Tripartite genuine non-gaussian entanglement in three-mode spontaneous parametric downconversion

A. Agustí, 1 C.W. Sandbo Chang, 2 F. Quijandría, 3 G. Johansson, 3 C.M. Wilson, 2 and C. Sabin 1

1 Instituto de Física Fundamental, CSIC, Serrano, 113-bis, 28006 Madrid, Spain
2 Institute for Quantum Computing and Electrical and Computer Engineering, University of Waterloo, Waterloo, Canada
3 Microtechnology and Nanoscience, MC2, Chalmers University of Technology, SE-412 96 Göteborg, Sweden

We show that the states generated by a three-mode spontaneous parametric downconversion (SPDC) interaction Hamiltonian possess tripartite entanglement of a different nature to other paradigmatic three-mode entangled states generated by the combination of two-mode SPDC interactions. While two-mode SPDC generates gaussian states whose entanglement can be characterized by standard criteria based on two-mode quantum correlations, these criteria fail to capture the entanglement generated by three-mode SPDC. We use criteria built from three-mode correlation functions to show that the class of states recently generated in a superconducting-circuit implementation of three-mode SPDC ideally have tripartite entanglement, contrary to recent claims in the literature. These criteria are suitable for triple SPDC but we show that they fail to detect tripartite entanglement in other states which are known to possess it, which illustrates the existence of two fundamentally different notions of tripartite entanglement in three-mode continuous variable systems.

Parametric amplification of the quantum vacuum in superconducting-circuit architectures [1] can be a useful resource for quantum technologies. For instance, the modulation of a SQUID terminating a superconducting transmission line can generate pairs of photons out of the vacuum - a particular realisation of the dynamical Casimir effect [2] which exhibit entanglement and other forms of quantum correlations [3-6]. These correlations can be swapped to superconducting qubits [7,8] or can be used for continuous-variable quantum technological applications [9,10].

The extension of the above scheme to a multipartite setting can be approached in several ways. For instance, a two-tone modulation of the SQUID with frequencies addressing two different pairs of modes a-b and b-c, do not only generate quantum correlations between a-b and b-c, but also in the pair a-c, giving rise to genuine multipartite entanglement [11-13]. While this entanglement is indeed tripartite, it is generated by the simultaneous action of a pair of two-mode SPDC interactions, therefore the Hamiltonian is merely bilinear. As a consequence of this, the evolution preserves the gaussian nature of initial states (e.g. the vacuum) and can be fully characterised by its covariance matrix, which contains two-mode correlation functions at most. Indeed, the tripartite entanglement in [13] is detected by standard separability criteria based on two-mode correlation functions [14].

A very different picture emerges if one considers a three-mode SPDC Hamiltonian acting on the quantum vacuum, which in superconducting circuits can be engineered by suitably flux-pumping an asymmetric SQUID terminating a coplanar waveguide resonator, as recently demonstrated experimentally in [15]. As this scheme includes a genuine direct three-mode interaction, the relevant physical features cannot be captured by two-mode correlations and it is necessary to analyse the three-mode correlations [15]. Indeed, common criteria such as the ones in [14] fail to detect multipartite entanglement in these states, as have been noted in [16]. However, this only points out the necessity of using higher-order criteria to detect the generated tripartite entanglement. Therefore, the claim in [16] that there is no entanglement in these three-mode SPDC states is overly broad. The correct statement is that there is no Gaussian entanglement.

In this work, we use entanglement criteria based on three-mode and four-mode correlations to detect the tripartite entanglement of the states produced experimentally in [15]. We show that the states exhibit both full inseparability and genuine tripartite entanglement. We also show that the same criteria fail to detect tripartite entanglement in states produced by Hamiltonians in which the multimode interaction is induced by the combination of two-mode interactions, such as in a double two-mode SPDC. Since we know the latter are also tripartite entangled states, as shown in [13], our results clearly suggest that higher-order SPDC interactions generate a different kind of multipartite entanglement, characterised by its fully multipartite non-gaussian nature. We will refer to this novel notion of entanglement as tripartite genuine non-gaussian entanglement. Let us now start with the description of our results.

