sum_{j=1}^n \|X_h^j - X_h^{j-1}\|_1^2
\]

\[
+ \sum_{j=1}^n |\Delta W^j|_1^2 P_{4/3}(\|X_h^j\|_{L^2(\Omega)}), \|X_h^{j-1}\|_{L^2(\Omega)}), \|X_h^{j-1}\|_{L^4(\Omega)})
\]

\[
+ \sum_{j=1}^n \langle Y_h^{j-1}, \Delta W^j \rangle \right).
\]

It follows in a similar way as in the proof of Lemma 4.1 using also (4.12) with \( \epsilon = 1 \) and \( \delta > 0 \) so small that the third term on the right hand side above can be
absorbed into the third term in the left hand side below, that

\( (4.21) \)

\[
\mathbf{E} \sup_{1 \leq n \leq N} J(X_h^n)^p + \mathbf{E} \left( \sum_{j=1}^{N} k|Y_h^{j,1}|^2 \right)^p \leq C \left\{ 1 + J(X_h^n)^p \right\}
\]

\[
+ N^{p-1} \mathbf{E} \sum_{j=1}^{N} \left( \|\Delta W^{j}\|_{L_2(D)}^{2p} P_{4p/3}(\|X_h^j\|_{L_2(D)}, \|X_h^{j-1}\|_{L_2(D)}) \right)
\]

\[
\times P_{8p/3}(\|X_h^j\|_{L_4(D)}, \|X_h^{j-1}\|_{L_4(D)}) + \mathbf{E} \left( \sup_{1 \leq n \leq N} \left\| \sum_{j=1}^{N} (Y_h^{j-1}, \Delta W^{j}) \right\|_p \right). \]

The first three terms to the right of the inequality are bounded by assumption. For the fourth term we use that \( \mathbf{E} \Delta W^{j}_{2p} \leq C k_p ||A^{1/2}Q^{1/2}||_{HS}^{2p} \), see (4.7), together with Hölder’s inequality with conjugate exponents \( q_1, q_1’ > 1 \), to get

\[
N^{p-1} \mathbf{E} \sum_{j=1}^{N} \left( \|\Delta W^{j}\|_{L_2(D)}^{2p} P_{4p/3}(\|X_h^j\|_{L_2(D)}, \|X_h^{j-1}\|_{L_2(D)}) \right)
\]

\[
\times P_{8p/3}(\|X_h^j\|_{L_4(D)}, \|X_h^{j-1}\|_{L_4(D)}) \leq N^{p-1} \sum_{j=1}^{N} \left( \mathbf{E} \|\Delta W^{j}\|_{1}^{2p/q_1’} \right)^{1/q_1’} \left[ \mathbf{E} \left( P_{4p/q_1, 3}(\|X_h^j\|_{L_2(D)}, \|X_h^{j-1}\|_{L_2(D)}) \right) \right]^{1/q_1}
\]

\[
\leq C ||A^{1/2}Q^{1/2}||_{HS}^{2p} k_p N^{p-1} \sum_{j=1}^{N} \left[ \mathbf{E} \left( P_{4p/q_1, 3}(\|X_h^j\|_{L_2(D)}, \|X_h^{j-1}\|_{L_2(D)}) \right) \right]^{1/q_1}.
\]

Here \( ||A^{1/2}Q^{1/2}||_{HS}^{2p} \leq C \). Hölder’s inequality, now with \( q_2, q_2’ > 1 \), bounds the above quantity by

\[
\leq C T^{p-1} \sum_{j=1}^{N} \left[ \left( \mathbf{E} \left( P_{4p/q_1, q_2, 3}(\|X_h^j\|_{L_2(D)}, \|X_h^{j-1}\|_{L_2(D)}) \right) \right)^{1/q_2’} \right]^{1/q_1’}
\]

\[
\times \left( \mathbf{E} \left( P_{8p/q_1, q_2, 3}(\|X_h^j\|_{L_4(D)}, \|X_h^{j-1}\|_{L_4(D)}) \right) \right)^{1/q_2’} \right]^{1/q_1}
\]

\[
(4.22) \]

\[
\leq C \left\{ 1 + \sum_{j=1}^{N} k \left[ \mathbf{E} \left( P_{4p/q_1, q_2, 3}(\|X_h^j\|_{L_2(D)}, \|X_h^{j-1}\|_{L_2(D)}) \right) \right]^{1/q_2’} \right\}
\]

