SEMILINEAR EQUATIONS, THE $\gamma_k$ FUNCTION, AND GENERALIZED GAUDUCHON METRICS

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Abstract. In this paper, we generalize the Gauduchon metrics on a compact complex manifold and define the $\gamma_k$ functions on the space of its hermitian metrics.

1. Introduction

Let $X$ be a compact $n$-dimensional complex manifold. Let $g$ be a hermitian metric on $X$ and $\omega$ its hermitian form. It is well known that if $d\omega = 0$, then $g$ or $\omega$ is called a Kähler metric and therefore $X$ is called a Kähler manifold. When $X$ is a non-Kähler manifold, one can consider the other conditions on $\omega$ such as

$$d\omega^k = 0, \quad 2 \leq k \leq n - 1. \quad (1.1)$$

If $d(\omega^{n-1}) = 0$, then $g$ or $\omega$ is called a balanced metric and so $X$ is called a balanced manifold [19]. However, when $2 \leq k \leq n - 2$, $d\omega^k = 0$ automatically yields $d\omega = 0$ [15]. Instead of (1.1), one can consider the $k$-Kähler condition [1]. A complex manifold is called $k$-Kähler if it admits a closed complex transverse $(k,k)$–form.

By this definition, a complex manifold is 1-Kähler if and only if it is Kähler; it is $(n - 1)$-Kähler if and only if it is balanced.

One can also generalize the Kähler condition along other directions, for instance,

$$\partial \bar{\partial} \omega^k = 0, \quad 1 \leq k \leq n - 1. \quad (1.2)$$

When $k = n - 1$, the metric $\omega$ is called a Gauduchon metric. Gauduchon [11] proved an interesting result that, for any hermitian metric $\omega$ on a compact complex $n$-dimensional manifold $X$, there exists a unique (up to a constant) smooth function $v$ such that

$$\partial \bar{\partial} (e^v \omega^{n-1}) = 0 \quad \text{on } X. \quad (1.3)$$

Thus, the Gauduchon metric always exists on a compact complex manifold. It is important in complex geometry since one can use such a metric to define the degree, and then make sense of the stability of holomorphic vector bundles over a non-Kähler complex manifold (see [18]).

When $k = n - 2$, the metric $\omega$ satisfying (1.2) is called an astheno-Kähler metric. Jost and Yau [17] used this condition to study hermitian harmonic maps, and extended Siu’s rigidity theorem to non-Kähler complex manifolds.

When $k = 1$, the metric $\omega$ in (1.2) is called a pluriclosed metric, which is also called strong KT (Kähler with torsion) metric (see [13] and the references therein). Such a condition appeared in [6] as a technical condition. Recently, Streets and
Tian [21] introduced a hermitian Ricci flow under which the pluriclosed metric is preserved.

It is important to find specific hermitian metrics on non-Kähler complex manifolds. J. Li, S.-T. Yau and Fu [8] have constructed balanced metrics on complex structures of manifolds \( k \geq 2 \) \((S^3 \times S^3)\) which are obtained from the conifold transition of Calabi-Yau threefolds. As a corollary, there exists no pluriclosed metric on such manifolds. We note here that the specific hermitian geometry of threefolds \( k \) \((S^3 \times S^3)\) was first considered by Bozhkov [3, 4]. In this paper, we generalize (1.2) to weaker conditions:

\[
\partial \bar{\partial} \omega^k \wedge \omega^{n-k-1} = 0, \quad 1 \leq k \leq n-1.
\]

**Definition 1.** Let \( \omega \) be a hermitian metric on an \( n \)-dimensional complex manifold \( X \), and \( k \) be an integer such that \( 1 \leq k \leq n-1 \). We call \( \omega \) the \( k \)-th Gauduchon metric if \( \omega \) satisfies (1.4).

Note that an \((n-1)\)-th Gauduchon metric is the classic Gauduchon metric. The natural question is whether there exists any \( k \)-th Gauduchon metric, \( 1 \leq k < n-2 \), on a complex manifold. To answer this question, one way is to look for such a metric in the conformal class of a given hermitian metric \( \omega \) on \( X \):

\[
\partial \bar{\partial} (e^v \omega^k) \wedge \omega^{n-k-1} = 0.
\]

However, equation (1.5) in general needs not admit a solution (see below for reasons). In this paper, we solve the equation

\[
\partial \bar{\partial} (e^v \omega^k) \wedge \omega^{n-k-1} = \gamma_k e^v \omega^n
\]

for some constant \( \gamma_k \) satisfying the compatibility condition. The constant \( \gamma_k \), if nonzero, can be viewed as an obstruction for the existence of a \( k \)-th Gauduchon metric in the conformal class of \( \omega \), for \( 1 \leq k < n-1 \).

Equation (1.6) can be reformulated, in a slightly more general form, as follows: Let \((X, \omega)\) be an \( n \)-dimensional compact hermitian manifold, and \( B \) be a smooth real 1-form on \( X \). For any smooth function \( f \) on \( X \) satisfying

\[
\int_X f \omega^n = 0,
\]

we consider the following semilinear equation

\[
\Delta v + |\nabla v|^2 + \langle B, dv \rangle = f \quad \text{on } X.
\]

Here \( \Delta \) and \( \nabla \) are, respectively, the Laplacian and covariant differentiation associated with \( \omega \). Clearly, equation (1.8) needs not have a solution, due to the compatibility condition (1.7). For instance, let \( \omega \) be balanced and \( B = 0 \), then in order that (1.8) has a solution the function \( f \) has to be zero. Nonetheless, we shall show that, there is a smooth function \( v \) so that equation (1.8) holds up to a unique constant \( c \). More generally, we have the following result:

**Theorem 2.** Let \((X, \omega)\) be a compact hermitian manifold, \( B \) be a smooth real 1-form on \( X \), and \( \psi \in C^\infty(\mathbb{R}) \) satisfy

\[
\lim_{t \to +\infty} \frac{\psi(t)}{t^\mu} \geq \nu > 0, \quad \text{where } \mu > 1/2 \text{ and } \nu \text{ are constants}.
\]
Then, for each $f \in C^\infty(X)$ satisfying (1.1), there exists a unique constant $c$, and a smooth function $v$ on $X$, unique up to a constant, such that

\[
\Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle = f + c \quad \text{on } X.
\]

**Remark 3.** The compatibility condition of (1.1) implies that

\[
c = \frac{\int_X (\Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle) \omega^n}{\int_X \omega^n}.
\]

which in general is nonzero.

Letting $\psi(t) = t$ on $\mathbb{R}$, we obtain an application of Theorem [2]

**Corollary 4.** Let $(X, \omega)$ be an $n$-dimensional compact hermitian manifold. For any integer $1 \leq k \leq n - 1$, there exists a unique constant $\gamma_k$, and a function $v \in C^\infty(X)$ satisfying that

\[
(\sqrt{-1}/2) \partial \bar{\partial} (e^v \omega^k) \wedge \omega^{n-k-1} = \gamma_k e^v \omega^n.
\]

The solution $v$ of (1.11) is unique up to a constant. In particular, when $k = n - 1$ we have $\gamma_{n-1} = 0$. If $\omega$ is Kähler, then $\gamma_k = 0$ and $v$ is a constant, for each $1 \leq k \leq n - 1$.

**Remark 5.** When $k = n - 1$, this corollary recovers the classical result of Gauduchon [11].

By Corollary [4], we can associate each hermitian metric $\omega$ a unique constant $\gamma_k(\omega)$. Clearly, $\gamma_k = \gamma_k(\omega)$ is invariant under biholomorphisms. Furthermore, we will prove that $\gamma_k$ depends smoothly on the hermitian metric $\omega$ (see Proposition [3]); and that $\gamma_k(\omega) = 0$ if and only if there exists a $k$-th Gauduchon metric in the conformal class of $\omega$ (Proposition [4]).

