A Riesz-Thorin type interpolation theorem
in Euclidean Jordan algebras

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Abstract

In a Euclidean Jordan algebra \( \mathcal{V} \) of rank \( n \) which carries the trace inner product, to each element \( a \) we associate the eigenvalue vector \( \lambda(a) \) in \( \mathbb{R}^n \) whose components are the eigenvalues of \( a \) written in the decreasing order. For any \( p \in [1, \infty] \), we define the spectral \( p \)-norm of \( a \) to be the \( p \)-norm of \( \lambda(a) \) in \( \mathbb{R}^n \). In a recent paper, based on the \( K \)-method of real interpolation theory and a majorization technique, we described an interpolation theorem for a linear transformation on \( \mathcal{V} \) relative to the same spectral \( p \)-norm. In this paper, using complex function theory methods, we describe a Riesz-Thorin type interpolation theorem relative to two spectral \( p \)-norms. We illustrate the result by estimating the norms of certain special linear transformations such as Lyapunov transformations, quadratic representations, and positive transformations.

Key Words: Euclidean Jordan algebra, Riesz-Thorin type interpolation theorem

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1 Introduction

Consider a Euclidean Jordan algebra $V$ of rank $n$ which carries the trace inner product. To each element $a$ in $V$, we associate the eigenvalue vector $\lambda(a)$ whose components are the eigenvalues of $a$ written in the decreasing order. For any $p \in [1, \infty]$, we define the spectral $p$-norm on $V$ by

$$||a||_p := ||\lambda(a)||_p,$$

where the right-hand side is the usual $p$-norm of the vector $\lambda(a)$ in $\mathbb{R}^n$. This is a special case of a spectral function on $V$ which arises as the composition of a permutation invariant function on $\mathbb{R}^n$ and $\lambda$ [2, 9]. The spectral $p$-norm is analogous to the Schatten $p$-norm of a complex square matrix (which is the $p$-norm of the vector of its singular values); In fact, the two norms coincide in the setting of the algebras of $n \times n$ real/complex Hermitian matrices. While there is a large body of literature on the singular values and Schatten norms [3], the non-associative nature of the Jordan product in a Jordan algebra prevents one from routinely stating and proving results for spectral $p$-norms. Yet, using novel techniques, several authors have studied spectral $p$-norms in general Euclidean Jordan algebras. In an early paper on interior point algorithms over a symmetric cone (which is the cone of squares in a Euclidean Jordan algebra), Schmieta and Alizadeh [12] describe some properties of the (spectral) 2-norm (also called the Frobenius norm) and the $\infty$-norm. Using majorization ideas, Tao et al. [13] show that $|| \cdot ||_p$ is a norm on $V$; see [2, 9] for related results. In a recent paper [6], Gowda proves the Hölder type inequality $||x \circ y||_1 \leq ||x||_p ||y||_q$ (where $q$ is the conjugate of $p$) and provides several references dealing with spectral $p$-norms.

The present paper deals with an interpolation theorem for a linear transformation on $V$ relative to spectral $p$-norms. Given $r, s \in [1, \infty]$ and a linear transformation $T : V \to V$, we let

$$||T||_{r \to s} := \sup_{a \neq 0} \frac{||T(a)||_s}{||a||_r}.$$

In [6], based on the $K$-method of real interpolation theory [10], the following result was proved.

**Theorem 1.1** Suppose $1 \leq r, s, p \leq \infty$, $0 \leq \theta \leq 1$, and

$$\frac{1}{p} = \frac{1 - \theta}{r} + \frac{\theta}{s}. \tag{1}$$

Then, for any linear transformation $T : V \to V$,

$$||T||_{p \to p} \leq ||T||_{r \to r}^{1 - \theta} ||T||_{s \to s}^\theta. \tag{2}$$

In particular,

$$||T||_{p \to p} \leq ||T||_{\infty \to \infty}^{1 - \frac{1}{p}} ||T||_1^{\frac{1}{p}}. \tag{3}$$

A key idea in the proof of the above result is the use of a majorization result that connects a $K$-functional defined on $V$ with a $K$-functional on an $L_p$-space. In [6], the issue of proving an
inequality of the type (2) that deals with the norm of $T$ relative to two spectral norms (such as $||T||_{r \rightarrow s}$) was raised. In the present paper, based on standard complex function theory methods (especially, Hadamard’s three lines theorem) we prove the following Riesz-Thorin type interpolation result.

