ON POLYHARMONIC POLYNOMIALS

HUBERT GRZEBUŁA AND SŁAWOMIR MICHALIK

ABSTRACT. We study the orthogonal projection of homogeneous polynomials onto the space of homogeneous polyharmonic polynomials. To do this we derive the decomposition of homogeneous polynomials in terms of the Kelvin transform of derivatives of the fundamental solution $|x|^{2-n}$ or $\log |x|$. We consider also the vector bases of the space of homogeneous polyharmonic polynomials and study the problem of convergence of orthogonal series.

1. Introduction

The theory of polyharmonic functions has been investigated extensively. The basic work about this class of functions is due to Aronszajn, Creese and Lipkin [1], and it is still an active field of interest in mathematics, see for example [7] and references therein. These functions are also very useful in many branches in applied mathematics, such as approximation theory, polysplines, radial basis functions and wavelet analysis (see for example [5, 10, 13]).

In this paper we study the polyharmonic polynomials. A lot of information about them can be found in a classical paper [4]. We start our considerations with the decomposition of homogeneous polynomials. There are many works about the decomposition of the homogeneous polynomials. In the papers [11, 17] the Fischer decomposition is considered, and it is relevant to the well known Khavinson-Shapiro conjecture (see also [15, 16]). The decomposition given in this paper is a natural extension of the result from [3] (see also [2]). It is worth to mention that the decomposition given there allows to obtain the solution of many Dirichlet-type problems with polynomial data without using integration, namely mentioned results lead to the algorithm which involves only differentiation (see [2]).

Let $\mathcal{P}_m(\mathbb{R}^n)$ denote the space of the homogeneous polynomials of degree $m$ on $\mathbb{R}^n$ with $n > 2$. It is known that for any $u \in \mathcal{P}_m(\mathbb{R}^n)$ the following formula holds (see Lemma 5.17 in [3])

$$K[u(D)]|x|^{2-n} = c_m u + |x|^2 v,$$

2020 Mathematics Subject Classification. 31B30, 32A25.

Key words and phrases. homogeneous polynomials, polyharmonic polynomials, orthogonal projection.

1
where
\[ c_m = \prod_{k=0}^{m-1} (2 - n - 2k), \]
\[ u(D) = \sum_{\alpha} a_\alpha D^\alpha \text{ for } u = \sum_{\alpha} a_\alpha x^\alpha, \text{ and } K \text{ is the Kelvin transform.} \]
In this paper we derive an explicit formula for the polynomial \( v \in \mathcal{P}_{m-2}(\mathbb{R}^n) \):
\[ v = \left[ \frac{m}{2} \right] \sum_{k=1}^{[m/2]} \sum_{(k_1, \ldots, k_n)\in\mathbb{N}_0^n \atop k_1 + \cdots + k_n = k} \frac{c_{m-k}}{2^k k_1! \cdots k_n!} |x|^{2(k-1)} D_{x_1}^{2k_1} \cdots D_{x_n}^{2k_n} u. \]
As a corollary we find the mentioned decomposition of the homogeneous polynomials in terms of the Kelvin transform, namely
\[ u(x) = \left[ \frac{m}{2} \right] \sum_{k=0}^{[m/2]} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)] |x|^{2-n}, \]
where \( u_0 = u \) and
\[ u_k = \sum_{(k_1, \ldots, k_n)\in\mathbb{N}_0^n \atop k_1 + \cdots + k_n = k} d_{m,k,k_1,\ldots,k_n} D_{x_1}^{2k_1} \cdots D_{x_n}^{2k_n} u \quad \text{for } k = 1, \ldots, [m/2] \]
with \( d_{m,k,k_1,\ldots,k_n} \) being some constants independent of \( u \). Analogous results are obtained for \( n = 2 \) by changing the fundamental solutions, here we replace \( |x|^{2-n} \) by \( \log |x| \). From the above we easily find the orthogonal projection of homogeneous polynomials onto the space of polyharmonic polynomials and the orthogonal projection of homogeneous polynomials on \( \mathbb{C}^n \) onto the space of so-called spherical polyharmonics \( \mathcal{H}_p^m(\hat{S}_p) \). The spherical polyharmonics are a natural generalisation of the well known spherical harmonics. Here we restrict the polyharmonic polynomials to the union of rotated spheres (see [9]):
\[ \hat{S}_p := \bigcup_{k=0}^{p-1} c^\frac{k+1}{p} S. \]
The motivation for the study of spherical polyharmonics on the union of rotated balls comes from the Pizzetti formulas for the polyharmonic operator \( \Delta^p \) given in [12, 13] (see also [8]).
In the next part of this paper we deal with the vector bases of the space of the polyharmonic homogeneous polynomials of order \( p \) and of degree \( m \leq p \) on \( \mathbb{C}^n \).
A criterion for the convergence of some orthogonal series is considered in [6]. An explicit formula for the reproducing kernel of the Hilbert space \( \mathcal{H}_p^m(\mathbb{R}^n) \) of homogeneous polyharmonic polynomials endowed with the Fischer inner product is derived in [18]. By this formula, the condition given in [6] is improved in [18]. In our paper we study a similar problem. Using [8] we get the analogous result for the
space of polyharmonic polynomials, but endowed with the integral inner product induced from $L^2(\mathcal{S}_p)$. In this paper we also discuss the connection between these two kinds of inner products.

2. Preliminaries

In this part we recall some notations, definitions and facts that we will use in this paper. By

$$|x| = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2}$$

we denote the real norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We also will use the real norm for complex vectors $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$:

$$|z| = \left( \sum_{j=1}^{n} z_j^2 \right)^{1/2},$$

(1)

here by a square root we mean the principal square root, where a branch cut is taken along the non-positive real axis.

We will also use the complex norm of $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ defined by

$$|z|_{\mathbb{C}^n} = \left( \sum_{j=1}^{n} z_j \overline{z}_j \right)^{1/2}.$$

Let $B$ and $S$ be respectively the unit ball and sphere in $\mathbb{R}^n$ with a centre at the origin. For the angle $\varphi \in \mathbb{R}$ we will consider a rotated unit ball in $\mathbb{R}^n$ defined by

$$e^{i\varphi}B := \{e^{i\varphi}x : x \in B\}.$$

We define similarly a rotated sphere $e^{i\varphi}S$. Let $p \in \mathbb{N}$. We will denote unions of rotated balls and spheres as follows

$$\hat{B}_p := \bigcup_{l=0}^{p-1} e^{i\frac{\pi}{p}} B \quad \text{and} \quad \hat{S}_p := \bigcup_{l=0}^{p-1} e^{i\frac{\pi}{p}} S.$$

More generally, for any $r > 0$ we use the notation $\hat{B}_p(r) := \bigcup_{l=0}^{p-1} e^{i\frac{\pi}{p}} B(r)$, where $B(r)$ denotes the ball in $\mathbb{R}^n$ with a radius $r$ and a centre at the origin.

Let $G \subset \mathbb{R}^n$ be open. A function $u : G \to \mathbb{C}$ is called polyharmonic of order $p \in \mathbb{N}$ if $u \in C^2(G)$ and $\Delta^p u(x) = 0$ for $x \in G$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is the Laplace operator and $\Delta^p$ is its $p$-th iterate. A polyharmonic function of order 1 is called harmonic. It is well known that if $u$ is polyharmonic on $B$, then $u$ can be holomorphically extended to the
Lie ball ([20, Theorem D]), in particular \( u \) can be extended to \( e^{i\varphi}B \) for every angle \( \varphi \in \mathbb{R} \) (see also [8, Lemma 1]).

**Lemma 1** (Almansi expansion, [1, Proposition 1.2, Proposition 1.3]). A function \( u \) is polyharmonic on \( B \) if and only if there exist unique harmonic functions \( h_0, h_1, \ldots, h_{p-1} \) on \( B \) such that

\[
u(x) = h_0(x) + |x|^2 h_1(x) + \cdots + |x|^{2(p-1)} h_{p-1}(x) \quad \text{for} \quad x \in B.
\]

**Definition 1.** A map

\[
x \mapsto x^* := \begin{cases} \frac{x}{|x|^2} & \text{if} \quad x \neq 0, \infty \\ 0 & \text{if} \quad x = \infty \\ \infty & \text{if} \quad x = 0
\end{cases}
\]

is called the *inversion* of \( \mathbb{R}^n \cup \{\infty\} \) relative to the unit sphere.

