Geometry of (0,2) Landau-Ginzburg Orbifolds

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Abstract
Several aspects of (0,2) Landau-Ginzburg orbifolds are investigated. Especially the elliptic genera are computed in general and, for a class of models recently invented by Distler and Kachru, they are compared with the ones from (0,2) sigma models. Our formalism gives an easy way to calculate the generation numbers for lots of Distler-Kachru models even if they are based on singular Calabi-Yau spaces. We also make some general remarks on the Born-Oppenheimer calculation of the ground states elucidating its mathematical meaning in the untwisted sector. For Distler-Kachru models based on non-singular Calabi-Yau spaces we show that there exist ‘residue’ type formulas of the elliptic genera as well.
1 Introduction

\(N = 2\) supersymmetric theories in two dimensions have been widely investigated ever since the recognition was made as to their relevance to string compactifications down to four dimensions with \(N = 1\) space-time supersymmetry imposed.

Now we have an almost good handle on \((2,2)\) compactifications since efficient machineries like Gepner’s construction or Landau-Ginzburg descriptions are known.

On the other hand, the subject of \((0,2)\) compactifications, despite their relevance to more realistic unifications with gauge groups such as \(SO(10)\) or \(SU(5)\), has remained nearly dormant since several pioneering works appeared \([1–4]\). One reason to hamper its development was the so-called ‘instanton destabilization’ \([5]\). The other was a more technical one; although we can study \((0,2)\) compactifications in the sigma model formulation \([1,2,4]\) where the target space is a Calabi-Yau threefold and the left fermions couple to (the pullback of) a holomorphic vector bundle, preferably, of rank 4 or 5, we did not know non-trivial but nonetheless manageable \((0,2)\) conformal theories that lend themselves to exact calculations.

Recently it has been shown that not only \((2,2)\) models but also \((0,2)\) models admit Landau-Ginzburg descriptions. This was initiated by Witten \([6]\) as an extension of his Calabi-Yau/Landau-Ginzburg scheme and further investigated by Distler and Kachru \([7]\) using the method of \([8]\). In particular, the latter paper paved the way for producing a multitude of concrete examples. Thus we are now in a situation to seriously start exploring various aspects of \((0,2)\) compactifications.

In this article we wish to study the geometrical aspects of \((0,2)\) Landau-Ginzburg orbifolds and discuss the correspondence with \((0,2)\) sigma models. This, we suppose, is reasonable since the corresponding problem in the \((2,2)\) case has attracted much attention over the years. Our approach is based on the elliptic genus for \((0,2)\) models. The theory of elliptic genus was introduced some time ago \([9–11]\) and recently there has been renewal of interest in this subject in the particular context of \((2,2)\) models \([12–16]\). In \((2,2)\) models there exist left and right \(U(1)\) symmetry both of which arise from the left-right \(N = 2\) algebra. In theories like \((0,2)\) sigma model and \((0,2)\) Landau-Ginzburg model, though the left \(N = 2\) symmetry is lost, one still has left \(U(1)\) symmetry in addition to the right \(U(1)\) symmetry which is part of the right \(N = 2\) algebra. Accordingly the elliptic genus can be defined as

\[
Z(\tau, z) = \text{Tr}(-1)^F y^{(J^L)_0} q^{H^L} q^{H^R}, \quad y = e^{2\pi \sqrt{-1} \tau}, \quad q = e^{2\pi \sqrt{-1} \tau} \quad (\text{Im} \tau > 0) \quad (1.1)
\]

where \((J^L,R)_0\) are the left, right \(U(1)\) charge operators and \(H^{L,R}\) are the left, right Hamiltonians. We have set \((-1)^F = \exp[-\pi \sqrt{-1} \{(J^L)_0 - (J^R)_0\}]\). As usual, due to the right supersymmetry \(Z(\tau, z)\) is \(\bar{q}\) independent. We can read off various topological data from
the elliptic genus which can be explicitly constructed for (0,2) Landau-Ginzburg orbifolds as we will demonstrate in this work.

The organization of this article is as follows. In sect.2 the basic and relevant properties of (0,2) sigma model and its elliptic genus are summarized. Sect.3 studies general aspects of (0,2) Landau-Ginzburg orbifolds providing formulas for elliptic genera and related objects which we employ in sect.4 where we investigate Distler-Kachru models and their correspondence to (0,2) sigma models. We give a practical method of computing the generation numbers for a large class of Distler-Kachru models. We pay particular attention to singular Calabi-Yau cases which have so far not been discussed.

2 (0,2) sigma model and its elliptic genus

The purpose of this section is to summarize the properties of (0,2) sigma model which become relevant in later sections, especially when discussing the correspondence with (0,2) Landau-Ginzburg orbifolds. So the most if not all of the materials presented here are relatively standard.

The geometrical data needed for the construction of a (0,2) sigma model are a $D$ dimensional Kähler manifold $(X, g)$ and a rank $r$ holomorphic vector bundle $E$ over $X$ equipped with a Hermitian metric $h$. It is assumed throughout that $r \geq D$. We introduce (0,2) bosonic chiral superfields $\Phi^i (\Phi^{\bar{i}})$ for the local (anti-) holomorphic coordinates of $X$ and fermionic ones $\Lambda^a (\Lambda^{\bar{a}})$ for the local (anti-) holomorphic sections of $E (\bar{E})$. They have the following expansions (in our convention):

\[
\begin{align*}
\Phi^i &= \phi^i + \theta^+ \psi^i + \sqrt{-1} \bar{\theta}^+ \theta^+ \partial_+ \phi^i \\
\Phi^{\bar{i}} &= \phi^{\bar{i}} - \theta^+ \bar{\psi}^{\bar{i}} - \sqrt{-1} \bar{\theta}^+ \theta^+ \partial_+ \phi^{\bar{i}} \\
\Lambda^a &= \lambda^a - \theta^+ l^a + \sqrt{-1} \bar{\theta}^+ \theta^+ \partial_+ \lambda^a \\
\Lambda^{\bar{a}} &= \lambda^{\bar{a}} - \theta^+ \bar{l}^{\bar{a}} - \sqrt{-1} \bar{\theta}^+ \theta^+ \partial_+ \lambda^{\bar{a}},
\end{align*}
\]

where $l^a$ and $\bar{l}^{\bar{a}}$ are auxiliary fields. The Lagrangian density is then given by

\[
\mathcal{L} = 2\sqrt{-1} \int d\theta^+ d\bar{\theta}^+ \frac{\partial K}{\partial \Phi^i} \partial_- \Phi^i - \int d\theta^+ d\bar{\theta}^+ h_{a\bar{b}} \Lambda^a \Lambda^{\bar{b}},
\]

where $K = K(\Phi^i, \Phi^{\bar{i}})$ is a Kähler potential, i.e. $g = \partial_i \partial_\bar{j} K$. The (2.2) case corresponds to setting $E = T$ where $T$ is the holomorphic tangent bundle of $X$.

The elliptic genus is a quantity which conveniently summarize the topological properties of the model and it can be obtained by a standard calculation in the high temperature limit of the path integral $\mathcal{Z}$ [11]. First it must be noted that in order for the operator $(-1)^F$ to make sense we have to assume $c_1(E) - c_1(T) \equiv 0 \pmod{2}$, i.e. the existence of space-time spin structure. If we use the splitting principle and write the total chern classes
of $E$ and $T$ as $c(E) = \prod_{k=1}^{r}(1 + v_k)$ and $c(T) = \prod_{j=1}^{D}(1 + \xi_j)$ then the elliptic genus is given by

$$Z_E(\tau, z) = \int_X \prod_{k=1}^{r} P(\tau, v_k + z) \prod_{j=1}^{D} \frac{\xi_j}{P(\tau, -\xi_j)}, \quad (2.3)$$

where we have introduced the notation $P(\tau, z) = \vartheta_1(\tau, z)/\eta(\tau)$ and the Jacobi theta function $\vartheta_1$ is defined by

$$\vartheta_1(\tau, z) = \sqrt{-1} q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - y q^{n-1})(1 - y^{-1} q^n). \quad (2.4)$$

Our convention here is such that the Witten index in the (2,2) case is given by

$$Z_T(\tau, 0) = (-1)^D \chi, \quad (2.5)$$

where $\chi$ is the Euler characteristic of $X$. It is easy to observe that

$$Z_{E \oplus O^r}(\tau, z) = Z_E(\tau, z) P(\tau, z)^n. \quad (2.6)$$

As is readily shown from (2.3), if the conditions

$$c_1(E) = 0, \quad \text{ch}_2(E) = \text{ch}_2(T) \quad (2.7)$$

are met, the elliptic genus transforms as

$$Z_E \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = \epsilon \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{r-D} e^{2 \pi \sqrt{-1}(r/2)c z^2/(c \tau + d)} Z_E(\tau, z), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}), \quad (2.8)$$

under modular transformations and exhibits the double quasi-periodicity

$$Z_E(\tau, z + \lambda \tau + \mu) = (-1)^{r(\lambda+\mu)} e^{-2 \pi \sqrt{-1}(r/2)(\lambda^2 \tau + 2 \lambda z)} Z_E(\tau, z), \quad \lambda, \mu \in \mathbb{Z}. \quad (2.9)$$

