The hyperanalytic signal

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Abstract—The concept of the analytic signal is extended from the case of a real signal with a complex analytic signal to a complex signal with a hypercomplex analytic signal (which we call a hyperanalytic signal) The hyperanalytic signal may be interpreted as an ordered pair of complex signals or as a quaternion signal. The hyperanalytic signal contains a complex orthogonal signal and we show how to obtain this by three methods: a pair of classical Hilbert transforms; a complex Fourier transform; and a quaternion Fourier transform. It is shown how to derive from the hyperanalytic signal a complex envelope and phase using a polar quaternion representation previously introduced by the authors. The complex modulation of a real sinusoidal carrier is shown to generalize the modulation properties of the classical analytic signal. The paper extends the ideas of properness to deterministic complex signals using the hyperanalytic signal. A signal example is presented, with its orthogonal signal, and its complex envelope and phase.

I. INTRODUCTION

The analytic signal has been known since 1948 from the work of Ville [1] and Gabor [2]. It can be simply described, even though its theoretical ramifications are deep. Its use in non-stationary signal analysis is routine and it has been used in numerous applications. Simply put, given a real-valued signal \( f(t) \), its analytic signal \( a(t) \) is a complex signal with real part equal to \( f(t) \) and imaginary part orthogonal to \( f(t) \). The imaginary part is sometimes known as the quadrature \( f(t) \) signal – in the case where \( f(t) \) is a sinusoid, the imaginary part of the analytic signal is in quadrature, that is with a phase difference of \(-\pi/2\). The orthogonal signal is related to \( f(t) \) by the Hilbert transform \([3], [4]\). The analytic signal has the interesting property that its modulus \(|a(t)|\) is an envelope of the signal \( f(t) \). The envelope is also known as the instantaneous amplitude. Thus if \( f(t) \) is an amplitude-modulated sinusoid, the envelope \(|a(t)|\), subject to certain conditions, is the modulating signal. The argument of the analytic signal, \( \angle a(t) \) is known as the instantaneous phase. The analytic signal has a third interesting property: it has a one-sided Fourier transform. Thus a simple algorithm for constructing the analytic signal (algebraically or numerically) is to compute the Fourier transform of \( f(t) \), multiply the Fourier transform by a unit step which is zero for negative frequencies, and then construct the inverse Fourier transform.

In this paper we extend the concept of the analytic signal from the case of a real signal \( f(t) \) with a complex analytic signal \( a(t) \), to a complex signal \( z(t) \) with a hypercomplex analytic signal \( h(t) \), which we call the hyperanalytic signal. Just as the classical complex analytic signal contains both the original real signal (in the real part) and a real orthogonal signal (in the imaginary part), a hyperanalytic signal contains two complex signals: the original signal and an orthogonal signal. We have previously published partial results on this topic \([5]–[7]\). In the present paper we develop a clear idea of how to generalise the classic case of amplitude modulation to a complex signal, and show for the first time that this leads to a correctly extended analytic signal concept in which the (complex) envelope and phase have clear interpretations. We show how the orthogonal signal can be constructed using three methods, yielding the same result:

1) a pair of classical Hilbert transforms operating independently on the real and imaginary parts of \( z(t) \),
2) a complex Fourier transform pair operating on \( z(t) \) as a whole, with positive and negative frequency coefficients rotated by \( \pm\pi/2 \) using \( i \) times a signum function,
3) a quaternion Fourier transform with suppression of negative frequencies in the Fourier domain.

All three of these methods are based in one way or another on single-sided spectra, but it is only in the third (quaternion) case that a single one-sided spectrum is involved.

The construction of an orthogonal signal alone would not constitute a full generalisation of the classical analytic signal to the complex case: it is also necessary to generalise the envelope and phase concepts, and this can only be done by interpreting the original and orthogonal complex signals as a pair. In this paper we have only one way to do this: by representing the pair of complex signals as a quaternion signal. This arises naturally from the third method above for creating an orthogonal signal, but also from the Cayley-Dickson construction of a quaternion as a complex number with complex real and imaginary parts (with different roots of \(-1\) used in each of the two levels of complex number).

Extension of the analytic signal concept to 2D signals, that is images, with real, complex or quaternion-valued pixels is of interest, but outside the scope of this paper. Some work has been done on this, notably by Bülow, Felsberg, Sommer and Hahn \([8]–[11]\). The principal issue to be solved in the 2D case is to generalise the concept of a single-sided spectrum. Hahn considered a single quadrant or orthant spectrum, Sommer et al considered a spectrum with support limited to half the complex plane, not necessarily confined to two quadrants, but still with real sample or pixel values.

Recently, Lilly and Olhede \([12]\) have published a paper on bivariate analytic signal concepts without explicitly considering the complex signal case which we cover here. Their approach is linked to a specific signal model, the modulated elliptical signal, which they illustrate with the example of a drifting oceanographic float. The approach taken in the present paper is more general and without reference to a specific signal model.
II. CONSTRUCTION OF AN ORTHOGONAL SIGNAL USING CLASSICAL COMPLEX TRANSFORMS

We first review and examine the notion of orthogonality in the case of complex signals before we consider how to construct a signal orthogonal to a given complex signal.

Definition 1: Two complex signals \( x(t) \) and \( y(t) \) are orthogonal if: \( \int_{-\infty}^{\infty} x(t)y^*(t) \, dt = 0 \), where \(*\) denotes the complex conjugation.

Orthogonality is invariant to multiplication of either signal by any real or complex constant. This is evident from the definition, since if the integral is zero for a given pair of \( x(t) \) and \( y(t) \) it must also be zero for \( \zeta y(t) \) or \( \zeta x(t) \), where \( \zeta \) is a complex constant. Therefore, given a pair of orthogonal signals, one of the signals may be rotated by an arbitrary angle in the complex plane (constant with respect to time), and still remain orthogonal to the other. Obviously the same applies for a real constant, so scaling one or both of the signals does not alter their orthogonality.