Let \( u \) be a function defined on a set \( E \subseteq (\mathbb{R}^n \cup \{\infty\}) \setminus \{0\} \). A function \( K[u] \) defined on \( E^* := \{x^* : x \in E\} \) by

\[
K[u](x) := |x|^{2-n} u(x^*)
\]

is called the *Kelvin transform* of \( u \).

**Remark 1.** We can extend the definition of the Kelvin transform to the complex case when \( E \subseteq (\mathbb{C}^n \cup \{\infty\}) \setminus \{z \in \mathbb{C}^n : |z| = 0\} \).

Observe that the set \( \{z \in \mathbb{C} : |z| = 0\} \) contains not only the origin since \( |z| \) given by (1) is a complex valued entity.

Let us recall a definition of harmonicity at \( \infty \).

**Definition 2.** Let \( E \subseteq \mathbb{R}^n \) be compact. If \( u \) is harmonic on \( \mathbb{R}^n \setminus E \), then \( u \) is harmonic at \( \infty \) provided \( K[u] \) has a removable singularity at the origin.

**Remark 2** ([3, Theorem 4.7, Theorem 4.9]). Let \( E \subseteq \mathbb{R}^n \) be compact and \( u \) be harmonic on \( \mathbb{R}^n \setminus E \). Then \( u \) is harmonic at \( \infty \) if and only if

(i) \( \lim_{x \to \infty} u(x) \) is finite, when \( n = 2 \),

(ii) \( \lim_{x \to \infty} u(x) = 0 \), when \( n > 3 \).

**Lemma 2** ([1, Proposition 1.4]). If \( u \) is a harmonic function on \( B \), then its Kelvin transform \( K[u] \) is harmonic on \( (\mathbb{R}^n \setminus B) \cup \{\infty\} \).

Let \( m, p \in \mathbb{N} \). We denote by \( \mathcal{P}_m(\mathbb{R}^n) \) (resp. \( \mathcal{P}_m(\mathbb{C}^n) \)) the space of homogeneous polynomials on \( \mathbb{R}^n \) (resp. \( \mathbb{C}^n \)) of degree \( m \). By \( \mathcal{H}_m^p(\mathbb{C}^n) \) we mean the space of polynomials on \( \mathbb{C}^n \), which are homogeneous of degree \( m \) and are polyharmonic of order \( p \). It is easy to see that if \( m < 2p \), then \( \mathcal{H}_m^p(\mathbb{R}^n) = \mathcal{P}_m(\mathbb{R}^n) \) (resp. \( \mathcal{H}_m^p(\mathbb{C}^n) = \mathcal{P}_m(\mathbb{C}^n) \)).
3. Projection of the homogeneous polynomials onto the polyharmonic polynomials

In this section we prove the main result on the orthogonal projection of the homogeneous polynomials onto the space of polyharmonic polynomials. The result is based on a more precise version of [3, Lemma 5.17].

**Proposition 1.** Let \( u \in \mathcal{P}_m(\mathbb{R}^n) \) with \( m \geq 1 \).

(i) If \( n \neq 2 \), then

\[
K[u(D)|x|^{2-n}] = \sum_{k=0}^{[\frac{m}{2}]} \sum_{\substack{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \ \text{such that} \ \sum_{i=1}^n k_i = k \text{ and} \ k_1 + \cdots + k_n = k}} \frac{c_{m-k}}{2^k k_1! \cdots k_n!} |x|^{2k} D_{x_1}^{2k_1} \cdots D_{x_n}^{2k_n} u,
\]

where

\[
c_m = \prod_{k=0}^{m-1} (2 - n - 2k).
\]

(ii) If \( n = 2 \), then

\[
K[u(D) \log |x|] = \sum_{k=0}^{[\frac{m}{2}]} \sum_{\substack{(k_1, k_2) \in \mathbb{N}_0^2 \ \text{such that} \ k_1 + k_2 = k}} \frac{\tilde{c}_{m-k}}{2^k k_1! k_2!} |x|^{2k} D_{x_1}^{2k_1} D_{x_2}^{2k_2} u,
\]

where

\[
\tilde{c}_m = (-2)^{m-1}(m - 1)!. \]

**Proof.** We shall prove (i). Since the Kelvin transform is linear, it is enough to prove it for a monomial \( u \in \mathcal{P}_m(\mathbb{R}^n) \). Let \( u(x) = x^\alpha \), \( |\alpha| = m \). We proceed by induction on \( m \). If \( m = 1 \), then \( u(x) = x_i \) for some \( i = 1, \ldots, n \). So \( u(D) = D_{x_i} \),

\[
K[u(D)|x|^{2-n}] = K[(2-n)|x|^{-n} x_i] = (2-n)|x|^{2-n} \left| \frac{x_i}{|x|^2} \right|^{-n} x_i = (2-n)x_i
\]

and \( c_1 = 2 - n \). Hence the proposition is valid. Let us assume that the proposition holds for \( m \geq 1 \). Let \( u(x) = x^\alpha \), \( |\alpha| = m \). By the induction hypothesis we have

\[
K[D^\alpha|x|^{2-n}] = \sum_{k=0}^{[\frac{m}{2}]} \sum_{\substack{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \ \text{such that} \ \sum_{i=1}^n k_i = k \text{ and} \ k_1 + \cdots + k_n = k}} \frac{c_{m-k}}{2^k k_1! \cdots k_n!} |x|^{2k} D_{x_1}^{2k_1} \cdots D_{x_n}^{2k_n} x^\alpha.
\]

Since \( u \in \mathcal{P}_m(\mathbb{R}^n) \), so \( D_{x_1}^{2k_1} \cdots D_{x_n}^{2k_n} x^\alpha \in \mathcal{P}_{m-2k}(\mathbb{R}^n) \). Moreover the Kelvin transform is an involution and is linear, therefore using this transform
Let us note that for every $x$ and $\alpha$ we set $\tilde{\alpha}$ to the last equation we get
\[
D^\alpha|x|^{2-n} = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \setminus k_1 + \cdots + k_n = k} \frac{c_{m-k}}{2^k k_1! \ldots k_n!} K[[x]|^{2k} D^{2k} x_1^{2k_1}, \ldots, x_n^{2k_n} x^\alpha]
\]
= \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \setminus k_1 + \cdots + k_n = k} \frac{c_{m-k}}{2^k k_1! \ldots k_n!} |x|^{2-n-2m+2k} D^{2k} x_1^{2k_1}, \ldots, x_n^{2k_n} x^\alpha
\]

Set $\hat{x} := (x_2, \ldots, x_n)$, $\hat{\alpha} := (\alpha_2, \ldots, \alpha_n)$. Without loss of generality we may differentiate the last equality with respect to $x_1$:
\[
D_{x_1} D^\alpha|x|^{2-n} = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \setminus k_1 + \cdots + k_n = k} \frac{c_{m-k}}{2^k k_1! \ldots k_n!} \times \left(2 - n - 2m + 2k\right) |x|^{2-n-2(m+1)+2k} x_1 D^{2k} x_1^{2k_1}, \ldots, x_n^{2k_n} x^\alpha + |x|^{2-n-2m+2k} D^{2k+1} x_1^{2k_1+1}, \ldots, x_n^{2k_n} x^\alpha
\]
= $c_m(2 - n - 2m) |x|^{2-n-2m-2} x_1 x^\alpha + I_1 + I_2,$

where
\[
I_1 = \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \setminus k_1 + \cdots + k_n = k} \frac{c_{m-k}}{2^k k_1! \ldots k_n!} \times \left(2 - n - 2m + 2k\right) |x|^{2-n-2(m+1)+2k} x_1 D^{2k} x_1^{2k_1}, \ldots, x_n^{2k_n} x^\alpha
\]
and
\[
I_2 = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \setminus k_1 + \cdots + k_n = k} \frac{c_{m-k}}{2^k k_1! \ldots k_n!} |x|^{2-n-2m+2k} D^{2k+1} x_1^{2k_1+1}, \ldots, x_n^{2k_n} x^\alpha.
\]

