Nef Divisors on Moduli Spaces of Abelian Varieties

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Dedicated to the memory of Michael Schneider

0 Introduction

Let $A_g$ be the moduli space of principally polarized abelian varieties of dimension $g$. Over the complex numbers $A_g = \mathbb{H}_g/\Gamma_g$ where $\mathbb{H}_g$ is the Siegel space of genus $g$ and $\Gamma_g = \text{Sp}(2g, \mathbb{Z})$. We denote the toroidal compactification given by the second Voronoi decomposition by $A_g^*$ and call it the Voronoi compactification. It was shown by Alexeev and Nakamura \cite{A} that $A_g^*$ coarsely represents the stack of principally polarized stable quasiabelian varieties. The variety $A_g^*$ is projective \cite{A} and it is known that the Picard group of $A_g^*$, $g \geq 2$ is generated (modulo torsion) by two elements $L$ and $D$, where $L$ denotes the ($\mathbb{Q}$-)line bundle given by modular forms of weight 1 and $D$ is the boundary (see \cite{Mu2}, \cite{Fa} and \cite{Mu1} for $g = 2, 3$ and $\geq 4$). In this paper we want to discuss the following

Theorem 0.1 Let $g = 2$ or 3. A divisor $aL - bD$ on $A_g^*$ is nef if and only if $b \geq 0$ and $a - 12b \geq 0$.

The varieties $A_g$ have finite quotient singularities. Adding a level-$n$ structure one obtains spaces $A_g(n) = \mathbb{H}_g/\Gamma_g(n)$ where $\Gamma_g(n)$ is the principal congruence subgroup of level $n$. For $n \geq 3$ these spaces are smooth. However, the Voronoi compactification $A_g^*(n)$ acquires singularities on the boundary for $g \geq 5$ due to bad behaviour of the second Voronoi decomposition. There is a natural quotient map $A_g^*(n) \to A_g^*$. Note that this map is branched of order $n$ along the boundary. Hence Theorem 0.1 is equivalent to

Theorem 0.2 Let $g = 2$ or 3. A divisor $aL - bD$ on $A_g^*(n)$ is nef if and only if $b \geq 0$ and $a - 12b/n \geq 0$.

This theorem easily gives the following two corollaries.
Corollary 0.3 If $g = 2$ then $K$ is nef but not ample for $\mathcal{A}_2^*(4)$ and $K$ is ample for $\mathcal{A}_2^*(n)$, $n \geq 5$; in particular $\mathcal{A}_2^*(n)$ is a minimal model for $n \geq 4$ and a canonical model for $n \geq 5$.

This was first proved by Borisov [3d].

Corollary 0.4 If $g = 3$ then $K$ is nef but not ample for $\mathcal{A}_3^*(3)$ and $K$ is ample for $\mathcal{A}_3^*(n)$, $n \geq 4$; in particular $\mathcal{A}_3^*(n)$ is a minimal model for $n \geq 3$ and a canonical model for $n \geq 4$.

In this paper we shall give two proofs of Theorem (0.1). The first and quick one reduces the problem via the Torelli map to the analogous question for $\overline{M}_2$, resp. $\overline{M}_3$. Since the Torelli map is not surjective for $g \geq 4$ this proof cannot possibly be generalized to higher genus. This is the main reason why we want to give a second proof which uses theta functions. This proof makes essential use of a result of Weissauer [We]. The method has the advantage that it extends in principle to other polarizations as well as to higher $g$. We will also give some partial results supporting the

Conjecture For any $g \geq 2$ the nef cone on $\mathcal{A}_g^*$ is given by the divisors $aL - bD$ where $b \geq 0$ and $a - 12b \geq 0$.

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1 Curves meeting the interior

We start by recalling some results about the Kodaira dimension of $\mathcal{A}_g^*(n)$. It was proved by Freitag, Tai and Mumford that $\mathcal{A}_g^*$ is of general type for $g \geq 7$. The following more general result is probably well known to some specialists.

Theorem 1.1 $\mathcal{A}_g^*(n)$ is of general type for the following values of $g$ and $n \geq n_0$:

\[
\begin{array}{cccccccc}
g & n_0 & 2 & 3 & 4 & 5 & 6 & \geq 7 \\
\hline
2 & 4 & 3 & 2 & 2 & 1 & & \\
\end{array}
\]
Proof. One can use Mumford’s method from [Mu1]. First recall that away from the singularities and the closure of the branch locus of the map \( H_g \to A_g(n) \) the canonical bundle equals

\[
K \equiv (g + 1)L - D. \tag{1}
\]

This equality holds in particular also on an open part of the boundary. If \( g \leq 4 \) and \( n \geq 3 \) the spaces \( A_g(n) \) are smooth and hence (1) holds everywhere. If \( g \geq 5 \) then Tai [T] showed that there is a suitable toroidal compactification \( \tilde{A}_g(n) \) such that all singularities are canonical quotient singularities. By Mumford’s results from [Mu1] one can use the theta-null locus to eliminate \( D \) from formula (1) and obtains

\[
K \equiv \left((g + 1) - \frac{2g^2 - 2(2g + 1)}{n2^{2g-5}} \right)L + \frac{1}{n2^{2g-5}}[\Theta_{null}]. \tag{2}
\]

We then have general type if all singularities are canonical and if the factor in front of \( L \) is positive. This gives immediately all values in the above table with the exception of \((g, n) = (4, 2) \) and \((7, 1)\). In the latter case the factor in front of \( L \) is negative. The proof that \( A_7 \) is nevertheless of general type is the main result of [Mu1]. The difficulty in the first case is that one can possibly have non-canonical singularities. One can, however, use the following argument which I have learnt from Salvati Manni: An immediate calculation shows that for every element \( \sigma \in \Gamma_g(2) \) the square \( \sigma^2 \in \Gamma_g(4) \). Hence if \( \sigma \) has a fixed point then \( \sigma^2 = 1 \) since \( \Gamma_g(4) \) acts freely. But for elements of order 2 one can again use Tai’s extension theorem (see [T, Remark after Lemma 4.5] and [T, Remark after Lemma 5.2]).

Remark 1.2 The Kodaira dimension of \( A_6 \) is still unknown. All other varieties \( A_g(n) \) which do not appear in the above list are either rational or unirational: Unirationality of \( A_g \) for \( g = 5 \) was proved by Donagi [D] and by Mori and Mukai [MM]. For \( g = 4 \) the same result was shown by Clemens [C]. Unirationality is easy for \( g \leq 3 \). Igusa [I2] showed that \( A_2 \) is rational. Recently Katsylo [Ka] proved rationality of \( M_3 \) and hence also of \( A_3 \). The space \( A_3(2) \) is rational by work of van Geemen [vG] and Dolgachev and Ortlang [DO]. \( A_2(3) \) is the Burkhardt quartic and hence rational. This was first proved by Todd (1936) and Baker (1942). See also the thesis of Finkelnberg [F1]. The variety \( A_2(2) \) has the Segre cubic as a projective model [dG1] and is hence also rational. Yamazaki [Ya] first showed general type for \( A_2(n), n \geq 4 \).
We denote the Satake compactification of $\mathcal{A}_g$ by $\overline{\mathcal{A}}_g$. There is a natural map $\pi : \mathcal{A}_g^* \to \overline{\mathcal{A}}_g$ which is an isomorphism on $\mathcal{A}_g$. The line bundle $L$ is the pullback of an ample line bundle on $\overline{\mathcal{A}}_g$ which, by abuse of notation, we again denote by $L$. In fact the Satake compactification is defined as the closure of the image of $\mathcal{A}_g$ under the embedding given by a suitable power of $L$ on $\mathcal{A}_g$. In particular we notice that $L.C \geq 0$ for every curve $C$ on $\mathcal{A}_g^*$ and that $L.C > 0$ if $C$ is not contracted to a point under the map $\pi$.