In (2.7), $\text{ch}_2 = \frac{1}{2}(c_1^2 - 2c_2)$ and in (2.8) the phase $\epsilon \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ is determined by

$$P \left( \frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d} \right) = \epsilon \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) e^{2 \pi \sqrt{-1}(1/2)c z^2/(c \tau + d)} P(\tau, z), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}). \quad (2.10)$$

In particular (2.8) means that

$$Z_E(\tau, -z) = (-1)^{r-D} Z_E(\tau, z). \quad (2.11)$$

The sigma model has potentially two kinds of local anomalies. One is the ‘sigma model anomaly’ whose cancellation condition is precisely given by the second equation of
The other is the chiral $U(1)$ anomalies. In fact we have the left and right $U(1)$ currents $J^L, J^R$

\[
(J^L, J^L) = (h_{ab} \lambda^a \lambda^b, 0), \quad (J^R, J^R) = (0, g_{ij} \psi^i \psi^j).
\]

with an anomaly $c_1(E)$, $c_1(T)$ respectively. In particular the vector $U(1)$ current $J^V = J^L - J^R$ can have an anomaly as opposed to the case of $(2,2)$ sigma models. Thus if we demand the absence of the vector $U(1)$ anomaly in addition to (2.7) we are led to the conditions familiar in heterotic string compactifications

\[
c_1(E) = c_1(T) = 0, \quad c_2(E) = c_2(T),
\]

which actually means that all the local anomalies are cancelled. We will assume (2.13) in the remainder of this paper. In particular $X$ is a Calabi-Yau manifold.

The elliptic genus (2.3) can be expanded as

\[
Z_E(\tau, z) = (\sqrt{-1})^{-D} q^{\frac{D}{12}} y^{-\frac{1}{2}} \int_X \left( \prod_{n=1}^{\infty} \Lambda_{-y^a n^{-1}} E \otimes \prod_{n=1}^{\infty} \Lambda_{-y^{-1} q^n} E^* \right.
\]

\[
\left. \otimes \prod_{n=1}^{\infty} S_{q^n T} \otimes \prod_{n=1}^{\infty} S_{q^n T^*} \right) \text{td}(X)
\]

\[
= (\sqrt{-1})^{-D} q^{\frac{D}{12}} y^{-\frac{1}{2}} \left[ \chi_y(E) + q(\sum_{s=0}^{\infty} \{ (-y)^{s+1} \chi(\wedge^s E \otimes E) + (-y)^s \chi(\wedge^s E \otimes (T \otimes T^*)) \}) + \cdots \right],
\]

where

\[
\chi_y(E) = \sum_{s=0}^{r} (-y)^s \chi(\wedge^s E)
\]

and

\[
\Lambda_t E = \sum_{s=0}^{r} t^s(\wedge^s E), \quad S_t E = \sum_{s=0}^{\infty} t^s(S^s E), \quad \text{etc.}
\]

In the second equality of (2.14) we have used the Riemann-Roch-Hirzebruch theorem

\[
\chi(E) := \sum_{l=0}^{D} (-1)^l \dim H^l(X, E) = \int_X \text{ch}(E) \text{td}(X).
\]

Using the Born-Oppenheimer approximation, Distler and Greene [2] identified the ground states of the $(R,R)$ sector, which have energy $((r - D)/12, 0)$, with the cohomology groups $H^l(X, \wedge^s E)$. The correspondence between the two objects is as follows.

\[
q^L = s - \frac{r}{2}, q^R = l - \frac{D}{2} \quad \Leftrightarrow \quad H^l(X, \wedge^s E).
\]
The shift of $U(1)$ charges above is simply due to those of the vacuum $|0\rangle_{R,R}$. This is in agreement with the definition (1.1) and the expansion (2.14) of the elliptic genus.

The $\chi_y(E)$ defined by (2.15) provides us useful information about the ground states of the $(R,R)$ sector or equivalently (part of) massless space-time fermions in the context of string compactification. It will be called ‘$\chi_y$ genus’ following the terminology of Hirzebruch [18]. It should be noted that (2.11) implies that

$$\chi_y(E) = (-1)^{r-D}y^r\chi_y^{-1}(E),$$

which can also be derived from the Serre duality $H^l(X, \wedge^sE) \cong H^{D-l}(X, \wedge^{r-s}E)^{\ast}$. For an irreducible $X$ we list several formulas of $\chi_y$ genera under the conditions (2.13):

$$D = 1 : \quad \chi_y(E) = 0,$$

$$D = 2 : \quad \chi_y(E) = 2(1 + 10y + y^2)(1 - y)^{r-2},$$

$$D = 3 : \quad \chi_y(E) = -\chi(E)y(1 + y)(1 - y)^{r-3},$$

$$D = 4 : \quad \chi_y(E) = (2 + (2r - 8 - \chi(E))y + (8r + 12 - 4\chi(E))y^2 + (2r - 8 - \chi(E))y^3 + 2y^4)(1 - y)^{r-4},$$

$$D = 5 : \quad \chi_y(E) = -\chi(E)y(1 + y)(1 + 10y + y^2)(1 - y)^{r-5},$$

where $r \geq D$ as assumed and

$$D = 3 : \quad \chi(E) = \frac{1}{2} \int_X c_3(E),\quad D = 4 : \quad \chi(E) = 2r - \frac{1}{6} \int_X c_4(E),\quad D = 5 : \quad \chi(E) = \frac{1}{24} \int_X c_5(E).$$

The elliptic genus is uniquely characterized by (2.8), (2.9) and $\chi_y(E)$. Hence for instance

$$D = 1 : \quad Z_{E}(\tau, z) = 0,$$

$$D = 2 : \quad Z_{E}(\tau, z) = Z_T(\tau, z)P(\tau, z)^{r-2},$$

$$D = 3, 5 : \quad Z_{E}(\tau, z) = \frac{\chi(E)}{\chi(T)}Z_T(\tau, z)P(\tau, z)^{r-D}.$$

As is common in the theories of elliptic genera it is possible to consider the Neveu-
Schwarz elliptic genus as well:

\[ Z_E^{NS}(\tau, z) := (\sqrt{-1})^{-r} q^{r/8} y^{r/2} Z_E(\tau, z + \tau/2) \]

\[ = (\sqrt{-1})^{-D} q^{-\frac{2D+r}{24}} \int_X \ch\left( \bigotimes_{n=1}^{\infty} \Lambda_{-yq^{-n/2}} \ E \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-y^{-1}q^{n} - \frac{1}{2}} \ E^* \right) \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T^* \) td(X) \]

\[ = (\sqrt{-1})^{-D} q^{-\frac{2D+r}{24}} [\chi_{y,q}^{NS}(E) + O(q^{3/2})] \]  

where

\[ \chi_{y,q}^{NS}(E) = \chi(O) - q^{1/2}(\chi(E)y + \chi(E^*)y^{-1}) \]

\[ + q(\chi(\Lambda^2 E)y^2 + \chi(\Lambda^2 E^*)y^{-2} + \chi(E \otimes E^*) + \chi(T) + \chi(T^*)) \]

is the collection of terms relevant in considering the massless sectors of string theory in the Born-Oppenheimer approximation. Since \( \chi(O) = 1 + (-1)^D \) and \( \chi(E^*) = (-1)^D \chi(E) \) for an (irreducible) \( D \) dimensional Calabi-Yau manifold \( X \), it follows that

\[ \chi_{y,q}^{NS}(E) = -\chi(E)q^{1/2}(y - y^{-1}) + \chi(\Lambda^2 E)q(y^2 - y^{-2}), \]  

if \( D \) is odd.

### 3 (0,2) Landau-Ginzburg orbifolds and their elliptic genera

The (0,2) Landau-Ginzburg orbifolds were formulated and studied by Distler and Kachru as a natural extension of the (2,2) ones. As in the (2,2) case their constructions exhibit a gratifying correspondence with the (0,2) sigma models as we further elaborate later. Our intention in this section is to make provision for the next section by summarizing the basic properties of (0,2) Landau-Ginzburg orbifolds in a slightly general context while introducing tools we will use in the following.

