Gravitational energy

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Abstract

Observers at rest in a stationary spacetime flat at infinity can measure small amounts of rest-mass + internal energies + kinetic energies + pressure energy in a small volume of fluid attached to a local inertial frame. The sum of these small amounts is the total ‘matter–energy’, $E_M$, for those observers. If $Mc^2$ is the total mass energy, the difference $Mc^2 - E_M$ is the binding gravitational energy. Misner, Thorne and Wheeler (MTW) evaluated the gravitational energy of a spherically symmetric static spacetime. Here we show how to calculate gravitational energy in any static and stationary spacetimes with isolated sources with a set of observers at rest. The result of MTW is recovered and we find that electromagnetic and gravitational 3-covariant energy densities in conformastatic spacetimes are of opposite signs. Various examples suggest that gravitational energy is negative in spacetimes with special symmetries or when the energy–momentum tensor satisfies usual energy conditions.

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1. Introduction

This paper deals mainly with stationary spacetimes of isolated sources that are asymptotically flat. In classical physics, energy has different forms which are additive. If energy is conserved one can assess how much change and transfers of energy occur between the various forms. In Einstein’s theory of gravitation, different forms of matter–energy are mixed in inseparable ways with gravitational binding energy. Gravitational energy is diffuse, partly mixed up with other forms, with matter–energy, partly stored in the gravitational field itself. Alas, the matter is the source of gravity so that it is impossible to disentangle gravitational energy from other forms of energy with or without solving the field equations. This is a great loss that adds to that of ‘gravitational force’ and ‘gravitational energy density’ with which it shares a common origin: the principle of equivalence.
One may, however, define gravitational energy with respect to a set of observers. Take for instance the matter–energy tensor of a perfect fluid; in standard notation
\[ T_{\mu\nu} = (\rho c^2 + P)u^\mu u_\nu - \delta_\mu^\nu P \]
let \( w^\mu \) be the 4-velocity field of local observers \((g_{\mu\nu}w^\mu w^\nu = 1)\). If
\[ dV_\mu = \frac{1}{3!} \epsilon_{\mu \nu \rho \sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \]
is the three-dimensional coordinate volume element of a spacelike hypersurface (say, for instance, \( d^3x \) on \( t = 0 \) in adapted coordinates \((x^0 = ct, x^k)\)), the scalar element
\[
\begin{align*}
    dE_M &\equiv \hat{T}^{\mu\nu} w_\nu dV_\mu = \frac{d(M_0c^2)}{\sqrt{1 - v^2}} + dE_I, \\
    d(M_0c^2) &= \rho c^2 \sqrt{-g} u^\mu dV_\mu \\
    u_\nu w^\nu &= \frac{1}{\sqrt{1 - v^2}},
\end{align*}
\]
where \( d(M_0c^2) \) is the element of rest mass + internal energy of the source, \( \vec{v} \) is the velocity of the matter with respect to the local observer in a local inertial frame and \( dE_I \) is the external potential energy of the pressure on the small proper volume element \( \sqrt{-g} u^\mu dV_\mu \).

One ‘natural’ selection of observers in static spacetimes which have a field of timelike Killing vectors of translations \( \xi^\mu \) are observers with velocities \( w^\mu = \xi^\mu/\xi^\nu \). They are at rest in any system of coordinates that singles out the inertial time at infinity \( \xi^\mu = \{1, 0, 0, 0\} \). They are also at rest with respect to the matter. There is, however, no such well-defined set of observers in stationary spacetimes. For instance, in a Kerr spacetime one might prefer to choose the family of ‘zero angular momentum observers’, the ZAMOS of Bardeen [3]. It is of course desirable to have some unique way of choosing observers in stationary spaces. Such a qualifying choice is not considered in this work.

The sum \( E_M \) of \( dE_M \) can be regarded as the total ‘rest-mass-energy + internal energies + kinetic energy + pressure energy’ to which it reduces in the weak field limit. No gravitational energy is involved in \( dE_M \). Therefore if \( Mc^2 \) is the total mass energy of spacetime the difference \( E_G \equiv Mc^2 - E_M \) can be regarded as the total gravitational energy of spacetime for our set of observers. When gravity is a binding force, and this is not necessarily the case in general relativity, one expects that, as in Newtonian gravity, \( E_G \) is negative.

Misner, Thorne and Wheeler [17] evaluated \( E_G \) along precisely this line of thought for spacetimes created by static spherical ‘stars’. Here we extend the calculation of \( E_G \) to any static spacetime with localized sources. The formula also holds in stationary spacetimes but its application may generate less enthusiasm because, as we said, sets of reference observers are not as well defined. In the few examples we treat \( E_G < 0 \) when spacetimes have special symmetries or when the energy–momentum tensors of the source satisfy usual energy conditions.

### 2. Conservation laws and observers

#### 2.1. The conserved current for energy

To begin with we need a conserved covariant vector \( J^\mu \) involving the source of gravity \( T^{\mu\nu} \).