Let $F$ be a modular form with respect to the full modular group $\text{Sp}(2g, \mathbb{Z})$. Then the order $o(F)$ of $F$ is defined as the quotient of the vanishing order of $F$ divided by the weight of $F$.

**Theorem 1.3 (Weissauer)** For every point $\tau \in \mathcal{H}_g$ and every $\varepsilon > 0$ there exists a modular form $F$ of order $o(F) \geq 1/12 + \varepsilon$ which does not vanish at $\tau$.

**Proof.** See [We].

**Proposition 1.4** Let $C \subset \mathcal{A}_g^*$ be a curve which is not contained in the boundary. Then $(aL - bD).C \geq 0$ if $b \geq 0$ and $a - 12b \geq 0$.

**Proof.** First note that $L.C > 0$ since $\pi(C)$ is a curve in the Satake compactification. It is enough to prove that $(aL - bD).C > 0$ if $a - 12b > 0$ and $a, b \geq 0$. This is clear for $b = 0$ and hence we can assume that $b \neq 0$. We can now choose some $\varepsilon > 0$ with $a/b > 12 + \varepsilon$. By Weissauer’s theorem there exists a modular form $F$ of say weight $k$ and vanishing order $m$ with $F(\tau) \neq 0$ for some point $[\tau] \in C$ and $m/k \geq 1/(12 + \varepsilon)$. In terms of divisors this gives us that

$$kL = mD + D_F, \quad C \not\subset D_F$$

where $D_F$ is the zero-divisor of $F$. Hence

$$\left(\frac{k}{m}L - D\right) = \frac{1}{m}D_F.C \geq 0.$$ 

Since $a/b > 12 + \varepsilon \geq k/m$ and $L.C > 0$ we can now conclude that

$$\left(\frac{a}{b}L - D\right).C > \left(\frac{k}{m}L - D\right).C \geq 0.$$

$\square$
Remark 1.5 Weissauer’s result is optimal, since the modular forms of order $> 1/12$ have a common base locus. To see this consider curves $C$ in $A_1^* \times \{A\}$ where $X(1)$ is the modular curve of level 1 parametrizing elliptic curves and $A$ is a fixed abelian variety of dimension $g-1$. The degree of $L$ on $X(1)$ is $1/12$ (recall that $L$ is a $\mathbb{Q}$-bundle) whereas it has one cusp, i.e. the degree of $D$ on this curve is 1. Hence every modular form of order $> 1/12$ will vanish on $C$. This also shows that the condition $a - 12b \geq 0$ is necessary for a divisor to be nef.

2 Geometry of the boundary (I)

We first have to collect some properties of the structure of the boundary of $A_g^*(n)$. Recall that the Satake compactification is set-theoretically the union of $A_g^*(n)$ and of moduli spaces $A_k^*(n)$, $k < g$ of lower dimension, i.e.

$$\overline{A}_g(n) = A_g(n) \amalg \left( \amalg_{i_1} A_{g-1}^{i_1}(n) \amalg \left( \amalg_{i_2} A_{g-2}^{i_2}(n) \amalg \cdots \amalg \left( \amalg_{i_g} A_g^{i_g}(n) \right) \right) \right).$$

Via the map $\pi : A_g^*(n) \to \overline{A}_g(n)$ this also defines a stratification of $A_g^*(n)$:

$$A_g^*(n) = A_g(n) \amalg \left( \amalg_{i_1} D_{g-1}^{i_1}(n) \amalg \left( \amalg_{i_2} D_{g-2}^{i_2}(n) \amalg \cdots \amalg \left( \amalg_{i_g} D_0^{i_g}(n) \right) \right) \right).$$

The irreducible components of the boundary $D$ are the closures $\overline{D}_{g-1}^{i_1}(n)$ of the codimension 1 strata $D_{g-1}^{i_1}(n)$. Whenever we talk about a boundary component we mean one of the divisors $\overline{D}_{g-1}^{i_1}(n)$. Then the boundary $D$ is given by

$$D = \sum_{i_1} \overline{D}_{g-1}^{i_1}(n).$$

The fibration $\pi : D_{g-1}^{i_1}(n) \to A_{g-1}^{i_1}(n) = A_{g-1}(n)$ is the universal family of abelian varieties of dimension $g-1$ with a level-$n$ structure if $n \geq 3$ resp. the universal family of Kummer surfaces for $n = 1$ or 2 (see [Mu]). We shall also explain this in more detail later on. To be more precise we associate to a point $\tau \in \mathbb{H}_g$ the lattice $L_{\tau,1} = (\tau, 1)\mathbb{Z}/\mathbb{Z}$, resp. the principally polarized abelian variety $A_{\tau,1} = \mathbb{C}^g/L_{\tau,1}$. Given an integer $n \geq 1$ we set $L_{n\tau,n} = (n\tau, n1_g)\mathbb{Z}/\mathbb{Z}$, resp. $A_{n\tau,n} = \mathbb{C}^g/L_{n\tau,n}$. By $K_{n\tau,n}$ we denote the Kummer variety $A_{n,\tau,n}/\{\pm 1\}$.

Lemma 2.1 Let $n \geq 3$. Then for any point $[\tau] \in A_{g-1}^{i_1}(n)$ the fibre of $\pi$ equals $\pi^{-1}([\tau]) = A_{n,\tau,n}$.  

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Proof. Compare [Mu1]. We shall also give an independent proof below. □

This result remains true for \( n = 1 \) or 2, at least for points \( \tau \) whose stabilizer subgroup in \( \Gamma_g(n) \) is \( \{ \pm 1 \} \), if we replace \( A_{n,\tau n} \) by its associate Kummer variety \( K_{n,\tau n} \).

**Lemma 2.2** Let \( n \geq 3 \). Then for \( [\tau] \in A_g^{g-1}(n) \) the restriction of \( D_{g-1}^{g-1}(n) \) to the fibre \( \pi^{-1}([\tau]) \) is negative. More precisely

\[
D_{g-1}^{g-1}(n)|_{\pi^{-1}([\tau])} = -\frac{2}{n}H
\]

where \( H \) is the polarization on \( A_{n\tau,n} \) given by the pull-back of the principal polarization on \( A_{\tau,1} \) via the covering \( A_{n,\tau n} \to A_{\tau,1} \).

Proof. Compare [Mu1, Proposition 1.8], resp. see the discussion below. □

Again the statement remains true for \( n = 1 \) or 2 if we replace the abelian variety by its Kummer variety.

