Thermal Operator Representation of Finite-Temperature Amplitudes in the Presence of Chemical Potential

M. Inui*, H. Kohyama† and A. Niégawa‡

Graduate School of Science, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, JAPAN

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Abstract

In a recent paper [Phys. Rev. D 72, 085006 (2005)], Brandt et al. deduced the thermal operator representation for a thermal N-point amplitude, both in the imaginary-time and real-time formalisms. In the case when a chemical potential present, however, the representation is not as simple as in the case with vanishing chemical potential. We propose a much simpler and transparent representation for the case of non-zero chemical potential.

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*inui@sci.osaka-cu.ac.jp
†kohyama@sci.osaka-cu.ac.jp
‡niegawa@sci.osaka-cu.ac.jp
In a recent paper [1] (see also [2–4]), through introducing a thermal operator, it has been shown that any thermal amplitude in the imaginary-time as well as the closed-time path formalisms of real-time thermal field theory is written in terms of its vacuum-theory counterpart, which is referred to as the thermal operator representation. This is proved using the mixed space representation \((t, p)\) with \(t\) the time. According to the Feynman rules, a thermal \(N\)-point amplitude is computed through following procedure. (1) Draw relevant Feynman graphs. (2) For each of them make a product of propagators and vertices (in a \((t, p)\)-space) that constitute the graph and perform integrations over the internal times, which we write \(\gamma^{(T)}_N\). Then (3) perform integrations over the loop momenta. An important finding in [1] is that \(\gamma^{(T)}_N\) can be factorized as \(\gamma^{(T)}_N = O^{(T)} \gamma^{(T=0)}_N\), where \(O^{(T)}\), being independent of times, is a thermal operator and \(\gamma^{(T=0)}_N\) is the vacuum-theory counterpart of \(\gamma^{(T)}_N\). It should be noted that, in the case of the imaginary-time formalism, each internal-time \((\tau)\) integration in \(\gamma^{(T)}_N\) is carried out over \(\int_{0}^{1/T} d\tau\), while in \(\gamma^{(T=0)}_N\), each integration goes over \(\int_{-\infty}^{\infty} d\tau\). In the case of the closed-time formalism, each internal time \((t)\) is integrated over \(\int_{-\infty}^{\infty} dt\), both in \(\gamma^{(T)}_N\) and \(\gamma^{(T=0)}_N\). Since \(O^{(T)}\) does not depend on the time coordinates, integrations over the internal times can be performed within \(\gamma^{(T=0)}_N\). Thus, a beautiful factorization of the thermal amplitude into a \(T\)-dependent part (through \(O^{(T)}\)) and a zero-temperature part (through \(\gamma^{(T=0)}_N\)) is established.

In [1] a generalization of this result to the case where a chemical potential \(\mu\) exists is attempted. It is found that \(O^{(T)}\) involves derivatives with respect to the time coordinates, so that the above beautiful factorization property is spoiled.

In this note we present a thermal operator representation for a scalar \(2N\)-point amplitude at finite temperature and charge density \((T \neq 0 \neq \mu)\), which is much simpler than the one in [1] and enjoys the quasi factorizability.
Imaginary-time formalism

In the presence of the chemical potential \( \mu \), being conjugate to the conserved charge, the complex-scalar propagator in the imaginary-time formalism reads

\[
\Delta^{(T,\mu)}(\tau, E) = \frac{1}{2E} \left[ \theta(\tau) \left\{ (1 + n_-)e^{-(E-\mu)\tau} + n_+e^{(E+\mu)\tau} \right\} \right. \\
+ \left. \theta(-\tau) \left\{ n_-e^{-(E-\mu)\tau} + (1 + n_+)e^{(E+\mu)\tau} \right\} \right], \\
\quad (-1/T \leq \tau \leq +1/T),
\]

(1)

\[
\Delta^{(T=0=\mu)}(\tau, E) = \frac{1}{2E} \left[ \theta(\tau)e^{-E-\tau} + \theta(-\tau)e^{E+\tau} \right]_{E_\pm=E} \\
\quad (-\infty < \tau < +\infty),
\]

(2)

where \( \tau \) is the Euclidean time, \( T \) is the temperature, and

\[ n_\pm = n(E \pm \mu), \]

with \( n(x) = 1/(e^{x/T} - 1) \). For later convenience, in Eq. (2), we have introduced \( E_\pm \). Now, we introduce the operators \( R(E), N, \) and \( S(\mu) \): \( R(E) \) is a “twisted” reflection operator that changes \( E_\pm \rightarrow -E_\mp \) (namely, it gives a term with \( E_\pm \rightarrow -E_\mp \)), \( N \) is an operator that acts as \( N e^{\pm E_\pm \tau} = n(E_\pm)e^{\pm E_\pm \tau} \), and \( S(\mu) \) is a translation operator that changes \( E_\pm \rightarrow E \pm \mu \).

