Bifurcation of limit cycles of the nongeneric quadratic reversible system with discontinuous perturbations ∗

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Abstract By using the Picard-Fuchs equation and the property of Chebyshev space to the discontinuous differential system, we obtain an upper bound of the number of limit cycles for the nongeneric quadratic reversible system when it is perturbed inside all discontinuous polynomials with degree n.

Keywords quadratic reversible system; Melnikov function; Picard-Fuchs equation; Chebyshev space

MSC 34C05; 34C07

1 Introduction and main results

Stimulated by discontinuous phenomena in the real world, such as biology [8], nonlinear oscillations [13], impact and friction mechanics [1], a big interest has appeared for studying the number of limit cycles and their relative positions of discontinuous differential systems. Similar to the smooth differential system, one of the main problems in the qualitative theory of non-smooth differential systems is the study of their limit cycles, and many methodologies have been developed, such as Abelian integral method (or first order Melnikov function) [11,12,19,20], averaging method [23,10,13,15]. This problem can be seen as an extension of the infinitesimal Hilbert’s 16th Problem to the discontinuous world.

The list of quadratic center at (0,0), almost all the orbits of which are cubic, looks as follows [9,21]:

The Hamiltonian system $Q_3^H$: $\dot{z} = -iz - z^2 + 2|z|^2 + (b + ic)\bar{z}^2$.

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The Hamiltonian triangle: $\dot{z} = -iz + \bar{z}^2$.

The reversible system: $\dot{z} = -iz + (2b + 1)z^2 + 2|z|^2 + b\bar{z}^2, b \neq -1$.

The generic Lotka-Volterra system: $\dot{z} = -iz + (1 - ci)z^2 + ci\bar{z}^2, c = \pm \frac{1}{\sqrt{3}}$.

Under the perturbations of continuous polynomials of degree $n$, Horozov and Iliev [6] proved that the number of limit cycles for $Q_3^H$ and Hamiltonian triangle does not exceed $5n + 15$, and Zhao et al. [21] proved that the number of limit cycles for reversible and generic Lotka-Volterra systems does not exceed $7n$.

Let $z = x + iy$ and by a linear transformation, the reversible system can be written [21]:

\[
\begin{cases}
\dot{x} = xy, \\
\dot{y} = \frac{3}{2}y^2 + ax^2 - 2(a + 1)x + a + 2,
\end{cases}
\]

where $a \in \mathbb{R}$. When $a = -2$, system (1.1) corresponds to the nongeneric case of the reversible system (1.1):

\[
\begin{cases}
\dot{x} = xy, \\
\dot{y} = \frac{3}{2}y^2 - 2x^2 + 2x
\end{cases}
\]

whose first integral is

\[H(x, y) = x^{-3}\left(\frac{1}{2}y^2 - 2x^2 + x\right) = h, \quad h \in (-1, 0)\]

with the integrating factor $\mu(x, y) = x^{-4}$.

In the present paper, by using the Picard-Fuchs equation and the property of Chebyshev space, we investigate the number of limit cycles of system (1.2) under discontinuous polynomial perturbations of degree $n$. The system (1.2) has a center (1,0) and $h = -1$ corresponds to the center (1,0). The perturbed system of (1.2) is

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{cases}
xy + \varepsilon f^+(x, y), & y > 0, \\
\frac{3}{2}y^2 - 2x^2 + 2x + \varepsilon g^+(x, y), & y < 0,
\end{cases}
\]

where $0 < |\varepsilon| \ll 1$,

\[f^\pm(x, y) = \sum_{i+j=0}^{n} a^\pm_{i,j} x^i y^j, \quad g^\pm(x, y) = \sum_{i+j=0}^{n} b^\pm_{i,j} x^i y^j, \quad i, j \in \mathbb{N}.
\]

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From the Theorems 1.1 in [7,12], by linear transformations, we know that the first order Melnikov function \( M(h) \) of system (1.4) is

\[
M(h) = \int_{\Gamma^+_h} x^{-4}[g^+(x,y)dx - f^+(x,y)dy] \\
+ \int_{\Gamma^-_h} x^{-4}[g^-(x,y)dx - f^-(x,y)dy], \quad h \in (-1,0),
\]

(1.5)

where

\[
\Gamma^+_h = \{(x,y)|H(x,y) = h, h \in (-1,0), y > 0\}, \\
\Gamma^-_h = \{(x,y)|H(x,y) = h, h \in (-1,0), y < 0\},
\]

and its number of zeros gives an upper bound of the number of limit cycles of system (1.4) bifurcating from the period annulus.

Our main results are the following two theorems.

**Theorem 1.1.** Suppose that \( h \in (-1,0) \).

(i) If \( n = 2, 3 \), then the number of limit cycles of system (1.4) bifurcating from the period annulus is not more than 40 (counting multiplicity).

(ii) If \( 4 \leq n \leq 7 \), then the number of limit cycles of system (1.4) bifurcating from the period annulus is not more than \( 24n - 56 \) (counting multiplicity).

(iii) If \( n \geq 8 \), then the number of limit cycles of system (1.4) bifurcating from the period annulus is not more than \( 22n - 64 \) (counting multiplicity).

**Theorem 1.2.** Suppose that \( h \in (-1,0) \), \( a_{i,j}^+ = a_{i,j}^- \) and \( b_{i,j}^+ = b_{i,j}^- \).

(i) If \( n = 2, 3 \), then the number of limit cycles of system (1.4) bifurcating from the period annulus is not more than 4 (counting multiplicity).

(ii) If \( n \geq 4 \), then the number of limit cycles of system (1.4) bifurcating from the period annulus is not more than \( 3n - 8 \) (counting multiplicity).

**Remark 1.1.** (i) By using the Picard-Fuchs equation, we greatly simplified the computation of the first order Melnikov function. And then we can estimate the number of zeros of the first order Melnikov function which controls the number of limit cycles of the corresponding perturbed system benefited from the property of Chebyshev space. It is worth noting that these methods can be applied to study the bifurcation of limit cycles for other integrable differential systems.

(ii) The perturbation as in (1.4) can be found in many practical applications, such as in the slender rocking block model and nonlinear compliant oscillator, see [5,16,17] and the references quoted there.

(iii) If \( h \in (-1,0) \), \( a_{i,j}^+ = a_{i,j}^- \) and \( b_{i,j}^+ = b_{i,j}^- \), then Zhao et al. [21] obtained that the
number of limit cycles of system (1.4) bifurcating from the period annulus is not more than $3n - 4$ for $n \geq 4$; 8 for $n = 3$; 5 for $n = 2$ (counting multiplicity).