**First proof of Theorem (0.1).** We have already seen (see Remark 1.5) that for every nef divisor \( aL - bD \) the inequality \( a - 12b \geq 0 \) holds. If \( C \) is a curve in a fibre of the map \( A_g^g(n) \to \overline{A}_g(n) \), then \( L.C = 0 \). Lemma (2.2) immediately implies that \( b \geq 0 \) for any nef divisor. It remains to show that the conditions of Theorem (0.1) are sufficient to imply nefness. For any genus the Torelli map \( t : M_g \to A_g \) extends to a morphism \( \overline{t} : \overline{M}_g \to \overline{A}_g \) (see [Nam]). Here \( \overline{M}_g \) denotes the compactification of \( M_g \) by stable curves. For \( g = 2 \) and 3 the map \( \overline{t} \) is surjective. It follows that for every curve \( C \) in \( A^*_g \) there is a curve \( C' \) in \( \overline{M}_g \) which is finite over \( C \). Hence a divisor on \( A^*_g \), \( g = 2 \) or 3 is nef if and only if this holds for its pull-back to \( \overline{M}_g \). In the notation of Faber’s paper [Fa] \( \overline{t}^*L = \lambda \) where \( \lambda \) is the Hodge bundle and \( \overline{t}'D = \delta_0 \) where \( \delta_0 \) is the boundary \( (g = 2) \), resp. the closure of the locus of genus 2 curves with one node \( (g = 3) \) (cf also [vdG2]). The result follows since \( a\lambda - b\delta_0 \) is nef on \( \overline{M}_g \), \( g = 2 \) or 3 for \( a - 12b \geq 0 \) and \( b \geq 0 \) (see [Fa]). □

As we have already pointed out the Torelli map is not surjective for \( g \geq 4 \) and hence this proof cannot possibly be generalized to higher genus. The main purpose of this paper is, therefore, to give a proof of Theorem (0.1) which does not use the reduction to the curve case. This will also allow us to prove some results for general \( g \). At the same time we obtain an independent proof of nefness of \( a\lambda - b\delta_0 \) for \( a - 12b \geq 0 \) and \( b \geq 0 \) on \( \overline{M}_g \) for \( g = 2 \) and 3.
We now want to investigate the open parts $D^i_{g-1}(n)$ of the boundary components $\overline{D}^i_{g-1}(n)$ and their fibration over $A_{g-1}(n)$ more closely. At the same time this gives us another argument for Lemmas (2.1) and (2.2). At this stage we have to make first use of the toroidal construction. Recall that the boundary components $D^i_{g-1}(n)$ are in 1 : 1 correspondence with the maximal dimensional cusps, and these in turn are in 1 : 1 correspondence with the lines $l \subset \mathbb{Q}^g$ modulo $\Gamma_g(n)$. Since all cusps are equivalent under the action of $\Gamma_g/\Gamma_g(n)$ we can restrict our attention to one of these cusps, namely the one given by $l_0 = (0, \ldots, 0, 1)$. This corresponds to $\tau_{gg} \to i\infty$.

To simplify notation we shall denote the corresponding boundary stratum simply by $D^i_{g-1}(n) = \overline{D}^i_{g-1}(n)$. The stabilizer $P(l_0)$ of $l_0$ in $\Gamma_g$ is generated by elements of the following form (cf. [HKW, Proposition I.3.87]):

\[
g_1 = \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g-1},
\]

\[
g_2 = \begin{pmatrix} 1_{g-1} & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 1_{g-1} & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix},
\]

\[
g_3 = \begin{pmatrix} 1_{g-1} & 0 & 0 & tN \\ M & 1 & N & 0 \\ 0 & 0 & 1_{g-1} & -tM \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M, N \in \mathbb{Z}^{g-1},
\]

\[
g_4 = \begin{pmatrix} 1_{g-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & S \\ 0 & 0 & 1_{g-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S \in \mathbb{Z}.
\]

We write $\tau = (\tau_{ij})_{1 \leq i, j \leq g}$ in the form

\[
\begin{pmatrix}
\tau_{11} & \cdots & \tau_{1,g-1} & \tau_{1g} \\
\vdots & & \vdots & \\
\tau_{1,g-1} & \cdots & \tau_{g-1,g-1} & \tau_{g-1,g} \\
\tau_{1g} & \cdots & \tau_{g-1,g} & \tau_{gg}
\end{pmatrix} = \begin{pmatrix} \frac{\tau_1}{\tau_2} \\ \frac{\tau_2}{\tau_3} \end{pmatrix}.
\]
Then the action of $P(l_0)$ on $\mathbb{H}_g$ is given by (cf. [IKW, I.3.91]):

\[
\begin{align*}
g_1(\tau) &= \left( (A\tau_1 + B)(C\tau_1 + D)^{-1} \tau_2(C\tau_1 + D)^{-1} \tau_3 - \tau_2(C\tau_1 + D)^{-1}C^t\tau_2 \right)^*, \\
g_2(\tau) &= \left( \tau_1^{\pm} \tau_2 \tau_3 \right)^*, \\
g_3(\tau) &= \left( \tau_2 + M\tau_1 + N \tau_3\right)^* \\
g_4(\tau) &= \left( \tau_1 \tau_2 \tau_3 + S \right)
\end{align*}
\]

where $\tau'_3 = \tau_3 + M\tau_1^tM + M^t\tau_2 + (M^t\tau_2) + N^tM$,

\[
g_4(\tau) = \left( \tau_1 \tau_2 \tau_3 + S \right)
\]

The parabolic subgroup $P(l_0)$ is an extension

\[
1 \longrightarrow P'(l_0) \longrightarrow P(l_0) \longrightarrow P''(l_0) \longrightarrow 1
\]

where $P'(l_0)$ is the rank 1 lattice generated by $g_4$. To obtain the same result for $\Gamma_g(n)$ we just have to intersect $P(l_0)$ with $\Gamma_g(n)$. Note that $g_2$ is in $\Gamma_g(n)$ only for $n = 1$ or 2. The first step in the construction of the toroidal compactification of $A^g_0(n)$ is to divide $\mathbb{H}_g$ by $P'(l_0) \cap \Gamma(n)$ which gives a map

\[
\begin{align*}
\mathbb{H}_g &\longrightarrow \mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}^*
\end{align*}
\]

Partial compactification in the direction of $l_0$ then consists of adding the set $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}$. It now follows immediately from the above formulae for the action of $P(l_0)$ on $\mathbb{H}_g$ that the action of the quotient group $P''(l_0)$ on $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \mathbb{C}^*$ extends to $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \times \{0\}$. Then $D_{g-1}(n) = (\mathbb{H}_{g-1} \times \mathbb{C}^{g-1})/P''(l_0)$ and the map to $A_{g-1}(n)$ is induced by the projection from $\mathbb{H}_{g-1} \times \mathbb{C}^{g-1}$ to $\mathbb{H}_{g-1}$. This also shows that $D_{g-1}(n) \to A_{g-1}(n)$ is the universal family for $n \geq 3$ and that the general fibre is a Kummer variety for $n = 1$ and 2.

