On the Couplings of the Rho Meson in AdS/QCD

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(Dated: January 20, 2005)

We argue that in generic AdS/QCD models (confining gauge theories dual to string theory on a weakly-curved background), the couplings $g_{\rho HH}$ of any $\rho$ meson to any hadron $H$ are quasi-universal, lying within a narrow band near $m_{\rho}^2/f_\rho$. The argument relies upon the fact that the $\rho$ is the lowest-lying state created by a conserved current, and the detailed form of the integrals which determine the couplings in AdS/QCD. Quasi-universality holds even when rho-dominance is violated. The argument fails for all other hadrons, except for the lowest-lying spin-two hadron created by the energy-momentum tensor. Explicit examples are discussed.

The couplings of the $\rho$ meson to pions and to nucleons are remarkably similar, $g_{\rho\pi\pi} \sim g_{\rho NN}$ [1]. Is this accidental or profound? Most other couplings of the $\rho$ cannot be easily measured; even the coupling $g_{\rho\rho}$ is unknown. Lattice gauge theory at large number of colors $N$, where hadrons are more stable and the quenched approximation is valid, could potentially be used to obtain additional information, but there have been few if any efforts in this direction. In the absence of constraints from data or from numerical simulation, the issue of whether the $\rho$ has universal couplings to all hadrons has been left to theoretical speculation.

In this letter we will reexamine the long-standing “$\rho$–coupling universality” conjecture [1]. We view the conjecture as having two parts: (a) the $\rho$ has universal couplings, and (b) the universal coupling is equal to $m_{\rho}^2/f_\rho$. There is very little reason to expect this conjecture to be exact in QCD, but even attempts to explain its approximate validity have relied upon particular arguments which themselves are open to question. We will reexamine this web of arguments in AdS/QCD. AdS/QCD offers us the opportunity to compute all the couplings of an infinite number of hadrons in a four-dimensional confining gauge theory. This makes it an ideal setting for testing theoretical arguments concerning the properties of hadrons. We will examine the AdS/QCD calculation of the $\rho$’s couplings, and make estimates that show they lie in a narrow band near $m_{\rho}^2/f_\rho$. We will then check that this is actually true in explicit models.

We will consider the form factor $F_H(q^2)$ for a hadron $|H\rangle$ with respect to a conserved spin-one current. The same current, applied to the vacuum, creates a set of spin-one mesons $|n\rangle$, where $n = 0$, 1, ...; we will refer to the $n = 0$ state as the “$\rho$”. At large $N$, a form factor for a hadron $H$ can be written as a sum over vector meson poles,

$$F_H(q^2) = \sum_n \frac{f_n g_{nHH}}{q^2 + m_n^2},$$

where $m_n$ and $f_n$ are the mass and decay constants of the vector meson $|n\rangle$, and $g_{nHH}$ is its coupling to the hadron $H$. (Henceforth we will use both subscript-$\rho$ and subscript-0 to denote quantities involving the $\rho$.) Charge conservation normalizes the form factor exactly: we factor out the total charge of the hadron $H$ in our definition of $F_H(q^2)$, so that

$$F_H(0) = 1 = \sum_n \frac{f_n g_{nHH}}{m_n^2}.$$  (2)

A classic argument in favor of $\rho$–coupling universality rests upon an assumption, $\rho$–dominance, which is supported to some degree by QCD data. There is some ambiguity in the terminology, but by our definition, “$\rho$–dominance” means that the $\rho$ gives by far the largest contribution to the form factor at small $q^2$:

$$\frac{f_0 g_{0HH}}{q^2 + m_0^2} \gg \frac{f_n g_{nHH}}{q^2 + m_n^2} \quad (|q^2| \lesssim m_0^2, \ n > 0)$$

(3)

Combined with (2), this condition implies, subject to certain convergence criteria, that

$$1 = \frac{f_0 g_{0HH}}{m_0^2} + \sum_{n=1}^{\infty} \frac{f_n g_{nHH}}{m_n^2} \approx \frac{f_0 g_{0HH}}{m_0^2}$$

(4)

which in turn proves $g_{0HH} \approx m_0^2/f_0$ for all $H$ in short, $\rho$–coupling universality.

However, the convergence conditions for the sums over $n$ are not necessarily met. Convergence cannot, of course, be checked using data. Meanwhile, $\rho$–dominance, though a sufficient condition for $\rho$–coupling universality, is clearly not necessary. The individual terms in the sum in (4) could be large and of alternating sign, and still permit $\rho$–coupling universality. We will see later that this does happen in explicit AdS/QCD examples.