#### 3.1 General aspects

A (0,2) Landau-Ginzburg model has \( N \) bosonic chiral superfields \( \Phi^i \) and \( M \) fermionic ones \( \Lambda^k \) whose expansions to component fields are as in (2.1). The Lagrangian density is given by

\[ \mathcal{L} = 2\sqrt{-1} \int d\bar{\theta}^+ d\theta^+ \bar{\Phi}^i \partial_+ \Phi^i - \int d\theta^+ d\bar{\theta}^+ \Lambda^k \bar{\Lambda}^k + \int d\theta^+ F_k \Lambda^k + \int d\bar{\theta}^+ F_k^* \Lambda^k, \]  

(3.1)
where \( F_k \)'s are quasi-homogeneous polynomials of \( \Phi^i \)'s satisfying
\[
F_k(x^{\omega_1} \Phi^1, x^{\omega_2} \Phi^2, \ldots, x^{\omega_N} \Phi^N) = x^{1-\rho_k} F_k(\Phi^1, \Phi^2, \ldots, \Phi^N), \quad 1 \leq k \leq M, \tag{3.2}
\]
with \( \omega_i \)'s and \( \rho_k \)'s being suitable rational numbers.

Owing to the quasi-homogeneity of the potentials \( F_k \), this model has both left and right \( U(1) \) symmetries. The assignment of the left and right \( U(1) \) charges to each (non-auxiliary) component field is shown in Table 1.

| \( q^L \) | \( q^R \) | \( q^L - q^R \) |
|----------|----------|------------|
| \( \phi^i \) | \( \omega_i \) | \( \omega_i \) | 0 |
| \( \bar{\phi}^i \) | \( -\omega_i \) | \( -\omega_i \) | 0 |
| \( \psi^i \) | \( \omega_i \) | \( \omega_i - 1 \) | 1 |
| \( \bar{\psi}^i \) | \( -\omega_i \) | \( 1 - \omega_i \) | -1 |
| \( \lambda^k \) | \( \rho_k - 1 \) | \( \rho_k \) | -1 |
| \( \lambda^{\pi} \) | \( 1 - \rho \) | \( -\rho \) | 1 |

Table 1: \( U(1) \) charges of Landau-Ginzburg model

By repeating the arguments in refs. \[8, 12\] the problem of computing \( \bar{Q}_+ \) cohomology, where \( \bar{Q}_+ \) is one of the right \( N = 2 \) supercharge, for this Landau-Ginzburg model boils down to the BRS cohomology theory of the left-moving conformal field theory realized by free ghost system\[†\]:
\[
(\phi^i, \partial_- \bar{\phi}^i, \lambda^k, \lambda^{\pi}) \implies (\gamma^i, \beta^i, b^k, c^k), \quad Q_{\text{BRS}} = \oint dz \sum_{k=1}^{M} F_k(\gamma) b^k, \tag{3.3}
\]
where \( (\gamma^i, \beta^i) \) is a pair of bosonic ghosts while \( (b^k, c^k) \) is a pair of fermionic ghosts with their quantum numbers given by:

| \( q^L \) | \( q^R \) | \( q^L - q^R \) | conformal weight |
|----------|----------|------------|-----------------|
| \( \gamma^i \) | \( \omega_i \) | \( \omega_i \) | \( \omega_i / 2 \) |
| \( \beta^i \) | \( -\omega_i \) | \( -\omega_i \) | \( 1 - \omega_i / 2 \) |
| \( c^k \) | \( 1 - \rho_k \) | \( -\rho_k \) | \( (1 - \rho_k) / 2 \) |
| \( b^k \) | \( -(1 - \rho_k) \) | \( \rho_k \) | \( (1 + \rho_k) / 2 \) |

Table 2: quantum numbers of \( bc\beta\gamma \)

\[†\]In the case of \( (2,2) \) theories this kind of realization was first considered in \[21\].
The left $U(1)$ current and the energy-momentum tensor of the ghost system are given by

\begin{equation}
J = - \sum_{k=1}^{M} (1 - \rho_k) b^k c^k - \sum_{i=1}^{N} \omega_i \beta^i \gamma^i \tag{3.4}
\end{equation}

\begin{equation}
T = \sum_{k=1}^{M} \left[- \frac{1 + \rho_k}{2} b^k \partial c^k + \frac{1 - \rho_k}{2} \partial b^k c^k \right] + \sum_{i=1}^{N} \left[- \frac{\omega_i}{2} \beta^i \partial \gamma^i + \frac{\omega_i}{2} \partial \beta^i \gamma^i \right] \tag{3.5}
\end{equation}

The central charge of $T$ is

\begin{equation}
c = (M - N) + 3 \sum_{i=1}^{N} (1 - \omega_i)^2 - 3 \sum_{k=1}^{M} \rho_k^2 \tag{3.6}
\end{equation}

and the center of $J$ is

\begin{equation}
r = \sum_{k=1}^{M} (1 - \rho_k)^2 - \sum_{i=1}^{N} \omega_i^2. \tag{3.7}
\end{equation}

Since we are concerned with Calabi-Yau/Landau-Ginzburg correspondence we assume that both $c$ and $r$ are integers.

Our main concern is not simply the Landau-Ginzburg model but actually the orbifold theory [19, 20] of it with the relevant group being the $\mathbb{Z}_h$ generated by $\exp[2\pi \sqrt{-1}(J_L)_0]$ where $h$ is the least positive integer such that $h\omega_i$ and $h\rho_k$ are integers. Accordingly we are led to consider the orbifold theory of the above left free ghost system and the associated problem of finding the ground states in BRS cohomology theory.

In the untwisted $(R,R)$ sector, finding the ground states (with energy $((M-N)/12,0)$) in BRS cohomology is easy since we have only to take into account of zero modes. The truncated Fock space and truncated BRS operator are given respectively by

\begin{equation}
\mathcal{F}_* := \bigoplus_{s=0}^{M} \mathcal{F}_s, \quad \mathcal{F}_s := \left\{ \bigoplus_{k_1<k_2<\ldots<k_s} \mathbb{C}[\gamma_0^1, \ldots, \gamma_0^N] c_{k_1}^{k_2} \cdots c_{k_s}^0 |0\rangle \right\}, \tag{3.8}
\end{equation}

and $(Q_{\text{BRS}})_0 = \sum_k F_k (\gamma_0^i) b^k_0$. Note that the grading of $\mathcal{F}_*$ is that of the vector $U(1) q^L - q^R$, while physics uses a bigrading $(q^L, q^R)$. The complex $(\mathcal{F}_*, (Q_{\text{BRS}})_0)$ is known as the Koszul complex of the polynomial ring $R = \mathbb{C}[Z_1, ..., Z_N]$ and the sequence $F_1, ..., F_M \in R$. Thus the problem reduces to computing the Koszul homologies projected to integral values of $q^L$. We can easily see that the 0th homology of the Koszul complex

\begin{equation}
H_0(\mathcal{F}_*) \cong R/I, \quad I = \text{the ideal generated by } F_1, ..., F_M, \tag{3.9}
\end{equation}

is similar to the expression of the chiral ring in (2,2) theory which we have been acquainted with [22]. But this is not the end of the story. Because in general there exist higher homology groups, i.e. the ground states with fermionic ghost $c_0^k$ excitations. Define $\text{depth}_R$
to be the maximal length of regular sequences in $I$. Then it is known \cite{23} that
\begin{equation}
\max \{ n \mid H_n(F_n) \neq 0 \} = M - \text{depth}_I R.
\end{equation}

We assume
\begin{equation}
\text{depth}_I R = N,
\end{equation}
in this paper which is equivalent to the existence of a sequence $G_1, \ldots, G_N$ in $I$ such that
\begin{equation}
\dim \{ p \in C^N \mid G_1(p) = \cdots = G_N(p) = 0 \} = 0.
\end{equation}

In the case of the $(2,2)$ model this assumption reduces to the usual one of an isolated critical point of the superpotential. We emphasize again that the non-triviality of homologies of degrees $\leq M - N$ is notable distinction from the $(2,2)$ model.

| the lowest excitation mode | \gamma^i | \alpha \omega_i - \alpha \omega_i |
|---------------------------|----------------|-----------------
| \beta^i                  | \alpha \omega_i - \lceil \alpha \omega_i \rceil - 1 |
| \epsilon^k              | \lceil \alpha(1 - \rho_k) \rceil - \alpha(1 - \rho_k) |
| b^k                      | \alpha(1 - \rho_k) - \lceil \alpha(1 - \rho_k) \rceil - 1 |

Table 3: The lowest excitation modes in the $\alpha^{th}$ twisted sector

As for the twisted sectors things are more complicated and we have to take into account up to the lowest excitation modes (see Table 3) in order to find the ground states in BRS cohomology since the vacuum state of the $\alpha^{th}$ twisted sector $|0\rangle_{(\alpha)}$ acquires non-trivial quantum numbers:

\begin{align}
Q^L_\alpha &= \sum_k (1 - \rho_k) \ll \alpha(1 - \rho_k) \gg - \sum_i \omega_i \ll \alpha \omega_i \gg, \\
Q^R_\alpha &= \sum_i (1 - \omega_i) \ll \alpha \omega_i \gg - \sum_k \rho_k \ll \alpha(1 - \rho_k) \gg, \\
E_\alpha &= \frac{1}{2} \sum_k \ll \alpha(1 - \rho_k) \gg^2 - \frac{1}{2} \sum_i \ll \alpha \omega_i \gg^2 - \frac{M - N}{8},
\end{align}

where $\ll \theta \gg = \theta - \lceil \theta \rceil - \frac{1}{2}$ and $E_\alpha$ is the deviation from the vacuum energy of the untwisted sector $(M - N)/12$. We refer the reader to refs. \cite{7,8} and sect.4 for illustration of how we can find the BRS non-trivial ground states in the twisted sectors.