Consider the integral in Definition 1 separated into real and imaginary parts:

\[
\begin{align*}
\int_{-\infty}^{\infty} (x_r(t)y_r(t) + x_i(t)y_i(t)) \, dt &= 0 \quad (1) \\
\int_{-\infty}^{\infty} (x_i(t)y_r(t) - x_r(t)y_i(t)) \, dt &= 0 \quad (2)
\end{align*}
\]

where \( x_r(t) \) (resp. \( y_r(t) \)) is the real part of \( x(t) \) (resp. \( y(t) \)) and \( x_i(t) \) (resp. \( y_i(t) \)) is the imaginary part of \( x(t) \) (resp. \( y(t) \)). There are two significantly different ways in which these integrals can vanish. One way is for all four of the products within them to vanish when integrated:

\[
\begin{align*}
\int_{-\infty}^{\infty} x_r(t)y_r(t) \, dt &= 0 \\
\int_{-\infty}^{\infty} x_i(t)y_i(t) \, dt &= 0 \\
\int_{-\infty}^{\infty} x_i(t)y_r(t) \, dt &= 0 \\
\int_{-\infty}^{\infty} x_r(t)y_i(t) \, dt &= 0
\end{align*}
\]

This can only happen if both complex signals have an imaginary part orthogonal to their real part. In this case only, orthogonality is invariant to complex conjugation of either or both of the complex signals.

In more general cases the integrals in (1) and (2) vanish because the product terms within them integrate to equal values and appropriate signs (opposite signs in the first case, the same sign in the second).

Of course, a hybrid of these possibilities is also possible, and indeed occurs in the next section: one integral vanishes because each of the product terms within it integrates to zero, and the other vanishes because the product terms integrate to the same value with appropriate sign, and cancel.

![Fig. 1. Construction of the orthogonal signal using a pair of Hilbert transforms.](image)

A. Hilbert transform

An orthogonal signal may be constructed using a pair of classical Hilbert transforms as shown in Figure 1.

Lemma 1: Given two arbitrary real-valued functions \( f(t) \) and \( g(t) \) the following holds, where \( \mathcal{H}[x(t)] \) is the Hilbert transform of \( x(t) \), provided the Hilbert transforms exist:

\[
\int_{-\infty}^{\infty} f(t)\mathcal{H}[g(t)] \, dt + \int_{-\infty}^{\infty} \mathcal{H}[f(t)]g(t) \, dt = 0 \quad (3)
\]

Proof: We make use of the inner product form of Parseval’s Theorem [13, §27.9 equation (27.34)] or [14, Theorem 7.5]:

\[
\int_{-\infty}^{\infty} x(t)y^*(t) \, dt = \int_{-\infty}^{\infty} X(\nu)Y^*(\nu) \, d\nu \quad (4)
\]

where \( X(\nu) \) is the Fourier transform of \( x(t) \) and similarly for \( y(t) \). Note that, if \( y(t) \) is real, as it is in what follows, \( Y(\nu) \) must still be conjugated. We also make use of the Fourier transform of a Hilbert transform [15, (2.13-8, p 68)]:

\[
\mathcal{F}[\mathcal{H}[x(t)]] = -i\text{sign}(\nu)X(\nu) , \quad X(\nu) = \mathcal{F}[x(t)].
\]

Using (4), we rewrite (3) as:

\[
\int_{-\infty}^{\infty} F^*(\nu)\text{sign}(\nu)G(\nu) \, d\nu + \int_{-\infty}^{\infty} i\text{sign}(\nu)F(\nu)G^*(\nu) \, d\nu = 0
\]

which may be simplified to:

\[
i\int_{-\infty}^{\infty} \text{sign}(\nu) [F^*(\nu)G(\nu) + F(\nu)G^*(\nu)] \, d\nu = 0
\]

Since \( F^*(\nu)G(\nu) \) and \( F(\nu)G^*(\nu) \) are conjugates, their imaginary parts cancel and the term inside the brackets is real. Since \( F(\nu) \) and \( G(\nu) \) are the Fourier transforms of real signals, their real parts are even functions of \( \nu \). Therefore when multiplied by the signum function, they integrate to zero.

Proposition 1: A complex signal \( o(t) \) constructed from an arbitrary complex signal \( z(t) \) using two classical Hilbert transforms operating independently on the real and imaginary parts, as represented in Figure 1 and below:

\[
o(t) = \mathcal{H}[z_r(t)] - i\mathcal{H}[z_i(t)] \quad (5)
\]

is orthogonal to \( z(t) \), that is:

\[1\text{An example of such a signal can be created numerically by taking four singular vectors from a singular value decomposition of a real matrix, since the singular vectors are mutually orthogonal.}
of the orthogonal signal — generalising from the classical case, we must treat the original and orthogonal complex signals as a pair in order to define the generalisations of envelope and phase. In the next section we introduce the quaternion Fourier transform, and show that this permits us to construct a hyperanalytic signal from which both the original signal and the orthogonal signal defined in the current section may be extracted. Using the quaternion hyperanalytic signal we then show how to construct the envelope and phase in the complex case.

III. CONSTRUCTION OF AN ORTHOGONAL SIGNAL USING A QUATERNION FOURIER TRANSFORM

In this section, we will be concerned with the definition and properties of the quaternion Fourier transform (QFT) of complex valued signals. Before introducing the main definitions, we give some prerequisites on quaternion valued signals.