Let us note that
\[
c_{m-k}(2 - n - 2m + 2k) = c_{m+1-k}
\]
for every $k = 0, 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor$. Since $x^\alpha = x_1^{\alpha_1} \hat{x}^{\bar{\alpha}}$, we have
\[
x_1 D^{2k} x_1^{2k_1}, \ldots, x_n^{2k_n} x^\alpha = x_1 D^{2k} x_1^{2k_1} x_1^{\alpha_1} D^{2(k-k_1)} x_2^{2k_2}, \ldots, x_n^{2k_n} \hat{x}^{\bar{\alpha}}
\]
= $\frac{\alpha_1 - 2k_1 + 1}{\alpha_1 + 1} D^{2k_1} D^{2(k-k_1)} x_2^{2k_2}, \ldots, x_n^{2k_n} x_1^{\alpha_1+1} \hat{x}^{\bar{\alpha}}$
\[
= \frac{\alpha_1 - 2k_1 + 1}{\alpha_1 + 1} D^{2k} x_1^{2k_1}, \ldots, x_n^{2k_n} x_1 x^\alpha.
\]
By (2) and (3) we get

\[ I_1 = \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \atop k_1 + \cdots + k_n = k} \frac{c_{m+1-k}}{2^k k_1! \cdots k_n!} \times |x|^{2-n-2(m+1)+2k} \frac{\alpha_1 - 2k_1 + 1}{\alpha_1 + 1} D^{2k}_{x_1^{k_1} \cdots x_n^{k_n}} x_1^{\alpha_1}. \]

Now we can consider \( I_2 \). Let us note that

\[ D^{2k+1}_{x_1^{(k_1+1)} \cdots x_n^{2k_n}} x_1^{\alpha_1} = \frac{1}{\alpha_1 + 1} D^{2(k+1)}_{x_1^{k_1} \cdots x_n^{2k_n}} x_1^{\alpha_1}, \]

so

\[ I_2 = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \atop k_1 + \cdots + k_n = k} \frac{c_{m-k}}{2^k k_1! \cdots k_n!} |x|^{2-n-2m+2k} \frac{1}{\alpha_1 + 1} D^{2(k+1)}_{x_1^{(k_1+1)} \cdots x_n^{2k_n}} x_1^{\alpha_1}. \]

Replacing \( k_1 \) by \( k_1 - 1 \) in the last formula we obtain

\[ I_2 = \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor + 1} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \atop k_1 + \cdots + k_n = k} \frac{c_{m+1-k}}{2^{k-1} (k_1 - 1)! k_2! \cdots k_n!} |x|^{2-n-2m+2k} \frac{2k_1}{\alpha_1 + 1} D^{2k}_{x_1^{k_1} \cdots x_n^{2k_n}} x_1^{\alpha_1} + R_m, \]

where

\[ R_m = \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \atop k_1 + \cdots + k_n = \left\lfloor \frac{m}{2} \right\rfloor + 1} \frac{c_{m-\left\lfloor \frac{m}{2} \right\rfloor}}{2^{\left\lfloor \frac{m}{2} \right\rfloor + 1} k_1! k_2! \cdots k_n!} |x|^{2-n-2m+2\left\lfloor \frac{m}{2} \right\rfloor} \frac{2k_1}{\alpha_1 + 1} D^{2\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)}_{x_1^{k_1} \cdots x_n^{2k_n}} x_1^{\alpha_1}. \]

From (4) and (5) we get

\[ I_1 + I_2 = \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n \atop k_1 + \cdots + k_n = k} \frac{c_{m+1-k}}{2^k k_1! k_2! \cdots k_n!} |x|^{2-n-2(m+1)+2k} D^{2k}_{x_1^{k_1} \cdots x_n^{k_n}} x_1^{\alpha_1} + R_m. \]

It is obvious that if \( m \) is an even number, then \( R_m = 0 \). Let \( m \) be an odd number, then \( \left\lfloor \frac{m}{2} \right\rfloor + 1 = \left\lceil \frac{m+1}{2} \right\rceil \). Let us note that the elements of the sum \( R_m \) do not vanish if and only if \((k_1, k_2, \ldots, k_n) = (\frac{\alpha_1+1}{2}, \frac{\alpha_2}{2}, \ldots, \frac{\alpha_n}{2})\) and \( \alpha_1 + 1, \ldots, \alpha_n \) are even. Therefore, without loss of generality we
may assume that $\alpha_1 + 1, \ldots, \alpha_n$ are even. Then

$$R_m = \frac{c_m - \frac{m}{2}}{2^\left[\frac{m}{2}\right] + 1 \cdot (\frac{m+1}{2})! \cdot \ldots \cdot (\frac{m}{2})!} \cdot |x|^{2-n-2m+2\left[\frac{m}{2}\right]} \cdot 2^{\frac{\alpha_1 + 1}{2}} \cdot D_{x_1}^{\left[\frac{m}{2}\right] + 1} \cdot \frac{1}{\alpha_1 + 1} \cdot \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha$$

$$= \frac{c_{m+1} - \frac{m+1}{2}}{2^\left[\frac{m+1}{2}\right] + 1 \cdot (\frac{m+1}{2})! \cdot \ldots \cdot (\frac{m}{2})!} \cdot |x|^{2-n-2(m+1)+2\left[\frac{m+1}{2}\right]} \cdot \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha .$$

Finally we can write:

$$D_{x_1} D_\alpha |x|^{2-n} = c_m (2 - n - 2m) |x|^{2-n-2m-2} x_1^\alpha$$

$$+ \sum_{k=1}^{\left[\frac{m}{2}\right]} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n} \frac{c_{m+1-k}}{2^k k_1! k_2! \ldots k_n!} \cdot |x|^{2-n-2(m+1)+2k} D_{x_1}^{2k} \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha + R_m$$

$$= \sum_{k=0}^{\left[\frac{m+1}{2}\right]} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n, k_1 + \cdots + k_n = k} \frac{c_{m+1-k}}{2^k k_1! k_2! \ldots k_n!} \cdot |x|^{2-n-2(m+1)+2k} D_{x_1}^{2k} \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha .$$

Hence

$$K[D_{x_1} D_\alpha |x|^{2-n}] = \sum_{k=0}^{\left[\frac{m}{2}\right]} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n, k_1 + \cdots + k_n = k} \frac{c_{m+1-k}}{2^k k_1! k_2! \ldots k_n!} \times K[|x|^{2-n-2(m+1)+2k} D_{x_1}^{2k} \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha]$$

$$= \sum_{k=0}^{\left[\frac{m+1}{2}\right]} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n, k_1 + \cdots + k_n = k} \frac{c_{m+1-k}}{2^k k_1! k_2! \ldots k_n!} \cdot |x|^{2k} D_{x_1}^{2k} \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha ,$$

because $D_{x_1}^{2k} \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha \in \mathcal{P}_{m+1-2k} (\mathbb{R}^n)$ for $k = 0, 1, \ldots, \left[\frac{m+1}{2}\right]$ and for every $(k_1, \ldots, k_n) \in \mathbb{N}_0^n$ such that $k_1 + \cdots + k_n = k$. One can replace the differentiation with respect to $x_1$ by the differentiation with respect to any variable $x_j, j = 1, \ldots, n$:

$$K[D_{x_j} D_\alpha |x|^{2-n}] = \sum_{k=0}^{\left[\frac{m}{2}\right]} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n, k_1 + \cdots + k_n = k} \frac{c_{m+1-k}}{2^k k_1! k_2! \ldots k_n!} \cdot |x|^{2k} D_{x_1}^{2k} \sum_{\alpha = 0}^{\alpha_n} x_1^\alpha .$$

Since the expression $x_j x^\alpha$ determines any monomial of degree $m + 1$, the proof of (i) is finished by the mathematical induction.

A proof of (ii) is almost the same. Since $K[D_{x_j} \log |x|] = x_j$, we notice that $\tilde{c}_1 = 1$. Here we have $n = 2$, hence $\tilde{c}_{m+1} = -2m\tilde{c}_m$. Therefore we conclude that

$$\tilde{c}_{m+1} = (-2)^m m!.$$
From the proved proposition we get a following corollary, which extends the result [3, Corollary 5.20] formulated only for homogeneous harmonic polynomials to any homogeneous polynomials.

**Corollary 1.** Let \( u \in \mathcal{P}_m(\mathbb{R}^n) \) with \( m \geq 1 \).

(i) If \( n > 2 \), then

\[
  u(x) = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n} \binom{m-k}{2k} |x|^{2k} c_{m-2k} K[u_k(D)|x|^{2-n}] c_m|D_2 u| x_1^{2k_1} \cdots x_n^{2k_n} u,
\]

(ii) If \( n = 2 \), then

\[
  u(x) = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n} \binom{m-k}{2k} |x|^{2k} \log |x| c_m|D_2 u| x_1^{2k_1} \cdots x_n^{2k_n} u,
\]

In both cases \( u_0 = u \) and

\[
  u_k = \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n} d_{m,k,k_1,\ldots,k_n} D^{2k_1}_{x_1} \cdots D^{2k_n}_{x_n} u \quad \text{for} \quad k = 1, \ldots, [m/2],
\]

where \( d_{m,k,k_1,\ldots,k_n} \) are some constants being independent of \( u \).

