Quantum bounds for gravitational de Sitter entropy and the Cardy-Verlinde formula

S. Nojiri\(^\sp{\diamondsuit}\), O. Obregon\(^{\sp{\spadesuit}}\), S. D. Odintsov\(^{\sp{\spadesuit}\sp{\clubsuit}}\), H. Quevedo\(^{\sp{\spadesuit}\sp{\heartsuit}}\) and M.P. Ryan\(^{\sp{\spadesuit}}\)

\(^{\sp{\diamondsuit}}\)Department of Applied Physics
National Defence Academy, Hashirimizu Yokosuka 239-8686, JAPAN

\(^{\sp{\spadesuit}}\)Instituto de Fisica de la Universidad de Guanajuato, Lomas del Bosque
103, Apdo. Postal E-143, 37150 Leon, Gto., MEXICO

\(^{\sp{\spadesuit}}\)Tomsk State Pedagogical University, 634041 Tomsk, RUSSIA

\(^{\sp{\spadesuit}\sp{\heartsuit}}\)Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. Postal 70-543, México 04510 D.F., MEXICO

ABSTRACT

We analyze different types of quantum corrections to the Cardy-Verlinde entropy formula in a Friedmann-Robertson-Walker universe and in an (anti)-de Sitter space. In all cases we show that quantum corrections can be represented by an effective cosmological constant which is then used to redefine the parameters entering the Cardy-Verlinde formula so that it becomes valid also with quantum corrections, a fact that we interpret as a further indication of its universality. A proposed relation between Cardy-Verlinde formula and the ADM Hamiltonian constraint is given.
1 Introduction

In the recent seminal paper by Verlinde [1] a very interesting approach to studying the holographic principle in a Friedmann-Robertson-Walker (FRW) universe filled with CFT has been developed. Using the dual AdS-description [2] a remarkable relation between entropy (energy) of the CFT and the equations of motion of the FRW-universe has been found. An equation describing the entropy bounds during the evolution has been obtained [1]. This equation has been presented in the form of the Cardy formula [3] (the entropy formula for 2d CFT) which has subsequently been called the Cardy-Verlinde formula. This approach has been further discussed in Refs. [4, 5, 6, 7]. In particular, the Cardy-Verlinde formula has been generalized to the case of non-zero cosmological constant [6]. The interpretation of entropy from a brane-world perspective was given in Refs. [7], and the quantum corrections to entropy from 4d QFT have been discussed in Ref. [3].

In the present work we will further discuss the entropy bounds and the Cardy-Verlinde formula in de Sitter (Anti-de Sitter) space, taking into account quantum corrections. We present the analogous model in which quantum corrections are included as an effective cosmological constant. This gives us a way to formulate the quantum-corrected entropy bounds and the Cardy-Verlinde formula in a universal way (with “renormalized” parameters) where the formula takes its classical form [1]. As an explicit example (Anti)-de Sitter Universe is considered. The simple way to formulate quantum-corrected Cardy-Verlinde formula in a dilatonic Anti-de Sitter space is also outlined. We briefly discuss the possibility of considering and generalizing the original Verlinde formula as an energy equation following from the ADM action.

2 Quantum corrections in a FRW-universe

Let us consider the FRW-universe equation of motion with quantum corrections (taking into account conformal-anomaly-induced effective action). In other words, we suppose that the universe is filled with conformal matter which gives a contribution to the classical stress-energy tensor, as well as a quantum contribution. Gravity also makes a quantum contribution to the effective equation of motion. As it has been shown in [3], the FRW-equation...
has the form:

\[
H^2 = -\frac{1}{a^2} + \frac{8\pi G}{3} \frac{E}{V} \\
+ \frac{8\pi G}{3} \left[ -b \left( 4H_{,tt} + 12H_{,t}H^2 - 2H^2_t + 6H^4 + \frac{8}{a^2}H^2 \right) \\
+ \frac{1}{12} \left\{ b'' + \frac{2}{3} (b + b') \right\} \\
\times \left( -36H^2_t + 216H_tH^2 + 72HH_{,tt} - \frac{72}{a^2}H^2 + \frac{36}{a^4} \right) \\
+ \tilde{a} \right],
\]