A. Preliminary remarks

Quaternions were discovered by Sir W.R. Hamilton in 1843 [16]. Quaternions are 4D hypercomplex numbers that form a noncommutative division ring denoted $\mathbb{H}$. A quaternion $q \in \mathbb{H}$ has a Cartesian form: $q = a + bi + cj + dk$, with $a, b, c, d \in \mathbb{R}$ and $i, j, k$ roots of $-1$ satisfying $i^2 = j^2 = k^2 = ijk = -1$. The scalar part of $q$ is $a$: $S(q) = a$. The vector part of $q$ is: $V(q) = q - S(q)$. Quaternion multiplication is not commutative, so that in general $qp \neq pq$ for $p, q \in \mathbb{H}$. The conjugate of $q$ is $\overline{q} = a - bi - cj - dk$. The norm of $q$ is $||q|| = |q|^2 = (a^2 + b^2 + c^2 + d^2) = q\overline{q}$. A quaternion with $a = 0$ is called pure. If $|q| = 1$, then $q$ is called a unit quaternion. The inverse of $q$ is $q^{-1} = \overline{q}/||q||$. Pure unit quaternions are special quaternions, among which are $i, j$ and $k$. Together with the identity of $\mathbb{H}$, they form a quaternion basis: $(1, i, j, k)$. In fact, given any two unit pure quaternions, $\mu$ and $\xi$, which are orthogonal to each other (i.e. $S(\mu\xi) = 0$), then $(1, \mu, \xi, \mu\xi)$ is a quaternion basis.

Quaternions can also be viewed as complex numbers with complex components, i.e. one can write $q = z_1 + jz_2$ in the basis $(1, i, j, k)$ with $z_1, z_2 \in \mathbb{C}$, i.e. $z_\alpha = \Re(z_\alpha) + i\Im(z_\alpha)$ for $\alpha = 1, 2, \ldots, 4$. This is called the Cayley-Dickson form. Now, it is possible to define a generalized Cayley-Dickson

\[ z(t) \to \mathcal{F} \to Z(\nu) \to -i\text{sign}(\nu) \to \mathcal{F}^{-1} \to o^*(t) \]

\[ z(t) \to \mathcal{F} \to Z(\nu) \to -i\text{sign}(\nu) \to \mathcal{F}^{-1} \to o^*(t) \]

\[ \int_{-\infty}^{\infty} z(t) o^*(t) \, dt = 0 \]

\[ \mathcal{F} \to Z(\nu) \to -i\text{sign}(\nu) \to \mathcal{F}^{-1} \to o^*(t) \]

\[ \int_{-\infty}^{\infty} [z_r(t) o_r(t) + z_i(t) o_i(t)] \, dt = 0 \]

\[ \int_{-\infty}^{\infty} [z_i(t) o_r(t) - z_r(t) o_i(t)] \, dt = 0 \]

\[ \int_{-\infty}^{\infty} [z_i(t) o_r(t) - z_r(t) o_i(t)] \, dt = 0 \]

\[ \mathcal{F} [o^*(t)] = -i\text{sign}(\nu) Z_r(\nu) - i\text{sign}(\nu) Z_i(\nu) \]

\[ = -i\text{sign}(\nu) Z(\nu) \]

from the linearity of the Fourier transform.
form of a given quaternion using an arbitrary quaternion basis \((1, \mu, \xi, \mu \xi)\). A quaternion \(q\) can then be written as \(q = w_1 + \xi w_2\) where \(w_1, w_2 \in \mathbb{C}_\mu\).

Any quaternion \(q\) also has a unique polar Cayley-Dickson form \([17]\) given by: \(q = A \exp(\beta j)\), where \(A\) is the complex modulus of \(q\) and \(B\) is its complex phase. Expressions of \(A\) and \(B\) can be found in \([17]\). This notation will be of use in Section V.

B. 1D Quaternion Fourier transform

In this paper, we use a 1D version of the (right) QFT first defined in discrete-time form in \([18]\). Thus, it is necessary to specify the axis (a pure unit quaternion) of the transform. So, we will denote \(F_\mu\) a QFT with transformation axis \(\mu\). This will become clear from the definition below. We now present the special case of convolution of a complex signal with a QFT with the following properties hold:

**Property 1:** Given a complex signal \(z(t) = z_r(t) + \xi z_i(t)\), its quaternion Fourier transform with respect to axis \(\mu\) is:

\[
Z_\mu(\nu) = F_\mu[z(t)] = \int_{-\infty}^{\infty} z(t)e^{-\mu 2\pi \nu t} dt \tag{7}
\]

and the inverse transform is:

\[
z(t) = F_\mu^{-1}[Z_\mu(\nu)] = \int_{-\infty}^{\infty} Z_\mu(\nu)e^{\mu 2\pi \nu \nu} d\nu \tag{8}
\]

where in both cases, the transform axis \(\mu\) can be chosen to be any unit pure quaternion orthogonal to \(\xi\) so that \((1, \xi, \mu, \xi \mu)\) is a quaternion basis.

**Property 2:** A one-sided QFT \(X(\nu)\) with \(X(\nu) = 0\) \((i.e. \forall \nu < 0)\) corresponds to a quaternion valued signal in the time domain.

Proof: A complex signal with values in \(\mathbb{C}_\xi\) has a \(QFT_{\mu}\) with the symmetry properties listed in property 1. Now, cancelling its negative frequencies is equivalent to adding a signal with a QFT with the following properties: \(\Re\) part is odd, \(\Im\) \(\xi\) is even and \(\Im \xi \mu\) is even. By linearity of the inverse QFT, the former two parts of the QFT produce a complex signal in the time domain with \(\Re\) and \(\Im \xi\) parts, while the latter produces a quaternion signal in the time domain with \(\Im \mu\) and \(\Im \xi \mu\) parts. Together, and again by linearity, the resulting signal in the time domain is a full 4D quaternion signal.

**Property 3:** Given a complex signal \(x(t)\), one can associate to it a unique canonical pair corresponding to a modulus and phase. These modulus and phase are uniquely defined through the hyperanalytic signal, which is quaternion valued.

Proof: Cancelling the negative frequencies of the QFT leads to a quaternion signal in the time domain. Then, any quaternion signal has a modulus and phase defined using its CD polar form.