\[
\times \left[ \mathbf{E} \left( P_{8p/q_1, q_2, 3}(\|X_h^j\|_{L_4(D)}, \|X_h^{j-1}\|_{L_4(D)}) \right) \right]^{1/q_2’} \right\}
\]

\[
\leq C \left\{ 1 + \sum_{j=1}^{N} k \left[ \mathbf{E} \left( P_{4p/q_1, q_2, 3}(\|X_h^j\|_{L_2(D)}, \|X_h^{j-1}\|_{L_2(D)}) \right) \right]^{1/q_2’} \right\}
\]

\[
+ \mathbf{E} \left( P_{8p/q_1, q_2, 3}(\|X_h^j\|_{L_4(D)}, \|X_h^{j-1}\|_{L_4(D)}) \right) \right\}. \]
With \( q_1q_2 = 3/2 \) we use (1.2) to get
\[
(4.23) \quad P_{3p,q_1, q_2}(\|X^j_h\|_{L^2(D)}, \|X^{-1}_h\|_{L^2(D)}) \leq C(\|X^j_h\|_{L^2(D)}^{4p} + \|X^{-1}_h\|_{L^2(D)}^{4p} + 1)
\leq C(F(X^j_h)^p + F(X^{-1}_h)^p + 1).
\]
Furthermore,
\[
(4.24) \quad \sum_{j=1}^N k \left[ E \left( P_{4pq_1,q_2'/3}(\|X^j_h\|_{L^2(D)}, \|X^{-1}_h\|_{L^2(D)}) \right) \right]
\leq C E k \sum_{j=1}^N (\|X^j_h\|_{L^2(D)}^{4pq_1,q_2'/3} + \|X^{-1}_h\|_{L^2(D)}^{4pq_1,q_2'/3} + 1)
= CT + C\|X_0\|_{L^2(D)}^q + C E k \sum_{j=1}^N \|X^j_h\|_{L^2(D)}^q,
\]
where for brevity \( q = 4pq_1,q_2'/3 \). Let \( t, s > 1 \) be conjugate exponents. Then, by Hölder’s and Young’s inequalities,
\[
E k \sum_{j=1}^N \|X^j_h\|_{L^2(D)}^q \leq E \left[ \sup_{1 \leq j \leq N} \left( \|X^j_h\|_{L^2(D)}^{q-2} \right) k \sum_{j=1}^N \|X^j_h\|_{L^2(D)}^2 \right]^{1/q} \left[ E \left( k \sum_{j=1}^N \|X^j_h\|_{L^2(D)}^2 \right)^{1/2} \right]^{1/s} \leq C \left( E \left[ \sup_{1 \leq j \leq N} \|X^j_h\|_{L^2(D)}^{(q-2)/2} \right] + E \left( k \sum_{j=1}^N \|X^j_h\|_{L^2(D)}^2 \right)^{1/2} \right)^{1/s} \leq C \left( E \left[ \sup_{1 \leq j \leq N} \|X^j_h\|_{L^2(D)}^{(q-2)/2} \right] + E \left( k \sum_{j=1}^N \|X^j_h\|_{L^2(D)}^2 \right)^{1/2} \right)^{1/s}.
\]
Next, (4.12) from Lemma 4.2 and (1.2) from Lemma 4.1 implies that
\[
(4.25) \quad E k \sum_{j=1}^N \|X^j_h\|_{L^2(D)}^q \leq C + K_0 E \left( \sum_{j=1}^N k |\mathcal{Y}^j_h|_{L^2(D)}^{4(q_2'/3 - 2)} \right)^1/4.
\]
We will find \( \epsilon > 0, q_1, q_2, q_2', t > 1 \), such that \( q_2, q_2' \) are conjugate, \( q_1q_2 = 3/2 \) and
\[
(4.26) \quad t(4pq_1,q_2'/3 - 2)(1 + \epsilon)
\leq 4.
\]
Since \( q_2, q_2' \) are conjugate and \( q_1q_2 = 3/2 \) we have that \( q_1,q_2' = 3/2(q_2' - 1) \) and hence (4.26) becomes
\[
tp(q_2' - 1) \frac{1 + \epsilon}{2} - \frac{1 + \epsilon}{2} \leq p.
\]
Note that \( q_2 < 3/2 \) and hence \( q_2' > 3 \). If we set \( q_2' = 3 + \frac{1}{p^2} \) (and thus \( q_2 = \frac{3p^2+1}{2p^2+1} \)) and \( q_1 = \frac{6p^2+3}{q_2' + 2} \) we need to find \( \epsilon > 0 \) and \( t > 1 \) such that
\[
(4.27) \quad tp(1 + \epsilon) + t \frac{1 + \epsilon}{2p} - \frac{1 + \epsilon}{2} \leq p.
\]
Thus inserting (4.23) and (4.28) into (4.22) we get, with \( \varepsilon > 0 \) and \( t > 1 \) such that (4.27) and hence (4.26) holds. Therefore, we can conclude from (4.24) and (4.25) that there is \( \varepsilon > 0, q_1, q_2, q'_2, t > 1 \), such that \( q_2, q'_2 \) are conjugate, \( q_1 q_2 = 3/2 \) and