\footnote{A sequence $f_1, \ldots, f_k$ of elements of $R$ is called a regular sequence if $f_1$ is not a zero-divisor in $R$ and $f_i$ is not a zero-divisor in $R/(f_1, \ldots, f_{i-1})$ for all $i = 2, \ldots, k$.}
3.2 Elliptic genus

The elliptic genus of (0,2) Landau-Ginzburg orbifold is a straightforward generalization of the (2,2) case \[14, 15\] and is given by

\[
Z_{\text{LG}}(\tau, z) = \frac{1}{h} \sum_{\alpha, \beta = 0}^ {h-1} (-1)^{r(\alpha + \beta + \alpha \beta)} \begin{array}{c}
\beta \\
\alpha
\end{array}(\tau, z),
\]

where

\[
\begin{array}{c}
\beta \\
\alpha
\end{array}(\tau, z) = (-1)^{r \alpha \beta} e^{2\pi \sqrt{-1}(r/2)(\alpha^2\tau + 2\alpha z)} 0 \\
0
\end{array}(\tau, z + \alpha \tau + \beta), \quad \alpha, \beta \in \mathbb{Z},
\]

(3.14)

with

\[
0 \\
0
\end{array}(\tau, z) = \frac{\prod_{k=1}^M P(\tau, (1 - \rho_k)z)}{\prod_{i=1}^N P(\tau, \omega_i z)}.
\]

(3.15)

If the conditions

\[
\sum_{k=1}^M (1 - \rho_k)^2 \sum_{i=1}^N \omega_i^2 = \sum_{k=1}^M (1 - \rho_k) - \sum_{i=1}^N \omega_i
\]

(3.17)

are satisfied, then the elliptic genus obeys

\[
Z_{\text{LG}}\left(\frac{a \tau + b}{c \tau + d}, \frac{z}{c \tau + d}\right) = \varepsilon\left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right)^{M-N} e^{2\pi \sqrt{-1}(r/2)cz^2/(ct+d)} Z_{\text{LG}}(\tau, z), \quad \left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right) \in SL(2, \mathbb{Z}),
\]

(3.18)

\[
Z_{\text{LG}}(\tau, z + \lambda \tau + \mu) = (-1)^{r(\lambda + \mu)} e^{2\pi \sqrt{-1}(r/2)(\lambda^2\tau + 2\lambda z)} Z_{\text{LG}}(\tau, z), \quad \lambda, \mu \in \mathbb{Z}.
\]

These equations are very similar to the ones for (0,2) sigma model and actually they are related in the Distler-Kachru models as we will explain later. When we speak of (0,2) Landau-Ginzburg orbifolds we always assume in what follows that (3.17) is satisfied. Note that the Virasoro central charge (3.6) becomes

\[
c = 3r - 2(M - N),
\]

(3.19)

under (3.17).

Note also that in considering the elliptic genus, the orbifoldization and multiplication by powers of \(P(\tau, z)\)'s essentially commute since

\[
(-1)^{\alpha \beta} e^{2\pi \sqrt{-1}(1/2)(\alpha^2\tau + 2\alpha z)} P(\tau, z + \alpha \tau + \beta) = (-1)^{\alpha + \beta + \alpha \beta} P(\tau, z), \quad \alpha, \beta \in \mathbb{Z}.
\]

(3.20)
As in the sigma model case one may consider the $\chi_y$ genus of the Landau-Ginzburg orbifold by considering an expansion

$$Z_{\text{LG}}(\tau, z) = (\sqrt{-1})^{M-N} q^{M-N/2} y^{-r/2} \left[ \chi_y^{\text{LG}} + \mathcal{O}(q) \right].$$

(3.21)

Apparently $\chi_y^{\text{LG}}$ can be further decomposed into contributions from various twisted sectors:

$$\chi_y^{\text{LG}} = \sum_{\alpha=0}^{h-1} (\chi_y^{\text{LG}})^{(\alpha)}.$$

(3.22)

To compute $(\chi_y^{\text{LG}})^{(\alpha)}$ one may either start from (3.14) or proceed from the outset in the Born-Oppenheimer approximation where it suffices to include the lowest excitation modes; in any case we find after some manipulation that

$$(\chi_y^{\text{LG}})^{(\alpha)} = \left[ (\sqrt{-1})^{M-N} y^{-r/2} \right]^{-1} \exp \left[ -\pi \sqrt{-1} (Q_a^L - Q_a^R) \right] \frac{q^{E_\alpha} y^{Q_\alpha}}{(\prod_k (1 - y^{(1-\rho_k)(\alpha(1-\rho_k)+\alpha(1-\rho_k))})(1 - y^{-1-\rho_k} q^{\alpha(1-\rho_k)+\alpha(1-\rho_k)}) \prod_i (1 - y^{\omega_i q^\alpha} (\alpha + \omega_i) + (1-\alpha - \omega_i)}{\prod_k (1 - y^{\omega_i q^{\alpha(1-\rho_k)-1}} (\alpha + \omega_i) + (1-\alpha - \omega_i))} \right]_{*},$$

(3.23)

where $\{x\} = x - \lfloor x \rfloor$, $\nu_k = 1 - \rho_k$ and $|_*$ means that we extract only terms of the form $q^0 y^{\text{integer}}$ in the expansion. Formula (3.23) is a useful and manageable formula as we exemplify in the case of Distler-Kachru models in the next section and is a substitute for Vafa’s formula [19] which plays an important role in the (2,2) case.

As in the sigma model case the $\chi_y$ genus of the Landau-Ginzburg orbifold is invariant under the duality transformation:

$$\chi_y^{\text{LG}} = (-1)^{M-N} y^r \chi_y^{\text{LG}}.$$

(3.24)

Actually we can say more; for each contribution from the untwisted and twisted sectors we see that

$$(\chi_y^{\text{LG}})^{(0)} = (-1)^{M-N} y^r (\chi_y^{\text{LG}})^{(0)},$$

$$(\chi_y^{\text{LG}})^{(\alpha)} = (-1)^{M-N} y^r (\chi_y^{\text{LG}})^{(h-\alpha)}, \quad \alpha = 1, \ldots, h - 1.$$

(3.25)

### 4 Distler-Kachru Models

In [6] Witten gave an interesting perspective to understand the remarkable but somewhat mysterious correspondence between Calabi-Yau sigma models and Landau-Ginzburg orbifolds which had been known for some time [24, 26]. In his picture the two theories appear
as certain limits of two distinct phases of a unified theory which involves a $U(1)$ gauge connection. This picture can be extended to $(0,2)$ theories \[6, 7\]. In particular, the work of Distler and Kachru \[7\] introduced a concrete construction of $(0,2)$ Landau-Ginzburg models from geometrical data and several examples were analyzed by the method of \[8\].

In this section we wish to perform further analysis of Distler-Kachru models. In the above picture, although more subtle topological quantities will change during the process of ‘phase transition’ \[6,27\], the quantity like elliptic genus is expected to remain unchanged in the whole phase space. Thus we compute the elliptic genera of Distler-Kachru models and compare the results from $(0,2)$ sigma models summarized in sect.2. Since all the examples studied in the literature are for non-singular Calabi-Yau manifolds, we will particularly focus our attention to singular cases.