We will prove in Proposition [11] that the sign of $\gamma_k(\omega)$, denoted by $(\mathrm{sgn}\gamma_k)(\omega)$, is invariant in the conformal class of $\omega$. We denote by $\Xi_k(X)$ the range of $\mathrm{sgn}\gamma_k$. By definition $\Xi_k(X) \subset \{-1, 0, 1\}$ for each $k$, and by Corollary [4] we have $\Xi_{n-1}(X) = \{0\}$. A natural question is whether $\Xi_k(X) = \{-1, 0, 1\}$ for any $1 \leq k \leq n - 2$ on any compact complex manifold $X$. Indeed, if $\Xi_k(X) \supset \{-1, 1\}$ then the answer is positive, by Proposition [3]. Thus, there will be a $k$-th Gauduchon metric on $X$. We can also ask whether $\Xi_k(X)$ is invariant under the modification. These questions will be systematically studied later. As a first step, we obtain the following result.

**Theorem 6.** For $n = 3$, we have $1 \in \Xi_1(X)$. Namely, for any 3-dimensional hermitian manifold $X$, there exists a hermitian metric $\omega$ such that $\gamma_1(\omega) > 0$. In particular, there is no 1-st Gauduchon metric in the conformal class of $\omega$.

Then, we combine the above results to prove that, as an example, $\Xi_1 = \{-1, 0, 1\}$ on the three-dimensional complex manifolds constructed by Calabi [5]. As a consequence, there exists a 1-st Gauduchon metric on these manifolds. It is well-known that such manifolds are non-Kähler but admit balanced metrics. We do not know whether there exists any pluriclosed metric on them.

Another example we considered is $Y = S^5 \times S^1$, endowed with a complex structure so that the natural projection $\pi : S^5 \times S^1 \to \mathbb{P}^2$ is holomorphic. This would imply
that there is no balanced metrics on $S^5 \times S^1$. Moreover, we can prove that $S^5 \times S^1$ does not admit any pluriclosed metric. On the other hand, by considering a natural hermitian metric on $S^5 \times S^1$, we are able to show that $\Xi_1(S^5 \times S^1) = \{-1, 0, 1\}$. Thus, $S^5 \times S^1$ admits a 1-st Gauduchon metric.

We shall solve equation (1.10) by the continuity method. In Section 2, we set up the machinery and prove the openness. The closedness and a priori estimates are established in Section 3. In Section 4, we prove Theorem 2 and also prove Corollary 4. In Section 5, we discuss the relation between $\gamma_k$ and the $k$-th Gauduchon metric. In section 6, we prove Theorem 6 and explicitly construct a metric with positive $\gamma_1$ number. As another example, we show that the natural balanced metric on the Iwasawa manifold has a positive $\gamma_1$ number.

In section 7, we establish the existence of 1-st Gauduchon metric on Calabi’s 3-dimensional non-Kähler manifold, by using Theorem 6 and proving that the balanced metric on the manifold has a negative $\gamma_1$ number. In the last section, we prove the existence of a 1-st Gauduchon metric on $S^5 \times S^1$. We also show the nonexistence of balanced metric and pluriclosed metric on $S^5 \times S^1$.

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2. Notation and preliminaries

Throughout this note, we use the following convention: We write

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} g_{ij} dz^i \wedge d\bar{z}^j.$$ 

Let $(g^{ij})$ be the transposed inverse of the matrix $(g_{ij})$. For any two real 1-forms $A$ and $B$ on $X$, locally given by

$$A = \sum_{i=1}^{n} (A_i dz_i + A_i d\bar{z}_i) \quad \text{and} \quad B = \sum_{i=1}^{n} (B_i dz_i + B_i d\bar{z}_i),$$

we denote

$$\langle A, B \rangle_\omega = \frac{1}{2} \sum_{i,j=1}^{n} g^{ij} (A_i B_j + A_j B_i).$$

We may omit the subscript $\omega$ in $\langle \cdot , \cdot \rangle_\omega$ when it is understood from the context. In particular, we have

$$\langle dh, dh \rangle = \sum_{i,j=1}^{n} g^{ij} \frac{\partial h}{\partial z_i} \frac{\partial h}{\partial \bar{z}_j} \equiv |\nabla h|^2, \quad \text{for all} \ h \in C^1(X).$$

The Laplacian $\Delta$ associated with $\omega$ is given by

$$\Delta h = \frac{n \omega^{n-1} \wedge (\sqrt{-1}/2) \partial \bar{\partial} h}{\omega^n} = \sum_{i,j=1}^{n} g^{ij} h_{ij}, \quad \text{for all} \ h \in C^2(X).$$
We use the continuity method to solve (1.10). Fix an integer \( l \geq n + 4 \) and a real number \( 0 < \alpha < 1 \). We denote by \( C^{l,\alpha}(X) \) the usual Hölder space on \( X \). Let

\[
S(u) = \Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle - \frac{\int_X (\Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle) \omega^n}{\int_X \omega^n},
\]

for each \( u \in C^{l,\alpha}(X) \). Consider the following family of equations,

\[
S(v_t) = tf, \quad 0 \leq t \leq 1.
\]

Let \( I \) be the subset of \([0, 1]\) consisting of \( t \) for which the equation (2.1) has a solution \( v_t \in C^{l,\alpha}(X) \) satisfying

\[
\int_X v_t \omega^n = 0.
\]

Obviously, the set \( I \) is nonempty since \( 0 \in I \). The openness of \( I \) will follow from our previous results [9, Section 3]. Indeed, let

\[
\mathcal{E}^{l,\alpha}_\omega = \left\{ h \in C^{l,\alpha}(X) ; \int_X h \omega^n = 0 \right\}.
\]

Notice that \( S : \mathcal{E}^{l+2,\alpha}_\omega \to \mathcal{E}^{l,\alpha}_\omega \). The linearization of \( S \) is

\[
L_\omega(h) = \left. \frac{d}{dt} S(v + th) \right|_{t=0} = \Delta h + \langle \tilde{B}, dh \rangle - \frac{\int_X (\Delta h + \langle \tilde{B}, dh \rangle) \omega^n}{\int_X \omega^n},
\]

where

\[
\tilde{B} = B + 2\psi'(|\nabla v|^2) dv.
\]

It follows from the proof of Lemma 13 in [9] that \( L_\omega \) is a linear isomorphism from \( \mathcal{E}^{l+2,\alpha}(X) \) to \( \mathcal{E}^{l,\alpha}(X) \). Thus, by the implicit theorem we obtain the openness of \( I \).

For the closedness of \( I \) we need the a priori estimate, which will be established in Section 3.

### 3. A Prior estimates

Let \((X, \omega)\) be an \( n \)-dimensional hermitian manifold, \( B \) a smooth 1-form on \( X \), \( f \) a smooth function on \( X \), \( c \) a constant, and \( \psi \in C^\infty(\mathbb{R}) \) satisfy (1.9). Consider the following semi-linear equation:

\[
S(v) \equiv \Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle - c = f \quad \text{on} \ X,
\]

where \( v \in C^3(X) \) satisfies the normalization condition

\[
\int_X v \omega^n = 0.
\]

We shall first derive a uniform gradient estimate:

**Lemma 7.** Let \( v \in C^3(X) \) be a solution of (3.1). We have

\[
\sup_X |\nabla v| \leq C,
\]

where \( C > 0 \) is a constant depending only on \( B, f, \omega, \psi(0), \mu \) and \( \nu \).
Throughout this section, we always denote by $C > 0$ a generic constant depending only on $B$, $f$, $\omega$, $\psi(0)$, $\mu$, and $\nu$, unless otherwise indicated.