\[
\sum_{j=1}^{N} k \left[ \mathbb{E}(P_{4p_1/q'_2/3}(\|X^j_h\|_{L_2(D)}), \|X^j_h^{-1}\|_{L_2(D)}) \right] \leq C T + C \|X^0_h\|^{4p+2p}_p
\]

(4.28)

\[
+ K \delta \left( 1 + \mathbb{E} \left( \sum_{j=1}^{N} k|Y^j_h|_1^2 \right)^p \right)
\]

Thus inserting (4.23) and (4.28) into (4.22) we get, with \( C = C(T,p) > 0 \) and \( K = K(T,p) > 0 \) independent of \( \delta > 0 \),

\[
N^{p-1} \mathbb{E} \sum_{j=1}^{N} |\Delta W^j|^2 p P_{4p/3}(\|X^j_h\|_{L_2(D)}) P_{4p/3}(\|X^j_h\|_{L_2(D)})
\]

\[
\leq C \left( 1 + \|X^0_h\|^{4p+2p}_p + \sum_{j=0}^{N} k\mathbb{E} F(X^j_h)^p \right) + K \delta \left( 1 + \mathbb{E} \left( \sum_{j=1}^{N} k|Y^j_h|_1^2 \right)^p \right).
\]

It remains to treat the Itô integral in (4.21). For this we invoke the Cauchy inequality and Burkholder–Davis–Gundy inequality to conclude that

\[
\mathbb{E} \sup_{1 \leq n \leq N} \left| \sum_{j=1}^{n} (Y^j_h^{-1}, \Delta W^j) \right|^p
\]

\[
\leq C \left( 1 + \varepsilon^p \mathbb{E} \sup_{0 \leq n \leq N} \left| \sum_{j=1}^{n} (Y^j_h^{-1}, \Delta W^j) \right|^{2p} \right)
\]

(4.30)

\[
\leq C \left( 1 + \varepsilon^p \mathbb{E} \left( \sum_{j=1}^{N} k||Q^{1/2}Y^j_h^{-1}||^2 \right)^p \right)
\]

\[
\leq C \left( 1 + \varepsilon^p \mathbb{E} \left( \sum_{j=1}^{N} k||Q^{1/2}A^{-1/2}P||Y^j_h^{-1}||^2 \right)^p \right).
\]

Thus, since \( ||Q^{1/2}A^{-1/2}P|| < \infty \), if we take \( \varepsilon' > 0 \) small enough in (4.30) and \( \delta > 0 \) small enough in (4.29), from (4.21) we may conclude that

\[
\mathbb{E} \sup_{1 \leq n \leq N} J(X^0_h)^p + \mathbb{E} \left( \sum_{j=1}^{N} k|Y^j_h|_1^2 \right)^p
\]

(4.31)

\[
\leq C \left( 1 + J(X^0_h)^p + \|X^0_h\|^{4p+2p}_p + k\mathbb{E} F(X^0_h)^p + k|Y^0_h|_1^{2p} \right)
\]

\[
+ \sum_{j=1}^{N} \mathbb{E} \sup_{1 \leq n \leq j} F(X^n_h)^p.
\]
Thus, if \( Ck < 1 \), then the desired result follows from Gronwall’s lemma by noting that \( \mathcal{F}(u) \leq J(u) \).

\[ \square \]

5. Convergence

Recall from Theorem [5.1] that \( X \) satisfies the mild equation

\[ (5.1) \quad X(t) = E(t)X_0 - \int_0^t AE(t-s)f(X(s))\,ds + \int_0^t E(t-s)\,dW(s). \]

Similarly, equation \((1.6)\) has the mild formulation

\[ (5.2) \quad X^n_h = R^n_{k,h}P_hX_0 - \sum_{j=1}^n A_hR^{n-j+1}_{k,h}P_hf(X^n_j) + \sum_{j=1}^n R^{n-j+1}_{k,h}P_h\Delta W^j, \]

where \( R^n_{k,h} = (I + kA^2)^{-n} \).

