ENUMERATION OF PARTITIONS MODULO 4

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Abstract. The number of standard Young tableaux possible of shape corresponding to a partition \( \lambda \) is called the dimension of the partition and is denoted by \( f^\lambda \). Partitions with odd dimensions were enumerated by McKay and were further classified by Macdonald. Let \( a_i(n) \) be the number of partitions of \( n \) with dimension congruent to \( i \) modulo 4. In this paper, we refine Macdonald’s and McKay’s results by calculating \( a_1(n) \) and \( a_3(n) \) when \( n \) has no consecutive 1s in its binary expansion or when the sum of binary digits of \( n \) is 2 and giving a recursive formula to compute \( a_2(n) \) for all \( n \).

1. Introduction

1.1. Foreword. A partition of \( n \) is a tuple of natural numbers, \( \lambda := (\lambda_1, \ldots, \lambda_k) \), with non-increasing entries that add up to \( n \). The dimension of the partition \( \lambda \), denoted by \( f^\lambda \), is the number of standard Young tableaux of shape \( \lambda \) (ref. Section 2.1) and can be calculated by the famous hook-length formula (Remark 21). Denote by \( m_p(n) \), the number of partitions whose dimensions are not divisible by a given natural number, \( p \).

Macdonald, in his landmark paper [Mac71], gave a complete answer for \( m_p(n) \) which can be understood quite elegantly using the \( p \)-core tower approach (ref. Section 6). The case for \( p = 2 \) has a wonderful form. For \( n = 2^{k_1} + \ldots + 2^{k_\ell} \) with \( k_1 > \ldots > k_\ell \), we have,

\[
m_2(n) = 2^{k_1+\ldots+k_\ell},
\]

which is entirely dependent on the binary expansion of \( n \).

Note that \( m_2(n) \) enumerates partitions of \( n \) which have an odd dimension. We call these odd partitions and will denote their count by \( a(n) := m_2(n) \).

Recently, there has been some interest in extending Macdonald’s results. In their unpublished notes, Amrutha P and T. Geetha [AG22] tackle the problem of computing \( m_{2k}(n) \). They provide general recursive results and find \( m_4(2^k) \) and \( m_8(2^k) \). As we shall see, the enumeration results for \( m_4(n) \) follow from our analysis of “sparse numbers”, i.e., numbers with no consecutive 1s in their binary expansions.
This study of partitions has a representation theoretic context as partitions of $n$ index the irreducible representations of the symmetric group, $S_n$. There are results regarding the behaviour of irreducible representations corresponding to odd partitions under restriction ([Gia+16], [APS16]). There are results ([Pel20], [GPS20]) which show that the density of odd character values (not just degrees) goes to zero. Although, they give an idea about the asymptotic behaviour of odd partitions, they do not explore the modular properties further. Some papers ([OZ21], [Las08]) give explicit results for character values of symmetric groups but the formulae do not aid us in enumeration.

We aim to fill this gap by providing explicit enumeration results which extend the odd dimensional partition enumeration formula of Macdonald. We present our results now and data for our quantity of interest, $\delta$, can be found here.

1.2. Main Results. Let $a_i(n)$ denote the number of partitions of $n$ with dimension congruent to $i$ modulo 4. Then the number of odd partitions, $a(n) = m_2(n) = a_1(n) + a_3(n)$. We compute the values of $a_1(n)$ and $a_3(n)$ for some specific values of $n$. Define $\delta(n) := a_1(n) - a_3(n)$. In this paper, we present the following main theorem:

**Theorem 1.** Let $n, m, R \in \mathbb{N}$ with $R \geq 2$ and $m > 0$. Suppose, $n = 2R + m$ with $2R - 1 > m$. Then we have

$$\delta(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 4\delta(m), & \text{if } n \text{ is odd.} \end{cases}$$

Equivalently, for $k_1 > \ldots > k_\ell$ such that $k_1 > k_2 + 1$, we have

$$a_1(2^{k_1} + \ldots + 2^{k_\ell}) = \begin{cases} 2^{k_1+\ldots+k_\ell-1}, & \text{if } k_\ell > 0 \\ 4a_1(\sum_{i=2}^{\ell} 2^{k_i}) + (2^{k_1-1} - 2)2^{k_2+\ldots+k_\ell}, & \text{if } k_\ell = 0, \end{cases}$$

and

$$a_3(2^{k_1} + \ldots + 2^{k_\ell}) = \begin{cases} 2^{k_1+\ldots+k_\ell-1}, & \text{if } k_\ell > 0 \\ 4a_3(\sum_{i=2}^{\ell} 2^{k_i}) + (2^{k_1-1} - 2)2^{k_2+\ldots+k_\ell}, & \text{if } k_\ell = 0. \end{cases}$$

We call a positive integer sparse if it does not have any consecutive ones in its binary expansion, i.e., $k_i > k_{i+1} + 1$ in our notation. For instance, $(42)_{10} = (101010)_2$ is sparse. This case has the following pleasing corollary:
Corollary 2. If \( n \) is a sparse number, then
\[
\delta(n) = \begin{cases} 
2, & \text{if } n = 2 \\
0, & \text{if } n > 2 \text{ is even} \\
4^{\nu(n)-1}, & \text{if } n \text{ is odd},
\end{cases}
\]
where \( \nu(n) \) denotes the number of 1s (or equivalently the sum of digits) in the binary expansion of \( n \). Let \( \ell \geq 2 \), then for \( k_1 > \ldots > k_\ell \geq 0 \) with \( k_i > k_{i+1} + 1 \) for all \( i \), we have
\[
a_1(2^{k_1} + \ldots + 2^{k_\ell}) = \begin{cases} 
2^{k_1+\ldots+k_\ell-1}, & \text{if } k_\ell > 0 \\
2^{k_1+\ldots+k_\ell-1} + 4^{\ell-2}, & \text{if } k_\ell = 0,
\end{cases}
\]
and
\[
a_3(2^{k_1} + \ldots + 2^{k_\ell}) = \begin{cases} 
2^{k_1+\ldots+k_\ell-1}, & \text{if } k_\ell > 0 \\
2^{k_1+\ldots+k_\ell-1} - 4^{\ell-2}, & \text{if } k_\ell = 0,
\end{cases}
\]

Note that we recover explicit and recursion formulae for \( a_1(n) \) and \( a_3(n) \) by using \( \delta(n) = a_1(n) - a_3(n) \) and \( a(n) = a_1(n) + a_3(n) \).

In the case where the binary expansion is \( 1100 \ldots \), we have the following result:

Theorem 3. Let \( n = 2^R + 2^{R-1} \) with \( R \geq 1 \), then
\[
\delta(n) = \begin{cases} 
2, & \text{if } n = 3 \\
8, & \text{if } n = 6 \\
0, & \text{else}.
\end{cases}
\]

Explicitly, we have
\[
a_1(2^R + 2^{R-1}) = \begin{cases} 
2, & \text{if } R = 1 \\
8, & \text{if } R = 2 \\
4^{R-1}, & \text{else},
\end{cases}
\]
and
\[
a_3(2^R + 2^{R-1}) = \begin{cases} 
0, & \text{if } R = 1 \\
0, & \text{if } R = 2 \\
4^{R-1}, & \text{else}.
\end{cases}
\]

For \( a_2(n) \), which counts even dimensional partitions with \( f^\lambda \) not divisible by 4, we have recursive results for all natural numbers and closed form results for sparse \( n \):

Theorem 4. Let \( n = 2^R + m \) such that \( m < 2^R \). Then we have
\[
a_2(n) = \begin{cases} 
2^R \cdot a_2(m) + \left( \binom{2^{R-1}}{2} \right) \cdot a(m), & \text{if } m < 2^{R-1} \\
2^R \cdot a_2(m) + \left( \binom{2^{R-1}}{3} + 2^{R-1} \right) \cdot \frac{a(m)}{2^{R-1}}, & \text{if } 2^{R-1} \leq m < 2^R.
\end{cases}
\]
Corollary 5. When $n$ is sparse, we have

\[ a_2(n) = \begin{cases} \frac{a(n)}{8}(n - 2\nu(n)), & \text{if } n \text{ is even} \\ a_2(n - 1), & \text{if } n \text{ is odd}. \end{cases} \]

1.3. Structure of the paper. In Section 2, we recall the basic notions relating to partitions such as tableaux, cores and quotients, and the hook-length formula. We present a characterization of odd partitions due to Macdonald [Mac71] and introduce the notion of parents. We also define a function, Od, which extracts the odd part of the input and returns it modulo 4. In Section 3, we prove “the workhorse formula” that relates the Od values of the dimensions of cores and parents which allows us to enumerate $\delta(n)$ recursively. In Section 4 and Section 5, we analyse the workhorse formula to obtain enumeration results and prove Theorem 1 and Theorem 3 respectively. In Section 6, we use the theory of 2-core towers to prove Theorem 4. In Section 7, we present some related problems that remain unsolved and discuss why finding $\delta(n)$ for the case not covered in this paper is hard.

2. Notations and Definitions

We denote by $\mathbb{N}$ the set of natural numbers beginning at 1.

2.1. Partitions, Ferrers diagrams and Young Tableaux. Let $\Lambda \subset \bigcup_{i=1}^{\infty} \mathbb{N}^i$ denote the set of all tuples $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \ldots \geq \lambda_k$.

Definition 6 (Partition). An element of $\Lambda$ is known as a partition. We call $\lambda \in \Lambda$ a partition of $n$ if $\sum_{i=1}^{k} \lambda_i = n$.

Notation 7. We use the notation $\lambda \vdash n$ to denote $\lambda$ is a partition of $n$. Denote the size of $\lambda$ by $|\lambda| := \sum_{i=1}^{k} \lambda_i = n$.

We can represent a partition in the Cartesian plane by constructing a top-left justified array of boxes with the $i^{th}$ row containing $\lambda_i$ many boxes.

Example 8. The partition $\lambda = (4, 3, 3, 1)$ can be represented as

```
  □ □ □ □ □
  □ □ □ □ □
  □ □ □
  □ □
  □
```
This is called the *Ferrers diagram* of $\lambda$ and will be denoted by $\text{sh}(\lambda)$, short for shape on $\lambda$.

The boxes of the Ferrers diagram $\text{sh}(\lambda)$ can be filled with distinct numbers from 1 to $n := |\lambda|$. We impose the constraint that the numbers increase from top to bottom and from left to right. Such a filling is called a *standard Young tableau (SYT)* on the shape $\text{sh}(\lambda)$.

**Example 9.** Continuing with the above example, the following filling

\[
\begin{array}{cccc}
1 & 2 & 7 & 8 \\
3 & 5 & 10 & \\
4 & 9 & 11 & \\
6 & \\
\end{array}
\]

is an SYT on $\text{sh}((4, 3, 3, 1))$.

We can have multiple such SYT of a given shape, and the total number of SYT on $\text{sh}(\lambda)$ is denoted by $f^{\lambda}$. We call this number, the *dimension of the partition* $\lambda$.

### 2.2. Hooks and hook-lengths.

We call the boxes in Ferrers diagrams *cells*. We label the cell in the $i^{th}$ row from top and $j^{th}$ column from left by $(i, j)$. Write $(i, j) \in \text{sh}(\lambda)$ if the $(i, j)$ cell exists.

