Explicit formulae for the two way time-transfer in the T2L2 experiment including the $J_2$ contribution to the Earth potential in a relativistic framework

O. Minazzoli**, B. Chauvineau*, E. Samain†, P. Exertier†, P. Vrancken† and P. Guillemot‡

* UNS, OCA-ARTEMIS UMR 6162, Observatoire de la Cête d’Azur, Avenue Copernic, 06130 Grasse, France
† UNS, OCA-Geosciences Azur UMR 6526, Observatoire de la Cête d’Azur, Avenue Copernic, 06130 Grasse, France
‡ CNES, Avenue Edouard Belin, 31401 Toulouse cedex 9, France

The topic of this paper is to study the two way time-transfer problem between a ground based station and a low orbit Earth’s satellite, in the aim of an application to the T2L2 experiment. The study is driven in a fully relativistic framework. Because of the rapid increase in clock’s precision/measurements, the first term beyond the Earth’s potential monopolar term is explicitly taken into account. Explicit formulae, for both the distance and offset problems (definitions in the text) are proposed for the relevant applications.

Keywords:

I. INTRODUCTION

The idea of optical space-based time transfer had first been proposed to ESA in the seventies through the project which was named LASSO [1] (Laser Synchronization from stationary Orbit). Even though it was quite difficult to get some data with LASSO, the mission was a great success with some time transfers obtained within a time stability in the range of 100 ps. In 1994 OCA proposed to build a new generation of optical transfer based on some new technology: it was the beginning of the T2L2 project. As compared to LASSO, the objective was to improve the performances by at least 2 orders of magnitude. After several proposals on the Mir Space Station, ISS, GIOVE, and Myriade, T2L2 was finally accepted in 2005 as a passenger instrument on the Altimetry Jason-2 satellite [2, 3].

The full exploitation of the T2L2 experiment requires a relativistic description of the Earth gravitational field for both the definition of the proper timescales relevant for the (ground-based and on board) clocks and the photons’ trajectories involved by laser links.

The aim of the current paper is to propose explicit useful formulae dealing with two-way experiments, like T2L2. We derive explicit formulae giving the ratio $\nu_G^+/\nu_G^-$ or $\nu_G^−/\nu_G^+$ of the ground-based frequencies at reception and emission respectively as a function of the required quantities (positions, velocities, ...) at a ground-based time. We also give explicit expressions for the time dependance of (1) the ground proper time needed for a photon to go to the satellite and to come back and (2) the offset between the on board and the ground based clocks. All the quantities are explicitly written as a function of the time attached to a unique event (emission or reception). To anticipate the performance increase of future clocks, these formulae take the $J_2$ term in the Earth gravitational field effect on the photons’ orbits into account.

Blanchet et al [4] derive explicit formulae in both the one-way and the two-way problems. However, in the two-way case, the ratio $\nu_G^+/\nu_G^-$ of the ground-based frequencies at reception and emission respectively and other useful expressions are not explicitly given as a function of the required quantities at a unique time. The current paper proposes such formulae, which can be directly used in practical applications. Besides, the photon’s orbit is determined in [4] using the monopolar Earth gravitational term only. (A more complete gravitational modelization is used in [3], but for the link between proper and coordinate timescales only.) In the present paper, we include the $J_2$ gravitational contribution to light propagation, which will be useful as soon as the $10^{-14}$ s level of precision is required, as it should be in the future ACES experiment [5], for instance. Using the Syng world function approach [6], Linet and Teysssandier [7] have obtained explicit expressions of laser link in the one-way case for general axisymmetric gravitational field, but no formulae in the two-way case is given. This problem has been reconsidered by Le Poncin-Lafitte et al [8], but they give explicit formulae in the spherical case and for the one-way problem. Let us remark that, from a theoretical point of view, the applicability of the Syng world function approach can be limited to regions of spatial extension of the order of $2R_{Sun}(R_{Sun}/R_{Schw}) \sim 2000$ A.U. $\sim 0.01$ pc in the worse case (light rays grazing the Sun and considering points at typically $\sim 1000$ A.U.). The last remark is obviously not relevant for solar system applications, but is relevant for interstellar scales.

II. EXPERIMENTAL DESCRIPTION

A. Principe

The experiment principle is issued from satellite laser ranging techniques. T2L2 permits to synchronize remote ground clocks and compare their frequency stabilities with a performance never reached before. T2L2 allows synchronization of a ground and space clock and measurement of the stability of remote ground clocks over continental distances, itself having a time stability in the
FIG. 1: Ground to space time transfer with T2L2

FIG. 2: Global synoptic of the T2L2 instrument on Jason-2 together with the LRA and the DORIS USO

The principle is based on light pulses propagation between ground laser stations and a satellite equipped with a photo detection system and a time-tagging unit (fig. 1). The space instrument uses a Laser Ranging Array (LRA) which is also used for some laser ranging purposes and the Doris ultra-stable oscillator (USO) fundamentally used by the DORIS positioning system (fig. 2). The USO is linked to the time tagging unit. It is the onboard reference clock of the project.

