(2 + 1)-dimensional $f(R)$ gravity solutions via Hojman symmetry

F. Darabi*, M. Golmohammadi†, A. Rezaei-Aghdam‡
Department of Physics, Azarbaijan Shahid Madani University
53714-161, Tabriz, Iran

March 29, 2022

Abstract

In this paper, we use the Hojman symmetry approach to find new (2 + 1)-dimensional $f(R)$ gravity solutions, in comparison to Noether symmetry approach. In the special case of Hojman symmetry vector $X = R$, we recover (2 + 1)-dimensional BTZ black hole and generalized (2 + 1)-dimensional BTZ black hole solutions, obtained by Noether symmetry approach, and the interesting point is that the cosmological constant is appeared as the direct manifestation of Hojman symmetry.

Keywords: Hojman symmetry, $f(R)$ gravity, BTZ black hole.
Mathematics Subject Classification: 83C20, 83C15, 83D05, 83C57.

1 Introduction

The (2 + 1)-dimensional gravity has no black hole solutions for vanishing cosmological constant [1]. However, BTZ black hole solutions have been obtained for (2 + 1)-dimensional gravity with a negative cosmological constant, defined by the following action [2] and [3]

$$I = \frac{1}{2} \int dx^3 \sqrt{-g} (R - 2\Lambda),$$

(1)

where $\Lambda = -l^{-2}$ is characterized by a length scale $l$. The line element in $(t, r, \phi)$ coordinates is given by

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2,$$

(2)

$$f(r) = \left( -m + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \right),$$

(3)

where the mass $m$ and angular momentum $J$ are two parameters corresponding to time displacement and rotational symmetries associated with two Killing vectors $\partial_t$ and $\partial_\phi$, respectively. Unlike the asymptotically flat Schwarzschild and Kerr black hole solutions having curvature singularity at $r = 0$, the BTZ black hole is asymptotically anti-de-Sitter (AdS) having no curvature singularity at $r = 0$. A BTZ black hole with $J \neq 0$ describes a spacetime with a constant negative curvature having outer and inner horizons $r_+$ (event horizon) and $r_-$ (Cauchy horizon) respectively, given by

$$r_{\pm}^2 = \frac{l^2}{2} \left( m \pm \sqrt{m^2 - \frac{J^2}{l^2}} \right).$$

(4)

*Corresponding author. e-mail: f.darabi@azaruniv.ac.ir
†author. e-mail: golmohammadi@azaruniv.ac.ir
‡author. e-mail: rezaei-a@azaruniv.ac.ir
It is known that at spacetime dimensions lower than 4, alternative theories of gravity are needed to define a proper Newtonian limit \[4\] and \[5\]. This necessity becomes remarkable when one intends to study the lower dimensional black holes, like the well known BTZ black holes. Among the well-known generalization of General Relativity, like Brans-Dicke gravity \[6\] scalar-tensor gravity \[7\] and \[8\], \(f(T)\) gravity \[9\] and \[10\] and Lovelock gravity \[11\], the \(f(R)\) gravity \[12\]-\[20\] is one of those alternative theories of gravity which is under recent focus of attention. Using suitable conformal transformations, \(f(R)\) gravity becomes equivalent to a scalar-tensor gravity and this remarkable feature makes it more interesting among the other alternatives of general relativity. Motivated by the above mentioned features of \(f(R)\) gravity, a Noether symmetry approach has already been developed by the authors \[21\] to obtain \(2+1\) dimensional black hole solutions in the framework of \(f(R)\) gravity.

An alternative approach, so called Hojman symmetry approach, has recently been received atention by which one can find new exact solutions \[22\]-\[29\]. Unlike the Noether symmetry approach which needs Lagrangian and Hamiltonian functions, in the Hojman symmetry approach we just need the symmetry vectors and the corresponding conserved charges which are easily obtained by using the equations of motion.

In the present paper, we intend to revisit the previous problem, considered in \[21\], and obtain possible \(2+1\) dimensional BTZ black hole solutions in the context of \(f(R)\) gravity, using Hojman symmetry approach, as an alternative to the Noether symmetry approach. This study is of particular importance because some new \(2+1\) dimensional generalized BTZ black hole solutions of \(f(R)\) gravity, as well as corresponding symmetry vectors, may be obtained which are absent in the Noether symmetry approach. This motivates us to apply Hojman symmetry approach for other gravitational systems and enables us to find new features which are not reported in their Noether symmetry approach.

### 2 \((2+1)\)-dimensional \(f(R)\) gravity

The action for \((2+1)\)-dimensional \(f(R)\) gravity is given by

\[
I = \frac{1}{2} \int d^3x \sqrt{-g} f(R).
\]

We consider the line element in the following form \[2\]

\[
ds^2 = [-N^2(r) + r^2M^2(r)]dt^2 + N^{-2}(r)dr^2 + 2r^2M(r)dt d\phi + r^2 d\phi^2,
\]

where the radial functions \(N(r)\) and \(M(r)\) are considered as degrees of freedom. The Ricci scalar is obtained

\[
R = -\frac{1}{2r}(4rN'^2 + 4rNN'' - r^3M'^2 + 8NN'),
\]

where ' denotes the derivative with respect to \(r\). Using the method of Lagrange multipliers to set \(R\) as a constraint of the dynamics, generalizing the degrees of freedom and defining a canonical Lagrangian \(\mathcal{L} = \mathcal{L}(N, M, R, N', M', R')\), the action (5) casts in the following form

\[
S = \int d^3x \sqrt{-g}[f(R) - \lambda (R + \frac{1}{2r}(4rN'^2 + 4rNN'' - r^3M'^2 + 8NN'))].
\]

After variation with respect to \(R\), we find \(\lambda = f_R \equiv df/dR\) and the action is rewritten as

\[
S = \int d^3x \sqrt{-g}[f(R) - f_R(R + \frac{1}{2r}(4rN'^2 + 4rNN'' - r^3M'^2 + 8NN'))].
\]

Integrating by parts lead to the Lagrangian

\[
\mathcal{L} = r(f - Rf_R) + \frac{r^3}{2}f_RM'^2 - 2rf_RNN' + 2rf_{RR}R'NN',
\]
where $f_{RR} \equiv d^2 f/dR^2$. The Euler-Lagrange equations for $N$, $M$ and $R$ are derived respectively as

\begin{align*}
N(f_{RRR}R^2 + f_{RR}R'' &= 0, 
(\frac{r^3}{2}f_R'+f_{RR})' &= 0, 
-\gamma Rf_{RR} + \frac{r^3}{2}f_{RR}M^2 - 4f_{RR}NN' - 2rf_{RR}N^2 - 2rf_{RR}NN'' &= 0.
\end{align*}

3 Hojman symmetry approach

Consider a set of second-order ordinary differential equations

\begin{equation}
\ddot{q}_i = F_i(q_j, \dot{q}_j, t), \quad i, j = 1, 2, \ldots n
\end{equation}

where $q_i$ and $F_i$ denote the generalized coordinates and forces, respectively, and each over dot denotes derivative with respect to time $t$. If there exists an associated symmetry vector $X_i = X_i(q_j, \dot{q}_j, t)$, then it should satisfy the differential equation

\begin{equation}
\frac{d^2 X_i}{dt^2} - \frac{\partial F_i}{\partial q_j} X_j - \frac{\partial F_i}{\partial \dot{q}_j} \frac{dX_j}{dt} = 0,
\end{equation}

where

\begin{equation}
\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}_j \frac{\partial}{\partial q_j} + F_i \frac{\partial}{\partial \dot{q}_i}.
\end{equation}

The symmetry vector $X_i$ maps the solutions $q^i$ of Eq.(14) into the solutions $\hat{q}^i$ of the same equations (up to $\epsilon^2$ terms), under the infinitesimal transformation

\begin{equation}
\hat{q}^i = q^i + \epsilon X_i(q^j, \dot{q}^j, t).
\end{equation}

Using this property, the Hojman conserved quantities are defined by the following theorem [22] and [23]:

**Theorem:**

1. Provided that $F_i$ satisfies

\begin{equation}
\frac{\partial F_i}{\partial \dot{q}_i} = 0,
\end{equation}

then

\begin{equation}
Q = \frac{\partial X_i}{\partial q_i} + \frac{\partial}{\partial \dot{q}_i} \left( \frac{dX_i}{dt} \right),
\end{equation}

is a conserved quantity.