C. Convolution

We consider the special case of convolution of a complex signal by a real signal. Consider \(q\) and \(f\) such that: \(q : \mathbb{R}^+ \rightarrow \mathbb{C}_\xi\) and \(f : \mathbb{R}^+ \rightarrow \mathbb{R}\). Now, consider the \(QFT_{\mu}\) of their convolution:

\[
F_{\mu}[g \ast f(t)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(\tau)f(t - \tau)e^{-\mu 2\pi \nu t} d\tau dt \tag{9}
\]

Thus, the definition used for the QFT here verifies the convolution theorem in the considered case. This specific case will be of use in our definition of the hyperanalytic signal.

D. The quaternion Fourier transform of the Hilbert transform

It is straightforward to verify that the quaternion Fourier transform of a real signal \(x(t) = \frac{1}{\pi t}\) is \(-\mu \text{sign}(\nu)\), where \(\mu\)
is the axis of the transform. Substituting \( x(t) \) into (7), we get:

\[
\mathcal{F}_\mu \left[ \frac{1}{\pi t} \right] = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\mu \pi i \pi t} \, dt
\]

and this is clearly isomorphic to the classical complex case. The solution in the classical case is \(-i \text{sign}(\nu)\), and hence in the quaternion case must be as stated above.

It is also straightforward to see that, given an arbitrary real signal \( y(t) \), subject only to the constraint that its classical Hilbert transform \( \mathcal{H}[y(t)] \) exists, then one can easily show that the classical Hilbert transform of the signal may be obtained using a quaternion Fourier transform as follows:

\[
\mathcal{H}[y(t)] = \mathcal{F}_\mu^{-1} [-\mu \text{sign}(\nu)Y_\mu(\nu)]
\]

where \( Y_\mu(\nu) = \mathcal{F}_\mu[y(t)] \). This result follows from the isomorphism between the quaternion and complex Fourier transforms when operating on a real signal, and it may be seen to be the result of a convolution between the signal \( y(t) \) and the quaternion Fourier transform of \( x(t) = \frac{1}{\pi t} \). Note that \( \mu \) and \( Y_\mu(\nu) \) commute as a consequence of \( y(t) \) being real.

IV. THE HYPERANALYTIC SIGNAL AS A QUATERNION SIGNAL WITH A ONE-SIDED SPECTRUM

We define the hyperanalytic signal \( h(t) \) by a similar approach to that originally developed by Ville [1]. The following definitions give the details of the construction of this signal. Note that the signal \( z(t) \) is considered to be non-analytic in the classical (complex) sense, that is its real and imaginary parts are not orthogonal. However, the following definitions are valid if \( z(t) \) is analytic, as it can be considered as a degenerate case of the more general non-analytic case.

**Definition 3:** Consider a complex signal \( z(t) = z_r(t) + \xi z_i(t) \) and its quaternion Fourier transform \( \mathcal{Z}_\mu(\nu) \) as defined in Definition 2. Then, the hypercomplex analogue of the Hilbert transform of \( z(t) \), is as follows:

\[
\mathcal{H}_\mu[z(t)] = \mathcal{F}_\mu^{-1} [-\mu \text{sign}(\nu)Z_\mu(\nu)]
\]

where the Hilbert transform is thought of as: \( \mathcal{H}_\mu[z(t)] = p.v. \{ \mathcal{H}_\mu[z(t)] \} \) where the principal value (p.v.) is understood in its classical way. This result follows from equation (10) and the linearity of the quaternion Fourier transform. Notice that the result will take the same form as \( z(t) \), namely a quaternion signal \( o_r(t) + \xi o_i(t) \) isomorphic to the complex signal \( \sigma(t) = o_r(t) + i o_i(t) \). To extract the imaginary part, the vector part of the quaternion signal must be multiplied by \( -\xi \). An alternative is to take the scalar or inner product of the vector part with \( \xi \).

Thus, the \( \mu \) and \( \mathcal{Z}_\mu(\nu) \) anticommute because \( \xi \) is orthogonal to \( \xi \), the axis of \( \mathcal{Z}_\mu(\nu) \). Therefore the ordering is not arbitrary, but changing it conjugates the result.

**Definition 4:** Given a complex valued signal \( z(t) \) that can be expressed in the form of a quaternion as \( z(t) = z_r(t) + \xi z_i(t) \), and given a pure unit quaternion \( \mu \) such that \((1, \xi, \mu, \xi \mu)\) is a quaternion basis, then the hyperanalytic signal of \( z(t) \), denoted \( h(t) \) is given by:

\[
h(t) = z(t) + \mu \mathcal{H}_\mu[z(t)]
\]

where \( \mathcal{H}_\mu[z(t)] \) is the hypercomplex analogue of the Hilbert transform of \( z(t) \) defined in the preceding definition. The quaternion Fourier transform of this hyperanalytic signal is thus:

\[
\mathcal{H}_\mu(\nu) = Z_\mu(\nu) - \mu^2 \text{sign}(\nu)Z_\mu(\nu)
\]

which is a direct extension of the ‘classical’ analytic signal.

This result is unique to the quaternion Fourier transform representation of the hyperanalytic signal — the hyperanalytic signal has a one-sided quaternion Fourier spectrum. This means that the hyperanalytic signal may be constructed from a complex signal \( z(t) \) in exactly the same way that an analytic signal may be constructed from a real signal \( x(t) \), by suppression of negative frequencies in the Fourier spectrum. The only difference is that in the hyperanalytic case, a quaternion rather than a complex Fourier transform must be used, and of course the complex signal must be put in the form \( z(t) = z_r(t) + \xi z_i(t) \) which, although a quaternion signal, is isomorphic to the original complex signal.

A second important property of the hyperanalytic signal is that it maintains a separation between the different even and odd parts of the original signal.