**Theorem 1.2** Let $r_0, r_1, s_0, s_1 \in [1, \infty]$ and $\theta \in [0, 1]$. Consider $r_\theta$ and $s_\theta$ in $[1, \infty]$ defined by

$$\frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \quad \text{and} \quad \frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}.$$  

Then, for any linear transformation $T$ on $\mathcal{V}$,

$$||T||_{r_\theta \rightarrow s_\theta} \leq C ||T||_{r_0 \rightarrow s_0}^{1-\theta} ||T||_{r_1 \rightarrow s_1}^\theta,$$

(4)

where $C$ is a constant, $1 \leq C \leq 4$, that depends only on $r_0, r_1, s_0, s_1$.

Illustrating this result, we estimate the norms of some special linear transformations on $\mathcal{V}$ such as Lyapunov transformations, quadratic representations, and positive transformations.

## 2 Preliminaries

Throughout this paper $(\mathcal{V}, \circ, \langle \cdot, \cdot \rangle)$ denotes a Euclidean Jordan algebra of rank $n$ with unit element $e$ [4], [8]. We let letters $a, b, c, d$, and $v$ denote elements of $\mathcal{V}$, $x$ and $y$ denote elements of $\mathbb{R}^n$, and write $z$ for a complex variable. For $a, b \in \mathcal{V}$, we denote their Jordan product and inner product by $a \circ b$ and $\langle a, b \rangle$, respectively. It is known that any Euclidean Jordan algebra is a direct product/sum of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to one of five algebras, three of which are the algebras of $n \times n$ real/complex/quaternion Hermitian matrices. The other two are: the algebra of $3 \times 3$ octonion Hermitian matrices and the Jordan spin algebra.

According to the spectral decomposition theorem [4], any element $a \in \mathcal{V}$ has a decomposition

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n,$$

where the real numbers $a_1, a_2, \ldots, a_n$ are (called) the eigenvalues of $a$ and $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame in $\mathcal{V}$. (An element may have decompositions coming from different Jordan frames, but the eigenvalues remain the same.) Then, $\lambda(a)$, called the eigenvalue vector of $a$, is the vector of eigenvalues of $a$ written in the decreasing order. The trace and spectral $p$-norm of $a$ are defined by

$$\text{tr}(a) := a_1 + a_2 + \cdots + a_n \quad \text{and} \quad ||a||_p := ||\lambda(a)||_p,$$

where $||\lambda(a)||_p$ denotes the usual $p$-norm of a vector in $\mathbb{R}^n$. An element $a$ is said to be invertible if all its eigenvalues are nonzero. We note that the set of invertible elements is dense in $\mathcal{V}$. Throughout
In this paper, we assume that the inner product is the trace inner product, that is, \( \langle a, b \rangle = \text{tr}(a \circ b) \).

Given a spectral decomposition \( a = \sum_{j=1}^{n} a_j e_j \) and a real number \( \gamma > 0 \), we write
\[
|a| := \sum_{j=1}^{n} |a_j| e_j, \quad |a|^\gamma := \sum_{j=1}^{n} |a_j|^\gamma e_j \quad \text{and} \quad ||a||_1 = \sum_{j=1}^{n} |a_j| = \text{tr}(|a|).
\] (5)

In what follows, we say that \( q \) is the conjugate of \( p \in [1, \infty] \) if \( \frac{1}{p} + \frac{1}{q} = 1 \) and denote the conjugate of \( r \in [1, \infty] \) by \( r' \). Also, we use the standard convention that \( 1/\infty = 0 \).

Based on the Fan-Theobald-von Neumann type inequality [2]
\[ \langle a, b \rangle \leq \langle \lambda(a), \lambda(b) \rangle \quad (a, b \in V) \]
and majorization techniques, the following result was proved in [6].

**Theorem 2.1** Let \( p \in [1, \infty] \) with conjugate \( q \). Then the following statements hold in \( V \):

(i) \(|\langle a, b \rangle| \leq ||a \circ b||_1 \leq ||a||_p ||b||_q\).

(ii) \( \sup_{b \neq 0} \frac{|\langle a, b \rangle|}{||b||_q} = ||a||_p \).

