CONSTRAINTS FOR SEIBERG-WITTEN BASIC CLASSES OF
GLUED MANIFOLDS

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Abstract. We use rudiments of the Seiberg-Witten gluing theory for trivial circle bundles over a Riemann surface to relate de Seiberg-Witten basic classes of two 4-manifolds containing Riemann surfaces of the same genus and self-intersection zero with those of the 4-manifold resulting as a connected sum along the surface. We study examples in which this is enough to describe completely the basic classes.

1. Statement of results

Since their introduction nearly a year ago the Seiberg-Witten invariants have proved to be at least as useful as their close relatives the Donaldson invariants. These provide differentiable invariants of a smooth 4-manifold, whose construction is very similar in nature to the Donaldson invariants. Conjecturally, they give the same information about the 4-manifold, but they are much easier to compute in many cases, e.g. algebraic surfaces (see [12]).

Problems in 4-dimensional topology are far from solved with these invariants. Nonetheless it is intriguing to compute them for a general 4-manifold. The first step towards it is obviously to relate the invariants of a manifold with those of the manifold which results after some particular surgery on it. Much progress has been made [4] [12]. One natural case to think about is that of connected sum along a codimension 2 submanifold (see Gompf [6]). The typical case would be:

Let $\bar{X}_i$ be smooth oriented manifolds and let $\Sigma$ be a Riemann surface of genus $g \geq 1$. Suppose we have embeddings $\Sigma \hookrightarrow \bar{X}_i$ with image $\Sigma_i$ representing a non-torsion element in cohomology whose self-intersection is zero. We form $X = \bar{X}_1 \#_{\Sigma} \bar{X}_2$ removing tubular neighbourhoods of $\Sigma$ in both $\bar{X}_i$ and gluing the boundaries $Y$ and $Y'$ by some diffeomorphism $\phi$. These boundaries are diffeomorphic to $\Sigma \times S^1$. The

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diffeomorphism type of the resulting manifold depends on the homotopy class of $\phi$. There is an exact sequence

$$0 \to H^1(Y; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \xrightarrow{\pi} G \oplus H^1(\Sigma; \mathbb{Z})$$

with $G$ to the subgroup of $H^2(\bar{X}_1; \mathbb{Z})/Z[\Sigma_1] \oplus H^2(\bar{X}_2; \mathbb{Z})/Z[\Sigma_2]$ consisting of elements $(\alpha_1, \alpha_2)$ such that $\alpha_1 \cdot \Sigma_1 = \alpha_2 \cdot \Sigma_2$. There are two interpretations for this. The first one (reading the exact sequence in homology through Poincaré duality) says that the 2-homology of $X$ is composed out of the 2-homology of $Y$ plus those cycles which restrict to $X_1$ and $X_2$ having the same boundary 1-cycle in $Y$ (here to be in $\pi^{-1}(G)$ means to have intersection with $Y = \Sigma \times S^1$ a multiple of the $S^1$ factor). The second interpretation says that a line bundle in $X$ comes from gluing two line bundles in $X_1$ and $X_2$ and that the possible gluings are parametrised by $H^1(Y; \mathbb{Z})$. Then the first main result we aim to prove regarding the Seiberg-Witten basic classes is:

**Theorem 1.** Let $\bar{X}_i$ be smooth oriented manifolds as before and with $b_1 = 0$ and $b^+ > 1$ and odd. Construct $X = \bar{X}_1 \#_\Sigma \bar{X}_2$. Then every basic class of $X$ lies in $\pi^{-1}(G)$. Equivalently, the intersection of the basic class with $Y$ is $n\mathbb{S}^1$. Moreover, $n$ is an even integer between $-(2g-2)$ and $(2g-2)$.

Our second result gives more specific information about the values of the Seiberg-Witten invariant. It should be understood as constraints in the possible basic classes.

**Theorem 2.** Now suppose that $\bar{X}_i$ are of simple type and $g \geq 2$. Denote by $SW_X(L)$ the Seiberg-Witten invariant associated to the characteristic line bundle $L$ (here we identify a line bundle with its first Chern class). Fix a pair $(A_1, A_2) \in G$ such that $A_i \cdot \Sigma_i = \pm(2g-2)$. Then the sum $\sum_L SW_X(L)$, where $L$ runs over all characteristic line bundles in $\pi^{-1}(A_1, A_2)$ is the product of the two terms

$$\sum_{\{L/L \equiv A_i \in H^2(\bar{X}_i)/Z[\Sigma_i]\}} SW_{\bar{X}_i}(L)$$

Note that as $g \geq 2$ at most one of the line bundles can appear in that sum for each of the $\bar{X}_i$ because of the simple type condition.

In many cases these theorems are all that we need to find out some of the basic classes for $X$. As an example we will prove

**Corollary 3.** Suppose that either for every cycle $\gamma \in H^1(\Sigma; \mathbb{Z})$ there exists a $(-1)$-embedded disc in both $X_i$ bounding $\gamma$ or that both $\bar{X}_i$ are Kähler manifolds and $X$ is deformation equivalent to $\bar{X}_1 \cup_\Sigma \bar{X}_2$. Then the basic classes $\kappa$ of $X$ such that
The solutions of equation (1) parametrize the moduli space $\mathcal{W}_{X,g}(c)$, which has the structure of a finite dimensional

real analytic space with expected dimension \(d = \frac{1}{4}(c_1^2 - 2\chi - 3\sigma)\), where \(\chi\) is the Euler characteristic of \(X\) and \(\sigma\) its signature. A solution \((A, \Phi)\) is reducible (i.e. has non-trivial stabiliser) if and only if \(\Phi = 0\). The moduli space is always compact and, for a generic metric \(g\), is smooth away from reducibles. Whenever \(b^+ > 0\) and \(c_1(L)\) is not torsion reducible can be avoided for a generic metric. Moreover, if \(b^+ > 1\) reducibles are not present in generic 1-parameter families of metrics. Another way of obtaining a smooth moduli space is by perturbing the equations by adding a self-dual 2-form to \(F_A^+\) [7]. It is also a fact that the moduli space is orientable and that an orientation is determined by a choice of homology orientation for \(X\) (see [12], [4]).