A laser station at ground emits short asynchronous laser pulses towards the satellite. The LRA returns a fraction of the received photons back to the station. To describe roughly the main idea of the principle of the measurements, the station records the start ($t_{start}$) and return ($t_{return}$) times of each light pulse. The T2L2 payload records the arrival time ($t_{board}$) in the temporal reference frame of the onboard oscillator. These data are regularly downloaded every 2 hours to the ground via a regular microwave communication link.

For a given light pulse emitted from ground, the time offset $\theta_{GS}$ between ground clock $G$ and space clock $S$ is deduced from the measurement triplets $t_{start}$, $t_{board}$, $t_{return}$ with the following time equation:

$$\theta_{GS} = \frac{(t_{start} + t_{return})}{2} - t_{board} + \epsilon,$$

where $\epsilon$ is a corrective term, including atmospheric effects for instance. The time transfer between several laser stations at ground (A, B, C, ...) is then deduced from the difference between each ground to space time transfer.

However, a formulae like (1) is based on some inadequate physical concepts from the start. Obviously, it involves only a rough notion of time, leaving in the dark the fact that different proper times have to be considered in the problem, as it is well known from the relativistic gravity theory. In fact, (1) is correct as long as the newtonian description of (space-)time correctly describes the physics. However, at the level of precision reached by experiments from years ago, it is well known it is no longer the case. Of course, it is possible to put the so-called "relativistic corrections" into the $\epsilon$ corrective term in (1). However, it is better to reformulate the problem directly in the relativistic framework, this way ensuring that all the "relativistic effects" will be consistently taken into account at the considered order.

Depending on the distance between the laser ground stations (roughly 6000 km for Jason2) T2L2 can be operated in a common view mode or in a non-common view mode. In common view configuration, with two laser ranging stations A and B firing towards the satellite simultaneously, the noise of the on-board oscillator has to be considered over time interval equal to the delay between consecutive laser pulses. In such a way, the noise coming from the onboard oscillator can be considered negligible in the global error budget. In a non-common view mode, the temporal information is carried by the satellite local oscillator over the distance separating the two ground stations visibility and the USOs noise becomes an important part.

B. T2L2 on JASON2

Jason-2 is a French-American follow-on mission to Jason-1 and Topex/Poseidon. Conducted by NASA and CNES, its goal is to study the internal structure and dynamics of ocean currents mainly by radar altimetry. For the needs of precise determination of the satellite orbit, three independent positioning systems are also included: a Doris transponder, a GPS receiver and the LRA (Laser Ranging Array) retroreflector. The satellite was placed in a 1,336 km orbit with 66° inclination by a Delta launcher. The time interval between two passes varies from 2 to 14 hours with an average duration of about 1000 s. The T2L2 instrumentation contains two main subsystems: of a photo detection module and an event timer (fig. 3). The photo detection unit is made with 2 avalanche photo diodes. One is working in a special non-linear Geiger mode for precise chronometry. The other works in linear gain mode in order to trigger the whole detection chain and to measure the received optical energy and the
FIG. 3: Synoptic of the T2L2 space instrument

Continuous noise flux originating in the earth albedo. To minimize the false detection rate, the detection threshold may be adjusted either by remote control or automatically as a function of the continuous flux measurement. Some elements of the photo detection unit are located outside the main Jason-2 payload on a boom which supports also the LRA. These elements are oriented toward the earth surface. The rest of the detection is, together with event timer, inside an unique package located inside the Jason-2 payload. All this equipment has a mass of 10 kg and a power consumption of 42 W.

C. T2L2 Objectives

The objectives of the T2L2 experiment on Jason-2 are threefold:

- Validation of optical time transfer, including the validation of the experiment, its time stability and accuracy. It should further allow to demonstrate one-way laser ranging
- Scientific applications concerning time and frequency metrology allowing the calibration of radiofrequency time transfer (GPS and Two-Way), fundamental physics with the measurement of light speed anisotropy and alpha fine structure constant, Earth observation and very long baseline interferometry (VLBI).
- Characterization of the on-board Doris oscillator, especially above the South Atlantic Anomaly (SAA).

D. First results

T2L2 relies on the international laser ranging network that permanently contributes to the International Earth Reference System (terrestrial frame and Earth’s rotation), and to the tracking of several scientific space missions. Since 1998, this network has been organised as an International Service (named ILRS), as it was the case for the GPS and VLBI geodetic space techniques. Among 35 Satellite Laser Ranging (SLR) stations, 17 of them provide full rate ranging data on Jason-2 that are downloaded to the European Data Center (EDC in Munich, Germany) regularly; the data files are uploaded by the T2L2 Scientific Mission Center, located in Grasse, France. Among these stations, 10 of them use the new data format that permits to extract the start time of laser pulses at the ps level, whereas the others use the current format. This data format was fitted for laser ranging only and use a maximum of 12 digits to record the start epoch of the laser pulses (corresponding to 100 ns).

Several steps are necessary to proceed together ground SLR and on board T2L2 data sets (the T2L2 data files are uploaded from CNES, Toulouse to the Mission Center every day with a time delay of 1-2 days):

- selection of data relatively to the SLR stations and satellite passes,
- extraction of measurement triplets from both data sets,
- estimation of instrumental corrections,
- determination of the time of light travel between the SLR station and the T2L2 space instrument,
- computation of the ground to space time transfer.