A very useful tool in the study of partitions is the notion of a *hook*. For a cell $(i, j)$, consider all cells to its right, below it and the cell itself. This constitutes the *hook at $(i, j)$*. The number of cells is known as its *hook-length* and is denoted by $h_{i,j}$.

**Example 10.** In the previous example, the cell at $(1, 2)$ has the hook-length, $h_{1,2} = 5$. The hook is denoted in pink.

### 2.3. Hook removal and cores.

A hook of length $t$ is called a *$t$-hook*. If the Ferrers diagram has a $t$-hook, then remove all the cells contained in the hook. This gives us two (possibly empty) Ferrers diagrams. Slide the bottom diagram to the left and then upwards to reconnect and form a Ferrers diagram.

**Example 11.** Consider the partition $(5, 5, 5, 4, 2)$ and its Ferrers diagram.
In our case, let $t = 5$, that is, we wish to remove a 5-hook from the diagram. We undergo the following procedure:

![Diagram of hook removal process]

to obtain the partition $(5, 4, 3, 2, 2)$

**Definition 12** ($t$-core of a partition). If $\text{sh}(\mu)$ does not have a $t$-hook, then we call $\mu$ a $t$-core.

For a partition $\lambda$, if we keep applying this $t$-hook removal process on the subsequent partitions we obtain, we will eventually reach a $t$-core. The $t$-core of $\lambda$ is independent of the order of hook-removal and is thus unique (cf. Example 8(c) [Mac15] p. 12). We denote it by $\text{core}_t(\lambda)$.

**Notation 13.** The set of all $t$-cores is denoted by $\tilde{\Lambda}_t$.

2.4. $\beta$ sets. The set of first column hook-lengths of $\text{sh}(\lambda)$ for $\lambda = (\lambda_1, \ldots, \lambda_k)$ will be denoted by

$$H(\lambda) = \{h_{i,1} \mid 1 \leq i \leq k\},$$

which we conventionally order in a decreasing fashion. Also, we have the relation, $h_{i,1} = \lambda_i + k - i$.

For any finite $X \subset \mathbb{N} \cup \{0\}$, define the $r$-shift of $X$ to be

$$X^+ = \{x + r \mid x \in X\} \cup \{0, \ldots, r - 1\}.$$ 

We fix $X^+ = X$.

**Definition 14** ($\beta$-set). For a partition $\lambda$, all sets of the form $H(\lambda)^+\, r$ for $r \geq 0$ are known as the $\beta$-sets of $\lambda$.

We can impose the relation, $\sim_\beta$, on finite sets $X, Y \subset \mathbb{N}$ such that $X \sim_\beta Y$ if and only if $X = Y^+\, r$ or $Y = X^+\, r$, for some $r \in \mathbb{N} \cup \{0\}$. This is an equivalence relation on the set of $\beta$-sets, i.e., $\mathbb{N} \cup \{0\}$.
There is a natural way to understand $t$-hooks and $t$-cores through $\beta$-sets.

**Proposition 15.** Let $\lambda$ be a partition and $X$ be a $\beta$-set of $\lambda$. We have that $sh(\lambda)$ contains a $t$-hook if and only if there exists an $h$ in $X$ such that $h > t$ and $h - t$ is not in $X$. Furthermore, if $\mu$ is the partition obtained after removing the $t$-hook, then $H(\mu) \sim_\beta (H(\lambda) \cup \{h-t\}) \setminus \{h\}$.

**Proof.** The proof can be found in Example 8(a) of [Mac15] and Corollary 1.5 on page 7 of [Ols93]. \qed

**Remark 16.** The element $h$, as above, will be referred to as the affected hook-length.

**Notation 17.** Let $\mu$ be obtained from $\lambda$ by removing a $t$-hook. Then denote the affected hook-length by $h^\lambda_\mu$.

The recipe for hook removal is quite clear. Pick an element $h$ of the $\beta$-set, $X$, and subtract $t$ from it. If $h - t \not\in X$, then replace $h$ by $h - t$; otherwise discard $h$ and choose another element from $X$. If no “$x \rightarrow x - t$” replacements can be performed, declare the partition as a $t$-core.

**Example 18.** Let $\lambda = (6, 5, 5, 4, 2)$, then $H(\lambda) = \{10, 8, 7, 5, 2\}$. In this case, we have that $8 \in H(\lambda)$ but $3 \not\in H(\lambda)$. Thus, there exists a 5-hook in $sh(\lambda)$ which we can remove to obtain $\lambda'$. Following the above rules, we get $H(\lambda') = \{10, 7, 5, 3, 2\}$. Now, we can further replace 5 by 0 to obtain $H(\lambda'') = \{10, 7, 3, 2, 0\} \sim_\beta \{9, 6, 2, 1\}$. Finally, we can replace 9 by 4 to get $H(\lambda''') = \{6, 4, 2, 1\}$. No further replacements can be performed. Thus, $\lambda''' = (3, 2, 1, 1)$ is the 5-core of $\lambda$.

We can reconstruct the original partition from a $\beta$-set with $k$ elements by setting $\lambda_i = h_i + i - k$ and considering only the values where $\lambda_i > 0$.

**Notation 19.** If $X$ is a $\beta$-set of $\lambda$, then we define $\text{Part}(X) = \lambda$.

### 2.5. Odd dimensional partitions.

Using the notation, $f^\lambda$, for the dimension of a partition $\lambda$ (refer Section 2.1), we have the following proposition due to Frobenius:

**Proposition 20 (Hook-length formula).** Let $\lambda \in \Lambda$ and $X$ be a $\beta$-set of $\lambda$. If explicitly, $X = \{h_1, \ldots, h_k\}$ such that $h_1 > \ldots > h_k$, then we have

$$f^\lambda = \frac{|\lambda|! \prod_{1 \leq i < j \leq k} (h_i - h_j)}{\prod_{i=1}^{k} h_i!}.$$
Proof. Exercise 9 in [Ful96]. A straightforward manipulation of the formula in Remark 21.

Remark 21. The more famous hook-length formula [FRT54]

\[ f^\lambda = \frac{n!}{\prod h} \]

has a product of all hook-lengths of \( \lambda \) in the denominator. This can be used to derive the formula in Proposition 20.

We call \( \lambda \) an odd partition if \( f^\lambda \) is odd. Macdonald [Mac71] gave a characterization of odd partitions, which we restate in terms of hooks and cores.

Proposition 22. If \( \lambda \) is a partition of \( n = 2^R + m \) with \( m < 2^R \), then \( \lambda \) is an odd partition if and only if \( \lambda \) contains exactly one \( 2^R \)-hook and \( \text{core}_{2^R}(\lambda) \) is also an odd partition.

Proof. The above proposition is stated as Lemma 1 in [APS16] and a proof of it is given in section 6 of the same paper.

Definition 23. If \( \text{core}_{2^R}(\lambda) = \mu \), then we call \( \lambda \) a \( 2^R \)-parent of \( \mu \).

We can now give an explicit form for the \( 2^R \)-parents of odd partitions:

Proposition 24. Let \( 2^R > m \) and \( \mu \) be a partition of \( m \). If \( \lambda \) is a partition such that \( \text{core}_{2^R}(\lambda) = \mu \), then exactly one of the following holds:

Type I. \( H(\lambda) = (H(\mu) \cup \{x + 2^R\}) \setminus \{x\} \) for some \( x \in H(\mu) \).

Type II. \( H(\lambda) = (H(\mu)^+ + \{2^R\}) \setminus \{0\} \) for some \( 1 \leq r \leq 2^R \) such that \( 2^R \notin H(\mu)^+ \).

In particular, each partition of \( m \) has exactly \( 2^R \)-parents.

Proof. By Proposition 15, we get \( (H(\lambda) \cup \{h - 2^R\}) \setminus \{h\} \sim_{\beta} H(\mu) \), which implies \( (H(\lambda) \cup \{h - 2^R\}) \setminus \{h\} = H(\mu)^+ \).

The reader can show that for non-empty finite sets \( A, B, C \subset \mathbb{N} \cup \{0\} \), we have

(1) \( A \setminus B = C \iff A = B \cup C \) if and only if \( B \subset A \)

(2) \( A = B \cup C \iff A \setminus B = C \) if and only if \( B \cap C = \emptyset \).

Using this, we get Type I by letting \( r = 0 \), putting \( h - 2^R = x \). Similarly, we get Type II by putting \( h = 2^R \).

Example 25. Let \( \mu = (2, 2, 2) \) with \( 2^R = 8 \). We have \( H(\mu) = \{4, 3, 2\} \). If we let \( x = 3 \), then we obtain a Type I parent, \( \lambda \), with \( H(\lambda) = \{11, 4, 2\} \). For a Type II parent, we choose \( r = 2 \), which gives us \( \lambda' \) with \( H(\lambda') = \{8, 6, 5, 4, 1\} \). The corresponding Ferrers diagrams for \( \lambda \) (left) and \( \lambda' \) (right) with hooks added to \( \mu \) are:
Using this, we can recover the enumeration result for odd partitions:

**Proposition 26.** Let \( n = 2^{k_1} + \ldots + 2^{k_r} \) with \( k_1 > \ldots > k_r \). Then the number of odd partitions of \( n \) is given by \( a(n) := 2^{k_1 + \ldots + k_r} \).

**Proof.** Let \( n = 2^{k_1} + m \) and \( \mu \vdash m \) be an odd partition. There are exactly \( 2^{k_1} \) many \( 2^{k_1} \)-parents of \( \mu \) (by Proposition 26) and all of them are odd partitions (by Proposition 22). This gives us the recursion, \( a(n) = 2^{k_1} a(m) \) which when iterated gives the formula. \( \square \)

### 2.6. Od function.

**Notation 27.** For \( n \in \mathbb{N} \), let \( v_2(n) \) denote the largest power of 2 that divides \( n \).

**Definition 28.** Let \( Od : \mathbb{N} \to \{\pm 1\} \) be defined as follows

\[
Od(n) = \begin{cases} 
1, & \text{if } n/2^{v_2(n)} \equiv 1 \pmod{4} \\
-1, & \text{if } n/2^{v_2(n)} \equiv 3 \pmod{4}.
\end{cases}
\]

The Od function takes the largest odd number dividing \( n \) and outputs it modulo 4.

**Lemma 29.** For all \( m, n \geq 1 \), we have \( Od(mn) = Od(m) Od(n) \).

**Proof.** The largest odd number dividing \( mn \) is the product of the largest odd numbers dividing \( m \) and \( n \) respectively. Taking modulo 4 over multiplication gives the result. \( \square \)

It also follows that if \( k \) divides \( n \), then \( Od(n/k) = Od(n) Od(k) \).

**Notation 30 (Binary expansion).** Let \( n \in \mathbb{N} \cup \{0\} \).

1. Let \( n = \sum_{i=0}^{k} b_i2^i \) such that \( b_i \in \{0, 1\} \). We write \( n = b_k \ldots b_0 \) if \( b_k = 1 \) and \( b_i = 0 \) for \( i > k \). This is the binary expansion of \( n \).
2. Let \( \text{bin}(n) := \{i \mid b_i = 1\} \), i.e., the “positions” of 1s in the binary expansion of \( n \).
3. Denote the sum of digits by \( \nu(n) := \sum_{i \geq 0} b_i \).
4. Let \( s_2(n) := b_k + b_{k-1} \), i.e., the sum of the leftmost two digits of \( n \).
Remark 31. Let $n$ be as above. Denote $j := \min(\text{bin}(n))$. Then, $\text{Od}(n) \equiv b_{j+1}b_j \mod 4$. This corresponds to taking the binary expansion of $n$, removing all the rightmost zeros and returning the rightmost two digits of the newly obtained string modulo 4.