There are a number of available vectors; see in particular Abbott and Deser [1]. Grishchuk,
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Petrov and Popova [7], or [10]2. All conserved currents for energy have the same form in stationary spacetimes:

\[ J^\mu = (T^\mu_\nu + t^\mu_\nu) \xi^\nu, \]  

(2.1)

where \( t^\mu_\nu \), which depends on the metric and its derivatives, is associated with part3 of \( G \)-energy. \( t^\mu_\nu \) differs from author to author, reflecting the ambiguity in defining \( G \)-energy density.

One necessary ingredient of \( J^\mu \) is a background metric. In asymptotically flat spacetimes it is advantageous to introduce a flat background. The background is an extension from infinity inwards. It is an artefact to get a covariant description and is very useful when we want to use for instance spherical coordinates instead of Minkowski coordinates. Mappings on the background may be chosen at our convenience. The timelike Killing vectors associated with conserved energy are translations in the background, i.e.

\[ D_\mu \xi^\lambda = 0. \]  

(2.2)

An overbar refers to covariant derivatives in the background.

All covariant conserved vectors are also the divergences of antisymmetric tensors or superpotentials \( J^{\mu\nu} \):

\[ \hat{J}_\mu = \partial_\nu \hat{J}^{\mu\nu}. \]  

(2.3)

The superpotentials are not uniquely defined nor are they the same from author to author. However, the surface integral on a sphere at infinity of any valid superpotential must be the same, i.e. \( M c^2 \).

We shall here adopt4 the current given in [9] and more fully described in [10]. There is a uniqueness [5, 8] to the ‘KBL-superpotential’ which has been found satisfactory in all applications at spatial as well as at null infinity, on flat or non-flat backgrounds including in the calculation of the very exotic mass of Kerr black holes on anti-de Sitter backgrounds in \( D \)-dimensions [6]. And it is quadratic in first-order derivatives of the metric which may be a real boon as we shall see. In [10], \( t^\mu_\nu \) is the covariant form of the quadratic Einstein pseudotensor.

2.2. The KBL conserved current

Formulations in [1, 7, 10] are all on generally curved backgrounds. We shall start from the KBL identity \( \hat{I}_\mu = \partial_\nu \hat{J}^{\mu\nu} \) which holds not only for curved backgrounds but also for arbitrary vector fields which are still denoted by \( \xi^\mu \) but are not necessarily Killing vectors. Our \( \hat{J}^{\mu\nu} \) in (2.3) is a particular value of \( \hat{I}_\mu \). We shall first lay out the mathematical expression of \( I^\mu \) and \( J^{\mu\nu} \), which are somewhat complicated, and recall their physical content next. \( \hat{I}^\mu = \partial_\nu \hat{J}^{\mu\nu} \) is originally a Noether identity which turns into a real conservation law once Einstein’s equations \( G^\mu_\nu = \kappa T^\mu_\nu \) are taken into account, \( \kappa = \frac{8\pi G}{c^4} \).

The KBL current and superpotential were derived in terms of Christoffel symbols of both the spacetime \( \Gamma^\mu_{\rho\nu} \) and the background \( \bar{\Gamma}^{\rho\nu} \) or, more precisely, in terms of a tensor

---

2 The Ashtekar and Hansen [2] conformal mass formula, the Hamiltonian formulation of Regge and Teitelboim [21] and various formulae assembled by Szabados [24] to calculate energy in a finite volume (quasi-local energy) are integrals on closed surfaces. The integrand is an antisymmetric tensor whose diverge is a conserved vector which must be of the same form as equation (2.1). One may thus start from those formulae as well but this needs a bit of additional work.

3 It is well known, see for instance [12], that the integral of \( \hat{T}^\mu_\nu \xi^\nu dV_\mu \) in the weak field limit contains twice the Newtonian gravitational energy.

4 The superpotential given in [1] does not provide the Bondi mass at null infinity [19] and the \( t^\mu_\nu \) of [7] also given in [19] contains second-order derivatives of the metric which is not a priori favourable to analyse the sign of the integrant.
\[ \Delta_{\mu\nu}^\lambda \equiv \Gamma_{\mu\nu}^\lambda - \Gamma_{\mu\nu}^\lambda. \]

We found it somewhat simpler to write \( \hat{J}^\mu = \partial_\nu \hat{J}_\nu^\mu \) in terms of
\[ L_{\rho}^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \bar{D}_{\rho} \hat{g}^{\mu\nu}, \quad \Delta_{\rho} \equiv \frac{1}{2} \hat{g}_{\rho\mu} L_{\rho}^{\mu\nu} = \bar{D}_{\rho} \log \sqrt{-g} \]
and
\[ \bar{F}^{\mu} \equiv \hat{L}_\nu^{\mu\nu} = \bar{D}_\nu \hat{g}^{\mu\nu}. \]

(2.4)

\( \Delta_{\mu\nu}^\lambda \) and \( L_{\rho}^{\mu\nu} \) are related as follows:
\[ \Delta_{\mu\nu}^\lambda = \delta_{(\mu} \Delta_{\nu)} - L_{-(\rho\sigma)}^\mu + \frac{1}{2} \left( L_{\rho\sigma}^\mu - \delta_{\rho\sigma} \Delta^\mu \right). \]

(2.5)