**Definition 4.** If \(d < 0\) then for a generic metric the moduli space is empty. We define the Seiberg-Witten invariant \(SW_X(c)\) for the \(\text{Spin}^C\) structure \(c\) as follows: choose a generic metric \(g\) making the moduli space smooth, compact, oriented (fixing a homology orientation of \(X\)) and of dimension the expected dimension \(d\). Then

1. if \(d < 0\), put \(SW_X(c) = 0\).
2. if \(d = 0\), let \(SW_X(c)\) be the number of points of \(W_{X,g}(c)\) counted with signs.
3. if \(d > 0\) and odd, we set \(SW_X(c)\) to be zero. Note that this condition is equivalent to \(b^+\) being even (since \(b_1 = 0\)).
4. if \(d > 0\) and even, then we still can defined \(SW_X(c)\) cutting down the moduli space with \(d/2\) times the generator \(\mu\) of the cohomology ring of \(B\) (which in this case is homotopy equivalent to \(\mathbb{C}P^\infty\)). \(\mu\) also can be thought as the first Chern class of the \(U(1)\)-bundle \(W_{X,g}^o(c) \to W_{X,g}(c)\), where the superscript means framed moduli space.

**Definition 5.** One defines, for a characteristic line bundle \(L\), \(SW_X(L)\) to be the sum of \(SW_X(c)\) over all \(\text{Spin}^C\) structures \(c\) with associated line bundle \(L\) (note that there are a finite number of them).

**Definition 6.** Let \(X\) be a compact, oriented manifold with \(b_1 = 0\), and suppose that \(b^+ > 1\). Then we say that a cohomology class \(c_1 \in H^2(X; \mathbb{Z})\) is a Seiberg-Witten basic class (or a basic class for brevity) for \(X\) if \(c_1\) is characteristic with \(c_1^2 = 2\chi + 3\sigma\) and \(SW_X(c_1) \neq 0\).

One important remark in place is the fact that the set of basic classes is finite [12]

**Definition 7.** For \(X\) compact, oriented manifold with \(b_1 = 0\) with \(b^+ > 1\) and odd, we say that it is of (Seiberg-Witten) simple type if \(SW_X(L) = 0\) whenever \(d > 0\).
3. Splitting along $\Sigma \times S^1$

Our main interest is the study of the behaviour of the Seiberg-Witten invariants under elementary surgeries. This amounts to splitting $X$ along an embedded 3-manifold $Y \subset X$. So we have $X = X_1 \cup_Y X_2$, where $X_1$ and $X_2$ are manifolds with boundary. We orient $Y$ so that $\partial X_1 = Y$ and $\partial X_2 = \overline{Y}$, $Y$ with reversed orientation.

The simplest cases are those for which $Y$ admits a metric of positive scalar curvature. For instance, for $Y = S^3$ (i.e., $X$ is a connected sum) we have that the hypothesis $b^+(X_i) > 0$ for both $X_i$ leads in a straightforward way to the vanishing of all the invariants for $X$ [12]. The case $b^+(X_1) = 0$ and $b^+(X_2) > 0$ is also of interest and we have for instance the following theorem about the behaviour of the Seiberg-Witten invariants under blowing-ups [4]

Proposition 8. If $X$ is of simple type and $\{K_i\}$ is the set of basic classes of $X$, then the blow-up $\tilde{X} = X \# \mathbb{CP}^2$ is of simple type and (denoting by $E$ the exceptional divisor) the set of basic classes is $\{K_i \pm E\}$.

The case of relevance to us is when $Y$ is a trivial circle bundle over a Riemann surface, that is $Y = \Sigma \times S^1$. Suppose we have an embedded $\Sigma \hookrightarrow X$ of self-intersection $n = \Sigma \cdot \Sigma \geq 0$. Then we can blow-up $X$ $n$ times in points of $\Sigma$. Algebraically (when $X$ is complex) this makes perfect sense, differentiably this amounts to considering $X \# n \mathbb{CP}^2$ and substituting the original $\Sigma$ by its proper transform $\Sigma - \sum_{i=1}^n E_i$ (where $E_i$ denote the homology class coming from the $i$-th $\mathbb{CP}^2$ summand). Therefore we can assume that $n = 0$. In this case take $X_2$ to be a tubular neighbourhood of $\Sigma$ and $X_1$ to be the closure of the complement of it. We have a decomposition $X = X_1 \cup_Y X_2$. Here $X_2$ is diffeomorphic to $A = \Sigma \times D^2$.

More generally, consider the case of two manifolds $\tilde{X}_1$ and $\tilde{X}_2$ together with embeddings $\Sigma \hookrightarrow \tilde{X}_i$. Call the image $\Sigma_i$ and put $n_i$ for the self-intersection of $\Sigma_i$. If $n_1 + n_2 \geq 0$ we can blow-up $\tilde{X}_1$ or $\tilde{X}_2$ until we lower that quantity to zero. Then we can put $N_i$ for an open tubular neighbourhood of $\Sigma_i$, $X_i = \tilde{X}_i - N_i$, so we have an orientation reversing diffeomorphism $\phi$ between the boundaries of $X_1$ and $X_2$. This lets us construct $X = X_1 \cup_\phi X_2$ which we call connected sum of $\tilde{X}_1$ and $\tilde{X}_2$ along $\Sigma$ (obviously the diffeomorphism type of $X$ depends on the homotopy class of $\phi$). A simple extension of the arguments in [6] gives the following

Proposition 9. The diffeomorphism type does not depend on which points we blow-up. More concretely, if we blow-up $X_i$ in $s_i$ points with $s_1 + s_2 = n_1 + n_2$ and we choose a diffeomorphism $\phi$, there is a unique homotopy class of diffeomorphisms $\phi'$
such that blowing-up $X_i$ in $s'_i$ points with $s'_1 + s'_2 = n_1 + n_2$ the result is diffeomorphic to the former.