Whenever $n_1 | n_2$ we have a Galois covering

\[
\pi(n_1, n_2) : A^g_0(n_2) \longrightarrow A^g_0(n_1)
\]

whose Galois group is $\Gamma_g(n_1) / \Gamma_g(n_2)$. This induces coverings $\overline{D}_{g-1}(n_2) \to \overline{D}_{g-1}(n_1)$, resp. $D_{g-1}(n_2) \to D_{g-1}(n_1)$. In order to avoid technical difficulties we assume for the moment that $A^g_0(n)$ is smooth (this is the case if
\[ g \leq 4 \text{ and } n \geq 3 \]. In what follows we will always be able to assume that we are in this situation. Then we denote the normal bundle of \( \mathcal{D}_{g-1}(n) \) in \( \mathcal{A}_g(n) \) by \( N_{\mathcal{D}_{g-1}(n)} \), resp. its restriction to \( D_{g-1}(n) \) by \( N_{D_{g-1}(n)} \). Since the covering map \( \pi(n_1, n_2) \) is branched of order \( n_2/n_1 \) along the boundary, it follows that
\[
\pi^*(n_1, n_2)n_1N_{\mathcal{D}_{g-1}(n_1)} = n_2N_{\mathcal{D}_{g-1}(n_2)}.
\]
We now define the bundle
\[
\mathcal{M}(n) := -nN_{\mathcal{D}_{g-1}(n)} + L.
\]
This is a line bundle on the boundary component \( \mathcal{D}_{g-1}(n) \). We denote the restriction of \( \mathcal{M}(n) \) to \( D_{g-1}(n) \) by \( \mathcal{M}(n) \). We find immediately that
\[
\pi^*(n_1, n_2)\mathcal{M}(n_1) = \mathcal{M}(n_2).
\]
The advantage of working with the bundle \( \mathcal{M}(n) \) is that we can explicitly describe sections of this bundle. For this purpose it is useful to review some basic facts about theta functions. For every element \( m = (m', m'') \) of \( \mathbb{R}^{2g} \) one can define the theta-function
\[
\Theta_{m', m''}(\tau, z) = \sum_{q \in \mathbb{Z}^g} e^{2\pi i [(q+m')^t(q+m')/2+(q+m')^t(z+m'')]}.
\]
The transformation behaviour of \( \Theta_{m', m''}(\tau, z) \) with respect to \( z \mapsto z + u\tau + u' \) is described by the formulae (\( \Theta_1 \))–(\( \Theta_5 \)) of [1, pp. 49, 50]. The behaviour of \( \Theta_{m', m''}(\tau, z) \) with respect to the action of \( \Gamma_g(1) \) on \( \mathbb{H}_g \times \mathbb{C}^g \) is given by the theta transformation formula [1, Theorem II.5.6] resp. the corollary following this theorem [1, p. 85].

**Proposition 2.3** Let \( n \equiv 0 \mod 4p^2 \). If \( m', m'', \overline{m'}, \overline{m''} \in \frac{1}{2p}\mathbb{Z}^{g-1} \), then the functions \( \Theta_{m', m''}(\tau, z)\Theta_{\overline{m'}, \overline{m''}}(\tau, z) \) define sections of the line bundle \( \mathcal{M}(n) \) on \( D_{g-1}(n) \).

**Proof.** It follows from (\( \Theta_3 \)) and (\( \Theta_1 \)) that for \( k, k' \in n\mathbb{Z}^{g-1} \) the following holds:
\[
\Theta_{m', m''}(\tau, z + k\tau + k') = e^{2\pi i [\frac{1}{2}k\tau^t k - k'(z + k)']}\Theta_{m', m''}(\tau, z).
\]
Similarly, of course, for \( \Theta_{\overline{m'}, \overline{m''}}(\tau, z) \). Moreover the theta transformation formula together with formula (\( \Theta_2 \)) gives
\[
\Theta_{m', m''}^\#(\tau^#, z^#) = e^{2\pi i [\frac{1}{2}z(C\tau + D)^{-1}C^t z]} \det(C\tau + D)^{1/2} u\Theta_{m', m''}(\tau, z)
\]
for every element \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n) \) and

\[
\tau# = \gamma(\tau), \quad z# = z(C\tau + D)^{-1}.
\]

Here \( u^2 \) is a character of \( \Gamma_g^{-1} \) with \( u^2|_{\Gamma_g} = 1 \).

On the other hand the boundary component \( D_g^{-1}(n) \) is defined by \( t_3 = 0 \) with \( t_3 = e^{2\pi i r_3/n} \). We have already described the action of \( P^n(l_0) \) on \( \mathbb{H}_{g-1} \times \mathbb{C}^{g-1} \). The result then follows by comparing the transformation behaviour of \((t_3/t_3^2)^n\) with respect to \( g_1 \) and \( g_3 \) with the above formulae together with the fact that the line bundle \( L \) is defined by the automorphy factor \( \det(C\tau + D) \).

This also gives an independent proof of Lemma (2.2).

3 Geometry of the boundary (II)

So far we have described the stratum \( D_g^{-1}(n) \) of the boundary component \( \overline{D_g^{-1}}(n) \) and we have seen that there is a natural map \( D_g^{-1}(n) \to A_g^{-1}(n) \) which identifies \( D_g^{-1}(n) \) with the universal family over \( A_g^{-1}(n) \) if \( n \geq 3 \).

We now want to describe the closure \( \overline{D_g^{-1}}(n) \) in some detail. In order to do this we have to restrict ourselves to \( g = 2 \) and \( 3 \). First assume \( g = 2 \). Then the projection \( D_1(n) \to A_1(n) = X^0(n) \) extends to a projection \( \overline{D}_1(n) \to X(n) \) onto the modular curve of level \( n \) and in this way \( \overline{D}_1(n) \) is identified with Shioda’s modular surface \( S(n) \to X(n) \). The fibres are either elliptic curves or \( n \)-gons of rational curves (if \( n \geq 3 \)). Similarly the fibration \( D_2(n) \to A_2(n) \) extends to a fibration \( \overline{D}_2(n) \to A_2^* \) whose fibres over the boundary of \( A_2^*(n) \) are degenerate abelian surfaces. This was first observed by Nakamura \[Nak\] and was described in detail by Tsushima \[Ts \] whose paper is essential for what follows.

We shall now explain the toroidal construction which allows us to describe the fibration \( \overline{D}_2(n) \to A_2^*(n) \) explicitly. Here we shall concentrate on a description of this map in the most difficult situation, namely in the neighbourhood of a cusp of maximal corank.

The toroidal compactification \( A^*_g(n) \) is given by the second Voronoi decomposition \( \Sigma_g \). This is a rational polyhedral decomposition of the convex hull in \( \text{Sym}_{g}^{\geq 0}(\mathbb{R}) \) of the set \( \text{Sym}_{g}^{\geq 0}(\mathbb{Z}) \) of integer semi-positive \((g \times g)\)-matrices. For \( g = 2 \) and \( 3 \) it can be described as follows. First note that \( \text{Gl}(g, \mathbb{Z}) \) acts on \( \text{Sym}_{g}^{\geq 0}(\mathbb{R}) \) by \( \gamma \mapsto \begin{pmatrix} I_M \gamma M \end{pmatrix} \). For \( g = 2 \) we define the standard cone

\[
\sigma_2 = \mathbb{R}_{\geq 0} \gamma_1 + \mathbb{R}_{\geq 0} \gamma_2 + \mathbb{R}_{\geq 0} \gamma_3
\]
with 
\[ \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \]

Then 
\[ \Sigma_2 = \{ M(\sigma_2); \; M \in \text{Gl}(2, \mathbb{Z}) \}. \]

Similarly for \( g = 3 \) we consider the standard cone 
\[ \sigma_3 = \mathbb{R}_{\geq 0} \alpha_1 + \mathbb{R}_{\geq 0} \alpha_2 + \mathbb{R}_{\geq 0} \alpha_3 + \mathbb{R}_{\geq 0} \beta_1 + \mathbb{R}_{\geq 0} \beta_2 + \mathbb{R}_{\geq 0} \beta_3 \]
with 
\[ \alpha_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
\[ \beta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Then 
\[ \Sigma_3 = \{ M(\sigma_3); \; M \in \text{Gl}(3, \mathbb{Z}) \}. \]

We consider the lattices 
\[ N_3 = \mathbb{Z} \gamma_1 + \mathbb{Z} \gamma_2 + \mathbb{Z} \gamma_3 \]
\[ N_6 = \mathbb{Z} \alpha_1 + \mathbb{Z} \alpha_2 + \mathbb{Z} \alpha_3 + \mathbb{Z} \beta_1 + \mathbb{Z} \beta_2 + \mathbb{Z} \beta_3. \]

The fans \( \Sigma_2 \) resp. \( \Sigma_3 \) define torus embeddings \( T^3 \subset X(\Sigma_2) \) and \( T^6 \subset X(\Sigma_3) \).