The KBL superpotential, which in its original form, is
\[ \hat{J}^{\mu\nu} = \frac{1}{\kappa} D^\mu \hat{\xi}^\nu + \frac{1}{\kappa} \hat{\xi}^{(\mu} \hat{F}^{\nu)} \quad \text{with} \quad \hat{k}^\nu = \frac{1}{\sqrt{-g}} \bar{D}_\nu (-g \hat{g}^{\mu\nu}). \]

(2.6)

can be rewritten as follows5:
\[ \hat{J}^{\mu\nu} = \frac{1}{\kappa} \left( \delta^{\mu\nu} \bar{D}_\rho \hat{\xi}^\nu - L_{\rho\sigma}^{\mu\nu} \hat{\xi}^{\lambda} + \hat{\xi}^{(\mu} \bar{F}^{\nu)} \right) \quad \text{where} \quad \bar{l}^{\mu\nu} \equiv \hat{g}^{\mu\nu} - \hat{g}^{\mu\nu}. \]

(2.7)

The current density \( \hat{I}^\mu \) is linear in \( \hat{\xi}^\mu \), \( \bar{D}_\rho \hat{\xi}^\mu \) and covariant derivatives of6
\[ \bar{Z}^{\lambda\sigma} \equiv \frac{1}{2} \left( \bar{D}_{(\rho} \hat{\xi}_{\sigma)} + \bar{D}_{(\sigma} \hat{\xi}_{\rho)} \right) \quad \text{in which} \quad \hat{\xi}^\mu \equiv \bar{g}^{\mu\lambda} \bar{F}_{\lambda}. \]

(2.8)

\( \bar{Z}^{\lambda\sigma} = 0 \) if \( \hat{\xi}^\mu \) is a Killing vector of the background. Thus7,
\[ \hat{I}^\mu = \left( \hat{I}^{\mu\nu} - \hat{I}_\nu^{\mu\nu} \right) \hat{\xi}^\nu + \left( \frac{1}{2\kappa} \bar{I}^{\mu\rho\sigma} \bar{D}_{\rho} \hat{\xi}^\nu + \hat{q}^{\mu\nu\sigma} \bar{D}_\nu \bar{Z}_{\lambda}^{\sigma} \right) \hat{\xi}^\lambda + \hat{q}^{\mu\nu\rho\sigma} \bar{D}_\nu \bar{Z}_{\lambda}^{\rho} \hat{\xi}^\lambda. \]

(2.9)

The three undefined quantities in this vector are
\[ t_{\nu}^{\mu} \equiv S_{\nu}^{\mu} - \frac{1}{2} \hat{g}^{\mu\nu} S \quad \text{with} \quad S_{\nu}^{\mu} \equiv \frac{1}{2\kappa} \left[ \left( \frac{1}{2} L_{\mu\rho\sigma} - L_{\rho\sigma}^{\mu\nu} \right) L_{\nu\rho\sigma} - \Delta^\nu \Delta_\nu \right]. \]

(2.10)

\[ \sigma_{\mu\nu}^{\lambda} \equiv \frac{1}{2\kappa} \left( L_{\mu}^{\lambda\nu} - F^\nu \delta_{\lambda}^\nu - 2 L_{\lambda}^{\mu\nu} \right), \quad \hat{q}^{\mu\nu\rho\sigma} = \frac{1}{2\kappa} \left( \bar{q}^{\mu\nu\rho\sigma} \bar{g}^{\rho\sigma} + \bar{q}^{\mu\nu\rho\sigma} \bar{g}^{\rho\sigma} - 2 \bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \bar{g}^{\rho\sigma} \right). \]

(2.11)

The most significant properties of the conserved vector (2.9) were pointed out in [10].
(1) on a flat background \( \bar{R}_{\mu\nu\rho\sigma} = 0 \) with a timelike Killing vector of translations (2.2), equation (2.9) reduces to (2.1) where \( \hat{I}^\mu \) is the special value of \( \hat{I}^\mu \) in this case. The presence on the right-hand side of \( \bar{R}_{\mu\nu\rho\sigma} \) shows that we are dealing with conservation of energy. The total energy is given by a surface integral of the superpotential (2.7). (2) If the Killing vector is one of the rotations the surface integral gives the corresponding total angular momentum components. (3) On other backgrounds with various symmetries we obtain other conserved quantities.

A remarkable feature of the conserved vector is that it is not useful for the very reason indicated in the introduction. There are no coordinate- and background-independent properties which one can associate with the different terms of \( J^\mu \). This holds, however, change if we introduced a preferred set of local observers.

5 Our interest in the superpotential here is limited. For a detailed analysis of its structure and properties the reader is referred to the original paper.

6 \( Z_{\mu\nu\rho\sigma} \) in [10] is here \( 2 \bar{Z}_{\mu\nu\rho\sigma} \). Also indices in [10] are displaced with \( \bar{Z}_{\mu\nu\rho\sigma} \), never with \( g_{\mu\nu}\). Here we do not stick to that convention. If an index is displaced with the metric of the background \( \bar{Z}_{\mu\nu\rho\sigma} \), the index has an overhead bar like in \( \hat{X}_{\mu\nu} \equiv \bar{g}_{\mu\nu} X^\nu \). Otherwise we write \( \hat{X}_{\mu\nu} = g_{\mu\nu} X^\nu \).