**Proof.** We shall prove (i) of the statement, a proof of (ii) is similar. By Proposition [4] we know that

\[
  \frac{1}{c_m} K[u(D)|x|^{2-n}] = u + |x|^2 u_1
\]

\[
  + \sum_{k=2}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{(k_1, \ldots, k_n) \in \mathbb{N}_0^n} \binom{m-k}{2k} |x|^{2k} c_{m-k} D^{2k}_{x_1} \cdots D^{2k_n}_{x_n} u,
\]

where

\[
  u_1 = \sum_{i=1}^n \frac{c_{m-1}}{2c_m} D^{2}_{x_i} u \in \mathcal{P}_{m-2}(\mathbb{R}^n).
\]
Using again Proposition 1 to the function $u_1$, we get:

$$K[u_1(D)]|x|^{2-n}$$

$$= c_{m-2}u_1 + \sum_{k=1}^{[\frac{m-2}{2}]} \sum_{k_1+\ldots+k_n=k} \frac{c_{m-2-k}}{2^kk_1!\ldots k_n!} |x|^{2k} D_{x_1}^{2k} x_1^{2k_1} x_2^{2k_2} \ldots x_n^{2k_n} u_1$$

$$= c_{m-2}u_1 + \sum_{k=1}^{[\frac{m-2}{2}]} \sum_{k_1+\ldots+k_n=k} \frac{c_{m-2-k}c_{m-1}}{2^kk_1!\ldots k_n!c_m} |x|^{2k} D_{x_1}^{2k} x_1^{2k_1} x_2^{2k_2} \ldots x_n^{2k_n} \left( \sum_{i=1}^{n} D_{x_i}^2 u \right)$$

$$= c_{m-2}u_1 + \sum_{k=1}^{[\frac{m-2}{2}]} \sum_{k_1+\ldots+k_n=k} \frac{c_{m-2-k}c_{m-1}(k+1)}{2^kk_1!\ldots k_n!c_m} |x|^{2k} D_{x_1}^{2(k+1)} x_1^{2k_1} x_2^{2k_2} \ldots x_n^{2k_n} u$$

$$= c_{m-2}u_1 + \sum_{k=2}^{[\frac{m-2}{2}]+1} \sum_{k_1+\ldots+k_n=k} \frac{c_{m-1-k}c_{m-1-k}}{2^kk_1!\ldots k_n!c_m} |x|^{2(k-1)} D_{x_1}^{2k} x_1^{2k_1} x_2^{2k_2} \ldots x_n^{2k_n} u.$$ 

So

$$u_1 = \frac{1}{c_{m-2}} K[u_1(D)]|x|^{2-n} - \sum_{k=2}^{[\frac{m-2}{2}]} \sum_{k_1+\ldots+k_n=k} \frac{c_{m-1-k}c_{m-1-k}}{2^kk_1!\ldots k_n!c_m} |x|^{2(k-1)} D_{x_1}^{2k} x_1^{2k_1} x_2^{2k_2} \ldots x_n^{2k_n} u.$$ 

Putting the above to (6) we obtain

$$\frac{1}{c_m} K[u_0(D)]|x|^{2-n} - \frac{|x|^2}{c_{m-2}} K[u_1(D)]|x|^{2-n}$$

$$= u + \sum_{k=2}^{[\frac{m-2}{2}]} \sum_{k_1+\ldots+k_n=k} \tilde{d}_{m,k,k_1,\ldots,k_n} |x|^{2(k-1)} D_{x_1}^{2k} x_1^{2k_1} x_2^{2k_2} \ldots x_n^{2k_n} u,$$

where

$$\tilde{d}_{m,k,k_1,\ldots,k_n} = \frac{c_m-kc_{m-2} - c_{m-1-k}c_{m-1-k}}{2^kk_1!\ldots k_n!c_m}.$$ 

By the linearity of the Kelvin transform we may write after changing appropriate constants

$$\frac{1}{c_m} K[u_0(D)]|x|^{2-n} + \frac{|x|^2}{c_{m-2}} K[u_1(D)]|x|^{2-n}$$

$$= u + \sum_{k=2}^{[\frac{m-2}{2}]} \sum_{k_1+\ldots+k_n=k} \tilde{d}_{m,k,k_1,\ldots,k_n} |x|^{2k} D_{x_1}^{2k} x_1^{2k_1} x_2^{2k_2} \ldots x_n^{2k_n} u.$$
From the last equation we have

\[
\frac{1}{c_m}K[u_0(D)|x|^{2-n}] + \frac{|x|^2}{c_{m-2}}K[u_1(D)|x|^{2-n}]
\]

\[= u + |x|^4u_2 + \sum_{k=3}^{[\frac{m}{2}]} \sum_{k_1 + \ldots + k_n = k} \tilde{d}_{m,k,k_1,\ldots,k_n} |x|^{2k}D^{2k}_{x_1x_2\ldots x_{2k_n}}u,
\]

where

\[
u_2 = \sum_{k_1 + \ldots + k_n = 2} \frac{c_{m-2} - 2c_{m-3}c_{m-1}}{4k_1!\ldots k_n!c_{m-2}} D^{2}_{x_1x_2\ldots x_{2k_n}}u \in P_{m-4}(\mathbb{R}^n).
\]

Continuing the process with respect to the function \(u_2\) and then with respect to the functions \(u_3, \ldots, u_{[m/2]}\) we conclude finally that

\[
u(x) = \sum_{k=0}^{[\frac{m}{2}]} \frac{|x|^2}{c_{m-2k}}K[u_k(D)|x|^{2-n}].
\]

\[\square\]

**Corollary 2.** Let \(m \geq 2p\) and \(u \in P_m(\mathbb{R}^n)\).

(i) If \(n > 2\), then

\[
\sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)|x|^{2-n}] \in H^p_m(\mathbb{R}^n)
\]

and

\[
\sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)|x|^{2-n}] = u - |x|^{2p}v.
\]

(ii) If \(n = 2\), then

\[
\sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)|x|^{2-n}] \log |x| \in H^p_m(\mathbb{R}^n)
\]

and

\[
\sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)|x|^{2-n}] \log |x| = u - |x|^{2p}v
\]

where in the both cases \(v \in P_{m-2p}(\mathbb{R}^n)\) and

\[
v = \sum_{k=p}^{[\frac{m}{2}]} \sum_{(k_1,\ldots,k_n) \in N_0} \tilde{d}_{m,k,k_1,\ldots,k_n} |x|^{2(k-p)}D^{2k}_{x_1x_2\ldots x_{2k_n}}u,
\]

with some constants \(d_{m,k,k_1,\ldots,k_n} \in \mathbb{C}\).
Proof. We will prove (i). Repeating the proof of Corollary 1 for \( k = 0, \ldots, p - 1 \) we see that
\[
    u(x) = \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)]|x|^{2-n} + |x|^{2p} v(x),
\]
where \( v(x) \) is given by (7). Moreover, since \( u_k \in \mathcal{P}_{m-2k}(\mathbb{R}^n) \), by [3, Lemma 5.15] we get \( K[u_k(D)]|x|^{2-n} \in \mathcal{H}_{m-2k}(\mathbb{R}^n) \). Hence by Lemma 1 we conclude that
\[
    \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)]|x|^{2-n} \in \mathcal{H}_m^p(\mathbb{R}^n).
\]
\[\square\]

Now we are ready to state the main result about the orthogonal projection of homogeneous polynomials onto the space of homogeneous polyharmonic polynomials. The presented result is a polyharmonic version of [3, Theorem 5.18].

**Theorem 1.** Let \( m \geq 2p \) and \( u \in \mathcal{P}_m(\mathbb{R}^n) \). Then the mapping \( Q: \mathcal{P}_m(\mathbb{R}^n) \rightarrow \mathcal{H}_m^p(\mathbb{R}^n) \) of the form
\[
    Q[u](x) = \begin{cases}
        \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)]|x|^{2-n} & \text{for } n > 2, \\
        \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)] \log |x| & \text{for } n = 2
    \end{cases}
\]
is the canonical projection of \( \mathcal{P}_m(\mathbb{R}^n) \) onto \( \mathcal{H}_m^p(\mathbb{R}^n) \), which extends in a natural way to the canonical projection \( \tilde{Q} \) of \( \mathcal{P}_m(\mathbb{C}^n) \) onto \( \mathcal{H}_m^p(\mathbb{C}^n) \).