We analyse a system related to the experimental setup of [15], consisting of a superconducting resonator terminated by an asymmetric SQUID. We consider three field modes with frequencies \( \omega_i \), \( i = a, b, c \) and the corresponding creation and annihilation operators \( i, i^\dagger \) with standard bosonic commutation relations. We assume that initially each mode is in a weak thermal state \( \rho_i(n_{ih}) \), characterised by the corresponding low average number of thermal photons according to its frequency and temperature, as given by \( \langle n_i \rangle = 1/(e^{\beta_i} - 1) \), where \( \beta_i = \hbar \omega_i / (k_B T) \). The system evolves under the interaction Hamiltonian (\( \hbar = 1 \)):

\[
H_I = g_0 \cos \omega_0 I (e^{i\theta_a} a + e^{-i\theta_a} a^\dagger) (e^{i\theta_b} b + e^{-i\theta_b} b^\dagger) (e^{i\theta_c} c + e^{-i\theta_c} c^\dagger),
\]

where \( \theta_i \) are locally controllable phases and \( g_0 \) is the coupling strength. Choosing the coupling modulation \( \omega_0 \) as

\[
\omega_0 = \omega_a + \omega_b + \omega_c.
\]
gives rise to the effective Hamiltonian in the interaction picture:

$$H_I = \frac{\theta}{2} (e^{i\theta} abc + e^{-i\theta} a^\dagger b^\dagger c^\dagger),$$

(3)

where $\theta = \theta_a + \theta_b + \theta_c$. This is a third-order squeezing Hamiltonian for three different modes.

The standard criteria to detect tripartite entanglement, such as [14, 20], are based on inequalities concerning expectation values and correlations which involve of course the three modes but in a pairwise fashion, such as $\langle x_i x_j \rangle$ ($x_i, x_j$ being the position quadrature associated to the modes $i, j$ and $x = 1/\sqrt{2(a + a^\dagger)}$). However, looking at the Hamiltonian (3) it seems natural to think that these criteria are not suitable in this case. Indeed it was shown in [16] that some of these criteria were not able to detect tripartite entanglement for these states. Perhaps the most compelling evidence proving this point is that the covariance matrix of initial thermal states -including the vacuum- evolved under Hamiltonian (3) remains diagonal. But we must stress that the criteria used in [16] are sufficient but not necessary conditions on entanglement, and as such they are inconclusive when they fail. Thus, what is needed is to look for higher-order criteria able to capture the pure three-mode nature of the states generated by the Hamiltonian (3), nature that has been demonstrated, both theoretically and experimentally, by the absence of second-order correlations together with the existence of third-order ones [15].

A typical approach to tripartite entanglement is to consider all the possible bipartitions of the system. For instance, for each bipartition we can look at the inequalities developed in [21]. If $A_1$ and $A_2$ are operators acting respectively on the Hilbert spaces of two subsystems in which the total system is split, the total state is not separable with respect to this partition, namely $\rho \neq \sum_n P_n \rho_n^1 \otimes \rho_n^2$, if:

$$|\langle A_1 A_2 \rangle| > \sqrt{\langle A_1^2 A_1 \rangle \langle A_2^2 A_2 \rangle}.$$  

(4)

Therefore, in our case, choosing the annihilation operator as the reference operator in all cases, if:

$$|\langle abc \rangle| > \sqrt{|\langle N_a \rangle| |\langle N_b N_c \rangle|},$$

(5)

for the three possible $i - jk$ bipartitions (namely $a - bc, b - ac, c - ab$) of the system $N$ being the number operator- then we know that the state is not biseparable with respect to any bipartition, that is $\rho \neq \sum_n P_n \rho_n^{(i)} \otimes \rho_n^{(jk)}$ for the three bipartitions. If the state is not biseparable for the three bipartitions, then the state has full tripartite entanglement, in the sense that it is fully inseparable. Defining $I_i = |\langle abc \rangle| - \sqrt{|\langle N_i \rangle| |\langle N_j N_k \rangle|}$, we have that the state is fully inseparable if $I_i > 0$ for the three bipartitions.

