Post measurement bipartite entanglement entropy in conformal field theories

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We derive exact formulas for bipartite von Neumann entanglement entropy after partial measurement in 1 + 1 dimensional conformal field theories with periodic and open boundary conditions. After carefully defining the measurement set up we will check the validity of our results in the case of Klein-Gordon field theory (coupled harmonic oscillators) and spin-1/2 XX chain in a magnetic field. The agreement between analytical results and the numerical calculations is very good.

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Entanglement and measurement are two important and puzzling concepts in quantum mechanics [1]. In the last couple of decades the first one attracted a lot of attention in the high energy physics and the many body condensed matter physics mostly due to the area law property of bipartite von Neumann entanglement entropy which is the most celebrated measure of entanglement [2]. In one dimensional quantum systems while the area law is usually valid just for massive (gapped) systems for the critical systems the entanglement entropy of subsystem with the size \( l \) of the ground state follows the logarithmic formula \( S = c \log l \), where \( c \) is the central charge of the underlying conformal field theory (CFT) describing the low energy behavior of the critical system [3, 4]. Since the central charge of the system usually can fix the universality class of the system calculating entanglement entropy for a system can give a lot of insight about the possible universality class of the system. In particular, since numerical calculation of the entanglement entropy by using the technics of DMRG in one dimension is now a well-known method [5] one can easily study the critical and non-critical properties of the quantum systems in one dimension by studying the entanglement entropy. It is worth mentioning that entanglement entropy is not the only quantity which gives directly the central charge of the system. Among other measures one can name, mutual informations [6, 7] and quantum contours [8], see also other related works [9, 10]. The former one is based on local measurements in particular basis, so called conformal basis [7, 8]. One of the simple properties of the entanglement entropy which makes it more appealing in numerical calculations with respect to other measures such as Shannon and Rényi mutual informations is that it is completely independent of the basis that we write the wave function. However, since it is a very non-local quantity it is not an easy quantity to be measured in the experiment, for recent developments see [11, 12]. Since a measurement, by definition, is done in particular basis and von Neumann entanglement entropy of many body system is a basis independent quantity it seems hard to combine the two concepts and still have some universal features. Consider the ground state of a many body system, if we do local measurement of a local quantity in some part of the system that part will be disentangled from the rest and at the same time the rest of the system will collapse to a wave function which its entanglement content will be dependent on the basis that we chose for our our measurement and of course to the outcome of the measurement. In a recent development [7] it is shown that although, in general, doing measurement in arbitrary basis is not compatible with CFT set up there are some basis (conformal basis) that are closely related to boundary CFT. In other words if one makes a measurement in these basis one can still hope to preserve some of the universal properties of the system and be able to calculate the entanglement entropy. Analytical calculation of bipartite von Neumann entanglement entropy after measurement in conformal basis is the main purpose of this article. To do that we first define our set up and fix our assumptions and then we will analytically calculate entanglement entropy using twist operators of CFT. Exact formulas will be derived and then we will check the validity of our results by explicit examples in field theory such as Klein-Gordon field theory (coupled harmonic oscillators) and in quantum spin chains such
as, XX model in a magnetic field.

**Bipartite entanglement entropy after measurement in CFT.** The basic setup for our problem is as follows: consider the ground state of a periodic critical system in one dimension with the total size $L$. Then we make a measurement in the subsystem $A$ with the length $s$ of this system in particular local basis. After the measurement the subsystem $A$ will be decoupled (unentangled) from the complement $\bar{A}$. However, the wave function of the subsystem $A$ after measurement, i.e. $\psi_A$, is highly entangled. In other words if we take a subsystem $B \subset \bar{A}$ it will be entangled with $\bar{B}$, where $B \cup \bar{B} = \bar{A}$. From now on we will take the length of the subsystem $B$ equal to $l$ and for simplicity we will take $B$ and $\bar{B}$ in a way that they are adjacent simply connected domains. Here we are interested to calculate the entanglement entropy of the subsystem $B$ with respect to $\bar{B}$, i.e. $S_B$, in a critical quantum chain. To calculate such a quantity it is first useful to think about the Euclidean version of the system after measurement. For a periodic boundary conditions using the transfer matrix approach one can simply map the system after measurement to a cylinder with a slit on it as it is demonstrated in Figure 1, see [8]. In general this system is not necessarily an example of boundary CFT except for those cases that the subsystem $A$ after measurement picks up a configuration which renormalizes to a boundary CFT. We will discuss this important issue later in the case of critical spin chains. Now if we consider that the outcome of the measurement leads us to a boundary CFT, then based on the Cardy-Calabrese argument [9] the calculation of the entanglement entropy $S_B$ should boil down to the calculation of the one point correlation function of twist operator sitting on the boundary between $B$ and $\bar{B}$ on the cylinder. In other words we have:

