FUNDAMENTAL SOLUTIONS OF A CLASS OF HOMOGENEOUS INTEGRO-DIFFERENTIAL ELLIPTIC EQUATIONS

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Abstract. In this paper, we study a class of integro-differential elliptic operators \( L_\sigma \) with kernel \( k(y) = a(y)/|y|^{d+\sigma} \), where \( d \geq 2, \sigma \in (0,2) \), and the positive function \( a(y) \) is homogenous and bounded. By using a purely analytic method, we construct the fundamental solution \( \Phi \) of \( L_\sigma \) if \( a(y) \) satisfies a natural cancellation assumption and \( |a(y) - 1| \) is small. Furthermore, we show that the fundamental solution \( \Phi \) is \(-\alpha^\ast\) homogeneous and Lipschitz continuous, where the constant \( \alpha^\ast \in (0,d) \). A Liouville-type theorem demonstrates that the fundamental solution \( \Phi \) is the unique nontrivial solution of \( L_\sigma u = 0 \) in \( \mathbb{R}^d \setminus \{0\} \) that is bounded from below.

1. Introduction and main results. The fractional Laplacian arises in many branches of sciences such as phase transitions, particles propagation, stratified materials and others (see [1, 15, 30]). In particular, it can be understood as the infinitesimal generator of a stable Levy process. This work is devoted to study the following integro-differential equation like a fractional Laplacian

\[
L_\sigma u(x) = \text{P.V.} \int_{\mathbb{R}^d} [u(x) - u(x + y)] \frac{a(y)}{|y|^{d+\sigma}} dy,
\]

where P.V. stands for the Cauchy principal value. The measurable function \( a(y) \) satisfies

\[
(2 - \sigma)\lambda \leq a(y) \leq (2 - \sigma)\Lambda, \quad \forall \ y \in \mathbb{R}^d,
\]

here \( 0 < \sigma < 2 \) and \( 0 < \lambda \leq \Lambda \). The operator \( L_\sigma \) is a non-locally linear operator, corresponding to purely jump processes when diffusion and drift are neglected.

Before proceeding to the precise statements of our results, let us give some additional contexts. We use := to denote a definition.

\[
L_\sigma^0 u := \text{P.V.} \int_{\mathbb{R}^d} [u(x) - u(x + y)] \frac{dy}{|y|^{d+\sigma}},
\]

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As is well known, the operator $L^\sigma_0$ is called the fractional Laplace operator, which has the symbol $C(d,\sigma)|\xi|^{d-\sigma}$. In that case, $u(x) = C/|x|^{d-\sigma}$ is a fundamental solution of the equation $L^\sigma_0 u = 0$ in $\mathbb{R}^d \setminus \{0\}$, where $C$ depends on $d$ and $\sigma$.

In recent years, there has been a great amount of interest in nonlocal equations. Cafarelli and Silvestre [11] introduced the extension method which turns nonlocal problems involving the fractional Laplacian into local ones in higher dimensions, then the classical theories for local elliptic partial differential equations can be applied. We refer to [7] for broad applications of this method. Later, Cafarelli and Silvestre [12, 13, 26] produced a deep study of Hölder estimates of fully nonlinear nonlocal equations in a sequence of papers, and Dong and Kim [18, 19] made a study of Schauder and $L^p$ estimates of linear nonlocal equations using different methods. We also refer to [17, 32] for symmetry of solutions on nonlocal equations.

In this article we want to construct a fundamental solution of the following equation

$$(L^\sigma u)(x) = 0 \text{ in } \mathbb{R}^d \setminus \{0\}. \quad (4)$$

In the literature the term fundamental solution refers to a viscosity solution of (4), which goes to infinity at the origin and is locally bounded in $\mathbb{R}^d \setminus \{0\}$. It should be noted that, if $\Phi$ is a fundamental solution of the fractional Laplacian, then $L^\sigma_0(\Phi) = 0$ at the origin.

In addition to the elliptical condition (2), we assume that $a(y)$ is a homogeneous function. Moreover, we make a natural cancellation assumption about the integral kernel.

$$\begin{cases} a(ry) = a(y) & \forall r > 0, \\ \int_{\partial B_1} ya(y) dS(y) = 0. \end{cases} \quad (5)$$

In particular, we do not assume that $a(y)$ is continuous or symmetric.

There is a number of paper concerned with fundamental solutions of linear and nonlinear equations. For example, some results on linear equations appeared in [5], moreover, Gilbarg and Strrin [22] made a thorough study of fundamental solutions and isolated singularities of linear equations in the view of modern theories. We refer to [29] for more details on fundamental solutions of quasi-linear equations. Fundamental solutions of the extremal Pucci operator were defined by Labutin [24, 25] and were used to study the removability of singularities for these operators. Armstrong, Sirakov and Smart [2, 3] obtained fundamental solutions for fully nonlinear differential operators which are general, not necessarily radially invariant, and they proved Liouville type theorems for these differential operators. We also refer to [10] more results on singular solutions of the fully nonlinear equations. For nonlocal integro-differential operators, Felmer and Quaas [20, 21] generalized the results in [24]. They obtained the existence of fundamental solutions and Liouville-type theorems for a class of Isaacs integral operators with symmetric kernels. In [16], Chen and Zhang obtained the existence of the Gauss kernels for a class of linear parabolic equations involving the nonlocal elliptic operators.

After constructing the fundamental solution of (4), we show a Liouville-type theorem. We refer to [6, 14, 31] for more details on Liouville-type theorems of fractional Laplacian. The essence of all results on isolated singularity is that, if a function fails to be a solution at an isolated point and it is bounded from below or above in a neighborhood of this point, then it behaves like a fundamental solution of the elliptic operator near this point.
In this work, we assume that the spacial dimension is at least 2. We establish the existence and the main properties of the fundamental solution of (4) in Theorem 1.1.

**Theorem 1.1.** There exists a universal constant $\delta(d, \sigma) > 0$, if (2) and (5) hold with $1 - \delta \leq \lambda \leq \Lambda \leq 1 + \delta$, then there is a nonconstant viscosity solution $\Phi$ of (4), which satisfies

1. $\Phi \in C^{0,1}_{loc}(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d, \mu)$ ($d\mu = dx/(1 + |x|^{d+\sigma})$).
2. For all $r > 0$, the viscosity solution $\Phi$ satisfies the following homogeneous relation
   $$r^{\alpha^*} \Phi(rx) = \Phi(x) \quad \text{in} \quad \mathbb{R}^d \setminus \{0\},$$
   where the constant $\alpha^* \in (0, d)$ depends on $L_{\sigma}$.
3. There exists a universal constant $0 < c < 1$, such that
   $$c|x|^{-\alpha^*} \leq \frac{1}{M} \Phi(x) \leq |x|^{-\alpha^*} \quad (M = \sup_{\partial B_1} \Phi),$$
   where the constant $c$ depends on $d, \sigma$ and $\delta$.
4. The viscosity solution $\Phi$ is unique in the following sense, i.e. if $u \in C_0(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d, \mu)$ is also a viscosity solution with (6) holds, then $u = a\Phi$ for some $a \in \mathbb{R}$.

**Remark 1.** The function $\Phi$ that appears in Theorem 1.1 is said to be the fundamental solution of (4). It is clear that $\Phi$ is positive in $\mathbb{R}$.

Theorem 1.1 allows us to solve a Dirichlet problem on space of homogeneous functions. We define the constant $\alpha^*(L_{\sigma})$ as the scaling exponent of the operator $L_{\sigma}$. Informally, $\alpha^*(L_{\sigma})$ characterizes the intrinsic internal scaling of the operator $L_{\sigma}$, and we think the scaling exponent as a kind of principle eigenvalue of (4) on the unit sphere. As we see in section 3, $\alpha^*(L_{\sigma})$ is defined by

$$\alpha^*(L_{\sigma}) = \sup \left\{ \alpha > 0 : \text{there exists an}(-\alpha)\text{homogeneous supersolution of} \right\}$$

$$L_{\sigma}v \geq 0 \quad \text{such that} \quad v > 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}$$

In order to prove Theorem 1.1, we make use of the strategy developed in [2, 3], which discussed the existence and characterization of singularity of the fundamental solution to homogenous fully nonlinear equation $F(D^2u) = 0$ in $\mathbb{R}^d \setminus \{0\}$. Of course, some new difficulties arise since the integral operator $L_{\sigma}$ is nonlocal. In our proof, we use a technique based on the comparison principle and the Perron method, which permit us to solve a Dirichlet problem on space of homogeneous functions. We define the scaling number $\alpha^*(L_{\sigma})$ in section 3, and show that $L_{\sigma}$ satisfies a comparable principle with respect to $-\alpha$ homogeneous functions for any $0 < \alpha < \alpha^*$.