2. Provided that $F_i$ satisfies

\begin{equation}
\frac{\partial F_i}{\partial \dot{q}_i} = -\frac{d}{dt} \ln \gamma,
\end{equation}

then

\begin{equation}
Q = \frac{1}{\gamma} \frac{\partial (\gamma X_i)}{\partial q_i} + \frac{\partial}{\partial \dot{q}_i} \left( \frac{dX_i}{dt} \right),
\end{equation}

is a conserved quantity, where $\gamma$ is merely a function of $q^i$.  

3
4 (2+1)-dimensional $f(R)$ gravity solutions via Hojman Symmetry

The equations of motion that are obtained in section 2, can be rewritten in accordance with Hojman symmetry approach as follows

$$R'' = -h(R)R^2,$$

$$r^3 f_R M' = C,$$

$C$ being a constant, and

$$-rRf_{RR} + \frac{r^3}{2}f_{RR}M'^2 - 4f_{RR}N' - 2rf_{RR}N'^2 - 2rf_{RR}NN'' = 0,$$

where

$$h(R) = \frac{f_{RRR}}{f_{RR}}.$$  

Note that $h(R)$ should be well-defined for each proposed $f(R)$. By comparing (21) and (14), we can recognize that $F(R, R') = -h(R)R'^2$. Also from Eq. (19), we obtain

$$\gamma(R) = \gamma_0 e^{\int 2h(R) dR},$$

where $\gamma_0$ is a constant. If there is no explicit dependence of $X$ on $r$, namely $X = X(R, R')$, then we can rewrite Eqs. (15) and (20), respectively as

$$\left(\frac{\partial^2 X}{\partial R^2} + h_R X + h(R) \frac{\partial X}{\partial R}\right) + R'^2 h^2(R) \frac{\partial^2 X}{\partial R \partial R'} - R' \left(2h(R) \frac{\partial^2 X}{\partial R \partial R'} + h_R \frac{\partial X}{\partial R'}\right) = 0,$$

and

$$Q = 1 - \frac{\partial (\gamma X)}{\gamma \partial R} + \frac{\partial}{\partial R'} \left(\frac{dX}{dr}\right),$$

where $h_R \equiv \frac{dh}{dR}$. In general, solving the differential equation (26) for vector $X$ is difficult. In this regard, we will limit ourselves to some particular ansatz for vector $X$, proposed in Ref. [23], as follows.

4.1 $X \sim X(R)$

From Eqs. (26) and (27), we see that

$$h(R)X + \frac{dX}{dR} = \frac{Q}{2},$$

where $Q$ is a conserved quantity.

4.1.1 $X = R$

As a first step, if we simply consider $X = R \neq 0$ then from equation (28) we find a well defined

$$h(R) = \frac{Q}{2R} - \frac{1}{R'},$$

and from equations (21) and (24) we have

$$f(R) = C_1 + C_2 R + C_3 R^{\frac{Q}{2}+1},$$

and

$$R(r) = \frac{1}{2^{\frac{2}{Q}}} \left[Q \left(C_5 + C_4 r\right)\right]^\frac{1}{2}.$$
Finally, from equations (22), (30) and (31) for \( M(r) \) and setting \( C_5 = 0 \) for simplicity, we obtain
\[
M(r) = -\frac{CC_6^2C_4^2\ln(C_2 + C_6C_4r)}{C_2^2} + \frac{CC_6^2C_4^3\ln(r)}{C_2^2} + \frac{CC_6C_4}{C_2^2}r - \frac{1}{2} \frac{C}{C_3r^2} + C_7, \tag{32}
\]
where \( C_6 = \frac{Q}{2}C_3(\frac{1}{2}Q + 1) \). Note that the requirement \( R(r) \neq 0 \) for a well-defined \( h(R) \), excludes the origin \( r = 0 \) from the domain of \( r \) for \( Q > 0 \), and avoids \( r \to \infty \) for \( Q < 0 \). Therefore, at least there is no curvature singularity issue at the origin.

Using (30), (31) and (32) in the equation of motion (23) we obtain
\[
N^2(r) = -\frac{1}{4C_2^2r^2} \left[ 2C^2C_6C_4r \ln(-\frac{C_2}{C_6C_4r} - 1)(3C_6C_4r + 2C_2) \right] + 2C_8 \frac{1}{r} + \frac{3C^2C_6C_4}{C_2^2} \frac{1}{r} - \frac{Q^2}{2(Q + 1)(3Q + 2)} \left( \frac{C_4Q}{2} \right)^2 r^{2Q + 2} + \frac{C^2}{4C_2^2} \frac{1}{r^2} - 2C_9 \tag{33}
\]
where \( C_8 \) and \( C_9 \) are constants of integration. Now (32) and (33) determine the spherically symmetric solutions for the metric (6) subject to the Ricci scalar (31), provided that their comparison between the solutions (43\footnote{We mean the solution (43) for which \( D_1 = 0, D_2 = 1 \) are imposed.}) and (51) in [21] and the constant \( C_1 \) is not appeared in the solutions for \( M(r) \) and \( N(r) \) as a direct consequence of imposing the Hojman symmetry along \( X = R \).

In order to compare these solutions with those of obtained in [21], by imposing Noether symmetry, we may consider the solution (51) in [21] and set \( C_3 = 0 \) here to recover \( f(R) = C_1 + C_2R \) which is linear in terms of \( R \) similar to \( f(R) = R + D_3 \), in [21]. This provides us with the following solutions
\[
M(r) = -\frac{1}{2} \frac{C}{C_3r^2} + C_7, \tag{34}
\]
\[
N^2(r) = \frac{C^2}{4C_2^2} \frac{1}{r^2} + 2C_8 \frac{Q^2}{2r} \frac{1}{(Q + 1)(3Q + 2)} \left( \frac{C_4Q}{2} \right)^2 r^{2Q + 2} - 2C_9. \tag{35}
\]
where \( Q > 0 \) and \( C_4 > 0 \) are assumed. A comparison between the solutions (43\footnote{We mean the solution (43) for which \( D_1 = 0, D_2 = 1 \) are imposed.}) and (51) in [21] and the solutions (32) and (35) here, shows that both solutions are the same, up to some identifications between the constants in each solution.

In order to investigate the black hole property of these solutions, we write the metric (6) in the following form
\[
ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2[M^2(r)dt^2 + d\phi^2]. \tag{36}
\]
For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric (35). Therefore, the spherical solutions (32) and (33) are capable of being as a black hole solution for \( f(R) \) gravity (30), subject to the non-vanishing Ricci scalar (31), provided that their asymptotic behaviours are analyzed.

Now, we investigate on the possibility of recovering the well known BTZ black hole solution
\[
M(r) = -\frac{J}{2r^2}, \tag{37}
\]
\[
N^2(r) = -m + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \tag{38}
\]
where \( m \) and \( J \) are the mass and angular momentum of the black hole, respectively. In this regard, we may use the following initial identifications
\[
C = J, \quad C_1 = 2/l^2, \quad C_2 = 1, \quad C_7 = C_8 = 0, \quad C_9 = m/2. \tag{39}
\]
At this stage, it is left to identify the third term in (35) with the term $r^2/l^2$. To this end, we may consider $C_4 = 2Q^{-1}(-6Q^2)^{Q/2}$ and $Q \sim l$, assuming $l \to \infty$, which gives us the third term in (35) with the behavior $\pm r^2/l^2$. Therefore, the solutions (34) and (35) exhibit an asymptotically de Sitter and anti-de Sitter spacetimes, corresponding to $-r^2/l^2$ and $+r^2/l^2$, respectively.