3. The Workhorse Formula

In this section, we construct a relationship between partitions and their $2^k$-parents, which will allow us to analyse their dimensional behaviour modulo 4. We first define a statistic of consequence inspired by [DW89]:

Definition 32. For a natural number $n$ (with binary expansion $b_k \ldots b_0$), let the number of pairs of consecutive 1s (disjoint or overlapping) be denoted by $D(n)$. Notationally, $D(n)$ is the number of $i$ such that the product $b_i \cdot b_{i+1} = 1$.

Example 33. We have $D(7) = 2$, $D(42) = 0$ and $D(367) = 4$.

We start with the following lemma:

Lemma 34. For any natural number $n$, we have

$$\text{Od}(n!) = (-1)^{D(n)+\nu(\lfloor n/4 \rfloor)},$$

where $\nu(n)$ is the number of 1s in the binary expansion of $n$ as in Notation 30 and $\lfloor \cdot \rfloor$ denotes the integral part.

Proof. By Lemma 29, we have

$$\text{Od}(n!) = \prod_{r=1}^{n} \text{Od}(r) = \prod_{1 \leq r \leq n} \text{Od}(r) \prod_{1 \leq r \leq n} \text{Od}(r).$$

The above rearrangement allows us to use the following two facts to obtain a recursion. Firstly, $\text{Od}(2r) = \text{Od}(r)$ which gives

$$\prod_{1 \leq r \leq n \atop r \text{ even}} \text{Od}(r) = \text{Od}(\lfloor n/2 \rfloor!).$$

We have

$$\prod_{1 \leq r \leq n \atop r \text{ odd}} \text{Od}(r) \prod_{1 \leq r \leq n \atop r \in 4N+1} \text{Od}(r) \prod_{1 \leq r \leq n \atop r \in 4N-1} \text{Od}(r)$$

$$= 1 \cdot \prod_{1 \leq r \leq n \atop r \in 4N-1} (-1).$$

As there are exactly $\lfloor (n+1)/4 \rfloor$ terms of the arithmetic progression $3, 7, 11, \ldots$ less than or equal to $n$, we get
\[ \text{Od}(n!) = (-1)^{\lceil (n+1)/4 \rceil} \text{Od}(\lfloor n/2 \rfloor!). \]

Let \( n \) have the binary expansion \( b_k \ldots b_0 \) as in Notation 30. By using \( \left\lfloor \frac{b_k \ldots b_0}{2} \right\rfloor = b_k \ldots b_1 \) and iteratively simplifying Od(\( \bullet \)), we can rewrite our expression as

\[ \text{Od}(n!) = (-1)^{D(n)}(-1)^{\left\lfloor \frac{b_k \ldots b_0}{4} \right\rfloor + \left\lfloor \frac{b_k \ldots b_1}{4} \right\rfloor + \ldots + \left\lfloor \frac{b_k b_{k-1} \ldots b_2}{4} \right\rfloor + \left\lfloor \frac{b_k b_{k-1} \ldots b_3}{4} \right\rfloor} \text{Od}(b_k!). \]

Notice that \( \left\lfloor \frac{b_k \ldots b_{j+1} b_j + 1}{4} \right\rfloor = \left\lfloor \frac{b_k \ldots b_j}{4} \right\rfloor + 1 \) if and only if both \( b_j \) and \( b_{j+1} \) are 1. For all other cases, we have the equality, \( \left\lfloor \frac{b_k \ldots b_{j+1} b_j + 1}{4} \right\rfloor = \left\lfloor \frac{b_k \ldots b_j}{4} \right\rfloor \).

The inequality occurs exactly \( D(n) \) times, giving us

\[ \text{Od}(n!) = (-1)^{D(n)}(-1)^{b_k b_{k-2} \ldots b_3 + b_4 + \ldots + b_k + 0}. \]

When \((-1)\) is raised to a number, the parity of the number entirely decides the result, thus \((-1)^{b_k \ldots b_j} = (-1)^{b_j}\). Using this, we obtain,

\[ \text{Od}(n!) = (-1)^{D(n)+b_2+b_3+\ldots+b_k} \]

\[ = (-1)^{D(n)+\nu([n/4])}. \]

\[ \square \]

Before stating the main formula, we define an important quantity:

**Definition 35.** Let \( \lambda \) be a \( 2^R \)-parent of \( \mu \) with \( |\mu| < 2^R \). Let \( h^\lambda_\mu \in H(\lambda) \) be the affected hook-length (Remark 16). We define \( \eta^\lambda_\mu \in \mathbb{Z}/2\mathbb{Z} \) as follows:

\[ (-1)^{\eta^\lambda_\mu} = \prod_{x \in H(\lambda) \atop x \neq h^\lambda_\mu} \frac{\text{Od}(h^\lambda_\mu - x)}{\text{Od}([h^\lambda_\mu - 2^R - x])}. \]

Now, we have the arsenal to approach the hook-length formula and apply our Od function to it. This gives us the following proposition:

**Proposition 36** (Workhorse formula). Let \( n = 2^R + m > 3 \) with \( m < 2^R \). Let \( \mu \vdash m \) and \( \lambda \) be a \( 2^R \)-parent of \( \mu \). Let \( h^\lambda_\mu \in H(\lambda) \) be the affected hook-length. Then,

\[ \text{Od}(f^\lambda) = (-1)^{s_2(n)+s_2(h^\lambda_\mu)+\eta^\lambda_\mu} \text{Od}(f^\mu), \]

where \( \eta^\lambda_\mu \) is as defined above and \( s_2 \) is as in Notation 30.
Proof. By applying $\text{Od}$ on Proposition 20 for $\lambda$, we get
\[
\text{Od}(f^\lambda) = (-1)^{D(n)+\nu(\frac{n}{4})} \prod_{h \in H(\lambda)} (-1)^{D(h)+\nu(\frac{1}{4})} \prod_{x > y \atop x, y \in H(\lambda)} \text{Od}(|x-y|).
\]

Suppose, $|H(\mu)^+| = |H(\lambda)|$ for some $r$. Then, we obtain,
\[
\text{Od}(f^\mu) = (-1)^{D(m)+\nu(\frac{1}{4})} \prod_{h \in H(\mu)^+} (-1)^{D(h)+\nu(\frac{1}{4})} \prod_{x > y \atop x, y \in H(\mu)^+} \text{Od}(|x-y|).
\]

Let $n$ have the binary expansion $b_R \ldots b_0$. Then, $m = n - 2^R = b_{R-1} \ldots b_0$. Clearly, $D(m) = D(n) - 1$ if and only if $b_{R-1} = 1$. Also, $\nu([m/4]) = \nu([n/4]) - 1$. These two facts, give us
\[
(-1)^{D(n)+\nu(\frac{1}{4})-D(m)-\nu(\frac{1}{4})} = (-1)^{s_2(n)}.
\]

We have that $h^\lambda_\mu \in H(\lambda) \setminus H(\mu)^+$ whereas $h^\lambda_\mu - 2^R \in H(\mu)^+ \setminus H(\lambda)$. All the other elements of the sets $H(\lambda)$ and $H(\mu)^+$ are same. Carrying out a similar computation as above, we get
\[
\prod_{h \in H(\lambda)} (-1)^{D(h)+\nu(\frac{1}{4})} - \prod_{h \in H(\mu)^+} (-1)^{D(h)+\nu(\frac{1}{4})} = (-1)^{s_2(h^\lambda_\mu)}.
\]

Extending this argument to the remaining product and using Definition 35 gives us the formula. \hfill \Box

Although the above recursion is quite compact, it must be untangled to aid in enumeration. The $s_2$ terms are simple to deal with and so we must investigate the $\eta^\lambda_\mu$ term.

Lemma 37. For any natural number $a$, we have:

1. for $2^R < a < 2^{R+1}$, $\text{Od}(a - 2^R) \neq \text{Od}(a) \iff a = 2^R + 2^{R-1}$;
2. for $0 < a < 2^R$, $\text{Od}(a + 2^R) \neq \text{Od}(a) \iff a = 2^{R-1}$, and
3. for $0 < a < 2^R$, $\text{Od}(2^R - a) = \text{Od}(a) \iff a = 2^{R-1}$.

Proof. Let $a$ have the binary expansion $b_R \ldots b_j \ldots 0$. The results can be shown by noticing that $\text{Od}(a) \equiv b_{j+1}b_j \mod 4$ (see Remark 31) and performing elementary algebraic manipulations. \hfill \Box

Example 38. We have $\text{Od}(50 - 32) = \text{Od}(18) = 1$ as $9$ is $1$ modulo $4$ and $\text{Od}(50) = 1$. On the other hand, $\text{Od}(48 - 32) = \text{Od}(16) = 1$ whereas $\text{Od}(48) = -1$. 


Definition 39 (Indicator function). Let $\mathbb{I}_X(x)$ be equal to 1 if $x \in X$ otherwise 0. We call $\mathbb{I}_X$, the indicator function on $X$.

Notation 40. For a partition $\lambda$ of $n = 2^R + m$ with $m < 2^R$ and $h \in H(\lambda)$, let $N_\lambda(h) = \#\{y \in H(\lambda) \mid h - 2^R < y < h\}$.

Lemma 37 enables us to prove the following proposition which forms the crux of our enumeration endeavour:

Proposition 41. Let $\lambda$ be a $2^R$-parent of $\mu$ with $|\mu| < 2^R$. Then, we have

$$\eta^\lambda_\mu = N_\lambda(h^\lambda_\mu) - \mathbb{I}_{H(\lambda)}(h^\lambda_\mu - 2^{R-1}) + \mathbb{I}_{H(\lambda)}(h^\lambda_\mu + 2^{R-1}) + \mathbb{I}_{H(\lambda)}(h^\lambda_\mu - 3 \cdot 2^{R-1}).$$

Proof. We consider the ratio

$$\rho_h(x) := \frac{\text{Od}(|h - x|)}{\text{Od}(|h - 2^R - x|)}.$$ 

In this proof, we will put $h = h^\lambda_\mu$ and consider $x \in H(\lambda)$ except for $x = h$. The idea of this proof is to find values of $x$ such that $\rho_h(x) = -1$, which allows us to compute $\eta^\lambda_\mu$ using

$$(-1)^{\eta^\lambda_\mu} = \prod_{x \in H(\lambda), x \neq h} \rho_h(x).$$

We can now have three cases:

1. Let $x < h - 2^R < h$. The ratio $\rho_h(x)$ simplifies to $\frac{\text{Od}(h - x)}{\text{Od}(h - 2^R - x)}$. Put $h - x$ as $a$ and use (1) in Lemma 37, to give $x = h - 2^R - 2^{R-1}$. Thus,

$$\rho_h(h - 3 \cdot 2^{R-1}) = -1.$$ 

2. Let $h - 2^R < h < x$. Similarly we use (2) from the lemma to get

$$\rho_h(h + 2^{R-1}) = -1.$$ 

3. Let $h - 2^R < x < h$. Now, $\rho_h(x)$ becomes $\frac{\text{Od}(h - x)}{\text{Od}(h + 2^R + x)}$. Using (3) from the lemma, we get that in this range, $\rho_h(x) = 1$ if and only if $x = h - 2^{R-1}$. Thus, for exactly

$$N_\lambda(h) - \mathbb{I}_{H(\lambda)}(h - 2^{R-1})$$

many values, we have $\rho_h(x) = -1$.