In contrast to \( S(E) \) in [1], \( R(E) \) here does not act on the factor \( 1/(2E) \) in Eq. (2). Then, one can easily find the following representation (cf. Eqs. (1) and (2));

\[
\Delta^{(T,\mu)} = S(\mu) [1 + N(1 + R(E))] \Delta^{(T=0=\mu)}, \\
\equiv \mathcal{O}^{(T,\mu)}(E, \mu) \Delta^{(T=0=\mu)}.
\]

(3)

Let \( A^{(2N)} \) be a complex-scalar \( 2N \)-point amplitude. Following the same procedure as in [1], we arrive at

\[
A^{(2N)} = \int \prod_{i=1}^{I} \frac{d^3k_i}{(2\pi)^3} \prod_{v=1}^{V} (2\pi)^3 \delta_v(k, \mathbf{p}) \gamma^{(T,\mu)}_{2N}(\mathbf{k}, \mathbf{p})
\]

\]
\[
= \int \prod_{i=1}^{I} \frac{d^3 k_i}{(2\pi)^3} \prod_{v=1}^{V} (2\pi)^3 \delta_v(k, p) \mathcal{O}^{(T,\mu)} \gamma^{(T=0=\mu)}_{2N},
\]

(4)

where, with obvious notation,

\[
\mathcal{O}^{(T,\mu)} = \prod_{i=1}^{I} \mathcal{O}_i^{(T,\mu)}(E_i, \mu) \\
\equiv \prod_{i=1}^{I} S_i(\mu) \left[ 1 + N_i(1 + R(E_i)) \right].
\]

Here \( I \) (\( V \)) is the total number of internal propagators (vertices) in the graph under consideration. The internal and external three momenta are denoted generically by \((k, p)\), respectively, and \(\delta_v(k, p)\) enforces the three-momentum conservation at the vertex \(v\). As mentioned at the beginning, the range of the internal-time integration in \(\gamma^{(T,\mu)}_{2N}\) is \((0, 1/T)\), while the range of the integration in \(\gamma^{(T=0=\mu)}_{2N}\) is \((-\infty, +\infty)\).

Each term \(F \in \gamma^{(T=0=\mu)}_{2N}\) includes \(E_{i+}\) or \(E_{i-}\); \(F(\ldots, E_{i+}, \ldots)\) or \(F(\ldots, E_{i-}, \ldots)\). From the above construction, actions of \(\mathcal{O}_i^{(T,\mu)}\) on \(F\) are unambiguously defined:

\[
(1 + N_i)F(\ldots, E_{i\pm}, \ldots) = (1 + n(E_{i\pm}))F(\ldots, E_{i\pm}, \ldots),
\]

\[
N_i R(E_i) F(\ldots, E_{i\pm}, \ldots) = n(E_{i\mp})F(\ldots, -E_{i\mp}, \ldots). \tag{5}
\]

The heart of the proof of Eq. (4) resides in the proposition:

The function \(G\) defined by

\[
G \equiv \int_{-\infty}^{+\infty} d\tau \left( \theta(-\tau) + \theta(\tau - 1/T) \right) \\
\times \prod_{i=1}^{N} \Delta^{(T=0=\mu)}(\tau - \tau_i, E_i) \\
\times \prod_{j=N+1}^{2N} \Delta^{(T=0=\mu)}(\tau_j - \tau, E_j);
\]

which appears in complex-scalar \(\lambda(\phi^i \phi)^N\) theory, is annihilated by \(\mathcal{O}^{(T,\mu)}\), \(\mathcal{O}^{(T,\mu)} G = 0\).

Proof: Substituting the expression (2), we obtain
\( G = \prod_{i=1}^{2N} \frac{1}{2E_i} \left( \frac{\prod_{i=1}^{N} e^{-E_i + n} \prod_{j=N+1}^{2N} e^{-E_j - \tau_j}}{\sum_{i=1}^{N} E_i + \sum_{j=N+1}^{2N} E_j} \right) \). 

Straightforward manipulation using the rules (5) and the identity \( n(E_{\pm})e^{E_{\pm}/T} = 1 + n(E_{\pm}) \) shows that

\[
O(T,\mu)G = \prod_{i=1}^{N} O_i^{(T,\mu)}(E_i,\mu)G = 0 .
\]

(Q.E.D.)

Using this proposition, the “general proof” in [1] goes as it is in the present case, and Eq. (4) is proved.

