Mathematical models for inertial torques acting on a spinning ring

Abstract

The main component of the gyroscopic devices is the spinning rotor, which design can be different and represented by the disc, cylinder, ring, etc. forms. These rotating objects manifest gyroscopic effects which action is increased proportionally with the intensification of the angular velocity of their spinning. Recent publications demonstrate the applied torque on gyroscopic devices generates internal resistance and precession torques based on the action of the centrifugal, common inertial, and Coriolis forces, as well as the change in the angular momentum. This internal inertial torques act simultaneously and interdependently around axes of gyroscopic devices. This paper presents mathematical models for the internal inertial torques generated by the mass elements and centre mass of the spinning ring. These models enable for the computing the forces acting on the supports of the spinning ring and describe the motions of the gyroscopic devices in space. The content of the manuscript presents novelty for machine dynamics and engineering.

Keywords: scope theory, property, torque, ring, centrifugal

Introduction

Since the Industrial Revolution, researchers paid attention to the remarkable gyroscopic properties of gyroscopic devices and tried to develop the gyroscope theory. The simplified theories of gyroscopes started to publish a hundred years ago.1–4 Numerous publications described gyroscopic effects and applications in engineering but none of them explained the physics of gyroscope properties.5–7 The action of the gyroscopic effects is an important aspect of the science of classical mechanics. Fundamental textbooks and manuals have chapters of gyroscope theory.8,9 Many simplified approaches to gyroscope theory are dedicated to gyroscopic effects.10,11 All publications based on assumptions and described gyroscopic effects in terms only of the angular momentum.12–14 Researchers focused attention on the action of inertial forces on the gyroscope but did not represent mathematical models.15,16 The known theories of gyroscopic effects do not adequate to actual motions of rotating objects.17,18 Unsolved gyroscopic problems generated artificial terms as gyroscopic resistance, gyroscopic effects and attributed fantastical properties to rotating objects.19,20 Still, gyroscopic effects attract researchers to find true theory.21,22 New investigations in the area of gyroscopic theory discovered the action of the system of inertial torques on rotating objects. This work describes the application of new mathematical models for the system of inertial torques generated by the rotating mass of the spinning ring.

Methodology

Centrifugal torques acting on a rotating ring

The ring is a typical component of gyroscopic devices that manifest the gyroscopic effects presented in Figure 1. For computing of the inertial torques acting on the narrow spinning ring with the mass \( M \) and a constant angular velocity of \( \omega \) in a counter-clockwise direction around axis \( oz \), is accepted locations of its mass elements \( m \) on the middle radius \( R \). The centrifugal forces are generated by the mass elements of a spinning ring and resisted to the action of an external torque. The scheme for computing the resistance torque generated by the centrifugal forces is the same as presented for the spinning disc.22

\[
\Delta T_x = -f_{ct,z} y_m = -m a_y y_m
\]

Where \( a_y \) is the acceleration of the mass element and \( y y_m = R\sin a \) is the distance of the disposal of the mass element along axis \( oy \).

The change in the centrifugal force of the mass element is expressed by the following equation:

\[
f_{ct,z} = f_{ct} \sin \alpha \sin \Delta \gamma = \frac{MRo_2^2}{2\pi} \Delta \delta \sin \alpha \sin \Delta \gamma = \frac{MRo_2^2}{2\pi} \Delta \delta \gamma \sin \alpha
\]

Figure 1 Schematic of the spinning ring.
The resistance torque produced by the centrifugal forces of the mass element is expressed by substituting the defined parameters into Eq. (1).

\[ \Delta T_{ct} = - \frac{MR^2}{2\pi} \Delta \delta \times \Delta \gamma \times \frac{\sin a}{\sin \alpha} \times y_m \]  

(3)

Where all components are as specified above.