7 In its original form, see [10], the last two terms in (2.9) are grouped slightly differently; in the present notation we would have written it as \( \delta_{\mu\nu} \bar{g}_{\rho\sigma} \bar{Z}_{\lambda}^{\rho} + \delta_{\mu\rho} \bar{g}^{\lambda\sigma} \bar{Z}_{\lambda}^{\rho} + \bar{q}^{\mu\nu\rho\sigma} \bar{D}_\nu \bar{Z}_{\lambda}^{\rho} \). 

8) reduces to (2.1) where \( \hat{I}^\mu \) is the special value of \( \hat{I}^\mu \) in this case. The presence on the right-hand side of \( \bar{R}_{\mu\nu\rho\sigma} \) shows that we are dealing with conservation of energy. The total energy is given by a surface integral of the superpotential (2.7). (2) If the Killing vector is one of the rotations the surface integral gives the corresponding total angular momentum components. (3) On other backgrounds with various symmetries we obtain other conserved quantities.
2.3. Singling out the matter–energy vector

What we really want to see in \( J^\mu \) is \( T^\mu_\nu w^\nu \) rather than \( T^\mu_\nu \xi^\nu \). We thus rewrite the first term of (2.9) as

\[
(T^\mu_\nu - \bar{T}^\mu_\nu) \xi^\nu = (T^\mu_\nu - \bar{T}^\mu_\nu) w^\nu + (T^\mu_\nu - \bar{T}^\mu_\nu) \tau^\nu \quad \text{where} \quad \tau^\nu \equiv \xi^\nu - w^\nu. \tag{2.12}
\]

Ultimately \( w^\nu \) will be a field of timelike unit vectors and \( \xi^\nu \) a timelike Killing vector field but for the moment both are unspecified to leave open the possibility of using the formulae in non-stationary spacetimes and spacetimes with non-flat backgrounds. We then replace \((T^\mu_\nu - \bar{T}^\mu_\nu) \tau^\nu \) in (2.12) by its expression in terms of the Ricci tensor \( R_{\rho\sigma} \) using Einstein’s equations in the foreground as well as in the background. Thus we set respectively

\[
T^\mu_\nu \tau^\nu \equiv \frac{1}{\kappa} \left[ g^{\mu(\rho \tau \sigma)} - \frac{1}{2} \delta^\mu_\rho \tau^\nu \right] R_{\rho\sigma} \equiv \bar{\rho}^{\mu\rho\sigma} R_{\rho\sigma} \quad \text{and} \quad \bar{T}^\mu_\nu \tau^\nu \equiv \bar{\rho}^{\mu\rho\sigma} \bar{R}_{\rho\sigma}.
\]  

(2.13)

A bar over the \( \sigma \) indicates that \( g \) have been replaced by \( \bar{g} \). Next we use the expression of \( R_{\rho\sigma} \) in terms of \( \bar{D} \)-covariant derivatives and \( \Delta^\rho_{\rho\sigma} \) as in [10]:

\[
R_{\rho\sigma} = \bar{D}_\rho \Delta^\rho_{\rho\sigma} - \bar{D}_\sigma \Delta^\rho_{\rho\rho} + \Delta^\rho_{\rho\rho} - \frac{1}{2} \Delta^\rho_{\rho\rho} = \bar{D}_\rho \Delta^\rho_{\rho\rho} + \bar{R}_{\rho\rho}.
\]

(2.14)

We replace the \((T - \bar{T}) \tau \) in (2.12) by (2.13) using (2.14), change the \( \bar{D}(\bar{D} \Delta) \)-terms into \([\bar{D}(\bar{D} \Delta) - (\bar{D} \bar{D} \Delta)] \)-terms and obtain after straightforward but slightly tedious calculations the following expression for a modified conserved current \( \tilde{J}^\mu \) and superpotential \( \tilde{\rho}^{\mu\nu} \) defined as follows:

\[
\tilde{J}^{\mu\nu} \equiv \int^{\mu\nu} - \frac{1}{\kappa} \left( \hat{L}^{[\mu|} \xi^{\nu]} + \tau^{[\mu} \hat{F}^{\nu]} \right) = \frac{1}{\kappa} \left( \hat{L}^{[\mu|} \bar{D}_\rho \xi^{\nu]} - \hat{L}^{[\mu]} \xi^{\nu]} w^\rho + w^{[\mu} \hat{F}^{\nu]} \right).
\]

(2.15)

Note that \( \tilde{J}^{\mu\nu} \) is similar to \( J^{\mu\nu} \); \( \xi \) have been replaced by \( w \) in the last two terms but not in the first one. \( \tilde{J}^\mu \) is very similar to \( J^\mu \) with different coefficients:

\[
\tilde{J}^\mu = (T^\mu_\nu - \bar{T}^\mu_\nu) w^\nu + \tilde{R}^\mu + (\check{\iota}^\mu_\nu w^\nu + \check{\iota}^{\mu\nu}_\lambda \bar{D}_\nu w^\lambda) + \frac{1}{2\kappa} \left( \hat{L}^{\mu|\rho} - \delta^{\mu|}_\rho \hat{F}^{\nu} \right) \bar{D}_\rho \xi^\lambda + \hat{q}^{\mu\nu\sigma\rho} \bar{D}_\nu \bar{z}_{\rho\sigma}.
\]