**Proof.** Since $X$ is compact, we can assume that $|\nabla v|^2$ attains its maximum at some point $x_0 \in X$. Consider the following linear elliptic operator

$$L(h) = \Delta h + 2\psi'(|\nabla v|^2)(dh, dv)\omega = \Delta h + \psi'(|\nabla v|^2)g^{ij}(h_i v_j + h_j v_i),$$

Here the summation convention is used, and we denote

$$h_i = \frac{\partial h}{\partial z^i}, \quad g_{ij}^{\hat{\nu}} = \frac{\partial g^{\hat{\nu}j}}{\partial z^k}, \quad \ldots.$$ We compute that

$$L(|\nabla v|^2) = \Delta(|\nabla v|^2) + \psi'g^{ij}[(|\nabla v|^2)v_j + v_i(|\nabla v|^2)_j]$$

$$= g_{ij}^{pq}(v_p v_{\bar{q}j} + v_{\bar{p}j} v_q) + g_{ij}^{pq}[(\Delta v)_p v_q + v_p(\Delta v)_q] + g_{ij}^{pq} g_{ij}^{p\bar{q}} v_p v_q$$

$$+ g_{ij}^{pq} g_{ij}^{p\bar{q}} v_p v_{\bar{q}} + g_{ij}^{pq} g_{ij}^{p\bar{q}} v_p v_q$$

$$- g_{ij}^{pq}(g_{ij}^{pq}v_{ij} + g_{ij}^{pq}v_{j\bar{i}}) + \psi' g^{ij}[(|\nabla v|^2)v_j + v_i(|\nabla v|^2)_j].$$

Using equation (3.1) to the second term on the far right of above equalities and then using the Schwarz inequality, we find

$$L(|\nabla v|^2) \geq \frac{1}{2} g^{ij} g^{pq} (v_p v_{\bar{q}j} + v_{\bar{p}j} v_q) - C|\nabla v|^2 - C.$$

To see more clearly, let us take a normal coordinate system around $x_0$ such that

$$g_{ij}^{\hat{\nu}}(x_0) = \delta_{ij}, \quad \text{for all } i, j = 1, \ldots, n.$$ It follows that

$$L(|\nabla v|^2) \geq \frac{1}{2} \sum_{i,p=1}^{n} |v_{pi}|^2 - C|\nabla v|^2 - C$$

$$\geq \frac{1}{2} \sum_{i=1}^{n} |v_{ii}|^2 - C|\nabla v|^2 - C$$

$$\geq \frac{1}{2n} |\Delta v|^2 - C|\nabla v|^2 - C \quad \text{(by Cauchy’s inequality)}$$

$$\geq \frac{1}{2n} [\psi(|\nabla v|^2) + \langle B, dv \rangle - f - c|^2 - C|\nabla v|^2 - C \quad \text{(by (3.1))}$$

$$\geq \frac{1}{4n} [\psi(|\nabla v|^2)]^2 - C|\nabla v|^2 - C(1 + |c|^2).$$

We can assume, without loss of generality, that $|\nabla v|^2(x_0)$ is sufficiently large so that

$$\psi(|\nabla v|^2) \geq \frac{\nu}{2} |\nabla v|^{2\mu} \quad \text{at } x_0,$$

where $\mu > 1/2$ and $\nu > 0$ are constants, by (1.9). Now notice that

$$L(|\nabla v|^2) \leq 0 \quad \text{at } x_0,$$
because of
\[ \Delta(|\nabla v|^2)(x_0) \leq 0, \quad \text{and} \quad \nabla(|\nabla v|^2)(x_0) = 0. \]

Hence, we obtain that
\[ \sup_X |\nabla v|^2 = |\nabla v|^2(x_0) \leq C(1 + |c|^2). \]

It remains to bound the constant \( c \) in terms of \( f \) and \( \psi(0) \): Apply the usual maximum principle to (3.1) to obtain that
\[ (3.3) \quad \psi(0) - \sup_X f \leq c \leq -\inf_X f + \psi(0). \]

This finishes the proof. \( \square \)

Next, we establish the \( C^0 \) estimate: Noticing (3.2), there must exist some point \( y_0 \in X \) such that \( v(y_0) = 0 \). Then, for any point \( y \in X \), we take a geodesic curve \( \gamma \) connecting \( y_0 \) to \( y \). We have by Lemma 7 that,
\[ |v(y)| = |v(y) - v(y_0)| = \left| \int_0^1 \frac{d(v \circ \gamma)}{dt} dt \right| \leq \int_0^1 (|\nabla v| \circ \gamma) dt < C. \]

This settles the \( C^0 \) estimate of \( v \).

We rewrite equation (3.1) as
\[ \triangle v = -\psi(|\nabla v|^2) - \langle B, dv \rangle + f + c. \]

By \( W^{2,p} \) theory of elliptic equations, we have for any \( p > 1 \),
\[
\|v\|_{W^{2,p}} \leq C(\|v\|_{L^p} + \|f + c - \psi(|\nabla v|^2) - \langle B, dv \rangle\|_{L^p}) \\
\leq C_1,
\]
where in the last inequality we have used the \( C^0 \) and \( C^1 \) estimates of \( v \), and (3.3). Here and below, we denote by \( C_1 \) a generic constant depending on \( B, f, \omega, \mu, \nu \), and also \( p \), and \( \max\{|\psi(t)|; 0 \leq t \leq \max |\nabla v|^2 \leq C\} \).

Fix a sufficiently large \( p \) such that \( \alpha \equiv 2n/p < 1 \). It follows from the Sobolev embedding theorem that
\[ \|v\|_{C^{1, \alpha}} \leq C_1. \]

This allows us to apply Schauder’s theory to obtain that
\[ \|v\|_{C^{2, \alpha}} \leq C_1. \]

Thus, by the bootstrap argument, we have
\[ (3.4) \quad \|v\|_{C^{1, \alpha}} \leq C_1, \quad \text{for any } k \geq 1. \]

This implies that the set \( I \) defined in Section 2 is closed. As a consequence, we have shown the existence part in Theorem 2.
4. Uniqueness and Corollary

Let us prove the uniqueness in Theorem 2. Suppose that there exist \( c, v \) and \( \tilde{c}, \tilde{v} \) such that
\[
\Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle = f + c,
\]
\[
\Delta \tilde{v} + \psi(|\nabla \tilde{v}|^2) + \langle B, d\tilde{v} \rangle = f + \tilde{c}.
\]
Then,
\[
c = \frac{\int_X (\Delta v + \psi(|\nabla v|^2) + \langle B, dv \rangle) \omega^n}{\int_X \omega^n},
\]
\[
\tilde{c} = \frac{\int_X (\Delta \tilde{v} + \psi(|\nabla \tilde{v}|^2) + \langle B, d\tilde{v} \rangle) \omega^n}{\int_X \omega^n}.
\]
Recall that we denote
\[
S(u) = \Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle - \int_X (\Delta u + \psi(|\nabla u|^2) + \langle B, du \rangle) \omega^n \int_X \omega^n,
\]
for all \( u \in C^2(X) \). It follows that
\[
0 = S(v) - S(\tilde{v}) = \int_0^1 \left[ \frac{d}{dt} S(tv + (1 - t)\tilde{v}) \right] dt
\]
(4.3)
\[
= \Delta w + \langle \tilde{B}, dw \rangle - c_w.
\]
Here \( w = v - \tilde{v} \),
\[
\tilde{B} = B + 2 \int_0^1 \psi(|t \nabla v + (1 - t) \nabla \tilde{v}|^2) \left[ t dv + (1 - t) d\tilde{v} \right] dt,
\]
and \( c_w \) is a constant given by
\[
c_w = \frac{\int_X (\Delta w + \langle \tilde{B}, dw \rangle) \omega^n}{\int_X \omega^n}.
\]
Applying the maximum principle to (4.3) yields
\[
c_w = 0.
\]
Then, by the strong maximum principle we conclude that \( w \) is equal to a constant. This shows that the solution of (1.10) is unique up to a constant. By (4.1) and (4.2) we have \( c = \tilde{c} \). This completes the proof of Theorem 2.

Let us now prove Corollary 4. We define a smooth real 1-form on \( X \)
\[
B_1 = \frac{\sqrt{-1}}{2} \frac{nk}{n - 1} \frac{1}{n!} * (\partial (\omega^{n-1}) - \bar{\partial} (\omega^{n-1}))
\]
(4.4)
and a smooth function
\[
\varphi = \frac{n(\sqrt{-1}/2) \partial \bar{\partial} (\omega^k) \wedge \omega^{n-k-1}}{\omega^n}.
\]
(4.5)
Then (1.11) is equivalent to
\[
\Delta v + |\nabla v|^2 + \langle B_1, dv \rangle + \varphi = n \gamma_k.
\]
Letting $\psi(t) = t$ and $f = \int_X \varphi \omega^n - \varphi$,

Corollary 4 then follows readily from Theorem 2.