**Remark 5.1.** Preservation of mass. From \[(2.5),(5.2)\] and \[(5.3)\] it follows that if \( W(t) \) has zero average, i.e., \((I - P)W(t) = 0\), then \((I - P)X(t) = (I - P)X_0\) and \((I - P)X^n_h = (I - P)X_h\). This means that \( X(t) \) and \( X^n_h \) preserve the mass.

In order to prove convergence of \( X^n_h \), we first state a maximal type error estimate for the stochastic convolution. We define the backward Euler approximation of the stochastic convolution \( W_A(t) := \int_0^t E(t-s)\,dW(s) \) by

\[ W^n_{A_k} := \sum_{j=1}^n R^{n-j+1}_{k,h}P_h\Delta W^j = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} R^{n-j+1}_{k,h}P_h\,dW(s). \]

**Lemma 5.2.** Let \( \gamma \in (0, \frac{1}{2}] \), \( \beta \in [1, 2] \), and \( p \geq 1 \). Then there is \( C = C(p, \gamma, T) \) such that for \( h, k > 0 \)

\[ (5.3) \quad \left( \mathbb{E} \left( \sup_{0 \leq n \leq N} \| W_A(t_n) - W^n_{A_k} \|^p \right) \right)^{1/p} \leq C(h^\beta + k^{\beta/4})\| A^{(\beta-2)/2}+\gamma P^{1/2} \|_{HS}. \]

**Proof.** The proof is completely analogous to the proofs of \[20\] Proposition 5.1 and \[23\] Theorem 2.1], based on a discrete factorization method using the analyticity of the semigroup \( E \) and the deterministic error estimate

\[ (5.4) \quad \| (E(t_n) - R^n_{k,h})P_hv \| \leq C(h^\beta + k^{\beta/4})t_n^{-(\beta-\alpha)/4}|v|_\alpha, \quad t_n > 0, \]

for \( \beta \in [1, 2], \alpha \in [-1, 1] \), from \[12\] Lemma 5.5] and \[25\] Theorem 2.2].

**Remark 5.3.** In \[20\] Proposition 5.1, instead of the term \( \| A^{(\beta-2)/2}+\gamma P^{1/2} \|_{HS} \) which appears on the right hand side of \[(5.3)\], one has \( \| A^{(\beta-1)/2}+Q^{1/2} \|_{HS} \). The reason for this difference (besides the different boundary conditions) is that \[20\] considers the stochastic Allen–Cahn equation where the semigroup in the stochastic convolution is generated by the Laplacian \( \Delta \) which has a weaker smoothing effect than in the present case, where the semigroup is generated by \(-\Delta^2\). The proofs in \[20\] and \[23\] require that \( p \) is large, but the result is then valid for smaller \( p \geq 1 \) as well.
Lemma 5.4. Let $0 < \delta < 1$. The following deterministic error estimates hold for $v \in H$:

$$
\|A_h P_{k,h} v - A_h E_h(t_n) P_h v\| \leq C k^{2(1 - \delta)} t_n^{-1 + \frac{\delta}{2}} \|v\|, \quad t_n > 0 \text{ and } h, k > 0, \tag{5.5}
$$

$$
\|A_h E_h(t) P_h v - A E(t) v\| \leq C h^{2(1 - \delta)} t^{-1 + \frac{\delta}{2}} \|v\|, \quad t > 0 \text{ and } h > 0. \tag{5.6}
$$

Proof. Note that it is enough to consider $v \in \tilde{H}$, as for $v$ constant the above differences equal 0. The error bounds follow by a simple interpolation between the cases $\delta = 0$ and $\delta = 1$. For $\delta = 1$ we use the estimates from (2.6) and (2.14) to get

$$
\|A_h R_{k,h} P_h\| + \|A_h E_h(t_n) P_h\| \leq C t_n^{-1/2}, \quad \|A_h E_h(t) P_h\| + \|A E(t)\| \leq C t^{-1/2}. \tag{5.7}
$$