**Property 4:** The original signal \( z(t) \) is the simplex part [21] Theorem 1, [22] § 13.1.3] of its corresponding hyperanalytic signal. The perplex part is the orthogonal or ‘quadrature’ component, \( o(t) \). They are obtained by:

\[
z(t) = \frac{1}{2} (h(t) - \xi h(t) \xi)
\]

\[
o(t) = \frac{1}{2} (h(t) + \xi h(t) \xi)
\]

**Proof:** This follows from equation (12). Writing this in full by substituting the orthogonal signal for \( \mathcal{H}_\mu[z(t)] \):

\[
h(t) = z(t) + \mu o(t) = z_r(t) + \xi z_i(t) + \mu o_r(t) + \mu \xi o_i(t)
\]

and substituting this into equation (13), we get:

\[
z(t) = \frac{1}{2} \left( z_r(t) + \xi z_i(t) + \mu o_r(t) + \mu \xi o_i(t) \right)
\]

\[
o(t) = \frac{1}{2} \left( -\xi z_r(t) + \xi z_i(t) + \mu o_r(t) + \mu \xi o_i(t) \right)
\]

and since \( \xi \) and \( \mu \) are orthogonal unit pure quaternions, \( \xi \mu = -\mu \xi \):

\[
= \frac{1}{2} \left( z_r(t) + \xi z_i(t) + \mu o_r(t) + \mu \xi o_i(t) \right)
\]

from which the first part of the result follows. Equation (14) differs only in the sign of the second term, and it is straightforward to see that if \( h(t) \) is substituted, \( z(t) \) cancels out, leaving \( o(t) \).

V. PHASE AND ENVELOPE FROM QUATERNION REPRESENTATION

As was shown in §11-A and Figure 1, it is possible to construct an orthogonal signal using classical Hilbert transforms operating independently on the real and imaginary parts of a complex signal \( z(t) \). However, to extend the analytic
signal concept to the hyperanalytic case, we need a method to construct the complex envelope (also known as the instantaneous amplitude) and the complex phase (also known as the instantaneous phase) from a pair of complex signals (the original complex signal $z(t)$, and the complex orthogonal signal $o(t)$).

A complex envelope may be obtained, apart from the signs of the real and imaginary parts, and therefore the quadrant, from the moduli of the analytic signals of the real and imaginary parts of the original signal. The resulting envelope will always be in the first quadrant and therefore in general it will not be a correct instantaneous amplitude.

In this paper the method we present is based on quaternion algebra which is a natural choice given that a quaternion may be expressed as a pair of complex numbers in the so-called Cayley-Dickson form, and, as we have shown earlier, the hyperanalytic signal may be computed using a discrete quaternion Fourier transform.

### A. Complex envelope

In a previous paper [5] we showed how the envelope of the hyperanalytic signal could be obtained from a biquaternion Fourier transform, using simply the complex modulus of the biquaternion hyperanalytic signal. This approach has a limitation, namely that the envelope is always in the right half of the complex plane. (This is because the modulus of the biquaternion hyperanalytic signal is obtained as the square root of the sum of the squares of the components, and the complex square root is conventionally in the right half-plane.) A second, more serious, limitation of the biquaternion Fourier transform is that under some circumstances the complex envelope can vanish because the biquaternion hyperanalytic signal is nilpotent [23]: that is it has values with zero modulus, even though the biquaternion values are non-zero.

The mathematical key to constructing the complex envelope from a quaternion hyperanalytic signal is given in a paper written in 2008 [17]. In that paper we set out a new polar representation for quaternions that has a complex modulus and a complex argument:

$$ q = A \exp(Bj) $$

where $A$ and $B$ are complex (more accurately they are degenerate quaternions of the form $w + xi + yj + zk$).

The representation as published is based on the standard quaternion basis $(1, i, j, k)$, although clearly it could use any other basis. For simplicity of numerical computation, we use the standard basis.

This representation of a quaternion is not trivial to compute numerically and one of the main problems we had in arriving at the results presented in this paper was to refine the computation of the representation in (15) to minimise numerical problems. We do not discuss these issues here, as they are solely concerned with correct numerical implementation of (15). The code is open source, and the numerical issues are documented in the code [24] Function: cdpolar.m. It is also necessary to use phase unwrapping on $B$ to obtain a continuous envelope and phase without discontinuities when computing $A(t)$ and $B(t)$ using (15) from a hyperanalytic signal $h(t)$, due to an inherent ambiguity of sign in $A$ and $B$ [17] §2.1. (In a similar way the complex envelope obtained from a biquaternion hyperanalytic signal may be phase unwrapped to obtain an envelope in all four quadrants.)

**Definition 5:** Given a complex signal $z(t)$, and its hyperanalytic signal $h(t)$ constructed according to Definition [3] expressed in the standard quaternion basis $(1, i, j, k)$, the instantaneous amplitude, or complex envelope, $e(t)$, and the instantaneous phase, $\phi(t)$, are given by the polar form of a quaternion defined in (15):

$$ h(t) = z(t) + o(t)j = e(t)\exp(\phi(t)j) $$

where $z(t)$, $o(t)$, $e(t)$, and $\phi(t)$ are complex.

### B. Complex modulation of a real carrier

A classic result in the case of the analytic signal concerns amplitude modulation, where the envelope of the analytic signal (that is its modulus) allows the modulating signal to be recovered, subject to certain constraints such as the depth of modulation [4] §7.16. Figure 3 shows an example. In this section we extend this result to the complex case.

**Property 5:** Given a signal $z(t)$ consisting of a real sinusoidal ‘carrier’ $f(t) = A \cos(2\pi\omega t + \theta)$, and a complex ‘modulating’ signal $g(t)$:

$$ z(t) = g(t) f(t) $$

where the frequency content of $g(t)$ is below the frequency of the carrier, $\omega$, then the instantaneous phase $\phi(t)$ as defined in Definition [5] is real and equal to the instantaneous phase of $f(t)$, that is $2\pi\omega t + \theta$, and the envelope $e(t)$ is a scaled version of the complex modulating signal, $A g(t)$.