For each $1 \leq k \leq n - 1$, the constant $\gamma_k$ is given by

$$
\gamma_k = \frac{\int_X e^{-v(\sqrt{-1}/2)\partial\bar{\partial}(e^v \omega^k)} \wedge \omega^{n-k-1}}{\int_X \omega^n}
$$

On the other hand, directly integrating (1.11) over $X$ yields that

$$
\gamma_k = \frac{\int_X (\Delta v + |\nabla v|^2 + \langle B_1, dv \rangle + \varphi) \omega^n}{n \int_X \omega^n}.
$$

This together with (4.6) imposes some constraint on the constant $\gamma_k$. For instance, when $k = n - 1$, by (4.8) we know that

$$
\gamma_{n-1} = 0.
$$

Thus, in this case Corollary 4 recovers the classic result of Gauduchon [11]. When $\omega$ is Kähler, by (4.8) again we have

$$
\gamma_k = 0 \quad \text{for all} \ 1 \leq k \leq n - 1.
$$

Then, it follows from (4.7) that

$$
\int_X |\nabla v|^2 \omega^n = 0.
$$

This tells us that the solution $v$ of (1.11) has to be a constant.

5. **Generalized Gauduchon metrics and $\gamma_k$**

Let $X$ be an $n$-dimensional complex manifold. We recall by Definition 1 that a hermitian metric $\omega$ on $X$ is called $k$-th Gauduchon metric if

$$
\partial\bar{\partial}(\omega^k) \wedge \omega^{n-k-1} = 0 \quad \text{on} \ X.
$$

Then, the $(n-1)$-th Gauduchon metric is the Gauduchon metric in the usual sense. By Corollary 4 each hermitian metric $\omega$ on $X$ can be associated with a unique constant $\gamma_k(\omega)$, which is invariant under biholomorphisms. The induced function $\gamma_k = \gamma_k(\omega)$ can be used to characterize the $k$-th Gauduchon metric.

**Proposition 8.** The hermitian manifold $X$ admits a $k$-th Gauduchon metric if and only if that there exists a hermitian metric $\omega$ on $X$ such that

$$
\gamma_k(\omega) = 0.
$$

**Proof.** If there is some hermitian metric $\omega$ satisfying (5.1), then Corollary 4 implies that the conformal metric $e^{v/k}\omega$ is a $k$-th Gauduchon metric on $X$. Conversely, if $\omega$ is a $k$-th Gauduchon metric, then the uniqueness of Corollary 4 implies that $\gamma_k(\omega) = 0$ and that $v$ is a constant. \[\square\]
Let $\mathcal{M}$ be the set of all hermitian metrics on $X$. We shall prove that $\gamma_k$ is a smooth function on $\mathcal{M}$. Here $\mathcal{M}$ is viewed as an open subset in $C^{l+2,\alpha}(\Lambda^{1,1}(X))$, for a nonnegative integer $l$ and a real number $0 < \alpha < 1$. We denote by $C^{l,\alpha}(\Lambda^{m,m}(X))$ the Hölder space of real $(m,m)$-forms on $X$, in which $l$ and $m$ are nonnegative integers, and $0 < \alpha < 1$ is a real number. In particular, $C^{l,\alpha}(\Lambda^{0,0}(X)) = C^{l,\alpha}(X)$.

**Proposition 9.** The function $\gamma_k = \gamma_k(\omega)$ is a smooth function on $\mathcal{M}$, where $\mathcal{M}$ is viewed as an open subset in $C^{l+2,\alpha}(\Lambda^{1,1}(X))$.

**Proof.** It follows from Corollary 4 that, for each $\omega \in \mathcal{M}$, there exists a unique constant $\gamma_k$ and a function $v$ such that

$$e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v \omega^k) \wedge \omega^{n-k-1} - \gamma_k \omega^n = 0. \quad (5.2)$$

Then,

$$\gamma_k = \int_X e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v \omega^k) \wedge \omega^{n-k-1} \int_X \omega^n$$

depends smoothly on $v$ and $\omega$. Thus, to show the result, it suffices to show that the solution $v$ depends smoothly on $\omega$. We shall use the implicit function theorem.

For each $\omega \in \mathcal{M}$, the space $E^{l,\alpha}_0$ is defined by (2.3). Fix $\omega_0 \in \mathcal{M}$, for which we abbreviate $E^{l,\alpha}_{0} = E^{l,\alpha}_{\omega_0}$. We have two obvious linear isomorphisms from $E^{l,\alpha}_\omega$ to $E^{l,\alpha}_{0}$, given respectively by

$$h \mapsto h - \frac{\int_X h \omega^n_0}{\int_X \omega^n_0}, \quad \text{for all } h \in E^{l,\alpha}_\omega, \quad (5.3)$$

and

$$h \mapsto h \cdot \frac{\omega^n_0}{\omega^n_0}, \quad \text{for all } h \in E^{l,\alpha}_\omega. \quad (5.4)$$

Define a map $F : \mathcal{M} \times E^{l+2,\alpha}_0 \to E^{l,\alpha}_0$ by

$$F(\omega, v) = \frac{ne^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v \omega^k) \wedge \omega^{n-k-1}}{\int_X \omega^n_0} - n \frac{\int_X e^{-v}(\sqrt{-1}/2)\partial\bar{\partial}(e^v \omega^k) \wedge \omega^{n-k-1}}{\int_X \omega^n} \int_X \omega^n_0.$$

Obviously, $F$ is a smooth map. Note that any $(\omega, v) \in \mathcal{M} \times E^{l+2,\alpha}_0$ satisfies (5.2) if and only if

$$F(\omega, v) = 0.$$

The Fréchet derivative of $F$ with respect to the variable $v$ is

$$D_v F(\omega, v)(h) = L_\omega(h) \frac{\omega^n_0}{\omega^n_0}.$$

Here

$$L_\omega(h) = \Delta h + (B_1 + 2dv, dh) \omega - \frac{\int_X (\Delta h + (B_1 + 2dv, dh) \omega) \omega^n}{\int_X \omega^n}.$$
in which the Laplacian $\Delta$ is with respect to $\omega$, and $B_1$ is the smooth real 1-form given by $\left(4.4\right)$. By the proof of Lemma 13 in [9] and the isomorphism $\left(5.3\right)$, the operator $L_\omega : \mathcal{E}^{l+2, \alpha}_0 \to \mathcal{E}^{l, \alpha}_\omega$ is a linear isomorphism. This combining isomorphism $\left(5.4\right)$ imply that $D_tF(\omega, v) : \mathcal{E}^{l+2, \alpha}_0 \to \mathcal{E}^{l, \alpha}_\omega$ is a linear isomorphism. The result then follows by the Implicit Function Theorem. □

A direct corollary of Proposition 9 is as below.

**Corollary 10.** For $1 \leq k \leq n-2$, if there exists two hermitian metric $\omega_1, \omega_2$ on $X$ such that

$$
\gamma_k(\omega_1) > 0 \quad \text{and} \quad \gamma_k(\omega_2) < 0,
$$

then there exists a metric $\omega$ on $X$ satisfying $\gamma_k(\omega) = 0$, i.e., $\omega$ is a $k$-th Gauduchon metric.

**Proof.** Let $\omega_t = t\omega_1 + (1-t)\omega_2$, for all $0 \leq t \leq 1$. Then $\omega_t$ is a hermitian metric for each $t$. The result follows immediately by applying the Mean Value Theorem to the function $\phi(t) = \gamma_k(\omega_t)$.

□

**Proposition 11.** For any function $\rho \in C^2(M)$, we have

$$
\left(5.5\right) \quad e^{-\max X \rho} \gamma_k(\omega) \leq \gamma_k(e^\rho \omega) \leq e^{-\min X \rho} \gamma_k(\omega).
$$

In particular, the sign of the function $\gamma_k$ is a conformal invariant for hermitian metrics.