One may conclude that the regular BTZ black hole is a solution corresponding to $f(R) = C_1 + R$ equipped with a Hojman symmetry vector $X = R$, and the corresponding gauge freedom in choosing the constant term $C_1$ is fixed by a cosmological constant $\Lambda < 0$ as $C_1 = \pm 2l^{-2} = \mp 2\Lambda > 0$. Although $\Lambda$ appears as a gauge choice here, however, similar to the constants $m$ and $J$ which are interpreted as the conserved charges corresponding to the time translation and rotation symmetries, respectively, we may interpret the cosmological constant $\Lambda$ as the conserved charge (because of $Q \sim (-\Lambda)^{-1/2}$) corresponding to the Hojman symmetry of solutions under the infinitesimal displacement of $R$. This is an important result accounting for the fact that the cosmological constant, in the regular BTZ black hole, is nothing but the manifestation of Hojman symmetry which induces the symmetry vector $X = R$ and generates the conserved charge $Q \sim (-\Lambda)^{-1/2}$.

By assuming $C_7 = 0, C_8 \neq 0$ we find

$$M(r) = \frac{J}{2r^2},$$

$$N^2(r) = -m + \frac{r^2}{l^2} + 2C_8 - \frac{J^2}{4r^2},$$

which is the same generalized BTZ black hole solution which was obtained in [21] using Noether symmetry.

Now, we may analyze the asymptotic structure of the generalized BTZ solutions (34) and (35) in the limit $r \to \infty$ to check whether or not the solutions represent asymptotically anti-de Sitter spacetime. It is easy to show that by choosing $C_7 = 0$ and non-vanishing constants $C, C_2, C_4, C_8$ and $C_9$, the solutions (34) and (35) exhibit an asymptotically de Sitter and anti-de Sitter spacetime, provided that $C_4 = 2Q^{-1}(-6Q^2)^{Q/2}$ and $C_4 = 2Q^{-1}(+6Q^2)^{Q/2}$, respectively, assuming $Q \sim l, l \to \infty$. Therefore, we may interpret the solutions (34) and (35) as an almost generalized BTZ black hole which is asymptotically de Sitter and anti-de Sitter spacetime with asymptotic cosmological constants $\Lambda \sim l^{-2} > 0$ and $\Lambda \sim -l^{-2} < 0$, respectively.

4.1.2 $X = \lambda \tan(R), \quad \lambda = \text{const}$

According to the previous subsection, by fixing $Q = 2\lambda$, we find

$$h(R) = -\frac{\sin R}{\cos R},$$

$$f(R) = C_1 + C_2R + C_3\cos R,$$

and

$$R(r) = \arcsin(C_4r + C_5).$$

For simplicity, we fix $C_5 = 0$, for which there is no curvature singularity at the origin $r = 0$. Again, the requirement $R(r) \neq (2k + 1)\pi/2$ for a well-defined $h(R)$, excludes some definite values from the domain of $r$.

Then, we have

$$M(r) = -\frac{CC_3^2C_4^2}{C_2^2} \ln(-C_2 + C_3C_4r) + \frac{CC_3^2C_4^2}{C_2^2} \ln r - \frac{C}{2C_2} \frac{1}{r} - \frac{CC_3C_4}{C_2^2} \frac{1}{r} + C_6.$$  (45)
Using (43), (44) and (45) in the equation of motion (23) we obtain

\begin{equation}
N^2(r) = -2C_7 + C_2^2 \frac{1}{4C_2^2 r^2} + 2C_8 \frac{1}{r} - 3C_3^2 C_4 \frac{1}{2C_2^2} - \frac{1}{r} \sqrt{1 - C_4^2 r^2} + \frac{1}{36C_4^2 r}(5C_4^2 r^2 - 8) + \frac{1}{4C_2^2 r^2} \left[ 2C_4 C_3 r \ln(C_3 - \frac{C_2}{C_4 r})(-3C_3 C_4 r + 2C_2) \right] - \frac{1}{12C_4^2 r}(2C_4^2 r^2 - 3) \arcsin(C_4 r),
\end{equation}

where \( C_7 \) and \( C_8 \) are integration constants. For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric (35). Therefore, the spherical solutions (45) and (46) are capable of being as a black hole solution for \( f(R) \) gravity in [21].

Furthermore, the function \( h \) (44) satisfying the condition \( R(r) \neq (2k + 1)\pi/2 \). For the special case \( C_3 = 0 \), namely \( f(R) = C_1 + C_2 R \), we obtain

\begin{equation}
M(r) = -\frac{C}{2C_2 r^2} + C_6,
\end{equation}

\begin{equation}
N^2(r) = \frac{1}{12C_4^2 r}(2C_4^2 r^2 - 3) \arcsin(C_4 r) - \frac{1}{36C_4^2 r} \left[ 8 + 5C_2^2 r^2 \right] r \sqrt{1 - C_4^2 r^2} + \frac{C_2^2}{4C_2^2 r^2} + 2C_8 \frac{1}{r} - 2C_7,
\end{equation}

where the domain of \( r \) is limited by the inequality \( 1 - C_4^2 r \geq 0 \). Unlike the previous solutions (34) and (35), the solutions (47) and (48) are new in comparison to the solutions (43) and (51) obtained for the same \( f(R) = C_1 + C_2 R \) gravity in [21].

Now, we investigate on the possibility of recovering the well known BTZ black hole solution from the solutions (47) and (48). It turns out that the existence of first and second terms in (48), subject to \( C_4 \neq 0 \), does not allow for such recovery even if we set \( C_6 = C_8 = 0, C_7 = m/2 \). Therefore, rather than BTZ black hole or generalized BTZ black hole solutions, the Hojman symmetry provides us with the new \((2+1)\) dimensional \( f(R) \) gravity solution, in comparison to the solution which has already been obtained by Noether symmetry in [21].

The study of asymptotic structure shows that by choosing \( C_6 = 0 \), together with finite-value constants \( C, C_2, C_4, C_7 \) and \( C_8 \), the solutions (47) and (48) cannot exhibit an asymptotically anti-de Sitter spacetime, because of the ill-defined trigonometric function, in the first term, and also the imaginary behaviour of the square root, in the second term, for \( C_4 r > 1 \).

However, by choosing an infinitesimal \( C_4 \) and restricting the domain \( 0 \leq r \leq C_4^{-1} \), it is possible to find the asymptotic behaviour up to the limit \( r \to C_4^{-1} \) \((C_4 r \to 1)\) where the first and second terms goes like constant terms, while the third and fourth terms almost vanishes. Therefore, provided that \( C_4 \) is infinitesimal and the domain of \( r \) is restricted by \( 0 \leq r \leq C_4^{-1} \), we may interpret the solutions (47) and (48) as new generalized \((2+1)\) dimensional \( f(R) \) gravity solution which is asymptotically flat spacetime.