The rest of the paper is organized as follows: In Section 2, we will obtain the algebraic structure of the first order Melnikov function $M(h)$ and the Picard-Fuchs equations satisfied by the generators of $M(h)$ are also obtained. Finally, we will prove Theorems 1.1 and 1.2 in Section 3.

2 The algebraic structure of $M(h)$ and Picard-Fuchs equation

In this section, we obtain the algebraic structure of the first Melnikov function $M(h)$. For $h \in (-1, 0)$, we denote

$$I_{i,j}(h) = \int_{\Gamma_h^+} x^{i-4} y^j dx, \quad J_{i,j}(h) = \int_{\Gamma_h^-} x^{i-4} y^j dx.$$ 

We first prove the following results.

**Lemma 2.1.** Suppose that $h \in (-1, 0)$, $i = -1, 0, 1, \ldots$ and $j = 0, 1, 2, \ldots$.

(i) The following equalities hold:

$$\begin{cases}
I_{-1,1}(h) = \frac{1}{3}[hI_{1,1}(h) + 8I_{0,1}(h)], \\
I_{0,0}(h) = \frac{1}{3}[hI_{2,0}(h) + 4I_{1,0}(h)], \\
I_{-1,2}(h) = \frac{4}{3}(h + 1)I_{2,0}(h), \\
I_{1,0}(h) = I_{2,0}(h), \\
I_{-1,3}(h) = 12[I_{1,1}(h) - I_{0,1}(h)].
\end{cases} \tag{2.1}$$

$$\begin{cases}
I_{-1,4}(h) = 4[I_{1,2}(h) - I_{0,2}(h)], \\
I_{0,3}(h) = 4[I_{2,1}(h) - I_{1,1}(h)], \\
I_{1,2}(h) = \frac{1}{h}[2I_{0,2}(h) - 3I_{-1,2}(h)], \\
I_{2,1}(h) = \frac{1}{h}[4I_{1,1}(h) - 5I_{0,1}(h)], \\
I_{3,0}(h) = \frac{1}{h}[\frac{1}{2}I_{0,2}(h) - 2I_{2,0}(h) + I_{1,0}(h)].
\end{cases} \tag{2.2}$$

(ii) If $4 \leq n \leq 7$, then

$$\begin{cases}
I_{i,2j+1}(h) = \frac{1}{h^{i+1}}[\bar{\alpha}(h)I_{0,1}(h) + \bar{\beta}(h)I_{1,1}(h)], \quad i + 2j + 1 = n, \\
I_{i,2j}(h) = \frac{1}{h^{i+1}}[\bar{\gamma}(h)I_{2,0}(h) + \bar{\delta}(h)I_{0,2}(h)], \quad i + 2j = n,
\end{cases}$$
where $\bar{\alpha}(h)$, $\bar{\beta}(h)$, $\bar{\gamma}(h)$ and $\bar{\delta}(h)$ are polynomials of $h$ with deg $\bar{\alpha}(h)$, deg $\bar{\delta}(h) \leq n-4$ and deg $\bar{\beta}(h)$, deg $\bar{\gamma}(h) \leq n-3$.

(iii) If $n \geq 8$, then
\[
\begin{cases}
I_{i,2j+1}(h) = \frac{1}{h^{n-3}}[\bar{\alpha}(h)I_{0,1}(h) + \bar{\beta}(h)I_{1,1}(h)], & i + 2j + 1 = n, \\
I_{i,2j}(h) = \frac{1}{h^{n-3}}\bar{\gamma}(h)I_{2,0}(h), & i + 2j = n,
\end{cases}
\]
where $\bar{\alpha}(h)$, $\bar{\beta}(h)$ and $\bar{\gamma}(h)$ are polynomials of $h$ with deg $\bar{\alpha}(h) \leq n-5$ and deg $\bar{\beta}(h)$, deg $\bar{\gamma}(h) \leq n-4$.

**Proof.** Let $D$ be the interior of $\Gamma_h^+ \cup \overrightarrow{AB}$, see the black line in Fig. 1. Using the Green’s Formula, we have for $j \geq 0$
\[
\int_{\Gamma_h^+} x^i y^j dy = \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^i y^j dy - \int_{\overrightarrow{AB}} x^i y^j dy
= \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^i y^j dy = -i \int_D x^{i-1} y^j dxdy,
\]
\[
\int_{\Gamma_h^+} x^{i-1} y^{j+1} dx = \oint_{\Gamma_h^+ \cup \overrightarrow{AB}} x^{i-1} y^{j+1} dx = (j + 1) \int_D x^{i-1} y^j dxdy.
\]
Hence,
\[
\int_{\Gamma_h^+} x^i y^j dy = -\frac{i}{j + 1} \int_{\Gamma_h^+} x^{i-1} y^{j+1} dx, \ j \geq 0. \tag{2.3}
\]
In a similar way, we have
\[
\int_{\Gamma_h^-} x^i y^j dy = -\frac{i}{j + 1} \int_{\Gamma_h^-} x^{i-1} y^{j+1} dx, \ j \geq 0. \tag{2.4}
\]
By a straightforward calculation and noting that (2.3) and (2.4), we obtain

\[ M(h) = \int_{\Gamma_h^+} x^{-4}(g^+(x, y)dx - f^+(x, y)dy) \]
\[ + \int_{\Gamma_h^-} x^{-4}(g^-(x, y)dx - f^-(x, y)dy) \]
\[ = \int_{\Gamma_h^+} \sum_{i+j=0}^n b_{i,j}^+ x^{i-4}y^j dx - \int_{\Gamma_h^-} \sum_{i+j=0}^n a_{i,j}^+ x^{i-4}y^j dy \]
\[ + \int_{\Gamma_h^-} \sum_{i+j=0}^n b_{i,j}^- x^{i-4}y^j dx - \int_{\Gamma_h^+} \sum_{i+j=0}^n a_{i,j}^- x^{i-4}y^j dy \]
\[ = \sum_{i+j=0}^n b_{i,j}^+ \int_{\Gamma_h^+} x^{i-4}y^j dx + \sum_{i+j=0}^n \frac{i-4}{j+1} a_{i,j}^+ \int_{\Gamma_h^+} x^{i-5}y^{j+1} dx \]
\[ + \sum_{i+j=0}^n b_{i,j}^- \int_{\Gamma_h^-} x^{i-4}y^j dx + \sum_{i+j=0}^n \frac{i-4}{j+1} a_{i,j}^- \int_{\Gamma_h^-} x^{i-5}y^{j+1} dx \]
\[ = \sum_{i+j=0}^n \tilde{a}_{i,j} I_{i,j}(h) + \sum_{i+j=0, i \geq -1, j \geq 0}^n \tilde{b}_{i,j} J_{i,j}(h) \]
\[ := \sum_{i+j=0, i \geq -1, j \geq 0}^n \rho_{i,j} I_{i,j}(h), \]

where in the last equality we have used that \( J_{i,j}(h) = (-1)^{j+1} I_{i,j}(h) \).