**Proof.** Let $\tilde{\omega} = e^\rho \omega$. Then, there exists a function $\tilde{v}$ and a number $\tilde{\gamma}_k = \gamma_k(\tilde{\omega})$ satisfying

$$
\left(\sqrt{-1}/2\right)\partial\bar{\partial}(e^{\tilde{v}} \omega^k) \wedge \omega^{n-k-1} = \tilde{\gamma}_k e^{\tilde{v}} \omega^n,
$$

that is,

$$
\left(\sqrt{-1}/2\right)\partial\bar{\partial}(e^{\tilde{v}+kp} \omega^k) \wedge \omega^{n-k-1} = \tilde{\gamma}_k e^{\tilde{v}+kp} \omega^n.
$$

We can rewrite $\left(5.6\right)$ as

$$
\left(5.7\right) \quad \Delta(\tilde{v} + kp) + |\nabla(\tilde{v} + kp)|^2 + \langle B_1, d(\tilde{v} + kp) \rangle + \varphi = ne^{\rho} \tilde{\gamma},
$$

where the operators $\Delta$ and $\nabla$ are with respect to $\omega$, and $B_1$ and $\varphi$ are given by $\left(4.4\right)$ and $\left(4.5\right)$, respectively. Subtracting $\left(5.7\right)$ by

$$
\Delta v + |\nabla v|^2 + \langle B_1, dv \rangle + \varphi = n\gamma_k(\omega)
$$

and then applying the maximum principle yields $\left(5.5\right)$. □

**Proposition 12.** For a hermitian metric $\omega$, the number $\gamma_k(\omega) > 0 \ (= 0, \ or \ < 0)$ if and only if there exists a metric $\tilde{\omega}$ in the conformal class of $\omega$ such that

$$
\left(5.8\right) \quad \left(\sqrt{-1}/2\right)\partial\bar{\partial}\tilde{\omega}^k \wedge \tilde{\omega}^{n-k-1} > 0 \ (= 0, \ or \ < 0) \quad \text{on} \quad X.
$$
Proof. Suppose that \( \gamma_k(\omega) > 0 \) (= 0, or < 0). Let \( \tilde{\omega} = e^{\nu/k}\omega \), where \( \nu \) is the smooth function associated with \( \omega \) so that (1.11) holds. Then,

\[
(\sqrt{-1}/2)\bar{\partial}\partial \tilde{\omega}^k \wedge \tilde{\omega}^{n-k-1} = \gamma_k(\omega)\omega^ne(n-k)^{\nu} > 0 \quad (= 0, \text{ or } < 0).
\]

Conversely, if there is a metric \( \tilde{\omega} \) in the conformal class of \( \omega \) such that (5.8) holds, then we claim that \( \gamma_k(\tilde{\omega}) > 0 \) (= 0, or < 0). Indeed, by Corollary 4 there exists a smooth function \( \tilde{v} \) such that

\[
(\sqrt{-1}/2)\bar{\partial}\partial (e^{\tilde{v}}\tilde{\omega}^k) \wedge \tilde{\omega}^{n-k-1} = \gamma_k(\tilde{\omega})e^{\tilde{v}}\tilde{\omega}.
\]

This is equivalent to the following equation

(5.9)

\[
\Delta \tilde{v} + |\nabla \tilde{v}|^2 + \langle \tilde{B}_1, d\tilde{v} \rangle + \tilde{\gamma} = n\gamma_k(\tilde{\omega}),
\]

where the operators \( \Delta \) and \( \nabla \) are with respect to \( \tilde{\omega} \), and \( \tilde{B}_1 \) and \( \tilde{\gamma} \) are given by (4.4) and (4.5), respectively, with \( \tilde{\omega} \) replacing \( \omega \). By (5.8) we have \( \tilde{\gamma} > 0 \) (= 0, or < 0). The claim then follows immediately by applying the maximum principle to (5.9). By Proposition 11, we finish the proof.

Moreover, for the case of \( \gamma_k > 0 \), we have the following criteria on the integration, which is often easier to verify.

**Lemma 13.** Suppose that \( n \), the complex dimension of \( X \), is an odd number. Let \( k = (n-1)/2 \). Then, there is some metric \( \omega \) satisfying \( \gamma_k(\omega) > 0 \) if and only if there is some semi-metric \( \tilde{\omega} \) (i.e., semi-positive real \((1,1)\)-form on \( X \)) satisfying

\[
\frac{\sqrt{-1}}{2} \int_X \bar{\partial}\partial \tilde{\omega}^k \wedge \tilde{\omega}^{n-k-1} > 0
\]

**Proof.** By Proposition 12, the necessary part is obvious. For the sufficient part, let \( \tilde{\omega} \) be any hermitian metric. Let

\[
\omega_t = \tilde{\omega} + t\tilde{\omega}
\]

for \( t \in (0,1) \). Then we have

\[
\int_X e^{-\nu}(\sqrt{-1}/2)\bar{\partial}\partial (e^{\nu}\omega_t^k) \wedge \omega_t^{n-k-1}
\]

\[
= \frac{\sqrt{-1}}{2} \int_X (\bar{\partial}\partial \omega_t^k \wedge \omega_t^{n-k-1} + \partial v \wedge \bar{\partial}v \wedge \omega_t^{n-1})
\]

\[
+ \frac{\sqrt{-1}}{2} \int_X \left[ \bar{\partial}\partial v \wedge \omega_t^{n-1} + \frac{k}{n-1} (\partial \omega_t^{n-1} \wedge \bar{\partial}v + \partial v \wedge \bar{\partial}\omega_t^{n-1}) \right]
\]

\[
= \frac{\sqrt{-1}}{2} \int_X (\bar{\partial}\partial \omega_t^k \wedge \omega_t^{n-k-1} + \partial v \wedge \bar{\partial}v \wedge \omega_t^{n-1})
\]

\[
+ \frac{\sqrt{-1}}{2} \left( 1 - \frac{2k}{n-1} \right) \int_X v\bar{\partial}\omega_t^{n-1}.
\]
Since \( k = (n - 1)/2 \), the second integral on the right of (5.10) vanishes. It follows that
\[
\int_X e^{-v} \sqrt{-1/2} \partial \bar{\partial} (e^v \omega_i^k) \wedge \omega_i^{n-k-1} \geq \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega_i^k \wedge \omega_i^k
\]
\[
= \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega_i^k \wedge \hat{\omega}_i^k + t \frac{\sqrt{-1}}{2} \int_X (\partial \bar{\partial} \omega_i^k \wedge \Psi_t + \partial \bar{\partial} \Psi_t \wedge \hat{\omega}_i^k)
\]
\[
+ t^2 \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \Psi_t \wedge \Psi_t > 0, \quad \text{for sufficiently small} \ t,
\]
where \( \Psi_t = \hat{\omega} \wedge (\hat{\omega}_i^{k-1} + \hat{\omega}_i^{k-2} \wedge \omega_t + \cdots + \hat{\omega} \wedge \omega_t^{k-2} + \omega_t^{k-1}) \). This implies that \( \gamma_k (\omega_t) > 0 \) for the sufficiently small \( t \). \( \square \)

A similar argument works for the (classic) Gauduchon metrics, for any dimension \( n \), and for all \( 1 \leq k \leq n - 2 \).

Lemma 14. Let \( X \) be an \( n \)-dimensional hermitian manifold, \( k \) an integer such that \( 1 \leq k \leq n - 2 \). Then, a hermitian metric \( \omega \) on \( X \) satisfies \( \gamma_k (\omega) > 0 \) if the Gauduchon metric \( \tilde{\omega} \) in the conformal class of \( \omega \) satisfies
\[
(5.11) \quad \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega_i^k \wedge \tilde{\omega}_i^{n-k-1} > 0.
\]

Proof. By Proposition 11, we can assume that \( \omega = \tilde{\omega} \), without loss of generality. By (5.10) with \( \omega \) replacing \( \omega_t \), and applying \( \partial \bar{\partial} \omega^{n-1} = 0 \) yields
\[
\int_X e^{-v} \sqrt{-1/2} \partial \bar{\partial} (e^v \omega_i^k) \wedge \omega_i^{n-1-k} \geq \frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega_i^k \wedge \omega_i^{n-1-k} > 0.
\]
\( \square \)

Corollary 15. Let \( (X, \omega) \) be an \( n \)-dimensional balanced manifold. Then, for each \( 1 \leq k \leq n - 2 \), we have \( \gamma_k (\omega) > 0 \) if
\[
\frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \omega_i^k \wedge \omega_i^{n-1-k} > 0.
\]

6. CONSTRUCTIONS ON HERMITIAN THREE–MANIFOLDS

We shall apply previous results to construct a hermitian metric with \( \gamma_1 > 0 \) on a complex three dimensional manifold. Theorem 6 will follow from Proposition 12, together with the following theorem.