$$S_B = - \lim_{n \to 1} \frac{\partial}{\partial n} \text{tr} \rho^n_B < \mathcal{T}_n(P) >_{cyl/slit},$$

where $\mathcal{T}_n$ is the twist operator with conformal weight $2h_n = \frac{2}{2n}(1 - \frac{c}{2})$ and $c$ is the central charge. The one point function of the twist operator on the cylinder with a slit can be easily found by mapping the system to the upper half plane by using the map $z(w) = \frac{a}{a + ws}$. The full line in the function (3) with $c = 1$ and $\gamma_1 = 0.21$.

Because cylinder with a slit is topologically different from cylinder. To get the result for before measurement case one can simply put $s = a$ which is the smallest scale in the system, then we will have $S_B = \frac{c}{3} \ln (\frac{L}{2a})$, which is the well-known result for the bipartite entanglement entropy in CFT [9]. In the next sections we will verify the validity of the above equation in two important cases: Klein-Gordon field theory (harmonic oscillators) and the critical XX model.

**Harmonic oscillators.** In this section we would like to calculate bipartite entanglement entropy of the ground state of coupled harmonic oscillators after measuring the position of a string of oscillators. Consider the Hamiltonian of $L$-coupled harmonic oscillators, with coordinates $\phi_1, \ldots, \phi_L$ and conjugated momenta $\pi_1, \ldots, \pi_L$:

$$\mathcal{H} = \frac{1}{2} \sum_{n=1}^L \pi_n^2 + \frac{L}{2} \sum_{n \neq n'} \phi_n K_{nn'} \phi_{n'}. \quad (4)$$

The ground state of the above Hamiltonian has the following form

$$\Psi_0 = (\frac{\det K^{1/2}}{\pi^L})^{1/4} e^{-\frac{1}{2} \langle |K^{1/2} \phi\rangle}. \quad (5)$$

One can calculate the two point correlators $X_D = \text{tr} \rho_D \phi \phi_j$ and $P_D = \text{tr} (\rho_D \pi \pi_j)$ using the $K$ matrix, where $\rho_D$ is the reduced density matrix of domain $D$. The squared root of this matrix, as well as its inverse, can be splitted up into coordinates of the subsystems $D$ and $\bar{D}$, i.e.,

$$K^{-1/2} = \begin{pmatrix} X_D & X_{DD} \end{pmatrix}, \quad K^{1/2} = \begin{pmatrix} P_D & P_{DD} \\ P_{DD} & P_D \end{pmatrix}. \quad (6)$$

The spectra of the matrix $2C = \sqrt{X_D P_D}$ can be used to calculate the entanglement entropy, see [17] and reference therein.
after measurement on subsystem with length $s$ bottom we have netic outcome for XX model in a magnetic field. From top to oscillators in the subsystem $\bar{f}$ field theory. we consider crete Klein-Gordon field theory) is shown in the figure 2. The results for the short-range harmonic oscillators (dis- have a new Gaussian wave function and one can calculate $K$ Hamiltonian of XX model is as follows:

$$H = -\frac{1}{2} \sum (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - 2h\sigma_i^z).$$

After using simple Jordan-Wigner transformation, i.e. $c_i = \prod_{m<i} \sigma_m^z + \frac{\sigma_i^z}{2}$, the Hamiltonian will have the following form

$$H = -\sum (c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + 2h(c_i^\dagger c_i - \frac{1}{2})).$$