Estimate (5.5) with $\delta = 0$ follows by expansion in the eigenbasis of $A_h$ and Parseval’s identity. For estimate (5.6) with $\delta = 0$, we first write, for $v \in \tilde{H}$,

$$
\|A_h E_h(t) P_h v - A E(t) v\| \leq \|(A_h^2 E_h(t) P_h - A^2 E(t)) A^{-1} v\|
$$

$$
+ \|A_h^2 E_h(t) P_h (A_h^{-1} P_h - A^{-1}) v\|
$$

$$
= \|D(t) E_h(t) P_h - E(t)\| A^{-1} v\| + \|A_h^2 E_h(t) P_h (R_h - I) A^{-1} v\|,
$$

where we used the identity $R_h = A_h^{-1} P_h A$. The desired bound $C t^{-1} h^2 \|A(A^{-1} v)\| = C h^2 t^{-\delta} \|v\|$ for the last term follows immediately from (2.12) and (2.10). The first term is an error estimate for the time derivative of the solution of the linear Cahn–Hilliard equation with smooth initial-value $u_0 = A^{-1} v \in \tilde{H}^2$. To prove this we adapt the arguments in [27, Chapt. 3] and [12, Sect. 5], where error estimates with lower initial regularity are proved. Let $u(t) = E(t) u_0$, $u_h(t) = E_h(t) P_h u_0$. Then the error $e = u_h - u$ satisfies the equation, see [12, (5.4)],

$$
G_h^2 \dot{e} + e = \rho + G_h \eta, \quad t > 0; \quad P_h e(0) = 0, \tag{5.8}
$$

with

$$
G_h = A_h^{-1} P_h, \quad R_h = G_h A, \quad \rho = (R_h - I) u, \quad \eta = -(R_h - I) A^{-1} \dot{u}.
$$

Any solution of an equation of the form (5.8) satisfies the following bound, with arbitrary $\epsilon > 0$,

$$
\|e(t)\| \leq \epsilon \sup_{s \in [0,t]} s \|\dot{\rho}(s)\| + C_\epsilon \sup_{s \in [0,t]} \|\rho(s)\| + \left( \int_0^t \|\eta(s)\|^2 \, ds \right)^{1/2}. \tag{5.9}
$$

In order to prove this we let $e_1$ be the solution of (5.8), with only $\rho$ as the source term and $P_h e_1(0) = 0$. Moreover, we let $e_2$ solve the same equation but driven by $G_h \eta$ alone and with $e_2(0) = 0$. Then $e = e_1 + e_2$ solves (5.8). We quote a bound for $e_1$ from [27, Lemma 3.5]:

$$
\|e_1(t)\| \leq \epsilon \sup_{s \in [0,t]} s \|\dot{\rho}(s)\| + C_\epsilon \sup_{s \in [0,t]} \|\rho(s)\|.
$$

In order to quote this lemma we note that $G_h$ is selfadjoint, positive semidefinite on $\tilde{H}$ and that $G_h e_1(0) = A_h^{-1} P_h e_1(0) = 0$. For $e_2$ we have

$$
\|e_2(t)\| \leq \left( \int_0^t \|\eta(s)\|^2 \, ds \right)^{1/2}.
$$

This is proved by a simple energy argument, see the beginning of the proof of [12, Lemma 5.2]. The reason why we need different proofs for $e_1$ and $e_2$ is that $G_h$ in front of $\eta$ must not appear in (5.9) for we have good bounds for $\eta$ but not for $G_h \eta$. 
This proves (5.9), which can now be combined with bounds for $\rho$ and $\eta$, obtained from bounds for $R_h$ and regularity estimates for $u = E(t)u_0$, to get an error bound for $\|e(t)\|$. However, we aim for $\|\dot{e}(t)\|$ and therefore take the derivative of the equation in (5.7) and multiply by $t$ to obtain an equation for $t\ddot{e}(t)$:

$$tG_h^2\ddot{e} + \dot{e} = t\dot{\rho} + tG_h\dot{\eta},$$

which can be written as

$$G_h^2(t\ddot{e}) + (\dot{e}) = G_h^2\ddot{e} + t\dot{\rho} + G_h(t\dot{\eta}) = -e + \rho + t\dot{\rho} + G_h(\eta + t\dot{\eta}),$$

where we substituted $G_h^2\ddot{e} = -e + \rho + G_h\eta$ from (5.7). Thus, $t\ddot{e}$ satisfies an equation of the form (5.7) but with $\rho$ and $\eta$ replaced by $-e + \rho + t\dot{\rho}$ and $\eta + t\dot{\eta}$. An application of (5.9) with $\epsilon = \frac{1}{2}$, say, gives

$$t\|\dot{e}(t)\| \leq \frac{1}{2} \sup_{s \in [0,t]} \left( s\|\dot{e}(s)\| + 2s\|\dot{\rho}(s)\| + s^2\|\ddot{\rho}(s)\| \right) + C \sup_{s \in [0,t]} \left( \|e(s)\| + \|\rho(s)\| + s\|\dot{\rho}(s)\| \right) + \left( \int_0^t \left( \|\eta(s)\|^2 + s^2\|\dot{\eta}(s)\|^2 \right) ds \right)^{1/2}.$$