(1)

where \( V \) is the spatial volume of the universe, \( \tilde{a} = -8b' \) (a normalization choice [5]), \( b'' = 0 \) and

\[
b = \frac{N + 6N_{1/2} + 12N_1 - 8N_{HD} + 611N_2 + 796N_W}{120(4\pi)^2},
\]

\[
b' = \frac{-N + 11N_{1/2} + 62N_1 - 28N_{HD} + 1411N_2 + 1566N_W}{360(4\pi)^2}.
\]

(2)

Here \( N, N_{1/2}, N_1, N_{HD} \) are the number of scalars, (Dirac) spinors, vectors and higher derivative conformal scalars which are present in QFT. The quantity \( N_2 \) denotes the contribution to conformal anomaly from a spin-2 field (Einstein gravity) and \( N_W \) the contribution from higher-derivative Weyl gravity.

In the classical limit, only the first line of the FRW equation (1) remains and we obtain Einstein’s dynamics with no cosmological constant. However, as we will see in a moment, the quantum corrections play the role of an effective cosmological constant.

In the absence of classical matter energy \( (E = 0) \), the general FRW equation allows the quantum-induced de Sitter space solution [3]:

\[
a(t) = A \cosh Bt, \quad ds^2 = dt^2 + A^2 \cosh^2 \frac{t}{A} d\Omega_3^2,
\]

(3)

where \( A \) is a constant and \( B^2 = \frac{1}{A^2} = -\frac{1}{16\pi G b'} \). On the other hand, in classical gravity with cosmological constant

\[
H^2 = \frac{8\pi G}{3} \frac{E}{V} + \frac{\Lambda}{3} - \frac{1}{a^2},
\]

(4)
and $E = 0$, exactly the same solution (3) exists for $A^2 = 3/\Lambda$. The main consequence of this result is that one can consider a simplified model described by Eq.(4) where the effective cosmological constant is defined by $\Lambda_{\text{eff}} = \frac{3}{A^2} = -\frac{3}{16\pi G}$, and the quantum effects are encoded in the definition of $\Lambda_{\text{eff}}$. In such a de Sitter universe we can also add classical energy.

Now one can refer to Ref.[6], where Verlinde’s work [1] has been extended to include the presence of a cosmological constant. We define the Bekenstein entropy $S_B = \frac{2}{3} \pi E a$ (the corresponding bound is valid for systems with limited self-gravity $Ha < 1$) and the Hubble entropy $S_H = HV/2G$ (the corresponding bound is valid for strongly self-gravitating systems, $Ha > 1$).

Let us consider a black hole of the size of the universe described by a de Sitter space with metric

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_2^2 ,$$

where $f(r) = 1 - \frac{2MG}{r} - \frac{r^2\Lambda_{\text{eff}}}{3}$. For the maximal value of the BH mass $M$ one has the Nariai BH with cosmological horizon radius $r_c = a_c = \sqrt{\Lambda_{\text{eff}}}$. This is the largest BH which may be formed in a de Sitter universe.

Using Eq.(4) and supposing that the energy sufficient to create a SdS BH of the size of the universe is provided when the universe goes from a weakly to a strongly self-gravitating phase (i.e. $Ha_c = \frac{H}{\sqrt{\Lambda_{\text{eff}}}} = \sqrt{c}$ with $c$ being a constant of order one), one finds

$$H^2 = \frac{8\pi G}{3} \frac{E_{BH}}{V} + \frac{\Lambda_{\text{eff}}}{3} - \Lambda_{\text{eff}} ,$$

that is

$$E_{BH} = \frac{3\Lambda_{\text{eff}}V}{8\pi G} \left(c + \frac{2}{3}\right) \quad \text{or} \quad S_{BH} = \frac{3 \left(c + \frac{2}{3}\right) V \sqrt{\Lambda_{\text{eff}}}}{8G} .$$

Keeping $c$ explicitly, when $a = a_c = \frac{1}{\sqrt{\Lambda_{\text{eff}}}}$ and $\Lambda = \Lambda_{\text{eff}}$, one can rewrite the FRW equation (4) as the following relation between $S_B$, $S_H$ and $S_{BH}$

$$S_H^2 = \frac{8}{3 \left(c + \frac{2}{3}\right)} S_B S_{BH} - \frac{1}{6} \left\{ \frac{8}{3 \left(c + \frac{2}{3}\right)} \right\}^2 S_{BH}^2 .$$

This expression represents the quantum generalization of the Verlinde formula relating the Bekenstein, Hubble and Bekenstein-Hawking entropies.
throughout the evolution of the universe. For $c = 1$ and a classical cosmological constant this equation was obtained in Ref.[6]. For $c = 2/3$ and zero cosmological constant it coincides with the original Verlinde formula [1]. Despite the fact that we began from classical gravity without a cosmological constant, the quantum effects change the dynamical structure in such a way that the universe develops an effective cosmological constant.