**Proof:** Separate $z(t)$ into real and imaginary parts:

$$ z_r(t) = g_r(t)A \cos(2\pi\omega t + \theta) $$

$$ z_i(t) = g_i(t)A \cos(2\pi\omega t + \theta) $$

...
By Bedrosian’s Theorem \[25\] \[4, §7.12 and Table 7.7.1\], the Hilbert transforms of the real and imaginary parts of \(z(t)\) are:
\[
\mathcal{H}[z_r(t)] = g_r(t)A \sin(2\pi \omega t + \theta) \\
\mathcal{H}[z_i(t)] = g_i(t)A \sin(2\pi \omega t + \theta)
\]
We can therefore write the hyperanalytic signal as follows:
\[
h(t) = z_r(t) + iz_i(t) + j\mathcal{H}[z_r(t)] + k\mathcal{H}[z_i(t)]
\]
\[
= g(t)A[\cos(2\pi \omega t + \theta) + j\sin(2\pi \omega t + \theta)]
\]
\[
= g(t)A\exp(j(2\pi \omega t + \theta))
\]
and comparison with Definition 5 shows that \(e(t) = Ag(t)\) and \(\phi(t) = 2\pi \omega t + \theta\).

Note that we have considered above only the case of a real sinusoidal carrier modulated by a complex modulating signal, and that scaling of the carrier cannot be separated from scaling of the modulating signal. A more general case would be a complex carrier, for example, a complex exponential, or a single sinusoid rotated in the complex plane, modulated by a real or complex modulating signal, but we now show that this is ambiguous – the complex nature of the carrier cannot be separated from the complex nature of the modulating signal. Consider a real sinusoidal carrier \(f(t)\) modulated by a complex modulating signal \(g(t)\), yielding a modulated signal \(m(t)\):
\[
m(t) = f(t)g(t)
\]
Now rotate the carrier by an angle \(\psi\) in the complex plane so that it oscillates along a line in the complex plane at angle \(\psi\) to the real axis, instead of along the real axis itself:
\[
m(t) = [f(t)\exp(i\psi)]g(t)
\]
Clearly, this modulated signal is indistinguishable from the following, in which the complex modulating signal \(g(t)\) is instead rotated through the angle \(\psi\):
\[
m(t) = f(t)[g(t)\exp(i\psi)]
\]
We conclude that the hyperanalytic signal cannot distinguish between a real carrier modulated by a complex signal and a complex carrier modulated by a (different) complex signal: it suffices to consider the former case, and assume that the modulating signal only is complex.

VI. PROPERNESS

In this section, we study the properness properties of the hyperanalytic signal. We make use of the term proper here for deterministic signals, by extension of the concept used for quaternion random functions in \[26\]. The term proper is also widely used for complex random signals when the real and imaginary parts are orthogonal and have the same amplitude. Our definition of properness is based on \[27\]. We use the properties of the correlation matrix in the real (Cartesian) representation of quaternions. As explained in \[27\], a quaternion can be represented as a four dimensional vector \(q = [a, b, c, d]^T\). Properness can be observed through the pattern of the covariance matrix \(\Gamma_q\) that contains the cross-correlations between the four components of the quaternion.

Here, we are interested in the covariance (correlation at zero-lag) of the hyperanalytic signal \(h(t)\). Assuming that the original signal \(z(t)\) takes values in \(\mathbb{C}^4\), then \(h(t)\) (obtained using a quaternion Fourier transform with axis \(\mu\)) can be written as:
\[
h(t) = z(t) + o(t)\mu = z_r(t) + \xi z_i(t) + (o_r(t) + \xi o_i(t))\mu
\]
The covariance matrix of \(h(t)\), denoted \(\Gamma_h\), is thus given as:
\[
\Gamma_h = \begin{bmatrix}
C_{z_r z_r} & C_{z_r z_i} & C_{z_r o_r} & C_{z_r o_i} \\
C_{z_i z_r} & C_{z_i z_i} & C_{z_i o_r} & C_{z_i o_i} \\
C_{o_r z_r} & C_{o_r z_i} & C_{o_r o_r} & C_{o_r o_i} \\
C_{o_i z_r} & C_{o_i z_i} & C_{o_i o_r} & C_{o_i o_i}
\end{bmatrix}
\]
Now, given a complex signal \(z(t)\) taking values in \(\mathbb{C}^4\) and its hyperanalytic (quaternion valued) signal \(h(t)\), then the following stands.

Property 6: If \(z(t)\) with values in \(\mathbb{C}^4\) is an improper complex signal, then its hyperanalytic signal \(h(t)\), computed using a quaternion Fourier transform with axis \(\mu\), is \(\mu\)-proper, providing that \((1, \xi, \mu, \xi\mu)\) is a quaternion basis. As a consequence, the covariance of \(h(t)\) is:
\[
\Gamma_h = \begin{bmatrix}
\alpha & \gamma & 0 & \beta \\
\gamma & \omega & \beta & 0 \\
0 & \beta & \alpha & -\gamma \\
\beta & 0 & -\gamma & \omega
\end{bmatrix}
\]
with:
\[
\alpha = \int_{-\infty}^{\infty} z_r^2(t)dt = \int_{-\infty}^{\infty} o_r^2(t)dt \\
\omega = \int_{-\infty}^{\infty} z_i^2(t)dt = \int_{-\infty}^{\infty} o_i^2(t)dt \\
\beta = \int_{-\infty}^{\infty} z_i(t) o_r(t)dt = \int_{-\infty}^{\infty} z_r(t) o_i(t)dt \\
\gamma = \int_{-\infty}^{\infty} z_r(t) z_i(t)dt = -\int_{-\infty}^{\infty} o_r(t) o_i(t)dt
\]

Proof: We need to prove the zeroing of the terms of \(\Gamma_h\) as well as equalities between some of its terms. Obviously \(\Gamma_h\) is a real symmetric matrix, so we only need proofs for the diagonal and upper-diagonal elements. We start with the diagonal elements.