We denote the divisors of \( X(\Sigma_3) \) which correspond to the 1-dimensional simplices of \( \Sigma_3 \) by \( D^i \). Let \( D = D^1 \) be the divisor corresponding to \( \mathbb{R}_{\geq 0} \alpha_3 \). An open part of \( D \) (in the \( C \)-topology) is mapped to the boundary component \( D_2(n) \). In order to understand the structure of \( D \) we also consider the rank 5 lattice 
\[ N_5 = \mathbb{Z} \alpha_1 + \mathbb{Z} \alpha_2 + \mathbb{Z} \beta_1 + \mathbb{Z} \beta_2 + \mathbb{Z} \beta_3 \cong N_6/\mathbb{Z} \alpha_3. \]

The natural projection \( \rho : N_{6, \mathbb{R}} \to N_{5, \mathbb{R}} \) maps the cones of the fan \( \Sigma_3 \) to the cones of a fan \( \Sigma'_4 \subset N_{5, \mathbb{R}} \). This fan defines a torus embedding \( T^5 = (D \setminus \bigcup_{i \neq 1} D^i) \subset X(\Sigma'_4) = D \).

The projection 
\[ \lambda : \; N_{6, \mathbb{R}} \cong \text{Sym}_3(\mathbb{R}) \rightarrow N_{3, \mathbb{R}} \cong \text{Sym}_2(\mathbb{R}) \]
\[ \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \]
maps $\Sigma_3$ to $\Sigma_2$ and factors through $N_{5,\mathbb{R}}$. In this way we obtain an induced map

$$D = X(\Sigma_3') \to X(\Sigma_2) \bigcup_{T^5} \to T^3.$$ 

In order to describe this map we first consider the standard simplices $\sigma_3 \subset N_{6,\mathbb{R}}$ and $\sigma_2 \subset N_{3,\mathbb{R}},$ resp. $\sigma_3' = \rho(\sigma_3) \subset N_{5,\mathbb{R}}$. On the torus $T^6$ (and similarly on $T^5$ and $T^3$) we introduce coordinates by

$$t_{ij} = e^{2\pi i r_{ij}/n} \quad (1 \leq i, j \leq 3).$$

These coordinates correspond to the dual basis of the basis $U^*_{ij}$ of Sym$(3, \mathbb{Z})$ where the entries of $U^*_{ij}$ are 1 in positions $(i, j)$ and $(j, i)$ and 0 otherwise. One easily checks that $T_{\sigma_3} \cong \mathbb{C}^6 \subset X(\Sigma_3)$ and as coordinates on $T_{\sigma_3}$ one can take the coordinates which correspond to the dual basis of the generators $\alpha_1, \ldots, \beta_3$. Let us denote these coordinates by $T_1, \ldots, T_6$. A straightforward calculation shows that the inclusion $T^6 \subset T_{\sigma_3}$ is given by

$$T_1 = t_{11}t_{13}t_{12}, \quad T_2 = t_{22}t_{23}t_{12}, \quad T_3 = t_{33}t_{13}t_{23},$$
$$T_4 = t_{23}^1, \quad T_5 = t_{13}^1, \quad T_6 = t_{12}^1.$$ \hspace{1cm} (1)

Then $D \cap T_{\sigma_3} = \{T_3 = 0\}$. For genus 2 the corresponding embedding $T^3 \subset T_{\sigma_2}$ is given by

$$T_1 = t_{11}t_{12}, \quad T_2 = t_{22}t_{12}, \quad T_3 = t_{12}^{-1}.$$ 

Finally we consider $T_{\sigma_2} \cong \mathbb{C}^5 \subset X(\Sigma_3')$. The projection $D = X(\Sigma_3') \to X(\Sigma_2)$ map $T_{\sigma_2}$ to $T_{\sigma_2}$. We can use $T_1, T_2, T_3, T_4, T_5, T_6$ as coordinates on $T_{\sigma_2}$. Since $\lambda(\alpha_1) = \hat{\lambda}(\beta_2) = \gamma_1, \lambda(\alpha_2) = \lambda(\beta_1) = \gamma_2$ and $\lambda(\alpha_3) = \gamma_3$ we find that

$$T_{\sigma_2} \cong \mathbb{C}^5 \to T_{\sigma_2} \cong \mathbb{C}^3$$
$$T_1, T_2, T_3, T_4, T_5, T_6 \quad \to \quad T_1T_5, T_2T_4, T_6.$$ \hspace{1cm} (2)

Given any (maximal dimensional) cone $\sigma' = \rho(\sigma)$ in $\Sigma_3'$ we can describe the map $T_{\sigma'} \to T_{\lambda(\sigma)}$ in terms of coordinates by the method described above. In this way we obtain a complete description of the map $D \to X(\Sigma_2)$.

Let us now return to the toroidal compactification $A_3^*(n)$ of $A_3(n)$. Let $u_0 \subset \mathbb{Q}^6$ be a maximal isotropic subspace. Then we obtain the compactification of $A_3(n)$ in the direction of the cusp corresponding to $u_0$ as follows: The parabolic subgroup $P(u_0) \subset \Gamma_3(n)$ is an extension

$$1 \to P'(u_0) \to P(u_0) \to P''(u_0) \to 1$$

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where \( P'(u_0) \) is a lattice of rank 6. We have an inclusion \( \mathbb{H}_g/P'(u_0) \subset T^6 \subset X(\Sigma_3) \) and we denote the interior of the closure of \( \mathbb{H}_g/P'(u_0) \) in \( X(\Sigma_3) \) by \( X(u_0) \). Then \( P''(u_0) \) acts on \( X(u_0) \) and we obtain a neighbourhood of the cusp corresponding to \( u_0 \) by \( X(u_0)/P''(u_0) \). We have already described the partial compactification in the direction of a line (in our case \( l_0 \)). Similarly we can define a partial compactification in the direction of an isotropic plane \( h_0 \). The space \( \mathcal{A}_2^\ast(n) \) is then obtained by glueing all these partial compactifications.

The result of Nakamura and Tsushima can then be stated as follows: The restriction of the map \( \pi : \mathcal{A}_2^\ast(n) \to \overline{\mathcal{A}_3(n)} \) to the boundary component \( \overline{D}_2(n) \) admits a factorisation

\[
\begin{array}{ccc}
\overline{D}_2(n) & \xrightarrow{\pi'} & \mathcal{A}_2^\ast(n) \\
& \searrow \pi'' & \downarrow \\
& & \mathcal{A}_2(n)
\end{array}
\]

where \( \pi'' : \mathcal{A}_2^\ast(n) \to \mathcal{A}_2(n) \) is the natural map of the Voronoi compactification \( \mathcal{A}_2^\ast(n) \) of \( \mathcal{A}_2(n) \) to the Satake compactification \( \mathcal{A}_2(n) \). The map \( \pi' : \overline{D}_2(n) \to \mathcal{A}_2^\ast(n) \) is a flat family of surfaces extending the universal family over \( \mathcal{A}_2(n) \). In order to describe the fibres over the boundary points of \( \mathcal{A}_2^\ast(n) \) recall that every boundary component of \( \mathcal{A}_2^\ast(n) \) is isomorphic to the Shioda modular surface \( S(n) \). We explain the type of a point \( P \) in \( \mathcal{A}_2^\ast(n) \) as follows:

- \( P \) has type I if \( P \in \mathcal{A}_2(n) \)
- \( P \) has type II if \( P \) lies on a smooth fibre of a boundary component \( S(n) \)
- \( P \) has type IIIa if \( P \) is a smooth point on a singular fibre of \( S(n) \)
- \( P \) has type IIIb if \( P \) is a singular point of an \( n \)-gon in \( S(n) \).