Proof. We shall prove the case \( n > 2 \). A proof of the case \( n = 2 \) is similar. By Corollary 2 we have a unique decomposition of \( u \in \mathcal{P}_m(\mathbb{R}^n) \) given by
\[
    u(x) = \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)]|x|^{2-n} + |x|^{2p} v(x)
\]
for \( v(x) \) satisfying (7), where \( \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)]|x|^{2-n} \in \mathcal{H}_m^p(\mathbb{R}^n) \) and the last term \( |x|^{2p} v(x) \) does not belong to \( \mathcal{H}_m^p(\mathbb{R}^n) \). Hence the mapping \( Q \) is the canonical projection of \( \mathcal{P}_m(\mathbb{R}^n) \) onto \( \mathcal{H}_m^p(\mathbb{R}^n) \).

Since every polynomial \( u(x) \) for \( x \in \mathbb{R}^n \) extends in a natural way to its complexification \( u(z) \) for \( z \in \mathbb{C}^n \), the mapping \( Q \) extends in a natural way to the canonical projection \( \tilde{Q} \) of \( \mathcal{P}_m(\mathbb{C}^n) \) onto \( \mathcal{H}_m^p(\mathbb{C}^n) \). \[\square\]

4. **Projection of the Complex Homogeneous Polynomials onto the Spherical Polyharmonics**

In this section we derive the projection of the complex homogeneous polynomials onto the space of spherical polyharmonics. A spherical polyharmonic is a restriction of the homogeneous polynomial to the union of rotated spheres \( \hat{S}_p := \bigcup_{l=0}^{p-1} e^{il} S \). The motivation to study
polyharmonic functions on the union of rotated balls \(\hat{B}_p := \bigcup_{i=0}^{p-1} e^{\frac{in\pi}{p}} B\) comes from the Pizzetti-type formula for the operator \(\Delta^p\) given in [12] and [14] (see also [8] and [9]). By this formula, the integral mean of \(u\) over the rotated spheres \(x + \bigcup_{i=0}^{p-1} e^{\frac{in\pi}{p}} S(0, r)\) given by

\[
M_{\Delta^p}(u; x, r) := \frac{1}{p \omega_n} \sum_{l=0}^{p-1} \int_S u(x + e^{\frac{in\pi}{p}}r\zeta) \, dS(\zeta)
\]

has the expansion

\[
M_{\Delta^p}(u; x, r) = \sum_{j=0}^{\infty} \Delta^{pj} u(x) \frac{r^{2pj}}{4pj(n/2)|pj|!},
\]

where \((a)_k := a(a+1) \cdots (a+k-1)\), for \(k \in \mathbb{N}\), is the Pochhammer symbol and \(\omega_n\) denotes the area of the unit sphere in \(\mathbb{R}^n\). Hence, the mean value property holds for every polyharmonic function \(u\) of order \(p\) on the closed rotated balls \(x_0 + \bigcup_{i=0}^{p-1} e^{\frac{ki\pi}{p}} B(0, r)\):

\[
u(x) = \frac{1}{p \omega_n} \sum_{i=0}^{p-1} \int_S u(x + e^{\frac{in\pi}{p}}r\zeta) \, dS(\zeta).
\]

In particular, it means that the value \(u(0)\) of the polyharmonic function \(u\) is uniquely determined by the boundary values on the rotated unit spheres \(\hat{S}_p\):

\[
u(0) = \frac{1}{p \omega_n} \sum_{k=0}^{p-1} \int_S u(e^{\frac{ik\pi}{p}}r\zeta) \, dS(\zeta).
\]

**Definition 3** ([9, Definition 1]). The restriction to the set \(\hat{S}_p\) of an element of \(\mathcal{H}_m^p(C^n)\) is called a spherical polyharmonic of degree \(m\) and order \(p\).

The set of spherical polyharmonics is denoted by \(\mathcal{H}_m^p(\hat{S}_p)\).

The spherical polyharmonics of order 1 are called spherical harmonics and their space is denoted by \(\mathcal{H}_m(S) := \mathcal{H}_m^1(S)\) (see [3, Chapter 5]). Analogously we write \(\mathcal{H}_m(C^n)\) instead of \(\mathcal{H}_m^1(C^n)\).

Let us consider the Hilbert space \(L^2(\hat{S}_p)\) of square-integrable functions on \(\hat{S}_p\) with the inner product defined by

\[
\langle f, g \rangle_{\hat{S}_p} := \frac{1}{p} \int_{\hat{S}_p} \sum_{j=0}^{p-1} f(e^{\frac{ij\pi}{p}}\zeta) \overline{g(e^{\frac{ij\pi}{p}}\zeta)} \, d\sigma(\zeta),
\]

where \(d\sigma\) is a normalized surface-area measure on the unit sphere \(S\).

It is known that \(\mathcal{H}_m^p(\hat{S}_p)\) is finite dimensional [9, Proposition 4] and \(h_m^p := \dim \mathcal{H}_m^p(C^n) = \mathcal{H}_m^p(\hat{S}_p)\), moreover
\[ h_m^p = \begin{cases} \binom{n+m-1}{n-1} & \text{for } m < 2p, \\ \binom{n+m-1}{n-1} - \binom{n+m-2p-1}{n-1} & \text{for } m \geq 2p. \end{cases} \]

**Lemma 3 (\cite{9} Theorem 1).** The space \( L^2(\hat{S}_p) \) is the direct sum of spaces \( H_m^p(\hat{S}_p) \), which is written as

\[
L^2(\hat{S}_p) = \bigoplus_{m=0}^{\infty} H_m^p(\hat{S}_p).
\]

It means that

(i) \( H_m^p(\hat{S}_p) \) is a closed subspace of \( L^2(\hat{S}_p) \) for every \( m \).

(ii) \( H_m^p(\hat{S}_p) \) is orthogonal to \( H_k^p(\hat{S}_p) \) if \( m \neq k \).

(iii) For every \( x \in L^2(\hat{S}_p) \) there exist \( x_m \in H_m^p(\hat{S}_p) \), \( m = 0, 1, \ldots \), such that \( x = x_0 + x_1 + x_2 + \ldots \), where the sum is convergent in the norm of \( L^2(\hat{S}_p) \).

Now we are ready to formulate the counterpart of Theorem 1 for spherical polyharmonics.

**Theorem 2.** Let \( m \geq 2p \) and \( u \in \mathcal{P}_m(\mathbb{C}^n) \). Then the mapping \( \tilde{Q}|_{\tilde{S}_p} : \mathcal{P}_m(\mathbb{C}^n)|_{\tilde{S}_p} \to H_m^p(\hat{S}_p) \) is the orthogonal projection of \( \mathcal{P}_m(\mathbb{C}^n)|_{\tilde{S}_p} \) onto \( H_m^p(\hat{S}_p) \) given by

\[
\tilde{Q}|_{\tilde{S}_p}[u|_{\tilde{S}_p}](z) = \begin{cases} e^{im\pi/p} \frac{\sum_{k=0}^{p-1} u_k(D) |x|^{2-n}/c_{m-2k}}{} & \text{for } n > 2 \\
e^{im\pi/p} \frac{\sum_{k=0}^{p-1} u_k(D) \log |x|/c_{m-2k}}{} & \text{for } n = 2 \end{cases},
\]

where \( z \in \hat{S}_p \subset \mathbb{C}^n \) and \( z = e^{i\pi/p}x \) for some \( l \in \{0, \ldots, p - 1\} \) and \( x \in S \).