Theorem 16. There always exists a hermitian metric \( \omega \) on a complex three dimensional manifold \( X \) such that
\[
(\sqrt{-1}/2) \partial \bar{\partial} \omega \wedge \omega > 0.
\]

Proof. By Lemma 13 and Proposition 12, it suffices to construct a semi-metric \( \hat{\omega} \) such that
\[
\frac{\sqrt{-1}}{2} \int_X \partial \bar{\partial} \hat{\omega} \wedge \hat{\omega} > 0.
\]
Fix a point \( q \in X \) and a coordinate patch \( U \ni q \). Let \( (z_1, z_2, z_3) \) be coordinates on \( U \) centered at \( q \). Here \( z_j = x_j + \sqrt{-1}y_j \) for \( 1 \leq j \leq 3 \). We can assume \( N = B \times B \times R \subset U \), where \( B \) is the unit ball in \( \mathbb{C} \), and
\[
R = \{ z_3 \in \mathbb{C} \mid |x_3| \leq 1, |y_3| \leq 1 \}.
\]

Take a nonnegative cut-off function \( \eta \in C_0^\infty(B) \) and two nonnegative functions \( f, g \in C_0^\infty([-1, 1]) \) to be determined later. On \( N \), define
\[
\phi = \eta(z_1)\eta(z_2)f(x_3)f(y_3), \quad \psi = \eta(z_1)\eta(z_2)g(x_3)g(y_3),
\]
and then define
\[
(6.1) \quad \tilde{\omega} = \frac{\sqrt{-1}}{2} [\phi(z)dz_1 \wedge d\bar{z}_1 + \psi(z)dz_2 \wedge d\bar{z}_2].
\]

Obviously, \( \tilde{\omega} \) is semi-positive and with compact support in \( N \). So it can be viewed as a semi-metric on \( X \). Clearly,
\[
(6.2) \quad \frac{\sqrt{-1}}{2} \partial\bar{\partial}\tilde{\omega} = \left( \phi \frac{\partial^2 \psi}{\partial z_3 \partial \bar{z}_3} + \psi \frac{\partial^2 \phi}{\partial z_3 \partial \bar{z}_3} \right) dV,
\]
where
\[
(6.3) \quad dV = \left( \frac{\sqrt{-1}}{2} \right)^3 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge dz_3 \wedge d\bar{z}_3.
\]
Since
\[
\frac{\partial}{\partial z_3} = \frac{1}{2} \left( \frac{\partial}{\partial x_3} - \sqrt{-1} \frac{\partial}{\partial y_3} \right), \quad \frac{\partial}{\partial \bar{z}_3} = \frac{1}{2} \left( \frac{\partial}{\partial x_3} + \sqrt{-1} \frac{\partial}{\partial y_3} \right),
\]
we have
\[
\phi \frac{\partial^2 \psi}{\partial z_3 \partial \bar{z}_3} + \psi \frac{\partial^2 \phi}{\partial z_3 \partial \bar{z}_3} = \frac{\phi}{4} \left( \frac{\partial^2 \psi}{\partial x_3 \partial x_3} + \frac{\partial^2 \psi}{\partial y_3 \partial y_3} \right) + \frac{\psi}{4} \left( \frac{\partial^2 \phi}{\partial x_3 \partial x_3} + \frac{\partial^2 \phi}{\partial y_3 \partial y_3} \right)
\]
\[
= \frac{1}{4} \eta^2(z_1)\eta^2(z_2) f(y_3)g(y_3) \left[ f(x_3)g''(x_3) + g(x_3)f''(x_3) \right]
\]
\[
+ \frac{1}{4} \eta^2(z_1)\eta^2(z_2) f(x_3)g(x_3) \left[ f(y_3)g''(y_3) + g(y_3)f''(y_3) \right].
\]

We choose \( \eta \) so that
\[
\int_B \eta^2(z) \frac{\sqrt{-1}}{2} dz \wedge d\bar{z} = 1.
\]

Then it follows that
\[
\frac{\sqrt{-1}}{2} \int_X \partial\bar{\partial}\tilde{\omega} = \frac{1}{2} \int_{-1}^1 f(t)g(t)dt \int_{-1}^1 \left[ f(t)g''(t) + f''(t)g(t) \right] dt
\]
\[
= \int_{-1}^1 f(t)g(t)dt \int_{-1}^1 \left[ -f'(t)g'(t) \right] dt.
\]

The result follows immediately from the proposition below. \qed

Proposition 17. There exist nonnegative functions $f, g \in C^\infty_0([-1,1])$ such that

$$- \int_{-1}^{1} f'(t)g'(t)dt > 0.$$ 

Proof. For any two real numbers $a < b$, we denote

$$\chi_{a,b}(t) = \begin{cases} \exp \left( \frac{1}{t-b} - \frac{1}{t-a} \right), & \text{if } a < t < b, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, we have that $\chi_{a,b} \in C^\infty_0(\mathbb{R})$, that $\chi'_{a,b}(t) > 0$ for $a < t < (a+b)/2$, that $\chi'_{a,b}(t) < 0$ for $(a+b)/2 < t < b$, and that $\chi_{a,b}(t) = 0$ when $t = (a+b)/2$. Letting

$$f(t) = \chi_{-1/3,1/3}(t), \quad \text{and} \quad g(t) = \chi_{0,2/3}(t)$$

yields that $-f'(t)g'(t) > 0$ for $0 < t < 1/3$ and otherwise $f'(t)g'(t) = 0$. This in particular implies the result. □

Let us now consider some examples. We can directly construct a hermitian metric $\omega$ with $\gamma_1(\omega) > 0$ on $T^3$, the 3-dimensional complex torus.

Proposition 18. On the complex $T^3$, there is a metric $\omega$ satisfying

$$\sqrt{-1/2} \partial \bar{\partial} \omega \wedge \omega > 0.$$ 

Proof. Let $(z_1, z_2, z_3)$ be the coordinates of $T^3$ induced from $\mathbb{C}^3$. Let

$$\omega = \frac{\sqrt{-1}}{2} \left[ \xi(x_3) dz_1 \wedge d\bar{z}_1 + \eta(x_3) dz_2 \wedge d\bar{z}_2 + dx_3 \wedge d\bar{x}_3 \right],$$

where $\xi$ and $\eta$ are two positive smooth functions on $T^3$ only depending on $x_3$, which will be determined later. Then

$$(\sqrt{-1/2}) \partial \bar{\partial} \omega \wedge \omega = \left( \eta \frac{\partial^2 \xi}{\partial z_3 \partial \bar{z}_3} + \xi \frac{\partial^2 \eta}{\partial z_3 \partial \bar{z}_3} \right) dV > 0$$

if and only if

$$\eta \frac{\partial^2 \xi}{\partial z_3 \partial \bar{z}_3} + \xi \frac{\partial^2 \eta}{\partial z_3 \partial \bar{z}_3} = \frac{1}{4} \eta \frac{\partial^2 \xi}{\partial x_3^2} + \frac{1}{4} \xi \frac{\partial^2 \eta}{\partial x_3^2} > 0.$$ 

Here $dV$ is defined by (6.3). So we need to look for two smooth, positive, $2\pi$-periodic functions $\eta$ and $\xi$ such that

$$\frac{\eta''(t)}{\eta(t)} + \frac{\xi''(t)}{\xi(t)} > 0.$$

We define

(6.4) \hspace{1cm} \xi(t) = 1 + \kappa \sin t, \quad \text{for some } 0 < \kappa < 1.