A second and qualitatively different argument treats the $\rho$ meson as a gauge boson [1], with the hope that the broken four-dimensional gauge symmetry might assure $\rho$–coupling universality. Hidden local symmetry [2] is a consistent formulation of this idea, but does not have exact $\rho$–coupling universality as a consequence (except, possibly, in a limit when the $\rho$ becomes massless relative to all other vector mesons, as proposed in [2].) Instead, it is related [3] to a deconstructed version of AdS/QCD, and is subject to the arguments given below.
In AdS/QCD, where gauge theories with ’t Hooft coupling $\lambda = g^2 N$ (g the Yang-Mills coupling, N the number of colors) are related to string theories on spaces with curvature $\sim 1/\sqrt{\lambda}$, these issues take on a new light. Although the $\rho$ cannot be treated as a four-dimensional gauge boson, a $\rho$ meson in AdS/QCD is the lowest cavity mode of a five-dimensional gauge boson. In the limit $\lambda \to \infty$, the $\rho$ meson, along with the entire tower of vector mesons — the remaining cavity modes of the five-dimensional gauge boson — becomes massless. More precisely, the vector mesons become parametrically light compared to the inverse Regge slope of the theory, and thus to all higher-spin mesons. However, the universal properties of the five-dimensional gauge boson do not imply $\rho$-coupling universality; they simply ensure that the corresponding global symmetry charge is conserved and that $F(q^2 \to 0) = 1$.

Nevertheless, the AdS/QCD context offers a new and logically distinct argument for generic and approximate $\rho$-coupling universality, as suggested by [8]. This argument does not rely upon the existence of a limit in which $\rho$-coupling universality is exact, and indeed there is no such limit. Even at infinite $N$ and/or infinite $\lambda$, the couplings of the $\rho$ always remain quasi-universal.

We will consider the string-theoretic dual descriptions of four-dimensional confining gauge theories which are asymptotically scale-invariant in the ultraviolet. For large $\lambda$ and $N$ the dual description reduces to ten-dimensional supergravity, on a space with four-dimensional Minkowski coordinates $x^\mu$, a “radial” coordinate $z$, and five compact coordinates $\Omega$. The coordinates can be chosen to put the metric in the form $ds^2 = e^{2A(z)}\eta_{\mu\nu} dx^\mu dx^\nu + R^2 dz^2/z^2 + R^2 d\Omega^2$; here $R^2 d\Omega^2$ is the metric on the five compact directions, and $R \sim \lambda^{1/4}$ is the typical curvature radius of the space. The coordinate $z$ corresponds to 1/Energy in the gauge theory. The ultraviolet of the gauge theory corresponds to $z \to 0$; the gauge theory is nearly scale-invariant in this region, and the metric correspondingly is that of AdS$_5$ (with $e^{2A(z)} = R^2/z^2$) times a five-dimensional compact space $W$. The infrared, where confinement occurs, is more model-dependent, but generally the radial coordinate terminates, at $z = z_{\text{max}} \sim 1/\Lambda$ [7,8]. Importantly, the metric deviates strongly from AdS$_5$ — scale invariance is strongly broken in the gauge theory — only for $z$ on the order of $z_{\text{max}}$.

The form factor of a hadron $H$ associated to a conserved current $J^\mu$ can be computed easily in AdS/CFT. A conserved current in the gauge theory corresponds to a conserved current in the gauge theory. This gauge field satisfies Maxwell’s equation (or the linearized Yang-Mills equation) in the five-dimensional space, subject to Neumann boundary conditions at $z \to z_{\text{max}}$. For each four-momentum $q^\mu$ there is a corresponding five-dimensional non-normalizable mode $\Psi(q^2, z)e^{i q x}$ of the gauge boson. (The mode will also have some model-dependent structure in the remaining five dimensions, but this structure always factors out because the gauge boson arises from a symmetry — for examples see [7,8].) The form factor is obtained by integrating this mode against the current built from the incoming and outgoing hadron $H$. We take $H$ to be spin-zero; the generalization to higher spin is straightforward. From the ten-dimensional wave function of the incoming hadron of momentum $p$, $\Phi_H(x, z, \Omega) = e^{ip x} \phi(z, \Omega)$, and the corresponding mode for the outgoing hadron with four-momentum $p'$, we construct the current $J_H'(x, z)$:

$$J_H' = -i \int R^2 d^5 \Omega \sqrt{g_\perp} \left\{ \Phi_H^{*} \partial^\mu \Phi_H \right\}. \quad (5)$$

Here $\hat{g}$ is the metric of $W$. Equivalently, $J_H'(z) = (p + p')^\mu e^{i (p-p') x} \sigma_H(z)$ where

$$\sigma_H(z) = \int R^5 d^5 \Omega \sqrt{g_\perp} \phi_H^* (z, \Omega) \phi_H(z, \Omega) \geq 0.$$