As an application of Theorem 1.1, we are able to characterize the isolated singularity of a viscosity solution $\Phi$ of (4), where $\Phi$ is positive in $\mathbb{R}^d$ and locally bounded except the origin. The following result shows that $\Phi$ is the only fundamental solution of (4).

**Theorem 1.2.** Suppose that all the hypotheses of Theorem 1.1 hold, and $u \in C(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d, \mu)$ is a viscosity solution of the equation (4). If $u$ is bounded from below or above in $B_1 \setminus \{0\}$ and $u$ is bounded in $\mathbb{R}^d \setminus B_1$, then either $u \equiv c$, or $u \equiv a\Phi + c$ for some $a, c \in \mathbb{R}$.
Remark 2. Theorem 1.2 implies that a bounded viscosity solution of (4) must be a constant.

The paper is organized as follows. In next section, we recall some preliminary definitions and some standard results for integro-differential equations that we use later. The Theorem 1.1 is established in section 4 after we show the scaling number $\alpha^*(L_\sigma) \in (0, d)$ in section 3. Finally, we discuss the behavior of a viscosity solution of (4) near the origin in section 5.

2. Preliminaries. In this section, we prepare some important results for later use. We begin by introducing some notations.

(1) The open ball of radius $r$ with center $x$ is denoted by $B_r(x)$.
(2) For any measurable set $\Omega \subset \mathbb{R}^d$, we denote its measure by $|\Omega|$.
(3) If $f$ is a $L^1$ function, its Fourier transformation is written as $\mathcal{F}(f)$.
(4) The letter $C$ with or without subscripts will denote a positive finite constant, whose exact value is not important and may change in different places.
(5) Throughout this paper, we also write $k(y) = a(y)/|y|^{d+\sigma}$.

Definition 2.1. A function $\psi$ is said to be $C^{1,1}$ at the point $x$, written as $\psi \in C^{1,1}(x)$, if there is a vector $e \in \mathbb{R}^d$ and a number $M > 0$ such that
$$|\psi(x + y) - \psi(x) - e \cdot y| \leq M|y|^2 \quad \text{for any } |y| \text{ small enough.}$$

We say a function is $C^{1,1}$ in a set $\Omega$ if the previous definition holds at every point in $\Omega$ with the uniform constant $M$.

It is easily seen that the value of $L_\sigma u(x)$ is well defined as long as $u \in C^{1,1}(x) \cap L^1(\mathbb{R}^d, \mu)$ (see [11, 14]), where $\mu$ is a measure
$$d\mu = dx/(1 + |x|^{d+\sigma}).$$

In [8, 9, 12], Caffarelli and Cabré have defined viscosity solutions by testing non-divergence operators in $C^{1,1}$ functions. Thus a continuous function $u$ may be a viscosity solution of the integral equation (4). We state the definition.

Definition 2.2. Suppose $\Omega$ is a bounded domain in $\mathbb{R}^d$ and $f \in C(\Omega)$. A function $u : \mathbb{R}^d \to \mathbb{R}$, upper(lower) semi-continuous in $\Omega$, is said to be a viscosity sub-(super-)solution to $L_\sigma u = f$, and we write as $L_\sigma u \leq (\geq) f$ in $\Omega$ if all the following happen,

(a): $x_0$ is any point in $\Omega$, and $U$ is a neighborhood of $x_0$ in $\Omega$,
(b): $\psi \in C^2(\overline{\Omega}), \psi(x_0) = u(x_0)$, and $\psi(x) > (<) u(x)$ for any $x \in U \setminus \{x_0\}$,

then we have $L_\sigma v(x_0) \leq (\geq) f(x_0)$, where

$$v = \begin{cases} 
\psi & \text{in } U, \\
\begin{array}{cc}
u & \text{in } \mathbb{R}^d - U.
\end{array}
\end{cases}$$

A viscosity solution is a function that is both a sub-solution and a super-solution, which is the appropriate notion of weak solutions for elliptic equations in non-divergence form. Note that every equation and inequality in this paper is assumed to be satisfied in viscosity sense.

An important observation of the equation (4) is the following property.

Proposition 1. Suppose $L_\sigma u(x) = f(x)$ in $\mathbb{R}^d$. For any positive constant $r > 0$, if $u_r(x) = u(rx)$ and $f_r(x) = f(rx)$, then we have
$$L_\sigma u_r(x) = r^\sigma f_r(x).$$

(10)
There have considerable works concerning regularity issues of non-local equations, such as Harnack inequalities, Hölder estimates, and non-local versions of $L^p$ estimates. We collect some results on integro-differential equations, which will be used in our work.

**Lemma 2.3.** ([12]). Let $u_k$ be a sequence of functions that are uniformly bounded in $\mathbb{R}^d$ and continuous in $\Omega$ such that

(a): $u_k \to u$ locally uniformly in $\Omega$, and $u_k \to u$ a.e. in $\mathbb{R}^d$,

(b): $L_\sigma u_k = f_k$ in $\Omega$,

(c): $f_k \to f$ locally uniformly in $\Omega$ for some continuous function $f$,

then $L_\sigma u = f$ in $\Omega$.

**Lemma 2.4.** (Harnack inequality, [4, 28]). Suppose that the function $u$ is continuous and non-negative in $\mathbb{R}^d$. If $L_\sigma u(x) = 0$ in $B_2$, then

$$u(x) \leq Cu(x) \quad \forall x \in B_{1/2},$$

where $C$ is a constant depending only on $d, \sigma, \lambda, \Lambda$.

**Lemma 2.5.** (Hölder estimates, [18, 19]). Let $\gamma \geq 0$, and $f \in L^\infty(B_1)$. If $u \in C^2_{loc}(B_1) \cap L^1(\mathbb{R}^d, \mu)$ satisfies

$$L_\sigma u + \gamma u = f \quad \text{ in } B_1,$$

then for any $\beta \in (0, \min(\sigma, 1))$,

$$[u]_{C^\beta(B_{1/2})} \leq C\left(||u||_{L^1(\mathbb{R}^d, \mu)} + osc_{B_1} f\right).$$

Moreover, if $u$ is a bounded function in $\mathbb{R}^d$, then we have

$$[u]_{C^\beta(B_{1/2})} \leq C\left(||u||_{L^\infty(\mathbb{R}^d)} + osc_{B_1} f\right),$$

where $C$ is a constant depending only on $d, \sigma, \lambda, \Lambda$ and $\beta$.

**Definition 2.6.** ([27]). For $p \in (1, \infty)$ and $\sigma > 0$, we use $H^\sigma_p(\mathbb{R}^d)$ to denote the Bessel potential space

$$H^\sigma_p(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d) : (1 - \Delta)^{\sigma/2} u \in L^p(\mathbb{R}^d)\},$$

which is equipped with the norm

$$||u||_{H^\sigma_p(\mathbb{R}^d)} = ||(1 - \Delta)^{\sigma/2} u||_{L^p(\mathbb{R}^d)}.$$ 

Note that

$$||u||_{H^\sigma_p(\mathbb{R}^d)} \approx ||u||_{L^p(\mathbb{R}^d)} + ||\Delta^{\sigma/2} u||_{L^p(\mathbb{R}^d)}.$$ 

**Lemma 2.7.** (Lp estimate, [18]). Given $\gamma > 0$ and $1 < p < \infty$, for any $f \in L^p(\mathbb{R}^d)$, there exists a unique strong solution $u \in H^\sigma_p(\mathbb{R}^d)$ satisfying

$$L_\sigma u + \gamma u = f \quad \text{ in } \mathbb{R}^d.$$ 

Moreover, we have

$$||u||_{H^\sigma_p(\mathbb{R}^d)} \leq N||f||_{L^p(\mathbb{R}^d)},$$

where $N$ is a constant depending only on $d, \lambda, \Lambda, \sigma, p$ and $\gamma$.

**Lemma 2.8.** (Comparison principle, [14, 25]). Suppose $\Omega$ is a bounded open set, and $u, v$ are two bounded functions in $\mathbb{R}^d$, which satisfy

(a): $u \leq v$ in $\mathbb{R}^d \setminus \Omega$,

(b): $u(v)$ is upper (lower) semi-continuous at every point in $\overline{\Omega}$,
(c): $L_{\sigma}u \leq f$ and $L_{\sigma}v \geq f$ in $\Omega$ for some continuous function $f$, then $u \leq v$ in $\Omega$.

Next, we give maximum principles which are suitable to our analysis.