The above equation can be written in terms of $E$ and $E_{BH}$ as

$$S_H = \pi \sqrt{-\frac{16\pi Gb'}{3}} \left[ \frac{8}{3 \left( c + \frac{2}{3} \right)} E_{BH} \left\{ \frac{2}{3} E - \frac{1}{6} \frac{8}{3 \left( c + \frac{2}{3} \right)} E_{BH} \right\} \right].$$

(9)

If we now identify the Virasoro operator $L_0$ and the central charge $c_{CFT}$ as

$$L_0 = \frac{1}{3} E \sqrt{-8\pi Gb'}, \quad \frac{c_{CFT}}{2} = \frac{8}{3 \left( c + \frac{2}{3} \right)} E_{BH} \sqrt{-8\pi Gb'},$$

(10)

we can identify Eq.(9) as the Cardy formula [3]

$$S_H = 2\pi \sqrt{\frac{c_{CFT}}{6} \left( L_0 - \frac{c_{CFT}}{24} \right)}.$$

(11)

Here we have included the factor $Gb'$ in the definitions of $L_0$ and $c_{CFT}$. Since the radius $a_c$ of the cosmological horizon is given by $a_c^2 = -\frac{16\pi Gb'}{3}$, we can rewrite (14) as

$$L_0 = \frac{1}{\sqrt{6}} E a_c, \quad \frac{c_{CFT}}{2} = \frac{8}{\sqrt{6} \left( c + \frac{2}{3} \right)} E_{BH} a_c.$$

(12)

Thus, we are taking into account the quantum corrections of the Verlinde-Cardy entropy formula explicitly by just redefining the Virasoro operator and the central charge. This redefinition includes only a multiplicative constant term and, therefore, can be considered as a “renormalization” of the quantities entering the Cardy formula. This is a further indication of the universality of Cardy’s formula. It is also remarkable that 4d quantum dynamics appears in 2d quantum dynamics (the Cardy formula) via the corresponding renormalization of the Virasoro operator and the central charge.

Finally, let us consider a radiation-dominated universe where [4]

$$\frac{GE}{V} \gg \Lambda_{eff}.$$

(13)
If we now define the Bekenstein entropy $S_B$, the Bekenstein-Hawking entropy $S_{BH}$, the Hubble entropy $S_H$ and the quantum contribution to the entropy $S_{QC}$ by

$$S_B \equiv \frac{2\pi}{3} E a , \quad S_{BH} \equiv \frac{V}{2G a} , \quad S_H \equiv \frac{HV}{2G} , \quad S_{QC} \equiv \frac{V \sqrt{\Lambda_{\text{eff}}}}{2G} .$$

then the effective FRW equation

$$H^2 = \frac{8\pi G E}{3 V} + \frac{\Lambda_{\text{eff}}}{3} - \frac{1}{a^2}$$

(15)
can be written as

$$\frac{S_H^2}{S_B^2} + \left( 1 - \frac{S_{BH}}{S_B} \right)^2 - \frac{S_{QC}^2}{S_B^2} = 1 .$$

(16)

In the classical theory ($S_{QC} = 0$), we obtain the original Verlinde formula. Since in a radiative universe ($E \propto a$) the Bekenstein entropy is constant, Eq.(16) represents a hyperboloid with coordinates $S_H$, $S_B - S_{BH}$ and $S_{QC}$. In this graphical representation of the entropy bounds, any section $S_{QC} = \text{const}$ corresponds to a circle of radius $R_S = \sqrt{S_B^2 + S_{QC}^2}$ which changes as the volume of the universe ($S_{QC} \propto V$). Clearly, this dependence of $R_S$ is valid only within the range in which our quantum approximation holds. For instance, in the case of an expanding universe the radius $R_S$ will reach its maximum value when the quantum corrections become incompatible with the method applied for their derivation.