### 3 The proof of the interpolation theorem

The Riesz-Thorin interpolation theorem, stated in the setting of \( L_p \)-spaces, is well-known in classical analysis. There is also a Riesz-Thorin type result available for linear transformations on the space of complex \( n \times n \) matrices with respect to Schatten \( p \)-norms, see the interpolation theorem of Calderón-Lions ([11], Theorem IX.20). Our Theorem 1.2 is stated in the setting of Euclidean Jordan algebras relative to spectral \( p \)-norms. In the absence of an isomorphism type argument that immediately gives our result, we offer a proof that mimics the classical proof based on the Hadamard's three lines theorem of complex function theory ([5], Theorem 6.27). In the proof given below, we complexify the real inner product space \( V \) and define norms on this complexification in such a way to have a Hölder type inequality. This procedure results in a constant \( C \) in the Riesz-Thorin type inequality (4) that is different from 1. Possibly, a different argument may show that this constant can be replaced by 1.

Recall that \( a \) and \( b \) denote elements of \( V \) and \( z \) denotes a complex variable. For \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) in \( \mathbb{R}^n \), we write \( x + iy = (x_1 + iy_1, x_2 + iy_2, \ldots, x_n + iy_n) \in \mathbb{C}^n \). Let \( T \) be a linear transformation on \( V \). We consider complexifications of \( V \) and \( T \):
\[
\tilde{V} := V + iV \quad \text{and} \quad \tilde{T}(a + ib) := T(a) + iT(b) \quad (a, b \in V).
\]
We define the inner product and spectral p-norm on $\tilde{V}$ as follows. For $a, b, c, d \in V$,
\[
\langle a + ib, c + id \rangle := \left[\langle a, c \rangle + \langle b, d \rangle \right] + i \left[\langle b, c \rangle - \langle a, d \rangle \right] \quad \text{and} \quad ||a + ib||_p := ||a||_p + ||b||_p.
\]
It is easily seen that $\tilde{V}$ is a complex inner product space, $\tilde{T}$ is a (complex) linear transformation on $\tilde{V}$. We state the following simple lemma.

**Lemma 3.1** Consider $\tilde{V}$ and $\tilde{T}$ as above. Let $p \in [1, \infty]$ with conjugate $q$, and $r, s \in [1, \infty]$. Then,

(i) $|\langle a + ib, c + id \rangle| \leq ||a + ib||_p ||c + id||_q$ for all $a, b, c, d \in V$, and

(ii) $||\tilde{T}||_{r \to s} = ||T||_{r \to s}$.

**Proof.** (i) By the definition of inner product in $\tilde{V}$ and Theorem 2.1,

\[
|\langle a + ib, c + id \rangle| \leq ||a + c||_p + ||b, d||_q \leq ||a||_p ||c||_q + ||a||_p ||b||_p ||d||_q.
\]

Since the right-hand side is $||a + ib||_p ||c + id||_q$, the stated inequality follows.

(ii) For $a, b \in V$,

\[
||\tilde{T}(a + ib)||_s = ||T(a) + iT(b)||_s = ||T(a)||_s + ||T(b)||_s \leq ||T||_{r \to s}(||a||_r + ||b||_r) = ||T||_{r \to s} ||a + ib||_r.
\]

This implies that $||\tilde{T}||_{r \to s} \leq ||T||_{r \to s}$. The reverse inequality holds as $\tilde{T}$ is an extension of $T$ to $\tilde{V}$. Hence we have (ii).

We now come to the proof of Theorem 1.2. In what follows, for any $p \in [1, \infty]$ with conjugate $q$, we let

\[
C_p = \begin{cases} \sqrt{2} & \text{if } 1 \leq p \leq 2, \\ 2^{\frac{1}{q}} & \text{if } 2 \leq p \leq \infty. \end{cases}
\]

**Proof.** Let the assumptions of the theorem be in place. Recalling that $s'$ denotes the conjugate of (any) $s \in [1, \infty]$, we define

\[
C := \max\{C_{r_0}, C_{r_1}, C_{s_0}, C_{s_1}\}
\]

which is a number between 1 and 4, and depends only on $r_0, r_1, s_0, s_1$. We show that (4) holds for this $C$. Since (4) clearly holds when $\theta = 0$ or $\theta = 1$, from now on, we assume that $0 < \theta < 1$.