**Lemma 2.9.** (Maximum Principle). Suppose $u \in C_0(\mathbb{R}^d \setminus \{0\}) \cap L^1_{\text{loc}}(\mathbb{R}^d)$ and $\lim_{|x| \to 0} \sup u(x) \leq 0$. If $L_{\sigma}u \leq 0$ in $\mathbb{R}^d \setminus \{0\}$, then $u \leq 0$ in $\mathbb{R}^d \setminus \{0\}$.

**Proof.** On the contrary, we assume that $u(x_0) > 0$ and $x_0 \neq 0$. Then there exists a constant $0 < r < |x_0|$ such that $u(x) < u(x_0)$ for any $x \in B_r \setminus \{0\}$. We may choose $R \gg |x_0|$ such that $u(x) < u(x_0)$ for any $x \in \mathbb{R}^d \setminus B_R$. Let $u(x_1) = \sup_{B_R \setminus B_r} u(x) \geq u(x_0) > 0$. Then

$$L_{\sigma}u(x_1) = \int_{\mathbb{R}^d} [u(x_1) - u(x_1 + y)]k(y)dy > 0,$$

but which is a contradiction.

**Lemma 2.10.** (Strong Maximum Principle). Suppose $u \in C_0(\mathbb{R}^d \setminus \{0\}) \cap L^1_{\text{loc}}(\mathbb{R}^d)$ and $u \leq 0$ in $\mathbb{R}^d \setminus \{0\}$. If $L_{\sigma}u \leq 0$ in $\mathbb{R}^d \setminus \{0\}$, and there exists a $x_0 \neq 0$ such that $u(x_0) = 0$, then we have $u \equiv 0$ in $\mathbb{R}^d \setminus \{0\}$.

**Proof.** We prove it by contradiction. It follows that

$$L_{\sigma}u(x_0) = \int_{\mathbb{R}^d} [0 - u(x_0 + y)]k(y)dy > 0,$$

which is a contradiction.

3. **The boundedness of scaling exponent $\alpha^*$.** The following spaces of homogeneous functions will play an important role in the sequel.

**Definition 3.1.** For each $\alpha \in (0, \infty)$, we denote

$$A_\alpha := \{ u \in C(\mathbb{R}^d \setminus \{0\}) : u \geq 0, r^\alpha u(rx) = u(x) \quad \forall r > 0 \},$$

$$A_\alpha^+ := \{ u \in C(\mathbb{R}^d \setminus \{0\}) : u > 0, r^\alpha u(rx) = u(x) \quad \forall r > 0 \}.$$

We define the scaling exponent of $L_{\sigma}$ by

$$\alpha^*(L_{\sigma}) := \sup \left\{ \alpha \in (0, \infty) : \text{there exists } u \in A_\alpha^+ \text{ such that } L_{\sigma}u \geq 0 \text{ in } \mathbb{R}^d \setminus \{0\} \right\}. \quad (17)$$

**Lemma 3.2.** For any $\sigma \in (0, 2)$, we have $\alpha^*(L_{\sigma}) < d$.

**Proof.** We claim that if $\alpha > d - \epsilon_0$, then for any $u \in A_\alpha^+$, there exists $x_0 \in \partial B_1$, such that

$$L_{\sigma}u(x_0) < 0. \quad (18)$$
Lemma 3.3. There exists a universal constant $\delta(d, \sigma) > 0$, such that $\alpha^*(L_\sigma) > 0$ if $|a(y) - 1| \leq 1 + \delta$.

Proof. Let $\alpha = \min(\sigma/2, 1 - \sigma/2)$. Assuming $u(x) = |x|^{-\alpha} \in A^*_\alpha$, we claim that $L_\sigma u(x_0) \geq 0$ for any $x_0 \in \partial B_1$. Without loss of generality, let $x_0 = e_1$.

$$L_\sigma u(e_1) = L^0_\sigma u(e_1) - \left( \int_E + \int_F + \int_{(E \cup F)^c} \right) (1 - |e_1 + y|^{-\alpha}) (a(y) - 1) |y|^{-d-\sigma} dy$$

$$:= \tilde{C}(d, \sigma) - (I_1 + I_2 + I_3),$$

Without loss of generality, we assume $u(e_1) = \min_{\partial B_1} u = 1$. Others, consider $\bar{u} = u/\min_{\partial B_1} u$. Let $v(x) = |x|^{-\alpha}$, then $v(x) \leq u(x)(v(e_1) = u(e_1) = 1)$. Thus

$$L_\sigma u(e_1) = \int_{\mathbb{R}^d} [u(e_1) - u(e_1 + y)]k(y)dy$$

$$\leq \int_{\mathbb{R}^d} [1 - v(e_1 + y)]k(y)dy$$

$$= \left( \int_{B_{\frac{1}{2}}^d} + \int_{B_{\frac{1}{2}}^d(-e_1)} + \int_{\mathbb{R}^d - (B_{\frac{3}{4}} \cup B_{\frac{1}{2}}(-e_1))} \right) (1 - |e_1 + y|^{-\alpha})k(y)dy$$

$$:= I_1 + I_2 + I_3.$$

Since

$$|e_1 + y|^{-\alpha} = (1 + 2y_1 + |y|^2)^{-\alpha/2} = 1 - \alpha y_1 + O(|y|^2),$$

we conclude that

$$I_1 = \int_{B_{\frac{1}{2}}^d} O(|y|^2)k(y)dy \leq C_1(d, \sigma, \Lambda),$$

$$I_3 < \Lambda \int_{\mathbb{R}^d - B_{\frac{1}{2}}^d} |y|^{-d-\sigma}dy \leq C_3(d, \sigma, \Lambda).$$

Now we estimate $I_2$. One can easily seen that $I_2 < 0$. Since $|y| < 3/2$ for any $y \in B_{1/2}(-e_1)$, we have

$$I_2 \leq \lambda(2/3)^{d+\sigma} \int_{B_{\frac{3}{4}}(-e_1)} (1 - |e_1 + y|^{-\alpha})dy$$

$$= \lambda(2/3)^{d+\sigma} \int_{B_{\frac{3}{4}}(0)} (1 - |y|^{-\alpha})dy$$

$$= \lambda(2/3)^{d+\sigma}(1/2)^d \omega_{d-1}[1/d - 2^{\sigma}/(d-\sigma)],$$

where $\omega_{d-1}$ denote the volume of $d - 1$ dimension unit ball.

Taking $\epsilon_0$ small enough, we derive that

$$2^{\sigma}\lambda \omega_{d-1}\left(\frac{2^{\alpha}}{\epsilon_0} - \frac{1}{d}\right) > (C_1 + C_3)(d, \sigma, \Lambda).$$

Combing (20)-(22), we conclude that (18) holds. \qed

**Lemma 3.3.** There exists a universal constant $\delta(d, \sigma) > 0$, such that $\alpha^*(L_\alpha) > 0$ if $|a(y) - 1| \leq 1 + \delta$.
Proof. Let \( E = B_{1/2}(0), F = B_{1/2}(-e_1) \). By (19), we have

\[
|I_1| = \int_E O(\|y\|^2)(a(y) - 1)|y|^{-d-\sigma} dy \leq \delta \int_E O(\|y\|^{2-d-\sigma}) dy < C(d, \sigma) \delta;
\]

\[
|I_3| < \int_{E^c} (a(y) - 1)|y|^{-d-\sigma} dy \leq \delta \int_{E^c} |y|^{-d-\sigma} dy < C(d, \sigma) \delta;
\]

\[
|I_2| \leq \delta \int_F ((e_1 + y)^{-\alpha} - 1)|y|^{-d-\sigma} dy \leq \delta \left( \frac{2}{3} \right)^{d+\sigma} \int_{B_{1/2}} (|y|^{-\alpha} - 1) dy
\]

\[
< C(d, \sigma) \delta.
\]

Thus \( L_\sigma^* u(e_1) \geq 0 \) follows from (23) by taking \( \delta \) small enough.

Combining the above two lemmas, we obtain that the scaling exponent \( \alpha^*(L_\sigma) \) is bounded.

**Corollary 1.** There exists a universal constant \( \delta(d, \sigma) > 0 \), such that \( 0 < \alpha^*(L_\sigma) < d \) as long as \( |a(y) - 1| \leq \delta \).

4. Existence and Uniqueness of fundamental solutions. In this section we will construct the fundamental solution of (4). Our proof borrows some ideas from the arguments of Armstrong [2], who proved similar results for fully nonlinear elliptic equations \( F(D^2u) = 0 \) in \( \mathbb{R}^d \setminus \{0\} \). We begin by proving a comparison principle on spaces of homogeneous functions, suitable for our purposes.

