Gravitational waves and geometrical optics in scalar-tensor theories

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Abstract. The detection of gravitational waves (GWs) propagating through cosmic structures can provide invaluable information on the geometry and content of our Universe, as well as on the fundamental theory of gravity. In order to test possible departures from General Relativity, it is essential to analyse, in a modified gravity setting, how GWs propagate through a perturbed cosmological space-time. Working within the framework of geometrical optics, we develop tools to address this topic for a broad class of scalar-tensor theories, including scenarios with non-minimal, derivative couplings between scalar and tensor modes. We determine the corresponding evolution equations for the GW amplitude and polarization tensor. The former satisfies a generalised evolution equation that includes possible effects due to a variation of the effective Planck scale; the latter can fail to be parallely transported along GW geodesics unless certain conditions are satisfied. We apply our general formulas to specific scalar-tensor theories with unit tensor speed, and then focus on GW propagation on a perturbed space-time. We determine corrections to standard formulas for the GW luminosity distance and for the evolution of the polarization tensor, which depend both on modified gravity and on the effects of cosmological perturbations. Our results can constitute a starting point to disentangle among degeneracies from different sectors that can influence GW propagation through cosmological space-times.
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1 Introduction

The ΛCDM model of cosmology (see e.g. [1] for an introduction) offers a well defined, simple framework for analysing current cosmological data, and it agrees with observations. In this set up, a cosmological constant $\Lambda$ is responsible for the present-day acceleration of our universe. Nevertheless, the extreme smallness of the value of $\Lambda$ with respect to all other energy scales one encounters in particle physics [2], as well as anomalies in observational results (the most recent being the so-called $H_0$ tension, see e.g. the assessment in [3]), motivate the exploration of alternative scenarios, possibly based on modifications of General Relativity (GR). In modified gravity, additional degrees of freedom are introduced, whose dynamics drive present-day cosmological acceleration (see e.g. [4–6] for reviews). The minimal possibility for modifying gravity is to consider scalar-tensor theories, where a single scalar field is added to the spin-2 degrees of freedom characterising GR. Thanks to interactions with
itself and with spin-$2$ tensor modes (or matter fields), scalar-tensor theories can exhibit self-accelerating solutions, where the scalar profile sources accelerating cosmologies with no need of cosmological constant (see e.g. [6] for a review). Moreover, scalar fifth forces can be suppressed thanks to powerful screening mechanisms, which are able to hide the effects of a light scalar nearby sources (see e.g. [7, 8] for reviews on chameleon and Vainshtein screening mechanisms).

The rapidly developing field of gravitational wave cosmology offers new avenues for testing modified theories of gravity (see e.g. the specific analyses of [9–14]). The single event GW170817 [15–18] and its EM counterpart [19] impose strong constraints on modified theories of gravity predicting a speed of gravitational waves different than light [20–23] (see also the analysis in [24, 25]). In general, the physics of standard sirens offer great promises for cosmology [26–32], and studies of cosmological parameter estimation have been carried on for 2nd generation GW experiments (see e.g. [33]), for ET (see [34–36]), for LISA and space-based detectors [37–40]. In the future, GW observables related with GW frequency, chirp mass, and luminosity distance of coalescing events will be able to further constrain (or discover) modified gravity effects with GW observations, see e.g. [40–46] for works discussing the potential of GW cosmology in probing modified gravity.

The purpose of this work is to develop a systematic and consistent approach for studying the propagation of GWs around arbitrary space-times, based on the separation by high and low frequency modes pioneered by Isaacson [47, 48] (see also [49, 50] for early works on the subject). This formalism is at the basis of a geometrical optics approach to GW propagation; we extend it for accounting the possibility of having rapidly moving scalar fluctuations, with non minimal couplings to the metric. Within this framework, in Sections 2 and 3 we derive general formulas for describing the propagation of GWs, whose structure does not depend on the specific choice of the scalar-tensor theory under consideration, nor on the particular background over which the high-frequency GW is travelling. We only need to make (physically well founded) assumptions on the structure of the EMT controlling the propagation of high-frequency fluctuations\(^1\), and then apply our arguments within a geometrical optics approximation. Besides well-known modified gravity consequences for the evolution of the amplitude of tensor fluctuations – related with non-conservation of the effective Planck mass along the null GW geodesics – we also point out potentially interesting effects associated with the failure of parallelly transport the polarization tensors along the GW geodesics. Although these latter effects can be vanishingly small for GWs propagating through smooth, homogeneous space-times, they can instead contribute to the evolution of the polarization tensors through the inhomogeneous Universe. GW evolution equations within GR, which include effects of non-parallel transport for the polarization tensors have been studied in a framework beyond geometrical optics, see e.g. [11, 49–52], associating it with (higher order) GW lensing effects (see e.g. [53–55]). Interestingly, we find that these effects can be present also within a geometrical optics approximation in theories of modified gravity. Over the past decades, various classes scalar-tensor theories have been applied to cosmology: Brans-Dicke [56]; K-essence [57]; Galileons [58], Horndeski [59–61], beyond-Horndeski [62, 63], and DHOST [64–70]. These are increasingly complex theories with specific derivative couplings between scalar and tensor degrees of freedom. Our general formalism can be applied to any of these theories (and possibly beyond, if one admits Lorentz-violating scenarios). For definiteness, in Section 4 we apply our approach to the kinetic-gravity braiding

\(^1\)We focus on set-up with luminal speed for the spin-2 modes, as suggested by the GW170817 observations.
set-up developed in [71], analysing the differences with respect to GR for what respects the evolution of GWs. Finally, in Section 5 we apply these formulas to GW propagation through a perturbed space-time, to understand the role of modified gravity effects for propagation in an inhomogeneous universe.

In fact, we wish to close our Introduction with more detailed arguments for motivating our study. The physics of GWs travelling through cosmological distances can provide valuable information on gravitational interactions and on the properties of space-time between the emission and detection of GWs. In the literature, the effect of cosmological perturbations on the propagation of GWs has been often neglected. On the other hand, we find it timely to investigate these subtle effects to understand possible contaminations and degeneracies, so to obtain a more reliable estimates of cosmography. Obviously a similar analysis, for the correlated fluctuations in luminosity distance of the electromagnetic spectrum, has already and widely been discussed in literature, e.g. see [72–76]. Recently there have been several initial attempts to investigate the Integrated-Sachs Wolfe (ISW) effect on GWs from supermassive black hole mergers and in particular its impact on the system’s parameter estimation [77], the ISW of a primordial stochastic background [78–80] and for astrophysical stochastic background in [81–91]. In these works the authors consider the presence of inhomogeneities in the matter distribution and allow to probe GW’s sources on cosmological, galactic and sub-galactic scales, peculiar velocity [92–94] and lensing effects [27, 95–101]. It is worth noticing that environmental effects can also influence estimates of the luminosity distance [102]. Precisely, these sources are affected by coherent peculiar velocity of the merging at low redshift and weak gravitational lensing by intervening inhomogeneities in the cosmic mass distribution. Both these systematic errors could introduce a misidentification of the host’s redshift. Consequently, changes of typically a few percent (but occasionally much larger) in the flux are introduced which do not significantly affect the redshift, providing a source of noise in the luminosity distance - redshift relation [34, 95]. Very recently, in [103], the authors discuss the effect of cosmological perturbations and inhomogeneities on estimates of the luminosity distance of black hole (BH) or neutron star (NS) binary mergers through gravitational waves. They applied the “Cosmic Rulers” formalism [104] and considered the observer frame as reference system and they derived a different expression wrt [77], which is correct for the effect of large-scale structures on GW waveforms, accounting for lensing, Sachs-Wolfe, integrated Sachs-Wolfe, time delay and volume distortion effects, and evaluate their importance for future GW experiments. [103] showed that the amplitude of the corrections is important and cannot be negligible for future interferometers as the ET, DECIGO and Big Bang Observer (BBO). From [103], the additional luminosity distance uncertainty, arising because of the inclusion of perturbations, has a peak at low-z due to velocity contributions that surpasses the predicted measurement errors for all the experiments considered here. However, peculiar velocity effects rapidly decrease as z increases. On the other hand, lensing, which is an integrated effect, increases with the redshift of the source z, and its amplitude is of a factor ∼ 10 smaller than ET forecast precision (∼ 2 for DECIGO). For BBO, the correction to the luminosity distance is consistently twice the predicted errors, making it a very relevant correction, that one will need to take into account. Making use of the weak-lensing magnification effect on a GW from a compact binary object, in [32] is showed that it is possible to discriminate the concordance ΛCDM cosmological model and up-to-date competing alternatives as dynamical dark energy models (DE) or modified gravity theories (MG) parametrised with the usual two free phenomenological functions that modify the perturbed Einstein equations relating the matter density contrast to the lensing and the

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Newtonian potential (e.g. see [105]). Finally, for the effect of lensing magnification, in [95], the authors pointed out that for the uncertainty in the distance to an ensemble of GW’s sources is not completely correct to use the standard deviation of the lensing magnification distribution divided by the square root of the number of sources. They showed that by exploiting the non-Gaussian nature of the lensing magnification distribution, it is possible to improve this distance determination, typically by a factor of 2 to 3.

2 General formalism

In this Section we develop a general formalism aimed to describe the propagation of high-frequency gravitational waves (GWs), in the framework of scalar-tensor theories.

While the evolution of spin-2 and spin-0 degrees of freedom decouple around conformally flat space-times (as for example homogeneous FRW universes), more care is needed when describing GW propagation around perturbed backgrounds – above all in scalar-theories with kinetic mixing between different fields, motivated by recent approaches to the dark energy problem. To address this topic in general terms, in this Section we propose an approach based on an expansion of the evolution equations in an high-frequency parameter $\epsilon$. Such approach makes use of a convenient unitary gauge to deal with scalar fluctuations. It does not require us to specify the exact structure of the scalar-tensor theory we consider, but only at which order in the small parameter $\epsilon$ it contributes to the energy momentum tensor controlling the high-frequency fluctuations. The general set of evolution equations we obtain in this Section shall be then applied in Section 3 to a specific geometrical optics Ansatz.

2.1 Linearized equations of motion: high and low frequency modes

Geometrical optics for gravitational waves (GWs) is a well developed subject, which started with the systematic analysis of Isaacson [47, 48], and that is now part of well-known textbooks discussing GW physics (see e.g. [106]). It corresponds to the limit where GWs are characterized by a short wavelength, while the background space-time, over which the wave propagates, varies over much larger scales. Such distinction between two characteristic length scales allows one to define a hierarchy and a ‘small’ parameter used for a derivative expansion. Most works consider geometrical optics for scalar-tensor systems in which the scalar is minimally coupled with gravity (see e.g. the book [107] and references therein). On the other hand, for analysing more modern scalar-tensor theories aimed to explain dark energy, the standard formalism is generally not sufficient, since it does not satisfactorily catch possible effects associated with derivative self-interactions and with non-minimal couplings of scalar to tensor degrees of freedom. The latter scenarios are well motivated by theories of dark energy exhibiting self-accelerating solutions (see e.g. [6] for a review). Our purpose in this Section is to provide an extension able to eventually include also these models (and possibly additional ones), trying to be as general and flexible as possible.

In the system we consider, the metric $g_{\mu \nu}$ satisfies the gravitational field equations

$$R_{\mu \nu} = \frac{8\pi G_N}{c^4} \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right)$$

where the left hand side (LHS) contains only geometrical quantities depending on the metric (the Ricci tensor), while the right hand side (RHS) contributions from the extra scalar coupled with the spin-2 excitations, and possibly additional matter fields. We concentrate all the effects of ‘modified gravity’ in the RHS of the equation.
We assume that the tensor $g_{\mu\nu}$ can be separated into a slowly varying background $\bar{g}_{\mu\nu}$, and high frequency small fluctuations $h_{\mu\nu}$, which represent the GWs propagating over the smooth space-time:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \text{with} \quad |h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|.$$  \hfill (2.2)

The background metric is characterized by the length scale $L_B$, while the wavelength of the wave is $\lambda$, such that the derivatives of these quantities scale as

$$\partial \bar{g} \sim \frac{1}{L_B}, \quad \partial h \sim \frac{1}{\lambda} \quad \text{and} \quad \lambda \ll L_B.$$  \hfill (2.3)

In this context, the background metric $\bar{g}_{\mu\nu}$ is general as long as it satisfies Eq. (2.2) and Eq. (2.3). In particular, on cosmological scales it does not necessarily describe an isotropic and homogeneous Universe. We shall make use of this fact in Section 5.

Our system contains also a scalar field $\varphi$, whose value can be separated in a smooth contribution $\bar{\varphi}$, plus a rapidly varying fluctuation

$$\varphi = \bar{\varphi} + \varphi_r,$$  \hfill (2.4)

analogously to the metric decomposition in Eq. (2.2). We assume that the typical wavelength of the scalar fluctuation $\varphi_r$ of order $\lambda$ and the variation length scale of $\bar{\varphi}$ of order $L_B$. As discussed in the Introduction, for the class of scalar-tensor theories we are interested in, the background scalar field acquires a non-null vacuum expectation value ($\text{vev}$) $\bar{\varphi}$, associated with the physics governing the late-universe cosmological acceleration. The smooth $\text{vev}$ $\bar{\varphi}$ normally depends on coordinates, and its gradient allows us to define a vector

$$\bar{v}^\mu = \nabla^\mu \bar{\varphi},$$  \hfill (2.5)

slowly varying over the background. For cosmological applications such vector is typically time-like, $\bar{v}^\mu \bar{v}_\mu < 0$.

We plug the decomposition of the metric Eq. (2.2) in the field equations of Eq. (2.1). We organize the expansion of the Ricci tensor as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + O(h^3),$$  \hfill (2.6)

where $R_{\mu\nu}^{(1)}$ and $R_{\mu\nu}^{(2)}$ denote respectively linear and second order terms in $h_{\mu\nu}$ and are of the order $\sim h/\lambda^2$ and $\sim h^2/\lambda^2$ respectively in a derivative expansion. On the other hand, $\bar{R}_{\mu\nu}$ only depends on $\bar{g}_{\mu\nu}$. The RHS of Eq. (2.1) may depend on the metric and on the scalar and matter fields. We can apply the same reasoning as for the Ricci and expand it at second order in the metric fluctuations. In this way Eq. (2.1) becomes

$$\bar{R}_{\mu\nu} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} = \frac{8\pi G_N}{c^4} \left[ \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \bar{T} \right) + \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{(1)} + \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{(2)} \right].$$  \hfill (2.7)

### 2.2 On the averaging procedure

It is possible to separate between high and low-frequency modes by averaging Eq. (2.7) over length scales large compared to $\lambda$ and small compared to $L$, i.e. over volumes of size $l$ with

$$\lambda \ll l \ll L.$$
This procedure has been known in literature as ADM-averaging scheme and is well described in [47]. The effect of such averages is to extract the slowly varying part of a quantity as its contribution is almost constant throughout the volume of integration, while the rapidly oscillating one averages out to zero. We can find the equation of motion of the high-frequency part by subtracting from Eq. (2.7) the slow part extracted through the averaging. In particular, in Eq. (2.7) we have

i) Barred quantities that survive the averaging, e.g. \( \langle \bar{R}_{\mu\nu} \rangle_I = \bar{R}_{\mu\nu} \).

ii) Linear quantities in \( h_{\mu\nu} \) that do not survive the averaging, e.g. \( \langle R^{(1)}_{\mu\nu} \rangle_I = 0 \).

iii) Quadratic quantities in \( h_{\mu\nu} \) that may survive the averaging. For instance, when meeting a quadratic contribution as \( h_{\mu\nu} h_{\rho\sigma} \), a mode with a high frequency wave-vector \( k_1 \) from \( h_{\mu\nu} \) could combine with a mode with a high wave-vector \( k_2 \approx -k_1 \) from \( h_{\rho\sigma} \) to give a low frequency wave-vector mode.