### 4.1.3 \( \lambda = \text{tanh}(R), \quad \lambda = \text{const} \)

By this ansatz, again from the procedure mentioned in the previous subsections and setting \( Q = 2\lambda \), we have

\begin{equation}
h(R) = \frac{\sinh R}{\cosh R},
\end{equation}

\begin{equation}
f(R) = C_1 + C_2 R + C_3 \cosh R,
\end{equation}

and

\begin{equation}
R(r) = \arcsinh(C_4 r + C_5).
\end{equation}

By fixing \( C_5 = 0 \), for simplicity, we find that there is no curvature singularity at the origin \( r = 0 \). Moreover, the function \( h(R) \) is well defined for the whole range of coordinate \( r \). Finally, we obtain

\begin{equation}
M(r) = -\frac{CC_2^2 C_4^2}{C_2^3} \ln(C_2 + C_3 C_4 r) + \frac{CC_2^2 C_4^2}{C_2^3} \ln r - \frac{C}{2C_2 r^2} + \frac{CC_3 C_4}{C_2^2} \frac{1}{r} + C_6.
\end{equation}
Using (50), (51) and (52) in the equation of motion (23), we find

\[
N^2(r) = - \frac{1}{4C_2^2 r^2} \left[ 2C_2^2 C_3 C_4 r \ln \left( \frac{C_2}{C_4 r} + C_3 \right) (2C_2 + 3C_3 C_4) \right] + \frac{8 + 5C_2^2 r^2}{36C_4^2 r} \sqrt{1 + C_4^2 r^2} - \frac{3 + 2C_2^2 r^2}{12C_4^2} \arcsinh(C_4 r) + \frac{C_2}{4C_2^2} \frac{3C_2 C_4}{r^2} + \frac{2C_8}{r} - 2C_7,
\]

(53)
where \( C_7 \) and \( C_8 \) are integration constants. The horizons of black hole metric (56) are obtained by setting \( N^2(r) = 0 \), for the given constants in the shift function. Therefore, the spherical solutions (52) and (53) are capable of being as a black hole solution for \( f(R) \) gravity (50) subject to the Ricci scalar (51) with \( C_5 = 0 \). For the special case \( C_3 = 0 \), \( f(R) = C_1 + C_2 R \), we obtain

\[
N^2(r) = \frac{8 + 5C_2^2 r^2}{36C_4^2 r} \sqrt{1 + C_4^2 r^2} - \frac{3 + 2C_2^2 r^2}{12C_4^2} \arcsinh(C_4 r) + \frac{C_2}{4C_2^2} \frac{3C_2 C_4}{r^2} + 2C_8 \frac{1}{r} - 2C_7.
\]

(55)

These solutions are also new for \( f(R) = C_1 + C_2 R \) gravity, because of the first two new terms of \( N^2(r) \), in comparison to the solutions obtained by Noether symmetry in [21]. However, similar to the previous cases, the existence of first and second terms in (55), subject to \( C_4 > 0 \), does not allow for recovering the BTZ black hole, even if we set \( C_6 = C_8 = 0 \). Therefore, rather than BTZ black hole or generalized BTZ black hole solutions, the Hojman symmetry provides us with the new (2+1) dimensional \( f(R) \) gravity solution, in comparison to the solution which has already been obtained by Noether symmetry in [21].

The study of asymptotic structure at \( r \to \infty \) shows that the third, forth and fifth terms of \( N^2(r) \) can be ignored, the first term can behave as \( -\Lambda r^2 \equiv r^2 / l^2 \), and the second term represents an uncommon behaviour. Therefore this solution can partially exhibit an asymptotically anti-de Sitter spacetime, due to the first term.

However, the presence of second term including “\( \arcsinh(C_4 r) \)” may represent some other features of this solution which deserves attention. Actually, if a de Sitter term \( -\Lambda r^2 \equiv r^2 / l^2 \) dominates the other terms at \( r \to \infty \), it will give rise to an asymptotic anti-de Sitter spacetime with a cosmological constant having any given value \( \Lambda \sim -l^{-2} \), from \( \Lambda_{as} \to 0^- \) (\( l \to \infty \)) to \( \Lambda_{as} \to -\infty \) (\( l \to 0 \)). Similarly, if a de Sitter term \( \Lambda r^2 \equiv r^2 / l^2 \) dominates the other terms at \( r \to \infty \), it will give rise to an asymptotic de Sitter spacetime with a cosmological constant having any given value \( \Lambda \sim l^{-2} \), from \( \Lambda_{as} \to 0^+ \) (\( l \to \infty \)) to \( \Lambda_{as} \to \infty \) (\( l \to 0 \)). So, one may think that the second term at \( r \to \infty \) can be effectively considered as “\( -\Lambda(r)r^{2n} \)” where \( \Lambda(r) \equiv \arcsinh(C_4 r) \) plays the role of a positive cosmological constant with an asymptotic value \( \Lambda_{as} \to \infty \) at \( r \to \infty \), or as “\( \Lambda(r)r^{2n} \)” where \( \Lambda(r) \equiv -\arcsinh(C_4 r) \) plays the role of a negative cosmological constant with an asymptotic value \( \Lambda_{as} \to -\infty \) at \( r \to \infty \). Therefore, the solutions (51) and (55) exhibit an asymptotically de Sitter and anti-de Sitter spacetime with an effective asymptotic values of cosmological constant \( \Lambda + \Lambda_{as} > 0 \) and \( \Lambda + \Lambda_{as} < 0 \), respectively.

4.1.4 \( X = \lambda \coth(R), \quad \lambda = \text{const} \)

By this ansatz, and setting \( Q = 2\lambda \), we have

\[
h(R) = \frac{\cosh(R)}{\sinh(R)},
\]

(56)

\[
f(R) = C_1 + C_2 R + C_3 \sinh R,
\]

(57)
and

\[
R(r) = \arccosh \left( C_4 r + C_5 \right).
\]

(58)
By taking $C_5 = 0$, the function $h(R)$ is not well-defined at $C_4 r = 1$, hence the domain of $r$ is limited to $r > 1/C_4$ and the problem of curvature singularity at the origin $r = 0$ is completely removed. Then, we have

$$M(r) = -\frac{CC_2^2 C_4^2}{C_3^2} \ln(C_2 + C_3 C_4 r) + \frac{CC_2^2 C_4^2}{C_2^2} \ln r - \frac{C}{2C_2 r^2} + \frac{CC_2 C_4}{C_2} \frac{1}{r} + C_6. \quad (59)$$

Using (57), (58) and (59) in the equation of motion (28), we obtain

$$N^2(r) = -\frac{1}{4C_4^2 r^2} \left[ 2C_2^2 C_5 C_4 r \ln\left(\frac{C_2}{C_4 r} + C_3\right) (2C_2 + 3C_3 C_4 r) \right] + \frac{-8 + 5C_4^2 r^2}{36C_4^2 r^2} \sqrt{-1 + C_4^2 r^2}$$

$$+ \ln\left(C_4 r + \sqrt{C_4^2 r^2 - 1}\right) - \frac{r^2}{6} \arccosh(C_4 r) + \frac{C_2^2}{4C_2^2 r^2} + \frac{3C_2 C_5 C_4}{2C_2} \frac{1}{r} + 2C_8 \frac{1}{r} - 2C_7, \quad (60)$$

where $C_7$ and $C_8$ are integration constants. For the given constants in the shift function, we may set $N^2(r) = 0$ to find the horizons of the black hole metric (56). Therefore, the spherical solutions (59) and (60) are capable of being as a black hole solution for $f(R)$ gravity (57) subject to the Ricci scalar (58) with $C_5 = 0$. For the special case $C_3 = 0$, namely $f(R) = C_1 + C_2 R$, we obtain

$$M(r) = -\frac{C}{2C_2 r^2} + C_6, \quad (61)$$

$$N^2(r) = \frac{-8 + 5C_4^2 r^2}{36C_4^2 r^2} \sqrt{C_4^2 r^2 - 1} - \frac{r^2}{6} \arccosh(C_4 r) + \frac{\ln\left(C_4 r + \sqrt{C_4^2 r^2 - 1}\right)}{4C_4^2}$$

$$+ \frac{C_2^2}{4C_2^2 r^2} + 2C_8 \frac{1}{r} - 2C_7, \quad (62)$$

where the domain of $r$ is limited by the inequality $C_4 r \geq 1$. These solutions are also new for $f(R) = C_1 + C_2 R$ gravity, because of the first and second terms of $N^2(r)$, in comparison to the solutions obtained by Noether symmetry in [21]. Similar to the previous cases, the existence of first and second terms in (62), subject to $C_4 \neq 0$, does not allow for recovering the BTZ black hole, even if we set $C_6 = C_8 = 0$. Therefore, rather than BTZ black hole or generalized BTZ black hole solutions, the Hojman symmetry provides us with the new (2+1) dimensional $f(R)$ gravity solution, in comparison to the solution which has already been obtained by Noether symmetry in [21].