### 4.1 Distler-Kachru models and their elliptic genera

Distler-Kachru models are special cases of $(0,2)$ Landau-Ginzburg orbifolds discussed in the previous section and are constructed from the following geometrical data \[7\]. Suppose that $X$ is a $D = N - 1 - t$ dimensional complete intersection defined by

$$X = \{ p \in \mathbb{WP}_{w_1, \ldots, w_N}^{N-1} | W_1(p) = \cdots = W_t(p) = 0 \}$$

(4.1)

where $W_j$ is a degree $d_j$ polynomial in the coordinates $Z_1, \ldots, Z_N$ of the weighted projective space $\mathbb{WP}_{w_1, \ldots, w_N}^{N-1}$. Let $E$ be the coherent sheaf on $X$ defined by the following short exact sequence

$$0 \to E \to \bigoplus_{a=1}^{r+1} \mathcal{O}(n_a) \stackrel{f}{\to} \mathcal{O}(m) \to 0$$

(4.2)

with the $n_a$'s and $m$ being positive integers and

$$f(u_1, \ldots, u_{r+1}) = \sum_{a=1}^{r+1} u_a J_a$$

(4.3)

where $J_a$'s are degree $m - n_a$ polynomials without common zeros on $X$. Note that only in favorable situations $X$ is non-singular and $E$ becomes a holomorphic vector bundle over $X$. The anomaly cancellation conditions are tantamount to

$$\sum_i w_i - \sum_j d_j = 0$$

$$\sum_a n_a - m = 0$$

(4.4)

$$\sum_i w_i^2 - \sum_j d_j^2 = \sum_a n_a^2 - m^2.$$
Distler-Kachru models assume these conditions in general and are given by those (0,2) Landau-Ginzburg orbifolds with the following assignment:

\[(F_1, \ldots, F_M) = (J_1, \ldots, J_{r+1}, W_1, \ldots, W_t)\]

\[(\rho_1, \ldots, \rho_M) = \frac{1}{m}(n_1, \ldots, n_{r+1}, m - d_1, \ldots, m - d_t)\]  \hspace{1cm} (4.5)

\[(\omega_1, \ldots, \omega_N) = \frac{1}{m}(w_1, \ldots, w_N)\]

\[h = m.\]

Note that \(M - N = r - D\) and eqs.(4.4) imply (3.17). Thus the elliptic genus of a Distler-Kachru model obeys precisely the same modular transformation laws and the double quasi-periodicity as the ones for (0,2) sigma models. It is easy to see that the (left) central charge of Distler-Kachru model is given by \(c = r + 2D\).

Now that we have identified Distler-Kachru models as special cases of (0,2) Landau-Ginzburg orbifolds we can apply the machinery developed in sect.3. Let us write \(\chi_y^{\text{LG}}\) of the Distler-Kachru model with initial data \((E, X)\) as \(\chi_y(E)^{\text{LG}}\) and similarly for the contributions from twisted sectors. In general we append a suffix ‘LG’ for a quantity computed in the Landau-Ginzburg orbifold calculation. We have performed various calculations of \(\chi_y(E)^{\text{LG}}\) using (3.23) and have found that they always take the forms expected from (0,2) sigma models even if \(X\) is singular. This is the result analogous to the (2,2) case and suggests that in general Distler-Kachru models correspond to those (0,2) sigma models with data \((\tilde{E}, \tilde{X})\) where \((\tilde{E}, \tilde{X})\) is a suitable resolution of \((E, X)\). For instance we have calculated \(\chi_y(E)^{\text{LG}}\) for several Distler-Kachru models with \(D = 2\) and \(X\) singular to find that \(\chi_y(E)^{\text{LG}} = 2(1 + 10y + y^2)(1 - y)^{-2}\) in agreement with the sigma model result (2.20). As for string theoretically more interesting cases of (singular) Calabi-Yau threefolds, some of our results are shown in Table 4–8.
$$\alpha$$  |  $$\chi_y(E)_{LG}^{(\alpha)}$$
--- | ---
0 | $$1 + 72y - 72y^3 - y^4$$
1 | $$y^4$$
2 | 0
3 | 0
4 | $$y^2 - y^3$$
5 | $$-10y + 7y^2$$
6 | $$-7y^2 + 10y^3$$
7 | $$y - y^2$$
8 | 0
9 | 0
10 | -1

$$\chi_y(E)_{LG} = 63y(1 + y)(1 - y)$$

Table 4: $$\chi_y(E)_{LG}$$ of some Distler-Kachru models ($$r = 4$$, singular Calabi-Yau threefolds)
\[
\chi_y(E)_{\mathrm{LG}} = 102y(1+y)(1-y)^2
\]

| \(\alpha\) | \(\chi_y(E)_{\mathrm{LG}}^{(\alpha)}\) |
|---|---|
| 0 | \(1 + 105y - 105y^2 - 105y^3 + 105y^4 + y^5\) |
| 1 | \(-2y^3 + 2y^4 - y^5\) |
| 2 | \(2y^3 - 2y^4\) |
| 3 | \(y^3\) |
| 4 | \(-y^3\) |
| 5 | \(-3y + 6y^2 - 3y^3\) |
| 6 | \(-3y^2 + 6y^3 - 3y^4\) |
| 7 | \(-y^2\) |
| 8 | \(y^2\) |
| 9 | \(-2y + 2y^2\) |
| 10 | \(-1 + 2y - 2y^2\) |

\[
\chi_y(E)_{\mathrm{LG}} = 66y(1+y)(1-y)^2
\]

Table 5: \(\chi_y(E)_{\mathrm{LG}}\) of some Distler-Kachru models \((r = 5, \text{ singular Calabi-Yau threefolds})\)
### Table 6: $\chi(E)_{LG}$ of some Distler-Kachru models ($r = 4$, $t = 1$, singular Calabi-Yau threefolds)

| $(w_1, \ldots, w_5; d)$ | $(n_1, \ldots, n_5; m)$ | $\chi(E)_{LG}$ |
|-------------------------|-------------------------|----------------|
| $(1, 1, 1, 3, 3; 9)$    | $(1, 1, 1, 6; 10)$      | $-126$         |
| $(1, 2, 2, 3; 10)$      | $(1, 1, 2, 6; 11)$      | $-63$          |
| $(1, 2, 2, 5; 12)$      | $(1, 1, 2, 4, 4; 12)$   | $-84$          |
| $(1, 2, 3, 4; 12)$      | $(1, 1, 2, 7; 13)$      | $-72$          |
| $(1, 1, 3, 3, 4; 12)$   | $(1, 1, 3, 7; 13)$      | $-96$          |
| $(1, 2, 2, 7; 14)$      | $(2, 2, 3, 3, 3; 13)$   | $-77$          |
| $(1, 1, 2, 3, 7; 14)$   | $(1, 1, 5, 6; 14)$      | $-132$         |
| $(1, 2, 2, 7; 14)$      | $(1, 1, 2, 9; 15)$      | $-99$          |
| $(1, 1, 2, 3, 7; 14)$   | $(1, 1, 3, 9; 15)$      | $-144$         |
| $(1, 2, 3, 4, 5; 15)$   | $(1, 1, 2, 4, 8; 16)$   | $-72$          |
| $(1, 2, 3, 4, 5; 15)$   | $(1, 1, 1, 4, 10; 17)$  | $-76$          |
| $(1, 1, 1, 3, 9; 18)$   | $(1, 3, 5, 5; 17)$      | $-80$          |
| $(1, 1, 1, 6, 9; 18)$   | $(2, 2, 3, 8; 17)$      | $-243$         |
| $(1, 2, 3, 3, 9; 18)$   | $(2, 2, 3, 4, 6; 17)$   | $-77$          |

### Table 7: $\chi(E)_{LG}$ of some Distler-Kachru models ($r = 5$, $t = 1$, singular Calabi-Yau threefolds)

| $(w_1, \ldots, w_5; d)$ | $(n_1, \ldots, n_6; m)$ | $\chi(E)_{LG}$ |
|-------------------------|-------------------------|----------------|
| $(1, 1, 2, 2, 6; 12)$   | $(1, 1, 2, 2, 3; 11)$   | $-102$         |
| $(1, 1, 1, 3, 6; 12)$   | $(1, 1, 2, 3, 3; 11)$   | $-138$         |
| $(1, 2, 3, 3, 3; 12)$   | $(1, 1, 1, 2, 7; 13)$   | $-66$          |
| $(1, 2, 2, 7; 14)$      | $(1, 1, 2, 3, 4; 13)$   | $-66$          |
| $(1, 2, 3, 4, 5; 15)$   | $(1, 1, 2, 3, 6; 15)$   | $-63$          |
| $(1, 1, 1, 6, 9; 18)$   | $(1, 1, 3, 3, 8; 17)$   | $-240$         |
| $(1, 1, 1, 6, 9; 18)$   | $(1, 1, 1, 3, 12; 19)$  | $-282$         |
\[(w_i; d) = (1, 2, 2, 7; 14)\]
\[(n_i; m) = (1, 1, 2, 3, 4; 13)\]

\[\alpha \chi_0(y) = 80y - y^{-1} - 80y^2 + 2y^2 - 448\]
\[\alpha \chi_1(y) = y^{-1} - 2y^2 - 27\]
\[\alpha \chi_2(y) = y^{-1} - y^2 - 10\]
\[\alpha \chi_3(y) = y^2 - 5\]
\[\alpha \chi_4(y) = 2y^2 + 5\]
\[\alpha \chi_5(y) = 3y^2 + 10y^2 + 35\]
\[\alpha \chi_6(y) = -15y + 15y^{-1} - 23y^2 - 23y^{-2}\]
\[\alpha \chi_7(y) = y - 10y^2 - 3y^2 - 35\]
\[\alpha \chi_8(y) = 0 - 2y^2 - 5\]
\[\alpha \chi_9(y) = 0 - y^2 + 5\]
\[\alpha \chi_{10}(y) = 0 - y^2 + 10\]
\[\alpha \chi_{11}(y) = 0 - y - 2y^2 + 27\]
\[\alpha \chi_{12}(y) = -1 - y - 2y^2 + 80y^{-1} - 2y^2 + 80y^2 + 448\]

Table 8: An example of \(\chi_{NS}^y(E)_{LG} = \chi_0(y) + \chi_1(y)q^{1/2} + \chi_2(y)q\) and its twisted sector contributions for a Distler-Kachru model \((r = 5, \text{singular Calabi-Yau threefolds})\).