Therefore the only choice (up to diffeomorphism) involved in all this process is the identification between the boundaries (that is an element of $\pi_0(\text{Diff}^+(Y))$). Note that when $n_1 \geq 0$ and $n_2 \geq 0$ we can lower both quantities to zero to have $Y = \Sigma \times S^1$.

Suppose now that $b_1 = 0$ for both $X_1$ and $X_2$. Consider embeddings $\Sigma \hookrightarrow X_i$ with images $\Sigma_i$. If $[\Sigma_i] \in H^2(\bar{X}_i; \mathbb{Z})$ is a non-torsion element, then the cohomology exact sequence for the pair $(\bar{X}_i, X_i)$ gives

$$H^1(X_i; \mathbb{Z}) \cong H^1(\bar{X}_i; \mathbb{Z})$$
$$H^3(X_i; \mathbb{Z}) \cong H^3(\bar{X}_i; \mathbb{Z})$$
$$0 \to H^2(\bar{X}_i; \mathbb{Z})/\mathbb{Z}[\Sigma_i] \to H^2(X_i; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z}) \to 0$$

where the last map is the composition

$$H^2(X_i; \mathbb{Z}) \to H_2(X_i, \partial X_i; \mathbb{Z}) \xrightarrow{\partial} H_1(\Sigma \times S^1; \mathbb{Z}) \xrightarrow{i} H_1(\Sigma \times D^2; \mathbb{Z}) = H_1(\Sigma; \mathbb{Z})$$

Hence there is a (non-canonical) splitting $H^2(X_i; \mathbb{Z}) = H^2(\bar{X}_i; \mathbb{Z})/\mathbb{Z}[\Sigma_i] \oplus H_1(\Sigma; \mathbb{Z})$.

The Mayer-Vietoris sequence for $X = X_1 \cup X_2$ gives (we drop $\mathbb{Z}$ in the notation)

$$H^1(X) = H^1(X_1) \oplus H^1(X_2)$$
$$0 \to H^1(Y) \to H^2(X) \to H^2(X_1) \oplus H^2(X_2) \to H^2(Y) \cong H^1(\Sigma) \oplus H^2(\Sigma)$$

So $b_1(X) = 0$. Under the splitting above, we can describe the last map as

$$(H^2(\bar{X}_1)/\mathbb{Z}[\Sigma_1] \oplus H_1(\Sigma)) \oplus (H^2(\bar{X}_2)/\mathbb{Z}[\Sigma_2] \oplus H_1(\Sigma)) \to H^1(\Sigma; \mathbb{Z}) \oplus H^2(\Sigma; \mathbb{Z})$$

$$(\alpha_1, \beta_1, \alpha_2, \beta_2) \mapsto (\beta_1 - \beta_2, (\alpha_1 - \alpha_2) \cdot \Sigma)$$

(the identifications between homology and cohomology groups are through Poincaré duality). Calling $G$ to the subgroup of $H^2(\bar{X}_1; \mathbb{Z})/\mathbb{Z}[\Sigma_1] \oplus H^2(\bar{X}_2; \mathbb{Z})/\mathbb{Z}[\Sigma_2]$ consisting of elements $(\alpha_1, \alpha_2)$ such that $\alpha_1 \cdot \Sigma_1 = \alpha_2 \cdot \Sigma_2$ (note that these pairings make sense), we have that

$$0 \to H^1(Y; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \xrightarrow{\pi} G \oplus H^1(\Sigma; \mathbb{Z})$$

**Remark 10.** Equation (2) is the most explicit description we can get of the picture. This admits two different interpretations. If we think in terms of the homology, the first term corresponds to the 2-homology of $Y$, i.e. $\Sigma$ and those classes of the form $\gamma \times S^1$, where $\gamma \in H_1(\Sigma; \mathbb{Z})$. $\pi^{-1}(G)$ are the classes obtained by gluing cycles coming in $\bar{X}_1$ with cycles in $\bar{X}_2$ intersecting $\Sigma$ on the same number of points. This process is not well-defined and actually choosing different representatives of given homology classes...
in both $X_i$ we can get all of $\pi^{-1}(G)$. The preimage of $H^1(\Sigma; \mathbb{Z})$ corresponds to 2-cycles which intersection with $Y$ is a 1-cycle contained in $\Sigma$ (that is, cycles with part in $X_1$ and in $X_2$ going through the neck). This last bit is not canonically defined as explained above.

If we think in terms of line bundles and their first Chern classes, $\pi(L)$ is the restriction of $L$ to the two open manifolds $X_i$ and $H^1(Y; \mathbb{Z})$ express the different ways in which two line bundles on each of $X_i$ could be glued to give a line bundle on $X$.

**Remark 11.** The characteristic numbers are related as follows:

$$\chi_X = \chi_{X_1} + \chi_{X_2} + 4g - 4$$

$$\sigma_X = \sigma_{X_1} + \sigma_{X_2}$$

Therefore $b^+$ and $b^-$ are both increased by $2g - 1$ and $2\chi + 3\sigma$ by $8g - 8$.

**Remark 12.** Call $m_i$ the minimal number of $(\Sigma_i \cdot D_i)$, for $D_i \in H^2(X_i; \mathbb{Z})$. Put $m$ for the least common multiple of $m_1$ and $m_2$ and $d = m_1m_2/m$ for the greatest common divisor. Then $m$ is the minimal number of $(\Sigma \cdot D)$, for $D \in H^2(X; \mathbb{Z})$ and the map $\pi$ above is surjective if and only if $m_1$ and $m_2$ are coprime. In general the cokernel is isomorphic to $\mathbb{Z}/d\mathbb{Z}$.