Points of type IIIb are also often called deepest points.

**Proposition 3.1 (Nakamura, Tsushima)** Assume \( n \geq 3 \). Let \( P \) be a point in \( \mathcal{A}_2^\ast(n) \) and denote the fibre of the map \( \pi' : \overline{D}_2(n) \to \mathcal{A}_2^\ast(n) \) over \( P \) by \( A_P \). Then the following holds:

(i) If \( P = [\tau] \in \mathcal{A}_2(n) \) is of type I then \( A_P \) is a smooth abelian surface, more precisely \( A_P \cong A_{n,\tau n} \).

(ii) if \( P \) is of type II, then \( A_P \) is a cycle of \( n \) elliptic ruled surfaces.
(iii) If $P$ is of type IIIa, then $A_P$ consists on $n^2$ copies of $\mathbb{P}^1 \times \mathbb{P}^1$.

(iv) If $P$ is of type IIIb, then $A_P$ consists of $3n^2$ components. These are $2n^2$ copies of the projective plane $\mathbb{P}^2$ and $n^2$ copies of $\overline{\mathbb{P}}^2$, i.e. $\mathbb{P}^2$ blown up in 3 points in general position.

Proof. The proof consists of a careful analysis of the map $\overline{D}_2(n) \to A^*_2(n)$ using the description of the map $D \to X(\Sigma_2)$. For details see [Ts, section 4].

Remarks  (i) The degenerations of type IIIa and IIIb are usually depicted by the diagrams

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(IIIa)
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where each square stands for a $\mathbb{P}^1 \times \mathbb{P}^1$, resp.

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(IIIb)
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where the triangles stand for projective planes $\mathbb{P}^2$ and the hexagons for blown-up planes $\overline{\mathbb{P}}^2$. 

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(ii) The singular fibres are degenerate abelian surfaces (cf. \[Nak\], \[IKW\]).

(iii) This description must be modified for \(n = 1\) or \(2\). Then the general fibre is a Kummer surface \(K_{n,\tau n}\) and the fibres of type (IIIb) consist of 8 \((n = 2)\), resp. 2 copies of \(\mathbb{P}^2\).

The following is a crucial technical step:

**Proposition 3.2** Let \(n \equiv 0 \mod 8\). If \(m', m'', m', m'' \in \frac{1}{2p}\mathbb{Z}^2\) then the sections \(\Theta_{m'm''}(\tau, z)\Theta_{m'm''}(\tau, z)\) of the line bundle \(M(n)\) on \(D_2(n)\) extend to sections of the line bundle \(\overline{M}(n)\) on \(\overline{D}_2(n)\).

**Proof.** We have to prove that the sections in question extend to the part of \(D_2(n)\) which lies over the boundary of \(A_{2}^*(n)\). This is a local statement. Moreover it is enough to prove extension in codimension 1. Due to symmetry considerations we can restrict ourselves to one boundary component in \(A_{2}^*(n)\). We shall use the above description of the toroidal compactifications \(A_{2}^*(n)\) and \(A_{3}^*(n)\) and of the map \(\overline{D}_2(n) \to A_{2}^*(n)\). We consider the boundary component of \(A_{2}^*(n)\) given by \(\{T_2 = 0\} \subset T_{\sigma^2} \subset X(\Sigma_2)\). Recall the theta functions

\[
\Theta_{m'm''}(\tau, z) = \sum_{q \in \mathbb{Z}^2} e^{2\pi i \left[\frac{1}{2}(q+m')^t(q+m')+(q+m')^t(z+m'')\right]}
\]

In our situation

\[
\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad z = (z_1, z_2) = (\tau_{13}, \tau_{23}) \]

In level \(n\) we have the coordinates

\[
t_{ij} = e^{2\pi i \tau_{ij}/n}
\]

and \(\Theta_{m'm''}(\tau, z)\) becomes

\[
\Theta_{m'm''}(\tau, z) = \sum_{q=(q_1, q_2) \in \mathbb{Z}^2} \frac{1}{t_{11}}(q_1+m'_1)^2(n)(q_1+m'_1)(q_2+m'_2)n_1(q_2+m'_2)^2(t_{12}+t_{22})e^{2\pi i (q+m')^t(m'')}.
\]

We use the coordinates \(T_1, T_2, T_4, T_5, T_6\) on \(T_{\sigma^2}\). It follows from \([H]\) that

\[
t_{11} = T_1T_5T_6, \quad t_{22} = T_2T_4T_6, \quad t_{23} = T_4^{-1}, \quad t_{13} = T_5^{-1}, \quad t_{12} = T_6^{-1}.
\]
This leads to the following expression for the theta-functions

\[
\Theta_{m',m''}(\tau, z) = \sum_{q \in \mathbb{Z}^2} \frac{T_1^{1/2}(q_1+m_1')^2 n_1}{T_2^{1/2}(q_2+m_2')^2 n_2} T_4^{1/2}(q_2+m_2')(q_2+m_2'-2)n_4 e^{2\pi i (q+m')^t m''}.
\]

By (2) the locus over \( T_2 = 0 \subset T_{\sigma_2} \) in \( T_{\sigma_3}' \) is given by \( T_2 T_4 = 0 \). The equation for the boundary component \( D_2(n) \) is given by \( t_{33} = 0 \). Since by (1) we have \( t_{33} = T_3 T_4 T_5 \) we can assume that the normal bundle and hence \( M(n) \) (more precisely its pullback to \( X(\Sigma_3) \)) is trivial outside \( T_4 T_5 = 0 \). Since the exponent of \( T_2 \) is a non-negative integer (here we use \( n \equiv 0 \mod 8p^2 \)) this shows that the sections extend over \( T_2 = 0, T_4 \neq 0 \). To deal with the other components of \( T_{\Sigma_3}' \) which lie over \( \{ T_2 = 0 \} \) in \( T_{\sigma_2} \) we use the matrices

\[
\nu_{nm} = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} \quad (n, m \in \mathbb{Z})
\]

(cf. [Ts]) which act on \( \text{Sym}^0_3(\mathbb{Z}) \) by

\[
\gamma \mapsto \nu_{nm} \gamma 
u_{nm}.
\]

Via \( \lambda \) this action lies over the trivial action on \( \text{Sym}^0_2(\mathbb{Z}) \). This action also factors through \( \rho \). Let \( \sigma_3'_{nm} = \rho(\nu_{nm} \sigma_3 \nu_{nm}) \). We can then either argue with the symmetries induced by this operation or repeat directly the above calculation for \( T_{(\sigma_3')_{nm}} \). Acting with \( \nu_{0m}, m \in \mathbb{Z} \), we can thus treat all components in \( X(\Sigma_3) \) lying over \( \{ T_2 = 0 \} \) in \( X(\Sigma_2) \).