**Lemma 4.1.** Suppose \( 0 < \alpha < d \) and \( u, v \in A_\alpha \) such that

\[
L_\sigma u \leq 0 \leq L_\sigma v \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \tag{24}
\]

Then either \( u \equiv 0 \) or \( v \equiv 0 \) or \( u \equiv cv \) and \( L_\sigma u = 0 = L_\sigma v \) for some constant \( c > 0 \) in \( \mathbb{R}^d \setminus \{0\} \).

**Proof.** From Lemma 2.10, we have either \( v \equiv 0 \) or \( v > 0 \) in \( \mathbb{R}^d \setminus \{0\} \), that is, \( v \in A_\alpha^+ \).

If \( v \equiv 0 \), the lemma holds. In the latter case, let

\[
w_t = u - tv.
\]

Thus \( w_t \) is strictly negative in \( \mathbb{R}^d \setminus \{0\} \) for large \( t \), namely, for \( t > (\max_{\partial B_1} u)/ (\min_{\partial B_1} v) \).

We define

\[
c = \inf \{ t > 0 : w_t < 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\} \}. \tag{25}
\]

Formally, we have

\[
L_\sigma w_c = L_\sigma u - cL_\sigma v \leq 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \tag{26}
\]

We know (26) implies that ether \( u \equiv cv \), which proves the lemma, or \( w_c < 0 \) in \( \mathbb{R}^d \setminus \{0\} \). In the latter case, if \( u \equiv 0 \), the lemma holds. Now we assume \( w_c < 0 \) and \( u > 0 \) in \( \mathbb{R}^d \setminus \{0\} \). That is \( c > 0 \). However, we get \( w_{c-\delta} < 0 \) in \( \mathbb{R}^d \setminus \{0\} \) for \( \delta < (\min_{\partial B_1} w_c)/(\max_{\partial B_1} v) \), that contradicts with (25).

The next lemma establishes that the set \( \alpha > 0 \) for which there exists a supersolution \( u \in A_\alpha^+ \) is an interval.

**Lemma 4.2.** For any \( 0 < \alpha < \alpha^* \), there is a function \( u \in A_\alpha^+ \), such that

\[
L_\sigma u(x) \geq \frac{1}{|x|^{|\alpha+\sigma|}} \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \tag{27}
\]
Proof. We may select a constant $\beta$, such that $\alpha < \beta < \alpha^*$. By the definition of $\alpha^*$, there exists a function $v \in A^+_{\beta}$, satisfying

$$L_\sigma v(x) \geq 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \quad (28)$$

Let $\tau = \alpha / \beta < 1$, and $\omega = v^\tau \in A^+_{\tau \beta}$. We claim that there exists a positive constant $\varepsilon_0 > 0$, such that for any $x_0 \in \partial B_1$,

$$L_\sigma \omega(x_0) \geq \varepsilon_0 v(x_0) > 0. \quad (29)$$

Without loss of generality, we may assume that $v(x_0) = 1$. To verify (29) in viscosity sense, we select a small neighborhood of $U(x_0)$. For any test function $\varphi \in C^2(U(x_0))$, such that

\[
\begin{cases}
0 < \varphi(x) < \omega(x) & \text{in } U \setminus \{x_0\}, \\
\varphi(x) = \omega(x) & \text{in } \mathbb{R}^d - U, \\
\varphi(x_0) = \omega(x_0) & .
\end{cases}
\]

One can easily check that $\psi(x) = \varphi^{1/\tau}(x)$ is a test function to $v$, and $\psi(x_0) = \varphi(x_0) = 1$. By (28) we see that

$$L_\sigma \psi(x_0) \geq 0. \quad (30)$$

Denoting $t = \psi(x_0 + y) - \psi(x_0)$, we have

$$\varphi(x_0) - \varphi(x_0 + y) = 1 - (1 + t)^{\tau} = -\tau t - \frac{\tau(\tau - 1)}{2}(1 + \theta t)^{\tau-2}t^2,$$

where $0 < \theta < 1$. It follows that

$$L_\sigma \varphi(x_0) = \int_{\mathbb{R}^d} [1 - (1 + t)^\tau]k(y)dy$$

$$= \tau L_\sigma \psi(x_0) + \frac{1}{2}\tau(1 - \tau) \int_{\mathbb{R}^d} (1 + \theta t)^{\tau-2}t^2k(y)dy$$

$$\geq \frac{1}{2}\tau(1 - \tau)(\lambda/4)(2 - \sigma) \int_{\{\psi(x_0 + y) < 1/2\}} |y|^{-(d+\sigma)}dy \quad (31)$$

$$= \frac{1}{2}\tau(1 - \tau)(\lambda/4)(2 - \sigma) \int_{\{\psi(x_0 + y) < 1/2\}} |y|^{-(d+\sigma)}dy \geq \varepsilon_0,$$

where $\varepsilon_0$ depends only on $d, \sigma, \alpha, \beta, \lambda$, and $v$. Thus we estimate

$$L_\sigma \omega(x) \geq \varepsilon_0 (\min_{\partial B_1} \omega)|x|^{-\alpha} \quad \text{in } \mathbb{R}^d \setminus \{0\}, \quad (32)$$

since the homogeneity of $\omega$ implies that $\omega \geq (\min_{\partial B_1} \omega)|x|^{-\alpha}$. So dividing the $\omega$ by a constant we obtain (27).

From the previous two results we deduce a maximum principle in $A_\alpha$.

**Corollary 2.** Suppose $0 < \alpha < \alpha^*(L_\sigma)$. If $u \in A_\alpha$ satisfies

$$L_\sigma u \leq 0 \quad \text{in } \mathbb{R}^d \setminus \{0\}, \quad (33)$$

then $u \equiv 0$ in $\mathbb{R}^d \setminus \{0\}$.

**Proof.** According to the previous lemma, there exists a function $v \in A^+_{\alpha^*}$, such that

$$L_\sigma u \leq 0 \leq \frac{1}{|x|^\alpha} \leq L_\sigma v \quad \text{in } \mathbb{R}^d \setminus \{0\}. \quad (34)$$

Then we conclude $u \equiv 0$ from Lemma 4.1. \qed
In next lemma we show that the interval $\alpha > 0$ for which there exists a supersolution $u \in A^+_{\alpha}$ is open.

**Lemma 4.3.** Suppose $0 < \alpha \leq \alpha^*$. Suppose further $u \in A^+_{\alpha}$ satisfies

$$L_\sigma u(x) \geq \frac{1}{|x|^\alpha} \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \quad (35)$$

Then $\alpha < \alpha^*$.

**Proof.** Let $v(x) = |x|^{-\epsilon} u(x) \in A^+_{\alpha+\epsilon}$. We need to show there exists a small $\epsilon > 0$, such that

$$L_\sigma v(x) \geq 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \quad (36)$$

Suppose $x_0 \in \partial B_1$, and $m = u(x_0) > 0$. To verify (36) in viscosity sense, we select a test function $\varphi \in C^2(U)$, where $U$ is a neighborhood of $\{x_0\}$, such that

$$\left\{ \begin{array}{ll}
0 < \varphi(x) < v(x) & \text{in} \quad U \setminus \{x_0\}, \\
\varphi(x) = v(x) & \text{in} \quad \mathbb{R}^d - U, \\
\varphi(x_0) = v(x_0) & .
\end{array} \right.$$

Let $\psi(x) = |x|^{\epsilon} \varphi(x)$. Then $\psi$ is a test function to $u$ with $\psi(x_0) = \varphi(x_0) = m$. We claim that $L_\sigma \varphi(x_0) \geq 0$. In fact,

$$L_\sigma \varphi(x_0) = \int_{\mathbb{R}^d} \left( m - \psi(x_0 + y) + \psi(x_0 + y) - \varphi(x_0 + y) \right) k(y) dy$$

$$\geq L_\sigma \psi(x_0) - \left( \int_{B_\rho} + \int_{B_2(x_0) \setminus B_\rho} \right) \left( \varphi(x_0 + y) - \psi(x_0 + y) \right) k(y) dy \quad (37)$$

$$\geq 1 - (I_1 + I_2),$$

where $\rho \in (0, 1)$ is a small constant that will be chosen later. Since $\varphi, \psi \in C^2(U)$, we can choose a small $\rho > 0$, such that

$$I_1 = \int_{B_\rho} (\varphi(x_0 + y) - \psi(x_0 + y)) k(y) dy$$

$$= \int_{B_\rho} (\varphi(x_0 + y) - \varphi(x_0) + \psi(x_0) - \psi(x_0 + y)) k(y) dy \leq 1/2. \quad (38)$$

Fixing $\rho > 0$, the dominated convergence theorem implies that

$$I_2 = \int_{B_2(x_0) \setminus B_\rho} \left( 1 - |x_0 + y|^\alpha \right) \varphi(x_0 + y) k(y) dy \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0. \quad (39)$$

Thus (36) is a consequence of (37)-(39). \hfill \Box

The next lemma is the key to the existence of the fundamental solution.