Since $t$ is arbitrary here we may apply a standard kick-back argument to remove the term $s|\dot{e}(s)|$. Another application of (5.9), now with $\epsilon = 1$, takes care of the term $\|e(s)\|$, which leads to

$$t\|\dot{e}(t)\| \leq C \sup_{s \in [0,t]} \left( \|\rho(s)\| + s\|\dot{\rho}(s)\| + s^2\|\ddot{\rho}(s)\| \right) + C \left( \int_0^t \left( \|\eta(s)\|^2 + s^2\|\dot{\eta}(s)\|^2 \right) ds \right)^{1/2}.$$

Here we use (2.12) and recall the regularity estimates (2.6), (2.7) for $u(t) = E(t)A^{-1}v$:

$$s^2\|D^j\rho(s)\| = s^j\|\left( R_h - I \right)D^j_\rho u(s)\| \leq Ch^2s^j\|AD^j_\rho E(s)A^{-1}v\| \leq Ch^2\|v\|$$

and

$$\left( \int_0^t s^2j\|D^j\eta(s)\|^2 ds \right)^{1/2} = \left( \int_0^t s^2j\|\left( R_h - I \right)A^{-1}D^j\dot{\rho}(s)\|^2 ds \right)^{1/2}$$

$$\leq Ch^2\left( \int_0^t s^2j\|D^j+1\rho(s)\|^2 ds \right)^{1/2}$$

$$\leq Ch^2\left( \int_0^t s^2j\|A^2E(s)A^{-1}v\|^2 ds \right)^{1/2}$$

$$\leq Ch^2\left( \int_0^t s^2j\|A^{2j+1}E(s)v\|^2 ds \right)^{1/2} \leq Ch^2\|v\|.$$

This completes the proof. 

\[ \square \]

**Theorem 5.5.** Suppose that (1.13) holds, $\|A^{1/2}Q^{1/2}\|_{HS} < \infty$, $\beta \in [1,2]$, and that $|X_0|_{\beta} < \infty$. Let $h > 0$ and $k > 0$ be small and $0 < \epsilon, \delta < 1$. Then, there is $\Omega_{h,k}^\epsilon \subset \Omega$ with $P(\Omega_{h,k}^\epsilon) > 1 - \epsilon$, and $C = C(T, \epsilon, \delta)$ such that for all $\omega \in \Omega_{h,k}$,

$$\|X(t_n) - X_n\| \leq C \left( (h^2 + k^{3/4})|X_0|_{\beta} + h^{2(1-\delta)} + k^4(1-\delta) \right), \quad t_n \in [0,T].$$
Proof. It follows from Proposition 3.2 for (5.10), Theorem 3.1 for (5.11), Theorem 4.3 for (5.12), and Lemma 5.2 for (5.13) that for every $0 < \epsilon, \delta < 1$ and $h, k > 0$ small enough, there is $\Omega_{h,k}^\epsilon \subset \Omega$ with $P(\Omega_{h,k}^\epsilon) > 1 - \epsilon$ and $K_{T,\epsilon} > 0$ such that

\begin{align}
(5.10) \quad &\|X(t) - X(s)\| \leq K_{T,\epsilon}|t - s|^{\frac{1}{2}(1-\delta)}, \quad t, s \in [0, T], \omega \in \Omega_{h,k}^\epsilon, \\
(5.11) \quad &|X(t)|^2 + \|X(t)\|_{L^4} \leq K_{T,\epsilon}, \quad t \in [0, T], \omega \in \Omega_{h,k}^\epsilon, \\
(5.12) \quad &|X^n_h|^2 + \|X^n_h\|_{L^4} \leq K_{T,\epsilon}, \quad t_n \in [0, T], \omega \in \Omega_{h,k}^\epsilon, \\
(5.13) \quad &\|W_A(t_n) - W_{A_h}^n\| \leq K_{T,\epsilon}(h^2 + k^{1/2}), \quad t_n \in [0, T], \omega \in \Omega_{h,k}^\epsilon.
\end{align}