### 4.2 Some general remarks on the Born-Oppenheimer calculations

Computations of \(\chi_y(E)_{LG}\) genera are relatively straightforward as we have seen, but they miss finer information about the ground states since they are essentially index objects. If one wishes to know more about the theory one has to perform the Born-Oppenheimer calculations as originally done by [7, 8] although this of course requires much more labor and can only be done by a case-by-case analysis. We already made some comments about such calculations for the untwisted sectors of (0,2) Landau-Ginzburg orbifolds in general. Here we should like to give a few more remarks specific to Distler-Kachru models.

Let us denote the space of the \((R,R)\) ground states in the \(\alpha\)th twisted sector which have \(U(1)\) charges \((q^L, q^R) = (s - r/2, l - D/2)\) by \(\mathcal{H}^l(X, \wedge^s E)^{(\alpha)}\) and set \(\mathcal{H}^l(X, \wedge^s E) := \bigoplus_{\alpha=0}^{m} \mathcal{H}^l(X, \wedge^s E)^{(\alpha)}\). Obviously we have

\[\chi_y(E)_{LG} = \sum_{l=0}^{D} (-y)^l \sum_{s=0}^{r} (-1)^s \dim \mathcal{H}^l(X, \wedge^s E), \quad (4.6)\]

and the CPT invariance implies that the untwisted sector is closed under the Serre duality
transformation:
\[
\dim \mathcal{H}^l(X, \wedge^s E)^{(0)} = \dim \mathcal{H}^{D-l}(X, \wedge^{r-s} E)^{(0)},
\]
while the twisted sectors are related one another by
\[
\dim \mathcal{H}^l(X, \wedge^s E)^{(\alpha)} = \dim \mathcal{H}^{D-l}(X, \wedge^{r-s} E)^{(m-\alpha)}.
\]
These results are consistent with (3.25) and hence (3.24).

Notice that all the elements of \( \mathcal{H}^l(X, \wedge^s E) \) with \( l > s \) have to emerge from twisted sectors.

In sect.3 we related the ground states of the Landau-Ginzburg orbifold in the untwisted sector with the homology of Koszul complex (with integral \( q^L \)). Although Distler-Kachru models are particular examples of Landau-Ginzburg orbifolds, they are constructed from the geometrical data \((E, X)\) and hence it seems reasonable to expect that for Distler-Kachru models this Koszul homology calculations in the untwisted sector have intimate connections to some cohomology calculations in classical algebraic geometry. We should like to spend the rest of this subsection in favor of this expectation.

The ground states in the untwisted sector form the \( q^L \) integer space of the homology of the Koszul complex \((\mathcal{F}_*, Q_{\text{BRS}})\), i.e.
\[
\bigoplus_{l \leq s} \mathcal{H}^l(X, \wedge^s E)^{(0)} \cong H_*(\mathcal{F}_*, Q_{\text{BRS}})_{\text{int}}.
\]

There is a natural decomposition of \( Q_{\text{BRS}} \)
\[
Q_{\text{BRS}} = Q_E + Q_X,
\]
\[
Q_E = \sum_{a=1}^{r+1} J_a(\gamma) b^a, \quad Q_X = \sum_{j=1}^{t} W_j(\gamma) b^{j+r+1},
\]
and hence we have a double complex \((\mathcal{F}_*, Q_E, Q_X)\). Define \( S \) as the graded coordinate ring of \( X \)
\[
S = \bigoplus_l S_l = R/(W_1, \ldots, W_t), \quad R = \mathbb{C}[\gamma^1, \ldots, \gamma^N].
\]

Since the assumption that \( X \) is a complete intersection means that \( W_1, \ldots, W_t \) is a regular sequence in \( R \), we have
\[
H_0(\mathcal{F}_*, Q_X)_{\text{int}} \cong \mathcal{G}_* \equiv \bigoplus_{s=0}^{r+1} \mathcal{G}_s, \quad H_i(\mathcal{F}_*, Q_X) = 0, \quad i > 0,
\]
where
\[ \mathcal{G}_s = \left\{ \bigoplus_{l \in \mathbb{Z}} \omega_{(l)}^{a_1 \cdots a_s} c_{a_1} \cdots c_{a_s} \bigg| \omega_{(l)}^{a_1 \cdots a_s} \in S_{lm+n_{a_1} + \cdots + n_{a_s}} \right\}, \] (4.13)
with \(a_i\)'s running from 1 to \(r+1\). Then, from the standard argument of the spectral sequence\(\|\) we obtain
\[ H_*(\mathcal{F}_*, Q_{\text{BRS}})_{\text{int}} \cong H_*(H_*(\mathcal{F}_{s,*}, Q_X), Q_E)_{\text{int}} \cong H_*(\mathcal{G}_s, Q_E). \] (4.14)

Now introduce recursively the series of coherent sheaves \(E_k\) for \(k = 0, 1, 2, \ldots, \) and \(s = 1, 2, \ldots \) and \(s + k \leq r\) with \(E_0 = E\), \(1E = E\) and \(0E_k = E_k = O(km)\) by the following exact sequences
\[ 0 \to E_k \to \bigoplus_{a_1 < a_2 < \cdots < a_s} O(km + n_{a_1} + \cdots + n_{a_s}) \xrightarrow{f} s^{-1}E_{k+1} \to 0. \] (4.15)
If \(X\) is non-singular then \(E_k \cong \wedge^s E \otimes O(km)\). Thus we have the associated long exact sequence of the cohomology groups \(H^l(X, ^sE)\). Using that
\[ \bigoplus_{a_1 < \cdots < a_s} H^0(X, O(km + n_{a_1} + \cdots + n_{a_s})) \cong \left\{ \sum_{a_1 < \cdots < a_s} f_{a_1 \cdots a_s}(\gamma)c_{a_1} \cdots c_{a_s} \bigg| f_{a_1 \cdots a_s} \in S_{km+n_{a_1} + \cdots + n_{a_s}} \right\}, \] (4.16)
we can see that \(H^l(X, ^sE)\)'s coincide with the homology groups \(H_*(\mathcal{G}_s^{\text{res}}, Q_E)\) of the restricted Koszul complex \((\mathcal{G}_s^{\text{res}}, Q_E)\) defined by
\[ \mathcal{G}_s^{\text{res}} = \bigoplus_{s=0}^r \mathcal{G}_s^{\text{res}}, \]
\[ \mathcal{G}_s^{\text{res}} = \left\{ \bigoplus_{l \leq r-s} \omega_{(l)}^{a_1 \cdots a_s} c_{a_1} \cdots c_{a_s} \bigg| \omega_{(l)}^{a_1 \cdots a_s} \in S_{lm+n_{a_1} + \cdots + n_{a_s}} \right\}, \] (4.17)
\[ Q_E = \sum_{a=1}^{r+1} J_a(\gamma)b^a. \]

Thus we have seen that as a Landau-Ginzburg orbifold Distler-Kachru model in the untwisted sector calculates \(H_*(\mathcal{G}_s, Q_E)\) while the classical algebraic geometry calculates \(H_*(\mathcal{G}_s^{\text{res}}, Q_E)\). In general, it follows that
\[ H^l(X, \wedge^s E)^{(0)} \cong H^l(X, ^sE), \quad \text{for } (l, s) \neq (0, r)(D, 0). \] (4.18)
On the other hand $H^0(X, rE) \cong \mathbb{C}$ appears not in the $\alpha = 0$ sector but in the $\alpha = 1$ sector in the Landau-Ginzburg orbifold computation. For example consider the following ideal case $0 < \omega_i < 1$, for $1 \leq i \leq N$ and $0 < \rho_k < 1$, for $1 \leq k \leq M$. Since the quantum numbers of $|0\rangle_{(i)}$ are $(Q_{1}^{R}, Q_{1}^{I}, E_{1}) = (r/2, -D/2, 0)$, and there are no zero modes in this sector, $|0\rangle_{(i)}$ is the unique ground state of the first twisted sector corresponding to $H^0(X, rE) \cong \mathbb{C}$. In the algebro-geometric calculation $Q_{E}(c^{1}c^{2} \cdots c^{r+1})$ is not the BRS exact element but represents $H^0(X, rE)$ as $(c^{1}c^{2} \cdots c^{r+1})$ is not the element of $\mathcal{G}_{e}^{res}$.