Differentiating (1.3) with respect to \( x \), we obtain

\[ x^{-3} y \frac{\partial y}{\partial x} - \frac{3}{2} x^{-4} y^2 + 2x^{-2} - 2x^{-3} = 0. \] (2.6)

Multiplying (2.6) by \( x^i y^{j-2} dx \), integrating over \( \Gamma_h^+ \) and noting that (2.3), we have

\[ (2i + 3j - 6)I_{i,j} = 4j(I_{i+2,j-2} - I_{i+1,j-2}). \] (2.7)

Similarly, multiplying (1.3) by \( x^{i-4}y^j dx \) and integrating over \( \Gamma_h^+ \) yields

\[ hI_{i,j} = \frac{1}{2} I_{i-3,j+2} - 2I_{i-1,j} + I_{i-2,j}. \] (2.8)

Eliminating \( I_{i-3,j+2} \) by (2.7) and (2.8) gives

\[ (2i + 3j - 6)hI_{i,j} = (2i + j - 10)I_{i-2,j} - 4(i + j - 4)I_{i-1,j}. \] (2.9)

From (2.7) we have

\[ I_{1,0} = I_{2,0}, \quad I_{-1,3} = 12(I_{1,1} - I_{0,1}). \] (2.10)
From (2.8) we obtain
\[ hI_{2,0} = \frac{1}{2}I_{-1,2} - 2I_{1,0} + I_{0,0}. \]  
(2.11)

Taking \((i, j) = (2, 0), (1, 1)\) in (2.9) we have
\[ I_{0,0} = \frac{1}{3}(hI_{2,0} + 4I_{1,0}) , \ I_{-1,1} = \frac{1}{7}(hI_{1,1} + 8I_{0,1}). \]  
(2.12)

Hence,
\[ I_{0,0} = \frac{1}{3}(h + 4)I_{2,0}. \]  
(2.13)

From (2.10)-(2.12) we get
\[ I_{-1,2} = \frac{4}{3}(h + 1)I_{2,0}. \]  
(2.14)

(2.10) and (2.12)-(2.14) imply (2.1) holds. In a similar way, applying the equalities (2.7) and (2.9), we can obtain (2.2). Hence, the conclusion (i) holds. By some straightforward calculations according to (2.7) and (2.9), we can get the conclusion (ii).

(iii) Now we prove the conclusion (iii) by induction on \(n\). Without loss of generality, we only show the case \(i + 2j + 1 = n\). With the help of Maple, from (2.7) and (2.9) and noting that the conclusions (i) and (ii), we obtain
\[
\begin{align*}
I_{-1,9} &= -\frac{768}{361896}[(200h^3 + 3000h^2 + 2024h + 512)I_{0,1} \\
&\quad + (663h^4 + 326h^3 + 239h^2 + 64h)I_{1,1}], \\
I_{0,8} &= -\frac{2948}{315h^5}(h + 1)^4I_{2,0}, \\
I_{1,7} &= -\frac{64}{7293h^5}[(385h^3 + 1385h^2 + 1480h + 512)I_{0,1} \\
&\quad + (139h^3 + 171h^2 + 64h)I_{1,1}], \\
I_{2,6} &= -\frac{128}{35h^5}(h + 1)^3I_{2,0}, \\
I_{3,5} &= -\frac{16}{3003h^5}[(480h^2 + 1000h + 512)I_{0,1} + (39h^3 + 111h^2 + 64h)I_{1,1}], \\
I_{4,4} &= -\frac{32}{105h^5}(h + 1)^2(h + 8)I_{2,0}, \\
I_{5,3} &= -\frac{4}{1001h^5}[(77h^2 + 584h + 512)I_{0,1} + (59h^2 + 64h)I_{1,1}], \\
I_{6,2} &= -\frac{4}{15h^5}(h + 1)(3h + 8)I_{2,0}, \\
I_{7,1} &= -\frac{1}{231h^5}[(232h + 512)I_{0,1} + (15h^2 + 64h)I_{1,1}], \\
I_{8,0} &= -\frac{1}{6h^5}(h^2 + 12h + 16)I_{2,0},
\end{align*}
\]
which imply that the conclusion holds for \( n = 8 \). Now assume that (iii) holds for 
\[ i + 2l + 1 \leq k - 1 \quad (k \geq 9) \]. For \( i + 2l + 1 = k \), if \( k \) is an even number, then taking 
\((i, 2l + 1) = (-1, k + 1)\) in (2.7) and \((i, 2l + 1) = (1, k - 1), (3, k - 3), \ldots, (k - 3, 3), (k - 1, 1)\) in (2.9), respectively, we have

\[
\begin{pmatrix}
I_{-1,k+1} \\
I_{1,k-1} \\
I_{3,k-3} \\
\vdots \\
I_{k-3,3} \\
I_{k-1,1}
\end{pmatrix} = \frac{1}{h} \begin{pmatrix}
\frac{4(k+1)}{5-3k} hI_{0,k-1} \\
\frac{1}{3k-7} [(k-9)I_{-1,k-1} - 4(k-4)I_{0,k-1}] \\
\frac{1}{3k-9} [(k-7)I_{1,k-3} - 4(k-4)I_{2,k-3}] \\
\vdots \\
\frac{1}{2k-3} (2k-13)I_{k-5,3} - 4(k-4)I_{k-4,3} \\
\frac{1}{2k-3} (2k-11)I_{k-3,1} - 4(k-4)I_{k-2,1}
\end{pmatrix}, 
\tag{2.15}
\]

where

\[
A = \begin{pmatrix}
1 & \frac{4k+1}{5-3k} & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