Using these prescriptions, after taking the average of Eq. (2.7), we can split Eq. (2.7) into

\[
\bar{R}_{\mu\nu} = \left( \bar{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \bar{T} \right) - \left( \langle R^{(2)}_{\mu\nu} \rangle_I \right) + \left( \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{(2)} \right) \tag{2.8}
\]

\[
R^{(1)}_{\mu\nu} = \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{(1)} - \left[ R^{(2)}_{\mu\nu} \right]_{\text{high}} + \left( T_{\mu\nu}^{(2)} - \frac{1}{2} g_{\mu\nu} T^{(2)} \right)_{\text{high}} \tag{2.9}
\]

where the over-script \( \text{high} \) means \( [R^{(n)}_{\mu\nu}]_{\text{high}} = R^{(n)}_{\mu\nu} - \langle R^{(n)}_{\mu\nu} \rangle_I \).

2.3 The derivative expansion

At the light of these arguments, we can define separate expansions for the quantities entering in the field equations:

i) An expansion in small fluctuations \( h_{\mu\nu} \), and (independently)

ii) an expansion in derivatives of the high-frequency modes, controlled by the small parameter

\[ \epsilon \equiv \lambda/L_B \tag{2.10} \]

For simplicity, from now on we choose units of length such that \( L_B = 1 \).

The mutual relationship between the two parameters is set by Eq. (2.8). If the background curvature is dominated by the matter contribution, Eq. (2.8) sets

\[ \frac{1}{L_B^2} = \frac{h^2}{\lambda^2} + \text{(matter contribution)} + \text{(dark energy contribution)} \gg \frac{h^2}{\lambda^2} \tag{2.11} \]

meaning that the hierarchy between the amplitude of GW and ratio of GW/background characteristics wavelengths is \([106]\)

\[ 1 \gg \frac{\lambda}{L_B} \gg h \quad \text{thus} \quad \epsilon \gg h \tag{2.12} \]

Note that this is not always the case: for instance, we can have situations where the background curvature is determined by the gravitational waves content. In this case the relationship between the two parameters would be \( \epsilon = h \) \([106]\). However, one of our goals is to
study the propagation of GWs through the cosmic inhomogeneities where the background space-time is completely determined by the large-scale structures present in the Universe. In this case the hierarchy Eq. (2.12) holds. The $\epsilon$-expansion is meant to single out high-frequency contributions to the equations: each derivative of high-frequency fluctuations ($h_{\mu\nu}$, $\varphi_r$) collects a factor $1/\epsilon$, which allows one to clearly separate independent contributions to the evolution equations, which are controlled by powers of $\epsilon$.

The hierarchy Eq. (2.12) allows us to discard the quadratic terms in $h_{\mu\nu}$ of Eq. (2.8) and Eq. (2.9), and expand what is left in powers of $\epsilon$ because $\epsilon \gg h \epsilon \gg h^2 \gg \cdots \gg h^2$. We have the background metric field equation

\[
\bar{R}_{\mu\nu} = \frac{8\pi G_N}{c^4} \left( \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \right),
\]

which is the fully non-linear Einstein equation solved by the slowly-varying metric $\bar{g}_{\mu\nu}$, when turning off the high-frequency fluctuations. The linearized equations for the perturbations are

\[
\left[ R_{\mu\nu}^{(1)} \right]_{1/\epsilon^2} = 0,
\]

\[
\left[ R_{\mu\nu}^{(1)} \right]_{1/\epsilon} = \frac{8\pi G_N}{c^4} \left[ \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{(1)} \right]_{1/\epsilon},
\]

\[
\left[ R_{\mu\nu}^{(1)} \right]_{\epsilon^0} = \frac{8\pi G_N}{c^4} \left[ \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{(1)} \right]_{\epsilon^0},
\]

the subscript $\epsilon^p$ means that the equality holds at that order in the expansion in power of $\epsilon$.

The limit of geometrical optics consists in focusing on the previous equations at leading and next-to-leading order in an $\epsilon$ expansion: that is, on Eq. (2.14) and Eq. (2.15). For sake of clearness, we write the RHS of the equations above explicitly

\[
\left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)^{(1)} = T_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu} T^{(1)} - \frac{1}{2} h_{\mu\nu} \bar{T} + \frac{1}{2} g_{\mu\nu} h^{\alpha\beta} T_{\alpha\beta}
\]

(2.17)

where we defined

- $\bar{T}_{\mu\nu}$ is the background stress-energy tensor, i.e. made with $\bar{g}_{\mu\nu}$ and $\bar{T} = \bar{g}^{\mu\nu} \bar{T}_{\mu\nu}$,

- $T_{\mu\nu}^{(1)}$ is the linear in $h_{\mu\nu}$ stress-energy tensor and $T^{(1)} = \bar{g}^{\mu\nu} T_{\mu\nu}^{(1)}$.

However, the last two terms of Eq. (2.17) do not contain derivatives of the metric perturbation $h_{\mu\nu}$, therefore they will not contribute to Eq. (2.15) but only to Eq. (2.16), namely

\[
\left[ T_{\mu\nu}^{(1)} - \frac{1}{2} \bar{g}_{\mu\nu} T^{(1)} - \frac{1}{2} h_{\mu\nu} \bar{T} + \frac{1}{2} \bar{g}_{\mu\nu} h^{\alpha\beta} T_{\alpha\beta} \right]_{1/\epsilon^1} = \left[ T_{\mu\nu}^{(1)} - \frac{1}{2} \bar{g}_{\mu\nu} T^{(1)} \right]_{1/\epsilon^1},
\]

(2.18)

therefore we can rewrite Eq. (2.15) as

\[
\left[ R_{\mu\nu}^{(1)} \right]_{1/\epsilon^1} = \frac{8\pi G_N}{c^4} \left[ T_{\mu\nu}^{(1)} - \frac{1}{2} \bar{g}_{\mu\nu} T^{(1)} \right]_{1/\epsilon^1}.
\]

(2.19)
2.4 On the structure of the energy-momentum tensor $T_{\mu\nu}$

Usually, when describing propagation of GWs over cosmological distances, it is assumed that the energy-momentum tensor (EMT) $T_{\mu\nu}$ experienced by GWs is smooth, and does not affect the high-frequency GW evolution in the limit of geometrical optics (see e.g. [106]). Under such hypothesis, both the RHS of Eq. (2.14) and of Eq. (2.19) vanish, and the EMT does not contribute to the GW evolution in the geometrical optics limit. But this hypothesis can be too restrictive in scalar-tensor theories, where derivative scalar self-interactions can lead to derivative contributions to the EMT involving high-frequency modes. As an example, the EMT associated with a cubic Galileon model described by the Lagrangian density

$$L = -\frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi \right) \Box \phi$$

contains second covariant derivatives of the scalar, $\nabla_\mu \nabla_\nu \phi$ that – through the Christoffel symbols contained in the $\nabla_\mu \nabla_\nu$ operator – lead to first derivatives of the high-frequency metric fluctuations. (We will study in more detail this and other models in Section 4.)

In this case, as well as in other modified gravity scenarios, one then expects contributions to the EMT at next-to-leading order in an $\epsilon$-expansion, affecting a geometrical optics description. For the rest of this Section and in Section 3, therefore, we assume to have a non-vanishing contribution to the RHS of eq Eq. (2.19), associated with single derivatives of the high-frequency modes. Without further specifying the theory one considers, we can then derive general consequences of this hypothesis for what respects GW propagation in the framework of geometrical optics.

Eq. (2.16) at order $\mathcal{O}(\epsilon^0)$, as well as other equations characterized by higher powers of $\epsilon$, will not be considered, since they do not contribute in the limit of geometrical optics (where $\epsilon \ll 1$). When expanded at order $\mathcal{O}(\epsilon^0)$, Eq. (2.16) acquires extra contributions after performing a coordinate transformation, and it is not easily related to a physical observable. This fact is discussed in [47] for the case of GR. In next Section, we show that the same argument holds also in the more general scenario we study.

2.5 Gauge choice and evolution equations

We now discuss how to choose a convenient gauge choice for high frequency fluctuations, and the resulting equations of motion for the propagating modes. We shall adopt a unitary gauge that sets to zero high frequency fluctuations associated with the scalar $\phi_r$. This choice turns out to be convenient for physically motivating our Ansatz, and the applications we develop in Section 3. Note that, within this gauge, the dark energy scalar field can still be characterized by slowly varying perturbations.

2.5.1 A convenient gauge for theories with a single extra high-frequency mode

We make the hypothesis that the scalar-tensor system one considers transforms in the standard way under diffeomorphism transformations. As discussed in [47, 106], gauge transformations can be organized in inverse powers of an expansion in $\epsilon$. Calling $\xi^\mu$ the high-frequency part of the the coordinate transformation $x^\mu \rightarrow x^\mu + \xi^\mu$, the high frequency GW metric fluctuations transform in curved space as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \left( \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \right),$$

(2.21)

For simplicity we shall only allow for contributions to the EMT with a single derivative acting on high frequency fields, i.e. at order $1/\epsilon$. As it will become clear in section 3, the presence of two or higher derivatives in such fields could affect our hypothesis that gravitons propagate on null geodesics, and at the speed of light.
where the bars on $\nabla_\mu$ mean covariant derivatives with respect to the background curved metric $\bar{g}_{\mu\nu}$. From now on, space-time indexes are always raised and lowered with the background metric $\bar{g}_{\mu\nu}$. As the decomposition Eq. (2.2) has to hold also in the new coordinate system, we require that $|\nabla_\mu \xi_\nu| \lesssim h$, $|\xi_\nu| \lesssim hL$. If we restrict ourselves only to such diffeomorphisms, then under the transformation $x^\mu \to x^\mu + \xi^\mu$ the slowly varying background is left unchanged. It is convenient to work with the tensor

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} h,$$  

(2.22)

with $h_{\mu} = h$, transforming under gauge transformations as

$$h_{\mu\nu} \to \hat{h}_{\mu\nu} + \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu - \bar{g}_{\mu\nu} \nabla_\rho \xi_\rho.$$  

(2.23)

In the scalar sector, the high-frequency fluctuation transforms under diffeomorphisms as

$$\varphi_r \to \varphi_r + \bar{v}^\rho \xi_\rho,$$  

(2.24)

where we used the slowly varying vector associated with the space-time derivative of the background scalar profile, $\bar{v}_\mu = \nabla_\mu \bar{\varphi}$ (recall the definition in Eq. (2.5)).

The high-frequency part of the linearized Ricci tensor and of the EMT are gauge invariant up to next-to leading order in the $1/\epsilon$ expansion [47, 106]. Indeed, under a gauge transformation they transform as (if we consider the Ricci tensor for example, it would be the same with the EMT perturbations)

$$R^{(1)}_{\mu\nu} \to R^{(1)}_{\mu\nu} + L_\xi \bar{R}_{\mu\nu},$$  

(2.25)

where $\bar{R}_{\mu\nu}$ is the one related the background manifold and $L_\xi$ is the Lie derivative along the vector field $\xi$. In general the Lie derivatives will be different from zero, in fact $L_\xi \bar{R}_{\mu\nu} \sim \epsilon^0$, therefore $R^{(1)}_{\mu\nu} \to R^{(1)}_{\mu\nu}$ only at $1/\epsilon^2$ and $1/\epsilon$ orders. This means that Eq. (2.14) and Eq. (2.19) are gauge invariant, however Eq. (2.16) is not, thus is not related to any physical observable.

The reason behind this is that, on a scale of distance of order $\lambda$, space appears locally flat and the curvature is locally gauge invariant. As long as $\lambda \ll L_B$, perturbations do not have any long-wavelength component and this local behavior carries over to curved background, to give a global gauge invariance [47] which is a result of our high-frequency assumptions. This fact ensures that the form of Eq. s(2.14)–(2.19) does not depend on the used coordinate system.

Since we will never use quantities at $O(\epsilon^0)$, we adopt the following prescription in our computation: we keep only terms at $1/\epsilon^2$ and $1/\epsilon$ orders, and neglect those at $\epsilon^0$ onwards.

As anticipated, we work in a convenient unitary gauge, setting to zero the high frequency scalar fluctuations $\varphi_r$. We can do so making use of the scalar transformation rule Eq. (2.24) and choosing

$$\xi_\mu = - \left( \frac{\bar{v}_\mu}{\bar{v}^\rho \bar{v}_\rho} \right) \varphi_r + \chi_\mu,$$  

(2.26)

where $\chi_\mu$ is any vector transverse to the vector $v^\mu$:

$$\bar{v}^\rho \chi_\rho = 0.$$  

(2.27)

We can use $\chi_\rho$ to impose further conditions on $\hat{h}_{\mu\nu}$: in fact $\chi_\rho$ is associated with a residual gauge freedom after fixing a unitary gauge. In particular we can use it to fix the metric perturbation to be transverse. Namely, exploiting the following transformation

$$\bar{\nabla}_\nu \hat{h}_{\mu\nu}^{(B)} \Rightarrow \bar{\nabla}_\nu \hat{h}_{\mu\nu}^{(A)} = \bar{\nabla}_\nu \hat{h}_{\mu\nu}^{(B)} + \bar{\nabla}_\rho \bar{\nabla}_\nu \chi_\mu - \bar{R}_{\mu\nu} \chi_\nu,$$  

(2.28)
where the suffixes $B$ and $A$ indicate the fields before and after applying the gauge transformation, we can choose $\chi_\mu$ such that

$$\nabla^\mu \hat{h}^{(A)}_{\mu\nu} = 0.$$  \hfill (2.29)

Since we focus on high frequency modes, we can neglect the part proportional to $\bar{R}_{\mu\nu}$ in Eq. (2.28). In order to ensure condition Eq. (2.29), we impose

$$\nabla^\rho \nabla_\rho \chi_\mu = -\nabla^\nu \hat{h}^{(B)}_{\mu\nu}.$$  \hfill (2.30)

Notice that this condition leaves again a fraction or residual gauge: $\chi_\mu \rightarrow \chi_\mu + \sigma_\mu$ with $\chi_\mu$ a vector satisfying $\nabla^\mu \sigma_\mu = \Box \sigma_\mu = 0$.

By contracting both sides of Eq. (2.30) with $\bar{v}^\mu$, we discover that it is compatible with Eq. (2.27) only if

$$\nabla^\nu \left( \bar{v}^\mu \hat{h}^{(B)}_{\mu\nu} \right) = 0,$$  \hfill (2.31)

where again we use the fact that $\bar{v}^\mu$ is slowly varying and thus we can bring it inside the covariant derivative. The condition Eq. (2.31) naturally leads to the definition of the divergence-less vector $c^\mu$

$$c^\nu = \bar{v}^\mu \hat{h}_{\mu\nu}.$$  \hfill (2.32)

Using the residual gauge condition associated with the vector $\sigma_\mu$, we can make our desired choice for the vector $c_\mu$, fixing the last remaining gauge freedom. Notice, on the other hand, that whatever our initial choice for $c_\mu$ is, this vector shall have to satisfy conditions of compatibility with the metric equations of motion, implying that in general its value will not be covariantly preserved over the GW null geodesics. We shall concretely discuss in section 3 how the $c_\mu$-evolution is controlled by the equations of motion for the high-frequency metric fields.