(2.16)

in this

\[
\check{\iota}^{\mu\nu}_\lambda \equiv \frac{1}{\kappa} \left( \hat{L}^{\mu|\rho} \bar{D}_\rho \xi^{\nu]} - \hat{L}^{[\mu|} \xi^{\nu]} \right) \quad \text{and} \quad \tilde{R}^\mu \equiv \frac{1}{2\kappa} \left( \hat{L}^{\rho\sigma} \bar{R}_{\rho\sigma} w^\mu + \hat{L}^{\mu\sigma} \bar{R}_{\rho\sigma} \tau^\rho - \hat{L}^{\rho\sigma} \bar{R}_{\rho\sigma} \tau^\rho \right).
\]

(2.17)

2.4. Gravitational energy in stationary spacetimes

The very complicated current \( \hat{J}^\mu \) in (2.16) with its superpotential \( \tilde{J}^{\mu\nu} \) in (2.15) is considerably simpler on a flat background, \( \tilde{R}^\mu = 0 \), with a timelike Killing vector of translations, \( \bar{z}_{\rho\sigma} = 0 \) and \( \bar{D}_\nu \xi^\rho = 0 \); we denote it then by \( \hat{J}^\mu \) defined as:

\[
\hat{J}^{\mu\nu} = \frac{1}{\kappa} \left( -\hat{L}^{[\mu|} \xi^{\nu]} + \hat{L}^{\mu|} \hat{F}^{\nu]} \right) \quad \text{and} \quad \hat{J}^\mu \equiv \partial_\nu \hat{J}^{\nu\mu} = \hat{T}^\mu_\nu w^\nu + (\check{\iota}^\mu_\nu w^\nu + \check{\iota}^{\mu\nu}_\lambda \bar{D}_\nu w^\lambda).
\]

(2.18)

This new conserved current has the following properties. Let \( w^\mu \) be a field of observers at rest, i.e.

\[
w^\mu = \xi^\mu \quad \text{with} \quad w^\mu \longrightarrow \xi^\mu + O\left( \frac{1}{r} \right) \quad \text{for} \quad r \to \infty,
\]

(2.19)

8 Expression (2.14) given in [10] and to which we refer, for the facility of the reader since it is on the web, was, however, published in 1940 by Rosen [20].
where \( r \to \infty \) is the radius of the infinite sphere in the background whose metric in spherical coordinates is
\[
\text{d}s^2 = c^2 \text{d}r^2 - \text{d}r^2 - r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2) \equiv c^2 \text{d}r^2 - \text{d}r^2 - r^2 \text{d}\omega^2.
\] (2.20)

If \( w^\mu \) is replaced by the Killing vector \( \xi^\mu \) in \( \hat{J}^{\mu\nu} \), it is equal to the KBL superpotential. Therefore the surface integral of \( \hat{J}^{\mu\nu} \) at infinity is equal to \( Mc^2 \) because of (2.19):
\[
\oint_{r \to \infty} \frac{1}{2} \hat{J}^{\mu\nu} \text{d}S_{\mu\nu} = \frac{1}{\kappa} \oint_{r \to \infty} \frac{1}{2} \left( -\hat{L}^{[\mu|\nu}\xi^{\lambda]} + \xi^{[\mu} \hat{e}^{\nu]} \right) \text{d}S_{\mu\nu} = Mc^2,
\]
\[
\text{d}S_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \text{d}x^\rho \wedge \text{d}x^\sigma.
\] (2.21)

But the matter–energy
\[
E_M = \int_{\infty} \hat{\mathcal{T}}^{\mu\nu} w^\nu \text{d}V^\mu.
\] (2.22)

Therefore in a stationary spacetime, the gravitational field energy with respect to our set of observers is given by an integral whose integrand is quadratic in first-order derivatives of the field:
\[
E_G = Mc^2 - E_M = \int_{\infty} \left( \hat{\mathcal{T}}^{\mu\nu} w^\nu + \hat{\mathcal{T}}^{\mu\nu} \hat{\mathcal{D}}_\nu w^\lambda \right) \text{d}V^\mu.
\] (2.23)

This is an integral that gives the total gravitational energy in a stationary spacetime with localized sources. The integrand is quadratic in first-order derivatives and is covariant.

Now we examine the gravitational energy in some particular static and stationary spacetimes.

3. Spherically symmetric static spacetimes

Considered in isotropic coordinates, the metric is
\[
\text{d}s^2 = a^2 (\text{d}x^0)^2 - b^2 \text{d}r^2 \quad \text{with} \quad \text{d}r^2 \equiv \sum_k (\text{d}x^k)^2,
\] (3.1)

where \( a \) and \( b \) are the functions of \( r \) with \( a(\infty) = b(\infty) = 1 \). The background metric is, in Minkowski coordinates,
\[
\text{d}s^2 = (\text{d}x^0)^2 - \text{d}c^2.
\] (3.2)

The components of \( \hat{g}^{\mu\nu} \) and the nonzero \( L \) are as follows:
\[
\hat{g}^{00} = \frac{b^3}{a} \quad \text{and} \quad \hat{g}^{kl} = -ab^{kl}.
\] (3.3)