### 4. Seiberg-Witten equations for $\Sigma \times S^1$

During the last years there has been a great deal of work on developing the gluing theory for computing the Donaldson invariants of a manifold $X = X_1 \cup_Y X_2$ out of information from $X_1$ and $X_2$ (see [1], [2]). The standard technique is to pull apart $X_1$ and $X_2$ so that we are led to consider metrics giving $X_i$ a cylindrical end and $L^2$-solutions of the equations in these open manifolds. This process has an analogue in the Seiberg-Witten setting, first introduced in [7]. We refer there for the notations used here. The analogue of the Chern-Simons functional is $\frac{1}{4\pi^2}C$ on $\mathcal{B}_3 = \mathcal{A}_3 \times \Gamma(W_3)/\text{Map}(Y, S^1)$ taking values on $\mathbb{R}/\mathbb{Z}$ and given by

$$C(A, \Phi) = \int (A - B) \wedge F_A + \int \langle \Phi, \partial_A \Phi \rangle$$

where $B \in \mathcal{A}_3$ is a fixed connection on $Y$. The downward gradient flow equations for this functional are

$$\begin{cases} 
  dA/dt = - * F_A + i\tau(\Phi, \Phi) \\
  d\Phi/dt = - \partial_A \Phi 
\end{cases}$$

(3)
with $\tau : \tilde{W}_3 \times W_3 \to \Lambda^1(Y)$ from Clifford multiplication and $\vartheta_A$ the 3-dimensional Dirac operator coupled with $A$. The critical points correspond to translation invariant solutions on the tube. So they are solutions to

$$
\begin{align*}
*F_A &= i\tau(\Phi, \Phi) \\
\vartheta_A \Phi &= 0
\end{align*}
$$

The reducible solutions are those for which $\Phi = 0$ and therefore $F_A = 0$ and $c_1(L) = 0$.

Now we pass on to study the equations (4) for $Y = \Sigma \times S^1$. We choose for $Y$ a metric which is rotation invariant. Let $L \to \Sigma \times S^1$ be a line bundle. We can pull it back to $\Sigma \times [0, 1]$ under the map identifying $\Sigma \times \{0\}$ with $\Sigma \times \{1\}$ and denote it by $L$ again. Then topologically this line bundle is the pull back of some line bundle $M$ on $\Sigma$. Clearly $c_1(M)$ is the restriction of $c_1(L)$ to $\Sigma$ and $L$ is obtained from the pull back of $M$ to $\Sigma \times [0, 1]$ by gluing with some gauge transformation $g \in \text{Map}(\Sigma, S^1)$. The homotopy class of $g$ is the component of $c_1(L)$ in $H^1(\Sigma; \mathbb{Z}) \otimes H^1(S^1; \mathbb{Z})$ under the isomorphisms $[\Sigma; S^1] \cong H^1(\Sigma; \mathbb{Z}) \cong H^1(\Sigma; \mathbb{Z}) \otimes H^1(S^1; \mathbb{Z})$, where the last one is product with the fundamental class of the circle. Now denote by $\theta$ the coordinate in the $S^1$-directions. Any connection $A$ on $L$ has a representative in its gauge equivalence class with no $d\theta$ component. This is unique up to constant gauge transformation (i.e. a gauge pulled-back from $\Sigma$). So giving $A$ (up to gauge) is equivalent to giving a family $A_\theta$, $\theta \in [0, 1]$ of connections on $M$ (up to constant gauge) with the condition $A_1 = g^*(A_0)$, with $g$ in the homotopy class determined by $L$.

The Spin$^c$ structure on $Y$ restricts to a Spin$^c$ structure on $\Sigma$ and therefore $W_3$ restricted to $\Sigma$ decomposes as $(\Lambda^0 \oplus \Lambda^{0,1}) \otimes \mu$ for some line bundle $\mu$ on $\Sigma$. Note that $M = K_\Sigma \otimes \mu^2$, where $K_\Sigma$ is the canonical bundle of $\Sigma$. The Dirac operator $\vartheta_A : \Gamma(W_3) \to \Gamma(W_3)$ is identified with $i\frac{\partial}{\partial \theta} + \sqrt{2}(\bar{\partial}_{A_\theta} + \bar{\partial}_{A_\theta}^*)$. Recall that the connection $A_\theta$ is on the bundle $M$ and induces uniquely a connection on the bundle $\mu$.

**Lemma 13.** Let $(A, \Phi) \in \mathcal{A}_3 \times \Gamma(W_3)$. Consider the family $A_\theta$, $\theta \in [0, 1]$ determined by $A$ and write $\Phi = (\sigma_1, \sigma_2) \in \Gamma(W_3) = \Omega^0(\mu) \oplus \Omega^{0,1}(\mu)$. Then the solutions of (4) correspond to solutions of:

$$
\begin{align*}
\frac{\partial \sigma_2}{\partial \theta} &= -\sqrt{2}i \bar{\partial}_{A_\theta}^* \sigma_2 \\
\frac{\partial \sigma_1}{\partial \theta} &= -\sqrt{2}i \bar{\partial}_{A_\theta} \sigma_1 \\
i \frac{\partial A_\theta}{\partial \theta} &= \sigma_1 \sigma_2^* + \sigma_2 \sigma_1^* \\
-2i\Lambda F_{A_\theta} &= |\sigma_1|^2 - |\sigma_2|^2
\end{align*}
$$
In the third equation $\sigma_1\sigma_2^* + \sigma_2\sigma_1^* \in \Omega^1$ is a real form. Recall that the connection $A_\theta$ is equivalent to the holomorphic structure $\bar{\partial}A_\theta$, so we can rewrite that line as

$$\frac{\partial}{\partial \theta} (\bar{\partial} A_\theta) = -i \sigma_1^* \sigma_2$$

**Proposition 14.** Let $(\sigma_1, \sigma_2) \in \Omega^0(\mu) \oplus \Omega^0,1(\mu)$ and $A_\theta, \theta \in [0,1]$ such that

$$\begin{cases}
\frac{\partial\sigma_1}{\partial \theta} = -\sqrt{2}i \bar{\partial} \sigma_2 \\
\frac{\partial\sigma_2}{\partial \theta} = -\sqrt{2}i \bar{\partial} \sigma_1 \\
\frac{\partial}{\partial \theta} (\bar{\partial} A_\theta) = -i \sigma_1^* \sigma_2
\end{cases}$$

(6)

Then $A_\theta, \sigma_1$ and $\sigma_2$ are constant and either $\sigma_1 = 0$ and $\bar{\partial}A_0 \sigma_2 = 0$ or $\sigma_2 = 0$ and $\bar{\partial}A_0 \sigma_1 = 0$.