The entanglement entropy of a subsystem can be calculated easily, see [18], by first calculating the reduced density matrix $\rho_B = K e^{-\sum \tilde{H}_i c_i^\dagger c_i}$ and then diagonalizing it, the final formula is

$$S = -\text{tr} \left( (1 - C) \log(1 - C) + C \log C \right).$$

The matrix $\tilde{H}$ is related to the correlation matrix $C$ as $\tilde{H} = \log(C^{-1} - 1)$. In our case the correlation matrix for infinite system has the form $C_{ij} = c_i^\dagger c_j = \frac{\sin(\pi n_f (i-j))}{\pi (i-j)}$, where $n_f = \frac{\arccos|h|}{\pi}$. Looking to the Jordan-Wigner transformation it is easy to see that any measurement in the $\sigma^z$ basis on particular site can be translated to the measuring the number of fermions in that site. In other words if one measures the $\sigma^z$ in all the sites of a subsystem with size $s$ the outcome will be one of $2^s$ possible configurations which can be easily translated to the configurations made of presence or absence of fermions in the sites of the subsystem. For simplicity we first consider that the outcome of the measurement is a string of $s$ fermion occupied sites. This can be calculated easily by using Grassmann numbers by first calculating the reduced density matrix for a sub-system with length $l+s$ and then finding the reduced density matrix of the subsystem with length $l$ with the assumption that the outcome of the measurement in the subsystem $s$ is a string of filled sites. The method is explained with full detail in the supplementary material. The results for different filling factors are shown in the Figure 3. There are some comments in order: first of all because of the $U(1)$ symmetry of the XX model the number of particles in the system is conserved and for this reason as far as $n_f$ is small it is very difficult to have a string of sites with fermions. In other words for example in the XX model with $h = 0$ the most probable outcome is an antiferromagnetic string rather than ferromagnetic string. In [8] it is already conjectured that for the half filling case most probably the ferromagnetic configuration is not going to lead to a boundary conformal field theory and so the very

![FIG. 3: Entanglement entropy of subregion B with length $l$ after measurement on subsystem with length $s$ ferromagnetic outcome for XX model in a magnetic field.](image)

FIG. 4: Entanglement entropy of subregion B with length $l$ after measurement on subsystem with length $s$ with anti-ferromagnetic outcome for XX model in half filling. In our numerics $l+s = 40$ is fixed. The full line is the function [4] with $L \to \infty$ and $c = 1$ and $\gamma_1 = 0.61$. The results for the short-range harmonic oscillators (discrete Klein-Gordon field theory) is shown in the figure 2 which shows a perfect agreement with the equation [3] if we consider $c = 1$ which is the central charge of the free field theory. **XX model in a magnetic field.** In this section we would like to calculate bipartite entanglement entropy after measurement in XX model in a magnetic field. The method that we use can work equally for all the other spin chains that can be mapped to free fermions. The Hamiltonian of XX model is as follows:

$$H = -\frac{1}{2} \sum (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - 2h\sigma_i^z).$$

Now if we do measurement on the position of all the oscillators $\{\phi_i\} \in A$ they will take some definite values and eventually will get decoupled from the rest of the oscillators. In other words the final state will be the same as [13] but instead of $K^{1/2}$ we need to consider $(K^{1/2})_A$ which is a subblock of the matrix $K^{1/2}$ corresponding to the oscillators in the subsystem $A$. This means that we now have a new Gaussian wave function and one can calculate its bipartite entanglement entropy with the formula [3]. The results for the short-range harmonic oscillators (discrete Klein-Gordon field theory) is shown in the figure 2 which shows a perfect agreement with the equation [3] if we consider $c = 1$ which is the central charge of the free field theory.

![FIG. 3: Entanglement entropy of subregion B with length $l$ after measurement on subsystem with length $s$ ferromagnetic outcome for XX model in a magnetic field.](image)
FIG. 5: Entanglement entropy of subregion \( B \) for a system (short-range coupled harmonic oscillators) with total length \( L = 50 \) and the measurement region sizes \( s = 0, 10 \) and \( 20 \) with respect to \( \ln f(L, s, l) \), where \( f(L, s, l) = \frac{2L}{\pi} \cos \frac{\pi}{a} \cos \frac{\pi}{a} \cot \frac{\pi}{2 L} \). The full line is the function (10) with \( c = 1 \) and \( \gamma_2 = -0.02 \).

first assumption that we used is going to fail. Despite this argument we found surprising that for \( l > (\frac{1}{n_f} - 1)s \) the formula (3) works perfectly. The reason that for larger \( n_f \) the formula works better is because for larger \( n_f \) the probability of having ferromagnetic string increases and so one does not need to inject particles in the system.