is a \( \frac{k+2}{2} \times \frac{k+2}{2} \) matrix and \( \det A = 1 \). If \( k \) is an odd number taking \((i, 2l + 1) = (0, k)\) in (2.7) and \((i, 2l + 1) = (2, k - 2), (4, k - 4), \ldots, (k - 3, 3), (k - 1, 1)\) in (2.9), respectively, we have

\[
\begin{pmatrix}
I_{0,k} \\
I_{2,k-2} \\
I_{4,k-4} \\
\vdots \\
I_{k-3,3} \\
I_{k-1,1}
\end{pmatrix} = \frac{1}{h} \begin{pmatrix}
\frac{4k}{6-3k} hI_{1,k-2} \\
\frac{1}{3k-8} [(k-8)I_{0,k-2} - 4(k-4)I_{1,k-2}] \\
\frac{1}{3k-10} [(k-6)I_{2,k-4} - 4(k-4)I_{3,k-4}] \\
\vdots \\
\frac{1}{2k-3} (2k-13)I_{k-5,3} - 4(k-4)I_{k-4,3} \\
\frac{1}{2k-5} (2k-11)I_{k-3,1} - 4(k-4)I_{k-2,1}
\end{pmatrix}, 
\tag{2.16}
\]

where

\[
B = \begin{pmatrix}
1 & \frac{4k}{6-3k} & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]
is a $\frac{k+1}{2} \times \frac{k+1}{2}$ matrix and $\det \mathbf{B} = 1$. Hence, we can get that $I_{i,2l+1}$ can be expressed by $I_{0,1}$ and $I_{1,1}$ for $i + 2l + 1 = k$ by the induction hypothesis.

From (2.15) and (2.16), we have for $(i, 2l + 1) = (-1, k + 1)$ or $(i, 2l + 1) = (0, k)$

\[
I_{-1,k+1}(h) = \frac{1}{h^{k-3}} [h \alpha^{(k-1)}(h) I_{0,1} + h \beta^{(k-1)}(h) I_{1,1}]
\]

\[
I_{0,k}(h) = \frac{1}{h^{k-3}} [h \alpha^{(k-1)}(h) I_{0,1} + h \beta^{(k-1)}(h) I_{1,1}]
\]

where $\alpha^{(k-1)}(h)$ and $\beta^{(k-1)}(h)$ are polynomials in $h$. By the induction hypothesis we obtain that

\[
\deg \alpha^{(k-1)}(h) \leq k - 6, \quad \deg \beta^{(k-1)}(h) \leq k - 5.
\]

Therefore,

\[
\deg \alpha^{(k)}(h) \leq k - 5, \quad \deg \beta^{(k)}(h) \leq k - 4.
\]

In a similar way, we can prove the cases for $(i, 2l + 1) = (1, k - 1), (3, k - 3), \cdots, (k - 3, 3), (k - 1, 1)$ or $(i, 2l + 1) = (2, k - 2), (4, k - 4), \cdots, (k - 3, 3), (k - 1, 1)$. This ends the proof. \(\diamondsuit\)

**Lemma 2.2.** Suppose that $h \in (-1, 0)$.

(i) If $n = 2, 3$, then

\[
M(h) = \alpha(h) I_{0,1}(h) + \beta(h) I_{1,1}(h) + \gamma(h) I_{2,0}(h) + \delta(h) I_{0,2}(h), \tag{2.17}
\]

where $\alpha(h)$ is a constant, and $\beta(h)$, $\gamma(h)$ and $\delta(h)$ are polynomials of $h$ with $\deg \beta(h), \deg \gamma(h), \deg \delta(h) \leq 1$.

(ii) If $4 \leq n \leq 7$, then

\[
M(h) = \frac{1}{h^{n-3}} [\alpha(h) I_{0,1}(h) + \beta(h) I_{1,1}(h) + \gamma(h) I_{2,0}(h) + \delta(h) I_{0,2}(h)],
\]

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ and $\delta(h)$ are polynomials of $h$ with $\deg \alpha(h), \deg \delta(h) \leq n - 4$ and $\deg \beta(h), \deg \gamma(h) \leq n - 3$.

(iii) If $n \geq 8$, then

\[
M(h) = \frac{1}{h^{n-3}} [\alpha(h) I_{0,1}(h) + \beta(h) I_{1,1}(h) + \gamma(h) I_{2,0}(h) + \delta(h) I_{0,2}(h)],
\]

where $\alpha(h)$, $\beta(h)$, $\gamma(h)$ and $\delta(h)$ are polynomials of $h$ with $\deg \alpha(h) \leq n - 5$, $\deg \beta(h), \deg \gamma(h) \leq n - 4$ and $\deg \delta(h) \leq 3$.  

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Lemma 2.3. (i) The vector function \((I_{0,1}, I_{1,1})^T\) satisfies the following Picard-Fuchs equation

\[
\begin{pmatrix}
I_{0,1} \\
I_{1,1}
\end{pmatrix} = \begin{pmatrix}
\frac{4}{3}h + \frac{16}{15} \\
\frac{4}{3}h
\end{pmatrix} \begin{pmatrix}
I'_{0,1} \\
I'_{1,1}
\end{pmatrix}.
\] (2.18)

(ii) The vector function \((I_{2,0}, I_{0,2})^T\) satisfies the following Picard-Fuchs equation

\[
\begin{pmatrix}
I_{2,0} \\
I_{0,2}
\end{pmatrix} = \begin{pmatrix}
2h + 2 & 0 \\
4h + 4 & h
\end{pmatrix} \begin{pmatrix}
I'_{2,0} \\
I'_{0,2}
\end{pmatrix}.
\] (2.19)

Proof. From (1.3) we get

\[
\frac{\partial y}{\partial h} = \frac{x^3}{y},
\]

which implies

\[
I'_{i,j} = j \int_{\Gamma^+} x^{i-1}y^{-2} \, dx.
\] (2.20)

Hence,

\[
I_{i,j} = \frac{1}{j+2} I'_{i-3,j+2}.
\] (2.21)

Multiplying both side of (2.20) by \(h\), we have

\[
hI'_{i,j} = \frac{j}{2(j+2)} I'_{i-3,j+2} - 2I'_{i-1,j} + I'_{i-2,j}.
\] (2.22)