However, even if the state has full inseparability, there is still the possibility that:

$$\rho = P_1 \sum_n P_{1n} \rho_{1n}^{(a)} \otimes \rho_{1n}^{(bc)} + P_2 \sum_m P_{2m} \rho_{2m}^{(b)} \otimes \rho_{2m}^{(ac)} + P_3 \sum_l P_{3l} \rho_{3l}^{(c)} \otimes \rho_{3l}^{(ab)},$$

(6)

namely the state can be written as a convex sum of biseparable states, but with each biseparable state belonging to a different bipartition. If this is the case, we would say that the state does not possess genuine tripartite entanglement [14, 22, 23]. Note that the difference between full inseparability and genuine entanglement is only relevant for non-pure states.

We have not found in the literature a condition for genuine tripartite entanglement involving correlations of more than two modes. However, we can derive an inequality for $|\langle abc \rangle|$ if the state is of the form (6). Using (5) and the triangle inequality, it is straightforward to write:

$$|\langle abc \rangle| \leq P_1 |\langle abc \rangle|_{P_1} + P_2 |\langle abc \rangle|_{P_2} + P_3 |\langle abc \rangle|_{P_3},$$

(7)

where in the LHS the expectation value refers to the total state $\rho$ while in the RHS refer to the different elements of the convex sum, that we are denoting $\rho_1, \rho_2, \rho_3$, namely:

$$\rho_1 = \sum_n P_{1n} \rho_{1n}^{(a)} \otimes \rho_{1n}^{(bc)} + \sum_m P_{2m} \rho_{2m}^{(b)} \otimes \rho_{2m}^{(ac)},$$

$$\rho_2 = \sum_n P_{3n} \rho_{3n}^{(c)} \otimes \rho_{3n}^{(ab)}.$$  

(8)

Now we know that, by construction, $\rho_1, \rho_2$ and $\rho_3$ are biseparable and therefore they must violate the inequalities (5). Then:

$$|\langle abc \rangle| \leq P_1 \sqrt{|\langle N_a \rangle| \langle N_b N_c \rangle} + P_2 \sqrt{|\langle N_b \rangle| \langle N_c N_a \rangle} + P_3 \sqrt{|\langle N_c \rangle| \langle N_a N_b \rangle},$$

(9)

Finally, using again (6) we have that, for instance:

$$P_1 |\langle N_a \rangle|_{P_1} - P_2 |\langle N_a \rangle|_{P_2} - P_3 |\langle N_a \rangle|_{P_3} \leq |\langle N_a \rangle|,$$

(10)

and similarly with all the expectation values in (9). Putting everything together, we find that if the state is of the form (6) then:

$$|\langle abc \rangle| \leq \sqrt{|\langle N_a \rangle| \langle N_b N_c \rangle} + \sqrt{|\langle N_b \rangle| \langle N_c N_a \rangle} + \sqrt{|\langle N_c \rangle| \langle N_a N_b \rangle},$$

(11)

where we have let the subindex $\rho$ drop since it would be the same for all expectation values. Therefore, we conclude that if a state violates (11), then it possesses genuine tripartite entanglement, which implies fully inseparability. If it does not violate the inequality (11) but verifies (5) for the three bipartitions, then it is just fully inseparable. We define $G = |\langle abc \rangle| - \sqrt{|\langle N_a \rangle| \langle N_b N_c \rangle} - \sqrt{|\langle N_b \rangle| \langle N_c N_a \rangle} - \sqrt{|\langle N_c \rangle| \langle N_a N_b \rangle}$, and then $G > 0$ is the condition for genuine tripartite entanglement.