**Proof.** We shall prove the theorem for \( n > 2 \). Again, by Corollary 2 we have a unique decomposition of \( u \in \mathcal{P}_m(\mathbb{R}^n) \) given by

\[
u(x) = \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)]|x|^{2-n} + |x|^{2p}v(x)
\]

for \( v(x) \) satisfying \( (7) \), where \( \sum_{k=0}^{p-1} \frac{|x|^{2k}}{c_{m-2k}} K[u_k(D)]|x|^{2-n} \in \mathcal{H}_m^p(\mathbb{R}^n) \)

and the last term \( |x|^{2p}v(x) \) does not belong to \( \mathcal{H}_m^p(\mathbb{R}^n) \). We restrict both sides of the complexification of \( (3) \) to \( e^{i\pi/p}S \subset \mathbb{C}^n \) for some \( l \in \{0, \ldots, p - 1\} \). Since the Kelvin transform preserves points of the unit sphere and the elements of the sum \( (3) \) are homogeneous of degree \( m \), we get

\[
u(e^{i\pi/p}x) = e^{im\pi/p} \sum_{k=0}^{p-1} u_k(D)|x|^{2-n}/c_{m-2k} + |x|^{2p}v(e^{i\pi/p}x).
\]

By \( \cite{9} \) Propositions 2 and 5 \( u|_{\tilde{S}_p} \) is orthogonal to \( H_m^p(\hat{S}_p) \) in \( L^2(\hat{S}_p) \). \( \Box \)
5. Basis of the space $\mathcal{H}_m^n(\mathbb{C}^n)$

In this section we want to provide a basis for the space $\mathcal{H}_m^n(\mathbb{C}^n)$, the space of homogeneous polyharmonic polynomials of order $p$ and degree $m$. For the case $p = 1$ and $n = 2$ it is known that $\mathcal{H}_m(\mathbb{R}^2)$ is the complex linear span of $\{z^m, \bar{z}^m\}$, and for $n > 2$ the basis of $\mathcal{H}_m(\mathbb{R}^n)$ is given in [3, Theorem 5.25]. We now prove the following result:

**Theorem 3.** Set $v_{1,k} = \text{Re} \left( (x_1 + ix_2)^{m-2k}|x|^{2k} \right)$, $v_{2,k} = \text{Im} \left( (x_1 + ix_2)^{m-2k}|x|^{2k} \right)$. Then

$$\{v_{1,k}, v_{2,k} : k = 0, \ldots, p - 1\}$$

is a vector space basis of $\mathcal{H}_m^n(\mathbb{R}^2)$.

Set $v_{k,\alpha(k)} = |x|^{2k}K[D^{\alpha(k)}]|x|^{2-n}$. Then the set

$$\{v_{k,\alpha(k)} : \alpha(k) \in \mathbb{N}_0^n, |\alpha(k)| = m - 2k, \alpha(k)_1 \leq 1, k = 0, \ldots, p - 1\}$$

is a vector space basis of $\mathcal{H}_m^n(\mathbb{R}^n)$ with $n > 2$.

**Proof.** Let $n = 2$. It is known that (see [3, page 82])

$$\mathcal{H}_m(\mathbb{R}^2) = \text{lin}\{v_{1,0}, v_{2,0}\},$$

and (see [3, Theorem 5.25])

$$\mathcal{H}_m(\mathbb{R}^n) = \text{lin}\{K[D^{\alpha}|x|^{2-n}] : |\alpha| = m, \alpha_1 \leq 1\}$$

for $n > 2$.

So, by Lemma [1] we get that every polynomial $q \in \mathcal{H}_m^n(\mathbb{C}^n)$ can be written as a linear combination of the vectors $v_{1,k}, v_{2,k}$ and $v_{k,\alpha(k)}$ when $n = 2$ and $n > 2$, respectively. Moreover, by Lemma [1] the number of these vectors is equal to dimension $h_m^n$ (see also [3, Proposition 9]). Hence these vectors create a basis of $\mathcal{H}_m^n(\mathbb{R}^2)$ and $\mathcal{H}_m^n(\mathbb{R}^n)$ ($n > 2$), respectively.

Consequently, we conclude that

**Corollary 3.** If $n = 2$, then the set

$$\{w_{1,k}(z) = e^{iml\pi/p} \cos(m - 2k)\theta, w_{2,k}(z) = e^{iml\pi/p} \sin(m - 2k)\theta : k = 0, \ldots, p - 1, \text{ for } z = e^{il\pi/p}(\cos \theta, \sin \theta), l \in \{0, \ldots, p - 1\}, \theta \in [0, 2\pi]\}$$

is a vector space basis of $\mathcal{H}_m^n(\hat{\mathbb{S}}_p)$.

Moreover for $n > 2$ a vector space basis of $\mathcal{H}_m^n(\hat{\mathbb{S}}_p)$ is given by the set

$$\{w_{k,\alpha(k)}(z) = e^{iml\pi/p}D^{\alpha(k)}|z|^{2-n} : |\alpha(k)| = m - 2k, \alpha(k)_1 \leq 1, k = 0, \ldots, p - 1, \text{ for } z = e^{il\pi/p}x, l \in \{0, \ldots, p - 1\}, x \in S\}.$$
Proof. By Theorem 3 a basis of $H^p_m(\mathbb{R}^2)$ is given by following polynomials of two real variables $x = (x_1, x_2) \in \mathbb{R}^2$
\[ v_{1,k}(x) = \operatorname{Re} (x_1 + ix_2)^{m-2k}|x|^{2k} \quad \text{and} \quad v_{2,k}(x) = \operatorname{Im} (x_1 + ix_2)^{m-2k}|x|^{2k}, \]
where $k = 0, \ldots, p - 1$.

Using the complexification of polynomials $v_{j,k}(x)$ ($j = 1, 2$, $k = 0, \ldots, l - 1$) we conclude that the set

\[ \{v_{1,k}(z), v_{2,k}(z) : k = 0, \ldots, p - 1, z = (z_1, z_2) \in \mathbb{C}^2\} \]
is a vector space basis of $H^p_m(\mathbb{C}^2)$.

It means that a vector space basis of $H^p_m(\mathcal{S}_p)$ is given by
\[ \{v_{1,k}\mid_{\mathcal{S}_p}, v_{2,k}\mid_{\mathcal{S}_p} : k = 0, \ldots, p - 1\}. \]

Let $w_{1,k} = v_{1,k}\mid_{\mathcal{S}_p}$, $w_{2,k} = v_{2,k}\mid_{\mathcal{S}_p}$. Take any $z \in \mathcal{S}_p \subseteq \mathbb{C}^2$ and observe that $z = e^{il\pi/p}(\cos \theta, \sin \theta)$ for some $l \in \{0, \ldots, p - 1\}$ and $\theta \in [0, 2\pi)$. For such $z$ we get
\[ w_{1,k}(z) = e^{ilm\pi/p}w_{1,k}(\cos \theta, \sin \theta) = e^{ilm\pi/p}\operatorname{Re} e^{i(m-2k)\theta} = e^{ilm\pi/p} \cos(m-2k)\theta \]
and analogously $w_{2,k}(z) = e^{ilm\pi/p} \sin(m-2k)\theta$.

Analogously, for $n > 2$ we define polynomials in $H^p_m(\mathbb{R}^n)$
\[ v_{k,\alpha(k)}(x) = |x|^{2k}K[D^{\alpha(k)}|x|^{2-n}], \]
where $|\alpha(k)| = m - 2k$, $\alpha(k)_1 \leq 1$ and $k = 0, \ldots, p - 1$. Using their complexification $v_{k,\alpha(k)}(z)$ for $z \in \mathbb{C}^n$, by Theorem 3 we conclude that the set
\[ \{v_{k,\alpha(k)}(z) : |\alpha(k)| = m - 2k, \alpha(k)_1 \leq 1, k = 0, \ldots, p - 1\} \]
is a vector space basis of $H^p_m(\mathbb{C}^n)$.

Hence a vector space basis of $H^p_m(\mathcal{S}_p)$ is given by
\[ \{v_{k,\alpha(k)}\mid_{\mathcal{S}_p} : |\alpha(k)| = m - 2k, \alpha(k)_1 \leq 1, k = 0, \ldots, p - 1\}. \]
Any $z \in \mathcal{S}_p \subseteq \mathbb{C}^n$ may be written as $z = e^{il\pi/p}x$ for some $l \in \{0, \ldots, p - 1\}$ and $x \in S$. Let $w_{k,\alpha(k)} = v_{k,\alpha(k)}\mid_{\mathcal{S}_p}$. Then
\[ w_{k,\alpha(k)}(z) = e^{ilm\pi/p}v_{k,\alpha(k)}(x) = e^{ilm\pi/p}D^{\alpha(k)}|x|^{2-n}. \]
\[ \square \]

6. Convergence of the orthogonal series

We shall prove (see Theorem 3.1 in [6]):

**Theorem 4.** Let $\{e_{m,1}, \ldots, e_{m,h^m_m}\}$ be an orthonormal basis of $H^p_m(\mathbb{C}^n)$ with respect to the inner product $\langle \rangle$. Then the series
\[ f(x) = \sum_{m=0}^{\infty} \sum_{j=1}^{h^m_m} a_{m,j} e_{m,j}(x) \]
converges absolutely and uniformly on compact subsets of \( \hat{B}_p(R) \), where

\begin{equation}
R^{-1} = \limsup_{m \to \infty} (||a_m||)^{1/m} \quad \text{and} \quad ||a_m||^2 = \sum_{j=1}^{h_p^m} |a_{m,j}|^2.
\end{equation}