From (2.3) and (2.20) we have for \(j \geq 1\)

\[
I_{i,j} = \int_{\Gamma^+} x^{i-4}y^2 \, dx = -\frac{j}{i-3} \int_{\Gamma^+} x^{i-3}y^{-1} \, dy
\]

\[
= -\frac{j}{i-3} \int_{\Gamma^+} x^{i-3}y^{-1} \frac{3hx^2 + 4x - 1}{y} \, dx
\] (2.23)

\[
= -\frac{1}{i-3} (3hI'_{i,j} + 4I'_{i-1,j} - I'_{i-2,j}).
\]

(2.21)-(2.23) imply

\[
I_{i,j} = -\frac{4}{2i + j - 6} (hI'_{i,j} + I'_{i-1,j}), \quad j \geq 1.
\] (2.24)

From (2.21) and noting that (2.14) we obtain

\[
I_{2,0} = \frac{1}{2} I'_{-1,2} = \frac{2}{3} I_{2,0} + \frac{2}{3} (h + 1) I'_{2,0}.
\]
Hence,

\[ I_{2,0} = 2(h + 1)I'_{2,0}. \]  

(2.25)

From (2.24) we have

\[ I_{0,1} = \frac{4}{5}(hI'_{0,1} + I'_{-1,1}), \quad I_{1,1} = \frac{4}{3}(hI'_{1,1} + I'_{0,1}), \quad I_{0,2} = hI'_{0,2} + I'_{-1,2}, \]  

(2.26)

and noting that (2.12) and (2.14) we obtain the conclusions (i) and (ii). This ends the proof. ♦

**Lemma 2.4.** For \( h \in (-1,0), \)

\[ I_{2,0}(h) = c_1\sqrt{h+1}, \quad I_{0,2}(h) = 2c_1\sqrt{h+1} - c_1h\ln \frac{1-\sqrt{h+1}}{1+\sqrt{h+1}}, \]

where \( c_1 \) is a nonzero constant.

**Proof.** From (2.19) we have \( I_{2,0}(h) = c_1\sqrt{h+1}, \) where \( c_1 \) is a constant. Therefore, we have for \( h \in (-1,0) \)

\[ I_{0,2}(h) = c_2h + 2c_1\sqrt{h+1} - c_1h\ln \frac{1-\sqrt{h+1}}{1+\sqrt{h+1}}, \]

where \( c_2 \) is a constant. Since \( I_{0,2}(-1) = 0, \) we have \( c_2 = 0. \) Hence, \( I_{0,2}(h) = 2c_1\sqrt{h+1} - c_1h\ln \frac{1-\sqrt{h+1}}{1+\sqrt{h+1}}. \) This ends the proof. ♦

Taking \( (i,j) = (4,1), (3,1) \) in (2.9) respectively and bearing in mind that (2.2), we get

\[ I_{3,1}(h) = -\frac{1}{h}I_{1,1}(h), \quad I_{4,1}(h) = -\frac{1}{5h}[I_{2,1}(h) + 4I_{3,1}(h)]. \]

Hence, \( I_{0,1}(h) = h^2I_{4,1}(h). \) Using Green formula, we have

\[ I_{4,1}(h) = \int_{\Gamma^+} ydx = \int_{\Gamma^+ \cup \overrightarrow{AB}} ydx = \int_{D} dxdy \neq 0, \]

where \( D \) is the interior of \( \Gamma^+_h \cup \overrightarrow{AB}, \) see Fig. 1. Thus, \( I_{0,1}(h) \neq 0 \) for \( h \in (-1,0). \)

Noting that \( \frac{\partial y}{\partial m} = x^3y^{-1} \) and \( dx = xydt, \) we have

\[ I'_{0,1}(h) = \int_{\Gamma^+_h} x^{-4}\frac{\partial y}{\partial h}dx = \int_{0}^{t_0} dt \neq 0, \]

where \( t_0 \) is the time from the left end point to right end point of \( \Gamma^+_h. \) So we can get the following lemma.
Lemma 2.5. Let $\omega_1(h) = \frac{I_{1,1}^0(h)}{I_{0,1}^0(h)}$ and $\omega_2(h) = \frac{I_{1,1}^1(h)}{I_{0,1}^1(h)}$ for $h \in (-1, 0)$, then $\omega_1(h)$ and $\omega_2(h)$ satisfy the following Riccati equations

$$G(h)\omega'_1(h) = \frac{1}{4} h \omega^2_1(h) - \frac{1}{2} (h - 2) \omega_1(h) - \frac{5}{4}$$

and

$$G(h)\omega'_2(h) = -\frac{1}{4} h \omega^2_2(h) - \frac{1}{2} h \omega_2(h) - \frac{1}{4},$$

respectively, where $G(h) = h(h + 1)$.

Proof. From (2.18), we have

$$G(h) \begin{pmatrix} I_{0,1}^0(h) \\ I_{1,1}^0(h) \end{pmatrix} = \begin{pmatrix} \frac{5}{4} h & -\frac{1}{4} h \\ -\frac{3}{4} & \frac{3}{4} h + 1 \end{pmatrix} \begin{pmatrix} I_{0,1}(h) \\ I_{1,1}(h) \end{pmatrix}$$

and

$$G(h) \begin{pmatrix} I_{0,1}^1(h) \\ I_{1,1}^1(h) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} h & -\frac{1}{4} h \\ -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} I_{0,1}(h) \\ I_{1,1}(h) \end{pmatrix},$$

where $G(h) = h(h + 1)$. Noting that $G(h) \neq 0$ for $h \in (-1, 0)$ and

$$\omega'_1(h) = \frac{I_{1,1}^0(h)}{I_{0,1}(h)} - \omega_1(h) \frac{I_{1,1}^0(h)}{I_{0,1}(h)}, \quad \omega'_2(h) = \frac{I_{1,1}^1(h)}{I_{0,1}(h)} - \omega_2(h) \frac{I_{1,1}^1(h)}{I_{0,1}(h)},$$

we obtain (2.27) and (2.28). This ends the proof. ♦

3 Proof of the main results

In order to prove the Theorem 1.1, we first introduce some helpful results in the literature. Let $V$ be a finite-dimensional vector space of functions, real-analytic on an open interval $I$. 

Definition 3.1 [4]. We say that $S$ is a Chebyshev space, provided that each non-zero function in $S$ has at most $\dim(S) - 1$ zeros, counted with multiplicity.

Proposition 3.1 [4]. The solution space $S$ of a second order linear analytic differential equation

$$x'' + a_1(t)x' + a_2(t)x = 0$$

on an open interval $I$ is a Chebyshev space if and only if there exists a nowhere vanishing solution $x_0(t) \in S (x_0(t) \neq 0, \forall t \in I)$. 