First, recall that, due to Parseval’s theorem:
\[
\int_{-\infty}^{\infty} z_r(t)^2dt = \int_{-\infty}^{\infty} |Z_r(\nu)|^2d\nu
\]
Now by definition of the orthogonal signal \(o(t)\), the Fourier transform of its real part is:
\[
O_r(\nu) = \mathcal{F}[o_r(t)] = \mathcal{F}[\mathcal{H}[z_r(t)] = -\xi \text{sign}(\nu)Z_r(\nu)
\]
It follows directly that \(|O_r(\nu)|^2 = |Z_r(\nu)|^2\) and using Parseval’s theorem (which is also valid for \(o_r(t)\)), the following is true:
\[
\alpha = \int_{-\infty}^{\infty} z_r^2(t)dt = \int_{-\infty}^{\infty} o_r^2(t)dt = \int_{-\infty}^{\infty} \mathcal{H}[z_r(t)]^2 dt
\]
A similar reasoning is valid to show that:
\[ \omega = \int_{-\infty}^{\infty} z_i(t)^2 dt = \int_{-\infty}^{\infty} o_i^2(t) dt \]
with the small difference that \( z_i(t) = -\mathcal{H}[z_i(t)] \), but the sign has no effect on the reasoning due to the modulus.

We now turn to off-diagonal elements. As \( z(t) \) is an improper signal, its real and imaginary parts are not orthogonal to each other, so the following is true:
\[ \gamma = \int_{-\infty}^{\infty} z_i(t) z_i(t) dt \neq 0 \]

Now, Parseval’s theorem allows us to write:
\[ \int_{-\infty}^{\infty} z_r(t)(z^*_r(t))^* dt = \int_{-\infty}^{\infty} Z_r(\nu) Z_i(-\nu) d\nu \]
where \( Z_r(\nu) \) and \( Z_i(\nu) \) are the classical Fourier transforms of \( z_r(t) \) and \( z_i(t) \). We also have that:
\[
\begin{align*}
O_r(\nu) &= -\xi \text{sign}(\nu) Z_r(\nu) \\
o_i(\nu) &= \xi \text{sign}(\nu) Z_i(\nu)
\end{align*}
\]

Therefore, we can write the following equality:
\[
\int_{-\infty}^{\infty} O_r(\nu) O_i(-\nu) d\nu \\
= \int_{-\infty}^{\infty} \xi \text{sign}(\nu) Z_r(\nu) \xi \text{sign}(\nu) Z_i(-\nu) d\nu \\
= -\int_{-\infty}^{\infty} Z_r(\nu) Z_i(-\nu) d\nu = -\int_{-\infty}^{\infty} z_r(t) z_i(t) dt
\]

This shows, due to Parseval’s theorem, that:
\[
\int_{-\infty}^{\infty} o_r(t) o_i(t) dt = -\int_{-\infty}^{\infty} z_r(t) z_i(t) dt = -\gamma
\]

Then, we have already shown in the proof of Proposition \( \{1\} \) that:
\[
\int_{-\infty}^{\infty} z_r(t) o_r(t) dt = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} z_i(t) o_i(t) dt = 0
\]
due to orthogonality between each signal and its Hilbert transform.

Now, it was also shown in Proposition \( \{2\} \) that, due to orthogonality between \( z(t) \) and \( o(t) \) the following stands:
\[
\int_{-\infty}^{\infty} z_i(t) o_r(t) dt = \int_{-\infty}^{\infty} z_r(t) o_i(t) dt = \beta
\]

We now show that in the case where \( z(t) \) is improper, then \( \beta \neq 0 \). We start by recalling that:
\[
\int_{-\infty}^{\infty} z_i(t) o_r(t) dt = \int_{-\infty}^{\infty} Z_r(\nu) O_i^*(\nu) d\nu = \int_{-\infty}^{\infty} O_r(\nu) Z_i^*(\nu) d\nu
\]
\[
\int_{-\infty}^{\infty} z_r(t) o_i(t) dt = \int_{-\infty}^{\infty} Z_r(\nu) O_r^*(\nu) d\nu = \int_{-\infty}^{\infty} Z_r(\nu) O_i^*(\nu) d\nu
\]

Now, using the expressions for \( O_r(\nu) \) and \( O_i(\nu) \) given above, we have that:
\[
\int_{-\infty}^{\infty} z_i(t) o_r(t) dt = -\xi \int_{-\infty}^{\infty} \text{sign}(\nu) Z_i(\nu) Z_r^*(\nu) d\nu
\]
\[
\int_{-\infty}^{\infty} z_r(t) o_i(t) dt = -\xi \int_{-\infty}^{\infty} \text{sign}(\nu) Z_r(\nu) Z_i^*(\nu) d\nu
\]

But, remember that \( z_r(t) \), \( o_r(t) \), \( o_i(t) \) and \( z_i(t) \) are real valued, and thus \( \int_{-\infty}^{\infty} z_i(t) o_r(t) dt \in \mathbb{R} \) and \( \int_{-\infty}^{\infty} z_r(t) o_i(t) dt \in \mathbb{R} \). As a consequence, the two integrals equal to \( \beta \) may be written:
\[
\int_{-\infty}^{\infty} z_i(t) o_r(t) dt = -\Im \left[ \int_{-\infty}^{\infty} \text{sign}(\nu) Z_i(\nu) Z_r^*(\nu) d\nu \right]
\]
\[
\int_{-\infty}^{\infty} z_r(t) o_i(t) dt = \Im \left[ \int_{-\infty}^{\infty} \text{sign}(\nu) Z_r(\nu) Z_i^*(\nu) d\nu \right]
\]

As these integrals are all equal, we simply need to show that one of them is not equal to zero for our purpose. Now, let us consider the following quantity:
\[
W(\nu) = \Im \left[ \int_{-\infty}^{\infty} Z_r(\nu) Z_i^*(\nu) d\nu \right]
\]

One can see by direct calculation (using Hermitian symmetry of the classical Fourier transform) that:
\[
W(-\nu) = \Im \left[ \int_{-\infty}^{\infty} Z_r(-\nu) Z_i^*(-\nu) d\nu \right]
\]
\[
= \Im \left[ \int_{-\infty}^{\infty} Z_r^*(\nu) Z_i(\nu) d\nu \right]
\]
\[
= -\Im \left[ \int_{-\infty}^{\infty} Z_r(\nu) Z_i^*(\nu) d\nu \right]
\]
\[
= -W(\nu)
\]

which shows that \( W(\nu) \) is odd. Now, as a consequence, the following is true:
\[
\Im \left[ \int_{-\infty}^{\infty} \text{sign}(\nu) Z_r(\nu) Z_i^*(\nu) d\nu \right] \] is even
And this makes it impossible to cancel out, except in the very special case where \( Z_r(\nu) \) and/or \( Z_i(\nu) \) are equal to zero for any \( \nu \). Thus the following is true:

\[
\Im \left[ \int_{-\infty}^{\infty} \text{sign}(\nu) Z_r(\nu) Z^*_i(\nu) d\nu \right] = \int_{-\infty}^{\infty} z_r(t) o_i(t) dt = \beta \neq 0
\]

This completes the proof.