1. Data selection

The first step of the data treatment consists in determining the absolute frequency offset and the phase delay between the space clock and the UTC time scale, along the time. The goal is to compute the on board date of each detected optical event (i.e., arrival time of emitted laser pulses) in a time scale that is identical to UTC at the picosecond level. Because the second step of the data treatment (Triplets extraction), is based on a direct comparison of the dates of T2L2 data and ground SLR data.

The T2L2 space instrument being connected to the GPS of the Jason-2 platform, we use Pulse Per Second (PPS) that are permanently emitted by this system at $\pm 1 \mu \text{sec}$ and then recorded by T2L2 in the space clock time scale. That permits to establish a phase link and an absolute frequency offset between the space clock and the UTC-GPS time scale, respectively to 0.3 $\mu \text{sec}$ and to $2-3 \times 10^{-11}$ Hz (over 5000 sec).
2. **Triplets extraction**

The second step of the data treatment consists in recognizing from all the events recorded by T2L2 those corresponding to emitted laser pulses. Because the events coming from the SLR stations are blended together, it is necessary to find from which station a given event is coming up. On the other hand, due to the combination of several parameters such as (i) the low energy level used by current SLR systems (around 50-100 mJ and even lower), (ii) the energy dispersion due to atmospheric effects, (iii) and the aperture size of the T2L2 optical module relatively to the diameter of the laser beam, all the emitted laser pulses don’t have a corresponding on board event. As a statistical result, T2L2 is detecting more events from SLR stations that use a higher energy level for their laser pulses (there is a factor 2 to 3 between the stations).

3. **Instrumental corrections**

Several instrumental corrections must be estimated both for the ground and space segments. At ground, the accuracy of the range is obtained by a calibration process which uses a calibration target located at a known position by a classical survey at the mm level. At the satellite, several parameters have to be taken into account as (i) the geometrical delay between the reference points of the LRA and the T2L2 optical module, (ii) the time walk compensation of the photo detection module which is sensitive to the number of received photons, (iii) the event timer calibration based on an internal measurement process.

All these corrections are computed at the ps level, using the available mechanical, optical and electronical measurements that were determined for all Jason-2 scientific equipments before its launch. In addition, the stellar sensors of the satellite are providing the attitude informations needed to compute the geometrical effects.

4. **Ground to space time transfer first results**

The ground to space time transfers that have been selected use a set of SLR data from very performant ground stations that, in addition, are equipped by a Hydrogen Maser or a Cesium clock as a permanent system of time. Thus, the present analysis takes into account data sets from Wettzell (Germany), Matera (Italy), Changchun (China), and Grasse (France) stations.

Nevertheless, if all these selected SLR stations provide data with an optimal precision (range and start times), a noise is introduced in the time transfer by the time interval measured at ground. It mainly comes from the quality of the laser beam (i.e., the pulse width), the ground detection module (a photo detector), the optical behaviour of the on board LRA, and the atmosphere effects (two times); the current budget error is of 20 to 35 ps depending of each station. By taking into account the cinematics between the satellite and the ground station during a satellite pass, it is possible to compute a synthetic time of flight by averaging the row data locally over an integration duration of a few tens of seconds. The noise is thus decreased by a factor $1/\sqrt{N}$ where $N$ is the number of row data. If that computation is done properly from a data set to another it permits also to reduce the slight but significant differences between the laser stations.

The noise of the time of light travel being reduced to 4-5 ps inside the time transfer, the time stability can be represented by a computed time variance between roughly 1 s and 600 s. As an example, the time stability of the Wettzell’s Hydrogen Maser compared to the T2L2’s DORIS quartz oscillator is 40 ps at 1 s and 7 ps at 30 s. For time integration greater than 30 s this measurement is limited by the DORIS time stability which is 5 ps at 30 s and 10 ps at 100 s.

5. **Improvements**

Now, we can routinely compute time transfers between different ground clocks and the space DORIS oscillator. Nevertheless, some work is still required to improve the instrumental model (physical behaviour of the hardware) to the level of 1-2 ps at 30 s. As examples of this model, (1) the T2L2 photo detection module is sensitive to the number of incoming photons that should be properly corrected, and (2) there is still a data filtering to develop in order to avoid Solar events that are detected by T2L2 just 1-3 ns before the real pulse event.

But the current estimation of the T2L2 performances seem to be in a good accordance with the previous specifications of the overall project thus justifying the present relativistics equations to be applied.

III. THE SPACE-TIME MODEL USED FOR TIME TRANSFER

In GR, photons follow null geodesics of the space-time metric $g_{\mu\nu}$. We write the metric under the form

$$ds^2 = g_{00}c^2 dt^2 + 2g_{0i}cdt dx^i + g_{ij}dx^i dx^j,$$  \hspace{1cm} (2)

where $ct = x^0$. In the IAU2000 recommendations, terms in $c^{-4}$ are neglected in $g_{ij}$, while they are taken into account in both $g_{00}$ and $g_{0i}$ (with no contribution in $g_{00}$) \[.\] Since we are interested in light motion, $dx \sim c dt$, considering $c^{-4}$ terms in $g_{00}$ (and $g_{0i}$) would be meaningless. Hence, the metric to be considered writes

$$ds^2 = \left(-1 + \frac{2w}{c^2}\right) c^2 dt^2 - \frac{8w^i}{c^3} c dt dx^i + \left(1 + \frac{2w}{c^2}\right) \delta_{ij} dx^i dx^j$$  \hspace{1cm} (3)
where \[9\]

\[
w(t, x^i) = G \int \frac{\sigma(t, y^k)}{\|y - x\|} d^3y
\]  