Note that it is enough to establish the above four bounds individually with $\epsilon/4$ on $\Omega_{h,k}^{\epsilon/4}$, $i = 1, ..., 4$, and then set $\Omega_{h,k}^\epsilon = \cap_{i=1}^4 \Omega_{h,k}^{\epsilon/4}$. The estimate in (5.10) follows directly from the assumption on the initial data and Proposition 3.2. The remaining bounds can be proved by using Chebychev’s inequality together with bounds from Theorem 3.1, Theorem 4.3, and Lemma 5.2. For example, to prove (5.13), consider

\begin{align}
F_{h,k} := \sup_{0 \leq n \leq N} \frac{\|W_A(t_n) - W_{A_h}^n\|}{h^2 + k^{1/2}}.
\end{align}

Chebychev’s inequality and Lemma 5.2 for some $p \geq 1$, $\gamma = \frac{1}{2}$, and $\beta = 2$, give

\begin{align}
P\left( \{ \omega \in \Omega : F_{h,k} > \alpha \} \right) \leq \frac{1}{\alpha^p} E[F_{h,k}^p] \leq \frac{K_p}{\alpha^p}, \quad K = C\|A^{1/2}Q^{1/2}\|_{HS}.
\end{align}

We choose $\alpha = \epsilon^{-1/p}K$ and set $\Omega_{h,k}^\epsilon = \{ \omega \in \Omega : F_{h,k} \leq \epsilon^{-1/p}K \}$. Then

\begin{align}
P(\Omega_{h,k}^\epsilon) = 1 - P\left( \{ \omega \in \Omega : F_{h,k} > \epsilon^{-1/p}K \} \right) \geq 1 - \epsilon,
\end{align}

and (5.13) follows.

Now let $\omega \in \Omega_{h,k}^\epsilon$. We decompose the error $e_n := X(t_n) - X^n_h$ as

\begin{align}
e_n &= \left( E(t_n) - R^n_{h,k}P_h \right) X_0 \\
&\quad + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left( A_h R^{n-j+1}_{h,k}P_h f(X^j_h) - AE(t_n - s)f(X(s)) \right) ds \\
&\quad + W_A(t_n) - W_{A_h}^n =: e_1^n + e_2^n + e_3^n.
\end{align}

For the first error term we use (5.4) to get

\begin{align}
\|e_1^n\| \leq C(h^\beta + k^{\beta/4})|X_0|_\beta.
\end{align}
The term $e_n^2$ is most involved and we decompose it further as

$$
A_h R^{n-j+1}_{k,h} P_h f(X_h^t) - AE_h(t_n - s)f(X(s))
= \left(A_h R^{n-j+1}_{k,h} - A_h E_h(t_n - t_{j-1})\right) P_h f(X_h^t)
+ A E_h(t_n - t_{j-1}) P_h - AE(t_n - t_{j-1}) f(X_h^t)
+ A^{3/2} E(t_n - t_{j-1}) A^{-1/2} P \left(f(X_h^t) - f(X(t_j))\right)
+ A E(t_n - t_{j-1}) - E(t_n - s) f(X(t_j))
+ A^{3/2} E(t_n - s) A^{-1/2} P \left(f(X(t_j)) - f(X(s))\right)
= e_{n,j}^2 + e_{n,j}^3 + e_{n,j}^4 + e_{n,j}^5.
$$

Here we used (2.5) to obtain $AE(t) = A^{3/2} E(t) A^{-1/2} P$. Further, since $f$ is cubic, we have $\|f(x)\| \leq C(1 + |x|^2)$ by Hölder’s and Sobolev’s inequalities. Then by (5.10), and (5.12) it follows that, with $C = C(T, \varepsilon, \delta)$,

$$
\|e_{n,j}^1\| \leq C k^{\frac{1}{2}(1-\delta)} t_{n-j+1}^{1-\delta} \|f(X_h^t)\| \leq C k^{\frac{1}{2}(1-\delta)} t_{n-j+1}^{1-\delta} (1 + |X_h^t|^2)
\leq C k^{\frac{1}{2}(1-\delta)} t_{n-j+1}^{1-\delta}.
$$

Similarly, by (5.6) and (5.12),

$$
\|e_{n,j}^2\| \leq C h^{2(1-\delta)} t_{n-j+1}^{1-\delta}.
$$

Also, using the local Lipschitz bound $|P(f(x) - f(y))|_{t_{j-1}} \leq C(1 + |x|^2 + |y|^2)\|x - y\|$, cf. (4.18), together with (5.11), and (5.12), we obtain

$$
\|e_{n,j}^3\| \leq C(t_n - t_{j-1})^{-3/4} |P(f(X_h^t) - f(X(t_j)))|_{t_{n-j+1}}
\leq C t_{n-j+1}^{-3/4} (1 + |X_h^t|^2 + |X(t_j)|^2) |X_h^t - X(t_j)|
\leq C t_{n-j+1}^{-3/4} e_{n,j}^4.
$$