3 Quantum corrections in an AdS space

In this section we will analyze the case of classical gravity with a (small) negative cosmological constant, for which the action is given by

$$S_{\text{cl}} = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left( R + 6\Lambda \right) ,$$

(17)

were $\kappa^2 = 16\pi G$. Assuming that the spacetime metric is AdS$_4$

$$ds^2 = e^{2\sigma(y)} \left( -dt^2 + (dx^1)^2 + (dx^2)^2 + dy^2 \right) ,$$

(18)
the action (17) reduces to
\[ S_{cl} = -\frac{1}{\kappa^2} \int d^4x e^{4\sigma} \left( -6e^{-2\sigma}((\sigma')^2 + (\sigma'')^2) + 6\Lambda \right), \] (19)
where \( \sigma' = \frac{d\sigma}{dy} \). For the analysis of quantum corrections we add to the classical action (17) or (19) the trace-anomaly-induced action
\[ W = \int d^4x \left[ 2b'\sigma^2 (\nabla^2 \sigma - 2(b + b')(\nabla^\sigma + \eta^\mu\nu(\partial_\mu\sigma)(\partial_\nu\sigma))^2) \right] = V_3 \int dy \left[ 2b'\sigma^4 - 2(b + b')(\sigma'' + (\sigma')^2) \right]. \] (20)
Here \( V_3 \) is the space-like volume. Then by variation with respect to \( \sigma \), we obtain the field equation
\[ \frac{a''''}{a} - \frac{4a'a'''}{a^2} - \frac{3(a'')^2}{a^2} + \left( 6 - 6\frac{b'}{b} \right) \frac{a''(a')^2}{a^3} + \frac{6b'(a')^4}{b a^4} - \frac{a}{4b\kappa^2} (-12a'' - 24\Lambda a^3) = 0, \] (21)
with \( a = e^\sigma \). By assuming \( a = \tilde{c} \), one obtains the following algebraic equation from (21) (we take \( \Lambda < 0 \))
\[ \kappa^2 b' - \tilde{c}^2 - \Lambda c^4 = 0, \] (22)
which has the solutions:
\[ \tilde{c}_1^2 = -\frac{1}{2\Lambda} \left( 1 + \sqrt{1 + 4\kappa^2 b'\Lambda} \right) \] (23)
and
\[ \tilde{c}_2^2 = -\frac{1}{2\Lambda} \left( 1 - \sqrt{1 + 4\kappa^2 b'\Lambda} \right). \] (24)
The first solution corresponds to the quantum corrected anti-de Sitter universe. Here, starting from some bare (even very small!) negative cosmological constant, we get an anti-de Sitter universe with a smaller cosmological constant due to quantum corrections. In other words, quantum corrections alone cannot create a 4d anti-de Sitter universe (unlike the case of the de Sitter universe). Rather, the quantum annihilation of the AdS universe may occur [4].
From Eq. (23) one finds that the effective cosmological constant \( \Lambda_{\text{eff}} \) is given by

\[
\Lambda_{\text{eff}} = \frac{2\Lambda}{1 + \sqrt{1 + 4\kappa^2 b\Lambda}}.
\]  

(25)

for the AdS case. Then, as in the previous section, we can consider an effective FRW equation,

\[
H^2 = \frac{8\pi G E}{3V} + \frac{\Lambda_{\text{eff}}}{3} - \frac{1}{a^2}.
\]  

(26)

Note that \( \Lambda_{\text{eff}} \) is negative, which differs from the previous section. If we assume the metric to be of the black hole type, as in (4), we have the following AdS black hole solution:

\[
f(r) = 1 - \frac{2MG}{r} + \frac{r^2}{r_0^2}, \quad r_0^2 = \frac{3}{\Lambda_{\text{eff}}}.
\]  

(27)

For a large black hole \( r_0 \ll r < 2MG \), we find that the black hole radius \( r^+ \) (obtained from the condition \( f(r^+) = 0 \)) is given by

\[
r^+ = \left( \frac{6MG}{\Lambda_{\text{eff}}} \right)^{\frac{1}{3}} - \frac{1}{(6G\Lambda_{\text{eff}})^{\frac{1}{3}}},
\]  

(28)

or equivalently

\[
\Lambda_{\text{eff}} = \frac{6GM}{r^3} - \frac{3}{r^2}.
\]  