We observe that

$$\int_0^{2\pi} \frac{\xi''}{\xi} dt = - \int_0^{2\pi} \frac{\kappa \sin t}{1 + \kappa \sin t} dt = -2\pi \int_0^{2\pi} \frac{dt}{1 + \kappa \sin t}.$$
By Proposition 8 in [9], the value of above integral tends to \(+\infty\) monotonically, as \(\kappa \to 1^\pm\). Hence, for a constant \(C > 0\), there is a unique real number \(\kappa\), such that the function \(\xi\) given by (6.4) satisfies
\[
\int_0^{2\pi} \frac{\xi''}{\xi} dt = \int_0^{2\pi} C dt.
\]
It implies that equation
\[
\zeta'' + \frac{\xi''}{\xi} = C
\]
has a smooth \(2\pi\)-periodic solution \(\zeta\) on \(\mathbb{R}\). Let \(\eta = e^\xi\). Thus,
\[
\eta''(t) + \frac{\xi''(t)}{\xi(t)} = (\zeta'(t))^2 + \zeta''(t) + \frac{\xi''(t)}{\xi(t)} \geq C > 0.
\]
\[\Box\]

As another example, we show that the natural balanced metric on the Iwasawa manifold has positive \(\gamma_1\). Recall (for example, [16, p. 444] and [20, p. 115]) that the Iwasawa manifold is defined to be the quotient space \(G/\Gamma\), where
\[
G = \left\{ \begin{bmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{bmatrix} : z_1, z_2, z_3 \in \mathbb{C} \right\},
\]
\(\Gamma\) is the discrete subgroup of \(G\) consisting of matrices where \(z_1, z_2, z_3\) are Gaussian integers, i.e., \(z_i \in \{a + b\sqrt{-1} | a, b \in \mathbb{Z}\}\) for \(1 \leq i \leq 3\), and \(\Gamma\) acts on \(G\) by left multiplications. Clearly, the global holomorphic 1-forms
\[
\varphi_1 = dz_1, \quad \varphi_2 = dz_2, \quad \varphi_3 = dz_3 - z_1 dz_2
\]
on \(G\) are invariant under the action of \(\Gamma\), hence descend down to \(G/\Gamma\). Observe that \(G/\Gamma\) does not admit any Kähler metric, because \(d\varphi_3 = \varphi_2 \wedge \varphi_1 \neq 0\). Let
\[
\omega = (\sqrt{-1}/2)(\varphi_1 \wedge \bar{\varphi}_1 + \varphi_2 \wedge \bar{\varphi}_2 + \varphi_3 \wedge \bar{\varphi}_3).
\]
Then, \(\omega\) is a balanced hermitian metric on \(G/\Gamma\), for \(d\omega^2 = 0\). Furthermore, we have
\[
(\sqrt{-1}/2)\partial\bar{\partial}\omega \wedge \omega = (\sqrt{-1}/2)^3 \varphi_1 \wedge \bar{\varphi}_1 \wedge \varphi_2 \wedge \bar{\varphi}_2 \wedge \varphi_3 \wedge \bar{\varphi}_3 > 0
\]
on \(G/\Gamma\); hence, by Proposition 12 we conclude that \(\gamma_1(\omega) > 0\).

7. The first Gauduchon metric on Calabi’s manifolds

In this section, we shall establish the existence of the 1-st Gauduchon metric on the non-Kähler manifold introduced by Calabi [9]. In view of Theorem 9 and Corollary 10 we need to find a hermitian metric with negative \(\gamma_1\) value.

We first recall Calabi’s construction of non-Kähler complex three dimensional manifolds. Let \(\mathbb{O} \cong \mathbb{R}^8\) denotes the Cayley numbers. We fix a basis \(\{I_1, \cdots, I_7\}\) such that
(1) $I_i \cdot I_j = \delta_{ij}$ with respect to the inner product.

(2) The table of the multiplication of the cross product $I_j \times I_k$ is the following

\[
\begin{array}{cccccccc}
\times & I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 \\
I_1 & 0 & I_3 & -I_2 & I_5 & -I_4 & I_7 & -I_6 \\
I_2 & -I_3 & 0 & I_1 & I_6 & -I_7 & -I_4 & I_5 \\
I_3 & I_2 & -I_1 & 0 & -I_7 & -I_6 & I_5 & I_4 \\
I_4 & -I_5 & -I_6 & I_7 & 0 & I_1 & I_2 & -I_3 \\
I_5 & I_4 & I_7 & I_6 & -I_1 & 0 & -I_3 & -I_2 \\
I_6 & -I_7 & I_4 & -I_5 & -I_2 & I_3 & 0 & I_1 \\
I_7 & I_6 & -I_5 & -I_4 & I_3 & I_2 & -I_1 & 0 \\
\end{array}
\]

Via this basis, we have the isomorphism $\mathbb{R}^7 \cong \text{Im}(\mathcal{O})$.

Calabi considered a smooth oriented hypersurface $X^6 \rightarrow \mathbb{R}^7$. Fix a unit normal vector field $N$ of $X$. There is a natural almost complex structure $J : TX \rightarrow TX$ induced by Cayley multiplication as follows. For any $x \in X$ and any $V \in T_xX$, define $J : T_xX \rightarrow T_xX$ as

\[ J(V) = N \times V. \]

Calabi proved that $J$ is integrable if and only if $J$ anticommutes with the second fundamental form of $X$.

Calabi constructed compact complex manifolds as follows. Let $\Sigma$ be a compact Riemann surface which admits 3 holomorphic differentials $\phi_1, \phi_2, \phi_3$ with the following properties:

1. linear independent;
2. $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$;
3. $\phi_1 \wedge \phi_1 + \phi_2 \wedge \phi_2 + \phi_3 \wedge \phi_3 > 0$.

Lifting $\phi_1, \phi_2, \phi_3$ to the universal covering $\widetilde{\Sigma} \rightarrow \Sigma$ and setting

\[ x^j(p) = \text{Re} \int_{p'}^p \phi_j, \quad j = 1, 2, 3 \]

for a fixed point $p' \in \Sigma$, we obtain a conformal minimal immersion

\[ \psi = (x^1, x^2, x^3) : \widetilde{\Sigma} \rightarrow \mathbb{R}^3. \]

This mapping is regular, since the differentials $\phi_j$ satisfy (3); by the weierstrass representation, property (2) is equivalent to the statement that $\psi$ is minimal; finally, because of property (1), it follows that $\widetilde{\Sigma}$ is not mapped into a plane.

Calabi then considered the hypersurface of the type

\[ (\psi, id) : \Sigma \times \mathbb{R}^4 \rightarrow \mathbb{R}^3 \times \mathbb{R}^4 = \text{Im}(\mathcal{O}), \]

where $\mathbb{R}^3 = \text{span}_\mathbb{R}\{I_1, I_2, I_3\}$ and $\mathbb{R}^4 = \text{span}_\mathbb{R}\{I_4, I_5, I_6, I_7\}$. Since $\psi : \widetilde{\Sigma} \rightarrow \mathbb{R}^3$ is minimal, $\Sigma \times \mathbb{R}^4$ is the complex manifold. If $g : \Sigma \rightarrow \Sigma$ denotes a covering transformation, then $\psi(gp) = \psi(p) + tg$ for some vector $tg \in \mathbb{R}^3$. It follows that the complex structure on $\Sigma \times \mathbb{R}^4$ is invariant by the covering group of $\Sigma$ and so descends to $\Sigma \times \mathbb{R}^4$. On the other hand, for $\mathbb{R}^4$, we can further divide by a lattice $\Lambda$ of translation of $\mathbb{R}^4$, and thereby produce a compact complex manifold $X_\Lambda = \Sigma \times T^4$. We can view $X_\Lambda$ as a family of complex tori, parameterized by the Riemann surface.
Calabi showed that such complex manifolds $X_\Lambda$ are non-Kähler. However, there exists a balanced metric on these manifolds \[14, 19\]. Let us consider the nature metric.