Let

\[
\alpha_j := \frac{1}{r_j}, \quad \beta_j := \frac{1}{s_j}, \quad \text{and} \quad M_j := ||T||_{r_j \to s_j} \quad (j = 0, 1),
\]

\[
\alpha := \frac{1}{r_{\theta}}, \quad \beta := \frac{1}{s_{\theta}}, \quad \text{and} \quad M_{\theta} := ||T||_{r_{\theta} \to s_{\theta}},
\]

and for a complex variable $z$,

\[
\alpha(z) := (1 - z)\alpha_0 + z\alpha_1 \quad \text{and} \quad \beta(z) := (1 - z)\beta_0 + z\beta_1.
\]
We show that
\[ M_\theta \leq C M_1^{1-\theta} M_1^\theta. \] (7)

Now, using Theorem 2.1, Item (ii),
\[ M_\theta = ||T||_{r_\theta \to s_\theta} = \sup_{0 \neq a \in \mathcal{V}} \frac{||T(a)||_{s_\theta}}{||a||_{r_\theta}} = \sup_{0 \neq a,b \in \mathcal{V}} \frac{||\langle T(a), b \rangle||_{s_\theta}}{||a||_{r_\theta}||b||_{s_\theta}} = \sup_{||a||_{r_\theta} = 1} ||\langle T(a), b \rangle||_{s_\theta}. \]

To prove (7), it is enough to show that for any \( a \) and \( b \) in \( \mathcal{V} \) with \( ||a||_{r_\theta} = 1 = ||b||_{s_\theta} \),
\[ ||\langle T(a), b \rangle||_{s_\theta} \leq C M_1^{1-\theta} M_1^\theta. \] (8)

By continuity, it is enough to prove this for \( a \) and \( b \) invertible (that is, with all their eigenvalues nonzero). We fix such \( a \) and \( b \) and write their spectral decompositions:
\[ a = \sum_{j=1}^n a_j |e_j \rangle \langle e_j | \quad \text{and} \quad \quad b = \sum_{j=1}^n |b_j \rangle \langle \delta_j | f_j |, \]
where \( \{e_1, e_2, \ldots, e_n\} \) and \( \{f_1, f_2, \ldots, f_n\} \) are Jordan frames, \( \varepsilon_j, \delta_j \in \{-1,1\} \) for all \( j \), and \( a_j \varepsilon_j \) and \( b_j \delta_j \) are the eigenvalues of \( a \), etc. Now, with the observation that \( 0 < \alpha, \beta < 1 \), we define two elements in \( \tilde{\mathcal{V}} \):
\[ a_z := \sum_{j=1}^n a_j |\alpha_j \rangle \langle \varepsilon_j | e_j | \quad \text{and} \quad \quad b_z := \sum_{j=1}^n |b_j \rangle \langle \beta_j | \delta_j | f_j |, \]
where we consider only the principal values while defining the exponentials. Then the function
\[ \phi(z) := \langle \tilde{T}(a_z), b_z \rangle \]
is continuous on the strip \( \{ z : 0 \leq Re(z) \leq 1 \} \) and analytic in its interior. We estimate \( |\phi(z)| \) on the lines \( Re(z) = 0 \) and \( Re(z) = 1 \) and then apply Hadamard’s three lines theorem ([5], Theorem 6.27). First, suppose \( Re(z) = 0 \). Let
\[ |a_j|^{\alpha(z)} = x_j + i y_j, \quad x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad \text{and} \quad y := (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n. \]

Then, \( |x + iy||_{r_0} = 1 \). When \( r_0 = \infty \), that is, when \( \alpha = 0 \), \( |x + iy||_{r_0} = 1 \) for all \( j \) and hence (in \( \mathbb{C}^n \)), \( |x + iy||_{r_0} = 1 \). When \( r_0 < \infty \), \( |x + iy||_{r_0} = |a_j||r_0| \). So, because \( ||a||_{r_0} = 1 \), we have \( ||x + iy||_{r_0} = \sum_{j=1}^n |x_j + iy_j||_{r_0} = \sum_{j=1}^n |a_j||r_0| = 1 \). Thus, in both cases,
\[ ||x + iy||_{r_0} = 1. \] (9)