(4)

\[
w^i(t, x^k) = G \int \frac{\sigma^i(t, y^k)}{\|y - x\|} d^3y
\]  

(5)

\[\sigma\] and \[\sigma^i\] being the gravitational mass and mass current respectively. \[\|u\|\] is a short notation for \[\sqrt{u^i u^i}\]. Hence, \[w\] corresponds to the potential in Newtonian gravity. Let us write \[w\] under the form

\[
w = w_E + w_{ext}
\]  

where \[w_E\] is the Earth potential, written as the sum of the monopolar term \[w_{E,m}\], the oblateness term \[w_{E,J}^2\], and the other terms \[w_{E,up}\], when developed in spherical harmonics, and \[w_{ext}\] the external potentials, caused by the presence of the Moon and the Sun essentially. The orders of magnitude of these different terms are given by

\[
\begin{align*}
\frac{w_{E,m}}{c^2} & \sim \frac{G M_E}{r c^2} \sim 10^{-9} \\
\frac{w_{E,J_2}}{c^2} & \sim \frac{J_2 G M_E R_E^2}{r c^2} \sim 10^{-3} w_{E,m} \sim 10^{-12} \\
\frac{w_{E,up}}{c^2} & \sim 10^{-2} w_{E,J_2} \sim 10^{-14} \\
\frac{w_{ext}}{c^2} & \sim x^i x^j \partial_i U \sim \frac{GM r^2}{L c^2} = \frac{GM E}{r c^2} \frac{M_L}{E} \left(\frac{r}{L}\right)^3
\end{align*}
\]  

where \[M_E\] and \[R_E\] are Earth’s mass and radius, and \[U\] the Newtonian potential for a remote body of mass \[M\], at distance \(L\). It results that, for satellites at low altitude (say \(1000 km\)), the term \[w_{E,m}\] induces corrections which can reach \(\sim 10^{-11}\) s for the time transfer. The effect of the \(J_2\) term can then be of the order of \(10^{-14}\) s. The following terms in the Earth potential development results into corrections of order \(10^{-16}\) s. For both the Sun and the Moon, \[w_{ext}\] is of order \(10^{-19}\).

The time-space term in the metric (dragging effect) is of order

\[
\frac{8 w^i}{c^3} \sim \frac{8 G}{c^3} \int \rho V^i(t, y^k) d^3y
\]  

(6)

where \(\rho\) is the Earth’s density, and \(V\) the field of matter velocity inside the Earth. Assuming a spherical rigidly rotating Earth, with uniform density and angular velocity \(\omega\), the expression \[6\] leads to, for a satellite located in the equatorial plane at altitude \(h\)

\[
\frac{8 \|w^i\|}{c^3} = \frac{8 G M_E \omega R_E}{5 R_E c^2} \left(\frac{R_E}{R_E + h}\right)^2 \sim 2.10^{-15}
\]  

(7)

for a low altitude satellite. The resulting correction to the photon flying time is then expected to be of the order of \(10^{-17}\) s.

In a foreseeable future, time transfer precision of order \(10^{-15}\) s is expected. Hence, for low altitude satellites, one has to consider the \(J_2\) term in Earth potential while all the other terms may be discarded.

### IV. APPLICATION

#### A. The photon orbit

From the previous section, we will consider in this section the time transfer problem in a space-time described by the metric

\[
ds^2 = \left(-1 + \frac{2w}{c^2}\right) c^2 dt^2 + \left(1 + \gamma \frac{2w}{c^2}\right) \delta_{ij} dx^i dx^j
\]  

(8)

where \(\gamma\) is a PPN parameter including possible extensions of GR, like scalar-tensor (ST) theories \[10\]. The potential \(w\) is given by

\[
w = \nu + J_2 \nu
\]  

(9)

with

\[
\nu = \frac{GM_E}{r} \quad \text{and} \quad \nu = \frac{GM_E R_E^2}{r^3} \left(1 - \frac{z^2}{r^2}\right)
\]  

(10)

\(R\) being a parameter of the order of the Earth radius. Since we are interested in photon propagation, one has to solve the geodesic equation

\[
\frac{d}{d\lambda} \left(\alpha_{\mu\nu} \frac{dx^\nu}{d\lambda}\right) = \frac{1}{2} k^\mu k^\nu \alpha_{\mu\nu}
\]  

(11)

(\(\lambda\) being an affine parameter). The wave triad-vector \(k\) satisfies the null condition

\[
\alpha_{\mu\nu} k^\mu k^\nu = 0
\]  

(12)

expressing the motion occurs on light cones. The way chosen to solve equations \[11\] and \[12\], up to first order in \(w\), is the way chosen in \[11\], in which only the \(\nu\)-term was considered in the potential. The aim is to derive the trajectory of the photon \(x^i(x^0)\), obtained after the elimination of the affine parameter. (This elimination can be made both after the integration of the equations or at the level of the differential equation \[11\] itself.)