The integrated torque produced by a change in the centrifugal forces is defined by the presentation of components of Eq.(3) in a form appropriate for integration. The axial component of the centrifugal forces is applied to the pseudo centroid at the ring’s semi-circle, which is calculated by the known integrated equation.

\[ y_a = \frac{\int_{a=0}^{a=\pi} f_{ct,y} da}{\int f_{ct,x} da} \]  

(4)

Substituting Eq. (4) into Eq. (3) and transformation yields the following.

\[ y_a = \frac{\int_{a=0}^{a=\pi} f_{ct,y} da}{\int f_{ct,x} da} \times \frac{MR^2}{2\pi} \Delta \delta \times \Delta \gamma \times \frac{\sin a}{\sin \alpha} \times \frac{\int R \times \frac{1}{2} (1 - \cos 2a) da}{\int \sin ada} \]

(5)

Where the expression \( MR^2 \Delta \delta \Delta \gamma \) is accepted as constant for the Eq.(5); the expression \( \sin \alpha = (1 - \cos 2\alpha) / 2 \) is a trigonometric identity that replaced in the equation, and other parameters are as specified above.

Substituting Eq. (5) into Eq. (3), replacing \( \sin \alpha = \int \cos ada \) by the integral expression and converting by the integral form, the following equation emerges:

\[ \frac{\int dT_{ct}}{0} = - \frac{MR^2}{2\pi} \times \frac{\int \Delta \delta \times \Delta \gamma \times \frac{1}{2} (1 - \cos 2a) da}{\int \sin ada} \times \frac{R^2}{0} \times \frac{1}{2} \sin ada \]

(6)

Where the first integral of the cosines is increased twice due to limits of integration, for the second integral of the cosines remains the same due to the symmetrical location of the centroid.

Integral Eq.(6) is solved and yields the following result

\[ T_{ct} = - \frac{MR^2}{2\pi} \times \frac{\int \Delta \delta \times \Delta \gamma \times \frac{1}{2} (1 - \cos 2a) da}{\int \sin ada} \times \frac{R^2}{0} \times \frac{1}{2} \sin ada \]

Thus giving rise to the following:

\[ T_{ct} = - \frac{MR^2}{2\pi} \times (\pi - 0) \times (\gamma - 0) \times 2(1 - 0) \times \frac{R^2}{0} \times \frac{1}{2} \sin ada \]

(7)

The components of Eq.(7) is expressed by the differential equation of time derivative

\[ \frac{dT_{ct}}{dt} = - \frac{MR^2 \omega^2 \gamma \alpha}{8} \]

(8)

where \( t = \alpha / \omega \), \( \frac{dt}{d\alpha} \) = \( \omega \), \( \frac{d\gamma}{d\alpha} \) is the angular velocity of the ring’s precession around axis \( \alpha \).

The defined components are substituted into Eq(8) and yield the following differential equation:

\[ \frac{d\omega T_{ct}}{d\alpha} = - \frac{MR^2 \omega^2 \gamma \alpha}{8} \]

(9)

Separating variables of Eq.(9) and presenting by the integral form yield the following equation:

\[ \int_0^\alpha d\omega T_{ct} = - \int_0^\alpha \frac{MR^2 \omega^2 \gamma \alpha}{8} d\alpha \]

(10)

The integral solution of Eq.(10) yields the following

\[ T_{ct} = - \frac{2\pi^2 MR^2 \omega^2 \gamma \alpha}{8} \]

(11)

Substituting limits and increasing the result twice because of centrifugal forces act on the upper and lower sides of the ring, yield the expression of the total resistance torque \( T_{ct} \)

\[ T_{ct} = - \frac{2\pi^2 MR^2 \omega^2 \gamma \alpha}{8} \]

Where \( J = MR^2 \) is the thin ring’s mass moment of inertia.

**Common inertial forces acting on the spinning ring**

The solution for the common inertial forces acting on the spinning ring that generates the element of the precession torque \( \Delta T_{ct} \) is similar as for the rotating disc and represented by the following equation:

\[ \Delta T_{ct} = f_{ct} x_m = ma \times x_m \]

(12)

Where \( f_{ct} \) is the inertial force of the spinning ring’s mass element \( x_m \) is the distance to the mass element’s location along with axis \( \alpha \), other components are as specified above.

The distance \( x_m \) is expressed by Eq.(3) with a change in the index of axis. The acceleration \( a \) is defined by the first derivative of the change in tangential velocity and is expressed by the following.

\[ a_z = \frac{d(V \sin \alpha)}{dt} = \frac{d\Delta \gamma \sin \alpha}{dt} = V \cos \alpha \frac{dy}{dt} = R \omega \cos \alpha \]

(13)

where \( V = V \cos \gamma \Delta \gamma \) is the change in tangential velocity \( V, V = R \omega \), \( \omega = dy / dt, t \) is time, and the other components are as specified above.