### 2.6 Summary of the gauge conditions and of the relevant equations of motion

To briefly summarize, our system satisfies a unitary gauge for the high-frequency perturbations of the scalar field

$$\varphi_r = 0$$  \hfill (2.33)

and a transverse (but not traceless) gauge for the fast moving part of the metric fluctuations:

$$\nabla^\mu \hat{h}_{\mu\nu} = 0$$  \hfill (2.34)

As a final condition, that exhausts our gauge freedom, we impose

$$\bar{v}^\mu \hat{h}_{\mu\nu} = c_\nu \quad \text{with} \quad \nabla^\mu c_\mu = 0,$$  \hfill (2.35)

where the initial value of $c_\mu$ can be chosen arbitrarily and its evolution along GW geodesics will be controlled by the condition of compatibility with the equations of motion for the high-frequency metric fields.

The number of propagating degrees of freedom (dofs) can be counted as follows. We start from 11 dofs: 1 in the scalar, 10 in the symmetric tensor $h_{\mu\nu}$. We have 1 gauge condition Eq. (2.33), 4 gauge conditions Eq. (2.34), 3 independent gauge conditions Eq. (2.35) ($c_\nu$ is divergenceless). In total, we generically have $11 - 1 - 4 - 3 = 3$ independent propagating dofs, as expected in a scalar-tensor theory of gravity.
We shall use this gauge fixing and rewrite the relevant equations of motion Eqs. (2.14)–(2.19), they become

\[
\left[ \Box \hat{h}_{\mu\nu} \right]_{1/\epsilon^2} = 0, \quad (2.37)
\]
\[
\left[ \Box \hat{h}_{\mu\nu} \right]_{1/\epsilon^1} = -\frac{16\pi G_N}{c^4} \left[ T^{(1)}_{\mu\nu} \right]_{1/\epsilon^1}. \quad (2.38)
\]

These are the evolution equations for the high frequency modes in our setting. Given that the only high frequency fields are the metric fluctuations \( \hat{h}_{\mu\nu} \), the EMT appearing in Eq. (2.38) is built in terms of single derivatives of these quantities, that select the \( 1/\epsilon \) contributions to the equations. (The concrete structure of \( T^{(1)}_{\mu\nu} \) depends of course on the scalar-tensor theory one considers.)

Since we work in an unitary gauge for the high frequency scalar fluctuations, the metric fluctuations \( \hat{h}_{\mu\nu} \) contain the extra degree of freedom which we expect to generically propagate in our system. Consequently, it does not correspond to a pure spin-2, transverse-traceless fluctuations of GR.

The compatibility of equations of motion and gauge conditions leads to an additional relation. We apply a covariant derivative \( \bar{\nabla}^\mu \) on both sides of Eq. (2.38), and we select the \( 1/\epsilon^2 \) contributions. In the LHS we can interchange the order of covariant derivatives (the error being of order \( 1/\epsilon \) hence negligible) and find \( \bar{\nabla}^\mu \bar{\nabla}^\rho \hat{h}_{\rho\nu} = 0 \) for the transverse gauge condition Eq. (2.34). In the RHS, we obtain a condition corresponding to the conservation of the energy momentum tensor:

\[
\left[ \bar{\nabla}^\mu T^{(1)}_{\mu\nu} \right]_{1/\epsilon^2} = 0 \quad (2.39)
\]

As typical in scalar-tensor systems, we expect such condition to be equivalent to the conditions provided by the high-frequency contributions to the scalar field equation in unitary gauge. Therefore, the three dynamical equations, at separate orders in \( \epsilon \) expansion, are

\[
\left[ \nabla^\rho \nabla_\rho \hat{h}_{\mu\nu} \right]_{1/\epsilon^2} = 0, \quad (2.40)
\]
\[
\left[ \nabla^\rho \nabla_\rho \hat{h}_{\mu\nu} \right]_{1/\epsilon^1} = -\frac{16\pi G_N}{c^4} \left[ T^{(1)}_{\mu\nu} \right]_{1/\epsilon}, \quad (2.41)
\]
\[
\left[ \nabla^\mu T^{(1)}_{\mu\nu} \right]_{1/\epsilon^2} = 0. \quad (2.42)
\]

We now analyse all these equations in the framework of geometrical optics. Our considerations will be sufficiently general to be applied to any scalar-tensor system with a high-frequency contribution to its EMT, as described above.

\textsuperscript{3} The linearized Ricci tensor, neglecting its contribution at \( \epsilon^0 \) order, in this gauge reads

\[
R^{(1)}_{\mu\nu} = -\frac{1}{2} \left( \Box \hat{h}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \Box \hat{h} \right). \quad (2.36)
\]
3 Geometrical optics with a single extra degree of freedom

The analysis of Section 2 sets the stage to consistently treat the evolution of high frequency fluctuations in scalar-tensor settings, without specific hypothesis on the scalar-tensor theory and the background one considers. The effects of ‘modified gravity’ in the high-frequency field dynamics are enclosed in contributions to the EMT, built in terms of high-frequency fluctuations. The scope of this Section is to apply this formalism to a framework of geometrical optics. We adopt a specific Ansatz that allows us to follow the evolution of the GW amplitude, phase, and GW polarization tensor, as a function of certain combinations of the EMT. We determine modified evolution equations for these quantities with respect to GR, and discuss the potential physical consequences.

We find that, for scalar-tensor theories of the kind we are interested on, the GWs follow null geodesics, but the evolution equation for the GW amplitude is modified with respect to GR, leading to the phenomenon of non-conservation of the effective Planck mass. This is a well known property found in various works that study GW propagation in FRW space-times: our results generalize those findings to more general settings.

Interestingly, we also find that the polarization tensor is generally not parallely transported along the GW null geodesics. Possible implications of this fact for the evolution of GWs will then be discussed analysing specific examples in the next Sections.

3.1 Ansatz and evolution equations

We start defining the geometrical optic Ansatz we adopt. The metric fluctuations can be expressed in terms of an amplitude and a phase as

\[ \hat{h}_{\mu\nu} \equiv [A_{\mu\nu} + \epsilon \cdots + \ldots] e^{i\theta/\epsilon}, \]

where all the parts with dots are higher order in the \( \epsilon \) expansion described in Section 2, and will be neglected in the geometrical optics approximations we adopted here. The tensor \( A_{\mu\nu} \) contains both spin-2 and spin-0 contributions. Since they are part of the same metric tensor \( \hat{h}_{\mu\nu} \) in the unitary gauge we adopt, we make the hypothesis that they have the same phase, hence they follow the same geodesics. (In section 3.3 we also investigate at what extent, within this hypothesis, the evolution of tensor and scalar modes can be studied independently.)

The tensorial quantity \( A_{\mu\nu} \) is decomposed into an amplitude \( A \) and a tensor polarization \( e_{\mu\nu} \) with unit norm

\[ A_{\mu\nu} \equiv A e_{\mu\nu}, \]

with

\[ A = \sqrt{A_{\mu\nu} A^{\mu\nu}}, \quad e_{\mu\nu} = \frac{A_{\mu\nu}}{A}, \quad e_{\mu\nu} e^{\mu\nu} = 1. \]

In the unitary gauge we adopt, the high frequency contribution to the EMT is built with derivatives of the metric fluctuations, therefore we can assume it have the same phase as the metric fluctuation (we will give an explicit example in section 4). We then adopt the following Ansatz for the EMT at order \( 1/\epsilon \) (the overall coefficient is chosen such to simplify the following equations):

\[ \left[ T^{(1)}_{\mu\nu} \right]_{1/\epsilon} \equiv \frac{i}{\epsilon} \frac{c^4}{16 \pi G_N} t_{\mu\nu} e^{i\theta/\epsilon}. \]

\(^4\)In fact, we made the choice to include all the effects of modified gravity on the RHS of Einstein equations, while on the LHS we keep the traditional Einstein tensor.
As explained in the previous Sections, the contributions that scale as $1/\epsilon$ collect the terms proportional to first derivatives of the high-frequency fields. The tensor $t_{\mu \nu}$, consequently, is proportional to the coefficients of these terms. Importantly, notice that we collect on the coefficient in the RHS of Eq. (3.4) the quantity $G_N = \bar{M}_{Pl}^{-2}$ with $\bar{M}_{Pl}$ a reference, constant mass scale. The effective Planck scale experienced by the GWs can on the other hand vary along the GW trajectory – a phenomenon found in various examples of scalar-tensor theories. We make the hypothesis that the tensor $t_{\mu \nu}$ includes contributions containing this possibility, if realised in the scalar-tensor theories under consideration.

So far, we learned that the unitary gauge condition we adopted allow us to make the reasonable assumption that all the quantities we are dealing with in geometrical optics share the same phase. We will focus on this simplifying assumption throughout all this work. We also write

$$\vec{v}_\mu \hat{h}_{\mu \nu} = c_\nu = A C_\nu e^{i \theta / \epsilon}. \quad \quad (3.5)$$

Finally, the wave-vector of the propagating GW is defined as

$$k_\mu \equiv -\partial_\mu \theta, \quad \quad (3.6)$$

and identifies the GW geodesics as we now show.

The gauge conditions Eqs. (2.34)–(2.35), at leading and next-to-leading order, are given by

$$k_\mu e_{\mu \nu} = 0, \quad \quad (3.7)$$

$$\vec{v}_\mu e_{\mu \nu} = C_\nu, \quad \quad (3.8)$$

$$k_\mu C_\mu = 0, \quad \quad (3.9)$$

so that the polarization tensor $e_{\mu \nu}$ and the vector $C_\mu$ are transverse to the GW propagation. We can now plug the Ansatz Eq. (3.1) and Eq. (3.4) in the equations of motion Eqs. (2.40), (2.41) and (2.42).

From Eq. (2.40) we obtain

$$k_\mu k_\nu \bar{g}^{\mu \nu} = 0 \quad \quad (3.10)$$

which means that the GWs wave-vector is a null vector of the slowly varying background $\bar{g}_{\mu \nu}$. Moreover, using the definition of $k_\mu$, one can show that it also satisfies

$$k_\mu \bar{\nabla}^\mu k_\nu = 0. \quad \quad (3.11)$$

Thus the GWs wave-vector is a geodesic vector. These two results allow us to conclude that, even in the more general case where $T^{(1)}_{\mu \nu}$ contains contributions at $1/\epsilon$, GWs travel along null geodesics of the background manifold as in the case of GR with a smooth matter content. Our hypothesis that $T^{(1)}_{\mu \nu}$ only contribute starting at order $1/\epsilon$, and not $1/\epsilon^2$, is crucial to ensure this property. Our assumption is motivated by the fact that the multi-messenger event GW170817 established that, in excellent approximation, GWs travel at the speed of light. From Eq. (2.42) we obtain

$$k_\mu t_{\mu \nu} = 0, \quad \quad (3.12)$$

In our definition of wave vector there is a minus sign because we want to follow the GW’s geodesics from the observer to the source. The opposite convention is used in [77]. In this way we can use the comoving distance $\chi$ as affine parameter.
which states that the $1/\epsilon$ order of $T^{(1)}_{\mu\nu}$ is transverse. This property is a direct consequence of the fact that $t_{\mu\nu}$ is built by the high-frequency metric fluctuations, which are transverse.

Finally, from Eq. (2.41) we obtain

$$2k_\rho \bar{\nabla}^\rho (A_{\mu\nu}) + A_{\mu\nu} \left( \bar{\nabla}^\rho k_\rho \right) = t_{\mu\nu},$$

(3.13)

which can be separated\(^6\) into an equation for the amplitude $A$

$$\nabla_\rho \left( k_\rho A^2 \right) = A t_{\mu\nu} e^{\mu\nu}$$

(3.14)

and equation for the polarization $e_{\mu\nu}$:

$$k_\rho \bar{\nabla}^\rho e_{\mu\nu} = \frac{1}{2A} \left[ t_{\mu\nu} - e_{\mu\nu} (t_{\rho\sigma} e^{\rho\sigma}) \right]$$

(3.15)

by multiplying Eq. (3.13) by $e^{\mu\nu}$ and using Eq. (3.3). Eqs. (3.14)–(3.15) control the evolution of metric fluctuations in the limit of geometrical optics. The difference with respect to standard geometrical optics in GR lies in their potentially non-vanishing right-hand-side, controlled by the high-frequency contribution to the EMT $t_{\mu\nu}$, as defined in Eq. (3.4).

We propose the following interpretation of these results:

- Eq. (3.14) controls the evolution of the GW amplitude of the high-frequency GWs as they travel along a null geodesics. A non-vanishing RHS is associated with the non-conservation of the current $A^2 k^\mu$.

In scalar-tensor theories where the graviton number is conserved, the physical interpretation of this fact is related with the non-conservation of effective Planck mass, as experienced by travelling GWs. Indeed, we expect its RHS to be proportional to the rate of change of the effective Planck mass, a phenomenon already discussed in related settings, see e.g. [40, 45]. Eq. (3.14) provides a ‘covariant’ version of this result in the gauge we are adopting. This fact has interesting phenomenological consequences being related with observables associated with the GW luminosity distance, see e.g. [40–46]. In the next Sections, we shall make this connection more explicit when analysing specific models.

There can also be theories where the graviton number is not conserved (as theories with extra dimensions, see e.g. [10, 12, 108–110] for recent studies in the framework of GW cosmology) which might be described in terms of effective scalar-tensor systems; in this case, a non-vanishing RHS of Eq. (3.14) controls the amount of graviton number non-conservation.

- Eq. (3.15) controls the evolution of the polarization tensor. If its RHS is non-vanishing, it implies that this quantity is not parallely transported along GW null geodesics. This is a novel effect, which again depends on the structure of the high-frequency EMT $t_{\mu\nu}$.

Similar phenomena have been noticed in standard gravity working beyond a geometrical optics approximation in GR – see e.g. [49–52, 111, 112] – and might be related with

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\(^6\)The amplitude and polarization evolution equations can be separated as follows. One plugs the decomposition Eq. (3.2) into Eq. (3.13), and contracts the resulting equation with $e^{\mu\nu}$. Using the fact that $e_{\mu\nu} \bar{\nabla}_\rho e^{\mu\nu} = 0$, after few steps one obtains Eq. (3.14). Multiplying this equation by a polarization tensor, and subtracting the result from Eq. (3.13), it is then simple to obtain Eq. (3.15).
lensing of GWs. Interestingly, we find that these effects can be present also at leading order in geometrical optics for certain scalar-tensor theories. We shall return in the next Sections on the physical implications of this result, when specialising to particular models.

Solving the independent Eqs. (3.14)–(3.15) allows one to determine the most general solution for $A$ and for the transverse, normalized tensor $e_{\mu\nu}$. Moreover, contracting Eq. (3.15) with $\bar{v}^\mu$, and using the gauge conditions Eq. (3.8) and Eq. (3.9), we find the following condition on the transverse vector $C^\mu = \bar{v}^\nu e_{\nu\mu}$,

$$k_\rho \nabla^\rho C_\nu = k_\rho \left( \nabla^\rho \bar{v}^\mu \right) e_{\mu\nu} + \frac{1}{2A} \left[ \bar{v}^\mu t_{\mu\nu} - C_\nu \left( t_{\rho\sigma} e^{\rho\sigma} \right) \right]$$ (3.16)

This equation informs us that the vector $C^\mu$ is in general not parallel transported along GW null geodesics (unless its RHS vanish):

- The first term in its RHS of Eq. (3.16) does not depend on the EMT $t_{\mu\nu}$; hence we would have this contribution in common to GR, in case one wishes to impose gauge choices corresponding to Eq. (3.8) and Eq. (3.9). Similar topics are investigated classic books, see Exercise 35.13 of [49].

- The second term in its RHS of Eq. (3.16) does depend on the EMT $t_{\mu\nu}$, hence this contribution depends specifically on the scalar-tensor set-up one considers.