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Proposition 3.2. Suppose the solution space of the homogeneous equation
\[ x'' + a_1(t)x' + a_2(t)x = 0 \]
is a Chebyshev space and let \( R(t) \) be an analytic function on \( I \) having \( l \) zeros (counted with multiplicity). Then every solution \( x(t) \) of the non-homogeneous equation
\[ x'' + a_1(t)x' + a_2(t)x = R(t) \]
has at most \( l + 2 \) zeros on \( I \).

In the following we denote by \( \#\{\varphi(h) = 0, h \in (a, b)\} \) the number of isolated zeros of \( \varphi(h) \) on \( (a, b) \) taking into account the multiplicity, and we also denote by \( \Theta_k(h) \) the polynomial of degree at most \( k \).

Lemma 3.1. Suppose that \( h \in (-1, 0) \).
\( \text{(i)} \) If \( n = 2, 3 \), then there exist polynomials \( P_2^1(h), P_1^1(h) \) and \( P_0^1(h) \) of \( h \) with degree respectively 4, 3 and 2 such that \( L^1(h)\Phi(h) = 0 \).
\( \text{(ii)} \) If \( 4 \leq n \leq 7 \), then there exist polynomials \( P_2^n(h), P_1^n(h) \) and \( P_0^n(h) \) of \( h \) with degree respectively \( 2n - 4 \), \( 2n - 5 \) and \( 2n - 6 \) such that \( L^2(h)\Phi(h) = 0 \).
\( \text{(iii)} \) If \( n \geq 8 \), then there exist polynomials \( P_2^n(h), P_1^n(h) \) and \( P_0^n(h) \) of \( h \) with degree respectively \( 2n - 6 \), \( 2n - 7 \) and \( 2n - 8 \) such that \( L^3(h)\Phi(h) = 0 \), where
\[ \Phi(h) = \alpha(h)I_{0,1}(h) + \beta(h)I_{1,1}(h), \]
\[ L^i(h) = P_2^i(h)\frac{d^2}{dh^2} + P_1^i(h)\frac{d}{dh} + P_0^i(h), \quad i = 1, 2, 3. \quad (3.1) \]

Proof. Without loss of generality, we only prove (iii). (i) and (ii) can be shown similarly. By (2.18), we have
\[ V'(h) = (E - B)^{-1}(Bh + C)V''(h), \]
where \( V(h) = (I_{0,1}(h), I_{1,1}(h))^T \), and
\[ E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 4 \\ 0 & \frac{4}{3} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{16}{15} & 0 \\ \frac{4}{3} & 0 \end{pmatrix}. \]

Hence,
\[ \Phi(h) = \tau(h)V(h) = \tau(h)(Bh + C)V'(h) = \tau(h)(Bh + C)(E - B)^{-1}(Bh + C)V''(h) := \Theta_{n-3}(h)I''_{0,1}(h) + \Theta_{n-2}(h)I''_{1,1}(h), \]

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where \( \tau(h) = (\alpha(h), \beta(h)), \Theta_{n-3}(h) \) denotes a polynomial in \( h \) of degree at most \( n - 3 \) and etc.. For \( \Phi'(h) \), we have

\[
\Phi'(h) = \tau'(h)V(h) + \tau(h)V'(h)
\]

\[
= [\tau'(h)(Bh + C) + \tau(h)](E - B)^{-1}(Bh + C)V''(h)
\]

\[
: = \Theta_{n-4}(h)I''_{0,1}(h) + \Theta_{n-3}(h)I''_{1,1}(h).
\]

In a similar way, we have

\[
\Phi''(h) = \Theta_{n-5}(h)I''_{0,1}(h) + \Theta_{n-4}(h)I''_{1,1}(h).
\]

Next, suppose that

\[
P_2(h) = \sum_{k=0}^{2n-6} p_{2,k} h^k, \quad P_1(h) = \sum_{m=0}^{2n-7} p_{1,m} h^m, \quad P_0(h) = \sum_{l=0}^{2n-8} p_{0,l} h^l
\]

(3.2)

are polynomials of \( h \) with coefficients \( p_{2,k}, p_{1,m} \) and \( p_{0,l} \) to be determined such that \( L(h)\Phi(h) = 0 \) for

\[
0 \leq k \leq 2n - 6, \quad 0 \leq m \leq 2n - 7, \quad 0 \leq l \leq 2n - 8.
\]

(3.3)

By straightforward computation, we have

\[
L(h)\Phi(h) = P_2(h)\Phi''(h) + P_1(h)\Phi'(h) + P_0(h)\Phi(h)
\]

\[
= \left[ P_2(h)\Theta_{n-5}(h) + P_1(h)\Theta_{n-4}(h) + P_0(h)\Theta_{n-3}(h) \right] I''_{0,1}(h)
\]

\[
+ \left[ P_2(h)\Theta_{n-4}(h) + P_1(h)\Theta_{n-3}(h) + P_0(h)\Theta_{n-2}(h) \right] I''_{1,1}(h)
\]

\[
: = X(h)I''_{0,1}(h) + Y(h)I''_{1,1}(h),
\]

where \( X(h) \) and \( Y(h) \) are polynomials of \( h \) with deg \( X(h) \leq 3n - 11 \) and deg \( Y(h) \leq 3n - 10 \).

Let

\[
X(h) = \sum_{i=0}^{3n-11} x_i h^i, \quad Y(h) = \sum_{j=0}^{3n-10} y_j h^j,
\]

where \( x_i \) and \( y_j \) are expressed by \( p_{2,k}, p_{1,m} \) and \( p_{0,l} \) of (3.2) linearly, \( k, m \) and \( l \) satisfy (3.3). Let

\[
x_i = 0, \quad y_j = 0, \quad 0 \leq i \leq 3n - 11, \quad 0 \leq j \leq 3n - 10,
\]

(3.4)

then system (3.4) is a homogenous linear equations with \( 6n - 19 \) equations about \( 6n - 18 \) variables of \( p_{2,k}, p_{1,m} \) and \( p_{0,l} \) for \( k, m \) and \( l \) satisfy (3.3). It follows that
from the theory of linear algebra that there exist $p_{2,k}$, $p_{1,m}$ and $p_{0,l}$ such that (3.4) holds, which yields $L(h)\Phi(h) = 0$. This ends the proof.  

Lemma 3.2. Let $\Phi(h) = \alpha(h)I_{0,1}(h) + \beta(h)I_{1,1}(h)$.