**Lemma 4.4.** For any $0 < \alpha < \alpha^*$, there exists a function $u \in A^+_{\alpha}$, satisfying

$$L_\sigma u(x) = \frac{1}{|x|^\alpha} \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \quad (40)$$

**Proof.** **Step I.** Let $f(t) = 1/t^{(\alpha+\epsilon)} (t > 0)$. Consider a family of functions $f^n : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \quad (n = 1, 2, \cdots)$,

$$f^n(t) := \begin{cases} 
2^n/(\alpha+\epsilon) & t < 2^{-n}, \\
1/t^{(\alpha+\epsilon)} & 2^{-n} \leq t \leq 2^n, \\
0 & t > 2^n.
\end{cases}$$
In the view of Lemma 2.7, we can set \( u^n \in H^\sigma_p(\mathbb{R}^d) (\rho > d/\sigma) \) is the unique solution to the following equation
\[
L_\sigma u^n(x) + 2^{-n} u^n(x) = f^n(|x|) \quad \text{in} \quad \mathbb{R}^d. \tag{41}
\]
We know that \( u^n \) is a continuous function from the embedding \( C(\mathbb{R}^d) \subset H^\sigma_p(\mathbb{R}^d) \) for \( p > d/\sigma \). From Lemma 4.2, there exists a function \( v(x) \in A^+_{\alpha} \), such that
\[
L_\sigma v(x) \geq f(|x|) \quad \text{in} \quad \mathbb{R}^d \setminus \{0\} \quad (n = 1, 2, \cdots). \tag{42}
\]
By Lemma 2.9, we have
\[
0 \leq u^n \leq v \quad \text{in} \quad \mathbb{R}^d \setminus \{0\} \quad (n = 1, 2, \cdots). \tag{43}
\]
From Lemma 2.5, there is a universal constant \( \beta > 0 \), such that
\[
\|u^n\|_{C^\beta(K)} \leq C \left( \sup_K v + \|v\|_{L^1(\mathbb{R}^d, \mu)} + \|f\|_{L^\infty(\Omega)} \right) \tag{44}
\]
for any compact subset \( K \subset \Omega \subset \mathbb{R}^d \setminus \{0\} \). By taking a subsequence if necessary, we may assume that \( u^n \) converges locally uniformly in \( \mathbb{R}^d \setminus \{0\} \) to a nonnegative function \( u \in C_0(\mathbb{R}^d \setminus \{0\}) \). It is obvious that \( u \in C_0(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d, \mu) \) from (43) and (44).

**Step II.** We prove the function \( u \in A_\alpha \), that is, for any \( r > 0 \),
\[
r^\alpha u(rx) = u(x). \tag{45}
\]
Given \( r > 0 \), we may select a large \( k \), such that \( 2^{-k} \leq r, r^\sigma < 2^k \). Let \( v^n_r(x) = r^\alpha u^n(rx)(n > 3k) \). According to Proposition 1, we have
\[
L_\sigma v^n_r(x) + 2^{-n-k}v^n_r(x) \leq r^{\alpha+\sigma} L_\sigma u^n(rx) + 2^{-n-r^{\alpha+\sigma} u^n(rx)}
\]
\[
= r^{\alpha+\sigma} f^n(r|x|) \leq f^{n+k}(|x|) \tag{46}
\]
\[
= L_\sigma u^{n+k}(x) + 2^{-n-k} u^{n+k}(x).
\]
By Lemma 2.8, we see that
\[
r^\alpha u^n(rx) \leq u^{n+k}(x). \tag{47}
\]
By a similar argument, we obtain
\[
r^\alpha u^n(rx) \geq u^{n-k}(x). \tag{48}
\]
It is easily seen that (45) holds by taking \( n \to \infty \) since \( r \) is any positive number.

**Step III.** Now we prove that \( u \) satisfies (40). By Proposition 1 and \( u \in A_\alpha \), we only need to show
\[
L_\sigma u(x_0) = 1 \tag{49}
\]
for any \( x_0 \in \partial B_1 \). Consider a cutoff function \( \eta(t) : \mathbb{R}_+ \to \mathbb{R}_+ \), such that
\[
\eta = \begin{cases} 
1 & |x| \geq 1/8, \\
0 & |x| < 1/8. 
\end{cases}
\]
Let
\[
\overline{u^n} := (1-\eta)(u^n), \quad \underline{u} := (1-\eta)(u) 
\]
\[
g^n(x) := \int_{|x+y| < 1/8} u^n(x+y)k(y)dy, 
\]
\[
g(x) := \int_{|x+y| < 1/8} u(x+y)k(y)dy. 
\]
By (43) we see that \( \overline{u^n} \) are uniformly bounded and continuous in \( \mathbb{R}^d \). For any \( x_0 \in \partial B_1 \), we have
\[
\overline{u^n}/g^n \to u, g \quad \text{locally uniformly in } B_{1/2}(x_0),
\]
(50)
since the kernel \( k(y) \) is bounded in \( B_{1/2}(x_0) \). For any \( x \in B_{1/2}(x_0) \),
\[
L_\sigma \overline{u^n}(x) = \int_{\mathbb{R}^d} \left[ u^n(x) - u^n(x + y) + u^n(x + y) - \overline{u^n}(x +y) \right] k(y) dy
= f^n(|x|) - 2^{-n} u^n(x) + g^n(x).
\]
(51)
Thus
\[
L_\sigma \overline{u}(x) = f(|x|) + g(x) \quad \text{in } B_{1/2}(x_0)
\]
(52)
is a consequence of Lemma 2.3. It is easily seen that (40) holds from (52).

**Step IV.** We claim \( u \in A^+_\alpha \). It follows that either \( u > 0 \) or \( u \equiv 0 \) from Lemma 2.10. Then \( u \) must be positive.

**Lemma 4.5.** Suppose that \( u \in A^+_\alpha \) satisfies (40) with \( 0 < \alpha < \alpha^* \). Then
\[
\sup_{\partial B^1} u_\alpha \to +\infty \quad \text{as } \alpha \to \alpha^*.
\]
(53)

**Proof.** On the contrary, we assume that there exists a sequence \( 0 < \alpha_n < \alpha^* \), such that \( \alpha_n \to \alpha^* \), and
\[
\left\{ \begin{array}{l}
L_\alpha u_{\alpha_n}(x) = |x|^{-\alpha_n-\sigma} \quad \text{in } \mathbb{R}^d \setminus \{0\}, \\
\sup_{n \geq 1} \sup_{x \in \partial B^1} u_{\alpha_n}(x) \leq M_0,
\end{array} \right.
\]
(54)
where \( M_0 \) is a universal constant. By the homogeneity of \( u_{\alpha_n} \), it follows that
\[
\sup_{n \geq 1} \sup_K u_{\alpha_n} \leq C M_0
\]
(55)
for any compact set \( K \subset \subset \mathbb{R}^d \setminus \{0\} \). From Lemma 2.5, we derive that there is a universal constant \( \beta > 0 \), such that
\[
\sup_{n \geq 1} ||u_{\alpha_n}||_{C^\beta(K)} \leq C M_0.
\]
(56)
By taking a subsequence if necessary, we may assume that \( u_{\alpha_n} \to u_{\alpha^*} \) locally uniformly in \( \mathbb{R}^d \setminus \{0\} \). It is immediate that \( u_{\alpha^*} \in A_{\alpha^*} \). We get \( u_{\alpha^*} \in L^1(\mathbb{R}^d, \mu) \) since \( \alpha^* < d \).

Next we claim that \( u_{\alpha^*} \in A^+_\alpha \), and
\[
L_\sigma u_{\alpha^*}(x) = \frac{1}{|x|^{\alpha^*+\sigma}} \quad \text{in } \mathbb{R}^d \setminus \{0\}.
\]
(57)

We obtain (57) through a similar argument in the previous lemma (see the proof of (40)). But this contradicts with Lemma 4.3. Then we see (53) holds.

**Theorem 4.6.** There exists a function \( \Phi \in A^+_\alpha \) with \( \max_{\partial B^1} \Phi = 1 \), which is a viscosity solution of (4). Moreover, if \( u \in A^+_\alpha \) is a viscosity solution of (4), then \( \alpha = \alpha^* \) and \( u \equiv c \Phi \) for some \( c > 0 \).