4 Curves in the boundary

We can now treat curves contained in a boundary component. The following technical lemma will be crucial. Its proof uses the ideas of [We, Abschnitt 4] in an essential way and it can be generalized in a suitable form to arbitrary \( g \).

We consider the boundary component \( \overline{D}_2(n) \) which belongs to the line \( l_0 = (0, \ldots , 0, 1) \subset \mathbb{Q}^6 \). Recall that the open part \( D_2(n) \) of \( \overline{D}_2(n) \) is of the form \( D_2(n) = \mathbb{C}^2 \times \mathbb{H}_2/(P''(l_0) \cap \Gamma(n)) \) and that the group \( P''(l_0)/(P''(l_0) \cap \Gamma(n)) \) acts on \( \overline{D}_2(n) \). Recall also the fibration \( \pi' : \overline{D}_2(n) \to \mathcal{A}_2^g(n) \). We shall denote the boundary of \( \mathcal{A}_2^g(n) \) by \( B \).
Proposition 4.1 Let \((z, \tau) \in \mathbb{C}^2 \times \mathbb{H}_2\). For every \(\varepsilon > 0\) there exist integers \(n, k\) and a section \(s \in H^0(M(n)^k)\) such that

(i) \(s([z, \tau]) \neq 0\) where \([z, \tau] \in D_2(n) = \mathbb{C}^2 \times \mathbb{H}_2/(P^n(l_0) \cap \Gamma(n))\),

(ii) \(s\) vanishes on \(\pi^*B\) of order \(\lambda\) with \(\frac{12n}{k} \geq \frac{n}{12+\varepsilon}\).

Proof. Let \(p \geq 3\) be a prime number (which will be chosen later). For \(l = 2p\) we consider the set of characteristics \(\mathcal{M}\) in \((\frac{1}{2}\mathbb{Z}/\mathbb{Z})^6\) of the form \(m = (m_1, m_2)\) in \((\frac{1}{2}\mathbb{Z}/\mathbb{Z})^6 \oplus (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^6\) with \(m_p \notin \mathbb{Z}_6\). The group \(\Gamma_3(1)\) acts on \(\mathcal{M}\) with 2 orbits. Assume \(\varepsilon > 0\) is given and that \(\widetilde{\mathcal{M}}\) is a subset of \(\mathcal{M}\) with \(\#\widetilde{\mathcal{M}} < \varepsilon \#\mathcal{M}\).

Then set

\[
\Theta_{\mathcal{M},\widetilde{\mathcal{M}}}(\tau, z) = \prod_{m \in \mathcal{M} \setminus \widetilde{\mathcal{M}}} \Theta^r_m(\tau, z).
\]

Let \(n = 8p^2\). By Proposition (3.2) the functions \(\Theta^r_m(\tau, z)\) define sections in \(\mathcal{M}(n)\). Let \(M_1, \ldots, M_N \in \Gamma_2(1)\) be a set of generators of \(\Gamma_2(1)/\Gamma_2(n) \cong \text{Sp}(4, \mathbb{Z}/n\mathbb{Z})\). Then \(\mathcal{M}\) is considered as elements in \(P(l_0)\), act on the line bundle \(\mathcal{M}(n)\). We set

\[
F_r(\tau, z) = \sum_{i=1}^N M_i^\ast \Theta^r_{\mathcal{M},\widetilde{\mathcal{M}}}.
\]

This is a \(\Gamma_2/\Gamma_2(n)\)-invariant section of \(\mathcal{M}(n)^{pr}\).

Now consider the abelian surface \(A = A_{\tau,1} = \mathbb{C}^2/((\mathbb{Z}^2\tau + \mathbb{Z}^2))\). Then \(A_{\tau,n} = \mathbb{C}^2/((n\mathbb{Z})^2\tau + (n\mathbb{Z})^2)\) is the fibre of \(\pi\) over the point \([\tau] \in A_2(n)\). Let

\[
\widetilde{\mathcal{M}} = \{m \in \mathcal{M}; \Theta_m(\tau, z) = 0\}.
\]

The argument of Weissauer shows that

\[
\#\widetilde{\mathcal{M}} < \varepsilon \#\mathcal{M}
\]

for \(p\) sufficiently large. For some \(r\) the section \(F_r(\tau, z)\) does not vanish at \([z, \tau] \in D_2(n)\). Let \(B'\) be a boundary boundary component of \(A^2_2(n)\). The inverse image \(D'\) of \(B'\) under \(\pi'\) consists of several components. Using the matrices \(\nu_m\) which were introduced in the proof of Proposition (3.2) one can, however, show that the vanishing order of the sections \(\Theta^r_m(\tau, z)\) on the components of \(D'\) only depends on \(B'\). Hence one can argue as in [Wd] and finds that the vanishing order along \(\pi^*B\) goes to \(\frac{prn}{12}\) as \(p\) goes to infinity. Setting \(k = pr\) this gives (ii). \(\square\)
We can now start giving the proof of Theorem (0.1). Let 

\[ H = aL - bD \quad b > 0, \quad 12a - \frac{b}{n} > 0 \]

be a divisor on \( \mathcal{A}_g(n) \). In view of Proposition (1.4) it remains to consider curves \( C \) which are contained in the boundary. To simplify notation we write the decomposition of the boundary \( D \) as

\[ D = \sum_{i=1}^{N} \overline{D}_{g-1}(n) \]

where \( N = N(n, g) \) can be computed explicitly. Then

\[ H|_{\overline{D}_{g-1}(n)} = \left( aL - b \sum_{i \neq 1} \overline{D}_{g-1}(n) \right) - b \overline{D}_{g-1}(n)|_{\overline{D}_{g-1}(n)}. \]  

(4)

Now let \( g = 2 \) or \( 3 \) where we have the fibration

\[ \pi' : \overline{D}_{g-1}(n) \rightarrow \mathcal{A}_{g-1}(n). \]

We shall denote the boundary of \( \mathcal{A}_{g-1}(n) \) by \( B \). Also note that the restriction of \( L \) to the boundary equals \( \pi'^*L_{\mathcal{A}_{g-1}(n)} \) where we use the notation \( L \) for both the line bundle on \( \mathcal{A}_{g}(n) \) and \( \mathcal{A}_{g-1}(n) \). Thus we find that

\[ H|_{\overline{D}_{g-1}(n)} = \pi'^*(aL - bB) - b \overline{D}_{g-1}(n)|_{\overline{D}_{g-1}(n)}. \]  

(5)

In view of the definition of the line bundle \( \overline{M}(n) \) this gives

\[ H|_{\overline{D}_{g-1}(n)} = \pi'^* \left( \left( a - \frac{b}{n} \right) L - bB \right) + \frac{b}{n} \overline{M}(n). \]  

(6)

**Proof of Theorem (0.1) for \( g = 2 \).** In this case the boundary components \( \overline{D}_1(n) \) are isomorphic to Shioda’s modular surface \( S(n) \) and the projection \( \pi' \) is just projection to the modular curve \( X(n) \). The degree of \( L \) on \( X(1) \) is \( \frac{1}{12} \) and we have one cusp. Hence

\[ \deg_{X(n)}(aL - bB) = \mu(n) \left( a - \frac{b}{12} \right) \]

where \( \mu(n) \) is the degree of the Galois covering \( X(n) \rightarrow X(1) \), i.e. \( \mu(n) = |\text{PSL}(2, \mathbb{Z}/n\mathbb{Z})| \). This is non-negative if and only if \( a - 12\frac{b}{n} \geq 0 \). The
normal bundle of $\mathcal{D}_i^1(n)$ can also be computed explicitly. This can be done as follows: Using the degree 10 cusp form which vanishes on the reducible locus one finds the equality $10L = 2H_1 + D$ on $A_2^*(n)$ where $H_1$ is the Humbert surface parametrizing polarized abelian surfaces which are products. Hence we conclude for the canonical bundle on $A_2^*(n)$ that $K = (3 - \frac{10}{n})L + \frac{2}{n}H_1$. The restriction of the divisor $H_1$ to a boundary component $(n)$ is the sum of the $n^2$ sections $L_{ij}$ of $S(n)$. Hence we conclude for the canonical bundle on $A_2^*(n)$ that $K = (3 - 10n) + 2nH_1$.