3.2 On the decomposition of the energy-momentum tensor $t_{\mu\nu}$

Even without relying on any specific scalar-tensor theory, we now have sufficient ingredients to pin down the general structure of the EMT for the systems we are interested in. In fact, it has to be: symmetric, transverse (in the sense that it has to satisfy Eq. (3.12)) and built with suitable combinations of the vectors $(\bar{v}^\mu, k^\mu, C^\mu)$ and tensors $(e_{\mu\nu}, \bar{g}_{\mu\nu})$ as it is formed by metric fluctuations. Hence, it is necessarily of the form

$$t_{\mu\nu} \equiv A \left\{ \tau^{(A)} e_{\mu\nu} + \tau^{(B)} (k_\mu C_\nu + k_\nu C_\mu) + \tau^{(C)} [k_\mu \bar{v}_\nu + k_\nu \bar{v}_\mu - \bar{g}_{\mu\nu} (k_\rho \bar{v}_\rho)] \right\} ,$$ (3.17)

since the three contributions proportional to the parameters $\tau^{(i)}$ are the only tensors with the desired properties. In Eq. (3.17), $\tau^{(A)}$, $\tau^{(B)}$, $\tau^{(C)}$ are scalar functions of the space-time coordinates, that depend on the specific theory under consideration, on the smooth profiles of the quantities $\varphi$ and $g_{\mu\nu}$, and (possibly) on the trace of the polarization tensor and on the GW momentum $k^\mu$. In Section 4, when discussing explicit examples, we shall find realisations for each of the three contributions to Eq. (3.17).

Using the explicit decomposition Eq. (3.17), we can re-express the evolution equations for amplitude and polarization:

$$\nabla_\mu \left( k^\rho A_\rho \right) = A \left[ \tau^{(A)} - \tau^{(C)} \right] e \left( k \cdot \bar{v} \right)$$ (3.18)

$$2k_\mu \nabla^\rho e_{\mu\nu} = \tau^{(B)} (k_\mu C_\nu + k_\nu C_\mu) + \tau^{(C)} [k_\mu \bar{v}_\nu + k_\nu \bar{v}_\mu - (g_{\mu\nu} - e_{\mu\nu} e) (k_\rho \bar{v}^\rho)]$$ (3.19)

with the trace

$$e = \bar{g}^{\mu\nu} e_{\mu\nu} .$$

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In terms of this EMT decomposition, the evolution Eq. (3.16) for the vector $C_{\mu} = \bar{v}^\nu e_{\nu \mu}$ reads

\[ k_\rho \nabla^\rho C_\nu = k_\rho \left( \nabla^\rho \bar{v}^\mu \right) e_{\mu \nu} + \frac{1}{2} \left\{ \tau^{(B)} \left[ (\bar{v} \cdot k) C_\nu + (\bar{v} \cdot C) k_\nu \right] + \tau^{(C)} \left[ \bar{v}^2 k_\nu + e (\bar{v} \cdot k) C_\nu \right] \right\} \]

(3.20)

We then learn that the evolution equations for GW amplitude and polarization depend on different contributions to the EMT. The equation of evolution for the amplitude $A$ can be further decomposed in a tensor and scalar part, as we are going to discuss in what next.

3.3 Separating the spin-2 and spin-0 degrees of freedom

So far, we made use of a polarization tensor $e_{\mu \nu}$ that contains both spin-0 and spin-2 contributions in the chosen gauge. In Appendix A, building on the classic work of [113] (see also the textbook [107]) we explain how to build, using null-tetrads associated with the bar metric $\bar{g}_{\mu \nu}$, spin-2 and spin-0 polarization tensors, respectively called $(e_{\mu \nu}^+, e_{\mu \nu}^\times)$ and $e_{\mu \nu}^S$. Such tensors are transverse, as they have to satisfy Eq. (3.7), and their traces are

\[ \bar{g}_{\mu \nu} e_{\mu \nu}^+ = 0 \]  
\[ \bar{g}_{\mu \nu} e_{\mu \nu}^\times = 0 \]  
\[ \bar{g}_{\mu \nu} e_{\mu \nu}^S = \sqrt{2} \]  

(3.21) (3.22) (3.23)

Using these tools there are various ways to express the tensor $A_{\mu \nu}$. We find convenient to decompose it as follows (we assume not to violate parity, hence plus and cross polarizations have the same amplitude)

\[ A_{\mu \nu} = A e_{\mu \nu} = A^T (e_{\mu \nu}^+ + e_{\mu \nu}^\times) + A^S e_{\mu \nu}^S \]  

(3.24)

where

\[ A = \sqrt{2 \left( A^T \right)^2 + \left( A^S \right)^2} \]  

(3.25)

Moreover, taking the trace of Eq. (3.24) we find

\[ e = \sqrt{2} A^S / A \]  

(3.26)

which states that the trace of the polarization is proportional to the amplitude of the spin-0 mode, which we have hidden in the GW’s polarization content because of our gauge choice. By plugging the decomposition Eq. (3.24) into Eq. (3.13) we get

\[ 2k_\rho \nabla^\rho \left( A^T e_{\mu \nu}^+ + A^T e_{\mu \nu}^\times + A^S e_{\mu \nu}^S \right) + \left( A^T e_{\mu \nu}^+ + A^T e_{\mu \nu}^\times + A^S e_{\mu \nu}^S \right) \left( \nabla^\rho k_\rho \right) = t_{\mu \nu} \]  

(3.27)

In Appendix A, starting from the previous equation, we show how to separate the evolution of the scalar and tensor amplitudes, that satisfy the separate equations

\[ \nabla^\rho \left[ k_\rho \left( A^S \right)^2 \right] = A^S t_{\mu \nu} (e_{\mu \nu}^S) \]  

(3.28)

and

\[ \nabla^\rho \left[ k_\rho \left( A^T \right)^2 \right] = \frac{A^T}{2} t_{\mu \nu} (e_{\mu \nu}^\times + e_{\mu \nu}^+) \]  

(3.29)
So we obtain two separate equations for the evolution of the amplitudes of the spin-0 and spin-2 modes. Plugging the EMT decomposition discussed in Section 3.2, we find that the previous two equations read

\[ \bar{\nabla}_\rho \left( k^\rho \left( A^T \right)^2 \right) = \left( A^T \right)^2 \tau^{(A)} \] (3.30)
\[ \bar{\nabla}_\rho \left( k^\rho \left( A^S \right)^2 \right) = \left( A^S \right)^2 \tau^{(A)} - \sqrt{2} \left( k \cdot \bar{v} \right) \bar{A} A^S \tau^{(C)} \] (3.31)

Hence the EMT contributions to the evolution of the GW amplitude is in principle different for spin-2 and spin-0 modes. In both cases, we can have non-conservation of the spin-2 and spin-0 currents \( (A^T k^\mu) \) and \( (A^S k^\mu) \).

Of course, the solutions of Eq. (3.30) and Eq. (3.31) depend on the initial conditions. For example, one can expect that the initial amplitude of scalar components \( A^S \) to be smaller than the tensor one \( A^T \), thanks to screening mechanisms that reduce the size of scalar excitations around the source. For an explicit example where this happens, see [114]. This is an interesting topic to explore, that goes beyond the scope of this work.

It is instead not possible to find separate evolution equations for scalar and tensor polarizations for this system. The procedure explained in footnote 6 does not allow one to obtain separate evolution equations for the tensors \( e^+_{\mu\nu} \) and \( e^\times_{\mu\nu} \).

4 An explicit scalar-tensor realization

The geometrical optics framework developed in the previous Sections provides us with a general and flexible formalism that allows us to consistently treat the evolution of high-frequency GWs, and to acquire a transparent physical understanding of the results. Our formulas depend on the EMT tensor \( t_{\mu\nu} \) which is controlled by (first derivatives) of the high-frequency fields. To investigate our formalism in a concrete setting, we focus in this Section on a representative scalar-tensor model called kinetic gravity braiding [71] accompanied by a non-minimal coupling with the Ricci scalar, whose Lagrangian density can be expressed as

\[ L^{(tot)} = L^{(F)} + L^{(G)} + L^{(K)}, \] (4.1)

with

\[ L^{(F)} = F(\varphi) R, \] (4.2)
\[ L^{(G)} = G(\varphi, X) \Box \varphi, \] (4.3)
\[ L^{(K)} = K(\varphi, X), \] (4.4)

where \( X = -\left( \partial_\mu \varphi \partial^\mu \varphi \right)/2 \). This is the most general subset of the Horndeski theory that leads to GW propagation at the speed of light, consistent with observational findings associated with the GW170817 event [115]. The Lagrangian \( L^{(F)} \) is related with the classic Brans-Dicke theory [116], and is common to many scalar-tensor theories that allow for a kinetic mixing between scalar and tensor fields in the Jordan frame. (Demixing can be obtained through a conformal transformation, at the price of introducing non-minimal couplings between scalar and matter degrees of freedom, that we prefer to avoid.) The Lagrangian \( L^{(G)} \) is a generalized cubic Galileon [58]. The contribution \( L^{(K)} \) corresponds instead to K-essence [57]. It would be interesting to explore further generalizations to DHOST theories [64–70] with the same property, or to consider scalar-tensor scenarios aimed to avoid constraints from graviton decay into dark energy [117–119]. We plan to study such generalizations in future works.
The energy-momentum tensor $T_{\mu\nu}$ associated with the Lagrangian density Eq. (4.1) can be expressed as [61]

$$T_{\mu\nu}^{(\text{tot})} = T_{\mu\nu}^{(F)} + T_{\mu\nu}^{(G)} + T_{\mu\nu}^{(K)},$$

(4.5)

with

$$T_{\mu\nu}^{(F)} = -g_{\mu\nu} \left( \frac{F_{\varphi} X}{F} \square \varphi - 2X \frac{F_{\varphi\varphi}}{F} \right) + \frac{F_{\varphi}}{F} \nabla_\mu \nabla_\nu \varphi + \frac{F_{\varphi}}{F} \nabla_\mu \varphi \nabla_\nu \varphi,$$

(4.6)

$$T_{\mu\nu}^{(G)} = -\frac{1}{2F} G X \square \varphi \nabla_\mu \varphi \nabla_\nu \varphi + \frac{1}{2F} \left( \nabla_\lambda G \nabla^\lambda \varphi \right) g_{\mu\nu} - \frac{1}{F} \nabla_{(\mu} G \nabla_{\nu)} \varphi,$$

(4.7)

$$T_{\mu\nu}^{(K)} = \frac{1}{2F} K X \nabla_\mu \varphi \nabla_\nu \varphi + \frac{1}{2F} K g_{\mu\nu},$$

(4.8)

The dynamics of the system is also controlled by the scalar equations of motion. We have proved that the relevant scalar equations are satisfied once the gravitational and the EMT conservation Eq. (2.39) are satisfied, hence we do not discuss the scalar equation any further.

We notice that $L^{(G)}$ and $L^{(F)}$ contain, respectively, derivative scalar self-couplings and non-minimal couplings of scalar to gravity, which lead to derivative contributions to the EMT involving metric fields. Hence, we expect them to give relevant contributions to the system evolution in the limit of geometrical optics.

In fact, we can extract the high frequency, $1/\epsilon$ contributions to the linearised EMT tensor, that form the tensor $t_{\mu\nu}$ as defined in Eq. (3.4); as explained after that equation, the tensor $t_{\mu\nu}$ is proportional to first derivatives of high-frequency metric fluctuations entering in the EMT tensor. In the presence instance, the tensor $t_{\mu\nu}$ can be organized into the structure of Eq. (3.17), that we re-write here

$$t_{\mu\nu}^{(\text{tot})} = A \left\{ \tau^{(A)} e_{\mu\nu} + \tau^{(B)} (k_\mu C_\nu + k_\nu C_\mu) + \tau^{(C)} [k_\mu \bar{v}_\nu + k_\nu \bar{v}_\mu - g_{\mu\nu} (k_\rho \bar{v}^\rho)] \right\},$$

(4.9)

where for the model of this Section the three functions $\tau^{(A)}$, $\tau^{(B)}$ and $\tau^{(C)}$ read

$$\tau^{(A)} = -\frac{F_{\varphi}}{2F} (k_\mu \bar{v}^\rho),$$

(4.10)

$$\tau^{(B)} = \frac{F_{\varphi}}{2F},$$

(4.11)

$$\tau^{(C)} = -\frac{G X}{4F} \left( \bar{v}_\rho C^\rho - \frac{\bar{v}^2}{2} e \right) - \frac{F_{\varphi}}{4F} e,$$

(4.12)

Notice that, as anticipated above, only $L^{F}$ and $L^{G}$ contribute to the high-frequency EMT $t_{\mu\nu}$. We can then apply the evolution equations derived in the previous Section to the present instance. We find the following three relations:

i) An evolution equation for the tensor component of the GW amplitude:

$$\nabla_\rho \left( k^\rho (A^T)^2 \right) = (A^T)^2 \tau^{(A)}$$

$$= -\frac{F_{\varphi}}{2F} (k_\mu \bar{v}^\rho) (A^T)^2.$$ 

(4.13)

---

The total action for the system can contain additional smooth matter fields that can influence cosmological evolution. We do not consider them here, since we focus on GWs in the geometrical optics limit.
ii) An evolution equation for the scalar component of the GW amplitude:

\begin{equation}
\nabla_\rho \left( k^\rho \left( A^S \right)^2 \right) = \left( A^S \right)^2 \tau^{(A)} - \sqrt{2}(k_\rho \bar{v}^\rho) A A^S \tau^{(C)}
\end{equation}

\begin{equation}
= - \frac{F_\varphi}{2F} \left( A^S \right)^2 (k_\rho \bar{v}^\rho)
\end{equation}

\begin{equation}
+ \sqrt{2}(k_\rho \bar{v}^\rho) A \left[ \frac{G_X}{2F} \left( A^S \bar{v}^\rho - \frac{\bar{v}^2}{\sqrt{2}} A^S \right) - \frac{F_\varphi}{2\sqrt{2}F} A^S \right].
\end{equation}

(4.14)

iii) An evolution equation for the tensor polarization:

\begin{equation}
k_\rho \nabla^\rho e_{\mu\nu} = \frac{\tau^{(B)}}{2} \left( k_\mu C_\nu + k_\nu C_\mu \right) + \frac{\tau^{(C)}}{2} \left[ k_\mu \bar{v}_\nu + k_\nu \bar{v}_\mu \right.
- \left( \bar{g}_{\mu\nu} - e_{\mu\nu} \bar{v}^\rho \right) (k_\rho \bar{v}^\rho) \left] - \left[ k_{\mu\nu} \bar{C}_{\mu\nu} + k_{\nu\mu} \bar{C}_{\nu\mu} \right] \right.
+ \frac{G_X}{8F} \left\{ \left( \bar{v}^\rho C^\rho - \frac{\bar{v}^2}{\sqrt{2}} A^S \right) \left[ k_\mu \bar{v}_\nu + k_\nu \bar{v}_\mu \right.
- \left( \bar{g}_{\mu\nu} - \frac{\sqrt{2}A^S}{A} e_{\mu\nu} \right) (k_\rho \bar{v}^\rho) \left] \right. \right.
\end{equation}

(4.15)

The three equations above are the most general obtained in this work, as they describe the evolution of both the GW scalar and tensor modes in the considered theory of gravity. They show that the GW propagation depends on the assumed scalar-tensor theory (through the dependence on the functions $F_\varphi, G_X$), and on the background quantities $\bar{g}_{\mu\nu}$ and $\bar{v}^\rho = \nabla^\rho \bar{\varphi}$. Moreover, from Eq. (4.14) we note that the evolution of the scalar amplitude $A^S$ cannot be decoupled from that of the tensor amplitude $A^T$, due to the presence of the total amplitude $A$ in Eq. (4.14). However, Eq. (4.13) shows that the tensor amplitude $A^T$ does not depend on $A^S$. In addition, Eq. (4.15) implies that the tensor and scalar modes of polarization are in general coupled one to each other.