Now, \( a_z = \sum_{j=1}^n (x_j + iy_j) e_j e_j = (\sum_{j=1}^n x_j e_j e_j) + i(\sum_{j=1}^n y_j e_j e_j) \) and so,
\[ ||a_z||_{r_0} = || \sum_{j=1}^n x_j e_j e_j ||_{r_0} + || \sum_{j=1}^n y_j e_j e_j ||_{r_0} = ||x||_{r_0} + ||y||_{r_0}. \]
In view of (9), from Proposition 4.1 in the Appendix, we have,
\[ ||a_z||_{r_0} \leq C_{r_0}. \]
Similarly, \[ ||b_z||_{s'_0} \leq C_{s'_0}. \] Hence, when \( Re(z) = 0 \), Lemma 3.1 gives
\[ |\phi(z)| \leq ||\tilde{T}(a_z)||_{s_0} ||b_z||_{s'_0} \leq ||\tilde{T}||_{r_0 \rightarrow s_0} ||a_z||_{r_0} ||b_z||_{s'_0} \leq ||T||_{r_0 \rightarrow s_0} C_{r_0} C_{s'_0} = C_{r_0} C_{s'_0} M_0. \]
A similar computation shows that
\[ Re(z) = 1 \Rightarrow |\phi(z)| \leq C_{r_1} C_{s'_1} M_1. \]
By Hadamard’s three lines theorem,
\[ |\phi(\theta)| \leq \left( C_{r_0} C_{s'_0} M_0 \right)^{1-\theta} \left( C_{r_1} C_{s'_1} M_1 \right)^{\theta}. \]
We recall that \( C = \max\{C_{r_0} C_{s'_0}, C_{r_1} C_{s'_1}\} \). Now, \( a_\theta = a \) and \( b_\theta = b \), and so, \( \phi(\theta) = \langle T(a), b \rangle \). Hence,
\[ |\langle T(a), b \rangle| \leq C M_0^{1-\theta} M_1^{\theta}. \]
This gives (8) and the proof is complete.

Remarks. Instead of the constant \( C \) defined in (6), one may consider a slightly better constant, namely, \( \max\{(C_{r_0} C_{s'_0})^{1-\theta}, (C_{r_1} C_{s'_1})^{\theta}\} \). However, this constant depends on \( \theta \).

We now consider the problem of estimating the norms of certain special linear transformations on \( \mathcal{V} \) relative to spectral \( p \)-norms. First, we make two observations. Writing \( T^* \) for the adjoint of a linear transformation \( T \) on \( \mathcal{V} \), we note, thanks to Theorem 2.1, that
\[ ||T^*||_{r \rightarrow s} = ||T||_{s' \rightarrow r'}, \]
where \( r' \) denotes the conjugate of \( r \), etc. Also, knowing the norms \( ||T||_{1 \rightarrow 1}, ||T||_{\infty \rightarrow \infty}, ||T||_{1 \rightarrow p}, \) and \( ||T||_{p \rightarrow 1} \), etc., one can estimate \( ||T||_{r \rightarrow s} \) for various \( r \) and \( s \). When \( r = s \), (3) gives such an estimate. In the result below, we consider the case \( r \neq s \).

**Corollary 3.2** Let \( 1 \leq r \neq s \leq \infty \). Then, for any linear transformation \( T : \mathcal{V} \rightarrow \mathcal{V} \),
\[ ||T||_{r \rightarrow s} \leq \begin{cases} 2\sqrt{2} ||T||_{\infty \rightarrow \infty} \left( ||T||_{1 \rightarrow \infty} \right)^{\frac{1}{2}} & \text{if } r < s, \\ 2\sqrt{2} ||T||_{\infty \rightarrow \infty} \left( ||T||_{1 \rightarrow 1} \right)^{\frac{1}{2}} & \text{if } r > s. \end{cases} \]

**Proof.** The stated inequalities are obtained by specializing Theorem 1.2. When \( r < s \), we let
\[ r_0 = \infty, s_0 = \infty, r_1 = 1, s_1 = \frac{s}{r}, r_\theta = r, s_\theta = s, \text{ and } \theta = \frac{1}{r}. \]
In this case, \( C = \max\{C_{r_0} C_{s'_0}, C_{r_1} C_{s'_1}\} = 2\sqrt{2} \). When \( r > s \), we let
\[ r_0 = \infty, s_0 = \infty, r_1 = \frac{r}{s}, s_1 = 1, r_\theta = r, s_\theta = s, \text{ and } \theta = \frac{1}{s}. \]
In this case also, \( C = 2\sqrt{2} \).