Combining these three together gives us the proposition. \qed

Example 42. Pick $\mu$ such that $H(\mu) = \{4, 2, 1\}$ and its 8-parent $\lambda$ such that $H(\lambda) = \{9, 8, 7, 6, 4, 3, 2, 1\}$. Here $h^\lambda_\mu = 8$. We first compute $N_\lambda(H^\lambda_\mu)$ which is $\#\{y \in H(\lambda) \mid 0 < y < 8\} = 6$. Furthermore, $\mathbb{I}_{H(\lambda)}(8 - 4) = 1$ while other indicator functions give us 0. Thus, $\eta^\lambda_\mu = 6 - 1 = 5 = 1$ in this case.
Now consider the 16-parent of \( \lambda, \nu \) such that \( H(\nu) = \{17, 9, 8, 7, 6, 4, 3, 2\} \) and thus \( h_\nu = 17 \). Then, \( N_\lambda(h_\nu) = \#\{y \in H(\nu) \mid 1 < y < 17\} = 7 \). Also, \( I_{H(\nu)}(17 - 8) = 1 \) and the other indicator functions are zero. So, \( \eta_\nu = 7 - 1 = 6 = 1 \).

**Remark 43.** It is important to remember that the equality for \( \eta_\mu \) is always considered in \( \mathbb{Z}/2\mathbb{Z} \).

We are now ready to enumerate odd dimensional partitions.

### 4. Odd Partitions of Sparse Numbers

In this section, we prove Theorem 1 and Corollary 2 through heavy usage of Proposition 36 and 41.

**Definition 44 (Sparse number).** A natural number \( n \) with the binary expansion \( b_k \ldots b_0 \) is called sparse if for all \( 0 \leq i \leq k - 1 \), we have the product \( b_{i+1} \cdot b_i = 0 \).

A sparse number has no consecutive 1s in its binary expansion.

**Example 45.** The number \( 165 = 10100101 \) is sparse while \( 91 = 1011011 \) is not.

Recall that \( a_i(n) \) denotes the number of partitions of \( n \) whose dimensions are congruent to \( i \) modulo 4. Further, \( \delta(n) := a_1(n) - a_3(n) \), and the total number of odd partitions of \( n \) is denoted by \( a(n) = a_1(n) + a_3(n) \). We start by first defining hook partitions and looking into their dimensions, which turn out to be binomial coefficients.

**Definition 46.** A partition \( \lambda \vdash n \) is called a hook partition if it is of the form \((n - b, 1, \ldots, 1)\) for \( 0 \leq b \leq n - 1 \). Note that their first column hook-lengths are given by \( H(\lambda) = \{n, b, b - 1, \ldots, 1\} \).

**Lemma 47.** We have \( \delta(1) = 1 \), \( \delta(2) = 2 \) and \( \delta(2^R) = 0 \) for \( R \geq 2 \).

**Proof.** The cases of \( \delta(1) \) and \( \delta(2) \) are easy to see as they have 1 and 2 partitions respectively, all of dimension 1. By Proposition 22, we deduce that all odd partitions of \( 2^R \) are hook-partitions. Let \( \lambda \vdash 2^R \) such that \( H(\lambda) = \{2^R, b, b - 1, \ldots, 1\} \). Putting this into the hook-length formula (Proposition 20), we get \( f^\lambda = \binom{2^R - 1}{b} \). By using Proposition 48 (stated below) and noticing that \( 2^R - 1 \) contains only 1s in its binary expansion (in particular isn’t sparse), we get the lemma. \( \square \)

**Proposition 48** (Davis and Webb, [DW89]). Let \( n \in \mathbb{N} \) be not sparse, then

\[
\#\{0 \leq k \leq n \mid \binom{n}{k} \equiv 1 \pmod{4}\} = \#\{0 \leq k \leq n \mid \binom{n}{k} \equiv 3 \pmod{4}\}.
\]
Proof. Refer to Theorem 6 of [DW89] for the proof. □

Let \( n = 2^R + m \) with \( m < 2^{R-1} < 2^R \). We proceed with the following strategy: consider an odd partition \( \mu \vdash m \) as our \( 2^R \)-core. Consider all partitions of \( n \) which are the \( 2^R \)-parents of \( \mu \). These are all odd partitions by Proposition 22. By Proposition 36, we can determine \( \delta(n) \) in terms of \( \delta(m) \). Firstly, we see that Proposition 36 simplifies nicely in the \( m < 2^{R-1} \) case.

**Corollary 49.** Let \( n = 2^R + m \) with \( m < 2^{R-1} \). For odd partitions, \( \lambda \vdash n \) and \( \mu \vdash m \) such that \( \lambda \) is a \( 2^R \)-parent of \( \mu \), we have

\[
\Od(f^\lambda) = (-1)^{\eta^\mu_\lambda} \Od(f^\mu).
\]

**Proof.** Let \( n \) have the binary expansion \( b_R b_{R-1} \ldots b_0 \). By the choice of \( m \), it must be that \( b_{R-1} = 0 \) which gives \( s_2(n) = b_R + b_{R-1} = 1 \). Also, we have \( 2^R \leq h^\lambda_\mu < n < 2^R + 2^{R-1} \) as all elements of \( H(\lambda) \) are less than or equal to \( n \) and the affected hook-length is at least \( 2^R \). By the same logic as above, we get \( s_2(h^\lambda_\mu) = 1 \). Adding these quantities modulo 2 gives zero, and the result follows. □

This makes finding \( \eta^\mu_\lambda \) our primary concern. In our particular case of \( m < 2^{R-1} \), even this calculation is simplified. We obtain a corollary of Proposition 41:

**Corollary 50.** Let \( n = 2^R + m \) with \( m < 2^{R-1} \). For odd partitions, \( \lambda \vdash n \) and \( \mu \vdash m \) such that \( \lambda \) is a \( 2^R \)-parent of \( \mu \), we have

\[
\eta^\mu_\lambda = N_\lambda(h^\lambda_\mu) - \mathbb{I}_{H(\lambda)}(h^\lambda_\mu - 2^{R-1})
\]

with notation as in Proposition 41.

**Proof.** We have \( 2^R \leq h^\lambda_\mu < n < 3 \cdot 2^{R-1} \). Clearly, \( \mathbb{I}_{H(\lambda)}(h^\lambda_\mu - 3 \cdot 2^{R-1}) = 0 \). Along similar lines, \( h^\lambda_\mu + 2^{R-1} \geq 2^R + 2^{R-1} > n \) and so it cannot be an element of \( H(\lambda) \). Thus, \( \mathbb{I}_{H(\lambda)}(h^\lambda_\mu + 2^{R-1}) = 0 \). □

We enumerate Type I and Type II parents (Proposition 24) separately.

**Notation 51.** When \( 2^R \) is understood, we denote the set of all \( 2^R \)-parents of \( \mu \) by \( \mathcal{P}(\mu) \). Denote the set of Type I and Type II \( 2^R \)-parents of \( \mu \) by \( \mathcal{P}_1(\mu) \) and \( \mathcal{P}_2(\mu) \) respectively.

We have \( \mathcal{P}(\mu) = \mathcal{P}_1(\mu) \cup \mathcal{P}_2(\mu) \) with \( |\mathcal{P}_1(\mu)| = H(\lambda) \) and \( |\mathcal{P}_2(\mu)| = 2^R - H(\lambda) \).

**Notation 52** (Signed-sum). For any finite \( \Lambda' \subset \Lambda \), and a partition \( \mu \), define the signed-sum, \( S_{\Lambda'}(\mu) = \frac{1}{\Od(f^\mu)} \sum_{\lambda \in \Lambda'} \Od(f^\lambda) \).
We see that $S_{\Lambda'}(\mu)$ counts the number of partitions in $\Lambda'$ with same value of Od as $\mu$ minus the number of partitions in $\Lambda'$ with a different value of Od. For the following discussion, we will consider natural numbers $m$ and $R$ such that $m < 2^{R-1}$.

We now count Type I parents.

**Proposition 53.** Let $\mu$ be an odd partition of $m$. Then, for $\lambda \in \mathcal{P}_1(\mu)$, we have $\mathbb{I}_{H(\lambda)}(h^\lambda_{\mu} - 2^{R-1}) = 0$. Furthermore,

$$S_{\mathcal{P}_1(\mu)}(\mu) = \begin{cases} 0, & \text{if } |H(\mu)| \text{ is even} \\ 1, & \text{if } |H(\mu)| \text{ is odd}. \end{cases}$$

**Proof.** By definition, we have $H(\lambda) = (H(\mu) \cup \{h^\lambda_{\mu}\}) \setminus \{h^\lambda_{\mu} - 2^R\}$ where $h^\lambda_{\mu} > 2^R$. Thus, if $h^\lambda_{\mu} - 2^{R-1} \in H(\lambda)$ then $h^\lambda_{\mu} - 2^{R-1} \in H(\mu)$, which is not possible as $h^\lambda_{\mu} - 2^{R-1} > m$. This shows $\mathbb{I}_{H(\lambda)}(h^\lambda_{\mu} - 2^{R-1}) = 0$. We also have

$$S_{\mathcal{P}_1(\mu)}(\mu) = \sum_{\lambda \in \mathcal{P}_1(\mu)} \frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)}$$

$$= \sum_{\lambda \in \mathcal{P}_1(\mu)} (-1)^{\eta^\lambda_{\mu}}$$

$$= \sum_{\lambda \in \mathcal{P}_1(\mu)} (-1)^{N_{\lambda}(h^\lambda_{\mu})}.$$

If, explicitly, $H(\mu) = \{h_1, \ldots, h_k\}$ with $h_1 > \ldots > h_k$, then the map $\phi : H(\mu) \to \mathcal{P}_1(\mu)$ defined by

$$\phi(h_i) = \text{Part}((H(\mu) \cup \{h_i + 2^R\}) \setminus \{h_i\})$$

is a bijection such that $h^\phi(h_i) = h_i + 2^R$. As all elements of $H(\mu)$ are strictly smaller than $2^{R-1}$, the element $h^\phi(h_i)$ is the largest element in $H(\phi(h_i))$. Thus,

$$N_{\phi(h_i)}(h^\phi(h_i)) = \# \{ y \in H(\phi(h_i)) \mid h_i + 2^R > y > h_i \}$$

$$= \# \{ y \in H(\mu) \mid y > h_i \}$$

$$= i - 1.$$

This gives us,

$$S_{\mathcal{P}_1(\mu)}(\mu) = \sum_{i=1}^{k} (-1)^{i-1}.$$

By considering the parity of $k = |H(\mu)| = |H(\lambda)|$, we get the result. \(\square\)

**Example 54.** We start with the partition $\mu$ such that $H(\mu) = \{4, 2, 1\}$ with Od$(f^\mu) = -1$. Its type I 8-parents are partitions with $\beta$-sets,
\{12, 2, 1\}, \{10, 4, 1\} and \{9, 4, 2\}. The first two have dimensions 55 and 891 which are both \(-1\) mod 4 while the last one has dimension 1925 which is \(1\) mod 4. This would give us \(S_{\mathcal{P}_1(\mu)}(\mu) = -1(-1 + -1 + 1) = 1\) as the proposition tells us.