**Proof.**

$$\bar{\partial}^* \bar{\partial} \sigma_1 = -\frac{1}{\sqrt{2}} \bar{\partial}^* \left( \frac{\partial \sigma_2}{\partial \theta} \right) = \partial \theta \left( -\frac{1}{\sqrt{2}i} \bar{\partial} \sigma_2 \right) + \frac{1}{\sqrt{2}i} \partial \theta \left( \bar{\partial}^* \sigma_2 \right) = \frac{\partial^2 \sigma_1}{\partial \theta^2} - \frac{1}{\sqrt{2}} |\sigma_2|^2$$

where we drop subindices by convenience of notation. First integrate along $\Sigma$ by parts to get for every $\theta$

$$\int_{\Sigma} |\bar{\partial} \sigma_1|^2 - \int_{\Sigma} \left< \frac{\partial^2 \sigma_1}{\partial \theta^2}, \sigma_1 \right> + \frac{1}{\sqrt{2}} \int_{\Sigma} |\sigma_1^* \sigma_2|^2 = 0$$

This equation makes sense in $\mathbb{S}^1$ and we can integrate again by parts to get

$$||\bar{\partial} \sigma_1||^2 + ||\frac{\partial}{\partial \theta} \sigma_1||^2 + \frac{1}{\sqrt{2}} ||\sigma_1^* \sigma_2||^2 = 0$$

The result is immediate from this. □

**Corollary 15.** If the line bundle $L$ admits any solution to (4) then $L$ is pulled-back from $\Sigma$ and any solution is invariant under rotations in the $\mathbb{S}^1$ factor.

**Remark 16.** We can paraphrase corollary 15 saying that any basic class is orthogonal to $H_1(\Sigma;\mathbb{Z}) \otimes H_1(\mathbb{S}^1;\mathbb{Z}) \hookrightarrow H_2(Y;\mathbb{Z}) \hookrightarrow H_2(X;\mathbb{Z})$. This is in fact a very natural phenomenon, for if we put $T_\gamma = \gamma \times \mathbb{S}^1$ then $T_\gamma$ is a torus of self-intersection zero and hence $K \cdot T_\gamma = 0$ for any basic class $K$.

Now let $L$ be a characteristic line bundle which is the pull-back of a line bundle in $\Sigma$. Since $\Sigma \cdot \Sigma = 0$ we have that $c_1(L) \cdot \Sigma$ is even. Consider a Spin$^c$ structure with associated line bundle $L$ (this is the pull-back of a Spin$^c$ structure on $\Sigma$ of the same kind). The description of solutions of the Seiberg-Witten equations in algebraic
varieties [8] [9] gives an identification of the moduli space of solutions of equations (4) with the space of vortices.

**Corollary 17.** Let $2k = (2g - 2) - |c_1(L) \cdot \Sigma|$. Then the moduli space of solutions of the translation invariant equations (4) on $\Sigma \times S^1$ is empty if $k < 0$, $s^k(\Sigma)$ whenever $c_1(L) \cdot \Sigma \neq 0$ (here $s^k(\Sigma)$ stands for the symmetric power of $\Sigma$). If $c_1(L) \cdot \Sigma = 0$ then the moduli space consists uniquely of reducibles and is isomorphic to the Jacobian of line bundles of degree $g - 1$ over $\Sigma$.

**Remark 18.** When $m = c_1(L) \cdot \Sigma = 0$ we have to perturb the equations (1) by adding a self-dual form. Choosing a form invariant under translations and rotations, this produces the effect of perturbing the last equation in (5) by adding a small 2-form to the curvature. The proof of proposition 14 goes through and the methods for proving corollary 17 give a moduli space of solutions $s^{g-1}(\Sigma)$.

Theorem 1 is a consequence of corollary 17 and the following

**Corollary 19.** Let $c$ be a Spin$^C$ structure for $X$ with associated line bundle $L$. If $c$ is a basic class for $X$ then $L|_Y$ is pulled-back from $\Sigma$.

### 5. Gluing theory

In the Seiberg-Witten context there is a parallel of the usual Floer theory for the Donaldson invariants [2]. Some nice few remarks about the case relevant to us appear in [3]. Generally for a three-manifold $Y$ and a line bundle $L|_Y$ on $Y$ (we use this as a matter of notation as $L|_Y$ will be the restriction of a line bundle $L$ in a 4-manifold containing $Y$), we perturb the equations for the translation invariant solutions of (4) on the tube until the solutions are finite and non-degenerate. Then one defines Seiberg-Witten Floer homology groups $HFSW_* (Y; L|_Y)$ as the Floer groups in the usual context. The cohomology groups $HFSW^*(Y; L|_Y)$ are naturally identified with the homology groups of $Y$ with reversed orientation, hence giving a natural pairing between them.

In the case of $Y = \Sigma \times S^1$, we need to consider characteristic line bundles $L$ whose restrictions to $Y$ have $c_1(L|_Y) = 2mS^1$, for $|m| \leq g - 1$, as already established in theorem 1. Fix $m$, i.e. the topological type of $L|_Y$ and put $k = (g - 1) - m$. The picture is analogue to the instanton theory. Instead of using a perturbation of the equations (4), we can work with the moduli space of solutions itself, which by corollary 17 is $M_\Sigma = s^k(\Sigma)$, whenever $m \neq 0$ (when $M_\Sigma$ consists only of irreducibles).
When \( m = 0 \) we perturb the equations as explained before to be in the same situation. So \( HFSW_*(Y; L) \) is identified with the homology of \( M_\Sigma \).