**Remarks.** In the result above, by considering \( \max\{\langle Cr_0C_s'\rangle^{1-\theta}, \langle Cr_1C_s'\rangle^\theta\} \), one can replace the constant \( 2\sqrt{2} \) by the following:

\[
(2\sqrt{2})^{\max\{1-\frac{1}{r}, \frac{1}{s}\}} \quad \text{when } r < s \quad \text{and} \quad (2\sqrt{2})^{\max\{1-\frac{1}{r}, \frac{1}{s}\}} \quad \text{when } r > s.
\]

We now illustrate our results via some examples. For any \( a \in \mathcal{V} \), consider the Lyapunov transformation \( L_a \) and the quadratic representation \( P_a \) defined by

\[
L_a(v) := a \circ v \quad \text{and} \quad P_a(v) := 2a \circ (a \circ v) - a^2 \circ v \quad (v \in \mathcal{V}).
\]

These self-adjoint linear transformations appear prominently in the study of Euclidean Jordan algebras. The norms of these transformations relative to some spectral \( p \)-norms have been described in [6]. For \( r, s \in [1, \infty] \), we have

\[
||a||_\infty \leq ||L_a||_{r \to s} \quad \text{and} \quad ||a^2||_\infty = ||a||_\infty^2 \leq ||P_a||_{r \to s}.
\]

Additionally, for any \( p \in [1, \infty] \) with conjugate \( q \),

- \( ||L_a||_{p \to p} = ||L_a||_{p \to \infty} = ||L_a||_{1 \to q} = ||a||_\infty \) and \( ||L_a||_{p \to 1} = ||L_a||_{\infty \to q} = ||a||_q \),
- \( ||P_a||_{p \to p} = ||P_a||_{p \to \infty} = ||P_a||_{1 \to q} = ||a||_q^2 \) and \( ||P_a||_{p \to 1} = ||P_a||_{\infty \to q} = ||a^2||_q \).

We now come to the estimation of \( ||L_a||_{r \to s} \) and \( ||P_a||_{r \to s} \) for \( r \neq s \). First suppose \( 1 \leq r < s \leq \infty \). Then, using the above properties and the fact that for any \( x \in \mathbb{R}^n \), \( ||x||_p \) is a decreasing function of \( p \) over \( [1, \infty] \), we have

\[
||a||_\infty \leq ||L_a||_{r \to s} = \sup_{0 \neq v \in \mathcal{V}} \frac{||L_a(v)||_s}{||v||_r} \leq \sup_{0 \neq v \in \mathcal{V}} \frac{||L_a(v)||_r}{||v||_r} = ||L_a||_{r \to r} = ||a||_\infty.
\]

Thus,

\[
||L_a||_{r \to s} = ||a||_\infty \quad (1 \leq r < s \leq \infty).
\]

A similar argument shows that

\[
||P_a||_{r \to s} = ||a||_\infty^2 \quad (1 \leq r < s \leq \infty).
\]

When \( 1 \leq s < r \leq \infty \), Corollary 3.2 yields the following estimate:

\[
||L_a||_{r \to s} \leq 2\sqrt{2}||a||_\infty^\theta.
\]

For the same \( s \) and \( r \), we can get a different estimate

\[
||L_a||_{r \to s} \leq 2C_q||a||_p, \tag{10}
\]

where \( \frac{1}{p} = \frac{1}{s} - \frac{1}{r} \) (so that \( p = s(\frac{r}{s})' \)) and \( q \) is the conjugate of \( p \). To see this, we apply Theorem 1.2.
with
\[ r_0 = \infty, \quad s_0 = p, \quad r_1 = q, \quad s_1 = 1, \quad r_\theta = r, \quad s_\theta = s, \quad \text{and} \quad \theta = \frac{q}{r}. \]

Then,
\[
||L_a||_{r \to s} \leq C ||L_a||_{\infty \to p}^{\frac{1-\theta}{\theta}} ||L_a||_{\infty \to 1} = C ||a||_p,
\]
where \( C = \max\{C_{r_0}C_{s_0}', \ C_{r_1}C_{s_1}'\} = 2C_q \). To see an interesting consequence of (10), let \( 1 \leq r, s, p \leq \infty \) with \( r \neq s \) and \( \frac{1}{s} = \frac{1}{p} + \frac{1}{r} \). Then, using the inequality \( ||a \circ b||_s \leq ||L_a||_{r \to s} ||b||_r \), the estimate (10) leads to
\[
||a \circ b||_s \leq 2C_q ||a||_p ||b||_r \quad (a, b \in \mathcal{V}),
\]
which can be regarded as a \textit{generalized Hölder type inequality}. We remark that the special case \( s = 1 \) was already covered in Theorem 2.1 with 1 in place of \( 2C_q \). It is very likely that the inequality \( ||a \circ b||_s \leq ||a||_p ||b||_r \) holds in the general case as well.