**Proof.** Let \( x \in \hat{B}_p(r) \) for some \( r > 0 \). We have \( e_{m,j}(x) = r^m e_{m,j}(\frac{x}{r}) \), hence by the Cauchy-Schwarz inequality

\[
\left| \sum_{m=0}^{\infty} r^m \sum_{j=1}^{h_p^m} a_{m,j} e_{m,j}(x) \right| \leq \left( \sum_{m=0}^{\infty} r^m \sum_{j=1}^{h_p^m} |a_{m,j}|^2 \right)^{1/2} \left( \sum_{m=0}^{\infty} r^m \right)^{1/2} \leq \left( \sum_{m=0}^{\infty} r^m ||a_m|| \sqrt{h_p^m} \right).
\]

Since \( e_{m,j}(x) \) are homogeneous polynomials of order \( m > 0 \), by Lemma 4 and Lemma 5 in the last inequality we can use the estimation

\[
\sum_{j=1}^{h_p^m} |e_{m,j}(x)|^2 \leq \sum_{j=1}^{h_p^m} |e_{m,j}(\frac{x}{|x|})|^2 = Z_m^p \left( \frac{x}{|x|} \right) = h_p^m.
\]

By the Cauchy-Hadamard theorem we get the formula for the radius \( R \):

\[
R^{-1} = \limsup_{m \to \infty} \left( \sqrt{h_p^m} ||a_m|| \right)^{1/m}.
\]

Moreover

\[
h_p^m = \left( \frac{n+m-1}{n-1} \right) - \left( \frac{n+m-2p-1}{n-1} \right),
\]

so

\[
1 \leq h_p^m \leq \left( \frac{n+m-1}{n-1} \right) \leq (n+m)^n,
\]

hence

\[
\lim_{m \to \infty} \left( \sqrt{h_p^m} \right)^{\frac{1}{m}} = 1,
\]

which completes the proof. \( \square \)

**Remark 3.** The same result holds, if we replace in (10) \( \ell_2 \)-norm \( ||a_m|| \) by an equivalent norm, such as \( \ell_1 \)-norm \( ||a_m||_1 = \sum_{j=1}^{h_p^m} |a_{m,j}|_C \) or \( \ell_\infty \)-norm \( ||a_m||_\infty = \max \{|a_{m,j}|_C : j = 1, \ldots, h_p^m\} \).

**Remark 4.** A similar result given in [6] is improved in [18] by the explicit formula for the zonal polyharmonics studied in the mentioned paper.
7. Relation with the Fischer inner product

This section we start with zonal polyharmonics, let us recall some needed information about them.

By Lemma 3 we may treat $\mathcal{H}^p_m(\hat{S}_p)$ as a Hilbert space with the inner product (8) induced from $q(11)$ on monic polynomials $H$ needed information about them.

By Lemma 3 we may treat $\mathcal{H}^p_m(\hat{S}_p)$ as a Hilbert space with the integral inner product (8) inherited from $\mathcal{H}^p(\hat{S}_p)$ such that (see [9, formula (9)])

\[
(11) \quad q(\eta) = \langle q, Z^p_m(\cdot, \eta) \rangle_{\hat{S}_p} \quad \text{for every} \quad q \in \mathcal{H}^p_m(\hat{S}_p).
\]

**Definition 4** ([9, Definition 2]). The function $Z^p_m(\cdot, \eta)$ satisfying (11) is called a zonal polyharmonic of degree $m$ and of order $p$ with a pole $\eta$.

**Remark 5.** By the property (11), the function $\langle \zeta, \eta \rangle \mapsto Z^p_m(\zeta, \eta)$ defined on $\hat{S}_p \times \hat{S}_p$ is also called a reproducing kernel for the space $\mathcal{H}^p_m(\hat{S}_p)$.

Zonal polyharmonics of order $p = 1$ are called zonal harmonics and we denote them by $Z_m(\cdot, \eta)$ instead of $Z^1_m(\cdot, \eta)$ for $\eta \in S$.

**Lemma 4.** [9] Proposition 7 and Theorem 2. Let $\zeta, \eta \in \hat{S}_p$. Then

(i) $Z^p_m(\zeta, \eta) = \sum_{k=1}^{p^m} e_k(\zeta)e_k(\eta)$, where $\{e_1, \ldots, e_{h^n}\}$ is an orthonormal basis of $\mathcal{H}^p_m(\hat{S}_p)$.

(ii) $Z^p_m(\zeta, \eta) = \sum_{k=0}^{\min\left\{ [m/2], p-1 \right\}} |\zeta^k|_1|\eta|^{2k}Z_{m-2k}(\zeta, \eta)$.

**Lemma 5.** [9] Proposition 8. If $\eta \in \hat{S}_p$, then

(i) $Z^p_m(\cdot, \eta) = \dim \mathcal{H}^p_m(\mathbb{C}^n)$.

(ii) $|Z^p_m(\zeta, \eta)|_C \leq \dim \mathcal{H}^p_m(\mathbb{C}^n)$ for all $\zeta \in \hat{S}_p$.

H. Render in [18] studies the space of all homogeneous polynomials $\mathcal{H}^p_m(\mathbb{R}^n)$ of degree $m$ and polyharmonic order $p$ endowed with the Fischer inner product defined by

\[
(12) \quad \langle P, Q \rangle_F := \sum_{|\alpha| \leq N} \alpha l c_{\alpha} d_{\alpha}
\]

for polynomials $P(x) = \sum_{|\alpha| \leq N} c_{\alpha} x^{\alpha}$ and $Q(x) = \sum_{|\alpha| \leq N} d_{\alpha} x^{\alpha}$ in $\mathbb{R}^n$.

In this section we discuss the relation between the space of polyharmonic polynomials $\mathcal{H}^p_m(\mathbb{C}^n)$ endowed with the Fischer inner product (12), and endowed with the integral inner product (8) inherited from the space $L^2(\hat{S}_p)$.

In the special case $p = 1$ of harmonic polynomials $\mathcal{H}^p_m(\mathbb{C}^n)$ we have the following direct relation between these two inner products

**Lemma 6** ([9, Theorem 5.14], see also [18, Theorem 2.1]). For $u, v \in \mathcal{H}^p_m(\mathbb{C}^n)$ we have

\[
\langle u, v \rangle_F = C \langle u, v \rangle_{\widehat{S}_1},
\]

where $C = C(n, m) = n(n+2) \cdots (n+2m-2)$ is a constant which depends only on the dimension $n$ of the space $\mathbb{C}^n$, and on the degree $m$ of the polynomials in $\mathcal{H}^p_m(\mathbb{C}^n)$. 

It means that these both inner products are the same on the space $\mathcal{H}_m(\mathbb{C}^n)$ up to the constant $C = C(n, m)$. We cannot extend such type result to the space $\mathcal{H}^{p}_m(\mathbb{C}^n)$. To this end take any $u, v \in \mathcal{H}^{p}_m(\mathbb{C}^n)$ and observe that by the Almansi theorem (Lemma 1) we may write
\begin{equation}
(13) \quad u(x) = \sum_{j=0}^{p-1} u_j(x)|x|^{2j} \quad \text{with} \quad u_j(x) = \sum_{|\alpha| = m-2j} a_{j,\alpha} x^\alpha \in \mathcal{H}_{m-2j}(\mathbb{C}^n).
\end{equation}

Analogously
\begin{equation}
(14) \quad v(x) = \sum_{j=0}^{p-1} v_j(x)|x|^{2j} \quad \text{with} \quad v_j(x) = \sum_{|\alpha| = m-2j} b_{j,\alpha} x^\alpha \in \mathcal{H}_{m-2j}(\mathbb{C}^n).
\end{equation}

Calculating the Fischer and integral products we get

**Theorem 5.** Let $u, v \in \mathcal{H}^{p}_m(\mathbb{C}^n)$ be given by (13) and (14), respectively. Then
\begin{equation}
(15) \quad \langle u, v \rangle_{\hat{S}_p} = \sum_{j=0}^{p-1} \sum_{|\alpha| = m-2j} a_{j,\alpha} b_{j,\alpha} \frac{\alpha!}{n(n+2) \cdots (n+2m-2(2j+1))}.
\end{equation}

and
\begin{equation}
(16) \quad \langle u, v \rangle_F = \sum_{j,k=0}^{p-1} \sum_{|\beta| = j} \sum_{|\tilde{\beta}| = k} \sum_{|\alpha| = m-2j} \sum_{|\tilde{\alpha}| = m-2k} (\alpha + 2\beta)! \frac{j! k!}{\beta! \tilde{\beta}!} a_{j,\alpha} b_{k,\tilde{\alpha}}.
\end{equation}