Let \( X_1 \) be an open manifold with cylindrical end \( Y \). For a Spin\(^C\) structure over \( X_1 \) whose associated line bundle \( L \) is of the required type, the limit values of solutions to the Seiberg-Witten equations give the element \( \Phi_{X_1}(c) \) of \( HFSW_*(Y; L) \). As before \( \Phi_{X_1}(L) \) will denote the sum over all possible \( c \) giving rise to the same \( L \). When we have two open manifolds \( X_1 \) and \( X_2 \) which we want to glue along the common boundary \( Y \) (with a fixed diffeomorphism of the boundaries), we have in general an indeterminacy for choosing the identification of the line bundles over \( Y \) resulting in different line bundles for \( X = X_1 \cup_Y X_2 \). In the case of section 3, i.e. when \( b_1(X_i) = b_1(X_i) = 0 \), the possibilities are parametrised by \( H^1(Y; \mathbb{Z}) \). In this case we have to refine the groups \( HFSW_*(Y; L|_Y) \) to keep track of the homotopy class of this identification. For that, we mod out the space of solutions by the component of the identity of \( \text{Map}(Y, S^1) \), instead of using all of it (i.e. we work with \( \tilde{B}_3 = A_3 \times \Gamma(W_3)/\text{Map}(Y, S^1) \)). The result is a number of copies of \( M_\Sigma \) parametrised by \( H^1(Y; \mathbb{Z}) \), which is called \( \tilde{M}_\Sigma = \sum_{i \in H^1(Y)} M_\Sigma^{(i)} \), producing groups \( HFSW_*(Y; L|_Y) \cong \sum_{i \in H^1(Y)} H_*(M_\Sigma^{(i)}) \). Since \( b_1 = 0 \), the invariants lift to \( \tilde{\Phi}_{X_1}(L) \). Clearly there is an addition map

\[
HFSW_*(Y; L|_Y) \xrightarrow{\phi} HFSW_*(Y; L|_Y)
\]

which recuperates the original invariant, i.e. \( a(\tilde{\Phi}_{X_1}(L)) = \Phi_{X_1}(L) \).

Now when we glue solutions coming from \( X_1 \) to solutions from \( X_2 \), the first thing to have in mind is that the copies of \( M_\Sigma \) in which both of them live determine uniquely a gluing of the line bundles over the boundary (and hence the line bundle \( L \) on \( X \)). For instance, if \((A, \phi)\) is a solution of the Seiberg-Witten equations in \( X \) for the line bundle \( L \), which splits as \((A_1, \Phi_1)\) with limit \((a, \phi) \in M_\Sigma^{(i)} \) and \((A_2, \Phi_2)\) with limit \((a, \phi) \in M_\Sigma^{(j)} \), then \(i - j \in H^1(Y)\) determines \( L \) and \(i + t \) and \(j + t \) will determine the same \( L \) for any \( t \in H^1(Y; \mathbb{Z}) \) (this can be thought as transferring \( t \) from \( X_2 \) to \( X_1 \) through the neck). When we pull \( X_1 \) and \( X_2 \) apart introducing metric with larger and larger tube lengths, every solution on \( X \) to the equations (1) appears as solutions in \( X_1 \) and \( X_2 \) giving the same boundary value and such that the determined line bundle is the one we started with. So the pairing between \( \Phi_{X_1}(L|_{X_1}) \) and \( \Phi_{X_2}(L|_{X_2}) \) corresponds to using all possible gluings of \( L|_{X_2} \).

**Proposition 20.** Let \( L \) be a line bundle over \( X = X_1 \cup_Y X_2 \). Then the pairing

\[
\tilde{\Phi}_{X_1}(L|_{X_1}) \cdot \tilde{\Phi}_{X_2}(L|_{X_2})
\]
in \( HFSW^*_\ast(Y; L|_Y) \) is equal to the summation (see (2) for definition of \( \pi \))

\[
\sum_{\{L'/\pi(c_1(L'))=\pi(c_1(L))\}} SW_X(L')
\]

i.e. over all \( L' \) whose restrictions to both \( X_i \) are isomorphic to the ones of \( L \).

**Remark 21.** In the case of \( X_2 = A = D^2 \times \Sigma \) we cannot do the same thing as we have to take into account the homology of \( A \), which is \( H^1(\Sigma; \mathbb{Z}) \). Therefore we only can lift the Floer groups to \( \tilde{M}_\Sigma/H^1(\Sigma; \mathbb{Z}) \), which is formed by copies of \( M_\Sigma \) parametrised by \( \mathbb{Z}[\Sigma] \). We call this \( M'_\Sigma = \sum_{n \in \mathbb{Z}[\Sigma]} M^{(n)}_\Sigma \).

We also have a forgetful map

\[
HFSW^*_\ast(Y; L) \to HFSW'_\ast(Y; L) = HFSW^*_\ast(Y; L)/H^1(\Sigma; \mathbb{Z})
\]

**Proposition 22.** Let \( L \) be a line bundle over \( \tilde{X}_1 = X_1 \cup_Y A \). Then the pairing

\[
\alpha(\tilde{\Phi}_{X_1}(L|_{X_1})) \cdot \Phi'_A(L|_A)
\]

in \( HFSW'_\ast(Y; L|_Y) \) is equal to the summation

\[
\sum_{\{L'/c_1(L')\equiv c_1(L)\mod (\Sigma)} SW_{\tilde{X}_1}(L')
\]

Furthermore, if \( \tilde{X}_1 \) is of simple type and \( m \neq 0 \), then at most one of the \( L' \) can appear in the sum above, since at most one of them has \( c_1(L)^2 = 2\chi + 3\sigma \).