The element of torque \( \Delta T_{ct} \) is defined by substituting defined parameters into Eq.(12).
The resulting torque of Coriolis forces acting on the centroid calculated by substituting defined parameters into Eq.(5) and expressed by the following.

\[ \sum f_{cr} da \alpha \cos \alpha - \sin \alpha \frac{1}{2} \int \cos \alpha d\alpha = \sum M_{rot} \Delta \delta \cos \alpha \times \sin \alpha \frac{1}{2} \int \cos \alpha d\alpha \]

(19)

Where the expression \( \frac{M_{rot} \Delta \delta}{2\pi} \cos \alpha \) is constant for Eq.(19), \( 2\sin \alpha \cos \alpha = \sin 2\alpha \) is a trigonometric identity that is replaced and other parameters are as specified above.

Substituting Eq.(19) into Eq.(18), the component \( \cos \alpha = \frac{1}{2} \sin 2\alpha \) is replaced and presented by the integral form that yields the following equation:

\[ \sum f_{cr} da \alpha \cos \alpha - \sin \alpha \frac{1}{2} \int \cos \alpha d\alpha = \sum M_{rot} \Delta \delta \cos \alpha \times \sin \alpha \frac{1}{2} \int \cos \alpha d\alpha \]

(20)

The solution of integral Eq.(21) yields the following:

\[ T_{cr} = \frac{MR_{rot} \omega_{\theta}}{2\pi} \int \left( \frac{1}{2} \sin 2\alpha \right) \int \left( \cos \alpha \right) d\alpha \frac{1}{2} \int \left( \sin 2\alpha \right) d\alpha \]

(21)

Where the limits taken for the quarter of the circle are due to its symmetrical location and the value is the same.

The resultant torque generated by Coriolis forces increased twice due to the action on the upper and lower sides of the ring.

\[ T_r = -2 \frac{MR_{rot} \omega_{\theta}}{2\pi} \times (\pi - 0) \times (1 - 1) \times \frac{-2\pi \times 2\pi}{4(1 - 0)} = -MR_{rot} \omega_{\theta} = -J_{\omega_{\theta}} \]

(22)

The inertial torque generated by the change in the angular momentum of the ring is presented by the equation:

\[ T_{in} = MR_{rot} \omega_{\theta} = J_{\omega_{\theta}} \]

(23)

Where all components are specified as above.

Attributes of inertial torques acting on the spinning ring

The resistance torques generated by the centrifugal (Eq.(12)) and Coriolis forces\(^{22}\) act around axis \( \omega \alpha \) in the same direction. The total initial resistance torque acting around axis \( \alpha \) generated by the external torque presented as their sum, whose equation is as follows

\[ T_{r} = T_{cr} + T_{in} = \left( \frac{\pi^2 + 4}{4} \right) J_{\omega_{\theta}} \]

(24)

Where \( T_r \) is the total initial resistance torque acting around axis \( \alpha \).

The torques generated by the common inertial forces (Eq.(16)) and by the change in angular momentum (Eq. (23)) are acted around one axis \( \omega \) and present a total initial precession torque generated by the external torque presented as their sum, whose equation is as follows

\[ T_{p} = T_{cr} + T_{in} = \left( \frac{\pi^2 + 4}{4} \right) J_{\omega_{\theta}} \]

(25)

where \( T_p \) is the total initial precession torque acting around axis \( \omega \).

Table 1 presents the torques generated by the inertial pseudo forces of the spinning ring. The inertial torques acting on the spinning ring demonstrates differ in the results compare with a spinning disc.\(^{21}\)

Working example

The ring has a mass of 1.0 kg and a mean radius of 0.1 m, spinning at 3000 rpm and precessing with an angular velocity of 0.05 rpm. It is determined the values of the components for resistance and precession torques. This problem is solved based on the equations in Table 1.