Up to \(O(c^{-3})\) terms, the result reads

\[
x_{ph}^{i} (t, n^k, x_A^k (t_e)) = x_0^i + n^j c t + x_A^{i (j)} (t_e)
\]  

(13)

where, using the short notation \(m = GM_E/c^2\)

\[
x_{ph}^{i (j)} (t, n^k, x_A^k (t_e)) = \frac{1 + \gamma}{2} mn^i \ln \frac{r - \Lambda}{r + \Lambda} - (1 + \gamma) m \frac{c^2}{K^2 r} + (1 + \gamma) J_2 m R_E^2 \Psi^i (t) + Z^i
\]
Let us remark that 
\[
\Psi^i(t) = \left[ -\frac{\xi^i + 2\delta_{3i}\xi^3}{K^4} + \frac{n^3n^i\xi^3 + 2n^3n^i\xi^3}{K^4} + 4\frac{\xi^3\xi^3}{K^6} \right] r + \frac{n^i + 2\delta_{3i}n^3}{2K^2} - \frac{n^3n^i\xi^3 - 2n^3\xi^3}{2K^2} \Lambda \frac{\xi^3}{r} + \frac{\xi^3 + 2\delta_{3i}\xi^3}{2K^2} \frac{n^3n^i\xi^3 + 2n^3\xi^3}{2K^2} \frac{\xi^3}{2K^2} \frac{1}{r^3} \right] A
\]

where \( x^i_0 \) is the position at \( t = 0 \) (the integration constants \( Z^i \) being adjusted this way [12]). The three numbers \( n^i \) are also integration constants satisfying the normalization condition
\[
n^in^i = 1.
\]

In the first order terms, one has
\[
\Lambda = ct + n^ix^i_0
\]
\[
K^2 = x^i_0x^i_0 - (n^ix^i_0)^2
\]
\[
r = \sqrt{K^2 + \Lambda^2}
\]
\[
\xi^i = x^i_0 - n^ik^kx^k_0.
\]

Let us remark that \( \xi^i\xi^i = K^2 \) and \( n^i\xi^i = 0 \).

### B. Time transfer and useful formulae

#### 1. The time transfer

Let us now consider two photons emitted from the station A and received by the station B. Let \( d\tau_A \) and \( d\tau_B \) be the proper time intervals between, respectively, the two emissions and the two receptions, and \( d\tau_A, d\tau_B \) the corresponding coordinate time intervals. Let \( x^iA, v^iA, v^iB, v^iB \) be the positions and \( v^iA, v^iB \) the coordinate velocities of A and B at the coordinate time corresponding to the emission of the first photon by A.

Since both A and B have velocities corresponding to orbital free motions (for a satellite) or smaller (for a ground station), the velocities of both are at best of the order of \( \sqrt{GME/r} \), in such a way that the \( g_{0i} \) term in [3] contributes as a \( c^{-4} \) term in each station proper time. The ground station’s potential has to be modeled more precisely than in [9, 3]. Hence we generally denote it by \( W \) and not by \( w \) which is its approximation [9] (\( w \) is only suitable for a satellite and one can replace \( W \) by \( w \) in this case in practical calculations). One has
\[
d\tau_A = d\tau_A \left[ 1 - \frac{W_A}{c^2} - \frac{v^2_A}{2c^2} + O(c^{-4}) \right]
\]
and an analogous expression for \( d\tau_B \). Since \( t_B = t_A + t \), where \( t \) is the time transfer (to be determined later), this leads to
\[
d\tau_B \frac{d\tau_A}{d\tau_A} = \left[ 1 + \left( \frac{dt}{d\tau_A} \right)^{\epsilon} \right] \times \left[ 1 + \frac{W_{0A}}{c^2} - \frac{v^2_{0A} - v^2_B}{2c^2} + O(c^{-4}) \right]
\]
The indexes 0 recall that the corresponding quantities have to be computed at the earliest emission time. The function \( t(e) \) as a function of the emission time \( t \) (= \( t_A \)) is implicitly obtained from the equation defining the interception of the photon by the station B
\[
x^i \left( t, n^k, x^k_A(t) \right) = x^i_B \left( t, x^k_B(t), v^k_B(t), ... \right)
\]
where one has explicitly written the dependence of the station B in initial (i.e. at the emission time) orbital parameters. The function \( x^i \left( t, n^k, x^k_A(t) \right) \) is explicitly given by [13, 14] up to first order. The function \( x^i_B \left( t, x^k_B(t), v^k_B(t), ... \right) \) can be obtained from the geodesic equation if the station B is a satellite while it is given by an Earth rotation model in the case of a ground based station. Let us write these functions as
\[
x^i \left( t, n^k, x^k_A(t) \right) = x^i_A(t) + n^i ct
\]
\[
+ x^i \left( t, n^k, x^k_A(t) \right)
\]
\[
+ (3/2)^i \times x^i \left( t, n^k, x^k_A(t) \right)
\]
and
\[
x^i_B \left( t, x^k_B(t), v^k_B(t), ... \right) = \left( \frac{0}{3} \right)
\]
\[
x^i_B(t) + tv^i_B(t) + \frac{1}{2} t^2 a^i_B(t) + \frac{1}{6} t^3 b^i_B(t)
\]
where \( x^i_p \) corresponds to the (unknown) 3/2 order part of the photon orbit (contribution of the \( g_{0i} \) term). The coefficients \( a^i_B(t) \) and \( b^i_B(t) \) correspond to the acceleration of the station B and its derivative at the emission time. The four r.h.s. terms in [24] are respectively zeroth, first, second and third order terms in \( c^{-1} \) (since \( tv^i_B/c^2 \) and \( (ct)^2 a^i_B/c^2 \) and so on). Solving [24] leads to
\[
ct = \epsilon \theta + \phi + \epsilon \theta + O(c^{-3})
\]
\[
n^i = \epsilon \eta i + \phi (0)^i + (1)^i + \epsilon \eta + O(c^{-3})
\]
where $\epsilon = \pm$ and with (quantities at time $t_e$)