The evolution equation for the tensor amplitude, Eq. (4.13) is proportional to $F_\varphi$. This is a quantity that in a cosmological setting is associated with the time-dependence of the effective Planck mass. In theories as the ones we consider here – with second order equations of motion, and GW unit speed – the failure of conservation of the current $A^2 k^\rho$ is associated with the rate of change of the effective Planck mass, a result well known in the literature (see e.g. the discussion in the recent [40] and references therein).

The evolution equation for the scalar part of the amplitude, Eq. (4.14) is more complex, and depends both on $F_\varphi$ and $G_X$. On the other hand, in case screening is very effective and the amplitude of the scalar component vanishes or is very reduced at GW emission (see e.g. the explicit model of [114]), then Eq. (4.14) is identically satisfied for a vanishing scalar amplitude along the entire GW null geodesics.

For simplicity, let us focus from now on on the case $A^S = 0$. In this case $e = 0$ since the trace is proportional to the amplitude of the scalar modes as showed in Eq. (3.26). The evolution equation for the polarization simplifies considerably, and becomes

\begin{equation}
k_\rho \nabla^\rho e_{\mu\nu} = \frac{F_\varphi}{4F} \left[ k_\mu C_\nu + k_\nu C_\mu \right] + \frac{G_X}{8F} \left( \bar{v} \cdot C \right) \left[ k_\mu \bar{v}_\nu + k_\nu \bar{v}_\mu - \bar{g}_{\mu\nu} \left( k \cdot \bar{v} \right) \right].
\end{equation}

(4.16)
Its RHS quantifies the failure of the GW polarization tensor of being parallely transported along the GW geodesics. It is associated with the vector $C_{\nu} = \bar{v}^\mu e_{\mu\nu}$, whose evolution equation is

$$k_\rho \nabla^\rho C_{\nu} = k_\mu \left( \nabla^\mu \bar{v}^\nu \right) e_{\mu\nu} + \left( \frac{F_\varphi}{4F} \right) \left( (\bar{v} \cdot k) C_{\nu} + (\bar{v} \cdot C) k_\nu \right) - \frac{G_X}{8F} (\bar{v} \cdot C) \bar{v}^2 k_\nu ,$$

(4.17)

which can be obtained either from Eq. (3.20) or by contracting Eq. (4.16) by $\bar{v}^\mu$.

When specialising to GW evolution on homogeneous cosmological FRW space-times, it is possible to select a choice of vector $C^\mu$ so to ensure parallel transport for the polarization tensor. An analysis on a perturbed cosmological space-time is instead more delicate. We address these topics in the next Section.

5 GW propagation on a cosmological space-time

One feature of the geometrical optics formalism we developed in the previous Sections is that it can be applied to GWs travelling over an arbitrary space-time, as long as the scale of variation of the ‘background’ geometry is well larger than the GW wavelength. In this Section, we specialize to GWs travelling over a perturbed FRW universe, and apply the formulas derived in the previous Section for a specific scalar-tensor model. Using the Cosmic Rulers formalism (see the discussion in the Introduction) we will show how to derive an expression for the GW luminosity distance from the analysis of the amplitude of the tensor modes, whose dynamics is governed by Eq. (4.13).

To avoid excessively cumbersome formulas, we make the hypothesis to set to zero the cubic Galileon contribution, $G_X = 0$, and we also specialize to the evolution of tensor fluctuations assuming that $A^S$ at GW emission is suppressed. Under these assumptions, the evolution equations reduce to

$$\nabla_\rho \left( k^\rho \left( A^T \right)^2 \right) = \frac{F_\varphi}{2F} \left( k_\rho \bar{v}^\rho \right) \left( A^T \right)^2 ,$$

(5.1)

$$k_\rho \nabla^\rho e_{\mu\nu} = \frac{F_\varphi}{4F} \left[ k_\mu C_{\nu} + k_\nu C_\mu \right] ,$$

(5.2)

$$k_\rho \nabla^\rho C_{\nu} = k_\rho \left( \nabla^\rho \bar{v}^\nu \right) e_{\mu\nu} + \left( \frac{F_\varphi}{4F} \right) \left( (\bar{v} \cdot k) C_{\nu} + (\bar{v} \cdot C) k_\nu \right) ,$$

(5.3)

and we focus on them in what follows when studying the evolution of the polarization.

Eq. (5.2) and Eq. (5.3) inform us that for the case of an unperturbed FRW space-time, we are allowed to make the choice $C_\mu = 0$. Indeed, we now show that this condition is preserved along the GW geodesics. Indeed, in a homogeneous FRW space-time, the scalar depends on time only, and $\bar{v}^\mu \propto \delta^\mu_0$. Since $C^\mu = 0$, the polarization tensors satisfy the condition $e_{\mu0} = 0$, and they have non-vanishing spatial components only. Since $e_{ij}$ is transverse to the GW direction, we can without loss of generality choose a frame in which the GW propagates along the $z$-direction, and $e_{ij}$ has non-vanishing components in the $(x,y)$-directions only. A simple calculation shows that, in this case, the first term in the RHS of Eq. (5.3) vanishes. This implies that the choice $C_\mu = 0$ satisfies this equation, and also implies that the RHS of Eq. (5.2) vanishes. Hence, when focussing on an homogeneous, unperturbed FRW space-time, if only the GW tensor components propagate and $C^\mu = 0$, the geometrical optics equations...
prescribe that the polarization tensor is parallel propagated along geodesics as the RHS of Eq. (5.2) is zero in this case.\footnote{Notice that even keeping the cubic Galileon contribution, in the case that $\mathcal{A}^S = 0$, we would be able to make the same considerations as above. In fact the terms in Eq. (4.16) and Eq. (4.17) proportional to $G_X$, are multiplied by $C^\mu$. Therefore, on a unperturbed space-time, we can still choose $C^\mu = 0$ which leads to a zero RHS of Eq. (4.16) and the parallel transport of the polarization, but not in the case of a perturbed space-time.}

For the case of a perturbed space-time, instead, the situation is more complex, since all quantities depend on time and space, and the previous arguments do not hold.

We use as background (i.e. slowly varying) space-time quantities

\begin{equation}
\bar{g}^{\mu\nu} dx^\mu dx^\nu = a^2(\eta)\left[-(1 + 2\phi) d\eta^2 + (1 - 2\psi) \delta_{ij} dx^i dx^j \right],
\end{equation}

\begin{equation}
\bar{\varphi} = \phi(\eta) + \delta \varphi(x^\mu).
\end{equation}

All fluctuations appearing in the previous Ansatz are assumed to have long wavelengths, much larger than the high frequency GW modes that travel on such space-time. Cosmological perturbations have the effect to alter the estimate of astrophysical parameters derived for the binary system. For example, the inferred luminosity distance of the source of the GWs will differ from the actual one because of the presence of inhomogeneities. It is important clarify whether there are degeneracies between these effects and modified gravity.

5.1 Cosmic rulers for gravitational waves

In order to study how cosmological perturbations affect the propagation of gravitational waves we follow [103], but in the more general scalar-tensor theory presented in the previous section. In [103] the authors use the so called Cosmic Rulers formalism to compute the correction to the luminosity distance of the spiralling binary. This formalism has been first formulated for electromagnetic radiation [120], [104] but it can be immediately extended to gravitational radiation in the geometrical optics limit. In the Cosmic Rulers formalism, the observer frame, called the Redshift-GW frame (RGW), is used as reference system. Such frame is different from the physical one because in the RGW-frame we use coordinates that actually flatten our past gravitational wave-cone.

Since we will use as background metric Eq. (5.4), it is convenient to perform a conformal transformation,

\begin{equation}
\begin{aligned}
\bar{g}^{\mu\nu} &\rightarrow \tilde{g}^{\mu\nu} = \bar{g}^{\mu\nu}/a^2, \\
\tilde{k}^\mu &\rightarrow k^\mu = k^\mu a^2,
\end{aligned}
\end{equation}

In this way $\tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \delta g^{\mu\nu}$. Under Eq. (5.6) the connection coefficients transform as

\begin{equation}
\Gamma^u_{\nu\rho} = \frac{1}{2a^2} \tilde{g}^{u\alpha} \left( (a^2 \tilde{g}_{\alpha\rho})_{,\nu} + (a^2 \tilde{g}_{\rho\alpha})_{,\nu} - (a^2 \tilde{g}_{\nu\rho})_{,\alpha} \right) = \\
\tilde{\Gamma}^u_{\nu\rho} + \frac{1}{2} \left( \delta^u_{\nu} \partial_{\rho} \ln a^2 + \delta^u_{\rho} \partial_{\nu} \ln a^2 - \delta_{\nu\rho} \tilde{g}^{\mu\alpha} \partial_{\alpha} \ln a^2 \right),
\end{equation}

where $\tilde{\Gamma}^u_{\nu\rho}$ is the connection symbol associated to $\tilde{g}^{\mu\nu}$.

We define $x^u(\chi)$ as the comoving coordinate in the real frame, where $\chi$ is the comoving distance to the observer. In the RGW-space, the geodesic of the emitted gravitational waves

\textit{The end of the text.}
takes the form\(^9\)

\[ x^\mu = (\bar{\eta}, \bar{x}) = (\eta_0 - \bar{\chi}, \bar{\chi} \hat{n}), \]  

(5.8)

where \(\eta_0\) is the conformal time at observation, \(\bar{\chi}(z)\) is the comoving distance to the observed redshift in the observer frame and \(n\) is the observed direction of arrival in the sky, i.e. \(\hat{n}^i = \bar{x}^i/\bar{\chi} = \delta^{ij}(\partial \bar{\chi}/\partial \bar{x}^j)\). The observed coordinate will be different from the one in the real frame. The total derivative along the part gravitational wave-cone is

\[ \frac{d}{d\bar{\chi}} = -\frac{\partial}{\partial \bar{\eta}} + \hat{n}^i \frac{\partial}{\partial \bar{x}^i}, \]  

(5.9)

It is convenient to define parallel and perpendicular projectors operator with respect the observed line-of-sight direction. As shown in [103], for any spatial vectors and tensor, in the RGW we have:

1. \(A_{||} = \hat{n}^i \hat{n}^j A_{ij}\)
2. \(B_{\perp}^i = \mathcal{P}_{ij} B_j = B^i - \hat{n}^i B_{||}\)
3. \(\bar{\delta}_{||} = \hat{n}^i \bar{\delta}_i\)
4. \(\bar{\delta}^2_{||} = \bar{\delta}_{||} \bar{\delta}_i\)
5. \(\bar{\delta}_{\perp i} = \mathcal{P}_{ij} = \bar{\delta}_i - \hat{n}_i \bar{\delta}_{||}\)
6. \(\bar{\delta}_i \hat{n}^j = \bar{\delta}_i \left(\frac{\bar{x}^j}{\bar{\chi}}\right) = \frac{1}{\bar{\chi}} \left(\bar{\delta}_i - \frac{\bar{x}^j}{\bar{\chi}} \frac{\partial \bar{\chi}}{\partial \bar{x}^j}\right) = \frac{1}{\bar{\chi}} \left(\bar{\delta}_i - \hat{n}_i \hat{n}_i\right) = \frac{1}{\bar{\chi}} \mathcal{P}_{ij}^i\)
7. \(\frac{d}{d\bar{\chi}} \bar{\delta}_i^j = \bar{\delta}_{\perp i} \frac{d}{d\bar{\chi}} - \frac{1}{\bar{\chi}} \bar{\delta}_i^j\)
8. \(\frac{\partial B_{\perp}^i}{\partial \bar{x}^j} = \bar{\delta}_{j} B_{\perp}^i + \hat{n}_j \bar{\delta}_{||} B_{\perp}^i + \frac{1}{\bar{\chi}} \mathcal{P}_{ij} B_{||} + \hat{n}^i [\bar{\delta}_{\perp j} + \hat{n}_j \bar{\delta}_{||}] B_{||}\)

where we have used \(\bar{\delta}_i = \partial/\partial \bar{x}^i\) and \(\mathcal{P}_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j\).

We define \(\bar{k}^\mu\) as the null geodesic vector in the redshift frame at zeroth order,

\[ \bar{k}^\mu = \frac{d\bar{x}^\mu}{d\bar{\chi}} = (-1, \hat{n}), \]  

(5.10)

which satisfies \(d\bar{k}^\mu/d\bar{\chi} = 0\).

Our aim is to compute GW observables in the RGW-space. However, in the previous section we derived the evolution equations of the quantities related to the gravitational wave in the physical frame (e.g. Eq. (5.1)). Therefore we need to build a map between the two frames. At linear order such mapping is given by

\[ \chi = \bar{\chi} + \delta \chi, \quad x^\mu(\chi) = \bar{x}^\mu(\bar{\chi}) + \Delta x^\mu(\bar{\chi}) \]  

(5.11)

where

\[ \Delta x^\mu(\bar{\chi}) = \delta x^\mu(\bar{\chi}) + \bar{k}^\mu \delta \chi. \]  

(5.12)

---

\(^9\)In our conventions, barred quantities belong to the RGW-frame, while unbarred quantities belong to the physical space.
Such correction has two contributions: one at fixed observed-comoving distance, $\delta x^\mu(\bar{\chi})$, and the other one is proportional to $\delta \chi$ in the direction of $\bar{k}^\mu$. From now on, the symbol $\Delta$ accounts for the sum of these two kinds of contributions. Eq. (5.11) imply that quantities in the real and RGW-frame coincides at zero order, i.e. if the Universe was homogeneous and isotropic.

If $x^\mu(\chi)$ is the geodesic wave-vector in the physical-space, then
\[
\dot{k}^\mu = \frac{d x^\mu(\chi)}{d\chi} = \frac{d\chi}{d\bar{\chi}} d\bar{\chi}^\mu (\bar{x}^\mu(\bar{\chi}) + \Delta x^\mu(\bar{\chi})) = \bar{k}^\mu + \frac{d \delta x^\mu}{d\chi},
\]
at linear order. Thus we can define $\delta k^\mu$ through
\[
\dot{k}^\mu = \bar{k}^\mu + \delta k^\mu = \bar{k}^\mu + \frac{d \delta x^\mu}{d\chi},
\]
which leads to
\[
[k^\mu]^{(0)} = \bar{k}_\mu = -\partial_\mu \theta, \quad [k^\mu]^{(1)} = \delta k^\mu = \frac{d \delta x^\mu(\bar{\chi})}{d\chi} = (\delta \nu, \delta n^i),
\]
where we defined $\delta \nu$ and $\delta n^i$ so that
\[
\dot{k}^\mu(\bar{\chi}) = (-1 + \delta \nu, \dot{n}^i + \delta n^i).
\]
We can obtain $\delta x^\mu$ upon integration
\[
\delta x^0(\bar{\chi}) = \int_0^{\bar{\chi}} d\bar{\chi} \delta \nu(\bar{\chi}) \quad \text{and} \quad \delta x^i(\bar{\chi}) = \int_0^{\bar{\chi}} d\bar{\chi} \delta n^i(\bar{\chi}),
\]
with the boundary conditions at the observer $\delta x_0^\mu = 0$. The real-space scale factor is given by
\[
a(x^0(\chi)) = a(\bar{x}^0 + \Delta x^0) = \bar{a}(\bar{x}^0) + \Delta x^0 \bar{a}'(\bar{x}^0) = \bar{a}[1 + \mathcal{H} \Delta x^0] + \delta a(\bar{x}^0),
\]
from which we can see that $\Delta \ln a = \mathcal{H} \Delta x^0$.