(i) If $n = 2, 3$, then $\Phi(h)$ has at most 4 zeros on $(-1, 0)$, taking into account the multiplicity.

(ii) If $4 \leq n \leq 7$, then $\Phi(h)$ has at most $3n - 8$ zeros on $(-1, 0)$, taking into account the multiplicity.

(iii) If $n \geq 8$, then $\Phi(h)$ has at most $3n - 11$ zeros on $(-1, 0)$, taking into account the multiplicity.

Proof. We only prove (iii). (i) and (ii) can be proved in a similar way. Let $\chi_1(h) = \alpha(h) + \beta(h)\omega_1(h)$, so $\Phi(h) = I_{0,1}(h)\chi_1(h)$ which implies

$$\#\{\Phi(h) = 0, h \in (-1, 0)\} = \#\{\chi_1(h) = 0, h \in (-1, 0)\}.$$ 

By (2.27) we know that $\chi_1(h)$ satisfies

$$G(h)\beta(h)\chi_1'(h) = \frac{1}{4}h\chi_1(h)^2 + F_1(h)\chi_1(h) + F_0(h)$$

with $\deg F_0(h) \leq 2n - 8$. Recall that the inequality (4.8) in [22] is

$$\nu \leq \sigma + \lambda + 1,$$

where $\nu$, $\sigma$ and $\lambda$ correspond here to $\#\{\chi_1(h) = 0, h \in (-1, 0)\}$, $\#\{F_0(h) = 0, h \in (-1, 0)\}$ and $\#\{\beta(h) = 0, h \in (-1, 0)\}$, respectively. Hence, we have for $h \in (-1, 0)$

$$\#\{\chi_1(h) = 0\} \leq \#\{\beta(h) = 0\} + \#\{F_0(h) = 0\} + 1 \leq 3n - 11.$$

Hence,

$$\#\{\Phi(h) = 0, h \in (-1, 0)\} = \#\{\chi_1(h) = 0, h \in (-1, 0)\} \leq 3n - 11.$$ 

This completes the proof.

Proof of the Theorem 1.1. We only prove (iii). (i) and (ii) can be proved similarly.

Let $M_1(h) = h^{n-3}M(h)$, then $M_1(h)$ has the same zeros as $M(h)$ on $(-1, 0)$. For the sake of clearness, we split the proof into three steps.

(1) For $h \in (-1, 0)$, $L^3(h)M_1(h) = R(h)$, where $L^3(h)$ is defined by (3.1),

$$R(h) = \Theta_{2n-4}(h) \ln \frac{1 - \sqrt{h + 1}}{1 + \sqrt{h + 1}} + \Theta_{3n-9}(h) \frac{1}{h(h + 1)^{\frac{3}{2}}}.$$  (3.6)
In fact, from Lemma 2.4, we have

\[ \Psi(h) := \gamma(h)I_{2,0}(h) + \delta(h)I_{0,2}(h) = c_1[\gamma(h) + 2\delta(h)]\sqrt{h + 1} - c_1h\delta(h)\ln \frac{1 - \sqrt{h + 1}}{1 + \sqrt{h + 1}} \]

\[ := \Theta_{n-4}(h)\sqrt{h + 1} + h\Theta_3(h)\ln \frac{1 - \sqrt{h + 1}}{1 + \sqrt{h + 1}}, \tag{3.7} \]

\[ \Psi'(h) = \Theta_{n-4}(h) \frac{1}{\sqrt{h + 1}} + \Theta_3(h)\ln \frac{1 - \sqrt{h + 1}}{1 + \sqrt{h + 1}} \]

\[ \Psi''(h) = \Theta_{n-3}(h) \frac{1}{h(h + 1)^{\frac{3}{2}}} + \Theta_2(h)\ln \frac{1 - \sqrt{h + 1}}{1 + \sqrt{h + 1}}. \]

From Lemma 3.1 (iii), we have

\[ L^3(h)M_1(h) = L^3(h)\Psi(h) = P_2^3(h)\Psi''(h) + P_1^3(h)\Psi'(h) + P_0^3(h)\Psi(h). \tag{3.8} \]

Substituting (3.7) into (3.8) gives (3.6).

(2) Zeros of \( R(h) \) for \( h \in (-1, 0) \).

Denote that \( U = \{ h \in (-1, 0) | \Theta_{2n-4}(h) = 0 \} \). For \( h \in (-1, 0) \setminus U \), by detailed computations, we get

\[ \left( \frac{R(h)}{\Theta_{2n-4}(h)} \right)' = \frac{\Theta_{5n-12}(h)}{\Theta_{2n-4}(h)h^2(h + 1)^{\frac{3}{2}}}. \tag{3.9} \]

Since \( h^2(h + 1)^{\frac{3}{2}} \neq 0 \) for \( h \in (-1, 0) \), we have

\[ \#\{ R(h) = 0, h \in (-1, 0) \} \leq 7n - 15. \tag{3.10} \]

(3) Zeros of \( M(h) \) for \( h \in (-1, 0) \).

By Lemma 3.2, we have \( \Phi(h) \) has at most \( 3n - 11 \) zeros on \((-1, 0)\). We assume that

\[ P_2^3(\bar{h}_i) = 0, \Phi(\bar{h}_j) = 0, \bar{h}_i, \bar{h}_j \in (-1, 0), 1 \leq i \leq 2n - 6, 1 \leq j \leq 3n - 11. \]

Denote \( \bar{h}_i \) and \( \bar{h}_j \) as \( h_{m}^* \), and reorder them such that \( h_{m}^* < h_{m+1}^* \) for \( m = 1, 2, \cdots, 5n-17 \). Let

\[ \Delta_s = (h_s^*, h_{s+1}^*), s = 0, 1, \cdots, 5n - 17, \]

where \( h_0^* = -1, h_{5n-16}^* = 0 \). Then \( P_2^3(h) \neq 0 \) and \( \Phi(h) \neq 0 \) for \( h \in \Delta_s \) and \( L^3(h)\Phi(h) = 0 \). By Proposition 3.1, the solution space of

\[ L^3(h) = P_2^3(h)\left( \frac{d^2}{dh^2} + \frac{P_1^3(h)}{P_2(h)} \frac{d}{dh} + \frac{P_0^3(h)}{P_2(h)} \right) = 0 \]