The similar calculation for Type II parents is more involved. We will eventually prove the following proposition:

**Proposition 55.** Let \(\mu\) be an odd partition of \(m < 2^{R-1} > m\). Then,

\[
S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 
2 - 2(-1)^m, & \text{if } |H(\mu)| \text{ is even} \\
1 - 2(-1)^m, & \text{if } |H(\mu)| \text{ is odd}.
\end{cases}
\]

Fix \(\mu \vdash m\) and \(n = 2^R + m\) with \(m < 2^{R-1}\) as before. If \(\lambda\) is a Type II \(2^R\)-parent, i.e., \(\lambda \in \mathcal{P}_2(\mu)\), we have \(h^\lambda_\mu = 2^R\). Thus, Corollary 50 simplifies to give \(\eta^\lambda_\mu = N_\lambda(2^R) - \mathbb{1}_{H(\lambda)}(2^{R-1})\). Although it looks simpler on the surface, this calculation comes with its own caveat that not all \(r\)-shifts have a corresponding \(2^R\)-parent. To record that, we define a quantity

**Notation 56.** Define

\[
\mathcal{D} := \{1 \leq r \leq 2^R \mid 2^R \notin H(\mu)^{+r}\},
\]

wherein we assume that \(R\) and \(\mu\) are understood. For a set \(X \subset [1, 2^R] \subset \mathbb{N}\), we write \(\mathcal{D}_X := \mathcal{D} \cap X\).

For \(r \in \mathcal{D}\), let

\[
\lambda_{[r]} = \text{Part} \left( (H(\mu)^{+r} \cup \{2^R\}) \setminus \{0\} \right).
\]

One can understand \(\mathcal{D}\) to be the set of all values of \(r\) for which we can create \(2^R\)-parents by doing an \(r\)-shift.

We define a new quantity, which features in the next lemma.

**Definition 57** (Parity gap). For a finite \(X \subset \mathbb{N}\), define the **parity gap** of \(X\), \(G(X)\), to be the number of even elements of \(X\) minus the number of odd elements of \(X\). In notation,

\[
G(X) = \#\{x \in X \mid x \equiv 0 \text{ mod } 2\} - \#\{x \in X \mid x \equiv 1 \text{ mod } 2\}.
\]

**Example 58.** If \(X = \{13, 12, 8, 5, 3, 1, 0\}\) then \(G(X) = 3 - 4 = -1\).

**Notation 59.** Let

\[
\mathcal{P}_2^\uparrow(\mu) := \{\lambda_{[r]} \mid r \in \mathcal{D}_{[1, 2^{R-1}]}\}
\]

and

\[
\mathcal{P}_2^\downarrow(\mu) := \{\lambda_{[r]} \mid r \in \mathcal{D}_{[2^{R-1}+1, 2^R]}\}.
\]

Note that for \(\lambda \in \mathcal{P}_2^\uparrow(\mu)\), it is a \(2^R\)-parent, such that \(H(\lambda)\) may or may not contain \(2^{R-1}\) whereas if \(\lambda' \in \mathcal{P}_2^\downarrow(\mu)\) then \(H(\lambda')\) will always contain \(2^{R-1}\).
Lemma 60. Following the above notation, we claim that $D_{1,2^{R-1}} = [1, 2^{R-1}]$. Further, $S_{P_2^*(\mu)}(\mu) = 2(-1)^{|H(\mu)|} G(H(\mu))$.

Proof. The maximum possible element of $H(\mu)^{+r}$ is $\max(|H(\mu)| + r)$ which itself is strictly smaller than $2^R$ as $|H(\mu)| < m < 2^R - 1$ and $r \leq 2^{R-1}$. Thus, for $1 \leq r \leq 2^{R-1}$, we have $2^R \notin H(\mu)^{+r}$.

For the signed-sum part, $N_{\lambda[\tau]}(2^R) = \{y \in \lambda[\tau] \mid 2^R > y > 0\} = |H(\mu)| + r - 1$ which counts all elements except the largest one. Thus, we get the following summation:

$$S_{P_2^*(\mu)}(\mu) = \sum_{r=1}^{2^{R-1}} (-1)^{|H(\mu)|+r-1}(-1)^{H(\lambda[\tau])}(2^{R-1})$$

$$= (-1)^{|H(\mu)|-1} \sum_{r=1}^{2^{R-1}} (-1)^{r+H(\lambda[\tau])}(2^{R-1}).$$

For every $h \in H(\mu)$, we can choose an $r_h := 2^{R-1} - h$ which ensures that $2^{R-1} \in H(\mu)^{+r_h}$. Conversely, if $2^{R-1} \in H(\mu)^{+r}$, then as $r \leq 2^{R-1}$, we must have $h + r = 2^{R-1}$ for some $h \in H(\mu)$. Thus, there is an injective map $H(\mu) \to P_2^*(\mu)$ given by $h \mapsto \lambda[\tau_h]$. Further, if $h \in H(\mu)$ and $h + r_h = 2^{R-1}$, then $h$ and $r_h$ have the same parity. We now break the sum into two parts depending on whether $2^{R-1}$ belongs to $H(\lambda[\tau])$, equivalently, whether $\mathbb{I}_{H(\lambda[\tau])}(2^{R-1}) = 1$.

$$S_{P_2^*(\mu)}(\mu)$$

$$= (-1)^{|H(\mu)|-1} \left( \sum_{h \in H(\mu)} (-1)^{r_h+H(\lambda[\tau_h])}(2^{R-1}) + \sum_{\text{other } r} (-1)^{r+H(\lambda[\tau])}(2^{R-1}) \right)$$

$$= (-1)^{|H(\mu)|-1} \left( \sum_{h \in H(\mu)} (-1)^{r_h+1} + \sum_{\text{other } r} (-1)^r \right)$$

We add and subtract $\sum_{h \in H(\mu)} (-1)^{r_h}$ which gives us nice summations.

$$S_{P_2^*(\mu)}(\mu) = (-1)^{|H(\mu)|-1} \left( \sum_{h \in H(\mu)} (-1)^{r_h+1} - \sum_{h \in H(\mu)} (-1)^{r_h} + \sum_{h \in H(\mu)} (-1)^{r_h} + \sum_{\text{other } r} (-1)^r \right)$$

$$= (-1)^{|H(\mu)|-1} \left( \sum_{h \in H(\mu)} ((-1)^{r_h+1} - (-1)^{r_h}) + \sum_{r=1}^{2^{R-1}} (-1)^r \right)$$
The sum $\sum_{r=1}^{2^R-1} (-1)^r$ undergoes full cancellation and gives us zero.

$$S_{P_2(\mu)}(\mu) = 2(-1)^{|H(\mu)|-1} \left( \sum_{h \in H(\mu)} ((-1)^{r_{h+1}} + 0) \right) = 2(-1)^{|H(\mu)|} \left( \sum_{h \in H(\mu)} (-1)^{r_h} \right)$$

As $h$ and $r_h$ have the same parity,

$$S_{P_2(\mu)}(\mu) = 2(-1)^{|H(\mu)|} \left( \sum_{h \in H(\mu)} (\mu) (-1)^{r_h} \right) = 2(-1)^{|H(\mu)|} G(H(\mu)).$$

We now hand the reader the following proposition which explicitly states what values $G(X)$ can take where $X$ is a $\beta$-set of an odd partition of $n$.

**Lemma 61.** Let $\lambda$ be an odd partition of $n$ and $X$ be its $\beta$-set. Then, we have

$$G(X) = \begin{cases} 1 - (-1)^n, & \text{if } |X| \text{ is even} \\ (-1)^n, & \text{if } |X| \text{ is odd}. \end{cases}$$

**Proof.** Recall that if $\lambda \in P(\mu)$, then there exists an $0 \leq s \leq 2^R$ such that $H(\lambda) = (H(\mu)^{+s} \cup \{h^\lambda_\mu\}) \setminus \{h^\lambda_\mu - 2^R\}$ for $h^\lambda_\mu \in H(\lambda)$. The case $s = 0$ corresponds to Type I parents.

Using the fact that $h^\lambda_\mu$ and $h^\lambda_\mu - 2^R$ have the same parity, we can deduce that $G(H(\lambda)) = G(H(\mu)^{+s})$. As $X^{+1} = \{x + 1 \mid x \in X\} \cup \{0\}$, one can show that all odd entries become even and all even entries odd with an added even entry 0. Thus, $G(X^{+1}) = 1 - G(X)$. Repeating this, we get

$$G(X^{+s}) = \begin{cases} G(X), & \text{if } s \text{ is even} \\ 1 - G(X), & \text{if } s \text{ is odd}. \end{cases}$$

We can now work our way down hook-by-hook (taking successive $2^S$-cores) and calculate $G(H(\lambda))$ by finding $G(H(\alpha))$ where $\alpha = \emptyset$ when $n$ is even and $\alpha = (1)$ when $n$ is odd. Applying the above relation
repeatedly, we get,
\[
\mathcal{G}(H(\lambda)) = \begin{cases} 
\mathcal{G}(H(\alpha)), & \text{if } |H(\lambda)| - |H(\alpha)| \text{ is even} \\
1 - \mathcal{G}(H(\alpha)), & \text{if } |H(\lambda)| - |H(\alpha)| \text{ is odd.}
\end{cases}
\]

Intuitively, the quantity $|H(\lambda)| - |H(\alpha)|$ counts the total number of $r$-shifts required to reach $\lambda$.

Trivially, $\mathcal{G}(H(\emptyset)) = 0$. If $n$ is even, we obtain
\[
\mathcal{G}(H(\lambda)) = \begin{cases} 
0, & |H(\lambda)| \text{ is even} \\
1, & |H(\lambda)| \text{ is odd.}
\end{cases}
\]

On the other hand, we have $\mathcal{G}(\{1\}) = -1$ and if $n$ is odd, then
\[
\mathcal{G}(H(\lambda)) = \begin{cases} 
2, & |H(\lambda)| \text{ is even.} \\
-1, & |H(\lambda)| \text{ is odd}
\end{cases}
\]

The claim follows immediately from this computation. \qed

**Example 62.** Let $\lambda$, $\mu$, $\nu$ and $\pi$ be partitions such that
- $H(\lambda) = (H(\mu^+5) \cup \{2^R\})\backslash\{0\}$
- $H(\mu)$ is a Type I $2^S$-parent of $\nu$
- $H(\nu) = (H(\pi^+7) \cup \{2^T\})\backslash\{0\}$

By above, we have $\mathcal{G}(H(\nu)) = 1 - \mathcal{G}(H(\pi))$. Further, $\mathcal{G}(H(\mu)) = \mathcal{G}(H(\nu))$ and $\mathcal{G}(H(\lambda)) = 1 - \mathcal{G}(H(\mu))$. Thus, $\mathcal{G}(H(\lambda)) = 1 - (1 - \mathcal{G}(H(\nu))) = \mathcal{G}(H(\pi))$.