a. The torque generated by the centrifugal \( T_{cr} \) and common inertial \( T_{in} \) forces

\[ T_{cr} = \frac{1}{4} \pi^2 \times J_{\omega_{\theta}} = \frac{1}{4} \pi^2 \times 1.0 \times 0.1 \times \frac{3000 \times 2\pi}{60} \times 0.05 \times 2\pi = 0.04058712 \text{Nm} \]

b. The torque generated by Coriolis \( T_{cr} \) forces and the change in the angular momentum \( T_{in} \)

\[ T_{cr} = \frac{1}{4} \pi^2 \times J_{\omega_{\theta}} = 1.0 \times 0.1^2 \times \frac{3000 \times 2\pi}{60} \times 0.05 \times 2\pi = 0.01644934 \text{Nm} \]

c. The initial resistance \( T_{cr} \) and precession \( T_{p} \) torques

\[ T_{cr} = \frac{1}{4} \pi^2 \times J_{\omega_{\theta}} = 1.0 \times 0.1^2 \times \frac{3000 \times 2\pi}{60} \times 0.05 \times 2\pi = 0.01644934 \text{Nm} \]
Mathematical models for inertial torques acting on a spinning ring

\[ T_p = \left( \frac{\pi^2 + 4}{4} \right) J \omega \omega = \left( \frac{\pi^2 + 4}{4} \right) \times (1.0 \times 0.1^2 \times \frac{3000 \times 2\pi}{60} \times 0.05 \times 2\pi \times 60) = 0.057036461 \text{Nm} \]

Where \( J = MR^2 \) is the mass moment of inertia of the ring.\(^8,9\)

| Type of the torque generated by | Equation | Percentage of action (%) |
|--------------------------------|----------|--------------------------|
| Centrifugal forces, \( T_c \) | \( T_c = \frac{1}{4} \pi^2 J \omega \omega \) | 35.58 |
| Inertial forces, \( T_i \)    | \( T_i = T_m = J \omega \omega \) | 35.58 |
| Coriolis forces, \( T_{cr} \) | \( T_{cr} = T_{am} = J \omega \omega \) | 14.42 |
| Change in an angular momentum, \( T_{am} \) | \( T_{am} = J \omega \omega \) | 14.42 |
| Total                        |          | 100                      |

Resistance torque \( T_r = T_{cr} + T_{in} \)

Precession torque \( T_p = T_m + T_{am} \)

\( T_p = T_{cr} + T_{in} \)

\( T_r = T_m + T_{am} \)

\( T_r = \frac{\pi^2 + 4}{4} J \omega \omega \)

\( T_p = \frac{\pi^2 + 4}{4} J \omega \omega \)

\( 100 \%

Results and discussion

The spinning ring is the typical part of movable mechanisms that manifest gyroscopic properties. The gyroscopic effects are dependent on the action of the system of internal torques generated by the rotating mass the spinning ring. The mathematical models for internal torques acting on the spinning ring are derived and now can be used for engineering computing of gyroscopic effects. The inertial torques acting on the spinning ring are generated by known four components presented in publications acting on other rotating objects. The equations of the inertial torques have different expressions than for the spinning disc and reflect the geometry of the narrow ring. New mathematical models for the inertial torques enable for describing all gyroscopic effects solved this long-time problem presented the equation of the inertial torques and described the physics of gyroscopic effects. The new analytical approach clearly describes gyroscopic properties in a new light while setting forth new challenges for future studies of the gyroscopic theory.

Notation

\( f_c, f_o, f_w \)–centrifugal, Coriolis and inertial forces, respectively, generated by mass elements of a spinning ring

\( J \)–mass moment of inertia of a ring

\( M \)–mass of a ring

\( m \)–mass element of a ring

\( R \)–radius of a ring

\( T \)–load external torque

\( T_c, T_o, T_w \)–torque generated by centrifugal, Coriolis and inertial forces and a change in the angular momentum, respectively

\( t \)–time

\( \bar{y}, \bar{y}_c \)–centroid and distance of the location of the mass element along axis

\( \Delta \alpha, \alpha \)–increment angle and angle of the turn for a ring respectively

\( \Delta \delta \)–angle of the ring’s a mass element

\( \Delta \gamma \)–angle of inclination of a ring’s plane

\( \omega \)–angular velocity of a ring

\( \alpha \)–angular velocity of precession around axes \( \alpha \)

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Conflicts of interest

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