### 4.3 Untwisted Yukawa couplings

The ground states of the untwisted sector can be represented as

$$
\mathcal{H}^l(X, \wedge^{l+k}E)^{(0)} = \left\{ \omega^{a_1 \cdots a_k}(\gamma)c^{a_1} \cdots c^{a_k}|0\rangle_{(0)}|Q_{E}\omega^{a_1 \cdots a_k}(\gamma)c^{a_1} \cdots c^{a_k}|0\rangle_{(0)} = 0 \right\} / \\
\left\{ \omega^{a_1 \cdots a_k}(\gamma)c^{a_1} \cdots c^{a_k}|0\rangle_{(0)} = Q_{E}\omega^{a_1 \cdots a_k+1}(\gamma)c^{a_1} \cdots c^{a_k+1}|0\rangle_{(0)} \right\}.
$$

The product on the zero modes $\gamma^I$, $c^a$ naturally induces a ring structure on the ground states as

$$
\mathcal{H}^{l_1}(X, \wedge^{s_1}E)^{(0)} \otimes \mathcal{H}^{l_2}(X, \wedge^{s_2}E)^{(0)} \longrightarrow \mathcal{H}^{l_1+l_2}(X, \wedge^{s_1+s_2}E)^{(0)}.
$$

Thus we obtain the subring of the chiral ring restricted to the untwisted sector. Note that this ring is a natural extension with fermionic excitations of that of $(2,2)$ Landau-Ginzburg models and only depends on the complex structure of $(X, E)$, i.e. the form of $J_a$ and $W_j$.

### 4.4 Analysis of a rank 5 model

In order to confirm our calculations of $\chi_g$ genera we have performed the Born-Oppenheimer analyses of the $(R, R)$ ground states for several Distler-Kachru models using the methods of [8]. Here we present the result for a rank 5 model with the following data

$$(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = \left( \frac{1}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}, \frac{7}{13} \right)$$

$$(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7) = \left( \frac{1}{13}, \frac{1}{13}, \frac{2}{13}, \frac{2}{13}, \frac{3}{13}, \frac{4}{13}, -\frac{1}{13} \right)$$

$$W = Z_1^{14} + Z_2^7 + Z_3^7 + Z_4^7 + Z_5^7$$

$$(J_1, J_2, J_3, J_4, J_5, J_6) = (Z_2^6, Z_3^6, Z_4^{11}, Z_5^4, Z_5^5, Z_2Z_5).$$

This is the lower example of Table 5. Since the calculations are quite similar to the ones in [8][8], we omit the details.

The quantum numbers of the ground state of the twisted sectors are summarized in Table 5.
Table 9: Vacuum quantum numbers

Note that the quantum numbers of the \((13 - \alpha)\)th twisted sector is the CPT conjugates of those of the \(\alpha\)th twisted sector. Now we look into the ground states in each sector.

\(\alpha = 0\) sector

As already mentioned, in this sector the ground states correspond to the homology group of Koszul complex of \(R = \mathbb{C}[Z_1, .., Z_5]\) and the elements \(J_1, ..., J_6, W\) with integral left \(U(1)\) charge. According to the remarks given in the previous section, the only three homology groups \(H_0(\mathcal{F}_s), H_1(\mathcal{F}_s), H_2(\mathcal{F}_s)\) are nontrivial. The result is summarized in Table 10.

| \(\alpha\) | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|---------|----|----|----|----|----|----|----|
| \(Q_{\alpha}^L\) | -5/2 | 37/26 | 23/26 | -5/26 | -1/26 | -9/26 | -23/26 |
| \(Q_{\alpha}^R\) | -3/2 | -41/26 | -29/26 | -5/26 | -1/26 | 17/26 | 29/26 |
| \(E_\alpha\) | 0 | -1/13 | -3/13 | -1/13 | -2/13 | -3/13 | -4/13 |

Table 10: \(\alpha = 0\) sector

\(\alpha = 1\) sector

In this sector the ground states can be obtained by having the excitation modes \(\gamma_{-1/13}^1, b_{-1/13}^{1,2}, c_{-1/13}^7\) act on the vacuum as shown in Table 11.

| \(H_0(\mathcal{F}_s)\) | \(\mathcal{H}^0(X, \mathcal{O})\) | \(\mathcal{H}^1(X, E)\) | \(\mathcal{H}^2(X, \wedge^2 E)\) | \(\mathcal{H}^3(X, \wedge^3 E)\) |
|----------------|------------------|------------------|------------------|------------------|
| dimension | 1 | 81 | 80 | 0 |
| \(H_1(\mathcal{F}_s)\) | \(\mathcal{H}^0(X, E)\) | \(\mathcal{H}^1(X, \wedge^2 E)\) | \(\mathcal{H}^2(X, \wedge^3 E)\) | \(\mathcal{H}^3(X, \wedge^4 E)\) |
| dimension | 1 | 160 | 160 | 1 |
| \(H_2(\mathcal{F}_s)\) | \(\mathcal{H}^0(X, \wedge^2 E)\) | \(\mathcal{H}^1(X, \wedge^3 E)\) | \(\mathcal{H}^2(X, \wedge^4 E)\) | \(\mathcal{H}^3(X, \wedge^5 E)\) |
| dimension | 0 | 80 | 81 | 1 |

Table 11: \(\alpha = 1\) sector

\(\alpha = 2\) sector

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The allowed excitation modes to create the ground states are \( \gamma_{-2/13}, \gamma_{-1/13}, b_{-2/13}^1, c_{-2/13}^7 \) with \( Q_{BRS} = (\gamma_{-1/13})^2 b_{-2/13}^7 \). We find the ground states in this sector.

\[
\begin{array}{|c|c|c|}
\hline
\text{state} & \gamma_{-2/13}^1 \gamma_{-1/13}^5 |0\rangle (2) & b_{-2/13}^1 \gamma_{-1/13}^5 |0\rangle (1) \\
\hline
\mathcal{H}_l(X, \wedge^s E) & \mathcal{H}_l^1(X, \wedge^4 E) & \mathcal{H}_l^1(X, \wedge^3 E) \\
\hline
dimension & 1 & 2 \\
\hline
\end{array}
\]

Table 12: \( \alpha = 2 \) sector

\( \alpha = 3 \) sector

In this sector we have only one excitation mode \( c_{-1/13}^6 \) to create the ground state.

\[
\begin{array}{|c|c|}
\hline
\text{state} & c_{-1/13}^6 |0\rangle (3) \\
\hline
\mathcal{H}_l(X, \wedge^s E) & \mathcal{H}_l^1(X, \wedge^3 E) \\
\hline
dimension & 1 \\
\hline
\end{array}
\]

Table 13: \( \alpha = 3 \) sector

\( \alpha = 4 \) sector

The excitation modes to create the ground states are \( \gamma_{-2/13}^5 \) and \( c_{-1/13}^5 \).

\[
\begin{array}{|c|c|}
\hline
\text{state} & \gamma_{-1/13}^5 |0\rangle (4) \\
\hline
\mathcal{H}_l(X, \wedge^s E) & \mathcal{H}_l^2(X, \wedge^3 E) \\
\hline
dimension & 1 \\
\hline
\end{array}
\]

Table 14: \( \alpha = 4 \) sector

\( \alpha = 5 \) sector

In this sector, we must take care of the excitation modes \( \beta_{-3/13}^{2,3,4}, c_{-3/13}^{3,4} \) and \( b_{-1/13}^5 \).
The excitation modes to be used are $\beta_{-1/13}^{2,3,4}$, $c_{-1/13}^{3,4}$, $\gamma_{-3/13}^5$ and $c_{-2/13}^6$ and the truncated BRS operator is $Q_{\text{BRS}} = \gamma_{1/13}^2 \gamma_{-3/13}^5 b_{2/13}^6$. The ground states are listed in Table 16.