The hadron wave functions are normalized to unity:

$$\int dz \, \mu_H(z) \, \sigma_H(z) = 1. \quad (6)$$

where $\mu_H = e^{2A(z)} R/z$ (for a spin-zero hadron.) In terms of $\sigma_H$, the form factor then reduces to

$$F_H(q^2) = g_5 \int dz \, \mu_H(z) \, \Psi(q^2, z) \sigma_H(z). \quad (7)$$

As $q^2 \to 0$, the non-normalizable mode $\Psi(q^2, z)$ goes to a constant, namely $1/g_5$; then Eq. 7 becomes equal to Eq. 4, enforcing the condition $F_H(0) = 1$.

Meanwhile, the $\rho$, as the lowest-mass state created by the conserved current, appears in AdS/QCD as the lowest-normalizable four-dimensional cavity mode of the same five-dimensional gauge boson. The coupling $g_{0HH}$ is computed in almost the same way, by integrating the $\rho$’s ten-dimensional wave function $\varphi_0$ (which is trivial in the five compact dimensions) against the current of the ten-dimensional wave function $\Phi_H$:

$$g_{0HH} = g_5 \int dz \, \mu_H(z) \varphi_0(z) \sigma_H(z). \quad (8)$$

Couplings $g_{nHH}$ for the $n^{th}$ vector meson are computed by replacing $\varphi_0(z)$ with the $n^{th}$ mode function $\varphi_n(z)$.

The function $\varphi_0(z)$, as the lowest normalizable solution of a second-order differential equation, is typically structureless and has no nodes. It can be chosen to be positive definite. On general grounds it grows as $z^2$ at small $z$ (near the boundary) until scale-invariance is badly broken, in the region $z \sim z_{\text{max}}$. Also, because the $\rho$ is a mode of a conserved current, it must satisfy Neumann boundary conditions (so that $\Psi(0, z) = 1/g_5$ is an allowed solution.) Generically it will not vanish at $z_{\text{max}}$. We therefore expect that in the region $z \sim z_{\text{max}}$ the wave function $\varphi_0(z)$ has a finite typical size $\hat{\varphi}_0$. 
We will now use these properties to make some estimates of the above integrals. The normalizable and non-normalizable modes are related by
\[
\Psi(q^2, z) = \sum_n f_n \varphi_n(z) \frac{1}{q^2 + m_n^2}.
\]
(9)

Meanwhile, the $\rho$ meson is normalized:
\[
1 = \int dz \mu_0(z) \varphi_0(z)^2 \sim \mu_0 \varphi_0^2 \delta z ,
\]
(10)

where $\mu_0(z) = R/z$ is a slowly-varying measure factor, $\mu_0$ is the typical size of $\mu_0$ in the region $z \sim z_{max}$, and $\delta z \sim z_{max}/2$ is the region over which $\varphi_0(z) \sim \varphi_0$. Since $\Psi(0, z) = 1/g_5$, applying Eq. (10) to Eq. (9) and using orthogonality of the modes $\varphi_n$ implies
\[
f_0/m_0^2 = \frac{1}{g_5} \int dz \mu_0(z) \varphi_0(z) \sim \frac{1}{g_5} \mu_0 \varphi_0^2 dz .
\]
(11)

For a scalar hadron $H$ created by an operator of dimension $\Delta \geq 1$, the integrand of Eq. (10) is $\mu_H \varphi_0 \sigma_H \sim z^{2\Delta - 1}$, is small at small $z$. Thus the integral is dominated by large $z \sim z_{max}$, where the slowly-varying function $\varphi_0$ is of order $\varphi_0$. Therefore, using Eqs. (6), (8), (10) and (11),
\[
\frac{f_0 g_{0HH}}{m_0^2} \sim \frac{f_0 g_{0} \varphi_0}{m_0^2} \int dz \mu_H(z) \sigma_H(z) \sim 1 .
\]
(12)

This approximate but generic relation applies to all $H$: it can fail for individual hadrons or in extreme models, but will generally hold. This explains the observations in [6]. In no limit, according to this argument, is $\rho$–dominance. Thus $\rho$–coupling universality exact across the entire theory. Instead, our estimates show that approximate $\rho$–coupling universality is a general property to be expected of all $\rho$ mesons in all AdS/QCD models. Strictly we have only shown this for scalar hadrons, but the argument is easily extended to higher spin hadrons.

These estimates are invalid when the $\rho$ meson mode is replaced with the mode for any other hadron, unless that hadron is also the lowest mode (structureless and positive-definite) created by a conserved current (with model-independent normalization and boundary condition at $z = z_{max}$). The only other hadrons of this type are the lowest spin-two glueball created by the energy-momentum tensor and (if present) the lightest spin-3/2 hadron created by the supersymmetry current.