Define a 2-form on $X_\Lambda$ as
\[
\omega_0(V, W) = N \cdot (V \times W)
\]
for any $V, W \in T_xX_\Lambda$ at any $x \in X_\Lambda$. Then clearly we have
\[
\omega_0(V, W) = -\omega_0(W, V);
\]
and using the formula
\[
N \cdot (V \times W) = (N \times V) \cdot W,
\]
we also have
\[
\omega_0(JV, JW) = \omega_0(V, W);
\]
\[
\omega_0(V, JV) = (N \times V) \cdot (N \times V) > 0, \quad \text{if} \ V \neq 0.
\]
So $\omega_0$ is the positive $(1, 1)$-form on $X_\Lambda$ and therefore defines a hermitian metric.

Next we check that $\omega_0$ is a balanced metric. The unit normal vector field of $X$ in $\mathbb{R}^7$ can be written as
\[
N = \sum_{j=1}^{3} a_j I_j, \quad \sum_{j=1}^{3} a_j^2 = 1,
\]
where $a_j$ for $j = 1, 2, 3$ are functions on $\Sigma$. Let $(x_4, x_5, x_6, x_7)$ be the coordinates of $\mathbb{R}^4$. Then we can write the hermitian metric $\omega_0$ as
\[
\omega_0 = \omega_\Sigma + \varphi_0,
\]
where $\omega_\Sigma$ is a Kähler metric on $\Sigma$ and
\[
\varphi_0 = a_1 dx_4 \wedge dx_5 + a_2 dx_4 \wedge dx_6 - a_3 dx_4 \wedge dx_7 - a_3 dx_5 \wedge dx_6 - a_2 dx_5 \wedge dx_7 + a_1 dx_6 \wedge dx_7.
\]
By direct check, we have
\[
\varphi_0^2 = 2dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7.
\]
Therefore,
\[
d(\omega_0^2) = d(2\omega_\Sigma \wedge \varphi_0 + \varphi_0^2) = 2d\omega_\Sigma \wedge \varphi_0 + 2\omega_\Sigma \wedge d\varphi_0 = 0,
\]
since $\omega_\Sigma$ is a Kähler metric and all functions $a_j$ are defined on $\Sigma$.

At last we prove that there exists a 1-Gauduchon metric on $X_\Lambda$. By direct computation, we have
\[
\partial \bar{\partial} \omega_0 \wedge \omega_0 = \partial \bar{\partial} \varphi_0 \wedge \varphi_0 = 2 \sum_{j=1}^{3} a_j \partial \bar{\partial} a_j \wedge dx_4 \wedge dx_5 \wedge dx_6 \wedge dx_7.
\]
Condition \[7.2\] implies
\[
\sum_{j=1}^{3} a_j \partial \bar{\partial} a_j = - \sum_{j=1}^{3} \partial a_j \wedge \bar{\partial} a_j,
\]
Combining the above two equalities yields
\[
\sqrt{-1} \partial \bar{\partial} \omega_0 \land \omega_0 = -2\sqrt{-1} \sum_{j=1}^{3} \partial a_j \land \bar{\partial} a_j \land dx_4 \land dx_5 \land dx_6 \land dx_7 \\
= -4 \sum_{j=1}^{3} |\partial a_j|^2 \omega_0^3,
\]
and therefore,
\[
\sqrt{-1} \int_{X_{\Lambda}} \partial \bar{\partial} (e^v \omega_0) \land \omega_0 = \sqrt{-1} \int_{X_{\Lambda}} e^v \omega_0 \land \partial \bar{\partial} \omega_0 < 0.
\]
Hence, we have \( \gamma_1(\omega_0) < 0 \), by Corollary 4; so \(-1 \in \Xi_1(X_{\Lambda})\).

**Proposition 19.** \( \Xi_1(X_{\Lambda}) = \{-1, 0, 1\} \).

*Proof.* We have proven \(-1 \in \Xi_1(X_{\Lambda})\) and According to Theorem 6 we also have \(1 \in \Xi_1(X_{\Lambda})\). Then by Corollary 10, \(0 \in \Xi_1(X_{\Lambda})\). \qed

**Corollary 20.** There exists a 1-Gauduchon metric on \( S^5 \times S^1 \).

8. THE FIRST GAUDUCHON METRIC ON \( S^5 \times S^1 \)

Let \( S^5 \to \mathbb{P}^2 \) be the hopf fibration of the complex projective plane \( \mathbb{P}^2 \). \( S^5 \) can be viewed as the circle bundle over \( \mathbb{P}^2 \) twisted by \( \omega_{FS} \in H^2(\mathbb{P}^2, \mathbb{Z}) \). Here \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{P}^2 \). We let \( \pi : S^5 \times S^1 \to \mathbb{P}^2 \) be the natural projection. Then using a canonical way (c.f. [10, 12]), we can define a complex structure on \( S^5 \times S^1 \) such that \( \pi \) is a holomorphic map. We can define a natural hermitian metric on \( S^5 \times S^1 \) as follows:

\[
(8.1) \quad \omega_0 = \pi^* \omega_{FS} + \frac{\sqrt{-1}}{2} \theta \land \bar{\theta},
\]
where \( \theta = \theta_1 + \sqrt{-1} \theta_2 \) is a \((1,0)\)-form on \( S^5 \times S^1 \) such that \( d\theta_1 = \pi^* \omega_{FS} \) and \( d\theta_2 = 0 \). So \( \partial \bar{\theta} = \pi^* \omega_{FS} \) and \( \partial \theta = 0 \) which imply

\[
(8.2) \quad \frac{\sqrt{-1}}{2} \partial \bar{\partial} \omega_0 = -\frac{1}{4} \pi^* \omega_{FS}^2.
\]
Thus

\[
(8.3) \quad \frac{\sqrt{-1}}{2} \partial \bar{\partial} \omega_0 \land \omega_0 = \left( \frac{\sqrt{-1}}{2} \right)^3 \pi^* \omega_{FS}^2 \land \theta \land \bar{\theta} = -\frac{\omega_0^3}{3!},
\]
and therefore
\[
\sqrt{-1} \int_{S^5 \times S^1} \partial \bar{\partial} (e^v \omega_0) \land \omega_0 = \sqrt{-1} \int_{S^5 \times S^1} e^v \omega_0 \land \partial \bar{\partial} \omega_0 < 0.
\]
Hence, we have \( \gamma_1(\omega_0) < 0 \), by Corollary 4; so \(-1 \in \Xi_1(S^5 \times S^1)\). Then by Corollary 10, \(0 \in \Xi_1(S^5 \times S^1)\). That is we have

**Proposition 21.** There exists a 1-Gauduchon metric on \( S^5 \times S^1 \).

Using above natural metric \( \omega_0 \) on \( S^5 \times S^1 \), we can also prove
Proposition 22. There does not exist any pluri-closed metric on $S^5 \times S^1$.

Proof. If there would exist a pluri-closed metric $\omega$ on $S^5 \times S^1$, then

$$0 = \int_{S^5 \times S^1} \frac{\sqrt{-1}}{2} \partial \bar{\partial} \omega \wedge \omega_0 = -\frac{1}{4} \int_{S^5 \times S^1} \omega \wedge \pi^* \omega_{FS}^2 < 0$$

since $\omega \wedge \pi^* \omega_{FS}^2$ is the strictly positive definite $(3,3)$-form on $S^5 \times S^1$. That is a contradiction. $\square$

We also know that there does not exist any balanced metric on $S^5 \times S^1$. The proof is standard and is given here. There is an obstruction to the existence of a balanced metric on a compact complex manifold. Namely, on a compact complex manifold with a balanced metric no compact complex submanifold of codimension 1 can be homologous to 0 $\square$. Now for $\pi: S^5 \times S^1 \to \mathbb{P}^2$, since $\pi$ is a holomorphic, $\pi^{-1}(\mathbb{P}^2)$ is a complex hypersurface in $S^5 \times S^1$. Certainly $\pi^{-1}(\mathbb{P}^2)$ is homologous to zero in $S^5 \times S^1$ since $H^4(S^5 \times S^1, \mathbb{R}) = 0$. Therefor there exist no balanced metric on $S^5 \times S^1$.

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