The restriction of the divisor $H_1$ to a boundary component $D_i^1(n)$ is the sum of the $n^2$ sections $L_{ij}$ of $S(n)$. The canonical bundle of the surfaces $S(n)$ is equal to the pull-back via $\pi'$ of $3L$ minus the divisor of the cusps on the modular curve $X(n)$ (see also [BH]). Hence adjunction together with an easy calculation gives

$$-n\mathcal{D}_1(n)_{|\mathcal{D}_1(n)} = 2\pi^*L_X(n) + 2\sum L_{ij}$$

Since $L_{ij}|L_{ij} = -L_X(n)$ one sees immediately that this line bundle is nef and positive on the fibres of $\pi' : S(n) \to X(n)$. The result now follows directly from (6).

We shall now turn to the case $g = 3$. As we have remarked before it remains to consider curves which are contained in the boundary of $A_3^*(n)$. Among those curves we shall first deal with curves whose image under the map $\pi'$ meets the interior of $A_2(n)$.

**Proposition 4.2** Let $H = aL - bD$ be a divisor on $A_3^*(n)$ with $a - 12\frac{k}{n} > 0$, $b > 0$. For every curve $C$ in a boundary component $\mathcal{D}_2(n)$ with $\pi'(C) \cap A_2(n) \neq \emptyset$ the intersection number $H.C > 0$.

**Proof.** We shall use (6) and Proposition (4.1). If we replace $n$ by some multiple and consider the pull-back of $H$ the coefficient $b/n$ is not changed. The inverse image of $C$ may have several components. All of these are, however, equivalent under some finite symplectic group and it is sufficient to prove that the degree of $H$ is positive on one (and hence on every) component lying over $C$. After this reduction we can again assume that $C$ is irreducible and by Proposition (4.1) we can find for every $\varepsilon > 0$ a divisor $C$ not containing $C$ with

$$\overline{M}(n) = C + \frac{\lambda}{k} \pi^*B, \quad \frac{\lambda}{k} \geq \frac{n}{12 + \varepsilon}.$$  

By (6)

$$H|\mathcal{D}_2(n) = \pi^* \left( \left( a - \frac{b}{n} \right) L - b \left( 1 - \frac{\lambda}{nk} \right) B \right) + \frac{b}{n}C.$$
The assertion follows from the corresponding result for $g = 2$ provided
\[
\left( a - \frac{b}{n} \right) - 12 \frac{b}{n} \left( 1 - \lambda nk \right) \geq \left( a - 12 \frac{b}{n} \right) - \frac{b}{n} \left( 1 + \frac{12}{12 + \varepsilon} \right) > 0.
\]
Since $a - 12b/n > 0$ this is certainly the case for $\varepsilon$ sufficiently small. □

We are now left with curves in the boundary of $A^*_3(n)$ whose image under $\pi'$ is contained in the boundary of $A^*_2(n)$. These are exactly the curves which are contained in more than 1 boundary component of $A^*_3(n)$. Before we conclude the proof, we have to analyze the situation once more. First of all we can assume by symmetry arguments that $C$ is contained in $D^2(n) = D^2_1(n)$. Let $B'$ be a component of the boundary $B$ of $A^*_2(n)$ which contains $\pi'(C)$. Let $D' = (\pi')^{-1}(B')$. Then $D'$ consists of $n$ irreducible components and we have the following commutative diagram ($n \geq 3$):

\[
\begin{array}{ccc}
D^2_2(n) & \xrightarrow{\pi'} & A^*_2(n) \\
\cup & & \cup \\
D' & \xrightarrow{\pi'} & B' \cong S(n) \\
\downarrow & & \downarrow \pi'' \\
& & X(n).
\end{array}
\]

Altogether there are three possibilities:
1. $\pi'(C) = pt$, i.e. $C'$ is contained in a fibre of $\pi'$.
2. $\pi(C) = pt, \pi'(C) \neq pt$. Then $\pi'(C)$ is either a smooth fibre of $S(n)$ or a component of a singular $n$--gon.
3. $\pi(C) = X(n)$.

The final step in the proof of Theorem (0.1) is the following:

**Proposition 4.3** Let $C \subset D^2_2(n)$ be a curve whose image $\pi'(C)$ is contained in the boundary of $A^*_2(n)$. If $H = aL - bD$ is a divisor with $b > 0, a - 12\frac{b}{n} > 0$ then $H.C > 0$.

**Proof.** By induction on $g$ and formula (5) it is enough to prove that there is some $D^2_i(n)$ with $C.D^2_i(n) \leq 0$. Consider the inverse image $D'$ of $B'$ under $\pi'$. Then $D'$ consists of $n$ irreducible components each of which is of the form $D^2_2(n) \cap D^2_i(n)$ for some $i \neq 1$. We already know that $-B'|_{B'}$ is nef. Hence
\[
\left( \sum_{i \in I} D^2_2(n) \cap D^2_i(n) \right).C \leq 0
\]

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where $I$ is a suitable set of indices consisting of $n$ elements. In particular \( \bar{D}_2(n).C \leq 0 \) for some index $j$.

**Remarks**

(i) If $\pi'(C) = pt$, then one can give an alternative proof of $\bar{D}_2(n).C > 0$ by computing the normal bundle of $\bar{D}_2(n)$ restricted to the singular fibres of $\pi'$. The conormal bundle is ample as in the smooth case (cf. Lemma (2.2)).

(ii) If $\pi'(C) \neq pt$ one can also use the theta functions $\Theta_{m'm''}$ with $m', m'' \in \frac{1}{2}\mathbb{Z}^2$ to construct sections of $\bar{M}(n)$ which, after subtracting suitable components of the form $\bar{D}_2(n) \cap \bar{D}_2(n)$, do not vanish identically on $C$. In this way one can compute similarly to the proof of Proposition (4.2) that $H.C > 0$.

**Proof of Theorem (0.1)(g=3).** This follows now immediately from Proposition (1.4), Proposition (4.2) and Proposition (4.3).

**Proof of the corollaries.** These follow immediately from Theorem (1.1) since the moduli spaces are smooth and since

\[
K \equiv (g + 1)L - D.
\]

Obviously

\[
(g + 1) - \frac{12}{n} \geq 0 \iff \begin{cases} 
    n \geq 4 & \text{if } g = 2 \\
    n \geq 3 & \text{if } g = 3.
\end{cases}
\]

Hence $K$ is nef if $g = 2, n \geq 4$ and $g = 3, n \geq 3$, resp. numerically positive if $g = 2, n \geq 5$ and $g = 3, n \geq 4$. It follows from general results of classification theory that $K$ is ample in the latter case.

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