As seen above in conjunction with Eq. (5), actions of \( O^{(T,\mu)} \) on the “\( E_+ \) sectors” and “\( E_- \) sectors” of \( \gamma^{(T=0=\mu)}_{2N} \) are asymmetric and, in contrast to the case of vanishing chemical potential, Eq. (4) does not achieve the “complete factorization”. Nevertheless, as seen above, computational procedure on the basis of Eq. (4) of a thermal amplitude from its zero-temperature counterpart is unambiguously defined.

**Closed-time-path formalism**

The quasi factorizability established above for the imaginary-time formalism holds also in the closed-time-path formalism of real-time thermal field theory. The propagator enjoys \((2 \times 2)\)-matrix structure, whose \((i, j)\)-element reads

\[
\Delta^{(T,\mu)}_{11(22)}(t, E) = \frac{1}{2E} \left[ \theta(\pm t)e^{-i(E-\mu \mp i\epsilon)t} + \theta(\mp t)e^{i(E+\mu \mp i\epsilon)t} + n_-e^{-i(E-\mu)t} + n_+e^{i(E+\mu)t} \right],
\]

\[
\Delta^{(T,\mu)}_{12}(t, E) = \frac{1}{2E} \left[ n_-e^{-i(E-\mu)t} + (1 + n_+)e^{i(E+\mu)t} \right],
\]

\[
\Delta^{(T,\mu)}_{21}(t, E) = \frac{1}{2E} \left[ (1 + n_-)e^{-i(E-\mu)t} + n_+e^{i(E+\mu)t} \right].
\]

The suffices ‘1’ and ‘2’ are called thermal indices. The vacuum-theory counterparts are...
Δ^{(T=0=μ)}_{11(22)}(t, E) = \frac{1}{2E} \left[ \theta(±t)e^{-(E_±±i\epsilon)t} + \theta(±t)e^{i(E_±±i\epsilon)t} \right]_{E_±=E},

Δ^{(T=0=μ)}_{12(21)}(t, E) = \frac{1}{2E} e^{±iE±t} \bigg|_{E_±=E}.

We modify the definition of \( N \) as

\[ N e^{-i(E_±±i\epsilon)t} = n(E_±)e^{-(E_±±i\epsilon)t}, \]

\[ N e^{i(E_±±i\epsilon)t} = n(E_±)e^{i(E_±±i\epsilon)t}. \]

Then we have

\[ \Delta^{(T,μ)}_{ij}(t, E) = O^{(T,μ)}(E, μ)\Delta^{(T=0=μ)}_{ij}(t, E). \]

Here \( O^{(T,μ)} \) is as in Eq. (3) with the above modification for \( N \). This quasi factorizability of the propagator straightforwardly leads, in much the same manner as in [1], to the thermal operator representation for the contribution to the amplitude from any graph.

Comment on the general real-time contour

Finally, we make a comment on the formalism constructed employing a general real-time contour, \(-∞ → +∞ → +∞ − iσ/T → −∞ − iσ/T → −∞ − i/T (0 ≤ σ ≤ 1),\) in a complex time plane. Let \( A^{(σ)}_{i_1i_2...i_{2N}}(p_1, ..., p_N; p_{N+1}, ..., p_{2N}) \) be a \( 2N \)-point amplitude in an energy-momentum space computed in this formalism, where the suffices \( i \)'s, each of which is 1 or 2, are the thermal indices. The closed-time-path counterpart is \( A^{(σ=0)}_{i_1i_2...i_{2N}}(p_1, ..., p_N; p_{N+1}, ..., p_{2N}) \). Here all four-momenta are defined to be incoming. It can easily be shown that

\[ A^{(σ)}_{i_1i_2...i_{2N}}(p_1, ..., p_N; p_{N+1}, ..., p_{2N}) = e^σ \sum_j p_j^0/T A^{(σ=0)}_{i_1i_2...i_{2N}}(p_1, ..., p_N; p_{N+1}, ..., p_{2N}), \]

(6)
where the summations are taken over all “type 2” external vertices and $p_j^0$ is the 0th component of $p_j$. Then, $A^{(\sigma)}$ can always be directly obtained from $A^{(\sigma=0)}$ and separate analysis of $A^{(\sigma\neq0)}$ is practically not necessary. Incidentally, from Eq. (6), we see that $A^{(\sigma)}_{11\ldots1} = A^{(\sigma=0)}_{11\ldots1}$, as it should be. Using the energy conservation, $\sum_{i=1}^{2N} p_i^0 = 0$, we have $A^{(\sigma)}_{22\ldots2} = A^{(\sigma=0)}_{22\ldots2}$, as also it should be.

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