If we neglect the initial temperature and consider that we are in a perturbative regime where the system evolves to a pure state containing only the vacuum and a triplet with small
probability amplitude $\alpha$, then $\langle abc \rangle \simeq \alpha$, while the $\langle N_i \rangle$, $\langle N_i N_k \rangle$ are of order $|\alpha|^2$. Then, for very low temperatures and coupling strengths, the conditions for entanglement are expected to be satisfied. In order to confirm and generalise this analytical intuition, we present now numerical results for the above inequalities for the states generated by the evolution under the interaction Hamiltonian (3) for three modes $\omega_a = \omega$, $\omega_b = 2\omega$, $\omega_c = 3\omega$ in the parameter regime $\beta_i > 1$, $g_0/\omega < < 1$ (low temperature and low coupling). We find that the states possess both full tripartite entanglement and genuine tripartite entanglement for $g_0/\omega = 1/250$ and $g_0/\omega = 1/100$ for $\beta_a = 10$ (lowest temperature) and $\beta_a = 5$. In this regime, the amount of violation of the inequalities increases with time and with $g_0$ in the case of full inseparability. However, for $g_0/\omega = 1/100$ the amount of violation of the inequality associated with genuine tripartite entanglement eventually decreases, and in the case of $\beta_a = 5$ and $\beta_a = 2$ (largest temperature) it even reaches small negative values. Finally, for the largest value of the coupling $g_0/\omega = 1/40$ the inequality stops to detect genuine tripartite entanglement earlier and eventually reaches very large negative values. Of course, let us note again that this does not mean necessarily that there is no genuine tripartite entanglement, since there might be another condition more suitable to detect it in this case. However, it is important to note that also the inequalities related to full tripartite entanglement stop to grow linearly with the coupling strength at $g_0/\omega = 1/100$ and also start to decrease at large times, although still taking positive values.

While the detrimental effect of temperature –small at this low-temperature regime– is fully expected, it might come as a surprise that the relation with coupling strength is more subtle. At first sight, we might think that increasing the cou-
pling strength would always be beneficial for entanglement since it increases the probability of generating the triplet of photons. Indeed, this view is fully valid in the perturbative regime $g t << 1$. However, richer dynamics must emerge if $g t \simeq 1$, and this is what our results seem to suggest. Whether this means that there is no tripartite entanglement in the non-perturbative regime or that new criteria—perhaps of even higher order—are needed remains as an open question.

We have seen that the states generated by the action of a three-mode SPDC states acting on an initial weak thermal possess tripartite entanglement—contrary to the claim in [16]—which can be detected by our three-mode criteria, not by two-mode ones. We can compare these results with the case of double-SPDC—namely, simultaneous two-tone modulation of $g(t)$ at frequencies $\omega_a + \omega_b$ and $\omega_b + \omega_c$—where it is shown in [13] that the resulting state possesses not only full inseparability but genuine entanglement by means of two-mode-correlations-based entanglement conditions. We see in Figure 2 that even for $\beta_a = 10$ and $g_0 = \omega/100$—which would in principle be the most favorable conditions for entanglement—the double-SPDC state fails to violate the inequality in Eq. (5) and indeed does not satisfy (11) either. Therefore, we are now in the opposite scenario: tripartite entanglement is detected by two-mode criteria but not by our three-mode criteria. This suggests that the key to choosing the right inequalities lies in knowing the Hamiltonian that is producing the states.

The three- and two-modes SPDCs are not the only Hamiltonians attainable in [15], and for completeness we consider the 1-2 photon swapping case—modulation at $2\omega_b - \omega_a$ and effective Hamiltonian $H_I = g_0 (a(b^\dagger)^2 + h.c.)$ [24]. We see that it is clearly a two-mode interacting term and the conditions for full tripartite entanglement and genuine tripartite entanglement are not satisfied at $\beta_a = 2$ (see Figure 3). Note that this in this case temperature would be in principle positive since the above Hamiltonian cannot generate any dynamics when acting upon an initial vacuum.

Therefore, we see that the conditions that we have introduced are able to capture a different notion of tripartite entanglement—genuine non-gaussian tripartite entanglement—characterised by the generation of pure three-mode interactions, as opposed to the simultaneous combination of several two-mode interactions or a cubic interaction involving only two modes.

Summarising, we have shown that three-mode SPDC interaction Hamiltonians generate states with both full tripartite entanglement and genuine tripartite entanglement when acting upon an weak thermal state, contrary to previous claims in the literature. The type of tripartite entanglement displayed by these states is different from other paradigmatic three-mode states, and therefore needs to be captured by different entanglement criteria. We introduce entanglement criteria based on three-mode correlations and show that our states satisfy them in a promising parameter regime. However, we show that double-SPDC Hamiltonians acting on the quasi-vacuum state, which generate states that have been proven to also possess tripartite entanglement by means of different two-mode criteria, fail to satisfy our conditions. This points to two different classes of tripartite entanglement in three-mode systems. On the one hand, we have states generated by three-mode Hamiltonians consisting of the combination of several two-mode interactions. Those states are gaussian and their entanglement can be detected by means of entanglement criteria built on two-mode correlation functions. On the other hand, we have states generated by the action of pure three-mode interactions upon a weak thermal state, which gives rise to non-gaussian states whose entanglement needs to be characterised by higher-order correlation functions.