Moreover with \( n^k \equiv \frac{x^k}{r} \),
\[
L^{k00} = a^2 \left( \frac{3b'}{b} - \frac{a'}{a} \right) n^k, \quad L_{k0} = -b^2 \left( \frac{b'}{b} + \frac{a'}{a} \right) n^k \delta_{mn},
\] (3.4)

and
\[
L^{k00} = -\frac{1}{a^2 b^2} \left( \frac{3b'}{b} - \frac{a'}{a} \right) n^k, \quad L^{k0} = \frac{1}{b^4} \left( \frac{b'}{b} + \frac{a'}{a} \right) n^k \delta_{mn}.
\] (3.5)

With these expressions one can calculate the gravitational energy:
\[
E_G = -\frac{1}{\kappa} \int_{\infty} (b^{-1})^2 \text{d}V \quad \text{where} \quad \text{d}V = b^3 \text{d}^4x.
\] (3.6)
Here and later on, $dV$ represents the proper volume element. This simple formula shows that $E_G < 0$. Moreover it gives a nice 3-coordinate-independent gravitational energy density:

$$\epsilon_G \equiv -\frac{1}{\kappa} (b^{-1})^2.$$  \hspace{1cm} (3.7)

Using the Schwarzschild solution for empty space in isotropic coordinates, we also find that

$$E_G = -\frac{1}{\kappa} \int_{r \leq r_M} (b^{-1})^2 dV - \frac{1}{2} \frac{GM^2}{r_M},$$  \hspace{1cm} (3.8)

where $r_M$ is the radial isotropic coordinate boundary of the matter. If the matter is a thin hollow shell the integral is zero.

By definition $E_G$ is the same as the $\Omega$ of MTW [17]. To make the connection explicit, consider the same metric in Schwarzschild coordinates,

$$ds^2 = a^2 dt^2 - B^2 dR^2 - R^2 d\omega^2$$ \hspace{1cm} (3.9)

where $b \frac{dr}{dR} = B$ and $br = R$.

Consider also one of Einstein’s equations in isotropic coordinates $G^0_0 = \kappa T^0_0$ or, see [25],

$$-\frac{4\sqrt{b}}{b} r^2 [r^2 (\sqrt{b})']' = \kappa \rho c^2.$$  \hspace{1cm} (3.10)

One easily finds, using (3.9), (3.10) and the well-known solution$^9$ of (3.10) for $\rho = 0$, i.e.

$$\sqrt{b} = 1 + \frac{GM/c^2}{2r},$$  \hspace{1cm} (3.11)

that (3.8) can be rewritten in the form given in [17] which in our notation reads

$$E_G = -\int (1 - B^{-1}) \rho c^2 dV.$$  \hspace{1cm} (3.12)

4. $E_G$ for conformastationary metrics

The gravitational energy can also be explicitly evaluated for the far less symmetric conformastationary metrics [22] which may be written as follows [11]:

$$ds^2 = f (dx^0 - A_k dx^k)^2 - f^{-1} dr^2.$$  \hspace{1cm} (4.1)

It has been noted in [4] that this metric is in harmonic coordinates provided, here at least, that $A_k$ is divergenceless. This leaves no gauge freedom except for global translations and Lorentz rotations.

Solutions of the Einstein–Maxwell equations with this form of metric have been the object of much scrutiny in empty$^{10}$ and non-empty spaces [11]. The background metric is again $c^2 \, dt^2 - dr^2$. The elements we need to calculate $E_G$ are $\hat{g}^{\mu\nu}$, $w^{\mu}$ and $L_{\mu}^{\mu\nu}$. First we write $g_{\mu\nu}$ and $\hat{g}^{\mu\nu}$:

$^9$ Which we shall only require for $r \to \infty$.

$^{10}$ See [22] for a review of the subject.
\[ g_{00} = f, \quad g_{0l} = -f A_l, \quad g_{kl} = -f^{-1} \delta_{kl} + f A_k A_l, \quad \sqrt{-g} = f^{-1}, \quad (4.2) \]
\[ \hat{g}^{00} = f^{-2} - |\vec{A}|^2, \quad \hat{g}^{0l} = -A_l, \quad \hat{g}^{kl} = -\delta^{kl} \quad \text{with} \quad |\vec{A}|^2 = \sum_k A_k^2. \quad (4.3) \]

Next, following (2.19),
\[ w^0 = \frac{1}{\sqrt{f}}, \quad w^k = 0. \quad (4.4) \]

We shall considerably simplify the formulae for the \( L \) by using where possible a vector notation for 3-vectors such as \( \vec{A} = \{A_k\} \). Thus the nonzero \( L \) are
\[ L^{00} = -f^2 \vec{A} \cdot \vec{\nabla} (f^{-2} - |\vec{A}|^2), \quad L^{0l} = -f^2 \partial_k (f^{-2} - |\vec{A}|^2), \quad (4.5) \]
\[ L^{00} = f^2 \vec{A} \cdot \vec{\nabla} A_l, \quad L^{kl} = f^2 \partial_k A_l, \quad (4.6) \]
\[ L_{k00} = -2 \partial_k f, \quad L_{k0l} = -f^{-1} \partial_k (f^2 A_l), \quad L_{k00} = -f^{-1} \partial_k (f^2 A_m A_n). \quad (4.7) \]

With these expressions one can easily calculate \( E_G \), which, as we shall see, is negative definite.