Analogous to the above norm estimates of \( L_a \), we can estimate \( ||P_a||_{r \to s} \) when \( r > s \) (with \( p \) and \( q \) defined above):
\[
||P_a||_{r \to s} \leq 2\sqrt{2}||a^2||_{(\frac{r}{s})'} \quad \text{and} \quad ||P_a||_{r \to s} \leq 2C_q ||a^2||_p.
\]

We now consider a \textit{positive linear transformation} \( P \) on \( \mathcal{V} \), which is a linear transformation on \( \mathcal{V} \) satisfying the condition
\[
a \geq 0 \Rightarrow P(a) \geq 0,
\]
where \( a \geq 0 \) means that \( a \) belongs to the symmetric cone of \( \mathcal{V} \) (or, equivalently, it is the square of some element of \( \mathcal{V} \)). Examples of such transformations include:

- Any nonnegative matrix on the algebra \( \mathcal{R}^n \).
- Any quadratic representation \( P_a \) on \( \mathcal{V} \) [4].
- The transformation \( P_A \) defined on \( S^n \) (the algebra of \( n \times n \) real symmetric matrices) by \( P_A(X) = AXA^T \), where \( A \in \mathcal{R}^{n \times n} \).
- The transformation \( P = L^{-1} \) on \( \mathcal{V} \), where \( L : \mathcal{V} \to \mathcal{V} \) is linear, positive stable (which means that all eigenvalues of \( L \) have positive real parts) and satisfies the \( Z \)-property:
\[
a \geq 0, b \geq 0, \quad \langle a, b \rangle = 0 \Rightarrow \langle L(a), b \rangle \leq 0.
\]
In particular, on the algebra \( \mathcal{H}^n \) (of \( n \times n \) complex Hermitian matrices), \( P = L_A^{-1} \), where \( A \) is a complex \( n \times n \) positive stable matrix and \( L_A(X) := AX + XA^* \).
- Any \textit{doubly stochastic transformation} on \( \mathcal{V} \) [7]: It is a positive linear transformation \( P \) with \( P(e) = e = P^*(e) \).
For any positive linear transformation $P$ on $V$, and $p \in [1, \infty]$ with conjugate $q$, we have the following from [6]:

(i) $||P||_{\infty \to p} = ||P(e)||_p$ and $||P||_{p \to 1} = ||P^*(e)||_q$.

(ii) $||P||_{p \to \infty} \leq ||P(e)||_{\infty}$ and $||P||_{1 \to p} \leq ||P^*(e)||_{\infty}$.

(iii) $||P||_{p \to p} \leq ||P(e)||_{\infty}^{1-\frac{1}{p}} ||P^*(e)||_{\infty}^{\frac{1}{p}}$.

So, for a positive $P$, an application of Corollary 3.2 gives the following inequalities:

(i) $||P||_{r \to s} \leq 2\sqrt{2} ||P(e)||_{\infty}^{1-\frac{1}{p}} ||P^*(e)||_{\infty}^{\frac{1}{p}}$ when $r < s$.

(ii) $||P||_{r \to s} \leq 2\sqrt{2} ||P(e)||_{\infty}^{1-\frac{1}{p}} ||P^*(e)||_{\infty}^{\frac{1}{p}}$ when $r > s$.

Additionally, when $P$ is also self-adjoint and $r > s$, analogous to (10), one can get the following estimate:

$$||P||_{r \to s} \leq 2C_q ||P(e)||_p.$$ 

4 Appendix

Proposition 4.1 Given $p \in [1, \infty]$ with conjugate $q$, consider the following real valued functions defined over $\mathbb{R}^n \times \mathbb{R}^n$, $n \geq 2$:

$$f(x, y) = ||x||_p + ||y||_p \quad \text{and} \quad g(x, y) = ||x + iy||_p.$$ 

Then,

$$\max \left\{ f(x, y) : g(x, y) = 1 \right\} = C_p,$$

where

$$C_p = \begin{cases} \sqrt{2} & \text{if } 1 \leq p < 2, \\ 2^{\frac{1}{p}} & \text{if } 2 \leq p \leq \infty. \end{cases}$$

Proof. By the continuity of $f$ and $g$, and the compactness of the constraint set, the maximum in (11) is attained.