The study of asymptotic structure at $r \to \infty$ shows that the first term of $N^2(r)$ behaves as $-\Lambda r^2 \equiv r^2/2$, but the second and third terms represent uncommon behaviours. Therefore this solution can partially exhibit an asymptotically anti-de Sitter spacetime, due to the first term, though the presence of the second and third terms with uncommon asymptotic behaviours may represent some other features of this solution.

The presence of negatively signed second term with “$\arccosh(C_4 r)$” may be considered at $r \to \infty$ as $-\Lambda r^2 \equiv r^2/2$, but the second and third terms represent uncommon behaviours. Therefore this solution can partially exhibit an asymptotically anti-de Sitter spacetime, due to the first term, though the presence of the second and third terms with uncommon asymptotic behaviours may represent some other features of this solution.

The presence of negatively signed second term with “$r \to \infty$” may be considered at $r \to \infty$ as $-\Lambda r^2 \equiv r^2/2$, where $\Lambda(r) \equiv \arccosh(C_4 r)$ plays the role of an asymptotic positive cosmological constant with a value $\Lambda_{as} \to \infty$. The presence of positively signed third term including “$\ln(C_4 r)$” may be considered at $r \to \infty$ as $-\Lambda(r) r^2$, where $\Lambda'(r) \equiv -\ln(2C_4 r)/r^2$ plays the role of an effective asymptotic negative cosmological constant with a value $\Lambda'_{as} \to 0^-$. Therefore, the solutions (61) and (62) exhibit an asymptotically de Sitter spacetime with an effective asymptotic value of cosmological constant $\Lambda + \Lambda_{as} + \Lambda'_{as} > 0$.

### 4.1.5 $X = \lambda \sinh(R), \quad \lambda = \text{const}$

By this ansatz, again from the procedure mentioned at the previous subsections and setting $Q = 2\lambda$, we have

$$h(R) = -\frac{\cosh(R) - 1}{\sinh(R)}, \quad (63)$$

9
By using Eqs. (24) and (73) we have
\[ f(R) = C_1 + C_2R + C_3 \left[ -\ln(\tanh(\frac{1}{2}R) - 1) - \ln(\tanh(\frac{1}{2}R) + 1) \right], \]  
(64)

and
\[ R(r) = -2 \text{arctanh} \left( C_4r + C_5 \right). \]  
(65)

Finally, by fixing \( C_5 = 0 \) for simplicity, we obtain
\[ M(r) = -\frac{CC_3C_2^2}{C_2^3} \ln(-C_2 + C_3C_4r) + \frac{CC_3C_2^2}{C_2^3} \ln r - \frac{C}{2C_2^2} \frac{1}{r^2} - \frac{CC_3C_4}{C_2^2} \frac{1}{r} + C_6. \]  
(66)

Note that by choosing \( C_5 = 0 \), the requirement \( R(r) \neq 0 \) for a well-defined \( h(R) \), excludes \( r = 0 \) and the problem of curvature singularity at the origin is completely removed. Using (64), (65) and (66) in the equation of motion \( \Box f = 0 \), we obtain
\[ N^2(r) = -\frac{1}{4C_4^2r^2} \left[ 2C^2C_3C_4r \ln \left( -\frac{C_2}{C_4r} + C_3 \right) (-2C_2 + 3C_3C_4r) \right] - \frac{1}{3C_4^2} \ln \left( C_4^2r^2 - 1 \right) \frac{1}{r} + \frac{r^2}{3} \text{arctanh}(C_4r) - \frac{1}{C_4^2} \text{arccoth}(C_4r) + \frac{2}{3C_4^2} \ln \left( C_4^2r^2 - 1 \right) \frac{1}{r} + \frac{2C^2}{4C_2^2r^2} - \frac{C_3C_4}{C_2} \frac{1}{r} + 2C_8 \frac{1}{r} - 2C_7, \]  
(67)

where \( C_7 \) and \( C_8 \) are integration constants. For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric (65). Therefore, the spherical solutions (65) and (66) are capable of being as a black hole solution for \( f(R) \) gravity (64) subject to the Ricci scalar (65) with \( C_5 = 0 \). For the special case \( C_3 = 0 \), namely \( f(R) = C_1 + C_2R \), we obtain
\[ M(r) = -\frac{C}{2C_2^2} \frac{1}{r^2} + C_6, \]  
(68)

\[ N^2(r) = -\frac{1}{3C_4^2} \ln \left( C_4^2r^2 - 1 \right) \frac{1}{r} - \frac{1}{C_4^2} \text{arccoth}(C_4r) + \frac{r^2}{3} \text{arctanh}(C_4r) + \frac{2}{3C_4^2} \ln \left( C_4^2r^2 - 1 \right) \frac{1}{r} + \frac{2C^2}{4C_2^2r^2} + 2C_8 \frac{1}{r} - 2C_7. \]  
(69)

These solutions are not physical because the domains of “\( \ln \)”, “\( \text{arccoth} \)” are not compatible with the domain of “\( \text{arctanh} \)”, in the solution \( N^2(r) \).

4.2 \( X = X(R') \)

For the choice \( X = X(R') \), the equation of symmetry vector \( X \), (26), reads as Euler equation
\[ h_RX + R'^2h^2(R) \frac{d^2X}{dR'^2} - R'h_R \frac{dX}{dR'} = 0, \]  
(70)

from which \( X \) and \( h(R) \) are obtained respectively as follows
\[ X = A_1R' + A_2R'^n, \]  
(71)

\[ h(R) = -\frac{1}{nR + h_0}, \]  
(72)

where \( A_1, A_2, h_0 \) and \( n \) are constant parameters, and the Hojman conserved quantity reads as
\[ Q = 2h(R)R'^n - h(R)n(n + 1)R'^n. \]  
(73)

By using Eqs. (24) and (73) we have
\[ f(R) = C_1 + C_2 \left( R + \frac{h_0}{n} \right) + C_3 \left( R + \frac{h_0}{n} \right)^{\frac{2n-1}{n}}. \]  
(74)
and

\[ R(r) = Q_0 \frac{r^{-1}}{n} - \frac{h_0}{n}, \]

(75)

which \(Q_0 = -\frac{Q}{2^{-(n+1)} n} \). Note that by choosing \(C_4 = 0\), the requirement \(R(r) \neq -h_0/n\) for a well-defined \(h(R)\), excludes \(r = 0\) and the problem of curvature singularity at the origin is completely removed.

From equations (22) and (23) we have

\[ M(r) = -\frac{CC_3C_5^2}{C_2^3} \ln(C_2 + C_3C_5r) + \frac{CC_3C_5^2}{C_2^3} \ln(r) - \frac{C}{2C_2 r^2} + \frac{CC_3C_5}{C_2^3} \frac{1}{r} + C_6, \]

(76)

where \(C_5 = \frac{2n-1}{n} Q_0^{1/n} (n-1)\). Moreover, we may discard \(n = 1, -2\) cases which result in vanishing conserved charge \(Q\). Again, using equations (24), (75) and (23) we obtain

\[ N^2(r) = \frac{C^2}{4C_2^2 r^2} + (2C_7 + \frac{C^2 C_5^2 C_5}{2C_2^3}) \frac{1}{r} - \frac{3C_3^2 C_5^2 C_2^2}{2C_2^3} - 2C_8 - \frac{Q_0^{1/n}}{n(3n-2)(4n-3)} ((n-1)r)^{\frac{2n-2}{n-1}} \]

(77)

\[ -\frac{1}{4C_2^2 r^2} \left[ 2C_2^2 C_5 C_5 r \ln \left( \frac{C_2}{n-1}r + \frac{C_3 C_5}{n-1} (3C_3 C_5 r + 2C_2) \right) \right]. \]

For the given constants in the shift function, we may set \(N^2(r) = 0\) to find the horizons of the black hole metric (36). Therefore, the spherical solutions (76) and (77) are capable of being as a black hole solution for \(f(R)\) gravity (34) subject to the Ricci scalar (75).