\[
\begin{align*}
(0)^i\theta &= x_B^i - x_A^i \\
(0)^j \eta &= \frac{x_B^j - x_A^j}{d} \\
(1/2)^i \theta &= d \eta^i e_B^i \\
(1/2)^j \eta &= \frac{p^{ij} v_B^j}{c} \\
(1)^i \theta &= \frac{1}{2} \delta^{ij} \left( \frac{\partial^2 V_B^i}{\partial x^j} \right) + \frac{1}{2} \frac{\partial^2 V_B^i}{\partial x^j} \\
(1)^j \eta &= \frac{1}{2} \frac{\partial^2 V_B^j}{\partial x^i} + 1 \frac{\partial^2 V_B^j}{\partial x^i} \\
\end{align*}
\]

\[\epsilon = + \] corresponds to a photon travelling from A to B ($t_A < t_B$) while $\epsilon = -$ corresponds to a photon travelling from B to A ($t_A > t_B$).

Considering now two emissions, at times $t_e$ and $t_e + dt_e$, the transfer times are $t$ and $t + dt$, while the "directions" are $n^i$ and $n^i + dn^i$. Differentiating $v_B^i (t_e)$ with respect to $t_e$ leads to a $c^{-4}$ term, since B has at best a satellite like velocity. Differentiating (22), (23), (24) and retaining only terms up to $c^{-3}$ leads to

\[
\begin{align*}
\begin{bmatrix}
 cn^i - v_B^i - t B^i + \frac{\partial (1)^i}{\partial t} - \frac{1}{2} t^2 B^i + \frac{\partial (3/2)^i}{\partial t} \\
 + \left[ v_A^i - v_B^i - t A^i + \frac{\partial (3/2)^i}{\partial t} \right] dt_e \\
 + \left[ c t \delta^{ik} + \frac{\partial (1)^i}{\partial n_k} + \frac{\partial (3/2)^i}{\partial n_k} \right] dn^k = 0. & (26)
\end{bmatrix}
\end{align*}
\]

All the coefficients are taken at the emission time $t_e$. Since both $n^i$ and $n^i + dn^i$ are normalized, one has also

\[n^i dn^k = 0. & (27)\]

Solving the four equations (26), (27) gives the four quantities $dt_e/dt$ and $dn^k/dt_e$. In the time transfer problem we are dealing with, only $dt_e/dt$ is required.

Making the scalar product of (26) by $n^i$, it turns out that, using (27)

\[
\begin{align*}
\begin{bmatrix}
 c + n^i \left( -v_B^i - t A^i + \frac{\partial (1)^i}{\partial t} \right) dt \\
 + n^i \left[ v_A^i - v_B^i - t A^i - \frac{1}{2} t^2 B^i + v_B^i \frac{\partial (3/2)^i}{\partial t} \right] dt_e \\
 + n^i \left[ \frac{\partial (1)^i}{\partial n_e} + \frac{\partial (3/2)^i}{\partial n_e} \right] \partial n^k = 0 & (28)
\end{bmatrix}
\end{align*}
\]

where the third order terms in the coefficient of $dt$ have been discarded since there is no zeroth order term in the coefficients of both $dt_e$ and $dn^k$. In (28), $dn^k$ has to be known up to the $c^{-1}$ order only, since there is no zeroth nor first order term in $c^{-1}$ in its coefficient. This is given by equation (29), written at the relevant order

\[
 cn^i - v_B^i \right) dt + [v_A^i - v_B^i] dt_e + c t d n^i = 0. & (29)
\]

Inserting (29) into (28) to eliminate $dn^k$ leads to an equation which can easily be solved in $dt_e/dt$. It turns out that $(3/2)^i$ leads to terms of order $c^{-2}$ or more. Hence, this term is not needed. Finally

\[
\begin{align*}
\begin{bmatrix}
 (1/2)^i \\
 (1)^i \\
 (3/2)^i \\
\end{bmatrix} dt_e = \frac{v_A^i - v_B^i}{c} + c t d n^i = 0 & (31)
\end{align*}
\]

(classical Doppler effect)

\[
\begin{align*}
\begin{bmatrix}
 (1)^i \\
\end{bmatrix} dt_e = n^i \frac{v_B^i - v_A^i}{c} + n^i \frac{ta_B^i}{c} & (32)
\end{align*}
\]