In the most ‘classical’ case (\( z(t) \) taking values in \( \mathbb{C}^4 \)), this means that if one computes the corresponding hyperanalytic signal using a quaternion Fourier transform with axis \( j \), then the hyperanalytic signal \( h(t) \) will be \( \mathbb{C}^j \)proper.

A way to figure out the meaning of the \( \mathbb{C} \)-properness of the hyperanalytic signal is to see it as a pair of complex signals, both \textit{improper} and orthogonal to each other (in the sense of the complex scalar product) but not jointly proper.

We now consider the case when the original signal is already a complex analytic signal. This is a degenerate case, and the consequences for properness properties are given as follows.

Property 7: If \( z(t) \) with values in \( \mathbb{C}^\mu \) is a proper complex signal, then its hyperanalytic signal \( h(t) \), computed using a quaternion Fourier transform with axis \( \mu \) is \( \mathbb{C}^\mu \)-superproper provided that \( (1, \xi, \mu, E \mu) \) is a quaternion basis. As a consequence, the covariance of \( h(t) \) reads:

\[
\Gamma_h = \begin{bmatrix}
\alpha & 0 & 0 & \beta \\
0 & \alpha & \beta & 0 \\
0 & \beta & \alpha & 0 \\
\beta & 0 & 0 & \alpha
\end{bmatrix}
\]

with:

\[
\alpha = \int_{-\infty}^{\infty} z^2_r(t) dt = \int_{-\infty}^{\infty} o_i^2(t) dt = \int_{-\infty}^{\infty} z^2_i(t) dt = \int_{-\infty}^{\infty} o_r^2(t) dt
\]

and \( \beta \) being as defined in Property 6.

Proof: Referring to Property 6 we need to prove that:

\[
\alpha = \omega \quad \text{and} \quad \gamma = 0.
\]

First, as \( z(t) \) is a complex analytic signal, it is \textit{proper} and so:

\[
\int_{-\infty}^{\infty} z_r(t) z_i(t) dt = \gamma = 0
\]

Then, it is also a well-known property of analytic signals that their real and imaginary parts have the same magnitude. Here, this means that:

\[
\int_{-\infty}^{\infty} z^2_r(t) dt = \int_{-\infty}^{\infty} z^2_i(t) dt
\]

and consequently \( \alpha = \omega \). This completes the proof.

Note that we have only highlighted here the properness properties of the hyperanalytic signal and left for future work the possibility of taking advantage of these properties in its processing.

\footnote{We use the term \( \mathbb{C}^\mu \)-superproper for a \( \mathbb{C}^\mu \)-proper quaternion valued signal obtained from a complex proper signal, \textit{i.e.} for which \( \gamma = 0 \) in the covariance matrix.}

VII. EXAMPLES

We present here an example signal, with its complex envelope and phase computed using a hyperanalytic signal. The hyperanalytic signal is computed from a one-sided quaternion Fourier transform, with the negative frequencies suppressed using the discrete-time scheme first published by Marple [28].

Figure 4 shows a complex analogue of the signal shown in Figure 3. The sinusoidal carrier is amplitude modulated by two Gaussian pulses, and modulated in angle (in the complex plane) by multiplication with a complex exponential. The figure shows an orthogonal signal and complex envelope, computed from a hyperanalytic signal constructed using a quaternion Fourier transform, with the envelope extracted as described in Definition 5. Figure 5 shows the real and imaginary components of the signals shown in Figure 4 as well as the original sinusoidal carrier and the extracted complex phase \( \phi(t) \) (in this case the imaginary part is negligible, as expected from Property 5 and is not shown). The envelope can be seen to pass through all four quadrants. Phase unwrapping was used to eliminate discontinuities in the complex envelope, operating on the angle of the complex envelope and constructing the final result from the modulus of the complex envelope combined with the phase-unwrapped angle. The QTFT toolbox \cite{24} was used for the computations, and in particular, the function \texttt{cdpolar.m} was used to extract the complex envelope from the hyperanalytic signal.

VIII. CONCLUSIONS

We have shown in this paper how the classical analytic signal concept may be extended to the case of the hyperanalytic signal of an original complex signal, constructing an orthogonal complex signal by one of three equivalent methods. The quaternion method yields an interpretation of the hyperanalytic signal as a quaternion signal which leads naturally to the definition of the complex envelope. We have shown that the quaternion polar representation developed by the authors in 2008 permits us to recover the phase of a sinusoidal carrier modulated by a complex signal with frequency content lower than the carrier frequency, extending the classical result of amplitude modulation.
We have presented the properness of the hyperanalytic signal, showing that for a given improper complex signal, its hyperanalytic signal is C-proper.

As we stated earlier in the paper, our aim in this work has been to make some progress towards an analytic signal concept applicable to vector or quaternion signals. We believe the present paper indicates some of the problems to be overcome in doing this, including the need to work in a higher dimensional algebra than the quaternions, just as the results in this paper are most simply expressed using quaternion concepts rather than the concept of complex pairs.

There is also scope to combine the ideas presented here with the ideas of monogenic signals (that is analytic signals in two dimensions, or the analytic signal of images) [10].

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