We are restricting ourselves in the local wave zone where the GW’s wavelength is small with respect to the comoving distance from the observer. In this portion of the space-time we are allowed to define the tetrad, $e^\mu_\alpha$, at the source position. The time-like vector of the basis can be chosen as the four-velocity $u^\mu$ of the observer, then
\[
u_\mu = e^0_\mu = aE^0_\mu \quad \text{and} \quad u^\mu = e_\mu^0 = a^{-1} E_\mu^0,
\]
where $E^0_\mu$ is the tetrad in the comoving frame. The other components are constructed using the definitions of the tetrad itself. Up to linear order, in Poisson’s gauge Eq. (5.4), these are
\[
E^\alpha_0 = \left(\begin{array}{c}
-1 - \phi \\
v_i
\end{array}\right), \quad E_{a0} = -v_a \quad \text{and} \quad E_{ai} = \delta_{ai}(1 - \psi).
\]
We can relate $\Delta \ln a$ to the components of the tetrad using the fact that the observed redshift is given by\(^{10}\) ($f$ denotes the GW frequency)
\[
1 + z = \frac{f_e}{f_o} = \left. \frac{(e^0_0 k^\mu)_e}{(e_0^0 k^\mu)_o} \right|_e = \frac{a_0}{a_e} \left. \frac{(E^0_0 \dot{k}^\mu)_e}{(E^0_0 \dot{k}^\mu)_o} \right|_e.
\]
\(^{10}\)“e” and “o” are the emitted and observed positions, respectively.
We will assume\(^ {11}\) \(a_o = \bar{a}_o = a(\bar{x}^0) = 1\) in such a way that \(\bar{a}\) is the scale factor in the redshift space, therefore \(\bar{a} = 1/(1 + z)\). We choose also \((E_{0\mu} \dot{k}^\mu)|_o = 1\). With this conditions Eq. (5.21) becomes

\[
(1 + z) = \frac{(E_{0\mu} \dot{k}^\mu)^{(0)}|_e + (E_{0\mu} \dot{k}^\mu)^{(1)}|_e}{\bar{a}(1 + \Delta \ln a)} = (1 + z) \frac{1 + (E_{0\mu} \dot{k}^\mu)^{(1)}|_e}{1 + \Delta \ln a},
\]

hence,

\[
\Delta \ln a = \mathcal{H} \Delta x^0 = (E_{0\mu} \dot{k}^\mu)^{(1)}|_e = -E^{(1)}_{00} + \bar{n}^i E_{0i}^{(1)} - \delta \nu.
\]

5.2 Gravitational waves in the observer frame

In this Section we follow the same prescription developed in [103], and we extend their results by including modified gravity effects. In particular, we compute the effects of large-scale structures on GW waveforms (e.g. the phase and the amplitude) of the GW in MG accounting for lensing, Sachs-Wolfe, integrated Sachs-Wolfe, time delay and volume effects.

5.2.1 The GW phase

We showed that in this theory of gravity the wave-vector is still tangent to null geodesics of the background space-time, i.e.

\[
k^\mu k_\mu = 0.
\]

After a conformal transformation Eq. (5.6) the latter can be written as

\[
\frac{d\theta}{d\chi} = 0,
\]

where we used \(\dot{k}^\mu = d/\chi\). We can expand it up to linear order

\[
\frac{d}{d\chi} \theta(\bar{x}^\mu + \Delta x^\mu) = \left( 1 - \frac{d \delta \chi}{d \chi} \right) \left( \frac{d}{d \chi} (\bar{\theta}(\bar{x}) + \Delta \theta(\bar{x})) \right) = \frac{d\bar{\theta}}{d\chi} + d\delta \theta + \delta x^\mu \frac{d}{d\chi} \bar{k}_\mu \bar{\theta} = 0.
\]

The only term at zero order in the latter equation is the first one, thus we can extract

\[
\frac{d\bar{\theta}}{d\chi} = 0 \quad \text{and} \quad \frac{d\delta \theta}{d\chi} = \delta k^\mu \dot{k}_\mu.
\]

The last equation allows us to get \(\delta \theta\) from \(\delta k_\mu\) by integration

\[
\delta \theta(\bar{\chi}) = \delta \theta_o + \int_0^{\bar{\chi}} d\chi \left( \delta \nu + \delta n_\parallel \right),
\]

where \(\delta \theta_o\) is the value of \(\delta \theta\) at the observer position who is located at \(\bar{\chi} = 0\). The full correction is

\[
\Delta \theta(\bar{x}(\bar{\chi})) = \delta \theta(\bar{x}(\bar{\chi})) + \Delta x^\mu \dot{\bar{\theta}}(\bar{x}(\bar{\chi})) = \delta \theta(\bar{x}(\bar{\chi})) - \Delta x^\mu \dot{k}_\mu = \delta \theta(\bar{x}(\bar{\chi})) - \Delta x^0 \dot{k}_0 - \Delta x^i \dot{k}_i = \delta \theta - (\Delta x^0 + \Delta x_\parallel).
\]

\(^{11}\)In general this is not true. Indeed, in principle, we should include the perturbation of the scale factor at observation \(\delta a_o = a_o - 1\), because we have a correction the physical coordinate time which does not coincide with the proper time of the observer in an inhomogeneous universe (for details, e.g. see [91]).
since \( k_\mu = -\partial_\mu \bar{\theta} \) and \( \bar{k}_0 = \eta_{\mu 0} \bar{k}^\mu = +1 \). We can define \( T \) as

\[
T = -(\Delta x^0 + \Delta x_{||}) = -(\delta x^0 + \delta x_{||}),
\]

therefore

\[
\Delta \theta(\bar{\chi}) = \delta \theta(\bar{\chi}) + T = \delta \theta|_0 + \int_0^{\bar{\chi}} d\chi (\delta \nu + \delta n_{||}) + T = \delta \theta|_0,
\]

where we used Eq. (5.17) and Eq. (5.29) in the last equality. This result is in agreement with [103].

5.2.2 The GW amplitude

In the previous section we derived the evolution equation of the amplitude of the tensor part of the gravitational wave in the most general Horndeski theory satisfying the constraint \( c_T = 1 \). The amplitude of those waves is inversely proportional to the luminosity distance of the spiralling binary, and thus we are interested in deriving the expression of the observed amplitude. To this aim, it is useful to rewrite Eq. (4.13) as

\[
k^\rho \nabla_\rho \ln(A_T^a) = -\frac{1}{2} \left( \nabla^\rho k_\rho + k^\rho \nabla_\rho \ln F[\bar{\varphi}] \right),
\]

where we have used \( v_\mu = \bar{\nabla}_\mu \bar{\varphi} \). After the transformation Eq. (5.6), it becomes

\[
\hat{k}^\rho \hat{\nabla}_\rho \ln(A_T^a) = -\frac{1}{2} \left( \hat{\nabla}_\rho \hat{k}_\rho + \hat{k}^\rho \hat{\nabla}_\rho \ln F \right),
\]

where \( \hat{\nabla}_\rho \) is the covariant derivative respect \( \hat{g}_{\mu \nu} \). At the zero order Eq. (5.32) is

\[
\bar{k}^\rho \bar{\nabla}_\rho \ln(A_T^a) = -\frac{1}{2} \left( \bar{\nabla}_\rho \bar{k}_\rho + \bar{k}^\rho \bar{\nabla}_\rho \ln F_0 \right) = -\frac{1}{2} \left( \bar{\partial}_\rho \bar{k}^0 + \bar{k}^\rho \bar{\partial}_\rho \ln F_0 \right) = -\frac{1}{2} \left( \bar{\partial}_i \bar{n}^i + \frac{d}{d\bar{\chi}} \ln F_0 \right) = -\frac{1}{2} \left( \frac{P^i}{\bar{\chi}} + \frac{d}{d\bar{\chi}} \ln F_0 \right) = -\frac{d}{d\bar{\chi}} \left( \ln \bar{\chi} + \ln \sqrt{F_0} \right),
\]

where we called \( F_0 = F[\bar{\varphi}_0] \). We can rewrite the latter as

\[
\frac{d}{d\bar{\chi}} \ln(A_T^a \sqrt{F_0}) = 0 \quad \rightarrow \quad A_T^a(x^0, \bar{\chi}) = \frac{Q}{a(x^0) \sqrt{F_0}}.
\]

where \( Q \) is an integration constant determined by the local-wave zone solution and constant along the null geodesic. The GR equivalent of Eq. (5.33) is\(^{12}\)

\[
(A_T^{GR}(x^0, \bar{\chi}) = \frac{Q}{a(x^0) \bar{\chi}}.
\]

where there is a factor \( 1/\sqrt{F_0} \) of difference with respect to the result in modified gravity.

\(^{12}\)To be precise, the integration constant \( Q \) could be different in modified gravity with respect to GR.
Now we proceed in expanding at the first order Eq. (5.32). It is built by four different contributions, three of them do not depend explicitly on the extra scalar field and can be found also in the case of GR [103]. These are

1) \[
\frac{d}{d\chi} \ln(A^T(\bar{x}^\mu + \Delta x^\mu)) = \left(1 - \frac{d\delta\chi}{d\chi}\right) \left[ \frac{d}{d\chi} \ln A^T + \frac{d}{d\chi} \delta \ln A^T + \frac{d}{d\chi} (\Delta x^\mu \partial_\mu \ln A^T) \right] = \]
\[
= \frac{d}{d\chi} \ln A^T + \frac{d}{d\chi} \delta \ln A^T + \delta A^T \delta \chi \frac{d^2}{d\chi^2} + \delta k^\mu \partial_\mu \ln A^T + \delta x^\mu \frac{d}{d\chi} \partial_\mu \ln A^T,
\]

2) \[
\frac{d}{d\chi} \ln a(\bar{x}^0 + \Delta x^0) = \left(1 - \frac{d\delta\chi}{d\chi}\right) \left[ \frac{d}{d\chi} \ln[a(1 + \mathcal{H} \Delta x^0)] = \right] \]
\[
= -\mathcal{H} \left(1 - \frac{d\delta\chi}{d\chi}\right) + \frac{d}{d\chi} \left(\mathcal{H} \Delta x^0\right) = -\mathcal{H} \left[1 - \delta k^0\right] - \mathcal{H}' \left[\delta x^0 - \delta \chi\right],
\]

3) \[
-\frac{1}{2} \bar{\nabla}_\rho \bar{k}^\rho = -\frac{1}{2} \left[ \frac{\partial x^\mu}{\partial x^\rho} \frac{\partial}{\partial x^\sigma} \bar{\partial}_\mu (\bar{k}^\sigma + \delta k^\sigma) + \delta \bar{\Gamma}^\sigma_{\rho\mu} \bar{k}^\mu \right] = \]
\[
= -\frac{1}{2} \left[ \bar{\delta} \bar{\Gamma}^\sigma_{\rho\mu} \bar{k}^\mu + \frac{2}{\chi} (1 + \delta k||) + \frac{d\delta k||}{d\chi} + \bar{\partial}_0 (\delta k^0 + \delta k||) \bar{\partial}_{iL} \delta k^i_L - \frac{2}{\chi^2} (\delta \chi + \delta x||) - \frac{1}{\chi} \bar{\partial}_{iL} \delta x^i_L \right],
\]
where \(\delta \bar{\Gamma}^\sigma_{\rho\mu}\) is the linear order connection coefficient associated to \(\bar{\partial}_{\mu}\). The new part with respect to GR is

4) \[
-\frac{1}{2} \frac{d}{d\chi} \ln F(\varphi_0 + \Delta \varphi) = -\frac{1}{2} \left[ \left(1 - \frac{d\delta\chi}{d\chi}\right) \right] \ln \left[ F_0 \left(1 + \frac{F_{\varphi_0}}{F_0} \Delta \varphi\right) \right] = \]
\[
= -\frac{1}{2} \left[ \left( \frac{F_{\varphi_0}}{F_0} + \Delta \varphi \left( \frac{F_{\varphi_0}}{F_0} - \frac{F_{\varphi_0}^2}{F_0^2} \right) \right) \right] \frac{d\varphi_0}{F_0} + \frac{F_{\varphi_0}}{F_0} \left( \frac{d\varphi}{d\chi} + \delta \chi \frac{d^2\varphi_0}{d\chi^2} + \delta k^0 \partial_\mu \varphi_0 + \delta x^0 \frac{d\varphi_0'}{d\chi} \right),
\]

were we used the short notations \((dF/d\varphi)|_{\varphi_0} = F_{\varphi_0}, (d^2F/d\varphi^2)|_{\varphi_0} = F_{\varphi_0}, \varphi_0' = \partial_0 \varphi_0\) and \(\Delta \varphi = \delta \varphi + \Delta x^\mu \partial_\mu \varphi_0\).

Combining together the four contributions yields

\[
\frac{d}{d\chi} \delta \ln(A^T)(\bar{\chi}) = -\frac{1}{2} \left[ \delta \bar{\Gamma}^\sigma_{\rho\mu} \bar{k}^\mu + \frac{d\delta k||}{d\chi} + \bar{\partial}_0 (\delta k^0 + \delta k||) - 2 \frac{d\kappa}{d\chi} + \frac{d}{d\chi} \left( \frac{F_{\varphi_0}}{F_0} \delta \varphi \right) \right],
\]

where \(\kappa\) is the weak lensing convergence term

\[
\kappa = -\frac{1}{2} \bar{\partial}_{iL} \Delta x^i_L.
\]

Eq. (5.35) can be integrated in order to obtain \(\delta \ln A^T(\bar{\chi})\). The full correction to the GW amplitude, under the assumption that \(A^S = 0\), then reads

\[
\Delta \ln A^T = \delta \ln A^T(\bar{\chi}) + \Delta x^0 \partial_0 \ln A^T + \Delta x^i \partial_i \ln A^T = \]
\[
= \delta \ln A^T(\bar{\chi}) - \Delta \ln a \left(1 + \frac{F_{\varphi_0} \varphi_0'}{2 F_0 \mathcal{H}} - \frac{1}{\bar{\chi} \mathcal{H}}\right) + \frac{T}{\bar{\chi}}, \tag{5.36}
\]
where \( T \) is given by Eq. (5.29). Besides contributions depending on cosmological fluctuations, we find an explicit additional term due to modified gravity.

### 5.3 Perturbations in the Poisson gauge

The expressions obtained so far never actually used the explicit form of the metric Eq. (5.4), therefore they are valid for cosmological perturbations in any gauge. However, by using the Poisson gauge we can rewrite Eq. (5.35) in a more physically transparent and understandable way. As far as the phase is concerned, its corrections will not change in this particular gauge, hence we focus on the GW amplitude only.