It is easy to see that the pair $(\overline{x}, \overline{y})$ with $\overline{x} = 2^{-\frac{1}{p}} (1, 0, 0, \ldots, 0)$ and $\overline{y} = 2^{-\frac{1}{p}} (0, 1, 0, \ldots, 0)$ satisfies the (constraint) equation $g(x, y) = 1$. Hence

$$C_p \geq f(\overline{x}, \overline{y}) = 2^{\frac{1}{p}}.$$ 

Consider any pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $g(x, y) = 1$. Writing $x = (x_1, x_2, \ldots, x_n)$, etc., by Hölder’s inequality, we have

$$||x||_p + ||y||_p \leq 2^\frac{1}{p} \left(||x||_p^p + ||y||_p^p\right)^\frac{1}{p} = 2^\frac{1}{p} \left(\sum_{j=1}^{n} |x_j|^p + |y_j|^p\right)^\frac{1}{p}. \tag{13}$$
We consider three cases.

Case 1: $p = \infty$. By (12), $C_\infty \geq 2^{\frac{1}{2}} = 2$ (as $q = 1$). Since $|x_j + iy_j| \leq 1$ for all $j$ (from our constraint), we get $||x||_\infty, ||y||_\infty \leq 1$; hence $C_\infty \leq 2$. We conclude that $C_\infty = 2$.

Case 2: $2 \leq p < \infty$.
In this case, we use the well-known Clarkson inequality for complex numbers $z$ and $w$ (see [1], page 163):

$$2(|z|^p + |w|^p) \leq |z + w|^p + |z - w|^p.$$ 

Then, for each $j$, with $z = x_j$ and $w = iy_j$, we have

$$2(|x_j|^p + |y_j|^p) \leq |x_j + iy_j|^p + |x_j - iy_j|^p.$$ 

Summing over $j$ and noting $|x_j + iy_j| = |x_j - iy_j|$, we get

$$\sum_{j=1}^{n} (|x_j|^p + |y_j|^p) \leq \sum_{j=1}^{n} |x_j + iy_j|^p = g(x, y)^p = 1.$$ 

It follows from (13) that $||x||_p + ||y||_p \leq 2^{\frac{1}{p}}$. As this holds for all $(x, y)$ with $g(x, y) = 1$, we have $C_p \leq 2^{\frac{1}{p}}$. From (12) we conclude that $C_p = 2^{\frac{1}{p}}$.

Case 3: $1 \leq p < 2$.
Let $\delta := n^{-\frac{1}{p}}2^{-\frac{1}{q}}$. It is easy to see that the pair $(\overline{x}, \overline{y})$ with $\overline{x} = \delta(1, 1, \ldots, 1) = \overline{y}$ satisfy the constraint equation $g(x, y) = 1$. As $f(\overline{x}, \overline{y}) = \sqrt{2}$ we have, $C_p \geq \sqrt{2}$.
Now, as $1 \leq p < 2$, we use a refined version of Clarkson inequality presented in [1], Theorem 2.3:

$$2^{p-1}(|z|^p + |w|^p) + (2 - 2^{\frac{q}{p}}) \min\{|z + w|^p, |z - w|^p\} \leq |z + w|^p + |z - w|^p.$$ 

Then, for each $j$, with $z = x_j$ and $w = iy_j$, we have

$$2^{p-1}(|x_j|^p + |y_j|^p) + (2 - 2^{\frac{q}{p}}) \min\{|x_j + iy_j|^p, |x_j - iy_j|^p\} \leq |x_j + iy_j|^p + |x_j - iy_j|^p.$$ 

Simplifying this expression and summing over $j$, we get

$$\sum_{j=1}^{n} (|x_j|^p + |y_j|^p) \leq 2^{1-\frac{q}{p}} \left( \sum_{j=1}^{n} |x_j + iy_j|^p \right) = 2^{1-\frac{q}{p}} g(x, y)^p = 2^{1-\frac{q}{p}}.$$ 

This leads, via (13), to

$$||x||_p + ||y||_p \leq 2^{\frac{1}{p}} \left( 2^{1-\frac{q}{p}} \right) = \sqrt{2}.$$ 

Now, taking the maximum of $||x||_p + ||y||_p$ over $(x, y)$, we get $C_p \leq \sqrt{2}$. Thus, when $1 \leq p < 2$,

$$C_p = \sqrt{2}.$$
This completes our proof.

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