Set
\[ E_G = \int_{\infty} \epsilon_G dV \quad \text{with} \quad dV = \left(1 - f^2 |\vec{A}|^2\right)^{\frac{1}{2}} f^{-\frac{3}{2}} d^3x. \quad (4.8) \]

Then the gravitational energy density \( \epsilon_G \), a scalar in 3-space, is
\[ \epsilon_G = -\frac{1}{8\pi G} \left[ f^{-1} |\vec{\nabla} f|^2 + f^3 |\vec{\nabla} \times \vec{A}|^2 + f^2 \vec{\nabla} f \cdot \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \right] (1 - f^2 |\vec{A}|^2)^{-\frac{1}{2}}. \quad (4.9) \]

To gain some insight into this formula let us consider a few special cases.

(i) **Static weak field.** This is not a particularly interesting case except to show that in the weak field approximation we recover the classical Newton \( G \)-energy. Indeed in this case \( \vec{A} = 0 \) and, setting \( f \approx 1 + 2\phi/c^2 \), we have
\[ \epsilon_G = -\frac{1}{8\pi G} |\vec{\nabla} \phi|^2, \quad (4.10) \]
where \( \phi \) is the Newtonian potential. This formula can also be recovered from (3.7) with (3.11).

(ii) **Static fields.** If \( \vec{A} = 0 \) but \( f \) is not nearly equal to one, the metric is that of Papapetrou [18] and Majumbar [16], and
\[ \epsilon_G = -\frac{1}{8\pi G} |\vec{E}|^2 \quad \text{where} \quad \vec{E} = c^2 \vec{\nabla} \ln \sqrt{f}. \quad (4.11) \]
\( \vec{E} \) acts as a potential force or what is also referred to as a ‘gravoelectric’ force per unit mass on test particles (see below). Synge [23], who called those metrics ‘conformastatic’, showed that solutions of Einstein’s equations of this form appear for static charged dust with electric and gravitational forces in exact balance; the electric charge density \( \sigma = \sqrt{G} \rho \). It is interesting to note that the electric field in this case\(^\text{12}\) is \( \vec{E} = \frac{c^2}{\sqrt{G}} \vec{\nabla} \sqrt{f} \) and the corresponding electromagnetic energy density
\[ \epsilon_{EM} = \frac{1}{8\pi} |\vec{E}|^2 = -\epsilon_G. \quad (4.12) \]

\(^{11}\) With one exception, see (4.14).
\(^{12}\) See for instance [15]. Incidentally on page 8 of that paper there is a typographical mistake in the expression for \( \vec{E} \); there must be a \( 1/\sqrt{G} \).
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There is no local force in the matter. There is also no local field energy density in this case. The total energy is equal to the mass energy of the dust. All this seems to make good sense.

(ii) Non-static fields. In the general case, $\epsilon_G$ can be written in terms of the gravoelectric field $\vec{E}$ and a ‘gravomagnetic’ field

$$\vec{B} = c^2 \sqrt{f} \vec{\nabla} \times \vec{A}.$$  

The reason for this electromagnetic analogy is that the equation for slow motions of test particles, in strong fields as seen in a 3-space with metric $f^{-1} \delta_{kl}$ [14], with the velocity $\vec{v}$ and the momentum $\vec{p}$,

$$\vec{v} = \left\{ v^k = c \frac{dx^k}{d\tau} = \frac{dx^k}{d\tau} \right\} \quad \text{and} \quad \vec{p} = \left\{ p_k = f^{-1} m v^k \right\} \quad \text{is} \quad \frac{d\vec{p}}{d\tau} = m \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right).$$

In terms of $\vec{E}$, $\vec{B}$ and $\vec{A}$, expression (4.9) can be written as

$$\epsilon_G = -\frac{1}{8\pi G} \left[ f \left( 1 - \frac{1}{4} f^{-1} |\vec{A}|^2 \right) |\vec{E}|^2 + \frac{1}{4} |\vec{B} + \sqrt{f} \vec{\nabla} \times \vec{A}|^2 + \frac{1}{4} f^{-1} (\vec{A} \cdot \vec{E})^2 \right] \left( 1 - f^{-2} |\vec{A}|^2 \right)^{-1/2}.$$  

One can see from this expression that $\epsilon_G$ is manifestly negative. For comparison we note that electromagnetic energy density in this spacetime has a similar form. Denoting by $\vec{E} = \{ E_k \equiv F_{k0} \equiv \partial_0 A_k \}$ and $\vec{B} = \{ B_k \equiv \sum \epsilon_{kmn} \partial_m A_n \}$ the electromagnetic vector fields,

$$\epsilon_{EM} = \frac{1}{8\pi} \left[ (1 - f^{-2} |\vec{A}|^2) |\vec{E}|^2 + f^2 |\vec{B}|^2 + f^2 (\vec{A} \cdot \vec{E})^2 \right] \left( 1 - f^{-2} |\vec{A}|^2 \right)^{-1/2}.$$  