**Lemma 63.** For $2^{R-1} + 1 \leq r \leq 2^R$, we have $\lambda_{[r]} \in \mathcal{P}_{2^R}(\mu)$ if and only if $2^{R-1} \notin H(\lambda_{[r-2^{R-1}]})$. Further,
\[
S_{\mathcal{P}_{2^R}(\mu)}(\mu) = \begin{cases} 
0, & \text{if } |H(\mu)| \text{ is even} \\
1, & \text{if } |H(\mu)| \text{ is odd.}
\end{cases}
\]

**Proof.** It is clear that $2^{R-1} \in \lambda_{[r-2^{R-1}]}$ if and only if $2^R \in H(\mu)^{+r}$ for $2^{R-1} + 1 \leq r \leq 2^R$. In this range, the values of $r \notin \mathcal{D}$ are exactly the values of $r$ such that $2^{R-1} \in H(\lambda_{[r]})$. By considering these values as $r_h$ (refer Lemma 60), we get that $|\mathcal{P}_{2^R}(\mu)| = 2^{R-1} - |H(\mu)|$.

For $r > 2^{R-1}$, we have $2^{R-1} \notin H(\lambda_{[r]})$ and thus, $\eta_\mu^\lambda = N_{\lambda_{[r]}}(2^R) + 1$.

Suppose, $\lambda_{[r]}, \lambda_{[r+s]} \in \mathcal{P}_{2^R}(\mu)$ and $r + 1, \ldots, r + s - 1 \notin \mathcal{D}$. This gives us that $H(\mu) = \{h_1, \ldots, h_k, h_{k+1}, \ldots, h_{k+s-1}, \ldots h_l\}$ where for $1 \leq i \leq s - 1$, we have $h_{k+i} = h_{k+i-1} - i + 1$ and $h_{k+i} + r + i = 2^R$. It follows that $h_k + r > 2^R$ and $h_{k+s} + r + s < 2^R$.

We have $N_{\lambda_{[r]}}(2^R) = |H(\mu)| + r - 1 - j_r$ where $j_r$ is the number of elements in $H(\lambda_{[r]})$ strictly greater than $2^R$. As $h_{k+i} + r + s > 2^R$ for $1 \leq i \leq s - 1$, we have $j_{r+s} = j_r + s - 1$. This gives us $N_{\lambda_{[r+s]}}(2^R) =$
Lemma 61

Lemma 60

to obtain

Corollary 2

Theorem 1

Theorem 1,

Let \( r_1 < \ldots < r_{2^{R-1} - |H(\mu)|} \) such that \( \lambda_{[r_1]}, \ldots, \lambda_{[r_{2^{R-1} - |H(\mu)|}]} \in \mathcal{P}_2(\mu) \).

By the above discussion, we have \( N_{\lambda_{[r_1]}}(2R) = N_{\lambda_{[r_1]}}(2^R) + i - 1 \). Thus,

\[
S_{\mathcal{P}_2(\mu)}(\mu) = \sum_{i=1}^{2^{R-1} - |H(\mu)|} (-1)^{N_{\lambda_{[r_1]}(2^R)+1} \cdot 2^{R-1} - |H(\mu)|} \]

\[
= (-1)^{N_{\lambda_{[r_1]}(2^R)+1}} \sum_{i=1}^{2^{R-1} - |H(\mu)|} (-1)^{i-1} \]

\[
= (-1)^{|H(\mu)| + r_1 - j_{r_1}} \sum_{i=1}^{2^{R-1} - |H(\mu)|} (-1)^{i-1} \]

\[
= (-1)^{|H(\mu)| + 2^{R-1} - 2^{R-1} - |H(\mu)|} \sum_{i=1}^{2^{R-1} - |H(\mu)|} (-1)^{i}. \]

The last equality follows by noticing that if \( r_1 = 2^{R-1} + x \), then \( j_{r_1} = x - 1 \) as \( 2^R \in H(\mu) + (2^{R-1} + 1), \ldots, H(\mu) + (2^{R-1} + x - 1) \). Putting the proper parity of \( |H(\mu)| \) and evaluating the expression gives us our result.

We are now ready to prove the result for Type II parents.

Proof of Proposition 55. We use the value of \( G(H(\mu)) \) from Lemma 61 and apply it to the result in Lemma 60 to obtain

\[
S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 
2 - 2(-1)^m, & \text{if } |H(\mu)| \text{ is even} \\
2(-1)^{m+1}, & \text{if } |H(\mu)| \text{ is odd}.
\end{cases}
\]

As \( S_{\mathcal{P}_2(\mu)}(\mu) = S_{\mathcal{P}_1(\mu)}(\mu) + S_{\mathcal{P}_2(\mu)}(\mu) \), we add the result of Lemma 63, to obtain the result of Proposition 55 which is

\[
S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 
2 - 2(-1)^m, & \text{if } |H(\mu)| \text{ is even} \\
1 - 2(-1)^m, & \text{if } |H(\mu)| \text{ is odd}.
\end{cases}
\]

We now have all the tools in our hands to give a proof of Theorem 1 and Corollary 2.

Proof of Theorem 1. If \( \mathcal{P}(\mu) \) denotes the set of \( 2^R \) parents of \( \mu \), then

\[
S_{\mathcal{P}(\mu)}(\mu) = S_{\mathcal{P}_1(\mu)}(\mu) + S_{\mathcal{P}_2(\mu)}(\mu).
\]

Clearly, \( S_{\mathcal{P}(\mu)}(\mu) = 2 - 2(-1)^m \). The theorem follows from the following computation and the observation that \( n \) and \( m \) have the same parity:

\[
\delta(n) = \sum_{\lambda \text{ is odd}} \text{Od}(f^\lambda)
\]
\[ \sum_{\mu \vdash m} S_{P(\mu)}(\mu) \text{ Od}(f^\mu) \]

\[ = (2 - 2(-1)^m) \sum_{\mu \vdash m} \text{ Od}(f^\mu) \]

\[ = (2 - 2(-1)^m)\delta(m). \]

This gives us the result:

\[ \delta(n) = \begin{cases} 
0, & \text{if } n \text{ is even} \\
4\delta(m), & \text{if } n \text{ is odd.} 
\end{cases} \]

\[ \square \]

**Proof of Corollary 2.** Repeatedly apply Theorem 1 and use Lemma 47. The \( \nu(n) - 1 \) comes from \( \delta(1) = 1 \). This gives us the result in the sparse case and we find \( n = 2 \) separately:

\[ \delta(n) = \begin{cases} 
2, & \text{if } n = 2 \\
0, & \text{if } n > 2 \text{ is even} \\
4^{\nu(n)-1}, & \text{if } n \text{ is odd.} 
\end{cases} \]

\[ \square \]

5. Odd Partitions of \( n = 2^R + 2^{R-1} \)

We wish to extend the results of the previous section beyond the sparse case. Although, a general formula remains elusive; we can take confident steps towards it. We consider all numbers whose binary expansions are of the form 1100\ldots. In notation, we consider \( n \) with \( \nu(n) = 2 \) and \( D(n) = 1 \). We now prove Theorem 3.

We use similar methods as we did in the previous section. Recall that for \( \Lambda' \subset \Lambda \) (the set of all partitions), we have

\[ S_{\Lambda'}(\mu) := \frac{1}{\text{ Od}(f^\mu)} \sum_{\lambda \in \Lambda'} \text{ Od}(f^\lambda). \]

We also state a corollary of Proposition 36 which will be pertinent in this case.

**Corollary 64.** Let \( n = 2^R + 2^{R-1} \) for some \( R \geq 2 \). Let \( \lambda \vdash n \) and \( \mu \vdash 2^{R-1} \) be odd partitions such that \( \lambda \) is a \( 2^R \)-parent of \( \mu \). Then,

\[ \text{ Od}(f^\lambda) = (-1)^{s_2(h_\lambda^\mu) + \eta^\lambda_\mu} \text{ Od}(f^\mu). \]
The proof of the above corollary is trivial as the sum of first two digits of \( n \) is 2. Notice that we start from \( R = 2 \) as the case of \( R = 1 \) is slightly different and can be handled independently.

Throughout our discussion, we will take \( \mu \) to be a hook partition such that \( H(\mu) = \{2^{R-1}, b, b-1, \ldots, 1\} \) where the case \( b = 0 \) corresponds to \( H(\mu) = \{2^{R-1}\} \). Now, we first count the \( 2^R \)-parents of Type I which, as before, we denote by \( P_1(\mu) \).

**Lemma 65.** With the above notation, we have

\[
S_{P_1(\mu)}(\mu) = \begin{cases} 
1, & \text{if } b \text{ is even} \\
2, & \text{if } b \text{ is odd.}
\end{cases}
\]

*Proof.* As in the proof of Proposition 53, define the bijection \( \phi : H(\mu) \rightarrow P_1(\mu) \) such that

\[
\phi(h) = \text{Part}((H(\mu) \cup \{h + 2^R\}) \setminus \{h\}).
\]

Firstly, consider \( \lambda := \phi(2^{R-1}) \). In this case, \( s_2(h_\mu^\lambda) = 2 \). Further, using Proposition 41 regarding \( \eta_\mu^\lambda \), we have

\[
\eta_\mu^\lambda = N_\lambda(2^R + 2^{R-1}) - \mathbb{H}(\lambda)(2^R) + \mathbb{H}(\lambda(2^{R+1}) + \mathbb{H}(\lambda))_.
\]

As \( 2^{R-1} \) is the largest entry in \( H(\mu) \), we have \( N_\lambda(2^R + 2^{R-1}) = \#\{y \in H(\lambda) \mid 2^R + 2^{R-1} > y > 2^{R-1}\} \). Further, \( H(\lambda) = \{2^R + 2^{R-1}, b, \ldots, 1\} \) and we can see that it does not contain 0, \( 2^R \) or \( 2^{R+1} \). Thus, \( \eta_\mu^\lambda \) in this case is equal to 0, which gives us \((-1)^0 = 1\).

On the other hand, for \( 1 \leq x \leq b \), define \( \lambda_{[x]} := \phi(x) \). In this case, \( s_2(h_\mu^{\lambda_{[x]}}) = 1 \) and

\[
\eta_\mu^{\lambda_{[x]}} = N_{\lambda_{[x]}}(2^R + x) - \mathbb{H}(\lambda_{[x]}(x + 2^{R-1}) + \mathbb{H}(\lambda_{[x]})(x + 2^R + 2^{R-1}) + \mathbb{H}(\lambda_{[x]})(x - 2^{R-1}).
\]

We have \( H(\lambda_{[x]}) = \{x + 2^R, 2^{R-1}, \ldots\} \) and \( N_{\lambda_{[x]}}(2^R + x) = b + 1 - x \).

As \( 1 \leq x < 2^{R-1} \) and \( |\lambda| = 2^R + 2^{R-1} \), \( H(\lambda_{[x]}) \) cannot contain \( x - 2^{R-1} \) or \( x + 2^R + 2^{R-1} \). From the explicit form of \( H(\lambda_{[x]}) \), we can see that the only element greater than \( 2^{R-1} \) is \( x + 2^R \). Thus, \( \eta_\mu^{\lambda_{[x]}} = b + 1 - x \).