Theorem 2 is a consequence of the following

**Theorem 23.** Now suppose that \( \tilde{X}_i \) are of simple type and \( g \geq 2 \). Fix a pair \( (\kappa_1, \kappa_2) \in G \) such that \( \kappa_i \cdot \Sigma_i = \pm(2g - 2) \) and \( \kappa_i \) are characteristic. Then the sum \( \sum_{L \in \pi^{-1}(\kappa_1, \kappa_2)} SW_X(L) \) is the product of the two terms

\[
\sum_{\{L/c_1(L)\equiv \kappa_i \in H^2(X_i/\mathbb{Z}[\Sigma_i])\}} SW_{\tilde{X}_i}(L) = SW_{\tilde{X}_i}(\kappa_i)
\]

**Proof.** In the case \( c_1(L) = \pm(2g - 2)S^1 \), one has \( k = 0 \) so \( M_\Sigma \) is a point and \( H_*(M_\Sigma) \cong \mathbb{Z} \). Now the solutions for \( A = \Sigma \times D^2 \) are all pulled-back from \( \Sigma \). So \( \Phi_A(L|_A) \) consists of a 1 in one \( \mathbb{Z} \) and zero in the rest and hence \( a(\Phi_A(L|_A)) = 1 \). Now proposition 22 says

\[
n_{\kappa_i} = SW_{\tilde{X}_i}(\kappa_i) = \alpha(\tilde{\Phi}_{X_i}(L|_{X_i})) \cdot \Phi'_A(L|_A) = \Phi_{X_i}(L|_{X_i})
\]

Now the results comes from applying proposition 20. \( \square \)
Remark 24. Note that the same argument is not applicable to the case $k > 0$ as plugging in $A$ we only get (knowing $SW_{X_i}(L)$) the values of $\Phi_{X_i}(L|_{X_i})$ in $H_0(M^j_\Sigma))$, but we need the higher homology. One expects that no new basic classes are going to appear from pairs $(\kappa_1, \kappa_2) \in G$ such that $|\kappa_i \cdot \Sigma_i| < 2g - 2$.

6. Examples

This section is devoted to examples in which the information already gathered in propositions 20 and 22 is enough to find the basic classes of the glued manifold.

Theorem 25. Suppose that for a basis of homology cycles $\gamma \in H^1(\Sigma; \mathbb{Z})$ there are embedded $(-1)$-discs in both $X_i$ bounding some embedded 1-cycle representing $\gamma$, then the basic classes $\kappa$ of $X$ such that $\kappa \cdot \Sigma = \pm (2g - 2)$ are in one-to-one correspondence with pairs of basic classes $(\kappa_1, \kappa_2)$ for $\bar{X}_1$ and $\bar{X}_2$ respectively, such that $\kappa_1 \cdot \Sigma_1 = \kappa_2 \cdot \Sigma_2 = \pm (2g - 2)$. Moreover, $\kappa$ is determined in an explicit way.

Proof. By a $(-1)$-embedded disc as above we understand an embedding $(\mathbb{D}^2, \partial \mathbb{D}^2) \hookrightarrow (X_i, \partial X_i)$ with the boundary going to an embedded curve representing $\gamma$ and such that a small perturbation has only one point of intersection with the original disc, lying in the interior and with sign $-1$. So when the situation of the theorem is given, we can glue together the discs (in a generally not unique way) to get an embedded sphere of self intersection $-2$, say $D_\gamma$. Now call $T_\gamma$ to the torus $\gamma \times S^1 \hookrightarrow \Sigma \times S^1$. We have the obvious fact $T_\beta \cdot T_\gamma = 0$ and $D_\beta \cdot T_\gamma = (\beta \cdot \gamma)$. So there is a torus of self-intersection zero and a sphere of self-intersection $-2$ intersecting in one point. This implies [4] that $X$ is of simple type and that the basic classes vanish on all of these cohomology classes. What is more, the $T_\gamma$ and $D_\gamma$ generate a primitive sublattice $V \subset H^2(X; \mathbb{Z})$ hence $V \oplus V^\perp = H^2(X; \mathbb{Z})$ and

\begin{equation}
0 \to \mathbb{Z}[\Sigma] \to V^\perp \xrightarrow{\pi} G
\end{equation}

Now let $\kappa$ be a basic class for $X$ such that $\kappa \cdot \Sigma = 2g - 2$. We have argued that $\kappa \in V^\perp$, theorem 1 tells us how the image of $\kappa$ under $\pi$ is. So there are basic classes $\kappa_1$ and $\kappa_2$ in $X_1$ and $X_2$ such that $\kappa \cdot \Sigma = \kappa_1 \cdot \Sigma_1 = \kappa_2 \cdot \Sigma_2 = 2g - 2$. For $g \geq 2$, we have that $\kappa^2 \neq (\kappa + n\Sigma)^2$ for $n \neq 0$, so at most one of the $\kappa + n\Sigma$ can be basic class. Thus the statement of the theorem. \[\square\]

Remark 26. To come to terms with theorem 25 we need to split up the sequence (7). Suppose $\Sigma_i$ are both primitive. In the words of remark 12, $m_1 = m_2 = m = 1$. We choose cohomology classes $D \in H^2(X; \mathbb{Z})$, $D_1 \in H^2(\bar{X}_1; \mathbb{Z})$, $D_2 \in H^2(\bar{X}_2; \mathbb{Z})$ with $D \cdot \Sigma = D_1 \cdot \Sigma_1 = D_2 \cdot \Sigma_2 = 1$. Let $W$ be the primitive sublattice in $V^\perp$.
containing $\Sigma$ and $D$, and analogously for $W_1$ and $W_2$. Then $\pi : W^\perp \cong W_1^\perp + W_2^\perp$ and $W/\mathbb{Z}[\Sigma] \cong \Delta \subset W_1/\mathbb{Z}[\Sigma_1] + W_2/\mathbb{Z}[\Sigma_2]$, taking $[D]$ to $([D_1], [D_2])$. Also $D$ provides an isomorphism $W \cong W/\mathbb{Z}[\Sigma] \oplus \mathbb{Z}[\Sigma]$ and the same for $D_1$ and $D_2$, so

$$H^2(\tilde{X}_i) \cong W_i^\perp / \mathbb{Z}[\Sigma_i] \oplus \mathbb{Z}[\Sigma_i]$$

$$H^2(X) \cong V \oplus W_1^\perp \oplus W_2^\perp \oplus \Delta \oplus \mathbb{Z}[\Sigma]$$