(29)

As in [1], one gets the following expression for the Hubble bound

\[
S_H^2 = \frac{8\pi^2 ER^3}{9G} - \frac{4\pi^2 R^4}{27G^2},
\]  

(30)

where \( R \) is the universe size that in this case is given by \( R = r^+ \). We also choose \( E = M \). If we then define the Bekenstein-Hawking entropy as \( S_{BH} = \frac{\pi R^2}{G} \) and the Bekenstein entropy as \( S_B = 2\pi ER \), we get

\[
S_H^2 = \frac{4}{9} S_{BH} \left( S_B - \frac{1}{3} S_{BH} \right).
\]  

(31)

Defining the Virasoro operator as \( L_0 = 3ER \) and the central charge as \( c = 24E_{BH}R \), we have

\[
S_H = \frac{2\pi}{3\sqrt{3}} \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)}.
\]  

(32)
Thus, we have again arrived at the quantum-corrected Cardy-Verlinde formula which is exactly the same as in Ref.\[4\]. However, now the quantum corrections are taken into account in the effective cosmological constant. As in the previous case, the parameters entering the Cardy formula have been “renormalized” in order to include the quantum corrections.

When quantum effects are small, we have

\[ \Lambda_{\text{eff}} = \Lambda \left(1 - k^2 b' \Lambda \right) \] (33)

from Eq.(25), and

\[ R = r_+ = R^c \left(1 + \frac{1}{3} k^2 b' \Lambda \right) \] (34)

from Eq.(28). Here \( R^c \) is the black hole radius without quantum corrections:

\[ R^c \equiv \left(\frac{6 M G}{\Lambda} \right)^{\frac{1}{3}} + \frac{1}{\left(6 G M A \right)^{\frac{1}{3}}} \] . (35)

In this case the expressions for the entropies can be divided into a classical part and a part containing quantum corrections:

\[ S^2_H = (S^c_H)^2 + \left(\frac{8 \pi^2 E (R^c)^3}{3 G} - \frac{16 \pi^2 (R^c)^4}{27 G^2}\right) \frac{1}{3} k^2 b' \Lambda \]

\[ S_{BH} = S^c_{BH} + \frac{2 \pi (R^c)^2}{G} \frac{1}{3} k^2 b' \Lambda \]

\[ S_H = S^c_H + \frac{2 \pi}{3} R^c k^2 b' \Lambda \] , (36)

where

\[ (S^c_H)^2 = \frac{8 \pi^2 E (R^c)^3}{9 G} - \frac{4 \pi^2 (R^c)^4}{27 G^2} \]

\[ S_{BH}^c = \frac{\pi (R^c)^2}{G} \]

\[ S_H^c = 2 \pi R^c \] . (37)

The quantum-corrected Cardy-Verlinde formula (31) can easily be rewritten with the help of Eqs.(36) so that quantum corrections are given explicitly.
We should note, however, that the classical parts of the entropy satisfy the same relation as in Eq. (31):

\[
(S^c_H)^2 = \frac{4}{9} S^c_{BH} \left( S^c_B - \frac{1}{3} S^c_{BH} \right).
\] (38)

4 Quantum corrections in an AdS space with a dilatonic field

We now consider the case where the dilaton \( \phi \) appears in the conformal anomaly. For simplicity we consider an \( \mathcal{N} = 4 \) \( SU(N) \) super YM theory. The one-loop effective action is given by

\[
\Gamma = V_3 \int d\eta \left[ 2b' \sigma'''' - 2(b + b')(\sigma'' + (\sigma')^2) + (C \sigma + A) \text{Re}(\phi^* \phi''') \right].
\] (39)

Here \( A \) is a constant depending on the regularization and

\[
C = \frac{N^2 - 1}{(4\pi)^2}.
\] (40)