This expression has now to be inserted in (21). However, $v_B^i$ and $W_B$ are considered at the reception time in (21) (where $dt_A = dt_e$), while all the coefficients in (31)-(33) are considered at the emission time. It turns out

\[
v_B^2 (t_e + t) = v_B^2 + 2 v_B t_B a_B + O(c^{-2}) & (34)
\]
\[ W_B(t_c + t) = W_B + tv_B^i \partial_i W_B + O(c^{-2}) \] (35)

all the terms being considered at time \( t_c \) when not precised. One obtains finally (suppressing the zero indexes)

\[
\frac{d\tau_G}{d\tau_A} = 1 + n_i \frac{v_i^A - v_i^B}{c} + \frac{v_i^2 - v_i^2_B}{2c^2} + \frac{W_A - W_B}{c^2} + n_i \frac{v_i^B}{c} + n_i \frac{v_i}{c} + \frac{\partial_i W_B}{c^2} + \frac{dt}{dt_e} + O(c^{-4})
\] (36)

where the \( dt/dt_e \) terms of orders \((1/2)\) and \((1)\) are written explicitly to exhibit first and second order terms, corresponding to classical Doppler effect, special relativistic contributions and Einstein’s gravitational effects. All the \( c^{-3} \) terms have been grouped on the third and fourth line of equation (36). Remark that \( \partial_i W_B \) in (35) and (36) corresponds to \( a_B \) in the case of free fall motion (i.e. if B is a satellite).

2. Useful formulae

Let us consider measurements made from a ground-based station (G), emitting a photon towards a satellite (S). This photon is reflected at (S) and come back to (G). To each event (emission, reflexion, return) is associated a proper time measured on the involved clock. Let \( \tau_G^+ \) be the proper time on (G) at emission, \( \tau_S \) the proper time on (S) at reflexion, and \( \tau_G^- \) the proper time on (G) when the photon comes back. Two problems may be tackled :

- the "distance problem", involving \( \tau_G^+ - \tau_G^- \);
- the "offset problem", involving \( (\tau_G^+ + \tau_G^-)/2 - \tau_S \).

Obviously, the (gravitational) theory doesn’t give the values of this last quantity, but allows to calculate \( d\tau_G^+ \) and \( d\tau_S \) as soon as \( d\tau_G^- \) and the orbits of the ground-based and the onboard clocks are known. Reciprocally, the measurements of these quantities lead to constraints on both the orbits and the real (unperfect) clocks used in the time-measurement process.

Hence, let us define the "distance problem" and "offset problem" quantities (observables) by, respectively

\[
DP = \frac{d}{d\tau_G^-} (\tau_G^+ - \tau_G^-)
\]
\[
OP = \frac{d}{d\tau_G^-} \left( \frac{\tau_G^+ + \tau_G^-}{2} - \tau_S \right).
\]

For exhaustivity, let us point out the ratio of the ground emitted \( \nu_G^- \) and the ground received \( \nu_G^+ \) frequencies is simply related to \( DP \) by

\[
\frac{\nu_G^-}{\nu_G^+} = \frac{d\tau_G^+}{d\tau_G^-} = 1 + DP.
\]

Remark that since both (G) and (S) data are available, these quantities can be evaluated as functions of both \( \tau_G^+, \tau_G^- \) or \( \tau_S \). However, in relevant applications, the (G) clock has a precision sensitively better than the (S) one. Hence, it is well-suited using the (G) clock as reference time.

3. Explicit expressions of \( DP \) and \( OP \)

Let us calculate \( DP \) and \( OP \) as functions of the involved quantities at the emission (\( \epsilon = - \)) or reception (\( \epsilon = + \)) time \( t_G^\epsilon \) :

\[
DP(t_G^\epsilon) = \frac{d\tau_G^+}{d\tau_G^-}(t_G^\epsilon) - 1
\] (37)
\[
OP(t_G^\epsilon) = \frac{1}{2} + \frac{1}{2} \frac{d\tau_G^+}{d\tau_G^-}(t_G^\epsilon) - \frac{d\tau_S}{d\tau_G^-}(t_G^\epsilon).
\] (38)

In the relevant applications, the ground based station is the Earth. It is legitimate to consider that, during the time transfer, (1) the Earth is rigid and uniformly rotating, (2) the Earth gravitational field only is acting (see discussion in section III). Hence, one has, from (26)

\[
\frac{d\tau_G^+}{d\tau_G^-} = \frac{dt}{dt_c} + O(c^{-4}).
\]

This shows the shift between the emitted and coming back frequencies of the photon doesn’t require an Earth model (for both potential and rotation). The interpretation is obvious, but this doesn’t explicitly appear in formulas available in the Blanchet et al paper [4].