Then if we choose $D^2 = D_1^2 + D_2^2$ (easy to arrange), we have (in the decomposition above) that for $\kappa_i = \alpha_i + (2g - 2)D_i + r_i\Sigma_i$ basic classes for $X_i$, the corresponding basic class for $X$ is $\kappa = 0 + \alpha_1 + \alpha_2 + (2g - 2)D + (r_1 + r_2 + 2)\Sigma$ (the coefficient of $\Sigma$ is found out using the fact that $\kappa^2 = 2\chi + 3\sigma$). Formally

$$\kappa = \kappa_1 + \kappa_2 + 2\Sigma$$

Remark 27. When both $\tilde{X}_i$ are symplectic manifolds and $\Sigma_i$ are symplectic submanifolds, Gompf [6] has proved that $X$ can be given a symplectic structure (regardless of the homotopy class of the chosen gluing $\phi$). Taubes [10] [11] has proved that the canonical class $K = -c_1(TX)$ is a basic class and that for any other basic class $\kappa \neq \pm K$, one has $|\kappa[\omega]| < K[\omega]$. Since, in the notation of the last theorem, $T_\gamma[\omega] = 0$, none of the $K + \sum n_\gamma T_i$ can be basic classes unless all $n_\gamma = 0$. Then in the formula of proposition 20 only one term appears in either side. Notice that Taubes also proves that this number is $\pm 1$.

Remark 28. The result of the last remark falls very short since it does not even tell us about the other basic classes that might appear when we glue two basic classes $K_i$ for $\tilde{X}_i$ with $K_i \cdot \Sigma = 2g - 2$ but $K_i$ are not the canonical classes. In some situations we get more: suppose that (both) $\tilde{X}_i$ have come out as the blow-up of some symplectic manifolds $M_i$ at points on $\Sigma'_i \hookrightarrow M_i$ (and $\Sigma_i$ is the proper transform of $\Sigma'_i$) and that the cohomology class defined by the symplectic forms in both $M_i$ where Poincaré dual to $[\Sigma'_i]$ (i.e. $[\Sigma'_i]$ are ample classes). Then one has $|\kappa_i[\Sigma_i]| < K[\Sigma_i] = 2g - 2$ for every basic class $\kappa_i$ in $\tilde{X}_i$. So we conclude that the only basic classes with $\kappa \cdot \Sigma = \pm(2g - 2)$ for $X$ are $\kappa = \pm K$.

When the manifolds involved are complex surfaces and $\Sigma_i$ are embedded complex curves, there is a preferred identification between the boundaries of $X_i$ given by the only $\phi$ which identifies holomorphically the holomorphic normal bundles of $\Sigma_i$ in $\tilde{X}_i$.

Proposition 29. Let $\mathcal{Z} \xrightarrow{\pi} \Delta$ be a family of complex surfaces. Suppose that $Z_t = \pi^{-1}(t)$ are smooth for $t \neq 0$ and that $X = Z_0$ is the union of two surfaces $\tilde{X}_1$ and $\tilde{X}_2$ intersecting in a normal crossing (see [5]). Then the diffeomorphism type of $X$ is
obtained by the necessary blow-ups and a connected sum of $\bar{X}_1$ and $\bar{X}_2$ along $\Sigma$ with the preferred identification alluded above.

Proof. To prove this result first we construct another deformation family which general member is $Z_t$ for $t \neq 0$ but the fibre over 0 is the union of the blow-ups of $\bar{X}_i$ at $s_i$ arbitrary points on $\Sigma_i$ (with $s_1 + s_2 = n_1 + n_2$). Without loss of generality, we can suppose the case of one point in $\bar{X}_1$. The problem is local, so we pick a small (Zariski) neighbourhood $U$ of the point in $Z$ such that $U_t = Z_t \cap U$ is embedded in $\mathbb{C}^3$. The intersection $C_t = U_t \cap U_0$ is a reducible curve which can be written as $C_{t,1} \cup C_{t,2}$ with $C_{t,i} \subset \bar{X}_i$ (reducing even further $U$ we can suppose that the only intersection of $C_{t,i}$ with $\Sigma$ is the given point and that this intersection is transverse). Consider $\bigcup_t U_t \times \{t\}$ and its divisor $\bigcup_t C_{t,2} \times \{t\}$. We blow-up such divisor, which does not affect any of $U_t$, $t \neq 0$ nor $\bar{X}_2$ and has the result of blowing-up $\bar{X}_1$ at the given point. Now glue this new (Zariski) open affine to the complement of the point in $Z$. We have the required deformation.

For completing the proof we note that when $n_1 + n_2 = 0$ the deformation of $\bar{X}_1 \cup \bar{X}_2$ is diffeomorphic to the connected sum along $\Sigma$ alluded in the statement.

Theorem 30. Suppose that both $\bar{X}_i$ are Kähler manifolds and that $X$ is deformation equivalent to $\bar{X}_1 \cup_\Sigma \bar{X}_2$. Then the basic classes $\kappa$ of $X$ such that $\kappa \cdot \Sigma = \pm(2g - 2)$ are in one-to-one correspondence with pairs of basic classes $(\kappa_1, \kappa_2)$ for $X_1$ and $X_2$ respectively, such that $\kappa_1 \cdot \Sigma_1 = \kappa_2 \cdot \Sigma_2 = \pm(2g - 2)$.

Proof. Firstly, it is known after Witten [12] that all Kähler manifolds with $b^+ > 1$ are of simple type. As in the proof of theorem 25 we just need to prove that if $\kappa$ is a basic class for $X$ and $T = \sum n_\beta T_\beta \neq 0$ in $H^1(Y; \mathbb{Z})$ then $\kappa + T$ is not basic class. In the Kähler case we know that the basic classes are in $H^{1,1}$, so it is enough to show that $T \notin H^{1,1}$. But $T^2 = 0$ and $T[\omega] = 0$, for the symplectic form $\omega$. If $T$ were in $H^{1,1} \cap H^2(X; \mathbb{Z})$, it would represent a divisor with $T^2 = 0$ and orthogonal to an ample class, but this is impossible.

Remark 31. Last two theorems combined with proposition 8 give information on the basic classes of a Kähler manifold which can be deformed to an algebraic variety with normal crossings, knowing the basic classes of the irreducible components.

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