One can also check the results for those cases that the outcome of the measurement is antiferromagnetic string. It is expected that this case leads to Dirichlet boundary condition in the bosonization language and so it is related the boundary CFT. The results presented in the Figure 4 indeed show that the formula (8) works perfectly also in this case. Since the ferromagnetic and antiferromagnetic configurations are at the two extreme sides of the all possible configurations based on the above numerical results one can conjecture that independent of the outcome of the measurement in the \( \sigma^z \) basis the formula (3) will work if the \( l \) is bigger enough than \( s \). Note that the number of configurations increases exponentially this conjecture is beyond what we can check numerically.

Open boundary conditions One can also do all the CFT calculations in the case of open boundary condition, the only difference is that now we have a strip with a slit instead of a cylinder. One can simply map a strip with a slit to upper half plane by using the map

\[
z(w) = \sqrt{1 - \frac{\tan^2 \frac{\pi w}{2L}}{\tan^2 \frac{\pi s}{2L}}}.\]

After some algebra we have a strip with a slit instead of a cylinder. One can simply map

\[
S_B = \frac{c}{6} \ln \left( \frac{2L \cos \frac{\pi s}{a} - \cos \frac{\pi l + s}{a}}{\cos^2 \frac{\pi s}{2L}} \cot \frac{\pi (l + s)}{2L} \right) + \gamma_2(10)
\]

The limit \( s \to 0 \) simply gives us the well-known formula \( S_B = -\frac{c}{2} \ln \left( \frac{\sin \frac{\pi l}{2L}}{\sqrt{\sin \frac{\pi l}{2L}}} \right) \), see [3]. We have checked the validity of this formula in the case of short-range harmonic oscillators using the same method that we hired for the periodic BC. The results shown in Figure 5 are fairly compatible with the formula (10). We noticed that although the log formula is perfectly compatible there is almost 8 percent deviation from central charge that might be due to finite size effects.

Conclusions. We described a general set up for calculating bipartite entanglement entropy after local measurement in critical one dimensional quantum systems. Exact formulas were derived for bipartite entanglement entropy after “conformal measurements” in the case of periodic and open boundary conditions. The formulas were checked in explicit examples of free bosonic system and XX model in a magnetic field. We noticed that since bosonization of XXZ model in the \( \sigma^z \) basis leads to free bosonic system the antiferromagnetic outcome of the measurement in this basis should be compatible with the results of free bosonic system [8]. In the case of XX model in a magnetic field we showed that if we do our measurements in the \( \sigma^z \) basis independent of having ferromagnetic or antiferromagnetic outcome for our measurements the bipartite entanglement entropy in particular regimes can be described with CFT formulas. There are interesting questions remained to be answered: first of all in our free fermion approach we were able to handle just \( \sigma^x \) basis, it is important to look to the other basis especially \( \sigma^x \) basis by using exact diagonalization methods. Another interesting question is investigating the same questions in the case of non-critical quantum systems and the validity of the area law. Some of these questions will be discussed in a forthcoming paper [19]. Finally understanding the problem in the holographic set up [20] will surely help to extend some of these results to higher dimensions.

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[1] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80, 517 (2008); K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. 84, 1655 (2012)
[2] M. Srednicki, Phys. Rev. Lett. 71, 666 (1993) and J. Eisert, M. Cramer, M. B. Plenio, Rev. Mod. Phys. 82, 277 (2010)
[3] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B, 424, 443 (1994).
[4] G. Vidal, J. I. Latorre, E. Rico, A. Kitaev, Phys. Rev. Lett. 90, 227902 (2003)
[5] P. Calabrese and J. Cardy, J. Stat. Mech. (2004) P06002 and J. Phys. A, 42, 504005 (2009)
[6] U. Schollwöck, Rev. Mod. Phys. 77, 259 (2005)
[7] F. C. Alcaraz, M. A. Rajabpour, Phys. Rev. Lett. 111, 017201 (2013) and Phys. Rev. B, 90, 075132 (2014)
[8] J-M Stéphan, J. Stat. Mech. (2014) P05010 and Phys. Rev. B 90, 045424 (2014)
[9] F. C. Alcaraz, M. A. Rajabpour, arXiv:1409.6558
[10] Y. Chen, G. Vidal, J. Stat. Mech. (2014) P10011
[11] P. Calabrese, J. Cardy, E. Tonni, Phys. Rev. Lett. **109**, 130502 (2012).
[12] J-M Stéphan, S. Inglis, P. Fendley, R. G. Melko, Phys. Rev. Lett. **112**, 127204 (2014).
[13] J. Cardy, Phys. Rev. Lett. **106**, 150404 (2011).
[14] D. A. Abanin and E. Demler, Phys. Rev. Lett. **109**, 020504 (2012).
[15] A. J. Daley, H. Pichler, J. Schachenmayer, and P. Zoller, Phys. Rev. Lett. **109**, 020505 (2012).
[16] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B **241** (1984) 333.
[17] H. Casini and M. Huerta, J. Phys. A, **42**, 504007 (2009).
[18] M. -C. Chung and I. Peschel, Phys. Rev. B, **62**, 4191 (2000) and I. Peschel, J. Phys. A: Math. Gen. **36**, L205 (2003).
[19] M. A. Rajabpour, *in preparation*.
[20] S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006).
Supplemental Material for
post measurement bipartite entanglement entropy in conformal field theories