For \(C_3 = 0\), namely \(f(R) = C_1 + C_2 (R + \frac{h_0}{n})\), we obtain

\[ M(r) = -\frac{C}{2C_2 r^2} + C_6, \]

(78)

\[ N^2(r) = \frac{C^2}{4C_2^2 r^2} - \frac{Q_0^{1/n}}{n(3n-2)(4n-3)} ((n-1)r)^{\frac{2n-2}{n-1}} + 2C_7 \frac{1}{r} - 2C_8. \]

(79)

These solutions are not capable of being either BTZ black hole or generalized BTZ black hole, with asymptotically anti-de Sitter spacetime, because the power \(\frac{2n-2}{n-1}\) in the second term cannot be equal to 2, unless for \(n = 0\) which itself diverges the second term. Therefore, rather than BTZ black hole or generalized BTZ black hole solutions, the Hojman symmetry provides us with the new (2+1) dimensional \(f(R)\) gravity solution, in comparison to the solution which has already been obtained by Noether symmetry in (21).

The asymptotic behaviour of this solution is determined by the second term including \(r^{\frac{2n-2}{n-1}}\) which depends on the value of \(n \neq 0\) as follows:

- For \(n = 2/3\) a constant term arises which contributes to the constant mass term \(-2C_8\),
- For \(n = 3/4\) a term “\(r^{-1}\)” arises which contributes to the third term,
- For \(n = 4/5\) a term “\(r^{-2}\)” arises which contributes to the first term.

For \(n = 1/2\) a novel linear “\(r^n\)” term arises which may describe the flatness of the galaxy rotation curves (30) and (31) as well as the non-trivial contribution of the characteristic feature of the surrounding quintessence field (32), (33).
4.3 $X(R, R') \sim R'g(R)$

We consider another ansatz $X(R, R') \sim R'g(R)$, where $g(R)$ is an arbitrary function. By this assumption, in order for $X$ to be the symmetry vector, we obtain

$$h(R) = \frac{g_{RR}}{g_R}.$$  \hfill (80)

Also, the Hojman conserved quantity is obtained as

$$Q_0 = R'g_R.$$  \hfill (81)

Now, by considering some ansatzs for $g(R)$, we find some exact solutions in the following.

4.3.1 $g(R) = \lambda e^{\alpha R}$

By this choice and using equation (24), (80) and (81), we obtain

$$h(R) = \alpha,$$  \hfill (82)

$$f(R) = C_1 + C_2 R + C_3 e^{\alpha R},$$  \hfill (83)

and

$$R(r) = \frac{1}{\alpha} \ln \left( \frac{Q_0(r + C_4)}{\lambda} \right),$$  \hfill (84)

where $\alpha$ is constant. By fixing $C_4 = 0$, for simplicity, it turns out that there is no curvature singularity at the origin $r = 0$. We have

$$M(r) = -\frac{C C_3^2 \alpha^2 Q_0^2}{\lambda^2 C_2^2} \ln \left( \frac{C_2 \lambda}{r} + C_3 \alpha Q_0 \right) - \frac{1}{2} \frac{C}{C_2} \frac{1}{r^2} + \frac{C C_3 \alpha Q_0}{\lambda C_2^2} \frac{1}{r} + C_5,$$  \hfill (85)

and

$$N^2(r) = -\frac{1}{4C_2^2 \lambda^2 r^2} \left[ 2\alpha Q_0 C^2 C_3 r \ln \left( \frac{C_2 \lambda}{Q_0 r} + C_3 \alpha \right) (2C_2 \lambda + 3C_3 \alpha Q_0 r) \right] + \frac{r^2}{6 \alpha} \ln \left( \frac{Q_0 r}{\lambda} \right) + \frac{5}{36 \alpha} r^2 + \frac{3Q_0 C^2 C_3 \alpha}{2C_2^2} \frac{1}{r} + \frac{C^2}{4C_2^2} \frac{1}{r^2} + 2C_7 \frac{1}{r} - 2C_6.$$  \hfill (86)

For the given constants in the shift function, we may set $N^2(r) = 0$ to find the horizons of the black hole metric (36). Therefore, the spherical solutions (85) and (86) are capable of being as a black hole solution for $f(R)$ gravity (83) subject to the Ricci scalar (84).

For $C_3 = 0$, namely $f(R) = C_1 + C_2 R$, we obtain

$$M(r) = -\frac{1}{2} \frac{C}{C_2} \frac{1}{r^2} + C_5,$$  \hfill (87)

$$N^2(r) = -\frac{r^2}{6 \alpha} \ln \left( \frac{Q_0 r}{\lambda} \right) + \frac{C^2}{4C_2^2} \frac{1}{r^2} + 2C_7 \frac{1}{r} + \frac{5}{36 \alpha} r^2 - 2C_6.$$  \hfill (88)

The above solutions are new, because of the extra first term $r^2 \ln \left( \frac{Q_0 r}{\lambda} \right)$, in comparison to the solutions obtained by Noether symmetry in [21], for $f(R) = C_1 + C_2 R$ gravity. Similar to the previous cases, the existence of “ln” term in (88) does not allow for recovering the BTZ black hole, even if we set $C_5 = C_7 = 0$. Therefore, rather than BTZ black hole or generalized BTZ black hole solutions, the Hojman symmetry provides us with the new (2+1) dimensional $f(R)$ gravity solution, in comparison to the solution which has already been obtained by Noether symmetry in [21].
The study of asymptotic structure at \( r \to \infty \) shows that the forth term of \( N^2(r) \) behaves as \(-\Lambda r^2 \equiv r^2/\ell^2\), but the first term represents an uncommon asymptotic behaviour. Therefore, the existence of the forth term (assuming \( \alpha > 0 \)) can partially exhibit an asymptotically anti-de Sitter spacetime.

The presence of first term with \( \alpha > 0 \) may be considered at \( r \to \infty \) as \(-\Lambda(r) r^{2\alpha}\) where \( \Lambda(r) \equiv (1/6\alpha)\ln (Q_0 r/\lambda) \) plays the role of an effective asymptotic positive cosmological constant with a behaviour \( \Lambda_{as} \to \infty \). Therefore, the solutions (87) and (88) exhibit an asymptotically de Sitter spacetime with an effective value of asymptotic cosmological constant \( \Lambda + \Lambda_{as} > 0 \).

\[ 4.3.2 \] \( g(R) = \frac{(g_0 + R)^{1+\alpha}}{1+\alpha} \)

Here \( g_0 \) and \( \alpha \) are constants parameters. Again, from Eqs. (80), (81), (22) and (23) we find respectively \( f(R) \), \( R(r) \), \( M(r) \) and \( N^2(r) \) for the case \( C_4 = 0 \), as follows

\[ h(R) = \frac{\alpha}{g_0 + R}, \quad (89) \]

\[ f(R) = C_1 + C_2(g_0 + R) + C_3(g_0 + R)^{\alpha+2}, \quad (90) \]

\[ R(r) = [Q_0(1 + \alpha) r]^{1+\alpha} + \frac{1}{g_0} - g_0. \quad (91) \]

Note that the requirement \( R(r) \neq -g_0 \) for a well-defined \( h(R) \) excludes \( r = 0 \), and the problem of curvature singularity at the origin is completely removed. We have

\[ M(r) = -\frac{CC_3 C_5}{C_2^2} \ln \left( \frac{C_2}{r} + C_3 C_5 \right) - \frac{1}{2} C \left( \frac{1}{C_2} - \frac{C_3 C_5}{C_2^2} + \frac{C_6}{r} \right), \quad (92) \]

and

\[ N^2(r) = -\frac{1}{4C_2 r^2} \left[ 2C^2 C_3 C_5 r \ln \left( \frac{C_2}{Q_0(1 + \alpha) r} + \frac{C_3 C_5}{Q_0(1 + \alpha)} \right) \left( 2C_2 + 3C_3 C_5 r \right) \right] \]

\[ -\frac{Q_0^{1+\alpha}}{(2\alpha + 3)(3\alpha + 4)} \left[ (1 + \alpha) r^{\frac{3+2\alpha}{1+\alpha}} + \frac{g_0}{6} r^2 + \frac{C^2}{4C_2^2} \left( \frac{1}{C_2} + \frac{C_2 C_5}{2C_2^2} \right) \right] \]

\[ -2C_7 - \frac{3C_2 C_3 C_5}{2C_2^2}, \quad (93) \]

where \( C_5 = (\alpha + 1)(\alpha + 2)Q_0 \). For the given constants in the shift function, we may set \( N^2(r) = 0 \) to find the horizons of the black hole metric (94). Therefore, the spherical solutions (92) and (93) are capable of being as a black hole solution for \( f(R) \) gravity (90) subject to the Ricci scalar (91).