The corrections \( \delta k^\mu \) are related to the two scalar potentials \( \phi \) and \( \psi \) via the geodesic equation satisfied by the wave-vector. At first order it is

\[
\frac{d}{d\bar{\chi}} \delta k^\mu(\bar{\chi}) + \delta \Gamma^\mu_{\alpha\beta}(\bar{\chi}) \bar{k}^\alpha(\bar{\chi}) \bar{k}^\beta(\bar{\chi}) = 0,
\]

(5.37)

where \( \delta \Gamma^\mu_{\alpha\beta} \) are given by

\[
\begin{align*}
\delta \Gamma_{00}^0 &= \phi', \\
\delta \Gamma_{ij}^0 &= -\delta_{ij} \psi', \\
\delta \Gamma_{ij}^i &= -\delta_i^j \psi - \delta_j^i \bar{\partial}_k \psi + \delta_{jk} \bar{\partial}^\mu \psi.
\end{align*}
\]

(5.38)

We can then derive the set of equations

\[
\frac{d}{d\bar{\chi}}(\delta \nu - 2\phi) = \phi' + \psi', \\
\frac{d}{d\bar{\chi}}(\delta n^i - 2n^i \psi) = -\bar{\partial}^i(\phi + \psi).
\]

(5.39)

In the observer frame

\[
E_{0\mu} \hat{k}^\mu = -(1 + \phi)(-1 + \delta \nu) + v_i(n^i + \delta n^i) = 1 - \delta \nu + \phi + v_\parallel,
\]

(5.40)

therefore the condition \((E_{0\mu} \hat{k}^\mu)_o = 1\) sets

\[
\delta \nu_o = \phi_o + v_\parallel,
\]

(5.41)

which we need as initial condition in order to integrate Eq. (5.39). The initial condition for the spatial part of the wave vector is

\[
\delta n^a_o = -v^a_o + n^a \psi_o \rightarrow \delta n^i_o = -v^i_o + n^i \psi_o.
\]

(5.42)

Using Eqs. (5.41)–(5.42), Eq. (5.39) can be integrated

\[
\begin{align*}
\delta \nu &= 2\phi - (\phi_o - v_\parallel) + \int_0^\bar{\chi} d\bar{\chi} (\phi' + \psi'), \\
\delta n^i &= -v_o^i - n^i \psi_o + 2n^i \psi - \int_0^\bar{\chi} d\bar{\chi} \bar{\partial}^i(\phi + \psi) = \hat{n}^i \delta n_\parallel + \delta n^i_\perp,
\end{align*}
\]

and

\[
\begin{align*}
\delta n_\parallel &= \phi_o - v_\parallel - \phi + \psi + 2I, \\
\delta n^i_\perp &= -v^i_\perp + 2S^i_\perp,
\end{align*}
\]

(5.45)

where we defined

\[
I = -\frac{1}{2} \int_0^\bar{\chi} d\bar{\chi} (\phi' + \psi'), \\
S^i_\perp = -\frac{1}{2} \int_0^\bar{\chi} d\bar{\chi} \bar{\partial}^i_\perp(\phi + \psi).
\]

(5.46)
Moreover
\[ \bar{\partial}_0 (\delta k^0 + \delta k^\perp) = \phi' + \psi', \quad \delta \Gamma^\mu_{\rho\mu} \bar{k}^\mu = -\phi' + 3\psi' + n^i \bar{\partial}_i \phi - 3n^i \bar{\partial}_i \psi = \frac{d\phi}{d\bar{\chi}} - 3\frac{d\psi}{d\bar{\chi}}. \] (5.47)

With this expressions we can find \( \delta \ln \mathcal{A}^T \) in the Poisson gauge by integrating Eq. (5.35)
\[ \delta \ln \mathcal{A}^T = \delta \ln \mathcal{A}_0 + \kappa + \psi - \psi_o - \frac{1}{2} \left[ \left( \frac{F_{x^0}}{F_0} \delta \varphi \right) (\bar{\chi}) - \left( \frac{F_{x^0}}{F_0} \delta \varphi \right) o \right], \] (5.48)

where we used \( \kappa_o = I_o = 0 \). In order to obtain \( \Delta \ln \mathcal{A}^T \) as in Eq. (5.36), we need also \( \Delta \ln a \) and \( T \), respectively they are
\[ \Delta \ln a = \mathcal{H} \Delta x^0 = E^{(1)} \delta k^\mu + E^{(0)} \delta k^\mu = (\phi_o - v^\perp) - (\phi - v^\perp) + 2I, \] (5.49)
\[ T = -\int_0^\bar{\chi} d\tilde{\chi} (\phi + \psi). \] (5.50)

In this way, the total correction to the GW amplitude in the Poisson gauge, under the assumption that \( \mathcal{A}^0 \) is suppressed at the emission, is\(^{13}\)
\[ \Delta \ln \mathcal{A}^T (\bar{\chi}) = \kappa + \psi - \frac{1}{2} \left( \frac{F_{x^0}}{F_0} \delta \varphi \right) (\bar{\chi}) - \frac{1}{\bar{\chi}} \int_0^\bar{\chi} d\tilde{\chi} (\phi + \psi) \]
\[ - (\phi_o - v^o - \phi + v^\perp) \left( 1 + \frac{F_{x^0} v^0}{2F_0 \mathcal{H}} - \frac{1}{\bar{\chi} \mathcal{H}} \right). \] (5.51)

The weak-lensing convergence term was \( \kappa \) can be also written in the more familiar way
\[ \kappa = -\frac{1}{2} \bar{\partial}_\perp \Delta x^i_\perp = \frac{1}{2} \int_0^\bar{\chi} d\tilde{\chi} \frac{(\bar{\chi} - \tilde{\chi})}{\bar{\chi} \tilde{\chi}} \Delta \Omega \bar{\partial}^i_\perp (\phi + \psi), \] (5.52)

where \( \Delta \Omega = \bar{\chi}^2 \bar{\nabla}^2 \), using
\[ \Delta x^i_\perp = \delta x^i_\perp = \int_0^\bar{\chi} d\chi \delta n^i_\perp = -v^i_\perp \bar{\chi} - \int_0^\bar{\chi} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \bar{\partial}^i_\perp (\phi + \psi). \] (5.53)

### 5.4 The luminosity distance

We can use the above results to write the full form of the gravitational wave in the geometrical optics limit
\[ \bar{h}_{\mu\nu}^{TT} = \mathcal{A}^T e^{TT} e^{i\theta/\epsilon} = \bar{A}^T (\bar{x}^0, \bar{\chi}) \left( 1 + \Delta \ln \mathcal{A}^T \right) e^{TT} e^{i(\theta(\bar{\chi}) + \Delta \theta)/\epsilon}. \] (5.54)

\(^{13}\)We have also used that by construction \( \delta \ln \mathcal{A}^T = \psi_o + \frac{1}{2} \left( \frac{F_{x^0}}{F_0} \delta \varphi \right) o \)
The amplitude at emission is given by

\[ A^T(\bar{\eta}_e, \bar{x}_i^e) = \bar{A}^T \left( 1 + \Delta \ln A^T \right) = \frac{Q(1 + z)^2}{\bar{D}^{GR}_L \sqrt{F_0}} \left( 1 + \Delta \ln(A^T) \right), \]  

where we used Eq. (5.33) and \( \bar{a}(\eta_e) = (1 + z)^{-1} \). \( \bar{D}^{GR}_L = (1 + z)\bar{\chi} \) is the observed average luminosity distance taken over all the sources with the same observed redshift in GR. We can define the following quantities

\[ \bar{D}^{gw}_L = \bar{D}^{GR}_L \sqrt{F_0}, \]  

\[ D^{gw}_L = \frac{\bar{D}^{GR}_L \sqrt{F_0}}{(1 + \Delta \ln(A^T))} = \bar{D}^{gw}_L (1 - \Delta \ln(A^T)), \]  

therefore, the relative correction to the gravitational luminosity distance is

\[ \frac{\Delta D^{gw}_L}{D^{gw}_L} = \frac{D^{gw}_L - \bar{D}^{gw}_L}{D^{gw}_L} = -\Delta \ln A^T(\bar{\chi}). \]  

The gravitational wave at the detector is also red-shifted. We can use Eq. (5.58) and Eq. (5.51) to write

\[ \frac{\Delta D^{gw}_L}{D^{gw}_L} = \left( 1 - \frac{1}{\bar{\chi} H} + \frac{F \varphi_0 \varphi_0'}{2F_0 H} \right) v_\parallel - \frac{1}{2} \int_0^{\bar{\chi}} d\bar{\chi} \left( \frac{\bar{\chi} - \bar{\chi}}{\bar{\chi}} \right) \Delta \Omega_{\perp} \delta_\perp (\phi + \psi) \]  

\[ + \phi \left( \frac{1}{\bar{\chi} H} - \frac{F \varphi_0 \varphi_0'}{2F_0 H} \right) - \left( 1 + \frac{F \varphi_0 \varphi_0'}{2F_0 H} - \frac{1}{\bar{\chi} H} \right) \int_0^{\bar{\chi}} d\bar{\chi} (\phi' + \psi') \]  

\[ - \left( \phi + \psi - \frac{F \varphi_0}{2F_0} \delta(\bar{\chi}) \right) + \frac{1}{\bar{\chi}} \int_0^{\bar{\chi}} d\bar{\chi} (\phi + \psi), \]  

where we dropped the unobservable constant contribution evaluated at the observer position.

If we choose \( F = \text{const} \), we recover the correct limit to GR, which can be found in [103]; on the other hand, Eq (5.59) shows how modified gravity alters the GW luminosity distance.

In fact, in Eq. (5.59) it is possible to distinguish six different contributions:

- a peculiar velocity contribution, explicitly modified by dynamical dark energy;
- a weak-lensing contribution, not explicitly altered;
- a Sachs-Wolfe effect, explicitly modified;
- an integrated Sachs-Wolfe effect, explicitly modified;
- volume effects, explicitly modified;
- a contribution from Shapiro time delay, not explicitly altered dynamical dark energy.

We stress that, on top of the explicit modifications in Eq. (5.59), due to the presence of the extra degree of freedom, also the dynamics of \( \phi \) and \( \psi \) is also potentially different with respect to GR.
5.5 The polarization tensor

We turn now to the evolution equation of the polarization tensor $e_{\mu\nu}$. In the case $A^S = 0 = G_X$, its dynamics is governed by Eq. (5.2). The gauge conditions are given by Eqs. (3.7)–(3.9).

It is possible to rewrite Eq. (5.2) in a more convenient way:

$$k^\mu \nabla_\rho e_{\mu\nu} = \frac{F_\rho}{4F} \bar{\psi} [e_{\sigma\mu} k_\nu + e_{\sigma\nu} k_\mu] = g^{\lambda\sigma} \frac{1}{4} \partial_\lambda \ln F [e_{\sigma\mu} k_\nu + e_{\sigma\nu} k_\mu],$$

(5.60)

where we used $C_\mu = \nu^\alpha e_{\alpha\mu} = \nabla^\alpha \bar{\psi} e_{\alpha\mu}$ and the fact that $F$ only depends on $\bar{\psi}$. After the conformal transformation, Eq. (5.6), Eq. (5.60) becomes

$$\hat{k}^\mu \nabla_\rho (e_{\mu\nu} a^{-2}) = \hat{g}^{\lambda\sigma} \frac{1}{4a^2} \left[ e_{\sigma\mu} \hat{k}_\nu + e_{\sigma\nu} \hat{k}_\mu \right] \left[ \partial_\lambda \ln F - 2\partial_\lambda \ln a^2 \right],$$

(5.61)

ad the three gauge conditions read

$$\hat{k}^\mu e_{\mu\nu} = 0,$$

(5.62)

$$\hat{k}^\mu C_\mu = 0,$$

(5.63)

$$\hat{g}^{\alpha\mu} \hat{v}_\alpha e_{\mu\nu} = \hat{C}_\nu,$$

(5.64)

where we have defined $\hat{C}_\nu = C_\nu a^2$. We also define the following decompositions

$$e_{\mu\nu} = \bar{e}_{\mu\nu} + \delta e_{\mu\nu}, \quad \bar{v}_\mu = \varphi'_0 \delta_{\mu0} + \delta v_\mu, \quad \hat{C}_\mu = \hat{C}^0_\mu + \delta \hat{C}_\mu,$$

(5.65)

where $\bar{e}_{\mu\nu}$, $\varphi'_0 \delta_{\mu0}$ and $\hat{C}^0_\mu$ are respectively the values that $e_{\mu\nu}$, $\bar{v}_\mu$ and $\hat{C}_\mu$ would have on an unperturbed space-time, while $\delta e_{\mu\nu}$, $\delta v_\mu$, $\delta \hat{C}_\mu$ represent their corresponding linear corrections. Note that we can make the choice $\hat{C}^0_\mu = 0$ since, as previously shown, we can choose a vanishing $C_\mu$ on a unperturbed space-time, compatibly with the transport equations.

The expansion of Eq. (5.61) at linear order is made of two contributions, namely the left hand side is

$$1) \hat{k}^\mu \nabla_\rho (e_{\mu\nu} a^{-2}) = \frac{d}{d\chi} (e_{\mu\nu} a^{-2}) = \left( 1 - \frac{d\Delta}{d\chi} \right) \frac{d}{d\chi} ((\bar{e}_{\mu\nu} + \Delta e_{\mu\nu}) a^{-2} (1 - 2\Delta \ln a)) =$$

$$= \frac{d(\bar{e}_{\mu\nu} a^{-2})}{d\chi} + \frac{d(\delta e_{\mu\nu} a^{-2})}{d\chi} + \frac{d(\delta \varphi'_0 \delta_{\mu0} a^{-2})}{d\chi} + \delta \chi \frac{d^2(\bar{e}_{\mu\nu} a^{-2})}{d\chi^2},$$

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where $\Delta e_{\mu\nu} = \delta e_{\mu\nu} + \Delta x^\rho \nabla_\rho e_{\mu\nu}$, and the the right hand side is

$$
2) \frac{\dot{g}^\lambda}{2 \dot{a}^2} [e_{\sigma\mu} \dot{k}_\nu + e_{\sigma\nu} \dot{k}_\mu] \partial_\lambda \left( \ln F - 2 \ln a^2 \right) = \nonumber
= \frac{\dot{g}^\lambda}{2 \dot{a}^2} [e_{\sigma\mu} \dot{k}_\nu + e_{\sigma\nu} \dot{k}_\mu] [\delta_\lambda^\alpha - \partial_\lambda \Delta x^\alpha] \partial_\alpha \left( \ln \left[F_0 \left(1 + \Delta \varphi \frac{F'}{F_0}\right)\right] - 4 \ln [a(1 + \mathcal{H} \Delta x^0)] \right) = 0
$$

with $\mathcal{B} = -4\mathcal{H} \left(1 - \frac{\dot{F}_0 \varphi' \dot{\varphi}}{F_0} \right)$.

Therefore, by equating the expansions of left and right hand side of Eq. (5.61) we obtain

$$
\frac{d(e_{\mu\nu} a^2)}{d\chi} + \frac{d((\delta e_{\mu\nu} a^2))}{d\chi} + \partial \frac{d^2(e_{\mu\nu} a^2)}{d\chi^2} = 0
$$

From Eq. (5.67) we can extract the zero order contribution, which reads

$$
\frac{d(e_{\mu\nu} a^2)}{d\chi} = \frac{\mathcal{B}}{4 \dot{a}^2} (\dot{k}_\mu \dot{e}_{0\nu} + \dot{k}_\nu \dot{e}_{0\mu}).
$$

In the GR limit, $\mathcal{B} = -4\mathcal{H}$, and Eq. (5.68) reduces to the parallel transport equation of the zero order polarization tensor along the unperturbed geodesics, written in the RGW-frame and conformal coordinates. Moreover, choosing $\dot{e}_{\mu} = 0$, the gauge condition Eq. (5.64) at the leading order yields

$$
\varphi' \eta^{\mu0} \dot{e}_{\mu\nu} = 0 \quad \text{hence} \quad \dot{e}_{0\nu} = 0,
$$

and Eq. (5.68) becomes

$$
\frac{d(e_{\mu\nu} a^2)}{d\chi} = 0 \quad \Rightarrow \quad e_{\mu\nu} = \ddot{a}^2 Q_{\mu\nu}
$$

where $Q_{\mu\nu}$ is a constant, transverse tensor\footnote{In Sec. 5, we have shown that the assumption $\dot{e}_{\mu} = 0$ holds for an unperturbed FRW space-time. In the GR limit we cannot use this equation to impose $\dot{e}_{0\nu} = 0$; however in this case, for an unperturbed FRW space-time, we are always allowed to choose a purely spatial polarization tensor with respect to the observer frame.}.
Using Eq. (5.70), the first order term of Eq. (5.67) yields

$$\frac{d(\delta e_{\mu\nu}\tilde{a}^{-2})}{d\bar{\chi}} = \frac{1}{4\tilde{a}^2} \eta^{\lambda\alpha}(\tilde{k}_\mu \delta e_{\lambda\nu} + \tilde{k}_\nu \delta e_{\lambda\mu}) \partial_\alpha \left( \frac{F_{\varphi}^0}{F_0} \delta \varphi \right)$$

$$+ \frac{B}{4\tilde{a}^2} \left( \tilde{k}_\mu \delta e_{\nu\mu} + \tilde{k}_\nu \delta e_{\nu\mu} + \delta g^{\lambda\alpha}(\tilde{k}_\mu \delta e_{\lambda\nu} + \tilde{k}_\nu \delta e_{\lambda\mu}) \right).$$

(5.71)

This equation allows us to obtain $\delta e_{\mu\nu}\tilde{a}^{-2}$ by integration. Then, in the case of a vanishing $A^5$, the full correction to the GW polarization, i.e. $\Delta e_{\mu\nu}(\bar{\chi})$ is

$$\Delta e_{\mu\nu} = \delta e_{\mu\nu} + 2\mathcal{H} \Delta x^0 \epsilon_{\mu\nu}.$$  

(5.72)

where we used Eq. (5.70) and that Christoffel symbols are null at zero order.