Let us put

\[
X^i = x_G^i - x_S^i
\]
\[
V^i = v_G^i - v_S^i
\]
\[
A^i = a_G^i - a_S^i
\]
\[
B^i = b_G^i - b_S^i
\]
\[
D = \sqrt{X^b X^b}
\]
\[
N^i = X^i / D
\]
\[
P^{ik} = \delta^{ik} - N^i N^k
\]

One finds, for the involved quantities

\[
\frac{d\tau_G^+}{d\tau_G^-}(t_G^\epsilon) = 1 + e^{-1(1/2)} D + e^{-2} D + e^{-3(3/2)} D
\]
\[
\frac{d\tau_S}{d\tau_G^-}(t_G^\epsilon) = 1 + e^{-1(1/2)} E + e^{-2} E + e^{-3(3/2)} E
\]
with
\[ D = \begin{cases} \frac{1}{2} \varepsilon^2 \text{OP} \quad \text{(1)} \\ 2(1 + \varepsilon)(N^i V^i)^2 - 2V^i V^i - 2\varepsilon D N^i A^i \quad \text{(3/2)} \end{cases} \]

\[ E = \begin{cases} N^i V^i \quad \text{(1)} \\ \left( \frac{1}{2} + \varepsilon \right) V^i V^i + (1 + \varepsilon) \left[ (N^i V^i)^2 - D N^i a_G^i \right] + \varepsilon D N^i a_S^i + W_G - W_S \quad \text{(3/2)} \end{cases} \]

\[ \Delta \tau (t_1, t_2) = t_2 - t_1 + \frac{1}{c^2} \int_{t_1}^{t_2} \left( w - \frac{v^2}{2} \right) dt + O(c^{-4}). \]

leading to the spacecraft proper time interval

\[ \left( \frac{d\tau}{dt} \right)^2 = 1 + \frac{2w}{c^2} - \frac{v^2}{c^2} + O(c^{-4}) \]

The quantities \( D, N, v_G, v_S, a_G, a_S, b_G, b_S \) are taken at the time \( t_G \), while the photon first order term \( \tau \) and its derivatives have to be computed for \( t = t_G \), \( \mu = +\Delta x / D \), \( x_S \) (at time \( t_G \)) for both \( \varepsilon = \pm \). Let us recall that \( c^2 \varepsilon^3 \) and \( c^2 \varepsilon \) are zeroth order quantities.

From these expressions, it turns out that

\[ OP (t_G) = e^{-2(1)e} \text{OP} + e^{-3(3/2)e} \]

V. CONCLUSION AND DISCUSSION

In this paper, one has proposed formulae involved in two way time transfer problems in a form directly adapted for practical uses. All these quantities are expressed as functions of quantities defined at a unique (emission or reception) ground-based time. Considering foreseeable increase of clock’s accuracy, the formulae are given including the \( J_2 \) contribution, but can be restricted to the spherical case making \( J_2 = 0 \), which is sufficient for the time being applications.

In this paper, the chosen unique time is a ground time because ground-based clocks have generally precisions sensitively better than on board clocks. However, in the case where one needs to make some time transfer between two different ground-based stations, distant in such a way there is no spacecraft in common view, the link will involve an on board clock, with the resulting limitation in precision. Let \( t_1 \) and \( t_2 \) be the times of the links between the spacecraft and the first and second ground stations. Since the spacecraft motion is considered during the time interval \( [t_1, t_2] \), the crossed \( cdtdx^i \) term in \( (2) \) is a \( c^{-4} \) order term. Hence, the onboard proper time formula reduces to

\[ \left( \frac{d\tau}{dt} \right)^2 = 1 + \frac{2w}{c^2} - \frac{v^2}{c^2} + O(c^{-4}) \]

leading to the spacecraft proper time interval

\[ \Delta \tau (t_1, t_2) = t_2 - t_1 + \frac{1}{c^2} \int_{t_1}^{t_2} \left( w - \frac{v^2}{2} \right) dt + O(c^{-4}). \]

Let us enforce the fact that, since the time interval \( [t_1, t_2] \) is finite, the orbit used in the previous r.h.s. member integral cannot be developed around the emission time, and should even include all the perturbations acting on the satellite’s motion. On the other hand, a classical description of this orbit is sufficient, since relativistic corrections generate \( c^{-4} \) contributions to \( \Delta \tau \).

Acknowledgments

Olivier Minazzoli wants to thank the Government of the Principality of Monaco for their financial support.
[1] P. Fridelance, C. Veillet, Metrologia 32, 27 (1995)
[2] P. Fridelance, E. Samain, C. Veillet, Exp. Astron. 7, 191 (1997)
[3] E. Samain & al., IJMPD 17, 1043 (2008)
[4] L. Blanchet, C. Salomon, P. Teyssandier, P. Wolf, Astron. Astrophys. 370, 320 (2001).
[5] Laurent, Ph. & al., Proceedings of the SPIE, Volume 6673, pp. 667308 (2007).
[6] J. L. Synge, Relativity: The General Theory, (Amsterdam, North-Holland, 1964)
[7] B. Linet, P. Teyssandier, Phys. Rev. D 66, 024045 (2002).
[8] C. Le Poncin-Lafitte, B. Linet, P. Teyssandier, Class. Quant. Grav. 21, 4463 (2004).
[9] M. Soffel et al, Astron. J. 126, 2687 (2003).
[10] C.M. Will, The confrontation between general relativity and experiments in http://relativity.livingreviews.org/Articles/ (2006).
[11] B. Chauvineau, T. Regimbau, J.-Y. Vinet, S. Pireaux, Phys. Rev. D 72, 122003 (2005).
[12] O. Minazzoli, B. Chauvineau, SF2A Colloquium, Grenoble, France, 2-6 July 2007, http://arxiv.org/abs/0709.4604