For the case \( C_3 = 0 \), namely namely \( f(R) = C_1 + C_2 R \), we obtain

\[ M(r) = -\frac{1}{2} C \left( \frac{1}{C_2} + C_6 \right), \quad (94) \]

\[ N^2(r) = -\frac{Q_0^{1+\alpha}}{(2\alpha + 3)(3\alpha + 4)} \left[ (\alpha + 1) r^{\frac{2\alpha+3}{1+\alpha}} + \frac{C^2}{4C_2^2} \left( \frac{1}{C_2} + \frac{C_6}{r} \right) \right] \]

\[ + \frac{g_0}{6} r^2 - 2C_7. \quad (95) \]

These solutions are considered as new, because of the first term, in comparison to the solutions obtained by Noether symmetry in [21] for \( f(R) = C_1 + C_2 R \) gravity. These solutions are not capable of being either BTZ black hole or generalized BTZ black hole because the power \( \frac{2\alpha+3}{1+\alpha} \) in the first term cannot be equal to \( 2 \). Therefore, rather than BTZ black hole or generalized BTZ black hole solutions, the Hojman symmetry provides us with the new (2+1) dimensional \( f(R) \) gravity solution, in comparison to the solution which has already been obtained by Noether symmetry in [21].
The study of asymptotic structure at $r \to \infty$ shows that the existence of the fourth $r^2$ term in $N^2(r)$ can partially exhibit asymptotically de Sitter and anti-de Sitter spacetimes according to $g_0 < 0$ and $g_0 > 0$, respectively. The first term of $N^2(r)$ represents an $r$ dependent behaviour which is determined by the chosen value of $\alpha$, some examples of which are given as follows:

- For $\alpha = -3/2$ a constant term arises which contributes to the constant mass term $-2C_7$,
- For $\alpha = -4/3$ a term $\cdot r^{-1}\cdot$ arises which contributes to the third term,
- For $\alpha = -5/3$ a term $\cdot r^{-2}\cdot$ arises which contributes to the second term.

These three cases are in agreement with the asymptotic behaviour exhibiting an asymptotically de Sitter or anti-de Sitter spacetime. Note that no value of $\alpha$ can give rise to a term $\cdot r^{-2}\cdot$ to contribute to the fourth $r^2$ term.

For $\alpha = -2$ a novel linear “$r$” term arises which may describe the flatness of the galaxy rotation curves [30] and [31] as well as the non-trivial contribution of the characteristic feature of the surrounding quintessence field [32]-[34].

4.3.3 $g(R) = \lambda \ln R$

For this ansatz we have

\[ h(R) = -\frac{1}{R}, \quad \text{(96)} \]

\[ f(R) = C_1 + C_2 R + C_3 R \ln R, \quad \text{(97)} \]

\[ R(r) = C_4 e^{\frac{Q_0 r}{\lambda}}. \quad \text{(98)} \]

The requirement $R(r) \neq 0$ for a well-defined $h(R)$, does not impose any restriction on the domain of $r$. Also, there is no curvature singularity at the origin $r = 0$. We have

\[ M(r) = -\frac{CC_2^2 Q^2_0}{\lambda^2 C^2_5} \ln\left(\frac{C_5 \lambda}{r} + C_3 Q_0\right) - \frac{C}{2C_5^2 r^2} + \frac{CC_2^2 Q_0}{\lambda C^2_5} \ln \left(\frac{C_5 \lambda}{r} + C_3 Q_0\right) + C_6, \quad \text{(99)} \]

where $C_5 = C_2 + C_3(1 + \ln C_4)$, and

\[ N^2(r) = -\frac{1}{4C_5^4 \lambda^2 r^2} \left[ 2Q_0 C^2_5 C_3 \ln \left(\frac{C_5 \lambda}{Q_0 r} + C_3\right) (3C_3 Q_0 r + 2C_5 \lambda) \right] - \frac{C_4 \lambda^2 e^{\frac{Q_0 r}{\lambda}}}{Q^2_0} \left( Q_0 - \frac{2\lambda}{r} \right) \quad \text{(100)} \]

For the given constants in the shift function, we may set $N^2(r) = 0$ to find the horizons of the black hole metric [36]. Therefore, the spherical solutions [99] and [100] are capable of being as a black hole solution for $f(R)$ gravity [97] subject to the Ricci scalar [98].

For the case $C_3 = 0$, namely namely $f(R) = C_1 + C_2 R$, we obtain

\[ M(r) = -\frac{C}{2C^2_5} \frac{1}{r^2}, \quad \text{(101)} \]

\[ N^2(r) = -\frac{C_4 \lambda^2}{Q^2_0} \frac{(-2\lambda + Q_0 r) e^{\frac{Q_0 r}{\lambda}}}{r} + \frac{C^2}{4C^2_5} \frac{1}{r^2} + 2C_7 \frac{1}{r} - 2C_8. \quad \text{(102)} \]

These solutions are considered as new, because of the first term, in comparison to the solutions obtained by Noether symmetry in [21] for $f(R) = C_1 + C_2 R$ gravity. These solutions are not capable of
being either BTZ black hole or generalized BTZ black hole, due to the absence of \( r^2 \) term. Therefore, the Hojman symmetry provides us with the new \((2+1)\) dimensional \(f(R)\) gravity solution.

The first term of \( N^2(r) \) represents an \( r \) dependent behaviour and is determined by the chosen values of \( C_4, \lambda, Q_0 \) which may lead this term to be positive or negative. A positive term may be considered as \( \Lambda'_{(r)} r^2 \) where \( \Lambda'(r) \equiv e^{\frac{2\lambda}{r^2}} \) plays the role of an effective asymptotic negative cosmological constant with a value \( \Lambda'_a \rightarrow -\infty \). A negative term may be considered as \( \Lambda'(r) r^2 \) where \( \Lambda'(r) \equiv e^{\frac{-2\lambda}{r^2}} \) plays the role of an effective asymptotic positive cosmological constant with a value \( \Lambda'_a \rightarrow \infty \). Therefore, the solutions (101) and (102) may exhibit asymptotically de Sitter and anti-de Sitter spacetimes, with an effective value of asymptotic cosmological constant \( \Lambda'_a > 0 \) and \( \Lambda'_a < 0 \), respectively.

5 Conclusions

We have obtained \((2+1)\)-dimensional spherically symmetric solutions in the context of \((2+1)\)-dimensional \(f(R)\) gravity by using the Hojman symmetry. These solutions are new in comparison to those obtained by Noether symmetry approach [21]. In the special case of Hojman symmetry along \( X = R \), these solutions cast in the form of \((2+1)\) dimensional BTZ black hole and generalized \((2+1)\) dimensional BTZ black holes which have already been obtained by Noether symmetry approach [21].

The interesting point of Hojman symmetry approach is that the cosmological constant is appeared as the direct manifestation of Hojman symmetry. We aim to study the T-Duality problem [35] of the obtained \((2+1)\) dimensional BTZ black hole solutions, in the next future.

References

[1] D. Ida, No Black Hole Theorem in Three-Dimensional Gravity, Phys. Rev. Lett. 85 (2000) 3758-3760, arXiv: gr-qc/0005129v2, doi:10.1103/PhysRevLett.85.3758.

[2] M. Banados, C. Teitelboim and J. Zanelli, The Black Hole in three- dimensional space- time, Phys. Rev. Lett. 69 (1992) 1849-1851, arXiv: hep-th/9204099v3, doi:10.1103/PhysRevLett.69.1849.