**Proof.** In the view of Lemma 4.4, we can choose a sequence \( 0 < \alpha_n < \alpha^* \), \( \alpha_n \to \alpha^* \), and \( u_{\alpha_n} \in A^+_\alpha \) which satisfies (54). Define the function \( v_{\alpha_n} \) as follows
\[
v_{\alpha_n}(x) := \frac{1}{M_{\alpha_n}} u_{\alpha_n}(x), \quad M_{\alpha_n} = \max_{\partial B^1} u_{\alpha_n}(x).
\]
(58)
It is clear that $v_{\alpha_n} \in A_{\alpha_n}^+$ and $\max_{\partial B_1} v_{\alpha_n} = 1$. We have

$$L_{\sigma} v_{\alpha_n}(x) = \frac{1}{M_{\alpha_n}} \cdot \frac{1}{|x|^{\alpha_n + \sigma}} \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}. \quad (59)$$

Hence for every compact subset $K \subset \subset \mathbb{R}^d \setminus \{0\}$, we obtain the estimates

$$\sup_{n \geq 1} \|v_{\alpha_n}\|_{C^s(K)} \leq C \quad (60)$$

from Lemma 2.5. Then there exists a function $\Phi \in A_{\alpha^*}$, such that, up to a subsequence,

$$v_{\alpha_n} \to \Phi \quad \text{locally uniformly in} \quad \mathbb{R}^d \setminus \{0\}. \quad (61)$$

By Lemma 2.3, we conclude that

$$L_{\sigma} \Phi = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}, \quad (62)$$

since $M_{\alpha_n}$ tend to $+\infty$. It is easily seen that $\Phi \in A_{\alpha^*}^+$ and $\max_{\partial B_1} \Phi = 1$. Thus $\min_{\partial B_1} \Phi \geq c$ is a consequence of the Harnack inequality (see Lemma 2.4), where the constant $c$ depends only on $\sigma, \lambda, \Lambda$.

Now we prove the uniqueness. Assume that $u \in A_{\alpha}^+$ ($\alpha > 0$) is a viscosity solution of (4). Then we get $\alpha = \alpha^*$ from Corollary 2. So Lemma 4.1 implies that $u = C\Phi$, where $C > 0$ is a positive constant.

We have concluded the existence of the fundamental solution of (4) in the previous theorem. The following corollary states that the fundamental solution is $C^{0,1}$ except the origin.

**Corollary 3.** If $\Phi$ is given by Theorem 4.6, then $\Phi \in C^{0,1}_{loc}(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d, \mu)$.

**Proof.** In order to show the corollary, we need to show that $\Phi$ is $C^{0,1}$ on $\partial B_1$ since $\Phi$ is homogeneous of $-\alpha^*$ order. It is clear that $\Phi \in L^1(\mathbb{R}^d, \mu)$ since $\Phi \in A_{\alpha^*}$. Choosing $0 < \frac{1}{n} < \min(\frac{1}{2}, \frac{2}{\sigma})(n \in \mathbb{N}^+)$, we have $\Phi \in C^1/n(\partial B_1)$ by Lemma 2.5. Assuming $\Phi \in C^k/n(\partial B_1)(k = 1, 2, \ldots, n - 1)$, we claim that $\Phi \in C^{(k+1)/n}(\partial B_1)$.

Considering

$$\overline{\Phi} = \begin{cases} \Phi(x) & \Phi \leq 8^{\alpha^*}, \\ 0 & \Phi > 8^{\alpha^*}, \end{cases}$$

it is easily seen that

$$\left\{ x \in \mathbb{R}^d : \Phi(x) \neq \overline{\Phi}(x) \right\} \subset B_{1/8}(0), \quad (63)$$

since $\max_{\partial B_1} \Phi = 1$. Fixing a small constant $h > 0$ and a unit vector $e \in \mathbb{R}^d$, we define a different quotient as the following

$$D^h_e f(x) := \frac{f(x + he) - f(x)}{|h|^{k/n}}. \quad (64)$$

Since $\Phi \in C^{k/n}_{loc}(\mathbb{R}^d \setminus \{0\})$, there exists a constant $C_0$ such that

$$|D^h_e \Phi| \leq C_0 \quad \text{in} \quad \mathbb{R}^d \setminus B_{1/8}(0), \quad (65)$$

where $C_0$ does not depend on $e, h$.

On the other hand, setting $x_0 \in \partial B_1$, for any $x \in B_{1/2}(x_0)$, let

$$f(x) := L_{\sigma} \overline{\Phi}(x) = \int_{E(x)} \Phi(x + y)k(y)dy, \quad (66)$$

where $E(x)$ is the set of all $y$ such that $x + y \in \partial B_1$. Thus

$$\left| \frac{\Phi(x + he) - \Phi(x)}{|h|^{k/n}} \right| \leq C_0 \quad \text{in} \quad \mathbb{R}^d \setminus B_{1/8}(0). \quad (67)$$

By the Harnack inequality (Lemma 2.4), we conclude that $\Phi \in C^{0,1}(\mathbb{R}^d \setminus \{0\})$.

We have shown that $\Phi \in C^{0,1}(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d, \mu)$, which completes the proof of the corollary.
where
\[ E(x) = \{ y \in \mathbb{R}^d : \Phi(x + y) > 8^{\alpha^*} \} \subset B_{1/8}(x). \]  
(67)
Thus \( E(x) \cap B_{1/8}(0) = \emptyset. \) For any \( x \in B_{3/8}(x_0), \) we have
\[
D^h_\epsilon f(x) = \int_{\Omega_1} \frac{\Phi(x + y + he)}{|h|^{k/n}} k(y) dy + \int_{\Omega_2} \frac{\Phi(x + y + he)}{|h|^{k/n}} \frac{k(y)}{h} dy - \int_{\Omega_3} \frac{\Phi(x + y)}{|h|^{k/n}} \frac{k(y)}{h} dy,
\]
(68)
where
\[
\Omega_1 = E(x) \cap E(x + he),
\]
\[
\Omega_2 = E(x + he) - E(x),
\]
\[
\Omega_3 = E(x) - E(x + he).
\]
By (65) and
\[
|\Omega_2| + |\Omega_3| \leq Ch,
\]
we derive that
\[
|D^h_\epsilon f| \leq C_1 \text{ in } B_{3/8}(x_0),
\]
(69)
where the constant \( C_1 \) does not depend on \( \epsilon, h. \)

It is clear that
\[
L\alpha(D^h_\epsilon \Phi) = D^h_\epsilon f \text{ in } B_{3/8}(x_0).
\]
(70)
Thus \( D^h_\epsilon \Phi \in C^{1/\alpha}(B_{1/4}(x_0)) \) follows from Lemma 2.5 and (69). We obtain the claim by taking \( h \to 0. \) Consequently, \( \Phi \in C^{0,1}(\partial B_1) \) by an iteration method.

Proof of Theorem 1.1. Our results is immediately concluded from Corollary 2, Theorem 4.6 and Corollary 3.

5. Characterization of singularities. Throughout this section, assuming that \( u \) is a viscosity solution of (4), we study its behavior near the origin. Here we always suppose that \( u \in C(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d, \mu) \) is positive in \( \mathbb{R}^d \setminus \{0\} \) and bounded in \( \mathbb{R}^d \setminus B_1. \) Let \( \Phi \) be the fundamental solution of \( L\alpha. \) We can define the quantities
\[
m(r) = \min_{\partial B_r} u \quad \text{and} \quad M(r) = \max_{\partial B_r} u,
\]
\[
\rho(r) = \min_{\partial B_r} \frac{u}{\Phi} \quad \text{and} \quad \bar{\rho}(r) = \max_{\partial B_r} \frac{u}{\Phi},
\]
(71)
since \( \Phi \) does not vanish on \( \partial B_r. \)

We divide the proof of Theorem 1.2 into several lemmas. The first two lemmas state \( \rho(r) = \bar{\rho}(r). \)

Lemma 5.1. Suppose \( u \geq 0 \) in \( \mathbb{R}^d \setminus \{0\}. \) Then there exists a universal constant \( C \) depending only on \( \lambda, \Lambda \) and \( d, \) such that for each \( r > 0, \)
\[
m(r) \leq CM(r) \quad \text{and} \quad \bar{\rho}(r) \leq C^2 \rho(r),
\]
(72)
where \( C \) is a constant depending only on \( d, \lambda, \Lambda \) and \( \sigma. \)

Proof. (72) is a simple consequence of the Harnack inequality and the fact that if a function \( u(x) \) is a solution of (4), then so is \( u(x/r). \)