Now let us see that this argument applies in the hard-wall model and D3/D7 model [14,15], reviewed in the appendices of [16]. (These should be viewed as toy models, as neither is precisely dual to a confining gauge theory. Calculations in specific gauge theories, such as the duality cascade [17], are often straightforward but cannot be done analytically.) Our statements about the $\rho$ meson modes in these models can be checked using [16], while those regarding the couplings $g_{0HH}$ are illustrated in the figures below. Figure 1 shows the $\rho$'s couplings $g_{n0HH}$ to the scalar states $|\Delta, a\rangle$ of the hard-wall model, as a function of the excitation level $a$, for all $\Delta < 40$. As expected, the couplings lie in a narrow range near 1. That this is true only of the $\rho$ is indicated in Fig. 2, where the couplings of all hadrons to the $n^{th}$ vector meson are shown. Only the $\rho$'s couplings lie in a narrow range near 1 and are always positive. These properties are also true in the D3/D7 model, shown in the next two figures (with the additional feature that the large-$\Delta$ limit and large-$a$ limit of $g_{0HH}$ are equal.)

As is clear from the figures, both models include hadrons for which $\rho$–coupling universality is approximately true but $\rho$–dominance fails badly. For instance, for the $|2, 9\rangle$ state in the D3/D7 model, $f_0 g_{0,9,9}/m_0^2 \sim 1$, but the terms in Eq. (4) fall off slowly: for $n = 0, 1, 2, 3, 4 \ldots$

\[
f_0 g_{n,9,9}/m_n^2 = 1.49, -1.21, 1.23, -1.12, 1.08, \ldots
\]
(13)

Yet a naive test (Fig. 5) of Eq. (3) moderately supports $\rho$–dominance. Thus $\rho$–coupling universality can lead to apparent $\rho$–dominance even if $\rho$–dominance is false. This might be relevant for the success of the $\rho$–dominance conjecture in QCD, which cannot be directly checked without measuring the higher $g_{nHH}$.  

FIG. 1: The combination $f_0 g_{\Delta n0}/m_0^2$ in the hardwall model, plotted as a function of $a$ for all $\Delta < 40$.

FIG. 2: The combination $f_n g_{\Delta n0}/m_n^2$ in the hardwall model, plotted as a function of $n$ for all $a$ and $\Delta < 40$.

FIG. 3: As in Fig. 1 for the D3/D7 model.
The pattern of couplings thus depends on the pattern of masses; both \( \rho \)-dominance and \( \rho \)-coupling universality can fail, and \( \rho \)-coupling universality can be true even when \( \rho \)-dominance is false. If, for large \( k \), \( m_k \sim k^p \) with \( p \leq 1/2 \), then \( \rho \)-coupling universality fails at large \( \Delta \). A flat-space stringy spectrum \( m_k \sim \sqrt{k} \) is a borderline case, while the low supergravity modes of AdS/QCD have \( p = 1 \). Meanwhile examples in which \( \rho \)-dominance breaks down are easily found. If \( m_k \sim (k + 1)m_0 \), then \( |f_n g_{nHH}/m_n^2| \) is largest for \( n \geq 0 \) but remains large for moderate \( n \). If \( m_k \sim \sqrt{(k + 2)(k + 1)/2} m_0 \), as in the D3/D7 model, then, for large \( \Delta \), \( |f_n g_{nHH}/m_n^2| \) is largest for \( n > 0 \) and \( \rho \)-dominance is badly violated. Some spin-one modes in the D3/D7 model are maximally truncated, including the \( \rho \), and for them \( \rho \)-dominance indeed fails.

In sum, neither \( \rho \)-dominance, nor the hypothesis that the \( \rho \) is somehow a four-dimensional gauge boson, are logically necessary for \( \rho \)-coupling universality, exact or approximate. An independent AdS/QCD argument, applicable only to the lightest mesons created by conserved currents, ensures the \( \rho \)'s couplings are quasi-universal at large \( \lambda \). Could this argument be generalized to QCD? With suitable definition of hadronic wave-functions, using operator matrix elements, it might be possible to directly extend this line of thinking to theories at arbitrary \( \lambda \), including QCD. Alternatively, the applicability of our arguments at smaller \( \lambda \) could be tested using numerical simulations of large-\( N \) (quenched) QCD-like theories.

We thank D.T. Son and L.G. Yaffe for useful conversations. This work was supported by U.S. Department of Energy grants DE-FG02-96ER40956 and DOE-FG02-95ER40893, and by an Alfred P. Sloan Foundation award.

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