**Proof.** To prove (15), observe that by [9, Proposition 5] $\langle u_j, v_k \rangle_{\hat{S}_p} = 0$ for $j \neq k$. Using this observation, Lemma 6 and (12) we conclude that
\begin{equation}
\langle u, v \rangle_{\hat{S}_p} = \sum_{j,k=0}^{p-1} \langle u_j(x)|x|^{2j}, v_k(x)|x|^{2k} \rangle_{\hat{S}_p} = \sum_{j=0}^{p-1} \langle u_j, v_j \rangle_{\hat{S}_i} = \sum_{j=0}^{p-1} \sum_{|\alpha| = m-2j} a_{j,\alpha} b_{j,\alpha} \frac{\alpha!}{n(n+2) \cdots (n+2m-2(2j+1))}.
\end{equation}

To derive (16), first we calculate
\begin{equation}
|x|^{2j} = (x_1^2 + \cdots + x_n^2)^j = \sum_{\beta_1+\cdots+\beta_n = j} \frac{j!}{\beta_1! \cdots \beta_n!} x_1^{2\beta_1} \cdots x_n^{2\beta_n} = \sum_{|\beta| = j} \frac{j!}{\beta!} x^{2\beta}.
\end{equation}

Hence by (13) and (14) obtain
\begin{equation}
\langle u_j, v \rangle_{F} = \sum_{|\alpha| = m-2j} \sum_{|\beta| = j} \frac{j!}{\beta!} a_{j,\alpha} x^{\alpha + 2\beta}.
\end{equation}
and

\[ v_k(x)|x|^{2k} = \sum_{|\alpha|=m-2k} \sum_{|\beta|=k} \frac{k!}{\beta!} b_{k,\alpha} x^{\alpha+2\beta}. \]

So, by Lemma 6 and (12) we conclude that

\[
\langle u, v \rangle_F = \sum_{j,k=0}^{p-1} \langle u_j(x)|x|^{2j}, v_k(x)|x|^{2k} \rangle_F \\
= \sum_{j,k=0}^{p-1} \sum_{|\beta|=j} \sum_{|\beta|=k} \sum_{|\alpha|=m-2j} \sum_{|\alpha|=m-2k} (\alpha + 2\beta)! \frac{j! k!}{\beta! \alpha!} a_{j,\alpha} \overline{b_{j,\alpha}}.
\]

Directly from the above theorem we get

**Corollary 4.** If \( p > 1 \), then there does not exist a constant \( C = C(n, m, p) \) depending only on \( n, m \) and \( p \) such that

\[
\langle u, v \rangle_F = C\langle u, v \rangle_{\mathcal{S}_p} \quad \text{for every } u, v \in \mathcal{H}_m^p(\mathbb{C}^n).
\]

In particular, if \( u(x) = u_j(x)|x|^{2j} \) and \( v(x) = v_j(x)|x|^{2j} \), then

\[
\langle u, v \rangle_{\mathcal{S}_p} = \sum_{|\alpha|=m-2j} a_{j,\alpha} \overline{b_{j,\alpha}} n(n+2) \cdots (n+2m-2(2j+1))
\]

and

\[
\langle u, v \rangle_F = \sum_{|\alpha|=m-2j} \sum_{|\alpha|=m-2k} \sum_{|\alpha|=m-2k} (\alpha + 2\beta)! \frac{j! k!}{\beta! \alpha!} a_{j,\alpha} \overline{b_{j,\alpha}}.
\]

**Remark 6.** In [18] Render also considers the reproducing kernel \( \tilde{Z}_m^p \) of \( \mathcal{H}_m^p(\mathbb{R}^n) \) with respect to the Fischer product (12), which is defined by

\[
\tilde{Z}_m^p(x, y) := \sum_{j=1}^{h_m^p} Q_m^j(x) Q_m^j(y),
\]

where \( \{Q_m^j(x)\}_{j=1}^{h_m^p} \) is an orthonormal basis of \( \mathcal{H}_m^p(\mathbb{R}^n) \) with respect to the Fischer product (12). In particular he showed the relation between the reproducing kernel \( \tilde{Z}_m^p \) and zonal harmonics \( Z_{m-2k}(x, y) \) (see [18, Theorem 2.4]):

\[
\tilde{Z}_m^p(x, y) = \sum_{k=0}^{\min\{\lfloor m/2 \rfloor, p-1\}} \frac{|x|^{2k} |y|^{2k} Z_{m-2k}(x, y)}{2^k k! n(n+2) \cdots (n+2m-2k-2)}.
\]

Observe that \( \tilde{Z}_m^p(x, y) \) one can treat as a version of zonal polyharmonic \( Z_m^p(\eta, \zeta) \), but with respect to the Fischer product (12) instead of the integral product (8). In particular (17) and (18) corresponds
to properties of zonal polyharmonic $Z^p_m(\eta, \zeta)$ collected in Lemma 4.

Differences between formulas given in Lemma 4 (ii) and in (18) are consequences of the differences between the Fischer and integral products, which are described in Theorem 5.

ACKNOWLEDGEMENT

The authors are grateful to the anonymous referees for the valuable comments and suggestions to improve the paper.

REFERENCES

[1] N. Aronszajn, T. M. Creese, L. J. Lipkin, Polyharmonic Functions, Clarendon Press, Oxford 1983.
[2] S. Axler, W. Ramey, Harmonic Polynomials and Dirichlet-Type Problems, Proceedings of the American Mathematical Society 123 (1995), 3765–3773.
[3] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory, Second edition, Springer-Verlag, New York 2001.
[4] M. Brelot, G. Choquet, Polynômes harmoniques et polyharmoniques, Second colloque sur les équations aux dérivées partielles, Bruxelles 1954.
[5] M. D. Buhmann, Radial Basis Functions. Theory and Implementations, Cambridge University Press, Cambridge 2003.
[6] A. Fryant, M. K. Vemuri, Pythagorean identity for polyharmonic polynomials, Int. J. Math. Math. Sci. 29 (2002), 115–119.
[7] F. Gazzola, H. Ch. Grunau, G. Sweers, Polyharmonic Boundary Value Problems, Springer-Verlag, New York 2010.
[8] H. Grzebula, S. Michalik, A Dirichlet type problem for complex polyharmonic functions, Acta Math. Hungar. 153 (2017), 216–229.
[9] H. Grzebula, S. Michalik, Spherical polyharmonics and Poisson kernels for polyharmonic functions, Complex Var. Elliptic Equ. 64 (2019), 420–442.
[10] O. Kounchev, Multivariate polysplines: applications to numerical and wavelet analysis, Academic Press, San Diego 2001.
[11] E. Lundberg, H. Render, The Khavinson–Shapiro conjecture and polynomial decompositions, J. Math. Anal. Appl. 376 (2011), 506–513.
[12] G. Lysik, On the mean-value property for polyharmonic functions, Acta Math. Hungar. 133 (2011), 133–139.
[13] W.R. Madych, S.A. Nelson, Polyharmonic cardinal splines, J. Approx. Theory 60 (1990), 141–156.
[14] S. Michalik, Summable solutions of some partial differential equations and generalised integral means, J. Math. Anal. Appl. 444 (2016), 1242–1259.
[15] H. Render, A Characterization of the Khavinson-Shapiro Conjecture Via Fischer Operators, Potential Anal. 45 (2016), 539–543.
[16] H. Render, Cauchy, Goursat and Dirichlet Problems for Holomorphic Partial Differential Equations, Comput. Methods Funct. Theory 10 (2011), 519–554.
[17] H. Render, Real Bargmann spaces, Fischer decompositions, and sets of uniqueness for polyharmonic functions, Duke Math. J. 142 (2008), 313–352.
[18] H. Render, Reproducing kernels for polyharmonic polynomials, Arch. Math. 91 (2008), 136–144.
[19] H. S. Shapiro, An algebraic theorem of E. Fischer, and the holomorphic Goursat problem, Bull. London Math. Soc. 21 (1989), 513–537.
[20] J. Siciak Holomorphic continuation of harmonic functions, Ann. Polon. Math. 29 (1974), 67–73.
Faculty of Mathematics and Natural Sciences, College of Science, Cardinal Stefan Wyszyński University, Wóycickiego 1/3, 01-938 Warszawa, Poland

Email address: h.grzebula@uksw.edu.pl

URL: http://www.impan.pl/~slawek