Now we can specialize this result to the case of a background metric in the Poisson gauge, where $\delta g^{\lambda\alpha} = 2\phi \delta^{\lambda\alpha}$. Eq. (5.72) becomes

$$\frac{d(\delta e_{\mu\nu}\tilde{a}^{-2})}{d\bar{\chi}} = \frac{1}{4\tilde{a}^2} \eta^{\lambda\alpha}(\tilde{k}_\mu \delta e_{\lambda\nu} + \tilde{k}_\nu \delta e_{\lambda\mu}) \partial_\alpha \left( \frac{F_{\varphi}^0}{F_0} \delta \varphi \right) + \frac{B}{4\tilde{a}^2} \left( \tilde{k}_\mu \delta e_{\nu\mu} + \tilde{k}_\nu \delta e_{\nu\mu} \right).$$

(5.73)

We have derived a general formula, Eq. (5.73), which proves that perturbations of the polarization tensor are generally not parallely transported in a perturbed universe. The failure of parallel transport depends both on scalar fluctuations, and on modified gravity effects. The GR limit is recovered for $F = \text{const}$, and therefore $B = -4\mathcal{H}$. We plan to further study physical consequences of these results in future works.

6 Conclusions

The detection of gravitational waves (GWs) propagating over cosmological distances will offer new opportunities for probing cosmological structures in our Universe, as well as for testing possible departures from GR. In this work we established tools for studying GW propagation in scalar-tensor theories of gravity, adopting a geometrical ansatz, and focussing on high frequency spin-2 and spin-0 modes travelling on slowly-varying backgrounds. Our approach can apply to scenarios with non-minimal couplings between scalar and tensor degrees of freedom: we took particular care in discussing and motivating our gauge choices—see Eqs. (2.33)–(2.35)—and to develop general arguments that do not rely on the particular choice of the scalar-tensor theory.

We determined the general structure of the evolution equations for the GW amplitude and polarization tensor. We found that they can be different with respect to GR – see in particular Eqs. (3.14)–(3.15), and Eqs. (3.30)–(3.31). In theories which preserve the graviton number, the equation for the amplitude of GWs can differ from GR if the effective Planck scale varies over the GW geodesics. Interestingly, we also find that the GW polarization tensor can fail to be parallely transported along the GW geodesics, and discussed physical interpretations of our results.

We then applied our general formulas to representative examples of scalar-tensor scenarios with GWs travelling with unit speed. In this context, Eqs. (4.13)–(4.15) show that the GW propagation depends both on background quantities and the assumed scalar-tensor theory, that the tensor and scalar modes of polarization are generally coupled one to each other, and that the amplitude $A^T$ of the tensor mode does not depend on its scalar counterpart $A^S$. 

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An important advantage of our approach is that it can be applied to study the propagation of GWs on perturbed cosmological space-times, provided that the wavelength of cosmological fluctuations is well larger than the GW wavelength. In fact, under the simplified assumption that \( A^S \) is suppressed at the GW emission, we derived the general evolution equations controlling GW propagation in such a set-up, distinguishing and identifying the distinct contributions associated with cosmological fluctuations, and with scalar-tensor effects.

More precisely, at first order in perturbation theory, we determined several corrections to standard formulas for the GW luminosity distance – see Eq. (5.59). We found that the contribution to the GW luminosity distance from the peculiar velocity, Sachs-Wolfe, integrated Sachs-Wolfe and volume effects are explicitly modified in the presence of scalar-tensor interactions. On the other hand, weak-lensing and Shapiro time delay are only implicitly altered, via the dynamics of \( \phi \) and \( \psi \), due to the presence of the extra degree of freedom.

Finally, under the further assumption \( G_X = 0 \), we obtained the evolution of the transverse polarisation tensor, Eq. (5.72), in the linear regime, proving that the failure of its parallel transport depends both on scalar fluctuations, and on modified gravity effects. This effect is absent at first order in GR, and arises only beyond the linear regime.

Starting from the general results presented here, much work is left for the future. It would be interesting to apply our method to more general scalar-tensor theories than the representative examples discussed here, or to extend our approach to modified gravity scenarios with multiple fields. It would be interesting to study in more detail the physical consequences of our findings for GWs propagating over cosmological distances, and analyse in detail degeneracies between modified gravity and effects of non-linearities. We hope to return soon to these topics.

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A Defining spin-2 and spin-0 polarization tensors using tetrads

This Appendix aims to explain the method we use in the main text for disentangling the evolution equations for the amplitude of tensor and scalar modes. We introduce null tetrads as [52, 111] in the form of four four-vectors

\[
z^\mu_a = (k^\mu, n^\mu, m^\mu, \bar{m}^\mu) ;
\]

\( k^\mu \) and \( n^\mu \) are real, \( m^\mu \) complex, and \( \bar{m}^\mu \) its complex conjugate. These vectors all null with respect to background metric \( \bar{g}_{\mu\nu} \). The vector \( k^\mu \) is identified with the GW direction. That is (raising/lowering with metric \( \bar{g} \)):

\[
0 = k_\mu k^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = n_\mu n^\mu
\]

Moreover they all orthogonal one with the other, besides \( m_\mu \) and \( \bar{m}_\mu \) that satisfy:

\[
\bar{g}_{\mu\nu} m^\mu \bar{m}^\nu = 1
\]
In what follows, to treat tensor and scalar polarizations we use only the complex $m^\mu$ and real $k^\mu$. The latter is parallely transported: $k^\mu \nabla_\rho k^{\nu} = 0$. The vector $n^\mu$ is used to treat vector polarizations [52, 111], we shall not need it here.

These vectors allow us to define spin-2 and spin-0 polarization tensors. Following [52, 111], a spin-2 (complex) transverse-traceless tensor can be defined as

$$m_\mu m_\nu$$  \hspace{1cm} (A.4)

This tensor is complex, but can be easily separated into a real and imaginary part

$$m_\mu m_\nu = \frac{1}{\sqrt{2}} e^+_{\mu \nu} + \frac{i}{\sqrt{2}} e^\times_{\mu \nu}$$  \hspace{1cm} (A.5)

with

$$e^+_{\mu \nu} = \frac{1}{\sqrt{2}} (m_\mu m_\nu + \bar{m}_\mu \bar{m}_\nu)$$  \hspace{1cm} (A.6)

$$e^\times_{\mu \nu} = \frac{i}{\sqrt{2}} (\bar{m}_\mu \bar{m}_\nu - m_\mu m_\nu)$$  \hspace{1cm} (A.7)

These are spin-2 real tensors, that correspond to plus and cross polarizations. They are normalized to unity, and they’re transverse each other.

Moreover, using the results of [113], we can build a spin-0 transverse (but not traceless!), real polarization tensor as

$$e^S_{\mu \nu} = \sqrt{2} \bar{m}_\mu m_\nu = \frac{1}{\sqrt{2}} (m_\mu \bar{m}_\nu + m_\nu \bar{m}_\mu)$$  \hspace{1cm} (A.8)

That has also norm one.

It is convenient to rewrite the previous results using the vectors

$$\alpha_\mu = \frac{1}{\sqrt{2}} (m_\mu + \bar{m}_\mu)$$  \hspace{1cm} (A.9)

$$\beta_\mu = \frac{i}{\sqrt{2}} (\bar{m}_\mu - m_\mu)$$  \hspace{1cm} (A.10)

These are orthogonal vectors of unit length. With these, one has

$$e^+_{\mu \nu} = \frac{1}{\sqrt{2}} (\alpha_\mu \alpha_\nu - \beta_\mu \beta_\nu)$$  \hspace{1cm} (A.11)

$$e^\times_{\mu \nu} = \frac{1}{\sqrt{2}} (\alpha_\mu \beta_\nu + \alpha_\nu \beta_\mu)$$  \hspace{1cm} (A.12)

$$e^S_{\mu \nu} = \frac{1}{\sqrt{2}} (\alpha_\mu \alpha_\nu + \beta_\mu \beta_\nu)$$  \hspace{1cm} (A.13)

Some results we are going to use next. We find the equalities

$$\sqrt{2} \alpha^\mu \nabla_\rho e^+_{\mu \nu} = \nabla_\rho \alpha_\nu - \alpha^\mu \beta_\nu \nabla_\rho \beta_\mu$$  \hspace{1cm} (A.14)

$$\sqrt{2} \alpha^\mu \nabla_\rho e^\times_{\mu \nu} = \nabla_\rho \beta_\nu + \alpha^\mu \alpha_\nu \nabla_\rho \beta_\mu$$  \hspace{1cm} (A.15)

$$\sqrt{2} \alpha^\mu \nabla_\rho e^S_{\mu \nu} = \nabla_\rho \alpha_\nu + \alpha^\mu \beta_\nu \nabla_\rho \beta_\mu$$  \hspace{1cm} (A.16)
The implied relations from these relations, immediately:
\[
\alpha^\mu \alpha^\nu \nabla^\rho e^+_{\mu \nu} = \alpha^\mu \alpha^\nu \nabla^\rho e^S_{\mu \nu} = 0
\]  
(20)
\[
\alpha^\mu \alpha^\nu \nabla^\rho e^\times_{\mu \nu} = \sqrt{2\alpha^\mu \nabla^\rho \beta^\mu}
\]  
(21)
\[
\beta^\mu \beta^\nu \nabla^\rho e^+_{\mu \nu} = \beta^\mu \beta^\nu \nabla^\rho e^S_{\mu \nu} = 0
\]  
(22)
\[
\beta^\mu \beta^\nu \nabla^\rho e^\times_{\mu \nu} = \sqrt{2\beta^\mu \nabla^\rho \alpha^\mu}
\]  
(23)
\[
\alpha^\mu \beta^\nu \nabla^\rho e^+_{\mu \nu} = \frac{1}{\sqrt{2}} (\beta^\mu \nabla^\rho \alpha^\mu - \alpha^\mu \nabla^\rho \beta^\mu)
\]  
(25)
These imply the relations
\[
0 = e^+_{\mu \nu} \nabla^\rho e^+_{\mu \nu} = e^S_{\mu \nu} \nabla^\rho e^S_{\mu \nu} = e^+_{\mu \nu} \nabla^\rho e^S_{\mu \nu} = e^S_{\mu \nu} \nabla^\rho e^S_{\mu \nu}
\]  
(26)
\[
0 = e^\times_{\mu \nu} \nabla^\rho e^\times_{\mu \nu}
\]  
(27)
\[
0 = e^\times_{\mu \nu} \nabla^\rho e^\times_{\mu \nu} = e^\times_{\mu \nu} \nabla^\rho e^S_{\mu \nu}
\]  
(28)
\[
0 = e^\times_{\mu \nu} \nabla^\rho e^+_{\mu \nu} + e^\times_{\mu \nu} \nabla^\rho e^\times_{\mu \nu}
\]  
(29)
Using these tools, we can decompose the tensor \( A_{\mu \nu} \) as (we assume we do not violate parity, hence plus and cross polarizations have the same amplitude)
\[
A_{\mu \nu} = A e^S_{\mu \nu} = A^T (e^+_{\mu \nu} + e^\times_{\mu \nu}) + A^S e^S_{\mu \nu}
\]  
(30)
where \( A^2 = 2 \left( A^T \right)^2 + (A^S)^2 \).
We plug decomposition Eq. (30) into Eq. (3.13) and get
\[
2k^\rho \nabla^\rho (A^T e^S_{\mu \nu} + A^T e^\times_{\mu \nu} + A^S e^S_{\mu \nu}) + (A^T e^+_{\mu \nu} + A^T e^\times_{\mu \nu} + A^S e^S_{\mu \nu}) \left( \nabla^\rho k^\rho \right) = t_{\mu \nu}
\]  
(31)
This can be rewritten as
\[
\frac{1}{A^T} \left( \nabla^\rho \left[ k^\rho A^T \right] \right) e^+_{\mu \nu} + \frac{1}{A^T} \left( \nabla^\rho \left[ k^\rho A^S \right] \right) e^\times_{\mu \nu} + \frac{1}{A^S} \left( \nabla^\rho \left[ k^\rho A^S \right] \right) e^S_{\mu \nu} + 2k^\rho A^T \left( \nabla^\rho e^\times_{\mu \nu} \right) + 2k^\rho A^T \left( \nabla^\rho e^\times_{\mu \nu} \right) + 2k^\rho A^S \left( \nabla^\rho e^S_{\mu \nu} \right) = t_{\mu \nu}
\]  
(32)
From Eq. (32) we can extract independent equations for the amplitudes \( A^S, A^T \). Contracting Eq. (32) with \( e^S_{\mu \nu} \), and using Eqs. (A.26)-(A.28) we get
\[
\nabla^\rho \left[ k^\rho (A^S)^2 \right] = A^S t_{\mu \nu} (e^S_{\mu \nu})
\]  
(33)
Now, we first contract Eq. (32) with \( e^+_{\mu \nu} \), and get
\[
\frac{1}{A^T} \left( \nabla^\rho \left[ k^\rho (A^T)^2 \right] \right) + 2k^\rho e^+_{\mu \nu} A^T \left( \nabla^\rho e^\times_{\mu \nu} \right) = t_{\mu \nu} e^+_{\mu \nu}
\]  
(34)
We then contract Eq. (A.32) with $e^x_{\mu\nu}$, and get

$$
\frac{1}{\mathcal{A}^T} \left( \nabla^\rho \left[ k_\rho \left( A^T \right)^2 \right] \right) + 2k_\rho e^x_{\mu\nu} A^T \left( \nabla^\rho e^x_{\mu\nu} \right) = t_{\mu\nu} e^x_{\mu\nu}
$$

(A.35)

Summing Eqs. (A.34)–(A.35), and using Eq. (A.29), we get:

$$
\left( \nabla^\rho \left[ k_\rho \left( A^T \right)^2 \right] \right) = \frac{A^T}{2} \left( t_{\mu\nu} e^x_{\mu\nu} + t_{\mu\nu} e^x_{\mu\nu} \right)
$$

(A.36)

So we obtain independent equations for the amplitudes. Unfortunately, it is instead not possible to find separate evolution equations for scalar and tensor polarizations.

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