In Eq. (39) we assume that the spacetime metric is given by (18) and that \( \phi \) also depends only on the radial coordinate \( y \). The prime means derivative with respect to \( y \). In Eq. (39) \( \phi \) can be complex,

\[
\phi = \chi + i e^{-\varphi},
\] (41)

where \( \varphi \) is the dilaton and \( \chi \) is the R-R scalar (axion) of type IIB supergravity. We will consider the simple case where \( \chi = 0 \) and the kinetic term for the dilaton \( \varphi \) in the classical action is absent. We also choose the regularization-dependent parameter \( A \) to be zero. The effective equations of motion then take the following form:

\[
\frac{a''''}{a} - \frac{4a' a'''}{a^2} - \frac{3a''^2}{a^2} + \frac{6a'' a'^2}{a^3} \left( 1 - \frac{b'}{b} \right) + \frac{6b' a'^4}{ba^4} + \frac{3a a''}{\kappa^2 b} - \frac{C}{4b} \varphi'''' = 0,
\ln a \varphi'' + (\ln a \varphi)'''' = 0.
\] (42)
We now make the following change of variables:

\[ dz = a(y) \, dy . \]  

(43)

In terms of \( z \) the first of Equations (42) can be rewritten as

\[ a^2 \dddot{a} + 3a \ddot{a} \dddot{a} + 6a \dot{a} \dddot{a} - \left( 5 + \frac{6b'}{b} \right) \dot{a} \dddot{a} + \frac{3}{\kappa^2 b} \left( a^2 \dddot{a} + a \dddot{a} \right) - \frac{C \varphi Y[\varphi, a]}{4b} = 0 . \]  

(44)

Here \( \dot{a} = \frac{da}{dz} \) and \( Y[\varphi, a] \) is given in [9],

\[ Y[\phi, a] = a^3 \dddot{\phi} + 6a^2 \dot{a} \dddot{\phi} + 4a^2 \dddot{\phi} + 7a \dddot{a} \dddot{\phi} + a^2 \dddot{a} \dddot{\phi} + a \dddot{a} \dddot{\phi} + \dot{a} \dddot{\phi} . \]  

(45)

The second of Eqs. (42) (in terms of \( z \)) is also given in [9] (Eq.(10)). We now search for special solutions of the form

\[ a(z) \simeq a_0 e^{Hz}, \quad \varphi(z) \simeq \varphi_0 e^{-\alpha Hz} , \]  

(47)

Analyzing the second of Eqs. (42) and dropping the logarithmic term (using the same arguments as in Ref. [9]), one arrives at the solution:

\[ \varphi(z) = \varphi_1 e^{-\frac{3}{2} \frac{\varphi}{z}} + \varphi_2 e^{-2.02 \frac{\varphi}{z}} + \varphi_3 e^{-0.38 \frac{\varphi}{z}} , \]  

(48)

where \( \varphi_1, \varphi_2, \) and \( \varphi_3 \) are constants. Substituting the particular solution \( \varphi(z) = \varphi_0 e^{-\alpha z} \) into Eq.(44), one obtains:

\[ \frac{1}{l^2} \simeq \frac{1}{\kappa^2} \left[ b' + \frac{C}{24} \varphi_0^2 \left( \alpha^4 - 6\alpha^3 + 11\alpha^2 - 6\alpha \right) \right]^{-1} . \]  

(49)

The first term in the denominator is always negative, while the second term may be positive only at \( \alpha = 3/2 \). It is only for the special dilaton solution \( \varphi(z) = \varphi_1 e^{-\frac{3}{2} Hz} \) (i.e. \( \varphi_2 = \varphi_3 = 0 \)) and for the condition \( \varphi_1^2 > 12 \) that one obtains a positive \( l^2 \) and, hence, a non-imaginary scale factor for the AdS universe. Note that the corresponding AdS scale factor is:

\[ a(y) = -\frac{l}{y} . \]  

(50)
The effective cosmological constant $\Lambda_{\text{eff}}$ can now be identified with

$$\Lambda_{\text{eff}} = \frac{6}{l^2} = -\kappa^2 \left[ b' + \frac{C}{24} \varphi_0^2 \left( \alpha^4 - 6\alpha^3 + 11\alpha^2 - 6\alpha \right) \right]. \quad (51)$$

As one can see, the only role of the dilaton contribution is to change the effective cosmological constant! As in the previous section, we can consider an effective FRW equation (26) again and find the black hole solution (27) and finally obtain the Cardy-like formula (32) taking into account the dilaton contribution. Similarly, using results of Refs. [9] one can obtain the effective cosmological constant for de Sitter space with non-trivial dilaton contribution. The formulas of the second section may again be used in this case.