Lemma 5.2. Suppose \( u \geq 0 \) in \( \mathbb{R}^d \setminus \{0\}. \) If \( \liminf_{r \to 0} \rho(r) = 0, \) then \( \limsup_{r \to 0} \bar{\rho}(r) = 0. \)
Proof. It is easily seen that \( \bar{\rho}(r) \geq \rho(r) \geq 0 \). On the contrary, we may assume that
\[
\limsup_{r \to 0} \bar{\rho}(r) = 1. \tag{73}
\]
Note that adding a constant to \( \Phi \) modifies neither the hypotheses nor the conclusion of the lemma. Without loss of generality, we may suppose that
\[
u(x) < 3/4 \Phi(x) \quad \forall \quad x \in \mathbb{R}^d \setminus B_1. \tag{74}
\]
We may choose \( \rho(r_k) \to 0 \) as \( r_k \to 0 \). For every \( r_k \), select a \( x_k \in \partial B_{r_k} \) such that
\[
u(x_k) = \rho(r_k) \Phi(x_k). \tag{75}
\]
According to the Harnack inequality, there exists a universal constant \( C > 0 \), depending only on \( d, \sigma, \lambda, \Lambda \), such that for any \( \epsilon > 0 \),
\[
u(x) \leq C \rho(r_k) \Phi(x) < \epsilon \Phi(x) \quad \text{for all} \quad 5^{-1} r_k < |x| < 5 r_k, \tag{76}
\]
as long as \( r_k \) is sufficiently small. By making \( r_k \) smaller, if necessary, we obtain
\[
u(x) < (1 + \epsilon) \Phi(x) \quad \text{for all} \quad 0 < |x| < 5^{-1} r_k. \tag{77}
\]
Let \( M := \max_{5 r_k \leq r \leq 1} \bar{\rho}(r) \). Then we have \( 1 - \epsilon < M < +\infty \) from (73) and (74). Choose a point \( z \in \mathbb{R}^d \setminus B_{5 r_k} \) such that \( \nu(z) = M \Phi(z) \). Since \( u \leq M \Phi \) in \( B_1 - B_{5 r_k} \), by (74), (76) and (77), we have
\[
L_{\sigma} u(z) = \int_{\mathbb{R}^d} \left( M \Phi(z) - M \Phi(z + y) + M \Phi(z + y) - u(z + y) \right) k(y) dy
= \left( \int_{|z + y| < r_k/5} + \int_{|z + y| \geq r_k/5} \right) (M \Phi(z + y) - u(z + y)) k(y) dy
\geq \int_{r_k/5 < |y| < 5 r_k} \lambda(1 - 2 \epsilon)(2 - \sigma) \Phi(y) |z - y|^{-d - \sigma} dy
\geq \int_{|y| < r_k/5} \epsilon \Lambda(2 - \sigma) \Phi(y) |z - y|^{-d - \sigma} dy
> (2 - \sigma) \int_{r_k/5 < |y| < 5 r_k} \lambda(1 - 2 \epsilon)(2|z|)^{-d - \sigma} \Phi(y) dy
\geq (2 - \sigma) \int_{|y| < r_k/5} \epsilon \Lambda(|z|/2)^{-d - \sigma} \Phi(y) dy > 0,
\]
by taking \( \epsilon \) small enough. But this contradicts with \( L_{\sigma} u = 0 \).

The next result tells us that \( u/\Phi \) is bounded near the origin.

Lemma 5.3. If \( u > 0 \) in \( \mathbb{R}^d \setminus \{0\} \), then \( 0 \leq \limsup_{r \to 0} \bar{\rho}(r) < \infty \).

Proof. On the contrary, we suppose that \( \limsup_{r \to 0} \bar{\rho}(r) = +\infty \). Similarly to the previous lemma, we have \( \liminf_{r \to 0} \rho(r) = +\infty \). Consequently, for any large number \( M > 1 \), there exists a \( r_0 > 0 \), such that \( u > M \Phi \) in \( B_{r_0} \setminus \{0\} \). It is clear that
\[
u(x) \geq M \left( \Phi(x) - \max_{\partial B_1} \Phi \right) \quad \text{in} \quad (B_{r_0} \setminus \{0\}) \cup B_1^r. \tag{79}
\]
Denoting
\[ \bar{u}(x) := \begin{cases} u(x) & |x| \geq r_0, \\ 0 & |x| < r_0, \end{cases} \]
and
\[ v(x) := \begin{cases} M(\Phi(x) - \max_{\partial B_1} \Phi) & |x| \geq r_0, \\ 0 & |x| < r_0, \end{cases} \]
we obtain that
\[ L_\sigma \bar{u}(x) = \int_{|x+y|<r_0} u(x+y)k(y)dy \geq \int_{|x+y|<r_0} v(x+y)k(y)dy = L_\sigma v(x) \quad (80) \]
in \( B_1 - B_{r_0} \). The comparison principle implies (79) holds in \( B_1 \setminus B_{r_0} \). Choosing a point \( x_0 \in \partial B_1/2 \) such that \( \Phi(x_0) = \max_{\partial B_1/2} \Phi \), we obtain
\[ M \leq u(x_0) - \Phi(x_0) \]
(81)
But (81) contradicts with that \( M \) is sufficiently large.

In the following three lemmas, we prove that a nonnegative viscosity solution of (4) must be a constant under the condition of that \( \rho(r) \) converges to zero at the origin.

**Lemma 5.4.** Suppose \( u > 0 \) in \( \mathbb{R}^d \setminus \{0\} \). If \( \lim inf_{r \to 0} \rho(r) = 0 \), then \( u \) is bounded in \( \mathbb{R}^d \setminus \{0\} \).

**Proof.** We have \( \lim r \to 0 \rho(r) = \lim r \to 0 \rho(r) = 0 \) from Lemma 5.2. Choosing a sequence of \( r_k \) such that \( \rho(2r_k) \to 0 \), and
\[ 0 < u(x) \leq \frac{1}{k} \Phi(x) \quad \text{for all} \quad 0 < |x| < 2r_k. \]
(82)
Let
\[ \bar{u}_k(x) := \begin{cases} u(x) & |x| \geq r_k, \\ 0 & 0 < |x| < r_k, \end{cases} \]
\[ \bar{\Phi}_k(x) := \begin{cases} \Phi(x)/k + \sup_{\mathbb{R}^d \setminus B_1} u & |x| \geq r_k, \\ 0 & 0 < |x| < r_k, \end{cases} \]
Thus \( \bar{u}_k, \bar{\Phi}_k \) are bounded in \( \mathbb{R}^d \). One can easily check that
\[ \left\{ \begin{array}{ll}
L_\sigma (\bar{u}_k - \bar{\Phi}_k) = \int_{B_{r_k}} [(u - \Phi/k)(y) - \sup_{\mathbb{R}^d \setminus B_1} u]k(x-y)dy < 0 & \text{in} \ \Omega_k, \\
0 \leq \bar{u}_k(x) \leq \bar{\Phi}_k(x) <+\infty & \text{in} \ \Omega^c_k,
\end{array} \right. \]
(83)
where \( \Omega_k = B_1 \setminus B_{2r_k} \). From the comparison principle, we obtain
\[ u(x) \leq \Phi(x)/k + \sup_{\mathbb{R}^d \setminus B_1} u \quad \text{for all} \quad r_k < |x| < 1. \]
(84)
Passing to the limit \( k \to \infty \), we derive that
\[ 0 < u \leq \sup_{\mathbb{R}^d \setminus B_1} u \quad \text{in} \ \mathbb{R}^d \setminus \{0\}. \]
(85)
Thus $u$ is bounded.