Bipartite entanglement entropy after partial measurement of fermion occupation numbers in free fermions

In this supplementary note we would like to present a method which can be used to measure the entanglement entropy after measuring the fermion occupation numbers in a subsystem in a generic free fermion system. The method is based on Grassmann numbers and it is a generalization of the work [1]. Although the method can be also used for the most generic free fermion system [2] here we will concentrate on a system with the following reduced density matrix:

\[ \rho_B = Ke^{-\sum \tilde{H}_{ij}c_i^†c_j} \]  

(S1)

The matrix \( \tilde{H} \) is related to the correlation matrix \( C \) as \( \tilde{H} = \log(C^{-1} - 1) \). Finally one can find the entanglement entropy using the following formula, see for example [3];

\[ S_B = -\text{tr}((1 - C) \log(1 - C) + C \log C). \]  

(S2)

To calculate the reduced density matrix after measurement we need to first define fermionic coherent states. It can be defined as follows

\[ |\xi\rangle := |\xi_1\xi_2...\xi_N⟩ = e^{-\sum_{i=1}^{N} \xi_ic_i^†}|0\rangle, \]  

(S3)

where \( \xi_i \)'s are Grassmann numbers following the properties: \( \xi_n\xi_m + \xi_m\xi_n = 0 \) and \( \xi_n^2 = \xi_m^2 = 0 \). Then it is easy to show that

\[ c_i|\xi⟩ = -\xi_i|\xi⟩. \]  

(S4)

Using the above formula one can derive the following useful formula:

\[ <\xi|e^{\sum_{ij}A_{ij}c_i^†c_j}|\xi’⟩ = e^{\sum_{ij}(e^A)_{ij}\xi_i^*\xi’_j}. \]  

(S5)

With the same method one can also define another kind of fermionic coherent state as

\[ |\eta⟩ := |\eta_1\eta_2...\eta_N⟩ = e^{-\sum_{i=1}^{N} \eta_ic_i}|1\rangle, \]  

(S6)

where \( \eta_i \)'s are Grassmann numbers. Then it is easy to show that

\[ c_i^†|\eta⟩ = -\eta_i|\eta⟩. \]  

(S7)

and consequently

\[ <\eta|e^{\sum_{ij}A_{ij}c_i^†c_j}|\eta’⟩ = e^{\sum_{ij}(e^{-A})_{ij}\eta_i^*\eta’_j}. \]  

(S8)

The two formulas (S5) and (S8) are the main formulas one can use to calculate reduced density matrix after measurement. We first discuss how the method works for a string of empty sites. A string of empty sites means that in the equation (S5) we need to put all the \( \xi_i \)'s with \( i \) inside the measuring subsystem equal to zero. This is equivalent to keeping just all the elements of the matrix \( e^A \) outside the subsystem. Then we take the logarithm of this matrix and find a new matrix \( A \) which is going to be our new entanglement Hamiltonian and then one can easily calculate the entanglement entropy of that system with the method we explained at the beginning of this section. To do the same calculation for an string of occupied sites instead of equation (S5) one needs to use equation (S8) and follow the same method as we just explained. In principle this method can be applied for any possible configuration by just appropriate using the equations (S5) and (S8).

[1] M. -C. Chung and I. Peschel, Phys. Rev. B 62, 4191 (2000)
[2] The most general case and especially the Ising model will be discussed in a forthcoming paper.
[3] Casini H. and Huerta M., J.Phys.A 42, 504007 (2009)