[3] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Geometry of the 2+1 Black Hole, Phys. Rev. D. 48 (1993) 1506-1525, arXiv: gr-qc/9302012v1, doi:10.1103/PhysRevD.48.1506.

[4] C. Romero and F. Dahia, Theories of gravity in 2+1 dimensions, Int. J. Theor. Phys. 33 (1994) 2091-2098, doi:10.1007/BF00675174.

[5] S. Islam, P. Kumar, G. S. Khadekar and T. K. Das, \((2+1)\) dimensional cosmological models in \(f(R,T)\) gravity with \( \Lambda(R,T) \), Journal of Physics: Conference Series 1258 (2019) 012026, arXiv: gen-ph/2003.11355, doi:10.1088/1742-6596/1258/1/012026.

[6] C. H. Brans, R. H. Dicke, Mach’s Principle and a Relativistic Theory of Gravitation, Physical Review 124(3) (1961) 925-935, doi:10.1103/PhysRev.124.925.

[7] Y. Fujii and K-i, Maeda, Book review: The Scalar-Tensor Theory of Gravitation, Class. Quantum Grav. 20 (2003) 4503, doi:10.1088/0264-9381/20/20/061.

[8] M. Demianski, R. de Ritis, C. Rubano and P. Scudellaro, Scalar fields and anisotropy in cosmological models, Phys. Rev. D 46(4) (1992) 1391-1398, doi:10.1103/PhysRevD.46.1391.

[9] H. Wei, X.-J. Guo, L.-F. Wang, Noether symmetry in \(f(T)\) theory, Phys. Lett. B 707 (2012) 298-304, arXiv: gr-qc/1112.2270v3, doi:10.1016/j.physletb.2011.12.039.

[10] K. Atazadeh, F. Darabi, \(f(T)\) cosmology via Noether symmetry, Eur. Phys. J. C 72(2016) (2012) 1-7, arXiv: physics.gen-ph/1112.2824v3, doi:10.1140/epjc/s10052-012-2016-z.
[11] D. Lovelock, The Einstein tensor and its generalizations, J. Math. Phys. 12 (1971) 498-501, doi:10.1063/1.1665613.

[12] S. Capozziello, A. De Felice, f(R) cosmology from Noether's symmetry, JCAP 2008(016) (2008) 1-32, arXiv: gr-qc/0804.2163v3, doi:10.1088/1475-7516/2008/08/016.

[13] S. Capozziello, N. Frusciante, D. Vernieri, New Spherically Symmetric Solutions in f(R)-gravity by Noether Symmetries, arXiv:1204.4650.

[14] S. Nojiri and S. D. Odintsov, Introduction to modified gravity and gravitational alternatives for Dark energy, Int. J. Geom. Meth. Mod. Phys. 4(1) (2007) 115-145, arXiv: hep-th/0601213, doi:10.1142/S0219887807001928.

[15] S. Nojiri and S. D. Odintsov, Unified cosmic history in modified gravity: from f(R) theory to Lorentz non-invariant models, Phys. Rep. 505 (2011) 59-144, arXiv: gr-qc/1011.0544, doi:10.1016/j.physrep.2011.04.001.

[16] B. Vakili, Noether symmetric f(R) quantum cosmology and its classical correlations, Phys. Lett. B 669 (2008) 206-211, arXiv:gr-qc/0809.4591, doi:10.1016/j.physletb.2008.09.058.

[17] T. P. Sotiriou and V. Faraoni, f(R) theories of gravity, Rev. Modern. Phys. 82 (2010) 451-497, arXiv: gr-qc/0805.1726v4, doi:10.1103/RevModPhys.82.451.

[18] A. De Felice and S. Tsujikawa, f(R) theories, Living Reviews in Relativity 13 (2010) 3, doi:10.12942/lrr-2010-3.

[19] S. Tsujikawa, Modified gravity models of dark energy, Lectures on Cosmology 800 (2010) 99-145, arXiv: gr-qc/1101.0191v1, doi:10.1007/978-3-642-10598-2-3.

[20] S. Capozziello and M. De Laurentis, Extended theories of gravity, Phys. Rep. 509 (2011) 167-321, arXiv: gr-qc/1108.6266v2, doi:10.1016/j.physrep.2011.09.003.

[21] F. Darabi, K. Atazadeh, A. Rezaei-Aghdam, Generalized (2+1) dimensional black hole by Noether symmetry, Eur. Phys. J. C. 73(2657) (2013) 1-7, arXiv: gr-qc/1304.2926, doi:10.1140/epjc/s10052-013-2657-6.

[22] S. A. Hojman, A new conservation law constructed without using either Lagrangians or Hamiltonians, J. Phys. A: Math. Gen 25 (1992) L291-L295, doi:10.1088/0305-4470/25/7/002.

[23] S. Capozziello and M. Roshan, Exact cosmological solutions from Hojman conservation quantities, Phys. Lett. B 726 (2013) 471-480, arXiv: gr-qc/1308.3910v1, doi:10.1016/j.physletb.2013.08.047.

[24] M. Paolella and S. Capozziello, Hojman symmetry approach for scalar-tensor cosmology, Phys. Lett. A 379 (2015) 1304-1308, arXiv: gr-qc/1503.00098v1, doi:10.1016/j.physleta.2015.02.044.

[25] A. Paliathanasis, P. G. L. Leach. Comment on the Hojman conservation quantities in Cosmology, arXiv:1503.08466.

[26] Hao Wei, Ya-Nan Zhou, Hong-Yu Li, Xiao-Bo Zou. Hojman symmetry in f(T) theory, Astrophys. Space Sci. 360(6) (2015) 1-7, arXiv: gr-qc/1505.07546v3, doi:10.1007/s10509-015-2518-x.

[27] I. A. Bizyaev, A. V. Borisov, I. S. Mamaev, The Hojman construction and Hamiltonization of non-holonomic systems, SIGMA 12 (2016) 012-030, arXiv: 1510.00181, doi:10.3842/SIGMA.2016.012.

[28] Hao Wei, Hong-Yu Li, Xiao-Bo Zou, Exact cosmological solutions of f(R) theories via Hojman symmetry, Nucl. Phys. B 903 (2016) 132-149, arXiv: gr-qc/1511.00376, doi:10.1016/j.nuclphysb.2015.12.006.
[29] A. Myrzakul, R. Myrzakulov, On the Hojman conservation quantities in FRW Cosmology, arXiv:1603.01611.

[30] P. D. Mannheim and D. Kazanas, Exact vacuum solution to conformal Weyl gravity and galactic rotation curves, Astrophysical Journal, 342 (1989) 635-638.

[31] M. Gürses, Y. Heydarzade, Ç. Sentürk, NAT black holes, Eur. Phys. J. C. 79(942) (2019) 1-14, arXiv: gr-qc/1907.09584, doi:10.1140/epjc/s10052-019-7455-3.

[32] Y. Heydarzade, F. Darabi, Surrounded Vaidya black holes: apparent horizon properties, Eur. Phys. J. C 78(342) (2018) 1-18, arXiv: gr-qc/1805.01022, doi:10.1140/epjc/s10052-018-5842-9.

[33] Y. Heydarzade, F. Darabi, Surrounded Vaidya solution by cosmological fields, Eur. Phys. J. C 78(582) (2018) 1-27, arXiv: gr-qc/1710.04485, doi:10.1140/epjc/s10052-018-6041-4.

[34] Y. Heydarzade, H. Hadi, C. Corda, F. Darabi, Braneworld black holes and entropy bounds, Phys. Lett. B 776 (2018) 457-463, arXiv: gr-qc/1706.04434, doi:10.1016/j.physletb.2017.11.061.

[35] A. Eghbali, L. Mehran-nia and A. Rezaei-Aghdam, BTZ black hole from Poisson-Lie T-dualizable sigma models with spectators, Phys. Lett. B 772 (2017) 791-799, arXiv: hep-th/1705.00458, doi:10.1016/j.physletb.2017.07.044.