Furthermore, for $s \in [t_{j-1}, t_j]$, by (2.6) and $\|(E(t) - I)x\| \leq C t^{\frac{1}{2}(1-\delta)} \|A^{1-\delta} x\|$, we have

$$
\|e_{n,j}^4\| = \|(E(s - t_{j-1}) - I) A^{-1+\delta} A^{2-\delta} E(t_n - s) f(X(t_j))\|
\leq C (s - t_{j-1})^{\frac{1}{2}(1-\delta)} (t_n - s)^{-1+\delta} \|f(X(t_j))\|
\leq C k^{\frac{1}{2}(1-\delta)} (t_n - s)^{-1+\delta}.
$$

Using also (5.10), for $s \in [t_{j-1}, t_j]$, we have

$$
\|e_{n,j}^5\| \leq C (t_n - s)^{-3/4} (1 + |X(t_j)|^2 + |X(s)|^2) |X(t_j) - X(s)|
\leq C (t_n - s)^{-3/4} (t_j - s)^{\frac{1}{2}(1-\delta)} \leq C (t_n - s)^{-3/4} k^{\frac{1}{2}(1-\delta)}.
$$

Finally, by (5.13), $\|e_{n,j}^6\| \leq C(h^2 + k^{1/2})$. Collecting all the above terms and applying a generalized version of Gronwall’s lemma, (12 Lemma 7.1), finishes the proof if $k$ is small enough.

**Theorem 5.6.** Under the hypothesis of Theorem 5.5 with $\beta = 1$, we have

$$
\lim_{h,k \to 0} \mathbb{E} \sup_{0 \leq n \leq N} \|X(t_n) - X_h^n\|^2 = 0.
$$
Proof. It follows from Theorem 5.1 and Theorem 4.3 that there is $K > 0$ such that
\[
\mathbb{E} \sup_{0 \leq n \leq N} (\|X(t_n)\|_{L_t}^2 + \|X_h^n\|_{L_t}^2) \leq K.
\]
Let $\epsilon > 0$, $0 < h, k < 1$ small enough, and let $C_\epsilon = C(T, \epsilon, \delta)$ and $\Omega_{h_k}$ as in Theorem 5.5. Then, by using Theorem 5.5 with $\beta = 1$ and $\delta = \frac{\eta}{2}$, we get
\[
\mathbb{E} \sup_{0 \leq n \leq N} \|X(t_n) - X_h^n\|^2 \leq \int_{\Omega_{h_k}} \sup_{0 \leq n \leq N} \|X(t_n) - X_h^n\|^2 \, d\mathbb{P}
\]
\[
+ 2 \int_{(\Omega_{h_k})^c} \sup_{0 \leq n \leq N} (\|X(t_n)\|^2 + \|X_h^n\|^2) \, d\mathbb{P}
\]
\[
\leq C_\epsilon (h^2 + k^{1/2}) + 4\epsilon^{1/2} \left( \int_{(\Omega_{h_k})^c} \sup_{0 \leq n \leq N} \|X(t_n)\|^4 + \|X_h^n\|^4 \right) \, d\mathbb{P}
\]
\[
\leq C_\epsilon (h^2 + k^{1/2}) + 4\epsilon^{1/2} \left( \mathbb{E} \sup_{0 \leq n \leq N} \|X(t_n)\|^4 + \|X_h^n\|^4 \right)^{1/2}
\]
\[
\leq C_\epsilon (h^2 + k^{1/2}) + 4\epsilon^{1/2} |\mathcal{D}|^{1/2} \left( \mathbb{E} \sup_{0 \leq n \leq N} \|X(t_n)\|_{L_t}^4 + \|X_h^n\|_{L_t}^4 \right)^{1/2}
\]
\[
\leq C_\epsilon (h^2 + k^{1/2}) + 4\epsilon^{1/2} |\mathcal{D}|^{1/2} K.
\]
Let $\eta > 0$. Choose $0 < \epsilon < 1$ such that $8\epsilon^{1/2} |\mathcal{D}|^{1/2} K < \frac{\eta}{2}$. Therefore, if $\max(h, k) < \left( \frac{\eta}{4C_\epsilon} \right)^2$, then
\[
\mathbb{E} \sup_{0 \leq n \leq N} \|X(t_n) - X_h^n\|^2 < \eta,
\]
and the proof is complete. \(\square\)

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