5. $E_G$ of a weak stationary field

In Minkowski coordinates in the background, the metric of the foreground is usually written as

$$dx^2 = (\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu \, dx^\nu,$$

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad \text{and} \quad |h_{\mu\nu}| \ll 1.$$

Then

$$\tilde{g}^{\mu\nu} = \eta^{\mu\nu} - (h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h) \equiv \eta^{\mu\nu} - \tilde{h}^{\mu\nu} \quad \text{with} \quad h = \eta^{\rho\sigma} h_{\rho\sigma} \quad \text{and} \quad L_{\rho}^{\mu\nu} = -\partial_\rho \tilde{h}^{\mu\nu}.\quad (5.2)$$

In harmonic coordinates, $\partial_\rho \tilde{h}^{\mu\nu} = 0$, there is no further gauge freedom available except uniform translations and Lorentz rotations. Stationarity implies $\partial_0 \tilde{h}^{\mu\nu} = 0$ while Einstein’s equations become

$$\nabla^2 \tilde{h}^\mu_\nu = 2\kappa T^\nu_\mu.$$  

Static observers have velocity components

$$w^0 = 1 - \frac{1}{2} h^0_0 \quad \text{and} \quad w^k = 0.$$  

If we regard the $h^0_0$ as components of a vector $\vec{h}$ in the background whose curl ($\vec{\nabla} \times \vec{h}$) = $\left\{ \sum \epsilon_{mkl} \partial_k h^l_0 \right\}$, and further decompose $\tilde{h}^0_0$ into a traceless part $\tilde{r} \tilde{h}^0_0$ and a trace $\tilde{h}^0_0$,

13 This is the 3-space in which proper lengths are measured.
we find that
\[ \epsilon_G = -\frac{1}{4\kappa} |\nabla \times \tilde{h}|^2 - \frac{1}{16\kappa} \left( \frac{1}{3} |\nabla \tilde{h}_k|^2 + \frac{1}{3} |\nabla \tilde{h}_k|^2 - 2 \nabla T \tilde{h}_m \cdot \nabla T \tilde{h}_m \right) \]
where
\[ \tilde{T} \tilde{h}_m \equiv \tilde{h}_m - \frac{1}{3} \delta_m^k \tilde{h}_k. \]
(5.5)

Note that the expression in terms of \( h \) instead of \( \tilde{h} \) is exactly the same. We see that \( \epsilon_G \) is not manifestly negative. However, if \( \nabla_T \tilde{h}_m = 0 \) or is negligible, then \( G \)-energy is indeed negative. This is, for instance, the case if the source of gravity is a perfect fluid for which \( \tilde{T}_0^0 \equiv \rho c^2 \) and \( \tilde{T}_m^n = -P \delta_m^k \). In this case, Einstein’s equations (5.3) imply that \( \nabla^2 (\tilde{T} \tilde{h}_m) = 0 \) because \( T_{\tilde{m}}^m \) is now traceless. From this follows, with usual boundary and regularity conditions, that
\[ \nabla_T \tilde{h}_m = 0 \]
and,
\[ \epsilon_G = -\frac{1}{4\kappa} |\nabla \times \tilde{h}|^2 - \frac{1}{16\kappa} \left( \frac{1}{3} |\nabla \tilde{h}_0|^2 + \frac{1}{3} |\nabla \tilde{h}_k|^2 \right) < 0. \]
(5.6)

The equation of slow motion of a test particle in the weak field has the form (4.14) with
\[ \vec{E} = -c^2 \vec{\nabla} \tilde{h}_0^0 \] and \[ \vec{B} = c^2 \vec{\nabla} \times \tilde{h}. \]
(5.7)

If \( |\nabla \tilde{h}_k| \ll |\nabla \tilde{h}_0| \), as in a perfect fluid with non-relativistic pressure or tension, \( |P| \ll \rho c^2 \), the gravitational energy density is
\[ \epsilon_G = -\frac{1}{8\pi G} \left( |\vec{E}|^2 + \frac{1}{4} |\vec{B}|^2 \right). \]
(5.8)

The source of \( E_G \) may be a fluid in fast, but not too fast and not necessarily uniform rotational motion, with vorticity, for instance, as in a Dedekind ellipsoid. 

Note that Landau and Lifshitz [14] prefer to regard the gravomagnetic force as a Coriolis force produced in an orthogonal frame rotating with the absolute angular velocity \( \Omega = \frac{c}{2} \nabla \times \tilde{h} \).

6. Comments

In a similar vein we can calculate gravitational energy of cosmological perturbations on a Robertson–Walker background. These backgrounds all have a conformal timelike Killing field. Thus one may be facing ‘conformal matter–energy’ and ‘conformal \( G \)-energy’. Conservation laws and additivity of different forms of energy remain valuable knowledge but the conserved vector may not have the simple form it has here in (2.17) and the interpretation of globally conserved quantities is quite different from asymptotically flat spacetimes.

It is possible that \( E_G \) is negative for any stationary metric with reasonable physical sources, that is, those whose energy–momentum tensor satisfies reasonable energy conditions. It is, however, interesting to note that \( E_G \) is negative in some spacetimes with special symmetries irrespective of any energy condition.

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