We omit the computations of the ground states of the remaining sectors which are CPT conjugates of those computed above. We summarize in Table 17 the whole ground states of this Landau-Ginzburg orbifold.
4.5 Residue Formulas

For Distler-Kachru models corresponding to non-singular $X$'s we found another kind of formula for the elliptic genus which we will explain now. It would be nice if we could find a path-integral derivation of this formula.

Let us, for simplicity of presentation, restrict ourselves to the non-singular hypersurface case in which the integers $w_i$ are pairwise coprime. Then the formula is

$$Z_E(\tau, z) = -2\pi \eta(\tau)^2 \text{Res}_{J=0} \left[ \prod_a \frac{P(\tau, n_a J + z)}{P(\tau, m J + z)} \cdot \frac{P(\tau, -d J)}{\prod_i P(\tau, -w_i J)} \right].$$

The reader may easily check that this has the right properties as an elliptic genus under the anomaly free conditions. In particular, we have

$$Z_{T\oplus O}(\tau, z) = -2\pi \eta(\tau)^2 \text{Res}_{J=0} \left[ \prod_i \frac{P(\tau, w_i J + z)}{P(\tau, d J + z)} \cdot \frac{P(\tau, -d J)}{\prod_i P(\tau, -w_i J)} \right].$$

Note that the (2,2) elliptic genus $Z_T(\tau, z)$ can be obtained from this since

$$Z_T(\tau, z) = Z_{T\oplus O}(\tau, z)/P(\tau, z).$$

By introducing the notation (with $x = \exp(2\pi \sqrt{-1} J)$)

$$\text{TD}_x[f(x)] := \text{Res}_{x=1} \left[ f(x) \frac{(1 - x^{-d})}{x \prod_i (1 - x^{-w_i})} \right],$$

the $\chi_y$ genera can be obtained from the above formulas as

$$\chi_y(E) = \text{TD}_x \left[ \frac{\prod_a (1 - y x^{n_a})}{1 - y x^m} \right]$$

and

$$\chi_y(T) = \frac{1}{1 - y} \text{TD}_x \left[ \frac{\prod_i (1 - y x^{w_i})}{1 - y x^d} \right].$$

This type of formulas for the $\chi_y$ genera earlier appeared in [23]. It is an easy calculation to derive the well-known formula

$$\chi = \text{Res}_{J=0} \left[ \frac{\prod_i (1 + w_i J)}{1 + d J} \cdot \frac{d J}{\prod_i (w_i J)} \right],$$

from (4.24) and $2\pi \eta(\tau)^2 = P'(\tau, 0)$.
It may be of interest to consider the connection between these formulas and the ones from the Landau-Ginzburg orbifolds. In (2,2) Landau-Ginzburg orbifold theories, we have the following formulas:

\[
\chi(T) = (-1)^N \prod_{\alpha,\beta=0}^{d-1} y^{\omega_{\alpha}} \prod_{\omega_i \not\in \mathbb{Z}} y^{-\omega_i} \frac{\sin \pi ((\omega_i - 1) t + \beta \omega_i)}{\sin \pi (\omega_i t + \beta \omega_i)}
\]

(4.29)

On the other hand, (4.27) can be rewritten as

\[
\chi(T) = \frac{-1}{1-y} \left[ \prod_{\alpha=0}^{d-1} \text{Res}_{y=x^d=1} \left[ \prod_{\alpha=0}^{d-1} y^{1-(\alpha \omega)} \prod_{\omega_i \in \mathbb{Z}} (1 - y x^{\omega_i}) \right] \right].
\]

(4.30)

In this expression we observe that the last term is the contribution from the untwisted sector while the remaining terms, which arise from the residues at \( J = \pm \sqrt{-1}\infty \), i.e. where the Kähler form takes infinitely large imaginary values, must correspond to the twisted sectors. For instance if \( X \) is given by

\[
X = \{(Z_1, \ldots, Z_d) \in \mathbb{CP}^{d-1} \mid Z_1^d + \cdots + Z_d^d = 0\} =: X_d,
\]

(4.31)

then

\[
\chi(T)^{(\alpha)} = \begin{cases} 
\sum_{p=0}^{d-2} y^p \sum_{m=0}^{p} (-1)^m \binom{d}{m} \binom{d-1 + dp - (d-1)m}{dp - (d-1)m}, & \alpha = 0, \\
(-1)^d y^{d-\alpha-1}, & 1 \leq \alpha \leq d - 1.
\end{cases}
\]

(4.32)

Three more examples are given in Table 18.
Starting from (4.22) one can compute $Z_{NS}(\tau,z)$ and hence $\chi_{y,q}^{NS}$. One easily confirms the agreement between thus obtained $\chi_{y,q}^{NS}$ and (2.24) using

\[
\begin{align*}
\chi(O) &= \text{TD}_x[1] = 1 + (-1)^D, \\
\chi(E) &= \text{TD}_x[\sum_a x^{na} - x^m], \\
\chi(E^*) &= \text{TD}_x[\sum_a x^{-na} - x^{-m}], \\
\chi(\wedge^2 E) &= \text{TD}_x[\sum_{a<b} x^{na+nb} - x^{2m}], \\
\chi(\wedge^2 E^*) &= \text{TD}_x[\sum_{a<b} x^{-na-nb} - x^{-2m}], \\
\chi(E \otimes E^*) &= \text{TD}_x[(\sum_a x^{na} - x^m)(\sum_a x^{-na} - x^{-m})], \\
\chi(T) &= \text{TD}_x[\sum_i x^{wi} - x^{-d} - 1], \\
\chi(T^*) &= \text{TD}_x[\sum_i x^{-wi} - x^{-d} - 1],
\end{align*}
\]  

(4.33)

For instance, if $X = X_d$ one finds from the Landau-Ginzburg orbifold computation that

Table 18: $\chi_y(T)$ of Landau-Ginzburg orbifolds corresponding to non-singular Calabi-Yau threefolds

| $\alpha$ | $\chi_y(T)^{(\alpha)}$ |
|---------|-----------------------|
| 0       | $1 + 149y + 149y^2 + y^3$ |
| 1       | $-y^3$ |
| 2       | 0 |
| 3       | $-y^2$ |
| 4       | 0 |
| 5       | $-y$ |
| 6       | 0 |

$\chi_y(T) = 102y(1+y)$

| $\alpha$ | $\chi_y(T)^{(\alpha)}$ |
|---------|-----------------------|
| 0       | $1 + 145y + 145y^2 + y^3$ |
| 1       | $-y^3$ |
| 2       | 0 |
| 3       | $-y^2$ |
| 4       | 0 |
| 5       | 0 |
| 6       | $-y$ |
| 7       | 0 |
| 8       | 0 |
| 9       | $-1$ |

$\chi_y(T) = 148y(1+y)$

$\chi_y(T) = 144y(1+y)$
the nonvanishing contributions to $\chi_{y,q}^{NS}$ are given by

$$\chi_{y,q}^{NS(0)} = (-1)^d \chi_{y^{-1},q}^{NS(d-1)}$$

$$= 1 + \left( \frac{2d - 1}{d} - d^2 \right) yq^{1/2}$$

$$+ \left\{ \left( \frac{3d - 1}{2d} - d \left( \frac{2d}{d+1} \right) + \frac{d^2(d^2 - 1)}{4} \right) y^2 - d \left( \frac{2d^2 - 2}{d-1} \right) + 2d^2 \right\} q$$

(4.34)

$$\chi_{y,q}^{NS(1)} = (-1)^d \chi_{y^{-1},q}^{NS(d-2)} = y^{-1} q^{1/2} - d^2 q$$

$$\chi_{y,q}^{NS(2)} = (-1)^d \chi_{y^{-1},q}^{NS(d-3)} = y^{-2} q .$$

The interested reader may compare this result with the above residue formulas.

5 Concluding Remarks

In this paper we calculated the elliptic genera of (0,2) Landau-Ginzburg orbifolds and the associated $\chi_y$ genera. We found that they are precisely in the forms expected from (0,2) sigma models even if $X$ is singular. As mentioned in the text this leads to a natural question: is there a suitable resolution $(\tilde{X}, \tilde{E})$ of $(X, E)$ so that the (0,2) Landau-Ginzburg orbifold actually describes a sigma model with data $(\tilde{X}, \tilde{E})$? To answer this we have to know if there is, for at least $D \leq 3$, a resolution $(\tilde{X}, \tilde{E})$ preserving the anomaly cancellation conditions:

$$c_1(\tilde{E}) = c_1(\tilde{T}) = 0 , \quad c_2(\tilde{E}) = c_2(\tilde{T}) .$$

Regarding a similar problem in the (2,2) case, see [30].

In (2,2) compactifications there have been remarkable exact calculations of Yukawa couplings by the discovery of mirror symmetry [31]. It would be nice if we can see an equally exciting development for the (0,2) case as well in the near future.

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