Lemma 5.5. Suppose that $u$ is bounded in $B_1 \setminus \{0\}$. Then $u$ can be defined at origin so that $u \in C(\mathbb{R}^d)$. Moreover, $u$ is a viscosity solution of

$$L_\sigma u = 0 \quad \text{in} \quad \mathbb{R}^d.$$  \hspace{1cm} (86)

Proof. Suppose $u > 0$ (if necessary, adding a constant). On the contrary, we assume that $\liminf_{|x| \to 0} u \neq \limsup_{|x| \to 0} u$. Let

$$u(0) = \left( \liminf_{|x| \to 0} u + \limsup_{|x| \to 0} u \right)/2.$$

Thus $u$ is bounded in $\mathbb{R}^d$. Consider $u^\epsilon = \eta^\epsilon \ast u$, where $\eta^\epsilon(x) = \epsilon^{-d} \eta(x/\epsilon)$. Note that $\eta \geq 0$ is a smooth function such that $\int_{\mathbb{R}^d} \eta(x) dx = 1$. For any $0 < \gamma < 1$, we derive that

$$\sup_{x \in B_{1/2}} \frac{|u^\epsilon(x) - u^\epsilon(0)|}{|x|^{\gamma}} \to \infty \quad \text{as} \quad \epsilon \to 0.$$ \hspace{1cm} (87)

On the other hand, since $L_\sigma u(x) = 0$ for all $x \neq 0$, then for any $r > 0$, we obtain

$$\int_{B_r} |L_\sigma u(x)| dx = \int_{B_r} \int_{\mathbb{R}^d} [u(x) - u(x + y)] k(y) dy dx$$

$$\leq 2(2 - \sigma) \Lambda ||u||_{L^\infty(\mathbb{R}^d)} \left( \int_{B_r} \int_{B_r} + \int_{B_r} \int_{\mathbb{R}^d \setminus B_r} \right) ||y||^{-d - \sigma} dy dx$$ \hspace{1cm} (88)

$$= 0,$$

by passing to the limit $r \to 0$. Then $L_\sigma u \in L^1(B_1)$. We deduce that

$$L_\sigma u^\epsilon = \eta^\epsilon \ast L_\sigma u = 0 \quad \text{in} \quad \mathbb{R}^d.$$ \hspace{1cm} (89)

According to Lemma 2.5, there is a constant $\alpha > 0$, such that

$$|u^\epsilon|_{C^\alpha(B_{1/2})} \leq C \sup_{B_{3/4}} |u^\epsilon| \leq C \sup_{B_1} |u|$$ \hspace{1cm} (90)

for any $\epsilon > 0$, where $C$ is a universal constant. But this contradicts with (87). Therefore, we conclude that $u$ is continuous by redefining $u(0) = \lim_{|x| \to 0} u(x)$. \hfill $\square$

Theorem 5.6. Suppose that $u \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a viscosity solution of (86), and $\liminf_{r \to 0} \rho(r) = 0$. Then $u \equiv \text{constant}.$

Proof. Let $v(x) = u(rx)(r > 0)$. By Lemma 2.5, we have

$$r^{-\beta}[u]_{C^\beta(B_1)} = \frac{[v]_{C^\beta(B_{1/r})}}{L^\infty(\mathbb{R}^d)} \leq C ||v||_{L^\infty(\mathbb{R}^d)} = C ||u||_{L^\infty(\mathbb{R}^d)},$$ \hspace{1cm} (91)

where $\beta, C$ are universal positive constants. Then we have $[u]_{C^\beta(B_1)} = 0$ by making $r \to 0^+$. Hence we deduce that $u$ must be a constant. \hfill $\square$

The next result says that $u/\Phi$ converges to a constant in the case of $\liminf_{r \to 0} \rho(r) > 0$.

Lemma 5.7. Suppose that $u$ is positive in $\mathbb{R}^d \setminus \{0\}$ with $\liminf_{r \to 0} \rho(r) > 0$. Then we have

$$\liminf_{r \to 0} \rho(r) = \limsup_{r \to 0} \rho(r).$$ \hspace{1cm} (92)
Proof. Denoting \( a = \liminf_{r \to 0} \rho(r) \) and \( \bar{a} = \liminf_{r \to 0} \bar{\rho}(r) \), it follows that \( 0 < a \leq \bar{a} < +\infty \) from Lemma 5.3. We claim that

\[
\liminf_{x \to 0} \frac{u}{\Phi(x)} = a
\]

(93) in \( \mathbb{R}^d \setminus \{0\} \). In fact, for any \( \epsilon > 0 \), we can select \( r_0 \in (0,1/4) \) and \( R_0 > 1 \) such that

\[
\sup_{B_{r_0}(0)} u > \bar{\rho}(r_0) \Phi \quad \text{and} \quad \sup_{\mathbb{R}^d \setminus B_{R_0}} u + a > a \Phi,
\]

(94) since \( \Phi(x) \to 0 \) as \( |x| \to +\infty \). We deduce that \( u + a > (a - \epsilon) \Phi \) in \( B_{R_0} \setminus \overline{B_{r_0}} \) from the comparison principle. Then (93) holds by taking \( \epsilon \to 0 \).

We now employ a rescaling argument to show that \( a = \bar{a} \). That is, we want to show

\[
\lim_{x \to 0} \frac{u}{\Phi(x)} = a.
\]

(95)

Note that the limit \( a \) do not change when we add a constant to \( u \). So we may assume that

\[
a \Phi \leq u \leq 2\bar{a} \Phi + 1 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}
\]

(96) since \( u \) is bounded in \( \mathbb{R}^d \setminus B_1 \). Let \( r_0 < 1 \). For each \( 0 < r < r_0 \), select \( x_r \) with \( |x_r| = r \) such that \( u(x_r) = \rho(r) \Phi(x_r) \). Let

\[
v_r(x) = r^{\alpha} u(rx) \quad 0 < r < r_0.
\]

(97)

By the homogeneity of \( \Phi \), we conclude that

\[
\begin{cases}
0 < v_r(x) \leq 2\bar{a} r^{\alpha} \Phi(rx) + 1 = 2\bar{a} \Phi(x) + 1 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}, \\
L_o v_r(x) = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}.
\end{cases}
\]

(98)

Using the Hölder estimate (13), we can find a function \( v \in C(\mathbb{R}^d \setminus \{0\}) \) and a sequence \( r_j \to 0 \) such that

\[
v_{r_j} \to v \quad \text{locally uniformly in} \quad \mathbb{R}^d \setminus \{0\} \quad \text{as} \quad j \to \infty.
\]

(99)

By taking a further subsequence, we may also assume that \( x_{r_j}/r_j \to y \) as \( j \to \infty \) for some \( y \in \partial B_1 \). We have \( v \geq a \Phi \) in \( \mathbb{R}^d \setminus \{0\} \) since \( \Phi \) is homogeneous. It is clear that \( v \) is a viscosity solution of (4) from Lemma 2.3. Since

\[
v(y) = \lim_{j \to \infty} r_j^{\alpha} u(x_{r_j}) = \lim_{j \to \infty} r_j^{\alpha} \rho(r_j) \Phi(x_{r_j})
\]

\[= \lim_{j \to \infty} \rho(r_j) \Phi(x_{r_j}/r_j) = a \Phi(y),
\]

(100)

we obtain \( v \equiv a \Phi \) by Lemma 2.10. For any \( r_{j+1} < r < r_j \),

\[
v_r(x) = r^{\alpha} u(rx) = (\frac{r}{r_j})^{\alpha} v_{r_j} (\frac{r}{r_j} x)
\]

\[= (\frac{r}{r_j})^{\alpha} a \Phi (\frac{r}{r_j} x) = a \Phi(x) \quad (\forall x \neq 0).
\]

(101)

Thus the full sequence \( v_r \to a \Phi \) locally uniformly in \( \mathbb{R}^d \setminus \{0\} \) as \( r \to 0 \). From this we deduce that

\[
\limsup_{x \to 0} \frac{u}{\Phi(x)} = \lim_{r \to 0} \max_{x \in \partial \overline{B_r}} \frac{r^{\alpha} u(x)}{\Phi(x/r)} = \lim_{r \to 0} \max_{x \in \partial \overline{B_r}} \frac{v_r(x)}{\Phi(x)} = a.
\]

(102)

This verifies (95). \( \square \)

**Theorem 5.8.** If \( u > 0 \) in \( \mathbb{R}^d \setminus \{0\} \) with \( a = \liminf_{r \to 0} \rho(r) > 0 \), then \( u = a \Phi + b \) in \( \mathbb{R}^d \setminus \{0\} \) for some \( b \in \mathbb{R} \).
Proof. We obtain that \( \liminf_{r \to 0} \rho(r) = a = \limsup_{r \to 0} \rho(r) \) from the last lemma. The comparison principle implies that
\[
(a - \epsilon)\Phi - B \leq u \leq (a + \epsilon)\Phi + B \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}
\]
for every small \( \epsilon > 0 \), where \( B \) is a constant. Now we let \( \epsilon \to 0 \) to deduce that
\[
a\Phi - B \leq u \leq a\Phi + B \quad \text{in} \quad \mathbb{R}^d \setminus \{0\}.
\]
Since \( u - a\Phi \) is a bounded viscosity solution of (4), we conclude that \( b = u - a\Phi \) is a constant by Theorem 5.6.

Proof of Theorem 1.2. We may assume \( u(\text{or } -u) \) is positive in \( \mathbb{R}^d \setminus \{0\} \) (if necessary, adding a constant) since \( u \) is bounded on one side. Thus Theorem 1.2 is immediately obtained from Theorem 5.6 and 5.8.

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