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is a Chebyshev space on $\Delta_s$. By Proposition 3.2, $M_1(h)$ has at most $2 + l_s$ zeros for $h \in \Delta_s$, where $l_s$ is the number of zeros of $R(h)$ on $\Delta_s$. Therefore, we obtain for $h \in (-1, 0)$

$$\#\{M(h) = 0\} = \#\{M_1(h) = 0\}$$

$$\leq \#\{R(h) = 0\} + 2 \cdot \text{the number of the intervals of } \Delta_s$$

$$+ \text{the number of the end points of } \Delta_s$$

$$\leq 22n - 64.$$ 

**Proof of the Theorem 1.2.** If $a^+_i = a^-_i$ and $b^+_i = b^-_i$, that is, the systems (1.4) is smooth. Since $\Gamma_h$ is symmetric with respect to $x$-axis for $h \in (-1, 0)$, $A_{i,2l}(h) = \oint_{\Gamma_h} x^{i-4} y^{2l} dx = 0$, $l = 0, 1, 2, \cdots$, where

$$\Gamma_h = \Gamma^+_h \cup \Gamma^-_h, \quad A_{i,j}(h) = I_{i,j}(h) + J_{i,j}(h).$$

Hence, from Lemma 2.2 we have

$$M(h) = \begin{cases}
\frac{1}{n-3} [\hat{\alpha}(h) A_{0,1}(h) + \hat{\beta}(h) A_{1,1}(h)], & n = 2, 3, \\
\frac{1}{n-3} [\alpha(h) A_{0,1}(h) + \beta(h) A_{1,1}(h)], & n \geq 4,
\end{cases}$$

where $\hat{\alpha}(h)$ is a constant, and $\hat{\beta}(h)$, $\alpha(h)$ and $\beta(h)$ are polynomials of $h$ with $\deg \hat{\beta}(h) \leq 1$, $\deg \alpha(h) \leq n - 4$ and $\deg \beta(h) \leq n - 3$. By the same proof of Lemma 3.2, we have

$$\#\{M(h) = 0, h \in (-1, 0)\} \leq \begin{cases}
4, & n = 2, 3, \\
3n - 8, & n \geq 4.
\end{cases}$$

**Acknowledgment**

Supported by Higher Educational Science Program of Ningxia(NGY201789), National Natural Science Foundation of China(11701306,11671040,11601250), Construction of First-class Disciplines of Higher Education of Ningxia(pedagogy)(NXYLXK2017B11), Key Program of Ningxia Normal University(NXSFZD1708) and Science and Technology Pillar Program of Ningxia(KJ[2015]26(4)).

**References**

[1] M. di Bernardo, C. Budd, A. Champneys, P. Kowalczyk, Piecewise-smooth dynamical systems, theory and applications, Springer-Verlag, London, 2008.
[2] X. Cen, S. Li, Y. Zhao, On the number of limit cycles for a class of discontinuous quadratic differential systems, J. Math. Anal. Appl. 449 (2017) 314–342.

[3] G. Dong, C. Liu, Note on limit cycles for $m$-piecewise discontinuous polynomial Liénard differential equations, Z. Angew. Math. Phys. 68 (2017) 97.

[4] L. Gavrilov, I. Iliev, Quadratic perturbations of quadratic codimension-four centers, J. Math. Anal. Appl. 357 (2009) 69–76.

[5] S. Hogan, Heteroclinic bifurcations in damped rigid block motion, Proc. R. Soc. Lond. A 439 (1992) 155–162.

[6] E. Horozov, I. Iliev, Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians, Nonlinearity 11 (1998) 1521–1537.

[7] M. Han, L. Sheng, Bifurcation of limit cycles in piecewise smooth systems via Melnikov function, J. Appl. Anal. Comput. 5 (2015) 809–815.

[8] V. Krivan, On the Gause predator-prey model with a refuge: a fresh look at the history, J. Theoret. Biol. 274 (2011) 67–73.

[9] I. Iliev, Perturbations of quadratic centers, Bull. Sci. Math. 122 (1998) 107–161.

[10] S. Li, C. Liu, A linear estimate of the number of limit cycles for some planar piecewise smooth quadratic differential system, J. Math. Anal. Appl. 428 (2015) 1354–1367.

[11] F. Liang, M. Han, V. Romanovski, Bifurcation of limit cycles by perturbing a piecewise linear Hamiltonian system with a homoclinic loop, Nonlinear Anal. 75 (2012) 4355–4374.

[12] X. Liu, M. Han, Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems, Internat. J. Bifur. Chaos Appl. Sci. Engrg, 20 (2010) 1379–1390.

[13] J. Llibre, A. Mereu, Limit cycles for discontinuous quadratic differential systems, J. Math. Anal. Appl. 413 (2014) 763–775.

[14] J. Llibre, A. Mereu, D. Novaes, Averaging theory for discontinuous piecewise differential systems, J. Differential Equations 258 (2015) 4007–4032.

[15] J. Llibre, M. Teixeira, Limit cycles for $m$-piecewise discontinuous polynomial Liénard differential equations, Z. Angew. Math. Phys. 66 (2015) 51-66.

[16] Z. Peng, Z. Lang, S. Billings, Y. Lu, Analysis of bilinear oscillators under harmonic loading using nonlinear output frequency response functions, International Journal of Mechanical Sciences, 49 (2007) 1213-1225.
[17] J. Shen, Z. Du, Heteroclinic bifurcation in a class of planar piecewise smooth systems with multiple zones, Z. Angew. Math. Phys. 67 (2016) 42.

[18] M. Teixeira, Perturbation theory for non-smooth systems, in: encyclopedia of complexity and systems science, Springer, New York, 2009.

[19] Y. Xiong, Limit cycle bifurcations by perturbing a piecewise Hamiltonian system with a double homoclinic loop, International J. Bifurcation and Chaos 26 (2016) 1650103(16pages).

[20] J. Yang, L. Zhao, Limit cycle bifurcations for piecewise smooth Hamiltonian systems with a generalized eye-figure loop, International J. Bifurcation and Chaos 26 (2016) 1650204(14pages).

[21] Y. Zhao, W. Li, C. Li, Z. Zhang, Linear estimate of the number of zeros of Abelian integrals for quadratic centers having almost all their orbits formed by cubics, Sci. China Ser. A: Math. 15 (8) (2002) 964–974.

[22] Y. Zhao, Z. Zhang, Linear estimate of the number of zeros of Abelian integrals for a kind of quartic Hamiltonians, J. Differential Equations 155 (1999) 73–88.