Our results pave the way for multipartite entanglement tests in the experimental setup of [15] and could be a guide for the characterisation and measurement of entanglement in three-mode SPDC in other platforms [25, 26].
Acknowledgements

AA and CS have received financial support through the Postdoctoral Junior Leader Fellowship Programme from la Caixa Banking Foundation (LCF/BQ/LR18/11640005). CMW and CWSC acknowledge funding from NSERC of Canada and the Canada First Research Excellence Fund (CFREF). F. Q. and G. J. acknowledge support from the Wallenberg Center for Quantum Technology (WACQT).

[1] P. D. Nation, J. R. Johansson, M. P. Blencowe, and F. Nori, Rev. Mod. Phys. 84, 1 (2012).
[2] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, Nature (London) 479, 376 (2011).
[3] J. R. Johansson, G. Johansson, C. M. Wilson, P. Delsing, and F. Nori, Phys. Rev. A 87, 043804 (2013).
[4] C. Sabin, I. Fuentes, and G. Johansson, Phys. Rev. A 92, 012314 (2015).
[5] C. Sabin and G. Adesso, Phys. Rev. A 92 042107 (2015).
[6] D. N. Samos-Saénz de Buruaga and C. Sabin, Phys. Rev. A 95, 022307 (2017).
[7] S. Felicetti, M. Sanz, L. Lamata, G. Romero, G. Johansson, P. Delsing, and E. Solano, Phys. Rev. Lett. 113, 093602 (2014).
[8] A. Agustí, E. Solano, C. Sabin, Phys. Rev. A 99, 052328 (2019).
[9] D. E. Bruschi, C. Sabin, P. Kok, G. Johansson, P. Delsing, I. Fuentes, Sci. Rep. 6, 18349 (2016).
[10] B. Peropadre, J. Huh, C. Sabin, Sci. Rep. 8, 3751 (2018).
[11] P. Lähteenmäki, G. S. Paraoanu, J. Hassel, P. J. Hakonen, Nature Comm. 7, 12548 (2016).
[12] D. E. Bruschi, C. Sabin, G. S. Paraoanu, Phys. Rev. A 95, 062324 (2017).
[13] C. W. Sandbo Chang et al. Phys. Rev. Appl. 10, 044019 (2018).
[14] R. Y. Teh, M. D. Reid, Phys. Rev. A 90, 062337 (2014).
[15] C. W. Sandbo Chang et al. Phys. Rev. X 10, 011011 (2020).
[16] E. A. Rojas González, A. Borne, B. Boulanger, J. A. Levinson and K. Bencheikh Phys. Rev. Lett. 120, 043601 (2018).
[17] S. L. Braunstein, R. I. McLachlan, Phys. Rev. A 35, 1659 (1987).
[18] S. L. Braunstein, C. M. Caves, Phys. Rev. A 42, 4115 (1990).
[19] M. Hillery, Phys. Rev. A 42, 498 (1990).
[20] P. van Loock, A. Furusawa, Phys. Rev. A 67, 052315 (2003).
[21] M. Hillery, Ho Trung Dong and H. Zheng, Phys. Rev. A 81, 062322 (2010).
[22] S. Gerke et al. Phys. Rev. Lett. 117, 773 (2016).
[23] E. Shchukin, P. van Loock, Phys. Rev. A 92, 042328 (2015).
[24] Z. Leghtas et al. Science 347, 853 (2015).
[25] N. A. Borschchevskaya, K. G. Katamadze, S. P. Kulik, M. V. Fedorov, Laser Phys. Lett. 12, 11404 (2015).
[26] M. Corona, K. Garay-Palmett, A. B. URen, Opt. Lett. 36, 190 (2011).