This gives us

\[
S_{P_1(\mu)}(\mu) = 1 + \sum_{x=1}^{b} (-1)^{b+1-x+1}
\]

\[
= 1 + \sum_{i=0}^{b-1} (-1)^{x}
\]

which with appropriate values of \( b \) simplifies to give the intended result. \( \square \)
Notice that as $b = |H(\mu)| - 1$, the parity of $b$ is opposite to the parity of $H(\mu)$. As one gives the other, we use $b$ for convenience.

Now, we move on to the results for Type II $2^R$-parents.

**Lemma 66.** With the above notation, we have

$$S_{P_2(\mu)}(\mu) = \begin{cases} 
3, & \text{if } b \text{ is even} \\
2, & \text{if } b \text{ is odd}.
\end{cases}$$

**Proof.** As before, let $1 \leq r \leq 2^R$ and $\lambda_{[r]}$ denote the $2^R$-parent corresponding to that $r$-shift. In notation,

$$\lambda_{[r]} = \text{Part} \left((H(\mu)^+ \cup \{2^R\}) \setminus \{0\}\right)$$

if and only if $r \in D$, where $D$ is as in Notation 56. For $r \in D$, we have $\lambda_{[r]} \in P_2(\mu)$.

As for all $\lambda \in P_2(\mu)$, $h_\lambda = 2^R$, we have $s_2(h_\lambda) = 1$. Thus,

$$S_{P_2(\mu)}(\mu) = \sum_{\lambda \in P_2(\mu)} (-1)^{\eta^\lambda + 1}$$

$$= (-1) \cdot \sum_{\lambda \in P_2(\mu)} (-1)^{\eta^\lambda}$$

where

$$\eta^\lambda = N_\lambda(2^R) - I_{H(\lambda)}(2^{R-1}) + I_{H(\lambda)}(2^R + 2^{R-1}).$$

We divide the values taken by $r$ into 6 (possibly empty, depending on $b$) intervals, which although overkill, helps illuminate the process better. It is recommended that the reader works out the assertions in the upcoming discussion using the basic definitions. As before, $H(\mu) = \{2^{R-1}, b, b - 1, \ldots, 1\}$. Here, $b$ can take values from 0 to $2^{R-1}$ where $b = 0$ implies $H(\mu) = \{2^{R-1}\}$.

(1) $1 \leq r \leq 2^{R-1} - b - 1$:

In this case, $\eta^\lambda_{[r]} = N_{\lambda_{[r]}}(2^R) = |H(\mu)^+| - 1 = b + r$.

(2) $2^{R-1} - b \leq r \leq 2^{R-1} - 1$:

We have $\eta^\lambda_{[r]} = N_{\lambda_{[r]}}(2^R) - 1$ as $2^{R-1} \in H(\mu)^+$. Thus, $\eta^\lambda_{[r]} = b + r - 1$.

(3) $r = 2^{R-1}$:

As $2^{R-1} + 2^{R-1} = 2^R \in H(\mu)^{+2^{R-1}}$, $2^{R-1} \notin D$.

(4) $2^{R-1} + 1 \leq r \leq 2^R - b - 1$:

For this, we have $\eta^\lambda_{[r]} = (b + r - 1) - 1 = b + r - 2 = b + r$.

(5) $2^R - b \leq r \leq 2^{R-1}$:

For all these values, $r \notin D$. 
(6) \( r = 2^R \):

In this case, \( \eta_{\mu}^{\lambda_{[2^R]}} = N_{\eta_{[2^R]}}^{\lambda_{[2^R]}}(2^R) = 2^{R-1} - 1 \) as \( I_{H(\lambda_{[2^R]})}(2^{R-1}) = I_{H(\lambda_{[2^R]})}(2^R + 2^{R-1}) = 1 \).

We can now combine the above information to give

\[
S_{P_2(\mu)}(\mu) = (-1) \left( \sum_{r=1}^{2R-1-b-1} (-1)^{b+r} + \sum_{r=2R-1-b}^{2R-1-1} (-1)^{b+r-1} + \sum_{r=2R-1+1}^{2R-b-1} (-1)^{b+r} + (-1)^{2R-1} \right).
\]

With some sleight-of-hand, this can be simplified to give

\[
S_{P_2(\mu)}(\mu) = 1 + 2(-1)^{b} \sum_{r=1}^{2R-1-b-1} (-1)^{r+1} + \sum_{r=0}^{b-1} (-1)^{r}.
\]

When \( b \) is even, we get \( S_{P_2(\mu)}(\mu) = 1 + 2 \cdot 1 + 0 = 3 \). When \( b \) is odd, we get \( S_{P_2(\mu)}(\mu) = 1 + 0 + 1 = 2 \).

With all of these ingredients in our hands, we are ready to give a Proof of Theorem 3. Combining the results of Lemma 65 and Lemma 66, we see that \( S_{P(\mu)}(\mu) = 4 \) when \( \mu \vdash 2^{R-1} \). Thus, for \( n = 2^R + 2^{R-1} \), we can do the following computation:

\[
\delta(n) = \sum_{\lambda \vdash n, \lambda \text{ is odd}} \text{Od}(f^\lambda) = \sum_{\mu \vdash 2^R-1, \mu \text{ is odd}} S_{P(\mu)}(\mu) \text{Od}(f^\mu) = 4 \sum_{\mu \vdash 2^R-1, \mu \text{ is odd}} \text{Od}(f^\mu) = 4 \delta(2^{R-1}).
\]

By the results of Lemma 47, we have our theorem, except for \( R = 1 \), which can be done by hand. □

6. Partitions with Dimension 2 modulo 4

We now consider partitions whose dimensions are congruent to 2 modulo 4, i.e., \( v_2(f^\lambda) = 1 \) for all such partitions \( \lambda \).

To tackle the problem of enumeration, we use the machinery of 2-core towers which requires us to introduce the notion of a 2-quotient. Recall that \( \bar{\Lambda}_t \) (Notation 13) denotes the set of all \( t \)-cores. It is well-known (refer Proposition 3.7 of [Ols93] p. 20) that there exists a bijection between the sets, \( \Lambda \) and \( \Lambda^2 \times \bar{\Lambda}_2 \), given by \( \lambda \mapsto (\text{quo}_2(\lambda), \text{core}_2(\lambda)) \).
Here, $\text{quo}_2(\lambda)$ (known as the 2-quotient of $\lambda$) is a pair $(\lambda^{(0)}, \lambda^{(1)})$ with the property,

$$|\lambda| = 2(|\lambda^{(0)}| + |\lambda^{(1)}|) + |\text{core}_2(\lambda)|.$$  

**Notation 67.** We simplify the notation by writing $\lambda^{(ij)}$ for $(\lambda^{(i)})^{(j)}$ for any binary string $i$ and $j \in \{0, 1\}$. Further, $\lambda^{(\emptyset)} = \lambda$ and $(\lambda^{(\emptyset)})^{(j)} = \lambda^{(j)}$.

We recall the construction of 2-core towers as presented in [Ols93]. For $\lambda \in \Lambda$, construct an infinite rooted binary tree with nodes labelled by 2-cores as follows:

- Label the root node with $\text{core}_2(\lambda) = \text{core}_2(\lambda^{(\emptyset)})$.
- If the length of the unique path from the root node to a node $v$ is $i$, then we say that the node $v$ is in the $i^{\text{th}}$ row.
- Every node in the $i^{\text{th}}$ row is adjacent to two nodes in the $(i+1)^{\text{st}}$ row. Define a recursive labelling as follows: if the label of some node, $v$, in the $i^{\text{th}}$ row, is $\text{core}_2(\lambda^{(b)})$ for some binary string $b$, then the two nodes in the $(i+1)^{\text{st}}$ row adjacent to $v$ have labels $\text{core}_2(\lambda^{(b0)})$ and $\text{core}_2(\lambda^{(b1)})$ respectively.

This tree is known as the 2-core tower of $\lambda$.

**Example 68.** We start with the partition $(6, 5, 4, 2, 1, 1)$. In order to compute its 2-core tower, we first compute the 2-quotients at each stage. The reader may refer to the third chapter of [Ols93] to get a refresher on how to take $p$-quotients.

We draw a tree such that for a node labelled by $\lambda$, the left child is labelled by $\lambda^{(0)}$ and the right child is labelled by $\lambda^{(1)}$. We have

The corresponding 2-core tower is given by taking the 2-core of every node.
Every partition has a unique 2-core tower and every 2-core tower comes from a unique partition. This bijection is inherited from the core-quotient bijection above.

We state a well-known classification result for 2-cores.

**Lemma 69.** A partition $\lambda$ is a 2-core if and only if $\lambda = (n, n-1, \ldots, 1)$ for some $n \geq 0$.

**Proof.** Let $\lambda$ be a 2-core. If $x \in H(\lambda)$, then $x - 2 \in H(\lambda)$ otherwise there will be a 2-hook. If $x$ is even, then $x, x-2, \ldots, 2 \in H(\lambda)$. As $2 \in H(\lambda)$, there is a 2-hook, so $H(\lambda)$ cannot contain any even entries. If $x$ is odd, then it must contain all consecutive odd numbers from 1 to $x$, which corresponds to the fact that $\lambda = (n, n-1, \ldots, 1)$ where $n = \frac{x+1}{2}$. The converse is straightforward to check. □

**Notation 70.** Let $T_k(w)$ denote the number of solutions $(\mu[1], \ldots, \mu[2^k])$ to

$$\sum_{i=1}^{2^k} |\mu[i]| = w,$$

where each $\mu[i]$ is a 2-core.

**Lemma 71.** With the above notation, we have,

$$T_k(w) = \begin{cases} 1, & \text{if } w = 0 \\ 2^k, & \text{if } w = 1 \\ \binom{2^k}{2}, & \text{if } w = 2 \\ \binom{2^k}{3} + 2^k, & \text{if } w = 3. \end{cases}$$

**Proof.** When $w = 0$, the only solution is $\mu[i] = \emptyset$ for all $i$. Thus, $T_k(0) = 1$.

When $w = 1$, we have $2^k$ possible options for $\mu[i] = (1)$ as $1 \leq i \leq 2^k$ which gives $T_k(1) = 2^k$.

For $w = 2$, we must choose $\mu[i] = \mu[j] = (1)$, for some $i$ and $j$, and the rest $\emptyset$. Thus, $T_k(2) = \binom{2^k}{2}$.
There does exist a 2-core of size 3, which is \((2, 1)\). If \(w = 3\), we have two options, either we choose \(i_1, i_2, i_3\) such that \(\mu[i_1] = \mu[i_2] = \mu[i_3] = (1)\) or \(j\) such that \(\mu[j] = (2, 1)\). For the former, we get \(\binom{3}{3} = 1\) ways and the latter can be done in \(2^k\) ways, thus giving \(T_k(3) = \binom{2^k}{3} + 2^k\).

**Definition 72** (Weight in a 2-core tower). The weight of the \(k\)th row of the 2-core tower of a partition \(\lambda\) is given by

\[
w_k(\lambda) := \sum_{b \in \{0, 1\}^k} |\text{core}_2(\lambda^{(b)})|,
\]

where we define \(w_0(\lambda) = |\text{core}_2(\lambda)|\).

We state some important properties of 2-core towers and how they relate to dimensions. For the subsequent discussion, we will let \(n = \sum_{i \geq 0} b_i 2^i\), where \(b_i \in \{0, 1\}\). Further, recall that \(\text{bin}(n) = \{i \mid b_i = 1\}\).