5 Discussion

In summary, we have discussed a simple model where quantum-corrected entropy bounds and a quantum-corrected (renormalized) Cardy-Verlinde formula are obtained. However, if we study Eq. (44), we see that the parts of the equation that we have replaced by $\Lambda_{\text{eff}}$ are actually a complicated combination of terms that contain up to second order time derivatives of $H$. This leads us to the question of how one might handle such terms if one is to go beyond the simple $\Lambda_{\text{eff}}$ model. A possibility that merits further study is to assume that we can construct a Hamiltonian formulation of the problem which can be used to identify terms in the equivalent of the formula for the Hubble parameter with the proper saturated entropy bounds.

An illustration of this idea is the ADM formulation for an isotropic $k = +1$ model with cosmological constant filled with CFT matter. The ADM action for this problem is

$$I = \frac{1}{16\pi G} \int \left[ \pi^{ij} \dot{g}_{ij} - N \sqrt{g} \left( \frac{1}{g} \left[ \pi^{ij} \pi_{ij} - \frac{1}{2} (\pi_k^k)^2 \right] \right) - 3R \right.$$

$$\left. + \frac{16\pi G E}{V} - 2\Lambda \right] dtd^3x, \quad (52)$$

where $g_{ij}$ is the three-metric on $t = \text{const.}$ surfaces and $^3R$ is the Ricci scalar of such a surface. If we assume that the metric has the form $g_{ij} = a^2(t)\tilde{g}_{ij}$,
where $\tilde{g}_{ij}$ is the metric of a unit three-sphere, the action now becomes (we take $N = 1$ to make $t$ cosmic time)

$$I = \pi \int \left[ p_a \dot{a} - \left( -\frac{Gp_a^2}{24a} - \frac{6a}{G} + \frac{16\pi Ea^3}{V} - \frac{2\Lambda a^3}{G} \right) \right] dt,$$

(53)

where we have used $\int \sqrt{g}d^3x = 16\pi^2$. We have also chosen units where $p_a$ has dimensions of ordinary momentum. The Hamiltonian constraint (from varying $N$) results from putting the quantity in parentheses equal to zero,

$$-\frac{Gp_a^2}{24a} - \frac{6a}{G} + \frac{16\pi Ea^3}{V} - \frac{2\Lambda a^3}{G} = 0.$$

(54)

Each of the terms in this equation has dimensions of energy, and we can write it as

$$-E_K - E_C - E_\Lambda + E_M = 0,$$

(55)

where $-E_K$ is the kinetic energy the gravitational field (a negative quantity, as is usual for a metric where the only time dependence is in a conformal factor), $E_C$ is the curvature energy (also negative), $E_\Lambda$ the energy due to $\Lambda$, and $E_M$ is the matter energy.

A simple way to convert this to an entropy equation would be to saturate the Bekenstein bound for each energy, but this leads to a linear relation that is inconsistent with the Cardy-Verlinde formula. However, it is not difficult to show that the gravitational kinetic energy term is $-\frac{6H^2a^4}{G}$, where $H$ is the Hubble parameter. In a similar way we can identify the sum of the curvature and $\Lambda$ energies $E_C + E_\Lambda$ with $S_{BH}$ of an SdS BH and the matter energy as proportional to $S_B S_{BH}$. These identifications allow us to conjecture that in any theory of gravity with a Hamiltonian formulation and a Hamiltonian constraint associated with possible redefinitions of time, we can use the Hamiltonian constraint rewritten in terms of energies to construct the equivalent of the Cardy-Verlinde formula for cosmological models in the theory by identifying the gravitational kinetic energy part with $S_H^2$, the curvature part with $S_{BH}$, and the matter part with $S_B$. Of course, for higher derivative theories such as the one given in Eq. (1) the construction of a Hamiltonian formalism is difficult, and it may not be possible to identify the

$^6$Quantum effective action typically contains the higher derivative terms.
terms mentioned above in such a way as to achieve a simple version of the Cardy-Verlinde formula, but for any cosmological model in ordinary gravity (extension of the ADM formalism to $n + 1$ dimensions is straightforward), one should be able to identify the Cardy-Verlinde formula in the way we have outlined above. Notice that for models with $t =$ constant surfaces that are not conformal to an isotropic metric, the $E_K$ term may be positive, but it can be decomposed into a negative and a positive part, and one must be careful in identifying the new version of $S_H$. The Bianchi type I (Kasner) models should be a good testing ground for our conjecture in this case.

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