**Notation 73.** Let \(\text{bin}'(n) = \{i > 0 \mid b_i = 1\}\).

Note the strict inequality in the definition. Furthermore, we have the relation, \(\text{bin}'(n) = \text{bin}(n) \setminus \{0\}\).

**Proposition 74.** Let \(\lambda\) be a partition of \(n\). We write \(w_i := w_i(\lambda)\). In this case, the following hold:

1. \(\lambda\) is an odd partition if and only \(w_i = b_i\) for all \(i \geq 0\).
2. \(v_2(f^\lambda) = 1\) if and only if for some \(R \in \text{bin}'(n)\), we have \(w_{R-1} = b_{R-1} + 2\), \(w_R = 0\) and \(w_i = b_i\) otherwise.

**Proof.** The proof of (1) can be found in section 4 of [Mac71]. Also, through section 4 of [Mac71], we know that there exists a sequence of non-negative integers \((z_i)_{i \geq 0}\) with \(z_0 = 0\) such that

\[
w_i + z_i = b_i + 2z_{i+1}.
\]

From Section 3 of [Mac71], we know that if \(v_2(f^\lambda) = 1\), then we have \(\sum_{i \geq 0} w_i = \sum_{i \geq 0} b_i + 1\). Combining these, we get that we must have \(z_i = 1\) for exactly one \(i > 0\) and zero for the rest. Putting \(z_k = 1\) gives us the equations,

\[
w_{k-1} = b_{k-1} + 2
w_k + 1 = b_k.
\]

Notice that the second equation above tells us that \(z_k = 1\) is possible only when \(k \in \text{bin}'(n)\). Thus, \(w_k = 0\) and \(w_{k-1} = b_{k-1} + 2\). The rest is clear as \(z_i = z_{i+1} = 0\).

**Notation 75.** Fix an \(n\) with binary expansion \(b_i\). For \(k \in \text{bin}'(n)\), define a sequence of non-negative integers, \(w_k(n) := (w_i(n))_{i \geq 0}\) with the following properties:
(1) \(w_k^{k-1}(n) = b_{k-1} + 2\),
(2) \(w_k^k(n) = 0\), and

(3) \(w_i^k(n) = b_i\) for all other values of \(i\).

Let \(T(w_k(n)) := \prod_{i \geq 0} T_i(w_i^k(n))\).

Note that the above product is the number of ways of constructing 2-core towers corresponding to partitions \(\lambda\) such that \(w_i(\lambda) = w_i^k(n)\), for some \(k \in \text{bin}'(n)\), and \(v_2(f^\lambda) = 1\).

Proof of Theorem 4. Fix an \(n\). By the above discussion (Proposition 74), we have that the number of partitions with dimensions congruent to 2 modulo 4, \(a_2(n)\), is given by

\[
a_2(n) = \sum_{k \in \text{bin}'(n)} T(w_k(n)) = \sum_{k \in \text{bin}'(n)} \prod_{i \geq 0} T_i(w_i^k(n)).
\]

Denote the binary expansion of \(n\) with \(b_i\) as before. If we suppose \(n = 2^R + m\) with \(m < 2^R\), then \(b_R = 1\) and \(b_i = 0\) for \(i > R\). Note that \(\text{bin}'(n) = \text{bin}'(m) \setminus \{R\}\). So, except for \(k = R\), we have \(w_k(n) = w_k(m)\).

Now, we can break the sum up as,

\[
a_2(n) = \left( \sum_{k \in \text{bin}'(n) \setminus \{R\}} \prod_{i \geq 0} T_i(w_i^k(n)) \right) + T(w_R(n)).
\]

For the term on the left of the plus sign, we have

\[
\sum_{k \in \text{bin}'(n) \setminus \{R\}} \prod_{i \geq 0} T_i(w_i^k(n)) = \sum_{k \in \text{bin}'(m) \setminus \{R\}} T_R(w_R^k(n)) \prod_{i \neq R} T_i(w_i^k(n))
\]

\[
= 2^R \sum_{k \in \text{bin}'(m) \setminus \{R\}} \prod_{i \neq R} T_i(w_i^k(n)) \quad (\text{As } w_R^k(n) = b_R = 1)
\]

\[
= \frac{2^R}{T_R(0)} \sum_{k \in \text{bin}'(m) \setminus \{R\}} T_R(0) \prod_{i \neq R} T_i(w_i^k(m)) \quad (\text{As } w_k(n) = w_k(m))
\]

\[
= \frac{2^R}{T_R(0)} \sum_{k \in \text{bin}'(m) \setminus \{R\}} \prod_{i \geq 0} T_i(w_i^k(m))
\]

\[
= 2^R a_2(m).
\]

Now, we deal with the term on the right of the plus sign. We see that

\[
T(w_R(n)) = T_{R-1}(w_{R-1}^R(n)) T_R(w_R^R(n)) \prod_{i \neq R, R-1} T_i(w_i^R(n)).
\]
\[ T_R^{-1}(w_R^{-1}(n)) T_R(w_R^{R}(n)) \prod_{i \neq R, R^{-1}} T_i(b_i) = \frac{T_R^{-1}(w_R^{-1}(n)) T_R(w_R^{R}(n))}{T_R^{-1}(b_{R-1}) T_R(b_R)} a(n). \]

The \( a(n) \) appearing is a consequence of (1) in Proposition 74. We have \( b_R = 1, w_R^{R}(n) = 0 \) and \( w_R^{R-1}(n) = b_{R-1} + 2 \). This simplifies the above expression to
\[ T_R^{-1}(b_{R-1} + 2) \frac{2R a(m)}{2R T_R^{-1}(b_{R-1}) a(n)}. \]

By Lemma 71 and \( a(n) = 2R a(m) \), we can complete the proof to get
\[ a_2(n) = \begin{cases} 2R \cdot a_2(m) + \left( \frac{2^{R-1}}{2} \right) \cdot a(m), & \text{if } m < 2^{R-1} \\ 2R \cdot a_2(m) + \left( \left( \frac{2^{R-1}}{3} \right) + 2^{R-1} \right) \cdot \frac{a(m)}{2^{R-1}}, & \text{if } 2^{R-1} < m < 2^R. \end{cases} \]

The first case corresponds to \( b_{R-1} = 0 \) and the second one corresponds to \( b_{R-1} = 1 \). \( \square \)

In the sparse case, this theorem takes a nice form as evident in Corollary 5. Although, we can prove the corollary using the recursive relations in Theorem 3, we choose to do perform direct computations as it is more illuminating.

**Proof of Corollary 5.** Let \( n \) be a sparse number, i.e., it has no consecutive 1s in its binary expansion. The summation as before is:
\[ a_2(n) = \sum_{k \in \text{bin}'(n)} T(w_k(n)) \]
\[ = \sum_{k \in \text{bin}'(n)} \frac{T_{k-1}(w_{k-1}^k(n)) T_k(w_k^k(n))}{T_{k-1}(b_{k-1}) T_k(b_k)} a(n). \]

As \( n \) is sparse, for \( k \in \text{bin}'(n) \), we have \( b_{k-1} = 0 \). Further, \( w_k^k(n) = 0 \) and \( w_{k-1}^k(n) = b_{k-1} + 2 = 2 \).

This gives us the following summation,
\[ a_2(n) = \sum_{k \in \text{bin}'(n)} \frac{T_{k-1}(2)}{T_k(1)} a(n). \]

By using the results from Lemma 71, this simplifies to
\[ a_2(n) = a(n) \sum_{k \in \text{bin}'(n)} \frac{2^{k-1}}{2^k}. \]
\[
= a(n) \sum_{k \in \text{bin}'(n)} \frac{(2^{k-1})(2^{k-1} - 1)}{2 \cdot 2^k} = a(n) \sum_{k \in \text{bin}'(n)} 2^k - 2.
\]

If \(n\) is even, then this summation becomes \(n - 2\nu(n)\). If \(n\) is odd, the summation becomes \((n - 1) + 2(\nu(n) - 1) = (n - 1) + 2(\nu(n - 1))\). This gives us the final answer as,

\[
a_2(n) = \begin{cases} 
a(n)(n - 2\nu(n)), & \text{if } n \text{ is even} \\
a_2(n - 1), & \text{if } n \text{ is odd.} \end{cases}
\]

\[\square\]

**Remark 76.** The theorem above immediately leads us to the value of \(m_4(n)\). We have \(m_4(n) = a_1(n) + a_2(n) + a_3(n) = a(n) + a_2(n)\) by definition.

### 7. Closing Remarks and Open Problems

It is natural to ask why the case of \(n = 2^R + m\) for \(2^{R-1} < m < 2^R\) is so difficult that it evades our methods. As we saw before, the values of \(S_{P(\mu)}(\mu)\) depend only on \(|\mu|\) (Section 4), thus, independent of the partition \(\mu\) chosen as our \(2^R\)-core. In the remaining case, we do not have this luxury. The values of \(S_{P(\mu)}(\mu)\) do depend on the chosen \(2^R\)-core \(\mu\). For instance, consider two partitions of 21, \(\mu = (17, 3, 1)\) and \(\nu = (12, 3, 3, 2, 1)\). Considering these as 32-cores, one can check that \(S_{P(\mu)}(\mu) = 8\) and \(S_{P(\nu)}(\nu) = 0\).

The difference between the case we tackle in the paper and the unsolved one differs in a few ways. Most importantly, in Lemma 60, when \(2^{R-1} < m < 2^R\) the set \(D_{[1,2^R-1]}\) is a strict subset of \([1,2^{R-1}]\). The injective map defined in the lemma is no longer valid and calculation now depends on the elements of \(H(\mu)\). One runs into a similar problem in Lemma 63 and also has to consider whether \(3 \cdot 2^{R-1} \in H(\mu)\). Even the computation in Proposition 53 starts depending on the entries of \(H(\mu)\).

In the \(2^{R-1} < m < 2^R\) case, one can see that for \(\mu \vdash m\) it might be necessary to consider the further \(2^{R-1}\) core. So, one may have to look at Type I/II parents of Type I/II parents thus greatly increasing the cases.

Furthermore, we saw that \(\delta(n)\) was a power of 4 for sparse numbers. If one looks through the initial run of the data, one may form the
hypothesis that every non-zero $\delta(n)$ is a power of 2, which soon is falsified as $\delta(118) = -384$ which is not a power of 2 and not even positive. Thus, for non-sparse numbers, the values of $\delta(n)$ begin to deviate from “nice” values. The following run from 122 to 127 shows how egregious this deviation can get.

| $n$  | $\delta(n)$ |
|------|-------------|
| 122  | -256        |
| 123  | -256        |
| 124  | 768         |
| 125  | 640         |
| 126  | 168         |
| 127  | 256         |

Along with this roadblock, we present some relevant problems that remain unsolved.

1. What is $\delta(n)$ in the case when $n = 2^R + m$ for $2^R-1 < m < 2^R$?
2. Is it possible to provide a reasonable bound for $S_{P(\mu)}(\mu)$?
3. Can we say something about the dimension of a partition by just looking at its 2-core tower?
4. Is there a characterization of odd partitions modulo 4 in terms of $\beta$-sets?

It is possible to answer (3) and (4) in the case of hook partitions as the analysis simplifies to looking at binomial coefficients.

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